Combinatorial Heat and Wave Equations on Certain Classes of Infinite Cayley and Coset Graphs

S. Mohanty* and A. K. Lal*

Abstract

The combinatorial heat and wave equations on all finite Cayley and coset graphs with discrete time variable was solved by Lal et al. In this paper, the results of the above paper are extended for infinite Cayley and coset graphs, whenever the associated groups are discrete, abelian and finitely generated. Furthermore, we study the solution of the combinatorial heat and wave equations on a $k$-regular tree whose associated group is a non-abelian free group on $k$ generators, each of order 2. It turns out that in case of Cayley graphs the solutions to combinatorial heat and wave equations are weighted sum of the initial functions over balls of certain radius which are dependent on the discrete time variable.

Keywords: combinatorial Laplacian, combinatorial heat equation, combinatorial wave equation, Cayley graph, $k$-regular tree.

1 Introduction

Let $G = (V(G), E(G))$, in short $(V, E)$, be an undirected connected graph without loops or multiple edges, with $V$ as the set of vertices and $E$ as the set of edges in $G$. We write $x \sim y$ to indicate that an undirected edge $\{x, y\} \in E$, i.e., the vertices $x, y \in V$ are adjacent in $G$. The degree of the vertex $x$, denoted by $m(x)$, is the number of vertices in $V$ that are adjacent to $x$. A graph is said to be $k$-regular if $m(x) = k$ for all $x \in V$ and is called locally finite if $m(x) < \infty$ for all $x \in V$.

A connected graph $G$ without loops or multiple edges is a metric space with respect to the metric $d_G(x, y)$, the length of the shortest path from $x$ to $y$ for all $x, y \in G$.

Let $G = (V, E)$ be a connected locally finite graph. For $1 \leq p < \infty$, let us consider the normed linear space $L^p(V) = \{ f : V \to \mathbb{C} : \|f\|_{L^p(V)} < \infty \}$, where $\|f\|_{L^p(V)} = \left( \sum_{x \in V} |f(x)|^p \right)^{\frac{1}{p}}$. Note

---

*Department of Mathematics and Statistics, IIT Kanpur, Kanpur, India - 208016.
Emails: sumitmth@iitk.ac.in, arlal@iitk.ac.in FAX No: 91-512-2597500.
that, for \( p = 2 \), \( L^2(V) \) is a Hilbert space with \( \langle f, g \rangle_{L^2(V)} = \sum_{x \in V} f(x) \overline{g(x)} \) as its inner product. Given a function \( f : V \to \mathbb{C} \), the combinatorial Laplacian operator on \( G \) is defined by
\[
\Delta_G f(x) = m(x) f(x) - \sum_{y \sim x} f(y) \quad \text{for all } x \in V.
\]

Observe that \( \Delta_G \) is bounded on \( L^2(V) \) if and only if \( m(x) \) is uniformly bounded. In case \( G \) is a finite graph, the operator \( \Delta_G \) represents a \( |V| \times |V| \) matrix (where \( |S| \) denotes the cardinality of the set \( S \)), called the Laplacian matrix of \( G \). It is well known that \( \Delta_G \) is a positive semi-definite matrix with 0 as the smallest eigenvalue. Moreover, if \( 0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{|V|}(G) \) are the eigenvalues of \( \Delta_G \) then \( \lambda_2(G) > 0 \) if and only if \( G \) is connected (for details see [1]).

**Remark 1.1.** Recall that, if we define the classical Laplacian operator on \( \mathbb{R}^n \) by \( \Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \), then its eigenvalues \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \) form a discrete subset (with multiplicities) of \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \} \) (for details see [3, pages 192-195]). Therefore from the previous paragraph it is clear that \( \Delta_G \) may be viewed as the discrete analogue of the classical Laplacian \( \Delta \).

We now recall the definition of a Cayley and coset graph.

**Definition 1.2 (Cayley Graph).** Let \( \Gamma \) be a group with \( e_\Gamma \) as its identity element. Let \( S \subseteq \Gamma \) such that \( S \) generates \( \Gamma \) (\( \langle S \rangle = \Gamma \) ), \( S = S^{-1} \), \( e_\Gamma \notin S \) and \( |S| < \infty \). The Cayley graph \( \text{Cay}(\Gamma,S) \) on \( \Gamma \) with respect to \( S \) has \( V = \Gamma \) and \( E = \{ \{x, xs\} : x \in \Gamma, s \in S \} \).

**Remark 1.3.** Cayley graphs as defined above are \( |S| \)-regular, undirected, connected and have no loops nor multiple edges (see [9]). The metric \( d_G \) on the Cayley graph, \( G = \text{Cay}(\Gamma,S) \), is given by
\[
d_G(x,y) = \min \{ r : y = xs_{i_1}s_{i_2} \cdots s_{i_r}, \text{ where } s_{i_1}, s_{i_2}, \ldots, s_{i_r} \in S \}.
\]

**Definition 1.4 (Coset Graph).** Let \( \Gamma \) be a group, \( H \) a subgroup of \( \Gamma \) and \( S \) be a subset of \( \Gamma \) such that \( S \subseteq \Gamma \setminus H \), \( S^{-1} = S \) and \( H \cup S \) generates \( \Gamma \). The coset graph \( G = \text{Coset} (\Gamma,H,S) \) on \( \Gamma \) with respect to \( H \) and \( S \) is defined to be the graph having \( V(G) \) as the set of all distinct left cosets of \( H \) in \( \Gamma \) and \( E(G) = \{ \{xH, xHS\} : x \in \Gamma, s \in S \} \), i.e., for \( xH, yH \in V(G) \), \( xH \neq yH, xH \sim yH \) if \( x^{-1}y \in HSH \).

Observations similar to that of Cayley graphs in Remark 1.3 can also be made for the coset graphs. The next remark is an important observation on coset graphs.

**Remark 1.5.** In the above definition of a coset graph, the set \( S \) may have the property that \( Hs_i = Hs_j \) for two distinct elements \( s_i, s_j \in S \). In this case, the contribution of \( s_i \) and \( s_j \) to the edge set of the coset graph remains the same. Therefore, from the set \( S \), we extract a set \( \tilde{S} \) such that the elements of \( \tilde{S} \) give all the distinct right cosets of \( H \) in \( S \).
We now define the difference operator, the discrete time analogue of the differentiation operator.

**Definition 1.6.** Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Then, for any complex valued function $v : \mathbb{Z}^+ \to \mathbb{C}$, let

$$\partial_nv(n) = v(n+1) - v(n).$$

Let $G = (V, E)$ be a Cayley graph. Then, in this paper, we are interested in solving the combinatorial heat equation on $G$ given by

$$\Delta_G u(x, n) + \partial_n u(x, n) = 0 \text{ on } V(G) \times \mathbb{Z}^+,$$

$$u(x, 0) = f(x),$$

and the combinatorial wave equation on $G$ given by

$$\Delta_G u(x, n) + \partial_n^2 u(x, n) = 0 \text{ on } V(G) \times \mathbb{Z}^+,$$

$$u(x, 0) = f(x), \quad \partial_n u(x, 0) = g(x).$$

The above equations were first studied in [2] for Hamming graphs on the vertex set $\mathbb{Z}_2^N$. The results of [2] related to the above equations were generalized for all finite Cayley and coset graphs (i.e., for all finite vertex transitive graphs) in [8]. Note that both [2] and [8] used the theory of Fourier analysis on finite groups to solve these equations.

In this paper, we use techniques from Fourier analysis on locally compact groups, discussed in Section 1.1, to extend the results of [8] to solve the combinatorial heat and wave equations on infinite Cayley and coset graphs whenever the associated groups are discrete, abelian and finitely generated (see Sections 2 and 3). Finally, in Section 4, we also solve the combinatorial heat and wave equations on $k$-regular trees which is a Cayley graph having a non-abelian free group on $k$ generators, each of order 2, as its associated group.

### 1.1 Fourier Transform on Locally Compact Abelian Groups

A group $\Gamma$ is said to be a locally compact abelian group if $\Gamma$ is an abelian group and there exists a topology on $\Gamma$ with respect to which $\Gamma$ is locally compact. An important result on locally compact abelian groups is stated next.

**Proposition 1.7.** [6, 11] Let $\Gamma$ be a locally compact abelian group. Then there exists a nonnegative regular translation-invariant measure $m_\Gamma$, called the Haar measure on $\Gamma$, i.e., for every $x \in \Gamma$ and every Borel set $Y$ in $\Gamma$, one has $m_\Gamma(xY) = m_\Gamma(Y)$.

Now, recall that a character of a locally compact abelian group $\Gamma$ is a continuous group homomorphism $\gamma : \Gamma \to \mathbb{T}$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. Then, the characters of $\Gamma$ form an abelian
group $\hat{\Gamma}$, called the dual group of $\Gamma$, with binary operation $(\gamma_1 \circ \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$ and the trivial character $\gamma_0$ as its identity element. Furthermore, there exists a topology on $\hat{\Gamma}$ with respect to which $\hat{\Gamma}$ is locally compact. Hence, using Proposition 1.7, there exists a Haar measure $m_{\hat{\Gamma}}$ on $\hat{\Gamma}$. For more details on the dual group $\hat{\Gamma}$ and its topology, the readers can refer to [6, 11, 12]. We now state a result which gives an interesting relation between the topology of $\Gamma$ and that of $\hat{\Gamma}$.

**Proposition 1.8.** [11] If $\Gamma$ is a discrete abelian group then the dual group $\hat{\Gamma}$ is a compact abelian group. In case $\Gamma$ is a compact abelian group then $\hat{\Gamma}$ is a discrete abelian group.

**Remark 1.9.** Given a discrete abelian group $\Gamma$, the corresponding Haar measure $m_\Gamma$ is a counting measure. The dual group $\hat{\Gamma}$ is a compact group and hence the corresponding Haar measure $m_{\hat{\Gamma}}$ on $\hat{\Gamma}$ is a finite measure, i.e., $m_{\hat{\Gamma}}(\hat{\Gamma}) < \infty$. Hence, for $1 \leq p < \infty$,

$$L^p(\Gamma) = \{ f : \Gamma \to \mathbb{C} : \|f\|_{L^p(\Gamma)} < \infty \}, \quad L^p(\hat{\Gamma}) = \{ f : \hat{\Gamma} \to \mathbb{C} : \|f\|_{L^p(\hat{\Gamma})} < \infty \},$$

where $\|f\|_{L^p(\Gamma)} = \left( \sum_{x \in \Gamma} |f(x)|^p \right)^{1/p}$ and $\|f\|_{L^p(\hat{\Gamma})} = \left( \frac{1}{m_{\hat{\Gamma}}(\hat{\Gamma})} \int_{\hat{\Gamma}} |f(\gamma)|^p \, dm_{\hat{\Gamma}}(\gamma) \right)^{1/p}$.

We are now ready to state a well known result which will be used subsequently.

**Proposition 1.10.** [11, page 10] Let $\Gamma$ be a discrete abelian group. Then

$$\frac{1}{m_{\hat{\Gamma}}(\hat{\Gamma})} \int_{\hat{\Gamma}} \gamma(x) \, dm_{\hat{\Gamma}}(\gamma) = \begin{cases} 0, & \text{if } x \neq e_{\Gamma}, \\ 1, & \text{if } x = e_{\Gamma}. \end{cases}$$

With the above background, we recall the following definitions.

**Definition 1.11.** Let $\Gamma$ be a discrete abelian group. Then, the Fourier transform of $f \in L^1(\Gamma)$ is given by

$$(\mathfrak{F} f)(\gamma) = \hat{f}(\gamma) = \sum_{x \in \Gamma} f(x)\gamma(x) \quad \text{for each character } \gamma.$$ 

Further, if $\hat{f} \in L^1(\hat{\Gamma})$ then, its inverse Fourier transform is given by

$$\mathfrak{F}^{-1}(\hat{f})(x) = f(x) = \frac{1}{m_{\hat{\Gamma}}(\hat{\Gamma})} \int_{\hat{\Gamma}} \hat{f}(\gamma)\gamma(x^{-1}) \, dm_{\hat{\Gamma}}(\gamma) \text{ for each } x \in \Gamma.$$ 

**Remark 1.12.** Recall that $L^1(\Gamma)$ is dense in $L^2(\Gamma)$ whenever $\Gamma$ is a discrete abelian group. Further, by Plancherel Theorem [11], the notion of Fourier transform can be extended to $L^2(\Gamma)$ using the density argument.
Further recall that for a compact abelian group $\Gamma$ and for a finite number of characters $\gamma_1, \gamma_2, \ldots, \gamma_t \in \hat{\Gamma}$, a function of the form $P(x) = \sum_{i=1}^t b_i \gamma_i(x)$, where $x \in \Gamma$ and $b_i \in \mathbb{C}$, is called a trigonometric polynomial on $\Gamma$. Now we state a result that is an application of the Stone-Weierstrass theorem.

**Proposition 1.13.** [11] Let $\Gamma$ be a compact abelian group. Then the trigonometric polynomials on $\Gamma$ form a dense subalgebra of $\mathcal{C}(\Gamma)$, the set of all continuous functions on $\Gamma$.

**Remark 1.14.** Note that if $\Gamma$ is a discrete abelian group then by Proposition 1.8, the dual group $\hat{\Gamma}$ is compact. Thus, Pontryagin Duality Theorem (see [11, page 27]) implies that the trigonometric polynomials on $\hat{\Gamma}$ are of the form $P(\gamma) = \sum_{i=1}^n b_i \gamma(x_i)$, where $x_i \in \Gamma$, for $1 \leq i \leq n$. If $f \in L^1(\Gamma)$ then by Proposition 1.13, $\sum_{x \in \Gamma} f(x) \gamma(x)$, the Fourier transform of $f$, converges uniformly to $\hat{f}$. Hence $\hat{f}$ is continuous on $\hat{\Gamma}$ and $\hat{f} \in L^1(\hat{\Gamma})$ as $\hat{\Gamma}$ is compact.

**Example 1.15.** For $1 \leq p < \infty$ and $\Gamma = \mathbb{Z}$, recall that $L^p(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{C} : \| f \|_{L^p(\mathbb{Z})} < \infty \}$ with $\| f \|_{L^p(\mathbb{Z})} = (\sum_{r \in \mathbb{Z}} |f(r)|^p)^{1/p}$. Also, recall that $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \cong \mathbb{R}/2\pi \mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers. Hence the functions on $\mathbb{T}$ are identifiable with $2\pi$-periodic functions on $\mathbb{R}$ and thus, for $1 \leq p < \infty$,

$$L^p(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} : \| f \|_{L^p(\mathbb{T})} < \infty \} \text{ with } \| f \|_{L^p(\mathbb{T})} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p},$$

where $dt$ is the Lebesgue measure on the interval $[0, 2\pi)$.

Then, it can be easily deduced that $\hat{\mathbb{T}} = \hat{\mathbb{Z}} \cong \mathbb{T}$, i.e., the set of characters can be parameterized as $\{ \gamma_t : t \in [0, 2\pi) \}$ with $\gamma_t(r) = e^{irt}$ for all $r \in \mathbb{Z}$ (for details, see [11, 12]). Thus, the Fourier transform of $f \in L^1(\mathbb{Z})$ is given by

$$(\mathfrak{F} f)(\gamma_t) = (\mathfrak{F} f)(t) = \hat{f}(t) = \sum_{r \in \mathbb{Z}} f(r) \gamma_t(r) = \sum_{r \in \mathbb{Z}} f(r) e^{irt}.$$

By Remark 1.14, one also has $\hat{f} \in L^1(\mathbb{T})$ and hence its inverse Fourier transform is given by

$$(\mathfrak{F}^{-1} (\hat{f}))(r) = f(r) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(t) \gamma_t(-r) dt = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(t) e^{-irt} dt.$$

Hence, note that the Fourier transform on $\mathbb{Z}$ is nothing but the Fourier series.

**Remark 1.16.** Let $\mathcal{P}_N(t) = \sum_{r=-N}^N a_r e^{irt}$ be a trigonometric polynomial on $\mathbb{T}$. Then, by Proposition 1.10 on $\mathbb{Z}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{irt} dt = \begin{cases} 1, & \text{if } r = 0, \\ 0, & \text{if } r \neq 0. \end{cases}$$

Thus, $\mathfrak{F}^{-1}(\mathcal{P}_N)(r) = \begin{cases} a_r, & \text{if } |r| \leq N, \\ 0, & \text{otherwise}. \end{cases}$
The convolution of two functions, which in a sense replaces the idea of point wise multiplication of two functions, plays a crucial role in Fourier analysis and is recalled next.

**Definition 1.17.** Let \( \Gamma \) be a finite group and \( f, g \in L^1(\Gamma) \). The convolution \( f \ast g \) is defined by \( f \ast g(x) = \sum_{y \in \Gamma} f(y) g(xy^{-1}) \).

We now state few properties of Fourier transform that will be referred in subsequent results.

**Proposition 1.18.** Let \( \Gamma \) be a discrete abelian group and let \( f, g \in L^1(\Gamma) \). Then

1. \( \mathfrak{F}(f \ast g) = (\mathfrak{F}f)(\mathfrak{F}g) \).
2. for each fixed \( y \in \Gamma \), if \( f^y(x) = f(xy^{-1}) \) for each \( x \in \Gamma \) then \( (\mathfrak{F}f^y)(\gamma) = \gamma(y) (\mathfrak{F}f)(\gamma) \).
3. \( \mathfrak{F}^{-1}(1) = \chi_{\{e\}} \), where \( 1 \) is the constant function on \( \hat{\Gamma} \) that takes the value \( 1 \).

**Proof.** See \([11, Theorem 1.2.4, page 9]\), for the proofs of first and second parts. For the third part, note that by Remark 1.9, the Haar measure \( m_{\hat{\Gamma}} \) on the dual group \( \hat{\Gamma} \) is a finite measure and hence \( 1 \in L^1(\hat{\Gamma}) \). Thus, using Proposition 1.10, we have

\[
\mathfrak{F}^{-1}(1)(x) = \frac{1}{m_{\hat{\Gamma}}(\hat{\Gamma})} \int_{\hat{\Gamma}} \chi_{\{e\}}(\gamma)(x^{-1}) \, dm_{\hat{\Gamma}}(\gamma) = \frac{1}{m_{\hat{\Gamma}}(\hat{\Gamma})} \int_{\hat{\Gamma}} \chi_{\{e\}}(x^{-1}) \, dm_{\hat{\Gamma}}(\gamma) = \chi_{\{e\}}(x)
\]

and hence the required result follows. \( \square \)

### 2 Results on Some Infinite Cayley Graphs whose Associated Group is Abelian and Finitely Generated

Let \( \Gamma \) be an infinite discrete abelian group generated by finitely many generators. In this section, we solve the combinatorial heat and wave equations on the infinite Cayley graph \( G = \text{Cay}(\Gamma, S) \), whenever \( |S| < \infty \). We start with the combinatorial heat equation.

**Theorem 2.1.** Let \( \Gamma \) be an infinite discrete abelian group and let \( S = \{s_1, \ldots, s_k\} \subset \Gamma \) such that \( e_\Gamma \notin S \), \( \langle S \rangle = \Gamma \) and \( S = S^{-1} \). Also, let \( G = \text{Cay}(\Gamma, S) \) be the Cayley graph on \( V(G) = \Gamma \) with respect to \( S \). Then, for \( f \in L^2(\Gamma) \) the combinatorial heat equation (2) on \( G \) admits a unique solution \( u(x,n) = K_n \ast f(x) \), where

\[
K_n(x) = \chi_{\{e\}}(x) + \sum_{j=1}^{n} (-1)^j {n \choose j} \mathfrak{F}^{-1}(a) \ast \cdots \ast \mathfrak{F}^{-1}(a)(x)
\]

with \( \mathfrak{F}^{-1}(a)(x) = |S| \chi_{\{e\}}(x) - \chi_B(x) \) and \( B \subset \Gamma \) is the boundary of the unit ball centered at \( e_\Gamma \).
Before coming to the proof of Theorem 2.1, we state a result that gives information about the solution of (2) on the Cayley graph (as stated in Theorem 2.1) whenever it exists.

**Lemma 2.2.** Suppose the hypothesis of Theorem 2.1 holds and \( f \in L^2(\Gamma) \). If (2) admits a solution \( u(x, n) \) on \( G = \text{Cay} \ (\Gamma, S) \) then \( u(\cdot, n) \in L^2(\Gamma) \) for all \( n \in \mathbb{Z}_+ \).

**Proof.** We will use the principle of mathematical induction on the discrete time variable \( n \in \mathbb{Z}_+ \) to prove this result. If \( n = 0 \), then \( u(\cdot, 0) = f(\cdot) \in L^2(\Gamma) \) and hence the result holds trivially for \( n = 0 \). So, let us assume that \( u(\cdot, t) \in L^2(\Gamma) \) for all \( t \leq n \) and compute \( u(x, n+1) \).

Now, let \( t = n + 1 \). Since \( G \) is a \( k \)-regular graph, using (1), we can re-write (2) as

\[
  k u(x, n) - \sum_{i=1}^{k} u(xs_i, n) + u(x, n+1) - u(x, n) = 0 \quad \text{and} \quad u(x, 0) = f(x).
\]

Now applying the Fourier transform (see Remark 1.12) and using Proposition 1.18, we have

\[
  \hat{u}(\cdot, n) = [1 - a(\cdot)]^n \hat{f}(\cdot), \quad \text{where} \quad a(\gamma) = k - \sum_{i=1}^{k} \gamma(s_i^{-1}).
\]

By Lemma 2.2 and Plancherel Theorem, \( u(\cdot, n) \in L^2(\Gamma) \) and hence \( \hat{u}(\cdot, n) \in L^2(\hat{\Gamma}) \) for all \( n \in \mathbb{Z}_+ \). Thus, applying the inverse Fourier transform and Proposition 1.18, we get

\[
  u(x, n) = K_n * f(x),
\]
where \( K_n (x) = \mathfrak{F}^{-1}((1-a)^n)(x) = \chi_{\{e_T\}}(x) + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a) \) and

\[
\mathfrak{F}^{-1}(a) = \frac{1}{m_\Gamma(\Gamma)} \int_\Gamma a(\gamma) \gamma(x^{-1}) \, dm_\Gamma(\gamma) = \frac{1}{m_\Gamma(\Gamma)} \int_\Gamma (k - \sum_{j=1}^{k} \gamma(s_j^{-1})) \gamma(x^{-1}) \, dm_\Gamma(\gamma)
\]

\[
= k \mathfrak{F}^{-1}(1)(x) - \sum_{j=1}^{k} \frac{1}{m_\Gamma(\Gamma)} \int_\Gamma \gamma(s_j^{-1}) \gamma(x^{-1}) \, dm_\Gamma(\gamma)
\]

\[
= k \chi_{\{e_T\}}(x) - \sum_{j=1}^{k} \frac{1}{m_\Gamma(\Gamma)} \int_\Gamma \gamma((xs_j)^{-1}) \, dm_\Gamma(\gamma).
\]

(4)

Note that for \( x = e_T \), \( \mathfrak{F}^{-1}(a)(e_T) = k \). But for \( x \neq e_T \), using Proposition 1.10 and (4), we have \( \mathfrak{F}^{-1}(a)(x) \neq 0 \) if and only if \( x \in S \). Thus,

\[
\mathfrak{F}^{-1}(a)(x) = k \chi_{\{e_T\}}(x) - \chi_S(x) = k \chi_{\{e_T\}}(x) - \chi_B(x) = |S| \chi_{\{e_T\}}(x) - \chi_B(x).
\]

Since the Fourier inversion is unique, the required result follows.

We now state the main result on the combinatorial wave equation.

**Theorem 2.3.** Suppose the hypothesis of Theorem 2.1 holds. Then, for \( f, g \in L^1(\Gamma) \), the combinatorial wave equation (3) on the Cayley graph \( G = \text{Cay}(\Gamma, S) \) has a solution if and only if \( \hat{g}(\gamma_0) = 0 \). Moreover, if this is the case, then this solution is unique and is expressed by

\[
u(x, n) = f(x) + ng(x) + \sum_{i=1}^{\lceil \frac{n}{2} \rceil} (-1)^i \binom{n}{2i} \mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a) * f(x)
\]

\[
+ \sum_{i=1}^{\lceil \frac{n}{2i+1} \rceil} (-1)^i \binom{n}{2i+1} \mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a) * g(x)
\]

with \( \mathfrak{F}^{-1}(a)(x) = |S| \chi_{\{e_T\}}(x) - \chi_B(x) \) and \( B \subset \Gamma \) is the boundary of the unit ball centered at \( e_T \).

Before proving the above theorem, we first prove the following result which gives information about the solution of (3) on \( G \), whenever it exists. The proof is similar to the proof of Lemma 2.2 but is presented here for the sake of completeness.

**Lemma 2.4.** Suppose the hypothesis of Theorem 2.3 holds and \( f, g \in L^1(\Gamma) \). If the combinatorial wave equation (3) on \( \text{Cay}(\Gamma, S) \) admits a solution \( u(x, n) \) then \( u(\cdot, n) \in L^1(\Gamma) \) for all \( n \in \mathbb{Z}_+ \).

**Proof.** As \( G \) is a \( k \)-regular graph, by using (1), we can re-write (3) on \( G \) as

\[
u(x, n + 2) = 2u(x, n + 1) + \sum_{i=1}^{k} u(xs_i, n) - (k + 1)u(x, n),
\]

\[
u(x, 0) = f(x) \quad \text{and} \quad u(x, 1) = u(x, 0) + g(x).
\]

(5)
We again use the principle of mathematical induction on the discrete time variable \( n \in \mathbb{Z}_+ \) to complete the proof. Note that if \( n \) is either 0 or 1 then from (5), we have \( u(\cdot, 0) = f(\cdot) \in L^1(\Gamma) \) and \( u(\cdot, 1) = f(\cdot) + g(\cdot) \in L^1(\Gamma) \). The result thus holds trivially for \( n = 0, 1 \). Assume that the result is true whenever the discrete time variable \( t \leq n + 1 \), i.e., \( u(\cdot, t) \in L^1(\Gamma) \) for all \( t \leq n + 1 \).

Now, for \( t = n + 2 \), using (5), the triangle inequality and the induction hypothesis, we get

\[
\|u(\cdot, n + 2)\|_{L^1(\Gamma)} \leq 2 \sum_{x \in \Gamma} |u(x, n + 1)| + \sum_{i=1}^{k} \sum_{x \in \Gamma} |u(xs_i, n)| + (k + 1) \sum_{x \in \Gamma} |u(x, n)|
\]

\[
= 2\|u(\cdot, n + 1)\|_{L^1(\Gamma)} + (2k + 1)\|u(\cdot, n)\|_{L^1(\Gamma)} < \infty.
\]

Hence, by the principle of mathematical induction, the desired result follows. 

We are now ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** As \( G \) is a \( k \)-regular graph with generating set \( S = \{s_1, \ldots, s_k\} \), the application of the Fourier transform on the combinatorial wave equation (3) gives

\[
\hat{u}(\gamma, n + 2) - 2\hat{u}(\gamma, n + 1) - [-a(\gamma) - 1]\hat{u}(\gamma, n) = 0, \quad \hat{u}(\gamma, 0) = \hat{f}(\gamma) \quad \text{and} \quad \hat{u}(\gamma, 1) - \hat{u}(\gamma, 0) = \hat{g}(\gamma),
\]

where \( a(\gamma) = k - \sum_{i=1}^{k} \gamma(s_i^{-1}) \). Solving the above recurrence equation, one has

\[
\hat{u}(\gamma, n) = \lambda_{\gamma} \left(1 + \sqrt{-a(\gamma)}\right)^n + \mu_{\gamma} \left(1 - \sqrt{-a(\gamma)}\right)^n.
\]

Now, using the initial conditions, we have

\[
\lambda_{\gamma} + \mu_{\gamma} = \hat{f}(\gamma) \quad \text{and} \quad \lambda_{\gamma}(1 + \sqrt{-a(\gamma)}) + \mu_{\gamma}(1 - \sqrt{-a(\gamma)}) = \hat{f}(\gamma) + \hat{g}(\gamma). \tag{6}
\]

Note that, \( a(\gamma) \) is a trigonometric polynomial on \( \hat{\Gamma} \) and hence continuous. Therefore, using Remark 1.14, \( \hat{f} \) and \( \hat{g} \) are also continuous functions on \( \hat{\Gamma} \). Thus, the point-wise estimate of (6) is well defined. Note that (6) is consistent if and only if \( \hat{g}(\gamma_0) = 0 \). We also observe that

\[
\hat{u}(\gamma, n) = \left[\sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n}{2i} (-a)^i(\gamma)\right] \hat{f}(\gamma) + \left[\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} (-a)^i(\gamma)\right] \hat{g}(\gamma).
\]

By Lemma 2.4, the solution \( u(\cdot, n) \in L^1(\Gamma) \) for all \( n \in \mathbb{Z}_+ \). Therefore, using Remark 1.14, \( \hat{u}(\cdot, n) \) is continuous on the compact group \( \hat{\Gamma} \) and hence \( \hat{u}(\cdot, n) \in L^1(\hat{\Gamma}) \). Taking the inverse Fourier transform and using Proposition 1.18, we get \( u(x, n) = F_n * f(x) + G_n * g(x) \), where

\[
F_n(x) = \chi(e\Gamma)(x) + \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{2i} \delta^{-1}(a) * \cdots * \delta^{-1}(a)(x), \quad \text{and}
\]

\[
G_n(x) = \sum_{i=1}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{2i+1} \delta^{-1}(a) * \cdots * \delta^{-1}(a)(x).
\]
\[ G_n(x) = n \chi_{\{e\}}(x) + \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^i \left( \frac{n}{2i+1} \right) \tilde{\mathcal{S}}^{-1}(a) \ast \cdots \ast \tilde{\mathcal{S}}^{-1}(a)(x) \]

with \( \tilde{\mathcal{S}}^{-1}(a)(x) = |S| \chi_{\{e\}}(x) - \chi_B(x) \) and hence the required result follows. \( \Box \)

**Remark 2.5.** One can verify that given a Cayley graph:

(1) the solution to combinatorial heat equation \( u(x,n) \) is a weighted sum of the initial function \( f \) over the ball of radius of \( n \) centered at \( x \).

(2) the solution to combinatorial wave equation \( u(x,n) \) is a weighted sum of the initial functions \( f \) and \( g \) over the ball of radius of \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( \left\lfloor \frac{n-1}{2} \right\rfloor \) centered at \( x \) respectively.

### 3 Results on Some Infinite Coset Graphs whose Associated Group is Abelian and Finitely Generated

Let \( \Gamma \) be an infinite discrete abelian group generated by finitely many generators and also let \( \Gamma \) contain a finite subgroup \( H \). In this subsection, we will solve the combinatorial heat and wave equations on the coset graph \( G = \text{Coset}(\Gamma, H, S) \), whenever the group \( \Gamma \) has the above mentioned property and \( |S| < \infty \). Therefore, in this section, we assume that \( \Gamma \) is an infinite discrete abelian group generated by finitely many generators and it also contains a finite subgroup \( H \). To proceed further, we also assume that \( S \) is a finite subset of \( \Gamma \) such that \( S \subset \Gamma \setminus H \), \( S^{-1} = S \) and \( H \cup S \) generates \( \Gamma \).

Proceeding in a manner similar to the case of finite coset graphs, we construct a new graph \( \tilde{G} \) with \( V(\tilde{G}) = \Gamma \) as the vertex set. Two elements \( x, y \in \Gamma \) are adjacent if there exist \( b_iH, b_jH \in V(G) \) such that \( x \in b_iH, y \in b_jH, b_iH \neq b_jH \) and \( b_iH \sim b_jH \). Thus, \( \tilde{G} \) is a \( \delta \)-regular graph with \( \delta = k |H| \), where \( k = |\tilde{S}| \) with \( \tilde{S} \) defined as in Remark 1.5. Further, for any complex valued function \( f : V(G) \rightarrow \mathbb{C} \), we define a function \( \tilde{f} : V(\tilde{G}) \rightarrow \mathbb{C} \) by

\[ \tilde{f}(x) = f(bH), \text{ whenever } x \in bH, \] (7)

i.e., fix a left coset \( bH \) of \( H \) in \( \Gamma \) and let \( y, z \in bH \). Then \( \tilde{f}(y) = \tilde{f}(z) = f(bH) \). Using Theorem 2.1 and proceeding as in the proof of [8, Lemma 2.3], we obtain the next lemma.

**Lemma 3.1.** Let \( \Gamma \) be an infinite discrete abelian group, \( H \) be a finite subgroup of \( \Gamma \) and let \( S \) be a finite subset of \( \Gamma \) such that \( S \subset \Gamma \setminus H \), \( S^{-1} = S \) and \( H \cup S \) generates \( \Gamma \). Let \( G = \text{Coset}(\Gamma, H, S) \)
be the coset graph of \( \Gamma \) with respect to \( H \) and \( S \). Then, for \( \tilde{f} \in L^2(\Gamma) \), the initial value problem

\[
\frac{1}{|H|} \Delta_G \tilde{u}(x, n) + \partial_n \tilde{u}(x, n) = 0 \quad \text{on} \quad V(\tilde{G}) \times \mathbb{Z}_+,
\]

\[
\tilde{u}(x, 0) = \tilde{f}(x)
\]

admits a unique solution with \( \tilde{u}(x, n) = \text{constant on each left coset of} \ H \).

**Proof.** Let \( \{Hs_1, Hs_2, \ldots, Hs_k\} \) be the set of all distinct right cosets of \( H \) in \( \Gamma \), where \( s_i \in S, 1 \leq i \leq k \). Using (1), we can rewrite (8) as

\[
\frac{1}{|H|} \left( k|H| \tilde{u}(x, n) - \sum_{h \in H} \sum_{i=1}^k \tilde{u}(xhs_i, n) \right) + \tilde{u}(x, n + 1) - \tilde{u}(x, n) = 0,
\]

\[
\tilde{u}(x, 0) = \tilde{f}(x).
\]

Using an argument similar to that in Theorem 2.1, we get

\[
\tilde{u}(\gamma, n) = [1 - \tilde{a}(\gamma)]^n \tilde{f}(\gamma),
\]

(9)

where \( \tilde{a}(\gamma) = k - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^k \gamma((hs_i)^{-1}) \). Taking the inverse Fourier transform on (9), we get \( \tilde{u}(x, n) = \tilde{K}_n \ast \tilde{f}(x) \), where

\[
\tilde{K}_n(x) = \sum_{j=0}^n (-1)^j \left( \binom{n}{j} \tilde{3}^{-1}(\tilde{a}) \ast \cdots \ast \tilde{3}^{-1}(\tilde{a}) \right)(x).
\]

The Fourier inversion of \( \tilde{a}(\gamma) = k - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^k \gamma((hs_i)^{-1}) \) is given by

\[
\tilde{3}^{-1}(\tilde{a})(x) = \frac{1}{m_\Gamma(\Gamma)} \int_{\Gamma} \left( k - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^k \gamma((hs_i)^{-1}) \right) \gamma(x^{-1}) \ dm_\Gamma(\gamma)
\]

\[
= k \tilde{3}^{-1}(1)(x) - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^k \left( \frac{1}{m_\Gamma(\Gamma)} \int_{\Gamma} \gamma(x^{-1}) \gamma((hs_i)^{-1}) \ dm_\Gamma(\gamma) \right)
\]

\[
= k \tilde{3}^{-1}(1)(x) - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^k \left( \frac{1}{m_\Gamma(\Gamma)} \int_{\Gamma} \gamma(x^{-1}(hs_i)^{-1}) \ dm_\Gamma(\gamma) \right).
\]

Now, note that for \( x = id_\Gamma \), \( \tilde{3}^{-1}(\tilde{a})(id_\Gamma) = k \). But, for \( x \neq id_\Gamma \), from Proposition 1.10 we observe that \( \tilde{3}^{-1}(\tilde{a})(x) \neq 0 \) if and only if \( x = (hs_i)^{-1} \) for some \( h \in H \) and some \( s_i \in S, 1 \leq i \leq k \). Thus, using the above argument and the condition \( S = S^{-1} \), we have

\[
\tilde{3}^{-1}(\tilde{a})(x) = k \chi_{(id_\Gamma)}(x) - \frac{1}{|H|} \sum_{i=1}^k \chi_{s_iH}(x).
\]
Since $S = S^{-1}$, it can easily be seen that \( \{s_1H, \ldots, s_kH\} = \{s_1^{-1}H, \ldots, s_k^{-1}H\} \). Therefore, whenever \( x = bh_0 \in bH \), we get

\[
\tilde{f}(x) = \sum_{y \in \Gamma} \tilde{f}(xy^{-1}) = \sum_{y \in \Gamma} \left( k \chi_{\{id\}} - \frac{1}{|H|} \sum_{i=1}^{k} \chi_{iH} \right) (y) \tilde{f}(xy^{-1})
\]

\[
= k \tilde{f}(x) - \frac{1}{|H|} \sum_{i=1}^{k} \sum_{y \in \pi_{iH}} \tilde{f}(xy^{-1}) = k \tilde{f}(x) - \frac{1}{|H|} \sum_{i=1}^{k} \sum_{h \in H} \tilde{f}(bh_{iH}^{-1}).
\]

The above equation implies that \( \tilde{f}(x) = \tilde{f}(z) \) for all \( x, z \in bH \) and hence the required result follows.

Thus, we have established the initial result which helps us in proving the main result, stated next as Theorem 3.2, on the combinatorial heat equation on infinite coset graphs.

**Theorem 3.2.** Let \( \Gamma \) be an infinite discrete abelian group, \( H \) a finite subgroup of \( \Gamma \) and \( S \) a finite subset of \( \Gamma \) such that \( S \subset \Gamma \setminus H \), \( S^{-1} = S \) and \( H \cup S \) generates \( \Gamma \). Let \( G = \text{Coset} (\Gamma, H, S) \) be the coset graph of \( \Gamma \) with respect to \( H \) and \( S \). Then, for any \( f \in L^2(V) \), the combinatorial heat equation (2) on \( G \) admits a unique solution if and only if the initial value problem represented by (8) admits a unique solution.

**Proof.** By Lemma 3.1, (8) admits a unique solution \( \tilde{u}(x, n) \) with \( \tilde{u}(x, n) \) being a constant on every left coset of \( H \). Hence, (8) can be rewritten as

\[
1 \over |H| \left( k|H| \tilde{u}(x, n) - |H| \sum_{i=1}^{k} \tilde{u}(xhs_i, n) \right) + \tilde{u}(x, n + 1) - \tilde{u}(x, n) = 0,
\]

(10)

\[
\tilde{u}(x, 0) = \tilde{f}(x).
\]

To proceed further, we define \( u : V(G) \times \mathbb{Z}_+ \rightarrow \mathbb{C} \) by \( u(bH, n) = \tilde{u}(x, n) \), whenever \( x \in bH \). Then, (10) can be rewritten in the form

\[
k u(\mathcal{X}, n) - \sum_{i=1}^{k} u(\mathcal{X}s_i, n) + u(\mathcal{X}, n + 1) - u(\mathcal{X}, n) = 0, \quad \text{and } u(\mathcal{X}, 0) = f(\mathcal{X}),
\]

where \( \mathcal{X}(= bH, \text{say}) \) represents some left coset of \( H \) in \( \Gamma \), i.e. \( u(\mathcal{X}, n) \) is a solution to (2) on \( G \). Since, \( \tilde{u}(x, n) \) is a unique solution to (8), so uniqueness of \( u(\mathcal{X}, n) \) follows.

Similarly, for a given solution \( u(\mathcal{X}, n) \) to (2) on \( G \), we define \( \tilde{u} : V(\tilde{G}) \times \mathbb{Z}_+ \rightarrow \mathbb{C} \) by \( \tilde{u}(x, n) = u(bH, n) \), whenever \( x \in bH \). Then, it is easily seen that \( \tilde{u}(x, n) \) gives a solution to (8) and the uniqueness follows from Lemma 3.1.

Before proceeding further we first state and prove the following lemma.
Lemma 3.3. Let $\Gamma$ be an infinite discrete abelian group, $H$ a finite subgroup of $\Gamma$ and $S$ a finite subset of $\Gamma$ such that $S \subset \Gamma \setminus H$, $S^{-1} = S$ and $H \cup S$ generates $\Gamma$. Let $G = \text{Coset} (\Gamma, H, S)$ be the coset graph of $\Gamma$ with respect to $H$ and $S$. Let $\tilde{G}$ be the graph as defined in the paragraph leading to Lemma 3.1 and let $\tilde{f}, \tilde{g} \in L^1(\Gamma)$. Then, the initial value problem

$$\frac{1}{|H|} \Delta_{\tilde{G}} \tilde{u}(x, n) + \partial^2 \tilde{u}(x, n) = 0 \quad \text{on } V(\tilde{G}) \times \mathbb{Z}_+, \quad (11)$$

admits a unique solution if and only if $\tilde{g}(\gamma_0) = 0$, where $\gamma_0$ is the trivial character of the group $\Gamma$. In case the solution exists, one has $\tilde{u}(x, n) = \text{constant}$ on each left coset of $H$.

Proof. Let $\{Hs_1, Hs_2, \ldots, Hs_k\}$ be the set of all distinct right cosets of $H$ in $\Gamma$, where $s_i \in S, 1 \leq i \leq k$. Using (1) and applying the Fourier transform on (11) with respect to group $\Gamma$, we get

$$\tilde{u}(\gamma, n + 2) - 2\tilde{u}(\gamma, n + 1) - [-\tilde{a}(\gamma) - 1] \tilde{u}(\gamma, n) = 0,$$

$$\tilde{u}(\gamma, 0) = \tilde{f}(\gamma) \quad \text{and} \quad \tilde{u}(\gamma, 1) - \tilde{u}(\gamma, 0) = \tilde{g}(\gamma),$$

where $\tilde{a}(\gamma) = k - \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^{k} \gamma((hs_i)^{-1})$. Using arguments similar to those in the proof of Theorem 2.3, the above equation has a unique solution if and only if $\tilde{g}(\gamma_0) = 0$. In case the solution exists, it has the form

$$\tilde{u}(x, n) = \tilde{F}_n \ast \tilde{f}(x) + \tilde{G}_n \ast \tilde{g}(x),$$

where

$$\tilde{F}_n(x) = \chi_{\{id_e\}}(x) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left( \begin{array}{c} n \\ 2i \end{array} \right) \tilde{3}^{-1}((\bar{a}) \ast \cdots \ast \tilde{3}^{-1}((\bar{a}))(x),$$

$$\tilde{G}_n(x) = n \chi_{\{id_e\}}(x) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \left( \begin{array}{c} n \\ 2i+1 \end{array} \right) \tilde{3}^{-1}((\bar{a}) \ast \cdots \ast \tilde{3}^{-1}((\bar{a}))(x).$$

Now, from Lemma 3.1, we see that $\tilde{3}^{-1}((\bar{a}))(x) \ast \cdots \ast \tilde{3}^{-1}((\bar{a})) = \chi_{\{id_e\}}(x) - \frac{1}{|H|} \sum_{i=1}^{k} x_{s_i} H(x)$ and for any $x, z \in bH$, $\tilde{3}^{-1}((\bar{a} \ast \tilde{f}))(x) = \tilde{3}^{-1}((\bar{a} \ast \tilde{f}))(z)$ and hence the desired result follows. \hfill \Box

This brings us to the final result of this section which deals with the combinatorial wave equation on infinite coset graphs. We omit the proof as the proof is similar to the proof of Theorem 3.2 and also in view of Lemma 3.3 which forms the initial step in the proof.

Theorem 3.4. Let $\Gamma$ be an infinite discrete abelian group, $H$ be a finite subgroup of $\Gamma$ and let $S$ be a finite subset of $\Gamma$ such that $S \subset \Gamma \setminus H$, $S^{-1} = S$ and $H \cup S$ generates $\Gamma$. Let $G = \text{Coset} (\Gamma, H, S)$ be the coset graph of $\Gamma$ with respect to $H$ and $S$. Given $f, g \in L^1(V)$ the combinatorial wave equation

$$\Delta_G u(\mathcal{X}, n) + \partial^2_n u(\mathcal{X}, n) = 0 \quad \text{on } V(G) \times \mathbb{Z}_+, \quad u(\mathcal{X}, 0) = f(\mathcal{X}), \quad \partial_n u(\mathcal{X}, 0) = g(\mathcal{X});$$

13
has a solution if and only if $\widehat{g}(\gamma_0) = 0$, where $\gamma_0$ is the trivial character of the group $\Gamma$, and $\widehat{g} : \Gamma \rightarrow \mathbb{C}$ is defined by $\widehat{g}(x) = g(bH)$, whenever $x \in bH$.

4 Combinatorial Heat and Wave Equations on Regular Trees

In this Section, we solve the combinatorial heat and wave equations on a $k$-regular tree, denoted $T$, which can be identified with an infinite Cayley graph $G = \text{Cay}(\Gamma, S)$, where $\Gamma$ is a non-abelian free group with generators $s_1, \ldots, s_k$, each of order 2.

A similar problem related to wave equation on $k$-regular trees have also been in [4, 10]. In particular, the authors in [10] looked at the normalized Laplacian operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \frac{1}{k} \sum_{y \sim x} [f(x) - f(y)]$$

for all $x \in V(T)$

and studied the problem $-(\mathcal{L} - b)u(x, t) = \partial_t^2 u(x, t)$ on $V(T) \times [0, \infty)$ with initial conditions $u(x, 0) = f(x), \partial_t u(x, 0) = g(x)$, where $b \geq 0$ and $\partial_t$ denotes the partial derivative with respect to the continuous variable $t$. In [4], the authors studied a similar problem with discrete time variable. They considered the problem $-2\mathcal{L}u(x, n) = u(x, n + 1) + u(x, n - 1) - 2u(x, n)$ on $V(T) \times [0, \infty)$ with initial conditions $u(x, 0) = f(x), u(x, 1) = g(x)$.

Our approach to (2) and (3) is analogous to the approach in the classical case of solving the wave equation on Euclidean space $\mathbb{R}^n$. The structure of $T$ and the techniques we have used in Section 2, allow us to adopt such an approach. For details of the classical approach, see [5, pages 166 – 169].

4.1 Notations and Preliminary Results

Let $P = (V(P), E(P))$ be the infinite path with $V(P) = \mathbb{Z}$ and $E(P) = \{\{r, r + 1\} : r \in \mathbb{Z}\}$. Thus, for $k = 2$ the tree $T$ is same as the path $P$. Also, note that $T$ is a metric space with respect to the metric $d_T$, in short $d$. If $\text{Aut}(T)$ is the automorphism group of $T$ then $\text{Aut}(T)$ corresponds to the group of isometries of the metric space $T$. Also, the symmetry of $T$ implies that for $x, y \in V(T)$, there exist a $\varphi \in \text{Aut}(T)$ such that $\varphi(x) = y$, i.e., $T$ is a vertex-transitive graph.

Now, for a fixed vertex $x \in V(T)$ and $r \in \mathbb{Z}_+$, let $S(x, r) = \{z \in V(T) : d(z, x) = r\}$ be the boundary of the ball of radius $r$ centered at $x$. Then, symmetry of $T$ implies that $|S(x, r)|$ is independent of $x$ and hence, we write $|S(x, r)| = S(r)$. Thus, for $x \in V(T)$,

$$S(r) = |S(x, r)| = \left| \left\{ z \in V(T) : d(z, x) = r \right\} \right| = \begin{cases} 1, & r = 0, \\ k(k - 1)^{r-1}, & r > 0. \end{cases}$$
Now, for each \( x \in V(T) \), \( r \in \mathbb{Z}_+ \) and \( \varphi : V(T) \to \mathbb{C} \), the spherical mean of \( \varphi \) is defined as

\[
M_{\varphi}(x, r) = \frac{1}{S(r)} \sum_{d(z,x)=r} \varphi(z) \quad \text{for all } x \in V(T).
\]

Now, by Holder’s inequality, we have

\[
|\varphi(z)| \leq \left( \sum_{d(x,z)=r} 1 \right)^{1/p'} \left( \sum_{d(x,z)=r} |\varphi(z)|^p \right)^{1/p} = (S(r))^{1/p'} \left( \sum_{d(x,z)=r} |\varphi(z)|^p \right)^{1/p}.
\]

Thus, from (13), we obtain

\[
\sum_{r \in \mathbb{Z}} |M_{\varphi}(x, r)|^p \leq 2 \sum_{r=0}^{\infty} |M_{\varphi}(x, r)|^p = 2 \sum_{r=0}^{\infty} \frac{1}{S(r)} \sum_{d(z,x)=r} \varphi(z) \left| \frac{1}{S(r)} \right|^{p'} \sum_{d(x,z)=r} |\varphi(z)|^p
\]

\[
= 2 \sum_{r=0}^{\infty} \frac{1}{S(r)} \sum_{d(x,z)=r} |\varphi(z)|^p \leq 2 \sum_{r=0}^{\infty} \frac{1}{d(x,z)} \sum_{z \in V(T)} |\varphi(z)|^p < \infty.
\]

Similarly, for \( p = 1 \), we have

\[
\sum_{r \in \mathbb{Z}} |M_{\varphi}(x, r)| \leq 2 \sum_{r=0}^{\infty} \frac{1}{S(r)} \sum_{d(z,x)=r} |\varphi(z)| \leq 2 \sum_{r=0}^{\infty} \frac{1}{d(z,x)} \sum_{z \in V(T)} |\varphi(z)| = 2 \sum_{z \in V(T)} |\varphi(z)| < \infty.
\]

Hence the desired result follows.

To proceeding further, we extend the definition of the difference operator \( \partial_n \) (see Definition 1.6) to an operator \( \partial_n : \mathbb{Z} \to \mathbb{C} \) by \( \partial_n v(n) = v(n+1) - v(n) \) for all \( n \in \mathbb{Z} \). The next result establishes an important relation between \( \Delta_T \) and \( \Delta_P \). This result can be viewed as an analogue of “Darboux equation” (for the classical case, see [7, Theorem 6.2.1]).

**Theorem 4.2.** Let \( T \) and \( P \) be defined as above. If \( \varphi \) is a complex valued function on \( V(T) \) then the spherical mean \( M_{\varphi} \) satisfies the following equation

\[
(\Delta_T M_{\varphi})(x, r) = ((\Delta_P + (2-k)\partial_r)M_{\varphi})(x, r) \quad \text{for all } x \in V(T), r \in \mathbb{Z}_+.
\]
Proof. Fix \( x \in V(T) \) and \( r \in \mathbb{Z}_+ \). Then, using (1) and the definition of \( M_\phi \), we have

\[
(\Delta_T M_\phi)(x, r) = k M_\phi(x, r) - \sum_{y \sim x} \left( \frac{1}{|S(r)|} \sum_{d(z, y) = r} \varphi(z) \right).
\] (14)

To compute the second term of (14), we observe the following. Let \( y_1, \ldots, y_k \) be the vertices of \( T \) that are adjacent to \( x \). Then, for each fixed \( i, 1 \leq i \leq k \), a vertex \( z \in S(y_i, r) \), if the path from \( y_i \) to \( z \) passes through the vertex \( x \) and \( z \in S(x, r - 1) \) or the path from \( y_i \) to \( z \) does not pass through \( x \) and \( z \in S(x, r + 1) \), i.e., for each fixed \( i, 1 \leq i \leq k \), one has

\[
S(y_i, r) = \{S(x, r - 1) \setminus S(y_i, r - 2)\} \cup \{S(x, r + 1) \setminus (\cup_{j \neq i} S(y_j, r))\}.
\]

Thus, as we sum over all \( y \sim x \) in the second term of (14), we get exactly \((k - 1)\) copies of \( S(x, r - 1) \) and one copy of \( S(x, r + 1) \). Hence, with these observations (14) reduces to

\[
\Delta_T (M_\phi(x, r)) = k M_\phi(x, r) - \frac{1}{S(r)} \left[ \sum_{d(z, x) = r + 1} \varphi(z) + (k - 1) \sum_{d(z, x) = r - 1} \varphi(z) \right]
= k M_\phi(x, r) - \frac{1}{S(r)} [S(r + 1) M_\phi(x, r + 1) + (k - 1) S(r - 1) M_\phi(x, r - 1)]
= k M_\phi(x, r) - (k - 1) M_\phi(x, r + 1) - M_\phi(x, r - 1)
= [2M_\phi(x, r) - M_\phi(x, r + 1) - M_\phi(x, r - 1)] + (k - 2) [M_\phi(x, r) - M_\phi(x, r + 1)]
\]

\[
= \Delta_p (M_\phi(x, r)) - (k - 2) \partial_r (M_\phi(x, r)).
\] (15)

Hence, we have obtained the desired result. \( \square \)

Corollary 4.3. Suppose \( u(x, n) \) is a function on \( V(T) \times \mathbb{Z}_+ \) and \( M_u(x, r, n) \) denotes the spherical mean of the function \( x \mapsto u(x, n) \). Then, for all \( x \in V(T) \), \( u \) satisfies

(1) the combinatorial heat equation (2) on \( T \) if and only if \( M_u \) satisfies

\[
\left( \Delta_p + (2 - k) \partial_r \right) (M_u(x, r, n)) + \partial_n M_u(x, r, n) = 0,
\] (16)

(2) the combinatorial wave equation (3) on \( T \) if and only if \( M_u \) satisfies

\[
\left( \Delta_p + (2 - k) \partial_r \right) (M_u(x, r, n)) + \partial_n^2 M_u(x, r, n) = 0.
\] (17)

Proof. For any \( x \in V(T) \) and \( r, n \in \mathbb{Z}_+ \), note that using (1) and the definition of \( M_\phi \), we have

\[
M_{(\Delta_T u)}(x, r, n) = \frac{1}{S(r)} \sum_{d(z, x) = r} \Delta_T u(z, n) = \frac{1}{S(r)} \sum_{d(z, x) = r} \left( k u(z, n) - \sum_{y \sim z} u(y, n) \right)
= \frac{k}{S(r)} \sum_{d(z, x) = r} u(z, n) - \frac{1}{S(r)} \sum_{d(z, x) = r} \sum_{y \sim z} u(y, n).
\]
Now an argument that is similar to the argument in the proof of Theorem 4.2 implies that as we sum over all \( y \sim z \) as \( z \) varies over \( S(x, r) \), we get exactly \((k - 1)\) copies of \( S(x, r - 1) \) and one copy of \( S(x, r + 1) \). Therefore, the above equation can be rewritten as

\[
M_{(\Delta u)}(x, r, n) = kM_u(x, r, n) - \frac{1}{S(r)} \left[ \sum_{d(y, x) = r+1} u(y, n) + (k - 1) \sum_{d(y, x) = r-1} u(y, n) \right]
\]

Thus, using (15) and the above, we obtain \( \Delta_T M_u(x, r, n) = M_{(\Delta u)}(x, r, n) \). On similar lines, it can be verified that \( \partial_n M_u(x, r, n) = M_{(\partial_n u)}(x, r, n) \) and \( \partial^2_n M_u(x, r, n) = M_{(\partial^2_n u)}(x, r, n) \). Hence, using Theorem 4.2, the desired results follow.

In the next section, we proceed to solve the combinatorial heat and wave equations on \( T \) which as stated earlier corresponds to a Cayley graph of an infinite non-abelian free group.

### 4.2 Results on Combinatorial Heat and Wave Equations on \( T \)

We are now interested in solving the combinatorial heat and wave equations on \( T \). In view of Corollary 4.3, we observe that if \( u \) is a solution to (2) on \( T \) then the spherical mean \( M_u \) satisfies (16) with the initial condition \( M_u(x, r, 0) = M_f(x, r) \). Similarly, if \( u \) is a solution to (3) on \( T \) then the spherical mean \( M_u \) satisfies (17) with the initial conditions \( M_u(x, r, 0) = M_f(x, r), \partial_n M_u(x, r, 0) = M_g(x, r) \). To simplify our notations, for any fixed \( x \in V(T) \), let

\[
\tilde{u}(r, n) = M_u(x, r, n), \quad \text{for } r \in \mathbb{Z}, n \in \mathbb{Z}_+.
\]

Accordingly, we re-write the initial conditions as

\[
\tilde{f}(r) = M_f(x, r) = M_u(x, r, 0) = \tilde{u}(r, 0) \quad \text{and} \quad \tilde{g}(r) = M_g(x, r) = M_{\partial_n u}(x, r, 0) = \partial_n \tilde{u}(r, 0).
\]

Then, it can easily be observed that solving the combinatorial heat equation (2) on \( T \) is equivalent to solving the initial value problem (IVP) on the infinite path \( P \)

\[
(\Delta_P + (2 - k)\partial_r)\tilde{u}(r, n) + \partial_n \tilde{u}(r, n) = 0 \quad \text{on } V(P) \times \mathbb{Z}_+,
\]

\[
\tilde{u}(r, 0) = \tilde{f}(r),
\]

and solving the combinatorial wave equation (3) on \( T \) is equivalent to solving the IVP

\[
(\Delta_P + (2 - k)\partial_r)\tilde{u}(r, n) + \partial^2_n \tilde{u}(r, n) = 0 \quad \text{on } V(P) \times \mathbb{Z}_+,
\]

\[
\tilde{u}(r, 0) = \tilde{f}(r), \quad \partial_n \tilde{u}(r, 0) = \tilde{g}(r).
\]

We now state a result that gives information about the solution of (19).
Lemma 4.4. Let \( f \in L^2(V(T)) \) and let (19) admit a solution \( \tilde{u}(r,n) \). Then \( \tilde{u}(\cdot, n) \in L^2(V(P)) \) for all \( n \in \mathbb{Z}_+ \).

Proof. The result is clearly true for \( n = 0 \). Now, using (15), we can re-write (19) as

\[
\tilde{u}(r, n+1) = (k-1)\tilde{u}(r+1, n) + \tilde{u}(r-1, n) - (k-1)\tilde{u}(r, n) \quad \text{and} \quad \tilde{u}(r, 0) = \bar{f}(r).
\]

Since \( f \in L^2(V(T)) \) by Lemma 4.1, \( \bar{f} \in L^2(V(P)) \). Therefore, by mathematical induction on \( n \), the desired result follows.

Now we are ready to solve the initial value problem (19).

Lemma 4.5. Let \( f \in L^2(V(T)) \). Then, the initial value problem (19) admits a unique solution \( \tilde{u}(r,n) = K_n \ast \bar{f}(r) \), where \( K_n(r) = 3^{-n} \left( \sum_{j=0}^{n} (-a)^j \right)(r) \) with \( a(t) = -(k-1)e^{-it} + k - e^{it} \).

Proof. Using (15), we can re-write (19) as

\[
k\tilde{u}(r, n) - (k-1)\tilde{u}(r+1, n) - \tilde{u}(r-1, n) + \tilde{u}(r, n+1) - \tilde{u}(r, n) = 0 \quad \text{with} \quad \tilde{u}(r, 0) = \bar{f}(r).
\]

Since \( \bar{f} \in L^2(V(P)) \), by Lemma 4.4, \( \tilde{u}(\cdot, n) \in L^2(V(P)) \) for all \( n \in \mathbb{Z}_+ \). Thus, applying Fourier transform with respect to \( r \in \mathbb{Z} \) and proceeding similar to the proof of Theorem 2.1, we get

\[
\tilde{u}(\cdot, n) = [1 - a(\cdot)]^n \tilde{f}(\cdot), \quad \text{where} \quad a(t) = -(k-1)e^{-it} + k - e^{it}.
\]

Using the inverse Fourier transform, we obtain the desired result.

Now we compute the trigonometric polynomials \( a^j(t) \), for \( j \geq 0 \).

\[
a^j(t) = \left[-(k-1)e^{-it} + k - e^{it}\right]^j = (1 - e^{-it})^j \left[(k-1) - e^{it}\right]^j
\]

\[
= \left[\sum_{m=0}^{j} \binom{j}{m} (-1)^{j-m} e^{-i(j-m)t}\right] \left[\sum_{l=0}^{j} \binom{j}{l} (-1)^{j-m} (k-1)^l e^{i(l-t)t}\right] = \sum_{s=-j}^{j} \alpha_s^j e^{ist},
\]

where

\[
\alpha_s^j = \begin{cases} 
(-1)^s \sum_{l=0}^{j-s} \binom{j}{l+s} \binom{j}{l} (k-1)^l, & \text{if } s \geq 0, \\
(-1)^{-s} \sum_{l=0}^{j-s} \binom{j}{l-s} \binom{j}{l} (k-1)^{-s-l}, & \text{if } s < 0.
\end{cases}
\]

(21)

Now we state and prove the combinatorial heat equation on \( T \).
Theorem 4.6. Let $T$ be a $k$-regular tree and let us denote $\beta_s = \frac{1 + (k-1)^s}{k(k-1)^{s-1}}$. If $f \in L^2(V(T))$ then the combinatorial heat equation (2) on $T$ has a unique solution $u(x,n) = \sum_{s=0}^{n} \left( w_s^{(n)} \sum_{d(x,y)=s} f(y) \right)$, where

$$w_s^{(n)} = \begin{cases} \beta_s \sum_{l \geq 0} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j+s}{l} \binom{j+s}{l+s}(k-1)^l \right], & \text{if } s > 0, \\ \sum_{l \geq 1} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l} \right] (k-1)^l, & \text{if } s = 0. \end{cases}$$

Proof. Using (18), for a fixed $x \in V(T)$, we obtain $\tilde{u}(0,n) = M_u(x,0,n) = u(x,n)$. In view of Corollary 4.3 and Lemma 4.5, the solution to the combinatorial heat equation on $T$ is

$$u(x,n) = \tilde{u}(0,n) = K_n \ast \tilde{f}(0).$$

Since $\tilde{f}(s) = \tilde{f}(-s)$ for all $s \in \mathbb{Z}$, we have

$$K_n \ast \tilde{f}(0) = \sum_{s \in \mathbb{Z}} \tilde{f}(s)K_n(-s) = K_n(0)\tilde{f}(0) + \sum_{s=1}^{\infty} [K_n(s) + K_n(-s)]\tilde{f}(s).$$

Further, if $a^j(t) = \sum_{s=-j}^{j} \alpha_s^j e^{ist}$ then $K_n(s) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \tilde{T}^{-1}(a^j)(s)$. Thus, using Remark 1.16, we get

$$K_n(s) = \begin{cases} \sum_{j \geq s} \binom{n}{j} (-1)^j \alpha_s^j, & \text{if } -n \leq s \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, by (21), for $0 < s \leq n$, we have

$$K_n(s) + K_n(-s) = \sum_{j \geq s} \binom{n}{j} (-1)^j (\alpha_s^j + \alpha_{-s}^j)$$

$$= \sum_{j \geq s} \binom{n}{j} (-1)^j \sum_{l=0}^{j-s} \binom{j}{l+s} \binom{j+s}{l+s}(k-1)^l[1 + (k-1)^s]$$

$$= [1 + (k-1)^s] \sum_{l \geq 1} \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j+s}{l} \binom{j+s}{l+s}(k-1)^l,$$

and for $s = 0$ with the convention $\binom{n}{0} = 1$, we have

$$K_n(0) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \alpha_0^j = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \sum_{l=0}^{j} \binom{j}{l} \binom{j}{l+s}(k-1)^l = \sum_{l \geq 0} \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l}^2(k-1)^l$$

$$= \sum_{j \geq 0} (-1)^j \binom{n}{j} \sum_{l \geq 1} \binom{n}{j} \binom{j}{l} \binom{j}{l+s}(k-1)^l$$

$$= \sum_{j \geq 0} (-1)^j \binom{n}{j} \sum_{l \geq 1} \binom{n}{j} \binom{j}{l}^2(k-1)^l$$

$$= \sum_{l \geq 1} \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l} \binom{j}{l+s}(k-1)^l$$

$$= \sum_{l \geq 1} \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l}^2(k-1)^l.$$
Therefore,

\[ K_n \ast \tilde{f}(0) \]
\[ = \left[ \sum_{l \geq 1} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l}^2 (k - 1)^l \right] \tilde{f}(0) \right. \]
\[ + \sum_{s=1}^{n} \left[ 1 + (k - 1)^s \right] \sum_{l \geq 0} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j + s} \binom{j + s}{l + s} \right] (k - 1)^l \tilde{f}(s) \]
\[ = \left[ \sum_{l \geq 1} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j} \binom{j}{l}^2 (k - 1)^l \right] f(x) \right. \]
\[ + \sum_{s=1}^{n} \left[ 1 + (k - 1)^s \right] \sum_{l \geq 0} \left[ \sum_{j \geq l} (-1)^j \binom{n}{j + s} \binom{j + s}{l + s} \right] (k - 1)^l \left. \frac{1}{S(s)} \sum_{d(x,y)=s} f(y) \right] \]
\[ = \sum_{s=0}^{n} \left( w_s^{(n)} \sum_{d(x,y)=s} f(y) \right). \]

Hence the desired result follows. \( \square \)

Before proceeding further, we state and prove a property of the solution of (20) that will be useful for solving the combinatorial wave equation on \( T \).

**Lemma 4.7.** Let \( f, g \in L^1(V(T)) \) and let (20) admit a solution \( \tilde{u}(r,n) \). Then \( \tilde{u}(\cdot, n) \in L^1(V(P)) \) for all \( n \in \mathbb{Z}_+ \).

**Proof.** Using (15), we can re-write (20) as

\[ \tilde{u}(r, n + 2) = 2\tilde{u}(r, n + 1) - (k + 1)\tilde{u}(r, n) + (k - 1)\tilde{u}(r + 1, n) + \tilde{u}(r - 1, n), \]
\[ \tilde{u}(r, 0) = \tilde{f}(r) \quad \text{and} \quad \tilde{u}(r, 1) = \tilde{u}(r, 0) + \tilde{g}(r). \]

As \( f, g \in L^1(V(T)) \), using Lemma 4.1, we have \( \tilde{f}, \tilde{g} \in L^1(V(P)) \). Therefore, by applying mathematical induction on discrete time variable \( n \), the desired result follows. \( \square \)

We now solve the initial value problem (20) and also obtain a necessary and sufficient condition for the existence of the solution.

**Lemma 4.8.** Let \( f, g \in L^1(V(T)) \). Then, the initial value problem (20) admits a unique solution if and only if \( \tilde{g}(0) = 0 \). In case the solution exists, it is unique and is expressed by

\[ \tilde{u}(r, n) = F_n \ast \tilde{f}(r) + G_n \ast \tilde{g}(r), \]

where \( F_n(r) = \tilde{S}^{-1} \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} (-a)^j \right)(r) \) and \( G_n(r) = \tilde{S}^{-1} \left( \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2j+1} (-a)^j \right)(r) \) with \( a(t) = -(k-1)e^{-it} + k - e^{it} \).
Theorem 4.9. Let $T$ be a $k$-regular tree. Then, for $f, g \in L^1(V(T))$, the combinatorial wave equation (3) on $T$ has a solution if and only if $\hat{g}(0) = 0$. Moreover, the solution is unique and if we denote \( \beta_s = \frac{1}{\frac{1}{k} + 1} \), then

\[
\hat{u}(x, n) = \sum_{s=0}^{[\frac{n}{s}]} \left( w_s^{(1,n)} \sum_{d(x,y)=s} f(y) \right) + \sum_{s=0}^{[\frac{n}{s}] - 1} \left( w_s^{(2,n)} \sum_{d(x,y)=s} g(y) \right),
\]

where

\[
w_s^{(1,n)} = \begin{cases} \beta_s \sum_{l \geq 0} \left[ \sum_{\ell \geq l} (-1)^\ell \binom{n}{\ell} \binom{j}{\ell+s} \binom{j+s}{l+s} \right] (k-1)^l, & \text{if } s > 0, \\ \sum_{\ell \geq 0} (-1)^\ell \binom{n}{2} \binom{j}{l} \binom{2}{l} (k-1)^l, & \text{if } s = 0, \end{cases}
\]

Proof. Using (15), we can re-write (20) as

\[
\begin{align*}
k\hat{u}(r, n) - (k-1)\hat{u}(r+1, n) - \hat{u}(r-1, n) + \hat{u}(r, n+2) - 2\hat{u}(r, n+1) + \hat{u}(r, n) &= 0, \\
\hat{u}(r, 0) &= \hat{f}(r) \quad \text{and} \quad \hat{u}(r, 1) - \hat{u}(r, 0) = \hat{g}(r).
\end{align*}
\]

Since $\hat{f}, \hat{g} \in L^1(V(\mathbb{P}))$, by Lemma 4.7, $\hat{u}(\cdot, n) \in L^1(V(\mathbb{P}))$ for all $n \in \mathbb{Z}_+$. Thus, taking the Fourier transform on both sides with respect to the variable $r \in \mathbb{Z} = V(\mathbb{P})$, we get

\[
\begin{align*}
k\hat{u}(t, n) - (k-1)e^{-it}\hat{u}(t, n) - e^{it}\hat{u}(t, n) + \hat{u}(t, n+2) - 2\hat{u}(t, n+1) + \hat{u}(t, n) &= 0, \\
\hat{u}(t, 0) &= \hat{f}(t) \quad \text{and} \quad \hat{u}(t, 1) - \hat{u}(t, 0) = \hat{g}(t),
\end{align*}
\]

By Remark 1.14, $\hat{u}(t, n)$ for all $n \in \mathbb{Z}_+$, is a continuous function on the unit circle $T$ with respect to the variable $t$. Hence, the point-wise calculation of the above equations is well defined. Now, re-writing the above equations, we have

\[
\begin{align*}
\hat{u}(t, n+2) - 2\hat{u}(t, n+1) - \left( (-a)(t) - 1 \right) \hat{u}(t, n) &= 0, \\
\hat{u}(t, 0) &= \hat{f}(t) \quad \text{and} \quad \hat{u}(t, 1) - \hat{u}(t, 0) = \hat{g}(t),
\end{align*}
\]

where $a(t) = -(k-1)e^{-it} + k - e^{it}$. Note that, the above recurrence relation (with respect to the variable $n \in \mathbb{Z}_+$) is of a form that is same as the recurrence relation that appeared in Theorem 2.3. Since, $\mathbb{Z}$ is also a finitely generated discrete abelian group thus, proceeding similar to the proof of Theorem 2.3, we obtain the desired result.

Now we state and prove the combinatorial wave equation on $T$. The ideas and calculations of the proof of the next theorem are similar to the proof of Theorem 4.6, but we provide the same, for the sake of completion.

Theorem 4.9. Let $T$ be a $k$-regular tree. Then, for $f, g \in L^1(V(T))$, the combinatorial wave equation (3) on $T$ has a solution if and only if $\hat{g}(0) = 0$. Moreover, the solution is unique and if we denote $\beta_s = \frac{1}{\frac{1}{k} + 1}$, then

\[
u(x, n) = \sum_{s=0}^{[\frac{n}{s}]} \left( w_s^{(1,n)} \sum_{d(x,y)=s} f(y) \right) + \sum_{s=0}^{[\frac{n}{s}] - 1} \left( w_s^{(2,n)} \sum_{d(x,y)=s} g(y) \right),
\]

where

\[
w_s^{(1,n)} = \begin{cases} \beta_s \sum_{l \geq 0} \left[ \sum_{\ell \geq l} (-1)^\ell \binom{n}{\ell} \binom{j}{\ell+s} \binom{j+s}{l+s} \right] (k-1)^l, & \text{if } s > 0, \\
\sum_{\ell \geq 0} (-1)^\ell \binom{n}{2} \binom{j}{l} \binom{2}{l} (k-1)^l, & \text{if } s = 0, \end{cases}
\]

21
and

\[ \begin{align*}
    w_s^{(2,n)}(n) &= \\
    &= \begin{cases} 
        \beta_s \sum_{l \geq 0} \left[ \sum_{j \geq l} (-1)^j \binom{n}{2(j+s)+1} \binom{j+s}{l} (k-1)^{l}, & \text{if } s > 0, \\
        \sum_{j \geq 0} (-1)^j \binom{n}{2j+1} + \sum_{l \geq 1} \left[ \sum_{j \geq l} (-1)^j \binom{n}{2j+1} \binom{j}{l} \right] (k-1)^{l}, & \text{if } s = 0. 
    \end{cases}
\end{align*} \]

Proof. Using (18), for a fixed \( x \in V(T) \), we obtain \( \bar{u}(0,n) = M_u(x,0,n) = u(x,n) \). In view of Corollary 4.3 and Lemma 4.8, the solution to the combinatorial wave equation on \( T \) is given by

\[ u(x,n) = \bar{u}(0,n) = F_n * \bar{f}(0) + G_n * \bar{g}(0). \]

Since \( \bar{f}(s) = \bar{f}(-s) \) for all \( s \in \mathbb{Z} \), we have

\[ F_n * \bar{f}(0) = \sum_{s \in \mathbb{Z}} \bar{f}(s) F_n(-s) = F_n(0) \bar{f}(0) + \sum_{s=1}^{\infty} [F_n(s) + F_n(-s)] \bar{f}(s). \]  \hfill (23)

Further, if \( a^j(t) = \sum_{s=-j}^{j} \alpha_s^j e^{ist} \) then \( F_n(s) = \sum_{j=0}^{\left[ \frac{n}{2} \right]} \binom{n}{2j} (-1)^j \tilde{\mathcal{F}}^{-1}(a^j)(s) \). Thus, using Remark 1.16, we get

\[ F_n(s) = \begin{cases} 
    \sum_{j \geq s} \binom{n}{2j} (-1)^j \alpha_s^j, & \text{if } - \left[ \frac{n}{2} \right] \leq s \leq \left[ \frac{n}{2} \right], \\
    0, & \text{if } s = 0.
\end{cases} \]

Therefore, for \( s > 0 \), using (21) we have

\[ F_n(s) + F_n(-s) = \sum_{j \geq s} \binom{n}{2j} (-1)^{j} (\alpha_s^j + \alpha_{-s}^j) \]

\[ = \sum_{j \geq s} \binom{n}{2j} (-1)^{j+s} \sum_{l=0}^{j-s} \binom{j}{l+s} \binom{j}{l} (k-1)^{l}[1 + (k-1)^{s}] \]

\[ = [1 + (k-1)^{s}] \sum_{l \geq 0} \left[ \sum_{j \geq l} (-1)^j \binom{n}{2(j+s)} \binom{j+s}{l} \right] (k-1)^{l}, \]

and for \( s = 0 \) with the convention \( \binom{0}{0} = 1 \), we have

\[ F_n(0) = \sum_{j=0}^{\left[ \frac{n}{2} \right]} \binom{n}{2j} (-1)^j \alpha_0^j = \sum_{j=0}^{\left[ \frac{n}{2} \right]} \binom{n}{2j} (-1)^j \sum_{l=0}^{j} \binom{j}{l}^2 (k-1)^{l} \]

\[ = \sum_{j \geq 0} (-1)^j \binom{n}{2j} + \sum_{l \geq 1} \left[ \sum_{j \geq l} (-1)^j \binom{n}{2j} \binom{j}{l}^2 \right] (k-1)^{l}. \]

22
Thus, (23) reduces to $F_n \ast \tilde{f}(0) = \sum_{s=0}^{n} \left( w_s^{(1,n)} \sum_{d(x,y)=s} f(y) \right)$. A similar calculation leads us to get

$G_n \ast \tilde{f}(0) = \sum_{s=0}^{n} \left( w_s^{(2,n)} \sum_{d(x,y)=s} g(y) \right)$. Hence the desired result follows.

We conclude this section with the following observation: Let $x_0$ be a fixed but arbitrary reference point in $V(T)$. Let $\Gamma = Aut(T)$ be the automorphism group of $T$ and $H = \{ \varphi \in Aut(T) : \varphi(x_0) = x_0 \}$ be the stabilizer of $x_0$. Since $T$ is vertex-transitive, so the orbit of $x_0$, namely $\{ \varphi(x_0) : \varphi \in Aut(T) \}$ is equal to $V(T)$. By the Orbit-Stabilizer theorem, we can identify $V(T)$ with $\Gamma/H$. If we choose $S = \{ \varphi \in \Gamma : \varphi(x_0) \sim x_0 \}$, it can be verified that $T$ can also be identified with the coset graph $G = \text{Coset}(\Gamma, H, S)$. It is interesting to observe that in this case, the subgroup $H$ is an infinite subgroup of the non-abelian group $\Gamma$. Therefore, in this section, we have solved the heat and wave equations on a class of graphs which can be identified with Cayley graphs as well as coset graphs, whenever the associated group is a non-abelian group.

References

[1] R. B. Bapat. Graphs and matrices. Hindustan Book Agency, New Delhi, 2010.

[2] E. Barletta and S. Dragomir. Combinatorial PDEs on Hamming graphs. Discrete Math., no. 1-3, 254:1-18, 2002.

[3] I. Chavel. Isoperimetric inequalities. Differential geometric and analytic perspectives. Cambridge Tracts in Mathematics, 145. Cambridge University Press, Cambridge, 2001.

[4] J. M. Cohen and M. Pagliacci. Explicit solutions for the wave equation on homogeneous trees. Adv. in Appl. Math. no. 4, 15:390-403, 1994.

[5] G. B. Folland. Introduction to partial differential equations. Second edition. Princeton University Press, Princeton, NJ, 1995.

[6] G. B. Folland. A course in abstract harmonic analysis. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[7] J. Jost. Partial differential equations. Second edition. Graduate Texts in Mathematics, 214. Springer, New York, 2007.

[8] A. K. Lal, S. Mohanty and N. Nilakantan. Combinatorial PDEs on Cayley and coset graphs. Discrete Math., no. 22, 311:2587-2592, 2011.
[9] J. Lauri and R. Scapellato. Topics in graph automorphisms and reconstruction. London Mathematical Society, Student Texts 54, Cambridge University Press, 2003.

[10] G. Medolla and A. G. Setti. The wave equation on homogeneous trees. Ann. Mat. Pura Appl. no. 4 176:127, 1999.

[11] W. Rudin. Fourier analysis on groups. Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

[12] M. Sugiura. Unitary representations and harmonic analysis. An introduction. Second edition. North-Holland Mathematical Library 44, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1990.