Two-state Markovian theory of input-output frequency and phase synchronization

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Abstract

A Markovian dichotomic system driven by a deterministic time-periodic force is analyzed in terms of the statistical properties of the switching events between the states. The consideration of the counting process of the switching events leads us to define a discrete phase. We obtain expressions for the instantaneous output frequency and phase diffusion associated to the dichotomic process, as well as for their cycle averages. These expressions are completely determined by the rates of escape from both states. They are a convenient starting point for the study of the stochastic frequency and phase synchronization in a wide range of situations (both classical and quantum) in which two metastable states are involved.

Key words: Synchronization, Markovian, two-state, phase, dichotomic, frequency

PACS: 05.40.-a, 05.45.Xt, 05.10.Gg, 02.50.-r

1 Introduction

The phenomenon of synchronization in stochastic systems has attracted much interest in recent years [1,2,3,4,5,6,7]. In the case of a periodically driven bistable stochastic system, it has been shown to be particularly useful to introduce a discrete phase associated to the output signal [1,2,5,6]. Then, the synchronization between the periodic input signal and the stochastic output signal is characterized in terms of certain quantities as, for example, the average output frequency and the average phase diffusion. Noise regulates the input-output synchronization in such a way that the value of the average output frequency might match the value of the input frequency for a range

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of noise strength values. Furthermore, even though the output phase is a stochastic quantity, it is characterized by a diffusion coefficient with a sharp minimum for values of the noise strength where frequency locking exists. Noise induced synchronization is one of the diverse type of phenomena arising from the interplay between nonlinearity and noise in systems externally driven by time-periodic forces. Stochastic resonance [8,9,10,11,12,13,14,15] and Brownian motors [16,17,18,19] provide other instances of this rich phenomenology.

In this paper we focus on the calculation of explicit expressions for the instantaneous output frequency and phase diffusion, as well as for their cycle averages. We restrict our study to the case of a Markovian two-state system, driven by an arbitrary deterministic time-periodic force, in contrast to Ref [1], where only dichotomic forces are considered. In our analysis, we do not specify the underlying continuous dynamics which leads, by contraction, to the two-state description. In this sense, our results can be applied to the study of the stochastic frequency and phase synchronization in a wide range of situations (both classical and quantum), whenever the system presents two metastable states and the contracted two-state dynamics is (approximately) Markovian. As it has been shown elsewhere [20], in the case of a rocked overdamped bistable stochastic system, such a situation takes place in the weak-noise and low-frequency limit.

The paper is organized as follows: We begin with a description of the Markovian two-state model in terms of the switching times and define the discrete phase associated to this dichotomic process. Subsequently, in Sect. 3, we analyze the statistical properties of the two-state process and the discrete phase. Making use of the previous results, in Sect. 4 we obtain expressions for the instantaneous output frequency and phase diffusion, as well as for their cycle averages. The main conclusions of this work are presented in Sect. 5, and some mathematical details are reported in the Appendix.

2 The two-state model: switching times and definition of the output discrete phase

We consider a two-state Markovian stochastic process $\chi^{\alpha_0,t_0}(t)$ which only takes the values $+1$ and $-1$. The notation used emphasizes the fact that at an initial instant of time $t_0$ the system is in the state $\alpha_0$, i.e., $\chi^{\alpha_0,t_0}(t_0) = \alpha_0$ with $\alpha_0 = +1$ or $-1$. Due to some physical mechanism (noise in the case of a classical system, or the combined action of tunneling and noise in a quantum problem), there are switches between those two states which are also modulated by the action of an external, periodic, time-dependent force with period $T$. The instant of time at which the $n$-th switch of state takes place is a random variable which will be denoted by $T_n^{\alpha_0,t_0}$, with $n = 1, 2, \ldots$. 

The analysis of the statistics of switching times $T_{n}^{\alpha_{0},t_{0}}$, can be carried out along the lines of the renewal theory detailed by Cox in Ref. [21], with the necessary extensions to explicitly time dependent situations. The random variable $T_{n}^{\alpha_{0},t_{0}}$ will be characterized by its probability density function

$$g_{n}^{\alpha_{0},t_{0}}(t) = \lim_{\Delta t \to 0^{+}} \frac{\text{Prob}[t < T_{n}^{\alpha_{0},t_{0}} \leq t + \Delta t]}{\Delta t},$$

or equivalently, by the cumulative distribution function

$$G_{n}^{\alpha_{0},t_{0}}(t) = \text{Prob}[T_{n}^{\alpha_{0},t_{0}} \leq t] = \int_{t_{0}}^{t} g_{n}^{\alpha_{0},t_{0}}(t') \, dt',$$

or its complementary,

$$G_{n}^{\alpha_{0},t_{0}}(t) = \text{Prob}[T_{n}^{\alpha_{0},t_{0}} > t] = 1 - G_{n}^{\alpha_{0},t_{0}}(t).$$

If we introduce the stochastic process

$$N_{\alpha_{0},t_{0}}(t) = \max \left[ n : T_{n}^{\alpha_{0},t_{0}} \leq t \right],$$

characterizing the number of switches of state in the interval $(t_{0}, t]$, then the two-state stochastic process $\chi_{\alpha_{0},t_{0}}(t)$ can be expressed as

$$\chi_{\alpha_{0},t_{0}}(t) = \alpha_{0} \cos \left[ \pi N_{\alpha_{0},t_{0}}(t) \right].$$

Taking into account the above expression, we will define a discrete phase $\varphi_{\alpha_{0},t_{0}}(t)$ associated to the two-state stochastic process as

$$\varphi_{\alpha_{0},t_{0}}(t) = \pi N_{\alpha_{0},t_{0}}(t).$$

3 Statistical characterization of the discrete phase $\varphi_{\alpha_{0},t_{0}}(t)$ and the two-state process $\chi_{\alpha_{0},t_{0}}(t)$

The one-time statistical properties of the discrete phase $\varphi_{\alpha_{0},t_{0}}(t)$ can be evaluated from the probability distribution of the number of switches of state

$$\rho_{n}^{\alpha_{0},t_{0}}(t) = \text{Prob}[N_{\alpha_{0},t_{0}}(t) = n],$$

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with \( n = 0, 1, 2, \ldots \). Making use of the definition of \( N^\alpha_{t_0}(t) \) in Eq. (4) it is easy to see that

\[
\text{Prob}\left[ N^\alpha_{t_0}(t) \geq n \right] = G^\alpha_{n_{t_0}}(t),
\]

with \( G^\alpha_{0_{t_0}}(t) = 1 \). Consequently, the probability distribution of the number of switches of state and its derivative with respect to \( t \) can be expressed, respectively, as

\[
\rho^\alpha_{n_{t_0}}(t) = G^\alpha_{n_{t_0}+1}(t) - G^\alpha_{n_{t_0}}(t),
\]

and

\[
\dot{\rho}^\alpha_{n_{t_0}}(t) = g^\alpha_{n_{t_0}+1}(t) - g^\alpha_{n_{t_0}}(t),
\]

with \( g^\alpha_{0_{t_0}}(t) = g^\alpha_{0_{t_0}+1}(t) = 0 \).

In order to express the above equation in a more transparent form, it is convenient to introduce the probability of an almost immediate switch of state after \( n \) switches

\[
\Gamma^\alpha_{n_{t_0}}(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}\left[ t < T^\alpha_{n_{t_0}+1} \leq t + \Delta t \mid N^\alpha_{t_0}(t) = n \right]}{\Delta t},
\]

as well as the probability of an almost immediate switch of state from state \( \beta \)

\[
\gamma^\alpha_{\beta_{t_0}}(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}\left[ t < T^\alpha_{n_{t_0}+1} \leq t + \Delta t \mid \chi^\alpha_{n_{t_0}}(t) = \beta \right]}{\Delta t}.
\]

Due to the Markovian character of the stochastic process \( \chi^\alpha_{n_{t_0}}(t) \), it is clear that the rate \( \gamma^\alpha_{\beta_{t_0}}(t) \) is independent of the initial preparation \( \alpha_0 \) at \( t_0 \) and, for this reason, henceforth it will be denoted by \( \gamma_(t) \). Analogously, the rate \( \Gamma^\alpha_{n_{t_0}}(t) \) only depends on the value of \( \chi^\alpha_{n_{t_0}}(t) \) after the \( n \)-th switch of state. Thus, noting that after an even number of switches of state the system ends up in the same state as it was initially, whereas for an odd number of switches the system ends up in the other state, it follows that

\[
\Gamma^\alpha_{n_{t_0}}(t) = \gamma^\alpha_{\alpha_{n}}(t),
\]

where \( \alpha_n = (-1)^n \alpha_0 \). The time-periodicity of the external force becomes manifest in the fact that the rate \( \gamma_{\beta}(t) \) is also a periodic function of time with the
same period $T$, i.e.,

$$\gamma_\beta(t + T) = \gamma_\beta(t). \quad (14)$$

Multiplying and dividing the right-hand side of Eq. (11) by $\rho^{\alpha_0,t_0}(t)$, assuming that the probability of more than one switch of state between $t$ and $t + \Delta t$ is $O[(\Delta t)^2]$, and taking into account Eqs. (1) and (13), it is easy to see that

$$\gamma_\alpha(t) = \frac{g^{\alpha_0,t_0}(t)}{\rho^{\alpha_0,t_0}(t)}. \quad (15)$$

Then, from Eq. (10) one obtains the following hierarchy of differential equations for $\rho^{\alpha_0,t_0}(t)$

$$\dot{\rho}^{\alpha_0,t_0}(t) = -\gamma_\alpha(t)\rho^{\alpha_0,t_0}(t) + \gamma_{\alpha-1}(t)\rho^{\alpha_0,t_0}_{n-1}(t) \quad \text{for } n \geq 1, \quad (16)$$

$$\dot{\rho}^{\alpha_0,t_0}_0(t) = -\gamma_\alpha(t)\rho^{\alpha_0,t_0}_0(t), \quad (17)$$

which must be solved with the initial condition $\rho^{\alpha_0,t_0}(t_0) = \delta_{n,0}$.

We will also introduce the probability of $\chi^{\alpha_0,t_0}(t)$ to take on the value $\beta$

$$p_\beta(t|\alpha_0,t_0) = \text{Prob}[\chi^{\alpha_0,t_0}(t) = \beta]$$

$$= \delta_{\alpha_0,\beta} \sum_{n=0}^{\infty} \rho^{\alpha_0,t_0}_{2n}(t) + \delta_{\alpha_0,\beta} \sum_{n=0}^{\infty} \rho^{\alpha_0,t_0}_{2n+1}(t), \quad (18)$$

with $\beta = 1$ or $-1$. Notice that in the notation we have made explicit the conditional dependence of this probability on the initial preparation.

Differentiating the above expression with respect to $t$ and taking into account Eq. (16), it is straightforward to obtain the master equation for the Markovian stochastic process $\chi^{\alpha_0,t_0}(t)$

$$\dot{p}_\beta(t|\alpha_0,t_0) = -\gamma_\beta(t)p_\beta(t|\alpha_0,t_0) + \gamma_{-\beta}(t)p_{-\beta}(t|\alpha_0,t_0), \quad (19)$$

which must be solved with the initial condition $p_\beta(t_0|\alpha_0,t_0) = \delta_{\alpha_0,\beta}$. The solution of the above equation can be easily obtained taking into account that $p_{-\beta}(t|\alpha_0,t_0) = 1 - p_\beta(t|\alpha_0,t_0)$, and the result is

$$p_\beta(t|\alpha_0,t_0) = \delta_{\alpha_0,\beta} \exp \left[ - \int_{t_0}^{t} dt' \gamma(t') \right]$$
\[ + \int_{t_0}^{t} dt' \gamma_{-\beta}(t') \exp \left[ - \int_{t'}^{t} dt'' \gamma(t'') \right], \]  
(20)

where
\[ \gamma(t) = \gamma_1(t) + \gamma_{-1}(t). \]  
(21)

In order to obtain an expression independent of the initial preparation, it is necessary to take the limit \( t_0 \to -\infty \) of Eq. (20) and introduce the long-time populations
\[ p_\beta(t) = \lim_{t_0 \to -\infty} p_\beta(t|\alpha_0, t_0) = \int_{-\infty}^{t} dt' \gamma_{-\beta}(t') \exp \left[ - \int_{t'}^{t} dt'' \gamma(t'') \right]. \]  
(22)

From the results contained in Appendix A, it follows that the long-time population \( p_\beta(t) \) is a periodic function of \( t \) with period \( T \) which can be expressed, for \( t \in [0, T] \), as
\[ p_\beta(t) = \frac{1}{2} \text{csch} \left( \frac{\bar{\gamma} T}{2} \right) \int_{0}^{T} dt' \gamma_{-\beta}(t') \times \exp \left[ \text{sgn}(t-t') \frac{\bar{\gamma} T}{2} - \int_{t'}^{t} dt'' \gamma(t'') \right], \]  
(23)

with \( \bar{\gamma} \) given by Eq. (A.5).

4 The output frequency and the phase diffusion

The instantaneous output frequency and phase diffusion are, respectively, defined as [1]
\[ \Omega_{\text{out}}^{\alpha_0,t_0}(t) = \frac{\partial}{\partial t} \left< \varphi^{\alpha_0,t_0}(t) \right> = \pi \frac{\partial}{\partial t} \left< N^{\alpha_0,t_0}(t) \right>, \]  
(24)

and
\[ D_{\text{out}}^{\alpha_0,t_0}(t) = \frac{\partial}{\partial t} \left\{ \left< \left[ \varphi^{\alpha_0,t_0}(t) \right]^2 \right> - \left< \varphi^{\alpha_0,t_0}(t) \right>^2 \right\} = \pi^2 \frac{\partial}{\partial t} \left\{ \left< N^{\alpha_0,t_0}(t) \right>^2 - \left< N^{\alpha_0,t_0}(t) \right>^2 \right\}, \]  
(25)
where \( \langle \ldots \rangle \) denotes the average with respect to the distribution \( \rho_{\alpha_0,t_0}^{n}(t) \), i.e.,

\[
\langle K \left[ N_{\alpha_0,t_0}^{n}(t) \right] \rangle = \sum_{n=0}^{\infty} K(n) \rho_{\alpha_0,t_0}^{n}(t),
\]

(26)

\( K \left[ N_{\alpha_0,t_0}^{n}(t) \right] \) being an arbitrary one-time function of \( N_{\alpha_0,t_0}^{n}(t) \).

Multiplying Eq. (16) by \( n \), summing up the series \( \sum_{n=1}^{\infty} n \dot{\rho}_{\alpha_0,t_0}^{n}(t) \), and taking into account Eq. (18), it is easy to obtain that

\[
\Omega_{\alpha_0,t_0}^{\mathrm{out}}(t) = \pi \left[ \gamma_1(t)p_{+1}(t|\alpha_0,t_0) + \gamma_{-1}(t)p_{-1}(t|\alpha_0,t_0) \right].
\]

(27)

Analogously, if one sums up the series \( \sum_{n=1}^{\infty} n^2 \dot{\rho}_{\alpha_0,t_0}^{n}(t) \) by using Eq. (16), it results after some simplifications that

\[
\frac{\partial}{\partial t} \left\langle \left[ N_{\alpha_0,t_0}^{n}(t) \right]^2 \right\rangle = \pi^{-1} \Omega_{\alpha_0,t_0}^{\mathrm{out}}(t) + 2 \sum_{\beta=\pm \alpha_0} \gamma_{\beta}(t) \Phi_{\alpha_0,t_0}^{\alpha_0,\beta}(t),
\]

(28)

where

\[
\Phi_{\alpha_0,t_0}^{\alpha_0,\beta}(t) = \delta_{\beta,\alpha_0} \sum_{n=0}^{\infty} 2n \rho_{2n}^{\alpha_0,t_0}(t) + \delta_{\beta,-\alpha_0} \sum_{n=0}^{\infty} (2n + 1) \rho_{2n+1}^{\alpha_0,t_0}(t).
\]

(29)

Replacing Eq. (28) into Eq. (25) and taking into account Eqs. (24), (27), and (29), it is straightforward to see that

\[
D_{\alpha_0,t_0}^{\alpha_0,\beta}(t) = \pi \Omega_{\alpha_0,t_0}^{\mathrm{out}}(t) + 2\pi^2 \Delta \gamma(t) \Psi_{\alpha_0,t_0}(t),
\]

(30)

with

\[
\Delta \gamma(t) = \gamma_{+1}(t) - \gamma_{-1}(t),
\]

(31)

and

\[
\Psi_{\alpha_0,t_0}(t) = \alpha_0 \left[ \Phi_{\alpha_0,t_0}^{\alpha_0,0}(t)p_{-\alpha_0}(t|\alpha_0,t_0) - \Phi_{-\alpha_0,t_0}^{\alpha_0,0}(t)p_{\alpha_0}(t|\alpha_0,t_0) \right].
\]

(32)

Making use of Eqs. (16) and (19) and after some lengthy calculations, it is possible to prove that \( \Psi_{\alpha_0,t_0}(t) \) satisfies the differential equation

\[
\dot{\Psi}_{\alpha_0,t_0}(t) = -\gamma(t) \Psi_{\alpha_0,t_0}(t) - \sum_{\beta=\pm 1} \beta \gamma_{\beta}(t) \left[ p_{\beta}(t|\alpha_0,t_0) \right]^2,
\]

(33)
with the initial condition $\Psi_{\alpha_0,t_0}(t_0) = 0$. The above equation can be formally solved, yielding

$$\Psi_{\alpha_0,t_0}(t) = -\sum_{\beta=\pm 1} \beta \int_{t_0}^{t} dt' \gamma_{\beta}(t') [p_{\beta}(t'|\alpha_0, t_0)]^2 \exp \left[ -\int_{t'}^{t} dt'' \gamma(t'') \right]. \quad (34)$$

Inserting Eq. (34) into Eq. (30), one obtains

$$D_{\alpha_0,t_0}^{\text{out}}(t) = \pi \Omega_{\alpha_0,t_0}^{\text{out}}(t) - 2\pi^2 \Delta\gamma(t)$$

$$\times \sum_{\beta=\pm 1} \beta \int_{t_0}^{t} dt' \gamma_{\beta}(t') [p_{\beta}(t'|\alpha_0, t_0)]^2 \exp \left[ -\int_{t'}^{t} dt'' \gamma(t'') \right]. \quad (35)$$

Equations (27) and (35) for the instantaneous output frequency and phase diffusion, respectively, still depend on the initial preparation. In order to obtain expressions independent on the initial preparation, it is necessary to take the limit $t_0 \to -\infty$. In this limit, one obtains

$$\Omega_{\text{out}}(t) = \lim_{t_0 \to -\infty} \Omega_{\alpha_0,t_0}^{\text{out}}(t) = \pi \left[ \gamma_{+1}(t)p_{+1}(t) + \gamma_{-1}(t)p_{-1}(t) \right] \quad (36)$$

and

$$D_{\text{out}}(t) = \lim_{t_0 \to -\infty} D_{\alpha_0,t_0}^{\text{out}}(t) = \pi \Omega_{\text{out}}(t) - 2\pi^2 \Delta\gamma(t)$$

$$\times \sum_{\beta=\pm 1} \beta \int_{-\infty}^{t} dt' \gamma_{\beta}(t') [p_{\beta}(t')]^2 \exp \left[ -\int_{t'}^{t} dt'' \gamma(t'') \right]. \quad (37)$$

The functions $\Omega_{\text{out}}(t)$ and $D_{\text{out}}(t)$ are periodic functions of the time $t$ (see Appendix A). Thus, one can perform a cycle average and define the average output frequency

$$\Omega_{\text{out}} = \frac{1}{T} \int_{0}^{T} dt \Omega_{\text{out}}(t) = \frac{\pi}{T} \int_{0}^{T} dt \left[ \gamma_{+1}(t)p_{+1}(t) + \gamma_{-1}(t)p_{-1}(t) \right], \quad (38)$$

and the average phase diffusion

$$D_{\text{out}} = \frac{1}{T} \int_{0}^{T} dt D_{\text{out}}(t) = \pi \Omega_{\text{out}} - \frac{\pi^2}{T} \text{csch} \left( \frac{\gamma T}{2} \right) \sum_{\beta=\pm 1} \beta \int_{0}^{T} dt \int_{0}^{T} dt' \Delta\gamma(t).$$
\[ \times \gamma_\beta(t') [p_\beta(t')]^2 \exp \left[ \text{sgn}(t - t') \frac{\bar{\gamma} T}{2} - \int_{t'}^t dt'' \gamma(t'') \right], \quad (39) \]

where \( \bar{\gamma} \) is given by Eq. (A.5) and we have made use of Eq. (A.6) with \( h(t) = \gamma_\beta(t) [p_\beta(t)]^2 \). Equations (36), (37), (38), and (39), complemented by Eq. (23), are the main results of this work.

5 Concluding remarks

In this paper, we have put forward a Markovian two-state theory to describe the phenomenon of frequency and phase synchronization in noisy systems driven by deterministic time-periodic forces. The fundamentals of our strategy are based on an extension of the renewal theory to time-dependent situations. We have obtained explicit expressions for the instantaneous output frequency and phase diffusion, as well as for their cycle averages. These expressions, which are completely determined by the rates of escape from both states, are quite general and should apply to realistic situations with two metastable configurations. Thus, they represent a convenient starting point for the study of the frequency and phase synchronization in a wide range of situations.

Acknowledgements

We acknowledge the support of the Dirección General de Enseñanza Superior of Spain (BFM2002-03822) and the Junta de Andalucía. We also want to gratefully acknowledge Prof. Peter Hänggi for suggesting the problem and for fruitful discussions.

A Some mathematical details

The aim of this Appendix is to analyze some interesting properties of integrals of the form

\[ H(t) = \int_{-\infty}^t dt' h(t') \exp \left[ - \int_{t'}^t dt'' \gamma(t'') \right], \quad (A.1) \]

with \( h(t) \) being a periodic function of \( t \) with period \( T \). Examples of integral of this type are those appearing in Eqs. (22) and (37).
First, we will prove that $H(t)$ is also a periodic function of $t$ with the same period $T$ as the functions $h(t)$ and $\gamma(t)$. In order to do so, let us consider its value at $t + T$,

$$H(t + T) = \int_{-\infty}^{t+T} dt' h(t') \exp \left[ - \int_{t'}^{t+T} dt'' \gamma(t'') \right]. \tag{A.2}$$

After making the change of variables $\tilde{t}' = t' - T$ and $\tilde{t}'' = t'' - T$ and using the periodicity of the functions $h(t)$ and $\gamma(t)$, it is straightforward to see that $H(t + T) = H(t)$.

Once that the periodicity of $H(t)$ has been proved, it is possible to express $H(t)$ in a more compact form which does not involve an improper integral. To do so, let us split the integral in Eq. (A.1) into the two pieces

$$H(t) = H(0) \exp \left[ - \int_0^{t} dt' \gamma(t') \right] + \int_0^{t} dt' h(t') \exp \left[ - \int_{t'}^{t} dt'' \gamma(t'') \right]. \tag{A.3}$$

Setting $t = T$ in the above expression and taking into account that $H(T) = H(0)$, it follows that

$$H(0) = \frac{e^{-\bar{\gamma}T}}{1 - e^{-\bar{\gamma}T}} \int_0^{T} dt' h(t') \exp \left[ \int_0^{t'} dt'' \gamma(t'') \right], \tag{A.4}$$

with

$$\bar{\gamma} = \frac{1}{T} \int_0^{T} dt \gamma(t). \tag{A.5}$$

Then, inserting Eq. (A.4) into Eq. (A.3), one obtains after some simplifications that

$$H(t) = \frac{1}{2} \text{csch} \left( \frac{\bar{\gamma}T}{2} \right) \int_0^{T} dt' h(t') \exp \left[ \text{sgn}(t - t') \frac{\bar{\gamma}T}{2} - \int_{t'}^{t} dt'' \gamma(t'') \right], \tag{A.6}$$

for $t \in [0, T]$. 

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