NEUMANN ISOPERIMETRIC CONSTANT ESTIMATE
FOR CONVEX DOMAINS

XIANZHE DAI, GUOFANG WEI, AND ZHENLEI ZHANG

Abstract. We present a new and direct proof of the local Neumann isoperimetric inequality on convex domains of a Riemannian manifold with Ricci curvature bounded below.

1. Introduction

Isoperimetric and Sobolev inequalities are equivalent inequalities (see e.g. Theorem 1.3 below) which play important role in geometric analysis on manifolds. Indeed, doing analysis on manifolds usually depends on the estimate of the Sobolev constant which could then be obtained via the isoperimetric constant. For closed manifolds there are extensive work on isoperimetric constant estimates. For domains, one usually needs to distinguish between two types of isoperimetric constants, the Dirichlet and the Neumann type. A subtle point is that the Neumann type isoperimetric inequality is harder to obtain and requires extra conditions on the boundary. For star-shaped domain in a manifold with Ricci curvature bounded from below, Buser [2] obtained a Neumann isoperimetric constant (the Cheeger constant) estimate (depends on in and out radius) using comparison geometry. For domain with smooth and convex boundary, the known estimate for Neumann isoperimetric constant is very analytic and surprisingly indirect. Namely it is through Li-Yau gradient estimate for heat kernel [10] and the equivalence of heat kernel bounds, Sobolev inequality, isoperimetric inequality, see [12, Page 448]. In this short note we give a geometric and direct proof of a Neumann isoperimetric inequality for convex domains (whose boundaries need not be smooth).

First we recall some definitions.

Definition 1.1. When $M$ is compact (with or without boundary), the Neumann $\alpha$-isoperimetric constant of $M$ is defined by

$$\text{IN}_\alpha(M) = \sup_H \frac{\min\{\text{vol}(M_1), \text{vol}(M_2)\}^{1-\frac{\alpha}{n}}}{\text{vol}(H)},$$

where $H$ varies over compact $(n - 1)$-dim submanifold of $M$ which divides $M$ into two disjoint open submanifolds $M_1, M_2$ (with or without boundary).

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Definition 1.2. The Neumann $\alpha$-Sobolev constant of $M$ is defined by

$$SN_\alpha(M) = \sup_{f \in C^\infty(M)} \inf_{a \in \mathbb{R}} \frac{\|f - a\|_{\alpha + 1}}{\|\nabla f\|_1}.$$ 

The isoperimetric constant and Sobolev constant are equivalent.

Theorem 1.3 ([3], see also [9]). For all $n \leq \alpha \leq \infty$,

$$IN_\alpha(M) \geq SN_\alpha(M) \geq \frac{1}{2} IN_\alpha(M).$$

For convenience we consider the normalized Neumann $\alpha$-isoperimetric and $\alpha$-Sobolev constant:

$$IN_\alpha^*(M) = IN_\alpha(M) \frac{\text{vol}(M)^{1/\alpha}}{\text{vol}(M)} \quad \text{and} \quad SN_\alpha^*(M) = SN_\alpha(M) \frac{\text{vol}(M)^{1/\alpha}}{\text{vol}(M)}.$$

Using comparison geometry and Vitali covering we give an estimate on the normalized Neumann isoperimetric constant for convex domain in terms of the Ricci curvature lower bound and the diameter of the domain.

Theorem 1.4. Let $(M, g)$ be a complete Riemannian manifold of dimension $n$, with $\text{Ric} \geq -(n - 1)K$ for some $K \geq 0$. Let $\Omega$ be a bounded convex domain. Then

$$IN_\alpha^*(\Omega) \leq 40^n e^{11(n-1)\sqrt{Kd}} \cdot d$$

where $d$ is the diameter of the domain $\Omega$. In particular, if $M$ is closed with diameter $d$, then

$$IN_\alpha^*(M) \leq 40^n e^{11(n-1)\sqrt{Kd}} \cdot d.$$ 

Corollary 1.5. Let $(M, g)$ be a complete Riemannian manifold of dimension $n$, with nonnegative Ricci curvature. Let $\Omega$ be a bounded convex domain. Then

$$IN_\alpha^*(\Omega) \leq 40^n \cdot d$$

where $d$ is the diameter of the domain $\Omega$. In particular, if $M$ is closed with diameter $d$, then

$$IN_\alpha^*(M) \leq 40^n \cdot d.$$ 

Remark 1.6. The case when $\Omega$ equals the whole manifold is well-known. The reference we mentioned earlier for convex domain in the literature deals with domains with (smooth) convex boundary which is a stronger condition.

Remark 1.7. For balls we can obtain both Dirichlet and Neumann isoperimetric constant estimates even under the much weaker integral Ricci lower bound assumption [6] [13]. On the other hand it is not clear if that will remain true for convex domains.

Remark 1.8. Using the mean curvature estimate from [13] one gets similar estimate when the Bakry-Emery Ricci curvature is bounded from below and oscillation of the potential function is bounded.
2. Proof of Theorem 1.4

The proof goes by a covering argument of Anderson [1], combined with an observation of Gromov [8]. See [1] or [6] for a similar argument of estimating the local Dirichlet isoperimetric constant. First of all we recall a lemma whose proof is a slight modification of Gromov’s observation [8, 5(C)].

**Lemma 2.1.** Let $M^n$ be a complete Riemannian manifold. Let $\Omega$ be a convex domain of $M$ and $H$ be any hypersurface dividing $\Omega$ into two parts $\Omega_1, \Omega_2$. For any Borel subsets $W_i \subset \Omega_i$, there exists $x_1$ in one of $W_i$, say $W_1$, and a subset $W$ in another one, $W_2$, such that

$$\text{(2.1)} \quad \text{vol}(W) \geq \frac{1}{2} \text{vol}(W_2)$$

and any $x_2 \in W$ has a unique minimal geodesic connecting to $x_1$ which intersects $H$ at some $z$ such that

$$\text{(2.2)} \quad \text{dist}(x_1, z) \geq \text{dist}(x_2, z).$$

The convexity assumption of $\Omega$ is essential. It implies that any minimal geodesic with endpoints in different parts must intersects $H$. The Bishop-Gromov relative volume comparison theorem gives

**Lemma 2.2.** Let $H, W$ and $x_1$ be as in the lemma above. Then

$$\text{(2.3)} \quad \text{vol}(W) \leq 2^{n-1} De^{(n-1)\sqrt{KD}} \text{vol}(H')$$

where $D = \sup_{x \in W} \text{dist}(x_1, x)$ and $H'$ is the set of intersection points with $H$ of geodesics $\gamma_{x_1, x}$ for all $x \in W$.

**Proof.** Let $\Gamma \subset S_{x_1}$ be the set of unit vectors such that $\gamma_v = \gamma_{x_1, x_2}$ for some $x_2 \in W$. We compute the volume in the polar coordinate at $x_1$. Write $dv = A(\theta, t) d\theta \wedge dt$ in the polar coordinate $(\theta, t) \in S_{x_1} \times \mathbb{R}^+$. For any $\theta \in \Gamma$, let $r(\theta)$ be the radius such that $\exp_{x_1}(r\theta) \in H$. Then $W \subset \{\exp_{x_1}(r\theta) | \theta \in \Gamma, r(\theta) \leq r \leq 2r(\theta)\}$. So, by relative volume comparison,

$$\text{vol}(W) \leq \int_{\Gamma} \int_{r(\theta)}^{2r(\theta)} A(\theta, t) dt d\theta \leq \frac{\sinh^{n-1}(2\sqrt{KD})}{\sinh^{n-1}(\sqrt{KD})} \int_{\Gamma} \int_{r(\theta)}^{2r(\theta)} A(\theta, r(\theta)) dt d\theta \leq D \frac{\sinh^{n-1}(2\sqrt{KD})}{\sinh^{n-1}(\sqrt{KD})} \text{vol}(H').$$

The required estimate follows from $\frac{\sinh(2t)}{\sinh t} = 2 \cosh t \leq e^t$ whenever $t \geq 0$. □

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Corollary 2.3. Let $H$ be any hypersurface dividing a convex domain $\Omega$ into two parts $\Omega_1, \Omega_2$. For any ball $B = B_r(x)$ we have

\[
\min\left( \text{vol}(B \cap \Omega_1), \text{vol}(B \cap \Omega_2) \right) \leq 2^{n+1}re^{(n-1)\sqrt{Kd}} \text{vol}(H \cap B_{2r}(x))
\]

where $d = \text{diam}(\Omega)$. In particular, if $B \cap \Omega$ is divided equally by $H$, we have

\[
\text{vol}(B_r(x) \cap \Omega) \leq 2^{n+2}re^{(n-1)\sqrt{Kd}} \text{vol}(H \cap B_{2r}(x))
\]

Proof. Put $W_i = B \cap \Omega_i$ in the above lemma and notice that $D \leq 2r$ and $H' \subset H \cap B_{2r}(x)$. \hfill \Box

Now we are ready to prove our main theorem.

Proof of Theorem 1.4. We may assume that $\text{vol}(\Omega_1) \leq \text{vol}(\Omega_2)$. For any $x \in \Omega_1$, let $r_x$ be the smallest radius such that

\[
\text{vol}(B_{r_x}(x) \cap \Omega_1) = \text{vol}(B_{r_x}(x) \cap \Omega_2) = \frac{1}{2} \text{vol}(B_{r_x}(x) \cap \Omega).
\]

Let $d = \text{diam}(\Omega)$. By above corollary,

\[
\text{vol}(B_{r_x}(x) \cap \Omega) \leq 2^{n+2}re^{(n-1)\sqrt{Kd}} \text{vol}(H \cap B_{2r}(x)).
\]

The domain $\Omega_1$ has a covering

\[
\Omega_1 \subset \bigcup_{x \in \Omega_1} B_{2r_x}(x).
\]

By Vitali Covering Lemma, cf. [11, Section 1.3], we can choose a countable family of disjoint balls $B_i = B_{2r_x}(x_i)$ such that $\bigcup_i B_{10r_x}(x_i) \supset \Omega_1$. Applying the relative volume comparison theorem and the convexity of $\Omega$ we have

\[
\text{vol}(\Omega_1) \leq \sum_i \int_0^{10r_x} \frac{\sinh^{-1}(\sqrt{Kt})dt}{\sinh^{-1}(\sqrt{Kr_x})} \text{vol}(B_{r_x}(x) \cap \Omega_1)
\]

\[
\leq 10 \sum_i \frac{\sinh^{-1}(10\sqrt{Kd})}{\sinh^{-1}(\sqrt{Kd})} \text{vol}(B_{r_x}(x) \cap \Omega_1)
\]

\[
\leq 10 \frac{\sinh^{-1}(10\sqrt{Kd})}{\sinh^{-1}(\sqrt{Kd})} \sum_i \text{vol}(B_{r_x}(x) \cap \Omega_1)
\]

\[
\leq 10^n e^{9(n-1)\sqrt{Kd}} \sum_i \text{vol}(B_{r_x}(x) \cap \Omega_1)
\]

\[
= 2^{-1} \cdot 10^n \cdot e^{9(n-1)\sqrt{Kd}} \sum_i \text{vol}(B_{r_x}(x) \cap \Omega_1).
\]

Moreover, since the balls $B_i$ are disjoint, (2.6) gives,

\[
\text{vol}(H) \geq \sum_i \text{vol}(B_i \cap H) \geq 2^{-n-2}e^{-(n-1)\sqrt{Kd}} \sum_i r_x^{-1} \text{vol}(B_{r_x}(x) \cap \Omega).
\]
These two estimates lead to

\[
\frac{\text{vol}(\Omega_1)}{\text{vol}(H)} \leq 2 \cdot 20^n e^{10(n-1)\sqrt{K_d}} \left( \sum_i \frac{\text{vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{\text{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \right)^{\frac{n}{n-1}}
\]

\[
\leq 40^n e^{10(n-1)\sqrt{K_d}} \sum_i \frac{\text{vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{\text{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \left( \frac{r_{x_i}^n}{\text{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \right)^{\frac{1}{n}}
\]

On the other hand, since \(\text{vol}(\Omega_1) \leq \text{vol}(\Omega_2)\), we have \(r_x \leq d\) for any \(x \in \Omega_1\). Thus, by the relative volume comparison and convexity of \(\Omega\) again, we have

\[
\text{vol}(\Omega) \leq \frac{\int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^r \sinh^{n-1}(\sqrt{K}t)dt} \text{vol}(B_r(x) \cap \Omega).
\]

Therefore,

\[
\frac{\text{vol}(\Omega_1)}{\text{vol}(H)} \leq 40^n e^{10(n-1)\sqrt{K_d}} \sup_{0 < r \leq d} \left( \frac{r^n \int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^r \sinh^{n-1}(\sqrt{K}t)dt} \right)^{\frac{1}{n}}.
\]

The last term on the right hand side has the estimate

\[
\frac{r^n \int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^r \sinh^{n-1}(\sqrt{K}t)dt} \leq r^n \cdot \frac{d \cdot \sinh^{n-1}(\sqrt{K}d)}{\sinh^{n-1}(\sqrt{Kr})} \leq d^n \cdot \frac{\sinh^{n-1}(\sqrt{K}d)}{(\sqrt{K}d)^{n-1}} \leq d^n e^{(n-1)\sqrt{Kd}}.
\]

The required normalized Neumann isoperimetric constant estimate now follows. \(\Box\)

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(Xianzhe Dai) Department of Mathematics, ECNU, Shanghai and UCSB, Santa Barbara CA 93106, email:dai@math.ucsb.edu

(Guofang Wei) Department of Mathematics, University of California, Santa Barbara CA 93106, email: wei@math.ucsb.edu

(Zhenlei Zhang) Department of Mathematics, Capital Normal University, China, email: zhleigo@aliyun.com