Generalized Ulam–Hyers–Rassias Stability Results of Solution for Nonlinear Fractional Differential Problem with Boundary Conditions

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Abstract

The problem of existence and generalized Ulam–Hyers–Rassias stability results for fractional differential equation with boundary conditions on unbounded interval is considered. Based on Schauder’s fixed point theorem, the existence and generalized Ulam–Hyers–Rassias stability results are proved, and then some examples are given to illustrate our main results.

1. Introduction and Position of Problem

There are various, not equivalent, definitions of fractional derivatives according to Grunwald Letnikov, Weil, Caputo, and Riemann–Liouville, etc. Ordinary and partial differential order equations (with fractional derivatives of Caputo and Riemann–Liouville) have awakened in recent years with considerable interest both in mathematics and in applications. Let us describe the abstract Cauchy problem:

\[
D^\gamma u(t) = Au(t), \quad m - 1 < \gamma \leq m \in \mathbb{N},
\]

where the corresponding solutions are represented through the Mittag–Leffler function. In mathematical papers on fractional differential equations, the Riemann–Liouville approach to the concept of a fractional order derivative \( \gamma \geq 0 \) is usually used as follows:

\[
D^\gamma u(t) = \left( \frac{d}{dt} \right)^m \frac{1}{\Gamma(m - \gamma)} \int_0^t (t - r)^{m - \gamma - 1} u(r) dr,
\]

\( m - 1 < \gamma \leq m \in \mathbb{N}. \)

The fractional Riemann-Liouville derivative is the left inverse to the corresponding fractional integral, which is a natural generalization of the Cauchy formula for the antiderivative function \( u(t) \). The initial conditions, of the initial value problem for ordinary differential equations of fractional order \( \gamma \) with fractional derivatives in the Riemann–Liouville form, are given in terms of fractional integrals:

\[
\frac{1}{\Gamma(m - \kappa - \gamma)} \int_0^t (t - r)^{m - \kappa - \gamma - 1} u^{(\kappa)}(r) dr,
\]

\( \kappa = 0, \ldots, m - 1. \)

To satisfy the physical requirements, Caputo introduced an alternative definition of the fractional differential derivative. It was adopted by Caputo and Mainardi as...
\[ cD_t^\gamma u(t) = \frac{1}{\Gamma(m - \gamma)} \int_0^t (t - r)^{m-\gamma-1} u^{(m)}(r) dr, \quad m - 1 < \gamma \leq m. \tag{4} \]

The advantage of this definition is a more natural solution for the problem of initial conditions for solving integro-differential equations of noninteger orders.

The cases of Caputo derivative for \(0 < \gamma < 1\) was called the regularized fractional derivative of order \(\gamma\).

This paper concerns the existence with Ulam stability for the following equation:

\[ D_0^\beta u(t) + f(t, u(t)) + \theta(t) u(t) = 0, \quad t \geq 0. \tag{5} \]

For a continuous function \(u(t)\) together with boundary conditions,

\[ u(0) = 0, \quad u'(0) = 0, \quad D_0^{-1} u(+\infty) = b u(\xi) + \lambda \int_0^\sigma u(s) ds, \tag{6} \]

where \(2 < \beta \leq 3, 0 \leq \lambda, b < \infty\), we fix \(0 < \xi < \sigma < \infty\), the functions \(f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}\) and \(g: \mathbb{R} \rightarrow \mathbb{R}\) are continuous and \(\theta\) is a continuous decreasing positive function such that \(0 < \theta(t) \leq 1\), for all \(t \in [0, +\infty)\). \(D_0^\beta\) is the standard Riemann–Liouville fractional derivative of order \(\beta\).

### 2. Literature Overview

Fractional differential equations, which are often encountered in mathematical modeling of various processes in natural and technical sciences, play an important role in describing many phenomena in physics, bioengineering, and engineering applications. The properties of such equations were investigated in many reviews (among them, we refer [1–6]).

Regarding the existence, we mention the work by Zhao and Ge [7], where the authors used the well-known Leray Schauder nonlinear alternative theorem to prove the existence of positive solutions to the problem

\[
\begin{cases}
D_0^\alpha u(t) + f(t, u(t)) = 0, & t \in [0, +\infty), 1 < \alpha \leq 2, \\
u(0) = 0, \\
D_0^{\alpha-1} u(+\infty) = \beta u(\xi),
\end{cases}
\tag{7}
\]

where \(f \in C([0, +\infty) \times \mathbb{R}, [0, +\infty))\), \(0 \leq \xi, \beta < +\infty\). Next, Wang et al. [8] extended the above results and discussed the question of existence for solutions of (7) with condition:

\[
D_0^{\alpha-1} u(+\infty) = \lambda \int_0^\tau u(s) ds, \tag{8}
\]

where \(0 \leq \lambda, \tau < +\infty\). Shen et al. [9] considered the existence of solution for boundary value problem of nonlinear multipoint fractional differential equation:

\[
\begin{align*}
\frac{D_0^\gamma u(t)}{1 + t_1^{\beta-1}} - \frac{u(t)}{1 + t_2^{\beta-1}} &< \epsilon, \\
t_1, t_2 &\geq T(\epsilon) > 0 \text{ and } u \in U. \tag{12}
\end{align*}
\]

SM Ulam in 1940 was the first to raise the question of stability for functional equations. After his lecture, this question became popular for many specialists in mathematical analysis. It became an area of in-depth research (see for more details [10–12]). Next, many mathematicians turned in their studies to two types of stability—according to Ulam–Hyers and according to Ulam–Hyers–Rassias. This kind of study has become one of the central and most important in the fields of fractional differential equations. Details of recent advances in Ulam–Hyers sustainability and according to Ulam–Hyers for differential equations can be found in [13, 14] and in articles [15–17]. However, as far as we know, most authors discussed Ulam stability of some fractional differential problem on bounded/unbounded intervals, while the present paper discusses the existence of solutions and stability in the sense of Ulam–Hyers–Rassias for nonlinear fractional differential equations boundary conditions, for which research is just beginning, please see [18–23].

### 3. Preliminaries

Here, we present some notations, definitions, auxiliary lemmas concerning fractional calculus, fixed point theorems, and some preliminary concepts of fractional calculus.

**Definition 1** (see [4, 24]). The Riemann–Liouville fractional integral of order \(\beta\) for a function \(f\) is defined as

\[
I_0^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \tag{11}
\]

provided the right side is pointwise defined on \((0; +\infty)\).

**Lemma 1** (see [25]). Let \(U \subset X\) be a bounded set. \(U\) is said to be relatively compact in a space \(E\) if

(i) \(\forall u \in U, \text{ the function } u(t)/(1 + t^{\beta-1})\) is equicontinuous on any compact subinterval of \(I\).

(ii) \(\forall \varepsilon > 0, \text{ there exists a constant } T = T(\varepsilon) > 0\) such that

\[
\left| \frac{u(t)}{1 + t_1^{\beta-1}} - \frac{u(t)}{1 + t_2^{\beta-1}} \right| < \varepsilon, \quad t_1, t_2 \geq T(\varepsilon) > 0 \text{ and } u \in U. \tag{12}
\]
Lemma 2 (see [4, 24]). Assume that \( x \in C (J) \cap L^1 (J) \) with a fractional derivative of order \( \beta > 0 \). Then,
\[
I_0^\beta D_0^\beta x (t) = x (t) - C_1 t^{\beta - 1} - C_2 t^{\beta - 2} - \cdots - C_n t^{\beta - n},
\]
where \( C_1, C_2, \ldots, C_N \in \mathbb{R} \) with \( n = [\beta] + 1 \).

Lemma 3. Let us define the following space:
\[
E = \left\{ u \in C [0, +\infty) : \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\beta - 1}} < +\infty \right\},
\]
equipped with the norm
\[
\|u\|_E = \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\beta - 1}}.
\]

Lemma 4. \( u \) is a solution of the problem (5)-(6) if and only if \( u \) satisfies the following integral equation:
\[
u(t) = \int_0^t H(t, s) [f(s, u(s)) + \theta(s) g(u(s))] ds,
\]
where
\[
H(t, s) = \begin{cases}
  -\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \frac{[\Gamma(\beta + 1) - b\beta(\xi - s)^{\beta-1} - \lambda(\sigma - s)^\beta]}{\Gamma(\beta)} t^{\beta-1}, & s \leq t, s \leq \xi, \\
  \frac{[\Gamma(\beta + 1) - b\beta(\xi - s)^{\beta-1} - \lambda(\sigma - s)^\beta]}{\Gamma(\beta)} t^{\beta-1}, & t \leq s \leq \xi \leq \sigma, \\
  -\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \frac{[\Gamma(\beta + 1) - \lambda(\sigma - s)^\beta]}{\Gamma(\beta)} t^{\beta-1}, & \xi \leq s \leq t, s \leq \sigma, \\
  -\frac{\beta t^{\beta-1}}{\Gamma(\beta + 1) - b\beta\xi^{\beta-1}}, & s \geq t, s \geq \sigma.
\end{cases}
\]

Proof. Using Lemma 2, we have
\[
u(t) = -I_0^\beta [f(t, u(t)) + \theta(t) g(u(t))]
  + c_1 t^{\beta-1} + c_2 t^{\beta-2} + c_3 t^{\beta-3},
\]
By the first and second conditions, we get
\[
c_3 = 0, \\
c_2 = 0.
\]
Consequently,
\[
u(t) = -I_0^\beta [f(t, u(t)) + \theta(t) g(u(t))] + c_1 t^{\beta-1}.
\]

From the third boundary condition, it follows that
\[
D_0^\beta u(t) = -I_0^{\beta-1} [f(t, u(t)) + \theta(t) g(u(t))] + c_1 \Gamma(\beta)
  = -\int_0^t [f(s, u(s)) + \theta(s) g(u(s))] ds + c_1 \Gamma(\beta).
\]
\[ bu(\xi) + \lambda \int_0^\sigma u(s) ds = -bI_0^\beta [f(\xi, u(\xi)) + \theta(\xi)g(u(\xi))] + c_1b^\beta - 1 \]
\[ - \lambda \int_0^\sigma I_0^\beta [f(s, u(s)) + \theta(s)g(u(s))] ds + \lambda \int_0^\sigma c_1^\beta - 1 ds \]
\[ = -bI_0^\beta [f(\xi, u(\xi)) + \theta(\xi)g(u(\xi))] + c_1b^\beta - 1 \]
\[ - \lambda I_0^\beta [f(\sigma, u(\sigma)) + \theta(\sigma)g(u(\sigma))] + \frac{\lambda b^\beta}{\beta} c_1. \]

Then, we deduce

\[ c_1 = \frac{\beta}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b\beta^\beta - 1} \left[ \int_0^\infty [f(s, u(s)) + \theta(s)g(u(s))] ds \right. \]
\[ - bI_0^\beta [f(\xi, u(\xi)) + g(u(\xi))] - \lambda I_0^\beta [f(\sigma, u(\sigma)) + \theta(\sigma)g(u(\sigma))] \right]. \]

By substituting the values of \( c_1, c_2, \) and \( c_3 \) in (18), we get the following integral equation:

\[ u(t) = -\int_0^t \frac{(t-s)^\beta - 1}{\Gamma(\beta)} [f(s, u(s)) + \theta(s)g(u(s))] ds \]
\[ + \frac{b\beta^\beta - 1}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b\beta^\beta - 1} \int_0^\infty [f(s, u(s)) + \theta(s)g(u(s))] ds \]
\[ - \frac{b\beta^\beta - 1}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b\beta^\beta - 1} \int_0^\xi \frac{(\xi-s)^\beta - 1}{\Gamma(\beta)} [f(s, u(s)) + \theta(s)g(u(s))] ds \]
\[ - \frac{\lambda b^\beta - 1}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b\beta^\beta - 1} \int_0^\sigma \frac{(\sigma-s)^\beta}{\Gamma(\beta + 1)} [f(s, u(s)) + \theta(s)g(u(s))] ds. \]

Then, we get (16).

Conversely, suppose that (16) is satisfied. To get (5), we use the following appropriate relationships:

\[ D_0^\beta \int_0^\beta [f(t, u(t)) + \theta(t)g(u(t))] = f(t, u(t)) + \theta(t)g(u(t)), \]
\[ D_0^\beta t^\beta - 1 = 0. \]

Thus, we deduce

The present paper is organized as follows. In Section 4, we prove the existence of the solution for problem (5)-(6) in the Banach space. The generalized Ulam–Hyers stable is stated and proved in Section 5. Finally, an illustrative example is given.

4. Existence Result

In order to prove the existence of the solution for problem (5)-(6), we transform problem (5)-(6) into the fixed point problem \( Pu = u, \) where \( P \) is an operator defined on

\[ \mathfrak{B}(r) = \{ u \in E, \| u \|_E \leq r \}, \]

by

\[ Pu(t) = \int_0^\infty H(t, s) [f(s, u(s)) + \theta(s)g(u(s))] ds. \]

Theorem 1. Let \( f: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) and \( g: \mathbb{R} \rightarrow \mathbb{R} \) are two functions such that
\((A_1) \) \( \Gamma (\beta + 1) \beta > \lambda \sigma^\beta + b\beta \xi^\beta - 1 \).

\((A_2) \) There exist a nonnegative measurable function \( \psi \), defined on \([0, +\infty)\) and a real constant \( L > 0 \) such that:

\[
\begin{align*}
|f(t, u(t)) - f(t, v(t))| & \leq \psi_1(t)\|u(t) - v(t)\|, \quad u, v \in \mathbb{R}, \\
g(u(t)) - g(v(t)) & \leq Lu(t) - v(t), \quad u, v \in \mathbb{R}, \\
\beta \int_0^\infty (1 + t^\beta - 1) [\psi_1(t) + \psi_2(t)] dt & < \Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1.
\end{align*}
\]

with

\[
\psi_2(t) = (t - \theta(t)L, \quad \text{for each} \ t \in [0, +\infty).
\]

\((A_3) \) Let \( \phi_1(t) = |f(t, 0)| \) and \( \phi_2(t) = \theta(t)|g(0)|, \ t \in [0, +\infty) \) such that

\[
\int_0^\infty [\phi_1(t) + \phi_2(t)] dt < +\infty.
\]

Then, problem (5)-(6) has at least one solution in \([0, +\infty)\).

**Lemma 5.** If \((A_1)\) holds, then the Green function \( H(t, s) \) satisfies for all \( \xi, \sigma, \xi, t \in [0, +\infty) \), we have

\[
H(t, s) \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} 
\]

**Proof.** If \( s \leq t \), and \( s \leq \xi \), we get

\[
\frac{H(t, s)}{1 + t^\beta - 1} \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \left( 1 + \beta \right) 
\]

All other cases of \( H(t, s) \) are simple. This completes the proof of Lemma 5.

**Proof.** of Theorem 1. We shall use Schauder’s fixed point theorem, which is divided into three steps.

**Step 1.** Let \( r > 0 \) such that

\[
\frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\phi_1(p) + \phi_2(p)] dp 
\]

If \( u \) is a continuous function on \( J \), then \( Pu \in C(J) \). In order to show \( P(\mathfrak{B}_r) \in \mathfrak{B}_r \), let \( u \in \mathfrak{B}_r, t \in \mathbb{R}^+ \). Then,

\[
\begin{align*}
\|Pu(t)\| & \leq \int_0^\infty H(t, s) \left| f(s, u(s)) + \theta(s)g(u(s)) \right| ds \\
& \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\phi_1(s) + \phi_2(s)] ds \\
& \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\psi_1(s) + \phi_2(s)] ds \\
& \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\psi_1(s) + \phi_2(s)] ds \\
& \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\psi_1(s) + \phi_2(s)] ds \\
& \leq \frac{\beta}{\Gamma (\beta + 1) \beta - \lambda \sigma^\beta - b\beta \xi^\beta - 1} \int_0^\infty [\psi_1(s) + \phi_2(s)] ds \\
& \leq r.
\end{align*}
\]
Therefore, $\|P\|_E \leq r$, thus, $P(\mathcal{B}_r) \in \mathcal{B}_r$.

Step 2. $P: \mathcal{B}_r \to \mathcal{B}_r$ is continuous. Let $\{u_n\}$ be a sequence which converges to $u$ in $\mathcal{B}_r$. Then, for all $t \in [0, +\infty)$,

$$
\frac{|Pu_n(t) - Pu(t)|}{1 + t^{\beta-1}} = \left| \int_0^{\infty} \frac{H(t, s)}{1 + t^{\beta-1}} \left( [f(s, u_n(s)) - f(s, u(s))] + t\theta_n(s)g(u_n(s)) - g(u(s)) \right) ds \right| 
\leq \frac{\beta}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b \rho \xi^\beta - 1} \int_0^{\infty} \left[ [f(s, u_n(s)) - f(s, u(s))] + t\theta_n(s)g(u_n(s)) - g(u(s)) \right] ds
\leq \frac{\beta}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b \rho \xi^\beta - 1} \int_0^{\infty} \left[ \psi_1(s) + \psi_2(s) \right](1 + s^{\beta-1}) ds
\leq \frac{\|u_n - u\|_E}{\Gamma(\beta + 1) - \lambda \sigma^\beta - b \rho \xi^\beta - 1} \int_0^{\infty} \left[ \psi_1(s) + \psi_2(s) \right](1 + s^{\beta-1}) ds
\leq \|u_n - u\|_E
$$

So, we conclude that $\|Pu_n - Pu\|_E \to 0$ as $n \to +\infty$. Hence, $P$ is a continuous operator on $E$.

Step 3. We have two claims to verify that $P(\mathcal{B}_r)$ is a relatively compact set.

First claim: let $I \subset J$ be a compact interval, $t_1, t_2 \in I$ with $t_1 < t_2$. Then, for any $u \in \mathcal{B}_r$, we have

$$
\frac{|Pu(t_2) - Pu(t_1)|}{1 + t_2^{\beta-1}} \leq \int_0^{\infty} \frac{H(t_2, s)}{1 + t_2^{\beta-1}} - \frac{H(t_1, s)}{1 + t_1^{\beta-1}} \left| [f(s, u(s)) + \theta(s)g(u(s))] ds \right|
\leq \int_0^{\infty} \frac{H(t_2, s)}{1 + t_2^{\beta-1}} - \frac{H(t_1, s)}{1 + t_1^{\beta-1}} \left[ [\psi_1(s) + \psi_2(s)](1 + s^{\beta-1}) \right] ||u||_E + \phi_1(s) + \phi_2(s) ds.
$$

Since it is continuous on $I \times J$, we have that $H(t, s)/(1 + t^{\beta-1})$ is a uniformly continuous function on the compact set $I \times I$.

For $s \geq t$, the function depends only on $t$, then it is uniformly continuous on $I \times (J/I)$. Therefore, we have $\forall s \in J$ and $t_1, t_2 \in I$; the next property holds.

$\forall \varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that, if $|t_1 - t_2| < \delta$, then

$$
\frac{|H(t_2, s) - H(t_1, s)|}{1 + t_2^{\beta-1}} \leq \varepsilon.
$$

This property, together with (36) and the fact that

$$
\int_0^{\infty} \left[ (1 + s^{\beta-1}) \left[ [\psi_1(s) + \psi_2(s)](1 + s^{\beta-1}) \right] ||u||_E + \phi_1(s) + \phi_2(s) \right] ds < \infty,
$$

means that $Pu(t)/(1 + t^{\beta-1})$ is equicontinuous on $I$.

Second claim: in order to achieve (ii) of Lemma 1, we use
For any $\epsilon > 0$ and a solution $u \in C([0, +\infty), E)$ of (5)-(6) such that
\[
\|u - v\|_E \leq \epsilon.
\]
and there exists a real number $C_{f, \beta} > 0$ and a solution
\[
\|u - v\|_E \leq \epsilon C_{f, \beta} \Phi(t), \quad t \in [0, +\infty).
\]
Definition 5 (see [26, 27]). For any $\epsilon > 0$ and for each solution $v$ of (5)-(6), problem (5)-(6) is called generalized Ulam–Hyers–Rassias stable with respect to $\Phi \in C([0, +\infty), \mathbb{R}^+)$ if
\[
\|\mathcal{F}v\|_E \leq \Phi(t) \quad t \in [0, +\infty),
\]
and there exists $C_{f, \beta} > 0$ and a solution $u \in C([0, +\infty), E)$ of (5)-(6) such that
\[
\|u - v\|_E \leq C_{f, \beta} \Phi(t), \quad t \in [0, +\infty).
\]

Theorem 2. If the assumptions $(A_1)$ and $(A_2)$ hold, then problems (5)-(6) are generalized Ulam–Hyers stable.

Proof. By the equivalence between the operators $(I \mathcal{d} - P)$ and $\mathcal{F}$ and the assumptions $(A_1), (A_2)$, we find
\[ |v(t) - u(t)| \leq |v(t) - P_v(t)| + |P_v(t) - u(t)| \]
\[ = |(I - \tilde{P})v(t)| + |P_v(t) - Pu(t)| \]
\[ = |\mathcal{F}v(t)| + \left| \int_{0}^{\infty} H(t, s)[f(s, v(s)) - f(s, u(s))]ds \right| \]
\[ - \left| \int_{0}^{\infty} H(t, s)\theta(s)[g(v(s)) - g(u(s))]ds \right| \]
\[ \leq |\mathcal{F}v(t)| + \left| \int_{0}^{\infty} H(t, s)[f(s, v(s)) - f(s, u(s))]ds \right| \]
\[ + \left| \int_{0}^{\infty} H(t, s)\theta(s)[g(v(s)) - g(u(s))]ds \right| . \] (50)

Then,
\[ \|v - u\|_E \leq \|\mathcal{F}v\|_E + \frac{\beta\|v - u\|_E}{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}} \cdot \int_{0}^{\infty} \left[ \psi_1(s) + \psi_2(s) \right](1 + s^{-1-\lambda})ds \]
\[ \leq \varepsilon + \frac{\beta\|v - u\|_E}{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}} \cdot \int_{0}^{\infty} \left[ \psi_1(s) + \psi_2(s) \right](1 + s^{-1-\lambda})ds . \] (51)

Consequently,
\[ \|v - u\|_E \leq \frac{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}}{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}} \int_{0}^{\infty} \left[ \psi_1(s) + \psi_2(s) \right](1 + s^{-1-\lambda})ds . \] (52)

Thus, we get the Ulam–Hyers stability of (5)-(6). Then, if we take $C_f(\varepsilon)$ equal to the right hand side of (52), we obtain the generalized Ulam–Hyers stability of (5)-(6).

**Theorem 3.** Assume that the hypotheses $(A_1)$ and $(A_2)$ hold. In addition, the following hypotheses hold:

(A3) There exist two positive constants $p$ and $q$ such that

\[ \frac{\beta}{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}} \int_{0}^{\infty} \left| H(t, s)[f(s, v(s)) - f(s, u(s))]ds \right| \]
\[ + \left| \int_{0}^{\infty} H(t, s)\theta(s)[g(v(s)) - g(u(s))]ds \right| \]
\[ \leq \Phi(t) + \frac{\beta}{\Gamma(\beta + 1) - \lambda_\sigma - b\mathcal{D}_\beta^{1-\lambda}} \int_{0}^{\infty} \left| f(s, v(s)) - f(s, u(s)) \right|ds \]
\[ + \left| \int_{0}^{\infty} H(t, s)\theta(s)[g(v(s)) - g(u(s))]ds \right| . \] (53)

Then, problems (5) and (6) are generalized Ulam–Hyers–Rassias stable.

**Proof.** By exploiting the assumptions $(A_2)$, $(A_3)$, $(A_4)$, and $(A_5)$, then we get
\[ |v(t) - u(t)| \leq |v(t) - P_v(t)| + |P_v(t) - u(t)| \]
\[ \leq |\mathcal{F}v(t)| + |P_v(t) - Pu(t)| \]
\[ \leq |\mathcal{F}v(t)| + \left| \int_{0}^{\infty} H(t, s)[f(s, v(s)) - f(s, u(s))]ds \right| \]
\[ + \left| \int_{0}^{\infty} H(t, s)\theta(s)[g(v(s)) - g(u(s))]ds \right| . \] (55)
\begin{align*}
\leq & \int_{0}^{\infty} (1 + s^{\beta - 1}) \Phi(s) ds + \frac{\beta}{\Gamma(\beta + 1) - \lambda s^{\beta} - b \xi s^{\beta - 1}} \int_{0}^{\infty} \theta(s) |g(v(s)) - g(u(s))| ds \\
& + \frac{\beta}{\Gamma(\beta + 1) - \lambda s^{\beta} - b \xi s^{\beta - 1}} \int_{0}^{\infty} \theta(s) |f(s, v(s)) - f(s, u(s))| ds \\
\leq & C_{q} \Phi(t) + \frac{\beta (p + q)}{\Gamma(\beta + 1) - \lambda s^{\beta} - b \xi s^{\beta - 1}} \int_{0}^{\infty} (1 + s)^{-\beta} \Phi(s) ds \\
& \leq \left( 1 + \frac{\beta (p + q)}{\Gamma(\beta + 1) - \lambda s^{\beta} - b \xi s^{\beta - 1}} \right) C_{q} \Phi(t) \\
& = C_{f, \Phi} \Phi(t).
\end{align*}

Hence, problems (5) and (6) are generalized Ulam–Hyers–Rassias stable. □

**Example 1.**

\[
\begin{cases}
D^{3/2}_{0+} u(t) = \frac{e^{-t} + \sin(u(t))}{100(1 + t^2)(1 + t^{3/2})} + \frac{1 + \sin(u(t))}{100(1 + t)(1 + t^{3/2})}, & t \in [0, +\infty), \\
u(0) = 0, u'(0) = 0, & D^{3/2}_{0+} u(+\infty) = bu(1) + \lambda \int_{0}^{\infty} u(s) ds.
\end{cases}
\]

In this example, we have

\[
\begin{align*}
f(t, u(t)) &= \frac{e^{-t} + \sin(u(t))}{100(1 + t^2)(1 + t^{3/2})}, \\
g(u(t)) &= 1 + \sin(u(t)), \\
|f(t, u(t)) - f(t, v(t))| &\leq \frac{1}{100(1 + t^2)(1 + t^{3/2})} |u(t) - v(t)|, \\
|g(u(t)) - g(v(t))| &\leq |u(t) - v(t)|, \\
L &= 1,
\end{align*}
\]

\[
\begin{align*}
\theta(t) &= \frac{1}{100(1 + t^2)(1 + t^{3/2})}, \\
\psi_1(t) &= \frac{1}{100(1 + t^2)(1 + t^{3/2})}, \\
\psi_2(t) &= \frac{1}{100(1 + t^2)(1 + t^{3/2})}.
\end{align*}
\]

Since \( \xi \) and \( \sigma \) are fixed, then \( \lambda \) and \( b \) are chosen so that hypothesis (\( A_1 \)) is satisfied. So, we have
\begin{align}
\begin{cases}
    b < \frac{\Gamma (\beta + 1) - \lambda \sigma^\xi}{\beta \xi^{\beta - 1}}, \\
    \lambda < \frac{\Gamma (\beta + 1)}{\sigma^\beta}.
\end{cases} \tag{59}
\end{align}

In our example, we have \(\beta = (5/2), \xi = 1, \sigma = 2\). Then,

\begin{align}
\beta \int_0^{\infty} \left(1 + t^{\beta - 1}\right)[\psi_1 (t) + \psi_2 (t)] \, dt = \frac{5}{200} \int_0^{\infty} \frac{dt}{1 + t^2} + \frac{5}{200} \int_0^{\infty} \frac{dt}{(1 + t)^2} \\
= \frac{\pi}{2} + 1 < + \infty.
\end{align}

By simple computation, we get

\begin{align}
\Gamma (\beta + 1) - \lambda \sigma^\xi - b \beta \xi^{\beta - 1} \approx 0.25.
\end{align}

Thus, \((A_3)\) is satisfied.

Now, it remains to verify \((A_3)\). We have

\begin{align}
\int_0^{\infty} \left[\phi_1 (t) + \phi_2 (t)\right] \, dt \leq \int_0^{\infty} \left(\frac{1}{1 + t^2} + \frac{1}{(1 + t)^2}\right) \, dt \\
= \frac{\pi}{2} + 1 < + \infty.
\end{align}

All hypotheses of Theorem 1 are satisfied. Therefore, boundary value problems \((5)-(6)\) has at least one solution in \(\mathbb{R}\).

**Data Availability**

No data were used in this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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