Dynamics of Josephson junction systems in the computational subspace

Wang Xiangbin, Matsumoto Keiji, Fan Heng
Imai Quantum Computation and Information project, ERATO, Japan Sci. and Tech. Corp.
Daini Hongo White Bldg. 201, 5-28-3, Hongo, Bunkyo, Tokyo 113-0033, Japan

Y. Nakamura
NEC Fundamental Research Lab., Tsukuba, Ibaraki 305-8051, Japan

Abstract

The quantum dynamics of the Josephson junction system in the computational subspace is investigated. A scheme for the controlled not operation is given for two capacitively coupled SQUIDs. In this system, there is no systematic error for the two qubit operation. For the inductively coupled SQUIDs, the effective Hamiltonian causes systematic errors in the computational subspace for the two qubit operation. Using the perturbation theory, we construct a more precise effective Hamiltonian. This new effective Hamiltonian reduces the systematic error to the level much lower than the threshold of the fault resilient quantum computation.

I. INTRODUCTION

It has been shown that, a quantum computer, if available, can perform certain tasks much more efficiently than a classical Turing machine. The realization of the basic constituent of quantum computer, fault tolerant quantum logic gate, is a central issue in the subject. One can make a fault tolerant quantum computation through the quantum error
correction. It has been pointed out that, if the error rate of each operation is smaller than certain threshold, the error correction method works for arbitrary large scale computation. The threshold is estimated to be $10^{-6}$.

Recently, it is proposed [4–7] to implement the quantum gate by superconducting interference device (SQUID). The single qubit operation has been experimentally demonstrated in ref [4]. This implementation has a number of advantages. One can address the single qubit (Josephson Junction) instead of the bulk material in NMR. One can even control the interaction between the two qubit system through the external parameters (voltages and magnetic flux). However, strictly speaking, the Josephson junction is not a two level system. Only when we choose certain specific parameters we can approximately have a two level computational subspace. In general there is a small transition probability between the computational space and the outside space, this causes the leakage [8]. The leakage error can be minimized by the specifically designed fabrication [8]. Also, one may detect and correct the leakage error by taking the measurement on the space outside computational space. Besides the leakage, the approximate effective Hamiltonian causes another type of error, the phase shift error. If the rate of all accumulatable errors are lower than certain threshold [17], a large reliable quantum computation can be done, in principle. Because in such cases we can always take certain appropriate error correction procedure to reduce the error exponentially [17]. In this paper we analyse the phase shift error caused by the approximate effective Hamiltonian. We give a new effective Hamiltonian through the perturbation method. Using this new effective Hamiltonian, the phase shift error rate is lower than the quantum computation threshold [17].

Consider a superconducting electron box formed by a symmetric SQUID (see fig. 1), piece by a magnetic flux $\Phi$ and with an applied gate voltage $V_x$. The device is operated in the charging regime, i.e., the Josephson couplings $E_{J0}$ are much smaller than the charging energy $E_{ch}$. Also a temperature much lower than the Josephson coupling is assumed. The Hamiltonian for this system is [1317]
\[ H = E_{ch}(\hat{n} - n_x)^2 - E_J(\Phi) \cos(\theta), \]  

(1)

\[ E_J(\Phi) = 2E_{J0} \cos \left( \frac{\Phi}{2\Phi_0} \right), \]

\[ E_{ch} \] is the charging energy and \( n_x \) can be tuned by the applied voltage \( V_x \) through \( V_x = 2en_x/C_x \) (see figure 1). The phase difference across the junction \( \chi \) and the cooper pair number \( \hat{n} \) canonical conjugate variables \( [\theta, \hat{n}] = i \). \( \Phi_0 = \hbar/2e \) is the quantum of flux. So here \( n_x \) and \( \Phi \) can be tuned externally.

If the value of parameter \( n_x \) is close to \( 1/2 \), the energy gap between the ground state \(|0\rangle\) and the first excited state \(|1\rangle\) is much smaller than the gaps among any other states. Thus the basis \(|0\rangle, |1\rangle\) approximately make a computational subspace. The transition between the states in the subspace and the state outside the subspace is small. Through calculation of the matrix element i.e., \( \langle n_1 H | n_2 \rangle \) (\( n_1, n_2 = 0, 1 \)), the matrix form of the Hamiltonian \( H \) in the subspace \(|0\rangle, |1\rangle\) is

\[ H_e = -\frac{1}{2}[B_x \sigma_x + B_y \sigma_y + B_z \sigma_z], \]  

(2)

where \( (B_x, B_y, B_z) = (E_J(\Phi), -E_J(\Phi), E_{ch}(1 - 2n_x)) \) and \( \sigma_{x,y,z} \) are Pauli matrices. Intuitively, we may regard this \( H_e \) as the effective Hamiltonian in the computation subspace. We can take the rotating operation through the time evolution property of this effective Hamiltonian \( H_e \).

However, this \( H_e \) may not be the best choice in simulating the real time evolution. To reduce the systematic error, one can use another form of the effective Hamiltonian which simulates the time evolution in the computational subspace more precisely. In this paper, we calculate the matrix element \( e^{-iHt} \) by the perturbation expansion. We know the time evolution operator generated by \( H \) can be expressed in the form \( e^{-iHt} = \sum_n e^{-iE'_nt} |\Psi_n\rangle \langle \Psi_n| \), we still have difficulty in obtaining the precise time evolution properties for state \(|0\rangle\) and \(|1\rangle\). Here \( E'_n \) and \(|\Psi_n\rangle\) are eigenvalue and eigenstate of \( H \), respectively. \(|\Psi_n\rangle\) can be expressed in Mathieu function, which is the infinite summation of trigonometric functions. But in the real calculation, we have to use the truncated Mathieu function. As we will show it latter, after taking the first non-zero perturbation term into consideration, the error is much smaller than the threshold for fault resilient quantum computation \([17]\). In the following sections,
after the we study the single qubit case by the perpurbation method, we will investigate the
two qubit dynamics, which is the heart of the elementary quantum logic device. We will
study both the capasitively coupled SQUIDs and the inductively coupled SQUIDs. For the
former one, we find the effective Hamiltonian in the subspace works exactly. For the latter
one, we give a new form for the effective Hamiltonian so that it works more precisely in the
computational subspace, especially, the error rate is much lower than the threshold for a
large, reliable quantum computation.

II. TIME EVOLUTION FROM THE PERTURBATION TREATMENT FOR ONE
QUBIT

We first write the Hamiltonian $H$ in the following equivalent form

$$H = E_{ch} \sum_n (n - n_x)^2 |n\rangle \langle n| - \frac{E_J}{2} \sum_{n=0}^{\infty} (|n\rangle \langle n+1| + |n+1\rangle \langle n|).$$

(3)

Furthermore, we decompose the Hamiltonian into two parts as

$$H = H_0 + H_c$$

and

$$H_0 = E_{ch} \sum_n (n - n_x)^2 |n\rangle \langle n| - \frac{E_J}{2} (|0\rangle \langle 1| + |1\rangle \langle 0|) = H_e + E_{ch} \sum_{n>1} (n - n_x)^2 |n\rangle \langle n|,$$

(4)

$$H_c = \frac{E_J}{2} \sum_{n \neq 1}^{\infty} (|n\rangle \langle n+1| + |n+1\rangle \langle n|).$$

(5)

We regard this $H_c$ as the perturbation term. $H_0$ part can be solved exactly. The first two
eigenvalues($E_0(0)$ and $E_1(0)$) and first two eigenstates($|\phi_0\rangle$ and $|\phi_1\rangle$) are just that of $H_e$. The
rest eigenvalues and eigenstates are are just that of operator $E_{ch} \sum_n (n - n_x)^2 |n\rangle \langle n|$. They
have already been denoted as $E_n$ and $|n\rangle$, respectively. If we ignore $H_c$, this $H_0$ is identical

4
to eigenstates here. (For more exact result, one can do it similarly through higher order approximation.) We have the following results:

\[
E_0 = \frac{E_{\text{ch}}(1 - 2n_x + 2n_x^2) - \sqrt{E_{\text{ch}}^2(1 - 2n_x) + E_J^2}}{2} + \Delta E_0, \tag{6}
\]

\[
E_1 = \frac{E_{\text{ch}}(1 - 2n_x + 2n_x^2) + \sqrt{E_{\text{ch}}^2(1 - 2n_x) + E_J^2}}{2} + \Delta E_1 \tag{7}
\]

Here \(\Delta E_1 = \frac{E_J^4}{4E_{\text{ch}}(3 - 2n_x)}\) and \(\Delta E_0 = \frac{E_J^2}{16E_{\text{ch}}n_x}\). Take the perturbation result above into consideration, we obtain the time evolution operator in the computational subspace

\[
U(t) = \sum_{n=0}^{2} e^{-iE_n t} |\phi_n\rangle \langle \phi_n|. \tag{8}
\]

Therefore, the new effective Hamiltonian with this timel evolution property is

\[
H'_e = E_0|\phi_0\rangle \langle \phi_0| + E_1|\phi_1\rangle \langle \phi_1| = H_e + \Delta E_0|\phi_0\rangle \langle \phi_0| + \Delta E_1|\phi_1\rangle \langle \phi_1| \tag{9}
\]

This new effective Hamiltonian gives the evolution operator as Eq. (8). If \(n_x\) is close to 1/2, the modification in the effective Hamiltonian is insignificant. Especially, if \(n_x = 1/2\), \(H'_e\) is same with \(H_e\), up to a constant term. But they may differ obviously when \(n_x\) is far from 1/2. With this new effective Hamiltonian, if we omit the leakage error, the distance between the state under real time evolution(\(\psi_r(t)\)) and the state given by evolution of Eq. (8) (\(\psi(t)\)) is smaller than \(\frac{E_J^4}{E_{\text{ch}}}\). Typically, if \(E_J/E_{\text{ch}} = 0.02\), the error is is between \(10^{-8}\) to \(10^{-7}\), smaller than the threshold for resilient quantum computation, \(10^{-6}\) to \(10^{-5}\) [17].

However, if we use the old effective Hamiltonian which is the first order perturbation in the computational subspace, the error rate is in the magnitude order of \(10^{-4}\), larger than the threshold. However, the error rate can be larger than this threshold if we use the old effective Hamiltonian \(H_e\), provided that \(n_x\) is far from 1/2. If the value \(E_J/E_{\text{ch}}\) is not so small in certain case, we can take a higher order perturbation calculation.

III. TWO QUBIT DYNAMICS
A. capacitive coupling

For the capacitively coupled SQUID, the interacting Hamiltonian is

\[ H_I = \delta(n_1 - n_{x,1})(n_2 - n_{x,2}), \]  

(10)

\[ \delta = 2E_{ch}E_K/C. \]  

Now we regard qubit 1 as the control bit and qubit 2 as the target bit. Our goal is to make a conditional flip to qubit 2, i.e., if state of qubit 1 is \( |0\rangle \), nothing happens to qubit 2; if state of qubit 1 is \( |1\rangle \), then qubit 2 is flipped. To make the C-NOT gate, we set \( n_{x,1} \) to 0 and \( \Phi_1 = \Phi_0/2 \) for qubit 1. For qubit 2 we set the magnetic flux \( \Phi_2 = \Phi_0 \) so that \( E_J = 0 \); and \( n_{x,2} = 1/2 \). Just wait for a period of

\[ \Delta t = \pi/\delta. \]  

(11)

We obtain the following conditional unitary transformation for qubit 2

\[ U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]  

(12)

if state of qubit 1 is \( |0\rangle \); and

\[ U = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \]  

(13)

if state of qubit 1 is \( |1\rangle \). This is equivalent to C-NOT gate through Hardmard transformation. Since \( E_J \) is set to 0 here, so the above two qubits operation is exact. Thus if we use the capacitively coupled SQUIDS, there is no phase shift or leakage error caused by the states outside the computational subspace in the two qubits operation.

B. Inductively coupled case

The C-NOT gate of inductively coupled SQUID system is given in [6] by using the effective Hamiltonian in the computational system. Here we use the time evolution operator
in the whole system and then to calculate the time $t$ at which the fidelity between time evolution operator and a C-NOT matrix has the maximum value.

The Hamiltonian for the two interacting SQUIDs is

$$H = H_1 + H_2 - \frac{1}{E_L}(E_{J1}(\Phi_1) \sin \Theta_1 + E_{J2}(\Phi_2) \sin \Theta_2)^2$$  \hspace{1cm} (14)

We know $\sin \Theta_i = (|n_i\rangle\langle n_i + 1| - |n_i + 1\rangle\langle n_i|)/(2i)$. We denote

$$H_O = H_0 \otimes 1 + 1 \otimes H_0 + \frac{E_{J1}E_{J2}}{2E_L} \left[(|0\rangle\langle 1| - |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| - |1\rangle\langle 0|)\right]$$  \hspace{1cm} (15)

and

$$H_P = H_e \otimes I + I \otimes H_e + \frac{E_{J1}E_{J2}}{2E_L} \left(\sum_{n \neq 0} |n\rangle\langle n + 1| - |n + 1\rangle\langle n|\right) \otimes \left(\sum_{n \neq 0} |n\rangle\langle n + 1| - |n + 1\rangle\langle n|\right) + \frac{E_{J1}^2}{E_L} \left(\sum_{n=-\infty}^{+\infty} |n+2\rangle\langle n+2| + |n+2\rangle\langle n|\right) + \frac{E_{J2}^2}{E_L} \left(\sum_{n=-\infty}^{+\infty} |n\rangle\langle n| + |n+2\rangle\langle n|\right).$$  \hspace{1cm} (16)

Previously, the $H_O$ above was used for the effective Hamiltonian in the computational sub-space for two qubits operation \cite{3,4} It is easy to see the total Hamiltonian for the two SQUIDS is

$$H = H_O + H_P.$$  \hspace{1cm} (17)

We can regard $H_P$ as the perturbation term. In the two qubit operation, we can always set $n_{x1} = n_{x2} = 1/2$. Thus $H_e$ is simplified to $E_J(\Phi) (|0\rangle\langle 1| + |1\rangle\langle 0|)$. Under this condition we have

$$H_O = U \begin{pmatrix} E_{J1} + E_{J2} & 0 & 0 & \frac{E_{J1}E_{J2}}{E_L} \\ 0 & E_{J1} - E_{J2} - \frac{E_{J1}E_{J2}}{E_L} & 0 \\ 0 & \frac{E_{J1}E_{J2}}{E_L} & -E_{J2} - E_{J1} & 0 \\ \frac{E_{J1}E_{J2}}{E_L} & 0 & 0 & -E_{J1} - E_{J2} \end{pmatrix} U^\dagger,$$  \hspace{1cm} (18)

where we have used the basis of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and $U = e^{-\frac{i}{2}\sigma_y} \otimes e^{-\frac{i}{2}\sigma_y}$. Explicitly,

$$e^{-\frac{i}{2}\sigma_y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$ We can obtain the eigenvalues and eigenstates of $H_O$ exactly. The
four eigenvalues are \( \pm \sqrt{(E_{J1} + E_{J2})^2 + \frac{E_{J1}^2}{E_L}}, \pm \sqrt{(E_{J1} - E_{J2})^2 + \frac{E_{J1}^2}{E_L}} \), corresponding to the following four eigenstates respectively

\[
|\psi_{00}\rangle = U \left( \cos \frac{\theta_1}{2} |00\rangle + \sin \frac{\theta_1}{2} |11\rangle \right) = \frac{1}{2} [\chi_+ (|00\rangle + |11\rangle) + \chi_- (|01\rangle + |10\rangle)],
\]

\[
|\psi_{01}\rangle = U \left( \cos \frac{\theta_2}{2} |01\rangle + \sin \frac{\theta_2}{2} |10\rangle \right) = \eta_+ (-|00\rangle + |11\rangle) + \eta_- (|01\rangle - |10\rangle),
\]

\[
|\psi_{10}\rangle = U \left( \sin \frac{\theta_2}{2} |01\rangle - \cos \frac{\theta_2}{2} |10\rangle \right) = \eta_+ (-|00\rangle + |11\rangle) - \eta_- (|01\rangle - |10\rangle),
\]

and

\[
|\psi_{11}\rangle = U \left( \sin \frac{\theta_1}{2} |00\rangle - \cos \frac{\theta_1}{2} |11\rangle \right) = \frac{1}{2} [\chi_+ (|00\rangle + |11\rangle) - \chi_- (|01\rangle + |10\rangle)].
\]

Here

\[
\chi_{\pm} = \sin \frac{\theta_1}{2} \pm \cos \frac{\theta_2}{2}, \eta_{\pm} = \sin \frac{\theta_2}{2} \pm \cos \frac{\theta_2}{2}, \cos \theta_1 = (E_{J1} + E_{J2})/(E_{J1} + E_{J2})^2 + \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2, \sin \theta_1 = E_{J1}E_{J2}/\left( E_L (E_{J1} + E_{J2})^2 + \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2 \right), \cos \theta_2 = E_{J1} - E_{J2}/\sqrt{(E_{J1} - E_{J2})^2 + \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2}
\]

and

\[
\sin \theta_2 = -E_{J1}E_{J2}/\sqrt{(E_{J1} + E_{J2})^2 + \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2}. E_L.
\]

Now we take the first non-zero modification in the perturbation to the 4 states. The modifications to the 4 energy levels above are

\[
\Delta_+, 0, 0\Delta_-,
\]

where

\[
\Delta_{\pm} = \chi_+ \Delta E_{11} \pm \chi_- \Delta E_{01},
\]

and

\[
\Delta E_{11} = \frac{1}{8E_{ch}} \left[ E_{J1}^2 + E_{J2}^2 + \frac{1}{2} \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2 + \frac{2}{3} \left( \frac{E_{J1}^2 + E_{J2}^2}{E_L} \right)^2 \right] \tag{19}
\]

\[
\Delta E_{01} = \frac{1}{8E_{ch}} \left[ E_{J1}^2 + E_{J2}^2 - \frac{1}{2} \left( \frac{E_{J1}E_{J2}}{E_L} \right)^2 + \frac{2}{3} \left( \frac{E_{J1}^2 + E_{J2}^2}{3E_L} \right)^2 \right] \tag{20}
\]

Up to an unimportant constant term, the new effective Hamiltonian is
\[ H_E = H_O + \Delta_+ |\psi_{00}\rangle\langle \psi_{00}| + \Delta_- |\psi_{11}\rangle\langle \psi_{11}| \]  

Suppose initially we have a state \(|\Psi_i\rangle\), after time \(t\), the state in the computational subspace is \(\rho(t) = \Pi e^{-iHt}|\Psi_i\rangle\langle \Psi_i| e^{iHt}\Pi\), \(\Pi = \sum |kj\rangle\langle kj|\) and \(k, j\) can take values of 0 and 1. However, if we use the effective Hamiltonian, the state we supposed in the computational subspace is \(\rho_e(t) = \Pi e^{-iH_E t}|\Psi_i\rangle\langle \Psi_i| e^{iH_E t}\Pi\). To estimate the error rate caused by the phase shift error due to the effective Hamiltonian, we just calculate distance between \(\rho(t)\) and \(\rho_e(t)\). Suppose the values of \(E_{J1}\), \(E_{J2}\) are close, \(E_L\) is not larger than \(E_{J1}\) or \(E_{J2}\). After calculation we know, if we use the \(H_O\) for the effective Hamiltonian, the magnitude order of the distance can be \(\frac{E_J^2}{10E_{ch}}\). However, if we use the modified one, \(H_E\) for the effective Hamiltonian, the error rate is reduced to the magnitude order of \(\frac{E_J^4}{10E_{ch}}\). This is to say, if \(H_O\) is used as the effective Hamiltonian, the error rate is larger than threshold for the fault resilient quantum computation, \(10^{-6}\) to \(10^{-5}\), provided that \(E_J/E_{ch}\) is larger than 0.01. However, if we use \(H_E\), the one proposed in this paper, the error rate is much smaller than the threshold value for the fault resilient quantum computation.

In summary, the properties of the effective Hamiltonian of the Josephson junction system in the computational subspace is investigated. For capacitively coupled Josephson junction system, there is no systematic error due to the effective Hamiltonian in the computational subspace for the two qubits operation. If the inductively coupled SQUIDs are used, the effective Hamiltonian in the computational subspace causes phase shift error. But the this effective Hamiltonian can be modified to a more exact form. Our new effective Hamiltonian given by the perturbation theory reduces the systematic error to a range much lower than the threshold of the fault tolerate quantum computation.

**Acknowledgement:** We thank Prof Imai for support. We thank Dr. Huang W.Y for discussions.
REFERENCES

[1] S. Lloyd *Science*, **261**, 1569(1993).

[2] C. H. Bennett, *Phys. Today* **48**, 24(1995).

[3] D. P. DiVincenzo, *Science*, **269**, 255(1995).

[4] Nakamura, Y., Pashkin, Yu and Tsai, J.S., *Nature*, **398**, 786-188(1999).

[5] Makhlin Y, Schon G and Shnirman A, Rev. of Mod Phys., **73**, 357(2001).

[6] Makhlin, Y., Schon, G. and Shnirman, A., *Nature*, **398**, 305-307(1999)

[7] Shnirman, A., Schon, G. and Hermon, Z., *Phys. Rev. Lett.*, **79**, 2371-2374(1997).

[8] R. Fazio, G. M. Palma and J. Siwert, *Phys. Rev. Lett.*, **83**, 5385(1999).

[9] S. Pancharatnam, Proc. India Acad. Sci. A **44**, 247(1956).

[10] M.V. Berry, Proc. R. Soc. London, Ser. A **392**, 45(1984).

[11] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593(1987).

[12] J A Jones, V Vedral, A Ekert, and G Castagnoli, *Nature* **403**, 869(2000)

[13] Falcl G, Fazio R, Palma G.M., Siewert J and Verdal V, *Nature* **407**, 355(2000)

[14] Deutsch, D., Barenco, A. and Ekert, A. *Proc. R. Soc. Lond. A**449**, 669-677(1995)

[15] Alexandre B and Andre-Marie S M, quant/ph-0105006.

[16] D. Suter, K. T. Muller, and A. Pines, *Phys. Rev. Lett.* **60**, 1216(1988).

[17] For example, E. Knill, R. Laflamme and W. H. Zurek, Science 279, 342(1998), J. Preskill, Proc. R. Soc. London A, 454, 385(1998), M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
FIG. 1. SQUID with symmetric Josephson junctions
FIG. 2. The capacitively coupled SQUIDs.