THE LINES OF THE KONTSEVICH INTEGRAL AND
ROZANSKY’S RATIONALITY CONJECTURE

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Abstract. This work develops some technology for accessing the loop expansion of the Kontsevich integral of a knot. The setting is an application of the LMO invariant to certain surgery presentations of knots by framed links in the solid torus. A consequence of this technology is a certain recent conjecture of Rozansky’s. Rozansky conjectured that the Kontsevich integral could be organised into a series of “lines” which could be represented by finite \( \mathbb{Q} \)-linear combinations of diagrams whose edges were labelled, in an appropriate sense, with rational functions. Furthermore, the conjecture requires that the denominator of the rational functions be at most the Alexander polynomial of the knot. This conjecture is obtained from an Aarhus-style surgery formula for this setting which we expect will have other applications.

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1. Introduction

In some lectures at the Joseph Fourier Institute in June of 1999, Lev Rozansky formulated an important conjecture concerning the structure of the Kontsevich integral [Roz], and mentioned something of a related program for a “finite-type theory of knots’ complements”.

By means of some notation let us now describe what of this will be proved in this paper.

A generating diagram is a diagram with oriented trivalent vertices (which is to say that the incident edges at a trivalent vertex are cyclically ordered) and edges decorated with oriented bivalent vertices labelled by elements of $\mathbb{Q}[[k]]$, the ring of formal power series in a variable $k$. A generating diagram represents a series of elements of $B(k)$ by expanding these power series into series of diagrams, as follows. If $f(k) \in \mathbb{Q}[[k]]$ is

$$f = f_0 + f_1 k + f_2 k^2 + f_3 k^3 \ldots,$$

where $f_i \in \mathbb{Q}$, then an edge labelled with $f(k)$ is to be expanded as follows.

$$f(k) = f_0 + f_1 k + f_2 k^2 + f_3 k^3 \ldots$$

Remark 1.0.1. The incoming edges at the label (which Definition 3.0.2 will introduce as a “winding coupon”) are ordered, which determines the orientation of the introduced trivalent vertices, as shown. The opposite ordering with the label $f(-k)$ gives the same series.

Definition 1.0.2. Define the gd-degree of a generating diagram to be half the number of trivalent vertices of the (original) diagram.

Definition 1.0.3. Take a knot $K$ in an integral homology three-sphere $M$. Let $A_{(M,K)}(t)$ denote the Alexander polynomial of the pair $(M, K)$ fixed by the requirement that it satisfy:

1. $A_{(M,K)}(t) = A_{(M,K)}(\frac{1}{t}),$
2. $A_{(M,K)}(1) = 1.$

Let $Q^1(t)$ be the ring of rational functions in $t$ that are non-singular at 1. Denote the inclusion

$$t : Q^1(t) \hookrightarrow \mathbb{Q}[[k]],$$

defined by substituting $e^k$ into $t$. 

Definition 1.0.4. Let $L_{(M,K)}$ be the $\mathbb{Q}$-vector subspace of $\mathbb{Q}[[k]]$ that is the image under $\iota$ of rational functions of the form
\[ \frac{P(t)}{A_{(M,K)}(t)} \]
where $P(t) \in \mathbb{Q}[t, t^{-1}]$. This notation can be read as the space of labels.

The next two definitions require the “wheel with $2n$ spokes”:
\[ \omega_2 = \quad \omega_4 = \quad \omega_6 = \quad \ldots \]

Definition 1.0.5. If the rational numbers $b_{2n}$ are determined by the equality:
\[ \sum b_{2n}x^{2n} = \frac{1}{2}\log \left( \frac{\sinh(x^2)}{x^2} \right), \]
then the series $\nu(k) \in B(k)$ is defined to be
\[ \exp_u(\sum b_{2n}\omega_{2n}). \]

Remark 1.0.6. This has recently been shown to be $\hat{Z}(U)$, the Kontsevich integral of the unknot, by Bar-Natan, Le and Thurston [TW].

Definition 1.0.7. Let $Wh(M, K)$ be defined by
\[ Wh(M, K) = \exp_u \left( \left[ \frac{1}{2}\log \left( A_{(M,K)}(e^h) \right) \right]_{h^{2n} \to \omega_{2n}} \right) \sqcup \nu(k), \]
where the operation indicated is to expand the term inside the square brackets into a power series in $u$, and then to replace terms like $ch^{2n}$ by $c\omega_{2n}$.

The $\text{LMO}$ invariant was introduced by Thang Le, Jun Murakami and Tomotada Ohtsuki [LMO] (following an earlier investigation also with Hitoshi Murakami [LMMO]). Our (perhaps non-standard) normalisation of the non-surgered component specialises to the three-sphere as follows:
\[ (1.0.1) \]
\[ Z^{\text{LMO}}(S^3, K) = \hat{Z}(K). \]

This brings us to the rationality conjecture. In his lectures Rozansky conjectured the $M \simeq S^3$ case of the following (see also the new paper by Garoufalidis and Rozansky [GR]).

Theorem 1.0.8. Let $K$ be a zero-framed knot in an integral homology three-sphere $M$. The $\text{LMO}$ invariant of this pair may be represented
\[ Z^{\text{LMO}}(M, K) = Wh(M, K) \sqcup \exp_u(r) \in B(k), \]
where $r = \sum_{m=1}^{\infty} r^{(m)}$ with $r^{(m)}$ a finite $\mathbb{Q}$-linear combination of connected generating diagrams of $\text{gd}$-degree $m$ whose edges are labelled from $L_{(M,K)}$.

This conjecture is motivated by an analogous property of the coloured Jones function, as shown by Rozansky (see, for example, [Roz2]).

The structures we describe which lead to the proof of this depend on a delicate assembly of results from the literature. A significant debt is to the papers of the
Aarhus group, who are Bar-Natan, Garoufalidis, Rozansky and Thurston [A1, A2, A3].

This theory was described at the workshop “Art of Low-Dimensional Topology VII”, January 8th, 2000, for which invitation the author thanks Toshitake Kohno.

A sequel to this paper [KS] will develop some technical issues raised within this work.

Note 1.0.9. Between versions of this paper, a new paper by Garoufalidis and Rozansky appeared [GR]. We (the author) suggest reading these papers in tandem, as their concerns are somewhat complementary.

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2. The outline

2.1. Special surgery presentations.

The strategy of the calculation to be presently described is to apply the LMO surgery formula to a special surgery presentation which exists for any knot in a $\mathbb{Z}HS^3$. The following is well-known.

**Lemma 2.1.1.** A zero-framed knot in a $\mathbb{Z}HS^3$ may be obtained from the zero-framed unknot $U$ in $S^3$ by performing surgery on some framed link which has the property that every component of it has linking number 0 with $U$.

It may be useful to keep the following example in mind. The figure of 8 knot is obtained by performing surgery on the (blackboard-framed) component marked with a $\ast$ below:

```
\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{figure_of_8_knot}
\caption{Figure of 8 knot obtained by surgery on a component marked with a $\ast$.}
\end{figure}
```

Thus a knot in a $\mathbb{Z}HS^3$ can be presented by a framed link in a solid torus (fixed in $S^3$) such that every component has linking zero with the core of the torus.

It will prove technically advantageous to work with a slightly different object: a **framed string link in the solid torus**. For this definition, realise the solid torus $ST$ as the complement in the cube $\{(x, y, z) \in \mathbb{R}^3; 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ of the hole $\{(x, y, z) \in \mathbb{R}^3; \frac{1}{4} < x < \frac{3}{4}, \frac{1}{4} < y < \frac{3}{4}, 0 \leq z \leq 1\}$.

**Remark 2.1.2.** We thus use the definite article “the”, as in “the solid torus”, to remind that we are referring to a particular solid torus embedded in $S^3$.

**Definition 2.1.3.** A $\mu$-string string link in the solid torus is a proper embedding $[0, 1] \sqcup \ldots \sqcup [0, 1] \hookrightarrow ST$ such that the $i$th $\{0\}$ is mapped to $\left(\frac{i}{\mu+1}, 0, \frac{1}{2}\right)$, such that the $i$th $\{1\}$ is mapped to $\left(\frac{i}{\mu+1}, 1, \frac{1}{2}\right)$, and with a framing in the familiar sense of a framed tangle. These are identified up to framed isotopies in the solid torus.

**Remark 2.1.4.** We will refer to the $y = 0$ plane as the base, and the $y = 1$ plane as the top. In this work string links in the solid torus will always be oriented from the base to the top.
Definition 2.1.5. We will call the meridional disc the disc \( \{ x \in \left[ \frac{3}{4}, 1 \right], y = \frac{1}{2}, z \in [0, 1] \} \) in the solid torus.

To draw a diagram of a string link in a solid torus, we will take a projection in general position onto the \( x - y \) plane, in the familiar sense. It is convenient to represent the “hole” as a fixed dashed loop, or to represent the “meridional disc” as a short dashed line segment. A diagram will be called in \textbf{general position with respect to the meridional disc} if it intersects that dashed line transversally. For example:

![Diagram of a string link in a solid torus](image)

Definition 2.1.6.

1. Let a marked framed tangle (resp. link) be a framed tangle (resp. link), possibly with some distinguished components.
2. For a string link in the solid torus \( T \), let \( \text{Thr}(T) \) denote the marked framed tangle obtained by marking all components, threading the hole with an unmarked zero-framed unknot (to fix this let us say we thread on the \( x > 1 \) side), and then forgetting the hole.
3. For a marked framed tangle \( T' \), let \( \text{Clos}(T') \) denote the marked framed link obtained by closing the tangle.
4. For a marked framed link \( L \), let \( \text{KII}(L) \) denote the class of \( L \) modulo Kirby move IIs, where markings indicate to-be-surgered components (slides of unmarked components over marked components are also allowed).

We now restrict to the presentations guaranteed by Lemma 2.1.1. In the following, the adjective special will just mean one of examples in question. In particular:

Definition 2.1.7. Let a \textbf{special string link in the solid torus} be a string link in the solid torus \( T \) such that:

1. Every component has zero algebraic intersection number with the meridional disc, for diagrams in general position with respect to the meridional disc.
2. The determinant of the linking matrix of the marked components of \( \text{Clos}(\text{Thr}(T)) \) is \( \pm 1 \).
**Definition 2.1.8.** Let a **special tangle** be a marked, framed tangle with one closed, unmarked component; forgetting that component leaves the tangle a string link whose linking matrix has determinant ±1.

**Definition 2.1.9.** Let a **special link** be a marked, framed link with one unmarked component; the determinant of the linking matrix of marked components is ±1.

### 2.2. The master diagram.

Here is the plan. The basic idea is that formulae for the Kontsevich integral of a knot can be obtained by applying the LMO invariant to the surgery presentations just considered. We will see that the resulting factorisation of the calculation through a certain invariant of string links in the solid torus has important implications for the result.

The following diagram records this factorisation, as we will now explain.

**Diagram 2.2.1.**

\[
\begin{array}{ccc}
\{ \text{Special string links in the solid torus.} \} & \xrightarrow{\text{Thr}} & \{ \text{Special tangles.} \} \\
\sigma \circ \tilde{Z} & & \sigma \circ \tilde{Z}
\end{array}
\]

\[
B^{ST}(X)^{Int} \xrightarrow{\text{S} \mapsto \text{Thr}^{D}(S) \sqcup \nu(k)} B(X,k) \xrightarrow{\text{KII} \circ \text{Clos}}
\]

\[
\int_{\text{FG in ST}} dX \times \sigma \times \det \xrightarrow{\int^{(n)} dX} \{ \text{Kirby move IIs.} \}
\]

\[
B^{QST}(\phi) \times \mathbb{Z} \times \mathbb{Z}[t,t^{-1}] \xrightarrow{\sigma \circ \tilde{Z} \circ \Gamma} B^{ST}(X)^{Int}
\]

\[
(S,\sigma,P(t)) \mapsto \text{Thr}^{D}(S) \sqcup (-1)^{n \sigma} \text{Wh}'(P(t))
\]

The invariant

\[
\sigma \circ \tilde{Z}^{ST} : \{ \text{Special string links in the solid torus.} \} \to B^{ST}(X)^{Int}
\]

is introduced in Section 3. This is an enhancement of the usual Kontsevich integral, taking values in a space of *winding diagrams*. A winding diagram is, in an appropriate sense, a uni-trivalent diagram decorated by *winding coupons*.

Intuitively speaking, this decoration (modulo some relations) describes a homotopy class of proper mappings of that diagram into the solid torus. In this intuitive picture winding coupons record intersections of the edges of some representative in general position with respect to some fixed meridional disc, with that disc. This is illustrated in the next figure.
This invariant is more or less pre-existent in the literature, though our approach and the structures we describe depart from existent works in certain ways. Our formal presentation, with labelled edges, is closest to that of Goryunov [G], and the intuitive picture is closest to that of Andersen-Mattes-Reshetikhin [AMR] (see also Suetsugue [S]).

The destination indicated, $B^{ST}(X)$, is a space of symmetrised winding diagrams. This plays the part of Bar-Natan’s algebra $B$: legs on skeletons are to be symmetrised. This space is introduced in Section 3.4. The map $\sigma$ is the appropriate version of Bar-Natan’s “formal Poincare-Birkhoff-Witt” map.

The utility of this invariant is expressed by the top face of the cube. Namely, the Kontsevich integral of one of the tangles of interest, $Thr(T)$, factors through $Z^{ST}$. The map which completes the square is expressed in terms of a map $Thr^D$, threading diagrams. This is introduced in Section 5. This map,

$$Thr^D : B^{ST}(X) \to B(X, k)$$

is the operation of replacing winding coupons (“intersections with the meridional disc”) with exponentials of legs. The commutativity of this face depends crucially on a recent calculation due to Bar-Natan, Le and Thurston, as is indicated in that section.

$$Thr^D \left( \begin{array}{c} t \\ \end{array} \right) = \begin{array}{c} k \\ \end{array} + \begin{array}{c} k \\ \end{array} + \frac{1}{2!} \begin{array}{c} k \\ \end{array} + \frac{1}{3!} \begin{array}{c} k \\ \end{array} + \frac{1}{4!} \begin{array}{c} k \\ \end{array} + \ldots$$

The subspace $B^{ST}(X)^{int} \subset B^{ST}(X)$ is the subspace of “integrable” elements (adapting a concept of the Aarhus papers to the present context), as is defined in Section 4. In this case it refers to the subspace of elements of the form
where $W(t) \in M_{\mu}(\mathbb{Z}[t, t^{-1}])$ is a Hermitian matrix of Laurent polynomials, such that $\det(W(1)) = \pm 1$, and $R$ is a series of diagrams without chords. In the case at hand ($T$ a string link in the solid torus), $\sigma(\mathcal{Z}^{ST}(T))$ is of this form with matrix $W(T, t)$, the winding matrix of $T$, introduced in Section 3.5. This is a generalisation of the notion of linking matrix which incorporates winding information of the link around the hole of the solid torus.

The mapping $\int_{\mathcal{F} \in \mathcal{ST} \mathcal{d}X}$ (again, an adaption of a concept from the Aarhus papers) is defined in the following way. Take an element $S \in B^{ST}(X)^{\text{int}}$, with decomposition as above.

$$
\int_{\mathcal{F} \in \mathcal{ST} \mathcal{d}X} S = \left\langle \exp_{\sqcup} \left( \frac{1}{2} \sum_{i,j} W_{ij}(t) \right), R \right\rangle.
$$

This takes values in the space $B^{QST}(\phi)$ of rational winding diagrams. This space is in some sense an extension of $B^{ST}(\phi)$ which admits rational functions as labels on winding coupons. This mapping and space are introduced in Section 3.5.

The front face of the master diagram is detailed in Section 7. The commutativity of the front face indicates that this formula calculates the LMO invariant (in the event, an extension of a theorem due to the Aarhus group [A3]). Note that $\sigma_+$ and $\det$ are just some normalisation factors that need to be carried along for the diagram to make sense.

The theorem to take home is the following.

**Theorem 2.2.2** (Surgery formula). Let a pair of a zero-framed knot $K$ in an integral homology three-sphere $M$ be presented by $T$, some special string link in the solid torus. Then $Z_{n, LMO}^{ST}(M, K)$ is equal to

$$
W h(M, K) \sqcup \Theta^{D} \left( \int_{\mathcal{F} \in \mathcal{ST} \mathcal{d}X} \sigma(\mathcal{Z}^{ST}(T)) \right) \left( \int_{\mathcal{F} \in \mathcal{D} \mathcal{d}U} \sigma(\mathcal{Z}(U^+)) \right)^{\sigma_+(W(T, t))} \left( \int_{\mathcal{F} \in \mathcal{D} \mathcal{d}U} \sigma(\mathcal{Z}(U^-)) \right)^{\sigma_-(W(T, t))} \in B_{\leq n}(k).
$$

Observe that in this setting ($\mathcal{Z}HS^3$s), $Z_{n, LMO}^{LMO}$ is the degree less or equal to $\lambda$ truncation of the full LMO invariant $Z_{n, LMO}^{LMO}$. We may alternatively present this formula as follows.

**Theorem 2.2.3** (Surgery formula, $\beta$-version.). Let a pair of a zero-framed knot $K$ in an integral homology three-sphere $M$ be presented by $T$, some special string link in the solid torus. Then $Z_{n, LMO}^{LMO}(M, K)$ is equal to

$$
W h(M, K) \sqcup \Theta^{D} \left( \int_{\mathcal{F} \in \mathcal{ST} \mathcal{d}X} \sigma(\mathcal{Z}^{ST}(T)) \right) \left( \int_{\mathcal{F} \in \mathcal{D} \mathcal{d}U} \sigma(\mathcal{Z}(U^+)) \right)^{\sigma_+(W(T, t))} \left( \int_{\mathcal{F} \in \mathcal{D} \mathcal{d}U} \sigma(\mathcal{Z}(U^-)) \right)^{\sigma_-(W(T, t))} \in B(k).
$$
2.3. Conjecture - a winding diagram valued invariant of knots. Clearly we are only seeing half of a cube in Diagram 2.2.1. It will be interesting to describe the other vertex and faces. More immediately, the dashed line

\[
\begin{align*}
\left\{ \text{Special links} \mid \text{Kirby move HI} \right\} \to B_{QST}^i(\phi) \times \mathbb{Z} \times \mathbb{Z}^3 \left[ t, t^{-1} \right]
\end{align*}
\]

would be a consequence of the following conjecture.

**Conjecture 2.3.1.** \( T h r^D : B_{QST}(\phi) \to B(k) \) is injective.

Actually, this seems clear; we defer a careful explanation of this to the sequel, which will also discuss the relation with some normalisation and other technical issues (see Section 3.7). Can this corollary be proved without reference to the Kontsevich integral?

**Corollary 2.3.2** (to the conjecture.). Take a pair \((M, K)\). Choose \( T, a \) special string link in the solid torus presenting \((M, K)\). Then

\[
\int_{F \in \text{ST}} F dX_{\sigma}(\hat{Z}_{ST}(T)) \left( \int_{U} F dU_{\sigma}(\hat{Z}(U_{+})) \right)^{\sigma_{+}(W(T, 1))} \left( \int_{U} F dU_{\sigma}(\hat{Z}(U_{-})) \right)^{\sigma_{-}(W(T, 1))} \in B_{QST}(\phi)
\]

is an invariant of the pair \((M, K)\).

Strictly speaking, this does not increase our pool of knot invariants. It may, however, be a presentation more appropriate for topological applications.

2.4. Notation and conventions.

Spaces of diagrams

We use standard definitions for spaces of uni-trivalent diagrams [BN]. There is one point that may be unfamiliar to some readers. To introduce this, we note that in generality a space may be denoted:

\[
A_n(SK; x_1, \ldots, x_p, w_1, \ldots, w_q)
\]

which indicates that univalent vertices may:

1. be located, up to orientation-preserving diffeomorphisms, on a skeleton \( SK \),
2. or be labelled from \( \{x_1, \ldots, x_p, w_1, \ldots, w_q \} \).

The underlining of a variable indicates that *link relations for that variable* are to be included in the definition. These were identified in Section 5.2 of [A2]. The point is that link relations are what must be included to obtain an isomorphism:

\[
A_n(SK; x_1, \ldots, x_p, w_1, \ldots, w_q) \simeq A_n(SK \cup \biguplus_{p} \bigcup_{q} \bigcup_{\text{labels}}).
\]

On occasions when there is no skeleton (that is to say that all univalent vertices are to be labelled), we may use a \( B \) in place of the \( A \), following the conventions of [BN]. When there is only one label, then \( B(x) \simeq B(x) \).

In this work, we usually work with a set of labels \( X = \{x_1, \ldots, x_\mu\} \) corresponding to string link components (ultimately providing surgery components), and/or a label \( k \) corresponding to the (closed) knot component. Such spaces will be denoted, for example, \( B(X, k) \).

An element of this space may be indicated \( S(\mathcal{F}, k) \), where \( \mathcal{F} \) is thought of as a vector of variables. The logic of this notation should be clear after reading Section 7.1.
The invariants

The notation $\check{Z}$ denotes precisely the functorial representation of the category of framed $q$-tangles, according to the definition of [LM] (see also [BN]). We use, unless otherwise stated, the associator with rational coefficients.

Our definition of $\check{Z}$ differs slightly from existent usage. In general we will be considering tangles which have a closed component, which, when forgotten, leaves the tangle a string link. Take such a tangle $T$, which has $n$ such string link components. Our usage of $\check{Z}(T)$ is:

$$\check{Z}(T) = (\nu \otimes \ldots \otimes \nu) \circ \Delta^{n-1}(\nu) \circ \check{Z}(T).$$

The LMO invariant

The LMO group are Thang Le, Jun Murakami and Tomotada Ohtsuki. In Theorem 6.2 of [LMO] there is defined an invariant of a link in a three-manifold, which is denoted there $\Omega(M, L)$. The invariant we use is related to the definition given there as follows:

$$Z^{LMO}(M, K) = \Omega(M, K)\#\nu^{-1}.$$

Thus, following Proposition 6.5 of [LMO], we have the restriction

(2.4.1) $$Z^{LMO}(S^3, K) = \check{Z}(K).$$
3. A diagram-valued invariant of string links in the solid torus

Definition 3.0.1. A skeleton is an oriented one-manifold whose boundary points are separated into an ordered pair of ordered sets.

Our theory employs a certain enhancement of the familiar notion of uni-trivalent diagram, which will be called a winding diagram. These will possibly contain a certain new type of vertex which we will call a winding coupon. This may conveniently be thought of as some decoration of an underlying uni-trivalent diagram.

Definition 3.0.2. A winding coupon is a bivalent vertex whose incoming edges are ordered.

This is depicted as follows. The edges are ordered so that the edge incoming at the base of $t$ is first, and the edge outgoing at the top of $t$ is second. We make the convention that a coupon labelled by $t^{-1}$ is given the reverse orientation.

Definition 3.0.3. A winding diagram on a skeleton $SK$ is a graph with univalent vertices, trivalent vertices and winding coupons, such that:

1. (Boundary points.) The set of univalent vertices is separated into an ordered pair of ordered sets.
2. (Skeleton.) There are distinguished disjoint oriented cycles and oriented paths between univalent vertices which are labelled by components of the skeleton $SK$; forgetting not distinguished edges leaves one with $SK$.
3. (Internal vertices.) Trivalent vertices which are not met by distinguished edges are vertex-oriented (have their incoming edges cyclically ordered).

Two winding diagrams are identified if there is a graph isomorphism between them respecting orientations at vertices (including winding coupons) and respecting skeleton information (orientations and labels of distinguished edges, ordering of boundary points).

Definition 3.0.4. The grade of a winding diagram is half the number of trivalent vertices.

The space in question will be a quotient of the space of finite Q-linear combinations of winding coupons of a fixed grade. The quotient will be by the span of the following classes of vectors. In the relations MULT and PUSH below, edges may be part of the skeleton.
Remark 3.0.5. Looking ahead to the definition of the map $Thr^D$, Definition 5.0.4, may give some feeling for these orientation conventions and relations.

Definition 3.0.6. Let $SK$ be a skeleton. Let

$$A^{ST}_m(SK) = \left\{ \text{Finite } \mathbb{Q} \text{-linear combinations of degree } m \text{ winding diagrams on } SK \right\} / \mathbb{Q} - \text{span of above relations}$$

Let $A^{ST}(SK)$ denote the completion of $\oplus_{m=0}^{\infty}A^{ST}(SK)$ with respect to degree.

Remark 3.0.7. If $SK$ has $\mu$ components, then $A^{ST}_0(SK)$ is isomorphic, as a $\mathbb{Q}$-vector space, to $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_{\mu}^{\pm 1}]$.

Remark 3.0.8. If the bottom boundary configuration (regarded as a word in the symbols $\uparrow$ and $\downarrow$, in the familiar sense) of the skeleton $K$ matches the top boundary configuration of a skeleton $L$, then an operation $\circ : A^{ST}_n(K) \times A^{ST}_m(L) \to A^{ST}_{n+m}(K \circ L)$ is obviously defined, and is extended to the completions.
Definition 3.0.9. For some skeleton $SK$ let
\[\gamma : A(SK) \rightarrow A^{ST}(SK),\]
be the mapping defined by linearly extending the operation of mapping the element represented by some diagram in $A(SK)$ to the element that that diagram represents in $A^{ST}(SK)$.

3.1. Some notation. A coupon labelled by a polynomial represents an element of $A^{ST}$ via the following expansion. At this stage this is best regarded as a notation; later a space will be introduced ($B^{QST}(X)$, Definition 4.1) which will admit such labels in its definition.

If the polynomial is
\[p(t) = p_0 + p_1 t + \ldots + p_n t^n\]
then the following expansion is to be understood.

\[
\begin{align*}
\cdots & = p_0 + p_1 t + \ldots + p_n t^n \\
\cdots & \quad \vdots \\
\end{align*}
\]

It will also prove helpful to have the following diagrammatic. An oriented edge, labelled with a polynomial, indicates that a coupon with that label is to be introduced, in the sense just introduced. The orientation of the coupon is specified by the orientation of the edge.

\[
p(t) \quad \rightarrow \quad p(t)
\]

3.2. The invariant.

Definition 3.2.1. A 4-tuple $(A, B, w_1, w_2)$, where
- $A$ and $B$ are q-tangles,
- the bottom boundary word of $A$ is equal to the top boundary word of $B$ is equal to $(w_1)(w_2)$,
- the top boundary word of $A$ is equal to the bottom boundary word of $B$ is equal to $(\ldots (↑↑) ↑ \ldots ↑)$,

is called a presentation for $T$, a $\mu$-string string link in the solid torus, if the result of composing $A$ with $B$, while drilling the hole at some point on the mutual bounding line between $w_1$ and $w_2$, gives $T$. It is clear that every string link in the solid torus has such a presentation.
For example, the string link in the solid torus associated to the previously considered surgery presentation of the figure of 8 knot, has the following presentation:

For a boundary word $w$, let $G_w$ be the winding diagram obtained from the identity diagram $I_w$ by attaching a winding coupon to each strand. This notation can be read as the gluing diagram. For example,

$$G_{(\uparrow\downarrow)} = t t t .$$

**Definition 3.2.2.** Let $\uparrow^\mu$ denote the skeleton underlying a $\mu$-string string link.

**Definition 3.2.3.** If $T$ is a $\mu$-string string link in the solid torus, then let

$$Z^{ST}(T) = \gamma(\check{Z}(A_T)) \circ (I_{w_1} \otimes G_{w_2}) \circ \gamma(\check{Z}(B_T)) \in A^{ST}(\uparrow^\mu),$$

where $(A_T, B_T, w_1, w_2)$ is a presentation for $T$.

The most pressing issue is, of course, to show that this is well-defined. For the time being, then, indicate the dependence on the presentation $Z^{ST}(A_T, B_T, w_1, w_2)$. The well-definedness will follow from the following, clear, observation.

**Lemma 3.2.4.** Let $A \in A^{ST}(SK)$, take a word $w$ such that $G_w \circ A$ is well-defined, and take a word $w'$ such that $A \circ G_{w'}$ is well-defined. Then

$$G_w \circ A = A \circ G_{w'} .$$

**Lemma 3.2.5.** $Z^{ST}(A_T, B_T, w_1, w_2)$ is independent of the choice of presentation, and hence is an invariant of $T$.

**Proof.**

It is clear that any two presentations can be related by a finite sequence of the following moves:

1. $(A \circ (1_{w_1} \otimes C), B, w_1, w_2) \leftrightarrow (A, (1_{w_1} \otimes C) \circ B, w_1, w'_2)$,
2. $(A \circ (C \otimes 1_{w_2}), B, w_1, w_2) \leftrightarrow (A, (C \otimes 1_{w_2}) \circ B, w'_1, w_2)$.

The lemma then follows from the functoriality of $\check{Z}$ and Lemma 3.2.4.
Thus we revert to the notation $Z^{ST}(T)$.

The normalisation of this invariant that is appropriate for surgery considerations is the following. This is the normalisation of $[\text{LMMO}]$.

**Definition 3.2.6.** Let $T$ be an $\mu$-string string link in a solid torus. Define:

$$\hat{Z}^{ST}(T) = \gamma((\nu \otimes \ldots \otimes \nu) \circ \Delta^{-1}(\nu)) \circ Z^{ST}(T),$$

in the space $A^{ST}(\uparrow^\mu)$, recalling that $\nu = \hat{Z}(U) \in A(\uparrow)$.

### 3.3. The co-product.

We now equip $A^{ST}(K)$ with a co-product. The presence of winding coupons does not affect the following familiar definition.

**Definition 3.3.1.** Take a diagram $D$ such that its dashed graph has connected components indexed by the set $I$. If $J \subset I$ let $D_J$ indicate the diagram obtained by forgetting those components in the subset $J$. Then, define the mapping $\Delta$ as the linear extension of

$$\Delta(D) = \sum_{J \subset I} D_J \otimes D_{I-J}.$$

**Remark 3.3.2.**

1. If $D$ has an empty dashed graph then this operation is defined to be $\Delta(D) = D \otimes D$.
2. This defines a co-product on the graded completions:

$$\Delta : A^{ST}(K) \to A^{ST}(K) \hat{\otimes} A^{ST}(K).$$

**Lemma 3.3.3.** This is well-defined, co-commutative, co-associative, and commutes with compositions.

To see that it is well-defined we must show that relations are mapped to relations. The only novelty is a PUSH relation when two of the involved edges are part of the skeleton; this relation is easily checked. Observe that it commutes with compositions by construction. All other properties are standard.

**Lemma 3.3.4.** For a string link in a solid torus $T$,

$$\Delta(\hat{Z}^{ST}(T)) = \hat{Z}^{ST}(T) \otimes \hat{Z}^{ST}(T).$$

**Proof.** This follows for the usual reasons: that is, from the corresponding property for $\hat{Z}$, the corresponding property for the normalisation factors of Definition 3.2.6, from the obvious property that $\Delta(G_w) = G_w \otimes G_w$ and from the commutation of composition with the co-product.

### 3.4. The Hopf algebra $B^{ST}(X)$.

We turn to the case of special string links in the solid torus. These are, remember, string links in the solid torus such that a representative in general position with respect to the meridional disc has algebraic intersection zero with it. For such an $\mu$-string string link in the solid torus $T$, $Z^{ST}(T)$ lies in a special subspace of $A^{ST}(\uparrow^\mu)$. 

\[\square\]
Definition 3.4.1. Let $A^{ST,\text{spec}}(\uparrow \mu)$ denote the subspace of $A^{ST}(\uparrow \mu)$ spanned by diagrams with the property that the product of all the labels on the winding coupons labelling some component of the skeleton is 1 (that is, using a factor of $t^{-1}$ if some coupon is oriented against the orientation of that component); for each component.

Observation 3.4.2. If $T$ is a special $\mu$-string string link in the solid torus then $\tilde{Z}(T) \in A^{ST,\text{spec}}(\uparrow \mu)$.

Remark 3.4.3. If a diagram is in this subspace, then repeated applications of the $PUSH$ relation can be used to make all of the labels on the skeleton 1 (say, by pushing all the labels to one end). Then all the labels (all the winding) will be carried by the dashed graph.

We now introduce an isomorphic description of this subspace. This is an enhancement of the familiar algebra $B$. Let $X = \{x_1, \ldots, x_\mu\}$ denote a labelling set for the skeleton $\uparrow \mu$.

Definition 3.4.4. Let a winding diagram on $X$ be a graph with oriented trivalent vertices, winding coupons, and univalent vertices labelled from $X$.

We may alternatively call this a symmetrised winding diagram on $\uparrow \mu$.

Definition 3.4.5. Define

$$B^{ST}_m(X) = \left\{ \frac{\text{Finite } \mathbb{Q}\text{-linear combinations of degree } m}{\text{winding diagrams on } X} \right\} \mathbb{Q} - \text{span of AS, IHX, OR, MULT and PUSH relations}$$

Let $B^{ST}(X)$ denote the graded completion of $\oplus_{m=0}^{\infty} B^{ST}_m(X)$. Equip this with the obvious analogs of the “disjoint-union” product, the “sum over partitions into two sets” co-product, the “empty set” unit and co-unit, and the “(−1) for every component” antipode.

Lemma 3.4.6. $B^{ST}(X)$ is a commutative, co-commutative Hopf algebra.

Definition 3.4.7. Let $\chi : B^{ST}(X) \to A^{ST,\text{spec}}(\uparrow \mu)$ be the operation defined on some symmetrised diagram of taking the average of all diagrams obtained by locating all univalent vertices labelled with $x_1$ on the first component, etc.; linearly extended to each $B^{ST}_m$, and to $B^{ST}$.

Lemma 3.4.8. The mapping $\chi$ describes a $\mathbb{Q}$-vector space isomorphism at each grade

$$B^{ST}_m(X) \simeq A^{ST,\text{spec}}_m(\uparrow \mu),$$

commuting with coproducts

$$(\chi \otimes \chi) \circ \Delta = \Delta \circ \chi.$$
Lemma 3.4.9. Let $T$ be a special $\mu$-string string link in the solid torus. Then $\sigma(\check{Z}^{ST}(T))$ is a group-like element in the Hopf algebra $B^{ST}(X)$. Thus it is an exponential of a series of connected diagrams, a finite $\mathbb{Q}$-linear combination at each grade.

To see this, note that Lemma 3.3.4 indicates that $\check{Z}^{ST}(T)$ is group-like in $A^{ST,\text{spec}}$. Thus its image in $B^{ST}(X)$ is also group-like because of the commuting of the co-product with the map $\sigma$. Thus it is an exponential of a primitive element (for example, \cite{Q1}, Appendix A): at each grade this will be a finite $\mathbb{Q}$-linear combination of connected diagrams.

3.5. The winding matrix. We now introduce $W(T, t) \in M_\mu(\mathbb{Z}[t, t^{-1}])$, the winding matrix of $T$, where $T$ is a $\mu$-string string link in the solid torus. Number the components of $T$ and choose a diagram for $T$ that is in general position with respect to the meridional disc. We consider paths on this diagram. The “algebraic intersection of a path with the meridional disc” is the sum over all crossings of that path with the disc of: a plus one if the tangent vector of the path points in the direction of increasing $y$ at the intersection; and a minus one otherwise (according to the model of the solid torus described in Section 2.1).

Definition 3.5.1. For a crossing $c$ of strands $i$ and $j$, let $\epsilon(i, j, c)$ denote the algebraic intersection with the meridional disc of the path obtained by travelling from the base along $i$ to $c$, crossing to $j$, and then travelling to the top along $j$. (To be precise, if $i = j$ then change strands the first time the crossing is encountered).

Definition 3.5.2. For $T$, an $\mu$-string string link in the solid torus, choose a (blackboard-framed) diagram for $T$ in general position with respect to the meridional disc. If $c$ denotes a crossing then let $\text{sgn}(c)$ denote the sign of that crossing. Let:

$$W_{ij}(T, t) = \begin{cases} \sum_{c, \text{a crossing of } i \text{ and } j} \frac{1}{2} \text{sgn}(c) t^{(i,j,c)} & \text{if } i \neq j, \\ \sum_{c, \text{a self-crossing of } i} \frac{1}{2} \text{sgn}(c) (t^{(i,i,c)} + t^{-c(i,i,c)}) & \text{otherwise.} \end{cases}$$

Remark 3.5.3.
- The winding matrix is Hermitian: $W_{ij}(T, t) = W_{ji}(T, t^{-1})$.
- The winding matrix specialises to the linking matrix of the underlying tangle: $W_{ij}(T, 1) = \text{Lk}_{ij}(T)$.

Lemma 3.5.4. $W_{ij}(T, t)$ is an isotopy invariant of $T$, regarded as a framed string link in a solid torus.

We will return to the topological interpretation of $W_{ij}$ at the end of this section when we make an important connection with the Alexander polynomial. Besides, its isotopy invariance follows from its appearance in the following theorem (the unique Hermitian matrix with the following property).

Let $X = \{x_1, \ldots, x_\mu\}$ denote a labelling set for $T$.

**Theorem 3.5.5.** If $T$ is a string link in a solid torus then

$$\sigma(\check{Z}^{ST}(T)) = \exp_{\mathbb{Q}} \left( \frac{1}{2} \sum_{i,j} W_{ij}(t) \right) \sqcup R,$$
where $R$ is a series of $X$-substantial diagrams.

**Proof.**

Lemma 3.4.9 indicated that $\sigma(\tilde{Z}^{ST}(T))$ is of the form $\exp_\omega(S)$ where $S$ is a series of connected diagrams. Thus to prove the theorem it is sufficient to calculate the degree one part of $\tilde{Z}^{ST}$.

Take a diagram of $T$ in general position with respect to the meridional disc, and an associated presentation of $T$. Examining the definition of $\tilde{Z}^{ST}$, we see that there will be a contribution of one chord from every crossing in this diagram. The introduced complexity is that there will be a distribution of winding coupons on the skeleton. (The reader is invited to consider the example that follows this proof).

Consider first a crossing between two different components, $i$ and $j$. Use the relation $PUSH$ to push the coupons onto the chord in the following manner: coupons that occur before the crossing as $i$ is traversed from the base should be pushed past the chord (where they will all cancel to 1); coupons that occur after the crossing as $j$ is traversed to the top should be pushed back past the chord (where they will all cancel to 1).

The reader can check that the label on the chord coming from that crossing is precisely as follows. Note that in this section we will use the notational convention described in Section 3.1.

Now consider self-crossings of components. Choose some self-crossing $c$ of some component $x_i$. Its contribution is as follows:
\[ \frac{1}{2} \text{sgn}(c) \]

Now, use the following relation:

\[
\begin{array}{c}
1 \\
\cdots \\
1
\end{array}
\]

\[ t \epsilon \left( i,i,c \right) \]

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where $r$ and $r'$ are series of diagrams that are either of grade greater than 1 or X-substantial.

3.6. **A topological interpretation of** $W(T, t)$. The attentive reader will have noticed the appearance of the Alexander polynomial for the figure of 8 knot in the previous calculation. Let us examine the meaning of the matrix $W(T, t)$ more closely.

Choose a base point close to the base of the strings, and choose paths from that basepoint to the bases of the strings (so that the bases, paths, and basepoint all lie in some ball). Then close the string link on the left, obtaining some link in the solid torus $L$, say with components $\{K_1, \ldots, K_\mu\}$, with a path from some basepoint to some point on each component.

Take the universal cyclic cover of the solid torus:

$$p : \tilde{ST} \to ST,$$

and lift the link $\{K_1, \ldots, K_\mu\}$ to $\tilde{ST}$. This can be done as we are restricting to special string links (that is, string links whose algebraic intersection with any meridional disc is zero). The group of translations is $\mathbb{Z}$: choose an action such that a path which starts at some point $p$; crosses the meridional disc in the direction on increasing $y$; and the returns to the $p$ (without again crossing the meridional disc) is lifted to a path starting at some $p$ and finishing at some $tp$. The lifted link $\tilde{L}$ can be identified as the set of translates

$$\{\ldots, t^{-1}\tilde{K}_1, \tilde{K}_1, t\tilde{K}_1, \ldots, t^{-1}\tilde{K}_\mu, \tilde{K}_\mu, t\tilde{K}_\mu, \ldots\},$$

where the components $\{\tilde{K}_1, \ldots, \tilde{K}_\mu\}$ can be fixed by choosing $\tilde{K}_1$ and then following the lifts of the arcs introduced when the closure was taken.

We define an invariant of $T$ using this arrangement, as follows.

**Definition 3.6.1.** Let $\tilde{Lk}(T) \in M_\mu(\mathbb{Z}[t, t^{-1}])$ be defined by

$$\tilde{Lk}_{ij}(T) = \sum_{m=-\infty}^{\infty} t^m lk(\tilde{K}_i, t^m \tilde{K}_j).$$

**Lemma 3.6.2.**

$$W(T, t) = \tilde{Lk}(T).$$

**Proof.**

Take a blackboard-framed diagram for $T$ that is in general position with respect to the projection of the meridional disc. A fundamental domain for a diagram for the pair $(\tilde{ST}, \tilde{L})$ can be obtained by cutting such a diagram along the projection of
the meridional disc, obtaining a rectangle, and then gluing a countable infinity of copies of that rectangle in the appropriate way.

Observe that a crossing between two different components, say \( x_i \) and \( x_j \), in the diagram for \( T \) lifts to a crossing between \( K_i \) and some translate of \( K_j \). Which translate is decided by counting intersections with the meridional disc as in the definition of \( W \).

The slightly different definition of \( W \) along the diagonal is accounted for by the fact that self-crossings of a component \( x_i \) in the diagram for \( T \) will either lift to a self-crossing of \( K_i \), or to a pair of crossings, between \( K_i \) and some \( t^n K_i \) and between \( K_i \) and \( t^{-n} K_i \). Observe that the factors give the right weights in both situations.

\[ \text{Example 3.6.3.} \] Continuing the example of the figure of 8 knot:

Take \( T \), a special string link in the solid torus, presenting some pair \((M, K)\). Remember that, according to Definition 1.0.3, \( A_{(M,K)}(t) \) denotes the canonical Alexander polynomial of a knot in a \( \mathbb{Z}HS^3 \).

\[ \text{Lemma 3.6.4.} \]

\[ A_{(M,K)}(t) = \pm \det(W(T, t)). \]

\[ \text{Proof.} \]

Surgery on the framed link \( \tilde{L} \) in \( D^2 \times \mathbb{R} \) recovers the universal cyclic cover of the complement of \( K_T \).

The Mayer-Vietoris sequence then indicates that the matrix \( \tilde{L} \), and hence \( W(T, t) \), is a presentation matrix for the \( \mathbb{Z}[t] \)-module \( H_1(M - K; \mathbb{Z}) \).

The Alexander polynomial is defined as a generator of the order ideal of that module. In the situation at hand, given a square presentation matrix, this is calculated by the determinant of that matrix, as in the statement of the lemma. Note that this only specifies the polynomial up to multiplications of the form \( \pm t^n \), and the statement of the lemma asserts that the recovered polynomial is symmetric under \( t \rightarrow t^{-1} \).

To see that this is true note that \( W(T, t) \) is a Hermitian matrix, according to Remark 3.5.3.

\[ \det(W(T, t)) = \det(W(T, t)^{Tr}) = \det(W(T, t)) \]
Remark 3.6.5. $W(T,t)$ is, presumably, the matrix referred to in Exercise C.13 of Rolfsen [Rol].

3.7. Alternative normalisations. We take this opportunity to draw attention to a certain subtle choice of normalisation that has been made in the construction of $Z^{ST}$ given here.

Let $\alpha$ denote a group-like element of $A(\uparrow)$, the space of uni-trivalent diagrams on a single strand.

**Definition 3.7.1.** Let

$$Z^{ST}[\alpha] : \left\{ \begin{array}{c}
\text{String links in the solid torus.}
\end{array} \right\} \rightarrow A^{ST}(\uparrow^\mu),$$

be defined in exactly the same way as given in Definition 3.2.3, except that in Equation 3.2.1 $I_{\omega_1} \otimes G_{\omega_2}$ should be replaced by

$I_{w_1} \otimes (\Delta_{\omega_2}(\alpha) \circ G_{\omega_2}),$

where $\Delta_{\omega_2}(\alpha)$ is the paralleling operation across the strands described by $\omega_2$, applied to $\alpha$. Let $\breve{Z}^{ST}[\alpha]$ denote the normalisation corresponding to Definition 3.2.6.

**Remark 3.7.2.** Lemma 3.4.9 and Theorem 3.5.5 still hold if $\breve{Z}^{ST}$ is replaced by $\breve{Z}^{ST}[\alpha]$.

Certain choices of $\alpha$ will prove appropriate for certain applications. For example, setting $\alpha = \nu^{-1}$ gives a normalisation that is better adapted to questions involving covering spaces.

Applying $\int F^G_{in ST} dX$, followed by $Thr^D$, presumably gives knots invariants which are different normalisations of the Kontsevich integral, in some sense. We will return to this question in the sequel.
4. Formal Gaussian integration

In this section we focus on the leftmost edge of Diagram 2.2.1.

\[ B^{ST}(X)^{Int} \]

\[ \int_{FG \text{ in } ST}^+ dX \times \sigma_+ \times \det \]

\[ B^{QST}(\phi) \times \mathbb{Z} \times \mathbb{Z}^1[t, t^{-1}] \]

**Definition 4.0.3.** An element \( S \in B^{ST}(X) \) is said to be integrable if it is of the following form.

\[
S = \exp_{\perp} \left( \frac{1}{2} \sum_{ij} W_{ij}(t) \right) \sqcup R,
\]

1. \( W_{ij}(t) \) is a Hermitian matrix \( (W_{ij}(t) = W_{ji}(t^{-1})) \) with the property that \( \det(W(1)) = \pm 1 \),
2. \( R \) is a series of \( X \)-substantial diagrams: that is, a diagram will have no chords (ignoring winding coupons).

**Remark 4.0.4.** Observe that the above decomposition is unique. In particular, a Hermitian matrix satisfying Equation 4.0.1 (for some given \( S \)) will be unique. The matrix will be called the Gaussian matrix of \( S \).

**Definition 4.0.5.** Let \( B^{ST}(X)^{Int} \) denote the subspace of integrable elements of \( B^{ST}(X) \).

**Remark 4.0.6.** As the associated Hermitian matrix is uniquely specified, we will freely apply any function of such matrices to the set \( B^{ST}(X)^{Int} \) with the understanding that the function is to be applied to the Gaussian matrix of the element. An example of such use is the use of the functions \( \sigma_+ \) and \( \det \) in the diagram above. These factors will be taken up again in Section 7.

4.1. Rational winding diagrams. The Aarhus calculation of the LMO invariant has an associated philosophy of formal Gaussian integration. The idea is that given an integrable element of \( B(X)^{Int} \) one “integrates” by splitting off the quadratic part, inverting it, and contracting the result with the remainder.

We would like to introduce an analog of this in the situation at hand, that is, for the space \( B^{ST}(X)^{Int} \). Whilst we have a clear definition for an integrable element, it remains for us to introduce a suitable space in which to “invert” the given matrix of polynomials.

The definition that follows generalises the definition of the space of winding diagrams, Definition 3.0.6. We will differentiate this space in our vocabulary by inserting the adjective “rational”.

Remark 4.1.1. We should point out that for the proof of Theorem 2.2.2, and hence Rozansky’s conjecture, the following definition is unnecessary. The reader who feels the following is an unnecessarily cumbersome space may happily evade this space by observing that, in this work, every appearance of \( \int_{F \in S}^T dX \) is followed by a \( Th \). (See Remark 5.1). The point of including such a definition here is Conjecture 2.3.1.

Definition 4.1.2. A rational winding coupon is a vertex of some even valency. The incoming edges are ordered up to reorderings of the form \((\sigma, \sigma)\) for some \(\sigma \in \Sigma_m\) (if the valency is \(2m\)). A winding coupon is labelled from \(Q_1(t)\), the ring of rational functions in a single variable which are non-singular at 1.

This is depicted as follows, the idea being that an edge leads through the coupon to the opposite edge. The ellipsis used as below will indicate the possibility of a number of other edges. The ordering of the edges is the bottom row, from left to right, followed by the top row, from left to right.

\[
\begin{array}{c}
q(t) \\
\cdots \\
\end{array}
\]

Definition 4.1.3. A rational winding diagram labelled from the set \(X\) is a graph with univalent vertices, trivalent vertices and winding coupons, such that:

1. (Univalent vertices.) Univalent vertices are labelled from \(X\).
2. (Trivalent vertices.) Trivalent vertices are vertex-oriented (have their incoming edges cyclically ordered).

Two rational winding diagrams are identified if there is a graph isomorphism between them respecting orientations at trivalent vertices, orientations and labels of winding coupons, and labels of univalent vertices.

Definition 4.1.4. The grade of a rational winding diagram is half the number of trivalent vertices.

The space in question will be a quotient of the space of finite \(\mathbb{Q}\)-linear combinations of rational winding diagrams of a fixed grade. The quotient will be by the span of the following classes of vectors.

\[
AS : \quad \begin{array}{c}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array} + \quad \begin{array}{c}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array}
\]

\[
IHX : \quad \begin{array}{c}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array} - \quad \begin{array}{c}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array} - \quad \begin{array}{c}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array}
\]
\(STU:\)

\[
\begin{align*}
\text{−} & \quad \text{−} & \quad \text{+}
\end{align*}
\]

\(OR:\)

\[
\begin{align*}
q(t) & \quad \text{−} & \quad q(t^{-1})
\end{align*}
\]

\(ADD:\)

\[
\begin{align*}
aq(t) + br(t) & \quad \text{−} & \quad aq(t) & \quad \text{−} & \quad br(t)
\end{align*}
\]

\(MULT:\)

\[
\begin{align*}
q(t) & \quad \text{−} & \quad q(t) r(t)
\end{align*}
\]

\(SPLIT:\)

\[
\begin{align*}
t & \quad \text{−} & \quad t \quad t
\end{align*}
\]

\(COMM:\)

\[
\begin{align*}
q(t) r(t) & \quad \text{−} & \quad r(t) q(t)
\end{align*}
\]
Definition 4.1.5. Let $X$ be a set of labels. Let

$$B^{QST}_m(X) = \left\{ \text{Finite} \mathbb{Q}\text{-linear combinations of degree } m \right. \text{rational winding diagrams labelled from } X \left. \right\}$$

$\mathbb{Q}$-span of above relations

Let $B^{QST}(X)$ denote the completion of $\oplus_{m=0}^{\infty} B^{QST}_m(X)$ with respect to degree.

4.2. Formal Gaussian integration in the solid torus. We can now introduce the map

$$\int^{FG in ST}_{dX} : B^{ST}(X)^{Int} \to B^{QST}(\phi).$$

Definition 4.2.1. If the unique decomposition of an element $S \in B^{ST}(X)^{Int}$ is

$$S = \exp_\sqcup \left( \frac{1}{2} \sum_{ij} W_{ij}(t) \right) \sqcup R,$$

then

$$\int^{FG in ST}_{dX} S = \left\langle \exp_\sqcup \left( -\frac{1}{2} \sum_{ij} W_{ij}^{-1}(t) \right), R \right\rangle_x \in B^{QST}(\phi).$$
5. Threading

In this section we will introduce $Thr^D$, the operation of threading (rational) winding diagrams. This is used in the following square from the master diagram. In this section we will show it commutes.

Diagram 5.0.2.

\[
\begin{array}{ccc}
\{ \text{Special string links} \} & \xrightarrow{Thr} & \{ \text{Special tangles} \} \\
\sigma \circ \tilde{Z}^{ST} & & \sigma \circ \tilde{Z} \\
B^{ST}(X)^{Int} & \xrightarrow{Thr^D} & B(X, k)
\end{array}
\]

$S \mapsto Thr^D(S) \sqcup \nu(k)$

- “Special string link in the solid torus” is defined in Definition 2.1.7, and “Special tangle” is defined in Definition 2.1.8.
- $Thr$ is the operation introduced in Definition 2.1.6 which threads the hole in the solid torus with a zero-framed unknot.
- The space $B^{ST}(X)^{Int}$ is defined in Definition 4.0.3, and $B(X, k)$ is recalled in Section 2.4.
- The operation $Thr^D$ will presently be introduced.

This operation will be defined on any of the spaces introduced so far which involve (possibly rational) winding coupons. If the labelling information is denoted $L$ (so maybe a skeleton and a set of labels, or possibly $\phi$) then the map will be between spaces as follows. The $Q$ is in brackets because it may, or may not, be present.

$Thr^D : B^{(Q)^{ST}}(L) \to B(L, k)$.

The definition of this map will be introduced via an intermediate construction, a generating diagram.

**Definition 5.0.3.** A **generating diagram** is precisely the same as a winding diagram, except that its coupons are labelled with formal power series in a variable $k$. A generating diagram denotes a particular series of uni-trivalent diagrams.

1. (One edge) If there is only one edge going through the coupon, then the association is as follows.

   If $f(k) \in \mathbb{Q}[[k]]$ is written $f_0 + f_1 k + f_2 k^2 + f_3 k^3 + \ldots$ then a coupon labelled as follows, is to be expanded as shown. See below for the meaning of the vertex.

\[
f(k) = f_0 + f_1 k + f_2 k^2 + f_3 k^3 + \ldots
\]
2. (More than one edge) In this case, one takes the above expansion and then, for each diagram, takes the sum of diagrams obtained by lifting each introduced leg to each edge going through the coupon.

This vertex depends on whether the edge is internal or part of the skeleton, and then also on the orientation of the skeleton, as follows.

\[
\begin{array}{c}
\text{k} \rightarrow \text{k} \\
\text{k} \rightarrow \text{k} \\
\text{k} \rightarrow (-1) \rightarrow \text{k}
\end{array}
\]

**Definition 5.0.4.** The operation \( \text{Thr}^D \) is defined by mapping a (rational) winding diagram to the series of uni-trivalent diagrams represented by making the substitution \( t \mapsto e^k \) in the label of every winding coupon.

**Lemma 5.0.5.** This is a well-defined operation.

For starters, the substitution makes sense because we are restricting labels to the subspace of rational functions that are non-singular at \( t = 1 \). Furthermore, the relations involving winding coupons (OR, ADD, MULT, SPLIT, COMM, PUSH1 and PUSH2) are easily checked.

**Theorem 5.0.6.** Diagram 5.0.2 commutes.

This theorem depends crucially on a certain consequence of the Kontsevich integral proof of the “Wheels Conjectures” that has recently been given by Bar-Natan, Le and Thurston [TW]. See the forthcoming paper of Bar-Natan and Lawrence [BNL] for the following calculation. (See that paper also for references to other proofs of the “Wheeling Conjecture” that are in the literature.)

**Theorem 5.0.7.**

\[
\hat{Z}(x, k) = \exp(x) \sqcup (x) \sqcup (v(k)) \in B(x, k).
\]

We use the following corollary. The tangle below is equipped with some choice of bracketting which is the same on both the top and the bottom boundary words.

**Corollary 5.0.8.**
in the space $A(\uparrow \ldots \uparrow \downarrow \ldots \downarrow ; k)$.

Proof of corollary. This is proved with Le and Murakami’s paralleling formula \(\text{LM}_2\). The small point to observe is that an application of the paralleling formula gives

$$\exp_x \bigcup_{x_1 k} \ldots \cup \exp_x \bigcup_{x_r k} \ldots \cup \exp_x \bigcup_{x_{r+1} k} \ldots \cup \exp_x \bigcup_{x_{r+s} k} \cup \nu(k).$$

This requires the averaged sum of orderings of legs on the components \(\{x_i\}\), whereas the given statement requires the composition exponential. But the other ends of these chords are unordered. So the corollary follows as stated, for \(\hat{Z}\).

(Observe that there are no problems with the choice of associator for this scenario).

\[\Box\]

Proof of Theorem 5.0.6. Take \(T\), an \(\mu\)-string string link in the solid torus, given by some presentation \((\hat{A}_T, B_T, w_1, w_2)\). Without loss of generality we can assume that \(w_2\) is some bracketting of some word of the form \(\uparrow \ldots \uparrow \downarrow \ldots \downarrow \).

Then, the functoriality of \(\hat{Z}\) indicates that \(\tilde{Z}(\text{Thr}(T))\) is equal to the following expression, in the space \(A(\uparrow \mu, k)\).

\[
\left(\nu \otimes \ldots \otimes \nu\right) \circ \Delta^{-1}(\nu) \circ \tilde{Z}(A_T) \circ \left( I_{w_1} \otimes e^k \quad e^k \quad e^k \quad e^k \right) \circ \tilde{Z}(B_T) \cup \nu(k)
\]

Alternatively, \(\text{Thr}^D(\tilde{Z}^{ST}(T)) \cup \nu(k)\) may be calculated, as follows.

\[
\text{Thr}^D\left(\left(\nu \otimes \ldots \otimes \nu\right) \circ \Delta^\mu(\nu) \circ \gamma(\tilde{Z}(A_T)) \circ (I_{w_1} \otimes G_{w_2}) \circ \gamma(\tilde{Z}(B_T))\right) \cup \nu(k).
\]

This is exactly the same thing, in the space \(A(\uparrow \mu, k)\). We have just proved the commutation of the top half of the following diagram. The precise statement of the theorem then follows from the commutation of the bottom half.
5.1. Evading $B^{QST}$. The reader who wishes to evade the space $B^{QST}$ may do so by redefining the map $\text{Thr}^D \circ \int^{FG \text{ in } ST}$ as follows:

**Alternative Definition 5.1.1.** If the unique decomposition of an element $S \in B^{ST}(X)^{\text{Int}}$ is

\begin{equation}
S = \exp_{\int} \left( \frac{1}{2} \sum_{ij} W_{ij}(t) \right) \sqcup R,
\end{equation}

then

\begin{equation}
\text{Thr}^D \left( \int^{FG \text{ in } ST} dX(S) \right) = \left\langle \exp_{\int} \left( -\frac{1}{2} \sum_{ij} W_{ij}^{-1}(e^S) \right), \text{Thr}^D(R) \right\rangle_x \in B(\mathbb{k}).
\end{equation}
6. The LMO Invariant

The LMO invariant was introduced by Thang Le, Jun Murakami and Tomotada Ohtsuki [LMO]. We will specialise our presentation to the setting in question, knots in integral homology three-spheres.

Let the pair of a knot \( K \) in an integral homology three sphere \( M \) be presented by some special framed tangle \( T \). That is, \( T \) has one closed component, and forgetting that closed component leaves the tangle a string link \( T' \) whose components are to be surgered, after closure; the determinant of the linking matrix of those components is \( \pm 1 \). Let \( X = \{x_1, \ldots, x_\mu\} \) be an index set for \( T' \), and let \( \text{lk}(T') \) denote the linking matrix of those components.

The LMO invariant is constructed as a sequence

\[
Z_{LMO}^n(M, K) \in B_{\leq n}(k)
\]

with the property that

\[
\text{Grad}_{\leq n}(Z_m^{LMO}(M, K)) = Z_n^{LMO}(M, K) \in B_{\leq n}(k)
\]

when \( m \geq n \). This sequence can thus be regarded as approximations to an invariant defined by

\[
Z^{LMO}(M, K) = 1 + \text{Grad}_1(Z_1^{LMO}(M, K)) + \text{Grad}_2(Z_2^{LMO}(M, K)) + \ldots \in B(k).
\]

It is convenient to introduce the operation that is the heart of this definition over a more general space.

**Definition 6.0.2.** For some labelling set \( L \), the space \( B^o(L) \) is defined with exactly the same definition as \( B(L) \), except that generating diagrams may also have a finite number of closed dashed loops.

**Remark 6.0.3.** The space \( B(L) \) clearly injects into \( B^o(L) \).

**Definition 6.0.4.** The mapping

\[
\int^{(n)} dX : B^o(X, X', k) \to B_{\leq n}(X', k),
\]

is defined on a diagram \( D \) as

\[
\left( \prod_{i=1}^{\mu} \left( \frac{1}{n!} \left( \frac{1}{2} \bigcap_{x_i} \bigcap X_i \right)^n \right) , D \right),
\]

followed by the exchange of each dashed loop component (some extra may arise) for a multiplicative factor of \(-2n\).

With this operation, \( Z_{LMO}^n(M, K) \) is defined as follows. Let \( \sigma_{\pm}(\text{lk}(T')) \) denote the number of positive (resp. negative) eigenvalues of \( \text{lk}(T') \). Let \( U_{\pm} \) be a \( \pm 1 \)-framed unknot.

**Definition 6.0.5.** The invariant \( Z_{LMO}^n(M, K) \) is defined by

\[
Z_{LMO}^n(M, K) = \frac{\int^{(n)} dX \sigma(\bar{Z}(T))}{\left( \int^{(n)} dU \sigma(\bar{Z}(U_+)) \right)^{\sigma_+(\text{lk}(T'))} \left( \int^{(n)} dU \sigma(\bar{Z}(U_-)) \right)^{\sigma_-(\text{lk}(T'))} },
\]

in the space \( B_{\leq n}(k) \).
Notation 6.0.6. Let the numerator in the expression above be denoted $Z^{LMO,o}(T)$. Note that this $o$ is of a different nature to the $o$ used in Definition 6.0.3 above.

Theorem 6.0.7. This definition gives a well-defined invariant of knots in integral homology three-spheres, assembled via Formula 6.0.4, with the specialisation

$$Z^{LMO}(S^3, K) = \hat{Z}(K).$$

Remark 6.0.8. Observe that our normalisation of the non-surgered components does not affect surgery formula.
7. The surgery formula

This section is the crux of the paper. It will prove that the following square from the master diagram commutes.

Diagram 7.0.9.

\[
\begin{array}{ccc}
B^{ST}(X)^{Int} & \xrightarrow{S \mapsto Thr^{D}(S) \sqcup \nu(k)} & B(X, k) \\
\downarrow & & \downarrow \\
\int_{FG \in ST}^{FG \in ST} dX \times \sigma_{+} \times det & & \int^{(n)} dX \\
\downarrow & & \downarrow \\
B^{QST}(\phi) \times \mathbb{Z} \times \mathbb{Z}^{1}[t, t^{-1}] & \xrightarrow{(S, \sigma, P(t)) \mapsto Thr^{D}(S) \sqcup (-1)^{n_{\sigma}} Wh'(P(t))} & B_{\leq n}(k) \\
\end{array}
\]

There are some components of this that are yet to be introduced.

**Definition 7.0.10.** Let \( \mathbb{Z}^{1}[t, t^{-1}] \) be the subring of \( \mathbb{Z}[t, t^{-1}] \) of polynomials such that if \( p(t) \in \mathbb{Z}^{1}[t, t^{-1}] \) then
1. \( p(t) = p(t^{-1}) \),
2. \( p(1) = \pm 1 \).

Clearly \( det \), evaluated on \( B^{ST}(X)^{Int} \) (according to Remark 4.0.6) takes values in this ring.

**Definition 7.0.11.** The mapping \( \sigma_{+} \) on \( S \in B^{ST}(X)^{Int} \) is the number of positive eigenvalues of \( W(1) \), where \( W(t) \) is the Gaussian matrix of \( S \).

**Definition 7.0.12.** The mapping

\[ Wh' : \mathbb{Z}^{1}[t, t^{-1}] \to B(k), \]

is defined by

\[
Wh'(P(t)) = \exp_{\sqcup} \left( \left[ -\frac{1}{2} \log \left( \frac{P(e^{h})}{P(1)} \right) \right] \right)_{h^{2n} \to \omega_{2n}} \sqcup \nu(k),
\]

where the operation indicated is to expand the term inside the square brackets into a power series in \( h \), and then to replace terms like \( ch^{2n} \) by \( c\omega_{2n} \), in exactly the same fashion as Definition 1.0.7.

Observe that the \( P(1) \) factor just adjusts the sign of the polynomial.

**Remark 7.0.13.**

\[ Wh(M, K) = Wh'(A_{(M, K)}(t)). \]
7.1. Translation by power series.

Before we consider the details of this proof, we introduce a certain operation on diagrams. For \( F(x_1, \ldots, x_\mu, k) \in B(X, k) \), the notation \( F(x_1 + z, x_2, \ldots, x_\mu, k) \), according to Aarhus, denotes the series in \( B(X, k, z) \) obtained by replacing every diagram with \( l \) legs labelled by \( x_1 \) by the \( 2^l \) diagrams obtained by relabelling each such leg with either \( x_1 \) or \( z \). This may be extended linearly to simultaneous "translations" of other variables.

Now we introduce something novel. If \( f(k) \in \mathbb{Q}[[k]] \) then \( F(f(k)x_1, x_2, \ldots, x_\mu, k) \) denotes the element obtained by replacing every diagram in the expansion of \( F \) with a generating diagram obtained by labelling (simultaneously) every leg marked \( x_1 \) as follows. Observe that the added coupon is oriented according to the position of the univalent vertex.

\[
\begin{array}{c}
\vdots \\
 x_1 \\
\vdots \\
\text{→} \\
\vdots \\
\text{f(k)} \\
\vdots
\end{array}
\]

We will want to combine these operations. For example, take some \( f(k) \in \mathbb{Q}[[k]] \):

\[
F(x_1 + f(k)z, x_2, \ldots, x_\mu, k) = F(x_1, x_2, \ldots, x_\mu, k) + F(f(k)z, x_2, \ldots, x_\mu, k).
\]

**Notation 7.1.1.** Take a matrix \( M(k) \in M_\mu(\mathbb{Q}[[k]]) \). The notation \( F(\mathfrak{F} + M(k)\mathfrak{F}', k) \) represents the element

\[
F(x_1 + \sum_{i_1} M_{i_1}(k)x'_{i_1}, x_2 + \sum_{i_2} M_{2i_2}(k)x'_{i_2}, \ldots, x_\mu + \sum_{i_\mu} M_{\mu i_\mu}(k)x'_{i_\mu}, k)
\]

in \( B(X, X', k) \).

7.2. Translation invariance. The core of our proof (our adaption of [A3]) is the following property.

**Theorem 7.2.1.** Take some matrix of power series \( M(k) \in M_\mu(\mathbb{Q}[[k]]) \).

\[
\int^{(n)} dXF(\mathfrak{F}, k) = \int^{(n)} dXF(\mathfrak{F} + M(k)\mathfrak{F}', k) \in B_{\leq n}(X', k).
\]

The proof of this lemma requires a certain subspace of \( B^\circ(X, X', k) \). This is the subspace generated by \( C_n \) vectors, which were introduced in [A3].

**Definition 7.2.2 (Cn vectors).** Consider some uni-trivalent diagram drawn to include some box with \( n \) dashed edges attached (in an ordered fashion) to its top, and \( n \) dashed edges attached (ditto.) to its base. If the box is labelled by some permutation \( \sigma \in \Sigma_n \), then this diagram is defined to represent the diagram obtained by joining up the edges according to \( \sigma \). A \( C_n \) relation is a linear combination of diagrams obtained from such a diagram by summing over all the diagrams obtained by labelling that box with all possible permutations.
Lemma 7.2.3 ([LMO, A3]). A $C_m$ vector is in the kernel of $\int^n dX$, if $m \geq 2n + 1$.

Remark 7.2.4. This is a slightly different viewpoint than is taken in the literature. LMO works with a different relation $P_{n+1}$. This was shown to be equivalent to $C_{2n+1}$ in [A3]. LMO introduces the relation $P_{n+1}$ and then shows that it is “redundant” in the image (that is, the quotient by diagrams of grade greater than $n$). Here we are in a more general situation than is explicitly found in the literature: namely, we are allowing non-surgered univalent vertices. The proof that $P_{n+1}$ is “redundant”, which is accessibly described in Section 2.5.4 of [LeGr], adapts immediately to this generality. Following the discussion there, one sees that every leg on a $P_{n+1}$ must still meet a separate vertex.

Proof of Theorem 7.2.1.
Consider some diagram $D(\varpi, k)$ appearing in the expansion of $F(\varpi, k)$. There are three cases which cover the possibilities. Case 1: there is some component $x_i$ which has less than $2n$ legs labelled with it. In this case both $D(x, k)$ and $D(x + Mx', k)$ are in the kernel of $\int^n dX$. Case 2: each component of $X$ has exactly $2n$ legs labelled with it. In this case there is exactly one contributing diagram on the right hand side, because if any leg $x_i$ is relabelled $M_{ij}x'_j$ then there will then be less than $2n$ legs labelled with that component (such a diagram will be in the kernel.) The one contributing term is the same as that obtained from the left hand side. Case 3: some component has more than $2n$ legs labelled with it. This vanishes on the left hand side by definition. This also vanishes on the right hand side because this is expressible as a series of $C_m$ vectors, for some $m \geq 2n + 1$.

7.3. Diagram 7.0.9 commutes. Take some element $S(\varpi)$ of $B^{ST}(X)^{int}$, with canonical decomposition:

$$S(\varpi) = \exp_{\uplus} \left( \frac{1}{2} \sum_{i,j} W_{ij}(t) \right) \sqcup R(\varpi).$$

We shall calculate $\int^n dX (\text{Thr}^D(S(\varpi)) \sqcup \nu(k))$. All expressions in the following are to be evaluated in the space $B_{\leq n}(X)$.

The first step is to split off the remainder, $\text{Thr}^D(R(\varpi))$. Denote this $R'(\varpi, k)$. We can do the following because $R'(\varpi, k)$ is a series of $X$-substantial diagrams.

$$\int^n dX \left( \exp_{\uplus} \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^k) \right) \sqcup R'(\varpi, k) \sqcup \nu(k) \right)$$

$$= \left( R'(\varpi', k), \int^n dX \exp_{\uplus} \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^k) + \sum_i \right) \right) \bigg|_{X', \sqcup \nu(k)}.$$
Theorem 7.2.1 tells us that making the “translation”
\[ x_i \rightarrow x_i - \sum_j W^{-1}_{ij}(e^k)x'_j \]
inside the integrand will not affect the result. [The matrix \( W(T, e^k) \) is invertible in \( M_\mu(Q[[k]]) \) by assumption (Definition 4.0.3.)]

Making that substitution transforms the integral into the following. [Note that this requires the property that \( W^{-1}(T, e^k) \) is Hermitian, which is again by assumption (Definition 4.0.3.])

\[
\int dX \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^k) - \frac{1}{2} \sum_{i,j} W^{-1}_{ij}(e^k) \right).
\]

Substituting this expression back into the above gives the following.

\[
\left( \nu(k) \cup \int dX \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^k) \right) \right)
\]

\[
\cup \left( \exp \left( -\frac{1}{2} \sum_{i,j} W^{-1}_{ij}(e^k) \right), R'(\pi, k) \right)
\]

evaluated in \( B_{\leq \mu}(k) \). The theorem follows from the calculation of the leading term that is performed in the following section.

\(\square\)
The wheels line. Examining the structure of Equation 7.3.1, we see that the projection onto the wheels line of the element \( f^{(n)} dX (Th_{D} S(\tau)) \cup \nu(k) \) is precisely the following function of the entries of the matrix \( W(t) \).

It is calculated as follows.

**Theorem 7.4.1.** Let \( W(t) \) be a Hermitian matrix satisfying \( \det(W(1)) = \pm 1 \). Then

\[
\left( \nu(k) \cup \int f^{(n)} dX \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^{k}) \right) \right) = (-1)^{n\sigma_{+}(W(1))} W h'(\det(W(t))).
\]

**Proof.**

We begin by observing that every such matrix \( W(t) \) can be realised as the winding matrix of some special string link in the solid torus \( T \), presenting some pair \((M_{T}, K_{T})\). The calculation is performed by calculating the wheels line of \( Z_{LMO}(M_{T}, K_{T}) \) in two different ways.

On the one hand, this “wheels line” has already been calculated by other means. Many authors contributed to this result. Let us highlight the original conjecture of Melvin–Morton [MM] and the Kontsevich integral proof given by Bar-Natan–Garoufalidis [BNG] (acknowledging other contemporaneous proofs [RMM, KM]). See Garoufalidis–Habegger [GH] for the following formula in the setting of \( \mathbb{Z} HS^{3} \)’s (recently extended to null-homologous knots in \( \mathbb{Q} HS^{3} \), [L]).

**Theorem 7.4.2.** Let \( T \) be a \( \mu \)-string string link in the solid torus presenting some pair \((M_{T}, K_{T})\). Then,

\[
Z_{LMO}(M_{T}, K_{T}) = Wh'(\det(W(T, t))) \cup (1 + R),
\]

where \( R \) is a series of diagrams whose dashed graphs have negative Euler characteristic.

On the other hand, we have just seen (Equation 7.3.1) that \( Z_{LMO}(M_{T}, K_{T}) \) may be calculated

\[
\left[ \text{Wheels bit} \right] \cup \left[ \text{The rest} \right] = \left( \int f^{(n)} dU \sigma(\tilde{Z}(U_{+})) \right)^{\sigma_{+}(W(1))} \left( \int f^{(n)} dU \sigma(\tilde{Z}(U_{-})) \right)^{\sigma_{-}(W(1))}
\]

in the space \( B_{\leq n}(k) \), where

\[
\left[ \text{Wheels bit} \right] = \left( \nu(k) \cup \int f^{(n)} dX \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^{k}) \right) \right),
\]
\[ \text{[The rest]} = \left\langle \exp \left( -\frac{1}{2} \sum_{i,j} W_{ij}(e^k) \right) \right. \quad R(\vec{\tau}, k) \right\rangle. \]

Given that \[ \int_{n=1}^{\infty} \sum \text{terms of negative Euler characteristic}, \]

the theorem follows by equating wheels lines.

**Exercise 7.4.3.** Prove Theorem 7.4.2 directly. That is, calculate

\[ \int_{n=1}^{\infty} dX \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(e^k) \right). \]

This is a recommended, non-trivial, exercise; it provides some perspective on what lies behind the commutativity of Diagram 7.0.9 and gives a particular sense in which wheels tangibly become cycles of something. Hint: Step A is the Aarhus formula; Step B is an appropriate identity from determinant theory, being careful with combinatorial factors.

8. **Rozansky’s rationality conjecture**

We start by recalling the surgery formula. Just say the pair \((M, K)\) is presented by some \(T\), a \(\mu\)-string string link in the solid torus. If the associated decomposition is denoted

\[ \sigma(\tilde{Z}^{ST}(T)) = \exp \left( \frac{1}{2} \sum_{i,j} W_{ij}(t) \right) \quad R(\vec{\tau}) \in B^{ST}(X), \]

where \(R(\vec{\tau})\) is a series of \(X\)-substantial terms, then \(Z_n^{LMO}(M, K)\) is equal to

\[ W h(M, K) \sqcup \left\langle \exp \left( -\frac{1}{2} \sum_{i,j} W_{ij}(e^k) \right) \quad Th r^D(R(\vec{\tau})) \right\rangle \]

\[ \left( -1 \right)^n \int_{n=1}^{\infty} dU \sigma(\tilde{Z}(U_+)) \right)^{\sigma_+(W(1))} \left( \int_{n=1}^{\infty} dU \sigma(\tilde{Z}(U_-)) \right)^{\sigma_-(W(1))} \in B_{\leq n}(\vec{\ell}). \]
We already know that this is a group-like element. In fact, we know that

\[ Z^{LMO}(M, K) = Wh(M, K) \sqcup \exp(q), \]

where \( q \) is a series of connected diagrams of degree greater than one whose dashed graphs have Euler characteristic less than zero. Thus we can read \( q \) directly from the expression above. Write \( q = \sum_{i=1}^{\infty} q^{(i)} \) where \( q^{(i)} \) is a series of diagrams of Euler characteristic minus \( i \). Comparing the two expressions, it is clear that \( q^{(i)} \) is calculated from precisely those diagrams appearing in \( R(M) \) which have \( 2i \) trivalent vertices, plus some finite contribution from the signature corrections.

In other words, the term \( q^{(i)} \) is a sum over those ways of gluing labelled chords into pairs of legs of some finite combination of polynomial generating diagrams which result in connected diagrams.

The correspondence is completed by noting that the determinant of the matrix \( W(T, e^k) \) is \( \pm A(M,K)(e^k) \) so that the entries of the matrix \( W^{-1}(T, e^k) \) lie in \( L(M,K) \). Thus the edge-labels on the chords gluing the legs fall into that subspace, and the labels on the edges of the diagrams that are assembled according to the gluing formula also lie in this subspace.

It is an informative exercise to prove this theorem without appealing to the prior-known fact that the result is group-like.

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