RELATIVISTIC TODA CHAIN AT ROOT OF UNITY

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Abstract. We declare briefly several interesting features of the quantum relativistic Toda chain at N-th root of unity. We consider the finite dimensional representation of the Weyl algebra. The origin of the features mentioned is that we consider simultaneously the quantum finite dimensional part and the classical dynamics of N-th powers of Weyl’s elements. As the main result, using the technique of Q-operators, we establish a correspondence between the separation of variables in the quantum model and the Bäcklund transformations of its classical counterpart.

Introduction

The quantum relativistic Toda chain was the subject of paper [1]. There it was considered for Weyl’s parameter \( q \) inside the unit circle. In this paper we collect several results concerning this model with \( q = \omega \) – N-th primitive root of unity, so that it is possible to realize Weyl’s elements as unitary \( N \times N \) matrices. The quantum state space of the model becomes the finite dimensional one. The relativistic Toda chain at the root of unity is a relative of, say, chiral Potts model \([2,3]\) etc., being the most simple model in the hierarchy of integrable models, associated with the cyclic representations of the affine Lie algebras.

In this notes we collect several interesting features of the relativistic Toda chain at the root of unity. The main result is the proof of the possibility to construct explicitly the eigenvectors of the off-diagonal element \( b(\lambda) \) of the monodromy matrix, that is the starting point of the separation of the variables method \([4,5,6,7]\). The main object in this construction is a kind of Baxter’s Q-operator \([10,3]\). The method we use resembles the idea of \([3]\) to get a projector to an eigenstate of \( b(\lambda) \) as a product of operators \( Q \) with special values of the spectral parameter. Actually we prove this hypothesis, but it appear that in the projector one has to take modified \( Q \)-operators which do not commute with the transfer matrix and do not form the commutative family.

1991 Mathematics Subject Classification. 82B23.

Key words and phrases. Toda chain, spin chain, integrable models.

This work was supported by the grants RFBR N 01-01-00201, INTAS OPEN 97-01312, RFBR-CNRS PICS N 608/RFB 98-01-22033 and RFBR N 00-01-00299.
Usually at the root of unity one tries to fix the primitive centers of a representation, working in the pure quantum space. Our method implies the consideration of the classical dynamics of these centers. This approach is natural when one formulates a model in the terms of mappings, and then makes the reduction \( q \mapsto \omega \). The results of this note is the application of more general scheme, derived for the three dimensional models, to the most simple spatial structure, see [8]. In the more convenient language, we construct a projector to an eigenstate of \( b(\lambda) \) in terms of modified quantum \( Q \)-operators, whose classical counterparts correspond to the Bäcklund transformations of the classical model [5].

Mainly we do not prove the propositions and do not fall into the details. The details are rather tedious, they are connected with a lot of extra definitions and parameterizations. The aim of this note is just to describe a general scheme of the investigations. The details are the subjects of the forthcoming papers.

1. Formulation of the model

1.1. Lax matrices. Define the quantum Lax matrix of \( m \)-th site of the quantum relativistic Toda chain as follows:

\[
\ell_m(\lambda) = \begin{pmatrix}
1 + \frac{\kappa}{\lambda} u_m w_m & -\frac{\omega^{1/2}}{\lambda} u_m \\
0 & w_m
\end{pmatrix}.
\]

(1)

Here \( \lambda \) is the spectral parameter, and \( \kappa \) is an extra complex parameter, common for all sites. Elements \( u_m \) and \( w_m \) form the ultra-local Weyl algebra

\[
u_m w_m = \omega w_m u_m,
\]

and \( u_m, w_m \) for different sites commute. Here Weyl’s parameter \( \omega \) is the primitive root of unity,

\[
\omega = e^{2\pi i/N}, \quad \omega^{1/2} = e^{i\pi/N}.
\]

(3)

This means, the \( N \)-th powers of the Weyl elements are the centers. We will deal with the finite dimensional unitary representation of the Weyl algebra, i.e.

\[
u = u \, x, \quad w = w \, z,
\]

where \( u \) and \( w \) are \( \mathbb{C} \)-numbers, and the popular representation of \( x \) and \( z \) is e.g.

\[
x |\alpha\rangle = \omega^{\alpha} |\alpha\rangle, \quad z |\alpha\rangle = |\alpha + 1\rangle, \quad \langle \alpha | \beta \rangle = \delta_{\alpha, \beta}.
\]

(5)
Thus \( x \) and \( z \) are normalized to the unity \( N \times N \) dimensional matrices, and the \( N \)-th powers of the local Weyl elements are \( \mathbb{C} \)-numbers

\[
\begin{align*}
&u^N_m = u^N_m \overset{\text{def}}{=} U_m, \quad w^N_m = w^N_m \overset{\text{def}}{=} W_m.
\end{align*}
\]

In general, all \( u_m \) and \( w_m \) are different, so we deal with the inhomogeneous chain. Index \( m \) means the number of Weyl algebra in the tensor product of \( M \) copies, and the state of this tensor product we denote as

\[
|\alpha_1 \rangle \otimes |\alpha_1 \rangle \otimes \cdots \otimes |\alpha_M \rangle \overset{\text{def}}{=} |\alpha_1, \ldots, \alpha_M \rangle \equiv |\vec{\alpha} \rangle.
\]

The variables \( U_m \) and \( W_m \) form the classical counterpart of the quantum relativistic Toda chain, and it is useful to consider the classical Lax matrix

\[
L_m \overset{\text{def}}{=} \left( \begin{array}{cc} 1 + \frac{\kappa}{\Lambda} U_m W_m & \frac{U_m}{\Lambda} \\
W_m & 0 \end{array} \right).
\]

Here in the spirit of (6) we have implied

\[
\kappa^N \overset{\text{def}}{=} \kappa, \quad \Lambda^N \overset{\text{def}}{=} \lambda.
\]

1.2. **Integrability.** The quantum Lax matrices are to be multiplied in the natural order,

\[
\hat{\mathfrak{t}}(\lambda) \overset{\text{def}}{=} \ell_1(\lambda) \ell_2(\lambda) \cdots \ell_M(\lambda) = \left( \begin{array}{cc} a(\lambda) & b(\lambda) \\
c(\lambda) & d(\lambda) \end{array} \right),
\]

and the quantum transfer matrix for the periodic chain of the length \( M \) is the trace of the monodromy matrix \( \hat{\mathfrak{t}}(\lambda) \):

\[
\mathfrak{t}(\lambda) = \text{tr} \hat{\mathfrak{t}}(\lambda) = \sum_{k=0}^{M} \lambda^{-k} \ell_k.
\]

In particular, \( t_0 = 1 \) and \( t_M = \kappa^M \prod_{m=1}^{M} u_m w_m \). The set of \( t_k \) is commutative, this is provided by the intertwining relation for the Lax operators,

\[
R(\lambda, \mu) \ell(\lambda) \otimes \ell(\mu) = (1 \otimes \ell(\mu)) (\ell(\lambda) \otimes 1) R(\lambda, \mu)
\]

where it is implied the tensor product of two \( 2 \times 2 \) matrices with the identical Weyl algebra entries \( u, w, \kappa \), different spectral parameters \( \lambda \).
and $\mu$, and
\begin{equation}
R(\lambda, \mu) = \begin{pmatrix}
\lambda - \omega \mu & 0 & 0 & 0 \\
0 & \lambda - \mu & \mu(1 - \omega) & 0 \\
0 & \lambda(1 - \omega) & \omega(\lambda - \mu) & 0 \\
0 & 0 & 0 & \lambda - \omega \mu \\
\end{pmatrix}
\end{equation}
is the almost usual six-vertex $R$-matrix.

We have formulated the inhomogeneous chain in general, the transfer matrix depends on $\lambda$, $\kappa$ and on set $\{u_m, w_m\}_{m=1}^M$: 
\begin{equation}
t(\lambda) = t(\lambda, \kappa; \{u_m, w_m\}_{m=1}^M) .
\end{equation}
From (12) it follows
\begin{equation}
\left[ t(\lambda, \kappa; \{u_m, w_m\} ), t(\mu, \kappa; \{u_m, w_m\} ) \right] = 0 .
\end{equation}
For the classical counterpart one may also define the monodromy matrix
\begin{equation}
\hat{T} = L_1 L_2 \cdots L_M = \begin{pmatrix}
A(\Lambda) & B(\Lambda) \\
C(\Lambda) & D(\Lambda) \\
\end{pmatrix},
\end{equation}
and its trace
\begin{equation}
T(\Lambda) = A(\Lambda) + D(\Lambda) = 1 + \sum_{k=1}^M \Lambda^{-k} T_M .
\end{equation}
Integrability of the classical model is provided by
\begin{equation}
\{ T(\Lambda), T(\Lambda') \} = 0 ,
\end{equation}
where $\{ , \}$ means the Poisson braces, defined by
\begin{equation}
\{U_n, W_m\} = \delta_{nm} U_n W_m .
\end{equation}
This Poisson bracket can be written in terms of $L$-operators (8)
\begin{equation}
\{L_n(\Lambda) \otimes L_m(\Lambda')\} = \delta_{nm}[r(\Lambda, \Lambda'), L_n(\Lambda) \otimes L_m(\Lambda')] 
\end{equation}
with classical trigonometric $r$-matrix
\begin{equation}
r(\Lambda, \Lambda') = \frac{1}{2(\Lambda - \Lambda')} \begin{pmatrix}
\Lambda + \Lambda' & 0 & 0 & 0 \\
0 & \Lambda - \Lambda' & 2\Lambda & 0 \\
0 & 2\Lambda' & \Lambda' - \Lambda & 0 \\
0 & 0 & 0 & \Lambda + \Lambda' \\
\end{pmatrix}
\end{equation}
Further in this letter we will never use the Hamilton formalism and the symbol $\{...,\}$ will be reserved for the notion of a set.
1.3. **Eigenvalues of** $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ **and** $d(\lambda)$. Admit now the following useful notation for the product over $\mathbb{Z}_N$:

\[
\prod_\lambda f(\lambda) \overset{\text{def}}{=} \prod_{n \in \mathbb{Z}_N} f(\omega^n \lambda) .
\]

The following proposition clarifies the relation between the quantum model and its classical counterpart:

**Proposition 1.**

\[
\prod_\lambda a(\lambda) = A(\Lambda) , \quad \prod_\lambda b(\lambda) = B(\Lambda) ,
\]

\[
\prod_\lambda c(\lambda) = C(\Lambda) , \quad \prod_\lambda d(\lambda) = D(\Lambda) .
\]

Thus the spectra of the quantum operators $a(\lambda) \ldots d(\lambda)$ can be expressed through the classical objects $A(\Lambda) \ldots D(\Lambda)$.

1.4. **Gauge fixing.** For the subsequent considerations we need to introduce a couple of notations and to fix some gauge degree of freedom. It is useful to consider following three operators for the whole spin chain:

\[
X = \prod_{m=1}^M x_m , \quad Z = \prod_{m=1}^M z_m , \quad Y = \prod_{m=1}^M (\omega^{-1/2} x_m z_m) .
\]

$X$ and $Z$ obey the following relations:

\[
t(\lambda) X = X t(\omega \lambda) , \quad t(\lambda) Z = Z t(\omega^{-1} \lambda) .
\]

One can see that the corresponding to $T_M$ flow results to the trivial re-scaling of the classical amplitudes $U_m$ and $W_m$: $U_m \mapsto CU_m , W_m \mapsto C^{-1}W_m$. Since $t_N = (-)^{M(N-1)}T_M$ we may always re-define $\lambda$ so that

\[
\prod_{m=1}^M \omega^{1/2} u_m w_m \equiv 1 , \quad \prod_{m=1}^M (-U_m W_m) = 1 .
\]

Also the gauge may be fixed in the most useful way. Define the homogeneous chain via ($\epsilon = (-)^N$)

\[
u_m = -\omega^{-1/2} , \quad w_m = -1 , \quad U_m = -\epsilon , \quad W_m = \frac{1}{\epsilon} ,
\]

whereas the inhomogeneous chain’s $u_m$ and $w_m$ we parameterize via sets of $\tau_m$ and $\theta_m$:

\[
u_m = -\omega^{-1/2} \frac{\tau_{m-1}}{\tau_m} , \quad w_m = -\frac{\theta_m}{\theta_{m-1}} , \quad m \in \mathbb{Z}_M .
\]
1.5. Spectral curves. Consider the spectral curve for the classical part of our model. By definition, and taking into account (26),

$$J(\Lambda, \mu) \overset{\text{def}}{=} -\mu \det(\mu^{-1} - \widehat{T}(\Lambda)) = T(\Lambda) - \frac{1}{\mu} - \frac{\mu}{\Lambda M}. \quad (29)$$

Condition $J = 0$ in terms of $\Lambda, \mu$ defines the genus $g_M = M - 1$ hyperelliptic classical spectral curve.

Our interest is the quantum model, and the quantum curve $\Gamma_M$ is to be defined as

$$(\lambda, \mu) \in \Gamma_M \Leftrightarrow J(\lambda^N, \mu^N) = 0. \quad (30)$$

Quantum curve is not hyperelliptic, and its genus in general position is $g_{N,M} = N^2M - 2N + 1$. The quantum curve will appear in Baxter’s equation.

1.6. Homogeneous chain. Now consider the homogeneous quantum and classical chains. The values of $u_m, u_m$ are given by (27). Due to proposition (1), the spectrum of $b(\lambda)$ is defined by the value of $B(\Lambda)$. We can calculate

$$B(\Lambda) = -\epsilon \frac{1}{\Lambda} \prod_{k=1}^{M-1} \left(1 - \frac{\Lambda(\phi_k)}{\Lambda} \right), \quad (31)$$

where

$$\phi_k \in \left\{ \frac{\pi}{M}, \frac{2\pi}{M}, \ldots, \frac{(M-1)\pi}{M} \right\}, \quad (32)$$

and the function $\Lambda(\phi)$ is defined by

$$\left\{ \begin{array}{l} 
\Delta(\phi) = e^{i\phi} \left( \sqrt{\cos^2 \phi + \kappa} + \cos \phi \right), \\
\Delta^*(\phi) = e^{-i\phi} \left( \sqrt{\cos^2 \phi + \kappa} + \cos \phi \right), \\
\Lambda(\phi) = \Delta(\phi) \Delta^*(\phi). 
\end{array} \right. \quad (33)$$

From (31) it follows that up to inessential $N$-dimensional block the spectrum of $b(\lambda)$ is described by the set of $M - 1$ phases of

$$\lambda_{\phi_k} : \lambda_{\phi_k}^N = \Lambda(\phi_k). \quad (34)$$

The reason for the above mentioned block which correspond to the spectrum of $x_M$ is the same as in the usual quantum Toda chain.

The curve $\Gamma_M$ now factorizes into $\Gamma_1^M$:}

$$J(\Lambda, \mu) = -\frac{1}{\mu \Lambda M} \prod_{\Delta^M = \mu} (\Delta^2 - \Delta(\Lambda - \kappa) + \Lambda), \quad (35)$$
where the product is taken over all $M$-th roots of unity. In particular, $J(\Lambda, \mu) = 0$ if there exists $\phi$: $\Lambda = \Lambda(\phi)$ and $\mu = \Delta(\phi)^M$.

2. ISOSPECTRALITY PROBLEM

This paper is devoted actually to a special class of inhomogeneous chains. In this section we will give some notations and formulate a proposition, which may be proved by the methods, described in the subsequent sections.

At first, define "solitonic tau - functions" recursively as

\[
\tau_m^{(0)} = 1, \\
\tau_m^{(1)} = 1 + f_1 e^{2i m \phi_1}, \\
\tau_m^{(2)} = 1 + f_1 e^{2i m \phi_1} + f_2 e^{2i m \phi_2} + d_{1,2} f_1 f_2 e^{2i m (\phi_1 + \phi_2)},
\]

and so on,

\[
\tau_m^{(n)} = \tau_m^{(n-1)}(\{f_k, \phi_k\}_{k=1}^{n-1}) + f_n e^{2i m \phi_n} \tau_m^{(n-1)}(\{d_{k,n} f_k, \phi_k\}_{k=1}^{n-1}),
\]

where all $\phi_k$ are different and belong to the set (32). The phase shift $d_{j,k} = d(\phi_j, \phi_k)$ is given by (see (33))

\[
d(\phi, \phi') = \frac{(\Delta(\phi) - \Delta(\phi')) (\Delta^*(\phi) - \Delta^*(\phi'))}{(\Delta(\phi) - \Delta^*(\phi')) (\Delta^*(\phi) - \Delta(\phi'))}.
\]

Complex number $f_k$ we will call the amplitude of $k$-th solitonic wave. Set (32) is finite, so the number of solitons can't exceed its maximal value $M - 1$. The complete $(M - 1)$-solitonic tau-function is

\[
\tau_m \overset{\text{def}}{=} \tau_{m}^{(M-1)} = \tau_m(\{f_k\}_{k=1}^{M-1}).
\]

Obviously, the less then $(M - 1)$-solitonic tau-functions are just particular cases of the complete one, which can be obtained by the vanishing the corresponding amplitudes.

Define a couple of useful functions:

\[
c(\phi) = \frac{\Delta^*(\phi)}{\Delta(\phi)} \frac{1 - \Delta(\phi)}{1 - \Delta^*(\phi)},
\]

and

\[
s(\phi, \phi') = \frac{\Delta^*(\phi)}{\Delta(\phi)} \frac{\Delta^*(\phi') - \Delta(\phi)}{\Delta^*(\phi') - \Delta^*(\phi)}.
\]

For given $\{f_k\}_{k=1}^{M-1}$ besides $\tau_m$ we need

\[
\Theta_m(\{f_k\}_{k=1}^{M-1}) \overset{\text{def}}{=} \tau_m(\{c_k f_k\}_{k=1}^{M-1}),
\]

where $c_k = c(\phi_k)$. 


Consider now the inhomogeneous relativistic Toda chain, whose parameters are given by (28) with \( \tau^N_m = \tau_m(\{f_k\}_{k=1}^{M-1}) \), \( \Theta^N_m = \Theta_m(\{f_k\}_{k=1}^{M-1}) \) with arbitrary set of \( \{f_k \in \mathbb{C}\}_{k=1}^{M-1} \). For the sake of shortness we shall use the symbolic notation \( s \), implying the definition of the solitonic data \( \{f_k\}_{k=1}^{M-1} \), construction of \( \tau_m, \Theta_m \) via eqs. (37,42), choice of the phases for \( \tau_m, \Theta_m \), and finally – the definition of \( u_m, w_m \), eq. (28):

\[ s: \left( \{f_k\}_{k=1}^{M-1} \mapsto \{\tau_m, \Theta_m\} \mapsto \{\tau_m, \Theta_m\} \mapsto \{u_m, w_m\} \right). \]

We will use the symbol \( s \) instead of sets \( \{f_k\}, \{\tau_m, \Theta_m\} \) and so on. The quantum transfer matrix we will denote as \( t(\lambda, \kappa; s) \). In part, when all \( f_k = 0 \), this \( t \) becomes the transfer matrix for the homogeneous chain. The state, corresponding to the homogeneous chain, we will denote as \( s_0 \).

**Proposition 2.** For the values of \( \{f_k\} \) being in general position, the eigenvalues of \( t(\lambda, \kappa; s) \) do not depend neither on \( \{f_k\} \) nor on the phases of \( \tau_m, \Theta_m \). In other words, for two states \( s \) and \( s' \) there exists a matrix \( \mathcal{S} \), invertible in general position, such that

\[ t(\lambda, \kappa; s) \mathcal{S} = \mathcal{S} t(\lambda, \kappa; s'). \]

Actually the proof of this proposition is based on the analogous proposition for the classical counterpart \( T(\Lambda) \). We have solved the problem of the isospectrality for the homogeneous chain, i.e. for the rational classical spectral curve. If one starts from the classical spectral curve in general position, then isospectral \( \tau_m, \Theta_m \) would become theta-functions on the jacobian of the spectral curve. Actually our “solitons” are the rational limit of the finite gap solution.

### 3. Auxiliary Lax operator

In this section we give a scheme for the construction of operators \( \mathcal{S} \), intertwining the states \( s \) and \( s' \).

3.1. “Dimer Self Trapping” Lax matrix. Define the quantum auxiliary lax matrix, acting in the space $\phi$ as follows:

$$\tilde{\ell}_\phi(\lambda, \lambda_\phi) =$$

$$(46)$$

$$= \left( 1 - \omega^{1/2} \kappa_\phi \frac{\lambda_\phi}{\lambda} w_\phi, \quad \frac{\omega^{1/2}}{\lambda} (1 - \omega^{1/2} \kappa_\phi w_\phi) u_\phi \right) - \omega^{1/2} \lambda_\phi u^{-1}_\phi w_\phi, \quad w_\phi$$

Here as previously

$$(47)$$

$$u_\phi = u_\phi x_\phi, \quad w_\phi = w_\phi z_\phi.$$

Also, as previously, define the classical counterpart $\tilde{L}$ of $\tilde{\ell}$. $U_\phi = u^N_\phi$, $W_\phi = w^N_\phi$, $\Lambda_\phi = \lambda^N_\phi$ etc:

$$\tilde{L}_\phi = \left( 1 + \kappa_\phi \frac{\Lambda_\phi}{\Lambda} W_\phi, \quad \frac{U_\phi}{\Lambda} (1 + \kappa_\phi W_\phi) \right) \Lambda_\phi \frac{W_\phi}{U_\phi}, \quad W_\phi$$

$$(48)$$

3.2. Quantum Darboux transformation. Darboux transformation for the classical chain is the following relation:

$$\tilde{L}(U_{\phi,m}, W_{\phi,m}) L(U_m, W_m) = L(U'_m, W'_m) \tilde{L}(U_{\phi,m+1}, W_{\phi,m+1}) ,$$

$$(49)$$

where all contents of $\Lambda, \Lambda_\phi, \kappa, \kappa_\phi$ are implied. Recall, (49) must be identity with respect to $\Lambda$, while $\Lambda_\phi$ enters into the mapping

$$u_m, w_m, u_{\phi,m}, w_{\phi,m} \mapsto u'_m, w'_m, u_{\phi,m+1}, w_{\phi,m+1} .$$

The quantum Darboux transformation is the following intertwining relation:

$$\tilde{\ell}_\phi(u_{\phi,m}, w_{\phi,m}) \cdot \ell_m(u_m, w_m) R_{m,\phi} =$$

$$= R_{m,\phi} \ell_m(u'_m, w'_m) \cdot \tilde{\ell}_\phi(u_{\phi,m+1}, w_{\phi,m+1}) .$$

$$(51)$$

Proposition 3. Eq. (49) is the admissibility condition for eq. (51). In other words, if mapping (50) solves (49), then there exists unique $N^2 \times N^2$ matrix $R_{m,\phi}$ solving eq. (51).

We do not give the explicit form of the matrix elements of $R_{m,\phi}$ here, it is not necessary in this brief notes.
Eq. (51) is to be iterated for the whole chain. It arises the monodromy of $\ell_m$ and the monodromy of $R_{m,\phi}$.

\[ \hat{Q}_\phi \overset{\text{def}}{=} R_{1,\phi} R_{2,\phi} \cdots R_{M,\phi}, \]

The monodromies obey the following relation:

\[ \tilde{\ell}_\phi(\lambda; u_{\phi,1}, w_{\phi,1}) \cdot \hat{t}(\lambda; \{ u_m, w_m \}) \hat{Q}_\phi = \]

\[ \hat{Q}_\phi \hat{t}(\lambda; \{ u'_m, w'_m \}) \cdot \tilde{\ell}_\phi(\lambda; u_{\phi,M+1}, w_{\phi,M+1}) . \]

Cyclic boundary conditions imply

\[ u_{\phi,M+1} = u_{\phi,1}, \quad w_{\phi,M+1} = w_{\phi,1} . \]

In general (54) have many solutions (they are subject of the next proposition). Suppose we have chosen an appropriate branch. The trace of the monodromy (52)

\[ Q_\phi = \text{tr}_\phi \hat{Q}_\phi \]

remembers the branch mentioned, and the reminder is the subscribe $'\phi$ of $Q_\phi$. Moreover we will regard $\phi$ as the spectral parameter of $Q_\phi$, implying $\lambda_\phi^N = \Lambda(\phi)$, see eq. (33), etc. Also one may show, parameter $K_\phi$ is responsible for a gauge transformation for $u'_m, w'_m$. This may be neglected by the choice $K_\phi = K/\Delta_\phi$. Matrix elements of $Q_\phi$ may be written out explicitly, and their parameterization implies all the information about $\lambda_\phi$ and $\{ u_m, w_m \}$ and $\{ u'_m, w'_m \}$ etc. Equation (53) with the cyclic boundary conditions (54) provides

\[ t(\lambda; \{ u_m, w_m \}) \cdot Q_\phi = Q_\phi \cdot t(\lambda; \{ u'_m, w'_m \}) . \]

It is useful to write this relation symbolically in the form

\[ t(\lambda; s) Q_\phi \overset{\leftrightarrow}{s} t(\lambda; s') . \]

3.3. Solutions of the cyclic boundary conditions. Careful and rather tedious investigations of the quantum Darboux relations (49, 51) and cyclic boundary conditions (54) allows us to formulate the following

**Proposition 4.** Let the state $s$ is $n$-solitonic one with the data $\{ f_k \}_{k=1}^n$. Then all possible solutions of (54) gives the following mappings $s \mapsto s'$, classified by the values of the spectral parameter $\phi$ of $Q_\phi$:

a). if $\phi \notin \{ \phi_k \}_{k=1}^n$, but $\phi$ belongs to set (52), then $s'$ is $n+1$-solitonic state with the data $\{ f'_k, \phi_k \}_{k=1}^{n+1}$, where

\[ f'_k = f_k(s(\phi_k, \phi)), \quad k = 1..n , \]
s is given by (41), and

\( \phi_{n+1} = \phi, \quad f'_{n+1} \) arbitrary complex number.

b). If the cyclicity condition is not satisfied, i.e. \( e^{2iM \phi} \neq 1 \), then the previous formulae are valid, but with \( f'_{n+1} \equiv 0 \).

c). If \( \phi = -\phi_n \), then \( s' \) is the \( n-1 \) solitonic state with the data \( \{ f_k s(\phi_k, \phi), \phi_k \}_{k=1}^{n-1} \). There is an annihilation of the soliton due to zero of the function \( s(\phi_n, -\phi_n) \).

d). If \( \phi = \phi_n \), then the state \( s' \) corresponds to \( n-1 \) solitonic state with the data \( \{ f_k s(\phi_k, \phi) d(\phi_k, \phi), \phi_k \}_{k=1}^{n-1} \) up to a gauge multipliers.

As it should be, (49) defines actually the action of the vertex operator of the Bäcklund transformation, creating or annihilating a soliton. Note also, not all the four items of proposition (4) are useful. Namely, we may ignore item 'd)'; it is equivalent to the item 'c)'. One may show, 'd)' and 'c)' produces the same \( U'_m \) and \( W'_m \) due to \( \Lambda(\phi) = \Lambda(-\phi) \) and \( \Delta^*(\phi) = \Delta(-\phi) \).

3.4. Permutation of \( Q_{\phi} \)-operators. Consider now the scenario 'a)' of the proposition – the creation of the new soliton. Besides the quantization of \( \phi \) this means also the additional degree of freedom in the final state: an arbitrary amplitude \( f \) of the \( \phi \)-th soliton. Following is the visual notation for it:

\[
\mathfrak{s} \xrightarrow{f, \phi} s'.
\]

It is useful to exhibit the parametric dependence on \( f \) : \( Q_{\phi}(f) \).

Compare now two successive creations of solitons: the first one

\[
\mathfrak{s} \xrightarrow{f, \phi} s' \xrightarrow{f', \psi} s'',
\]

and the second one

\[
\mathfrak{s} \xrightarrow{g, \psi} \tilde{s'} \xrightarrow{g', \phi} \tilde{s''}.
\]

The initial and the final states are the same, this provides

\[
g' = f s(\phi, \psi), \quad f' = g s(\psi, \phi).
\]

It follows the equivalence:

\[
Q_{\phi}(f) \cdot Q_{\psi}(g s(\psi, \phi)) \sim Q_{\psi}(g) \cdot Q_{\phi}(fs(\phi, \psi)).
\]

The sign ‘\( \sim \)’ means the existence of some extra multipliers in such pseudocommutation relations.
Consider now the successive creation of the maximal solitonic state. Let
\[ g_1 = f_1, \quad g_k = f_k \prod_{j<k} s(\phi_k, \phi_j). \tag{65} \]
Consider
\[ \mathcal{S}(\{f_k\}_{k=1}^{M-1}) \overset{\text{def}}{=} Q_{\phi_1}(g_1) Q_{\phi_2}(g_2) \cdots Q_{\phi_{M-1}}(g_{M-1}), \tag{66} \]
where the initial state \( s_0 \) is supposed to be the homogeneous one. Up to multipliers, operator-valued function \( \mathcal{S} \) is symmetrical with respect to any permutation of \( \{f_k\}_{k=1}^{M-1} \), this follows from (64). The final state \( s_{M-1} \) of \( \mathcal{S} \) has the solitonic data \( \{F_k\}_{k=1}^{M-1} \), where
\[ F_k = f_k \prod_{j \neq k} s(\phi_k, \phi_j). \tag{67} \]
In general all \( Q_{\phi} \) are invertible, and so invertible \( \mathcal{S} \) obeys
\[ t(\lambda; s_0) \mathcal{S} = \mathcal{S} t(\lambda; s_{M-1}), \tag{68} \]
this proves proposition (2) for the given initial and final states.

3.5. Baxter’s equation for operator \( Q_{\phi} \). Auxiliary quantum Lax operator (46) factorizes when \( \lambda = \lambda_{\phi} \):
\[ \tilde{\ell}_{\phi}(\lambda_{\phi}) = \begin{pmatrix} 1 - \omega^{1/2}K_{\phi}w_{\phi} & \omega^{1/2}v_{\phi}^{-1}u_{\phi} \\ -\omega^{1/2}v_{\phi}^{-1}u_{\phi}^{2} & 1 \end{pmatrix}. \tag{69} \]
This degeneration leads to the Baxter equation in the operator form by rather usual procedure of the triangulization of the monodromy matrix \[ [10, 3]. \] Details are inessential here, and the final answer reads
\[ t(\lambda_{\phi}; s) Q_{\phi} = Q_{\phi} t(\lambda_{\phi}; s') = Q_{\phi}^{(1)} + Q_{\phi}^{(2)}, \tag{70} \]
where in the operator form
\[ Q_{\phi}^{(1)} = \frac{1}{\mu_{\phi}} X Q_{\phi} X^{-1}, \quad Q_{\phi}^{(2)} = \frac{\mu_{\phi}}{\lambda_{\phi}^{2}} Z Q_{\phi} X. \tag{71} \]
\( X \) and \( Z \) are given by (24). Multiplier \( \mu_{\phi} \) is originated from the detailed consideration of the recursion (51) and (54), providing
\[ (\lambda_{\phi}, \mu_{\phi}) \in \Gamma_M. \tag{72} \]
In the special case when \( s = s' = s_0 \) (this is the case ‘b’ of proposition (4)), \( \lambda_{\phi} \) is arbitrary complex number, and eq. (70) becomes usual Baxter’s relation.
3.6. **Meaning of the modified $Q$-operators.** Consider now \(53\) in the degeneration point \(69\) for the whole chain. It is easy to rewrite \(53\) in the form

\[
\begin{align*}
\left(a(\lambda) - \omega^{1/2} \frac{u_{\phi,1}}{\lambda} x_{\phi} c(\lambda) \right) \cdot \hat{Q}_\phi = \hat{Q}_\phi^{(1)}, \\
\left(b(\lambda) - \omega^{1/2} \frac{u_{\phi,1}}{\lambda} x_{\phi} d(\lambda) \right) \cdot \hat{Q}_\phi = -\omega^{1/2} \frac{u_{\phi,M+1}}{\lambda} \hat{Q}_\phi^{(1)} x_{\phi},
\end{align*}
\]

or, in the equivalent form

\[
\begin{align*}
a(\lambda) \hat{Q}_\phi \omega^{1/2} \frac{u_{\phi,M+1}}{\lambda} x_{\phi} + b(\lambda) \hat{Q}_\phi = \omega^{1/2} \frac{u_{\phi,1}}{\lambda} x_{\phi} \hat{Q}_\phi^{(2)}, \\
c(\lambda) \hat{Q}_\phi \omega^{1/2} \frac{u_{\phi,M+1}}{\lambda} x_{\phi} + d(\lambda) \hat{Q}_\phi = \hat{Q}_\phi^{(2)},
\end{align*}
\]

It is interesting to consider these relations for the mapping \(s_0 \mapsto s_1\), where \(s_0\) is the homogeneous case, and \(s_1\) is one-solitonic state with the wave number \(\phi\) and the amplitude \(f\). Note, \(\mu_{\phi}\) and \(\lambda_{\phi}\) do not depend on \(f\). Consider now the limit

\[
f \mapsto -1.
\]

In this limit state \(s_1\) has the remarkable feature: \(u_{\phi,1} \equiv u'_{1} = 0\).

Then from \(73\) and \(74\) it follows:

\[
b(\lambda) \cdot Q_{\phi} = 0.
\]

Due to the symmetry of \(\mathcal{G}(\{f_k\}_{k=1}^{M-1})\) with respect to any permutation of \(\{f_k\}\), one may conclude

\[
b(\lambda_{\phi_n}) \cdot \mathcal{G}(\{f_k = -1\}_{k=1}^{M-1}) = 0 \quad \forall \ n = 1, ..., M - 1.
\]

This means that in the special point \(f_k = -1\) operator \(\mathcal{G}\) becomes rather degenerative, and in the \(N\)-dimensional projector decomposition of \(\mathcal{G}\) all left vectors may be regarded as the eigenvectors of operator \(b(\lambda)\) with zeros \(\lambda_{\phi_k}\), see eq. \(74\) and the discussion around it.

4. **Discussion**

The main object, appeared in this note, is the operator \(\mathcal{G}(\{f_k = -1\}_{k=1}^{M-1})\). Recall once more, we have the explicit form of the matrix elements of each \(Q_{\phi}\)-operator, entering to \(\mathcal{G}\). Note, there are some arbitrariness in their definition. The technical reason of it is that one has to deal with the ambiguities 0/0, and the principal reason is that the right vectors of \(\mathcal{G}\) may be chosen in rather arbitrary way. There still exists the problem of the most useful choice of these right vectors.
All the details concerning the explicit form of $Q_\phi$ and $S$, as well as the exact form of the pseudocommutation relation (64) will be the subjects of the subsequent papers. The other subject to be mentioned is the strict investigation of the classical counterpart in the application to the quantum technique. The final aim is the exact solution of the model, i.e. the solution of Baxter’s equation an the construction of the eigenvectors of $t(\lambda)$, at least with the help of the separation of the variables method.

What is to be mentioned else. The “dimer self-trapping” Lax matrix at the root of unity belongs to the class of the chiral Potts model Lax matrices [3]. The quantum intertwiner of DST Lax matrices is a particular case of the chiral Potts model $S$-matrix. The method of applying a nontrivial classical dynamics to the chiral Potts (or DST - it is much more simple) model will give the analogous scheme for the separation of the variables method, including the exact construction for the eigenvectors of the elements of the CPM monodromy matrix. These we will do separately.

Acknowledgements The authors are grateful to R. Baxter, V. Bazhanov, V. Mangazeev, G. Pronko, E. Sklyanin, A. Belavin, Yu. Stroganov and A. Isaev for useful discussions and comments.

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