Inflaton potential reconstruction without slow-roll

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We describe a method of obtaining the inflationary potential from observations which does not use the slow-roll approximation. Rather, the microwave anisotropy spectrum is obtained directly from a parametrized potential numerically, with no approximation beyond linear perturbation theory. This permits unbiased estimation of the parameters describing the potential, as well as providing the full error covariance matrix. We illustrate the typical uncertainties obtained using the Fisher information matrix technique, studying the $\lambda\phi^4$ potential in detail as a concrete example.

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I. INTRODUCTION

The determination of the initial power spectrum of perturbations in the Universe is an essential step in using the cosmic microwave background to constrain cosmological parameters. Indeed, without an understanding of these initial perturbations, the cosmic microwave background in isolation says nothing about the values of parameters such as the Hubble parameter $h$ and the density parameter $\Omega_0$. The reason is that the effect of the cosmology is on the dynamics of the perturbations, and a single timeslice, such as the perturbations at last scattering, says nothing about the dynamics. Usually, this problem is circumvented by assuming a parametrization of the initial conditions; the cosmological parameters then enter via the dynamics converting these initial conditions into the conditions at last scattering. For example, a common assumption is that there is a power-law spectrum of gaussian adiabatic scalar perturbations. This is a popular choice because it fits current observations, and because it is a good approximation to the perturbations produced by the simplest inflation models. But is clearly quite specific since it requires four descriptive qualifiers.

Given a set of microwave anisotropy measurements, one must determine both the cosmological parameters and the parameters describing the initial perturbations simultaneously: it is not possible to do one and not the other. Given the description of the initial perturbations, one can then try to determine the model which gave rise to them. If the best-fit model proves to have passive perturbations, meaning that the perturbations are observed to be entirely in their growing mode and hence are inferred to have existed since early in the Universe’s evolution (in particular, to have already existed when their scale was considerably larger than the Hubble radius), then it is reasonable to believe that they arose via the inflationary mechanism, rather than being induced by topological defects or other causal mechanism. It would be further encouraging if the perturbations proved to be adiabatic and gaussian, because although inflation models exist which violate those conditions, they are properties of the simplest inflation models.

Attempting to derive the underlying inflation model from observations has become known as inflaton potential reconstruction. Studies have focused on models where inflation is driven by a single scalar field $\phi$, moving in a potential $V(\phi)$. Such models indeed give perturbations which are passive, adiabatic and gaussian, though they come in two types, scalar (density perturbations) and tensor (gravitational waves) which need not be perfect power-laws. Even then they are not the most general class of models leading to that set of properties, because models where there is more than one scalar field can also give rise to that outcome. However, the single-field case appears to be the largest class of models which can be dealt with as a single set, where one aims to identify the member of the set responsible for the observations. If it turns out that there is no such member, then the net must be cast wider to include more complicated models. Amongst more general inflation models, there is no problem in generating the required predictions of the spectra to test them one by one against the data, but there is no known way of looking at the observations and constructing an inflationary model which will generate the desired predictions. We stress again that unless a valid model for the initial perturbations can be found, one cannot obtain the cosmological parameters as there would be no way to compute a microwave anisotropy spectrum to compare with observations.

Even under the single-field paradigm, the situation has not been wholly satisfactory, the reason being that the analytic results available for the spectra are only approximate, having been calculated using the slow-roll approximation, within which results are known only to

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*In fact, as shortly discussed, there’s actually a fifth qualifier as it is normally assumed that these perturbations are entirely in the growing mode.
second-order. Therefore, even if unbiased estimates of the parameters describing the perturbation spectra are obtained from observations (such as the amplitude and the spectral index $n$), these are not translated into unbiased estimators of the inflation potential. This paper describes how this shortcoming can be overcome.

II. DIRECT ESTIMATION OF INFLATIONARY PARAMETERS

Throughout, we work within the single field paradigm. The traditional technique for obtaining observational predictions from an inflationary model is the following. The potential is specified as an analytical function. The perturbations are then computed using the slow-roll approximation, to give the perturbation spectra in parametrized form. For example, the density perturbation spectrum $\delta_H(k)$ (following the notation of Liddle and Lyth [2]) can be expanded as a Taylor series in $\ln k$ as

$$\ln \delta_H^2(k) = \ln \delta_H^2(k_*) + (n_* - 1) \ln \frac{k}{k_*} + \frac{1}{2} \frac{dn}{d\ln k} \Bigg|_{k_*} \ln^2 \frac{k}{k_*} + \cdots,$$

where $k$ is the comoving wavenumber and $k_*$ is an arbitrary scale where the coefficients are evaluated. The slow-roll approximation gives the coefficients as functions of the potential, but only approximately, and there is a further approximation when the series is truncated at some level. Further, because the expression relating the scalar field value $\phi$ to the scale $k$ crossing the horizon is also approximate, there is a problem of a ‘drift of scales’: as we move from the expansion scale $k_*$, we begin to misidentify the $\phi$ value corresponding to a $k$ value by more and more.

Often all these errors are unimportant, especially for observations of the current quality. That for certain models they give an error which will be significant for future observations has been noted by several authors [3, 4]. One can attempt to improve things by going to the highest possible order in the slow-roll expansion, which is unfortunately only second-order, or by taking more and more terms in Eq. (1) [10, 11], which does not require going to higher order in slow-roll.

One does the best one can with the scalar perturbations, and also carries out a similar process for the tensor spectrum, which is less demanding theoretically as tensors are harder to detect observationally. These parametrized spectra are then fed into a numerical code (e.g. cmbfast [2]) possibly enhanced to allow non-power-law spectra) to compute the microwave anisotropy spectrum, the $C_\ell$, as a function of those and the cosmological parameters.

In going in the reverse direction, starting with observations, the standard procedure is to use the observations to estimate the coefficients in Eq. (1), along with the cosmological parameters. An example using current data is the analysis by Tegmark [3]. As long as the parametrization of the perturbations was adequate, that’s the job done as far as the cosmological parameters are concerned. However, to obtain the inflationary potential, the approximate slow-roll results which give the spectra in terms of the inflationary potential are inverted. This procedure is reviewed in Ref. [3], but does not yield unbiased estimates of the inflation potential.

The numerical technology now exists to circumvent this problem. The key is to immediately abandon any attempt to make the calculations analytically. Instead, the perturbation spectra are obtained by numerical solution of the relevant mode equations, which give the perturbation amplitude at a particular wavenumber. The best formalism is that of Mukhanov [14], and the only assumption is that linear perturbation theory is valid, which is more or less guaranteed by the fact that the observed (dimensionless) perturbations are order $10^{-5}$. The necessary ingredients to proceed are

1. A program which can numerically solve the mode equations wavenumber by wavenumber. We described such a code in Ref. [13]. This must be able to compute both scalar and tensor perturbations.

2. A version of cmbfast which is capable of taking arbitrary power spectra as input to produce the $C_\ell$ curve. The output $C_\ell$ is the sum of the scalar and tensor parts. Polarization anisotropies should be computed as well as temperature ones.

We have assembled these codes into an IDL pipeline. The input step is to supply a parametrization of the potential, rather than an analytic form. In this paper we use the simplest version, a Taylor expansion about some scalar field value $\phi_*$, with the slight subtlety of pulling out the overall normalization as a prefactor for later convenience. This leads to the $C_\ell$ as a function of the cosmological parameters and the potential parameters, i.e. $C_\ell(V_*, V'_*/V_*, V''*/V_*, \ldots, h, \Omega_m, \Omega_B, \Omega_\Lambda, \ldots)$ where primes are derivatives with respect to $\phi$, evaluated at $\phi_*$.

The inversion is now direct: the observed anisotropy spectrum is used to directly estimate the potential parameters, which can be done in an unbiased way to generate the best possible reconstruction. If at this stage one were to find that the overall best-fit model was a poor fit to the data, the first thing would be to try an improved parametrization of the potential and/or inclusion of extra cosmological parameters, and if that still fails it would be time to suspect that the single field paradigm is not correct.

However, optimistically assuming that the best fit is adequate, we have our best-fit inflationary potential. But
that’s not all; a further advantage of this direct method is that it immediately gives us the covariance of the uncertainties on the potential parameters. For example, it is known that the errors on $V_*$ and $V'_*$ will be highly correlated. Using the old approach, these correlations would have to be carried through the complicated reconstruction equations, an unpleasant enough task that in Ref. [8] we instead used a Monte Carlo method to illustrate the uncertainties of a reconstruction from simulated data.

In this approach, the consistency equation relating scalar and tensor perturbations (see Ref. [8] for a discussion) is automatically incorporated, being tested by whether there is a potential offering a satisfactory absolute fit to the data. It could of course also be tested in the traditional way by power spectrum fitting, but anyway it is unlikely that observations will be good enough to say anything significant.

The two strategies are contrasted in Fig. 1. We do not view our new approach as replacing the traditional one, but rather as a next step that one would take if the traditional fitting proves successful, in order to obtain optimal results.

![Diagram](image)

**FIG. 1.** The traditional route from model to observables and back is the two-stage process at the top. The procedure outlined in this paper enables a direct route without approximations beyond linear perturbation theory.

### III. UNCERTAINTY AND COVARIANCE OF INFLATON POTENTIAL PARAMETERS

#### A. Parametrizing the potential

Although in principle a solution of the mode equation runs from an early initial time until the scale is well outside the horizon, when considering perturbations on a given scale $k$, the details of how the Universe expands are only important for a fairly brief interval around the time $k = aH$ when the scale crosses outside the horizon. The reason is that while the scale is well inside the horizon the relevant timescales are much less than the expansion timescale and expansion can be neglected, while when scales are above the horizon the perturbations are frozen in at fixed values (in the appropriate variables) whatever the evolution is. As the observations cover a limited range of wavenumbers $k$, we need only know $V(\phi)$ for a limited range of $\phi$ values about the time when the relevant scales cross outside the horizon during inflation. Our input assumption (ultimately to be tested against the observations) is that there is a potential $V(\phi)$, and we will simply need a parametrization of it which is accurate enough over the desired range.

There is one subtlety to this. The scalar wave equation is second order, so, in addition to the value of $\phi$, it looks as if $\dot{\phi}$ is an arbitrary initial condition which needs to be considered as an extra parameter. However it has long been known that this is not the case, because scalar field cosmologies have an attractor behaviour whereby all initial conditions quickly converge [16,17] (indeed, during inflation convergence is at least exponentially fast with $\phi$, but even non-inflationary expansion exhibits this behaviour). However, this does mean that we have to be sure that the simulation has run for long enough that the attractor is attained before the perturbations on observable scales are generated, exactly as is believed to have happened in the real Universe.

As in known inflation models all observable scales cross outside the horizon over a very narrow range of $\phi$, the simplest approach is a simple Taylor series expansion

$$\frac{V(\phi)}{V_*} = 1 + \frac{V'_*}{V_*} (\phi - \phi_*) + \frac{1}{2} \frac{V''_*}{V_*} (\phi - \phi_*)^2 + \cdots , \quad (2)$$

where $\phi_*$ is arbitrary and can be set to zero if desired. We pulled out the normalization before expanding, as then the normalization of the $C_\ell$ depends only on $V_*$ and not the other terms. In principle one could consider a more sophisticated expansion to try and improve the convergence properties such as a Padé approximant, but that can be assessed once actual data is available.

#### B. Parameter uncertainty

Having obtained the $C_\ell$ as a function of the potential and cosmological parameters, we can assess the likely accuracy with which those parameters can be found by a given experiment. This is carried out using the well-established Fisher matrix technique [18,19], which amounts to taking the derivative of the $C_\ell$ with respect to each of the parameters. The parameter uncertainties depend on the choice of ‘correct’ model and on the number of parameters allowed to vary. For illustration, we vary cosmological parameters about an underlying model with Hubble parameter $h = 0.65$, density parameter $\Omega_0 = 0.3$, cosmological constant $\Omega_\Lambda = 0.7$, baryon density $\Omega_B = 0.05$ and reionization optical depth $\tau = 0.1$.

The potential we choose is the $\lambda \phi^4$ potential, and we take the epoch where the present Hubble radius equaled the Hubble radius during inflation as being 60 $e$-foldings.
before the end of inflation. Numerical solution of the equations of motion gives this as $\phi_∗ = 4.37 m_{Pl}$. In fact the slow-roll approximation will work well for this potential, and for instance can be used to show that gravitational waves should contribute about twenty percent of the signal at large angular scales.

Following Zaldarriaga et al. [19], we consider a version of the PLANCK satellite which measures both temperature and polarization anisotropies, as described in Ref. [11]. We do not attempt to include the effects of foregrounds, as extensively studied recently by Tegmark et al. [20], but choose to consider only one polarized Planck channel (in effect assuming that the polarized foregrounds can be removed using all the other channels) which, while rather approximate, yields similar results. The actual data, when available, will of course merit more sophisticated treatment. Our numbers are therefore indicative only, and more importantly they would vary significantly if the assumed underlying model were changed — the quality of information available from reconstruction depends strongly on which model (if any, of course) proves to be correct.

The results are shown in Table I. The higher derivatives are not detected, but it is interesting to note that even with this very flat potential, the variation of the potential during inflation, $V'$, is detected at 7-sigma. However $V''$ is not detected; this may seem a little surprising given that both the gravitational wave amplitude and the scalar spectral index (which depend on different combinations of $V'/V$ and $V''/V$) are in fact detectable for this potential [8,19,11,20], but it turns out that the combination of these giving $V''/V$ is not distinguishable from zero. In our approach one never needs to make the separation of scalars and tensors explicitly.

**C. Parameter uncertainty covariance**

The Fisher matrix technique also generates the covariances of the error estimates, and these are crucial in interpreting the observational constraints. In particular, the covariance matrix is essential in illustrating the reconstructions graphically; if correlations are ignored then the reconstruction deteriorates much more quickly with $\phi$ than the true picture (the correlations allow for the fact that scale of the expansion $\phi_*$ need not be the scale at which the observations are the most powerful). That our method gives the full correlation matrix of the reconstructed potential directly is its first key advantage over earlier techniques.

To illustrate the quality of the reconstruction, we carry out Monte Carlo reconstructions with errors drawn according to the covariance matrix, and plot them against the true potential in Fig. 2. These reconstructions include up to the fourth derivative; though as seen from Table I the higher derivatives are not detected, they can still be assigned values according to their upper limit. We note that this potential is much less favourable for reconstruction than ones explored previously, as it is much closer to the scale-invariant limit.

The reconstructions indicate the second key advantage of the method proposed here over previous ones (e.g. Refs. [13]) — the reconstructed potentials are unbiased estimates of the true potential, being as likely to be too high as too low. In the upper panel, we see that the uncertainty in the overall normalization is quite large (ultimately due to a degeneracy in the effect of scalars and tensors on large angular scales). The lower panel shows the combination $V'/V^{3/2}$ which is primarily sensitive to density perturbations alone (indeed perfectly so in the slow-roll approximation), and which is much better determined (in particular, better than $V$ or $V'$ separately). This figure allows us to see directly the range of $\phi$ which is constrained by the data; to highlight this we have indicated the values which $\phi$ takes while the microwave anisotropies are being generated. At larger $\phi$, corresponding to larger scales, the perturbations are unobservable, and even some way within the current horizon scale cosmic variance contributes significantly to the spread. Near the centre of the data the determination is at its best, and on short scales the information again becomes poor, partly because of the dependence on all the cosmological parameters and partly because Silk damping erases the perturbations as one goes beyond $\ell \sim 1000$.

This figure highlights once again that the information available from reconstruction constrains only a tiny portion of the scalar field potential. Nevertheless, the information available there is of good accuracy, and can be highly constraining in instances where theoretical motivation suggests a potential containing few unspecified

| parameter | underlying model value | relative uncertainty |
|-----------|------------------------|----------------------|
| $\tau$    | 0.1                    | 6.1%                 |
| $\Omega_{h}^2$ | 0.021               | 1.2%                 |
| $\Omega_{CDM}h^2$ | 0.11               | 2%                   |
| $\Omega_{\Lambda}h^2$ | 0.30              | 5%                   |
| $10^{12}V_*/m_{Pl}^4$ | 2.3              | 22%                  |
| $m_{Pl}V'/V_*$ | 0.92              | 14%                  |
| $m_{Pl}V''/V_*$ | 0.63              | 2×                   |
| $m_{Pl}V'''/V_*$ | 0.29              | 60×                  |
| $m_{Pl}V^{(4)}/V_*$ | 0.066             | 400×                 |

**TABLE I.** Uncertainties for each parameter, marginalizing over the remaining parameters. We stress that these are specifically for the $\lambda \phi^2$ model. These values correspond to the diagonal entries of the covariance matrix. There are substantial correlations between parameters, especially those describing the potential, so the off-diagonal entries of the covariance matrix are significant. Of the potential parameters, only the magnitude and gradient of the potential are detected with any significance.
FIG. 2. Twenty Monte Carlo reconstructions of the potential, compared against the true potential which is shown as a dashed line. The upper panel shows the potential itself, and the lower one the combination \( V' / V^{3/2} \) which is a combination coming primarily from the density perturbations alone. The dotted vertical lines indicate the region of the potential directly probed by the microwave background, ranging from the current horizon scale to the horizon scale when the \( \ell = 1500 \) mode was generated (evaluated in the underlying model). The upper panel shows that the gradient is quite well recovered but the overall amplitude much less so, while the lower highlights the obvious fact that the reconstruction is accurate only where there is data available to constrain it.

The quartic potential is an interesting test case because it is not far from the slow-roll limit, and this is the first time such a potential has been used to test reconstruction methods. However the true strength of the method would be unveiled if the true model does not satisfy slow-roll well, despite the potential being smooth. An example is the potential introduced by Wang et al. [7], which was used to test the traditional reconstruction technique in Ref. [8]. It was shown in that latter paper that traditional reconstruction could still work well, but led to a bias in the estimate of the potential (albeit within observational errors). Figure 3 illustrates test reconstructions of this potential, using the techniques of the present paper. The uncertainties on the cosmological parameters, and on \( V \) and \( V' \), are very similar to those of the quartic case. However, in addition \( V'' \) is detected at around 3-sigma and the next two derivatives have uncertainties comparable to their values.

FIG. 3. Twenty Monte Carlo reconstructions of the combination \( V'/V^{3/2} \), as in the lower panel of Fig. 2, but for the model investigated by Wang et al. [7].

IV. DISCUSSION

It may well be that one of the simplest models of inflation is correct, and the perturbation spectra are perfectly satisfactorily approximated by a power-law (or at least some low-order truncation of Eq. (1)). If so, then cosmological parameter estimation can proceed as described in previous works. However, given the intellectual and financial investment in pursuing cosmological parameters, it is vital to be aware of possible difficulties, and to analyze ways of dealing with them. We have considered one such possibility — that the single field paradigm is correct but slow-roll is not very good — and explained that this is readily dealt with using existing numerical technology. In using this technique, as with others, it is imperative to make an overall goodness-of-fit test to ensure that the class of models being considered is capable of adequately explaining the data. Even if the power-law approximation proves valid (and certainly this is the method which should be tried first), one will want to use these techniques to ensure that estimates of the inflationary potential are unbiased ones, and to obtain the fullest possible information about the inflaton potential from observations.
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