The Point Interpolation Method for Static Analysis Using Quadratically Consistent Three-Point Integration Scheme

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Abstract. The element free Galerkin method (EFG) is much interest in recent years due to its merits of only node information needed in constructing shape functions and high precision. However, its low computational efficiency becomes a technical issue in the simulation of realistic problems. To develop a more efficient and accurate method, the point interpolation method using quadratically consistent three-point integration scheme (PIM-QC3) is proposed, based on consistency framework for meshfree nodal shape function and its derivatives. And the shape functions along with corrected nodal derivatives at integration points, adopt quadratic basis, meet the differentiation of the approximation consistency (DAC) and the discrete divergence consistency (DDC). In addition, they possess Kronecker delta property and could enforce displacement boundary conditions simply. The T6-Scheme is utilized for node selection in constructing approximation, to avoid the singular moment matrix and make the computation of shape functions simple. Numerical results show the excellence of the proposed method in accuracy and efficiency of all tested methods. Specially, it is more efficient than three-point integration scheme based on element-free Galerkin method. The research in this paper could provide an efficient and reliable tool for analysis of engineering structure.

1. Introduction

Meshfree methods have attracted intensive interesting in computational mechanics field in the past two decades because of their features that they do not need mesh informations to construct approximation fields, have super convergence and could be easy to construct high order approximation. Many kinds of meshfree methods have been proposed such as the reproducing kernel particle method (RKPM) [1], point interpolation method (PIM) [2], Meshless Local Petrov–Galerkin (MLPG) [3], element-free Galerkin (EFG) [4], etc. Among them, the element-free Galerkin (EFG) [4] method is most widely studied and extensively applied to varied problems such as crack propagation, plate structure etc.

However, EFG method also shows two main disadvantages. Firstly, the shape function of EFG using moving least-square (MLS) approximation does not have the property of Kronecker delta and this makes difficulty in imposition of displacement boundary conditions. Some strategies, such as penalty method [4], Lagrange multipliers method [5], were proposed for addressing this problem. Secondly, computational efficiency is relatively low, which is probable a more difficult issue. Nevertheless, high-order Gauss quadrature is necessary to calculate the stiffness matrix for EFG since its shape functions possesses non-polynomial property and supports domains are overlapped. As a result, this leads to a lot
of CPU time. For example, each background triangle cell must have 16 quadrature points to obtain a stable solution in standard EFG with quadratic MLS approximation [6], while in the finite element analysis, only 3 evaluation points needed in each element can make the 6-node triangular element to get stable solutions for elastostatic problems.

To improve computation efficiency of meshfree methods, some approaches were presented in the literatures. The ideology of nodal integration was first proposed by Beissel and Belytschko [7], which could be performed integrating the Galerkin weak form only at nodes. However, the accuracy will be worsened by the stabilization term introduced in this approach and the value of the stabilization parameter usually depends on numerical experiments. Nagashima [8] proposed a nodal integration approach by using a Taylor’s expansion of the displacement fields and stabilization terms introduced do not have artificial parameters. But, computation time increases due to higher-order derivatives in Taylor’s expansion and substantial oscillations present in the resulting stress fields [9]. The stabilized conforming nodal integration (SCNI) developed by Chen et al. [10], is stable and gives even better accuracy results than that of Gauss integration. Wang and Wu [11] developed a scheme of two-level nesting triangular sub-domains utilizing the gradient smoothing method [10] and achieved satisfactory results in both efficiency and accuracy. However, there are still oscillations occurred near the domain boundary for SCNI [12]. According to the divergence theorem, the consistency requirements regarding relationship between meshfree nodal shape function and its derivatives were reported in [6]. A quadratically consistent three-point integration scheme (QC3) based on EFG (EFG-QC3) is developed for two dimensional problems. Comparing with those of SCNI, this method is more accurate, convergent and stable.

Although EFG-QC3 shows well performance in accuracy, convergence and stability, its computation efficiency is still not satisfactory due to property of MLS approximation employed by it. The construction of shape function at every integration point using MLS needs to calculate inverse matrix which includes large number of nodes at influence domain. This will consume lots of CPU time especially for large engineering structures. To overcome the shortcoming, this paper develops point interpolation method using six nodes quadratically consistent three-point integration scheme (PIM-QC3) that still has the advantages of high precision and high stability. Meanwhile PIM-QC3 is more efficient. In addition, its shape function possesses the desirable Kronecker delta property, and the imposition of displacement boundary conditions is as simple as FEM.

2. Point Interpolant Method

As a type of series representation meshfree method, the point interpolation method (PIM) establishes function approximation by utilizing scattered nodes within the local support domain of a point of interest. The displacement function \( u(x) \) of any point \( x \) in the domain is defined by

\[
u(x) = \sum_{i=1}^{n} p_i(x)a_i = \mathbf{p}^T(x)\mathbf{a}
\]

where \( p_i(x) \) is the basis function of monomials at coordinates \( x = [x, y]^T \), \( a_i \) is the unknown coefficient. The complete linear and quadratic basis functions are given by

\[
p(x) = \begin{bmatrix} x & y \end{bmatrix} \quad \text{linear basis}
\]

\[
p(x) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \quad \text{quadratic basis}
\]

To obtain unknown coefficients, let displacement of node \( i \) inside the local support domain equal to the displacement function. Then the displacement of node \( i \) can be expressed as

\[
u_i = \mathbf{p}^T(x^i)\mathbf{a}
\]

where \( u_i \) is the displacement value of node \( i \) at coordinates \( x = x_i \). Considering the displacements of \( n \) nodes, we derive
where $\mathbf{U}_s = P_u \mathbf{a}$

By solving Eq. (5), the coefficient vector $\mathbf{a}$ is obtained by

\[ \mathbf{a} = P_u^{-1} \mathbf{U}_s \tag{6} \]

Eq. (1) can be rewritten by substituting Eq. (6)

\[ u(x) = P^T(x) P_u^{-1} \mathbf{U}_s = \sum_{i=1}^{n} \phi_i u_i = \Phi(x) \mathbf{U}_s \tag{7} \]

where $\Phi(x)$ is the shape function of PIM

\[ \Phi(x) = [\varphi_1(x) \varphi_2(x) \cdots \varphi_n(x)] \tag{8} \]

Quadratic basis is selected in this paper. And T6-Scheme [13] is employed for node selection in constructing approximation of an interested point. The procedure of establishing the PIM shape function is straightforward and addresses the singularity issue of moment matrix for special cases, and makes construction of PIM shape function efficient. Also PIM shape functions generated using quadratic basis are quadratic consistent [13]. In addition, it is easy to impose the prescribed nodal essential boundary conditions due to Kronecker delta property of shape functions.

Consider a 2D elastostatic problem defined in domain $\Omega$, the equilibrium equation can be expressed as

\[ \nabla \cdot \sigma + \mathbf{b} = 0 \quad \text{in} \quad \Omega \tag{9} \]

where $\sigma$ and $\mathbf{b}$ are the stress tensor and the body force vector respectively. The essential and natural boundary conditions are:

\[ \sigma \cdot \mathbf{n} = \mathbf{f} \quad \text{on} \quad \Gamma_t \tag{10} \]

\[ \mathbf{u} = \mathbf{u}^* \quad \text{on} \quad \Gamma_u \tag{11} \]

where $\mathbf{u}^*$ and $\mathbf{f}$ are the prescribed displacement and traction along boundaries $\Gamma_u$ and $\Gamma_t$ respectively, and $\mathbf{n}$ is vector of the unit outward normal to the boundary.

By using the standard Galerkin procedure, we have

\[ \mathbf{K} \mathbf{u} = \mathbf{f} \tag{12} \]

where

\[ \mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j \, d\Omega \quad \mathbf{B}_j = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \tag{13} \]

\[ \mathbf{f}_j = \int_{\Omega} N_{i} \mathbf{b} \, d\Omega + \int_{\Gamma_t} N_{i} \mathbf{f} \, d\Gamma \tag{14} \]

3. Quadratically Consistent Three-point Integration Scheme Based on Point Interpolant Method (PIM-QC3)

Based on the consistency requirements for the derivatives of nodal shape functions presented in [6], the differentiation of the approximation consistency (DAC), as well as the discrete divergence consistency (DDC) should be met. The requirement of DAC is given by

\[ p_{i,j}(x) = \sum_{j} p(x_j) N_{i,j}(x) \tag{15} \]
As PIM shape functions possess quadratic consistency with regard to the quadratic basis $p(x) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T$ [2][13], the quadratic DAC requirement could be satisfied like MLS shape functions [6]. Virtually the DDC is the divergence theorem about the nodal shape function and its derivatives, whose weak form can be described as

$$\int_{\Omega_s} N_{ij}(x) q(x) d\Omega = \int_{\Gamma_s} N_{ij}(x) q(x) n_i d\Gamma - \int_{\Omega_s} N_{ij}(x) \partial_{ij} q(x) d\Omega$$

where $\Omega_s$ is background integration cell and $\Gamma_s$ is the cell boundary, $q(x) = p_{\partial \Omega}(x) \cup p_{\partial \Gamma}(x)$, and for the quadratic basis $p(x) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T$, $q(x) = p_{\partial \Omega}(x) \cup p_{\partial \Gamma}(x) = \begin{bmatrix} 1 & x & y \end{bmatrix}^T$, $n_i$ is the outward normal on the boundary $\Gamma_s$.

To solve Eq. (16) numerically, six nodes quadratically consistent three-point integration scheme on PIM was proposed, as illustrated in Fig. 1. In the scheme, there are three integral points per background triangle cell which is constructed by connecting the scattered nodes. And the dark dots denote the field nodes which are utilized for establishing displacement field function by using the point interpolation method. The dark six nodes $i_i, -i_i$ for cell $i$ are support nodes based on T6-Scheme: three nodes which connect the cell $i$ and the other three nodes that are remote vertexes of the three adjacent cells. The three blue crosses denote integral points for the domain integrations in the background triangle cell. The red dots on the edges of the cell $i$ denote the boundary integration points of one dimension Gauss integration. In fact, equation (16) could be subdivided into three equations and from which the derivatives of the shape functions at three integral points can be obtained.

![Figure 1. Six nodes quadratically consistent integration scheme based on PIM](image)

The discrete version of Eq. (16) can be written as

$$\sum_{H=1}^{3} W_{H} N_{ij}(x_H) q(x_H) = \sum_{L=1}^{2} N_{iL}(x_L) q(x_L) n_i^L w_G - \sum_{H=1}^{3} W_{H} N_{ij}(x_H) \partial_{ij} q(x_H)$$

To be distinguished from the classical shape function derivative $N_{ij}(x)$, $\tilde{N}_{ij}(x)$ is used to denote corrected derivative. $W_{H}$ is the domain integration weight of integration point $x_H$, and $w_G$ is boundary integration weight of integration point $x_G$. Taking x-derivative for an example, and substituting $q(x) = \begin{bmatrix} 1 & x & y \end{bmatrix}^T$, $q_{ij}(x) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $q_{ij}(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ into Eq. (17), we have

$$W_{ij} d_x = f_i$$

Where,
By solving Eq. (18), the corrected x-derivatives at three integral points \( \hat{N}_{1,x}(x) \) are obtained. The y-derivatives \( \hat{N}_{1,y}(x) \) can be obtained in the same way.

Then, the stiffness matrix for PIM-QC3 can be obtained by substituting \( \hat{N}_{1,x}(x) \) and \( \hat{N}_{1,y}(x) \) into Eq. (13), the stiffness

\[
K_{ij} = \int_{\Omega} B_i^T D B_j \, \mathrm{d}\Omega = \sum_{H=1}^{3} W_H B_i^T (x_H) D B_j (x_H)
\]

(21)

where \( B_i = \left[ \hat{N}_{1,x}, 0, \hat{N}_{1,y}, 0, \hat{N}_{1,x} \right]^T \).

As the quadratic basis \( \mathbf{p}(x) = [1, x, y, x^2, xy, y^2]^T \) is adopted for determining the nodal derivatives, the PIM-QC3 integration scheme meets the quadratic DDC. Meanwhile the PIM-QC3 integration scheme can also fulfill the requirement of quadratic DAC [6]. And if only one integration point is adopted in each cell at the its center, and \( q(x) = [1]^T \), then the discrete version of Eq. (16) will be simplified into one equation as follows:

\[
\hat{N}_{1,x}(x) = \frac{1}{A_h} \sum_{L=1}^{2} \sum_{G=1}^{3} N_f(x_G) n^G_x w_G
\]

(22)

where \( A_h \) and \( x_c \) are the cell area and cell center respectively. This only meets the linear DDC, like SCNI [10].

4. Numerical Examples

To assess accuracy of the proposed method, two types of relative errors in displacement and energy are used, and defined respectively by

\[
E_{\text{disp}} = \left[ \int_{\Omega} (\mathbf{u}^h - \mathbf{u})^T (\mathbf{u}^h - \mathbf{u}) \, \mathrm{d}\Omega \right]^2 / \int_{\Omega} \mathbf{u}^T \mathbf{u}^h \, \mathrm{d}\Omega
\]

\[
E_{\text{eng}} = \left[ \int_{\Omega} (\mathbf{\varepsilon}^h - \mathbf{\varepsilon})^T D (\mathbf{\varepsilon}^h - \mathbf{\varepsilon}) \, \mathrm{d}\Omega \right]^2 / \int_{\Omega} \mathbf{\varepsilon}^T D \mathbf{\varepsilon} \, \mathrm{d}\Omega
\]

(23)

where the superscript e denotes the analytical or exact solutions and the superscript h represents the numerical solutions.

The proposed six nodes quadratically consistent integration scheme based on point interpolant method is denoted as PIM-QC3, one quadrature point per cell for linear DDC is called PIM-LC1, QC3 integration scheme based on MLS approximation [6] is denoted as EFG-QC3. While LFEM refers to linear finite element method, using 1 and 3 integration points in the standard triangle (ST) integration
which is adopted in point interpolant method with classical derivatives of the shape functions, are denoted as PIM-ST1 and PIM-ST3 respectively.

4.1. Cook’s Skew Beam Problem
A skew cantilever beam given by Cook et al. [14] subjected to vertical distributed load $P$ is used for example, and the dimensions are provided in Fig. 2.

![Figure 2. Cook’s skew beam](image)

Four nodal distributions, that are, $3\times3$, $5\times5$, $9\times9$ and $17\times17$, are employed for analysis. The Poisson’s ratio is $1/3$, Young’s modulus is $1$ and the thickness is $1$. The computed vertical displacement at point A as well as the minimum principal stress at point B are listed in Tab. 1 and Tab. 2 respectively. It can be seen that the PIM-QC3 gives more accurate results than LFEM, PIM-LC1, PIM-ST1 and PIM-ST3.

| Method          | $v_\Delta$ |
|-----------------|------------|
|                 | $3\times3$ | $5\times5$ | $9\times9$ | $17\times17$ |
| LFEM            | 6.743      | 11.25      | 17.33      | 21.59        |
| PIM-LC1         | 26.14      | 26.01      | 24.66      | 24.19        |
| PIM-ST1         | 28.38      | 25.51      | 24.59      | 24.53        |
| PIM-ST3         | 19.24      | 23.80      | 24.22      | 24.08        |
| PIM-QC3         | 20.13      | 23.61      | 24.11      | 24.05        |
| Reference Solution [14] | 23.9 |

| Method          | $\sigma_{\sigma_{\min}}$ |
|-----------------|----------------------------|
|                 | $3\times3$ | $5\times5$ | $9\times9$ | $17\times17$ |
| LFEM            | -0.0427    | -0.0639    | -0.1163    | -0.1678      |
| PIM-LC1         | -0.1868    | -0.2462    | -0.2057    | -0.2046      |
| PIM-ST1         | -0.2154    | -0.2131    | -0.1958    | -0.2166      |
| PIM-ST3         | -0.1474    | -0.2144    | -0.2070    | -0.2036      |
| PIM-QC3         | -0.1537    | -0.2070    | -0.1997    | -0.2034      |
| Reference Solution [14] | -0.201 |

4.2. Cantilever Beam
Considering the cantilever beam geometry shown in Fig. 3, the beam is now subjected to traction at the free end. The analytical displacement solution is available [15] for the parabolic traction $P$ and can be described as
where I is the moment of inertia, written as \( I = \frac{D^3}{12} \). The stresses solutions is available in [15].

In this paper, \( D = 40 \), \( L = 10 \), \( E = 3.0 \times 10^7 \), \( v = 0.3 \), and \( P = 1000 \). The displacements and traction in accordance with the exact solutions are applied at \( x = 0 \) and \( x = L \) respectively. For the beam discretization, \( 27(9 \times 3) \), \( 85(17 \times 5) \), \( 175(25 \times 7) \), \( 297(33 \times 9) \) and \( 451(41 \times 11) \) nodal distributions are used for analysis. The displacement and energy errors are shown in Fig. 4 and Fig. 5 separately. The value \( h \) is chosen to be the nodal spacing in the model.

It is observed that for regular nodal distributions the PIM-QC3 performs the best among all these methods and gives a much more accurate displacement and energy solution, as shown in Fig. 4. And PIM-LC1 is better than PIM-ST1 but poorer than PIM-ST3 on the whole both in displacement and energy solution. It is also found from Fig. 4 that all the methods could be convergent in both displacement and energy. For irregular nodal distributions as shown in Fig. 5, once again the PIM-QC3 performs the best in both displacement and energy solution. Furthermore, comparing with results for regular nodal distributions, the PIM-QC3 still show high precision and its results are not affected by nodal irregularity. Although effects of nodal irregularity on PIM-LC1 and LFEM are not significant but also their accuracy is evidently lower than PIM-QC3. However, PIM-ST1 and PIM-ST3 are affected by nodal irregularity remarkably since their precision for irregular nodal distributions dropped drastically comparing with other methods. It is found that PIM-QC3 is also convergent well in both displacement and energy while PIM-ST1 and PIM-ST3 are not of convergence.
The computational efficiency of PIM-QC3 method is also investigated as shown Fig. 6 where N is number of nodes. Obviously, PIM-QC3 is the most efficient method of all the methods. Compared with EFG-QC3, the CPU time cost by EFG-QC3 is far more than PIM-QC3 for the same number of nodes. And for the same error, PIM-QC3 is better than EFG-QC3.

5. Conclusion
A three-point integration scheme based on point interpolation method (PIM-QC3) is developed, which is derived from consistency framework for nodal shape function and its derivatives. The PIM-QC3 method could meet both quadratic DDC and DAC, moreover, has the advantage of Kronecker delta property so simply enforce essential boundary conditions like FEM. Numerical examples of two dimension solids demonstrate that the PIM-QC3 is much more accurate and efficient than LFEM, PIM-LC1, PIM-ST1 and PIM-ST3 in both displacement and energy fields. Specially, it also performs well for irregular nodal distributions. Compared with EFG-QC3, the PIM-QC3 is more efficient. This paper could provide a potential efficient and reliable tool for analysis of various engineering structures.
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References
[1] W K Liu, S Jun, Y F Zhang, Reproducing kernel particle methods. International Journal for Numerical Methods in Engineering, 1995, 20(8-9): 1081-1106.
[2] G R Liu, Y T Gu, A point interpolation method for two-dimensional solids. International Journal for Numerical Methods in Engineering, 2001, 50(4): 937-951.
[3] J Sladek, V Sladek, M Repka, et al., Static and dynamic behavior of porous elastic materials based on micro-dilatation theory: A numerical study using the MLPG method. International Journal of Solids and Structures, 2016, 96: 126-135.
[4] T Belytschko, Y Y Lu, L Gu, Crack propagation by element free Galerkin methods. Engineering Fracture Mechanics, 1995, 51(2): 295-315.
[5] G Ventura, An augmented Lagrangian approach to essential boundary conditions in meshless methods. International Journal for Numerical Methods in Engineering, 2001, 53(4): 825-842.
[6] Q L Duan, X K Li, H W Zhang, et al., Second-order accurate derivatives and integration schemes for meshfree methods. International Journal for Numerical Methods in Engineering, 2012, 92(4): 399-424.
[7] S Beissel, T Belytschko, Nodal integration of the element-free Galerkin method. Computer Methods in Applied Mechanics and Engineering, 1996, 139(1-4): 49-74.
[8] T Nagashima, Node-by-node meshless approach and its applications to structural analyses. International Journal for Numerical Methods in Engineering, 1999, 46(3): 341-385.
[9] Q L Duan, T Belytschko, Gradient and dilatational stabilizations for stress-point integration in the element-free Galerkin method. International Journal for Numerical Methods in Engineering, 2009, 77(6): 776-798.
[10] J S Chen, C T Wu, S Yoon, et al., A stabilized conforming nodal integration for Galerkin meshfree methods. International Journal for Numerical Methods in Engineering, 2001, 50(2): 435-466.
[11] D D Wang, J C Wu, An efficient nesting sub-domain gradient smoothing integration algorithm with quadratic exactness for Galerkin meshfree methods. Computer Methods in Applied Mechanics and Engineering, 2016, 298: 485-519.
[12] M A Puso, J S Chen, E Zywicz, et al., Meshfree and finite element nodal integration methods. International Journal for Numerical Methods in Engineering, 2008, 74(3): 416-446.
[13] G R Liu, Meshfree Methods: Moving Beyond the Finite Element Method. 2nd Ed. Boca Raton: Taylor & Francis, 2009.
[14] R D Cook, D S Malkus, M E Plesha, Concepts and applications of finite element analysis. 4rd Ed. Hoboken: John Wiley & Sons Inc, 2001.
[15] S P Timoshenko, J N Goodier, Theory of Elasticity. 3rd Ed. New York: McGraw-Hill, 1970.