Abstract

The conformal compactification is considered in a hierarchy of hypercomplex projective spaces with relevance in physics including Minkowski and Anti-de Sitter space. The geometries are expressed in terms of bicomplex Vahlen matrices and further broken down into their structural components. The relation between two subsequent projective spaces is displayed in terms of the complex unit and three additional hypercomplex numbers.

Keywords: Clifford algebras, bicomplex numbers, AdS/CFT, twistor methods, Laplace equation

PACS: 02.20.Sv, 02.30.Fn, 11.25.Hf, 04.20.Gz, 41.20.Cv

2010 MSC: 15A66, 30G35, 22E70, 53C28, 81T40

1. Introduction

The complex numbers are central for the representation of physical processes in terms of mathematical models. However, they cannot cover all aspects alone and generalizations are necessary. One of the possible generalizations with potentially underestimated relevance in physics is the bicomplex number system \[ 1, 2, 3, 4, 5 \]. Recent investigations in this area have been provided for example by \[ 6, 7, 8, 9, 10 \]. The number system is also known under the name of Segre numbers \[ 11 \]. The bicomplex numbers coincide with the combination of hyperbolic and complex numbers formed by two commutative imaginary units, \( i = \sqrt{-1} \) and \( j = \sqrt{+1} \). The hyperbolic unit carries here the second complex number of the bicomplex number system. More details on hyperbolic numbers have been provided beside many other authors by Yaglom \[ 12 \], Sobczyk \[ 13 \], Gal \[ 14 \], or in correspondence with the bicomplex numbers by Rochon and Shapiro \[ 15 \]. The hypercomplex number systems are strongly connected to the theory of Clifford algebras and Lie groups, see Porteous \[ 16 \] or Ablamowicz et al. \[ 17, 18 \]. Algebraic properties of higher dimensional geometric spaces can be investigated in terms of hypercomplex matrix representations of Clifford algebras. The generalizations can be applied also to functional calculus. The properties of holomorphic
functions of one complex variable can be extended to functions with values in a
Clifford algebra, consider here for example Brackx, Delanghe, and Sommen [19].
Further details can be found also in Gürlebeck et al. [20] or Colombo et al. [21].

The most prominent hypercomplex number system, representing the Clifford
algebra $\mathbb{R}_{0,2}$, is given by the quaternions. Girard shows in a short summary
that quaternions provide spin representations of the most important equations
in quantum physics and classical field theory [22], see also Majerník and Nagy
[23]. An extension is given by the spinor of conformal space, the twistor with its
inherent null-plane geometry introduced by Penrose [24]. Mathematically such
an extension is described in terms of complex projective spaces [27]. In the sixties
of the last century conformal physics became popular in the context of the strong
interaction. Conformal field theories play a central role within string theories
and the AdS/CFT (Anti-de Sitter space/conformal field theory) correspondence
of Maldacena [26], Gubser et al. [27], and Witten [28]. In this area is situated also
the higher spin holography [29, 30, 31, 32], which could potentially benefit from
the spin representations discussed in the following sections. Möbius geometries,
conformal transformations, and their action on cycles have been studied with
focus on hypercomplex variables by Kisil [33]. Conformal extensions in relation
with Clifford algebras and quaternionic analysis have been studied beside others
by Sobczyk [34] and Frenkel and Libine [35].

The mentioned publications provide the context for a work about confor-
mal relativity with hypercomplex variables, which has been published recently
[36]. The motivation for a follow-up article was initially the consolidation of
the method discussed in [36]. The objective was to provide easy to use mathem-
atical tools within conformal physics in order to proceed with the calculation
of scattering amplitudes for comparison with experiment. From a conceptual
point of view it turned out that important geometries in physics, like the plane,
Minkowski space, and the Anti-de Sitter space, should be aligned and connected
with a common set of rules. Such connections between different geometries have
been described in mathematical generality in terms of the conformal compact-
ification with $2 \times 2$ Clifford matrices, a method which traces back to Vahlen
[34] and Ahlfors [38, 39]. Consider here also Porteous [16] or Hertrich-Jeromin
[40]. Maks [41] investigated explicitly the hierarchy of Möbius geometries, which
will be used also in the following representation. Each level in this hierarchy
introduces an additional hypercomplex projective space based on the preceding
group. On all levels Möbius transformations can be applied to the considered
hypercomplex variables. The hypercomplex projective line and Möbius spaces co-
incide throughout this hierarchy, which is in its complex restriction only the case
within two-dimensional geometries [42]. A similar generalization for quaternions
is of relevance in physics with respect to the concept of instantons, see Atiyah
and Ward [43].
The hierarchy of Möbius geometries is introduced in the following sections based on the complexified null-plane numbers \[36\]. These numbers refer to the idempotents, which appear typically in the context of hyperbolic numbers. The hierarchy of projective geometries begins with the two-dimensional plane as a non-trivial base manifold represented by complex numbers. This brings the method in \[36\] closer to the twistor programme with its interpretation of space-time points as derived objects \[44\]. Furthermore, it reminds of the dimensional reduction of Minkowski space discussed by ’t Hooft \[45\]. These investigations led to what is known today as the holographic principle, see also Susskind \[46\]. It has been suggested to consider the holographic principle as a foundation of a quantum gravity theory, in the same way as the equivalence principle is a foundation of general relativity \[47\].

One may understand the hierarchy of Möbius geometries also from the perspective of AdS/CFT. General relativity on, e.g., AdS$_5$ is dual to the conformal field theory on its conformal boundary, which is given by Minkowski space. The isometries within AdS$_5$ act as conformal transformations in Minkowski space \[48\].

2. Complex numbers

As mentioned in the introduction the two-dimensional Euclidean plane, represented by the complex numbers, is the first non-trivial geometry to be considered. The intention of this section is also to provide simple examples for the notation used in this article. In the context of Clifford algebras a complex number corresponds to a Clifford paravector, which is expanded in terms of the basis $e_\mu = (1, e_k)$. In this case there is only the single non-trivial basis element

$$e_1 = i. \quad (1)$$

As usual $i$ denotes the complex unit. For the sake of completeness it is worth to note that the imaginary unit squares to the negative identity element and changes sign with respect to conjugation

$$i^2 = -1, \quad \bar{i} = -i. \quad (2)$$

The complex unit can be considered as the single basis element of the Clifford algebra $\mathbb{R}_{0,1}$.

Two fundamental products can be defined for the complex numbers, the real and the complex product. Consider here Andreescu and Andrica \[49\] especially with respect to the analogy of these products to the scalar and the cross product. The real product is defined by its action on the basis elements $e_\mu = (1, e_k)$ of the paravector algebra

$$e_\mu \cdot e_\nu = \frac{e_\mu e_\nu + e_\nu e_\mu}{2} = g_{\mu\nu}. \quad (3)$$
The metric tensor \( g_{\mu\nu} \) has been introduced in this equation. Inserting the definition of the paravector one finds the explicit result
\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (4)

As expected the metric of the plane \( \mathbb{R}^{2,0} \) is obtained.

The complex product is defined in analogy to the real product, but with a negative sign between the two contributions
\[
e_\mu \wedge e_\nu = \frac{e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu}{2} = \sigma_{\mu\nu}.
\] (5)

The anti-symmetric tensor \( \sigma_{\mu\nu} \) has been introduced, which will be denoted in the following as spin tensor. An explicit calculation based on the preceding definitions is leading to the following result
\[
\sigma_{\mu\nu} = \begin{pmatrix} 0 & -e_1 \\ e_1 & 0 \end{pmatrix}.
\] (6)

The spin tensor is directly related to the spin angular momentum operator, where the spin tensor is just divided by a factor of two
\[
s_{\mu\nu} = \frac{\sigma_{\mu\nu}}{2}.
\] (7)

In the context of Clifford algebras rotations can be defined with the spin angular momentum operator in a notation, which is still valid in higher dimensional spaces
\[
r = \exp \left( \frac{1}{2} s_{\mu\nu} \omega^{\mu\nu} \right), \quad \omega^{\mu\nu} = -\omega^{\nu\mu}.
\] (8)

The rotation is acting on a complex number by \( r z r^\dagger \) or equivalently by \( r z \tilde{r}^{-1} \), where \( r^\dagger \) indicates reversion and \( \tilde{r} \) the main involution \([16, 20]\).

3. Complex null-plane

The complex null-plane numbers have appeared in \([36]\) as a key structure in a representation of Möbius geometries based on hyperbolic and complex units. In a real form null-plane numbers were considered before for example by Zhong \([50]\) and Hucks \([51]\) in the context of the hyperbolic number system. The complexified null-plane numbers can be defined by the following equations
\[
o o = i o, \quad \bar{o} \bar{o} = -i \bar{o}, \quad o \bar{o} = 0.
\] (9)
The bar symbol denotes conjugation. One may express the null-plane numbers in an alternative notation with the complex unit and a second hypercomplex structure

\[ i = o - \bar{o}, \quad j = o + \bar{o}. \]  

(10)

It should be noted that the hypercomplex unit \( j \) corresponds to \( ij \) in the notation of [36]. The square of \( j \) can be calculated with the preceding definitions

\[ j^2 = -1, \quad \bar{j} = j. \]  

(11)

The two imaginary units \( i \) and \( j \) show a different behaviour with respect to conjugation, which can be derived from Eq. (10) with the rule \( \bar{o} = o \). The imaginary unit \( j \) is invariant with respect to conjugation, whereas the complex unit \( i \) changes sign. The combination of these units results in an algebra, which is among the six representations of bicomplex numbers investigated by Alpay et al. [52].

4. Conformal compactification

The complex space is by its nature unlimited. However, with a conformal compactification it is possible to enclose the unlimited geometry by adding infinity. This leads to the projective space \( \mathbb{P}_1^C = \mathbb{C} \cup \{\infty\} \). In order to perform the compactification the algebra introduced in the preceding section has to be multiplied by additional \( 2 \times 2 \) matrix structures, see for example Obolashvili [53]. Therefore two additional units are introduced, which are based on explicit matrix representations

\[ i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

(12)

The matrix \( i \) can be seen as the counterpart of \( i \) as it changes sign under conjugation, which is represented by transposition of the matrix. In contrast, \( j \) remains invariant with respect to conjugation, but squares to the identity element. Multiplication of these matrices results in

\[ ij = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -ju. \]  

(13)

The three matrices correspond to the Lie algebra of \( SL(2, \mathbb{R}) \).

With these matrices and the complex null-plane numbers the conformal compactification can be represented more compact and generalized compared to [36]. The base geometry is considered to have an even number of \( 2m = n \) dimensions. The \( n - 1 \) basis elements of the Clifford algebra \( \mathbb{R}_{m-1,m} \) are transformed to the basis elements of the projective space by

\[ e_k = ije_k, \quad k = 1, \ldots, n - 1. \]  

(14)
On the right hand side of the equation are the basis elements of the source geometry. The two additional basis elements of the conformal algebra are given by
\[ e_n = ij, \quad e_{n+1} = ij. \] (15)

The resulting basis elements generate the Clifford algebra \( \mathbb{R}_{m,m+1} \). The argument to enclose the original space by adding infinity applies to arbitrary even dimensional spaces in this hierarchy. Thus there is an infinite series of conformal compactifications. One may understand the infinite series of Möbius geometries as representation spaces of a dimension independent projective scheme.

5. Minkowski space

The method discussed in the previous section can now be applied explicitly to the complex numbers. With Eqs. (14) and (15) one can derive the following three basis elements, which can be used to introduce a paravector model \( e_\mu = (1, e_k) \) of Minkowski space
\[ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}. \] (16)

The algebra corresponds to the Clifford algebra \( \mathbb{R}_{1,2} \) and is thus isomorphic to the Pauli algebra [16]. The Clifford algebra \( \mathbb{R}_{1,2} \) as introduced above will replace the algebra \( \mathbb{R}_{3,0} \) in [36].

The metric tensor can be calculated with Eq. (3) using the basis elements introduced above
\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \] (17)

The metric convention of Minkowski space \( \mathbb{R}^{3,1} \) has been reversed compared to [36]. The spin tensor \( \sigma_{\mu\nu} \) can be calculated with Eq. (5)
\[ \sigma_{\mu\nu} = \begin{pmatrix} 0 & -e_1 & -e_2 & -e_3 \\ e_1 & 0 & -je_3 & -je_2 \\ e_2 & je_3 & 0 & je_1 \\ e_3 & je_2 & -je_1 & 0 \end{pmatrix}. \] (18)

The spin angular momentum operator is given again by Eq. (7). With the multiplication rules of the hypercomplex variables the commutation relation of the relativistic spin angular momentum can be calculated
\[ [s_{\mu\nu}, s_{\rho\sigma}] = g_{\mu\rho}s_{\nu\sigma} - g_{\mu\sigma}s_{\nu\rho} + g_{\nu\sigma}s_{\mu\rho} - g_{\nu\rho}s_{\mu\sigma}. \] (19)
Thus the spin matrices provide a representation of the Lorentz group $SO(3,1,\mathbb{R})$. In comparison to [36] the time coordinate has to switch to $e_3$. In this sense pure rotations are represented within the paravector model $e_\mu = (1,e_1,e_2)$. One can see in the matrix representations of the spin tensor, that the rotations are free of the imaginary unit $j$. Boosts and time thus come in relation with $j$.

This becomes more obvious if one breaks up the above $SO(3,1,\mathbb{R})$ spin representation with Eqs. (14) and (15) in terms of the single basis element of the complex numbers, $e_1 = i$. This leads to the spin tensor

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & -ije_1 & -ije_1 & -ije_1 \\ ije_1 & 0 & -ije_1 & -ije_1 \\ ije_1 & ije_1 & 0 & ije_1 \\ ije_1 & jje_1 & -ije_1 & 0 \end{pmatrix}.$$  \hspace{1cm} (20)

This representation is equivalent to Eq. (18). The time coordinate $t$ is attached to $e_3$ as mentioned before. The space dimensions $(x,y,z)$ can be attributed in arbitrary rotated form to $e_\mu = (1,e_1,e_2)$. The base algebra $(1,e_1)$ is still included. It can be interpreted as a geometric polarization plane and is potentially related to interacting forces and masses. In order to indicate a possible relation between geometry and particle masses one can introduce the following equation, which is inspired by the Regge trajectories in the sense that the angular momentum is related to squared masses [54]

$$4\pi \exp(4\pi) = \left(\frac{m_p}{m_e}\right)^2.$$  \hspace{1cm} (21)

The experimental proton to electron mass ratio is calculated with a deviation of 3.4%. The question arises whether this equation can be derived from geometric properties of manifolds, which describe single protons and electrons [55].

6. Anti-de Sitter space

One can apply the conformal compactification to Minkowski space and reaches the ambient space $\mathbb{R}^{4,2}$, which includes the Anti-de Sitter space $AdS_5$. Equations (14) and (15) are applied to the basis elements of $\mathbb{R}_{1,2}$ representing the base manifold. The resulting new basis elements of the Clifford algebra $\mathbb{R}_{2,3}$ are used to set up the paravector model $e_\mu = (1,e_k)$. The metric tensor is calculated with Eq. (8)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (22)
The spin tensor is computed with Eq. (5). The result can be displayed in terms of the three basis elements of the Clifford algebra $\mathbb{R}_{1,2}$ and the four hypercomplex units, which have been introduced in the preceding sections:

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & -\imath e_1 & -\imath e_2 & -\imath e_3 & -\imath j & -\imath j \\ \imath e_1 & 0 & -\imath e_2 & -\imath e_3 & -\imath e_1 & -\imath e_1 \\ \imath e_2 & \imath e_3 & 0 & \imath e_1 & -\imath e_2 & -\imath e_2 \\ \imath j & \imath e_1 & \imath e_2 & \imath e_3 & 0 & \imath ij \\ \imath j & \imath j e_1 & \imath j e_2 & \imath j e_3 & -\imath ij & 0 \end{pmatrix}. \quad (23)$$

The spin angular momentum operator is defined again by Eq. (7) and satisfies the commutation relations of Eq. (19). The first column in the spin tensor includes the basis elements of the Clifford algebra $\mathbb{R}_{2,3}$, which will replace $\mathbb{R}_{4,1}$ in [36]. The algebra is equivalent to the Dirac algebra [16].

7. Conformal spin in the base space

The elements of the spin algebra of an $n+2$ dimensional ambient space, can be represented within the $n$ dimensional base manifold. For example the spin angular momentum defined by Eqs. (7) and (23) can be restricted to indices $\mu = 0, \ldots, n-1$, in this case with $n = 4$. The reduced spin angular momentum operator then still satisfies Eq. (19) with the corresponding metric tensor. The remaining operators can be reorganized using the definitions of Kastrup [56], which result in spin representations of the conformal group:

$$p_{\mu} = -s_{\mu n} - s_{\mu n+1},$$
$$q_{\mu} = s_{\mu n} - s_{\mu n+1},$$
$$d = s_{nn+1}. \quad (24)$$

Here $p_{\mu}$ labels the spin representation of the momentum operator. The notation $q_{\mu}$ is used for the spin representation of the special conformal transformations and $d$ for the scale transformation.

Based on these definitions explicit expressions for the spin representation of these operators can be calculated:

$$2p_0 = \imath j + ij,$$
$$2q_0 = \imath j - ij,$$
$$2p_k = e_k(jj + ii),$$
$$2q_k = e_k(jj - ii),$$
$$2d = \imath jj. \quad (25)$$
The basis elements $e_k$ refer to the base manifold, this means to the right hand side of Eq. (14). Beside the already mentioned commutation relation of the spin angular momentum one finds the following commutation relations

$$
[s_{\mu\nu}, p_\sigma] = g_{\sigma\alpha} p_\mu - g_{\mu\alpha} p_\nu,
$$

$$
[s_{\mu\nu}, q_\sigma] = g_{\sigma\alpha} q_\mu - g_{\mu\alpha} q_\nu,
$$

$$
[d, p_\mu] = -p_\mu,
$$

$$
[d, q_\mu] = q_\mu,
$$

$$
[q_\mu, p_\nu] = 2(g_{\mu\nu} d + s_{\mu\nu}).
$$

(26)

All other commutators vanish. Thus the spin operators form a representation of the conformal group. Due to the relation between conformal transformations and Möbius geometries, it should be noted that the Möbius space is identified with the homogeneous space defined by the Lie algebras

$$
g = \{s_{\mu\nu}, p_\mu, q_\mu, d\},
$$

$$
h = \{s_{\mu\nu}, q_\mu, d\}.
$$

(27)

More detailed information about Möbius geometries can be found in the textbook of Sharpe [42].

With respect to physics one finds that the Minkowski space is situated within the sequence of projective geometries. Therefore Eq. (24) points to the dimensional reduction considered by ’t Hooft [45] and furthermore to the AdS/CFT correspondence. The isometries on a sphere in Minkowski space $\mathbb{R}^{3,1}$ are dual to conformal transformations in the Euclidean planar limit [57]. The conformal transformations refer to the symmetries of the Laplace equation in $\mathbb{R}^{2,0}$, which can be chosen to define the conformal field theory

$$
\Delta \psi = 0.
$$

(28)

The solutions give rise to geometric fields, which can be transformed with the above symmetry operators represented in the corresponding function space.

8. Summary

A system of hypercomplex units has been defined for the representation of relativistic physics in terms of paravector models. The representation is used to attach a light cone to a given base manifold by virtue of the conformal compactification. The method is applied to the complex numbers representing the initial non-trivial base space, which results in a hypercomplex representation of Minkowski space.

The system of hypercomplex units is used furthermore to create hypercomplex spin representations within Möbius geometries. This is leading to a finite
dimensional representation of the conformal symmetry operators of the two-dimensional Laplace equation. Higher dimensional geometries of physical relevance can be derived around the base manifold, which can be accessed through holomorphic functions of ordinary complex numbers.

References

[1] M. Futagawa, On the theory of functions of a quaternary variable, Tôhoku Math. Journal 29, (1928) 175.
[2] G. Scorza Dragoni, Sulle funzioni olomorfe di una variabile bicomplessa, Reale Accademia d’Italia, Mem. Classe Sci. Nat. Fis. Mat. 5, (1934) 597.
[3] K. Spampinato, Sulla rappresentazione delle funzioni di variabile bicomplessa totalmente derivabili, Annali di Matematica pura ed applicata 14, (1936) 305.
[4] J. D. Riley, Contributions to the theory of functions of a bicomplex variable, Tôhoku Math. Journal 5, (1953) 132.
[5] G. B. Price, An introduction to multicomplex spaces and functions, Monographs and Textbooks in Pure and Applied Mathematics 140, (Marcel Dekker Inc., New York, 1991).
[6] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa, A. Vajiac, Complex Laplacian and derivatives of bicomplex functions, Complex Analysis and Operator Theory 7, (2013) 1675.
[7] D. C. Struppa, A note on analytic functionals on the complex light cone, in Advances in Hypercomplex Analysis, eds. G. Gentili, I. Sabadini, M. Shapiro, F. Sommen, D. C. Struppa, Springer INdAM Series 1, (2013) 119.
[8] L. Chen, G. Ren, H. Wang, Bicomplex Hermitian Clifford analysis, Frontiers of Mathematics in China 10, (2015) 523.
[9] Ji Eun Kim, Kwang Ho Shon. Properties of regular functions with values in bicomplex numbers, Bulletin of the Korean Mathematical Society 53, (2016) 507.
[10] M. Mursaleen, Md. Nasiruzzaman, H. M. Srivastava, Approximation by bicomplex beta operators in compact BC-disks, Math. Meth. Appl. Sci. 39, (2016) 2916.
[11] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Math. Ann. 40, (1892) 413.
[12] I. M. Yaglom, A Simple Non-Euclidean Geometry and its Physical Basis, (Springer, New York, 1979).
[13] G. Sobczyk, The hyperbolic number plane, Coll. Maths. Jour. 26, (1995) 268.
[14] S. G. Gal, Introduction to Geometric Function Theory of Hypercomplex Variables, (Nova Science Publishers, New York, 2002).
[15] D. Rochon, M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, Anal. Univ. Oradea, fasc. math. 11, (2004) 71.
[16] I. R. Porteous, Clifford Algebras and the Classical Groups, (Cambridge University Press, Cambridge, 1995).
[17] R. Ablamowicz, P. Lounesto, J.M. Parra (eds.), Clifford Algebras with Numeric and Symbolic Computations, (Birkhäuser, Basel, 1996).
[18] R. Ablamowicz, G. Sobczyk (eds.), Lectures on Clifford (Geometric) Algebras and Applications, (Birkhäuser, Basel, 2004).
[19] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, (Pitman, London, 1982).
[20] K. Gürlebeck, K. Habetha, W. Sprößig, Holomorphic functions in the plane and n-dimensional space, (Birkhäuser, Basel, 2008).
[21] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, Analysis of Dirac Systems and Computational Algebra, Progress in Mathematical Physics 39, (Birkhäuser, Boston, 2004).
[22] P. R. Girard, The quaternion group and modern physics, Eur. J. Phys. 5, (1984) 25.
[23] V. Majerník, M. Nagy, Quaternionic form of Maxwell's equations with sources, Lettere Nuovo Cimento 16, (1976) 265.
[24] R. Penrose, Twistor Algebra, J. Math. Phys. 8, (1967) 345.
[25] W. M. Goldman, Complex Hyperbolic Geometry, (Oxford University Press, Oxford, 1999).
[26] J. M. Maldacena, The large $N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2, (1998) 231.
[27] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B 428, (1998) 105.
[28] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2, (1998) 231.
[29] J. M. Maldacena, The large $N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2, (1998) 231.
[30] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B 428, (1998) 105.
[31] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2, (1998) 231.
[32] J. M. Maldacena, The large $N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2, (1998) 231.
[33] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B 428, (1998) 105.
[34] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2, (1998) 231.
with Bicomplex Scalars, and Bicomplex Schur Analysis, (Springer, Cham, 2014).

[53] E. Obolashvili, Partial differential equations in Clifford analysis, Pitman Monographs and Surveys in Pure and Applied Mathematics 96, (Longman, Harlow, 1998).

[54] M. Guidry, Gauge Field Theories: An Introduction with Applications, (Wiley, New York, 1991).

[55] M. Atiyah, N. S. Manton, B. J. Schroers, Geometric models of matter, Proc. R. Soc. A 468, (2012) 1252.

[56] H. A. Kastrup, Zur physikalischen Deutung und darstellungstheoretischen Analyse der konformen Transformationen von Raum und Zeit, Annalen der Physik 7, (1962) 388.

[57] G. ’t Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72, (1974) 461.