Charged spherically symmetric black holes in \( f(R) \) gravity

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A new class of analytic charged spherically symmetric black hole solutions, which behave asymptotically as flat or (A)dS spacetimes, is derived for specific classes of \( f(R) \) gravity, i.e., \( f(R) = R - 2\alpha \sqrt{R} \) and \( f(R) = R - 2\alpha \sqrt{R} - 8\Lambda \), where \( \Lambda \) is the cosmological constant. These black holes are characterized by the dimensional parameter \( \alpha \) that makes solutions deviate from the standard solutions of general relativity. The Kretschmann scalar and squared Ricci tensor are shown to depend on the parameter \( \alpha \) which is not allowed to be zero. Thermodynamical quantities, like entropy, Hawking temperature, quasi-local energy and the Gibbs free energy are calculated. The interesting result of these calculations is the possibility of a negative black hole entropy. Furthermore, present calculations show that for negative energy, particles inside a black hole, behave as if they have a negative entropy. This fact gives rise to instability for \( f_{RR} < 0 \).

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I. INTRODUCTION

Challenging problems ranging from quantum gravity to dark energy (DE) and dark matter (DM) give support to search for other gravitational theories beyond the standard Einstein general relativity (GR). Actually, GR has many unsolved issues like singularities, the nature of DE and DM, etc. All these issues encourage scientists to modify GR or extend it in view of addressing shortcomings at UV and IR scales [1]. In other words, viable modified/extended theories should be compatible with the current experimental constraints and should give motivations on issues in quantum gravity and cosmology. Thus, it is straightforward to extend directly GR considering it as a limit of a more general theory of gravitation\(^1 \). Among the possible extensions of GR, the so-called \( f(R) \) gravity generalizes the Einstein-Hilbert action by substituting the Ricci scalar \( R \) by an analytic differentiable function. The fundamental reasons for this approach come out from the formulation of any quantum field theory on curved spacetime [3]. \( f(R) \) gravity have some important applications, like the Starobinsky model, \( f(R) = R + \alpha R^2 \), \( \alpha > 0 \), which is successful to explain inflationary behavior of early universe [4–6]. Furthermore \( f(R) \) gravity is capable of explaining the observed cosmic acceleration without assuming the cosmological constant. Possible toy models have the form \( f(R) = R - \frac{\beta}{R^n} \), where \( \alpha \) and \( n \) have positive values [7]. Nevertheless, this model suffers from instability problems because of the second derivative of the function \( f \) that has a negative value, i.e., \( f_{RR} < 0 \) [8, 9]. Later this problem has been tackled [10] and cosmological stable models have been achieved using some limitations on the parameter space. There are many viable cosmological models constructed using \( f(R) \) [11, 12]. Finally \( f(R) \) models give interesting results for structure formation, as the modification of the spectra of galaxy clustering, CMB, weak lensing, etc. [13]. There are many application of \( f(R) \) from the astrophysical point of view [14], for general reviews of \( f(R) \) gravity, see [1, 15].

From the viewpoint of mathematics, modified/extended gravity poses the issue to establish or modified well-known facts of GR like the stability of solutions, initial value problem and the problem of deriving new black hole solutions [16]. As it s well-
known, in addition to the cosmological solutions, there exist axially symmetric as well as spherical ones that could have a main role in several astrophysical problems spanning from black hole solutions to galactic nuclei. Modified gravitational theories must include black hole solutions like Schwarzschild-like in order to be compatible with GR results and, in principle, must give new black hole solutions that might have physical interest. According to this fact, the way to find out exact or approximate black hole solutions is highly important to investigate if observations can be matched to modified/extended gravity [17].

In the framework of \( f(R) \) gravity there is a specific interest for spherically symmetric black hole solutions. They have been derived using constant Ricci scalar [18]. Moreover, spherically symmetric black hole solutions, including perfect fluid matter, have been analyzed [19]. Additionally, by using the method of Noether symmetry, many spherically symmetric black holes have been derived [17]. Hollenstein and Lobo [20] derived exact solutions of static spherically symmetric spacetimes in \( f(R) \) coupled to non-linear electrodynamics. For the readers interested in the static black holes we refer to [21]–[54] and the references therein. Using the Lagrangian multiplier, new analytic solutions with dynamical Ricci scalar have been derived [55]. It is the purpose of the present study, by using the field equation of \( f(R) \), to generalize these black hole solutions [55] and derive new charged black hole solutions with dynamical Ricci scalar asymptotically converging towards flat or (A)dS spacetimes.

The paper is organized as follows. In Sec. II, a summary of Maxwell-\( f(R) \) gravity is provided. In Sec. III, a spherically symmetric ansatz is applied to the field equation of Maxwell-\( f(R) \) theory and an exact solution is derived. In Sec. IV, the same spherically symmetric ansatz is applied to the field equation of Maxwell-\( f(R) \) theory that includes a cosmological constant. Solving the resulting differential equations, we derive a new black hole solution that behaves as (A)dS. In Sec. V, the characteristic properties of these black holes are analyzed. In Sec. VI, thermodynamical quantities like entropy, quasi-local energy, Hawking temperature, Gibbs energy are calculated. We show that the entropy of the derived black hole solutions are not proportional to the horizon area and show some regions of the parameter space where the entropy becomes negative. The main reason for this result is due to the parameter \( \alpha \) related to the higher order correction. In Sec. VII, we discuss the main results of the present study and draw conclusions.

II. MAXWELL–\( f(R) \) GRAVITY

The theory of gravity that will be considered in this work is the \( f(R) \) gravity which was first taken into account in [56]. See also [1, 7, 57]:

\[
S := S_g + S_{E.M},
\]

where \( S_g \) is the gravitational action given by:

\[
S_g := \frac{1}{2\kappa} \int d^4x \sqrt{-g}(f(R) - \Lambda),
\]

where \( \Lambda \) represents the cosmological constant, \( R \) is the Ricci scalar, \( \kappa \) is the gravitational constant, \( g \) is the determinant of the metric and \( f(R) \) is an analytic differentiable function. In this study \( S_{E.M.} \) is the action of the non-linear electrodynamics field which takes the form:

\[
S_{E.M.} := -\frac{1}{2} F^{2s},
\]

where \( s \geq 1 \) is an arbitrary parameter that is equal to one for the standard Maxwell theory and \( F^2 = F_{\mu\nu} F^{\mu\nu} \), where \( F_{\mu\nu} = 2A_{[\mu,\nu]} \) with \( A_\mu \) being the gauge potential 1-form and the comma denotes the ordinary differentiation\(^2 \) [54].

The field equations of \( f(R) \) gravitational theory can be obtained by carrying out the variations of the action given by Eq. (1) with respect to the metric tensor \( g_{\mu\nu} \) and the strength tensor \( F \) that yield the following form of the field equations [58, 59]:

\[
I_{\mu\nu} = R_{\mu\nu} f_R - \frac{1}{2} g_{\mu\nu} f(R) - 2 g_{\mu\nu} \Lambda + g_{\mu\nu} \Box f_R - \nabla_\mu \nabla_\nu f_R - 8\pi T_{\mu\nu} \equiv 0,
\]

\[
\partial_\nu \left( \sqrt{-g} \Gamma^{\alpha\beta\gamma} F_{\alpha\beta} \right) = 0,
\]

with \( R_{\mu\nu} \) being the Ricci tensor defined by

\[
R_{\mu\nu} = R^\rho_{\rho\mu\nu} = 2 \Gamma^\rho_{\rho[\nu;\mu]} + 2 \Gamma^\rho_{\rho\mu\beta} \Gamma^\beta_{\nu;\mu},
\]

\(^2\) The square brackets represent the anti-symmetrization, i.e. \( A_{[\mu,\nu]} = \frac{1}{2}(A_{\mu,\nu} - A_{\nu,\mu}) \) and the symmetric one is represented by \( A_{(\mu,\nu)} = \frac{1}{2}(A_{\mu,\nu} + A_{\nu,\mu}) \).
where $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbols of second kind. The d’Alembert operator $\Box$ is defined as $\Box = \nabla_\alpha \nabla^\alpha$ where $\nabla_\alpha V^\beta$ is the covariant derivatives of the vector $V^\beta$ and $f_R = \frac{df(R)}{dR}$. In this study $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} := \frac{1}{4\pi} \left( \epsilon_{\mu\nu\rho\sigma} F^\rho_{\phantom{\rho}F}^\sigma F^{\nu-1} - \frac{1}{4} g_{\mu\nu} F^2 \right),$$

which is the energy momentum-tensor of the non-linear electrodynamic field. When $s = 1$, we get the standard energy-momentum tensor of Maxwell field.

The trace of equation of (4) is:

$$R f_R - 2 f(R) - 8\Lambda + 3\Box f_R = T$$

where $T = F^\gamma (s F - F^\gamma)$.

It is worth noticing that, for $s = 1$, it is $T = 0$. This property means that the Maxwell field is conformally invariant. In the following, we are going to assume some form of the field Eqs. (4) without and with a cosmological constant to derive exact solutions that asymptotically behave as flat or (A)dS spacetimes.

III. AN EXACT CHARGED BLACK HOLE SOLUTION

Let us derive a charged black hole solution adopting the model $f(R) = R - 2\alpha \sqrt{R}$. To this aim, we are going to use the following spherically symmetric spacetime:

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta$ is the line element on the unit sphere. The Ricci scalar of the metric (8) has the form

$$R = \frac{2 - \frac{1}{r^2} B'' - 4r B' - 2B}{r^2}.$$
Applying the ansatz (8) to Eqs. (4), (5) and (7), after using (9) and putting the parameter \( s = 1 \), we get the following non-vanishing field equations:

\[
I'_r = \frac{1}{2 \alpha^{10} \sqrt{R}} \left[ r^6 \sqrt{R} \left( r^4 B B'''' + r^3 B'''(1/2 r^4 B' + 6 r^3 B) + 2 r^2 B'(r + r B) - r^2 B' + 2 B(r - 3 r B) + 4 B(B - 1) \right) + r^4 \sqrt{R^2} |4 r^2 B''^2 + 1/2 r^3 B''(r - 4 B - 2 r B) + 2 B(29 r B - r) + 40 B(B - 1) + 2 r^4 B^2(r - 4 B + 1) + r^2 B'' \times (23 r^2 B^2 + 2 r B(23 B + 8 r^2 B^2 - 13) + 8 B(B - 1)(6 r B + r B^2 - 1) + 28 r^3 B^2 + B^2(34 r^2 B + 32 r^2 B^2 - 54 r^2) + 4 r B(B - 1) \times (7 r^2 B^2 - 9) + 8 B(B - 1)(r^2 B^2 - 1) \right) + r^4 R^2 \left[ B \sqrt{R} \left( r^3 B'' + r^2 B''' - 2 r B' + 4(1 - B)^2 a \right) + r^6 B B'''' - 3/2 r^5 B B''^2 + 1/2 B''(r B' - 12 B) + 4 r^2 B^2 + 2 B(31 r B - r) + 48 B(B - 1) - r^5 B^3 - 100 r^3 B^3 + 2 r^2 B^2(96 B - 85) + B''(8 r^4 - 12 r^3 B - 30 r^2 B^2 + 14 r B(3 - 5) + 20 + B(4 - 3 B) - 4 r B'(B - 1)(27 B - 23) - 16(B - 1)^2(2 B - 1) \right) \right] = 0,
\]

\[
I''_r = -\frac{1}{4 \alpha^{10} \sqrt{R^2}} \left[ r^4 \sqrt{R} \left[ (r B + 4 B'[4(1 - B) - 2 r B' + r^2 B''(r B + 4 B)] + r^2 B'' B[5 B + 2 r^2 B^2 + 2 r B - 1) + 14 r^2 B^2 + 2 B'[16 r^2 B^2 + 12 r B - 20 r B(8 B - 1)(r^2 B^2 - 1) \right] + r^4 R^2 \left[ r^3 B'' B[4 B + r B'] - 2 r^4 B^2 + B''[4 r^2 B + 3) - 16 r B' - 50 r^2 B^2 + 4 r B'(15 - 17 B) - 16[B - 1](B - 1) \right] = 0,
\]

\[
I''_\theta = I''_\theta = -\frac{1}{2 \alpha^{10} \sqrt{R^2}} \left[ r^6 B B''' + r^3 B'''(r + B) + 2 r^2 B''(r B' - B) + 2 r B'(2 - 3 B) - 2 r^2 B^2 + 8 B(B - 1) \right]
\[
-5 \sqrt{R} \left[ r^6 B B'''' + r^5 B''''(r B' - 3 B) + 4 r^2 B^2 + 2 r B'(13 B - 2) + 18 B(B - 1) r + r^5 B^3 - r^3 B^3 + 2 r^2 B^2 \times 1/2 B' + 7 B) + 2 r B'(23 r^2 B^2 - r B'[14 + 8 r^2 B^2 - 17 B] + 2 B(B - 1)(1 - 10 B + 2 r^2 B^2) + 24 r^2 B^3 - 4 r B^2(3 + 8 r^2 B^2)
\[
+4 r^2 B(1)(B - 1) + 3 B - 2 r^2 B^2) - 8 r B^2(1 + 2 B - 1) + r^4 R^2 B' + 2 r B')^2 + a(r^6 B B'''' + 3/2 r^5 B B''^2 + 3 B'[r B' - 7 B] + 4 r^2 B^2 + 2 r B'[14 B - 1] + 22 B(B - 1) + 2 r B + 4 r^2 B^218 B + 9 B - 5 + 2 r B + 33 r^2 B^2
\[
+r B'(27 B - 34 - 2(9 B^2 - 5 B - 4)) + 104 r^2 B^3 + 2 r^2 B^2(74 - 81 B) + 4 r B'(15 B - 15 B - 16) - 8B(1 - B^3 - 3 B + 5 B)] \right] = 0,
\]

\[
I = -\frac{3}{2 \alpha^{10} \sqrt{R^2}} \left[ r^6 \sqrt{R} \left( r^4 B B''' + r^3 B'''(r B' + 6 B) + 2 r B' B + 2 r B') - 2 r B^2 + 4 r B'(1 - 2 B) + 4 B(B - 1) r^4 \sqrt{R^2} r^5 B B''''
\]

\[
+ r^6 B B'''' - r^5 B''''(r B' - B) + 2 B(15 B - 1) + 20 B(B - 1) + 2 /3 r^5 B^3 + 2 r^4 B^2(6 r B^2 - 5 B - 2)
\]

\[
+ 2 r^2 B(23 r^2 B^2 + 2 r B'(14 B - 9) + 4(6 B + 7 B) + 1) + 104 /3 r^3 B^3 + 2 r B^2(6 B - 11) + 8 r B'(B - 1)(B - 3) - 8 /3 B^2)
\]

\[
+ 8 B) + r^4 R^2 B[4 r^3 B'' - 4 B + 2 r B'- 2 r B') - a(r^6 B B'''' + 3 /2 r^5 B B''^2 - r^3 B'' B'(r B' - B) + 4 r^2 B^2
\]

\[
+ 2 r B'(16 B - 1) + 24(2 B - 1) + 2 r B^2 + 2 r B'(10 B' + 17 B) + 2 r B' + 3 r B^2(11 r^2 B^2 + 2 r B'[16 B + 23) + 4(3 - 4 B^2 + B)
\]

\[
+ 136 r^2 B^3 - 2 r^2 B^2(106 - 117 B) + 8 B B'(B - 1)(B - 13) + 16(2 B - 1)(B - 1) \right] = 0,
\]

(10)

where \( q \) is the gauge potential which is defined as

\[
A := q(r) dt.
\]

(11)

If we subtract \( I'_r \) from \( I'_r \) and solving the system \( I'_r - I'_r \) and \( I'_\theta \), which is a closed system for the two unknowns functions \( B(r) \) and \( q(r) \), we get the following exact solution

\[
B(r) = \frac{1}{2} - \frac{1}{3 a r^2} - \frac{1}{3 a r^2}, \quad A = \frac{1}{\sqrt{3 a r}}.
\]

(12)

The analytic solution (12) satisfy the system of differential equations (10) including the trace equation \( I \). Using Eq. (9) we get the Ricci scalar in the form

\[
R = \frac{1}{r^2},
\]

(13)

---

3 Here and through all this study \( B \equiv B(r), B' = \frac{dB(r)}{dr}, B'' = \frac{d^2 B(r)}{dr^2}, \) etc. Also in this application we put \( A = 0 \)
which is also a consistency check for the whole procedure. The metric of the above solution takes the form

\[ ds^2 = \left( \frac{1}{2} - \frac{1}{3ar} - \frac{1}{3ar^2} \right) dt^2 - \left( \frac{1}{2} - \frac{1}{3ar} - \frac{1}{3ar^2} \right)^{-1} dr^2 - r^2 d\Omega^2, \]

(14)

which asymptotically behaves as a flat space-time. We have to stress the fact that solution (12) is different from that obtained in [55] due to the fact that they derived their solution using the form \( f(R) = R + \alpha \sqrt{R} \). Therefore, our solution is identical with their one when we neglect the term \( \frac{1}{3ar^2} \), which is responsible for the electric charge, and reverse the negative sign to be positive to satisfy the field equation of \( f(R) = R + \alpha \sqrt{R} \).
IV. AN EXACT (A)DS CHARGED BLACK HOLE SOLUTION

Let us derive now a charged (A)ds black hole solution for the model $f(R) = R - 2\alpha \sqrt{R} - 8\Lambda^4$. Applying the anzatz (8) to the field Eqs. (4), (5) and (7), after using (9) and putting $s = 1$, we get the following non-vanishing field equations

\[ I = \frac{1}{2\sqrt{10}} \sqrt{R} \left[ \rho^6 \sqrt{R} \left( r^4 B B'''' + B''''(1/2r B' + 6r^2 B) + 2r B''(B + r B') - r^2 B'^2 + 2r B'(1 - 3 B) + 4B(B - 1) \right) + r^4 \sqrt{R} \left( 4r^2 B'^2 + r^2 B B'''' - \rho^6 \sqrt{R} \left( r^4 B B'''' + B''''(1/2r B' + 6r^2 B) + 2r B'''(r B' - 4 B) + 2 B'(29r B - r + 4r^2 B) + 8B(5B + 12r^2 B - 5) \right) + 4r^3 B'^2(r B' - 2 B') - 2 B' + 2r B'(23 B + 8r^2 q^2 - 13 + 40r^2 B) + 8r^2 q^2 (B - 1 + 4r^2 B) + 48B^2 \right) + 8B(8r^2 B - 7) + 162 B^2 (4r^2 B - 3) + 8 \right) + 28r B^3 + r^2 B^2 (34 B + 32r^2 q^2 + 184r^2 B - 54) + 4r B'(B - 1 + 4r^2 B) \right) + 7 B + 8r^2 q^2 - 9 + 24r^2 B \right) + 8r^2 q^2 (B - 1 - 4r^2 B) + 8B^2(16r^2 B - 12r^2 + 1) + 8(2r^2 + 1)(4r^2 - 1) \right] + r^4 R^2 B B'' + r^3 B'' - 2 B' + 4(1 - B)^2 + \alpha \left[ \rho^6 R B B'''' - 3/2 \rho^6 B B'''' + 1/2 \rho^6 B''''(r B' - 12 B) + 4 \right) \right] + r^2 B'(31 B - 1 + 4r^2 B) + 48B(B - 1 + 2r^2 B) - r^2 B'^2 \right) + 100r^3 B^3 + 22r^2 B^2 [96B - 85 + 324r^2 B] + 2r^2 B'^2 (4 - 6r B' - 15r B + 16r^2 B) + r^2 B''(57r B^2 + 2r B'[21B - 35 + 68r^2 B] + 4B(4 - 3B - 4r^2 B) + (4r^2 B - 1)^2 - 4r B'[27B^2 + 4B(47r^2 B - 50) - 23 - 4r^2 B(88r^2 B - 45)] - 16(1 + 1)(B - 1) + 2r^2 B^2 + 5B^2 + 3(2r^2 B - 11) + (4r^2 B - 1)^2) \right] \right] = 0, \]

\[ I' = - \frac{1}{4 \sqrt{10}} \sqrt{R} \left[ \rho^6 \sqrt{R} \left( r^4 B B'''' + r^3 B''''(r B' + 5 B) + 2r B''(2r B' - B) + 2r B'(2 - 2 B) - r^2 B^2 + 8B(B - 1) \right) + r^3 B B''''(r B' - 3 B) + 4r^2 B^2 + 2r B'(13 B - 1 + 2r^2 B) + 2B(9B - 9 + 20r^2 B) \right] + r^3 B'^3 \right] - 2r^3 B'^2(r^2 q^2 - 2 - 7r B' + 7B - 10r^2 B) + 2r B'''(23r^2 B^2 - r B'[14 + 8r^2 q^2 - 17B - 80r^2 B] + 4B(1 - 1 - 4r^2 B)^2 q^2 \right] - B[10B + 8r^2 B - 11] + 4r^2 [3 + 8r^2 B + 1] + 24r^2 B'^3 + 4r B^2(3 + 8r^2 q^2 - 44r^2 B) + 4B \right) \right] = 0. \]

\[ I^3 = I^2 \left[ \rho^6 \sqrt{R} \left[ r^4 B B'''' + r^3 B''''(r B' + 5 B) + 2r B''(2r B' - B) + 2r B'(2 - 2 B) - r^2 B^2 + 8B(B - 1) \right] + r^3 B B'''' + r^2 B B'''' + r^2 B''''(r B' - 3 B) + 4r^2 B^2 + 2r B'(13 B - 1 + 2r^2 B) + 2B(9B - 9 + 20r^2 B) \right] + r^3 B'^3 \right] - 2r^3 B'^2(r^2 q^2 - 2 - 7r B' + 7B - 10r^2 B) + 2r B'''(23r^2 B^2 - r B'[14 + 8r^2 q^2 - 17B - 80r^2 B] + 4B(1 - 1 - 4r^2 B)^2 q^2 \right] - B[10B + 8r^2 B - 11] + 4r^2 [3 + 8r^2 B + 1] + 24r^2 B'^3 + 4r B^2(3 + 8r^2 q^2 - 44r^2 B) + 4B \right) \right] = 0. \]

\[ I = - \frac{3}{2 \sqrt{10}} \sqrt{R} \left[ \rho^6 \sqrt{R} \left[ r^4 B B'''' + r^3 B''''(r B' + 5 B) + 2r B''(2r B' - B) + 2r B'(2 - 2 B) - r^2 B^2 + 8B(B - 1) \right] + r^3 B B'''' + r^2 B B'''' + r^2 B''''(r B' - 3 B) + 4r^2 B^2 + 2r B'(13 B - 1 + 2r^2 B) + 2B(9B - 9 + 20r^2 B) \right] + r^3 B'^3 \right] - 2r^3 B'^2(r^2 q^2 - 2 - 7r B' + 7B - 10r^2 B) + 2r B'''(23r^2 B^2 - r B'[14 + 8r^2 q^2 - 17B - 80r^2 B] + 4B(1 - 1 - 4r^2 B)^2 q^2 \right] - B[10B + 8r^2 B - 11] + 4r^2 [3 + 8r^2 B + 1] + 24r^2 B'^3 + 4r B^2(3 + 8r^2 q^2 - 44r^2 B) + 4B \right) \right] = 0. \]
If we subtract the component $I^i_r$ from the component $I^r_i$ and solve the system $I^i_r - I^r_i$ and $I^a_b$ which is a closed system for the two unknowns functions $B(r)$ and $q(r)$, we get the exact solution

$$B(r) = \frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3ar} - \frac{1}{3ar^2}, \quad A = \frac{1}{\sqrt{3ar}},$$

(16)

Using Eq. (16) in (9) we get the Ricci scalar in the form

$$R = \frac{8r^2\Lambda + 1}{r^2}. \quad (17)$$

The metric of the above solution takes the form

$$ds^2 = \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3ar} - \frac{1}{3ar^2}\right)dr^2 - \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3ar} - \frac{1}{3ar^2}\right)^{-1}dr^2 - r^2d\Omega^2. \quad (18)$$

which behaves asymptotically as (A)dS spacetime. Solution (16) is different from that derived in [55] due to the reason discussed for solution (12).

V. PHYSICAL PROPERTIES OF THE BLACK HOLES

The metric of solution (12) can be rewritten in the form

$$ds^2 = \left(\frac{1}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(\frac{1}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 - r^2d\Omega^2, \quad \text{where} \quad M = \frac{1}{6\alpha}, \quad q = \frac{1}{\sqrt{3\alpha}},$$

(19)

which shows clearly that the dimensional parameter $\alpha$ cannot be equal zero and, in that case, the line element coincides with the Reissner-Nordström spacetime. Also the metric of solution (16) may be rewritten as

$$ds^2 = \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 - r^2d\Omega^2, \quad \text{where, again} \quad M = \frac{1}{6\alpha} \quad \text{and} \quad q = \frac{1}{\sqrt{3\alpha}},$$

(20)

which shows that line element coincides with the (A)dS Reissner-Nordström spacetime.

Let us study now the regularity of the solutions (12) and (16) when $B(r) = 0$. For solution (12), we evaluate the scalar invariants and get

$$R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda} = \frac{56 + 9r^4\alpha^2 + 12\alpha r^3 + 12r^2[\alpha + 1] + 48r}{9\alpha^2r^8},$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{9r^4\alpha^2 - 12\alpha r^2 + 8}{18\alpha^2r^8},$$

$$R = \frac{1}{r^2},$$

(21)

where $R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda}$, $R^{\mu\nu}R_{\mu\nu}$, $R$ are the Kretschmann scalars, the Ricci tensor square, the Ricci scalar, respectively. Eqs. (21) show that the solutions, at $r = 0$, have true singularities and the dimensional parameter $\alpha \neq 0$. Also Eq. (12) as well as Eq. (14) show clearly that the dimensional parameter $\alpha$ cannot be equal to zero which insure that solution (12) cannot reduce to GR. This means that this solution is a new exact charged one in the frame of $f(R)$ gravitational theory.

Using Eq. (16) we get the scalar invariants in the form

$$R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda} = \frac{96r^8\alpha^2 + 24r^6\Lambda\alpha^2 + 9r^4\alpha^2 + 12\alpha r^3 + 12r^2[\alpha + 1] + 48r + 56}{9\alpha^2r^8},$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{288r^8\Lambda^2\alpha^2 + 72r^6\Lambda\alpha^2 + 9a^2r^4 - 12\alpha r^2 + 8}{18\alpha^2r^8},$$

$$R = \frac{8r^2\Lambda + 1}{r^2}.$$  

(22)

The same considerations carried out for solution (12) can also be applied for solution (16) which insure also that solution (16) is a novel charged one in the framework of $f(R)$ gravity that cannot reduce to GR.
VI. BLACK HOLE THERMODYNAMICS

Now we are going to explore the thermodynamics of the new black hole solutions derived in the previous sections. The Hawking temperature is defined as

$$T_+ = \frac{B'(r_+)}{4\pi},$$

(23)

where the event horizon is located at \( r = r_+ \) which is the largest positive root of \( B(r_+) = 0 \) that fulfill \( B'(r_+) \neq 0 \). The Bekenstein-Hawking entropy in the framework of \( f(R) \) gravity is given as \[61, 62\]

$$S(r_+) = \frac{1}{4} A f_R(r_+),$$

(24)

where \( A \) is the area of the event horizon. The form of the quasi-local energy in the framework of \( f(R) \) gravity is defined as \[61, 62\]

$$E(r_+) = \frac{1}{4} \int \left[ 2 f_R(r_+) + r_+^2 \left( f(R(r_+)) - R(r_+) f_R(r_+) \right) \right] dr_+.$$  

(25)

At the horizon, one has the constraint \( B(r_+) = 0 \) which gives

$$r_+ = \frac{1}{3\alpha} \left[ 1 \pm \sqrt{1 + 6\alpha} \right],$$

$$r_+ = \text{Root} \left( 4\alpha^4 \Lambda - 3\alpha x^2 + 2x + 2 = 0 \right),$$

where \( \text{Root} \left( 4\alpha^4 \Lambda - 3\alpha x^2 + 2x + 2 = 0 \right) \) is the roots of the equation \( (4\alpha^4 \Lambda - 3\alpha x^2 + 2x + 2 = 0) \). For the black hole (12) we must have \(-\frac{1}{6} \leq \alpha \leq \frac{1}{3}\) to get a real value of the radial coordinate \( r \). The relation between the radial coordinate \( r \) and the dimensional parameter \( \alpha \) of the black hole (12) is represented in Figure 1. From this figure, one can see that the branch \( \frac{1}{3\alpha} (1 - \sqrt{1 + 6\alpha}) \) has a limit equal -1 as \( \alpha \to 0 \) and the branch \( \frac{1}{3\alpha} (1 + \sqrt{1 + 6\alpha}) \) has a phase transition from \(-ve\) to \(+ve\) at \( \alpha = 0 \).

Using Eq. (24), the entropy of the black holes (12) and (16) are computed as

$$S_{+ \text{Eq.(12)}} = \frac{\pi r_+^2}{3} \left[ 2 \mp \sqrt{1 + 6\alpha} \right] = \frac{\pi}{27\alpha^2} \left[ 1 \pm \sqrt{1 + 6\alpha} \right] \left[ 2 \mp \sqrt{1 + 6\alpha} \right],$$

$$S_{+ \text{Eq.(16)}} = \frac{\pi r_+^2}{4} \left[ 1 - \alpha r_+ \right].$$

(27)
The entropy of the black hole solution (12) 

$S_+ = \frac{\pi (1 + \sqrt{1 + 6\alpha})^2 (2 - \sqrt{1 + 6\alpha})}{27\alpha^2}$

The entropy of the black hole solution (16) 

$S_+ = \frac{\pi (1 - \sqrt{1 + 6\alpha})^2 (2 + \sqrt{1 + 6\alpha})}{27\alpha^2}$

FIG. 2: Schematic plot of the entropy of the two black holes (12) and (16) versus $r$.

The first equation of (27) shows that, for a real value of entropy, we have $0.5 < \alpha < 0.167$. The second equation of (27) informs us that we must have $\alpha < \frac{e}{r}$ for positive entropy. Eqs. (27) are drawn in figure 2. As one can see from figure 2 (a) that for the black hole (12), we have $+ve$ entropy, for the branch $[1 - \sqrt{1 + 6\alpha}]^2 [2 + \sqrt{1 + 6\alpha}]$ and a phase transition from $+ve$ to $-ve$ one for the branch $[1 + \sqrt{1 + 6\alpha}]^2 [2 - \sqrt{1 + 6\alpha}]$ and then a negative value of the entropy. For the black hole (16) as Figure 2 (b) shows, we have a double phase at $\pm 2.738612788$ then the entropy has a $-ve$ value and it evolves to $+ve$ at $r = -2.187$ and then a $-ve$ value at $r \to 0$. The following remarks must taken into account: It is remarkable that the entropy $S$ is not proportional to the area of the horizon due to Eq. (24). We should also note that the entropy $S$ is proportional to the area if there is no Ricci scalar squared term i.e., $f_R = 1$. The Hawking temperatures associated with the black hole solutions (12) and (16) are

$$T_{+_{Eq.(12)}} = \frac{3\alpha (1 \pm \sqrt{1 + 6\alpha} + 6\alpha)}{4\pi (1 \pm \sqrt{1 + 6\alpha})^3},$$

$$T_{+_{Eq.(16)}} = \frac{r_+ - 4\alpha \Lambda r_+^3 + 2}{12\pi r_+^3}.$$  

(28)

where $T_+$ is the Hawking temperature at the event horizon. We represent the Hawking temperature in Figure 3. Figure 3 (a), which is related to the black hole (12), shows that the branch $\frac{3\alpha (1 + \sqrt{1 + 6\alpha} + 6\alpha)}{4\pi (1 + \sqrt{1 + 6\alpha})^3}$ always has a zero or positive temperature and the branch $\frac{3\alpha (1 - \sqrt{1 + 6\alpha} + 6\alpha)}{4\pi (1 - \sqrt{1 + 6\alpha})^3}$ has a $+ve$ value then it has a phase transition at $\alpha = 0$ then at $-ve$ value. As for figure 3 (b), which is related to black hole (16), the temperature has always $+ve$ value.

From Eq. (25), the quasi-local energy of the two black holes (12) and (16) are calculated as

$$E_{+_{Eq.(12)}} = \frac{r_+}{8} (4 - 3\alpha r_+) = \frac{(1 \pm \sqrt{1 + 6\alpha} - 3\alpha)}{12\alpha},$$

$$E_{+_{Eq.(16)}} = \frac{r_+}{8} (4 - 3\alpha r_+ + 4\Lambda r_+^3).$$  

(29)

The first equation of (29) shows that, if the dimensional parameter is $\alpha = 0$, we get $E_+ = \frac{r_+}{2}$ which is the energy of the Schwarzschild black hole. We plot the energy in figure 4. Figure 4 (a) shows that the quasi-local energy, for the branch $\frac{1 + \sqrt{1 + 6\alpha} - 3\alpha}{12\alpha}$, has a $-ve$ value then phase transition at $\alpha = 0$ then the energy becomes $+ve$. However, for the branch,
The Hawking temperature of the black hole solution (a) $T_+ = \frac{3 \alpha (1 + \sqrt{1 + 6 \alpha})}{4 \pi (1 + \sqrt{1 + 6 \alpha})^3}$ and (b) $T_+ = \frac{3 \alpha (1 - \sqrt{1 + 6 \alpha}) + 6 \alpha}{4 \pi (1 - \sqrt{1 + 6 \alpha})^3}$.

FIG. 3: Schematic plot of the Hawking temperature of the two black holes (12) and (16) versus $r$.

The quasilocal energy of the black hole solution (a) $E_+ = \frac{1 + \sqrt{1 + 6 \alpha} + 3 \alpha}{12 \alpha}$ and (b) $E_+ = \frac{1 - \sqrt{1 + 6 \alpha} + 3 \alpha}{12 \alpha}$.

FIG. 4: Schematic plot of the Hawking temperature of the black holes (12) versus $r$ and Hawking temperature of the black holes (16) versus the dimensional parameter $\alpha$.

$\frac{1 - \sqrt{1 + 6 \alpha} - 3 \alpha}{12 \alpha}$, the quasi-local energy has always a negative value. As for the black hole (16), we have a negative value for the quasi-local energy till $r = 1$ and then the energy becomes positive as figure 4 (b) shows.

The free energy in the grand canonical ensemble, also called Gibbs free energy, can be defined as [62, 63]

$$G(r_+) = E(r_+) - T(r_+)S(r_+)$$  \hspace{1cm} (30)

where $E(r_+)$, $T(r_+)$ and $S(r_+)$ are the quasilocal energy, the temperature and entropy at the event horizons, respectively. Using
The free energy of the black hole solution (12) and the free energy of the black hole solution (16) are given by:

\[ G_{+}^{(12)} = \frac{1}{12\alpha} \left( 1 + \sqrt{1 + 6\alpha} - 3\alpha - 3(1 + \sqrt{1 + 6\alpha})(2 - \sqrt{1 + 6\alpha}) \right), \]

\[ G_{+}^{(16)} = \frac{r_+(4 - 3ar_+ + 4\Lambda r_+^3)}{8} - \frac{(r - 4\Lambda r_+ + 2)(1 - ar)}{48ar_+}, \]

The behaviors of the Gibbs energy of our black holes are presented in figures 5(a), 5(b) for particular values of the model parameters. As figure 5(a) shows, for the black hole solution (12), the Gibbs energy is always positive when \( \alpha > 0 \) for the branch which means that it is more globally stable than the other case. For the black hole solution (16), the Gibbs energy has a triple phase transition. It has a negative value when \( r < -2.73 \). It becomes positive when \(-2.73 < r < -2.55 \). It becomes negative when \(-2.55 < r < -1 \). It becomes positive when \(-1 < r < -0.82 \). When \(-0.82 < r < 2.73 \), it has a negative value and, finally, for \( r > 2.73 \), the Gibbs energy becomes positive.

VII. DISCUSSION AND CONCLUSIONS

Spherically symmetric spacetimes constitute an essential part of black hole physics because all the fundamental properties of the black holes can be explained and can further be used to recognize and hence generalize in any eligible more general scenario [64]. In this paper, we discussed two main issues. In the first part, we focused on a spherically symmetric spacetime in the framework of \( f(R) \) gravitational theories. We derived new black hole charged solutions for the specific forms \( f(R) = R - 2\alpha \sqrt{R} \) and \( f(R) = R - 2\alpha \sqrt{R} - 8\alpha \). The main merits of these black holes are the fact that they depend on the dimensional parameter \( \alpha \) and have dynamical Ricci scalar, i.e., \( R = \frac{1}{\alpha} \) for the first model of \( f(R) \) and \( R = \frac{2\alpha + 1}{\alpha} \) for the second model. These solutions are new and cannot reduce to the standard solutions of GR due to the fact that the parameter \( \alpha \) is not allowed to have a zero value. We calculate the scalar invariant of those black holes and found that the Kretschmann and Ricci tensor square invariants depending on \( \alpha \) with a singularity in \( \alpha = 0 \).

In the second part, we study the thermodynamical properties of these black holes to extract more physical information from them. The first important thing in \( f(R) \) gravity is the fact that entropy is not always proportional to the area of the horizon [65, 66]. We have shown that, for some constraints on the parameter \( \alpha \), we have a positive value of the entropy. However, there are some regions for which the parameter \( \alpha \) makes the entropy have negative values [65, 67, 68]. This is not the first time that a black hole with negative entropy is found. Several black holes with negative entropy have been found as well as in charged Gauss-Bonnet (A)dS gravity [65, 67]. As our calculations show, negative entropy may be interpreted as a region where the
parameter \( \alpha \) has transitions into forbidden regions related to some phase transition. The complete understanding of gravitational entropy of non-trivial solution in the framework of \( f(R) \) gravitational theories remains the subject of future research.

We also calculated the thermodynamical quasi-local energy and showed that it is not the same as the ADM mass. It depends on the parameter \( \alpha \). It is worth noticing that when the parameter \( \alpha = 0 \), we get the ADM mass. Moreover, we calculated the Hawking temperature and have shown that it also depends on the parameter \( \alpha \). It is possible to show, for the spherically symmetric solution which asymptotically behaves as the flat spacetime, there is a phase transition for \( \alpha = 0 \). Also, we have shown that there is a negative value for the temperature for \( \alpha > 0 \) and vice versa. The negative value of temperature means that it goes below the absolute zero forming an ultra-cold black hole. Davies [69] noted earlier that there is no clear reasons, from the viewpoint of thermodynamics, to prevent the black hole temperature to go below the absolute zero and to turn it into a naked singularity. In fact, this is the case presented in Fig. 3(a) for the \( \alpha > 0 \) regime. However, the case of the ultra-cold black hole is justified in the presence of phantom energy field [70], and explains the decreasing energy pattern and Gibbs free energy as shown in Figs. 4(a) and 5(a). The results obtained here, together with other results in the literature, seem to indicate that the thermodynamical origin of \( f(R) \) gravitational theories, when horizons are present, has a broad of validation. To confirm this statement we need to know more about the novel black holes derived in this paper. This will be done in future studies.

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[1] S. Capozziello and M. De Laurentis, Phys. Rept. 509 (2011), 167.
[2] Y. F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, Rept. Prog. Phys. 79 (2016), 106901.
[3] N. D. Birrell and P. C. W. Davies, 1984 Quantum Fields in Curved Space (Cambridge: Cambridge University Press).
[4] A. A. Starobinsky, Phys. Lett. B 91 (1980), 99.
[5] S. Nojiri and S. D. Odintsov, Phys. Rev. D 68 (2003), 123512.
[6] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. 36 (2004), 1765.
[7] S. Capozziello, Int. J. Mod. Phys. D 11 (2002), 483491; S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, Int. J. Mod. Phys. D 12 (2003), 19691982; S. Capozziello, S. Carloni, and A. Troisi, Recent Research Developments in Astronomy and Astrophysics 1, p. 625, (Research Signpost, Trivandrum, India, 2003); S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D 70 (2004), 043528; S. Capozziello and M. Francaviglia, Gen. Rel. Grav. 40 (2008) 357.
[8] A. D. Dolgov and M. Kawasaki, Phys. Lett. B 573 (2003), 14; V. Faraoni, Phys. Rev. D 74 (2006), 104017; I. Sawicki and W. Hu, Phys. Rev. D 75 (2007), 127502; Y. S. Song, W. Hu, and I. Sawicki, Phys. Rev. D 75 (2007), 044004.
[9] T. Chiba, Phys. Lett. B 573 (2003), 1; G. J. Olmo, Phys. Rev. Lett. 95 (2005), 261102; G. J. Olmo, Phys. Rev. D 72 (2005), 083505; A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D 74 (2006), 121501(R); S. Jana. and S. Mohanty, arXiv:1807.04060v1 [gr-qc].
[10] W. Hu and I. Sawicki, Phys. Rev. D 76 (2007), 064004.
[11] A. A. Starobinsky, J. Exp. Theor. Phys. Lett. 86 (2003), 157.
[12] S. Nojiri and S. D. Odintsov, Phys. Rev. D 74 (2006), 086005.
[13] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D 77 (2008), 046009; L. Pogosian and A. Silvestri, Phys. Rev. D 77 (2008), 023503; P. Zhang, Phys. Rev. D 73 (2006), 123504; B. Li and J. D. Barrow Phys. Rev. D 75 (2007), 084010; Y.-S. Song, H. Peiris, and W. Hu Phys. Rev. D 76 (2007), 063517; F. Schmidt, Phys. Rev. D 78 (2008), 043002; S. Nojiri and S.D. Odintsov, Phys. Rev. D77 (2008), 026007; S. Nojiri and S.D. Odintsov, Phys. Lett. B657 (2007), 238; S. Capozziello, C. A. Manica and L. G. Molinari, Int. J. Geom. Meth. Mod. Phys. 16 (2018), 1950008.
[14] A. V. Frolov, Phys. Rev. Lett. 101 (2008), 061103; A. Upadhye and W. Hu, Phys. Rev. D 80 (2009), 064002; A. Cooney, S. DeDeo, and D. Psaltis, Phys. Rev. D 82 (2010), 064033; D. D. Doneva, S. S. Yazadjiev, and K. D. Kokotov, Phys. Rev. D 92 (2015), 064015; K. Bamba, S. Nojiri and S.D. Odintsov, JCAP 0810 (2008), 045; S. Capozziello, S. Nojiri, S.D. Odintsov and A. Troisi Phys. Lett. B639 (2006), 135; S. Capozziello, M. De Laurentis, R. Farinelli and S. D. Odintsov, Phys. Rev. D 93 (2016), 023501; S. Capozziello and M. De Laurentis, Annalen Phys. 524 (2012) 545.
[15] S. Nojiri and Sergei D. Odintsov, Phys. Rept. 505 (2011), 59; S. Nojiri and Sergei D. Odintsov, eConf C0602061 (2006) 06; Int. J. Geom. Meth. Mod. Phys. 4 (2007), 115; S. Nojiri, S.D. Odintsov and V.K. Oikonomou Phys. Rept. 692 (2017), 1; Phys. Rept 692 (2017), 1.
[16] S. Capozziello, S. Vignolo, Class Quantum Grav 26 (2009), 168001; V. Faraoni, Class. Quantum Grav. 26 (2009), 168002; G.J. Olmo, P. Singh, JCAP 030 (2008), 0901; V. Faraoni, N. Lanahan-Tremblay, Phys. Rev. D 78 (2008), 064017; S. Capozziello, S. Vignolo, Class. Quantum. Grav. 26 (2009), 175013.
[17] S. Capozziello, A. Stabile, A. Troisi, Class. Quant. Grav. 24 (2007), 2153; S. Capozziello, M. De Laurentis, A. Stabile, Class. Quant. Grav. 27 (2010), 165008.
[18] T. Multamki and I. Vilja, Phys. Rev. D 74 (2006), 064022.
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The sign is represented by the green curve and the sign by the yellow curve.
\[ S_+ = \frac{\pi \left( 1 \pm \sqrt{1 + 6\alpha} \right)^2 \left( 2 + \sqrt{1 + 6\alpha} \right)}{27\alpha^2} \]

+ sign represented by green curve and the
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