Mathematical programming for influence diagrams

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February 20, 2019

Abstract

Key words: Influence diagrams, Partially Observed Markov Decision Processes, Probabilistic graphical models, Linear Programming.

Influence Diagrams (ID) are a flexible tool to represent discrete stochastic optimization problems, including Markov Decision Process (MDP) and Partially Observable MDP as standard examples. More precisely, given random variables considered as vertices of an acyclic digraph, a probabilistic graphical model defines a joint distribution via the conditional distributions of vertices given their parents. In ID, the random variables are represented by a probabilistic graphical model whose vertices are partitioned into three types: chance, decision and utility vertices. The user chooses the distribution of the decision vertices conditionally to their parents in order to maximize the expected utility. Leveraging the notion of rooted junction tree, we present a mixed integer linear formulation for solving an ID, as well as valid inequalities, which lead to a computationally efficient algorithm. We also show that the linear relaxation yields an optimal integer solution for instances that can be solved by the “single policy update”, the default algorithm for addressing IDs.

1 Introduction

In this paper we want to address stochastic optimization problems with structured information and discrete decision variables, through mixed integer linear reformulations. We start by recalling the framework of influence diagrams (more details can be found in \cite{KollerFriedman2009}), and show classical linear formulation for some special cases.

1.1 The framework of parametrized influence diagram

Let $G = (V, E)$ be a directed graph, and for each vertex $v$ in $V$, let $X_v$ be a random variable taking value in a finite state space $\mathcal{X}_v$. We say that, the random vector $X_V := \{X_v \mid v \in V\}$ factorizes as a directed graphical model on $G$ if, for all $x_V \in \prod_{v \in V} \mathcal{X}_v$, we have

$$\mathbb{P}(X_V = x_V) = \prod_{v \in V} p(x_v \mid x_{\text{prt}(v)}),$$

where $\text{prt}(v)$ is the set of parents of $v$, that is, the set of vertices $u$ such that $(u, v)$ belongs to $E$, and $p(x_v \mid x_{\text{prt}(v)}) = \mathbb{P}(X_v = x_v \mid X_{\text{prt}(v)} = x_{\text{prt}(v)})$. Further, given an arbitrary collection of conditional distribution $\{p(x_v \mid x_{\text{prt}(v)})\}_{v \in V}$, Equation (1) uniquely defines a probability distribution on $\mathcal{X}_V = \prod_{v \in V} \mathcal{X}_v$. 
Let \((V^a, V^c, V^r)\) be a partition of \(V\) where \(V^c\) is the set of chances vertices, \(V^a\) is the set of decision vertices, and \(V^r\) is the set of utility vertices (the ones with no descendants). For ease of notation we denote \(V^s = V^c \cup V^r\). Letters \(a, r,\) and \(s\) respectively stand for action, reward, and state in \(V^a, V^r,\) and \(V^s,\) and are used to avoid conflict of notations. We say that \((V^s, V^a, E)\), sometimes simply denoted by \(G\) is an Influence Diagram (ID). Consider a set of conditional distributions \(p = \{p(x_v|x_{\text{prt}(v)})\}_{v \in V^s, \delta v}\), and a collection of reward functions \(r = \{r_v\}_{v \in V^r}\), with \(r_v : X_v \to \mathbb{R}\). Then we call \((G, X_V, p, r)\) a Parametrized Influence Diagram (PID). We sometimes denote the parameters \((X_V, p, r)\) by \(\rho\) for conciseness.

Let \(\Delta_a\) denote the set of conditional distributions \(\delta_{v|\text{prt}(v)}\) on \(X_v\), given \(X_{\text{prt}(v)}\). Given the set of conditional distributions \(p\), the policy \(\delta \in \Delta = \prod_{v \in V^s} \Delta_v\), uniquely defines a distribution \(\mathbb{P}_\delta\) on \(X_V\) through

\[
\mathbb{P}_\delta(X_V = x_V) = \prod_{v \in V^s} p(x_v|x_{\text{prt}(v)}) \prod_{v \in V^r} \delta_{v|\text{prt}(v)}(x_v|x_{\text{prt}(v)}). \tag{2}
\]

Let \(\mathbb{E}_\delta\) denote the corresponding expectation. Finally, we can define the Maximum Expected Utility (MEU) problem associated to the PID \((G, X_V, p, r)\), as

\[
\max_{\delta \in \Delta} \mathbb{E}_\delta \left( \sum_{v \in V^r} r_v(X_v) \right). \tag{3}
\]

A deterministic policy \(\delta \in \Delta^d \subset \Delta\), is such that for every \(v \in V^a\), and any \(x_v, x_{\text{prt}(v)} \in X_v \times X_{\text{prt}(v)}\), \(\delta_{v|\text{prt}(v)}(x_v|x_{\text{prt}(v)})\) is a Dirac measure. It is well known that there exists an optimal solution to MEU \(\triangleright\) that is deterministic (see e.g., [Liu, 2014, Lemma C.1] for a proof).

We conclude this section with some classical examples of IDs, shown in Figure 1.

**Example 1.** Consider a maintenance problem where at time \(t\) a machine is in state \(s_t\). The action \(a_t\) taken by the decision maker according to the current state is typically maintaining it (which is costly) or not (which increase the probability of failure). State and decision yields a new (random) state \(s_{t+1}\), and the triple \((s_t, a_t, s_{t+1})\) induce a reward \(r_t\). This is an example of Markov decision processes (MDP) which are probably the simplest IDs, represented in Figure 1A.

In practice, the actual state \(s_t\) of the machine is not known, but we can only have some observation \(o_t\), which leads to a more complex ID known as Partially observed Markov decision processes (POMDP). In theory an optimal decision should be taken knowing all past observations and decisions (which is the perfect recall case). However, this would leads to untractable decision strategies which requires long memory. It is usual to restrict the decision \(a_t\) to be taken only with respect to observation \(o_t\), as illustrated in Figure 1B. \(\triangleright\)

**Example 2.** Consider two chess players : Bob and Alice. They are used to play chess and for each game they bet a symbolic coin. However, they can refuse to play. Suppose that Alice wants to play chess every day. At each time step, she has a current confidence level \(s_t\). The day of the game, her current mental fitness is denoted \(o_t\). When Bob meets Alice, he takes the decision to play depending on her attitude and her appearance of the day, denoted \(u_t\). Then Bob can accept or decline the challenge, and his decision is denoted \(a_t\). Let \(v_t\) denote the winner (getting a reward \(r_t\)). Then, Alice’s next confidence level is affected by the result of the game and her previous confidence level. This stochastic decision problem can be modeled by an influence diagram as shown in Figure 2. \(\triangleright\)

1.2 Solving MDP through linear programs

We recall here a well known linear programming formulation for MDP (see e.g., [Puterman, 2014]), which is a special case of the MILP formulation introduced in the paper. Indeed,
for \( t \in [T] := \{1, \ldots, T\} \), let \( \mu^t_s \) represent the probability of being in state \( s \) at time \( t \), let \( \mu^t_{sa} \) represent the probability of being in state \( s \) and taking action \( a \) at time \( t \), and let \( \mu^t_{sas'} \) represent the probability of being in state \( s \) and taking action \( a \) at time \( t \), while transiting to state \( s' \) at time \( t + 1 \). This leads to the following linear program.

\[
\begin{align*}
\max_\mu & \quad \sum_{t=1}^{T-1} \sum_{s,a,s'} \mu^t_{sas'} r(s, a, s') \\
\text{s.t.} & \quad \mu^t_{sas'} = p(s'|s, a) \mu^t_s, \quad \forall t, s, s' \quad (4b) \\
& \quad \mu^0_s = 1 \quad (4c) \\
& \quad \sum_{s,a} \mu^t_{sas'} = \mu^{t+1}_s, \quad \forall t, s' \quad (4d) \\
& \quad \sum_s \mu^t_s = 1, \quad \forall t \quad (4e) \\
& \quad \mu^t_s, \mu^t_{sa}, \mu^t_{sas'} \geq 0, \quad \forall t, a, s, s' \quad (4f)
\end{align*}
\]

where the objective \((4a)\) is simply the expected reward, Constraint \((4b)\) represent the state dynamics, Constraint \((4c)\) fix the initial state of the system, and Constraints \((4d)-(4f)\) ensure that \( \mu \) represent marginals laws. Recall that an MDP always admits a deterministic optimal solution, that is that, knowing current state \( s_t \) at time \( t \) the optimal action \( a_t \) is chosen deterministically. Consequently, in this linear program, there exists an optimal solution such that, for all state \( s \), action \( a \) and time \( t \), \( \mu^t_{sa} \in \{0, 1\} \). Adding this integrity constraints is useless.
here, but this idea is used in Section 4 to ensure that McCormick’s relaxation of non-linear constraints are tight.

1.3 Literature

Influence diagrams were introduced by [Howard and Matheson 1984] (see also [Howard and Matheson 2005]) to model stochastic optimization problems using a probabilistic graphical model framework. Originally, the decision makers were assumed to have perfect recall [Shenoy, 1992; Shachter, 1986; Jensen et al., 1994]. Lauritzen and Nilsson [2001] relaxed this assumption and provided a simple (coordinate descent) algorithm to find a good policy: the Single Policy Update (SPU) algorithm. The same authors also introduced the notion of soluble ID as a sufficient condition for SPU to converge to an optimal solution. This notion has been generalized by Koller and Milch [2003] to obtain a necessary and sufficient condition. SPU finds a locally optimal policy, but performs exact inference, and is therefore limited by the treewidth.

More recently, Mauá and Campos [2011] and Mauá and Cozman [2016] have introduced a new algorithm, Multiple Policy Update, which has both an exact and a heuristic version and relies on dominance to discard partial solutions. It can be interpreted as a generalization of SPU where several decisions are considered simultaneously. Later on Khaled et al. [2013] proposed a similar approach, with a Branch-and-Bound flavor, while Liu [2014] has introduced heuristics based on approximate variational inference. Finally, Mauá [2016] have recently shown that the problem of solving an ID can be polynomially transformed into a maximum a posteriori (MAP) problem, and hence can be solved using popular MAP solvers such as toulbar2 [Hurley et al., 2016].

Finding an optimal policy for an ID has been shown to be NP-hard even when restricted to ID of treewidth non-greater than two, or to trees with binary variables Mauá et al. [2012a, 2013]. Note that even obtaining an approximate solution is also NP-hard Mauá et al. [2012a].

Credal networks are generalizations of probabilistic graphical model where the parameters of the model are not known exactly. MILP formulations for credal networks that could be applied to IDs have been introduced by de Campos and Cozman [2007], de Campos and Ji [2012]. However, the number of variables they require is exponential in the pathwidth, which is non-smaller and can be arbitrarily larger than the width of the tree we are using (follows from Schefter, 1990, Theorem 4), and the linear relaxation of their MILP is not as good as the one of the MILP we propose, and does not give an integer solution on soluble IDs. Our technique on RJT can naturally be extended to credal networks.

1.4 Contributions

In Section 2, we recall some definitions for graphical models, that are used to extend the notion of junction tree to rooted junction tree in Section 3. With these tools, Section 4 gives a bilinear formulation that can be rewritten as a mixed integer linear programming (MILP) formulation to the MEU Problem (3). In Section 5 we give efficient valid independance inequalities for the MILP formulation, and gives interpretation in terms of graph relaxations. Section 6 studies the polynomial case of soluble ID, showing that the ID that can be solved to optimality through SPU can be solved by (continuous) linear programming by our formulation. Finally Section 7 synthesizes our numerical experiments.

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1 They used the name limited memory influence diagrams when relaxing the perfect recall assumption, but we follow the convention of Koller and Friedman [2009] who use the “ID” terminology.
2 Tools from Probabilistic graphical model theory

In this section we present notation and tools used in the following sections to solve the MEU Problem.

2.1 Graph notation

This sections introduces our notations for graphs, which mostly follows terminology commonly used in the combinatorial optimization community [Schrijver, 2003]. A directed graph \( G \) is a pair \((V, E)\) where \( V \) is the set of vertices and \( E \subseteq V^2 \) the set of arcs. We write \( u \rightarrow v \) when \((u, v) \in A\). A path is a sequence of vertices \( v_1, \ldots, v_k \) such that \( v_i \rightarrow v_{i+1} \) for any \( i \in [k] \). A path between two vertices \( u \) and \( v \) is called a \( u\rightarrow v \) path. We write \( u \leftrightarrow v \) to denote the existence of a \( u\rightarrow v \) path in \( G \), or simply \( u \rightarrow v \) when \( G \) is clear from context. We write \( u \equiv v \) if there is an arc \( u \rightarrow v \) or \( v \rightarrow u \). A trail is a sequence of vertices \( v_1, \ldots, v_k \) such that \( v_i \equiv v_{i+1} \).

A parent (resp. child) of a vertex \( v \) is a vertex \( u \) such that \( (u, v) \) (resp. \((u, u)\)) belongs to \( E\); we denote by \( \text{prt}(v) \) the set of parents vertices (resp. \( \text{cl}(v) \) the set of children vertices). A vertex \( u \) is a coparent of a vertex \( v \) if \( v \) and \( u \) have a child in common, and write \( \text{cpt}(v) \) for this set. The family of \( v \), denoted by \( \text{fa}(v) \), is the set \( \{v\} \cup \text{prt}(v) \). A vertex \( u \) is an ascendant (resp. a descendant) of \( v \) if there exists a \( u\rightarrow v \) path. We denote respectively \( \text{asc}(v) \) and \( \text{dsc}(v) \) the set of ascendants and descendants of \( v \). Finally, let \( \overrightarrow{\text{asc}}(v) = \{v\} \cup \text{asc}(v) \), and \( \overleftarrow{\text{dsc}}(v) = \{v\} \cup \text{dsc}(v) \).

A set of vertices \( C \), the parent set of \( C \), again denoted by \( \text{prt}(C) \), is the set of vertices \( u \) that are parents of a vertex \( v \in C \). We define similarly \( \text{fa}(C) \), \( \text{cl}(C) \), \( \text{asc}(C) \), and \( \text{dsc}(C) \). Note that we sometimes indicate in subscript the graph according to which the parents, children, etc., are taken. For instance, \( \text{prt}_G(v) \) denotes the parents of \( v \) in \( G \). We drop the subscript when the graph is clear from the context.

A cycle is a path \( v_1, \ldots, v_k \) such that \( v_1 = v_k \). A graph is connected if there exists a path between any pair of vertices. An undirected graph is a tree if it is connected and has no cycles. A directed graph is a directed tree if its underlying undirected graph is a tree. A rooted tree is a directed tree such that all vertices have a common ascendant referred to as the root of the tree.

In a rooted tree, all vertices but the root have exactly one parent.

2.2 Directed graphical model

In this paper, we manipulate several distributions on the same random variables. Given three random variables \( X, Y, Z \), the notation \( (X \perp Y \mid Z)_\mu \) stands for “\( X \) is independent from \( Y \) given \( Z \) according to \( \mu \)”. The parenthesis \( (\cdot)_\mu \) are dropped when \( \mu \) is clear from context. The same notation is used for independence of events.

A well-known sufficient condition for a distribution to factorize as a probabilistic graphical model is that each vertex is independent from its non-descendants given its parents.

**Proposition 1.** [Koller and Friedman, 2009, Theorem 3.1, p. 62] Let \( \mu \) be a distribution on \( \mathcal{X}_V \). Then \( \mu \) factorizes as a directed graphical model on \( G \), that is

\[
\mathbb{P}_\mu(X_V = x_v) = \prod_{v \in V} \mathbb{P}_\mu(X_v = x_v \mid X_{\text{prt}(v)} = x_{\text{prt}(v)}),
\]

if and only if

\[
\left( X_v \perp \overrightarrow{X}_{\text{dsc}(v)} \mid X_{\text{prt}(v)} \right)_\mu \quad \text{for all } v \in V.
\]

The probabilistic graphical model community sometimes calls a directed tree what we call here a rooted tree, and a polytree what we call here a directed tree.
2.3 Cluster graph and junction trees

When dealing with the MEU Problem 3, one needs to deal with distributions $\mu_V$ on $\mathcal{X}_V$ that factorize as in [2] for some policy $\delta$. In theory, it suffices to consider distributions $\mu$ satisfying the conditional independences given by Equation (4) and such that $\mathbb{P}_\mu(X_v|X_{\text{net}(v)}) = p_{\text{net}(v)}$ for each vertex $v$ that is not a decision. However, the joint distribution $\mu_V$ on all the variables is too large to be manipulated in practice as soon as $V$ is moderately large. In that case, it is handy to work with a vector of moments $\tau = (\tau_C)_{C \subseteq V}$, where $V \subseteq 2^V$, that is, a vector of distributions $\tau_C$ on subsets of variables $C$ of tractable size. A vector of moment $(\tau_C)_{C \subseteq V}$ derives from a distribution $\mu_V$ on $\mathcal{X}_V$ if each moment is the marginal of $\mu_V$, i.e., $\sum_{x_C \subseteq C} \mu_V(x_C, x_{V \setminus C}) = \tau_C(x_C)$ for all $C \subseteq V$ and $x_C$ in $\mathcal{X}_C$. We use the notation $\mu$ for vector of moments deriving from a distribution, and $\mathbb{P}_\mu$ for the corresponding distribution.

A necessary condition for a vector of moments $(\tau_C)_{C \subseteq V}$ to derive from a distribution is to be \textit{locally consistent}, that is to have matching marginals on intersections of all $C \subseteq V$. More precisely, a vector of moments $(\tau_C)_{C \subseteq V}$ is locally consistent if it belongs to the \textit{local polytope} $\mathcal{L}^0_\mathcal{G}$ defined by

$$\mathcal{L}^0_\mathcal{G} = \left\{ (\tau_C)_{C \subseteq V} : \begin{array}{l}
\tau_C \geq 0 \text{ and } \sum_{x_C} \tau_C(x_C) = 1 \text{ for all } C \in \mathcal{V}, \text{ and } \\
\sum_{x_{C_1 \setminus C_2}} \tau_{C_1} = \sum_{x_{C_2 \setminus C_1}} \tau_{C_2}, \quad \forall (C_1, C_2) \in \mathcal{V}^2, \quad \forall x_{C_1 \cap C_2} \in \mathcal{X}_{C_1 \cap C_2}
\end{array} \right\}$$

where $\sum_{x_{C_1 \setminus C_2}} \tau_{C_1}$ is a useful notation for $\sum_{x_{C_1 \setminus C_2}} \tau_{C_1}(x_{C_1 \setminus C_2}, x_{C_1 \cap C_2})$.

If $(\tau_C)_{C \subseteq V}$ derives from a distribution, then $\tau \in \mathcal{L}^0_\mathcal{G}$. Graphical model theory [Wainwright and Jordan, 2008, Proposition 2.1] ensures that this condition is sufficient if $\mathcal{V}$ is the set of nodes of a \textit{junction tree}, a notion we now recall.

A \textit{cluster graph} $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ over a graph $G = (V, E)$ is an undirected graph such that $\mathcal{V} \subseteq 2^V$, and there is a mapping $v \mapsto C_v$ from $V^s$ to $\mathcal{V}$ such that $\text{fa}(v) \subseteq C_v$. A \textit{junction tree} is a cluster graph $\mathcal{G}$ that is a tree, and satisfies the \textit{running intersection property}: given two vertices $C_1$ and $C_2$ in $\mathcal{G}$, their intersection $C_1 \cap C_2$ is a subset of each vertices on the (unique) undirected path from $C_1$ to $C_2$. See Figure 3 for an illustration of this notion.

3 Rooted junction trees

To solve the MEU Problem (3), we work on vectors of moments $(\tau_C)_{C \subseteq V}$ that correspond to the moments of distributions $\mu_C$ induced by policies $\delta \in \Delta$. Hence, we are interested in vectors of moments that factorize as a directed graphical model on $G$. Such vectors of moments necessarily satisfy the following “local” version of the sufficient condition (5).

$$X_v \independent X_{C \setminus \text{dec}(v)} | X_{\text{net}(v)} \text{ for all } C \subseteq \mathcal{V}, \text{ for all } v \in V : \text{fa}(v) \subseteq C.$$

Given a vector of moment $\tau_C$ in the local polytope of a junction tree $(\mathcal{V}, \mathcal{A})$, satisfying (7) is not a sufficient condition for $\tau_C$ to be the moments of a distribution $\mu_V$ that factorizes on $G$.

But it becomes a sufficient condition under the additional assumption that $(\mathcal{V}, \mathcal{A})$ is a “rooted junction tree”, a notion that we introduce in this section.

3.1 Definition and main properties

Let $\mathcal{G} = (V, E)$ be a junction tree on $G = (\mathcal{V}, \mathcal{A})$ and $v \in V$ a vertex of $G$. Then, thanks to the running intersection property, the subgraph $\mathcal{G}_v$ of $\mathcal{G}$ made of all nodes $C \subseteq \mathcal{V}$ containing $v$ is a
Figure 3: a) A directed graph $G$, b) a junction tree on $G$, and c) a rooted junction tree on $G$, where, for each cluster $C$, we indicate on the left part of the labels the vertices of $C \setminus \hat{C}$, and on the right part the vertices of $\hat{C}$.

Moreover, any orientation of the edges of $\mathcal{G}$ that makes it a rooted tree, also makes $\mathcal{G}_v$ a rooted tree, and we denote $C_v$ its root node.

**Definition 1.** A rooted junction tree (RJT) on $G = (V, E)$ is a rooted tree with nodes in $2^V$, such that

(i) its underlying undirected graph $\mathcal{G} = (V, A)$ is a junction tree,

(ii) for all $v \in V$, we have $fa(v) \subseteq C_v$,

where $C_v$ is the root clique of $v$ defined as the root node of the subgraph $\mathcal{G}_v$ of $\mathcal{G}$ induced by the nodes $C \in V$ containing $v$.

See Figure 3 for a graphical example of this notion. Note that an RJT always exists. Indeed, the cluster graph composed of a single vertex $C = V$ is an RJT. Algorithms to build interesting RJTs are provided in Section 3.2.

Let $\mathcal{G}$ be an RJT on $G$, and $v$ a vertex of $V$. Given $C \in \mathcal{G}$, let $\hat{C} = \{v \in V : C_v = C\}$ be the offspring of $C$. Here are some properties of RJTs.

**Proposition 2.** Let $\mathcal{G}$ be an RJT on $G$.

1. If there is a path from $u$ to $v$ in $G$, then there is a path from $C_u$ to $C_v$ in $\mathcal{G}$.

2. If $\overline{dsc}_G(u) \cap \overline{dsc}_G(v) \neq \emptyset$, then either there is a unique path from $C_u$ to $C_v$ or from $C_v$ to $C_u$ in $\mathcal{G}$.

**Proof.** Let $\mathcal{G}$ be an RJT on $G$. Consider a vertex $v$ of $G$ and a node $C$ of $\mathcal{G}$ containing $v$. By definition $C$ is a node of $\mathcal{G}_v$, and by definition of $C_v$ there exists a $C_v$-C path in $\mathcal{G}$. Now consider $u \in \text{prt}(v)$, as $fa(v) \subseteq C_v$, we have $u \in C_v$. Thus there exists a $C_u$-$C_v$ path in $\mathcal{G}$. The first statement follows by induction along a $u$-$v$ path in $G$.

We now show the second statement. Let $w$ be a vertex in $\overline{dsc}_G(u) \cap \overline{dsc}_G(v)$, then by the first statement there exists both a $C_u$-$C_w$ and a $C_v$-$C_w$ path in $\mathcal{G}$. As $\mathcal{G}$ is a rooted tree, this implies the existence of either a $C_u$-$C_v$ path or of a $C_v$-$C_u$ path in $\mathcal{G}$.

**Theorem 3.** Ensures that given a vector of moments on an RJT satisfying local independences, we can construct a distribution on the initial directed graphical model.
Theorem 3. Let $\mu$ be a vector of moments in the local polytope of an RJT $\mathcal{G}$ on $G = (V, E)$. Suppose that for each vertex $v$, according to $\mu_{C_v}$, the variable $X_v$ is independent from its non-descendants in $G$ that are in $C_v$, conditionally to its parents. Then there exists a distribution $P_\mu$ on $X_V$ factorizing on $G$ with moments $\mu$.

Theorem 3 is a natural generalization of the well-known Proposition 1. Its proof relies on the following lemma.

Lemma 4. Let $C, D$ be subsets of $V$ such that $\text{fa}(D) \subseteq C$ and $\text{dsc}(D) \cap C = D$. Any distribution $\mu_C$ on $C$ such that each $v$ in $D$ is independent from its non-descendants given its parents factorizes as

$$
\mu_C = \mu_{C \setminus D} \prod_{v \in D} P_\mu(X_v | X_u, u \in C, u \prec v) = \mu_{C \setminus D} \prod_{v \in D} P_\mu(X_v | X_{\text{prt}(v)}),
$$

where the first equality is the chain rule and the second follows from the hypothesis of the lemma.

Proof. Let $\preceq$ be a topological order on $C$ such that $u \in C \setminus D$ and $v \in D$ implies $u \preceq v$. Such a topological order exists as $\text{dsc}(D) \cap C = D$. We have

$$
\mu_C = \mu_{C \setminus D} \prod_{v \in D} P_\mu(X_v | X_u, u \in C, u \prec v) = \mu_{C \setminus D} \prod_{v \in D} P_\mu(X_v | X_{\text{prt}(v)}),
$$

where the first equality is the chain rule and the second follows from the hypothesis of the lemma.

Proof of Theorem 3. Let $\mathcal{G}$ be an RJT on $G$. Let $C_1, \ldots, C_n$ be a topological ordering on $\mathcal{G}$, and $C_{\leq i} = \bigcup_{j \leq i} C_j$. Let $\tau$ be a vector of moments satisfying the hypothesis of the theorem, and for each $v$ in $V$, let $q_v(\text{prt}(v)) = \frac{\sum_{C \in \text{fa}(v)} \mu_C}{\sum_{C \in \text{prt}(v)} \mu_C} \tau_C$. We show by induction on $i$ that $\mu_{C_{\leq i}} = \prod_{v \in C_{\leq i}} q_v(\text{prt}(v))$ is such that $\tau_{C_i} = \sum_{C \subseteq C_{\leq i} \setminus C_i} \mu_C$. Suppose the result true for all $j < i$, with the convention that $\mu_0 = 1$. We only have to prove $\tau_{C_i} = \sum_{C \subseteq C_{\leq i}} \mu_C$, where $C_{\leq i} = C_{\leq i} \setminus C_i$. By definition of an RJT, we have $\text{fa}(C_i) \subseteq C_i$. Proposition 1 implies that $\text{dsc}(C_i) \cap C_i \subseteq C_i$. Indeed let $u$ be in $\text{dsc}(C_i) \cap C_i$. Then there is a $C_i$-$C_u$ path as $u \in \text{dsc}(C_i)$, and a $C_u$-$C_i$ path as $u \in C_i$. Hence $C_u = C_i$ and $u \in C_i$. By Lemma 4 we have $\tau_{C_i} = \tau_{C_i} \prod_{v \in C_i} q_v(\text{prt}(v))$. Let $C_j$ be the parent of $C_i$ in $\mathcal{G}$, we have $\mu_{C \setminus C_i} = \sum_{C \subseteq C_{\leq i} \setminus C_i} \tau_{C_j} \mu_{C_i} = \sum_{C \subseteq C_{\leq i} \setminus C_i} \tau_{C_j} \mu_{C_i}$, the first equality coming from the fact that $(\tau_{C_j})_{C \subseteq V}$ belongs to the local polytope, and the second from the induction hypothesis. Thus,

$$
\sum_{C \subseteq C_{\leq i}} \mu_{C \subseteq i} = \sum_{C \subseteq C_{\leq i}} \prod_{v \in V_{\leq i}} q_v(\text{prt}(v)) = \left( \sum_{C \subseteq C_{\leq i} \setminus C_i} \mu_{C \subseteq i} \right) \prod_{v \in C_i} q_v(\text{prt}(v)) = \tau_{C_i} \prod_{v \in C_i} q_v(\text{prt}(v)) = \tau_{C_i},
$$

which gives the induction hypothesis, and the theorem.

Remark 1. By adding nodes to an RJT, we can always turn it into an RJT satisfying $\hat{C}_v = \{v\}$ for each vertex $v$ in $V$. Indeed, suppose that $C = \{v_1, \ldots, v_k\}$, where $v_1, \ldots, v_k$ are given along a topological order. It suffices to replace the node $C$ by $C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_k$, where $C_i = C \setminus \{v_{i+1}, \ldots, v_k\}$.

Remark 2. Jensen et al. [1994, beginning of Section 4] introduces a similar notion of strong junction tree. It relies on the notion of elimination ordering for a given influence diagram with perfect recall. The main difference is that a strong junction tree is a notion on an influence diagram, where the set of decision vertices and their orders play a role, when RJTs rely on
the underlying digraph. The notion of strong junction tree is obtained by replacing (ii) in the
definition of an RJT by: “given an elimination ordering if \((C_u, C_v)\) is an arc, there exists an
ordering of \(C_v\) that respects the elimination ordering such that \(C_u \cap C_v\) is before \(C_v \setminus C_u\)
in that ordering”. An RJT is a strong junction tree. Conversely, a strong junction tree is not neces-sarly
an RJT. Indeed, Jensen et al. [1994, Figure 4] shows an example of strong junction where there
is \(v \in V\) such that \(\text{fa}(v) \subset C_v\). As strong junction trees is a notion on influence
diagram and not on graphs, Theorem 3 is not naturally generalized to strong junction trees.

△

Remark 3. The reader familiar with probabilistic graphical model theory will note that, as we
are considering local polytopes only for junction trees, the above defined local polytope is equal
to the marginal polytope [Wainwright and Jordan, 2008], that is, the set

\[ \{(\tau_C)_{C \in V} : \text{there exists a distribution } \mu \text{ on } X_V \text{ satisfying } \sum_{x_{V \setminus C}} \mu = \tau_C \text{ for all } C \} \].

\[ \triangle \]

3.2 Building an RJT

Although \(\{V, \emptyset\}\) is a rooted junction tree, the concept has only practical interest if it is possible
to construct RJTs with small cluster nodes. In that respect, note that any RJT must satisfy,
for all \(u, v \in V\), the implication

\[ \exists w \in V \text{ s.t. } C_v \rightarrow C_w \text{ and } u \in \text{fa}(w) \Rightarrow C_u \rightarrow C_v \]  (10)

where \(C \rightarrow C'\) denotes the existence of a \(C-C'\) path in the RJT \(G\) considered. This notation
is used in the remaining of the section. Indeed, since \(u \in C_u\) and \(\text{fa}(w) \subset C_w\) by definition,
and since \(C_u \rightarrow C_v \rightarrow C_w\), the running intersection property implies \(u \in C_v\). This motivates
Algorithm 1 a simple RJT construction algorithm which propagates iteratively elements present
in each cluster node to their parent cluster node. Let \(\preceq\) be an arbitrary topological order on
\(G\), and \(\max \preceq C\) denote the maximum of \(C\) for the topological order \(\preceq\). As we will show,
Algorithm 1 produces an RJT \(G = (V, A)\) which is minimal for \(\preceq\), in the sense that it satisfies
a converse of (10). We recall that \(\hat{C}_v\) is the set \(C_v \setminus \{v\}\).

Algorithm 1 Create an RJT given a topological order

1: Input \(G = (V, E)\) and a topological order \(\preceq\) on \(G\)
2: Initialize \(C_v = \emptyset\) for all \(v \in V\) and \(A' = \emptyset\)
3: for each node \(v\) of \(V\) taken in reverse topological order \(\preceq\) do
4: \(C_v \leftarrow \text{fa}(v) \cup \bigcup_{(u,v) \in A'} C_u \setminus \{u\}\)
5: if \(\hat{C}_v \neq \emptyset\) then
6: \(u \leftarrow \max \preceq (\hat{C}_v)\)
7: \(A' \leftarrow A' \cup (u, v)\)
8: end if
9: end for
10: \(A \leftarrow \{(C_u, C_v) \mid (u, v) \in A'\}\)
11: Return \(G = (\{C_v\}_{v \in V}, A)\)

Remark 4. Algorithm 1 takes in input a topological order on \(G\). For a practical use, we
recommend to use Algorithm 3 in Appendix 3 which builds simultaneously the RJT and a
“good” topological order.
Indeed, if \( u \in C_v \) then either Step 3 of the algorithm ensures that \( u \in fa(v) \) and \( u \preceq v \) or \( u \notin fa(v) \) and there exists \( x \) such that \( u \in C_x \) and \( C_x \rightarrow C_v \). But by Step 6 of Algorithm 1 the fact that \( C_x \rightarrow C_v \) entails that \( v \) is the maximal element of \( C_x \setminus \{x\} \) for the topological order, so that \( u \prec v \). Furthermore, Step 4 ensures that \( (C_u, C_v) \in \mathcal{A} \) implies \( u \in C_v \). We deduce from the previous result that \( (C_u, C_v) \in \mathcal{A} \) implies \( u \preceq v \), and \( \preceq \) is a topological order on \( G \).

Then we show that (11) holds. We first show that \( u \in C_v \) \( \Rightarrow \) \( C_u \rightarrow C_v \) and \( u \in C' \) for any \( C' \) on path \( C_u \rightarrow C_v \). Either \( u = v \) and this is obvious, or \( u \in \text{prt}_G(C_v') \) and by recursion \( C_u \rightarrow C_v \) or \( u \in C_r \) with \( C_r \) the root of the tree which is also the first element in the topological order. But, unless \( u = r \), this is excluded given that \( u \in C_r \) implies \( u \preceq r \). Note that this shows that \( C_u \) is indeed the unique minimal element of the set \( \{C: u \in C\} \) for the partial order defined by the arcs of the tree. To show the first part of (11), we just need to note that either \( u \in fa(v) \) and the result holds, or there must exist \( x \) such that \( C_v \rightarrow C_x \) and \( u \in C_x \) and by recursion, there exists \( w \) such that \( C_u \rightarrow C_w \) and \( u \in fa(w) \).

Finally, we prove that we have constructed an RJT. Indeed, if two vertices \( C_v \) and \( C_{v'} \) contain \( u \) then since \( G \) is singly connected, the trail connecting \( C_v \) and \( C_{v'} \) must be composed of vertices on the paths \( C_v \rightarrow C_u \) and \( C_u \rightarrow C_{v'} \), and we have shown in the previous paragraph that \( u \) belongs to any \( C' \) on \( C_v \rightarrow C_u \) and \( C_u \rightarrow C_{v'} \), and so the running intersection property holds. Finally, property (ii) of Definition 1 must hold because the fact that \( C_u \) is minimal among all cluster vertices containing \( u \) together with the running intersection property entails that the cluster vertices containing \( u \) are indeed a subtree of \( G \) with root \( C_u \).

The previous result provides a justification for Algorithm 1, but it characterizes the content of the cluster vertices based on the topology of the obtained RJT, which is itself produced by the algorithm (note that the composition of cluster vertices depends only on \( \leq \) via the partial order of the tree). The cluster nodes of any RJT and those produced by Algorithm 1 admit however more technical characterizations using only \( \leq \) and the information in \( G \), which we present next. These characterizations will be useful in Section 6. For each vertex \( v \) in \( V \), let

\[
T_{\geq v} = \{w \in V_{\geq v} \mid \text{there is a v-w trail in } V_{\geq v}\}.
\]
Figure 5: Example of RJT produced by Algorithm 1 on a graphical model representing a POMDP.

**Proposition 6.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be an RJT satisfying $\hat{C}_v = \{v\}$, and $\preceq$ be a topological order on $\mathcal{G}$. Then $\preceq$ induces a topological order on $\mathcal{G}$ and

$$w \in T_{\mathcal{G}}(v) \implies C_v \rightarrow C_w,$$

(12a)

$$\text{cld}(u) \cap \{v \in T_{\mathcal{G}}(v) : u \preceq v\} = \emptyset \implies u \in C_v.$$  

(12b)

**Proof of Proposition 6.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be an RJT satisfying $\hat{C}_v = \{v\}$, and $\preceq$ be a topological order on $\mathcal{G}$. Property 1 in Proposition 2 ensures that $\preceq$ induces a topological order on $\mathcal{G}$.

We start by proving (12a). Let $v$ and $w$ be vertices such that $w \preceq v$ and that there is a $v$-$w$ trail $Q$ in $T_{\mathcal{G}}(v)$. Let $s_0, \ldots, s_k$ be the nodes where $Q$ has a $v$-structure and $t_1, \ldots, t_k$ the nodes with diverging arcs in $Q$. Note that, since the trail is included in $T_{\mathcal{G}}(v)$, the first nodes of the trail have to be immediate descendants of $v$ in $\mathcal{G}$ so that the trail takes the form $v \rightarrow s_0 \rightarrow t_1 \rightarrow s_1 \ldots t_k \rightarrow s_k \leftarrow w$, where possibly $s_k = w$ and the last arc is not present. Then, given that $v \prec s_0$, and that $\preceq$ is topological for $\mathcal{G}$, Proposition 2 implies that $C_v \rightarrow C_{s_0}$. But by the same argument, property Item 2 in Proposition 2 implies $C_{t_1} \rightarrow C_{s_0}$, since $\mathcal{G}$ is a tree and $v \prec t_1$, we must have $C_v \rightarrow C_{t_1} \rightarrow C_{s_1}$. By induction on $i$, we have $C_v \rightarrow C_{s_i}$ and thus $C_v \rightarrow C_w$, which shows Equation (12a).

We now prove (12b). Let $u$ and $v$ be two vertices such that $u \preceq v$ and there is a $u$-$v$ trail $P$ with $P \setminus \{u\} \subseteq T_{\mathcal{G}}(v)$. Let $w$ be the vertex right after $u$ on $P$. We have $u \in \text{fa}(w)$, and there is a $v$-$w$ trail in $T_{\mathcal{G}}(v)$, which implies $C_v \rightarrow C_w$ by (12a). But, since $u \preceq v$, the $u$-$v$ trail is also in $T_{\mathcal{G}}(u)$, which similarly shows that $C_u \rightarrow C_v$. So by (10) we have proved (12b).

**Proposition 7.** The graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ produced by Algorithm 1 is the unique RJT satisfying $\hat{C}_v = \{v\}$ such that the topological order $\preceq$ on $\mathcal{G}$ taken as input of Algorithm 1 induces a topological order on $\mathcal{G}$ and the implications in (12) are equivalences.

**Proof of Proposition 7.** It is sufficient to prove the following inclusions

$$\overline{\text{dsc}}_{\mathcal{G}}(C_v) \subseteq \{C_w : w \in T_{\mathcal{G}}(v)\},$$

(13a)

$$C_v \subseteq \{u \preceq v : \exists w \in T_{\mathcal{G}}(v), u \in \text{fa}(w)\}.$$  

(13b)
Indeed, note that by Proposition 5, the obtained tree is an RJT so that, by Proposition 6, the reverse inequalities hold.

We prove the result by backward induction on (13b) and (13a). For a leaf \( C_v \) of \( G \), \( dsc_G(C_v) = \{ C_v \} \) so that (13a) holds trivially and \( C_v = \text{fa}(v) \) so that (13b) holds because \( u \in \text{fa}(v) \) implies \( u \preceq v \). Then, assume the induction hypothesis holds for all children of a node \( C_v \).

We first show (13a) for \( C_v \), i.e. that \( (C_v \xrightarrow{\text{ } } C_w) \Rightarrow (w \in T_{x} \supseteq v) \) (see Figure 6). Let \( C_x \) be the child of \( C_v \) on the path \( C_v \xrightarrow{\text{ } } C_w \). By Proposition 5, we have \( v \prec x \), so that \( V_{x} \supseteq v \subset V_{x} \supseteq v \). Then, using the induction hypothesis, by (13b), \( (v \in C_x) \) implies that there is a \( v \)-\( x \) trail in \( V_{x} \supseteq v \), and by (13a), \( C_x \xrightarrow{\text{ } } C_w \) implies there is a trail \( x \rightarrow w \) in \( V_{x} \supseteq v \), so there is a \( v \)-\( w \) trail in \( v \prec x \) in \( V_{x} \supseteq v \), which shows the result.

We then show (13b) for \( C_v \) (see Figure 6). Indeed if \( u \in C_v \), either \( u \in \text{fa}(v) \) and \( u \) is in the RHS of (13b), or there exists a child of \( C_v \), say \( C_x \) such that \( u \in C_x \) and \( u \prec v \), because the algorithm imposes \( v = \max_{x}(C_x \setminus \{x\}) \). Since \( C_v \xrightarrow{\text{ } } C_x \) there exists a path \( v \rightarrow x \) in \( V_{x} \supseteq v \), and using induction, by (13b), \( (u \in C_x) \) implies that \( \exists w \) such that \( u \in \text{fa}(w) \) and there exists a trail \( w \rightarrow x \) in \( T_{x} \supseteq v \). But we have shown in Proposition 3 that \( (v \in C_x) \Rightarrow (v \preceq x) \), so \( T_{x} \supseteq v \supseteq v \) and we have shown that there exists a \( v \)-\( w \) trail in \( T_{x} \supseteq v \) with \( u \preceq v \) and \( u \in \text{fa}(w) \), which shows the result.

4 MILP formulation for influence diagrams

Given that Algorithm 1 produces an RJT such that \( \hat{C}_v = \{ v \} \) for all \( v \in V \), we will assume in the rest of the paper that all the RJT considered satisfy this property. As noted in Remark 1, any RJT can be transformed in an RJT satisfying this property by adding more nodes. We denote \( \hat{C}_v \) the set \( C_v \setminus \hat{C}_v = C_v \setminus \{ v \} \). In the rest of the paper, we work with the following variant of the local polytope \( \mathcal{L}_G \) defined in Equation (6)

\[
\mathcal{L}_G = \left\{ (\mu_{C_v}, \mu_{\hat{C}_v})_{v \in V} : (\mu_{C_v})_{v \in V} \in \mathcal{L}_G^0 \text{ and } \mu_{\hat{C}_v} = \sum_{x \in \hat{C}_v} \mu_{C_x} \right\},
\]

where moments \( \mu_{\hat{C}_v} \) have been introduced. This is for convenience, and all the results could have been written using \( \mathcal{L}_G^0 \).

On graphical models, the inference problem, which is hard in general, becomes easy on junction trees. Since problem (3) is NP-hard even when restricted to graphs of treewidth 2 [Maná et al., 2012], unless \( P = NP \), the situation is strictly worse for the MEU problem associated with influence diagrams. However, we will see in this section that, given a rooted junction tree, we can obtain mathematical programs to solve the MEU problem with a tractable number of variables an constraints provided that cliques are of reasonable size. We first obtain an
NLP formulation in Section 4.1 and then linearize it into an exact mixed integer linear program (MILP) in Section 4.2.

4.1 An exact Non Linear Progam formulation

Consider a Parameterized Influence Diagram (PID) encoded as the quadruple \((G, \mathcal{X}, p, r)\), where \(G = (V, E)\) is a graph with set of vertices \(V\) partitioned into \((V^a, V^s)\), with \(\mathcal{X} = \prod_{v \in V} X_v\) is the support of the vector of random variables attached to all nodes of \(G\), \(p = \{p_{v|\text{prt}(v)}\}_{v \in V^s}\) is the collection of fixed and assumed known conditional probabilities, and \(r = \{r_v\}_{v \in V^a}\) is the collection of reward functions.

For \((G, \mathcal{X}, p, r)\) a given PID, and \(G\) a given RJT, we introduce the following polytope

\[
\overline{\mathcal{P}}(G, \mathcal{X}, p, G) = \{ \mu \in \mathcal{L}_G : \mu_{C_v} = \mu_{C_v} p_{v|\text{prt}(v)} \mu_{C_v} \text{ for all } v \in V^a \}.
\] (14)

Moments \(\mu_{C_v}\) are introduced only for notational ease and not listed in the local polytope \(\mathcal{L}_G\) or in \(\overline{\mathcal{P}}(G, \mathcal{X}, p, G)\). We omit the dependence of \(\overline{\mathcal{P}}\) in \((G, \mathcal{X}, p, G)\) when the context is clear.

Consider the following Non Linear Program (NLP)

\[
\begin{align*}
\max_{\mu, \delta} & \quad \sum_{v \in V^a} \langle r_v, \mu_v \rangle \\
\text{s.t.} \quad & \mu \in \overline{\mathcal{P}}(G, \mathcal{X}, p, G) \\
& \delta \in \Delta \\
& \mu_{C_v} = \delta_{v|\text{prt}(v)} \mu_{C_v}, \quad \forall v \in V^a,
\end{align*}
\] (15)

where the inner product notation \(\langle r_v, \mu_v \rangle\) stands for \(\sum_{x_v} \mu_v(x_v) r_v(x_v)\). Note that the constraints \(\delta \in \Delta\) are implied by the other ones.

Given \((G, p)\), a parametrized ID, and \(G\), an RJT, we introduce the set

\[
\mathcal{S}(G) = \{ \mu \in \overline{\mathcal{P}} : \exists \delta \in \Delta, \mu_{C_v} = \mu_{C_v} \delta_{v|\text{prt}(v)} \text{ for all } v \in V^a \}
\] (16)

of moments corresponding to distributions induced by feasible policies: \(\mu\) is in \(\mathcal{S}(G)\) if there exists \(\delta\) in \(\Delta\) such that \(\mu_{X_{C_v}}(x_{C_v}) = P_{\delta}(X_{C_v} = x_{C_v})\) for all \(C\) and \(x_{C_v}\). It will sometimes be convenient to use the following equivalent of (15)

\[
\max_{\mu \in \mathcal{S}(G)} \sum_{v \in V^a} \langle r_v, \mu_v \rangle.
\] (17)

\(\mathcal{S}(G)\) is non-convex in general as shown by the examples in the proof of Theorem 8. However, we show in Section 9 that \(\mathcal{S}(G)\) is a polytope if \(G\) is soluble, a property identifying “easy” IDs.

**Theorem 8.** The (NLP) Problems (15) and (17) are equivalent to the MEU Problem (3), in the sense that they have the same value and that, if \((\mu, \delta)\) is a feasible solution for Problem (15), then \(\delta\) defines an admissible policy for Problem (3), and \(\mu\) characterizes the moments of the distribution induced by \(\delta\).

**Theorem 8** and the following lemma, whose proof is straightforward, play a key role in the proof of Theorem 8.

**Lemma 9.** Let \(A, P,\) and \(D\) be disjoint subsets of \(V\), let \(\mu\) be a distribution on \(X_V\) and \(\mu_{A \cup P \cup D}\) be the distribution induced by \(\mu\) on \(X_{A \cup P \cup D}\). Then

\[
\mu_{A \cup P \cup D} = \mu_{A \cup P} p_D |_{P} \quad \iff \quad X_D \perp \!
\!
\!
\perp X_A | X_P,
\]

where the independence is according to \(\mu\).

---

\(^3\) we remind that \(V^s\) is the set of utility vertices as introduced in Section 1.3
Proof of Theorem 8. If \((\mu, \delta)\) is a solution of (15), then \(\mu\) is a solution of (17), and conversely, if \(\mu\) is a solution of (17), by definition of \(\mathcal{S}(G)\), there exists \(\delta\) such that \((\mu, \delta)\) is a solution of (15), which gives the equivalence between (15) and (17).

Let now \((\mu, \delta)\) be an admissible solution of Problem (15). Lemma 9 ensures that the vector \(\mu\) satisfies the conditions of Theorem 8 and hence corresponds to a distribution \(\mathbb{P}_\mu\) that factorizes on \(G\). Furthermore, we have \(\mathbb{E}_\mu \left( \sum_{v \in V^s} r_v(X_v) \right) = \sum_{v \in V^s} \langle r_v, \mu_v \rangle\), and constraint (14) ensures that \(\mathbb{P}_\mu(X_v|X_{\text{prt}(v)}) = p_v|_{\text{prt}(v)}\) for all \(v \in V^s\), which yields the result.

4.2 MILP formulation

The NLP (15) is hard to solve due to the non-linear constraints (15d). But by Theorem 8, Problems (3) and (15) are equivalent, and in particular admit the same optimal solutions in terms of \(\delta\).

We can therefore add integrality constraint (18) to (15). With this integrality constraint, Equation (15d) becomes a logical constraint, i.e., a constraint of the form \(\lambda g = z\) with \(\lambda\) binary and continuous \(g\) and \(z\). Such constraints can be handled by modern MILP solvers such as CPLEX or Gurobi, that can therefore directly solve Problem (15). Alternatively, by a classical result in integer programming we can turn Problem (15) into an equivalent MILP by replacing constraint (15d) by its McCormick relaxation [McCormick, 1976]. For a given \(p\), let \(b\) be a vector of upper bounds \(b_{\mathcal{C}_v}(x_{\mathcal{C}_v})\) satisfying

\[
P_\delta(X_{\mathcal{C}_v} = x_{\mathcal{C}_v}) \leq b_{\mathcal{C}_v}(x_{\mathcal{C}_v}) \quad \forall \delta' \in \Delta, \quad \forall v \in V, \quad \forall x_{\mathcal{C}_v} \in \mathcal{X}_{\mathcal{C}_v}.
\]

(19)

For such a vector \(b\), we say that, for a given node \(v\), \((\mu_{\mathcal{C}_v}, \delta_{\text{prt}(v)})\) satisfies McCormick’s inequalities (see appendix A) if

\[
\left\{ \begin{array}{l}
\mu_{\mathcal{C}_v} \geq \mu_{\mathcal{C}_v} + (\delta_{\text{prt}(v)} - 1)b_{\mathcal{C}_v}, \\
\mu_{\mathcal{C}_v} \leq \delta_{\text{prt}(v)}b_{\mathcal{C}_v}, \\
\mu_{\mathcal{C}_v} \leq \mu_{\mathcal{C}_v}.
\end{array} \right. \quad \text{(McCormick}(C_v, b))
\]

Note that the last inequality \(\mu_{\mathcal{C}_v} \leq \mu_{\mathcal{C}_v}\), can be omitted in our case as it is implied by the marginalization constraint \(\mu_{\mathcal{C}_v} = \sum_{x_v \in \mathcal{X}_v} \mu_{\mathcal{C}_v}\) in the definition of \(\mathcal{L}_G\). Given the upper bounds provided by \(b\), we introduce the polytope

\[
Q^\delta(G, \mathcal{X}, p, G) = \left\{ (\mu, \delta) \in \mathcal{L}_G \times \Delta : \text{McCormick}(C_v, b_v) \text{ is satisfied for all } v \in V^s \right\}.
\]

With the previously introduced notation the MEU Problem (3) is equivalent to the following MILP:

\[
\max_{\mu, \delta} \sum_{v \in V^s} \langle r_v, \mu_v \rangle \quad \text{(21a)}
\]

subject to \(\mu \in \overline{\mathcal{P}}(G, \mathcal{X}, p, G)\)

\[
\delta \in \Delta^d \quad \text{(21c)}
\]

\[
(\mu, \delta) \in Q^\delta \quad \text{(21d)}
\]

where \(\Delta^d\) is the set of deterministic policies and contains the integrality constraints (18).
Remark 5. The strength of the McCormick constraints \( \text{McCormick}(C, b) \) depends on the quality of the bounds \( b_{\tilde{C}_v} \) on \( \mu_{\tilde{C}_v} \). As for a solution \( \mu \) of Problem (21), \( \mu_{\tilde{C}_v} \) corresponds to a probability distribution, the simplest admissible bound over \( \mu_{\tilde{C}_v} \) is simply \( b = 1 \). Unfortunately, McCormick’s constraints are loose in this case: we show in Appendix C.1 that, for any \( \mu \) in \( \mathcal{P} \), there exists \( \delta \) in \( \Delta \) such that \( (\mu, \delta) \) satisfies the McCormick constraints. Hence, when \( b = 1 \), McCormick constraints fail to retain any information about the conditional independence statements encoded in the associated nonlinear constraints. Since \( \delta \) does not appear outside of the McCormick constraints, their sole interest in that case is to enable the branching decisions on \( \delta \) to have an impact on \( \mu \). Appendix C.2 gives example showing that McCormick constraints do retain information about the conditional independence if bounds \( b_{\tilde{C}_v} \) smaller than 1 are used. Finally, Appendix C.3 provides a dynamic programming algorithm that efficiently computes such a \( b \).

5 Valid cuts

Classical techniques in integer programming such as branch and bound algorithms rely on solving the relaxation of the MILP to obtain lower bound on the value of the objective. For Problem (21) the relaxation is likely to be poor, and hence the MILP is not well solved by off-the-shelf solvers: indeed as explained above, when \( b = 1 \), the McCormick inequalities fail completely to enforce in the linear relaxation the conditional independences that are encoded in the nonlinear constraints the McCormick inequalities. In this section, we introduce valid cuts to strengthen its relaxation and ease its resolution. A valid cut for a MILP is an (in)equality that is satisfied by any solution of the MILP, but not necessarily by solutions of its linear relaxation. A family of valid cuts is stronger than another when the former yields a polytope strictly included in the latter.

We start by recalling some useful notions from the theory of probabilistic graphical models.

5.1 d-separation, Markov blanket, and augmented graph

In the theory of probabilistic graphical models, a central result is that when a distribution factorizes according to a certain graph it must satisfy a number of conditional independence statements, which provide more abstract characterizations of the properties of the distribution. In particular, leveraging conditional independence will allow us to identify maximal conditional independence statements which are not immediately readable from the graph, although they can be computed from it, and which entail useful factorizations that can be added as constraints in our optimization problem.

We first briefly review key relevant concepts from graphical model theory, in particular to characterize conditional independence from properties of the graph.

Let \( D \subset V \) be a set of vertices. A trail \( v_1 \rightleftharpoons \cdots \rightleftharpoons v_n \) is active given \( D \) if, whenever there is a v-structure \( v_{i-1} \rightarrow v_i \leftarrow v_{i+1} \), then \( v_i \) or one of its descendant is in \( D \), and no other vertex of the trail is in \( D \). Two sets of vertices \( B_1 \) and \( B_2 \) are d-separated by \( D \) in \( G \) if there is no active trail between \( B_1 \) and \( B_2 \) given \( D \). We have \( X_{B_1} \perp X_{B_2} | X_D \) for any distribution that factorizes on \( G \) if and only if \( B_1 \) and \( B_2 \) are d-separated by \( D \) [Koller and Friedman, 2009, Theorem 3.4]. The Markov blanket of a set \( C \), denoted \( \text{mb}(C) \) is the smallest set \( D \) of \( V \setminus C \) such that \( C \) is d-separated from \( V \setminus (C \cup D) \) given \( D \). It admits the following characterization [Pearl, 1988]:

\[
\text{mb}(C) = \text{cpt}(C) \cup \text{prt}(C) \cup \text{cld}(C).
\]

The authors Cohen and Parmentier [2019] recently introduced a generalization of the notion of Markov blanket.
Figure 7: The Markov blanket of $t$ is $\{u, v\}$, and its Markov blanket in $C$ and $C'$ is $\{u, w\}$.

Figure 8: Example of augmented graph $G^\dagger$ on a POMDP.

**Proposition 10.** Given two sets of vertices $B$ and $C$ such that $B \subseteq C$, there exists a unique subset $M$ of $C$ such that

$$X_B \perp\!\!\!\perp X_{C \setminus (B \cup M)} | X_M$$

for any distribution that factorizes on $G$, which is minimal for the inclusion.

We denote this minimal set by $mb_C(B)$ and call it the Markov blanket of $B$ in $C$. It admits the following characterization based on $d$-separation:

$$mb_C(B) = \{v \in C: v \text{ is not } d\text{-separated from } B \text{ given } C \setminus (B \cup \{v\})\}.$$ 

Furthermore, any $M \subseteq C$ satisfying (23) contains $mb_C(B)$, and any set $M \subseteq C \setminus B$ containing $mb_C(B)$ satisfies (23).

Note that the usual Markov blanket of $B$ is the Markov blanket of $B$ in $V$ with this new definition. Note that $mb_C(B)$ also depends on the structure of the graph outside $C$. In particular, $mb_C(B)$ is not the Markov blanket in the graph induced on $C$. The characterization of $mb_C(B)$ with $d$-separation shows that it can be computed in $O(|V| + |E|)$ operations. Figure 7 provides an example illustrating the difference between the Markov blanket and the Markov blanket in a set.

We finally introduce the notion of augmented model [Koller and Friedman, 2009, Chapter 21]. Consider $(G, \rho)$, a PID with $G = (V^s, V^a, E)$, and let $V = V^s \cup V^a$. For each $v \in V^a$, we introduce a vertex $\vartheta_v$ and a corresponding random variable $\theta_v$. The variable $\theta_v$ takes its value in the space $\Delta_v$ of conditional distributions on $X_v$ given $X_{\text{prev}(v)}$. Let $G^\dagger$ be the digraph with vertex set $V_{G^\dagger} = V \cup \vartheta_{V^a}$, where $\vartheta_{V^a} = \{\vartheta_v\}_{v \in V^a}$, and arc set $E_{G^\dagger} = E \cup \{((\vartheta_v, v), v) \vee v \in V^a\}$. Such a graph $G^\dagger$ is illustrated on Figure 8 where vertices in $G^\dagger \setminus G$ are illustrated by rectangle with rounded corners. The augmented model of $(G, \rho)$ is the collection of distributions factorizing on
$G^\dagger$ such that $\mathcal{X}_v$ is defined as in $\rho$ for each $v \in V$, $\mathcal{X}_{\theta_v} = \Delta_v$, and

$$F \left( X_v = x_v | X_{\text{part}}(v) = x_{\text{part}}(v), \theta_v = \theta_v^\rho \right) = \begin{cases} \theta_v^\rho(x_v | x_{\text{part}}(v)) & \text{if } v \in V^a, \\ p_{\text{part}}(x_v, x_{\text{part}}(v)) & \text{if } v \in V^s. \end{cases} \quad (24)$$

A distribution of the augmented model is specified by choosing the distributions of the $\theta_v$. In the remaining of the paper, we denote by $F G^\dagger$ the distribution of the augmented model with uniformly distributed $\theta_v$ for each $v \in V^a$.

With these definitions, a policy $\delta$ can now be interpreted as a value taken by $\theta_{V_a}$, and we have

$$F( X_B = x_B | X_D = x_D ) = F G^\dagger( X_B = x_B | X_D = x_D, \theta_{V_a} = \delta ). \quad (25)$$

Note that in general $F G^\dagger( X_B = x_B | X_D = x_D )$ is the expected value over $\theta_{V_a}$ of $F \theta_{V_a}( X_B = x_B | X_D = x_D )$. The following is an immediate consequence of (25).

**Proposition 11.** We have $F( X_B | X_D ) = F G^\dagger( X_B | X_D )$ for any PID on $G$, any policy $\delta$, and any $D$ such that $F( X_D ) > 0$ if and only if $B$ is d-separated from $V_a$ given $D$ in $G^\dagger$.

Note that this is a particular case of a result known in the causality theory for graphical model [see e.g., Koller and Friedman 2009, Proposition 21.3].

**5.2 Constructing valid cuts**

By restricting ourselves to distributions $\mu \in \mathcal{F}$, we impose for all $v \in V^a$ that

$$F( X_v | X_{\text{desc}}(v) ) = F( X_v | X_{C_v} ) = p_{\text{part}}(v),$$

using equalities $\mu_{C_v} = \mu_{C_v} p_{\text{part}}(v)$ for $v \in V^s$. If we could impose the nonlinear constraints $\mu_{C_v} = \mu_{C_v} \delta_{\text{part}}(v)$ for $v \in V^a$, we would be able to impose as well that decisions encoded in $\mu$ at the nodes $a \in V^a$ would satisfy $F( X_a | X_{C_a} \setminus \{a\} ) = F( X_a | X_{\text{part}}(a) )$. Unfortunately, in general, McCormick’s relaxation is not sufficient to enforce $\mu_{C_v} = \mu_{C_v} \delta_{\text{part}}(v)$ for $v \in V^a$ for solutions of the linear relaxation. The constraint $\mu_{C_v} = \mu_{C_v} p_{\text{part}}(v)$ for $v \in V^s$ is linear only because $p_{\text{part}}(v)$ is a constant that does not depend on $\delta$. But, as an indirect consequence of setting these conditional distribution $p_{\text{part}}(v)$ for $v \in V^s$, there are other conditional distributions that do not depend on $\delta$. Indeed, for some pairs of sets of vertices $C, D$ with $D \subseteq C$, the conditional probabilities $F( X_D = x_D | X_{C \setminus D} = x_{C \setminus D} )$ are identical for any policy $\delta$. We can therefore introduce valid cuts of the form

$$\mu_C = \mu_{C \setminus D} p_{D | C \setminus D} \quad (26)$$

While these constraints are not needed to fix the value of the conditionals on $v \in V^s$ and the conditional independences of the form $X_v \perp X_{\text{desc}}(v) | X_{\text{part}}(v)$, for $v \in V^s$ they can be useful to enforce some of the conditional independences that should be satisfied by $\mu$ at decision nodes. In particular, if there exists a subset $M$ of $C \setminus D$ such that $p_{D | C \setminus D} = p_{D | M}$, then (26) enforces that for any $v \in V^a \cap (C \setminus (D \cup M))$, we have $F( X_a | X_{D \cup M} ) = F( X_a | X_M )$. Clearly, the larger $D$, the stronger the valid cut.

**Definition 2.** Given a set of vertices $C$, we denote

$$C^\parallel = \text{mb} C \cup \theta_{V_a}( \theta_{V_a} ),$$

and

$$C^\perp = \text{ml} C \setminus D, \quad (27)$$

**Lemma 12.** The set $C^\perp$ is the largest subset $B$ of $C$ such that $F( X_B | X_{C \setminus B} ) = p_{B | C \setminus B}$ holds for any policy $\delta$ and any parametrization of $G$.  

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Figure 9: Valid cut (28) with $C = \{a, u, v, b\}$ with $C^\perp = \{u\}$ is not implied by the linear inequalities of (21). Indeed, suppose that $X_a = X_v = \{0\}$, while $X_u = X_b = \{0, 1\}$. Then the solution defined by $\mu_{aub}(0, i, 0, i) = 0.5$ and $\mu_{aub}(0, i, 0, 1 - i) = 0.5$ for $i \in \{0, 1\}$ is in the linear relaxation of (21) but does not satisfy (28).

Proof. It is an immediate corollary of Propositions 10 and 11: by definition $m_{bC\cup\theta_{V^a}}(\theta_{V^a})$ is the smallest set $M$ such that its complement in $C$ is independent of $\theta_{V^a}$ given $M$.

The previous lemma shows that the equalities

$$\mu_C = \mu_{C \cup \theta_{V^a}} \quad \forall C \in V,$$  

(28)

are the strongest valid cuts of the form (26) that we can obtain for Problem (21). Thus, as we defined $Q^b$ (see (20)) by strengthening $\mathcal{P}$ with McCormick’s inequalities, we now leverage Equation (28) to strengthen $\mathcal{P}$ and obtain the following polytope

$$\mathcal{P}^\perp(G, X, p, G) = \left\{ \mu \in \mathcal{P}: \mu_{C_v} = p_{C_v \mid |C_v|^\perp} \sum_{x_{C_v} \mid |C_v|^\perp} \mu_{C_v} \text{ for all } v \in V^a \right\}. \quad (29)$$

In the definition of $\mathcal{P}^\perp$, we decided to introduce valid cuts of the form (28) only for sets of vertices $C$ of the form $C_v$ with $v \in V^a$. The aim of this choice is to strike a balance between the number of constraints added and the number of independences enforced. Our choice is however heuristic, and it could notably be relevant to introduce constraints of the form (28) for well chosen $C \subsetneq C_v$.

Figure 9 provides an example of ID where valid cuts (28) reduce the size of the initial polytope. We show in Section 7 that these inequalities numerically do help to solve the MILP. Practically, using these inequalities requires to compute $p_{C_v \mid |C_v|^\perp}$. Proposition 11 ensures that, if we solve the inference problem on the RJT for an arbitrary policy, e.g., one where decisions are taken with uniform probability, we can deduce $p_{C_v \mid |C_v|^\perp}$ from the distribution $\mu_C$ obtained.

5.3 Strength of the relaxations and their interpretation in terms of graph

Consider $(G, \rho)$, a PID with $G = (V^s, V^a, E)$ and $\rho = (\mathcal{X}, p, r)$. Let $\mathcal{G}$ be an RJT on $G$, and $b$ an admissible bound satisfying (19). The valid cuts of Section 5.2 enable to introduce the following strengthened version of the MILP (21).

$$\max_{\mu, \delta} \sum_{v \in V^r} (r_v, \mu_v) \text{ subject to } \mu \in \mathcal{P}^\perp(G, \mathcal{X}, p, \mathcal{G}), \delta \in \Delta^d, (\mu, \delta) \in Q^b. \quad (30)$$

The following proposition summarizes the results of Section 5.2.

Proposition 13. Any feasible solution of the MILP (30) is such that $\mu$ is the vector of moments of the distributions $\mathbb{P}_s$. Hence, $(\mu, \delta)$ is an optimal solution of (30) if and only if $\delta$ is an optimal solution of the MEU problem (3) on $(G, \rho)$.
In this section we give interpretations of the linear relaxations of (21) and (30) in terms of graphs. We introduce the sets of edges and IDs

$$\mathcal{E} = E \cup \{ (u, v) : v \in V^a \text{ and } u \in C_v \setminus \text{fa}(v) \} \quad \text{ and } \quad \mathcal{G} = (V^a, V^a, \mathcal{E}),$$

$$E^\perp = E \cup \{ (u, v) : v \in V^a \text{ and } u \in \mathcal{C}_v \setminus \text{fa}(v) \} \quad \text{ and } \quad G^\perp = (V^a, V^a, E^\perp).$$

Figure 10 illustrate $\mathcal{G}$ and $G^\perp$ on the ID of Figure 2. Note that $E \subseteq E^\perp \subseteq \mathcal{E}$, and remark the three following facts on $\mathcal{G}$ and $G^\perp$. First, the definition of both IDs depends on $G$ and $\mathcal{G}$. Second, $\mathcal{G}$ is still an RJT on $\mathcal{G}$ and $G^\perp$. And third, any parametrization $(\mathcal{X}_v, p, r)$ of $G$ is also a parametrization of $\mathcal{G}$ and of $G^\perp$. The second and third results are satisfied by any ID $G' = (V^a, V^a, \mathcal{E} \cup E')$, where $E'$ contains only arcs of the form $(u, v)$ with $v \in V^a$ and $u \in C_v$. Hence, if we denote by $\Delta_{G'}$ the set of feasible policies for $(G', \mathcal{X}_v, p, r)$, we can extend the definition of $S(G')$ in Equation (10) to such $G'$.

$$S(G') = \{ \mu \in \mathcal{F} : \exists \delta \in \Delta_{G'}, \mu_{C_v} = \mu_{C_v \delta \mid \text{pt}_{G'}(v)} \text{ for all } v \in V^a \}. $$

**Theorem 14.** We have

$$\mathcal{F} = S(\mathcal{G}) \quad \text{ and } \quad \max_{\mu \in \mathcal{F}} \sum_{v \in V^r} \langle r_v, \mu_v \rangle = MEU(\mathcal{G}, \rho),$$

and

$$\mathcal{F}^\perp = S(G^\perp) \quad \text{ and } \quad \max_{\mu \in \mathcal{F}^\perp} \sum_{v \in V^r} \langle r_v, \mu_v \rangle = MEU(G^\perp, \rho).$$

Hence, if $(\mu, \delta)$ is a solution of the linear relaxation of (21), then $\delta$ is a policy on $\mathcal{G}$, while if $(\mu, \delta)$ is a solution of the linear relaxation of (30), then $\delta$ is a policy on $G^\perp$.

Remark furthermore that $S(G')$ is generally not a polytope. Indeed, when $G' = G$, this is the reason why (14) is not a linear program. An important result of the theorem is that $S(\mathcal{G})$ and $S(G^\perp)$ are polytopes, and $MEU(\mathcal{G}, \rho)$ and $MEU(G^\perp, \rho)$ can therefore be solved using the linear programs $\max_{\mu \in \mathcal{F}} \sum_{v \in V^r} \langle r_v, \mu_v \rangle$ and $\max_{\mu \in \mathcal{F}^\perp} \sum_{v \in V^r} \langle r_v, \mu_v \rangle$ respectively.

The proof of the theorem uses the following lemma.

**Lemma 15.** Let $v$ be a vertex in $V^a$. Then $x_{C_v} \mapsto p_{C_v\mid |C_v \setminus v}(x_{C_v\mid |C_v \setminus v}, x_v)$ is a function of $(x_{C_v \mid |C_v \setminus v})$ only. Hence, if a distribution $\mu_{C_v}$ satisfies $\mu_{C_v} = \mu_{C_v \parallel p_{C_v\mid |C_v \setminus v}}$, then $C_v \perp v \mid v_{C_v \setminus \{v\}}$ according to $\mu_{C_v}$.
Proof. Consider the augmented model \( \mathbb{P}^{G'} \). Let \( P \) be a \( C_{v}^{-} \) v trail. Let \( Q \) be trail \( P \) followed by arc \((v, \vartheta_{v})\). As \( v \) has no descendants in \( C_{v} \) by hypothesis \( \vartheta = \{v\} \), vertex \( v \) is a \( v \)-structure of \( Q \).

As \( v \in C_{v}^{\perp} \), if \( P \) is active given \( M_{C_{v}} \setminus v \), then \( P \) is active given \( M_{C_{v}} \), which contradicts the definition of \( C_{v}^{\perp} \). Hence, \( C_{v}^{\perp} \perp_{G} v|C_{v}^{\perp} \{v\} \) according to \( \mathbb{P}^{G'} \), and \( x_{C_{v}} \mapsto p_{C_{v}^{-}}(x_{C_{v}^{\perp}}|x_{C_{v}^{\perp}} \setminus v, x_{v}) \) is a function of \( (x_{C_{v}^{\perp}}|x_{C_{v}^{\perp}} \setminus v) \) only. The second part of the lemma is an immediate corollary.

Proof of Theorem \[14.\] First, remark that, once we have proved \( \overline{P} = S(G) \) and \( \mathcal{P}^\perp = S(G^\perp) \), the result follows from Theorem \[8\].

We now prove \( \overline{P} = S(G) \). Let \( \mu \) be in \( \overline{P} \). Then \( \mu \) is a vector of moments in the local polytope of the RJT \( G \) on \( \overline{G} \). Furthermore, as, first, \( fa_{\overline{G}}(v) = C_{v} \) for \( v \in V^a \), and second, \( \mu_{C_{v}} = \mu_{\tilde{C}_{v}, p_{\varepsilon, \pi(t)}} \) implies that \( X_{v} \) is independent from its non-descendants in \( C_{v} \) given its parents according to \( \mu_{C_{v}} \). Theorem \[3\] ensures that \( \mu \) is a vector of moments of a distribution that factorizes on \( \overline{G} \), which gives \( \overline{P} = S(G) \).

Consider now a vector of moments \( \mu \) in \( \mathcal{P}^\perp \). Given \( v \in V^a \), lemma \[13\] and the definition of \( G^\perp \) ensures that, according to \( \mu_{C_{v}} \), variables \( X_{v} \) is independent from its non-descendants in \( G^\perp \) in \( C_{v} \), i.e., \( C_{v}^{\perp} \), given its parents in \( G^\perp \), i.e., \( C_{v}^{\perp} \setminus v \). If \( v \in V^a \), constraints \( \mu_{C_{v}} = \mu_{\tilde{C}_{v}, p_{\varepsilon, \pi(t)}} \) still implies that \( X_{v} \) is independent from its non-descendants in \( C_{v} \) given its parents according to \( \mu_{C_{v}} \). Theorem \[3\] again enables to conclude that \( \mathcal{P}^\perp = S(G^\perp) \).

\section{Soluble influence diagrams}

In this section, we make the assumption that IDs are such that any vertex \( v \in V \) has a descendant in the set of utility vertices \( V^{\tau} \), i.e., \( V^{a} \cup V^{a} = \overline{\text{scc}}(V^{\tau}) \). The following remark explains why we can make this assumption without loss of generality.

Remark 6. Consider a parametrized ID \((G, \rho)\) where \( G = (V^{a}, V^{a}, E) \) and \( V^{a} \) is the union of chance vertices \( V^{c} \) and utility vertices \( V^{\tau} \). Let \((G', \rho')\) be the ID obtained by removing any vertex that is not in \( V^{\tau} \) and has no descendant in \( V^{\tau} \) and restrict \( \rho \) accordingly. If a random vector \( X_{V} \) factorizes as a directed graphical model on \((V, E)\) and \( V^{\tau} \subseteq V \) is such that \( \overline{\text{scc}}(V^{\tau}) = V^{\tau} \), then \( X_{V} \) factorizes as a directed graphical model on the subgraph induced by \( V^{\tau} \) with the same conditional probabilities \( p_{\varepsilon, \pi(t)} \). Hence, given a policy \( \delta \) on \((G, \rho)\) and its restriction \( \delta' \) to \((G', \rho')\), we have \( E_{\delta}(\sum_{v \in V^{a}} r_{v}(X_{v})) = E_{\delta'}(\sum_{v \in V^{\tau}} r_{v}(X_{v})) \) where the first expectation is taken in \((G, \rho)\) and the second in \((G', \rho')\), and the two IDs model the same MEU problem. \[\triangle\]

\subsection{Linear program for soluble influence diagrams}

Consider an ID \( G = (V^{a}, V^{a}, E) \) with \( V^{a} \) being the union of chance vertices \( V^{c} \) and utility vertices \( V^{\tau} \). Given a policy \( (\delta_{u})_{u \in V^{a}} \) and a decision vertex \( v \), we denote \( \delta_{-v} \) the partial policy \( (\delta_{u})_{u \in V^{a} \setminus v} \). A policy \( (\delta_{v})_{v \in V^{a}} \) is called a local optimum if

\[
\delta_{v} \in \arg \max_{\delta'_{v}, \delta_{-v}} \mathbb{E}_{\delta'_{v}, \delta_{-v}} \left( \sum_{u \in V^{a}} r_{u}(X_{u}) \right) \text{ for each vertex } v \in V^{a}.
\]

It is a global optimum if it is an optimal solution of \[3\]. Two concepts, s-reachability and the relevance graph have been introduced in the literature to characterize when a local minimum is also global [see e.g., \cite{Koller2009}, Chapter 23.5]. A decision vertex \( u \) is s-reachable from a decision vertex \( v \) if \( \vartheta_{u} \) is not d-separated from dsc \((v)\) given fa \((v)\):

\[
\vartheta_{u} \not\perp \perp \text{dsc}(v) \mid \text{fa}(v).
\]
The usual definition is $\vartheta_u \not\perp_{G^a} \text{dsc}(v) \cap V^r \mid \text{fa}(v)$, but these definitions coincide in our setting, since we have assumed that $\text{dsc}(v) \cap V^r \neq \emptyset$ for any $v \in V^a$. Intuitively, the definition of this concept is motivated by the fact that the choice of policy $\delta_u$ given $(\delta_w)_{w \neq v}$ depends on $\delta_u$ only if $u$ is s-reachable from $v$. Note that, for example, if $u \in \text{dsc}(v)$, then $u$ is s-reachable from $v$.

The relevance graph of $G$ is the digraph $H$ with vertex set $V^a$, and whose arcs are the pairs $(v, u)$ of decision vertices such that $u$ is s-reachable from $v$. Finally, the single policy update algorithm (SPU) [Lauritzen and Nilsson, 2001] is the standard “coordinate descent” heuristic for IDs. It iteratively improves a policy $\delta$ as follows: at each step, a vertex $v$ is picked, and $\delta_v$ is replaced by an element in $\text{argmax}_{\delta' \in \Delta_v} E_{\delta', \delta - v} \left( \sum_{u \in V^r} r_u(X_u) \right)$.

The following proposition [Koller and Friedman, 2009, Theorem 23.5] characterizes a subset of IDs, called soluble IDs, which are easily solved, and provides several equivalent criteria to identify them.

**Proposition 16.** Given an influence diagram $G$, the following statements are equivalent and define a soluble influence diagram.

1. For any parametrization $\rho$ of $G$, any local optimum is a global optimum.

2. For any parametrization $\rho$ of $G$, SPU converges to a global optimum.

3. The relevance graph is acyclic.

Given a parametrized influence diagram $G$ and an RJT $G$, we introduced in Equation (16) the notation $S(G)$ for the subset of the local polytope $L_G$ corresponding to moments of policies.

The following theorem introduces a new characterization of soluble IDs in terms of convexity. If the graph is soluble, and if the decision nodes are ordered in reverse topological order for the relevance graph, then SPU converges after exactly one pass over the nodes.

**Theorem 17.** If $G$ is soluble, Algorithm 2 returns an RJT such that $\mathcal{P}^\perp = S(G)$ for any parametrization $\rho$.

If $G$ is not soluble then there exists a parametrization $\rho$ such that, for any junction tree $\mathcal{G}$, the set of achievable moments $S(\mathcal{G})$ is not convex.

The property of being soluble characterizes “easy” ID that can be solved by SPU. Theorems 8 and 17 imply that, if $G$ is soluble, our MILP formulation 30 can be simplified to the linear program

$$\max_{\mu \in \mathcal{P}^\perp} \sum_{v \in V^r} \langle r_v, \mu_v \rangle$$

and is therefore “easy” to solve. Of course, this property of being “easy” refers only to the decision part of the ID. If a soluble ID is such that, given a policy, solving the inference problem is not tractable, both SPU and our MILP formulation will not be tractable in practice. Theorem 17 is a corollary of Theorem 14 and the following lemma, and both results are proved in Section 6.3.

**Lemma 18.** There exists an RJT $\mathcal{G}$ such that $G^\perp = G$ if and only if $G$ is soluble. Such an RJT can be computed using Algorithm 2.

### 6.2 Comparison of soluble and linear relaxations

MILP solvers are based on (much improved) branch-and-bound algorithms that use the linear relaxation to obtain bounds. Their ability to solve formulation (30) therefore depends on the quality of the bound provided by the linear relaxation. As SPU solves efficiently soluble IDs, we
could imagine alternative branch-and-bounds schemes that use bounds computed using SPU on “soluble graph relaxation” of influence diagrams. We now formalize this notion and compare the two approaches.

A **soluble graph relaxation** of an ID $G = (V^s, V^a, E)$ is a soluble ID $G' = (V^s, V^a, E')$ where $E'$ is the union of $E$ and a set of arcs with head in $V^a$. Remark that Theorem 14 can be reinterpreted as the link between soluble graph relaxation and linear relaxations. And since $S(G) = \mathcal{P}$ and $S(G^\mu) = \mathcal{P}^\mu$, by Theorem 17, $G^\mu$ and $G$ are soluble, and therefore soluble graph relaxations of $G$.

Since any feasible policy for the ID $G$ is a feasible policy for a soluble graph relaxation $G'$, for any parametrization $\rho$, the value of $\text{MEU}(G', \rho)$, which can be computed by SPU, provides a tractable bound on $\text{MEU}(G, \rho)$. Soluble relaxations can therefore be used in branch-and-bound schemes for IDs, as proposed in Khaled et al. 2013. To compare the interest of such a scheme to our MILP approach we need to compare the quality of the soluble graph relaxation and linear relaxation bounds. Let $G'$ be a soluble graph relaxation of $G$, applying Algorithm 2 on $G'$ provides an RJT such that $E^\mu \subseteq E'$. Indeed, by Lemma 18, $v$ is d-separated from $C_u \backslash \{\text{fa}_{G'}(v)\}$ given $\text{pr}_{G'}(v)$ in $G'$, and therefore also in $G$, which implies $E^\mu \subseteq E'$. Thus, by Theorem 14, the bound provided by the linear relaxation of the MILP 2 is at least as good as the soluble graph relaxation bound, and sometimes strictly better thanks to constraints $(\mu, \delta) \in Q^b$.

### 6.3 Proofs

For any set $C$ and binary relation $R$, we denote $C_{Ra}$ the set of vertices $u$ in $C$ such that $u R v$. The following lemma will be useful in the proof of Lemma 18.

**Lemma 19.** If $G$ is soluble and $\preceq$ is a topological order on $G$, then its restriction $\preceq_H$ to $V^a$ is a topological order on the relevance graph $H$.

**Proof.** Suppose that $u$ is strategically reachable from $v$, that is $(v, u)$ is an arc in $H$. Then $u$ and $v$ have common descendants. Indeed, otherwise any trail from $\text{dsc}(v)$ to $\varnothing_u$ has a v-structure. Consider a trail $T$ from $\text{dsc}(v)$ to $\varnothing_u$ and let $x$ be the last v-structure starting from $\text{dsc}(v)$. If $x = u$, then $T$ is not active given $\text{fa}(v)$, else $x \in \text{dsc}(u)$, hence $x \notin \text{asc}(v)$ and $T$ is not active. It contradicts the assumption on $u$ and $v$. Therefore, $u$ and $v$ have common descendants. If $u$ is not a descendant of $v$, then $v$ is s-reachable from $u$, which is not possible as $H$ is acyclic. Hence $u \in \text{dsc}(v)$, which implies $v \preceq u$, and gives the result. 

**Proof of Lemma 18.** Let $G$ be a soluble influence diagram. We start by proving that Algorithm 2 with $G$ as an input returns an RJT $G$. It suffices to show that it is possible to compute topological orderings in step 5 that is, to prove that $G'$, defined in Algorithm 2, is acyclic. Suppose that there is a cycle in $G'$. Let $\preceq_G$ be a topological order on $G$, and let $v_h$ be the smallest vertex $v$ for $\preceq_G$ in the cycle such that there is an arc $(u, v)$ in $E' \backslash E$ in the cycle. And let $(u_h, v_h)$ be the corresponding arc in the cycle. Let $(u_l, v_l)$ be the arc of $E$ right before $(u_h, v_h)$ in the cycle. By definition of $v_h$, we have $v_h \preceq v_l$. And since all the arcs in the $v_l-u_h$ subpath of the cycle...
are in $E$, we have $u_h \in \overline{\operatorname{dsc}_G(v_h)}$. Hence $u_h \in \operatorname{dsc}_G(v_h)$, which contradicts the definition of $E'$ in Step 3. Hence, Algorithm 2 always returns an RJT, which we denote by $G$.

It remains to prove that $G$ is such that $C_i \subseteq \operatorname{fa}(v)$ for each decision vertex $v \in V_a$. We start with two preliminary results. Remark that $E \subseteq E'$ implies that $\preceq$ is a topological order on $G$. Let $\preceq_H$ denotes its restriction to $V_a$. Lemma 19 ensures that $\preceq_H$ is a topological order on $H$. Hence, we have

$$\partial V_{\preceq_H} \perp \operatorname{dsc}(v) \mid \operatorname{fa}(v), \quad \text{for all } v \in V_a.$$  

Furthermore, if $u \in V^a$ and $v \in V^a_{\preceq_H}$, the definition of $G'$ implies the existence of a path from $u$ to $v$ in $G$, and hence the existence of a path from $V^a_{\preceq_H}$ to $v$ in $G$.

We now prove $C_i \subseteq \operatorname{fa}(v)$ for each $v \in V_a$. This part of the proof is illustrated on Figure 11

Let $v$ be a vertex in $V^a$, let $u \in C_v \setminus \operatorname{fa}(v)$, and let $b \in V^a_{\preceq_H}$. Suppose that there is an $u \vartriangleleft v(b)$ trail $P$ that is active given $\operatorname{fa}(v)$. Proposition 4 guarantees that (12b) is an equivalence. Hence, there exists a $u \vartriangleleft v$ trail $Q$ with a minimum number of $v$-structures. Starting from $u$, let $w_1, \ldots, w_k$ be the $v$-structures of $Q$. For each $i$ in $\{1, \ldots, k\}$, and let $u_i$ be vertex with diverging arcs inbetween $w_i$ and $w_{i+1}$. Using the result at the end of the previous paragraph, we have $w_i \in \overline{\operatorname{dsc}(V^a_{\preceq_H})}$. As $Q$ has been chosen with a minimal number of $v$-structures, we obtain $u_i \in \overline{\operatorname{dsc}(V^a_{\preceq_H})}$. Let $a_i$ denote an ascendant of $u_i$ in $V^a_{\preceq_H}$. We prove by iteration on $i$ that $u_i \in \operatorname{dsc}(v)$. Suppose that it is true for $j < i$. Then $w_i \in \operatorname{dsc}(v)$.

Following (33), we have $w_i \perp \vartriangleleft v(\operatorname{fa}(a_i))$. Hence, the $v \vartriangleleft w_i$ path is not active given $\operatorname{fa}(a_i)$, and it therefore necessarily intersects $\operatorname{prt}(a_i)$, which gives the induction hypothesis and the result. Hence $w_k \in \operatorname{dsc}(v)$. Let $P'$ be the trail composed of the $w_k \vartriangleleft u$ subtrail of $Q$ followed by $P$. As the $w_k \vartriangleleft u$ subtrail of $Q$ is included in $V^a_{\preceq_H}$ and does not contain a $v$-structure, trail $P'$ is active given $\operatorname{fa}(v)$. This contradicts the fact that $\operatorname{dsc}(v) \perp \vartriangleleft v(\operatorname{fa}(v))$. Hence, path $P$ is not active, $u \in C_i \setminus \operatorname{fa}(v)$, and $C_i \subseteq \operatorname{fa}(v)$, which gives the result and the proposition.

Conversely, suppose that $G$ is such that $C_i \subseteq \operatorname{fa}(v)$ for $v \in V^a$. Suppose that there exists two vertices $u \in V^a$ and $v \in V^a$ such that $\operatorname{dsc}(v) \vartriangleleft \vartriangleleft u \mid \operatorname{prt}(v)$ and $\operatorname{dsc}(u) \vartriangleleft \vartriangleleft v \mid \operatorname{prt}(u)$. Without loss of generality, we assume that if there is a path between $C_u$ and $C_v$, it is from $C_v$ to $C_u$. There exists an active trail $Q$ from $w \in \operatorname{dsc}(u)$ to $\vartriangleleft v$ given $\operatorname{prt}(u)$. Let $x$ be the first
A structure of $Q$ if $Q$ contains such a structure, and be equal to $v$ otherwise. And let $P$ be the $w$-$x$ subtrail of $Q$. Remark that $P$ is an $x$-$w$ path in $G$. Path $P$ contains no $v$-structure, and is active given $fa(u)$. Hence, it does not intersect $fa(u)$. As $x$ and $w$ have $w$ as common descendant, Proposition 2 ensures that $C_z$ and $C_u$ are on the same branch of $G$. And as $x = v$ or $x \in asc(u)$, there is a path in $G$ from $C_y$ to $C_w$ which contains $C_u$ and all the vertices of $P$. Starting from $x$, let $y$ be the last vertex of $P$ such that $C_y$ is above $C_u$ in $G$, and $z$ be the child of $y$ in $P$. As $y$ belongs to $C_z$ and to $C_y$, by running intersection property, $y$ belongs to $C_u$. As $Q$ is active, the $y$-$w$ subtrail of $Q$ is active given $fa(u)$. As as $y$ belongs to $P$, it does not belong to $fa(u)$. This contradicts $C_u \not\subseteq fa(u)$, and gives the lemma.

Proof of Theorem 17. If $G$ is soluble, Lemma 13 ensures that Algorithm 2 builds an RJT $G$ such that $G^\mu = G$, and Theorem 14 ensures that $P^\mu = S(P)$.

Consider now the result for non-soluble IDs. Let $G$ be a non-soluble influence diagram. Let $a$ and $b$ be two decision vertices that are both strategically dependent on the other one.

First, we suppose that $a \not\in asc(b)$ and $b \not\in asc(a)$. Let $P$ be a path from $a$ to $dsc(b)$ with a minimum number of arc, and $Q$ be a $b$-$w$ path with a minimum number of arc. Then $w$ is the unique vertex in the intersection of $P$ and $Q$. Let $u$ and $v$ be the parents of $w$ in $P$ and $Q$ respectively. Consider a parametrization where all the variables that are not in $P$ or $Q$ are unary, all the variables in $P$ and $Q$ are binary, all the variables in the $a$-$u$ subpath of $P$ are equal to $X_u$, all the variables in the $b$-$v$ subpath of $P$ are equal to $X_b$, and $p_{w|prt(w)}$ is defined arbitrarily. Let $G$ be an arbitrary junction tree, $C$ be its cluster containing $fa(w)$. Then choosing a distribution $\mu_a$ as policy $\delta_a$ and a distribution $\mu_b$ as policy $\delta_b$ implies that the restriction of $\mu_C$ to $X_{uw}$ is $\mu_{uw} = \mu_a\mu_b$. Hence, the marginalization of $X_{uw}$ of the set of distributions $\mu_C$ that can be reached for different policy is the set of independent distributions, which is not convex. Hence, $S(G)$ is not convex.

We now consider the case where $a \in asc(b)$ or $b \in asc(a)$. W.l.o.g., we suppose $a \in asc(b)$. There exits a trail form $a$ to $w$ in $dsc(b)$ that is active given $prt(b)$. Let $Q$ be such a trail with a minimum number of $v$-structures. And let $P$ be a $b$-$w$ path. W.l.o.g., we suppose that $w$ is the only vertex in both $P$ and $Q$. Let $w_b$ be the parent of $w$ on $P$ and $w_a$ its parent in $Q$. Starting from $w$, let $s_0, \ldots, s_{k-1}$ denote the divergent vertices in $Q$, let $t_1, \ldots, t_k$ the $v$-structures, and $p_i$
denote the parent of \( b \) that is below \( t_i \). Finally let \( s'_i \) (resp. \( s''_i \)) denote the parent of \( t_i \) (resp \( t_{i+1} \)) on the \( s_i-t_i \) subpath (resp. \( s_{i+1}-t_i+1 \) subpath) of \( Q \). These notions are illustrated on Figure 12.

We have drawn the case where \( a \neq t_k \). Paths \( P \) and \( Q \) are respectively drawn in red and blue. Plain arrows correspond to arcs an dashed arrows to paths whose extremities are potentially equal.

We now introduce a parametrization of the influence diagram that enables to encode the following dice game with two players \( a \) and \( b \). Before rolling a die, player \( a \) chooses to play 1 or 2, where playing \( i \) means observing if the die is equal to \( i \), and passing this information to player \( b \). The die \( s_0 \) is rolled. If \( a \) has played 1 (resp. 2), he passes the information \textit{true} to \( b \) if the die \( s_0 \) is equal to 1 (resp. 2), and \textit{false} if it is equal to 2 (resp. 1), or something else \( e \). Player \( b \) does not know what \( a \) has played. Based on the information he receives, player \( b \) decides to play 1, 2, or joker, that we denote \( j \). If he plays \( j \), the both players neither earn nor lose money. If the plays \( i \) in \{1, 2\}, then both players earn \( i \) euros if die \( s_0 \) is equal to \( i \), and lose 10 euros otherwise. This game has two locally optimal strategies \( \delta^1 \) and \( \delta^2 \). In strategy \( \delta^1 \), player \( a \) plays 1 and \( b \) plays \( i \) if he receives \textit{true} and \( j \) otherwise. Both strategy are locally optimal: each players decision is the best possible given the other ones. But only strategy \( \delta^2 \) is globally optimal.

The parametrization of the influence diagram that enables to encode this game is indicated on the right part of Figure 12. For any \( x \), mapping \( 1_x (\cdot) \) is the indicator function of \( x \). All the variables that are not on Figure 12 or on the paths on Figure 12 are unary. All the variables along paths represented by dashed arrows are equal. The parametrization ensures that variable \( X_{p_i} \) is equal to 1 if and only if \( X_{s_{i-1}} \) and \( X_{s_i} \) are not equal. Hence \( X_{a} \) and \( X_{s_0} \) are equal if and only if \( \sum_{i=1}^{k} X_{p_i} = 0 \) mod 2. Policies \( \delta^i \) can therefore be defined as

\[
\delta^i_a = 1_i \quad \text{and} \quad \delta^i_b(x_{p_1}, \ldots, x_{p_k}) = \begin{cases} i & \text{if } \sum_{i=1}^{k} x_{p_i} = 0 \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}
\]

where \( 1_i \) is the Dirac in \( i \). If \( a = t_k \), we define \( X_a = \{0, 1\} \) and \( \delta^i_a(x_{s_{k-1}}) = 1_i(x_{s_{k-1}}) \).

Consider now a junction tree on \( G \). As \( b \) is on an \( s_0-w \) path that is disjoint from \( Q \) except at its extremities, due to the running intersection property, there is a vertex \( u \) of the \( s_0-w_u \) subpath of \( Q \) that is in the cluster \( C \) where \( f_a(b) \) is included. Let \( \mu^1_C \) and \( \mu^2_C \) be the distributions induced by \( \delta^1 \) and \( \delta^2 \) on \( X_C \), and \( \mu^1_{ba} \) and \( \mu^2_{ba} \) their marginalizations on \( X_{ba} \). Let \( \mu_{ba} = \frac{\mu^1_{ba} + \mu^2_{ba}}{2} \). Denoting again \( 1_x \) the Dirac in \( x \), we have

\[
\mu^1_{ba} = \frac{1_{11} + 1_{j1} + 1_{je}}{3}, \quad \mu^2_{ba} = \frac{1_{j1} + 1_{22} + 1_{je}}{3}, \quad \text{and} \quad \mu_{ba} = \frac{1_{11} + 1_{j1} + 1_{j2} + 1_{22} + 21_{je}}{6}.
\]

We claim that there is no policy \( \delta \) that induces distribution \( \mu_{ba} \) on \( X_{ba} \). Indeed, in a distribution induced by a policy \( \delta \), it follows from the parametrization that if \( \mathbb{P}(X_a = 1) < 1 \) and \( \mathbb{P}(X_b = 1) > 0 \), then \( \mathbb{P}(X_b = 1, X_a = e) > 0 \). And, if \( \mathbb{P}(X_a = 2) < 1 \) and \( \mathbb{P}(X_b = 2) > 0 \), then \( \mathbb{P}(X_b = 2, X_a = e) > 0 \). (In both claims, "if \( \mathbb{P}(X_a = 1) < 1 \)" must be replaced by "if \( \delta_a(x_{s_{k-1}}) \neq 1_i(x_{s_{k-1}}) \) when \( a = t_k \)."") As \( \mu_{ab} \) is such that \( \mathbb{P}(X_b = 1) > 0, \mathbb{P}(X_b = 2) > 0, \) and \( \mathbb{P}(X_b = 1, X_a = e) > 0 \), it cannot be induced by a policy. Hence, \( S(G) \) is not convex. \( \square \)

7 Numerical experiments

In this section we provide numerical experiments showcasing the results of the paper. In particular, on two examples of varying size, we study the impact of the valid inequalities. On such examples, we solve the MILP formulation (21) with improved McCormick bounds relying on Appendix C.3 and valid inequalities from Section 5 obtained from the RJT of Algorithm 1.
More precisely we solve \( \max \{ \sum_{v \in V} (r_v, \mu_v) \mid (\mu, \delta) \in Q, \delta \in \Delta^d \} \) where \( Q \) is one of the four following polytopes: \( Q^1 = (\mathcal{P} \times \Delta) \cap Q^1 \) (no cuts), \( Q^b = (\mathcal{P} \times \Delta) \cap Q^b \) (McCormick only), \( Q^{\perp,1} = (\mathcal{P}^{\perp} \times \Delta) \cap Q^1 \) (independence cuts only), \( Q^{\perp,b} = (\mathcal{P}^{\perp} \times \Delta) \cap Q^b \) (McCormick and independence cuts).

An instance’s difficulty can be measured by the number of feasible deterministic policies, i.e., \( |\Delta^d| \) (Mauá et al. [2012b]). We have \( |\Delta^d| = \prod_{v \in V^a} |X_v| \prod_{v \in \text{pr}(v)} |X_v| \). Therefore, the difficulty depends on \( |X_v| \) for \( v \in \text{fa}(V^a) \). In our examples, we assume that \( \omega_a = |X_v| \) for all \( v \in \text{fa}(V^a) \) and \( \omega_s = |X_v| \) for all \( v \in V \setminus \text{fa}(V^a) \). Each instance is generated by first choosing \( \omega_a \) and \( \omega_s \). We then draw uniformly on [0, 1] the conditional probabilities \( p_{v|\text{pr}(v)} \) for all \( v \in V \setminus V^a \) and on [0, 10] the rewards \( r_v \) for all \( v \in V^r \).

The results are reported in Table 1 and Table 2. The first column specifies the size of the problem, the second the approximate number of admissible strategies. The third column indicates the cuts used. In the last four columns, we report the integrity gap (i.e., the relative difference between the linear relaxation and the best optimal integer solution), the final gap (between best admissible solution and best lower bound), the improvement obtained over the solution given by SPU and the computation time for each instance. All gaps are given in percent, time in seconds.

All mixed-integer linear programs have been written in Julia with JuMP interface and solved using Gurobi 7.5.2. Experiments have been run on a server with 192Gb of RAM and 32 cores at 3.30GHz. For each program, we use a warm start solution obtained by running SPU algorithm of Lauritzen and Nilsson [2001] on the instances.

For notational simplicity, and since it is unambiguous, in this section we use the same notation to refer to a given node of the graph and to refer to the random variable associated with this node.

### 7.1 Bob and Alice daily chess game

We consider the chess game example represented in Figure 2. The RJT built by Algorithm 1 for this example is represented in Figure 13. Since \( \psi_{a_t-1} \notin \mathcal{G}_t \) for all \( t \in [T] \), the chess game example is not a soluble ID, thus cannot be solved to optimality by SPU. Table 1 reports results on the generated instances.

In this problem we see that we can tackle large size problems: we can reach optimality in less than one hour for a strategy set of size \( 10^{44} \), and find a small provable gap on even bigger instances. Moreover, we see that the independance cuts reduce the computation time by a factor 100, whereas the improved McCormick bounds yield less impactfull improvements.

However, on this problem the SPU heuristic yields good results that are marginally improved by our MILP formulation. On this problem the main interest of our formulation is the bounds obtained. In the next problem we show better improvement.
Table 1: Results on generated instances for the chess game example, with a time limit TL=3600s. All gaps are relative and given in %. We note TL when the time limit is reached, in which case the gap between the best admissible solution and the lower bound is reported, otherwise we write Opt to specify that the solver reached optimality.
### 7.2 Partially Observed Markov Decision Process (POMDP) with limited memory

Another classical examples of ID are Partially Observed Markov Decision Processes introduced in Section 1. Figure 14 provides the graph representation of the POMDP with limited information. Since $\vartheta_{a_{t-1}} \mathcal{G}_t : \text{disc}(a_t) \cup \text{prt}(a_t)$ for all $t \in [T]$, this ID is not soluble. Figure 5b represents the RJT built by Algorithm 1.

However, for $v \in V^a$, $C_v \perp = \emptyset$ and the linear relaxation of Problem (21) does not enforce all the conditional independences that are entailed by the graph structure. Indeed, Theorem 14 ensures that the linear relaxation of MILP (21) corresponds to solving Problem (3) on the graph $\mathcal{G}$. For this example $\mathcal{G}$ corresponds to the MDP relaxation, which implies that the decision maker knows the state when he takes his decision. Therefore, the conditional independences $s_t \perp a_t | o_t$ is no more satisfied. Even if we cannot enforce these independences with linear constraints, we propose slightly weaker independences. We propose an extended formulation corresponding to a bigger RJT represented in Figure 15 to enforce $s_t$ to be conditionally independent from $a_t$ given $(s_{t-1}, a_{t-1}, o_t)$. In such a RJT, we have $C_{a_t} \perp = \{s_t\}$. Therefore, we can derive valid equalities (28) in $\mathcal{P}$. We demonstrate the efficiency of such inequalities by solving the different formulation on a set of instances, summed up in Table 2.

This example is harder to solve to optimality as we only reach strategy set of size $10^{72}$. Further, we can see that there are some instances where SPU seems to reach a local maximum that is improved by our MILP formulation. Once again the valid cuts significantly reduce the root linear relaxation gap and the solving time, even on large instances.

### Conclusion

This paper introduces linear and mixed integer linear programming approaches for the MEU problem on influence diagrams. The variables of the programs correspond to the moments of
| $(\omega_1, \omega_2, T)$ | $|\Delta|$ | Polytope | Int. Gap | Final Gap | SPU Gap | Time (s) |
|----------------------|--------|----------|---------|----------|---------|----------|
| $(4, 4, 10)$ | $10^{24}$ | $\mathcal{O}$ | 9.87 | Opt | 0.21 | 1004 |
| | | $\mathcal{O}$ | 9.11 | Opt | 0.21 | 971 |
| | | $\perp . . 1$ | 3.31 | Opt | 0.21 | 34 |
| | | $\perp . . b$ | 2.65 | Opt | 0.21 | 33 |
| $(4, 3, 20)$ | $10^{28}$ | $\mathcal{O}$ | 0.07 | Opt | 0.00 | 0.19 |
| | | $\mathcal{O}$ | 0.07 | Opt | 0.00 | 0.21 |
| | | $\perp . . 1$ | 0.00 | Opt | 0.00 | 0.19 |
| | | $\perp . . b$ | 0.00 | Opt | 0.00 | 0.20 |
| $(3, 5, 10)$ | $10^{34}$ | $\mathcal{O}$ | 8.88 | Opt | 0.58 | 12.19 |
| | | $\mathcal{O}$ | 7.72 | Opt | 0.58 | 7.31 |
| | | $\perp . . 1$ | 2.61 | Opt | 0.58 | 3.58 |
| | | $\perp . . b$ | 1.76 | Opt | 0.58 | 3.70 |
| $(4, 6, 10)$ | $10^{46}$ | $\mathcal{O}$ | 5.54 | Opt | 0.28 | 2527.48 |
| | | $\mathcal{O}$ | 5.20 | Opt | 0.28 | 1912.72 |
| | | $\perp . . 1$ | 0.37 | Opt | 0.28 | 1.26 |
| | | $\perp . . b$ | 0.03 | Opt | 0.28 | 1.11 |
| $(3, 5, 20)$ | $10^{69}$ | $\mathcal{O}$ | 9.06 | 2.56 | 0.66 | TL |
| | | $\mathcal{O}$ | 8.49 | 2.88 | 0.66 | TL |
| | | $\perp . . 1$ | 2.45 | Opt | 0.66 | 116.54 |
| | | $\perp . . b$ | 2.03 | Opt | 0.66 | 105.99 |
| $(4, 8, 10)$ | $10^{72}$ | $\mathcal{O}$ | 7.68 | 1.56 | 6.68 | TL |
| | | $\mathcal{O}$ | 6.13 | 1.54 | 6.68 | TL |
| | | $\perp . . 1$ | 2.51 | Opt | 6.68 | 518.44 |
| | | $\perp . . b$ | 1.31 | Opt | 6.68 | 396.50 |
| $(3, 6, 20)$ | $10^{93}$ | $\mathcal{O}$ | 10.45 | 7.17 | 0.76 | TL |
| | | $\mathcal{O}$ | 9.77 | 6.96 | 0.76 | TL |
| | | $\perp . . 1$ | 2.41 | 1.83 | 0.76 | TL |
| | | $\perp . . b$ | 2.34 | 1.75 | 0.76 | TL |
| $(4, 7, 20)$ | $10^{118}$ | $\mathcal{O}$ | 9.96 | 5.61 | 0.11 | TL |
| | | $\mathcal{O}$ | 9.15 | 5.07 | 0.10 | TL |
| | | $\perp . . 1$ | 2.92 | 1.84 | 0.11 | TL |
| | | $\perp . . b$ | 2.48 | 1.84 | 0.10 | TL |
| $(3, 9, 20)$ | $10^{171}$ | $\mathcal{O}$ | 11.21 | 6.73 | 4.60 | TL |
| | | $\mathcal{O}$ | 10.86 | 5.72 | 4.60 | TL |
| | | $\perp . . 1$ | 1.96 | 1.47 | 4.60 | TL |
| | | $\perp . . b$ | 1.78 | 1.46 | 4.60 | TL |
| $(3, 10, 20)$ | $10^{200}$ | $\mathcal{O}$ | 7.52 | 5.68 | 0.18 | TL |
| | | $\mathcal{O}$ | 7.30 | 5.63 | 0.19 | TL |
| | | $\perp . . 1$ | 1.32 | 1.11 | 0.05 | TL |
| | | $\perp . . b$ | 1.02 | 1.02 | 0.15 | TL |

Table 2: Results on generated instances for the POMDP example.
the measure induced by policies on the nodes of a new kind of junction trees, we coined rooted
junction trees. We introduce algorithms to build rooted junction trees tailored for our linear
and integer programs.

Soluble IDs are IDs for which the MEU problem is easy, in the sense that it can be solve
by the single policy update algorithm. We have introduced a linear program to solve soluble
IDs. Furthermore, we have shown a new characterization of soluble IDs as those for which there
exists a junction tree such that the vector of moments on the nodes of the tree is convex for
any parametrization of an influence diagram.

Finally, we provide a mixed integer linear programming approach to solve the MEU problem
on non-soluble IDs, as well as valid cuts. The bound provided by the linear relaxation is
better than the bound that could be obtained using SPU on a soluble relaxation. Numerical
experiments show that the bound is indeed better in practice.

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A McCormick Relaxation

For the sake of completeness we briefly recall the McCormicks relaxation, and their exactness if all but one are binary.

**Proposition 20.** Consider the variables \((x, y, z) \in [0, 1]^3\) and the following constraint

\[ z = xy \]  \hspace{1cm} (34)

Further, assume that we have an upper bound \(y \leq b\). We call McCormick the following set of constraints

\[ z \geq y + xb - b \] \hspace{1cm} (35a)

\[ z \leq y \] \hspace{1cm} (35b)

\[ z \leq bx \] \hspace{1cm} (35c)

If \(x, y\) and \(z\) satisfy eq. (34), then they also satisfy eq. (35). If \(x\) is a binary variable (that is \(x \in \{0, 1\}\)) and eq. (35) is satisfied, then so is eq. (34).

**Proof.** Consider \(x \in [0, 1], y \in [0, b]\) and \(z \in [0, 1]\), such that \(z = xy\). Noting that \((1-x)(b-y) \geq 0\) we obtain Constraint (35a). Constraints (35b) and (35c) are obtained by upper bounding by bounding one variable. Now assume that \(x \in \{0, 1\}, y \in [0, b]\) and \(z \in [0, 1]\) satisfy eq. (35). Then, if \(x = 1\), constraints (35a) and (35b) yield \(z = y\). Otherwise, as \(z \geq 0\), we have \(z = 0\) by (35c).

B Algorithm to build a small RJT

In this section we present an algorithm to build a RJT without considering a topological ordering on the initial graph \(G = (V, A)\).

**Algorithm 3** Build a RJT

```
1: Input \(G = (V, E)\)
2: Initialize \(C = \emptyset\) and \(A' = \emptyset\)
3: \(L = \{v\text{ s.t } \text{cl}_G(v) = \emptyset\}\)
4: while \(L \neq \emptyset\) do
5: if \(V^a \cap L \neq \emptyset\) then
6: Pick \(v \in V^a \cap L\)
7: else
8: Pick \(v \in L\)
9: end if
10: \(C_v \leftarrow \text{fa}(v)\)
11: for \(C_x \in C : v \in C_x\) do
12: \(C_v \leftarrow C_v \cup (C_x \setminus \{x\})\)
13: Remove \(C_x\) from \(C\)
14: Add \(\{v, x\}\) in \(A'\)
15: end for
16: Add \(C_v\) to \(C\)
17: Remove \(v\) from \(G\) and recompute \(L\)
18: end while
19: \(A \leftarrow \{(C_u, C_v) | (u, v) \in A'\}\)
20: Return \(\mathcal{G} = ((C_v)_{v \in V}, A)\)
```
The only difference between Algorithms 1 and 3 is that the for loop along a reverse topological ordering of Algorithm 1 is replaced in Algorithm 3 by a breadth first search that computes online this reverse topological ordering. Hence, if we denote $\leq$ this ordering, Algorithm 3 builds the same RJT as the one we obtain when we use Algorithm 1 with $\leq$ in input. Therefore, the RJT built by Algorithm 3 satisfies 11, and is such that the implications in (12) are equivalence.

Furthermore, Steps 5 and 6 enable to ensure that, when there is no path between a vertex $u \in V^a$ and a vertex $v \in V^s$, then $u$ is placed before $v$ in the reverse topological ordering computed by the breadth first search. Therefore, $\leq$ is a topological ordering on the graph $G''$ used as Step 5 of Algorithm 2. Hence, if $G$ is soluble, Algorithm 3 builds a RJT such that $G'\dagger = G$.

Remark that on non-soluble IDs, Steps 5 and 6 are a heuristic aimed at minimizing the size of $C_v$ for each $v \in V^a$. Such a heuristic is not relevant if valid cuts (28) are not used. In that case, an alternative strategy could be to add as few variable as possible to $C_v$ in order to improve the quality of the soluble relaxation $\bar{G}$. This could be done by putting vertices $u \in V^s$ unrelated to $v \in V^a$ after in this topological order, i.e., by replacing $V^a$ by $V^s$ in Steps 5 and 6.

### C Choice of the bounds in McCormick inequalities

#### C.1 Using $b_{C_v} = 1$ leads to loose constraints

As $\mu_{C_v}$ is a probability distribution, 1 is an immediate upper bound on $\mu_{C_v}$. Let $Q^1$ be the polytope $b\bar{Q}$ obtained using bounds vector $b$ defined by $b_{C_v} = 1$ for all $v \in V^a$.

**Proposition 21.** Let $\mu$ be in $\overline{\mathcal{P}}$. Then there exists $\delta$ in $\Delta$ such that $(\mu, \delta)$ belongs to $Q^1$, and the linear relaxation of (21) is equal to $\max_{\mu \in \overline{\mathcal{P}}} \sum_{v \in V^r} \langle r_v, \mu_v \rangle$.

**Proof.** Let $v$ be a vertex in $V^a$, and let

$$
\delta_{v|\text{prt}(v)}(x_{fa(v)}) = \begin{cases} 
\frac{\sum_{x'_{V\setminus fa(v)}} \mu_{C_v}(x_{fa(v)}, x'_{V\setminus fa(v)})}{\sum_{x'_{\text{prt}(v)}} \mu_{C_v}(x_{\text{prt}(v)}, x'_{V\setminus \text{prt}(v)})} & \text{if } \sum_{x'_{V\setminus \text{prt}(v)}} \mu_{C_v}(x_{\text{prt}(v)}, x'_{V\setminus \text{prt}(v)}) \neq 0, \\
1 & \text{otherwise},
\end{cases}
$$

where $e_v$ is an arbitrary element of $\mathcal{X}_v$. To prove the result, we show that $[\text{McCormick}(C_v, b)_{C_v}]$ is satisfied for this well-chosen $\delta_{v|\text{prt}(v)}$ and $b_{C_v} = 1$.

We have

$$
\mu_{C_v}(x_{C_v}) - \mu_{C_v}(x_{C_v}) \geq \sum_{x'_{C_v\setminus fa(v)}} \mu_{C_v}(x'_{C_v\setminus fa(v)}, x_{fa(v)}) - \mu_{C_v}(x'_{C_v\setminus fa(v)}, x_{fa(v)}) \leq 0
$$

$$
= \mu_{fa(v)}(x_{fa(v)}) - \mu_{\text{prt}(v)}(x_{\text{prt}(v)})
$$

$$
= \frac{1}{\mu_{\text{prt}(v)}} (\delta_{v|\text{prt}(v)}(x_{fa(v)}) - 1)
$$

$$
\geq \delta_{v|\text{prt}(v)}(x_{fa(v)}) - 1
$$

which gives $\mu_{C_v} \geq \mu_{C_v} + (\delta_{v|\text{prt}(v)} - 1)b_{C_v}$.

Besides, if $\mu_{C_v}(x_{C_v}) \geq 0$, following the definition of the $\delta$ and as $\sum_{x'_{V\setminus \text{prt}(v)}} \mu_{C_v}(x'_{\text{prt}(v)}, x'_{V\setminus \text{prt}(v)}) \leq 1$, we have

$$
\delta_{v|\text{prt}(v)}(x_{fa(v)}) \geq \sum_{x'_{V\setminus fa(v)}} \mu_{C_v}(x_{fa(v)}, x'_{V\setminus fa(v)}) \geq \mu_{C_v}(x_{C_v})
$$

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and the constraint $\mu_{C_v} \leq \delta_{\text{opt}(v)} b_{\tilde{C}_v}$ is satisfied.

Finally, $\mu_{C_v} \leq \mu_{C_v}$ follows from the marginalization constraint $\mu_{C_v} = \sum_{x_v} \mu_{C_v}$ in the definition of the local polytope.

\section{McCormick inequalities with well-chosen bounds are useful}

This section provides examples of IDs where McCormick inequalities improve the linear relaxation of MILPs \ref{eq:milp} and \ref{eq:milp2}. Consider the ID on Figure 16.a, and assume that we have a bound $\mu_{st} \leq b_{st}$. Then, the McCormick relaxation of $\mu_{sta} = \mu_{st} \delta_{a|t}$ reads

$$
\begin{align*}
\mu_{sta} & \geq \mu_{st} + b_{st}(\delta_{a|t} - 1) \\
\mu_{sta} & \leq b_{st}\delta_{a|t}
\end{align*}
$$

Suppose that all variables are binary, that $s$ is Bernoulli with parameter $\frac{1}{2}$, that $\mathbb{P}(X_t = 1 | X_s) = 1 + \varepsilon X_s - \varepsilon (1 - X_s)$, that $X_w$ indicates if $X_s = X_t$, and that the objective is to maximize $E_{\delta}(X_w)$, and has value $\frac{1}{2} + \varepsilon$. An optimal policy consists in choosing $X_s = X_t$. An optimal solution of the linear relaxation of \ref{eq:milp} on $\mathbb{P}$ without McCormick inequalities, has value $1$. Whereas an optimal solution with McCormick inequality and $b_{st}(x_s, x_t) = \frac{1}{2} + \varepsilon 1_{x_s = x_t}$ has value $\frac{1}{2} + \varepsilon$. However, on this simple example, the McCormick inequalities are implied by the valid inequalities of Section 3.

This is no more the case on the ID of Figure 16.b, where $r$ is a Bernoulli of parameter $0.5$ and $X_s = X_r X_b + (1 - X_r)(1 - X_b)$, and the remaining of the parameters are defined as previously. Using the same bounds, this new example leads to exactly the same results as before.

\section{Algorithm to choose good quality bounds}

This section provides a dynamic programming equation to compute bounds $b_{C_v}$ on $\mu_{C_v}$ that are small than $1$. Let $G$ be a RJT, and $C_1, \ldots, C_n$ be a topological order on $G$. We define inductively on $i$ the functions $\tilde{b}_i : X_{C_i} \to [0, 1]$ as follows.

$$
\begin{align*}
\tilde{b}_1(x_{C_1}) &= \prod_{v \in C_1 \cap V^s} p(x_v | x_{\text{prt}(v)}) \\
\tilde{b}_i(x_{C_i}) &= \min \left\{ 1, \sum_{x_{\text{prt}(V^s \cap C_k) \setminus C_i}} \max_{x_{V^s \cap (C_k \cup \text{prt}(V^s)) \setminus (C_k \cap \text{prt}(V^s))}} \tilde{b}_k(x_{C_k \cup \text{prt}(C_k \cap V^s)}), \prod_{v \in C_i \cap V^s} p(x_v | x_{\text{prt}(v)}) \right\}
\end{align*}
$$

for $i > 1$

where $C_k$ is the unique parent of $C_i$ in $G$. W.l.o.g. we can assume that $\text{prt}(C_k \cap V^s) \subseteq C_k$.

\begin{proposition}
Let $\mu$ be in $S(G)$. We have $\mu_{C_i}(x_{C_i}) \leq \tilde{b}_i(x_{C_i})$ for all $i$ and $x_{C_i}$ in $X_{C_i}$.
\end{proposition}

As a consequence, $b_{C_v}$ defined as $\sum_{x_v} \tilde{b}_{C_v}$ provides an upper bound on $\mu_{C_v}$ that can be used in McCormick constraints.