Experimental Quantum Cloning with Prior Partial Information

Jiangfeng Du,1,2,3 Thomas Durt,4 Ping Zou,1 Hui Li,1 L.C. Kwek,5 C.H. Lai,2 C.H. Oh,2 and Artur Ekert3,2

1Hefei National Laboratory for Physical Sciences at Microscale & Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, PR China
2Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542
3Centre for Quantum Computation, DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA U.K.
4TONA–TENA Free University of Brussels, Pleinlaan 2, B-1050 Brussels, Belgium.
5Department of Natural Sciences, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616

When prior partial information about a state to be cloned is available, it can be cloned with a fidelity higher than that of universal quantum cloning. We experimentally verify this intriguing relationship between the cloning fidelity and the prior information by reporting the first experimental optimal quantum state-dependent cloning, using nuclear magnetic resonance techniques. Our experiments may further have important implications into many quantum information processing protocols.

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One of the most striking and important implication of quantum mechanical laws to the information theory is that an unknown quantum state cannot be copied exactly, as observed by Wootters and Zurek [1], and Ghi- radini and Weber [2]. However, if it is not demanded to copy the state exactly, then it is possible to clone it approximately [3]. Since Bužek and Hillery proposed their universal quantum cloning machine (UQCM) [4], lots of efforts had been made to the theoretical investigation of this kind of cloner [5]. Also a number of experimental implementations of 1 → 2 qubit universal cloning had been proposed [6].

A universal cloning machine for qubits produces copies of equal quality for all possible input states, with a fidelity being 5/6 ≈ 0.833 for a 1 → 2 optimal UQCM. However, if partial prior knowledge of the state to be copied is available, or it is known to belong to a subset of all possible input states, then it is possible to clone the state with a higher fidelity. For instance this occurs in the case of the quantum phase covariant cloning machine (QPCCM) [2, 3, 7, 11]. A QPCCM clones an equatorial qubit state, which is in the form of $|\psi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$, with an optimal fidelity being 0.854, higher than that of a UQCM.

In a previous paper [11], we have shown why two qubits are enough for 1 → 2 optimal phase-covariant cloning in agreement with Niu-Griffiths scheme [8]. In the present letter, we constructed the two-qubit quantum logic circuit for the optimal quantum state-dependent cloning, and experimentally observe the interesting relationship between the cloning fidelity and the prior information available. In our case, for a qubit state of the form

$$|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle,$$  (1)

the prior knowledge is $\theta \in [0, \pi]$, the polar angle of the corresponding vector in the Bloch sphere, while the azimuthal angle is completely unknown, i.e. uniformly distributed between $[0, 2\pi]$. When $\theta$ is fixed to be $\pi/2$, our quantum state-dependent cloning reduced to the conventional quantum optimal phase-covariant cloning. Our cloning, as will be presented in the following, indeed achieves the optimal fidelities for any given polar angle $\theta$, which is presented in Ref. [10].

We experimentally implement the 1 → 2 cloning of qubit states with a priori known value of the polar angle $\theta$. The observed fidelities are higher than that of the UQCM, with a minimal fidelity being exactly that of the QPCCM, which is, however, still higher than that of the UQCM. The experimental observation agrees well with theoretical predictions, showing the interesting relationship between the cloning fidelities and the prior information of the input state. As quantum cloning is intimately related to many quantum information processing protocols, our experimental observation may further have important implications into them.

The quantum logic circuits for our optimal quantum state-dependent cloning is described in Fig. 1. We use two different circuits for the case where the state resides in the northern and the southern hemisphere, respectively. Qubit $a$ contains the state to be cloned as in Eq. (1), while qubit $b$ is the blank one initially in the state $|0\rangle$.

The unitary operator representing the quantum logic circuit for the northern hemisphere, i.e. for $\theta \in [0, \pi/2]$, is

$$U_n = |00\rangle\langle 00| + |11\rangle\langle 11| + \frac{1}{\sqrt{2}}(|01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| - |10\rangle\langle 01|).$$  (2)

While for the southern hemisphere, i.e. $\theta \in [\pi/2, \pi]$, the unitary operator representing the corresponding circuit

$$U_s = \cos \frac{\theta}{2}|0\rangle\langle 0| + e^{i\phi} \sin \frac{\theta}{2}|1\rangle\langle 1|.$$  (3)
easily verified to be described by density matrices

\[ U_s = |00\rangle\langle 01| + |11\rangle\langle 10| + \frac{1}{\sqrt{2}} (|01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 11| - |01\rangle\langle 11|) \]  

(3)

With the initial state being \(|\psi\rangle\) in Eq. (1) for qubit \(a\) and \(|0\rangle\) for qubit \(b\), the two identical clones of \(|\psi\rangle\) can be easily verified to be described by density matrices

\[ \rho_a = \rho_b = \left( \begin{array}{cc} \cos^2 \frac{\theta}{2} + \frac{1}{2} \sin^2 \frac{\theta}{2} & \frac{1}{2} e^{-i\phi} \sin \theta \\ \frac{1}{2} e^{i\phi} \sin \theta & \cos^2 \frac{\theta}{2} + \frac{1}{2} \sin^2 \frac{\theta}{2} \end{array} \right) \]  

(4)

for \(\theta \in [0, \pi/2]\), while

\[ \rho_a = \rho_b = \left( \begin{array}{cc} \frac{1}{2} \cos^2 \frac{\theta}{2} & \frac{1}{2} e^{-i\phi} \sin \theta \\ \frac{1}{2} e^{i\phi} \sin \theta & \frac{1}{2} \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \end{array} \right) \]  

(5)

for \(\theta \in [\pi/2, \pi]\). It then immediately follows that the fidelity of our quantum state-dependent cloning is

\[ F(\theta) = \text{Tr}(\rho_a|\psi\rangle\langle \psi|\rho_a) = \text{Tr}(\rho_b|\psi\rangle\langle \psi|\rho_b) \]

\[ = \begin{cases} \frac{1}{2} \sin^2 \theta + \cos^2 \frac{\theta}{2} + \frac{1}{2} \sin^2 \theta, & \theta \in \left[0, \frac{\pi}{2}\right], \\ \frac{1}{2} \cos^2 \theta + \frac{1}{2} \sin^2 \theta + \frac{1}{4} \sin^2 \theta, & \theta \in \left[\frac{\pi}{2}, \pi\right]. \end{cases} \]  

(6)

which has already been expressed as a function of the polar angle \(\theta\), for all \(\theta \in [0, \pi]\). This fidelity is exactly the optimal cloning fidelity which could be achieved for our state-dependent cloning, as proved in Ref. [10]. It would also be interesting to note that the cloning fidelity is independent on the azimuthal angle \(\phi\), which implies that the cloning is equally efficient for all the states with the same value of polar angle \(\theta\). In our case, it is interesting to investigate the relationship between the fidelity \(F(\theta)\) and the corresponding entropy of the input state (a state with fixed known polar angle \(\theta\) but completely unknown azimuthal angle \(\phi\)). Here, for states \(|\psi\rangle\) in Eq. (1) with fixed \(\theta\) but unknown \(\phi\), the average state can be represented by a density matrix \(\rho = \frac{1}{2\pi} \int |\psi\rangle\langle \psi|\,d\phi\), and its uncertainty can be expressed by the entropy, i.e.

\[ S(\rho) = -\int \rho(\phi) \log_2 \rho(\phi) \, d\phi = -p \log_2 p - (1-p) \log_2 (1-p) \]

where \(p\) and \(1-p\) are eigenvalues of \(\rho\). In terms of the polar angle \(\theta\), the entropy can then be expressed as a function of \(\theta\):

\[ S(\theta) = -\cos^2 \frac{\theta}{2} \log_2 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \log_2 \sin^2 \frac{\theta}{2}. \]  

(7)

In our nuclear magnetic resonance (NMR) experimental implementation of the above quantum state-dependent cloning, we use \(^{13}\text{C}\) nucleus in Carbon-13 labeled chloroform (Cambridge Isotopes) dissolved in \(^{1}\text{H}\) acetone as qubit \(a\) and qubit \(b\) correspondingly. Experimentally, the reduced Hamiltonian of the two-spin ensemble is given by

\[ H = \omega_a I^a_z + \omega_b I^b_z + 2\pi J I^a_z I^b_z \]

where the first two terms describe the free procession of spin \(a\) (\(^{13}\text{C}\)) and spin \(b\) (\(^1\text{H}\)) around the static magnetic field (along direction \(z\)) with frequencies 100MHz and 400MHz, respectively. \(I^a_z = \frac{1}{2}\sigma^a_z\) is the angular momentum operator of \(a\) (\(b\)) in direction \(z\), and the third term is the \(J\) coupling of the two spins with \(J = 214.5\)Hz. \(^{13}\text{C}\) nucleus’s \(T_1\) relaxation time is 17.2s and its \(T_2\) relaxation time is 0.35s. \(^1\text{H}\) nucleus’s \(T_1\) relaxation time is 4.8s and it’s \(T_2\) relaxation time is 3.3s. The experiments must be implemented within the relax-
ation time before the quantum coherence is lost due to the unavoidable decoherence.

We prepare the pseudo-pure state $|00\rangle$ following the same proposal as presented in Ref. 12. Then, by rotating qubit $a$, we get the two-qubit initial state as $(\cos \theta |0\rangle + e^{i\phi} \sin \theta |1\rangle) / \sqrt{2} \otimes |0\rangle$. In the experiments, we set $\theta = n \cdot \pi / 12$, which determines the entropy of the input state, for $n = 0, 1, \cdots, 12$ and $\phi = m \cdot \pi / 4$ for $m = 0, 1, \cdots, 7$. For each set of $(\theta, \phi)$ the full process of the cloning in Fig. 3 is implemented and the fidelity is extracted. The NMR pulse sequences for our quantum state-dependent cloning are developed by replacing unitary operations with idealized sequences of NMR pulses and delays. The resulting sequences are further simplified by appropriately combining radio-frequency (rf) pulses when convenient. All the rf pulses on a single qubit are hard pulses which give nearly no affection on the state of the other qubit due to the heteronuclear sample we used.

The specific pulse sequences for preparing the pseudo-pure state $|00\rangle$ is

$$R^b_y(\pi/2) - \tau_1 - R^b_x(\pi/2) - R^b_y(\pi/2) - \tau_2 - R^a_y(\pi/2) - R^a_x(\pi/4) - R^a_y(\pi/2) - \tau_1 - R^b_y(\pi/2)$$

and for the southern hemispheres is

$$R^b_y(\pi/2) - \tau_1 - R^b_x(\pi/2) - R^a_y(\pi/2) - \tau_2 - R^a_y(\pi/2) - R^a_x(\pi/4) - R^a_y(\pi/2) - \tau_1 - R^b_y(\pi/2)$$

with the same meaning of notions as in the sequence in Eq. 3 but $\tau_2$ representing a time interval of $1/(4J)$. Alert readers may argue that the operation realized by the pulse sequences as in Eqs. 4, 10 are actually different from Eqs. 2, 3 by certain rotations around the $z$-axis which should be added before we implement the pulse sequences. Specifically, $R^+_z(\pi) \otimes R^b_z(\pi)$ should be added before the sequence in Eq. 9 is implemented, while $R^+_z(\pi)$ before that in Eq. 10. However, those rotations can always be absorbed into the abstract reference frames in NMR experiments, hence no actual rotations are required in our case.

In the language of Bloch sphere, the state of a single qubit can be represented by a density matrix of the form $\rho = \frac{1}{2}(\sigma_0 + \vec{r} \cdot \vec{\sigma})$, where $\sigma_0$ is the identity operator, $\sigma_\mu$ ($\mu = x, y, z$) are the usual Pauli matrices, and $\vec{r} = (r_x, r_y, r_z)$ is a real vector, which is of length less than one for mixed states while equal to one for pure states. Let $\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{2}(\sigma_0 + \vec{r} \cdot \vec{\sigma})$ be the density matrix for the initial state $|\psi\rangle$ in Eq. 1 while $\rho = \rho_a = \rho_b = \frac{1}{2}(\sigma_0 + \vec{r} \cdot \vec{\sigma})$ for the copy. The fidelities of the cloning in Eq. 4 can then be expressed as $F(\theta) = \text{Tr}(\rho_0 \cdot \rho) = \frac{1}{2}(1 + \vec{r} \cdot \vec{r})$. To experimentally determine the fidelities of the cloning, we need to measure the $x$, $y$, $z$ part of $\vec{r}$. This can be accomplished by phase sensitive detectors for measuring the $x$ and $y$ parts, while by adding a gradient pulse and a $\pi/2$ pulse for measuring the $z$ parts of the copies. And then the fidelities can be obtained directly from the experimental data.

All our experiments are conducted at room temperature and normal pressure on a Bruker AV-400 spectrometer. The experimental fidelities $F(\theta)$ are presented in Fig. 2 as a function of the entropy $S(\theta)$ of the input state. We observe that as the prior information increases, i.e. the entropy decreases, the fidelity monotonously increases. Our experimental results coincides with the theoretical predictions. The small errors are mainly due to imperfect pulses, the variability over time of the measurement process, and inhomogeneity of magnetic field.

It is worth noting that the fidelity of our state-dependent cloning are higher than either that of the UQCM or of the QPCCM, while with a minimal value of fidelity being exactly that of the QPCCM. Furthermore, when $\theta$ approaches 0 or $\pi$, the corresponding fidelity increases. This is an interesting observation from an information prospective. Indeed, when $\theta$ approaches 0 or $\pi$, there are less states in the Bloch sphere with the same polar angle of $\theta$. Hence there is less uncertainty of the input state, i.e. more prior information is known about the input state, and consequently higher fidelity is attainable in cloning this subset of states. Moreover, in our present letter, the optimal cloning happens to be exactly
the square root of a SWAP operation. This is an interesting observation when taking into account the interesting connection between quantum cloning and quantum communication. Roughly speaking, this is because a SWAP permutes Bob’s state and Eve’s state, which gives all the information to Eve and nothing to Bob, while not doing anything sends all the information to Bob and nothing to Eve: optimal cloning is in-between: the same information for Bob and for Eve.

In conclusion, we experimentally implement the $1 \rightarrow 2$ quantum optimal state-dependent cloning for qubits. Our experimental observation highlights the interesting relationship between the optimal cloning fidelities and the prior information of the input states. The more is the prior information available for the input states, the higher optimal cloning fidelity is attainable. We further expect our experimental observation have important implications into many quantum information processing protocols.

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* Electronic address: dfj@ustc.edu.cn

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