Large $N$-wormhole approach to spacetime foam

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Abstract

A simple model of spacetime foam, made by $N$ wormholes in a semiclassical approximation, is taken under examination. We show that the qualitative behaviour of the fluctuation of the metric conjectured by Wheeler is here reproduced.

I. INTRODUCTION

One of the most fascinating problem of our century is the possibility of combining the principles of Quantum Mechanics with those of General Relativity. The result of this combination is best known as Quantum Gravity. However such a theory has to be yet developed, principally due to the UV divergences that cannot be kept under control by any renormalization scheme. J.A. Wheeler [1] was the first who conjectured that fluctuations of the metric have to appear at short scale distances. The collection of such fluctuations gives the spacetime a kind of foam-like structure, whose topology is constantly changing. In this
foamy spacetime a fundamental length comes into play: the Planck length. Its inverse, the Planck mass $m_p$, can be thought as a natural cut-off. It is believed that in such spacetime, general relativity can be renormalized when a density of virtual black holes is taken under consideration coupled to $N$ fermion fields in a $1/N$ expansion [2]. It is also argued that when gravity is coupled to $N$ conformally invariant scalar fields the evidence that the ground-state expectation value of the metric is flat space is false [3]. However instead of looking at gravity coupled to matter fields, we will consider pure gravity. In this context two metrics which are solutions of the equations of motion without a cosmological constant are known with the property of the spherical symmetry: the Schwarzschild metric and the Flat metric. We will focus our attention on these two metrics with the purpose of examining the energy contribution to the vacuum fluctuation generated by a collection of $N$ coherent wormholes.

A straightforward extension to the deSitter and the Schwarzschild-deSitter spacetime case is immediate. The paper is structured as follows, in section II we briefly recall the results reported in Ref. [9], in section III we generalize the result of section II to $N_w$ wormholes. We summarize and conclude in section IV.

II. ONE WORMHOLE APPROXIMATION

The reference model we will consider is an eternal black hole. The complete manifold $\mathcal{M}$ can be thought as composed of two wedges $\mathcal{M}_+$ and $\mathcal{M}_-$ located in the right and left sectors of a Kruskal diagram whose spatial slices $\Sigma$ represent Einstein-Rosen bridges with wormhole topology $S^2 \times R^1$. The hypersurface $\Sigma$ is divided in two parts $\Sigma_+$ and $\Sigma_-$ by a bifurcation two-surface $S_0$. We begin with the line element

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

and we consider the physical Hamiltonian defined on $\Sigma$

$$H_p = H - H_0 = \frac{1}{l_p^2} \int_\Sigma d^3x \left( N\mathcal{H} + N_i\mathcal{H}^i \right) + H_{\partial \Sigma^+} + H_{\partial \Sigma^-}$$
\[
\frac{1}{l_p^2} \int_{\Sigma} d^3 x \left( N \mathcal{H} + N_i \mathcal{H}^i \right) + \frac{2}{l_p^2} \int_{S_+} d^2 x \sqrt{\sigma} \left( k - k^0 \right) - \frac{2}{l_p^2} \int_{S_-} d^2 x \sqrt{\sigma} \left( k - k^0 \right),
\] (2)

where \( l_p^2 = 16\pi G \). The volume term contains two constraints

\[
\begin{aligned}
\mathcal{H} &= G_{ijkl} \pi^i_j \pi^k_l \left( \frac{l^2}{\sqrt{g}} \right) - \left( \frac{\sqrt{\sigma}}{l_p} \right) R^{(3)} = 0, \\
\mathcal{H}^i &= -2\pi^i_j = 0,
\end{aligned}
\] (3)

where \( G_{ijkl} = \frac{1}{2} (g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) \) and \( R^{(3)} \) denotes the scalar curvature of the surface \( \Sigma \). By using the expression of the trace

\[
k = -\frac{1}{\sqrt{h}} \left( \sqrt{h} n^\mu \right)_{,\mu},
\] (4)

with the normal to the boundaries defined continuously along \( \Sigma \) as \( n^\mu = (h^{\gamma\gamma})^{\frac{1}{2}} \delta^\mu_\gamma \). The value of \( k \) depends on the function \( r, y \), where we have assumed that the function \( r, y \) is positive for \( S_+ \) and negative for \( S_- \). We obtain at either boundary that

\[
k = \frac{-2r_{,y}}{r}.
\] (5)

The trace associated with the subtraction term is taken to be \( k^0 = -2/r \) for \( B_+ \) and \( k^0 = 2/r \) for \( B_- \). Then the quasilocal energy with subtraction terms included is

\[
E_{\text{quasilocal}} = E_+ - E_- = \left( r [1 - |r| y] \right)_{y=y_+} - \left( r [1 - |r| y] \right)_{y=y_-}.
\] (6)

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to the bifurcation surface \( S_0 \) and this is the necessary condition to obtain instability with respect to the flat space. A little comment on the total Hamiltonian is useful to further proceed. We are looking at the sector of asymptotically flat metrics included in the space of all metrics, where the Wheeler-DeWitt equation

\[
\mathcal{H} \Psi = 0
\] (7)
is defined. In this sector the Schwarzschild metric and the Flat metric satisfy the constraint equations \( (3) \). Here we consider deviations from such metrics in a WKB approximation and we calculate the expectation value following a variational approach where the WKB functions are substituted with trial wave functionals. Then the Hamiltonian referred to the line element \( (1) \) is

\[
H = \int_{\Sigma} d^3x \left[ G_{ijkl} \pi^{ij} \pi^{kl} \left( \frac{l_p^2}{\sqrt{g}} \right) - \left( \frac{\sqrt{g}}{l_p^2} \right) R^{(3)} \right].
\]

Instead of looking at perturbations on the whole manifold \( M \), we consider perturbations at \( \Sigma \) of the type \( g_{ij} = \bar{g}_{ij} + h_{ij} \). \( \bar{g}_{ij} \) is the spatial part of the background considered in eq.\( (1) \)In Ref. \[9\], we have defined \( \Delta E (m) \) as the difference of the expectation value of the Hamiltonian approximated to second order calculated with respect to different backgrounds which have the asymptotic flatness property. This quantity is the natural extension to the volume term of the subtraction procedure for boundary terms and is interpreted as the Casimir energy related to vacuum fluctuations. Thus

\[
\Delta E (m) = E (m) - E (0) = \frac{\langle \Psi \mid H^{Schw.} - H^{Flat} \mid \Psi \rangle}{\langle \Psi \mid \Psi \rangle} + \frac{\langle \Psi \mid H_{quasilocal} \mid \Psi \rangle}{\langle \Psi \mid \Psi \rangle},
\]

(8)

By restricting our attention to the graviton sector of the Hamiltonian approximated to second order, hereafter referred as \( H_{\mid 2} \), we define

\[
E_{\mid 2} = \frac{\langle \Psi^\perp \mid H_{\mid 2} \mid \Psi^\perp \rangle}{\langle \Psi^\perp \mid \Psi^\perp \rangle},
\]

where

\[
\Psi^\perp = \Psi \left[ h_{ij}^\perp \right] = \mathcal{N} \exp \left\{ -\frac{1}{4l_p^2} \left[ \langle (g - \bar{g}) K^{-1} (g - \bar{g}) \rangle_{x,y}^\perp \right] \right\}.
\]

After having functionally integrated \( H_{\mid 2} \), we get

\[
H_{\mid 2} = \frac{1}{4l_p^2} \int_{\Sigma} d^3x \sqrt{g} G^{ijkl} \left[ K^{-1 \perp} (x, x)_{ijkl} + (\Delta g)^a_{ij} K^\perp (x, x)_{ijkl} \right]
\]

(9)
The propagator $K^\perp(x,x)_{iakl}$ comes from a functional integration and it can be represented as

$$K^\perp(x,x)_{iakl} := \sum_N h^\perp_{ia} (x) h^\perp_{kl} (y),$$

where $h^\perp_{ia} (x)$ are the eigenfunctions of

$$(\triangle_2)^a := -\triangle^a + 2R^a_j.$$

This is the Lichnerowicz operator projected on $\Sigma$ acting on traceless transverse quantum fluctuations and $\lambda_N(p)$ are infinite variational parameters. $\triangle$ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and $R^a_j$ is the mixed Ricci tensor whose components are:

$$R^a_j = \text{diag} \left\{ \frac{-2m}{r^3}, \frac{m}{r^3}, \frac{m}{r^3} \right\}.$$  

After normalization in spin space and after a rescaling of the fields in such a way as to absorb $l^2_p$, $E_{|2}$ becomes in momentum space

$$E_{|2}(m,\lambda) = \frac{V}{2\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^\infty dp p^2 \left[ \lambda_i(p) + \frac{E^2_{i}(p,m,l)}{\lambda_i(p)} \right],$$

where

$$E^2_{1,2}(p,m,l) = p^2 + \frac{l(l+1)}{r_0^2} + \frac{3m}{r_0^3},$$

and $V$ is the volume of the system. $r_0$ is related to the minimum radius compatible with the wormhole throat. We know that the classical minimum is achieved when $r_0 = 2m$. However, it is likely that quantum processes come into play at short distances, where the wormhole throat is defined, introducing a quantum radius $r_0 > 2m$. The minimization with respect to $\lambda$ leads to $\bar{\lambda}_i(p,l,m) = \sqrt{E^2_{i}(p,m,l)}$ and eq.(13) becomes

$$E_{|2}(m,\lambda) = 2 \frac{V}{2\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^\infty dp p^2 \sqrt{E^2_{i}(p,m,l)},$$

with $p^2 + \frac{l(l+1)}{r_0^2} > \frac{3m}{r_0^3}$. Thus, in presence of the curved background, we get
\[ E_{|2}\ (m) = \frac{V}{2\pi^2} \frac{1}{2} \sum_{l=0}^{\infty} \int_0^\infty dp \, p^2 \left( \sqrt{p^2 + c_-^2} + \sqrt{p^2 + c_+^2} \right) \] (16)

where

\[ c_+^2 = \frac{l(l+1)}{r_0^2} \mp \frac{3m}{r_0^3}, \]

while when we refer to the flat space, we have \( m = 0 \) and \( c^2 = \frac{l(l+1)}{r_0^2} \), with

\[ E_{|2} (0) = \frac{V}{2\pi^2} \frac{1}{2} \sum_{l=0}^{\infty} \int_0^\infty dp \, p^2 \left( 2\sqrt{p^2 + c^2} \right). \] (17)

Since we are interested in the \( UV \) limit, we will use a cut-off \( \Lambda \) to keep under control the \( UV \) divergence

\[ \int_0^\infty \frac{dp}{p} \sim \int_0^{\Lambda} \frac{dx}{x} \sim \ln \left( \frac{\Lambda}{c} \right), \] (18)

where \( \Lambda \leq m_p \). Note that in this context the introduction of a cut-off at the Planck scale is quite natural if we look at a spacetime foam. Thus \( \Delta E (m) \) for high momenta becomes

\[ \Delta E (m) \sim -\frac{V}{2\pi^2} \left( \frac{3m}{r_0^3} \right)^2 \frac{1}{16} \ln \left( \frac{r_0^3 \Lambda^2}{3m} \right). \] (19)

We now compute the minimum of \( \tilde{\Delta E} (m) = E (0) - E (m) = -\Delta E (m) \). We obtain two values for \( m \): \( m_1 = 0 \), i.e. flat space and \( m_2 = \Lambda^2 e^{-\frac{1}{2}} r_0^3 / 3 \). Thus the minimum of \( \tilde{\Delta E} (m) \) is at the value \( \tilde{\Delta E} (m_2) = \frac{V \Lambda^4}{64\pi^2 e} \). Recall that \( m = MG \), thus

\[ M = G^{-1} \Lambda^2 e^{-\frac{1}{2}} r_0^3 / 3. \] (20)

When \( \Lambda \to m_p \), then \( r_0 \to l_p \). This means that an Heisenberg uncertainty relation of the type \( l_p m_p = 1 \) (in natural units) has to be satisfied, then

\[ M = m_p^2 e^{-\frac{1}{2}} m_p^{-1} / 3 = \frac{m_p}{3\sqrt{e}}. \] (21)
III. $N_w$ Wormholes Approximation

Suppose to consider $N_w$ wormholes and assume that there exists a covering of $\Sigma$ such that $\Sigma = \bigcup_{i=1}^{N_w} \Sigma_i$, with $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$. Each $\Sigma_i$ has the topology $S^2 \times R^1$ with boundaries $\partial \Sigma_i^\pm$ with respect to each bifurcation surface. On each surface $\Sigma_i$, quasilocal energy gives

$$E_i^{\text{quasilocal}} = \frac{2}{l_p^2} \int_{S_i^+} d^2x \sqrt{\sigma} \left(k - k^0\right) - \frac{2}{l_p^2} \int_{S_i^-} d^2x \sqrt{\sigma} \left(k - k^0\right), \quad (22)$$

and by using the expression of the trace

$$k = -\frac{1}{\sqrt{\hbar}} \left(\sqrt{\hbar n^\mu}\right)_{,\mu}, \quad (23)$$

we obtain at either boundary that

$$k = -\frac{2r_y}{r}, \quad (24)$$

where we have assumed that the function $r_y$ is positive for $S_{i+}$ and negative for $S_{i-}$. The trace associated with the subtraction term is taken to be $k^0 = -2/r$ for $B_{i+}$ and $k^0 = 2/r$ for $B_{i-}$. Here the quasilocal energy with subtraction terms included is

$$E_i^{\text{quasilocal}} = E_{i+} - E_{i-} = (r \left[1 - \left|r_y\right]\right])_{y=y_{i+}} - (r \left[1 - \left|r_y\right]\right])_{y=y_{i-}}.$$ \quad (25)

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to each bifurcation surface $S_{0,i}$. We are interested to a large number of wormholes, each of them contributing with a Hamiltonian of the type $H_{\text{2}}$. If the wormholes number is $N_w$, we obtain (semiclassically, i.e., without self-interactions)

$$H_{\text{tot}}^{N_w} = H^1 + H^2 + \ldots + H^{N_w}. \quad (26)$$

Thus the total energy for the collection is

$$E_{\text{tot}}^{\text{2}} = N_w H_{\text{2}}.$$ 

The same happens for the trial wave functional which is the product of $N_w$ t.w.f.. Thus
\[ \Psi_{tot}^\perp = \Psi_1^\perp \otimes \Psi_2^\perp \otimes \ldots \otimes \Psi_{N_w}^\perp = \mathcal{N} \exp N_w \left\{ -\frac{1}{4l_p^2} \left[ \langle (g - \bar{g}) K^{-1} (g - \bar{g}) \rangle_{x,y} \right] \right\} \]

\[ = \mathcal{N} \exp \left\{ -\frac{1}{4} \left[ \langle (g - \bar{g}) K^{-1} (g - \bar{g}) \rangle_{x,y} \right] \right\}, \]

where we have rescaled the fluctuations \( h = g - \bar{g} \) in such a way to absorb \( N_w/l_p^2 \). Of course, if we want the trial wave functionals be independent one from each other, boundaries \( \partial \Sigma^\pm \) have to be reduced with the enlarging of the wormholes number \( N_w \), otherwise overlapping terms could be produced. Thus, for \( N_w \)-wormholes, we obtain

\[ H_{tot} = N_w H = \int_\Sigma d^3x \left[ G_{ijkl} \pi^{ij} \pi^{kl} \left( l_w^2 \frac{\sqrt{g}}{l_p^2} \right) - \left( N_w \frac{\sqrt{g}}{l_p^2} \right) R^{(3)} \right] \]

\[ = \int_\Sigma d^3x \left[ G_{ijkl} \pi^{ij} \pi^{kl} \left( \frac{l_w^2}{\sqrt{g}} \right) - \left( N_w \frac{\sqrt{g}}{l_p^2} \right) R^{(3)} \right] , \]

where we have defined \( l_w^2 = l_p^2 N_w \) with \( l_w^2 \) fixed and \( N_w \to \infty \). Thus, repeating the same steps of section 2 for \( N_w \) wormholes, we obtain

\[ \Delta E_{N_w} (m) \sim -N_w^2 V \left( \frac{3m}{r_0^3} \right)^2 \frac{1}{16} \ln \left( \frac{r_0^3 \Lambda^2}{3m} \right). \quad (27) \]

Then at one loop the cooperative effects of wormholes behave as one macroscopic single field multiplied by \( N_w^2 \); this is the consequence of the coherency assumption. We have just explored the consequences of this result in Ref. [9]. Indeed, coming back to the single wormhole contribution we have seen that the black hole pair creation probability mediated by a wormhole is energetically favored with respect to the permanence of flat space provided we assume that the boundary conditions be symmetric with respect to the bifurcation surface which is the throat of the wormhole. In this approximation boundary terms give zero contribution and the volume term is nonvanishing. As in the one-wormhole case, we now compute the minimum of \( \Delta \tilde{E}_{N_w} (m) = (E(0) - E(m))_{N_w} = -\Delta E_{N_w} (m) \). The minimum is reached for \( \bar{m} = \Lambda^2 e^{-\frac{\Lambda}{2r_0^3}/3} \). Thus the minimum is

\[ \Delta \tilde{E} (\bar{m}) = N_w^2 \frac{V}{64\pi^2} \frac{\Lambda^4}{e} . \quad (28) \]
The main difference with the one wormhole case is that we have $N_w$ wormholes contributing with the same amount of energy. Since $m = MN_w G = ML_{N_w}^2$, thus

$$M = \left( \frac{l_{N_w}^2}{N_w} \right)^{-1} \Lambda^2 e^{-\frac{1}{2} r_0^2/3}. \tag{29}$$

When $\Lambda \to m_p$, then $r_0 \to l_p$ and $l_p m_p = 1$. Thus

$$M = \left( \frac{l_{N_w}^2}{N_w} \right)^{-1} \frac{m_p^{-1}}{3\sqrt{e}} = N_w m_{N_w} \frac{m_{N_w}}{3\sqrt{e}} \tag{30}$$

So far, we have discussed the stable modes contribution. However, we have discovered that for one wormhole also unstable modes contribute to the total energy \[4,9\]. Since we are interested to a large number of wormholes, the first question to answer is: what happens to the boundaries when the wormhole number is enlarged. In the one wormhole case, the existence of one negative mode is guaranteed by the vanishing of the eigenfunction of the operator $\Delta_2$ at infinity, which is the same space-like infinity of the quasilocal energy, i.e. we have the $ADM$ positive mass $M$ in a coordinate system of the universe where the observer is present and the anti-$ADM$ mass in a coordinate system where the observer is not there. When the number of wormholes grows, to keep the coherency assumption valid, the space available for every single wormhole has to be reduced to avoid overlapping of the wave functions. This means that boundary conditions are not fixed at infinity, but at a certain finite radius and the $ADM$ mass term is substituted by the quasilocal energy expression under the condition of having symmetry with respect to each bifurcation surface. As $N_w$ grows, the boundary radius $\bar{r}$ reduces more and more and the unstable mode disappears. This means that there will exist a certain radius $r_c$ where below which no negative mode will appear and there will exist a given value $N_w c$ above which the same effect will be produced. In rigorous terms: $\forall N \geq N_w c \exists r_c$ s.t. $\forall r_0 \leq r \leq r_c$, $\sigma (\Delta_2) = \emptyset$. This means that the system begins to be stable. In support of this idea, we invoke the results discovered in Ref. [10] where, it is explicitly shown that the restriction of spatial boundaries leads to a stabilization of the system. Thus at the minimum, we obtain the typical energy density behavior of the foam
IV. CONCLUSIONS AND OUTLOOKS

According to Wheeler’s ideas about quantum fluctuation of the metric at the Planck scale, we have used a simple model made by a large collection of wormholes to investigate the vacuum energy contribution needed to the formation of a foamy spacetime. Such investigation has been made in a semiclassical approximation where the wormholes are treated independently one from each other (coherency hypothesis). The starting point is the single wormhole, whose energy contribution has the typical trend of the gravitational field energy fluctuation. The wormhole considered is of the Schwarzschild type and every energy computation has to be done having in mind the reference space, i.e. flat space. When we examine the wormhole collection, we find the same trend in the energy of the single case. This is obviously the result of the coherency assumption. However, the single wormhole cannot be taken as a model for a spacetime foam, because it exhibits one negative mode. This negative mode is the key of the topology change from a space without holes (flat space) to a space with an hole inside (Schwarzschild space). However things are different when we consider a large number of wormholes $N_w$. Let us see what is going on: the classical vacuum, represented by flat space is stable under nucleation of a single black hole, while it is unstable under a neutral pair creation with the components residing in different universes divided by a wormhole. When the topology change has primed by means of a single wormhole, there will be a considerable production of pairs mediated by their own wormhole. The result is that the hole production will persist until the critical value $N_w^c$ will be reached and spacetime will enter the stable phase. If we look at this scenario a little closer, we can see that it has the properties of the Wheeler foam. Nevertheless, we have to explain why observations measure a flat space structure. To this purpose, we have to recall that the foamy spacetime structure should be visible only at the Planck scale, while at greater scales it is likely that

\[
\frac{\Delta E}{V} \sim -N_w^2\Lambda^4
\]
the flat structure could be recovered by means of averages over the collective functional describing the \textit{semiclassical} foam. Indeed if $\eta_{ij}$ is the spatial part of the flat metric, ordinarily we should obtain

$$\langle \Psi | g_{ij} | \Psi \rangle = \eta_{ij}, \quad (32)$$

where $g_{ij}$ is the spatial part of the gravitational field. However in the foamy representation we should consider, instead of the previous expectation value, the expectation value of the gravitational field calculated on wave functional representing the foam, i.e., to see that at large distances flat space is recovered we should obtain

$$\langle \Psi_{foam} | g_{ij} | \Psi_{foam} \rangle = \eta_{ij}, \quad (33)$$

where $\Psi_{foam}$ is a superposition of the single-wormhole wave functional

$$\Psi_{foam} = \sum_{i=1}^{N_w} \Psi_i^\perp. \quad (34)$$

This has to be attributed to the semiclassical approximation which render this system a non-interacting system. However, things can change when we will consider higher order corrections and the other terms of the action decomposition, i.e. the spin one and spin zero terms. Nevertheless, we can argue that only spin zero terms (associated with the conformal factor) will be relevant, even if the part of the action which carries the physical quantities is that discussed in this text, i.e., the spin two part of the action related to the gravitons.

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