SEMICALSSICAL LIMIT FOR THE VARYING-MASS
SCHRÖDINGER EQUATION WITH RANDOM INHOMOGENEITIES

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Abstract.
The varying-mass Schrödinger equation (VMSE) has been successfully applied to model electronic properties of semiconductor heterostructures, for example, quantum dots and quantum wells. In this paper, we consider VMSE with small random heterogeneities, and derive radiative transfer equations for its solutions. The main tool is to systematically apply the Wigner transform in the semiclassical regime (the rescaled Planck constant \( \varepsilon \ll 1 \)), and then expand the resulted Wigner equation to proper orders of \( \varepsilon \). As a proof of concept, we numerically compute both VMSE and radiative transfer equations, and show that their solutions agree well.

AMS subject classifications. 81Q20, 35Q40

Key words. Varying-mass Schrödinger equation, random inhomogeneities, semiclassical limit, radiative transfer

1. Introduction. The Schrödinger equation with varying mass has gained great attention in solid state physics, and been successfully used to study electronic properties of semiconductor heterostructures [11, 24, 27, 29]. For example, it can describe localized defects in crystalline media, which may yield bound states localized to the defect. It is also related to describe non-compact line defect (edge) perturbations, e.g., by interpolating two-dimensional honeycomb structures via domain walls [12, 13]. Edge modes may be produced by such perturbations, which propagate in the direction parallel to the edge and localize transverse to the edge.

We are interested in deriving the asymptotic limit of the following varying-mass Schrödinger equation

\[
i \varepsilon \partial_t u^\varepsilon(t, x) + \frac{1}{2} \varepsilon^2 \nabla_x \cdot (m(t, x) \nabla_x u^\varepsilon(t, x)) = 0, \tag{1.1}
\]

where \( t > 0, x \in \mathbb{R}^d \) with \( d \geq 1 \) and \( \varepsilon \ll 1 \) is the rescale Planck constant. The varying mass \( m_0 \), which can be time-dependent, is assume to be random and highly oscillatory, with a given covariance matrix in time and space. We shall assume that \( u^\varepsilon \) decays fast enough at infinity to validate all the derivations. One goal of the paper is to show that in the \( \varepsilon \to 0 \) regime, the Wigner transform of the solution converges to a special radiative transfer equation.

The problem is motivated by a fact that simulating (1.1) is extremely challenging in the semi-classical regime (\( \varepsilon \ll 1 \)). The challenges are two-folded. In deterministic regime (meaning \( m(t, x) \) is a deterministic highly oscillatory function in \( (t, x) \)), standard numerical solvers require to the small wavelength of \( u^\varepsilon(t, x) \) to be resolved, for example, a mesh size and time step of order \( o(\varepsilon) \) is required when finite difference methods are used [22, 23]. The time-splitting spectral method [26] can improve the mesh size to be of order \( O(\varepsilon) \), however, has limitations to compute the Schrödinger

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equation with varying mass. A bigger problem comes from the randomness in $m$. Since only the covariance of $m$ is given, numerically one has to find many realizations and compute the deterministic Schrödinger equation before finding the ensemble mean/variance of the solution. The number of realizations, however, increases as $\varepsilon \to 0$, as details in the random fluctuation become more and more important.

Disregarding the challenges from the stochasticity, merely for the deterministic system, alternative approaches have been developed. These included the WKB-type methods, e.g., Gaussian beam methods [17,18] and frozen Gaussian approximation [15,19,20]. The idea is to apply the WKB-type ansatz

$$u^\varepsilon(t, x) = A(t, x) \exp\left(\frac{i S(t, x)}{\varepsilon}\right),$$

and derive the eikonal equation for $S(t, x)$ and transport-like equation for $A(t, x)$, where both $S(t, x)$ and $A(t, x)$ are functions of large scale, i.e., $\varepsilon$-independent. To our best knowledge, no such types of methods have been applied to efficiently solve (2.1) in the literature yet. And even for standard Schrödinger equation with random potential term, the application of the methods have not been fully understood.

In the paper, we shall systematically derive asymptotic equations for (2.1) by the Wigner transform [14], which is a main tool in semiclassical theory parallel to the WKB-type methods mentioned above. The literature on deriving the asymptotic equations for wave propagation in random media [1–3,7–10,21,26] is rich, most of which started with the Schrödinger equation with constant mass, and the randomness and high oscillations are introduced through the potential term. When it is the effective mass term that is random and highly oscillatory, the process of the derivation is rather similar but much more delicate, as will be detailed later in our paper. As a proof of concept, we numerically verify the derived radiative transfer equation by carefully computing and comparing its solution to the one of VMSE (1.1), and show that the two solutions agree.

The rest of the paper is organized as follows. To better illustrate the derivation, we start with a simpler case with $m_0$ being deterministic and only spatially dependent, and derive the limiting radiative transfer equation by the Wigner transform in Section 2. In Section 3, we systematically introduce the derivation of the limiting equation for the varying-mass Schrödinger equation (1.1) with random heterogeneities. We present our numerical validation in Section 4 and make conclusive remarks in Section 5.

2. Wigner transform of VMSE in the deterministic setting. As a preparation, we first investigate the semi-classical limit for (1.1) with deterministic mass in this section, which is to consider:

$$i\varepsilon \partial_t u^\varepsilon(t, x) + \frac{1}{2}\varepsilon^2 \nabla_x \cdot (m_0(x) \nabla_x u^\varepsilon(t, x)) = 0.$$  

(2.1)

The varying mass $m_0$ is a real function of $x$. It is assumed to be deterministic and time-independent. We study the Cauchy problem, i.e., the $u^\varepsilon$ is assumed to decay at infinite. $u^\varepsilon$ is a complex function and we typically care about its physical observables such as the energy density $\rho^\varepsilon$ and the energy flux $J^\varepsilon$, defined respectively as

$$\rho^\varepsilon(t, x) = |u^\varepsilon(t, x)|^2, \quad J^\varepsilon(t, x) = \varepsilon \text{Im}(\overline{u^\varepsilon(t, x)} \nabla_x u^\varepsilon(t, x)).$$

Wigner transform is a technique explored in [25] for the Schrödinger equation with random potential, and has been demonstrated as a very powerful tool for investigating
the semi-classical limit. It defines a function on the phase space:

\[
W^\varepsilon(t, x, k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iky} u^\varepsilon \left( t, x - \frac{\varepsilon}{2} y \right) \overline{u^\varepsilon} \left( t, x + \frac{\varepsilon}{2} y \right) dy. \tag{2.2}
\]

Here \(\overline{u^\varepsilon}\) is the complex conjugate of \(u^\varepsilon\). This definition is essentially the Fourier transform of

\[
\langle x - \frac{\varepsilon}{2} y | u \rangle \langle u | x + \frac{\varepsilon}{2} y \rangle
\]
on the \(y\) variable.

The Wigner transform loses phase information: if \(u^\varepsilon\) is perturbed to \(u^\varepsilon e^{iS}\), the Wigner transform is kept the same. However, the physical observables can be recovered, namely, the first and second moments of \(W^\varepsilon\) provide the energy density and the energy flux:

\[
\int_{\mathbb{R}^d} W^\varepsilon(t, x, k) dk = \rho^\varepsilon(t, x), \quad \int_{\mathbb{R}^d} k W^\varepsilon(t, x, k) dk = J^\varepsilon(t, x). \tag{2.3}
\]

It is not guaranteed that \(W^\varepsilon\) is positive, and thus it does not serve directly as the energy density on the phase space. By plugging in the Schrödinger equation, we derive the equation satisfied by \(W^\varepsilon\) in the following Lemma.

**Lemma 2.1.** Let \(u^\varepsilon\) satisfy the VMSE \([2.1]\), then its Wigner transform \([2.2]\) satisfies:

\[
\partial_t W^\varepsilon + \frac{1}{\varepsilon} Q_1^\varepsilon W^\varepsilon + Q_2^\varepsilon W^\varepsilon = \varepsilon Q_3^\varepsilon W^\varepsilon, \tag{2.4}
\]

where the three operators are defined to be:

\[
Q_1^\varepsilon W^\varepsilon = \frac{|k|^2}{2} \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \tilde{m}_1(t, p) i \left[ W^\varepsilon \left( t, x, k - \frac{\varepsilon}{2} p \right) - W^\varepsilon \left( t, x, k + \frac{\varepsilon}{2} p \right) \right] dp, \tag{2.5}
\]

\[
Q_2^\varepsilon W^\varepsilon = \frac{k}{2} \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \tilde{m}_1(t, p) \left[ \nabla_x W^\varepsilon \left( t, x, k - \frac{\varepsilon}{2} p \right) + \nabla_x W^\varepsilon \left( t, x, k + \frac{\varepsilon}{2} p \right) \right] dp, \tag{2.6}
\]

\[
Q_3^\varepsilon W^\varepsilon = \frac{1}{8} \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \tilde{m}_1(t, p) i \left[ \Delta_x W^\varepsilon \left( t, x, k - \frac{\varepsilon}{2} p \right) - \Delta_x W^\varepsilon \left( t, x, k + \frac{\varepsilon}{2} p \right) \right] dp + \frac{1}{8} \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \tilde{m}_1(t, p) |p|^2 \left[ W^\varepsilon \left( t, x, k - \frac{\varepsilon}{2} p \right) - W^\varepsilon \left( t, x, k + \frac{\varepsilon}{2} p \right) \right] dp. \tag{2.7}
\]

where the Fourier transform of \(m_0\) in space is defined by

\[
\tilde{m}_0(t, p) = \int_{\mathbb{R}^d} e^{-ipz} m_0(t, z) dz. \tag{2.8}
\]

**Proof.** The proof is direct derivation. Notice that

\[
\partial_t W^\varepsilon = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iky} \partial_t u^\varepsilon \overline{u^\varepsilon} dy + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iky} \partial_t \overline{u^\varepsilon} \overline{u^\varepsilon} dy, \tag{2.9}
\]
we have, plugging in (2.1):
\[
\partial_t W^\varepsilon = \frac{i\varepsilon}{2(2\pi)^d} \int_{\mathbb{R}^d} e^{ik_y \cdot \nabla_x} \cdot \left(m_0 \left(x - \frac{\varepsilon}{2} y\right) \nabla_x u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right)\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right) dy
\]
\[
- \frac{i\varepsilon}{2(2\pi)^d} \int_{\mathbb{R}^d} e^{ik_y \cdot \nabla_x} \cdot \left(m_0 \left(x + \frac{\varepsilon}{2} y\right) \nabla_x \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right) u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) dy
\]
\[
:= \frac{i\varepsilon}{2(2\pi)^d} M_1 - \frac{i\varepsilon}{2(2\pi)^d} M_2.
\]
(2.10)

Since the two terms \(M_1\) and \(M_2\) are conjugate with \(y \to -y\) for the second term, we only study the first one. With integration by parts:
\[
M_1 = \frac{2}{\varepsilon} \int_{\mathbb{R}^d} \left[\nabla_y (e^{ik_y}) \cdot \nabla_x u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right)\right] m_0 \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right) dy
\]
\[
+ \frac{2}{\varepsilon} \int_{\mathbb{R}^d} e^{ik_y} m_0 \left(x - \frac{\varepsilon}{2} y\right) \nabla_x u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \cdot \nabla_y \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right) dy
\]
(2.11)
\[
:= I_1 + I_2.
\]

We treat the \(I_1\) and \(I_2\) respectively in the following. Perform integration by parts again to \(I_1\)
\[
I_1 = \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \Delta_y (e^{ik_y}) m_0 \left(x - \frac{\varepsilon}{2} y\right) \left[u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] dy
\]
\[
+ \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \nabla_y (e^{ik_y}) \cdot \nabla_y m_0 \left(x - \frac{\varepsilon}{2} y\right) \left[u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] dy
\]
\[
+ \frac{2}{\varepsilon} \int_{\mathbb{R}^d} \left[\nabla_y (e^{ik_y}) \cdot \nabla_x u^\varepsilon \left(x + \frac{\varepsilon}{2} y\right)\right] m_0 \left(x - \frac{\varepsilon}{2} y\right) u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) dy
\]
\[
:= I_{11} + I_{12} + I_{13}.
\]

Note that \(I_1\) and the last term \(I_{13}\) can be combined so that a complete \(x\)-gradient of \(u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\) is available, namely one arrives at a formula for \(I_1\)
\[
I_1 = \frac{1}{\varepsilon^2} I_1 + \frac{1}{2} (I_{11} + I_{12} + I_{13}) = \frac{1}{2} (I_1 + I_{13}) + \frac{1}{2} (I_{11} + I_{12})
\]
\[
= \frac{2}{\varepsilon^2} \int_{\mathbb{R}^d} \Delta_y (e^{ik_y}) m_0 \left(x - \frac{\varepsilon}{2} y\right) \left[u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] dy
\]
\[
+ \frac{2}{\varepsilon^2} \int_{\mathbb{R}^d} \nabla_y (e^{ik_y}) \cdot \nabla_y m_0 \left(x - \frac{\varepsilon}{2} y\right) \left[u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] dy
\]
\[
+ \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \nabla_y (e^{ik_y}) \cdot \nabla_x \left[u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] m_0 \left(x - \frac{\varepsilon}{2} y\right) dy.
\]
(2.13)

For \(I_2\) in (2.11), integration by parts against \(\nabla_x u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right)\) produces
\[
I_2 = \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \left[\nabla_y (e^{ik_y}) \cdot \nabla_y \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] m_0 \left(x - \frac{\varepsilon}{2} y\right) u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) dy
\]
\[
+ \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} e^{ik_y} \left[\nabla_y m_0 \left(x - \frac{\varepsilon}{2} y\right) \cdot \nabla_y \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right)\right] u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) dy
\]
\[
+ \int_{\mathbb{R}^d} e^{ik_y} m_0 \left(x - \frac{\varepsilon}{2} y\right) u^\varepsilon \left(x - \frac{\varepsilon}{2} y\right) \Delta_x \overline{u^\varepsilon} \left(x + \frac{\varepsilon}{2} y\right) dy := I_{21} + I_{22} + I_{23}.
\]
(2.14)
Hence using (2.14)-(2.17) and the trick in (2.13), one derives the formula for
\[ I_2 = \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \left[ \nabla_y \bar{u}^{\varepsilon} \cdot \nabla_y u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \right] m_0 \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \, dy \]
\[ + \frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} e^{iky} \left[ \nabla_y m_0 \left( x - \frac{\varepsilon}{2} y \right) \cdot \nabla_y u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \right] \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \, dy \]
\[ + \int_{\mathbb{R}^d} e^{iky} m_0 \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \Delta_x u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \, dy \quad \text{for} \quad I_{21} + I_{22} + I_{23}. \]

Note that \( I_{21} \) and \( I_{21}' \) can be combined after another integration by parts
\[ I_{21} + I_{21}' = -\frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \nabla_y \left( e^{iky} \cdot \nabla_y m_0 \right) \left( x - \frac{\varepsilon}{2} y \right) \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ -\frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} m_0 \left( x - \frac{\varepsilon}{2} y \right) \nabla_y \left( e^{iky} \cdot \nabla_y \bar{u}^{\varepsilon} \right) \left( x + \frac{\varepsilon}{2} y \right) \, dy. \]

\( I_{22} \) and \( I_{22}' \) can be combined similarly
\[ I_{22} + I_{22}' = -\frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} \nabla_y \left( e^{iky} \cdot \nabla_y m_0 \right) \left( x - \frac{\varepsilon}{2} y \right) \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ -\frac{4}{\varepsilon^2} \int_{\mathbb{R}^d} m_0 \left( x - \frac{\varepsilon}{2} y \right) \nabla_y \left( e^{iky} \cdot \nabla_y \bar{u}^{\varepsilon} \right) \left( x + \frac{\varepsilon}{2} y \right) \, dy. \]

Hence using (2.14)-(2.17) and the trick in (2.13), one derives the formula for \( I_2 \) in (2.11)
\[ I_2 = \frac{1}{4} (I_{21} + I_{22} + I_{23} + I_{24}) + \frac{1}{2} I_2 + \frac{1}{4} (I_{21}' + I_{22}') + I_{23} \]
\[ = \left( \frac{1}{4} I_{23} + \frac{1}{2} I_2 + \frac{1}{4} I_{23}' \right) + \frac{1}{4} (I_{21} + I_{21}') + \frac{1}{4} (I_{22} + I_{22}') \]
\[ = \frac{1}{4} \int_{\mathbb{R}^d} e^{iky} m_0 \left( x - \frac{\varepsilon}{2} y \right) \Delta_x \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ -\frac{2}{\varepsilon^2} \int_{\mathbb{R}^d} \nabla_y \left( e^{iky} \cdot \nabla_y m_0 \right) \left( x - \frac{\varepsilon}{2} y \right) \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} m_0 \left( x - \frac{\varepsilon}{2} y \right) \nabla_y \left( e^{iky} \cdot \nabla_y \bar{u}^{\varepsilon} \right) \left( x + \frac{\varepsilon}{2} y \right) \, dy \]
\[ -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} e^{iky} \Delta_y m_0 \left( x - \frac{\varepsilon}{2} y \right) \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy. \]

Finally from (2.13) and (2.18), one gets
\[ M_1 = \frac{1}{4} \int_{\mathbb{R}^d} e^{iky} m_0 \left( x - \frac{\varepsilon}{2} y \right) \Delta_x \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ +\frac{1}{\varepsilon} \int_{\mathbb{R}^d} m_0 \left( x - \frac{\varepsilon}{2} y \right) \nabla_y \left( e^{iky} \cdot \nabla_x u^{\varepsilon} \right) \left( x + \frac{\varepsilon}{2} y \right) \, dy \]
\[ +\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} m_0 \left( x - \frac{\varepsilon}{2} y \right) \Delta_y \left( e^{iky} \cdot \nabla_x \bar{u}^{\varepsilon} \right) \left( x + \frac{\varepsilon}{2} y \right) \, dy \]
\[ -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} e^{iky} \Delta_y m_0 \left( x - \frac{\varepsilon}{2} y \right) \left[ u^{\varepsilon} \left( x - \frac{\varepsilon}{2} y \right) \bar{u}^{\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] \, dy \]
\[ := T_1 + T_2 + T_3 + T_4. \]
All the $T_i$ terms can be explicitly expressed by the Wigner transform (2.2). In particular:

$$
T_1 = \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) \Delta_x W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp ,
$$

$$
T_2 = \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) i k \cdot \nabla_x W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp ,
$$

$$
T_3 = \int_{\mathbb{R}^d} -|k|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp ,
$$

$$
T_4 = \varepsilon^2 \int_{\mathbb{R}^d} |p|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp .
$$

We use $T_1$ as an example to show this. Recalling:

$$
\Delta_x W^\varepsilon(x, k) = \frac{1}{(2\pi)^d} \int e^{iky} \Delta_x \left[ u^\varepsilon \left( x - \frac{\varepsilon}{2} y \right) \overline{u^\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] dy ,
$$

we have

$$
\int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) \Delta_x W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp
$$

$$
= \frac{1}{(2\pi)^d} \int \int \int e^{ipx} e^{-ipz} m_0(z) e^{i(k - \frac{\varepsilon}{2} p)y} \Delta_x \left[ u^\varepsilon \left( x - \frac{\varepsilon}{2} y \right) \overline{u^\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] dz dp dy ,
$$

$$
= \int_{\mathbb{R}^d} e^{iky} m_0 \left( x - \frac{\varepsilon}{2} y \right) \Delta_x \left[ u^\varepsilon \left( x - \frac{\varepsilon}{2} y \right) \overline{u^\varepsilon} \left( x + \frac{\varepsilon}{2} y \right) \right] dy = \frac{1}{4} T_1 ,
$$

where we used the fact that

$$
\delta(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ixz} dz , \quad \text{and} \quad \frac{1}{(2\pi)^d} \int \int f(x) e^{ixz} dz dx = f(0) .
$$

Using (2.20), we get

$$
M_1 = \frac{1}{4} \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) \Delta_x W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp 
$$

$$
+ \frac{1}{\varepsilon} \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) ik \cdot \nabla_x W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp 
$$

$$
+ \int_{\mathbb{R}^d} |p|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp 
$$

$$
- \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |k|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k - \frac{\varepsilon}{2} p \right) dp .
$$

By the conjugate argument, one gets, setting $p \rightarrow -p$:

$$
M_2 = \frac{1}{4} \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) \Delta_x W^\varepsilon \left( x, k + \frac{\varepsilon}{2} p \right) dp 
$$

$$
- \frac{1}{\varepsilon} \int_{\mathbb{R}^d} e^{ipx} \tilde{m}_0(p) ik \cdot \nabla_x W^\varepsilon \left( x, k + \frac{\varepsilon}{2} p \right) dp 
$$

$$
+ \int_{\mathbb{R}^d} |p|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k + \frac{\varepsilon}{2} p \right) dp 
$$

$$
- \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |k|^2 e^{ipx} \tilde{m}_0(p) W^\varepsilon \left( x, k + \frac{\varepsilon}{2} p \right) dp .
$$
Finally, substitute (2.23) and (2.24) into (2.10), and we arrive at the Wigner equation in (2.4).

**Proposition 2.2 (Formal).** With certain regularity, the formal $\varepsilon \to 0$ limit of (2.4) is a transport equation:

$$
\partial_t W + m_0(x)k \cdot \nabla_x W - \frac{|k|^2}{2} \nabla_x m_0(x) \cdot \nabla_k W + O(\varepsilon^2) = 0.
$$

(2.25)

The trajectory of particles follows:

$$
\dot{x} = k m_0(t, x), \quad \dot{k} = -\frac{|k|^2}{2} \nabla_x m_0(t, x),
$$

(2.26)

and

$$
|k|^2 m_0(x) = \text{const}.
$$

(2.27)

**Proof.** To prove the limiting equation (2.25), one only needs to derive the limiting behavior of $Q_i W^\varepsilon$. Indeed, formally, assuming $W^\varepsilon \in C^2$:

$$
\frac{1}{\varepsilon} \frac{Q_1}{2} W^\varepsilon = -\frac{|k|^2}{2} \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \bar{m}_1(t, p) i p \cdot \nabla_k W^\varepsilon(t, x, k) dp + O(\varepsilon^2) = -\frac{|k|^2}{2} \nabla_x m_0 \nabla_k W^\varepsilon + O(\varepsilon^2),
$$

and that

$$
Q_2 W^\varepsilon = k \cdot \int_{\mathbb{R}^d} \frac{e^{ipx}}{(2\pi)^d} \bar{m}_1(t, p) \nabla_x W^\varepsilon(t, x, k) dp + O(\varepsilon^2) = m_0 k \cdot \nabla_x W^\varepsilon + O(\varepsilon^2).
$$

One then arrives at (2.25) by plugging them in the original equation (2.4). To show (2.27) with $m_0 = m_0(x)$, one simply takes the time derivative along the trajectory:

$$
\frac{d}{dt} [\frac{1}{2} |k(t)|^2 m_0(x(t))] = 2 k \cdot \dot{k} m_0 + |k|^2 \nabla_x m_0 \cdot \dot{x} = 0,
$$

according to the trajectory equation (2.26), and thus the quantity $|k|^2 m_0$ is conserved along the trajectory.

This proposition essentially guarantees the positivity of the solution to the limiting equation (2.25).

### 3. Semi-classical limit for VMSE with random perturbation.

We consider the VMSE where the effective mass involves random perturbation, namely:

$$
\frac{i}{\varepsilon} \partial_t u^\varepsilon + \frac{1}{2} \varepsilon^2 \nabla_x \cdot (m^\varepsilon(t, x) \nabla_x u^\varepsilon) = 0,
$$

(3.1)

where the effective mass is

$$
m^\varepsilon(t, x) = m_0(t, x) + \sqrt{\varepsilon} m_1(t/\varepsilon, x/\varepsilon).
$$

(3.2)

While the leading order $m_0$ is assumed to be deterministic and smooth, we allow the random perturbation $m_1(t, x)$ to present small scales at $\varepsilon$. Furthermore we assume it is mean-zero and stationary in both $t$ and $x$ with the correlation function $R(t, x)$:

$$
R(t, x) = \mathbb{E}[m_1(s, z) m_1(t + s, x + z)] \quad \forall x, z \in \mathbb{R}^d \quad \text{and} \quad t, s \in \mathbb{R}.
$$

(3.3)
Taking the Fourier transform of the function in both time and space, one has:

\[
\hat{R}(\omega, p) = \int_{\mathbb{R}^d+1} e^{-i\omega s - ipz} R(s, z) ds dz,
\]

then it is straightforward to show:

\[
\mathbb{E}[\hat{m}_1(\tau, p) \hat{m}_1(\omega, q)] = (2\pi)^d e^{-i\omega \tau} \hat{R}(\omega, p) \delta(p + q),
\]

and

\[
\hat{R}(-\omega, p) = \hat{R}(\omega, p), \quad \text{and} \quad \hat{R}(\omega, -p) = \hat{R}(\omega, p).
\]

We dedicate this section to the derivation of the semi-classical limit of the equation above. We will show that

**Theorem 3.1.** In the zero limit of \(\varepsilon\), the Wigner transform of \(u^\varepsilon\), the solution to the VMSE \((2.1)\) with varying random mass \((3.2)\), solves the radiative transfer equation:

\[
\partial_t W + m_0 k \cdot \nabla_x W - \frac{k^2}{2} \nabla_x m_0 \cdot \nabla_k W = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{4} (p \cdot k)^2 \hat{R} \left( \frac{m_0}{2} (p^2 - k^2), p - k \right) [W(p) - W(k)] dp.
\]

**Proof.** In view of \((2.4)\) in Lemma 2.1 noting that \(m_0 \rightarrow m_0 + \sqrt{\varepsilon} m_1\), the Wigner equation \((3.1)\) is transformed to

\[
\partial_t W^\varepsilon + \frac{1}{\varepsilon} Q_1^\varepsilon W^\varepsilon + Q_2^\varepsilon W^\varepsilon + \frac{1}{\sqrt{\varepsilon}} P_1^\varepsilon W^\varepsilon + \sqrt{\varepsilon} P_2^\varepsilon W^\varepsilon = \varepsilon Q_3^\varepsilon W^\varepsilon + \frac{1}{\sqrt{\varepsilon}} P_3^\varepsilon W^\varepsilon,
\]

where the operators \(Q_i^\varepsilon\) are defined in \((2.5)-(2.7)\), and \(P_i^\varepsilon\) are their counterparts defined by \(m_1\):

\[
P_1^\varepsilon W^\varepsilon = \frac{|k|^2}{2} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{\hat{m}}_0(\tau, p) i \left[ W^\varepsilon \left( k - \frac{1}{2} p \right) - W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp,
\]

\[
P_2^\varepsilon W^\varepsilon = \frac{k}{2} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{\hat{m}}_0(\tau, p) \left[ \nabla_x W^\varepsilon \left( k - \frac{1}{2} p \right) + \nabla_x W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp,
\]

and

\[
P_3^\varepsilon W^\varepsilon = \frac{\varepsilon^2}{8} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{\hat{m}}_0(\tau, p) i \left[ \Delta_x W^\varepsilon \left( k - \frac{1}{2} p \right) - \Delta_x W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp + \frac{1}{8} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{\hat{m}}_0(\tau, p) |p|^2 \left[ W^\varepsilon \left( k - \frac{1}{2} p \right) - W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp,
\]

where we use the fast variables

\[
\tau = \frac{t}{\varepsilon}, \quad \xi = \frac{x}{\varepsilon}.
\]
Explicitly spelling out the fast variables in $W$, one has:

$$W(t, x, k) \rightarrow W(t, \tau, x, \xi, k), \quad \nabla_x \rightarrow \nabla_x + \frac{1}{\varepsilon} \nabla_\xi, \quad \partial_t \rightarrow \partial_t + \frac{1}{\varepsilon} \partial_\tau,$$

and thus the leading orders in (3.7) become:

$$\frac{1}{\varepsilon} Q^1 \varepsilon W = -\frac{|k|^2}{2} \nabla_x m_0 \cdot \nabla_k W + O(\varepsilon^2),$$

$$Q^2 \varepsilon W = \frac{1}{\varepsilon} m_0 k \cdot \nabla_\xi W + m_0 k \cdot \nabla_x W + O(\varepsilon),$$

$$\varepsilon Q^3 \varepsilon W = -\frac{1}{8} \nabla_x m_0 \cdot \nabla_k (\Delta_\xi W) + O(\varepsilon),$$

and

$$\frac{1}{\sqrt{\varepsilon}} P^1 \varepsilon W = \frac{1}{\sqrt{\varepsilon}} \left( \frac{|k|^2}{2} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{m}_1(\tau, p) i \left[ W^\varepsilon \left( k - \frac{1}{2} p \right) - W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp, \right.$$ 

$$\sqrt{\varepsilon} P^2 \varepsilon W = \frac{1}{\sqrt{\varepsilon}} \left( \frac{1}{2} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{m}_1(\tau, p) \left[ \nabla_\xi W^\varepsilon \left( k - \frac{1}{2} p \right) + \nabla_\xi W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp + O(\sqrt{\varepsilon}), \right.$$ 

$$\frac{1}{\sqrt{\varepsilon}} P^3 \varepsilon W = \frac{1}{\sqrt{\varepsilon}} \left( \frac{1}{8} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{m}_1(\tau, p) \left[ \Delta_\xi W^\varepsilon \left( k - \frac{1}{2} p \right) - \Delta_\xi W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp \right.$$

$$+ \frac{1}{8} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{m}_1(\tau, p) |p|^2 \left[ W^\varepsilon \left( k - \frac{1}{2} p \right) - W^\varepsilon \left( k + \frac{1}{2} p \right) \right] dp + O(\sqrt{\varepsilon}).$$

To perform the asymptotic expansion of the equation, we first write the ansatz

$$W^\varepsilon(t, x, k) = W^{(0)} + \sqrt{\varepsilon} W^{(1)} + \varepsilon W^{(2)} + \cdots. \quad (3.11)$$

By plugging the expansion above into (3.7), we have, at the order of $O(1/\varepsilon)$:

$$\partial_{\tau} W^{(0)} + m_0 k \cdot \nabla_\xi W^{(0)} = 0, \quad (3.12)$$

which suggests $W^{(0)}$ having no dependence on $\tau$ and $\xi$, the fast variables. The next order is $O(1/\sqrt{\varepsilon})$, and the equation writes:

$$\partial_{\tau} W^{(1)} + m_0 k \cdot \nabla_\xi W^{(1)}$$

$$= \frac{1}{i} \int_{\mathbb{R}^d} \frac{e^{ip\xi}}{(2\pi)^d} \bar{m}_1(\tau, p) \left( \frac{|k|^2}{2} - \frac{|p|^2}{8} \right) \left[ W^{(0)} \left( k - \frac{1}{2} p \right) - W^{(0)} \left( k + \frac{1}{2} p \right) \right] dp. \quad (3.13)$$

Since the only $\tau$ dependence on the right hand side is in $\bar{m}_1(\tau, p)$, the equation can be solved explicitly using the Fourier transform:

$$i(2\pi)^{d+1} W^{(1)}(t, \tau, x, \xi, k)$$

$$= \int_{\mathbb{R}^d} e^{i\omega \tau} \bar{m}_1(\omega, p) (4|k|^2 - |p|^2) \frac{1}{8(i\omega + im_0k \cdot p + \theta)} \left[ W^{(0)} \left( k - \frac{p}{2} \right) - W^{(0)} \left( k + \frac{p}{2} \right) \right] dp d\omega, \quad (3.14)$$

where $\theta$ is a regularization parameter, to be sent to 0 in the end, and $\bar{m}_1(\omega, p)$ is the space-time Fourier transform of $m_1$:

$$\bar{m}_1(\omega, p) = \int_{\mathbb{R}} e^{-i\tau \omega} \bar{m}_1(\tau, p) d\tau. \quad (3.15)$$
The following order is $O(1)$ and is the order we use to close:

\[
\begin{align*}
\partial_t W^{(0)} + m_0 k \cdot \nabla_x W^{(0)} - \frac{k^2}{2} \nabla_x m_0 \cdot \nabla_k W^{(0)} \\
+ \partial_t W^{(2)} + m_0 k \cdot \nabla_x W^{(2)} \\
+ \frac{|k|^2}{2} \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) i \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \\
+ \frac{k}{2} \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) \left[ \nabla_\xi W^{(1)} \left( k - \frac{1}{2} p \right) + \nabla_\xi W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \\
= \frac{1}{8} \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) \left[ \nabla_\xi W^{(1)} \left( k - \frac{1}{2} p \right) - \nabla_\xi W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \\
+ \frac{1}{8} \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) i|p|^2 \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp.
\end{align*}
\] (3.16)

Noticing

\[
E[\partial_t W^{(2)} + m_0 k \cdot \nabla_x W^{(2)}] = 0,
\] (3.17)

we eliminate the dependence on $W^{(2)}$ in the equation and arrive at:

\[
\begin{align*}
\partial_t W^{(0)} + m_0 k \cdot \nabla_x W^{(0)} - \frac{k^2}{2} \nabla_x m_0 \cdot \nabla_k W^{(0)} \\
= -\frac{|k|^2}{2} E \left( \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) i \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right) \\
- \frac{k}{2} \cdot E \left( \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) \left[ \nabla_\xi W^{(1)} \left( k - \frac{1}{2} p \right) + \nabla_\xi W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right) \\
+ \frac{1}{8} E \left( \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) \left[ \nabla_\xi W^{(1)} \left( k - \frac{1}{2} p \right) - \nabla_\xi W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right) \\
+ \frac{1}{8} E \left( \int_{R^d} \frac{e^{ip\xi}}{(2\pi)^d} \tilde{m}_1(\tau, p) i|p|^2 \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right) \\
= \text{TermI} + \text{TermII} + \text{TermIII} + \text{TermIV}.
\end{align*}
\] (3.18)

Getting the simplified version of the equation and showing the radiative transfer equation limit amounts to analyzing the four terms respectively. Since they are quite similar, we only present the calculation of TermIII below. It essentially comes from
plugging in $W^{(1)}$ formula in [3.14] into it. For example, the first part of TermIII is:

$$
\begin{align*}
\mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i p \xi} \bar{m}_1 (\tau, p) i \Delta \xi W^{(1)} \left( k - \frac{1}{2} p \right) dp \right] \\
= \int_{\mathbb{R}^{2d+1}} -|q|^2 e^{i p \xi + i q \cdot \omega + \omega \tau} \mathbb{E} \left[ \bar{m}_1 (\tau, p) \bar{m}_1 (\omega, q) \right] 4 \frac{|k - \frac{1}{2} p|^2 - |q|^2}{8(i\omega + i m_0 (k - \frac{1}{2} p) \cdot q + \theta)} W^{(0)} \left( k - \frac{1}{2} p - \frac{1}{2} q \right) dq d\omega dp \\
- \int_{\mathbb{R}^{2d+1}} -|q|^2 e^{i p \xi + i q \cdot \omega + \omega \tau} \mathbb{E} \left[ \bar{m}_1 (\tau, p) \bar{m}_1 (\omega, q) \right] 4 \frac{|k - \frac{1}{2} p|^2 - |q|^2}{8(i\omega + i m_0 (k - \frac{1}{2} p) \cdot q + \theta)} W^{(0)} \left( k - \frac{1}{2} p + \frac{1}{2} q \right) dq d\omega dp \\
= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} -|p|^2 \bar{R} (\omega, p) \frac{4 |k - \frac{1}{2} p|^2 - |q|^2}{8(i\omega + i m_0 (k - \frac{1}{2} p) \cdot p + \theta)} W^{(0)} (k) dp \\
- \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} -|p|^2 \bar{R} (\omega, p) \frac{4 |k - \frac{1}{2} p|^2 - |q|^2}{8(i\omega + i m_0 (k - \frac{1}{2} p) \cdot p + \theta)} W^{(0)} (k) dp \\
= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} -|p - k|^2 \bar{R} (\omega, p - k) \frac{1}{2(i\omega + i m_0 (p^2 - k^2) + \theta)} [W^{(0)} (p) - W^{(0)} (k)] dp.
\end{align*}
$$

Similarly, the second half is

$$
\begin{align*}
\mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i p \xi} \bar{m}_1 (\tau, p) i \Delta \xi W^{(1)} \left( k + \frac{1}{2} p \right) dp \right] \\
= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} -|p - k|^2 \bar{R} (\omega, p - k) \frac{1}{2(i\omega + i m_0 (p^2 - k^2) + \theta)} [W^{(0)} (p) - W^{(0)} (k)] dp.
\end{align*}
$$

These combined gives the simplification to TermIII:

$$
\begin{align*}
\frac{1}{8} \mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i p \xi} \bar{m}_1 (\tau, p) i \Delta \xi W^{(1)} \left( k - \frac{1}{2} p \right) - \Delta \xi W^{(1)} \left( k + \frac{1}{2} p \right) dp \right] \\
= \frac{1}{8} \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} -|p - k|^2 (p \cdot k) \bar{R} (\omega, p - k) \frac{\theta}{(\omega - m_0 \theta (p^2 - k^2))^2 + \theta^2} [W^{(0)} (k) - W^{(0)} (p)] dp,
\end{align*}
$$

where we used

$$
\lim_{\theta \to 0} \frac{\theta}{x^2 + \theta^2} = \pi \delta(x).
$$

Other terms in [3.18] can be similarly treated. TermI becomes:

$$
- \frac{|k|^2}{2} \mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i p \xi} \bar{m}_1 (\tau, p) i \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right] \\
= - \frac{|k|^2}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d+1}} \frac{1}{2} (p \cdot k) \bar{R} \left( m_0 \theta (p^2 - k^2), p - k \right) [W^{(0)} (k) - W^{(0)} (p)] dp,
$$

TermII becomes:

$$
- \frac{k}{2} \cdot \mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i p \xi} \bar{m}_1 (\tau, p) \left[ \nabla \xi W^{(1)} \left( k - \frac{1}{2} p \right) + \nabla \xi W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp \right] \\
= - \frac{k}{2} \cdot \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d+1}} (p - k) \frac{1}{2} (p \cdot k) \bar{R} \left( m_0 \theta (p^2 - k^2), p - k \right) [W^{(0)} (k) - W^{(0)} (p)] dp.
$$
and Term IV becomes:
\[
\frac{1}{8} E \int_{\mathbb{R}^d} e^{i p \xi} \hat{m}_4(\tau, p) |p|^2 \left[ W^{(1)} \left( k - \frac{1}{2} p \right) - W^{(1)} \left( k + \frac{1}{2} p \right) \right] dp
\]
\[
= \frac{1}{8} \left( \frac{1}{2 \pi} \right)^d \int_{\mathbb{R}^d} |p - k|^2 \left( \frac{m_0}{2} |p|^2 - k^2 \right) \left( W^{(0)}(k) - W^{(0)}(p) \right).
\]
(3.22)

Inserting (3.19)-(3.22) into (3.18) and simplify, we have the leading order asymptotic limit of (3.7), concluding the proposition.

4. Numerical result. As a proof of concept, we provide some numerical evidences for Proposition 2.2 and Theorem 3.1, the two results respectively for \(m\) being completely deterministic and \(m\) having random fluctuations.

4.1. Verification of Proposition 2.2. We present numerical evidence for Proposition 2.2 in this subsection. According to the proposition, the physical observables of the solution to VMSE satisfy, in the leading order, the Liouville equation, with trajectory of particles following equation (2.26).

To compare the physical observables of VMSE and the Wigner limit, we evaluate the following two quantities:
\[
\rho^0 = \int W dk \quad \text{v.s.} \quad \rho^\varepsilon(t, x) = |u^\varepsilon(t, x)|^2.
\]
(4.1)

As a computational setup, we set \(\Omega = [0, L] \times [0, T]\), and choose the spatial mesh size \(\Delta x = L/M\) with \(M\) being an even integer and the time step \(\Delta t\). The grid points and the time step are denoted by \(x_j = j \Delta x, j = 0, 1, \cdots, M\), and \(t_n = n \Delta t, n = 0, 1, 2, \cdots\). The initial data for Schrödinger equation has a Gaussian form:
\[
u_0(x/\varepsilon, x) = \exp \left( -A(x - x_0)^2 + i \varepsilon p_0 x \right).
\]
(4.2)

The periodic boundary conditions are imposed
\[
u^\varepsilon(t, 0) = u^\varepsilon(t, L), \quad \partial_x u^\varepsilon(t, 0) = \partial_x u^\varepsilon(t, L).
\]
(4.3)

In computation we will set \(L\) to be large enough and the periodic boundary condition plays minimum role.

Correspondingly the transport equation (2.25) has the initial data:
\[
W_0(x, k) = \exp(-2A(x - x_0)^2) \delta(k - p_0).
\]
(4.4)

For Schrödinger equation (2.1), we use the standard Finite Difference method with the discretization resolved, namely \(\Delta x = O(\varepsilon)\) and \(\Delta t = o(\varepsilon)\). The Crank-Nicolson is applied in time, and spectral method is applied to treat spacial discretization \([5]\), namely: let \(U^{\varepsilon,n}_j\) be the approximation of \(u^\varepsilon(x_j, t_n)\), then
\[
\frac{U^{\varepsilon,n+1}_j - U^{\varepsilon,n}_j}{\Delta t} = i \varepsilon \left( D_x^s(m_0^{n+1/2} D_x^s U^{\varepsilon,n+1})|_{x_j} + D_x^s(m_0^{n+1/2} D_x^s U^{\varepsilon,n})|_{x_j} \right)
\]
(4.5)

with
\[
U^{\varepsilon,n+1}_0 = U^{\varepsilon,n+1}_M, \quad U^{\varepsilon,n+1}_1 = U^{\varepsilon,n+1}_{M+1}, \quad U^{\varepsilon,0}_j = u_0^\varepsilon(x_j), \quad \forall j.
\]
Here the super-index \( n \) is for time step, while the lower-index \( j \) is for the grid point. We sample \( M \) grid points in the domain \([0, L]\). The differential-operator \( D_x \) is computed through spectral method:

\[
D_x^s U|_{x=x_j} = \frac{1}{M} \sum_{l=-M/2}^{M/2-1} \sum_{l=-M/2}^{M/2-1} i\mu_l \hat{U}_l e^{i\mu_l x_j},
\]

(4.6)

with

\[
\hat{U}_l = \sum_{j=0}^{M-1} U_j e^{-i\mu_l x_j}, \quad l = \frac{M}{2}, \ldots, \frac{M}{2} - 1.
\]

(4.7)

To compute the deterministic Liouville equation, we use the particle method, that is to compute a large number of ODE systems:

\[
\begin{cases}
\dot{x} = -km_0(T-t, x) \\
\dot{k} = \frac{|k|^2}{2} \partial_x m_0(T-t, x),
\end{cases}
\]

(4.8)

\[0 \leq t \leq T, \, x(0) = x_0, k(0) = k_0.
\]

The final solution is \( W(T, x_0, k_0) = W_0(x(T), k(T)) \). The equations (4.8) for trajectory can be efficiently solved with typical ODE solvers. See also [28,31] for the discussions of the regularized delta function.

In the examples, we set \( L = 1 \) and \( T = 0.5 \). For the initial data (4.2) and (4.4), we take \( A = 2^7 \), \( p_0 = 1 \) and \( x_0 = 0.25 \), and set

\[m_0(t, x) = (1 + 0.2 \sin(2\pi x))(1 + 0.2 \cos(2\pi t)).\]

To compute Liouville equation, we set the spatial size \( \Delta x = 2^{-10} \) and the frequency step \( \Delta k = 2^{-10} \). The system (4.8) are computed using MATLAB adaptive ODE solver with a prescribed error accuracy \( 10^{-8} \). To compute VMSE, we set \( \varepsilon = 2^{-n} \) and we use the discretization:

\[
\Delta t = 2^{-1.2n-3}, \quad \Delta x = 2^{-n-2}
\]

(4.9)

that resolves the sales.

In Figure 4.1 we show the solution to the transport equation (2.25). In Figure 4.2 we compare \( \rho^0 \) and \( \rho^\varepsilon \) with different \( \varepsilon \).

The convergence to Liouville equation as \( \varepsilon \to 0 \) can be observed in Figure 4.2. Such convergence can also be quantified by the error:

\[
\text{Err}^\varepsilon(\rho) = \int_{\mathbb{R}} |\rho^0 - \rho^\varepsilon| dx.
\]

(4.10)

In Figure 4.3 we show the convergence of \( \text{Err}^\varepsilon(\rho) \) as a function of \( \varepsilon \). According to the plot, the error decays at a rate of \( O(\varepsilon^2) \).

4.2. Verification of Theorem 3.1 We show the numerical evidence to Theorem 3.1 here, namely, we will compute VMSE with small \( \varepsilon \) and random potential, and compare the numerical results, when taking expectation values, with that of the limiting radiative transfer equation.

As a set-up, we take the computational domain to be \( \Omega = [0, L] \times [0, T] \), and set the correlation function to be:

\[
R(t, x) = \mathbb{E}[m_1(s, z)m_1(t+s, x+z)] = D^2 \exp(-t/a - x/b),
\]

(4.11)
Fig. 4.1: The left column shows the contour of $W$ in phase space and the right column shows the energy density $\rho = \int W \, dk$. 

(a) $t = 0.05$  

(b) $t = 0.15$  

(c) $t = 0.4$
Fig. 4.2: The plot compares energy density $\rho^\varepsilon$ with $\rho^0$ at $T = 0.5$ for different $\varepsilon$.

Fig. 4.3: The plot shows the $L^1$-error as a function of $\varepsilon$. The decay rate suggests that $\text{Err}^\varepsilon(\rho)$ is of $O(\varepsilon^2)$.

where $a > 0, b > 0$ and $D^2$ is the variance of $m_1$.

We choose the initial data to have a Gaussian form:

$$u_0^\varepsilon(x/\varepsilon, x) = \exp \left( -A(x - x_0)^2 + \frac{i}{\varepsilon} p_0 x \right). \quad (4.12)$$

The periodic boundary conditions are imposed

$$u^\varepsilon(t, 0) = u^\varepsilon(t, L), \quad \partial_x u^\varepsilon(t, 0) = \partial_x u^\varepsilon(t, L). \quad (4.13)$$
Correspondingly the transport equation (3.6) has the initial data:

\[ W_0(x, k) = \exp(-2A(x - x_0)^2)\delta(k - p_0), \quad (4.14) \]

and it is equipped with periodic conditions:

\[ W(t, 0, k) = W(t, L, k), \quad \text{for } t > 0 \text{ and all } k \in \mathbb{R}. \quad (4.15) \]

Similar to the previous subsection, \( L \) is set to be large enough and the periodic boundary condition plays minimum role.

The computation of the limiting radiative transfer equation is rather straightforward. Due to the form of the correlation function (4.11), one has

\[ \hat{R}(\omega, p) = \frac{4abD^2}{(1 + a^2\omega^2)(1 + b^2p^2)}. \quad (4.16) \]

Since \( m_0 \) is the deterministic slow-varying function, the equation composes of two transport terms, which we use a fifth-order WENO scheme [16], and a collision operator, which we apply the trapezoidal rule to approximate.

There are more numerical difficulties regarding the computation of VMSE. The challenge is two-folded: dealing with the randomness, and resolving the high oscillation. To handle the randomness, we perform the Karhunen-Loeve expansion by setting

\[ m_1(t/\varepsilon, x/\varepsilon) = D \sum_{i,j=1}^{\infty} \sqrt{\lambda_\varepsilon^i \sigma_\varepsilon^j} \psi_\varepsilon^i(t) \phi_\varepsilon^j(x) \xi_{ij}, \quad (4.17) \]

where \( \xi_{ij} \) are i.i.d. random variables with

\[ \mathbb{E}[\xi_{ij}] = 0, \quad \mathbb{E}[\xi_{i}^2] = 1, \quad \forall i, j = 1, 2, \ldots. \]

The form of \( \xi \) depends on the field, and we numerically use either uniformly distributed random variable or Gaussian random variable. \( \lambda_\varepsilon^i \) and \( \sigma_\varepsilon^j \) are descending eigenvalues corresponding eigenfunctions \( \psi_\varepsilon^i \) and \( \phi_\varepsilon^j \):

\[ \int_0^T e^{-\frac{\xi_{i}(t)}{\varepsilon}} \psi_\varepsilon^i(s) ds = \lambda_\varepsilon^i \psi_\varepsilon^i(t), \quad \int_0^L e^{-\frac{\xi_{j}(x)}{\varepsilon}} \phi_\varepsilon^j(z) dz = \sigma_\varepsilon^j \phi_\varepsilon^j(x). \quad (4.18) \]

For the particular form of \( R \) defined in (4.11), it is shown in [30] that

\[ \lambda_\varepsilon^i = \frac{2a\varepsilon}{1 + a^2\varepsilon^2 \omega_i^2}, \quad \sigma_\varepsilon^j = \frac{2b\varepsilon}{1 + b^2\varepsilon^2 v_j^2}, \]

\[ \psi_\varepsilon^i(t) = \begin{cases} \sin(w_i(t - T/2))/\sqrt{T/2 - \frac{\sin(w_iT)}{2w_i}}, & \text{if } i \text{ is even}, \\
\cos(w_i(t - T/2))/\sqrt{T/2 + \frac{\sin(w_iT)}{2w_i}}, & \text{if } i \text{ is odd}, \end{cases} \quad (4.19) \]

\[ \phi_\varepsilon^j(x) = \begin{cases} \sin(v_j(x - L/2))/\sqrt{L/2 - \frac{\sin(v_jL)}{2v_j}}, & \text{if } j \text{ is even}, \\
\cos(v_j(x - L/2))/\sqrt{L/2 + \frac{\sin(v_jL)}{2v_j}}, & \text{if } j \text{ is odd}. \end{cases} \]
where \( w_i \) and \( v_j \) are solutions to

\[
\begin{align*}
\left\{ \begin{array}{ll}
aw_i + \tan(w_i T_i^2) = 0, & \text{for even } i, \\
1 - aw_i \tan(w_i T_i^2) = 0, & \text{for odd } i,
\end{array} \right. \\
\left\{ \begin{array}{ll}
bv_j + \tan(v_j T_j^2) = 0, & \text{for even } j, \\
1 - bv_j \tan(v_j T_j^2) = 0, & \text{for odd } j.
\end{array} \right.
\] (4.20)

Numerically we perform Monte Carlo, that is to sample a large number of \( N \) configurations of \( \xi_{ij} \) which give rise to \( N \) configuration of \( m_1 \). For these deterministic \( m_1 \), we compute the deterministic VMSE, and take the ensemble mean and variance in the end. For Schrödinger equation, the Crank-Nicolson and spectral method are applied as in the previous section with the scales resolved: \( \Delta x = O(\varepsilon) \) and \( \Delta t = o(\varepsilon) \). Note that \( m \) is already deterministic for each Monte Carlo sample. Numerically to verify Theorem 3.1, we mainly compare the physical observables. In particular we will compare the energy density, that is to compare

\[
\rho^0 = \int W dk \quad \text{v.s.} \quad \mathbb{E}[\rho^\varepsilon(t,x)] = \mathbb{E}[|u^\varepsilon(t,x)|^2] \approx \frac{1}{N} \sum_{i=1}^{N} |u_i^\varepsilon(t,x)|^2 .
\] (4.21)

In the example, we set \( \Omega = [0, 1.625] \times [0, 0.4] \), and the parameters in \( R(t,x) \) (defined in (4.11)) are \( a = b = 100 \). For the initial data (4.12) and (4.14), we take \( A = 2^8 \) and \( x_0 = 0.3 \), and set \( m_0 = 1 \). To compute RTE, we set \( \Delta x = 2^{-10} \), and \( \Delta t = 2^{-12} \). To compute VMSE, we set \( \varepsilon = 2^{-n} \) and we use the discretization:

\[
\Delta t = 2^{-1.2n-3}, \quad \Delta x = 2^{-n-2}.
\] (4.22)

The KL series is truncated at \( N_{KL}^\varepsilon \) finite terms with

\[
\sqrt{(\lambda_i^\varepsilon \sigma^\varepsilon_i/N_{KL}^\varepsilon)} \lesssim 2^{-9}.
\] (4.23)

As \( i \) and \( j \) increase, the oscillations in the associated eigenfunctions \( \phi \) and \( \psi \) also increase, but the choice of \( \Delta x, \Delta t \) ensures that these oscillations are resolved.

In Figure 4.4 we show the solution to the transport equation (3.6) at three specific time for \( D = 1.5 \) and \( p = 1.5 \).

In Figure 4.5 we show that for different pairs of \((D,p_0)\), the numerical solution to RTE and numerical solution to VMSE are rather close for \( \varepsilon = 2^{-10} \). VMSE is computed using \( N = 10000 \) Monte Carlo samples and \( N_{KL}^\varepsilon = 27968 \) in the examples.

In Figure 4.6 we compare \( \rho^0 \) and \( \mathbb{E}[\rho^\varepsilon(t,x)] \) with different \( \varepsilon \), with \( D = 1.5 \) and \( p_0 = 1.5 \). VMSE is computed using 10000 Monte Carlo samples and \( N_{KL}^\varepsilon = 663, 3157, 27968 \) for \( \varepsilon = 2^{-6}, 2^{-8}, 2^{-10} \) respectively to ensure (4.23).

It is fairly straightforward to observe the convergence of VMSE to RTE as \( \varepsilon \to 0 \). Such convergence can also be quantified. Define the error:

\[
\text{Err}^\varepsilon(\rho) = \int_{\mathbb{R}} |\rho^0 - \mathbb{E}[\rho^\varepsilon]| \, dx .
\] (4.24)

In Figure 4.7 we show the convergence of \( \text{Err}^\varepsilon(\rho) \) as a function of \( \varepsilon \) for both Gaussian and uniform distributed variable \( \xi_{ij} \). According to the plot, the error decays at a rate
of $O(\varepsilon)$ – this is stronger than our ansatz where $W^{(1)}$ is assumed to be at the order of $\sqrt{\varepsilon}$. This implies that $E(W^{(1)})$ is of higher order than $\sqrt{\varepsilon}$. To generate the plot, we use $N_{KL}^{\varepsilon} = 663, 1342, 3157, 8727, 27968$ for $\varepsilon = 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}$ respectively to ensure (4.23).

Although we do not derive the equation for the standard deviation, we do numerically investigate the statistics of $\rho$. In particular, we set $\xi_{ij}$'s to be Gaussian random variables, and we plot, in Figure 4.8 the standard deviation $\sigma[\rho^\varepsilon]$ for different $\varepsilon$, and in Figure 4.9, the covariance at $t = 0.4$. The two quantities are defined as follows:

$$\sigma[\rho^\varepsilon(t, x)] \approx \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (\rho^\varepsilon_i(t, x) - E[\rho^\varepsilon(t, x)])^2},$$

and

$$\text{Cov}(\rho^\varepsilon(t, x), \rho^\varepsilon(t, y)) \approx \frac{1}{N-1} \sum_{i=1}^{N} (\rho^\varepsilon_i(t, x) - E[\rho^\varepsilon(t, x)])(\rho^\varepsilon_i(t, y) - E[\rho^\varepsilon(t, y)]).$$

In the computation we set $D = 1.5$ in the correlation function (4.3) and $p_0 = 1.5$ in the initial data. For $\varepsilon = 2^{-6}, 2^{-8}, 2^{-10}$, we take $N_{KL}^{\varepsilon} = 663, 3157, 27968$ to ensure (4.23). 10000 samples are used for the ensemble mean and covariance. Numerically we observe that with smaller $\varepsilon$ we have high standard deviation at the wave-packet center. We leave the mathematical justification to the future research.

5. Conclusion. In this paper, we systematically derived the radiative transfer equation for the solution to the varying-mass Schrödinger equation (VMSE) with random heterogeneities. In specific, we consider VMSE in the semiclassical regime (the rescaled Planck constant $\varepsilon \ll 1$), and expand the corresponding Wigner equation to proper orders to obtain the asymptotic limit. We verify the derivation by numerically computing both VMSE and radiative transfer equations, and showing that the two solutions agree well.

Acknowledgment. The research of S.C. and Q.L. are supported in part by NSF under DMS-1619778, DMS-1750488, and Wisconsin Data Science Initiatives. X.Y. was partially supported by the NSF grant DMS 1818592. Q.L. would like to thank Josselin Garnier and Guillaume Bal for valuable discussions.

REFERENCES

[1] Guillaume Bal, Tomasz Komorowski, and Lenya Ryzhik, Asymptotics of the solutions of the random schrödinger equation, Archive for rational mechanics and analysis, 200 (2011), pp. 613–664.

[2] Guillaume Bal, George Papanicolaou, and Leonid Ryzhik, Radiative transport limit for the random schrödinger equation, Nonlinearity, 15 (2002), p. 513.

[3] Guillaume Bal and Olivier Pinaud, Accuracy of transport models for waves in random media, Wave Motion, 43 (2006), pp. 561–578.

[4] W. Bao, S. Jin, and P. A. Markowich, On time-splitting spectral approximations for the Schrödinger equation in the semiclassical regime, J. Comput. Phys., 175 (2002), pp. 487–524.

[5] Weizhu Bao, Shi Jin, and Peter A Markowich, On time-splitting spectral approximations for the Schrödinger equation in the semiclassical regime, Journal of Computational Physics, 175 (2002), pp. 487–524.

[6] W. Bao, S. Jin, and P. A. Markowich, Numerical study of time-splitting spectral discretizations of nonlinear Schrödinger equations in the semi-classical regimes, SIAM J. Sci. Comput., 25 (2003), pp. 27–64.
[7] Liliana Borcea and Josselin Garnier, Laser beam imaging from the speckle pattern of the off-axis scattered intensity, SIAM Journal on Applied Mathematics, 78 (2018), pp. 677–704.
[8] Liliana Borcea, Josselin Garnier, and Knut Solna, Wave propagation and imaging in moving random media, Multiscale Modeling & Simulation, 17 (2019), pp. 31–67.
[9] Lihui Chai, Shi Jin, and Qin Li, Semi-classical models for the Schrödinger equation with periodic potentials and band crossings, Kinetic & Related Models, 6 (2013), p. 505.
[10] Lihui Chai, Shi Jin, Qin Li, and Omar Morandi, A multiband semiclassical model for surface hopping quantum dynamics, Multiscale Modeling & Simulation, 13 (2015), pp. 205–230.
[11] G.T. Einevoll, Operator ordering in effective-mass theory for heterostructures. II. Strained systems, Phys. Rev. B, 42 (1990), pp. 3497–3502.
[12] C. Fefferman, J.P. Lee-Thorp, and M.I. Weinstein, Bifurcations of edge states—topologically protected and non-protected—in continuous 2D honeycomb structures, 2D Materials, (2016).
[13] C. Fefferman, J.P. Lee-Thorp, and M.I. Weinstein, Edge states in honeycomb structures, Annals of PDE, (2016).
[14] Patrick Gérard, Peter A Markowich, Norbert J Mauser, and Frédéric Poupaud, Homogenization limits and Wigner transforms, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 50 (1997), pp. 323–379.
[15] M. F. Herman and E. Kluk, A semiclassical justification for the use of non-spreading wavepackets in dynamics calculations, Chem. Phys., 91 (1984), pp. 27–34.
[16] Guang-Shan Jiang and Chi-Wang Shu, Efficient implementation of weighted ENO schemes, Journal of Computational Physics, 126 (1996), pp. 202–228.
[17] S. Jin, P. A. Markowich, and C. Sparber, Mathematical and computational methods for semiclassical Schrödinger equations, Acta Numer., 20 (2011), pp. 211–289.
[18] S. Jin, H. Wu, and X. Yang, Gaussian beam methods for the Schrödinger equation in the semi-classical regime: Lagrangian and Eulerian formulations, Commun. Math. Sci., 6 (2008), pp. 995–1020.
[19] J. Lu and X. Yang, Frozen Gaussian approximation for high frequency wave propagation, Commun. Math. Sci., 9 (2011), pp. 663–683.
[20] J. Lu and X. Yang, Convergence of frozen Gaussian approximation for high frequency wave propagation, Comm. Pure Appl. Math., 65 (2012), pp. 759–789.
[21] J. Lukkari and H. Spohn, Kinetic limit for wave propagation in random medium, Arch. Rational Meth. Anal., 183 (2007), pp. 93–162.
[22] P. A. Markowich, P. Pietra, and C. Pohl, Numerical approximation of quadratic observables of Schrödinger-type equations in the semiclassical limit, Numer. Math., 81 (1999), pp. 595–630.
[23] P. A. Markowich, P. Pietra, C. Pohl, and H. P. Stimming, A Wigner measure analysis of the Dufort-Frankel scheme for the Schrödinger equation, SIAM J. Numer. Anal., 40 (2002), pp. 1281–1310.
[24] R.A. Morrow and K.R. Brownstein, Model effective-mass Hamiltonians for abrupt heterojunctions and the associated wave-function-matching conditions, Phys. Rev. B, 30 (1984), pp. 670–680.
[25] Leonid Ryzhik, George Papanicolaou, and Joseph B Keller, Transport equations for elastic and other waves in random media, Wave motion, 24 (1996), pp. 327–370.
[26] H. Spohn, Derivation of the transport equation for electrons moving through random impurities, J. Stat. Phys., 17 (1977), pp. 385–412.
[27] J. Thomsen, G.T. Einevoll, and P.C. Hemmer, Operator ordering in effective-mass theory, Phys. Rev. B, 39 (1989), pp. 12783–12788.
[28] Anna-Karin Tornberg and Björn Engquist, Numerical approximations of singular source terms in differential equations, Journal of Computational Physics, 200 (2004), pp. 462–488.
[29] O. Von Roos, Position-dependent effective masses in semiconductor theory, Phys. Rev. B, 27 (1983), pp. 7547–7552.
[30] Dongbin Xiu, Numerical methods for stochastic computations: a spectral method approach, Princeton university press, 2010.
[31] Sara Zahedi and Anna-Karin Tornberg, Delta function approximations in level set methods by distance function extension, Journal of Computational Physics, 229 (2010), pp. 2199–2219.
Fig. 4.4: The left column shows the contour of $W$ in phase space and the right column shows the energy density $\rho = \int W dk$. 
Fig. 4.5: The plot shows the energy density $E[\rho^\varepsilon]$ ($\varepsilon = 2^{-10}$) and $\rho^0 = \int W dk$ at $T = 0.4$ with different $(D, p_0)$ pairs.

Fig. 4.6: The plot compares energy density $E[\rho^\varepsilon]$ with $\rho^0$, defined in (4.21) at $T = 0.4$ for different $\varepsilon$ and different random distribution of $\xi_{ij}$. 

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Fig. 4.7: The plot shows the $L^1$-error as a function of $\varepsilon$. Both Gaussian and Uniform distributions are used to sample $\xi_{ij}$. The decay rate suggests that $Err^\varepsilon(\rho)$ is of $O(\varepsilon)$.

Fig. 4.8: The plot shows the standard deviation of energy density $\sigma[\rho^\varepsilon]$ at $T = 0.4$ for different $\varepsilon$. 
Fig. 4.9: The graphs from left to right show the covariance $\text{Cov}(\rho^\varepsilon(x), \rho^\varepsilon(y))$ at $T = 0.4$ for $\varepsilon = 2^{-6}, 2^{-8}, 2^{-10}$, respectively. The random variables $\xi_{ij}$’s are chosen to be standard Gaussian random variables.