Magnetic fields from inflation?

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Abstract

We consider the possibility of generation of the seeds of primordial magnetic field on inflation and show that the effect of the back reaction of this field can be very important. Assuming that back reaction does not spoil inflation we find a rather strong restriction on the amplitude of the primordial seeds which could be generated on inflation. Namely, this amplitude recalculated to the present epoch cannot exceed $10^{-32} G$ in Mpc scales. This field seems to be too small to be amplified to the observable values by galactic dynamo mechanism.

1 Introduction

Astronomical observations show that all celestial bodies carry magnetic fields. From planets to interstellar medium, fields of varying strength and extension have been measured. A particular interesting case are galaxies, galaxy clusters and beyond, the intergalactic medium (IGM) and the Universe at large. These fields are of order of a few micro Gauss and they extend over kiloparsecs or more. Unfortunately their structure is not always simple. Besides a constant component they have complex structure with varying symmetry, that shows that processing has taken place since their appearance.

The origin of magnetic fields is unknown and many scenarios have been proposed to explain them. Until recently the most accepted idea for the formation of large-scale magnetic fields was the exponentiation of a seed field as suggested by Zeldovich and collaborators long ago. This seed mechanism is known as galactic dynamo, the idea is the amplification of a tiny field created early enough by differential rotation of the galaxies and the subsequent generation of the galactic and cluster fields.

However recent observational developments have cast serious doubts on this possibility. In fact there are already many reasons to believe, tough this is a possible mechanism in some cases, it cannot be universal. Some of the reasons to think that seeding cannot be an answer are simple. First, the very existence of high z galaxies with fields comparable to the Milky Way is incompatible with the necessary number of turns. Second, the narrowness of the distribution, most galaxies and clusters have fields of a few micro Gauss, and this is not compatible with the different number of rotations and the parameters involved in every galaxy. Furthermore, magnetic fields seem to increase with redshift, though the evidence is not overwhelming, the sample of Faraday rotations measured is now consistent with an increase and the set includes tens of galaxies showing
this pattern. Finally, as pointed out by Dolgov, it is difficult to create the fields in clusters since even the most efficient ejection from point bodies in galaxies like supernovas would have difficulty creating them. All put together seeding seems to be ruled out and moreover, even if the galactic dynamo was effective, one should justify the presence of a seed field which started the process.

This is why the mechanism responsible for the origin of large-scale magnetic fields is looked in the Early Universe [3], [4], [5]. In this paper we will consider the generation of large-scale magnetic fields during inflation, which as it was noted by Turner and Widrow is a natural candidate for doing this job [6].

It is known that in the Friedmann universe the conformal vacuum is preserved if the theory is conformally invariant [7]. Classical electrodynamics is conformally invariant, so that photons should not be produced in cosmological background. Thus the conformal invariance of the electromagnetism must be broken to produce long wave magnetic fields via excitation of the vacuum fluctuations.

Different mechanisms to break the conformal invariance of electromagnetic field were proposed in ref. [6]. All of them are effectively reduced either to the appearance of the effective mass or time dependent coupling constant [6], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. Here we will consider the broad class of models where conformal invariance is broken during inflation and investigate the back reaction of the generated magnetic field on the background. We show that this back reaction is very important and leads to rather strong bounds on the maximal value of the strength of primordial magnetic fields which seems not enough to explain the observed fields as a result of amplification of these primordial seeds by dynamo mechanism.

2 Models

The action for the massless vector field is

$$ S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4 x = -\frac{1}{4} \int F_{\mu\rho} F_{\nu\sigma} g^{\mu\nu} g^{\rho\sigma} \sqrt{-g} d^4 x, \quad (1) $$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ is conformally invariant. It is easy to see that under conformal transformation $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$ the determinant transforms as $g \rightarrow \Omega^8 g$ and $g^{\mu\nu} \rightarrow \Omega^{-2} g^{\mu\nu}$. This is the reason why in the Friedmann universe with the metric

$$ ds^2 = a^2(\eta) \left( d\eta^2 - \delta_{ik} dx^i dx^k \right) \quad (2) $$

the conformal vacuum is preserved. Therefore if we want to amplify quantum fluctuations on inflation and thus explain the origin of primordial magnetic fields we have to assume that either electromagnetic field is massive or its effective coupling is time-dependent during inflation. Both of these options are taken into account if we write the action in the form

$$ S = \int \left( \frac{1}{4} I^2 F_{\mu\nu} F^{\mu\nu} + M^2 A_\mu A^\mu \right) \sqrt{-g} d^4 x. \quad (3) $$

Here $I(t) = I(\phi(t),...)$, where $\phi$ can be the inflaton, dilaton or some other scalar field and the dots can be anything, for instance, invariants of the curvature (see [11], [16], [14], [15]). The appearance of time dependence of the coefficient in front of $F^2$ term is naturally interpreted as time-dependent coupling constant of the vector field. In fact if we write the Lagrangian density of the vector field coupled with a charged fermion in the standard form as

$$ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu + ig A_\mu) \psi, \quad (4) $$

where $g$ is the coupling constant, then after rescaling the vector potential by the coupling constant $A_\mu \rightarrow g A_\mu$ we bring this Lagrangian to the form

$$ L = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu + i A_\mu) \psi, \quad (5) $$

2
which is “ready” for introducing a time-dependent coupling constant. Note that \( I \) is an inverse coupling constant and small values of \( I \) correspond to a large coupling constant \( g \), which in turn would mean that we are in uncontrollable strong coupling regime. Only if \( I \) is large we can trust the theory. For our purposes we do not need to specify in more details the origin of the time dependence of \( I \) here. Note that the time-dependent effective coupling leaves the Lagrangian to be \( U(1) \) gauge-invariant.

The mass term introduced by “hand” spoils gauge invariance. Only when it is generated via Higgs mechanism the gauge invariance is preserved. On the other hand as it was noticed already in [6] large enough magnetic fields can be obtained only if \( M^2 \) is negative during inflation. However, to generate negative mass squared term via Higgs mechanism one needs a ghost scalar field with negative kinetic energy [19], [20]. As it is well known ghosts lead to catastrophic instabilities and therefore we will not exploit this possibility any further here. Instead we introduce the effective negative mass square terms considering the non-minimal coupling of the vector field to gravity, so that,

\[
M^2 = m^2 + \xi R,
\]

where for generality we also keep “hard” mass term \( m^2 \) assuming that it is positive.

Let us now rewrite the action (3) in terms of the vector potential \( A_\alpha = (A_0, A_i) \). It is convenient to decompose the spatial part of the vector potential in terms of its transverse and longitudinal components \( A_i = A_i^T + \partial_i \chi \), where \( \partial_i A_i^T = 0 \) (we will be assuming summation over repeated indices irrespective of their position). In the homogeneous flat universe with metric (2), the action (3) then becomes

\[
S = \frac{1}{2} \int \left[ I^2 \left( A_i^{T\prime} A_i^{T\prime} + A_i^T \Delta A_i^T + 2 A_0 \Delta A_0 - A_0 \Delta A_0 - \chi' \Delta \chi' \right) + M^2 a^2 \left( A_i^2 + \chi^2 \Delta \chi - A_i^T A_i^T \right) \right] d^4x,
\]

where prime denotes derivative with respect to the conformal time \( \eta \). We will consider different cases separately.

### 2.1 Time dependent coupling

Let us first consider the case when \( M^2 = 0 \) and \( I = I(t) \). Then the variation of the action (7) with respect to \( A_0 \) gives \( A_0 = \chi' \), and the action simplifies to

\[
S = \frac{1}{2} \int I^2 \left( A_i^{T\prime} A_i^{T\prime} + A_i^T \Delta A_i^T \right) d^4x.
\]
It describes two real scalar fields with time-dependent effective masses in terms of their Fourier components.

We are interested in the correlation functions of the transverse components of the vector potential and magnetic field assuming that initially the field was in its vacuum state. The quantization of the fields with action \((12)\) is standard and we will simply summarize here the results referring the reader to \([21], [22]\) for the details. Taking into account \((11)\) and \((9)\), we immediately find the correlation function

\[
<0|\hat{A}_i^T(\eta, x)\hat{A}^{T j}(\eta, y)|0> = -\frac{1}{4\pi^2 a^2 F^2} \sum_{\sigma=1,2} \int |v_k^{(\sigma)}(\eta)|^2 k^3 \frac{\sin k|x-y|}{k|x-y|} \, dk, \tag{13}
\]

where \(v_k^{(\sigma)}(\eta)\) satisfy the equations

\[
v_k^{(\sigma)\mu} + \omega^2(\eta) v_k^{(\sigma)} = 0, \quad \omega^2(\eta) \equiv \left(k^2 - \frac{I''}{I}\right), \tag{14}
\]

which immediately follow from action \((12)\). The initial conditions for these equations corresponding to the initial vacuum state at \(\eta_i\) are

\[
v_k^{(\sigma)}(\eta_i) = \frac{1}{\sqrt{\omega(\eta_i)}}, \quad v_k^{(\sigma)\prime}(\eta_i) = i \sqrt{\omega(\eta_i)}. \tag{15}
\]

These initial conditions make sense only if \(\omega^2 > 0\). Anyway we will need them only for the short-wavelength modes for which \(\omega^2 \approx k^2\). The power spectrum characterizing the typical amplitude squared of the invariant magnitude of the vector potential, \(A = \sqrt{-A_i A^i}\), in the appropriate comoving scale \(\lambda = 2\pi/k\) is

\[
\delta_A^2(k, \eta) = \sum_{\sigma=1,2} \frac{|v_k^{(\sigma)}(\eta)|^2 k^3}{4\pi^2 a^2 F^2}. \tag{16}
\]

Taking into account that the magnitude of the magnetic field is

\[
B^2 = -B_i B^i = \frac{1}{2a^4} F_{ik} F_{ik} = \frac{1}{a^4} (\partial_i A_k \partial_i A_k - \partial_k A_i \partial_i A_k), \tag{17}
\]

we obtain for the power spectrum of the magnetic field

\[
\delta_B^2(k, \eta) = \delta_A^2(k, \eta) \frac{k^2}{a^2} = \sum_{\sigma=1,2} \frac{|v_k^{(\sigma)}(\eta)|^2 k^5}{4\pi^2 a^4 F^2}, \tag{18}
\]

that is, its amplitude decays faster by an extra power of the scale compared to the amplitude of the vector potential. For example, a flat spectrum for magnetic field \((\delta_B(k) = \text{const})\) corresponds to the linearly growing towards large scales spectrum for the vector potential, that is, \(\delta_A(k, \eta) \propto k^{-1}\).

We will need to control the back reaction of the generated electromagnetic field on the background. With this purpose let us calculate the expectation value of the energy density equal to \(T^{00}\) component of the energy-momentum tensor:

\[
T^{00}_0 = I^2 \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - F_{0\alpha} F^{0\alpha}\right) = \frac{I^2}{2a^4} \left(A_i A^i + \partial_i A_i \partial_i A^i\right). \tag{19}
\]

Taking into account \((9)\) and \((11)\) we obtain

\[
\varepsilon_{EM} = <0|T^{00}_0|0> = -\frac{1}{8\pi^2 a^4} \sum_{\sigma=1,2} \int \left[|v_k^{(\sigma)\prime}(\eta)|^2 - \frac{I'}{I} |v_k^{(\sigma)}(\eta)|^2 + \left(\frac{I''}{I} + k^2\right) |v_k^{(\sigma)}(\eta)|^2\right] k^3 \frac{dk}{k}. \tag{20}
\]

Let us assume that the function \(I\) depends on time during inflation and find the resulting spectrum of the magnetic field at the end of inflation. For short waves with \(k|\eta| \gg 1\) we can neglect
$I''/I$ compared to $k^2$ in (14) and the solution of this equation with vacuum initial conditions (15) then becomes

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} e^{i k (\eta - \eta_i)}.$$

Because $|\eta|$ decreases during inflation at some moment $|\eta_k| \simeq 1/k$ the physical scale of the wave with comoving wavenumber $k$ begins to exceed the curvature scale and taking into account that $k^2 \ll I''/I$ we can write the general longwave solution of (14) as

$$v_k^{(\sigma)}(\eta) \simeq C_1 I + C_2 I \int \frac{dn}{T^2},$$

where $C_1$ and $C_2$ are the constants of integration which have to be fixed by matching this solution to (21) at $|\eta_k| \simeq 1/k$. Let us assume that $I$ is a power-law function of the scale factor during inflation

$$I = I_f \left( \frac{a}{a_f} \right)^n,$$

where $a_f$ is the scale factor at the end of inflation. Taking into account that

$$dn = \frac{da}{Ha^2},$$

and the Hubble constant $H$ does not change significantly during inflation we obtain from (22)

$$v_k^{(\sigma)}(\eta) \simeq C_1 a^n + C_2 a^{-n-1}.$$

### 2.1.1 Strong coupling case

In case $n > -1/2$ the first mode dominates and, matching solutions (21) and (24) at $|\eta_k| \simeq 1/k$, we find

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} \left( \frac{a}{a_k} \right)^n \simeq \frac{1}{\sqrt{k}} \left( \frac{H_f a}{k} \right)^n,$$

where we have taken into account that at the moment $\eta_k$, when the corresponding wave crosses the Hubble scale, the scale factor is $a_k = k/H_I$. Substituting (25) into (18) we obtain at the end of inflation

$$\delta_B (\lambda_{ph}, \eta_f) \simeq \frac{H_f^2}{\sqrt{2} \pi I_f} \left( \frac{\lambda_{ph}}{H_I} \right)^{n-2},$$

where $\lambda_{ph} = a_f/k$ is the physical wavelength and $H_f$ is the Hubble constant on inflation. This formula is valid for $H_f^{-1} (a_f/a_i) > \lambda_{ph} > H_f^{-1}$, where $a_i$ is the value of the scale factor at the beginning of inflation. If $n = 2$ the spectrum of the magnetic field is flat. For $H_f^2 \approx 10^{-12}$ (in Planck units), required by primordial inhomogeneities (21), and $I_f \approx O(1)$, the amplitude of the field is the same in all scales and it is equal to $\delta_B \approx 10^{-12}$ Planck units or $\sim 10^{46} G$ immediately after inflation. Later on the magnetic field is frozen and decays inversely proportional to the scale factor squared. To estimate how much the scale factor decreases after inflation we can use the entropy conservation law. Assuming that inflation is followed but the dust dominated stage we obtain

$$\frac{a_0}{a_f} \approx g^{1/12} \frac{H_I^{1/2}}{T_0} \left( \frac{a_R}{a_f} \right)^{1/4},$$

where $g$ is the number of relativistic degrees of freedom of those particles which later on transfer their entropy to the photons, $T_0$ is the temperature of the background radiation today and $a_f$ the scale factor at the moment of reheating. The lower bound on this ratio is obtained assuming that reheating happens immediately after inflation. In this case for $H_I \approx 10^{-6}$ we have $a_0/a_f \approx 10^{-29}$ and correspondingly the strength of the generated magnetic field cannot exceed $10^{-12} G$.

Let us calculate the energy density of the generated magnetic field. The main contribution to the energy density comes from the scales exceeding $H_f^{-1}$ because the contribution from the
In this case the result follows immediately by substituting in the formulae (25) and (26) for \( v \propto I \) and \( A^2 \propto v/I \propto \text{const} \) the time derivatives of the vector potential in (19) contribute only in subleading \( k^2 \) order and their contribution is comparable to the contribution of the magnetic field itself given by the last term in (20). Thus we obtain

\[
\varepsilon_{EM} = \frac{O(1)}{a^4} \int_{H_1 a}^{H_f a} |v_k(\eta)|^2 k^4 dk,
\]

where \( a_i \) is the value of the scale factor at the beginning of inflation. Substituting (25) into (28) we find that at the end of inflation when \( a = a_f \)

\[
\varepsilon_{EM} = O(1) H_f^4 \times \begin{cases} 
\frac{1}{2-n}, & n < 2, \\
\ln \left( \frac{a_f}{a_i} \right), & n = 2, \\
\frac{1}{n-2} \left( \frac{a_f}{a_i} \right)^{2(n-2)}, & n > 2.
\end{cases}
\]

We see that the magnetic field energy can be comparable with the energy density of the background only for \( n \geq 2 \). Requiring that inflation should last at least 75 e-folds we obtain that the contribution of the magnetic field energy density does not spoil inflation, that is, \( \varepsilon_{EM} \) is smaller that \( H_f^2 \) until the end of inflation, only if \( n - 2 < 0.2 \). Thus, we can have slightly growing toward large scales spectrum of the magnetic field. In particular, for \( n \approx 2.2 \) the amplitude of the magnetic field in Mpc scales can be larger by a factor \( 10^5 \) compared to the considered above case of the flat spectrum, that is, \( \delta_B \approx 10^{-7} \) \( G \) today. This is the greatest amplitude of the primordial magnetic field which we can obtain in the considered above case. Note that the theory where \( I \) grows with the scale factor corresponds to the case when the effective coupling constant, which is inversely proportional to \( I \), is incredibly large at the beginning of inflation and becomes of the order of one at the end of inflation. Hence at the beginning we are in strongly coupled regime where such theory is not trustable at all.

The case considered above is the only one in which we can generate strong enough fields on inflation. Let us show that in all other cases there is very strong bound on the possible value of the generated field due to the back reaction of this field on the background.

### 2.1.2 Weak coupling case

For \( n < -1/2 \) the second term in (24) dominates and

\[
v_k(\eta) \propto a^{-n-1}.
\]

In this case the result follows immediately by substituting in the formulæ (25) and (26) \(-n - 1\) instead of \( n \), so that

\[
v_k(\sigma)(\eta) \simeq \frac{1}{\sqrt{k}} \left( \frac{a}{a_k} \right)^{n+1-n-1} \simeq \frac{1}{\sqrt{k}} \left( \frac{H_f a}{k} \right)^{n-1},
\]

and

\[
\delta_B(\lambda_{ph}, \eta_f) \simeq \frac{H_f^2}{\sqrt{2\pi I_f}} \left( \frac{\lambda_{ph}}{H_f^{-1}} \right)^{-n-3}.
\]

Thus the spectrum of the magnetic field is flat for \( n = -3 \). This case corresponds to the coupling constant growing as \( I^{-1} \propto a^3 \), that is, it changes from extremely small values at the beginning of inflation to values of order of unity at the end of inflation. Thus the theory is trustable everywhere. However here the back reaction of the field is very large because \( A \propto v/I \propto a^{-2n-1} \) changes very fast and the main contribution to the energy density comes from the time derivative of the vector potential in (19), that is, from the electric field. Substituting (31) in (20) we obtain that at the end of inflation

\[
\varepsilon_{EM} \simeq \frac{4n^2 + 4n + 1}{8\pi^2} H_f^4 \times \begin{cases} 
\frac{1}{n+2}, & n > -2, \\
\ln \left( \frac{a_f}{a_i} \right), & n = -2, \\
\frac{1}{n+2} \left( \frac{a_f}{a_i} \right)^{-2(n+2)}, & n < -2.
\end{cases}
\]
Taking the Fourier transform, we obtain

\[ \Delta \chi' - \Delta A_0 + M^2 a^2 A_0 = 0. \] (34)

Taking the Fourier transform

\[ \chi(x, \eta) = \int \chi_k(\eta) e^{i k \cdot x} \, \frac{d^3 k}{(2\pi)^3/2}, \quad A_0(x, \eta) = \int A_{0k}(\eta) e^{i k \cdot x} \, \frac{d^3 k}{(2\pi)^3/2}, \] (35)

we obtain from here

\[ A_{0k} = \frac{k^2}{k^2 + M^2 a^2} \chi'_k \equiv F_k \chi'_k. \] (36)

Substituting into action (7) the expansions (9), (35) and using (30) to express \( A_{0k} \) in terms \( \chi_k \) we obtain

\[ S = \frac{1}{2} \sum_{\sigma=1,2} \int \left( v_k^{(\sigma)} v_k^{(\sigma)} - (k^2 + M^2 a^2) v_k^{(\sigma)} v_k^{(\sigma)} \right) d\eta d^3 k \]
\[ + \frac{1}{2} \int \text{sign} (1 - F_k) \left( \chi'_k \chi'_k - \left( k^2 + M^2 a^2 - \frac{\sqrt{1 - |F_k|}}{\sqrt{1 - F_k}} \right) \chi_k \chi_k \right) d\eta d^3 k, \] (37)

where \( v_k^{(\sigma)} \) is defined in (11) \( (I = 1) \), and

\[ \chi_k = k \sqrt{|1 - F_k|} \chi_k. \] (38)

Thus we see that in the case of massive field the longitudinal degree of freedom \( \chi \) becomes dynamical. In the case of positive mass squared \( F_k \) is always smaller than unity and therefore the sign in front of the longitudinal part of the action is positive. However, if \( M^2 \) is negative then \( 1 - F_k \) is negative for high momentum modes with \( k^2 > M^2 a^2 \) and these modes have negative kinetic energy. The low momentum modes with \( k^2 < M^2 a^2 \) have positive kinetic energy because \( F_k \) is negative for them. Thus, introducing a tachyonic mass for the vector field in a “hard” way seems to lead inevitably to the appearance of ghost for high momentum longitudinal modes. Therefore if we want to avoid catastrophic instabilities related with ghost fields we have to consider tachyonic vector field only as a low energy effective field theory description of some unknown yet theory with “safe” ultraviolet completion. On the other hand if negative effective mass appears as interaction with the curvature, \( M^2 = \xi R \), then the field is massless on scales smaller than the typical distance between particles inducing the average curvature and thus there is a natural ultraviolet cutoff in the theory. Note that this argument is not directly applicable in the presence of the cosmological constant. Let us assume that the problem of ghosts can be somehow solved and proceed with the calculation of the magnetic field from inflation in the theory with \( M^2 = m^2 + \xi R \). In the case \( m = 0 \) the photon mass is \( m_\gamma \sim R^{1/2} \), where \( R^{1/2} \sim H \). Today it would be \( m_\gamma = H_{\text{today}} \sim 10^{-33} \text{eV} \), well below the available experimental limits on the photon mass. The breaking of charge conservation also manifests itself only on scales of the horizon or larger \( (\geq H^{-1} \sim 10^{28} \text{cm}) \) and hence has no observable consequences.
The equations of motion for transverse and longitudinal modes follow immediately from the action (37):
\[ v_k^{(\sigma)\mu} + (k^2 + M^2 \alpha^2) v_k^{(\sigma)} = 0, \]  
(39)
and
\[ \ddot{\chi}^\nu + \left( k^2 + M^2 \alpha^2 - \frac{\sqrt{1-F_{11}}}{\sqrt{1-F_{11}}} \right) \dot{\chi}_k = 0. \]  
(40)

Let us consider de Sitter universe where
\[ a = -\frac{1}{H_1\eta}. \]

Taking into account that \( R = -12H_1^2 \), for \( m^2 = 0 \) equation (39) becomes
\[ v_k^{(\sigma)\mu} + \left( k^2 - \frac{12\xi}{\eta^2} \right) v_k^{(\sigma)} = 0. \]
(41)

For short waves with \( k |\eta| \gg 1 \) the solution of this equation corresponding to vacuum initial conditions is
\[ v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} e^{\pm ik(\eta-t)}. \]
(42)

For \( k |\eta| \ll 1 \) we can neglect the \( k^2 \) term in (41) and the dominating longwavelength solution of this equation is
\[ v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} \left( H_1\alpha \right)^n, \]
(43)
where we use the matching conditions at \( |\eta_k| \simeq 1/k \) to fix the constant of integration. Since here the calculations are very similar to those in the previous section we can immediately write the result for the magnetic field
\[ \delta_B(\lambda_{ph},\eta_f) \simeq O(1) H_1^2 \left( \frac{\lambda_{ph}}{H_1} \right)^{n-2}. \]
(44)

For \( \xi = 1/6 \) we have \( n = 1 \) and the spectrum linearly decays with the scale. In this case its value today is about \( 10^{-37} G \) in \( Mpc \) scales. The flat spectrum is obtained for \( \xi = 1/2 \). However, to find out whether this case is possible we have to verify that the back reaction of the magnetic field will not spoil inflation too early. In the energy density also contributes the longitudinal mode and to determine its contribution we will need a longwavelength solution for \( \chi_k \). It is easy to check that the term which is different in the equations (39) and (40) can be neglected for both shortwave and longwave solutions and hence
\[ \ddot{\chi}_k(\eta) \simeq \frac{1}{\sqrt{k}} \left( H_1\alpha \right)^n, \]
(45)

Variation of action (3), where \( I = 1 \) and \( M^2 = m^2 + \xi R \), with respect to the metric gives
\[ T_{\mu}^{\phi} = \frac{1}{4} \delta_{\mu}^{\rho} F_{\alpha\beta} F^{\alpha\beta} - F^{\rho\beta} F_{\mu\beta} - \frac{1}{2} \eta_{\mu} (m^2 + \xi R) A_\alpha A^\alpha + (m^2 + \xi R) A_\mu A^\mu + \xi R^{\mu}_{\rho} A_\alpha A^\alpha + \xi [\delta_{\mu}^{\rho} \nabla_\alpha (A^\beta A_\beta) - \nabla_\mu \nabla_\alpha (A^\beta A_\beta)]. \]
(46)

As a result of straightforward but rather lengthy calculations we obtain
\[ <0|\hat{T}_0^{\phi}|0> = \varepsilon_T + \varepsilon_L, \]

where
\[ \varepsilon_T = \frac{1}{8\pi^2 a^4} \sum_{\sigma = 1,2} \int \left[ |v_k^{(\sigma)}|^2 - 6\xi a H |v_k^{(\sigma)}|^2 + (k^2 + m^2 a^2 + 6\xi H^2 a^2) |v_k^{(\sigma)}|^2 \right] k^3 \frac{dk}{k} \]
(47)
is the contribution of the transverse modes, $H = a'/a^2$ is the Hubble constant. The contribution of the longitudinal mode is given by

$$\varepsilon_L = \frac{1}{8\pi^2 a^4} \int (1 - F) \left( (1 - 6\xi bF)|\tilde{\chi}_k|^2 - 6\xi aH \left( \frac{1 + F}{1 - F} \right) |\tilde{\chi}_k|^2 + \left( \frac{m^2 a^2 + 6\xi H^2 a^2}{1 - F} \right) |\tilde{\chi}_k|^2 \right) k^3 \frac{dk}{k},$$

where

$$\tilde{\chi} = \tilde{\chi}/\sqrt{|1 - F_k|}, \quad b = \frac{H + 7H^2 + 4H M_\lambda}{M^2}.$$  \hspace{1cm} (48)

For the longwave modes with $k^2 \ll |M^2 a^2|$ we have $F_k \ll 1$, $\tilde{\chi} \simeq \tilde{\chi}$ and their contribution to the total energy density is the same as the contribution from the transverse mode. It is interesting to note that the longitudinal mode is the ghost in de Sitter background. However, in Friedmann universe filled by matter with positive pressure it is not ghost in spite of the fact that the effective mass squared is negative.

Substituting (43) into (47) we find that in the leading order the contribution of the longwave modes into the energy density in the case $m = 0$ is

$$\varepsilon_L \simeq O(1) \frac{H_f^2}{a^2} (n^2 - 12n\xi + 6\xi) \int_{H_f a_i}^{H_f a_f} |\phi_k(\eta)|^2 k^2 dk,$$

and calculating the integral we obtain

$$\varepsilon_L \simeq O(1) H_f^2 (n^2 - 12n\xi + 6\xi) \left\{ \begin{array}{ll}
\frac{1}{n-1} \left( \frac{a}{a_f} \right)^{(n-1)/2}, & \text{for } n < 1, \\
\frac{1}{n-1} \left( \frac{a}{a_f} \right)^{(n-1)/2}, & \text{for } n > 1. \end{array} \right.$$

(51)

In the case $\xi = 1/6$ and when $n = 1$ the contribution is canceled in the leading order and $k^2$ terms give a contribution of the order of $H_f^2$, that is the same as for $n < 1$. However, for $\xi > 1/6$, and correspondingly $n > 1$, the energy density of the longwavelengh electromagnetic waves grows with time rather fast. It is negative and therefore when it becomes of order $H_f^2$ inflation is over. Requiring that inflation should last at least 75 e-fold we find that the contribution of electromagnetic field does not spoil inflation only if $n - 1 < 0.2$. Thus, in the most favorable case of $n \approx 1.2$, the amplitude of the magnetic field decays as $\delta_B \propto \lambda^{-0.8}$ and its value does not exceed $10^{-32} G$ in Mpc scales today.

## 3 Conclusions

In this paper we have studied the generation of large-scale magnetic fields in a two classes of models. In the first case the conformal invariance of the Maxwell field was broken by a non-minimal coupling of the form $RA^2$, this gives a non-zero time-dependent mass to the photon. In the second case the conformal invariance is violated because of the time-dependent coupling constant, $I(t)F^\mu \nu F_\mu \nu$, where $I(t) = I(\phi(t), ...)$ is a general function of non-trivial background fields and $\phi$ can be for instance inflaton or dilaton.

In principle it looks like inflation can strongly amplify the vacuum quantum fluctuations and therefore can lead to sizable magnetic fields. However, if we take into account the back reaction of the electromagnetic field and require that inflation lasts at least 75 e-folds, the strength of the primordial field cannot exceed $10^{-32} G$ on Mpc scales and it is not clear whether such a small field can work as a seed for a possible dynamo mechanism.

Only in the strong coupling case, $I(t)F^\mu \nu F_\mu \nu$, where $I = I_f (a/a_f)^n$ and $n \approx 2.2$, the amplitude can reach the interesting value of $10^{-7} G$ today. However, this case corresponds to the situation when the effective coupling constant is extremely large at the beginning of inflation and becomes of the order of one at the end of inflation and hence the theory is not trustable.

We conclude therefore that the models considered above are not efficient in producing primordial
magnetic fields during inflation and, even if the galactic dynamo was effective, the field produced seems to be too small to play the role of a seed for this mechanism.

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