Codes Associated with Special Linear Groups and Power Moments of Multi-dimensional Kloosterman Sums

Dae San Kim, Member, IEEE

Abstract—In this paper, we construct the binary linear codes $C(SL(n, q))$ associated with finite special linear groups $SL(n, q)$, with both $n, q$ powers of two. Then, via Pless power moment identity and utilizing our previous result on the explicit expression of the Gauss sum for $SL(n, q)$, we obtain a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in $C(SL(n, q))$. In particular, when $n = 2$, this gives a recursive formula for the power moments of Kloosterman sums. We illustrate our results with some examples.

Index Terms—Kloosterman sum, finite special linear group, Pless power moment identity, weight distribution, Gauss sum.

I. INTRODUCTION

Let $\psi$ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements ($p$ a prime), and let $m$ be a positive integer. Then the $m$-dimensional Kloosterman sum $K_m(\psi; a)$ ([9]) is defined by

$$K_m(\psi; a) = \sum_{\alpha_1, \ldots, \alpha_m \in \mathbb{F}_q^*} \psi(\alpha_1 + \cdots + \alpha_m + a\alpha_1^{-1} \cdots \alpha_m^{-1})$$

(a $\in \mathbb{F}_q^*$).

In particular, if $m = 1$, then $K_1(\psi; a)$ is simply denoted by $K(\psi; a)$, and is called the Kloosterman sum. The Kloosterman sum was introduced in 1926 [7] to give an estimate for the Fourier coefficients of modular forms.

For each nonnegative integer $h$, by $MK_m(\psi; a)^h$ we will denote the $h$-th moment of the $m$-dimensional Kloosterman sum $K_m(\psi; a)$. Namely, it is given by

$$MK_m(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K_m(\psi; a)^h$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK_m(\lambda)^h$ will be simply denoted by $MK_m^h$. If further $m = 1$, for brevity $MK_1^h$ will be indicated by $MK^h$. The power moments of Kloosterman sums can be used, for example, to give an estimate for the Kloosterman sums and have also been studied to solve a variety of problems in coding theory over finite fields of characteristic two.

If $q = p$ is an odd prime, for $h \leq 4$, $MK^h$ was evaluated by Salié [15]. For details about these, the reader is referred to Section IV.

From now on, let us assume that $q = 2^r$. Carlitz [1] evaluated $MK^h$, for $h \leq 4$, while Moisio computed $MK^6$ in [14]. Recently, Moisio was able to find explicit expressions of $MK^h$, for $h \leq 10$ (cf. [11]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the binary Zetterberg code of length $q+1$, which were known by the work of Schoof and Vlugt in [16].

In this paper, we adopt Moisio’s idea to show the following theorem giving a recursive formula for the power moments of multi-dimensional Kloosterman sums. To do that, we construct the binary linear code $C(SL(n, q))$ associated with the special linear group $SL(n, q)$, and express those power moments in terms of the frequencies of weights in the code. Here, in addition to the assumption $q = 2^r$, we restrict $n$ to be $n = 2^s$. Then, thanks to our previous result on the explicit expression of “Gauss sum” for the special linear group [6], we can express the weight of each codeword in the dual $C^+(SL(n, q))$ of $C(SL(n, q))$, in terms of $(n - 1)$-dimensional Kloosterman sums. Then our formula follows immediately from the Pless power moment identity.

**Theorem 1:** Let $n = 2^s$, $q = 2^r$. Then, for all positive integers $h$, we have the following recursive formula for the moments of multi-dimensional Kloosterman sums $MK^n_{n-1}$:

$$q(\mathbb{Z})^h MK^n_{n-1} = \sum_{i=0}^{h-1} (-1)^{h+i+1} \binom{h}{i} N^{h-i} q(\mathbb{Z})^i MK^i_{n-1}$$

$$+ q \sum_{i=0}^{h} (-1)^{h+i} C_i \sum_{t=i}^{h} t! S(h,t) 2^{h-t} \binom{N-t}{N}.$$

(1)

Here $N = q(\mathbb{Z}) \prod_{j=2}^{n} (q^j - 1)$ is the order of $SL(n, q)$, and $S(h,t)$ indicates the Stirling number of the second kind given by

$$S(h,t) = \frac{1}{h!} \sum_{j=0}^{h} (-1)^{h-j} \binom{h}{j} j^h.$$

(2)

In addition, $\{C_i\}_{i=0}^{N}$ denotes the weight distribution of the code $C = C(SL(n, q))$, which is given by

$$C_i = \sum_{\beta \in \mathbb{F}_q} \prod_{\nu} \binom{n_\beta}{\nu_\beta} (0 \leq i \leq N),$$
where the sum runs over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta \in \mathbb{F}_q} \) satisfying \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta = i \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0 \) (an identity in \( \mathbb{F}_q \)), and

\[
n_\beta = |\{g \in SL(n, q) | \text{tr}(g) = \beta\}| = q(\nu)\cdot(1) + 1 + q\theta(\beta),
\]

with

\[
\theta(\beta) = \begin{cases} K_{n-2}(\lambda; \beta^{-1}), & \beta \neq 0, \\ 0, & \beta = 0. \end{cases}
\]

Here we understand that \( K_0(\lambda; \beta^{-1}) = \lambda(\beta^{-1}). \) In addition, from now on we agree that \( \binom{b}{a} = 0 \), if \( b < a \).

**Corollary 2:** Let \( q = 2^r \). Then, for all positive integers \( h \), we have the following recursive formula for the moments of Kloosterman sums \( MK^h \):

\[
q^HMK^h = \sum_{i=0}^{h-1} (-1)^{h+i+1} \binom{h}{i} N^{h-i}q^i MK^i + q \sum_{i=0}^{h-1} (-1)^{h+i} C_i \sum_{t=i}^{h} tS(h, t) 2^{h-t}(N-i)_{N-t}, \tag{3}
\]

Here \( N = q(q^2 - 1) \) is the order of \( SL(2, q) \), \( S(h, t) \) indicates the Stirling number of the second kind as in (2), and \( \{C_i\}_{i=0}^{h} \) denotes the weight distribution of the code \( C = C(SL(2, q)) \), which is given by

\[
C_i = \sum_{(q^2)_{\nu_0}} \prod_{\nu_\beta = 0} \frac{q^2 + \nu_\beta}{q^2 - \nu_\beta} \prod_{\nu_\beta = 1} \frac{(q^2 - q)}{q^2 - \nu_\beta},
\]

where the sum runs over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta \in \mathbb{F}_q} \) satisfying \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta = i \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0 \), and the first and second product run respectively over the elements \( \beta \in \mathbb{F}_q \), with \( \text{tr}(\beta^{-1}) = 0 \) and \( \text{tr}(\beta^{-1}) = 1 \).

**II. Preliminaries**

The following notations will be used throughout this paper except in Section IV, where \( q \) is allowed to be any prime powers.

\[
n = 2^s (s \in \mathbb{Z}_{>0}),
\]

\[
q = 2^r (r \in \mathbb{Z}_{>0}),
\]

\( SL(n, q) \) = the special linear group,

\( N = q^{\binom{n}{2}} \prod_{j=2}^{n} (q^j - 1) \) the order of \( SL(n, q) \),

\( \text{Tr}(g) \) = the matrix trace for \( g \in SL(n, q) \),

\( tr(x) = x + x^2 + \cdots + x^{q^2 - 1} \) the trace function \( \mathbb{F}_q \rightarrow \mathbb{F}_2 \),

\( \lambda(x) = (-1)^{tr(x)} \) the canonical additive character of \( \mathbb{F}_q \).

Let \( g_1, g_2, \cdots, g_N \) be a fixed ordering of the elements in \( SL(n, q) \). Let \( C = C(SL(n, q)) \) be the binary linear code of length \( N \), defined by:

\[
C = C(SL(n, q)) = \{u \in \mathbb{F}_2^N | u \cdot v = 0\}, \tag{4}
\]

where

\[
v = (\text{Tr}(g_1), \text{Tr}(g_2), \cdots, \text{Tr}(g_N)) \in \mathbb{F}_q^N. \tag{5}
\]

**Theorem 3 (Delsarte, [10]):** Let \( B \) be a linear code over \( \mathbb{F}_q \). Then \( (B|_{\mathbb{F}_2}) \rightarrow tr(B^{-1}) \).

From Delsarte’s theorem, the next result follows immediately.

**Proposition 4:** The dual \( C^\perp = C^\perp(SL(n, q)) \) of \( C = C(SL(n, q)) \) is given by

\[
C^\perp = \{c(a) = (\text{tr}(a \text{Tr}(g_1)), \text{tr}(a \text{Tr}(g_2)), \cdots, \text{tr}(a \text{Tr}(g_N))) | a \in \mathbb{F}_q\}.
\]

The next Proposition is stated in Theorem 6.1 of [6]. But we slightly modified the expression there.

**Proposition 5:** Let \( n_\beta = |\{g \in SL(n, q) | \text{Tr}(g) = \beta\}|, \) for each \( \beta \in \mathbb{F}_q \). Then

\[
n_\beta = q^{\binom{\nu}{2}} \prod_{j=2}^{n} (q^j - 1) - (q - 1)^{n-1} + q\delta(n, 1, q; \beta),
\]

where

\[
\delta(n-1, q; \beta) = |\{\alpha_1, \cdots, \alpha_{n-1} \in (\mathbb{F}_q^n - 1) | \alpha_1 + \cdots + \alpha_{n-1} = \beta\}|.
\]

The following corollary is immediate from Proposition 5.

**Corollary 6:** The map \( Tr : SL(n, q) \rightarrow \mathbb{F}_q \) given by \( g \mapsto \text{Tr}(g) \) is surjective.

**Proposition 7:** The map \( \mathbb{F}_q \rightarrow C^\perp(SL(n, q)) \) given by \( a \mapsto c(a) \) is an \( \mathbb{F}_2 \)-linear isomorphism.

**Proof:** It is \( \mathbb{F}_2 \)-linear and surjective. Let \( a \in \mathbb{F}_q \) be in the kernel of the map. Then \( \text{tr}(a Tr(q)) = 0 \), for all \( g \in SL(n, q) \). In view of Corollary 6, \( tr(\alpha a) = 0 \), for all \( \alpha \in \mathbb{F}_q \). As \( tr : \mathbb{F}_q \rightarrow \mathbb{F}_2 \) is surjective, \( a = 0 \).

The next theorem is about the Gauss sum for \( SL(n, q) \), and is one of the main results of the paper [6].

**Theorem 8:** Let \( \psi \) be any nontrivial additive character of \( \mathbb{F}_q \). Then

\[
\sum_{g \in SL(n, q)} \psi(\text{Tr}(g)) = q^{\binom{\nu}{2}} K_{n-1}(\psi; 1).
\]

For the following lemma, observe that \( (n, q - 1) = 1 \).
Lemma 9: The map $a \mapsto a^n : \mathbb{F}_q^* \to \mathbb{F}_q^*$ is a bijection.

For the proof of the next proposition and the following, we borrowed an idea from the proof of Theorem 6.1 in [13].

Proposition 10: For $a \in \mathbb{F}_q^*$, the Hamming weight of the codeword

$$c(a) = (tr(aT(g_1)), tr(aT(g_2)), \ldots, tr(aT(g_N)))$$

is given by(cf. Proposition [4]):

$$w(c(a)) = \frac{1}{2} (N - q \binom{\lambda}{a} K_{n-1}(\lambda; a)).$$

Proof:

$$w(c(a)) = \frac{1}{2} \sum_{i=1}^N (1 - (-1)^{tr(aT(g_i))})$$

$$= \frac{N}{2} - \frac{1}{2} \sum_{g \in SL(n,q)} \lambda(aT(g))$$

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} K_{n-1}(\psi; 1)$$

(Theorem [8] with $\psi(x) = \lambda(ax)$)

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} \sum_{x_1, \ldots, x_{n-1} \in \mathbb{F}_q^*} \lambda(x_1 + \cdots + x_{n-1}$$

$$+ a x_1^{-1} \cdots x_{n-1}^{-1})$$

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} \sum_{x_1, \ldots, x_{n-1} \in \mathbb{F}_q^*} \lambda(x_1 + \cdots + x_{n-1}$$

$$+ a^n x_1^{-1} \cdots x_{n-1}^{-1})$$

(by Lemma [9])

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} \sum_{x_1, \ldots, x_{n-1} \in \mathbb{F}_q^*} \lambda((x_1 + \cdots + x_{n-1}$$

$$+ a x_1^{-1} \cdots x_{n-1}^{-1})^n)$$

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} \sum_{x_1, \ldots, x_{n-1} \in \mathbb{F}_q^*} \lambda(x_1 + \cdots + x_{n-1}$$

$$+ a x_1^{-1} \cdots x_{n-1}^{-1})$$

([9], Theorem 2.23(v))

$$= \frac{N}{2} - \frac{1}{2} q \binom{\lambda}{a} K_{n-1}(\lambda; a).$$

We are ready to determine $\delta(n-1, q; \beta)$, which appears in Proposition [5].

Proposition 11: For each $\beta \in \mathbb{F}_q$, let

$$\delta(n-1, q; \beta) = \{ (\alpha_1, \ldots, \alpha_{n-1}) \in (\mathbb{F}_q^*)^{n-1} | \alpha_1 + \cdots + \alpha_{n-1} + \alpha_1^{-1} \cdots \alpha_{n-1}^{-1} = \beta \}.$$

Then

$$\delta(n-1, q; 0) = q^{-1}(q - 1)^{n-1} + 1,$$

and, for $\beta \in \mathbb{F}_q^*$,

$$\delta(n-1, q; \beta) = K_{n-2}(\lambda; \beta^{-1}) + q^{-1}(q - 1)^{n-1},$$

where $K_{0}(\lambda; \beta^{-1}) = \lambda(\beta^{-1})$ by convention.

Proof:

$$q \delta(n-1, q; \beta)$$

$$= \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \sum_{\alpha \in \mathbb{F}_q} \lambda(\alpha(\alpha_1 + \cdots + \alpha_{n-1}$$

$$+ \alpha_1^{-1} \cdots \alpha_{n-1}^{-1} - \beta))$$

$$= \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha \beta) \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda((\alpha \alpha_1 + \cdots + \alpha \alpha_{n-1}$$

$$+ \alpha_1^{-1} \cdots \alpha_{n-1}^{-1}) + \alpha \alpha_1^{-1} \cdots \alpha_{n-1}^{-1}))$$

$$= \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha \beta) \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda((\alpha_1 + \cdots + \alpha_{n-1}$$

$$+ \alpha_1^{-1} \cdots \alpha_{n-1}^{-1} + (q - 1)^{n-1})$$

(following the steps in [7] - [9])

$$= \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda((\alpha_1 + \cdots + \alpha_{n-1}$$

$$+ \alpha \alpha_1^{-1} \cdots \alpha_{n-1}^{-1}) + (q - 1)^{n-1})$$

$$= \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1}$$

$$+ \alpha \alpha_1^{-1} \cdots \alpha_{n-1}^{-1})$$

$$= \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1}$$

$$+ \alpha_1^{-1} \cdots \alpha_{n-1}^{-1})$$

$$= q \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + 1 + (q - 1)^{n-1})$$

The sum in (10) runs over all $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*$ satisfying $\alpha_1^{-1} \cdots \alpha_{n-1}^{-1} = \beta$, so that it is given by

$$\begin{cases} 0, & \text{if } \beta = 0, \\ K_{n-2}(\lambda; \beta^{-1}), & \text{if } \beta \neq 0, \text{ and } n > 2, \\ \lambda(\beta^{-1}), & \text{if } \beta \neq 0, \text{ and } n = 2. \end{cases}$$

So we get the desired result.
Combining Propositions 5 and 11 we get the following corollary.

**Corollary 12:** Let
\[ n_β = |\{g ∈ SL(n, q) | Tr(g) = β\}|, \]
for each $β ∈ \mathbb{F}_q$. Then
\[
n_β = q^\binom{2}{2} - 1 \{\prod_{j=2}^{n} (q^j - 1) + 1 + qθ(β)\}, \tag{11}\]
where
\[ θ(β) = \begin{cases} K_{n-2}(λ; β^{-1}), & β ≠ 0, \\
0, & β = 0, \end{cases} \]
with the convention that $K_0(λ; β^{-1}) = λ(β^{-1})$.

## III. Proof of Main Results

In this section, we will derive the recursive formula (11) for the power moments of multi-dimensional Kloosterman sums which is expressed in terms of the frequencies $C_i$ of weights in the code $C = C(SL(n, q))$.

**Theorem 13 (Pless power moment identity, [10]):** Let $B$ be an $q$-ary $[n, k]$ code, and let $B_i$ (resp. $B_i^+$) denote the number of codewords of weight $i$ in $B$ (resp. in $B^+$). Then, for $h = 0, 1, 2, ⋯,
\[
\sum_{i=0}^{n} i^h B_i = \sum_{i=0}^{\min\{n, h\}} (-1)^i B_i^+ \sum_{t=i}^{h} t! S(h, t) q^{h-t} \times (q-1)^{1-t} \binom{n-i}{n-t}, \tag{12}\]
where $S(h, t)$ is the Stirling number of the second kind defined in (2).

**Theorem 14 ([18]):** Let $q = 2^r$, with $r ≥ 2$. Then the range $R$ of $K(λ; α)$, as $α$ varies over $\mathbb{F}_q^*$, is given by
\[ R = \{t ∈ \mathbb{Z} | |t| < 2\sqrt{q}, t ≡ -1(\text{mod } 4)\}. \]
In addition, each value $t ∈ R$ is attained exactly $H(t^2 - q)$ times, where $H(d)$ is the Kronecker class number of $d$.

**Theorem 15 ([2]):** For the canonical additive character $λ$ of $\mathbb{F}_q$, and $α ∈ \mathbb{F}_q^*$,
\[ K_2(λ; α) = K(λ; α)^2 - q. \tag{13}\]
Let $u = (u_1, ⋯, u_N) ∈ \mathbb{F}_q^N$, with $ν_β$ 1’s in the coordinate places where $Tr(g_j) = β$, for each $β ∈ \mathbb{F}_q$. Then we see from the definition of the code $C = C(SL(n, q))$ (cf. (4), (5)) that $u$ is a codeword with weight $i$ if and only if $\sum β ∈ \mathbb{F}_q ν_β = i$ and $\sum β_1 ∈ \mathbb{F}_q ν_β β = 0$ (an identity in $\mathbb{F}_q$). As there are $\prod β ∈ \mathbb{F}_q ν_β$ many such codewords with weight $i$, we obtain the following theorem.

**Theorem 16:** Let $\{C_i\}_{i=0}^{N}$ be the weight distribution of the code $C = C(SL(n, q))$. Then, for $0 ≤ i ≤ N$,
\[ C_i = \prod_{β ∈ \mathbb{F}_q} \binom{n_β}{ν_β}, \tag{14}\]
where $n_β$ is as in (11), and the sum runs over all the sets of nonnegative integers $\{ν_β\}_{β ∈ \mathbb{F}_q}$ satisfying
\[
\sum_β ν_β = i \text{ and } \sum_β ν_β β = 0 \text{ (an identity in } \mathbb{F}_q). \tag{15}\]

**Corollary 17:** Let $\{C_i\}_{i=0}^{N}$ be the weight distribution of the code $C = C(SL(n, q))$. Then, for $0 ≤ i ≤ N$,
\[ C_i = C_{N-i}. \]

**Proof:** Under the replacements $ν_β → n_β - ν_β$, for all $β ∈ \mathbb{F}_q$, the first sum in (15) is changed to $N - i$, while the second one in (15) and the summands in (14) are left unchanged. Here the second sum in (15) is left unchanged, since $\sum_β n_β β = 0$, as one can see by using the explicit expression of $n_β$ in (11).

**Corollary 18:** Let $\{C_i\}_{i=0}^{N}$ be the weight distribution of the code $C = C(SL(2, q))$. Then, for $0 ≤ i ≤ N$,
\[ C_i = \prod_{β ∈ \mathbb{F}_q} \binom{q^2}{ν_β}, \tag{16}\]
where the sum runs over all the sets of nonnegative integers $\{ν_β\}_{β ∈ \mathbb{F}_q}$ satisfying $\sum_β ν_β = i$ and $\sum_β ν_β β = 0$, and the first and second product run respectively over the elements $β ∈ \mathbb{F}_q^*$ with $tr(β^{-1}) = 0$ and $tr(β^{-1}) = 1$.

**Proof:** For $n = 2$, we see from (11) that $n_β$ is given by
\[
n_β = \begin{cases} q^2, & \text{if } β = 0, \\
q^2 + q, & \text{if } tr(β^{-1}) = 0, \\
q^2 - q, & \text{if } tr(β^{-1}) = 1. \end{cases} \]

**Corollary 19:** Assume that $r ≥ 2$, and that $\{C_i\}_{i=0}^{N}$ is the weight distribution of the code $C = C(SL(4, q))$. Then, for $0 ≤ i ≤ N$,
\[ C_i = \prod_{β ∈ \mathbb{F}_q} \binom{m_0}{ν_β}, \tag{16}\]
where the sum runs over all the sets of nonnegative integers $\{ν_β\}_{β ∈ \mathbb{F}_q}$ satisfying $\sum_β ν_β = i$ and $\sum_β ν_β β = 0$.
\[
m_0 = n_0 = q^4 \{\prod_{j=2}^{4} (q^j - 1) + 1\},
\]
and
\[
m_i = q^6 \{(q^2(q^2 - 1)(q^4 - q - 1) + t^2)\}
\]
for all integers $t$ satisfying $|t| < 2\sqrt{q}$ and $t ≡ -1(4)$.
Proof: Note here that, for $n = 4$, and $\beta \in \mathbb{F}_q^*$,
\[
n_\beta = q^5 \left\{ \prod_{j=2}^{4} (q^j - 1) + 1 + qK_2(\lambda; \beta^{-1}) \right\}
\]
\[
= q^5 \left\{ \prod_{j=2}^{4} (q^j - 1) + 1 + q(K(\lambda; \beta^{-1})^2 - q) \right\} \quad \text{(cf. (13)) (17)}
\]
\[
= q^6 \{ q^2(q^2 - 2)(q^4 - 1) + K(\lambda; \beta^{-1})^2 \}.
\]
Now, invoking Theorem 14, we obtain the result.

We are now ready to prove Theorem 1 which is the main result of this paper. To do that, we apply Pless power moment identity in (12), with $B = C^+(SL(n, q))$. Then, in view of Proposition 7 and utilizing (6), the left hand side of (12) is given by

\[
\sum_{a \in \mathbb{F}_q^*} w(c(a))^h = \frac{1}{2^n} \sum_{a \in \mathbb{F}_q^*} (N - q(\alpha))^h
\]
\[
= \frac{1}{2^n} \sum_{h=0}^{h} \binom{h}{i} N^{h-i} q(\alpha)^i MK_{n-1}^i
\]
\[
= \frac{1}{2^n} \sum_{h=0}^{h} \binom{h}{i} q(\alpha)^i MK_{n-1}
\]
\[
+ \frac{1}{2^n} \sum_{i=0}^{h-1} \binom{h}{i} N^{h-i} q(\alpha)^i MK_{n-1}^i.
\]

On the other hand, noting that $dim \ C^+(SL(n, q)) = r$ (cf. Proposition 7), the right hand side of (12) is given by

\[
q \sum_{i=0}^{min\{N, h\}} (-1)^i C_i \sum_{t=1}^{h} t! S(h, t) 2^{-t} \left( \frac{N-i}{N-t} \right).
\]

Here the frequencies $C_i$ of codewords with weight $i$ in $C = C(SL(n, q))$ are given by (14).

Now, Corollary 2 follows from Theorem 1 and Corollary 18 and Corollary 20 from Theorem 1 and Corollary 19.

Corollary 20: For all positive integers $h$, we have the following recursive formula for the moments of the 3-dimensional Kloosterman sums $MK_n^h$,

\[
q^{6h} MK_n^h = \sum_{i=0}^{h-1} (-1)^{h+i+1} \binom{h}{i} N^{h-i} q^{6i} MK_3^i
\]
\[
+ q \sum_{i=0}^{min\{N, h\}} (-1)^{h+i} C_i \sum_{t=1}^{h} t! S(h, t) 2^{h-t} \left( \frac{N-i}{N-t} \right).
\]

Here $N = q^8 \prod_{j=2}^{4} (q^j - 1)$ is the order of $SL(4, q)$, $\{C_i\}_{i=0}^{N}$ denotes the weight distribution of the code $C = C(SL(4, q))$ given by (16), and $S(h, t)$ indicates the Stirling number of the second kind as in (2).

IV. REMARKS

Here we will briefly review the previous results on power moments of Kloosterman sums $MK_n^h$, and make some comments on our result in (3). For any $q = p^r$ ($p$ a prime),

\[
MK_n^h = \frac{q^2}{q-1} A_h - (q-1)^{h-1} + 2(-1)^{h-1},
\]
where

\[
A_h = \{(\alpha_1, \cdots, \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 0 = \sum_{j=1}^{h} \alpha_j^{-1}\}.
\]

For $h \in \mathbb{Z}_{\geq 0}$, define $M_h$ as:

\[
M_h = \{(\alpha_1, \cdots, \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1}\},
\]
for $h > 0$, and $M_0 = 0$.

Then, as one can see, $(q-1)M_{h-1} = A_h$, for any positive integer $h$. So (18) can be rewritten as

\[
MK_n^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1}(h \geq 1).
\]

Salié obtained this form of expression for $MK_n^h$ already in [15], for any odd prime $q$. Iwaniec [5] showed the expression (18) for any prime $q$. However, the proof given there works for any prime power $q$, without any restriction. Also, this is a special case of Theorem 1 in [3], as mentioned in Remark 2 there.

Let $q = p$ be any prime. Then

\[
MK_1 = 1,
\]
\[
MK_2 = p^2 - p - 1,
\]
\[
MK_3 = (\frac{-3}{p}) p^2 + 2p + 1
\]
(with the understanding $\frac{-3}{p} = -1, \frac{-3}{3} = 0$),

\[
MK_4 = \begin{cases} 2p^3 - 3p^2 - 3p - 1, & \text{if } p \geq 3, \\ 1, \quad & \text{if } p = 2. \end{cases}
\]

Salié obtained these results in [15] by determining $M_1, M_2, M_3,$ and Iwaniec got these ones in [5] by computing $A_2, A_3, A_4$.

Except [1] for $1 \leq h \leq 4$ and [14] for $h = 6$, not much progress had been made until Moisio succeeded in evaluating $MK_n^h$, for the other values of $h$ with $h \leq 10$ over the finite fields of characteristic two (Similar results exist also over the finite fields of characteristic three [4],[12]). His results are as follows:
MK^1 = 1,
MK^2 = q^2 - q - 1,
MK^3 = -(1 - i)^2 q^2 + 2q + 1,
MK^4 = 2q^2 - 2q^2 - 3q - 1,
MK^5 = (u_1 + (1 - i)^4 q^3 + 5q^2 + 4q + 1,
MK^6 = 5q^4 - (5 + (1 - i)) q^3 - 9q^2 - 5q - 1,
MK^7 = (u_2 + 6u_1 + (1 - i)^14 + 1) q^4 + 14q^3 + 14q^2 + 6q + 1,
MK^8 = 14q^5 - (15 + (1 - i)^7) q^4 - 28q^3 - 20q^2 - 7q - 1,
MK^9 = (u_3 + 8u_2 + 27u_1 + 8 + (1 - i)^{48}) q^5 + 42q^4 + 48q^3 + 27q^2 + 8q + 1,
MK^{10} = 42q^6 - (51 + (1 - i)^{35}) q^5 - 90q^4 - 75q^3 - 35q^2 - 9q - 1 - u_4.
(20)

Here u_1, u_2, u_3, u_4 are the following numbers which are dependent upon the extension degree r of \( \mathbb{F}_q \) over \( \mathbb{F}_2 \):

\[
\begin{align*}
u_1 & = (1 + \sqrt{-15})/4 + (1 - \sqrt{-15})/4, \\
u_2 & = (5 + \sqrt{-39})/8 + (5 - \sqrt{-39})/8, \\
u_3 & = ((3 + \sqrt{505}) + \sqrt{-510 - 6\sqrt{505}})/32, \\
& + ((3 - \sqrt{505}) - \sqrt{-510 - 6\sqrt{505}})/32, \\
& + ((3 + \sqrt{505}) - \sqrt{-510 + 6\sqrt{505}})/32, \\
& + ((3 - \sqrt{505}) + \sqrt{-510 + 6\sqrt{505}})/32, \\
u_4 & = (12 + 4\sqrt{-119}) + (12 - 4\sqrt{-119}).
\end{align*}
\]

As we mentioned earlier, these were obtained, via Pless power moment identity, by expressing power moments of Kloosterman sums in terms of the frequencies of weights in the binary Zetterberg code of length \( q + 1 \). In fact, Moisio used the frequencies \( B_i \) in the Zetterberg code for \( i \leq 12 \), which were available in Table 6.2 of [16].

Even though it was a breakthrough, it had a few drawbacks. Firstly, the way it is proved is too indirect, since the frequencies are expressed in terms of the Eichler Selberg trace formulas for the Hecke operators acting on certain spaces of cusp forms for \( \Gamma_1(4) \). Secondly, the power moments of Kloosterman sums are obtained only for \( h \leq 10 \) and not for any higher order moments. On the other hand, our formula in [3] allows one, at least in principle, to compute moments of all orders for any given \( q \). Moreover, it gives a recursive formula not only for power moments of Kloosterman sums but also for those of multi-dimensional Kloosterman sums(cf. [1]). Nevertheless, obviously it is good to have explicit formulas like the ones presented in (20). In the next section, we will give some numerical examples demonstrating that our formula in [3] is quite useful for evaluating power moments of Kloosterman sums for each given \( q \).

### V. Examples

In this section, for small values of \( i \), we compute, by using Corollary 2 and MAGMA, the frequencies \( C_i \) of weights in \( C(SL(2, 2^1)) \) and \( C(SL(2, 2^4)) \), and the power moments \( MK^i \) of Kloosterman sums over \( \mathbb{F}_{2^3} \) and \( \mathbb{F}_{2^4} \). In particular, our results confirm those of Moisio’s given in [20], when \( q = 2^5 \) and \( q = 2^4 \).

#### TABLE I

| w | frequency | \( w \) | frequency |
|---|---|---|---|
| 0 | 1 | 11 | 149542406326242956416 |
| 1 | 64 | 12 | 61437053467599526740 |
| 2 | 15844 | 13 | 232515435619775717374208 |
| 3 | 260560 | 14 | 8154684929991910122364729 |
| 4 | 332067914 | 15 | 266385184075218923500482944 |
| 5 | 3320770816 | 16 | 81413978183092952553734637429 |
| 6 | 276256975973 | 17 | 2337059897741406887869645824 |
| 7 | 196480443747136 | 18 | 63323459273593401393172170526526 |
| 8 | 1220634764256355 | 19 | 16173684530938435842527341156928 |
| 9 | 672705382262671680 | 20 | 39221184987526914463644771778098622 |
| 10 | 3320891564305363704 | 21 | 90395403188175503873640492641088 |

#### TABLE II

The power moments of Kloosterman sums over \( \mathbb{F}_{2^3} \)

| \( i \) | \( MK^i \) | \( i \) | \( MK^i \) |
|---|---|---|---|
| 0 | 7 | 10 | 99427275 | 20 | 9537789199381 |
| 1 | 1 | 11 | -4829687 | 21 | -476806577143519 |
| 2 | 55 | 12 | 245734951 | 22 | 23842799341944545 |
| 3 | -47 | 13 | -121920159 | 23 | -11920646525541647 |
| 4 | 871 | 14 | 6117864535 | 24 | 59605492064000071 |
| 5 | -2399 | 15 | -30474531407 | 25 | -298020862011124799 |
| 6 | 17815 | 16 | 15271700791 | 26 | 1490123734802256015 |
| 7 | -71567 | 17 | -762552032639 | 27 | -745055770131537617 |
| 8 | 410311 | 18 | 381559527095 | 28 | 3722971614966505511 |
| 9 | -1894079 | 19 | -190699955437272 | 29 | -18626340903196368479 |

#### TABLE III

The weight distribution of \( C(SL(2, 2^3)) \)

| w | frequency |
|---|---|
| 0 | 1 | 6 | 398943240589823720 |
| 1 | 256 | 7 | 232184965775802188544 |
| 2 | 520072 | 8 | 1182117069839415200320 |
| 3 | 70962176 | 9 | 5348398745381696122698324 |
| 4 | 720560617320 | 10 | 2177331292444581822006776 |
| 5 | 587401078798592 | 11 | 8056132578206330016084726166784 |

#### TABLE IV

The power moments of Kloosterman sums over \( \mathbb{F}_{2^4} \)

| \( i \) | \( MK^i \) | \( i \) | \( MK^i \) |
|---|---|---|---|
| 0 | 15 | 4 | 7631 | 8 | 13118351 |
| 1 | 1 | 5 | 22981 | 9 | 72973441 |
| 2 | 239 | 6 | 300719 | 10 | 604249199 |
| 3 | 289 | 7 | 1343239 | 11 | 3760049569 |
ACKNOWLEDGMENT

I would like to thank Mr. Dong Chan Kim for providing me with the above tables.

REFERENCES

[1] L. Carlitz, “Gauss sums over finite fields of order $2^n$.” Acta Arith., vol. 15, pp. 247-265, 1969.
[2] L. Carlitz, “A note on exponential sums,” Pacific J. Math., vol. 30, pp. 35-37, 1969.
[3] Hi-joon Chae and D. S. Kim, “A generalization of power moments of Kloosterman sums,” Arch. Math.(Basel), vol. 89, pp. 152-156, 2007.
[4] G. van der Geer, R. Schoof and M. van der Vlugt, “Weight formulas for ternary Melas codes,” Math. Comp., vol. 58, pp. 781-792, 1992.
[5] H. Iwaniec, Topics in Classical Automorphic Forms, Amer. Math. Soc., Providence, R. I., 1997.
[6] D. S. Kim, “Gauss sums for general and special linear groups over a finite field.” Arch. Math.(Basel), vol. 69, pp. 297-304, 1997.
[7] H. D. Kloosterman, “On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$.” Acta. Math. vol. 49, pp. 407-464, 1926.
[8] G. Lachaud and J. Wolfmann, “The weights of the orthogonals of the extended quadratic binary Goppa codes,” IEEE Trans. Inform. Theory, vol. 36, pp. 686-692, 1990.
[9] R. Lidl and H. Niederreiter, Finite Fields, 2nd ed. Cambridge, U. K.:Cambridge University Pless, 1997, vol. 20, Encyclopedia of Mathematics and Its Applications.
[10] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error Correcting Codes. Amsterdam, The Netherlands: North-Holland, 1998.
[11] M. Moisio, “The moments of a Kloosterman sum and the weight distribution of a Zetterberg-type binary cyclic code,” IEEE Trans. Inform. Theory, vol. 53, pp. 843-847, 2007.
[12] M. Moisio, “On the moments of Kloosterman sums and fibre products of Kloosterman curves,” Finite Field Appl., vol.14, pp. 515-531, 2008.
[13] M. Moisio, “Kloosterman sums, elliptic curves, and irreducible polynomials with prescribed trace and norm,” Acta Arith., to appear.
[14] M. Moisio and K. Ranto, “Kloosterman sum identities and low-weight codewords in a cyclic code with two zeros,” Finite Fields Appl., vol.13, pp. 922-935, 2007.
[15] H. Salié, “Über die Kloostermanschen Summen $S(u, v; q).$” Math. Z., vol. 34, pp. 91-109, 1931.
[16] R. Schoof and M. van der Vlugt, “Hecke operators and the weight distributions of certain codes,” J. Combin. Theory Ser. A, vol. 57, pp.163-186, 1991.