A 3-LOCAL CHARACTERIZATION OF THE HARADA–NORTON SPORADIC SIMPLE GROUP

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ABSTRACT. We provide 3-local characterizations of the Harada–Norton sporadic simple group and its automorphism group. Both groups are examples of groups of parabolic characteristic three and we identify them from the structure of the normalizer of the centre of a Sylow 3-subgroup.

1. Introduction

In [8] in 1975, Harada introduced a new simple group. He proved that a group with an involution whose centralizer is a double cover of the automorphism group of the Higman–Sims sporadic simple group is simple of order $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7.11.19$. In 1976, in his PhD thesis, Norton proved such a group exists and thus we have the Harada–Norton sporadic simple group, HN. The simple group was not proved to be unique until 1992. In [16], Segev proves that there is a unique group $G$ (up to isomorphism) with two involutions $u$ and $t$ such that $C_G(u) \cong (2 \cdot HS) : 2$ and $C_G(t) \cong 2^{1+8} \cdot (\text{Alt}(5) \wr 2)$ with $C_G(O_2(C_G(t))) \leq O_2(C_G(t))$. We can therefore define the group HN by the structure of two involution centralizers in this way.

In the ongoing project to understand groups of local and parabolic characteristic $p$ (see for example [12]) it has been observed that both HN and Aut(HN) are example of groups of parabolic characteristic 3. This is to say that every 3-local subgroup, $H$, containing a Sylow 3-subgroup satisfies $C_H(O_p(H)) \leq O_p(H)$. The aim of this paper is therefore to characterize HN and HN : 2 in terms of their 3-structure. The hypothesis we consider and the theorem we prove are as follows.

Hypothesis 1.1. Let $G$ be a group and let $Z$ be the centre of a Sylow 3-subgroup of $G$ with $Q := O_3(N_G(Z))$. Suppose that

(i) $Q \cong 3^{1+4}$;
(ii) $C_G(Q) \leq Q$;
(iii) $Z \neq Z^x \leq Q$ for some $x \in G$; and
(iv) $N_G(Z)/Q$ has shape $4 \cdot \text{Alt}(5)$ or $4 \cdot \text{Sym}(5)$.

Theorem 1.2. If $G$ satisfies Hypothesis 1.1 then $G \cong \text{HN}$ or $G \cong \text{Aut(HN)}$.

For a large part of this proof we work under the hypothesis that $N_G(Z)/Q$ has shape either $4 \cdot \text{Alt}(5)$ or $4 \cdot \text{Sym}(5)$. We will refer to these two possibilities as
Case I and Case II respectively. In Section 3, we determine the possibilities for the structure of certain 3-local subgroups of $G$. This allows us to see the fusion of elements of order three in $G$. In particular, it allows us to identify a distinct conjugacy class of elements of order three. In 3-local recognition results, it is often necessary to determine $C_G(x)$ for each element of order three in $G$. In this case, we have just one further centralizer to determine which is isomorphic to $3 \times \text{Alt}(9)$ or $3 \times \text{Sym}(9)$ and we use a theorem due to Prince [15] to recognize this centralizer.

In Section 4 we determine the structure of $C_G(t)$ where $t$ is a 2-central involution. This requires a great deal of 2-local analysis, in particular, we must take full advantage of our knowledge of the 2-local subgroups in $\text{Alt}(9)$ and use a theorem due to Goldschmidt about 2-subgroups with a strongly closed abelian subgroup. The determination of $C_G(t)$ is much more difficult than similar recognition results (in [3] for example). A reason for this may be that the 3-rank of $C_G(t)/O_2(C_G(t))$ is just two whilst the 2-rank is four. An easier example may have greater 3-rank and lesser 2-rank. We also show in Section 4 that in Case II of the hypothesis, $G$ has a proper normal subgroup which satisfies Case I. Once we have made this observation, our calculations are simplified significantly as we can reduce to a Case I hypothesis only.

One conjugacy class of involution centralizer is not enough to recognize $\text{HN}$ and so in Section 5 we prove that $G$ also has an involution centralizer which has shape $(2\cdot \text{HS}) : 2$ by making use of a theorem of Aschbacher. The results of Sections 4 and 5 allow us to apply the uniqueness theorem by Segev to prove that in Case I $G \cong \text{HN}$. It then follows easily that in Case II, $G \cong \text{Aut(} \text{HN})$.

All groups in this article are finite. We note that $\text{Sym}(n)$ and $\text{Alt}(n)$ denote the symmetric and alternating groups of degree $n$ and $\text{Dih}(n)$ and $\text{Q}(n)$ denote the dihedral group and quaternion groups of order $n$. Notation for classical groups follows [1]. All other groups and notation for group extensions follows the ATLAS [4] conventions. In particular we mention that the shape of a group is some description of its normal structure and we use the symbol $\sim$ (for example if $G \cong \text{Sym}(4)$, we may choose to write $G \sim 2^2\cdot \text{Sym}(3)$). If $H$ is a group acting on a set containing $x$ then $x^H$ is the orbit of $x$ under $H$. If a group $A$ acts on a group $B$ and $a \in A$ and $b \in B$ then $[b, a] = b^{-1}b^a$. Further group theory notation and terminology is standard as in [1] and [11] except that $\mathbf{Z}(H)$ denotes the centre of a group $H$.

2. Preliminary Results

**Theorem 2.1** (Aschbacher). [2] Let $G$ be a group with an involution $t$ and set $H := C_G(t)$. Let $V \leq G$ such that $V \cong 2 \times 2 \times 2$ and set $M := N_G(V)$. Suppose that

(i) $O_2(H) \cong 4 \times 2_1^{1+4}$ and $H/O_2(H) \cong \text{Sym}(5)$; and

(ii) $V \leq O_2(H)$, $O_2(M) \cong 4 \times 4 \times 4$ and $M/O_2(M) \cong \text{GL}_3(2)$.

Then $G \cong \text{HS}$.
Theorem 2.2 (Segev). [16] Let $G$ be a finite group containing two involutions $u$ and $t$ such that $C_G(u) \cong (2\, HS) : 2$ and $C_G(t) \sim 2^{1+8}$. (Alt(5);2) with $C_G(O_2(C_G(t))) \leq O_2(C_G(t))$. Then $G \cong \text{HN}$.

Recall that given a $p$-group $S$, we set $\Omega(S) = \langle x | x^p = 1 \rangle$.

Theorem 2.3 (Goldschmidt). [11, p370] Let $S$ be a Sylow $2$-subgroup of a group $G$ and let $A$ be an abelian subgroup of $S$ such that $A$ is strongly closed in $S$ with respect to $G$. Suppose that $G = \langle A^G \rangle$ and $O_2^c(G) = 1$. Then $G = F^*(G)$ and $A = O_2(G)\Omega(S)$.

Theorem 2.4 (Hayden). [9, 3.3, p545]. Let $G$ be a finite group and let $T \in \text{Syl}_3(G)$ be elementary abelian of order nine. Assume that

(i) $N_G(T)/C_G(T) \cong 2 \times 2$;
(ii) $C_G(T) = T$; and
(iii) $C_G(t) \leq N_G(T)$ for each $t \in T^\#$.

Then $G = N_G(T)$.

Theorem 2.5 (Feit–Thompson). [5] Let $G$ be a finite group containing a subgroup, $X$, of order three such that $C_G(X) = X$. Then one of the following holds:

(i) $G$ contains a nilpotent normal subgroup, $N$, such that $G/N \cong \text{Sym}(3)$ or $C_3$;
(ii) $G$ contains an elementary abelian normal $2$-subgroup, $N$, such that $G/N \cong \text{Alt}(5)$; or
(iii) $G \cong \text{PSL}_2(7)$.

The result can be found in [5] however the additional information in conclusion (ii) that $N$ is elementary abelian uses a theorem of Higman [10].

Theorem 2.6 (Prince). [15] Let $G$ be a group and suppose $x \in G$ has order $3$ such that $C_G(x) \cong C_{\text{Sym}(9)}((1,2,3)(4,5,6)(7,8,9))$ and there exists $J \leq C_G(x)$ which is elementary abelian of order $27$ and normalizes no non-trivial $3'$-subgroup of $G$. Then either $J \lhd G$ or $G \cong \text{Sym}(9)$.

Lemma 2.7. Let $G$ be a group of order $3^4 2$ with $S \in \text{Syl}_3(G)$ and $T \in \text{Syl}_2(G)$ and $J \triangleleft G$ elementary abelian of order $27$. Suppose that $Z := Z(S)$ has order three and $Z \leq C_S(T) \neq S$. Then $G \cong C_{\text{Sym}(9)}((1,2,3)(4,5,6)(7,8,9))$.

Proof. We have that $T$ normalizes $Z$ and $J$ and so by Maschke’s Theorem, there exists a subgroup $K \leq J$ such that $K$ is a $T$-invariant complement to $Z$ in $J$. Set $L := KT$ then $K \lhd L$ and $[G : L] = 9$. Suppose that $N \lhd L$ and that $N$ is normal in $G$. If $3 \mid |N|$ then $N \cap Z(S) \neq 1$ which is a contradiction since $Z \nsubseteq K$. So $N$ is a $2$-group which implies $N = 1$ otherwise $G$ has a central involution. Hence there is an injective homomorphism from $G$ into $\text{Sym}(9)$. Moreover there is a map from $G$ into the centralizer in $\text{Sym}(9)$ of the centre of a Sylow $3$-subgroup. Since $|C_{\text{Sym}(9)}((1,2,3)(4,5,6)(7,8,9))| = |G|$, we have an isomorphism. \hfill \Box

Theorem 2.8 (Parker–Rowley). [14] Let $G$ be a finite group with $R := (a,b)$ an elementary abelian Sylow $3$-subgroup of $G$ of order nine. Assume the following hold.

(i) $C_G(R) = R$ and $N_G(R)/C_G(R) \cong \text{Dih}(8)$. 
(ii) $C_G(a) \cong 3 \times \text{Alt}(5)$ and $N_G((a))$ is isomorphic to the diagonal subgroup of index two in $\text{Sym}(3) \times \text{Sym}(5)$.

(iii) $C_G(b) \leq N_G(R)$, $C_G(b)/R \cong 2$ and $N_G((b))/R \cong 2 \times 2$.

Then $G$ is isomorphic to $\text{Alt}(8)$.

**Corollary 2.9.** Let $G$ be a group and $\text{Alt}(8) \cong H \leq G$ such that for $R \in \text{Syl}_3(H)$ and each $r \in R^\#$, $C_G(r) \leq H$. Then $G = H$.

**Proof.** Suppose $R$ is not a Sylow 3-subgroup of $G$. Then there exists $R < S \in \text{Syl}_3(G)$. Therefore $R < N_S(R)$ and $1 \neq r \in Z(N_S(R)) \cap R$. Therefore $N_S(R) \leq C_G(r) \leq H$ which is a contradiction. Thus $R \in \text{Syl}_3(G)$. Pick $a, b \in R$ such that $C_H(a) \cong 3 \times \text{Alt}(5)$ and $C_H(b) \leq N_H(R)$. Now we check the hypotheses of Theorem 2.8. We have that for any $r \in R^\#$, $C_G(r) \leq C_G(r) \leq H$ and so $C_G(R) = C_H(R) = R$. So consider $N_G(R)/C_G(R)$ which is isomorphic to a subgroup of $\text{GL}_2(3)$. Since $R \in \text{Syl}_3(G)$, $N_G(R)/R$ is a 2-group. Also $N_H(R)/R \cong \text{Dih}(8)$. Suppose $N_G(R)/R \cong \text{SDih}(16)$. Then $N_G(R)$ is transitive on $R^\#$ which is a contradiction. Therefore $N_G(R) = N_H(R)$ and $N_G(R)/C_G(R) \cong \text{Dih}(8)$ so (i) is satisfied. Now $C_G(a) = C_H(a)$ and there exists some $x \in H$ that inverts $a$. Therefore $N_H((a)) = C_H(a)(x) \leq H$. Similarly $C_G(b) = C_H(b)$ and there exists some $y \in H$ that inverts $b$. Therefore $N_H((b)) = C_H(b)(y) \leq H$. Thus (ii) and (iii) are satisfied so $G = H \cong \text{Alt}(8)$. 

**Lemma 2.10.** [13, 3.20 (iii)] Let $X \cong \text{SL}_2(3)$ and $S \in \text{Syl}_3(X)$. Suppose that $X$ acts on an elementary abelian 3-group $V$ such that $V = \langle C_V(S)^X \rangle$, $C_V(X) = 1$ and $[V, S, S] = 1$. Then $V$ is a direct product of natural modules for $X$.

**Lemma 2.11.** Let $G$ be a group, $p$ be a prime and $S \in \text{Syl}_p(G)$. Suppose $J(S)$ is abelian and suppose $a, b \in J(S)$ are conjugate in $G$. Then $a$ and $b$ are conjugate in $N_G(J(S))$.

**Proof.** Suppose $a^g = b$ for some $g \in G$. Notice first that it follows immediately from the definition of the Thompson subgroup that $J(S)^g = J(S^g)$. Now $J(S), J(S^g) \leq C_G(b)$. Let $P, Q \in \text{Syl}_p(C_G(b))$ such that $J(S) \leq P$ and $J(S^g) \leq Q$. Again, by the definition of the Thompson subgroup, it is clear that $J(S) \leq P$ implies $J(S) = J(P)$ and similarly $J(S^g) = J(Q)$. By Sylow’s Theorem, there exists $x \in C_G(b)$ such that $Q^x = P$ and so $J(S) = J(P) = J(Q)^x = J(S)^{g^x}$. Thus $gx \in N_G(J(S))$ and $a^{gx} = b^x = b$ as required. 

**Lemma 2.12.** Let $X$ be a group with an elementary abelian subgroup $E < X$ of order $2^{2n}$ such that $C_X(E) = E$. Let $S \in \text{Syl}_2(X)$ and suppose that whenever $E < R \leq S$ with $R/E$ elementary abelian and $|R/E| = 2^n$ we have $|C_E(R)| \leq 2^{n-a-1}$. Then $E$ is characteristic in $S$.

**Proof.** First observe that since $C_X(E) = E$, $X/E$ is a group of outer automorphisms of $E$. Let $\alpha$ be an automorphism of $S$ such that $E^\alpha \neq E$. Then $R := EE^\alpha \leq S$. Since $E^\alpha$ is elementary abelian, we have that $E^{\alpha}/(E \cap E^{\alpha}) \cong EE^{\alpha}/E = R/E$ is elementary abelian and $E \cap E^{\alpha}$ is central in $R$. If $|R/E| = 2^n$ then $|E \cap E^{\alpha}| = 2^{n-a}$ so $|C_E(R)| \geq |E \cap E^{\alpha}| = 2^{2n-a}$ which is a contradiction.
Lemma 2.13. Let $E \leq \text{GL}_4(3)$ such that $|E| = 2^5$, and $|\Phi(E)| \leq 2$. Furthermore let $S \leq \text{GL}_4(3)$ be elementary abelian of order nine such that $S$ acts faithfully on $E$. If $Q(8) \cong A \cong B$ with $A \neq B$ both $S$-invariant subgroups of $E$, then $E \cong 2_+^{1+4}$ and $E$ is uniquely determined up to conjugation in $\text{GL}_4(3)$.

Proof. Note that $E$ is non-abelian since $A, B \leq E$. Therefore $|E/\Phi(E)| = 2^4$ is acted on faithfully by $S$. Hence, $S$ is isomorphic to a subgroup of $\text{GL}_4(2)$. Now observe that $\text{GL}_4(2)$ has Sylow 3-subgroups of order nine which contain an element of order three which acts fixed-point-freely on the natural module. Thus any $S$-invariant subgroup of $E$ properly containing $\Phi(E)$ has order $2^3$ or $2^5$. Since $A$ and $B$ are distinct and normalized by $S$, we have $E = AB$. Suppose $|Z(E)| > 2$. Then $Z(E)| = 8$ is $S$-invariant. By coprime action, $Z(E) = \langle C_{Z(E)}(s) \mid s \in S^0 \rangle$. Thus there exists $s \in S^0$ such that $C_{Z(E)}(s) > \Phi(E)$. Since $E = AB$, we find $a \in A$ and $b \in B$ such that $ab \in C_{Z(E)}(s)\Phi(E)$. Then, as $s$ normalizes $A$ and $B$, $s$ must centralize $a$ and $b$. Now $C_E(s)$ is $S$-invariant with $|C_E(s) \cap A| \geq 4$ and $|C_E(s) \cap B| \geq 4$. It follows that $[E, s] = 1$ which is a contradiction. Thus $Z(E) = \Phi(E)$ and so $E$ is extraspecial and $E \cong 2^{1+4}$.

Since $E$ is extraspecial, $[E : E'] = 2^4$. Therefore there are sixteen 1-dimensional representations of $E$ over GF(3). Moreover there is a 4-dimensional representation of $E$ since $E \leq \text{GL}_4(3)$. Since $16 + 4^2 = 32 = |E|$, this accounts for all the irreducible representations of $E$ over GF(3). Hence there is a unique 4-dimensional representation of $E$ and so there is one conjugacy class of such subgroups in $\text{GL}_4(3)$.

The following lemma is an application of Extremal Transfer (see [7, 15.15, p92] that will be needed in Section 5.

Lemma 2.14. Let $G$ be a group and $4 \times 4 \times 4 \cong A \leq G$ with $C_G(A) = A$. Set $X := N_G(A)$ and assume that $X \sim 4^3 : (2 \times \text{GL}_4(2))$ contains a Sylow 2-subgroup of $G$. Furthermore suppose that there exists an involution $u \in X \setminus O^2(X)$ such that $C_G(u) \cong 2 \times \text{Sym}(8)$. Then $u \notin O^2(G)$. In particular, $O^2(G) \neq G$.

Proof. Let $Y := O^2(X)$ then $Y/A \cong \text{GL}_3(2)$ and $u \notin Y$. We assume for a contradiction that for some $g \in G$, $r := u^g \in Y$ and so we apply [7, 15.15, p92] to see that $C_X(r)$ contains a Sylow 2-subgroup of $C_G(r) \cong 2 \times \text{Sym}(8)$. Observe first that $r \notin A$ because no element of order four in $G$ squares to $r$.

Set $V := \Omega(A) \cong 2^3$ and let $S \in \text{Syl}_2(C_X(r))$. Then $|S| = 2^8$ and therefore $|S \cap A| \geq 2^4$. It follows that $S \cap A \cong 4 \times 4$ since $Ar \leq Y/A \cong \text{GL}_3(2)$ acts faithfully on $V$ and therefore $|C_V(r)| \leq 2^2$. In particular, $|C_A(r)| = 2^4$ and so $SA \in \text{Syl}_2(X)$.

Since $X/A \cong 2 \times \text{GL}_3(2) \cong 2 \times \text{Dih}(8) \cong SA/A \cong S/(S \cap A) \cong S/C_A(r)$. Set $S_0 := S \cap Y$ then $r \in S_0$ and we have that $\text{Dih}(8) \cong S_0A/A \cong S_0/(S \cap S_0) = S_0/C_A(r)$. Since $r \in Z(S)$, $C_A(r)r \in Z(S_0/C_A(r))$. Therefore $S_0/C_A(r), r) \cong 2 \times 2$. Let $C_A(r) \leq R < S$ such that $|R/C_A(r)| = 2$ and $S = S_0R$ and $|R, S_0| \leq C_A(r)$. This is possible as $S/C_A(r) \cong 2 \times \text{Dih}(8)$. We have therefore that $[R, S_0], S_0 \cap R \leq C_A(r) \leq C_A(r), r) \cong 2 \times 2$. Suppose $S/C_A(r), r) \cong 4 \times 4$. Hence, $S/C_A(r) \cong 4 \times 4, (2 \times 2 \times 2)$ which is a subgroup of $C_G(r)/r) \cong \text{Sym}(8)$. Therefore $E \cong 2^{1+4}$ and $E$ is uniquely determined up to conjugation in $\text{GL}_4(3)$. 

However, a 2-subgroup of Sym(8) has non-abelian derived subgroup which supplies us with a contradiction. Thus $u \notin O^2(G)$. □

**Lemma 2.15.** Let $G$ be a group with a normal 2-subgroup $V$ which is elementary abelian of order $2^n$. Suppose $t$ and $w$ are in $G$ such that $Vt$ has order two and $Vw$ has order three and $Vt$ inverts $Vw$. If $|C_V(w)| = 2^a$ then $|C_V(t)| \leq 2^{(n+a)/2}$.

**Proof.** Since $Vt$ inverts $Vw$, we have that $Vw = Vtw^2t$ and so $Vw^2 = Vtw^2tw = VtV$. Therefore $C_V(t) \cap C_V(tw) = C_V(w) = C_V(w)$. We have that $|V| \geq |C_V(t)C_V(tw)| = |C_V(t)||C_V(tw)|/|C_V(t) \cap C_V(tw)|$ and so $2^a \geq |C_V(t)|^2/2^a$ which implies $|C_V(t)| \leq 2^{(n+a)/2}$. □

**Lemma 2.16.** Let $G$ be a finite group and $V \leq G$ be an elementary abelian 2-group. Suppose that $r \in G$ is an involution such that $C_V(r) = [V, r]$. Then

(i) every involution in $Vr$ is conjugate to $r$; and
(ii) $|C_G(r)| = |C_V(r)||C_{G/V}(Vr)|$.

**Proof.** (i) Let $t \in Vr$ be an involution. Then $t = qr$, for some $q \in V$. Since $t^2 = 1$, we have that $1 = qrqr = [q, r]$ as $r$ and $q$ have order at most two. So $q \in C_V(r) = [V, r]$. So $q = q_1r_1r$, for some $q_1 \in V$, and therefore $t = q_1r_1rr = r^a$ and so $t$ is conjugate to $r$ by an element of $V$.

(ii) Define a homomorphism, $\phi : C_G(r) \rightarrow C_{G/V}(Vr)$ by $\phi(x) = Vx$. Then ker $\phi = C_V(r)$. Moreover, if $Vy \in C_{G/V}(Vr)$ then $Vyr = Vr$. Hence, using (i) we see that there exists $q \in V$ such that $r^q = r^a$. Therefore $q^{-1}y \in C_G(r)$ and of course $Vq^{-1}y = Vy$ and so $\phi(q^{-1}y) = Vy$. Therefore $\phi$ is surjective. Thus, by an isomorphism theorem, $C_G(r)/C_V(r) \cong C_{G/V}(Vr)$ and $|C_G(r)| = |C_V(r)||C_{G/V}(Vr)|$, as required.

### 3. Determining the 3-Local Structure of $G$

We assume Hypothesis 1.1. Let $x \in G\setminus N_G(Z)$ such that $Z^x \leq Q$ and set $Y := ZZ^x \leq Q$. We begin by making some easy observations in particular noting that $Z \leq Q^x$ and so our hypothesis is symmetric. For a large part of this proof we work under the hypothesis that $N_G(Z)/Q$ has shape either $4{:}\text{Alt}(5)$ or $4{:}\text{Sym}(5)$. We will refer to these two possibilities as Case I and Case II respectively. At the end of Section 4 we are able to see that in Case II $G$ has an proper normal subgroup which satisfies the hypothesis of Case I and so from that point we consider Case I only.

**Lemma 3.1.** (i) $|Z| = 3$.
(ii) $C_{C_G(Z)}(Q/Z) = Q$.
(iii) $Z \leq Q^x$.
(iv) $Q \cap Q^x$ is elementary abelian.

**Proof.** (i) By hypothesis, $Z$ is the centre of a Sylow 3-subgroup, $S$ say, of $G$, therefore $\text{Syl}_3(C_G(Z)) \leq \text{Syl}_3(G)$. We have that $Q = O_3(C_G(Z))$ and $C_G(Q) \leq Q$. Therefore $[Q, Z] = 1$ implies that $Z \leq Q$. Thus $Z \leq Z(Q)$ and $|Z(Q)| = 3$ since $Q$ is extraspecial. Hence $Z = Z(Q) = Z(S)$ has order three.
(ii) Suppose that $p$ is a prime and $g \in C_G(Z)$ is a $p$-element such that $|Q/Z, g| = 1$. Then if $p \neq 3$ we may apply coprime action to say that $Q/Z = C_G(Z)(g) = C_Q(g)Z/Z = C_Q(g)/Z$ and so $[Q, g] = 1$ which is a contradiction as $C_Q(Z) \leq Q$. Therefore $C_{C_G(Z)/Z}(Q/Z)$ is a 3-group and the preimage in $C_G(Z)$ is a normal 3-subgroup of $C_G(Z)$ and so must be contained in $O_3(C_G(Z)) = Q$. Therefore $Q \leq C_{C_G(Z)(Q/Z)} \leq Q$.

(iii) Suppose $Z \not\leq Q^x$. Notice that $C_Q(Y)$ normalizes $Q$ and $Q^x$ and therefore $Q \cap Q^x \leq C_Q(Y)$. This implies that $Q \cap Q^x = Z^x$ for if we had $Q \cap Q^x > Z^x$ then $|C_Q(Y)/(Q \cap Q^x)| \leq 9$ and so $Z = C_Q(Y)' \leq Q \cap Q^x$. Therefore $Q^xC_Q(Y)/Q^x \cong C_Q(Y)/Z^x$ which must be non-abelian of exponent three and order 3 and so $Q^xC_Q(Y)/Q^x$ is a subgroup of $C_G(Z^x)/Q^x$ of order 3 which is a contradiction.

(iv) Since $|Q, Q| = Z \neq Z^x = [Q^x, Q^x]$, we immediately see that $[Q \cap Q^x, Q \cap Q^x] \leq Z \cap Z^x = 1$. Therefore $Q \cap Q^x$ is abelian and since $Q$ has exponent three, $Q \cap Q^x$ is elementary abelian.

Let $t \in N_G(Z)$ be an involution such that $Qt \in Z(N_G(Z)/Q)$.

**Lemma 3.2.** (i) $N_G(Z)/Q$ acts irreducibly on $Q/Z$ and in Case I, $C_G(Z)/Q \cong 2\text{Alt}(5) \cong \text{SL}(2, 5)$ whilst in Case II, it has shape $2\text{Sym}(5)$.

(ii) $C_Q(t) = Z = C_Q(f)$ for every element of order five $f \in C_G(Z)$. In particular, in Case I, $C_G(Y)$ is a 3-group and in Case II, $C_G(Y)$ is a $\{2, 3\}$ group with 2-part at most 2.

(iii) There exists a group $X < C_G(Z)$ with $X/Q \cong 2\text{Alt}(4) \cong \text{SL}_2(3)$ and such that $X/Q$ has no central chief factors on $Q/Z$.

**Proof.** It is clear that $N_G(Z)/Q$ acts irreducibly on $Q/Z$ (for example, from the fact that $5 \nmid |\text{GL}_3(3)|$) and so has no non-trivial modules of dimension less than four over GF(3). Since $Qt \in Z(N_G(Z)/Q)$, it follows from coprime action that $C_{Q/Z}(Qt) = C_Q(t)/Z$. Hence $C_Q(t)$ is a normal subgroup $N_G(Z)$ which is not equal to $Q$ since $C_G(Z) \leq Q$. Since $C_Q(t)/Z$ is a proper $N_G(Z)/Q$-submodule of $Q$, we have $C_Q(t) = Z$ and $[Q, t] = Q$. In particular notice that this means that the normal subgroup of order four in $N_G(Z)/Q$ does not centralize $Z$. Thus $C_G(Z)/Q \cong 2\text{Alt}(5) \cong \text{SL}(2, 5)$ or has shape $2\text{Sym}(5)$ which proves (i). Now, for $f \in N_G(Z)$ of order five, by coprime action, $Q/Z = C_{Q/Z}(f) \times [Q/Z, f]$ and $C_{Q/Z}(f)/Z \neq Q/Z$. Since $f$ acts fixed-point-freely on $[Q/Z, f]$, $[Q/Z, f]^\#$ has order a multiple of five. Therefore $Q/Z = [Q/Z, f]$ and so $C_Q(f) = Z$.

Now $C_G(Y) \leq C_G(Z)$ and $C_G(Y)$ contains no involution or element of order five. In the case that $N_G(Z)/Q \cong 4\text{Alt}(5)$ (and so $C_G(Z)/Q \cong 2\text{Alt}(5)$) since the Sylow 2-subgroups are quaternion, we see that $C_G(Y)$ is a 3-group. Otherwise, $Y$ is centralized by a 2-group of order at most 2. This proves (ii).

(iii) Observe (using [1, 33.15, p170] for example) that a group of shape $2\text{Alt}(5)$ is uniquely defined up to isomorphism and that in either case $C_G(Z)/Q$ has such a subgroup which necessarily contains $Qf$. Moreover, $2\text{Alt}(5)$ has Sylow 2-subgroups isomorphic to $Q(8)$ with normalizer isomorphic to $\text{SL}_2(3)$. Let $X \leq C_G(Z)$ be such a subgroup. There can be no central chief factor of $X$ on $Q/Z$ because $Qt \in Z(X/Q)$ inverts $Q/Z$. 

\[\square\]
Lemma 3.3.  (i) $W := C_Q(Y)C_{Q^x}(Y) = O_3(C_G(Y))$ is a group of order $3^5$ and there exists $S \in \text{Syl}_3(N_G(Z))$ such that $Y < S$.
(ii) $L := \langle Q, Q^x \rangle \leq N_G(Y)$, $W = C_L(Y)$, $L/W \cong \text{SL}_2(3)$ and $N_G(Y)/C_G(Y) \cong \text{GL}_2(3)$.
(iii) $Y$ and $W/(Q \cap Q^x)$ are natural $L/C_L(Y)$-modules.
(iv) $Z(W) = Y$.
(v) $W$ has exponent three.

Proof. (i) Notice that $C_Q(Y)$ normalizes $C_{Q^x}(Y) = Q^x \cap C_G(Z)$ and so $W$ is a 3-group. Since $Z$ is the centre of a Sylow 3-subgroup of $G$ we clearly have that $|W| < 3^6$. Now, $Q$ is extraspecial and so $|C_Q(Y)| = 3^4$ and since $Y = ZZ^x \leq Q^x$, we similarly have $|C_{Q^x}(Y)| = 3^4$. Of course, $C_{Q^x}(Y) \neq C_Q(Y)$ as their derived subgroups are not equal and so we have that $|W| = 3^5$. Moreover, $|C_G(Y) : W| = 2$ and so $W = O_3(C_G(Y))$ and since $Q \not\subseteq W$ we have that $Y < QW \in \text{Syl}_3(N_G(Z))$.

(ii) We have that $L := \langle Q, Q^x \rangle \leq N_G(Y)$ and $L/C_L(Y)$ is isomorphic to a subgroup of $\text{GL}_2(3)$. Since $L$ is generated by two distinct subgroups, $Q$ and $Q^x$, it follows that $L/C_L(Y)$ is generated by two distinct subgroups of order three and so $L/C_L(Y) \cong \text{SL}_2(3)$. Also, we have seen that $C_Q(t) = Z$, therefore $t$ inverts $Y/Z$ and of course centralizes $Z$. Hence $(L, t)/C_L(Y) \cong \text{GL}_2(3)$ and in particular $N_G(Y)/C_G(Y) \cong \text{GL}_2(3)$. If $W \neq C_L(Y)$ then $|C_L(Y)/W| = 2$. Notice that $L/W$ has a normal, non-abelian Sylow 2-subgroup, $P$ say, of order $2^4$ and that any smaller normal 2-subgroup of $L/W$ must centralizes $QW/W$ (else they would together generate $L/W$). It follows therefore that $P$ has center of order four however there can be no such group. Thus $W = C_L(Y)$.

(iii) We clearly have that $Y$ is a natural $L/W$-module. Now suppose that $Y = Q \cap Q^x = C_Q(Y) \cap C_{Q^x}(Y)$. Then $|W| = 3^43^1/3^2 = 3^6$ which is a contradiction hence $Y < Q \cap Q^x$ and since $Q \cap Q^x$ is elementary abelian, $|Q \cap Q^x| = 3^3$ and so $|W/(Q \cap Q^x)| = 9$. Notice that $Q \cap Q^x$ is normalized by $L$ and that $L/W$ acts on $W/(Q \cap Q^x)$, which is the direct product of the groups $C_Q(Y)/(Q \cap Q^x)$ and $C_{Q^x}(Y)/(Q \cap Q^x)$. Therefore it must be a natural $L/W$-module.

(iv) Clearly $Y \leq Z(W)$ so suppose $Y < Z(W)$ then $Z(W)$ has index at most nine in $W$. Since the non-abelian group $C_Q(Y)$ is contained in $W$, $W$ is non-abelian. Therefore $[W : Z(W)] = 9$. Notice that $Z(W) \neq Q \cap Q^x$ otherwise $C_Q(Y)$ is abelian. So we have that $(Q \cap Q^x)/Z(W)/(Q \cap Q^x)$ is a proper and non-trivial $L/W$ invariant subgroup of the natural $L/W$-module, $W/(Q \cap Q^x)$. This is a contradiction. Thus $Y = Z(W)$.

(v) Since $Q$ has exponent three, $Q \cap Q^x$ does also. Choose $a \in Q \setminus (Q \cap Q^x)$ then $(Q \cap Q^x)a$ is a non-identity element of the natural $N_G(Y)/W$-module, $W/Q \cap Q^x$. Moreover, every element in the coset has order dividing three since $Q$ has exponent three. Since $N_G(Y)/W$ is transitive on the non-identity elements of the natural module $W/(Q \cap Q^x)$, every element of $W$ has order dividing three.

We continue to use the notation $W$ and $L$ as in the previous lemma. Furthermore, let $s$ be an involution such that $Ws \in Z(L/W)$ then $s$ inverts $Y$ and so $s \in N_G(Z)$. Since $t \in N_G(Y)$ we may choose $s$ such that $s$ and $t$ are in a Sylow 2-subgroup of $N_G(Y)$ and therefore $[s,t] = 1$. Furthermore set $J := [W,s]$ and we also now
set $S := QW \in \text{Syl}_3(C_G(Z)) \cap \text{Syl}_3(L)$ and let $Z_2$ be the second centre of $S$ (so $Z_2/Z = Z(S/Z)$).

**Lemma 3.4.** (i) We have that $Y \leq Z_2 \leq C_G(Y)$ and $Z_2$ is abelian of order $3^3$ but distinct from $Q \cap Q^x$.

(ii) $W' = Y$.

(iii) $Q \cap Q^x = YC_{C_G(Y)}(s)$ and $C_{C_G(Y)}(s) \triangleleft C_L(s) \cong 3.SL_2(3)$.

(iv) $J$ is an elementary abelian subgroup of $C_G(Y)$ of order $3^4$ that is inverted by $s$ and $Q \cap J = Z_2$.

(v) $J = J(S) = J(W)$ and $Y \leq S' \leq Q \cap J$.

(vi) $t$ and $st$ are conjugate in $N_G(Y)$.

*Proof.* (i) By Lemma 3.1 (ii), $C_G(Z)/Q$ acts faithfully on $Q/Z$. Therefore $S/Q$ is isomorphic to a cyclic subgroup of $GL(Q/Z) \cong GL_4(3)$ of order three, we may consider the Jordan blocks of elements of order three to see that any such cyclic subgroup centralizes a subgroup of $Q/Z$ of order at least $3^2$. Therefore $|Z_2/Z| \geq 9$ and so $|Z_2| \geq 27$. Since $[Q/Z, Z_2] = 1$, $Z_2 < Q$ by Lemma 3.1 (ii). Now suppose $Z_2 \not\leq C_G(Y)$. Then $S = Z_2C_G(Y) \in \text{Syl}_3(G)$. Since $Z_2/Z = Z(S/Z)$, $[S, Z_2] \leq Z$ and so $|W/Z_2| \leq Z \leq Q \cap Q^x$. Therefore $|W/(Q \cap Q^x), Z_2| = 1$, however this implies that $S/W$ acts trivially on the natural $L/W$-module $W/(Q \cap Q^x)$ which is a contradiction. So $Z_2 \leq C_G(Y) \cap Q$. Suppose $Q \cap Q^x \not\leq Z_2$. Then $Z^x = [C_Q(Y), Q \cap Q^x] \leq [C_Q(Y), Z_2] \leq Z$ which is a contradiction. Therefore $|Z_2| = 27$ and in particular, $Z_2 \not\leq Q \cap Q^x$. Furthermore $Y < S$ and so $Y/Z$ is central in $S/Z$. Therefore $Y \leq Z_2$ and since $Y$ is central in $Z_2 \leq C_G(Y)$, $Z_2$ is abelian.

(ii) Now $Z = C_Q(Y)' \leq W'$ and $Z^x = C_{Q^x}(Y)' \leq W'$ and so $Y \leq W'$. Moreover, we have just observed that $Z_2 \not\leq Q \cap Q^x$ and so $Q \cap Q^x$ and $Z_2$ are distinct normal subgroups of $W$ both of index nine. Thus $Y \leq W' \leq Q \cap Q^x \cap Z_2$. It follows from the group orders that $Y = C_G(Y)' = Q \cap Q^x \cap Z_2$.

(iii) By coprime action on an abelian group, $W/Y = C_{W/Y}(s) \times [W/Y, s]$. By Lemma 3.3 (ii), $Y$ and $W/(Q \cap Q^x)$ are natural $L/W$-modules. Therefore $s$ inverts $Y$ and $W/(Q \cap Q^x)$. It follows from coprime action that $|C_{W}(s)| = 3$ and that $C_{W}(s) \leq Q \cap Q^x$ with $Q \cap Q^x = YC_W(s)$. Furthermore, $L/W \cong SL_2(3)$ and $Ws \in Z(L/W)$ so it follows from coprime action that $C_{W}(s) \leq C_L(s) \cong 3.SL_2(3)$.

(iv) We have that $Y$ is inverted by $s$ and so $Y = [Y, s] \leq J$. We also have that $|[W/Y, s]| = 9$ and by coprime action we see, $[W/Y, s] = Y[W, s]/Y = YJ/Y \cong J/(J \cap Y) = J/Y$. This implies that $J$ has order $3^4$ and furthermore we have that $1 = J \cap C_W(s)$. This implies that $s$ acts fixed-point-freely on $J$ and so $J$ is abelian and inverted by $s$. By Lemma 3.3 (v), $W$ has exponent three and so $J$ is elementary abelian. Now $Z_2$ is a characteristic subgroup of $S$ and so is normalized by $s$. Moreover $Y \leq Z_2$. If $C_{Z_2}(s) \neq 1$ then $2Z_2 = Q \cap Q^x = YC_W(s)$ which is a contradiction. Thus $Z_2 = J \cap Q$.

(v) Suppose there was another abelian subgroup of $W$ of order $3^4$, $J_0$ say. Then $|J \cap J_0| = 3^3$ and $J \cap J_0$ is central in $W$. This contradicts Lemma 3.3 which says that $Z(W) = Y$. It follows therefore that $J(W) = J$. Clearly $3^4$ is the largest possible order of an abelian subgroup of $S$ (else $Q$ would contain abelian subgroups of order $3^4$). So suppose $J_1$ is an abelian subgroup of $S$ distinct from $J$. Then $J_1 \not\leq W$ and $J_1 \not\leq Q$. Therefore, $S/Z$ contains three distinct abelian subgroups
We must have that $S = QJ = QJ_1$. Hence, $(Q/Z) \cap (J/Z)$ and $(Q/Z) \cap (J_1/Z)$ both have order nine and are both central in $S/Z$. We must have that $Q/Z \cap J/Z = Q/Z \cap J_1/Z = Z_2/Z$. Thus $Y \leq Z_2 \leq J_1$ and so $J_1 \leq C_S(Y)$ which we have seen is not possible. Thus $J = J(S)$. In particular, $J$ is a normal subgroup of $S$ of index nine and so $Y = W' \leq S' \leq Q \cap J$.

Finally, we have seen that $L(t)/W \cong \text{GL}_2(3)$ and so $Wt$ is conjugate to $Wst$ and since $W$ is a 3-group, an element of $L$ conjugates $t$ to $st$. \qed

We now choose a subgroup $S \subseteq X \subseteq C_G(Z)$ such that $X/Q \cong \text{SL}_2(3)$ as in Lemma 3.2.

**Lemma 3.5.** $Q/Z = \langle C_{Q/Z}(S)^{X/Q} \rangle$ and $S/Q$ acts quadratically on $Q/Z$.

**Proof.** First observe that since $X/Q \cong \text{SL}_2(3)$ and there is no central chief factor of $X/Q$ on $Q/Z$, any proper $X/Q$-submodule of $Q/Z$ is necessarily a natural $X/Q$-module. Let $Z < V < Q$ such that $V/Z$ is an $X/Q$-submodule and is therefore a natural module. Thus $S/Q$ acts non-trivially on $V/Z$. In particular this means $V/Z \neq Z(S/Z) = C_{Q/Z}(S)$. So $Z(S/Z)$ is not contained in any proper $X$-invariant subgroup of $Q$. Thus $Q = \langle C_{Q/Z}(S)^{X/Q} \rangle$.

Now, $J = J(S)$ is abelian and normalized by $S$ and so $[Q, J] \leq J$ and then $[Q, J, J] = 1$. Now $J \neq Q$ (as $Q$ has no abelian subgroups of order $3^4$) and so $S/Q = JQ/Q$ and therefore $[Q/Z, JQ/Q, JQ/Q] = 1$ and so $S/Q$ acts quadratically on $Q/Z$. \qed

We have now satisfied the conditions of Lemma 2.10 and so we have the following results.

**Lemma 3.6.**

(i) $Q/Z$ is a direct product of natural $X/Q$-modules.

(ii) There are exactly four $X$-invariant subgroups $N_1, N_2, N_3, N_4 < Q$ properly containing $Z$ such that for $i \neq j$, $N_i \cap N_j = Z$.

(iii) $N_i \cap J$ has order nine for each $i$ and $S' = J \cap Q = \langle N_i \cap J \rangle |1 \leq i \leq 4\rangle$.

(iv) For some $i \in \{1, 2, 3, 4\}$, $Y = N_i$ and $N_i$ is abelian.

(v) For each $i \in \{1, 2, 3, 4\}$, $X$ is transitive on $N_i \backslash Z$.

**Proof.** Part (i) follows immediately from Lemma 2.10 which says that $Q/Z$ is a direct product of natural $X/Q$-modules. Let $N_1$ and $N_2$ be the corresponding subgroups of $Q$. View $N_1/Z$ and $N_2/Z$ as vector spaces over $GF(3)$. Since $N_1/Z$ and $N_2/Z$ are isomorphic as $X$-modules, it follows that there are two additional isomorphic submodules in $Q/Z$. Let $N_3$ and $N_4$ be the corresponding normal subgroups of $Q$. Then $N_3/Z$ and $N_4/Z$ are natural $X$-modules and for $i \neq j$, $N_i \cap N_j = Z$. This proves (ii).

By Lemma 3.4, if $Z_2/Z = Z(S/Z)$ then $Y < Z_2$ and $Z_2 = J \cap Q$ is elementary abelian of order 27. Now for each $i \in \{1, 2, 3, 4\}$, $C_{N_i/Z}(S) \neq 1$ and so $Z_2 \cap N_i = J \cap N_i$ has order at least nine. In fact the order must be exactly nine for were it greater then for some $i$, $N_i = Z_2$ and then $N_i \cap N_j$ would have order at least nine for each $j \neq i$. Now for each $i \neq j$, $N_i \cap N_j = Z$ and so $N_i \cap J \neq N_j \cap J$ and so $Z_2 = \langle N_i \cap J \rangle |1 \leq i \leq 4\rangle$. In particular we must have (without loss of generality)
that $N_1 \cap J = Y$. By Lemma 3.4 (v), $Y \leq S' \leq Q \cap J$. Suppose $S' = Y$. Then for any $2 \leq i \leq 4$, $Y \not\leq N_i$ and so $[N_i, S] \subseteq N_i \cap Y = Z$. Therefore $N_i \leq Z_2$ which is a contradiction. Thus $Y < S' = J \cap Q$ which proves (iii).

We already have that (without loss of generality) $N_1 \cap J = Y$. Suppose that $N_1$ is non-abelian. Then $C_Q(N_1) \cong N_1 \cong \mathbb{Z}^{1+2}$. Since $N_1$ is $X$-invariant, $C_Q(N_1)$ is also $X$-invariant. Therefore, without loss of generality, we can assume that $N_2 = C_Q(N_1)$ and so $N_2 \leq C_S(Y)$ and $N_2$ is also non-abelian with $Y \not\leq N_2$. Since $N_1 \not\subseteq C_S(Y)$, $S = C_S(Y)N_1$ and so we have that $S' \leq [C_S(Y), N_1]C_S(Y) \cap N_1 \leq (C_S(Y) \cap N_1)YZ = Y$ (using Lemma 3.4) which is a contradiction since $S' > Y$. This proves (iv).

Finally, since each $N_i/Z$ is a natural $X/Q$-module, $X$ is transitive on the non-identity elements of $N_i/Z$. So let $Z \neq Zn \in N_i/Z$. Then $(Z, n) \lhd Q$ however $|C_Q(n)| = 3^4$. Therefore $n$ lies in a $Q$-orbit of length three in $Zn$. Hence every element in $Zn$ is conjugate in $X$. Thus $X$ is transitive on $N_i/Z$ which completes the proof. □

For the rest of this section we continue the notation from Lemma 3.6 with $N_1, N_2, N_3, N_4$ chosen such that $Y \leq N_1$ and satisfying the notation set in the following lemma also.

Lemma 3.7. Without loss of generality we may assume that $N_1 \cong N_2$ is elementary abelian and $N_3 \cong N_4$ is extraspecial with $[N_3, N_4] = 1$.

Proof: By Lemma 3.6, $N_1$ is abelian. So suppose $N_i$ is non-abelian for some $i \in \{2, 3, 4\}$. Then $C_Q(N_i) \cong N_i \cong \mathbb{Z}^{1+2}$ is $X$-invariant and we may assume $C_Q(N_i) = N_i$ for some $i \neq j \in \{2, 3, 4\}$. Now it follows that either $N_j$ is abelian for every $i \in \{1, 2, 3, 4\}$ or without loss of generality $N_1 \cong N_2$ and $N_3 \cong N_4$ are non-abelian. So we assume for a contradiction that $N_2$, $N_3$ and $N_4$ are all abelian.

Since $N_1/Z$ is isomorphic as a $\mathbb{GF}(3):X$/$Q$-module to $N_2/Z$, for any $m \in N_1 \setminus Z$ there is an $n \in N_2 \setminus Z$ such that $Zn$ is the image of $Zm$ under a module isomorphism. It then follows (without loss of generality) that $Znm$ is an element of $N_3/Z$ and $Zn^2m$ is an element of $N_4/Z$. In particular $x_1 := nm \in N_3$ and $x_2 := n^2m \in N_4$. Let $g \in X$ have order four then $Qg^2 = Ql$ inverts $Q/Z$ and so

$$Zn^{2^k} = Zn^2 \text{ and } Zm^{2^k} = Zm^2.$$  \hspace{1cm} (1)

Also if $Z \neq Za \in N_1/Z$ and $g$ and $h$ are elements of order four in $X$ such that $Q(g) \neq Q(h)$ then $N_1/Z = \langle Za^a, Za^b \rangle$ and so $N_i = Z(a^a, a^b)$.

So consider $[x_1, x_2^2]$. We calculate the following using commutator relations and using that all commutators are in $Z$ and therefore central.
Thus \([x_1, x_2^g] = [x_1, x_2]^g\) and so \([x_1, x_2^g] = 1\). This holds for any element of order four in \(X\). Thus \(mn \in N_3\) commutes with \(N_4 = \langle (n^2)^g \rangle\), \((n^2m)^h\) where \(g\) and \(h\) are elements of order four as above. Furthermore this argument works for any element of \(N_i \setminus Z\) and so \([N_3, N_4] = 1\). However this contradicts our assumption that \(N_3\) and \(N_4\) are abelian. 

**Lemma 3.8.** For \(i \in \{3, 4\}\), elements in \(N_i \setminus Z\) are not conjugate into \(Z\). In particular, there are 12 elements of order three in \(S'\) which are not \(G\)-conjugate into \(Z\).

**Proof.** Let \(\{i, j\} = \{3, 4\}\) and let \(a \in N_i \setminus Z\). Since \(a \in Z_2\), \(|C_S(a)| = 3^5\). Suppose that \(a \in Z^G\). Then we must have that \(C_S(a)\) \(\triangleleft O_3(C_G(a, Z))\) and we must similarly have that \(\langle a, Z \rangle = C_S(a)\). Now \(S = C_S(a)N_j\) and so \(S' \leq C_S(a)N_i \leq C_S(a)\cap N_i \leq \langle a, Z \rangle\) which is a contradiction. Thus \(a \notin Z^G\). Every element in \(N_i \setminus Z\) is conjugate to \(a\) and therefore no element in \(N_i \setminus Z\) is conjugate into \(Z\).

Furthermore, by Lemma 3.6 (iii), we see that \(S' = J\cap Q\) contains twelve elements of order three which are not conjugate into \(Z\). These are contained in \(N_3 \cap J = N_3 \cap S'\) and \(N_4 \cap J = N_4 \cap S'\).

**Lemma 3.9.** (i) Let \(i \in \{1, 2, 3, 4\}\) and set \(S_i := C_S(J \cap N_i)\) then \(|S_i| = 3^5\) and \(|C(S_i)| = 9\).

(ii) \(S'_1 = Z(S_1) = J \cap N_1 = Y, S'_2 = Z(S_2) = J \cap N_2, S'_3 = Z(S_3) = J \cap N_3, S'_4 = Z(S_4) = J \cap N_3\).

In particular \(S_i \neq S_j\) for each \(i \neq j\).

**Proof.** By Lemma 3.6, \(|J \cap N_i| = 9\) for each \(i \in \{1, 2, 3, 4\}\) and since \(J\) is elementary abelian of order 81, \(J \leq S_i\). Hence \(S_i \geq \langle J, C_Q(J \cap N_i)\rangle\). Since \(C_Q(J \cap N_i)\) has order \(3^4\) and is non-abelian, \(|\langle J, C_Q(J \cap N_i)\rangle| \geq 3^5\). Moreover, since \(|S| = 3^6\) and \(Z(S) = Z\) has order three, it follows that \(S_i = \langle J, C_Q(J \cap N_i)\rangle\) has order \(3^5\). Now for each \(i \in \{1, 2, 3, 4\}, Z(S_i) \leq Q\) otherwise \(S = QZ(S_i)\) and then \([S_i \cap Q, S] \leq [S_i \cap Q, Q][S_i \cap Q, Z(S_i)] \leq Z\) which implies that \(S_i \cap Q \leq Z_2\) which is a contradiction. Thus \(Z(S_i) \leq Q\) has order nine and \(Z(S_i) = N_i \cap J\).
Now for \( i \in \{1, 2, 3, 4\} \), we have that \( Z \leq S'_i \). If \( S'_i = Z \) then \( Q/Z \) and \( S_i/Z \) are two distinct abelian subgroups of \( S/Z \) of index three. This implies that \( S/Z \) has centre of order at least \( 3^3 \). However by Lemma 3.4 (iii), \( Z(S/Z) \) has order nine. Thus \( S'_i > Z \). Now for \( i = 1 \), by Lemma 3.6, \( Y = N_1 \cap J \) and so \( Z(S_1) = N_1 \cap J = Y \). Furthermore, for \( i \in \{1, 2\} \), \( N_i \) is abelian and so \( N_i \leq S_i \). Therefore \( S'_i \leq S' \cap N_i = J \cap N_i \) since \( N_i \leq S_i \). For \( \{i, j\} = \{3, 4\} \), \( [N_i, N_j] = 1 \) and so \( N_j \leq S_i \). Therefore \( S'_i \leq S' \cap N_j = J \cap N_j \) since \( N_j < S_j \).

**Lemma 3.10.** Every element of order three in \( S \) lies in the set \( Q \cup S_1 \cup S_2 \) and the cube of every element of order nine in \( S \) is in \( Z \).

**Proof.** By hypothesis, \( Q \) has exponent three and by Lemma 3.4 (vi), so does \( J \). So let \( g \in S \) such that \( g \notin Q \cup J \). Then \( g = cb \) for some \( c \in Q \setminus J = Q \setminus S' \) and \( b \in J \setminus Q \). We calculate using the equality \( c[b, c][b, c, c] = [b, c]c \) and that \( b \in J \) so commutes with all commutators in \( S' \leq J \).

\[
\begin{align*}
chenb b &= c^2b[b, c]bcb \\
&= c^2b^2[b, c]c \\
&= c^2b^2c[b, c][b, c, c] \\
&= c^2b^2c[b, c][b, c, c] \\
&= [c, b][b, c][b, c, c] \\
&= [b, c, c].
\end{align*}
\]

Since \( c \in Q \setminus J = Q \setminus S' \), \( S'(c) \) is a proper subgroup of \( Q \) properly containing \( S' \). As \( S' \cap N_i = J \cap N_i \) has order nine for each \( i \in \{1, 2, 3, 4\} \), \( S'N_i \) has order 81. Thus \( S'(c) = S'N_i \) for some \( i \in \{1, 2, 3, 4\} \).

If \( S'(c) = S'N_1 = C_Q(Y) \) then \( cb \in C_S(Y) = W \) and \( W \) has exponent three. Suppose \( S'(c) = S'N_2 \). Then \( S_2 = C_S(S' \cap N_2) = J(c) \) and \( S'_2 = S' \cap N_2 \) therefore \([b, c] \in S' \cap N_2 \) is central in \( S'(c) = S'N_2 \). Therefore \([b, c, c] = 1 \) and so \( cb \) has order three.

Now suppose \( S'(c) = S'N_3 \) (and a similar argument holds if \( S'(c) = S'N_4 \)). Then \( S_4 = C_S(S' \cap N_4) = J(c) \) and \([b, c] \in S'_4 \cap N_3 \). Suppose \( c[b, c][b, c, c] = 1 \). Then \([b, c] \) commutes with \( J(c) = S_4 \) and so \([b, c] \in S'_4 \cap Z(S_4) = Z \). Thus \( S_4 = J(c) = S'(c) \) and so \([b, c] \in \langle s', c \rangle, [S', b], [c, b] \rangle \). However \([S', b] \leq Z \), \([S', b] = 1 \) and \([c, b] \in Z \) which is a contradiction since \( S_4 = N_3 \cap S' > Z \). Thus \([b, c, c] \neq 1 \) and \( cb \) has order nine (no element can order 27 since \( Q \) has exponent three). Furthermore, \( (cb)^3 = [b, c, c] \in [S' \cap N_4, c] \leq [Q, Q] = Z \) and so the cube of every such element of order nine is in \( Z \). \( \square \)

**Lemma 3.11.** For each \( i \in \{3, 4\} \), if \( a \in Z(S_i) \) then \( Z(S_i/\langle a \rangle) = Z(S_i)/\langle a \rangle \).

**Proof.** Let \( \{i, j\} = \{3, 4\} \) then by Lemma 3.9, we have that \( S'_i = Z(S_j) \) and \( S'_j = Z(S_i) \). So let \( a \in Z(S_i) \) and suppose \( Z(S_i/\langle a \rangle) \) is not \( Z(S_i)/\langle a \rangle \). Let \( V \leq S_i \) such that \( a \in V \) and \( Z(S_i/\langle a \rangle) = V/\langle a \rangle \) then \(|V| > 3^3 \). Therefore \( S_i/V \) is abelian and so \( S'_i \leq V \). Therefore \( [S'_i, S_i] \leq \langle a \rangle \). However \( S_i \) normalizes \( Z(S_j) = S'_j \) and so \( [S'_i, S_i] \leq \langle a \rangle \cap S'_i = \langle a \rangle \cap Z(S_j) = 1 \) since \( Z(S_i) \cap Z(S_j) \leq N_i \cap N_j = Z \). However
this implies that \( S_j' \leq Z(S_j) \) and so \( N_j \cap J \leq N_i \cap J \) which is a contradiction. Therefore \( Z(S_i/(a)) = Z(S_i)/(a) \).

We fix an element of order three \( a \) in \( Q \) such that \( a \in (N_3 \cap J) \setminus Z \) and therefore \( a \notin Z^G \) by Lemma 3.8. Let \( 3A := \{a^g | g \in G \} \) and \( 3B := \{z^g | g \in G \} \) where \( Z = \langle z \rangle \). We show in the rest of this section that these are the only conjugacy classes of elements of order three in \( G \).

**Lemma 3.12.**

(i) \(|C_S(a)| = 3^3\);
(ii) \(|a^G \cap Q| = |a^{C_G(Z)} \cap Q| = 120 \) and \(|z^G \cap Q| = |z^{x_{C_G(Z)}} \cap Q| + 2 = 122 \); and
(iii) \( Q^Z \subseteq 3A \cup 3B \).

Furthermore, in Case II, \( C_G(Z)/Q \) is isomorphic to the group of shape \( 2 \text{Sym}(5) \) which has semi-dihedral Sylow 2-subgroups and in either case \( (s,C_G(Z)/Q) / 4 \text{Alt}(5) \).

**Proof.** We have chosen \( a \in N_3 \cap J \) and so by Lemma 3.9, \( C_S(a) = C_S(\langle z, a \rangle) = C_S(N_3 \cap J) = S_3 \) which has order 3^3. Now let \( q \in Q \setminus Z \) and consider \([C_G(Z) : C_{C_G(Z)}(q)]\). By Lemma 3.2 (ii), an element of order five acts fixed-point-freely on \( Q/Z \) so we have that \( 5 \mid [C_G(Z) : C_{C_G(Z)}(q)] \). Let \( R \) be a Sylow 2-subgroup of \( C_{C_G(Z)}(q) \). Recall that \( C_G(Z)/Q \) has subgroups isomorphic to \( Q(8) / Q(8) \) with \( Q(8) \) in the centre. The preimage of any such subgroup in \( C_G(Z) \) intersects trivially with \( R \) as \( Q(8) \) inverts \( Q/Z \). So \( 8 \mid [C_G(Z) : C_{C_G(Z)}(q)] \). Furthermore \( q \) is not 3-central in \( C_G(Z) \) and so \( 3 \mid [C_G(Z) : C_{C_G(Z)}(q)] \). Therefore \([C_G(Z) : C_{C_G(Z)}(q)]\) is a multiple of 120. Now there exists \( z^x \in Q \setminus Z \) which lies in \( C_G(Z) \)-orbit in \( Q \) of length at least 120 and also there exists \( a \in Q \) which is not conjugate to \( z \) and lies in \( C_G(Z) \)-orbit in \( Q \) of length at least 120. Since \( a \) is not conjugate to \( z^x \), these orbits are distinct. Thus \(|a^G \cap Q| = |a^{C_G(Z)} \cap Q| = 120 \) and \(|z^G \cap Q| = |z^{x_{C_G(Z)}} \cap Q| + 2 = 122 \).

We may now observe that in Case II, when \( N_G(Z)/Q \sim 4 \text{Sym}(5) \), every subgroup of index two must contain an involution centralizing \( z^2 \in Q \setminus Z \). In particular, \( t \) can not be the unique involution in any such index two subgroup. It follows then that \( C_G(Z)/Q \) is isomorphic to the group of shape \( 2 \text{Sym}(5) \) which has semi-dihedral Sylow 2-subgroups as claimed.

The final comment in the statement of this lemma is clear in Case I so suppose we are in Case II. Recall that \( J \) lies in the centre of \( N_G(J)/J \) and that \( s \) centralizes \( S/J \). We have that \( (s,C_G(Z)/Q \sim 4 \text{Alt}(5)) \) or \( 2 \text{Sym}(5) \). Thus a Sylow 2-subgroup of the normalizer of \( S \) is isomorphic to \( C_4 \times C_2 \) or \( \text{Dih}(8) \) respectively. If \( (s,C_G(Z)/Q \sim 2 \text{Sym}(5)) \) then a Sylow 2-subgroup of the normalizer of \( S \) must act faithfully on \( S/J = QJ/J \cong Q/Q \cap J \) as the centre of the dihedral group is a conjugate of \( t \) and so inverts \( S/J \). Thus \( s \) cannot lie in such a subgroup and we may conclude that \( (s,C_G(Z)/Q \sim 4 \text{Alt}(5)) \).

**Lemma 3.13.**

(i) \(|C_J(t)| = |C_S(t)| = 3^2 \) and \( t \) inverts \( S/J \).
(ii) In Case I, we have that \(|N_G(S) \cap C_G(Z)| = 3^62^2 \) and \(|N_G(S)| = 3^62^3\).
(iii) In Case II, we have that \(|N_G(S) \cap C_G(Z)| = 3^62^3 \) and \(|N_G(S)| = 3^62^4\).
(iv) There exists an element of order four \( e \in N_G(S) \cap C_G(Z) \) such that \( e^2 = t \) and \( e \) does not normalize \( Y \).
Proof. We have that \(C_G(t) = Z\) and so \(t\) inverts \(Q/Z\). We also have that \(t\) centralizes \(S/Q = QJ/Q \cong J/(J \cap Q)\). Since \(C_G(t) = Z\), we see using coprime action and an isomorphism theorem that \(C_{J/(J \cap Q)}(t) = C_J(t)\) and so \(|C_J(t)| = 3^2\). We also see that \(t\) inverts \(Q/(Q \cap J) \cong QJ/J = S/J\) which proves (i).

Now, in Case I, the normalizer of a Sylow 3-subgroup has order 2\(^3\)3 with a cyclic Sylow 2-subgroup and in Case II, it has order 2\(^3\)3. Recall that \(s \in N_G(Y)\) inverts \(Z\) and normalizes \(S \leq N_G(Y)\). Thus (ii) and (iii) follow immediately. Furthermore, in either case, we may choose an element of order four \(e \in C_G(Z)\) that squares to \(t\) and normalizes \(S\). Suppose \(e\) normalizes \(Y\). Then \(e^3 = t\) centralizes \(Y\) which is impossible. This completes the proof. □

Lemma 3.14. (i) \(J^# \subseteq 3A \cup 3B\).
(ii) \(N_j^# \subseteq 3B\), \(|C_J(t) \cap 3A| = |C_J(t) \cap 3A| = 4\) and \(|C_{G(S)}(s)^#| \subseteq 3A\).
(iii) Every element of order three in \(S\) is in the set \(3A \cup 3B\).
(iv) For every \(q \in Q\) there exists \(P \in \text{Syl}_3(C_G(Z))\) such that \(q \in J(P)\).

Proof. (i) Since \(N_G(Y)/G(Y) \cong GL_2(3)\) and \(J\) is characteristic in \(G(Y)\) and inverted by \(C_G(Y)\), we have that \(J=Y\) is a natural \(N_G(Y)/G(Y)\)-module. Hence there are four \(N_G(Y)\)-images of \(S\) in \(J\) with pairwise intersection equal to \(Y\). By Lemma 3.12, \(Q^# \subseteq 3A \cup 3B\). Therefore \([S, S]^# \subseteq 3A \cup 3B\) which implies that \(J \setminus \{1\} \subseteq 3A \cup 3B\).

(ii) We have that for \(i \in \{1, 2, 3, 4\}\), by Lemma 3.6 (v), \(X\) is transitive on \(N_i \setminus Z\) and so either \(N_i \setminus Z \subseteq 3A\) or \(N_i \setminus Z \subseteq 3B\). By Lemma 3.13 (iii), there exists \(e \in N_G(S)\) such that \(Y^e \neq Y\). Since \(e\) normalizes \(S\), \(e\) normalizes \(Z_2 = \langle J \cap N_i \mid i \in \{1, 2, 3, 4\}\rangle\). Therefore \(Y^e = N_i \cap J\) for some \(i \in \{1, 2, 3, 4\}\). We have that \(N_i \setminus Z \subseteq 3A\) for \(i = 3, 4\) and so \(Y^e = N_2 \cap J\). Thus \(N_2^# \subseteq 3B\). Notice now that \(C_{Z_2}(st)\) has order 9 and is a complement to \(Z_2\). Therefore \(|C_J(st) \cap 3A| = |C_J(st) \cap 3A| = 4\). By Lemma 3.4 (vi), \(t\) is conjugate to \(st\) by an element of \(N_G(Y) = N_G(J)\) and so the same count holds for \(t\).

Now there are five conjugates of \(X\) in \(C_G(Z)\) and therefore five images of \(N_1\) and of \(N_2\) in \(C_G(Z)\) (since \(N_1\) was normal in two distinct conjugates of \(X\) then \(N_i\) would be normal in \(C_G(Z)\)). For each \(i \in \{1, 2\}\), \(N_i \setminus Z\) contains 24 conjugates of \(z\). Since \(Q' \setminus Z\) contains 120 conjugates of \(Z\), there exists \(i \in \{1, 2\}\) and \(g \in C_G(Z)\) such that \(Y \leq N_i^g \leq X^g\) and \(N_i^g \neq N_1\). Now consider \(C_Q(Y)\) which is normalized by \(s\) (as \(s\) normalizes \(Q\) and \(Y\) by Lemma 3.4 (iii), \(C_{G(S)}(s) \leq Q \cap Q^x\) has order three. Now there are four proper subgroups of \(C_Q(Y)\) properly containing \(Y\). These include \(Q \cap Q^x, S', N_1\) and \(N_i^g\) (we can not yet exclude the possibility that \(Q \cap Q^x = N_1\) or \(N_i^g\)). We have that \(s\) normalizes at least two subgroups: \(S' \neq Q \cap Q^x\) (since \(S' = J \cap Q\) and using 3.4 (i) and (iv)). Suppose that \(s\) normalizes \(N_1\) and \(N_i^g\). If \(s\) inverts \(N_1\) then \(N_1 \leq [C_S(Y), s] = J\) which is a contradiction (as \(|N_1 \cap J| = 9\)). Therefore \(N_1 = Y C_{C_S(Y)}(s) = Q \cap Q^x\) and by the same argument \(N_i^g = Q \cap Q^x\) which is a contradiction since \(N_i^g \neq N_1\). Therefore at least one of \(N_1\) and \(N_i^g\) is not normalized by \(s\). We assume that \(N_i^g \neq N_1\) (and the same argument works if \(N_i^g \neq N_i^g\)) and so the four proper subgroups of \(C_Q(Y)\) properly containing \(Y\) are \(Q \cap Q^x, S', N_1\) and \(N_i^g\). Now consider \(\{C_Q(Y) \setminus 3A\}\). Since \(Q/N_1\) is a natural \(X/Q\)-module, there are four \(X\)-conjugates of \(C_Q(Y)\) in \(Q\) intersecting at \(N_1\). Each
must contain exactly $120/4 = 30$ conjugates of $a$. Thus $|C_Q(Y) \cap 3A| = 30$. Clearly $N_1 \cap 3A = N_1^s \cap 3A = \emptyset$ and $|S\cap 3A| = 12$ by Lemma 3.8. Therefore we have $|Q \cap Q^g \cap 3A| = 18$. In particular this implies $C_{C_S(Y)}(s)^# \subseteq 3A$.

(iii) By Lemma 3.10, every element of order three in $S$ lies in $Q \cup C_S(N_1 \cap J) \cup C_S(N_2 \cup J)$ and the cube of every element of nine is in $Z$. Recall that $N_1 \cap J = Y$ and since $N_2^# \subseteq 3B$ and $C_G(Z)$ is transitive on $Q \cap 3B \setminus Z$, $N_1 \cap J$ is conjugate in $C_G(Z)$ to $N_2 \cap J$. Therefore $S_2 = C_S(N_2 \cap J)$ is conjugate to $C_S(Y) = S_1$. Now, by Lemma 3.3, $C_S(Y)/(Q \cap Q^g)$ is a natural $SL_2(3)$-module and so there are four $N_G(Y)$-conjugates of $C_S(Y)$ in $C_S(Y)$ and this accounts for every element of $C_S(Y)$.

Since $C_Q(Y)^# \subseteq Q^# \subseteq 3A \cup 3B$, $C_S(Y)^# \subseteq 3A \cup 3B$ and therefore every element of order three in $S$ is in $3A \cup 3B$.

(iv) Since $z^x, a \in J = J(S)$ and every element in $Q \setminus Z$ is $C_G(Z)$-conjugate to one of these, every element in $Q$ lies in the Thompson subgroup of a Sylow 3-subgroup of $C_G(Z)$.

Lemma 3.15. $C_G(a) \not\subseteq N_G(J)$.

Proof. By Lemma 3.14 (ii) and (iv), there exists $g \in Q \cap Q^g \cap 3A$ and there exists $R \in Syl(Z(C_G(Z)))$ such that $g \in J(R)$. The same lemma applied to $C_G(Z^g)$ says that there exists $P \in Syl(C_G(Z))$ such that $g \in J(P)$. If $Q \cap Q^g \not\subseteq J(R)$ then $Y \leq J(R) \leq C_G(Y)$. Hence $J(R) = J(C_G(Y)) = J$ (see Lemma 3.4 (vi)) however $Q \cap Q^g \not\subseteq J$ (by Lemma 3.4 (iii) since $1 \neq C_{C_G(Y)}(s) \leq Q \cap Q^g$ but $J$ is inverted by $s$). Therefore $C_R(g) = J(R)(Q \cap Q^g)$ and similarly $C_P(g) = J(P)(Q \cap Q^g)$.

Suppose $J(P) = J(R)$. Then $J(R)$ is normalized by $(Q, Q^g) = L$. However $O_3(L) = C_S(Y)$ and so $g \in J(R) = J(C_S(Y)) = J$ which is a contradiction and so $J(P) \neq J(R)$. This implies that $C_G(g)$ has two distinct Sylow 3-subgroups with distinct Thompson subgroups. Since $a$ is conjugate to $g$, it follows that $C_G(a) \not\subseteq N_G(J)$.

Lemma 3.16. Let $A \in Syl_3(C_G(Z))$ such that $t \in A$ and suppose that $f \in A$ such that $f^2 = t$. Then $Z \in Syl_3(C_G(f)) \cap Syl_3(C_G(A))$.

Proof. We have that $C_G(A) \leq C_G(f) = Z$ since $f^2 = t$ and $C_G(t) = Z$. In either case for the structure of $C_G(Z)$, we have that every element of order four in $A$ (which is isomorphic to either $SD_{16}$ or $Q(8)$) lies in the subgroup of $A$ isomorphic to $Q(8)$ which in turns lies in $O^2(C_G(Z))$. Thus it follows from the structure of $2^3\cdot Alt(5)$ that no element of order three in $C_G(Z)/Q$ centralizes $Qf$. Therefore, by coprime action, we have that $Z \in Syl_3(C_G(f)) \cap Syl_3(C_G(A))$.

Lemma 3.17. We have that $[N_G(J) : C_{N_G(J)}(a)] = 48$ and $[N_G(J) : C_{N_G(J)}(Z)] = 32$. Furthermore, $|N_G(J)| = 3^62^7$ or $3^62^8$ in Case I and II respectively.

Proof. Since $J/Y$ is a natural $N_G(Y)/C_G(Y)$-module, $J$ contains four $N_G(Y)$-conjugates of $S'$ with pairwise intersection $Y$. By Lemma 3.6, $S' = \langle N_i \cap S' \mid 1 \leq i \leq 4 \rangle$. Since the conjugates of $z$ lie in $N_1 \cup S'$ and $N_2 \cup S'$, $|S' \cap 3B| = 8 + 6 = 14$ and so $|J \cap 3B| = 8 + (4 \times 6) = 32$. Therefore, by Lemma 2.11, $[N_G(J) : C_{N_G(J)}(z)] = 32$. Now by Lemma 3.13, $|C_{N_G(J)}(z)| = 3^62^2$ in Case I and $|C_{N_G(J)}(z)| = 3^62^3$ in Case II. Note that $C_{N_G(J)}(z) \leq C_{N_G(J)}(z) \leq C_G(z) \cap N_G(QJ) = C_{N_G(J)}(z)$.
We have that 

\[ |N_G(J)| = 3^6 2^{27} \text{ or } 3^6 2^{28} \text{ respectively. Since } J^# \subseteq 3^A \cup 3^B, |J \cap 3^A| = 48 \text{ and so } [N_G(J) : C_{N_G(J)}(a)] = 48. \]

\[ \text{Lemma 3.18. If } r \text{ is any involution in } C_G(Z)/Q, \text{ then } C_G(r) \cong [Q, r] \cong 3^{1+2}. \]

Furthermore, we have that \( C_G(J) = J \) and \( J \) normalizes no non-trivial \( 3' \)-subgroup of \( G \).

**Proof.** We may assume that \( \langle t, r \rangle \leq C_G(Z) \) is elementary abelian of order four. By coprime action, \( Q = \langle C_Q(t), C_Q(r), C_Q(tr) \rangle \). It follows from the fact that \( C_Q(t) = Z \) that \( C_Q(r), C_Q(tr) > Z \). By the three subgroup lemma we have that \( [[Q, r], C_Q(r)] = 1 \) and it therefore follows that \( C_Q(r) \cong [Q, r] \cong 3^{1+2} \) as claimed.

So now suppose that an involution in \( C_G(Z) \) centralizes \( J \) then \( Z_2 = C_Q(r) \) is elementary abelian and it follows then that \( C_G(J) = J \). If \( N \) is a \( 3' \)-subgroup of \( G \) normalized by \( J \) then by coprime action, \( N = \langle C_N(y) : y \in Y^# \rangle \). Since \( Y \neq O_3(C_N(y)) \) for each \( y \in Y^# \), we have that \( |C_N(y), Y| \leq C_N(y) \cap O_3(C_N(y)) = 1. \)

Thus \( N \trianglelefteq C_G(Y) \) and in particular, \( N \) normalizes \( J(O_3(C_G(Y))) = J(W) = J. \)

Thus \( [J, N] \leq N \cap J = 1. \) We thus have that \( N = 1. \)

Recall that a group \( H \) is said to be \( 3 \)-soluble of length one if \( H/O_3^*(H) \) has a normal Sylow \( 3 \)-subgroup which is to say that \( H = O_{3,3'}(H) \).

**Lemma 3.19.** We have that \( O_3(N_G(Y)/J) \leq O_3(N_G(J)/J) \cong 2^{1+4} \) and \( N_G(J)/J \) is \( 3 \)-soluble of length one.

**Proof.** Set \( K := N_G(J) \) and \( K = K/J. \) Then \( K \) has order \( 3^2 2^7 \) or \( 3^2 2^8 \) and \( S \in \text{Syl}_3(K). \) Clearly \( O_3(K) \leq S \) so recall Lemma 3.9. If \( O_3(K) \succ J \) then it is clear from the order that \( O_3(K) \neq S \) and so we must have \( O_3(K) = S_i \) for some \( i \in \{1, 2, 3, 4\} \). Therefore \( K \) normalizes \( Z(S_i). \) However, this leads to a contradiction since \( K \) is transitive on \( 3 \) and \( 4 \) and on \( 3 \) and \( 5 \) so we have that \( O_3(K) = J. \)

By Burnside’s \( p^a q^b \)-Theorem [6, 4.3.3, p131], \( F \) is solvable. Let \( N \) be a subgroup of \( K \) such that \( J \subseteq N \) and \( N = O_3(K). \) Then \( N \neq 1 \) since \( K \) is solvable and \( O_3(K) = 1. \) Recall that \( s \) inverts \( J \) and so \( \pi \in Z(F) \), in particular, \( \pi \in N. \) Moreover \( N \) is the Fitting subgroup of \( K \), \( F(K), \) and so by [11, 6.5.8, \( C_F(N) \leq N. \) If any element in \( S \) centralizes \( \Phi(N) \) then by a theorem of Burnside [6, 5.1.4, p174], such an element centralizes \( \Phi(N) \) and is the identity. Therefore \( S \) acts faithfully on \( \Phi(N) \) and so by calculating the order of a Sylow \( 3 \)-subgroup in \( GL_n(2) \) for \( n = 1, 2, 3 \) we see that \( |N/\Phi(N)| \geq 2^4. \) Moreover, since \( \pi \) centralizes \( K \), we have that \( \pi \in \Phi(N) \) and so \( |\Phi(N)| \geq 2^5. \) We use Lemma 3.13 (iii) to find \( e \in N(G(S)) \) such that \( e^2 = 1 \) and \( e \) does not normalize \( Y \). Since \( t \) inverts \( S \), by Lemma 3.13 (i), \( (\pi, t) \in N \). So, when \( |K| = 3^2 2^7 \) we have that \( |N| \leq 2^5 \) and so we have that \( |N| = 2^5 \) and then that \( N = NS(\pi) \) is \( 3 \)-soluble of length one. So suppose instead that \( K = 3^2 2^8 \) and let \( P \in \text{Syl}_3(N_{CG}(Z)(\pi)) \) then \( |P| = 2^3. \) We have seen that a subgroup of order four in \( P \) intersects trivially with \( N \) and so \( |P \cap N| \leq 2. \) Suppose for a contradiction that \( R := P \cap N \) has order two. We have that \( |S/J, R, J/J| \leq S/J \cap N/J = 1 \) and so \( R \) acts trivially on \( S/J \cong Q/(Q \cap J). \) Clearly \( QR/Q \neq Q(\pi)/Q = Z(C_G(Z)/Q) \) and so by Lemma 3.18, \( C_{Q(R)} \cong [Q, R] \cong 3^{1+2}. \) However we have seen that \( R \)
centralizes $Q/Z_2$ and so $Z_2 = [Q, R]$ which is a contradiction. Thus we again have that $|\overline{N}| = 2^5$ and $\overline{K} = \overline{NSP}$ is 3-solvable of length one.

Recall that $L = \langle Q, Q^r \rangle \leq N_G(Y)$ and $W = O_3(L) = C_L(Y)$ and $L/W \cong SL_2(3)$. It follows then that $L/J \cong 3 \times SL_2(3)$ and so there exists a group $J \leq K$ such that $\overline{A} \cong Q(8)$ is normalized by $\overline{S}$ and thus necessarily contained in $\overline{N}$. Recall that we have $e \in N_G(S) \leq K$ such that $e^2 = t$ and $e$ does not normalize $Y$. If $\overline{A} = \overline{A}$ then $\overline{B} = \overline{L}$ and it would follow that $Y^e = Y$. Thus if we set $\overline{B} = \overline{A}$ then we may apply Lemma 2.13 to see that $\overline{N} \cong 2_+^{1+4}$ which completes the proof. \hfill $\square$

**Lemma 3.20.** $C_S(s) = \langle \alpha_1, \alpha_2 \rangle \cong 3 \times 3$ where $\alpha_1, \alpha_2 \in 3A$ and there exist $\langle \alpha_1, \alpha_2 \rangle$-invariant subgroups $Q(8) \cong X_i \leq C_S(s) \cap C_S(\alpha_i)$ for $i \in \{1, 2\}$ such that $s \in X_i$ and $[X_1, X_2] = 1$.

**Proof.** Consider $D := C_{N_G(J)}(s) \cong C_{N_G(J)}(s)/J = C_{N_G(J)/J} = N_G(J)/J$.

This is a group of order $2^3 3^2$ or $2^8 3^2$ in which $O_2(D) \cong 2_+^{1+4}$. Since $s$ centralizes $S/J \cong Q'(Q \cap J)$ (which is elementary abelian), we see that $P := C_S(s)$ is an elementary abelian of order nine. By Lemma 3.14 (ii), $C_{C_S(Y)}(s) \cong 3A$ so let $\langle \alpha_1 \rangle = C_{C_S(Y)}(s) \leq P$ and by Lemma 3.4 (iii), $\langle \alpha_1 \rangle \cong C_S(s) \cong 3SL_2(3)$. This extension is split and thus a direct product by Gaschütz’s Theorem as $P$ is elementary abelian. Thus $\alpha_1$ commutes with a group $X_1 \leq C_S(s)$ isomorphic to $Q(8)$ which is normalized by $P$.

Recall that using Lemma 3.13 (iii) there is an element of order four $e \in C_G(Z)$ which normalizes $S$ (and therefore $J$ and $J(s)$) but not $Y$ and so $C_S(Y) \neq C_S(Y)^e$ and $J \leq C_S(Y^e) \leq S$. Since $s$ centralizes $S/J$, we have $\alpha_i = 2: \alpha_2 \in C_{C_S(Y^e)}(s)$ and $P = \langle \alpha_1, \alpha_2 \rangle$. Moreover, $\langle \alpha_1 \rangle \leq C_S(s) \cong 3 \times SL_2(3)$. Let $X_2 \leq C_S(s)$ isomorphic to $Q(8)$ which is normalized by $P$. Now $D$ is 3-solvable of length one, and since for $i \in \{1, 2\}$, $\langle X_i, P \rangle \cong 3 \times SL_2(3)$, it follows that $X_1 \leq O_2(D) \cong 2_+^{1+4}$. Since $2_+^{1+4}$ contains just two subgroups isomorphic to $Q(8)$, we have that $X_1X_2 \cong 2_+^{1+4}$ and $[X_1, X_2] = 1$. \hfill $\square$

**Lemma 3.21.** (i) In Case I, $C_G(a) \cong 3 \times \text{Alt}(9)$ and $N_G(\langle a \rangle)$ is isomorphic to the diagonal subgroup of index two in $\text{Sym}(3) \times \text{Sym}(9)$.

(ii) In Case II, $C_G(a) \cong 3 \times \text{Sym}(9)$ and $N_G(\langle a \rangle) \cong \text{Sym}(3) \times \text{Sym}(9)$.

**Proof.** We will apply Theorem 2.6 in Case I to $N_G(\langle a \rangle)/\langle a \rangle$ and in Case II to $C_G(\langle a \rangle)/\langle a \rangle$ to see that each is isomorphic to $\text{Sym}(9)$. Let $N_a := N_G(\langle a \rangle)$, $C_a := C_G(\langle a \rangle)$, $S_a := C_S(s) \in Syl_3(N_a)$ and $\overline{N_a} := N_a/\langle a \rangle$.

We first restrict ourselves to Case I. Consider $C_{\overline{N_a}}(Z)$. If $g \in N_a$ and $\overline{Z}^g = \overline{Z}$, then $[Z, g] = 1$. This is clear since $(Z, a) \cap 3B \subset Z$. Recall that $t$ inverts $Q/Z$ and so by swapping $a$ with some appropriate conjugate from $(Z, a)$, we may assume that $t$ inverts $a$. By Lemma 3.12, $|Q \cap 3A| = 120$ and $C_G(Z)$ is transitive on the set. We have therefore that $[C_G(Z) : C_G(Z)(a)] = 120$. Hence $|C_G(Z) \cap N_a| = 3^5 2$. Therefore $C_{\overline{N_a}}(Z)$ has order $3^2 2$ with Sylow 3-subgroup $\overline{S_a}$ and Sylow 2-subgroup $\langle t \rangle$.  


Now \( J < C_{\overline{\text{Sym} (3)}} \) and \( J \) is elementary abelian of order 27. Consider \( Z(\overline{S}_3) \).

Since \( a \in N_3 \cap J \), we may apply Lemma 3.11 to say that \( Z(\overline{S}_3) = Z(\overline{S}_a) = Z \) which has order three. Clearly \( J \) commutes with \( Z \) but not \( \overline{S}_a \). Thus we may apply Lemma 2.7 to see that \( C_{\overline{\text{Sym} (9)}}(\overline{Z}) \cong C_{\text{Sym} (9)}((1, 2, 3)(4, 5, 6)(7, 8, 9)) \).

So it remains to show that \( J \) normalizes no non-trivial 3'-subgroup of \( \overline{N}_a \). However this follows from Lemma 3.18. Hence we may apply Theorem 2.6 to see that either \( N_a \leq N_G(J) \) or \( \overline{N}_a \cong \text{Sym} (9) \). Thus we use Lemma 3.15 to see that \( \overline{N}_a \cong \text{Sym} (9) \). It follows of course that \( C_G(a)/\langle a \rangle \cong \text{Alt}(9) \) and using [1, 33.15, p170], for example, we see that the Schur Multiplier of \( \text{Alt}(9) \) has order two. Therefore \( C_G(a) \) splits over \( \langle a \rangle \) and so \( C_G(a) \cong 3 \times \text{Alt}(9) \). To see the structure of the normalizer we need only observe that an involution \( s \) inverts \( J \) and therefore inverts \( a \) whilst acting non-trivially on \( O^3(C_G(a)) \). Therefore since \( \text{Aut}(\text{Alt}(9)) \cong \text{Sym}(9) \), we see that \( N_G(\langle a \rangle) \) is isomorphic to the diagonal subgroup of index two in \( \text{Sym}(3) \times \text{Sym}(9) \).

Now in Case II, we consider \( C_{\overline{\text{Sym} (9)}}(\overline{Z}) \). Arguing as before, if \( g \in C_a \) and \( \overline{Z} = \overline{Z} \), then \( [Z, g] = 1 \). In this case, we conclude from \( |C_G(Z) : C_{CG}(a)| = 120 \) that \( |C_G(Z) \cap C_a| = 3^2 \) and so \( C_{\overline{\text{Sym} (9)}}(\overline{Z}) \) has order \( 3^2 \) with Sylow 3-subgroup \( \overline{S}_a \) and a Sylow 2-subgroup \( \langle r \rangle \) say, where \( r \) is an involution in \( C_G(Z) \). If \( \overline{[r, S_a]} = 1 \) then \( [r, S_a] = 1 \). By Lemma 3.18, \( C_Q(r) \cong 3^{1+2} \) however \( Q \cap J \leq Q \cap S_a \leq C_Q(r) \) gives us a contradiction. Thus \( \overline{[r, S_a]} \neq 1 \) and we may again apply Lemma 2.7 to see that \( C_{\overline{\text{Sym} (9)}}(\overline{Z}) \cong C_{\text{Sym} (9)}((1, 2, 3)(4, 5, 6)(7, 8, 9)) \). Of course we again have that \( J \) normalizes no non-trivial 3'-subgroup of \( \overline{C_a} \) so we may apply Theorem 2.6 to see that \( \overline{C_a} \cong \text{Sym}(9) \). It follows then that \( C_G(a) \cong 3 \times \text{Sym}(9) \) and \( N_G(\langle a \rangle) \cong \text{Sym}(3) \times \text{Sym}(9) \).

4. The Structure of the Centralizer of \( t \)

We now have sufficient information concerning the 3-local structure of \( G \) to determine the centralizer of \( t \) and to show that in one case \( G \) has a non-trivial 2-quotient. We set \( H := C_G(t) \), \( P := C_J(t) \) and \( H/\langle t \rangle := H/\langle t \rangle \). We will show that \( H \) has shape \( 2^{1+8}.(\text{Alt}(5) \times 2) \) (possibly extended by a cyclic group of order two) and so we must first show that \( H \) has an extraspecial subgroup of order \( 2^9 \). We then show that \( H \) has a subgroup, \( K \), of the required shape and then finally we apply a theorem of Goldschmidt to prove that \( K = H \). Along the way we gather several results which will be useful in Section 5.

**Lemma 4.1.** In Case I, \( C_H(Z) \cong 3 \times 2 \cdot \text{Alt}(5) \) and \( N_H(Z) \cong 3 \cdot 4 \cdot \text{Alt}(5) \) and \( C_H(P) = P \langle t \rangle \). In Case II, \( C_H(Z) \cong 3 \times 2 \cdot \text{Sym}(5) \) and \( N_H(Z) \cong 3 \cdot 4 \cdot \text{Sym}(5) \) and \( C_H(P) \cong 3^2 \times 2^2 \). Furthermore, \( |P \cap 3 \mathcal{A}| = |P \cap 3 \mathcal{B}| = 4 \) and \( P \in \text{Syl}_3(H) \).

**Proof.** By coprime action and an isomorphism theorem, we have that \( C_{C_G(Z)/Q}(t) \cong C_{C_G(Z)}(t)/C_Q(t) \) and \( C_{N_G(Z)/Q}(t) \cong C_{N_G(Z)}(t)/C_Q(t) \). By Lemma 3.13, \( |P| = 9 \) and since \( P \leq J \) is elementary abelian, \( P \) splits over \( Z \). Thus \( C_{C_G(Z)}(t) \) splits over \( Z \) by Gaschütz’s Theorem and so \( C_H(Z) \) and \( N_H(Z) \) are as claimed.
Lemma 3.14 gives us that \(|P \cap 3A| = |P \cap 3B| = 4\) and then it is immediate that \(N_H(P) \leq N_H(Z)\) and so \(P \in \text{Syl}_3(H)\). 

We fix notation such that \(P = \{1, z_1, z_1^2, z_2, z_2^2, a_1, a_1^2, a_2, a_2^2\}\) where \(z_1 \in Z\), \(P \cap 3B = \{z_1, z_1^2, z_2, z_2^2\}\) and \(P \cap 3A = \{a_1, a_1^2, a_2, a_2^2\}\).

**Lemma 4.2.** Let \(\{i, j\} = \{1, 2\}\) then \(P \cap O^3(C_G(a_i)) = \langle a_j \rangle\) and \(N_H(P)/C_H(P) \cong \text{Dih}(8)\) acts transitively on \(3A \cap P\) and \(3B \cap P\).

**Proof.** By Lemma 4.1, \(|P \cap 3A| = |P \cap 3B| = 4\) and so it is clear that \(N_H(P)/C_H(P)\) is isomorphic to a subgroup of \(\text{Dih}(8)\). Observe that every element of order three in \(O^3(C_G(a_i)) \cong \text{Alt}(9)\) or \(\text{Sym}(9)\) is conjugate to its inverse. Therefore an element in \(O^3(C_G(a_i))\) inverts \(P \cap O^3(C_G(a_i))\) and so we must have that element inverting \(a_j\) and permuting \(\langle z_1 \rangle\) and \(\langle z_2 \rangle\). Thus \(P \cap O^3(C_G(a_i)) = \langle a_j \rangle\). Furthermore an element of order four in \(N_H(Z)\) inverts \(Z\) whilst centralizing \(P/Z\). Hence an element in \(N_H(Z)\) permutes \(\langle a_1 \rangle\) and \(\langle a_2 \rangle\). We have that \(s\) inverts \(P\) and so we have that \(N_H(P)\) is transitive on \(3A \cap P\) and \(3B \cap P\). 

**Lemma 4.3.** Let \(\{i, j\} = \{1, 2\}\).

(i) The image in \(O^3(C_G(a_i))\) of elements of cycle type 3 and 32 are in \(3A\) and those of cycle type 33 are in \(3B\). In particular, \(a_1 \in P \cap O^3(C_G(a_i))\) has cycle type 32.

(ii) \(t\) has cycle type 21 and is not \(G\)-conjugate to involutions of cycle type 22 in \(O^3(C_G(a_i))\).

(iii) In Case II when \(O^3(C_G(a_i)) \cong \text{Sym}(9)\), even involutions are not \(G\)-conjugate to odd involutions.

**Proof.** We have that \(C_G(a_i) \cong 3 \times \text{Alt}(9)\) or \(3 \times \text{Sym}(9)\) and so \(|P \cap O^3(C_G(a_i))| = 3\). Consider representatives for the three conjugacy classes of elements of order three in \(\text{Alt}(9)\). An element of cycle type 3 must clearly be in \(3A\) and an element of cycle type 33 is the cube of an element of order nine and so by Lemma 3.10 must be in \(3B\). Consider now the image of \(P \cap O^3(C_G(a_i))\) (which we have seen is equal to \(\langle a_j \rangle\)) in \(\text{Alt}(9)\). If it is conjugate to \(\langle 1, 2, 3 \rangle\) then \(P\) commutes with a subgroup isomorphic to \(3 \times 3 \times \text{Alt}(6)\). However \(z \in P\) and \(C_G(z)\) has no such subgroup. So, since elements of cycle type 33 are in \(3B\), we must have that the image in \(\text{Alt}(9)\) of \(P \cap O^3(C_G(a_i))\) is conjugate to \(\langle 1, 2, 3 \rangle(4, 5, 6)\).

We see easily that the image of \(t\) is an even permutation since when \(O^3(C_G(a_i)) \cong \text{Sym}(9)\), the odd involutions commute with a group of order nine and so in \(C_G(a_i)\) they centralize a group of order 27. Let \(r\) be an involution of cycle type 22. Then \(r\) commutes with a single three cycle \(x\) say in \(O^3(C_G(a_i))\). Thus \(r \in C_G(\langle a_i, x \rangle) \cong 3^2 \times \text{Alt}(6)\) or \(3^2 \times \text{Sym}(6)\). Clearly then \(\langle a_i, x \rangle^\# \subseteq 3A\) and so \(r\) commutes with no conjugate of \(Z\). Thus \(r\) is not \(G\)-conjugate to \(t\).

Now suppose we are in Case II and so \(O^3(C_G(a_i)) \cong \text{Sym}(9)\). Continue to let \(r \in C_G(\langle a_i, x \rangle) \cong 3^2 \times \text{Sym}(6)\) have cycle type 22. There is an element of order four \(f\) in \(C_G(\langle a_i, x \rangle)\) which squares to \(r\). Hence for every \(y \in \langle a_i, x \rangle, f \in C_G(y)\) and so the image of \(r\) in \(O^3(C_G(y))\) is an even involution and thus of type 22. Thus \(C_G(r) \cap C_G(y)\) has Sylow 3-subgroups of order nine. It follows that \(C_G(r)\) has
Lemma 4.6. Sylow 3-subgroups of order nine. Now any odd involution in $O^3(C_G(a_i))$ commutes with a group of order 27 and so $r$ is not $G$-conjugate to any odd involution in $O^3(C_G(a_i))$. □

Set 2A to be all $G$-conjugates of an involution whose image in $O^3(C_G(a_i))$ has cycle type $2^2$ and $2B = \{ t^g | g \in G \}$. We now introduce some further notation by first fixing an injective homomorphism from $N_G(a_i)$ into $\text{Sym}(12)$ such that $O^3(C_G(a_i))$ maps into $\text{Sym}(\{1, \ldots, 9\})$ and $a_i$ maps to $(10, 11, 12)$ ($\{i, j\} = \{1, 2\}$).

We define subgroups and elements of $G$ by their image.

**Notation 4.4.** Let $\{i, j\} = \{1, 2\}$.

- $a_i \mapsto (10, 11, 12)$.
- $a_j \mapsto (1, 3, 5)(2, 4, 6)$.
- $t \mapsto (1, 2)(3, 4)(5, 6)(7, 8)$.
- $Q_i \mapsto \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 3)(2, 4)(5, 8)(6, 7), (1, 5)(3, 8)(2, 6)(7, 4), (1, 2)(3, 4), (3, 4)(5, 6) \rangle$.
- $r_i \mapsto (1, 3)(2, 4)$.
- When $i = 1$, $Q_1 > E \mapsto \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 3)(2, 4)(5, 8)(6, 7), (1, 5)(3, 8)(2, 6)(7, 4) \rangle$.
- When $i = 2$, $Q_2 \ni u \mapsto (1, 2)(3, 4)$ and $Q_2 > F \mapsto \langle (1, 2)(3, 4), (3, 4)(5, 6) \rangle$.

We observe the following by calculating directly in the image of $N_G(\langle a_i \rangle)$ in $\text{Sym}(12)$.

**Lemma 4.5.** (i) In Case I, $C_H(a_i) \simeq 3 \times (2^{1+4}_+ : \text{Sym}(3))$. In Case II, $C_H(a_i) \simeq 3 \times (2^{1+4}_+ : (2 \times \text{Sym}(3)))$. In either case, $Q_i \cong 2^{1+4}_+$ is normal in $C_H(a_i)$ with $r_i \in C_H(a_i) \setminus Q_i$.

(ii) $2 \times 2 \times 2 \cong E < H \cap O^2(C_G(a_i))$ and there exists $\text{GL}_3(2) \cong C \subseteq C_G(a_i)$ such that $a_2 \in C$ and $C$ is a complement to $C_{C_G(a_i)}(E)$ in $N_{C_G(a_i)}(E)$. Furthermore, $N_G(E) \cap C_G(P) = \langle t, P \rangle$.

(iii) If $\langle t \rangle < V < Q_i$, such that $V < C_H(a_i)$ then $V$ is elementary abelian.

(iv) $C_{C_G(a_i)}(Q_i) = \langle t, a_i \rangle$.

(v) $C_{C_G(a_i)}(E) = \langle E, a_i \rangle$.

(vi) $s, t, st \in 2B$.

**Proof.** These can mostly be checked by direct calculation in the permutation group.

To prove (vi) we observe that having fixed the image of $a_j$ we see that the image of $J \cap O^3(C_G(a_i))$ is $\langle (1, 3, 5), (2, 4, 6), (7, 8, 9) \rangle$ and since $s$ inverts $J$ we may assume the image of $s$ is $C_G(a_i)$-conjugate to $(1, 3)(2, 4)(7, 8)(10, 11)$. Thus it becomes clear that $s$ is conjugate to $st$. Now by Lemma 3.4 (vi), $t$ is conjugate to $st$ and therefore to $s$ also. □

For the sake of simplifying language in the following lemma, we define the following for $g \in G$, $\mathcal{V}_P(g) := \{ M \leq C_H(g) \ | \ (|M|, 3) = 1, \ [M, P] \leq M, \ C_M(P) \leq \langle t \rangle \}$.

**Lemma 4.6.** Let $i \in \{1, 2\}$, $\mathcal{V}_P(z_i) = \{ 1, \langle t \rangle, A_i, B_i \}$ where $A_i \cong B_i \cong \text{mathrm}Q(8)$ are distinct 2-subgroups of $C_G(z_i)$ with $z_j \in \langle A_i, B_i \rangle = C_H(z_i)' \cong 2 \times \text{Alt}(5)$. Meanwhile, $M \in \mathcal{V}_P(a_i)$ only if $M \leq Q_i$. 
Proof. We have that $C_H(a_i) \sim 3 \times 2^{1+4} : \text{Sym}(3)$ or $3 \times 2^{1+4} : (2 \times \text{Sym}(3))$ which are subgroups of $3 \times \text{Alt}(9)$ and $3 \times \text{Sym}(9)$ respectively. It is clear in the first case that if $M$ is any normal $3'$-subgroup of $C_H(a_i)$ then $M \leq Q_i$. In the second case we must check within our permutation group that any such $M$ with $C_M(P) \leq \langle t \rangle$ must satisfy $M \leq Q_i$.

We have that $C_H(z_i) \cong 3 \times 2 \cdot \text{Alt}(5)$ or $3 \times 2 \cdot \text{Sym}(5)$. Let $M$ be a $3'$-subgroup of $C_H(z_i)$ that is normalized by $P$. It is clear that $5 \nmid |M|$ so $M$ must be a 2-group. Now $C_H(z_i)$ has Sylow 2-subgroups isomorphic to $Q(8)$ or $\text{SDih}(16)$. Since $C_M(P) \leq \langle t \rangle$, we must have that $M \leq \langle t \rangle$ or $M \cong Q(8)$ and $MP \cong 3 \times \text{SL}_2(3)$. Notice that $P$ is involved in precisely two subgroups of $C_H(z_i)$ isomorphic to $MP \cong 3 \times \text{SL}_2(3)$. We define $A_i$ and $B_i$ to be the two distinct 2-radical subgroups. It then follows that $\langle A_i, B_i \rangle = C_H(z_i) \cong 2 \cdot \text{Alt}(5)$ and since an element of order four centralizes $z_i$ and inverts $P \cap \langle A_i, B_i \rangle$, we must have that $z_i \in P \cap \langle A_i, B_i \rangle$. □

We continue the notation for the $P$-invariant subgroups from the previous lemma. The subgroups $\{A_i, B_i\}$ and $Q_j$ for $i, j \in \{1, 2\}$ play key roles in this section as our building blocks for $H$.

Lemma 4.7. Let $\{i, j\} = \{1, 2\}$. The following hold.

(i) $N_H(P) \cap C_H(a_i)$ acts transitively on the set $\{\langle z_1 \rangle, \langle z_2 \rangle\}$.

(ii) $N_H(P) \cap C_H(z_i)$ acts transitively on the set $\{\langle a_1 \rangle, \langle a_2 \rangle\}$.

(iii) $N_H(P)$ acts as $\text{Dih}(8)$ on $\{A_1, B_1, A_2, B_2\}$ with blocks of imprimitivity $\{A_i, B_i\}$.

In particular, $N_H(P) \cap N_H(\langle a_i \rangle)$ acts transitively.

Proof. By Lemma 4.2, $N_H(P)/C_H(P) \cong \text{Dih}(8)$ and $N_H(P)$ is transitive on $P \cap 3A$ and $P \cap 3B$ which both have order four and so (i) and (ii) are clear.

Now by Lemma 4.6, $N_H(P)$ acts imprimitively on the set $\{A_1, B_1, A_2, B_2\}$ swapping $\{A_1, B_1\}$ with $\{A_2, B_2\}$. Recall that $N_H(\langle z_2 \rangle) \sim 3 : 4 \cdot \text{Alt}(5)$ or $3 : 4 \cdot \text{Sym}(5)$. In either case an element of order four, $g$, say, inverts $z_2$ whilst centralizing $\langle A_2, B_2 \rangle$. Therefore $g \in C_H(z_1)$. This element necessarily permutes $A_1$ and $B_1$ since Sylow 2-subgroups of $C_H(Z)$ are either quaternion of order eight of semi-dihedral of order 16. Thus $N_H(P)$ acts transitively and imprimitively on $\{A_1, B_1, A_2, B_2\}$ and contains a transposition and so (iv) follows. If the subgroup $N_H(P) \cap N_H(\langle a_i \rangle)$ acts as a non-transitive subgroup of $\text{Dih}(8)$ then it preserves each $\{A_i, B_i\}$ but therefore normalizes $\langle z_1 \rangle$ and $\langle z_2 \rangle$ which we have seen is not the case and so we have (iii). □

The following lemma is a key step in determining the structure of $H$ since it proves that $H$ contains a subgroup which is extraspecial of order $2^9$.

Lemma 4.8. Let $\{i, j\} = \{1, 2\}$ then $Q_i \cap Q_j = \langle t \rangle$ and $O_H(C_G(Q_i)) = Q_j$. In particular $\langle t \rangle$ is the centre of a Sylow 2-subgroup of $G$ and $Q_1Q_2 \cong 2^{1+8}$ with $C_G(Q_1Q_2) = \langle t \rangle$ and $C_{Q_1Q_2}(z_i) = \langle t \rangle$.

Proof. Let $\{i, j\} = \{1, 2\}$. Recall Lemma 4.5 (iv) which says that $C_{G \langle a_i \rangle}(Q_i) = \langle t, a_i \rangle$. Therefore $C_{\langle t \rangle}(Q_i)$ has a self-centralizing element of order three and so we may use Theorem 2.5.
Notice also that \( \langle a_i \rangle \in \text{Syl}_2(C_G(Q_i)) \) and \( N_H(P) \cap N_H(\langle a_i \rangle) \) normalizes \( C_G(Q_i) \).
By Lemma 4.5 (iii), \( N_H(P) \cap N_H(\langle a_i \rangle) \) is transitive on \( \{ A_1, A_2, B_1, B_2 \} \). Thus no group from this set commutes with \( Q_i \) else we would have \( z_1 \in \langle A_2, B_2 \rangle \leq C_G(Q_i) \), which is a contradiction.

Set \( N = O_3(C_G(Q_i)) \). Then \( N \) is normalized by \( P \) and by the comment above we see that \( C_N(z_1) = C_N(z_2) = \langle t \rangle \). Moreover, \( C_N(a_i) = \langle t \rangle \) and so by coprime action we have,
\[
N = \langle C_N(z_1), C_N(z_2), C_N(a_1), C_N(a_2) \rangle = C_N(a_j).
\]
By Lemma 4.6, since \( C_N(P) \leq \langle t \rangle \), we have that \( N \leq Q_j \). Suppose that \( \langle t \rangle < N < Q_j \). Since \( a_i \) acts fixed-point-freely on \( N/\langle t \rangle \), \( |N| = 2^4 \). By Lemma 4.5 (iii), \( N \) is elementary abelian. Now by Lemma 4.5 (vi), \( s \) is conjugate to \( t \) in \( G \). Recall Lemma 3.20. This, together with the fact that \( P = \langle a_1, a_2 \rangle \in \text{Syl}_3(H) \), implies that for \( k \in \{ 1, 2 \} \) there exists a \( P \)-invariant subgroup \( Q(8) \cong X_k \leq C_H(a_k) \) with \( |X_1, X_2| = 1 \). Now by Lemma 4.6, \( X_i \leq Q_i \) and \( X_j \leq Q_j \). We have that \( X_j \) and \( N \) are both \( P \)-invariant and furthermore we have that \( X_j \cong Q(8) \) where as \( N \) is elementary abelian. Therefore \( |X_j \cap N| = 2 \) and so \( Q_j = N X_j \). Similarly, \( Q_i = O_2(C_G(Q_j)) X_i \). Therefore \( X_j \) commutes with \( Q_i \), which is a contradiction.

So we have that either \( N = Q_i \) or \( N = \langle t \rangle \). Assume the latter for a contradiction.

In Case I, we have that \( N_G(\langle a_i \rangle) \) is the diagonal subgroup of index two in \( \text{Sym}(3) \times \text{Sym}(9) \). It follows that \( C_G(Q_i) \) has a self-normalizing Sylow 3-subgroup and so a normal 3-complement. Therefore, \( C_G(Q_i) = N(a_i) = \langle t \rangle \times \langle a_i \rangle \). Hence \( N_G(Q_i) \leq N_G(\langle a_i \rangle) \) has Sylow 2-subgroups of order \( 2^7 \) isomorphic to a Sylow 2-subgroup of \( \text{Sym}(9) \). We check that \( Q_i \) is characteristic in such a 2-group to see that \( N_G(\langle a_i \rangle) \) contains a Sylow 2-subgroup of \( G \). Therefore there exists \( g \in G \) such that \( A_1^g \in N_G(\langle a_i \rangle) \). However, using Lemma 3.16 we now get a contradiction since no element of order four in \( A_1 \) commutes with an element of \( 3A \). Thus in Case I we have that \( O_3(C_G(Q_i)) = Q_j \).

In Case II we instead have that \( N_G(\langle a_i \rangle) \cong \text{Sym}(3) \times \text{Sym}(9) \) and using Theorem 2.5 we see that \( C_G(Q_i)/\langle t \rangle \cong \text{Alt}(5), \text{PSL}_2(7) \) or \( \text{Sym}(3) \). Notice that an element of order three in \( P \) must therefore commute with \( C_G(Q_i) \). That element must be in \( \langle a_j \rangle \). However, \( t \in C_G(a_j) \) and it follows from Lemma 4.3 that \( t \) is 2-central in \( C_G(a_j) \cong 3 \times \text{Sym}(9) \). A 2-central involution in \( \text{Sym}(9) \) does not commute with subgroups isomorphic to \( \text{Alt}(5), \text{PSL}_2(7) \) or their double covers. It follows that we must have \( C_G(Q_j)/\langle t \rangle \cong \text{Sym}(3) \). We now again have that \( N_G(Q_i) \) is contained in \( N_G(\langle a_i \rangle) \) and we get a contradiction as before.

Hence we have in both cases that \( N = Q_j \) and also that \( C_G(Q_i)/N \) acts faithfully on \( N/\langle t \rangle \). So \( |Q_1, Q_2| = 1 \) and furthermore \( Q_1 \cap Q_2 \leq C_{Q_1}(Q_i) = \langle t \rangle \) and so we get that \( Q_1 Q_2 \cong 2^{1+8} + \) and clearly \( C_{Q_1 Q_2}(z_1) = \langle t \rangle \). Now let \( Q_1 Q_2 \leq T \in \text{Syl}_2(G) \) then \( T(G) \leq C_T(Q_1) \leq C_T(Q_1) \cap C_T(Q_2) \). Since \( C_T(Q_1) \cap C_T(Q_2) \) acts faithfully on \( N/\langle t \rangle \), it is clear that \( C_T(Q_1) \) and \( C_T(Q_2) \) act faithfully on \( N/\langle t \rangle \). Hence \( T(G) = C_G(Q_1) \cap C_G(Q_2) = \langle t \rangle \). □

Set \( Q_{12} := \langle Q_1 Q_2 \rangle \equiv 2^{1+8} + \) and recall that in Notation 4.4 we defined \( E \leq C_G(a_1) \) such that \( t \in E \leq C_H(a_1) \) is elementary abelian of order eight. We now consider \( C_G(E) \) and \( N_G(E) \).
Lemma 4.9. (i) We have that \(C_G(E)/O_2(C_G(E)) \cong C_3\) in Case I and \(C_G(E)/O_2(C_G(E)) \cong \text{Sym}(3)\) in Case II.

(ii) Without loss of generality (on choices of \(A_i\)) we may assume that \(O_2(C_G(E)) = \langle E, Q_2, A_1, A_2 \rangle\) which is normalized by \(P\).

(iii) \(N_G(E)/C_G(E) \cong \text{GL}_3(2)\) where the extension is split and there is a complement to \(C_G(E)\) in \(C_H(a_1)\) containing \(a_2\).

(iv) \(\langle Q_{12}, A_1, A_2 \rangle\) is a 2-group which is normalized by \(P\).

Proof. By Lemma 4.5 (v), \(C_{C_H(a_1)}(E) = \langle E, a_1 \rangle\) and so \(C_G(E)/E\) satisfies Theorem 2.5. Set \(N = O_V(C_G(E))\) then \(C_N(a_1) = E\) and \(a_1\) acts fixed-point-freely on \(N/E\).

A theorem of Thompson says that \(N/E\) is nilpotent and therefore \(N\) is nilpotent. By Theorem 2.5, \(C_G(E)/N \cong C_3\), \(\text{Sym}(5)\), \(\text{Alt}(5)\) or \(\text{PSL}_2(7)\) (in which case \(N = E\)). Also by Lemma 4.5 (ii), there exists a complement to \(C_G(E)\) in \(N_G(E)\) which in particular is a subgroup of \(C_H(a_1)\) containing \(a_2\).

Since \(P\) normalizes \(N\), we may apply coprime action to see that

\[N = \langle C_N(z_1), C_N(z_2), C_N(a_1), C_N(a_2) \rangle.\]

Since \(C_N(P) \leq C_E(a_2) = \langle t \rangle\), we use Lemma 4.6 to see that \(N\) is generated by 2-groups and as \(N\) is nilpotent, \(N\) is a 2-group.

We have that \(E \leq Q_1\) and by Lemma 4.8, \([Q_1, Q_2] = 1\) and \(Q_1 \cap Q_2 = \langle t \rangle\) so \(Q_2 \cap E = \langle t \rangle\) and \(Q_2 \cap N > \langle t \rangle\) and so has order \(2^3\) or \(2^5\). Notice also that this implies that \(N\) does not split over \(E\). Let \(g \in N_G(E) \cap C_G(a_1)\) be an element of order seven. Then \(g\) acts fixed-point-freely on \(E\). If \([N/E, g] = 1\) then \(N = C_N(g) \times E\) which is a contradiction. Thus \([N/E, g] \neq 1\) and so \(|N/E| \geq 2^3\). Since \(a_1\) acts fixed-point-freely on \(N/E\) and preserves \(|N/E, g|\), we have \(|[N/E, g]| \geq 2^6\).

If \(z_1\) and \(z_2\) act fixed-point-freely on \(N/E\) then \(N = Q_2E\) and so \(|N/E| = 2^2\) or \(2^4\) which we have seen is not the case. Therefore at least one of \(C_{N/E}(z_1)\) and \(C_{N/E}(z_2)\) is non-trivial. Since \(E \leq C_H(a_1)\) we may apply Lemma 4.7 (i) which says that \(N_{H}(P) \cap C_H(a_1)\) acts transitively on the set \(\{z_1, z_2\}\). Therefore \(C_{N/E}(z_1)\) and \(C_{N/E}(z_2)\) are both non-trivial. So we may assume, without loss of generality, that \(A_1 \leq N\) and \(A_2 \leq N\) and so \(N = \langle E, Q_2 \cap N, A_1, A_2 \rangle\).

Now suppose that \(C_G(E)/N \cong \text{Alt}(5)\). Since \(N_G(E)/C_G(E) \cong \text{GL}_3(2)\), we have that \(N_G(E) \cong \text{Alt}(5) \times \text{GL}_3(2)\). We have that \(Na_2\) is an element of order three in \(N_G(E)/N\) and \(C_{N_G(E)/N}(Na_2)\) contains a subgroup isomorphic to \(\text{Alt}(5)\). By coprime action,

\[C_{N_G(E)/N}(a_2) = C_{N_G(E)}(a_2)N/N \cong C_{N_G(E)}(a_2)/C_N(a_2)\]

which is a 2-group of order 8 or 32 extended by \(\text{Alt}(5)\) which does not exist in \(\text{Sym}(9)\). Hence \(C_G(E)/N \cong \text{Sym}(3)\) or \(C_3\) and it follows that \(N = \langle E, A_1, A_2 \rangle\). Additionally since \(Q_1\) normalizes \(E\) and so normalizes \(N\), we see that \(\langle Q_{12}, A_1, A_2 \rangle\) is a 2-group which is clearly normalized by \(P\).

We continue the notation from this lemma for the rest of this section such that \(A_1\) and \(A_2\) commute with \(E\). Set \(K := N_G(Q_{12}) \leq H\). We show in the rest of this section that \(K = H\).
Lemma 4.10.  
(i) \(N_H(P) \leq K\) and \(|N_H(P)Q_{12}/Q_{12}| = 3^22^3\) in Case I or \(3^22^4\) in Case II.
(ii) Suppose that \(v \in K\) such that \(Q_{12}v\) is an involution which inverts \(Q_{12}z_i\) for some \(i \in \{1, 2\}\). Then \(C_{Q_{12}}(v) = \langle Q_{12}, v \rangle\) has order 24.
(iii) \(C_{Q_{12}}(A_1) \neq C_{Q_{12}}(A_2)\).
(iv) For \(i \in \{1, 2\}\), \(C_H(a_i) \leq K\).
(v) For \(i \in \{1, 2\}\), \(C_H(z_i) \leq K\).
(vi) \(\langle A_1, B_1 \rangle\) commutes with \(\langle A_2, B_2 \rangle\) modulo \(Q_{12}\) and in particular, \(\langle A_1, A_2 \rangle Q_{12}/Q_{12} \cong 2^4\).

Proof.  
(i) First observe that \(N_H(P)\) acts on the set \(\{a_1, a_2, a_1^2, a_2^2\} = P \cap 3A\) and therefore it preserves \(Q_{12} = Q_{12}\) so \(N_H(P) \leq K\). Clearly \(N_{Q_{12}}(P)\) commutes with \(P\) and \(\langle t \rangle \leq C_{Q_{12}}(P) \leq Q_{1} \cap Q_{2} = \langle t \rangle\) so the order of the quotient is clear given Lemmas 4.1 and 4.2.

(ii) Observe that \(\overline{Q_{12}}\) is elementary abelian and \(\overline{Q_{12}v}\) has order two and inverts \(\overline{Q_{12}z_i}\) which has order three. Therefore we may use Lemma 2.15 and since \(|C_{\overline{Q_{12}}}(z_i)| = 1\), we have that \(|C_{\overline{Q_{12}}}(v)| \leq 2^4\). Of course we always have that \(|C_{\overline{Q_{12}}}(v)| \geq 2^4\) and so we get equality.

(iii) Suppose that \(C_{Q_{12}}(A_1) = C_{Q_{12}}(A_2)\). Recall that \(N_H(P) \sim 3 : 4.Alt(5)\) acts as Dih(8) on \(\{A_1, B_1, A_2, B_2\}\). Therefore there exists \(g \in N_H(P)\) permuting \(A_1\) and \(B_1\) and fixing \(A_2\) and \(B_2\). Hence

\[
C_{Q_{12}}(A_1) = C_{Q_{12}}(A_2) = C_{Q_{12}}(A_2)^g = C_{Q_{12}}(A_1)^g = C_{Q_{12}}(B_1).
\]

Therefore \(E \leq C_{Q_{12}}(A_1) = C_{Q_{12}}(\langle A_1, B_1 \rangle) \leq C_{Q_{12}}(z_2)\). This is a contradiction.

(iv) This is clear since \(C_H(a_i) = Q_i N_{C_H(a_i)}(P)\) and so \(C_H(a_i) \leq K\).

(v) By Lemma 4.9, \(T := \langle Q_{12}, A_1, A_2 \rangle\) is a 2-group which is normalized by \(P\). We consider \(N_T(Q_{12}) \leq K\). Since \(T\) is normalized by \(P\), we apply coprime action to see that

\[
N_T(Q_{12}) = \langle C_{N_T(Q_{12})}(r) \mid r \in P^# \rangle.
\]

We have that \(T \leq N_G(E)\) and by Lemma 4.5 (ii), \(C_T(P) = \langle t \rangle\) and so we may use Lemma 4.6. Since \(Q_{12}\) is normalized by \(N_H(P)\) which is transitive on \(\{A_1, B_1, A_2, B_2\}\) (by Lemma 4.7 (iii)), it is clear that \(A_i \not\leq Q_{12}\). Thus \(N_T(Q_{12}) > Q_{12}\). Now we use Lemma 4.6 to see that for \(j \in \{1, 2\}\) \(C_{N_T(Q_{12})}(a_j) = Q_j\) and to see that for some \(i \in \{1, 2\}\), \(C_{N_T(Q_{12})}(z_i) \in \{A_i, B_i\}\). However we again apply Lemma 4.7 (iii) to see that since one of \(A_i\) or \(B_i\) is in \(K\) and \(N_H(P) \leq K\) is transitive on \(\{A_1, B_1, A_2, B_2\}\), \(\langle A_1, A_2, B_2 \rangle \leq K\). Moreover, \(C_H(P) \leq K\) and so for \(i \in \{1, 2\}\), \(\langle A_i, B_i, C_H(P) \rangle = C_H(z_i) \leq K\).

(vi) Let \(\widehat{K} := K/Q_{12}\) Since \(\widehat{T} = \langle \widehat{A_1}, \widehat{A_2} \rangle\) is a 2-subgroup of \(\widehat{K}\). We apply the same coprime action arguments again to \(\widehat{T}\) and \(Z(\widehat{T})\) to see that \(\widehat{T}\) is elementary abelian of order 16. Thus \([\widehat{A_1}, \widehat{A_2}] = 1\). However we have seen that we can choose an element of order four \(g \in N_G(Z)\) that fixes \(A_2\) whilst permuting \(\{A_1, B_1\}\). Thus \(1^g = [\widehat{A_1}, \widehat{A_2}]^g = [\widehat{B_1}, \widehat{A_2}]\). So \([\widehat{A_1}, \widehat{B_1}, \widehat{A_2}]\). Repeating these arguments we get to \([\widehat{A_1}, \widehat{B_1}]\), \(\langle \widehat{A_2}, \widehat{B_2} \rangle = 1\). \(\square\)

Lemma 4.11. Set \(D = \langle N_H(P), C_H(p) \mid p \in P^# \rangle\).
(i) In Case I, \( D/Q_{12} \) is isomorphic to \( \text{Alt}(5) \circ 2 \). In Case II, \( D/Q_{12} \) is isomorphic to a subgroup of \( \text{Sym}(5) \circ 2 \) of shape \( \text{Alt}(5) \circ 2 \).

(ii) A Sylow 2-subgroup of \( D/Q_{12} \) has a unique elementary abelian subgroup of order 16.

(iii) If \( Q_{12}v \) is an involution in \( (DQ_{12})/(Q_{12} \cong \text{Alt}(5) \times \text{Alt}(5)) \), then either \( v \) has order four and squares to \( t \) with \( C_H(v) \) containing a conjugate of \( Z \) or \( Q_{12}v \) is diagonal in which case \( v \in 2B \) and \( [Q_{12}, v] = C_{Q_{12}}(v) \) has order 24.

Proof. Set \( \hat{K} = K/Q_{12} \) then \( \hat{D} = \langle \hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, N_H(P) \rangle \). We have seen that \( \langle \hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2 \rangle \) is isomorphic to \( \text{Alt}(5) \times \text{Alt}(5) \). It is normalized by \( N_H(P) \) which has order 32^2 or 32^4 by Lemma 4.10 (i). Thus we have that either \( \hat{D} \sim \text{Alt}(5) \times \text{Alt}(5) \) which in fact is isomorphic to \( \text{Alt}(5) \circ 2 \) since an element in \( N_H(P) \) swaps \( \langle \hat{A}_1, \hat{B}_1 \rangle \) and \( \langle \hat{A}_2, \hat{B}_2 \rangle \). Otherwise we have that \( \hat{D} \sim \text{Alt}(5) \times \text{Alt}(5) : (2 \times 2) \). Note that \( \hat{D} \) is a subgroup of the automorphism of \( \text{Alt}(5) \times \text{Alt}(5) \) otherwise \( C_{\hat{D}}(\hat{A}_1) \) contains an abelian subgroup of \( C_{\hat{D}}(\hat{A}_1) \) of order 2^4 which is not possible. So \( \hat{D} \) is an index two subgroup of \( \text{Sym}(5) \o 2 \). It is not isomorphic to \( \text{Sym}(5) \times \text{Sym}(5) \) as an element in \( N_H(P) \) swaps \( \langle \hat{A}_1, \hat{B}_1 \rangle \) and \( \langle \hat{A}_2, \hat{B}_2 \rangle \). It remains to calculate in the remaining two groups that a Sylow 2-subgroup has a unique elementary abelian subgroup of order 16. This proves (i) and (ii).

Now \( \hat{D}' \) has two conjugacy classes of involutions: diagonal and non-diagonal. If \( \hat{v} \) is an involution in \( \hat{D}' \) then it inverts a conjugate of \( \hat{v} \) and therefore by Lemma 4.10 (ii), \( C_{Q_{12}}(v) = [Q_{12}, v] \) has order 24. Moreover \( [Q_{12}v] \) is an involution and by Lemma 2.16, every involution in \( Q_{12}v \) is conjugate to \( v \). Thus if \( \hat{w} = \hat{v} \) is conjugate to \( \hat{w} \) or to \( \hat{w} \hat{v} \). We may choose an element of order four, \( f_i \in \langle \hat{A}_i, \hat{B}_i \rangle \cap N_H(P) \) with \( f_i^2 = t \). Then \( \hat{f}_i \) represents every non-diagonal involution in \( \hat{D}' \). Now \( \hat{f}_1 \hat{f}_2 \) is a diagonal involution and \( f_1f_2 \in N_H(P) \). In fact \( f_1f_2 \in N_H(Z) \) and inverts \( P \). We can see in \( N_H(\langle a_1 \rangle) \), for example, that no element of order four inverts \( P \) and so \( f_1f_2 \) is an involution. Moreover, \( f_1f_2, A_1, B_1 = \langle f_2, A_1, B_1 \rangle \cong 2 \times \text{Alt}(5) \) so \( f_1f_2, A_1, B_1 \cong 4 \text{ Alt}(5) \). It therefore follows from Lemma 3.12 that \( f_1f_2 \) is conjugate to \( s \). Thus if \( \hat{v} \) is an involution in \( \hat{D}' \) then \( \hat{v} \) is either an element of order four conjugate to \( f_i \) or \( f_i^3 \) or is conjugate to \( s \) or \( st \) and so in \( 2B \) which proves (iii).

\( \square \)

Lemma 4.12. Assume that we are in Case I. For \( V < E \) such that \( t \in V \cong 2 \times 2 \), \( C_G(V) \leq \langle C_H(p) \mid p \in P^\# \rangle \).

Proof. Notice that \( E = \langle t \rangle \times [E, P] \) and that the image of \( [E, P] \) in \( O^2(C_G(a_1)) \) is \( \langle 1, 5 \rangle(2, 6)(3, 8)(4, 7), (1, 8)(2, 7)(3, 5)(4, 6) \). Since we are in Case I, we have that \( N_{G}(\langle a_1 \rangle) \) is the diagonal subgroup of index two in \( \text{Sym}(3) \times \text{Sym}(9) \). We calculate the centralizer in \( \text{Sym}(9) \) and \( \text{Alt}(9) \) of such a fours group to see it has order 32. Thus \( C_G([E, P]) \) has a Sylow 3-subgroup \( \langle a_1 \rangle \) with centralizer \( \langle E, a_1 \rangle \) equal to its normalizer. Hence \( C_G([E, P]) \) has a normal 3-complement, \( N \) say. It is clear that \( N \) is normalized by \( P \) with \( C_N(P) = \langle t \rangle \) and contains \( C_{Q_1}(\langle E, P \rangle)Q_2, A_1, A_2 \) which has order \( 2^{12} \). Now by coprime action, \( N = \langle C_N(a_1), C_N(a_2), C_N(z_1), C_N(z_2) \rangle \). It is clear that for \( i \in \{1, 2\} \), \( A_i \) is a maximal \( 3' \)-subgroup of \( C_G(z_i) \), so \( C_N(z_i) = A_i \). Also \( Q_2 \) is a maximal \( 3' \)-subgroup of \( C_G(a_2) \) normalized by \( P \) and we have calculated
Lemma 4.13.  
(i) $K$ has a subgroup $K_0$ such that $K_0/Q_{12} \cong \text{Alt}(5) \wr 2$ acts faithfully on $Q_{12}$.
(ii) Every involution in $Q_{12}$ is in $2A \cup 2B$ and $K_0$ is transitive on $Q_{12} \cap 2A$ and on $(Q_{12} \setminus \{t\}) \cap 2B$. The orbit lengths are 120 and 150 respectively.
(iii) Diagonal subgroup of $K_0$ isomorphic to Alt(5) either centralizes a subgroup of $Q_{12}$ isomorphic to $C_4 \times C_2$ containing an involution in $2A$ or contain an element of order five acting fixed-point-free on $Q_{12}$.

In particular, in Case I, $K_0 = K$.

Proof. Again we set $K = K/Q_{12}$. We have seen in Lemma 4.3 and Notation 4.4, that $Q_1$ contains non-conjugate involutions from $2A$ and from $2B$. We have seen that $K$ has a subgroup, $K_0$ say, such that $K_0$ is isomorphic to Alt(5)$\wr 2$. We consider the action of this group on $Q_{12}$. The action is clearly faithful as $C_G(Q_{12}) = \langle t \rangle$.

Now for $\{i,j\} = \{1,2\}$, $(\hat{A}_i, \hat{B}_i) \cong \text{Alt}(5)$ acts on $Q_{12}$ with an element of order three, $\hat{z}_j$, acting fixed-point-free. It therefore follows that $Q_{12}$ is a sum of two natural $\text{GF}(2)\text{Alt}(5)$-modules. In particular, an element of order five in $(\hat{A}_1, \hat{B}_1)$ acts fixed-point-free on $Q_{12}$.

Now $K_0$ contains two further conjugacy classes of subgroups isomorphic to Alt(5); the diagonal subgroups. One of these lies in a Sym$(5)$ the other in an Alt(5)$\times 2$ and furthermore $K_0$ contains two further conjugacy classes of subgroups of order five. Let $F$ be a Sylow 5-subgroup of $K_0$ then by coprime action $Q_{12} = \langle C_{Q_{12}}(f) : f \in F' \rangle$. Since $C_G(Q_{12}) = \langle t \rangle$, no element in $F$ acts trivially. However $F$ has six subgroups of order five and for one of these, $(f)$ say we must have $\langle t \rangle < C_{Q_{12}}(f) < Q_{12}$ and $Q_{12} = [Q_{12}, f]C_{Q_{12}}(f)$ where $[Q_{12}, f]$ and $C_{Q_{12}}(f)$ are both extraspecial and intersect at $\langle t \rangle$. Moreover, for any other element $f' \in F \setminus \{f\}$ we have that $C_{Q_{12}}(f') \cap C_{Q_{12}}(f) = \langle t \rangle$ so $C_{Q_{12}}(f)$ has an automorphism of order five. Of course $[Q_{12}, f]$ has an automorphism $f$ of order five also. It follows that $C_{Q_{12}}(f) \cong [Q_{12}, f] \cong 2^2:4$. Moreover, exactly two of the subgroups of $F$ of order five act non-trivially on $Q_{12}$ and so it follows that one class of diagonal Alt(5) subgroups of $K_0$ contain such an element of order five and the other contains a fixed-point-free element of order five.

Recall that $E \leq Q_1$ commutes with $\langle A_1, A_2 \rangle$ so consider an involution, $v$ say in $E$ distinct from $t$. Suppose that $v$ is fixed by an element of order five as well as $\langle A_1, A_2 \rangle$ in $K_0$. Then it follows that $C_{K_0}(v)$ contains $K_0'$ which is a contradiction. Thus $v$ lies in a $K_0$-orbit which is a multiple of 25. Clearly $v$ does not commute with $P$ and so $v$ lies in an orbit which is a multiple of 3. Also $v$ is conjugate to $vt$ in $Q_{12}$ and so $|v^{K_0}| \geq 150$. 







Now, let $D$ be a diagonal subgroup of $\bar{K}_0$ isomorphic to $\text{Alt}(5)$ and let $F$ be a Sylow 5-subgroup of $D$. We choose $D$ such that $Q_{12}\alpha_2$ generates a Sylow 3-subgroup of $D$. Note that $F$ acts non-trivially on a subgroup of $Q_{12}$ of order $2^4$ and that $[Q_{12}, F] \cong 2^{1+4} \not\cong 2^{1+4} \cong [Q_{12}, a_2] = Q_1$. In fact $|[Q_{12}, F] \cap Q_1| \leq 2^3$ because $[Q_{12}, F]$ is centralized by an element of order five. Thus, if $|[Q_{12}, F] \cap Q_1| = 2^4$ then $|[Q_{12}, F] \cap E| = 2^2$ but we have seen no element of $E \langle t \rangle$ commutes with an element of order five. Thus, $|[Q_{12}, F] \cap Q_1| \leq 2^3$. Now suppose that $|C_{Q_{12}}(F) \cap C_{Q_{12}}(a_2)| = |C_{Q_{12}}(F) \cap Q_2| = 4$ then, as a $D$-module, $Q_{12}$ has no submodule which is a sum of two trivial modules. However this implies that $\langle Q_{12}, F, Q_1 \rangle$ is a sum of a 4-dimensional and a trivial module and so has order 2^5 however that means that $|[Q_{12}, F] \cap Q_1| \geq 2^4$ which is a contradiction. Thus $|C_{Q_{12}}(F) \cap Q_2| = 8$ (it can be no larger without containing a conjugate of $v \in E \langle t \rangle$). Since $2^{1+4}$ has 2-rank 2, $C_{Q_{12}}(F) \cap Q_2$ has two rank at most 2. If it had 2-rank 1 then it would necessarily be isomorphic to $Q(8)$. However, $Q_2$ has just two subgroups isomorphic to $Q(8)$ both of which are normalized by $a_1$. Any subgroup of $C_{Q_{12}}(F) \cap Q_2$ normalized by $a_1$ is normalized by $\langle a_1, D \rangle = \bar{K}_0$, which is again a contradiction. Thus $C_{Q_{12}}(F) \cap Q_2 \cong 4 \times 2$. This implies that an involution in $Q_2 \langle t \rangle$ is centralized by $D$. Call this involution $w$ and observe that $w \notin v^{K_0}$. Now $K_0$ acts faithfully on $Q_{12}$ and so $C_{K_0}(w) = D$ or is a maximal subgroup of $K_0$ of shape $2 \times \text{Alt}(5)$ or $\text{Sym}(5)$. In particular, $w$ lies in a $K_0$-orbit of length a multiple of 60. Thus $|w^{K_0}| \geq 120$. Now $Q_{12}$ has 270 involutions and so every involution lies in $v^{K_0} \cup w^{K_0}$. Since $Q_{12}$ contains representatives from $2A$ and $2B$ we must have that $w^{K_0} = Q_{12} \cap 2A$ and $w^{K_0} = Q_{12} \cap 2B$.

Finally, in Case I, by Lemma 4.11, $K_0$ contains $C_H(p)$ for each $p \in P^*$ and by Lemma 4.12, an involution in $Q_{12} \langle t \rangle$ has centralizer contained in $K_0$ so we may conclude that $K = \bar{K}_0$.

**Lemma 4.14.** $N_H(E) \leq K$ and contains a Sylow 2-subgroup of $G$. In Case I, a Sylow 2-subgroup has order $2^{14}$. In Case II, it has order $2^{15}$ and is self-normalizing with derived subgroup contained in $Q_{12}A_1A_2$.

**Proof.** It follows from Lemma 4.9 that $C_G(E) \leq K$. So we consider $N_H(E)$. By Lemma 4.9, there exists a complement, $C$, to $C_G(E) \leq N_G(E)$ such that $C \leq C_G(a_1)$. Now, by Dedekind’s Modular Law, $N_G(E) \cap H = C_G(E)C \cap H = C_G(E)(C \cap H)$. Furthermore, $\text{Sym}(4) \cong C \cap H \leq C_H(a_1) \leq K$ by Lemma 4.10 (iv). Thus $N_H(E) \leq K$.

In Case I, we have seen that $K/Q_{12} \cong \text{Alt}(5) \langle 2 \rangle$ and so $K$ has Sylow 2-subgroups of order $2^{14}$. To see these are Sylow 2-subgroups of $H$ we must show that $Q_{12}$ is characteristic in any such. First we consider Case II.

We have seen in Lemma 4.9 that $O_2(C_G(E)) = \langle E, Q_2, A_1, A_2 \rangle$ and $C_G(E)/O_2(C_G(E)) \cong \text{Sym}(3)$. We have seen also that $A_1$ and $A_2$ commute modulo $Q_{12}$, however it is clear that they must in fact commute modulo $EQ_2$. Thus $|O_2(C_G(E))| = 2^{11}$ and $|N_H(E)| = 2^{13}3^2$. Since $N_H(E)$ normalizes $Q_{12}$, it is clear that $Q_{12}A_1A_2 \leq N_H(E)$ and has order $2^{13}$. It follows that $Q_{12}A_1A_2 = O_2(N_H(E))$ and since $C \cap H$ is complement commuting with $a_1$, we additionally see that $N_H(E)/O_2(N_H(E)) \cong \text{Sym}(3) \times \text{Sym}(3)$. 

Now set \( \hat{K} = K/Q_{12} \) and consider \( M := N_K(Q_{12}A_1A_2) \supseteq N_H(E) \). This has \( P \) as a Sylow 3-subgroup. We have seen in Lemma 4.7 \( (iii) \) that \( N_H(P) \) acts as \( \text{Dih}(8) \) on \( \{ A_1, B_1, A_2, B_2 \} \). Thus the subgroup of \( N_H(P) \) which preserves \( \{ A_1, A_2 \} \) has index four. Using Lemmas 4.1 and 4.2 we therefore see that \( |N_M(P)| = 3^22^3 \). We have, using Lemma 4.1, that a Sylow 2-subgroup of \( C_H(P) \) is elementary abelian and if an involution, \( r \) say, in \( C_H(P) \) normalizes \( A_1 \) and \( A_2 \) then it must be in \( A_1 \) and \( A_2 \) otherwise \( \langle A_1, r \rangle \) is a 2-subgroup of \( C_G(Z) \) normalized by \( P \) which is not possible. Thus \( C_M(P) = \langle t \rangle \). Now we may apply Lemma 4.6 together with coprime action to argue that \( Q_{12}A_1A_2 = O_2(M) \) and so \( A_1A_2 = O_2(M) \). Now \( |N_{\hat{K}}(\hat{P})| = 3^22^2 \) with elementary abelian Sylow 2-subgroups (as seen in \( N_H(E) \)). We also see that \( C_M(p) \subset N_M(P) \) for any \( p \in P^\# \). Thus \( M/A_1A_2 \) satisfies Theorem 2.4 and we may conclude that \( M/A_1A_2 \) has a normal Sylow 3-subgroup. Thus \( M = Q_{12}A_1A_2N_M(P) = N_H(E) \). Let \( T \) be a Sylow 2-subgroup of \( N_H(E) \). Then \( \hat{T} \) is a Sylow 2-subgroup of the subgroup \( \hat{D} \) in Lemma 4.11 and therefore \( A_1A_2 \) is characteristic in \( \hat{T} \). Hence \( T \) is a Sylow 2-subgroup of \( K \) and has order \( 2^{15} \).

We now show that in both Cases I and II that if \( T \in \text{Syl}_2(N_H(E)) \) then \( Q_{12} \) is characteristic in \( T \) to conclude that \( T \) is a Sylow 2-subgroup of \( H \). We continue the notation that \( \hat{K} = K/Q_{12} \) and use Lemma 2.12 by considering the action of \( \hat{T} \) on \( \hat{Q}_{12} \). Now any involution in \( A_1A_2 \) inverts a conjugate of \( Z \) and so by Lemma 4.10 \( (ii) \) has centralizer of order \( 2^4 \) in \( \hat{Q}_{12} \). Now if \( R \) is any elementary abelian normal subgroup of \( \hat{T} \) then \( R \cap A_1A_2 \) has order at least two and so for \( R \) of order \( 2^48 \) we have satisfied the requirements of Lemma 2.12. It remains to check that if \( |R| = 2^4 \) then \( |C_{\hat{Q}_{12}}(R)| \leq 2^3 \). However we have seen in Lemma 4.11 that such an \( R \) must be conjugate to \( A_1A_2 \) and now we may use Lemma 4.10 \( (iii) \) to see that \( C_{\hat{Q}_{12}}(A_1) \neq C_{\hat{Q}_{12}}(A_2) \) and since each \( C_{\hat{Q}_{12}}(A_i) \) has centralizer in \( \hat{Q}_{12} \) of order at most \( 2^4 \), we can conclude that \( |C_{\hat{Q}_{12}}(A_1A_2)| \leq 2^3 \). Hence Lemma 2.12 gives us that \( Q_{12} \) is characteristic in \( T \) and since \( T \in \text{Syl}_2(N_G(K)) \), we must have that \( T \in \text{Syl}_2(H) \) and then by Lemma 4.8, \( T \in \text{Syl}_2(G) \).

Finally it is clear that \( T' \subseteq Q_{12}A_1A_2 \) and since \( Q_{12} \) is characteristic in \( T \), \( N_G(T) \subseteq K \) and therefore, \( A_1A_2 \) char \( \hat{T} \subseteq N_G(T) \). Thus \( N_G(T) \subseteq N_H(E) \) and so, in Case II, is self-normalizing as claimed.

\( \square \)

**Lemma 4.15.** In Case II, \( G \) has proper normal subgroup \( G_0 \) which satisfies the hypotheses in Case I.

**Proof.** In Case II we have that \( O^3(C_G(a_1)) \cong \text{Sym}(9) \). We choose an involution, \( r \) say, in \( O^3(C_G(a_1)) \) whose image is the transposition \( (7,8) \) and so \( r \) is in the subgroup of \( K \) described in Lemma 4.11. Let \( T \) be a Sylow 2-subgroup of \( N_H(E) \). Then by Lemma 4.14, \( T \in \text{Syl}_2(G) \) and \( T' \subseteq Q_{12}A_1A_2 \). Now, using Lemma 4.11 \( (iii) \) and Lemma 4.13, we see that every involution in \( T' \) lies in \( 2A \cup 2B \). By Lemma 4.3, \( r \) is not in \( 2A \) or \( 2B \) and so no \( G \)-conjugate of \( r \) lies in \( T' \). Now we may apply Grün’s Theorem to see that a Sylow 2-subgroup of \( G' \) is equal to \( \langle N_G(T), T \cap R' \mid R \in \text{Syl}_2(G) \rangle \). Hence \( r \not\in G' \) and \( G/G' \) has even order. Now by Lemma 3.12, it is clear that \( \langle A_1, A_2, s \rangle \leq Q' \) and so we see that \( G \) has a proper
normal subgroup $G_0$ such that $N_{G_0}(Z) \sim 3^{1+4} : 4 \cdot \text{Alt}(5)$. We must check that $Z$ is conjugate to $Z^z$ in $G_0$ however this is clear as $Z$ and $Z^z$ are conjugate in $\langle Q, Q^z \rangle \leq G_0$. □

In light of Lemma 4.15, we may simplify our working significantly by assuming from now on that we are in Case I only and so by Lemma 4.13, $K/Q_{12} \cong \text{Alt}(5)/2$.

**Lemma 4.16.** $K$ is strongly 3-embedded in $H$.

**Proof.** Let $h \in H$ and $y \in K \cap K^h$ be an element of order three. By Lemma 4.10, the centralizer in $H$ of every element of order three in $K$ is contained in $K$. Thus $C_H(y) \leq K \cap K^h$. Therefore $K \cap K^h$ contains a Sylow 3-subgroup of $H$. So assume $P \leq K \cap K^h$. Then $Q_{12} = O_2(K) = \prod_{p \in P} O_2(C_H(p)) = O_2(K^h) = Q_{12}^h$. Therefore $h \in N_G(Q_{12}) = K$ and so $K = K^h$. □

Recall we fixed an involution $r_1 \in C_H(a_1)$ in Notation 4.4.

**Lemma 4.17.** $r_1$ is not in $O^2(H)$. In particular, $H \neq O^2(H)$ and $O^2(H) \cap K \sim 2_+^{1+8}(\text{Alt}(5) \times \text{Alt}(5))$.

**Proof.** Given the cycle type of the images of $r_1$ and $t$ in $\text{Alt}(9) \cong O^4(C_G(a_1))$ and by Lemma 4.3, we see that $r_1$ is not conjugate to $t$ in $G$ however the product $r_1t$ is conjugate to $t$ in $O^3(C_G(a_1))$ and therefore $r_1$ is not conjugate to $r_1t$ in $G$.

Observe that $r_1$ inverts $a_2$ therefore $r_1 \notin Q_{12}$. Since $r_1$ centralizes $a_1$ whilst inverting $a_2$, we have that $r_1$ permutes $\langle z_1 \rangle$ and $\langle z_2 \rangle$ and therefore permutes $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ and so $r_1 \notin O^2(K)$. Let $T \in \text{Syl}_2(K)$ such that $r_1 \in T$ and suppose that for some $h \in H$, $r_1^h \in O^2(K) \cap T$. Suppose that $r_1^h \in Q_{12}$. Then $\langle r_1^h, t \rangle \lhd Q_{12}$ but is not central in $Q_{12}$ as $Q_{12}$ is extraspecial. Therefore $r_1^h$ conjugate to $r_1^h = (r_1t)^h$ in $Q_{12}$ and so $r_1$ is conjugate to $r_1t$ which is a contradiction. So $r_1^h \notin Q_{12}$. So consider $Q_{12} \neq Q_{12}r_1^h$. By Lemma 4.11, either $r_1^h \in 2B$ or has order four. However $r_1$ is an involution and is not conjugate to $t$ in $G$ and so we have a contradiction.

Thus no $H$-conjugate of $r_1$ lies in $T \cap O^2(K)$ which is a maximal subgroup of $T \in \text{Syl}_2(H)$. By Thompson Transfer, $r_1 \notin O^2(H)$ and so $H \neq O^2(H)$. Since $[K : O^2(K)] = 2$, we must have $O^2(K) = O^2(H) \cap K \sim 2_+^{1+8}(\text{Alt}(5) \times \text{Alt}(5))$. □

**Lemma 4.18.** Let $f \in Q_{12} \setminus \{t\}$. Then either $f$ has order four or one of the following occurs.

(i) $f \in 2B$, $C_H(f) \leq K$ has order $2^{13}3$ and $T$ is 2-central in $K$.

(ii) $f \in 2A$, $|C_K(f)| = 2^{11}35$ and $C_K(f)Q_{12}/Q_{12} \cong \text{Alt}(5) \times 2$ or $\text{Sym}(5)$.

In particular, $K$ acts irreducibly on $\overline{Q_{12}}$, $C_H(f) \cap 3A \neq 1$ and if $T \in Z(T)$ then $f \in 2B$ and $C_H(f) \leq K$.

**Proof.** Lemma 4.13 tells us that every involution in $Q_{12} \setminus \{t\}$ lies in one of two $K$-conjugacy classes. Meanwhile, using Lemma 4.12 we see that if such an involution $f \in 2B$ then $C_H(f) \leq K$ and lies in a $K$-orbit of length 150 which means that
$$|C_H(f)| = 2^{13}3$$ and so $\bar{f}$ is 2-central in $\overline{K}$. If $f \in 2A$ then $f$ commutes with a diagonal subgroup of $K/Q_{12}$ isomorphic to $\text{Alt}(5)$ and lies in a $K$-orbit of length 120. It follows from the structure of the maximal subgroups of $\text{Alt}(5) \wr 2$ that $C_{K/Q_{12}}(f) \cong 2 \times \text{Alt}(5)$ or $\text{Sym}(5)$.

We now suppose that $f \in Q_{12}$ has order four. In Lemma 4.13 we saw that an element of order four in $Q_{12}$ also commutes with a diagonal subgroup of $K/Q_{12}$ isomorphic to $\text{Alt}(5)$ and so lies in a $K$-orbit of length 120 or 240. Suppose there is more than one $K$-orbit of elements of order four. Any $K$-orbit has length a multiple of 30 (because $Q_{12}$ is non-abelian and because there exists elements of order three and five which act fixed-point-freely on $Q_{12}$). However no orbit can have length 30 or 60 because $K/Q_{12}$ has no subgroups of order $2^335$ or $2^435$. Since $Q_{12}$ has 240 involutions we have either one orbit of length 240 or two of length 120. In particular, no element of order four is 2-central in $K$.

Now if $f$ is any element in $Q_{12}$ then $f \neq \bar{f} \in \overline{Q_{12}}$ commutes with an element of order three in $\overline{K}$. Since each $z_i$ acts fixed-point-freely on $Q_{12}$, we have that $\bar{f}$ is centralized by a conjugate of $Q_{12}$. Therefore $f$ commutes with a conjugate of $a_i$. Furthermore we observe that if $f$ has order four or $f \in 2A$ then $\bar{f}$ is not 2-central in $\overline{K}$ whereas if $f \in 2B$ then $\bar{f}$ is 2-central in $\overline{K}$. Finally, suppose that $W \triangleleft K$ with $t \in W \triangleleft K$. Then $W$ must be a union of $K$-orbits. However the $K$-orbits on $Q_{12}$ have lengths in $\{1, 150, 120, 240\}$ and no union of orbits is a power of 2 greater than 2 and less than $2^3$. Thus $K$ acts irreducibly on $Q_{12}$. \(\square\)

Lemma 4.19. Let $h \in H$. If $(Q_{12} \cap Q_{12}^h)\langle t \rangle$ contains an involution in $2B$ then $Q_{12} = Q_{12}^h$.

Proof. We may suppose that for some $1 \neq \bar{f} \in Z(T)$, $\bar{f} \in \overline{Q_{12} \cap Q_{12}^h}$. By Lemma 4.18, $f \in 2B$ and $C_H(f) \leq K$ and also $C_H(f) \leq K^h$. However this implies that $3 \mid |K \cap K^h|$ and so $K = K^h$ and $Q_{12} = Q_{12}^h$ by Lemma 4.16. \(\square\)

Lemma 4.20. Let $T \in \text{Syl}_2(K)$. Then $\overline{Q_{12}}$ is strongly closed in $T$ with respect to $\overline{T}$.

Proof. Let $1 \neq \bar{f} \in \overline{Q_{12}}$ such that $\bar{f} \in \overline{T^h \backslash Q_{12}}$ for some $h \in H$. Since $f \in Q_{12} \leq O^2(K) \leq O^2(H)$, we must have that $f \in O^2(K^h) = O^2(H) \cap K^h$. By Lemma 4.11 (iii) applied to $K^h$, either $f$ is an element of order four squaring to $t$ and commuting with a conjugate of $Z$ or $f \in 2B$ and $C_{Q_{12}}(f) = [Q_{12}^h, f]$ has order $2^4$.

Suppose first that $f$ has order four. Then $f^2 = t$ and $Q_{12}^h f$ is an involution in $O^2(K/Q_{12})^h$. By Lemma 4.11 (iii), $C_G(f)$ contains a conjugate of $Z$ and then by Lemma 3.16, a Sylow 3-subgroup of $C_G(f)$ is conjugate to $Z$. However, by Lemma 4.18, since $f \in Q_{12}$, $C_H(f) \cap 3A \neq 1$ which is a contradiction.

So we suppose instead that $f$ is an involution. Then $Q_{12}^h f$ is a diagonal involution with $f \in 2B$ and it follows from Lemma 4.13 (iii) that $f$ commutes with an element of order four in $Q_{12}^h$. Now, by Lemma 4.18, $C_H(f) \leq K$. The element of order four in $Q_{12}^h$ commuting with $H$ is necessarily in $Q_{12}$ else we have a contradiction as before. Let $D, V \leq K^h$ such that $\overline{D} := C_{K^h}(f)$ and $\overline{V} := C_{Q_{12}^h}(f)$. Then we
have seen that \( V \cap Q_{12} > \langle t \rangle \). By Lemma 2.16, since \( \overline{V} = \langle Q_{12}^4, f \rangle \), \( |C_{\overline{V}}(f)| = |\overline{V}||C_{\overline{K}/Q_{12}^4}(f)| = 2^9 \). Thus \( Q_{12}^4D \) is a Sylow 2-subgroup of \( \overline{K} \).

Since \( 1 \neq \overline{V} \cap Q_{12} \trianglelefteq \overline{D} \), we get that \( \overline{V} \cap \overline{Q}_{12} \cap Z(\overline{D}) \neq 1 \). However, \( \overline{V} \cap Z(\overline{D}) \trianglelefteq Z(\overline{Q}_{12}^4D) \) the preimage of which contains only involutions in \( 2B \). Therefore \( V \cap Q_{12} \leq Q_{12}^4 \cap Q_{12} \) contains involutions in \( 2B \) distinct from \( t \). This contradicts Lemma 4.19. Thus \( Q_{12}^4 \) is strongly closed in \( \overline{T} \) with respect to \( \overline{H} \).

Lemma 4.21. \( K = H \).

Proof. Assume for a contradiction that \( K < H \) then \( Q_{12} \not\trianglelefteq H \). Consider \( O_{3'}(H) \). By Lemma 4.18, the only proper non-trivial subgroup of \( Q_{12} \) which is normalized by \( K \) is \( \langle t \rangle \). So we have that \( O_{3'}(H) \cap K \leq O_{3'}(H) \cap Q_{12} = \langle t \rangle \). Since \( O_{3'}(H) \) is normalized by \( P \), by coprime action, \( O_{3'}(H) \) is generated by elements commuting with elements of \( P^\# \). However by Lemma 4.10, for every \( p \in P^\# \), \( C_H(p) \leq K \). Therefore \( O_{3'}(H) \leq K \) and so \( O_{3'}(H) = \langle t \rangle \).

Set \( M := \langle Q_{12}^H \rangle \leq H \) then \( M \leq O^2(H) \). Moreover \( O_{3'}(M) \leq O_{3'}(H) \) and so \( O_{3'}(M) = \langle t \rangle \). Therefore we have \( P \cap M \neq 1 \). Now \( M \cap K \) is a normal subgroup of \( K \) and contained in \( O^2(H) \cap K = O^2(K) \). Hence \( M \cap K = O^2(K) \).

Set \( N := O_{3'}(M) \). If \( N = 3' \) then \( N \leq O_{3'}(H) = \langle t \rangle \) and so \( N = 1 \). Otherwise \( P \cap N \neq 1 \) and then \( |P \cap N, Q_{12}| \leq N \cap Q_{12} = 1 \) which is a contradiction as \( C_G(Q_{12}) \leq Q_{12} \). Therefore \( O_{3'}(M) = 1 \). Now, since \( P \leq M < H \), \( H = MN_H(P) \) by a Frattini argument and so \( M = \langle Q_{12}^P \rangle = \langle Q_{12}^{N_H(P)M} \rangle = \langle Q_{12}^M \rangle \) since \( N_H(P) \leq K = N_G(Q_{12}) \). Finally, we may apply Theorem 2.3 to \( \overline{T} = \langle Q_{12}^M \rangle \). As required, we have that for \( T \in Syl_2(K) \), \( Q_{12} \) is strongly closed in \( \overline{T} \) with respect to \( \overline{H} \). Hence \( Q_{12} \) is strongly closed in \( M \cap \overline{T} \) with respect to \( \overline{M} \). We have also that \( O_{3'}(M) = 1 \). Thus \( Q_{12} = O_{2}(M) \Omega(T \cap M) \). Since \( Q_{12} \) is not a Sylow 2-subgroup of \( M \leq O^2(H) \) we may find \( e \in (M \cap T) \setminus Q_{12} \). Then by Lemma 4.11 (iii), \( Q_{12}e \) contains either involutions or elements of order four squaring to \( t \). In either case \( \overline{Q_{12}^t} \cap \Omega(T \cap M) \neq 1 \) and so \( Q_{12} \not\leq \Omega(T \cap M) \). This contradiction proves that \( H = K \).

5. The Structure of the Centralizer of \( u \)

We continue to assume that we are in Case I only. We now know the structure of the centralizer of an involution in \( G \)-conjugacy class \( 2B \) and so we must determine the structure of the centralizer of an involution in \( 2A \). We continue notation from Section 4. Recall that in Notation 4.4 we fixed an involution \( u \in Q_2 \leq C_G(a_2) \) and we defined \( 2A \) to be the conjugacy class of involutions in \( G \) containing \( u \). By Lemma 4.3, \( 2A \not= 2B \). Let \( L := C_G(u) \) and \( \overline{L} = L/\langle u \rangle \) and we continue to set \( H = C_G(t) \) and \( \overline{H} = H/\langle t \rangle \). We will show that \( L \sim (2\text{ HS}) : 2 \) and so we must identify that \( \overline{L} \) has an index two subgroup isomorphic to the sporadic simple group HS. We first show that \( \overline{L} \) has a subgroup \( 2 \times \text{Sym}(8) \) and later that the centre of this subgroup does not live in \( O^2(\overline{L}) \). We will use the information we have about \( C_G(t) = H \) and \( N_G(E) \) to see the structure of some 2-local subgroups of \( \overline{L} \). Once we have used extremal transfer to find the index two subgroup of \( \overline{L} \) we are then
able to use this 2-local information to apply a theorem due to Aschbacher [2] to recognize HS. The Aschbacher result requires us to find 2-local subgroups of shape $(4 \ast 2_{1+4}^+).\text{Sym}(5)$ and $(4^3).\text{GL}_3(2)$.

Recall using Notation 4.4 that $u \in F \leq Q_2 \subseteq C_G(a_2)$ and that $a_1$ normalizes $F$.

Lemma 5.1. \(C_G(F) \cong 2 \times 2 \times \text{Alt}(8)\) with \(C_G(F) \supset C_G(u) \cap C_G(a_1) \cong \text{Alt}(8)\) and \(C_L(F) \cong 2 \times \text{Sym}(8)\). Moreover if \(F_0\) is any fours subgroup of \(C_G(a_2)\) such that \(F_0^\# \leq 2A\) then \(C_G(F_0) \cong C_G(F)\).

**Proof.** Set \(M := C_G(F)\). First observe that \(F \leq O^3(C_G(a_2)) \cong \text{Alt}(9)\) and the image of \(F^\#\) in \(\text{Alt}(9)\) consists of involutions of cycle type \(2^2\). Notice also that \(\text{Alt}(9)\) has two classes of such fours groups with representatives \(\langle(1,2)(3,4), (1,3)(2,4)\rangle\) and \(\langle(1,2)(3,4), (3,4)(5,6)\rangle\). These subgroups of \(\text{Alt}(9)\) have respective centralizers isomorphic to \(2^2 \times \text{Alt}(5)\) and \(2^2 \times \text{Sym}(3)\) and respective normalizers \((\text{Alt}(4) \times \text{Alt}(5)) : 2\) and \(\text{Sym}(4) \times \text{Sym}(3)\).

Given the image of \(F\) in \(O^3(C_G(a_2))\), we have that \(M \cap C_G(a_2) \cong 3 \times 2^2 \times \text{Sym}(3)\). Let \(R \in \text{Syl}_3(M \cap C_G(a_2))\) such that \((R, a_1)\) is a Sylow 3-subgroup of \(N_G(F)\). Then \(a_2 \in R\) and \((R, a_1)\) is abelian and \(R^\# \subseteq 3B\) since no element of order three in \(3B\) commutes with a fours group. Therefore by the earlier argument for each \(r \in R^\#,\) \(C_G(r) \cap M \cong 3 \times 2^2 \times \text{Alt}(5)\) or \(3 \times 2^2 \times \text{Sym}(3)\).

Consider \(M \cap C_G(a_1)\) which is isomorphic to a subgroup of \(\text{Alt}(9) \cong O^3(C_G(a_1))\). Notice that \(F\) does not commute with \(O^3(C_G(a_1))\) (for we would then have \(F\) commuting with an element of \(3B\) and such elements do not commute with a fours group) and so \(M \cap C_G(a_1)\) is a proper subgroup of \(O^3(C_G(a_1))\). By Lemma 4.8, we have that \(F \leq Q_2\) commutes with \(Q_1 \leq C_G(a_1)\). Also \(F\) commutes with \(R \leq C_G(a_1)\) and so \(|M \cap C_G(a_1)|\) is a multiple of \(2^3\). Moreover \(M \cap C_G(a_1)\) contains the subgroup \(Q_1(a_2) \sim 2_{1+4}^+\).

We check the maximal subgroups of \(\text{Alt}(9)\) (see [4]) to see that \(M \cap C_G(a_1)\) is isomorphic to either a subgroup of \(\text{Alt}(8)\) or the diagonal subgroup of index two in \(\text{Sym}(5) \times \text{Sym}(4)\). The latter possibility leads to a Sylow 2-subgroup of order \(2^5\) with centre of order four which is impossible as \(2_{1+4}^+ \cong Q_2 \leq M \cap C_G(a_1)\). So \(M \cap C_G(a_1)\) is isomorphic to a subgroup of \(\text{Alt}(8)\). Suppose it is isomorphic to a proper subgroup of \(\text{Alt}(8)\). We again check the maximal subgroups of \(\text{Alt}(8)\) ([4]) to see that \(M \cap C_G(a_1)\) is isomorphic to a subgroup of \(N_{\text{Alt}(8)}((1,2)(3,4), (1,3)(2,4), (5,6)(7,8), (5,7)(6,8))) \sim 2^4 : (\text{Sym}(3) \times \text{Sym}(3))\). This subgroup can be seen easily in \(\text{GL}_4(2)\) as the subgroup of matrices of shape

\[
\begin{pmatrix}
* & * & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}
\]

We calculate in this group that an extraspecial subgroup of order \(2^5\) is not normalized by an element of order three. Therefore \(M \cap C_G(a_1)\) is not isomorphic to a subgroup of this matrix group. Thus \(M \cap C_G(a_1) \cong \text{Alt}(8)\). In particular \(M\) has a subgroup isomorphic to \(2^2 \times \text{Alt}(8)\).
Now we have that for every $r \in R^\#$, $C_M(r) \cong 3 \times 2^2 \times \text{Sym}(3)$ or $3 \times 2^2 \times \text{Alt}(5)$. Now $R \leq C_M(a_1) \cong \text{Alt}(8)$ and so $R \in \text{Syl}_2(C_M(a_1))$. Moreover, $\text{Alt}(8)$ has two conjugacy classes of elements of order three. So we may set $R = \langle 1, a_2, a_2, a_3, a_2^2, b_1, b_2, b_2^2 \rangle$ where $a_2$ is conjugate to $a_3$ in $C_M(a_1)$ and $b_1$ is conjugate to $b_2$ in $C_M(a_1)$ such that $C_{C_M(a_1)}(b_1) \cong 3 \times \text{Alt}(5)$ ($i \in \{1, 2\}$) and $C_{C_M(a_1)}(a_j) \cong 3 \times \text{Sym}(3)$ ($j \in \{2, 3\}$). Now we already have that $C_M(a_3) \cong C_M(a_2) \cong 3 \times 2^2 \times \text{Sym}(3)$ and we have two possibilities for the structure of the other 3-centralizer. Therefore we must have that $C_M(b_i) \cong 3 \times 2 \times 2^\# \times \text{Alt}(5)$. Now by coprime action $C_{M/F}(b_i) \cong C_M(b_i)/F$ and $C_{M/F}(F a_i) \cong C_M(a_i)/F$. Hence we may apply Corollary 2.9 to $M/F$ to say that $M/F \cong \text{Alt}(8)$. Therefore $M \cong 2^2 \times \text{Alt}(8)$.

Consider $N_L(F)$. We have seen that $N_G(F)/M \cong \text{Sym}(3)$ and so $[N_L(F) : M] = 2$. It follows that $N_L(F)/F \cong 2 \times \text{Alt}(8)$ or $\text{Sym}(8)$. For $b_1 \in R$, $C_M(b_1) \cong 3 \times 2^2 \times \text{Alt}(5) and so $C_{N_L(F)}(b_1) \cong 3 \times (2^2 \times \text{Alt}(5)) : 2'$ and $C_{N_L(F)}(F) \cong 2 \times \text{Sym}(5)$ which is not a subgroup of $\text{Alt}(8) \times 2$. Thus we must have that $N_L(F)/F \cong \text{Sym}(8)$ and so $C_L(F) \cong 2 \times \text{Sym}(8)$.

Now let $F_0 = C_G(a_2)$ have image $(1, 2)(3, 4), (1, 3)(2, 4)$ in $\text{Alt}(9) \cong O^3(C_G(a_2))$. Then $C_G(a_2) \cap C_G(F_0) \cong 3 \times 2^2 \times \text{Alt}(5)$ and so $N_{C_G(F_0)}(b_1) \cong 3 \times (2^2 \times \text{Alt}(5)) : 2'$ and $C_{N_{C_G(F_0)}}(F_0) \cong 2 \times \text{Sym}(5)$ which is not a subgroup of $\text{Alt}(8) \times 2$. Thus we must have that $N_L(F)/F \cong \text{Sym}(8)$ and so $C_L(F_0) \cong C_G(F_0)$.

Recall from Notation 4.4 that $r_2$ is an involution in $O^3(C_G(a_2))$ which is conjugate to $u$ and $r_2u$. In light of Lemma 5.1, the following result is a calculation in a group isomorphic to $2 \times 2 \times \text{Alt}(8)$.

**Lemma 5.2.** $O_2(C_{H \cap \langle r_2 \rangle}(r_2))$ has order $2^6$.

**Proof.** It is clear from Notation 4.4 that $\langle r_2, u \rangle^\# \subseteq 2A$. Set $F_0 := \langle r_2, u \rangle$ then by Lemma 5.1, $C_G(F_0) \cong 2 \times 2 \times \text{Alt}(8)$. Notice also from Notation 4.4 that $t \in C_G(F_0) \cap C_G(a_2) \cong 3 \times 2^2 \times \text{Alt}(5)$ which has an abelian subgroup containing $t$ isomorphic to $3 \times 2^4$. Consider $(F_0, t) \cap C_G(F_0)$’ (of course $C_G(F_0) \cong \text{Alt}(8)$) which has order two. If $(F_0, t) \cap C_G(F_0)$’ is 2-central in $C_G(F_0)$’ then $a_2 \in C_G(F_0)$’ isomorphic to the subgroup of $\text{Alt}(8)$ of shape $2^{1+4} \times \text{Sym}(3)$. However this implies that $C_G((F_0, t)) \cap C_G(a_2) \cong 2 \times 2 \times 2 \times 3$ which is not the case. Thus $(F_0, t) \cap C_G(F_0)$’ is not 2-central in $C_G(F_0)$’ and so $C_G(F_0)$’ isomorphic to a subgroup of $\text{Alt}(8)$ of shape $(2^2 \times \text{Alt}(4)) : 2$. Thus $C_{H \cap \langle r_2 \rangle}(r_2) \sim 2^2 \times (2^2 \times \text{Alt}(4)) : 2$ and so the order of the 2-radical is clear.

**Lemma 5.3.** $H \cap L$ contains a Sylow 2-subgroup of $L$ which has order $2^{11}$ and centre $\langle t, u \rangle$.

**Proof.** Let $S_u$ be a Sylow 2-subgroup of $C_L(t)$. We have that $u \in Q_2 \leq Q_{12}$ and since $u \in 2A$, we may apply Lemma 4.18 to see that $|C_H(u)| = 2^{11}$. Therefore $|S_u| = 2^{11}$. Now, $u \in Q_2$ and $|Q_1, Q_2| = 1$ (by Lemma 4.8) so we have that $Q_1 \leq C_G(H)(u) \leq S_u$. Moreover, $Z(S_u) \leq C_{S_u}(Q_1) \leq C_{S_u}(Q_2)$. Therefore $Z(S_u) \leq Z(C_{Q_2}(u)) \subseteq \langle t, u \rangle$ since $Q_2$ is extraspecial of order $2^5$. Hence $Z(S_u) = \langle t, u \rangle$. Since $\langle t, u \rangle \leq Q_{12}$ and $Q_{12}$ is extraspecial, $u$ is conjugate to $ut$ in $Q_{12}$. Therefore
$N_G((t,u)) \leq C_G(t)$. So let $S_u \leq T_u \in \text{Syl}_2(H \cap L)$ then $N_{T_u}(S_u) \leq N_L((t,u)) \leq H \cap L$. Thus $S_u$ is a Sylow 2-subgroup of $L$. \hfill \Box$

**Lemma 5.4.** $(H \cap L)/(Q_{12} \cap L) \cong \text{Sym}(5)$.

**Proof.** Using Lemma 4.18 we have that $C_H(u)/C_{Q_{12}}(u) \cong \text{Alt}(5) \times 2$ or $\text{Sym}(5)$. We suppose for a contradiction that $(H \cap L)/(Q_{12} \cap L) = C_H(u)/C_{Q_{12}}(u) \cong C_H(u)Q_{12}/Q_{12} \cong 2 \times \text{Alt}(5)$. Now set $V := C_{Q_{12}}(u)$ then $|V| = 2^8$ and $V$ is normalized by $C_H(u)/V \cong 2 \times \text{Alt}(5)$.

Recall from Notation 4.4 that $r_2$ is an involution in $O^3(C_G(a_2))$ and from Lemma 4.5 that $r_2 \in C_H(a_2) \cap Q_2$. Since $[r_2,a_2] = 1$, $[Vr_2,Va_2] = 1$ and therefore $Vr_2 \in Z(C_H(u)/V)$. In particular, $C_{r_2}(r_2)$ is preserved by $O^2(C_H(u)/V)$ and so $C_{r_2}(r_2)$ is necessarily a sum of trivial $O^2(C_H(u)/V)$-modules. Moreover, this chief factor has order $2^4$.

Now Lemma 5.2 gives us that $|O_2(C_{H \cap L}(r_2))| = 2^6$. Clearly $C_{r_2}(r_2)$ is a normal 2-subgroup of $C_{H \cap L}(r_2)$ as is $(r_2) \not\leq V$. Therefore $|C_{r_2}(r_2)| \leq 2^5$ and so $|C_{r_2}(r_2)| = 2^4$ or $2^5$. Suppose first that $|C_{r_2}(r_2)| = 2^4$ then $\pi \in C_{r_2}(r_2)$ is normalized by $O^2(C_H(u)/V)$ and so $C_{r_2}(r_2)$ is necessarily a sum of trivial $O^2(C_H(u)/V)$-modules. Moreover $V/C_{r_2}(r_2)$ has dimension three and is therefore also a sum of trivial $O^2(C_H(u)/V)$-modules. This is a contradiction.

So suppose instead that $|C_{r_2}(r_2)| = 2^5$. Then $|V,r_2| = 2^2$. Furthermore $V,r_2$ is preserved by $O^2(C_H(u)/V)$. Thus $V,r_2$ is a sum of two trivial $O^2(C_H(u)/V)$-modules. Since $[V,r_2] \leq C_{r_2}(r_2)$, it follows that $C_{r_2}(r_2)$ is also a sum of trivial $O^2(C_H(u)/V)$-modules as is $V/C_{r_2}(r_2)$. Again this gives us a contradiction. Hence we may conclude that $C_H(u)/C_{Q_{12}}(u) \cong C_H(u)Q_{12}/Q_{12} \cong \text{Sym}(5)$. \hfill \Box

**Lemma 5.5.** There exists an element of order four $d \in C_{Q_2}(u)$ such that $d^2 = t$ and $4 \times 2 \cong \langle d, u \rangle \triangleleft H \cap L$.

**Proof.** This is clear once we recall Lemma 4.13 (iii) which tells us that diagonal subgroups of $H/Q_{12}$ isomorphic to $\text{Alt}(5)$ which centralize an involution in $Q_{12}$, in fact centralize a subgroup of $Q_{12}$ isomorphic to $C_4 \times C_2$. We have that $(H \cap L)Q_{12}/Q_{12} \cong (H \cap L)/(Q_{12} \cap L) \cong \text{Sym}(5)$ contains such a subgroup. Finally we must check that the element of order four in $Q_2$ however this is clear as $a_2 \in H \cap L$ and $C_{Q_{12}}(a_2) = Q_2$. \hfill \Box

**Lemma 5.6.** There exists a complement $C \cong \text{GL}_3(2)$ to $C_L(E)$ in $N_L(E)$ such that $EC \leq C_G(F)$.

**Proof.** Recall that $u \in F \leq Q_2$ and by Lemma 5.1, $2 \times 2 \times \text{Alt}(8) \cong C_G(F) \supset C_G(u) \cap C_G(a_1) \cong \text{Alt}(8)$. Notice that $E \leq Q_1 \leq C_G(F)$ since $[Q_1,Q_2] = 1$. Notice also that $t \in C_G(u) \cap C_G(a_1)$. From notation 4.4, the image of $t$ in $\text{Alt}(9) \cong O^3(C_G(a_1))$ is $(1,2)(3,4)(5,6)(7,8)$ and so clearly $t$ lies in exactly one subgroup of $O^3(C_G(a_1))$ isomorphic to $\text{Alt}(8)$. By Lemma 4.5, $O^3(C_G(a_1))$ contains a complement, $C$ say, to $C_G(E)$ in $N_G(E)$. Moreover the image of $EC$...
in $O^3(C_G(a_1))$ lies in a subgroup isomorphic to $\text{Alt}(8)$ containing $t$. Therefore $EC \leq C_G(u) \cap C_G(a_1) \leq C_G(F).$ □

**Lemma 5.7.** We have that $L = FO^2(L)$ with $|L : O^2(L)| = 2$, $N_{O^2(L)}(E) \cong 4^3 : \text{GL}_3(2)$ and $C_{O^2(L)}(t) \sim 2^{1+4} \ast 4.\text{Sym}(5)$.

**Proof.** By Lemma 4.5 (v), a Sylow 3-subgroup of $C_G(E)/E$ is self-centralizing in $G$ and therefore $3 \nmid |C_G(u) \cap C_G(E)|$. Hence $C_L(E)$ is a 2-group. Notice that $|C_L(E)| \leq 2^8$ as, by Lemma 5.6, a complement to $C_G(E)$ in $N_G(E)$ isomorphic to $\text{GL}_3(2)$ is in $L$ and by Lemma 5.3, $|L|_2 = 2^{11}$. By Lemma 5.5, there exists an element of order four for $d \in C_{Q_2}(u)$ such that $d^2 = t \in E$ and $(d, u) \leq H \cap L$ which implies that $4 \equiv (d) \leq H \cap L$. Moreover, $d \in Q_2 \leq C_G(E)$ and so $(d) \leq N_L(E) \cap H$. So consider $(d)^{N_L(E)} \leq C_L(E)$. Since $36$ SARAH ASTILL

Recall that $\tilde{E}$ normalizes the sporadic simple group HS. Now we have that $\tilde{E}$ acts non-trivially on $A$ and so $\tilde{AC} \sim 4^3 : \text{GL}_3(2)$. Since $\text{C}_L(E)$ is a 2-group, we must have that $\tilde{ACF} = N_L(E)$. We now apply Lemma 2.14 to $E$ to say that $O^2(\tilde{E}) \neq \tilde{E}$ and clearly $[L : O^2(L)] = 2$. Notice that $u \in O^2(L)$ because $u$ in $\text{Alt}(9) \cong O^3(C_G(a_2))$ and recalling Notation 4.4, $u \mapsto (1, 2)(3, 4)$, we see that an element of order four with image $1, 3, 2, 4)(5, 6)$ to $u$. Thus $[L : O^2(L)] = 2$. Let $C_{L_0}(E)$ with $O^2(L) \cong 4^3 : \text{GL}_3(2)$).

Since $F \leq Q_{12} \cap L$ and $F \not\leq L_0$, $Q_{12} \cap L \leq Q_{12} \cap L$. Since $[Q_1, a_2] = Q_1$, and so $Q_1 \leq L_0 \leq Q_{12} \cap L$. Also $(d, u) \equiv 4 \times 2$ is normal in $L \cap H$ and clearly $d \in L_0$. Thus $(2^{1+4} \ast 4) \times 2 \sim Q_{12} \cap L$. Now $(H \cap L)/(Q_{12} \cap O^2(L)) \cong \text{Sym}(5)$ follows from an isomorphism theorem since

$$\frac{H \cap L_0}{Q_{12} \cap L} = \frac{H \cap L_0}{(H \cap L_0) \cap (Q_{12} \cap L)} \cong \frac{(H \cap L_0)(Q_{12} \cap L)}{Q_{12} \cap L} = \frac{H \cap L}{Q_{12} \cap L} \cong \text{Sym}(5).$$

Thus $C_{L_0}(t)$ has 2-radical, $Q_{12} \cap L \cong 2^{1+4} \ast 4 \times 2$ with quotient $\text{Sym}(5).$ □

**Lemma 5.8.** $L \cong 2 \times \text{HS} : 2$.

**Proof.** We must prove that $O^2(\tilde{E})$ satisfies the hypotheses of Theorem 2.1 to recognize the sporadic simple group HS. Now we have that $t$ is an involution in $\tilde{L}$ and since $ut$ is not conjugate to $t$, we have that $C_{L_0}(\tilde{t}) = C_E(t \cap L) \sim 2^{1+4} \ast 4.\text{Sym}(5)$.

Suppose that $g \in O^2(L)$ and $g$ normalizes $\tilde{E}$. Then $g$ normalizes $E\langle u \rangle$. Since $N_L(E)$ is transitive on $E^\#$ and we have seen that $tu \in 2A$, we have that $\{eu : e \in E^\#\} \subseteq 2A$. Therefore $E\langle u \rangle \cap 2B = E^\#$. Hence $g$ normalizes $E$. Thus $N_{O^2(L)}(E) = N_{O^2(L)}(E) \cong 4^3 : \text{GL}_3(2))$. 


We therefore apply Theorem 2.1 to $\overline{O^2(L)}$ to see that $\overline{O^2(L)} \cong HS$ and so $O^2(L) \cong 2 \times HS$. We have seen that $L$ does not split over $\langle u \rangle$ and we have also seen that $L = FO^2(L)$. Thus we must have that $L \cong 2 \cdot \text{Aut}(HS) \cong 2 \times HS$. We have seen that $L$ does not split over $\langle u \rangle$ and we have also seen that $L = F O^2(L)$. Thus we must have that $L \cong 2 \cdot \text{Aut}(HS) \cong 2 \times HS$. □

Lemma 5.9. In Case I, $G \cong \text{HN}$.

Proof. We have that $G$ is a finite group with two involutions $u$ and $t$ and $L = C_{G}(u) \cong (2 \cdot \text{HS}) : 2$. Also $C_{G}(t) \cong 2^{1+8}.(\text{Alt}(5) \wr 2)$ and $O_{2}(H) = Q_{12}$ and by Lemma 4.8, $C_{G}(Q_{12}) \leq Q_{12}$. Thus, by Theorem 2.2, $G \cong \text{HN}$. □

Now we recall Case II and Lemma 4.15 to see that in Case II, $G$ has proper subgroup $G_0$ of even index and we have proved that $G_0 \cong \text{HN}$. By a Frattini argument, $G = G_0 N_G(S)$ ($S \in \text{Syl}_3(G)$) and so it follows using Lemma 4.8 that $[G : G_0] = 2$. We check finally that $G \not\cong 2 \times \text{HN}$ however we can use, for example, $C_{G}(J) \leq J$ (Lemma 3.18). Thus, in Case II, $G \cong \text{Aut}(\text{HN})$.

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