A CHARACTERISATION OF COMPACT, FRAGMENTABLE LINEAR ORDERS

R. J. SMITH

ABSTRACT. We give a characterisation of fragmentable, compact linearly order spaces. In particular, we show that if $K$ is a compact, fragmentable, linearly ordered space then $K$ is a Radon-Nikodým compact. In addition, we obtain some corollaries in topology and renorming theory.

1. INTRODUCTION

Throughout this note, all topological spaces are assumed to be Hausdorff.

Definition 1.1. Let $K$ be a compact space.

(1) We say that $K$ is fragmentable if there exists a metric $d : K \times K \rightarrow [0, \infty)$ with the property that given any non-empty set $M \subseteq K$ and $\varepsilon > 0$, there exists an open set $U \subseteq K$ satisfying $M \cap U \neq \emptyset$ and $d\text{-diam} (M \cap U) < \varepsilon$.

(2) We say that $K$ is a Radon-Nikodým compact, or RN compact, if there exists a metric that is lower semicontinuous on $K \times K$ and satisfies the conditions in (1).

Fragmentable and RN compact spaces have been the subject of enduring study. The paper of Namioka [2] contains many of the fundamental results on RN compact spaces. For example, Namioka showed that the definition of RN compacta given above is equivalent to the original definition of RN compact spaces, namely that $K$ is RN compact if it is homeomorphic to a $w^*$-compact subset of an Asplund Banach space.

The most well known unsolved problem in the theory of RN compacta is the question of whether the continuous image of an RN compact space is again RN compact. In [1], it is proved that if $K$ is a linearly ordered compact space and the continuous image of a RN compact, then $K$ is RN compact. This result uses a necessary condition for $K$ to be fragmentable. We say that a compact space $M$ is almost totally disconnected if it is homeomorphic to some $A \subseteq [0, 1]^\Gamma$ in the pointwise topology, with the property that if $f \in A$ then $f(\gamma) \in (0, 1)$ for at most countably many $\gamma \in \Gamma$. The class of almost totally disconnected spaces contains all Corson compact spaces and all totally disconnected spaces.
Theorem 1.2. [1, Theorem 3] Let $K$ be a linearly ordered fragmentable compact. Then $K$ is almost totally disconnected.

In [1], Avilés asks whether a linearly ordered compact space $K$ is RN compact whenever $K$ is fragmentable. In this note, we characterise compact, fragmentable, linearly ordered spaces.

Theorem 1.3. Let $K$ be a compact, linearly ordered space. Then the following are equivalent.

1. $K$ is fragmentable;
2. there is a family $L_n$, $n \in \mathbb{N}$, of compact, scattered subsets of $K$, with union $L$, such that whenever $u, v \in K$ and $u < v$, there exist $x, y \in L$ satisfying $u \leq x < y \leq v$.
3. $K$ is RN compact.

In doing so, we obtain Avilés’s result concerning continuous images.

Corollary 1.4. [1, Corollary 4] Let $K$ be a compact, linearly ordered space that is also a continuous image of a RN compact. Then $K$ is a RN compact.

In fact, Corollary 1.4 is originally stated in [1] in terms of quasi-Radon-Nikodým compact spaces, which we won’t define here. All we need to know is that if $K$ is a continuous image of a RN compact then it is a quasi-RN compact, which in turn implies that $K$ is fragmentable. With this in mind, we will see that the original statement also follows from Theorem 1.3.

The proof of Theorem 1.3, (1) $\Rightarrow$ (2), is the subject of Sections 2 and 3. As a byproduct of this investigation, we obtain more results. We denote the first uncountable cardinal by $\omega_1$.

Proposition 1.5. Let $K$ be a compact, fragmentable, linearly ordered space, and assume that $K$ contains no order-isomorphic copy of $\kappa$, where $\kappa$ is a regular, uncountable cardinal. Then the topological weight of $K$ is strictly less than $\kappa$. In particular, if $K$ contains no copy of $\omega_1$ then $K$ is metrisable.

The next corollary extends a theorem in [6], which states that $C(\omega_1 + 1)$ admits no equivalent norm with a strictly convex dual norm.

Corollary 1.6. Let $K$ be a compact, linearly ordered set, and suppose that $C(K)$ admits an equivalent norm with strictly convex dual norm. Then $K$ is metrisable.

This leads directly to the final result.

Corollary 1.7. Let $K$ be a compact, linearly ordered, Gruenhage space. Then $K$ is metrisable.

It is worth noting that we cannot simply demand that the union $L$ in Theorem 1.3 part (2), is topologically dense in $K$. If we let $K = [0, 1] \times \{0, 1\}$ be the lexicographically ordered ‘split interval’, then $K$ is separable. However, it is well known not to be fragmentable. We provide a proof of this at the beginning of section 2. Alternatively, we can use Proposition 1.5, since it is easy to show that $K$ is not
second countable and does not contain any uncountable well ordered or conversely well ordered subsets. Incidentally, since $K$ is 0-dimensional, this example shows that the necessary condition of Theorem 1.2 is not sufficient. Moreover, it is not possible to deduce Proposition 1.5 from Theorem 1.2.

We conclude this section by proving Theorem 1.3, (2) ⇒ (3). The argument is a straightforward elaboration of Namioka’s proof that the ‘extended long line’ is RN compact; see [2, Example 3.9, (b)] for details. Of course, Theorem 1.3, (3) ⇒ (1), follows immediately from Definition 1.1. We shall use $(x, y)$ to denote both ordered pairs and open intervals. The interpretation of the notation should be clear from the context. We make use of the following characterisation of RN compacta.

**Theorem 1.8** ([2, Corollary 3.8]). A compact Hausdorff space $K$ is RN compact if and only if there exists a norm bounded set $\Gamma \subseteq C(K)$ such that

1. $\Gamma$ separates points of $K$, and
2. for every countable set $A \subseteq \Gamma$, $K$ is separable, relative to the pseudo-metric $d_A$, given by

$$d_A(x, y) = \sup\{|f(x) - f(y)| \mid f \in A\}$$

for $x, y \in K$.

**Proof of Theorem 1.8.** (2) ⇒ (3). Let $L_n$ be as in (2). We shall assume that $\min K, \max K \in L_1$ and $L_n \subseteq L_{n+1}$ for all $n$. Let

$$\Delta_n = \{(x, y) \in L_n^2 \mid x < y \text{ and } (x, y) \cap L_n \text{ is empty}\}.$$ 

For each $(x, y) \in \Delta_n$, take an increasing function $f_{x, y} \in C(K)$ such that $f(w) = 0$ for $w \leq x$ and $f(z) = n^{-1}$ for $z \geq y$. We claim that the family $f_{x, y}, (x, y) \in \Delta_n$, $n \in \mathbb{N}$ separates points of $K$ and, moreover, if $A \subseteq \bigcup_{n=1}^\infty \Delta_n$ is countable, then $K$ is $d_A$-separable, where $d_A$ is the pseudo-metric defined by

$$d_A(u, v) = \sup\{|f_{x, y}(u) - f_{x, y}(v)| \mid (x, y) \in A\}.$$ 

First, let $u, v \in K$, with $u < v$. By the hypothesis and the fact that $L_n \subseteq L_{n+1}$, there exists $n$ and $x, y \in L_n$ such that $u \leq x < y \leq v$. If we take an isolated point $w$ of $[x, y] \cap L_n$ then at least one of the points

$$\inf\{z \in [x, y] \cap L_n \mid w < z\}, \quad \sup\{z \in [x, y] \cap L_n \mid z < w\}$$

is in $[x, y] \cap L_n$ and necessarily not equal to $w$. Therefore, we can find $(x', y') \in \Delta_n$ with $x \leq x' < y' \leq y$. It follows that $f_{x', y'}$ separates $u$ and $v$.

Now let $A \subseteq \bigcup_{n=1}^\infty \Delta_n$ be countable. We set $A_n = A \cap \Delta_n$ and

$$M_n = \bigcup_{(x, y) \in A_n} \{x, y\}.$$ 

Since $M_n$ is compact, separable, scattered and linearly ordered, it is countable. If $(x, y) \in \Delta_n$ then observe that, for every $i \leq n$, there is a unique $(x_i, y_i) \in \Delta_i$ satisfying $x_i \leq x < y \leq y_i$. Define

$$\Gamma_{x, y, n} = \{((q_1, r_1), \ldots, (q_n, r_n)) \in (\mathbb{Q}^2)^n \mid q_k < r_k \text{ and } \bigcap_{i=1}^n f_{x_i, y_i}^{-1}(q_i, r_i) \neq \emptyset\}$$
and, for every \((q, r) = ((q_1, r_1), \ldots, (q_n, r_n)) \in \Gamma_{x,y,n}\), take
\[ z_{x,y,n,q,r} \in L \cap \bigcap_{i=1}^{n} f_{x,y_i}^{-1}(q_i, r_i). \]

We claim that the countable set
\[ D = \{ \min K, \max K \} \cup \bigcup_{n=1}^{\infty} M_n \cup \{ z_{x,y,n,q,r} \mid (x, y) \in A_n, (q, r) \in \Gamma_{x,y,n} \text{ and } n \in \mathbb{N} \} \]
is \(d_A\)-dense in \(K\).

If \(w \in K\) and \(n \in \mathbb{N}\), we find \(z \in D\) such that \(d_A(w, z) < n^{-1}\). We assume \(w \notin D\) and let
\[ M = \{ \min K, \max K \} \cup \bigcup_{k=1}^{n} M_k. \]

If \(w \in (x, y)\) and \((x, y) \in A_k\) for some \(k \leq n\) then let \(k\) be maximal. Otherwise, let \(k = 0\). Since \(M\) is closed and \(\min K, \max K \in M\), we can find \(u, v \in M\), such that \((u, v) \cap M\) is empty and \(w \in (u, v)\). Assume that \(k < j \leq n\) and \((x, y) \in A_j\). We must have \(f_{x,y}(u) = f_{x,y}(w) = f_{x,y}(v)\). Indeed, if \((x, y) \in A_j\) then \(x, y \in M\). Since \((u, v) \cap M\) is empty, either \(x \leq u\) and \(v \leq y\), or \(y \leq u\), or \(v \leq x\). However, the first possibility cannot hold because \(w \notin (x, y)\).

Now let \(w \in (x, y)\), where \((x, y) \in A_k\). For \(i \leq k\), take rationals \(q_i\) and \(r_i\) such that \(q_i < f_{x,y_i}(u) < r_i\) and \(r_i - q_i < n^{-1}\). Let \((x', y') \in A_i\), where \(i \leq k\). If \((x', y') \neq (x_i, y_i)\) then since \(w, z_{x,y,n,q,r} \in (x, y) \subseteq (x_i, y_i)\), we have
\[ f_{x',y'}(w) = f_{x',y'}(z_{x,y,n,q,r}). \]

On the other hand, if \((x', y') = (x_i, y_i)\) then we have ensured that
\[ |f_{x',y'}(w) - f_{x',y'}(z_{x,y,n,q,r})| < n^{-1}. \]

If \(z_{x,y,n,q,r} \leq w\) then set \(z = \max\{u, z_{x,y,n,q,r}\}\), and if \(w < z_{x,y,n,q,r}\) then set \(z = \min\{v, z_{x,y,n,q,r}\}\). By the construction, and the fact that the \(f_{x,y}\) are increasing, we have made sure that \(|f_{x,y}(w) - f_{x,y}(z)| < n^{-1}\) whenever \((x, y) \in \bigcup_{n=1}^{\infty} A_n\). If \((x, y) \in A_m\) and \(n < m\), then \(|f_{x,y}(w) - f_{x,y}(z)| \leq m^{-1} < n^{-1}\). Therefore \(d_A(w, z) < n^{-1}\).

That \(K\) is RN compact now follows from Theorem [1,8].

2. Simple subsets of trees

It is a standard result in elementary analysis that every uncountable set \(H \subseteq \mathbb{R}\) contains an uncountable subset \(E\), with the property that each \(x \in E\) is a two-sided condensation point of \(E\); that is, given \(x \in E\) and \(\varepsilon > 0\), both \((x - \varepsilon, x) \cap E\) and \((x, x + \varepsilon) \cap E\) are uncountable. As explained in [1], the abundance of condensation points in each uncountable subset of \(\mathbb{R}\) can be used to show that the split interval \(K = [0, 1] \times \{0, 1\}\) is not fragmentable. It is worth repeating the argument here. If \(d\) is a metric on \(K\), then there exists an uncountable subset \(H \subseteq [0, 1]\) with the property that \(d((x, 0), (x, 1)) \geq n^{-1}\) for all \(x \in H\). If \(E \subseteq H\) is as above, then for every open subset \(U \subseteq K\) such that \(U \cap (E \times \{0, 1\})\) is non-empty, \(d\)-diam \((U \cap (E \times \{0, 1\})) \geq n^{-1}\).
In this note, we shall consider subsets of more general linear orders which behave similarly to uncountable subsets of $\mathbb{R}$ in this way. In order to do so, we first define and investigate a family of subsets of trees. Linear orders share a long and close relationship with trees, and trees will feature strongly in what follows. For convenience, we lay down some of the basic definitions. A partially ordered set $(T, <)$ is a tree if the set of predecessors of any given element of $T$ is well ordered. If $x, z \in T$, we define the interval $[x, z] = \{ y \in T \mid x < y \leq z \}$. We introduce elements $0$ and $\infty$, not in $T$, with the property that $0 < x < \infty$ for all $x \in T$, and define the intervals $(0, z)$, $[x, \infty)$ and $[x, z]$ in the obvious manner. In general, we say that $s \subseteq T$ is an interval if $y \in s$ whenever $x \leq y \leq z$ and $x, z \in s$. We let $ht(x)$ be the order type of $(0, x)$ and, if $\alpha$ is an ordinal, we let $T_\alpha$ be the level of $T$ of order $\alpha$; that is, the set of all $x \in T$ satisfying $ht(x) = \alpha$. The interval topology of $T$ takes as a basis all sets of the form $(x, z]$, where $x \in T \cup \{0\}$ and $z \in T$. This topology is locally compact and scattered. We shall say that a subset of $T$ is open if it is so with respect to this topology. If $x \in T$, we let $x^+$ be the set of immediate successors of $x$. We say that $y \in T$ is a successor if $y \in x^+$ for some $x$ or, equivalently, $y \in T_{\xi+1}$ for some $\xi$. In this note, we shall only consider trees with the Hausdorff property, i.e., if $A \subseteq T$ is non-empty and totally ordered, then $A$ has at most one minimal upper bound. If $T$ is Hausdorff then the interval topology on $T$ is Hausdorff in the usual, topological sense. If $H$ is a subset of a tree $T$, the map $\pi : H \rightarrow T$ is regressive if $\pi(x) < x$ for all $x \in H$. If $\alpha$ is a limit ordinal with cofinality $\kappa = \text{cf}(\alpha)$, we say that $(\alpha_\xi)_{\xi < \kappa}$ is a cofinal sequence of ordinals if it is strictly increasing and converges to $\alpha$.

**Definition 2.1.** Let $\alpha$ be a limit ordinal. We say that $H \subseteq T_\alpha$ is simple if, for some cofinal sequence $(\alpha_\xi)_{\xi < \kappa}$, there is a injective, regressive map $\pi : H \rightarrow \bigcup_{\xi < \kappa} T_{\alpha_\xi}$.

We will tie trees to compact linear orders in the next section. Before doing so, we explore the properties of simple sets. Firstly, it is worth noting that the choice of cofinal sequence in the definition of simple sets is not important.

**Lemma 2.2.** If $(\alpha_\xi)_{\xi < \kappa}$ and $(\alpha'_\xi)_{\xi < \kappa}$ are cofinal sequences, and $\pi : H \rightarrow \bigcup_{\xi < \kappa} T_{\alpha_\xi}$ is an injective, regressive map, then there exists an injective, regressive map $\pi' : H \rightarrow \bigcup_{\xi < \kappa} T_{\alpha'_\xi}$.

**Proof.** By taking a subsequence of $(\alpha'_\xi)_{\xi < \kappa}$ if necessary, we can assume that $\alpha_\xi \leq \alpha'_\xi$ for all $\xi < \kappa$. If $x \in H$ and $\pi(x) \in T_{\alpha_\xi}$, let $\pi'(x)$ be the unique element of $[\pi(x), x] \cap T_{\alpha'_\xi}$. Now suppose that $\pi'(x) = \pi'(y)$. It follows that $\pi(x)$ and $\pi(y)$ are comparable, and since $\pi'(x)$ and $\pi'(y)$ share the same level, so do $\pi(x)$ and $\pi(y)$. Therefore, $\pi(x) = \pi(y)$, giving $x = y$. 

The next result reveals an important permanence property of simple sets.

**Proposition 2.3.** Let $\alpha$ be a limit ordinal. Suppose that $H \subseteq T_\alpha$, $(\alpha_\xi)_{\xi < \kappa}$ is a cofinal sequence, and $\pi : H \rightarrow \bigcup_{\xi < \kappa} T_{\alpha_\xi}$ is a regressive mapping, with the property that every fibre of $\pi$ is simple. Then $H$ is simple.

**Proof.** For each $w \in \bigcup_{\xi < \kappa} T_{\alpha_\xi}$, let $\pi_w : \pi^{-1}(w) \rightarrow \bigcup_{\xi < \kappa} T_{\alpha_\xi}$ be a regressive, injective map. By Lemma 2.2, we can assume that $\pi(x) \leq \pi_{\pi(x)}(x)$ for all $x \in H$. For
\[ \xi \leq \xi' < \kappa, \text{ define} \]
\[ \Gamma_{\xi, \xi'} = \{ x \in H \mid \pi(x) \in T_\alpha \text{ and } \pi_\alpha(x) \in T_\alpha \} \]
The sets \( \Gamma_{\xi, \xi'}, \xi \leq \xi' < \kappa \), are pairwise disjoint, and for every \( x \in H \), there exist such ordinals with the property that \( x \in \Gamma_{\xi, \xi'} \). Take \( \eta \leq \eta' < \kappa \). We define \( \sigma(x) \) for \( x \in \Gamma_{\eta, \eta'} \) in the following way. First, observe that the set
\[ \bigcup \{ (0, x] \cap [\pi_\alpha(y), y] \mid y \in \Gamma_{\xi, \xi'}, \xi \leq \xi' \leq \eta' \text{ and } (\xi, \xi') \neq (\eta, \eta') \} \]
has an upper bound \( w < x \). Indeed, suppose first that \( \xi \leq \xi' \leq \eta' \), \( (\xi, \xi') \neq (\eta, \eta') \) and \( y, z \in \Gamma_{\xi, \xi'} \) are such that
\[ (0, x] \cap [\pi_\alpha(y), y] \text{ and } (0, x] \cap [\pi_\alpha(z), z] \]
are both non-empty. Then \( \pi(y), \pi(z) < x \), so they are comparable. As they occupy the same level \( T_\alpha \), we have \( \pi(y) = \pi(z) \). Moreover, \( \pi_\alpha(y), \pi_\alpha(z) < x \) and occupy the same level \( T_\alpha \), thus \( y = z \), because \( \pi_\alpha \) is injective. So \( (0, x] \cap [\pi_\alpha(y), y] \) is non-empty for at most one \( y \in \Gamma_{\xi, \xi'} \). Because \( (\xi, \xi') \neq (\eta, \eta') \) and \( T \) is Hausdorff, this intersection has an upper bound \( w_{\xi, \xi'} < x \). Since \( \xi \leq \xi' \leq \eta' < \kappa = \text{cf}(\alpha) \), the required upper bound \( w \) exists. Thus we can take \( \sigma(x) \in \bigcup_{\xi < \kappa} T_\alpha \), satisfying \( w, \pi_\alpha(x) < \sigma(x) \).

Now let \( x, y \in H \). We claim that if \( [\sigma(x), x] \cap [\sigma(y), y] \) is non-empty, then \( x = y \). Indeed, take such \( x \) and \( y \), and let \( \xi \leq \xi' \), \( \eta \leq \eta' \) satisfy \( x \in \Gamma_{\eta, \eta'} \) and \( y \in \Gamma_{\xi, \xi'} \). Without loss of generality, we can assume that \( \xi' < \eta' \). If \( (\xi, \xi') = (\eta, \eta') \) then, as above, \( \pi(x) \) and \( \pi(y) \) are comparable, as are \( \pi_\alpha(x) \) and \( \pi_\alpha(y) \). Since both pairs occupy the same levels respectively, we get \( x = y \). Instead, if \( (\xi, \xi') \neq (\eta, \eta') \) then from the construction above, it follows that
\[ [\sigma(x), x] \cap [\sigma(y), y] \subseteq [\sigma(x), x] \cap [\pi_\alpha(y), y] \]
is empty. Thus we must have \( (\xi, \xi') = (\eta, \eta') \) and \( x = y \).

The next corollary follows immediately from the proof of Proposition \ref{prop:2.3}

**Corollary 2.4.** Let \( H \subseteq T_\alpha \) be simple. Then for every cofinal sequence \( (\alpha_\xi)_{\xi < \kappa} \), there exists a regressive map \( \pi : H \to \bigcup_{\xi < \kappa} T_\alpha \) with the property that \( x = y \) whenever
\[ [\pi(x), x] \cap [\pi(y), y] \]
is non-empty.

Proposition \ref{prop:2.3} yields another straightforward corollary.

**Corollary 2.5.** If \( \alpha \) is a limit ordinal, \( \kappa = \text{cf}(\alpha) \) and \( H_\xi \subseteq T_\alpha \) is simple for all \( \xi < \kappa \), then so is the union \( H = \bigcup_{\xi < \kappa} H_\xi \). In particular, countable unions of simple subsets of \( T_\alpha \) are simple.

**Proof.** Assume the \( H_\xi \) are pairwise disjoint. Let \( (\alpha_\xi)_{\xi < \kappa} \) be a cofinal sequence and define \( \pi : H \to \bigcup_{\xi < \kappa} T_\alpha \) by letting \( \pi(x) \) be the unique element of \( (0, x] \cap T_\alpha \), whenever \( x \in H_\xi \). Then \( \pi^{-1}(w) \subseteq H_\xi \) whenever \( w \in T_\alpha \).
We finish this section with a final result concerning simple subsets. We say that \( W \subseteq T \) is an \textit{initial part} of \( T \) if \( x \in W \) whenever \( x \in T \), \( x \leq y \) and \( y \in W \).

**Proposition 2.6.** Let \( T \) be a tree and suppose that the level \( T_\alpha \) is simple for every limit ordinal \( \alpha \). Then there is a partition of \( T \) consisting entirely of open intervals.

\textit{Proof.} The proof comes in two parts. In the first part, we prove the following claim. Let \( \mathcal{W} \) be a family of initial parts of \( T \), with union \( T \), and such that, given \( V, W \in \mathcal{W} \), either \( V \) is an initial part of \( W \), or vice-versa. Suppose further that each \( W \) has a partition \( P_W \) consisting wholly of open intervals of \( W \), with the property that if \( V \) is an initial segment of \( W \) then, for every \( s \in P_V \), there exists \( t \in P_W \) with \( s \subseteq t \). Then \( T \) has a partition \( P \) consisting of open intervals only.

Define \( \sim \) on \( T \) by declaring that \( x \sim y \) if and only if \( x, y \in s \) for some \( s \in P_W \), \( W \in \mathcal{W} \). This is an equivalence relation. Symmetry is immediate. If \( x \in T \) then \( x \in s \) for some \( s \in P_W \) and \( W \in \mathcal{W} \), so \( \sim \) is reflexive. If \( x \sim y \) and \( y \sim z \) then take \( s \in P_V \), \( t \in P_W \) with \( x, y \in s \) and \( y, z \in t \). Without loss of generality, we may assume that \( V \) is an initial segment of \( W \), and so there is \( u \in P_V \) with \( s \subseteq u \). Since \( y \in t \cap u \), we have \( x, z \in t \), so transitivity holds. If \( P \) is the corresponding partition of \( T \) then it is clear that each \( s \in P \) is an open interval. This completes the first part of the proof.

For the second part, for each initial segment \( W_\alpha = \bigcup_{\xi<\alpha} T_\xi \), we construct a partition \( P_{W_\alpha} \) in such a way that the resulting family satisfies the property above. Assume \( (P_{W_\xi}, \xi<\alpha) \) have been constructed with the property in question. If \( \alpha \) is a limit ordinal then we simply define \( P_{W_\alpha} \) as in part one of the proof. If \( \alpha \) is a successor ordinal \( \eta+1 \) then there are two cases. If \( \eta \) is itself a successor ordinal then all we need to do is set

\[
P_{W_\alpha} = P_{W_\eta} \cup \{ \{x\} \mid x \in T_\eta \}.
\]

Instead, if \( \eta \) is a limit ordinal, we use Corollary 2.4 to obtain a regressive map \( \sigma : T_\eta \to \bigcup_{\xi<\eta} T_{\xi+1} \) with the property that

\[
[\sigma(x), x] \cap [\sigma(y), y] = \emptyset
\]

whenever \( x \neq y \). Notice that each \( \sigma(x) \) is a successor, so \([\sigma(x), x]\) is an open interval. For each \( x \in T_\eta \), let \( s_x \) be the unique element of \( P_{W_\eta} \) containing \( \sigma(x) \). Now define

\[
P_{W_\alpha} = \{ s_x \cup [\sigma(x), x] \mid x \in T_\eta \} \cup \{ s \in P_\alpha \mid s \cap \bigcup_{x \in T_\eta} [\sigma(x), x] = \emptyset \}.
\]

It is straightforward to check that \( P_{W_\alpha} \) has the required property. This completes the induction and the proof. \( \square \)

3. Compact linear orders and partition trees

As mentioned Section 2, linear orders and trees enjoy a close relationship. We will employ the established notion of a partition tree of a linear order. Let \( K \) be a compact, linear order. We let an interval \( a \subseteq K \) be called \textit{trivial} if it contains at most one point.
Definition 3.1. We shall say that a tree \( T \) with level \( T_\alpha \) of order \( \alpha \) is an admissible partition tree of \( K \) if it satisfies the following properties

(1) every \( a \in T \) is a non-trivial, closed interval of \( K \);
(2) \( a \leq b \) if and only if \( b \subseteq a \);
(3) \( T_0 = \{ K \} \);
(4) if \( a \in T \) contains two elements then \( a^+ \) is empty;
(5) if \( a \in T \) contains at least three elements then \( a \) has exactly two immediate successors \( b \) and \( c \), satisfying

\[
\min a = \min b < \max b = \min c < \max c = \max a;
\]
(6) if \( \alpha \) is a limit ordinal then \( a \in T_\alpha \) if and only if there exist \( \alpha_\xi \in T_\xi \), \( \xi < \alpha \), with \( a = \bigcap_{\xi < \alpha} a_\xi \).

The next lemma states some obvious consequences of the definition above.

Lemma 3.2. Let \( T \) be an admissible partition tree of \( K \).

(1) if \( a, b \in T_\alpha \) are distinct then \( a \cap b \) is trivial;
(2) if \( a \cap b \) is non-trivial then \( a, b \in T \) are comparable;
(3) if \( L = \bigcup_{a \in T} \{ \min a, \max a \} \) then given \( u, v \in K \), \( u < v \), there exist \( x, y \in M \) such that \( u \leq x < y \leq v \). In particular,

\[
\{ (x, y) \mid x, y \in L, x < y \}
\]

is a basis for the topology of \( K \);
(4) if \( s \subseteq T \) is totally ordered then \( \{ \min a \mid a \in s \} \) and \( \{ \max a \mid a \in s \} \) are well ordered and conversely well ordered respectively. Moreover, if the cardinality \( \kappa \) of \( s \) is infinite, then either the cardinality of \( \{ \min a \mid a \in s \} \), or that of \( \{ \max a \mid a \in s \} \), is equal to \( \kappa \).

Proof. We prove (1) by induction on \( \alpha \). Suppose that the result holds for all \( \xi < \alpha \). First, assume \( \alpha = \xi + 1 \) and take distinct \( a, b \in T_\alpha \). Let \( a \in a_0^+ \) and \( b \in b_0^+ \) for some \( a_0, b_0 \in T_\xi \). If \( a_0 = b_0 \) then by Definition 3.1 part (5), \( a \cap b \) is trivial. Otherwise, by the inductive hypothesis, \( a \cap b \subseteq a_0 \cap b_0 \) is trivial. Now assume that \( \alpha \) is a limit ordinal, with \( a, b \in T_\alpha \) distinct. Take sequences \( (a_\xi), (b_\xi)_{\xi < \alpha} \) as in Definition 3.1 part (6). If \( a_\xi = b_\xi \) for all \( \xi \) then \( a = b \), thus \( a_\xi \neq b_\xi \) for some \( \xi \). It follows that \( a \cap b \subseteq a_\xi \cap b_\xi \). This completes the proof of (1).

To prove (2), take \( a, b \in T \) with \( a \cap b \) non-trivial. Now let \( a_0 \leq a \) and \( b_0 \leq b \) with \( a_0 \) and \( b_0 \) in the same level, and either \( a_0 = a \) or \( b_0 = b \). Then \( a_0 \cap b_0 \) is non-trivial, so they must be equal. (2) follows.

For (3), let \( u, v \in K \) with \( u < v \). Let \( a \) be the greatest element of \( T \) containing \( u \) and \( v \). The existence of \( a \) follows by compactness. If \( a = \{ u, v \} \) is a two-element set then we are done. Otherwise, by Definition 3.1 part (5), \( a \) has distinct immediate successors \( b \) and \( c \), with \( x = \max b = \min c \). The maximality of \( a \) ensures that \( u < x < v \). Now repeat with \( x \) and \( v \).

(4) If \( s \subseteq T \) is totally ordered then it is well ordered. Put

\[
E = \{ \xi \mid s \cap T_\xi \neq \emptyset \}
\]
and given $\xi \in E$, let $a_\xi$ be the unique element of $s \cap T_\xi$. Define also

$$F = \{\eta \in E \mid \min a_\xi < \min a_\eta \text{ whenever } \xi < \eta\}$$

and

$$G = \{\eta \in E \mid \max a_\xi > \max a_\eta \text{ whenever } \xi < \eta\}.$$

It is clear that $\{\min a \mid a \in s\}$ and $F$ share the same order type, and likewise $\{\max a \mid a \in s\}$ has converse order type equal to that of $G$. Moreover, $E = F \cup G$. Indeed, if there exist $\xi, \xi' < \eta$ such that $\min a_\xi = \min a_\eta$ and $\max a_{\xi'} = \max a_\eta$ then, assuming as we can that $\xi \geq \xi'$, we have $a_\xi = a_\eta$, which is not allowed in an admissible partition tree. The cardinality assertion follows immediately.

By recursion, it is clear that an admissible partition tree exists for every compact linear order $K$ with at least two elements. Moreover, for any such $T$ and any branch $s \subseteq T$, we have that $\bigcap s$ contains at most two elements. From now on, we shall assume that all trees $T$ are admissible partition trees of compact linear orders. Recall the discussion of the split interval at the beginning of Section 2. Given a compact linear order $K$, an admissible partition tree $T$ of $K$, a limit ordinal $\alpha$ and a non-simple subset $H \subseteq T_\alpha$, we show that

$$\bigcup_{a \in H} \{\min a, \max a\}$$

behaves similarly to $[0, 1] \times \{0, 1\}$. This allows us to find a necessary condition for $K$ to be fragmentable. In the next three lemmas, we shall assume that $\alpha$ is a limit ordinal and $(\alpha_\xi)_{\xi < \kappa}$ is a cofinal sequence.

**Lemma 3.3.** Let $H \subseteq T_\alpha$. Then there exists a regressive map $\pi : H \to \bigcup_{\xi < \kappa} T_{\alpha_\xi}$ with the property that, for all $w \in \bigcup_{\xi < \kappa} T_{\alpha_\xi}$, we can find $b \in H$ such that the set

$$\{\min a \mid a \in \pi^{-1}(w)\}$$

is bounded above by $\min b$.

**Proof.** If $\{\min a \mid a \in H\}$ has a maximum element then there is nothing to prove. Now suppose otherwise. Then, given $a \in H$, there exists $b \in H$ with $\max a < \min b$. Since $a$ is a limit element, we can take $\pi(a) < a$, $\pi(a) \in \bigcup_{\xi < \kappa} T_{\alpha_\xi}$, such that $\max \pi(a) < \min b$. Now let $w \in \bigcup_{\xi < \kappa} T_\xi$, with $\pi(a) = w$. By definition, there exists $b \in H$ such that $\max \pi(a) < \min b$. If $a' \in \pi^{-1}(w)$ then

$$\min a' < \max a' \leq \max \pi(a') = \max \pi(a) < \min b.$$

**Lemma 3.4.** Let $H \subseteq T_\alpha$. Define

$L = \{b \in H \mid \text{there is } x_b < \min b \text{ such that } \{a \in H \mid a \subseteq (x_b, \min b]\} \text{ is simple}\}.$

Then $L$ is simple. Similarly, if

$R = \{b \in H \mid \text{there is } x_b > \max b \text{ such that } \{a \in H \mid a \subseteq [\max b, x_b)\} \text{ is simple}\}$

then $R$ is simple.
Proof. Suppose $L$ is not simple. For each $b \in L$, we can find $\pi(b) \in \bigcup_{\xi<\kappa} T_\alpha^{\xi}$ such that $\pi(b) < b$ and $x_b < \min \pi(b)$. Since $\pi$ is regressive and $L$ is not simple, by Proposition 2.3 there exists $w \in \bigcup_{\xi<\kappa} T_\alpha^{\xi}$ such that $E = \pi^{-1}(w)$ is not simple. Observe that whenever $a, b \in E$ satisfy $\min a < \min b$, we have

$$x_b < \min \pi(b) = \min \pi(a) \leq \min a < \max a \leq \min b$$

whence $a \subseteq (x_b, \min b]$. It follows that whenever $b \in E$, the set

$$\{a \in E \mid \min a \leq \min b\}$$

is simple. By Lemma 3.3 there exists a regressive map $\sigma : E \to \bigcup_{\xi<\kappa} T_\alpha^{\xi}$ with the property that whenever $w \in \bigcup_{\xi<\kappa} T_\alpha^{\xi}$, the set

$$\{\min a \mid a \in \sigma^{-1}(w)\}$$

bounded above by $\min b$, for some $b \in E$. Hence $\sigma^{-1}(w)$ is simple for all $w$. Therefore $E$ is simple by Proposition 2.3, which is a contradiction. Consequently, $L$ is simple. It is clear that the ‘right hand’ version of Lemma 3.3 holds, thus $R$ is also simple. □

**Lemma 3.5.** Suppose that $H \subseteq T_\alpha$ is non-simple. Then there exists a subset $C \subseteq H$ with the property that whenever $c \in C$, $x, y \in K$ and $x < \min c < \max c < y$, the sets

$$\{a \in C \mid a \subseteq (x, \min c]\} \quad \text{and} \quad \{a \in C \mid a \subseteq [\max c, y)\}$$

are both non-simple.

**Proof.** If $M \subseteq T_\alpha$ is non-simple, let $C_M = M \setminus (L \cup R)$, where $L$ and $R$ are defined as in Lemma 3.4. We know that $C_M$ is non-simple by Lemma 3.4 and Corollary 2.5. Put $C = C_H$. Let $c \in C$ and $x < \min c$. If we set

$$M = \{a \in H \mid a \subseteq (x, \min c]\}$$

then $M$ is non-simple. We can see that $C_M$, which is non-simple, is a subset of $C \cap M = \{a \in C \mid a \subseteq (x, \min c]\}$. Likewise, if $\max c < y$ then $\{a \in C \mid a \subseteq [\max c, y)\}$ is non-simple. □

This allows us to give a necessity (and sufficient) condition for the fragmentability of $K$ in terms of admissible partition trees.

**Proposition 3.6.** If $K$ is a compact, fragmentable, linearly ordered set and $T$ is any admissible partition tree of $K$, then there is a partition of $T$ consisting entirely of open intervals.

**Proof.** The first thing to show is that if $K$ is fragmentable then $T_\alpha$ is simple for every admissible partition tree $T$ of $K$ and limit ordinal $\alpha$. Assume that $H \subseteq T_\alpha$ is non-simple and let $d$ be a metric on $K$. Set

$$H_n = \{a \in H \mid d(\min a, \max a) \geq n^{-1}\}.$$  

By Corollary 2.5, $G = H_n$ is non-simple for some $n$. Let

$$E = \bigcup_{c \in C} \{\min c, \max c\}$$

This is a partition of $T$ consisting entirely of open intervals. □
where $C = C_G \setminus \{\min K, \max K\}$ and $C_G$ is as in Lemma 3.5. Suppose that $U \cap E$ is non-empty. If $\min c \in U$ for some $c \in C$, then there exists $x < \min c$ with $(x, \min c] \subseteq U$. From Lemma 3.5, we know that $b \subseteq U$ for some $b \in C$, whence $\text{diam}(U \cap E) \geq n^{-1}$. If $\max c \in U$ for some $c \in C$ then we reach the same conclusion. Therefore $d$ does not fragment $K$. Since the metric was arbitrary, we deduce that $K$ is not fragmentable. To finish the proof, use Proposition 2.6. □

This result allows us to complete the proof of Theorem 1.3.

Proof of Theorem 1.3, (1) ⇒ (2). Let $K$ be a compact, fragmentable, linearly ordered space. We show that if $T$ is any admissible partition tree and

\[ L = \bigcup_{a \in T} \{\min a, \max a\} \]

is as in Lemma 3.2, part (3), then $L$ is a countable union of compact, scattered subsets.

Let $T$ be an admissible partition tree. By Proposition 3.6, let $P$ be a partition of $T$ consisting entirely of open intervals. If $a \in T$ then the set

\[ \{ s \in P \mid (0, a] \cap s \text{ is non-empty} \} \]

is non-empty and finite by compactness, and the fact that $P$ is a partition. We define rank $a$ to be the cardinality of this set. Define

\[ L_n = \bigcup_{\text{rank } a \leq n} \{\min a, \max a\} \]

For convenience, we set $L_0 = \{\min K, \max K\}$. We prove by induction on $n \geq 0$ that $L_n$ is closed and scattered. For each $n$, define

\[ \Delta_n = \{(x, y) \in L_n^2 \mid x < y \text{ and } (x, y) \cap L_n \text{ is empty}\} \]

as in the proof of Theorem 1.3 (2) ⇒ (3). Assuming that $L_n$ is closed and scattered, and prove that $L_{n+1}$ shares these properties by showing that

\[ L_{n+1} \setminus L_n = \bigcup_{(x, y) \in \Delta_n} (x, y) \setminus L_n \]

and that each such set $(x, y) \cap L_{n+1}$ is scattered and closed in $(x, y)$. Let $w \in L_{n+1} \setminus L_n$. There exists $b \in T$ with rank $b = n + 1$, such that $w$ is an endpoint of $b$. If $b \in s \in P$ then $(0, b] \setminus s$ is a closed, bounded interval, so has a greatest element $a$, of rank $n$. Let $a$ have immediate successors $c$ and $d$, with

\[ \min a = \min c < \max c = \min d < \max d = \max a. \]

Without loss of generality, we can assume that $c \leq b$. Necessarily, rank $c = n + 1$. There are two cases: rank $d = n$ or rank $d = n + 1$. If rank $d = n$ then let $x = \min a = \min c$ and $y = \max c = \min d$. Since $b \subseteq c$ and $w \notin L_n$, we have $x < w < y$. If $v \in (x, y)$ and $v$ is an endpoint of some $e \in T$, then by Lemma 3.2, part (2), $c$ and $e$ are comparable, and moreover $c \leq e$. Since rank $c = n + 1$, we have $v \notin L_n$. This means that $(x, y) \in \Delta_n$. Moreover, if $v \in L_{n+1}$ then since $c \leq e$ and
rank \( e = n + 1 = \text{rank } c \), we have \( e \in s \). Conversely, if \( v \in (x, y) \) is an endpoint of some \( e \in s \), then \( v \in L_{n+1} \). Therefore

\[
(x, y) \cap L_{n+1} = (x, y) \cap \bigcup_{e \in s} \{\min e, \max e\}.
\]

Since \( s \) is a closed interval, we see from Lemma 3.2 part (1) and Definition 3.1 part (3), that \( (x, y) \cap L_{n+1} \) is scattered and closed in \( (x, y) \). If \( \text{rank } d = n + 1 \) then let \( x = \min a \) and \( y = \max a \). We use a similar argument to show that \( (x, y) \in \Delta_n \) and \( (x, y) \cap L_n \) is scattered and closed in \( (x, y) \).

We finish this section with proofs of Proposition 1.5 and Corollaries 1.4, 1.6 and 1.7.

**Proof of Proposition 1.5.** Let \( T \) be an admissible partition tree of \( K \). Suppose that \( T_\alpha \) is non-empty for all \( \alpha < \kappa \). If \( \alpha < \kappa \) is a limit ordinal, note that, as \( T_\alpha \) is simple by Proposition 3.6, we have \( \text{card } T_\alpha \leq \text{card } \bigcup_{\xi < \alpha} T_\xi \). By a simple transfinite induction, it follows that \( \text{card } T_\alpha < \kappa \) for all \( \alpha < \kappa \). Now, for every limit \( \alpha < \kappa \), choose \( a_\alpha \in T_\alpha \). Since \( T \) splits into a partition \( P \) of open sets, there exists \( \sigma(a_\alpha) < a_\alpha \) with \( \sigma(a_\alpha), a_\alpha \in s_\alpha \in P \). We obtain a regressive map \( \tau : L \rightarrow \kappa \) by setting \( \tau(\alpha) = \text{ht}(\sigma(a_\alpha)) \), where \( L \) is the set of limit ordinals in \( \kappa \). By the pressing down lemma, there is an ordinal \( \xi \) and a stationary set \( E \subseteq L \) such that \( \tau(\alpha) = \xi \) for all \( \alpha \in E \). It is well known in set theory that the union of strictly less than \( \kappa \) non-stationary subsets of \( \kappa \) is again non-stationary. Thus, as \( \text{card } T_\xi < \kappa \), we conclude that there is \( w \in T_\xi \) and a stationary subset \( F \subseteq E \) such that \( \sigma(a_\alpha) = w \) for all \( \alpha \in F \). Being stationary, \( F \) is unbounded in \( \kappa \), and since \( w \in s_\alpha \) for all \( \alpha \in F \), we obtain an interval of length \( \kappa \) in \( T \). Therefore \( K \) contains a copy of \( \kappa \) by Lemma 3.2 part (3). It follows that if \( K \) contains no copy of \( \kappa \), every admissible partition tree \( T \) of \( K \) has height strictly less than \( \kappa \). This, together with the fact that \( \text{card } T_\alpha < \kappa \) for all \( \alpha < \kappa \), means that \( \text{card } T < \kappa \), so the weight of \( K \) is strictly less than \( \kappa \) by Lemma 3.2 part (3).

**Proof of Corollary 1.4.** If \( K \) is the continuous image of a RN compact then it is the continuous image of a fragmentable compact and thus fragmentable by [3, Proposition 2.8]. Hence, by Theorem 1.3, \( K \) is RN compact.

**Proof of Corollary 1.6.** By [4, Theorem 1.1], \( K \) is fragmentable. By [6], \( K \) contains no copy of \( \omega_1 \). Thus \( K \) is metrisable by Proposition 1.5.

**Proof of Corollary 1.7.** By [5, Theorem 7], if \( K \) is Gruenhage then \( C(K) \) admits an equivalent norm with a strictly convex dual norm. Now apply Corollary 1.6.

**References**

1. A. Avilés *Linearly ordered Radon-Nikodým compact spaces*. Toplogy Appl. 154 (2007), 404–409.
2. I. Namioka *Radon-Nikodým compact spaces and fragmentability*. Mathematika 34 (1987), 258–281.
3. N. K. Ribarska *Internal characterization of fragmentable spaces*. Mathematika 34 (1987), 243–257.
4. N. K. Ribarska \textit{The dual of a Gâteaux smooth space is weak star fragmentable.} Proc. Amer. Math. Soc. \textbf{114} (1992), 1003–1008.

5. R. J. Smith, \textit{Gruenhage compacta and strictly convex dual norms.} Forthcoming in J. Math. Anal. App. doi:10.1016/j.jmaa.2008.07.017

6. M. Talagrand \textit{Renormages de quelques }\textit{C}(K) Israel J. Math. \textbf{54} (1986), 327–334.

\textbf{Institute of Mathematics of the AS CR, Žitná 25, CZ - 115 67 Praha 1, Czech Republic}

\textit{E-mail address: smith@math.cas.cz}