REGULARITY OF SYMBOLIC POWERS OF CERTAIN GRAPHS

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ABSTRACT. Let $G_{n,r}$ denote the graph with $n$ vertices $\{x_1, \ldots, x_n\}$ in cyclic order and for each vertex $x_i$ consider the set $A_i = \{x_{i-r}, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+r}\}$, where $x_{i-j}$ is the vertex $x_{n+i-j}$, whenever $i < j$ and $0 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. In $G_{n,r}$, every vertex $x_i$ is adjacent to all the vertices of $V(G_{n,r})/A_i$. Let $I = I(G_{n,r})$ be the edge ideal of $G_{n,r}$. We show that Minh’s conjecture is true for $I$, i.e. regularity of ordinary powers and symbolic powers of $I$ are equal. We compute the Waldschmidt constant and resurgence for the whole class.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. Let $M$ be a finitely generated graded $R$-module. The Castelnuovo-Mumford regularity (or simply, regularity) of $M$, denoted by $\text{reg}(M)$, is defined as $\text{reg}(M) = \max\{a_i(M) + i | i \geq 0\}$, where $a_i(M)$ denotes the largest non-vanishing degree of the $i$-th local cohomology module of $M$ with respect to the graded maximal ideal of $R$. It is an important invariant in commutative algebra and algebraic geometry. There is a correspondence between quadratic square free monomial ideals of $R$ and finite simple graphs with $n$ vertices corresponding to the variables of $R$. To every finite simple graph $G$ with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$, we associate its edge ideal by $I(G) = \{x_ix_j | x_ix_j \in E(G)\} \subseteq R$. For $n \geq 1$, the $n$-th symbolic power of $I$ is defined as $I^{(n)} = \bigcap_{p \in \text{Ass}I} (P^nR_p \cap R)$. In general computing the generators of symbolic powers of an ideal is a very difficult job so as to compute the regularity of the symbolic powers of ideals. Apart from regularity, it is interesting to study for which values of $r$ and $m$ the containment $I^{(r)} \subseteq I^m$ holds. To answer this question C. Bocci and B. Harbourne in [2] defined an asymptotic quantity known as resurgence which is defined as $\rho(I) = \sup\{\frac{s}{t} | I^{(s)} \not\subseteq I^t\}$ and showed that it exists for radical ideals. Since computing the exact value of resurgence is difficult, another asymptotic invariant $\tilde{\alpha}(I) = \lim_{s \to \infty} \frac{\alpha(I^{(s)})}{s}$, known as Waldschmidt

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constant was introduced by Waldschmidt in [10], where $\alpha(I)$ denotes the least generating degree of $I$.

This paper is mainly motivated by the conjecture of N. C. Minh which predicts that if $G$ is a finite simple graph, then $\text{reg } I(G)^{(s)} = \text{reg } I(G)^s$ for all $s \geq 1$. By the result of Simis, Vasconcelos and Villarreal in [14], the conjecture is true for bipartite graphs as $I(G)^{(s)} = I(G)^s$ for all $s \geq 1$. Thus it is interesting to study the conjecture for non-bipartite graphs. In this direction in [8], Gu et al. have proved the conjecture for odd cycles. Recently many researchers are working in this direction and in [12], Jayanthan and Kumar have proved the conjecture for certain class of unicyclic graph, in [3] we proved it for complete graph and in [6], Seyed Fakhari has solved the conjecture for unicyclic graph and chordal graph respectively. In [4], DiPasquale et al. have studied resurgence and asymptotic resurgence.

In this paper, we investigate the invariants for the class of graph $G_{n,r}$ described in the abstract. One description of such graph is given in section 2 in figure 1. We see that for different values of $r$, $G_{n,r}$ covers a large class of graphs. For example if $r = 0$ and $r = 1$, $G_{n,r}$ gives the class of complete graphs and anticycles respectively. If $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$, then $G_{n,r}$ become bipartite graph or odd cycles depending on $n$ is even or odd. Here we describe a unified way to compare the regularity of the symbolic powers and ordinary powers of edge ideals of $G_{n,r}$ for different values of $n$ and $r$. Instead of studying separately for the individual classes of graphs we show that Minh’s conjecture is true for the whole class of $G_{n,r}$. In section 2, we define the terminologies and recall the basic results to be used in the rest of the paper. In section 3, we describe the condition when $G_{n,r}$ become unmixed and in Lemma 3.7 we show that for $r = 1, 2$, $G_{n,r}$ is always unmixed. In proposition 3.9 we show that for $r = 1$ and 2, $G_{n,r}$ is gap-free graph. In section 4, we give an explicit description of the elements of the symbolic powers of edge ideals of $G_{n,r}$ when it is unmixed. Using the local cohomology technique in Theorem 4.4 we show that Minh’s conjecture is true for the graph $G_{n,r}$ for all values of $n$ and $r$. In section 5, we compute the invariants like Waldschmidt constant and resurgence for the edge ideal of $G_{n,r}$.

2. Preliminaries

In this section, we collect the notations and terminologies used in the rest of the paper.

Definition 2.1. Let $V' \subseteq V(G) = \{x_1, x_2, \ldots, x_n\}$ be a set of vertices. For a monomial $x^a \in k[x_1, \ldots, x_n]$ with exponent vector $a = (a_1, a_2, \ldots, a_n)$ define the
vertex weight $W_{V'}(x^2)$ to be

$$W_{V'}(x^2):= \sum_{x_i \in V'} a_i.$$ 

**Definition 2.2.** The vertex independence number of a graph is the cardinality of the largest independent vertex set and it is denoted by $\alpha(G)$. Formally,

$$\alpha(G) = \max \{ |U| : U \subseteq V(G) \text{ is independent set} \}.$$ 

Now we recall some results which describe the symbolic powers of the edge ideal in terms of minimal vertex covers of the graph.

**Lemma 2.3.** [15, Corollary 3.35] Let $G$ be a graph on vertices $\{x_1, \ldots, x_n\}$, $I = I(G) \subseteq k[x_1, \ldots, x_n]$ be the edge ideal of $G$ and $V_1, \ldots, V_r$ be the minimal vertex covers of $G$. Let $P_j$ be the monomial prime ideal generated by the variables in $V_j$. Then

$$I = P_1 \cap \cdots \cap P_r$$

and

$$I^{(m)} = P_1^m \cap \cdots \cap P_r^m.$$ 

**Lemma 2.4.** [1, Lemma 2.6] Let $I \subseteq R$ be a square free monomial ideal with minimal primary decomposition $I = P_1 \cap \cdots \cap P_r$ with $P_j = (x_{j_1}, \ldots, x_{j_{s_j}})$ for $j = 1, \ldots, r$. Then $x_1^{a_1} \cdots x_n^{a_n} \in I^{(m)}$ if and only if $a_{j_1} + \cdots + a_{j_{s_j}} \geq m$ for $j = 1, \ldots, r$.

Using Lemma 2.4 and the concept of vertex weight Janssen et al. in [11] described the elements of symbolic powers of edge ideals as follows

$$I^{(t)} = \{ x^2 \mid \text{for all minimal vertex covers } V', W_{V'}(x^2) \geq t \}.$$ 

Further they have divided the elements of the symbolic powers of edge ideals into two sets written as $I^{(t)} = (L(t)) + (D(t))$, where

$$L(t) = \{ x^2 \mid \deg(x^2) \geq 2t \text{ and for all minimal vertex covers } V', W_{V'}(x^2) \geq t \}$$

and

$$D(t) = \{ x^2 \mid \deg(x^2) < 2t \text{ and for all minimal vertex covers } V', W_{V'}(x^2) \geq t \}.$$ 

Thus for any graph, if we are able to identify the elements in $L(t)$ and $D(t)$ then we will be able to describe $I^{(t)}$.

**Definition 2.5.** Let $x^2 \in k[x_1, \ldots, x_n]$ be a monomial and $G$ be a finite simple connected graph on the set of vertices $\{x_1, \ldots, x_n\}$. Let $\{e_1, e_2, \ldots, e_r\}$ denote the set of edges in the graph. We may write $x^2 = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} e_1^{b_1} e_2^{b_2} \cdots e_r^{b_r}$, where $b(x^2) = \sum b_j$ is as large as possible. When $x^2$ is written in this way, we call this an optimal form of $x^2$ or we say that $x^2$ is expressed in optimal form, or simply
$x^2$ is in optimal form. In addition, each $x_i^{a_i}$ with $a_i > 0$ in this form is called an ancillary factor of the optimal form, or just ancillary for short and $x_i$'s are called ancillary vertices.

Note that optimal form of a monomial need not be unique but $b(x^2)$ is unique. Let us give an example to understand the definition of optimal form.

**Example 2.6.** Let $I = I(C_5)$ be the edge ideal on the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$, i.e, $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle$. Consider the monomial $x^2 = x_1^3x_2^4x_3x_4x_5^3$. Note that we can write $x^2 = x_2^2(x_1x_2)^3(x_2x_3)(x_4x_5)$ and also observe that $b(x^2)$ is maximum in this factorization, so this is an optimal form of the monomial.

3. Description of the Graph

In this section, we try to analyze the properties of the graph $G_{n,r}$ for different values of $n$ and $r$. We are considering the vertex set $V = \{x_1, \ldots, x_n\}$ in cyclic order to ensure that the vertices $x_1, x_n$ are consecutive. The following figure describe one such graph when $n = 7, r = 2$.

![Figure 1](image)

**Figure 1.** Example with $n = 7$ and $r = 2$

From the description of the graph $G_{n,r}$, it is clear that any set of consecutive $n - (r + 1)$ vertices is always a minimal vertex cover. So we might think that $G_{n,r}$ is always unmixed but not all the graph of $G_{n,r}$ are unmixed. Now we characterize the graphs.

**Remark 3.1.** Note that for $n \leq 3$ there are only two graphs $G_{2,0}$ and $G_{3,0}$, one is just an edge and another one is a triangle. So we can assume $n \geq 4$.

**Observation 3.2.** There will be no edge between two vertices $\{x_{i_j}, x_{i_i}\}$ (where $i_j < i_i$) in the graph $G_{n,r}$ if and only if one of the following conditions hold

1. $i_i - i_j < r + 1$
2. $i_i - i_j > n - (r + 1)$.

**Observation 3.3.** The set of vertices $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is an independent set if and only if there exist a solution for the following equations

$$i_p - i_q < r + 1 \text{ or } i_p - i_q > n - (r + 1) \text{ for } p, q \in \{i_1, i_2, \ldots, i_k\}, \text{ with } q < p.$$ (1)
Lemma 3.4. The cardinality of a maximal independent set in $G_{n,r}$ is at most $r+1$.

Proof. Observe that any set of consecutive $r+1$ vertices always give a maximal independent set. Let us assume that $V' = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is an independent set containing at least two non-consecutive vertices. We will show that $k \leq r$. Let $V_{ij}$ denotes set of vertices which are not connected with $x_{i_j}$. We choose the vertices $x_{i_1}$ and $x_{i_2}$ such a way that they are not connected. Then we have to choose $x_{ij} (i_j \geq 3)$ from $\bigcap_{s=1}^{j-1} V_{is}$. So in this way after choosing $k$ vertices, we show that $\bigcap_{j=1}^{k} V_{ij}$ will be empty set. Note that $x_{i_1}$ is connected with $n-(2r+1)$ vertices. Since at least two vertices in $V'$ are not consecutive so remaining $k-1$ vertices are connected with at least other $k$ vertices apart from $n-(2r+1)$ vertices which are connected with $x_{i_1}$. Therefore after choosing $k$ independent vertices, the number vertices remain to choose is at most $n - \{(n-2r-1) + k + (k-1) + 1\} = 2r+1-2k$, which implies that the maximum value of $k$ is $r$.

Note that the size of a maximal independent set containing at least two non-consecutive vertices is at most $r$ and the cardinality of a maximal independent set is $r+1$ if and only if all the vertices in that set are consecutive. So we can conclude the following corollary and remark.

Corollary 3.5. Maximal independent sets containing at least two non-consecutive vertices exist if and only if the graph $G_{n,r}$ is not unmixed.

Remark 3.6. The independence number of the graph $G_{n,r}$ is $r+1$.

In the next Lemma we give a class of unmixed graphs arising from $G_{n,r}$.

Lemma 3.7. The graph $G_{n,r}$ is unmixed for $r = 1, 2$.

Proof. The proof is clear for $r = 1$. Now consider the case for $r = 2$. If the graph is not unmixed then there exists a maximal independent set containing two non-consecutive vertices. Suppose $\{x_{i_1}, x_{i_2}\}$ is an independent set and the vertices are not consecutive, then $i_2 - i_1 < 3$ and $i_2 - i_1 > 1$ which implies $i_2 = i_1 + 2$. But then the set is not maximal independent set as the set $\{x_{i_1}, x_{i_1+1}, x_{i_2}\}$ is independent. Therefore the graph is unmixed.

The next examples shows that $G_{n,r}$ is not always unmixed.

Example 3.8. Consider the graph corresponding to $n = 11, r = 4$. Note that the set $\{x_3, x_6, x_7, x_{10}\}$ is a solution of $[1]$, therefore the set is an independent set and also it is maximal independent set. But the graph is not unmixed as $\{x_3, x_4, x_5, x_6, x_7\}$ is also an maximal independent set.
Lemma 3.9. The graph $G_{n,r}$ is gap-free for $r = 1, n \geq 5$ and $r = 2, n \geq 9$.

Proof. Let $x_i x_j$ and $x_k x_l$ are edges where $i < j$ and $k < l$. We can assume that $i < l$. We will show that if $x_i x_k, x_i x_l, x_j x_k$ are not edges then $x_j x_l$ is an edge. Without loss of generality, we can assume that $i = 1$.

Case 1: First assume that $j < l$. Then there are two possibilities, one is $1 < k < j < l$ and the other is $1 < j < k < l$.

Subcase 1: Let us assume that $1 < k < j < l$. As $l \leq n$ and $j \geq r + 2$ (since $x_1 x_j$ is an edge), so we have

$$l - j \leq n - (r + 2) \leq n - (r + 1).$$

(2)

Now $x_1 x_j$ is edge and $x_1 x_k$ not an edge imply that $k \leq r + 1$. Also $x_1 x_j$ edge and $x_1 x_l$ not edge imply that

$$l - 1 > n - (r + 1) \text{ i.e. } l \geq n - r + 1.$$  

(3)

Since $x_j x_k$ is not an edge, we have either $j - k < r + 1$ or $j - k > n - (r + 1)$. But note that $j - k > n - (r + 1)$ is not possible for $r = 1, 2$. Suppose possible then $j - k > n - 3$, which implies that $j \geq k + n - 2$, also $k \geq 2$, hence $j \geq n$ which contradicts the fact that $j < l \leq n$. So we need to prove only for $j - k < r + 1$. Then $j \leq k + r \leq 2r + 1$, which together with (3) give $l - j \geq n - 3r$. Observe that for $r = 1, n \geq 5$ and $r = 2, n \geq 9$ the inequality $l - j \geq r + 1$ satisfied, hence by (2) $x_j x_l$ is an edge.

Subcase 2: Now consider the case $1 < j < k < l$ and also $x_1 x_j$, $x_k x_l$ are edges. Assume that $x_1 x_k$ is not an edge then $k \geq n - r + 1$, which implies that $l - k \leq n - (n - r + 1) = r - 1$. Hence we arrived at a contradiction that $x_k x_l$ is an edge. Then $x_1 x_k$ is an edge.

Case 2: If $l < j$, then $1 < k < l < j$. Since $r \leq 2$, if $x_1 x_l$ is not edge then $k$ has to be 2 and $l$ has to be 3, but this contradicts that $x_k x_l$ is an edge. Thus $x_1 x_l$ is an edge.

□

4. Regularity comparison

The main aim of this section is to compare the regularities of the ordinary powers and symbolic powers of $I = I(G_{n,r})$. Using the local cohomology technique we show that Minh’s Conjecture is true for the graph $G_{n,r}$.

Theorem 4.1. Let $I = I(G_{n,r})$ be the edge ideal for the graph $G_{n,r}$, then $I^t = (L(t))$.

Proof. Observe that any set of consecutive $n - (r + 1)$ vertices forms a minimal vertex cover for $G_{n,r}$. As the relation $I^t \subset (L(t))$ always holds, we need to show...
that \((L(t)) \in I^t\). We divide the proof into 3 cases depending on the number of ancillaries present in the optimal form of the monomial. Observe that there can be at most \((r + 1)\) ancillaries in the optimal form of a monomial.

**Case 1:** Let \(x^a \in (L(t))\). Suppose there is no ancillary or only one ancillary with degree 1 is present, then we can write \(x^a = x_{i_1}^{c_1}p(x)\), where \(c_i \leq 1\) and \(\deg(p(x)) \geq 2t - 1\). Note that \(p(x)\) can be factored into edges only, which implies that \(x^a \in I^t\) and hence in this case \((L(t)) \in I^t\).

We prove case 2 and 3 by contrapositive. Let \(x^a = x_1^{c_1}\cdots x_n^{c_n} \notin I^t\) which implies that \(b(x^a) < t\), now if we can find a minimal vertex cover \(V'\) such that \(W_{V'}(x^a) \leq b(x^a)\), then \(x^a \notin (L(t))\).

**Case 2:** Assume that \(i\) ancillaries are present in the optimal form, where \(2 \leq i \leq r\). Let \(x_1^{c_1}, \cdots, x_i^{c_i}\) are the ancillaries. Consider the following set of minimal vertex covers \(L = \{\{x_{i+1}, \cdots, x_{n-r+i-1}\}, \cdots, \{x_{r+2}, \cdots, x_n\}\}\). The aim here is to find a minimal vertex cover \(V'\) from the set \(L\) such that \(W_{V'}(x^a) \leq b(x^a)\). Our claim is that we can find a minimal vertex cover \(V'\) from the set \(L\) such that there will be no edge between the vertices of \(V'\) in the optimal form of the monomial and if so then \(V'\) will be the suitable candidate for minimal vertex cover. Thus it is enough to prove that if at least one edge is present from each vertex set of \(L\) then the monomial will not be optimal form. Without loss of generality, we can write the monomial in the following way \(x^a = x_1^{c_1}x_2^{c_2}\cdots x_i^{c_i}(x_{k(i+1)}^{(i+1)})x_{k(i+1)}x_{k(i+2)}\cdots x_{k(r+2)}x_{k(r+2)}\) and assume that it is the optimal form, where \(x_1^{c_1}, x_2^{c_2}, \cdots, x_i^{c_i}\) are the ancillaries and \((x_{k(i+1)}^{(i+1)})x_{k(i+1)}x_{k(i+2)}\) are the edges between the vertices of \(\{x_{i+1}, \cdots, x_{n-r+i-1}\}, \cdots, \{x_{r+2}, \cdots, x_n\}\) respectively. If two edges belong to the same vertex set of \(L\) then we will keep their positions according to the sum of their indices. For example if \((x_{k(i+1)}x_{k(i+2)})_{(i+1)}, (x_{k(i+2)}x_{k(i+2)})_{(i+2)}\) are the edges from the vertex set \(\{x_{i+1}, \cdots, x_{n-r+i-1}\}\) and suppose that \(k_{(i+1)} + k_{(i+1)} + k_{(i+2)} + k_{(i+2)}\) then we keep the edge \((x_{k(i+1)}x_{k(i+1)})\) before \((x_{k(i+2)}x_{k(i+2)})\). So by rearranging we can assume that if they belong to same the vertex set then \(k_{(i+1)} + k_{(i+1)} + k_{(i+2)} + k_{(i+2)}\). From Observation 3.2 it follows that \(x_ix_j(i < j)\) is an edge if and only if \(r + 1 \leq j - i \leq n - (r + 1)\). Therefore the following equations be satisfied
\[ i + 1 \leq k_{(i+1)1} < k_{(i+1)2} \leq n - r + i - 1 \text{ and } r + 1 \leq k_{(i+1)2} - k_{(i+1)1} \leq n - (r + 1) \]
\[ i + 2 \leq k_{(i+2)1} < k_{(i+2)2} \leq n - r + i \text{ and } r + 1 \leq k_{(i+2)2} - k_{(i+2)1} \leq n - (r + 1) \]  
\[ \vdots \]
\[ r + 2 \leq k_{(r+2)1} < k_{(r+2)2} \leq n \text{ and } r + 1 \leq k_{(r+2)2} - k_{(r+2)1} \leq n - (r + 1). \]

Now
\[ x^2 = x_1^2 x_2^2 \cdots x_i^2 \big(x_{k(i+1)1}, x_{k(i+1)2}\big)(x_{k(i+2)1}, x_{k(i+2)2})\cdots(x_{k(r+2)1}, x_{k(r+2)2}) \]
\[ = x_1^{c_1-1} x_2^{c_2-1} \cdots x_i^{c_i-1} x_i (x_{k(i+1)1}, x_{k(i+1)2})(x_{k(i+2)1}, x_{k(i+2)2})\cdots(x_{k(r+2)1}, x_{k(r+2)2}) \]
\[ = x_1^{c_1-1} x_2^{c_2-1} \cdots x_i^{c_i-1} x_i (x_{k(i+1)1}, x_{k(i+1)2})(x_{k(i+1)1}, x_{k(i+1)2})\cdots(x_{k(r+1)1}, x_{k(r+2)2}). \]

It is enough to show that all the pairs of the last expression of \( x^2 \) are edges. From the equations (4), it follows that \( r + 1 \leq k_{(r+2)2} - 1 \) and \( k_{(r+1)2} - 1 \leq k_{(r+2)2} - (r + 1) \leq n - (r + 1). \) Hence \( r + 1 \leq k_{(r+2),1} - 1 \leq n - (r + 1). \) Therefore \( x_1 x_{k_{(r+2)1}} \) is an edge. Again from the equations (4), we can write \( k_{(i+1)2} \geq k_{(i+1)1} + (r + 1) \geq (i + 1) + (r + 1) \), which implies that
\[ k_{(i+1)2} - i \geq r + 1. \]  

Also note that \( k_{(i+1)2} - i \leq n - (r + 1). \) Therefore \( r + 1 \leq k_{(i+1)2} - i \leq n - r + 1, \) hence \( x_i x_{k_{(i+1)2}} \) is an edge. Next we show that \( x_{k_{(i+1)1}}, x_{k_{(i+2)2}} \) is an edge. First assume that
\[ (x_{k_{(i+1)1}}, x_{k_{(i+1)2}}), \]
\[ (x_{k_{(i+2)1}}, x_{k_{(i+2)2}}) \] are the edges from the vertex set \( \{x_{i+1}, \ldots, x_{n-r+i-1}\} \) then \( k_{(i+1)1} + k_{(i+1)2} \leq k_{(i+2)1} = k_{(i+2)2} \) hold which implies that \( k_{(i+2)1} - k_{(i+1)1} \geq k_{(i+2)2} - k_{(i+1)2} \geq r + 1 \). Also observe that the maximum difference between the indices from the vertex set of \( L \) is \( n - r - 2 \) hence \( (x_{k_{(i+1)1}}, x_{k_{(i+2)2}}) \) is an edge. The remaining case is that if they do not belong to the same vertex set, in that case \( k_{(i+2)2} = n - r + i \), then it is obvious that \( (x_{k_{(i+1)1}}, x_{k_{(i+2)2}}) \) is an edge as \( k_{(i+2)2} - k_{(i+1)1} \leq n - (r + 1). \) In the same procedure, we can show that other pairs are also edges. Then \( x^2 \) is not in optimal form.

If only one ancillary is present, then \( i = 1 \). Assume that \( x_1^{c_1} \) is the ancillary with \( c_1 \geq 2 \). Then take \( x_i = x_1 \) and follow the process described above. In this case we have to show only that \( x_1 x_{k_{(i+1)2}} \) is an edge. From equation (5) it follows that \( k_{(i+1)2} - 1 \geq r + 1 \). By putting \( i = 1 \) in equations (4) we get that \( k_{(i+2)2} \leq n - r, \) i.e. \( k_{(i+1)2} - 1 \leq n - (r + 1) \). Therefore \( x_1 x_{k_{(i+1)2}} \) is an edge. Hence we can find a minimal vertex cover \( V' \) from the set \( L \) such that no edges of \( V' \) will be present in the optimal form of \( x^2 \), so \( W_{V'}(x^2) \leq b(x^2) < t. \)
Case 3: Let us assume that \((r + 1)\) ancillaries are present in the optimal form and \(x_1^{c_1}, \ldots, x_{r+1}^{c_{r+1}}\) are the ancillaries. Then we can write the optimal form of the monomial in the following way, \(x^{d_2} = x_1^{c_1} \cdots x_{r+1}^{c_{r+1}} e_1 e_2 \cdots e_{n-r} e_{2(n-r+1)} e_{(r+1)(2r+2)} e_{(r+1)n}\).

Consider the vertex set \(V' = \{x_{r+2}, \ldots, x_n\}\). Clearly \(V'\) is a minimal vertex cover.

Observe that in the optimal form of \(x^{d_2}\) there is no edge between the vertices of \(V'\). Thus \(W_{V'}(x^{d_2}) = a_{r+2} + \cdots + a_n\) and \(b(x^{d_2}) = b_1(b_{r+2} + \cdots + b_1 + b_{2r+3} + \cdots + b_{2(n-r+1)} + \cdots + b_{3r+4}) = a_{r+4}, \ldots, b_1(b_{n-r} + b_{2(n-r)} + \cdots + b_{(n-2r-1)(n-r)} = a_{n-r}, \ldots, b_{(1+1)n} = a_n\). Therefore \(W_{V'}(x^{d_2}) = b(x^{d_2}) < t\).

\[\square\]

**Lemma 4.2.** Let \(H_{n,r}\) be the class of unmixed graphs of the form \(G_{n,r}\) and let \(I = I(G)\) be the edge ideal of some graph in \(H_{n,r}\). Then we have
\[D(t) = \{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} | a_{i_1} + a_{i_2} + \cdots + a_{i_{n-(r+1)}} \geq t \text{ for consecutive } n - (r + 1) \text{ tuple with } \{i_1, i_2, \ldots, i_{n-(r+1)}\} \subseteq \{1, 2, \ldots, n\} \text{ and } a_1 + \cdots + a_n \leq 2t - 1\}\]

**Proof.** The set of all consecutive \(n - (r + 1)\) vertices forms the set of minimal vertex cover for \(H_{n,r}\). Let \(x^{d_2} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in D(t)\) then \(a_1 + \cdots + a_n \leq 2t - 1\) and since \(x^{d_2} \in I^{(t)}\), so for any minimal vertex cover \(V\), \(W_{V'}(x^{d_2}) \geq t\) which implies that \(a_{i_1} + a_{i_2} + \cdots + a_{i_{n-(r+1)}} \geq t\) for all consecutive \(n - (r + 1)\) tuple with \(\{i_1, i_2, \ldots, i_{n-(r+1)}\} \subseteq \{1, 2, \ldots, n\}\). Hence the result follows. \(\square\)

**Lemma 4.3.** Let \(I = I(G_{n,r})\) be the edge ideal of a graph \(G_{n,r}\), and let \(m\) be the homogeneous maximal ideal of \(R\). Then \(m^t I^{(t)} \subseteq I^t\).

**Proof.** Let \(x^{d_2} \in m^t\) and \(x^{d_2} \in I^{(t)}\), then \(\deg(x^{d_2}) \geq t\) as well as \(\deg(x^{d_2}) \geq t\). Therefore \(\deg(x^{d_2 + b}) \geq 2t\), which implies that \(x^{d_2 + b} \in (L(t))\), hence by Theorem 4.1 \(x^{d_2 + b} \in I^t\). Thus the result follows. \(\square\)

**Theorem 4.4.** Let \(I = I(G_{n,r})\) be the edge ideal for the graph \(G_{n,r}\). Then for all \(t \geq 1\) we have
\[\text{reg} I^{(t)} = \text{reg} I^t\]

**Proof.** From Lemma 4.3 it follows that \(\dim I^{(t)}/I^t = 0\), therefore \(H^i_m(I^{(t)}/I^t) = 0\) for \(i > 0\). Now consider the following short exact sequence
\[0 \to I^{(t)}/I^t \to R/I^t \to R/I^{(t)} \to 0.\]

Applying local cohomology functor we get \(H^i_m(R/I^{(t)}) \cong H^i_m(R/I^t)\) for \(i \geq 1\) and the following short exact sequence
\[0 \to H^0_m(I^{(t)}/I^t) \to H^0_m(R/I^t) \to H^0_m(R/I^{(t)}) \to 0.\]
From (7), it follows that \( a_0(R/I(t)) \leq a_0(R/I^t) \). So we can conclude that \( \text{reg } R/I(t) = \max\{a_i(R/I(t)) \}
+ i \mid i \geq 0\} \leq \max\{a_i(R/I^t) + i \mid i \geq 0\} = \text{reg}(R/I^t) \). Therefore \( \text{reg } I(t) \leq \text{reg } I^t \). Now we prove the other inequality.

Since \( \dim(I(t)/I^t) = 0 \), it follows that \( \text{reg } I(t)/I^t = a_0(I(t)/I^t) \). If a monomial \( x_2 \in I(t) \) with \( \deg(x_2) \geq 2t \), then \( x_2 \in (L(t)) \) and hence by Theorem 4.1, it follows that \( x_2 \in I^t \). Thus \( a_0(I(t)/I^t) \leq 2t - 1 \), which implies \( \text{reg}(I(t)/I^t) \leq 2t - 1 \). Since \( I^t \subseteq I(t) \), we have \( 2t \leq \text{reg } I(t) \). Therefore \( \text{reg } I(t)/I^t \leq \text{reg } R/I(t) \). Now by applying \[5, Corollary 20.19\] to \[6\] we get \( \text{reg } R/I^t \leq \max\{\text{reg } I(t)/I^t, \text{reg } R/I(t)\} \), which implies \( \text{reg } I^t \leq \text{reg } I(t) \) and hence \( \text{reg } I^t = \text{reg } I(t) \).

\[\Box\]

**Remark 4.5.** If \( r = \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and \( n \) is odd then \( G_{n,r} \) gives the class of odd cycles which has been studied by Gu et al. in \[8\]. If \( r = 0 \) then \( G_{n,r} \) gives the class of complete graph which we have studied in \[3\]. Both of the above results follow as a special case of our result.

### 5. Invariants of Edge Ideals

In this section, we calculate Waldschmidt constant and resurgence for the edge ideal \( I = I(G_{n,r}) \) of the whole class.

**Definition 5.1.** An automorphism of a graph \( G \) is a bijection \( \pi : V(G) \to V(G) \) with the property that \( e \) is an edge of \( G \) if and only if \( \pi(e) \) is an edge as well.

**Definition 5.2.** A graph \( G \) is vertex-transitive if for all \( u, v \in V(G) \) there is an automorphism \( \pi \) of \( G \) with \( \pi(u) = v \).

**Lemma 5.3.** The graph \( G_{n,r} \) is vertex-transitive graph.

**Proof.** For every vertex \( x_i, x_j \in V(G) \) we have to produce an automorphism \( \pi \) of \( G \) such that \( \pi(x_i) = x_j \). Let \( k \equiv j - i + k \mod n \), where \( x_0 = x_n \). Now consider the map \( \pi(x_k) = x_k \). Observe that it is an automorphism. \[\Box\]

**Proposition 5.4.** Let \( I = I(G_{n,r}) \) be the edge ideal of a graph \( G_{n,r} \). Then \( \alpha(I(t)) < \alpha(I^t) \) if and only if \( I(t) \notin I^t \).

**Proof.** By Theorem 4.1, we have \( (L(t)) = I^t \). Then the proof follows similarly from \[11, Lemma 5.5\]. \[\Box\]

Let us recall some notions used in section 5 of the article \[11\]. Note that the graph \( G_{n,r} \) contains unmixed graphs as well as non-unmixed graphs. For unmixed graphs, there are only \( n \) minimal vertex covers and they are precisely consecutive \( n - (r + 1) \)
vertices. But for non-unmixed graph there are some extra minimal vertex covers and they contain more than \( n - (r + 1) \) vertices. Now fix the minimal vertex covers \( V_1, \ldots, V_k \) and order them so that \(|V_1| \leq |V_{i+1}|\). We define the minimal vertex cover matrix \( A = (a_{ij}) \) to be the matrix of 0’s and 1’s defined by:

\[
a_{ij} = \begin{cases} 
0 & \text{if } x_j \notin V_i \\
1 & \text{if } x_j \in V_i
\end{cases}
\]

We first seek a lower bound of \( \alpha(I^{(t)}) \) using linear programming. Consider the following linear program, where \( A \) is the minimal vertex cover matrix,

\[
\begin{align*}
\min & \quad b^T y \\
\text{subject to} & \quad Ay \geq c \\
& \quad y \geq 0
\end{align*}
\]  

(8)

Observe that if \( y^* \) is the value which realizes (8), we have \( \alpha(I^{(t)}) \geq b^T y^* \).

Now consider the submatrix \( A' \) of \( A \) consisting of first \( n \) rows of \( A \), then it is a \( n \times n \) matrix. Thus we create a sub-program of (8).

\[
\begin{align*}
\min & \quad b^T y \\
\text{subject to} & \quad A'y \geq c \\
& \quad y \geq 0
\end{align*}
\]  

(9)

**Lemma 5.5.** Let \( I = I(G_{n,r}) \) be the edge ideal of the graph \( G_{n,r} \), then \( \alpha(I^{(t)}) \geq \left\lfloor \frac{nt}{n-(r+1)} \right\rfloor \).

**Proof.** It is easy to check that

\[
y^* = \begin{bmatrix} 
t \\
\vdots \\
\frac{t}{n-(r+1)}
\end{bmatrix}
\]

is a feasible solution of (9). In this case \( b^T y^* = \frac{nt}{n-(r+1)} \). To show that this is the value of (9), we make use of the fundamental theorem of linear programming by showing the existence of an \( x^* \) which produces the same value for the dual linear program:

\[
\begin{align*}
\max & \quad c^T x \\
\text{subject to} & \quad A'^T x \leq b \\
& \quad x \geq 0
\end{align*}
\]  

(10)
Let
\[
x^* = \begin{bmatrix}
\frac{1}{n-(r+1)} \\
\vdots \\
\frac{1}{n-(r+1)}
\end{bmatrix}.
\]

It is straightforward to check that \(x^*\) is a feasible solution of the dual problem and
\[
c^T x^* = b^T y^* = \frac{nt}{n-(r+1)}.
\]

Observe that (8) is obtained from (9) by introducing additional constraints and also \(\alpha(I(t))\) is an integer. Hence the result follows.

In order to compute Waldschmidt constant we now introduce the fractional chromatic number.

**Definition 5.6.** [13] A \(b\)-fold coloring of a graph \(G\) assigns to each vertex of \(G\) a set of \(b\) colors so that adjacent vertices receive disjoint sets of colors. We say that \(G\) is \(a : b\)-colorable if it has a \(b\)-fold coloring in which the colors are drawn from a palette of \(a\) colors. We sometimes refer to such a coloring as an \(a : b\)-coloring. The least \(a\) for which \(G\) has a \(b\)-fold coloring is the \(b\)-fold chromatic number of \(G\), denoted \(\chi_b(G)\).

The fractional chromatic number is defined as
\[
\chi^*(G) = \lim_{b \to \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.
\]

**Theorem 5.7.** Let \(I = I(G_{n,r})\) be the edge ideal of a graph \(G_{n,r}\) then the Waldschmidt constant \(\hat{\alpha}(I) = \frac{n}{n-(r+1)}\).

**Proof.** By [13, Proposition 3.1.1], it follows that the fractional chromatic number \(\chi^*(G_{n,r}) = \frac{n}{n-(r+1)}\), as independence number is \(r+1\). Thus by [1, Theorem 4.6] we have \(\hat{\alpha}(I)) = \frac{n}{n-(r+1)}\).

**Theorem 5.8.** Let \(I = I(G_{n,r})\) be the edge ideal then \(\rho(I) = \frac{2(n-(r+1))}{n}\).

**Proof.** Consider the set \(T = \{ s \mid I(s) \notin I(t) \}\). For any \(s \in T\) by Proposition 5.4, we have \(\alpha(I^{(s)}) < \alpha(I^{(t)})\). Then by applying Lemma 5.5, we can write \(n s < \frac{2(n-(r+1))}{n}\), which implies \(s < \frac{2(n-(r+1))}{n}\). Hence \(\rho(I) \leq \frac{2(n-(r+1))}{n}\). As \(\frac{\alpha(I)}{\hat{\alpha}(I)} = \frac{2(n-(r+1))}{n}\), by [9, Theorem 1.2], we have \(\alpha(I)/\hat{\alpha}(I) \leq \rho(I)\), therefore \(\frac{2(n-(r+1))}{n} \leq \rho(I)\) and hence \(\rho(I) = \frac{2(n-(r+1))}{n}\).

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