Non-Hermitian Floquet topological superconductors with multiple Majorana edge modes

Longwen Zhou\textsuperscript{1,2} \textsuperscript{*}

\textsuperscript{1}Department of Physics, College of Information Science and Engineering, Ocean University of China, Qingdao 266100, China
\textsuperscript{2}Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

(Dated: 2019-11-28)

Majorana edge modes are candidate elements of topological quantum computing. In this work, we purpose a Floquet engineering approach to generate arbitrarily many non-Hermitian Majorana zero and \( \pi \) modes at the edges of a one-dimensional topological superconductor. Focusing on a Kitaev chain with periodically kicked superconducting pairings and gain/losses in the chemical potential or nearest neighbor hopping terms, we found rich non-Hermitian Floquet topological superconducting phases, which are originated from the interplay between drivings and non-Hermitian effects. Each of the phases is characterized by a pair of topological winding numbers, which can in principle take arbitrarily large integer values thanks to the applied driving fields. Under open boundary conditions, these winding numbers also predict the number of degenerate Majorana edge modes with quasienergies zero and \( \pi \). Our findings thus expand the family of Floquet topological phases in non-Hermitian settings, with potential applications in realizing environmentally robust Floquet topological quantum computations.

I. INTRODUCTION

Non-Hermitian topological states of matter have attracted great attention in recent years \cite{1-12}. These exotic phases could appear in open systems with gain/loss \cite{13-15}, bosonic systems with dynamical instability \cite{16-22}, electronic systems with finite-lifetime quasiparticles \cite{23-29} and mechanical metamaterials \cite{30-32}, with potential applications in achieving unidirectional invisibility \cite{33-36}, enhanced sensitivity \cite{37-41} and topological lasers \cite{42-49}. Theoretically, various schemes of classifying non-Hermitian topological phases in the absence or presence of interactions have been proposed \cite{50-62}, and the related bulk-boundary correspondence have been investigated in detail \cite{63-73}. Experimentally, non-Hermitian topological states of matter have been observed in mechanical \cite{74-75}, optical \cite{76-78}, photonic \cite{79-81}, optomechanical \cite{82-83} systems and topologic circuits \cite{84}. Recently, non-Hermitian topological phases with large topological invariants \cite{85-87}, many edge states \cite{88-90}, non-Hermitian skin effects \cite{91} and quantized pumps \cite{92} have also been found in periodically driven (Floquet) systems.

In applications, non-Hermitian topological states of matter in superconducting systems is one of the most intriguing candidates, as they may possess degenerate Majorana modes at their boundaries. The Majorana edge modes have been investigated intensively in the past decade due to their potential in the realization of topological quantum computations (see \cite{93-99} for reviews). The emergence and persistence of Majorana edge modes in non-Hermitian settings could further provide evidence for their robustness to environmental effects.

Recently, several studies have touched upon the topic of Majorana modes in non-Hermitian systems \cite{100,113}, which possess unique phenomena like the nonlocal particle transport \cite{102} due to the interplay between non-Hermiticity and superconductivity and coalescing \cite{104} due to the presence of exceptional points. It was also suggested that putting topological superconductors in the non-Hermitian setting could be helpful to resolve the controversy between Andreev and Majorana bound states \cite{109}.

However, most of the above mentioned studies focus on the generation and characterization of Majorana edge modes in static non-Hermitian systems. In periodically driven (Floquet) systems, due to the long-range hopping induced by the driving fields, superconducting phases characterized by large topological invariants and multiple topological edge states could appear \cite{115-123}. In one-dimension (1d), the resulting Floquet Majorana edge modes have been further employed in a recent proposal of Floquet topological quantum computation \cite{124,127}. In this work, we extend the Floquet engineering approach to non-Hermitian topological superconducting systems in 1d, with an aim of finding Floquet topological phases with multiple Majorana edge modes that are unique to non-Hermitian systems. As the Kitaev chain (KC) \cite{128} is a paradigmatic model in the study of 1d topological superconductors and Majorana edge modes, we start with a brief review of the non-Hermitian version of KC. After that, we introduce our Floquet version of the non-Hermitian KC (NHKC) via periodically kicking the superconducting pairing terms. In Sec. III we investigate the bulk properties of the periodically kicked NHKC systematically, focusing on the symmetries that protects its topological phases and the invariants that characterize its topological states. We establish the bulk topological phase diagram of the periodically kicked NHKC, where rich non-Hermitian Floquet topological superconducting
phases are found. Furthermore, we show in Sec. [X] the Floquet Majorana edge modes with quasienergies zero and $\pi$ in the system under the open boundary condition (OBC), and relate their numbers to the bulk topological invariants of the model. In Sec. [Y] we summarize our results and discuss potential future directions.

II. MODEL AND SYMMETRY

In this section, we first introduce the non-Hermitian variant of KC and discuss its known physical properties. This is followed by a presentation of our periodically driven NHKC model in both lattice and momentum representations. The non-Hermitian Floquet topological phases and Majorana edge modes that can appear in this model are the focus of our study in the later parts of this work.

A. Non-Hermitian Kitaev chain

In this subsection, we briefly review the non-Hermitian extension of KC model, with a focus on its symmetry and topological properties. The KC provides a paradigmatic example for the study of topological superconductors and Majorana edge modes [128]. It describes spinless electrons in a 1d tight-binding lattice with $p$-wave pairings. The model Hamiltonian is given by

$$H = \frac{1}{2} \sum_n \left[ \mu (2c_n^\dagger c_n - 1) + J (c_n^\dagger c_{n+1} + \text{H.c.}) \right] + \frac{1}{2} \sum_n \Delta (c_n^\dagger c_{n+1}^\dagger + \text{H.c.}),$$

(1)

where $c_n^\dagger$ ($c_n$) is the creation (annihilation) operator of an electron on the lattice site $n$, $\mu$ is the chemical potential and $J$ is the nearest neighbor hopping amplitude. $\Delta \in \mathbb{R}$ characterizes the strength of $p$-wave superconducting pairing and $\phi \in [0, 2\pi)$ is the superconducting pairing phase.

For a chain of length $L$ and under the periodic boundary condition, the Hamiltonian $H$ in Eq. (1) can be expressed in the momentum representation in terms of Fourier transforms $c_n = \frac{1}{\sqrt{2}} \sum_k c_k e^{i k n}$ and $c_n^\dagger = \frac{1}{\sqrt{2}} \sum_k c_k^\dagger e^{-i k n}$, yielding

$$H = \frac{1}{2} \sum_{k \in BZ} \Psi_k^\dagger H_\phi(k) \Psi_k.$$  

(2)

Here $k \in [-\pi, \pi)$ is the quasimomentum, $\Psi_k \equiv (c_k^\dagger, c_{-k}^\dagger)$ is the Nambu spinor, and the Hamiltonian matrix

$$H_\phi(k) = (\mu + J \cos k) \sigma_z - \Delta \sin(k) [\sin(\phi) \sigma_x + \cos(\phi) \sigma_y]$$

(3)

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices in their usual representations. The dispersion relation of the system $E_\pm(k) = \pm E(k)$ can further be obtained by performing the Bogoliubov transformation [129], yielding

$$E(k) = \sqrt{\Delta^2 \sin^2(k) + (\mu + J \cos k)^2}.$$  

(4)

Note in passing that the excitation energy $E(k)$ is independent of the pairing phase $\phi$. From now on, we choose the pairing phase $\phi = \pi$, which leads to the most frequently studied version of KC. The Hamiltonian matrix $H_\phi(k)$ then simplifies to

$$H_\pi(k) \equiv h_y(k) \sigma_y + h_z(k) \sigma_z,$$  

(5)

where the two components in front Pauli matrices $\sigma_y, \sigma_z$ are

$$h_y(k) = \Delta \sin(k), \quad h_z(k) = \mu + J \cos k.$$  

(6)

It is clear that the Hamiltonian matrix $H_\pi(k)$ of KC possesses the sublattice symmetry $\Gamma = \sigma_x$, in the sense that

$$\Gamma H_\pi(k) \Gamma = -H_\pi(-k), \quad \Gamma^2 = \sigma_0,$$  

(7)

where $\sigma_0$ is the $2 \times 2$ identity matrix. When $\mu$ and $J$ take real values, $H_\pi(k)$ also possesses the time-reversal symmetry $T = \sigma_0$ and particle-hole symmetry $C = \sigma_x$, in the sense that

$$\sigma_0 H_\pi^*(k) \sigma_0 = -H_\pi(-k),$$  

(8)

$$\sigma_z H_\pi^*(k) \sigma_x = -H_\pi(-k),$$  

(9)

with $\sigma_0 \sigma_0^* = \sigma_0$ and $\sigma_z \sigma_x^* = \sigma_0$, respectively. The Hamiltonian $H_\pi(k)$ then belongs to the topological class BDI [130]. Each of its superconducting phases is characterized by an integer topological winding number $w$, defined as

$$w = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{h_z(k) \partial_k h_y(k) - h_y(k) \partial_k h_z(k)}{h_y^2(k) + h_z^2(k)}.$$  

(10)

Under the OBC, the number of Majorana zero modes localized at each boundary of the chain is equal to $|w|$, which reflects the general principle of bulk-edge correspondence. The degeneracy of these Majorana modes is protected by the particle-hole symmetry $\sigma_x$.

An NHKC can be obtained by setting the hopping amplitude $J$ or the chemical potential $\mu$ to complex values, yielding $H_\pi^*(k) \neq H_\pi(k)$ in Eq. (5). Physically, these choices may be realized by introducing nonreciprocal effects to the nearest neighbor hopping amplitude ($J \in \mathbb{C}$), or adding onsite particle gain/losses ($\mu \in \mathbb{C}$). However, in these cases the time-reversal and particle-hole symmetries as defined in Eqs. (8) and (9) are not hold. A simple reason behind this anomaly is that while the transpose ($\top$) and complex conjugate ($\ast$) of a Hermitian matrix are the same, they are different for a non-Hermitian matrix, i.e., $H_\pi^\ast(k) \neq H_\pi(k) \iff H_\pi^\top(k) \neq H_\pi^\ast(k)$. Recently, progresses have been made in the symmetry and
topological classification of non-Hermitian matrices. In Ref. [56], a classification scheme for non-Hermitian topological phases is introduced by generalizing the “tenfold way” of topological insulators and superconductors [130] to a 38-fold “periodic table”. In this enlarged periodic table, the matrix transposition and complex conjugation are treated as independent operations, each generating its own flavor of “time-reversal” and “particle-hole” symmetries. According to this new classification scheme, the KC Hamiltonian $H_e(k)$ in Eq. (5), with $J \in \mathbb{C}$ or $\mu \in \mathbb{C}$, belongs to a generalized BDI class. It possesses the modified version of time-reversal ($\mathcal{T} = \sigma_0$) and particle-hole ($\mathcal{C} = \sigma_x$) symmetries, defined with respect to the matrix transposition as [56] [101]

$$\sigma_0 H^\dagger_x(k) \sigma_0 = H_x(-k),$$  
$$\sigma_x H^\dagger_x(k) \sigma_x = - H_x(-k).$$  

The topological phases of the NHKC can still be characterized by the winding number $w$ in Eq. (10) under the periodic boundary condition.

In recent studies, several variants of the NHKC have been investigated in detail [106–114]. Some topological phases unique to non-Hermitian superconducting systems were identified, and degenerate Majorana zero modes were found numerically in the KC with losses. Potential applications of these findings include the realization of topological quantum computing, and the resolution of the controversy between Andreev and Majorana modes [109]. However, most of these studies are limited to static non-Hermitian systems carrying a limited number of Majorana edge modes. This motivates us to introduce the Floquet extension of the NHKC in the following subsection, which will be shown to possess an enriched topological phase diagram together with a large number of coexisting Majorana zero and $\pi$ edge modes.

### B. Periodically kicked non-Hermitian Kitaev chain

In this subsection, we introduce our Floquet extension of the NHKC, whose topological properties will be investigated in detail in later sections. Our model describes a KC, whose superconducting pairing term is kicked periodically by a delta-shaped pulse, with the pairing phase fixed at $\phi = \pi$. The model Hamiltonian then takes the form

$$H(t) = \frac{1}{2} \sum_n [\mu (2c_n^\dagger c_n - 1) + J(c_{n+1}^\dagger c_n + \text{H.c.})],$$
$$- \frac{1}{2} \sum_n \Delta \delta_T(t)(c_{n+1}^\dagger c_n + \text{H.c.}),$$

where $\delta_T(t) \equiv \sum_{\ell \in \mathbb{Z}} \delta(t/T - \ell)$ describes delta kickings with the period $T$, and the system is made non-Hermitian by assuming $J \in \mathbb{C}$ or $\mu \in \mathbb{C}$. Under the periodic boundary condition, the Hamiltonian $H(t)$ can be expressed in momentum space as $H(t) = \frac{1}{2} \sum_{k \in \text{BZ}} \Psi_k^\dagger H(k,t) \Psi_k$, where $\Psi_k$ is the same Nambu spinor operator as defined in the last subsection, and

$$H(k,t) = h_y(k) \delta_T(t) \sigma_y + h_z(k) \sigma_z.$$  

Here $h_{x,y}(k)$ are given explicitly by Eq. (6). The Floquet operator of the system, which describes its time evolution from $t = \ell T + 0^-$ to $(\ell + 1)T + 0^-$ is $U(k) = e^{-i \frac{\pi}{\hbar} h_y(k) \sigma_y} e^{-i \frac{\pi}{\hbar} h_y(k) \sigma_y}$. Choosing the unit of energy to be $\hbar/T$ and setting $\hbar = T = 1$, we can express the Floquet operator in dimensionless units as

$$U(k) = e^{-i h_z(k) \sigma_z} e^{-i h_y(k) \sigma_y}.$$  

By expanding each exponential term in $U(k)$ with the Euler formula, we can further obtain the dispersion relation of Floquet quasienergy bands as $\varepsilon_{\pm}(k) = \pm \varepsilon(k)$, with

$$\varepsilon(k) = \arccos \{ \cos[h_y(k)] \cos[h_z(k)] \}.$$  

Note that $\varepsilon(k)$ could become complex if the chemical potential $\mu$ or hopping amplitude $J$ in $h_z(k)$ take complex values. In the following section, we will analyze in detail the bulk topological properties of this periodically kicked NHKC.

### III. TOPOLOGICAL INVARIANTS AND PHASE DIAGRAM

The periodically kicked KC introduced in the last section possesses rich non-Hermitian Floquet topological superconducting phases. In this section, we first reveal the underlying symmetries of $U(k)$ and the winding numbers that characterize its topological phases. The bulk phase diagram of the periodically kicked NHKC are then presented in some typical parameter regimes, with each of the phases being characterized by the introduced winding numbers.

#### A. Symmetry and topological winding numbers

The topological phases that can appear in the Floquet system described by $U(k)$ in Eq. (15) depend on the symmetries that $U(k)$ possesses. Following earlier studies on the topological classification of Floquet systems [121], we first express $U(k)$ in a pair of symmetric time frames as

$$U_1(k) = e^{-i h_y(k) \sigma_y} e^{-i h_z(k) \sigma_z} e^{-i h_y(k) \sigma_y},$$
$$U_2(k) = e^{-i h_y(k) \sigma_y} e^{-i h_y(k) \sigma_y} e^{-i h_y(k) \sigma_y}.$$  

It is clear that the Floquet operators \{ $U(k), U_1(k), U_2(k)$ \} are related by $k$-dependent similarity transformations. Therefore, they share the same quasienergy spectrum $\varepsilon(k)$ as given by Eq. (16), but
may possess different topological properties. Moreover, employing again the Euler formula, we can express \( U_1(k) \) and \( U_2(k) \) in terms of effective Floquet Hamiltonians as

\[
U_1(k) = e^{-iH_1(k)}, \quad U_2(k) = e^{-iH_2(k)}.
\]

The effective Hamiltonians \( H_1(k) \) and \( H_2(k) \) take the form

\[
H_\alpha(k) = h_{\alpha y}(k)\sigma_y + h_{\alpha z}(k)\sigma_z \quad (20)
\]

for \( \alpha = 1, 2 \), where the components \( h_{\alpha y}(k) \) and \( h_{\alpha z}(k) \) are explicitly given by:

\[
\begin{align*}
h_{1y}(k) &= \varepsilon(k) \sin[h_y(k)] \cos[h_z(k)]/|\varepsilon(k)|, \\
h_{1z}(k) &= \varepsilon(k) \sin[h_y(k)]/|\varepsilon(k)|, \\
h_{2y}(k) &= \varepsilon(k) \sin[h_y(k)]/|\varepsilon(k)|, \\
h_{2z}(k) &= \varepsilon(k) \cos[h_y(k)] \sin[h_z(k)]/|\varepsilon(k)|.
\end{align*}
\]

According to Eqs. (6) and (16), we have \( h_y(-k) = -h_y(k), h_z(-k) = h_z(k) \) and \( \varepsilon(-k) = \varepsilon(k) \). Therefore, following the discussions in Subsec. 11A both the effective Hamiltonians \( H_1(k) \) and \( H_2(k) \) possess the generalized time-reversal symmetry \( T = \sigma_0 \), particle-hole symmetry \( C = \sigma_x \) and sublattice symmetry \( \Gamma = \sigma_x \), i.e.,

\[
\begin{align*}
\sigma_0 H^\dagger_\alpha (k) \sigma_0 &= H_\alpha(-k), \\
\sigma_x H^\dagger_\alpha (k) \sigma_x &= -H_\alpha(-k), \\
\sigma_z H_\alpha(k) \sigma_x &= -H_\alpha(k),
\end{align*}
\]

where \( \alpha = 1, 2 \). \( H_1(k) \) and \( H_2(k) \) are then belong to the non-Hermitian extension of symmetry class BDI, with their topological phases being characterized by the winding number \( w \) in Eq. (10), i.e.,

\[
w_\alpha = \int_{-\pi}^\pi \frac{dk}{2\pi} \frac{\partial h_{\alpha z}(k)/\partial \varepsilon}{h_{\alpha y}(k) h_{\alpha y}(k) + h_{\alpha z}(k) h_{\alpha z}(k)}
\]

for \( \alpha = 1, 2 \). The topological phases of Floquet operator \( U(k) \) in Eq. (15) then admit a \( \mathbb{Z} \times \mathbb{Z} \) topological characterization. Namely, each of its Floquet topological phases is characterized by a pair of winding numbers \((w_0, w_\pi)\), defined as

\[
w_0 \equiv \frac{w_1 + w_2}{2}, \quad w_\pi \equiv \frac{w_1 - w_2}{2}.
\]

In the following subsection, we will demonstrate that these topological winding numbers indeed characterize all the possible Floquet topological phases of the periodically kicked NHKC. Moreover, in the next section we will see that the values of \( w_0 \) and \( w_\pi \) determine the number of degenerate Floquet Majorana edge modes at quasienergies zero and \( \pi \) under the OBC, respectively.

### B. Topological phase diagram of the periodically kicked NHKC

In this subsection, we present the topological phase diagrams of the periodically kicked NHKC. As shown in Eq. (16), the system possesses two quasienergy bands \( \pm \varepsilon(k) \), and the real part of quasienergy \( \varepsilon(k) \) is a phase factor defined modulus \( 2\pi \). Therefore, the periodically kicked NHKC in general holds two band gaps around quasienergies \( 0 \) and \( \pm \pi \). The Floquet spectrum of the system would become gapless when \( \varepsilon(k) = 0 \) or \( \varepsilon(k) = \pi \).

Assuming the chemical potential \( \mu \in \mathbb{C} \) and hopping amplitude \( J \in \mathbb{R} \), our model describes a periodically kicked KC with onsite gain/loss. Its quasienergy spectrum becomes gapless under the condition

\[
\mu_i = \pm \arccosh \left\{ \frac{1}{|\cos \left[ \sqrt{\Delta (1 - (n\pi - \mu)^2/J^2) \right]}|} \right\},
\]

where \( n \in \mathbb{Z}, |n\pi - \mu| \leq |J| \), and \( J_r (J_i) \) is the real (imaginary) part of the hopping amplitude \( J \) (see Appendix A for the derivation details). Similarly, when the hopping amplitude \( J \in \mathbb{C} \) and chemical potential \( \mu \in \mathbb{R} \) our model describes a periodically kicked KC with nonreciprocal hopping amplitudes. Its quasienergy spectrum would be gapless under the condition

\[
J_i = \pm \left| \frac{J_r}{n\pi - \mu} \right| \arccosh \left\{ \frac{1}{|\cos \left[ \sqrt{\Delta (1 - (n\pi - \mu)^2/J^2) \right]}|} \right\},
\]

where \( n \in \mathbb{Z}, |n\pi - \mu| \leq |J_r| \), and \( J_r (J_i) \) is the real (imaginary) part of the hopping amplitude \( J \) (see Appendix A for the derivation details). Eqs. (30) and (31) determine the gap closing lines/surfaces in the parameter space for two typical situations of our interest (i.e., non-Hermiticity due to onsite gain/loss or nonreciprocal hopping). These lines/surfaces are expected to form the boundaries between different Floquet topological superconducting phases of the periodically kicked NHKC. Moreover, each of the non-Hermitian Floquet topological phases is characterized by the topological winding numbers \((w_0, w_\pi)\) in Eq. (29).

In Fig. 1 we present the bulk topological phase diagram of the periodically kicked NHKC with respect to the (real) hopping amplitude \( J \) and the imaginary part of the chemical potential \( \mu_i \). The real part of chemical potential \( \mu_r \) and the superconducting pairing strength are fixed at \( \mu_r = 0.3\pi \) and \( \Delta = 0.5\pi \), respectively. In Eq. (30), we also notice that the phase boundaries possess reflection symmetries with respect to the axis \( \mu_i = 0 \) and \( J = 0 \). Therefore, we only show the phase diagram in a regime with \( \mu_i, J > 0 \). In Fig. 1 each region with a uniform color corresponds to a non-Hermitian Floquet topological phase, characterized by the winding numbers \((w_0, w_\pi)\) shown explicitly therein. The black solid lines denote the boundaries between different topological phases, which
are calculated analytically from the gap closing condition Eq. (30). We can see from this “rainbow-shaped” phase diagram that at each imaginary chemical potential $\mu_i$, new Floquet topological superconducting phases emerge progressively with the increase of the hopping amplitude $J$, which are characterized by larger and larger topological winding numbers $(w_0, w_\pi)$ as denoted on the patch. The black solid lines separating different patches are the topological phase boundaries determined by Eq. (30).

In Fig. 2, similar to Fig. 1, each region in Fig. 2 with a uniform color corresponds to a topological phase. The phase boundaries separating different regions are denoted by black solid lines, which are obtained analytically from Eq. (31). Again, we observe that with the increase of the real hopping amplitude $J_r$, a series topological phase transitions happen progressively, accompanied by gradually increased winding numbers $(w_0, w_\pi)$ from small to large values, as denoted within each region. Furthermore, with the change of the imaginary hopping amplitude $J_i$, topological phase transitions unique to non-Hermitian Floquet systems also happen successively. These observations further exemplify the universality and richness of non-Hermitian Floquet topological phases and phase transitions in the periodically kicked NHKC.

In the next section, we will establish the connection between the bulk topological invariants $(w_0, w_\pi)$ and the Majorana zero and $\pi$ edge modes that can appear in the periodically kicked NHKC under the OBC.

**IV. MAJORANA EDGE MODES AND BULK-EDGE CORRESPONDENCE**

Majorana edge modes are one of the most important manifestations of the topological properties of KC. They appear in the system under the OBC when the system parameters are tuned to the topological nontrivial regime.
For the periodically kicked KC, its Hamiltonian can be expressed under the OBC as:

\[
H(t) = \frac{1}{2} \sum_{n=1}^{L} \mu (2c_n^\dagger c_n - 1) + \frac{1}{2} \sum_{n=1}^{L-1} J (c_n^\dagger c_{n+1} + H.c.) \\
- \frac{i}{2} L \sum_{n=1}^{L-1} [\Delta \delta_T(t)] c_n^\dagger c_{n+1}^\dagger + H.c.]
\]

(32)

where \(\delta_T(t) \equiv \sum_{t \in \mathbb{Z}} \delta(t/T - \ell)\), \(L\) is the total number of lattice sites, and \(H(t)\) is non-Hermitian if either the hopping amplitude \(J\) is non-Hermitian or the chemical potential \(\mu \in \mathbb{C}\).

To find the Floquet Majorana edge modes of the system, we need to express \(H(t)\) in the Majorana representation and obtain the corresponding equation of motion. To achieve this, we first express fermionic annihilation and creation operators \(a_n, c_n^\dagger\) in terms of Majorana operators \(\gamma_n^a, \gamma_n^b\) as

\[
c_n = \frac{\gamma_n^a - i \gamma_n^b}{2}, \quad c_n^\dagger = \frac{\gamma_n^a + i \gamma_n^b}{2},
\]

(33)

where \(n = 1, 2, ..., L\). The Majorana operators are Hermitian \([\gamma_n^{a,b}]^\dagger = \gamma_n^{a,b}\) and satisfy the anti-commutation relation

\[
\{\gamma_n^a, \gamma_{m}^b\} = 2 \delta_{\ell m} \delta_{ss'},
\]

(34)

where \(\ell, m = 1, 2, ..., L\) and \(s, s' = a, b\). In terms of these Majorana operators, \(H(t)\) in Eq. (32) takes the form:

\[
H(t) = \frac{i}{4} \sum_{n=1}^{L} (-2\mu) \gamma_n^{a,b} \\
+ \frac{i}{4} \sum_{n=1}^{L-1} J (\gamma_n^{b} \gamma_{n+1}^a - \gamma_{n+1}^b \gamma_n^a) \\
- \frac{i}{4} \sum_{n=1}^{L-1} \Delta \delta_T(t) (\gamma_n^{b} \gamma_{n+1}^a + \gamma_{n+1}^b \gamma_n^a).
\]

(35)

The Floquet operator of the system can be derived by solving the Heisenberg equation for Majorana modes, i.e.,

\[
\frac{d\gamma_n^a(t)}{dt} = i[H(t), \gamma_n^a(t)]
\]

(36)

for \(\ell = 1, ..., L\) and \(s = a, b\), where \(t\) is the scaled time under the unit choice \(\hbar = T = 1\). Plugging Eq. (36) into Eq. (35) and using the commutation relation (34), we could express the Floquet operator \(U\) of the system in Majorana basis in matrix form:

\[
U = e^{-iH_2} e^{-iH_1},
\]

(37)

where the explicit expressions of \(H_1\) and \(H_2\) are given in Appendix B (alternatively, one can express the Floquet operator in the Bogoliubov-de Gennes (BdG) representation, as elaborated in the Appendix C).

To clearly show the Floquet Majorana edge states at both quasienergies and \(\pi\) in the spectrum, we introduce a pair of “gap functions” \(F_0\) and \(F_\pi\), defined as:

\[
F_0 \equiv \frac{1}{\pi} \sqrt{(\text{Re}\varepsilon)^2 + (\text{Im}\varepsilon)^2},
\]

(38)

\[
F_\pi \equiv \frac{1}{\pi} \sqrt{(|\text{Re}\varepsilon| - \pi)^2 + (\text{Im}\varepsilon)^2},
\]

(39)

where \(\varepsilon\) corresponds to all the quasienergy eigenvalues, obtained by diagonalizing \(U\) in Eq. (37) and taking the logarithm of its eigenvalues. It is clear that the eigenmodes satisfying \(F_0 = 0 \quad (F_\pi = 0)\) have the quasienergy \(\varepsilon = 0 \quad (\varepsilon = \pi)\). If there is a gap (on the complex plane) between these modes and all the other eigenmodes of \(U\), we could identify the eigenstates with \(F_0 = 0 \quad (F_\pi = 0)\) as Floquet Majorana zero \((\pi)\) edge modes in the periodically kicked NHKC.

In Fig. 3, we show the gap functions \(F_0\) (red solid lines) and \(F_\pi\) (blue dashed lines) of the periodically kicked NHKC versus the imaginary part of the chemical potential \(\mu_\ell\) under OBC. The real part of chemical potential, hopping amplitude, superconducting pairing strength and the number of lattice sites are fixed at \(\mu_\ell = 0.3\pi\), \(J = 4\pi\), \(\Delta = 0.3\pi\) and \(L = 1000\), respectively. The values of \(\mu_\ell\) at grids \(y_1 \sim y_8\) are computed analytically from the gapless condition (30). They correspond to the bulk topological phase transition points.
tem parameters are fixed at $F_0$, $F_e$, and the real part of hopping amplitude $J$ under OBC. The other system parameters are fixed at $\mu = 0.4\pi$, $\Delta = 0.9\pi$, $J_s = 1$, and $L = 500$ for both panels (a) and (b). The solid (dashed) lines appearing at $F_0 = 0$ ($F_e = 0$) in panel (a) correspond to Floquet Majorana zero ($\pi$) modes, whose number is determined by the bulk winding number in Eq. (29). The ticks $y_1 \sim y_8$ along the $J_y$ axis in both panels correspond to the phase transition points obtained from Eq. (30). Their numerical values are approximately given by $J_y/\pi = (0.42, 0.63, 1.47, 1.68, 2.51, 2.72, 3.56, 3.77)$, respectively. The numerical values in panel (b) denote the number of Floquet Majorana edge modes at quasienery zeros and $\pi$ in each gapped regions.

of the periodically kicked NHKC. We notice that across each transition point, a pair of Floquet Majorana zero modes (red solid lines) or $\pi$ modes (blue dashed lines) merge into the bulk. Referring to the phase diagram Fig. 1, we further observe that each of the above mentioned transitions is accompanied by a quantized change of winding number $w_0$ or $w_\pi$ by 1. Between each pair of adjacent topological transitions, the winding numbers $(w_0, w_\pi)$ also predict the number of degenerate Majorana edge modes $(N_0, N_\pi)$ at quasienergy zeros and $\pi$, as $N_0 = 2|w_0|$ and $N_\pi = 2|w_\pi|$. These relations establish the correspondence between the bulk topological invariants and the Floquet topological edge states in the periodically kicked NHKC. Note that the deviation between theory and numerical results around $\mu_i = y_8$ is a finite-size effect, which is going to vanish in the thermodynamic limit $L \rightarrow \infty$.

For completeness, in Fig. 4 we also presented the edge states and bulk-edge correspondence of the periodically kicked NHKC with nonreciprocal hopping amplitudes ($J \neq 0$), and observed similar results as in the case of onsite gain/loss ($\mu_i \neq 0$). Note in passing that in Fig. 4(b), multiple pairs of degenerate Majorana zero and $\pi$ modes could appear and coexist in the same parameter regime, as denoted in the figure. These Majorana modes could have applications in the recently proposed concept of Floquet topological quantum computing [124, 126], with a potential advantage due to their persistence in the non-Hermitian regions. From Figs. 3 and 4 we see that changing gain/loss in the onsite chemical potential or nonreciprocity of the hopping amplitude can indeed induce topological phase transitions in the periodically kicked NHKC, and create non-Hermitian topological superconducting phases that are unique to Floquet systems. These phases are further characterized by the appearance of non-Hermitian Floquet Majorana zero and $\pi$ edge modes under the OBC. One example of the spatial profiles of these Majorana modes is shown explicitly in Fig. 5, where we see that the Majorana zero modes [red columns in Figs. 5(a,c,e)] and $\pi$ modes [blue columns in Figs. 5(b,d,f)] are indeed localized exponentially around the left and right edges of the chain. In Fig. 6 we further show the largest quasienergy splittings of the Majorana zero and $\pi$ modes versus the length $L$ of the chain. We observe that both the maximal splittings $\delta \varepsilon_0$ and $\delta \varepsilon_\pi$ of the Majorana zero and $\pi$ modes decrease exponentially with the increase of the system size $L$ before reaching the machine precision, which is expected for Majorana edge modes.

Putting together, we conclude that the periodically kicked NHKC could indeed possess multiple Floquet Majorana zero and $\pi$ edge modes under OBC. The number of these modes in each topological phase is correctly counted by the bulk winding numbers we introduced in Eq. (29).
V. SUMMARY AND DISCUSSION

In this work, we found rich non-Hermitian Floquet topological superconducting phases in a periodically kicked non-Hermitian Kitaev chain. Our system belongs to an extended BDI class in non-Hermitian Floquet systems. Each of its topological phases is characterized by a pair of integer winding numbers, which can take large values due to the interplay between driving and non-Hermitian effects. Under OBC, multiple Floquet Majorana modes with quasienergies zero and \( \pi \) appear at the edges of the chain, with their numbers determined by the topological invariants of the bulk Floquet operator. The degeneracy of these edge modes is protected by a particle-hole symmetry that is unique to non-Hermitian systems. These results establish a new class of Floquet topological superconductors in non-Hermitian settings.

A potential application of our discovery resides in the recently proposed Floquet topological quantum computing [124,126]. There, time is utilized as an extra dimension to assist the braiding of Majorana zeros and \( \pi \) modes at the edges or corners of Floquet topological superconductors. The non-Hermitian Floquet Majorana modes found in this work might be able to make the Floquet topological quantum computations more robust to environmentally induced nonreciprocity and losses. On the other hand, the existence of more pairs of non-Hermitian Floquet Majorana modes could create stronger signals at the edges of the system, making it easier for the experimental detection of their topological signatures in open system settings.

ACKNOWLEDGEMENT

This work is supported by the National Natural Science Foundation of China (Grant No. 11905211), the China Postdoctoral Science Foundation (Grant No. 2019M602444), the Fundamental Research Funds for the Central Universities (Grant No. 841912009), the Young Talents Project at Ocean University of China (Grant No. 861801013196), and the Applied Research Project of Postdoctoral Fellows in Qingdao (Grant No. 861905040009).

Appendix A: Analytical derivation of the gapless conditions

In this Appendix, we give the derivation details for the gapless conditions of the periodically kicked NHKC. According to the main text, the Floquet spectrum of the periodically kicked NHKC becomes gapless at the center or boundary of the quasienergy Brillouin zone if the dispersion relation \( \varepsilon(k) = 0 \) or \( \pi \), respectively. Putting together, we find the bulk’s spectrum gap closes if

\[
\cos[h_y(k)] \cos[h_z(k)] = \pm 1 \tag{A1}
\]

where \( h_y(k) = \Delta \sin k \) and \( h_z(k) = \mu + J \cos k \) as defined in the main text. When both the chemical potential \( \mu \) and hopping amplitude \( J \) take real values, it straightforward to see that the condition (A1) is met if

\[
\left( \frac{m\pi}{\Delta} \right)^2 + \left( \frac{n\pi - \mu}{J} \right)^2 = 1, \tag{A2}
\]

where \( m, n \in \mathbb{Z}, |m\pi| \leq |\Delta| \) and \( |n\pi - \mu| \leq |J| \). Eq. (A2) determines the gapless conditions of the periodically kicked Hermitian KC in the space of system parameters \( \Delta, \mu \) and \( J \). When either \( \mu \in \mathbb{C} \) or \( J \in \mathbb{C} \), the gapless conditions are different from Eq. (A2). We discuss the cases with \( \mu \in \mathbb{C}, J \in \mathbb{R} \) and \( J \in \mathbb{C}, \mu \in \mathbb{R} \) separately below.

1. \( \mu \in \mathbb{C} \) and \( J \in \mathbb{R} \)

In this case, the system is subject to onsite gain/loss. We denote the real and imaginary parts of the chemical potential \( \mu \) as \( \mu_r \) and \( \mu_i \), such that \( \mu = \mu_r + i\mu_i \). The gapless condition (A1) then becomes

\[
\cos(\Delta \sin k) \cos(\mu_r + J \cos k) \cos \mu_i = \pm 1 \tag{A3}
\]
\[
\cos(\Delta \sin k) \sin(\mu_r + J \cos k) \sin \mu_i = 0 \tag{A4}
\]

As \( \mu_i \neq 0 \), we have \( \sin \mu_i \neq 0 \) in Eq. (A4). Moreover, the condition Eq. (A3) cannot be satisfied when \( \cos(\Delta \sin k) = 0 \). Therefore, to satisfy both Eqs. (A3) and (A4), we must have \( \sin(\mu_r + J \cos k) = 0 \), yielding \( \mu_r + J \cos k = n\pi \) for \( n \in \mathbb{Z} \). The solution of this equa-
tion determines the quasimomentum $k$ at which the band touches, i.e.,
\[
\cos k = \frac{n\pi - \mu_r}{J} \quad \text{for} \quad |n\pi - \mu_r| \leq |J|. \tag{A5}
\]
In the meantime, $\sin(\mu_r + J\cos k) = 0$ also implies $\cos(\mu_r + J\cos k) = \pm 1$. Plugging this and Eq. (A5) into Eq. (A3) finally yields
\[
\mu_i = \pm \arccosh \left\{ \frac{1}{\cos \left[ \Delta \sqrt{1 - (n\pi - \mu_r)^2/J^2} \right]} \right\}. \tag{A6}
\]
Therefore, under the condition $n \in \mathbb{Z}$ and $|n\pi - \mu_r| \leq |J|$, the Floquet spectrum of periodically kicked NHKC is gapless when the Eq. (A6) for the system parameters is satisfied, as given by Eq. (30) in the main text.

2. $J \in \mathbb{C}$ and $\mu \in \mathbb{R}$

In this case, the system is subject to nonreciprocal hopplings. We denote the real and imaginary parts of hopping amplitude $J$ as $J_r$ and $J_i$, such that $J = J_r + iJ_i$. The gapless condition \[A1\] then becomes
\[
\cos(\Delta \sin k) \cos(\mu + J_r \cos k) \cosh(J_i \cos k) = \pm 1 \quad \text{for} \quad |n\pi - \mu_r| \leq |J_r|. \tag{A7}
\]
\[
\cos(\Delta \sin k) \sin(\mu + J_r \cos k) \sinh(J_i \cos k) = 0 \tag{A8}
\]
Since $J_i \neq 0$, we must have $k = \pm\pi/2$ if $J_i \cos k = 0$. According to Eq. \[A7\], this further implies that $\Delta = m\pi$ and $\mu = n\pi$ for $m, n \in \mathbb{Z}$. These conditions only give isolated points in the parameter space, instead of continuous boundary lines between possibility different phases. Also the imaginary part of hopping amplitude $J_i$ becomes irrelevant to these gapless points, which is not the situation of our interest. When $J_i \cos k \neq 0$, we need to have $\sin(\mu + J_r \cos k) = 0$ for both conditions \[A7\] and \[A8\] to be satisfied, yielding $\mu + J_r \cos k = n\pi$ for $n \in \mathbb{Z}$. The solution of this equation determines the quasimomentum $k$ at which the band touches, i.e.,
\[
\cos k = \frac{n\pi - \mu_r}{J_r} \quad \text{for} \quad |n\pi - \mu_r| \leq |J_r|. \tag{A9}
\]
Furthermore, $\sin(\mu + J_r \cos k) = 0$ also implies $\cos(\mu + J_r \cos k) = \pm 1$. Combining this and Eq. \[A9\] into Eq. \[A7\] finally leads to
\[
J_i = \pm \frac{J_r}{n\pi - \mu} \arccosh \left\{ \frac{1}{\cos \left[ \Delta \sqrt{1 - (n\pi - \mu_r)^2/J_r^2} \right]} \right\}. \tag{A10}
\]
Therefore, under the condition $n \in \mathbb{Z}$ and $|n\pi - \mu_r| \leq |J_r|$, the Floquet spectrum of periodically kicked NHKC is gapless when Eq. \[A10\] for the system parameters is satisfied, as given by Eq. (31) in the main text.

\section*{Appendix B: Floquet operator in Majorana representation}

In this appendix, we present the derivation details for the Floquet operator of the periodically kicked NHKC in the Majorana representation. We first write the Hamiltonian $H(t)$ in Eq. (35) as $H(t) = \frac{1}{2} H(t)$, where
\[
H(t) \equiv (-2\mu) \sum_n \gamma_n a_n^b - J \sum_n (\gamma_n \gamma_n^b - \gamma_n^a \gamma_n^b) - \Delta \delta(t) \sum_n (\gamma_n^a \gamma_n^a + \gamma_n^b \gamma_n^b). \tag{B1}
\]
Using the commutation relation \[B4\] and the commutator formula
\[
[AB, C] = A[B, C] - \{A, C\} B, \tag{B2}
\]
it is straightforward to show that
\[
[\gamma_n^a \gamma_n^b, \gamma_{\ell}^a] = 2(\delta_{\ell, n} \delta_{\ell, b} \gamma_n^a - \delta_{\ell, n} \delta_{\ell, a} \gamma_n^b), \tag{B3}
\]
\[
[\gamma_n^a \gamma_{n+1}^b, \gamma_{\ell}^a] = 2(\delta_{\ell, n+1} \delta_{\ell, a} \gamma_n^a - \delta_{\ell, b} \gamma_{n+1}^a), \tag{B4}
\]
\[
[\gamma_n^a \gamma_{n+1}^b, \gamma_{\ell}^b] = 2(\delta_{\ell, n+1} \delta_{\ell, b} \gamma_n^a - \delta_{\ell, a} \gamma_{n+1}^b). \tag{B5}
\]
Combining these and Eq. \[B1\] with Eq. \[35\], we obtain:
\[
\frac{d\gamma_n^a(t)}{dt} = \mu s \delta_n^a \gamma_n^a(t) - \mu s \delta_n^a \gamma_n^b(t) - \frac{J}{2} \delta_{s, a} \left[ \gamma_{\ell-1}^b(t) + \gamma_{\ell+1}^b(t) \right] + \frac{J}{2} \delta_{s, b} \left[ \gamma_{\ell-1}^a(t) + \gamma_{\ell+1}^a(t) \right] + \frac{\Delta}{2} \delta_{T}(t) \delta_{s, a} \left[ \gamma_{\ell-1}^a(t) - \gamma_{\ell+1}^a(t) \right] + \frac{\Delta}{2} \delta_{T}(t) \delta_{s, b} \left[ \gamma_{\ell-1}^b(t) - \gamma_{\ell+1}^b(t) \right]
\]
for $s = a, b$, or separately
\[
\frac{d\gamma_n^a}{dt} = -\mu \gamma_n^a - \frac{J}{2} (\gamma_{\ell-1}^a + \gamma_{\ell+1}^a) + \frac{\Delta}{2} \delta_{T}(t) (\gamma_{\ell-1}^a - \gamma_{\ell+1}^a), \tag{B6}
\]
\[
\frac{d\gamma_n^b}{dt} = \mu \gamma_n^b + \frac{J}{2} (\gamma_{\ell-1}^b + \gamma_{\ell+1}^b) + \frac{\Delta}{2} \delta_{T}(t) (\gamma_{\ell-1}^b - \gamma_{\ell+1}^b). \tag{B7}
\]
The Floquet operator of the system in the Majorana basis $(\gamma_1^a, \gamma_2^a, ..., \gamma_n^a, \gamma_1^b, \gamma_2^b, ..., \gamma_n^b)$ is then obtained by integrating Eqs. \[B6\] and \[B7\] over a driving period, i.e., from $t = mT + 0^{-}$ to $(m + 1)T + 0^{+}$. The resulting Floquet matrix can be cast in the form
\[
U = e^{-i\mathcal{H}_2} e^{-i\mathcal{H}_1}, \tag{B8}
\]
where \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are both \( 2L \times 2L \) block off-diagonal matrices in the Majorana basis, given by

\[
\mathcal{H}_1 = \frac{\Delta}{2t} \sigma_x \otimes \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & 1 & 0 & \\
0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0
\end{pmatrix}_{L \times L},
\]

\[
\mathcal{H}_2 = \sigma_y \otimes \begin{pmatrix}
\mu & J/2 & 0 & 0 & \cdots & 0 \\
J/2 & \mu & J/2 & 0 & \cdots & 0 \\
0 & J/2 & \ddots & \ddots & \ddots & \cdots \\
0 & 0 & \ddots & J/2 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \ddots & J/2 & \mu \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \mu
\end{pmatrix}_{L \times L},
\]

where \( \sigma_{x,y} \) are Pauli matrices in their usual representations. The quasienergy spectrum and Floquet eigenstates can then be obtained by diagonalizing the Floquet operator \( \mathcal{U} \) in the Majorana representation.

### Appendix C: Floquet operator in BdG representation

Instead of Majorana basis, it is also possible to express the Floquet operator of the periodically kicked NHKC described by (32) in fermionic basis. This is achieved by performing the following Bogoliubov transformation [52]:

\[
c_n = \sum_{j=1}^{L} (u_{jn} f_j + v_{jn} f_j^\dagger),
\]

where \( f_j^\dagger \) and \( f_j \) are creation and annihilation operators of normal fermions, satisfying the anticommutation relations

\[
\{ f_i, f_j \} = \{ f_i^\dagger, f_j^\dagger \} = 0, \quad \{ f_i, f_j^\dagger \} = \delta_{ij}.
\]

The coefficients \( u_{jn}, v_{jn} \) are chosen to be real, satisfying the normalization condition

\[
\sum_{j=1}^{L} (u_{jn}^2 + v_{jn}^2) = 1.
\]

The Hamiltonian \( H(t) \) of the periodically kicked NHKC is assumed to be diagonalized in the basis \( \{ f_1, \ldots, f_L, f_1^\dagger, \ldots, f_L^\dagger \} \), such that it can be expressed as

\[
H(t) = \sum_{\ell} E_{\ell} f_{\ell}^\dagger f_{\ell}.
\]

Using the relation [52], it is straightforward to show that

\[
[H(t), f_j] = -E_j f_j, \quad [H(t), f_j^\dagger] = E_j f_j^\dagger, \tag{C5}
\]

and

\[
[H(t), c_n] = -\sum_j E_j (u_{jn} f_j - v_{jn} f_j^\dagger) \tag{C6}
\]

In the meantime, we can also compute the commutator \([H(t), c_n]\) directly with the \( H(t) \) in Eq. (C5) and \( c_n \) in Eq. (C1). The result is

\[
[H(t), c_n] = -\sum_j \mu(u_{jn} f_j + v_{jn} f_j^\dagger) \tag{C7}
\]

\[
-\sum_j J_2 [(u_{jn-1} + u_{jn+1}) f_j + (v_{jn-1} + v_{jn+1}) f_j^\dagger] \\
-\sum_j J_2 [(u_{jn-1} - u_{jn+1}) f_j + (u_{jn-1} - u_{jn+1}) f_j^\dagger].
\]

Comparing Eqs. (C6) and (C7), we obtain the following BdG self-consistent equations (after dropping the redundant index \( j \)):

\[
Eu_n = + \mu u_n + \frac{J_2}{2} (u_{n-1} + u_{n+1}) + \frac{\Delta}{2} \delta_T(t) (v_{n-1} - v_{n+1}) \tag{C8}
\]

\[
Ev_n = - \mu v_n - \frac{J_2}{2} (v_{n-1} + v_{n+1}) - \frac{\Delta}{2} \delta_T(t) (u_{n-1} - u_{n+1}) \tag{C9}
\]

The Floquet operator of the system in the BdG basis can then be obtained by integrating Eqs. (C8) and (C9) over a driving period, i.e., from \( t = mT \) to \( (m+1)T \), leading to the Floquet matrix:

\[
\mathcal{U} = e^{-iH_2} e^{-iH_1}. \tag{C10}
\]

Here \( H_1 \) is a block off-diagonal matrix of the form:

\[
H_1 = \frac{\Delta}{2t} \sigma_y \otimes \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & 1 & 0 & \\
0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0
\end{pmatrix}_{L \times L}, \tag{C11}
\]
and $H_2$ is a $2L \times 2L$ tridiagonal matrix of the form:

$$H_2 = \sigma_x \otimes \begin{pmatrix} \mu & J/2 & 0 & 0 & \cdots & 0 \\ J/2 & \mu & J/2 & 0 & \cdots & 0 \\ 0 & J/2 & \mu & J/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & J/2 & \mu \\ 0 & 0 & \cdots & \cdots & 0 & J/2 \mu \end{pmatrix}_{L \times L}, \quad (C12)$$

with $\sigma_{y,z}$ being Pauli matrices in their usual representations. So the quasienergy spectrum and Floquet eigenstates of the periodically kicked NHKC under OBC can also be obtained by diagonalizing the Floquet operator $U$ in the BdG basis. In fact, the Floquet operators $U$ in Majorana basis and $U$ in BdG basis can be simply mapped into each other by swapping the Pauli matrices $\sigma_x \leftrightarrow \sigma_y$ and $\sigma_y \leftrightarrow \sigma_z$, which does not change the quasienergy spectrum as expected.

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