Approximate transmission conditions for a Poisson problem at mid-diffusion

Khaled El-Ghaouti BOUTARENE
USTHB, Faculty of Mathematics, AMNEDP Laboratory,
PO Box 32, El Alia 16111, Bab Ezzouar, Algiers , Algeria
kboutarene@usthb.dz, boutarenekhaled@yahoo.fr

May 12, 2014

Abstract

This work consists in the asymptotic analysis of the solution of Poisson equation in a bounded domain of $\mathbb{R}^P$ ($P = 2, 3$) with a thin layer. We use a method based on hierarchical variational equations to derive asymptotic expansion of the solution with respect to the thickness of the thin layer. We determine the first two terms of the expansion and prove the error estimate made by truncating the expansion after a finite number of terms. Next, using the first two terms of the asymptotic expansion, we show that we can model the effect of the thin layer by a problem with transmission conditions of order two.

Keywords: Asymptotic analysis; Asymptotic expansion; Approximate transmission conditions; Thin layer; Poisson equation.

1 Introduction

This paper deals with the study of the asymptotic behavior of the solution of Poisson equation in a bounded domain $\Omega$ of $\mathbb{R}^P$ ($P = 2, 3$) consisting of two sub-domains separated by a thin layer of thickness $\delta$ (destined to tend to 0). The mesh of these thin geometries presents numerical instabilities that can severely damage the accuracy of the entire process of resolution. To overcome this difficulty, we adopt asymptotic methods to model the effect of the thin layer by problems with either appropriate boundary conditions when we consider a domain surrounded by a thin layer (see for instance [2, 4, 10, 11]) or, as in this paper, with suitable transmission conditions on the interface (see for instance [6, 8, 15, 16, 18, 19]). Although this type of conditions has been widely studied, there is still a lot to be understood concerning the effects of thin shell and their modelisation. Our motivation comes from [17, 18], in which the authors have worked on problems of electromagnetic and biological origins. We cite for example that of Poignard [17, Chapter 2]. He considered a cell immersed in an ambient medium and studied the electric field in the transverse magnetic (TM) mode at mid-frequency and from which our problem was inspired.
Let us give now precise notations. Let $\Omega$ be a bounded domain of $\mathbb{R}^P$ ($P = 2, 3$) consisting of three smooth sub-domains: an open bounded subset $\Omega_{i,\delta}$ with regular boundary $\Gamma_{\delta,1}$, an exterior domain $\Omega_{e,\delta}$ with disjoint regular boundaries $\Gamma_{\delta,2}$ and $\partial \Omega$, and a membrane $\Omega_\delta$ (thin layer) of thickness $\delta$ separating $\Omega_{i,\delta}$ from $\Omega_{e,\delta}$ (see Fig. 1).

![Figure 1: Geometric data](image)

Define the piecewise regular function $\alpha$ by

$$
\alpha(x) = \begin{cases} 
\alpha_e & \text{if } x \in \Omega_{e,\delta}, \\
\alpha_d & \text{if } x \in \Omega_{d}, \\
\alpha_i & \text{if } x \in \Omega_{i,\delta}, 
\end{cases}
$$

where $\alpha_e$, $\alpha_d$ and $\alpha_i$ are strictly positive constants satisfying $\alpha_i < \alpha_d < \alpha_e$ or $\alpha_e < \alpha_d < \alpha_i$ which correspond to the case of mid-diffusion. For a given $f$ in $C^\infty(\Omega)$, we are interested in the unique solution $u_\delta := (u_{i,\delta}, u_{d,\delta}, u_{e,\delta})$ in $H^1(\Omega)$ of the following diffusion problem

$$
\begin{cases}
-\text{div}(\alpha \nabla u_\delta) = f & \text{in } \Omega, \\
\alpha \partial_n u_\delta = 0 & \text{on } \partial \Omega,
\end{cases}
$$

with transmission conditions on the interfaces

$$
\begin{align*}
\alpha_{\delta} \partial_{n_{\delta,2}} u_{d,\delta} |_{\Gamma_{\delta,2}} &= \alpha_e \partial_{n_{\delta,2}} u_{e,\delta} |_{\Gamma_{\delta,2}} & \text{on } \Gamma_{\delta,2}, \\
\alpha_{\delta} \partial_{n_{\delta,1}} u_{e,\delta} |_{\Gamma_{\delta,1}} &= \alpha_e \partial_{n_{\delta,1}} u_{e,\delta} |_{\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1}, \\
\alpha_{\delta} \partial_{n_{\delta,1}} u_{i,\delta} |_{\Gamma_{\delta,1}} &= \alpha_i \partial_{n_{\delta,1}} u_{d,\delta} |_{\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1},
\end{align*}
$$

where $\partial_{n_{e}}$, $\partial_{n_{\delta,2}}$ and $\partial_{n_{\delta,1}}$ denote the derivatives in the direction of the unit normal vectors $n_e$, $n_{\delta,2}$ and $n_{\delta,1}$ to $\partial \Omega$, $\Gamma_{\delta,2}$ and $\Gamma_{\delta,1}$ respectively (see Fig. 1).

The main result of this paper is to approximate the solution $u_\delta$ of Problem (1) by a solution of a problem involving Poisson equation in $\Omega$ with two sub-domains separated by an arbitrary interface $\Gamma$ between $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$ (see Fig. 2 and Fig. 3), with transmission conditions of order two on $\Gamma$, modeling the effect of the thin layer. However, it seems that the existence and uniqueness of the solution of this problem are not obvious. Therefore, we rewrite the problem into a pseudodifferential equation (cf. [5]) and show that in the case of mid-diffusion, we can find the appropriate position of the surface $\Gamma$ to solve this equation. The cases 3D and 2D are similar. We treat the three-dimensional case and the two dimensional one comes as a remark.
The present paper is organized as follows. In Section 2, we give the statement of the model problem considered. In Section 3, we collect basic results of differential geometry of surfaces. Sections 4 and 5 are devoted to the asymptotic analysis of our problem. We present, in Section 4, hierarchical variational equations suited to the construction of a formal asymptotic expansion up to any order, while Section 5 focuses on the convergence of this ansatz. With the help of the asymptotic expansion of the solution $u_\delta$, we model, in the last section, the effect of the thin layer by a problem with appropriate transmission conditions.

2 Problem setting

![Diagram of the studied problem](image1)

Figure 2: Geometry of the studied problem

![Diagram of the thin layer](image2)

Figure 3: The thin layer $\Omega_\delta$

We consider a parallel surface $\Gamma$ to $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$ dividing $\Omega_\delta$ into two thin layers $\Omega_{\delta,1}$ and $\Omega_{\delta,2}$ of thickness respectively $p_1\delta$ and $p_2\delta$, where $p_1$ and $p_2$ are nonnegative real numbers satisfying $p_1 + p_2 = 1$ and such that $p_1$ and $p_2$ belong to a small neighborhood of $1/2$ (see Fig. 2 and Fig. 3). The term small neighborhood means that the constants $p_1$ and $p_2$ are not too close to 1 or 0, in order to avoid having a layer too thin compared to the other because the following analysis does not lend itself to this case. Under the aforementioned assumptions, we investigate in $H^1(\Omega)$ the solution $u_\delta := (u_{i,\delta}, u_{d,\delta,1}, u_{d,\delta,2}, u_{e,\delta})$ of the following problem

$$-\text{div}(\alpha \nabla u_\delta) = f \quad \text{in } \Omega, \quad u_\delta|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega,$$

(2a)

with transmission conditions

$$u_{d,\delta,1}|_{\Gamma_{\delta,1}} = u_{d,\delta,2}|_{\Gamma_{\delta,2}} \quad \text{on } \Gamma_{\delta,2},$$

(2c)

$$\alpha_{\delta} n_{\delta,2} u_{d,\delta,2}|_{\Gamma_{\delta,2}} = \alpha_{\delta} n_{\delta,1} u_{d,\delta,1}|_{\Gamma_{\delta,1}} \quad \text{on } \Gamma, \quad u_{d,\delta,1}|_{\Gamma_{\delta,1}} = u_{d,\delta,2}|_{\Gamma_{\delta,2}} \quad \text{on } \Gamma \delta,$$

(2d)

$$\alpha_{\delta} \partial_n u_{d,\delta,1}|_{\Gamma_{\delta,1}} = \alpha_{\delta} \partial_n u_{d,\delta,2}|_{\Gamma_{\delta,2}} \quad \text{on } \Gamma \delta,$$

(2e)

$$u_{i,\delta}|_{\Gamma_{\delta,1}} = u_{i,\delta}|_{\Gamma_{\delta,2}} \quad \text{on } \Gamma \delta, \quad \alpha_{\delta} n_{\delta,1} u_{i,\delta}|_{\Gamma_{\delta,1}} = \alpha_{\delta} n_{\delta,2} u_{i,\delta}|_{\Gamma_{\delta,2}} \quad \text{on } \Gamma \delta,$$

(2f)

where $\partial_n$ denotes the derivative in the direction of the unit normal vector $n$ to $\Gamma$ (outer for $\Omega_{\delta,1}$ and inner for $\Omega_{\delta,2}$).
3 Notations and definitions

The goal of this section is to define and to collect the main features of differential geometry \([9]\) (see also \([13]\)) in order to formulate our problem in a fixed domain (independent of \(\delta\)) which is a key tool to determine the asymptotic expansion of the solution \(u_\delta\).

In the sequel, Greek indice \(\beta\) takes the values 1 and 2. Let \(I_{\delta,1} = (-\delta,0)\) and \(I_{\delta,2} = (0,\delta)\).

We parameterize the thin shell \(\Omega_{\delta,\beta}\) by the manifold \(\Gamma \times I_{\delta,\beta}\) through the mapping \(\psi_{\beta}\) defined by
\[
\left\{ \begin{array}{l}
\Gamma \times I_{\delta,\beta} \xrightarrow{\psi_{\beta}} \Omega_{\delta,\beta} \\
(m, \eta_{\beta}) \rightarrow x := m + p_{\beta} \eta_{\beta} n(m).
\end{array} \right.
\]

As well-known \([9]\), if the thickness of \(\Omega_{\delta,\beta}\) is small enough, \(\psi_{\beta}\) is a \(C^\infty\)-diffeomorphism of manifolds and it is also known \([15, \text{Remark 2.1}]\) that the normal vector \(n_{\delta,\beta}\) to \(\Gamma_{\delta,\beta}\) can be identified to \(n\). To each function \(v_{\beta}\) defined on \(\Omega_{\delta,\beta}\), we associate the function \(\tilde{v}_{\beta}\) defined on \(\Gamma \times I_{\delta,\beta}\) by
\[
\left\{ \begin{array}{l}
\tilde{v}_{\beta}(m, \eta_{\beta}) := v_{\beta}(x), \\
x = \psi_{\beta}(m, \eta_{\beta}),
\end{array} \right.
\]

then, we have
\[
\nabla v_{\beta} = (I + p_{\beta} \eta_{\beta} R)^{-1} \nabla_{\Gamma} \tilde{v}_{\beta} + p_{\beta}^{-1} \partial_{\eta_{\beta}} \tilde{v}_{\beta} n,
\]

where \(\nabla_{\Gamma} \tilde{v}_{\beta}(m)\) and \(R\) are respectively the surfacic gradient of \(\tilde{v}\) at \(m \in \Gamma\) and the curvature operator \(R\) of \(\Gamma\) at point \(m\). The volume element on the thin shell \(\Omega_{\delta,\beta}\) is given by
\[
d\Omega_{\delta,\beta} = p_{\beta} \det (I + p_{\beta} \eta_{\beta} R) \ d\Gamma \ d\eta_{\beta}.
\]

Now, we introduce the scaling \(s_{\beta} = \eta_{\beta}/\delta\), and the intervals \(I_1 = (-1,0)\) and \(I_2 = (0,1)\) such that the \(C^\infty\)-diffeomorphism \(\Phi_{\beta}\), defined by
\[
\left\{ \begin{array}{l}
\Omega^{\beta} := \Gamma \times I_{\beta} \xrightarrow{\Phi_{\beta}} \Omega_{\delta,\beta} \\
(m, s_{\beta}) \rightarrow x := m + \delta p_{\beta} s_{\beta} n(m),
\end{array} \right.
\]

parameterizes the thin shell \(\Omega_{\delta,\beta}\). To any function \(v_{\beta}\) defined on \(\Omega_{\delta,\beta}\), we associate the function \(v^{[\beta]}\) defined on \(\Omega^{\beta}\) through
\[
\left\{ \begin{array}{l}
v^{[\beta]}(m, s_{\beta}) := v_{\beta}(x), \\
x = \Phi_{\beta}(m, s_{\beta}),
\end{array} \right.
\]

then the gradient takes the form
\[
\nabla v_{\beta} = (I + \delta p_{\beta} s_{\beta} R)^{-1} \nabla_{\Gamma} v^{[\beta]} + p_{\beta}^{-1} \delta^{-1} \partial_{s_{\beta}} v^{[\beta]} n.
\] \((3)\)

The volume element on the thin shell \(\Omega_{\delta,\beta}\) becomes
\[
d\Omega_{\delta,\beta} = p_{\beta} \delta \det J_{\delta,\beta} \ d\Gamma ds_{\beta}, \quad (4)
\]

where
\[
J_{\delta,\beta} := I + p_{\beta} \delta s_{\beta} R.
\]
Let $u_\beta$ and $v_\beta$ be two regular functions defined on $\Omega_{\delta,\beta}$. From (3) and (4), we get the change of variables formula

$$\int_{\Omega_{\delta,\beta}} \nabla u_\beta \cdot \nabla v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^3} J_{\delta,\beta}^{-2/3} \Gamma u^{[\beta]} \cdot \nabla v_\beta \, d\Gamma ds_\beta$$

$$+ p_\beta^{-1} \delta^{-1} \int_{\Omega^3} \partial s_\beta u^{[\beta]} \partial s_\beta v_\beta \, d\Gamma ds_\beta.$$  

(5)

**Remark 1** In the two-dimensional case, if $m \in \Gamma$, we parameterize the curve $\Gamma$ by $m(t)$ where $t \in (0, l_{\Gamma})$ is the curvilinear abscissa and $l_{\Gamma}$ is the length of the curve $\Gamma$, then formula (2) turns into

$$\int_{\Omega_{\delta,\beta}} \nabla u_\beta \cdot \nabla v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^3} (1 + p_\beta \delta s_\beta R) \partial s_\beta u^{[\beta]} \partial s_\beta v_\beta \, d\Gamma ds_\beta$$

$$+ p_\beta \delta \int_{\Omega^3} (1 + p_\beta \delta s_\beta R)^{-1} \partial t u^{[\beta]} \partial t v_\beta \, d\Gamma ds_\beta.$$  

(6)

**Remark 2** For any function $u$ defined in a neighborhood of $\Gamma$, we denote, for convenience, by $u|_{\Gamma}$ the trace of $u$ on $\Gamma$ indifferently in local coordinates or in Cartesian coordinates.

# 4 The asymptotic analysis

This section is devoted to the asymptotic analysis of the solution of Problem (2). We show that this latter is equivalent to a variational equation from which we derive the asymptotic expansion of $u_\delta$. We give a hierarchy of variational equations needed to determine the terms of the expansion and we calculate the first two terms of the expansion.

Let $v_\delta$ be in $H^1(\Omega_\delta)$. We denote by $v_{d,\delta}$ its restriction to $\Omega_{\delta,\beta}$. Multiplying Equation $-div (\alpha_\delta \nabla v_{d,\delta}) = f_{\Omega_\delta}$ in $\Omega_\delta$, by test functions $v_\delta$, using (2a), (2b), (2c) and Green’s formula, we get

$$\langle \alpha_\delta \partial_{n_{\delta,1}} u_{i,\delta} |_{\Gamma_{\delta,1}}, v_{d1} \rangle_{H^{-1/2}(\Gamma_{\delta,1}) \times H^{1/2}(\Gamma_{\delta,1})} + \alpha_\delta \int_{\Omega_\delta} \nabla u_{d,\delta} \cdot \nabla v_\delta \, d\Omega_\delta$$

$$- \langle \alpha_\delta \partial_{n_{\delta,2}} u_{e,\delta} |_{\Gamma_{\delta,2}}, v_{d2} \rangle_{H^{-1/2}(\Gamma_{\delta,2}) \times H^{1/2}(\Gamma_{\delta,2})} = \int_{\Omega_\delta} f_{\Omega_\delta} v_\delta \, d\Omega_\delta;$$

in which $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma_{\delta,\beta}) \times H^{1/2}(\Gamma_{\delta,\beta})}$ denotes the duality pairing between $H^{-1/2}(\Gamma_{\delta,\beta})$ and $H^{1/2}(\Gamma_{\delta,\beta})$. We use the dilation in the thin layer and Formula (5), to obtain

$$\langle \alpha_\delta \partial_{n_{\delta,1}} u_{i,\delta} |_{\Gamma_{\delta,1}} \circ \Phi_1(m, -1), v^{[1]}(m, -1) \rangle_{H^{-1/2}(\Gamma \times \{-1\}) \times H^{1/2}(\Gamma \times \{-1\})}$$

$$- \langle \alpha_\delta \partial_{n_{\delta,2}} u_{e,\delta} |_{\Gamma_{\delta,2}} \circ \Phi_2(m, 1), v^{[2]}(m, 1) \rangle_{H^{-1/2}(\Gamma \times \{1\}) \times H^{1/2}(\Gamma \times \{1\})}$$

$$+ \sum_{\beta=1}^2 \left[ \alpha_\delta \delta a_\delta^{[\beta]}(u^{[\beta]}_{d,\delta}, v^{[\beta]}_\delta) \right] = \int_{\Omega_\delta} f_{\Omega_\delta} v_\delta \, d\Omega_\delta,$$  

(8)
which is the starting point for the asymptotic analysis, where the bilinear form \( a^{[\beta]}(\ldots) \) is defined by

\[
a^{[\beta]}_\delta(u^{[\beta]}, v^{[\beta]}) := p_\beta \int_{\Omega^\delta} J_{\delta,\delta}^{-2} \nabla u^{[\beta]} \cdot \nabla v^{[\beta]} \det J_{\delta,\delta} d\Gamma ds_\beta
\]

\[
+ p_\beta^{-1} \delta^{-2} \int_{\Omega^\delta} \partial_{s_\delta} u^{[\beta]} \partial_{s_\delta} v^{[\beta]} \det J_{\delta,\delta} d\Gamma ds_\beta,
\]

for every \( u^{[\beta]} \) and \( v^{[\beta]} \) in \( H^1(\Omega^\delta) \).

### 4.1 Hierarchy of the variational equations

In the spirit of \([15, 18]\), we will consider two asymptotic expansions. Exterior expansions corresponding to the asymptotic expansion of \( u_\delta \) restricted to \( \Omega_{e,\delta} \) and to \( \Omega_{i,\delta} \) and characterized by the ansatz

\[
u_{e,\delta} = \nu_{e,0} + \delta \nu_{e,1} + \cdots, \quad u_{i,\delta} = u_{i,0} + \delta u_{i,1} + \cdots,
\]

where the terms \( \nu_{e,n} \) and \( u_{i,n} \) (\( n \in \mathbb{N} \)) are independent of \( \delta \) and defined on \( \Omega_{e} := \Omega_{e,\delta} \cup \Gamma_{\delta,2} \cup \Omega_{\delta,2} \), and on \( \Omega_{i} := \Omega_{i,\delta} \cup \Gamma_{\delta,1} \cup \Omega_{\delta,1} \) which are respectively the limits of \( \Omega_{e,\delta} \) and \( \Omega_{i,\delta} \) for \( \delta \to 0 \). They fulfill

\[
\begin{cases}
-\text{div}(\alpha_e \nabla \nu_{e,n}) = \delta_{0,n} f_{\Omega_e} & \text{in } \Omega_e, \\
-\text{div}(\alpha_i \nabla u_{i,n}) = \delta_{0,n} f_{\Omega_i} & \text{in } \Omega_i, \\
u_{e,n}\mid_{\partial \Omega} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \delta_{0,n} \) indicates the Kronecker symbol, and an interior expansion corresponding to the asymptotic expansion of \( u_{d,\delta} \) written in a fixed domain and defined by the ansatz

\[
u_{d,\delta}^{[\beta]} = \nu_{e,0}^{[\beta]} + \delta \nu_{1,0}^{[\beta]} + \cdots, \quad \text{in } \Omega^\delta,
\]

where the terms \( \nu_{n}^{[\beta]} \), \( n \in \mathbb{N} \), are independent of \( \delta \). Using a Taylor expansion in the normal variable, we infer formally

\[
u_{i,0} \circ \Phi_1(m, s_1) = u_{i,0}\mid_{\Gamma} + \delta(u_{i,1}\mid_{\Gamma} + s_1 p_1 \partial_n u_{i,0}\mid_{\Gamma})
\]

\[
+ \delta^2(u_{i,2}\mid_{\Gamma} + s_1 p_1 \partial_n u_{i,1}\mid_{\Gamma} + \frac{s_1^2}{2} p_1^2 \partial_n^2 u_{i,0}\mid_{\Gamma}) + \cdots ,
\]

\[
:= U_{i,0} + \delta U_{i,1} + \delta^2 U_{i,2} + \cdots .
\]

\[
u_{e,0} \circ \Phi_2(m, s_2) = u_{e,0}\mid_{\Gamma} + \delta(u_{e,1}\mid_{\Gamma} + s_2 p_2 \partial_n u_{e,0}\mid_{\Gamma})
\]

\[
+ \delta^2(u_{e,2}\mid_{\Gamma} + s_2 p_2 \partial_n u_{e,1}\mid_{\Gamma} + \frac{s_2^2}{2} p_2^2 \partial_n^2 u_{e,0}\mid_{\Gamma}) + \cdots ,
\]

\[
:= U_{e,0} + \delta U_{e,1} + \delta^2 U_{e,2} + \cdots .
\]

and Transmission Conditions \([2c]\) and \([2g]\) become

\[
u_{e,0}\mid_{\Gamma} + \delta(u_{e,1}\mid_{\Gamma} + p_2 \partial_n u_{e,0}\mid_{\Gamma}) + \cdots = u_{0,0}^{[2]} + \delta u_{0,1}^{[2]} + \cdots ,
\]

\[
u_{i,0}\mid_{\Gamma} + \delta(u_{i,1}\mid_{\Gamma} - p_1 \partial_n u_{i,0}\mid_{\Gamma}) + \cdots = u_{0,0}^{[1]} + \delta u_{0,1}^{[1]} + \cdots .
\]
As  
\[-\text{div} \left( \alpha_e \nabla \left( \sum_{n \geq 0} \delta^n u_{e,n} \right) \right) = f_{\Omega_{\delta,2}},\]

thanks to Green’s formula, we get

\[
\begin{align*}
\left\langle \alpha_e \partial_n \left( \sum_{n \geq 0} \delta^n u_{e,n|\Gamma} \right), v_{d_2|\Gamma} \right\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \\
- \left\langle \alpha_e \partial_{n,\delta} \left( \sum_{n \geq 0} \delta^n u_{e,n|\Gamma_{\delta,2}} \right), v_{d_2|\Gamma_{\delta,2}} \right\rangle_{H^{-1/2}(\Gamma_{\delta,1}) \times H^{1/2}(\Gamma_{\delta,1})} \\
+ \alpha_e \int_{\Omega_{\delta,2}} \nabla \left( \sum_{n \geq 0} \delta^n u_{e,n} \right) \cdot \nabla v_{d_2} \, d\Omega_{\delta,2} = \int_{\Omega_{\delta,2}} f_{\Omega_{\delta,2}} v_{d_2} \, d\Omega_{\delta,2}.
\end{align*}
\]

Using the the scaling \(s_2 = \eta_2/\delta\), we obtain

\[
\begin{align*}
\int_\Gamma \alpha_e \partial_n \left( \sum_{n \geq 0} \delta^n u_{e,n|\Gamma} \right) \, v^{[2]}(m, 0) \, d\Gamma &+ \alpha_e \delta a_\delta^{[2]} \left( \sum_{n \geq 0} \delta^n U_{e,n}, v^{[2]} \right) \\
- \left\langle \alpha_e \partial_{n,\delta} \left( \sum_{n \geq 0} \delta^n u_{e,n|\Gamma_{\delta,2}} \right), \Phi_2(m, 1), v^{[2]}(m, 1) \right\rangle_{H^{-1/2}(\Gamma \times \{1\}) \times H^{1/2}(\Gamma \times \{1\})} \\
= \int_{\Omega_{\delta,2}} f_{\Omega_{\delta,2}} v_{d_2} \, d\Omega_{\delta,2}. \tag{18}
\end{align*}
\]

In the same way, we obtain the equation for \(\alpha_i \partial_{n,\delta} \left( \sum_{n \geq 0} \delta^n u_{i,n|\Gamma_{\delta,1}} \right) \circ \Phi_1(m, -1):

\[
\begin{align*}
\alpha_i \delta a_\delta^{[1]} \left( \sum_{n \geq 0} \delta^n U_{i,n}, v^{[1]} \right) &- \int_\Gamma \alpha_i \partial_n \left( \sum_{n \geq 0} \delta^n u_{i,n|\Gamma} \right) v^{[1]}(m, 0) \, d\Gamma \\
+ \left\langle \alpha_i \partial_{n,\delta} \left( \sum_{n \geq 0} \delta^n u_{i,n|\Gamma_{\delta,1}} \right), \Phi_1(m, -1), v^{[1]}(m, -1) \right\rangle_{H^{-1/2}(\Gamma \times \{-1\}) \times H^{1/2}(\Gamma \times \{-1\})} \\
= \int_{\Omega_{\delta,1}} f_{\Omega_{\delta,1}} v_{d_1} \, d\Omega_{\delta,1}. \tag{19}
\end{align*}
\]

Inserting expansions (10), (11) and (13) in (8), using (14)-(15) and (18)-(19), we get

\[
\begin{align*}
\int_\Gamma \alpha_i \partial_n \left( \sum_{n \geq 0} \delta^n u_{i,n|\Gamma} \right) v^{[1]}(m, 0) \, d\Gamma &- \alpha_i \delta a_\delta^{[1]} \left( \sum_{n \geq 0} \delta^n U_{i,n}, v^{[1]} \right) \\
+ \sum_{\beta = 1}^2 \left[ \alpha_\delta \delta a_\delta^{[\beta]} \left( \sum_{n \geq 0} \delta^n u_{e,n}^{[\beta]}, v^{[\beta]} \right) \right] &- \alpha_e \delta a_\delta^{[2]} \left( \sum_{n \geq 0} \delta^n U_{e,n}, v^{[2]} \right) \\
- \int_\Gamma \alpha_e \partial_n \left( \sum_{n \geq 0} \delta^n u_{e,n|\Gamma} \right) v^{[2]}(m, 0) \, d\Gamma = 0. \tag{20}
\end{align*}
\]
Now, we use the identity (see [4, p. 1680])

\[ J_{\delta,\beta}^{-2} := I - 2s_{\beta}p_{\beta}\delta R + 3 (p_{\beta}s_{\beta}\delta R)^2 + \cdots + n (p_{\beta}s_{\beta}\delta R)^n + (-s_{\beta}p_{\beta}\delta R)^n [nJ_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}] . \]

Since

\[ \det J_{\delta,\beta} = 1 + 2p_{\beta}s_{\beta}\delta H + (p_{\beta}s_{\beta}\delta)^2 \mathcal{K}, \]

where \( 2\mathcal{H} := tr\mathcal{R} \) and \( \mathcal{K} := \det \mathcal{R} \) are respectively the mean and the Gaussian curvatures of the surface \( \Gamma \), the bilinear form \( a_{\delta}^{[\beta]} (\ldots) \) admits the expansion

\[ a_{\delta}^{[\beta]} (\ldots) = \delta^{-2}a_{0,2}^{[\beta]} + \delta^{-1}a_{1,2}^{[\beta]} + (a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]}) + \delta a_{1,1}^{[\beta]} + \cdots \]

\[ + \delta^{n-1}a_{n-1,1}^{[\beta]} + \delta^n r_n^{[\beta]} (\delta; \ldots), \quad (21) \]

where the forms \( a_{k,l}^{[\beta]} \) are independent of \( \delta \) and are given by

\[ a_{0,2}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} p_{\beta}^{-1} \partial_{s_{\beta}} u^{[\beta]} \partial_{s_{\beta}} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{1,2}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} 2\mathcal{H} s_{\beta} \partial_{s_{\beta}} u^{[\beta]} \partial_{s_{\beta}} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{2,2}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} p_{\beta} \mathcal{K} s_{\beta}^2 \partial_{s_{\beta}} u^{[\beta]} \partial_{s_{\beta}} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{0,1}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} p_{\beta} \nabla_{\Gamma} u^{[\beta]} \nabla_{\Gamma} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{1,1}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} \frac{2p_{\beta}^2 s_{\beta} (\mathcal{H} I - \mathcal{R}) \nabla_{\Gamma} u^{[\beta]} \nabla_{\Gamma} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{2,1}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} \frac{p_{\beta}^3 (\mathcal{K} I - 4\mathcal{H} \mathcal{R} + 3\mathcal{R}^2) s_{\beta}^2 \nabla_{\Gamma} u^{[\beta]} \nabla_{\Gamma} v^{[\beta]} d\Gamma ds_{\beta}, \]

\[ a_{n-1,1}^{[\beta]} (u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} p_{\beta}^n \left( (n - 2) \mathcal{K} \mathcal{R}^{n-3} - (n - 1) 2\mathcal{H} \mathcal{R}^{n-2} + n \mathcal{R}^{n-1} \right) (-s_{\beta})^{n-1} \nabla_{\Gamma} u^{[\beta]} \nabla_{\Gamma} v^{[\beta]} d\Gamma ds_{\beta}. \]

The form \( r_n^{[\beta]} (\delta; \ldots) \) is the remainder of Expansion (21) and is expressed by

\[ r_n^{[\beta]} (\delta; u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^{\beta}} (B_{n,\delta} + 2\mathcal{H} B_{n-1,\delta} + \mathcal{K} B_{n-2,\delta}) s_{\beta}^n \nabla_{\Gamma} u^{[\beta]} \nabla_{\Gamma} v^{[\beta]} d\Gamma ds_{\beta}, \]

with

\[ B_{n,\delta} := \begin{cases} (-\mathcal{R})^n (nJ_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}) & \text{if } n \geq 0, \\ J_{\delta,\beta}^{-2} & \text{otherwise}. \end{cases} \]

**Remark 3** In the two-dimensional case, with the help of (7), Expansion (21) turns into

\[ a_{\delta}^{[\beta]} (\ldots) = \delta^{-2}a_{0,2}^{[\beta]} + \delta^{-1}a_{1,2}^{[\beta]} + a_{0,1}^{[\beta]} + \delta a_{1,1}^{[\beta]} + \cdots + \delta^{n-1}a_{n-1,1}^{[\beta]} + \delta^n r_n^{[\beta]} (\delta; \ldots), \]
with
\[
\begin{align*}
a_{n,2}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^{\beta}} p_{\beta}^{n-1} (s_{\beta} R)^n \partial_{s_{\beta}} u^{[\beta]} \partial_{s_{\beta}} v^{[\beta]} \, d\Gamma \, ds_{\beta}, \\
a_{n,1}^{[\beta]}(u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^{\beta}} p_{\beta}^{n+1} (-s_{\beta} R)^n \partial_{t} u^{[\beta]} \partial_{t} v^{[\beta]} \, d\Gamma \, ds_{\beta}, \\
r_n^{[\beta]}(\delta; u^{[\beta]}, v^{[\beta]}) &:= \int_{\Omega^{\beta}} J_{\delta,\beta}^{-1} (-s_{\beta} R)^n \partial_{t} u^{[\beta]} \partial_{t} v^{[\beta]} \, d\Gamma \, ds_{\beta}.
\end{align*}
\]

Inserting Expansion (21) in (20) and matching the same powers of \(\delta\), we obtain the following variational equations, which hold for all \(v = (v^{[1]}, v^{[2]})\) in \(H^1(\Gamma \times (-1, 1))\),

\[
a_{0,2}^{[1]}\left( \alpha_{\delta} u_{0}^{[1]} - \alpha_i U_{i,0}, v^{[1]} \right) + a_{0,2}^{[2]}\left( \alpha_{\delta} u_{0}^{[2]} - \alpha_e U_{e,0}, v^{[2]} \right) = 0,
\] (22)

\[
a_{1,2}^{[1]}\left( \alpha_{\delta} u_{0}^{[1]} - \alpha_i U_{i,0}, v^{[1]} \right) + a_{0,2}^{[1]}\left( \alpha_{\delta} u_{1}^{[1]} - \alpha_i U_{i,1}, v^{[1]} \right) \\
+ a_{1,2}^{[2]}\left( \alpha_{\delta} u_{0}^{[2]} - \alpha_e U_{e,0}, v^{[2]} \right) + a_{0,2}^{[2]}\left( \alpha_{\delta} u_{1}^{[2]} - \alpha_e U_{e,1}, v^{[2]} \right) \\
= \alpha_e \int_{\Gamma} \partial_n u_{e,0} v^{[2]} (m, 0) \, d\Gamma - \alpha_i \int_{\Gamma} \partial_n u_{i,0} v^{[1]} (m, 0) \, d\Gamma,
\] (23)

\[
a_{0,2}^{[1]}\left( \alpha_{\delta} u_{2}^{[1]} - \alpha_i U_{i,2}, v^{[1]} \right) + a_{1,2}^{[1]}\left( \alpha_{\delta} u_{1}^{[1]} - \alpha_i U_{i,1}, v^{[1]} \right) \\
+ \left( a_{2,2}^{[1]} + a_{0,1}^{[1]} \right)\left( \alpha_{\delta} u_{0}^{[1]} - \alpha_i U_{i,0}, v^{[1]} \right) \\
+ a_{0,2}^{[2]}\left( \alpha_{\delta} u_{2}^{[2]} - \alpha_e U_{e,2}, v^{[2]} \right) + a_{1,2}^{[2]}\left( \alpha_{\delta} u_{1}^{[2]} - \alpha_e U_{e,1}, v^{[2]} \right) \\
+ \left( a_{2,2}^{[2]} + a_{0,1}^{[2]} \right)\left( \alpha_{\delta} u_{0}^{[2]} - \alpha_e U_{e,0}, v^{[2]} \right) \\
= \alpha_e \int_{\Gamma} \partial_n u_{e,1} v^{[2]} (m, 0) \, d\Gamma - \alpha_i \int_{\Gamma} \partial_n u_{i,1} v^{[1]} (m, 0) \, d\Gamma,
\] (24)

\[
\alpha_i \int_{\Gamma} \partial_n u_{i,2} v^{[1]} (m, 0) \, d\Gamma + a_{0,2}^{[1]}\left( \alpha_{\delta} u_{3}^{[1]} - \alpha_i U_{i,3}, v^{[1]} \right) \\
+ a_{1,2}^{[1]}\left( \alpha_{\delta} u_{2}^{[1]} - \alpha_i U_{i,2}, v^{[1]} \right) + \left( a_{2,2}^{[1]} + a_{0,1}^{[1]} \right)\left( \alpha_{\delta} u_{1}^{[1]} - \alpha_i U_{i,1}, v^{[1]} \right) \\
+ a_{1,1}^{[1]}\left( \alpha_{\delta} u_{1}^{[1]} - \alpha_i U_{i,0}, v^{[1]} \right) + a_{0,2}^{[2]}\left( \alpha_{\delta} u_{3}^{[2]} - \alpha_e U_{e,3}, v^{[2]} \right) \\
+ a_{1,2}^{[2]}\left( \alpha_{\delta} u_{2}^{[2]} - \alpha_e U_{e,2}, v^{[2]} \right) + \left( a_{2,2}^{[2]} + a_{0,1}^{[2]} \right)\left( \alpha_{\delta} u_{1}^{[2]} - \alpha_e U_{e,1}, v^{[2]} \right) \\
+ a_{1,1}^{[2]}\left( \alpha_{\delta} u_{0}^{[2]} - \alpha_e U_{e,0}, v^{[2]} \right) - \alpha_e \int_{\Gamma} \partial_n u_{e,2} v^{[2]} (m, 0) \, d\Gamma = 0,
\] (25)
\[\alpha_i \int_{\Gamma} \partial_n u_{i,n} [V^{[1]}] (m, 0) \, d\Gamma + a_{0,2}^{[1]} (\alpha_\delta u_{n+1}^{[1]} - \alpha_i U_{i,n+1}, v^{[1]}) + a_{1,2}^{[1]} (\alpha_\delta u_{n}^{[1]} - \alpha_i U_{i,n}, v^{[1]}) + a_{0,1}^{[1]} (\alpha_\delta u_{n-1}^{[1]} - \alpha_i U_{i,n-1}, v^{[1]})
\]
\[+ a_{1,1}^{[1]} (\alpha_\delta u_{n-2}^{[1]} - \alpha_i U_{i,n-2}, v^{[1]}) + a_{2,1}^{[1]} (\alpha_\delta u_{n-3}^{[1]} - \alpha_i U_{i,n-3}, v^{[1]})
\]
\[+ \sum_{l=3}^{n} a_{l-1,2}^{[1]} (\alpha_\delta u_{n-l}^{[2]} - \alpha_i U_{i,n-l}, v^{[2]}) + a_{2,2}^{[2]} (\alpha_\delta u_{n}^{[2]} - \alpha_i U_{i,n}, v^{[2]}) + a_{0,2}^{[2]} (\alpha_\delta u_{n+1}^{[2]} - \alpha_i U_{i,n+1}, v^{[2]})
\]
\[+ a_{1,2}^{[2]} (\alpha_\delta u_{n}^{[2]} - \alpha_i U_{i,n}, v^{[2]}) + a_{2,1}^{[2]} (\alpha_\delta u_{n-1}^{[2]} - \alpha_i U_{i,n-1}, v^{[2]})
\]
\[+ \sum_{l=3}^{n} a_{l-1,2}^{[2]} (\alpha_\delta u_{n-l}^{[2]} - \alpha_i U_{i,n-l}, v^{[2]}) - \alpha_e \int_{\Gamma} \partial_n u_{e,n} |v^{[2]}| (m, 0) \, d\Gamma = 0, \quad n \geq 3. \quad (26)
\]

4.2 Calculation of the first terms

In this paragraph, we first recall some theoretical results needed for our calculation. After, we calculate explicitly the first two terms of Expansions (10)-(11) and (13) in order to present a recursive method to define successively the terms of these expansions.

Let \( s \) be a nonnegative real number. We define \( PH^s(\Omega) \) (see [14]) the space of functions in \( \Omega \), with \( H^s \)-regularity in \( \Omega_e \) and \( \Omega_i \) as follows

\[PH^s(\Omega) := \{ V = (V_i, V_e); V_i \in H^s(\Omega_i) \text{ and } V_e \in H^s(\Omega_e) \},\]

equipped with the norm

\[\|V\|_{PH^s(\Omega)} := \left( \|V_i\|_{H^s(\Omega_i)} + \|V_e\|_{H^s(\Omega_e)} \right)^{1/2}.
\]

We need the following theorem. Its proof [17, p. 122] is an application of the reflection principle [12, p. 147].

**Theorem 4** Let \( G \) belongs to \( H^s(\Gamma) \), \( s \geq -1/2 \). Then the following problem

\[
\begin{cases}
- \text{div} (\alpha_i \nabla U_i) = 0 & \text{in } \Omega_i, \\
- \text{div} (\alpha_e \nabla U_e) = 0 & \text{in } \Omega_e, \\
U_{i|\Gamma} = U_{e|\Gamma} & \text{on } \Gamma, \\
\alpha_i \partial_n U_{i|\Gamma} - \alpha_e \partial_n U_{e|\Gamma} = G & \text{on } \Gamma, \\
U_{e|\partial \Omega} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

admits a unique solution \( U = (U_i, U_e) \) in \( PH^{s+3/2}(\Omega) \). Moreover, if \( m \) is a nonnegative integer, and \( s > m + \frac{P-1}{2} \). Then

\[U_i \in C^m(\overline{\Omega_i}) \quad \text{and} \quad U_e \in C^m(\overline{\Omega_e}).\]

We also need the following technical lemma. This construction was motivated by [4] and its proof is a straightforward verification.
Lemma 5 For $\beta = 1, 2$, let $q^{[\beta]}$ be a given function in $L^2(\Gamma)$ and let $k^{[\beta]}$ be a vectorial function in $L^2(\Omega^\beta, \mathbb{R}^3)$ such that the partial application $s_\beta \rightarrow k^{[\beta]}(., s_\beta)$ is valued in the space of vectorial fields tangent to $\Gamma$ and also $\text{div}_\Gamma k^{[\beta]} \in L^2(\Omega^\beta)$. Then the solution $h^{[\beta]}$ of the variational equation

$$
\mathcal{L}^{[\beta]} v^{[\beta]} := \int_{\Omega^\beta} h^{[\beta]} \partial_{s_\beta} v^{[\beta]} \ d\Gamma s_\beta + \int_{\Omega^\beta} k^{[\beta]} \cdot \nabla v^{[\beta]} \ d\Gamma s_\beta = 0;
$$

is explicitly given by

$$
h^{[\beta]} (m, s_\beta) = \int_{s_\beta} (-1)^\beta \text{div}_\Gamma k^{[\beta]} (m, \lambda) \ d\lambda.
$$

Moreover, if $v^{[\beta]}(., 0) \neq 0$, we have

$$
\mathcal{L}^{[\beta]} v^{[\beta]} = (-1)^{\beta+1} \int_{\Gamma} h^{[\beta]}(m, 0) v^{[\beta]}(m, 0) \ d\Gamma
$$

$$
= \int_{\Gamma} \left[ (-1)^{\beta+1} \int_{0}^{(-1)^\beta} \text{div}_\Gamma k^{[\beta]} (m, s_\beta) \ ds_\beta \right] v^{[\beta]} (m, 0) \ d\Gamma.
$$

4.2.1 Term of order 0

Equation (22) implies that $\partial_{s_\beta} u_0^{[\beta]} = 0$. Using (16), (17) and (26), we obtain

$$
u_{i,0}\Gamma = u_0^{[1]}(m, s_1) = u_0^{[2]}(m, s_2) = u_{i,0}\Gamma, \ m \in \Gamma.
$$

(27)

The choice of $v$ such that $v^{[2]} = 0$ in (23) gives

$$
a_{0,2}^{[1]} \left[ \alpha_\delta u_1^{[1]} - \alpha_i U_{i,1}^{[1]} \right] = 0.
$$

(28)

An intégration by parts in $s_1$ leads to

$$
p_{1}^{-1} \alpha_\delta \partial_{s_1} u_1^{[1]} = \alpha_i \partial_{n} u_{i,0}\Gamma.
$$

Similarly, the choice of $v$ such that $v^{[1]} = 0$ in (23) gives

$$
a_{0,2}^{[2]} (\alpha_\delta u_1^{[2]} - \alpha_e U_{e,0}^{[2]} v^{[2]} = 0.
$$

We obtain

$$
p_{2}^{-1} \alpha_\delta \partial_{s_2} u_2^{[2]} = \alpha_e \partial_{n} u_{e,0}\Gamma.
$$

(29)

Therefore

$$\alpha_i \int_{\Gamma} \partial_{n} u_{i,0}\Gamma v^{[1]}(m, 0) d\Gamma = \alpha_e \int_{\Gamma} \partial_{n} u_{e,0}\Gamma v^{[2]}(m, 0) d\Gamma.
$$

As $v^{[1]}(m, 0) = v^{[2]}(m, 0), \alpha_i \partial_{n} u_{i,0}\Gamma = \alpha_e \partial_{n} u_{e,0}\Gamma.
$$

(30)

Let us define $\alpha_0$ and $u_n (\forall n \in \mathbb{N})$ by

$$
\alpha_0(x) = \begin{cases} 
\alpha_e & \text{if } x \in \Omega_e, \\
\alpha_i & \text{if } x \in \Omega_i
\end{cases}
$$

and $u_n = \begin{cases} 
u_{e,n} & \text{in } \Omega_e, \\
u_{i,n} & \text{in } \Omega_i
\end{cases}$.
Therefore, with (12), (27) and (30), \( u_0 \) satisfies the following problem
\[
\begin{aligned}
-\text{div} (\alpha \nabla u_0) &= f & \text{in } \Omega, \\
u_0|_{\partial \Omega} &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

Elliptic regularity results (see e.g. [1]) show that if \( f \) belongs to \( C^\infty(\Omega) \), then \( (u_{i,0}, u_{e,0}) \) is a well-defined element of \( C^\infty(\overline{\Omega_1}) \times C^\infty(\overline{\Omega_e}) \). As a consequence the first term is determined.

### 4.2.2 Term of order 1

Integrating Relations (28) and (29) in \( s_\beta \), yields
\[
\begin{align*}
u_1^1(m, s_1) &= u_{i,1}\Gamma + p_1 \left[ (s_1 + 1) \alpha_1 \alpha_\delta^{-1} - 1 \right] \partial_n u_{i,0}\Gamma, \quad \forall (m, s_1) \in \Omega^1, \\
u_2^1(m, s_2) &= u_{e,1}\Gamma + p_2 \left[ (s_2 - 1) \alpha_2 \alpha_\delta^{-1} + 1 \right] \partial_n u_{e,0}\Gamma, \quad \forall (m, s_2) \in \Omega^2.
\end{align*}
\]

By identifying terms of order 1 in (17) and (16), we obtain the first transmission condition on \( \Gamma \)
\[
u_{i,1}\Gamma - u_{e,1}\Gamma = p_1 (1 - \alpha_1 \alpha_\delta^{-1}) \partial_n u_{i,0}\Gamma + p_2 (1 - \alpha_2 \alpha_\delta^{-1}) \partial_n u_{e,0}\Gamma. \tag{31}
\]
The second one follows the same lines as for order 0. The choice of \( v \) such that \( v^2 = 0 \) in (24) gives
\[
a_{0,2} \left( \alpha \delta u_2^1 - \alpha_1 U_{i,2}, v^1 \right) + a_{0,1} \left( \alpha \delta u_0^1 - \alpha_1 U_{i,0}, v^1 \right) = 0.
\]

We apply Lemma 5 with
\[
\begin{align*}
h^1 &= p_1^{-1} \alpha_\delta \partial_s u_2^1 - p_1^{-1} \alpha_1 U_{i,2} - p_1^{-1} \alpha_\delta \partial_s u_2^1 - \alpha_1 \partial_n u_{i,1}\Gamma - s_1 p_1 \partial_n^2 u_{i,0}\Gamma, \\
k^1 &= p_1 \nabla_{\Gamma} \left( p_1^{-1} u_0^1 - \alpha_1 U_{i,0} \right) = p_1 \left( \alpha \delta - \alpha_1 \right) \nabla_{\Gamma} u_{i,0}\Gamma,
\end{align*}
\]

we find
\[
p_1^{-1} \alpha_\delta \partial_s u_2^1(m, s_1) - \alpha_1 \partial_n u_{i,1}\Gamma - s_1 p_1 \partial_n^2 u_{i,0}\Gamma = - (s_1 + 1) p_1 \left( \alpha \delta - \alpha_1 \right) \Delta_{\Gamma} u_{i,0}\Gamma.
\]

Moreover, for all \( v^1 \) we obtain
\[
\mathcal{L}^1 v^1 = - \int_{\Gamma} p_1 \left( \alpha \delta - \alpha_1 \right) \Delta_{\Gamma} u_{i,0}\Gamma \ v^1(m, 0) \ d\Gamma.
\]

Again Similarly, the choice of \( v \) such that \( v^1 = 0 \) in (24) gives
\[
a_{0,2}^1 \left( \alpha \delta u_2^1 - \alpha_1 U_{e,2}, v^1 \right) + a_{0,1}^1 \left( \alpha \delta u_0^1 - \alpha_1 U_{e,0}, v^1 \right) = 0.
\]

We apply Lemma 5 with
\[
\begin{align*}
h^2 &= p_2^{-1} \alpha_\delta \partial_s u_2^1 - p_2^{-1} \alpha_1 U_{e,2} - p_2^{-1} \alpha_\delta \partial_s u_2^1 - \alpha_1 \partial_n u_{e,1}\Gamma - s_2 p_2 \partial_n^2 u_{e,0}\Gamma, \\
k^2 &= p_2 \nabla_{\Gamma} \left( \alpha \delta u_0^1 - \alpha_1 U_{e,0} \right) = p_2 \left( \alpha \delta - \alpha_1 \right) \nabla_{\Gamma} u_{e,0}\Gamma,
\end{align*}
\]
we find
\[ p_2^{-1} \alpha_\delta \partial_n u_2[2](m, s_2) - \alpha_e \partial_n u_{e,1}\Gamma - s_2 p_2 \partial_n^2 u_{e,0}\Gamma = (1 - s_2) (\alpha_\delta - \alpha_e) \Delta_r u_{e,0}\Gamma. \]

Moreover, for all \( v^{[2]} \) we obtain
\[ \mathcal{L}^{[2]} v^{[2]} = - \int_{\Gamma} p_2 (\alpha_\delta - \alpha_e) \Delta_r u_{e,0}\Gamma. \]

As a consequence,
\[ \int_{\Gamma} [\alpha_i \partial_n u_{i,1}\Gamma - p_1 (\alpha_\delta - \alpha_1) \Delta_r u_{i,0}\Gamma] v^{[1]}(m, 0) \, d\Gamma \]
\[ = \int_{\Gamma} [\alpha_i \partial_n u_{e,1}\Gamma + p_2 (\alpha_\delta - \alpha_e) \Delta_r u_{e,0}\Gamma] v^{[2]}(m, 0) \, d\Gamma. \]

As \( v^{[1]}(m, 0) = v^{[2]}(m, 0) \),
\[ \alpha_i \partial_n u_{i,1}\Gamma - \alpha_e \partial_n u_{e,1}\Gamma = p_1 (\alpha_\delta - \alpha_1) \Delta_r u_{i,0}\Gamma + p_2 (\alpha_\delta - \alpha_e) \Delta_r u_{e,0}\Gamma. \]

It follows from (12), (31), (32) and Theorem 4 that \( u_1 \) is the unique solution of the following problem
\[
\begin{cases}
-\text{div}(\alpha_i \nabla u_{i,1}) = 0 & \text{in } \Omega_i, \\
-\text{div}(\alpha_e \nabla u_{e,1}) = 0 & \text{in } \Omega_e, \\
u_{e,1}\partial\Omega = 0 & \text{on } \partial\Omega,
\end{cases}
\]

with transmission conditions on \( \Gamma \)
\[ u_{i,1}\Gamma - u_{e,1}\Gamma = p_1 (1 - \alpha_i \alpha_\delta^{-1}) \partial_n u_{i,0}\Gamma + p_2 (1 - \alpha_e \alpha_\delta^{-1}) \partial_n u_{e,0}\Gamma, \]
\[ \alpha_i \partial_n u_{i,1}\Gamma - \alpha_e \partial_n u_{e,1}\Gamma = p_1 (\alpha_\delta - \alpha_i) \Delta_r u_{i,0}\Gamma + p_2 (\alpha_\delta - \alpha_e) \Delta_r u_{e,0}\Gamma. \]

or
\[ u_{i,1}\Gamma - u_{e,1}\Gamma = [p_1 (1 - \alpha_i \alpha_\delta^{-1}) + p_2 (\alpha_i \alpha_e^{-1} - \alpha_i \alpha_\delta^{-1})] \partial_n u_{i,0}\Gamma, \]
\[ \alpha_i \partial_n u_{i,1}\Gamma - \alpha_e \partial_n u_{e,1}\Gamma = [p_1 (\alpha_\delta - \alpha_i) + p_2 (\alpha_\delta - \alpha_e)] \Delta_r u_{i,0}\Gamma. \]

5 Convergence Theorem

The process described in the previous section can be continued up to any order provided that the data are sufficiently regular. We can also estimate the error made by truncating the series after a finite number of terms. Let \( n \) be in \( \mathbb{N} \), we set
\[ u_{i,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{i,j}, \quad u_{e,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{e,j} \quad \text{and} \quad u_{d_1,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_1,j} \quad \text{in } \Omega_{\delta,1}, \]
\[ u_{d_2,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_2,j} \quad \text{in } \Omega_{\delta,2}, \]

where \( u_{d_\beta,j}(x) := \tilde{u}_{d_\beta,j}(m, \delta s_\beta) := u^{[\beta]}_j(m, s_\beta); \forall x = \Phi_\beta(m, s_\beta) \in \Omega_{\delta,\beta}. \)
Theorem 6 (Convergence Theorem)  For all integers $n$, there exists a constant $c$ independent of $\delta$ such as

$$\|u_{i,\delta} - u_{i,\delta}^{(n)}\|_{H^1(\Omega_{i,\delta})} + \delta^{1/2}\|u_{d,\delta} - u_{d,\delta}^{(n)}\|_{H^1(\Omega_{\delta})} + \|u_{e,\delta} - u_{e,\delta}^{(n)}\|_{H^1(\Omega_{e,\delta})} \leq c\delta^{n+1}.$$  

Proof. Since $f$ is $C^\infty$, all terms in Expansions (10), (11) and (13) up to order $n + 1$ may be obtained from Equations [22]-[26]. Let us define the remainders $R_{D_{1,n}}, R_{D_{2,n}}, R_{1,n}$ and $R_{2,n}$ of Taylor expansions in the normal variable with respect to $\delta$ up to order $n$ of $u_{i,\delta}, u_{e,\delta}, u_{i,\delta}^{(n)}$ and $u_{e,\delta}^{(n)}$ respectively by

$$R_{D_{1,n}} := u_{i,\delta}/\Gamma_{i,\delta} - \sum_{j=0}^{n} \sum_{l=0}^{n-j} \frac{(-1)^{j+l}}{l!} p_{1}^{(j)} \partial_{n}^{l} u_{i,j}|_{\Gamma},$$

$$R_{D_{2,n}} := u_{e,\delta}/\Gamma_{\delta,\delta} - \sum_{j=0}^{n} \sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_{2}^{(j)} \partial_{n}^{l} u_{e,j}|_{\Gamma},$$

$$R_{1,n}^{[1]} := (u_{i,\delta})^{[1]} - \sum_{j=0}^{n} \sum_{l=0}^{n-j} \frac{(s_{1})^{j+l}}{l!} p_{1}^{(j)} \partial_{n}^{l} u_{i,j}|_{\Gamma} := (u_{i,\delta})^{[1]} - \sum_{j=0}^{n} \delta^{j} U_{i,j},$$

$$R_{2,n}^{[2]} := (u_{e,\delta})^{[2]} - \sum_{j=0}^{n} \sum_{l=0}^{n-j} \frac{(s_{2})^{j+l}}{l!} p_{2}^{(j)} \partial_{n}^{l} u_{e,j}|_{\Gamma} := (u_{e,\delta})^{[2]} - \sum_{j=0}^{n} \delta^{j} U_{e,j},$$

where $s_{\beta} \in I_{\beta}$. We shall rely on the following proposition to show the estimates of the remainders $R_{D_{\beta,n}}$ and $R_{\delta,n}$. The steps of the proof are very similar to those given in [18, Section 5]. We refer the reader to this paper.

Proposition 7 There exists a constant $c > 0$, independent of $\delta$, such as

$$\|\nabla R_{\delta,n}\|_{L^2(\Omega_{\delta,\delta})} \leq c\delta^{n+1/2},$$

$$\|\nabla^{(j)} R_{\delta,n}\|_{L^2(\Gamma)} \leq c\delta^{n+1/2}, \text{ for } j = 0, 1.$$  

Moreover, there exists an extension $\mathcal{P}R$ of $R_{D_{\beta,n}}$ into $\Omega_{\delta}$ with

$$\partial_{\eta_{\beta}} \mathcal{P}R (m, \eta_{\beta})|_{\eta_{\beta} = (-1)^{\beta}\delta} = 0 \text{ and } \|\mathcal{P}R\|_{H^1(\Omega)} \leq c\delta^{n}.$$  

Continuation of the proof of Theorem 6. Let $r_{i,\delta}, r_{d,\delta}$ and $r_{e,\delta}$ be the remainders made by truncating Series [10], [11] and [13]

$$r_{i,\delta} := u_{i,\delta} - u_{i,\delta}^{(n)}, \quad r_{d,\delta} := u_{d,\delta} - u_{d,\delta}^{(n)}, \quad r_{e,\delta} := u_{e,\delta} - u_{e,\delta}^{(n)},$$

and $L_{\delta}$ be the linear form defined on $H^1_{0}(\Omega)$

$$L_{\delta} v := \alpha_{i} \int_{\Omega_{i,\delta}} \nabla r_{i,\delta} \cdot \nabla v_{i} \, d\Omega_{i,\delta} + \alpha_{d} \int_{\Omega_{\delta}} \nabla(r_{d,\delta} - \mathcal{P}R) \cdot \nabla v_{d} \, d\Omega_{\delta}$$

$$+ \alpha_{e} \int_{\Omega_{e,\delta}} \nabla r_{e,\delta} \cdot \nabla v_{e} \, d\Omega_{e,\delta},$$

(37)
in which $\mathcal{P}R$ is the extension function of $R_{D,\beta,n}$ into $\Omega_\delta$. Using Green’s formula in $\Omega_i$ and in $\Omega_e$ with the help of (12), we obtain

$$\mathcal{L}_\delta v = -\alpha_i \int_{\Gamma} \left( \partial_n u_{i,0}\Gamma + \cdots + \delta^n \partial_n u_{i,n}\Gamma \right) v_{i}\Gamma d\Gamma$$

$$- \alpha_i \int_{\Delta_\delta} \nabla u_{i,\delta} \nabla v_{1} d\Omega_{\delta_1} + \alpha_e \int_{\Delta_{\delta,2}} \nabla u_{e,\delta} \nabla v_{2} d\Omega_{\delta,2}$$

$$+ \alpha_e \int_{\Gamma} \left( \partial_n u_{e,0}\Gamma + \cdots + \delta^n \partial_n u_{e,n}\Gamma \right) v_{e}\Gamma d\Gamma$$

It follows, from (33)-(36), that

$$\mathcal{L}_\delta v = \alpha_i \int_{\Omega_{i,\delta}} \nabla R_{i,n} \nabla v_{i} d\Omega_{i,\delta} + \alpha_e \int_{\Omega_{e,\delta}} \nabla R_{e,n} \nabla v_{e} d\Omega_{e,\delta}$$

$$- \alpha_{\delta} \int_{\Omega_\delta} \nabla \mathcal{P}R. \nabla v_{d} d\Omega_{\delta} - \sum_{\beta=1}^{2} \mathcal{L}_\delta v_{\beta}$$

$$\alpha_{\delta} \delta a_{[\delta]} (U_{\delta,0} + \cdots + \delta^n U_{\delta,n}) + \alpha_e \delta a_{[\delta]} (U_{e,0} + \cdots + \delta^n U_{e,n})$$

$$- \alpha_i \int_{\Gamma} \left( \partial_n u_{i,0}\Gamma + \cdots + \delta^n \partial_n u_{i,n}\Gamma \right) v_{i}\Gamma d\Gamma$$

$$+ \alpha_e \int_{\Gamma} \left( \partial_n u_{e,0}\Gamma + \cdots + \delta^n \partial_n u_{e,n}\Gamma \right) v_{e}\Gamma d\Gamma \right).$$

Now, we use the fact that $u_{[\beta]} \cdots, u_{[\beta]} + 1$ are solutions of Equations (22)-(26), we obtain

$$\mathcal{L}_\delta v = \alpha_{\delta} \delta_{\beta} + 1 \sum_{\beta=1}^{2} \left\{ \right.$$

$$a_{[\beta]} \left( u_{[\beta]} + \delta u_{[\beta]} + \delta^2 u_{[\beta]} \right) - \cdots$$

$$- a_{[\beta]} \left( u_{[\beta]} + \delta u_{[\beta]} + \delta^2 u_{[\beta]} \right)$$

$$+ \alpha_i \int_{\Omega_{i,\delta}} \nabla R_{i,n} \nabla v_{i} d\Omega_{i,\delta} + \alpha_e \int_{\Omega_{e,\delta}} \nabla R_{e,n} \nabla v_{e} d\Omega_{e,\delta}$$

By the estimates based on the explicit expressions of the bilinear form $a_{[\beta]}(\cdots)$ and those of Propositions [7] we have

$$|\mathcal{L}_\delta v| \leq c \delta_{\beta} + 1 \sum_{\beta=1}^{2} \left( \right.$$
Since \( \delta \) is small enough, we have, \( \forall v \in H^1_0(\Omega) \),
\[
|\mathcal{L}_\delta v| \leq c\delta^{n+\frac{1}{2}} \sum_{\beta=1}^{2} \left( \delta^{\frac{1}{2}} \left\| \nabla v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{\frac{1}{2}} \left\| \partial_{s\beta} v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{\frac{1}{2}} \left\| v^{[\beta]} \right\|_{L^2(\Omega^\beta)} \right) 
+ c\delta^{n-1/2} \left( \left\| v_i \right\|_{H^1(\Omega_{i,\delta})} + \sum_{\beta=1}^{2} \left\| v_{\beta} \right\|_{H^1(\Omega_{\beta,\delta})} + \left\| v_e \right\|_{H^1(\Omega_{e,\delta})} \right).
\]
Therefore
\[
|\mathcal{L}_\delta v| \leq c\delta^{n-\frac{1}{2}} \left\| v \right\|_{H^1(\Omega)}. \tag{38}
\]
Since \( r^n := (r^n_{i,\delta}, r^n_{d,\delta} - \mathcal{P} R, r^n_{e,\delta}) \) is in \( H^1_0(\Omega) \), we set in (37) \( v = r^n \), we obtain
\[
\left\| r^n_{i,\delta} \right\|_{H^1(\Omega_{i,\delta})} + \left\| r^n_{d,\delta} - \mathcal{P} R \right\|_{H^1(\Omega_{d,\delta})} + \left\| r^n_{e,\delta} \right\|_{H^1(\Omega_{e,\delta})} \leq c\delta^{n-\frac{1}{2}}. \tag{39}
\]
Thanks to Proposition 7 we find
\[
\left\| r^n_{i,\delta} \right\|_{H^1(\Omega_{i,\delta})} + \left\| r^n_{d,\delta} \right\|_{H^1(\Omega_{d,\delta})} + \left\| r^n_{e,\delta} \right\|_{H^1(\Omega_{e,\delta})} \leq c\delta^{n-\frac{1}{2}}. \tag{39}
\]
Moreover, since \( f_i \) and \( f_e \) are \( C^\infty \) and for every integer \( j \), we have \( \left\| u_{\delta,j} \right\|_{H^1(\Omega_{\delta,\beta})} = O(1) \) and \( \left\| u_{i,j} \right\|_{H^1(\Omega_{i,\delta})} = O(\delta^{-1/2}) \), therefore (see 20)
\[
\left\| r^n_{i,\delta} \right\|_{H^1(\Omega_{i,\delta})} = \left\| \delta^{n+1} u_{i,n+1} + \delta^{n+2} u_{i,n+2} + r^n_{i,\delta} \right\|_{H^1(\Omega_{i,\delta})} \leq c\delta^{n+1} + c\delta^{n+2} + c\delta^{n+3/2} \leq c\delta^{n+1}, \tag{39}
\]
\[
\left\| r^n_{e,\delta} \right\|_{H^1(\Omega_{e,\delta})} = \left\| \delta^{n+1} u_{e,n+1} + \delta^{n+2} u_{e,n+2} + r^n_{e,\delta} \right\|_{H^1(\Omega_{e,\delta})} \leq c\delta^{n+1} + c\delta^{n+2} + c\delta^{n+3/2} \leq c\delta^{n+1}, \tag{39}
\]
\[
\left\| r^n_{d,\delta} \right\|_{H^1(\Omega_{d,\delta})} = \left\| r^n_{d,\delta} + \delta^{n+1} u_{d,n+1} \right\|_{H^1(\Omega_{d,\delta})} \leq c\delta^{n+1/2} + c\delta^{n+1/2} \leq c\delta^{n+1/2}.
\]
This completes the proof. \( \blacksquare \)

## 6 Approximate transmission conditions

This section is devoted to the approximation of \( u_\delta \) by a solution of a problem modelling the effect of the thin layer with a precision of order two in \( \delta \). We truncate the series defining the asymptotic expansions, keeping only the first two terms
\[
u_{i,\delta} \simeq u_{i,0}^{(1)} = u_{i,0} + \delta u_{i,1} \text{ in } \Omega_i,
\]
\[
u_{e,\delta} \simeq u_{e,0}^{(1)} = u_{e,0} + \delta u_{e,1} \text{ in } \Omega_e,
\]
\[
u_{d,\delta}(x) \simeq u_{d_{1,\delta}}^{(1)}(x) = u_{0}^{[1]}(m, s_1) + \delta u_{1}^{[1]}(m, s_1), \quad \forall x = \Phi_1(m, s_1) \in \Omega_{\delta,1},
\]
\[
u_{d,\delta}(x) \simeq u_{d_{2,\delta}}^{(1)}(x) := u_{0}^{[2]}(m, s_2) + \delta u_{1}^{[2]}(m, s_2), \quad \forall x = \Phi_2(m, s_2) \in \Omega_{\delta,2}.
\]
where $U^{(1)}_{\delta} := (u_{i,\delta}^{(1)}, u_{e,\delta}^{(1)})$ is the solution of

$$
\left\{
\begin{array}{l}
-d\text{div} \left( \alpha_i \nabla u_{i,\delta}^{(1)} \right) = f_{\Omega_i} \\
-d\text{div} \left( \alpha_e \nabla u_{e,\delta}^{(1)} \right) = f_{\Omega_e} \\
u_{i,\delta}^{(1)}|_\Gamma - u_{e,\delta}^{(1)}|_\Gamma = \delta A \left( u_{i,\delta}^{(1)} \right) - \delta^2 \xi_\delta \\
\alpha_i \partial_n u_{i,\delta}^{(1)} - \alpha_e \partial_n u_{e,\delta}^{(1)} = \delta B \left( u_{i,\delta}^{(1)} \right) - \delta^2 \rho_\delta \\
U_{\delta|\partial\Omega} = 0
\end{array}\right. \\
(40)
$$

with

$$
A(u) := \left[ p_1(1 - \alpha_i \alpha_\delta^{-1}) + p_2(\alpha_i \alpha_e^{-1} - \alpha_i \alpha_\delta^{-1}) \right] \partial_n u_{\Gamma},
$$

$$
B(u) := \left[ p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e) \right] \Delta u_{\Gamma},
$$

$$
\xi_\delta := \left[ p_1(1 - \alpha_i \alpha_\delta^{-1}) + p_2(\alpha_i \alpha_e^{-1} - \alpha_i \alpha_\delta^{-1}) \right] \partial_n u_{i,1|\Gamma},
$$

$$
\rho_\delta := \left[ p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e) \right] \Delta u_{i,1|\Gamma}.
$$

Let $U_{\delta}^{op} := (u_{i,\delta}^{op}, u_{e,\delta}^{op})$ be the solution of (40) with $\rho_\delta = 0$ and $\xi_\delta = 0$. We obtain a problem $(P_{\delta}^{op})$ with transmission conditions of order equal to that of the differential operator. The new transmission conditions on $\Gamma$ are defined by

$$
\left\{
\begin{array}{l}
u_{i,\delta|\Gamma}^{op} - u_{e,\delta|\Gamma}^{op} = \delta \left[ p_1(1 - \alpha_i \alpha_\delta^{-1}) + p_2(\alpha_i \alpha_e^{-1} - \alpha_i \alpha_\delta^{-1}) \right] \partial_n u_{i,\delta|\Gamma}^{op}, \\
\alpha_i \partial_n u_{i,\delta|\Gamma}^{op} - \alpha_e \partial_n u_{e,\delta|\Gamma}^{op} = \delta \left[ p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e) \right] \Delta u_{i,\delta|\Gamma}^{op}.
\end{array}\right. \\
(41)
$$

However, the bilinear form associated to Problem $(P_{\delta}^{op})$ is neither positive nor negative. Then the existence and uniqueness of the solution are not ensured by the Lax–Milgram lemma. Therefore, we reformulate Problem $(P_{\delta}^{op})$ into a nonlocal equation on the interface $\Gamma$ (cf. [5]). A direct use of transmission conditions (41) leads to an operator which is not self-adjoint. So, we choose the position of $\Gamma$ in such a way that the jump of the trace of the solution on $\Gamma$ is null. We put

$$
p_1(1 - \alpha_i \alpha_\delta^{-1}) + p_2(\alpha_i \alpha_e^{-1} - \alpha_i \alpha_\delta^{-1}) = 0,
$$

we obtain

$$
p_1 = \frac{\alpha_i (\alpha_e - \alpha_\delta)}{\alpha_\delta (\alpha_e - \alpha_i)} \quad \text{and} \quad p_2 = \frac{\alpha_e (\alpha_\delta - \alpha_i)}{\alpha_\delta (\alpha_e - \alpha_i)},
$$

which is valid only when $\alpha_i < \alpha_\delta < \alpha_e$ or $\alpha_e < \alpha_\delta < \alpha_i$. This corresponds to the case of mid-diffusion. Transmission conditions (41) become

$$
\left\{
\begin{array}{l}
u_{i,\delta|\Gamma}^{op} - u_{e,\delta|\Gamma}^{op} = 0, \\
\alpha_i \partial_n u_{i,\delta|\Gamma}^{op} - \alpha_e \partial_n u_{e,\delta|\Gamma}^{op} = \delta \frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta u_{i,\delta|\Gamma}^{op}.
\end{array}\right.
$$
After, we remove the right-hand side of Problem \((P_{\delta}^{ap})\) by a standard lift: let \(G\) be in \(H_0^1(\Omega)\) such that \(-\text{div}(\alpha\nabla G) = f\). Then the function \(\Psi = U_{\delta}^{ap} - G\) solves the following problem

\[
\begin{cases}
-\text{div}(\alpha_i \nabla \Psi_i) = 0 & \text{in } \Omega_i, \\
-\text{div}(\alpha_e \nabla \Psi_e) = 0 & \text{in } \Omega_e, \\
\Psi_{\mid\Gamma} = \Psi_{e\mid\Gamma} = 0 & \text{on } \Gamma, \\
\alpha_i \partial_n \Psi_{\mid\Gamma} - \alpha_e \partial_n \Psi_{e\mid\Gamma} - \frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta_{\Gamma} \Psi_{\mid\Gamma} = g & \text{on } \Gamma, \\
\Psi_{e\mid\partial\Omega} = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(g = (\alpha_e - \alpha_i) \partial_n G_{\mid\Gamma} + \frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta_{\Gamma} G_{\mid\Gamma}\).

We introduce the Steklov-Poincaré operators \(S_i\) and \(S_e\) (called also Dirichlet-to-Neumann operators) defined from \(H^{1/2}(\Gamma)\) onto \(H^{-1/2}(\Gamma)\) by \(S_i \varphi := \partial_n u_{i\mid\Gamma}\), where \(u_i\) is the solution of the boundary value problem

\[
\begin{cases}
-\Delta u_i = 0 & \text{in } \Omega_i, \\
u_{i\mid\Gamma} = \varphi & \text{on } \Gamma,
\end{cases}
\]

and by \(S_e \psi := \partial_n u_{e\mid\Gamma}\), where \(u_e\) is the solution of the boundary value problem

\[
\begin{cases}
-\Delta u_e = 0 & \text{in } \Omega_e, \\
u_{e\mid\Gamma} = \psi & \text{on } \Gamma, \\
u_{e\mid\partial\Omega} = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Then \((P_{\delta}^{ap})\) is equivalent to the boundary equation

\[
\alpha_i S_i \omega + \alpha_e S_e \omega - \frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta_{\Gamma} \omega = g \text{ on } \Gamma,
\]

where \(\omega\) is the trace of \(\Psi\) on the surface \(\Gamma\). We are now in position to state the existence and uniqueness theorem, which proof is similar to that of Theorem 2.5 in [5].

**Theorem 8** The operator \(P_\delta := \frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta_{\Gamma} - \alpha_i S_i - \alpha_e S_e\) is an elliptic self-adjoint semi-bounded from below pseudodifferential operator of order 2. Moreover, there exists series \((\lambda_n)_{n \in \mathbb{N}}\) growing to infinity such that for any \(F \in H^s(\Gamma)\) with \(s \in \mathbb{R}\), we have the following:

1. If \(0 \notin (\lambda_n)_{n \in \mathbb{N}}\), then equation \(-P_\delta \omega = g\) admits a unique solution in \(S'(\Gamma)\) which, in addition, belongs to \(H^{s+2}(\Gamma)\);

2. If \(0 \in (\lambda_n)_{n \in \mathbb{N}}\), then there is either no solution or a complete affine finite dimensional space of \(H^{s+2}(\Gamma)\) solutions.

Finally, we give an error estimate between the solution \(u_\delta\) of \((\Pi)\) and the approximate solution \(u_{\delta}^{ap}\). Let us define \(u_{d_\delta,\delta}^{ap}\) on \(\Omega_{\delta,\beta}\)

\[
u_{d_\delta,\delta}^{ap}(x) := u_{d_1,\delta}^{ap}(m, s_1) := u^{ap}_{i,\delta\mid\Gamma} + \frac{\delta \alpha_i (\alpha_e - \alpha_\delta)}{\alpha_\delta (\alpha_e - \alpha_i)} [(s_1 + 1) \alpha_i \alpha_\delta^{-2} - 1] \partial_n u_{i,\delta\mid\Gamma},
\]

\[
u_{d_\delta,\delta}^{ap}(x) := u_{d_2,\delta}^{ap}(m, s_2) := u^{ap}_{e,\delta\mid\Gamma} + \frac{\delta \alpha_e (\alpha_\delta - \alpha_i)}{\alpha_\delta (\alpha_e - \alpha_i)} [(s_2 - 1) \alpha_e \alpha_\delta^{-1} + 1] \partial_n u_{e,\delta\mid\Gamma},
\]

18
and let us denote by $u^{ap}_\delta$ the approximate solution defined on $\Omega$

$$u^{ap}_\delta := \begin{cases} 
  u^{ap}_{i,\delta} & \text{in } \Omega_{i,\delta}, \\
  u^{ap}_{d,\delta} & \text{in } \Omega_{d,\delta}, \\
  u^{ap}_{e,\delta} & \text{in } \Omega_{e,\delta}.
\end{cases}$$

We can now formulate our main result.

**Theorem 9** There exists a constant $c$ independent of $\delta$ such as

$$\| u_{i,\delta} - u^{ap}_{i,\delta} \|_{H^1(\Omega_{i,\delta})} + \delta^{1/2} \sum_{\beta=1}^2 \| u_{d,\delta} - u^{ap}_{d,\delta} \|_{H^1(\Omega_{d,\beta})} + \| u_{e,\delta} - u^{ap}_{e,\delta} \|_{H^1(\Omega_{e,\delta})} \leq c \delta^2.$$ 

**Proof.** According to the Convergence Theorem, it is sufficient to estimate the error $U^{ap}_\delta - U^{(1)}_\delta$. Therefore, as in [21], we perform an asymptotic expansion for $U^{ap}_\delta$. The ansatz

$$U^{ap}_\delta = \sum_{j \geq 0} \delta^j w^j,$$  \hspace{1cm} (42)

where $w_{j|\Omega_e} := w_{e,j}$ and $w_{j|\Omega_i} := w_{i,j}$, gives the recurrence relations

$$\begin{cases}
-\text{div} (\alpha_i \nabla w_{i,j}) = f_{i|\Omega_i} \delta_{j,0} & \text{in } \Omega_i, \\
-\text{div} (\alpha_e \nabla w_{e,j}) = f_{e|\Omega_e} \delta_{j,0} & \text{in } \Omega_e, \\
w_{i,j}|\Gamma - w_{e,j}|\Gamma = 0 & \text{on } \Gamma, \\
\alpha_i \partial_n w_{i,j}|\Gamma - \alpha_e \partial_n w_{e,j}|\Gamma = \frac{(\alpha_e - \alpha_{\delta})}{\alpha_{\delta}} \Delta_\Gamma w_{i,j-1}|\Gamma & \text{on } \partial \Omega,
\end{cases}$$

with the convention that $w_{-1} = 0$. A simple calculation shows that the two first terms ($w_{i,0}, w_{e,0}$) and ($w_{i,1}, w_{e,1}$) coincide with the two first terms of (10) and (11). Furthermore, since $f_e$ and $f_i$ are $C^\infty$, each term of (42) is bounded in $H^1(\Omega)$. Then, by setting $R_w := U^{ap}_\delta - w_0 - \delta w_1 - \delta^2 w_2$, there exists $c > 0$, such as $\| R_w \|_{H^1(\Omega)} \leq c \delta^2$, which gives the desired result. 

**References**

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, Comm. Pure Appl. Math. 17 (1964) 35-92.

[2] H. Ammari and J. C. Nédélec, Sur les conditions d’impédance généralisées pour les couches minces, C. R. Acad. Sci. Paris Sér. I Math. 322 (1996) 995-1000.

[3] A. Bendali and K. Lemrabet, Asymptotic analysis of the scattering of a time-harmonic electromagnetic wave by a perfectly conducting metal coated with a thin dielectric shell, Asymptot. Anal. 57 (2008) 199-227.
[4] A. Bendali and K. Lemrabet, The effect of a thin coating on the scattering of the time-harmonic wave for the Helmholtz equation, SIAM J. Appl. Maths. 56 (1996) 1664-1693.

[5] V. Bonnaillie-Noël, M. Dambrine, F. Hérau and G. Vial, On generalized Ventcel’s type boundary conditions for Laplace operator in a bounded domain, SIAM J. Math. Anal. 42, n°2 (2010) 931–945.

[6] K.E. Boutarene, Asymptotic analysis for a diffusion problem, C. R. Math. Acad. Sci. Paris 349 (2011) 57-60.

[7] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, NewYork 2011.

[8] B. Delourme, H. Haddar and P. Joly, Approximate models for wave propagation across thin periodic interfaces, J. Math. Pures Appl. 98 (2012) 28-71.

[9] M.P. Do Carmo, Differential geometry of curves and surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.

[10] M. Duruflé, H. Haddar and P. Joly, High order generalized impedance boundary conditions in electromagnetic scattering problems, C. R. Physique 7 (2006) 533-542.

[11] B. Engquist and J.C. Nédélec, Effective boundary conditions for acoustic and electromagnetic scattering in thin layers, Research Report CMAP 278, Ecole Polytechnique, France 1993.

[12] Y.Y. Li and M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Ration. Mech. Anal. 153 (2000) 91-151.

[13] J.C. Nédélec, Acoustic and electromagnetic equations, integral Representations for Harmonic Problems, Springer 2001.

[14] R. Perrussel and C. Poignard, Asymptotic Expansion of Steady-State Potential in a High Contrast Medium with a Thin Resistive Layer, Applied Mathematics and Computation. 221 (2013) 48-65

[15] C. Poignard, About the transmembrane voltage potential of a biological cell in time-harmonic regime, in: Mathematical methods for imaging and inverse problems, volume 26 of ESAIM Proc. EDP Sci. Les Ulis (2009) 162-179.

[16] C. Poignard, Approximate transmission conditions through a weakly oscillating thin layer, Math. Methods Appl. Sci. 32 (2009) 603-626.

[17] C. Poignard, Méthodes asymptotiques pour le calcul des champs électromagnétiques dans des milieux à couches minces. Application aux cellules biologiques, Thèse de Doctorat, Université Claude Bernard-Lyon 1, 2006.

[18] K. Schmidt and S. Tordeux, Asymptotic modelling of conductive thin sheets, Z. Angew. Math. Phys. 61 (2010) 603-626.
[19] K. Schmidt and S. Tordeux, High order transmission conditions for thin conductive sheets in magneto-quasistatics, ESAIM, Math. Model. Numer. Anal. 45 No. 6 (2011) 1115-1140.

[20] S. Tordeux, Méthodes asymptotiques pour la propagation des ondes dans les milieux comportant des fentes, Ph.D. thesis, Université de Versailles-Saint-Quentin, Yvelines, France 2004.

[21] G. Vial, Analyse multi-échelle et conditions aux limites approchées pour un problème avec couche mince dans un domaine à coin, Ph.D. thesis, Université de Renne 1, France 2003.