On the inverse problem of fractional Brownian motion and the inverse of infinite Toeplitz matrices

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Abstract

The inverse problem of fractional Brownian motion and other Gaussian processes with stationary increments involves inverting an infinite hermitian positively definite Toeplitz matrix (a matrix that has equal elements along its diagonals). The problem of inverting Toeplitz matrices is interesting on its own and has various applications in physics, signal processing, statistics, etc. A large body of literature has emerged to study this question since the seminal work of Szegö on Toeplitz forms in 1920’s. In this paper we obtain, for the first time, an explicit general formula for the inverse of infinite hermitian positive definite Toeplitz matrices. Our formula is explicitly given in terms of the Szegö function associated to the spectral density of the matrix. These results are applied to the fractional Brownian motion and to m-diagonal Toeplitz matrices and we provide explicit examples.

1. Introduction

The statistical inverse solution to the inverse problem involves quite often the inverse of a large or infinite matrix. Generally, the measurement vector is modeled as a realization of a stochastic process with unknown parameters and the problem is to recover these parameters from the measurements. In this paper we are mainly concerned with the fractional Brownian motion which is a Gaussian process \( \{X(t): t \geq 0\} \) defined by the covariance function

\[
\langle X(t), X(s) \rangle = \frac{a}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0
\]

where \( H \in (0, 1) \) is a parameter called the Hurst index and \( a > 0 \) is a scale parameter. The fractional Brownian motion has stationary increments but these increments are not independent. It is an important model that has been successfully used to model real data that appear in different contexts such as anomalous diffusion, single-le dynamics, volatility in financial markets, hydrology, geophysics, queueing networks such as internet traffic, etc ([1–4]). In what follows we shall assume that \( a = 1 \) and in this case \( X(t) \) is the standard fractional Brownian motion. The inverse problem of fractional Brownian motion deals with the recovery of the Hurst parameter \( H \) from a given vector of measurements. For this it is necessary to consider the increments \( \Delta_n = X(n + 1) - X(n), n \in \mathbb{N} \), of the fractional Brownian motion \( X(t) \) and the corresponding covariance matrix \( G = (g_{m,n}) \) given by

\[
g_{m,n} = \langle \Delta_m, \Delta_n \rangle = \frac{1}{2}|n - m + 1|^{2H} + \frac{1}{2}|n - m - 1|^{2H} - |n - m|^{2H}, \quad n, m = 1, 2, 3, \ldots
\]

The process \( \{\Delta_n: n \in \mathbb{N}\} \) is called the (discrete) fraction Gaussian noise. The statistical inverse method for fractional Brownian motion requires the computation of the inverse of the matrix \( G \). In fact assume that \( X_n = X(t_1), X(t_2), \ldots, X(t_n) \) is the vector of the observed values of the process at times \( 0 \leq t_1 < t_2 < \ldots < t_n \leq 1 \). Take \( t_i = i/n \) for all \( i \). Set \( W_n = (W_1, W_2, \ldots, W_n) \) where \( W_i = X(t_i) - X(t_{i-1}), i = 1, 2, \ldots, n \) with \( X(t_0) = 0 \). The unknown parameter \( H \) is considered as a random variable. The conditional distribution of the random vector \( W_n \) under the condition that the parameter \( H \) is equal to \( \hat{H} \) is:
The joint distribution $D_{\text{post}}(\hat{H}, W_n)$ of $\hat{H}$ and $W_n$ is given by:

$$D_{\text{post}}(\hat{H}, W_n) = \frac{1}{(2\pi)^{n/2}(\text{det}(M_n))^{1/2}} \exp\left(-\frac{1}{2} W_n^T(M_n)^{-1}W_n\right)$$

where $M_n$ is the covariance matrix of $W_n$ (increments of the fractional Brownian motion with parameter $\hat{H}$). Clearly,

$$M_n = \frac{1}{nH} G_{n \times n}$$

where $G_{n \times n}$ is the covariance matrix of the fractional Gaussian noise $(\Delta_1, \Delta_2, \ldots, \Delta_n)$. Of course it is necessary to take $n$ very large for a better approximation and ultimately $n \to \infty$ for the exact value. To recover the value of the unknown parameter $H$, it is important to have an analytic expression of the elements of the inverse matrix $(G_{n \times n})^{-1}$ for large $n$ and ultimately for $G^{-1}$ as functions of $H$. However such analytic forms are not known.

Up to so far in the literature, because no closed form for $G^{-1}$ is known, the matrix $G$ has to be approximated by some simpler matrices. One example is to consider the inverse of the matrix $\hat{G} = (\hat{g}_{n,m})$ given by

$$\hat{g}_{n,m} \text{ form } - m \in \{0, 1, -1\} \text{ and otherwise.}$$

(See for example [5].) A more widely used approximation in the literature for the matrix $G^{-1}$ is the asymptotic approximation introduced by Whittle [6] in 1953. It consists of the matrix $\Gamma$ given by:

$$\Gamma_{kj} = \int_0^1 e^{-2\pi i(k-j)t} \varphi_H(t) \, dt, \quad k, j = 1, 2, 3, \ldots$$

where $\varphi_H(t)$ is the spectral density function of the fractional Gaussian noise of index $H$ (introduced in section 2). (We refer to Beran [7] for more details.) This matrix $\Gamma$ corresponds to the inverse $G^{-1}$ only asymptotically in the sense that

$$\lim_{k \to \infty} \lim_{n \to \infty} \Gamma_{kj} = \lim_{k \to \infty} (G^{-1})_{kj} \text{ for each } j.$$
We have that the Taylor coefficients \( (a_n) \) of \( \psi(z) \) are given by:

\[
a_{n+1} = \frac{\psi^{(n+1)}(0)}{(n + 1)!} = \frac{1}{n + 1} \sum_{k=0}^{n} (k + 1)u_{k+n-k}, \quad n = 1, 2, 3, \ldots
\]

with

\[
a_0 = \psi(0) = e^{\mu_0}.
\]

The upper left-hand \( n \times n \) bloc of \( G^{-1} \), denoted \( (G^{-1})_{n \times n} \), is given by:

\[
(G^{-1})_{n \times n} = \\
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & a_0 & a_1 & \ldots & a_{n-2} \\
0 & a_1 & a_2 & \ldots & a_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-2} & a_{n-3} & a_{n-4} & \ldots & a_0
\end{pmatrix}
\]

These general results are then applied to the case of the covariance matrix of the fractional Gaussian noise and other matrices.

Toeplitz matrices appear in various contexts in physics such as condensed matter physics, entanglement theory, electrical engineering and chemical physics and has many important applications. (See [8, 9] and references therein for more details.) The problem of inversion of Toeplitz matrices has been studied for many years since the seminal algorithm of Trench [10]. It has important applications and it is interesting on its own (see for example [11–14]).

From the point of view of numerical applications, the problem can be considered as solved as there exist fast algorithms to compute the inverse of these matrices (see for example [15]). However as mentioned earlier, in terms of parameter estimation, it is desirable to obtain closed analytic expressions for inverses of Toeplitz matrices. An analytic expression can also be used to analyse the performance of numerical algorithms.

Furthermore from the mathematical modelling point of view, it is important to see how the elements of the inverse matrix are directly related to elements of the initial matrix or other associated characteristics such as eigenvalues. Many Toeplitz matrices that arise in Physics such as the sine-kernel

\[
a_{ij} = \frac{\sin K(i - j)}{i - j} \quad \text{for } i \neq j \quad \text{and } a_{ii} = K/\pi
\]

are given in analytical forms and it is important to find the analytic form for the corresponding inverse (in the case where it exists).

The rest of the paper is organised as follows. Section 2 discusses the notion of spectral density function section 3, the notion of Szegő function of a Toeplitz matrix. The link between the inverse of a Toeplitz matrix and its Szegő function is given section 4 and the explicit formula for the elements of the inverse matrix is given in sections 5 and 6. The case of the inverse of the covariance matrix of the fractional Gaussian noise is given in section 7 and further illustrative examples are given in section 8. Concluding remarks are given in section 9.

2. Spectral density function

Let \( G = (g_{kj})_{k,j = 1,2,\ldots} \) be an infinite, Hermitian and positive definite Toeplitz matrix. We assume that \( G \) is such that

\[
g_{kj} = \gamma(j - k), \quad k, j = 1, 2, 3, \ldots
\]

where \( \gamma \) is a positive definite function defined on the set of integers \( \mathbb{Z} \) and taking values in the set of complex numbers \( \mathbb{C} \) and satisfies

\[
\gamma(0) = 1, \quad \text{and } \gamma(-k) = \overline{\gamma(k)}, \quad \text{for all } k \in \mathbb{Z}.
\]

This implies (by the classical Bochner theorem) that there exists a probability measure \( \mu \) on the unit interval \([0, 1]\) such that

\[
\gamma(k) = \int_0^1 e^{2\pi ikt}d\mu(t)
\]

for all \( k \in \mathbb{Z} \). We shall assume that \( \mu \) admits a probability density function \( \varphi \geq 0 \) that is integrable (more precisely \( \varphi^p \) is integrable for some number \( p > 1 \)) so that its Fourier series

\[
\varphi(t) \sim \sum_{k \in \mathbb{Z}} \gamma(k)e^{-2\pi ikt}
\]

converges for almost all \( t \in [0, 1] \) (with respect to the Lebesgue measure on \([0, 1]\)). The function \( \varphi \) is the spectral density function of the matrix \( G \). We shall assume that \( G \) is invertible in the sense that there exists
another positive definite matrix $G^{-1}$ such that $GG^{-1} = G^{-1}G = I$ where $I$ is the infinite identity matrix. Our goal is to compute the inverse matrix $G^{-1}$.

If $G$ is the covariance matrix of fractional Gaussian noise given by relation (1), then clearly $G$ is the Toeplitz matrix corresponding to the function

$$
\gamma(k) = \frac{1}{2}|k + 1|^{2H} + \frac{1}{2}|k - 1|^{2H} - |k|^{2H}.
$$

It is an important result obtained by Sinai [16] that $G$ admits a spectral density function $\varphi_H$ given by

$$
\varphi_H(t) = C(H)|e^{2\pi it} - 1|^2\left(\sum_{n=-\infty}^{\infty} \frac{1}{|n|^{2H+1}}\right),
$$

where $C(H)$ is a normalising constant given by

$$
C(H) = -\zeta(-2H)/(2\zeta(1+2H))
$$

Here $\zeta(.)$ is the Riemann zeta function. Clearly,

$$
\varphi_H(t) = 4C(H)(\sin^2\pi t)\sum_{n=0}^{\infty} \left(\frac{1}{(n+t)^{2H+1}} + \frac{1}{(n+1-t)^{2H+1}}\right) = 4C(H)(\sin^2\pi t)(\zeta(2H+1, t) + \zeta(2H+1, 1-t))
$$

where $\zeta(.,.)$ is the classical Hurwitz zeta function. Moreover it is well-known that

$$
\varphi_H(t) = O(t^{1-2H}(1-t)^{1-2H})
$$

for $t$ near 0 or near 1. (See [17] for details.) This implies that for $0 < H \leq 1/2$, the function $\varphi_H$ is continuous on the interval $[0, 1]$ and for $1/2 < H < 1$, it is not continuous but $\varphi_H$ is integrable on the interval $[0, 1]$ for all $1 < p < 1/(2H - 1)$. The Fourier series of $\varphi_H$ (given by relation (7)) yields the following representation:

$$
\varphi_H(t) = (\cos(2\pi t) - 1)\text{Li}(-2H, e^{-2\pi it}) + \text{Li}(-2H, e^{2\pi it}),
$$

where $\text{Li}$ is the analytic continuation of the classical polylogarithm function $\text{Li}(s, z) = \sum_{k=1}^{\infty} e^{-sk}z^k$ in the complex plane (except at the single point $z = 1$). This representation is amenable to calculations (for example the function $\text{Li}$ is implemented in Wolfram Mathematica).

3. Szegö function of a Toeplitz matrix

Assume that the matrix $G$ admits a spectral density function $\varphi$. Under the condition that the spectral density function $\varphi$ satisfies the Szegö condition

$$
\int_{0}^{1} \log(\varphi(t))dt > -\infty,
$$

one can associate to the matrix $G$ the function $S$: $D \rightarrow \mathbb{C}$ (where $D = \{z \in \mathbb{C}: |z| < 1\}$ is the unit disc) defined by

$$
S(z) = \exp\left(\frac{1}{2}\int_{0}^{1} \left(\frac{e^{2\pi it} + z}{e^{2\pi it} - z}\right)\log(\varphi(t))dt\right), \text{ for all } |z| < 1.
$$

The function $S(z)$ is known as the Szegö function associated to the matrix $G$ (or equivalently associated to $\varphi$). (We refer the reader to Grenander and Szegö [18] and Simon [19] for more details.)

We shall mainly use the inverse $\psi(z) = 1/S(z)$ given by (3). It is also known that if the Szegö function $S(z)$ is written in the form,

$$
S(z) = c_0 + c_1z + c_2z^2 + \ldots, \text{ } |z| < 1
$$

where $c_0, c_1, c_2, \ldots$ are complex numbers, then these coefficients satisfy the system:

$$
\int_{0}^{1} e^{-2\pi i k t} \varphi(t)dt = c_0^0c_k + c_1c_{k+1} + c_2c_{k+2} + \ldots, \text{ } k = 0, 1, 2, \ldots
$$

together with the condition that $c_0$ is a positive real number.

Let $P_1(z), P_2(z), P_3(z), \ldots$ be a sequence of polynomials of a complex variable $z$ which are orthonormal on the unit circle $z = e^{\pi i t}$, $t \in [0, 1]$ with respect to the weight function $\varphi(t)$. That is

$$
\int_{0}^{1} P_n(e^{2\pi it})\bar{P}_m(e^{2\pi it})\varphi(t)dt = \delta_{nm}, \text{ } n, m = 1, 2, 3, \ldots
$$

Assume moreover that degree($P_n(z)$) $= n - 1$ for all $n = 1, 2, \ldots$ and further that the leading coefficient of $z^{n-1}$ is a positive real number. These conditions uniquely determine the sequence $(P_n(z))$. Szegö proved that the
function $\psi$ is such that for all $z, w \in D$,
\[
\sum_{n=1}^{\infty} P_n(z)\overline{P_n(w)} = \frac{\psi(z)\overline{\psi(w)}}{1-z\overline{w}}.
\] (12)

(See the book by Grenander and Szegö [18], pp. 37-51.)

For the case of the fractional Gaussian noise, since as mentioned before, the spectral density function $\varphi_H(t)$ is continuous in the open interval $(0, 1)$ and satisfies
\[
\varphi_H(t) = O(t^{1-2H}(1-t)^{1-2H})
\]
at the boundary of the interval, that is for $t$ near 0 and $t$ near 1, then it is clear that $\varphi_H$ satisfies the Szegö condition (8) and hence the fractional Gaussian noise admits a Szegö function. We shall denote its inverse by $\psi_H$.

4. Szegö function and the inverse of an infinite Toeplitz matrix

Let $Q_1(z)$, $Q_2(z)$, $Q_3(z)$, ... be arbitrary sequence of orthonormal polynomials on the unit circle with respect to the weight function $\varphi$ such that $\deg(Q_n) = n - 1$. Here it is not necessary to impose any extra condition on the coefficient of $z^{n-1}$ in $Q_n(z)$ as it is the case for the polynomials $P_n(z)$. We have that any such general sequence of orthogonal polynomials satisfies:
\[
\sum_{n=1}^{\infty} Q_n(z)\overline{Q_n(w)} = \frac{\psi(z)\overline{\psi(w)}}{1-z\overline{w}}.
\]
This is a consequence of an important result about the structure of the space $H$ of holomorphic functions $f: D \rightarrow \mathbb{C}$ of the form:
\[
f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}, \quad |z| < 1
\]
where $(a_n)$ is a sequence of complex numbers satisfying the following two conditions:
\[
\sum_{n=1}^{\infty} |a_n|^2 < \infty \text{ and } \int_0^1 |f(e^{2\pi it})|^2 \varphi(t)dt < \infty.
\]
In fact it is shown in [20] that the space $\mathcal{H}$ is a Hilbert space with respect to the inner product
\[
\langle f, g \rangle = \int_0^1 f(e^{2\pi it})\overline{g(e^{2\pi it})} \varphi(t) dt.
\]
(Here $f(e^{2\pi it})$ is the left limit $\lim_{t \uparrow 1} f(re^{2\pi it})$.) Moreover $\mathcal{H}$ is a reproducing kernel Hilbert space associated to the kernel
\[
\mathbb{K}: D \times D \rightarrow \mathbb{C}, \quad \mathbb{K}(z, w) = \sum_{n,m=1}^{\infty} (G^{-1})_{k,j} z^{k-1} \overline{(\varphi)}^{j-1}.
\]
This means that for every function $f \in \mathcal{H}$ and for all $|z| < 1$,
\[
f(z) = \int_0^1 f(e^{2\pi it})\overline{\mathbb{K}(e^{2\pi it}, z)} \varphi(t) dt.
\]
Or equivalently,
\[
f(z) = \int_0^1 f(e^{2\pi it})\overline{\mathbb{K}(z, e^{2\pi it})} \varphi(t) dt
\]
because obviously
\[
\mathbb{K}(y, z) = \overline{\mathbb{K}(y, z)}
\]
by the fact that $G$ is hermitian. We shall write
\[
\mathbb{K}(z, w) = \sum_{n,m=1}^{\infty} (G^{-1})_{k,j} z^{k-1} \overline{(\varphi)}^{j-1} = Z^T G^{-1} W
\]
where
\[
Z^T = (1, z, z^2, z^3, \ldots) \text{ and } W^T = (1, w, w^2, w^3, \ldots).
\]
A key property of a reproducing kernel Hilbert space is that the quantity $\sum_{n=1}^{\infty} \phi(z)\overline{\phi(w)}$ is invariant for all possible choices of orthonormal basis $\phi_1$, $\phi_2$, $\phi_3$, ... and in fact it is equal to the kernel of the space. Now it is again proven in [20] that the monomials $f(z) = z^{n-1}$ for all $n = 1, 2, \ldots$ are elements of $\mathcal{H}$ and indeed the sequence $1, z, z^2, z^3, \ldots$ forms a basis in $\mathcal{H}$ (but in general it is not an orthonormal basis). This implies that any
infinite sequence of polynomials \(Q_1(z), Q_2(z), Q_3(z), \ldots\) such that \(\deg Q_n(z) = n - 1\) for all \(n = 1, 2, 3, \ldots\) is a basis of \(\mathcal{H}\). In particular, the sequence \(P_1(z), P_2(z), \ldots\) (of the previous section) is an orthonormal basis of \(\mathcal{H}\).

Since this particular orthonormal basis satisfies (12), it follows that the kernel \(K\) of \(\mathcal{H}\) is such that:

\[
K(z, w) = \sum_{n=1}^{\infty} P_n(z)\overline{P_n(w)} = \frac{\psi(z)\overline{\psi(w)}}{1 - zw}, \quad \text{for all} |z| < 1, |w| < 1.
\]

That is,

\[
Z^T G^{-1} W = \frac{\psi(z)\overline{\psi(w)}}{1 - zw}.
\]  

(13)

This is a key result that will now be used to compute the matrix \(G^{-1}\).

5. Explicit expression of the inverse matrix \(G^{-1}\)

Since

\[
Z^T G^{-1} W = \frac{\psi(z)\overline{\psi(w)}}{1 - zw}
\]  

for all \(z, w\) in the unit disc, then taking \(z = w = 0\) yields that \(Z^T = W^T = (1, 0, 0, 0, \ldots)\) and hence

\[
Z^T G^{-1} W = (G^{-1})_{1,1}
\]

that is the element of the first row and first column of \(G^{-1}\). Hence

\[
(G^{-1})_{1,1} = \psi(0)\overline{\psi(0)} = |\psi(0)|^2 = e^{-\int_0^1 \log(\psi(t)) dt}.
\]

To obtain \((G^{-1})_{2,1}\) one can take differentiate (14) with respect to \(z\) at the point \(z = w = 0\) and obtain

\[
\frac{\partial}{\partial z} (Z^T G^{-1} W)_{|z=w=0} = (G^{-1})_{2,1}.
\]

Hence

\[
(G^{-1})_{2,1} = \frac{\partial}{\partial z} \left( \frac{\psi(z)\overline{\psi(w)}}{1 - zw} \right)_{|z=w=0} = \psi'(0)\overline{\psi(0)}.
\]

Also

\[
(G^{-1})_{3,1} = \frac{1}{2} \frac{\partial^2}{\partial z^2} (Z^T G^{-1} W)_{|z=w=0} = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( \frac{\psi(z)\overline{\psi(w)}}{1 - zw} \right)_{|z=w=0} = \frac{1}{2} \psi''(0)\overline{\psi(0)}.
\]

In general for all \(k = 1, 2, 3, \ldots\)

\[
(G^{-1})_{k,1} = \frac{1}{(k-1)!} \psi^{(k-1)}(0)\overline{\psi(0)}.
\]

Since the matrix \(G^{-1}\) is also hermitian, then clearly

\[
(G^{-1})_{j,k} = \frac{1}{(j-1)!} \psi^{(j-1)}(0)\overline{\psi(0)}.
\]

All other elements can also be obtained by taking successive partial derivatives with respect to \(z\) and \(w\), that is,

\[
(G^{-1})_{k,j} = \frac{1}{(k-1)!(j-1)!} \frac{\partial^{k+j-2}}{\partial z^{k-1}\partial \overline{w}^{j-1}} \left( \frac{\psi(z)\overline{\psi(w)}}{1 - zw} \right)_{|z=w=0}.
\]  

(15)

For example,

\[
(G^{-1})_{2,2} = \frac{\partial^2}{\partial z\partial \overline{w}} \left( \frac{\psi(z)\overline{\psi(w)}}{1 - zw} \right)_{|z=w=0} = |\psi'(0)|^2 + |\psi'(0)|^2.
\]
Also
\[
(G^{-1})_{3,3} = |\psi(0)|^2 + |\psi'(0)|^2 + \frac{1}{4}|\psi''(0)|^2,
\]
\[
(G^{-1})_{4,4} = |\psi(0)|^2 + |\psi'(0)|^2 + \frac{1}{4}|\psi''(0)|^2 + \frac{1}{36}|\psi'''(0)|^2,
\]
\[
(G^{-1})_{5,5} = |\psi(0)|^2 + |\psi'(0)|^2 + \frac{1}{4}|\psi''(0)|^2 + \frac{1}{36}|\psi'''(0)|^2 + \frac{1}{576}|\psi^{(4)}(0)|^2.
\]
An induction argument yields that
\[
(G^{-1})_{n,n} = \sum_{k=0}^{n-1} \frac{|\psi(k)(0)|^2}{(k!)^2}
\]
and hence
\[
\lim_{n \to \infty} (G^{-1})_{n,n} = \sum_{k=0}^{\infty} \frac{|\psi(k)(0)|^2}{(k!)^2}.
\]
Since the Szegö function \(S(z)\) is analytic in the unit disc \(D = \{ z \in \mathbb{C}; |z| < 1 \}\) and the Szegö condition implies that it does not vanish anywhere in \(D\), then its inverse \(\psi(z) = 1/S(z)\) is also an analytic function in \(D\). Assume then that
\[
\psi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D, \quad a_n \in \mathbb{C}.
\]
Then
\[
\frac{\psi^{(k)}(0)}{(k!)} = a_k
\]
and hence \(16\) yields
\[
\lim_{n \to \infty} (G^{-1})_{n,n} = \sum_{k=0}^{\infty} |a_k|^2.
\]
In particular \(\sum_{k=0}^{\infty} |a_k|^2 < \infty\). By the Szegö theorem, since \(1/\varphi(t)\) is also integrable and satisfies Szegö condition, then
\[
\int_0^1 e^{-2\pi it} \frac{1}{\varphi(t)} - dt = \sum_{k=0}^{\infty} \frac{1}{\varphi'_{n+k}}
\]
for all \(k = 0, 1, 2, \ldots\) Taking \(k = 0\) yields
\[
\int_0^1 \frac{1}{\varphi(t)} - dt = \sum_{k=0}^{\infty} |a_n|^2,
\]
and therefore
\[
\lim_{n \to \infty} (G^{-1})_{n,n} = \sum_{k=0}^{\infty} |a_k|^2 = \int_0^1 \frac{1}{\varphi(t)} - dt = \frac{1}{\varphi(0)}.
\]
In general taking
\[
\psi(z) = \sum_{n=0}^{\infty} a_n z^n
\]
in the main formula \(15\) yields that for all \(k, j\) with \(j \leq k\),
\[
(G^{-1})_{k,j} = \sum_{\ell=0}^{k} \tilde{a}_{\ell} a_{\ell+j+1} + \sum_{\ell=0}^{k} \tilde{a}_{\ell} a_{\ell+j+2} + \ldots + \tilde{a}_{k-j-1} a_{k-j-1}, \quad \text{where} \quad \tilde{a} = k - j, j \leq k.
\]
Thus the \(G^{-1}\) is fully determined by the coefficients of the function \(\psi(z)\). Hence
\[
\lim_{n \to \infty} (G^{-1})_{k+n,k} = \sum_{\ell=0}^{\infty} \tilde{a}_{\ell} a_{\ell+n} = \int_0^1 e^{-2\pi it} \frac{1}{\varphi(t)} - dt = \frac{1}{\varphi(-n)},
\]
(this is the Whittle approximation of the matrix \(G^{-1}\)).

6. The upper left-hand bloc of \(G^{-1}\)

From the discussion above, it is clear that once the elements of the first row of \(G^{-1}\) are determined, the other elements can easily be determined. Let \((G^{-1})_{n \times n}\) be the upper left-hand \(n \times n\) bloc of \(G^{-1}\) (that is the bloc of \(G^{-1}\) consisting of the first \(n\) rows and first \(n\) columns of \(G^{-1}\)). Then relation \(18\) implies that all the elements of
\((G^{-1})_{n \times n}\) are explicitly determined by the first \(n - 1\) derivatives of the function \(\psi(z)\) at the origin (or the first \(n\) coefficients of \(\psi(z)\) in its Taylor expansion at zero). In fact,

\[
(G^{-1})_{n \times n} = \begin{pmatrix}
\bar{a}_0 & 0 & 0 & \cdots & 0 \\
\bar{a}_1 & \bar{a}_0 & 0 & \cdots & 0 \\
\bar{a}_2 & \bar{a}_1 & \bar{a}_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{a}_n-1 & \bar{a}_n-2 & \bar{a}_n-3 & \cdots & \bar{a}_0
\end{pmatrix}
\]

This is yields an LU decomposition of the matrix \(G^{-1}\). This decomposition also yields a system of orthonormal polynomials with respect to the weight function \(j(t)\). Indeed it is immediately seen that the system 
\[
\{Q_0(z), Q_1(z), \ldots, Q_n(z)\}
\]

is orthonormal on the unit circle with respect to the weight function \(j(t)\) (the inverse of the spectral density of \(G\)).

7. Explicit expression for the inverse matrix

We can calculate the derivatives of \(\psi(z)\) at the origin with respect to \(z\). Note that

\[
\frac{d}{dz} \left( \int_0^1 \frac{e^{2\pi it} + z}{e^{2\pi it} - z} \log(\varphi(t)) \, dt \right) = \int_0^1 \frac{d}{dz} \left( \frac{e^{2\pi it} + z}{e^{2\pi it} - z} \right) \log(\varphi(t)) \, dt.
\]

Set

\[
u_0 = -\frac{1}{2} \int_0^1 \log(\varphi(t)) \, dt
\]

and

\[
u_k = -\int_0^1 e^{-2\pi ikz} \log(\varphi(t)) \, dt, \quad k = 1, 2, \ldots
\]

Then we can express the quantities \(a_n = \psi^{(n)}(0)/(n!)\) as functions of \(u_0, u_1, u_2, \ldots, u_n\). We shall write

\[
\psi(z) = e^{U(z)}, \quad U(z) = -\frac{1}{2} \int_0^1 \left( \frac{e^{2\pi it} + z}{e^{2\pi it} - z} \right) \log(\varphi(t)) \, dt.
\]

Then

\[
\psi'(z) = U'(z)\psi(z).
\]

Using Leibniz formula this yields

\[
\psi^{(n+1)}(0) = \sum_{k=0}^{n} \binom{n}{k} U^{(k+1)}(0) \psi^{(n-k)}(0)
\]

where

\[
U^{(0)}(0) = U(0) = -\frac{1}{2} \int_0^1 \log(\varphi(t)) \, dt
\]

and for \(k = 1, 2, \ldots\),

\[
U^{(k)}(0) = -k! \int_0^1 e^{-2\pi ikz} \log(\varphi(t)) \, dt.
\]

Then

\[
U^{(k)}(0) = -k! \int_0^1 e^{-2\pi ikz} \log(\varphi(t)) \, dt = ku_k, \quad k = 1, 2, \ldots
\]

Hence

\[
a_{n+1} = \frac{\psi^{(n+1)}(0)}{(n+1)!} = \frac{1}{n+1} \sum_{k=0}^{n} (k+1)u_k a_{n-k}
\]

with

\[
a_0 = \psi(0) = e^{u_0}.
\]

These relations yield the values of the coefficients \(a_n\) for all \(n\) and finally we can compute the bloc \((G^{-1})_{n \times n}\) for any number \(n\).
8. Illustrating example: fractional Gaussian noise

For the fractional Gaussian noise, taking for example \( n = 5 \) and the Hurst index \( H = 0.75 \) yields that the 5 values of sequence \((u_k)\) are:

\[
u_0 = 0.113994, \quad u_1 = -0.333504, \quad u_2 = -0.123701, \quad u_3 = -0.0838558, \quad u_4 = -0.0626411
\]

and the corresponding first 5 values of \((a_k)\) are:

\[
a_0 = 1.12075, \quad a_1 = -0.373773, \quad a_2 = -0.0763097, \quad a_3 = -0.0546738, \quad a_4 = -0.0374192.
\]

Hence this yields using relation (19) that the upper left-hand 5 \( \times \) 5 bloc of \(G^{-1}\) (that is the bloc corresponding of the first 5 rows and 5 columns) is:

\[
(G^{-1})_{5 \times 5} = \begin{pmatrix}
1.25607 & -0.418904 & -0.0855238 & -0.0612754 & -0.0419375 \\
-0.418904 & 1.39578 & -0.390382 & -0.0650882 & -0.0472891 \\
-0.0855238 & -0.390382 & 1.4016 & -0.386209 & -0.0622327 \\
-0.0612754 & -0.0650882 & -0.386209 & 1.4059 & -0.384164 \\
-0.0419375 & -0.0472891 & -0.0622327 & -0.384164 & 1.40599
\end{pmatrix}
\]

One can compare that with the inverse of the bloc \((G_m)_{m \times m}\) of the original covariance matrix \(G\) for \(m\) sufficiently large. Taking \(m = 1000\) yields the following:

\[
([(G_m)_{m \times m}]^{-1})_{5 \times 5} = \begin{pmatrix}
1.25599 & -0.418956 & -0.0855708 & -0.0613187 & -0.0419781 \\
-0.418956 & 1.39574 & -0.390413 & -0.0651171 & -0.0473163 \\
-0.0855708 & -0.390413 & 1.40157 & -0.386235 & -0.0622571 \\
-0.0613187 & -0.0651171 & -0.386235 & 1.40456 & -0.384186 \\
-0.0419781 & -0.0473163 & -0.0622571 & -0.384186 & 1.40597
\end{pmatrix}
\]

These two matrices are very close and the equality holds when \(m \to \infty\).

9. Illustrating example: finite-diagonal Toeplitz matrices

In this section we now consider the particular case where the Toeplitz matrix \(G\) has only a finite number \(2m + 1\) of non-zero diagonals (such matrices are also called \(2m + 1\)-diagonal Toeplitz matrices). It is also an interesting problem to compute the inverse of such matrices. Assume as previously that \(G = (g_{k,j})\) with \(g_{k,j} = \gamma(j - k) = \gamma(k - j), \ k, j = 1, 2, 3,\ldots\)

and \(\gamma(0) = 1\). Moreover we assume that \(\gamma(k) = 0\) for the integers \(|k| \geq m + 1\). This means that the matrix \(G\) is a \(2m + 1\)-diagonal Toeplitz matrix. Set \(\gamma(k) = q_k\) for all \(k = 1, 2, 3,\ldots, m\).

The first row of \(G\) is:

\[1, \ q_1, \ q_2, \ldots, \ q_m, \ 0, \ 0, \ldots\]

The corresponding spectral density if obviously given by

\[
\varphi(t) = 1 + \sum_{k=1}^{m} e^{2\pi i t q_k} + \sum_{k=1}^{m} e^{-2\pi i t \bar{q}_k}, \ t \in [0, 1].
\]

The condition that \(G\) is positive definite implies that \(\varphi(t) \geq 0\) everywhere everywhere in \([0, 1]\). If the function \(\log(\varphi(t))\) is integrable, then the corresponding Szegö function \(S(z)\) and its inverse \(\psi(z)\) exist as already discussed. The formulas of the previous section can be used to compute the elements \((u_n)\) and from which the inverse matrix \(G^{-1}\) follows.

9.1. Case of infinite 3-diagonal Toeplitz matrix

This corresponds to \(m = 1\). Assume that the matrix \(G\) is defined by the function

\[
\gamma(k) = \begin{cases} 
1 & \text{if } k = 0 \\
q & \text{if } k = 1 \\
\bar{q} & \text{if } k = -1 \\
0 & \text{otherwise}
\end{cases}
\]

where \(q\) is a complex number such that \(|q| < 1/2\). The corresponding spectral density is

\[
\varphi(t) = 1 + q e^{2\pi i t} + \bar{q} e^{-2\pi i t}, \ t \in [0, 1].
\]

We can explicitly compute the corresponding function \(\psi(z)\) as follows. First consider the Szegö function \(S(z)\) associated to \(\phi(t)\) and write as in (10): Write
\[ S(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \]

where \( c_0, c_1, c_2, \ldots \) are complex numbers. We can compute the values of these parameters of \( S(z) \) using similar argument as for the parameters \( a_0, a_1, a_2, \ldots \). Because

\[
S(z) = \psi(z)^{-1} = \exp \left( \frac{1}{2} \int_0^1 \left( \frac{e^{2\pi i t} + z}{e^{2\pi i t} - z} \right) \log(\varphi(t)) \, dt \right),
\]

then

\[ c_{n+1} = \frac{S^{n+1}(0)}{(n + 1)!} = \frac{1}{n + 1} \sum_{k=0}^n (k + 1) v_{k+1} c_{n-k} \]

with \( c_0 = S(0) \) and

\[ v_k = -u_k = \int_0^1 e^{-2\pi i t} \log(\varphi(t)) \, dt, \quad k = 1, 2, \ldots. \]

This yields that

\[ c_0 = \sqrt{1 + \sqrt{1 - 4|q|^2}} \quad \text{and} \quad c_k = \frac{q}{c_0} \]

and \( c_k=0 \) for all \( k = 2, 3, \ldots \) Hence the Szegő function associated to \( \varphi \) is:

\[ S(z) = c_0 + \frac{q}{c_0} z, \quad |z| < 1. \]

The corresponding function \( \psi \) is:

\[ \psi(z) = \frac{1}{S(z)} = \frac{c_0}{c_0^2 + qz} \quad \text{where} c_0 = \sqrt{1 + \sqrt{1 - 4|q|^2}}. \]

That is:

\[ \psi(z) = \frac{\sqrt{(1 + \sqrt{1 - 4|q|^2})/2}}{(1 + \sqrt{1 - 4|q|^2})/2 + qz} \]

In the case where \( q \) is a real number, this function was obtained in [20] where it is expressed as:

\[ \psi(z) = \left( \frac{2}{|q|} \right)^{1/2} \left( \frac{1}{a + bz} \right) \quad \text{where} \quad a = \sqrt{|q|^{-1} + \sqrt{|q|^{-2} - 4}} \quad \text{and} \quad b = (2/a) \text{sign}(q). \]

We can now explicitly compute the inverse of the matrix \( G \). Using (21), it is clear that the Taylor expansion of \( \psi(z) \) is given by:

\[ \psi(z) = \sum_{n=0}^{\infty} \left( -1 \right)^n \left( \frac{q^n}{c_0^{2n+1}} \right) z^n \]

and hence the coefficients \( a_0, a_1, a_2, \ldots \) of \( \psi(z) \) are explicitly given by:

\[ a_n = (-1)^n \left( \frac{q^n}{c_0^{2n+1}} \right), \quad n = 0, 1, 2, \ldots. \]

Therefore the element of order \((k, j)\) of the inverse matrix \( G^{-1} \) (with \( j \leq k \)) is explicitly given by:

\[ (G^{-1})_{kj} = \sum_{\ell=1}^j \frac{\alpha_{k-j} \alpha_{j-\ell}}{\alpha_{k-j}} a_{j-\ell} = \frac{(-1)^{j+k} c_0^{2j-k} (q^{j-k} c_0^4 - |q|^{2j})}{c_0^4 - |q|^2} \]

where

\[ c_0 = \sqrt{1 + \sqrt{1 - 4|q|^2}}. \]

This yields an explicit analytic expression of the inverse of the matrix \( G \). Summarising we have the following result:
Proposition 1. For any complex number \( q \) such that \( |q| < 1/2 \), the infinite matrix \( G \) given by
\[
G_{kj} = \begin{cases} 
1 & \text{if } k = j \\
q & \text{if } j - k = 1 \\
q^{-1} & \text{if } j - k = -1 \\
0 & \text{otherwise}
\end{cases}
\]
is such that its inverse is the matrix \( G^{-1} \) given explicitly by:
\[
(G^{-1})_{kj} = \frac{(-1)^{j+k} q^{2(j-k)} (q^2 - |q|^2)}{c_0^4 - |q|^2}, \quad \text{for } j \leq k
\]
and
\[
(G^{-1})_{kj} = (G^{-1})_{j,k} \quad \text{for } j > k
\]
where
\[
c_0 = \sqrt{1 + \sqrt{1 - 4|q|^2}}.
\]
To illustrate this result, take \( q = -1/5 \). Then the upper left-hand \( 5 \times 5 \) bloc of \( G^{-1} \) is:
\[
(G^{-1})_{5 \times 5} = \begin{pmatrix}
\frac{550 - 120\sqrt{2}}{2} & \frac{255 - 120\sqrt{2}}{2} & \frac{25527 - 115\sqrt{2}}{2} & \frac{52527 - 115\sqrt{2}}{2} \\
\frac{25527 - 115\sqrt{2}}{2} & \frac{550 - 120\sqrt{2}}{2} & \frac{25527 - 115\sqrt{2}}{2} & \frac{52527 - 115\sqrt{2}}{2} \\
\frac{25527 - 115\sqrt{2}}{2} & \frac{25527 - 115\sqrt{2}}{2} & \frac{550 - 120\sqrt{2}}{2} & \frac{25527 - 115\sqrt{2}}{2} \\
\frac{52527 - 115\sqrt{2}}{2} & \frac{52527 - 115\sqrt{2}}{2} & \frac{25527 - 115\sqrt{2}}{2} & \frac{550 - 120\sqrt{2}}{2}
\end{pmatrix}
\]
We can again compare this exact matrix with the inverse of the bloc \((G)_{m \times m}\) of \( G \) for \( m \) sufficiently large. Taking \( m = 10 \) yields the following:
\[
((G)_{m \times m}^{-1})_{5 \times 5} = \begin{pmatrix}
1.04356 & 0.217804 & 0.0454583 & 0.0094877 & 0.0019802 \\
0.217804 & 1.08902 & 0.227292 & 0.0474385 & 0.0099009 \\
0.0454583 & 0.227292 & 1.09109 & 0.227705 & 0.0475248 \\
0.0094877 & 0.0474385 & 0.227705 & 1.09109 & 0.227723 \\
0.0019802 & 0.0099009 & 0.0475248 & 0.227723 & 1.09109
\end{pmatrix}
\]
which is very close to the exact matrix.

9.2. Case of infinite 5–diagonal Toeplitz matrices

Assume \( G \) is given by the function:
\[
\gamma(k) = \begin{cases} 
1 & \text{if } k = 0 \\
q_1 & \text{if } k = 1 \\
q_2 & \text{if } k = 2 \\
q_3 & \text{if } k = -1 \\
q_4 & \text{if } k = -2 \\
0 & \text{otherwise}
\end{cases}
\]
Write
\[
\varphi(t) = 1 + e^{2\pi it} q_1 + e^{-2\pi it} \overline{q_1} + e^{4\pi it} q_2 + e^{-4\pi it} \overline{q_2}.
\]
Write
\[
S(z) = c_0 + c_1 z + c_2 z^2 + \ldots
\]
and we obtain that
\[
c_0 = \exp \left( \frac{1}{2} \int_0^1 \log(\varphi(t)) \, dt \right)
\]
\[
c_1 = \frac{c_0 (e_0^2 q_1 - e_2^4 \overline{q_1})}{c_0^4 - |q_2|^2}, \quad c_2 = q_2/c_0 \text{ and } c_3 = \ldots = 0.
\]
In particular case where \( q_1, q_2 \) are real numbers, then
\[
\zeta_1 = \frac{c_0 q_1}{c_0^2 + q_2}.
\]
Therefore the Szegö function \( S(z) \) is the polynomial:
\[
S(z) = c_0 + \zeta_1 z + \zeta_2 z^2 = c_0 + \left( \frac{c_0 (c_0^2 q_1 - q_2 i q_1)}{c_0^2 - \|q_2\|^2} \right) z + \left( \frac{q_1}{c_0} \right) z^2, \quad |z| < 1.
\]
So we only need to compute one integral to obtain the value of \( c_0 \). We do not know a closed form for the integral giving \( c_0 \) so one will need to a numerical approximation of \( c_0 \) which is very easy to obtain. Finally the inverse Szegö function \( T(z) \) is:
\[
T(z) = c_0 - c_0 \zeta_1 z + c_0 \zeta_2 z^2 - c_0 \zeta_3 z^3 + \ldots
\]
\[
S(z) = c_0 + \zeta_1 z + \zeta_2 z^2 = c_0 + \left( \frac{c_0 (c_0^2 q_1 - q_2 i q_1)}{c_0^2 - \|q_2\|^2} \right) z + \left( \frac{q_1}{c_0} \right) z^2, \quad |z| < 1.
\]
So we only need to compute one integral to obtain the value of \( c_0 \). We do not know a closed form for the integral giving \( c_0 \) so one will need to a numerical approximation of \( c_0 \) which is very easy to obtain. Finally the inverse Szegö function \( T(z) \) is:
\[
T(z) = c_0 - c_0 \zeta_1 z + c_0 \zeta_2 z^2 - c_0 \zeta_3 z^3 + \ldots
\]
\[
S(z) = c_0 + \zeta_1 z + \zeta_2 z^2 = c_0 + \left( \frac{c_0 (c_0^2 q_1 - q_2 i q_1)}{c_0^2 - \|q_2\|^2} \right) z + \left( \frac{q_1}{c_0} \right) z^2, \quad |z| < 1.
\]
So we only need to compute one integral to obtain the value of \( c_0 \). We do not know a closed form for the integral giving \( c_0 \) so one will need to a numerical approximation of \( c_0 \) which is very easy to obtain. Finally the inverse Szegö function \( T(z) \) is:
\[
T(z) = c_0 - c_0 \zeta_1 z + c_0 \zeta_2 z^2 - c_0 \zeta_3 z^3 + \ldots
\]
\[
S(z) = c_0 + \zeta_1 z + \zeta_2 z^2 = c_0 + \left( \frac{c_0 (c_0^2 q_1 - q_2 i q_1)}{c_0^2 - \|q_2\|^2} \right) z + \left( \frac{q_1}{c_0} \right) z^2, \quad |z| < 1.
\]
\[ S(z) = c_0 + a z + c_2 z^2 + \ldots + c_m z^m, \quad z \in D \]

where \( c_0, c_1, \ldots, c_m \) are complex numbers. Then under that assumption one only needs to compute these coefficients from relations (20) and deduce that

\[ \psi(z) = \frac{1}{S(z)} = \frac{1}{c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m}. \]

For example, consider the matrix \( G = ( \gamma(j - k))_{i,j} \) with

\[
\gamma(k) = \begin{cases} 
1 & \text{if } k = 0 \\
3/10 & \text{if } k = 1 \\
2(1 + i)/10 & \text{if } k = 2 \\
(1 + i)/10 & \text{if } k = 3 \\
3/10 & \text{if } k = -1 \\
2(1 - i)/10 & \text{if } k = -2 \\
(1 - i)/10 & \text{if } k = -3 \\
0 & \text{otherwise}
\end{cases}
\]

Then

\[
\varphi(t) = (1/10)(10 + 3e^{2\pi it} + (2 + 2i)e^{4\pi it} + (1 + i)e^{6\pi it} + 3e^{-2\pi it} + (2 - 2i)e^{-4\pi it} + (1 - i)e^{-6\pi it})
\]

\[= (1/5)(5 + 3 \cos(2\pi t) + 2 \cos(4\pi t) + \cos(6\pi t) - 2 \sin(4\pi t) - \sin(6\pi t))\]

We compute

\[ c_0 = e^i \int_{-1}^{1} \log(\varphi(t))dt = 0.917429 \]

and

\[ a_1 = 0.242589 - 0.0634194i, \quad a_2 = 0.196713 + 0.181643i, \quad a_3 = 0.109 + 0.109i \]

and \( c_n = 0 \) for all \( n \geq 4 \). Then

\[ S(z) = 0.917429 + (0.242589 - 0.0634194i)z + (0.196713 + 0.181643i)z^2 + (0.109 + 0.109i)z^3 \]

and

\[ \psi(z) = (0.917429 + (0.242589 - 0.0634194i)z + (0.196713 + 0.181643i)z^2 + (0.109 + 0.109i)z^3)^{-1}. \]

From this explicit inverse Szegö function \( \psi(z) \) one can now compute every element of the inverse matrix \( G^{-1} \). For example using (15), we have that

\[ (G^{-1})_{10,9} = -0.282433 + 0.183806i. \]

We can also compute the first 10 coefficients \( a_{00}, a_{10}, \ldots, a_{09} \) of the Taylor expansion of \( \psi(z) \) at \( z = 0 \) and using (18) obtain

\[ (G^{-1})_{10,9} = \sum_{k=0}^{8} \pi^2 a_{k+1} = -0.282433 + 0.183806i. \]

The first 5 × 5 bloc of \( G^{-1} \) is:

\[ (G^{-1})_{5 \times 5} = A + iB \]

where

\[
A = \begin{pmatrix}
1.18811 & -0.314162 & -0.177357 & 0.00862465 & 0.0300867 \\
-0.314162 & 1.27685 & -0.286529 & -0.182067 & 0.0098166 \\
-0.177357 & -0.286529 & 1.36869 & -0.279574 & -0.217594 \\
0.00862465 & -0.182067 & -0.279574 & 1.36979 & 0.283269 \\
0.0300867 & 0.0098166 & -0.217594 & -0.283269 & 1.38529
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
0 & 0.0821306 & -0.278669 & -0.0351419 & 0.132323 \\
-0.0821306 & 0 & 0.168077 & -0.269973 & -0.0722108 \\
0.278669 & -0.168077 & 0 & 0.175346 & -0.282669 \\
0.0351419 & 0.269973 & -0.175346 & 0 & 0.177196 \\
-0.132323 & 0.0722108 & 0.282669 & -0.177196 & 0
\end{pmatrix}.
\]

Again this exact value is very close to the first 5 × 5 bloc of the inverse of \( (G)_{m \times m} \) for \( m \) large. In fact \( m = 100 \) yields an approximation error \( \leq 10^{-6} \).
10. Concluding remark

This paper studies the inverse of infinite Toeplitz matrices \( G \) that are hermitian and positive definite. We have obtained an exact formula that expresses the elements of the inverse in terms of the coefficients of the inverse Szegö function associated to the matrix \( G \). We provided explicit calculations throughout in order to clearly show how the method can be used in practice to compute the inverse of an infinite matrix. We applied the results to the covariance matrix of the classical fractional Gaussian noise and this is the first time the inverse of this matrix is explicitly computed. This will be of interest in the study of the inverse problem of fractional Brownian motion. We also considered the particular case of infinite Toeplitz matrices with only a finite number \( 2m + 1 \) of non-zero diagonals. For the case \( m = 1 \) (tridiagonal matrices) we obtained an explicit analytical formula for the inverse. We also considered the case where \( m = 2 \) (pentadiagonal matrices) but here no explicit analytic solution is known. An interesting observation for the inverse of \( 2m + 1 \)-diagonal Toeplitz matrices is that, possibly, the associated Szegö function is a polynomial of the form \( S(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m \). This means the following: If \( q_1, q_2, \ldots, q_m \) are complex numbers such that the function

\[
\varphi(t) = 1 + \sum_{k=1}^{m} e^{2k\pi it} q_k + \sum_{k=1}^{m} e^{-2k\pi it} q'_k
\]

is strictly positive for all \( t \in [0, 1] \), then there exist complex coefficients \( c_1, c_2, \ldots, c_m \) such that

\[
\exp\left( \frac{1}{z} \int_{0}^{1} \left( \frac{e^{2k\pi it} + z}{e^{2k\pi it} - z} \right) \log(\varphi(t)) \, dt \right) = c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m
\]

for all complex numbers \( z \) such that \( |z| < 1 \). This needs to be proven. In that case the inverse of the matrix is obtained by simply determining the Taylor coefficients \( (a_0, a_1, a_2, \ldots) \) of \( (c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m)^{-1} \) and making use of formula (19).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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