Dynamics of the modified emden and pseudo-modified emden equations: position-dependent mass, invariance and exact solvability

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Abstract
We consider the modified Emden equation (MEE) and introduce its most general solution, using the most general solution for the simple harmonic oscillator’s linear dynamical equation (i.e. the initial conditions shall be identified by the PDM-MEE problem at hand). We use a general nonlocal point transformation and show that modified Emden dynamical equation is transformed to describe position-dependent mass (PDM) classical particles. Two PDM-MEE-type classical particles are used as illustrative examples, and their exact solutions are reported. Under specific parametric considerations, the phase-space trajectories are reported for the MEE-type and for PDM-MEE-type classical particles.

1. Introduction

A classical particle, with mass \( m_0 \) (\( m_0 = 1 \) throughout unless otherwise mentioned), moving under the influence of a conservative quartic anharmonic potential \( V(u) = \frac{1}{2} \omega^2 u^2 + \frac{1}{4} \beta u^4 \) is described by the standard Lagrangian

\[
L(u, \dot{u}, t) = \frac{1}{2} \dot{u}^2 - \left( \frac{1}{2} \omega^2 u^2 + \frac{1}{4} \beta u^4 \right),
\]

(i.e. \( L = T - V \)). Moreover, if this particle is subjected to a non-conservative dissipative Rayleigh force field \( \mathcal{R}(u, \dot{u}) = \frac{1}{2} \alpha u \dot{u} \), then the corresponding Euler–Lagrange dynamical equation reads

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}} \right) - \frac{\partial L}{\partial u} + \frac{\partial \mathcal{R}}{\partial \dot{u}} = 0 \iff \ddot{u} + \alpha u \dot{u} + \omega^2 u + \beta u^3 = 0.
\]

This equation is known as the modified Emden equation (MEE), which describes a cubic anharmonic oscillator with a linear forcing term \( \omega^2 u(t) \) and an additional position-dependent nonlinear damping force \( \alpha u(t) \dot{u}(t) \) (e.g. [1, 2]). Where \( \alpha, \omega > 0 \) and \( \alpha \) is the frictional coefficient, \( \omega = \sqrt{k/m_0} \) is the frequency of the standard simple harmonic oscillator, with the force constant \( k > 0 \), and \( \beta \) is a coupling constant. Such a nonlinear differential equation finds its applicability in different fields of study. Amongst are, the equilibrium configuration of spherical gas clouds under mutual attraction of their molecules and admit the thermodynamical laws [3], the spherically symmetric expansion or collapse of relativistically gravitating mass [4], the one-dimensional analog of the Yang-Mills boson gauge theory [5], etc. Throughout the current proposal, we shall refer to it as MEE, in short.

The MEE, nevertheless, admits an exact solution [2, 6] in the form of

\[
u(t) = \frac{C \sin(\omega t + \varphi)}{1 - \eta C \cos(\omega t + \varphi)}; \quad 0 \leq \eta < \frac{1}{C},
\]

Where \( \eta = \alpha/3 m_0 \omega \geq 0 \) is a damping related parameter, and \( 0 < C < 1/\eta \) is the amplitude. Then, the corresponding canonical momentum is simply given by
When equation (3) is substituted in equation (2), it implies that $\alpha = 3 \omega \eta, \beta^2 = \omega^2 \eta^2 \rightarrow \beta = \omega \eta$. As such, equation (2) can be rewritten as

$$\ddot{u}(t) + 3 \omega \eta u(t) \dot{u}(t) + \omega^2 u(t) + \omega^2 \eta^2 u(t)^3 = 0, \quad (5)$$

Obviously, the parametric structure of each term identifies the dynamical correlation between the forces involved in the problem at hand. Yet, the solution in equation (3) is an immediate consequence of the linearization of equation (2) into a linear harmonic oscillator

$$\ddot{U} + \omega^2 U = 0; \quad U(t) = A \sin(\omega t + \varphi), \quad (6)$$

using a nonlocal transformation of the form

$$U(t) = u(t) \exp \left( \beta \int u(t) \, dt \right) \quad (7)$$

(for more details on this issue the reader may refer to Chandrasekar and coworkers [1, 2, 6]). It is clear that, the solution of the typical harmonic oscillator $U(t) = A \sin(\omega t + \varphi)$ belongs to the initial conditions $U(0) = 0$ and $\dot{U}(0) \not= 0$ at $\varphi = 0$. Moreover, the dynamical equation (5) is a second-order autonomous differential equation and is known to exhibit certain unusual nonlinear properties (e.g. [11]). The frequency of oscillation, for example, is an amplitude-independent one (i.e. isochronic oscillations similar to the linear harmonic oscillator) documenting in effect that an amplitude-dependent frequency is not always a consequential property of the nonlinear dynamical systems.

On the other hand, it has been asserted that a point mass moving within the curved space transforms into an effective position-dependent mass (PDM) in Euclidean space [7]. Exploring/studying the effect of such coordinate transformation (that manifestly introduces PDM metaphoric concept) on some dynamical systems deserves some attention, and is the main objective of our methodical proposal, therefore. This PDM concept has, in fact, inspired research activities on PDM (both in classical and quantum mechanical models) ever since the introduction of the prominent Mathews-Lakshmanan oscillator back in 1974 [8]. For example, a point particle moving within a specific set of coordinates may very well be transformed into an effectively PDM particle in the new deformed coordinates (c.f., e.g. the sample of references [7, 9–14]). Such research activities/studies have used both non-standard Lagrangians/Hamiltonians (c.f., e.g. [8, 10, 15–27] and standard Lagrangians/Hamiltonians. Whilst the intimate long-standing gain-loss correlation between the kinetic and potential energies is ignored in the non-standard Lagrangians/Hamiltonians, it has been kept intact in the standard Lagrangians/Hamiltonians (i.e. $L = T - V$ and $H = T + V$), so that the total energy of the dynamical system is an integral of motion [14, 26, 27].

In classical mechanics, a PDM-particle with $M(x) = m_0 \, G(x)$ (where $G(x)$ is a positive valued dimensionless scalar multiplier) is shown to have a canonical momentum $p(x) = M(x) \dot{x} = m_0 \, G(x) \dot{x}$ for a PDM-Lagrangian $L = M(x) \dot{x}^2 / 2 - V(x)$. In quantum mechanics, nevertheless, the focus was on the prominent van Roos PDM-Hamiltonian [28–40], where the PDM-operator [12] is shown to be given (in $\hbar = 2m_0 = 1$ units) by

$$\hat{p}(x) = -i \left( \partial_x - \frac{\partial_G(x)}{4G(x)} \right).$$

Which, in turn, corresponds to the PDM kinetic energy operator

$$\hat{T}(x) = -G(x)^{-1/4} \partial_x G(x)^{-1/2} \partial_x G(x)^{-1/4},$$

that is known in the literature as Mustafa and Mazharimousavi’s ordering [13, 14], and belongs the set of von Roos kinetic energy operators [28]. It would be interesting, therefore, to study the modified Emden equation (5) for PDM particles, which is in fact the focal point of the current methodical proposal. Throughout, we shall identify a classical-state by $\{x(t), p(t)\}$ [41] which corresponds to a specific phase-space trajectory. To the best of our knowledge, the current study has never been reported elsewhere.

In this article, we recollect (in section 2) the most general solution for the harmonic oscillator in equation (6) and map it report the most general solution for the MEE (2), where the initial conditions are to be identified for the corresponding PDM systems at hand. In the same section, we report some MEE classical-states at some specific parametric structures. In section 3, we introduce the PDM-settings for the MEE equation (2) and show that the dynamical equations for PDM-MEE systems are invariant with that of equation (2) and, therefore, are quasi-MEE that inherit the solutions of MEE. Two PDM quasi-MEE classical particles are used as illustrative examples. We give our concluding remarks in section 4.
2. MEE solution revisited and generalized

Let us consider a more general nonlocal point transformation

\[ U = u(t) \exp \left( \int [g(t) + \beta f(t)] \, dt \right); \quad g(t), f(t) \in \mathbb{R}, \]

(8)

for the harmonic oscillator of equation (6), where \( g(t), \) and \( f(t) \) are two arbitrary real functions, and \( \alpha \) and \( \beta \) are as defined in equation (2) (for more details on the nonlocal point transformation, the reader may refer to, e.g. [42, 43] and related references cited therein). This assumption, when substituted in equation (6), would yield

\[ \ddot{u}(t) + u(t)[2g(t) + 2\beta f(t)] + u(t)[\alpha g(t) + \beta f(t)] + u(t)[\alpha g(t) + \beta f(t)]^2 + \omega^2 u(t) = 0. \]

(9)

Comparing this equation with the MEE in equation (2), the parametric correlations arise in the process so that

\[ 2\alpha g(t) + 2\beta f(t) = \frac{2}{3} \alpha u(t) \iff \alpha g(t) + \beta f(t) = \frac{\alpha}{3} \ddot{u}(t); \quad [\alpha g(t) + \beta f(t)]^2 = \beta^2 u(t)^2 \iff \beta = \frac{\alpha}{3}, \]

(10)

and

\[ \alpha g(t) = \beta [u(t) - f(t)] \iff U(t) = u(t) \exp \left( \beta \int u(t) \, dt \right), \]

(11)

which is indeed the nonlocal transformation in equation (7) as in [2, 6]. Yet, \( \beta = \omega \eta \) not only retrieves the MEE in equation (5) (readily reported in [2, 6]), but also it retrieves the more general solution

\[ U(t) = A \sin(\omega t + \varphi) + B \cos(\omega t + \varphi) \]

(12)

for the linear harmonic oscillator dynamical equation in equation (6) (i.e. no initial boundary conditions, like \( U(0) = 0 \) as in [1, 2, 6], on equation (6) are imposed). Such general solution equation (12) would, in a straightforward manner, yield

\[ u(t) = \frac{A \cos(\omega t + \varphi) + B \sin(\omega t + \varphi)}{1 + A\eta \sin(\omega t + \varphi) - B\eta \cos(\omega t + \varphi)}; \quad 0 \leq (A, B) < \frac{1}{\eta}, \]

(13)

which is the most general solutions for equation (5). Moreover, the corresponding canonical momentum is given by equation (4).

In figure 1(a) we show \( u(t) \) of equation (13) as it evolves in time for different values of the damping parameter \( \eta \), where \( \omega, A, \) and \( B \) are fixed. An obvious isochronic oscillatory trend is observed. In figure 1(b) we show the corresponding phase-space trajectories (using equation (13) and equation (4)) for different values of \( A \), where \( B = 1, \omega = 1, \) and \( \eta = 0.15 \). The effect of the inherited amplitude \( A \) (i.e. inherited from the standard harmonic oscillator general solution in equation (12)) exhibits the regular phase-space trajectories trend for the MEE equation (5). Similar effect is also observed for different values of the inherited amplitude \( B \) for a fixed value of \( A \). In figure 1(c), however, we show the effect of the damping related parameter \( \eta \) on the phase-space trajectories as they evolve in time.

In the forthcoming sections we shall introduce and report on the PDM counterpart of MEE equation (5). We shall, therefore, leave the initial conditions to be identified by the nature of the corresponding PDM-MEE problem at hand.

3. PDM-Modified emden equation

In this section we discuss the MEE equation (2) within position-dependent mass settings. In so doing, we may assume that the coordinate \( u \) is deformed in such a way that

\[ u \longrightarrow u(x) = \int \sqrt{G(x)} \, dx = \sqrt{F(x)} \, x \iff \sqrt{G(x)} = \sqrt{F(x)} \left( 1 + \frac{F'(x)}{2F(x)} \right) \]

(14)

One should notice that \( u = \sqrt{F(x)} \, x \) suggests that \( F(x) \) and \( G(x) \) are both positive valued dimensionless scalar multipliers. This would, in turn, imply that \( -\infty \leq (u, x) \leq \infty \). Under such deformation/ transformation, therefore, one would find that

\[ \dot{u} \longrightarrow \dot{u}(x) = \sqrt{F(x)} \left( 1 + \frac{F'(x)}{2F(x)} \right) \dot{x} = \sqrt{G(x)} \ddot{x}. \]

(15)

Which when substituted in equation (1) and equation (2), respectively, yields

\[ L(u, \dot{u}, t) \longrightarrow L(x, \dot{x}, t) = \frac{1}{2} G(x) \dot{x}^2 - \left[ \frac{1}{2} \omega^2 F(x) x^2 + \frac{1}{4} \beta^2 F(x)^2 x^4 \right] \]

(16)
and

\[ \ddot{x} + \frac{G'(x)}{2G(x)} \dot{x}^2 + \alpha \sqrt{F(x)} x \, \dot{x} + \omega^2 x \sqrt{\frac{F(x)}{G(x)} \left( 1 + \frac{\beta^2}{\omega^2} F(x)x^2 \right)} = 0. \]

It should be noted here that the PDM-MEE equation (17) is a dynamical equation that describes the motion of a PDM-particle moving under the influence of a conservative potential force field

\[ V(x) = \frac{1}{2} \omega^2 F(x)x^2 + \frac{1}{2} \beta^2 F(x)x^4 \]

and subjected to a non-conservative Rayleigh dissipation force field

\[ \mathcal{R}(x, \dot{x}) = \frac{1}{2} \alpha \sqrt{F(x)} \, G(x) x \, \dot{x}^2. \]

Moreover, equation (17) resembles the well known mixed Liénard-type differential equation

\[ \ddot{x} + a(x) \dot{x}^2 + b(x) \dot{x} + c(x) = 0, \]

with,

\[ a(x) = \frac{G'(x)}{2G(x)}, \quad b(x) = \alpha \sqrt{F(x)} x, \quad c(x) = \omega^2 x \sqrt{\frac{F(x)}{G(x)} \left( 1 + \frac{\beta^2}{\omega^2} F(x)x^2 \right)}. \]

Hence, the transition from a mixed Liénard-type differential equation (18) into a MEE type equation (2) (the other way round is also viable) is feasible through the transformation equation (14). Moreover, the PDM quasi-MEE equation (17) consequently inherits solution equation (3) of MEE equation (2) through the point transformation equation (14). That is, once the relation between \( u(t) \) and \( x(t) \) is identified, for a given PDM particle \( G(x) = G(x(t)) \), then the structure of equation (17) collapses into that of equation (2) and consequently

\[ \text{Figure 1.} \]
the exact solution of the PDM-MEE equation (17) would be obtained in the process, therefore. We illustrate this issue in the following examples.

3.1. MEE-type mathews-lakshmanan PDM \( G(x) = 1/(1 + \lambda x^2) \)

A Mathews-Lakshmanan [8] PDM particle, with a positive valued dimensionless scalar multiplier \( G(x) = 1/(1 + \lambda x^2) \), would lead, through \( F(x) \) and \( G(x) \) correlation equation (14), to

\[
u = \sqrt{F(x)} x \Rightarrow \sqrt{F(x)} x = \frac{1}{\lambda} \ln (\lambda x + \sqrt{1 + \lambda^2 x^2}), \tag{19}\]

At this point one should notice that \( \lambda \) is a coupling parameter that renders the scalar multipliers \( F(x) \) and \( G(x) \) dimensionless and positive-valued (i.e. \( \lambda \) can never be zero, otherwise the dynamical system will collapse). The corresponding PDM quasi-MEE dynamical equation (17) reads

\[
\ddot{x} - \frac{\lambda^2 x}{1 + \lambda^2 x^2} \dot{x}^2 + \frac{3\eta \omega}{\lambda} Z(x) \dot{x} + \frac{\omega^2}{\lambda} \sqrt{1 + \lambda^2 x^2} Z(x) \left[ 1 + \frac{\eta^2}{\lambda^2} Z(x)^2 \right] = 0, \tag{20}\]

where,

\[
Z(x) = \ln (\lambda x + \sqrt{1 + \lambda^2 x^2}),
\]

is used. Hence,

\[
u(t) = \sqrt{F(x)} x = \frac{Z(x)}{\lambda} \Leftrightarrow x(t) = \frac{1}{\lambda} \sinh (\lambda u(t)), \tag{21}\]

where \( u(t) \) is given by equation (13). Hereby, one should notice that \( \lambda \) is a coupling parameter that renders the positive-valued scalar multipliers \( G(x) \) and \( F(x) \) dimensionless. Nevertheless, one should be advised that the substitution of the solution \( x(t) \) of equation (21) would transform equation (20) into the MEE equation (5) (with its solution readily given in equation (3)). The exact solvability of equation (20) is secured, therefore.

Moreover, the dynamical equation (20) represents a Mathews-Lakshmanan PDM-particle, \( G(x) = 1/(1 + \lambda x^2) \), moving under the influence of a conservative PDM-deformed potential force field

\[
V(x) = \frac{1}{2} \omega^2 \left( \frac{1}{\lambda} \ln (\lambda x + \sqrt{1 + \lambda^2 x^2}) \right)^2, \tag{22}\]

and a non-conservative PDM-deformed dissipative force field

\[
\mathcal{R}(x, \dot{x}) = \frac{3\eta \omega}{2\lambda} \frac{Z(x)}{(1 + \lambda^2 x^2)} \dot{x}^2. \tag{23}\]

The standard PDM Lagrangian and PDM-Hamiltonian for this system, respectively read,

\[
L = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 \left( \frac{1}{\lambda} \ln (\lambda x + \sqrt{1 + \lambda^2 x^2}) \right)^2, \tag{24}\]

\[
H = \frac{\dot{p}}{2} (1 + \lambda^2 x^2) + \frac{1}{2} \omega^2 \left( \frac{1}{\lambda} \ln (\lambda x + \sqrt{1 + \lambda^2 x^2}) \right)^2, \]

where the PDM canonical momentum in this case reads

\[
p(t) = \frac{\dot{x}(t)}{1 + \lambda^2 x^2} = \frac{\dot{u}(t)}{\cosh (\lambda u(t))}. \tag{25}\]

In figure 2, we plot \( x(t) \) of equation (21) as it evolves in time with \( \omega = 2 \) and \( \varphi = 0 \), for different values of \( \eta \) at \( A = B = \lambda = 1 \) in figure 2(a), for different values of \( \lambda \) at \( \eta = 0.5 \) and \( A = B = 1 \) in figure 2(b), and for different values of \( B \) at \( \eta = 0.25, \lambda = 2, \) and \( A = 1 \) in figure 2(c). We observe that the oscillations are still isochronic, documenting again that an amplitude-dependent frequency is not a consequential property of the nonlinear dynamics even under PDM-settings. We, moreover, show the corresponding phase-space trajectories for different values of \( \eta \) at \( A = 1, B = 0.25, \) and \( \lambda = 2 \) in figure 2(d), for different values of \( \lambda \) at \( \eta = 0.5 \) and \( A = B = 0.25 \) in figure 2(e), and for different values of \( B \) at \( \eta = 0.05, \lambda = 2, \) and \( A = 1 \) in figure 2(f).

3.2. An exponential MEE-type PDM \( G(x) = e^{2\lambda x} \)

A PDM with a positive valued dimensionless scalar multiplier in the form of \( G(x) = e^{2\lambda x} \), where \( \lambda \) is as defined above, would lead to

\[
F(x) = \frac{e^{2\lambda x}}{\lambda x^2} (1 - e^{-\lambda x})^2. \tag{26}\]
With $\varphi = 0$, we show (a) $x(t)$ of (21) for different values $\eta$, with $\omega = 2$, $A = B = 1$, and $\lambda = 1$, (b) $x(t)$ of (21) for different values $\lambda$, with $\omega = 2$, $A = B = 1$, and $\eta = 0.5$, (c) $x(t)$ of (21) for different values $B$, with $\omega = 2$, $A = B = 1$, and $\eta = 0.25$, and $\lambda = 2$. The corresponding phase-space trajectories are shown in (d) for different values of $\eta$ with $A = 1$, $B = 0.25$, $\lambda = \omega = 2$, (e) for different values of $\lambda$ with $A = B = 0.25$, $\omega = 2$, $\eta = 0.5$, and (f) for different values of $B$ with $A = 1$, $\eta = 0.05$, $\lambda = \omega = 2$.

Figure 2.
and results
\[ x(t) = \frac{1}{\lambda} \ln (\lambda u(t) + 1), \] (27)

where \( u(t) \) is given by equation (3). The substitution of \( x(t) \) of equation (27) in the corresponding Euler–Lagrange equation
\[ \ddot{x} + \lambda \dot{x}^2 + \frac{3\eta \mu}{\lambda} e^{\lambda x} (1 - e^{-\lambda x}) \dddot{x} + \frac{\omega^2}{\lambda} (1 - e^{-\lambda x}) \left[ 1 + \frac{\eta^2}{\lambda^2} e^{2\lambda x} (1 - e^{-2\lambda x}) \right] = 0, \] (28)

implies the dynamical equation (17) with its readily existing exact solution in equation (3). The dynamical equation (28) represents a PDM-particle, \( G(x) = e^{2\lambda x} \), moving under the influence of a conservative PDM-deformed potential force field
\[ V(x) = \frac{1}{2} \omega^2 \frac{e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2 + \frac{1}{4} \beta^2 \frac{e^{4\lambda x}}{\lambda^4} (1 - e^{-\lambda x})^4, \] (29)

and a non-conservative PDM-deformed Rayleigh dissipative force field
\[ R(x, \dot{x}) = \frac{3\mu \dot{x}}{2\lambda} e^{2\lambda x} (e^{\lambda x} - 1) \dot{x}^2. \] (30)

Moreover, the standard PDM-Lagrangian and PDM-Hamiltonian that describe this PDM-particle are given, respectively by
\[ L = \frac{1}{2} e^{2\lambda x} \dot{x}^2 - \left[ \frac{1}{2} \omega^2 \frac{e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2 + \frac{1}{4} \beta^2 \frac{e^{4\lambda x}}{\lambda^4} (1 - e^{-\lambda x})^4 \right], \] (31)
\[ H = \frac{1}{2} e^{-2\lambda x} p(x)^2 + \frac{1}{2} \omega^2 \frac{e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2 + \frac{1}{4} \beta^2 \frac{e^{4\lambda x}}{\lambda^4} (1 - e^{-\lambda x})^4, \] (32)

where
\[ p(t) = e^{2\lambda x(t)} \dot{x}(t), \] (33)

is the PDM canonical momentum.

In figure 3, we plot \( x(t) \) of equation (27) with \( \omega = 1 \), and \( \varphi = 0 \) for different values of \( \eta \) at \( A = B = 0.25 \), \( \lambda = 1.5 \) in figure 3(a), for different values of \( \lambda \) at \( \eta = 0.05 \) and \( A = B = 0.5 \) in figure 3(b), and for different values of \( A \) at \( \eta = 0.25 \), \( \lambda = 0.5 \), and \( B = 0.5 \) in figure 3(c). The oscillations are still isochronic even under PDM-settings. The corresponding phase-space trajectories for different values of \( \eta \) at \( A = B = 0.25 \), and \( \lambda = 1.5 \) in figure 3(d), for different values of \( \lambda \) at \( \eta = 0.05 \) and \( A = B = 0.5 \) in figure 3(e), and for different values of \( A \) at \( \eta = 0.05 \), \( \lambda = 1.2 \), and \( B = 0.5 \) in figure 3(f).

4. Concluding remarks

In this article, we have used the most general solution for the simple harmonic oscillator’s dynamical equation (i.e. the initial conditions to be identified by the PDM quasi-MEE problem at hand) and reported the most general solution for the modified Emden equation (3). This solution is valid as long as the MEE is linearized into the HO dynamical equation (6) through the nonlocal transformation equation (8). Moreover, We have introduced the corresponding PDM quasi-MEE dynamical systems equation (17) through the transformation equation (14). Two illustrative examples are considered and their exact solutions, equation (21) and equation (27), as well as the corresponding phase-space trajectories are reported.

In connection with the certain unusual nonlinear properties of the MEE, we have observed that the corresponding PDM quasi-MEE continues to exhibit such unusual properties and isochronic oscillations similar to the linear harmonic oscillator.

On the linearizability side of the MEE equation (2), one may start with the damped harmonic oscillator (DHO) dynamical equation
\[ \ddot{U} + 2\zeta \dot{U} + \omega^2 U = 0, \] (34)

where \( \zeta = \omega \eta \) as in [44] and use a nonlocal transformation (a special case of the ones reported by Chandrasekar et al in [1]) of the form
\[ U = u(t) e^{-\zeta t} \exp \left( \beta \int u(t) dt \right), \] (35)
Figure 3. With $\varphi = 0$, we show (a) $x(t)$ of (27) as it evolves in time for different values $\eta$, with $\omega = 1$, $A = B = 0.25$, and $\lambda = 1.5$, (b) $x(t)$ of (27) for different values $\lambda$, with $\omega = 1$, $A = B = 0.5$, and $\eta = 0.05$, (c) $x(t)$ of (27) for different values $A$, with $\omega = 1$, $B = 0.5$, $\eta = 0.25$, and $\lambda = 0.5$. The corresponding phase-space trajectories are shown in (d) for different values of $\eta$ with $A = B = 0.25$, $\lambda = 1.5$, $\omega = 1$, (e) for different values of $\lambda$ with $A = B = 0.5$, $\omega = 1$, $\eta = 0.05$, and (f) for different values of $A$ with $B = 0.5$, $\eta = 0.05$, $\lambda = 1.2$, $\omega = 1$. 
to obtain
\[ \ddot{u}(t) + 3\beta \dot{u}(t) + \Omega^2 u(t) + \beta^2 u(t)^3 = 0, \tag{36} \]
where \( \Omega^2 = \omega^2(1 - \eta^2) \) and \( \beta = \eta \Omega \). This would, in turn, suggest a solution for the new MEE equation (9) as
\[ u(t) = \left( \frac{A \cos(\Omega t + \varphi) + B \sin(\Omega t + \varphi)}{1 + A r^2 \sin(\xi t + \phi) + B r^2 \cos(\xi t + \phi)} \right); \quad 0 < \left( A, B \right) \frac{1}{\eta}. \tag{37} \]

Obviously, the frequency of oscillation \( \Omega \) is now \( \eta \)-dependent (i.e. frictional related parameter) and consequently would yield non-isochronic oscillations as it evolve in time.

Finally, such a nonlinear PDM quasi-MME equation (17) may find its applicability for more complicated nonlinear dynamical systems, for example, in the equilibrium configuration of spherical gas clouds [3], in the spherically symmetric expansion or collapse of relativistically gravitating mass [4], in the one-dimensional analog of the Yang-Mills boson gauge theory [5], etc. To the best of our knowledge, this work has never been discussed elsewhere.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files). Data will be available from 28 February 2023.

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References

[1] Chandrasekar V K, Santhivelan M, Kundu A and Lakshmanan M 2006 J. Phys. A: Math. Gen. 39 9743
[2] Chandrasekar V K, Sheeba J H, Pradeep R G, Divyasree R S and Lakshmanan M 2012 Phys. Lett. A 376 2188
[3] Leach P G L 1985 J. Math. Phys. 26 2510
[4] Vittie G C M 1984 Ann. Inst. H Poincare 40 231
[5] Yang C N and Mills R L 1954 Phys. Rev. 96 191
[6] Chandrasekar V K, Santhivelan M and Lakshmanan M 2005 Phys. Rev. E 72 066203
[7] Kilevicius A and Tytmchchina V 2018 J. Math. Phys. 59 082901
[8] Mathews P M and Lakshmanan M 1974 Quart. Appl. Math. 32 215
[9] Cariñena J F, Rañada M F, Santander M and Senthilvelan M 2004 Nonlinearity 17 1941
[10] Mustafa O 2015 J. Phys. A; Math. Theor. 48 225206
[11] Mustafa O 2019 J. Phys A: Math. Theor. 52 148001
[12] Mustafa O and Algadhi Z 2019 Eur. Phys. J. Plus 134 228
[13] Mustafa O and Mazharimousavi S H 2007 Int. J. Theor. Phys. 46 1786
[14] Mustafa O 2020 Phys. Lett. A 384 126265
[15] Quesne C 2015 J. Math. Phys. 56 012903
[16] Tiwari A K, Pandey S N, Santhivelan M and Lakshmanan M 2013 J. Math. Phys. 54 053506
[17] Lakshmanan M and Chandrasekar V K 2013 Eur. Phys. J. ST 222 665
[18] Chandrasekar V K, Santhivelan M and Lakshmanan M 2007 J. Phys. A 40 032701
[19] Musielak Z E 2008 J. Phys. A: Math. Theor. 41 055205
[20] Bhuvaneswari A, Chandrasekar V K, Santhivelan M and Lakshmanan M 2012 J. Math. Phys. 53 073504
[21] Cariñena J F and Raniya M F 2005 J. Math. Phys. 46 062703
[22] Mustafa O 2020 Phys. Scr. 95 065214
[23] Ranada M 2016 J. Math. Phys. 57 052703
[24] Cariñena J F, Herranz F J and Raniya M F 2017 J. Math. Phys. 58 022701
[25] Mustafa O 2013 J. Phys. A: Math. Theor. 46 368001
[26] Bender C M, Gianfreda M and Jones H F 2016 J. Math. Phys. 57 084101
[27] Mustafa O 2021 Eur. Phys. J. Plus 136 249
[28] von Roos O 1983 Phys. Rev. B 27 2574
[29] de Souza Dutra A and Almeida C A S 2000 Phys. Lett. A 275 25
[30] dos Santos M A F, Gomez I S and Da Costa B G and Mustafa O 2021 Eur. Phys. J. Plus 136 96
[31] El-Nabulsi R A 2020 Few-Body Syst. 61 37
[32] El-Nabulsi R A 2020 J. Phys. Chem. Solids 140 109384
[33] Mustafa O and Mazharimousavi S H 2006 Phys. Lett. A 358 259
[34] Bagchi B, Banerjee A, Quesne C and Tkachuk V M 2005 J. Phys. A: Math. Gen. 38 2929
[35] Mustafa O 2011 J. Phys. A: Math. Theor. 44 355303
[36] Cruz y Cruz S and Rosas-Ortiz O 2009 J Phys A: Math. Theor. 42 185205
[37] Mazharimousavi S H and Mustafa O 2013 Phys. Scr. 87 055006
[38] Quesne C and Tkachuk V M 2004 J. Phys. A 37 4267
[39] Quesne C 2019 Eur. Phys. J. Plus 134 391
[40] da Costa B G, Gomez I S and Borges E P 2020 Phys. Rev. E 102 062105
[41] Shankar R 1980 Principles of Quantum Mechanics (Plenum Press)
[42] Muriel C and Romero J L 2010 J. Phys. A: Math. Theor. 43 434025
[43] Sinelshchikov D I 2020 Chaos Solitons Fractals 141 110318
[44] Mustafa O 2021 Phys. Scr. 96 065205