GLOBAL EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR ONE-DIMENSIONAL REACTION-INTERFACE SYSTEMS

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ABSTRACT. In this paper, we provide a mathematical framework in studying the wave propagation with the annihilation phenomenon in excitable media. We deal with the existence and uniqueness of solutions to a one-dimensional free boundary problem (called a reaction–interface system) arising from the singular limit of a FitzHugh–Nagumo type reaction–diffusion system. Because of the presence of the annihilation, interfaces may intersect each other. We introduce the notion of weak solutions to study the continuation of solutions beyond the annihilation time. Under suitable conditions, we show that the free boundary problem is well-posed.

1. INTRODUCTION

A variety of wave patterns can be triggered in excitable media such as traveling front, pulse waves, periodic wave trains, rotating spirals, and so on. Reaction-diffusion systems have been successfully modeling these spatio-temporal patterns. A wide class of spatio-temporal patterns has been discussed in, for example, [2, 3, 4, 15, 18, 19, 21, 24, 25, 26, 28, 29] and the references cited therein. A fundamental phenomenon observed in the experiments is that chemical waves propagating at a roughly constant speed may collide with each other, and they annihilate one another. Though it might be a relatively simple phenomenon in excitable media, it is still challenging to be proved theoretically. To better understand this issue, considering interface problems is one of possible ways to model this phenomenon (e.g., [13]). In general, solutions of interface problems produce transition layers (called interfaces), which separate the domain into different phase regions. Interface problems have been deduced from nonlinear reaction-diffusion equations such as Allen–Cahn type equation, Belousov–Zhabotinsky (BZ) systems, competition-diffusion systems or FitzHugh–Nagumo type systems with the diffusion rate being sufficiently small and/or the reaction term being large enough (see, e.g., [1, 6, 12, 16, 20, 22, 27]). Although there have been many studies regarding interface problems, it is rather difficult to investigate the global dynamics in presence models. For this, we shall consider a simplified model but still exhibit abundant patterns. As a part of our series works, we aim to provide a mathematical framework to study the wave propagation, including colliding of waves, based on a FitzHugh–Nagumo type reaction-diffusion system proposed in [8].

More precisely, we are concerned with a one-dimensional free boundary problem arising from the following system ([8]):

\[
\begin{aligned}
  u_t &= \Delta u + \frac{1}{\varepsilon^2} (f_\varepsilon(u) - \varepsilon \beta v), \quad x \in \mathbb{R}^2, \ t > 0, \\
  v_t &= g(u, v), \quad x \in \mathbb{R}^2, \ t > 0,
\end{aligned}
\]

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where $\varepsilon > 0$ is assumed to be a small parameter; $\beta > 0$; $f_\varepsilon$ and $g$ take the following form:

$$f_\varepsilon(u) := u(1 - u)\left(u - \frac{1}{2} + \varepsilon\alpha\right), \quad g(u, v) = g_1u - \frac{g_2v}{g_3v + g_4}$$

for some $\alpha > 0$ and $g_j > 0$ for $j = 1, 2, 3, 4$. Let $(u^\varepsilon, v^\varepsilon)$ be a solution of (1.1) with a suitable initial data. By a formal analysis in [8], $u^\varepsilon$ converges to either 1 or 0 as $\varepsilon \downarrow 0$. Denote the region on which $u^\varepsilon$ converges to 1 by $\Omega(t)$. Then, under the assumption

(A) $g_1g_3 > g_2$,

the limiting problem of (1.1) reduces to

\[
\begin{align*}
V &= W(v) - \kappa, & (x, y) &\in \partial\Omega(t), & t > 0, \\
v_t &= g(1_{\Omega(t)}, v), & (x, y) &\in \mathbb{R}^2, & t > 0,
\end{align*}
\]

where $1_{\Omega(t)}$ stands for the characteristic function having the value 1 in $\Omega(t)$; $\kappa$ is the curvature function of $\partial\Omega(t)$; $V$ is the outer normal velocity of $\partial\Omega(t)$, and

$$W(v) = a - bv, \quad a = \sqrt{2}\alpha, \quad b = 6\sqrt{2}\beta.$$ 

We refer to [8, Appendix] for further details.

The problem (1.2) supports many fundamental patterns appearing in excitable media. In [8], traveling spots are considered. The convergence of traveling spots to planar traveling waves is studied in [9]. Traveling curved waves and their stability are investigated in [23]. To understand the global dynamics of (1.2), it is natural to begin with the one-dimensional spatial problem so that the curvature vanishes completely. Therefore, in this paper, we will focus on the following problem:

\[
\begin{align*}
V &= W(v), & x &\in \partial\Omega(t), & t > 0, \\
v_t &= g(1_{\Omega(t)}, v), & x &\in \mathbb{R}, & t > 0.
\end{align*}
\]

In this paper, we always assume (A). In fact, we only require that $g_1g_3 > g_2$ in our analysis. Before studying the global dynamical behavior of (1.3), it is necessary to establish the global existence and uniqueness of solutions of (1.3) with suitable initial conditions, which will be the primary purpose of this paper. The global dynamics issue is studied in [10] as a companion paper. The weak entire solutions are discussed in [11].

For initial data of (1.3), we assume that $\Omega(0) = \Omega_0$ contains finitely many disjoint bounded intervals. Namely,

$$\Omega_0 := \bigcup_{j=1}^{m} (x_{2j-1}, x_{2j})$$

for some $m \in \mathbb{N}$;

while the initial function $v_0$ is assumed to be a bounded Lipschitz function defined in $\mathbb{R}$ and $W(v_0)$ is non-zero on the boundary $\partial\Omega_0$. Then we will show that the solution exists in the classical sense until interfaces collide with each other. The time that two interfaces collide is called an *annihilation time*. This phenomenon has been discussed by Chen and Gao [7], who studied the singular limit problem of the following problem

\[
\begin{align*}
\begin{cases}
\varepsilon u_t &= \varepsilon \Delta u + \frac{1}{\varepsilon}(F(u) - v), & x \in \mathbb{R}, & t > 0, \\
v_t &= G(u, v), & x \in \mathbb{R}, & t > 0,
\end{cases}
\]

\[
\begin{align*}
\begin{cases}
\varepsilon u_t &= \varepsilon \Delta u + \frac{1}{\varepsilon}(F(u) - v), & x \in \mathbb{R}, & t > 0, \\
v_t &= G(u, v), & x \in \mathbb{R}, & t > 0,
\end{cases}
\]
where

\[ F(u) = \left( \frac{3}{\sqrt{2}} - 2u^2 \right)u, \quad G(u, v) = u - \gamma v - b \quad \text{for some } \gamma > 0 \text{ and } b \in \mathbb{R}. \]

For each \( v \in (-1, 1) \), the equation \( F(u) - v = 0 \) has three real roots: \( h_-(v), h_0(v) \) and \( h_+(v) \) satisfying \( h_-(v) < h_0(v) < h_+(v) \), where \( h_\pm(v) \) are the stable equilibria solution of the ODE \( u_t = \varepsilon^{-1}(F(u) - v) \). The singular limit problem of (1.4) is described by

\[
\begin{cases}
V = R(v), & \text{in } \partial \Omega_\pm(t), \\
v_t = G^\pm(v) := G(h_\pm(v), v), & \text{in } \Omega_\pm(t),
\end{cases}
\]

where \( \Omega_\pm(t) \) denotes the region on which \( u^\varepsilon \) converges to \( \pm 1 \).

The notion of solutions of (1.5) is extended as follows: let \( D \) be a closed domain in \( \mathbb{R} \times [0, \infty) \), \((v, Q^+, Q^-)\) is called a weak solution of (1.5) in \( D \) if \((v, Q^+, Q^-)\) satisfies the following conditions:

(i) \( v \in C^0(D) \) and \( v_t \in L^\infty(D) \), \( v_t = G^\pm(v) \) in \( Q^\pm \),

(ii) \( Q^\pm \) are open and disjoint such that \( m(\Gamma) = 0 \), where \( \Gamma := D \setminus (Q^+ \cup Q^-) \) and \( m(\cdot) \) denotes the Lebesgue measure in \( \mathbb{R}^2 \),

(iii) (Propagation) If \( B(x_0, r_0) \times \{t_0\} \subset Q^\pm \) and \( \pm v < 1 \) in \( B(x_0, r_0 + c^\pm \delta) \times [t_0, t_0 + \delta] \subset D \) for some \( \delta > 0 \), then \( B(x_0, r_0 + c^\pm \delta) \times \{t_0 + \delta\} \subset Q^\pm \), where

\[ c^\pm := \min_{t_0 \leq t \leq t_0 + \delta} \{ \sup_{-1 < s < 1} |R(s)| \}. \]

(iv) (Nucleation condition) \( \{(x, t) \in D| \pm v > 1\} \subset Q^\pm \).

To avoid the confusion of our definition of weak solutions, it may be called a “switching” solution. Based on this setting, they proved that (1.5) admits a unique “switching” solution \((v, Q^+, Q^-)\) that satisfies \( v(x, 0) = v_0(x) \) with \( \Omega_0 \) consisting of a finite number of bounded intervals such that

\[ R(v_0(x)) \neq 0 \quad \text{for any } x \in \partial \Omega(0). \]

Without the condition given by (1.6), they also showed the ill-posedness of the problem (1.5). Similarly, our problem will be ill-posed with a similar condition (see \( \text{(H2)} \) below). We remark that (1.5) exhibits the nucleation phenomenon of interfaces and will not appear in our problem. To study the continuation of solutions after the annihilation time for our problem, one possible way to discuss weak solutions is to follow the idea of Chen and Gao [7]. However, we introduce a different way to define weak solutions, which is more likely to follow a PDE approach. We also refer to [5, 6, 14, 17] for theoretical works on the existence and uniqueness of solutions with diffusion term appearing in the \( v \)-equation.

The rest of the paper is organized as follows. In Section 2, we introduce the notion of classical and weak solutions for our model, and state the main results. In Section 3, we establish the global existence and uniqueness of weak solutions. Some tedious or straightforward proofs are provided in the Appendix.

## 2. Main results

We consider the following initial value problem:

\[
\begin{cases}
V = W(v) := a - bv, & x \in \partial \Omega(t), \ t > 0, \\
v_t = g(\mathbf{1}_{\Omega(t)}, v), & x \in \mathbb{R}, \ t > 0, \\
\Omega(0) = \Omega_0, \ v(x, 0) = v_0(x), & x \in \mathbb{R}.
\end{cases}
\]
We assume that \((\Omega_0, v_0)\) satisfies

**H1** (Boundedness) \(\Omega_0 := \bigcup_{i=1}^{m}(x_i^0, x_{i+1}^0)\) for some \(m \in \mathbb{N}\) and \(x_i < x_{i+1}\) for \(i = 1, \cdots, 2m - 1\), and \(v_0 \geq 0\) is a bounded Lipschitz function with

\[
M := \|v_0\|_{L^\infty(\mathbb{R})}.
\]

**H2** (Well-posedness) \(W(v_0(x)) \neq 0\) for all \(x \in \partial\Omega(0)\).

We note that condition \((H2)\) is similar to (1.6) used in [7]. This condition is used to guarantee the well-posedness of (2.1). More precisely, the uniqueness of the initial value problem (2.1) may not hold without \((H2)\). See Remark 3.11 below for the details.

Hereafter, we always define \(Q_T := \mathbb{R} \times [0,T]\). The definition of classical solutions is given as follows.

**Definition 2.1.** (i) We say that \((\Omega, v)\) is a classical solution of (2.1) for \(0 \leq t \leq T\) if there exist \(x_k \in C^1([0,T]), k = 1, \cdots, 2m\), and

\[
v \in C(Q_T) \cap C^1\bigg(\mathbb{R} \times [0,T] \setminus \{x = x_k(t), \ t \in [0,T], \ k = 1, \cdots, 2m\}\bigg)
\]

such that \(x_i(\cdot) < x_{i+1}(\cdot)\) in \([0,T]\) for \(i = 1, \cdots, 2m - 1\), and \(\Omega := \bigcup_{0 \leq t \leq T} [\Omega(t) \times \{t\}]\), where

\[
\Omega(t) := \bigcap_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)),
\]

and the following equations hold pointwisely:

\[
x_k^\prime(t) = (-1)^k W\big(v(x_k(t), t)\big) := (-1)^k \left(a - bv(x_k(t), t)\right), \ 0 \leq t \leq T, \ k = 1, \cdots, 2m, \tag{2.3}
\]

\[
v_t = g(1_{\Omega(t)}, v) \quad \text{in} \ Q_T, \tag{2.4}
\]

\[
x_k(0) = x_k^0, \quad v(x,0) = v_0(x). \tag{2.5}
\]

(ii) \((\Omega, v)\) is called a classical solution of (2.1) for \(0 \leq t < T\) if it is a classical solution for \(0 \leq t \leq \tau\) for each \(\tau \in (0,T)\).

(iii) \((\Omega, v)\) is called a classical solution of (1.3) for \(\tau \leq t \leq T\) for some \(\tau > 0\) if (i) holds with \(t = 0\) replaced by \(t = \tau\).

(iv) \((\Omega, v)\) is called a non-negative classical solution of (1.3) for \(\tau \leq t \leq T\) for some \(\tau > 0\) if (iii) holds and \(v \geq 0\).

Under \((H1)\) and \((H2)\), we will establish the local existence of a classical solution to problem (2.1), where each interface can be represented by a strictly monotone function of \(t\). The classical solution can be extended until an annihilation occurs and thus the notion of weak solutions is needed. Let us define the annihilation time \(T_A\) depending on \((\Omega_0, v_0)\) by

\[
T_A := \sup\{\tau > 0 \mid x_i(t) < x_{i+1}(t) \ \forall \ t \in [0,\tau) \text{ and } i = 1, \cdots, 2m - 1\} \in (0,\infty].
\]

We see that the classical solution exists globally in time if \(T_A = \infty\).

Next, we introduce the definition of weak solutions, which is different from the one given in [7]. Before we state the definition of weak solutions, we denote the interior of \(\Lambda\) in \(\mathbb{R}\) (resp. \(\mathbb{R}^2\)) by \(\text{int}\mathbb{R} \Lambda\) (resp. \(\text{int}\mathbb{R}^2 \Lambda\)). Define the space \(X_T\) consisting of \((\Omega, v)\) that satisfies the following:

1. \(v \in C(Q_T)\) and is Lipschitz continuous in \(x\),
2. \(\Omega \subset Q_T, \partial\Omega\) is Lipschitz,
3. \(v \neq a/b\) on \(\bigcup_{0 \leq t \leq T} \partial\Omega(t) \times \{t\}\).
\((4)\) \(v(x, 0) = v_0(x), \Omega(0) = \Omega_0,\)

where

\[ \Omega(t) := \text{int}_\mathbb{R}\{x \in \mathbb{R} \mid (x, t) \in \mathcal{O}\}. \]

We remark that \(\bigcup_{0 \leq t \leq T} \partial \Omega(t) \times \{t\}\) represents the set of all interfaces in \([0, T]\) when it is a classical solution. Since \(\partial \Omega\) is Lipschitz, the unit outer normal vector \(n := (n_1, n_2)\) to \(\partial \Omega\) exists almost everywhere.

**Definition 2.2.** (i) We say that a pair \((\Omega, v) \in X_T\) is a weak solution of (2.1) for \(0 \leq t \leq T\) if the following two conditions \((C1)\) and \((C2)\) hold:

\(\textbf{(C1)}\) For any \(\varphi, \psi \in H^1((0, T); L^2(\mathbb{R}))\) and \(\psi\) has a compact support in \(Q_T\),

\[
\left[ \int_{\mathbb{R}} 1_{\Omega(t)} \varphi dx \right]_0^T = \int_0^T \int_{\Omega(t)} \varphi_t dx dt + \int_{\partial \mathcal{O}\cap(\mathbb{R} \times (0, T))} W(v)\varphi|n_1|d\sigma,
\]

\[
\left[ \int_{\mathbb{R}} v\psi dx \right]_0^T = \int_0^T \int_{\mathbb{R}} (v\psi_t + g(1_{\Omega(t)}, v)\psi) dx dt.
\]

\(\textbf{(C2)}\) If \(B(x_0, r_0) \times \{t_0\} \subset \Omega\) (resp. \(\subset \Omega^c := Q_T \setminus \Omega\) for some \(r_0 > 0\) and \(t_0 \in [0, T]\)), then there exists \(\tau_0 \in (0, T)\) depending only on \(r_0\) such that

\[ \{x_0\} \times [t_0, t_0 + \tau_0] \subset \Omega \text{ (resp. } \subset \Omega^c). \]

(ii) We say that a pair \((\Omega, v)\) is a weak solution of (2.1) for \(0 \leq t < T\) if it is a weak solution for \(0 \leq t \leq \tau\) for all \(\tau \in (0, T)\).

Note that (2.1) could be ill-posed without the condition \((C2)\). To prevent the nucleation of interfaces, we need an extra condition \((C2)\) similar to condition (iii) of the weak solution in [7]. If \((x_0, t_0) \in \Omega\), then \((x_0, t_0 + \varepsilon) \in \Omega\) for sufficiently small \(\varepsilon\) owing to the openness of \(\Omega\). The above condition means that the slope of the interface \(\partial \Omega\) has a positive lower bound in \((x, t)\) space. Clearly, if any new interface generates from some time \(\tau \in (0, T)\), we can choose \(\tau_0 \ll 1\) and \(t_0\) close to \(\tau\) such that \(t_0 + \tau_0 > \tau > t_0\) and then

\[ \left(\{x_0\} \times [t_0, t_0 + \tau_0]\right) \cap \Omega \neq \emptyset \quad \text{and} \quad \left(\{x_0\} \times [t_0, t_0 + \tau_0]\right) \cap \Omega^c \neq \emptyset, \]

which contradicts \((C2)\).

**Remark 2.3.** Let \((\Omega, v)\) be a weak solution of (2.1) for \(0 \leq t \leq T\). Then it is also a weak solution for \(0 \leq t \leq \tau\) for any \(\tau \in (0, T)\) (see Lemma 3.12). Moreover, for any \(0 \leq t_1 < t_2 \leq T\) and \(-\infty \leq x_1 < x_2 \leq \infty\), a similar argument used in the proof of Lemma 3.12 given in the Appendix can imply

\[
\left[ \int_{x_1}^{x_2} 1_{\Omega(t)} \varphi dx \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Omega(t) \cap (x_1, x_2)} \varphi_t dx dt + \int_{\partial \mathcal{O}\cap((x_1, x_2) \times (t_1, t_2))} W(v)\varphi|n_1|d\sigma,
\]

\[
\left[ \int_{\mathbb{R}} v\psi dx \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{x_1}^{x_2} (v\psi_t + g(1_{\Omega(t)}, v)\psi) dx dt
\]

for any \(\varphi, \psi \in H^1((t_1, t_2); L^2(x_1, x_2))\) with \(\psi\) has a compact support if \(|x_1 - x_2| = \infty\).

We now state the main results as follows.

**Theorem 2.4.** Assume \((H1)\) and \((H2)\). Then problem (2.1) has a unique local in time non-negative classical solution.
Theorem 2.5. Assume (H1) and (H2). Then there is a unique global in time weak solution to problem (2.1).

3. CLASSICAL SOLUTIONS AND WEAK SOLUTIONS

We shall divide this section into two subsections. In the first subsection, we study the local existence and uniqueness of classical solutions and prove Theorem 2.4. In the second subsection, we establish the global existence and uniqueness of weak solutions (Theorem 2.5). Some tedious proofs are put in the Appendix. Hereafter, (H1) and (H2) are always assumed.

3.1. The local existence and uniqueness of classical solutions. First, we deal with the local existence and uniqueness of classical solutions and prove Theorem 2.4. In the second subsection, we study the local existence and prove Proposition 3.1.

To simplify the proof, we only consider $m = 1$, i.e., $\Omega_0 = (x_1^0, x_2^0)$. The following process can apply to $m > 1$ with some simple modifications but tedious details. Because of (H2), we can divide our discussion into four cases:
(1) $W(v_0(x_1^0)) > 0$ and $W(v_0(x_2^0)) > 0$,
(2) $W(v_0(x_1^0)) > 0$ and $W(v_0(x_2^0)) < 0$,
(3) $W(v_0(x_1^0)) < 0$ and $W(v_0(x_2^0)) > 0$,
(4) $W(v_0(x_1^0)) < 0$ and $W(v_0(x_2^0)) < 0$.

For the case (1), first we assume in advance that $x_k(t)$ exists ($k = 1, 2$). By the continuity of $v$ and $W$, we see that $x_1'(t) < 0$ and $x_2'(t) > 0$ for $t \in [0, \tau)$ for some $\tau > 0$ sufficiently small. By (2.1), we have $v_t = g(0, v)$ for $0 < t < \tau$ and $x \in (-\infty, x_1(t)) \cup [x_2(t), \infty)$. By dividing both sides by $g(0, v)$ and integrating it over $[0, t]$, we can easily solve $v$ as

$$v(x, t) = G_0(G_0^{-1}(v_0(x)) + t) \quad \text{for } 0 < t < \tau \text{ and } x \in (-\infty, x_1(t)) \cup [x_2(t), \infty),$$

where $G_0^{-1}$ is defined in (3.1). Thus (2.3) reduces to

$$\frac{dx_k}{dt} = (-1)^k W(G_0(G_0^{-1}(v_0(x_k(t)) + t)), \quad x_k(0) = x_0^k.$$

With the help of Lemma 3.4 and the Lipschitz continuity of $v_0$, the above initial value problem allows us to define the position of $x_k$ ($k = 1, 2$) uniquely for all small $t \in [0, \tau')$ for some $\tau' < \tau$.

Finally, for $(x, t) \in (x_1(t), x_2(t)) \times (0, \tau')$, $v$ can be solved by integrating $v_t/g(1, v) = 1$. Namely,

$$v(x, t) = \begin{cases} G_1 \left(G_1^{-1}(v_0(x)) + t\right), & x_0^1 \leq x \leq x_2^k, \ t \in (0, \tau'), \\ G_1 \left(G_1^{-1}(v_x(t), T_1(x)) + t - T_1(x)\right), & x_1(t) < x < x_1^0, \ t \in (0, \tau'), \\ G_1 \left(G_1^{-1}(v_x(t), T_2(x)) + t - T_2(x)\right), & x_2^0 < x < x_2(t), \ t \in (0, \tau'), \end{cases}$$

where $T_k(x)$ is the arrival time of $x_k(t)$ to $x$.

Hence we obtain the local existence and of a classical solution of (2.1) for the case (1). The similar process can apply to cases (2), (3) and (4) respectively as well as the case where $m \geq 2$. We omit the details. This completes the proof. \hfill \Box

Next, we deal with the uniqueness and continuation of solutions. To extend the local in time solution uniquely, we need the Lipschitz continuity of $v$. For this, we prepare several lemmas.

**Lemma 3.5.** Let $(\Omega, v)$ be a classical solution of (2.1) for $0 \leq t \leq T$. Furthermore, assume that there exists $\delta > 0$ such that

$$|x_i'(t)| \geq \delta \quad \text{for } 0 \leq t \leq T \text{ and } k = 1, \ldots, 2m.$$

Then $v$ is Lipschitz continuous on $Q_T := \mathbb{R} \times [0, T]$, where the Lipschitz constant depends on $\delta$.

**Proof.** The proof is involved because each point $x$ may be passed through by several interfaces during a period of time. We simply separate $Q_T$ into finitely many adjacent closed regions such that at most one interface can pass through any points and then show Lipschitz continuity on each closed region.

Recall the definition of classical solutions, we have $x_k(t) < x_{k+1}(t)$ for $t \in [0, T]$ and $x_k \in C^1([0, T])$ for $k = 1, \ldots, 2m - 1$. Hence we can take $A = \min_{t \in [0, T]} x_1(t)$ and $B = \max_{t \in [0, T]} x_{2m}(t)$ such that $\Omega \subset [A, B] \times [0, T]$. In other words, we have

$$v_t = g(0, v) \quad \text{in } D_T := Q_T \setminus ([A, B] \times [0, T]),$$

which gives (see the proof of Proposition 3.1)

$$v(x, t) = G_0(G_0^{-1}(v_0(x)) + t), \quad (x, t) \in D_T.$$
We now show that
\[ (3.4) \quad v \text{ is Lipschitz continuous on } D_T. \]
Clearly, \( v \) is Lipschitz continuous in \( t \). From (3.3) and (H1), we can use Lemma 3.4 and the Lipschitz continuity of \( v_0 \) to assert
\[ |v(x,t) - v(\bar{x},t)| \leq |v_0(x) - v_0(\bar{x})| \leq L_0|x - \bar{x}| \]
for all \( (x,t), (\bar{x},t) \in D_T \) and for some \( L_0 > 0 \). Hence (3.4) follows.

Next, we show that
\[ (3.5) \quad v \text{ is Lipschitz continuous on } [A, B] \times [0, T]. \]
To simplify our discussion, we write \( [0, T] = [0, \tau] \cup [\tau, 2\tau] \cup \cdots \cup [(N - 1)\tau, T] \), where \( \tau := T/N \) for some \( N \in \mathbb{N} \) large enough such that
\[ (3.6) \quad x_j([(n - 1)\tau, n\tau]) \cap x_k([(n - 1)\tau, n\tau]) = \emptyset \quad \text{for all } n = 1, \ldots, N \text{ and } j \neq k. \]
To prove (3.5), it suffices to show
\[ (3.7) \quad v \text{ is Lipschitz continuous on } [A, B] \times [(n - 1)\tau, n\tau] \text{ for } n = 1, \ldots, N. \]
By Lemma 3.6, we see that (3.7) follows for \( n = 1 \) with the Lipschitz constant depending on \( \delta \). Repeating the same argument used in the proof of Lemma 3.6, we obtain (3.7) and then (3.5) holds.

Moreover, the solution can be extended until \( x_k' \) vanishes at some time for some \( k \) or an annihilation occurs.

The proofs of the above lemmas are put in the Appendix.

**Proposition 3.8.** The classical solution of problem (2.1) can be extended uniquely until an annihilation occurs. Moreover, \( x_k(t) \) is strictly monotone in \( [0, T_A) \), where \( T_A \) is defined in (2.6). If \( T_A < \infty \), there exists a positive constant \( \delta \) such that \( |x_k'(t)| \geq \delta \) for all \( t \in [0, T_A) \) and \( k = 1, \ldots, 2m \).

**Proof.** By Lemma 3.7, the classical solution can be extended uniquely until \( x_k'((n - 1)\tau) = 0 \) (or \( W(v(x_k((n - 1)\tau), \tau)) = 0 \)) for some \( \tau_1 > \tau_0 \) (non-uniqueness will occur) or \( x_k \) intersects \( x_{k+1} \) for some \( k \) at some time (an annihilation occurs).

We now prove the existence of \( \delta \) when \( T_A < \infty \) by using a contradiction argument. Assume that there exist an increasing sequence \( \{t_j\} \) and some \( k \) such that \( W(v(x_k(t_j)), t_j) \to 0 \) as \( j \to \infty \). Then we can divide our discussion into two cases:

**Case 1:** \( t_j \uparrow \tau \) for some \( \tau \in (0, T_A) \) as \( j \to \infty \), \hspace{1cm} **Case 2:** \( t_j \uparrow T_A \) as \( j \to \infty \).

We now consider **Case 1.** In this case, we can assume that there exists \( k \in \{1, \ldots, 2m\} \) such that
\[ W(v(x_K((n - 1)\tau), \tau)) = 0, \quad W(v(x_k(t), t)) \neq 0 \quad \text{for } t \in [0, \tau). \]

We shall divide our discussion into four subcases:

(1-a) \( k \) is odd and \( W(v_0(x_k^0)) < 0 \),

(1-b) \( k \) is odd and \( W(v_0(x_k^0)) > 0 \),

...
with arrival time $t$ to be extended continuously to $t^\ast$. Since each interface $T_k$ is in $\Omega$ (excitation region), which means $(3.9)$

$$v(x,t)$$ is strictly increasing in $t$ for all $(x,t) \in D_\varepsilon$.

In particular,

$$v(y,t) \uparrow \frac{a}{b} \text{ as } t \uparrow \tau \text{ for } \tau_\varepsilon \leq t < \tau.$$ It follows that $v(y,\tau_\varepsilon) < a/b$. On the other hand, by (3.8), we have $v(y - \varepsilon,\tau_\varepsilon) > a/b$. By the continuity of $v$, there exists $x_0 \in (y - \varepsilon, y)$ such that $v(x_0,\tau_\varepsilon)$ must attain at $a/b$. Hence we can define the following point in $(y - \varepsilon, y)$:

$$x_\varepsilon := \sup \left\{ x \in (y - \varepsilon, y) \mid v(x,\tau_\varepsilon) = \frac{a}{b} \right\}.$$ Because of (3.9), we can introduce the notion of the attaining time $\beta(x) \leq T_k(x)$ satisfying

$$v(x,\beta(x)) = \frac{a}{b} \text{ for each } x \in [x_\varepsilon, y).$$ See Figure 3.1. Then we can show that $\beta$ is Lipschitz on $[x_\varepsilon, y]$. To do so, using $v_t/g(1,v) = 1$ in $D_\varepsilon$ and integrating it over $[\tau_\varepsilon, \beta(x)]$ give

$$\int_{v(x,\tau_\varepsilon)}^{a/b} \frac{ds}{g(1,s)} = \beta(x) - \tau_\varepsilon.$$ Since $v(\cdot, \tau_\varepsilon)$ is Lipschitz on $[x_\varepsilon, y]$ (because of Lemma 3.5), we see that $\beta(\cdot)$ is Lipschitz on $[x_\varepsilon, y]$ with the Lipschitz constant, say $L$. Thus using $T_k(y) = \beta(y)$ and $T_k(x) > \beta(x)$ for $x \in [x_\varepsilon, y)$, we have

$$(3.10) \quad 0 \leq T_k(y) - T_k(y - \varepsilon') \leq \beta(y) - \beta(y - \varepsilon') \leq L\varepsilon'$$ for any small $\varepsilon' \in (0, \varepsilon)$. On the other hand, since $x_k'(\tau) = 0$ and $x_k'(t) > 0$ for $0 \leq t < \tau$, we have $T_k(y^-) = +\infty$. This reaches a contradiction with (3.10). Hence we have shown the existence of $\delta$ for the subcase (1-a). The argument used in the proof of (1-a) can apply to subcases (1-b), (1-c), (1-d). We omit the details.

Next, we deal with Case 2. Because of Case 1, there exists $k \in \{1, \ldots, 2m\}$ such that

$$(1) \quad W(v(x_k(T_k^{-}), T_A^-)) = 0, \quad W(v(x_k(t), t)) \neq 0 \text{ for } t \in [0, T_A).$$ Since each interface $x = x_k(t)$ is monotone in $t$ and bounded for $[0, T_A)$ because of $T_A < \infty$, $x_k(T_A^-)$ exists and is finite, then limit $t \to T_A^-$ $v(x, t)$ exists and is finite for each $x \in \mathbb{R}$, which means that $v$ can be extended continuously to $t = T_A$ as in the proof of Proposition 3.1. Hence we can define the arrival time $T_k(\cdot)$ on $(x_k^0, x_k(T_A^-))$ for each $k$. This allows us to use the same argument as in Case 1 with $\tau$ replaced by $T_A$ to complete the proof of Case 2.
Finally, note that if \( T_A = \infty \), from the argument of **Case 1**, we see that \( x_k' \) never vanishes in \([0, \infty)\) and then \( x_k \) is strictly monotone. Hence the proof of Proposition 3.8 is complete. \( \square \)

Next we show that a classical solution becomes a weak one. Since \( x_i(t) \) is monotone in time for each \( i \), we can define

\[
\lim_{t \to T^-} \Omega(t) := \bigcup_{j=1}^{m} \left( \lim_{t \to T^-} x_{2j-1}(t), \lim_{t \to T^-} x_{2j}(t) \right),
\]

when \( \Omega(t) = \bigcup_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)) \).

**Proposition 3.9.** Let \((\Omega, v)\) be a classical solution of (2.1) for \( 0 \leq t < T_A \) with \( T_A < \infty \). Then \((\tilde{\Omega}, \tilde{v})\) is a weak solution for \( 0 \leq t \leq T_A \), where

\[
\tilde{\Omega}(t) := \begin{cases} 
\Omega(t) & \text{for } t \in [0, T_A), \\
\text{int}_{\mathbb{R}} \left( \lim_{t \to T^-} \Omega(t) \right) & \text{for } t = T_A,
\end{cases}
\]

\[
\tilde{v}(\cdot, t) := \begin{cases} 
v(\cdot, t) & \text{for } t \in [0, T_A), \\
\lim_{t \to T^-} v(\cdot, t) & \text{for } t = T_A.
\end{cases}
\]

In particular, (H1) and (H2) holds with \((\Omega_0, v_0)\) replaced by \((\tilde{\Omega}(T_A), \tilde{v}(x, T_A))\). Hence there exists a unique classical solution with initial time \( t = T_A \) and the solution can be extended until the next annihilation occurs.

**Proof.** Recall that \( \Omega(t) := \bigcup_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)) \). Since \( x_k(\cdot) \) is monotone and bounded for each \( k \), we see that \( x_k(T_A^-) \) exists and thus \( \tilde{\Omega}(T_A) \) is well-defined. Since \( g(1_{\Omega(t)}, v) \leq g_1, \tilde{v}(\cdot, T_A) \) is bounded in \( \mathbb{R} \). Note that, for each fixed \( x \), there exists small \( \epsilon > 0 \) such that \( \tilde{v}(x, t) \) is monotone in \( t \) for \( t \in (T_A - \epsilon, T_A) \). Thus, \( \tilde{v}(\cdot, T_A) := \lim_{t \to T^-} v(\cdot, t) \) is well-defined. Moreover, \( \tilde{v} \) is Lipschitz in \( x \) by using the proof of Proposition 3.8 and Lemma 3.5. Also, from the definition of (3.11), we see that \( \tilde{\Omega}(t) \) consists of finitely many disjoint bounded intervals for \( t \in [0, T_A] \). It follows that \((\tilde{\Omega}, \tilde{v}) \in X_{T_A} \). Clearly, (2.7) and (2.8) hold with \( T \) replaced by \( T_A \). Also, since \((\tilde{\Omega}, \tilde{v})\) is a classical solution for \( 0 \leq t < T_A < \infty \), the condition (C2) is satisfied with \( T \) replaced by \( T_A \). Hence, \((\tilde{\Omega}, \tilde{v})\) is a weak solution for \( 0 \leq t \leq T_A \). In particular, (H1) and (H2) holds with \((\Omega_0, v_0)\) replaced by \((\tilde{\Omega}(T_A), \tilde{v}(x, T_A))\). By taking \( t = T_A \) as an initial time and by applying Proposition 3.8, we can assert that there exists a unique classical solution with initial time \( t = T_A \). Namely, the solution can be extended until the next annihilation occurs. Therefore, the proof is completed. \( \square \)
Lemma 3.10. Let \((\Omega, v)\) be a weak solution of (2.1) for \(0 \leq t \leq T\). If \(v_0(x) \geq 0\) for all \(x \in \mathbb{R}\), then \(v(x, t) \geq 0\) in \(\mathbb{R} \times [0, T]\).

The proof of this lemma is stated in the Appendix. We are ready to prove Theorem 2.4.

Proof of Theorem 2.4. By Proposition 3.1 and Proposition 3.8, we obtain the local existence and uniqueness of a classical solution. Moreover, by Lemma 3.10, the classical solution (also a weak solution) is non-negative. This complete the proof. \(\square\)

Remark 3.11. The condition (H2) is necessary for the uniqueness. If there is a point \(x_0 \in \partial \Omega(0)\) such that \(W(v_0(x_0)) = 0\), then the uniqueness does not hold. Indeed, we can construct the multiple solutions starting from the initial condition \((\Omega(0), v_0(x))\) as follows:

\[
\Omega(0) = \{x \in \mathbb{R} \mid x > 0\}, \quad v_0(x) = \frac{a}{b} - \arctan x.
\]

Then we have two solutions:

\[
\begin{align*}
\Omega_1(t) &= \{x \in \mathbb{R} \mid x > s_1(t)\}, \\
v_1(x, t) &= \begin{cases} 
G_1 \left(G_1^{-1}(v_0(x)) + t\right), & (x > s_1(t)), \\
G_0 \left(G_0^{-1}(v_0(x)) + t\right), & (x \leq s_1(t)), \\
s_1'(t) &= a - bG_1 \left(G_1^{-1}(v_0(s_1(t))) + t\right),
\end{cases}
\]

and

\[
\begin{align*}
\Omega_2(t) &= \{x \in \mathbb{R} \mid x > s_2(t)\}, \\
v_2(x, t) &= \begin{cases} 
G_1 \left(G_1^{-1}(v_0(x)) + t\right), & (x > s_2(t)), \\
G_0 \left(G_0^{-1}(v_0(x)) + t\right), & (x \leq s_2(t)), \\
s_2'(t) &= a - bG_0 \left(G_0^{-1}(v_0(s_2(t))) + t\right).
\end{cases}
\]

We regard the interface as a front for the first solution and a back for the second solution. Note that \(g(1, a/b) > 0\) by the assumption (A). Thus \(s_1(t)\) is decreasing in \(t\) and \(s_2(t)\) is increasing in \(t\) for \(t\) close to zero. We can also construct other solutions. Thus the condition (H2) is required for the uniqueness of solutions as well as the well-posedness of solutions. We refer to [7] for more detailed discussion.

3.2. The global existence and uniqueness of weak solutions. In this subsection, we shall establish the global existence and uniqueness of weak solutions to problem (2.1). Since \(T_A = \infty\) means the classical solution exists globally in time (so does the weak solution), we only discuss weak solutions when \(T_A < \infty\).

Lemma 3.12. Let \((\Omega, v)\) be a weak solution of (2.1) for \(0 \leq t \leq T\). Then it is also a weak solution of (2.1) for \(0 \leq t \leq \tau\) for any \(\tau \in (0, T)\).

The converse of Lemma 3.12 does not hold in general because \(v\) may attain \(a/b\) at \(t = T\) or \(\partial \Omega\) is not Lipschitz continuous at \(t = T\) (such that \((\Omega, v) \notin X_T\)). However, in our problem, the weak solution exists globally in time (see Proposition 3.14).

Lemma 3.13. Let \((\Omega_1, v_1)\) (resp. \((\Omega_2, v_2)\)) be a weak solution for \(0 \leq t \leq T_1\) (resp. \(T_1 \leq t \leq T_2\)). Define

\[
\begin{align*}
\tilde{\Omega} &:= \text{int}_{\mathbb{R}^2} (\Omega_1 \cup \Omega_2), \\
\tilde{v} &:= \begin{cases} 
v_1 & (0 \leq t \leq T_1), \\
v_2 & (T_1 \leq t \leq T_2).
\end{cases}
\end{align*}
\]
If 
\[ \int_{\mathbb{R}} \lim_{t \to T_1^-} \Omega_1(t) = \Omega_2(T_1), \quad v_1(x, T_1) = v_2(x, T_1) \quad \text{for any } x \in \mathbb{R}, \]
then \((\tilde{\Omega}, \tilde{v})\) is a weak solution for \(0 \leq t \leq T_2\).

The proofs of two lemmas are put in the Appendix. We now provide a simple example to illustrate Lemma 3.13. Let \((\Omega_1, v_1)\) (resp. \((\Omega_2, v_2)\)) a classical solution for \(0 \leq t < T_1\) (resp. \(T_1 \leq t < T_2\)) such that
\[
\begin{align*}
\Omega_1(t) &:= (x_1(t), x_2(t)) \cup (x_3(t), x_4(t)) \quad (0 \leq t < T_1), \\
\Omega_2(t) &:= (x_1(t), x_4(t)) \quad (T_1 \leq t < T_2).
\end{align*}
\]
Also, we assume that
\[
x_1(T_1) < x_2(T_1) = x_3(T_1) < x_4(T_1) \quad \text{and} \quad v_1(x, T_1) = v_2(x, T_1) \quad \text{for any } x \in \mathbb{R}.
\]
By Proposition 3.9, two classical solutions then become two weak solutions defined on \([0, T_1]\) and \([T_1, T_2]\), respectively. Then Lemma 3.13 implies that if \(\tilde{\Omega}\) is defined by (3.13), which is given by
\[
\left\{ (x, t) \in \mathbb{R} \times (0, T_2) \left| \begin{array}{l} x_1(t) < x < x_2(t), \quad x_3(t) < x < x_4(t) \quad \text{for } 0 < t < T_1, \\
\hspace{1cm} x_1(t) < x < x_4(t) \quad \text{for } T_1 \leq t < T_2 \end{array} \right. \right\},
\]
and \(\tilde{v}\) is defined by (3.14), then \((\tilde{\Omega}, \tilde{v})\) is a weak solution of (1.3) for \(0 \leq t \leq T_2\).

We are ready to show the existence of global weak solutions of (2.1).

**Proposition 3.14.** There is a global in time weak solution to problem (2.1).

**Proof.** By Proposition 3.8, the classical solution \((\Omega, v)\) exists for \(t \in [0, T_A]\), where \(T_A\) is the annihilation time. If \(T_A = \infty\), then the proof is done. If \(T_A < \infty\), we define \((\tilde{\Omega}, \tilde{v})\) as in Proposition 3.9 and apply Proposition 3.9, \((\tilde{\Omega}, \tilde{v})\) becomes a weak solution for \(0 \leq t \leq T_A\). In particular, (H1) and (H2) holds with replacing \((\Omega_0, v_0)\) by \((\tilde{\Omega}(T_A), \tilde{v}(x, T_A))\). Then, using Proposition 3.8 again, there exists a unique classical solution \((\Omega_2, v_2)\) for \(T_A \leq t < T_2\) for some \(T_2 > T_A\) with initial data
\[
\Omega_2(T_A) := \Omega(T_A), \quad v_2(x, T_A) := \tilde{v}(x, T_A).
\]
By Proposition 3.9, \((\Omega_2, v_2)\) can be extended to a weak solution for \(T_A \leq t < T_2\) (still denoted by \((\Omega_2, v_2)\)).

Next, we define \((\tilde{\Omega}, \tilde{v})\) as in Lemma 3.13 with \((\Omega_1, v_1) := (\tilde{\Omega}, \tilde{v})\). Then it follows from Lemma 3.13 that \((\tilde{\Omega}, \tilde{v})\) is a weak solution for \(t \in [0, T_2]\). If \(T_2 = \infty\), then Proposition 3.14 follows. Otherwise, \(T_2 < \infty\) implies that \(T_2\) is the second annihilation time (Proposition 3.8). We can repeat the above process to extend the weak solution. Because of (C2), the number of interfaces cannot increase in time, which implies that the annihilation only occurs finitely many times. Therefore, by repeating the above process finitely many times, we thus find a globally in time weak solution by gluing classical solutions. This completes the proof. \(\square\)

Next, we deal with the uniqueness of the weak solutions. The uniqueness result is given as follows:

**Proposition 3.15.** Suppose that \((\Omega_1, v_1)\) and \((\Omega_2, v_2)\) are two weak solutions. Then \(\Omega_1 = \Omega_2\) and \(v_1 = v_2\).
The proof of Proposition 3.15 is quite involved. We need prepare several lemmas. The first two lemmas show that any interface of the weak solutions can be locally viewed as a smooth function of $t$.

**Lemma 3.16.** Let $(\Omega, v)$ be a weak solution of (2.1) for $t \in [0,T]$. Also, assume that $(x_0, t_0) \in \partial \Omega(t_0)$ for some $t_0 \in (0,T)$ such that

(a) $W(v(x_0, t_0)) > 0$,

(b) $(x_0 - \varepsilon, t_0) \in \Omega(t_0)$ (resp. $(x_0 + \varepsilon, t_0) \in \Omega(t_0)$) for any sufficiently small $\varepsilon > 0$.

Then, one of the following cases holds:

(i) there are positive constants $\delta, \varepsilon$ and a function $x(\cdot) \in C^1((t_0-\delta, t_0+\delta))$ such that $x'(t_0) > 0$ (resp. $x'(t_0) < 0$) and

$$\{(x(t), t) \mid t \in (t_0 - \delta, t_0 + \delta)\} = \partial \Omega \cap \left[ (x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \delta, t_0 + \delta) \right].$$

(ii) there are positive constants $\delta, \varepsilon$ and two functions $x_i(\cdot) \in C^1((t_0 - \delta, t_0))$ ($i = 1, 2$) such that $x_i(t_0) = x_0$,

$$\lim_{t \to t_0} x_1(t) = \lim_{t \to t_0} x_2(t) = x_0, \quad \lim_{t \to t_0} x_1'(t) > 0, \quad \lim_{t \to t_0} x_2'(t) < 0,$$

$$\bigcup_{i=1}^2 \{(x_i(t), t) \mid t_0 - \delta < t \leq t_0\} = \partial \Omega \cap \left[ (x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \delta, t_0) \right].$$

Namely, $(x_0, t_0)$ is an intersection point of two interfaces.

**Proof.** Since $\partial \Omega$ is Lipschitz continuous (from $(\Omega, v) \in X_T$), there exists a Lipschitz continuous function (upon relabeling and reorienting the coordinates axis if necessary) whose graph passes through $(x_0, t_0)$. Also, by (C2), no new interface can generate. Hence, by taking $\varepsilon > 0$ and $\delta > 0$ sufficiently small and defining

$$D_{\varepsilon, \delta} := (x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \delta, t_0 + \delta),$$

we may assume that $\partial \Omega \cap D_{\varepsilon, \delta}$ contains only one Lipschitz curve passing through $(x_0, t_0)$. Because of (a) and the continuity of $W$, we may also assume (if necessary, we may choose $\varepsilon$ and $\delta$ smaller)

$$W(v(x, t)) > 0 \quad \text{on} \ D_{\varepsilon, \delta}. \quad (3.15)$$

**Claim 1:** we show

$$|\Omega(t_2) \cap (\xi_1, \xi_2)| > |\Omega(t_1) \cap (\xi_1, \xi_2)| \quad (3.16)$$

for all $x_0 - \varepsilon < \xi_1 < \xi_2 < x_0 + \varepsilon$ and $t - \delta < t_1 < t_2 < t_0 + \delta$. Here we recall that $\Omega(t) \cap (\xi_1, \xi_2) = (\xi_1, x(t))$.

By Remark 2.3, we get

$$\left[ \int_{\xi_1}^{\xi_2} 1_{\Omega(t)}(x) dx \right]_{t_1}^{t_2} = \int_{\partial \Omega \cap [(\xi_1, \xi_2) \times (t_1, t_2)]} W(v)|n_1| d\sigma \quad (3.17)$$

for $x_0 - \varepsilon < \xi_1 < \xi_2 < x_0 + \varepsilon$ and $t_0 - \delta < t_1 < t_2 < t_0 + \delta$. If $\partial \Omega \cap ((\xi_1, \xi_2) \times (t_1, t_2)) \neq \emptyset$, by (3.15), we have either the right hand side is zero if $|n_1| \equiv 0$ or the right hand side is positive if $|n_1| \neq 0$. If the former case happens, we have

$$|\Omega(t_2) \cap (\xi_1, \xi_2)| = |\Omega(t_1) \cap (\xi_1, \xi_2)|$$
for \( x_0 - \varepsilon < \xi_1 < \xi_2 < x_0 + \varepsilon \) and \( t_0 - \delta < t_1 < t_2 < t_0 + \delta \). This contradicts with \( |n_1| \equiv 0 \). Hence the latter case must hold and then Claim 1 is completed.

Claim 2: we show the Lipschitz curve can be represented as a Lipschitz function of \( x \) locally. Namely, there exists a Lipschitz \( \varphi(\cdot) \) such that \( t = \varphi(x) \) for \( x \in (x_0 - \varepsilon, x_0 + \varepsilon) \) with \( t_0 = \varphi(x_0) \).

For contradiction, suppose that the Lipschitz curve passes through two points \((x_3, t_3)\) and \((x_3, t_4)\) in \((x_0 - \varepsilon, x_0 + \varepsilon) \times I_3\). If \( x_3 \leq x_0 \), taking \( \xi_1 = x_3 - \kappa \) for some sufficiently small \( \kappa > 0 \), \( \xi_2 = x_3 \), \( t_1 = t_3 \) and \( t_2 = t_4 \) it follows from (b) that

\[
|\Omega(t_2) \cap (\xi_1, \xi_2)| = |\Omega(t_1) \cap (\xi_1, \xi_2)|,
\]

which contradicts (3.16). Similarly, if \( x_3 \geq x_0 \), we can reach a contradiction. Hence Claim 2 is completed.

We now complete the proof of this lemma. First we observe that \( t = \phi(\cdot) \) cannot have a local minimum point on \((x_0 - \varepsilon, x_0 + \varepsilon)\). Otherwise, by (C2) we reach a contradiction immediately. It follows that one of the following cases must occur:

1. \( t = \phi(\cdot) \) is monotone decreasing on \((x_0 - \varepsilon, x_0 + \varepsilon)\).
2. \( t = \phi(\cdot) \) is monotone increasing on \((x_0 - \varepsilon, x_0 + \varepsilon)\).
3. \( t = \phi(\cdot) \) is monotone increasing on \((x_0 - \varepsilon, {\eta})\) and is monotone decreasing on \((\eta, x_0 + \varepsilon)\) for some \( \eta \in (x_0 - \varepsilon, x_0 + \varepsilon) \).

We shall show that (1) cannot occur. Indeed, from (b) and (3.17) with \( t_i := \phi(\xi_i) \) for \( i = 1, 2 \), we obtain

\[
0 = |\Omega(t_2) \cap (\xi_2, \xi_1)| > |\Omega(t_1) \cap (\xi_2, \xi_1)|,
\]

which is impossible. This shows that (1) never occurs.

For (2), we furthermore show that \( t = \phi(\cdot) \) is strictly increasing. If it is not true, then its graph contains a flat piece, say \( \phi(x) = \kappa \) on \((p_1, p_2)\) for some \( \kappa > 0 \) and \( p_i \in (x_0 - \varepsilon, x_0 + \varepsilon) \). Then by (C2), we reach a contradiction. Hence \( t = \phi(\cdot) \) is strictly increasing. So its inverse function \( x = x(t) \) is well defined. Moreover, by (3.17) with \( \xi_i = x(t_i) \) for \( i = 1, 2 \), we have

\[
(3.18) \quad x(t_2) - x(t_1) = \int_{t_1}^{t_2} W(v(x(t), t))dt.
\]

Differentiating (3.18) in \( t_2 \) and using (3.15), we obtain the conclusion (i) of Lemma 3.16.

For (3), if \( x_0 < \eta \), then we can shrink \( \varepsilon > 0 \) and \( \delta > 0 \) sufficiently small and reduces this case into case (2). Then by the same process we can obtain the conclusion (i) of Lemma 3.16. If \( x_0 = \eta \), then following the process of (2) we see that \( t = \phi(\cdot) \) is strictly increasing (resp. decreasing) on \((x_0 - \varepsilon, x_0)\) (resp. \([x_0, x_0 + \varepsilon)\)). Hence there exist two continuous functions \( x_1(t) \) and \( x_2(t) \) such that \( x_i(t_0) = x_0 \) for \( i = 1, 2 \). Using (3.17), we can obtain the conclusion (ii) of Lemma 3.16. This completes the proof.

Similar result holds for \( W(v(x_0, t_0)) < 0 \). We state the result as follows without repeating a proof.

Lemma 3.17. Let \((\Omega, v)\) be a weak solution of (2.1) for \( t \in [0, T] \). Also, assume that \((x_0, t_0) \in \partial \Omega(t_0)\) for some \( t_0 \in (0, T) \) such that

(a) \( W(v(x_0, t_0)) < 0 \),

(b) \((x_0 - \varepsilon, t_0) \in \Omega(t_0)\) (resp. \((x_0 + \varepsilon, t_0) \in \Omega(t_0)\)) for any sufficiently small \( \varepsilon > 0 \).
Then one of the following cases holds:

(i) there are positive constants \( \delta, \varepsilon \) and a function \( x(\cdot) \in C^1((t_0-\delta, t_0+\delta)) \) such that \( x'(t_0) < 0 \) (resp. \( x'(t_0) > 0 \)) and

\[
\{(x(t), t) \mid t \in (t_0 - \delta, t_0 + \delta)\} = \partial \Omega \cap \left[(x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \delta, t_0 + \delta)\right],
\]

(ii) there are positive constants \( \delta, \varepsilon \) and two functions \( x_i(\cdot) \in C^1((t_0 - \delta, t_0)) \) \( (i = 1, 2) \) such that \( x_i(t_0) = x_0 \),

\[
\lim_{t \to t_0^-} x_1(t) = \lim_{t \to t_0^+} x_2(t) = x_0, \quad \lim_{t \to t_0^-} x_1'(t) > 0, \quad \lim_{t \to t_0^+} x_2'(t) < 0,
\]

\[
\bigcup_{i=1}^2 \{(x_i(t), t) \mid t_0 - \delta < t \leq t_0\} = \partial \Omega \cap \left[(x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \delta, t_0)\right].
\]

Namely, \((x_0, t_0)\) is an intersection point of two interfaces.

**Remark 3.18.** (i) We remark that Lemmas 3.16 and 3.17 still hold for \( t_0 = 0 \). Indeed, following the same process in the proof therein but replace \((t_0 - \delta, t_0 + \delta)\) by \([0, \delta]\) we can show that conclusion (i) with \((t_0 - \delta, t_0 + \delta)\) replaced by \([0, \delta]\) in Lemmas 3.16 and 3.17 always occurs when \( t_0 = 0 \).

(ii) Lemmas 3.16 and 3.17 show that if \((\Omega, v)\) is a weak solution for \( t \in [t_0, t_1] \), then there exists an integer \( N \) such that \((\Omega, v)\) becomes a classical solution for \( t \in [\tau_i, \tau_{i+1}] \) for some \( i = 0, 1, \ldots, N \) with \( \tau_0 = t_0 \) and \( \tau_N = t_1 \), where \( \tau_i \) is an annihilation time for \( 1 \) \( \geq \) \( N - 1 \).

Suppose that \((\Omega_1, v_1)\) and \((\Omega_2, v_2)\) are two weak solutions of (2.1) for \( t \in [0, T] \). The following lemma shows that \( v_1 = v_2 \) over a rectangle lying in either \( \Omega_1 \cap \Omega_2 \) or \( \Omega_1^c \cap \Omega_2^c \), and its bottom edge lies on the \( x \)-axis.

**Lemma 3.19.** Let \((\Omega_1, v_1)\) and \((\Omega_2, v_2)\) be two weak solutions of (2.1) for \( t \in [0, T] \). Then \( v_1 = v_2 \) in \( \overline{Q} \), where

\[
Q := \left\{(x, t) \mid \exists \varepsilon > 0 \text{ such that } [x - \varepsilon, x + \varepsilon] \times [0, t] \text{ lies in either } \Omega_1 \cap \Omega_2 \text{ or } \Omega_1^c \cap \Omega_2^c\right\}.
\]

**Proof.** Let \((x_0, t_0) \in Q\). Let \( L_0 > 0 \) (resp. \( L_1 > 0 \)) be the Lipschitz constant for \( g(0, \cdot) \) (resp. \( g(1, \cdot) \)). First, we choose \( 0 < t_1 < t_0 \) such that

\[
\max\{L_0, L_1\}t_1 < \sqrt{2}.
\]

Set \( z := 1_{\Omega_1} - 1_{\Omega_2} \) and \( w := v_1 - v_2 \). Since \((x_0, t_0) \in Q\), we can find an interval \( I_\varepsilon = [x_0 - \varepsilon, x_0 + \varepsilon] \) such that \( I_\varepsilon \times [0, t_0] \) lies in either \( \Omega_1 \cap \Omega_2 \) or \( \Omega_1^c \cap \Omega_2^c \). It follows that \( z = 0 \) in \( I_\varepsilon \times [0, t_0] \). In particular, \( z = 0 \) in \( I_\varepsilon \times [0, t_1] \).

By Lemma 3.12, \((\Omega_1, v_1)\) and \((\Omega_2, v_2)\) are two weak solutions of (2.1) for \( 0 < t < t_1 \). By definition of weak solutions and choosing a test function \( \psi \) satisfying

\[
\psi := 1_{I_\varepsilon}(x) \int_t^{t_1} w^+(x, s) ds, \quad w^+ := \max\{w, 0\},
\]

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we have
\[
0 = \left[ \int_{\mathbb{R}} w\psi dx \right]_{0}^{t_1} = \int_{0}^{t_1} \int_{\mathbb{R}} \left( w\psi_t + (g(1, v_1) - g(1, v_2))\psi \right) dx dt \\
\leq - \int_{0}^{t_1} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} (w^+)^2 dx dt + \max\{L_0, L_1\} \int_{0}^{t_1} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} w \left( \int_{t}^{t_1} w^+(x, s) ds \right) dx dt \\
\leq - \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} (w^+)^2 dx dt + \frac{1}{\sqrt{2}} \max\{L_0, L_1\} t_1 \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} (w^+)^2 dx dt,
\]
where we have used the Hölder inequality to obtain the last inequality.

By (3.19), from the above estimate we must have \( w^+ = 0 \) and so \( w \leq 0 \) in \( I_\varepsilon \times [0, t_1] \). Similarly, we have \( w \geq 0 \) and so \( w = 0 \) in \( I_\varepsilon \times [t_1, \min\{2t_1, t_0\}] \). Using a bootstrap argument, we can obtain that \( w = 0 \) in \( I_\varepsilon \times [0, t_0] \). This completes the proof. \( \square \)

**Remark 3.20.** From the proof of Lemma 3.19, we see that the conclusion still hold when the interval \( [x - \varepsilon, x + \varepsilon] \) in \( Q \) is replaced by \( [x, x + \varepsilon] \) or \([x-\varepsilon, x]\).}

**Lemma 3.21.** Let \((\Omega_1, v_1)\) and \((\Omega_2, v_2)\) be two weak solutions of (2.1) for \( t \in [0, T] \). Suppose that there exist a rectangle \( D_T := [\xi_1, \xi_2] \times [0, T] \) and continuous functions \( y_k(t), k = 1, 2, \) such that
\begin{itemize}
  \item[(i)] \( \xi_1 < y_1(t_1) \leq y_2(t_1) < \xi_2 \) for \( 0 \leq t \leq T \) and \( y_1(0) = y_2(0) \);
  \item[(ii)] either \( \Omega_k \cap D_T = \{(x, t) \mid \xi_1 < x < y_k(t), 0 \leq t \leq T\} \) for \( k = 1, 2 \) or \( \Omega_k \cap D_T = \{(x, t) \mid y_k(t) < x < \xi_2, 0 \leq t \leq T\} \) for \( k = 1, 2 \).
\end{itemize}
Then \( \Omega_1 \cap D_T = \Omega_2 \cap D_T \) and \( v_1 = v_2 \) in \( D_T \).

**Proof.** Since the proof is similar, we only consider the case \( \Omega_k \cap D_T = \{(x, t) \mid \xi_1 < x < y_k(t), 0 \leq t \leq T\} \) for \( k = 1, 2 \). Because of assumptions (i) and (ii), we can apply Lemma 3.19 (and Remark 3.20) so that \( v_1(\xi_j, t) = v_2(\xi_j, t) \) for \( 0 < t < T \) and \( j = 1, 2 \) and \( v_1(y_2(t), t) = v_2(y_2(t), t) \) for \( 0 < t < T \).

Taking test functions \( 1_{[\xi_1, \xi_2]} \varphi \) and \( 1_{[\xi_1, \xi_2]} \psi \) implies
\[
\int_{\xi_1}^{\xi_2} v_k \psi dx = \int_{0}^{T} \int_{\xi_1}^{\xi_2} v_k \psi dx dt + \int_{0}^{T} W(v_k(y_k(t), t))\varphi(y_k(t), t) dt,
\]
for \( k = 1, 2 \). Set \( z := 1_{\Omega_2} - 1_{\Omega_1} \) and \( w := v_2 - v_1 \). Subtracting each equality for \( k = 1, 2 \), we get
\begin{align*}
(3.20) & \quad \int_{y_1(t)}^{y_2(t)} \varphi dx = \int_{0}^{T} \int_{y_1(t)}^{y_2(t)} \varphi_t dx dt \\
& \quad + \int_{0}^{T} \left( W(v_2(y_2(t), t))\varphi(y_2(t), t) - W(v_1(y_1(t), t))\varphi(y_1(t), t) \right) dt, \\
(3.21) & \quad \int_{\xi_1}^{\xi_2} w \psi dx = \int_{0}^{T} \int_{\xi_1}^{\xi_2} \left( w\psi_t + (g(1, v_2) - g(1, v_1))\psi \right) dx dt.
\end{align*}
First, we show that \( v_1 \leq v_2 \) in \( D_T \). Plugging
\[
\psi(x,t) = \begin{cases} 
0 & \text{if } w(x,t) \geq 0, \\
-1 & \text{if } w(x,t) < 0,
\end{cases}
\]
we get from (3.21) that
\[
\int_{\xi_1}^{\xi_2} w^-(x,T)dx = \int_0^T \int_{\xi_1}^{\xi_2} \left( g(1_{\Omega_2(t)}, v_2) - g(1_{\Omega_1(t)}, v_1) \right) \psi dx dt,
\]
where \( w^- := \max\{-w,0\} \geq 0 \). Also, we see from assumptions (i) and (ii) that
\[
\left( g(1_{\Omega_2(t)}, v_2) - g(1_{\Omega_1(t)}, v_2) \right) \psi \leq 0 \quad \text{in } D_T.
\]
This implies
\[
\int_{\xi_1}^{\xi_2} w^-(x,T)dx = \int_0^T \int_{\xi_1}^{\xi_2} \left( g(1_{\Omega_2(t)}, v_2) - g(1_{\Omega_1(t)}, v_2) \right) \psi dx dt
\]
\[
+ \int_0^T \int_{\xi_1}^{\xi_2} \left( g(1_{\Omega_1(t)}, v_2) - g(1_{\Omega_1(t)}, v_1) \right) \psi dx dt
\]
\[
\leq C \int_0^T \int_{\xi_1}^{\xi_2} w^- dx dt.
\]
The Gronwall inequality implies that \( w^- \equiv 0 \). Namely, \( v_1 \leq v_2 \) in \( D_T \).

Next, we show that \( \Omega_1 \cap D_T = \Omega_2 \cap D_T \). Plugging
\[
\varphi(t) = \int_t^T (y_2(s) - y_1(s))ds \geq 0, \quad \psi(x,t) = 1_{[\xi_1, \xi_2]}(x) \int_t^T w(x,s)ds
\]
into (3.20) and (3.21) implies
\[
(3.22) \quad 0 = -\int_0^T (y_2 - y_1)^2 dt - b \int_0^T \left( v_2(y_2(t), t) - v_1(y_1(t), t) \right) \varphi(t) dt,
\]
\[
(3.23) \quad 0 = -\int_0^T \int_{\xi_1}^{\xi_2} w^2 dx dt + \int_0^T \int_{\xi_1}^{\xi_2} \left( g(1_{\Omega_2(t)}, v_2) - g(1_{\Omega_1(t)}, v_1) \right) \psi dx dt.
\]
Recall that \( v_1(y_2(t), t) = v_2(y_2(t), t) \) for \( 0 < t < T \). Then from (3.22) we have
\[
0 \leq -\int_0^T (y_2 - y_1)^2 dt - b \int_0^T \left( v_2(y_2(t), t) - v_1(y_1(t), t) \right) \varphi(t) dt
\]
\[
- b \int_0^T \left( v_1(y_2(t), t) - v_1(y_1(t), t) \right) \varphi(t) dt
\]
\[
\leq -\int_0^T (y_2 - y_1)^2 dt + b \varphi(0) \int_0^T |v_1|_{\text{Lip}} (y_2(t) - y_1(t)) dt
\]
\[
\leq -\int_0^T (y_2 - y_1)^2 dt + b|v_1|_{\text{Lip}} \varphi(0)^2
\]
\[
\leq -\left( 1 - bT|v_1|_{\text{Lip}} \right) \int_0^T (y_2 - y_1)^2 dt,
\]
by using the Schwarz inequality. We conclude that \( \Omega_1 \cap D_T = \Omega_2 \cap D_T \) for small \( T > 0 \). Thus from
(3.23) we have
\[
0 = -\int_0^T \int_{\xi_1}^{\xi_2} w^2 dx dt + \int_0^T \int_{\xi_1}^{\xi_2} \left( g(1_{\Omega_1(t)}, v_2) - g(1_{\Omega_1(t)}, v_1) \right) \psi dx dt.
\]
From this identity, by a similar argument as above leads that
\[
0 \leq -(1 - T|g_v|_{L^\infty}) \int_0^T \int_{\xi_1}^{\xi_2} |w|^2 dx dt.
\]
Thus we get \( v_1 = v_2 \) in \([\xi_1, \xi_2] \times [0, T]\) for small \( T > 0 \).

Therefore, by a bootstrap argument as in Lemma 3.19, we can complete the proof. \qed

We are ready to show Proposition 3.15.

**Proof of Proposition 3.15.** First, we claim that \( \Omega_1 = \Omega_2 \) for \( 0 < t < \tau \) where \( \tau \) is small enough. Given any \( x_0 \in \partial \Omega(0) \), let us consider the case of Lemma 3.16 (i) or Lemma 3.17 (i). Then together with Remark 3.18, there are positive constants \( \delta_i, \varepsilon_i \) and functions \( x_i(\cdot) \in C^1([0, \delta_i]), i = 1, 2 \) such that
\[
\{(x_i(t), t) \mid t \in [0, \delta_i]\} = \partial \Omega_i \cap \left[(x_0 - \varepsilon_i, x_0 + \varepsilon_i) \times [0, \delta_i]\right], \quad i = 1, 2.
\]
Taking \( \xi_1 = x_0 - \varepsilon, \xi_2 = x_0 + \varepsilon \) with \( \varepsilon = \max\{\varepsilon_1, \varepsilon_2\} \) and \( T = \min\{\delta_1, \delta_2\} \). In view of Lemma 3.21 we have \( x_1(t) = x_2(t) \) for \( t \in [0, T] \). Since \( x_0 \) is given arbitrarily and \( \Omega(0) \) consists of a finite number of bounded intervals, there exists a \( \tau \) small enough such that \( \Omega_1(t) = \Omega_2(t) \) for \( 0 \leq t \leq \tau \).

Next, for \( i = 1, 2 \), we set \( T_i \) as an annihilation time of \( \Omega_i \). Without loss of generality, we assume \( T_1 \leq T_2 \). By a bootstrap argument as in Lemma 3.19, we have \( \Omega_1(t) = \Omega_2(t) \) for \( 0 \leq t \leq \tau \) for any \( \tau \leq T_1 \). By the definition of the annihilation time, we know that \( T_1 = T_2 \). Applying Lemma 3.16 (ii), Lemma 3.17 (ii) and Lemma 3.21, we obtain that \( \Omega_1(t) = \Omega_2(t) \) for \( 0 \leq t \leq T_1 \). Again, by a bootstrap argument as in Lemma 3.19, we have \( \Omega_1 = \Omega_2 \).

Finally, set \( \Omega_1 = \Omega_2 = \Omega \). If \( (x_0, t_0) \in \partial \Omega \), we have \( v_1(x_0, t_0) = v_2(x_0, t_0) \) by Lemmas 3.16, 3.17 and 3.21; if \( (x_0, t_0) \notin \partial \Omega \), we obtain \( v_1(x_0, t_0) = v_2(x_0, t_0) \) by Lemma 3.19. The proof of this proposition is done. \qed

We end this section with the proof of Theorem 2.5.

**Proof of Theorem 2.5.** Theorem 2.5 follows from Proposition 3.14 and Proposition 3.15. \qed

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**APPENDIX**

**Proof of Lemma 3.6.** We divide \([A, B] \times [0, \tau]\) into \[
\left( \bigcup_{j=1}^{2m} x_j([0, \tau]) \times [0, \tau] \right), \quad [A, B] \times [0, \tau] \setminus \left( \bigcup_{j=1}^{2m} x_j([0, \tau]) \times [0, \tau] \right).
\]
By (3.6), we see that $I_k := x_k([0,\tau]) \times [0,\tau]$ contains exactly one interface $x = x_k(t), t \in [0,\tau].$

This allows us to represent $v$ in terms of functions in (3.1). Indeed, by some simple computations, we have the following:

(i) For $j \in \{1, 2, ..., m\}$, if $x_{2j}(\cdot)$ is increasing (resp. $x_{2j-1}(\cdot)$ is decreasing), then

\[
v(x,t) = \begin{cases} G_0^{-1}(v_0(x)) + t, & t \leq T_k(x), (x,t) \in I_k, \\ G_1^{-1}(v(x,T_k(x))) + t - T_k(x), & t > T_k(x), (x,t) \in I_k,
\end{cases}
\]

where $k = 2j$ (resp. $k = 2j - 1$).

(ii) For $j \in \{1, 2, ..., m\}$, if $x_{2j}(\cdot)$ is decreasing (resp. $x_{2j-1}(\cdot)$ is increasing), then

\[
v(x,t) = \begin{cases} G_1^{-1}(v_0(x)) + t, & t \leq T_k(x), (x,t) \in I_k, \\ G_0^{-1}(v(x,T_k(x))) + t - T_k(x), & t > T_k(x), (x,t) \in I_k,
\end{cases}
\]

where $k = 2j$ (resp. $k = 2j - 1$).

Let us first deal with (i). For $t \leq T_k(x)$, since

\[
v(x,t) = G_0^{-1}(v_0(x)) + t,
\]

as in deriving (3.4), Lemma 3.4 and the Lipschitz continuity of $v_0$ imply the Lipschitz continuity of $v$ for $(x,t) \in I_k$ with $t \leq T_k(x)$. For $t > T_k(x)$, since $T_k$ is strictly monotone (because of (3.2)), there exist a unique $z \in (x,x_k(\tau)) \subset I_k$ such that $t = T_k(z)$ (see Figure A.2).

Then, if $x < z \leq y < x_k(\tau)$, namely, $T_k(x) < t \leq T_k(y)$, we have

\[
|v(x,t) - v(y,t)| = |G_1^{-1}(v(x,T_k(x))) + T_k(z) - T_k(x) - G_0^{-1}(v(y,0)) + T_k(z)|
\]

\[
\leq |G_1^{-1}(v(x,T_k(x))) + T_k(z) - T_k(x)| + |G_0^{-1}(v(z,0)) + T_k(z)|
\]

\[
=: J_1 + J_2.
\]

Let us define

\[
C_1 := \sup_{(x,t) \in [A,B] \times [0,T]} |v(x,t)|, \quad C_2 := G_1^{-1}(C_1) + T.
\]

Using $v(z,T_k(z)) = G_1^{-1}(v(z,T_k(z)))$ and the mean value theorem, we have

\[
J_1 \leq G_1^{-1}(v(x,T_k(x))) - G_1^{-1}(v(z,T_k(z))) + T_k(z) - T_k(x)
\]

\[
\leq \left(\|G_1^{-1}\|_{L^\infty([0,C_2])}|v(x,T_k(x)) - v(z,T_k(z))| + |T_k(z) - T_k(x)|\right).
\]
Note that \((G_1^{-1})'(v) = 1/g(1, v)\) and \(0 < g_1 - g_2/g_3 \leq g(1, v) \leq g_1\). Hence \(\|G_1'\|_{L^\infty([0, C_2])} < \infty\).

Next, using \(v(\zeta, T_k(\zeta)) = G_0 \left( G_0^{-1}(v_0(\zeta)) + T_k(\zeta) \right)\) for \(\zeta = x, z\) and Lemma 3.4, we have

\[
J_1 \leq C_3 \left( |v_0(z) - v_0(x)| + |T_k(z) - T_k(x)| \right)
\]

\[
\leq C_3 \left( L_0|x - z| + \frac{1}{\delta}|x - z| \right) \leq C_4|x - y|
\]

for some positive constant \(C_3\) and \(C_4\). We can also get

\[
J_2 \leq C_5|y - z| \leq C_5|x - y|
\]

for some positive constant \(C_5\) by the same argument to the proof of outside of \(D_T\). Combining the estimates for \(J_1\) and \(J_2\), from (A.1), we see that \(v\) is Lipschitz continuous when \(T_k(x) < t\) and \(T_k(y) \geq t\). For the case where \(T_k(x) \leq T_k(y) < t\), we directly obtain

\[
|v(x, t) - v(y, t)| = |G_1 \left( G_1^{-1}(v(x, T_k(y))) + t - T_k(y) \right) - G_1 \left( G_1^{-1}(v(y, T_k(y))) + t - T_k(y) \right)|
\]

\[
\leq \|G_1'\|_{L^\infty([0, C_2])} \|G_1^{-1}\|_{L^\infty([0, C_1])} |v(x, T_k(y)) - v(y, T_k(y))|.
\]

Since this reduces to the previous case, we see that \(v\) is Lipschitz continuous on \(I_k\) when (i) holds. The similar process is applicable to assert the Lipschitz continuity of \(v\) on \(I_k\) when (ii) occurs. We omit the detailed proof.

**Proof of Lemma 3.7.** By Proposition 3.1, there exists a positive constant \(\tau_0\) such that problem (2.1) has a classical solution for \(t \in [0, \tau_0]\) and \(x_k(\cdot)\) is strictly monotone on \([0, \tau_0]\) for each \(k = 1, \ldots, 2m\).

We show that the local in time classical (\(\Omega, v\)) is unique. Let (\(\Omega, v\)) and (\(\tilde{\Omega}, \tilde{v}\)) be two local in time classical solutions of (2.1) for \(t \in [0, \tau_0]\), where

\[
\tilde{\Omega}(t) = \bigcup_{j=1}^m (\tilde{x}_{2j-1}(t), \tilde{x}_{2j}(t)).
\]

Moreover, we may assume that \(x_k'(\cdot)\) and \(\tilde{x}_k'(\cdot)\) never vanish in \([0, \tau_0]\) by choosing \(\tau_0\) sufficiently small. We shall show that \(\Omega = \tilde{\Omega}\) and \(v = \tilde{v}\) for \(t \in [0, \tau_0]\). For this, we first show \(x_{2m}(t) = \tilde{x}_{2m}(t)\) for \(t \in [0, \tau_0]\). Without loss of generality, we may assume that \(W(v_0(x_{2m}^0)) > 0\) i.e., \(x_{2m}(\cdot) > 0\) and \(\tilde{x}_{2m}(\cdot) > 0\). Without loss of generality, we may assume that

\[
(A.2) \quad x_{2m}(t) > \tilde{x}_{2m}(t), \quad t \in (0, \tau_0).
\]

In fact, the following proof with slight modifications works for the case that \(x_{2m} \geq \tilde{x}_{2m}\).

Let \(T_k(\xi)\) (resp. \(\tilde{T}_k(\xi)\)) be the arrival time of \(x_k(t)\) (resp. \(\tilde{x}_k(t)\)) at \(\xi\). For each \(\xi \in (x_{2m}^0, \tilde{x}_{2m}(\tau_0^-))\),

\[
(T.3) \quad \begin{cases}
T_{2m}(\xi) = \frac{1}{x_{2m}(T_{2m}(\xi))} = \frac{1}{a - bv(\xi, T_{2m}(\xi))}, \\
\tilde{T}_{2m}(\xi) = \frac{1}{\tilde{x}_{2m}(T_{2m}(\xi))} = \frac{1}{a - b\tilde{v}(\xi, \tilde{T}_{2m}(\xi))}.
\end{cases}
\]

By (A.2), we have

\[
(A.4) \quad T_{2m}(\xi) < \tilde{T}_{2m}(\xi), \quad \xi \in [x_{2m}^0, \tilde{x}_{2m}(\tau_0^-)).
\]

Note that \(\tilde{v}_t(\xi, t) = g(0, \tilde{v}) < 0\) for \(t \in [T_{2m}(\xi), \tilde{T}_{2m}(\xi)]\), and \(\tilde{v}(\xi, T_{2m}(\xi)) = v(\xi, T_{2m}(\xi))\), we have

\[
v(\xi, T_{2m}(\xi)) > \tilde{v}(\xi, \tilde{T}_{2m}(\xi)), \quad \xi \in (x_{2m}^0, \tilde{x}_{2m}(\tau_0^-)).
\]
Together with (A.3), we see that
\[ \tilde{T}_m^j(x, \xi) < M^j_m(x, \xi), \quad \xi \in (x_2^m, \tilde{x}_2^m(\tau_0^+)). \]
By integrating over \([x_2^m, \tilde{x}_2^m]\), we reach a contradiction with (A.4). Thus, we must have \(x_2^m(t) = \tilde{x}_2^m(t)\) for \(t \in [0, \tau_0]\). The same argument can be applied to prove \(x^j(t) = \tilde{x}^j(t)\) for \(t \in [0, \tau_0]\) for \(j = 1, \ldots, 2m - 1\), but the details are tedious. We safely omit the details here. Thus, we obtain \(\Omega = \tilde{\Omega}\). Moreover, \(v\) and \(\tilde{v}\) can be represented in terms of the functions in (3.1) as in the proof of Proposition 3.1. Note that \(\Omega = \tilde{\Omega}\) implies that \(T_i(x) = \tilde{T}_i(x)\) for any \(x \in \mathbb{R}\) and \(i = 1, \ldots, 2m\). It follows that \(v = \tilde{v}\). From the above discussion, one can use a bootstrap argument to extend the local in time classical solution uniquely until \(x^j_k\) vanishes for some \(k\) or an annihilation occurs. This completes the proof.

**Proof of Lemma 3.10.** Take \(\psi := \phi_\epsilon 1_{(-\infty, K)\times[0, t_0]}\) as a test function, where \(\phi_\epsilon := \max\{-\phi, 0\}\) and we also used \(1_{\Omega}\) as the characteristic function of \(\Omega \subset \mathbb{R}^2\). Since \((\nu^-)_t = -\nu_1 1_{\{\nu < 0\}}\), we have
\[ \int_0^T \int_{\mathbb{R}} v\phi dx dt = -\frac{1}{2} \int_{-K}^K v^2(x, t_0) \cdot 1_{\{\nu(x, t_0) < 0\}} dx. \]
Therefore, for any large \(K\) and any \(t_0 \in (0, T)\), it follows from (2.8) that \(v^-(x, t_0) = 0\) for all \(x \in (-K, K)\).

**Proof of Lemma 3.12.** For any given \(\tau \in (0, T)\), we replace \(\varphi\) in (2.7) by \(\varphi \eta_\epsilon(t)\), where
\[ \eta_\epsilon(t) := \begin{cases} 1, & 0 \leq t \leq \tau - \epsilon, \\ \frac{\tau - t}{\epsilon}, & \tau - \epsilon \leq t \leq \tau, \\ 0, & \tau \leq t \leq T. \end{cases} \]
It follows from (2.7) that
\[ (A.5) \quad \int_{\mathbb{R}} 1_{\Omega(0)} \varphi(x, 0) dx = \int_0^T \int_{\Omega(t)} (\varphi \eta_\epsilon) dx dt + \int_{\partial\Omega(\mathbb{R} \times (0, T))} W(v) \varphi \eta_\epsilon |n_1| d\sigma. \]
Note that
\[ \int_0^T \int_{\Omega(t)} (\varphi \eta_\epsilon) dx dt = \int_0^T \int_{\Omega(t)} \varphi \eta_\epsilon dx dt - \frac{1}{\epsilon} \int_{\tau - \epsilon}^\tau \int_{\Omega(t)} \varphi dx dt \]
\[ \quad \rightarrow \int_0^T \int_{\Omega(t)} \varphi dx dt - \int_{\mathbb{R}} 1_{\Omega(\tau)} \varphi(x, \tau) dx \quad \text{as} \ \epsilon \rightarrow 0. \]
Therefore, taking \(\epsilon \rightarrow 0\) in (A.5), we obtain
\[ \left[ \int_{\mathbb{R}} 1_{\Omega(t)} \varphi dx \right]_0^\tau = \int_0^\tau \int_{\Omega(t)} \varphi dx dt + \int_{\partial\Omega(\mathbb{R} \times (0, T))} W(v) \varphi |n_1| d\sigma. \]
Similarly, if we replace \(\psi\) in (2.8) by \(\psi \eta_\epsilon(t)\) with \(\eta_\epsilon(t)\) defined above, then the above process can be applied to obtain
\[ \left[ \int_{\mathbb{R}} v\psi dx \right]_0^\tau = \int_0^\tau \int_{\mathbb{R}} (v\psi_t + g(1_{\Omega(t)}, v)\psi) dx dt. \]
Hence the proof of Lemma 3.12 is completed. \qed
Proof of Lemma 3.13. By the assumption, we have
\[
\left[ \int_\Omega \varphi dx \right]_0^{T_1} = \int_0^{T_1} \varphi_1 dt + \int_{\partial \Omega_1 \cap (\mathbb{R} \times (0,T))} W(v_1) n_1 d\sigma,
\]
\[
\left[ \int v_1 \psi dx \right]_0^{T_1} = \int_0^{T_1} \int_\Omega \left( v_1 \psi_t + g(1_{\Omega_1(t)} v_1) \right) dx dt,
\]
\[
\left[ \int 1_{\Omega_2(t)} \varphi dx \right]_0^{T_2} = \int_0^{T_2} \int_{\Omega_2(t)} \varphi_1 dt + \int_{\partial \Omega_2 \cap (\mathbb{R} \times (0,T))} W(v_2) n_1 d\sigma,
\]
\[
\left[ \int v_2 \psi dx \right]_0^{T_2} = \int_0^{T_2} \int_\Omega \left( v_2 \psi_t + g(1_{\Omega_2(t)} v_2) \right) dx dt.
\]
Let \((\hat{\Omega}, \hat{v})\) be as in (3.13) and (3.14). By adding the above equalities, we obtain (2.7) and (2.8) for \(0 \leq t \leq T_2\) and for \((\hat{\Omega}, \hat{v})\). By the assumptions, it is not hard to obtain \((\hat{\Omega}, \hat{v}) \in X_{T_2}\) and satisfies (C2). Hence \((\hat{\Omega}, \hat{v})\) is a weak solution for \(0 \leq t \leq T_2\) and then the proof is completed.

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