A General Maximum Principle for Stochastic Systems with Delay

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Abstract: In this paper, we consider optimal control problems derived by stochastic systems with delay, where control domains are non-convex and the diffusion coefficients depend on control variables. By an estimate of the integral of $x_1(t)x_1(t - \delta)$ term, we obtain a general maximum principle for the optimal control problems with a standard spike variational technique and duality method. The maximum principle is applied to study a delayed linear-quadratic optimal control problem with a non-convex control domain; an optimal solution is obtained.

Keywords: General maximum principle, Stochastic differential equation with delay, Anticipated backward stochastic differential equations, Linear-quadratic optimal control.

Mathematics Subject Classification: 93E20, 60H10, 34K50

1 Introduction

It is well known that maximum principle, namely, necessary condition for optimality, is an important approach in solving optimal control problems. In 1956, Boltyanski, Gamkrelidze and Pontryagin [3] announced the Pontryagin’s maximum principle for deterministic control systems. They introduced the spike variation and studied the first-order term in a sort of Taylor expansion with respect to this perturbation. But for stochastic control systems, if the diffusion terms depend on the controls, then one can’t follow this idea. Peng [11] first introduced the second-order term in the Taylor expansion of the variation and obtained the global maximum principle for the classical stochastic optimal control problem. Since then, many researchers investigate this kind of optimal control problems for various stochastic systems (see e.g. the references [8], [15], [14], [2] and [7]).
The stochastic optimal control problems with delay have received a lot of attention recently. Some examples can be found in \[5\]. One of the reasons is that many real-world systems evolve according to not only their current state but also essentially their previous history. Although there exist lots of works about maximum principle for SDDEs control system (see e.g. the references \[4\], \[10\], \[16\], \[1\] and \[13\]), yet the related general case, i.e., the control domain is not necessarily convex and the diffusion coefficient explicitly depends on some control variables, is far from being solved and worth further study. Notice that without delay, the equation for the square of the variation of the state has itself the structure of a linear equation, with other terms that can be handled. In the case with delay, the square of the variation does not solve an equation written in a closed form since a cross term \(x_1(t)x_1(t - \delta)\) appears. From the author’s viewpoint, the main difficulty in solving this problem is how to deal with this "extra" term. We mention the arxiv preprint arXiv:1805.07957v2, by Guatteri and Masiero \[6\], where the authors themselves declare there is a mistake in the calculation of the second-order adjoint equation that they are not able to fix. Zhang \[17\] introduced a five-coupled backward stochastic differential equations system as second-order adjoint equations to deal with the cross terms in stochastic systems with pointwise and average time delays. However, the solution of the five-coupled backward stochastic differential equations system is required to satisfy that \((P_4(t), Q_4(t)) = (P_5(t), Q_5(t)) = 0, t \in [0, T]\). This is a strong requirement, and we don’t know whether and when the equations system has such a solution. Similar to \[17\], Meng and Shi \[9\] introduced a coupled backward stochastic differential equation and backward random differential equation \(K(t)\) as second-order adjoint equations to deal with the cross terms in stochastic systems with pointwise time delay. Also, \(K(t)\) is required to satisfy that \(K(t) = 0, t \in [0, T]\). So a certain new idea is expected to be found.

The rest of the paper is organized as follows. In Section 2, the problem is formulated. In Section 3, along the line of Peng \[11\], we give the estimates of the first- and second-order expansions of the perturbed state variable \(x^\varepsilon(t)\), then obtain the variational inequality. Section 4 gives an explicit estimate of the integral of \(x_1(t)x_1(t - \delta)\) term. By means of the estimate, we transform the integral of \(x_1(t)x_1(t - \delta)\) term into the integral of \(x_1(t)x_1(t)\) term and establishes a general maximum principle. Section 5 studies an LQ case with a non-convex control domain. The general maximum principle is applied to find a optimal control. Finally, we end this paper with some concluding remarks.

2 Formulation of the optimal control problem

Let \(\mathbb{R}\) be the 1-dimensional Euclidean space and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space equipped with a natural filtration

\[
\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\},
\]
where \( \{ B_s \}_{s \geq 0} \) is a standard Brownian motion valued in \( \mathbb{R} \). Let \( T > 0 \) be the finite time duration and \( 0 < \delta \leq T \) be constant time delay. We define \( \mathcal{F}_t \equiv \mathcal{F}_0 \) for all \( t \in [-\delta, 0] \), \( \mathcal{F} = \mathcal{F}_{T+\delta} \), and \( \mathbb{E} \) denotes the expectation on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). For any Euclidean spaces or sets of matrices \( S \) and a real number \( p \in [1, \infty) \), the following notations will be used throughout the paper:

- \( C([-\delta, 0]; S) \), the space of \( S \)-valued continuous functions;
- \( L^p(\Omega, \mathcal{F}, \mathbb{P}; S) \), the space of square integrable \( S \)-valued \( \mathcal{F}_t \)-measurable random variable \( \xi \) such that \( \mathbb{E}[|\xi|^p] < +\infty \);
- \( L^p_c(t_1, t_2; S) \), the space of \( S \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( \{x(s), t_1 \leq s \leq t_2\} \) such that \( \mathbb{E} \int^t_{t_1} |x(t)|^p \, dt < +\infty \).
- \( S^0_c(t_1, t_2; S) \), the space of \( S \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( \{x(s), t_1 \leq s \leq t_2\} \) such that which have right-continuous paths with left limits such that \( \mathbb{E} \sup_{s \in [t_1, t_2]} |x(s)|^p < +\infty \).

Now consider the following stochastic differential equation with delay:

\[
\begin{cases}
    dx(t) = b(t, x(t), x(t-\delta), v(t), v(t-\delta)) \, dt + \sigma(t, x(t), x(t-\delta), v(t), v(t-\delta)) \, dB_t, \ t \in [0, T], \\
    x(t) = \xi_t, \ v(t) = \eta_t, \ t \in [-\delta, 0],
\end{cases}
\]

where \( b : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \times U \rightarrow \mathbb{R} \), \( \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \times U \rightarrow \mathbb{R} \). \( \xi_t \in C([-\delta, 0]; \mathbb{R}) \) and \( \eta_t \in C([-\delta, 0]; U) \) are the initial path of \( x \) and \( v \), respectively. An admissible control \( v(\cdot) \) is an \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted process with value in \( U \) such that

\[
\mathbb{E} \sup_{t \in [0, T]} |v(t)|^m < \infty, \quad \forall m = 1, 2, \cdots,
\]

where \( U \) is a nonempty subset of \( \mathbb{R} \), which is not necessary convex. We denote the set of all admissible controls by \( U_{ad} \). For any \( v(\cdot) \in U_{ad} \), the cost functional is in the form of

\[
J(v(\cdot)) = \mathbb{E} \left[ \int_0^T L(t, x(t), x(t-\delta), v(t), v(t-\delta)) \, dt + h(x(T)) \right],
\]

where \( L : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \times U \rightarrow \mathbb{R} \) and \( h : \mathbb{R} \rightarrow \mathbb{R} \). The optimal control problem under consideration in this paper is to find an admissible \( u(\cdot) \) such that

\[
J(u(\cdot)) = \inf_{v(\cdot) \in U_{ad}} J(v(\cdot)).
\]

We need the following Assumption (H1):

(\textbf{H1.1}) for any \( x, x_\delta \in \mathbb{R} \), \( v, v_\delta \in U \), \( b(\cdot, x, x_\delta, v, v_\delta), \sigma(\cdot, x, x_\delta, v, v_\delta) \) and \( L(\cdot, x, x_\delta, v, v_\delta) \) are \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted; for any \( x \in \mathbb{R} \), the mapping \( h(x) \) is \( \mathcal{F}_T \)-measurable;

(\textbf{H1.2}) \( b, \sigma \) and \( L \) are twice continuously differentiable with respect to \( x \) and \( x_\delta \); They and all their derivatives \( l_x, l_{x_\delta}, l_{xx}, l_{x_\delta x_\delta} \) and \( l_{x_\delta x_\delta} \) of \( l \) are uniformly bounded, where \( l = b, \sigma, L \); \( h \) is twice continuously differentiable with respect to \( x \) and the derivatives \( h_x \) and \( h_{xx} \) are bounded.

(\textbf{H1.3}) \( \sigma^{-1}_{x_\delta} \) is uniformly bounded with respect to \( (t, \omega) \).

Now, if Assumption (H1) holds, equation (2.1) admits a unique solution and the cost functional (2.2) is well defined for any admissible control \( v \in U_{ad} \).
3 Variational inequality

Suppose that \((x(\cdot), u(\cdot))\) is the solution to the optimal control problem (2.1)-(2.3). We introduce the spike variation as follows:

\[
u^\varepsilon(t) = \begin{cases} v(t), & \text{if } t \in [\tau, \tau + \varepsilon], \\ u(t), & \text{otherwise}, \end{cases}
\]

(3.1)

where \(0 \leq \tau < T\) is fixed, \(\varepsilon > 0\) is sufficiently small such that \([\tau, \tau + \varepsilon] \subset [0, T]\), and \(v\) is an arbitrary \(F^\tau\)-measurable random variable with values in \(U\), such that

\[
\sup_{\omega \in \Omega} |v(\omega)| < +\infty.
\]

Suppose that \(x^\varepsilon(\cdot)\) is the trajectory of control system (2.1) corresponding to the control \(u^\varepsilon(\cdot)\). We would like to derive the variational inequality from the fact that

\[
J(u^\varepsilon(\cdot)) - J(u(\cdot)) \geq 0.
\]

Firstly, we introduce the following first-order and second-order variational equations:

\[
\begin{aligned}
&dx_1(t) = \left[ b_x(\Theta(t))x_1(t) + b_{xx}(\Theta(t))x_1(t - \delta) + (b(\Theta^\varepsilon(t)) - b(\Theta(t)))x_1(t) \\
&+ [\sigma_x(\Theta(t))x_1(t) + \sigma_{xx}(\Theta(t))x_1(t - \delta) + (\sigma(\Theta^\varepsilon(t)) - \sigma(\Theta(t)))x_1(t)] dB_t, \ t \in [0, T],
\end{aligned}
\]

(3.2)

\[
\begin{aligned}
&dx_2(t) = \left[ b_x(\Theta(t))x_2(t) + b_{xx}(\Theta(t))x_2(t - \delta) + (b(\Theta^\varepsilon(t)) - b(\Theta(t)))x_1(t) \\
&+ (b_{xx}(\Theta(t)) - b_{xx}(\Theta(t)))x_1(t - \delta) + \frac{1}{2}b_{xxx}(\Theta(t))(x_1(t))^2 \\
&+ [\sigma_x(\Theta(t))x_2(t) + \sigma_{xx}(\Theta(t))x_2(t - \delta) + (\sigma(\Theta^\varepsilon(t)) - \sigma(\Theta(t)))x_1(t)] dB_t, \ t \in [0, T],
\end{aligned}
\]

(3.3)

where the notations \((\Theta(t)) := (t, x(t), x(t - \delta), u(t), u(t - \delta))\) and \((\Theta^\varepsilon(t)) := (t, x(t), x(t - \delta), u^\varepsilon(t), u^\varepsilon(t - \delta))\). It is easy to know that (3.2) and (3.3) admit unique adapted solutions \(x_1(t)\) and \(x_2(t)\), respectively. We want to give the estimates of the variational state processes \(x_1(t)\) and \(x_2(t)\).

Lemma 3.1 If Assumption (H1) holds, then

\[
\begin{aligned}
&\sup_{0 \leq t \leq T} \mathbb{E}|x_1(t)|^2 \leq C\varepsilon, \\
&\sup_{0 \leq t \leq T} \mathbb{E}|x_2(t)|^2 \leq C\varepsilon^2,
\end{aligned}
\]

(3.4)

\[
\sup_{0 \leq t \leq T} \mathbb{E}|x^\varepsilon(t) - x(t) - x_1(t) - x_2(t)|^2 = o(\varepsilon^2).
\]
**Proof.** (I) The first estimation \( \sup_{0 \leq t \leq T} \mathbb{E}|x_1(t)|^2 \leq C \varepsilon^2 \):

Applying Itô’s formula to \( |x_1(t)|^2 \), we have

\[
\sup_{0 \leq t \leq T} \mathbb{E}|x_1(t)|^2 \leq C \left[ \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_1(s)|^2 \, ds + \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_1(s - \delta)|^2 \, ds 
\right. \\
\left. + \mathbb{E} \int_{T+\delta}^{T+\varepsilon} \left| b(\Theta^\varepsilon(t)) - b(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) \right|^2 \, ds 
\right. \\
\left. + \mathbb{E} \int_T^{T+\delta + \varepsilon} \left| b(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) - b(\Theta(t)) \right|^2 \, ds 
\right. \\
\left. + \mathbb{E} \int_T^{T+\delta + \varepsilon} \left| \sigma(\Theta^\varepsilon(t)) - \sigma(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) \right|^2 \, ds 
\right. \\
\left. + \mathbb{E} \int_T^{T+\varepsilon} \left| \sigma(\Theta^\varepsilon(t)) - \sigma(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) - \sigma(\Theta(t)) \right|^2 \, ds \right].
\]

Moreover, we can get that

\[
\mathbb{E} \int_0^t |x_1(s - \delta)|^p \, ds \leq \mathbb{E} \int_0^t |x_1(s)|^p \, ds,
\]

for any \( p \geq 0 \). Then the estimation follows from Assumption (H1) and Gronwall’s inequality.

(II) The second estimation \( \sup_{0 \leq t \leq T} \mathbb{E}|x_2(t)|^2 \leq C \varepsilon^2 \):

Applying Itô’s formula to \( |x_2(t)|^2 \), we have

\[
\sup_{0 \leq t \leq T} \mathbb{E}|x_2(t)|^2 \\
\leq C \left[ \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_2(s)|^2 \, ds + \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_2(s) - x_2(s - \delta)|^2 \, ds 
\right. \\
\left. + \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_2(s) - x_2(s - \delta)|^4 \, ds + \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t |x_2(s)|^4 \, ds 
\right. \\
\left. + \sup_{0 \leq t \leq T} \mathbb{E} \int_{T+\delta}^{T+\varepsilon} \left| b_x(\Theta^\varepsilon(t)) - b_x(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) \right|^2 (x_1(t))^2 \, ds 
\right. \\
\left. + \sup_{0 \leq t \leq T} \mathbb{E} \int_T^{T+\delta + \varepsilon} \left| b_x(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) - b_x(\Theta(t)) \right|^2 (x_1(t))^2 \, ds 
\right. \\
\left. + \sup_{0 \leq t \leq T} \mathbb{E} \int_T^{T+\delta + \varepsilon} \left| \sigma_x(\Theta^\varepsilon(t)) - \sigma_x(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) \right|^2 (x_1(t))^2 \, ds 
\right. \\
\left. + \sup_{0 \leq t \leq T} \mathbb{E} \int_T^{T+\varepsilon} \left| \sigma_x(t, x(t), x(t - \delta), u^\varepsilon(t), u(t - \delta)) - \sigma_x(\Theta(t)) \right|^2 (x_1(t))^2 \, ds \right]
\]

Then the estimation can be obtained by the Gronwall’s inequality.
(III) The last estimation \( \sup_{0 \leq t \leq T} E|x^e(t) - x(t) - x_1(t) - x_2(t)|^2 = o(\varepsilon^2) \):

Set \( x_3 = x_1 + x_2 \). For simplification, we use the following notations in this paper:

\[ x_{1\delta}(t) = x_1(t - \delta), x_{2\delta}(t) = x_2(t - \delta), x_{3\delta}(t) = x_3(t - \delta). \]

According to Taylor expansion, it yields (for simplification we omit the time subscript in some places)

\[
\begin{align*}
&\int_0^t b(x + x_3, x_\delta + x_{3\delta}, u^e, u_\delta^S) \, ds + \int_0^t \sigma(x + x_3, x_\delta + x_{3\delta}, u^e, u_\delta^S) \, dB_s \\
= &\int_0^t \left[ b(x, x_\delta, u^e, u_\delta^S) + b_x(x, x_\delta, u^e, u_\delta^S)x_3 + b_{xx}(x, x_\delta, u^e, u_\delta^S)x_{3\delta} + \frac{1}{2} b_{xx\delta}(x, x_\delta, u^e, u_\delta^S)(x_{3\delta})^2 \right] \, ds \\
&+ \int_0^t \left[ \sigma(x, x_\delta, u^e, u_\delta^S) + \sigma_x(x, x_\delta, u^e, u_\delta^S)x_3 + \frac{1}{2} \sigma_{xx}(x, x_\delta, u^e, u_\delta^S)(x_{3\delta})^2 \right] \, dB_s \\
= &\int_0^t \left[ b(x, x_\delta, u, u_\delta) + b_x(x, x_\delta, u, u_\delta)x_3 + b_{xx}(x, x_\delta, u, u_\delta)x_{3\delta} + (b(x, x_\delta, u^e, u_\delta^S) - b(x, x_\delta, u, u_\delta)) \right] \, ds \\
&+ \frac{1}{2} b_{xx}(x, x_\delta, u, u_\delta)(x_{3\delta})^2 + \frac{1}{2} b_{xx\delta}(x, x_\delta, u, u_\delta)(x_{3\delta})^2 \\
&+ \int_0^t \left[ \sigma(x, x_\delta, u, u_\delta) + \sigma_x(x, x_\delta, u, u_\delta)x_3 + \frac{1}{2} \sigma_{xx}(x, x_\delta, u, u_\delta)(x_{3\delta})^2 \right] \, dB_s \\
&= x(t) + x_3(t) - \xi_0 + \int_0^t G^e(s) \, ds + \int_0^t \Lambda^e(s) \, dB_s \\
&= x(t) + x_3(t) - \xi_0 + \int_0^t G^e(s) \, ds + \int_0^t \Lambda^e(s) \, dB_s,
\end{align*}
\]

(3.7)

where

\[
\begin{align*}
G^e(s) &= [b_x(x, x_\delta, u^e, u_\delta^S) - b_x(x, x_\delta, u, u_\delta)]x_3 + [b_{xx}(x, x_\delta, u^e, u_\delta^S) - b_{xx}(x, x_\delta, u, u_\delta)]x_{3\delta} \\
&+ \frac{1}{2} b_{xx}(x, x_\delta, u, u_\delta)(x_{3\delta})^2 + 2x_1 x_2 + \frac{1}{2} b_{xx\delta}(x, x_\delta, u, u_\delta)(x_{3\delta})^2 + 2x_1 x_{1\delta} x_{2\delta} \\
&+ b_{xx\delta}(x, x_\delta, u, u_\delta)(x_{2\delta})^2 + x_1 x_{2\delta} + x_2 x_{1\delta} \\
&+ \frac{1}{2} \sigma_x(x, x_\delta, u^e, u_\delta^S) x_3 + \frac{1}{2} \sigma_{xx}(x, x_\delta, u^e, u_\delta^S)(x_{3\delta})^2 \\
&+ \int_0^t \left[ \sigma_x(x, x_\delta, u^e, u_\delta^S) + \frac{1}{2} \sigma_{xx}(x, x_\delta, u^e, u_\delta^S)(x_{3\delta})^2 \right] \, dB_s \\
&= x(t) + x_3(t) - \xi_0 + \int_0^t G^e(s) \, ds + \int_0^t \Lambda^e(s) \, dB_s,
\end{align*}
\]

(3.8)
\[
\Lambda_{\varepsilon}(s) = |\sigma_x(x, x_\delta, u^\varepsilon, u_\delta^\varepsilon) - \sigma_x(x, x_\delta, u, u_\delta)|x_2 + |\sigma_{xx}(x, x_\delta, u^\varepsilon, u_\delta^\varepsilon) - \sigma_{xx}(x, x_\delta, u, u_\delta)||x_2\delta \\
+ \frac{1}{2}\sigma_{xx}(x, x_\delta, u, u_\delta)(x_2)^2 + 2x_1x_2 + \frac{1}{2}\sigma_{xx}(x, x_\delta, u, u_\delta)(x_2\delta)^2 + x_1x_\delta \\
+ L_1 \int_0^1 \int_L^1 \lambda [\sigma_{xx}(x, x_\delta, u^\varepsilon, u_\delta^\varepsilon) - \sigma_{xx}(x, x_\delta, u, u_\delta)] \, d\lambda \, d\mu(x_3)^2 \\
+ L_1 \int_0^1 \int_L^1 \lambda [\sigma_{xx}(x, x_\delta, u^\varepsilon, u_\delta^\varepsilon) - \sigma_{xx}(x, x_\delta, u, u_\delta)] \, d\lambda \, d\mu(x_3\delta)^2 \\
+ L_1 \int_0^1 \int_L^1 \sigma_{xx}(x, x_\delta, u^\varepsilon, u_\delta^\varepsilon) - \sigma_{xx}(x, x_\delta, u, u_\delta) \, d\lambda \, d\mu(x_3\delta).
\]

From the first and second estimations, we can check that

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \int_0^t G^\varepsilon(s) \, ds \right]^2 + \int_0^t \Lambda^\varepsilon(s) \, dB_s^\varepsilon \right) = o(\varepsilon^2),
\]

where \(o(\varepsilon^2)\) denotes an element such that \(\lim_{\varepsilon \to 0} \frac{o(\varepsilon^2)}{\varepsilon^2} = 0\). Since

\[
x^\varepsilon = \xi_0 + \int_0^t b(x^\varepsilon, x_\delta^\varepsilon, u^\varepsilon, u_\delta^\varepsilon) \, ds + \int_0^t \sigma(x^\varepsilon, x_\delta^\varepsilon, u^\varepsilon, u_\delta^\varepsilon) \, dB_s,
\]

we can derive

\[
(x^\varepsilon - x - x_3)(t) = \int_0^t A^\varepsilon(s)(x^\varepsilon - x - x_3)(s) + A_\delta^\varepsilon(s)(x_\delta^\varepsilon - x_\delta - x_3\delta)(s) \, ds \\
+ \int_0^t D^\varepsilon(s)x^\varepsilon - x - x_3\delta)(s) + D_\delta^\varepsilon(s)(x_\delta^\varepsilon - x_\delta - x_3\delta)(s) \, dB_s \\
+ \int_0^t G^\varepsilon(s) \, ds + \int_0^t \Lambda^\varepsilon(s) \, dB_s,
\]

where

\[
|A^\varepsilon(s)| + |A_\delta^\varepsilon(s)| + |D^\varepsilon(s)| + |D_\delta^\varepsilon(s)| \leq C, \quad \forall (s, \omega) \in [0, T] \times \Omega.
\]

From the above relation and (3.10), we can use Itô’s formula and Gronwall’s inequality to obtain the estimation. The proof is completed.

Now we can present the following variational inequality.

**Lemma 3.2** (variational inequality) Under Assumption (H1), we have

\[
\mathbb{E} \int_0^T \left[ L_x(x, x_\delta, u, u_\delta)(x_1 + x_2) + L_{xx}(x, x_\delta, u, u_\delta)(x_1 + x_2) \right] \, dt \\
+ \frac{1}{2} L_{xx}(x, x_\delta, u, u_\delta)(x_1 x_2) + \frac{1}{2} L_{xx}(x, x_\delta, u, u_\delta)(x_1 x_2) \\
+ \mathbb{E} \left[ h_x(x(T))(x_1(T) + x_2(T)) + \frac{1}{2} h_{xx}(x(T))(x_1(T))^2 \right] \geq o(\varepsilon).
\]

**Proof.** Since \(x(\cdot)\) is optimal, we have

\[
\mathbb{E} \int_0^T L(x^\varepsilon, x_\delta^\varepsilon, u^\varepsilon, u_\delta^\varepsilon) \, ds + \mathbb{E} h(x(T)) - \mathbb{E} \int_0^T L(x, x_\delta, u, u_\delta) \, ds - \mathbb{E} h(x(T)) \geq 0.
\]
It follows from Lemma 3.1,

\[
\begin{align*}
\mathbb{E} \int_0^T L(x^ε, x_δ^ε, u^ε, u_δ^ε) \, ds &+ \mathbb{E} h(x^ε(T)) - \mathbb{E} \int_0^T L(x, x_δ, u, u_δ) \, ds - \mathbb{E} h(x(T)) \\
= \mathbb{E} \int_0^T L(x + x_1 + x_2, x_δ + x_1δ + x_2δ, u^ε, u_δ^ε) - L(x, x_δ, u, u_δ) \, dt \\
+ \mathbb{E}[h((x + x_1 + x_2)(T)) - h(x(T))] + o(ε) \\
= \mathbb{E} \int_0^T L(x + x_1 + x_2, x_δ + x_1δ + x_2δ, u, u_δ) - L(x, x_δ, u, u_δ) \, dt \\
+ \mathbb{E}[h((x + x_1 + x_2)(T)) - h(x(T))] + o(ε) \\
= \mathbb{E} \int_0^T \left[ L_x(x, x_δ, u, u_δ)(x_1 + x_2) + L_{xx}(x, x_δ, u, u_δ)(x_1δ + x_2δ) + \frac{1}{2} L_{xx}(x, x_δ, u, u_δ)(x_1 + x_2)^2 \\
+ L_{xx}(x, x_δ, u, u_δ)(x_1 + x_2)(x_1δ + x_2δ) + \frac{1}{2} L_{xx}(x, x_δ, u, u_δ)(x_1δ + x_2δ)^2 \\
+ L(x, x_δ, u^ε, u_δ^ε) - L(x, x_δ, u, u_δ) + (L_x(x, x_δ, u^ε, u_δ^ε) - L_x(x, x_δ, u, u_δ))(x_1 + x_2) \\
-(L_{xx}(x, x_δ, u^ε, u_δ^ε) - L_{xx}(x, x_δ, u, u_δ))(x_1δ + x_2δ) \right] \, dt \\
+ \mathbb{E}[h_x(x(T))(x_1(T) + x_2(T)) + \frac{1}{2} h_{xx}(x(T))(x_1(T))^2] + o(ε) \\
= \mathbb{E} \int_0^T \left[ L_x(x, x_δ, u, u_δ)(x_1 + x_2) + L_{xx}(x, x_δ, u, u_δ)(x_1δ + x_2δ) \\
+ \frac{1}{2} L_{xx}(x, x_δ, u, u_δ)(x_1)^2 + L_{xx}(x, x_δ, u, u_δ)x_1x_1δ \\
+ \frac{1}{2} L_{xx}(x, x_δ, u, u_δ)(x_1δ)^2 + L(x, x_δ, u^ε, u_δ^ε) - L(x, x_δ, u, u_δ) \right] \, ds \\
+ \mathbb{E}[h_x(x(T))(x_1(T) + x_2(T)) + \frac{1}{2} h_{xx}(x(T))(x_1(T))^2] + o(ε)
\end{align*}
\]

(3.15)

Our proof is completed.

4 Adjoint Equation and Maximum Principle

To establish a maximum principle, we introduce a Hamilton function \( H \) by

\[
H(Θ(t), p(t), q(t)) = L(Θ(t)) + p(t)b(Θ(t)) + q(t)σ(Θ(t)),
\]

(4.1)

where \((p(·), q(·))\) satisfies the following first-order adjoint equation

\[
\begin{align*}
-dp(t) &= [b_x(Θ(t))p(t) + σ_x(Θ(t))q(t) + L_x(Θ(t))] + \mathbb{E} \mathcal{F}_t [b_x(Θ(t + δ))p(t + δ) \\
&+ σ_x(Θ(t + δ))q(t + δ) + L_x(Θ(t + δ))] \, dt - q(t)dB_t, \ t \in [0, T], \\
np(T) &= h_x(x(T)), \ p(0) = 0, \ t \in [T, T + δ], \\
q(t) &= 0, \ t \in [T, T + δ].
\end{align*}
\]

(4.2)

We can see that equation (4.13) is a linear anticipated backward stochastic differential equation and it admits a unique adapted solution \((p(·), q(·)) \in S^2_P(0, T; \mathbb{R}) × L^2_P(0, T; \mathbb{R})\) (see [12]).
Next we introduce the following second-order adjoint equation,
\begin{equation}
\begin{cases}
dP(t) = \left[ -2b_x(\Theta(t))P(t) - \sigma_x^2(\Theta(t))P(t) - 2\sigma_x(\Theta(t))Q(t) - \frac{1}{2}H_{xx}(\Theta(t), p(t), q(t)) \\
-\mathbb{E}[\sigma_x^2(\Theta(t))P(t + \delta) + \frac{1}{2}H_{xx}(\Theta(t + \delta), p(t + \delta), q(t + \delta))] \\
-\left(\frac{b_x(\Theta(t))}{\sigma_x(\Theta(t))} - \sigma_x(\Theta(t))) + (2P(t) - 2\sigma_x(\Theta(t)) + 2Q(t) \\
+ \frac{H_{xx}(\Theta(t), p(t), q(t))}{\sigma_x(\Theta(t))}\right) dt + Q(t) dB_t, \ t \in [0, T] \\
P(T) = \frac{1}{2}h_{xx}(x(T)), \ P(t) = 0, \ t \in (T, T + \delta], \\
Q(t) = 0, \ t \in [T, T + \delta],
\end{cases}
\end{equation}

which also admits a unique solution \((P(\cdot), Q(\cdot))\) (see [12]).

Because variational inequality (3.13) contains process \(L_{xx}x_1x_\delta\), which will bring us more difficulties to deal with, especially when we use the duality technique to derive the maximum principle since \(x_1\delta\) has an expression in terms of \( B(\cdot - \delta)\), and it does not admit an Itô stochastic process with respect to Brownian motion \(B(\cdot)\). It is essentially necessary to look for more explicit estimate of \(x_1(\cdot)x_1\delta(\cdot)\), and we have the following lemma.

**Lemma 4.1** Let \((\Phi(t))_{t \in [0, T]}\) be \( \{F_t\}_{t \geq 0}\)-adapted and bounded, then we have
\begin{equation}
\mathbb{E} \int_0^T \sigma_x^2(\Phi x_1x_\delta) dt = \mathbb{E} \int_0^T \left( \frac{b_x}{\sigma_x^2(\Phi x)} - \sigma_x \right) \Phi x_\delta^2 dt + o(\varepsilon). \tag{4.4}
\end{equation}

**Proof.** We introduce a process \(\varphi(t)\) such that
\begin{equation}
\varphi(t) = -\int_t^T P_0^{-1}\Phi x_1 dB_s, \tag{4.5}
\end{equation}
where \(P_0(\cdot)\) satisfies
\begin{equation}
\begin{cases}
dP_0(t) = -(b_x - \frac{b_x}{\sigma_x^2(\Phi)})P_0(t)dt - \frac{b_x}{\sigma_x^2(\Phi)}P_0(t)dB_t, \\
P_0(0) = 1,
\end{cases} \tag{4.6}
\end{equation}
and it is obvious that \(P_0^{-1}(\cdot)\) exists and \(P_0(\cdot), P_0^{-1}(\cdot) \in L^2_T(0, T; \mathbb{R})\). Note that \(\varphi(T) = 0\) and \(x_1(0) = 0\). Using Itô’s formula to \(\varphi(t)x_1(t)\), we have
\begin{equation}
d\varphi_1 = \left[ \varphi b_x + \varphi \Phi x_1 x_\delta + \varphi(b(\Theta^\varepsilon) - b(\Theta)) + P_0^{-1}\Phi x_1 x_\delta + P_0^{-1}\Phi \sigma_x x_1 x_\delta + P_0^{-1}\Phi x_1(\sigma(\Theta^\varepsilon) - \sigma(\Theta)) + [P_0^{-1}\Phi x_1^2 + \varphi \sigma_x x_1 + \varphi \sigma_x x_1 x_\delta + \varphi(\sigma(\Theta^\varepsilon) - \sigma(\Theta))] dB_t. \tag{4.7}
\end{equation}

Applying Itô’s formula to \(P_0(t)\varphi(t)x_1(t)\) and taking expectation, we get
\begin{equation}
\mathbb{E} \int_0^T \left( \sigma_x^2(\Phi x_1x_\delta) + \sigma_x^2(\Phi x_1x_\delta) + P_0\varphi(b(\Theta^\varepsilon) - b(\Theta)) - \frac{b_x}{\sigma_x^2(\Phi)}P_0\varphi(\sigma(\Theta^\varepsilon) - \sigma(\Theta)) \right) dt + o(\varepsilon) = 0. \tag{4.8}
\end{equation}

Note that
\begin{equation}
\mathbb{E} \int_0^T P_0\varphi(b(\Theta^\varepsilon) - b(\Theta)) dt \\
= -\mathbb{E} \int_0^T P_0(\int_t^T P_0^{-1}\Phi x_1 dB_s)(b(\Theta^\varepsilon) - b(\Theta)) dt \\
\leq \mathbb{E} \left[ \sup_{0 \leq \varepsilon \leq T} ||\int_t^T P_0(\Phi x_1 dB_s)|| \int_0^T P_0(b(\Theta^\varepsilon) - b(\Theta)) dt \right] \tag{4.9}
\end{equation}

\begin{equation}
= o(\varepsilon),
\end{equation}

\begin{equation}
\begin{cases}
d\varphi_1 = \left[ \varphi b_x + \varphi \Phi x_1 x_\delta + \varphi(b(\Theta^\varepsilon) - b(\Theta)) + P_0^{-1}\Phi x_1 x_\delta + P_0^{-1}\Phi \sigma_x x_1 x_\delta + P_0^{-1}\Phi x_1(\sigma(\Theta^\varepsilon) - \sigma(\Theta)) + [P_0^{-1}\Phi x_1^2 + \varphi \sigma_x x_1 + \varphi \sigma_x x_1 x_\delta + \varphi(\sigma(\Theta^\varepsilon) - \sigma(\Theta))] dB_t. \tag{4.7}
\end{equation}

Applying Itô’s formula to \(P_0(t)\varphi(t)x_1(t)\) and taking expectation, we get
\begin{equation}
\mathbb{E} \int_0^T \sigma_x^2(\Phi x_1x_\delta) + \sigma_x^2(\Phi x_1x_\delta) + P_0\varphi(b(\Theta^\varepsilon) - b(\Theta)) - \frac{b_x}{\sigma_x^2(\Phi)}P_0\varphi(\sigma(\Theta^\varepsilon) - \sigma(\Theta)) dt + o(\varepsilon) = 0. \tag{4.8}
\end{equation}

Note that
\begin{equation}
\mathbb{E} \int_0^T P_0\varphi(b(\Theta^\varepsilon) - b(\Theta)) dt \\
= -\mathbb{E} \int_0^T P_0(\int_t^T P_0^{-1}\Phi x_1 dB_s)(b(\Theta^\varepsilon) - b(\Theta)) dt \\
\leq \mathbb{E} \left[ \sup_{0 \leq \varepsilon \leq T} ||\int_t^T P_0(\Phi x_1 dB_s)|| \int_0^T P_0(b(\Theta^\varepsilon) - b(\Theta)) dt \right] \tag{4.9}
\end{equation}

\begin{equation}
= o(\varepsilon),
\end{equation}
This gives the maximum principle immediately.

Therefore, from (4.12), (4.13) and Lemma 4.1, we get

\[ \mathbb{E} \int_0^T \frac{b_{x\delta}}{\sigma_{x\delta}} P_0\varphi(\sigma(\Theta^\delta) - \sigma(\Theta))dt = o(\varepsilon). \] (4.10)

Then by (4.20), we obtain (4.4).

Now we give the main result of this paper.

**Theorem 4.1** Suppose (H1) holds, \( u(-) \) is an optimal control of the control problem (2.1)-(2.3), then we have

\[ H(\tau,x(\tau), x(\tau - \delta), v, v_\delta, p(\tau), q(\tau)) - H(\tau,x(\tau), x(\tau - \delta), u(\tau), u(\tau - \delta), p(\tau), q(\tau)) \]
\[ + P(\tau) [\sigma(\tau,x(\tau), x(\tau - \delta), v, v_\delta) - \sigma(\tau,x(\tau), x(\tau - \delta), u(\tau), u(\tau - \delta))]^2 \]
\[ \geq 0, \forall v \in U, \text{ a.e., a.s.} \]

**Proof.** Applying Itô’s formula to \( P(t)(x_1(t) + x_2(t)) \), taking expectation and by Lemma 3.2, we have

\[ \mathbb{E} \left[ \int_0^T \left\{ H(\Theta^\varepsilon(t),p(t),q(t)) - \nabla_{x,v}^2 H(\Theta(t),p(t),q(t)) \right. \]
\[ \left. + \frac{1}{2} \frac{b_{x\delta}}{\sigma_{x\delta}} P_0 \varphi(\sigma(\Theta^\varepsilon(t)) - \sigma(\Theta(t))) \right\} dt + \frac{1}{2} h_{xx}(x_1(T))(x_1(T))^2 \right] = o(\varepsilon). \] (4.12)

Applying Itô’s formula to \( P(t)(x_1(t))^2 \) and taking expectation, we get

\[ \mathbb{E} \left[ h_{xx}(x(T))(x_1(T))^2 \right] \]
\[ = \mathbb{E} \left[ \int_0^T \left\{ -\frac{1}{2} H_{xx}(\Theta(t),p(t),q(t))(x_1(t))^2 \right. \]
\[ - \left. \frac{1}{2} \frac{b_{x\delta}}{\sigma_{x\delta}} P_0 \varphi(\sigma(\Theta^\varepsilon(t)) - \sigma(\Theta(t))) \right\} dt \right] + o(\varepsilon). \] (4.13)

Therefore, from (4.12), (4.13) and Lemma 4.1, we get

\[ \mathbb{E} \left[ \int_0^T \left\{ H(\Theta^\varepsilon(t),p(t),q(t)) - H(\Theta(t),p(t),q(t)) + P(t)[\sigma(\Theta^\varepsilon(t)) - \sigma(\Theta(t))]^2 \right\} dt \right] \geq o(\varepsilon). \] (4.14)

This gives the maximum principle immediately.

Next, we will study two special cases of the general maximum principle.

**Case I:** In this case, \( \sigma \) does not contain \( v \) and \( v_\delta \), i.e., (2.1) is reduced to the following stochastic system:

\[ \begin{cases}
  dx(t) = b(t,x(t),x(t - \delta),v(t),v(t - \delta)) dt + \sigma(t,x(t),x(t - \delta)) dB_t, t \in [0,T], \\
  x(t) = \xi_t, \ v(t) = \eta_t, \ t \in [-\delta,0].
\end{cases} \] (4.15)

In this case, the second-order adjoint equation (4.3) is not necessary, and Theorem 4.1 is reduced to the following result.
**Proposition 4.1** Suppose \( u(\cdot) \) is an optimal control subject to (4.15), (2.2) and (2.3), then we have

\[
H(\tau, x(\tau), x(\tau - \delta), v, v_\delta, p(\tau), q(\tau)) - H(\tau, x(\tau), x(\tau - \delta), u(\tau), u(\tau - \delta), p(\tau), q(\tau))
\geq 0, \forall v \in U, a.e., a.s.,
\]

where

\[
H(t, x, x_\delta, v, v_\delta, p, q) = L(t, x, x_\delta, v, v_\delta) + pb(t, x, x_\delta, v, v_\delta) + q\sigma(t, x, x_\delta),
\]

and \((p(\cdot), q(\cdot))\) satisfies

\[
\begin{cases}
-\frac{dp}{dt} = [b_x(\Theta(t))p + \sigma_x(t, x, x_\delta(t))q + L_x(\Theta(t)) + \mathbb{E}^F[b_{x_\delta}(\Theta(t + \delta))p(t + \delta)]
+ \sigma_{x_\delta}(t + \delta, x(t + \delta), x(t))q(t + \delta) + L_{x_\delta}(\Theta(t + \delta)))] dt - q(t) dB_t, \quad t \in [0, T],
\end{cases}
\]

\[
p(T) = h_x(x(T)), \quad p(t) = 0, \quad t \in [T, T + \delta],
\]

and \(q(t) = 0, \quad t \in [T, T + \delta].\) \hspace{1cm} (4.18)

**Case II:** Consider the case with delay in just control variable, i.e., \( b(t, x, x_\delta, v, v_\delta) = b(t, x, v, v_\delta), \quad \sigma(t, x, x_\delta, v, v_\delta) = \sigma(t, x, v, v_\delta) \) and \( L(t, x, x_\delta, v, v_\delta) = L(t, x, v, v_\delta). \) That is, (2.1) is reduced to the following stochastic system:

\[
\begin{cases}
dx(t) = b(t, x(t), v(t), v(t - \delta)) dt + \sigma(t, x(t), v(t), v(t - \delta)) dB_t, \quad t \in [0, T],
x(0) = \xi_0, \quad v(t) = \eta_t, \quad t \in [-\delta, 0].
\end{cases}
\]

and the cost functional is in the following form

\[
J(v(\cdot)) = \mathbb{E} \left[ \int_0^T L(t, x(t), v(t), v(t - \delta)) dt + h(x(T)) \right].
\]

Now we have the following result.

**Proposition 4.2** Suppose \( u(\cdot) \) is an optimal control subject to (4.19), (4.20) and (2.3), then we have

\[
H(\tau, x(\tau), v, v_\delta, p(\tau), q(\tau)) - H(\tau, x(\tau), v, v_\delta, \sigma(\tau, x(\tau), v, v_\delta), u(\tau), u(\tau - \delta), p(\tau), q(\tau))
\]

\[+ P(\tau) [\sigma(\tau, x(\tau), v, v_\delta) - \sigma(\tau, x(\tau), u(\tau), u(\tau - \delta))]^2 \geq 0, \forall v \in U, a.e., a.s.,
\]

where

\[
H(t, x, v, v_\delta, p, q) = L(t, x, v, v_\delta) + pb(t, x, v, v_\delta) + q\sigma(t, x, v, v_\delta),
\]

\((p(\cdot), q(\cdot))\) satisfies

\[
\begin{cases}
-\frac{dp}{dt} = [b_x(t, x, v, v_\delta)p + \sigma_x(t, x, v, v_\delta)q + L_x(t, x, v, v_\delta)] dt - q dB_t, \quad t \in [0, T],
p(T) = h_x(x(T)).
\end{cases}
\]

and \((P(\cdot), Q(\cdot))\) satisfies

\[
\begin{cases}
dP = \left[ -2b_x(t, x, v, v_\delta)P - \sigma_x^2(t, x, v, v_\delta)P - 2\sigma_x(t, x, v, v_\delta)Q + \frac{1}{2} H_{xx}(t, x, v, v_\delta, p, q) \right] dt + Q dB_t, \quad t \in [0, T],
P(T) = \frac{1}{2} h_{xx}(x(T))
\end{cases}
\]

(4.24)
5 Application to a delayed LQ problem

Let us consider the following stochastic control system with delay:

\[
\begin{align*}
dx(t) &= \left[A_1 x(t) + A_2 x(t - \delta) + B v(t)\right]dt + \left[C_1 x(t) + C_2 x(t - \delta) + D v(t)\right]dB_t, \quad t \in [0, T] \\
x(t) &= a, \quad t \in [-\delta, 0],
\end{align*}
\]

(5.1)

where \(x(\cdot) \in \mathbb{R}\) and \(\delta > 0\) is the time delay. A constant \(a\) is the given initial capital. An admissible control \(v(\cdot)\) is an \(\mathcal{F}_t\)-adapted process with values in \(U \subset \mathbb{R}\). In practice, the range of control variable is limited for some reasons such as \(U = (-\infty, -1] \cup [1, +\infty)\). The optimal control problem is to minimize

\[
J(v(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (R_1 x^2(t) + R_2 x^2(t - \delta) + L v^2(t))dt + H x^2(T) \right],
\]

(5.2)

subject to \(U_{ad}\), where \(A_1, A_2, B, C_1, C_2, D, R_1, R_2, L\) and \(H\) are constants satisfying \(C_2 \neq 0, R_1 \geq 0, R_2 \geq 0, L > 0\) and \(H \geq 0\). Since \(U\) is not convex, then the existing maximum principles cannot be used here. However, Theorem 4.1 is still workable in this case. The Hamiltonian function is given by

\[
H(t, x(t), x(t - \delta), v(t), p(t), q(t)) = \frac{1}{2} R_1 x^2(t) + \frac{1}{2} R_2 x^2(t - \delta) + \frac{1}{2} L v^2(t) + p(t) [A_1 x(t) + A_2 x(t - \delta) + B v(t)] + q(t) [C_1 x(t) + C_2 x(t - \delta) + D v(t)].
\]

(5.3)

For any \(v \in U_{ad}\), it is necessary that the optimal control \(v^*\) and the optimal solution \(x^*\) of (5.1) satisfy the maximum condition in Theorem 4.1. Here the corresponding first-order and second-order adjoint processes \((p(\cdot), q(\cdot))\) and \((P(\cdot), Q(\cdot))\) are determined by:

\[
\begin{align*}
-dp(t) &= \left[A_1 p(t) + C_1 q(t) + R_1 x^*(t) + \mathbb{E} \mathcal{F}_t^r [A_2 p(t + \delta) + C_2 q(t + \delta)]\right]dt - q(t)dB_t, \quad t \in [0, T], \\
p(T) &= H x^*(T), \quad p(t) = 0, \quad t \in (T, T + \delta], \\
q(t) &= 0, \quad t \in [T, T + \delta],
\end{align*}
\]

(5.4)

\[
\begin{align*}
dP(t) &= \left[-2A_1 P(t) - 2C_1 Q(t) - C_2^2 P(t) - \frac{1}{2} R_1 - \frac{1}{2} R_2 - \mathbb{E} \mathcal{F}_t [C_2^2 P(t + \delta)]\right]dt + Q(t)dB_t, \quad t \in [0, T], \\
P(T) &= \frac{H}{2}, \quad P(t) = 0, \quad t \in (T, T + \delta], \\
Q(t) &= 0, \quad t \in [T, T + \delta].
\end{align*}
\]

(5.5)

Consider the following ABSDE:

\[
\begin{align*}
dP_0(t) &= \left[-2A_1 P_0(t) - 2C_1 Q_0(t) - C_2^2 P_0(t) - \mathbb{E} \mathcal{F}_t^r [C_2^2 P_0(t + \delta)]\right]dt + Q_0(t)dB_t, \quad t \in [0, T], \\
P_0(T) &= 0, \quad P_0(t) = 0, \quad t \in (T, T + \delta], \\
Q_0(t) &= 0, \quad t \in [T, T + \delta].
\end{align*}
\]

(5.6)
Since \( \frac{1}{2}R_1 + \frac{1}{2}R_2 \geq 0, t \in [0, T] \) and \( \frac{H}{t} \geq 0 \), then from the comparison Theorem of ABSDEs, we obtain \( P(t) \geq P_0(t) = 0, t \in [0, T] \).

Noting \( U = (-\infty, -1] \cup [1, +\infty) \), we know that the candidate optimal control \( v^*(t) \) should satisfy \( \frac{1}{2}Lv^2(t) + p(t)Bv(t) + q(t)Dv(t) - \frac{1}{2}L(v^*(t))^2 - p(t)Bv^*(t) - q(t)Dv^*(t) + (v(t) - v^*(t))P(t)D^2(v(t) - v^*(t)) \geq 0, \forall v(\cdot) \in U_{ad} \). Since \( P(t) \geq 0 \), then we have

\[
v^*(t) = \begin{cases} 
   u(t), & u(t) \in (-\infty, -1] \cup [1, +\infty), \\
   1, & 0 \leq u(t) < 1, \\
   -1, & -1 < u(t) < 0,
\end{cases}
\]

with \( u(t) = -\frac{p(t)B + q(t)D}{L} \).

In what follows we can prove that \( v^*(\cdot) \) in (5.7) is indeed an optimal control.

**Proposition 5.1** \( v^*(\cdot) \) defined by (5.7) is an optimal control for the delayed LQ optimal control problem.

**Proof.** Suppose \( x(\cdot) \) is the trajectory of system (5.1) for any \( v(\cdot) \in U_{ad} \). Since \( R_1 \geq 0, R_2 \geq 0 \) and \( H \geq 0 \), we have

\[
J(v(\cdot)) - J(v^*(\cdot)) \\
= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( R_1 x^2(t) + R_2 x^2(t - \delta) + Lv^2(t) - R_1(x^*(t))^2 - R_2(x^*(t - \delta))^2 - L(v^*(t))^2 \right) dt + H(x(T))^2 - H(x^*(T))^2 \right] \\
\geq \mathbb{E} \left[ \int_0^T \left( R_1 x^*(t)(x(t) - x^*(t)) + R_2 x^*(t - \delta)(x(t - \delta) - x^*(t - \delta)) + \frac{1}{2}Lv^2(t) - \frac{1}{2}L(v^*(t))^2 \right) dt + Hx^*(T)(x(T) - x^*(T)) \right] \\
\geq \mathbb{E} \left[ \int_0^T \left( R_1 x^*(t)(x(t) - x^*(t)) + R_2 x^*(t - \delta)(x(t - \delta) - x^*(t - \delta)) + \frac{1}{2}Lv^2(t) - \frac{1}{2}L(v^*(t))^2 \right) dt + Hx^*(T)(x(T) - x^*(T)) \right] \\
\geq \mathbb{E} \left[ \int_0^T \left( \frac{1}{2}Lv^2(t) + p(t)Bv(t) + q(t)Dv(t) - \frac{1}{2}L(v^*(t))^2 - p(t)Bv^*(t) - q(t)Dv^*(t) + (v(t) - v^*(t))P(t)D^2(v(t) - v^*(t)) \right) dt \right].
\]

Applying Itô’s formula to \( p(t)(x(t) - x^*(t)) \) and taking expectation, we get

\[
\mathbb{E}[Hx^*(T)(x(T) - x^*(T))] \\
= \mathbb{E} \int_0^T [p(t)B(v(t) - v^*(t)) + q(t)D(v(t) - v^*(t)) - R_1 x^*(t)(x(t) - x^*(t)) dt \\
- \mathbb{E} \int_0^T \delta R_2 x^*(t)(x(t) - x^*(t)) dt.
\]

Substitution of (5.9) into (5.8) gives

\[
J(v(\cdot)) - J(v^*(\cdot)) \\
\geq \mathbb{E} \int_0^T \left[ \frac{1}{2}Lv^2(t) - \frac{1}{2}L(v^*(t))^2 + p(t)B(v(t) - v^*(t)) + q(t)D(v(t) - v^*(t)) \right] dt.
\]

Now from \( L > 0 \) and (5.7), \( \left[ \frac{1}{2}Lv^2(t) + p(t)Bv(t) + q(t)Dv(t) \right] - \left[ \frac{1}{2}L(v^*(t))^2 + p(t)Bv^*(t) + q(t)Dv^*(t) \right] \geq 0, \forall v(\cdot) \in U_{ad} \). The inequalities above imply that

\[
J(v(\cdot)) - J(v^*(\cdot)) \geq 0, \forall v(\cdot) \in U_{ad}.
\]

Thereby, \( v^*(\cdot) \) is the optimal control and \( x^*(\cdot) \) is the optimal trajectory. \( \blacksquare \)
6 Conclusion

Motivated by the lack of the theory, we study the optimal control problem with delay in this paper, where the control domain need not be convex and the diffusion coefficient contains control variable. A necessary condition for optimality called general maximum principle is obtained. It is remarkable that Lemma 4.1 which gives an explicit estimate of integral of $x_1(t)x_1(t-\delta)$ term plays an important role on our maximum principle, and we can extend this result to multidimensional systems whenever we impose some heavy assumptions on the derivatives of the coefficients. The theoretical result established here is applied to solve a delayed LQ problem. An optimal control of the LQ control problem is obtained explicitly. Since LQ models are usually applied to describe some economic and insurance phenomena, we hope that the established LQ optimal control theory has a broad range of applications in these fields.

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