Scaling of the Sasamoto-Spohn model in equilibrium*

Milton Jara† Gregorio R. Moreno Flores‡

Abstract

We prove the convergence of the Sasamoto-Spohn model in equilibrium to the energy solution of the stochastic Burgers equation on the whole line. The proof, which relies on the second order Boltzmann-Gibbs principle, follows the approach of [9] and does not use any spectral gap argument.

Keywords: KPZ equation; Burgers equation; Sasamoto-Spohn model.
AMS MSC 2010: Primary 60K35, Secondary 82B20; 60H15.
Submitted to ECP on October 29, 2018, final version accepted on December 20, 2018.

1 Model and results

The goal of this note is to show the convergence of a certain discretization of the stochastic Burgers equation:

\[ \partial_t u = \frac{1}{2} \partial_x^2 u + \partial_x u^2 + \partial_x \mathcal{W}, \tag{1.1} \]

where \( \mathcal{W} \) is a space-time white noise. This equation can be seen as the evolution of the slope of solutions to the KPZ equation [15] which is itself a model of an interface in a disordered environment. The KPZ/Burgers equation has been subject to an extensive body of work in the last years. It appears as the scaling limit of a wide range of particle systems [4, 8], directed polymer models [1, 20] and interacting diffusions [6], and constitutes a central element in a vast family of models known as the KPZ universality class [5, 21].

Due to the nonlinearity, a lot of care has to be taken to obtain a notion of solution for (1.1). There are today several alternatives, for instance, regularity structure [14], paracontrolled distributions [11] and energy solutions [8, 10, 12], which is the approach we will follow.

The discretization we consider corresponds to

\[ du_j = \frac{1}{2} \Delta u_j + \gamma B_j(u) + d\xi_j - d\xi_{j-1}, \tag{1.2} \]

*This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovative programme (grant agreement No 715734)
†Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brasil. Partially supported by CNPq and FAPERJ. E-mail: mjara@impa.br
‡Pontificia Universidad Católica de Chile, Santiago, Chile. Partially supported by Fondecyt grant 1171257 and Núcleo Milenio ‘Modelos Estocásticos de Sistemas Complejos y Desordenados’. E-mail: grmoreno@mat.uc.cl
Scaling of the Sasamoto-Spohn model in equilibrium

where \((\xi_j)_j\) is an i.i.d. family of standard one-dimensional Brownian motions,

\[
\begin{align*}
\Delta u_j &= u_{j+1} + u_{j-1} - 2u_j, \\
B_j(u) &= w_j - w_{j-1} \quad \text{with} \quad w_j = \frac{1}{3}(u_j^2 + u_j u_{j+1} + u_{j+1}^2).
\end{align*}
\]

This model, introduced in [16] (see also [17]) and further studied in [22], is nowadays often referred to as the Sasamoto-Spohn model.

While the discretization of the second derivative and noise are quite straightforward, there are a priori several ways to discretize the nonlinearity in Burgers equation. This particular choice is motivated by two reasons: first, it only involves nearest neighbor sites and, second, it yields the explicit invariant measure \(\mu = \rho \otimes \mathbb{Z}\), where \(d\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\) (see Section 3).

Our result states the convergence of the discrete equations (1.2) to Burgers equation in the sense of energy solutions (see Section 2 for a precise definition).

**Theorem 1.1.** For each \(n \geq 1\), let \(u^n\) be the solution to the system (1.2) for \(\gamma = n^{-1/4}\) and initial law \(\mu\), and let

\[
\mathcal{X}_n^\pi(\varphi) = \frac{1}{n^{1/4}} \sum_j u^n_j(tn) \varphi(\frac{j}{\sqrt{n}}).
\]

The sequence of processes \((\mathcal{X}_n^\pi)_{n \geq 1}\) converges in distribution in \(C([0, T], \mathcal{S}'(\mathbb{R}))\) to the unique energy solution of the Burgers equation.

A similar result was shown in [11] for much more general initial conditions although restricted to the periodic setting.

At the technical level, our approach relies on the techniques of [9] and avoids the use of any spectral gap estimate. The core of the proof consists in deriving certain dynamical estimates among which the so-called second order Boltzmann-Gibbs principle plays a major role. A key ingredient is a certain integration-by-parts satisfied by the model.

The paper is organized as follows: in Section 2, we recall the notion of energy solution from [8]. We show the invariance of the measure \(\mu\) in Section 3. In Section 4, we prove the dynamical estimates. Finally, in Sections 5 and 6, we show, respectively, tightness and convergence to the energy solution. The construction of the dynamics (1.2) is given in the appendix.

Notations: We denote by \(\mathcal{S}(\mathbb{R})\) the space of Schwarz functions on \(\mathbb{R}\). For \(n \geq 1\) and a smooth function \(\varphi\), we define \(\varphi_j^n = \varphi(\frac{j}{\sqrt{n}})\), \(\nabla^n \varphi_j^n = \sqrt{n}(\varphi_{j+1}^n - \varphi_j^n)\) and \(\Delta^n \varphi_j^n = n(\varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi_j^n)\). We also define

\[
\mathcal{E}(\varphi) = \int \varphi^2(x) \, dx, \quad \mathcal{E}_n(\psi) = \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{Z}} \psi_j^2,
\]

respectively, for \(\varphi \in L^2(\mathbb{R})\) and \(\psi \in l^2(\mathbb{Z})\).

## 2 Energy solutions of the Burgers equation

We will introduce the notion of an energy solution for Burgers equation [8]. We start with two definitions:

**Definition 2.1.** We say that a process \(\{u_t : t \in [0, T]\}\) satisfies condition (S) if, for all \(t \in [0, T]\), the \(\mathcal{S}'(\mathbb{R})\)-valued random variable \(u_t\) is a white noise of variance 1.

For a stationary process \(\{u_t : t \in [0, T]\}\), \(0 \leq s < t \leq T\), \(\varphi \in \mathcal{S}(\mathbb{R})\) and \(\varepsilon > 0\), we define

\[
\mathcal{A}_{s,t}^\varepsilon(\varphi) = \int_s^t \int_{\mathbb{R}} u_r(i\varepsilon(x))^2 \partial_x \varphi(x) dx dr
\]
where \( i_\varepsilon(x) = \varepsilon^{-1} 1_{(x, x+\varepsilon]} \).

**Definition 2.2.** Let \( \{u_t : t \in [0, T]\} \) be a process satisfying condition (S). We say that \( \{u_t : t \in [0, T]\} \) satisfies the energy estimate if there exists a constant \( \kappa > 0 \) such that:

**(EC1)** For any \( \varphi \in S(\mathbb{R}) \) and any \( 0 \leq s < t \leq T \),

\[
E \left[ \left| \int_s^t u_r (\partial^2_x \varphi) \, dr \right|^2 \right] \leq \kappa (t-s) \mathcal{E}(\partial_x \varphi)
\]

**(EC2)** For any \( \varphi \in S(\mathbb{R}) \), any \( 0 \leq s < t \leq T \) and any \( 0 < \delta < \epsilon < 1 \),

\[
E \left[ \left| A_{s,t}^\epsilon (\varphi) - A_{s,t}^\delta (\varphi) \right|^2 \right] \leq \kappa (t-s) \epsilon \mathcal{E}(\partial_x \varphi)
\]

We state a theorem proved in [8]:

**Theorem 2.3.** Assume \( \{u_t : t \in [0, T]\} \) satisfies (S) and (EC2). There exists an \( \mathcal{S}'(\mathbb{R}) \)-valued stochastic process \( \{A_t : t \in [0, T]\} \) with continuous paths such that

\[
A_t(\varphi) = \lim_{\epsilon \to 0} A_{0,t}^\epsilon(\varphi).
\]

in \( L^2 \), for any \( t \in [0, T] \) and \( \varphi \in S(\mathbb{R}) \).

We are now ready to formulate the definition of an energy solution:

**Definition 2.4.** We say that \( \{u_t : t \in [0, T]\} \) is a stationary energy solution of the Burgers equation if

- \( \{u_t : t \in [0, T]\} \) satisfies (S), (EC1) and (EC2).
- For all \( \varphi \in S(\mathbb{R}) \), the process

\[
u_t(\varphi) - u_0(\varphi) - \frac{1}{2} \int_0^t u_s (\partial^2_x \varphi) \, ds - A_t(\varphi)
\]

is a martingale with quadratic variation \( t \mathcal{E}(\partial_x \varphi) \), where \( A \) is the process from Theorem 2.3.

Existence of energy solutions was proved in [8]. Uniqueness was proved in [12].

### 3 Generator and invariant measure

The construction of the dynamics given by (1.2) is detailed in Appendix A. We denote by \( \mathcal{C} \) the set of cylindrical functions \( F \) of the form \( F(u) = f(u_{-n}, \ldots, u_0) \), for some \( n \geq 0 \), with \( f \in C^2(\mathbb{R}^{2n+1}) \) with polynomial growth of its partial derivatives up to order 2. The generator of the dynamics (1.2) acts on \( \mathcal{C} \) as

\[
L = \sum_j \left\{ \frac{1}{2} (\partial_{j+1} - \partial_j)^2 - \frac{1}{2} (u_{j+1} - u_j)(\partial_{j+1} - \partial_j) + \gamma B_j(u) \partial_j \right\},
\]

where \( \partial_j = \frac{\partial}{\partial u_j} \). Let us introduce the operators

\[
S = \sum_j \left\{ \frac{1}{2} (\partial_{j+1} - \partial_j)^2 - \frac{1}{2} (u_{j+1} - u_j)(\partial_{j+1} - \partial_j) \right\}, \quad A = \sum_j \gamma B_j(u) \partial_j,
\]

which formally correspond to the symmetric and anti-symmetric parts of \( L \) with respect to \( \mu = \rho \otimes \mathcal{L} \), where \( d\rho(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \). We note that our model satisfies the Gaussian integration-by-parts formula:

\[
\int u_j f \, d\mu = \int \partial_j f \, d\mu.
\]
Scaling of the Sasamoto-Spohn model in equilibrium

which will be heavily used in the sequel.

We will also consider the periodic model \( u^M \) on \( \mathbb{Z} / M \mathbb{Z} \) and denote by \( L_M, S_M \) and \( A_M \) the corresponding generator and its symmetric and anti-symmetric parts respectively. Finally, denote \( \mu_M = \rho^\otimes \mathbb{Z} / M \mathbb{Z} \) and let \( \rho_M \) be its density.

**Lemma 3.1.** The measure \( \mu_M \) is invariant for the periodic dynamics \( u^M \).

**Proof.** The lemma follows from Echeverría’s criterion ([7], Thm 4.9.17) once we show

\[
\int L_M f \, d\mu_M = 0,
\]

for all \( f \in C^2(\mathbb{R}^\mathbb{Z} / M \mathbb{Z}) \) with polynomial growth of its derivatives up to order 2. By standard integration-by-parts,

\[
\int S_M f \, d\mu_M = \int f(u)S_M^\dagger \rho_M(u) \, du_{-M} \cdots du_M,
\]

where

\[
S_M^\dagger = \frac{1}{2} \sum_{j \in \mathbb{Z} / M \mathbb{Z}} \left\{ (\partial_{j+1} - \partial_j)^2 + (u_j - u_{j+1})(\partial_j - \partial_{j+1}) + 2 \right\}.
\]

It is a simple computation to show that \( S_M^\dagger \rho_M \equiv 0 \). It then remains to verify that

\[
\int A_M f \, d\mu_M = \int \sum_{j \in \mathbb{Z} / M \mathbb{Z}} (w_j - w_{j+1}) \partial_j f(u) \rho_M(u) \, du_{-M} \cdots du_M = 0.
\]

But, using standard integration-by-parts once again, we can verify that there exists a degree three polynomial in two variables \( p(\cdot, \cdot) \) such that

\[
\int A_M f \, d\mu_M = \int \sum_{j \in \mathbb{Z} / M \mathbb{Z}} f(u) \left\{ p(u_j, u_{j+1}) - p(u_{j-1}, u_j) \right\} \, d\mu_M.
\]

Finally, Gaussian integration-by-parts yields a degree two polynomial in two variables \( \tilde{p}(\cdot, \cdot) \) such that

\[
\int A_M f \, d\mu_M = \int \sum_{j \in \mathbb{Z} / M \mathbb{Z}} \left\{ \tilde{p}(\partial_j, \partial_{j+1}) - \tilde{p}(\partial_{j-1}, \partial_j) \right\} f(u) \, d\mu_M,
\]

which is telescopic. This ends the proof.

By construction of the infinite volume dynamics and taking the limit \( M \to \infty \), we obtain

**Corollary 1.** The measure \( \mu \) is invariant for the dynamics (1.2).

### 4 The second-order Boltzmann-Gibbs principle

We recall the Kipnis-Varadhan inequality: there exists \( C > 0 \) such that

\[
E \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t F(u(sn)) \, ds \right\|^2 \right] \leq CT\|F(\cdot)\|_{-1,n}^2 ds,
\]

where the \( \| \cdot \|_{-1,n} \)-norm is defined through the variational formula

\[
\|F\|^2_{-1,n} = \sup_{f \in \mathcal{E}} \left\{ 2 \int F(u) f \, d\mu + n \int f L f \, d\mu \right\}.
\]
The proof of this inequality in our context follows from a straightforward modification of the arguments of [12], Corollary 3.5. In our particular model, we have

\[- \int fLfd\mu = \frac{1}{2} \sum_j \int ((\partial_{j+1} - \partial_j)f)^2 d\mu\]

so that the variational formula becomes

\[
||F||^2_{1,n} = \sup_{f \in \mathcal{C}} \left\{ 2 \int F(u)fd\mu - \frac{n}{2} \sum_j \int ((\partial_{j+1} - \partial_j)f)^2 d\mu \right\}.
\]

Denote by \(\tau_j\) the canonical shift \(\tau_j u_i = u_{j+i}\) and let \(\overline{u}^l_j = \frac{1}{l} \sum_{k=1}^l u_{j+k}\).

**Lemma 4.1.** Let \(l \geq 1\) and let \(g\) be a function with zero mean with respect to \(\mu\) which support does not intersect \(\{1, \cdots, l\}\). Let \(g_j(s) = g(\tau_j u(s))\). There exists a constant \(C > 0\) such that

\[
E \left[ \left| \int_0^t ds \sum_j g_j(s)(u_{j+1}(sn) - \overline{u}^l_j(sn))\varphi_j \right|^2 \right] \leq C \frac{tl}{\sqrt{n}} ||g||^2_{L^2(\mu)} E_n(\varphi) \tag{4.2}
\]

**Proof.** Let \(\psi_i = \frac{u_{j+i}}{l}, i = 0, \cdots, l-1\). Then,

\[
u_{j+1} - \overline{u}^l_j = \sum_{i=1}^{l-1} (u_{j+i} - u_{j+i+1})\psi_i.
\]

Hence,

\[
\sum_j \varphi_j g_j(u_{j+1} - \overline{u}^l_j) = \sum_j \varphi_j g_j \sum_{i=0}^{l-1} (u_{j+i} - u_{j+i+1})\psi_i \\
= \sum_k \left( \sum_{i=1}^{l-1} \varphi_{k-i} g_{k-i} \psi_i \right)(u_k - u_{k+1}) \\
= \sum_k F_k(u_k - u_{k+1})
\]

Now, for \(f \in \mathcal{C}\), using integration-by-parts,

\[
2 \int \sum_j \varphi_j g_j(u_{j+1} - \overline{u}^l_j)fd\mu = 2 \int \sum_k F_k(u_k - u_{k+1})fd\mu \\
= 2 \int \sum_k F_k(\partial_k - \partial_{k+1})fd\mu \\
\leq \int \sum_k \left\{ \alpha F_k^2 + \frac{1}{\alpha}((\partial_k - \partial_{k+1})f)^2 \right\} d\mu,
\]

by Young’s inequality. Taking \(\alpha = 2/n\), we find that the above is bounded by

\[
\frac{2}{n} \sum_k \int F_k^2 d\mu + \frac{n}{2} \sum_k \int ((\partial_k - \partial_{k+1})f)^2 d\mu,
\]

which, thanks to the Kipnis-Varadhan inequality, shows that the left-hand-side of (4.2) is bounded by

\[
C \frac{t}{n} \sum_k \int F_k^2 d\mu.
\]
Finally, as $g$ is centered,
\[
\sum_{k} F_k^2 d\mu \leq \sum_{k} \sum_{i=1}^{l-1} \varphi_{k-i}^2 g^2 d\mu \leq l\sqrt{n} \int g^2 d\mu \mathcal{E}_n(\varphi). \]

We now state the second-order Boltzmann-Gibbs principle: let $Q(l, u) = (\overline{u}_l^l)^2 - \frac{1}{l}$.

**Proposition 4.2.** Let $l \geq 1$. There exists a constant $C > 0$ such that
\[
E \left[ \left\| \int_0^t ds \sum_j \left\{ \varphi_j^l(sn)u_j(s) - \tau_j Q(l, u(s)) \right\} \varphi_j \right\|^2 \right] \leq C \frac{l}{\sqrt{n}} \mathcal{E}_n(\varphi)
\]

**Proof.** We use the factorization
\[
u_j u_{j+1} - \tau_j Q(l, u) = \nu_j(u_{j+1} - \overline{u}_j^l) + \overline{u}_j^l(u_j - \overline{u}_j^l) + \frac{1}{l}.
\]
We handle the first term with Lemma 4.1. The second term is treated in the following lemma.

**Lemma 4.3.** Let $l \geq 1$. There exists a constant $C > 0$ such that
\[
E \left[ \left\| \int_0^t ds \sum_j \left\{ \overline{u}_j^l(sn)[u_j(s) - \overline{u}_j^l(sn)] + \frac{1}{l} \right\} \varphi_j \right\|^2 \right] \leq C \frac{l}{\sqrt{n}} \mathcal{E}_n(\varphi)
\]

**Proof.** Let $\psi_i = \frac{l-i}{l}$. Then,
\[
\overline{u}_j^l[u_j - \overline{u}_j^l] = \sum_{i=0}^{l-1} \psi_i(u_{j+i} - u_{j+i+1}) \overline{u}_j^l.
\]
For $f \in \mathcal{C}$, using integration-by-parts,
\[
\int \overline{u}_j^l[u_j - \overline{u}_j^l] f d\mu = \int \sum_{i=0}^{l-1} \psi_i(u_{j+i} - u_{j+i+1}) \overline{u}_j^l f d\mu = \int \left\{ \sum_{i=0}^{l-1} \psi_i \overline{u}_j^l(\partial_{j+i} - \partial_{j+i+1}) f - \frac{1}{l} f \right\} d\mu
\]
\[
The second summand comes from the term $i = 0$. Hence,
\[
2 \int \sum_j \varphi_j \left\{ \overline{u}_j^l[u_j - \overline{u}_j^l] + \frac{1}{l} \right\} f d\mu = 2 \int \sum_j \varphi_j \sum_{i=0}^{l-1} \psi_i \overline{u}_j^l(\partial_{j+i} - \partial_{j+i+1}) f d\mu
\]
\[
By Young’s inequality, this last expression is bounded by
\[
\int \sum_{j} \sum_{i=0}^{l-1} \left\{ \alpha \varphi_j^2(\overline{u}_j^l)^2 + \frac{1}{\alpha} \psi_i^2((\partial_{j+i} - \partial_{j+i+1}) f)^2 \right\} d\mu \leq \alpha l \int \sum_j \varphi_j^2(\overline{u}_j^l)^2 d\mu + \frac{1}{\alpha} \int \sum_j ((\partial_j - \partial_{j+1}) f)^2 d\mu \]
\[
Taking $\alpha = 2l/n$, this is further bounded by
\[
\frac{2l^2}{n} \int (\overline{u}_j^l)^2 d\mu \sum_j \varphi_j^2 + \frac{n}{2} \int \sum_j ((\partial_j - \partial_{j+1}) f)^2 d\mu \leq \frac{l}{\sqrt{n}} \mathcal{E}_n(\varphi) + \frac{n}{2} \int \sum_j ((\partial_j - \partial_{j+1}) f)^2 d\mu
\]
\[
The result then follows from the Kipnis-Varadhan inequality. \qed

ECP 24 (2019), paper 3.  
Page 6/12  
http://www.imstat.org/ecp/
5 Tightness

In the sequel, we let $\varphi \in S$ be a test function. Remember the fluctuation field is given by

$$X_t^n(\varphi) = \frac{1}{n^{1/4}} \sum_j u_j(tn) \varphi^n_j.$$ 

Recalling the definition of the operators $S$ and $A$ from Section 3, the symmetric and anti-symmetric parts of the dynamics are given by

$$dS_t^n(\varphi) = \frac{1}{n^{1/4}} n \sum_j u_j(tn) \Delta^n \varphi^n_j dt = \frac{1}{n^{1/4}} \sum_j u_j(tn) \Delta^n \varphi^n_j dt$$

$$dB_t^n(\varphi) = -\frac{1}{n^{1/4}} n \sum_j w_j(tn) (\varphi^n_{j+1} - \varphi^n_j) dt = \sum_j w_j(tn) \nabla^n \varphi^n_j dt$$

where we used $\gamma = n^{-1/4}$. Then, the martingale part of the dynamics corresponds to

$$M_t^n(\varphi) = X_t^n(\varphi) - X_0^n(\varphi) - S_t^n(\varphi) - B_t^n(\varphi) = n^{1/4} \int_0^t \sum_j (\varphi_j - \varphi_{j+1}) d\xi_j(s)$$

and has quadratic variation

$$\langle M_t^n(\varphi) \rangle_t = n^{1/2} \sum_j (\varphi^n_j - \varphi^n_{j+1})^2 = t \mathcal{E}_n(\nabla^n \varphi^n).$$

We will use Mitoma’s criterion [19]: a sequence $\mathcal{Y}^n$ is tight in $C([0, T], S'(\mathbb{R}))$ if and only if $\mathcal{Y}^n(\varphi)$ is tight in $C([0, T], \mathbb{R})$ for all $\varphi \in S(\mathbb{R})$.

5.1 Martingale term

We recall that $\langle M_t^n(\varphi) \rangle = t \mathcal{E}_n(\nabla^n \varphi^n)$. From the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E} \left[ |M_t^n(\varphi) - M_s^n(\varphi)|^p \right] \leq C |t-s|^{p/2} \mathcal{E}_n(\nabla^n \varphi^n)^{p/2},$$

for all $p \geq 1$. Tightness then follows from Kolmogorov criterion by taking $p$ large enough.

5.2 Symmetric term

Tightness is obtained via a second moment computation and Kolmogorov criterion:

$$\mathbb{E} \left[ |S_t^n(\varphi) - S_s^n(\varphi)|^2 \right] \leq |t-s| \frac{1}{\sqrt{n}} \sum_j \mathbb{E}[u_j^2](\Delta^n \varphi^n_j)^2 = |t-s| \mathcal{E}_n(\Delta^n \varphi^n).$$

5.3 Anti-symmetric term

We study the tightness of the term

$$B_t^n(\varphi) = \int_0^t \sum_j w_j(sn) \nabla^n \varphi^n_j ds$$

$$= \int_0^t \sum_j \frac{1}{3} [u_{j+1}(sn) + u_j(sn) u_{j+1}(sn) + u_j^2(sn)] \nabla^n \varphi^n_j ds.$$
Lemma 5.1. The process

\[ Y_t^n(\varphi) = \int_0^t ds \sum_j \varphi_j \{ (u_j(sn)u_{j+1}(sn) - u_j^2(sn)) + 1 \} \]

goes to zero in the ucp topology.

Proof. Using integration by parts,

\[
\int \sum_j \varphi_j (u_j u_{j+1} - u_j^2) f d\mu = \int \sum_j \varphi_j (u_{j+1} - u_j) u_j f d\mu = \int \sum_j \varphi_j (\partial_{j+1} - \partial_j) (u_j f) d\mu = \int \sum_j \varphi_j \{ u_j (\partial_{j+1} - \partial_j) f - f \}
\]

Hence,

\[
\int \sum_j \varphi_j \{ (u_j u_{j+1} - u_j^2) + 1 \} f d\mu = \int \sum_j \varphi_j u_j (\partial_{j+1} - \partial_j) f d\mu
\]

Using Young’s inequality,

\[
2 \int \sum_j \varphi_j \{ (u_j u_{j+1} - u_j^2) + 1 \} f d\mu \leq \int \sum_j \left\{ \alpha \varphi_j^2 u_j^2 + \frac{1}{\alpha} ((\partial_{j+1} - \partial_j) f)^2 \right\} d\mu \leq \frac{2}{\sqrt{n}} E_n(\varphi) + \frac{n}{2} \sum_j \int ((\partial_{j+1} - \partial_j) f)^2 d\mu,
\]

by taking \( \alpha = 2/n \). Into the Kipnis-Varadhan inequality, this yields

\[
E \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t ds \sum_j \varphi_j \{ (u_j(sn)u_{j+1}(sn) - u_j^2(sn)) + 1 \} \right\|^2 \right] \leq \frac{CT}{\sqrt{n}} E_n(\varphi)
\]

which shows that this process goes to zero in the ucp topology. \( \Box \)

This means we can switch the term \( w_j \) in the anti-symmetric part of the dynamics by \( u_j u_{j+1} \) modulo a vanishing term. Note that, as we apply the previous lemma to a gradient, the constant term \( 1 \) will disappear. We are then left to prove the tightness of

\[
\tilde{B}_t^n(\varphi) = \int_0^t \sum_j u_j(sn)u_{j+1}(sn)\nabla^n \varphi_j^2 ds.
\]

From Proposition 4.2, we have

\[
E \left[ \left\| \tilde{B}_t^n(\varphi) - \int_0^t \sum_j \tau_j Q(l, u(sn))\nabla^n \varphi_j^2 ds \right\|^2 \right] \leq C \frac{t l}{\sqrt{n}} E_n(\nabla^n \varphi^n)
\]

where, here and below, \( C \) denotes a constant which value can change from line to line. On the other hand, a careful \( L^2 \) computation, taking dependencies into account, shows that

\[
E \left[ \left\| \int_0^t \sum_j \tau_j Q(l, u(sn))\nabla^n \varphi_j^2 ds \right\|^2 \right] \leq C \frac{l^2 \sqrt{n}}{l} E_n(\nabla^n \varphi^n).
\]
Observe that \(\lim_{n \to \infty} \mathcal{E}_n(\nabla^n \varphi^n) = \int \partial_x \varphi(x)^2 \, dx < \infty\). Summarizing,

\[
E \left[ \left| \tilde{B}_t^n(\varphi) \right|^2 \right] \leq Ct^3/2.
\]

For \(t \geq 1/n\), we take \(l \sim \sqrt{tn}\) and get

\[
E \left[ \left| \tilde{B}_t^n(\varphi) \right|^2 \right] \leq Ct^3/2.
\]

For \(t \leq 1/n\), a crude \(L^2\) bound gives

\[
E \left[ \left| \tilde{B}_t^n(\varphi) \right|^2 \right] \leq Ct^2 \sqrt{n} \leq Ct^3/2.
\]

This gives tightness.

6 Convergence

From the previous section, we get processes \(X, S, B\) and \(M\) such that

\[
\lim_{n \to \infty} X_n = X, \quad \lim_{n \to \infty} S_n = S, \quad \lim_{n \to \infty} B_n = B, \quad \lim_{n \to \infty} M_n = M,
\]

along a subsequence that we still denote by \(n\). We will now identify these limiting processes.

6.1 Convergence at fixed times

A straightforward adaptation of the arguments in [6], Section 4.1.1, shows that \(X_t^n\) converges to a white noise for each fixed time \(t \in [0, T]\). This in turns proves that the limit satisfies property (S).

6.2 Martingale term

The quadratic variation of the martingale part satisfies

\[
\lim_{n \to \infty} \langle M^n(\varphi) \rangle_t = t ||\partial_x \varphi||_{L^2}^2.
\]

By a criterion of Aldous [2], this implies convergence to the white noise.

6.3 Symmetric term

A second moment bound shows that

\[
E \left[ \left| S_t^n(\varphi) - \int_0^t X_s^n(\partial_x^2 \varphi) \, ds \right|^2 \right] \leq C t^2 / n,
\]

which shows that

\[
S(\varphi) = \lim_{n \to \infty} S^n(\varphi) = \int_0^\cdot X_s(\partial_x^2 \varphi) \, ds.
\]

6.4 Anti-symmetric term

We just have to identify the limit of the process \(\tilde{B}_t^n(\varphi)\). Remembering the definition of the field \(X^n\), we observe that

\[
\sqrt{n}Q(\varepsilon \sqrt{n}, u(0)) = X_t^n(i_\varepsilon(0))^2 - \frac{1}{\varepsilon}.
\]
Scaling of the Sasamoto-Spohn model in equilibrium

from where we get the convergences

$$\lim_{n \to \infty} \sqrt{n}Q(\varepsilon \sqrt{n}, u(nt)) = A_t(i_{\varepsilon}(0))^2 - \frac{1}{\varepsilon}$$

and

$$A_{s,t}^\varepsilon(\varphi) := \lim_{n \to \infty} \int_s^t \sum_j \tau_j Q(\varepsilon \sqrt{n}, u(rn)) \nabla^n \varphi_j^r dr.$$ 

The second limit follows by a suitable approximation of $i_{\varepsilon}(x)$ by $S(R)$ functions (see [8], Section 5.3 for details). Now, by the second-order Boltzmann-Gibbs principle and stationarity,

$$E \left[ \left\| \int_0^t \nabla^n \varphi_j^r \partial_x^2 \varphi \right\|^2 \right] \leq \kappa t. \quad (6.1)$$

Taking $l \sim \varepsilon \sqrt{n}$ and the limit as $n \to \infty$ along the subsequence,

$$E \left[ \left\| B_t(\varphi) - B_s(\varphi) - A_{s,t}^\varepsilon(\varphi) \right\|^2 \right] \leq C(t - s)\varepsilon. \quad (6.1)$$

The energy estimate (EC2) then follows by the triangle inequality. Theorem 2.3 yields the existence of the process

$$A_t(\varphi) = \lim_{\varepsilon \to 0} A_{0,t}^\varepsilon(\varphi).$$

Furthermore, from (6.1), we deduce that $B = A$.

It remains to check (EC1). It is enough to check that

$$E \left[ \left\| \int_0^t \nabla^n \varphi_j^r \partial_x \varphi \right\|^2 \right] \leq \kappa t. \quad (6.2)$$

Using the smoothness of $\varphi$ and a summation by parts, it is further enough to verify that

$$E \left[ \left\| \int_0^t n^{1/4} \sum_j [u_{j+1}(sn) - u_j(sn)] \nabla^n \varphi_j^r \right\|^2 \right] \leq \kappa t. \quad (6.2)$$

For that purpose, we will use Kipnis-Varadhan inequality one last time: let $f \in \mathcal{C}$,

$$2 \int n^{1/4} \sum_j (u_{j+1} - u_j) \nabla^n \varphi_j^r f d\mu = 2 \int n^{1/4} \sum_j \nabla^n \varphi_j^r (\partial_{j+1} - \partial_j) f d\mu \leq \sum_j \left\{ \alpha \sqrt{n} (\nabla^n \varphi_j^r)^2 + \frac{1}{\alpha} \int ((\partial_{j+1} - \partial_j) f)^2 d\mu \right\} \leq 2\varepsilon_n (\nabla^n \varphi^n) + \frac{n}{2} \sum_j \int ((\partial_{j+1} - \partial_j) f)^2 d\mu,$$

with $\alpha = 2/n$, from where (6.2) follows.
A Construction of the dynamics

The system of equations (1.2) can be reformulated as

\[ u_j(t) = \frac{1}{2} \int_0^t \Delta u_j(s) \, ds + \gamma \int_0^t B_j(u(s)) \, ds + \xi_j(t) - \xi_{j-1}(t). \]

We consider the system \( u^M \) on \( \mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z} \) evolving under its invariant distribution. We first check that, for all \( j \) and \( T > 0 \)

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |u^M_j(t)|^2 \right] < \infty, \]

so that the dynamics is well-defined. Everything boils down to estimates of type

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t u^M_j(s) \, ds \right|^2 \right] \leq T \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_0^t |u^M_j(s)|^2 \, ds \right] \]

\[ \leq T \mathbb{E} \left[ \int_0^T |u^M_j(s)|^2 \, ds \right] \leq T^2, \]

where we used invariance in the last step.

Next, we show tightness of the processes (in \( M \)) where we now identify \( u^M \) with a periodic system on the line. This follows from Kolmogorov’s criterion. It is enough to control expressions of type

\[ \mathbb{E} \left[ \left| \int_s^t u^M_j(r) \, dr \right|^4 \right] \leq |t-s|^3 \mathbb{E} \left[ \int_s^t |u^M_j(r)|^4 \, dr \right] \leq C |t-s|^3. \]

Together with a standard estimate on the increments of the Brownian motion, this yields

\[ \mathbb{E} \left[ |u^M_j(t) - u^M_j(s)|^2 \right] \leq C |t-s|^2. \]

Hence, each coordinate is tight. By diagonalization, we can extract a subsequence of \( M_k \) such that \( (u^M_{j_k}) \) converges in law in \( C[0,T] \) for each \( j \). This gives a meaning to the system (1.2).

References

[1] Alberts, T., Khanin, K. and Quastel, J. (2014) The intermediate disorder regime for directed polymers in dimension 1 + 1, Ann. Probab. 42, 1212–1256 MR-3189070
[2] Aldous, D. (1981) Weak convergence and the general theory of processes, Unpublished notes
[3] Amir, G., Corwin, I. and Quastel, J. (2010) Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions, Comm. Pure. Appl. Math. 64, (4), 466–537 MR-2796514
[4] Bertini, L. and Giacomin, G. (1997) Stochastic Burgers and KPZ equations from particle systems, Comm. Math. Phys. 183, (3), 571–607 MR-1462228
[5] Corwin, I. (2012) The Kardar-Parisi-Zhang equation and universality class, Random Matrices Theory Appl. 1, 1130001 MR-2930377
[6] Diehl, J., Gubinelli, M. and Perkowski, N. (2016) The Kardar-Parisi-Zhang equation as scaling limit of weakly asymmetric interacting Brownian motions, Comm. Math. Phys. 354, no. 2, 549–589 MR-3663617
[7] Ethier, S., Kurtz, T. (2009) Markov processes: characterization and convergence, vol. 282, Wiley, Hoboken MR-0838085

ECP 24 (2019), paper 3. http://www.imstat.org/ecp/
Scaling of the Sasamoto-Spohn model in equilibrium

[8] Goncalves, P. and Jara, M. (2014) Nonlinear fluctuations of weakly asymmetric interacting particle systems, Arch. Ration. Mech. Anal. 212, no. 2, 597–644 MR-3176353

[9] Goncalves, P., Jara, M. and Simon, M. (2017) Second order Boltzmann-Gibbs principle for polynomial functions and applications, J. Stat. Phys. 166, no. 1, 90–113 MR-3592852

[10] Gubinelli, M. and Jara, M. (2013) Regularization by noise and stochastic Burgers equations, Stoch. Partial Differ. Equ. Anal. Comput. 1, no. 2, 325–350 MR-3327509

[11] Gubinelli, M. and Perkowski, N. (2017) KPZ reloaded Comm. Math. Phys. 349, no. 1, 165–269 MR-3592748

[12] Gubinelli, M. and Perkowski, N. (2018) Energy solutions of KPZ are unique, J. Amer. Math. Soc. 31, 427–471 MR-3758149

[13] Gubinelli, M. and Perkowski, N. (2018) Probabilistic approach to the stochastic Burgers equation, In: Eberle A., Grothaus M., Hoh W., Kassmann M., Stannat W., Trutnau G. (eds) Stochastic Partial Differential Equations and Related Fields. SPDERF 2016. Springer Proceedings in Mathematics & Statistics, vol 229. Springer, Cham MR-3828193

[14] Hairer, M. (2013) Solving the KPZ equation, Annals of Mathematics 178, 559–664 MR-3071506

[15] Kardar, M., Parisi, G and Zhang, Y-C. (1986) Dynamic scaling of growing interfaces Phys. Rev. Lett., 56(9):889–892

[16] Krug, J. and Spohn, H. (1991) Kinetic roughening of growing surfaces, In: Godreéche, C. (ed.) Solids Far from Equilibrium, pp. 412–525. Cambridge University Press, Cambridge MR-1163829

[17] Lam, C.-H. and Shin, F.G. (1998) Improved discretization of the Kardar-Parisi-Zhang equation, Phys. Rev. E 58, 5592–5595

[18] Liggett, T. (2005) Interacting Particle Systems, Classics in Mathematics, Springer-Verlag Berlin Heidelberg MR-2108619

[19] Mitoma, I. (1983) Tightness of probabilities in $C([0, 1], Y')$ and $D([0, 1], Y')$, Ann. of Probability, 11, 4, 989–999 MR-0714961

[20] Moreno Flores, G., Quastel, J. and Remenik, D., in preparation

[21] Quastel, J. (2012) Introduction to KPZ, Curr. Dev. Math. 2011, 125–194, Int. Press, Somerville, MA MR-3098078

[22] Sasamoto, T. and Spohn, H. (2009) Superdiffusivity of the 1D Lattice Kardar-Parisi-Zhang Equation, J. Stat. Phys., 137: 917–935 MR-2570756