PARA-CR STRUCTURES
ON ALMOST PARACONTACT METRIC MANIFOLDS

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Almost paracontact metric manifolds are the famous examples of almost para-CR manifolds. We find necessary and sufficient conditions for such manifolds for to be para-CR. Next we examine these conditions in certain subclasses of almost paracontact metric manifolds. Especially, it is shown that the normal almost paracontact metric manifolds are para-CR. We establish necessary and sufficient conditions for paracontact metric manifolds as well as for almost paracosymplectic manifolds to be para-CR. We find also basic curvature identities for para-CR paracontact metric manifolds and study their consequences. Among others, we prove that any para-CR paracontact metric manifold of constant sectional curvature and of dimension greater than three must be para-Sasakian and its curvature equal to minus one. The last assertion do not hold in dimension tree. Moreover, we show that a conformally flat para-Sasakian manifold is of constant sectional curvature equal to minus one. New classes of examples of para-CR manifolds are established.

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1. Preliminaries

Let $M$ be an almost paracontact manifold and $(\varphi, \xi, \eta)$ its almost paracontact structure (e.g. [9], [17]). This means that $M$ is an $(2n + 1)$-dimensional differentiable manifold and $\varphi, \xi, \eta$ are tensor fields on $M$ of type $(1, 1), (1, 0), (0, 1)$, respectively, such that

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$ 

Moreover the tensor field $\varphi$ induces an almost paracomplex structure on the paracontact distribution $\mathcal{D} = \text{Ker}\eta$, i.e. the eigendistributions $\mathcal{D}^\pm$ corresponding to the eigenvalues $\pm 1$ of $\varphi$ are both $n$-dimensional. A pseudo-Riemannian metric $g$ on $M$ satisfying the condition

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

is said to be compatible with the structure $(\varphi, \xi, \eta)$. If $g$ is such a metric, then the quadruplet $(\varphi, \xi, \eta, g)$ is called an almost paracontact metric structure and $M$ an almost paracontact metric manifold. For such a manifold, we additionally have $\eta(X) = g(X, \xi)$, and we define the (skew-symmetric) fundamental 2-form $\Phi$ by $\Phi(X, Y) = g(X, \varphi Y)$. 


In the above and in the sequel, \( W, X, Y, Z, \ldots \) indicate arbitrary vector fields on the considered manifold if it is not otherwise stated.

An almost paracontact metric manifold is called to be

(a) normal if \((12, \, 13)\)

\[
N(X, Y) - 2\, d\eta(X, Y)\xi = 0,
\]

where \( N \) is the Nijenhuis torsion tensor of \( \varphi \),

\[
N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi([\varphi X, Y] + [X, \varphi Y]);
\]

(b) paracontact metric if \( \Phi = d\eta \) \((9, \, 17)\);

(c) para-Sasakian if is normal and paracontact metric;

(d) almost para-cosympletic if the forms \( \eta \) and \( \Phi \) are closed, that is, \( d\eta = 0 \) and \( d\Phi = 0 \) \((7)\).

2. Para-CR Manifolds

Almost paracontact metric manifolds can be interpreted as almost para-CR manifolds. Following \([1, \, 2, \, 4, \, 11, \, 17]\), we will resume the demanded details. It is also important to mention certain analytic and geometric studies on generalizations of para-CR structures which have occurred in the papers \([10, \, 14, \, 15]\).

Let \( M \) be an almost paracontact metric manifold and \( (\varphi, \xi, \eta, g) \) its almost para-contact metric structure. Then \( \dim D = 2n \) and \( \varphi \) (precisely, \( \varphi|_D \)) is a field of endomorphisms of \( D \) such that \( \varphi^2 = \text{Id} \), and the eigendistributions \( D^\pm \) corresponding to the eigenvalues \( \pm 1 \) of \( \varphi \) are both \( n \)-dimensional. Thus, the pair \( (D, \varphi) \) becomes an almost para-CR structure on \( M \).

We say that \( (D, \varphi) \) is a para-CR structure if it is formally integrable, that is, the following two conditions are satisfied

\[
[S(X, Y) + [\varphi X, \varphi Y] - \varphi([\varphi X, Y] + [X, \varphi Y])] = 0 \quad (1)
\]

for all \( X, Y \in D \). Equivalently, the formal integrability means that the eigendistributions \( D^+ \) and \( D^- \) are involutive, that is,

\[
[D^+, D^+] \subset D^+, \quad [D^-, D^-] \subset D^-.
\]

\( M \) will be called to be a para-CR manifold if \( (D, \varphi) \) is a para-CR structure.

In the sequel, we need certain new shapes of the condition \((1)\).

**Lemma 1.** Any of the following conditions is equivalent to \((1)\)

\[
d\eta(X, \varphi Y) - d\eta(Y, \varphi X) = 0, \quad (4)
\]

\[
(\nabla_X \eta)(\varphi Y) + (\nabla_{\varphi X} \eta)(Y) = (\nabla_Y \eta)(\varphi X) + (\nabla_{\varphi Y} \eta)(X) \quad (5)
\]

for any \( X, Y \in D \).
Proof. Indeed, note that (1) is fulfilled if and only if $\eta([\varphi X, Y] + [X, \varphi Y]) = 0$. Now, since $d\eta(U, V) = -(1/2)\eta([U, V])$ for any $U, V \in \mathcal{D}$, we claim easily that the condition (4) is equivalent to (1). On the other hand, since in general $d\eta(U, V) = \frac{1}{2}((\nabla_U \eta)(V) - (\nabla_V \eta)(U))$, the condition (4) can be equivalently written as (5).

When we define a $(0, 2)$-tensor field $L$ on $\mathcal{D}$ by $L(X, Y) = -d\eta(X, \varphi Y)$, $X, Y \in \mathcal{D}$, (6) then the condition (4) can be interpreted as the symmetry of $L$. By an analogy to the theory of CR structures (cf. e.g. [8]), the tensor $L$ defined by (6) can be called the Levi form corresponding to the para-CR structure.

The following theorem gives a necessary and sufficient condition for $M$ to be para-CR.

**Theorem 1.** An almost paracontact metric manifold $M$ is a para-CR manifold if and only if the following condition

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)\varphi Y = -((\nabla_Y \eta)(\varphi X) + (\nabla_X \eta)(\varphi Y))\xi$$

(7)

is satisfied for any $X, Y \in \mathcal{D}$, with $\nabla$ being the Levi-Civita connection of $M$.

**Proof.** Before we start with the proof, it will be useful to find the following expression

$$\varphi S(X, Y) = \varphi([X, Y] + [\varphi X, \varphi Y]) - [X, \varphi Y] - [\varphi X, Y]$$

$$+ \eta([X, \varphi Y] + [\varphi X, Y])\xi$$

$$= \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X - (\nabla_X \varphi)Y + (\nabla_Y \varphi)X$$

$$+ \eta([X, \varphi Y] + [\varphi X, Y])\xi$$

$$= -(\nabla_{\varphi X} \varphi)\varphi Y + (\nabla_{\varphi Y} \varphi)\varphi X - (\nabla_X \varphi)Y + (\nabla_Y \varphi)X$$

$$+ \eta([X, \varphi Y] + [\varphi X, Y])\xi$$

(8)

for any $X, Y \in \mathcal{D}$.

Let us assume that (1) and (2) are satisfied. Define an auxiliary $(0, 3)$-tensor field $A$ on $\mathcal{D}$ by

$$A(X, Y, Z) = g((\nabla_{\varphi X} \varphi)\varphi Y + (\nabla_X \varphi)Y, Z)$$

(9)

for any $X, Y, Z \in \mathcal{D}$. Applying (8) with $S = 0$, we claim that

$$A(X, Y, Z) = A(Y, X, Z).$$

(10)
Moreover, by a simply calculation we show that

\[
A(X, Y, Z) + A(X, Z, Y) = g((\nabla_\varphi X \varphi) Y + (\nabla_X \varphi) Y, Z) + g((\nabla_\varphi X \varphi) Z + (\nabla_X \varphi) Z, Y)
\]

\[
= -g((\nabla_\varphi X \varphi) Z, \varphi Y) - g((\nabla_X \varphi) Y, \varphi Z) = 0.
\]

(11)

Using (10) and (11), we can compute

\[
A(X, Y, Z) = -A(X, Z, Y) = -A(Z, X, Y) = A(Z, Y, X) = A(Y, Z, X) = -A(Y, X, Z) = A(Y, Z, X).
\]

Hence, \(A(X, Y, Z) = 0\) for any \(X, Y, Z \in \mathcal{D}\). By the definition (9), this implies the following

\[
(\nabla_\varphi X \varphi) Y + (\nabla_X \varphi) Y = \lambda(X, Y) \xi \quad \text{for} \ X, Y \in \mathcal{D},
\]

(12)

for a certain \((0, 2)\)-tensor field \(\lambda\) on \(\mathcal{D}\). The projection (12) onto \(\xi\) leads to

\[
\lambda(X, Y) = g((\nabla_\varphi X \varphi) Y + (\nabla_X \varphi) Y, \xi)
\]

\[
= -g((\nabla_\varphi X \varphi) \xi, \varphi Y) - g((\nabla_X \varphi) \xi, Y)
\]

\[
= g(\varphi \nabla_X \xi, \varphi Y) - g((\nabla_\varphi X \varphi) \xi, Y)
\]

\[
= -g((\nabla_\varphi X \varphi) \xi, Y) + g(\varphi \nabla_X \xi, Y)
\]

\[
= g(\varphi \nabla_X \xi - \nabla_\varphi X \xi, Y)
\]

\[
= - (\nabla_X \eta) \varphi Y - (\nabla_\varphi X \eta) Y.
\]

(13)

By applying (5) (which is equivalent to (11)) into (13), we see that \(\lambda(X, Y)\) can be written as

\[
\lambda(X, Y) = - (\nabla_Y \eta) \varphi X - (\nabla_\varphi Y \eta) X,
\]

which substituted into (12), gives (7).

Conversely, assume that (7) is fulfilled. Projecting (7) onto the vector field \(\xi\), we obtain (5), and consequently (11) too. But using (5) and (7), we have

\[
(\nabla_\varphi X \varphi) Y + (\nabla_X \varphi) Y = (\nabla_\varphi Y \varphi) \varphi X + (\nabla_Y \varphi) X.
\]

Now, using the above and (11) into (8), we have \(\varphi S(X, Y) = 0\). Hence, \(S(X, Y) = 0\). Thus, we get (2), completing the proof.

\[\square\]

**Theorem 2.** Any 3-dimensional almost paracontact metric manifold is a para-CR manifold.

**Proof.** Recall that it is proved by the author in [16] that for any 3-dimensional almost paracontact metric manifold, it holds

\[
(\nabla_X \varphi) Y = g(\varphi \nabla_X \xi, Y) \xi - \eta(Y) \varphi \nabla_X \xi
\]
for any \(X, Y \in \mathfrak{X}(M)\). Now, using the above formula, we get for such a manifold,

\[
(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - g(\nabla_{\varphi X} \xi, Y)\xi
\]

\[
= -(\langle \nabla_{\varphi X}(\varphi Y) + (\nabla_{\varphi X}Y) \rangle, Y - (\varphi(\nabla_X \varphi)Y + (\varphi(\nabla_X \eta))(\varphi Y))\xi
\]

for any \(X, Y \in \mathcal{D}\). In view of the above and Theorem 1, the condition (7) reduces to (4), or equivalently, to (14). But since \(\dim \mathcal{D} = 2\), the verification of (5) is easy. In fact it is sufficient to take \(X = E_1\) and \(Y = E_2\), where \(E_1, E_2\) form a local basis for \(\mathcal{D}\) such that \(\varphi E_1 = -E_1\) and \(\varphi E_2 = E_2\). \(\square\)

### 3. Normal Almost Paracontact Metric Manifolds

We start with recalling the theorem proved by the author in [16]: An almost paracontact metric manifold is normal if and only if

\[
\varphi(\nabla_X \varphi)Y - (\nabla_{\varphi X} \varphi)Y + (\nabla_X \eta)(\varphi Y)\xi = 0
\]  

(14)

for any \(X, Y \in \mathfrak{X}(M)\).

Assume that \(M\) is a normal almost paracontact metric manifold. For such a manifold, we additionally have the following relations

\[
\nabla_{\xi} \xi = 0, \quad \nabla_{\xi} \eta = 0, \quad \nabla_{\varphi X} \xi = \varphi \nabla_X \xi, \quad \nabla_{\xi} \varphi = 0.
\]  

(15) \hspace{1cm} (16) \hspace{1cm} (17)

Indeed, by putting \(X = Y = \xi\) into the formula (14), we obtain 0 = \(\varphi(\nabla_{\xi} \varphi)\xi = -\varphi^2 \nabla_{\xi} \xi\). Hence, \(\nabla_{\xi} \xi = 0\) and \(\nabla_{\xi} \eta = 0\). Putting \(Y = \xi\) into (14), we get

\[
0 = \varphi(\nabla_X \varphi)\xi - (\nabla_{\varphi X} \varphi)\xi = -\nabla_X \xi + \varphi \nabla_{\varphi X} \xi.
\]

Hence, (16) follows. Putting \(X = \xi\) into (14) and using (15), we obtain \(\varphi(\nabla_{\xi} \varphi)Y = 0\), which implies (17).

The following theorem is the main result of this section:

**Theorem 3.** Any normal almost paracontact metric manifold is a para-CR manifold.

**Proof.** Let \(M\) be a normal almost paracontact metric manifold. Let us suppose that \(X, Y \in \mathcal{D}\). From (14), we deduce

\[
\varphi(\nabla_X \varphi) \varphi Y - (\nabla_{\varphi X} \varphi) \varphi Y + (\nabla_X \eta)(\varphi Y)\xi = 0.
\]  

(18)

On the other hand, it is easy to see that

\[
\varphi(\nabla_X \varphi) \varphi Y = - (\nabla_X \varphi) Y - (\nabla_X \eta)(\varphi Y)\xi,
\]
which together with (18) leads to

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)Y = 0.$$  \hspace{2cm} (19)

Moreover, we can show that

$$(\nabla_Y \eta)(X) + (\nabla_X \eta)(\varphi Y) = 0.$$  \hspace{2cm} (20)

In fact, using (16), the left hand side of (20) can be transformed in the following way

$$g(\nabla_Y \xi, X) + g(\nabla_X \xi, \varphi Y) = g(\varphi \nabla_Y \xi, X) + g(\nabla_Y \xi, \varphi Y) = 0.$$

Finally, having (19) and (20), we claim that the relation (7) is obviously fulfilled. In view of Theorem 1, this completes the proof. \hspace{2cm} \square

4. Paracontact Metric Manifolds

We recall certain facts about paracontact metric manifolds from the paper [17]. Let $M$ be a paracontact metric manifold. Define $h = \frac{1}{2} \mathcal{L}_{\xi} \varphi$, where $\mathcal{L}$ indicates the Lie derivation operator. Then,

$$g(hX, Y) = g(hY, X) \quad (h \text{ is a symmetric operator}),$$  \hspace{2cm} (21)

$$\varphi h + h \varphi = 0, \quad \text{Tr} \ h = 0, \quad h \xi = 0, \quad \eta \circ h = 0.$$  \hspace{2cm} (22)

It is very important that on a paracontact metric manifold $M$, the following relations hold

$$\nabla_X \xi = - \varphi X + \varphi h X,$$  \hspace{2cm} (23)

$$(\nabla_{\varphi X} \varphi)Y - (\nabla_X \varphi)Y = 2g(X, Y)\xi - \eta(Y)(X - hX + \eta(X)\xi).$$  \hspace{2cm} (24)

A paracontact metric manifold is a para-Sasakian one if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X,$$  \hspace{2cm} (25)

and on a para-Sasakian manifolds, it always holds

$$\nabla_X \xi = -\varphi X, \quad h = 0.$$  \hspace{2cm} (26)

**Theorem 4.** A paracontact metric manifold is a para-CR manifold if and only if

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi.$$  \hspace{2cm} (27)

In [17, Th. 4.10], S. Zamkovoy proved that the paracontact metric manifold is para-CR manifold if and only if

$$(\nabla_X \varphi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX).$$  \hspace{2cm} (28)

In view of (23), the condition (28) is equivalent to (27).
However, using our convention we shall give a short prove of this fact.

**Proof.** Let $M$ be a paracontact manifold. For such a manifold, using (23) and (22), we obtain for $X, Y \in \mathcal{D}$,

$$(\nabla_Y \eta)(\varphi X) + (\nabla_{\varphi Y} \eta)(X) = -2g(hY, X)\xi = -2d(hX, Y).$$

(29)

Moreover, from (24), for $X, Y \in \mathcal{D}$, we get

$$(\nabla_X \varphi)Y = (\nabla_X \varphi)Y + 2g(X, Y)\xi.$$ 

(30)

By applying (29) and (30) into (7), we claim from Theorem 1 that the paracontact metric manifold $M$ is a para-CR manifold if and only if

$$(\nabla_X \varphi)Y = -g(X - hX, Y)\xi$$

(31)

for any $X, Y \in \mathcal{D}$.

It is clear that (28) implies (31). Conversely, take arbitrary vector fields $X, Y \in \mathcal{X}(M)$. Then $X - \eta(X)\xi \in \mathcal{D}$ and $Y - \eta(Y)\xi \in \mathcal{D}$. Put $X - \eta(X)\xi$ instead of $X$ and $Y - \eta(Y)\xi$ instead of $Y$ into the formula (31). After some calculations, in which the above listed properties of $\nabla \varphi$ and $h$ should be used, one obtains (28). This completes the proof. □

A para-Sasakian manifold is obviously para-CR paracontact metric.

**Corollary 1.** A para-CR paracontact metric manifold is para-Sasakian if and only if $h = 0$.

**Proof.** Comparing (25) with (28), we get

$$g(hX, Y)\xi - \eta(Y)hX = 0,$$

which is equivalent to $h = 0$. □

**Theorem 5.** For a para-CR paracontact metric manifold, we have the following curvature identity

$$(R(W, X)\varphi)Y = R(W, X)\varphi Y - \varphi R(W, X)Y$$

$$= g((\nabla_W h)X - (\nabla_X h)W, Y)\xi$$

$$+ g(hX - X, Y)\varphi(W) - g(hW - W, Y)\varphi(hX - X)$$

$$- g(\varphi(hW - W), Y)(hX - X) + g(\varphi(hX - X), Y)(hW - W)$$

$$- \eta(W)((\nabla_W h)X - (\nabla_X h)W).$$

(32)
Proof. Using (27) and (23), we calculate
\[
\begin{aligned}
(\nabla^2_{WX}\varphi)_Y &= \nabla_W((\nabla_X\varphi)_Y) - (\nabla_{\nabla_W X}\varphi)_Y - (\nabla_X\varphi)\nabla_W Y \\
&= \nabla_W(g(hX - X, Y)\xi - \eta(Y)(hX - X)) \\
&\quad - g(h(\nabla_W X) - \nabla_W X, Y)\xi + \eta(Y)(h\nabla_W X - \nabla_W X) \\
&\quad - g(hX - X, \nabla_W Y)\xi + \eta(\nabla_W Y)(hX - X) \\
&= g((\nabla_W h)X, Y) + g(hX - X, Y)\varphi(hW - W) \\
&\quad - g(Y, \varphi(hW - W))(hX - X) - \eta(Y)(\nabla_W h)X.
\end{aligned}
\]

We obtain the final assertion by applying the above formula into the below identity
\[
(R(W, X)\varphi)_Y = (\nabla^2_{WX}\varphi)_Y - (\nabla^2_{XY}\varphi)_Y.
\]

\[\square\]

Corollary 2. For a paracontact metric para-CR manifold, we have
\[
g(R(W, X)\varphi Y, \xi) = g((\nabla_W h)X - (\nabla_X h)W, Y) - 2\eta(Y)g(\varphi h^2 W, X) \\
+ \eta(X)g(\varphi(hW - W), Y) - \eta(W)g(\varphi(hX - X), Y).
\] (33)

Proof. The projection of (32) onto \(\xi\) leads to
\[
g(R(W, X)\varphi Y, \xi) = g((\nabla_W h)X - (\nabla_X h)W, Y) \\
+ \eta(X)g(\varphi(hW - W), Y) - \eta(W)g(\varphi(hX - X), Y) \\
- \eta(Y)g((\nabla_W h)X - (\nabla_X h)W, \xi). (34)
\]

On the other hand, using (23), we find
\[
g((\nabla_W h)X - (\nabla_X h)W, \xi) = -g(h\nabla_W \xi, X) + g(h\nabla_X \xi, W) \\
= 2g(\varphi h^2 W, X),
\]

which turns (34) into (33). \[\square\]

In [17, Th. 3.12], S. Zamkovoy proved that if a paracontact metric manifold \(M\) is of constant sectional curvature \(k\) and of dimension \(2n + 1 \geq 5\), then \(k = -1\) and \(|h|^2 = 0\). From the proof of his theorem it even more follows that \(h^2 = 0\). Unfortunately, \(h^2 = 0\) does not imply \(h = 0\), and therefore the manifold does not have to be a para-Sasakian one (see Example). We will strengthen the assertion of Zamkovoy’s theorem making additional assumption that the paracontact metric manifold is para-CR.

Theorem 6. A para-CR paracontact metric manifold of constant sectional curvature \(k\) and of dimension \(2n + 1 \geq 5\) is para-Sasakian and \(k = -1\).
Proof. Since the manifold is of constant sectional curvature $k$, we have

$$R(W, X)\varphi Y = k(g(X, \varphi Y)W - g(W, \varphi Y)X),$$

which applied into (33) gives

$$g((\nabla_W h)X - (\nabla_X h)W, Y) = (k + 1)(\eta(W)g(X, \varphi Y) - \eta(X)g(W, \varphi Y))$$

$$+ \eta(W)g(\varphi hX, Y) - \eta(X)g(\varphi hW, Y) + 2\eta(Y)g(\varphi h^2W, X).$$

Hence

$$(\nabla_W h)X - (\nabla_X h)W = -(k + 1)(\eta(W)\varphi X - \eta(X)\varphi W)$$

$$+ \eta(W)\varphi hX - \eta(X)\varphi hW + 2g(\varphi h^2W, X)\xi. \tag{36}$$

Now, we use (35) and (36) in (32) and get

$$k(g(X, \varphi Y)W - g(W, \varphi Y)X - g(X, Y)\varphi W + g(W, Y)\varphi X)$$

$$= \eta(X)(g((k + 1)\varphi W - \varphi hW, Y)\xi - \eta(Y)((k + 1)\varphi W - \varphi hW))$$

$$- \eta(W)(g((k + 1)\varphi X - \varphi hX, Y)\xi - \eta(Y)((k + 1)\varphi X - \varphi hX))$$

$$+ g(hX - X, Y)\varphi (hW - W) - g(hW - W, Y)\varphi (hX - X)$$

$$+ g(\varphi (hX - X), Y)(hW - W) - g(\varphi (hW - W), Y)(hX - X). \tag{37}$$

We calculate the trace of (37) with respect to the arguments $Y, W$ and the metric $g$. Then we obtain

$$(2n - 2)(k + 1)\varphi X = (2n - 2)\varphi hX - \text{Tr}(\varphi h)(hX - \varphi^2 X). \tag{38}$$

The left hand side of formula (38) is an antisymmetric linear operator, whereas in view of (21) and (22) the right hand side of (38) is a symmetric operator. Hence, we infer that

$$(2n - 2)(k + 1)\varphi X = 0, \tag{39}$$

$$\text{Tr}(\varphi h)(hX - \varphi^2 X) = 0. \tag{40}$$

Because of $n > 1$, from (39), we have $k = -1$. Moreover, calculating the trace of (40) and using (22), we get $\text{Tr}(\varphi h) = 0$. Therefore, (40) takes the form $\varphi hX = 0$. Hence $h = 0$ and by Collorary 1. the manifold is para-Sasakian. \hfill \square

As it follows from the following example, the assertion of the above theorem does not hold in dimension 3.

Example 1. Define an almost paracontact metric structure $(\varphi, \xi, \eta, g)$ on $\mathbb{R}^3$ by
assuming
\[
\frac{\partial \varphi}{\partial x} = \cosh(2z) \frac{\partial}{\partial z}, \quad \frac{\partial \varphi}{\partial y} = \sinh(2z) \frac{\partial}{\partial z},
\]
\[
\frac{\partial \varphi}{\partial z} = \cosh(2z) \frac{\partial}{\partial x} - \sinh(2z) \frac{\partial}{\partial y},
\]
\[
\xi = -\sinh(2z) \frac{\partial}{\partial x} + \cosh(2z) \frac{\partial}{\partial y}, \quad \eta = \sinh(2z)dx + \cosh(2z)dy,
\]
g = \frac{1}{2} \left( (\xi, \frac{\partial}{\partial z}) - \varphi (\xi, \frac{\partial}{\partial z}) \right) = -\frac{\partial}{\partial z},
\]
which shows that h \neq 0, and therefore the structure is not para-Sasakian.

A para-Sasakian structure of any dimension 2n + 1 \geq 3 and of constant sectional curvature equal to minus one can be constructed in the following way.

**Example 2.** Let \((J, G)\) be the standard flat para-Kähler structure on \(R^{2n+2}_\alpha\),
\[
J \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^{\alpha+n+1}}, \quad J \frac{\partial}{\partial x^{\alpha+n+1}} = \frac{\partial}{\partial x^\alpha},
\]
\[
G \left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = 1, \quad G \left( \frac{\partial}{\partial x^{\alpha+n+1}}, \frac{\partial}{\partial x^{\alpha+n+1}} \right) = -1
\]
for \(\alpha = 1, \ldots, n\), where \((x^1, \ldots, x^{2n+2})\) are the Cartesian coordinates on \(R^{2n+2}_n\).

Consider the hypersurface \(H^{2n+1}_n\) in \(R^{2n+2}_n\) given by the equation
\[
H^{2n+1}_n = \left\{ x \in R^{2n+2}_n : \sum_{\alpha=1}^{n} x_\alpha^2 + \sum_{\alpha=n+2}^{2n+2} x_\alpha^2 = -1 \right\}.
\]

Let
\[
N = \sum_{\alpha=1}^{2n+2} x_\alpha \frac{\partial}{\partial x^\alpha}
\]
be the normal vector field of \(H^{2n+1}_n\). Then \(G(N, N) = -1\). Define a vector field \(\xi\), a tensor field \(\varphi\) of type \((1,1)\), a \(1\)-form \(\eta\) and a pseudo-Riemannian metric \(g\) on \(H^{2n+1}_n\) by assuming
\[
\xi = -JN, \quad JX = \varphi X - \eta(X)\nu, \quad g = G|H^{2n+1}_n.
\]
We get an almost paracontact metric structure $(\varphi, \xi, \eta, g)$ on $H^{2n+1}_n$. We shall show that this structure is para-Sasakian and $H^{2n+1}_n$ is of constant sectional curvature $(-1)$.

For our hypersurface, the Weingarten formula is $D_X N = X$, $D$ being the Levi-Civita connection of $G$. Hence, we obtain the shape operator $A = -I$ and the second fundamental form $h(X,Y) = g(X,Y)$. Using the parallelity of $J$ and the Gauss formula, we have

$$0 = (D_X J)\xi = D_X J\xi - J(D_X \xi) = -D_X N - J(\nabla_X \xi + h(X,\xi)N) = -X - \varphi \nabla_X \xi + \eta(X)\xi,$$

where $\nabla$ is the Levi-Civita connection of $g$. Applying $\varphi$ we get $\nabla X \xi = -\varphi X$. This yields

$$d\eta(X,Y) = \frac{1}{2}(g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)) = g(X, \varphi Y)$$

and the structure $(\varphi, \xi, \eta, g)$ is paracontact metric. Moreover,

$$0 = (D_X J)Y = D_X (\varphi Y - \eta(Y)N) - J(\nabla_X Y + g(X,Y)N)$$

$$= \nabla_X \varphi Y + g(X,\varphi Y)N - (X\eta(Y)N - \eta(Y)X)$$

$$- \varphi \nabla_X Y + \eta(\nabla_X Y)N + g(X,Y)\xi$$

$$= (\nabla_X \varphi)Y + g(X,Y)\xi - \eta(Y)X - ((\nabla_X \eta)Y - g(X,\varphi Y))N.$$

Taking the tangential part we see that $(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X$, and in view of (25) the structure is para-Sasakian.

From the Gauss equation we see that $H^{2n+1}_n$ is the hypersurface of constant curvature equal to minus one. Indeed, we have

$$R(X,Y)Z = h(Y,Z)AX - h(X,Z)AY = -(g(Y,Z)X - g(X,Z)Y).$$

In [17, Th. 3.10], S. Zamkovoy established a certain relationship for the values of the Ricci curvature of a conformally flat para-Sasakian manifold of dimension $2n+1 \geq 5$. We shall generalize his result and prove the following theorem.

**Theorem 7.** A conformally flat para-Sasakian manifold is of constant sectional curvature $k = -1$.

**Proof.** Denote by $\text{Ric}$ and $\text{Ric}^*$ the Ricci and $*$-Ricci operators, and by $r$ and $r^*$ the scalar and $*$-scalar curvatures,

$$g(\text{Ric} Y, Z) = Tr \{ X \to R(X,Y)Z \},$$

$$g(\text{Ric}^* Y, Z) = Tr \{ X \to -\varphi R(X,Y)\varphi Z \},$$

$$r = Tr \text{Ric},$$

$$r^* = Tr \text{Ric}^*.$$
For a para-Sasakian manifold, the following identities are known (cf. 17)

\[ R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \]
\[ g(\text{Ric} \xi, Y) = -2n\eta(Y), \]
\[ r + r^* + 4n^2 = 0. \]

At first, consider the case \( n > 1 \) (\( \dim M > 3 \)). The Riemann curvature tensor field of a conformally flat manifold is given as

\[ R(X, Y)Z = \frac{1}{2n-1}(g(Y, Z) \text{Ric} X + g(\text{Ric} Y, Z)X - g(X, Z) \text{Ric} Y - g(\text{Ric} X, Z)Y) - \frac{r}{2n(2n-1)}(g(Y, Z)X - g(X, Z)Y). \]

Using (44) and (42), we get

\[ r^* = -\frac{1}{2n-1}(r + 4n). \]

Now, from (45) and (43), we obtain

\[ r = -2n(2n + 1). \]

Taking \( Y = Z = \xi \) in (44) and using (41), (42) and (46), we have \( \text{Ric} X = -2nX \). Putting the last relation and (46) into (44) gives

\[ R(X, Y)Z = -(g(Y, Z)X - g(X, Z)Y), \]

which completes the proof in this case.

Now, consider the case \( n = 1 \) (\( \dim M = 3 \)). Since \( M \) is conformally flat, we have

\[ g((\nabla_X \text{Ric})Y, Z) - g((\nabla_Z \text{Ric})Y, X) - \frac{1}{4}(\nabla_X r)g(Y, Z) - (\nabla_Z r)g(Y, X)) = 0. \]

Since \( \xi \) is a Killing vector field, we have

\[ 0 = g((L_\xi \text{Ric})Y, Z) = g((\nabla_\xi \text{Ric})Y, Z) + g(\text{Ric} \nabla_Y \xi, Z) + g(\text{Ric} Y, \nabla_Z \xi), \]
\[ L_\xi r = \nabla_\xi r = 0. \]

Using (47), (49) and (26), we find

\[ g((\nabla_\xi \text{Ric})Y, Z) = g((\nabla_Z \text{Ric})\xi, Y) - \frac{1}{4}(\nabla_Z r)\eta(Y) = -2ng(\varphi Y, Z) + g(\text{Ric} \varphi Z, Y) - \frac{1}{4}(\nabla_Z r)\eta(Y). \]}
On the other hand, by (48) and (26), we get
\[ g((\nabla_{\xi} \text{Ric})Y, Z) = g(\text{Ric} \varphi Y, Z) + g(\text{Ric} Y, \varphi Z). \] (51)

Comparing (50) and (51) gives
\[ g(\text{Ric} \varphi Y, Z) = -2ng(\varphi Y, Z) - \frac{1}{4}g(Y)\nabla_Z r. \] (52)

Putting \( \varphi Y \) instead of \( Y \) in the above equality and using (42), we obtain
\[ g(\text{Ric} Y, Z) = -2ng(Y, Z). \]

So \( M \) is an Einstein space, and consequently, it is of constant sectional curvature. In view of (41), the sectional curvature \( k = -1 \), which completes the proof. \( \square \)

To finish this section, we give an example of a class of new paracontact metric structures which are para-CR and not normal in general.

**Example 3.** Let \( (x^\alpha, y_\alpha, z), n \geq 2, 1 \leq \alpha \leq n \), be the Cartesian coordinates in \( \mathbb{R}^{2n+1} \). Define a frame of vector fields \( (e_i; 1 \leq i \leq 2n+1) \) on \( \mathbb{R}^{2n+1} \) by
\[ e_\alpha = \frac{\partial}{\partial x^\alpha}, \quad e_{n+\alpha} = -f \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial y_\alpha} - 2x^\alpha \frac{\partial}{\partial z}, \quad e_{2n+1} = \frac{\partial}{\partial z}, \]
f being an arbitrary function. The dual frame of 1-forms \( (\theta^i) \) is given by
\[ \theta^\alpha = dx^\alpha + fdy_\alpha, \quad \theta^{n+\alpha} = dy_\alpha, \quad \theta^{2n+1} = 2 \sum_{\omega=1}^n x^\omega dy_\omega + dz. \]

Define an almost paracontact structure \( (\varphi, \xi, \eta, g) \) on \( \mathbb{R}^{2n+1} \) by assuming
\[ \varphi e_\alpha = -e_\alpha, \quad \varphi e_{n+\alpha} = e_{n+\alpha}, \quad \varphi e_{2n+1} = 0, \quad \xi = e_{2n+1}, \quad \eta = \theta^{2n+1}, \quad g(e_\alpha, e_{n+\alpha}) = g(e_{n+\alpha}, e_\alpha) = g(e_{2n+1}, e_{2n+1}) = 1, \]
\[ g(e_i, e_j) = 0 \quad \text{otherwise}. \]

Here, we have \( \Phi(e_\alpha, e_{n+\alpha}) = -\Phi(e_{n+\alpha}, e_\alpha) = 1 \) and \( \Phi(e_i, e_j) = 0 \) otherwise. Therefore the fundamental 2-form \( \Phi \) has the shape
\[ \Phi = \sum_{\alpha=1}^n (\theta^\alpha \otimes \theta^{n+\alpha} - \theta^{n+\alpha} \otimes \theta^\alpha) = 2 \sum_{\alpha=1}^n \theta^\alpha \wedge \theta^{n+\alpha} = 2 \sum_{\alpha=1}^n dx^\alpha \wedge dy_\alpha = d\eta. \]

Thus, \( (\varphi, \xi, \eta, g) \) is a paracontact metric structure. Let us calculate the structure
We see that \( h^2 = 0 \), and \( h \neq 0 \Leftrightarrow \frac{\partial f}{\partial z} \neq 0 \). Therefore, if \( \frac{\partial f}{\partial z} \neq 0 \), the structure is not para-Sasakian one. To find necessary and sufficient conditions for this structure to be para-CR, we are going to use the condition (3). We see that the eigendistribution \( D^- \) is spanned by the vector fields \((e_\alpha)\). Since \([e_\alpha, e_\beta] = 0\), \( D^- \) is involutive. The eigendistribution \( D^+ \) is spanned by the vector fields \((e_{n+\alpha})\). Since

\[
[e_{n+\alpha}, e_{n+\beta}] = \left( f \frac{\partial f}{\partial x^{\alpha}} - \frac{\partial f}{\partial y_\alpha} + 2x^{\alpha} \frac{\partial f}{\partial z} \right) e_\beta - \left( f \frac{\partial f}{\partial x^{\beta}} - \frac{\partial f}{\partial y_\beta} + 2x^{\beta} \frac{\partial f}{\partial z} \right) e_\alpha,
\]

\( D^+ \) is involutive if and only if the function \( f \) fulfills the following system of partial differential equations

\[
f \frac{\partial f}{\partial x^{\alpha}} - \frac{\partial f}{\partial y_\alpha} + 2x^{\alpha} \frac{\partial f}{\partial z} = 0, \quad 1 \leq \alpha \leq n.
\]

(53)

Therefore, \((\varphi, \xi, \eta, g)\) is a para-CR structure if and only if (53) holds.

To see that this structure is not normal in general, at first we find

\[
[e_{n+\alpha}, \xi] = \frac{\partial f}{\partial z} e_\alpha,
\]

and next we compute

\[
N(e_{n+\alpha}, \xi) - 2d\eta(e_{n+\alpha}, \xi)\xi = \varphi^2[e_{n+\alpha}, \xi] - \varphi[\varphi e_{n+\alpha}, \xi] - 2d\eta(e_{n+\alpha}, \xi)\xi = [e_{n+\alpha}, \xi] - \eta([e_{n+\alpha}, \xi])\xi - \varphi[e_{n+\alpha}, \xi] = 2\frac{\partial f}{\partial z} e_\alpha,
\]

which is non-zero if \( \frac{\partial f}{\partial z} \neq 0 \).

Finally, note that the function

\[
f(x^{\alpha}, y_\alpha, z) = \frac{1}{z} \left( c + \sum_{\omega=1}^{n} (x^{\omega})^2 \right), \quad c = \text{const.},
\]

is an example of a particular solution of (53), and moreover \( \frac{\partial f}{\partial z} \neq 0 \).
5. Almost Para-Cosymplectic Manifolds

Before we study almost paracosymplectic manifolds, recall the notion of para-Kählerian manifolds and almost para-Kählerian manifolds; cf. [5, 6].

An almost para-Kählerian manifold \( \tilde{M} \) by definition is a \( 2n \)-dimensional differentiable manifold endowed with an almost para-Kählerian structure \( (\tilde{J}, \tilde{g}) \). The structure is formed by a \((1,1)\)-tensor field \( \tilde{J} \) such that \( \tilde{J}^2 = \tilde{I} \), and a pseudo-Riemannian metric \( \tilde{g} \) satisfying \( \tilde{g}(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{Y}) \), and the fundamental form \( \tilde{\Phi} \), \( \tilde{\Phi}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}\tilde{Y}) \), is closed. An almost para-Kählerian manifold with integrable almost para-complex structure \( \tilde{J} \) (equivalently, \( \tilde{\nabla}\tilde{J} = 0 \)) is said to be para-Kählerian.

Let \( M \) be a paracosymplectic manifold. Since \( d\eta = 0 \), the paracontact distribution \( \mathcal{D} \) is completely integrable and the manifold \( M \) possesses a foliation \( \mathcal{F} \) generated by \( \mathcal{D} \). Any leaf \( \tilde{M} \) of \( \mathcal{F} \) is a submanifold of \( M \) of codimension 1. Since \( \xi|_{\tilde{M}} \) is a vector field normal to \( \tilde{M} \), we may treat \( \tilde{M} \) as a pseudo-Riemannian hypersurface.

Let \( \tilde{J} \) be the \((1,1)\)-tensor field defined by \( \tilde{J} = \varphi|_{\tilde{M}} \) and \( \tilde{g} \) the induced metric on \( \tilde{M} \). Then the pair \( (\tilde{J}, \tilde{g}) \) is an almost para-Kählerian structure on \( \tilde{M} \) (its fundamental form is closed since it is the pullback of the fundamental form of \( M \)). We say that \( M \) has para-Kählerian leaves if on every leaf \( \tilde{M} \), the induced structures \( (\tilde{J}, \tilde{g}) \) are para-Kählerian.

In the paper ([7]), P. Dacko shows that an almost paracosymplectic manifold satisfies the following conditions

\[
\begin{align*}
\nabla_\xi \xi &= 0, \quad \nabla_\xi \varphi = 0, \quad \nabla_{\varphi \xi} \xi = -\varphi \nabla_X \xi, \\
(\nabla_{\varphi X} \varphi)\varphi Y - (\nabla_X \varphi)Y &= \eta(Y)\varphi \nabla_X \xi. 
\end{align*}
\]

(54) (55)

Moreover, he proves that an almost para-cosymplectic manifold has para-Kählerian leaves if and only if

\[
(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi.
\]

(56)

Using the above equality, we obtain

\textbf{Theorem 8.} An almost para-cosymplectic manifold is a para-CR manifold if and only if it has para-Kählerian leaves.

\textbf{Proof.} We prove that for a para-cosymplectic manifold, formulas (56) and (7) are equivalent to each other.

To do it, we check at first that under (54), (55) and the general formula \( (\nabla_V \eta)(V) = (\nabla_V \eta)(U) \) (which is a consequence of \( d\eta = 0 \)), formula (7) reduces equivalently to

\[
(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi
\]

(57)

for any \( X, Y \in \mathcal{D} \). We see that (56) implies (57). Conversely, let us assume that (57) is fulfilled. Let \( X, Y \in \mathfrak{X}(M) \). Then \( X - \eta(X)\xi \in \mathcal{D} \) and \( Y - \eta(Y)\xi \in \mathcal{D} \).
Replacing \( X \) by \( X - \eta(X)\xi \) and \( Y \) by \( Y - \eta(Y)\xi \) in \((57)\), and applying \((54)\), we find

\[
(\nabla_X \varphi)Y - \eta(Y)(\nabla_X \varphi)\xi = g(\varphi \nabla_X \xi, Y)\xi,
\]
which leads to \((56)\). \(\square\)

Below, we give a class of new almost paracosymplectic structures which are para-CR and not normal in general.

**Example 4.** Let \((x^\alpha, y_\alpha, z)\) \((n \geq 1, 1 \leq \alpha \leq n)\) be the Cartesian coordinates in \(\mathbb{R}^{2n+1}\). Let for \(1 \leq \alpha, \beta \leq n\), \(F^\alpha_\beta: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}\) be functions depending on the coordinates \(x^\alpha\) and \(z\) only, and such that \(F^\beta_\alpha = F^\alpha_\beta\). Define a frame of vector fields \((e_i; 1 \leq i \leq 2n+1)\) in \(\mathbb{R}^{2n+1}\) by

\[
e_\alpha = \frac{\partial}{\partial x^\alpha} - \sum_{\omega=1}^{n} F^\omega_\alpha \frac{\partial}{\partial y_\omega}, \quad e_{n+\alpha} = \frac{\partial}{\partial y_\alpha}, \quad e_{2n+1} = \frac{\partial}{\partial z}.
\]

Then the dual frame of 1-forms \((\theta^i; 1 \leq i \leq 2n+1)\) is given by

\[
\theta^\alpha = dx^\alpha, \quad \theta^{n+\alpha} = dy_\alpha + \sum_{\omega=1}^{n} F^\alpha_\omega dx^\omega, \quad \theta^{2n+1} = dz.
\]

Define an almost paracontact structure \((\varphi, \xi, \eta, g)\) on \(\mathbb{R}^{2n+1}\) by assuming

\[
\varphi e_\alpha = -e_\alpha, \quad \varphi e_{n+\alpha} = e_{n+\alpha}, \quad \varphi e_{2n+1} = 0, \quad \xi = e_{2n+1}, \quad \eta = \theta^{2n+1},
\]

\[
g(e_\alpha, e_{n+\alpha}) = g(e_{n+\alpha}, e_\alpha) = g(e_{2n+1}, e_{2n+1}) = 1.
\]

Here, we have \(\Phi(e_\alpha, e_{n+\alpha}) = -\Phi(e_{n+\alpha}, e_\alpha) = 1\) and \(\Phi(e_i, e_j) = 0\) otherwise. Therefore the fundamental 2-form \(\Phi\) has the shape

\[
\Phi = 2 \sum_{\alpha=1}^{n} \theta^\alpha \wedge \theta^{n+\alpha} = 2 \sum_{\alpha=1}^{n} dx^\alpha \wedge dy_\alpha + 2 \sum_{\alpha,\omega=1}^{n} F^\alpha_\omega dx^\alpha \wedge dx^\omega.
\]

By the symmetry \(F^\alpha_\omega = F^\omega_\alpha\), it must be that

\[
\Phi = 2 \sum_{\alpha=1}^{n} dx^\alpha \wedge dy_\alpha,
\]

and consequently, \(d\Phi = 0\). Since also \(d\eta = 0\), the structure \((\varphi, \xi, \eta, g)\) is almost paracosymplectic.

As in the previous example, to find necessary and sufficient conditions for this structure to be para-CR, we shall use the condition \((3)\). The eigendistribution \(\mathcal{D}^+\) is spanned by the vector fields \((e_{n+\alpha})\). Since \([e_{n+\alpha}, e_{n+\beta}] = 0\), \(\mathcal{D}^+\) is involutive. The eigendistribution \(\mathcal{D}^-\) is spanned by the vector fields \((e_\alpha)\). Since

\[
[e_\alpha, e_\beta] = \sum_{\omega} \left( \frac{\partial F^\omega_\beta}{\partial x^\alpha} - \frac{\partial F^\omega_\alpha}{\partial x^\beta} \right) \frac{\partial}{\partial y^\omega},
\]
\( \mathcal{D}^{-} \) is involutive if and only if
\[
\frac{\partial F_\omega}{\partial x^\beta} - \frac{\partial F_\beta}{\partial x^\alpha} = 0,
\]
or equivalently,
\[
F_\omega = \frac{\partial G_\omega}{\partial x^\alpha}
\]
for certain functions \( G_\omega \) on \( \mathbb{R}^{2n+1} \) depending on \( x^\alpha \) and \( z \) only. By \( F_\alpha = F_\omega \),
\[
\frac{\partial G_\omega}{\partial x^\alpha} - \frac{\partial G_\omega}{\partial x_\alpha} = 0
\]
and consequently,
\[
G_\omega = \frac{\partial H}{\partial x^\omega}
\]
for a certain function \( H \) on \( \mathbb{R}^{2n+1} \) depending on \( x^\alpha \) and \( z \) only. Thus, \( \mathcal{D}^{-} \) is involutive if and only if
\[
F_\alpha = \frac{\partial^2 H}{\partial x^\alpha \partial x^\beta},
\]
\( H \) being a function \( H \) on \( \mathbb{R}^{2n+1} \) depending on \( x^\alpha \) and \( z \) only.

We check that this structure is not normal. At first, we find
\[
[e_\alpha, \xi] = \sum_{\omega=1}^{n} \frac{\partial^2 H}{\partial x^\omega \partial x^\alpha \partial z} e_{n+\omega},
\]
and next we compute
\[
N(e_\alpha, \xi) - 2d\eta(e_\alpha, \xi)\xi = \phi^2 [e_\alpha, \xi] - \phi[\phi e_\alpha, \xi] - 2d\eta(e_\alpha, \xi)\xi
\]
\[
= [e_\alpha, \xi] - \eta([e_\alpha, \xi])\xi + \phi[e_\alpha, \xi]
\]
\[
= 2 \sum_{\omega=1}^{n} \frac{\partial^2 H}{\partial x^\omega \partial x^\alpha \partial z} e_{n+\omega},
\]
which is non-zero in general.

**Note**

Certain of the presented results can be treated as paracontact analogies of those known in contact geometry. For contact metric manifolds, we refer the beautiful book [3].

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