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Non-Inertial Frames in Minkowski Space-Time, Accelerated either Mathematical or Dynamical Observers and Comments on Non-Inertial Relativistic Quantum Mechanics

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Abstract

After a review of the existing theory of non-inertial frames and mathematical observers in Minkowski space-time we give the explicit expression of a family of such frames obtained from the inertial ones by means of point-dependent Lorentz transformations as suggested by the locality principle. These non-inertial frames have non-Euclidean 3-spaces and contain the differentially rotating ones in Euclidean 3-spaces as a subcase.

Then we discuss how to replace mathematical accelerated observers with dynamical ones (their world-lines belong to interacting particles in an isolated system) and of how to define Unruh-DeWitt detectors without using mathematical Rindler uniformly accelerated observers. Also some comments are done on the transition from relativistic classical mechanics to relativistic quantum mechanics in non-inertial frames.
I. INTRODUCTION

The theory of non-inertial frames in special relativity (SR) is a topic rarely discussed and till recently there was no attempt to develop a consistent general theory of it. All the results of the standard model of elementary particles are defined in the inertial frames of Minkowski space-time. Only at the level of nuclear, atomic and molecular physics one needs a local study of non-inertial frames in SR, for instance the rotating ones for the Sagnac effect.

As can be seen in the review paper [1] most of the papers treating accelerated observers are concentrated on the Rindler uniformly accelerated ones due to their relevance for the Unruh-DeWitt effect. However these observers form a peculiar set existing only due to the Lorentz signature of Minkowski space-time: since they are asymptotic to the light-cone at past and future time infinity, they disappear in the non-relativistic (NR) limit together with the light-cone and do not identify any accelerated observer in Galilei space-time.

A first consistent theory of global non-inertial frames and accelerated observers in Minkowski space-time, together with their limit to Galilei space-time, was developed in the papers of Refs.[2]. It was motivated by relativistic metrology \(^1\) with its problem of clock synchronization, by the problem of the elimination of the relative times in relativistic bound states (absence of time-like excitations in spectroscopy) so as to arrive at a consistent formulation of relativistic quantum mechanics (RQM) [4] and by the necessity to have a formulation of non-inertial frames extendible to general relativity (GR) at least in the Post-Newtonian approximation used by space physics around the Earth and by astronomical conventions in the Solar System and outside it (see Ref.[3]). See Ref.[5] for a review on the use of this theory of non-inertial frames in special (SR) and general (GR) relativity.

The description of non-inertial frames in SR is highly non trivial because, due to the Lorentz signature of Minkowski space-time, time is no longer absolute and there is no notion of instantaneous 3-space: the only intrinsic structure is the conformal one, i.e. the light-cone as the locus of incoming and outgoing radiation. A convention on the synchronization of clocks is needed to define an instantaneous 3-space. For instance the 1-way velocity of light from one observer A to an observer B has a meaning only after a choice of a convention for synchronizing the clock in A with the one in B. Therefore the crucial quantity in SR is the 2-way (or round trip) velocity of light \(c\) involving only one clock. It is this velocity which is isotropic and constant in SR and replaces the standard of length in relativistic metrology [3].

The Einstein convention for the synchronization of clocks in Minkowski space-time uses the 2-way velocity of light to identify the Euclidean 3-spaces of the inertial frames centered on an inertial observer A by means of only his/her clock. The inertial observer A sends a ray of light at \(x_i^0\) towards the (in general accelerated) observer B; the ray is reflected towards A at a point P of B’s world-line and then reabsorbed by A at \(x_f^0\); by convention P is synchronous with the mid-point between emission and absorption on A’s world-line, i.e. \(x_P^0 = x_i^0 + \frac{1}{2} (x_f^0 - x_i^0) = \frac{1}{2} (x_i^0 + x_f^0)\). This convention selects the Euclidean instantaneous 3-spaces \(x^0 = ct = const.\) of the inertial frames centered on A. Only in this case does the one-way velocity of light between A and B coincides with the two-way one, \(c\). However if the

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\(^1\) See Ref.[3] for an updated review on relativistic metrology on Earth and in the Solar System.
observer A is accelerated, the convention can break down due to the possible appearance of coordinate singularities.

As reviewed in Ref.[2], the existing coordinatizations centered on accelerated observers, like either Fermi or Riemann-normal coordinates, hold only locally. They are based on the 1+3 point of view [6], in which only the world-line of a time-like observer is given. In each point of the world-line the observer 4-velocity determines an orthogonal 3-dimensional space-like tangent hyper-plane, which is identified with an instantaneous 3-space. However, these tangent planes intersect at a certain distance from the world-line (the so-called acceleration length depending upon the 4-acceleration of the observer [7]), where 4-coordinates of the Fermi type develop a coordinate singularity. Another type of coordinate singularity is developed in rigidly rotating coordinate systems at a distance \( r = c \) (\( \omega \) is the angular velocity and \( c \) the two-way velocity of light). This is the so-called "horizon problem of the rotating disk": a time-like 4-velocity becomes a null vector at \( \omega r = c \), like it happens on the horizon of a black-hole.

As a consequence, a theory of global non-inertial frames in Minkowski space-time has to be developed in a metrology-oriented way to overcome the pathologies of the 1+3 point of view. This has been done in the papers of Ref.[2] by using the 3+1 point of view in which, besides the world-line of a time-like observer, one gives a global nice foliation of the space-time with instantaneous 3-spaces. In Section II we give a review of the status of the theory.

In Ref.[2] there is the list of conditions needed to avoid every kind of pathology in the definition of non-inertial frames. Since they are complicated non-linear restrictions, till now only differentially rotating non-inertial frames in Euclidean 3-spaces are completely under control.

The aim of this paper is to extend this class to a family of non-inertial frames with non-Euclidean 3-spaces obtainable from inertial frames by means of point-dependent Lorentz transformations as suggested by the locality principle [7] (at each instant an accelerated detector gives the same data of an instantaneously comoving inertial detector). This will be done in Section III and in Appendix A.

Then in Section IV we will make some comments on the nature (either mathematical or dynamical) of the observers on which the non-inertial frames are centered. In particular we will study how to compare the descriptions given two mathematical observers (Alice and Bob), starting from the case in which one of them is the origin of the inertial rest frame of an isolated system. We will also show that at the classical level it is possible to have dynamical observers by using the world-lines of dynamical particles contained in the isolated system as origin of the non-inertial frame.

In Section V we will make some comments on how to extend RQM and relativistic entanglement (in the formulation of Refs. [4, 8]) to non-inertial frames, on what could be the meaning of a "quantum observer" and on the description of Unruh-DeWitt detectors [1] in our framework.

Finally in the Conclusions we will delineate some open problems.
II. REVIEW ON NON-INERTIAL FRAMES

After a review of global non-inertial frames and the description of isolated systems in them by means of parametrized Minkowski theories (Subsections A and B), we introduce the inertial rest frame of isolated systems, whose relativistic collective variables and Wigner covariant 3-variables in the rest Wigner 3-spaces are then given (Subsections C and D). In Subsection E there is the expression of differentially rotating frames, while in Subsection F there is the form of the rest-frame conditions in the non-inertial rest frames of isolated systems. The isolated systems described here consist only of scalar massive positive-energy particles, because in Section IV we will use one of them as a dynamical observer origin of the non-inertial frame.

A. Global Non-Inertial Frames from the 3+1 Point of View

Assume that the world-line \( x^\mu(\tau) \) of an arbitrary time-like observer \(^2\) carrying a standard atomic clock is given: \( \tau \) is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible 3+1 splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces \( \Sigma_\tau \). It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of \( \Sigma_\tau \) sign the same time of the atomic clock of the observer \(^3\). The observer and the foliation define a global non-inertial reference frame after a choice of 4-coordinates. On each 3-space \( \Sigma_\tau \) one chooses curvilinear 3-coordinates \( \sigma^r \) having the observer as origin. The quantities \( \sigma^A = (\tau; \sigma^r) \) are the Lorentz-scalar and observer-dependent radar 4-coordinates, first introduced by Bondi \(^9\).

Giving the whole world-line of an arbitrary time-like observer and moreover a nice foliation with 3-spaces is a non-factual necessity required by the Cauchy problem. Once we have given the Cauchy data on the initial Cauchy surface (a un-physical process), we can predict the future with every observer receiving the information only from his/her past light-cone (retarded information from inside it; electromagnetic signals on it) \(^4\). For non-relativistic observers the situation is simpler, but the non-factual need of giving the Cauchy data on a whole initial absolute Euclidean 3-space is present also in this case for non-relativistic field equations like the Euler equation for fluids.

If \( x^\mu \mapsto \sigma^A(x) \) is the coordinate transformation from the Cartesian 4-coordinates \( x^\mu \) of an inertial frame centered on a reference inertial observer to radar coordinates, its inverse \( \sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r) \) defines the embedding functions \( z^\mu(\tau, \sigma^r) \) describing the 3-spaces \( \Sigma_\tau \) as embedded 3-manifolds into Minkowski space-time. The induced 4-metric on \( \Sigma_\tau \) is the

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\(^2\) An observer, or better a mathematical observer, is an idealization of a measuring apparatus containing an atomic clock and defining, by means of gyroscopes, a set of spatial axes (and then a, maybe orthonormal, tetrad with a convention for its transport) in each point of the world-line.

\(^3\) Actually the physical protocols (think of GPS) can establish a clock synchronization convention only inside future light-cone of the physical observer defining the local 3-spaces only inside it.

\(^4\) As far as we know the theorem on the existence and unicity of solutions has not yet been extended starting from data given only on the past light-cone.
following functional of the embedding: $4g_{AB}(\tau, \sigma^r) = [z^A_\mu \eta_{\mu \nu} z^B_\nu](\tau, \sigma^r)$, where $z^A_\mu = \partial z^A / \partial \sigma^A$ and $4\eta_{\mu \nu} = \epsilon (+ - - -)$ is the flat metric.  

While the 4-vectors $z^\mu(\tau, \sigma^u)$ are tangent to $\Sigma_\tau$, so that the unit normal $l^\mu(\tau, \sigma^u)$ is proportional to $\epsilon^\mu_{\alpha \beta \gamma} [z^\alpha_1 z^\beta_2 z^\gamma_3](\tau, \sigma^u)$, one has $z^\mu(\tau, \sigma^r) = [N l^\mu + N^r z^\mu](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon [z^\mu_\tau l_\mu](\tau, \sigma^r)$ and $N_r(\tau, \sigma^r) = -\epsilon 4g_{\tau r}(\tau, \sigma^r)$ being the lapse and shift functions respectively. The unit normal $l^\mu(\tau, \sigma^u)$ and the space-like 4-vectors $z^\mu(\tau, \sigma^u)$ identify a (in general non-ortho-normal) tetrad in each point of Minkowski space-time. The tetrad in the origin $\left(l^\mu(\tau, 0)\right)$ (in general non parallel to the observer 4-velocity), $z^\mu(\tau, 0)$ is a set of axes carried by the observer; their $\tau$-dependence implies a convention of transport along the world-line. See Ref.[5] for the two congruences of time-like observers (the Euclidean one with 4-velocity field equal to the unit normal and the other, not surface-forming, with velocity field proportional to $z^\mu(\tau, \sigma^r)$) associated with each global non-inertial frame.

Therefore starting from the four independent embedding functions $z^\mu(\tau, \sigma^r)$ one obtains the ten components $4g_{AB}$ of the 4-metric, which play the role of the inertial potentials generating the relativistic apparent forces in the non-inertial frame. For instance the shift functions $N_r(\tau, \sigma^u) = -\epsilon 4g(\tau, \sigma^u)$ describe inertial forces of the gravito-magnetic type induced by the global non-inertial frame. It can be shown [2] that the usual NR Newtonian inertial potentials are hidden in these functions. The extrinsic curvature tensor $3K_{rs}(\tau, \sigma^u) = \left[\frac{1}{2N} (N_{r|x} + N_{s|r} - \partial_r \epsilon^4 g_{rs})\right](\tau, \sigma^u)$, describing the shape of the instantaneous 3-spaces of the non-inertial frame as embedded 3-sub-manifolds of Minkowski space-time, is a secondary inertial potential, functional of the ten inertial potentials $4g_{AB}$.

Now a relativistic positive-energy scalar particle with world-line $x^\mu_\tau(\tau)$ is described by 3-coordinates $\eta^\nu(\tau)$ defined by $x^\mu_\tau(\tau) = z^\mu(\tau, \eta^\nu(\tau))$, satisfying equations of motion containing relativistic inertial forces with the correct non-relativistic limit as shown in Ref.[2, 10]. Fields have to be redefined so as to know the clock synchronization convention: for instance a Klein-Gordon field $\phi(\tau)$ has to be replaced with $\phi(\tau, \sigma^r) = \phi(z^\mu(\tau, \sigma^r))$.

The foliation is nice and admissible if it satisfies the conditions:

1) $N(\tau, \sigma^r) > 0$ in every point of $\Sigma_\tau$ so that the 3-spaces never intersect, avoiding the coordinate singularity of Fermi coordinates;

2) $\epsilon 4g_{\tau r}(\tau, \sigma^r) = (N^2 - N_u N^u)(\tau, \sigma^r) > 0$, so to avoid the coordinate singularity of the rotating disk, and with the positive-definite 3-metric $h_{rs}(\tau, \sigma^u) = -\epsilon 4g_{rs}(\tau, \sigma^u)$ ($h = det h_{rs}$) having three positive eigenvalues (these are the Møller conditions [11]);

3) all the 3-spaces $\Sigma_\tau$ must tend to the same space-like hyper-plane at spatial infinity with a unit normal $\epsilon^\mu_\tau$, which is the time-like 4-vector of a set of asymptotic ortho-normal tetrads $\epsilon^\mu_A$. These tetrads are carried by asymptotic inertial observers and the spatial axes $\epsilon^\mu_A$ are identified by the fixed stars of star catalogues. At spatial infinity the lapse function tends to 1 and the shift functions vanish.

By using the asymptotic tetrads $\epsilon^\mu_A$ one can give the following parametrization of the embedding functions

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5 $\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention.

6 In the 1+3 point of view usually the tetrad carried by the observer has the unit 4-velocity as time-like vector and often the Fermi-Walker transport of the tetrad is used.
\[ z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon_\mu^A F^A(\tau, \sigma^r), \quad F^A(\tau, 0) = 0, \]

\[ x^\mu(\tau) = x^\mu_o + \epsilon_\mu^A f^A(\tau), \quad (2.1) \]

where \( x^\mu(\tau) \) is the world-line of the observer. The functions \( f^A(\tau) \) determine the 4-velocity \( u^\mu(\tau) = \tfrac{dx^\mu(\tau)}{\sqrt{\epsilon x^2(\tau)}} \) (\( \epsilon = \tfrac{dx^\mu(\tau)}{dT} \)) and the 4-acceleration \( a^\mu(\tau) = \tfrac{d\omega^\mu(\tau)}{dT} \) of the observer. For an inertial frame centered on the inertial observer \( x^\mu(\tau) = x^\mu_o + \epsilon_\mu^\# \tau \) the embedding is \( z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon_\mu^r \sigma^r \), with \( \epsilon_\mu^r \) being an ortho-normal tetrad identifying the Cartesian axes.

The Møller conditions are non-linear differential conditions on the functions \( f^A(\tau) \) and \( F^A(\tau, \sigma^r) \), so that it is very difficult to construct explicit examples of admissible 3+1 splittings. When these conditions are satisfied Eqs.(2.1) describe a global non-inertial frame in Minkowski space-time.

**B. Dynamics in Non-Inertial Frames: Parametrized Minkowski Theories for Isolated Systems**

In this framework one can describe every isolated system (particles, fields, strings, fluids) admitting a Lagrangian description with parametrized Minkowski theories [2, 12]. One couples the Lagrangian to an external gravitational field and then replaces the 4-metric with the 4-metric \( g_{AB}(\tau, \sigma^r) \) induced by an admissible foliation. The new Lagrangian, a functional of the matter described in radar 4-coordinates and of the embedding \( z^\mu(\tau, \sigma^r) \) (through the 4-metric), allows us to define an action which is invariant under frame preserving diffeomorphisms [13]. As a consequence, if \( T^{\mu\nu} \) is the energy-momentum tensor of the matter and \( \rho_\mu(\tau, \sigma^r) \) the canonical momentum conjugate to the embedding, we get the following four first-class constraints and the following form of the Poincaré generators (\( T_{\perp\perp} = l_\mu l_\nu T^{\mu\nu}, T_{\perp r} = l_\mu z_{r\nu} T^{\mu\nu}, T_{rs} = z_{r\mu} z_{s\nu} T^{\mu\nu}; h^{\mu\nu} g_{us} = \delta^{\mu}_{s} \))

\[
\mathcal{H}_\mu(\tau, \sigma^r) = \rho_\mu(\tau, \sigma^u) - \sqrt{h(\tau, \sigma^u)} \left[l_\mu T_{\perp\perp} - z_{r\mu} h^{rs} T_{\perp s}\right](\tau, \sigma^u) \approx 0,
\]

\[
\{\mathcal{H}_\mu(\tau, \sigma^r_1), \mathcal{H}_\nu(\tau, \sigma^r_2)\} = 0,
\]

\[
P^{\mu} = \int d^3\sigma \rho^{\mu}(\tau, \sigma^u), \quad J^{\mu\nu} = \int d^3\sigma (z^{\mu}\rho^\nu - z^{\nu}\rho^\mu)(\tau, \sigma^u). \quad (2.2)
\]

These constraints imply that the transition among different non-inertial frames is described as a gauge transformation (so that only the appearances of phenomena change, not the physics) [2, 4, 5, 12, 14]. The canonical Hamiltonian is zero and the Dirac Hamiltonian is \( H_D = \int d^3\sigma \lambda^\mu(\tau, \sigma^r) \mathcal{H}_\mu(\tau, \sigma^r) + S_M \) with \( \lambda^\mu(\tau, \sigma^r) \) arbitrary Dirac multipliers and \( S_M \) a surface term at spatial infinity needed to define the functional derivatives so that the variation \( \delta H_D \) is proportional to the Hamilton equations. This term is the analogue of the DeWitt surface term in canonical ADM GR: as shown in Ref.[5] in GR this term is the strong ADM energy, which is equal to the weak ADM energy, i.e. to a volume integral over the 3-space of the energy density, modulo the first-class constraints of GR. Here we have \( S_M = Mc + constraints(2.2) \) with \( Mc = \sqrt{\epsilon P^2} \) the mass of the isolated system.
C. The Inertial Rest Frame of Isolated Systems, Their Relativistic Collective Variables, the Inertial Wigner Rest 3-Space and the External Poincare’ Generators

Inertial frames with Euclidean 3-spaces are a special case of this theory. For isolated systems there is a special family of inertial systems, the intrinsic inertial rest frames, in which the space-like 3-spaces are orthonormal to the conserved time-like 4-momentum of the isolated system 7.

The internal rest 3-space, named Wigner 3-space, is defined in such a way that it is the same for all the reference inertial systems describing it (modulo a Wigner rotation) and its 3-vectors are Wigner spin-1 3-vectors, so that the covariance under Poincaré transformations is under control.

As a consequence it turns out [2, 4, 5] that at the Hamiltonian level every isolated system can be described by a decoupled canonical non-covariant relativistic center of mass (whose spatial part is the classical counterpart of the Newton-Wigner position operator) carrying a pole-dipole structure, namely an internal 3-space with a well defined total invariant mass $M$ and a total rest spin $\vec{S}$ and a well defined realization of the Poincaré algebra (the external Poincaré group for a free point particle, i.e. for the external center of mass, whose mass $M$ and spin $\vec{S}$ are Casimir invariants describing the matter of the isolated system in a global way).

The canonical non-covariant (a pseudo 4-vector) relativistic center of mass $\tilde{x}^\mu(\tau)$, the non-canonical covariant (a 4-vector) Fokker-Pryce center of inertia $Y^\mu(\tau)$ and the non-canonical non-covariant (a pseudo 4-vector) Møller center of energy $R^\mu(\tau)$ are the only three relativistic collective variables which can be built only in terms of the Poincaré generators of the isolated system [4, 15] 8 so that they depend only on the system and nothing external to it. All of them have the same constant 4-velocity $h^\mu = P^\mu/Mc$ and collapse onto the Newton center of mass of the system in the non-relativistic limit. As shown in Ref. [14] these three variables can be expressed as known functions of the Lorentz scalar rest time $\tau$, of canonically conjugate Jacobi data (frozen (fixed $\tau = 0$) Cauchy data) $\vec{z} = Mc\vec{x}_{NW}(0)$, $\vec{h} = \vec{P}/Mc$, ($\vec{x}_{NW}(\tau) = \tilde{x}(\tau)$ is the standard Newton-Wigner 3-position; $P^\mu$ is the external 4-momentum) 9, and of the invariant mass $M$ and rest spin $\vec{S}$. The external Poincaré generators are then expressed in terms of these variables.

The rest frame embedding has the following definition [4, 14]

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7 The non-inertial rest frames are a special class of non-inertial frames, in which the space-like hyper-planes at spatial infinity are orthogonal to the conserved 4-momentum of the isolated system. As shown in Ref.[5] they are important in GR in asymptotically Minkowskian space-times without super-translations.

8 Since the Poincare’ generators are integrals of the components of the energy-momentum tensor of the isolated system over the whole rest 3-space, these three collective variables are non-local quantities which cannot be determined with local measurements.

9 The use of $\vec{z}$ avoids taking into account the mass spectrum of the isolated system at the quantum kinematical level and allows one to avoid the Hegerfeldt theorem (the instantaneous spreading of wave packets with violation of relativistic causality) in the relativistic quantum mechanics (RQM) developed in Ref.[4] using this formalism.
\[
\begin{align*}
  z_W^\mu(\tau, \vec{\sigma}) &= Y^\mu(\tau) + e^\mu_r(\vec{h}) \sigma^r, \\
  e^\mu_r(\vec{h}) &= \left( h_r; \delta_r^i + \frac{h^i h_r}{1 + \sqrt{1 + \vec{h}^2}} \right), \\
  Y^\mu(\tau) &= \left( \sqrt{1 + \vec{h}^2}(\tau + \frac{\vec{h} \cdot \vec{z}}{M c}) + (\tau + \frac{\vec{h} \cdot \vec{z}}{M c}) \vec{h} + \frac{\vec{S} \times \vec{h}}{M c (1 + \sqrt{1 + \vec{h}^2})} \right), \\
  \tilde{x}^\mu(\tau) &= Y^\mu(\tau) + \left( 0; \frac{-\vec{S} \times \vec{h}}{M c (1 + \sqrt{1 + \vec{h}^2})} \right),
\end{align*}
\]

where \( Y^\mu(\tau) \) is the Fokker-Pryce center of inertia and \( \tilde{x}^\mu(\tau) \) is the canonical center of mass of the isolated system. For the inertial rest frame the asymptotic tetrads are \( \epsilon^\mu_r(\vec{h}) \) with \( \epsilon^\mu_r(\vec{h}) = h^\mu \) and with \( \epsilon^\mu_r(\vec{h}) \) of Eq.(2.3)\(^{10}\). The external Poincaré group has the generators

\[
\begin{align*}
  P^\mu &= M c h^\mu = M c \left( \sqrt{1 + \vec{h}^2}; \vec{h} \right), \\
  J^{ij} &= z^i h^j - z^j h^i + \tilde{e}^{ijk} S^k, \\
  K^i &= J^{oi} = -\sqrt{1 + \vec{h}^2} z^i + \frac{(\vec{S} \times \vec{h})^i}{1 + \sqrt{1 + \vec{h}^2}},
\end{align*}
\]

as a consequence of Eqs.(2.2). The last term in the boost is responsible for the Wigner covariance of the 3-vectors in the rest Wigner 3-space \( \tau = \text{const.} \).

**D. The Relative 3-Variables of Isolated Systems of Relativistic Particles in the Inertial Rest Frame and the Internal Poincare’ Generators**

As already said, the particles of an isolated system are identified by Wigner-covariant 3-vectors \( \eta^\mu_i(\tau) \). The world-lines of the particles (and their 4-momenta) are derived notions, which can be rebuilt given the 3-coordinates, the time-like observer and the axes of the inertial rest frame \([16]\).

Let us consider the simple two-particle system of Ref.[16]. The Wigner-covariant 3-positions and 3-momenta inside the rest Wigner 3-space are \( \vec{\eta}_1(\tau), \vec{\eta}_2(\tau), i = 1, 2 \). The world-lines and the 4-momenta of the two particles are (\( V \) is an arbitrary action-at-a-distance potential)

\[
\begin{align*}
  x^\mu_i(\tau) &= z_W^\mu(\tau, \vec{\eta}_i(\tau)) = Y^\mu(\tau) + e^\mu_r(\vec{h}) \eta^r_i(\tau), \\
  p^\mu_i(\tau) &= h^\mu \sqrt{m_i^2 c^2 + \vec{\eta}_i^2(\tau) + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2)} - e^\mu_r(\vec{h}) \kappa_{ir}(\tau), \\
  \epsilon p_i^2 &= m_i^2 c^2 + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2). \tag{2.5}
\end{align*}
\]

\(^{10}\) They are the columns of the standard Wigner boost for time-like orbits: this is the source of the Wigner covariance of the Wigner 3-spaces.
They are 4-vectors but not canonical like in most of the approaches: there is a *non-commutative structure* induced by the Lorentz signature of Minkowski space-time [4, 16].

In the Wigner 3-space there is another realization of the Poincaré algebra (*the internal Poincaré group*) built with the rest 3-coordinates and 3-momenta of the matter of the isolated system starting from its energy-momentum tensor: the internal energy is the invariant mass $Mc$ (as said $Mc$ is the Hamiltonian inside the rest 3-space, because we have $H_D = Mc + constraints$) and the internal angular momentum is the rest spin $\vec{S}$. Since we are in rest frames the internal 3-momentum must vanish. Moreover, to avoid a double counting of the center of mass, the internal center of mass, conjugate to the vanishing 3-momentum, has to be eliminated: this can be done by fixing the value of the internal Poincaré boost. If we put it equal to zero, this implies [2] that the time-like observer has to be an inertial observer coinciding with the non-canonical 4-vector describing the Fokker-Pryce center of inertia of the isolated system. Therefore the internal realization of the Poincaré algebra is unfaithful and inside the Wigner rest 3-spaces the matter is described by *relative* 3-positions and 3-momenta.

For the two-particle system the conserved internal Poincaré generators are ($T^{\mu\nu} = e_{\lambda}^{\mu}(\vec{h}) e_{\nu}^{\nu}(\vec{h}) T^{A\beta}$ is the matter energy-momentum tensor)

\[
Mc = \int d^3\sigma T^{\tau\tau}(\tau, \sigma^u) = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2),
\]

\[
\vec{P} = \left( \int d^3\sigma T^{\tau\tau}(\tau, \sigma^u) \right) = \sum_{i=1}^{2} \vec{\kappa}_i(\tau) \approx 0,
\]

\[
\vec{S} = \left( \frac{1}{2} \epsilon^{\mu\nu\lambda} \int d^3\sigma \sigma^{\mu} T^{\nu\tau}(\tau, \sigma^\tau) \right) = \sum_{i=1}^{2} \vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau),
\]

\[
\vec{K} = \left( - \int d^3\sigma \sigma T^{\tau\tau}(\tau, \sigma^u) \right) = - \sum_{i=1}^{2} \vec{\eta}_i(\tau) \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2) \approx 0.
\]

(2.6)

The rest-frame conditions $\vec{P} \approx 0$, $\vec{K} \approx 0$, imply that the 3-variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ are un-physical: the physical canonical variables in the rest 3-space are $\vec{\rho}(\tau) = \vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)$ and $\vec{\pi}(\tau) = \frac{m_2}{m} \vec{\kappa}_1(\tau) - \frac{m_1}{m} \vec{\kappa}_2(\tau)$, ($m = m_1 + m_2$). Using these relative variables and imposing the rest frame condition gives for the internal center of mass $\vec{\eta}(\tau)$ (conjugate to $\vec{P} \approx 0$)

\[
\vec{\eta}(\tau) = \frac{m_1 \vec{\eta}_1(\tau) + m_2 \vec{\eta}_2(\tau)}{m} \approx \frac{m_1 \sqrt{m_2^2 c^2 + H(\tau)} - m_2 \sqrt{m_1^2 c^2 + H(\tau)}}{m (\sqrt{m_1^2 c^2 + H(\tau)} + \sqrt{m_2^2 c^2 + H(\tau)})} \vec{\rho}(\tau),
\]

\[
\downarrow \quad H(\tau) = \vec{\pi}^2(\tau) + V(\rho^2(\tau)),
\]

\[
Mc \approx \sqrt{m_1^2 c^2 + H(\tau)} + \sqrt{m_2^2 c^2 + H(\tau)}, \quad \vec{S} \approx \vec{\rho}(\tau) \times \vec{\pi}(\tau),
\]

\[
x_1^\mu(\tau) \approx Y^\mu(\tau) + e_{\nu}^\mu(\vec{h}) \frac{\sqrt{m_2^2 c^2 + H(\tau)}}{Mc} \rho^\nu(\tau),
\]

9
\[ x_2^\mu(\tau) \approx Y^\mu(\tau) - \epsilon^\mu_\nu(\bar{h}) \sqrt{m_1^2 c^2 + H(\tau)} \frac{\rho(\tau)}{M c}. \] (2.7)

Therefore besides the non-local features of the relativistic collective variables there is an intrinsic spatial non-separability (only internal relative 3-variables for the whole isolated system due to the elimination of relative times) forbidding the identification of subsystems at the physical level: as shown in Refs. [4, 8] this fact generates a notion of relativistic entanglement in RQM very different from the non-relativistic one.

See Subsection F for the inertial rest frames centered on inertial observers different from the Fokker-Pryce center of inertia.

E. Well Defined Global Non-Inertial Frames

Till now [2] the solution of the Møller conditions given in Subsection A is known in the following two cases in which the instantaneous 3-spaces are parallel Euclidean space-like hyper-planes not equally spaced due to a linear acceleration.

A) Rigid non-inertial reference frames with translational acceleration. An example are the following embeddings

\[
\begin{align*}
    z^\mu(\tau, \sigma^u) &= x^\mu_o + \epsilon^\mu_\nu f(\tau) + \epsilon^\mu_\nu \sigma^r, \\
    4g_{rr}(\tau, \sigma^u) &= \epsilon \left( \frac{df(\tau)}{d\tau} \right)^2, \\
    4g_{rr}(\tau, \sigma^u) &= 0, 4g_{rs}(\tau, \sigma^u) = -\epsilon \delta_{rs}. \\
\end{align*}
\] (2.8)

This is a foliation with parallel hyper-planes with normal \( l^\mu = \epsilon^\mu_\tau = \text{const.} \) and with the time-like observer \( x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau f(\tau) \) as origin of the 3-coordinates. The hyper-planes have translational acceleration \( \ddot{x}^\mu(\tau) = \epsilon^\mu_\tau \ddot{f}(\tau) \), so that they are not uniformly distributed like in the inertial case \( f(\tau) = \tau \).

B) Differentially rotating non-inertial frames without the coordinate singularity of the rotating disk. The embedding defining these frames is

\[
\begin{align*}
    z^\mu(\tau, \sigma^u) &= x^\mu(\tau) + \epsilon^\mu_\nu R^r_s(\tau, \sigma^\alpha) \sigma^\alpha \rightarrow_{\sigma \rightarrow \infty} x^\mu(\tau) + \epsilon^\mu_\nu \sigma^r, \\
    R^r_s(\tau, \sigma) &= R^r_s(\alpha_i(\tau, \sigma)) = R^r_s(f(\sigma) \tilde{\alpha}_i(\tau)), \\
    0 < f(\sigma) < \frac{A^2}{\sigma}, \quad - \frac{d f(\sigma)}{d\sigma} \neq 0 \ (\text{Moller conditions}), \\
    z^\mu(\tau, \sigma^w) &= \dot{x}^\mu(\tau) - \epsilon^\mu_\nu R^r_s(\tau, \sigma) \delta^sw \epsilon_{wuv} \sigma^u \Omega^w_s(\tau, \sigma), \\
    z^\mu(\tau, \sigma^u) &= \epsilon^\mu_k R^k_v(\tau, \sigma) \left( \delta^v_r + \Omega^v_r(\tau, \sigma) \sigma^u \right),
\end{align*}
\] (2.9)
where $\sigma = |\vec{\sigma}|$ and $R_{i}^{s}(\alpha_{i}(\tau, \sigma))$ is a rotation matrix satisfying the asymptotic conditions $R_{i}^{s}(\tau, \sigma) \to \sigma \to \infty \delta_{i}^{s}$, $\partial_{\sigma}^{2} R_{i}^{s}(\tau, \sigma) \to \sigma \to \infty 0$, whose Euler angles have the expression $\alpha_{i}(\tau, \vec{\sigma}) = f(\sigma) \hat{\alpha}_{i}(\tau)$, $i = 1, 2, 3$. The unit normal is $l^{\mu} = \epsilon_{\nu}^{\mu} = \text{const.}$ and the lapse function is $1 + n(\tau, \sigma^{u}) = \epsilon(\dot{z}_{\mu} l_{\mu})(\tau, \sigma^{u}) = \epsilon \epsilon_{\sigma}^{\mu} \dot{x}_{\mu}(\tau) > 0$. In Ref.[2] there is an indirect demonstration that the three eigenvalues of the 3-metric $h_{rs}(\tau, \sigma^{u})$ are positive so that the Møller conditions are satisfied.

In Eq.(2.9) one uses the notations $\Omega_{(r)}(\tau, \sigma)_{uv} = \left( R^{-1}(\tau, \vec{\sigma}) \partial_{r} \partial_{\sigma} R(\tau, \sigma) \right)_{uv}$ and $\epsilon_{uvr} \Omega^{(r, \sigma)} c = \left( R^{-1}(\tau, \vec{\sigma}) \partial_{r} \partial_{\sigma} R(\tau, \sigma) \right)_{uv} = \Omega_{(r)}(\tau, \sigma)_{uv}$, with $\Omega^{(r, \sigma)} = f(\sigma) \Omega(\tau, \sigma) \dot{n}^{(r, \sigma)}$ being the angular velocity. The angular velocity vanishes at spatial infinity and has an upper bound proportional to the minimum of the linear velocity $v_{l}(\tau) = \dot{x}_{\mu} l_{\mu}$ orthogonal to the space-like hyper-planes. When the rotation axis is fixed and $\Omega(\tau, \sigma) = \omega = \text{const.}$, a simple choice for the function $f(\sigma)$ is $f(\sigma) = 1 / \left(1 + \frac{\sigma^{2}}{c^{2}}\right)$ \to $c \to \infty$. To evaluate the non-relativistic limit for $c \to \infty$, where $\tau = ct$ with $t$ the absolute Newtonian time, one chooses the gauge function $f(\sigma) = \frac{1}{1 + \frac{\sigma^{2}}{c^{2}}} \to c \to \infty 1 - \frac{\sigma^{2}}{c^{2}} + O(c^{-4})$. This implies that the corrections to rigidly-rotating non-inertial frames coming from Møller conditions are of order $O(c^{-2})$ and become important at the distance from the rotation axis where the horizon problem for rigid rotations appears.

As shown in the first paper in Refs.[2], global rigid rotations are forbidden in relativistic theories, because, if one uses the embedding $z^{\mu}(\tau, \sigma^{u}) = x^{\mu}(\tau) + \epsilon^{\mu}_{\nu} R^{r}_{s}(\sigma) \sigma^{s}$ describing a global rigid rotation with angular velocity $\Omega^{r} = \Omega^{r}(\tau)$, then the resulting $g_{rr}(\tau, \sigma^{u})$ violates Møller conditions, because it vanishes at $\sigma = \sigma_{R} = \frac{1}{\Omega^{r}} \left[ \dot{x}^{2}(\tau) + \dot{x}_{\mu}(\tau) \epsilon^{\mu}_{\nu} R^{s}_{r}(\sigma) (\dot{\sigma} \times \Omega(\tau))^{s} \right]^{2} - \dot{x}_{\mu}(\tau) \epsilon^{\mu}_{\nu} R^{s}_{r}(\sigma) (\dot{\sigma} \times \Omega(\tau))^{s} (\sigma^{u} = \sigma \dot{\sigma}^{u}, \Omega^{r} = \Omega^{r}, \dot{\sigma}^{2} = \dot{\Omega}^{2} = 1)$. At this distance from the rotation axis the tangential rotational velocity becomes equal to the velocity of light. This is the horizon problem of the rotating disk (the horizon is often named the light cylinder). Let us remark that even if in the existing theory of rotating relativistic stars one uses differential rotations, notwithstanding that in the study of the magnetosphere of pulsars often the notion of light cylinder is still used.

\section{The Rest-Frame Conditions in Non-Inertial Frames}

In admissible either inertial or non-inertial frames described by the embedding (2.1) with asymptotic tetrads $\epsilon_{\nu}^{\mu}$, we must consider the Lorentz transformation connecting them to the tetrads $\epsilon^{\mu}_{A}(\vec{h})$ of the rest frame: $\epsilon^{\mu}_{A} = A_{B}^{A} B(\vec{h}) \epsilon^{\mu}_{B}(\vec{h})$. An isolated system is still described as a non-local non-covariant decoupled external center of mass with Jacobi data $\vec{z}$, $\vec{h}$, carrying a pole-dipole structure with an invariant mass and a spin, whose expression has been found in Ref.[2] and is $(l^{\mu}(\tau, \sigma^{u}) = \epsilon_{A}^{\mu} l^{A}(\tau, \sigma^{u})$ is the unit normal to the 3-space).

---

11 $\dot{\Omega}(\tau, \sigma)$ defines the instantaneous rotation axis and $0 < \dot{\Omega}(\tau, \sigma) < 2 \max \left( \hat{\alpha}(\tau), \hat{\beta}(\tau), \hat{\gamma}(\tau) \right)$ as shown in Ref.[2].

12 Nearly rigid rotating systems, like a rotating disk of radius $\sigma_{o}$, can be described by using a function $f(\sigma)$ approximating the step function $\theta(\sigma - \sigma_{o})$. 

11

12
\[ M_c \approx \int d^3 \sigma \sqrt{h(\tau, \sigma^u)} \left[ T_{\perp l}^A - T_{\perp s} h^{s r} \partial_r F^A \right](\tau, \sigma^u) \Lambda_A^r(\vec{h}), \]

\[ S^\tau \approx \frac{1}{2} \epsilon^{\tau uv} \int d^3 \sigma \sqrt{h(\tau, \sigma^u)} \left[ F^C(\tau, \sigma^u) \left( T_{\perp l}^D - T_{\perp s} h^{s r} \partial_r F^D \right)(\tau, \sigma^u) - F^D(\tau, \sigma^u) \left( T_{\perp l}^C - T_{\perp s} h^{s r} \partial_r F^C \right)(\tau, \sigma^u) \right] \Lambda_C^u(\vec{h}) \Lambda_D^v(\vec{h}). \] (2.10)

As shown in Eqs. (5.7) and (5.13) of the first paper in Ref.[2] and in Eq.(3.4) of Ref.[17], the three pairs of second class constraints eliminating the internal center of mass in arbitrary non-inertial rest frames have the form

\[ \mathcal{P}^r = \int d^3 \sigma \sqrt{h(\tau, \sigma^u)} \left[ T_{\perp l}^A - T_{\perp s} h^{s r} \partial_r F^A \right](\tau, \sigma^u) \Lambda_A^r(\vec{h}) \approx 0, \]

\[ \mathcal{K}^r = \int d^3 \sigma \sqrt{h(\tau, \sigma^u)} \left[ F^C(\tau, \sigma^u) \left( T_{\perp l}^D - T_{\perp s} h^{s r} \partial_r F^D \right)(\tau, \sigma^u) - F^D(\tau, \sigma^u) \left( T_{\perp l}^C - T_{\perp s} h^{s r} \partial_r F^C \right)(\tau, \sigma^u) \right] \Lambda_C^r(\vec{h}) \Lambda_D^v(\vec{h}) \approx \]

\[ \approx M_c h^r \left( x_0^o + f^B(\tau) \Lambda_B C(\vec{h}) \epsilon_C^o(\vec{h}) - \frac{\sum_u h^r (x_0^u - z^u + f^B(\tau) \Lambda_B C(\vec{h}) \epsilon_C^u(\vec{h}))}{1 + \sqrt{1 + \vec{h}^2}} \right) - \]

\[ - \left( x_0^r - z^r + f^B(\tau) \Lambda_B C(\vec{h}) \epsilon_C^r(\vec{h}) + \frac{\delta^{rm} \epsilon_{mnk} h^m \tilde{S}^k}{M_c (1 + \sqrt{1 + \vec{h}^2})} \right). \] (2.11)

Let us remark that if we put \( \Lambda_A^B(\vec{h}) = \delta_A^B \) and \( x_0^\mu + f^B(\tau) \Lambda_B C(\vec{h}) \epsilon_C^\mu(\vec{h}) = Y^\mu(0) + h^\mu \tau \), then we recover the results for the inertial rest frame centered on the Fokker-Pryce inertial observer when \( F^A(\tau, \sigma^u) = \sigma^A \).

Instead the conditions \( \Lambda_A^B(\vec{h}) = \delta_A^B \) and \( f^B(\tau) \Lambda_B C(\vec{h}) \epsilon_C^\mu(\vec{h}) = h^\mu \tau \), identifying the inertial rest frame centered on the inertial observer \( x_0^\mu + h^\mu \tau \), have the constraints \( \mathcal{K}^r \approx 0 \) replaced by the second of Eqs.(2.11).

For an inertial frame with \( \epsilon_A^\mu = \Lambda_A^B(\vec{h}) \epsilon_B^\mu(\vec{h}) \) centered on the inertial observer with world-line \( x_0^\mu + \epsilon_t^\mu \tau \) (\( \epsilon_t^\mu = l^\mu \), the normal to the Euclidean 3-space) one has \( F^A(\tau, \sigma^u) = \sigma^A \)

\( f^A(\tau) = \delta_t^A \tau \), and the second-class constraints (2.11) but with \( \mathcal{P}^r \approx M c h^r = P^r \).

However, in the non-inertial case it is highly non-trivial to find the relative variables inside the internal 3-space due to its non-Euclidean structure. For instance, if we have particles with radar 3-coordinates \( \eta_i^\mu(\tau) \) in the non-Euclidean 3-space \( \Sigma_\tau \) and interacting through action-at-a-distance potentials \( V((\eta_i(\tau) - \eta_j)^2) \) in the inertial rest frame, their transcription in \( \Sigma_\tau \) must use a bi-scalar like the Synge world function \( \Omega(i, j) \) [18] built for the space-like 3-geodesic joining the particles \( i \) and \( j \) in \( \Sigma_\tau \).\(^{13}\)

\(^{13}\) This quantity is a 3-scalar in both points, its gradient with respect the end points gives the 3-vectors tangent to the 3-geodesic in the end points. In the Euclidean limit one recovers the quantity \((\vec{\eta}_i - \vec{\eta}_j)^2\).
III. A NEW FAMILY OF ADMISSIBLE GLOBAL NON-INERTIAL FRAMES

Let us now look at a family of admissible global non-inertial frames wider than the ones quoted in Subsection E of Section II. These new non-inertial frames will have in general non-Euclidean 3-spaces. They are motivated by the locality principle [7], according to which at each instant an accelerated detector gives the same data as an instantaneously comoving inertial detector. This inertial detector is connected to a standard reference inertial frame by a Lorentz transformation whose boost part is determined by the instantaneous velocity of the accelerated detector.

This suggests replacing the embedding (2.1) with the following one (\(\epsilon_A\) asymptotic tetrad, \(\epsilon^\mu_A, \epsilon^\nu_B = \eta_{AB}, \eta_{\mu\nu} = \epsilon (+ - - -) = \eta_{AB}; \sigma = |\bar{\sigma}| = \sqrt{\sum_r (\sigma^r)^2}, \partial_r \sigma = \frac{\sigma^r}{\sigma} = \bar{\sigma}^r\))

\[
z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon^\mu_A \Lambda^A_r(\tau, \sigma^r) \sigma^r \rightarrow_{\sigma \rightarrow \infty} x^\mu(\tau) + \epsilon^\mu \sigma^r,
\]

\[
\Lambda^A_r(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^A_r, \quad \Lambda^A_r(\tau, 0) \ finite,
\]

(3.1)

with the Lorentz matrices \(\Lambda^A_B(\tau, \sigma)\) satisfying \(\Lambda^A_C(\tau, \sigma) \eta_{AB} \Lambda^B_D(\tau, \sigma) = \eta_{CD}\).

The origin of the 3-coordinates \(\sigma^r\) in the 3-spaces \(\Sigma_\tau\) is a time-like (not uniformly accelerated like the Rindler ones) observer with world-line

\[
z^\mu(\tau, 0) = x^\mu(\tau) = x^\mu_0 + \epsilon^\mu_A f^A(\tau),
\]

\[
\dot{x}^\mu(\tau) = \epsilon^\mu_A \dot{f}^A(\tau) = \epsilon^\mu_A \alpha(\tau) \gamma_x(\tau) \left( \frac{1}{\bar{\beta}_x^r(\tau)} \right), \quad \epsilon \dot{x}^2(\tau) = \alpha^2(\tau) > 0,
\]

\[
f^A(\tau) = \int_0^\tau d\tau_1 \alpha(\tau_1) \gamma_x(\tau_1) \left( \frac{1}{\bar{\beta}_x^r(\tau)} \right), \quad \gamma_x(\tau) = \frac{1}{\sqrt{1 - \bar{\beta}_x^2(\tau)}}...
\]

(3.2)

In this equation \(\bar{\beta}_x(\tau) (|\bar{\beta}_x(\tau)| < 1)\) is the instantaneous 3-velocity, divided by \(c\), of the observer and \(\tau\) is the proper time of the observer when \(\alpha(\tau) = 1\).

The Lorentz matrix is parametrized as the product of a boost by a rotation matrix,

\[
\Lambda^A_B(\tau, \sigma) = B^A_C(\tau, \sigma) \bar{R}^C_B(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^A_B, \quad \text{with the following notation (R is an ordinary 3 \times 3 rotation matrix, } R^{-1} = R^T; \bar{\beta} \text{ is a 3-velocity divided by } c; \bar{\gamma}(\tau, \sigma) = \frac{1}{\sqrt{1 - \beta^2(\tau, \sigma)}}; \bar{\gamma} = \frac{\bar{\gamma}}{\sqrt{1 - \beta^2(\tau, \sigma)}})
\]

\[
(B^{-1})^A_B = B^C_D \eta_{CB} \eta^{BA}
\]

\[
B^A_C(\tau, \sigma) = \left( \begin{array}{c} \bar{\gamma} \\ \bar{\gamma} \bar{\beta}^r \\ \bar{\gamma} \bar{\beta}^s + \frac{\bar{\gamma}^2 \bar{\beta}^r \bar{\beta}^s}{\bar{\gamma} + 1} \end{array} \right)(\tau, \sigma),
\]
\[
\begin{align*}
\tilde{R}^C_B(\tau, \sigma) &= \begin{pmatrix} 1 & 0 \\ 0 & R_{rs} \end{pmatrix} (\tau, \sigma), \\
A^A_B(\tau, \sigma) &= \begin{pmatrix} \tilde{\gamma} & \tilde{\beta}^u R_{us} \\ \tilde{\gamma} \tilde{\beta}^r (\delta^r u + \frac{\tilde{\gamma}^2 \tilde{\beta}^u}{\tilde{\gamma} + 1}) R_{us} \end{pmatrix} (\tau, \sigma),
\end{align*}
\]

(3.3)

The Lorentz matrix becomes the identity at spatial infinity and is finite at the origin, where the rotation matrix is assumed to become the identity and where the 3-velocity may or may not become the one of the observer, \(\tilde{\beta}(\tau, 0) = \tilde{\beta}_x(\tau)\) or \(\tilde{\beta}(\tau, 0) \neq \tilde{\beta}_x(\tau)\).

The angles in the rotation matrix have the same parametrization as in Subsection E of Section II

\[
R_{rs}(\tau, \sigma) = R_{rs}(\tilde{\alpha}_i(\tau, \sigma)), \quad i = 1, 2, 3,
\]

\[
\tilde{\alpha}_i(\tau, \sigma) = f(\sigma) \alpha_i(\tau), \quad f(\sigma) \rightarrow \sigma \rightarrow \infty 0, \quad f(0) = 1.
\]

(3.4)

The parameters in the Lorentz boosts have the following parametrization

\[
B^A_B(\tau, \sigma) = B^A_B(\tilde{\beta}^r(\tau, \sigma)), \quad r = 1, 2, 3,
\]

\[
\tilde{\beta}^r(\tau, \sigma) = g(\sigma) \beta^r(\tau), \quad g(\sigma) \rightarrow \sigma \rightarrow \infty 0, \quad g(0) = 1,
\]

\[
\tilde{\gamma}(\tau, \sigma) = \frac{1}{\sqrt{1 - g^2(\sigma) \tilde{\beta}^2(\tau)}}.
\]

(3.5)

In terms of the quantities (A1), (A2), defined in Appendix A and of Eqs. (A4), (A5), (A6), we get the following results for the metric \(\epsilon^4 g_{AB}(\tau, \sigma^u) = (z^u_A \eta_{\mu \nu z^u_B})(\tau, \sigma^u)\)

\[
\epsilon^4 g_{\tau \tau}(\tau, \sigma^u) = \epsilon \left( z^u_\mu \eta_{\mu \nu} z^u_\tau \right)(\tau, \sigma^u) = \alpha^2(\tau) +
\]

\[
+ 2 \sigma \alpha(\tau) \gamma(\tau) \left( \frac{1 - g(\sigma) \beta_x(\tau) \cdot \tilde{\beta}(\tau)}{\sqrt{1 - g^2(\sigma) \tilde{\beta}^2(\tau)}} g(\sigma) \tilde{\beta}(\tau) \cdot \sum_n \tilde{\Omega}(B \cdot \tau)^{n \cdot \tilde{\sigma}^n} \right) R_{uv} +
\]

\[
+ \sum_{uv} \left[ \frac{g(\sigma) \beta_u(\tau)}{1 + \sqrt{1 - g^2(\sigma) \tilde{\beta}^2(\tau)}} \left( \frac{1 - g(\sigma) \beta_x(\tau) \cdot \tilde{\beta}(\tau)}{\sqrt{1 - g^2(\sigma) \tilde{\beta}^2(\tau)}} \right) \beta_u(\tau) \right] R_{uv}.
\]

\[
\left[ f(\sigma) \frac{\Omega(R)}{c} (\tilde{\sigma} \times \tilde{n})^v + g(\sigma) \tilde{\beta}(\tau) \cdot \sum_m \tilde{\Omega}(B \cdot \tau)^{v \cdot m \cdot \tilde{\sigma}^m} \right] (\tau, \sigma) -
\]

\[
- \sigma^2 \left( f^2(\sigma) \frac{\Omega^2(R)}{c^2} (\tilde{\sigma} \times \tilde{n})^2 \right) +
\]

14
\[ + f(\sigma) g(\sigma) \frac{\Omega(R)}{c} \sum_v (\dot{\sigma} \times \dot{n})^\nu \dot{\beta}(\tau) \cdot \sum_n \tilde{\Omega}_{(B)}^\nu \cdot \dot{\sigma}^n + \\
\] 
\[ + g^2(\sigma) \sum_{nm} \left[ \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^\nu m \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^\nu n - \right. \\
\] 
\[ - \sum_v \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^\nu m \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^\nu n \right] \dot{\sigma}^n \dot{\sigma}^n(\tau, \sigma). \] (3.6)

\[ \epsilon^4 g_{rr}(\tau, \sigma^u) = \epsilon \left( z^\mu \eta_{\mu\nu} z^\nu \right)(\tau, \sigma^u) = -N_r(\tau, \sigma^u) = \\
= \alpha(\tau) \gamma_x(\tau) \left[ \Delta^r - \sum_v \beta^u_x(\tau) \Lambda^u + \\
+ \sigma \dot{\sigma}^r \left( f'(\sigma) \sum_u \left[ \Lambda^r - \sum_v \beta^u_x(\tau) \Lambda^u \right] (\dot{\sigma} \times \Omega(R))^u \right. \\
+ g'(\sigma) \left[ \Delta^r - \sum_v \beta^u_x(\tau) \Lambda^u \right] \ddot{\beta}(\tau) \cdot \sum_n \tilde{\Omega}_{(B)}^r n \dot{\sigma}^n \right) (\tau, \sigma) - \\
- \sigma \left[ f(\sigma) \frac{\Omega(R)}{c} (\dot{\sigma} \times \dot{n})^r + g(\sigma) \beta(\tau) \cdot \sum_n \tilde{\Omega}_{(B)}^r n \dot{\sigma}^n + \\
+ \sigma \dot{\sigma}^r \left( f(\sigma) f'(\sigma) \frac{\Omega(R)}{c} (\dot{\sigma} \times \dot{n}) \cdot (\dot{\sigma} \times \Omega(R))^r \right) + \\
+ g(\sigma) f'(\sigma) \sum_{vn} \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^r n \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^r n \dot{\sigma}^n \dot{\sigma}^n \right] (\tau, \sigma), \] (3.7)

\[ -\epsilon^4 g_{ss}(\tau, \sigma^u) = \h_{rs}(\tau, \sigma^u) = -\epsilon \left( z^\mu \eta_{\mu\nu} z^\nu \right)(\tau, \sigma^u) \]
\[ = \delta_{rs} + \sigma f'(\sigma) \left( \dot{\sigma}^r (\dot{\sigma} \times \Omega(R))^s + \dot{\sigma}^s (\dot{\sigma} \times \Omega(R))^r \right) (\tau, \sigma) + \\
+ \sigma^2 \dot{\sigma}^r \dot{\sigma}^s \left( f'^2(\sigma) (\dot{\sigma} \times \Omega(R))^2 - \\
- g'^2(\sigma) \sum_{nm} \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^r m \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^r n \ddot{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^r n \dot{\sigma}^m \dot{\sigma}^n \right) \] (3.8)

From Eq.(A8) of Appendix A, after a lengthy calculation, we get the following expression for the lapse function \( N(\tau, \sigma^u) = 1 + n(\tau, \sigma^u) \)

\[ \left( \sqrt{h} (1 + n) \right)(\tau, \sigma^u) = \left[ \alpha(\tau) \gamma_x(\tau) \frac{1 - g(\sigma) \dot{\beta}(\tau) \cdot \dot{\beta}(\tau)}{\sqrt{1 - g'^2(\sigma) \dot{\beta}^2(\tau)}} \right. \\
+ \sigma g(\sigma) \dot{\beta}(\tau) \cdot \sum_n \tilde{\Omega}_{(B)}^r n \dot{\sigma}^n \right] (\tau, \sigma) - \\
\left. \sigma g'(\sigma) \dot{\beta}(\tau) \cdot \sum_n \tilde{\Omega}_{(B)}(\tau, \sigma)^r n \dot{\sigma}^n \right] \] 
\[ \sum_v \dot{\sigma}^v \left[ \alpha(\tau) \gamma_x(\tau) \sum_u R^T_{uvu} \right. \]

15
\[
\left( \beta_x^u(\tau) - \frac{g(\sigma) \beta^u(\tau)}{1 + \sqrt{1 - g^2(\sigma) \bar{\beta}^2(\tau)}} \left[ 1 + \frac{1 - g(\sigma) \bar{\beta}_x(\tau) \cdot \bar{\beta}(\tau)}{\sqrt{1 - g^2(\sigma) \bar{\beta}^2(\tau)}} \right] \right) + \\
+ \sigma g(\sigma) \dot{\bar{\beta}}(\tau) \cdot \sum_m \bar{\Omega}(\tau) \cdot \hat{m} \cdot \hat{m}](\tau, \sigma).
\]

(3.9)

The positivity requirement for the quantities of Eqs. (3.6) and (3.9) together with the positivity of the three eigenvalues \( \lambda_i(\tau, \sigma^u) > 0 \) of the matrix (3.8) are the conditions on the observer and on the Lorentz matrix of Eq.(3.1) for having a nice foliation, i.e. a well defined global non-inertial frame described by the embedding (3.1). Therefore one must have

\[ 1 + n(\tau, \sigma^u) > 0, \quad \epsilon^4 g_{\tau \tau}(\tau, \sigma^u) > 0, \]

\[
\left( \lambda_1 \lambda_2 \lambda_3 \right)(\tau, \sigma^u) = h(\tau, \sigma^u) > 0,
\]

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 \right)(\tau, \sigma^u) = \left( h_{11} + h_{22} + h_{33} \right)(\tau, \sigma^u) > 0,
\]

\[
\left( \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \right)(\tau, \sigma^u) = \left( h_{11} h_{22} - h_{12} h_{21} +
\right.
\]

\[ + h_{22} h_{33} - h_{23} h_{32} + h_{33} h_{11} - h_{13} h_{31} \right)(\tau, \sigma^u) > 0.
\]

(3.10)

A. Some Solutions to the Positivity Requirements.

By using Eqs.(A2) we get the following explicit expressions for Eqs. (3.6), (3.8) and (3.9)

\[ \epsilon^4 g_{\tau \tau}(\tau, \sigma^u) = \alpha^2(\tau) + 2\sigma \alpha(\tau) \gamma_x(\tau) \left( \frac{1 - g(\sigma) \bar{\beta}_x(\tau) \cdot \bar{\beta}(\tau)}{\sqrt{1 - g^2(\sigma) \bar{\beta}^2(\tau)}} \right) g(\sigma) \sum_{su} \beta^u(\tau) R_{su}(\tau, \sigma) \hat{s}^u + \]

\[ + \sum_{uv} \left[ \frac{g(\sigma) \beta^u(\tau)}{1 + \sqrt{1 - g^2(\sigma) \bar{\beta}^2(\tau)}} \left( 1 + \frac{1 - g(\sigma) \bar{\beta}_x(\tau) \cdot \bar{\beta}(\tau)}{\sqrt{1 - g^2(\sigma) \bar{\beta}^2(\tau)}} \right) \right] R_{uv}(\tau, \sigma) \]

\[ \left[ f(\sigma) \Omega_{(R)}(\tau, \sigma) \left( \hat{s} \times \hat{n}(\tau, \sigma) \right) \right]^v + \]

\[ + g^2(\sigma) \sum_{mn} R^T_{nm}(\tau, \sigma) \left( \hat{\beta}^m(\tau) \beta^n(\tau) - \hat{\beta}^m(\tau) \beta^m(\tau) \right) R_{sn}(\tau, \sigma) \hat{s}^n \right] - \]

\[ - \sigma^2 \left( f^2(\sigma) \frac{\Omega_{(R)}^2(\tau, \sigma)}{c^2} \left( \hat{s} \times \hat{n}(\tau, \sigma) \right)^2 \right) + \]

\[ + f(\sigma) g^2(\sigma) \frac{\Omega_{(R)}(\tau, \sigma)}{c} \sum_{vnmn} \left( \hat{s} \times \hat{n}(\tau, \sigma) \right)^v \]

\[ R^T_{vm}(\tau, \sigma) \left( \hat{\beta}^m(\tau) \beta^n(\tau) - \hat{\beta}^m(\tau) \beta^m(\tau) \right) R_{su}(\tau, \sigma) \hat{a}^u + \]

\[ + g^2(\sigma) \sum_{mn} \left[ \sum_{rs} \hat{\beta}^r(\tau) R_{rm}(\tau, \sigma) \hat{\beta}^s(\tau) R_{sn}(\tau, \sigma) \right] \]

\[ - \sum_{vrm} R^T_{vu}(\tau, \sigma) \left( \hat{\beta}^u(\tau) \beta^r(\tau) - \hat{\beta}^r(\tau) \beta^u(\tau) \right) R_{rm}(\tau, \sigma) \]

\[ R^T_{vu}(\tau, \sigma) \left( \hat{\beta}^u(\tau) \beta^u(\tau) - \hat{\beta}^u(\tau) \beta^u(\tau) \right) R_{sn}(\tau, \sigma) \hat{a}^m \hat{a}^n \right] > 0, \]
\[ h_{rs}(\tau, \sigma^u) = \delta_{rs} + \sigma f'(\sigma) \left( \hat{\sigma}^r (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^s + \hat{\sigma}^s (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^r \right) + \]
\[ + \sigma^2 \hat{\sigma}^r \hat{\sigma}^s \left[ f'^2(\sigma) (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^2 - g^2(\sigma) \left( \sum_{nm} \beta^m(\tau) R_{nm}(\tau, \sigma) \hat{\sigma}^m \right)^2 \right], \]
\[ \sqrt{h(\tau, \sigma^u)} (1 + n(\tau, \sigma^u)) = \alpha(\tau) \gamma_x(\tau) \frac{1 - g(\sigma) \beta(\tau) \cdot \beta(\tau)}{\sqrt{1 - g^2(\sigma) \beta^2(\tau)}} + \sigma g(\sigma) \sum_{sn} \beta^s(\tau) R_{sn}(\tau, \sigma) \hat{\sigma}^n - \]
\[ - \sigma \frac{g'(\sigma)}{g(\sigma)} \sum_{snw} \beta^s(\tau) R_{snw}(\tau, \sigma) \hat{\sigma}^n \hat{\sigma}^w \left[ \alpha(\tau) \gamma_x(\tau) \sum_{u} R_{vx}(\tau, \sigma) \right] \]
\[ + \sigma g^2(\sigma) \sum_{wuv} R_{wuv}(\tau, \sigma) (\beta^u(\tau) \beta^v(\tau) - \beta^w(\tau) \beta^v(\tau)) R_{uv}(\tau, \sigma) \hat{\sigma}^w > 0. \]
\[ (3.11) \]

The positivity conditions are the restrictions on the form factors \( f(\sigma) \) and \( g(\sigma) \) implying that the global non-inertial frame with its non-Euclidean 3-spaces is well defined, i.e. it has no pathology.

Due to the complicated form of the positivity conditions, we give explicitly only two special families of solutions.

1. **Boosts with Small Velocities**

When the boost parameter \( \beta^r(\tau) \) of Eq.(3.5) and its time variation \( \dot{\beta}^r(\tau) \) are small quantities of order \( \epsilon \) \( (|\beta(\tau)|, |\dot{\beta}(\tau)| \approx O(\epsilon) << 1 ; \text{nearly non-relativistic small velocities}) \), Eqs.(3.11) become

\[ \epsilon^4 g_{rr}(\tau, \sigma^u) = \alpha^2(\tau) + 2\sigma \alpha(\tau) \gamma_x(\tau) f(\sigma) \frac{\Omega_{(R)}(\tau, \sigma)}{c} \sum_{uv} \beta^u(\tau) R_{uv}(\tau, \sigma) (\hat{\sigma} \times \hat{n}(\tau, \sigma))^v - \]
\[ - \sigma^2 f^2(\sigma) \frac{\Omega_{(R)}^2(\tau, \sigma)}{c^2} (\hat{\sigma} \times \hat{n}(\tau, \sigma))^2 + O(\epsilon) > 0, \]
\[ \sqrt{h(\tau, \sigma^u)} (1 + n(\tau, \sigma^u)) = \alpha(\tau) \gamma_x(\tau) + O(\epsilon) > 0, \]
\[ \sqrt{h(\tau, \sigma^u)} (1 + n(\tau, \sigma^u)) = \alpha(\tau) \gamma_x(\tau) + O(\epsilon) > 0, \]
\[ h_{rs}(\tau, \sigma^u) = \delta_{rs} + \sigma f'(\sigma) \left( \hat{\sigma}^r (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^s + \hat{\sigma}^s (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^r \right) + \]
\[ + \sigma^2 \hat{\sigma}^r \hat{\sigma}^s f'^2(\sigma) (\hat{\sigma} \times \hat{\Omega}_{(R)}(\tau, \sigma))^2 + O(\epsilon^2). \]
\[ (3.12) \]

Eqs.(3.12) are the conditions for differentially rotating non-inertial frames in Euclidean 3-spaces. Now the 3-spaces have deviations of order \( O(\epsilon) \) from Euclidean 3-spaces and there is no restriction on \( g(\sigma) \). Instead \( f(\sigma) \) must satisfy Eq.(2.9) and this also implies the positivity of the three eigenvalues of the 3-metric.
2. Time-Independent Boosts

Let us now consider time-independent boosts: \( \beta(\tau) = 0 \), i.e. \( \vec{\beta} = \vec{\beta} = \text{const.} \) At every time there is the same non-Euclidean 3-space. Since Eqs.(3.11) remain complicated, let us also put the restriction \( \vec{\beta} \cdot \vec{\beta} = 0 \) on the world-line \( x^\mu(\tau) \) of the observer. Then Eqs.(3.11) become

\[
\varepsilon^4 g_{\tau\tau}(\tau, \sigma^u) = \alpha^2(\tau) + 2\sigma \alpha(\tau) \gamma_x(\tau) f(\sigma) \frac{\Omega(R)(\tau, \sigma)}{c} - \sum_{uv} \left( \beta_u^x(\tau) - \frac{g(\sigma) \beta^u}{\sqrt{1 - g^2(\sigma) \beta^2}} \right) R_{uv}(\tau, \sigma) (\hat{\sigma} \times \hat{n}(\tau, \sigma))^u - \sigma^2 f^2(\sigma) \frac{\Omega^2(R)(\tau, \sigma)}{c^2} (\hat{\sigma} \times \hat{n}(\tau, \sigma))^2 > 0,
\]

\[
h_{rs}(\tau, \sigma^u) = \delta_{rs} + \sigma f'(\sigma) \left( \hat{\sigma}^r (\hat{\sigma} \times \Omega(R)(\tau, \sigma))^s + \hat{\sigma}^s (\hat{\sigma} \times \Omega(R)(\tau, \sigma))^r \right) + \sigma^2 \hat{\sigma}^r \hat{\sigma}^s \left[ f'^2(\sigma) (\hat{\sigma} \times \Omega(R)(\tau, \sigma))^2 - g^2(\sigma) \left( \sum_{nm} \beta^m R_{nm}(\tau, \sigma) \hat{\sigma}^m \right)^2 \right],
\]

\[
\sqrt{h(\tau, \sigma^u)} (1 + n(\tau, \sigma^u)) = \alpha(\tau) \gamma_x(\tau) \left[ \frac{1}{\sqrt{1 - g^2(\sigma) \beta^2}} - \sigma g'(\sigma) \sum_{snvu} \beta^v R_{sn}(\tau, \sigma) \hat{\sigma}^n \hat{\sigma}^v R^T_{vu}(\tau, \sigma) \left( \beta_u^x(\tau) - \frac{g(\sigma) \beta^u}{\sqrt{1 - g^2(\sigma) \beta^2}} \right) \right] > 0.
\]

(3.13)

If \(|\beta^2|\) is small, the 3-metric \( h_{rs}(\tau, \sigma^u) \) has small deviations from the pure rotational case and its three eigenvalues remain positive if \( f(\sigma) \) satisfies Eq.(2.9). Moreover if for every \( \tau \) and \( \sigma \) we have the following restriction on \( g(\sigma) \)

\[
\beta_u^x(\tau) > \frac{g(\sigma) \beta^u}{\sqrt{1 - g^2(\sigma) \beta^2}},
\]

(3.14)

then also the condition \( \varepsilon^4 g_{\tau\tau}(\tau, \sigma) > 0 \) is satisfied by \( f(\sigma) \) of Eq.(2.9) with a different constant \( A^2 \).

Finally the condition \( 1 + n(\tau, \sigma^u) > 0 \) implies

\[
-C^2 \sigma < g'(\sigma) < C^2 \sigma,
\]

\[
C^2 = \min_{\tau, \sigma} \left| \sum_{snvu} \beta^v R_{sn}(\tau, \sigma) \hat{\sigma}^n \hat{\sigma}^v R^T_{vu}(\tau, \sigma) \left( \sqrt{1 - g^2(\sigma) \beta^2} \beta_u^x(\tau) - g(\sigma) \beta^u \right) \right|.
\]

(3.15)

In conclusion we now have control on some families of global non-inertial frames with non-Euclidean 3-spaces. When there will be experimental reasons for studying other families of such frames, one will deepen the study of Eqs.(3.11).
IV. COMPARISON OF INERTIAL AND NON-INERTIAL REFERENCE FRAMES CENTERED ON EITHER MATHEMATICAL OR DYNAMICAL OBSERVERS: DYNAMICAL INERTIAL ALICE VERSUS DYNAMICAL NON-INERTIAL BOB

Both in Galilei and Minkowski space-times the description of isolated systems is ideally done by an either inertial or accelerated observer origin of an either inertial or non-inertial frame. The primary role of these “mathematical” observers is to define a 4-coordinate system and a system of axes (a tetrad). They are considered as mathematical idealizations of “dynamical” observers (Alice, Bob, Charlie,...) endowed with macroscopic apparatuses with which they perform measurements on the given isolated system (breaking its isolation) with a subsequent mutual communication of the results obtained. But already at the classical level this transition from mathematical to dynamical observers requires the inclusion of the observers in the isolated system to have control of the either Galilei or Poincaré generators (so that, for instance, the total energy of the new system is finite) and to have the world-lines of the observers dynamically determined (and not given by hand like with mathematical observers). In SR the spatial non-separability due to the elimination of relative times and the non-measurability of the relativistic center of mass induce an ever bigger difference between mathematical and dynamical observers.

This type of problems becomes extremely complicated at the quantum level due to its unsolved foundational problems (interpretation; theory of measurement) and due to the absence of a notion of “reality” of quantum systems (see Ref. [8] for a discussion of these problems in the framework presented in this paper). Usually the quantum system is described with respect to an inertial reference frame centered on a classical mathematical observer; this observer either carries or describes the location and orientation of a measuring apparatus. A dynamical observer should be identified with such an apparatus considered as either a macroscopic classical object or quantum system (with some semi-classical description of its quantum many-body structure). As a consequence a distinction between macroscopic dynamical observers and a microscopic quantum system becomes extremely problematic already at the non-relativistic level even before taking into account the spatial non-separability of SR (see for instance Ref.[19]).

Let us come back to classical SR. Having found families of admissible 3+1 splittings (global non-inertial frames) of Minkowski space-time centered on arbitrary time-like mathematical observers, we can face the following two problems:

A) how to compare the description of an isolated system given by two different mathematical observers and in particular how to rewrite all the results which can be obtained in an inertial rest frame in an arbitrary non-inertial frame;

B) whether it is possible to get the description of an isolated system in a non-inertial rest frame centered on a particle of the system used as a dynamical observer.

A. Alice and Bob: Accelerated Mathematical Observers

Let us consider the two world-lines \( x_1^\mu(\tau) \) and \( x_2^\mu(\tau) \) of two time-like observers (Alice and Bob) in Minkowski space-time in a given inertial frame with Cartesian coordinates \( x^\mu \) centered on a mathematical inertial observer.
To each observer we can associate a non-inertial frame giving two nice foliations centered on the two observers and defining the radar 4-coordinates \((\tau_1, \sigma_1^\tau)\) and \((\tau_2, \sigma_2^\tau)\) for parametrizing their respective 3-spaces. There will be two embeddings \(x^\mu = z_1^\mu(\tau_1, \sigma_1^\tau)\) and \(x^\mu = z_2^\mu(\tau_2, \sigma_2^\tau)\) for the 3-spaces of the two foliations with the following equations allowing one to express one set of radar 4-coordinates in terms of the other

\[
x^\mu = z_1^\mu(\tau_1, \sigma_1^\tau) = z_2^\mu(\tau_2, \sigma_2^\tau),
\]

\[
\downarrow
\]

\[
\tau_1 = f_1(\tau_2, \sigma_2^\tau), \quad \sigma_1^\tau = f_1^\tau(\tau_2, \sigma_2^\tau),
\]

\[
\tau_2 = f_2(\tau_1, \sigma_1^\tau), \quad \sigma_2^\tau = f_2^\tau(\tau_1, \sigma_1^\tau).
\] (4.1)

Therefore we get the following identification of the two world-lines

\[
x_1^\mu(\tau_1) = z_1^\mu(\tau_1, 0) = z_2^\mu(\tau_2, \tilde{\eta}_1^\mu(\tau_2)) = \tilde{x}_1^\mu(\tau_2) = x_1^\mu(\tau_1 = f_1(\tau_2, \tilde{\eta}_1^\mu(\tau_2))),
\]

\[
x_2^\mu(\tau_1) = z_1^\mu(\tau_1, \eta_2^\mu(\tau_1)) = z_2^\mu(\tau_2, 0) = \tilde{x}_2^\mu(\tau_2) = x_2^\mu(\tau_1 = f_2(\tau_1, \eta_2^\mu(\tau_1))).
\] (4.2)

If we have an isolated system of N dynamical particles, their world-lines will have the following expression

\[
y_i^\mu(\tau_1) = z_1^\mu(\tau_1, y_{i\eta}^\mu(\tau_1)) = z_2^\mu(\tau_2, \tilde{\eta}_{i\eta}^\mu(\tau_2)) = \tilde{y}_i^\mu(\tau_2) = z_2^\mu(f_2(\tau_1, y_{i\eta}^\mu(\tau_1))), \tilde{\eta}_{i\eta}^\mu(\tau_2) = \tilde{\eta}_{i\eta}^\mu(f_2(\tau_1, \eta_{i\eta}^\mu(\tau_1))),
\]

\[
\tilde{y}_{i\eta}^\mu(\tau_2) \equiv \tilde{y}_{i\eta}^\mu(f_2(\tau_1, \eta_{i\eta}^\mu(\tau_1))) = \eta_i^\tau(\tau_1) = \eta_{i\eta}^\tau(f_1(\tau_2, \tilde{\eta}_{i\eta}^\mu(\tau_2))), \quad i = 1, \ldots, N,
\] (4.3)

in the two non-inertial frames.

In general the 3-spaces \(\Sigma_{\tau_1}\) and \(\Sigma_{\tau_2}\) of the two foliations will intersect each other, since they correspond to different clock synchronization conventions. Therefore, in general the two foliations associated with the two observers do not have a common 3-space to be used as a common Cauchy surface. We can only transcribe the solutions of the equations of motion of an isolated system with Cauchy data on a 3-space of observer 1 in the radar 4-coordinates of observer 2 (or vice versa).

As said in Subsection F of the Introduction the second-class constraints eliminating the internal center of mass and the form of the relative variables are very complicated, so that it is very difficult to rewrite Eqs.(4.3) in terms of the Jacobi data \(\tilde{z}, \tilde{h}\) (the same for Alice and Bob) of the external center of mass and of the relative variables.

\[^{14}\] \(\eta_{i\eta}^\mu(\tau_1)\) are the 3-coordinates of \(x_2^\mu(\tau)\) if we use the foliation of \(x_1^\mu(\tau)\); \(\tilde{\eta}_{i\eta}^\mu(\tau_2)\) are the 3-coordinates of \(x_1^\mu(\tau)\) if we use the foliation of \(x_2^\mu(\tau)\).
B. Alice Inertial and Bob Accelerated Mathematical Observers

Let observer 1 (Alice) be in the inertial rest frame with embedding \( z_1^\mu(\tau_1, \sigma_1^\tau) = z_W^\mu(\tau_1, \sigma_1^\tau) = Y^\mu(\tau_1) + \epsilon^\mu_\tau(\vec{h}) \sigma_1^\tau \). Instead the observer 2 (Bob) is the origin of the non-inertial frame with given embedding \( z_2^\mu(\tau_2, \sigma_2^\mu) \) and has the world-line \( x_2^\mu(\tau_2) = z_2^\mu(\tau_2, 0) \). The world-line of Alice is the Fokker-Planck center of inertia, \( x_W^\mu(\tau_1) = Y^\mu(\tau_1) \): in its expression (2.3) \( M \) and \( \vec{S} \) are the mass and the spin of the isolated system expressed in terms of the relative variables of Alice by means of Eqs.(2.6).

Since Eq.(2.3) implies \( Y^\mu(\tau_1) + \epsilon^\mu_\tau(\vec{h}) \sigma_1^\tau = h^\mu \tau_1 + \epsilon^\mu_\tau(\vec{h}) \left( \sigma_1^\tau + \frac{z^\tau_y}{Mc} + \frac{h^\tau \vec{h} \cdot \vec{z} + (\vec{S} \times \vec{h})^r}{Mc(1 + \sqrt{1 + \vec{h}^2})} \right) \), Eq.(4.1) can be written in the form \( z_2^\mu(\tau_1, \sigma_2^\mu) = z_2^\mu(\tau_2, \sigma_2^\mu) \triangleq h^\mu \bar{z}_2(\tau_2, \sigma_2^\mu) + \epsilon^\mu_\tau(\vec{h}) \bar{z}_2^\tau(\tau_2, \sigma_2^\mu) \). Then we get

\[
\tau_1 = \bar{z}_2(\tau_2, \sigma_2^\mu) = \epsilon h_\mu z_2^\mu(\tau_2, \sigma_2^\mu),
\sigma_1^\tau = z_2^\mu(\tau_2, \sigma_2^\mu) - \frac{z^\tau_y}{Mc} - \frac{h^\tau \vec{h} \cdot \vec{z} + (\vec{S} \times \vec{h})^r}{Mc(1 + \sqrt{1 + \vec{h}^2})} = \epsilon \tau_0(\vec{h}) z_2^\mu(\tau_2, \sigma_2^\mu) - \frac{z^\tau_y}{Mc} - \frac{h^\tau \vec{h} \cdot \vec{z} + (\vec{S} \times \vec{h})^r}{Mc(1 + \sqrt{1 + \vec{h}^2})}.
\] (4.4)

As a consequence for the world-lines \( y_1^\mu(\tau_1) = Y^\mu(\tau_1) + \epsilon^\mu_\tau(\vec{h}) \eta_1^\tau(\tau_1) \) of the N particles of the isolated system we get

\[
\Rightarrow \eta_1^\tau(\tau_1) = z_2^\tau(\tau_2, \eta_1^\mu(\tau_2)) - \frac{h^\tau \vec{h} \cdot \vec{z} + (\vec{S} \times \vec{h})^r}{Mc(1 + \sqrt{1 + \vec{h}^2})}.
\] (4.5)

Therefore all the results in the inertial rest frame centered on Alice with radar 4-coordinates \((\tau_1, \sigma_1^\tau)\) can be rewritten in the accelerated frame centered on the accelerated observer 2 (Bob) using the radar 4-coordinates \((\tau_2, \sigma_2^\mu)\) of the non-inertial frame.

Let us remark that, as shown in Ref.[2], the equations of motion for the matter of the isolated system are very complicated in non-inertial frames due to the presence of the relativistic inertial forces. The results of this Subsection allow to avoid the study of the equations of motion in non-inertial frames: the solution of these equations can be recovered from the solution of the equations of motion in the inertial rest frame, where the 3-space is Euclidean and there are not inertial forces (in general relativity this is not possible). The only problem is to solve the inertial equations of motion with inertial rest-frame Cauchy data and to impose the second-class constraints eliminating the internal center of mass.

C. Alice and Bob Dynamical Observers

We can now take as the world-line of the accelerated observer 2 (Bob) the solution \( x_2^\mu(\tau_1) \) of the equations of motion of a dynamical particle of an isolated N-particle system described
by the mathematical observer 1 (i.e. $x^i_2(\tau_1) = y^i_2(\tau_1)$ for some $i$): in this way we can get the description of the physics from the point of view of a dynamical observer which is always accelerated.

If $x^i_3(\tau_1) = y^i_j(\tau_1)$ with $j \neq i$ is the world-line of another dynamical particle, we can also get the description from the point of view of this second dynamical observer (Charlie).

Therefore, by using the inertial observer Alice to solve the equations of motion, we can get the description of the same physics given by two dynamical observers (Bob and Charlie) and we can compare their descriptions by using Eqs.(4.1).

D. Unruh-DeWitt Detectors

There is a big literature concerning the uniformly accelerated Rindler observers in connection with the Unruh radiation and the entanglement of field modes (see the review in Ref. [1]). These observers are not considered in our framework because their world-lines are asymptotically tangent to a light-cone at $\tau = \pm \infty$ and therefore they disappear with the light-cone in the non-relativistic limit (only time-like observers become Newtonian observers).

Many papers in this area [20] study global field-mode entanglement by using either point-like inertial or non-inertial Unruh-DeWitt detectors. The simplest Unruh-DeWitt detector is a two-level atom with a hybrid description: a) it moves along a classical given world-line and b) it has quantum interactions with a field implying transitions between the two levels.

In Ref. [21] we gave a pseudo-classical description of a relativistic two-level atom in the inertial rest frame and its quantization. Therefore either Alice or Bob (or both) can be described as a relativistic two-level atom interacting with the other matter components of the isolated system and we can have (and compare) the description given by two two-level atom dynamical observers.

V. COMMENTS ON RELATIVISTIC QUANTUM MECHANICS IN NON-INERTIAL FRAMES

After a review of relativistic quantum mechanics (RQM) in the inertial rest frame we make some comments on how to extend it to non-inertial frames and on the problem of what could be the meaning of a quantum observer.

A. Relativistic Quantum Mechanics in the Inertial Rest Frame

In Ref.[4] there is a consistent formulation of RQM of an isolated system of scalar particles in the inertial rest frame with the correct non-relativistic limit. As it is shown in this paper one must quantize the canonically conjugate frozen Jacobi data $\vec{z}$, $\vec{h}$, of the external center of mass and the set of relative variables and relative momenta in the Wigner-covariant Euclidean 3-space after the elimination of the internal center of mass with the second-class constraints corresponding to the rest-frame conditions (see Eq.(2.6) for a two-body case). The solution of the problem of the relativistic collective variables (leading to the non-local
and non-measurable notion of relativistic center of mass), the elimination of the relative times in relativistic bound states and the avoidance of causality problems (like the instantaneous spreading of wave-packets shown by the Hegerfeldt theorem) imply a spatial non-separability according to which the only allowed presentation of the Hilbert space is \( H = H_{H,\text{com}} \otimes H_{\text{rel}} \). While \( H_{\text{rel}} \) is the Hilbert space of relative variables, \( H_{H,\text{com}} \) is the Hilbert space of the frozen external center of mass \(^{15}\). The Hamiltonian is the quantum version of the invariant rest mass \( M \).

Due to the non-local non-measurable nature of the relativistic center of mass, there is the open problem of the type of operator to be used in \( H_{H,\text{com}} \) for the Jacobi position \( \vec{z} = M c \vec{x}_{NW}(0) \). See Ref.\(^{[8]}\) for a discussion of the localization problems in RQM and on the possibility that the center-of-mass position operator be a non-self-adjoint operator. In that paper there is also a discussion on relativistic entanglement and on the implications of the spatial non-separability forbidding the identification of subsystems.

Since the non-separability is due to the elimination of the internal center of mass in the 3-space by means of the three pairs of second-class constraints implementing the rest-frame conditions, one could start with a quantization with a separable un-physical Hilbert space \( H_{\text{unphy}} = H_{H,\text{com}} \otimes H_{1} \otimes H_{2} \otimes ... \), where \( H_{i} \) is the Hilbert space of particle \( i \) described by the Wigner-covariant 3-vectors \( \vec{n}_{i}(\tau), \vec{n}_{i}(\tau) \). Then the reduction to the physical Hilbert space \( H = H_{H,\text{com}} \otimes H_{\text{rel}} \) could be realized by imposing a quantum version of the second-class constraints by means of the Gupta-Bleuler method, namely by selecting as physical states those vectors in \( H_{\text{unphy}} \) which imply a vanishing expectation value for the three pairs of quantum operators corresponding to the second-class constraints. Even if for free particles the procedure works, there is the possibility that in general it leads to an inequivalent quantization. Also the determination of the physical scalar product is not trivial in this case.

This procedure seems the most useful one to extend the rest-frame RQM to arbitrary inertial frames by restricting to inertial frames the second-class constraints discussed in Subsection F of Section II (see Eqs.\((2.11)\)) for the elimination of the inner center of mass of general non-inertial frames.

### 2. Relativistic Quantum Mechanics in Non-Inertial Frames

To extend RQM to non-inertial frames is a highly non-trivial problem. Here we list some possibilities:

A) Even if the 3-spaces are in general non-Euclidean, let us assume that we have found a canonical basis of relative variables in the given non-inertial frame. Then, modulo ordering problems in the terms containing the interactions and using as Hamiltonian the inertial rest

\(^{15}\) In the non-relativistic limit three presentations are unitarily equivalent: \( H = H_{1} \otimes H_{2} \otimes ... = H_{\text{com}} \otimes H_{\text{rel}} = H_{H,\text{com}} \otimes H_{\text{rel}} \). While \( H_{i} \) are the Hilbert spaces of the single particles (non-relativistic separability of subsystems as the zeroth postulate of quantum mechanics) and \( H_{\text{com}} \) is the Hilbert space of the Newtonian center of mass, \( H_{H,\text{com}} \) is the Hilbert space of its frozen Jacobi data obtained with a Hamilton-Jacobi transformation. At the relativistic level the presentation \( H_{1} \otimes H_{2} \otimes ... \) is forbidden by the problem of relative times, while the presentation \( H_{\text{com}} \otimes H_{\text{rel}} \) has the causality problems of the Hegerfeldt theorem.
mass $M$ augmented by suitable inertial potentials (see Eq.(5.32) of the first paper in Ref.[2] for the case of non-inertial rest frames), one can make a quantization like in the case of the inertial rest frame. However, besides the problem of finding the scalar product, there is the generic problem that the physical Hilbert spaces corresponding to different non-inertial frames could be non unitarily equivalent, namely one could have inequivalent quantizations for different non-inertial frames, maybe also inequivalent to the rest-frame RQM.

B) Like in the previous Subsection, one could make an unphysical quantization of the particle 3-variables and then impose the quantum second-class constraints eliminating the internal center of mass with the Gupta-Bleuler method. Again there are the problems of the physical scalar product and of the possibility of inequivalent quantizations.

C) Since at the classical level the descriptions in different non-inertial frames are gauge equivalent in the framework of parametrized Minkowski theories as said in Subsection B of Section II, one could think of making a quantization preserving the gauge equivalence (replaced with some type of unitary equivalence) at the quantum level. This was done in the first paper of Ref.[10] (with the non-relativistic limit studied in the second paper) for the case of Euclidean 3-spaces with the rotating coordinates of Eq.(2.9) by means of the multi-temporal quantization scheme of Refs. [22].

While in A) and B) one quantizes only matter after having fixed the embedding $z^\mu(\tau, \sigma^r)$ (a gauge variable) to a given function identifying a well defined non-inertial frame (first reduce, then quantize), now we consider the enlarged phase space containing the matter and the conjugate variables $z^\mu(\tau, \sigma^r)$, $\rho^\mu(\tau, \sigma^r)$. In this phase space there are the four first-class constraints (2.2), i.e. $\mathcal{H}_\mu(\tau, \sigma^r) \overset{\text{def}}{=} \rho^\mu(\tau, \sigma^r) - \mathcal{G}_\mu(\tau, \sigma^r) \approx 0$ (with $\mathcal{G}_\mu$ depending upon both the matter and the embedding), implying that the embeddings $z^\mu(\tau, \sigma^r)$ are gauge variables, besides the second-class ones eliminating the internal center of mass.

The idea of the multi-temporal quantization is to quantize only the physical degrees of freedom of the particles, but not the gauge variables $z^\mu(\tau, \sigma^r)$: they are considered as c-number generalized times in analogy to the treatment of time in the non-relativistic Schrödinger equation, $i\hbar \frac{\partial}{\partial \tau} \psi(t, q) = \hat{H}(q, \hat{p}) \psi(t, q)$. In this theory we have the c-number time $t$ and the classical equality $E = H$ is realized with $E \mapsto i\hbar \frac{\partial}{\partial \tau}$ and $H \mapsto \hat{H}(q, \hat{p})$. Therefore we send $\rho^\mu(\tau, \sigma^r) \mapsto i\hbar \frac{\partial}{\partial z^\mu(\tau, \sigma^r)}$ and we replace $\mathcal{G}_\mu$ with a suitably ordered self-adjoint operator $\hat{G}_\mu$ depending upon the matter operators and the c-number embeddings. Then the wave functional $\Psi(\tau; z^\beta(\tau, \sigma^r)|\eta^\mu_r)$ must satisfy the following equations (as said in Subsection A of the Section II the canonical Hamiltonian for the $\tau$-evolution is $\dot{M}c$)

$$i\hbar \frac{\partial}{\partial \tau} \Psi = \dot{M}c \Psi,$$

---

16 The multi-temporal approach is different by quantization methods like BRST, in which one firstly quantizes all the variables, also the gauge ones, and then makes the reduction to the physical ones at the quantum level by selecting the states annihilated by a quantum version of the first-class constraints (assuming that there is an ordering such that the quantum constraints satisfy the same algebra as in the classical case, with the quantum constraints located at the extreme right in the results of commutators).

17 $E$, the energy, is the generator of the kinematical Poincaré group identified by the relativity principle, while $H$ is the Hamiltonian governing the time evolution.
\[
\frac{i \hbar}{\delta z^\mu(\tau, \sigma^\tau)} \Psi = \hat{G}_\mu(\tau, \sigma^\tau) \Psi.
\] (5.1)

The theory is consistent if the quantum constraint operators \( \hat{H}_\mu(\tau, \sigma^\tau) = i \hbar \frac{\delta}{\delta z^\mu(\tau, \sigma^\tau)} - \hat{G}_\mu(\tau, \sigma^\tau) \) are still Abelian like in the classical case, i.e. if we find an ordering such that the integrability conditions for Eqs. (5.1) \([\hat{H}_\mu(\tau, \sigma^\tau), \hat{H}_\nu(\tau, \sigma^\nu)] = 0\) hold \(^{18}\). The other integrability condition for Eqs. (5.1) is \([\hat{M}, \hat{G}_\mu(\tau, \sigma^\tau)] = 0\).

In this functional Hilbert space one has to implement the second-class constraints with the Gupta-Bleuler method. A non trivial problem is to find the scalar product in the final physical Hilbert space.

The restriction of the solution of the coupled equations (5.1) to the surface \(z^\mu(\tau, \sigma^\tau) = z^\mu_F(\tau, \sigma^\tau)\) of the functional space of generalized times, with \(z^\mu_F(\tau, \sigma^\tau)\) an admissible 3+1 splitting of Minkowski space-time, gives the RQM in the non-inertial frame classically defined by the gauge-fixings \(z^\mu(\tau, \sigma^\tau) - z^\mu_F(\tau, \sigma^\tau) \approx 0\) to the first-class constraints \(\hat{H}_\mu(\tau, \sigma^\tau) \approx 0\). The integrability conditions imply that we can go from a non-inertial frame to a different one preserving the quantum equivalence of the two descriptions.

When this program can be implemented, we have unitary equivalence of RQM in every admissible non-inertial frame. In particular it should be possible to implement the transformations (4.4) as time-dependent unitary transformations connecting the rest-frame RQM to its non-inertial version. Till now this has been achieved only for rotating coordinates and their non-relativistic limit in Refs.[10].

C. Alice and Bob Quantum Observers?

As already said, the definition of a quantum dynamical observer is quite problematic already at the non-relativistic level. At the relativistic level the spatial non-separability implied by the second-class constraints eliminating the internal center of mass inside the 3-space forces us to include dynamical observers inside the isolated system as said in connection with the Unruh-DeWitt detectors.

To avoid hybrid descriptions in which the trajectory of the detector is classical but its interactions with the other objects are quantum, one has to describe the detectors as macroscopic quantum many-body systems included in the isolated quantum system. If one would be able to quantize these systems, then the hybrid view would emerge due to notions like decoherence, suggesting that the macroscopic quantum system has a quasi-classical collective variable (the Pointer) following a semi-classical Newton-like trajectory. See Ref.[23] for a discussion of the emergence of this classical regime from the quantum one.

However, in the framework of non-relativistic quantum information theory the problem of reference frames and of observers is very important [24] (see these papers and Ref. [25] for the relativistic extension), because for many tasks one needs information on clock synchronization, on the alignment of distinct Cartesian axes and on the determination of global positions. Connected problems are how two unrelated observers Alice and Bob can compare

\(^{18}\) See Ref.[10] for the modifications of the equations (5.1) when one has the more general integrable case \([\hat{H}_\mu(\tau, \sigma^\tau), \hat{H}_\nu(\tau, \sigma^\nu)] = \int d^3 \sigma \hat{C}_\mu(\sigma^\nu, \sigma^\tau) \hat{H}_\nu(\tau, \sigma^\nu)\)
measures of spin when they do not share a common reference frame. In these cases the lack of a reference frame (or its level of imprecision [26]) is treated as a kind of decoherence (or quantum noise) which can wash out all the quantum features of a measurement. See Refs. [24] for an attempt to define quantum reference frames by quantizing the measuring apparatus associated with a mathematical observer (the previously quoted hybrid description).

VI. FINAL REMARKS

We have described the status of the theory of non-inertial frames in Minkowski space-time developed by taking into account the problem of relative times in relativistic bound states and the implications of Lorentz signature for the relativistic collective variables.

In particular we have defined a new family of global non-inertial frames with non-Euclidean instantaneous 3-spaces, which can be obtained from an inertial frame by means of a point-dependent Lorentz transformation as suggested by the locality principle.

We have discussed properties of inertial and non-inertial either mathematical or dynamical relativistic observers.

Already at the classical level we get a non-locality (and non-measurability) of the canonical non-covariant relativistic external center of mass and a spatial non-separability implied by the second-class constraints eliminating the internal center of mass in the 3-spaces. At the quantum level this non-locality and non-separability are at a deeper level with respect to the standard discussion about the violation (Bell’s inequalities) of the local separable realism of Einstein in ordinary non-relativistic quantum mechanics.

The main open problems, besides the explicit construction of particle RQM in non-inertial frames, are connected with the quantization of fields in non-inertial frames:

A) find the rest-frame quantization of free scalar and transverse electro-magnetic fields (see the second paper in Ref.[14]);

B) find the non-inertial frames in which the evolution of a massive scalar field is unitary because the Bogoliubov transformation is of the Hilbert-Schmidt type (solution of the Torre-Varadarajan no-go theorem [27, 28]) and try to understand what happens to the notion of particle (see Ref.[8]).

Appendix A: Calculations

Given the parametrization of Eqs. (3.3)-(3.5) of the Lorentz transformation appearing in the embedding (3.1), we must introduce new quantities needed in the evaluation of the gradients of the embeddings and then of the 4-metric of the non-inertial frame.

For the rotations we define the following quantities (in accord with the notation of Subsection E of Section II)

\[ \tilde{\Omega}_{(R_i)}(\tau, \sigma) = \left( \tilde{R}^{-1} \frac{\partial \tilde{R}}{\partial \tilde{\alpha}_i} \right)(\tau, \sigma) = \left( \begin{array}{cc} 0 & 0 \\ 0 & \Omega_{(R_i)} = R^{-1} \frac{\partial R}{\partial a_i} \end{array} \right)(\tau, \sigma), \]
\[ \tilde{\Omega}_{(R_i)^B}(\tau, \sigma) = \sum_u \delta_u^B \tilde{\Omega}_{(R_i)uv}(\tau, \sigma), \quad \tilde{\Omega}_{(R_i)uv} = -\tilde{\Omega}_{(R_i)vu} \]

\[ \tilde{\Omega}_{(R_i)v}(\tau, \sigma) = \sum_i \alpha_i(\tau) \tilde{\Omega}_{(R_i)v}(\tau, \sigma) \]

\[ \tilde{\Omega}_{(R_i)}(\tau, \sigma) = (\tilde{\Omega}_{(R_i)v}(\tau, \sigma) = \sum_i \alpha_i(\tau) \tilde{\Omega}_{(R_i)v}(\tau, \sigma)), \]

\[ \Omega_{(r)}(\tau, \sigma)_{uv} = (R^{-1} \partial_r R_{uv}(\tau, \sigma) = \sum_i \partial_r \tilde{\alpha}_i(\tau, \sigma) \Omega_{(R_i)uv}(\tau, \sigma) = \]

\[ = \partial_r f(\sigma) \sum_i \alpha_i(\tau) \Omega_{(R_i)uv}(\tau, \sigma) = \partial_r f(\sigma) \sum_i \epsilon_{uvw} \tilde{\Omega}_{(R_i)uw}(\tau, \sigma), \]

\[ \Omega_{(r)}(\tau, \sigma)_{uv} = (R^{-1} \partial_r R_{uv}(\tau, \sigma) = f(\sigma) \sum_i \tilde{\alpha}_i(\tau) \Omega_{(R_i)uv}(\tau, \sigma) = \]

\[ = f(\sigma) \sum_i \epsilon_{uvw} \frac{\Omega_{(R_i)}(\tau, \sigma)}{c} \tilde{n}^w(\tau, \sigma), \quad (A.1) \]

where \( \Omega'(\tau, \sigma) = f(\sigma) \Omega_{(R)}(\tau, \sigma) \tilde{n}^w(\tau, \sigma) \) is the instantaneous angular velocity in the point \( (\tau, \sigma^r) \) \( (\tilde{\alpha}_i(\tau) = \frac{d\alpha_i(\tau)}{d\tau} = \frac{1}{c} \frac{d\alpha_i(ct)}{dt} \) \) and the unit 3-vector \( \tilde{n}(\tau, \sigma) \) is the instantaneous rotation axis there.

For the Lorentz boosts we define the following derived quantities (we use \( \frac{\partial \hat{\beta}^u}{\partial \beta^u} = \tilde{\gamma} \tilde{\beta}^u \),

\[ \frac{\partial \tilde{\gamma} \tilde{\beta}^r}{\partial \beta^u} = \tilde{\gamma} (\delta^{ur} + \tilde{\gamma}^2 \tilde{\beta}^u \tilde{\beta}^r), \quad \frac{\partial^2 \tilde{\gamma} \tilde{\beta}^r}{\partial \beta^u \partial \beta^u} = \tilde{\gamma} \frac{\tilde{\gamma}^2 - 1}{\tilde{\gamma} + 1} \tilde{\beta}^u, \quad \tilde{\gamma}^2 \tilde{\beta}^u = \tilde{\gamma}^2 - 1 \]

\[ \tilde{\Omega}_{(B)}(\tau, \sigma)^A_B = (\tilde{\Omega}_{(B)uv}(\tau, \sigma))^A_B = (\tilde{R}^{-1} B^{-1} \frac{\partial B}{\partial \beta^u} \tilde{R})(\tau, \sigma))^A_B = \]

\[ = (\tilde{R}^{-1} \tilde{\Omega}_{(B)u} \tilde{R})(\tau, \sigma))^A_B, \]

\[ B^{-1} = \left( \begin{array}{cc} \tilde{\gamma} \tilde{\beta}^u \\ -\tilde{\gamma} \tilde{\beta}^r \delta^{rs} + \frac{\tilde{\gamma}^2 \tilde{\beta}^s \tilde{\beta}^s}{\tilde{\gamma} + 1} \end{array} \right), \]

\[ \frac{\partial B}{\partial \beta^u} = \left( \begin{array}{c} \tilde{\gamma} \tilde{\beta}^u \\ \tilde{\gamma} (\delta^{su} + \tilde{\gamma}^2 \tilde{\beta}^u \tilde{\beta}^s) \end{array} \right) \frac{\tilde{\gamma}^2 \tilde{\beta}^t + \tilde{\beta}^s \delta^{tu} + \delta^{su} \tilde{\beta}^t + \tilde{\beta}^s \delta^{su} \frac{\tilde{\gamma}^2 (\tilde{\beta}^s + \tilde{\beta}^u \tilde{\beta}^s)}{\tilde{\gamma} + 1} \right) \]

\[ \tilde{\Omega}_{(B)u} = B^{-1} \frac{\partial B}{\partial \beta^u} = \left( \begin{array}{c} 0 \\ \tilde{\gamma} (\delta^{um} + \frac{\tilde{\gamma}^2}{\tilde{\gamma} + 1} \tilde{\beta}^u \tilde{\beta}^m) \end{array} \right) \]

\[ 19 \text{ We have } 0 < \frac{\Omega_{(uv)}}{c} \leq 2 \max (\dot{\alpha}_1(\tau), \dot{\alpha}_2(\tau), \dot{\alpha}_3(\tau)) \text{ as in Subsection E of Section II.} \]
\[
\gamma \left( \delta^{us} + \frac{\gamma^2}{\gamma+1} \bar{\beta}^u \bar{\beta}^s \right),
\]

\[
\Omega_{(B)u} = \tilde{R}^{-1} \tilde{\Omega}_{(B)u} \tilde{R} = \left( \begin{array}{cc} 0 & \tilde{\gamma} R^T_{vm} \left( \delta^{um} + \frac{\gamma^2}{\gamma+1} \bar{\beta}^u \bar{\beta}^m \right) \\ -\frac{\gamma}{\gamma+1} R^T_{vm} \left( \delta^{um} \bar{\beta}^s - \delta^{us} \bar{\beta}^m \right) \end{array} \right),
\]

\[
\bar{\beta} \cdot \tilde{\Omega}_{(B)} = \left( \begin{array}{cc} 0 & \tilde{\gamma}^2 \tilde{\beta}^s R_{sn} \\ \tilde{\gamma} R^T_{vm} \left( \tilde{\beta}^m + \frac{\tilde{\gamma}}{\gamma+1} \tilde{\beta} \cdot \tilde{\beta} \tilde{\beta}^m \right) \end{array} \right).
\]

As a consequence of the previous notations, we get the following expression for the gradients of the Lorentz matrix appearing in Eq.(3.1) \((\hat{\sigma}^s = \sigma^s/\sigma, \ f'(\sigma) = \frac{d f(\sigma)}{d \sigma})\)

\[
\partial_r \Lambda^A_r(\tau, \sigma) = \Lambda^A_B(\tau, \sigma) \left( \Lambda^{-1} \partial_r \Lambda \right)^B_r(\tau, \sigma) =
\]

\[
= \Lambda^A_B(\tau, \sigma) \left[ \tilde{R}^{-1} B^{-1} \left( f(\sigma) B \sum_i \hat{\alpha}_i(\tau) \frac{\partial \tilde{R}}{\partial \hat{\alpha}_i} + g(\sigma) \tilde{\beta}(\tau) \cdot \frac{\partial B}{\partial \tilde{\beta}} \tilde{R} \right) \right]^B_r(\tau, \sigma) =
\]

\[
= \Lambda^A_B(\tau, \sigma) \left[ f(\sigma) \sum_i \hat{\alpha}_i(\tau) \tilde{\Omega}_{(Ri)}^B_r + g(\sigma) \tilde{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^B_r \right](\tau, \sigma) =
\]

\[
= \Lambda^A_B(\tau, \sigma) \left[ f(\sigma) \frac{\tilde{\Omega}_{(B)}}{c} \sum_{uv} \delta^B_u \epsilon_{ur} \hat{n}^r + g(\sigma) \tilde{\beta}(\tau) \cdot \tilde{\Omega}_{(B)}^B_r \right](\tau, \sigma),
\]

\[
\partial_s \Lambda^A_r(\tau, \sigma) = \Lambda^A_B(\tau, \sigma) \left( \Lambda^{-1} \partial_s \Lambda \right)(\tau, \sigma)^B_r =
\]

\[
= \Lambda^A_B(\tau, \sigma) \hat{\sigma}^s \left[ f'(\sigma) \sum_i \hat{\alpha}_i(\tau) \left( \tilde{R}^{-1} \frac{\partial \tilde{R}}{\partial \hat{\alpha}_i} \right) +
\]

\[
+ g'(\sigma) \tilde{\beta}(\tau) \cdot \left( \tilde{R}^{-1} B^{-1} \frac{\partial B}{\partial \tilde{\beta}} \tilde{R} \right) \right](\tau, \sigma)^B_r =
\]

28
\[ \begin{align*}
&= \Lambda^A_B(\tau, \sigma) \hat{s}^* \left[ f'(\sigma) \sum_i \alpha_i(\tau) \hat{\Omega}_{(R)i}^B r + g'(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B \right](\tau, \sigma) = \\
&= \Lambda^A_B(\tau, \sigma) \hat{s}^* \left[ f'(\sigma) \sum_{uv} \delta_u^B \epsilon_{uv} \hat{\Omega}_{(R)uv} + g(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B \right](\tau, \sigma). \quad (A3)
\end{align*} \]

Therefore we get the following expression for the gradients of the embedding (3.1)

\[ \begin{align*}
z^\mu_r(\tau, \sigma^u) &= \hat{x}^\mu(\tau) + \epsilon^\mu_A \partial_\tau \Lambda^A_r(\tau, \sigma) \sigma^r = \\
&= \epsilon^\mu_A \left( \hat{f}^A(\tau) + \sigma \Lambda^A_B \left[ f(\sigma) \Omega_{(R)} \sum_u \delta_u^B (\hat{\sigma} \times \hat{n})^u + \\
&+ g(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B \right] \right)(\tau, \sigma) = \\
&\overset{\text{def}}{=} \epsilon^\mu_A \Lambda^A_B(\tau, \sigma) \left( \Lambda^{-1} B C \hat{f}^C(\tau) + F^B \right)(\tau, \sigma), \quad (A4)
\end{align*} \]

with the following identifications

\[ \begin{align*}
\left( \Lambda^{-1}(\tau, \sigma) \right)^r H \hat{f}^H(\tau) + F^r(\tau, \sigma) &= \\
= (\alpha(\tau) \hat{\gamma}_x(\tau) \hat{\gamma} \left[ 1 - \hat{\beta} \cdot \hat{\beta}_x(\tau) \right] + \sigma g(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B \hat{n} \hat{\sigma}^n)(\tau, \sigma),
\end{align*} \]

\[ \begin{align*}
\left( \Lambda^{-1}(\tau, \sigma) \right)^r H \hat{f}^H(\tau) + F^r(\tau, \sigma) &= \\
= (\alpha(\tau) \hat{\gamma}_x(\tau) \sum_v R^T_{tv} \left[ \beta^v_x(\tau) - \hat{\gamma} \beta^v \left( 1 - \frac{\hat{\gamma} \hat{\beta} \cdot \hat{\beta}_x(\tau)}{\hat{\gamma} + 1} \right) \right] + \\
+ \sigma \left[ f(\sigma) \Omega_{(R)} \sum_u \hat{\delta}^B (\hat{\sigma} \times \hat{n})^u + g(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B \hat{n} \hat{\sigma}^n \right] \right)(\tau, \sigma),
\end{align*} \]

\[ \begin{align*}
G^s_r(\tau, \sigma) &= \delta^s_r + \sigma \hat{\sigma}^r \left[ f'(\sigma) (\hat{\sigma} \times \hat{\Omega}_{(R)r})^s + g'(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)s}^B \hat{n} \hat{\sigma}^n \right](\tau, \sigma), \\
G^r_r(\tau, \sigma) &= \sigma \hat{\sigma}^r \left[ g'(\sigma) \hat{\beta}(\tau) \cdot \hat{\Omega}_{(B)r}^B(\tau, \sigma) \hat{n} \hat{\sigma}^n \right]. \quad (A5)
\end{align*} \]

For the evaluation of the 4-metric we also need
\[\Lambda_{\tau B}(\tau, \sigma) = \sum_{\tau} \beta_{x}^{\tau}(\tau) \Lambda_{\beta B}(\tau, \sigma) = \left(\tilde{\gamma} \left(1 - \tilde{\beta}_{x}(\tau) \cdot \tilde{\beta}\right)\delta_{B}^{\tau} + \right.\]
\[\left. + \sum_{uv} \left[\tilde{\gamma} \left(1 - \frac{\tilde{\gamma}}{\tilde{\gamma} + 1}\tilde{\beta}_{x}(\tau) \cdot \tilde{\beta} - \beta_{x}^{u}(\tau)\right) R_{uv} \delta_{B}^{uv}\right]\right) (\tau, \sigma). \quad (A6)\]

The unit normal \(l^{\mu}(\tau, \sigma^{u})\) to the instantaneous 3-space \(\Sigma_{\tau}\) is

\[l^{\mu}(\tau, \sigma^{u}) = \epsilon_{D}^{\mu} l^{D}(\tau, \sigma^{u}) = \left(\frac{1}{\sqrt{h}} \epsilon_{\alpha \beta \gamma} z_{1}^{\alpha} z_{2}^{\beta} z_{3}^{\gamma}\right)(\tau, \sigma^{u}) = \]
\[\left(\frac{1}{\sqrt{h}} \epsilon_{\alpha \beta \gamma} \epsilon_{\lambda}^{A} \epsilon_{\mu}^{B} \epsilon_{\nu}^{C} \Lambda^{A}_{\lambda} \Lambda^{B}_{\mu} \Lambda^{C}_{\nu} G^{E}_{1} G^{F}_{2} G^{G}_{3}\right)(\tau, \sigma^{u}). \quad (A7)\]

Using Eq.(A7) for the unit normal to the 3-space, the lapse function \(N = 1 + n\) defined in Subsection A of Section II turns out to be (we have \(\epsilon_{\mu \alpha \beta \gamma} \epsilon_{A}^{k} \epsilon_{B}^{\beta} \epsilon_{C}^{C} = \epsilon_{DABC}\) and \(\epsilon_{DABC} \Lambda^{A}_{W} \Lambda^{B}_{E} \Lambda^{C}_{F} G_{G} = \epsilon_{WEGF}\)

\[1 + n(\tau, \sigma^{u}) = e \left(l^{\mu} \eta_{\mu \nu} z_{\nu}(\tau, \sigma^{u})\right) = \]
\[\left(\frac{1}{\sqrt{h}} \epsilon_{\mu \alpha \beta \gamma} \epsilon_{D}^{\mu} \epsilon_{A}^{\alpha} \epsilon_{B}^{\beta} \epsilon_{C}^{C} \Lambda^{D}_{W} \Lambda^{A}_{E} \Lambda^{B}_{F} \Lambda^{C}_{G}\right) \left[\left(\Lambda^{-1}\right)^{W}_{H} \dot{f}_{H}(\tau) + F^{W}\right] G^{E}_{1} G^{F}_{2} G^{G}_{3}\right)(\tau, \sigma^{u}) = \]
\[\left(\frac{1}{\sqrt{h}} \epsilon_{WEGF} \left[\left(\Lambda^{-1}\right)^{W}_{H} \dot{f}_{H}(\tau) + F^{W}\right] G^{E}_{1} G^{F}_{2} G^{G}_{3}\right)(\tau, \sigma^{u}). \quad (A8)\]
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