Relative types and extremal problems
for plurisubharmonic functions

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Abstract
A type $\sigma(u, \phi)$ of a plurisubharmonic function $u$ relative to a maximal plurisubharmonic weight $\phi$ with isolated singularity at $\zeta$ is defined as $\liminf u(x)/\phi(x)$ as $x \to \zeta$. We study properties of the relative types as functionals $u \mapsto \sigma(u, \phi)$; it is shown that they give a general form for upper semicontinuous, positive homogeneous and tropically additive functionals on plurisubharmonic singularities. We consider some extremal problems whose solutions are Green-like functions that give best possible bounds on $u$, given the values of its types relative to some of (or all) weights $\phi$; in certain cases they coincide with known variants of pluricomplex Green functions. An analyticity theorem is proved for the upper-level sets for the types with respect to exponentially Hölder continuous weights, which leads to a result on propagation of plurisubharmonic singularities.

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1 Introduction
If a holomorphic mapping $F$ vanishes at a point $\zeta$, then the asymptotic behaviour of $|F|$ near $\zeta$ completely determines such fundamental characteristics of $F$ at $\zeta$ as the multiplicity of the zero or the integrability index. On the other hand, in most cases the values of such characteristics can just give certain bounds on the asymptotics of $F$ rather than recover it completely.

The transformation $F \mapsto \log |F|$ puts this into the context of pluripotential theory, which leads to a question of characteristics of singularities of plurisubharmonic functions and their relations to the asymptotic behaviour of the functions. Since our considerations are local, we assume the functions to be defined on domains of $\mathbb{C}^n$, $n > 1$.

Let $u$ be a plurisubharmonic function near a point $\zeta$ of $\mathbb{C}^n$, such that $u(\zeta) = -\infty$. The value $\nu_u(\zeta)$ of the Lelong number of $u$ at $\zeta$ gives some information on the asymptotic
behaviour near $\zeta$: $u(x) \leq \nu_u(\zeta) \log |x - \zeta| + O(1)$. A more detailed information can be obtained by means of its directional Lelong numbers $\nu_u(\zeta, a)$, $a \in \mathbb{R}^n_+$, due to Kiselman: $u(x) \leq \nu_u(\zeta, a) \max_k a_k^{-1} \log |x_k - \zeta_k| + O(1)$.

In addition, these characteristics of singularity are well suited for the tropical structure of the cone of plurisubharmonic functions, namely $\nu_{u+v} = \nu_u + \nu_v$ (tropical multiplicativity) and $\nu_{\max\{u,v\}} = \min\{\nu_u, \nu_v\}$ (tropical additivity). These properties play an important role, for example, in investigation of valuations on germs of holomorphic functions [6]. Note that the tropical operations $u \oplus v := \max\{u, v\}$ and $u \otimes v := u + v$, when applied to plurisubharmonic singularities, can be viewed as Maslov’s dequantization of usual addition and multiplication of holomorphic functions.

A general notion of Lelong numbers $\nu(u, \varphi)$ with respect to plurisubharmonic weights $\varphi$ was introduced and studied by Demailly [3], [5]. Due to their flexibility, the Lelong–Demailly numbers have become a powerful tool in pluripotential theory and its applications. They still are tropically multiplicative, however tropical additivity is no longer true for $\nu(u, \varphi)$ with arbitrary plurisubharmonic weights $\varphi$, even if they are maximal outside $\varphi^{-1}(-\infty)$. In addition, the value $\nu(u, \varphi)$ gives little information on the asymptotics of $u$ near $\varphi^{-1}(-\infty)$.

The good properties of the classical and directional Lelong numbers result from the fact that they can be evaluated by means of the suprema of $u$ over the corresponding domains (the balls and polydiscs, respectively). This makes it reasonable to study the asymptotics of the suprema of $u$ over the corresponding domains $\{\varphi(x) < t\}$ for a maximal weight $\varphi$ with an isolated singularity at $\zeta$ and consider the value $\sigma(u, \varphi) = \lim \inf u(x)/\varphi(x)$ as $x \to \zeta$, the type of $u$ relative to $\varphi$. The relative type is thus an alternative generalization of the notion of Lelong number.

Unlike the Lelong–Demailly numbers, the relative types need not be tropically multiplicative, however they are tropically additive. Moreover, they are the only ”reasonable” tropically additive functionals on plurisubharmonic singularities (for a precise statement, see Theorem 4.3).

Maximality of $\varphi$ gives the bound $u \leq \sigma(u, \varphi)\varphi + O(1)$ near the pole of $\varphi$. We are then interested in best possible bounds on $u$, given the values of its types relative to some of (or all) the weights $\varphi$ with fixed $\varphi^{-1}(-\infty)$. Tropical additivity of the relative types makes them a perfect tool for dealing with upper envelopes of families of plurisubharmonic functions, constructing thus extremal plurisubharmonic functions with prescribed singularities. In certain cases these Green-like functions coincide with known variants of pluricomplex Green functions. In particular, this gives a new representation of the Green functions with divisorial singularities (Theorems 6.6 and 6.7). We study relations between such extremal functions; one of the relations implies a complete characterization of holomorphic mappings $f$ with isolated zero at $\zeta$ of the multiplicity equal to the Newton number of $f$ at $\zeta$ (Corollary 6.5).
We also prove that the upperlevel sets for the types relative to exponentially Hölder continuous weights are analytic varieties (an analogue to the Siu theorem). As an application, we obtain a result on propagation of plurisubharmonic singularities (Corollary 7.3) that results in a new representation of the Green functions with singularities along complex spaces (Corollary 7.5).

The paper is organized as follows. Section 2 recalls basic facts on Lelong numbers and Green functions. In Section 3 we present the definition and elementary properties of the relative types. A representation theorem for tropically additive functionals on plurisubharmonic singularities is proved in Section 4. In Sections 5 and 6 we consider extremal problems for plurisubharmonic functions with given singularities. An analyticity theorem for the upperlevel sets and its applications are presented in Section 7.

2 Preliminaries

2.1 Lelong numbers

The Lelong number $\nu_T(\zeta)$ of a closed positive current $T$ of bidimension $(p, p)$ at a point $\zeta \in \mathbb{C}^n$ is the residual mass of $T \wedge (dd^c \log |x - \zeta|)^p$ at $\zeta$:

$$\nu_T(\zeta) = \lim_{r \to 0} \int_{|x - \zeta| < r} T \wedge (dd^c \log |x - \zeta|)^p; \quad (2.1)$$

here $d = \partial + \bar{\partial}$, $d^c = (\partial - \bar{\partial})/2\pi i$.

The Lelong number $\nu_u(\zeta)$ of a plurisubharmonic function $u$ is just the Lelong number of the current $dd^c u$. It can also be calculated as

$$\nu_u(\zeta) = \lim_{r \to -\infty} r^{-1} \int_{S_1} u(\zeta + xe^r) dS_1(x), \quad (2.2)$$

where $dS_1$ is the normalized Lebesgue measure on the unit sphere $S_1$, as well as

$$\nu_u(\zeta) = \lim_{r \to -\infty} r^{-1} \sup\{u(x) : |x - \zeta| < e^r\} = \lim_{z \to \zeta} \inf \frac{u(z)}{\log |z - \zeta|}, \quad (2.3)$$

see [7]. Since the function $\sup\{u(x) : |x - \zeta| < e^r\}$ is convex in $r$, representation (2.3) implies the bound $u(x) \leq \nu_u(\zeta) \log |x - \zeta| + O(1)$ near $\zeta$.

Lelong numbers are independent of the choice of coordinates. Siu’s theorem states that the set $\{\zeta : \nu_T(\zeta) \geq c\}$ is analytic for any $c > 0$. 

3
2.2 Directional Lelong numbers

A more detailed information on the behaviour of \( u \) near \( \zeta \) can be obtained by means of the \textit{directional Lelong numbers} due to Kiselman [8]: given \( a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \),

\[
\nu_a(\zeta, a) = \lim_{r \to -\infty} r^{-1} \sup \{ u(x) : |x_k - \zeta_k| < e^{r a_k}, 1 \leq k \leq n \},
\]

or equivalently, in terms of the mean values of \( u \) over the distinguished boundaries of the polydiscs, similarly to (2.2). Namely,

\[
u_a(\zeta, a) \phi_{a, \zeta}(x) + O(1) \text{ with } \phi_{a, \zeta}(x) = \max_k a^{-1} k \log |x_k - \zeta_k|.
\]

(2.5)

Analyticity of the upperlevel sets \( \{ \zeta : \nu_a(\zeta, a) \geq c \} \) was established in [8], [9]. Directional Lelong numbers give rise to the notion of local indicators of plurisubharmonic functions [13]. Given a plurisubharmonic function \( u \), its (local) indicator at a point \( \zeta \) is a plurisubharmonic function \( \Psi_{u, \zeta} \) in the unit polydisc \( D^n \) such that for any \( y \in D^n \) with \( y_1 \cdots y_n \neq 0 \),

\[
\Psi_{u, \zeta}(y) = -\nu_a(\zeta, a), \quad a = -(\log |y_1|, \ldots, \log |y_n|) \in \mathbb{R}_+^n.
\]

(2.6)

It is the largest nonpositive plurisubharmonic function in \( D^n \) whose directional Lelong numbers at 0 coincide with those of \( u \) at \( \zeta \), see the details in [13], [14].

2.3 Lelong–Demailly numbers

A general notion of Lelong numbers with respect to plurisubharmonic weights was introduced and studied by J.-P. Demailly [3], [5]. Let \( T \) be a closed positive current of bidimension \( (p, p) \) on a domain \( \Omega \subset \mathbb{C}^n \), and let \( \varphi \) be a continuous plurisubharmonic function \( \Omega \to [-\infty, \infty) \), semiexhaustive on the support of \( T \), that is, \( B_R^\varphi \cap \text{supp } T \subset \Omega \) for some real \( R \), where \( B_R^\varphi := \{ x : \varphi(x) < R \} \), and let \( S^x_\varphi := \varphi^{\underline{\underline{\varphi}}}(-\infty) \neq 0 \). The value

\[
\nu(T, \varphi) = \lim_{r \to -\infty} \int_{B_r^\varphi} T \wedge (dd^c \varphi)^p = T \wedge (dd^c \varphi)^p(S^x_\varphi)
\]

(2.7)

is called the \textit{generalized Lelong number}, or the \textit{Lelong–Demailly number}, of \( T \) with respect to the weight \( \varphi \). When \( \varphi(x) = \log |x - \zeta| \), this is just the classical Lelong number of \( T \) at \( \zeta \).

For a plurisubharmonic function \( u \), we use the notation \( \nu(u, \varphi) = \nu(dd^c u, \varphi) \).

The generalized Lelong numbers have the following semicontinuity property.

\textbf{Theorem 2.1} ([5], Prop. 3.11) \textit{If } \( T_k \to T \) \textit{and } \( \varphi \) \textit{is semiexhaustive on a closed set containing the supports of all } \( T_k \), \textit{then } \( \limsup_{k \to \infty} \nu(T_k, \varphi) \leq \nu(T, \varphi) \).
The following comparison theorems describe variation of the Lelong–Demailly numbers with respect to the weights and currents.

**Theorem 2.2** ([5], Th. 5.1) Let $T$ be a closed positive current of bidimension $(p, p)$, and let $\varphi$, $\psi$ be two weights semiexhaustive on $\text{supp} T$ such that $\lim \sup \psi(x)/\varphi(x) = l < \infty$ as $\varphi(x) \to -\infty$. Then $\nu(T, \psi) \leq l^p \nu(T, \varphi)$.

**Theorem 2.3** ([5], Th. 5.9) Let $u$ and $v$ be plurisubharmonic functions such that $(dd^c u)^q \wedge T$ and $(dd^c v)^q \wedge T$ are well defined near $S^c_{\varphi-\infty}$ and $u = -\infty$ on $\text{supp} T \cap S^c_{\varphi-\infty}$, where $T$ is a closed positive current of bidimension $(p, p)$, $p \geq q$. If $\lim \sup v(x)/u(x) = l < \infty$ as $\varphi(x) \to -\infty$, then $\nu((dd^c v)^q \wedge T, \varphi) \leq l^q \nu((dd^c u)^q \wedge T, \varphi)$.

The directional Lelong numbers can be expressed in terms of the Lelong–Demailly numbers with respect to the directional weights $\phi_a, \zeta$ (2.5):

$$\nu_u(\zeta, a) = a_1 \cdots a_n \nu(u, \phi_a, \zeta).$$

Siu’s theorem on analyticity of upperlevel sets was extended in [3] to generalized Lelong numbers with respect to weights $\varphi_\zeta(x) = \varphi(x, \zeta)$ that are exponentially Hölder continuous with respect to $\zeta$.

### 2.4 Green functions

Let $D$ be a hyperconvex domain in $\mathbb{C}^n$ and let $PSH^-(D)$ denote the class of all negative plurisubharmonic functions in $D$.

The pluricomplex Green function $G_{\zeta, D}$ of $D$ with logarithmic pole at $\zeta \in D$ (introduced by Lempert, Zahariuta, Klimek) is the upper envelope of the class $\mathcal{F}_{\zeta, D}$ of $u \in PSH^-(D)$ such that $u(x) \leq \log |x - \zeta| + O(1)$ near $\zeta$. The class $\mathcal{F}_{\zeta, D}$ can be also described as the collection of $u \in PSH^-(D)$ such that $\nu_u(\zeta) \geq 1$. The function satisfies $G_{\zeta, D}(x) = \log |x - \zeta| + O(1)$ near $\zeta$ and $(dd^c G_{\zeta, D})^n = \delta_{\zeta}$.

A more general construction was presented by Zahariuta [19, 20]. Given a continuous plurisubharmonic function $\varphi$ in a neighbourhood of $\zeta \in D$ such that $\varphi^{-1}(-\infty) = \zeta$ and $(dd^c \varphi)^n = 0$ outside $\zeta$, let

$$G_{\varphi, D}(x) = \sup\{u(x) : u \in PSH^-(D), u \leq \varphi + O(1) \text{ near } \zeta\}.$$  (2.9)

Then $G_{\varphi, D}$ is maximal in $D \setminus \{\zeta\}$ and $G_{\varphi, D} = \varphi + O(1)$ near $\zeta$. We will refer to this function as the Green–Zahariuta function with respect to the singularity $\varphi$.
A Green function with prescribed values of all directional Lelong numbers at \( \zeta \) (the Green function with respect to an indicator \( \Psi \)) was introduced in [13] as
\[
G_{\Psi, \zeta, D}(x) = \sup \{ u(x) : u \in PSH^-(D), \ \nu_u(\zeta, a) \geq \nu_\Psi(0, a) \quad \forall a \in \mathbb{R}^n \}, \tag{2.10}
\]
where \( \Psi \) is a negative plurisubharmonic function in the unit polydisc \( D^n \) such that
\[
\Psi(z_1, \ldots, z_n) = \Psi(|z_1|^c, \ldots, |z_n|^c), \quad c > 0, \quad z \in \mathbb{D}^n; \tag{2.11}
\]
such a function \( \Psi \) coincides with its own indicator (2.6), so \( \nu_\Psi(0, a) = -\Psi(e^{-a_1}, \ldots, e^{-a_n}) \).

The above Green functions were also considered for several isolated poles. In the case of non-isolated singularities, a variant of Green functions was introduced by Lárusson–Sigurdsson [11] by means of the class of negative plurisubharmonic functions \( u \) satisfying \( \nu_u(x) \geq \alpha(x) \), where \( \alpha \) is an arbitrary nonnegative function on \( D \). In [12] it was specified to the case when \( \alpha(x) = \nu_A(x) \) is the Lelong number of a divisor \( A \); then
\[
F_{A,D} = \{ u \in PSH^-(D) : \nu_u(x) \geq \nu_A(x) \quad \forall x \in D \}, \tag{2.12}
\]
and
\[
G_{A,D}(x) = \sup \{ u(x) : u \in F_{A,D} \} \tag{2.13}
\]
is the Green function for the divisor \( A \). It was shown that if \( A \) is the divisor of a bounded holomorphic function \( f \) in \( D \), then \( G_{A,D} = \log |f| + O(1) \) near points of \( |A| = f^{-1}(0) \).

This was used in [16], [17] for consideration of Green functions with arbitrary analytic singularities. Given a closed complex subspace \( A \) on \( D \), let \( \mathcal{I}_A = \mathcal{I}_{A,D} = (\mathcal{I}_{A,x})_{x \in D} \) be the associated coherent sheaf of ideals in the sheaf \( \mathcal{O}_D = (\mathcal{O}_x)_{x \in D} \) of germs of holomorphic functions on \( D \), and let \( |A| = \{ x \in D : \mathcal{I}_{A,x} \neq \mathcal{O}_x \} \). A Green function \( G_{A,D} \) for the complex space \( A \) on \( D \) was constructed as
\[
G_{A,D}(x) = \sup \{ u(x) : u \in PSH^-(D), \ u \leq \log |f| + O(1) \text{ locally near } |A| \}, \tag{2.14}
\]
where \( f = (f_1, \ldots, f_p) \) and \( f_1, \ldots, f_p \) are local generators of \( \mathcal{I}_A \). The function \( G_{A,D} \) is plurisubharmonic and satisfies
\[
G_{A,D} \leq \log |f| + O(1) \tag{2.15}
\]
locally near points of \( |A| \); if \( \mathcal{I}_A \) has bounded global generators, then \( G_{A,D} = \log |f| + O(1) \).
3 Definition and elementary properties of relative types

Given \( \zeta \in \mathbb{C}^n \), let \( PSH_\zeta \) stand for the collection of all (germs of) plurisubharmonic functions \( u \not\equiv -\infty \) in a neighbourhood of \( \zeta \).

Let \( \varphi \in PSH_\zeta \) be locally bounded outside \( \zeta \), \( \varphi(\zeta) = -\infty \), and maximal in a punctured neighbourhood of \( \zeta \): \( (dd^c \varphi)^n = 0 \) on \( B^c_R \setminus \{\zeta\} \) for some \( R > -\infty \); we recall that \( B^c_R = \{z : \varphi(z) < R\} \). The collection of all such maximal weights (centered at \( \zeta \)) will be denoted by \( MW_\zeta \). If we want to specify that \( (dd^c \varphi)^n = 0 \) on \( \omega \setminus \{\zeta\} \), we will write \( \varphi \in MW_\zeta(\omega) \).

Given a function \( u \in PSH_\zeta \), its singularity at \( \zeta \) can be compared to that of \( \varphi \in MW_\zeta \); the value
\[
\sigma(u, \varphi) = \liminf_{z \to \zeta} \frac{u(z)}{\varphi(z)} \quad (3.1)
\]
will be called the \( \varphi \)-type, or the relative type of \( u \) with respect to \( \varphi \).

Both the Lelong–Demailly numbers and the relative types are generalizations of the classical notion of Lelong number, however they use different points of view on the Lelong number: while the Lelong–Demailly numbers correspond to (2.1) (and (2.2) in the case of functions), the relative types are based on (2.3). As we will see, the two generalizations have much in common, however some features are quite different.

Note that \( \sigma(u + h, \varphi) = \sigma(u, \varphi) \) if \( h \) is pluriharmonic (and hence bounded) near \( \zeta \), so \( \sigma(u, \varphi) \) is actually a function of \( dd^c u \).

The following properties are direct consequences of the definition of the relative type.

**Proposition 3.1** Let \( \varphi, \psi \in MW_\zeta \) and \( u, u_j \in PSH_\zeta \). Then

(i) \( \sigma(cu, \varphi) = c\sigma(u, \varphi) \) for all \( c > 0 \);

(ii) \( \sigma(\max\{u_1, u_2\}, \varphi) = \min_j \sigma(u_j, \varphi) \);

(iii) \( \sigma(u_1 + u_2, \varphi) \geq \sigma(u_1, \varphi) + \sigma(u_2, \varphi) \);

(iv) \( \sigma(u, \psi) \geq \sigma(u, \varphi) \sigma(\varphi, \psi) \); in particular, if there exists \( \lim_{z \to \zeta} \varphi(z)/\psi(z) = l < \infty \), then \( \sigma(u, \varphi) = l \sigma(u, \psi) \);

(v) \( \text{if } \liminf_{z \to \zeta} \frac{u_1(z)}{u_2(z)} = l < \infty, \text{ then } \sigma(u_1, \varphi) \geq l \sigma(u_2, \varphi) \);

(vi) \( \text{if } u = \log \sum_{j=1}^m e^{u_j}, \text{ then } \sigma(u, \varphi) = \sigma(\max_j u_j, \varphi) = \min_j \sigma(u_j, \varphi) \).

**Proof.** Properties (i) – (v) are direct consequences of the definition of the relative type, and (vi) follows from (ii) and (iv) together. Note that (iv) makes sense because \( \sigma(u, \varphi) < \infty \) for any \( u \in PSH_\zeta \) and \( \varphi \in MW_\zeta \); this follows, for example, from an alternative description of \( \sigma(u, \varphi) \) given by (3.3–3.5) below. \( \square \)
Remarks 3.2 (1) The relative types need not be tropically multiplicative functionals on $\text{PSH}_\zeta$. Take $\varphi = \max \{3 \log |z_1|, 3 \log |z_2|, \log |z_1 z_2|\} \in MW_0$ and $u_j = \log |z_j|$, $j = 1, 2$, then $\sigma(u_j, \varphi) = 1/3$, while $\sigma(u_1 + u_2, \varphi) = 1$.

(2) Properties (iv) and (v) are analogues to Comparison Theorems 2.2 and 2.3.

(3) For holomorphic functions $f_j$ and positive numbers $p_j$, $j = 1, \ldots, m$, property (vi) gives the relation

$$\sigma(\log \sum_j |f_j|^{p_j}, \varphi) = \sigma(\max_j p_j \log |f_j|, \varphi) = \min_j p_j \sigma(\log |f_j|, \varphi). \quad (3.2)$$

Given a weight $\varphi \in MW_\zeta$ and a function $u \in \text{PSH}_\zeta$, consider the growth function

$$\Lambda(u, \varphi, r) = \sup \{u(z) : z \in B^\varphi_r\}. \quad (3.3)$$

Proposition 3.3 (see also [2], Corollary 6.6) Let $\varphi \in MW_\zeta(B^\varphi_R)$ such that $B^\varphi_R$ is bounded, and $u \in \text{PSH}(B^\varphi_R)$. Then the function $\Lambda(u, \varphi, r)$ is convex in $r \in (-\infty, R)$.

Proof. Given $-\infty < r_1 < r_2 < R$ and $0 < \epsilon < R - r_2$, let $u_\epsilon = u * \chi_\epsilon$ be a standard regularization (smoothing) of $u$. Take $c \geq 0$ and $d \in \mathbb{R}$ such that $cr_j + d = \Lambda(u_\epsilon, \varphi, r_j)$, $j = 1, 2$. Since $\varphi \geq r$ on $\partial B^\varphi_r$, we have $c\varphi + d \geq u_\epsilon$ on $\partial(B^\varphi_r \setminus B^\varphi_{r_1})$ and thus on the set $B^\varphi_{r_2} \setminus B^\varphi_{r_1}$ because of the maximality of the function $c\varphi + d$ on $B^\varphi_R \setminus \{0\}$. In other words, $\Lambda(u_\epsilon, \varphi, r) \leq cr + d$ on $(r_1, r_2)$, which means that $\Lambda(u_\epsilon, \varphi, r)$ is convex on $(r_1, r_2)$. Since $\Lambda(u_\epsilon, \varphi, r) \to \Lambda(u, \varphi, r)$ as $\epsilon \to 0$, this implies the assertion. \qed

Since $\Lambda(u, \varphi, r)$ is increasing and convex, the ratio

$$g(u, \varphi, r, r_0) := \frac{\Lambda(u, \varphi, r) - \Lambda(u, \varphi, r_0)}{r - r_0}, \quad r < r_0 < R, \quad (3.4)$$

is increasing in $r \in (-\infty, r_0)$ and, therefore, has a limit as $r \to -\infty$; it is easy to see that the limit equals $\sigma(u, \varphi)$. We have, in particular,

$$\sigma(u, \varphi) \leq g(u, \varphi, r, r_0), \quad r < r_0, \quad (3.5)$$

which implies the following basic bound.

Proposition 3.4 Let $\varphi \in MW_\zeta(B^\varphi_R)$, $u \in \text{PSH}^-(B^\varphi_{r_0})$, $r_0 < R$. Then $u \leq \sigma(u, \varphi)(\varphi - r_0)$ in $B^\varphi_{r_0}$. In particular, every function $u \in \text{PSH}_\zeta$ has the bound

$$u(z) \leq \sigma(u, \varphi) \varphi(z) + O(1), \quad z \to \zeta. \quad (3.6)$$
Next statement is an analogue to Theorem 2.1.

**Proposition 3.5** Let \( u, u_j \in PSH^-(\Omega), \varphi \in MW_\zeta, \zeta \in \Omega \). If \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega) \), then \( \sigma(u, \varphi) \geq \limsup \sigma(u_j, \varphi) \).

**Proof.** Take any \( r_0 \) such that \( \Lambda(u, \varphi, r_0) < 0 \), then for any \( \epsilon > 0 \) and \( r < r_0 \) there exists \( j_0 \) such that \( \Lambda(u_j, \varphi, r) \geq \Lambda(u, \varphi, r) - \epsilon \) for all \( j > j_0 \). Using (3.5) we get

\[
\sigma(u_j, \varphi) \leq g(u_j, \varphi, r, r_0) \leq \Lambda(u, \varphi, r) - \epsilon \frac{r}{r_0}, \tag{3.7}
\]

which implies the assertion. \( \square \)

Let us compare the values of relative types with some known characteristics of plurisubharmonic singularities. Denote

\[
\nu_\varphi := \nu_\varphi(\zeta) \tag{3.8}
\]

the Lelong number of \( \varphi \in MW_\zeta \) at \( \zeta \);

\[
\tau_\varphi := (dd^c \varphi)^n(\zeta) \tag{3.9}
\]

the residual Monge–Ampère mass of \( \varphi \) at \( \zeta \);

\[
\alpha_\varphi := \limsup_{z \to \zeta} \frac{\varphi(z)}{\log |z - \zeta|}. \tag{3.10}
\]

By Theorem 2.2 \( \nu_\varphi^n \leq \tau_\varphi \leq \alpha_\varphi^n \) and, by Proposition 3.4

\[
\alpha_\varphi \log |z - \zeta| + O(1) \leq \varphi(z) \leq \nu_\varphi \log |z - \zeta| + O(1), \quad z \to \zeta. \tag{3.11}
\]

If \( \varphi \) has analytic singularity, that is, \( \varphi = \log |f| + O(1) \) near \( \zeta \), where \( f = (f_1, \ldots, f_n) \) is a holomorphic map with isolated zero at \( \zeta \), then \( \nu_\varphi \) equals the minimum of the multiplicities of \( f_k \) at \( \zeta \), \( \tau_\varphi \) is the multiplicity of \( f \) at \( \zeta \), and \( \alpha_\varphi = \gamma_f \), the Lojasiewicz exponent of \( f \) at \( \zeta \), i.e., the infimum of \( \gamma > 0 \) such that \( |f(z)| \geq |z - \zeta|^\gamma \) near \( \zeta \). Therefore, \( \nu_\varphi > 0 \) and \( \alpha_\varphi < \infty \) in this case.

In the general situation, since \( \varphi \) is locally bounded and maximal on \( B^c_\zeta \setminus \{\zeta\} \), the condition \( \varphi(\zeta) = -\infty \) implies \( 0 < \tau_\varphi < \infty \).

We do not know if \( \nu_\varphi > 0 \) for every weight \( \varphi \in MW_\zeta \). It is actually equivalent to the famous problem of existence of a plurisubharmonic function which is locally bounded outside \( \zeta \) and has zero Lelong number and positive Monge-Ampère mass at \( \zeta \) (see a discussion in [13] or the remark after Proposition 6.1).
Furthermore, (3.11) implies \( \alpha \varphi > 0 \), however we do not know if the “Lojasiewicz exponent” \( \alpha \varphi \) is finite for every maximal weight \( \varphi \). It is worth noting that, by Proposition 3.1 (v), \( \sigma(\log |f|, \varphi) \leq \gamma_f \alpha^{-1} \) for any holomorphic map \( f \) with isolated zero at \( \zeta \), where \( \gamma_f \) is the Lojasiewicz exponent of \( f \), so \( \sigma(\log |f|, \varphi) = 0 \) if \( \alpha \varphi = \infty \).

By Theorem 2.3 the condition \( \alpha \varphi < \infty \) implies \( \tau \varphi \leq \alpha^{-1} \nu \varphi \) and so, \( \nu \varphi > 0 \) for such a weight \( \varphi \). In other words, denote

\[
SMW_\zeta = \{ \varphi \in MW_\zeta : \nu \varphi > 0 \}
\]

(3.12)

(the weights with “strong” singularity) and

\[
LMW_\zeta = \{ \varphi \in MW_\zeta : \alpha \varphi < \infty \}
\]

(3.13)

(the weights with finite Lojasiewicz exponent), then

\[
LMW_\zeta \subset SMW_\zeta \subset MW_\zeta,
\]

(3.14)

and it is unclear if the inclusions are strict.

**Proposition 3.6** The type \( \sigma(u, \varphi) \) of \( u \in PSH_\zeta \) with respect to \( \varphi \in SMW_\zeta \) is related to the Lelong number \( \nu_u(\zeta) \) of \( u \) at \( \zeta \) as

\[
\alpha^{-1} \nu_u(\zeta) \leq \sigma(u, \varphi) \leq \nu^{-1}_u \nu(\zeta).
\]

(3.15)

If, in addition, \( \varphi \) is continuous, then

\[
\sigma(u, \varphi) \leq \tau^{-1} \nu(u, \varphi),
\]

(3.16)

where \( \nu(u, \varphi) \) is the Lelong–Demailly number of \( u \) with respect to \( \varphi \).

**Proof.** Bounds (3.15) follow from (3.11) by Theorem 2.2 and relation (3.16) follows from Proposition 3.1 by Theorem 2.3. \( \square \)

**Remark 3.7** Due to (2.4) and (2.8), there is always an equality in (3.16) if \( \varphi = \phi_{a, \zeta} \), a directional weight (2.5). On the other hand, for general weights \( \varphi \) the inequality can be strict. For example, let \( u_1, u_2 \) and \( \varphi \) be as in Remarks 3.2 (1), and let \( u = \max \{2u_1, u_2\} \). Then \( \sigma(u, \varphi) = 1/3, \nu(u, \varphi) = 3 \) and \( \tau \varphi = 6 \), so the right hand side of (3.16) equals \( 1/2 > 1/3 \).
4 Representation theorem

It was shown in Section 3 that relative types \( \sigma(u, \varphi) \) are positive homogeneous, tropically additive (in the sense \( \sigma(\max u_k, \varphi) = \min \sigma(u_k, \varphi) \)) and upper semicontinuous functionals on \( \text{PSH}_\zeta \) that preserve ordering of the singularities, i.e., \( u \leq v + O(1) \) implies \( \sigma(u, \varphi) \geq \sigma(v, \varphi) \).

Here we show that any such functional on \( \text{PSH}_\zeta \) can be represented as a relative type, provided it does not vanish on a function that is locally bounded outside \( \zeta \).

**Lemma 4.1** Let \( D \) be a bounded hyperconvex neighbourhood of a point \( \zeta \), and let a function \( \sigma : \text{PSH}^- (D) \rightarrow [0, \infty) \) be such that \( \sigma(u) < \infty \) if \( u \neq -\infty \) and

(i) \( \sigma(cu) = c \sigma(u) \) for all \( c > 0 \);
(ii) if \( u_1 \leq u_2 + O(1) \) near \( \zeta \), then \( \sigma(u_1) \geq \sigma(u_2) \);
(iii) \( \sigma(\max_k u_k) = \min_k \sigma(u_k) \), \( k = 1, 2 \);
(iv) if \( u_j \rightarrow u \) in \( L^1_{\text{loc}} \), then \( \limsup \sigma(u_j) \leq \sigma(u) \);
(v) \( \sigma(w_0) > 0 \) for at least one \( w_0 \in \text{PSH}^- (D) \cap L^\infty_{\text{loc}}(D \setminus \zeta) \).

Then there exists a unique function \( \varphi \in \text{MW}_\zeta(D) \), \( \varphi(z) \rightarrow 0 \) as \( z \rightarrow \partial D \), such that

\[
\sigma(u) = \sigma(u, \varphi) \quad \forall u \in \text{PSH}^- (D). 
\]

If, in addition, (v) is true with \( w_0(z) = \log|z - \zeta| + O(1) \), then \( \varphi \in \text{LMW}_\zeta \).

**Proof.** Denote \( \varphi(z) = \sup \{ u(z) : u \in \mathcal{M} \} \), where \( \mathcal{M} = \{ u \in \text{PSH}^- (D) : \sigma(u) \geq 1 \} \).

By the Choquet lemma, there exists a sequence \( u_j \in \mathcal{M} \) increasing to a function \( v \) such that \( v^* = \varphi^* \in \text{PSH}^- (D) \). Properties (iii) and (iv) imply \( v^* \in \mathcal{M} \), so \( v^* \leq \varphi \). Therefore, \( \varphi = v^* \in \mathcal{M} \). Evidently, \( \sigma(\varphi) = 1 \).

If \( v \in \text{PSH}^- (D) \) satisfies \( v \leq \varphi \) outside \( \omega \subset D \setminus \{ \zeta \} \), then \( v \in \mathcal{M} \) and so, \( v \leq \varphi \) in \( D \). Therefore, the function \( \varphi \) is maximal on \( D \setminus \{ \zeta \} \). Furthermore, \( \varphi \in L^\infty_{\text{loc}}(D \setminus \{ \zeta \}) \) because \( \varphi \geq w_0/\sigma(w_0) \). It is not hard to see that \( \varphi(\zeta) = -\infty \). Indeed, assuming \( \varphi(\zeta) = A > -\infty \), the maximality of \( \varphi \) on \( \{ \varphi(z) < A \} \) gives \( \varphi \geq A \) everywhere, which contradicts \( \sigma(\varphi) > 0 \) in view of (ii). So, \( \varphi \in \text{MW}_\zeta(D) \).

Standard arguments involving a negative exhaustion function of \( D \) show that \( \varphi(z) \rightarrow 0 \) as \( z \rightarrow \partial D \).

By Proposition 3.1 \( u \leq \sigma(u, \varphi) \varphi + O(1) \) and thus, by (i) and (ii), \( \sigma(u) \geq \sigma(u, \varphi) \) for every \( u \in \text{PSH}^- (D) \). This gives, in particular, \( \sigma(u, \varphi) = 0 \) if \( \sigma(u) = 0 \). Let \( \sigma(u) > 0 \), then \( u/\sigma(u) \in \mathcal{M} \), so \( u \leq \sigma(u) \varphi \) and consequently, \( \sigma(u, \varphi) \geq \sigma(u) \). This proves (4.1).
If ψ is another weight from $MW_\zeta(D)$ with zero boundary values on $\partial D$, representing the functional $\sigma$, then $\psi \leq \varphi$. On the other hand, the relation $\sigma(\varphi, \psi) = 1$ implies that for any $\epsilon \in (0, 1)$ we have $\varphi \leq (1 - \epsilon)\psi + \epsilon$ on a neighbourhood of $\zeta$ and near $\partial D$ and thus, by the maximality of $\psi$ on $D \setminus \{\zeta\}$, everywhere in $D$.

Finally, the last assertion follows from the relation $\varphi \geq w_0/\sigma(w_0) \in LMW_\zeta$. □

**Remarks 4.2** (1) Note that for the functional $\sigma(u) = \sigma(u, \phi)$ with a continuous weight $\phi \in MW_\zeta$, the function $\varphi$ constructed in the proof of Lemma [4.1] is just the Green–Zahariuta function for the (continuous) singularity $\phi$ (2.9). We will keep this name for the case of Green functions with respect to arbitrary singularities $\phi \in MW_\zeta$,

$$G_{\phi,D}(z) = \sup\{u(z) \in PSH^{-}(D) : u \leq \phi + O(1) \text{ near } \zeta\}. \quad (4.2)$$

We have thus $G_{\phi,D} \in MW_\zeta(D)$, $G_{\phi,D} = \phi + O(1)$ because $\sigma(G_{\phi,D}, \phi) = 1$, and $G_{\phi,D}(z) \to 0$ as $z \to \partial D$ if $D$ is hyperconvex.

(2) Let $\preceq$ be a natural partial ordering on $MW_\zeta$: $\phi \preceq \psi$ if $\sigma(u, \phi) \leq \sigma(u, \psi)$ for any $u \in PSH_\zeta$. It is easy to see that $\phi \preceq \psi \iff \sigma(\phi, \psi) \geq 1 \iff G_{\phi,D} \leq G_{\psi,D}$ for some (and, consequently, for any) hyperconvex neighbourhood $D$ of $\zeta$.

The following representation theorem is an easy consequence of Lemma 4.1.

**Theorem 4.3** Let a function $\sigma : PSH_\zeta \to [0, \infty)$ satisfy conditions (i)–(v) of Lemma 4.1 for $u, u_k \in PSH_\zeta$ and $D$ a bounded hyperconvex neighbourhood of $\zeta$. Then there exists a weight $\varphi \in MW_\zeta$ such that $\sigma(u) = \sigma(u, \varphi)$ for every $u \in PSH_\zeta$. The representation is essentially unique: if two weights $\varphi$ and $\psi$ represent $\sigma$, then $\varphi = \psi + O(1)$ near $\zeta$. If, in addition, (v) is true with $w_0(z) = \log|z - \zeta|$, then $\varphi \in LMW_\zeta$.

**Proof.** We may assume $\sigma(w_0) = 1$. Let $\varphi \in MW_\zeta$ be the function constructed in Lemma 4.1. Exactly as in the proof of the lemma, we get $\sigma(u) \geq \sigma(u, \varphi)$ for every $u \in PSH_\zeta$. To prove the reverse inequality, take any $u \in PSH_\zeta$. The function $v_0 = \max\{u, \sigma(u)w_0\}$ can be extended from a neighbourhood of $\zeta$ to a plurisubharmonic function $v$ on a neighbourhood of $\overline{D}$; by (iii), $\sigma(u) = \sigma(v_0) = \sigma(v - \sup_D v) = \sigma(v - \sup_D v, \varphi) = \sigma(v_0, \varphi) \leq \sigma(u, \varphi)$, so $\sigma(u) = \sigma(u, \varphi)$.

If $\psi \in MW_\zeta$ is another weight representing the functional $\sigma$, then $\sigma(\varphi, \psi) = \sigma(\psi, \varphi) = 1$ and $\psi = \varphi + O(1)$ by Proposition 3.3. □

**Remark 4.4** Recall that a valuation on the local ring $R_\zeta$ of germs of analytic functions $f$ at $\zeta$ is a nonconstant function $\mu : R_\zeta \to [0, +\infty]$ such that

$$\mu(f_1f_2) = \mu(f_1) + \mu(f_2), \quad \mu(f_1 + f_2) \geq \min\{\mu(f_1), \mu(f_2)\}, \quad \mu(1) = 0; \quad (4.3)$$
a valuation $\mu$ is centered if $\mu(f) > 0$ for every $f$ from the maximal ideal $m_\zeta$, and normalized if $\min \{\mu(f) : f \in m_\zeta\} = 1$. Every weight $\varphi \in MW_\zeta$ generates a functional $\sigma_\varphi$ on $R_\zeta$, $\sigma_\varphi(f) = \sigma(\log |f|, \varphi)$, with the properties

$$\sigma_\varphi(f_1 f_2) \geq \sigma_\varphi(f_1) + \sigma_\varphi(f_2), \quad \sigma_\varphi(f_1 + f_2) \geq \min \{\sigma_\varphi(f_1), \sigma_\varphi(f_2)\}, \quad \sigma_\varphi(1) = 0. \quad (4.4)$$

Such a functional is thus a valuation if the weight $\varphi$ satisfies the additional condition

$$\sigma(u + v, \varphi) = \sigma(u, \varphi) + \sigma(v, \varphi) \quad (4.5)$$

for any $u, v \in PSH_\zeta$ — in other words, if $u \mapsto \sigma(u, \varphi)$ is tropically linear (both additive and multiplicative); $\sigma_\varphi$ is centered if and only if $\varphi \in LMW_\zeta$, and normalized iff $\alpha_\varphi = 1$.

The weights $\phi_{a,\zeta}$ (2.5) satisfy (4.5), and the corresponding functionals $\sigma_{\phi_{a,\zeta}}$ are monomial valuations on $R_\zeta$; they are normalized, provided $\min_k a_k = 1$. It was shown in [6] that an important class of valuations in $C^2$ (quasimonomial valuations) can be realized as $\sigma_\varphi$ with certain weights $\varphi \in LMW_\zeta$ satisfying (4.5) and $\alpha_\varphi = 1$, and all other normalized valuations in $C^2$ can be realized as limits of increasing sequences of the quasimonomial ones. We believe the relative types with respect to weights satisfying (4.5) can be used in investigation of valuations in higher dimensions.

## 5 Greenifications

In this section we consider some extremal problems for plurisubharmonic functions with singularities determined by a given plurisubharmonic function $u$. Solutions to these problems resemble various Green functions mentioned in Section 2 (and in some cases just coincide with them), and we will call them greenifications of the function $u$. This reflects the point of view on Green functions as largest negative plurisubharmonic functions with given singularities; different types of the Green functions arise from different ways of measuring the singularities (or different portions of information on the singularities used). Note that the tropical additivity makes relative types an adequate tool in constructing extremal plurisubharmonic functions as upper envelopes.

### 5.1 Type-greenifications

Let a bounded domain $D$ contain a point $\zeta$ and let $u \in PSH_\zeta$. Given a collection $P$ of weights $\phi \in MW_\zeta$, denote

$$\mathcal{M}_u^P = \mathcal{M}_{u,\zeta,D} = \{v(x) : v \in PSH(D), \quad \sigma(v, \phi) \geq \sigma(u, \phi) \quad \forall \phi \in P\} \quad (5.1)$$
and define the function
\[ h^P_u(x) = h^P_{u,\zeta,D}(x) = \sup \{ v(x) : v \in \mathcal{M}^P_{u,\zeta,D} \}. \] (5.2)

We will write simply \( \mathcal{M}_{u,\zeta,D} \) and \( h_{u,\zeta,D} \) if \( P = MW_\zeta \). The function \( h^P_{u,\zeta,D} \) will be called the type-greenification of \( u \) with respect to the collection \( P \), and \( h_{u,\zeta,D} \) will be called just the type-greenification of \( u \) at \( \zeta \).

Consideration of the functions \( h^P_u \) with \( P \neq MW_\zeta \) can be useful in situations where the only information on the singularity of \( u \) available is the values of \( \sigma(u,\varphi) \) for certain selected weights \( \varphi \). One more reason is that for some collections \( P \), the functions \( h^P_u \) are quite easy to compute and, at the same time, they can give a reasonably good information on the asymptotic behaviour of \( u \) (see Examples 2 and 3 after Proposition 5.2). Note that \( h^P_u \geq h^Q_u \geq h_u \) if \( P \subset Q \subset MW_\zeta \).

**Proposition 5.1** Let \( u \in PSH(D) \) be bounded above in \( D \ni \zeta \), and \( P \subseteq MW_\zeta \). Then
\begin{enumerate}[(i)]
  \item \( h^P_{u,\zeta,D} \in PSH^-(D) \);
  \item \( u \leq h^P_{u,\zeta,D} + \sup_D u \);
  \item \( \sigma(u,\phi) = \sigma(h^P_{u,\zeta,D},\phi) \) for any weight \( \phi \in P \);
  \item \( h^P_{u,\zeta,D} \) is maximal on \( D \setminus \{ \zeta \} \);
  \item if \( D \) has a strong plurisubharmonic barrier at a point \( z \in \partial D \) (i.e., if there exists \( v \in PSH(D) \) such that \( \lim_{x \to z} v(x) = 0 \) and \( \sup_{D \setminus U} v < 0 \) for every neighbourhood \( U \) of \( z \)) and if \( u \) is bounded below near \( z \), then \( h^P_{u,\zeta,D}(x) \to 0 \) as \( x \to z \).
\end{enumerate}

**Proof.** Let \( w \) denote the right hand side of (5.2), then its upper regularization \( w^* \) is plurisubharmonic in \( D \). By the Choquet lemma, there exists a sequence \( v_j \in \mathcal{M}^P_u \) such that \( w^* = (\sup_j v_j)^* \). Denote \( w_k = \sup_j v_j \). By Proposition 3.1(ii), \( w_k \in \mathcal{M}^P_u \). Since the functions \( w_k \) converge weakly to \( w^* \), Proposition 3.3 implies then \( \sigma(w^*,\phi) \geq \sigma(w_k,\phi) \geq \sigma(u,\phi) \) for any weight \( \phi \in P \) and so, \( w^* \in \mathcal{M}^P_u \), which proves (i) and gives, at the same time, the inequality \( \sigma(u,\phi) \leq \sigma(h^P_u,\phi) \). The reverse inequality, even for arbitrary weights \( \phi \in MW_\zeta \), follows from the (evident) assertion (ii) and completes the proof of (iii).

If a function \( v \in PSH(D) \) satisfies \( v \leq h_{u,\zeta,D}^P \) on \( D \setminus \omega \) for some open set \( \omega \subseteq D \setminus \zeta \), then \( \max \{v, h_{u,\zeta,D}^P\} \in \mathcal{M}^P_u \) and therefore \( v \leq h_{u,\zeta,D}^P \) on \( \omega \), which proves (iv).

Finally, to prove (v), take a neighbourhood \( U \) of \( z \in \partial D \) such that \( u > t > -\infty \) on \( U \setminus D \) and choose \( c > 0 \) such that \( v < t/c + D \setminus U \), so \( u > cv \) on \( \partial U \setminus D \). Let
\[ w(x) = \begin{cases} \max\{u(x),cv(x)\}, & x \in D \cap U \\ u(x), & x \in D \setminus U, \end{cases} \] (5.3)

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then \( w \in \mathcal{M}^P_{u,\zeta,D}(u) \) and \( \lim_{x \to z} w(x) = 0. \)

If \( v \in PSH(D) \), then the relation \( \sigma(v, \varphi) \geq \sigma(u, \varphi) \) is equivalent to \( v \leq \sigma(u, \varphi) G_{\varphi,D}. \)
Therefore, these Green-like functions \( h^P_{u,\zeta,D} \) can be described by means of the Green–Zahariuta functions \( G_{\varphi,D} \) (4.2) as follows.

**Proposition 5.2** The function \( h^P_{u,\zeta,D}, P \subset MW_\zeta, \) is the largest plurisubharmonic minorant of the family \( \{ \sigma(u, \varphi) G_{\varphi,D} : \varphi \in P \} \). In particular, if \( \varphi \in MW_\zeta \), then \( h_{\varphi,\zeta,D} = h_{\varphi,\zeta,D} = G_{\varphi,D}. \)

**Examples 5.3**

(1) If \( P \) consists of a single weight \( \varphi \), then \( h^P_{u,\zeta,D} = \sigma(u, \varphi) G_{\varphi,D}. \) In particular, if \( \varphi(x) = \log |x - \zeta| \), then \( h^P_{u,\zeta,D} = \nu_u(\zeta) G_{\zeta,D}. \)

(2) Let \( A \) be a finite subset of \( \mathbb{R}^n_+ \) and let \( P \) be the collection of the weights \( \phi_a = \phi_{a,0} \) (2.5) with \( a \in A \). According to Proposition 5.2 and (2.4), the function \( h^P_{u,0,D} \) is the largest plurisubharmonic minorant of the family \( \{ \nu_u(0,a) G_{\phi_a,D} : a \in A \} \). Using methods from [13] and [14], it can then be shown that the minorant is the Green–Zahariuta function for the singularity \( \varphi_{u,A}(x) = \psi_{u,A}(\log |x_1|, \ldots, \log |x_n|) \), where

\[
\psi_{u,A}(t) = \sup \{ \langle b, t \rangle : b \in H_{u,A} \}, \quad t \in \mathbb{R}^n_-, \quad H_{u,A} = \bigcap_{a \in A} \{ b \in \mathbb{R}^n_+ : \sum_k b_k a_k \geq \nu_u(0,a) \}. \tag{5.4}
\]

Thus, if the only information on \( u \) is that it is locally bounded on \( D \setminus \{0\} \) and the values \( \nu_u(0,a) \) for \( a \in A \), then its residual Monge–Ampère mass at 0 can be estimated as

\[
(dd^c u)^n(0) \geq (dd^c h^P_{u,0,D})^n(0) = (dd^c \varphi_{u,A})^n(0) = n! Vol(H_{u,A}). \tag{5.5}
\]

(3) If \( P \) consists of all the directional weights \( \phi_{a,\zeta} \) (2.6), then \( h^P_{u,\zeta,D} = G_{\Psi,D} \), the Green function (2.10) with respect to the indicator \( \Psi = \Psi_{u,\zeta} \) of the function \( u \) at \( \zeta \).

(4) Let \( u = \log |z_1| \) in the unit polydisc \( D^n \) and let \( P \) be the collection of the weights \( \phi_{a,0} \) (2.5) with \( a_1 = 1 \). Then the type of \( u \) with respect to any \( \phi_{a,0} \in P \) equals 1 and thus, \( v \leq \phi_{a,0} \) for every \( v \in \mathcal{M}_{u,0,D^n} \) and any such direction \( a \). Therefore \( h^P_{u,0,D^n} = u. \) (For a more general statement, see Theorem 6.6.)

**Remarks 5.4**

(1) As follows from Example 4, the functions \( h^P_{u,\zeta,D} \) need not be locally bounded outside \( \zeta \).

(2) The same example shows that the condition of strong plurisubharmonic barrier cannot be replaced by hyperconvexity in general. However this can be done if \( u \in L^\infty_{loc}(\Omega \setminus \{\zeta\}) \), see Proposition 5.6.
5.2 Complete greenifications

Another natural extremal function determined by the singularity of \( u \) can be defined as follows. Let \( u \in PSH(\Omega) \) be such that \( u(\zeta) = -\infty \) for some \( \zeta \in \Omega \). Given a domain \( D \subset \Omega, \zeta \in D \), consider the class

\[ \mathcal{F}_{u,\zeta,D} = \{ v \in PSH^{-}(D) : v(z) \leq u(z) + O(1), \ z \to \zeta \}, \quad (5.6) \]

then the upper regularization of its upper envelope is a plurisubharmonic function in \( D \); we will call it the complete greenification of \( u \) at \( \zeta \) and denote by \( g_{u,\zeta,D} \):

\[ g_{u,\zeta,D}(x) = \limsup_{y \to x} \sup \{ v(y) : v \in \mathcal{F}_{u,\zeta,D} \}. \quad (5.7) \]

If \( \varphi \in MW_{\zeta} \), then \( g_{\varphi,\zeta,D} = G_{\varphi,D} \), the Green–Zahariuta function \([1,2]\).

It follows from the definition that if \( u \) is bounded above on \( D \), then

\[ u \leq g_{u,\zeta,D} + \sup_{D} u. \quad (5.8) \]

It is easy to see that the function \( g_{u,\zeta,D} \) need not belong to the class \( \mathcal{F}_{u,\zeta,D} \) (take, for example, \( u = -|\log |z||^{1/2} \), then \( g_{u,\zeta,D} \equiv 0 \)).

**Proposition 5.5** If a plurisubharmonic function \( u \) is bounded above on \( D \ni \zeta \), then

(i) \( g_{u,\zeta,D} \) is maximal on any open \( \omega \subset D \) such that \( g_{u,\zeta,D} \in L_{loc}^{\infty}(\omega) \);

(ii) \( \nu(u, \varphi) = \nu(g_{u,\zeta,D}, \varphi) \) for any continuous weight \( \varphi \) with \( \varphi^{-1}(-\infty) = \zeta \);

(iii) \( \sigma(u, \varphi) = \sigma(g_{u,\zeta,D}, \varphi) \) for any \( \varphi \in MW_{\zeta} \);

(iv) if \( D \) has a strong plurisubharmonic barrier at a point \( z \in \partial D \) and if \( u \) is bounded below near \( z \), then \( \lim_{x \to z} g_{u,\zeta,D}(x) = 0 \).

**Proof.** Take a sequence of pseudoconvex domains \( D_j \) such that \( D_{j+1} \Subset D_j \Subset D \), \( \cap_{j} D_j = \{ \zeta \} \), and let

\[ u_j(x) = \sup \{ v(x) : v \in PSH^{-}(D), \ v \leq u - \sup_{D} u \text{ in } D_j \}, \ x \in D. \quad (5.9) \]

Since its upper regularization \( u_{j}^{*} \) belongs to \( PSH^{-}(D) \) and coincides with \( u - \sup_{D} u \) in \( D_j \), the function \( u_j = u_{j}^{*} \in \mathcal{F}_{u,\zeta,D} \) and is maximal on \( D \setminus D_j \). When \( j \to \infty \), the functions \( u_j \) increase to a function \( v \) such that \( v^{*} \in PSH^{-}(D) \) and is maximal where it is locally bounded. Evidently, \( g_{u,\zeta,D} \geq v^{*} \).
By the Choquet lemma, there exists a sequence \( w_k \in F_{u,\zeta,D} \) that increases to \( w \) such that \( w^* = g_{u,\zeta,D} \). Take any \( \epsilon > 0 \), then for each \( k \) there exists \( j = j(k) \) such that \( w_k \leq (1 - \epsilon)u_j \) on \( D_j \). Therefore \( w_k \leq (1 - \epsilon)u_j \leq (1 - \epsilon)g_{u,\zeta,D} \) in \( D \), which gives \( g_{u,\zeta,D} \leq (1 - \epsilon)v^* \) for all \( \epsilon > 0 \) and thus, \( g_{u,\zeta,D} = v^* \), and (i) follows now from the maximality of \( v^* \).

To prove (ii), consider again the functions \( u_j (5.9) \), then \( \nu(u_j, \varphi) = \nu(u, \varphi) \) for any \( \varphi \). By Theorem 2.1,

\[
\nu(g_{u,\zeta,D}, \varphi) \geq \limsup_{j \to \infty} \nu(u_j, \varphi) = \nu(u, \varphi),
\]

(5.10) while the reverse inequality follows from (5.8) by Theorem 2.3.

Similar arguments (but now using Propositions 3.5 and 3.1 (v) instead of Theorems 2.1 and 2.3) prove (iii). Finally, (iv) can be proved exactly as assertion (v) of Proposition 5.1. □

More can be said if \( u \) is locally bounded outside \( \zeta \). Note that then it can be extended (from a neighbourhood of \( \zeta \)) to a plurisubharmonic function in the whole space, and none of its greenifications at \( \zeta \) depend on the choice of the extension.

**Proposition 5.6** If \( D \) is a bounded hyperconvex domain, \( u \in PSH(D) \cap L^\infty_{loc}(D \setminus \zeta) \), then

\[
\lim_{x \to z} g_{u,\zeta,D}(x) = 0, \quad z \in \partial D,
\]

\[
(dd^cu)^n(\zeta) = (dd^cg_{u,\zeta,D})^n(\zeta).
\]

(5.11)

**Proof.** The first statement follows exactly as in the case of the pluricomplex Green function with logarithmic singularity.

To prove (5.11), observe first that relation (5.8) implies \( (dd^cu)^n(\zeta) \geq (dd^cg_{u,\zeta,D})^n(\zeta) \). On the other hand, the functions \( u_j (5.9) \) belong to the Cegrell class \( F \) and increase a.e. to \( g_{u,\zeta,D} \). By Theorem 5.4 of [1], \( (dd^cu_j)^n \to (dd^cg_{u,\zeta,D})^n \). Therefore, \( (dd^cg_{u,\zeta,D})^n(\zeta) \geq \limsup_{j \to \infty} (dd^cu_j)^n(\zeta) = (dd^cu)^n(\zeta) \), which completes the proof. □

**Remark 5.7** In spite of the relations in Proposition 5.5 and (5.11), some important information on the singularity can be lost when passing to the function \( g_{u,\zeta,D} \). For example, if we take \( u(z) = \max\{ \log |z_1|, - \log |z_2|^{1/2} \} \), then \( g_{u,0,D} = 0 \) for any \( D \subset \mathbb{C}^2 \) containing 0, while \( \nu(u, \log |z_2|) = 1 \) (note that the function \( \log |z_2| \) is semiexhaustive on the support of \( dd^cu \)).

6 Greenifications and Green functions

Now we turn to relations between the extremal functions considered above. We will write \( M^S_{u,\zeta,D} \) and \( h^S_{u,\zeta,D} \) if \( P = SMW_\zeta \) (3.12), and \( M^L_{u,\zeta,D} \) and \( h^L_{u,\zeta,D} \) if \( P = LMW_\zeta \) (3.13); in view of (3.14), \( h^L_{u,\zeta,D} \geq h^S_{u,\zeta,D} \).
According to Proposition 5.5, we have \( \sigma(g_{u,\zeta,D}, \varphi) = \sigma(u, \varphi) \) for all \( \varphi \in MW_\zeta \), so
\[
g_{u,\zeta,D} \leq h_{u,\zeta,D} \leq h^P_{u,\zeta,D} \tag{6.1}
\]
for any \( u \in PSH(D) \) and \( P \subset MW_\zeta \).

By Proposition 5.2, the condition \( g_{u,\zeta,D} \equiv 0 \) implies \( h^P_{u,\zeta,D} \equiv 0 \) for every \( P \subseteq MW_\zeta \). So let us assume \( g_{u,\zeta,D} \not\equiv 0 \).

When \( \varphi \in MW_\zeta \), the function \( g_{\varphi,\zeta,D} \) is the Green–Zahariuta function for the singularity \( \varphi \) in \( D \); by Proposition 5.2, the same is true for \( h_{\varphi,\zeta,D} \), so \( g_{\varphi,\zeta,D} = h_{\varphi,\zeta,D} \). More generally, we have the following simple

**Proposition 6.1** Let \( u \in PSH_\zeta \) be locally bounded outside \( \zeta \), then \( g_{u,\zeta,D} = h_{u,\zeta,D} \). If, in addition, \( \nu_u(\zeta) > 0 \), then \( g_{u,\zeta,D} = h^S_{u,\zeta,D} \).

**Proof.** The equalities result from Proposition 5.2 and (5.8) because \( g_{u,\zeta,D} \) belongs to \( MW_\zeta \) and, in case of \( \nu_u(\zeta) > 0 \), to \( SMW_\zeta \). □

**Remark 6.2** We do not know if \( g_{u,\zeta,D} = h^S_{u,\zeta,D} \) when \( \nu_u(\zeta) = 0 \). The condition \( \nu_u(\zeta) = 0 \) implies, by (3.15), \( h^S_{u,\zeta,D} \equiv 0 \). As follows from Theorem 6.4 below, for functions \( u \) locally bounded outside \( \zeta \) the relation \( g_{u,\zeta,D} = h^S_{u,\zeta,D} \) is thus equivalent to \( (dd^cw)^n(\zeta) = 0 \), and we are facing the problem of existence of plurisubharmonic functions with zero Lelong number and positive Monge-Ampère mass. It can be reformulated as follows: is it true that \( g_{u,\zeta,D} \equiv 0 \) if \( u \) is locally bounded outside \( \zeta \) and \( \nu_u(\zeta) = 0 \)? Equivalently: is it true that \( SMW_\zeta = MW_\zeta \)?

To study the situation with the type-greenifications with respect to arbitrary subsets \( P \) of \( MW_\zeta \), we need the following result on "incommensurability" of Green functions.

**Lemma 6.3** Let \( D \) be a bounded hyperconvex domain and let \( v, w \in PSH(D) \cap L^\infty_{\text{loc}}(D \setminus \zeta) \) be two solutions of the Dirichlet problem \( (dd^cu)^n = \tau \delta_\zeta, \ u|_{\partial D} = 0 \) with some \( \tau > 0 \). If \( v \geq w \) in \( D \), then \( v \equiv w \).

**Proof.** We use an idea from the proof of Theorem 3.3 in [21]. Choose \( R > 0 \) such that \( \rho(x) = |x|^2 - R^2 < 0 \) in \( D \). Given \( \epsilon > 0 \), consider the function \( u_\epsilon = \max\{v + \epsilon \rho, w\} \). Since \( u_\epsilon \) is \( 0 \) near \( \partial D \), we have
\[
\int_D (dd^cu_\epsilon)^n = \int_D (dd^cw)^n = \tau. \tag{6.2}
\]
On the other hand, \( u_\epsilon \leq v \) and thus, by Theorem 2.3
\[
(dd^cu_\epsilon)^n(\zeta) \geq (dd^cv)^n(\zeta) = \tau. \tag{6.3}
\]
Therefore (6.2) implies \((dd^c u_\epsilon)^n = 0\) on \(D \setminus \{\zeta\}\). The functions \(v + \epsilon \rho\) and \(w\) are locally bounded outside \(\zeta\), so
\[
(dd^c u_\epsilon)^n \geq \chi_1 (dd^c (v + \epsilon \rho))^n + \chi_2 (dd^c w)^n
\]
on \(D \setminus \{\zeta\}\) (\cite{14}, Proposition 11.9), where \(\chi_1\) and \(\chi_2\) are the characteristic functions of the sets \(E_1 = \{ w \leq v + \epsilon \rho \} \setminus \{ \zeta \}\) and \(E_2 = \{ w > v + \epsilon \rho \} \setminus \{ \zeta \}\), respectively. This gives
\[
\epsilon^n \int_{E_1} (dd^c \rho)^n \leq \chi_1 (dd^c (v + \epsilon \rho))^n = 0.
\]Hence, the set \(\{ w < v \}\) has zero Lebesgue measure, which proves the claim. □

**Theorem 6.4** Let \(D\) be bounded and hyperconvex, \(u \in PSH(D) \cap L^\infty_{loc}(D \setminus \zeta)\), and let \(P \subseteq MW_\zeta\). Then \(g_{u,\zeta,D} = h_{u,\zeta,D}^P\) if and only if
\[
(dd^c u)^n(\zeta) = (dd^c h_{u,\zeta,D}^P)^n(\zeta).
\]

**Proof.** If \(h_{u,\zeta,D}^P = g_{u,\zeta,D}\), then (6.6) follows from Proposition 5.6. The reverse implication follows from Lemma 6.3 (by Proposition 5.6 and (6.1), the functions \(v = h_{u,\zeta,D}^P\) and \(w = g_{u,\zeta,D}\) satisfy the conditions of the lemma). □

This can be applied to evaluation of the multiplicity of an equidimensional holomorphic mapping by means of its Newton polyhedron. Let \(\zeta\) be an isolated zero of an equidimensional holomorphic mapping \(f\). Denote by \(\Gamma_+(f,\zeta)\) the Newton polyhedron of \(f\) at \(\zeta\), i.e., the convex hull of the set \(E_\zeta^+ + \mathbb{R}_+^n\), where \(E_\zeta^+ \subset \mathbb{Z}_+^n\) is the collection of the exponents in the Taylor expansions of the components of \(f\) about \(\zeta\), and let \(N_\zeta\) denote the Newton number of \(f\) at \(\zeta\), i.e., \(N_\zeta = n! \text{Vol}(\mathbb{R}_+^n \setminus \Gamma_+(f,\zeta))\). Kouchnirenko’s theorem \cite{10} states that the multiplicity \(m_\zeta\) of \(f\) at \(\zeta\) can be estimated from below by the Newton number,
\[
m_\zeta \geq N_\zeta,
\]
with an equality under certain non-degeneracy conditions. An application of Theorem 6.4 gives a necessary and sufficient condition for the equality to hold.

As was shown in \cite{14} and \cite{15}, \(N_\zeta = (dd^c \Psi_{u,\zeta})^n(0)\), where \(\Psi_{u,\zeta}\) is the indicator (2.6) of the plurisubharmonic function \(u = \log |f|\) at \(\zeta\). Let \(P\) consist of all the directional weights \(\phi_{a,\zeta}\), \(a \in \mathbb{R}_+^n\), and let \(D\) be a ball around \(\zeta\). Then (see Example 3 after Proposition 5.2) the function \(h_{u,\zeta,D}^P\) coincides with the Green function (2.10) with respect to the indicator \(\Psi_{u,\zeta}\), which in turn equals the Green–Zahariuta function \(G_{\varphi,D}\) (2.9) for the singularity \(\varphi(x) = \Psi_{u,\zeta}(x - \zeta)\). Therefore \(h_{u,\zeta,D}^P = g_{\varphi,\zeta,D}\) and
\[
N_\zeta = (dd^c h_{u,\zeta,D}^P)^n(\zeta).
\]
Since \( m_{\zeta} = (dd^c u)^n(\zeta), \) the equality \( m_{\zeta} = N_{\zeta} \) is equivalent to (6.6) and thus, by Theorem 6.4 to \( g_{u,\zeta,D} = g_{\varphi,\zeta,D}. \) Finally, as \( u, \varphi \in MW_{\zeta}, \) we have \( u = g_{u,\zeta,D} + O(1) \) and \( \varphi = g_{\varphi,\zeta,D} + O(1) \) near \( \zeta, \) which gives \( u = \varphi + O(1). \) We have just proved the following

**Corollary 6.5** The multiplicity of an isolated zero \( \zeta \) of an equidimensional holomorphic mapping \( f \) equals its Newton number at \( \zeta \) if and only if \( \log |f(x)| = \Psi(x - \zeta) + O(1) \) as \( x \to \zeta, \) where \( \Psi = \Psi_{\log |f|, \zeta} \) is the indicator (2.7) of the function \( \log |f| \) at \( \zeta. \)

The situation with non-isolated singularities looks more complicated. Observe, for example, that \( h_{u,\zeta,D} \) is maximal on the whole \( D \setminus \{\zeta\}, \) however we do not know if the same is true for \( g_{u,\zeta,D}. \)

We can handle the situation in the case of analytic singularities. In this section, we prove the equality \( g_{u,\zeta,D} = h_{L,\zeta,D} \) for \( u = \log |f|, \) where \( f : D \to \mathbb{C} \) is a holomorphic function; we recall that \( h_{L,\zeta,D} \) is the type-greenification with respect to the class \( LMW_{\zeta} \) (3.13). In this case, the greenifications coincide with the Green function \( G_{A,D} \) (2.13) in the sense of Lárusson–Sigurdsson. Note that the function \( G_{A,D} \) is defined as the upper envelope of functions \( u \) with \( \nu_u(a) \geq \nu_A(a) \) for all \( a \in |A| \). It turns out that one can consider only one point (or finitely many ones) from \( |A|, \) but then an infinite set of weights should be used. The case of mappings \( f : D \to \mathbb{C}^p, p > 1, \) will be considered in Section 7.

**Theorem 6.6** Let \( u = \log |f|, \) where \( f \) is a holomorphic function on \( \Omega. \) Given \( \zeta \in f^{-1}(0), \) let \( f = ab \) with \( a = a_1^{m_1} \cdots a_k^{m_k} \) such that \( a_j \) are irreducible factors of \( f \) vanishing at \( \zeta \) and \( b(\zeta) \neq 0. \) Then for any hyperconvex domain \( D \Subset \Omega \) that contains \( \zeta, \)

\[
h_{u,\zeta,D} = g_{u,\zeta,D} = G_{A,\zeta,D},
\]

where \( G_{A,\zeta,D} \) is the Green function (2.13) for the divisor \( A_{\zeta} \) of the function \( a. \) Moreover, there exists a sequence \( P \) of continuous weights \( \varphi_j \in LMW_{\zeta} \) such that the Green–Zahariuta functions \( G_{\varphi_j,D} \) decrease to \( h_{u,\zeta,D} = G_{A,\zeta,D}. \)

**Proof.** Let \( F_{A,\zeta,D} \) be the class defined by (2.12) for \( A = A_{\zeta}, \) then \( F_{A,\zeta,D} \subset F_{u,\zeta,D} \subset M_{u,\zeta,D}, \) so

\[
G_{A_{\zeta},D} \leq g_{u,\zeta,D} \leq h_{u,\zeta,D}.
\]

Choose a sequence of domains \( D_j \) such that \( D_{j+1} \Subset D_j \Subset D, \cap_j D_j = \{\zeta\}, \) and \( fa^{-1} \) does not vanish in \( D_1, \) then the functions \( u_j \) defined by (5.9) satisfy

\[
u_j \leq \log |a| + C_j \quad \text{in} \quad D_j.
\]
By Siu’s theorem, the set \( \{ x \in D : \nu(u_j, x) \geq m_k \} \) is analytic; by (6.11), it contains the support \(|A_\zeta|\) of the divisor \(A_\zeta\) of \(a_\zeta\). Therefore, \(\nu(u_j, x) \geq \nu(\log |a|, x)\) for all \(x \in |A_\zeta|\). Since \(u_j\) converge to \(g_{u, \zeta, D}\), this implies \(\nu(g_{u, \zeta, D}, x) \geq \nu(\log |a|, x)\) and thus, \(g_{u, \zeta, D} \in F_{A_\zeta, D}\). This proves the second equality in (6.9).

Let \(f_2, \ldots, f_n\) be holomorphic functions in a neighbourhood \(\omega \in D\) of \(\zeta\) such that \(\zeta\) is the only point of the zero set of the mapping \((f_1, \ldots, f_n)\) in \(\omega\) and \(\Omega_0 = \{ z \in \omega : |f_k| < 1, 1 \leq k \leq n \} \in \omega\), where \(f_1 = 2a(\sup_\omega |a|)^{-1}\). Denote

\[
\varphi := \sup \{ \log |f_1|, j \log |f_k| : 2 \leq k \leq n \}, \quad j \in \mathbb{Z}^+.
\]

We have \(\varphi_j \in \text{LMW}_{\zeta}(\Omega_0)\) and \(\varphi_j = 0\) on \(\partial \Omega_0\), so \(\varphi_j = G_{\varphi_j, \Omega_0}\), the Green–Zahariuta function for the singularity \(\varphi\) in \(\Omega_0\). Since \(h_{u, \zeta, D}^L < 0\) in \(\Omega_0\) and \(\sigma(h_{u, \zeta, D}^L, \varphi_j) = \sigma(u, \varphi_j) = 1\), we have \(h_{u, \zeta, D}^L \leq \varphi_j\) in \(\Omega_0\) for each \(j\). Therefore \(h_{u, \zeta, D}^L \leq \log |f_1| = \lim_{j \to \infty} \varphi_j\) in \(\Omega_0\). This implies \(h_{u, \zeta, D}^L \in F_{u, \zeta, D}\) and thus, \(h_{u, \zeta, D}^L = g_{u, \zeta, D}\).

Finally, the Green–Zahariuta functions \(G_{\varphi_j, D}\) dominate \(h_{u, \zeta, D}^L\) and decrease to some function \(v \in \text{PSH}^-(D)\). Since \(\sigma(v, \varphi_k) \geq \lim \sup_{j \to \infty} \sigma(\varphi_j, \varphi_k) = 1\), we get \(v \leq h_{u, \zeta, D}^L\), which completes the proof. \(\square\)

One can also consider the greenifications with respect to arbitrary finite sets \(\mathcal{Z} \subset D\),

\[
h_{u, \mathcal{Z}, D}^P(x) = \sup \{ v(x) : v \in \mathcal{M}_{u, \zeta, D}^P, \zeta \in \mathcal{Z} \}
\]

and

\[
g_{u, \mathcal{Z}, D}(x) = \limsup_{y \to x} \sup \{ v(y) : v \in F_{u, \zeta, D}, \zeta \in \mathcal{Z} \}.
\]

They have properties similar to those of the functions \(h_{u, \zeta, D}^P\) and \(g_{u, \zeta, D}\), stated in Propositions 5.4, 5.5 (with obvious modifications). In particular, if \(\varphi \in \text{PSH}(D)\) is such that \(\varphi^{-1}(-\infty) = \mathcal{Z}\) and the restriction of \(\varphi\) to a neighbourhood of \(\zeta\) belongs to \(\text{MW}_{\zeta}\) for each \(\zeta \in \mathcal{Z}\), then \(h_{\varphi, \mathcal{Z}, D} = g_{\varphi, \mathcal{Z}, D} = G_{\varphi, D}\), the Green–Zahariuta function with the singularities defined by \(\varphi\). They are also related to the Green functions \(\{2, 13\}\) as follows (cf. Theorem 6.6).

**Theorem 6.7** Let \(A\) be the divisor of a holomorphic function \(f\) in \(\Omega\) and let \(u = \log |f|\). If a finite subset \(\mathcal{Z}\) of a hyperconvex domain \(D \in \Omega\) is such that each irreducible component of \(|A| \cap D\) contains at least one point of \(\mathcal{Z}\), then \(h_{u, \mathcal{Z}, D}^L = g_{u, \mathcal{Z}, D} = G_{A, D}\).

**7 Analyticity theorem**

We let \(\Omega\) be a pseudoconvex domain in \(\mathbb{C}^n\), and let \(R : \Omega \to (-\infty, \infty]\) be a lower semicontinuous function on \(\Omega\). We consider a continuous plurisubharmonic function \(\varphi : \Omega \times \Omega \to [\infty, \infty)\) such that:
(i) \( \varphi(x, \zeta) < R(\zeta) \) on \( \Omega \times \Omega \);

(ii) \{ x : \varphi(x, \zeta) = -\infty \} = \{ \zeta \};

(iii) for any \( \zeta \in \Omega \) and \( r < R(\zeta) \) there exists a neighbourhood \( U \) of \( \zeta \) such that the set \( \{(x, y) : \varphi(x, y) < r, y \in U\} \in \Omega \times \Omega \);

(iv) \( (dd^c \varphi)^n = 0 \) on \( \{ \varphi(x, \zeta) > -\infty \} \);

(v) \( e^{\varphi(x, \zeta)} \) is H"older continuous in \( \zeta \):

\[ \exists \beta > 0 : \quad |e^{\varphi(x, \zeta)} - e^{\varphi(x, y)}| \leq |\zeta - y|^\beta, \quad x, y, \zeta \in \Omega. \] (7.1)

It then follows that \( \varphi_\zeta(x) := \varphi(x, \zeta) \in SMW_\zeta \) [3.12]. Similarly to [3.3] and [3.1] we introduce the function \( \Lambda(u, \varphi_\zeta, r) \) and the relative type \( \sigma(u, \varphi_\zeta) \). By Theorem 6.8 of [2], \( \Lambda(u, \varphi_\zeta, r) \) is plurisubharmonic on each connected component of the set \( \{ \zeta : R(\zeta) > r \} \).

As was shown (in a more general setting) by Demailly [3], the sets \( \{ \zeta : \nu(u, \varphi_\zeta) \geq c \} \) are analytic for all \( c > 0 \). By an adaptation of Kiselman’s and Demailly’s proofs of Siu’s theorem, we prove its analogue for the relative types. Denote

\[ S_c(u, \varphi, \Omega) = \{ \zeta \in \Omega : \sigma(u, \varphi_\zeta) \geq c \}, \quad c > 0. \] (7.2)

**Theorem 7.1** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^n \), a function \( \varphi(x, \zeta) \) satisfy the above conditions (i)–(v), and \( u \in PSH(\Omega) \). Then \( S_c(u, \varphi, \Omega) \) is an analytic subset of \( \Omega \) for each \( c > 0 \).

**Proof.** By Theorem 6.11 of [2], the function \( U(\zeta, \xi) = \Lambda(u, \varphi_\zeta, \text{Re} \xi) \) is plurisubharmonic in \( \{(\zeta, \xi) \in \Omega \times \mathbb{C} : \text{Re} \xi < R(\zeta)\} \). Fix a pseudoconvex domain \( D \subseteq \Omega \) and denote \( R_0 = \inf \{R(\zeta) : \zeta \in D\} > -\infty \). Given \( a > 0 \), the function \( U(\zeta, \xi) - a \text{Re} \xi \) is then plurisubharmonic and independent of \( \text{Im} \xi \), so by Kiselman’s minimum principle [7], the function

\[ U_a(\zeta) = \inf\{\Lambda(u, \varphi_\zeta, r) - ar : r < R_0\} \] (7.3)

is plurisubharmonic in \( D \).

Let \( \zeta \in D \). If \( a > \sigma(u, \varphi_\zeta) \), then \( \Lambda(u, \varphi_\zeta, r) > ar \) for all \( r \leq r_0 < \min \{R_0, 0\} \). If \( r_0 < r < R_0 \), then \( \Lambda(u, \varphi_\zeta, r) - ar > \Lambda(u, \varphi_y, r_0) - aR_0 \). Therefore \( U_a(\zeta) > -\infty \).

Now let \( a < \sigma(u, \varphi_\zeta) \). H"older continuity (7.1) implies

\[ \Lambda(u, \varphi_y, r) \leq \Lambda(u, \varphi_\zeta, \log(e^r + |y - \zeta|^\beta)) \leq \sigma(u, \varphi_\zeta) \log(e^r + |y - \zeta|^\beta) + C \] (7.4)
in a neighbourhood of \( \zeta \). Denote \( r_y = \beta \log |y - \zeta| \), then
\[
U_a(y) \leq \Lambda(u, \varphi_y, r_y) - ar_y \leq (\sigma(u, \varphi_\zeta) - a)\beta \log |y - \zeta| + C_1 \tag{7.5}
\]
near \( \zeta \).

Finally, let \( Z_{a,b} (a, b > 0) \) be the set of points \( \zeta \in D \) such that \( \exp(-b^{-1}U_a) \) is not integrable in a neighbourhood of \( \zeta \). As follows from the Hörmander–Bombieri–Skoda theorem, the sets \( Z_{a,b} \) are analytic.

If \( \zeta \notin S_c(u, \varphi, \Omega) \), then the function \( b^{-1}U_a \) with \( \sigma(u, \varphi_\zeta) < a < c \) is finite at \( \zeta \) and so, by Skoda’s theorem, \( \zeta \notin Z_{a,b} \) for all \( b > 0 \). If \( \zeta \in S_c(u, \varphi, \Omega) \), \( a < c \), and \( b < (c-a)\beta(2n)^{-1} \), then \( 7.5 \) implies \( \zeta \in Z_{a,b} \). Thus, \( S_c(u, \varphi, \Omega) \) coincides with the intersection of all the sets \( Z_{a,b} \) with \( a < c \) and \( b < (c-a)\beta(2n)^{-1} \), and is therefore analytic. \( \square \)

**Remark 7.2** The result can be reformulated in the following way: under the conditions of Theorem 7.1 the set \( S(u, \varphi, \Omega) = \{ \zeta \in \Omega : u(x) \leq \varphi(x, \zeta) + O(1) \text{ as } x \to \zeta \} \) is analytic. As was noticed by the referee, condition (iv) is actually necessary. Take, for example, the function \( \varphi(x, \zeta) = \max\{\log |x_1 - \zeta_1| + \log |(x_1 - \zeta_1)x_2|, \log |x_2 - \zeta_2|\} \) in \( \mathbb{C}^2 \times \mathbb{C}^2 \); it has all the properties except for (iv), while the set \( S(\log |x_1|, \varphi, \mathbb{C}^2) = \{(0, \zeta_2) : \zeta_2 \neq 0\} \) is not analytic.

As an application, we present the following result on propagation of plurisubharmonic singularities. We will say that a closed complex space \( A \) is a **locally complete intersection** if \( |A| \) is of pure codimension \( p \) and the associated ideal sheaf \( \mathcal{I}_A \) is locally generated by \( p \) holomorphic functions.

**Corollary 7.3** Let a closed complex space \( A \) be locally complete intersection on a domain \( D \subset \mathbb{C}^n \), and let \( \omega \) be an open subset of \( D \) that intersects each irreducible component of \( |A| \). If a function \( u \in PSH(D) \) satisfies \( u \leq \log |f| + O(1) \) locally in \( \omega \), where \( f_1, \ldots, f_p \) are local generators of \( \mathcal{I}_A \), then it satisfies this relation near every point of \( |A| \).

**Proof.** Denote by \( Z_l, l = 1, 2, \ldots \), the irreducible components of \( |A| \), and
\[
Z_l^* = (\text{Reg } Z_l) \setminus \bigcup_{k \neq l} Z_k. \tag{7.6}
\]

Every point \( z \in Z_l^* \) has a neighbourhood \( V \) and coordinates \( x = (x', x'') \in \mathbb{C}^p \times \mathbb{C}^{n-p} \), centered at \( z \), such that \( V \cap |A| = V \cap Z_l^* = V \cap \{x' = 0\} \) and \( f_1, \ldots, f_p \) are global generators of \( \mathcal{I}_A \) on \( V \). Let \( U \subset D \) be a pseudoconvex neighbourhood of \( z \) such that \( U - U \subset V \). For any \( \zeta \in U \) and \( N > 0 \), the function
\[
\varphi^N(x, \zeta) = \max \{\log |f(x - (\zeta', 0))|, N \log |x'' - \zeta''|\} \tag{7.7}
\]
satisfies the conditions of Theorem 7.1 with $\Omega = U$. Therefore, $S_1(u, \varphi^N, U)$ is an analytic subset of $U$. Note that $S_1(\log |f|, \varphi^N, U) = U \cap |A|$.

Let $\{U^j\}$ be a denumerable covering of $Z_K^*$ by such neighbourhoods, and let $\{\varphi_{i,j}^{j,N}\}$ be the corresponding weights. We may assume $u \leq \log |f| + O(1)$ near a point $\zeta_0 \in U^1 \cap \omega \cap |A|$, so $\sigma(u, \varphi_{\zeta, j}^{j,N}) \geq \sigma(\log |f|, \varphi_{\zeta, j}^{1,N}) = 1$ for every $\zeta \in U^1 \cap \omega \cap |A|$ and thus for every $\zeta \in U^1 \cap |A|$.

Take any $\zeta \in U^1 \cap Z_K^*$ and choose constants $a, b > 0$ such that

$$V_\zeta = \{ x \in U^1 : a|f(x)| < 1, b|x'' - \zeta''| < 1 \} \subset U^1. \quad (7.8)$$

Assuming $u \leq C$ in $U^1$, the relation $\sigma(u, \varphi_{\zeta, j}^{j,N}) \geq 1$ implies $u \leq g_{j,N} + C$ in $V_\zeta$, where

$$g_{j,N}(x) = \max \{ \log |af(x)|, N \log b|x'' - \zeta''| \} \quad (7.9)$$

is the Green–Zahariuta function in $V_\zeta$ for the singularity $\varphi_{\zeta, j}^{j,N}$. Observe that $g_{j,N}$ decrease to $\log |af|$ as $N \to \infty$, so $u \leq \log |f| + C_1$ near $\zeta$. Therefore we have extended the hypothesis of the theorem from $\omega$ to $\omega \cup U^1$. By repeating the argument, we get $u \leq \log |f| + O(1)$ near every point of $Z_K^*$ (since it is connected) and so, of $\text{Reg}|A|$. Finally, the bound near irregular points of $|A|$ can be deduced by using Thie’s theorem and the equation $(dd^c \log |f|)^p = 0$ outside $|A|$ (see, for example, the proof of Lemma 4.2 in [17]). \qed

Remarks 7.4

1. In particular, if $F$ is a holomorphic function in $D$ whose restriction to $\omega$ belongs to the integral closure $\mathcal{I}_{A,\omega}$ of the ideal sheaf $\mathcal{I}_{A,D}$ on $\omega$, then $F \in \mathcal{I}_{A,D}$. This follows from Corollary 7.3 because $F \in \mathcal{I}_A$ if and only if $\log |F| \leq \log |f| + O(1)$.

2. Corollary 7.3 fails when the complete intersection assumption is removed. Take, for example, $f = (z_1^2, z_1 z_2)$ in $\mathbb{C}^2$, then $\log |z_1| \leq \log |f(z)| + O(1)$ near every point $(0, \xi)$ with $\xi \neq 0$, but not near the origin.

A consequence of Corollary 7.3 is the following analogue to Theorems 6.6 and 6.7 for the Green functions (2.14) with singularities along complex spaces.

Corollary 7.5

Let a closed complex space $B$ be locally complete intersection on a pseudo-convex domain $\Omega \subset \mathbb{C}^n$, and let $A$ be the restriction of $B$ to a hyperconvex domain $D \subset \Omega$. Further, let $F_1, \ldots, F_m$ be global sections of $\mathcal{I}_B$ generating $\mathcal{I}_A$, and $u = \log |F|$. If a finite subset $\mathcal{Z}$ of $D$ is such that each irreducible component of $|A|$ contains at least one point of $\mathcal{Z}$, then

$$h_{u,\mathcal{Z},D}^L = \delta_{u,\mathcal{Z},D} = G_{A,D}, \quad (7.10)$$

where the functions $h_{u,\mathcal{Z},D}^L$ and $\delta_{u,\mathcal{Z},D}$ are defined by (6.13) and (6.14), $L$ denotes the collection of maximal weights with finite Lojasiewicz exponents (3.14) at the points of $\mathcal{Z}$, and $G_{A,D}$ is the Green function (2.14) for the space $A$.  

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Proof. We use the same arguments as in the proof of Theorems 6.6 and 6.7, the only difference being referring to Corollary 7.3 instead of Siu’s theorem.

By (2.15), we have $G_{A,D} \leq g_{u,Z,D}$. Let $Z = \{\zeta_1, \ldots, \zeta_s\}$. For $k = 1, \ldots, s$ take a sequence of pseudoconvex domains $D_{k,j}$ such that $D_{k,j+1} \Subset D_{k,j} \Subset D$, $\cap_j D_{k,j} = \{\zeta_k\}$, and denote $D_j = \cup_k D_{k,j}$. As in the proof of Proposition 5.5, the functions $u_j$ defined by (5.9) with such a choice of $D_j$ are plurisubharmonic in $D$, the sequence increases to $g_{u,Z,D}$ a.e. in $D$, and $u_j \leq \log |F| + C_j$ near each $\zeta_k$. Since $\log |F| = \log |f| + O(1)$, where $f_1, \ldots, f_p$ are local generators of $\mathcal{I}_A$, $p = \text{codim} \ |A|$, this gives, by Corollary 7.3 the relations $u_j \leq \log |f| + O(1)$ locally near $|A|$ and thus, $g_{u,Z,D} \leq G_{A,D}$. This proves the second equality in (7.10).

The rest is proved as in Theorem 6.6.

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References

[1] U. Cegrell, The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 1, 159–179.

[2] J.-P. Demailly, Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines, Mém. Soc. Math. France (N. S.) 19 (1985), 1–124.

[3] J.-P. Demailly, Nombres de Lelong généralisés, théorèmes d’intégralité et d’analyticité, Acta Math. 159 (1987), 153–169.

[4] J.-P. Demailly, Potential theory in several complex variables, manuscript. ICPAM, Nice, 1989.

[5] J.-P. Demailly, Monge-Ampère operators, Lelong numbers and intersection theory, Complex Analysis and Geometry (Univ. Series in Math.), ed. by V. Ancona and A. Silva, Plenum Press, New York 1993, 115–193.

[6] C. Favre and M. Jonsson, Valuative analysis of planar plurisubharmonic functions, Invent. Math. 162 (2005), 271–311.

[7] C.O. Kiselman, Densité des fonctions plurisousharmoniques, Bull. Soc. Math. France 107 (1979), 295–304.

[8] C.O. Kiselman, Un nombre de Lelong raffiné, In: Séméinaire d’Analyse Complexe et Géométrie 1985-87, Fac. Sci. Monastir Tunisie 1987, 61–70.
[9] C.O. Kiselman, *Attenuating the singularities of plurisubharmonic functions*, Ann. Polon. Math. **LX.2** (1994), 173–197.

[10] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.

[11] F. Lárusson and R. Sigurdsson, *Plurisubharmonic functions and analytic discs on manifolds*, J. reine angev. Math. **501** (1998), 1–39.

[12] F. Lárusson and R. Sigurdsson, *Plurisubharmonic extremal functions, Lelong numbers and coherent ideal sheaves*, Indiana Univ. Math. J. **48** (1999), 1513–1534.

[13] P. Lelong and A. Rashkovskii, *Local indicators for plurisubharmonic functions*, J. Math. Pures Appl. **78** (1999), 233–247.

[14] A. Rashkovskii, *Newton numbers and residual measures of plurisubharmonic functions*, Ann. Polon. Math. **75** (2000), no. 3, 213–231.

[15] A. Rashkovskii, *Lelong numbers with respect to regular plurisubharmonic weights*, Results Math. **39** (2001), 320–332.

[16] A. Rashkovskii and R. Sigurdsson, *Green functions with analytic singularities*, Comptes Rendus Acad. Sci. Paris **340** (2005), Série I, 479–482.

[17] A. Rashkovskii and R. Sigurdsson, *Green functions with singularities along complex spaces*, Internat. J. Math. **16** (2005), no. 4, 333–355.

[18] J. Wiklund, *Pluricomplex charge at weak singularities*, preprint, 2005, [http://arxiv.org/abs/math.CV/0510671](http://arxiv.org/abs/math.CV/0510671)

[19] V.P. Zahariuta, *Spaces of analytic functions and maximal plurisubharmonic functions*. D.Sci. Dissertation, Rostov-on-Don, 1984.

[20] V.P. Zahariuta, *Spaces of analytic functions and Complex Potential Theory*, Linear Topological Spaces and Complex Analysis **1** (1994), 74–146.

[21] A. Zeriahi, *Pluricomplex Green functions and the Dirichlet problem for the complex Monge–Ampère operator*, Michigan Math. J. **44** (1997), 579–596.

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