TILT-STABILITY, VANISHING THEOREMS AND BOGOMOLOV-GIESEKER TYPE INEQUALITIES

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ABSTRACT. We investigate the tilt-stability of stable sheaves on projective varieties with respect to certain tilt-stability conditions depending on two parameters constructed by Bridgeland [12] (see also [1, 7, 6]). For a stable sheaf, we give effective bounds of these parameters such that the stable sheaf is tilt-stable. These allow us to prove new vanishing theorems for stable sheaves and an effective Serre vanishing theorem for torsion free sheaves. Using these results, we also prove Bogomolov-Gieseker type inequalities for the third Chern character of a stable sheaf on $\mathbb{P}^3$.

1. Introduction

Let $X$ be a complex smooth projective variety of dimension $n$ with a fixed ample divisor $H$ and a fixed $\mathbb{Q}$-divisor $B$ on it. For any real numbers $\alpha > 0$ and $\beta$, the $\mathbb{R}$-divisors $\alpha H$ and $\beta H + B$ determine a weak Bridgeland stability condition on $X$ (see Section 2.2 for the precise definition). We also call it $\nu_{\alpha, \beta}$-stability (or tilt-stability). In recent years, this stability has drawn a lot of attentions, and has been investigated intensively.

When $X$ is a surface, $\nu_{\alpha, \beta}$-stability is a Bridgeland stability condition introduced by Bridgeland [11, 12], Arcara and Bertram [1]. There are many and fruitful applications of this stability to birational geometry of moduli spaces of stable sheaves on surfaces (cf. [1], [2], [4], [5], [10], [25], [26], · · · ). For higher dimensional $X$, $\nu_{\alpha, \beta}$-stability appears in the construction of Bridgeland stability on $X$ by Bayer, Macrì and Toda [7], and it has been systematically investigated by Bayer, Macrì and Stellari [6].

The prototypical example of a $\nu_{\alpha, \beta}$-stability result is Bridgeland’s large volume limit theorem [12, 21]:

**Theorem 1.1** (Bridgeland). Suppose that dim $X = 2$. For $E \in \text{Coh}^{\beta H + B}(X) \cap \text{Coh}(X)$ and $\alpha \gg 0$, we have $E$ is $\nu_{\alpha, \beta}$-(semi)stable if and only if $E$ is $(H, \beta H + B - \frac{1}{2}K_X)$-twisted Gieseker (semi)stable.

A parallel result also holds for higher dimensional case. In that case, the large volume limit ($\alpha \gg 0$) of $\nu_{\alpha, \beta}$-stability for a coherent sheaf is the same as the $p_{H, \beta H + B}$-stability (see Proposition 2.10).

Bridgeland’s arguments are non-effective, and there is no known bound on how large $\alpha$ must be in order to obtain the conclusion of the theorem. In light of this theorem, it is natural to ask:
Question 1.2. For which finite value of \( \alpha \) and \( \beta \) does \( p_{H, \beta H+B} \)-stability become \( \nu_{\alpha, \beta} \)-stability for a coherent sheaf?

The goal of this paper is to answer this question for a \( \mu_{H, B} \)-stable torsion free sheaf (see Definition \[21\]).

**Theorem 1.3** (\( = \) Theorem \[5,1\]). Suppose that \( E \) is a \( \mu_{H, B} \)-stable torsion free sheaf on \( X \), and \( \mu \) is a rational number satisfies \( \mu_{H, B}^{\text{max}}(E) \leq \mu < \mu_{H, B}(E) \). Let \( \beta_0 = \mu_{H, B}(E) - \frac{\Sigma_0(E)(H^n \cdot E)^2}{\mu_{H, B}(E) - \mu} \) and \( \beta_1 = \mu_{H, B}(E) - \frac{\sqrt{(rk E + 1) \Sigma(E)}}{H^n \cdot E} \).

1. If \( \mu > \mu_{H, B}(E) - \frac{1}{H^n \cdot E} \sqrt{\Sigma(E)} \), then \( E \) is \( \nu_{\alpha, \beta} \)-stable for any \( \alpha > 0 \) and \( \beta \leq \beta_0 \).
2. If \( \mu \leq \mu_{H, B}(E) - \frac{1}{H^n \cdot E} \sqrt{\Sigma(E)} \) and \( \Sigma^B(E) > 0 \), then \( E \) is \( \nu_{\alpha, \beta_1} \)-stable for any \( \alpha > 0 \).
3. If \( \Sigma^B(E) = 0 \), then \( E \) is \( \nu_{\alpha, \beta} \)-stable for any \( \alpha > 0 \) and \( \beta < \mu_{H, B}(E) \).

**Theorem 1.4** (\( = \) Theorem \[5,4\]). Suppose that \( E \) is a \( \mu_{H, B} \)-stable reflexive sheaf on \( X \), and \( \bar{\mu} \) is a rational number satisfies \( \mu_{H, B}(E) - \frac{\mu_{H, B}^B(E)}{1} \leq \bar{\mu} \leq \mu_{H, B}^B(E) \). Let \( \bar{\beta}_0 = \mu_{H, B}(E) + \frac{\Sigma_0(E)(H^n \cdot E)^2}{\bar{\mu} - \mu_{H, B}(E)} \) and \( \bar{\beta}_1 = \mu_{H, B}(E) + \frac{\sqrt{(rk E + 1) \Sigma_0(E)}}{H^n \cdot E} \).

1. If \( \bar{\mu} < \mu_{H, B}(E) + \frac{1}{H^n \cdot E} \sqrt{\Sigma_0(E)} \), then \( E[1] \) is \( \nu_{\alpha, \bar{\beta}} \)-stable for any \( \alpha > 0 \) and \( \beta \geq \bar{\beta}_0 \).
2. If \( \bar{\mu} \geq \mu_{H, B}(E) + \frac{1}{H^n \cdot E} \sqrt{\Sigma_0(E)} \), then \( E[1] \) is \( \nu_{\alpha, \bar{\beta}_1} \)-stable for any \( \alpha > 0 \).
3. If \( \Sigma^B(E) = 0 \), then \( E[1] \) is \( \nu_{\alpha, \beta} \)-stable for any \( \alpha > 0 \) and \( \beta \geq \mu_{H, B}(E) \).

These answers can help us to understand the \( \nu_{\alpha, \beta} \)-stability more explicitly. They also give some interesting applications to the positivity of coherent sheaves, such as vanishing theorems of stable sheaves, effective Serre vanishing theorem, and Bogomolov-Gieseker type inequalities for the third Chern character of a stable sheaf on \( \mathbb{P}^3 \).

The slopes \( \mu_{H, B}^{\text{max}}(E) \) and \( \mu_{H, B}^{\text{min}}(E) \) in the above theorems are defined in Section \[5\]. The strategy of the proof is essentially the same as that of \[21\].

We now explain the strategy in greater detail. Given a \( \mu_{H, B} \)-stable sheaf \( E \), we define an ellipse \( C_E \) on the \( (\beta, \alpha) \) half plane. By the Bogomolov-Gieseker type inequality in \[7\] and \[8\], we can show that for a point \( (\beta, \alpha) \) outside the ellipse \( C_E \), if a subobject \( F \) of \( E \) has large \( \nu_{\alpha, \beta} \)-slope, then it must have small rank (Lemma \[4,1\] and Lemma \[4,2\]). When \( F \) has small rank, by the \( \mu_{H, B} \)-stability of \( E \) and the Bogomolov-Gieseker type inequality, \( \nu_{\alpha, \beta}(F) \) can be bounded above. Hence we could obtain the \( \nu_{\alpha, \beta} \)-stability of \( E \) by computing the intersection of the ellipse and the wall \( W(F, E) \).

**Applications to vanishing theorems.** By the basic properties of \( \nu_{\alpha, \beta} \)-stability, Theorem \[1,3\] and \[1,4\] can immediately give the following vanishing theorems for \( \mu_{H, B} \)-stable sheaves.

**Corollary 1.5.** Let \( E \) be a \( \mu_{H} \)-stable torsion free sheaf on \( X \), and \( \mu \) be a rational number satisfies \( \mu_{H}^{\text{max}}(E) \leq \mu < \mu_{H}(E) \).
(1) If \( \mu > \mu_H(E) - \frac{1}{H^n \text{rk} E} \sqrt{\frac{\Delta_H(E)}{\text{rk} E + 1}} \), then \( H^{n-1}(X, E(K_X + lH)) = 0 \) for any integer \( l > \frac{\Delta_H(E)}{H^n \text{rk} E} - \mu_H(E) \).

(2) If \( \mu \leq \mu_H(E) - \frac{1}{H^n \text{rk} E} \sqrt{\frac{\Delta_H(E)}{\text{rk} E + 1}} \), then \( H^{n-1}(X, E(K_X + lH)) = 0 \) for any integer \( l > \frac{\sqrt{(rk E + 1) \Delta_H(E)}}{H^n \text{rk} E} - \mu_H(E) \).

**Corollary 1.6.** Let \( E \) be a \( \mu_H \)-stable reflexive sheaf on \( X \), and \( \bar{\mu} \) be a rational number satisfies \( \mu_H(E) < \bar{\mu} \leq \mu_{\min}^\text{tr}(E) \).

(1) If \( \bar{\mu} < \mu_H(E) + \frac{1}{H^n \text{rk} E} \sqrt{\frac{\Delta_H(E)}{\text{rk} E + 1}} \), then \( H^1(X, E(-lH)) = 0 \) for any integer \( l > \mu_H(E) + \frac{\sqrt{(rk E + 1) \Delta_H(E)}}{H^n \text{rk} E} \).

(2) If \( \bar{\mu} \geq \mu_H(E) + \frac{1}{H^n \text{rk} E} \sqrt{\frac{\Delta_H(E)}{\text{rk} E + 1}} \), then \( H^1(X, E(-lH)) = 0 \) for any integer \( l > \mu_H(E) + \frac{\sqrt{(rk E + 1) \Delta_H(E)}}{H^n \text{rk} E} \).

These vanishing theorems generalize the Kodaira vanishing. To see this, just taking \( E = \mathcal{O}_X(H) \) in Corollary 1.5 and Corollary 1.6 then one obtains the Kodaira vanishing:

\[
H^{n-1}(X, \mathcal{O}_X(K_X + H)) = H^1(X, \mathcal{O}_X(-H)) = 0.
\]

These vanishing theorems can be used to give an effective Serre vanishing theorem for \( H^{n-1} \). In order to state them explicitly, we need the following function.

**Definition 1.7.** Let \( r \) be a real number, and \( m \) be a positive integer, we define \( [r]_m := \max\{a/b : a, b \in \mathbb{Z}, \frac{a}{b} < r, 1 \leq b \leq m \} \).

**Theorem 1.8 (Effective Serre vanishing for \( H^{n-1} \)).** Let \( \mathcal{F} \) be a coherent torsion free sheaf on \( X \), and let \( 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F} \) be its Harder-Narasimhan filtration. Set \( \mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1} \). Then \( H^{n-1}(X, \mathcal{F}(K_X + lH)) = 0 \) as soon as

\[
l > M(\mathcal{F}) := \max_{1 \leq i \leq k} \left\{ \frac{\Delta_H(\mathcal{G}_i)/(H^n \text{rk} \mathcal{G}_i)}{H^n \mu_H(\mathcal{G}_i) - \lceil H^n \mu_H(\mathcal{G}_i) \rceil_{\text{rk} \mathcal{G}_i} - \mu_H(\mathcal{G}_i)}, \sqrt{\frac{2\Delta_H(\mathcal{G}_i)}{(H^n)^2 \text{rk} \mathcal{G}_i} - \mu_H(\mathcal{G}_i)} \right\}.
\]

To the best of our knowledge, no explicit bounds for such an \( l \) in Theorem 1.8 are known except for some special cases. Under the stronger assumption that \( H \) is very ample, Langer [17] also gives an effective bounds for Serre vanishing theorem in the surface case. See [28], [8] and [32] for such an effective bound for rank one torsion free sheaves on a surface. See also [13] Example 10.2.9 for the effective bound for an ample line bundle on a projective variety of any dimension.

The constant \( M(\mathcal{F}) \) in the above theorem can have a simpler but weaker form (see Remark 5.3). In particular, for a \( \mu_H \)-semistable torsion free sheaf one has:

**Corollary 1.9.** Let \( \mathcal{F} \) be a \( \mu_H \)-semistable torsion free sheaf on \( X \) with \( \text{rk} \mathcal{F} \geq 2 \). Then \( H^{n-1}(X, \mathcal{F}(K_X + lH)) = 0 \) if \( l > \frac{\Delta(\mathcal{F})}{H^n} - \mu_H(\mathcal{F}) \).

**Applications to stable sheaves on \( \mathbb{P}^3 \).** From the Bogomolov-Gieseker type inequality on \( \mathbb{P}^3 \) proved by Macri (Theorem 2.13), one could obtain another application of Theorem 1.3 to stable sheaves on \( \mathbb{P}^3 \):
Theorem 1.10. Suppose $H$ is a plane in $\mathbb{P}^3$. Let $E$ be a $\mu_H$-stable torsion free sheaf on $\mathbb{P}^3$, and let $l(E) = \frac{c_1(E)-3c_1(E)\sum H(E)}{6(rk E)^2}$.

1. If $\mu_H^{\max}(E) > \mu_H(E) - \frac{1}{rk E} \sqrt{\frac{\sum H(E)}{rk E+1}}$, then

$$\frac{\sum H(E)}{6rk E} \left(\mu_H(E) - |\mu_H(E)|rk E + \frac{\sum H(E)/(rk E)^2}{\mu_H(E) - |\mu_H(E)|rk E}\right) + l(E) \geq ch_3(E).$$

2. If $\mu_H^{\max}(E) \leq \mu_H(E) - \frac{1}{rk E} \sqrt{\frac{\sum H(E)}{rk E+1}}$, then

$$\frac{(rk E+2)(\sum H(E))^2}{6(rk E)^2\sqrt{rk E+1}} + l(E) \geq ch_3(E).$$

In particular, when $rk E = 2$, we have:

Corollary 1.11. Under the situation in the above theorem, we further assume that $E$ is of rank two.

1. If $\mu_H^{\max}(E) > \sqrt{\frac{c_1(E)}{4}}$ and $c_1(E) = 0$, then $c_3(E) \leq \frac{1}{4}c_2^2(E) + \frac{3}{4}c_2(E)$.

2. If $\mu_H^{\max}(E) \leq \sqrt{\frac{c_1(E)}{4}}$ and $c_1(E) = 0$, then $c_3(E) \leq \frac{1}{4}c_2^2(E)$.

3. If $\mu_H^{\max}(E) > -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{4c_2(E)-1}{3}}$ and $c_1(E) = -1$, then $c_3(E) \leq \frac{1}{4}c_2^2(E) - \frac{3}{4}c_2(E)$.

4. If $\mu_H^{\max}(E) \leq -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{4c_2(E)-1}{3}}$ and $c_1(E) = -1$, then $c_3(E) \leq \left(\frac{4c_2(E)-1}{3}\right)^{\frac{3}{2}}$.

One can compare Corollary 1.11 with Hartshorne’s bounds:

Theorem 1.12 (Hartshorne [15]). Let $E$ be a rank two $\mu_H$-stable reflexive sheaf on $\mathbb{P}^3$, where $H$ is a plane in $\mathbb{P}^3$.

1. If $c_1(E) = 0$, then $c_3(E) \leq c_2^2(E) - c_2(E) + 2$.

2. If $c_1(E) = -1$, then $c_3(E) \leq c_2^2(E)$.

We notice that when $\mu_H^{\max}(E) > \mu_H(E) - \frac{1}{2} \sqrt{\frac{\sum H(E)}{3}}$, Corollary 1.11 is weaker than Hartshorne’s result, but it works for the more general torsion free case.

The bounds in Theorem 1.10 can have a simpler but weaker form:

Corollary 1.13. Let $E$ be a $\mu_H$-stable torsion free sheaf on $\mathbb{P}^3$ with $rk E \geq 3$, where $H$ is a plane in $\mathbb{P}^3$. We have

$$\frac{(\sum H(E))^2}{6rk E} + \frac{\sum H(E)}{6(rk E)^3} + \frac{c_1^2(E)-3c_1(E)\sum H(E)}{6(rk E)^2} \geq ch_3(E).$$

Organization of the paper. Our paper is organized as follows. In Section 2 we review basic notions and properties of some classical stabilities for coherent sheaves, tilt-stability, the conjectural inequality proposed in [7] [6] and variants of the classical Bogomolov-Gieseker inequality satisfies by tilt-stable objects. Then in Section 3 we recall the properties of walls for tilt-stability. In Section 4 we introduce the extremal ellipses, and study the intersections of the walls and the extremal ellipses. We prove Theorems 1.3 and 1.4 in Section 5. Vanishing theorems in Corollary 1.5. Corollary 1.6 and their applications will be showed in Section 6. In Section 7 we give the application of Theorem 1.3 to the Chern classes of a $\mu_H$-stable sheaf on $\mathbb{P}^3$. 
Notation. We work over the complex numbers in this paper. We will always denote by $X$ a smooth projective variety of dimension $n \geq 2$ and by $D^b(X)$ its bounded derived category of coherent sheaves. $K_X$ and $\omega_X$ denote the canonical divisor and canonical sheaf of $X$, respectively. For a triangulated category $\mathcal{D}$, we write $K(\mathcal{D})$ for the Grothendieck group of $\mathcal{D}$.

We write $H^j(E)$ ($j \in \mathbb{Z}$) for the cohomology sheaves of a complex $E \in D^b(X)$. We also write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the sheaf cohomology groups of a sheaf $F \in \text{Coh}(X)$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively.

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2. Preliminaries

In this section, we review some basic notions of stability for coherent sheaves, the weak Bridgeland stability conditions and Bogomolov-Gieseker type inequalities.

2.1. Stability for sheaves. For any $\mathbb{Q}$-divisor $D$ on $X$, we define the twisted Chern character $\text{ch}^D = e^{-D} \text{ch}$. More explicitly, we have

\[
\begin{align*}
\text{ch}^D_0 &= \text{rk} \\
\text{ch}^D_1 &= \text{ch} - D \text{ch} + \frac{D^2}{2} \text{ch} \\
\text{ch}^D_2 &= \text{ch} - D \text{ch}_1 + \frac{D D}{2} \text{ch}_0 \\
\text{ch}^D_3 &= \text{ch} - 2D \text{ch}_2 + \frac{D^2}{2} \text{ch}_1 - \frac{D^3}{6} \text{ch}_0.
\end{align*}
\]

The first important notion of stability for a sheaf is slope stability, also known as Mumford stability. We define the slope $\mu_{H,D}$ of a coherent sheaf $E \in \text{Coh}(X)$ by

\[
\mu_{H,D}(E) = \begin{cases} 
+\infty, & \text{if } \text{ch}^D_0(E) = 0, \\
\frac{H^{n-1} \text{ch}^D_0(E)}{H^n \text{ch}^D_0(E)}, & \text{otherwise}.
\end{cases}
\]

Definition 2.1. A coherent sheaf $E$ on $X$ is $\mu_{H,D}$-(semi)stable (or slope-(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have

\[
\mu_{H,D}(F) < (\leq) \mu_{H,D}(E/F).
\]

Note that $\mu_{H,D}$ only differs from $\mu_H := \mu_{H,0}$ by a constant, thus $\mu_{H,D}$-stability and $\mu_H$-stability coincide. Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to $\mu_{H,D}$-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_k = E
\]

such that: $G_i := E_i/E_{i-1}$ is $\mu_{H,D}$-semistable, and $\mu_{H,D}(G_1) > \cdots > \mu_{H,D}(G_k)$. We set $\mu_{H,D}^+(E) := \mu_{H,D}(G_1)$ and $\mu_{H,D}^-(E) := \mu_{H,D}(G_k)$.

Another well-know stability for a sheaf is Gieseker stability. To define it, write the reduced twisted Hilbert polynomial of a positive rank sheaf $E$ as

\[
G_{H,D}(E,m) = \frac{\chi(E(mH - D))}{\text{rk} E},
\]
where the Euler characteristic is computed formally. A simple Riemann-Roch computation shows that

\[
G_{H,D}(E, m) = \frac{m^n H^n}{n!} \left( \frac{\chi_1^D(E)) - \frac{K}{2} \text{rk} E}{\text{rk} E} \right) + \frac{m^{n-2} H^{n-2}}{(n-2)!} \left( \frac{\chi_2^D(E) - \frac{K}{2} \chi_1^D(E)}{\text{rk} E} + \frac{K_2}{12} + c_2(X) \right) + \ldots
\]

**Definition 2.2.** A coherent torsion free sheaf \( E \) on \( X \) is \((H, D)\)-twisted Gieseker (semi)stable (or \( G_{H,D}\)-(semi)stable) if, for all non-zero proper subsheaves \( F \rightarrow E \), we have

\[
G_{H,D}(F, m) < (\leq) G_{H,D}(E, m)
\]

for all \( m \gg 0 \).

When \( D = 0 \), we recover usual \( H \)-Gieseker stability.

Now we introduce the \( p_{H,D}\)-stability mentioned in the introduction. The polynomial slope \( p_{H,D} \) of a sheaf \( E \in \text{Coh}(X) \) is

\[
p_{H,D}(E, m) = \begin{cases} (+\infty)m + (+\infty), & \text{if } \chi_0^D(E) = 0, \\ \frac{H^{n-1} \chi_0^D(E)}{H^n \chi_0^0(E)} m + \frac{H^{n-2} \chi_0^D(E)}{H^n \chi_0^0(E)}, & \text{otherwise}. \end{cases}
\]

**Definition 2.3.** A coherent sheaf \( E \) on \( X \) is \( p_{H,D}\)-(semi)stable (or polynomial slope-(semi)stable) if, for all non-zero subsheaves \( F \rightarrow E \), we have

\[
p_{H,D}(F, m) < (\leq)p_{H,D}(E/F, m)
\]

for all \( m \gg 0 \).

There are obvious relations among those stabilities. One can easily proves

**Lemma 2.4.** For any coherent torsion free sheaf \( E \) on \( X \), one has the following chain of implications

\[
\mu_H\text{-stable} \Rightarrow p_{H,D+\frac{K_X}{2}}\text{-stable} \Rightarrow G_{H,D}\text{-stable} \Rightarrow G_{H,D}\text{-semistable} \Rightarrow p_{H,D+\frac{K_X}{2}}\text{-semistable} \Rightarrow \mu_H\text{-semistable}.
\]

In particular, \( G_{H,D}\)-stability and \( p_{H,D+\frac{K_X}{2}}\)-stability are equivalent for any coherent torsion free sheaf on a surface.

2.2. Weak Bridgeland stability conditions. The notion of weak Bridgeland stability condition and its variant have been introduced in \[33\] Section 2 and \[6\] Definition B.1. We will use a slightly different notion in order to adapt our situation.

**Definition 2.5.** A weak Bridgeland stability condition on \( X \) is a pair \( \sigma = (Z, A) \), where \( A \) is the heart of a bounded \( t \)-structure on \( D^b(X) \), and \( Z : K(D^b(X)) \rightarrow \mathbb{C} \) is a group homomorphism (called central charge) such that

- \( Z \) satisfies the following positivity property for any \( E \in A \):

\[
Z(E) \in \left\{ re^{i\pi \phi} : r \geq 0, 0 < \phi \leq 1 \right\}.
\]
the following central charge
of a bounded t-structure on \( D^b \) generated by
\[
\mu(2.1)
\]
and the second map is defined by
\[
T = \text{Equivalently, } \mathcal{T}_{\beta H + B} \text{ and } \mathcal{F}_{\beta H + B} \text{ are the extension-closed subcategories of } \text{Coh}(X)
generated by } \mu_{H, \beta H + B} \text{ for positive and non-positive slope, respectively.}

**Definition 2.6.** We let \( \text{Coh}^{\beta H + B}(X) \subset D^b(X) \) be the extension-closure
\[
\text{Coh}^{\beta H + B}(X) = \langle \mathcal{T}_{\beta H + B}, \mathcal{F}_{\beta H + B}[1] \rangle.
\]

By the general theory of torsion pairs and tilting [13], \( \text{Coh}^{\beta H + B}(X) \) is the heart of a bounded t-structure on \( D^b(X) \); in particular, it is an abelian category. Consider the following central charge
\[
Z_{\alpha, \beta}(E) = H^{n-2} \left( \frac{\alpha^2 H^2}{2} \text{ch}_0^{\beta H + B}(E) - \text{ch}_2^{\beta H + B}(E) + iH \text{ch}_1^{\beta H + B}(E) \right).
\]
We think of it as the composition
\[
Z_{\alpha, \beta} : K(D^b(X)) \xrightarrow{\text{ch}_H} \mathbb{Q}^3 \xrightarrow{z_{\alpha, \beta}} \mathbb{C},
\]
where the first map is given by
\[
\text{ch}_H(E) = (H^n \text{ch}_0^B(E), H^{n-1} \text{ch}_1^B(E), H^{n-2} \text{ch}_2^B(E)),
\]
and the second map is defined by
\[
(2.1) \quad z_{\alpha, \beta}(e_0, e_1, e_2) = \frac{1}{2}(\alpha^2 - \beta^2)e_0 + \beta e_1 - e_2 + i(e_1 - \beta e_0).
\]

**Theorem 2.7.** For any \((\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}, \quad \sigma_{\alpha, \beta} = (Z_{\alpha, \beta}, \text{Coh}^{\beta H + B}(X)) \) is a weak
Bridgeland stability condition.

**Proof.** The argument is proved in [12] [1] for the surface case. For the threefold case, the conclusion is showed in [7] [6]. But the proof in [6] Appendix B] still works for the general case. \( \square \)

We write \( \nu_{\alpha, \beta} \) for the slope function on \( \text{Coh}^{\beta H + B}(X) \) induced by \( Z_{\alpha, \beta} \). Explicitly, for any \( E \in \text{Coh}^{\beta H + B}(X) \), one has
\[
\nu_{\alpha, \beta}(E) = \begin{cases} 
+\infty, & \text{if } H^{n-1} \text{ch}_1^{\beta H + B}(E) = 0, \\
H^{n-2} \text{ch}_2^{\beta H + B}(E) - \frac{\alpha^2 H^n \text{ch}_0^{\beta H + B}(E)}{H^{n-1} \text{ch}_1^{\beta H + B}(E)}, & \text{otherwise}.
\end{cases}
\]
Theorem 2.7 gives the notion of tilt-stability:
Definition 2.8. An object $E \in \text{Coh}^{\beta H+B}(X)$ is tilt-(semi)stable (or $\nu_{\alpha,\beta}$-(semi)stable) if, for all non-trivial subobjects $F \hookrightarrow E$, we have

$$\nu_{\alpha,\beta}(F) < (\leq) \nu_{\alpha,\beta}(E/F).$$

For any $\mathcal{E} \in \text{Coh}^{\beta H+B}(X)$, the Harder-Narasimhan property gives a filtration in $\text{Coh}^{\beta H+B}(X)$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$
such that: $\mathcal{F}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\nu_{\alpha,\beta}$-semistable with $\nu_{\alpha,\beta}(\mathcal{F}_1) > \cdots > \nu_{\alpha,\beta}(\mathcal{F}_n)$.

Definition 2.9. In the above filtration, we call $\mathcal{F}_1$ the $\nu_{\alpha,\beta}$-maximal subobject of $\mathcal{E}$ in $\text{Coh}^{\beta H+B}(X)$, and call $\mathcal{F}_n$ the $\nu_{\alpha,\beta}$-minimal quotient of $\mathcal{E}$.

The following well-known proposition establishes the relation between $\nu_{\alpha,\beta}$-stability and $p_{H,B}$-stability.

Proposition 2.10. For $E \in \text{Coh}^{\beta H+B}(X) \cap \text{Coh}(X)$ and $\alpha \gg 0$, we have $E$ is $\nu_{\alpha,\beta}$-(semi)stable if and only if $E$ is $p_{H,B}$-(semi)stable.

Proof. See [12, Section 14] and [21, Appendix A].

2.3. Bogomolov-Gieseker type inequality. We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [7, 6].

Definition 2.11. We define the discriminant

$$\Delta_H := H^{n-2}(ch_1^2 - 2ch_0ch_2) = H^{n-2}\left((ch_1^{B})^2 - 2ch_0^{B}ch_2^{B}\right),$$

and the generalized discriminant

$$\Sigma_H^{\beta H+B} := (H^{n-1}ch_1^{\beta H+B})^2 - 2H^n ch_0^{\beta H+B} \cdot (H^{n-2}ch_2^{\beta H+B}).$$

A short calculation shows

$$\Sigma_H^{\beta H+B} = (H^{n-1}ch_1^B)^2 - 2H^n ch_0^B \cdot (H^{n-2}ch_2^B) = \Sigma_H^B.$$

Hence the generalized discriminant is independent of $\beta$.

Theorem 2.12 (Bogomolov, Gieseker). Assume $E$ is a $\mu_H$-semistable torsion free sheaf on $X$. Then $\Delta_H(E) \geq 0$.

Theorem 2.13. Assume $E \in \text{Coh}^{\beta H+B}(X)$ is $\nu_{\alpha,\beta}$-semistable, then $\Sigma_H^B(E) \geq 0$.

Proof. This inequality was proved in [7] and [6] on threefolds, but their proof works for the general case.

Conjecture 2.14 ([6, Conjecture 4.1]). Assume $n = 3$, $B = 0$ and $E \in \text{Coh}^{\beta H}(X)$ is $\nu_{\alpha,\beta}$-semistable. Then

$$(2.2)\quad \alpha^2\Sigma_H^{\beta H}(E) + 4\left(Hch_0^{\beta H}(E)\right)^2 - 6H^2ch_1^{\beta H}(E)ch_3^{\beta H}(E) \geq 0.$$

Such an inequality provides a way to construct Bridgeland stability conditions on threefolds, and it was proved to be hold in the some cases:

Theorem 2.15. The inequality (2.2) holds for $\nu_{\alpha,\beta}$-semistable objects on $\mathbb{P}^3$, quadric threefolds, abelian threefolds and Fano threefolds of Picard number one.

Proof. Please see [24], [29], [6] and [19].

Remark 2.16. Recently, Schmidt [30] found a counterexample to Conjecture 2.14 when $X$ is the blowup at a point of $\mathbb{P}^3$. Therefore, the inequality (2.2) needs some modifications in general setting. See [27] and [9] for the recent progress.
3. Types of walls

In this section, we recall some basic properties of walls for the weak Bridgeland stability $\sigma_{\alpha, \beta}$ in Theorem 2.7. They are completely analogous to the case of walls for Bridgeland stability on surfaces, treated most systematically by Lo and Qin [22] and Maciocia [23]. We freely use the notations in Section 2.

3.1. Numerical walls and actual walls. Let $\mathbf{v} = (v_0, v_1, v_2)$ and $\mathbf{w} = (w_0, w_1, w_2)$ be two vectors in $\mathbb{Q}^3$ with

$$\Delta_H^B(\mathbf{v}) = v_1^2 - 2v_0v_2 \geq 0, \quad \Delta_H^B(\mathbf{w}) = w_1^2 - 2w_0w_2 \geq 0.$$  

Assume that $\mathbf{w}$ does not have the same $\nu_{\alpha, \beta}$-slope as $\mathbf{v}$ everywhere in the $(\beta, \alpha)$-half plane $\mathbb{R} \times \mathbb{R}_{>0}$.

**Definition 3.1.** The numerical wall $W(\mathbf{w}, \mathbf{v})$ is the set of points $(\beta, \alpha)$ such that $\mathbf{w}$ and $\mathbf{v}$ have the same $\nu_{\alpha, \beta}$-slope. A numerical wall is an actual wall if there exists a point $(\beta, \alpha) \in W(\mathbf{w}, \mathbf{v})$ and two objects $E, F \in \text{Coh}^{\beta H+B}(X)$ with $\text{ch}_H(E) = \mathbf{v}$, $\text{ch}_H(F) = \mathbf{w}$, such that $F$ is a subobject of $E$, or $E$ is a quotient of $F$ in $\text{Coh}^{\beta H+B}(X)$. In this situation, we also write $W(F, E) = W(\mathbf{w}, \mathbf{v})$.

We will frequently use the following facts about the walls [23], [13], [10]:

**Proposition 3.2.** Keep the above notation.

- The numerical walls $W(\mathbf{w}, \mathbf{v})$ in the $(\beta, \alpha)$-half plane are disjoint.
- Let $\mathbf{v}$ and $\mathbf{w}$ have positive rank. If $\mu_{H,B}(\mathbf{v}) = \mu_{H,B}(\mathbf{w})$, i.e., $\frac{\mu_{H,B}(\mathbf{v})}{\mu_{H,B}(\mathbf{w})} = \frac{w_0}{w_0}$, then $W(\mathbf{w}, \mathbf{v})$ is a line $\beta = \mu_{H,B}(\mathbf{v})$. If $\mu_{H,B}(\mathbf{v}) \neq \mu_{H,B}(\mathbf{w})$, then $W(\mathbf{w}, \mathbf{v})$ is a semicircle defined by $(\beta - s(\mathbf{w}, \mathbf{v}))^2 + \alpha^2 = r^2(\mathbf{w}, \mathbf{v})$, where

$$s(\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mu_{H,B}(\mathbf{v}) + \mu_{H,B}(\mathbf{w})) - \frac{1}{2} \frac{\Delta_H^B(\mathbf{v})/v_0^2 - \Delta_H^B(\mathbf{w})/w_0^2}{\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})},$$

$$r^2(\mathbf{w}, \mathbf{v}) = (s(\mathbf{w}, \mathbf{v}) - \mu_{H,B}(\mathbf{v}))^2 - \Delta_H^B(\mathbf{v})/v_0^2. \tag{3.1}$$

When $r^2(\mathbf{w}, \mathbf{v}) < 0$, the wall is empty.

- Let $W_1$, $W_2$ be two numerical walls to the left of $\beta = \mu_{H,B}(\mathbf{v})$ with centers $(s_1, 0)$, $(s_2, 0)$. Then $W_1$ is nested inside $W_2$ if and only if $s_1 > s_2$.
- Let $\mathbf{v}$ and $\mathbf{w}$ have positive rank and $\mu_{H,B}(\mathbf{v}) > \mu_{H,B}(\mathbf{w})$. If $\mu_{H,B}(\mathbf{v}) > \beta$ or $\mu_{H,B}(\mathbf{v}) < \beta$, then $\nu_{\alpha, \beta}(\mathbf{w}) > \langle \nu_{\alpha, \beta}(\mathbf{w}) \rangle$ if and only if the point $(\beta, \alpha)$ is outside (inside) $W(\mathbf{w}, \mathbf{v})$. If $\mu_{H,B}(\mathbf{v}) > \beta > \mu_{H,B}(\mathbf{w})$, then $\nu_{\alpha, \beta}(\mathbf{w}) > \langle \nu_{\alpha, \beta}(\mathbf{w}) \rangle$ if and only if the point $(\beta, \alpha)$ is inside (outside) $W(\mathbf{w}, \mathbf{v})$.

Without loss of generality, we now assume $v_0 > 0$, $w_0 > 0$, $\mu_{H,B}(\mathbf{v}) > \mu_{H,B}(\mathbf{w})$, and the wall $W(\mathbf{w}, \mathbf{v})$ is a non-empty semicircle.

If $s(\mathbf{w}, \mathbf{v}) \leq \mu_{H,B}(\mathbf{v})$, i.e.,

$$\left(\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})\right)^2 \geq \Delta_H^B(\mathbf{w})/w_0^2 - \Delta_H^B(\mathbf{v})/v_0^2, \tag{3.3}$$

since $r(\mathbf{w}, \mathbf{v})^2 \geq 0$, one sees that $\mu_{H,B}(\mathbf{v}) - s \geq \sqrt{\Delta_H^B(\mathbf{v})/v_0}$. This and (3.1) imply

$$\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w}) + \frac{\Delta_H^B(\mathbf{w})/v_0^2 - \Delta_H^B(\mathbf{w})/w_0^2}{\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})} \geq 2 \sqrt{\Delta_H^B(\mathbf{v})/v_0}.$$
i.e.,
\[
\left( \mu_{H,B}(v) - \mu_{H,B}(w) - \sqrt{\Delta_B^H(v)/v_0} \right)^2 \geq \Delta_B^H(w)/w_0^2.
\]
Hence one obtains
\[
\mu_{H,B}(v) - \mu_{H,B}(w) \leq \sqrt{\Delta_B^H(v)/v_0} - \sqrt{\Delta_B^H(w)/w_0}
\]
or
\[
\mu_{H,B}(v) - \mu_{H,B}(w) \geq \sqrt{\Delta_B^H(v)/v_0} + \sqrt{\Delta_B^H(w)/w_0}
\]
If \( s(w,v) \geq \mu_{H,B}(v) \), a similar computation gives
\[
\mu_{H,B}(v) - \mu_{H,B}(w) \leq \sqrt{\Delta_B^H(w)/w_0} - \sqrt{\Delta_B^H(v)/v_0}
\]

\textbf{Definition 3.3.} The wall \( W(w,v) \) is called of Type 1, if it satisfies (3.4). If \( W(w,v) \) satisfies (3.5) (respectively, (3.6)), we call it of Type 2 (respectively, Type 3).

Direct computations show us that the wall of Type 1 lies to the left of \( \beta = \mu_{H,B}(w) < \mu_{H,B}(v) \), the wall of Type 2 lies between \( \beta = \mu_{H,B}(w) \) and \( \beta = \mu_{H,B}(v) \), and the wall of Type 3 is to the right of \( \beta = \mu_{H,B}(v) \) (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Three types of walls}
\end{figure}

\textit{Remark 3.4.} By the definition of \( \text{Coh}^{\beta H+B}(X) \), one sees that an actual wall can not be of Type 2.

3.2. Modifications of walls.

\textbf{Definition 3.5.} Given a vector \( u = (u_0, u_1, u_2) \in \mathbb{Q}^3 \) with \( u_0 \neq 0 \), we define its discriminant free vector \( \tilde{u} \) to be \( (u_0, u_1, u_2^2/2u_0) \in \mathbb{Q}^3 \).

The motivation behind this definition is that \( \tilde{u} \) satisfies \( \tilde{\Delta}_H^B(\tilde{u}) = 0 \) and \( \mu_{H,B}(u) = \mu_{H,B}(\tilde{u}) \).

For a numerical wall \( W(w,v) \) of Type 1, we consider the wall \( W(\tilde{w},v) \) to be its modification. Since
\[
\mu_{H,B}(v) - \mu_{H,B}(\tilde{w}) = \mu_{H,B}(v) - \mu_{H,B}(w)
\]
\[
\leq \sqrt{\Delta_B^H(v)/v_0} - \sqrt{\Delta_B^H(w)/w_0}
\]
\[
\leq \sqrt{\Delta_B^H(v)/v_0} - \sqrt{\Delta_B^H(\tilde{w})/w_0},
\]
and

\[
\left(\mu_{H,B}(v) - \mu_{H,B}(w)\right)^2 \geq -\frac{\Delta^B_H(v)/v_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)},
\]

one sees the wall \(W(\bar{w}, v)\) is also of Type 1. Similarly, if \(W(w, v)\) is of Type 2, then \(W(\bar{w}, v), W(w, \bar{v})\) and \(W(\bar{w}, \bar{v})\) are still of Type 2. If \(W(w, v)\) is of Type 3, then \(W(w, \bar{v})\) is of Type 3.

We can compute the center and radius for the modifications of \(W(w, v)\) more explicitly. Let \((\bar{s}_1, 0)\) and \(\bar{r}_1\) be the center and radius of the circle \(W(\bar{w}, v)\) of Type 1, respectively. Equalities (3.1) and (3.2) give

\[
\bar{s}_1 = \frac{1}{2}(\mu_{H,B}(v) + \mu_{H,B}(w)) - \frac{\Delta^B_H(v)/2v_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)},
\]

\[
\bar{r}_1 = \frac{\Delta^B_H(v)/2v_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)} - \frac{1}{2}(\mu_{H,B}(v) - \mu_{H,B}(w)),
\]

\[
\bar{s}_1 + \bar{r}_1 = \mu_{H,B}(w),
\]

\[
\bar{s}_1 - \bar{r}_1 = \mu_{H,B}(v) - \frac{\Delta^B_H(v)/v_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)}.
\]

If \(W(w, \bar{v})\) is of Type 3, we let \((\bar{s}_3, 0)\) and \(\bar{r}_3\) be its center and radius, respectively. Similarly, one has

\[
\bar{s}_3 = \frac{1}{2}(\mu_{H,B}(v) + \mu_{H,B}(w)) + \frac{\Delta^B_H(w)/2w_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)},
\]

\[
\bar{r}_3 = \frac{-\Delta^B_H(w)/2w_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)} - \frac{1}{2}(\mu_{H,B}(v) - \mu_{H,B}(w)),
\]

\[
\bar{s}_3 + \bar{r}_3 = \mu_{H,B}(w) + \frac{-\Delta^B_H(w)/w_0^2}{\mu_{H,B}(v) - \mu_{H,B}(w)},
\]

\[
\bar{s}_3 - \bar{r}_3 = \mu_{H,B}(v).
\]

From the above equalities, one sees:

**Lemma 3.6.** If \(W(w, v)\) is of Type 1 (respectively, 3), then the semicircle \(W(w, v)\) is inside the semicircle \(W(\bar{w}, v)\) (respectively, \(W(w, \bar{v})\)).

A similar conclusion holds for walls of Type 2, but we do not need it in this paper. See Figure 2 for the Modifications of walls of Type 1 and Type 3.

**Figure 2.** Modifications of walls
4. Extremal ellipses

Throughout this section, we let \( E \neq 0 \) be a torsion free sheaf on \( X \) with \( \text{ch}_H(E) = v = (v_0, v_1, v_2) \) and \( \sum v_i \geq 0 \). We will define the extremal ellipse \( C_E \) for such \( E \). It can bound the rank of the subobject or quotient of \( E \). We keep the same notations as that in the previous sections.

4.1. Extremal ellipses. The following lemmas are our main tools to study the tilt-stability of \( E \) and \( E[1] \). They can be considered as a generalization of [31, Lemma 4.1].

**Lemma 4.1.** Let \( F \) be the \( \nu_{\alpha,\beta} \)-maximal subobject of \( E \in \text{Coh}^{\beta H + B}(X) \) for some \((\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0} \). If

\[
(4.1) \quad v_0 (\beta - \mu_{H, B}(E))^2 + (v_0 + H^n)\alpha^2 \geq \frac{v_0 + H^n \sum B}{v_0 H^n} \beta H(E),
\]

then \( \text{rk}(F) \leq \text{rk}(E) \).

**Proof.** By the long exact sequence of cohomology sheaves induced by the short exact sequence

\[
0 \to F \to E \to Q \to 0
\]

in \( \text{Coh}^{\beta H + B}(X) \), one sees that \( F \) is a torsion free sheaf. If \( E \) is \( \nu_{\alpha,\beta} \)-semistable, then \( F = E \). Thus we are done.

Now we assume that \( E \) is not \( \nu_{\alpha,\beta} \)-semistable. One deduces

\[
\nu_{\alpha,\beta}(F) = \frac{H^{n-2} \text{ch}_2^{\beta H + B}(F) - \frac{1}{2} \alpha^2 H^n \text{ch}_0(F)}{H^{n-1} \text{ch}_1^{\beta H + B}(F)} > \nu_{\alpha,\beta}(E),
\]

i.e.,

\[
(4.2) \quad H^{n-2} \text{ch}_2^{\beta H + B}(F) > \nu_{\alpha,\beta}(E) H^{n-1} \text{ch}_1^{\beta H + B}(F) + \frac{1}{2} \alpha^2 H^n \text{ch}_0(F).
\]

By Theorem 2.13, we obtain

\[
(4.3) \quad \frac{(H^{n-1} \text{ch}_1^{\beta H + B}(F))^2}{2 H^n \text{ch}_0(F)} \geq H^{n-2} \text{ch}_2^{\beta H + B}(F).
\]

Combining (4.2) and (4.3), one sees that

\[
\alpha^2 (H^n \text{ch}_0(F))^2 + 2 \nu_{\alpha,\beta}(E) H^{n-1} \text{ch}_1^{\beta H + B}(F) H^n \text{ch}_0(F) < \left(H^{n-1} \text{ch}_1^{\beta H + B}(F)\right)^2.
\]

This implies

\[
(4.4) \quad H^n \text{ch}_0(F) < \left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 - \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \text{ch}_1^{\beta H + B}(F)}{\alpha^2}.
\]

Since \( F \) is a subobject of \( E \) in \( \text{Coh}^{\beta H + B}(X) \), by the definition of \( \text{Coh}^{\beta H + B}(X) \), we deduce that

\[
0 < H^{n-1} \text{ch}_1^{\beta H + B}(F) \leq H^{n-1} \text{ch}_1^{\beta H + B}(E) = v_1 - \beta v_0.
\]

From (4.4), it follows that

\[
(4.5) \quad H^n \text{ch}_0(F) < \left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 - \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \text{ch}_1^{\beta H + B}(E)}{\alpha^2}.
\]
Hence $\text{rk}(F) \leq \text{rk}(E)$, if one can show that
\[
\left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 - \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \chi_1^{\beta H + B}(E)}{\alpha^2} \leq H^n \chi_0(E) + H^n,
\]
i.e.,
\[(\nu_{\alpha,\beta}(E))^2 + \alpha^2 \leq \left(\frac{\alpha^2 H^n(\chi_0(E) + 1)}{H^{n-1} \chi_1^{\beta H + B}(E)} + \nu_{\alpha,\beta}(E)\right)^2.
\]

On the other hand, a direct computation shows that inequality (4.6) is equivalent to
\[(4.7) \quad (\chi_0(E) + 1)H^n \alpha^2 + 2H^{n-2} \chi_2^{\beta H + B}(E) \geq \sum_H \frac{(\beta H + B)(E)}{H^n} = \frac{\sum_B}{H^n}.
\]
Expanding $\chi_2^{\beta H + B}(E)$, one sees that inequality (4.7) is equivalent to our assumption (4.1). Thus the lemma follows. \hfill \Box

The dual result holds for $E[1]$:

**Lemma 4.2.** Let $F[1]$ be the $\nu_{\alpha,\beta}$-minimal quotient of $E[1] \in \text{Coh}^{\beta H + B}(X)$ for some $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$. If (4.1) holds, i.e.,
\[v_0(\beta - \mu_{H,B}(E))^2 + (v_0 + H^n)\alpha^2 \geq \frac{v_0 + H^n}{v_0 H^n} \sum_B(E),\]
then $\text{rk}(F) \leq \text{rk}(E)$.

**Proof.** The proof follows in the same way as that of Lemma 4.1.

We assume that $E[1]$ is not $\nu_{\alpha,\beta}$-semistable. In this case, one sees that $F$ is a torsion free sheaf with
\[H^{n-1} \chi_1^{\beta H + B}(E) \leq H^{n-1} \chi_1^{\beta H + B}(F) < 0\]
and $\nu_{\alpha,\beta}(E[1]) > \nu_{\alpha,\beta}(F[1])$. One can still obtain (4.2) and (4.3). Since
\[H^{n-1} \chi_1^{\beta H + B}(F) < 0,
\]
(4.4) becomes
\[H^n \chi_0(F) < -\left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 + \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \chi_1^{\beta H + B}(F)}{\alpha^2}.
\]
This implies
\[H^n \chi_0(F) < -\left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 + \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \chi_1^{\beta H + B}(E)}{\alpha^2}.
\]
Therefore Lemma 4.2 follows, if
\[-\left(\sqrt{(\nu_{\alpha,\beta}(E))^2 + \alpha^2 + \nu_{\alpha,\beta}(E)}\right) \frac{H^{n-1} \chi_1^{\beta H + B}(E)}{\alpha^2} \leq H^n \chi_0(E) + H^n.
\]
A direct computation shows that the above inequality is equivalent to (4.1) in the situation of this lemma. Hence we are done. \hfill \Box
**Definition 4.3.** We call the curve in the $(\beta, \alpha)$ half plane defined by the equality of (4.11), i.e.,

$$v_0(\beta - \mu_{H,B}(E))^2 + (v_0 + H^n)\alpha^2 = \frac{v_0 + H^n}{v_0 H^n} \Delta_H^B(E)$$

the extremal ellipse of $E$, and denote it by $C_E$.

### 4.2. The intersection of the wall and the extremal ellipse.

Let $\mathbf{w} = (w_0, w_1, w_2) \in \mathbb{Q}^3$ be a vector with $w_0 > 0$ and $\Delta_H^B(\mathbf{w}) = w_1^2 - 2w_0 w_2 \geq 0$.

**Lemma 4.4.** Assume that $\mu_{H,B}(E) > \mu_{H,B}(\mathbf{w})$ and the wall $W(\mathbf{w}, \mathbf{v})$ is of Type 1. Let $W(\bar{\mathbf{w}}, \mathbf{v})$ be the modification of $W(\mathbf{w}, \mathbf{v})$. Then $C_E \cap W(\bar{\mathbf{w}}, \mathbf{v}) \neq \emptyset$ if and only if

$$\mu_{H,B}(\mathbf{w}) > \mu_{H,B}(E) - \frac{1}{H^n \operatorname{rk} E} \sqrt{\frac{\Delta_H^B(E)}{\operatorname{rk} E + 1}}.$$

**Proof.** Recall that $W(\bar{\mathbf{w}}, \mathbf{v})$ is defined by $(\beta - \bar{s}_1)^2 + \alpha^2 = \bar{r}_1^2$, where

$$\bar{s}_1 = \frac{1}{2}(\mu_{H,B}(\mathbf{v}) + \mu_{H,B}(\mathbf{w})) - \frac{\Delta_H^B(\mathbf{v})/2v_0^2}{\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})},$$

$$\bar{r}_1 = \frac{\Delta_H^B(\mathbf{v})/2v_0^2}{\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})} - \frac{1}{2}(\mu_{H,B}(\mathbf{v}) - \mu_{H,B}(\mathbf{w})).$$

Eliminating $\alpha$ from the equations of $C_E$ and $W(\bar{\mathbf{w}}, \mathbf{v})$, one obtains

$$(\beta - \bar{s}_1)^2 - \frac{v_0}{v_0 + H^n}(\beta - \mu_{H,B}(E))^2 = \bar{r}_1^2 - \frac{\Delta_H^B(E)}{v_0 H^n},$$

i.e.,

$$\frac{H^n}{v_0 + H^n} \beta^2 + 2\left(\frac{\mu_{H,B}(E)}{v_0 + H^n} - \bar{s}_1\right) \beta + \bar{s}_1^2 - \frac{(\mu_{H,B}(E))^2v_0}{v_0 + H^n} - \bar{r}_1^2 + \frac{\Delta_H^B(E)}{v_0 H^n} = 0.$$

We consider (4.11) to be a quadratic equation with variable $\beta$. Let $\delta$ be its discriminant. Then one has

$$\frac{1}{4} \delta = \left(\frac{\mu_{H,B}(E)}{v_0 + H^n} - \bar{s}_1\right)^2 - \frac{H^n}{v_0 + H^n} \left(\bar{s}_1^2 - \frac{(\mu_{H,B}(E))^2v_0}{v_0 + H^n} - \bar{r}_1^2 + \frac{\Delta_H^B(E)}{v_0 H^n}\right)$$

$$= \frac{v_0}{v_0 + H^n} \left(\mu_{H,B}(E) - \bar{s}_1\right)^2 + \frac{H^n \bar{r}_1^2 - \Delta_H^B(E)/v_0}{v_0 + H^n}$$

Since $(\mu_{H,B}(E) - \bar{s}_1)^2 = \bar{r}_1^2 + \Delta_H^B(E)/v_0^2$, one sees $\delta = 4\bar{r}_1^2$. Thus the two solutions of the quadratic equation (4.11) are

$$\beta_\pm = \frac{v_0 + H^n}{H^n} (\bar{s}_1 + \bar{r}_1) - \frac{v_0}{H^n} \mu_{H,B}(E).$$

Since the wall $W(\bar{\mathbf{w}}, \mathbf{v})$ is of Type 1, one deduces

$$\beta_+ - (\bar{s}_1 - \bar{r}_1) = \frac{v_0}{H^n} (\bar{s}_1 - \bar{r}_1 - \mu_{H,B}(E)) < 0.$$
and
\[
\beta_+ - (\tilde{s}_1 + \tilde{r}_1) = \frac{v_0}{H^n}(\tilde{s}_1 + \tilde{r}_1 - \mu_{H,B}(E)) < 0.
\]
These imply that \(C_E \cap W(\tilde{w}, v) \neq \emptyset\) if and only if \(\beta_+ > \tilde{s}_1 - \tilde{r}_1\) (see Figure 3).

On the other hand, one has
\[
\beta_+ - (\tilde{s}_1 - \tilde{r}_1) = \frac{v_0}{H^n}(\tilde{s}_1 - \tilde{r}_1 - \mu_{H,B}(E) + 2\tilde{r}_1)
\]
\[
= \left(\frac{v_0}{H^n} + 1\right)(\mu_{H,B}(w) - \mu_{H,B}(E)) + \frac{\Delta^B_E/v_0^2}{\mu_{H,B}(E) - \mu_{H,B}(w)}.
\]
Therefore \(\beta_+ > \tilde{s}_1 - \tilde{r}_1\) if and only if
\[
\mu_{H,B}(w) > \mu_{H,B}(E) - \frac{1}{H^n \text{rk } E} \sqrt{\frac{\Delta^B_E}{\text{rk } E + 1}}.
\]
This completes the proof. \(\square\)

![Figure 3. Intersection of \(W(\tilde{w}, v)\) and \(C_E\)](image)

For walls of Type 3, we have a similar lemma:

**Lemma 4.5.** Assume that \(\mu_{H,B}(E) < \mu_{H,B}(w)\) and the wall \(W(v, w)\) is of Type 3. Let \(W(\tilde{w}, \tilde{v})\) be the modification of \(W(v, w)\). Then \(C_E \cap W(\tilde{v}, \tilde{w}) \neq \emptyset\) if and only if
\[
\mu_{H,B}(w) < \mu_{H,B}(E) + \frac{1}{H^n \text{rk } E} \sqrt{\frac{\Delta^B_E}{\text{rk } E + 1}}.
\]

**Proof.** The proof is the same as that of Lemma 4.4. \(\square\)

5. **Tilt-stability of \(\mu_{H,B}\)-stable sheaves**

The aim of this section is to establish the tilt-stability for a \(\mu_{H,B}\)-stable torsion free sheaf via computing the intersection of the wall and the extremal ellipse. We always assume that \(E\) is a \(\mu_{H,B}\)-stable torsion free sheaf on \(X\) in this section.

We define
\[
\mu_{H,B}^{\text{max}}(E) = \max \left\{ \mu_{H,B}(F) : F \text{ is a subsheaf of } E, \mu_{H,B}(F) \neq \mu_{H,B}(E) \right\},
\]
and let \(\mu\) be a rational number satisfies \(\mu_{H,B}^{\text{max}}(E) \leq \mu < \mu_{H,B}(E)\).
Theorem 5.1. Let $\beta_0 = \mu_{H, B}(E) - \frac{\sum^B_{H}(E)/(H^n \text{rk} E)^2}{\mu_{H, B}(E) - \mu}$ and $\beta_1 = \mu_{H, B}(E) - \frac{\sqrt{(\text{rk} E + 1)\sum^B_{H}(E)}}{H^n \text{rk} E}$.

1. If $\mu > \mu_{H, B}(E) - \frac{1}{H^n \text{rk} E} \frac{\sum^B_{H}(E)}{\text{rk} E + 1}$, then $E$ is $\nu_{\alpha, \beta}$-stable for any $\alpha > 0$ and $\beta \leq \beta_0$.

2. If $\mu \leq \mu_{H, B}(E) - \frac{1}{H^n \text{rk} E} \frac{\sum^B_{H}(E)}{\text{rk} E + 1}$ and $\sum^B_{H}(E) > 0$, then $E$ is $\nu_{\alpha, \beta_1}$-stable for any $\alpha > 0$.

3. If $\sum^B_{H}(E) = 0$, then $E$ is $\nu_{\alpha, \beta}$-stable for any $\alpha > 0$ and $\beta < \mu_{H, B}(E)$.

Proof. (1) We prove the first statement firstly. Choose a vector $u = (u_0, u_1, u_2) \in \mathbb{Q}^3$ such that $u_0 > 0$, $\frac{u_1}{u_0} = \mu$ and $u_1^2 - 2u_0u_2 = 0$. By (3.1) and (3.2), one sees that the left intersection point of $W(u, v)$ and $\beta$-axis is $(\beta_0, 0)$. We assume $\alpha > 0$ and $\beta \leq \beta_0$. The condition

$$\mu > \mu_{H, B}(E) - \frac{1}{H^n \text{rk} E} \frac{\sum^B_{H}(E)}{\text{rk} E + 1}$$

implies that

$$\beta_0 = \mu_{H, B}(E) - \frac{\sum^B_{H}(E)/(H^n \text{rk} E)^2}{\mu_{H, B}(E) - \mu} < \mu_{H, B}(E) - \frac{1}{H^n \text{rk} E} \sqrt{(\text{rk} E + 1)\sum^B_{H}(E)} = \beta_1 < \mu_{H, B}(E).$$

Since the left intersection point of $C_E$ and $\beta$-axis is just

$$\left( \mu_{H, B}(E) - \frac{1}{H^n \text{rk} E} \sqrt{(\text{rk} E + 1)\sum^B_{H}(E), 0} = (\beta_1, 0), \right.$$

the point $(\beta_0, \alpha)$ is outside the ellipse $C_E$. By Lemma 1.1, the $\nu_{\alpha, \beta}$-maximal subobject $F$ of $E \in \text{Coh}^{H+B}(X)$ satisfies $\text{rk} F \leq \text{rk} E$. Set $\text{ch}_H(F) = w$.

Step 1. $\nu_{\alpha, \beta}$-semistability of $E$.

From the definition of $\text{Coh}^{H+B}(X)$, one sees that $\beta < \mu_{H, B}(F)$. If $\mu_{H, B}(F) \leq \mu$, one can assume that $W(w, v)$ is an actual wall of Type 1. We have $W(w, v)$ is inside its modification $W(w, v)$. By Proposition 3.2, one sees $W(w, v)$ is inside the wall $W(u, v)$. It follows that the point $(\beta, \alpha)$ is outside the wall $W(w, v)$. Therefore we conclude that $\nu_{\alpha, \beta}(F) < \nu_{\alpha, \beta}(E)$. This contradicts our assumption that $F$ is the $\nu_{\alpha, \beta}$-maximal subobject of $E \in \text{Coh}^{H+B}(X)$.

Hence we obtain $\mu_{H, B}(F) > \mu \geq \mu_{H, B}(E)$. Considering the corresponding exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0,$$

in $\text{Coh}^{H+B}(X)$, we get a long exact sequence in $\text{Coh}(X)$:

$$0 \rightarrow \mathcal{H}^{-1}(Q) \rightarrow F \rightarrow E \rightarrow \mathcal{H}^0(Q) \rightarrow 0.$$

When $\mathcal{H}^{-1}(Q) \neq 0$, one has $\mu_{H, B}(\mathcal{H}^{-1}(Q)) \leq 0 < \mu_{H, B}(F)$ and $\text{rk}(F/\mathcal{H}^{-1}(Q)) < \text{rk} E$. This implies

$$\mu_{H, B}(F) < \mu_{H, B}(F/\mathcal{H}^{-1}(Q)) \leq \mu_{H, B}(E) \leq \mu,$$

which is absurd. Thus $\mathcal{H}^{-1}(Q) = 0$, and $F$ is a subsheaf of $E$. By the definition of $\mu_{H, B}(E)$, one sees that $\mu_{H, B}(F) = \mu_{H, B}(E)$ and $\text{rk} F = \text{rk} E$. Hence $Q$ is a torsion sheaf, and the codimension of the support of $Q$ is $\geq 2$. It follows that $\nu_{\alpha, \beta}(F) \leq \nu_{\alpha, \beta}(E)$. Therefore we conclude that $E$ is $\nu_{\alpha, \beta}$-semistable.
Step 2. $\nu_{\alpha, \beta}$-stability of $E$.

We argue by contradiction to show the $\nu_{\alpha, \beta}$-stability of $E$. Suppose that there is a subobject $K \subset E$ in $\text{Coh}^{\beta_{1}H+B}(X)$ such that $\nu_{\alpha, \beta}(K) \geq \nu_{\alpha, \beta}(E/K)$. One sees that $\nu_{\alpha, \beta}(K) = \nu_{\alpha, \beta}(E)$, $0 < \text{ch}_{1}^{\beta_{1}H+B}(K) < \text{ch}_{1}^{\beta_{1}H+B}(E)$, and $K$ is $\nu_{\alpha, \beta}$-semistable. This implies that (1.5) in the proof of Lemma 4.1 holds for $K$. Hence Lemma 4.1 holds also for $K$. The proof of Step 1 shows us that $K$ is a subsheaf of $E$, $\mu_{H, B}(K) = \mu_{H, B}(E)$ and $\text{rk} K = \text{rk} E$. It contradicts that $\text{ch}_{1}^{\beta_{1}H+B}(K) < \text{ch}_{1}^{\beta_{1}H+B}(E)$. Thus we conclude that $E$ is $\nu_{\alpha, \beta}$-stable.

(2) Now we prove the second statement in the same way as above. We assume $\alpha > 0$.

The assumption $\Delta_{H}^{B}(E) > 0$ makes sure that $E$ is an object in $\text{Coh}^{\beta_{1}H+B}(X)$. Since the point $(\beta_{1}, \alpha)$ is outside the ellipse $C_{E}$, by Lemma 4.1, the $\nu_{\alpha, \beta_{1}}$-maximal subobject $F'$ of $E$ in $\text{Coh}^{\beta_{1}H+B}(X)$ satisfies $\text{rk} F' \leq \text{rk} E$. Set $\text{ch}_{H}(F') = w'$. As in Step 1, if $\mu_{H, B}(F') \leq \mu$, we can assume that $W(w', v)$ is an actual wall of Type 1. Hence $W(w', v)$ is inside its modification $W(\tilde{w}', \tilde{v})$. By Lemma 4.1 and the condition

$$\mu \leq \mu_{H, B}(E) - \frac{1}{H \times \text{rk} E} \sqrt{\frac{\Delta_{H}^{B}(E)}{\text{rk} E + 1}},$$

it follows that $W(\tilde{w}', \tilde{v}) \cap C_{E} = \emptyset$. On the other hand, from the definition of $\text{Coh}^{\beta_{1}H+B}(X)$, one sees $\beta_{1} < \mu_{H, B}(F')$. Hence $W(w', v)$ is inside $C_{E}$. This implies that $(\beta_{1}, \alpha)$ is outside the wall $W(w', v)$. Thus $\nu_{\alpha, \beta_{1}}(F') < \nu_{\alpha, \beta_{1}}(E)$ which is absurd. When $\mu_{H, B}(F') > \mu \geq \mu_{H, B}^{\text{max}}(E)$, the proof in Step 1 still works here. It turns out that $E$ is $\nu_{\alpha, \beta}$-semistable. The same argument in Step 2 shows that $E$ is $\nu_{\alpha, \beta}$-stable.

(3) To show the third statement, one can just replace $\beta_{1}$ in the proof of the second statement to any $\beta < \mu_{H, B}(E)$. \hfill \square

Remark 5.2. The above theorem can be improved when $\text{rk} E = 1$. In that case, if $(\beta, \alpha)$ is outside $C_{E}$, the $\nu_{\alpha, \beta}$-maximal subobject of $E$ is just a subsheaf of $E$. Hence One can consider the wall $W(G, E)$ which satisfies

(1) $G$ is a subsheaf of $E$.
(2) $W(G, E)$ is of Type 1.
(3) $W(G, E)$ is as large as possible, subject to (1) and (2).

Computing the intersection of $C_{E}$ and $W(G, E)$, one can obtain a better result.

Now we consider the tilt-stability of $E[1]$. Set

$$Q = \{ Q \in \text{Coh}(X) : Q \text{ is a torsion free quotient sheaf of } E \}$$

and

$$T = \{ G \in \text{Coh}(X) : G \text{ is a torsion free extension of } E \text{ and a torsion sheaf} \}.$$ 
One sees that any $Q \in Q \cup T$ satisfies $\mu_{H, B}(Q) \geq \mu_{H, B}(E)$.

We define

$$\mu_{H, B}^{\text{min}}(E) = \min \{ \mu_{H, B}(Q) : Q \in Q \cup T \text{ and } \mu_{H, B}(Q) \neq \mu_{H, B}(E) \}.$$
Remark 5.3. The definition of \( \mu_{\text{min}}(E) \) is not like that of \( \mu_{\text{max}}(E) \). We must consider the sheaf in \( T \) here, when we investigate the tilt-stability of \( E[1] \). The reason is that a torsion free sheaf has no torsion subsheaf, but can have a torsion quotient.

Theorem 5.4. Assume that \( E \) is a \( \mu_{H,B} \)-stable reflexive sheaf, and \( \bar{\mu} \) is a rational number satisfies \( \mu_{H,B}(E) < \bar{\mu} \leq \mu_{H,B}^{\text{min}}(E) \). Let \( \bar{\beta}_0 = \mu_{H,B}(E) + \frac{\sqrt{\frac{\nu(H^\alpha(E))(H^\beta(E)^2}{\mu - \mu_{H,B}(E)}}}{\nu + \mu_{H,B}(E)} \).

\[ \bar{\beta}_1 = \mu_{H,B}(E) + \frac{\sqrt{\nu(H^\alpha(E))\nu(H^\beta(E))}}{\nu + \mu_{H,B}(E)}. \]

1. If \( \bar{\mu} < \mu_{H,B}(E) + \frac{1}{\nu + \mu_{H,B}(E)} \sqrt{\frac{\nu(H^\alpha(E))}{\nu + \mu_{H,B}(E)}} \), then \( E[1] \) is \( \nu_{\alpha, \beta}\)-stable for any \( \alpha > 0 \) and \( \beta \geq \bar{\beta}_0 \).
2. If \( \bar{\mu} > \mu_{H,B}(E) + \frac{1}{\nu + \mu_{H,B}(E)} \sqrt{\frac{\nu(H^\alpha(E))}{\nu + \mu_{H,B}(E)}} \), then \( E[1] \) is \( \nu_{\alpha, \beta}\)-stable for any \( \alpha > 0 \).
3. If \( \Delta_H^B(E) = 0 \), then \( E[1] \) is \( \nu_{\alpha, \beta}\)-stable for any \( \alpha > 0 \) and \( \beta \geq \mu_{H,B}(E) \).

Proof. The proof is almost the same as that of Theorem 5.1. The only difference is that in our situation, when we consider the \( \nu_{\alpha, \beta}\)-minimal quotient \( Q[1] \) of \( E[1] \) with \( \mu_{H,B}(Q) < \mu \leq \mu_{H,B}^{\text{min}}(E) \), we have \( \mu_{H,B}(E) = \mu_{H,B}(Q) \) and \( \nu(E) = \nu(Q) \). One gets a short exact sequence in \( \text{Coh}^{\beta H+B}(X) \):

\[ 0 \to T \to E[1] \to Q[1] \to 0, \]

where \( T \) is a torsion sheaf with \( \text{codim}(\text{Supp}(T)) \geq 2 \). When \( T \neq 0 \), one sees that \( \nu_{\alpha, \beta}(T) = +\infty \), hence \( E[1] \) can not be \( \nu_{\alpha, \beta}\)-stable. In order to exclude this case, we need the reflexive assumption of \( E \).

To see this, we take the generic point \( x \) of an irreducible component of \( \text{Supp}(T) \), if \( T \neq 0 \). Then one has \( \dim \mathcal{O}_{x, x} \geq 2 \) and \( H^1_x(T_\mathcal{O}) \neq 0 \). By considering the long exact sequence of local cohomology for \( H^1_x(B) \) over the local ring, one sees that \( H^1_x(E_x) \neq 0 \). Thus \( \text{depth}(E_x) \leq 1 \). By 15 Proposition 1.3, this contradicts that \( E \) is reflexive. \( \square \)

6. Vanishing Theorem for \( \mu_{H,B} \)-stable sheaves

In this section we prove Corollary 1.5, Corollary 1.6, Theorem 1.8 and Corollary 1.9. Corollary 1.3 and Corollary 1.9 follow from the lemmas below. We always assume \( B = 0 \) in this section.

Lemma 6.1. Let \( \mathcal{F} \) be an object in \( \text{Coh}^{\beta H}(X) \). If \( \mathcal{F} \) is \( \nu_{\alpha, \beta}\)-stable for any \( \alpha > 0 \), then \( H^{n-1}(X, \mathcal{F}(\mathcal{K}_S + lH)) = 0 \) for any integer \( l > -\beta \).

Proof. Serre duality implies

\[ H^{n-1}(X, \mathcal{F}(\mathcal{K}_S + lH))^\vee = \text{Ext}^1(\mathcal{F}, \mathcal{O}_X(-lH)) \cong \text{Hom}(\mathcal{F}, \mathcal{O}_X(-lH)[1]). \]

Since \( \Delta_H(\mathcal{O}_X(-lH)) = 0 \), by Theorem 5.4 one sees that \( \mathcal{O}_X(-lH)[1] \) is \( \nu_{\alpha, \beta}\)-stable for any \( \alpha > 0 \) and \( l \geq -\beta \).

On the other hand, the wall \( W(\mathcal{F}, \mathcal{O}_X(-lH)[1]) \) is of Type 2, and is defined by \( (\beta - s)^2 + \alpha^2 = r^2 \), where \( s - r = -l \). Hence the point \((\beta, 0)\) is inside the wall \( W(\mathcal{F}, \mathcal{O}_X(-lH)[1]) \) if \( l > -\beta \). This implies that \( \nu_{\alpha, \beta}(\mathcal{F}) > \nu_{\alpha, \beta}(\mathcal{O}_X(-lH)[1]) \) for \( l > -\beta \) and some \( \alpha > 0 \). From the \( \nu_{\alpha, \beta}\)-stability of \( \mathcal{F} \), it follows that \( \text{Hom}(\mathcal{F}, \mathcal{O}_X(-lH)[1]) = 0 \). \( \square \)
Lemma 6.2. Let $F[1]$ be an object in $\text{Coh}^\ast_{X}(X)$. If $F[1]$ is $H_{1,\ast}$-stable for any $\alpha > 0$, then $H^1(X,F(-lH)) = 0$ for any integer $l > \beta$.

Proof. We consider the Type 2 wall $W(O_X(lH),F[1])$; $(\beta - s)^2 + \alpha^2 = r^2$, where $s + r = l$ and $\mu_{H,B}(F) \leq \beta < l$. Hence the point $(\beta,0)$ is inside the wall $W(O_X(lH),F[1])$, if $l > \beta$. It follows that $\nu_{\alpha,\beta}(O_X(lH)) > \nu_{\alpha,\beta}(F[1])$ for some $\alpha > 0$ and $l > \beta$. By the $H_{1,\ast}$-stability of $O_X(lH)$ and $F[1]$, we get our conclusion. \hfill \Box

Proof of Corollary 1.5 and 1.6 Corollary 1.5 follows from Theorem 6.1 and Lemma 6.2 and Corollary 1.6 follows from Theorem 5.4 and Lemma 6.2. \hfill \Box

Proof of Theorem 5.3 Since $h^{n-1}(X,F(K_X + lH)) \leq \sum_{i=1}^k h^{n-1}(X,G_i(K_X + lH))$, one can assume that $F$ is $\mu$-$H$-semistable. Consider its Jordan-Hölder filtration

$$0 = F'_0 \subset F'_1 \subset \cdots \subset F'_{m-1} \subset F'_m = F,$$

and set $Q_i$ be the $\mu$-$H$-stable sheaf $F'_i/F'_{i-1}$.  

For any torsion free sheaf $E$ on $X$, one sees that $H^n\mu_{H}^{\text{max}}(E) \leq [H^n\mu_{H}(E)]_{\text{rk}E}$. Thus applying Corollary 1.5 for $E = Q_i$ and $\mu = [H^n\mu_{H}(Q_i)]_{\text{rk}Q_i}/H^n$, one deduces that $h^{n-1}(X,F(K_X + lH)) \leq \sum_{i=1}^k h^{n-1}(X,Q_i(K_X + lH)) = 0$ for

$$l > \max_{1 \leq i \leq m} \left\{ \frac{\Delta_H(Q_i)/H^n(\text{rk}Q_i)^2}{H^n(\mu_{H}(Q_i) - [H^n\mu_{H}(Q_i)]_{\text{rk}Q_i}) - \mu_{H}(Q_i)} \right\}.$$  

From

$$\frac{\Delta_H(F)}{H^n(\text{rk}F)} = \mu_{H}(F)H^{n-1}ch_1(F) - 2H^{n-2}ch_2(F) = \mu_{H}(F)\sum_{i=1}^m H^{n-1}ch_1(Q_i) - 2\sum_{i=1}^m H^{n-2}ch_2(Q_i) = \sum_{i=1}^m \left( \mu_{H}(Q_i)H^{n-1}ch_1(Q_i) - 2H^{n-2}ch_2(Q_i) \right) = \sum_{i=1}^m \frac{\Delta_H(Q_i)}{H^n(\text{rk}Q_i)},$$

it follows that $\frac{\Delta_H(Q_i)/H^n(\text{rk}Q_i)^2}{H^n(\mu_{H}(Q_i) - [H^n\mu_{H}(Q_i)]_{\text{rk}Q_i}) - \mu_{H}(Q_i)} \leq \frac{\Delta_H(F)}{H^n(\text{rk}F)}$. Hence one sees that

$$\frac{\Delta_H(Q_i)/H^n(\text{rk}Q_i)^2}{H^n(\mu_{H}(Q_i) - [H^n\mu_{H}(Q_i)]_{\text{rk}Q_i}) - \mu_{H}(Q_i)} \leq \frac{\Delta_H(F)/H^n(\text{rk}F)}{H^n(\mu_{H}(F) - [H^n\mu_{H}(F)]_{\text{rk}F}}.$$  

and

$$\frac{\sqrt{(\text{rk}Q_i + 1)\Delta_H(Q_i)}}{H^n(\text{rk}Q_i)} - \mu_{H}(Q_i) \leq \frac{1}{H^n} \sqrt{\frac{2\Delta_H(F)}{\text{rk}F} - \mu_{H}(F)}.$$
It turns out that $H^{n-1}(X, \mathcal{F}(K_X + lH)) = 0$, if

$$l > \max \left\{ \frac{\Sigma_H(\mathcal{F})/H^n}{\mu_H(\mathcal{F})} - \frac{\mu_H(\mathcal{F})}{H^n}, \sqrt{\frac{2 \Sigma_H(\mathcal{F})}{(H^n)^2 \text{rk} \mathcal{F}}} - \frac{\mu_H(\mathcal{F})}{H^n} \right\}.$$ 

This completes the proof. \ensuremath{\Box}

Remark 6.3. The constant $M(\mathcal{F})$ in Theorem 1.8 can have a simpler but weaker form. In fact, by the following lemma, one sees that (6.1) in the proof above becomes

$$\frac{\Sigma_H(\mathcal{Q}_i)/H^n(\text{rk} \mathcal{Q}_i)^2}{H^n \mu_H(\mathcal{Q}_i) - |H^n \mu_H(\mathcal{Q}_i)|_{\text{rk} \mathcal{Q}_i}} \leq \frac{\Sigma_H(\mathcal{Q}_i)}{H^n} \leq \frac{\Sigma_H(\mathcal{F})}{H^n}.$$ 

It follows that

$$M(\mathcal{F}) \leq \max_{1 \leq i \leq k} \left\{ \frac{\Sigma_H(\mathcal{G}_i)}{H^n} - \mu_H(\mathcal{G}_i), \sqrt{\frac{2 \Sigma_H(\mathcal{G}_i)}{(H^n)^2 \text{rk} \mathcal{G}_i}} - \mu_H(\mathcal{G}_i) \right\}.$$ 

Lemma 6.4. Let $r \geq 1$ and $d$ be two integers. Then $\left[\frac{d}{r}\right]_r < \frac{d}{r} - \frac{1}{r}$. If the equality holds, we have $r = 1$.

Proof. We argue by contradiction. Assume that there are two integers $a$ and $b$ satisfy $\frac{a}{b} < \frac{d}{r}, 1 \leq b \leq r$ and $\frac{a}{b} > \frac{d}{r} - \frac{1}{r}$. Then one sees that

$$ar > bd - \frac{b}{r} \geq bd - 1.$$ 

This implies $ar \geq bd$. Hence $\frac{a}{b} \geq \frac{d}{r}$ which is absurd.

If $[\frac{d}{r}]_r = \frac{d}{r} - \frac{1}{r}$, we can write $\frac{a}{b} = \frac{d}{r} - \frac{1}{r}$, where $p$ and $q$ are integers with $1 \leq p \leq r$. It turns out that

$$q^2 = (dr - 1)p.$$ 

Since $r$ and $dr - 1$ are coprime, one sees that $r$ divides $p$. Hence $p = r$. We deduce that $qr = dr - 1$. This implies $r = 1$. \ensuremath{\Box}

Proof of Corollary 1.9. By Remark 6.3, one sees that $H^{n-1}(X, \mathcal{F}(K_X + lH)) = 0$, if

$$l > \max \left\{ \frac{\Sigma_H(\mathcal{F})/H^n}{\mu_H(\mathcal{F})} - \frac{\mu_H(\mathcal{F})}{H^n}, \sqrt{\frac{2 \Sigma_H(\mathcal{F})}{(H^n)^2 \text{rk} \mathcal{F}}} - \frac{\mu_H(\mathcal{F})}{H^n} \right\}.$$ 

From $\text{rk} \mathcal{F} \geq 2$, it follows

$$\frac{\Sigma_H(\mathcal{F})}{H^n} - \mu_H(\mathcal{F}) \geq \sqrt{\frac{2 \Sigma_H(\mathcal{F})}{(H^n)^2 \text{rk} \mathcal{F}}} - \mu_H(\mathcal{F}).$$ 

Thus we are done. \ensuremath{\Box}
7. Chern classes of $\mu_H$-stable sheaves on $\mathbb{P}^3$

In this section we exhibit the applications of Theorem 1.3 to the Chern classes of $\mu_H, B$-stable sheaves on $\mathbb{P}^3$, and prove Theorem 1.10. From now on, we assume that $B = 0$, $X = \mathbb{P}^3$, $H$ is a plane on $\mathbb{P}^3$, and $E$ is a $\mu_H$-stable torsion free sheaf on $\mathbb{P}^3$.

Proof of Theorem 1.10. If $E$ is $\nu_{\alpha, \beta}$-semistable for any $\alpha > 0$, by Theorem 2.15, we have

\[
\frac{2}{3} \left( \frac{\mu E}{\text{rk } E} \right)^2 \geq \frac{1}{\text{rk } E} (\chi(E) - \beta \cdot \text{rk } E).
\]

Substituting

\[
\begin{align*}
\chi_1^\beta(E) & = \chi_1(E) - \beta \cdot \text{rk } E \\
\chi_2^\beta(E) & = \chi_2(E) - \beta \cdot \chi_1(E) + \frac{\beta^2}{2} \cdot \text{rk } E \\
\chi_3^\beta(E) & = \chi_3(E) - \beta \cdot \chi_2(E) + \frac{\beta^2}{2} \cdot \chi_1(E) - \frac{\beta^3}{6} \cdot \text{rk } E
\end{align*}
\]

into (7.1), we have

\[
4 \chi_2^\beta(E) + \beta^2 \Delta_H(E) - 2 \beta \chi_1(E) \chi_2(E) \geq 6 \chi_1(E) \chi_2(E) \chi_3(E).
\]

This implies

\[
\Delta_H(E) \geq 6 \chi_3(E),
\]

(7.2)

where $l(E) = \frac{c_1(E) - 3c_1(E) \Delta_H(E)}{6 \beta \chi_1(E)}$.

From Theorem 1.3, one sees that if $\mu_H^\text{max}(E) > \mu_H(E) - \frac{1}{\text{rk } E} \sqrt{\frac{\Delta_H(E)}{\text{rk } E}}$, then

\[
\Delta_H(E) \geq 6 \chi_3(E).
\]

Since $\mu_H(E) \geq \mu_H^\text{max}(E)$, we deduce

\[
\Delta_H(E) \geq 6 \chi_3(E),
\]

If $\mu_H^\text{max}(E) \leq \mu_H(E) - \frac{1}{\text{rk } E} \sqrt{\frac{\Delta_H(E)}{\text{rk } E}}$, by Theorem 1.3 and (7.2), one has

\[
\frac{(\text{rk } E + 2) \sqrt{\Delta_H(E)} + 2}{\text{rk } E} \geq 6 \chi_3(E).
\]

Thus Theorem 1.10 follows. \qed

In particular, when $\text{rk } E = 2$, one sees $[\mu_H(E)]_2 = \frac{c_1(E)_2}{2}$. Hence by Theorem 1.3 and (7.2), Theorem 1.10 gives Corollary 1.11. From $[0]_3 = -\frac{1}{3}$, $[-\frac{1}{3}]_3 = -\frac{1}{3}$ and $[-\frac{2}{3}]_3 = -1$, we can also bound $c_3$ for a rank 3 stable sheaf on $\mathbb{P}^3$ (compare it with the bounds got by Ein, Hartshorne and Vogelaar [14]).
Proof of Corollary 1.13. By Lemma 6.4, one deduces
\[ \mu_H(E) - \mu_H^\text{max}(E) \geq \mu_H(E) - [\mu_H(E)]_{\text{rk} E} \geq \frac{1}{(\text{rk} E)^2}. \]

If \( \mu_H^\text{max}(E) > \mu_H(E) - \frac{1}{\text{rk} E} \sqrt{\frac{\Delta_H(E)}{\text{rk} E + 1}} \), then
\[ \mu_H(E) - [\mu_H(E)]_{\text{rk} E} + \frac{\Delta_H(E)}{\text{rk} E} \leq \frac{1}{(\text{rk} E)^2} + \Delta_H(E). \]

Hence Theorem 1.10 implies
\[ \Delta_H(E) \geq \frac{1}{6 \text{rk} E} \left( \frac{1}{(\text{rk} E)^2 + \Delta_H(E)} + \mu_H(E) - [\mu_H(E)]_{\text{rk} E} \right) \geq \frac{(\text{rk} E + 2)(\Delta_H(E))^2}{6(\text{rk} E)^2 \sqrt{\text{rk} E + 1}}. \]

This completes the proof. □

Since the Bogomolov-Gieseker type inequality for tilt-stable objects also holds on quadric threefolds, abelian threefolds and Fano threefolds of Picard number one, one can deduces similar results in this section for \( \mu_H \)-stable torsion free sheaves on such threefolds.

References

1. D. Arcara and A. Bertram, Bridgeland-stable moduli spaces for K-trivial surfaces. J. Eur. Math. Soc. 15 (2013), no. 1, 1–38. With an appendix by Max Lieblich.
2. D. Arcara, A. Bertram, I. Coskun and J. Huizenga, The minimal model program for the Hilbert scheme of points on \( \mathbb{P}^2 \) and Bridgeland stability. Adv. Math. 235 (2013), 580–626.
3. A. Bayer, A. Bertram, E. Macrì and Y. Toda, Bridgeland stability conditions on threefolds II: An application to Fujita’s conjecture. J. Algebraic Geom. 23 (2014), 693–710.
4. A. Bayer and E. Macrì, MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. Invent. Math. 198 (2014), no. 3, 505–590.
5. A. Bayer and E. Macrì, Projectivity and birational geometry of Bridgeland moduli spaces. J. Amer. Math. Soc. 27 (2014), no. 3, 707–752.
6. A. Bayer, E. Macrì and P. Stellari, Stability conditions on abelian threefolds and some Calabi-Yau threefolds. Invent. Math. 206 (2016), no. 3, 869–933.
7. A. Bayer, E. Macrì and Y. Toda, Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. J. Algebraic Geom. 23 (2014), 117–163.
8. M. C. Beltrametti and A. J. Sommese, Zero cycles and the k-th order embeddings of smooth projective surfaces, (with an appendix by L. Göttsche), in: Problems in the Theory of Surfaces and Their Classification, (eds: F. Catanese, C. Ciliberto and M. Cornalba), Sympos. Math. 32 (1991), Academic Press, London, 33–48.
9. M. Bernardara, E. Macrì, B. Schmidt and X. Zhao, Bridgeland Stability Conditions on Fano Threefolds. [arXiv:1607.08199]
10. B. Bolognese, J. Huizenga, Y. Lin, E. Riedl, B. Schmidt, M. Woolf, and X. Zhao, Nef cones of Hilbert schemes of points on surfaces. [arXiv:1509.04722]
11. T. Bridgeland, Stability conditions on triangulated categories. Ann. of Math. 166 (2007), no. 2, 317–345.
12. T. Bridgeland, Stability conditions on K3 surfaces. Duke Math. J. 141 (2008), no. 2, 241–291.
13. I. Coskun and J. Huizenga, Interpolation, Bridgeland stability and monomial schemes in the plane. J. Math. Pures Appl. 102 (2014), 930–971.
14. L. Ein, R. Hartshorne and H. Vogelaar, Restriction theorems for rank 3 vector bundles on \( \mathbb{P}^n \). Math. Ann. 259 (1982), 541–569.
15. R. Hartshorne, Stable reflexive sheaves. Math. Ann. 254 (1980), 121–176.
16. D. Happel, I. Reiten, and S. Smalø, Tilting in abelian categories and quasitilted algebras. Mem. Amer. Math. Soc. 120 (1996), viii+ 88.
17. A. Langer, Moduli spaces and Castelnuovo-Mumford regularity of sheaves on surfaces. Am. J. Math. 128 (2006), no. 2, 373–417.
18. R. Lazarsfeld, Positivity in algebraic geometry I & II, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48 & 49, Springer-Verlag, Berlin, 2004.
19. C. Li, Stability conditions on Fano threefolds of Picard number one. arXiv:1510.04089
20. C. Li and X. Zhao, Birational models of moduli spaces of coherent sheaves on the projective plane. arXiv:1603.05035
21. W. Liu, Bayer-Macrì decomposition on Bridgeland moduli spaces over surfaces. arXiv:1501.06397
22. J. Lo and Z. Qin, Mini-wall for Bridgeland stability conditions on the derived category of sheaves over surfaces. Asian J. Math. 18 (2014), no. 2, 321–344.
23. A. Maciocia, Computing the walls associated to Bridgeland stability conditions on projective surfaces. Asian J. Math. 18 (2014), no. 2, 263–279.
24. E. Macrì, A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space. Algebra Number Theory 8 (2014), no. 1, 173–190.
25. H. Minamide, S. Yanagida, and K. Yoshioka, Some moduli spaces of Bridgeland’s stability conditions. Int. Math. Res. Not. 19 (2014), 5264–5327.
26. H. Nuer, Projectivity and birational geometry of Bridgeland moduli spaces on an Enriques surface. arXiv:1406.09008
27. D. Piyaratne, Generalized Bogomolov-Gieseker type inequalities on Fano 3-folds. arXiv:1607.07172
28. I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math., 127 (1988), 309–316.
29. B. Schmidt, A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold. Bull. Lond. Math. Soc. 46 (2014), no. 5, 915–923.
30. B. Schmidt, Counterexample to the generalized Bogomolov-Gieseker inequality for threefolds. arXiv:1609.05055
31. H. Sun, Arithmetic genus of integral space curves. arXiv:1605.06888
32. S.-L. Tan, Effective behavior of multiple linear systems, Asian J. Math. 7 (2003), 1–18.
33. Y. Toda, Curve counting theories via stable objects I. DT/PT correspondence. J. Amer. Math. Soc. 23 (2010), no. 4, 1119–1157.

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