Undamped Bloch Oscillations in the $U \to \infty$ one-dimensional Hubbard model

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I. INTRODUCTION

As is well-known, the charge current of noninteracting fermions in a periodic lattice shows an undamped oscillation behavior under a constant electric field, the so-called Bloch oscillations. It is believed that such oscillations become damped and ultimately give way to a direct current when interactions in the common sense are included; however, it is still not well studied that whether this picture is true or not. The Hubbard model is a basic model to address this problem.

Eckstein and Werner [1] have shown numerically that the correlation interaction plays no role on the response of the system to an electric field. As is well-known, the charge current operator [5] can be written as

$$\mathbf{j} = -e \sum_i n_i \mathbf{A}(t) \mathbf{C}_{i\sigma}^{\dagger} \mathbf{C}_{i+1\sigma} + \text{H.c.}$$

where $\mathbf{C}_{i\sigma}^{\dagger}$ ($\mathbf{C}_{i\sigma}$) is the creation (annihilation) operator of electron at site $i$ and spin $\sigma = \uparrow$ or $\downarrow$. And $n_i\sigma$ stands for the particle number operator as usual. We use the periodic boundary condition $C_{i+L\sigma} = C_{i\sigma}$ and $C_{i+L\sigma} = C_{i\sigma}$. For simplicity, we set the lattice constant to 1.

In the limit $U \to +\infty$, which we are interested in, the double occupation of a site by $\uparrow$ and $\downarrow$ electrons is prohibited, and the Hamiltonian is formally reduced to

$$H = -t \sum_{i,\sigma} \left[ e^{i\mathbf{A}(t)} \mathbf{C}_{i\sigma}^{\dagger} \mathbf{C}_{i+1\sigma} + \text{H.c.} \right] + U \sum_i n_i\uparrow n_i\downarrow,$$

where $C_{i\sigma}^{\dagger}$ ($C_{i\sigma}$) is the creation (annihilation) operator of electron at site $i$ and spin $\sigma = \uparrow$ or $\downarrow$. And $n_i\sigma$ stands for the particle number operator as usual. We use the periodic boundary condition $C_{i+L\sigma} = C_{i\sigma}$ and $C_{i+L\sigma} = C_{i\sigma}$. For simplicity, we set the lattice constant to 1.

In this paper, we want to show that, the one-dimensional (1D) Hubbard model in the $U \to +\infty$ limit permits an exact discussion of this problem. Our main conclusion is that even in such large $U$ case, there is no damping in Bloch oscillations.

II. MODEL AND TRANSFORMATION

We consider an $N$ electrons system on a periodic ring with $L$ sites, and we can treat the constant electric field as the temporal derivative of a time-dependent vector potential along the ring, $E = -1/c A_{z}(t)$ or $A(t) = -cEt$.

The model Hamiltonian then can be written as

$$H = -t \sum_{i,\sigma} \left[ e^{i\mathbf{A}(t)} \mathbf{C}_{i\sigma}^{\dagger} \mathbf{C}_{i+1\sigma} + \text{H.c.} \right] + U \sum_i n_i\uparrow n_i\downarrow,$$

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into a set of Hamiltonians, each of which corresponds to a system of spinless fermions and can be exactly solved \[\begin{align*} \tilde{\sigma} \end{align*}\]. Here, we also base our discussion on this point, but use a method of unitary transformation.

Firstly, we introduce a new kind of basis states, in which the spin and charge are completely independent,

\[|\mathbf{i}, \sigma\rangle = |i_1, i_2, \cdots, i_N\rangle \otimes |\sigma_1\sigma_2 \cdots \sigma_N\rangle, \quad (4)\]

where \(|\mathbf{i}\rangle = |i_1, i_2, \cdots, i_N\rangle\) denotes a state that \(N\) spinless but charged fermions occupy sites \(i_1, i_2, \cdots, i_N\), and it is required that \(1 \leq i_1 < i_2 < \cdots < i_N \leq L\), as before; by introducing the \(i\)-site creation (annihilation) operator \(f_i^\dagger (f_i)\) of such fermions, this state can be expressed as \(|\mathbf{i}\rangle = f_i^\dagger f_i^\dagger \cdots f_i^\dagger |0\rangle\). And \(|\sigma\rangle = |\sigma_1\sigma_2 \cdots \sigma_N\rangle\) is configuration of \(N\) sequenced spins. Obviously, the direct product state defined by Eq. (4) has a one-to-one correspondence with \(|\mathbf{i}, \sigma\rangle\). We can introducing a transformation

\[T = \sum_{i, \sigma} |\mathbf{i}, \sigma\rangle \langle \sigma, \mathbf{i}|, \quad (5)\]

of which the unitary, \(T^\dagger T = 1\), can be easily verified. Then, we have \(T|\mathbf{i}, \sigma\rangle = |\mathbf{i}\rangle \otimes |\sigma\rangle\). Under such transformation, the form of all the terms in Eq. (2), except the \(T^\daggerTI\) \(|\sigma\rangle\) part is concerned. What we need to do next is just to

\[|\sigma\rangle = |\sigma_1\sigma_2 \cdots \sigma_N\rangle, \quad \text{then} \quad |\tilde{\sigma}\rangle = |\sigma_2 \cdots \sigma_N\sigma_1\rangle. \quad \text{And the summations over all these configurations can be replaced by the left and right cyclic permutation operators,} \quad P \quad \text{and} \quad P^{-1}, \quad \text{respectively, as we have done.}\]

Then the final form of the Hamiltonian after the unitary transformation \(\tilde{H}\) is

\[\tilde{H} = -t \sum_{i=1}^{L-1} [e^{iA(t)} f_i^\dagger f_{i+1} + H.c.] - t e^{iA(t)} f_i^\dagger f_{i+1} P - t e^{-iA(t)} f_i^\dagger f_{i-1} P^{-1}. \quad (6)\]

We will find that our discussion can be substantially reduced with such new form of Hamiltonian.

And noting that \(T\) is independent of \(t\) or \(A(t)\), the charge current operator under this transformation becomes

\[\tilde{j} = -cT \frac{\delta H}{\delta A} T^\dagger = -c \frac{\delta \tilde{H}}{\delta A}. \quad (7)\]

### III. SOLUTIONS

In Eq. (6), the spin-part operators \(P, P^{-1}\) and \(I\) do not cause trouble for our discussion, since they commute with each other and the eigenstate can be obtained easily. For any spin configuration \(|\sigma_{s_i}\rangle = |\sigma_1\sigma_2 \cdots \sigma_N\rangle\), denoting \(|\sigma_{s_2}\rangle = P|\sigma_{s_1}\rangle, \quad |\sigma_{s_3}\rangle = P^2|\sigma_{s_1}\rangle, \quad \cdots\), till some integer \(m_s \leq N\), for which the resulted state repeats \(|\sigma_{s_1}\rangle\) for the first time, namely, \(|\sigma_{s_{m_s}}\rangle = P^{m_s}|\sigma_{s_1}\rangle = |\sigma_{s_1}\rangle\). Obviously, \(m_s\) is directly related to the detailed form of \(|\sigma_{s_1}\rangle\). These configurations form a subset as

\[s = \{|\sigma_{s_1}\rangle, |\sigma_{s_2}\rangle, \cdots, |\sigma_{s_{m_s}}\rangle\}, \quad (8)\]

from which we can obtain \(m_s\) eigenstates of \(P\) or \(P^{-1}\) as follows,

\[|\chi_{k_s}\rangle = 1/\sqrt{m_s} [\exp(ik_s)|\sigma_{s_1}\rangle + \exp(2ik_s)|\sigma_{s_2}\rangle + \cdots + \exp(im_s k_s)|\sigma_{s_{m_s}}\rangle], \quad (9)\]

where \(k_s = 2\pi j/m_s\), and \(j = 0, 1, 2, \cdots, m_s - 1\). And \(P|\chi_{k_s}\rangle = \exp(ik_s)|\chi_{k_s}\rangle, \quad P^{-1}|\chi_{k_s}\rangle = \exp(-ik_s)|\chi_{k_s}\rangle\). Furthermore, we obtain the representation \(P = \sum_s \sum_{k_s} \exp(ik_s)|\chi_{k_s}\rangle \langle \chi_{k_s}|, \quad P^{-1} = \sum_s \sum_{k_s} \exp(-ik_s)|\chi_{k_s}\rangle \langle \chi_{k_s}|\), and \(I = \sum_s \sum_{k_s} |\chi_{k_s}\rangle \langle \chi_{k_s}|\), where \(“\sum_s\”\) is a sum over all the possible existing subsets. Then, the Hamiltonian \(\tilde{H}\) is reduced to

\[\tilde{H} = \sum_s \sum_{k_s} \left[-te^{iA(t)} \left(\sum_{i=1}^{L-1} f_i^\dagger f_{i+1} + f_i^\dagger f_{i-1} e^{ik_s}\right) + H.c.\right] \otimes (|\chi_{k_s}\rangle \langle \chi_{k_s}|), \quad (10)\]

which is already in a diagonal form now as far as the spin part is concerned. What we need to do next is just to
diagonalize the spinless-fermion part inside “[ ]", associating with each \(|\chi_{k_i}\rangle\), which we can denote by \(h_i\). Introducing \(h_0 = -te^{i\pi(\tilde{A}(t))} (\sum_{i=1}^{L-1} f^+_i f_{i+1} + f^+_L f_1 e^{ik_x})\) to write \(h = h_0 + h_0^\dagger\), and noting \([h_0, h_0^\dagger] = 0\), we find that both \(h_0\) and \(h_0^\dagger\), and hence \(h\), can be written in a diagonalized form simultaneously. To do this, we assume a form of instantaneous eigenstates for \(h_0\) as \(|\varphi\rangle = \sum_{i=1}^L a_i f^+_i |0\rangle\). Letting \(h_0|\varphi\rangle = a|\varphi\rangle\) yields
\[-te^{i\pi(\tilde{A}(t))}a_{i+1} = a_i, \ i \neq N, \] (11)
and
\[-te^{i\pi(\tilde{A}(t))}\exp(ik_x) a_1 = a a_N. \] (12)

From Eq. (11),
\[a_N = \frac{a}{-te^{i\pi(\tilde{A}(t))}}, a_{N-1} = \cdots = \frac{a}{-te^{i\pi(\tilde{A}(t))}} a_1, \]
combining which with Eq. (12) yields \(a = a_q = -te^{i\pi(\tilde{A}(t))}\exp(ik_x)\), where
\[q = k_x/L + 2m\pi/L, \ m = 0, 1, 2, \cdots, L - 1. \] (13)

For each \(q\), repeatedly using (11) to determine the coefficient \(a_q\), we finally obtain an eigenstate which can be normalized as \(|\varphi_q\rangle = \frac{-1}{\sqrt{L}} \sum_{i=1}^L \exp(iqr_i) |i\rangle = f^+_q |0\rangle\), where
\[f^+_q = \frac{1}{\sqrt{L}} \sum_{i=1}^L \exp(iqr_i) f^+_i. \] (14)

It can be verified that \(h_0^\dagger|\varphi_q\rangle = a_q^* |\varphi_q\rangle\), and then \(h|\varphi_q\rangle = (a_q + a_q^*) |\varphi_q\rangle = \varepsilon_q(t) |\varphi_q\rangle\), with
\[\varepsilon_q(t) = -2t \cos[q + \frac{e}{hc} A(t)]. \] (15)

Further, we can write \(h\) as \(h = \sum_q \varepsilon_q(t) |\varphi_q\rangle \langle \varphi_q|\) or in an operator form
\[h = \sum_q \varepsilon_q(t) f^+_q f_q. \] (16)

And using Eq. (13), it can be verified that \(f^+_q\) or \(f^+_q f_q\) is still a Fermi operator. The eigenstate of \(h\) for \(N\) spinless fermions at each time \(t\), can be written as
\[|\mathbf{q}\rangle = f^+_q f^+_e \cdots f^+_q |0\rangle, \] (17)
where we have used a \(N\)-dimensional vector \(\mathbf{q} = (q_1, q_2, \cdots, q_N)\) to simplify the formulation. Due to the anticommuting property of \(f^+_j\)'s, \(q_1 \neq q_2 \neq \cdots \neq q_N\). Then the \(N\)-electron eigenstate of \(\tilde{H}\) at time \(t\) can be written as
\[|\Psi_{\mathbf{q}}\rangle = |\mathbf{q}\rangle \bigotimes |\chi_{k_e}\rangle, \] (18)
with a instantaneous eigenvalue
\[E_q(t) = \varepsilon_q(t) + \varepsilon_{q_2}(t) + \cdots + \varepsilon_{q_N}(t). \] (19)

The most important thing is that, though \(\tilde{H}\) and the instantaneous eigenvalue \(E_q(t)\) both depend on time, the eigenstate \(|\Psi_{\mathbf{q}}\rangle\) is independent of time. This permits us an exact study of the time evolving behavior of the system. Since any initial state of the system can be represented by the instantaneous eigenstates at the initial time, we can only focus on the case in which the initial state is an instantaneous eigenstate of the system. Namely, we assume that at \(t = 0\), the state of the system, \(|\Psi(0)\rangle = |\Psi_{\mathbf{q_0}}\rangle\). Since \(|\Psi_{\mathbf{q}}\rangle\) is still an instantaneous eigenstate of \(\tilde{H}\) at any time latter, the evolving state \(|\Psi(t)\rangle\) differs from \(|\Psi(0)\rangle\) only by a time-dependent factor and is determined by
\[i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \tilde{H} |\Psi(t)\rangle = E_q(t)|\Psi(t)\rangle, \] (20)
following from which,
\[|\Psi(t)\rangle = \exp \left[ -i \int_0^t E_q(t) dt \right] |\Psi_{\mathbf{q_0}}\rangle. \] (21)

Then, the charge current
\[I(t) = \langle \Psi(t)| \overline{J} |\Psi(t)\rangle = -c \langle \Psi(t)| \frac{\delta \tilde{H}}{\delta A} |\Psi(t)\rangle \]
\[= -c \frac{\delta}{\delta A} \langle \Psi(t)| \tilde{H} |\Psi(t)\rangle = -c \frac{\delta E_q(t)}{\delta A}, \] (22)
where we have used the fact that \(\frac{\delta}{\delta A} (\Psi(t)|\Psi(t)\rangle) = 0\). And Eq. (22) is our key result.

**IV. DISCUSSION**

Using Eq. (22), we can in principle calculate the charge current at any time \(t\) for any given initial instantaneous eigenstate of the system. Noting Eqs. (19) and (15), we have
\[I(t) = -c \frac{\delta E_q(t)}{\delta A} \]
\[= B_1 \cos[\frac{2\hbar}{e} A(t)] + B_2 \sin[\frac{2\hbar}{e} A(t)], \] (23)
where
\[B_1 = \frac{2 e^2}{\hbar} \sum_{m=1}^N \sin q_m; \quad B_2 = \frac{2 e^2}{\hbar} \sum_{m=1}^N \cos q_m. \]

For the half-filling case, \(N = L\), one can find that both \(B_1\) and \(B_2\) are 0, due to the complete cancellations in the sums, and hence \(I(t) = 0\).

However, away from half-filling, nonzero \(B_1\) and \(B_2\) will lead to a nonzero \(I(t)\) which oscillates with a frequency \(\omega = eE/h\) and an amplitude \(\sqrt{B_1^2 + B_2^2}\), the so-called Bloch oscillation. Very strangely, the Bloch oscillation now, similar to that in the noninteracting case, is an
undamped one, in spite of the infinity on-site interaction $U$.

An interesting question is that whether such undamped Bloch oscillations only exist in the $U \to +\infty$ case. An exact discussion of this question in the general sense seems impossible. However, we find that there do exist undamped Bloch oscillations in finite $U$ cases. A simple example is given by an analysis of the current response of the special eigenstates of Hubbard model, which have been constructed by Yang and Zhang \cite{9}. The operator used, $\zeta^\dagger = \sum_{m=1}^N C_{m\uparrow}^\dagger C_{m\downarrow}$, which is found to be an eigen-operator of the Hamiltonian \cite{11}, for any magnitude of $U$, satisfies

$$[\zeta^\dagger, H] = 0. \quad (24)$$

The acting of $\zeta^\dagger$ on a state is only to change one of the spin $\downarrow$ electrons in the system to a spin $\uparrow$ one. It should be noted that if all the $N$ electrons in the system are spin $\downarrow$ ones, the on-site $U$ will play no role and the time-evolution states of Hamiltonian \cite{11} are just the same as that in the noninteracting case and would show the same Bloch oscillations. We introduce a state of this kind and denote it by $|\Psi_0(t)\rangle$. Then obviously, due to Eq. (24), the state $|\Psi_m(t)\rangle = (\zeta^\dagger)^m|\Psi_0(t)\rangle (m \leq N)$ is also a time-evolution state of Hamiltonian \cite{11}, but for a system with $m$ spin $\uparrow$ electrons and $N-m$ spin $\downarrow$ electrons. Since the acting of $\zeta^\dagger$'s on $|\Psi_0(t)\rangle$ only changes the spin of electrons from $\downarrow$ to $\uparrow$, the characteristic of Bloch oscillations will keep. That is, time-evolution states of this kind do show undamped Bloch oscillations as that in the noninteracting case, seemingly disregarding the on-site $U$ in the Hubbard model. Obviously, such result is also applicable to the two- or three-dimensional case.

These results are very strange, since it was almost a common sense that the strong correlation interaction of electrons, which in nature is a kind of particle-particle interactions, might lead to a damping current. However, we note that there have already existed some discussions about the issue that electrons in strong correlated systems could show dissipationless transport behaviors in the linear response level (see Refs. \cite{11, 11} for more details). Though more general and exact discussions are still lacking, in the appendix we give an exact discussion about the effect of correlation interaction of electrons on the charge current of the system, basing on a $\delta$-function interaction model, which is not the Hubbard model but can be viewed as a version of Hubbard model in a continuous space, and we find that the current is indeed similar to that in the noninteracting case.

\section{V. CONCLUSION}

In conclusion, basing on the exact discussion of the $U \to +\infty$ Hubbard model in a electric field, we find that undamped Bloch oscillations extensively exist. This result is very strange, since it seems that the role of strong correlation to dissipate a charge current might not be so significant as we expected. Though a rigorous discussion about such dissipationless transport behavior for the Hubbard model in general sense is still lack, the appearance of undamped Bloch oscillations has been confirmed for the special eigenstates constructed by Yang and Zhang \cite{9}, for any magnitude of $U$, in any dimensions. And an ideal conductivity is rigorously confirmed for the model of electrons with $\delta$-function interactions, which is believed to be closely related to the Hubbard model.

\section{Appendix: Ideal conductivity in a system of electrons with $\delta$-function interactions}

We consider a 1D system of electrons with $\delta$-function interactions, the Hamiltonian is formally written as

$$H = \sum_{k\sigma} \left[ p - eA(t)/c \right]^2/2m C_{k\sigma}^\dagger C_{k\sigma} + U \sum_{k_1,k_2,q} C_{k_1+q\downarrow}^\dagger C_{k_2-q\uparrow}^\dagger C_{k_1\downarrow} C_{k_2\uparrow}$$

where $p = \hbar k$. The electric field $E = -1/cA(t)$. The charge current operator now is

$$j(t) = \sum_{k\sigma} e p C_{k\sigma}^\dagger C_{k\sigma} - e^2/mc A(t) \sum_{k\sigma} C_{k\sigma}^\dagger C_{k\sigma}.$$

Then the charge current for an evolving state $|\Psi(t)\rangle$ is

$$I(t) = \langle \Psi(t)|j(t)|\Psi(t)\rangle,$$

and

$$\frac{dI}{dt} = \frac{1}{i\hbar} \langle \Psi(t)|[j(t), H]|\Psi(t)\rangle + \langle \Psi(t)|\frac{\partial j}{\partial t}|\Psi(t)\rangle = \frac{e^2}{m} E N,$$

where we have use the fact that $[j(t), H] = 0$, and $N$ the total number of electrons in the system. Then,

$$I(t) = I_0 + \frac{e^2}{m} EN t,$$

where $I_0$ is the current at $t = 0$. Namely, an undamped charge current appears with the electric field. This result can be easily extended to two- or three-dimensional case, or cases with more general particle-particle interactions.
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