Uniform Asymptotic Expansion for the Incomplete Beta Function

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Abstract. In [Temme N.M., Special functions. An introduction to the classical functions of mathematical physics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996, Section 11.3.3.1] a uniform asymptotic expansion for the incomplete beta function was derived. It was not obvious from those results that the expansion is actually an asymptotic expansion. We derive a remainder estimate that clearly shows that the result indeed has an asymptotic property, and we also give a recurrence relation for the coefficients.

Key words: incomplete beta function; uniform asymptotic expansion

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1 Introduction

For positive real numbers $a$, $b$ and $x \in [0, 1]$, the (normalised) incomplete beta function $I_x(a, b)$ is defined by

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} \, dt,$$

where $B(a, b)$ denotes the ordinary beta function:

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$$

(see, e.g., [2, Section 8.17(i)]). In this paper, we will use the notation of [2, Section 8.18(ii)].

The incomplete beta function plays an important role in statistics in connection with the beta distribution (see, for instance, [1, pp. 210–275]). Large parameter asymptotic approximations are useful in these applications. For fixed $x$ and $b$, one could use the asymptotic expansion

$$I_x(a, b) \sim \frac{x^a(1-x)^{b-1}}{aB(a, b)} 2F_1 \left( 1, 1-b; \frac{x}{x-1} \right) \sim \frac{x^a(1-x)^{b-1}}{aB(a, b)} \sum_{n=0}^\infty \frac{(1-b)_n}{(a+1)_n} \left( \frac{x}{x-1} \right)^n,$$  \hspace{1cm} (1)

as $a \to +\infty$. The right-hand side of (1) converges only for $x \in [0, \frac{1}{2})$, but for any fixed $x \in [0, 1)$ it is still useful when used as an asymptotic expansion as $a \to +\infty$. For more details, see [3, Section 11.3.3]. However, it is readily seen that (1) breaks down as $x \to 1$. Since this limit has significant importance in applications, Temme derived in [3, Section 11.3.3.1] an asymptotic expansion as $a \to +\infty$ that holds uniformly for $x \in (0, 1]$. His result can be stated as follows.
Theorem 1. Let $\xi = -\ln x$. Then for any fixed positive integer $N$ and fixed positive real $b$,

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)} \left( \sum_{n=0}^{N-1} d_n F_n + O\left(a^{-N}\right) F_0 \right),$$

(2)

as $a \to +\infty$, uniformly for $x \in (0,1]$. The functions $F_n = F_n(\xi, a, b)$ are defined by the recurrence relation

$$a F_{n+1} = (n + b - a \xi) F_n + n \xi F_{n-1},$$

(3)

with

$$F_0 = a^{-b} Q(b, a \xi), \quad F_1 = \frac{b - a \xi}{a} F_0 + \frac{\xi b e^{-a \xi}}{a \Gamma(b)},$$

and $Q(a, z) = \Gamma(a, z)/\Gamma(a)$ is the normalised incomplete gamma function (see [2, Section 8.2(i)]). The coefficients $d_n = d_n(\xi, b)$ are defined by the generating function

$$\left( \frac{1 - e^{-t}}{t} \right)^{b-1} = \sum_{n=0}^{\infty} d_n (t - \xi)^n.$$  

(4)

In particular,

$$d_0 = \left( \frac{1 - x}{\xi} \right)^{b-1}, \quad d_1 = \frac{x \xi + x - 1}{(1 - x) \xi} (b - 1) d_0.$$  

They satisfy the recurrence relation

$$\xi(n+1)(n+2) d_0 d_{n+2} = \xi \sum_{m=0}^{n} (m+1) \left( n - 2m + 1 + \frac{m - n - 1}{b - 1} \right) d_{m+1} d_{n-m+1}$$

$$+ \sum_{m=0}^{n} (m+1) \left( n - 2m - 2 - \xi + \frac{m - n}{b - 1} \right) d_{m+1} d_{n-m}$$

$$+ \sum_{m=0}^{n} (1 - m - b) d_m d_{n-m}.$$  

(5)

In the case that $b = 1$, we have $d_0 = 1$ and $d_n = 0$ for $n \geq 1$.

Our contribution is the remainder estimate in (2) and the recurrence relation (5). In fact, it is not at all obvious from (3) that the sequence $\{F_n\}_{n=0}^{\infty}$ has an asymptotic property as $a \to +\infty$. We will show that for any non-negative integer $n$,

$$0 < F_{n+1} \leq \frac{n + \beta}{a} F_n,$$

(6)

where $\beta = \max(1, b)$.

In [4, Section 38.2.8] the function $F_n$ is identified as a Kummer $U$-function:

$$F_n = \frac{\xi^{n+b} e^{-a \xi} n!}{\Gamma(b)} U(n + 1, n + b + 1, a \xi).$$
2 Proof of the main results

We proceed similarly as in [3, Section 11.3.3.1] and start with the integral representation

\[ I_x(a,b) = \frac{1}{B(a,b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at} \left( \frac{1-e^{-t}}{t} \right)^{b-1} dt. \]  

(7)

We substitute the truncated Taylor series expansion

\[ \left( \frac{1-e^{-t}}{t} \right)^{b-1} = \sum_{n=0}^{N-1} d_n (t - \xi)^n + r_N(t) \]

into (7) and obtain

\[ I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)} \left( \sum_{n=0}^{N-1} d_n F_n + R_N(a,b,x) \right), \]

where \( F_n \) is given by the integral representation

\[ F_n = \frac{1}{\Gamma(b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at}(t-\xi)^n dt = \frac{e^{-a\xi}}{\Gamma(b)} \int_{0}^{+\infty} (\tau + \xi)^{b-1} \tau^n e^{-a\tau} d\tau, \]  

(8)

and the remainder term \( R_N(a,b,x) \) is defined by

\[ R_N(a,b,x) = \frac{1}{\Gamma(b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at} r_N(t) dt. \]  

(9)

The recurrence relation (3) can be obtained from (8) via a simple integration by parts. Let, for a moment,

\[ c_n(a,b) = \int_{0}^{+\infty} (\tau + \xi)^{b-1} \tau^n e^{-a\tau} d\tau. \]

Then via integration by parts we find

\[ ac_{n+1}(a,b) = (n+b)c_n(a,b) + \xi(1-b)c_n(a,b-1). \]  

(10)

We make the observation that

\[ 0 \leq \xi c_n(a,b-1) = \xi \int_{0}^{+\infty} (\tau + \xi)^{b-2} \tau^n e^{-a\tau} d\tau \leq c_n(a,b). \]  

(11)

It follows from (10) and (11) that

\[ ac_{n+1}(a,b) \leq \begin{cases} 
(n+1)c_n(a,b) & \text{if } 0 < b \leq 1, \\
(n+b)c_n(a,b) & \text{if } b \geq 1.
\end{cases} \]

Since \( F_n = e^{-a\xi} c_n(a,b)/\Gamma(b) \), this inequality implies (6).

To obtain the remainder estimate in (2), we use the Cauchy integral representation

\[ r_N(t) = \frac{(t - \xi)^N}{2\pi i} \oint_{\{t,\xi\}} \frac{\left(1-{e^{-\tau}}\right)^{b-1}}{(\tau - t)(\tau - \xi)^N} d\tau, \]  

(12)
where the contour encircles the points $\xi$ and $t$ once in the positive sense. From the integral representation (9), we have that $0 \leq \xi \leq t$. Thus, in the case that $N \geq 1$, we can deform the contour in (12) to the path

$$\left[1 + \infty \im i, 1 + \pi \im i\right] \cup \left[1 + \pi \im i, -1 + \pi \im i\right] \cup \left[-1 + \pi \im i, -1 - \pi \im i\right] \cup \left[-1 - \pi \im i, 1 - \pi \im i\right] \cup \left[1 - \pi \im i, 1 - \infty \im i\right].$$

For the integrals along the final three portions of the path, we have the estimates

$$\left| \frac{1}{2\pi i} \int_{-1+\pi \im i}^{-1-\pi \im i} \frac{(1-e^{-\tau})^{b-1}}{(\tau - t)(\tau - \xi)^N} d\tau \right| \leq \frac{\max \left((e - 1)^{b-1}, \left(\frac{e+1}{\sqrt{\pi^2+1}}\right)^{b-1}\right)}{(1 + \xi)^{N+1}},$$

and

$$\left| \frac{1}{2\pi i} \int_{-1-\pi \im i}^{1-\pi \im i} \frac{(1-e^{-\tau})^{b-1}}{(\tau - t)(\tau - \xi)^N} d\tau \right| \leq \frac{\max \left((e+1)^{b-1}/\sqrt{\pi^2+1}\right)}{\pi^{N+2}},$$

and

$$\left| \frac{1}{2\pi i} \int_{1-\pi \im i}^{1-\infty \im i} \frac{(1-e^{-\tau})^{b-1}}{(\tau - t)(\tau - \xi)^N} d\tau \right| \leq \frac{1}{2\pi} \int_{\pi}^{+\infty} \frac{\max \left((1 \pm e^{-1})^{b-1}\right) (s^2 + 1)^{(1-b)/2}}{\sqrt{s^2 + 1-t^2} (s^2 + (1-\xi)^2)^{N/2}} ds \leq \frac{\max \left((1 \pm e^{-1})^{b-1}\right) \int_{\pi}^{+\infty} \frac{(s^2 + 1)^{(1-b)/2}}{s^{N+1}} ds}{2\pi},$$

respectively. The integrals along the first two portions can be estimated similarly to (13) and (14). Hence, for $0 \leq \xi \leq t$ and $N \geq 1$, we have

$$|r_N(t)| \leq C_N(b)(t - \xi)^N,$$

where the constant $C_N(b)$ does not depend on $\xi$. Using this result in the integral representation (9), we can infer that

$$|R_N(a, b, x)| \leq C_N(b)F_N.$$

Finally, combining this result with the inequalities (6), we obtain the required remainder estimate in (2).

The reader can check that the function $f(t) = \left(\frac{1-e^{-t}}{t}\right)^{b-1}$ is a solution of the nonlinear differential equation

$$tf(t)f''(t) - \frac{b-2}{b-1} tf'^2(t) + (t + 2)f(t)f'(t) + (b-1)f^2(t) = 0.$$

If we substitute the Taylor series (4) into this differential equation and rearrange the result, we obtain the recurrence relation (5).

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