Motivic interpretation of Milnor $K$-groups attached to Jacobian varieties

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Abstract

In the paper [Som90] p.105, Somekawa conjectures that his Milnor $K$-group $K(k, G_1, \ldots, G_r)$ attached to semi-abelian varieties $G_1, \ldots, G_r$ over a field $k$ is isomorphic to $\text{Ext}_{M_k}(\mathbb{Z}, G_1[−1] \otimes \ldots \otimes G_r[−1])$ where $M_k$ is a certain category of motives over $k$. The purpose of this note is to give remarks on this conjecture, when we take $M_k$ as Voevodsky’s category of motives $\text{DM}^e(k)$.

Key words: motivic cohomology, 1-motives, Milnor $K$-groups, Weil reciprocity law

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0 Introduction

To unify the Moore exact sequence and the Bloch exact sequence, K. Kato defined the generalized Milnor $K$-groups attached to finite family of semi-abelian varieties over a base field $\mathbb{k}$ in [Som90]. (See also [Akh00], [Kah92].) That is, for semi-abelian varieties $G_1, \ldots, G_r$ over $\mathbb{k}$, he associated the group $K(\mathbb{k}, G_1, \ldots, G_r)$. (For precise definition, see [4.1].) This group is a generalization of the Milnor $K$-group as the following example shows.

**Example 0.1.** In the notation above, if $G_1 = G_2 = \cdots = G_r = \mathbb{G}_m$, the following equality holds.

$$K(\mathbb{k}, \mathbb{G}_m, \ldots, \mathbb{G}_m) = K^M_r(\mathbb{k})$$

On the other hand this group is also a generalization of the Bloch group $V$.

**Example 0.2.** Let $C$ be a projective smooth curve over $\mathbb{k}$ such that $C(\mathbb{k}) \neq \phi$. We have the following equality

$$K(\mathbb{k}, \text{Jac} C, \mathbb{G}_m) = V(C).$$

where $V(C)$ is defined by S. Bloch (c.f. [Blo81]) as the following way

$$V(C) = \frac{\text{Ker}(\bigoplus_{x \in C^1} k(x)^{\times} \to \Sigma_{k(x)/k} k^x)}{\text{Im}(K_2(K(C)) \otimes_{\mathbb{Q}_{\text{alg}}} \bigoplus_{x \in C^1} k(x)^{\times})}.$$ 

As is explained in [Som90], there is the generalized Bloch-Moore exact sequence.

0.3. Let $k$ be a number field and $A$ a semi-abelian variety over $k$. We write $G = \text{Gal}(\bar{k}/k)$ and $G_v = \text{Gal}(\bar{k}_v/k_v)$ for a place $v$. Let $T(A)$ be the Tate module of $A$. $S$ is a finite set of places including all Archimedian and places where $A$ has bad reduction. Then $T(A)_G$ is a finite group by owing to [KL81]. Let $m$ be a nonzero integer divisible by the order of $T(A)_G$. Somekawa proves the following generalized Moore-Bloch exact sequence (c.f. [Som90] Theorem 4.1):

$$K(k, A, \mathbb{G}_m) \to \bigoplus_{v \notin S} T(A)_{G_v} \oplus \bigoplus_{v \in S} K(k_v, A_v, \mathbb{G}_m)/m \to T(A)_G \to 0$$
In the case of $A = \mathbb{G}_m$, the above exact sequence is proved by Moore (c.f. [Moo69])

$$K_2(k) \to \bigoplus_{v: \text{not complex}} \mu(k_v) \to \mu(k) \to 0.$$ 

In the case of $A = \text{Jac} C$ in the notation in [U2], the above exact sequence is proved by S. Bloch, K. Kato and S. Saito (c.f. [Blo81], [KS83])

$$V(C) \to \bigoplus_{v \in S} T(\text{Jac} C)_{G_v} \oplus \bigoplus_{v \in S} V(C \times_k k_v)/m \to T(\text{Jac} C)_G \to 0.$$ 

In [Som90], Somekawa conjectures that the Somekawa $K$-groups should be motivic cohomology groups attached to semi-abelian varieties. More precisely

**Conjecture 0.4. (Somekawa conjecture)**

Let $G_1, \ldots, G_r$ be semi-abelian varieties over $k$, then $K(k, G_1, \ldots, G_r)$ is isomorphic to $\text{Ext}_{\mathcal{M}_k}^r(\mathbb{Z}, G_1[-1] \otimes \cdots \otimes G_r[-1])$, where $\mathcal{M}_k$ is a certain category of motives over $k$ and $G_i[-1]$ means 1-motif (c.f. [Del74]).

In this paper we will examine this conjecture, if we take $\mathcal{M}_k$ as Voevodsky’s category of motives $\text{DM}^{\text{eff}}(k)$.

**Main Theorem 0.5. (Somekawa conjecture for Jacobian varieties)**

Let $(C_1, a_1), \ldots, (C_n, a_n)$ be pointed projective smooth curves over perfect field $k$ which admits resolution of singularities. Then

$$K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n) \xrightarrow{\sim} \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\mathbb{M}_\text{gm}(\text{Spec} k), \mathbb{Z}(\bigwedge_{i=1}^n (C_i, a_i))[n]).$$

0.6. In this paper, let $k$ be a perfect field which admits resolution of singularity.

1 Milnor $K$-groups attached to semi-abelian varieties

1.1 Extension of valuations and tame symbols

1.1. Suppose $k$ is a field and $G$ is a semi-abelian variety defined over $k$, that is, there is an exact sequence of group schemes (viewed as sheaves in the flat topology) over $k$:

$$0 \to T \to G \to A \to 0$$

where $T$ is a torus and $A$ is an abelian variety.

1.2. In the notation above, let $K/k$ be an algebraic function field and $v$ a place of $K/k$. Let $L/K_v$ be a finite unramified Galois extension such that $T \times_k F \xrightarrow{\sim} \mathbb{G}_m^n$ for the residue field $F$ of $L$ and some $n$; let $w$ be the unique extension of $v$ of $L$. We obtain the following commutative diagram of exact sequences defining a
map $r_w = (r^1_w, \ldots, r^n_w)$;

\[
\begin{array}{cccc}
0 & 0 & \downarrow & \\
T(O_w) & G(O_w) & A(O_w) & 0 \\
0 & T(L) & G(L) & A(L) & 0 \\
\downarrow_{ord_w} & \downarrow & \downarrow & \downarrow & \\
\mathbb{Z}^n & id & \mathbb{Z}^n & \\
0 & 0 & \\
\end{array}
\]

1.3. In the notation above, we are going to construct a map

$$\partial_v : G(K_v) \otimes K_v^\times \to G(k(v)).$$

Fix $g \in G(K_v)$ and $h \in K_v^\times$. For each $i = 1, \ldots, n$, we define $h_i \in T(L)$ to be the $n$-th tuple having $h$ in the $i$-th coordinate and 1 elsewhere. Then set

$$\varepsilon(g, h) = ((-1)^{ord_w(h)}r^1_w(g), \ldots, (-1)^{ord_w(h)}r^n_w(g)) \in T(O_w) \subset G(O_w)$$

and

$$\tilde{\partial}_v(g, h) = \varepsilon(g, h)g^{ord_w(h)} \prod_{i=1}^n h_i^{-r^i_w(g)} \in G(O_w).$$

We define the “extended tame symbol” $\partial_v(g, h)$ to be the image of $\tilde{\partial}_v(g, h)$ under the canonical map $G(O_w) \to G(F)$; Then $\partial_v(g, h)$ is invariant under the action of $\text{Gal}(F/k(v))$, so that it belongs to $G(k(v))$. This definition of $\partial_v$ is independent of the choice of $L$ and of the isomorphic from the torus to $\mathbb{G}_m^\oplus n$.

1.2 Definition of the Milnor $K$-groups attached to semi-abelian varieties

1.4. Let $k$ be a field and $G_1, \ldots, G_r$ a finite (possibly empty) family of semi-abelian varieties defined over $k$. We define Milnor $K$-groups attached to semi-abelian varieties $K(k, G_1, \ldots, G_r)$ as follows. If $r = 0$, we write $K(k, \phi)$ for our groups and set $K(k, \phi) = \mathbb{Z}$.

For $r \geq 1$, we define

$$K(k, G_1, \ldots, G_r) = F/R$$

where

$$F = \bigoplus_{E/k; \text{finite}} G_1(E) \otimes \cdots \otimes G_r(E)$$
and $R \subset F$ is the subgroup generated by the relation $\textbf{R1-R2}$ below.

**R1** For any finite extensions $k \hookrightarrow E_1 \overset{\phi}{\to} E_2$, let $g_{i_0} \in G_{i_0}(E_2)$ and $g_i \in G_i(E_1)$ for $i \neq i_0$, the relation

\[(\phi^*(g_1) \otimes \ldots \otimes g_{i_0} \otimes \ldots \otimes \phi^*(g_r))_{E_2} - (g_1 \otimes \ldots \otimes N_{E_2/E_1}(g_{i_0}) \otimes \ldots \otimes g_r)_{E_1}\]

(Here $N_{E_2/E_1}$ denotes the norm map on the group scheme $G_{i_0}$)

**R2** For every algebraic function field $K/k$ and all choices $g_i \in G_i(K)$, $h \in K^\times$ such that for each place $v$ of $K/k$, there exists $i(v)$ such that $g_i \in G_i(O_v)$ for all $i \neq i(v)$, the relation

\[\sum_{v:\text{place of } K/k} (g_1(v) \otimes \ldots \otimes \partial_v(g_{i(v)}, h) \otimes \ldots \otimes g_r(v))_{k(v)/k}\]

Here $g_i(v) \in G_i(k(v))$ ($i \neq i(v)$) denotes the reduction of $g_i \in O_v$ modulo $m_v$, and $\partial_v(g_{i(v)}, h)$ is the extended tame symbol as defined in [1.3]

The class in $F/R$ of an element $a_1 \otimes \ldots \otimes a_r \in G_1(E) \otimes \ldots \otimes G_r(E)$ will be denoted $\{a_1, \ldots, a_r\}_{E/k}$.

Remark 1.5. By the relation $\textbf{R1}$, if $\phi$ is a $k$-isomorphism $E_1 \overset{\sim}{\to} E_2$, then $\{g_1, \ldots, g_r\}_{E_1/k} = \{\phi^*(g_1), \ldots, \phi^*(g_r)\}_{E_2/k}$. This shows that symbols form a set.

## 2 Triangulated categories of motives

In this section, we will briefly review the definition of the triangulated categories of motives. (c.f. [15Ca]).

### 2.1 Triangulated category of effective geometric motives

First we will review the construction of the category of geometric motives.

1. Let $\text{Sm}/k$ be the category of schemes which are separated smooth, and of finite type over $k$.

2. Recall the definition of the category $\text{SmCor}(k)$: its objects are those of $\text{Sm}/k$. The set of morphism from $Y$ to $X$ is given by the group $\text{Cor}(Y, X)$ of finite correspondences from $Y$ to $X$, defined as the free abelian group on the symbol $(Z)$, where $Z$ runs through the integral closed subschemes of $Y \times_k X$ which are finite over $Y$ and surjective over a connected component of $Y$. We will denote the object of $\text{SmCor}(k)$ which corresponds to a smooth scheme $X$ by $[X]$.

2.2. The category $\text{SmCor}(k)$ is an additive category. Consider the homotopy category $\mathcal{H}^b(\text{SmCor}(k))$ of bounded complexes over $\text{SmCor}(k)$. Let $T$ be the class of complexes of the following two forms:

1. For any smooth scheme $X$ over $k$ the complex

\[ [X \times \mathbb{A}^1_k] \overset{[pr_1]}{\to} [X] \]
belongs to $\mathcal{T}$.

2. For any smooth scheme $X$ over $k$ and an open covering $X = U \cup V$ of $X$ the complex

$$[U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[i_U] \oplus ([i_V])} [X]$$

belongs to $\mathcal{T}$. (here $j_U$, $j_V$, $i_U$, $i_V$ are the obvious open embeddings.)

Denote by $\bar{T}$ the minimal thick subcategory of $H^b(SmCor(k))$ which contains $T$.

The triangulated category $DM^{\text{eff}}_{gm}(k)$ of effective geometric motives over $k$ is the pseudo-Abelian envelope of the localization of $H^b(SmCor(k))$ with respect to the thick subcategory $\bar{T}$. We denote the obvious functor $Sm/k \to DM^{\text{eff}}_{gm}(k)$ by $\mathbb{M}_{gm}$.

2.3. For a pair of smooth schemes $X, Y$ over $k$, we set

$$[X] \otimes [Y] := [X \times Y].$$

For any smooth schemes $X_1, Y_1, X_2, Y_2$ the external product of cycles defines a homomorphism:

$$\text{Cor}(X_1, Y_1) \otimes \text{Cor}(X_2, Y_2) \to \text{Cor}(X_1 \times X_2, Y_1 \times Y_2)$$

which gives us a definition of tensor product of morphisms in $SmCor(k)$. This structure defines in the usual way a tensor category structure on $H^b(SmCor(k))$ which can be descended to the category $DM^{\text{eff}}_{gm}(k)$ by the universal property of localization.

Note that the unit object our tensor structure is $\mathbb{M}_{gm}(\text{Spec } k)$. We will denote it by $\mathbb{Z}$.

**Example 2.4.** Let $x, y : \text{Spec } k \to \mathbb{P}^1_k$ be two $k$-rational points. Then $\mathbb{M}_{gm}(x) = \mathbb{M}_{gm}(y) : \mathbb{M}_{gm}(\text{Spec } k) \to \mathbb{M}_{gm}(\mathbb{P}^1_k)$.

**Proof.** We take an affine open set $A^1_k \xrightarrow{j} \mathbb{P}^1_k$ which contains $x$ and $y$. That is, there are $\hat{x}, \hat{y} : \text{Spec } k \to A^1_k$ such that $x = j \circ \hat{x}$ and $y = j \circ \hat{y}$. Then we have $\mathbb{M}_{gm}(x) = \mathbb{M}_{gm}(j) \circ \mathbb{M}_{gm}(\hat{x}) = \mathbb{M}_{gm}(j) \circ \mathbb{M}_{gm}(p)^{-1} = \mathbb{M}_{gm}(j) \circ \mathbb{M}_{gm}(\hat{y}) = \mathbb{M}_{gm}(y)$, where $p : A^1_k \to \text{Spec } k$ is the structure morphism.

2.2 Triangulated category of effective motivic complexes

To study the fundamental property of $DM^{\text{eff}}_{gm}(k)$, Voevodsky uses sheaf theoretic method in [HTCG]. More precisely, he constructs another category $DM^{\text{eff}}(k)$ using a sheaf category and he proves $DM^{\text{eff}}_{gm}(k)$ admits a natural full embedding as a tensor category and a triangulated category to the category $DM^{\text{eff}}(k)$. We will review the construction of $DM^{\text{eff}}(k)$.

2.5. 1. A presheaf with transfers on $Sm/k$ is an additive contravariant functor from the category $SmCor(k)$ to the category of abelian groups. We denote by $\text{PST}(k)$ the category of presheaf with transfers on $Sm/k$. 6
2. A presheaf with transfers on $\text{Sm}/k$ is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on $\text{Sm}/k$ is a sheaf in the Nisnevich topology. We denoted by $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ the category of Nisnevich sheaves with transfers.

**Example 2.6.** For any smooth scheme $X$ over $k$, a presheaf $Z_{tr}(X) := \text{Cor}(?, X)$ is a Nisnevich sheaf with transfers on $\text{Sm}/k$. (c.f. [TriCa] Lemma 3.1.2).

For a $k$-rational point $x : \text{Spec} k \rightarrow X$, we put

$$Z_{tr}(X, x) := \text{Coker}(Z_{tr}(\text{Spec} k) \to Z_{tr}(X)).$$

2.7. 1. A presheaf with transfers $F$ is called homotopy invariant if for any smooth scheme $X$ over $k$ the projection $X \times \mathbb{A}^1_k \to X$ induces the isomorphism $F(X) \to F(X \times \mathbb{A}^1_k)$.

2. A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

2.8. $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ is an abelian category. (c.f. [TriCa] Theorem 3.1.4) Inside the derived category $D^- (\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ of complexes bounded from above, one defines the full subcategory $\text{DM}^{\text{eff}}(k)$ of effective motivic complexes over $k$ as the one consisting of objects whose cohomology sheaves are homotopy invariant. This subcategory is triangulated. (Need the assumption of perfectness of $k$.) (c.f. [TriCa] Proposition 3.1.13).

2.9. 1. Let $F$ be a presheaf with transfers. There is a canonical surjection of presheaves

$$\bigoplus_{(X, x \in F(X))} Z_{tr}(X) \to F.$$

Iterating this construction we get a canonical left resolution $\mathcal{L}(F)$ of $F$ which consists of direct sums of presheaves of the form $Z_{tr}(X)$ for smooth schemes $X$ over $k$.

2. We set for two smooth schemes $X, Y$:

$$Z_{tr}(X) \otimes Z_{tr}(Y) := Z_{tr}(X \times Y)$$

and for two presheaves with transfers $F, G$:

$$F \otimes G := \mathbb{H}_0(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

3. This construction provides us with a tensor structure on the derived category $D^- (\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$.

To define the tensor structure on $\text{DM}^{\text{eff}}(k)$, we will need an alternative description of $\text{DM}^{\text{eff}}(k)$.

2.10. Let $\Delta^\bullet$ be the standard cosimplicial object in $\text{Sm}/k$. For any presheaf with transfers $F$ on $\text{Sm}/k$ let $C_\bullet(F)$ be the complex of presheaves on $\text{Sm}/k$ of the form $C_\bullet(F) = \text{Hom}(\Delta^\bullet, F)$ with differentials given by alternated sums of morphisms which correspond to the boundary morphisms of $\Delta^\bullet$. This complex
is called the singular simplicial complex of $F$.

The following properties are fundamental.

1. If $F$ is a presheaf with transfers (resp. a Nisnevich sheaf with transfers) then $C_*(F)$ is a complex of presheaves with transfers (resp. Nisnevich sheaves with transfers).

2. For any presheaf with transfers $F$ over $k$, the cohomology presheaf $h^i(F)$ of the complex $C_*(F)$ and its Nisnevich sheafication $h^{Nis}_i(F)$ are homotopy invariant. (Need the assumption of perfectness of $k$.) (c.f. [TriCa] Lemma 3.2.1).

3. In view of 1. and 2., $C_*(?)$ is a functor from the category of Nisnevich sheaves with transfers on $Sm/k$ to $DM_{eff}(k)$.

**Proposition 2.11.** (c.f. [TriCa] Proposition 3.2.3)

The functor $C_*(?)$ can be extended to a functor

$$RC : D^- (Shv_{Nis}(SmCor(k))) \to DM_{eff}(k)$$

which is left adjoint to the natural embedding. The functor $RC$ identifies $DM_{eff}(k)$ with localization of $D^- (Shv_{Nis}(SmCor(k)))$ with respect to the localizing subcategory $A$ generated by complexes of the form

$$Z_{tr}(X \times \mathbb{A}^1_k) \xrightarrow{Z_{tr}(pr_1)} Z_{tr}(X)$$

for smooth schemes $X$ over $k$.

2.12. 1. In the notation above, $A$ is a $\otimes$-ideal, that is, for any object $T$ of $D^- (Shv_{Nis}(SmCor(k)))$ and an object $S$ of $A$ the object $T \otimes S$ belongs to $A$. (c.f. [TriCa] Lemma 3.2.4).

2. We define tensor structure on $DM_{eff}(k)$ as the descent of the tensor structure on $D^- (Shv_{Nis}(SmCor(k)))$ with respect to the projector $RC$. Note that such a descent exists by the universal property of localization and 1.

**Theorem 2.13.** (c.f. [TriCa] Theorem 3.2.6)

There is a commutative diagram of functors of the form such that the following conditions hold:

$$
\begin{array}{ccc}
\mathcal{H}^b(SmCor(k)) & \xrightarrow{L} & D^- (Shv_{Nis}(SmCor(k))) \\
\downarrow & & \downarrow \text{RC} \\
DM_{eff}^{gm}(k) & \xrightarrow{i} & DM_{eff}(k)
\end{array}
$$

1. The functor $i$ is a full embedding with a dense image.

2. For any smooth scheme $X$ over $k$ the object $RC(L(X))$ is canonically isomorphic to the $C_*(Z_{tr}(X))$.

3. All functors preserve tensor and triangulated structures.
Example 2.14. Let $x : \text{Spec} \, k \to X$ be a $k$-rational point of a smooth scheme $X$. Then we have an identification

$$C_*(Z_{tr}(X, x)) \xrightarrow{\sim} \text{Cone}(\text{M}_{gm}(\text{Spec} \, k)^{M_{gm}(x)} \text{M}_{gm}(X))$$

defined by

$$
\begin{bmatrix}
Z_{tr}(\text{Spec} \, k) \\
Z_{tr}(x) \\
Z_{tr}(X)
\end{bmatrix} \to 
\begin{bmatrix}
0 \\
\downarrow \\
Z_{tr}(X, x)
\end{bmatrix}.
$$

2.3 Motives with compact support

In this subsection, we will briefly review the notation and fundamental result of [FV00], [RelCy] and [TriCa].

2.15. For any scheme of finite type $X$ over $k$ and any $r \geq 0$ we denote by $Z_{\text{equi}}(X, r)$ the presheaf on the category of smooth schemes over $k$ which takes a smooth scheme $Y$ to free abelian groups generated by closed integral subschemes $Z$ of $X \times Y$ which are equidimensional of relative dimension $r$ over $Y$.

This presheaf has the following property.
1. $Z_{\text{equi}}(X, r)$ is a sheaf in the Nisnevich topology.
2. It has a canonical structure of a presheaf with transfers.
3. The presheaf with transfers $Z_{\text{equi}}(X, r)$ is covariantly functorial with respect to proper morphisms of $X$ by means of the usual proper push-forward of cycles.
4. It is contravariantly functorial with an appropriate dimension shift with respect to flat equidimensional morphisms.
5. There is a pairing $\times : Z_{\text{equi}}(X, r) \otimes Z_{\text{equi}}(X', r') \to Z_{\text{equi}}(X \times X', r + r')$ of presheaves.

Let $U$ be a smooth scheme over $k$. For any pair of integral closed subschemes $Z \subset X \times U, Z' \subset X' \times U$ equidimensional over $U$. Sending $Z, Z'$ to the cycle associated to the subscheme $Z \times_U Z' \subset X \times X' \times U$ determines a pairing.
6. (Need the assumption of [TriCa]) The flat pull-back morphism induces a quasi-isomorphism

$$C_*(Z_{\text{equi}}(X, 0)) \to C_*(Z_{\text{equi}}(X \times \mathbb{A}^n, n)).$$

2.16. For any scheme of finite type over $k$ the object $C_*(Z_{\text{equi}}(X, 0))$ belongs to $\text{DM}^{\text{eff}}(k)$. Moreover it belongs to $\text{DM}_{gm}^{\text{eff}}(k)$. (Using the assumption [TriCa], Corollary 4.1.6). We will denote $M_{gm}(X) := C_*(Z_{\text{equi}}(X, 0))$ and call it a motivic complex of $X$ with compact support.

Since $Z_{tr}(X)$ is a subsheaf of $Z_{\text{equi}}(X, 0)$, the inclusion morphism induces the natural morphism $M_{gm}(X) \to M_{gm}^{c}(X)$.

The following properties are fundamental. (c.f. [TriCa], Proposition 4.1.5, Proposition 4.1.7)
1. If $X$ is proper then the canonical morphism $\text{M}_\text{gm}(X) \to \text{M}_\text{gm}(X)$ is the isomorphism.

2. (Need the assumption 0.6.) Let $Z$ be a closed subscheme of $X$. Then there is a canonical distinguished triangle of the form

$$\text{M}_\text{gm}(Z) \to \text{M}_\text{gm}(X) \to \text{M}_\text{gm}(X - Z) \to \text{M}_\text{gm}(Z)[1].$$

3. For $X, Y$ of finite type over $k$, the pairing $Z\text{equi}(X, 0) \otimes Z\text{equi}(Y, 0) \to Z\text{equi}(X \times_k Y, 0)$ induces an isomorphism

$$\text{M}_\text{gm}(X) \otimes \text{M}_\text{gm}(Y) \xrightarrow{\sim} \text{M}_\text{gm}(X \times_k Y).$$

### 2.4 Tate object

2.17. For any smooth scheme $X$ over $k$, the morphism $X \to \text{Spec } k$ gives us a morphism in $\text{DM}_\text{eff}^\text{gm}(k)$ of the form $\text{M}_\text{gm}(X) \to Z$. There is a canonical split distinguished triangle

$$\widehat{\text{M}}_\text{gm}(X) \to \text{M}_\text{gm}(X) \to Z \to \widehat{\text{M}}_\text{gm}(X)[1]$$

where $\widehat{\text{M}}_\text{gm}(X)$ is the reduced motif of $X$ represented in $\mathcal{H}^k(\text{SmCor}(k))$ by the complex $[X] \to [\text{Spec } k]$.

**Example 2.18.** In the notation above, for any $k$-rational point $x : \text{Spec } k \to X$, we have the canonical identification $\text{M}_\text{gm}(X, x) \xrightarrow{\sim} \widehat{\text{M}}_\text{gm}(X)$ as the following way.

$$\begin{bmatrix}
[\text{Spec } k] \\
\downarrow x \\
[\text{X}] \\
\downarrow \\
0
\end{bmatrix} \xrightarrow{\text{id - M}_{\text{gm}}(x \circ p)} \begin{bmatrix}
0 \\
\downarrow [\text{X}] \\
\downarrow p \\
[\text{Spec } k]
\end{bmatrix}$$

where $p : X \to \text{Spec } k$ is the structure morphism.

$x : \text{Spec } k \to X$ defines splitting $\text{M}_\text{gm}(X) \xrightarrow{\sim} \text{M}_\text{gm}(X, x) \oplus Z$.

2.19. We define the Tate object $\mathbb{Z}(1)$ of $\text{DM}_\text{eff}^\text{gm}(k)$ as $\widehat{\text{M}}_\text{gm}(\mathbb{P}^1)[-2]$. We further define $\mathbb{Z}(n)$ to be the $n$-th tensor power of $\mathbb{Z}(1)$.

For any object $A$ of $\text{DM}_\text{eff}^\text{gm}(k)$ we put

$$A(n) = A \otimes \mathbb{Z}(n)$$

$$A(n) = A \otimes \mathbb{Z}(n)[n]$$

$$A((n)) = A \otimes \mathbb{Z}(n)[2n].$$

By Example 2.18 for any $x : \text{Spec } k \to \mathbb{P}^1_k$, we have the canonical isomorphism $\text{M}_\text{gm}(\mathbb{P}^1_k, x) \xrightarrow{\sim} \mathbb{Z}(1)$. 

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2.20. Let \( x : \text{Spec} \ k \to \mathbb{P}^1_k \) be a \( k \)-rational point. Comparing the following split distinguished triangles

\[
\begin{array}{cccccc}
M_{gm}^c(\text{Spec} \ k) & \to & M_{gm}^c(\mathbb{A}^1) & \to & M_{gm}^c(\text{Spec} \ k)[1] \\
\id & \to & \id & \to & \id \\
M_{gm}(\text{Spec} \ k) & \to & M_{gm}(\mathbb{P}^1_k) & \to & Z((1)) & \to & M_{gm}(\text{Spec} \ k)[1],
\end{array}
\]

where \( M_{gm}(\mathbb{P}^1_k) \to Z((1)) \) is defined by

\[
M_{gm}(\mathbb{P}^1_k) \to \text{can} \ M_{gm}(\mathbb{P}^1, x) \sim \to Z((1)),
\]

we know that there is a natural isomorphism \( Z((1)) \sim \to M_{gm}^c(\mathbb{A}^1) \). It does not depend on the choice of a \( k \)-rational point by Example 2.4. Similarly using Mayer-Vietoris sequence for canonical covering of \( \mathbb{P}^1_k \), we know also that there is a natural isomorphism \( Z(\{1\}) \sim \to \tilde{M}_{gm}(\mathbb{A}^1 - \{0\}) \).

\[ \text{2.5 The triangulated category of geometric motives} \]

In this subsection, we will define the triangulated category \( \text{DM}_{gm}(k) \) of geometric motives over \( k \).

2.21. 1. We define the category \( \text{DM}_{gm}(k) \): its objects are pairs of the form \( (A, n) \) where \( A \) is an object of \( \text{DM}_{gm}^\text{eff}(k) \) and \( n \in \mathbb{Z} \) and morphisms are defined by the following formula

\[
\text{Hom}_{\text{DM}_{gm}(k)}((A, n), (B, m)) := \lim_{k \geq -n, -m} \text{Hom}_{\text{DM}_{gm}^\text{eff}(k)}(A(k + n), B(k + m)).
\]

2. The category \( \text{DM}_{gm}(k) \) with the obvious shift functor and class of distinguished triangles is a triangulated category.

3. The permutation involution on \( Z(1) \otimes Z(1) \) is identity in \( \text{DM}_{gm}^\text{eff}(k) \). (c.f. [TriCa] Corollary 2.1.5)

4. Using the fact of 3. and general theory, \( \text{DM}_{gm}(k) \) has a natural tensor structure.

Theorem 2.22. (c.f. [Voe02] The cancellation theorem) (Need the assumption of perfectness of \( k \).) For objects \( A, B \) in \( \text{DM}_{gm}^\text{eff}(k) \) the natural map

\[
? \otimes \id_{Z(1)} : \text{Hom}_{\text{DM}_{gm}^\text{eff}(k)}(A, B) \to \text{Hom}_{\text{DM}_{gm}^\text{eff}(k)}(A(1), B(1))
\]

is an isomorphism. Thus the canonical functor

\[
\text{DM}_{gm}^\text{eff}(k) \to \text{DM}_{gm}(k)
\]

is a full embedding.
3 Various morphisms between motives

In this section, we will briefly review [TriCa] and [MotGe]

3.1 Transpose for finite equidimensional morphisms

3.1. Let X, Y be smooth schemes over k and f : X → Y a finite equidimensional morphism. Then we have the transpose of f, \( t_f : Y \rightarrow X \) in SmCor(k). That is \( s(\Gamma_f) \in \text{Hom}_{\text{SmCor}(k)}(Y, X) \), where \( s : X \times Y \rightarrow Y \times X \) is the switch morphism.

Example 3.2. Let \( L/K/k \) be finite field extensions. Then we have a canonical morphism \( i : \text{Spec} \ L \rightarrow \text{Spec} \ K \). Hence we get \( M_{\text{gm}}(t_i) : M_{\text{gm}}(\text{Spec} \ K) \rightarrow M_{\text{gm}}(\text{Spec} \ L) \) in \( \text{DM}_{\text{gm}}(k) \). We shall write this map as \( N_{L/K} \) because there is the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(M_{\text{gm}}(\text{Spec} \ L), \mathbb{Z}\{n\}) & \xrightarrow{\cong} & K^M_n(L) \\
\text{Hom}(N_{L/K}, \mathbb{Z}\{n\}) & \downarrow & \downarrow N_{L/K} \\
\text{Hom}(M_{\text{gm}}(\text{Spec} K), \mathbb{Z}\{n\}) & \xrightarrow{\cong} & K^M_n(K).
\end{array}
\]

(c.f. [BKcon] Lemma 3.4.4. See also Theorem 5.16)

3.2 Pull back for flat equidimensional morphisms

3.3. Let X, Y be smooth schemes and f : X → Y a flat equidimensional morphism of relative dimension n. Then one can define a morphism \( f^* : M_{\text{gm}}^c(Y)((n)) \rightarrow M_{\text{gm}}^c(X) \) as follows. (This is slightly different from the definition in [TriCa] Corollary 4.2.4).

\[
M_{\text{gm}}^c(Y)((n)) = C_*(Z_{\text{equi}}(Y \times \mathbb{A}^n, 0))
\]

\[
C_*(f \times \text{id}_{\mathbb{A}^n})^* : C_*(Z_{\text{equi}}(X \times \mathbb{A}^n, n)) \leftarrow_{\text{qis}} C_*(Z_{\text{equi}}(X, 0)) = M_{\text{gm}}^c(X)
\]

where every quasi-isomorphisms are induced from flat pull backs.

3.4. The following properties are easily proved.

1. In the notation above, if f is an open immersion, this morphism coincides with the canonical morphism \( M_{\text{gm}}^c(Y) \rightarrow M_{\text{gm}}^c(X) \).
2. In the notation above, if X and Y are proper over Spec k and f is flat finite equidimensional, then \( M_{\text{gm}}^c(f) = f^* \). (c.f. [MotGe] Lemma 1.1.2)
3. Let X, Y and Z be smooth schemes and f : X → Y, g : Y → Z flat equidimensional morphisms of relative dimension n and m respectively. Then we have \( f^* \circ g^*((m)) = (g \circ f)^* \).

Lemma 3.5.

Let \( p : \mathbb{A}^1_k \rightarrow \text{Spec} k \) be the structure morphism. Then \( p^* : M_{\text{gm}}^c(\text{Spec} k)((1)) = M_{\text{gm}}^c(\mathbb{A}^1_k) \rightarrow M_{\text{gm}}^c(\mathbb{A}^1_k) \) coincides with \( \text{id}_{M_{\text{gm}}^c(\mathbb{A}^1_k)} \).

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Proof. By definition (See 3.3),

\[ p^* = C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \xrightarrow{C_*(pr_1^* \circ C_*(pr_2^*))^{-1}} C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \]

\[ \xrightarrow{\text{where } pr_1, pr_2 \text{ are two projections } pr_1, pr_2 : A^2_k \to A^1_k.} \]

We assert that two projections \( pr_1, pr_2 \) induce the same morphism \( pr_1^* = pr_2^* : C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \to C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 1)) \) in \( DM^{eff}(k) \). Since

\[ pr_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ pr_2, \]

it suffices to prove that the action of \( GL_2(k) \) on \( A^2_k \) induces trivial action on \( C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \) in \( DM^{eff}(k) \). On the other hand \( GL_2(k) \) is generated by the elements of conjugate of \( \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \). For \( A = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \in GL_2(k) \), considering the following diagram

\[ \begin{array}{ccc}
C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) & \xrightarrow{A^*} & C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \\
pr_1^* & \downarrow i & \downarrow i \\
C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) & \xrightarrow{id} & C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)),
\end{array} \]

we get the result.

\[ \square \]

3.3 Motives with closed support

3.6. We call \((X, Z)\) a closed pair if \(X\) is a smooth scheme over \(k\) and \(Z\) is a closed subscheme. If \(Z\) is smooth over \(k\), we call \((X, Z)\) a smooth pair. We call a pair of morphisms of scheme \((f, g) : (Y, T) \to (X, Z)\) a morphism of closed pair if a commutative square \[
\begin{array}{ccc}
T & \xrightarrow{f} & X \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{j} & Y
\end{array}
\]
is a Cartesian square as underlying topological spaces. Such a morphisms called Cartesian (resp. excisive) if the diagram above is a Cartesian square (resp. \(f\) is étale and \(g_{red}\) is an isomorphism.)

3.7. Let \(X\) be a smooth scheme and \(U\) its open subset. Then we define

\[ M_{\text{gm}}(X/U) := C^*(\text{Coker}(\mathbb{Z}_{\text{et}}(U) \to \mathbb{Z}_{\text{et}}(X))). \]

For any closed pair \((X, Z)\), we define relative motif associated \((X, Z)\) by \(M_Z(X) := M_{\text{gm}}(X/X - Z)\). By definition there is a canonical distinguished triangle of the form

\[ M_{\text{gm}}(X - Z) \xrightarrow{M_{\text{gm}}(j)} M_{\text{gm}}(X) \xrightarrow{i^*} M_Z(X) \to M_{\text{gm}}(X - Z)[1]. \]

where \(i : Z \hookrightarrow X\) is a closed immersion and \(j : X - Z \hookrightarrow X\) is an open immersion.
3.8. For any morphisms of closed pair \((f, g) : (Y, T) \to (X, Z)\), we associate a morphism \((f, g)_* : M_T(Y) \to M_Z(X)\) which makes the following diagram commute

\[
\begin{array}{ccl}
0 & \longrightarrow & M_{\text{gm}}(Y - T) \\
\downarrow_{M_{\text{gm}}(h)} & & \downarrow_{M_{\text{gm}}(f)} \\
0 & \longrightarrow & M_{\text{gm}}(X - Z) \\
\end{array}
\begin{array}{ccl}
M_{\text{gm}}(Y) & \longrightarrow & M_T(Y) & \longrightarrow & 0 \\
\downarrow_{(f, g)_*} & & & & \\
M_{\text{gm}}(X) & \longrightarrow & M_Z(X) & \longrightarrow & 0
\end{array}
\]

where \(h : Y - T \to X - Z\) is a induced morphism from \(f\).

3.9. In the notation above, if \(f\) is finite equidimensional, we associate a morphism \((f, g)^* : M_T(Y) \to M_Z(X)\) which makes the following diagram commute

\[
\begin{array}{ccl}
0 & \longrightarrow & M_{\text{gm}}(Y - T) \\
\downarrow_{M_{\text{gm}}('h)} & & \downarrow_{M_{\text{gm}}('f)} \\
0 & \longrightarrow & M_{\text{gm}}(X - Z) \\
\end{array}
\begin{array}{ccl}
M_{\text{gm}}(Y) & \longrightarrow & M_T(Y) & \longrightarrow & 0 \\
\downarrow_{(f, g)^*} & & & & \\
M_{\text{gm}}(X) & \longrightarrow & M_Z(X) & \longrightarrow & 0
\end{array}
\]

Proposition 3.10. (c.f. \textit{IntMo} Proposition 2.3)

(\textit{Red}) Reduction: If \((X, Z)\) is a closed pair, the canonical morphism \((X, Z_{\text{red}}) \to (X, Z)\) induces identity map \(M_{Z_{\text{red}}}(X) \to M_Z(X)\).

(\textit{Add}) Additivity: Let \(X\) be a smooth scheme, and \(Z, W\) disjoint closed subschemes of \(X\). Then induced morphism \(M_{Z_{\mid W}}(X) \to M_Z(X) \oplus M_W(X)\) is an isomorphism.

(\textit{Exc}) Excision: Any excisive morphism \((Y, T) \to (X, Z)\) induces an isomorphism \(M_T(Y) \to M_Z(X)\).

(\textit{MV}) Mayer-Vietoris: Let \(X\) be a smooth scheme over \(k\), \(U\) and \(V\) two open subsets of \(X\) such that \(X = U \cup V\), and \(Z\) a closed subscheme of \(X\). Then we have a distinguished triangle of the form

\[
M_{Z_{\mid (U \cap V)}}(U \cap V) \to M_{Z_{\mid U}}(U) \oplus M_{Z_{\mid V}}(V) \to M_Z(X) \xrightarrow{\pm 1}.
\]

(\textit{Htn}) Homotopy invariance: A Cartesian morphism \(\pi : (\mathbb{A}^1_X, \mathbb{A}^1_Z) \to (X, Z)\) induced from the canonical projection induces an isomorphism

\[
M_{\text{gm}}(\pi) : M_{\mathbb{A}^1_X}(\mathbb{A}^1_Z) \to M_Z(X).
\]

3.4 Thom isomorphism

3.11. Let \(X\) be a scheme and \(E/X\) a vector bundle. We consider \(X\) as a closed subscheme of \(E\) by zero section. We define the Thom motif of \(E/X\) by \(M_{\text{gm}}(\text{Th}E) := M_X(E)\).

3.12. In the notation above, if rank of \(E\) is \(n\), there is the Thom isomorphism \(\theta(E) : M_{\text{gm}}(\text{Th}E) \cong M_{\text{gm}}(X)((n))\). We will briefly review the construction of this isomorphism.
3.13. Let $X$ be a smooth scheme and $\Delta_X : X \to X \times_k X$ the diagonal immersion. Let $\mathcal{M}, \mathcal{N}$ be objects of $\text{DM}^{\text{eff}}(k)$ and

$$
\alpha : \text{M}_{\text{gm}}(X) \to \mathcal{M}
$$

$$
\beta : \text{M}_{\text{gm}}(X) \to \mathcal{N}
$$

are morphisms in $\text{DM}^{\text{eff}}(k)$. We define external cup product of $\alpha$ and $\beta$ over $X$ is composition of the following morphisms

$$
\text{M}_{\text{gm}}(X) \xrightarrow{\text{M}_{\text{gm}}(\Delta_X)} \text{M}_{\text{gm}}(X) \otimes \text{M}_{\text{gm}}(X) \xrightarrow{\alpha \otimes \beta} \mathcal{M} \otimes \mathcal{N}.
$$

We denote this morphism by $\alpha \boxtimes X \beta$ or $\alpha \boxtimes \beta$.

3.14. In the notation above, if $\mathcal{M} = \mathbb{Z}((m))$ and $\mathcal{N} = \mathbb{Z}((n))$, then we have the canonical isomorphism $\varepsilon : \mathbb{Z}((m)) \otimes \mathbb{Z}((n)) \xrightarrow{\sim} \mathbb{Z}((m+n))$. We put $\alpha \cup \beta := \varepsilon \circ \alpha \boxtimes \beta$ and call it internal cup product of $\alpha$ and $\beta$.

3.15. There is a natural isomorphism as a presheaf with transfers (c.f. [MotGa] Corollary 2.2.7)

$$
c_1 : \text{Pic}(?) \to \text{Hom}_{\text{DM}_{\text{gm}}(k)}(\text{M}_{\text{gm}}(?), \mathbb{Z}((1))).
$$

For any smooth scheme $X$ and $\mathcal{L} \in \text{Pic}(X)$, we will call $c_1(\mathcal{L})$ a motivic Chern class of $\mathcal{L}$.

3.16. Let $X$ be a smooth scheme, $E$ a vector bundle over $X$ of rank $n$, $\lambda_E$ the canonical invertible sheaf on $\mathbb{P}(E)$ and $p : \mathbb{P}(E) \to X$ the canonical projection. We define the $r$-th motivic Lefschetz projector of $E$ by

$$
l_r(E) := c_1(\lambda_E)^{-1-r} \boxtimes \text{M}_{\text{gm}}(p) : \text{M}_{\text{gm}}(\mathbb{P}(E)) \to \text{M}_{\text{gm}}(X)((r))
$$

We put the motivic Lefschetz operator as

$$
l(E) := \sum_{r=0}^{n-1} l_r(E).
$$

**Proposition 3.17.** (c.f. [TriCa] Proposition 3.5.1)

In the notation above, the morphism

$$
l(E) : \text{M}_{\text{gm}}(\mathbb{P}(E)) \to \bigoplus_{r=0}^{n-1} \text{M}_{\text{gm}}(X)((r))
$$

is an isomorphism.

3.18. Let $X$ be a smooth scheme and $E/X$ a vector bundle of rank $n$. Put $\bar{E} := E \times_X \mathbb{A}_X^1$. Then we have the canonical isomorphisms

$$
\text{M}_{\text{gm}}(ThE) \xrightarrow{1} \text{M}_X(\mathbb{P}(\bar{E})) \xrightarrow{2} \text{M}_{\text{gm}}(\mathbb{P}(E)/\mathbb{P}(E)).
$$
The first morphism is induced from an open immersion \( E \to \mathbb{P}^E \) which is an 

isomorphism by (Exc). The second morphism is induced from the projection 

\( \mathbb{P}(E) - X \to \mathbb{P}(E) \) which is an isomorphism by (MV) and (Htp). From this 

isomorphisms, we get the following distinguished triangle 

\[
\mathcal{M}_{gm}(\mathbb{P}(E)) \to \mathcal{M}_{gm}(\mathbb{P}(E)) \xrightarrow{\tau} \mathcal{M}_{gm}(ThE) \xrightarrow{\pm 1}. 
\]

Using this distinguished triangle and Proposition 3.17, we get the follow ing 

isomorphism 

\[
\mathcal{M}_{gm}(X)((n)) \to \bigoplus_{r=0}^{n} \mathcal{M}_{gm}(X)((r)) \xrightarrow{(l(E))^{-1}} \mathcal{M}_{gm}(\mathbb{P}(E)) \xrightarrow{\tau} \mathcal{M}_{gm}(ThE). 
\]

We call the inverse of this isomorphism the Thom isomorphism and denote it 

by \( \theta(E) \).

3.5 Normal cone deformation

3.19. Let \((X, Z)\) be a smooth pair of pure codimension \( c \) over \( k \) such that 

dimension of \( X \) is \( n \). We will write \( B_Z X \) by the blow up of \( X \) in \( Z \). Put 

\( D_Z X := B_0 \times Z (\mathbb{A}^1 \times X) - B_Z X \). There are canonical isomorphisms 

\[
\mathcal{M}_Z(X) \to \mathcal{M}_{A^1_Z}(D_Z X) \leftarrow \mathcal{M}_{gm}(ThN_Z X). 
\]

Hence we get the isomorphism \( \mathcal{M}_Z(X) \to \mathcal{M}_{gm}(ThN_Z X) \).

3.6 Gysin triangles

3.20. Let \((X, Z)\) be a smooth pair of pure codimension \( c \) over \( k \) and \( i : Z \to X \) a 

closed immersion. In [IntMo], Dèglise constructs the following funct orial Gysin 

triangle in \( DM_{gm}(k) \)

\[
\mathcal{M}_{gm}(X - Z) \to \mathcal{M}_{gm}(X) \xrightarrow{\tau} \mathcal{M}_{gm}(Z)((c)) \xrightarrow{\partial_{X,Z}[1]} \mathcal{M}_{gm}(X - Z)[1].
\]

This is constructed from the following distinguished triangle 

\[
\mathcal{M}_{gm}(X - Z) \to \mathcal{M}_{gm}(X) \to \mathcal{M}_Z(X) \to \mathcal{M}_{gm}(X - Z)[1]
\]

and the following isomorphisms 

\[
\mathcal{M}_Z(X) \xrightarrow{\sim} \mathcal{M}_{gm}(Th(N_Z(X))) \xrightarrow{\theta(N_Z(X))} \mathcal{M}_{gm}(Z)((c))
\]

where the first isomorphism is induced from the normal cone deformation.

3.21. Let \((f, g) : (Y, T) \to (X, Y)\) be a morphism of closed pairs. We assume \( Z \) 

(resp. \( T \)) is connected and smooth over \( k \) of codimension \( n \) in \( X \) (resp. \( m \) in \( Y \)).
Then we define Gysin morphism associated to \((f, g)\), denote \((f, g)\), by the following commutative diagram:

\[
\begin{array}{ccc}
M_T(Y) & \xrightarrow{(1)} & M_{\text{gm}}(Th(N_T Y)) \xrightarrow{\theta(N_T Y)} M_{\text{gm}}(T)((m)) \\
(f, g) & & (f, g) \\
M_Z(X) & \xrightarrow{(1)} & M_{\text{gm}}(Th(N_Z X)) \xrightarrow{\theta(N_Z X)} M_{\text{gm}}(Z)((n)) \\
\end{array}
\]

where morphisms (1) are isomorphisms induced from the morphisms of normal cone deformations.

3.22. In the notation above, consider \(i : Z \to X\), \(j : X - Z \to X\), \(k : T \to Y\) and \(l : Y - T \to T\) the canonical immersions. The following diagram is commutative:

\[
\begin{array}{ccc}
M_{\text{gm}}(Y - T) & \xrightarrow{M_{\text{gm}}(l)} & M_{\text{gm}}(Y) \\
M_{\text{gm}}(h) & & (f, g) \\
M_{\text{gm}}(X - Z) & \xrightarrow{M_{\text{gm}}(j)} & M_{\text{gm}}(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
M_{\text{gm}}(Y - T) & \xrightarrow{k^*} & M_{\text{gm}}(T)((m)) \\
M_{\text{gm}}(h) & & (f, g) \\
M_{\text{gm}}(X - Z) & \xrightarrow{\partial_{X,Z}[1]} & M_{\text{gm}}(X - Z)[1]
\end{array}
\]

\[
\begin{array}{ccc}
M_{\text{gm}}(Y - T) & \xrightarrow{k^*} & M_{\text{gm}}(T)((m)) \\
M_{\text{gm}}(h) & & (f, g) \\
M_{\text{gm}}(X - Z) & \xrightarrow{\partial_{X,Z}[1]} & M_{\text{gm}}(X - Z)[1]
\end{array}
\]

**Lemma 3.23.** Let \(x : \text{Spec } k \to \mathbb{A}_k^1 \to \mathbb{P}^1\) be a \(k\)-rational point. Then the following diagram is commutative.

\[
\begin{array}{ccc}
M_{\text{gm}}(\mathbb{P}^1) & \xrightarrow{z^*} & M_{\text{gm}}(\text{Spec } k)((1)) \\
\text{id} & & \text{id} \\
M_{\text{gm}}(\mathbb{P}^1) & \xrightarrow{j^*} & M_{\text{gm}}(\mathbb{A}_k^1)
\end{array}
\]

where the vertical isomorphism \(M_{\text{gm}}(\text{Spec } k)((1)) \sim M_{\text{gm}}(\mathbb{A}_k^1)\) is defined in \([2, 27]\).

**Proof.** It is just a matter of considering two split distinguished triangles below

\[
\begin{array}{ccc}
M_{\text{gm}}(\mathbb{A}_k^1) & \xrightarrow{1} & M_{\text{gm}}(\mathbb{P}^1) \\
M_{\text{gm}}(\text{Spec } k) & \xrightarrow{j^*} & M_{\text{gm}}(\mathbb{P}^1)
\end{array}
\]

\[
\begin{array}{ccc}
M_{\text{gm}}(\mathbb{A}_k^1) & \xrightarrow{1} & M_{\text{gm}}(\text{Spec } k)((1)) \\
M_{\text{gm}}(\text{Spec } k) & \xrightarrow{j^*} & M_{\text{gm}}(\text{Spec } k)[1]
\end{array}
\]

where the commutativity of 1 follows from Example [2, 21].

3.24. In the notation above, if \(f\) is finite equidimensional, then we define \((f, g)\)' by the following commutative diagram:

\[
\begin{array}{ccc}
M_T(Y) & \xrightarrow{(1)} & M_{\text{gm}}(Th(N_T Y)) \\
(f, g) & & (f, g) \\
M_Z(X) & \xrightarrow{(1)} & M_{\text{gm}}(Th(N_Z X)) \\
\end{array}
\]

\[
\begin{array}{ccc}
M_T(Y) & \xrightarrow{\theta(N_T Y)} & M_{\text{gm}}(T)((m)) \\
(f, g) & & (f, g) \\
M_Z(X) & \xrightarrow{\theta(N_Z X)} & M_{\text{gm}}(Z)((n))
\end{array}
\]
where the morphisms (1) are the isomorphisms induced from the morphisms of normal cone deformations.

The following diagram is commutative:

\[ \begin{array}{ccc}
M_{gm}(Y - T) & \xrightarrow{M_{gm}(l)} & M_{gm}(Y) \\
\downarrow_{M_{gm}(h)} & & \downarrow_{(f,g)^t} \\
M_{gm}(X - Z) & \xrightarrow{M_{gm}(j)} & M_{gm}(X)
\end{array} \]

\[ \begin{array}{ccc}
k^* & \xrightarrow{\partial_Y.T[1]} & M_{gm}(T)((m)) \\
\downarrow & & \downarrow \\
k^* & \xrightarrow{\partial_{X,Z}[1]} & M_{gm}(X - Z)[1]
\end{array} \]

**Proposition 3.25.** ([MotGe] Proposition 2.5.2)

In the notation above, if \((f, g)\) is Cartesian, then

\[ (f, g)^t = M_{gm}(g)(m). \]

Next we cite the some proposition in [MotGe]. This is needed to prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor \(K\)-groups.

**3.26.** By [2.20] we have the following distinguished triangle

\[ Z \to M_{gm}(\mathbb{G}_m) \xrightarrow{\ell} \mathbb{Z}\{1\} \xrightarrow{+1} \]

where \(Z \to M_{gm}(\mathbb{G}_m)\) is induced from the unit morphism \(\text{Spec } k \to \mathbb{G}_m\).

**Proposition 3.27.** (c.f. [MotGe] Proposition 2.6.6)

Let \((X, Z)\) be a smooth closed pair of codimension 1. We denote \(i : Z \to X\) a closed immersion and \(j : X - Z \to X\) a canonical open immersion.

Suppose there is a regular function \(\pi : X \to \mathbb{A}^1_k\) which parameterizes \(Z\). Hence we have a morphism \(\pi = \pi|_{X - Z} : X - Z \to \mathbb{G}_m\). Then the following diagram is commutative

\[ \begin{array}{ccc}
M_{gm}(Z)\{1\} & \xrightarrow{\partial_{X,Z}} & M_{gm}(X - Z) \\
\downarrow_{M_{gm}(i)\{1\}} & & \downarrow_{M_{gm}(\pi)\circ p_1 \circ \text{id}_{M_{gm}(X - Z)}} \\
M_{gm}(X)\{1\} & \xrightarrow{M_{gm}(j)\{1\}^{-1}} & M_{gm}(X - Z)\{1\}
\end{array} \]

4 Motivic cohomology groups attached to pointed smooth curves

4.1 Definition

4.1. For pointed smooth curves \((C_1, x_1), \ldots, (C_r, x_r)\) over field \(k\), we define a motivic complex \(Z((C_1, x_1) \wedge \ldots \wedge (C_r, x_r))\), or \(Z(C_1 \wedge \ldots \wedge C_r)\) in \(\text{DM}^{eff}(k)\) as follows

\[ Z(C_1 \wedge \ldots \wedge C_r) = C^*(Z_{tr}(C_1, x_1) \otimes \ldots \otimes Z_{tr}(C_r, x_r))[-r] \]
4.2. The restriction $Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))|_X$ of $Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))$ to the Zariski site of $X \in Sm/k$ is a complex of sheaves in Zariski topology and the motivic cohomology groups $H^n_\text{Mot}(X, \overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))$, or $H^n_\text{Mot}(X, \overset{\circ}{\Lambda}_{i=1}^t C_r)$ are defined to be the hyper cohomology of the motivic complexes $Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))$ with respect to Zariski topology:

$$H^n_\text{Mot}(X, \overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i)) = \mathbb{H}_\text{Mot}^n(X, Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))|_X).$$

4.3. As in [TriCa] we have

$$\mathbb{H}_\text{Mot}^n(X, Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))|_X) = \text{Hom}_{\text{Mot}_{\text{str}}(k)}(\text{M}_{\text{gm}}(X), Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))[n])$$

Moreover if $k$ is a perfect field as in [CohTh] Proposition 3.1.11, we have

$$\mathbb{H}_\text{Mot}^n(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))|_X) = \mathbb{H}_\text{Mot}^n(X, Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, x_i))|_X).$$

4.2 Fundamental properties

4.4. (Product structure) Let $(C_1, a_1), \ldots, (C_s, a_s), (D_1, b_1), \ldots, (D_t, b_t)$ be pointed smooth curves over field $k$. As usual motivic complexes, we have canonical morphism

$$Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, a_i)) \otimes Z(\overset{\circ}{\Lambda}_{j=1}^t (D_j, b_j)) \rightarrow Z(\overset{\circ}{\Lambda}_{i=1}^s (C_i, a_i) \wedge \overset{\circ}{\Lambda}_{j=1}^t (D_j, b_j)).$$

Hence we get for any $X \in Sm/k$ the pairing

$$H^p_\text{Mot}(X, \overset{\circ}{\Lambda}_{i=1}^s \wedge (C_i, a_i)) \otimes H^q_\text{Mot}(X, \overset{\circ}{\Lambda}_{j=1}^t (D_j, b_j)) \rightarrow H^{p+q}_\text{Mot}(X, \overset{\circ}{\Lambda}_{i=1}^s (C_i, a_i) \wedge \overset{\circ}{\Lambda}_{j=1}^t (D_j, b_j)).$$

4.5. Let $(C_1, a_1), \ldots, (C_r, a_r)$ be pointed smooth curves over field $k$. For any field extension $L/k$, we abbreviate $H^p_\text{Mot}(\text{Spec } L, \overset{\circ}{\Lambda}_{i=1}^s (C_i \times_k L, a_i \times_k \text{id}_L))$ as $H^p_\text{Mot}(L, \overset{\circ}{\Lambda}_{i=1}^s C_i)$. By definition, we have

$$H^p_\text{Mot}(L, \overset{\circ}{\Lambda}_{i=1}^s C_i) = H_{i-p}(\text{C} \otimes_{\text{mot}} (C_i \times_k L, a_i \times_k \text{id}_L))(\text{Spec } L).$$

4.6. (Norm map) In [TriCa] if we assume $L/k$ is a finite field extension, then using the description of [TriCa] and the proper push-forward of cycles induces a map

$$N_{L/k} : H^p_\text{Mot}(L, \overset{\circ}{\Lambda}_{i=1}^s C_i) \rightarrow H^p_\text{Mot}(k, \overset{\circ}{\Lambda}_{i=1}^s C_i).$$

From the corresponding properties of proper push-forward, the following properties are immediately verified.

For finite field extension $k \subset L \subset M$ and $x \in H^p_\text{Mot}(M, \overset{\circ}{\Lambda}_{i=1}^s C_i)$ and $y \in H^p_\text{Mot}(L, \overset{\circ}{\Lambda}_{i=1}^s C_i)$
then we have
(1) $N_{M/L}(y_M \cdot x) = y \cdot N_{M/L}(x)$ and $N_{M/L}(x \cdot y_M) = N_{M/L}(x) \cdot y$
(2) $N_{M/k}(x) = N_{M/L}(N_{L/k}(x))$
(3) If $M/k$ is a normal extension, we have
\[ N_{L/k}(x) = [L:k] \sum_{j:M \to L} j^*(x). \]

**Example 4.7.** Let $(C, x)$ be a pointed projective smooth curve over $k$. Then we have
\[ H^1_{\text{Mot}}(k, (C, x)) = \text{Ker}(\text{CH}_0(C) \xrightarrow{\text{deg}} \mathbb{Z}). \]

**Example 4.8.** Let $X$ be a smooth curve over $k$. A good compactification of $X$ is a pair $(\bar{X}, X_\infty)$ such that there is an open embedding $X \hookrightarrow \bar{X}$, $\bar{X}$ is proper non-singular curve over $k$ and $X_\infty = \bar{X} - X$ has an affine open neighborhood in $\bar{X}$.

Let $(C, x)$ be a pointed smooth affine curve with a good compactification $(\bar{X}, X_\infty)$, then
\[ H^1_{\text{Mot}}(k, (C, x)) = \text{Ker}(\text{Pic}(\bar{X}, X_\infty) \xrightarrow{\text{deg}} \mathbb{Z}). \]

where Pic$(\bar{X}, X_\infty)$ is the relative Picard group. The elements of Pic$(\bar{X}, X_\infty)$ are the isomorphism classes $(\mathcal{L}, t)$ of line bundle $\mathcal{L}$ on $\bar{X}$ with a trivialization $t$ on $X_\infty$.

4.9. In 4.7, for finite field extension $L/k$, using the property 4.6 (3), we know through isomorphisms in 4.7, norm maps in 4.6 and classical one are compatible.

5 Calculation of motivic cohomology groups attached to pointed smooth curves

5.1 Pro-motives

In this subsection, we will briefly review the result of [MotGe].

5.1. Let $\mathcal{A}$ be a tensor triangulated category. We consider Pro-$\mathcal{A}$ the pro-category of $\mathcal{A}$. Then the following facts are fundamental.
1. Pro-$\mathcal{A}$ is additive.
2. The shift functor of $\mathcal{A}$ induces an auto-functor of Pro-$\mathcal{A}$.
3. There is a unique tensor structure over Pro-$\mathcal{A}$ such that $\otimes$ commutes projective limits.

5.2. In the notation above, we call any triangle in Pro-$\mathcal{A}$ isomorphic to formal projective limit of distinguished triangles of $\mathcal{A}$ a pro-distinguished triangle. Let $H: A^{\text{op}} \to Ab$ be a cohomological functor. Then the functor
\[ \mathcal{T}: (\text{Pro-}A)^{\text{op}} \ni (X_i)_{i \in I} \mapsto \text{inj lim}_{i \in I^{op}} H(X_i) \in Ab \]
sends pro-distinguished triangles to long exact sequences.
5.3. Let \( \mathcal{O} \) be a \( k \)-algebra. We say \( \mathcal{O} \) is local smooth over \( k \) iff there is an formally smooth of finite type \( k \)-algebra \( A \), and a prime ideal \( x \) of \( A \) and an isomorphism \( \mathcal{O} \xrightarrow{\sim} A_x \).

Since \( k \) is perfect, \( \mathcal{O} \) is local smooth iff it is regular and essentially of finite type.

5.4. Let \( \mathcal{O} \) be a local smooth \( k \)-algebra. A model of \( \mathcal{O}/k \) is a pair \((X, x)\) consist of a smooth scheme \( X \) and a morphism \( x : \text{Spec} \mathcal{O} \to X \) such that, if we write the image of closed point of \( \text{Spec} \mathcal{O} \) to \( x \), induced morphism \( x^\#: \mathcal{O}_{X,x} \to \mathcal{O} \) is an isomorphism.

We put
\[
\mathcal{M}^{smo}(\mathcal{O}/k) := \{ A \subset \mathcal{O}; (\text{Spec} A, \text{Spec} \mathcal{O} \xrightarrow{\text{obvious map}} \text{Spec} A) \text{ is a model of } \mathcal{O}/k \}.
\]
\( \mathcal{M}^{smo}(\mathcal{O}/k) \) is not empty and filtrant for inclusion. (c.f. [MotGe] Lemma 3.1.5).

5.5. 1. Let \( \mathcal{O} \) be a local smooth \( k \)-algebra. We consider a pro-object of \( \text{Sm}/k \)
\[
(O) := \{ \text{Spec} A \}_{A \in \mathcal{M}^{smo}(\mathcal{O}/k)}.
\]

2. Let \( X \) be a smooth scheme and \( x \in X \), we define localization of \( X \) in \( x \) as a pro-object of \( \text{Sm}/k \).
\[
X_x := \{ U \}_{x \in U \subset X}
\]
where \( U \) runs through the open neighborhood of \( x \).

5.6. (c.f. [MotGe] Lemma 3.1.8) Let \( \mathcal{O} \) be a local smooth \( k \)-algebra, and \((X, x)\) a model of \( \mathcal{O} \). Then \( x \) induces a canonical isomorphism
\[
(O) \to X_x.
\]

5.7. Let \( \mathcal{O} \) be a local smooth \( k \)-algebra and \( n, m \in \mathbb{Z} \). We consider a pro-object of \( \text{Pro-DM}_{gm}(k) \)
\[
M_{gm}(\text{Spec} \mathcal{O})(n)[m] := \{ M_{gm}(\text{Spec} A)(n)[m] \}_{A \in \mathcal{M}^{smo}(\mathcal{O}/k)}.
\]

Next we define a residue morphism associated to a discrete valuation.

5.8. Let \( E/k \) be a field extension of finite type, \( v \) a valuation of \( E/k \), \( \mathcal{O}_v \) a valuation ring of \( v \) and \((X, t)\) a \( k \)-model of \( \mathcal{O}_v \). We say a special point of \((X, t)\) for image of closed point of \( \text{Spec} \mathcal{O}_v \) for \( t \) and denote by \( s \).

We say that \((X, t)\) is a strict \( k \)-model of \( \mathcal{O}_v \) iff closure \( \{ s \} \) in \( X \) is a smooth scheme.

Any discrete valuation ring \( \mathcal{O}_v \) essentially of finite type over \( k \) admits a strict \( k \)-model. (c.f. [MotGe] Lemma 4.5.3).

5.9. Let \( E/k \) be a field extension of finite type, \( v \) a valuation of \( E/k \), \( \mathcal{O}_v \) a valuation ring of \( v \) and \((X, t)\) a strict \( k \)-model of \( \mathcal{O}_v \). Put \( Z := \{ s \} \). Since \((X, Z)\) is a smooth closed pair of codimension 1. We have a distinguished triangle of the form
\[
M_{gm}(Z)\{1\} \xrightarrow{\delta_{X,Z}^*} M_{gm}(X - Z) \xrightarrow{\iota_1} M_{gm}(X) \xrightarrow{\iota_1}.
\]

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Since this triangle is natural for inclusions of open sets in $X$. (c.f. 3.22) Considering a cofiltrant system of open neighborhoods of $s$ in $X$, we get a pro-distinguished triangle

$$M_{gm}(Z_s\{1\}) \xrightarrow{\partial_{X_s,Z_s}} M_{gm}(X_s - Z_s) \xrightarrow{j_{s}} M_{gm}(X_s) \xrightarrow{+1}.$$ 

Since $(X,s)$ is a $k$-model of $\mathcal{O}_v$, a morphism $s : \text{Spec} \mathcal{O}_v \rightarrow X$ induces an isomorphism of pro-object $(\mathcal{O}_v) \rightarrow X_s$; Hence we get a pro-distinguished triangle isomorphic to the form

$$M_{gm}(\text{Spec } k(v))\{1\} \xrightarrow{\partial_{X_s,Z_s}} M_{gm}(\text{Spec } E) \xrightarrow{i^\ast} M_{gm}(\text{Spec } \mathcal{O}_v) \xrightarrow{+1}$$

where $E$ (resp. $k(v)$) is a fraction field (resp. residue field) of $v$, $i : \mathcal{O}_v \rightarrow E$ is a canonical inclusion.

Lemma 5.10. (c.f. [MotGe] Lemma 4.5.5.) Let $E/k$ be a field extension of finite type, $v$ a discrete valuation of $E/k$, $\mathcal{O}_v$ a valuation ring of $v$ and $k(v)$ is this residue field. Then adopting the notation above, if $(X,s)$ and $(Y,t)$ are two strict $k$-models of $\mathcal{O}_v$, put $Z := \{s\}$ closure in $X$ and $T := \{t\}$ closure in $Y$. Then we have

$$\partial_{X_s,Z_s} = \partial_{Y_t,T_t}.$$ 

5.11. Let $E/k$ be a field extension of finite type, $v$ a discrete valuation of $E/k$. We define a residue morphism associated to $v$, denoted by $\partial_v$ defined by $\partial_v := \partial_{(X_s,Z_s)}$ where $(X,s)$ is a strict $k$-model of valuation ring of $v$. By Lemma 5.10 this does not depend on a choice of a strict $k$-model of valuation ring of $v$. So we have the following pro-distinguished triangle of the form

$$M_{gm}(\text{Spec } k(v))\{1\} \xrightarrow{\partial_v} M_{gm}(\text{Spec } E) \xrightarrow{i^\ast} M_{gm}(\text{Spec } \mathcal{O}_v) \xrightarrow{+1}.$$ 

Having defined residue morphisms, we explain the connection of Milnor $K$-groups and Hom sets in $\text{Pro-DM}_{gm}(k)$.

5.12. Using the distinguished triangle in 3.20 and the definition of tensor structure in $\text{DM}_{eff}^c(k)$, for any $n \in \mathbb{N}$, there is a distinguished triangle of the form

$$\bigoplus_{i=1}^n M_{gm}(\mathbb{G}_m^{n-1}) \rightarrow M_{gm}(\mathbb{G}_m^n) \rightarrow Z\{n\} \xrightarrow{+1}.$$ 

where the first morphism is induced from sum of $n$ closed immersions

$$\iota_i := \text{id} \times \text{id} \times \ldots \times \text{id} \times 1 \times \ldots \times \text{id} \times \text{id}.$$ 

5.13. Let $E/k$ be a field extension of finite type. $f : M_{gm}(\text{Spec } E) \rightarrow \mathcal{M}$ and $g : M_{gm}(\text{Spec } E) \rightarrow \mathcal{N}$ are morphisms in $\text{Pro-DM}_{gm}(k)$. Then we can extend the definition of external cup product $f \boxtimes g : M_{gm}(\text{Spec } E) \rightarrow \mathcal{M} \otimes \mathcal{N}$. 

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If $M = \mathbb{Z}\{p\}$ and $N = \mathbb{Z}\{q\}$, we have a canonical isomorphism $\mathbb{Z}\{p\} \otimes \mathbb{Z}\{q\} \xrightarrow{\sim} \mathbb{Z}\{p + q\}$. Then we have the following identity

$$\alpha \boxtimes \beta = -\beta \boxtimes \alpha.$$  

(c.f. [MotGe] Remarque 4.4.2.)

5.14. Let $E/k$ be a field extension of finite type. Then we have a morphism

$$(E^\times)^n \xrightarrow{\sim} \text{Hom}(\text{Spec } E, \mathbb{G}_m^n)$$

$$\text{Hom}(\text{Spec } E, \mathbb{G}_m^n) \xrightarrow{\text{Hom}_{\text{Pro-DM}}(k)(M_{\text{gm}}(\text{Spec } E), M_{\text{gm}}(\mathbb{G}_m^n)))} \text{Hom}_{\text{Pro-DM}}(k)(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}).$$

This map induces a morphism

$$\alpha : K^M_n(E) \to \text{Hom}_{\text{Pro-DM}}(k)(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}).$$

5.15. In the notation above, for any $x \in (E^\times)^{\otimes n}$ and $y \in (E^\times)^{\otimes m}$, we have

$$\alpha(x \otimes y) = \alpha(x) \boxtimes \alpha(y).$$

**Theorem 5.16.** (c.f. [MotGe] Theorem 4.4.4) (Need the assumption of perfection of $k$.)

In the notation above,

$$\alpha : K^M_n(E) \to \text{Hom}_{\text{Pro-DM}}(k)(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\})$$

is an algebra isomorphism.

5.2 Motivic reciprocity law

The classical theorems “Weil reciprocity law” and “residue formula” are unified using Milnor K-groups. More precisely, the following statement is known. (c.f. [Sus82])

5.17. (Reciprocity law for Milnor $K$-groups)

Let $K$ be an algebraic function field over a field $k$. Then the following composition are the zero maps for all non-negative integers $n$.

$$K^M_{n+1}(K) \oplus \bigoplus_v K^M_n(k(v)) \xrightarrow{\Sigma N_{k(v)/k}} K^M_n(k)$$

In this subsection, we will prove more fundamental style of the following reciprocity law.

**Theorem 5.18.** (Motivic reciprocity law)

The following composition

$$M_{\text{gm}}(\text{Spec } k)^{\{1\}} \xrightarrow{\Sigma N_{k(e)/k}^{\{1\}}} \prod_v M_{\text{gm}}(\text{Spec } k(v))^{\{1\}} \xrightarrow{\prod \partial_v} M_{\text{gm}}(\text{Spec } K)$$

is the zero map in Pro-DM$_{\text{gm}}(k)$.  

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5.19. Let $K/k$ be a field extension of transcendental degree one. Let $C/k$ be a projective nonsingular curve such that $K(C) = K$. As in the previous subsection, we can construct the following pro-distinguished triangle in Pro-$\text{DM}_{\text{gm}}(k)$.

$$M_{\text{gm}}(\text{Spec } K) \to M_{\text{gm}}(C) \to \prod_{x \in C: \text{closed points}} M_{\text{gm}}(\text{Spec } k((x)))((1)) \to \prod_{x \in Z} \partial_{x} [1] M_{\text{gm}}(\text{Spec } K)[1]$$

This is constructed as follows: For any closed set $Z \subset C$, there is the Gysin triangle

$$M_{\text{gm}}(C - Z) \to M_{\text{gm}}(C) \to \bigoplus_{x \in Z} M_{\text{gm}}(\text{Spec } k((x)))((1)) \to \partial_{C,Z}[1] M_{\text{gm}}(C - Z)[1]$$

and we consider $M_{\text{gm}}(\text{Spec } K) = \{ M_{\text{gm}}(C - Z) \}_{Z \subset C: \text{closed subsets}} \in \text{Pro-DM}_{\text{gm}}(k)$

**Lemma 5.20.**
In the notation above, for any closed point $x \in C$, the diagram of structure morphisms

$$\text{Spec } k(x) \xrightarrow{i} C \xrightarrow{p} \text{Spec } k$$

induces the following commutative diagram:

$$M_{\text{gm}}(\text{Spec } k((x)))((1)) \xrightarrow{i^*} M_{\text{gm}}(C) \xrightarrow{p^*} M_{\text{gm}}(\text{Spec } k)((1)).$$

**Proof.** First choose a finite equidimensional morphism $C \xrightarrow{\pi} \mathbb{P}^1$ which is unramified at every points over $\pi(x)$. (This can be done by using Bertini theorem.) Next using 3.4 and Proposition 3.25, we may assume $C = \mathbb{P}^1$. Replacing $\mathbb{P}^1$ by $\mathbb{P}^1_{k(x)}$ and using Proposition 3.25 again, we may assume $k(x) = k$. In this case, $i^* \circ p^* = \text{id}$ by Lemma 3.20 and 3.26.

5.21. Hence we get the following diagram:

$$M_{\text{gm}}(\text{Spec } k)((1)) \xrightarrow{p^*} \bigoplus_{x \in Z} M_{\text{gm}}(\text{Spec } k((x)))((1)) \xrightarrow{\Sigma \partial_{x/k}[1]} M_{\text{gm}}(C - Z)[1].$$

Taking a limit with respect to $Z$, we get the motivic reciprocity law.
Next we prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor $K$-groups.

**Lemma 5.22.**

In the notation above, let $v$ be a valuation of $K/k$, $O_v$ a valuation ring of $v$, $\pi$ a uniformizer element of $O_v$. Then the following diagram is commutative.

\[
\begin{array}{c}
\text{M}_{\text{gm}}(\text{Spec } k(v))\{1\} \\ \downarrow \partial_v \\
\text{M}_{\text{gm}}(\text{Spec } O_v)\{1\}
\end{array}
\rightarrow
\begin{array}{c}
\text{M}_{\text{gm}}(\text{Spec } K) \\ \alpha(\pi) \xi \text{id}_{\text{M}_{\text{gm}}(\text{Spec } K)} \\
\text{M}_{\text{gm}}(\text{Spec } k(v))\{1\}
\end{array}.
\]

**Proof.** Take a strict $k$-model of $\text{Spec } O_v$. Denote it by $(X, s)$. If we take $X$ sufficiently small, $\pi \in O_v \rightarrow O_{X,s}$ determines a regular function $X \rightarrow A^1_k$ which parameterizes $Z$. Using Proposition 3.27 and considering a cofiltrant system of open neighborhoods of $s$ in $X$, we get the following commutative diagram of pro-motives:

\[
\begin{array}{c}
\text{M}_{\text{gm}}(Z_s)\{1\} \\ \downarrow \partial_{X_s,Z_s} \\
\text{M}_{\text{gm}}(X_s)\{1\}
\end{array}
\rightarrow
\begin{array}{c}
\text{M}_{\text{gm}}(X_s - Z_s) \\ \text{M}_{\text{gm}}(\pi) \circ \rho_1 \xi \text{id}_{\text{M}_{\text{gm}}(X_s - Z_s)} \\
\text{M}_{\text{gm}}(X_s - Z_s)\{1\}
\end{array}.
\]

Hence we get the result.

**Example 5.23.** In the notation above, for any discrete valuation $v$ of $K/k$, there is a commutative diagram

\[
\begin{array}{c}
\text{Hom}(\text{M}_{\text{gm}}(\text{Spec } k(v)\{1\}, Z\{n+1\}) \\ \downarrow \partial_v, Z\{n+1\}) \\
\text{Hom}(\text{M}_{\text{gm}}(\text{Spec } k(v)\{1\}, Z\{n+1\})
\end{array}
\rightarrow
\begin{array}{c}
K^M_{n+1}(K) \\ (-1)^n \partial_v \\
K^M_{n}(k(v)).
\end{array}
\]

**Proof.** For $u_1, \ldots, u_{n+1} \in O^*_v$ and a uniformizer $\pi$, it is enough to check the following two conditions.

1. $\text{Hom}(\partial_v, Z\{n+1\})(\alpha\{u_1, \ldots, u_{n+1}\}) = 0$
2. $\text{Hom}(\partial_v, Z\{n+1\})(\alpha\{u_1, \ldots, u_n, \pi\}) = (-1)^n \alpha(\{u_1, \ldots, u_n\})$

To prove 1: Notice that there is a pro-distinguished triangle as follows (c.f. 5.11):

\[
\begin{array}{c}
\text{M}_{\text{gm}}(\text{Spec } k(v))\{1\} \\ \partial_v \\
\text{M}_{\text{gm}}(\text{Spec } K) \\ \text{M}_{\text{gm}}(\text{Spec } k(v))\{1\}
\end{array} \rightarrow
\begin{array}{c}
\text{M}_{\text{gm}}(\text{Spec } K) \\ +1 \\
\text{M}_{\text{gm}}(\text{Spec } k(v))\{1\}.
\end{array}
\]

To prove 2: Anti-commutativity of $\xi$ (c.f. 5.13) and Lemma 5.22.

\[
\square
\]
Example 5.24. Let $K/k$ be a field extension of transcendental degree 1, $(C, x)$ be a pointed smooth curve and $v$ a place of $K/k$. There is a tame symbol $\partial_v : \text{Jac}(K_v) \otimes K_v^\times \to \text{Jac}(k(v))$. Then the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Jac}(K_v) \otimes K_v^\times & \to & \text{Hom}_{\text{DM}(k)}(\text{M}(\text{Spec } K_v), \mathbb{Z}(C, x)[1]) \\
\downarrow \partial_v & & \downarrow \otimes \\
\text{Jac}(k(v)) & \to & \text{Hom}_{\text{DM}(k)}(\text{M}(\text{Spec } k_v), \mathbb{Z}(C, x)(1)[2])
\end{array}
\]

This is proved in the same way as Example 5.23.

Corollary 5.25. The motivic reciprocity law implies the Weil reciprocity law for Milnor $K$-groups.

Proof. Take $\text{Hom}(\cdot, \mathbb{Z}\{n+1\})$ and use Theorem 2.22, Theorem 5.16 and notice Example 3.2 and Example 5.23. \hfill \Box

5.3 Main result

5.26. In this section, let $k$ be a perfect field which admits resolution of singularities and $(C_1, a_1), \ldots, (C_n, a_n)$ pointed projective smooth curves over $k$.

5.27. Let $p : Z \to \mathbb{A}^1_k$ be a finite surjective morphism and suppose that $Z$ is integral. Let $f_i \in \text{Hom}(Z, C_i)$ and

\[ p^{-1}(j) = \coprod n_i^j z_i^j \quad (j = 0, 1) \]

where $n_i^j$ are the multiplicities of points $z_i^j = \text{Spec } L_i^j$. Define:

\[ \phi_j = \Sigma n_i^j \{f_1, \ldots, f_n\}_{L_i^j/k} \]

then we have

\[ \phi_0 = \phi_1 \]

in $K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n)$.

The proof is similar to MVW02, p.45 Corollary 5.5.

5.28. As $\otimes_{i=1}^n \mathbb{Z}_{\text{tr}}(C_i, a_i)(\text{Spec } k))$ is a quotient of the free abelian groups generated by the closed points of $C_1 \times \ldots \times C_n$ modulo the subgroup generated by all points of the form $(x_1, \ldots, a_i, \ldots, x_n)$ where the $a_i$'s can be any position. If $x$ is a closed point of $C_1 \times \ldots \times C_n$ with residue field $L$ then $x$ is defined by a
canonical sequence \((x_1, \ldots, x_n) \in \text{Jac} C_1(L) \times \cdots \times \text{Jac} C_n(L)\).

Since
\[
H^n_{\Lambda}(k, \bigwedge^n C_i) = \text{Coker}(\bigotimes_{i=1}^n \text{tr}_C(C_i, a_i)(\mathbb{A}^1_k) \to \bigotimes_{i=1}^n \text{tr}_C(C_i, a_i)(\text{Spec} k))
\]
using 5.27 we have a natural map \(H^n_{\Lambda}(k, \bigwedge^n C_i) \to K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n)\).

5.29. Using 4.7 for every finite field extension \(L/k\) we have an isomorphism
\[
\bigotimes_{i=1}^n \text{Jac} C_i(L) \xrightarrow{\sim} \bigotimes_{i=1}^n H^1_{\Lambda}(L, C_i).
\]
Combining a natural pairing in 4.4
\[
\bigotimes_{i=1}^n H^1_{\Lambda}(L, C_i) \to H^n_{\Lambda}(L, \bigwedge^n C_i)
\]
and a norm map (cf 4.6)
\[
N_{L/k}: H^n_{\Lambda}(L, \bigwedge^n C_i) \to H^n_{\Lambda}(k, \bigwedge^n C_i)
\]
we get a canonical map
\[
\bigoplus_{L/k: \text{finite extension}} \bigotimes_{i=1}^n \text{Jac} C_i(L) \to H^n_{\Lambda}(k, \bigwedge^n C_i).
\]
If we use 4.6(1), Theorem 5.18 and Example 5.24 this map should factor through the map \(K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n) \to H^n_{\Lambda}(k, \bigwedge^n C_i)\).

5.30. Obviously the morphisms above are inverse to each other. Hence we get the following result.

**Theorem 5.31.** *(Somekawa conjecture for Jacobian varieties)*

Let \((C_1, a_1), \ldots, (C_n, a_n)\) be pointed projective smooth curves over perfect field \(k\) which admits resolution of singularities. Then
\[
K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n) \xrightarrow{\sim} \text{Hom}_{\text{DM-eff}}(k)(\text{M}(\text{Spec} k), \mathbb{Z}(\bigwedge^n C_i[n])).
\]

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**References**

[Akh00] Reza Akhtar, *Milnor K-theory and zero-cycles on algebraic varieties*, thesis.
[Akh02] Reza Akhtar, Milnor K-theory of smooth quasi projective varieties, preprint.

[BT73] H. Bass and J. Tate, The Milnor ring of global field, Springer Lecture Notes in Math. 342 (1973), p. 349-446.

[Blo76] S. Bloch, Some elementary theorems about algebraic cycles on abelian varieties, Inventions math. 37, (1976), p. 215-228.

[Blo81] S. Bloch, Algebraic $K$-theory and class field theory for arithmetic surface, Ann. of Math. 114, (1981), p. 229-266.

[IntMo] F. Dégilde, Interprétation motivique de la formule d’excès d’intersection, C. R. Acad. Sci. Paris, Ser. I 338 (2004) p. 41-46.

[MotGe] F. Dégilde, Motifs génériques, preprint.

[Del74] P. Deligne, Théorie de Hodge III, Publ. Math. I.H.E.S. 44 (1974), p.5-78.

[FV00] Eric M. Friedlander and V. Voevodsky, Bivariant cycle cohomology, in Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies, vol 143, Princeton University press, (2000), p. 138-187.

[Gil81] H. Gillet, Riemann-Roch theorems for higher algebraic $K$-theory, Adv. in Math. 40 (1981), no. 3, p. 203-289.

[Kah92] B. Kahn, Nullité de certains groupes attachés aux variétés semi-abéliennes sur corps fini, C. R. Acad. Sci. Paris Sér. I Math. 314 (1992), no. 13, p. 1039-1042.

[KS83] K. Kato and S. Saito, Unramified class field theory of arithmetic surfaces, Ann. of Math. 118 (1983), p. 241-275.

[KL81] N. Katz and S. Lang, Finiteness theorems in geometric class field theory, l’Enseignement Mathématique 27 (1981) p. 185-314.

[Lan59] S. Lang, Abelian variety, New York: Interscience-Wiley (1959)

[MVW02] C. Mazza, V. Voevodsky, C. Weibel Notes on Motivic Cohomology, preprint.

[Mil70] J. Milnor, Algebraic $K$ theory and quadratic forms, Inventions Math. (1970), p. 318-344.

[Moo69] C. Moore, Group extension of $p$-adic and adelic linear groups, Publ. Math. I.H.E.S. 35 (1969), p. 251-281.

[Som90] M. Somekawa, On Milnor $K$-groups attached at semi-Abelian varieties, $K$-theory, 4 (1990), p. 105-119.
[Sus82] A. Suslin, *Menicke symbols and their applications in the K-theory of fields*, Proceedings of a Conference held at Oberwolfach, June 16-20, 1980, Springer-Verlag, Berlin, 1982, p. 334-356.

[RelCy] A. Suslin and V. Voevodsky, *Relative cycles and Chow sheaves*, in *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol 143, Princeton University press, (2000), p. 10-86.

[BKcon] A. Suslin and V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The Arithmetic and Geometry of Algebraic Cycles, Nato ASI series C, vol. 548, Kluwer, (2000), p. 117-189.

[CohTh] V. Voevodsky, *Cohomological theory of presheaves with transfers*, in *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol 143, Princeton University press, (2000), p. 87-137.

[TriCa] V. Voevodsky, *Triangulated categories of motives over field*, in *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol 143, Princeton University press, (2000), p. 188-254.

[Voe02] V. Voevodsky, *Cancellation theorem*, preprint.

[Wei67] A. Weil, *Basic Number Theory*, Springer-Verlag (1967)

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