Curing singularities in cosmological evolution of $F(R)$ gravity

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Abstract: We study $F(R)$ modified gravity models which are capable of driving the accelerating epoch of the Universe at the present time whilst not destroying the standard Big Bang and inflationary cosmology. Recent studies have shown that a weak curvature singularity with $|R| \to \infty$ can arise generically in viable $F(R)$ models of present dark energy (DE) signaling an internal incompleteness of these models. In this work we study how this problem is cured by adding a quadratic correction with a sufficiently small coefficient to the $F(R)$ function at large curvatures. At the same time, this correction eliminates two more serious problems of previously constructed viable $F(R)$ DE models: unboundedness of the mass of a scalar particle (scalaron) arising in $F(R)$ gravity and the scalaron overabundance problem. Such carefully constructed models can also yield both an early time inflationary epoch and a late time de Sitter phase with vastly different values of $R$. The reheating epoch in these combined models of primordial and present dark energy is completely different from that of the old $R + R^2/6M^2$ inflationary model, mainly due to the fact that values of the effective gravitational constant at low and intermediate curvatures are different for positive and negative $R$. This changes the number of e-folds during the observable part of inflation that results in a different value of the primordial power spectrum index.
1. Introduction

Numerous recent observational data prove convincingly that the Universe is undergoing accelerated expansion at the present time, whilst decelerating in the past for redshifts larger than about $z \sim 0.7$. If interpreted in terms of the Einstein general theory of relativity, this acceleration requires the existence of some new component in the right-hand side of the Einstein equations, dubbed dark energy (DE), which remains practically non-clustered at all scales at which gravitational clustering of baryonic and dark non-baryonic matter is observed, and which has an effective pressure $p_{DE}$ approximately equal to minus its effective energy density $\rho_{DE}$. Thus, its properties are very close to those of a cosmological constant $\Lambda$ (see [1, 2, 3, 4, 5] for some reviews). The simplest possible DE model, $\Lambda$ combined with a non-relativistic non-baryonic dark matter (the standard spatially flat $\Lambda$CDM cosmological model), is completely self-consistent from the mathematical point of view and provides a good fit to all existing observational data [6]. In this case $\Lambda$ acquires the status of a
new fundamental physical constant. However, its required value is very small compared to known atomic and elementary particle scales (which are well below the Planck scale), so a firm theoretical prediction for this quantity from first principles is currently lacking.

On the other hand, in the second case when a component with qualitatively similar properties is assumed to exist – in the inflationary scenario of the early Universe, we are sure that this “primordial DE” may not be an exact cosmological constant since it should decay in the early Universe. Hence it is natural to seek non-stationary models of the current DE, too.

Among them, the simplest purely gravitational models in 3+1 space-time dimensions are provided by $F(R)$ gravity which modifies and generalizes Einstein gravity by incorporating a new phenomenological function of the Ricci scalar $R$, $F(R)$. They represent a self-consistent and non-trivial alternative to the $\Lambda$CDM model. The literature on these models is dense, and we direct the reader to [7] and references therein for a detailed recent review. The action of $F(R)$ gravity is given by

$$S = \frac{M^2_{\text{Pl}}}{2} \int d^4x \sqrt{-g} F(R) + S_m , \quad (1.1)$$

where $M^2_{\text{Pl}} = 1/8\pi G$, $\hbar = c = 1$ is assumed throughout the paper, and $S_m$ describes all non-gravitational matter including non-relativistic (cold) dark matter which is minimally coupled to gravity.\(^{1}\) The field equations following from (1.1) have the form

$$F'(R) R^\nu_\mu - \frac{1}{2} F(R) \delta^\nu_\mu + (\delta^\nu_\mu \Box - \nabla_\mu \nabla^\nu) F'(R) = M^2_{\text{Pl}} T^\nu_\mu , \quad (1.2)$$

and their trace reads

$$3 \Box F'(R) + R F'(R) - 2 F(R) = M^2_{\text{Pl}} T , \quad (1.3)$$

where $T$ is the trace of the matter energy-momentum tensor $T^\nu_\mu$ and the prime denotes the ordinary derivative with respect to an argument.

In the Jordan frame where the action (1.1) is written, fermion masses are constant and atomic clocks built from usual matter measure the proper time. However, equivalent description is possible in the Einstein frame where gravity resembles standard GR but free particles of usual matter do not follow space-time geodesics due to an interaction with a new scalar field. As will be seen below, sometimes it is easier to solve equations in the latter frame first. We also assume the metric variation of (1.1); the Palatini variation of formally the same action leads to a completely different theory, in which the number of degrees of freedom is not the same.

In the absence of matter, exact de Sitter (positive constant curvature) solutions of Eqs. (1.2) are given by real positive roots of the functional equation

$$RF'(R) - 2 F(R) = 0 . \quad (1.4)$$

\(^{1}\)The sign conventions are: the metric signature $(+++)$, the curvature tensor $R^\sigma_{\rho\sigma\nu} = \partial_\rho \Gamma^\sigma_{\mu\nu} - ...$, $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$, so that the Ricci scalar $R = R^\mu_\mu > 0$ for the de Sitter space-time and the matter-dominated cosmological epoch.
These solutions (and solutions close to them) are the basis for a description of primordial and present DE. They are future stable if

\[
F'(R_1)/F''(R_1) > R_1 ,
\]

where \( R_1 \) is a root of Eq. (1.4). This condition was first obtained in [9], the easiest way to derive it is to use the trace equation (1.3) for a small perturbation \( R - R_1 \) (in fact, \( F''(R_1) \) should be positive, too, as will be discussed below).

\( F(R) \) gravity is a special class of scalar-tensor gravity with a vanishing Brans-Dicke parameter \( \omega_{BD} \). If \( F''(R) \) is not zero identically, it contains a new scalar degree of freedom dubbed “scalaron” in [8], thus, it is a non-perturbative generalization of Einstein gravity. We will consider it as a purely phenomenological semiclassical macroscopic theory of gravity which arises from some more fundamental quantum microscopic theory after tracing out degrees of freedom which are not excited at sufficiently small space-time curvature. Thus, the resulting function \( F(R) \) need not necessarily be some simple (e.g. polynomial) function of \( R \). It may well have some complicated behaviour for small \( R \), too, as many examples from condensed matter physics teach us. So, we will not discuss which functional form of \( F(R) \) is “natural” in any sense. On the other hand, for a phenomenological \( F(R) \) model to be viable, it should satisfy a rather large list of viability conditions:

1) Classical and quantum stability in the region of \( R \) where we want to use this theory:

\[
F'(R) > 0, \quad F''(R) > 0 .
\]

The first condition means that gravity is attractive and the graviton is not a ghost. It was recognized long ago that its violation during the time evolution of a Friedmann-Robertson-Walker (FRW) background results in the immediate loss of homogeneity and isotropy and formation of a strong space-like anisotropic curvature singularity [10, 11]. The second condition on the flat background was also known since the first papers on \( F(R) \) gravity [12], and was assumed when constructing inflationary models in \( F(R) \) gravity [8]. However, in the case of \( F(R) \) models of present DE, the necessity to keep it valid for all values of \( R \) during the matter- and radiation-dominated stages in order to avoid the Dolgov-Kawasaki instability [13] has been realized rather recently [14, 15]. In addition, a weak (“sudden”) curvature singularity forms generically if \( F''(R) \) becomes zero for a finite value \( R = R_s \). This is also undesirable; see the discussion below in Sec. 2.4.

2) Existence of the stable Newtonian limit for all values of \( R \) where Newtonian gravity accurately describes observed inhomogeneities and compact objects in the Universe, i.e. for \( R \) exceeding the present Friedmann-Robertson-Walker (FRW) background value \( R_0 \equiv R(t_0) \), where \( t_0 \) is the present moment, and up to curvatures in the centre of neutron stars. The conditions required for this are

\[
|F(R) - R| \ll R , \quad |F'(R) - 1| \ll 1 , \quad RF''(R) \ll 1
\]

for \( R \gg R_0 \). Note that in this regime the effective scalaron mass squared is \( M^2_{\text{scalaron}}(R) = 1/(3F''(R)) \), as directly follows from Eq. (1.3). Then the second of the conditions (1.6) means that the scalaron is not a tachyon, while the last of the conditions (1.7) implies
that its Compton wavelength is much less than the radius of curvature of the background space-time. For more general backgrounds than matter-dominated FRW, for which General Relativity (GR) has to be used in full (in particular, if the pressure $P$ of matter is not small compared to its energy density $\rho$), the conditions (1.7) guarantee that non-GR corrections to a space-time metric remain small.

3) Absence of deviations from GR at the level of accuracy following from present laboratory and Solar system tests of gravity.

4) Existence of a future stable (or at least metastable) de Sitter asymptote. This is necessary for a description of the present DE, which behaves in a similar manner to that of a cosmological constant.

5) $F(R)$ cosmology should not destroy previous successes of present and early Universe cosmology in the scope of GR including the existence of the matter-dominated stage driven by non-relativistic matter preceded by the radiation-dominated stage with the correct Big Bang nucleosynthesis (BBN) of light elements and, as we shall see, some kind of inflation prior to this.

The first of these conditions is also the main reason why we do not include other invariants constructed from the Riemann tensor and its derivatives as additional arguments of the function $F$. Indeed, it has long been known [16, 17] that if $F$ also depends on $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, then generically a new massive spin-2 particle appears which is a ghost whenever the standard massless graviton is not a ghost. This is problematic from the quantum field theory point of view for many reasons, see e.g. [18]. Moreover, as argued in [19], cosmological models with ghosts are unsatisfactory even at the purely classical level. In particular, in such models we can no longer explain the observed approximate large-scale homogeneity and isotropy of the Universe without tremendous fine-tuning of initial conditions, even if we include a primordial inflationary stage. The only way to avoid this ghost (without including derivatives of the Riemann tensor in the action) is to consider $F = F(R, G)$ where $G$ is the Gauss-Bonnet invariant, $G = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. However, it was recently shown in [20] that in this case linear scalar perturbations have pathological behaviour in the ultra-violet regime, except in some special cases.\footnote{After the first variant of our paper was submitted to JCAP and archives, the further paper [21] appeared where this remaining special case was shown to possess an ultra-violet instability of scalar perturbations on a FRW background with matter in the form of a perfect fluid, too.} Invariants in $F$ containing derivatives of the Riemann tensor lead to new particles (in particular, scalar ones if $F$ depends on $R$ and its derivatives only [22]) among which ghosts are generic, too, see e.g. [23]. Thus, such terms may be considered when making perturbative expansion around solutions of the Einstein gravity, but (possibly apart from some exceptional cases still to be found) they are of no use in our approach since we want to use a (semi)classical modified theory of gravity in a fully non-perturbative regime.

For these reasons, we concentrate on $F(R)$ effective macroscopic gravity only. Note that this is in contrast to quantum-gravitational and string corrections to Einstein gravity, which generically produce terms with all possible invariants of the Riemann tensor. The required “$R$-dominance” presents a serious problem for such microscopic mechanisms to act as the origin of $F(R)$ gravity, however there exists a number of cases when just this form of
modified gravity appears in some limit. The simplest of them follows from above-mentioned fact that \( F(R) \) gravity is a particular case of more general scalar-tensor gravity with the Brans-Dicke parameter \( \omega_{BD} = 0 \). Therefore, it also yields a good approximate description for scalar-tensor gravity with \(|\omega_{BD}| \ll 1\), in particular, for a non-minimally coupled scalar field with a negative and large by modulus coupling constant \( \xi \) (we use the sign convention where conformal coupling corresponds to \( \xi = +1/6 \)), cf. [24]. For this reason, in particular, predictions of the Higgs inflationary model [25] (without loop corrections to the Higgs potential) for primordial power spectra of scalar (density) perturbations and gravitational waves are the same as those of the \( F(R) = R + R^2/6M^2 \) model – the simplified variant of the model [8]. Another, completely unrelated case where phenomenological \( F(R) \) gravity arises rather unexpectedly [26] is the so called emergent gravity approach using ideas and methods borrowed from quantum theory of condensed matter. This illustrates the well-known fact that an elegant and internally consistent mathematical model may appear multiple times from totally different physical foundations. Note that in both of these examples, the new scalar gravitational degree of freedom (scalaron) is present already at the underlying microscopic level; however, this does not mean that it is fundamental even at this level.

Although \( F(R) \) gravity can successfully pass the first requirement from the list above, it is evident from the beginning that only a very narrow subset of all possible \( F(R) \) functions may be of interest for cosmology. The situation is simpler in case of inflationary models. Here, the simplest variant with \( F''(R) \neq 0 \) identically, namely \( F = R + R^2/6M^2 \) (where \( M \) is the scalaron rest-mass at low curvature), presents an internally self-consistent inflationary model with slow-roll during inflation (see Sec. 4.4 below) and a graceful exit to a subsequent FRW matter-dominated stage driven by scalarons [8]. Reheating, creation of usual matter and transition to the radiation-dominated FRW stage are achieved by gravitational particle production due to strong oscillations of \( R \) during the scalaron-dominated stage [8], see [27, 28, 29, 30, 31] for more details regarding the background evolution and reheating in this model. In addition, in contrast to many other inflationary models proposed later which have been falsified by observational data, this model still remains viable since it predicts the value of the slope of the primordial power spectrum of scalar (density) perturbations \( n_s = 1 - 2/N \) [32, 33] in agreement with present observational data. Here \( N \) is the number of e-folds between the first Hubble radius crossing of the present inverse comoving scale 0.002 Mpc\(^{-1}\) and the end of inflation; \( N \approx (50-55) \) for the reheating mechanism mentioned above. The tensor-to-scalar ratio is rather small, \( r = 12/N^2 \), but not negligible [33]; see also [34, 35, 36] for further papers on equations and solutions for perturbations and [37] for the energy theorem in this model. To fit the observed amplitude of the power spectrum, the only free model parameter \( M \) should be chosen as \( M \approx 1.5 \times 10^{-5}(N/50)^{-1}M_{Pl} \). Furthermore, we will show in Sec. 4.4. that all viable inflationary models in \( F(R) \) gravity with other values of \( n_s \) and \( r \) have behaviour close to \( R^2 \) at large \( R \) (or around some large fixed value \( R = R_1 \)). Thus, the \( R^2 \) behaviour of \( F(R) \) is characteristic for inflation in \( F(R) \) gravity.

Due to the remarkable quantitative analogy between properties of primordial DE supporting inflation in the early Universe and present DE, it is tempting to use \( F(R) \) gravity
to also build models of the latter, to act as alternatives to the trivial case of a cosmological constant which corresponds to $F(R) = R - 2\Lambda$. Indeed, many such models have been proposed, beginning with the papers [38, 39, 40, 41]. However, most of these attempts remained unsuccessful since the conditions written above had not been fully satisfied (once more, we direct a reader to the review [7] for an extensive list of papers on the subject). As a result, there has arisen a widespread doubt if viable $F(R)$ models of present DE may exist at all (in contrast to primordial DE).

Still at last a rather narrow class of functional forms of $F(R)$ was found [42, 43, 44] which can satisfy the first four viability conditions from the list above and even partly the fifth one, with regards to the existence of a stable matter dominated epoch with $a(t) \propto t^{2/3}$ in the recent past which is driven by cold dark matter and baryons (not, we stress, by terms due to modified gravity.) Here, $a(t)$ is the scale factor in a FRW spacetime. It should be emphasized from the very beginning that in none of these viable $F(R)$ DE models is it possible to derive the energy scale of present DE from first principles. Instead, it has to be inserted into the action (1.1) as a free parameter, the value of which is taken from observational data. Thus, $F(R)$ DE models may not be superior to DE being an exact cosmological constant $\Lambda$, they are simply an alternative to it. In the models [42, 43, 44], $F(R)$ is analytic for $R = 0$, has a non-trivial structure for $R \sim R(t_0)$ and then quickly approaches Einstein gravity with an effective cosmological constant for $R \gg R_0$, see Eq. (2.3) below. Also, in these models the condition $F(0) = 0$ (dubbed the “disappearing cosmological constant” in [44]) is imposed by hand to ensure that there is no true cosmological constant in flat space-time (otherwise, why work with $F(R)$ gravity at all?) The main difference between these models is in the law of approach to the standard gravity for $R \to \infty$: an inverse power law in [42, 44] and an exponential in [43].

However, it was immediately recognized that the story of constructing at least one viable cosmological model of present DE in $F(R)$ gravity was not finished: three new problems related to the last, fifth viability condition arise when tracing small deviations of a FRW background from the standard ΛCDM model to the past. First, as was shown in [44] (see also [45]), for these models the frequency of small oscillations of the Ricci scalar $R$ (i.e. the scalaron rest-mass) around the general relativistic limit $R_{GR} = -T/M_{Pl}^2$, $\omega \equiv M_s(R) = 1/\sqrt{3F''(R)}$, grows quickly to the past, $t \to 0$, and exceeds the Planck value very soon, thus invalidating classical consideration of the theory (1.1). For example, for the model [43] it happens already for redshifts $z > 7$ and matter densities exceeding $\sim 10^{-27}$ g cm$^{-3}$.

The second problem is that the amplitude of these linear oscillations quickly grows back in time [44]. As a result, we get the scalaron overabundance problem which is actually a problem of initial conditions of the Universe; they should be such that the scalaron number density (which is proportional to the square of an amplitude of these oscillations) should be sufficiently small at the period of BBN. Third, due to the same reason, linear consideration of these oscillations may become inadequate even before their backreaction on a FRW background becomes important. A non-linear analysis of these models was undertaken in [46], where it was observed that the Ricci scalar would generically evolve to a weak singularity at some finite time in the past. This singular behaviour was predicted
independently in [47] for a more general class of models. These three problems once again raised the question as to whether viable DE $F(R)$ gravity models can be constructed.

The main aim of this paper is to find a way to avoid these three difficulties and to show that there indeed exist $F(R)$ DE models satisfying all five viability conditions. We shall show that for this purpose, as envisaged in [44], it is sufficient to change the behaviour of $F(R)$ at large $R$ by adding the term $\propto R^2$ with a sufficiently small coefficient to ensure the existence of an inflationary stage in the early Universe. Moreover, since this behaviour is characteristic for viable inflationary models in $F(R)$ gravity, we show additionally that it is possible to construct a combined model where both primordial and present DE are described in the scope of $F(R)$ gravity.\footnote{Of course, this model does not really unify primordial and present DE since we have to introduce two tremendously different curvature scales corresponding to inflation in the early Universe and to the present space-time curvature by hand.} Rather unexpectedly, such a construction also requires us to change the low curvature behaviour of $F(R)$ in the models [42, 43, 44] for $R < R_0$ and further to the region of negative values of $R$ which are not observable at the present time, in order to avoid violation of the conditions (1.6) during strong oscillations of $R$ after the end of inflation. Moreover, it will be shown that a non-trivial structure of $F(R)$ at low $R$, required for an alternative description of present DE (as opposed to a cosmological constant), greatly affects the stage of post-inflationary evolution and reheating in this combined model, and even results in the change of numerical values for parameters of primordial power spectra of perturbations generated after inflation.

The rest of the paper proceeds as follows. In section 2 we review the cosmological evolution of viable models of present DE in $F(R)$ gravity, and comment on the existence and properties of the weak singularity mentioned above. We then consider possible approaches to eliminate this singularity and bound the scalaron mass by introducing additional terms into the action in section 3. Finally, we consider the possibility that these additional terms may drive an early inflationary period of the Universe, and study the slow-roll and reheating epochs in section 4. Section 5 contains conclusions and discussion.

2. Review of cosmological evolution

We will be concerned with the following $F(R)$ functions describing present DE which satisfy the first four viability conditions listed above and possess a stable matter-dominated epoch at intermediate redshifts $z$ for some range of their parameters:

$$F_{HSS}(R) = R - \frac{R_{\text{vac}}}{2} \frac{c \left( \frac{R}{R_{\text{vac}}} \right)^{2n}}{1 + c \left( \frac{R}{R_{\text{vac}}} \right)^{2n}}, \quad (2.1)$$

$$F_{AB}(R) = \frac{R}{2} + \frac{\epsilon_{AB}}{2} \log \left[ \frac{\cosh \left( \frac{R}{\epsilon_{AB}} - b \right)}{\cosh b} \right], \quad (2.2)$$

where $b$ and $c$ are dimensionless constants, $\epsilon_{AB} = R_{\text{vac}}/(b + \log(2 \cosh b))$ and $n > 0$. The model (2.1) is the model introduced in [42] (with a slightly different notation of parameters).
it is similar to the model of [44]. The model (2.2) is from the paper [43] (a model with a similar behaviour was also introduced later in [45]). For large $R$, these models mimic General Relativity with a cosmological constant, in the sense that for $R \gg R_{\text{vac}}$, $F_{\text{HSS}}$ and $F_{\text{AB}}$ can be expanded in the region of interest to cosmology as

$$F(R) \approx R - \frac{R_{\text{vac}}}{2} + \chi(R). \quad (2.3)$$

Thus, $R_{\text{vac}}/4$ acts as a small effective cosmological constant induced by space-time curvature. Since $F(0) = 0$, there is no true cosmological constant in these models. $\chi(R)$, $\chi'(R)$ and $\chi''(R)$ are all small functions of $R$, which satisfy $\chi(R)/R \ll 1$, $\chi'(R) \ll 1$ and $R\chi''(R) \ll 1$ for $R \gg R_{\text{vac}}$. For the HSS and AB models, we have

$$\chi_{\text{HSS}} = \frac{\epsilon_{\text{HSS}}^{2n+1}}{R^{2n}}, \quad \chi_{\text{AB}} = \frac{\epsilon_{\text{AB}}}{2} e^{2b} \exp \left( -2R/\epsilon_{\text{AB}} \right), \quad (2.4)$$

where $\epsilon_{\text{AB}}$ and $\epsilon_{\text{HSS}} = R_{\text{vac}}/(2c)^{1/(2n+1)}$ are smaller than $R_{\text{vac}}$.

In this section we briefly review the cosmological evolution of the AB and HSS models, in particular the behaviour of the scalar degree of freedom (scalaron), and highlight the existence of a singularity in the evolution of the Ricci scalar. In ref.[44], the trace equation (1.3) was solved using a perturbative approach: an ansatz $R = R_{\text{GR}} + \delta R$ was taken, where $R_{\text{GR}} = -T/M_{\text{Pl}}^2$, and the equation linearized for $\delta R \ll R_{\text{GR}}$. It was found that the Ricci scalar for the HSS model will generically undergo rapid oscillations around its General Relativistic limit, a result also obtained for the AB model in ref.[45] (although see ref.[46] for a discussion of the applicability of the linearized approach in the case of the AB model.) The frequency and amplitude of these oscillations increase without bound for both models as $R$ grows to the past ($t \to 0$). Specifically, $\delta R$ contains an oscillating component that is given by

$$\delta R_{\text{osc}} = C a^{-3/2} (F''(R_{\text{GR}}))^{-3/4} \sin \left[ \int \frac{dt}{\sqrt{3F''(R_{\text{GR}})}} \right]. \quad (2.5)$$

It represents scalaron oscillations (particles) with the frequency (rest-mass) $M_s(R) = (3F''(R_{\text{GR}}))^{-1/2}$ in the regime when $M_s^2 \gg |R_{\text{GR}}|$. Typically, $M_s$ becomes $\gg M_{\text{Pl}}$ during the matter and radiation eras for both models.

2.1 Non-linear oscillations and existence of a “sudden” singularity with $|R| \to \infty$

In refs.[46], an alternative approach to solving (1.3) was considered. In this work it was noted that at curvatures of cosmological interest, we can use the expansions (2.4) to write (1.3) as a non-linear oscillator equation. By making the field redefinitions

$$R = R_{\text{GR}} - \frac{c}{2} \log(1 + x), \quad (2.6)$$

$$R = \frac{R_{\text{GR}}}{(1 + x)^{1/(2n+1)}}, \quad (2.7)$$
for the AB and HSS models respectively, and neglecting terms of order $O(\chi(R_{GR}))$, (1.3) can be written as

$$x_{\lambda\lambda} + \frac{\alpha(\lambda)}{\lambda} x_{\lambda} + \frac{\epsilon}{6} \log(1 + x) \approx 0 \quad (\text{AB}),$$

$$x_{\lambda\lambda} + \frac{\beta(\lambda)}{\lambda} x_{\lambda} - \frac{\epsilon}{6} \left( \frac{1}{(1 + x)^{1/3}} - 1 \right) \approx 0 \quad (\text{HSS}),$$

for $R \gg R_{\text{vac}}$, where we have taken $n = 1$ for simplicity in the HSS model. In (2.8) and (2.9) $x$ is a dimensionless field, $\epsilon$ has the same dimensions as the Ricci scalar, and $\alpha(\lambda)$ and $\beta(\lambda)$ are dimensionless damping terms that will be unimportant in the following discussion. $\lambda$ subscripts denote derivatives with respect to the ‘fast time’ $\lambda$, which is related to the cosmological time by

$$\frac{d\lambda}{dt} = \frac{1}{\sqrt{|f'_0|}},$$

where we have introduced the dimensionless function $f'_0 = F'(R_{GR}) - 1$. Both (2.8) and (2.9) are damped, non-linear oscillator equations, with potentials given by integrating the expressions $dV/dx = (\epsilon/6) \log(1 + x)$ and $dV/dx = (\epsilon/6)(1 - (1 + x)^{-1/3})$ for the AB and HSS models respectively.

Equations (2.8) and (2.9) can be solved numerically, and $R$ obtained from (2.6) and (2.7). By taking into account non-linear terms in (1.3), a number of effects were observed that were not apparent in the linearized analysis. Specifically, it was found the oscillations become asymmetric in the past (as expected, when the linearized approximation breaks down), and further that by evolving $R$ backwards in time, it will generically evolve to a singularity.

A singularity arises in both models as $x \to -1$ after a finite time. At this point, the potential has the asymptotic behaviour $V(x) \to \text{const}$, but its derivative diverges $dV/dx \to \infty$. From (2.6) and (2.7), it is clear that the Ricci scalar will diverge as $x \to -1$. The potentials for the AB and HSS models are given by

$$V_{\text{AB}}(x) = \alpha_1 + \frac{\epsilon}{6} (1 + x) \left[ \log(1 + x) - 1 \right],$$

$$V_{\text{HSS}}(x) = \alpha_2 + \frac{\epsilon}{6} \left[ (1 + x) - \frac{2n + 1}{2n}(1 + x)^{2n/(2n+1)} \right],$$

and are exhibited in fig.1. $\alpha_{1,2}$ are integration constants, which dictate the ground state of $x$. In the limit $x \to -1$, we have $V \to \text{const}$, as expected.

The divergence of the Ricci scalar represents a weak singularity in the sense that $\ddot{a}$ and $R$ diverge whilst $\rho$, $p$ and $\dot{a}$ remain finite. As a result, a small vicinity of space-time containing this space-like singularity (and from both sides of it) may be covered by the Minkowski metric with small perturbations which, however, are not $C^2$ continuous. Moreover, if the singularity occurs during the matter-dominated epoch when $p \ll \rho$, the Newtonian approximation is applicable around it. Let us obtain the expression for the
behaviour of a FRW scale factor near the singularity. It follows from the trace equation Eq. (1.3) that the analytic form of $a(t)$ in the vicinity of $t = t_s$ – the time at which $R$ diverges – is given by

$$a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 (\log |t - t_s| + \tilde{a}_2) + ...,$$

(2.13)

$$a(t) = a_0 + a_1(t - t_s) + a_2|t - t_s|^{(1+2n)/(1+n)} + ...,$$

(2.14)

for the AB and HSS models respectively, where $a_{0,1,2}$ and $\tilde{a}_2$ are constants. Such a singularity was considered in [48] from a kinematic viewpoint (i.e. not as a solution of any dynamical equations) where it was called “sudden”. It also appeared in a different dynamical setting in [49], where it was dubbed the “Big Boost”.

In contrast to strong curvature singularities occurring in cosmology and inside black holes, there is no geodesic incompleteness here [50]. However, this does not mean that the weak “sudden” singularities (2.13, 2.14) are harmless. Just the opposite, they are undesirable. Indeed, it can be checked that Eqs. (1.2) do not supply us with any information on how the coefficients $a_2$ for $t < t_s$ and $t > t_s$ are related to each other. Thus, the ”sudden” singularity results in the loss of predictability; from initial data given at any Cauchy
hypothesis with \( t = \text{const} < t_s \), it is not possible to predict the space-time metric for \( t > t_s \) unambiguously. Note also that in contrast to strong curvature singularities, we may not evade the problem by arguing that quantum-gravitational effects invalidate the very notion of deterministic classical space-time near such a singularity. It can be shown \cite{51} that such effects remain small as one approaches the weak singularity; the energy density and other components of an averaged energy-momentum tensor (EMT) of quantum fields in such curved space-time generically remain subdominant to the background (this also follows from general expressions for an EMT average value in a weakly curved background obtained in \cite{52}). So, the singularity has to be resolved at the classical level.

Therefore, the appearance of “sudden” singularities in a given \( F(R) \) DE model signals the internal incompleteness of said model. One way to resolve the singularity is to return from the effective macroscopic \( F(R) \) theory (1.1) to the underlying microscopic theory from which the former originated, and see what happens in the latter. Then, however, there is no guarantee that the resulting ansatz for the resolution of the “sudden” singularity will be universal and will not depend on the choice of underlying physics. Since we prefer to remain at the purely phenomenological level in this paper, we instead look for a way to modify the given \( F(R) \) model in such a way that the weak singularities do not appear at all, at least in solutions which are of interest for cosmology, an approach that we will consider in Sec. 3.

2.2 General viable \( F(R) \) models of present DE

Thus far, we have focussed attention on two specific models. We will now generalize the existence of a singularity to an arbitrary \( F(R) \) function for which \( F''(R) > 0 \) and \( R|F'(R) - 1| \to 0 \) as \( R \to \infty \). Whilst viable \( F(R) \) models do not have to satisfy this asymptotic limit at infinity, they do have to satisfy \( RF''(R) \ll 1 \) for \( R \gg R_{\text{vac}} \), as we demand that any modification to General Relativity should only become dominant at late times. We begin with the function \( F(R) = R - R_{\text{vac}}/2 + \chi(R) \), where \( \chi(R) \) satisfies \( \chi(R)/R \ll 1 \), \( \chi'(R) \ll 1 \) and \( R\chi''(R) \ll 1 \). We then define \( 1 + x = \chi'(R)/\chi'_0 \), where \( \chi'_0 \equiv \chi'(R_{\text{GR}}) \), and write (1.3) in terms of \( x \),

\[
3\Box\chi'_0(1 + x) - (R - R_{\text{GR}}) + \chi'_0(1 + x) - 2\chi(R) \simeq 0, \tag{2.15}
\]

where we have defined \( R_{\text{GR}} \equiv -T/M_{\text{Pl}}^2 + R_{\text{vac}} \). It should be understood that \( R = R(x, R_{\text{GR}}) \) in (2.15), since the Ricci scalar will generically be a function of both \( x \) and \( R_{\text{GR}} \). Using a flat FRW metric ansatz, we obtain

\[
\chi'_0 \left[ x_{\lambda\lambda} + \left[ 2(\log \chi'_0)_{\lambda} + 3\tilde{H} \right] x_{\lambda} \right] + (1+x)\Box\chi'_0 + \frac{R(x, R_{\text{GR}}) - R_{\text{GR}}}{3} + \frac{2\chi(R) - R(x)\chi'_0(1 + x)}{3} = 0, \tag{2.16}
\]

where \( \tilde{H} = a_\lambda/a \). Now by using the conditions \( \chi'(R) \ll 1 \) and \( \chi(R)/R \ll 1 \), Eq. (2.16) can be approximated as

\[
\chi'_0 \left[ x_{\lambda\lambda} + \left[ 2(\log \chi'_0)_{\lambda} + 3\tilde{H} \right] x_{\lambda} \right] + \frac{R(x, R_{\text{GR}}) - R_{\text{GR}}}{3} \simeq 0. \tag{2.17}
\]
The $\chi'_0$ function multiplying the $x_\lambda$ and $x_\lambda$ terms is unimportant and can be removed by a redefinition of the time coordinate. The final step is to invert the expression $1 + x = \chi'(R)/\chi'_0$ to obtain $R = R(x, R_{GR})$ and substitute this in (2.17). Then we can associate the last term on the left-hand side of (2.17) with a potential gradient $\partial V/\partial x = (R(x, R_{GR}) - R_{GR})/3$.

The two models considered in this paper have particularly simple $\chi(R)$ functions, and hence it is straightforward to invert the expression $1 + x = \chi'(R)/\chi'_0$ to obtain $R = R(x, R_{GR})$, giving an analytic form for $\partial V/\partial x$. However, in general we will not be able to write $\partial V/\partial x$ in terms of known functions. Another feature of the two models under consideration is that all time dependence in $\partial V/\partial x$ drops out, making $dV/dx$ a function of $x$ only (this requires a further redefinition of the time coordinate in the HSS model).

Generically, $\partial V/\partial x$ will depend on both $x$ and $R_{GR}$.

To show that the singularity is a generic feature of these models, we write $\partial V/\partial x$ as

$$
\frac{\partial V}{\partial R} = \frac{1}{3} \frac{\chi''(R)}{\chi'_0}(R - R_{GR}),
$$

(2.18)

which, by using the definition of $x$, can subsequently be written as

$$
\frac{\partial V}{\partial R} = \frac{1}{3} \frac{\chi''(R)}{\chi'_0}(R - R_{GR}).
$$

(2.19)

By integrating this expression, we obtain the potential as a function of $R$ and $R_{GR}$,

$$
V(R, R_{GR}) = \frac{\chi'(R)}{3\chi'_0}(R - R_{GR}) - \frac{\chi(R)}{3\chi'_0} + \lambda(R_{GR}),
$$

(2.20)

where $\lambda(R_{GR})$ is an arbitrary function of $R_{GR}$. From (2.20) and the expressions $\partial V/\partial x = (R - R_{GR})/3$ and $1 + x = \chi'(R)/\chi'_0$, it is clear that for any model in which $\chi'(R)R \rightarrow 0$ (more rigorously, $R\chi''(R)$ is integrable) and $\chi(R) \rightarrow 0$ as $R \rightarrow \infty$, the potential will possess the singular point $\partial V/\partial x \rightarrow \infty$ and $V \rightarrow \lambda(R_{GR})$ as $x \rightarrow -1$.

We note that the singularity occurs for models in which $\chi''(R) > 0$ at large curvatures, and hence is unrelated to the Dolgov-Kawasaki instability. The Dolgov-Kawasaki instability corresponds to exponentially growing scalaron modes that are generically present in models that satisfy $\chi''(R) < 0$ in some dynamically accessible regime. We only consider models for which $F''(R) > 0$ for $R > R_{vac}$.

### 2.3 Determination of the Hubble parameter

In obtaining $x$ and hence $R$ from (2.8) and (2.9), it was assumed that the Hubble parameter could be written as $H = H_{GR} + \delta H$, where $\delta H \ll H_{GR}$ is small throughout the evolution and hence can be neglected. To check that this assumption is valid, the $(0, 0)$ component of the Einstein equations should also be solved for $H$ to check that it does not diverge when $R \rightarrow \infty$.

The $(0, 0)$ component is given by

$$
18HF''(R)\left(\ddot{H} + 4H\dot{H}\right) + \frac{F'(R)}{2} - 3\left(\dot{H} + H^2\right)F'(R) = \frac{\rho}{M^2_{Pl}},
$$

(2.21)
which, for the AB and HSS models, can be expanded as

\[ 18H\chi''(R) \left( \ddot{H} + 4H\dot{H} \right) + 3H^2 + \frac{\chi'(R)}{2} - 3 \left( \dot{H} + H^2 \right) \chi'(R) - \frac{\rho_{\text{vac}}}{M_{\text{Pl}}^2} = \rho, \quad (2.22) \]

for \( R > R_{\text{vac}} \). Eqn (2.21) is the first integral of the trace of the gravitational field equations, and is a second order differential equation for the Hubble parameter. However, due to the oscillatory behaviour of \( R \), it is difficult to solve (2.22) over a significant dynamical range, since the divergence in \( R \) is extremely sensitive to the initial conditions of \( H \) and \( \dot{H} \). This is a manifestation of the singular behaviour of the Ricci scalar. However, we will be able to solve (2.21) for the regularized models presented in section 3.

2.4 Structure of a singularity with \( F''(R) = 0 \) for a finite \( R \)

Thus, we have shown that the “Big Boost” weak curvature singularity (2.13, 2.14) arises whenever \( RF''(R) \) is integrable for \( R \to \infty \) with \( F''(R) > 0 \). Now, for completeness, let us present the structure of an even weaker singularity arising when \( F''(R) \) becomes zero at some finite value of \( R \), \( R = R_s \), so that the second stability condition (1.6) is marginally violated. For a generic case with \( F''(R_s) \neq 0 \), as follows from Eq. (1.3), this occurs at a finite moment of time \( t = t_s \) when \( a, H \) and \( R \) remain finite, but \( R \) diverges \( \propto |t - t_s|^{-1/2} \).

Thus, the scale factor has the following behaviour for \( t \to t_s \):

\[ a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 + a_3|t - t_s|^{5/2} + \ldots. \quad (2.23) \]

The metric is \( C^2 \), but not \( C^3 \), continuous across this singularity, and there is no unambiguous relation between the coefficients \( a_3 \) for \( t < t_s \) and \( t > t_s \).

Therefore, all that was said in section 2.1 regarding the “Big Boost” singularity also applies to this weak singularity. Namely, its appearance in solutions of interest for cosmology results in the loss of predictability of future Cauchy evolution. Therefore, if we choose to remain inside the scope of \( F(R) \) gravity, models where the second of the conditions (1.6) is even marginally violated during evolution should be avoided.

3. Avoiding the weak singularity and solving the problems of \( F(R) \) DE models

In the previous section, we have reviewed the existence of the weak “Big Boost” singularity which occurs in general at a finite redshift in the HSS and AB models, as well as in any other \( F(R) \) model of present DE for which \( F''(R) > 0 \) and \( RF''(R) \) is integrable for \( R \to \infty \). Since the problem arises at large curvatures, it is clear that \( F(R) \approx R - \text{const} \) as \( R \to \infty \) is an inappropriate law for viable \( F(R) \) DE models.\(^4\) Under the less restrictive conditions \( F''(R) > 0 \) and \( RF''(R) \to 0 \) as \( R \to \infty \), another difficulty, the first point

\[^4\text{This refers also to combined models of primordial and present DE considered recently in \([53, 54]\) which have the same behaviour of } F(R) \text{ for } R \to \infty. \text{ Generically these models possess weak singularities either of the type (2.13, 2.14), or of the type (2.23), since the second of the stability conditions (1.6) is violated for them, too.}\]

\[ \]
mentioned in the Introduction, arises; an unlimited growth of the scalaron rest-mass $M_s = (3F''(R))^{-1/2}$ in the quasi-GR regime (1.7). If $M_s$ exceeds $M_{Pl}$, while $|R|$ remains less than $M_{Pl}^2$, particles (scalarons) should collapse to black holes from a naive physical point of view. More formally, this means that loop quantum-gravitational corrections to the tree action (1.1) become dominant, so it may not be used further in an effective quasi-classical theory. Thus, such models of present DE are in general not compatible with the standard early Universe cosmology including the correct BBN and recombination.

Since the origin of these two difficulties is that $F''(\infty) = 0$, they can both be cured by a very simple change in the HSS and AB models. Let us add an additional term, quadratic in the Ricci scalar, to their $F(R)$ functions, so that $F''(\infty)$ becomes non-zero. In particular, we consider the following functions,

$$
\hat{F}_{HSS} = F_{HSS}(R) + \frac{R^2}{6M^2}, \quad \hat{F}_{AB} = F_{AB}(R) + \frac{R^2}{6M^2},
$$

(3.1)

where $M$ is a mass scale coinciding with the scalaron rest-mass whenever low curvature modifications to GR can be neglected. There is no “Big Boost” singularity in solutions of the $R^2$-corrected HSS and AB models (3.1).

If we now return to the linearized analysis for these new models, in which $R = R_{GR} + \delta R$, then it follows from Eq. (2.5) that the quadratic term has two important effects on the dynamics of $\delta R_{osc}$ – the oscillating component of $\delta R$. The first is that it introduces an upper bound on the mass of the scalaron, $M_s \leq M$, and hence limits the frequency of oscillations of $R$ (as was noted in [44]). The second effect is to moderate the amplitude growth of $\delta R_{osc}$, which now goes $\propto a^{-3/2}$. Specifically, going to the past, $\delta R_{osc}/R_{GR}$ decreases throughout the matter and radiation eras beginning from the moment when the $R^2$ correction in Eq. (3.1) becomes larger than the non-GR term decaying with the growth of $R$. Hence $\delta R_{osc}$ remains a small perturbation throughout the cosmological evolution, subject to it being small at present. Vice versa, going to the future time direction, since $\delta R_{osc}/R_{GR}$ grows until recently for the models (3.1), it remains an open problem to explain its very small initial amplitude in the early Universe. It is clear that the scalaron overabundance problem noted in the Introduction has not yet been solved.

Let us now discuss possible values of the parameter $M$ in Eq. (3.1). It should be sufficiently large in order to satisfy the viability conditions presented in the Introduction. In particular, it may not be $\sim \sqrt{R_{osc}} \sim 10^{-33}$ eV as was considered in [41, 55], and even the values discussed in [56, 57] are not high enough to solve the overabundance problem. A non-oscillating part of $\delta R$ induced by the $R^2$ correction to $F(R)$ becomes important when $H(t) \sim M$. As a consequence of this, the lower limit $M > 10^{-2.5}$ eV which follows from the most recent laboratory Cavendish-type experiment [58] is already sufficient for this correction to GR to be negligible both during BBN and in the center of neutron stars.\footnote{Still, as we shall see from section 4, non-GR terms in Eq. (3.1) might become important should the trace $T$ of the matter energy-momentum tensor change sign and become positive inside neutron stars, as happens in idealized $\rho = const$ solutions considered in [59, 57]. However, it is argued in [60] (see also [61]) that this does not occur inside realistic neutron stars.}
However, taking $M$ close to this lower limit is incompatible with the existence of any kind of inflation (not specifically driven by $F(R)$ gravity) in the early Universe, which is required to solve many other cosmological problems. Indeed, if $M \ll H_{\text{inf}}$ where $H_{\text{inf}}$ is the Hubble parameter during the last part of inflation, quantum fluctuations of the scalaron, including long-wave ones, are generated during inflation that generically results in a large value of $\delta R_{\text{osc}}$ after the end of inflation, and can even lead to the existence of a second stage of inflation driven by the scalaron itself [62]. Of course, the scalaron is not stable and decays into pairs of particles and antiparticles of all non-conformal quantum fields (this is a particular case of the effect of particle creation in gravitational fields). However, this process is sufficiently slow. Even in the pure $R + R^2/6M^2$ model which does not describe the present DE, the characteristic scalaron decay time is $\tau \sim M^2_{\text{Pl}} M^{-3}$ [8, 27, 28, 31]. Thus, one would need $M \gg 10^5 \text{ GeV}$ to ensure the scalaron decay by the moment when $H \sim 1 \text{ s}^{-1}$ to avoid any problems with BBN. However, we shall see in the next section that a non-trivial low-$R$ structure of $F(R)$ models describing present DE results in slowdown of the scalaron decay after the end of inflation driven by the scalaron itself. Since the aim of the present paper is to find at least one $F(R)$ model of present DE satisfying all five viability conditions formulated in the Introduction, we choose the value of $N$ which is sufficient to solve the scalaron overabundance problem. Namely, we take $M$ either larger than $H_{\text{inf}}$ in case inflation is produced by some other scalar field, or $\approx 3.7 \times 10^{13} (50/N)$ GeV if the scalaron plays the role of an inflaton, as was noted in the Introduction. In the former case, scalarons are practically not generated during inflation, so $\delta R_{\text{osc}}$ is zero after its end. In the latter case, reheating after inflation appears to be very non-trivial; it is studied in section 4.

Let us now return to the evolution of the $R^2$-corrected HSS and AB models at recent redshifts. Although the above reasoning suggests that the perturbative analysis is sufficient, it is also interesting to investigate the effect of the quadratic term on the scalaron potential. Performing the same steps as in section 2, the potentials are exhibited in fig.2. We see that the singular point at which $V(x) \rightarrow \text{const}$, $dV/dx \rightarrow -\infty$ is no longer present, and we observe a regular potential for all values of $x$ of physical interest.

In fig.3, we have confirmed numerically that the introduction of the $R^2$ term bounds the oscillations, which now grow to the past at a slower rate than $R_{\text{GR}}$ in the AB model. To obtain these curves we have numerically evolved the trace of the gravitational field equations, using the $R^2$-corrected AB model and taking $\delta \equiv \epsilon/M^2 = 4 \times 10^{-8}$. We have evolved equation (1.3) through the matter era, using a dimensionless time coordinate $\hat{t} = R_{\text{vac}}^{1/2} t$ and using random initial conditions for $R$ and $dR/d\hat{t}$. In the AB and HSS models without the $R^2$ term, it was found that $R$ could only be evolved over very short timescales (evolving backwards over the matter era), and only if we fine tuned initial conditions to approximately $R \simeq R_{\text{GR}}$. Now, we can choose random initial conditions for $R_i$, $\dot{R}_i$ and $H_i$, $\dot{H}_i$ and evolve backwards. In all cases we observe that the oscillations of $R$ decay to the past, and no singularity is present. Unfortunately, we cannot directly compare the results obtained here with the corresponding functions $\delta R$ and $\delta H$ in the uncorrected AB and HSS models. This is due to the fact that the uncorrected models will generically evolve to a singularity over much shorter timescales than presented here (see ref.[46] for a discussion...
of the difficulties associated with numerically modelling the original AB and HSS models).

Summarizing, we have found a way to cure all three problems of the HSS and AB models of present, low-curvature, DE which does not destroy correct inflation, BBN and other advantages of the early Universe cosmology. The approach consists of changing the behaviour of $F(R)$ at $R \gg R_{\text{vac}}$ according to Eq. (3.1) with a very large value of the free parameter $M$ which should either exceed the scale of inflation, or be equal to the concrete value needed for scalaron driven inflation in $F(R)$ gravity. Still this not the end of the story, one loophole remains which requires further correction of the functions (3.1), now in the range $R < R_{\text{vac}}$ including negative values of $R$. This final step in constructing a viable $F(R)$ DE model will be made in section 4.

Let us finish this section with a comment on possible alternatives to the large-$R$ behaviour (3.1). If instead a more general term $M^2 - 2^m R^m$, $m > 1$, is added to $F_{\text{HSS}}$ or $F_{\text{AB}}$, it can be checked that $M_s$, while growing with $R$, never exceeds it, so it may not become larger than $M_{\text{Pl}}$ if $R$ is less than $M_{\text{Pl}}^2$. Thus, there is no problem with the unlimited growth of $M_s$. However, instead we face a new difficulty: a possibility of the formation of a new
Figure 3: The fractional difference $\delta R/R_{\text{GR}} \equiv (R - R_{\text{GR}})/R_{\text{GR}}$ for the $R^2$-corrected AB model, obtained by numerically evolving the trace of the gravitational field equations using $\delta \equiv \epsilon/M^2 = 4 \times 10^{-8}$, and taking $R_{\text{GR}} = 4/3t^2$ (that is, we evolve the Ricci scalar assuming matter domination). We take random initial conditions for $R$ and $dR/dt$, and use a dimensionless time coordinate $\tilde{t} = R_{\text{vac}}^{1/2}t$, where $t$ is the cosmological time. (a) and (b) differ only by the initial conditions placed on $R$ and $dR/dt$; in (a) we have perturbed $R$ significantly away from its General Relativistic value. We observe that the oscillatory component of the Ricci scalar decays to the past relative to $R_{\text{GR}}$, as predicted in the text.

space-like curvature singularity with

$$a(t) \propto (t_s - t)^q, \quad q = \frac{(m - 1)(2m - 1)}{2 - m}$$

(3.2)
during evolution to the future which destroys all subsequent evolution [63, 64, 65]. For $m > 2$, this singularity is of the “Big Rip” type, and the scale factor $a(t)$ becomes infinite at a finite moment of time. For $1 < m < 2$, the scale factor becomes zero at $t = t_s$. In both cases, it remains an open problem if the singularity (3.2) can be avoided in generic future Cauchy evolution after inflation. So, at present the large-$R$ behaviour (3.1) of $F(R)$, up to logarithmic in $R$ corrections, seems to be the only one free from dangerous pathologies.

4. Inflation and late time acceleration from one $F(R)$ function

As has been shown in the previous sections, to construct a viable model of present DE in the scope of $F(R)$ gravity which satisfies all the viability conditions and does not destroy previous successes of the early Universe cosmology, one has not only to choose the correct
non-GR structure of $F(R)$ at low curvatures $R \sim R_{\text{vac}}$, like that in the HSS or AB models, but also to modify high-$R$ behaviour of $F(R)$. Moreover, it appears that the only high-$R$ behaviour which does not lead to problems with new singularities is just that was originally proposed for the scalaron driven $R^2$ inflation in the early Universe. Therefore, it is natural to consider combined $F(R)$ models which describe both primordial and present DE using one $F(R)$ function, albeit one containing two greatly different characteristic mass scales.

It presents no problem to find a function $F(R)$ for which the equation (1.4) defining de Sitter solutions has two or more roots. The real issues are, first, to ensure that the inflationary de Sitter solution is metastable, slow-roll and leads to the correct spectrum of primordial perturbations and, second, that there exist a sufficiently effective mechanism of reheating after inflation which transfers energy from scalarons into ordinary matter and radiation and heats them to a high temperature long before the beginning of a second de Sitter stage which we observe now as the present acceleration of the Universe. This transition between the two accelerating epochs in $F(R)$ gravity must be carefully analyzed and the absence of singular points or instabilities in the cosmological evolution has to be proved.

4.1 New problem

When we begin to consider post-inflationary evolution in $F(R)$ gravity a new problem immediately arises, not only in the combined case when inflation is scalaron-driven but also when inflation is produced by a minimally coupled scalar field $\psi$, which requires further generalization of even the corrected models (3.1). Namely, while $R$ is positive during both inflation and the recent evolution of the Universe, it becomes negative (and large) during each post-inflationary oscillation of $\psi$. In particular, $T \equiv 3P - \rho = \dot{\psi}^2 > 0$ at the moment when $V(\psi) = 0$. It will be shown below that the same occurs after scalaron-driven inflation. Thus, the range of $R$ in the models (3.1), previously used for $R > R_0$ only, has to be extended to negative values, and the stability conditions (1.6) should be satisfied for negative $R$, too, at least up to values $R \sim -M^2$ which occur during post-inflationary evolution.

However, this is not possible to achieve without a further change of these models. In the case of the HSS model, both original and the $R^2$-corrected one, first, one has to assume additionally that $n$ is an integer to avoid non-analytical behaviour at $R = 0$ (note that there is no such problem in the variant of this model introduced in [44]). A more serious problem is the appearance of the weak singularity (2.23) at the points $R/R_{\text{vac}} = \pm ((2n - 1)/c(2n + 1))^{1/2n}$ where $F''(R) = 0$. As shown in section 2.4, it is not possible to predict deterministic Cauchy evolution through it in the generic case (when $a_3 \neq 0$).

In the case of the AB model, its function $F_{AB}(R)$ is analytic and satisfies the conditions (1.6) for all $R$. However, $F_{AB}'(-\infty) = 0$ and it approaches this limit exponentially fast for low values of $|R|$: $F'(R) \sim \exp[2(R/\epsilon - b)]$ for $R < 0$, $|R| \gg \epsilon$. As a consequence of this, the $R^2$-corrected AB model (3.1) acquires a point at which $\hat{F}'(R_0) = 0$ for $R_0 \sim -R_{\text{vac}}$. As explained in the Introduction, such points should be avoided because of the formation of a generic anisotropic curvature singularity [10, 11]. Since this occurs at very low curvatures in this case, this point is dynamically accessible during the post-inflationary phase of the
Universe evolution, where $R$ oscillates around the vacuum state. Hence, the $R^2$-corrected AB model is not viable either.

Moreover, integrating the viability condition $F''(R) > 0$ over the interval $(-R, R)$ with $R \ll M^2$, we obtain that any viable $F(R)$ DE model should have a non-zero $g$-factor

$$g = \frac{F'(R) - F'(-R)}{2F'(R)} , \quad R_0 \ll R \ll M^2 . \quad (4.1)$$

Physically this means that the value of the effective background Newton gravitational constant $G_{\text{eff}} = G/F'(R)$ in the quasi-GR regime (1.7) is larger for $R < 0$ than for $R > 0$. It follows from the stability conditions (1.6) that the $g$-factor always lies in the range $0 < g < 1/2$.

![Figure 4](image-url)

**Figure 4:** The allowed parameter range of $g$ and $b$. For any choice of $b$ and $g$ above the curve, the model (4.2) has a stable de Sitter vacuum state. The curve is obtained by calculating the values of $g$ and $b$ for which the functions $Q(R) \equiv RF''(R) - 2F(R)$ and $dQ/dR$ are both zero.

### 4.2 Resolution of the problem and the improved AB model

This new problem can be solved and the problematic point $F'(R) = 0$ for small $|R|$ can be avoided if we consider the improved, $g$-extended $R^2$-corrected AB model, more concisely, the $gR^2$-AB model,

$$F(R) = (1 - g)R + g\log \left[ \frac{\cosh (R/\epsilon - b)}{\cosh b} \right] + \frac{R^2}{6M^2} , \quad (4.2)$$

where we have introduced the new dimensionless parameter – $g$-factor, $0 < g < 1/2$. Note that $g = 1/2$ corresponds to the $R^2$-corrected AB model, and $g = 0$ to the $R^2$ inflationary model with $F(R) = R + R^2/6M^2$ [44] which does not present DE. The function (4.2) corresponds to an interpolation between two different gravitational constants, as $F(R) \sim R$ for $R_{\text{vac}} < R < M^2$ and $F(R) \sim (1 - 2g)R$ for $-M^2 < R < \epsilon$, with a step at $R \sim b\epsilon$.

### 4.3 de Sitter attractors

Like the AB and $R^2$-corrected AB models, this new model can be expanded as $F(R) \approx R - R_{\text{vac}}/2 + \chi(R)$ for $R_{\text{vac}} < R < M^2$, where now $R_{\text{vac}} = 2g(b + \log[2\cosh b])\epsilon_{\text{AB}}$ and
\[ \chi(R) = g e^{2b} e^{-2R/\epsilon} + R^2/6M^2. \] It has stable Minkowski and de-Sitter vacuum states for an appropriate choice of \( b \) and \( g \); in fig.4 we have exhibited the allowed parameter range. To obtain this curve we have used the fact that for particular values of \( b \) and \( g \), the function \( Q(R) \equiv RF'(R) - 2F(R) \) will possess three zeros corresponding to vacuum states of the model (Minkowski space, and two de Sitter vacua). The values of \( g \) and \( b \) below the curve yield only one zero in \( Q(R) \) (Minkowski space), and those above will yield three. The curve corresponds to the limiting case where \( Q(R) \) possesses a double zero, that is when \( Q(R) = 0 \) and \( Q'(R) = 0 \). Hence we find that for any parameter choices of \( b \) and \( g \) above the curve, there exists a stable de Sitter vacuum state. As we increase \( b \), the allowed range of \( g \) increases, and as \( b \to \infty \), \( g \) must satisfy \( g \geq 0.25 \). We also note that as \( g \to 0.5 \), the condition \( b \geq 1.6 \) is required for a de Sitter state. Finally, the model satisfies \( F'(R) > 0 \) for \( R > -3M^2(1 - 2g) \) and \( F''(R) > 0 \) for all \( R \), and hence possesses no known instabilities for \( R > -3M^2(1 - 2g) \).

### 4.4 Slow-roll inflation

We now study the evolution of the model (4.2), starting from a high curvature slow roll epoch and evolving forwards in time. We assume that the classical evolution of this model begins with \( H_i \approx M_{Pl} \) and \( |\dot{H}| \sim M^2 \ll H^2_i \). We analyze the (0,0) component of the Einstein field equations

\[ 18HF''(R)\ddot{H} + 72F''(R)H^2\dot{H} + \frac{F(R)}{2} - 3\left(\dot{H} + H^2\right)F'(R) = 0, \quad (4.3) \]

in order to study the behaviour of the Hubble parameter. In the regime \( M^2 < R < M_{Pl}^2 \), the function (4.2) has the form \( F(R) \approx R + R^2/6M^2 - R_{vac}/2 + g e^{-2(R/\epsilon - b)} \), and the last two terms are negligible (in particular, the low curvature correction is exponentially suppressed). Therefore it is an excellent approximation to use \( F(R) = R + R^2/6M^2 \) in (4.3), which gives

\[ \ddot{H} - \frac{1}{2} \frac{(\dot{H})^2}{H} + 3H\dot{H} + \frac{M^2}{2}H = 0. \quad (4.4) \]

The slow-roll evolution of (4.4) (and generalizations thereof) has been studied extensively [33, 28, 29, 30], and it has been shown that for \( H^2 \gg |\dot{H}|, H|\dot{H}| \gg |\ddot{H}| \), the Hubble parameter has the following form

\[ H(t) \approx H_i - \frac{M^2t}{6}, \quad (4.5) \]

where we have taken \( t = 0 \) at the beginning of the evolution and \( H_i \approx M_{Pl} \). Slow roll ends when \( H \sim M \). Following this, we have an epoch in which the both the Hubble parameter and Ricci scalar oscillate with high frequency around a stable ground state. In the following section, we consider the oscillations of the Ricci scalar in detail, and how they may reheat the Universe.

Let us show now that all other scalaron-driven inflationary models in \( F(R) \) gravity which produce primordial spectra with other values of \( n_s \) and \( r \) are, in a sense, close to the
$R^2$ model. Indeed, there may be two kinds of inflationary models. In the first one which is the analogue of chaotic inflation [66] in GR, inflation occurs over a wide range of $R$. Then it follows from Eq. (4.3) that we need $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of $R$, namely

$$|A'(R)| \ll \frac{A(R)}{R}, \quad |A''(R)| \ll \frac{A(R)}{R^2},$$

(4.6)

for this kind of inflation to take place. Thus, these models are indeed close to the $R^2$ one.

In the second case, inflation occurs around some fixed root $R = R_1$ of Eq. (1.4) - an analogue of the “new” inflationary model [67, 68] in GR. Then, taking into account that the inequality (1.5) should be satisfied only marginally for metastability of the de Sitter solution, we get

$$F'(R_1) = \frac{2F(R_1)}{R_1}, \quad F''(R_1) \approx \frac{2F(R_1)}{R_1^2}.$$  

(4.7)

Thus, these models are close to the $R^2$ one near the point $R = R_1$.

Finally in this section, we exhibit the potential of the scalaron in the Einstein frame in fig.5. We observe the metastable de-Sitter point, and the stable Minkowski vacuum.

### 4.5 Reheating

We now study the reheating epoch for this class of models (a discussion of reheating mechanisms for modified gravity models can be found in, for example, [27, 28, 31]). For the $R^2$ inflationary model, immediately following slow roll we have $H^2 \sim |\dot{H}| \sim M^2$, and the Ricci scalar undergoes damped harmonic oscillations around $R = 0$ with frequency $\omega = M$ and $|R| \ll M^2$. In this section, we consider the evolution of $H$ for the model in question, taking initially $H^2 \sim |\dot{H}| \sim M^2$. We find behaviour that is dramatically different to that of the $R^2$ model, suggesting that the modifications to General Relativity at $R \sim 0$, which were introduced to induce late time acceleration, also have a significant effect on the dynamics of the Ricci scalar in this early epoch of the Universe. In any unified model of inflation and present DE there must be an efficient reheating mechanism, otherwise the Universe will relax to its late time de Sitter attractor without first undergoing epochs of radiation and matter domination.

To study reheating for the model (4.2) we again consider the $(0,0)$ field equation, which can be written in terms of dimensionless parameters $\hat{H} = H/M$, $\hat{R} = R/M^2$, $\delta \equiv \epsilon/M^2 \ll 1$ and dimensionless time $\hat{t} = Mt$,

$$\hat{H}\hat{H}'' - \frac{(\hat{H}')^2}{2} + 3\hat{H}^2\hat{H}' + \frac{1-g}{2}\hat{H}^2 - \frac{g}{2}(\hat{H}' + \hat{H}^2) \tanh \left[\frac{\hat{R}}{\delta} - b\right]$$

$$+ \frac{g\delta}{12} \log \left[\frac{\cosh(\hat{R}/\delta - b)}{\cosh(b)}\right] + \frac{3g}{\delta} \text{sech}^2 \left[\frac{\hat{R}}{\delta} - b\right] (\hat{H}\hat{H}'' + 4\hat{H}^2\hat{H}') = 0,$$

(4.8)

where primes now denote derivatives with respect to $\hat{t}$. Initially, we will solve equation (4.8), neglecting any effects due to gravitational particle production (that is, we will neglect the backreaction of created particles on the dynamics of $\hat{H}(\hat{t})$). The issue of backreaction will be tackled in section 4.5.2.
Figure 5: The Einstein frame potential for the $gR^2$-AB model (solid) and the original AB model (dashed). To obtain the potential we have used $g = 0.47$, $\delta = \epsilon/M^2 = 10^{-4}$ and $b = 2$. We observe two vacuum states; one late time de Sitter attractor and a flat Minkowski vacuum state. Slow-roll occurs for $\phi \gg M_{\text{Pl}}$. We note that in the original model, the scalar field satisfies $\phi < 0$ and the singularity corresponds to the point $\phi \to 0$. This singularity is removed in the $gR^2$-AB model by mapping the singular point $R \to \infty$ in the Jordan frame to $\phi \to \infty$ in the Einstein frame.

4.5.1 Evolution of $H(t)$ without backreaction

To begin, we solve (4.8) numerically. We are considering the epoch immediately following slow roll, so we take as initial conditions $\dot{H}_1' = -1$ and $\ddot{H}_1 = 1$ and evolve in $\hat{t}$ (we will begin our evolution at $\hat{t} = 0$). The results are exhibited in fig.7 for the Hubble parameter and fig.8 for the absolute value of the Ricci scalar, where we have taken $\delta = 1.5 \times 10^{-8}$, $b = 3$ and $g = 0.45$. We note that for realistic choices of $M$ and $\epsilon$, $\delta$ will be many orders of magnitude smaller, however numerically we are restricted to $\delta \sim 10^{-8}$. As can be seen in fig.7, $\dot{H}$ evolves through a number of distinct regimes, which we discuss below.

Initially, $\dot{R} \sim 1$, so we can use $F(R) \approx R + R^2/6M^2$, and the Hubble parameter behaves as in the $R^2$ inflationary model (that is, $\dot{H}$ rapidly decreases to $\dot{H} \ll 1$ over timescales $\hat{t} \sim O(1)$). This region corresponds to $\hat{t} \lesssim 1$ in figs.7,8. In this regime $\dot{R} \gg \delta$, and hence
the value of $\delta$ will have no significant effect on the dynamics of $\dot{H}$ and $\dot{R}$.

After this (very short) initial period, we observe a significant change in the gradient of $\dot{H}$. In this regime, we have $\dot{H}^2 \gg \delta$, $|\dot{H}'| \gg \delta$ but $\dot{R} = 6\dot{H}' + 12\dot{H}^2 \gtrsim \delta$ (that is, $\dot{R} \approx -2\dot{H}^2$.) The exact value of $\dot{R}$ will depend on the specific function $F(R)$ being considered, and corresponds to its value when the large curvature contribution to the scalaron mass is approximately equal to the low curvature terms; for the model constructed here when $1/3M^2 \sim (g/\epsilon)\text{sech}^2[R/\epsilon - b]$.

To analytically model the behaviour of $\dot{H}$ in this regime, we look for a solution to (4.8) of the form $\dot{H}(t) = \dot{H}_0(t) + \delta\dot{H}_1(t) + \mathcal{O}(\delta^2)$. In doing so we obtain

\[
\delta \left( \dot{H}_0\dot{H}_0'' - \frac{(\dot{H}_0')^2}{2} + 3\dot{H}_0^2\dot{H}_0' + \frac{(1-g)}{2}\dot{H}_0^2 - \frac{g}{2}(\dot{H}_0' + \dot{H}_0^2) \tanh \left[6\dot{H}_1' + 24\dot{H}_0\dot{H}_1 - b\right] \right) + 3g\text{sech}^2 \left[6\dot{H}_1' + 24\dot{H}_0\dot{H}_1 - b\right] \left(\dot{H}_0\dot{H}_0'' + 4\dot{H}_0^2\dot{H}_0'\dot{H}_0'\right) + 3g\delta\text{sech}^2 \left[6\dot{H}_1' + 24\dot{H}_0\dot{H}_1 - b\right] \left(\dot{H}_0\dot{H}_1'' + \dot{H}_1\dot{H}_0'' + 8\dot{H}_0\dot{H}_1\dot{H}_0' + 4\dot{H}_0^2\dot{H}_1'\right) = 0, \tag{4.9}
\]

where we have anticipated that $\dot{R}_0 = 6\dot{H}_0' + 12\dot{H}_0^2 = 0$ at zeroth order in the $\tanh[\dot{R}/\delta - b]$ and $\text{sech}^2[\dot{R}/\delta - b]$ terms (we will confirm that this assumption is valid below). At order $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ we arrive at

\[
\mathcal{O}(1) \quad \dot{H}_0\dot{H}_0'' + 4\dot{H}_0^2\dot{H}_0' = 0, \tag{4.10}
\]

\[
\mathcal{O}(\delta) \quad \dot{H}_0\dot{H}_1'' + \dot{H}_1\dot{H}_0'' + 8\dot{H}_0\dot{H}_1\dot{H}_0' + 4\dot{H}_0^2\dot{H}_1' = \frac{\cosh^2 \left[6\dot{H}_1' + 24\dot{H}_0\dot{H}_1 - b\right]}{3g} \times \tag{4.11}
\]

\[
\left(\frac{(\dot{H}_0')^2}{2} - \dot{H}_0\dot{H}_0'' - 3\dot{H}_0^2\dot{H}_0' - \frac{(1-g)}{2}\dot{H}_0^2 + \frac{g}{2}(\dot{H}_0' + \dot{H}_0^2) \tanh \left[6\dot{H}_1' + 24\dot{H}_0\dot{H}_1 - b\right] \right),
\]

and hence at $\mathcal{O}(1)$, $\dot{H}_0(t) = 1/(2\dot{t} + \alpha_0)$, where $\alpha_0$ is an integration constant. At zeroth order we have $a(t) \propto t^{1/2}$, $\dot{H}(t) \sim 1/2t$ and $\dot{R} = 0$. This is in agreement with our numerical results; in fig.6 we have exhibited $\ddot{H}$ as calculated numerically and $\ddot{H} = 1/2t$ logarithmically as a function of $\log[t]$, over the range $\log[t] = (\log[1], \log[180])$, and we see a close agreement between our analytic and numerical results. The number of scale factor e-folds during this period is given by $N_1 \sim -\ln(1 - 2g)$.

Reheating in this class of models will begin at this stage in the evolution of $\dot{H}$, via gravitational particle production due to the abrupt change in the Ricci scalar (in our numerical calculation this occurs at $\dot{t} \sim 2$, as shown in fig.8). Prior to this, the Ricci scalar is approximately in a de Sitter phase, which then abruptly evolves to a radiation-like Universe with $R \sim 0$. The main contribution to the number of particles and antiparticles of all non-conformal matter quantum fields created after the end of inflation occurs during this (almost) discontinuous change in $R$. Using either the expression for the rate of creation of the density of scalar particles and antiparticles $n$ in the massless limit in a FRW background.
Figure 6: exhibits $\dot{H}$ (solid), obtained by numerically solving (4.8), and $1/2\dot{t}$ (dashed) plotted logarithmically against $\dot{t}$. We see that over the range $\dot{t} = (5, 180)$, $\dot{H} \sim 1/2\dot{t}$, in agreement with our analytic result.

Figure 7: The behaviour of $\dot{H}$ using the parameters $\delta = 1.5 \times 10^{-8}$, $b = 3$, $g = 0.45$, and neglecting backreaction. We observe a number of distinct regimes; for $0 < \dot{t} < 2$ the Hubble parameter is exiting the slow roll regime. Following this is an epoch for which $\dot{H} \sim 1/2\dot{t}$. The abrupt change in $\dot{H}$ at $\dot{t} \sim 180$ corresponds to a spike in the evolution of $R$. Following this $\dot{H} \sim \delta^{1/2}$ and $a(t) \sim$ const. Once $\dot{H}$ spikes once more at $\dot{t} \sim 5000$, the Hubble parameter has completed one full oscillation.

\[ \frac{d(a^3n)}{a^3 \, dt} = \frac{(1 - 6\xi)^2}{576\pi} R^2, \]  

which for $\xi = 0$ is valid for gravitons (although the rate should be doubled in this case due to two polarization states) and the longitudinal component of vector bosons, too, or

\[ \text{Strictly speaking, Eq. (4.12) is derived in the limit } R^2 \ll R_{\mu\nu}R^{\mu\nu}. \]
Figure 8: The behaviour of the Ricci scalar $\hat{R}$ using the parameters $\delta = 1.5 \times 10^{-8}$, $b = 3$, $g = 0.45$, and neglecting backreaction. We observe the Ricci scalar evolving from a slow-roll de Sitter state to $\hat{R} \gtrsim \delta$, where the low curvature corrections to GR dominate. $\hat{R}$ is typically $\sim \delta$ throughout this epoch, although it periodically exhibits spikes.

results of the papers [52, 70], we arrive to the following estimate of the total number and energy density of created ultrarelativistic particles:

$$n \sim x H_r^3 \left(\frac{a_r}{a}\right)^3, \quad \rho_{\text{rad}} = x H_r^4 \left(\frac{a_r}{a}\right)^4,$$

where $H_r$ is the value of the Hubble parameter at which the discontinuous change in $R$ occurs (for our model $H_r \sim M$), $x$ is a dimensionless parameter of order $x \sim 10^{-2}$ and $a_r$ is the value of the scale factor at this point. The energy density of created particles will be initially subdominant, but may grow to have a significant backreaction effect on the dynamics. For the remainder of this section we neglect the effect of $\rho_{\text{rad}}$ on the evolution of $\hat{H}$ and consider backreaction in the following section.

Over the regime $\hat{H} = 1/2\hat{t} + O(\delta)$ discussed above, the Ricci scalar is given by $\hat{R} \simeq 6\delta \left(\hat{H}_r' + 4\hat{H}_0 \hat{H}_1\right)$, which decreases from $\hat{R} \gtrsim \delta$ to $\hat{R} \sim 0$, at which time the $(1/\delta) \text{sech}^2(\hat{R}/\delta - b)$ term in (4.8) no longer dominates. At this point, we can use $\text{sech}^2(\hat{R}/\delta - b) \simeq 0$ and $\tanh(\hat{R}/\delta - b) \simeq -1$, in which case (4.8) becomes

$$\hat{H} \hat{H}'' - \frac{(\hat{H}')^2}{2} + 3\hat{H}^2 \hat{H}' + \frac{(1-2g)}{2} \hat{H}^2 \simeq 0,$$

which has solution $\hat{H} \simeq A_0 \sin^2(\sqrt{1-2g}/2)\hat{t}$ (that is, $\hat{H}$ oscillates on timescales of order $\hat{t} \sim 1$, as in the $R^2$ inflationary model). This corresponds to the spike in $\hat{R}$ at $\hat{t} \sim 200$, with amplitude $\hat{R} \gtrsim -1$. However, unlike in the $R^2$ inflationary model, $\hat{R}$ only completes one half of an oscillation before we once again enter a regime where the $(1/\delta) \text{sech}^2(\hat{R}/\delta - b)$ term dominates.

Following the spike in $\hat{R}$, $\hat{H}$ does not decay like $\hat{H} \propto \hat{t}^{-1}$. Instead, $\hat{H}^2, \hat{H}' \lesssim \delta$, and the scale factor $a(\hat{t})$ is approximately constant over this regime. This is confirmed numerically
in fig.9, where we observe a plateau in the evolution of the scale factor. During this period, the number e-folds will be $N_2 \approx 0$, and $\dot{H}$ and $\dot{R}$ will grow until $\dot{R} > \delta$.

Following this, $\dot{R}$ will once again produce a spike (with amplitude $\dot{R} \lesssim 1$), completing one full oscillation of $\dot{H}$. Numerically we can only observe one complete oscillation before $\dot{R}$ reaches its final state $\dot{R} \gtrsim \delta$, however this is simply a consequence of being unable to choose a sufficiently small value of $\delta$. Finally, we note that if we average $\dot{H}$ over an even number of oscillations, we find $\langle \dot{H} \rangle = 1/3\dot{t}$, as one might expect for a period of kination (domination of a massless scalar field).

We can summarize the dynamics of $\dot{H}$ as follows; the Hubble parameter periodically undergoes (almost) discontinuous jumps between periods when $\dot{H} = 1/2\dot{t}$ and $\dot{H} \sim \delta^{1/2}$. The duration of these periods in terms of $\ln t$ is determined by the non-zero $g$-factor (4.1) and is equal to $\approx -2\ln(1-2g)$ and $-\ln(1-2g)$ respectively. It is clear that such behaviour of $\dot{H}$ and $\dot{R}$ is markedly different to the standard reheating dynamics, and we expect that the low curvature modifications to General Relativity will leave unique observational imprints. In particular, our reheating mechanism is less efficient than the pure $F(R) = R + R^2/6M^2$ model, and we expect a significantly lower reheat temperature. Additionally, the average expansion rate after inflation is slower than $a(t) \propto t^{1/2}$, and therefore the number of e-foldings, $N$, should be $\sim 70$, larger than in standard inflationary models and the same that would occur if all inflation proceeded at $H = M_{Pl}$. Indeed, by comparing the energy density of created particles (4.13) to $H^2$, we see that they become equal at

$$t = t_{reh} \sim x^{-3/2}M^{-4}M_{Pl}^3 \sim 10^{-18} \text{ s}$$

after the end of inflation (assuming the value $M \approx 3 \times 10^{13}$ GeV needed to fit the amplitude of observed curvature fluctuations with $N = 70$). If $g = 0.45$, then about 7 complete non-linear oscillations of $R$ have occurred by this moment. If the radiation component has been already thermalized at $t = t_{reh}$ due to interactions between particles, then its temperature is $T(t_{reh}) \sim 10^6$ GeV which is sufficiently large (however, in principle, it may thermalize significantly earlier while being a sub-dominant component). At $t = t_{reh}$, the comoving scale which was equal to the Hubble radius at the end of inflation is given by $M^{-1}(M_{t_{reh}})^{1/3} = x^{-1/2}M^{-2}M_{Pl} = x^{1/4}M_{Pl}^{-1/2}t_{reh}^{1/2}$. Therefore, up to a factor of a few, it coincides with the comoving scale which is equal to the Planck length at the Planck time in a universe which is radiation-dominated at subsequent times. As a result, irrespective of the fact that if the thermodynamic equilibrium in the radiation component is reached before or after $t_{reh}$, this scale coincides with the characteristic thermal length of present CMB photons with temperature $T_\gamma = 2.725$ K, up to a purely numerical factor depending mainly on an effective number of species at the moment when the equilibrium has been achieved. This explains why $N \approx 70$ for our model. In the next subsection, we consider backreaction of created particles of a FRW background numerically and in more detail.

Hence, due to this change in $N$, the index (slope) of the power spectrum of primordial scalar (density) perturbations $n_s$ is slightly higher, $n_s = 1 - 2/N \approx 0.97$, in our model which combines both inflation and present DE using one $F(R)$ function (4.2), as compared to the $R + R^2/6M^2$ model describing inflation only. However, this distinctive and observable
prediction is degenerate, since it may be changed by introducing loop corrections to the large curvature $R^2$ term of another purely inflationary model, making it of the type (4.6).

Figure 9: The behaviour of $a(t)$, normalized such that $a(\hat{t}_i) = 2 \times 10^{-2}$. We observe a plateau in the evolution of the scale factor, as predicted in the text.

4.5.2 Effect of backreaction

As pointed out in the previous section, at the beginning of the reheating epoch $\hat{R}$ undergoes an almost discontinuous change from a de Sitter phase to $\hat{R} \gtrsim \delta$, and particle production occurs mainly during this period. To take into account the backreaction of these particles on the evolution of the Hubble parameter, we must now solve the equation

$$\hat{H}\hat{H}'' - \frac{\hat{H}'}{2} + 3\hat{H}^2\hat{H}' + \frac{1-g}{2}\hat{H}^2 - \frac{g}{2}(\hat{H}' + \hat{H}^2) \tanh \left[ \frac{\hat{R}}{\delta} - b \right]$$

$$+ \frac{g\delta}{12} \log \left[ \frac{\cosh(\hat{R}/\delta - b)}{\cosh(b)} \right] + \frac{3g}{\delta} \text{sech}^2 \left[ \frac{\hat{R}}{\delta} - b \right] (\hat{H}\hat{H}'' + 4\hat{H}^2\hat{H}') = \frac{\rho_{\text{rad}}}{6M^2M_{\text{Pl}}^2} \approx \frac{xM^2a^4}{6M_{\text{Pl}}^2a^4}.$$  

For future convenience we define the dimensionless parameter $\hat{\rho}_{\text{rad}} \equiv \rho_{\text{rad}}/M^4$. Initially, the energy density $\rho_{\text{rad}}$ of produced particles will be subdominant to the background evolution of $\hat{H}(\hat{t})$, driven by the scalaron. Numerically, we will take the radiation component to be $\rho_i/M^2M_{\text{Pl}}^2 \sim 10^{-4}\hat{H}_i^2$, and $xM^2/6M_{\text{Pl}}^2 = 2 \times 10^{-4}$ (one must be careful to preserve the hierarchy between $\hat{H}$, $\hat{\rho}_{\text{rad}}$ and $\delta$, as when $\hat{\rho}_{\text{rad}}$ is of order $\delta$, our solution will cease to be physically relevant.)

We begin our numerical evolution at the point where the radiation energy density is produced, so $M^2\hat{\rho}/6M_{\text{Pl}}^2 = xM^2/6M_{\text{Pl}}^2 = 2 \times 10^{-4}$, and use initial conditions $2\hat{H}^2 = -\hat{H}'$. Taking $\delta = 1.5 \times 10^{-8}$, $b = 3$ and $g = 0.45$, we present $\hat{H}^2/2$ and $M^2\hat{\rho}/6M_{\text{Pl}}^2$ in fig.10. We note that the observed behaviour of $\hat{H}$ is significantly different to that of the previous section, however our conclusions will remain essentially the same. As before, we observe
Figure 10: The behaviour of $\hat{H}^2/2$ (solid) and $M^2\hat{\rho}/6M_{Pl}^2$ (dashed). We see that $\hat{H}^2/2$ oscillates around $\hat{\rho}$, and satisfies $\hat{H}^2 \gg \delta$ throughout the reheating epoch ($\delta$ is also exhibited (dotted)).

Figure 11: The Hubble parameter during one complete oscillation (solid). We have also exhibited the two approximate analytic solutions as calculated in the text; $\hat{H} = 1/2\hat{t}$ (dashed) and $\hat{H} = 1/(2\hat{t} + \alpha)$ (dotted).

one complete oscillation of $\hat{H}$ before $\dot{\rho} \sim \delta$. We now discuss the various regimes over the course of this oscillation.

Initially, $\hat{H}$ behaves in a very similar manner to the previous section; the radiation is subdominant and $\hat{H} = 1/2\hat{t}$, $\hat{\rho} \propto \hat{t}^{-2}$. During this time $\hat{R}$ evolves from $\hat{R} \gtrsim \delta$ to $\hat{R} \sim 0$. Then, at the point $\hat{R} \sim 0$, the Ricci scalar undergoes half an oscillation, spiking at $\hat{R} \gtrsim -1$ (again, as before). However, $\hat{H}$ dies not oscillate to $\hat{H} = 0$ as in the previous section, but rather $\hat{H}^2$ oscillates around $M^2\hat{\rho}/3M_{Pl}^2$. We see that by incorporating $\hat{\rho}$ into the dynamics, $\hat{H}$ now satisfies $\hat{H} \gg \sqrt{\delta}$ throughout the reheating epoch.

All that remains is to consider the behaviour of $\hat{H}$ and $a(t)$ over the second half of the oscillation of $\hat{H}$. Following the (almost) discontinuous change in $\hat{H}$ at $\hat{t} \sim 10$, the Hubble
parameter satisfies $\hat{H} \gg \sqrt{\delta}$ and $\dot{\hat{H}} \gg \delta$, however $\hat{R} \gtrsim \delta$. By using the same perturbative analysis as before, we must conclude that $\hat{H}$ is given by $\hat{H} = 1/(2\hat{t} + \alpha)$, where $\alpha$ is an integration constant. The value of this constant can be obtained numerically, and depends on the ratio of $H^2$ and $M^2 \hat{\rho}/M_{Pl}^2$. For realistic values of $M$ and $M_{Pl}$ we expect that $\alpha$ will be much greater than $\hat{t}$, and hence $\hat{H} \sim \alpha^{-1}$ and $a(t) \approx \text{const}$ during this period. The number of e-foldings over this regime will be $N_2 \approx 0$, as before. During this time $\hat{R}$ grows from $\hat{R} \sim 0$ to $\hat{R} \gtrsim \delta$, at which point we observe another spike in $\hat{R}$, thus completing one full oscillation of $\hat{H}$. In fig.11 we have exhibited the Hubble parameter and the two analytic approximations $1/2\hat{t}$ and $1/\left(2\hat{t} + \alpha\right)$ derived above; we see that our results are in agreement with the numerical solution.

Although the evolution of $\hat{H}$ is significantly modified when we incorporate backreaction, our conclusions remain essentially unchanged. $\hat{H}$ now undergoes periodic oscillations around $3\dot{H}^2 \sim M^2 \hat{\rho}/M_{Pl}^2$; the first half of this oscillation is characterized by the behaviour $\hat{H} = 1/2\hat{t}$, whereas its behaviour over the second half is given by $\hat{H} = 1/(2\hat{t} + \alpha) \approx 1/\alpha \ll 1$. As in the previous section, we find that the reheating mechanism is less efficient than the standard $R + R^2/6M^2$ model, and the averaged Hubble parameter will evolve at a slower rate than $1/2\hat{t}$, so our conclusions regarding the total number of e-foldings and the consequent increase in the value of $n_s$ will persist.

### 4.6 Cosmological evolution

Following the reheating epoch of the Universe, we expect that the model (4.2) reproduces the standard cosmology, that is it evolves from a radiation dominated epoch to matter domination, with the final state of the Universe being the de Sitter vacuum. To see that the standard cosmology is reproduced, we rearrange the $(i,j)$ and $(0,0)$ gravitational field equations, assuming that the energy-momentum tensor is comprised of matter and radiation fluids only, obtaining

\begin{align}
3H^2 &= \frac{\rho_m + \rho_r + \rho_F(H, \dot{H}, \ddot{H}, \hat{R})}{M_{Pl}^2}, \tag{4.17} \\
2\dot{H} + 3H^2 &= -\frac{P_m + P_r + P_F(H, \dot{H}, \ddot{H}, \hat{R})}{M_{Pl}^2}, \tag{4.18}
\end{align}

where the subscripts $m$ and $r$ represent the matter and radiation components, and $F$ denotes effective density and pressure terms due to the $F(R)$ modified gravity function (we have made it clear that $\rho_F$ and $P_F$ depend on $H$ and its derivatives, to stress that we are dealing with a system of fourth order differential equations). $\rho_F$ and $P_F$ are given by

\begin{align}
\frac{\rho_F}{M_{Pl}^2} &= -3H \left[ \frac{1}{3M^2} + \frac{g}{\epsilon} \text{sech}^2(R/\epsilon - b) \right] \dot{R} - \frac{R^2}{12M^2} + (\dot{H} + H^2) \frac{R}{M^2} + \frac{g_2}{2} \log(\cosh b) \\
&\quad - \frac{g}{2} \left[ \epsilon \log \cosh(R/\epsilon - b) - R \right] + 3g(\dot{H} + H^2) (\tanh(R/\epsilon - b) - 1), \tag{4.19}
\end{align}
as an excellent approximation we may simply use
$R \approx \frac{2c\dot{R}^2}{\epsilon^2} \tanh(R/\epsilon - b) \text{sech}^2(R/\epsilon - b)$ (4.20)

\begin{align*}
&\quad + \left( \frac{1}{3M^2} + \frac{g}{\epsilon} \text{sech}^2(R/\epsilon - b) \right) (\ddot{R} + 2H \dot{R}) - g(\dot{\dot{H}} + 3H^2) (\tanh(R/\epsilon - b) - 1) \\
&\quad - \frac{g}{2} \left( R - \epsilon \log \left[ \frac{\cosh(R/\epsilon - b)}{\cosh b} \right] \right) - (\dot{\dot{H}} + 3H^2) \frac{R}{3M^2} + \frac{R^2}{12M^2}.
\end{align*}

It is clear that $\rho_F$ and $P_F$ will act as dark energy components, and by solving the gravitational field equations we can obtain the equation of state parameter $w \equiv P_F/\rho_F$.

To begin, we note that previous studies [44] have shown that the Ricci scalar can be written as $R = R_{GR} + \delta R_{osc} + \delta R_{ind}$ during the cosmological evolution of models such as (4.2), where $\delta R_{osc}$ is the oscillatory, scalaron component, and $\delta R_{ind}$ is given by $\delta R_{ind} \simeq \Box F'(R_{GR}) + R_{GR} F'(R_{GR}) - 2F(R_{GR}) + R_{GR}$. The scalaron oscillations $\delta R_{osc}$ have been discussed in previous sections; they are well behaved and regular for the model (4.2), and we can assume that the energy density of these oscillations has decayed during reheating and can be neglected. $\delta R_{ind}$ also satisfies $\delta R_{ind} \ll R_{GR}$ for $R > R_{vac}$, and hence as an excellent approximation we may simply use $\dot{R} \simeq R_{GR}$ and $H \simeq H_{GR}$ in $\rho_F$ and $P_F$.

Typically during the matter and radiation era’s, we have $\epsilon \ll R_{GR} \ll M^2$, in which case (4.19) and (4.20) can be written as

$$
\frac{\rho_F}{M_{Pl}^2} \simeq \frac{g\epsilon}{2} \left[ b + \log(e^b + e^{-b}) \right] + \frac{1}{M^2} \left[ (\dot{H} + H^2) R - H \dot{R} - \frac{R^2}{12} \right] - e^{-2(R/\epsilon - b)} \left[ \frac{12gH\dot{R}}{\epsilon} + \frac{g\epsilon}{2} + 6g(\dot{H} + H^2) \right],
$$

$$
\frac{P_F}{M_{Pl}^2} \simeq - \frac{g\epsilon}{2} \left[ b + \log(e^b + e^{-b}) \right] + \frac{1}{12M^2} \left[ 4(\ddot{R} + 2H \dot{R}) + R^2 - 4(\dot{H} + 3H^2) \dot{R} \right] + e^{-2(R/\epsilon - b)} \left[ \frac{g\epsilon}{2} + 2g(\dot{H} + 3H^2) + \frac{4g}{\epsilon} (\dot{R} + 2H \dot{R}) - \frac{8g}{\epsilon^2} (\dot{R})^2 \right].
$$

Since last two terms in (4.21) and (4.22) are suppressed by factors of $1/M^2$ and $e^{-2(R/\epsilon - b)}$ respectively, they are completely subdominant, and throughout the matter and radiation era’s we have $w \approx -1$. This simply reflects the fact that for $R \gg R_{vac}$, the mass of the scalaron $M_{scalon}$ is very large, and as an effective field theory below energy scales $E \sim M_{scalon}$ the model (4.2) reduces to GR with a cosmological constant.

However, at late times, when $R_{GR} \sim \mathcal{O}(R_{vac})$, the mass of the scalaron is small, and we expect that there may be significant deviations from $w = -1$. The change in $w$ is due to the $\tanh(R/\epsilon - b) - 1$, $\text{sech}^2(R/\epsilon - b)$ and $\log[\cosh(R/\epsilon - b)] - R$ terms in (4.19, 4.20), which are no longer exponentially suppressed. This behaviour is exhibited in fig.12, where $w$ is shown as a function of redshift. $w(z)$ was calculated by solving the full gravitational field equations numerically, starting at $z = 4$ and evolving to $z = 0$. As initial conditions,
we have assumed that $H$ and its derivatives are exactly their GR counterparts at $z = 4$, and $\Omega_m = 0.3$, and have taken $\delta = 10^{-7}$. With this choice, we observe that $w$ oscillates around $-1$ in the past as expected, but at redshifts $z \sim 1$, $w$ drifts from its GR value and presents phantom behaviour, $w < -1$. We note that the size of this ‘drift’ depends almost entirely on our choice of $b$; the larger we take $b$, the smaller $R/\epsilon$ is and the more suppressed the $\tanh(R/\epsilon - b)$ and sech$^2(R/\epsilon - b)$ terms become. This is clearly shown in fig.12; as we increase $b$, the late time drift from $w = -1$ becomes increasingly suppressed; $|\Delta w| \simeq 0.06$ for $b = 1.2$ and $|\Delta w| \simeq 2 \times 10^{-3}$ for $b = 4$.

Finally, we note that $w$ crosses the phantom boundary and satisfies $w > -1$ for $z < 1$ and moderate values of $b$. Thus, our model naturally exhibits phantom behaviour at recent redshifts during the matter dominated stage. Deviations of $w(z)$ from $-1$ are small, less than several percent, hence there is a good agreement with present observational upper bounds on $|w_{\text{DE}} + 1|$, see e.g. [6].

![Figure 12: The ‘dark energy’ equation of state parameter $w$ as a function of redshift for the $gR^2$-AB model, with $b = 1.2$ (top) and $b = 4$ (bottom). We note that $w$ oscillates around $w = -1$ at earlier times, as can clearly be seen in the $b = 4$ case, however for redshifts $z \sim 0$ the oscillations are small but the ‘drift’ terms in $\rho$ and $P_k$ become important. We see the deviation at $z \sim 0$ depends on $b$; as $b$ increases, deviations from GR become increasingly suppressed.](image-url)
5. Conclusions and discussion

To summarize, in this paper we have shown for the first time that it is possible to construct at least one self-consistent model of present DE in the scope of $F(R)$ gravity which satisfies all five viability conditions presented in the Introduction, is free of new singularities and does not destroy any of the previous successes of cosmology. To achieve this aim, it has been necessary to extend the range of $R$ over which previous $F(R)$ models were defined to both large positive and negative values, and change the behaviour of the models correspondingly; see Eq. (4.2) for the improved $gR^2$-AB model.

Furthermore, since the large-$R$ behaviour (3.1) of the $F(R)$ function, which is needed to avoid new singularities and to solve two other problems of previous models, appears to be just the same as needed for scalaron-driven inflation in $F(R)$ gravity, we have shown that the model Eq. (4.2) can describe inflation (primordial DE), the present acceleration of the Universe (present DE) and the intermediate epochs of radiation and matter domination for the unique choice of its parameter $M$, determined by the observed power of scalar (density) perturbations. Unexpectedly, we have found that the low-curvature modification of $F(R)$ from its GR value, which is needed to describe present DE, strongly affects processes at very high values of curvature, specifically during reheating after inflation, through its non-zero $g$-factor (4.1). As a result, in contrast to pure inflationary models of $F(R)$ gravity which have $g = 0$, scalaron oscillations after the end of inflation become strongly non-linear and the Universe evolution passes through the sequence of interchanging periods with $a(t) \propto \sqrt{t}$ and $a \approx \text{const}$, with the number of time e-folds $\Delta \ln t$ equal to $2 \ln(1/(1 - 2g))$ and $\ln(1/(1 - 2g))$ for them correspondingly. We find that on average, the Universe expands as $a(t) \propto t^{1/3}$ during this period; this is the most significant mathematical result of the paper.

Creation of particles and antiparticles of usual matter and final reheating are achieved by taking into account the process of gravitational particle creation by these non-linear oscillations of $R$. This mainly occurs at the end of inflation, so reheating is less efficient than in the purely inflationary $R + R^2/6M^2$ model, although still viable. Due to the different average law of expansion after the end of scalaron-driven inflation, predictions for parameters of primordial spectra of scalar (density) perturbations and gravitational waves generated during inflation in this combined model of primordial and present DE are slightly different to those for the inflationary model only. The difference is due to the change in the number of scale factor e-folds $N$ used in the corresponding formulas: from $N \approx (50 - 55)$ to $N = 70$. This specific prediction of the combined model is observable, however, it is degenerate with a possible slow variation of the $R^2$ behaviour of $F(R)$ at large $R$ in the model (4.2), e.g. like in Eq. (4.6).

In this paper we have not addressed the issue of neutron star stability in $F(R)$ gravity raised in [47] and further considered in [59, 57], since the most recent results in [60, 61] suggest that there is no problem and that, for our choice of the parameter $M$ in the improved model (4.2), non-GR corrections are very small if the trace $T$ of the matter energy-momentum tensor remains non-positive inside neutron stars. However, due to a non-zero $g$-factor, this problem becomes highly non-trivial and requires special consideration if $T$ is
permitted to become positive at large matter densities.

The most critical observational prediction of DE models in $F(R)$ gravity remains the anomalous growth of density perturbations in the matter component at recent redshifts [44, 71, 72, 73, 74] which results, in particular, in a mismatch between parameters such as $\sigma_8$ and $n_s$ determined from CMB temperature anisotropy and galaxy clustering separately, assuming GR (as well as from the cluster abundance at different $z$). The absence of such an effect at the level of $\sim 5\%$ in the HSS model makes its background evolution practically indistinguishable from that in the standard $\Lambda$CDM. However, this effect is suppressed in the AB model as compared to the HSS one, so it is more difficult to falsify the former model. Future observational data will determine the fate of this whole class of models.

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