ESTIMATE OF SOME MEASURE-DIMENSIONS ASSOCIATED WITH HYPERBOLIC RECURRENT IFS

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Abstract. Let $\mu$ be a Borel probability measure generated by a hyperbolic recurrent iterated function system defined on a nonempty compact subset of $\mathbb{R}^d$. In this paper, we study the Hausdorff and the packing dimensions, and the quantization dimensions of $\mu$ with respect to the geometric mean error. The results in this paper establish the connections with various dimensions of the measure $\mu$, and generalize many known results about local dimensions and quantization dimensions of measures.

1. Introduction

Given a Borel probability measure $\mu$ on $\mathbb{R}^d$, the $n$th quantization error for $\mu$ with respect to the geometric mean error is given by

$$e_n(\mu) := \inf \left\{ \exp \int \log d(x, \alpha)d\mu(x) : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n \right\},$$

where $d(x, \alpha)$ denotes the distance between $x$ and the set $\alpha$ with respect to an arbitrary norm $d$ on $\mathbb{R}^d$. A set $\alpha$ for which the infimum is achieved and contains no more than $n$ points is called an optimal set of $n$-means for $\mu$, and the collection of all optimal sets of $n$-means for $\mu$ is denoted by $C_n(\mu)$. Under some suitable restriction $e_n(\mu)$ tends to zero as $n$ tends to infinity. Following [GL2] we write

$$\hat{e}_n := \hat{e}_n(\mu) = \log e_n(\mu) = \inf \left\{ \int \log d(x, \alpha)d\mu(x) : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n \right\}.$$

The numbers

$$\underline{D}(\mu) := \liminf \frac{\log n}{-\hat{e}_n(\mu)}$$

and

$$\overline{D}(\mu) := \limsup \frac{\log n}{-\hat{e}_n(\mu)},$$

are called the lower and the upper quantization dimensions of $\mu$ (of order zero), respectively. If $\underline{D}(\mu) = \overline{D}(\mu)$, the common value is called the quantization dimension of $\mu$ and is denoted by $D(\mu)$. The quantization dimension measures the speed at which the specified measure of the error tends to zero as $n$ tends to infinity. This problem arises in signal processing, data compression, cluster analysis, and pattern recognition, and it also has been studied in the context of economics, statistics, and numerical integration (see [BW, GG, GN, P, Z]). The quantization dimension with respect to the geometric mean error can be regarded as a limit state of that based on $L_r$-metrics as $r$ tends to zero (see [GL2, Lemma 3.5]). The following proposition gives a characterization of the lower and the upper quantization dimensions.

Proposition 1.1. (see [GL2, Proposition 4.3]) Let $\underline{D} = \underline{D}(\mu)$ and $\overline{D} = \overline{D}(\mu)$.

(a) If $0 \leq t < \underline{D} < s$, then

$$\lim_{n \to \infty} \left( \log n + t\hat{e}_n(\mu) \right) = +\infty,$$

and

$$\liminf_{n \to \infty} \left( \log n + s\hat{e}_n(\mu) \right) = -\infty.$$

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(b) If \( 0 \leq t < \overline{D} < s \), then
\[
\limsup_{n \to \infty} \left( \log n + t\hat{e}_n(\mu) \right) = +\infty, \quad \text{and} \quad \lim_{n \to \infty} \left( \log n + s\hat{e}_n(\mu) \right) = -\infty.
\]

For any \( \kappa > 0 \), the two numbers \( \liminf_n n^{1/\kappa} e_n(\mu) \) and \( \limsup_n n^{1/\kappa} e_n(\mu) \) are called the \( \kappa \)-dimensional lower and the upper quantization coefficients for \( \mu \) with respect to the geometric mean error. For every \( x \in \mathbb{R}^d \), the lower and the upper local dimensions of the measure \( \mu \) at \( x \) are defined, respectively, by
\[
d_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]
where \( B(x, r) \) is the ball of radius \( r \) centered at \( x \). We say that the local dimension exists at \( x \) if \( d_\mu(x) \) and \( \overline{d}_\mu(x) \) are equal, and write \( d_\mu(x) \) for the common value. These local dimensions, also known as pointwise dimensions, describe the power law behavior of \( \mu(B(x, r)) \) for small \( r \), with \( d_\mu(x) \) small if \( \mu \) is ‘highly concentrated’ near \( x \). Notice that \( d_\mu(x) = \infty \) if \( x \) is outside the support of \( \mu \) and \( d_\mu(x) = 0 \) if \( x \) is an atom of \( \mu \). The lower and the upper Hausdorff dimensions of \( \mu \) are defined, respectively, by
\[
\dim_* \mu = \inf \left\{ \dim H E : E \text{ is Borel with } \mu(E) > 0 \right\} \quad \text{and} \quad \dim^* \mu = \inf \left\{ \dim H E : E \text{ is Borel with } \mu(E) = 1 \right\}.
\]
Analogously, we define the lower and the upper packing dimensions of \( \mu \), respectively, by
\[
\dim_* \mu = \inf \left\{ \dim P E : E \text{ is Borel with } \mu(E) > 0 \right\} \quad \text{and} \quad \dim^* \mu = \inf \{ \dim P E : E \text{ is Borel with } \mu(E) = 1 \}.
\]
Clearly, \( \dim_* \mu \leq \dim^* \mu \) and \( \dim_* \mu \leq \dim^* \mu \). When the equality \( \dim_* \mu = \dim^* \mu \) and \( \dim_* \mu = \dim^* \mu \) are satisfied, we denote by \( \dim H \mu \) and \( \dim P \mu \), respectively, the Hausdorff and the packing dimensions of the measure \( \mu \). Hausdorff dimension and packing dimension of a measure are closely related to lower local dimension and upper local dimension of the measure. More precisely,
\[
(2) \quad \dim_* (\mu) = \sup \{ s : d_\mu(x) \geq s \text{ for } \mu \text{-a.e. } x \}, \quad \dim^* (\mu) = \inf \{ s : d_\mu(x) \leq s \text{ for } \mu \text{-a.e. } x \},
\]
and
\[
(3) \quad \dim_* (\mu) = \sup \{ s : \overline{d}_\mu(x) \geq s \text{ for } \mu \text{-a.e. } x \}, \quad \dim^* (\mu) = \inf \{ s : \overline{d}_\mu(x) \leq s \text{ for } \mu \text{-a.e. } x \}.
\]
Hence, for \( \mu \)-a.e. \( x \in \mathbb{R}^d \), it follows that
\[
0 \leq \dim_* (\mu) \leq d_\mu(x) \leq \dim^* (\mu) \leq d, \quad \text{and} \quad 0 \leq \dim_* (\mu) \leq \overline{d}_\mu(x) \leq \dim^* (\mu) \leq d.
\]
If \( d_\mu(x) \) and \( \overline{d}_\mu(x) \) are both constant for \( \mu \)-a.e. \( x \), then we say that \( \mu \) is ‘exact-dimensional’ or ‘unidimensional’. We say that a measure \( \mu \) has exact lower dimension \( s \) if \( d_\mu(x) = s \) for \( \mu \)-a.e. \( x \), and exact upper dimension \( s \) if \( \overline{d}_\mu(x) = s \) for \( \mu \)-a.e. \( x \). Thus, from (2) and (3), it follows that \( \mu \) has exact lower dimension \( s \) if and only if
\[
(4) \quad \dim H (\mu) = \dim_* (\mu) = \dim^* (\mu) = s,
\]
and \( \mu \) has exact upper dimension \( s \) if and only if
\[
(5) \quad \dim P (\mu) = \dim_* (\mu) = \dim^* (\mu) = s.
\]
For more details about the relationships between the different dimensions of measures, one is referred to [1, 2, 3, 4] and the references therein.

Let \( P = [p_{ij}]_{1 \leq i, j \leq N} \) be an \( N \times N \) irreducible row stochastic matrix, and \( \{ S_i : 1 \leq i \leq N \} \) be a system of contractive hyperbolic maps defined on a nonempty compact metric space \( X \subset \mathbb{R}^d \) such that \( \hat{s}_i d(x, y) \leq d(S_i(x), S_i(y)) \leq \overline{s}_i d(x, y) \) for all \( x, y \in X \) where \( 0 < \hat{s}_i \leq \overline{s}_i < 1, 1 \leq i \leq N \). Then, the collection \( \{ X ; S_i, p_{ij} : 1 \leq i, j \leq N \} \) is called a hyperbolic recurrent
iterated function system (hyperbolic recurrent IFS) (see [BEH]). Since $P$ is irreducible it follows that (see [F1]) there is a unique probability vector $p = (p_1, p_2, \ldots, p_N)$ such that
\[ \sum_{i=1}^{N} p_i p_{ij} = p_j. \]

Then, as shown in the next section, there exist unique nonempty compact sets $E_1, E_2, \ldots, E_N$ that satisfy $E_i = \bigcup_{j:p_{ji} > 0} S_i(E_j)$ for all $1 \leq i \leq N$. Let $E = \bigcup_{i=1}^{N} E_i$. Then, by Kolmogorov’s extension theorem, it can be proved that there exists a unique Borel probability measure $\mu$ on $\mathbb{R}^d$ with support $E$ such that $\mu$ satisfies:
\[ \mu = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ji} \mu_j \circ S_i^{-1}, \]

where $\mu_j := \mu|_{E_j}$, i.e., $\mu_j$ is the restriction of $\mu$ on $E_j$, i.e., for any Borel $B \subset \mathbb{R}^d$, we have $\mu_j(B) = \mu(B \cap E_j)$ for all $1 \leq j \leq N$. We say that the hyperbolic recurrent IFS satisfies the open set condition (OSC) if there exist bounded nonempty open sets $U_1, U_2, \ldots, U_N$ with the property that
\[ \bigcup_{\{j: p_{ji} > 0\}} S_i(U_j) \subset U_i \text{ and } S_i(U_j) \bigcap S_i(U_k) = \emptyset \text{ for } j \neq k \text{ with } p_{ji} p_{ki} > 0. \]

The hyperbolic recurrent IFS satisfies the strong separation condition (SSC) if
\[ d(S_i(E_k), S_j(E_l)) > 0 \text{ for all } k \neq l \text{ with } 1 \leq i, j, k, l \leq N. \]

In this paper, under the open set condition in Theorem 3.1, we have proved that for $\mu$-a.e. $x \in \mathbb{R}^d$,
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij} \leq \sum_{i=1}^{N} p_i \log \underline{s}_i. \]

Thus, by (2), (3), and (6), we have
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij} \leq \dim_s(\mu) \leq \underline{d}(\mu) \leq D(\mu) \leq \overline{D}(\mu) \leq \dim^s(\mu). \]

The following theorem is known.

**Theorem 1.2. (see [Z2, Theorem 2.1])** Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^d$. Assume that there exist constants $C > 0$ and $\eta > 0$ such that $\mu(B(x, \epsilon)) \leq C \epsilon^\eta$ for every $x \in \mathbb{R}^d$ and all $\epsilon > 0$. Then, $\dim_s(\mu) \leq D(\mu) \leq \overline{D}(\mu) \leq \dim^s(\mu)$.

In Lemma 2.1 we have proved that the conditions given in the statement of Theorem 1.2 are also true for the hyperbolic recurrent IFS considered in this paper. Thus, by (7) and Theorem 1.2 we see that
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij} \leq D(\mu) \leq \overline{D}(\mu) \leq \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}. \]

Under the strong separation condition in Theorem 3.2 we give an independent proof of it, and showed that
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij} \leq d_\mu(x) \leq \overline{D}(\mu) \leq \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}. \]

Notice that if we assume that $s_i = \underline{s}_i = s_i$ and $p_{ij} = p_j$ for all $1 \leq i, j \leq N$, then (6) reduces to
\[ d_\mu(x) = \overline{d}(\mu) = \sum_{i=1}^{N} \frac{p_i \log p_i}{\sum_{i=1}^{N} p_i \log s_i}. \]
which is the result of Geronimo and Hardin in \[ \text{[GH]} \], and \( \text{(8)} \) reduces to
\[
D(\mu) = \overline{D}(\mu) = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log s_i},
\]
which is the result of Graf-Luschgy in \[ \text{[GL2]} \]. In addition, Theorem 3.1 generalizes a similar result of Deliu et al. in \[ \text{[DGSH]} \], and Theorem 3.2 generalizes a similar result of Roychowdhury which is the result of Graf-Luschgy in \[ \text{[GL2]} \]. In addition, Theorem 3.1 generalizes a similar

2. Basic definitions and results

Let \( X \) be a nonempty compact set equipped with a metric \( d \), such that \( X = \text{cl}(\text{int}X) \). Let \( N \geq 2 \), and \( P = [p_{ij}]_{1 \leq i,j \leq N} \) be an \( N \times N \) irreducible row stochastic matrix, in other words, for any \( i \), \( \sum_{j=1}^{N} p_{ij} = 1 \), and for all \( i,j \) it follows that \( p_{ij} \geq 0 \), and there exists a series of indicators: \( i_1, i_2, \cdots, i_n \) which satisfy \( i_1 = i, i_n = j \) such that
\[
p_{i_1i_2p_{i_2i_3}\cdots p_{i_{n-1}i_n}} > 0.
\]
Let \( \{S_i : 1 \leq i \leq N\} \) be a system of contractive hyperbolic maps defined on the compact metric space \( X \) such that \( s_i d(x,y) \leq d(S_i(x),S_i(y)) \leq \bar{s}_i d(x,y) \) for all \( x, y \in X \), where \( 0 < s_i \leq \bar{s}_i < 1 \), and \( 1 \leq i \leq N \). Then, the collection \( \{X ; S_1, p_{ij} : 1 \leq i, j \leq N\} \) is called a hyperbolic recurrent iterated function system (hyperbolic recurrent IFS) (see \[ \text{[BEH]} \]). Define the Hausdorff metric \( h \) by
\[
h(E, F) = \inf\{\delta : d(x, F) \leq \delta \text{ for all } x \in E, \text{ and } d(y, E) \leq \delta \text{ for all } y \in F\}
\]
in the space \( C \) of all nonempty compact subsets of \( X \) (see \[ \text{[H, F]} \]). The mapping \( \Phi \) on the \( N \)-fold product space \( C^N \) given by
\[
\Phi(F_1, \cdots, F_N) = \left( \bigcup_{\{j : p_{j1} > 0\}} S_1(F_j), \cdots, \bigcup_{\{j : p_{jN} > 0\}} S_N(F_j) \right)
\]
is a contraction mapping. By Banach’s Fixed Point Theorem the contraction mapping \( \Phi \) has a fixed point in \( C^N \) (see \[ \text{[BEH]} \]), i.e., a vector of nonempty compact subsets of \( X \), \( (E_1, \cdots, E_N) \in C^N \), with
\[
E_i = \bigcup_{\{j : p_{ji} > 0\}} S_i(E_j),
\]
for all \( 1 \leq i \leq N \). The union \( E = \bigcup_{i=1}^{N} E_i \) is called the limit set of the hyperbolic recurrent IFS. Define \( \Omega \) and \( T : \Omega \rightarrow \Omega \) by
\[
\Omega = \{ x = (x_i)_{i=1}^{\infty} : 1 \leq x_i \leq N, p_{x_ix_{i+1}} > 0 \text{ for } i = 1, 2, \cdots \}
\]
and
\[
T : \Omega \ni (x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, \cdots) \in \Omega.
\]
Let \( d : \Omega \times \Omega \rightarrow \mathbb{R} \) be defined by
\[
d(x, y) = 2^{-n} \text{ if and only if } n = \min\{m : x_m \neq y_m\}
\]
for \( x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots) \in \Omega \). Then, \( d \) is a metric on \( \Omega \). With this metric \( \Omega \) becomes a compact metric space, and \( T \) is called a shift map on \( \Omega \).

For \( n \geq 2 \), let \( W_n \) denote the set of all \( n \)-tuples \( (i_1, i_2, \cdots, i_n) \) (called words of length \( n \)), which are admissible with respect to \( \Omega \), i.e., there exists a sequence \( (i'_1, i'_2, \cdots) \in \Omega \) such that
\[
i'_1 = i_1, i'_2 = i_2, \cdots, i'_n = i_n.
\]
Set \( W = \bigcup_{n \geq 2} W_n \). If \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \in W \), then the set \( \{y \in \Omega : y_i = \omega_i, 1 \leq i \leq n\} \) is called a cylinder in \( \Omega \) of length \( n \) generated by the word \( \omega \). A cylinder of length zero is called the empty cylinder. The set of all sequences in \( \Omega \) starting with the symbol \( i \) is denoted by \( C(i) \). For \( \omega = (\omega_1, \omega_2, \cdots) \in W \cup \Omega \), if \( n \) does not exceed
the length of $\omega$, by $\omega|_n$ we mean $\omega|_n = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\omega|_0 = \emptyset$, $\omega$ is called an extension of $\tau \in W$ if $\omega|_{|\tau|} = \tau$, where $|\tau|$ represents the length of $\tau$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_p)$ in $W$, if $p_{\omega_n\tau_1} > 0$ by $\omega \tau$ we mean $\omega \tau = (\omega_1, \omega_2, \ldots, \omega_n, \tau_1, \ldots, \tau_p)$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W$, $n \geq 2$, let us write

$$S_\omega = S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_{n-1}}; \quad p_\omega = p_{\omega_n}p_{\omega_n\omega_{n-1}} \cdots p_{\omega_2\omega_1}; \quad P_\omega = p_{\omega_n\omega_{n-1}} \cdots p_{\omega_2\omega_1},$$

and $E_\omega = S_\omega(E_{\omega_n})$.

From (9) it follows that the limit set of the recurrent hyperbolic IFS satisfies the following invariance equality (see [BEH]):

$$(10) \quad E = \bigcup_{(\omega_1, \ldots, \omega_n) \in W_n} S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_{n-1}}(E_{\omega_n}) \text{ for } n \geq 2.$$

Let $\mathcal{B}$ be the Borel sigma-algebra generated by the cylinders in $\Omega$. The matrix $P$ being irreducible, determines a unique probability vector $p = (p_1, p_2, \ldots, p_N)$ such that $pP = p$, i.e.,

$$\sum_{i=1}^N p_ip_{ij} = p_j \text{ for } 1 \leq j \leq N,$$

Define

$$\nu\left(\{\tau \in \Omega : \tau_i = \omega_i, 1 \leq i \leq n\}\right) = p_{\omega_n}p_{\omega_n\omega_{n-1}} \cdots p_{\omega_2\omega_1},$$

where $\{\tau \in \Omega : \tau_i = \omega_i, 1 \leq i \leq n\}$ is the cylinder set in $\Omega$ generated by the word $(\omega_1, \ldots, \omega_n)$. By the Kolmogorov’s extension theorem $\nu$ can be extended to a unique Borel probability measure, which is also identified as $\nu$, on $(\Omega, \mathcal{B})$. Clearly supp($\nu$) = $\Omega$ and $T$ is an ergodic transformation on $(\Omega, \mathcal{B}, \nu)$. The transformation $T$ is called the Markov shift with respect to the transition matrix $P$ and stationary distribution $\nu$, and $\nu$ is called the ergodic Markov measure on $(\Omega, \mathcal{B})$. Since given $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ the diameters of the compact sets $S_{\omega|n}(E_{\omega_n})$ converge to zero and since they form a descending family, the set

$$\bigcap_{n=2}^\infty S_{\omega|n}(E_{\omega_n})$$

is a singleton and therefore, denoting its element by $\pi(\omega)$, defines the coding map $\pi : \Omega \to E$, i.e., for $(x_1, x_2, x_3, \ldots) \in \Omega$ we have

$$\pi(x_1, x_2, x_3, \ldots) = \bigcap_{n=2}^\infty S_{x_1} \circ S_{x_2} \circ \cdots \circ S_{x_{n-1}}(E_{x_n}).$$

Let $\mu$ be the image measure of the probability measure $\nu$ under the coding map $\pi$ on the limit set $E$, i.e., $\mu = \nu \circ \pi^{-1}$. Let $C(i)$ represents the set of all sequences in $\Omega$ starting with the symbol $i$. Define

$$\pi_i := \pi|_{C(i)} \text{ and } \mu_i := \mu|_{E_i},$$

i.e., $\pi_i$ is the restriction of $\pi$ on $C(i)$, and $\mu_i$ is the restriction of $\mu$ on $E_i$, i.e., for any Borel $B \subset \mathbb{R}^d$, we have $\mu_i(B) = \mu(B \cap E_i)$. Thus, $\mu_i = \nu \circ \pi_i^{-1}$, and it satisfies:

$$\mu_i = \sum_{j=1}^N p_{ji}\mu_j \circ S_i^{-1}$$

for each $1 \leq i \leq N$. Hence, we get a Markov type measure $\mu := \sum_{i=1}^N \mu_i$ supported by $E$ such that

$$(11) \quad \mu = \sum_{i=1}^N \sum_{j=1}^N p_{ji}\mu_j \circ S_i^{-1}.$$
which implies \( \mu_i(E_i) = \sum_{j=1}^{N} p_{ji} p_{j} = p_i \), and

\[
\mu(E) = \sum_{i=1}^{N} \mu_i \left( \bigcup_{k=1}^{N} E_k \right) = \sum_{i=1}^{N} \mu_i(E_i) = \sum_{i=1}^{N} p_i = 1.
\]

The hyperbolic recurrent IFS satisfies the open set condition (OSC) if there exist bounded nonempty open sets \( U_1, U_2, \ldots, U_N \) with the property that

\[
\bigcup_{\{j: p_{ji} > 0\}} S_i(U_j) \subset U_i \text{ and } S_i(U_j) \bigcap S_i(U_k) = \emptyset \quad \text{for } j \neq k \text{ with } p_{ji} p_{ki} > 0.
\]

The hyperbolic recurrent IFS satisfies the strong separation condition (SSC) if

\[
d(S_i(E_k), S_j(E_i)) > 0 \quad \text{for all } k \neq i, j, k, \ell \leq N.
\]

It is a well-known fact that the hyperbolic recurrent IFS satisfies the open set condition if it satisfies the strong separation condition.

We call \( \Gamma \subset W \) a finite maximal antichain if \( \Gamma \) is a finite set of words in \( W \), such that every sequence in \( \Omega \) is an extension of some word in \( \Gamma \), but no word of \( \Gamma \) is an extension of another word in \( \Gamma \). Notice that as all words of \( W \) are of length at least two, for any \( \omega \in \Gamma \), we have \( |\omega| \geq 2 \).

We now prove the following lemma assuming the open set condition.

**Lemma 2.1.** There exist constants \( C > 0 \) and \( \eta > 0 \) depending on \( \mu \) such that

\[
\mu(B(x, \epsilon)) \leq C \epsilon^{\eta} \quad \text{for every } x \in \mathbb{R}^d \text{ and all } \epsilon > 0.
\]

As a consequence, \( e_n(\mu) < e_{n-1}(\mu) \) (\( e_0(\mu) := \infty \)) and \( C_n(\mu) \neq \emptyset \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( \epsilon_0 \in (0, 1] \) be arbitrary. By [G.1] Lemma 12.3, it suffices to show \((12)\) for every \( x \in E \) and all \( \epsilon \in (0, \epsilon_0) \). Write \( s_{\min} = \min \{ s_i : 1 \leq i \leq N \} \), and for \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W \) let \( \omega^{-} = (\omega_1, \omega_2, \ldots, \omega_{n-1}) \), i.e., \( \omega^{-} \) is the word obtained from \( \omega \) by deleting the last letter of \( \omega \), and \( X_\omega := S_\omega(X) \), \( s_\omega := s_{\omega_1} s_{\omega_2} \cdots s_{\omega_{n-1}} \). Without any loss of generality we assume that the diameter of \( X \) is one. Let

\[
\Gamma_\epsilon = \{ \omega \in W : s_{\omega^{-}} \geq \epsilon > s_{\omega} \} \text{ and } \Gamma_\epsilon(x) = \{ \omega \in \Gamma_\epsilon : X_\omega \cap B(x, \epsilon) \neq \emptyset \}.
\]

We claim that there exists a constant \( C > 0 \), independent of \( x \) and \( \epsilon \), such that \( |\Gamma_\epsilon(x)| < C \).

Since \( X \) has nonempty interior, \( X \) contains a ball of radius \( a \), where \( a > 0 \) is a constant, and so for each \( \omega \in \Gamma_\epsilon \), the set \( X_\omega \) contains a ball of radius \( as_{\omega^{-}} \geq as_{\omega^{-}} s_{\min} \geq as_{\min} \), and due to open set condition, all such balls are disjoint. Again all \( X_\omega \) for \( \omega \in \Gamma_\epsilon \) are contained in the ball \( B(x, 2\epsilon) \). Hence, comparing the volumes, we have

\[
|\Gamma_\epsilon(x)|(as_{\min} \epsilon)^d \leq (2\epsilon)^d \text{ which implies } |\Gamma_\epsilon(x)| \leq 2^d (as_{\min})^{-d},
\]

where \( d \) is dimension of the underlying space. Write \( C = 2^d (as_{\min})^{-d} \). Then, \( C > 0 \), and is independent of \( x \) and \( \epsilon \), and thus the claim is proved. Now to prove the lemma, write

\[
P_{\max} = \max \{ \max \{ p_{1}, p_{2}, \ldots, p_{N} \}, \max \{ p_{ij} : 1 \leq i, j \leq N \} \}.
\]

Then, from \( B(x, \epsilon) \subset \bigcup_{\omega \in \Gamma_\epsilon(x)} X_\omega \), we obtain

\[
\mu(B(x, \epsilon)) \leq \sum_{\omega \in \Gamma_\epsilon(x)} \mu(X_\omega) = \sum_{\omega \in \Gamma_\epsilon(x)} \mu(E_\omega) \leq C \max_{\omega \in \Gamma_\epsilon(x)} p_\omega \leq C \max_{\omega \in \Gamma_\epsilon(x)} \left( P_{\max} \right)^{|\omega|/\log s_{\min}}.
\]

Again, for \( \omega \in \Gamma_\epsilon(x) \), we have \( s_{\min}^{|\omega|} \leq s_{\omega} < \epsilon \), and so, \( |\omega| \geq \frac{\log \epsilon}{\log s_{\min}} \). Combining this facts leads to

\[
\mu(B(x, \epsilon)) \leq C \left( P_{\max} \right)^{\frac{\log \epsilon}{\log s_{\min}}} = C \epsilon^{\log(P_{\max})/\log s_{\min}}.
\]
The lemma follows by setting \( \eta = \log(P_{\text{max}})/\log s_{\text{min}} \). As in [GL2, Proposition 3.1], one can see that the condition in [GL2, Theorem 2.5] is satisfied. As a consequence, \( e_n(\mu) < e_{n-1}(\mu) \) \((e_0(\mu) := \infty)\) and \( C_n(\mu) \neq \emptyset \) for every \( n \in \mathbb{N} \).

In the next section we state and prove the main results of the paper.

3. Main Results

Theorem 3.1 and Theorem 3.2 contain the main results of the paper. First, we state and prove the following theorem about the lower and the upper local dimensions of the measure \( \mu \).

**Theorem 3.1.** Let \( \mu \) be the probability measure generated by the hyperbolic recurrent IFS \( \{X; S_i, p_{ij} : 1 \leq i, j \leq N\} \) satisfying the open set condition. Then, for \( \mu \text{-a.e. } x \in X \),

\[
\frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}}{\sum_{i=1}^{N} p_i \log \rho_s} \leq d_\mu(x) \leq \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}}{\sum_{i=1}^{N} p_i \log \overline{s}_i},
\]

where \( (p_1, p_2, \ldots, p_N) \) is the stationary distribution associated with \( [p_{ij}]_{1 \leq i, j \leq N} \).

**Proof.** Let \( s_i \) are the contractive ratios of the hyperbolic maps \( S_i, 1 \leq i \leq N \). Then, each \( s_i \) varies between the two numbers \( \overline{s}_i \) and \( \underline{s}_i \). To prove the theorem, in the first sight we assume that the contractive ratios \( s_i \) of \( S_i \) are fixed, i.e., we are assuming that the hyperbolic maps \( S_i \) are similarity mappings with contractive ratios \( s_i \) for all \( 1 \leq i \leq N \). Let \( E_1, E_2, \ldots, E_N \) denote the components of the attractor \( E \), such that \( E = \bigcup_{i=1}^{N} E_i \). Write,

\[
\alpha := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}}{\sum_{i=1}^{N} p_i \log s_i}.
\]

Fix \( x \in E \), and let \( \omega = (\omega_1, \omega_2, \ldots) \in \Omega \) be the code of \( x \). Consider the ball \( B_\rho(x(\omega)) \) of diameter \( \rho \) centered at \( x \). Let \( \ell \) be the least positive integer such that

\[
S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_{\ell-1}}(E_{\omega_{\ell}}) \subset B_\rho(x(\omega)),
\]

which implies

\[
\mu(B_\rho(x(\omega))) \geq p_{\omega_1} p_{\omega_2 \omega_{\ell-1}} \cdots p_{\omega_2 \omega_1}.
\]

Set

\[
s_{\text{min}} := \min\{s_i : 1 \leq i \leq N\},\quad p_{\text{min}} := \min\{p_1, p_2, \ldots, p_N\},
\]

\[
p_{\text{max}} := \max\{p_1, p_2, \ldots, p_N\},\quad L_{\text{min}} := \min\{\text{diam}(E_j) : 1 \leq j \leq N\},
\]

\[
L_{\text{max}} := \max\{\text{diam}(E_j) : 1 \leq j \leq N\}.
\]

Then, by the definition of \( \ell \), we have

\[
\prod_{i=1}^{\ell-1} s_{\omega_i} \text{diam}(E_{\omega_i}) \leq \rho, \quad \text{which yields} \quad \prod_{i=1}^{\ell-1} s_{\omega_i} \leq \rho L_{\text{min}}^{-1},
\]

and

\[
\prod_{i=1}^{\ell-2} s_{\omega_i} \text{diam}(E_{\omega_{\ell-1}}) \geq \rho, \quad \text{which yields} \quad \prod_{i=1}^{\ell-1} s_{\omega_i} \geq \rho s_{\text{min}} L_{\text{max}}^{-1}.
\]

Thus, by the definition of \( \ell \), we have

\[
\rho s_{\text{min}} L_{\text{max}}^{-1} \leq \prod_{i=1}^{\ell-1} s_{\omega_i} \leq \rho L_{\text{min}}^{-1}.
\]

Therefore,

\[
\mu(B_\rho(x(\omega))) \geq \frac{\prod_{i=1}^{\ell-1} p_{\omega_{i+1} \omega_i}}{\prod_{i=1}^{\ell-1} s_{\omega_i}^{\alpha}} \left( \frac{\rho s_{\text{min}}}{L_{\text{max}}} \right)^\alpha \geq \rho^\alpha p_{\text{min}} \left( \frac{s_{\text{min}}}{L_{\text{max}}} \right)^\alpha \prod_{i=1}^{\ell-1} \frac{p_{\omega_{i+1} \omega_i}}{s_{\omega_i}^{\alpha}}.
\]
Consequently,
\begin{equation}
\mu(B_\rho(x(\omega))) \geq C_1 \rho^\alpha \prod_{i=1}^{\ell-1} \frac{p_{\omega i+1 \omega i}^\alpha}{s_{\omega i}^\alpha},
\end{equation}
where $C_1 = \min(s_{\min}/L_{\max})^\alpha$. We next obtain an upper bound for $\mu(B_\rho(x(\omega)))$. For $\rho > 0$ and $(j_1, j_2, \cdots) \in \Omega$, let $q$ be the least integer such that
\begin{equation}
\prod_{m=1}^{q-1} s_{jm} < \rho.
\end{equation}
Let $S(\rho)$ be the set of such finite codes $(j_1, j_2, \cdots, j_q)$. By making the identification of $(j_1, j_2, \cdots, j_q)$ with the corresponding cylinder set in $\Omega$, we see that $S(\rho)$ generates a partition of $\Omega$. Let $U_1, U_2, \cdots, U_N$ be the open sets that arise in the open set condition. For each $(j_1, \cdots, j_q) \in S(\rho)$, set
\begin{equation*}
U_{j_1, \cdots, j_q} = S_{j_1} \circ S_{j_2} \circ \cdots \circ S_{j_q-1}(U_{j_q}).
\end{equation*}
For fixed $j_1 \in \{1, 2, \cdots, N\}$ the sets $C_{j_1}(\rho) = \{U_{j_1, j_2, \cdots, j_q} : (j_1, j_2, \cdots, j_q) \in S(\rho)\}$ are open disjoint sets. From (15) and noting the fact that, by assumption, the $S_i$’s are similitudes, there are constants $c_1 > 0$ and $c_2 > 0$ independent of $\rho$ and $j_1$ such that each $U_{j_1, \cdots, j_q}$ contains a ball of radius $c_1 \rho$ and is contained in a ball of radius $c_2 \rho$. Consequently, Lemma 5.3.1 of Hutchinson (see [H]) implies that there are at most $(1+2c_2)/c_1)^d$ elements of $C_{j_1}(\rho)$ whose closure meets $B_\rho(x(\omega))$ where $d$ is the dimension of the underlying space. Let $E_{j_1, j_2, \cdots, j_q} = S_{j_1} \circ S_{j_2} \circ \cdots \circ S_{j_q-1}(E_{j_q})$ and let $I_\omega(\rho)$ denote the set of $(j_1, \cdots, j_q) \in S(\rho)$ such that $E_{j_1, j_2, \cdots, j_q}$ meets $B_\rho(x(\omega))$. The open set condition implies that $E_{j_1, j_2, \cdots, j_q} \subset \bigcup_{j_1, j_2, \cdots, j_q} U_{j_1, j_2, \cdots, j_q}$ and thus $I_\omega(\rho)$ contains at most $N((1+2c_2)/c_1)^d$ elements. Write $\tilde{N} = N((1+2c_2)/c_1)^d$. Then, we have
\begin{align*}
\mu(B_\rho(x(\omega))) &= \nu(\pi^{-1}(B_\rho(x(\omega)))) \leq \nu(I_\omega(\rho)) = \sum_{(j_1, \cdots, j_q) \in I_\omega(\rho)} p_{j_1} p_{j_2, j_3, \cdots, j_q} \\
&= \sum_{(j_1, \cdots, j_q) \in I_\omega(\rho)} \frac{p_{j_1}^\alpha p_{j_2, j_3, \cdots, j_q}^\alpha}{s_{j_1}^\alpha s_{j_2}^\alpha \cdots s_{j_q-1}^\alpha} \left(s_{j_1}^\alpha s_{j_2}^\alpha \cdots s_{j_q-1}^\alpha\right) \leq p_{\max}^\alpha \sum_{(j_1, \cdots, j_q) \in I_\omega(\rho)} \left(\prod_{i=1}^{q-1} \frac{p_{j_{i+1} j_i}}{s_{j_i}^\alpha}\right) \text{ [by (15)]}
\end{align*}
\begin{align*}
&\leq p_{\max}^\alpha \tilde{N} \rho^\alpha \left(\prod_{i=1}^{q-1} \frac{p_{j_{i+1} j_i}}{s_{j_i}^\alpha}\right),
\end{align*}
which yields
\begin{equation}
\mu(B_\rho(x(\omega))) \leq C \rho^\alpha \left(\prod_{i=1}^{q-1} \frac{p_{j_{i+1} j_i}}{s_{j_i}^\alpha}\right),
\end{equation}
where $C = p_{\max} \tilde{N}$. Now, let us define two functions $f, g : \Omega \to \mathbb{R}$ as follows:
\begin{align*}
&f(j_1, j_2, \cdots) = \log p_{j_2 j_1} \quad \text{and} \quad g(j_1, j_2, \cdots) = \log s_{j_1}^\alpha,
&\text{for } j = (j_1, j_2, \cdots) \in \Omega.
\end{align*}
Then, by Birkhoff’s Ergodic Theorem, for $\nu$-a.e. $j \in \Omega$, we have
\begin{align*}
\lim_{q \to \infty} \frac{1}{q} \sum_{i=0}^{q-1} f(T^i(j)) = \int_{\Omega} f(j) d\nu \quad \text{and} \quad \lim_{q \to \infty} \frac{1}{q} \sum_{i=0}^{q-1} g(T^i(j)) = \int_{\Omega} g(j) d\nu.
\end{align*}
Notice that
\begin{align*}
\frac{1}{q} \sum_{i=0}^{q-1} f(T^i(j)) &= \frac{1}{q} \sum_{i=0}^{q-1} f(j_{i+1} j_{i+2} \cdots) = \frac{1}{q} \sum_{i=0}^{q-1} \log p_{j_{i+2} j_{i+1}} = \frac{1}{q} \log \prod_{i=1}^{q} p_{j_{i+1} j_i}.
\end{align*}
Using the same method, and recalling the fact that \( C(i) = \{(j_1, j_2, \cdots) \in \Omega : j_1 = i\} \), we have
\[
\int_{\Omega} f(j) d\nu = \sum_{i=1}^{N} p_i \int_{C(i)} f(j) d\nu = \sum_{i=1}^{N} p_i \sum_{j=1}^{N} p_{ij} \log p_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij}.
\]
Hence, for \( \nu \)-a.e. \( j \in \Omega \), we have
\[
\lim_{q \to \infty} \frac{1}{q} \log \prod_{i=1}^{q} p_{j_{i+1}j_i} = \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_{ij} \log p_{ij},
\]
and similarly,
\[
\lim_{q \to \infty} \frac{1}{q} \log \prod_{i=1}^{q} s_{j_i}^{\alpha} = \alpha \sum_{i=1}^{N} p_i \log s_i.
\]
By (15), we have
\[
\rho s_{\min} \leq s_{j_1} s_{j_2} \cdots s_{j_{q-1}} = \prod_{i=1}^{q-1} s_{j_i} < \rho,
\]
which implies \( q \to \infty \) if \( \rho \to 0 \), and thus for \( \nu \)-a.e. \( j \in \Omega \), we have
\[
\lim_{\rho \to 0} \frac{1}{q} \log \rho = \lim_{q \to \infty} \frac{1}{q} \log \prod_{i=1}^{q-1} s_{j_i} = \sum_{i=1}^{N} p_i \log s_i.
\]
Using (17), (18) and (19), for \( \nu \)-a.e. \( j \in \Omega \), we have
\[
\lim_{\rho \to 0} \log \left( \prod_{i=1}^{q-1} \frac{p_{j_{i+1}j_i}}{s_{j_i}^{\alpha}} \right) / \log \rho = 0.
\]
By (16) and (20), for \( \nu \)-a.e. \( \omega \in \Omega \), we have
\[
\lim_{\rho \to 0} \frac{\log \mu(B_\rho(x(\omega)))}{\log \rho} \geq \alpha.
\]
Equivalent to (19), by (13), for \( \nu \)-a.e. \( \omega \in \Omega \), we have
\[
\lim_{\rho \to 0} \frac{\log \rho}{q} = \lim_{\ell \to \infty} \frac{1}{\ell} \log \prod_{i=1}^{\ell-1} s_{\omega_i} = \sum_{i=1}^{N} p_i \log s_i.
\]
Thus, equivalent to (20), the following relation is also true, for \( \nu \)-a.e. \( \omega \in \Omega \),
\[
\lim_{\rho \to 0} \log \left( \prod_{i=1}^{\ell-1} \frac{p_{\omega_{i+1}\omega_i}}{s_{\omega_i}^{\alpha}} \right) / \log \rho = 0.
\]
By (14) and (23), for \( \nu \)-a.e. \( \omega \in \Omega \), we have
\[
\lim_{\rho \to 0} \frac{\log \mu(B_\rho(x(\omega)))}{\log \rho} \leq \alpha.
\]
Thus, by (21) and (24), for \( \nu \)-a.e. \( \omega \in \Omega \), we have
\[
\lim_{\rho \to 0} \frac{\log \mu(B_\rho(x(\omega)))}{\log \rho} = \alpha.
\]
Therefore, from the relationship between \( \nu \) and \( \mu \), for \( \mu \)-a.e. \( x \in X \), it follows that
\[
\lim_{\rho \to 0} \frac{\log \mu(B_\rho(x))}{\log \rho} = \alpha.
\]
Let us now vary \( s_i \) between \( \underline{s}_i \) and \( \overline{s}_i \), then \( \alpha \) will also vary between the two numbers

\[
\alpha_1 := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \underline{s}_i} \quad \text{and} \quad \alpha_2 := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \overline{s}_i}
\]

\( \alpha \) being continuous and strictly increasing in the closed interval \([\alpha_1, \alpha_2]\), for \( \mu \)-a.e. \( x \in X \), we have

\[
\frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \underline{s}_i} \leq d_\mu(x) \leq \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \overline{s}_i},
\]

which completes the proof of the theorem. \( \square \)

Now, we state and prove the following theorem, which gives the bounds of the lower and the upper quantization dimensions of the measure \( \mu \) with respect to the geometric mean error.

**Theorem 3.2.** Let \( \mu \) be the probability measure generated by the hyperbolic recurrent IFS \( \{X; S_i, p_{ij} : 1 \leq i, j \leq N\} \) satisfying the strong separation condition. Then,

\[
\frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \underline{s}_i} \leq D(\mu) \leq \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \overline{s}_i},
\]

where \((p_1, p_2, \ldots, p_N)\) is the stationary distribution associated with \( [p_{ij}]_{1 \leq i, j \leq N} \).

To prove Theorem 3.2 we need some lemmas and propositions. We set

\[
\alpha_1 := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \underline{s}_i} \quad \text{and} \quad \alpha_2 := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_ip_ij \log p_ij}{\sum_{i=1}^{N} p_i \log \overline{s}_i}.
\]

In the sequel for each \( 1 \leq i \leq N \), let \( \hat{\mu}_i \) be the conditional probability measure of \( \mu \) given that \( E_i \) has occurred, i.e., for any Borel \( B \subset \mathbb{R}^d \),

\[
\hat{\mu}_i(B) = \frac{\mu(B \cap E_i)}{\mu(E_i)} = \frac{1}{p_i} \mu(B \cap E).
\]

Notice that \( \hat{\mu}_i \) has the support \( E_i \) and \( \mu_i = p_i \hat{\mu}_i \) for all \( 1 \leq i \leq N \). Moreover,

\[
\mu = \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \mu_j \circ S_i^{-1} = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \hat{\mu}_j \circ S_i^{-1}.
\]

Let us first prove the following lemma.

**Lemma 3.3.** For every \( n \in \mathbb{N} \) and \( 1 \leq i \leq N \),

\[
\hat{e}_n(\hat{\mu}_i) \leq \log \overline{s}_i + \min \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_{ij} \hat{e}_{n_j}(\hat{\mu}_j) : n_j \geq 1, \sum_{j=1}^{N} n_j \leq n \right\}.
\]
Lemma 3.5. Let $\Gamma \subset \mathcal{W}$ be a finite maximal antichain. Let $C > \alpha_2$ be arbitrary. Then, for all $n \geq |\Gamma|$, 
\[ \hat{e}_n(\mu) \leq \frac{1}{C} \sum_{\sigma \in \Gamma} p_\sigma \log P_\sigma + \min \left\{ \sum_{\sigma \in \Gamma} p_\sigma \hat{e}_n(\mu_{\sigma|\sigma}) : n_\sigma \geq 1, \sum_{\sigma \in \Gamma} n_\sigma \leq n \right\}. \]

Proof. Write $\ell(\Gamma) = \max\{|\sigma| : \sigma \in \Gamma\}$. We will prove the lemma by induction on $\ell(\Gamma)$. If $\ell(\Gamma) = 2$, the lemma is true by Lemma 3.4. Next let $\ell(\Gamma) = k + 1$, and assume that the lemma
has been proved for all finite maximal antichains $\Gamma'$ with $2 \leq \ell(\Gamma') \leq k$ for some $k \geq 2$. Define

$$ \Gamma_1 = \{ \sigma \in \Gamma : |\sigma| < \ell(\Gamma) \}; $$

$$ \Gamma_2 = \{ \sigma^* : \sigma \in \Gamma \text{ and } |\sigma| = \ell(\Gamma) \}, $$

and

$$ \Gamma_0 = \Gamma_1 \cup \Gamma_2. $$

It is easy to see that $\Gamma_0$ is a finite maximal antichain with $2 \leq \ell(\Gamma_0) \leq k$. Let $\sigma^* j$ denotes the word $\sigma j$. Then, for $n \geq |\Gamma|$ and $(n_\sigma)_{\sigma \in \Gamma}$ with $n_\sigma \geq 1$ and $\sum_{\sigma \in \Gamma} n_\sigma \leq n$, we have

$$ a := \frac{1}{C} \sum_{\sigma \in \Gamma} p_\sigma \log P_\sigma + \sum_{\sigma \in \Gamma} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) $$

$$ = \frac{1}{C} \left[ \sum_{\sigma \in \Gamma_1} p_\sigma \log P_\sigma + \sum_{\sigma \in \Gamma_2} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) + \sum_{\sigma \in \Gamma_2} \sum_{j=1}^{N} p_j p_{j \sigma |\sigma| \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1} \left( \log(p_{j \sigma |\sigma|}) + \log(p_{\sigma_1 \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1}) \right) \right] $$

$$ + \sum_{\sigma \in \Gamma_1} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) + \sum_{\sigma \in \Gamma_2} \sum_{j=1}^{N} p_j p_{j \sigma |\sigma| \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1} \hat{e}_{n_{\sigma^* j}}(\hat{\mu}_j) $$

$$ = \frac{1}{C} \left[ \sum_{\sigma \in \Gamma_1} p_\sigma \log P_\sigma + \sum_{\sigma \in \Gamma_2} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) + \sum_{\sigma \in \Gamma_2} \sum_{j=1}^{N} p_j p_{j \sigma |\sigma| \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1} \hat{e}_{n_{\sigma^* j}}(\hat{\mu}_j) \right] + \sum_{\sigma \in \Gamma_1} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) + \sum_{\sigma \in \Gamma_2} \sum_{j=1}^{N} p_j p_{j \sigma |\sigma| \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1} \hat{e}_{n_{\sigma^* j}}(\hat{\mu}_j).$$

If we set $n_\sigma = \sum_{j=1}^{N} n_{\sigma^* j}$ for $\sigma \in \Gamma_2$, by Lemma 3.3 we obtain

$$ (25) \sum_{j=1}^{N} p_j p_{j \sigma |\sigma|} \hat{e}_{n_{\sigma^* j}}(\hat{\mu}_j) \geq p_{\sigma |\sigma|} \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) - p_{\sigma |\sigma|} \log \overline{s}_{\sigma |\sigma|}, $$

which yields

$$ \sum_{\sigma \in \Gamma_1} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) = \sum_{\sigma \in \Gamma_2} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) + \sum_{\sigma \in \Gamma_2} \sum_{j=1}^{N} p_j p_{j \sigma |\sigma| \sigma |\sigma| - 1} \cdots p_{\sigma_2 \sigma_1} \hat{e}_{n_{\sigma^* j}}(\hat{\mu}_j) - \sum_{\sigma \in \Gamma_1} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) \geq \sum_{\sigma \in \Gamma_2} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}) - \sum_{\sigma \in \Gamma_1} p_\sigma \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma |\sigma|}).$$

Moreover,

$$ C > \alpha_2 := \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} p_j p_{ji} \log p_{ji}}{\sum_{i=1}^{N} p_i \log \overline{s}_i} $$

which implies

$$ \sum_{i=1}^{N} \left( \sum_{j=1}^{N} p_j p_{ji} \log p_{ji} - C p_i \log \overline{s}_i \right) > 0. $$

To prove the lemma, for all $1 \leq i \leq N$, we assume that $\sum_{j=1}^{N} p_j p_{ji} \log p_{ji} - C p_i \log \overline{s}_i \geq 0$, and then for any $\sigma \in \Gamma_2$, we obtain

$$ (26) \sum_{j=1}^{N} p_j p_{j \sigma |\sigma|} \log p_{j \sigma |\sigma|} \geq C p_{\sigma |\sigma|} \log \overline{s}_{\sigma |\sigma|}. $$
Thus, using (25) and (26), we have
\[
a \geq \frac{1}{C} \left[\sum_{\sigma \in \Gamma_1} p_{\sigma} \log P_{\sigma} + C \sum_{\sigma \in \Gamma_2} p_{\sigma} \log \varphi_{\sigma|\sigma} + \sum_{\sigma \in \Gamma} p_{\sigma} \log P_{\sigma} \right] + \sum_{\sigma \in \Gamma_1} p_{\sigma} \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma|\sigma}) \\
+ \sum_{\sigma \in \Gamma_2} p_{\sigma} \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma|\sigma}) - \sum_{\sigma \in \Gamma_2} p_{\sigma} \log \varphi_{\sigma|\sigma}
\]
\[
= \frac{1}{C} \sum_{\sigma \in \Gamma_0} p_{\sigma} \log P_{\sigma} + \sum_{\sigma \in \Gamma} p_{\sigma} \hat{e}_{n_\sigma}(\hat{\mu}_{\sigma|\sigma}).
\]
Since
\[
\sum_{\sigma \in \Gamma_0} n_{\sigma} = \sum_{\sigma \in \Gamma_1} n_{\sigma} + \sum_{\sigma \in \Gamma_2} n_{\sigma} \leq n.
\]
by the induction hypothesis, we obtain
\[
\hat{e}_n(\mu) \leq \frac{1}{C} \sum_{\sigma \in \Gamma} p_{\sigma} \log P_{\sigma} + \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j).
\]
which yields the lemma.

The following lemma plays an important role in the paper.

**Lemma 3.6.** Let \( p_{\min} = \min \{ \min \{ p_j : 1 \leq j \leq N \}, \min \{ p_{ij} : p_{ij} > 0, 1 \leq i, j \leq N \} \} \). For \( 0 < \varepsilon < 1 \), write
\[
\Gamma(\varepsilon) = \{ \sigma \in W : p_{\sigma^-} \geq \varepsilon > p_{\sigma} \},
\]
where \( \sigma^- \) is the word obtained from \( \sigma \) by deleting the last letter of \( \sigma \). Let \( m, n \in \mathbb{N} \) with \( m \) fixed and \( \frac{m}{n} < p_{\min}^2 \). Write \( \varepsilon_n = \frac{m}{n} p_{\min}^{-1} \). Then,
\[
(27) \quad \hat{e}_n(\mu) \leq \frac{1}{C} \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \log P_{\sigma} + \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j).
\]

**Proof.** We have,
\[
1 = \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \geq \sum_{\sigma \in \Gamma(\varepsilon_n)} \varepsilon_n p_{\min} = \frac{m}{n} |\Gamma(\varepsilon_n)|,
\]
which implies \( n \geq m|\Gamma(\varepsilon_n)| \). Write \( \Gamma(\varepsilon_n, j) = \{ (\sigma_1, \ldots, \sigma_{|\sigma|}) \in \Gamma(\varepsilon_n) : \sigma_{|\sigma|} = j \} \), and then \( \Gamma(\varepsilon_n) = \bigcup_{j=1}^{N} \Gamma(\varepsilon_n, j) \). Choosing \( n_{\sigma} = m \) for every \( \sigma \in \Gamma(\varepsilon_n) \) in Lemma 3.5, we have
\[
\hat{e}_n(\mu) \leq \frac{1}{C} \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \log P_{\sigma} + \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \hat{e}_m(\hat{\mu}_{\sigma|\sigma}) = \frac{1}{C} \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \log P_{\sigma} + \sum_{j=1}^{N} \sum_{\sigma \in \Gamma(\varepsilon_n, j)} p_{\sigma} \hat{e}_m(\hat{\mu}_j)
\]
\[
\leq \frac{1}{C} \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \log P_{\sigma} + \left( \sum_{j=1}^{N} \sum_{\sigma \in \Gamma(\varepsilon_n, j)} p_{\sigma} \right) \left( \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j) \right)
\]
\[
= \frac{1}{C} \sum_{\sigma \in \Gamma(\varepsilon_n)} p_{\sigma} \log P_{\sigma} + \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j),
\]
and thus the lemma is proved.

Let us now prove the following proposition.

**Proposition 3.7.** Let \( C > \alpha_2 \) be arbitrary. Then,
\[
\limsup_{n \to \infty} n^{1/C} e_n(\mu) \leq p_{\min}^{-1/C} \inf_{m \geq 1} m^{1/C} \prod_{j=1}^{N} e_m(\hat{\mu}_j) < +\infty.
\]
Proof. It is enough to prove that

\[
\limsup_{n \to \infty} (\log n + C\hat{e}_n(\mu)) \leq -\log p_{\min} + \inf_{m \geq 1} \left( \log m + C \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j) \right).
\]

Given \(m \in \mathbb{N}\), by (27), for \(\varepsilon_n = \frac{m}{n}p_{\min}^{-1}\), we obtain

\[
\log n + C\hat{e}_n(\mu) \leq \sum_{\sigma \in \Gamma(\varepsilon_n)} p_\sigma \log P_\sigma - \log \varepsilon_n - \log p_{\min} + \log m + C \sum_{j=1}^{N} \hat{e}_m(\hat{\mu}_j)
\]

for all but finitely many \(n\). To prove (28), it is therefore, enough to prove that

\[
\limsup_{n \to \infty} \left[ \sum_{\sigma \in \Gamma(\varepsilon_n)} p_\sigma \log P_\sigma - \log \varepsilon_n \right] \leq 0.
\]

Since \(p_\sigma < \varepsilon_n\) for all \(\sigma \in \Gamma(\varepsilon_n)\), we have \(\log \varepsilon_n \geq \log P_\sigma\), and hence

\[
\sum_{\sigma \in \Gamma(\varepsilon_n)} p_\sigma \log P_\sigma \leq \log \varepsilon_n,
\]

which proves (28), and thus the proposition follows.

In order to prove Proposition 3.10 we need the following lemma.

Lemma 3.8. Let the hyperbolic recurrent IFS satisfy the strong separation condition. Then, for \(1 \leq i \leq N\),

\[
\hat{e}_n(\hat{\mu}_i) \geq \log s_i + \max \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \hat{e}_{n,j}(\hat{\mu}_j) : n_j \geq 1, \sum_{j=1}^{N} n_j \leq n \right\},
\]

for all but finitely many \(n \in \mathbb{N}\).

Proof. Let \(\delta = \min\{d(S_i(E_k), S_j(E_\ell)) : k \neq \ell \text{ with } 1 \leq i, j, k, \ell \leq N\}\) and let \(\alpha_n \in C_n(\hat{\mu}_i)\), \(n \in \mathbb{N}\). Then, \(\delta > 0\). Now proceeding in the similar lines as [GL2] Lemma 5.9, it can be proved that

\[
\lim_{n \to \infty} \max_{x \in E_i} d(x, \alpha_n) = 0,
\]

and so there exists a positive integer \(n_0\) such that \(\sup_{x \in E_i} d(x, \alpha_n) < \frac{\delta}{2}\) for all \(n \geq n_0\). For \(1 \leq k \leq N\), set \(\alpha_{n,k} = \{a \in \alpha_n : W(a|\alpha_n) \cap S_k(E_i) \neq \emptyset\}\), where \(W(a|\alpha_n)\) is the Voronoi region generated by \(a \in a_n\) (see [GL2] for more details on Voronoi regions). Then, \(\alpha_{n,k} \neq \emptyset\) and \(\alpha_{n,k} \cap \alpha_{n,\ell} = \emptyset\) for \(1 \leq k \neq \ell \leq N\) and \(n \geq n_0\). Using \(\hat{\mu}_i = \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \hat{\mu}_j \circ S_i^{-1}\), for all \(n \geq n_0\), we obtain

\[
\hat{e}_n(\hat{\mu}_i) = \int \log d(x, \alpha_n)d\hat{\mu}_i = \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \int \log d(S_i(x), \alpha_{n,j})d\hat{\mu}_j
\]

\[
= \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \int \log d(S_i(x), \alpha_{n,j})d\hat{\mu}_j
\]

\[
\geq \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \log s_i + \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \int \log d(x, S_i^{-1}(\alpha_{n,j}))d\hat{\mu}_j
\]

\[
\geq \log s_i + \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \hat{e}_{n,j}(\hat{\mu}_j),
\]

where \(n_j = \text{card}(\alpha_{n,j}) \geq 1\). Since \(n = \text{card}(\alpha_n) = \sum_{j=1}^{N} n_j\), this proves the lemma.

We now prove the following lemma.
Lemma 3.9. Since $\mu = \sum_{i=1}^{N} p_i \hat{\mu}_i$, we have

$$\hat{e}_n(\mu) \geq \sum_{i=1}^{N} p_i \hat{e}_n(\hat{\mu}_i).$$

Proof. Let $\alpha \in C_n(\mu)$. Then,

$$\hat{e}_n(\mu) = \int \log d(x, \alpha) d\mu = \sum_{i=1}^{N} p_i \int \log d(x, \alpha) d\hat{\mu}_i \geq \sum_{i=1}^{N} p_i \hat{e}_n(\hat{\mu}_i),$$

and thus the lemma follows. \qed

Proposition 3.10. Let the hyperbolic recurrent IFS satisfy the strong separation condition, and let $\alpha_1$ be defined as before. Then

$$\inf_{n \in \mathbb{N}} n^{1/\alpha_1} e_n(\mu) > 0.$$

Proof. The proposition will be proved if we can prove that

$$\inf_{n \in \mathbb{N}} (\log n + \alpha_1 \hat{e}_n(\mu)) > -\infty.$$

By Lemma 3.8, there is an $n_0 \in \mathbb{N}$, such that

$$\hat{e}_n(\hat{\mu}_i) \geq \log s_i + \min \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \hat{e}_{n_j}(\hat{\mu}_j) : n_j \geq 1, \sum_{j=1}^{N} n_j \leq n \right\},$$

for all $1 \leq i \leq N$ and all $n \geq n_0$. Since $\hat{e}_n(\hat{\mu}_i) > -\infty$ for all $n \in \mathbb{N}$, we have

$$c = \min \left\{ \frac{1}{\alpha_1} \log n + \hat{e}_n(\hat{\mu}_i) : n \leq n_0 \right\} > -\infty.$$

By induction, we now prove that

$$\hat{e}_n(\hat{\mu}_i) \geq c - \frac{1}{\alpha_1} \log n,$$

for all $n \in \mathbb{N}$. For $m \leq n_0$, this is true by the definition of $c$. Let $m > n_0$, and assume that the inequality holds for all $n < m$. Then,

$$\hat{e}_m(\hat{\mu}_i) \geq \log s_i + \min \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \hat{e}_{n_j}(\hat{\mu}_j) : n_j \geq 1, \sum_{j=1}^{N} n_j \leq m \right\} \geq \log s_i + \min \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} c - \frac{1}{\alpha_1} \sum_{j=1}^{N} p_j p_{ji} \log n_j : n_j \geq 1, \sum_{j=1}^{N} n_j \leq m \right\} = \log s_i + c - \frac{1}{\alpha_1} \log m - \frac{1}{\alpha_1} \max \left\{ \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \log \frac{n_j}{m} : n_j \geq 1, \sum_{j=1}^{N} n_j \leq m \right\}.$$

Using [T] page 23, Lemma 1] and the fact that $\frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} = 1$, we obtain

$$\frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \log \frac{n_j}{m} \leq \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \log \left( \frac{p_j p_{ji}}{p_i} \right),$$

for all $n_j \geq 1$ with $\sum_{j=1}^{N} n_j \leq m$. Thus we have,

$$\hat{e}_m(\hat{\mu}_i) \geq \log s_i + c - \frac{1}{\alpha_1} \log m - \frac{1}{\alpha_1} \frac{1}{p_i} \sum_{j=1}^{N} p_j p_{ji} \log \left( \frac{p_j p_{ji}}{p_i} \right),$$

for all $m \geq n_0$.\qed
which by Lemma 3.9 yields
\[ \hat{e}_m(\mu) \geq \sum_{i=1}^{N} p_i \hat{e}_m(\hat{\mu}_i) \geq \sum_{i=1}^{N} p_i \log \frac{1}{\alpha_1} + c - \frac{1}{\alpha_1} \log m - \frac{1}{\alpha_1} \sum_{i,j=1}^{N} p_j p_{ji} \log \left( \frac{p_j p_{ji}}{p_i} \right). \]

Notice that
\[ \sum_{i,j=1}^{N} p_j p_{ji} \log \left( \frac{p_j p_{ji}}{p_i} \right) = \sum_{i,j=1}^{N} p_j p_{ji} \log p_{ji} + \sum_{i,j=1}^{N} p_j p_{ji} \log p_j - \sum_{i,j=1}^{N} p_j p_{ji} \log p_i \]
and thus,
\[ \frac{1}{\alpha_1} \sum_{i,j=1}^{N} p_j p_{ji} \log \left( \frac{p_j p_{ji}}{p_i} \right) = \sum_{i=1}^{N} p_i \log \frac{1}{\alpha_1}. \]

Hence, by (29) and (30), we have
\[ \hat{e}_m(\mu) \geq c - \frac{1}{\alpha_1} \log m. \]

This implies
\[ \inf_{n \in \mathbb{N}} (\log n + \alpha_1 \hat{e}_n(\mu)) \geq c \alpha_1 > -\infty, \]
and hence the proposition is proved. \(\square\)

**Proof of Theorem 3.2.** Proposition 3.7 tells us that \( \limsup_{n \to \infty} n^{1/C} e_n(\mu) < \infty \), which by Proposition 1.1 implies that \( D(\mu) \leq C \). Since \( C > \alpha_2 \) is arbitrary, we have \( D(\mu) \leq \alpha_2 \). Proposition 3.10 tells us that \( \liminf_{n \to \infty} n^{1/\alpha_1} e_n(\mu) > 0 \), which by Proposition 1.1 implies \( D(\mu) \geq \alpha_1 \). Thus the proof of Theorem 3.2 is complete. \(\square\)

**Remark 3.11.** Let \( \alpha_2 := \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i,j=1}^{N} p_i p_{ij} \log p_{ij}} \) be the upper bound for the upper local dimension of the probability measure \( \mu \) generated by the hyperbolic recurrent IFS \( \{X; S_i, p_{ij} : 1 \leq i, j \leq N\} \). Then, the following problem remains open:

\[ \text{Is } \limsup_{n \to \infty} n^{1/\alpha_2} e_n(\mu) < +\infty? \]

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