EXTREMES OF THE 2D SCALE-INHOMOGENEOUS DISCRETE GAUSSIAN FREE FIELD: SUB-LEADING ORDER AND TIGHTNESS

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Abstract. This is the first of a three paper series in which we present a comprehensive study of the extreme value theory of the scale-inhomogeneous discrete Gaussian free field. This model was introduced by Arguin and Ouimet in [6] in which they computed the first order of the maximum. In this first paper we establish the log-correction to the order of the maximum and establish tightness of the centred maximum. Our proofs are based on the second moment method and Gaussian comparison techniques.

1. Introduction

In recent years, log-correlated (Gaussian) processes have received considerable attention, see e.g. [3, 4, 10, 15, 24, 42]. One of the reasons for this is that their correlation structure is such that it becomes relevant for the properties of the extremes of the processes. Some prominent examples that fall into this class are branching Brownian motion (BBM), the 2d discrete Gaussian free field (DGFF), local maxima of the randomised Riemann zeta function on the critical line and cover times of Brownian motion on the torus. The 2d DGFF is one of the well understood non-hierarchical log-correlated models (see [9, 10, 11, 19]). For simplicity, consider the 2d DGFF on a square lattice box of side length $N$. It turns out that the maximum can be written as a first order term which is the logarithm of the volume of the box, a second order correction which is proportional to the logarithm of the first order and stochastically bounded fluctuations. If one considers an uncorrelated Gaussian field on the same box with identical variances, a simple computation shows that the first order of the maximum coincides with the one of the DGFF, whereas the constant in front of the second order correction differs. In [6], Arguin and Ouimet introduced the scale-inhomogeneous 2d DGFF, the analogue model of the time-inhomogeneous BRW or the variable speed BBM [42] where the variance is a function of time. They determined the first order of the maximum. In this paper we continue the study of the maximum, find the second order correction and show tightness of the centred maximum.

1.1. The 2d discrete Gaussian free field. Let $V_N := ([0, N) \cap \mathbb{Z})^2$. The interior of $V_N$ is defined as $V_N^o := ([1, N - 1] \cap \mathbb{Z})^2$ and the boundary of $V_N$ is denoted by $\partial V_N := V_N \setminus V_N^o$. Moreover, for points $u, v \in V_N$ we write $u \sim v$, if and only if $\|u - v\|_2 = 1$, where $\|\cdot\|_2$ is the Euclidean norm. Let $\mathbb{P}_u$ be the law of a SRW $\{W_k\}_{k \in \mathbb{N}}$ starting at $u \in \mathbb{Z}^2$. The normalised Green kernel is given by

$$G_{V_N}(u, v) := \frac{\pi}{2} \mathbb{E}_u \left[ \sum_{i=0}^{\tau_{\partial V_N}} \mathbb{1}_{\{W_i = v\}} \right], \text{ for } u, v \in V_N. \quad (1.1)$$

Here, $\tau_{\partial V_N}$ is the first hitting time of the boundary $\partial V_N$ by $\{W_k\}_{k \in \mathbb{N}}$. For $\delta > 0$, we set $V_N^\delta := (\delta N, (1 - \delta)N)^2 \cap \mathbb{Z}^2$. By [22] Lemma 2.1, we have for $u, v \in V_N^\delta$,

$$G_{V_N}(u, v) = \log N - \log \|u - v\|_2 + O(1). \quad (1.2)$$
Definition 1.1. The 2d discrete Gaussian free field (DGFF) on \( V_N \), \( \phi^N := \{\phi^N_v\}_{v \in V_N} \), is a centred Gaussian field with covariance matrix \( G_{V_N} \) and entries \( G_{V_N}(x,y) = \mathbb{E}[\phi^N_x\phi^N_y] \), for \( x, y \in V_N \).

From [Definition 1.1] it follows that \( \phi^N_v = 0 \) for \( v \in \partial V_N \), i.e. we have Dirichlet boundary conditions.

1.2. The 2d scale-inhomogeneous discrete Gaussian free field.

Definition 1.2. (2d scale-inhomogeneous discrete Gaussian free field).

Let \( \phi^N = \{\phi^N_v\}_{v \in V_N} \) be a 2d DGFF on \( V_N \). For \( v = (v_1, v_2) \in V_N \), let \([v]_N\) be the box of side length \( N^{1-\lambda} \) centred at \( v \), namely

\[
[v]_N \equiv [v]_N^N := \left( v_1 - \frac{1}{2} N^{1-\lambda}, v_1 + \frac{1}{2} N^{1-\lambda} \right] \times \left( v_2 - \frac{1}{2} N^{1-\lambda}, v_2 + \frac{1}{2} N^{1-\lambda} \right] \cap V_N.
\]

(1.3)

and set \([v]_0^N := V_N \) and \([v]_1^N := \{v\} \). We define \( \phi^N_v(\lambda) \) by conditioning on the DGFF outside the box \([v]_X^N \), i.e.

\[
\phi^N_v(\lambda) = \mathbb{E} \left[ \phi^N_w | \sigma(\phi^N_v : w \notin [v]_X^N) \right], \quad \lambda \in [0,1].
\]

(1.4)

We denote by \( \nabla \phi^N_v(\lambda) \) the gradient of the DGFF at vertex \( v \) and scale \( \lambda \). Further, let \( s \mapsto \sigma(s) \) be a non-negative function such that \( I_{\sigma^2}(\lambda) := \int_0^1 \sigma^2(x)dx \) is a non-decreasing function on \([0,1]\) with \( I_{\sigma^2}(0) = 1 \) and \( I_{\sigma^2}(1) = 1 \). Then the 2d scale-inhomogeneous DGFF on \( V_N \) is a centred Gaussian field \( \psi^N := \{\psi^N_v\}_{v \in V_N} \) defined as

\[
\psi^N_v := \int_0^1 \sigma(s) \nabla \phi^N_v(s)ds.
\]

(1.5)

We consider the case when \( \sigma \) is a left-continuous step function taking \( M \in \mathbb{N} \) values. Thus, there are variance parameters \( (\sigma_1, \ldots, \sigma_M) \in [0,\infty)^M \) and scale parameters \( (\lambda_1, \ldots, \lambda_M) \in (0,1)^M \) with \( 0 =: \lambda_0 < \lambda_1 \ldots < \lambda_M = 1 \), such that

\[
\sigma(s) = \sum_{i=1}^M \sigma_i \mathbb{1}_{(\lambda_{i-1}, \lambda_i]}(s), \quad s \in [0,1].
\]

(1.6)

The discrete increment of the DGFF at scale \( \lambda_i \) is \( \phi^N_v(\lambda_i) - \phi^N_v(\lambda_{i-1}) \). In this case, the scale-inhomogeneous 2d DGFF or 2d \((\sigma, \lambda)\)–DGFF takes the form

\[
\psi^N_v = \sum_{i=1}^M \sigma_i (\phi^N_v(\lambda_i) - \phi^N_v(\lambda_{i-1})).
\]

(1.7)

By Green function estimates for \( v, w \in V_N^0 \) (see [44] Appendix), we have

\[
\mathbb{E} \left[ \psi^N_v \psi^N_w \right] = \log_2 N I_{\sigma^2} \left( \frac{\log_2 N - ||v - w||_1}{\log_2 N} \right) + O(\sqrt{\log(N)}).
\]

(1.8)

Remark 1.3. Let \( \mathcal{F}_{\partial[v]_N \cup [v]_N^N} := \sigma \left( \phi^N_v, v \notin [v]_N^N \right) \) be the \( \sigma \)–algebra generated by the variables outside \([v]_N^N \), which consists of those vertices that have a common edge with a vertex outside the box \([v]_N \). Note that \( \sigma \left( \mathcal{F}_{\partial[v]_N \cup [v]_N^N}, v \notin [v]_N^N \right) \) is a filtration as the neighbourhoods are shrinking with increasing \( \lambda \). In particular, it is a Gaussian field and thus for \( \lambda' > \lambda \) its increments \( \phi^N_v(\lambda') - \phi^N_v(\lambda) \), which are differences of conditional expectations, are independent. As a consequence, \( (\phi^N_v(\lambda))_{\lambda \in [0,1]} \) is a martingale for each \( v \in V_N \). Further, note that the scale-inhomogeneous DGFF is a martingale-transform of \( (\phi^N_v(\lambda), \lambda \in [0,1]) \) applied simultaneously to each \( v \in V_N \). In analogue to the 2d DGFF, we are able to give an alternative definition of the scale-inhomogeneous DGFF by prescribing a zero mean Gaussian fields that has correlations which can be written in terms of the parameters along with Green kernels and harmonic measures.
The main result of this paper is the identification of the correct second order correction of the maximum of the scale-inhomogeneous 2d DGFF when there are finitely many scales \( M \in \mathbb{N} \). We start with some notation.

Let \( \hat{I}_{\sigma^2}(s) \) be the concave hull of \( I_{\sigma^2}(s) \). There exists a unique non-increasing, left-continuous step function \( s \to \bar{\sigma}(s) \), which we call ‘effective variance’, such that

\[
\hat{I}_{\sigma^2}(s) = \int_0^s \bar{\sigma}^2(r) \, dr =: I_{\bar{\sigma}^2}(s) \quad \text{for all } s \in [0, 1].
\]

(2.1)

The points where \( \bar{\sigma} \) jumps on \([0, 1]\) we call

\[
0 =: \lambda^0 < \lambda^1 < \ldots < \lambda^m := 1,
\]

(2.2)

where \( m \leq M \). To be consistent with previous notation (cf.(1.6)), we write \( \bar{\sigma}^i = \bar{\sigma}(\lambda^i) \). It turns out that the concave hull of \( I_{\sigma^2} \), denoted \( \hat{I}_{\sigma^2} \), gives the desired control for the first order of the maximum. Arguin and Ouimet [6, Theorem 1.2] determined the correct first order behaviour, i.e. they showed that in probability,

\[
\lim_{N \to \infty} \frac{\psi^*_N}{2 \log(N)} = I_{\bar{\sigma}^2}(1) = \sum_{i=1}^m \bar{\sigma}_i \nabla \lambda^i =: \gamma^*.\]

(2.3)

In the following, the goal is to prove a second order correction and tightness of the maximum around its mean.

**Theorem 2.1.** Let \( \{\psi^*_N\}_{v \in V_N} \) be a 2d \((\sigma, \lambda)-\)DGFF on \( V_N \) with \( M \in \mathbb{N} \) scales. Assume that on each interval \([\lambda^{i-1}, \lambda^i]\) and \( i = 1, \ldots, m \), we have either \( I_{\sigma^2} = I_{\bar{\sigma}^2} \) or \( I_{\sigma^2} < I_{\bar{\sigma}^2} \). Then,

\[
\mathbb{E} \left[ \max_{v \in V_N} \psi^*_N \right] = 2 \gamma^* \log(N) - \sum_{j=1}^m \frac{w_j \bar{\sigma}_j \log(\nabla \lambda^j / \log(N))}{2} + O(1),
\]

(2.4)

where

\[
w_j = \begin{cases} 3, & I_{\sigma^2}|_{(\lambda^{j-1}, \lambda^j]} = I_{\bar{\sigma}^2}|_{(\lambda^{j-1}, \lambda^j]} \\ 1, & \text{else} \end{cases}
\]

(2.5)

**Remark 2.2.** Regarding the additional assumption in Theorem 2.1, we expect that in general there are essentially two events determining the logarithmic correction. For each interval \([\lambda^{j-1}, \lambda^j]\) one has to see whether the effective variance and the real variance coincide in a neighbourhood at the beginning or the end of the interval. If neither is the case we have the 1/2 correction. If it coincides
in a neighbourhood at exactly one end point, we expect the factor to be 2/2 and if it coincides in neighbourhoods at the beginning and the end, we expect the correction factor to be 3/2.

An interesting fact is that the profile of the variance matters both for the leading term and the logarithmic correction. This phenomenon was first observed in GREM \cite{32,17,18} and in the context of the time-inhomogeneous branching Brownian motion/branching random walk \cite{45}. The following theorem establishes tightness of the centred maximum.

**Theorem 2.3.** Let $N \in \mathbb{N}$ and $(\psi_N^{\star})_{v \in V_N}$ be a 2d $(\sigma, \lambda)$-DGFF on $V_N$ with $M \in \mathbb{N}$ scales. Then, the sequence of the centred maximum $|\psi_N^{\star} - \mathbb{E}[\psi_N^{\star}]|_{N=0}$ is tight. In particular, there exists a constant $c_N > 0$, depending solely on the variance parameters such that for any $x > 0$ and $N \in \mathbb{N}$, it holds that

$$
P\left(|\psi_N^{\star} - \mathbb{E}[\psi_N^{\star}]| \geq x \right) \leq (1 + x)e^{-c_N x}.
$$

By Theorem 2.1, the statement in Theorem 2.3 is equivalent to

$$
\psi_N^{\star} = 2y^* \log(N) - \sum_{j=1}^{m} \sum_{i=1}^{\sigma_j} \log(\sqrt{\lambda_j}) \log(N) + O_p(1),
$$

where $O_p(1)$ means that the sequence $\{\psi_N^{\star}\}_{N \in \mathbb{N}}$ is stochastically bounded, that is for any $\epsilon > 0$ there is an $M$ such that for all $N$ we have $\mathbb{P}(|\psi_N^{\star} - \mathbb{E}[\psi_N^{\star}]| > M) < \epsilon$.

**Remark 2.4.** We have two more papers in preparation that deal with the case when there are finitely many scales and variances are strictly increasing. In the first, we show convergence of the centred maximum to a randomly shifted Gumbel distribution. This result is then extended in a second paper, in which we prove convergence of the extremal process of local maxima to a Cox process.

**2.1. Overview of related results.** We want to mention that for the maximum of the DGFF much more precise information is available. We write $\phi_N^{\star} := \max_{v \in V_N} \phi_N^{\star}$ for the maximum of the DGFF. Through the works of Bolthausen, Deuschel and Giacomin \cite{11} as well as Bramson and Zeitouni \cite{20} one obtains,

$$
\phi_N^{\star} = 2 \log N - \frac{3}{4} \log \log N + Y,
$$

where $Y$ is random variable of order $o(\log \log N)$ in probability. They further deduced that the centred maximum $\phi_N^{\star} - \mathbb{E}[\phi_N^{\star}]$ is tight as a sequence of real random variables. Further refinements such as the convergence of the centred maximum \cite{19} and of the extremal process to a cluster Cox process \cite{9,10} are available as well.

Also, in the context of BBM there are analogues to ours and further results available. Branching Brownian motion (BBM) can be defined as a Gaussian field indexed by the leaves of an underlying Galton Watson tree with zero mean and covariance given by the overlap on the tree. Its hierarchical structure makes it easier to analyse and its extremes are particularly well understood (see \cite{2,5,15,21}). Further, in the context of BBM there is an analogue model to the scale-inhomogeneous DGFF which we consider, and which is called variable-speed BBM. In this model each particle performs a time-changed Brownian motion, where the time change is identical for every particle and fixed. To be more precise, let $A : [0, 1] \to [0, 1]$, strictly increasing with $A(0) = 0$, $A(1) = 1$ and bounded second derivatives. Further, let $X$ be a standard Brownian motion and fix a time horizon $t > 0$. Variable-speed BBM in time $t$ and with time change $tA(\cdot/t)$ can then be constructed as usual BBM. The only difference is that when particles split at some time $s < t$ their offspring perform independent time-changed Brownian motions, i.e. they are independent copies of the process \{$X_{tA(u/t)} - X_{tA(s/t)}\}_{s \leq s \leq t}$, starting at their parents position at time of splitting. For variable-speed BBM analogue results as achieved in this paper and further details are proved in \cite{13,14,26}. In particular, the first order and second order correction of the maximum in the regime of weak correlations, i.e. when $A(s) < s$ for $s \in (0, 1)$, is identical to the uncorrelated regime. In the case of decreasing speed with finitely many
changes in speed, the global maximum is a simple concatenation of the maximum at speed change.
When the speed is strictly decreasing, i.e. when $A'' < 0$, it is known that unlike in the weakly correlated
and the usual BBM, the second order correction is no longer logarithmic but proportional to $t^{1/3}$ (see
[18, 26, 27]).
In the discrete analogue model of (variable-speed) BBM, the (time-inhomogeneous) branching random
walk (BRW) on the Galton-Watson tree, there are analogue results available to ours (see [26, 42, 45]).
A notable major difference in this model is however that their increments do not need to be Gaussian
(see [26, 42]).

2.2. Idea of proof. The idea of the proof of Theorem 2.1 is similar to the one in the 2d DGFF [20], i.e.
one constructs auxiliary Gaussian fields, a time-inhomogeneous BRW (IBRW) and an inhomogeneous
modified branching random walk (MIBRW). The time-inhomogeneous BRW is constructed in such a
way that it is slightly less correlated than the scale-inhomogeneous DGFF, whereas the MIBRW has
correlations that differ from those of the scale-inhomogeneous DGFF inside the field only up to a unif-
formly bounded constant. Using Gaussian comparison, we can bound the expected maximum of the
scale-inhomogeneous DGFF from above by the expected maximum of the time-inhomogeneous BRW
which is explicitly known up to bounded fluctuations [42, Theorem 1.4] [45]. In a second step, we use
Slepian’s lemma (see Theorem A.2) to show that the expected maximum of the scale-inhomogeneous
DGFF can be bounded from below by the expected maximum of a truncated version of the MIBRW.
We then further reduce the lower bound on the expected maximum of the truncated MIBRW to a lower
bound of the expected maximum of the MIBRW on a subset. The corresponding lower bound is then
achieved by a second moment analysis.
The main idea to prove Theorem 2.3 is to use Slepian’s lemma to compare the distribution of the
centred maximum of the scale-inhomogeneous DGFF with the distribution of the centred maximum of the
MIBRW. In a second step, we prove tail estimates for the centred maximum of the MIBRW that
allow us to deduce tightness. The tail estimates are obtained using a modified second moment analysis.

Outline of the paper: In the next section we define two auxiliary Gaussian processes, the time-
inhomogeneous branching random walk (IBRW) and the modified time-inhomogeneous branching
random walk (MIBRW), and estimate their covariance structure. Section 4 comprises the proof of
Theorem 2.1 which we split into an upper and lower bound. In Section 5 we provide tail estimates
that allow us to deduce tightness of the centred maximum, which proves Theorem 2.3. In A we provide
the Gaussian comparison theorems we use in the proof and Borell’s Gaussian concentration inequality.
In B we prove the covariance estimates stated in Section 3.

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3. Auxiliary processes and covariance estimates
Consider $N = 2^n$ for some $n \in \mathbb{N}$. For $k = 0, 1, \ldots, n$ let $\mathcal{B}_k$ denote the collection of subsets of
$\mathbb{Z}^2$ consisting of squares of side length $2^k - 1$ with corners in $\mathbb{Z}^2$ and let $\mathcal{BD}_k$ denote the subset of $\mathcal{B}_k$
consisting of squares of the form $((0, 2^k - 1) \cap \mathbb{Z})^2 + (i2^k, j2^k)$. We remark that the collection $\mathcal{BD}_k$
partitions $\mathbb{Z}^2$ into disjoint squares. For $v \in V_N$, let $\mathcal{B}_k(v)$ denote those elements $B \in \mathcal{B}_k$ with $v \in B$.
Likewise define $\mathcal{BD}_k(v)$, i.e. for $v \in V_N$, $B \in \mathcal{BD}_k(v)$ if and only if $v \in B$. One should note that
$\mathcal{BD}_k(v)$ contains exactly one element, whereas $\mathcal{B}_k(v)$ contains $2^{2k}$ elements.

Definition 3.1 (Inhomogeneous branching random walk (IBRW)). Let $\{\xi_{L,B}^{(k)} \geq 0, B \in \mathcal{BD}_k \}$ be an i.i.d. family
of standard Gaussian random variables. We define the time-inhomogeneous branching random
walk \( \{R^N_t\}_{t \in V_N} \) by

\[
R^N_z(t) := \sum_{k=n-t}^{n} \sum_{B \in \mathcal{B}_k} \sigma \left( \frac{n-k}{n} \right) a_{k,B}, \tag{3.1}
\]

where \( 0 \leq t \leq n, t \in \mathbb{N} \) and \( \sigma \) is defined as in (1.6).

It turns out that IBRW is structurally much more independent than the scale-inhomogeneous DGFF. This makes it unsuitable to obtain sufficient lower bounds. We therefore introduce another auxiliary process which interpolates between the IBRW and the scale-inhomogeneous DGFF by taking uniform averages of IBRWs. For \( v \in V_N \), let \( \mathcal{B}_k(v) \) be the collection of subsets of \( \mathbb{Z}^2 \) consisting of squares of size \( 2^k \) with lower left corner in \( V_N \). Let \( \{b_{k,B}\}_{k \geq 0, B \in \mathcal{B}_k} \) denote an i.i.d. family of centred Gaussian random variables with unit variance and set

\[
b^N_{k,B} := \begin{cases} b_{k,B}, & B \in \mathcal{B}_k, \\ b_{k,B'}, & B \sim N B' \in \mathcal{B}_k. \end{cases}
\]

**Definition 3.2** (Modified inhomogeneous branching random walk (MIBRW)). The modified inhomogeneous branching random walk (MIBRW) \( \{S^N_v\}_{v \in V_N} \) is defined by

\[
S^N_z(t) := \sum_{k=n-t}^{n} \sum_{B \in \mathcal{B}_k} 2^{-k} \sigma \left( \frac{n-k}{n} \right) b^N_{k,B}, \tag{3.3}
\]

where \( 0 \leq t \leq n, t \in \mathbb{N} \) and \( \sigma \) is defined as in (1.6).

### 3.1. Covariance estimates

In order to be able to apply Gaussian comparison, we need to compare the correlations of the processes introduced previously. We write \( \log_+ (x) = \max(0, \log_+(x)) \). Further, let \( \| \cdot \|_2 \) be the usual Euclidean distance and \( \| \cdot \|_\infty \) the maximum distance. As we are working in two dimensions, they satisfy the relation \( \|x - y\|_\infty \leq \|x - y\|_2 \leq \sqrt{2} \|x - y\|_\infty \). In addition, we introduce for \( v, w \in V_N \) two distances on the torus induced by \( V_N \),

\[
d^N(v, z) := \min_{z \sim N w} \|v - z\|_2, \quad d^N_{w}(v, w) := \min_{z \sim N w} \|v - z\|_\infty. \tag{3.4}
\]

Note that the Euclidean distance on the torus is smaller than the standard Euclidean distance, i.e. for all \( v, w \in V_N \), it holds \( d^N(v, w) \leq \|v - w\|_2 \). However, equality trivially holds if one restricts oneself to a smaller box, e.g. if \( v, w \in (\mathcal{N}/4, \mathcal{N}/4) + V_{2n} \subset V_N \). In the following we call \( \{S^N_v\}_{v \in V_N} \) the homogeneous version of the process \( \{S^N_v\}_{v \in V_N} \) which was introduced in [20], i.e. we assume that there is only one scale \( \lambda_1 = 1 \) with variance parameter \( \sigma_1 = 1 \).

**Lemma 3.3.** There exists a constant C independent of \( N = 2^n \) such that for any \( v, w \in V_N \),

\[
i. \quad \mathbb{E} \left[ S^N_v S^N_w - (n - \log_+(d^N(v, x))) \right] \leq C \\
ii. \quad \mathbb{E} \left[ S^N_v S^N_w - \sum_{i=1}^M \sigma_i^2 \left( n \nabla \phi_{v, n} \mathbb{1}_{n - [\log_+(d^N(v, w))] < \lambda_n} \right) \right] \leq C \\
iii. \quad \mathbb{E} \left[ \phi_{v, n} \phi_{w, n} - \log_+ (\|v - w\|_2) \right] \leq C \\
iv. \quad \mathbb{E} \left[ \phi_{v, n} \phi_{w, n} - \log_+ (\|v - w\|_2) \right] \leq C.
\]

**Proof.** See [18] \( \square \)

**Remark 3.4.** The assumption \( N = 2^n \) for \( n \in \mathbb{N} \) mainly simplifies notation and also the proof, however without removing essential difficulties.
An important tool in the analysis of the scale-inhomogeneous DGFF is the Gibbs-Markov property of the DGFF. For two sets $U \subset V \subset \mathbb{Z}^2$ the DGFF on $V$ can be decomposed into a sum of a DGFF on $U$ and an independent Gaussian field, i.e.

$$
\phi^V_u \overset{d}{=} \phi^U_u + \mathbb{E}\left[\phi^V_v | \sigma \left(\phi^V_v : v \in V \setminus U^o \right)\right], \quad u \in V.
$$

(3.5)

Further, if $A, B \subset V$ such that $A^o \cap B^o = \emptyset$, then $\{\phi^V_u - \mathbb{E}[\phi^V_u | F_{AB}]\}_{u \in A}$ is a DGFF on $A$, independent of the DGFF on $B \{\phi^V_u - \mathbb{E}[\phi^V_u | F_{AB}]\}_{u \in B}$.

4. Proof of Theorem 2.1

4.1. Upper bound. The goal in this section is to prove the upper bound in Theorem 2.1. We first relate the maxima of the scale-inhomogeneous DGFF with those of the MIBRW.

**Lemma 4.1.** Let $\{g_v\}_{v \in V_N}$ denote a collection of independent identically distributed standard Gaussian random variables. Then, there is a finite constant $C_1$ such that

$$
\mathbb{E}\left[\max_{v \in V} \phi^N_v \right] \leq \sqrt{\log(2)} \mathbb{E}\left[\max_{v \in V} (S^N_{\phi} + C_1 g_v) \right].
$$

(4.1)

**Proof.** First, write $V^*_N = V_N + (2N, 2N) \subset V_{2N}$. By the Gibbs-Markov property for the underlying DGFF we can find a constant $\hat{C} > 0$, uniformly in $N$, such that for $v, w \in V^*_N$,

$$
\mathbb{E}\left[(\phi^N_v - \phi^N_w)^2\right] \leq \mathbb{E}\left[(\phi^N_v - \phi^N_{w,N})^2\right] + \hat{C}.
$$

(4.2)

Further, for $v, w \in V_N$ we write $v_N = v + (2N, 2N)$ and $w_N = w + (2N, 2N)$. By Lemma 3.3, we have

$$
\mathbb{E}\left[(\phi^N_{v,N} - \phi^N_{w,N})^2\right] \leq \log(2) \mathbb{E}\left[(S^N_v - S^N_w)^2\right] + \hat{C}.
$$

(4.3)

Thus, we can apply Sudakov-Fernique and get

$$
\mathbb{E}\left[\max_{v \in V_N} \phi^N_v \right] \leq \log(2) \mathbb{E}\left[\max_{v \in V_N} S^N_v + C_1 g_v \right] + \hat{C},
$$

(4.4)

where $\{g_v\}_{v \in V_N}$ form a family of independent standard Gaussian random variables and where $C_1 > 0$ is a constant independent of $N$.

**Proposition 4.2.** There is a finite constant $C_2$, such that

$$
\mathbb{E}[\max_{v \in V_N} \phi^N_v] \leq \sqrt{\log(2)} \mathbb{E}[\max_{v \in V_N} R^N_v] + C_2.
$$

(4.5)

**Proof.** For $v, w \in V_N$, we have

$$
\mathbb{E}\left[(R^N_v)^2\right] = \mathbb{E}\left[(S^N_v)^2\right] \quad \text{and} \quad \mathbb{E}\left[R^N_v R^N_w\right] \leq \mathbb{E}\left[S^N_v S^N_w\right] + C.
$$

Together they imply

$$
\mathbb{E}\left[(S^N_v - S^N_w)^2\right] \leq \mathbb{E}\left[(R^N_v - R^N_w)^2\right].
$$

(4.6)

By Sudakov-Fernique in combination with Lemma 4.1 we find

$$
\mathbb{E}\left[\max_{v \in V_N} \phi^N_v \right] \leq \mathbb{E}\left[\max_{v \in V_N} R^N_v + C_1 g_v \right],
$$

(4.7)

with $\{g_v\}_{v \in V_N}$ being a family of independent standard Gaussian random variables.

An application of [42, Theorem 1.4] yields that the time-inhomogeneous branching random walk $R^N$ satisfies

$$
\mathbb{E}\left[\max_{v \in V_N} R^N_v \right] = 2 \sqrt{\log(2)} \sum_{j=1}^{m} \hat{\sigma} \sqrt{\lambda^j n} - \frac{(w_j \hat{\sigma} \log(\sqrt{\lambda^j n}))}{4 \sqrt{\log(2)}} + O(1).
$$

(4.8)

Proposition 4.2 yields the desired upper bound in Theorem 2.1.
4.2. Lower bound. In this section our goal is to prove a corresponding lower bound on the expected maximum of the scale-inhomogeneous DGFF. In a first step, we prove that the expected maximum of \( \psi_N \) can be bounded from below by a truncated version of the MIBRW \( S_N \). This is done by applying Slepian’s lemma. The truncation of the first few largest scales gives us additional independence which we want to exploit to obtain a summable lower bound. In a second step, we prove a suitable lower bound on the MIBRW via a second moment computation on the number of particles reaching the desired level of height. Recall the definition of the MIBRW in (3.3).

**Definition 4.3** (Truncated MIBRW). Let \( 0 \leq k_0 \leq n \) be an integer. We define the truncated modified inhomogeneous branching random walk (TMIBRW) by

\[
S_{z,k_0}^N = \sum_{k=k_0}^{n} \sum_{B \in B_k(z)} 2^{-k} \sigma \left( \frac{n-k}{n} \right) p_k^N(B),
\]

and set

\[
S_{N,k_0}^* = \max_{v \in V_N} S_{z,k_0}^N.
\]

For \( v, w \in V_N \), let \( Q_{N,k_0}(v, w) = \mathbb{E} \left[ (S_{v,k_0}^N - S_{w,k_0}^N)^2 \right] \). We collect some basic properties of \( Q_{N,k_0} \).

**Lemma 4.4.** The function \( Q_{N,k_0} \) satisfies:

i. \( Q_{N,k_0} \) decreases in \( k_0 \).

ii. \( \lim_{k_0 \to \infty} \sup_{v, w \in V_N} Q_{N,k_0}(v, w) = 0 \).

iii. There is a function \( g: \mathbb{Z}_+ \to \mathbb{R}_+ \) such that \( g(k_0) \to \infty \) as \( k_0 \to \infty \), and for \( v, w \in V_N \) with \( d^N(v, w) \geq 2^{\sqrt{k_0}} \) and \( 2^n > k_0 \), we have

\[
Q_{N,k_0}(v, w) \leq |N|,0(v, w) - g(k_0).
\]

**Proof.** For \( v = (v_1, v_2) \neq w = (w_1, w_2) \in V_N \) and \( i = 1, 2 \), let \( r_i(v, w) = \min([v_i - w_i], |v_i - w_i - N_i|, |v_i - w_i + N_i|) \). Recall that in the definition of the MIBRW the number of common boxes of side length \( 2^k \) for \( v, w \in V_N \) is given by \([2^k - r_1(v, w)] [2^k - r_2(v, w)]\). And so we may compute the order of fluctuations by counting the number of unshared boxes, i.e.

\[
Q_{N,k_0}(v, w) = \mathbb{E} \left[ (S_{v,k_0}^N - S_{w,k_0}^N)^2 \right] = 2 \mathbb{E} \left[ (S_{v,k_0}^N)^2 \right] - 2 \mathbb{E} [S_{v,k_0}^N S_{w,k_0}^N]
\]

\[
= 2 \sum_{k=k_0}^{n} 2^{-k} 2^k \sigma^2 \left( \frac{n-k}{n} \right)
\]

\[
+ 2 \sum_{k=\lceil \log_2(d_{v,w}^N+1) \rceil}^{n} 2^{-k} \sigma^2 \left( \frac{n-k}{n} \right) \left( -2^k + r_1(v, w) \right) \left( -2^k + r_2(v, w) \right)
\]

\[
= 2 \sum_{k=k_0}^{n} \sigma^2 \left( \frac{n-k}{n} \right)
\]

\[
+ 2 \sum_{k=\lceil \log_2(d_{v,w}^N+1) \rceil}^{n} \sigma^2 \left( \frac{n-k}{n} \right) \left( r_1(v, w) + \frac{r_2(v, w)}{2^k} - \frac{r_1(v, w)r_2(v, w)}{4^k} \right).
\]

From this representation the first two properties are immediate. For the third statement, note that for \( \log_2(d_{v,w}^N(v, w)) \geq \sqrt{k_0} - 1 \),

\[
Q_{N,0}(v, w) - Q_{N,k_0}(v, w) \geq \sqrt{k_0} \min_{0 \leq k_0 \leq \sqrt{k_0}} \sigma^2 \left( \frac{n-k}{n} \right) - 1.
\]

□

In addition, we need a comparison between the maxima of the \((\sigma, \lambda)-DGFF\) and the truncated MIBRW.
**Corollary 4.5.** There is a constant $k_0 > 0$, such that for all $N = 2^n$ large enough and all $v, w \in V_N$,

$$\mathbb{E} \left[ \phi_N^v \right] \geq \log(2) \mathbb{E} \left[ S_{j,k_0}^N \right].$$

**Proof.** We have

$$\phi_N^v \geq \max_{v \in V_N} \psi_N^v.$$  

On the other hand, for $v, w \in V_N$, we get with statement iv. in Lemma 3.3

$$\mathbb{E} \left[ (\phi_N^v - \phi_N^w)^2 \right] \geq \log(2) \mathbb{E} \left[ (S_{i,j}^v - S_{i,j}^w)^2 \right] - C.$$  

Applying Lemma 4.6 and Sudakov-Fernique to (4.15), we obtain

$$\mathbb{E} \left[ \phi_N^v \right] \geq \max_{v \in V_N} \psi_N^v \geq \log(2) \mathbb{E} \left[ S_{j,k_0}^N \right].$$

where we compensated the constant $C > 0$ in (4.15) by a cut-off of the first $k_0$ contributions of $\{S_{v,i}^N\}_{v \in V_N}$.

**4.2.1. Lower bound for the truncated MIBRW.** Let $n_j$ be the unique index such that for $1 \leq j \leq m$ we have $\lambda_j = \lambda_n$. Moreover, we write $t^j = \lambda^j n$ as well as $t^j = \lambda_n$. Define

$$M_n^j(t) := \sum_{j=1}^m t^j / \sqrt{N} \left[ 2 \sqrt{\log 2} \sigma_j \sqrt{N} t^j - \frac{(w_j, \sigma_j \log(\sqrt{N} t^j))}{4 \sqrt{\log(2)}} \right], \quad t \in \mathbb{R}_+.$$  

As a short notation, we write $M_n^j = M_n^j(n)$.

**Proposition 4.6.** Let $N = 2^n$, then

$$\mathbb{E} \left[ \phi_N^v \right] \geq \log(2) \mathbb{E} \left[ M_n^j \right] + O(1).$$

**Proposition 4.7.** There is a function $f : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, such that for all $N \geq 2^{2k_0}$,

$$\mathbb{E} \left[ S_{j,k_0}^N \right] \geq \sum_{j=1}^m 2 \sqrt{\log 2} \sigma_j \sqrt{N} t^j - \frac{(w_j, \sigma_j \log(\sqrt{N} t^j))}{4 \sqrt{\log(2)}} - f(k_0).$$

In order to prove Proposition 4.7, it is convenient to restrict $S_{N,k_0}^j$ to a subset of $V_N$. Let $V_N = V_{N,j} + (\mathbb{N},\mathbb{N}) \subset V_N$ and define

$$\tilde{S}_{N,k_0}^N = S_{N,k_0}^N | V_N, \quad \tilde{S}^N = \tilde{S}_{N,0}^N.$$  

**Proposition 4.8.** There is a constant $\delta_0 \in (0, 1)$, such that for all $N \in \mathbb{N}$,

$$\mathbb{P}(S_{N,k_0}^N \geq M_n^j) \geq \delta_0.$$  

**Proof of Proposition 4.6** This follows by combining Corollary 4.5 and Proposition 4.7.

**Proof of Proposition 4.7 assuming Proposition 4.8** We start with the proof in the case of $k_0 = 0$. For $i = 1, 2, 3, 4$, we introduce the four sets $W_{N,i} = \{0, N/2 + z_i\}$, where $z_1 = (N/4, N/4)$, $z_2 = (2N/12, N/4)$, $z_3 = (N/4, 2N/12)$, and $z_4 = (23N/12, 23N/12)$. Note that $3/4 - 1/3 = 23/32$, $\cup_i W_{N,i} \subset V_N$ and that these sets are $N/4$-separated, i.e. for $i \neq j$

$$\min_{v \in W_{N,i}, w \in W_{N,j}} d_{\infty}(v, w) > N/4.$$  

For $n > 6$, let

$$S_{N,v}^n = \sum_{k=0}^{n-6} \sum_{B \in B_0(v)} 2^{-k} \sigma(n-k) b_{k,B}^N,$$

and note that

$$G_{N,v} = S_{N,v}^N - S_{N,v}^N = \sum_{k=0}^{5} \sum_{B \in B_0(v)} 2^{-n+k} \sigma(k) b_{n-k,B}^N.$$
We drop boxes of sizes bigger than $2^{n-6}$ and observe that only boxes of such size can cover particles in both $W_{N,i,j}$ and $W_{N,j,i}$ for $i \neq j$, which implies that $\{S^N_{\nu}\}_{\nu \in W_{N,i,j}}$ are independent for $i = 1, 2, 3, 4$. First, we show that this does not change the order of the maximum by bounding the probability of $\max_{v \in V_N} G^N_v$ being large. This is done using Fernique’s criterion ([1, Theorem 4.1]) together with Borell’s concentration inequality (Theorem A.1). Let $m_N = \frac{1}{|N|} \sum_{v \in V_N} \delta_v$ be the uniform probability measure on $V_N$ and $g : (0, 1] \to \mathbb{R}_+$ be the function defined as

$$g(t) = \sqrt{\log(t)},$$

and set

$$B(z, \epsilon) = \{w \in V_N : \mathbb{E} \left[(G^N_v - G^N_w)^2\right] \leq \epsilon^2\}.$$  

Next, we apply Fernique’s criterion ([1, Theorem 4.1]) to obtain an upper bound on the expected maximal difference $G^N_v = S^N_v - \bar{S}_v^N$, with a universal constant $K \in (1, \infty)$, i.e.

$$\mathbb{E} \left[\max_{v \in V_N} G^N_v\right] \leq K \sup_{v \in V_N} \int_0^\infty g(m(B(v, \epsilon)))d\epsilon.$$  

For $n \geq 6$ and with previous notation from Lemma 4.4 we observe

$$\mathbb{E} \left[(G^N_v - G^N_w)^2\right] = \mathcal{G}_{n-5}(v, w).$$

As in the proof of Lemma 4.4, we can find a constant $C$ such that for $\epsilon \geq 0$,

$$\left\{w \in V_N : d^N_{\infty}(v, w) \leq \frac{\epsilon^2 N}{C}\right\} \subset B(v, \epsilon).$$

In particular, for $v \in V_N$ and $\epsilon > 0$ we can bound the number of particles in $B(v, \epsilon)$ from below by $\left(\left(\frac{\epsilon^2}{C}\right)^N \right) \wedge 1$. This allows us to bound the right hand side of (4.27), i.e.

$$\sup_{v \in V_N} \int_0^\infty g(m(B(v, \epsilon)))d\epsilon \leq \int_0^C \sqrt{\frac{\epsilon}{\pi}} \sqrt{\log(N^2)}d\epsilon + \int_C^\infty \sqrt{\frac{\epsilon}{\pi}} \sqrt{\log(C^2/\epsilon^4)}d\epsilon < C_4,$$

where $C_4 > 0$ is some constant. In combination with the application of Fernique’s criterion in (4.27) we deduce

$$\mathbb{E} \left[\max_{v \in V_N} G^N_v\right] \leq C_4 K,$$

uniformly in $N$. Moreover, $\mathbb{E}[(S^N_v - \bar{S}^N_v)^2]$ is bounded in $N$. By Borell’s concentration inequality, it follows that for some constant $C_5$ and all $\beta > 0$,

$$\mathbb{P} \left(\max_{v \in V_N} (S^N_v - \bar{S}^N_v) \geq C_4 K + \beta\right) \leq 2 \exp(-C_5 \beta^2).$$

In addition, we can couple $b^N_{k,B}$ and $\tilde{b}^N_{k,B}$ such that $S^N_{\nu} = \sum_{k=0}^{n-4} \sum_{B \in \mathcal{B}_k(\nu)} 2^{-k} \sigma(\frac{u-k}{n-4}, b^N_{k,B})$ and thus, we can analogously find constants $C_6, C_7 > 0$ so that for all $\beta > 0$,

$$\mathbb{P} \left(\max_{v \in V^N_{n/16}} (S^N_{\nu} - \bar{S}^N_{\nu}) \geq C_6 \beta + \beta\right) \leq 2 \exp(-C_7 \beta^2).$$

For $\alpha, \beta > 0$ and using (4.32), we have

$$\mathbb{P} \left(\tilde{S}^N_v \geq M^N_v - \alpha\right) \geq \mathbb{P} \left(\max_{v \in V_N} \tilde{S}^N_v \geq M^N_v + C_4 - \alpha + \beta\right) - \mathbb{P} \left(\max_{v \in V_N} (S^N_v - \bar{S}^N_v) \geq C_4 + \beta\right) \geq \mathbb{P} \left(\max_{v \in V_N} \tilde{S}^N_v \geq M^N_v + C_4 - \alpha + \beta\right) - 2 \exp(-C_5 \beta^2).$$
On the other hand, for \(\gamma, \gamma' > 0\) and using independence on distinct \(W_{N,i}\),

\[
\mathbb{P}\left( \max_{v \in V_N^*} \tilde{S}^N_v \geq M_n^* - \gamma \right) \geq \mathbb{P}\left( \max_{i=1}^4 \max_{v \in W_{N,i}} \tilde{S}^N_v \geq M_n^* - \gamma \right) = 1 - \left( \mathbb{P}\left( \max_{v \in W_{N,i}} \tilde{S}^N_v < M_n^* - \gamma \right) \right)^4 \\
\geq 1 - \left( \mathbb{P}\left( \max_{i=1}^4 \tilde{S}^{N/\alpha}_{v_j} < M_n^* - \gamma + C_6 + \gamma' \right) + 2 \exp(-C_7(\gamma')^2) \right)^4. \tag{4.35}
\]

For the last inequality we used the identity \(\tilde{S}^N_v = \tilde{S}^{N/\alpha}_{v_j} - S^{N/\alpha}_{v_j} + S^{N/\alpha}_{v_j}\), as well as \((4.33)\). If we plug this estimate into \((4.34)\), we obtain that for \(\alpha, \beta, \gamma, \gamma' > 0\),

\[
\mathbb{P}\left( \tilde{S}^N_v \geq M_n^* - \alpha \right) \geq 1 - 2 \exp(-C_3\beta^2) \\
- \left[ 2 \exp(-C_7(\gamma')^2) + \mathbb{P}\left( \max_{v \in V_{N/\alpha}} S^{N/\alpha}_{v_j} < M_n^* + C_4 + C_6 + \beta + \gamma' - \alpha \right) \right]^4. \tag{4.36}
\]

This allows us to iterate the last estimate. Let \(\eta_0 = 1 - \delta_0 < 1\) and for \(j \geq 1\), choose a constant \(C_8 = C_8(\delta_0) > 0\) so that for \(\beta_j = \gamma_j = C_8 \sqrt{\log \left( \frac{1}{\eta_j} \right)}\) we set

\[
\eta_{j+1} = 2 \exp(-C_3\beta^2_j) + [\eta_j + 2 \exp(-C_7(\gamma_j)^2)]^4. \tag{4.37}
\]

The sequence \(\{\eta_j\}_{j \in \mathbb{N}}\) satisfies \(\eta_{j+1} < \eta_j(1 - \delta_0)\). With this choice of \(\beta_j\) and \(\gamma_j\), set \(\alpha_0 = 0\) and \(\alpha_{j+1} = \alpha_j + C_4 + C_6 + \beta_j + \gamma_j\). Note that for some constant \(C_9 = C_9(\delta_0)\), we have \(\alpha_j \leq C_9 \sqrt{\log \left( \frac{1}{\eta_j} \right)}\).

If we now substitute the indexed constants into \((4.36)\) and apply Proposition 4.8 to start the recursion, we get

\[
\mathbb{P}(\tilde{S}^N_v \geq M_n^* - \alpha_{j+1}) \geq 1 - \eta_{j+1}. \tag{4.38}
\]

Thus,

\[
\mathbb{E}\left[ \tilde{S}^N_v \right] \geq M_n^* - \int_0^\infty \mathbb{P}(\tilde{S}^N_v \leq x)dx \geq M_n^* - \sum_{j=0}^\infty \eta_j \mathbb{P}(\tilde{S}^N_v \leq M_n^* - \alpha_j)
\]

\[
\geq M_n^* - C_9 \sum_{j=0}^\infty \eta_j \sqrt{\log \left( \frac{1}{\eta_j} \right)}. \tag{4.39}
\]

Since \(\eta_j \leq (1 - \delta_0)^j\), \(\sum_{j=0}^\infty \eta_j \sqrt{\log \left( \frac{1}{\eta_j} \right)} \leq C \sum_{j=0}^\infty (1 - \delta_0)^j \sqrt{j} < \infty\), it exists a constant \(C_{10} > 0\), such that

\[
\mathbb{E}\left[ S^N_v \right] \geq \mathbb{E}\left[ \tilde{S}^N_v \right] \geq M_n^* - C_{10}. \tag{4.40}
\]

This proves Proposition 4.7 in the case \(k_0 = 0\).

For the case when \(k_0 > 0\), consider

\[
\tilde{S}_{N,k_0} = \max_{v \in V_N^* \cap (2k_0 \mathbb{Z})^2} S^{N,k_0}_v. \tag{4.41}
\]

As we are taking the maximum of the same object over a subset, we have \(\tilde{S}_{N,k_0}^* \leq \tilde{S}_{N,k_0}^*\). To compare \(\tilde{S}_{N,k_0}^*\) with \(\tilde{S}_{2^{-k_0}N,0}^*\), we need to be able to identify the underlying grids. We start with the grid for \(V_{2^{-k_0}N}\). We identify the origin in this with the origin in \(V_N \cap 2^k \mathbb{Z}\) and likewise for the boundaries. In this procedure, we ask to keep relative relations untouched, that is neighbouring particles stay neighbouring particles, even though being neighbours in \(V_N \cap 2^k \mathbb{Z}\) means that they are \(2^k\) away from each other. Under this restriction the relation naturally extends to all other points. Note that this gives a one-to-one relation since both grids have equally many particles. Then, any \(v, w \in V_{2^{-k_0}N}\) with
counterparts $2^{h_0}v$, $2^{h_0}w \in V_N \cap 2^{h_0}\mathbb{Z}$ satisfy $\log_+ (2^{2h_N} (v, w)) \leq \log_+ (2^N (2^{h_0}v, 2^{h_0}w))$. In combination with Lemma 3.3 and Slepian’s Lemma, we deduce
\[ P\left( S_{N,k_0}^* \geq x \right) \geq P\left( S_{2^{-h_0}N,0}^* \geq x \right), \tag{4.42} \]
and thus for any $x \in \mathbb{R}$,
\[ P\left( S_{N,k_0}^* \geq x \right) \geq P\left( S_{2^{-h_0}N,0}^* \geq x \right) \geq P\left( S_{2^{-h_0}N,0}^* \geq x \right). \tag{4.43} \]
Together with (4.40) we obtain
\[ \mathbb{E}\left[ S_{N,k_0}^* \right] \geq \mathbb{E}\left[ S_{2^{-h_0}N,0}^* \right] \geq \mathbb{E}\left[ S_{2^{-h_0}N,0}^* \right] \geq M_{n-k_0}^* - C_{11}, \tag{4.44} \]
where $C_{11} > 0$ is a constant. This concludes the proof of Proposition 4.7. □

In order to prove Proposition 4.8 we need to introduce additional notation. We further split our considerations into two steps. First, we treat the case of exactly one effective variance parameter and in a second step, we generalize to the multi-parameter case using the independence of increments. In the instance of exactly one effective variance parameter $\tilde{\sigma}_1$, we have $\lambda_1 = 1$ or equivalently $t^1 = n$. Write $\nabla S^N_v (t^i) = S^N_v (t^i) - S^N_v (t^{i-1})$ and $\nabla M^*_v (t^i) = M^*_v (t^i) - M^*_v (t^{i-1})$. We introduce suitable restricted events for $v \in V^*_N = V_\lambda + (N/\lambda, N/4) \subseteq V_N$, $x \in \mathbb{R}$, $0 \leq k \leq n$ and $0 \leq i \leq m$:

Let
\[ s_{k,a}(x) := \begin{cases} \frac{f_{a,2}^{(1)}(x)}{\frac{f_{a,2}^{(1)}(t^i)}{f_{a,2}^{(1)}(t^i)}}(x), & \text{if } 0 \leq k \leq t^1, \\ \frac{f_{a,2}^{(1)}(x)}{\frac{f_{a,2}^{(1)}(t^i)}{f_{a,2}^{(1)}(t^i)}}(x), & \text{if } t^{i-1} < k \leq t^i. \end{cases} \tag{4.45} \]
be the ‘optimal path’ followed by extremal particles and
\[ f_{k,a} := \begin{cases} C_f (I_{a,2} (\lambda, m), 0)^{\tilde{\sigma}_1}, & \text{if } 0 \leq k \leq t_1, \\ C_f (I_{a,2} (\lambda, t^i), 0)^{\tilde{\sigma}_1}, & \text{if } t_1 < k \leq t^1, \\ C_f (I_{a,2} (\lambda, t^{i+1}), 0)^{\tilde{\sigma}_1}, & \text{if } t^i < k \leq t_{i+1} : i \in \{1, \ldots, m - 1\}, \\ C_f (I_{a,2} (\lambda, t^{i+1}), 0)^{\tilde{\sigma}_1}, & \text{if } t_{i+1} < k \leq t^{i+1} : i \in \{1, \ldots, m - 1\}. \end{cases} \tag{4.46} \]
be the concave barrier. The constant $C_f$ depends on the parameters and will be fixed later in the proof. Further, define
\[ I_n (i) := [\nabla M^*_v (t^i), \nabla M^*_v (t^{i-1}) + 1], \tag{4.47} \]
\[ I_{k,a}(x) := [s_{k,a}(x) - f_{k,a}, s_{k,a}(x) + f_{k,a}], \]
\[ C_{v,p,q}^N (r) := [\nabla S^N_v (t^i) \in I_n (i), S^N_v (k + t^{i-1}) - S^N_v (t^{i-1}) \in I_{k,a}(\nabla S^N_v (t^i))] \]
\[ \forall 0 < k < t^{i+1} - t^i, \quad p < i \leq q \text{ such that } k + t^i \leq r, \tag{4.49} \]
\[ h_N := \sum_{v \in V^*_N} 1_{C_{v,p,q}^N (r)}. \tag{4.50} \]

One should note that the events $C_{v,p,q}^N (r)$ are corresponding concatenated Brownian bridge estimate events for each effective variance parameter $\tilde{\sigma}_i$. $I_n (i)$ denote the desired increments for each period in which there is exactly one effective variance. $\{I_{k,a}(x)\}_{0 \leq k \leq n}$ denote the admissible discrete paths to reach $x$ in $n$ steps with drift $x_{\frac{\nabla S^N_v (t^i)}{\frac{\nabla S^N_v (t^i)}{\frac{\nabla S^N_v (t^i)}}}}$ and concave barrier $f_{k,a}$. We want to prove Proposition 4.8 using a second moment method, i.e. we want to use the Paley-Zygmund inequality for the event $h_N$. We therefore need an upper bound on the first and a suitable lower bound on the second moment of $h_N$. We start with a lower bound on the first moment.

Lemma 4.9. There is a constant $C_1 > 0$ which is independent of $N \in \mathbb{N}$ and such that
\[ \mathbb{E}[h_N] \geq C_1. \tag{4.51} \]
Lemma 4.10. There is a constant $C_2 > 0$, independent of $N \in \mathbb{N}$ and such that
\[ \mathbb{E} \left[ h_N^2 \right] \leq \mathbb{E} \left[ h_N \right]^2 + (1 + C_2) \mathbb{E} \left[ |h_N| \right]. \] (4.52)

Proof of Lemma 4.9: We start the proof in the case of exactly one effective variance parameter, i.e. $m = 1$. We may further assume that $\bar{I}_{\sigma^2}(s) < \bar{I}_{\sigma^2}(s)$ for $s \in (0, 1)$, as the other case is covered by (bramson zeitouni paper). By the linearity of expectations,
\[ \mathbb{E} \left[ h_N \right] = \frac{1}{4} 2^{2n} \mathbb{P}(S_N^N(t^1) \in I_n(1), S_N^N(k) \in I_{k,n}(S_N^N(t^1)) \text{ for } 0 < k < t^1). \] (4.53)

Note that $\mathbb{E} \left[ s_{k,n}(S_N^N(t^1)) \left( S_N^N(k) - s_{k,n}(S_N^N(t^1)) \right) \right] = 0$, and so
\[ \text{Var} \left[ S_N^N(k) - s_{k,n}(S_N^N(t^1)) \right] = \text{Var} \left[ S_N^N(k) - s_{k,n}(S_N^N(t^1)) \right] = n I_{\sigma^2} \left( \frac{k}{n} \right) \left( 1 - \frac{I_{\sigma^2}(t/n)}{I_{\sigma^2}(t)} \right). \] (4.54)

In particular, $\mathbb{E} \left[ S_N^N(k) - s_{k,n}(S_N^N(t^1)) \right] = 0$. By conditioning the last event in (4.53) on $S_N^N(t^1)$ and using that this is independent of $\{S_N^N(k) - s_{k,n}(S_N^N(t^1))\}_{k=0}^n$,
\[ \mathbb{E} \left[ h_N \right] = \frac{1}{4} 2^{2n} \mathbb{P}(S_N^N(t^1) \in I_n(1)) \mathbb{P}(S_N^N(k) \in I_{k,n}(S_N^N(t^1)) \text{ for } 0 < k < n). \] (4.55)

Next, we estimate the last two probabilities separately and start with the first. Recall $M_n^N(t^1) = 2 \sqrt{\log(2)} \sigma_1 t^1 - \frac{1}{4 \sqrt{\log(2)}} \log(t^1) \sigma_1$ and $S_N^N(t^1) \sim N(0, \sigma_1^2 t^1)$. A standard Gaussian estimate yields
\[ \mathbb{P}(S_N^N(t^1) \in I_n(1)) = \int_{M_n^N(t^1)}^{M_n^N(t^1)+1} \frac{\exp(-z^2/(2\sigma_1^2 t^1))}{\sqrt{2\pi \sigma_1^2 t^1}} \, dz \geq \frac{c}{\sqrt{t}} \exp(- (M_n^N(t^1) + 1)^2/(2\sigma_1^2 t^1)) \]
\[ \geq c_1 N^{-2t^1} = c_1 2^{-2t^1}. \] (4.56)

We turn to the second probability in (4.55). By subadditivity of measures and using (4.54).
\[ \mathbb{P}(S_N^N(k) \in I_{k,n}(S_N^N(t^1)) \text{ for } 0 < k < t^1) \leq 1 - 2 \sum_{k=1}^{t^1-1} \mathbb{P}(S_N^N(k) - s_{k,n}(S_N^N(t^1)) > f_{k,n}) \]
\[ \geq 1 - 2 \sum_{k=1}^{t^1-1} C \exp \left( - \frac{1}{2} \frac{f_{k,n}^2}{I_{\sigma^2}(t/n) n (1 - I_{\sigma^2}(t^1)/I_{\sigma^2}(t))} \right). \] (4.57)

Taking into account our definition of the concave barrier in (4.46), we may split and bound the sum in (4.57) from above by
\[ \sum_{k=1}^{n} C \exp \left( - \frac{1}{2} C f_{k,n}^2 k^1 \right) 1_{\sigma_1 \neq 0} + \sum_{k=1}^{t^1-1} C \exp \left( - \frac{1}{2} C \min_{i \in [2, \ldots, t] \setminus \{\sigma_1 \neq 0\}} (\sigma_i) - 1) k^1 \right) < c/2, \] (4.58)

where $0 < c < 1$ is a constant independent of $n$, provided we choose $C_f > 0$ large enough. Inserting this into (4.57), gives
\[ \mathbb{P}(S_N^N(k) \in I_{k,n}(S_N^N(t^1)) \text{ for } 0 < k < t^1) > 1 - c = c_2 > 0. \] (4.59)

Inserting (4.56) and (4.59) into (4.55) implies the claim in the case of exactly one effective variance parameter.
The case of $m$ effective variance parameters can now be reduced to an iterative application of the bounds (4.56), (4.59) obtained in the one parameter case, i.e.

\[
\mathbb{E}[h_N] \geq \frac{2^{2n}}{2} \mathbb{P}(C_v^N(r^m)) = 2^{2n-1} \mathbb{P}(C_v^{N(1,t')}^1) \mathbb{P}(C_v^{N(1,m)}(t^m)) \geq \frac{1}{2} c_1(1)c_2(1)(2^{2n})^{1-l-t} \mathbb{P}(C_v^{N(1,m)}(t^m))
\]

\[
\geq \frac{1}{2} c_1(1)c_2(1)c_2(2)(2^{2n})^{1-l-t} \mathbb{P}(C_v^{N(2,m)}) \geq \frac{1}{2} \prod_{i=1}^m c_1(i)c_2(i) > 0,
\]

where the constants $c_1(i), c_2(i)$ for $i = 1, \ldots, m$ are the corresponding constants in (4.56), (4.59) obtained for each $[t^{i-1}, t^i]$.

Now, we can turn to prove a corresponding upper bound on the 2nd moment of $h_N$.

**Proof of Lemma 4.10**: As in the previous proof, we start with the case of one effective parameter, i.e. $m = 1$ and may again assume $I_{\rho^2}(s) < \tilde{I}_{\rho^2}(s)$ for $s \in (0, 1)$. For $v, w \in V_N$, let $r(v, w) = n - \lceil \log_2(d_v^N(v, w) + 1) \rceil$ denote the number of independent increments in the processes $S_v^N(k)$ and $S_w^N(k)$. Using this we may rewrite the second moment, i.e.

\[
\mathbb{E}[h_N^2] = \sum_{v, w \in V_N} \mathbb{P}(C_v^N(r^1) \cap C_w^N(r^1)) = \sum_{k=0}^n \sum_{v, w \in V_N} \mathbb{P}(C_v^N(r^1) \cap C_w^N(r^1)).
\]

The right hand side may be bounded from above by

\[
\mathbb{E}[h_N]^2 + \mathbb{E}[h_N] + \sum_{k=1}^{n-1} \sum_{v, w \in V_N} \mathbb{P}(C_v^N(r^1) \cap C_w^N(r^1))
\]

We need to bound the double sum in (4.62) which we do by bounding each summand. Fix $v, w \in V_N$ with $r(v, w) = r = k \in \{1, \ldots, n-1\}$. We set $B_{k,r}(x) = x - s_{k,r}(x)[-f_{r,n}, f_{r,n}]$. Dropping the constraint for $w$ up to time $r$, we have

\[
\mathbb{P}(C_v^N(r^1) \cap C_w^N(r^1)) \leq \mathbb{P}(C_v^N(r^m)) \max_{x \in I_1(1)} \mathbb{P}(S_v^N(t^1) - S_v^N(r) \in B_{r,n}(x))
\]

\[
\leq \mathbb{P}(C_v^N(r^m)) \max_{x \in I_1(1)} \mathbb{P}(S_v^N(t^1) - S_v^N(r) \in B_{r,n}(x)).
\]

For fixed $v \in V_N$, the number of points $w \in V_N$ satisfying $d_v^N(v, w) \in [2^k, 2^{k+1}]$, i.e. when $r(v, w) \in [n-k-1, n-k]$, is bounded by $c_1 2^{2k} = 2^{2(t-r)}$ for a $c_1 > 0$. Therefore, we can bound the last summand in (4.62) from above by

\[
c_1 \mathbb{E}[h_N] \sum_{r=1}^{n-1} 2^{2(t-r)} \max_{x \in I_1(1)} \mathbb{P}(S_v^N(t^1) - S_v^N(r) \in B_{r,n}(x)).
\]

Next, we set $A_{r,n, x} \equiv \mathbb{P}(S_v^N(t^1) - S_v^N(r) \in B_{r,n}(x))$ and bound this for any $x \in I_1(1)$, i.e.

\[
\mathbb{P}(S_v^N(t^1) - S_v^N(r) \in B_{r,n}(x)) = \frac{1}{\sqrt{2\pi I_{\rho^2}(\|/n, \lambda^1)n}} \int_{x-s_{r,n}(x)-f_{r,n}}^{x-s_{r,n}(x)+f_{r,n}} \exp\left(-\frac{1}{2} \frac{(z^2)}{I_{\rho^2}(\|/n, \lambda^1)n}\right)dz
\]

\[
\leq \frac{2 f_{r,n}}{\sqrt{I_{\rho^2}(\|/n, \lambda^1)n}} \exp\left(-\frac{1}{2} \frac{(M_v^N(t^1) - s_{r,n}(M_v^N(t^1)) - f_{r,n})^2}{I_{\rho^2}(\|/n, \lambda^1)n}\right).
\]

Regarding the definition of $f_{r,n}$, we split our considerations into two steps. First, we consider the case where $0 < r \leq t_1$. We may assume that $\sigma_1 > 0$, as else we could consider the case $0 < r \leq t_1$ for the
minimal \( i \) such that \( \sigma_i > 0 \). In this case and for \( v \in V^N \), we have \( \text{Var}[S^N_v(t^1) - S^N_v(r)] = I_{\sigma^2}([v, A^1])n \) and \( f_{r,n} = C_f(\sigma^2 r)^{1/2} \). To estimate the exponential on the right hand side of (4.65), we note that

\[
(M^0_n(t^1) - s_{r,n}(M^0_n(t^1))) - f_{r,n})^2 \geq M^0_n(t^1)^2 \left( 1 - \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right)^2 - 2 f_{r,n} M^0_n(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)}. \tag{4.66}
\]

Inserting this into (4.65) and computing the square, we obtain that (4.65) is bounded from above by

\[
\frac{2 f_{r,n}}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - \frac{1}{2} M^0_n(t^1)^2 \left( 1 - \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right)^2 - 2 f_{r,n} M^0_n(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right)
\]

\[
\leq \frac{2 f_{r,n}}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - 2 \log(2) t^1 \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} + \frac{1}{2} \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right) \leq \frac{2 f_{r,n}}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - 2 \log(2) t^1 \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} + \frac{1}{2} \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right)
\]

\[
\leq \frac{2 f_{r,n}}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - 2 \log(2) t^1 \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} + \frac{1}{2} \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \right) + \frac{C f (\sigma^1 r)^{1/2}}{(4 \log(2))^{1/2} \sigma_1}. \tag{4.67}
\]

Note, since \( r \leq t_1, \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} = \frac{\sigma^2}{\sigma_1^2} \in (0, 1) \) and so we have

\[
I_{\sigma^2}([v, A^1]) t^1 \leq t^1 - r t^1 \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} = t^1 - \frac{1}{2} \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} = t^1 - \eta_1 r, \tag{4.68}
\]

for a \( \eta_1 < 1 \), independent of \( r \) and \( n \). Inserting this into (4.67), we get

\[
A_{r,n,x} \leq C r^{1/2} \exp \left( \tilde{C} r \right) 2^{-2(1-\eta_1 r)} \frac{\exp \left( \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{2 I_{\sigma^2}(A^1)} \right)}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{32 \log(2) t^1 I_{\sigma^2}(A^1)} \right)
\]

\[
\leq C 2^{-2(1-\eta_1 r)+\sigma(t^1-r)}. \tag{4.69}
\]

As \( \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} < 1 \), the fraction involving the square root in (4.69) can be estimated as

\[
\exp \left( \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{\sqrt{I_{\sigma^2}([v, A^1])}} \right) < \frac{1}{\sqrt{I_{\sigma^2}([v, A^1])}}. \tag{4.70}
\]

Using this in (4.69), we obtain in the case of \( 0 < r \leq t_1 \),

\[
A_{r,n,x} \leq C r^{1/2} \exp \left( \tilde{C} r \right) 2^{-2(1-\eta_1 r)} \leq C 2^{-2(1-\eta_1 r)+\sigma(t^1-r)}. \tag{4.71}
\]

The same computation as in (4.65), now in the case of \( t_1 < r \leq t^1 \), \( f_{r,n} = C_f(\sigma^2 r)^{1/2} \) and \( x \in I_\alpha(1) \), yields

\[
A_{r,n,x} \leq C \frac{2 f_{r,n}}{\sqrt{I_{\sigma^2}([v, A^1])}} \exp \left( - \frac{1}{2} M^0_n(t^1) - s_{r,n}(M^0_n(t^1)) - f_{r,n})^2 \right)
\]

\[
\leq C 2^{-2(1-\eta_1 r)} \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} \exp \left( \frac{1}{2} \log(t^1) \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} + \frac{C_f(\sigma^1 r)^{1/2}}{(4 \log(2))^{1/2} \sigma_1} \right). \tag{4.72}
\]

Next, we observe that the second exponential is bounded by \( (I_{\sigma^2}([v, A^1])n)^{1/2} \), as seen in (4.70) and the fact that for \( t_1 < r < t^1 \),

\[
t^1 \frac{I_{\sigma^2}([v, A^1])}{I_{\sigma^2}(A^1)} = \frac{1}{\sqrt{I_{\sigma^2}(A^1)}} (t^1 - r) \geq \eta_2 (t^1 - r), \tag{4.73}
\]
for a constant $\eta_2 > 1$ that is independent of $r$ and $n$. Using these in (4.72), we get

$$A_{r,n,s} \leq C 2^{-2n_1 l(r)} (I_{\varphi}^2 I_{\varphi}^2 \{y_1, y_1^2\})^{n_2/2} \exp(C (I_{\varphi}^2 I_{\varphi}^2 \{y_1, y_1^2\})^{n_2/2}) \leq C 2^{-2n_2 l(r)} \delta_0 l(r).$$

(4.74)

Combining the bounds in (4.71) and (4.74) and observing that both $(1 - \eta_1) > 0$ and $(1 - \eta_2) < 0$ hold, allows us to bound the sum in (4.64) by an absolute constant $C_2 > 0$, i.e.

$$\sum_{r=1}^{n-1} 2^{(r-1)} \max_{x \in I_{r,s}(r)} A_{r,n,s} \leq C \left[ \sum_{r=1}^{t_1} 2^{-(1-\eta_1)\tilde{r} + \tilde{r}} + \sum_{r=t_1+1}^{t-1} 2^{(1-\eta_2)(1-\eta_2)\tilde{r} + \tilde{r}} \right] \leq C_2.$$

(4.75)

Inserting this into (4.62) finishes the proof in the one parameter case. We can bound $\mathbb{E}[h_N^2]$ from above as in the one parameter case (see (4.62)),

$$\mathbb{E}[h_N^2] = \mathbb{E}[h_N]^2 + \mathbb{E}[h_N] + \sum_{k=1}^{n-1} \sum_{v,w \in V_n \cap I_{r,s}(r)} \mathbb{P} \left( C_{v,0,m}(r') \cap C_{w,0,m}(r') \right) \leq \mathbb{E}[h_N]^2 + \mathbb{E}[h_N] + \sum_{k=1}^{n-1} \sum_{v,w \in V_n \cap I_{r,s}(r)} \mathbb{P} \left( C_{v,0,m}(r') \cap C_{w,0,m}(r') \right) \leq \mathbb{E}[h_N]^2 + \mathbb{E}[h_N] + \sum_{k=1}^{n-1} \sum_{v,w \in V_n \cap I_{r,s}(r)} \mathbb{P} \left( C_{v,0,m}(r') \cap C_{w,0,m}(r') \right).$$

(4.76)

We bound the last sum by bounding each summand. Let $v, w \in V_n \cap I_{r,s}(r)$ with $r = r(v,w) \in \{1, \ldots, n-1\}$ and $k(r) \in \{0, \ldots, n-1\}$ such that $k(r) - 1 < r < k(r)$. Let $B_{k,n}(x) := x - s_{k,n}(x) + [-f_{r,n}, r_{k,n}]$. Then by dropping the condition for $w$ up to time $r$,

$$\mathbb{P} \left( C_{v,0,m}(r') \cap C_{w,0,m}(r') \right) \leq \mathbb{P} \left( C_{v,0,m}(r') \right) \max_{x \in I_{r,s}(r)} \mathbb{P} \left( \tilde{S}_w^N(r') - \tilde{S}_w^N(r) \in B_{r,n}(x) \right) \times \prod_{k=1}^{m-1} \max_{x \in I_{r,s}(r)} \mathbb{P} \left( \tilde{S}_w^N(r^{k+1}) - \tilde{S}_w^N(r^k) \in B_{k,n}(x) \right).$$

(4.77)

Note, $\mathbb{P} \left( \tilde{S}_w^N(r^{k+1}) - \tilde{S}_w^N(r) \in B_{r,n}(x) \right)$ and $\mathbb{P} \left( \tilde{S}_w^N(r^{k+1}) - \tilde{S}_w^N(r^k) \in B_{k,n}(x) \right)$ are terms that we have already considered in the one parameter case, and that each event considered here is also involving exactly one effective parameter. Thus for each effective parameter, we have analogue estimates as in (4.69) and (4.74), now with constants $\eta_{1,k}, \eta_{2,k}$ as before, only that each now depends on the corresponding effective parameter. Further, for fixed $v \in V_n$, the number of points $w \in V_n$ with $d_{w,v} \in \{k, 2^{k+1}\}$, or equivalently $r(v,w) \in [n - k - 1, n - k]$, is bounded by $c_1 2^{2k}$ for some $c_1 > 0$. Hence, we can bound the double sum on the right-hand side of (4.76) from above by

$$c_1 \mathbb{E}[h_N] \sum_{r=1}^{n-1} 2^{(r-1)} \max_{x \in I_{r,s}(r)} \mathbb{P} \left( \tilde{S}_w^N(r^{k+1}) - \tilde{S}_w^N(r) \in B_{r,n}(x) \right) \prod_{k=1}^{m-1} \max_{x \in I_{r,s}(r)} \mathbb{P} \left( \tilde{S}_w^N(r^{k+1}) - \tilde{S}_w^N(r^{k}) \in B_{k,n}(x) \right) \leq C \mathbb{E}[h_N] \sum_{r=1}^{n-1} 2^{(r-1)} \left( 2^{-2(\tilde{r} - \eta_1 \tilde{r})} + 2^{-2(\tilde{r} - \eta_2 \tilde{r})} \right) \prod_{k=1}^{m-1} 2^{(r^{k+1} - \eta_1 \tilde{r})} \leq C \mathbb{E}[h_N].$$

(4.78)

In the last line we used that the last sum can be uniformly bounded by a constant and the last product is bounded from above by 1. Inserting this into (4.76) allows to bound the second moment by $\mathbb{E}[h_N]^2 + (1 + C_2) \mathbb{E}[h_N]$, which concludes the proof of Lemma 4.10. 

Proof of Proposition 4.8

Using the Paley-Zygmund inequality with Lemma 4.9 and Lemma 4.10 we obtain

$$\mathbb{P}(h_N \geq 1) \geq \frac{\mathbb{E}[h_N]^2}{\mathbb{E}[h_N]^2} \geq \frac{\mathbb{E}[h_N]^2}{\mathbb{E}[h_N]^2 + (1 + C_2) \mathbb{E}[h_N]} \geq \frac{1}{1 + \frac{C_2}{\mathbb{E}[h_N]^2}} \geq \delta_0.$$

(4.79)
for some constant $\delta_0 \in (0,1)$, uniformly in $N$.

5. Tail estimates and tightness

The following analysis provides the necessary estimates to conclude tightness of the centred maximum $[\psi^N_v - E[\psi^N_v]]_{N \geq 0}$. We use Borell’s concentration inequality and Slepian’s Lemma to reduce the necessary estimates to corresponding estimates for the MIBRW. These are obtained using a refined version of the second moment computation from the proof of Proposition 4.8.

**Lemma 5.1.** There is a constant $\alpha_0 > 0$ such that for sufficiently large $N \in \mathbb{N}$ and any $v, w \in V_N$, we have

$$\text{Var}[\psi^N_v] \leq \log(2)n \sum_{i=1}^{M} \sigma_i^2 \nabla \lambda_i + \alpha_0, \quad (5.1)$$

and

$$E \left[ (\psi^N_v - \psi^N_w)^2 \right] \leq 2 \log(2) \sum_{i=1}^{M} \sigma_i^2 \left[ n \nabla \lambda_i 1_{n - \lfloor \log_2 \|v - w\| \rfloor} < \lambda_i - (1 - \lambda_i - 1)n - (\lfloor \log_4 \|v - w\| \rfloor) \right]$$

$$\times 1_{\lambda_i - 1 < \log_2 \|v - w\|} \chi_{\lambda_i} - \left| \text{Var}[\psi^N_v] - \text{Var}[\psi^N_w] \right| + 4\alpha_0. \quad (5.2)$$

**Proof.** Recall [Definition 1.2] and note that we have an underlying discrete Gaussian free field $\{\phi^N_i\}_{i \in V_N}$ such that $\psi^N = \sum_{i=1}^{M} \lambda_i \phi^N_i$, $\phi^N_i(\lambda_i) = \phi^N_i(\lambda_{i-1})$ for $i = 1, \ldots, M$ are independent Gaussian free fields increments. A short computation shows that the variance of $\nabla \phi^N_i(\lambda_i)$ is up to constants given by the difference of Green kernels on the boxes, that is $G_{[v]}(v,v) - G_{[v]}(v,v)$, for which we have a sufficient bound (see [53, Lemma 3.10]), and (5.1) follows.

For (5.2), let $b_N(v,w) := \max(\lambda \in [0,1]; [v]_\lambda \cap [w]_\lambda \neq \emptyset)$ be the branching scale for particles $v,w \in V_N$. For scales $\mu_i > \mu_i' \geq b_N(v,w)$ and $i = 1,2$, increments $\phi^N_i(\mu_i) - \phi^N_i(\mu_i')$ are independent of $\phi^N_{i'}(\mu_2) - \phi^N_{i'}(\mu_2')$, where $\phi^N_i$ is a set of representatives at scale $\lambda \in [0,1]$, denoted $R_i$, such that it contains the centre of boxes that form a decomposition of $V_N$ into disjoint boxes with size $N^{1-\lambda}$. Now, fix $v,w \in V_N$. Then we can find a decomposition $R_i$, at scale $\lambda = b_N(v,w) - \frac{4}{\log N}$, such that there is a common representative for $v$ and $w$, which we call $u_i$. By [6, Lemma A.6], there is a universal constant $C > 0$ such that for $N$ large enough,

$$\max_{u \in [v,w]} E \left[ (\psi^N_u(\lambda) - \psi^N_v(\lambda))^2 \right] \leq C. \quad (5.3)$$

We further note that increments of $v$ and $w$ beyond $b_N(v,w)$ are independent and further, that by Cauchy-Schwarz

$$E \left[ (\psi^N_u(b_N(v,w)) - \psi^N_v(\lambda))(\psi^N_u(b_N(v,w)) - \psi^N_v(\lambda)) \right] \leq C \quad (5.4)$$

as well as

$$\max_{u \in [v,w]} E \left[ (\psi^N_u(b_N(v,w)) - \psi^N_v(\lambda))^2 \right] \leq \tilde{C}, \quad (5.5)$$

for some $\tilde{C} > 0$. Thus, if we now write

$$\psi^N_v - \psi^N_u = \psi^N_v(\lambda) - \psi^N_u(\lambda) + \psi^N_u(\lambda) - \psi^N_u(\lambda) + \psi^N_u(b_N) - \psi^N_u(\lambda) + \psi^N_u(b_N) - \psi^N_u(\lambda) + \psi^N_u(b_N) - \psi^N_u(b_N)$$

and compute using (5.3), (5.4), (5.5), Green kernel estimates as for the first statement, as well as independence of increments beyond $b_N(v,w)$, we obtain the upper bound in (5.2). □
Lemma 5.2. Let $N \in \mathbb{N}$ and $\{\psi_v^N\}_{v \in V_N}$ be a $(\sigma, \lambda)$-DGFF. For any $\delta > 0$, there is a constant $c_\sigma(\delta) \in (0, \infty)$, depending only on the variance parameter $\sigma$ and the constant $\delta$, such that
\[
P \left( \left| \psi_v^N - \sqrt{\log(2)} M_{n}^v \right| \geq \delta \sqrt{\log(N)} \right) \leq e^{-c_\sigma(\delta) \log(N)}.
\] (5.7)

Proof. Apply Borell’s concentration inequality with (3.1).

This allows us to focus on the decay for deviations less than $O\left(\sqrt{\log(N)}\right)$. We begin with an upper bound on the right tail.

Proposition 5.3. There is a constant $C = C(\alpha_0)$, independent of $N$ such that for all $N \in \mathbb{N}$ and $x > 0$,
\[
P \left( \max_{v \in V_N} \psi_v^N \geq \sqrt{\log(2)} M_{n}^v + x \right) \leq C(1 + x \mathbb{1}_{\sigma_1 = \sigma_2}) e^{-\frac{2\log(2)}{\kappa}}.
\] (5.8)

Before proving Proposition 5.3, we need one more lemma.

Lemma 5.4. There is an integer $\kappa = \kappa(\alpha_0) > 0$ such that for all $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $A \subset V_N$,
\[
P \left( \max_{v \in A} \psi_v^N \geq \lambda \right) \leq 2 \P \left( \max_{v \in 2^A} R_{2v}^{2N} \geq \lambda \right).
\] (5.9)

Proof. By Lemma 5.1, we can choose a sufficiently large constant $\kappa$ that depends only on $\alpha_0$, such that $\Var(\psi_v^N) \leq \log(2) \Var(R_{2v}^{2N})$ for all $v \in V_N$. Thus,
\[
a_v^2 := \log(2) \Var(R_{2v}^{2N}) - \Var(\psi_v^N)
\] (5.10)

are non-negative. Let $X$ be a standard Gaussian. Since the variance of the BRW $R_N$ is the same for all vertices, we get
\[
\mathbb{E} \left[ (\psi_v^N + a_v X - \psi_w^N - a_w X)^2 \right] = \mathbb{E} \left[ (\psi_v^N - \psi_w^N)^2 \right] + (a_v - a_w)^2
\]
\[
= \mathbb{E} \left[ (\psi_v^N - \psi_w^N)^2 \right] + \Var(\psi_v^N) - \Var(\psi_w^N)
\]
\[
\leq 2 \log(2) \sum_{i=1}^{M} \sigma_i^2 \left[ n \nabla \lambda_i \mathbb{1}_{n - \left[ \log_2(\|v - w\|_2) \right] < \lambda_i n} - (1 - \lambda_{i-1}) n \right.
\]
\[
\left. -[\log_2(\|v - w\|_2)] \mathbb{1}_{\lambda_{i-1} n < \left[ \log_2(\|v - w\|_2) \right] < \lambda_i n} \right] + 4a_0,
\] (5.11)

by Lemma 3.3. On the other hand by (3.1), $\Var(R_{2v}^{2N}) = n + \kappa$ grows linearly in $\kappa$, whereas the distance on the trees $d_{2v}^{2N}(2u, 2^k v) = d_{N}(u, v)$ stays invariant, and so $\mathbb{E}[R_{2v}^{2N} R_{2v}^{2N}] = (n + \kappa) \int_{0}^{\frac{d_{N}(u, v)}{d_0}} \sigma(s) ds$ does not grow with $\kappa$. By (3.1) and taking into account that for two particles $u, v$, $d_{N}(u, v) \geq \log_{\kappa}(\|u - v\|_2)$, there is a constant $C > 0$ which is independent of $N$ and $v, w \in V_N$, such that
\[
\mathbb{E} \left[ (R_{2v}^{2N} - R_{2v}^{2N})^2 \right] \geq 2 \sum_{i=1}^{M} \sigma_i^2 \left[ n + \kappa \nabla \lambda_i \mathbb{1}_{n - \left[ \log_2(\|v - w\|_2) \right] < \lambda_i n} - (1 - \lambda_{i-1}) (n + \kappa) \right.
\]
\[
\left. -[\log_2(\|v - w\|_2)] \mathbb{1}_{\lambda_{i-1} n < \left[ \log_2(\|v - w\|_2) \right] < \lambda_i n} \right] - C.
\] (5.12)

Combining (5.12) with the upper bound in (5.11), it follows that there is a constant $C(\alpha(0))$, uniformly in $N$ so that for any $v, w \in V_N$,
\[
\log(2) \mathbb{E} \left[ (R_{2v}^{2N} - R_{2v}^{2N})^2 \right] - \mathbb{E} \left[ (\psi_v^N + a_v X - \psi_w^N - a_w X)^2 \right] \geq \log(2) \mathbb{1}_{v \neq w} \sigma_0^2 \nabla \lambda_{MK} - C.
\] (5.13)

Therefore, we may choose $\kappa(\alpha_0)$ such that for all $v, w \in V_N$,
\[
\mathbb{E} \left[ (\psi_v^N - \psi_w^N)^2 \right] \leq \mathbb{E} \left[ (\psi_v^N + a_v X - \psi_w^N - a_w X)^2 \right] \leq \log(2) \mathbb{E} \left[ (R_{2v}^{2N} - R_{2v}^{2N})^2 \right].
\] (5.14)

Applying Slepian’s Lemma, we obtain for any $\lambda \in \mathbb{R}_+$ and $A \subset V_N$,
\[
P \left( \max_{v \in A} \psi_v^N + a_v X \geq \lambda \right) \leq \P \left( \max_{v \in 2^A} \sqrt{\log(2)} R_{2v}^{2N} \geq \lambda \right)
\] (5.15)
By independence and symmetry of $X$,

$$\mathbb{P}\left(\max_{v \in A} \psi_v^N \geq \lambda\right) \leq 2 \mathbb{P}\left(\max_{v \in 2^A} \sqrt{\log(2)} R_v^N \geq \lambda\right). \quad (5.16)$$

**Proof of Proposition 5.3.** [42, Theorem 4.1] gives us

$$\mathbb{P}\left(\max_{v \in V_N} R_v^N \geq M_n^* + x\right) \leq C(1 + x) e^{-\frac{\lambda^2}{2c_1}}, \quad \text{for any } x \geq 0. \quad (5.17)$$

The claim now follows from Lemma 5.4.

Next, we prove a corresponding lower bound on the right tail.

**Lemma 5.5.** There is an integer $\kappa > 0$ such that for all $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$,

$$\frac{1}{2} \mathbb{P}\left(\max_{v \in V_{2^{\kappa}}} \sqrt{\log(2)} S_v^{2^\kappa} \geq \lambda\right) \leq \mathbb{P}\left(\max_{v \in V_N} \psi_v^N \geq \lambda\right). \quad (5.18)$$

**Proof.** Note that $(N \frac{N}{4}) + 2^{-\kappa} V_{2^{\kappa}} \subset (N \frac{N}{4}) + V_N$. By Lemma 3.3 ii. and iv., there is a constant $C > 0$, independent of $N$, such that

$$\left|\text{Var}\left[\psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} u\right] - \text{Var}\left[\psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} v\right]\right| \leq C, \quad \text{for all } u, v \in V_{2^{\kappa}}. \quad (5.19)$$

Moreover, by iv. in Lemma 3.3

$$\text{Var}\left[\psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} V_{2^{\kappa}}\right] \geq \log(2) \text{Var}\left[\psi_v \right], \quad \text{for all } v \in V_{2^{\kappa}}. \quad (5.20)$$

for $\kappa > 0$ large enough, independent of $N$. Thus, we can find a family of positive real numbers $\{a_v : v \in V_{2^{\kappa}}\}$ that satisfy $|a_u - a_v| \leq \sqrt{C}$ for a constant $C > 0$, such that for $u, v \in V_N$ and an independent standard Gaussian random variable $X$,

$$\text{Var}\left[\psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} u\right] = \log(2) \text{Var}\left[S_v^{2^\kappa} + a_u X\right], \quad \text{for all } v \in V_{2^{\kappa}}. \quad (5.21)$$

Using Lemma 3.3 iv., and choosing $\kappa$ large enough, we have for $u, v \in V_{2^{\kappa}}$,

$$\mathbb{E}\left[\left(\psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} u - \psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} v\right)^2\right] \geq \log(2) \mathbb{E}\left[(S_u^{2^\kappa} - S_v^{2^\kappa} + (a_u - a_v) X)^2\right]. \quad (5.22)$$

Hence, by Slepian’s Lemma we have for any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v \in V_{2^{\kappa}}} \psi_v^N (\frac{N}{4}, \frac{N}{4}) + 2^{-\kappa} \geq \lambda\right) \geq \mathbb{P}\left(\sqrt{\log(2)} \max_{v \in V_{2^{\kappa}}} (S_v^{2^\kappa} + a_v X) \geq \lambda\right) \geq \frac{1}{2} \mathbb{P}\left(\sqrt{\log(2)} \max_{v \in V_{2^{\kappa}}} S_v^{2^\kappa} \geq \lambda\right), \quad (5.23)$$

as $X$ is an independent standard Gaussian.

**Lemma 5.6.** There are constants $C, c > 0$ such that for any $N \in \mathbb{N}$ and $y \in (0, \infty)$,

$$\mathbb{P}\left(\max_{v \in V_N} S_v^N > M_n^* + y\right) \geq C(1 + y) e^{-\frac{2y}{\sqrt{C}}} \frac{1}{\sqrt{R_{v_1}}}. \quad (5.24)$$
To prove Lemma 5.6 we use a second moment computation similar to one in the proof of Proposition 4.8. We introduce suitable events that control the paths that reach the maximum. For $v \in V_N$, $x \in \mathbb{R}$, $\infty > y > 0$, $0 \leq k \leq n$ and $0 < t \leq m$, let

$$I_k(t) := [\nabla_i M_n^*(t^t) + y - 1, \nabla_i M_n^*(t^t) + y], \quad (5.25)$$

$$I_{k,n}(x) := [s_{k,n}(x) - f_{k,n}, s_{k,n}(x) + f_{k,n}], \quad (5.26)$$

$$C_{v,N}(r) := (\nabla_i S_v^N(t^t) \in I_k(t), S_v^N(k + t^{t^t} - 1) \in \nabla_i S_v^N(t^t)) \quad \forall 0 < k < t^{t^t} - 1, 0 < t \leq m : k + t^{t^t} \leq r), \quad (5.27)$$

$$h_N(y) := \sum_{v \in V_N} 1_{C(N^N(y^N))}. \quad (5.28)$$

$f_{k,n}$ and $s_{k,n}(x)$ are defined as before (see (4.45) and (4.46)). As before, we can restrict the proof to the case of $m = 1$ and to the assumption that $I_{a^2}(s) < I_{a^2}(s)$ holds for all $0 < s < 1$. The statement in case of equality is given by [23] Theorem 1.1. The lower bound then follows using independence of increments and the fact that there is a constant $c > 0$, which depends on the number of effective scale $m$, such that

$$\prod_{i=1}^{m} e^{-\frac{2\sqrt{\log 2}}{\sigma_1}} \geq e^{-\frac{2\sqrt{\log 2}}{\sigma_1}}. \quad (5.29)$$

So from now on, we restrict the proof to the case of $m = 1$ and to the assumption that $I_{a^2}(s) < I_{a^2}(s)$ holds for all $0 < s < 1$.

Lemma 5.7. There are constants $C, c > 0$ such that it holds for all $N \in \mathbb{N}$ sufficiently large,

$$c \geq \mathbb{E}[h_N(y)] \geq C e^{-\frac{2\sqrt{\log 2}}{\sigma_1}y}. \quad (5.30)$$

Lemma 5.8. There is a constant $C > 0$ independent of $N$, such that

$$\mathbb{E}[h_N^2(y)] \leq \mathbb{E}[h_N(y)]^2 + (1 + C)\mathbb{E}[h_N(y)]. \quad (5.31)$$

Proof of Lemma 5.7. In the following, we write $M_n^*$ instead of $M_n^*(t^t)$. Further, note since we assume $m = 1$ and thus $t^t = n$. By conditioning on $S_v^N(t^t)$, using its independence of $\{S_v^N(k) - s_{k,n}(S_v^N(t^t))\}_{k=0}^{t}$ and linearity of expectations, we have

$$\mathbb{E}[h_N(y)] = 2^{2n}\mathbb{P}(S_v^N(t^t) \in [M_n^* + y - 1, M_n^* + y]) \mathbb{P}(S_v^N(k) \in I_{k,n}(S_v^N(t^t))). \quad (5.32)$$

To estimate the first probability, note that $S_v^N(t^t) \sim N(0, \sigma_1^2 t^t)$ and by a standard Gaussian estimate,

$$\mathbb{P}(S_v^N(t^t) \in I_{k,n}(1)) = \int_{I_{k,n}(1)} \frac{\exp(-x^2/(2\sigma_1^2 t^t))}{\sqrt{2\pi\sigma_1^2 t^t}} dx \geq \frac{\exp(-\frac{1}{2\sigma_1^2 t^t})}{\sqrt{2\pi\sigma_1^2 t^t}}. \quad (5.33)$$

By expanding the square in (5.33) as in (4.56) and bounding all terms in the exponential that tend to zero as $n \to \infty$ by a uniform constant, we can find a constant $C > 0$ such that

$$\mathbb{P}(S_v^N(t^t) \in I_{k,n}(1)) \geq C(N^{-2})^t e^{-\frac{2\sqrt{\log 2}}{\sigma_1}}. \quad (5.34)$$

The second probability in (5.32) can be bounded from below in the same way as in (4.57), which gives the lower bound of the claim. For the upper bound in (5.32) we can bound the second probability simply by 1 and for the first probability, we can compute analogously as for the lower bound, i.e. we have

$$\mathbb{P}(S_v^N(t^t) \in I_{k,n}(1)) \leq C(N^{-2})^t \exp\left(-\frac{2\sqrt{\log 2}}{\sigma_1}\right). \quad (5.35)$$

Inserting this into (5.32), we obtain the upper bound. □
Proof of Lemma 5.8: Recall that for \( v, w \in V_N, r(v, w) = n - \lfloor \log_2(d_N^{\lfloor v, w \rfloor} + 1) \rfloor \) denotes the number of scales of independent increments of the processes \( S_k^N(k) \) and \( S_w^N(k') \). As in \eqref{eq:4.62}, we obtain

\[
\mathbb{E}[h_N(y)] = \sum_{v,w \in V_N} \mathbb{P}(C_v^{N,y}(t^i) \cap C_w^{N,y}(t^i)) = \sum_{k=0}^{n} \sum_{v,w \in V_N \atop r(v,w) = k} \mathbb{P}(C_v^{N,y}(t^i) \cap C_w^{N,y}(t^i))
\]

\[
\leq \mathbb{E}[h_N(y)]^2 + \mathbb{E}[h_N(y)] + \sum_{k=1}^{n-1} \sum_{v,w \in V_N \atop r(v,w) = k} \mathbb{P}(C_v^{N,y}(t^i) \cap C_w^{N,y}(t^i)). \tag{5.36}
\]

We need to bound the double sum from above, which can be done analogously as in the proof of Lemma 4.10. Dropping the constraint for \( w \) until time \( r \), the double sum in \eqref{eq:5.36} can be bounded from above by

\[
c_1 \mathbb{E}[h_N(y)] \sum_{r=1}^{n-1} 2^{2(t^i-r)} \max_{x \in L_i(1)} \mathbb{P}(S_v^N(t^i) - S_v^N(r) = x + I_{r,n}(x)) \tag{5.37}
\]

We need to bound the probability in \eqref{5.37}. For any \( x \in L_i(1) \), we have

\[
A_{r,n,x} \triangleq \mathbb{P}(S_v^N(t^i) - S_v^N(r) = x + I_{r,n}(x)) \leq \frac{2f_{r,n}}{\sqrt{I_{\sigma^2}(x^n)} n} \exp \left( -\frac{1}{2} \left( \frac{M_n^r(t^i) + y - s_{r,n}(M_n^r(t^i) + y) - f_{r,n}^2}{I_{\sigma^2}(x^n)} \right)^2 \right) \tag{5.38}
\]

Writing \( M_n^r = M_n^r(t^i) \), noting that \( n = t^i \) and using \eqref{eq:4.45}, we bound from below the square in the exponential by

\[
(M_n^r + y)^2 \left( 1 - \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} \right)^2 - 2f_{r,n} \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} (M_n^r + y) + (M_n^r)^2 \left( 1 - \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} \right)^2 - 2f_{r,n} \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} M_n^r
\]

\[
(2M_n^r y + y^2) \left( 1 - \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} \right)^2 - 2y f_{r,n} \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} \tag{5.39}
\]

We further consider two cases, first when \( 0 < r \leq t_1 \) and the other case when \( t_1 < k < t^i \). Let \( 0 < r \leq t_1 \) and \( v \in V_N \). We may assume that \( \sigma_1 > 0 \), else we consider the cases \( 0 < r \leq t_1 \) and \( t_1 < k < t^i \) for the minimal \( \lambda \) such that \( \sigma_i > 0 \). We then have \( \text{Var}[S_y^N(t^i)] - S_y^N(r) = I_{\sigma^2}(\sigma_n, \lambda_1) n \) and \( f_{r,n} = C_f(\sigma_1^2 r^j) \).

So, when \( 0 < r \leq t_1 \) and \( \sigma_1 > 0 \), we have \( \lambda_1 n \frac{I_{\sigma^2}(\sigma_n, \lambda_1)}{I_{\sigma^2}(\lambda_1)} = 1 - \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} n \lambda_1 = t^i - \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} n \lambda_1 = t^i \eta_1 r \) with \( 0 < \eta_1 < 1 \), as by assumption \( \sigma_1 < \sigma_1 \) and similarly \( \frac{I_{\sigma^2}(\sigma_n, \lambda_1)}{I_{\sigma^2}(\lambda_1)} \geq 1 - \frac{\sigma_1^2}{\sigma_1^2} \lambda_1^2 \). We can estimate in analogy to \eqref{eq:4.69}. Comparing \eqref{5.39} to \eqref{4.66}, the only additional input we need, is the observation that for \( n \) sufficiently large \( 1 + \frac{y}{4 \sqrt{\log(2^i)}} \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} < 1 \), which replaces \( \frac{I_{\sigma^2}(\sigma_n)}{I_{\sigma^2}(\lambda_1)} < 1 \) in \eqref{4.70}. Thus,

\[
A_{r,n,x} \leq C r^j 2^{-2t_1 - \eta_1 r} \exp(C r^j) \exp \left[ \frac{2 \sqrt{\log(2^i)}}{\sigma_1} \left( 1 - \frac{\sigma_1^2}{\sigma_1^2} \lambda_1 \right) y \right]. \tag{5.40}
\]

Note that for the last factor in the exponent we know \( 0 < 1 - \frac{\sigma_1^2}{\sigma_1^2} \lambda_1 r < 1 \), which guarantees that we have the correct sign to have sufficient decay in \( y \). Next, we turn to the bound on \( A_{r,n,x} \) in the case when
$t_1 < r < t^1$ and deduce in analogy to (4.74), however now using \((5.39)\) instead of (4.66),
\[
A_{v,n,x} \leq C2^{-n(\ell'-\ell)(I_{\ell^2}(\ell/n, \lambda^1)n)^{\gamma/2}} \exp \left[ C\left(I_{\ell^2}(\ell/n, \lambda^1)n\right)^{\gamma/2} 2\sqrt{\log(2) - y} \left( \frac{2\sqrt{\log(2)}}{\bar{\sigma}_1} \right) \right],
\]
(5.41)

As in the proof of Lemma 4.10 (5.40) and (5.41) show that the sum in (5.37) is finite, which finishes the proof.

\[ \square \]

**Proof of Lemma 5.6** Combining Lemma 5.7 with Lemma 5.8 there are constants $C, C, c > 0$ such that
\[
\Pr \left( \max_{v \in V_N} S_N^v > M_n^* + y \right) \geq \Pr (h_N(y) \geq 1) \geq \frac{\mathbb{E} [h_N(y)]^2}{\mathbb{E} [h_N(y)]^2 + (1 + C)\mathbb{E} [h_N(y)]} \geq e^{-y^2\sqrt{\log(2)}}/c.
\]
(5.42)

The goal in this subsection is to provide an upper bound on the left tail of the centered maximum of the $(\sigma, \lambda)$–DGFF. We start with a bound on the left tail of $S_N^* - \mathbb{E} [S_N^*]$.  

**Lemma 5.9.** There exist constants $C, c > 0$, such that for all $N \in \mathbb{N}$, $n = \log_2(N)$ and $0 \leq \lambda \leq (\log(n))^{\gamma/3}$,
\[
\Pr \left( \max_{v \in V_N} S_N^v \leq M_n^* - \lambda \right) \leq Ce^{-c\lambda}.
\]
(5.43)

**Proof.** By Proposition 4.8 there are $\beta > 0$ and $\delta_0 \in (0, 1)$ such that for all $N \in \mathbb{N}$,
\[
\Pr \left( \max_{v \in V_N} S_N^v \geq M_n^* - \beta \right) \geq \delta_0.
\]
(5.44)

In particular, there is a $\kappa > 0$ such that for all $N \geq N' \geq 4$,
\[
2\sqrt{\log(2)I_{\ell^2}(1)} \log \left( \frac{N}{N'} \right) - \frac{3}{4\sqrt{\log(2)}} \sum_{j=1}^{m} \bar{\sigma}_j \log \left( \log \left( \frac{N}{N'} \right) \right) - \kappa \leq M_n^* - M_n^*
\]
(5.45)

We now pick $\lambda' = 4$, $N' = N \exp \left[ -\frac{2\sqrt{\log(2)I_{\ell^2}(1)}}{2\sqrt{\log(2)I_{\ell^2}(1)}} (\lambda' - \beta - \kappa - 4) \right]$ and set $n' = \log_2 N'$. F Inserting these into (5.45) we see that $M_n - M_n^* \leq \lambda' - \beta$. Divide $V_N$ into disjoint boxes of side length $N'$ and consider a maximal collection $B$ of $N'$–boxes such that all pairwise distances between two boxes are at least $2N'$. This implies independence of the processes $(S_i^N)_{i \in B}$ on pairwise disjoint boxes. A possible choice of such boxes is to put at each position $(3iN', 3jN')$ a box of size $N'$ for $1 \leq i, j \leq \frac{N}{N'}$. This allows us to bound the number of boxes $B \in B$ from below by
\[
\frac{N}{3N'} \geq \frac{1}{3} \exp \left( \frac{1}{2\sqrt{\log(2)I_{\ell^2}(1)}} (\lambda' - \beta - \kappa - 4) \right).
\]
(5.46)

Let $\tilde{S}_v = S_v^N + X$ for $v \in B$ and $B \in B$ where $X \sim \mathcal{N}(0, s^2)$ is an independent random variable and with $s^2$ such that $\text{Var}(S_i^N) = \text{Var}(S_i^N)$. For $u, v \in \bigcup_{B \in B} B$, we then have
\[
\mathbb{E} \left[ (\tilde{S}_u - \tilde{S}_v)^2 \right] = \mathbb{E} \left[ (S_u^N - S_v^N)^2 \right] \leq \mathbb{E} \left[ (S_u^N - S_v^N)^2 \right] = (S_u^N - S_v^N)^2.
\]
(5.47)
An application of Slepian’s Lemma gives that for any $t \in \mathbb{R}$,

$$\mathbb{P}\left( \max_{v \in V_N} S^N_v \leq t \right) \leq \mathbb{P}\left( \max_{v \in \cup_{B \in \mathcal{B}} B} S^N_v \leq t \right) \leq \mathbb{P}\left( \max_{v \in \cup_{B \in \mathcal{B}} B} \tilde{S}^N_v \leq t \right).$$ \hspace{1cm} (5.48)

Using $M'_n - \lambda' \leq M'_n - \beta$ and (5.44) one obtains for each $B \in \mathcal{B}$,

$$\mathbb{P}\left( \max_{v \in B} S^N_v' \geq M'_n - \lambda' \right) \geq \mathbb{P}\left( \max_{v \in B} S^N_v' \geq M'_n - \beta \right) \geq \delta_0.$$ \hspace{1cm} (5.49)

By (5.49) as well as independence of $\{S^N_v\}_{v \in B}$ and $\{S^N_v'\}_{v \in B'}$ for different $B, B' \in \mathcal{B}$,

$$\mathbb{P}\left( \max_{v \in \cup_{B \in \mathcal{B}} B} S^N_v' < M'_n - \lambda' \right) \leq (1 - \delta_0)^{|\mathcal{B}|}.$$ \hspace{1cm} (5.50)

As $\delta_0 \in (0, 1)$ and by (5.46), there are constants $C, c > 0$ such that

$$(1 - \delta_0)^{|\mathcal{B}|} \leq \exp\left[ \frac{\log(1 - \delta_0)}{3} \exp\left( \frac{1}{2 \log(2) \bar{L}_\rho(1)} (\lambda' - \beta - \kappa - 4) \right) \right] \leq Ce^{-c \lambda'}. $$ \hspace{1cm} (5.51)

Using (5.48) we can bound $\mathbb{P}\left( \max_{v \in V_N} S^N_v \leq M'_n - \lambda \right)$ from above by

$$\mathbb{P}\left( \max_{v \in \cup_{B \in \mathcal{B}} B} S^N_v' < M'_n - \lambda' \right) + \mathbb{P}(\theta \leq -\lambda') \leq Ce^{-c \lambda'},$$ \hspace{1cm} (5.52)

where the last bound follows from (5.51) and a Gaussian tail bound. \hfill \Box

The next Lemma provides a corresponding upper bound on the left tail of the centred maximum of the scale-inhomogeneous DGFF $\{\psi_N^v\}_{v \in V}$ by comparing it to the MIBRW $\{S_N\}_{N \in \mathbb{N}}$ and using Lemma 5.9.

**Lemma 5.10.** There exist constants $C, c > 0$ so that for all $N \in \mathbb{N}$, $n = \log_2(N)$ and $0 \leq \lambda \leq (\log(n))^b$,

$$\mathbb{P}\left( \max_{v \in V_N} \psi_N^v \leq \sqrt{\log(2)M'_n - \lambda} \right) \leq Ce^{-c \lambda}. $$ \hspace{1cm} (5.53)

**Proof.** We can bound the probability in (5.53) by considering the maximum only over a subset, which avoids the necessity to consider boundary effects. By Lemma 3.3 iv., there is a constant $k_0 > 0$ such that for all $\kappa \geq k_0$,

$$\text{Var}\left[ \psi_{2^{\kappa}N}^v \right] \leq \log(2) \text{Var}\left[ S^N_v \right] \quad \forall v \in V_N. $$ \hspace{1cm} (5.54)

Therefore, we can choose a collection of positive numbers $\{a_v : v \in V_N\}$ and an independent standard Gaussian random variable $X$ so that for any $N$ and $u, v \in V_N$,

$$\text{Var}\left[ \psi_{2^{\kappa}N}^{u,v} + a_vX \right] = \log(2) \text{Var}\left[ S^N_v \right] \quad \forall v \in V_N. $$ \hspace{1cm} (5.55)

As the MIBRW has the same variance along all vertices and by the uniform bound in Lemma 3.3 ii., there is a constant $C_1 > 0$ such that

$$|a_u - a_v| \leq C_1. $$ \hspace{1cm} (5.56)
Writing $\tilde{u} = 2^*u + (2^{k+1}N, 2^{k+1}N)$ and using Lemma 3.3 ii and iv., we get
\[
\mathbb{E} \left[ \psi_{\tilde{u}}^{2\kappa N} \psi_{\tilde{u}}^{2\kappa N} \right] \geq \log(2) \sum_{i=1}^{M} \sigma_i^2 \left[ (n + k) \nabla \lambda_i \mathbb{1} \mathbb{1}_{n+k-\log_\lambda(\|\tilde{u}-2^*v\|_2) \geq \lambda(n+k)} 
+ ((1 - \lambda_i)(n + k) - \log_\lambda(\|\tilde{u}-2^*v\|_2)) \mathbb{1} \mathbb{1}_{\lambda_i(n+k) < n+k-\log_\lambda(\|\tilde{u}-2^*v\|_2) < \lambda(n+k)} \right] - c
\]
(5.57)
\[
= \log(2) \sum_{i=1}^{M} \sigma_i^2 \left[ (n + k) \nabla \lambda_i \mathbb{1} \mathbb{1}_{n-\log_\lambda(\|u-v\|_2) \geq \lambda(n+k)} 
+ ((1 - \lambda_i)(n + k) - \log_\lambda(\|u-v\|_2)) \mathbb{1} \mathbb{1}_{\lambda_i(n+k) < n-\log_\lambda(\|u-v\|_2) < \lambda(n+k)} \right] - c,
\]
where $c > 0$ is a constant. Further, taking into account that the Euclidean distance on the torus is bounded by the usual Euclidean distance, we have by Lemma 3.3 ii.,
\[
\mathbb{E} \left[ S_{\tilde{u}}^{2\kappa N} S_{\tilde{u}}^{2\kappa N} \right] \leq \sum_{i=1}^{M} \sigma_i^2 \left[ (n + 2k) \nabla \lambda_i \mathbb{1} \mathbb{1}_{n+2k-\log_\lambda(\|u-v\|_2) \geq \lambda(n+2k)} 
+ ((1 - \lambda_i)(n + 2k) - \log_\lambda(\|u-v\|_2)) \mathbb{1} \mathbb{1}_{\lambda_i(n+2k) < n+2k-\log_\lambda(\|u-v\|_2) < \lambda(n+2k)} + C\right],
\]
(5.58)
where $C > 0$ is another constant. Comparing (5.57) and (5.58), one deduces using (5.55) that there is a $k_0$ such that for $k \geq k_0$,
\[
\mathbb{E} \left( \psi_{2^*u+(2^{k+1}N, 2^{k+1}N)} \psi_{2^*v+(2^{k+1}N, 2^{k+1}N)} + a_v X \right) \leq \log(2) \mathbb{E} \left[ S_{\tilde{u}}^{2\kappa N} S_{\tilde{u}}^{2\kappa N} \right].
\]
(5.59)
Using (5.59) and (5.55), we can apply Slepian’s lemma to obtain
\[
\mathbb{P} \left( \max_{v \in V} \psi_{2^*v+(2^{k+1}N, 2^{k+1}N)} \leq \sqrt{\log(2)} M_n^* - \lambda \right)
\]
\[
\leq \mathbb{P} \left( \max_{v \in V} \psi_{2^*v+(2^{k+1}N, 2^{k+1}N)} + a_v X \leq \sqrt{\log(2)} M_n^* - \frac{\lambda}{2} \right) + \mathbb{P} \left( X \leq - \frac{\lambda}{C \kappa} \right),
\]
(5.60)
where $C \kappa > 0$ is a constant that solely depends on $\kappa$. Note that there is a collection of boxes $V$ consisting of at most $2^{8\kappa}$ translated copies of $V_N$ such that $V_{2^*N} \subset \bigcup_{V \in V} V$. Since
\[
\left\{ \max_{v \in V_{2^*N}} S_{2^*v}^{2\kappa N} \leq M_n^* - x \right\} = \bigcap_{V \in V} \left\{ \max_{v \in V} S_{2^*v}^{2\kappa N} \leq M_n^* - x \right\},
\]
(5.61)
we have by the FKG inequality [30] Proposition 1] that
\[
\mathbb{P} \left( \max_{v \in V_{2^*N}} S_{2^*v}^{2\kappa N} \leq M_n^* - x \right) \geq \left( \mathbb{P} \left( \max_{v \in V} S_{2^*v}^{2\kappa N} \leq M_n^* - \frac{\lambda}{2 \sqrt{\log(2)}} \right) \right)^{8\kappa}.
\]
(5.62)
Using (5.62) and then Lemma 5.9 we bound (5.60) from above by
\[
\mathbb{P} \left( \max_{v \in V_{2^*N}} \psi_{2^*v+(2^{k+1}N, 2^{k+1}N)} \leq \sqrt{\log(2)} M_n^* - \lambda \right) \leq \mathbb{P} \left( \max_{v \in V} \psi_{2^*v+(2^{k+1}N, 2^{k+1}N)} \leq \sqrt{\log(2)} M_n^* - \lambda \right)
\]
\[
\leq \mathbb{P} \left( \max_{v \in V} S_{2^*v}^{2\kappa N} \leq M_n^* - \frac{\lambda}{2 \sqrt{\log(2)}} \right) + \mathbb{P} \left( X \leq - \frac{\lambda}{C \kappa} \right)
\]
\[
\leq \left( \mathbb{P} \left( \max_{v \in V_{2^*N}} S_{2^*v}^{2\kappa N} \leq M_n^* - \frac{\lambda}{2 \sqrt{\log(2)}} \right) \right)^{1/(8\kappa)}
\]
\[
+ \mathbb{P} \left( X \leq - \frac{\lambda}{C \kappa} \right) \leq \tilde{C} e^{-\frac{\lambda}{C}}.
\]
(5.63)
where \( \tilde{C}, \tilde{c} > 0 \) are some constants that are independent of \( N \). This concludes the proof. \( \square \)

Together with Proposition 5.3 this allows us to conclude tightness of \( \{\psi_N - \mathbb{E}[\psi_N]\}_{N \in \mathbb{N}} \).

**Proof of Theorem 2.3.** By Proposition 5.3, Lemma 5.10 and Lemma 5.2 there are constants \( C, \sigma > 0 \) such that for all \( N \in \mathbb{N} \) and \( x > 0 \),

\[
\mathbb{P}\left(\left|\psi_N - \mathbb{E}[\psi_N]\right| \geq x\right) \leq C(1 + x)e^{-\gamma x},
\]

which concludes the proof. \( \square \)

**Appendix A. Gaussian comparison**

**Theorem A.1** (Borell’s inequality, [40 Lemma 3.1]). Let \( T \) be compact and \( \{X_i\}_{i \in \mathbb{N}} \) a centred Gaussian process on \( T \) with continuous covariance. Further assume that almost surely, \( X^* := \sup_{t \in T} X_t < \infty \). Then,

\[
\mathbb{E}[X^*] < \infty,
\]

and

\[
\mathbb{P}\left(|X^*| > x\right) \leq 2e^{-\gamma x},
\]

where \( \sigma_T^2 := \max_{i \in \mathbb{T}} \mathbb{E}[X_i^2] \).

**Theorem A.2** (Slepian’s Lemma, [40 Theorem 3.11]). Let \( T = \{1, \ldots, n\} \) and \( X, Y \) be two centred Gaussian vectors. Assume that we have two subsets \( A, B \subset \mathbb{T} \times T \) satisfying

\[
\mathbb{E}[X_{i,j}] \leq \mathbb{E}[Y_{i,j}], \quad (i, j) \in A
\]

\[
\mathbb{E}[X_{i,j}] \geq \mathbb{E}[Y_{i,j}], \quad (i, j) \in B
\]

\[
\mathbb{E}[X_{i,j}] = \mathbb{E}[Y_{i,j}], \quad (i, j) \notin A \cup B.
\]

Further, suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth function with at most exponential growth at infinity of \( f \) itself, as well as its first and second derivatives, and that

\[
\partial_{ij}f \geq 0, \quad (i, j) \in A
\]

\[
\partial_{ij}f \leq 0, \quad (i, j) \in B.
\]

Then,

\[
\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)].
\]

We use Slepian’s Lemma in a particular setting, i.e. we assume that \( \mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2] \) and \( \mathbb{E}[X_i X_j] \geq \mathbb{E}[Y_i Y_j] \) for all \( i, j \in T \). We then have for any \( x \in \mathbb{R} \),

\[
\mathbb{P}\left(\max_{i \in T} X_i > x\right) \leq \mathbb{P}\left(\max_{i \in T} Y_i > x\right).
\]

In particular, \( \mathbb{E}[\max_{i \in T} X_i] \leq \mathbb{E}[\max_{i \in T} Y_i] \). For a reference see [52 Corollary 3]. If we only want to compare the expectation of maxima we do not need the equality of variances. This is a result due to Sudakov and Fernique.

**Theorem A.3** (Sudakov-Fernique, [28]). Let \( I \) be an arbitrary set of finite size \( n \), \( \{X_i\}_{i \in I}, \{Y_i\}_{i \in I} \) be two centred Gaussian vectors. Define \( \gamma_{ij}^X := \mathbb{E}[(X_i - X_j)^2], \gamma_{ij}^Y := \mathbb{E}[(Y_i - Y_j)^2] \). Let \( \gamma := \max_{i,j} |\gamma_{ij}^X - \gamma_{ij}^Y| \). Then,

\[
|\mathbb{E}[X^*] - \mathbb{E}[Y^*]| \leq \sqrt{\gamma \log(n)}.
\]

If \( \gamma_{ij}^X \leq \gamma_{ij}^Y \) for any \( i, j \in I \), then

\[
\mathbb{E}[X^*] \leq \mathbb{E}[Y^*].
\]
In particular, if \( \{X_i\}_{i \in I} \) and \( \{Y_i\}_{i \in I} \) are independent centred Gaussian fields without any additional assumptions on their correlations, one deduces
\[
\mathbb{E} \left[ \max_{i \in I} (X_i + Y_i) \right] \geq \mathbb{E} \left[ \max_{i \in I} X_i \right]. \tag{A.12}
\]

### Appendix B Covariance estimates

For particles \( v, w \in V_N \), let
\[
b_N(v, w) := \max \{ \lambda \in [0, 1] : [v]^N_\lambda \cap [w]^N_\lambda \neq \emptyset \} \tag{B.1}
\]
denote the branching scale. The key point is that beyond \( b_N(v, w) \), increments are independent, that is for \( 1 \geq \lambda' > \lambda > b_N(v, w) \), \( \phi^N_\lambda(\lambda') - \phi^N_\lambda(\lambda) \) is independent of \( \phi^N_\lambda(\lambda') - \phi^N_\lambda(\lambda) \), whereas increments before the branching scale are correlated. Further, for some \( B \subset V_N \), we set
\[
\phi^N_B := \mathbb{E} \left[ \phi^N_\lambda | \sigma (\phi^N_\lambda : w \in B^c) \right]. \tag{B.2}
\]
Recall that for \( \lambda \in [0, 1] \), we also write \( \phi^N_\lambda(\lambda) = \phi^N_\lambda([v]^N_\lambda) \).

**Lemma B.1.** Let \( \delta \in (0, 1/2) \) and \( N \in \mathbb{N} \) such that \( \min_{1 \leq i \leq M} 2^{-N^{\delta} \lambda_i} \leq N \), as well as \( N^{\lambda_i} > \delta^{-1} \).

Let \( v, w \in V_N^c \) and assume that the branching scale \( b_N(v, w) \) coincides with a scale parameter, i.e. \( b_N(v, w) = \lambda_i \) for some \( i \in \mathbb{N} \). Then for any \( 0 \leq i, j \leq M \) with \( \lambda_i, \lambda_j \leq b_N(v, w) \), we have
\[
\mathbb{E} \left[ \nabla \phi^N_\lambda(\lambda_i) \nabla \phi^N_\lambda(\lambda_j) \right] = \nabla \lambda_i \log(N) \mathbb{E} 1_{i = j} + O(1). \tag{B.3}
\]

**Proof.** For \( v = w \) the statement is contained in [6, Lemma A.2]. Let us assume \( v \neq w \) throughout the proof. We start with the case \( i = j \). More, we assume \([v]^N_{\lambda_i} \cap [w]^N_{\lambda_i} \neq \emptyset \), i.e. the boxes should intersect at least at the boundary. If this is not the case, we can subdivide the scales further and use that beyond \( b_N(v, w) \) the respective increments are independent. This implies that \( \|v - w\|_2 \leq \sqrt{2} N^{1-\lambda_i} \). We now pick a box \( B \) of side length \( 2 N^{1-\lambda_i} \), centred at the middle of the line connecting the vertices \( v \) and \( w \). This ensures the inclusion
\[
\sigma (\phi^N_\lambda : u \in B^c) \subset \sigma (\phi^N_\lambda : u \in [v]^N_{\lambda_i}), \quad \sigma (\phi^N_\lambda : u \in [w]^N_{\lambda_i}). \tag{B.4}
\]
Next we pick a box \( \tilde{B} \) of side length \( \frac{1}{2} N^{1-\lambda_i-1} \) with the same centre as \( B \). For \( N \) as in the assumption, this implies in particular that \( \sigma (\phi^N_\lambda : u \in \tilde{B}^c) \subset \sigma (\phi^N_\lambda : u \in B^c) \), as well as
\[
\sigma (\phi^N_\lambda : u \in [v]^N_{\lambda_i-1}), \quad \sigma (\phi^N_\lambda : u \in [w]^N_{\lambda_i-1}) \subset \sigma (\phi^N_\lambda : u \in \tilde{B}^c). \tag{B.5}
\]
We write \( \nabla \phi^N_B = \phi^N_B - \phi^N_\lambda(B) \) and compute,
\[
\mathbb{E} \left[ \nabla \phi^N_\lambda(\lambda_i) \nabla \phi^N_\lambda(\lambda_j) \right] = \mathbb{E} \left[ (\phi^N_\lambda(\lambda_i) - \phi^N_B + \phi^N_B(B) - \phi^N_B(\lambda_i)) \phi^N_\lambda(\lambda_i - \phi^N_B(\lambda_i)) \right] 
\]
\[
= \mathbb{E} \left[ \nabla \phi^N_B \phi^N_B(\lambda_i) \right] \tag{B.6}
\]
\[
+ \mathbb{E} \left[ (\phi^N_B(\lambda_i) - \phi^N_B(B)) \phi^N_B(\lambda_i) \phi^N_B(B) + \phi^N_B(B) - \phi^N_B(\lambda_i)) \right] \tag{B.7}
\]
\[
+ \mathbb{E} \left[ (\phi^N_B(\lambda_i) - \phi^N_B(B)) \phi^N_B(B) + \phi^N_B(B) - \phi^N_B(\lambda_i)) \right] \tag{B.8}
\]
\[
- \mathbb{E} \left[ (\phi^N_B(\lambda_i) - \phi^N_B(B)) \phi^N_B(B) + \phi^N_B(B) - \phi^N_B(\lambda_i)) \right]. \tag{B.9}
\]
Using the conditional covariance identity
\[
\mathbb{E} [\mathbb{E} [X | \mathcal{A}] \mathbb{E} [Y | \mathcal{A}]] = \mathbb{E} [XY] - \mathbb{E} [(X - \mathbb{E} [X | \mathcal{A}]) (Y - \mathbb{E} [Y | \mathcal{A}])], \tag{B.10}
\]
with $X = \phi_N^N(1) - \phi_N^N(B)$, $Y = \phi_N^N(1) - \phi_N^N(\tilde{B})$ and $\mathcal{A} = \sigma(\phi_u^N : u \notin B^c)$, along with noting that by the Gibbs-Markov property of the DGFF $\phi_N^N(1) - \phi_N^N(\tilde{B}) \overset{d}{=} \phi_B^B$, we can write the first term (B.6) as
\[
\mathbb{E} \left[ \phi^B_B \right] - \mathbb{E} \left[ \phi^B_B \right] = \log \left( N^{1/2 + \log(2)/\log(N)} \right) - \log \left( N^{1/2 + \log(2)/\log(N)} \right) + O(1) = \mathbb{V}_i \log(N) + O(1). \tag{B.11}
\]

For the remaining terms we need to show that they are at most of constant order. As the last two terms (B.8) and (B.9) can be estimated the same way, we only deal with (B.8). Using Cauchy-Schwarz,
\[
\mathbb{E} \left[ (\phi_N^N(\lambda_i) - \phi_N^N(B)) \left( \phi_N^N(\lambda_i) - \phi_N^N(B) \right) \right] \\
\leq \mathbb{E} \left[ (\phi_N^N(\lambda_i) - \phi_N^N(B))^2 \right] + \mathbb{E} \left[ (\phi_N^N(B) - \phi_N^N(\lambda_i-1))^2 \right] \\
= (\log(2) + c_1)(\log(2) + c_2 + \log(2) + c_3) = O(1). \tag{B.12}
\]

To estimate (B.7) we make exhaustive use of our choice of boxes and use the relations (B.4) and (B.5) along with the tower property for conditional expectations and the law of total expectation, i.e. we first observe that both $\mathbb{E} \left[ \phi_N^N(B) \phi_N^N(\lambda_i) \right] = \mathbb{E} \left[ \phi_N^N(B) \phi_N^N(\lambda_i) \right]$ and $\mathbb{E} \left[ \phi_N^N(B) \phi_N^N(\lambda_i) \right] = \mathbb{E} \left[ \phi_N^N(B) \phi_N^N(\tilde{B}) \right]$ hold. Using this, we reformulate (B.7), i.e.
\[
\mathbb{E} \left[ \nabla \phi_N^N(\lambda_i) \phi_N^N(\tilde{B}) \phi_N^N(\lambda_i) - \phi_N^N(\lambda_i) \right] \\
= \mathbb{E} \left[ \left( \phi_N^N(\lambda_i) - \phi_N^N(B) \right) \phi_N^N(\tilde{B}) \right] \\
= \mathbb{E} \left[ \phi_N^N(\tilde{B}) \phi_N^N(\lambda_i) - \phi_N^N(\lambda_i) \right] = 0. \tag{B.13}
\]

For the remaining case $i \neq j$, we note that for $|i - j| \geq 2$ increments are independent as the difference of the boxes do not intersect for any $v, w \in V_N$, as we assume $N$ to be sufficiently large. The only remaining case is $j = i - 1$. Note that in this case, the increment $\nabla \phi_N^N(\lambda_i)$ is independent of the increment $\phi_N^N(\lambda_i - 1 - \log(N)) - \phi_N^N(\lambda_{i-2})$, as the annuli of the corresponding boxes do not intersect. This gives,
\[
\mathbb{E} \left[ \nabla \phi_N^N(\lambda_i) \nabla \phi_N^N(\lambda_{i-1}) \right] = \mathbb{E} \left[ \nabla \phi_N^N(\lambda_i) \left( \phi_N^N(\lambda_i - 1) - \phi_N^N(\lambda_i - 1 - \log(N)) + \phi_N^N(\lambda_i - 1 - \log(N)) - \phi_N^N(\lambda_i - 2) \right) \right] \\
= \mathbb{E} \left[ \nabla \phi_N^N(\lambda_i) \left( \phi_N^N(\lambda_i - 1) - \phi_N^N(\lambda_i - 1 - \log(N)) \right) \right] \\
= \mathbb{E} \left[ \left( \phi_N^N(\lambda_i) - \phi_N^N([v]_i) \right) + \phi_N^N([w]_{i+1}) - \phi_N^N([w]_{i+1}) - \phi_N^N([w]_{i+1}) - \phi_N^N([v]_i) \right] \\
\times \phi_N^N(\lambda_i - 1 - \log(N)) - \phi_N^N(\lambda_i - 1 - \log(N)). \tag{B.14}
\]

Provided $N$ is large, we have $[v]_i \cup [w]_{i+1} \supset [w]_{i+1} \subset [w]_{i+1} \supset [w]_{i+1}$ and so by the tower property and the law of total expectation, we deduce
\[
\mathbb{E} \left[ (\phi_N^N(\lambda_i) - \phi_N^N([v]_i)) \left( \phi_N^N(\lambda_i - 1) - \phi_N^N(\lambda_i - 1 - \log(N)) \right) \right] \\
= \mathbb{E} \left[ \phi_N^N(\lambda_i - 1) - \phi_N^N(\lambda_i - 1 - \log(N)) \right] \mathbb{E} \left( \phi_N^N : u \in [v]_i \right) \\
- \mathbb{E} \left[ \phi_N^N(\lambda_i - 1) - \phi_N^N(\lambda_i - 1 - \log(N)) \right] \mathbb{E} \left( \phi_N^N : u \in [w]_{i+1} \right) = 0. \tag{B.15}
\]
As the annuli $[w]_{l-1} \setminus [w]_{l-1}$ and $[w]_{l-1} \setminus [w]_{l-1}$ do not intersect, we have independence of the corresponding increments, i.e.

$$E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v([w]_{l-1})) \left( \rho_N^v(\lambda_{l-1}) - \phi_N^v(\lambda_{l-1} - \log(4) / \log(N)) \right) \right] = 0. \quad (B.16)$$

The remaining term in (B.14) can be bounded in a first step by the Cauchy-Schwarz inequality,

$$E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v(\lambda_{l-1})) \left( \rho_N^v(\lambda_{l-1}) - \phi_N^v(\lambda_{l-1} - \log(4) / \log(N)) \right) \right] \leq c \sqrt{\log(4)} \E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v(\lambda_{l-1}))^2 \right]^{1/2}. \quad (B.17)$$

In order to bound the expectation on the right hand side, we consider a box $B$ centred at the middle of the line connecting $v$ and $w$ of side length $N^{1-l-1} - \sqrt{2}N^{1-l}$. The assumption $|v - w|_\infty \leq \sqrt{2}N^{1-l}$ ensures the inclusion $B \subset [v]_{l-1} \cap [w]_{l-1}$. This allows us to compute in a similar fashion as in the first case (B.6), i.e.

$$E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v(\lambda_{l-1}))^2 \right] = E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v(B) + \phi_N^v(B) - \phi_N^v(\lambda_{l-1}))^2 \right] \leq 4 \max \left( E \left[ (\phi_N^v([w]_{l-1}) - \phi_N^v(B))^2 \right], E \left[ (\phi_N^v(B) - \phi_N^v(\lambda_{l-1}))^2 \right] \right) \leq 4(c + \log(N^{1-l-1}) - \log(N^{1-l-1}(1 - \sqrt{2}N^{-w}))) \leq C. \quad (B.18)$$

The constants $c, C > 0$ can be chosen uniformly in $N$, however depending on the scale parameters. Altogether, we obtain

$$E \left[ \nabla \phi_N^v(\lambda_i) \nabla \phi_N^v(\lambda_j) \right] \leq C, \quad (B.19)$$

for some constant $C > 0$ that is uniform in $N$, which finishes the proof.

**Proof of Lemma 3.3** For a proof of the statements i. and iii., we refer to [20, Lemma 2.2]. We have that $\log_+ (d_N^v(v, w)) \leq \log_+ (d_N^w(v, w)) \leq \log_+ (d^N_N(v, w)) + 1$. We begin with the proof of the second statement. Note that if $1 \leq k < \log_+ (d^N_N(v, w)) + 1$, there are no boxes of size $2^k$ that cover both $v$ and $w$. Thus, if $B, \tilde{B}$ are boxes such that one covers $v$ but not $w$ and the other $w$ but not $v$, the associated random variables $b_{k, B}, b_{k, \tilde{B}}$ are independent. And so, only random variables $b_{k, B}, b_{k, \tilde{B}}$ associated to boxes of size $2^k$ with $k > \left[ \log_+ (d^N_N(v, w)) + 1 \right]$ contribute to the covariance. For $v = (v_1, v_2), w = (w_1, w_2)$ and $i = 1, 2$, we write $r_i(v, w) = \min(|v_i - w_i|, |v_i - w_i - N|, |v_i - w_i + N|)$. Using the fact that the number of common boxes for $v, w \in V_N$ is given by $[2^k - r_i(v, w)][2^k - r_i(v, w)]$,

$$E \left[ S^N_v S^N_w \right] = \sum_{k = \log_+ (d^N_N(v, w))}^n 2^{-2k} \sigma^2 \left( \frac{n - k}{n} \right) \left( 2^k - r_1(v, w) \right) \left( 2^k - r_2(v, w) \right). \quad (B.20)$$

We note that since $a + b - ab \geq 0$ for $0 \leq a, b \leq 1$, we get

$$E \left[ S^N_v S^N_w \right] \leq n \sum_{k = 1}^M \sigma^2 \left( \frac{n}{n - k} \right) \left( 2^k - r_1(v, w) \right) \left( 2^k - r_2(v, w) \right) \left( 1 - \frac{r_1(v, w)}{2^k} - \frac{r_2(v, w)}{2^k} + \frac{r_1(v, w)r_2(v, w)}{2^{2k}} \right) \sum_{k = n}^M \left( \frac{n}{n - k} \right) \left( 2^k - r_1(v, w) \right) \left( 2^k - r_2(v, w) \right). \quad (B.21)$$
On the other hand, since \(a + b - ab \leq a + b\) for \(a, b \geq 0\), we get
\[
\begin{align*}
\mathbb{E} \left[ S^N_x S^N_w \right] & \geq \sum_{k = \lfloor \log_x (d^N(v, w)) \rfloor}^n \sigma^2 \left( \frac{n-k}{n} \right) - \max_{1 \leq i \leq M} \sigma^2 \left( \frac{i}{n} \right) 2^{n-k+1} d^N_x(v, w) \\
& \geq \sum_{i=1}^M \sigma_i^2 \left[ n \nabla \lambda_i \mathbb{1}_{n-\left[ \log_x (d^N(v, w)) \right]} + ((1 - \lambda_i) n) \right] \\
& \geq \sum_{i=1}^M \sigma_i^2 \left[ n \nabla \lambda_i \mathbb{1}_{n-\left[ \log_x (d^N(v, w)) \right]} + ((1 - \lambda_i) n) \right] - C,
\end{align*}
\]
where in the last step we did a rescaling from \([0, n]\) onto the unit interval \([0, 1]\) and where \(C > 0\) is a constant independent of \(N\) with \(C > 2 \max_{1 \leq i \leq M} \sigma^2 (i/M)\) that deals with the second part of the sum. To prove the last statement iv., we note that beyond the branching scale, \(N\) being sufficiently large (see assumptions of Lemma B.1) and by the Gibbs-Markov property, increments are independent as the annuli of the corresponding boxes do not intersect (see for instance [6, Section 2]). Moreover, by a refinement of the scale parameters and possibly allowing for an additional uniformly bounded constant, we can assume that the branching scale coincides with a scale parameter. With this we can apply Lemma B.1 and obtain the result, i.e.
\[
\begin{align*}
\mathbb{E} \left[ \phi^N_x \phi^N_y \right] &= \mathbb{E} \left[ \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j \nabla \phi^N_x (\lambda_i) \nabla \phi^N_y (\lambda_j) \right] \\
&= \sum_{i=1}^M \sigma_i^2 \mathbb{E} \left[ \left( \nabla \phi^N_x (\lambda_i) \right)^2 + \phi^N_x (\lambda_i-1) \right] + O(1) \\
&= \log(2) \sum_{i=1}^M \sigma_i^2 \left[ n \nabla \lambda_i \mathbb{1}_{n-\left[ \log_x (\|v - w\|_2) \right]} + ((1 - \lambda_i) n) \right] + O(1),
\end{align*}
\]
where \(O(1)\) is uniform in \(N\). 

\[\square\]

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