EULER-LIKE RECURRENCES FOR SMALLEST PARTS FUNCTIONS

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In memory of Basil Gordon

Abstract. We obtain recurrences for smallest parts functions which resemble Euler’s recurrence for the ordinary partition function. The proofs involve the holomorphic projection of non-holomorphic modular forms of weight 2.

1. Introduction

Let $p(n)$ denote the unrestricted partition function. One of the fundamental results in partition theory is Euler’s recurrence, which states that for $n > 0$ we have

$$\sum_k (-1)^k p \left( n - \frac{k(3k+1)}{2} \right) = 0. \quad (1.1)$$

The smallest parts function $\text{spt}(n)$, which counts the number of smallest parts in the partitions of $n$, was introduced by Andrews [4]. This and other smallest parts functions have been studied widely in recent years from a number of perspectives (see, e.g. [1, 2, 5, 7, 8, 9, 10] and the many references therein). Many of the beautiful properties of these functions originate from the fact that the associated generating functions are components of mock modular forms of weight $3/2$.

Here we use the technique of holomorphic projection (as described by Sturm [14] and Gross-Zagier [12]) to derive analogues of (1.1) for smallest parts functions. The basic principle (also used recently in [3] and [11]) is that for a non-holomorphic modular form $f = f^+ + f^-$ written as a sum of holomorphic and non-holomorphic parts, we have $\pi_{\text{hol}}(f) = f^+ + \pi_{\text{hol}}(f^-)$. If one can identify the holomorphic modular form $\pi_{\text{hol}}(f)$ and can compute $\pi_{\text{hol}}(f^-)$ explicitly, then a formula for $f^+$ results. The simplest such analogue involves $\text{spt}(n)$. The associated holomorphic projection has been described (without proof) by Zagier [15, §6]; for completeness we give a brief account here.

Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

denote Dedekind’s eta function and let $E_2(z)$ be the quasimodular weight 2 Eisenstein series on $\text{SL}_2(\mathbb{Z})$. Define

$$F(z) := \sum_{n=1}^{\infty} \text{spt}(n) q^{n-\frac{1}{4}} - \frac{1}{12} \cdot \frac{E_2(z)}{\eta(z)} + \frac{\sqrt{3}i}{2\pi} \int_{-\infty}^{\infty} \frac{\eta(\tau)}{(z + \tau)^{\frac{3}{2}}} d\tau.$$

2010 Mathematics Subject Classification. 11F37, 11P84.

Key words and phrases. Smallest parts functions, holomorphic projection.

The first author was supported by a grant from the Simons Foundation (#208525 to Scott Ahlgren).
Let $\varepsilon$ be the multiplier on $SL_2(\mathbb{Z})$ associated to the eta function. It can be shown (see [6] or [1, §3]) that $F(z)$ is a weak harmonic Maass form of weight $3/2$ on $SL_2(\mathbb{Z})$ with multiplier $\varepsilon$, so the function $\eta(z)F(z)$ transforms like a modular form of weight $2$ on $SL_2(\mathbb{Z})$. For positive integers $n$, define
\[ a(n) := -\sum_{\substack{ab=6n \\ 0 < a < b}} \left( \frac{12}{b^2-a^2} \right) \cdot a. \]

We have
\[ \sum_{n=1}^{\infty} a(n)q^n = q + 2q^2 + q^3 + 2q^4 - q^5 + 3q^6 - 2q^7 + 2q^8 + q^9 + q^{10} + \ldots. \]

Letting $E^*_2(z)$ denote the non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$, it can be shown that the holomorphic projection of $\eta(z)F(z) + \frac{1}{12}E^*_2(z)$ is equal to $0$. By computing this projection directly (using an argument similar to those given below) one can deduce that
\[ \prod_{n=1}^{\infty} (1 - q^n) \cdot \sum_{n=1}^{\infty} \text{spt}(n)q^n = \sum_{n=1}^{\infty} a(n)q^n. \]

In other words, we have the following Euler-like recurrence for $\text{spt}(n)$, which is recorded in a slightly different form by Zagier [15] and Andrews-Rhoades-Zwegers [3, Thm. 11.1].

**Theorem 1.** For $n > 0$ we have
\[ \sum_k (-1)^k \text{spt} \left( n - \frac{k(3k+1)}{2} \right) = a(n). \]

We will derive similar recurrences for other smallest parts functions. An *overpartition* is a partition in which the first occurrence of each part may be overlined. Let $p(n)$ denote the number of overpartitions of $n$ and let $\overline{\text{spt}}(n)$ denote the number of odd smallest parts in the overpartitions of $n$ (see [7]). Define a divisor function $s(n)$ by
\[ s(n) := \sum_{d|n} \min \left( d, \frac{n}{d} \right), \]
with the convention that $s(n) = 0$ if $n \notin \mathbb{Z}$. Define
\[ b(n) := (-1)^{n+1} \begin{cases} 2s(n) & \text{if } n \text{ is odd}, \\ 4s(n/4) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}. \end{cases} \]

Then we have the following analogue of (1.1) for $\overline{\text{spt}}(n)$.

**Theorem 2.** For $n > 0$ we have
\[ \sum_k (-1)^k \overline{\text{spt}}(n - k^2) = b(n). \]

Theorem 2 is equivalent to the identity
\[ \sum_{n \in \mathbb{Z}} (-1)^n q^n \sum_{m=1}^{\infty} \overline{\text{spt}}(m) q^m = \sum_{n=0}^{\infty} b(n)q^n = 2q + 4q^3 - 4q^4 + 4q^5 + 4q^7 - 8q^8 + \ldots. \]
EULER-LIKE RECURRENCES FOR SMALLEST PARTS FUNCTIONS

Since we have
\[ \sum_{n=0}^{\infty} \overline{p}(n)q^n = \left( \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \right)^{-1} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \ldots, \]
we obtain the following

**Corollary 3.** For all \( N > 0 \) we have
\[ \text{spt}(N) = \sum_{n+m=N} \overline{p}(n)b(m). \]

Following \([8]\), let \( m_2(n) \) denote the number of partitions of \( n \) without repeated odd parts, and define \( M_2\text{spt}(n) \) as the restriction of \( \text{spt}(n) \) to these partitions whose smallest part is even. Define
\[ c(n) := \sigma(n) - \sigma(n/2) - \frac{1}{2} s(2n) + s(n/2), \]
where \( \sigma(n) \) denotes the usual sum of divisors function.

**Theorem 4.** For \( n > 0 \) we have
\[ \sum_{k \geq 0} (-1)^{k(k+1)/2} M_2\text{spt} \left( n - \frac{k(k+1)}{2} \right) = (-1)^n c(n). \]

We will prove the theorem by establishing the identity
\[ \sum_{n=0}^{\infty} q^n(n+1)/2 \sum_{m=1}^{\infty} (-1)^m M_2\text{spt}(m)q^m = \sum_{n=1}^{\infty} c(n)q^n = q^2 + q^3 + 3q^4 + 3q^5 + 4q^6 + \ldots. \]
Since
\[ \left( \sum_{n=0}^{\infty} q^n(n+1)/2 \right)^{-1} = \sum_{n=0}^{\infty} (-1)^n m_2(n)q^n = 1 - q + q^2 - 2q^3 + 3q^4 - 4q^5 + 5q^6 + \ldots, \]
we obtain the following

**Corollary 5.** For all \( n > 0 \) we have
\[ M_2\text{spt}(N) = \sum_{n+m=N} (-1)^m m_2(n)c(m). \]

2. **Preliminaries**

Let \( k \in \mathbb{Z} \). For matrices \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{Q}) \) and functions \( f \) on the upper half plane we define
\[ (f|_k \gamma)(z) := \det(\gamma)^k \overline{f}(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right). \]
We say that \( f \) has weight \( k \) for \( \Gamma_0(N) \) if \( f|_k \gamma = f \) for all \( \gamma \in \Gamma_0(N) \). Let \( E_2 \) denote the weight 2 quasi-modular Eisenstein series
\[ E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n. \]
Then the functions
\[ E_2^*(z) := E_2(z) - \frac{3}{\pi y} \quad \text{and} \quad E(z) := 2E_2(2z) - E_2(z) \]
have weight 2 for SL_2(\mathbb{Z}) and \Gamma_0(2), respectively. Letting \( W_2 := \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \) denote the Fricke involution, we have \( E|_2 W_2 = -E \) and

\[
(E|_2 W_2) (z) = 2E_2(2z).
\]

Define

\[
G(z) := \sum_{n \geq 1} \text{spt}_1(n) q^n + \frac{1}{12} \frac{\eta(2z)}{\eta^2(z)} (E_2(z) - 4E_2(2z)) + \frac{1}{2\sqrt{2\pi i}} \int_{-\tau}^{i\infty} \frac{\eta^2(\tau)/\eta(2\tau)}{(-i(\tau + z))^{3/2}} d\tau \tag{2.1}
\]

and

\[
H(z) := \sum_{n \geq 1} (-1)^n \text{M2spt}(n) q^{n-\frac{1}{4}}
\]

\[
+ \frac{1}{24} \frac{\eta(z)}{\eta^2(2z)} (E_2(2z) - E_2(z)) + \frac{1}{2\pi i} \int_{-\tau}^{i\infty} \frac{\eta^2(\tau)/\eta(\tau)}{(-i(\tau + z))^{3/2}} d\tau. \tag{2.2}
\]

By work of Bringmann, Lovejoy, and Osburn [8], these functions are harmonic weak Maass forms of weight 3/2 (see, for example, [13] for details). In the notation of [8], \( G(z) = -\frac{1}{4} M(z) \) and (correcting a sign error) \( H(z) = M_2(z/8) \). From the proof of Lemma 6.1 of [8], we have

\[
(-i\sqrt{2}z)^{-\frac{3}{2}} G(-1/2z) = -2^\frac{3}{4} H(z). \tag{2.3}
\]

We use this fact to obtain the following proposition.

**Proposition 6.** The functions

\[
g(z) := \frac{\eta^2(z)}{\eta(2z)} G(z) \quad \text{and} \quad h(z) := \frac{\eta^2(2z)}{\eta(z)} H(z)
\]

have weight 2 for \( \Gamma_0(2) \).

**Proof.** The group \( \Gamma_0(2)/\{\pm I\} \) is generated by the matrices \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \). By (2.1) and (2.2) we have \( g(z+1) = g(z) \) and \( h(z+1) = h(z) \). To check the transformation under \( \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \), we write

\[
\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = W_2 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_2^{-1}.
\]

Using (2.3) and the fact that \( \eta(-1/z) = \sqrt{-iz} \eta(z) \), we find that

\[
g(z)|_2 W_2 = 2h(z), \tag{2.4}
\]

from which

\[
g(z)|_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = g(z).
\]

The same is true for \( h(z) \), and the proposition follows. \( \square \)

We introduce the holomorphic projection operator. Let \( k \geq 2 \) be an even integer. Suppose that \( \phi(z) \) has weight \( k \) for \( \Gamma_0(N) \) and has Fourier expansion

\[
\phi(z) = \sum_{m \in \mathbb{Z}} \alpha(m, y) q^m.
\]

Define

\[
\pi_{\text{hol}}(\phi) := \sum_{m=1}^{\infty} a(m) q^m,
\]
Suppose that Lemma 7. ensures that the limit and integral at the bottom of page 296 may be interchanged. When \( k = 2 \) it follows from the proof of Proposition 6.2, loc. cit. (note that condition (2.6) ensures that the limit and integral at the bottom of page 296 may be interchanged).

**Lemma 7.** Suppose that \( k \geq 2 \). Suppose that \( \phi(z) \) has weight \( k \) for \( \Gamma_0(N) \) and satisfies

\[
(\phi|_k \gamma)(z) \ll y^{-\varepsilon} \quad \text{as} \quad y \to \infty
\]

for some \( \varepsilon > 0 \) and for all \( \gamma \in SL_2(\mathbb{Z}) \). If \( k = 2 \), suppose in addition that for some \( \varepsilon' > 0 \) we have

\[
\alpha(m, y) \ll_m y^{-1+\varepsilon'} \quad \text{as} \quad y \to 0 \quad \text{for all} \quad m > 0.
\]

Then \( \pi_{\text{hol}}(\phi) \) is a weight \( k \) cusp form on \( \Gamma_0(N) \).

3. **Proof of Theorem 2**

Write \( G = G^+ + G^- \), where

\[
G^-(z) = \frac{1}{2\sqrt{2\pi}i} \int_{i\tau}^{i\infty} \frac{\eta^2(\tau)/\eta(2\tau)}{(-i(\tau + z))^2} d\tau
\]

is the non-holomorphic part. Since \( \eta^2(z)/\eta(2z) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \), a computation gives

\[
G^-(z) = \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n n \beta(n^2 y) q^{-n^2},
\]

where \( \beta(y) := \Gamma(-1/2, 4\pi y) \) is the incomplete gamma function. Then

\[
g(z) = \frac{\eta^2(z)}{\eta(2z)} G(z) = \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^{\infty} \text{spt} \Gamma(n) q^n + \frac{1}{12} (E_2(z) - 4E_2(2z)) + \sum_{N \in \mathbb{Z}} B(N, y) q^N,
\]

where

\[
B(N, y) = \frac{(-1)^N}{\sqrt{\pi}} \left\{ \begin{array}{ll}
2 \sum_{\begin{subarray}{c}m > n \\ m, n \geq 1 \end{subarray}} m \beta(m^2 y) + \delta_{\square}(|N|) \sqrt{|N|} \beta(|N| y) & \quad \text{if} \quad N < 0, \\
\frac{1}{2\sqrt{\pi} y} + 2 \sum_{m=1} m \beta(m^2 y) & \quad \text{if} \quad N = 0, \\
2 \sum_{\begin{subarray}{c}m > n \\ m, n \geq 1 \end{subarray}} m \beta(m^2 y) + \delta_{\square}(N) \frac{1}{\sqrt{\pi} y} & \quad \text{if} \quad N > 0.
\end{array} \right.
\]

Here \( \delta_{\square}(N) = 1 \) if \( N \) is a square, and 0 otherwise. Since \( \beta(y) \sim (4\pi y)^{-3/2} e^{-4\pi y} \) as \( y \to \infty \), we have

\[
\sum_{\begin{subarray}{c}n^2 - m^2 = N \\ n, m \geq 1 \end{subarray}} m \beta(m^2 y) \ll y^{-3/2} \sum_{\begin{subarray}{c}n^2 - m^2 = N \\ n, m \geq 1 \end{subarray}} \frac{1}{m^2} e^{-4\pi m^2 y},
\]

where the implied constants here and in the rest of the paragraph are absolute. Since \( n^2 - (n-1)^2 = 2n - 1 \), the equation \( n^2 - m^2 = N \) implies that \( n, m \leq (|N| + 1)/2 \). If \( N > 0 \),
then this sum is $\ll N y^{-3/2}$. If $N < 0$ then we have $m^2 > -N$ for each term in the sum, from which it follows that the sum is $\ll |N| y^{-3/2} e^{4\pi N y}$. We conclude that as $y \to \infty$, we have

$$B(N, y) \ll \begin{cases} |N| y^{-\frac{3}{2}} e^{4\pi N y} & \text{if } N < 0, \\ y^{-\frac{1}{2}} + N y^{-\frac{3}{2}} & \text{if } N \geq 0. \end{cases} \quad (3.2)$$

Define

$$\hat{g}(z) := g(z) + \frac{1}{6} E(z) + \frac{1}{12} E_2^*(z)$$

$$= \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^{\infty} \text{spt}(n) q^n - \frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y) q^N.$$

By (3.2) we have $\hat{g}(z) \ll y^{-1/2}$ as $y \to \infty$. From (2.4) we obtain

$$\hat{g} \big|_2 W_2 = 2 h(z) + \frac{1}{6} (E_2(z) - E_2(2z)) - \frac{1}{4\pi y}.$$

Therefore $\hat{g} \big|_2 W_2 \ll y^{-1}$ as $y \to \infty$ since $h(z)$ decays exponentially at $\infty$.

For $N > 0$, we have the bound

$$B(N, y) \ll N y^{-\frac{3}{2}} \quad \text{as } y \to 0$$

since $\lim_{y \to 0} \beta(y) = -2\sqrt{\pi}$. Therefore we may apply Lemma 7 to obtain

$$\pi_{\text{hol}}(\hat{g}) = 0$$

since there are no nontrivial cusp forms of weight 2 on $\Gamma_0(2)$.

We may also compute $\pi_{\text{hol}}(\hat{g})$ using (2.5). Since $\pi_{\text{hol}}$ leaves holomorphic functions unchanged, we have

$$\pi_{\text{hol}}(\hat{g}) = \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^{\infty} \text{spt}(n) q^n + \pi_{\text{hol}} \left( -\frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y) q^N \right).$$

By (2.5) we have

$$\pi_{\text{hol}} \left( -\frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y) q^N \right) = \sum_{N=1}^{\infty} \left( 4\pi N \int_0^{\infty} B(N, y) e^{-4\pi N y} dy \right) q^N.$$

By (3.1), the coefficient of $q^N$ above is

$$(-1)^N 8\sqrt{\pi} N \sum_{\substack{n^2 - m^2 = N \\text{ coprime}}} m \int_0^{\infty} \beta(m^2 y) e^{-4\pi N y} dy + \delta_\square(N)(-1)^N 4N \int_0^{\infty} y^{-\frac{1}{2}} e^{-4\pi N y} dy. \quad (3.3)$$

The second integral evaluates to $\frac{1}{2\sqrt{\pi}}$ and the first is evaluated using the following lemma. The proof is routine (some care is required to justify the change in the order of integration).

**Lemma 8.** If $A, B > 0$ then

$$\int_0^{\infty} \beta(A y) e^{-4\pi B y} dy = \frac{1}{2\sqrt{\pi B}} \left( \sqrt{1 + \frac{B}{A}} - 1 \right). \quad (3.4)$$
Therefore (3.3) becomes
\[
(-1)^N 4 \sum_{n^2 - m^2 = N \atop n, m \geq 1} m \left( \sqrt{1 + \frac{N}{m^2}} - 1 \right) + \delta_{\square}(N)(-1)^N 2\sqrt{N} \
= (-1)^N 2 \left( \sum_{n^2 - m^2 = N \atop n, m \geq 1} (n - m) + \delta_{\square}(N)\sqrt{N} \right).
\]

It remains to show that this evaluates to \(-b(N)\). If \(N \equiv 2 \pmod{4}\), then the sum is empty and \(\delta_{\square}(N) = 0\). If \(N\) is odd, then \(n - m\) runs over all divisors of \(N\) which are less than \(\sqrt{N}\).

In this case we have
\[
2 \sum_{n^2 - m^2 = N \atop n, m \geq 1} (n - m) + \delta_{\square}(N)\sqrt{N} = \sum_{d \mid N} \min \left( d, \frac{N}{d} \right).
\]

Finally, if \(4 \mid N\) then each \(n - m\) is even. Letting \(r = \frac{n-m}{2}\) and \(s = \frac{n+m}{2}\), we find that
\[
\sum_{n^2 - m^2 = N \atop n, m \geq 1} (n - m) = \sum_{rs = N/4 \atop 0 < r < s} 2r = \sum_{d \mid \frac{N}{4}} \min \left( d, \frac{N}{d} \right) - \delta_{\square}(N)\sqrt{\frac{N}{4}}.
\]

\[
\square
\]

4. PROOF OF THEOREM 4

We proceed as in the proof of Theorem 2. Write \(H = H^+ + H^-\), where
\[
H^-(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\eta^2(2\tau)/\eta(\tau)}{(-i(z + \tau))^2} d\tau.
\]

Since \(\eta^2(2z)/\eta(z) = \sum_{\text{odd } n \geq 1} q^{n^2/8}\), we have
\[
H^-(z) = \frac{1}{4\sqrt{\pi}} \sum_{\text{odd } n \geq 1} n\beta (\frac{n^2 y}{8}) q^{-\frac{n^2}{\pi}}.
\]

Define \(\hat{h}(z) := h(z) - \frac{1}{24}(E(z) - E_2(z))\). Then (2.2) gives
\[
\hat{h}(z) = \frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n \text{M2pt}(n)q^{n-1} + \frac{1}{24}(E_2(z) - E_2(2z)) - \frac{1}{8\pi y} + \sum_N C(N, y)q^N,
\]

where
\[
C(N, y) = \frac{1}{4\sqrt{\pi}} \sum_{n^2-m^2=8N \atop n, m \geq 1 \text{ odd}} \beta (\frac{m^2 y}{8}).
\]

By an argument similar to that which gives (3.2), we find that as \(y \to \infty\) we have
\[
C(N, y) \ll \begin{cases} |N| y^{-\frac{1}{2}} e^{4\pi N y} & \text{if } N < 0, \\ y^{-\frac{3}{2}} & \text{if } N = 0, \\ N y^{-\frac{3}{2}} & \text{if } N > 0. \end{cases}
\]
Thus we have $h(z) \ll y^{-1}$ as $y \to \infty$. We have

$$
\hat{h} \big|_2 W_2 = \frac{1}{2} g + \frac{1}{24}(E(z) + 2E_2(2z)) - \frac{1}{8\pi y}.
$$

Therefore $\hat{h} \big|_2 W_2 \ll y^{-1/2}$ as $y \to \infty$ since the constant term of $g(z)$ is $-1/4$ and the constant term of $E(z) + 2E_2(2z)$ is 3. For $N > 0$ we have the bound $C(N, y) \ll_{N} 1$ as $y \to 0$. Therefore, we may apply Lemma 7 to conclude that $\pi_{hol}(\hat{h}) = 0$.

Using (2.5), we find that

$$
0 = \pi_{hol}(\hat{h}) = \frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n M2spt(n)q^{n-\frac{1}{8}} + \frac{1}{24}(E_2(z) - E_2(2z)) + \sum_{N=1}^{\infty} C(N)q^N,
$$

where

$$
C(N) = \sqrt{\pi N} \int_{0}^{\infty} \sum_{\substack{n^2 - m^2 = 8N \\ n, m \geq 1 \ \text{odd}}} m\beta \left( \frac{m^2y}{8} \right) e^{-4\pi Ny} dy.
$$

By Lemma 8 we obtain

$$
C(N) = \frac{1}{2} \sum_{\substack{n^2 - m^2 = 8N \\ n, m \geq 1 \ \text{odd}}} (n - m).
$$

Writing $u = \frac{n-m}{2}$ and $v = \frac{n+m}{2}$ gives

$$
C(N) = \sum_{\substack{uv=2N \\ u<\sqrt{N} \\ u+v \ \text{odd}}} u = \frac{1}{2}s(2N) - s(N/2).
$$

From (4.1) we conclude that

$$
\frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n M2spt(n)q^{n-\frac{1}{8}} = \sum_{n=1}^{\infty} c(n)q^n.
$$

\[\square\]

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