Abstract—The Wheeled Inverted Pendulum (WIP) is an underactuated, nonholonomic mechanical system, and has been popularized commercially as the Segway. Designing a control law for motion planning, that incorporates the state and control constraints, while respecting the configuration manifold, is a challenging problem. In this article we derive a discrete-time model of the WIP system using discrete mechanics and generate optimal trajectories for the WIP system by solving a discrete-time constrained optimal control problem, and describe a nonlinear continuous-time model with parameters for designing close loop LQ-controller. A dual control architecture is implemented in which the designed optimal trajectory is then provided as a reference to the robot with the optimal control trajectory as a feedforward control action, and an LQ-controller is employed to mitigate noise and disturbances for ensuing stable motion of the WIP system. While performing experiments on the WIP system involving aggressive maneuvers with fairly sharp turns, we found a high degree of congruence in the designed optimal trajectories and the path traced by the robot while tracking these trajectories; this corroborates the validity of the nonlinear model and the control scheme. Finally, these experiments demonstrate the highly nonlinear nature of the WIP system and robustness of the control scheme.

Index Terms—Wheeled inverted pendulum, optimal control, geometric control, discrete mechanics

I. INTRODUCTION

DESIGNING discrete-time control laws for mechanical systems subject to both state and control constraints, while preserving the configuration manifold of the system, is an extremely challenging problem. Existing control techniques, typically, use trial and error approaches based on prior experience to meet the constraints while the discretization procedure is (somewhat heuristically) a variant of RK4. A scheme for control synthesis in discrete-time that respects the manifold structure and is computationally tractable for the resulting discrete-time system while respecting the state and control constraints, is most desirable. This problem is addressed and implemented here in two steps on the wheeled inverted pendulum: First, the variational integrator for the mechanical system is derived using discrete mechanics [1] that preserves the manifold structure, and an open-loop control function is obtained by solving a discrete-time constrained optimal control problem using nonlinear programming techniques. Second, the optimal trajectory resulting from this open-loop strategy is tracked, in general, via a close loop tracking controller such as LQR, PID controller, etc.

The Wheel Inverted Pendulum (WIP) is a benchmark mechanical system that brings in considerable complexity due to its nonholonomic behavior and underactuation. In this article we derive a discrete-time model of the nonholonomic WIP system and synthesize an optimal control sequence considering both state and control constraints. The efficacy of the proposed control scheme is demonstrated through experiments.

The WIP, (see Figure 1), consists of a vertical body with two coaxial driven wheels. The system is underactuated since there are fewer actuating mechanisms (the drive on the wheels) than the number of configuration variables. In addition, the system has nonholonomic constraints that arise due to the pure rolling (without slipping) assumption on the wheels [2], [3] and the no side-slip condition. The WIP finds many applications that include baggage transportation, commuting and navigation [4]. The system has gained interest in the past several years due to its maneuverability and simple construction (see e.g. [5], [6]). Other robotic systems based on the WIP are fast becoming popular as well in the robotics community for human assistance and transportation as can be seen in the works of [7]–[10], and a commercially available...
model *Segway* for human transportation \[4\]. Various linear and nonlinear control techniques have been applied to the WIP ranging from LQR \[11\]–\[14\] to partial feedback linearization based nonlinear control \[2\], vision based tracking controller using partial feedback linearization \[15\], and vision based leader following control using adaptive control techniques \[16\]. A controllability analysis for the WIP kinematics is presented in \[17\] and filtering techniques to prevent the WIP leading to limit cycle while stabilizing are discussed in \[18\]. Recently, a nonlinear position and velocity stabilization controller using energy shaping technique has been proposed in \[19\], and modeling of the WIP as a linear system with time delays and its stabilization using an integral slide mode control may be found in \[20\]. A fairly detailed overview of the WIP modeling with various stabilization and tracking control techniques may be found in \[7\]. Existing control techniques mainly focus on stabilization of the system using some variant of linearization. These control techniques are inapplicable during aggressive and constrained maneuvers due to the fact that these techniques do not consider state and control constraints. Therefore, in challenging scenarios, the performance of these control schemes remains questionable. Note that the WIP models available in literature (see e.g. \[2\], \[3\], \[21\]) consider torques as control inputs instead of the physical inputs (voltage available to DC motors). During constrained motion planning scenarios, it is essential to consider voltage and current restrictions at the trajectory design stage which necessitates the modeling of the motor dynamics. In this article we address this issue by deriving a model of the WIP with motor dynamics in both continuous and discrete-time for constrained path planning. In contrast to our technique, constrained motion planning problems in stochastic framework are studied extensively in \[22\], \[23\] and references therein.

Our proposed technique differs from existing control techniques on the following accounts: A variational integrator is proposed that preserves the manifold structure, and in turn, it leads to accurate optimal trajectory design. In addition, we have included the motor dynamics of the system at the design stage to arrive at an accurate nonlinear model. In contrast to various stabilizing controllers proposed in literature, the main thrust of this article lies in the implementation of constrained reachability maneuvers on the WIP. We demonstrate through experiments that the proposed control technique for constrained maneuvers of WIP is efficient and easy to implement.

The article unfolds as follows: We present the model of the WIP system in Section \[II\]. Section \[III\] presents the discrete variational integrator of the WIP, and an optimal control problem is posed in discrete-time and solved using a nonlinear solver in Section \[III-A\]. Section \[IV\] is dedicated to setup description and followed by results and experiments.

A fairly detailed overview of nonholonomic systems in a geometric framework, in particular the nonholonomic connection, that bears particular relevance to the discrete Lagrange-D’Alembert-Pontryagin (LDAP) principle for deriving variational integrator of WIP is presented in Appendix \[A\] to Appendix \[D\]. System parameters of the experimental setup are in Appendix \[E\]. Nonlinear continuous-time WIP model and its discretization is discussed in Appendix \[F\] and Appendix \[G\] and design of the LQ-controller and observer may be found in Appendix \[H\].

II. A CONTINUOUS-TIME MODEL

The WIP consists of a body of mass \(m\) mounted on wheels of radius \(r_w\) and at a height \(l\) from the wheels axis of rotation. A pair of wheels, of mass \(m_w\) each, are mounted at the base of the body with a distance \(2d_w\) between them, and these wheels are able to rotate independently. The actuating mechanisms of the system, typically two separate motors, are fitted on the body in order to rotate the individual wheels and generate the tilting motion in the system. For these type of systems, one of the control objectives is to steer the system from a given initial configuration on \(x - y\) plane to a given final configuration with its body stabilized in the upward position.

The configuration variables of the system are:

- \((x, y) \in \mathbb{R}^2\): the coordinates of the origin of the body-fixed frame in the horizontal plane of the inertial frame, with \(x\) as the direction along the natural rolling motion;
- \(\theta \in \mathbb{S}^1\): the heading angle (angle of the wheel rotation axis with the \(x\)-axis or the \(y\)-axis in the inertial frame);
- \(\alpha \in \mathbb{S}^1\): the tilt angle of the body (angle of the body \(z\)-axis with the horizontal plane in the inertial frame);
- \(\phi_R \in \mathbb{S}^1\) and \(\phi_L \in \mathbb{S}^1\): the relative rotations of the individual wheels w.r.t. the body-fixed frame about the rotation axes of the corresponding wheels;
- \(q_R \in \mathbb{R}\) and \(q_L \in \mathbb{R}\): the charge on the right and left motor terminals. Their time derivatives are the currents flowing through the circuit of right and left electric motors fitted onboard to deliver torque to the wheels.

Based on this choice, the configuration space of the system is

\[
Q := SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R},
\]

with a state represented as

\[
q := (x, y, \theta, \alpha, \phi_R, \phi_L, q_R, q_L) \in Q.
\]

The system is subject to nonholonomic constraints that arise due to no-slip conditions on the wheels, i.e., no lateral sliding
and only pure rotation without slipping. Let \( (x_L, y_L) \in \mathbb{R}^2 \) be the left wheel’s position and \( (x_R, y_R) \in \mathbb{R}^2 \) be the right wheel’s position on the \( x - y \) plane in the spatial coordinates. Let \( \mathbb{R} \ni t \to q(t) \in Q \) denote a system trajectory. Then the pure rolling motion of the system is given by

\[
\begin{align*}
\dot{x}_R(t) \cos \theta(t) + \dot{y}_R(t) \sin \theta(t) &= r_w \phi_R(t), \\
\dot{x}_L(t) \cos \theta(t) + \dot{y}_L(t) \sin \theta(t) &= r_w \phi_L(t),
\end{align*}
\]

and the no side-slip constraints are given by

\[
\begin{align*}
-\dot{x}_R(t) \sin \theta(t) + \dot{y}_R(t) \cos \theta(t) &= 0, \\
-\dot{x}_L(t) \sin \theta(t) + \dot{y}_L(t) \cos \theta(t) &= 0.
\end{align*}
\]

For a system trajectory \( \mathbb{R} \ni t \to q(t) \in Q \), the pure rolling constraints (1) and no side-slip constraints (2) are defined in the configuration space by

\[
\begin{align*}
\dot{x}(t) \cos \theta(t) + \dot{y}(t) \sin \theta(t) + d_w \dot{\theta}(t) - r_w \phi_R(t) &= 0, \\
\dot{x}(t) \cos \theta(t) + \dot{y}(t) \sin \theta(t) - d_w \dot{\theta}(t) - r_w \phi_L(t) &= 0,
\end{align*}
\]

which are written in the compressed form as

\[
\begin{align*}
\dot{x}(t) - \frac{r_w}{2} \cos \theta(t) \left( \dot{\phi}_R(t) + \dot{\phi}_L(t) \right) &= 0, \\
\dot{y}(t) - \frac{r_w}{2} \sin \theta(t) \left( \dot{\phi}_R(t) + \dot{\phi}_L(t) \right) &= 0,
\end{align*}
\]

\[
\begin{align*}
\theta(t) - \frac{r_w}{2d_w} \left( \dot{\phi}_R(t) - \dot{\phi}_L(t) \right) &= 0.
\end{align*}
\]

A detailed discussion on the derivation of the nonlinear continuous-time model of the WIP is provided in Appendix D.

III. DISCRETE-TIME DYNAMICS OF THE WIP

For the purpose of emphasizing the key contribution of this article - the experimental validation of the discrete-time optimal control scheme - we move the theoretical details of arriving at the discrete variational integrator of the WIP to Appendix D. The key differential geometric notions related to nonholonomic systems are discussed there as well, and we apply tools from discrete mechanics to derive a discrete-time variational integrator of the WIP satisfying the nonholonomic constraints (3).

We launch directly into a discrete-time model of the WIP. Let \([N] := \{0, \ldots, N\}\). Let us define a path in discrete-time on the configuration space \( Q := G \times M \) as

\[ [N] \ni k \to \{ (s_k, v_{s_k}, g_k) := (s(t_k), v_s(t_k), g(t_k)) \in TM \times G, \]

where \( G = SE(2) \) is the Lie group and \( M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R} \) is the base manifold. The variational integrator for the WIP (derived by applying the LDAP principle (11) with the reduced Lagrangian \( L^* \) (17), local form of the connection \( \mathcal{A} \) (19), and the discrete control force \( F \) (20) in Appendix D) is

\[
\begin{align*}
g_{k+1} &= g_k e^{-h \mathcal{A}(s_k)v_{s_k}}, \\
s_{k+1} &= s_k + hv_{s_k}, \\
\frac{\partial L^*_k}{\partial v_s} - h \frac{\partial L^*_k}{\partial s} - (T e^{-1} ( - v_{s_k}) \circ \mathcal{A}(s_k))^* \left( \frac{\partial L^*_{k+1}}{\partial \xi} \right) &= 0, \\
\frac{\partial L^*_{k+1}}{\partial v_s} - (T e^{-1} (v_{s_{k+1}}) \circ \mathcal{A}(s_{k+1}))^* \left( \frac{\partial L^*_{k+1}}{\partial \xi} \right) + h \mathcal{F}_{k+1},
\end{align*}
\]

where \( h > 0 \) is the step length, \( e : g \to G \) is the exponential map, and

\[
T e^{-1} (v_{s_k}) := d e^{-1} (e(-h \mathcal{A}(s_k)v_{s_k})) \circ e(-h \mathcal{A}(s_k)v_{s_k}) \]

is the right translated tangent lift of the local diffeomorphism \( e^{-1} \), and

\[
L^*_{k+1} := L^*(s_{k+1}, s_k - s_k/h, e^{-1}(g_{k}^{-1}g_{k+1})/h)
\]

is the reduced Lagrangian. The selection procedure of \( h \) is discussed at length in Appendix B.

A few comments are in order here. (5a) governs the update of the system orientation and translation in the \( x - y \) plane for a motion in the base space \( M \), (5b) provides the update of the tilt, wheel angles, and the charge flowing in the motor circuit, and (5c) describes the dynamics on the base manifold \( M \). The calculations involved in the discrete-time model are as follows:

Let \( (g_k, s_k, v_{s_k}) \) be the states of the system at a discrete instant \( k \). Then the state at the \((k+1)\)th instant is computed in the following manner:

1. Compute the group (orientation and position of the system in \( x - y \) plane) update \( g_{k+1} \) using (5a) for given \( g_k \) and \( v_{s_k} \).
2. Compute the base configuration update \( s_{k+1} \) using (5b) for given \( s_k \) and \( v_{s_k} \).
3. If one substitutes \( s_{k+1} \) from (5b) in (5c), then (5c) is an implicit form in \( v_{s_{k+1}} \) for given states \( v_{s_k}, s_k \) and control torque \( \mathcal{F}_k \). This implicit form is further solved using Newton’s root finding algorithm.

The preceding discussion provides a discrete-time model of the controlled WIP. We move to a constrained optimal control problem in the context of (5).

A. Energy-optimal trajectory planning

As evident from the above, the system dynamics are highly nonlinear and the system is inherently unstable. Our objective is to generate an optimal trajectory of the system that passes through pre-specified points at pre-defined times while respecting state and control constraints along the way. Conventional path planning algorithms lack the ability to accommodate state and control constraints at the trajectory design stage while simultaneously minimizing a performance measure. In the technique proposed here, we design a constrained discrete-time optimal trajectory for the WIP system accounting for both
state and control constraints that are essential for fast nonlinear dynamics and safety critical systems. The constrained optimal trajectory designed offline is then tracked via an LQ-controller fine-tuned for the discrete-time model derived for the WIP system around zero. The dual controller architecture adopted here is better than conventional schemes due to the fact that it reduces the online computation time and is easy to implement.

We design a constrained trajectory by solving a discrete-time constrained optimal control problem in which the variational integrator of the WIP accounts for the system dynamics. This integrator is employed for the trajectory generation due to the fact that it is more accurate than conventional integration techniques \cite{19}, and it preserves system invariants like momentum, energy, etc. The optimal control objective is then to design a control-effort minimizing path to transport the WIP from a given fixed initial state to a given final state passing through \(N_m \leq N\) pre-specified configurations \(\{g_k\}_{k=1}^{N_m} \subset G\) and satisfying the following state and control constraints throughout its journey:

- (c-i) Input voltage \((u_R, u_L)\): \([-5, 5]\) V,
- (c-ii) Input voltage rate \((\dot{u}_R, \dot{u}_L)\): \([-2, 2]\) V s\(^{-1}\),
- (c-iii) Motor current \((v_{qL}, v_{qR})\): \([-3, 3]\) A,
- (c-iv) Tilt angle \((\alpha)\): \([-15, 15]\)°,
- (c-v) Turn angle rate \((v_{\theta})\): \([-120, 120]\) deg/s.

The pre-specified intermediate points on paths are uniformly chosen for the path planning. In our experiments, the WIP follows a figure eight knot and a certain zig-zag path. In particular, seven intermediate points are prescribed on the eight knot at a distance of 1.41 m between them and seventeen intermediate points are prescribed on the zig-zag path at a distance of 0.35 m between them.

The discrete-time optimal control problem for the variational integrator \((5)\) with control constraints \((c-i)-(c-ii)\) and state constraints \((c-iii)-(c-v)\) is given by

\[
\text{minimize } J(u) := \frac{1}{2} \sum_{k=0}^{N-1} (u_R)_k^2 + (u_L)_k^2
\]

subject to

\[
\text{system dynamics } (5) \quad \text{for } k \in [N - 1],
\]
\[
-5 \leq (u_R)_k, (u_L)_k \leq 5
\]
\[
-3 \leq (v_{qL}), (v_{qR}) \leq 3
\]
\[
-\frac{\pi}{12} \leq \alpha_k \leq \frac{\pi}{12}
\]
\[
\frac{2\pi}{3} \leq (v_{\theta})_k \leq \frac{2\pi}{3}
\]
\[
-0.05 \leq (u_R)_{k-1} - (u_R)_k \leq 0.05
\]
\[
-0.05 \leq (u_L)_{k-1} - (u_L)_k \leq 0.05
\]
\[
g_k = g_j, \text{ if } k = k_j, \text{ for any } j = 1, \ldots, N_m
\]
\[
\text{for } k = 1, \ldots, N - 1,
\]
\[
(\tilde{g}_0, \tilde{\alpha}_0, (\tilde{v}_N)_0) = (g_0, \alpha_0, (v_N)_0)
\]
\[
(\tilde{g}_N, \tilde{\alpha}_N, (\tilde{v}_N)_N) = (g_N, \alpha_N, (v_N)_N)
\]

where \(\{g_k\}_{k=1}^{N_m} \subset G\) are fixed.

**Remark 1.** Note that the optimal trajectories are synthesized by solving the discrete-time optimal control problem \((6)\) using IPOPT solver \cite{20} that is integrated into MATLAB with the help of a symbolic computation toolbox CasADi \cite{21}. Although, we are using an direct technique for solving the optimal control problem \((6)\) here, there is also the alternative of employing an indirect method by utilizing the techniques in \cite{22}; arguably, the latter may be more accurate \cite{23}.

The figures *eight knot* and *zig-zag* paths are employed as benchmarks to test our numerical algorithms those are used as benchmark to test algorithms in control literature. The following optimization parameters have been used for simulating the constrained optimal trajectories:

1. Step length \((h) = 25 \cdot 10^{-3}\) sec,
2. Final time \((T)\):
   1. The figure *eight knot*: 19.25 sec,
   2. The *zig-zag path*: 17.6 sec,
3. Total number of Steps \((N = T/h)\)
   1. The figure *eight knot*: 770,
   2. The *zig-zag path*: 704.

These simulated trajectories have been further validated by experiments.

**IV. Experiments**

The WIP (see Figure 1) is an experimental setup developed at TU Munich\(^2\) for research \cite{30}, teaching and demonstration purposes. The model parameters (see Table I) have been derived from the CAD model of the robot and further validated by conducting experiments on the setup. The simulation results of the identified model demonstrate a high degree of congruence with the experimental data.

**A. WIP system description**

The WIP weights 333 g, has a height of 195 mm and a width of 103 mm. Most of the parts of the robot are 3D printed. The wheels of the robot are driven by two brushed 6W DC electric motors mounted on the main body of the robot. The motors are connected to wheels via two stage gears with the total gear ratio from the motor shafts to the wheels equal to 50:28. The motors are connected to two H-Bridges motor driver DRV8835 from Texas Instruments, which limit the motor current to a maximum of 3 A. A lithium polymer battery of 7.4 V nominal voltage is fitted onboard to provide energy to the sensors, electronics, and motor drives. The electronic circuit board fitted onboard comes with a 32-bit microcontroller AT32UC3C1512C from Atmel that runs at 66 MHz, a Bluetooth\textsubscript{®} module to bridge the communication between the PC and the microcontroller, a 3-axis accelerometer ADXL345 from Analog Devices to measure the body acceleration and the acceleration due to gravity, and a 3-axis gyroscope ITG-3050 from InvenSense to measure the angular rate of the body. In addition, two optical encoders of 900 cpr are fitted on each wheel to measure the relative differences between the body tilt angle \(\alpha\) and the wheel angles \(\phi_R \text{ and } \phi_L\).

\textsuperscript{2}The WIP shown in Figure 1 has been developed, build and modelled by Klaus Albert together with several students writing their term- and masters thesis on the project.
and these encoders are evaluated by two quadrature decoders on microcontrollers which leads to an effective resolution of 3600 cpr. The onboard sensors provide the rate measurements and the body tilt angle measurements. However, the absolute measurements of the robot position and its orientation on the $x - y$ plane is not possible with the on-board sensors, and an optical tracking system, Vicon with 10 Vera v1.3 cameras covering a tracking area of $4 \times 6.5 \text{ m}$, provides the position and the orientation measurements via Bluetooth® to the robot controller. The Vicon system runs at a sampling rate of 40 Hz and the controller generates its digital control sequences for the motor drives at a sampling rate of 5 ms.

### B. Experimental results

The trajectory tracking system for the WIP consists of a feedforward input computed from an optimal state-action trajectory that is generated offline by solving a discrete-time optimal control problem (6), and an onboard closed loop input computed from an LQ-controller to mitigate unaccounted disturbances and to maintain stability of the system; this is shown in Figure 3. The derivation of the linear discrete-time model may be found in Appendix [F] and Appendix [G] and the design of the LQ-controller, the observer and the guidance algorithm may be found in Appendix [H].

We experimented with two trajectories: the figure eight knot and a zig-zag path. During the experiments, both trajectories were concatenated to allow the robot to transit from one trajectory to another smoothly. For demonstration purposes, we have truncated the state-action trajectory to $27.5 \text{ s}$ in which the robot moves along the figure eight knot starting at $(2, 0.5)$ in the $x - y$ plane and switches to the zig-zag path at the location $(5, 1.5)$ in $x - y$ plane as shown in Figure 4. We have included a supplementary MPEG-4 video file that contains a video of the experiments along with a synchronized animation of Figure 4 and it is available at https://youtu.be/Vw7vco-Rdrw.

The corresponding optimal control profile for the maneuver is shown in Figure 5. It is evident from Figure 4 that the robot follows the reference trajectory very well in linear motion and deviates from the reference while executing sharp turns at high speed due to unmodeled disturbances, sensor noise, and communication delays. The robot moves on the straight lines with velocities up to $0.7 \text{ m/s}$ followed by $90^\circ$ turns and zig-zag curves with high yaw rates up to $2\pi \text{ rad/s}$ at velocities around $0.4 \text{ m/s}$. The maximum error (observed during the experiments) in the position $(x, y)$ is $4 \text{ cm}$ and in orientation $(\theta)$ is $17^\circ$. The position and the orientation of the robot are estimated based on the data received from the Vicon tracking system. The difference in the sampling rate of the controller (5 ms) and the Vicon system (25 ms) necessitate the robot observer to predict the robot position and the orientation without correction steps unless the position and orientation measurements are received from the tracking system.

As mentioned above, a LQ-controller is employed for the stabilization of the system and compensation of disturbances due to system nonlinearities, friction, and communication delays over the network. Note that the system is highly nonlinear and stabilization of the tilt angle $\alpha$ (see Figure 6) requires sharp control responses from the controller as shown in Figure 8. In order to execute a fast forward motion, the robot tilts up to $6^\circ$ during maneuvers (see Figure 6) and the tilt rate goes up to $60^\circ \text{ s}^{-1}$. During the maneuvers with high tilt angle and tilt rate (see Figure 7) the controller response reaches $1.2 \text{ V}$ for maintaining stability, and the controller response remains nearly zero during the stable motion, i.e., when the tilt angle and tilt rate are nearly zero (see Figure 8).

### V. Conclusion

In this article, we derived a discrete-time model of the WIP system using a structure preserving discretization scheme and generated optimal trajectories for the robot by solving a discrete-time constrained optimal control problem. We then conducted experiments in which the optimal state trajectory
is provided as a reference to the robot with the optimal control trajectory as a feedforward control action and found a high degree of congruence in the optimal trajectory and the estimated trajectory of the robot. These experiments establish the validity of the proposed model and the proposed tracking control strategy. Finally, these experiments throw light on the validity of the proposed model and the proposed tracking control trajectory as a feedforward control action and found

**Fig. 7: Optimal tilt rate $\dot{v}_a$ and estimated tilt rate $\hat{\dot{v}}_a$.**

Lagrangians, followed by a discussion on the nonholonomic connection and its local form. We give a catalog of concepts from classical mechanics below; a wealth of information about the geometry of nonholonomic systems may be found in [31]–[33].

1) **Lagrangian and constrained distributions:** Let $Q$ be the configuration space of a nonholonomic mechanical system and let $G$ be a Lie group. Suppose

$$G \times Q \ni (\bar{g}, q) \mapsto \Phi_{\bar{g}}(q) \in Q$$

be a group action of the Lie group $G$ on the manifold $Q$. Then the space of symmetries at a given configuration $q \in Q$ is the orbit of $G$:

$$\text{Orb}_G(q) := \{ \Phi_{\bar{g}}(q) \mid \bar{g} \in G \},$$

and it is a submanifold [31] p. 107] of $Q$. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$ and

$$\xi_Q(q) := \frac{d}{de} \bigg|_{e=0} \Phi_{e \xi}(q)$$

be the infinitesimal generator of $\xi \in \mathfrak{g}$. Then the tangent space of the orbit at a point $q$ is given as

$$T_q \text{Orb}_G(q) = \{ \xi_Q(q) \mid \xi \in \mathfrak{g} \}.$$

Let $TQ \ni (q, v_q) \mapsto L(q, v_q) \in \mathbb{R}$ be the Lagrangian of the nonholonomic system with a regular distribution of satisfying nonholonomic constraints.

The following assumptions, standard in the literature [31]–[33], have been imposed throughout:

(A-i) The Lagrangian $L$ is invariant under the group action $\Phi$, i.e., for all $\bar{g} \in G$ and $q \in Q$,

$$L(q, v_q) = L(\Phi_{\bar{g}}(q), T_q \Phi_{\bar{g}}(v_q)).$$

(A-ii) The distribution $\mathcal{D}$ is invariant under the group action, i.e., the subspace $\mathcal{D}_q \subset T_q Q$ is translated under the tangent lift of the group action to the subspace $\mathcal{D}_{\Phi_{\bar{g}}(q)} \subset T_{\Phi_{\bar{g}}(q)} Q$ for all $\bar{g} \in G$ and $q \in Q$.

(A-iii) For each $q \in Q$, $T_q Q = \mathcal{D}_q + T_q \text{Orb}_G(q)$.

Assumption (A-i) is the key property needed to define the reduced Lagrangian below and (A-ii) is necessary to define the local form of the nonholonomic connection that is discussed in Appendix A2.

Let a principal fiber bundle [34] $Q := G \times M$ be the configuration space of a mechanical system with $G$ as a Lie group, and $M$ as a manifold that defines the shape space or the base manifold. Let $q := (g, s)$ be a configuration on the manifold $G \times M$. Then the reduced Lagrangian is defined as

$$TM \times \mathfrak{g} \ni (s, v_s, \xi) \mapsto L^g(s, v_s, \xi)$$

$$:= L((e, s), (T_g \Phi_{g^{-1}}(v_g), v_s)) \in \mathbb{R}, \quad (7)$$

where

$$\xi = T_g \Phi_{g^{-1}}(v_g) \in \mathfrak{g}.$$
2) Nonholonomic connection (see [34], [32] for details.): Let $\mathcal{V}_q$ be the space of tangent vectors parallel to the symmetric directions (i.e., the vertical space), $D_q$ be the space of velocities satisfying the nonholonomic constraints at a given configuration $q$, $S_q$ be the space of symmetric directions satisfying nonholonomic constraints, and $H_q$ be a space of tangent vectors satisfying nonholonomic constraints but not aligned with the symmetric directions. Then these subspaces of $T_q Q$ are identified as

$$V_q = T_q \text{Orb}_G(q), \quad S_q = \mathcal{V}_q \cap D_q, \quad D_q = S_q \oplus H_q.$$ 

**Definition.** A principal connection $\mathcal{A} : TQ \to g$ is a Lie algebra valued one form that is linear on each subspace and satisfies the following conditions:

1) $\mathcal{A}(q) \cdot \xi_Q(q) = \xi$, $\xi \in g$, and $q \in Q$,

2) $\mathcal{A}$ is equivariant:

$$\mathcal{A}(\Phi_g(q)) \cdot T_q \Phi_g(v_q) = \text{Ad}_g (\mathcal{A}(q) \cdot v_q)$$

for all $v_q \in T_q Q$ and $g \in G$, where $\Phi_g$ denotes the group action of $G$ on $Q$ and $\text{Ad}_g$ denotes the adjoint action of $G$ on $g$.

The principal connection determines a unique Lie algebra element corresponding to a tangent vector $v_q \in T_q Q$. For a given vertical space $\mathcal{V}_q$ and a horizontal space $H_q$, a vector $v_q \in T_q Q$ can be uniquely represented as $v_q = \text{ver}(v_q) + \text{hor}(v_q)$, where $\text{ver}(v_q) \in \mathcal{V}_q$ and $\text{hor}(v_q) \in H_q$. By the definition of the principal connection,

$$\mathcal{A}(q) \cdot \text{ver}(v_q) = \xi,$$

where $\xi \in g$ is the unique Lie algebra element associated with the vertical component $\text{ver}(v_q)$, i.e., $\text{ver}(v_q) = \xi_Q(q) \in T_q Q$ for some $\xi \in g$. Consequently, the connection evaluates to zero on the horizontal component $\text{hor}(v_q)$, i.e.,

$$\mathcal{A}(q) \cdot \text{hor}(v_q) = 0.$$

If the configuration space is a principle fiber bundle $Q = G \times M$, the principle connection admits a local form $\& : TM \to g$ such that the principle connection in terms of the local form is given by [34]

$$\mathcal{A}(q) \cdot v_q = \text{Ad}_g \left( g^{-1}(v_q) + \& (s) v_s \right)$$

for all $q := (g, s) \in Q$ and $v_q := (v_g, v_s) \in T_q Q$, where $g^{-1}(v_q)$ is the tangent lift of the left action of $g^{-1}$ on $g \in G$, and $\text{Ad}_g : g \to g$ is given by

$$\text{Ad}_g(\xi) := \frac{d}{de} \bigg|_{e=0} g e^{\xi} g^{-1} \quad \text{for all} \ \xi \in g.$$

For mechanical systems evolving on principle fiber bundles, in general, the base space $M$ corresponds to the set of configurations that are directly controlled by the control forces. Hence, a path on the base space can be followed by applying these forces. A path on the fiber space $G$ is constructed by fiber velocities at given fiber configurations. These fiber velocities are uniquely related to the nonholonomic momentum and the base velocities via a nonholonomic connection. Let us pick, for $q \in Q$, a vector subspace $U_q \subset \mathcal{V}_q$ such that

$$\mathcal{V}_q = S_q \oplus U_q,$$

where $S$ is a distribution consists of the symmetric horizontal directions.

**Definition** ([34], Definition 6.2 on p. 38]). Assume that the Assumption (A-iii) holds. Then the nonholonomic connection $A^{\text{nhc}} : TQ \to \mathcal{V}$ is a vertical valued one form whose horizontal space at $q \in Q$ is the orthogonal complement of the subspace $S_q$ in $D_q$ and satisfies the following:

$$A^{\text{nhc}} := A^{\text{kin}} + A^{\text{sym}},$$

where $A^{\text{kin}} : TQ \to U$ is the kinematic connection enforcing nonholonomic constraints and $A^{\text{sym}} : TQ \to S$ is the mechanical connection corresponding to symmetries in the constrained direction.

The kinematic connection $A^{\text{kin}}$ and the mechanical connection $A^{\text{sym}}$ satisfy the following conditions:

$$A^{\text{kin}}(q) \cdot v_q = 0 \quad \text{for all} \ v_q \in D_q,$$

$$A^{\text{sym}}(q) \cdot v_q = v_q \quad \text{for all} \ v_q \in S_q.$$

**Remark 2.** If the distribution $S_q$ and the horizontal distribution are invariant under the group action, then the nonholonomic connection is a principal connection.

In case the nonholonomic connection is a principal connection, the connection is represented as

$$g^{-1}(v_q) + \& (s) v_s = \text{Ad}_g (\Omega),$$

where

$$\Omega \in \{ \xi \in g \mid \xi_Q(q) \in S_q \}$$

is the **locked angular velocity**, i.e., the velocity achieved by locking the joints represented by the base configuration variable. This local form of the nonholonomic connection can be written as

$$g^{-1}(v_q) + \& (s) v_s = \Omega.$$  \quad (8)

For the principal kinematic case, i.e., $S_q = D_q \cap \mathcal{V}_q = \{0\}$ for all $q \in Q$, the local form of the nonholonomic connection [35] simplifies to

$$g^{-1}(v_q) + \& (s) v_s = 0.$$  \quad (9)

Therefore, for a smooth curve $\mathbb{R} \ni t \mapsto (g(t), v(t)) \in G \times M$, the group motion can be constructed by the nonholonomic connection for a given base trajectory as

$$\dot{g}(t) = -g(t)\& (s(t)) \dot{s}(t).$$

**B. Discrete-time variational integrator**

Equipped with necessary geometric notions, in particular, the reduced Lagrangian and the local form of the nonholonomic connection, we are in a position to state the discrete-time reduced Lagrange-D’Alembert-Pontryagin nonholonomic principle [32] to derive the variational integrators for nonholonomic principal kinematic systems.

Recall that $[N] := \{0, \ldots, N\}$. Define a discrete path

$$[N] \ni k \mapsto (s_k, v_{s_k}, g_k) := (s(t_k), v_s(t_k), g(t_k)) \in TM \times G,$$

where $\mathcal{S}$ is a distribution consists of the symmetric horizontal directions.

$$A^{\text{nhc}} := A^{\text{kin}} + A^{\text{sym}},$$

where $A^{\text{kin}} : TQ \to U$ is the kinematic connection enforcing nonholonomic constraints and $A^{\text{sym}} : TQ \to S$ is the mechanical connection corresponding to symmetries in the constrained direction.

The kinematic connection $A^{\text{kin}}$ and the mechanical connection $A^{\text{sym}}$ satisfy the following conditions:

$$A^{\text{kin}}(q) \cdot v_q = 0 \quad \text{for all} \ v_q \in D_q,$$

$$A^{\text{sym}}(q) \cdot v_q = v_q \quad \text{for all} \ v_q \in S_q.$$
on the reduced space that satisfies the following constraints
\[ s_{k+1} - s_k = hv_{sk}, \quad g_{k+1} = g_k \varphi(-h\dot{A}(s_k)v_{sk}), \]
where \( h \) is the time difference between any two consecutive configurations, i.e., \( t_{k+1} - t_k = h \), and the map \( \varphi : g \rightarrow G \) represents the difference between two system configurations defined by Lie group elements by a unique element in its Lie algebra.

In most of the cases, \( \varphi \) is taken to be the exponential map \( e : g \rightarrow G \) that is a diffeomorphism in the neighborhood \( O_e \subset G \) of the group identity \( e \in G \) [35, p. 256]. The map \( e \) serves the purpose of \( \varphi \) because the consecutive group configurations \( g_k \) and \( g_{k+1} \) do not differ by a large value, i.e., \( g_k^{-1}g_{k+1} \in O_e \subset G \) for any discrete-time instant \( k \). Further, the discrete control force
\[ [N] \ni k \mapsto \tau_k := \tau(t_k) \in T^* M \]
is an approximation of the continuous-time force \( \tau \) controlling the shape of the dynamics.

**Definition.** The Discrete Reduced LDAP Principle for Principal Kinematic Systems
\[ \delta \sum_{k=0}^{N-1} L^k \left( s_k, \frac{s_{k+1} - s_k}{h}, \varphi^{-1}(g_k^{-1}g_{k+1}) \right) + (\tau_k, s_k) = 0, \]
subject to
nonholonomic constraints: \( g_{k+1} = g_k \varphi(-h\dot{A}(s_k)v_{sk}), \)
and,
horizontal variations: \( (g_k^{-1}g_{k+1}) = (-\dot{A}(s_k)g_{sk}, \delta s_k). \)

The discrete reduced LDAP principle leads to the following sets of discrete-time equations:
\[ s_{k+1} = s_k + hv_{sk}, \]
\[ g_{k+1} = g_k \varphi(-h\dot{A}(s_k)v_{sk}), \]
\[ \frac{\partial L^k}{\partial v_{sk}} - h \frac{\partial L^k}{\partial s} - (\varphi_{v_{sk}} \circ \dot{A}(s_k)) \varphi^{-1}(g_{k+1}^{-1}g_k) \]
\[ = \frac{\partial L^k}{\partial v_{sk}} - (\varphi_{v_{sk}} \circ \dot{A}(s_k)) \varphi^{-1}(g_{k+1}^{-1}g_k) + h\tau_{sk-1}, \]
subject to
nonholonomic constraints: \( g_{k+1} = g_k \varphi(-h\dot{A}(s_k)v_{sk}), \)
and,
horizontal variations: \( (g_k^{-1}g_{k+1}) = (-\dot{A}(s_k)g_{sk}, \delta s_k). \)

\[ T_b(q, \dot{q}) = \frac{1}{2}(m_b v_b^T v_b + \omega_b^T I_b \omega_b) \]

Analogously, to calculate the kinetic energy of the wheels, let \( v_{w,R}, v_{w,L} \) be the translational velocity of the right and the left wheel’s center of mass respectively, and let \( \omega_{w,R}, \omega_{w,L} \) be the angular velocity of the wheels with \( I_w := \text{diag}(I_{wxx}, I_{wy}, I_{wzz}) \) the inertia of the wheels with respect to its center of mass in the body-fixed frame. Then the kinetic energy of the wheels is given by
\[ T_w(q, \dot{q}) = \frac{1}{2}(m_w v_{w,R}^T v_{w,R} + \omega_{w,R}^T I_{w} \omega_{w,R}) + \frac{1}{2}(m_w v_{w,L}^T v_{w,L} + \omega_{w,L}^T I_{w} \omega_{w,L}) \]

where
\[ v_{w,R} = \begin{pmatrix} \dot{x} - d_w \dot{\alpha} \cos \theta & \dot{y} - d_w \dot{\cos \theta} \\ 0 & 0 \end{pmatrix} \]
and
\[ v_{w,L} = \begin{pmatrix} \dot{x} + d_w \dot{\alpha} \cos \theta & \dot{y} + d_w \dot{\cos \theta} \\ 0 & 0 \end{pmatrix}. \]

1) Lagrangian of the WIP: In order to define the Lagrangian of the system, let us calculate its kinetic energy \( T \) and potential energy \( V \). We start by independently calculating the kinetic energy of each subsystem. Let \( v_b \) be the translational velocity of the center of mass and \( \omega_b \) be the angular velocity of the body with \( I_B := \text{diag}(I_{Bxx}, I_{Byy}, I_{Bzz}) \) as the inertia of the main body with respect to its center of mass in the body-fixed frame. Then the kinetic energy of the main body is given by
\[ T_b(q, \dot{q}) = \frac{1}{2}(m_b v_b^T v_b + \omega_b^T I_B \omega_b) \]

where
\[ v_b := \begin{pmatrix} \dot{x} + l \alpha \cos \alpha \cos \theta - l \dot{\alpha} \sin \alpha \sin \theta \\ \dot{y} + l \alpha \cos \alpha \sin \theta + l \dot{\alpha} \sin \alpha \cos \theta \\ -l \dot{\alpha} \sin \alpha \end{pmatrix}. \]

\[ \omega_b := \begin{pmatrix} \dot{\alpha} \cos \alpha \alpha & \dot{\alpha} \cos \alpha \theta \end{pmatrix}. \]

We know that the electric motors and the gears rotate at different angular speeds compared to the wheels. Therefore, the rotational energy arising from their relative motion with respect to the body has to be calculated in addition to the above. Let \( n_{w,R} \) be the transmission ratio from wheel shaft to the gear shaft and \( n_{w,L} \) be the transmission ratio from the wheel shaft to the motor shaft. The kinetic energy terms due to the relative motion of the gears and the rotors are given by
\[ T_g(q, \dot{q}) = \frac{1}{2}I_M (\dot{\alpha} + n_{w,R}(\dot{\phi}_R - \dot{\alpha}))^2 + \frac{1}{2}I_G (\dot{\alpha} - n_{w,R}(\dot{\phi}_R - \dot{\alpha}))^2 + \frac{1}{2}I_M (\dot{\alpha} + n_{w,L}(\dot{\phi}_L - \dot{\alpha}))^2 + \frac{1}{2}I_G (\dot{\alpha} - n_{w,L}(\dot{\phi}_L - \dot{\alpha}))^2, \]

where \( I_M \) and \( I_G \) are the moments of inertia of the rotor and the gear about their rotation axis respectively. To incorporate the motor dynamics, the kinetic energy of motor circuits [36] is defined by
\[ T_m(q, \dot{q}) = \frac{1}{2}L_m (\dot{q}_R^2 + \dot{q}_L^2), \]
where \( L_m \) is the rotor inductance. Therefore, the total kinetic energy of the system is given by

\[ T = T_b + T_w + T_g + T_m. \]

The potential energy of the system is due to the gravitational potential of the body and the potential due to the back EMF of the motor circuits, given by

\[ V(q, \dot{q}) = m_g g \cos \alpha + k_e u_{wm}(\dot{\omega}_R - \dot{\omega}) q_R + (\dot{\omega}_L - \dot{\omega}) q_L. \]

The Lagrangian of the WIP is the total kinetic energy minus the potential energy

\[ TQ \ni (q, \dot{q}) \rightarrow L(q, \dot{q}) = T(q, \dot{q}) - V(q, \dot{q}) \in \mathbb{R}. \quad (12) \]

2) Dissipative and external forces: The generalized dissipative forces are friction forces between the robot body and the wheels due to the gears and bearing, and the motors losses due to the resistive elements. Let \( F_{\text{inc}} \) be the dissipative force applied along the generalized coordinates \((\alpha, \phi_L, \phi_R)\) and is given by

\[ F_{\text{inc}}(q, \dot{q}) := (\text{fric}_R + \text{fric}_L - \text{fric}_R - \text{fric}_L) \top, \]

where

\[ \text{fric}_R := d_e (\dot{\omega}_R - \dot{\omega}) + d_c \tanh (d_0 (\dot{\omega}_R - \dot{\omega})), \]

and

\[ \text{fric}_L := d_e (\dot{\omega}_L - \dot{\omega}) + d_c \tanh (d_0 (\dot{\omega}_L - \dot{\omega})). \]

The friction loses of the gears and bearing are obtained by an approximation of a typical Coulomb and viscous friction curve. Let \( R_m \) be the resistance of the motor circuit. The potential drop in the motor circuits due to the resistance is the dissipative force along the generalized coordinates \( q_L, q_R \), and is given by

\[ F_{\text{loss}}(q, \dot{q}) := (-R_m\dot{q}_R - R_m\dot{q}_L) \top. \]

The external force applied to the system is the voltage available to the motors by the batteries. Let \( u_R, u_L \) be the voltage supplied by the batteries to right and left motors respectively. Therefore the external force applied along the generalized coordinates \((q_L, q_R)\) is given by

\[ F_{\text{ext}}(u_R, u_L) := (u_R - u_L) \top. \]

The net dissipative force and external force applied to the system is given by

\[ F(q, \dot{q}, u_R, u_L) = \begin{pmatrix} F_{\text{inc}}(q, \dot{q}) \\ F_{\text{loss}}(q, \dot{q}) + F_{\text{ext}}(u_R, u_L) \end{pmatrix}. \quad (13) \]

D. Variational integrator of WIP

We remind our readers that the key geometric concepts needed to derive variational integrators for principal kinematic systems are the reduced Lagrangian \([7]\) and the local form of the nonholonomic connection \([9]\). Therefore, to begin with, we establish that the WIP is a principal kinematic system, and further, derive the reduced Lagrangian and the local form of the nonholonomic connection to apply the LDAP principle \([11]\).

To establish that the WIP is a principle kinematic system, let us define a group action and prove that the vertical space and the constrained distribution at a given configuration have only zero in common as discussed in Appendix A. With \( G = \text{SE}(2) \) as the Lie group, the configuration space \( Q \) of the WIP system can be written in the trivial bundle form as

\[ Q = G \times M := \text{SE}(2) \times (S^1 \times S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}). \]

Therefore, with \( s := (\alpha, \phi_L, \phi_R, q_R, q_L) \in M \) and \( g := (x, y, \Theta) \in G \), the system configuration is defined by \( q := (g, s) \in Q \), and a tangent vector at \( q \) is defined by

\[ v_g = (v_g, v_s) \in T_Q q. \]

where \( v_g := (v_x, v_y, v_\Theta) \) and \( v_s := (v_\alpha, v_{\phi_L}, v_{\phi_R}, v_{q_R}, v_{q_L}) \).

- **Group action** (see Appendix A): The map \( \Phi : G \times Q \rightarrow Q \) is the group action of the Lie group \( G \) on the manifold \( Q \) and for \( \bar{g} := (X, Y, \Theta) \in G \), the group action \( \Phi \) is defined in (coordinates) by

\[ \Phi_{\bar{g}}(q) = (X + x \cos \Theta - y \sin \Theta, Y + x \sin \Theta + y \cos \Theta, \Theta + \theta, \alpha, \phi_L, \phi_R, q_R, q_L). \quad (14) \]

- **Vertical Space** (see Appendix A): The vertical space for the system is given by

\[ \mathcal{V}_q = \left\{ \left. \frac{d}{d\gamma(t)} \right|_{\gamma = 0} (\gamma(t), s) \right| \gamma(t) = e^{t\xi} g \in G, \gamma(0) = g, \gamma(0) = v_g, \xi \in \mathfrak{g}, s \in M \right\} = \{(v_g, 0) \in T_Q G \times T_s M\}. \]

For a given local representation of the tangent vectors \( v_g := (v_x, v_y, v_\Theta) \in T_Q G \), the local basis of the vertical space \( \mathcal{V}_q \) is given by

\[ \mathcal{V}_q = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi_L}, \frac{\partial}{\partial \phi_R}, \frac{\partial}{\partial q_R}, \frac{\partial}{\partial q_L} \right\}. \]

- **Constrained distribution**: The distribution \( \mathcal{D} \) satisfying nonholonomic constraints \([10]\) is called the constrained distribution. The local generator (a collection of linearly independent vector fields spanning the distribution) of the constrained distribution \( \mathcal{D}_q \) satisfying the nonholonomic constraints \([10]\) is given by

\[ \mathcal{D}_q = \text{span}\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\} \]

where

\[ \mathcal{X}_1 = \cos \theta \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y} + \frac{1}{r_w} \frac{\partial}{\partial \phi_R} + \frac{1}{r_w} \frac{\partial}{\partial \phi_L}, \]

\[ \mathcal{X}_2 = \frac{\partial}{\partial \phi_R}, \quad \mathcal{X}_3 = \frac{\partial}{\partial \phi_L} \]

Thus, it can be seen that

\[ \mathcal{S}_q = \mathcal{V}_q \cap \mathcal{D}_q = \{0\}. \]

The class of systems for which \( \mathcal{S}_q = \{0\} \) falls into a special category, known as principal kinematic systems, in which the tangential directions along symmetry are independent of the constrained tangential directions \([32]\).
1) **Reduced Lagrangian:** The tangent lift of the group action $\Phi_g$ is defined in coordinates by

$$T_g \Phi_g(v_q) = \left( v_x \cos \Theta - v_y \sin \Theta, v_x \sin \Theta + v_y \cos \Theta, v_\theta, v_{\phi R}, v_{\phi L}, v_{q_R}, v_{q_L} \right).$$  \hfill (15)

Let $TM \times g \ni (s,v_s,\xi) := \left( \Phi_{g^{-1}}(q), T_g \Phi_{g^{-1}}(v_q) \right)$ be a point on the reduced space, where

$$\xi := (v_x \cos \theta + v_y \sin \theta, -v_x \sin \theta + v_y \cos \theta, v_\theta),$$

$$v_s := (v_\alpha, v_{\phi R}, v_{\phi L}, v_{q_R}, v_{q_L}).$$  \hfill (16)

Then the reduced Lagrangian is defined by

$$L^s(s,v_s,\xi) := L(\Phi_{g^{-1}}(q), T_g \Phi_{g^{-1}}(v_q)) \in \mathbb{R}.$$  \hfill (17)

2) **Local nonholonomic connection:** We know from (16) that

$$\xi = (v_x \cos \theta + v_y \sin \theta, -v_x \sin \theta + v_y \cos \theta, v_\theta).$$  \hfill (18)

With our current convention, the $q(t) \in T_qQ$ in (11) is defined by

$$\dot{q}(t) := (\dot{x}(t), \dot{y}(t), \dot{\theta}(t), \dot{\phi}_R(t), \dot{\phi}_L(t)), \dot{q}_R(t), \dot{q}_L(t)),$$

and further, substituting the value of $v_x, v_y, v_\theta$ from (19) into (18) we obtain the local form of the nonholonomic connection as

$$\xi + \mathbf{A} v_s = 0,$$

where

$$\mathbf{A} := \frac{1}{2} \begin{pmatrix} 0 & -r_w & -r_w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_w & 0 \end{pmatrix},$$  \hfill (19)

$$v_s := (v_\alpha, v_{\phi R}, v_{\phi L}, v_{q_R}, v_{q_L})^T.$$

3) **Variational integrator of WIP:** Collecting the definitions of the reduced Lagrangian (17) and the local form of the nonholonomic connection (19), we apply the LDAP principle (11) with the discrete-time control force (see (13))

$$[N] \ni k \mapsto F_k := \mathcal{F}(q(t_k), v_q(t_k), u_R(t_k), u_L(t_k)) \in T^*M$$

(20)

to arrive at the variational integrator of the WIP defined by (5).

### E. Model parameters

#### F. Nonlinear State Space Model

The nonlinear state space model of the WIP is derived using first order modeling as discussed in [37]. We derive a more comprehensive WIP model than the existing literature [21], [37] based on the fact that we have included the dynamics of the currents at the modeling stage instead of modeling it as a separate system and then connecting it to the mechanical system. Let

$$A := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 & 0 \\ \cos(\theta) & \sin(\theta) & d_w & 0 & -r_w & 0 & 0 \\ \cos(\theta) & \sin(\theta) & -d_w & 0 & 0 & -r_w & 0 \end{pmatrix}.$$

#### TABLE I: WIP model parameters

| Symbol | Value | Description |
|--------|-------|-------------|
| $g$    | 9.81 kg s$^{-2}$ | gravity constant |
| $m_b$  | 277 · 10$^{-3}$ kg | body mass |
| $I_B$  | 543.108 · 10$^{-6}$ kg m$^2$ | body inertia |
| $I_{Bx}$ | 481.457 · 10$^{-6}$ kg m$^2$ | about the y-axis |
| $I_{Bz}$ | 153.951 · 10$^{-6}$ kg m$^2$ | about the z-axis |
| $m_w$  | 28 · 10$^{-3}$ kg | wheel mass |
| $I_w$  | 4.957 · 10$^{-6}$ kg m$^2$ | wheel inertia |
| $I_{wx}$ | 7.411 · 10$^{-6}$ kg m$^2$ | about the y-axis |
| $I_{wz}$ | 4.957 · 10$^{-6}$ kg m$^2$ | about the z-axis |
| $l$    | 48.67 · 10$^{-3}$ m | distance from wheel axis to body center of mass |
| $r_w$  | 33 · 10$^{-3}$ m | wheel radius |
| $d_w$  | 2 · 10$^{-3}$ m | distance between wheels |
| $d_v$  | 1.532 · 10$^{-3}$ N m rad$^{-1}$ | viscous damping coefficient |
| $d_e$  | 32.6 · 10$^{-3}$ N m | coulomb damping coefficient |
| $d_0$  | 8 | slope of the damping curve |
| $I_M$  | 268.528 · 10$^{-9}$ kg m$^2$ | inertia motor shaft |
| $I_G$  | 1.807 · 10$^{-9}$ kg m$^2$ | inertia gear stage |
| $n_{w_m}$ | (78/11)$^2$ | gear ratio wheel to motor |
| $k_c$  | 78/11 | gear ratio wheel to gear |
| $k_b$  | 3.76 · 10$^{-3}$ V/(trads) | motor back emf constant |
| $k_m$  | 3.76 · 10$^{-3}$ N m A$^{-1}$ | motor torque constant |
| $L_m$  | 4 · 10$^{-4}$ H | motor inductance |
| $R_m$  | 1.5Ω | motor resistance |

Note that the nonholonomic constraints (22) define permissible velocities of the WIP. Therefore, to define the equations of motion of the system in minimal coordinates without nonholonomic constraints, let us consider

$$\dot{q} = S \nu,$$

where

$$\nu := (v_\alpha, v_d, v_\theta, v_{q_e}, v_{q_L})^T,$$  \hfill (23)
lies in the null space of the matrix $A$. Now the Lagrange multipliers $\lambda$ in \((22)\) are eliminated by pre-multiplying both sides in \((22a)\) by $S^T$. Then substituting $\dot{q} = S\nu$ and $\ddot{q} = S\dot{\nu} + \dot{S}\nu$ to \((22)\) leads to

\[
\dot{\nu} = -(S^TMS)^{-1}S^T(M\dot{S}\nu + K + P - F).
\]

Finally, neglecting the kinematics for the states $\phi_L, \phi_R, q_L, q_R$ in \((25)\), the system dynamics is written on the reduced space in terms of the new states

\[
X := (q_r, \nu)^T = ((x, y, \theta, \alpha), (v_x, v_d, v_\theta, v_{q_R}, v_{q_L}))^T
\]

as

\[
\dot{X} := \left(- (S^TMS)^{-1}S^T(M\dot{S}\nu + K + P - F)\right) \tag{24}
\]

where the reduced matrix $S_r$ is set up by removing last four rows in $S$.

### G. Linear state space model

We derive a discrete-time linear state space model of the WIP system for designing the controller and the observer. The linear discrete-time model is derived via linearization of the nonlinear model \((24)\) around $X = 0, u_L = 0, u_R = 0$, and followed by the discretization of the resulting linear model at a sampling rate of 5 ms and under the assumption of constant inputs during the sampling interval. In order to avoid high damping due to nonlinear friction, the damping parameters for the linear model are set to $d_v = 4.25 \cdot 10^{-3}$ Nm rad$^{-1}$ and $d_c = 0$ Nm. With this set of damping parameters, the linear damping torque curve approximates the nonlinear damping curve well at the nominal operating speed of the system (0.4 m s$^{-1}$).

Note that due to the linearisation at the yaw angle $\theta = 0$, the state $y$ is decoupled from the other states and control inputs. Further, the state $x$ in the linear system is the integration of $v_d$, and hence, to avoid confusion with nonlinear model state $x$, we denote it by $d$. The linear discrete-time model with the states

\[
Y := (d, \theta, \alpha, v_x, v_d, v_\theta, v_{q_R}, v_{q_L})^T
\]

is given by

\[
Y_{k+1} = AY_k + Bu_k \tag{25}
\]

where $u_k := ((u_R)_k, (u_L)_k)^T$, and the matrices $A$ and $B$ are given in \((26)\) and \((27)\).

### H. Tracking controller, guidance algorithm and observer

The WIP is stabilized by an LQ-controller which is designed for the discrete-time linear model. We have discussed in Appendix \(G\) that the linearized discrete-time system \((25)\) does not have access to the states $(x, y)$, but only the traveled forward distance $d$ is available to the controller. Therefore, to facilitate motion of the system on the $(x, y)$ plane, a guidance algorithm is required for the controller to generate the necessary control actions. The guidance algorithm calculates the control errors for the controller in distance $e_d$ and angle $e\theta$ based on the difference between the current position-orientation $(x, y, \theta)$ and the desired position-orientation $(x_d, y_d, \theta_d)$ of a given trajectory. The LQ-controller is calculated using the weighting matrices

\[
Q_Y := \text{diag} \left( \begin{bmatrix} 1500 & 75 & 0.1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \right)
\]

for the states and

\[
R := I_{2\times2}
\]

for the inputs. The state $\dot{q}$ of the robot (see Figure 3) is estimated by sensor-fusion and a model-based observer that suppresses sensor noise besides dealing with communication delays. The position and the orientation of the robot are estimated based on the data received from the Vicon tracking system over the communication channel with communication time delays in the range between 5 ms to 125 ms with an average time delay of 45 ms. Furthermore, the difference in sampling rate of the controller (5 ms) and the Vicon system (25 ms) as well as possible data package losses requires the robot observer to predict the robot position and its orientation without correction steps unless the position and orientation measurements are available to the observer from the tracking system.

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\[
A = \begin{pmatrix}
1 & 0 & 1.55 \cdot 10^{-5} & 8.15 \cdot 10^{-6} & 4.75 \cdot 10^{-3} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 4.87 \cdot 10^{-3} & 3.88 \cdot 10^{-3} \\
0 & 0 & 3.18 \cdot 10^{-1} & 9.49 \cdot 10^{-1} & 1.58 \\
0 & 0 & 6.28 \cdot 10^{-3} & 3.32 \cdot 10^{-3} & 9 \cdot 10^{-1} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.53 \cdot 10^{-2} & 1.08 \cdot 10^{-1} & -3.27 \\
0 & 0 & 1.53 \cdot 10^{-2} & 1.08 \cdot 10^{-1} & -3.27 \\
\end{pmatrix}
\]

(26)

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