Multiphoton coherent states for bilayer graphene

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Abstract

The multiphoton coherent states, a generalization to coherent states, are derived for electrons in bilayer graphene placed in a constant homogeneous magnetic field which is orthogonal to the bilayer surface. For that purpose a generalized annihilation operator is constructed in order to determine the multiphoton coherent states as eigenstates of such operator with complex eigenvalue. In addition, some physical quantities are calculated for these states, as the Heisenberg uncertainty relation, the probability density and the mean energy value. Finally, in order to study the dynamics of the system the time evolution is explored and the time-correlation function is computed.

1. Introduction

In 1900, Max Planck introduced for the first time the concept of quantization to explain black-body radiation. The revolutionary idea that the exchange of energy between radiation and matter takes place in a discrete way, through quantum units of energy, was the first breakthrough that gave rise to quantum mechanics, a probabilistic and indeterministic theory describing the microscopic world [1]. Since then, quantum mechanics has became the basis of modern physics and has been used in different branches, giving place to many theoretical and technological developments.

Nevertheless, the efforts trying to establish a connection between quantum and classical theories keep constant, and they have contributed to the emergence of different semi-classical approaches. One of them is the so-called coherent states (CS) which were proposed first by Erwin Schrödinger in 1926 for the harmonic oscillator [2]. The standard coherent states (SCS) are quantum states that minimize the Heisenberg uncertainty relation, they evolve along the classical trajectory and are not deformed in time [3, 4]. These are the reasons for the SCS to be sometimes called quasi-classical states, since they provide a natural framework to analyze the connection between quantum and classical mechanics.

One of the most famous applications of the SCS happened in the early 1960s, when Glauber, Klauder and Sudarshan used them to describe coherent electromagnetic radiation [5–10], giving place to a new area in optics nowadays called quantum optics. The CS can be generalized, i.e., defined appropriately in order to describe other systems in different areas of physics, as condensed matter, particle, nuclear and atomic physics, among other [4, 11–13]. In particular, the so-called multiphoton coherent states (MCS) [14–16] are typically derived as eigenstates of powers of the annihilation operator; they have been addressed recently in [17] for the harmonic oscillator in the framework of polynomial Heisenberg algebras (PHA) [18–21]. Furthermore, the MCS were generated in [22] for the supersymmetric harmonic oscillator. Let us notice that the MCS have been used as non-classical states of light for multiphoton processes occurring in the matter-radiation interaction [23, 24].

It is worth noticing that coherent states and their generalizations have been employed recently to describe interesting physical systems that have attracted attention of the scientific community, the so-called 2D Dirac materials [25, 26]. In particular several works on the most conspicuous member of such a family, the graphene which is formed by carbon atoms arranged in a honeycomb hexagonal crystal lattice [27], have been addressed successfully through this semi-classical approach [28–33].

Motivated by the pioneer work about coherent states for monolayer graphene [28], a similar treatment was recently implemented to derive coherent states for electrons in bilayer graphene placed in a constant...
homogeneous magnetic field which is orthogonal to the bilayer surface [34] (see also [35]). With this in mind, the goal of this article is to extend and generalize the coherent states approach for bilayer graphene, by constructing now the corresponding multiphoton coherent states which turn out to be intrinsically quantum states. For that purpose, this work has been organized as follows. In section 2 a generalized annihilation operator is constructed and the MCS for bilayer graphene will be derived as eigenstates of such a matrix operator with complex eigenvalues $\tilde{\alpha}$. In order to describe and characterize the system, in section 3 several physical quantities as the Heisenberg uncertainty relation, probability density and mean energy value will be determined. In section 4 the time evolution is studied and the time-correlation function for the MCS will be obtained. Finally, in section 5 the conclusions of this work are presented.

2. Multiphoton coherent states

As mentioned before, the MCS are generalizations of the standard CS which can be defined as eigenstates of a generalized or deformed annihilation operator $A^- := (\hat{a}^-)^m$ with complex eigenvalues $\tilde{\alpha}$ [14–16]

$$\hat{A}^- |\tilde{\alpha}| = |\tilde{\alpha}|, \quad \tilde{\alpha} \in \mathbb{C}. \quad (1)$$

The generalized operators $A^+, A^- := (\hat{a}^\dagger)^m$ and the harmonic oscillator Hamiltonian generate an algebraic structure called polynomial Heisenberg algebra in which the Hilbert space $\mathcal{H}$ with states $|n\rangle$ is decomposed as a direct sum of $m$ orthogonal subspaces $\mathcal{H}_j$, i.e., $\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_j$ [18–21]. As a consequence, the MCS $|\tilde{\alpha}|_m$ will be built in each subspace $\mathcal{H}_j$ through equation (1), thus they can be expressed as superpositions of the Fock states $|mn+j\rangle$ as follows (see [22])

$$|\tilde{\alpha}|_m = c_j \sum_{n=0}^{\infty} \frac{2^n}{\sqrt{(mn+j)!}} |mn+j\rangle, \quad j = 0, \ldots, m-1, \quad (2)$$

with $c_j^m$ being normalization constants.

2.1. Generalized annihilation operator $\hat{A}^-$

Bilayer graphene coherent states (BGCS) were built recently in [34] as eigenstates of the simplest diagonal annihilation operator $\hat{A}^-$ of the form

$$\hat{A}^- = \begin{pmatrix} f_1(N) \hat{a}^- & 0 \\ 0 & f(N+1) \hat{a}^- \end{pmatrix}, \quad (3)$$

where the operators $\hat{a}^\pm$ and $\hat{N}$ are given by

$$\hat{a}^\pm = \frac{1}{\sqrt{2}} \left( \alpha D + \beta D^\dagger \right), \quad \hat{N} = \hat{a}^+ \hat{a}^- \quad (4)$$

with $\alpha = \sqrt{\frac{\omega}{2}} \left( x + \frac{2k}{\omega} \right)$ and $f, f_1, f_3$ are two arbitrary functions of the number operator $\hat{N}$.

Consider now a new generalized annihilation operator $\hat{A}^-_m$ defined as follows

$$\hat{A}^-_m := (\hat{A}^-)^m, \quad m \in \mathbb{Z}^+. \quad (5)$$

Using the explicit expression of $\hat{A}^-$ given in equation (3), the operator $\hat{A}^-_m$ turns out to be

$$\hat{A}^-_m = \begin{pmatrix} f_3(N) f_1(N+1) \cdots f_3(N+(m-1)1) (\hat{a}^-)^m \\ 0 \\ 0 \\ f(N+(m-1)1) f(N+2) \cdots f(N+m1)(\hat{a}^-)^m \end{pmatrix}, \quad (6)$$

where, $f, f_1, f_3$ will be used to ensure that

$$\hat{A}^-_m |\psi_n\rangle = a_n |\psi_{n-m}\rangle, \quad (7)$$

$a_n$ are coefficients to be determined later, and $|\psi_n\rangle_{n=0}^{\infty}$ are the eigenstates of the bilayer graphene effective Hamiltonian given by

$$|\psi_n\rangle = \exp(iky) \sqrt{2^{\delta_{nm}} \delta_{nm}} \left( 1 - \delta_{nm} \right) |\psi_{n-2}\rangle, \quad n = 0, 1, \ldots, \quad (8)$$

with $|\psi_n\rangle$ being the harmonic oscillator Fock states, whose explicit wave functions $\psi_n(x) = \langle x |\psi_n\rangle$ can be found in [34].

By applying $\hat{A}_m$ onto the eigenstates $|\psi_n\rangle$, in order to satisfy equation (7) the functions $f, f_1, f_3$ must fulfill the following constraint,
Consequently, the generalized annihilation operator $\hat{A}^-_g$ can be rewritten as follows

$$\hat{A}^-_g = \begin{pmatrix} \sqrt{(\bar{N} + 3) \cdots (\bar{N} + (m + 2))} f(\bar{N} + 3) \cdots f(\bar{N} + (m + 2)) (\hat{\alpha})^m & 0 \\ 0 & f(\bar{N} + 1) \cdots f(\bar{N} + m)(\hat{\alpha})^m \end{pmatrix},$$

thus the coefficients $a_n$ can be determined, i.e.,

$$\hat{A}^-_g |\Psi_i\rangle = \begin{cases} 0 & \text{for } n = 0, 1, \ldots, m - 1, \\ \frac{\sqrt{m!} [f(n)!]}{\sqrt{f(1)!}} |\Psi_i\rangle & \text{for } n = m, \\ \frac{n! [f(n)!]}{(n - m)! [f(n - m)!]} |\Psi_{n-m}\rangle & \text{for } n = m + 2, m + 3, \ldots, \\ \end{cases}$$

where $\delta_{ij}$ is the Kronecker delta and

$$|f(n)!| = \begin{cases} 1 & \text{for } n = 0, \\ f(1) \cdots f(n) & \text{for } n > 0. \end{cases}$$

### 2.2. Bilayer graphene MCS as eigenstates of $\hat{A}^-_g$

The bilayer graphene MCS (BGMCS) can be constructed as eigenstates $|\tilde{\alpha}\rangle_m$ of the generalized annihilation operator $\hat{A}^-_g$ of equation (10) with complex eigenvalue $\tilde{\alpha}$,

$$\hat{A}^-_g |\tilde{\alpha}\rangle_m = \tilde{\alpha} |\tilde{\alpha}\rangle_m, \quad \tilde{\alpha} \in \mathbb{C},$$

where the states $|\tilde{\alpha}\rangle_m$ are expressed as linear combinations of $|\Psi_0\rangle_{n=0}^\infty$, i.e.,

$$|\tilde{\alpha}\rangle_m = \sum_{n=0} C_n^m |\Psi_0\rangle.$$

From equations (12), (13), and using (11) as well as the linear independence of the states $|\Psi_0\rangle_{n=0}^\infty$, two recurrence relationships for the coefficients $C_n^m$ are obtained which lead to

$$C_n^m = \frac{\sqrt{2} \tilde{\alpha}}{\sqrt{(1 + \delta_{1m}) m! [f(m)]!}} C_0^m,$$

$$C_{n+1}^m = \frac{2^{2n} n! [f(n)]! \tilde{\alpha}}{(n + 1)! f(n + 1)!} C_n^m, \quad n = 1, 2, \ldots,$$

Note that there are $m$ free parameters $\{C_0^m, C_1^m, \ldots, C_{m-1}^m\} = \{C_m^m\}_{j=0}^{m-1}$, thus $m$ independent sets of BGMCS can be constructed, all of them depending on the particular choice of $f(n)$.

First of all, suppose that $f(n) \neq 0 \forall n = 1, 2, \ldots$ In particular, equations (14), (15) for $m = 1$ lead to

$$C_1^1 = \frac{\tilde{\alpha}}{f(1)} C_0^1, \quad C_{n+1}^1 = \frac{2^{2n} n! [f(n)]! \tilde{\alpha}}{(n + 1)! f(n + 1)!} C_n^1, \quad n = 1, 2, \ldots,$$

with $C_0^1$ being the only free parameter. Note that these recurrence relationships are the same as the ones obtained in [34] (see equation (26)). Thus, the BGMCS for $m = 1$ and $j = 0$ become

$$|\tilde{\alpha}\rangle_1 = C_0^1 |\Psi_0\rangle + \sum_{n=1} C_n^1 |\Psi_0\rangle_{f(n)}^n,$$

where $C_0^1$ will be used for normalizing them. These states are identical to the BGCS derived in [34] with $\tilde{\alpha} = \alpha$, i.e., for $m = 1$ the BGCS are recovered.
On the other hand, for \( m > 1 \) the \( m \) independent relations resulting from equations (14), (15) become

\[
C^m_{mn+j} = \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{\sqrt{(mn+j)!} [f(mn+j)]!} C_j^n, \quad n = 1, 2, \ldots, \tag{18}
\]

where \( j = \{0, 1, 2, \ldots, m-1\} \). From equations (13) and (18) the BGMCS turn out to be

\[
|\tilde{\alpha}_m\rangle = C_j^n \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{\sqrt{(mn+j)!} [f(mn+j)]!} |\Psi_{mn+j}\rangle \right]. \tag{19}
\]

The parameters \( C^m_j \) will be used for normalizing the BGMCS, which in general will depend on the values of the pair \( \{m, j\} \). Therefore, it is natural these states to be redefined as \(|\tilde{\alpha}_m\rangle := |\tilde{\alpha}, m, j\rangle\), and this notation will be used from now on. Some important explicit expressions of the states, \(|\tilde{\alpha}, m, j\rangle\), will be written next.

For \( m = 2 \) the index \( j \) can take two values, \( \{0, 1\} \), thus two sets of BGMCS will be obtained:

\[
|\tilde{\alpha}, 2, 0\rangle = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(2n+1)!(2n+1)! [f(2n+1)]!} \right]^{1/2} \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(2n+1)!(2n+1)!} |\Psi_{2n+1}\rangle \right], \tag{20}
\]

\[
|\tilde{\alpha}, 2, 1\rangle = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{[f(1)]! \sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(2n+1)!(2n+1)! [f(2n+1)]!} \right]^{1/2} \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(2n+1)!(2n+1)!} |\Psi_{2n+1}\rangle \right]. \tag{21}
\]

These states can be obtained also as even and odd superpositions of bilayer graphene coherent states. Note that, this approach has been implemented recently for the particular case when \( f(n) = 1 \) in [36].

On the other hand, for \( m = 3 \) the index \( j \) can take three values, \( \{0, 1, 2\} \), thus three sets of BGMCS are obtained:

\[
|\tilde{\alpha}, 3, 0\rangle = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+1)!(3n+1)! [f(3n+1)]!} \right]^{1/2} \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+1)!(3n+1)!} |\Psi_{3n+1}\rangle \right], \tag{22}
\]

\[
|\tilde{\alpha}, 3, 1\rangle = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{[f(1)]! \sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+2)!(3n+2)! [f(3n+2)]!} \right]^{1/2} \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+2)!(3n+2)!} |\Psi_{3n+2}\rangle \right], \tag{23}
\]

\[
|\tilde{\alpha}, 3, 2\rangle = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{[f(1)]! [f(2)]! \sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+2)!(3n+2)! [f(3n+2)]!} \right]^{1/2} \left[ |\Psi_0\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{ij} + \sqrt{(\delta_{ij} + j)!} [f(j)]! \hat{a}^n}{(3n+2)!(3n+2)!} |\Psi_{3n+2}\rangle \right]. \tag{24}
\]

Additional sets of BGMCS could be written explicitly, all of them depending on the particular choice of the function \( f(n) \) and the parameters \( \{m, j\} \). In order to analyze the electrons behavior in bilayer graphene, several physical quantities for the states \(|\tilde{\alpha}, m, j\rangle\) are going to be computed in the following sections.

### 3. Physical quantities for the BGMCS

The BGMCS belong to the Hilbert space \( \mathcal{H}_c \), and several physical quantities can be extracted from them in order to describe the system behavior in such approach.

#### 3.1. Heisenberg uncertainty relation

One of the most important physical quantities useful to characterize a quantum state is the Heisenberg uncertainty relation (HUR). In order to obtain this quantity for the BGMCS, the following matrix operators \( \hat{S}_k \) and their squares are defined as follows [25]:

\[
\hat{S}_k = \hat{b}_k \otimes \hat{1}, \quad \hat{S}_k^2 = \hat{b}_k^2 \otimes \hat{1}, \tag{25}
\]

where

\[
\hat{b}_k = \frac{1}{\sqrt{2^P}} [\hat{a}^- + (-1)^k \hat{a}^+], \tag{26}
\]

\[
\hat{b}_k^2 = \frac{1}{2} [2\hat{N} + \hat{1} + (-1)^k (\hat{a}^-)^2 + (\hat{a}^+)^2], \tag{27}
\]

with \( k = 0, 1 \), such that \( \langle \hat{S}_k \rangle_{k=0} = \langle \hat{q} \rangle \) and \( \langle \hat{S}_k \rangle_{k=1} = \langle \hat{p} \rangle \) (similarly for their squares). The explicit expressions for the mean values of these operators in the BGMCS are given in appendix A.
The standard deviation for $\hat{S}_k$ will be found through

$$\sigma_{\hat{S}_k} = \sqrt{\langle \hat{S}_k^2 \rangle - \langle \hat{S}_k \rangle^2} ,$$

thus the HUR can be obtained for the BGMCS, which must fulfill

$$\sigma_{\hat{q}_k} \sigma_{\hat{p}_k} \geq \frac{1}{2} .$$

Figures 1 and 2 show the resulting Heisenberg uncertainty product $\sigma_{\hat{q}_k} \sigma_{\hat{p}_k}$ for the BGMCS as function of $\tilde{\alpha}$ with $f(n) = 1$ and the two values $m = \{2, 3\}$ respectively.

### 3.2. Probability density

In this section the position probability density $\rho_{\tilde{\alpha}}(x)$ for the BGMCS will be addressed, such that $\rho_{\tilde{\alpha}}(x) \, dx$ \(^1\) is the probability of finding the electron between $x$ and $x + dx$, which is defined as follows [1]

$$\rho_{\tilde{\alpha}}(x) = \Psi_{\tilde{\alpha}}^*(x, y) \, \Psi_{\tilde{\alpha}}(x, y) ,$$

where $\Psi_{\tilde{\alpha}}(x, y) = \langle x, y | \tilde{\alpha}, m, j \rangle$. The explicit expression for $\rho_{\tilde{\alpha}}(x)$ is given in appendix B.

Some graphs of $\rho_{\tilde{\alpha}}(x)$ for the BGMCS with $m = 2$ and $m = 3$ are shown in figures 3–7 for the particular case when $f(n) = 1$.

\(^1\) Although the wave function $\Psi_{\tilde{\alpha}}(x, y)$ depends explicitly on $x$ and $y$, the associated probability density is independent of $y$ due to the translational symmetry along this direction; in addition, it will be time independent for stationary states.
3.3. Mean energy value
In order to characterize the energy of a system, the expected value of the Hamiltonian must be calculated. For the states $|\tilde{\alpha}, m, j\rangle$ the mean energy value $\langle \hat{H} \rangle_{\tilde{\alpha}}$ is obtained through

$$\langle \hat{H} \rangle_{\tilde{\alpha}} = \langle \tilde{\alpha}, m, j | \hat{H} | \tilde{\alpha}, m, j \rangle,$$  (31)
with $\hat{H}$ being the bilayer graphene Hamiltonian given by [34],

$$\hat{H} = -\hbar \omega_c^\pm \begin{pmatrix} 0 & \hat{a}^\pm \omega_c^\pm \omega_c^\pm \\
\hat{a}^\pm \omega_c^\pm & 0 \end{pmatrix}, \quad \omega_c^\pm = \frac{eB}{m^* c},$$  \hspace{1cm} (32)

where $\omega_c^\pm$ is the cyclotron frequency for non-relativistic electrons with effective mass $m^*$ and $\hat{a}^\pm$ are the ladder operators given in equation (4). Thus, the mean energy value $\langle \hat{H}_0 \rangle_\alpha$ turns out to be

$$\langle \hat{H}_0 \rangle_\alpha = |C_j|^2 \omega_c^\pm \left[ \sqrt{j(j-1)} + \sqrt{j+1} \right]^2 \sum_{n=1}^{\infty} \frac{[f(j)]^2 \sqrt{(mn+j)(mn+j-1)} \omega_c^\pm^2 (mn+j)!}{[f(mn+j)]^2 (mn+j)!}.$$

This quantity will be useful later on for analyzing the time evolution of the BGMCS, in the same way as in [34]. Figures 8–9 show the mean energy value (33) for the BGMCS as function of $|\alpha| = r$ with $f(m) = 1$ and the two values of $m = \{2, 3\}$.

### 3.4. Discussion

Several physical quantities have been calculated when the system is in a BGMCS. As can be seen in figures 1 and 2, the Heisenberg uncertainty relation for the BGMCS acquires a minimum when $\alpha \rightarrow 0$ which depends on the eigenstate $|\Psi_j\rangle$ with the minimum energy eigenvalue involved in the corresponding expansion. Thus, for the states $|\alpha, m, j\rangle$ with $m = 2, j = 0$ and $m = 3, j = 0$ the HUR takes the minimum value $1/2$ while for the other three cases ($m = 2, j = 1, m = 3, j = 1$ and $m = 3, j = 2$) this quantity tends to $3/2$ when $\alpha$ goes to zero.

On the other hand, the probability density for the BGMCS shows an oscillatory behavior around the point $x_0 = -2k/\omega_c^\pm$, which is similar to what happens in [34] for the BGCS ($\omega_c^\pm$ is the cyclotron frequency). Moreover,
this behavior becomes more evident as \( r \) increases and \( \tilde{\rho}_a(x) \) extends along the \( x \)-direction (see figures 3–7). Besides, when the phase of \( \tilde{\alpha} \) changes the maximum value of \( \tilde{\rho}_a(x) \) also changes. Therefore, by choosing a specific \( \tilde{\alpha} \) it is possible to find the electrons in a given region with the highest probability.

Finally, from figures 8–9 it can be seen that the mean energy value for the BGMCS is a continuous function of \( |\tilde{\alpha}| \) whose behavior above a certain \( |\tilde{\alpha}| \) is approximately linear. Moreover, when \( |\tilde{\alpha}| \to 0 \) the behavior of \( \langle \hat{H} \rangle_{\tilde{\alpha}} \) is different for each set \( |\tilde{\alpha}, m, j\rangle \), since in this limit the BGMCS tend to the state \( \Psi_j \) with minimum energy eigenvalue involved in the expansion, which is different for different \( j \) (see equations (20), (24)).

4. Evolution of the BGMCS

The time evolution operator \( \hat{U}(t) = \exp(-i\hat{H}t/\hbar) \) acting on the states \( |\tilde{\alpha}, m, j\rangle \) of equation (19) will induce the corresponding dynamical behavior. Thus, the evolved state is given by:

\[
|\tilde{\alpha}, m, j; t\rangle = C_j^m \left[ e^{-i\frac{\omega_{\tilde{\psi}}}{\hbar}(j-1)} |\Psi_j\rangle + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{n0} + \sqrt[(f(n)+j)!][f(j)!]}{(mn+j)! [f(mn+j)!]} |\tilde{\alpha}|^n \right.
\]

\[
\times \left. e^{-i\frac{\omega_{\tilde{\psi}}}{\hbar}(mn+j)(mn+j-1)} |\Psi_{mn+j}\rangle \right].
\]

(34)

In figures 10–14 the probability densities for the evolved states (34), whose explicit expressions are given in appendix C, are shown with \( f(n) = 0 \) and the two values \( m = \{2, 3\} \).

4.1. Auto-correlation function

In order to analyze further the dynamics of the BGMCS, the auto-correlation function \( C(t) \) will be computed, as in [30],

\[
C(t) = \langle \Psi(t = 0)|\Psi(t)\rangle.
\]

(35)

Using equations (19) and (34) such auto-correlation function becomes

\[
C(t) = |C_j^m|^2 \left[ e^{-i\frac{\omega_{\tilde{\psi}}}{\hbar}(j-1)} t + \sum_{n=1}^{\infty} \frac{\sqrt{2} \delta_{n0} + \sqrt[(f(n)+j)!][f(j)!]}{(mn+j)! [f(mn+j)!]} |\tilde{\alpha}|^n \right.
\]

\[
\times \left. e^{-i\frac{\omega_{\tilde{\psi}}}{\hbar}(mn+j)(mn+j-1)} t \right].
\]

(36)

In figures 15–19 the squared absolute value of this auto-correlation function \( |C(t)|^2 \) for the BGMCS is shown, with \( \omega_{\tilde{\psi}}^2 = 1, f(n) = 1 \) and several values of \( |\tilde{\alpha}| \).
4.2. Discussion

Since the energy levels for the harmonic oscillator are equally spaced the SCS are stable in time, i.e., an SCS evolves into another SCS and for a given $\alpha$ such evolution is cyclic, with the harmonic oscillator period $\tau = \frac{2\pi}{\omega}$. On the other hand, for bilayer graphene the Landau-levels are not equidistant for all $n$ (see [27, 34]) and thus the stability in time in general cannot be guaranteed. However, as it was shown in [34] starting from certain integer (for $n \geq 2$) the energy spectrum is essentially linear (see a similar approximation in [37]), thus the time stability of the BGMCS will appear when the contribution of the eigenstates $|\Psi_0\rangle$ and $|\Psi_1\rangle$ is either small or null compared with the contribution of all other states. This behavior can be seen clearly in figure 14, where the

**Figure 10.** Left: probability density $\rho_\alpha(x, t)$ for the BGMCS with $f(n) = 1$, $m = 2$ and $j = 0$. Right: probability density $\rho_\alpha(x, t)$ at some fixed times (suggested approximate period $\tau \approx \sqrt{2}\pi$ and some of its multiples). The values $|\alpha| = 1$, $\theta = 0$ and $\omega_\alpha = k = 1$ were taken.

**Figure 11.** Left: probability density $\rho_\alpha(x, t)$ for the BGMCS with $f(n) = 1$, $m = 2$ and $j = 1$. Right: probability density $\rho_\alpha(x, t)$ at some fixed times (suggested approximate period $\tau \approx \sqrt{2}\pi/\sqrt{3}$ and some of its multiples). The values $|\alpha| = 1$, $\theta = 0$ and $\omega_\alpha = k = 1$ were taken.

**Figure 12.** Left: probability density $\rho_\alpha(x, t)$ for the BGMCS with $f(n) = 1$, $m = 3$ and $j = 0$. Right: probability density $\rho_\alpha(x, t)$ at some fixed times (suggested approximate period $\tau \approx \sqrt{2}\pi/\sqrt{3}$ and some of its multiples). The values $|\alpha| = 1$, $\theta = 0$ and $\omega_\alpha = k = 1$ were taken.
evolved BGMCS for $m = 3$ and $j = 2$ are stable in time, with a period $\tau \simeq \pi / \sqrt{3}$ and some of its multiples. Moreover, as $|\tilde{a}|$ grows ($|\tilde{a}| \to \infty$) this condition is also fulfilled, thus the BGMCS in practice are stable in time for all $m$ and $j$, with the period $\tau \simeq 2\pi / m\omega^a_k$.

However, if the contribution of the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ is non-trivial compared with all other contributions the states $|\tilde{\alpha}, m, j\rangle$ will be only approximately stable in time, i.e., for some values of $t$ the probability density looks similar to what it was at $t = 0$ (see figures 10–13). In order to explain the evolution of these states an approximate period $\tau$ can be calculated as previously done for the BGCS [34]. Thus, by setting $\tilde{a}$ the mean

\[ \text{Figure 13. Left: probability density } \rho(x, t) \text{ for the BGMCS with } f(n) = 1, m = 3 \text{ and } j = 1. \text{ Right: probability density } \rho(x, t) \text{ at some fixed times (suggested approximate period } \tau \simeq \pi / \sqrt{3} \text{ and some of its multiples). The values } |\tilde{a}| = 1, \theta = 0 \text{ and } \omega^a_k = k = 1 \text{ were taken.} \]

\[ \text{Figure 14. Left: probability density } \rho(x, t) \text{ for the BGMCS with } f(n) = 1, m = 3 \text{ and } j = 2. \text{ Right: probability density } \rho(x, t) \text{ at some fixed times (suggested approximate period } \tau \simeq 2\pi / 3 \text{ and some of its multiples). The values } |\tilde{a}| = 1, \theta = 0 \text{ and } \omega^a_k = k = 1 \text{ were taken.} \]

\[ \text{Figure 15. Left: squared absolute value } |C(t)|^2 \text{ of the auto-correlation function for the BGMCS with several values of } |\tilde{a}|, f(n) = 1, m = 2 \text{ and } j = 0. \text{ Right: probability density } \rho(x, t) \text{ for } |\tilde{a}| = 1 \text{ and several fixed times, multiples of the first approximate period } \tau \simeq 2\sqrt{2} \pi \text{ obtained from } |C(t)|^2. \text{ The values of } \theta = 0 \text{ and } \omega^a_k = k = 1 \text{ were taken.} \]

\[ 2 \text{ When the evolved probability density adopts a shape similar to what it was at } t = 0 \text{ it is said that there are revivals [38].} \]
energy value is first computed, then the interval in which it lies is determined, which is bounded by two consecutive energies $E_{mn+j} + E_{mn+j}$ such that $E_{mn+j} < \langle \hat{H} \rangle_0 < E_{mn+j} + E_{mn+j}$. Thus, the possible approximate period is obtained as follows

$$\tau = \frac{2\pi \hbar}{E_{mn+j} + E_{mn+j}}.$$  (37)

As an example, the approximate period $\tau$ for the BGMCS (34) with $|\hat{a}| = 1$, $m = 2$ and $m = 3$ has been obtained. Figures 10–14 show the probability density $\rho_j(x, t)$ for these states evaluated at this suggested period and some of its multiples.

On the other hand, as it was explained before a useful tool to analyze the dynamics of a quantum system is the auto-correlation function. This function provides a qualitative way to know how long a BGMCS persists at two different times. More precisely, its squared absolute value indicates how close the evolved state is to the initial state at $t = 0$. For the BGMCS with $m = 2$ and $m = 3$ the auto-correlation function shows an oscillatory

Figure 16. Left: squared absolute value $|C(t)|^2$ of the auto-correlation function for the BGMCS with several values of $|\hat{a}|, f(n) = 1$, $m = 2$ and $j = 1$. Right: probability density $\rho_j(x, t)$ for $|\hat{a}| = 1$ and several fixed times, multiples of the first approximate period $\tau_t \simeq 2.6$ obtained from $|C(t)|^2$. The values of $\theta = 0$ and $\omega_p = k = 1$ were taken.

Figure 17. Left: squared absolute value $|C(t)|^2$ of the auto-correlation function for the BGMCS with several values of $|\hat{a}|, f(n) = 1$, $m = 3$ and $j = 0$. Right: probability density $\rho_j(x, t)$ for $|\hat{a}| = 1$ and several fixed times, multiples of the first approximate period $\tau_t \simeq 2.6$ obtained from $|C(t)|^2$. The values of $\theta = 0$ and $\omega_p = k = 1$ were taken.

Figure 18. Left: squared absolute value $|C(t)|^2$ of the auto-correlation function for the BGMCS with several values of $|\hat{a}|, f(n) = 1$, $m = 3$ and $j = 1$. Right: probability density $\rho_j(x, t)$ for $|\hat{a}| = 1$ and several fixed times, multiples of the first approximate period $\tau_t \simeq 1.8$ obtained from $|C(t)|^2$. The values of $\theta = 0$ and $\omega_p = k = 1$ were taken.
behavior, with an oscillation period which depends on the value of $|\tilde{\alpha}|$ (see figures 15–19). If the squared absolute value of the auto-correlation function is very close or equal to one, $|C(t)|^2 \approx 1$, the states $|\tilde{\alpha}, m; j; t\rangle$ and $|\tilde{\alpha}, m; j; 0\rangle$ are said to be almost completely correlated, i.e., the BGMCS at $t = 0$ is reconstructed for some $t > 0$. Therefore, the approximate evolution period for the BGMCS can be determined by looking for the time $t$ when $|C(t)|^2 \approx 1$. In figures 15–19 the squared absolute value of $C(t)$ is shown for several values of $|\tilde{\alpha}|$, and from these plots a suggested approximate period $\tau_c$ has been determined. Moreover, the probability densities for some fixed times (the suggested approximate period $\tau_c$ and some of its multiples) are also plotted.

Finally, as it was said before, when the contribution of the states $|\Psi_{\text{BGCS}}\rangle$ with $n = 0$ and $n = 1$ is small compared to the eigenstates with $n \geq 2$ the BGMCS turn out to be stable, as the BGCS derived in [34]. Therefore, the evolved BGMCS and BGCS are cyclic, showing the so-called revivals in both kind of states. Moreover, in this regime the evolution period of the BGMCS turns out to be a fraction of the evolution period of the BGCS (see figure 20), i.e., $\tau_{\text{BGMCS}} = \tau_{\text{BGCS}}/m = 2\pi/m\omega_0^2$ turning them into non-classical states. This fact is similar to what happens for the multiphoton coherent states of the harmonic oscillator [17].

5. Conclusions

In this work the multiphoton coherent states were derived, in order to describe the interaction of electrons in bilayer graphene placed in a constant homogeneous magnetic field which is perpendicular to the bilayer surface. Such states are an important generalization of the CS, and constitute an alternative description allowing the quantum systems to be addressed through a semi-classical approach. Based on [34] an appropriate generalized annihilation operator was first defined as $\hat{A}_k^* = (\hat{A})^m$, then the BGMCS were obtained as eigenstates of such operator with complex eigenvalue.

In addition, in order to analyze the system some physical quantities were obtained for such states, including the Heisenberg uncertainty relation, probability density and mean energy value. It was found that in this approach the complex eigenvalue $\tilde{\alpha}$ plays an important role in the description, since it defines the system initial conditions.

On the other hand, the time evolution of the BGMCS were studied as in [34]. It was found that the BGMCS in general are not stable in time (see figures 10–14), i.e., the shape of its probability density is not preserved in time since the energy spectrum of the bilayer graphene Hamiltonian $\hat{H}$ is not linear in $n$. However, since starting from certain integer ($n \geq 2$) the energy spectrum of $\hat{H}$ becomes practically equidistant, there are cases for which the probability density $\rho_0(x, t)$ shows as well revivals suggesting that the BGMCS could be quasi-stable. Hence, for the BGMCS where the contribution of the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ is small compared with the contribution of all other eigenstates, their time evolution will be quasi-stable, with a period of evolution $\tau = 2\pi/m\omega_0^2$ being a fraction of the bilayer graphene coherent states period. Meanwhile, for the BGMCS where the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are involved in a non-trivial way, just an approximate period of evolution can be obtained through equation (37).

In this work, the auto-correlation function $C(t)$ was also derived as an additional tool to analyze the dynamical behavior of the BGMCS. Through its modulus squared $|C(t)|^2$, also called fidelity, the times $\tau_C$ at which the revivals of $\rho_0(x, t)$ happen were obtained. Thus, despite the system does not have an equidistant

3 In quantum mechanics, notably in quantum information theory, a parameter called fidelity $F(\sigma, \rho)$ is defined as a measure of the distance between quantum states [39]. For pure states the fidelity is simply the squared absolute value of the scalar product between the two states, i.e., $F = |\langle \sigma | \rho \rangle|^2$. The reconstruction happens precisely if the fidelity is equal to one [40, 41].
energy spectrum, in this work two different ways to calculate the approximate period for the BGMCS have been implemented, in which the revivals of \( |\tilde{x}_t| \), \( r_{\text{BGCS}} \) arise. Finally, it seems possible to study similar quantum systems through this alternative approach, as the 2D Dirac materials, and to describe their dynamical behavior through the evolution of the corresponding BGMCS. Let us note that evolutions of this kind of systems have been recently addressed working in phase-space, by calculating the Wigner function for the corresponding coherent states

\[ |C(t)|^2 \approx 1 \]

From \( |C(t)| \approx 1 \) it can be seen the revivals taking place at \( \tau_{\text{BGCS}} = 2\pi \) and \( \tau_{\text{BGMCS}} = \pi \) respectively. The values \( \theta = 0 \) and \( \omega_0 = k = 1 \) were taken.

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**Data availability statement**

No new data were created or analysed in this study.
Appendix A. Heisenberg uncertainty relation

The mean values of the operators $\hat{S}_k$ and their square given in equations (25), (27) turn out to be

$$
\langle \hat{S}_k \rangle = 0 \quad \forall m,
$$

(38)

$$
\langle \hat{S}_k^2 \rangle = |C_m|^2 \left\{ \frac{(1 - \delta_{0j} - \delta_{1j})^2(2j - 3) + 2j + 1}{2^{1 - 2\delta_{0j} - 2\delta_{1j}}} + \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right]^2 \sum_{n=1}^{\infty} \left[ \frac{\tilde{a}^*_{n}^2}{(mn + j)!} \right]^2 \right. $$

$$
+ \left. \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right] \sum_{n=1}^{\infty} \frac{1}{(mn + j)!} \sum_{n=1}^{\infty} \sqrt{(mn + j + 2)!(mn + j - 2)!} \right\} \delta_{2m}. \right. $$

(39)

Appendix B. Probability densities

For the BGMCS $|\tilde{\alpha}, m, j\rangle$ of equation (19) the probability density defined in (30) turns out to be

$$
\rho_{\tilde{\alpha}}(x) = |C_m|^2 \left\{ \frac{(1 - \delta_{0j} - \delta_{1j})^2|\psi_j|^2 + |\tilde{\psi}_j|^2}{2^{1 - 2\delta_{0j} - 2\delta_{1j}}} + \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right] \sum_{n=1}^{\infty} \frac{r^n \cos(n\theta)}{(mn + j)!} 
$$

$$
\times \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos(k\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
+ \left. \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right] \sum_{n=1}^{\infty} \frac{r^n \cos((n - k)\theta)}{(mn + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
\times \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos((n - k)\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

(40)

where the polar form $\tilde{\alpha} = re^{i\theta} = r(\cos \theta + is\sin \theta)$ has been used.

Appendix C. Evolution of the BGMCS

The evolved probability density for the BGMCS with $\tilde{\alpha} = re^{i\theta}$ becomes

$$
\rho_{\tilde{\alpha}}(x, t) = |C_m|^2 \left\{ \frac{(1 - \delta_{0j} - \delta_{1j})^2|\psi_j|^2 + |\tilde{\psi}_j|^2}{2^{1 - 2\delta_{0j} - 2\delta_{1j}}} + \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right]^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{r^n \cos(n\theta)}{(mn + j)!} \right) \right. $$

$$
\times \left. \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos(k\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
+ \left. \left[ \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \right] \sum_{n=1}^{\infty} \frac{r^n \cos((n - k)\theta)}{(mn + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
\times \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos((n - k)\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
\times \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos((n - k)\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

$$
\times \frac{\Gamma(j)!!}{\Gamma((2j + 1)/2)!} \sum_{k=1}^{\infty} \frac{r^k \cos((n - k)\theta)}{(mk + j)!} \right) \right. $$

$$
\times \left( (1 - \delta_{0j} - \delta_{1j}) \psi_{mn+j-2}\psi_{mn+j}^* + \psi_{mn+j}\psi_{mn+j-2}^* \right)
$$

(41)
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References

[1] Zettili N 2009 Quantum Mechanics Concepts and Applications (Chichester: Wiley)
[2] Schrödinger E 1926 Naturwissenschaften 14 664
[3] Howard S and Roy S K 1987 Am. J. Phys. 55 1109
[4] Gazeau J P 2009 Coherent States in Quantum Physics (Berlin: Wiley-VHC)
[5] Glauber R J 1963 Phys. Rev. Lett. 10 84
[6] Glauber R J 1963 Phys. Rev. 130 2529
[7] Glauber R J 1963 Phys. Rev. 131 2766
[8] Klauder J R 1963 J. Math. Phys. 4 1055
[9] Klauder J R 1963 J. Math. Phys. 4 1058
[10] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277
[11] Klauder J and Skagerstam B 1983 Coherent States: Applications in Physics and Mathematical Physics (Singapore: World Scientific)
[12] Perelomov A 1986 Generalized Coherent States and Their Applications (Berlin: Springer-Verlag)
[13] Ali S T, Antoine J P and Gazeau J P 2014 Coherent States, Wavelets and Their Generalizations (New York: Springer)
[14] Bužek V, Jex I and Quang T 1990 J. Mod. Opt. 37 159
[15] Bužek V 1990 J. Mod. Opt. 37 303
[16] Jex I and Bužek V 1993 J. Mod. Opt. 40 771–83
[17] Castillo-Celeita M, Díaz-Bautista E and Fernández D J 2019 Phys. Scr. 94 045203
[18] Fernández D J and Hussin V 1999 J. Phys. A: Math. Gen. 32 5603–19
[19] Carballo J M, Fernández D J, Negro J and Nieto L M 2004 J. Phys. A: Math. Gen. 37 10349–62
[20] Bermudez D, Contreras-Astorga A and Fernández D J 2014 Ann. Phys. 350 615
[21] Fernández D J and Morales-Salgado V S 2016 J. Phys. A: Math. Theor. 49 195202
[22] Díaz-Bautista E and Fernández D J 2019 Eur. Phys. J. Plus 134 61
[23] Castaño S, López-Peña R and Man’Iko V I 1995 J. Russ. Laser. Res. 16 477
[24] Dell’Anno F, De Siena S and Illuminati F 2006 Phys. Rep. 428 53
[25] Díaz-Bautista E, Concha-Sánchez Y and Raya A 2019 J. Phys.: Condens. Matter 31 435702
[26] Díaz-Bautista E, Oliva-Leyva M, Concha-Sánchez Y and Raya A 2020 J. Phys. A: Math. Theor. 53 105301
[27] Katsnelson M I 2012 Graphene: Carbon in Two Dimensions (Cambridge: Cambridge University Press)
[28] Díaz-Bautista E and Fernández D J 2017 Eur. Phys. J. Plus 132 499
[29] Díaz-Bautista E, Negro J and Nieto L M 2019 J. Phys.: Conf. Ser. 1194 012025
[30] Díaz-Bautista E 2020 J. Math. Phys. 61 102101
[31] Castillo-Celeita M, Díaz-Bautista E and Oliva-Leyva M 2020 Ann. Phys. 421 168287
[32] Motamedinasab A and Anbaraki A 2020 Chinese J. Phys. 65 139
[33] Díaz-Bautista E, Negro J and Nieto L M 2021 Eur. Phys. J. Plus 136 505
[34] Fernández D J and Martínez-Moreno D I 2020 Eur. Phys. J. Plus 135 739
[35] Fernández D J and Ortiz-Campa D 2022 Eur. Phys. J. Plus 137 1012
[36] Motamedinasab A and Anbaraki A 2021 Phys. Scr. 96 065301
[37] Moya-Cessa H, Knight P L and Rosenhouse-Dantsker A 1994 Phys. Rev. A 50 1814
[38] Krueckl V and Kramer T 2009 New J. Phys. 11 093010
[39] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[40] Bogdanov Yu I, Bukeev I D and Gavrichenko A K 2011 arXiv:1102.3880
[41] Uhlmann A 2000 Phys. Rev. A 62 032307
[42] Betancur-Ocampo Y and Díaz-Bautista E 2020 Phys. Rev. B 101 125402
[43] Betancur-Ocampo Y, Díaz-Bautista E and Stegmann T 2022 Phys. Rev. B 105 045401