RIGIDITY OF PROPER COLORINGS OF \( \mathbb{Z}^d \)

RON PELED AND YINON SPINKA

Abstract. A proper \( q \)-coloring of a domain in \( \mathbb{Z}^d \) is a function assigning one of \( q \) colors to each vertex of the domain such that adjacent vertices are colored differently. Sampling a proper \( q \)-coloring uniformly at random, does the coloring typically exhibit long-range order? It has been known since the work of Dobrushin that no such ordering can arise when \( q \) is large compared with \( d \). We prove here that long-range order does arise for each \( q \) when \( d \) is sufficiently high. Our results further characterize all periodic maximal-entropy Gibbs states for the model. Ordering also emerges in low dimensions if the lattice \( \mathbb{Z}^d \) is replaced by \( \mathbb{Z}^d \times T^2 \) with \( d_1 \geq 2, \; d = d_1 + d_2 \) sufficiently high and \( T \) a cycle of even length. The results address questions going back to Berker–Kadanoff (1980), Kotecký (1985) and Salas–Sokal (1997).

1. Introduction and results

What does a typical proper coloring with \( q \) colors of the integer lattice \( \mathbb{Z}^d \) look like? By proper we mean that adjacent vertices must be colored differently. As the lattice \( \mathbb{Z}^d \) is bipartite, having an even and an odd sublattice, it admits proper \( q \)-colorings for any \( q \geq 2 \). The \( q = 2 \) case is degenerate with only two possible (proper) colorings – the chessboard coloring and its translation by one lattice site. For \( q \geq 3 \) the number of colorings of bounded domains is exponentially large in the volume of the domain, as witnessed by the following important construction: Partition the \( q \) colors into two subsets \( A, B \) and consider the family of colorings obtained by coloring sites in the even sublattice with colors from \( A \) and sites in the odd sublattice with colors from \( B \). On a domain \( \Lambda \) with an equal number of even and odd sites this gives \((|A|, |B|)|\Lambda|/2 \) colorings, and this quantity is maximized when \(|A| = \lfloor q/2 \rfloor, |B| = \lceil q/2 \rceil \). Certainly most colorings are not obtained this way, but could it be that most colorings coincide with such a ‘pure \((A, B)\)-coloring’ at most vertices? This is evidently not so in dimension \( d = 1 \) (when \( q \geq 3 \)) and, in fact, is not the case in any dimension provided the number of colors is large compared with the dimension \((q > 4d)\) suffices, see discussion after Theorem 1.1. The main result presented here deals with the opposite regime – when the dimension is large compared with the number of colors – where it is shown that coincidence at most vertices with a ‘pure \((A, B)\)-coloring’ does in fact take place. More precisely, when \( \{A, B\} \) partitions the \( q \) colors into sets of sizes \( \lfloor q/2 \rfloor \) and \( \lceil q/2 \rceil \), then picking a coloring uniformly among colorings of a domain which follow the \((A, B)\)-pattern on its boundary, the coloring at any vertex in the domain is very likely to follow the \((A, B)\)-pattern as well.

We proceed to state our main result, following required notation. A pattern is a pair \((A, B)\) of disjoint subsets of \([q] := \{1, \ldots, q\}\) (we stress that \((A, B)\) and \((B, A)\) are distinct patterns). It is called dominant if \(|A|, |B| = \{\lfloor q/2 \rfloor, \lceil q/2 \rceil\}\). A domain is a non-empty finite \( \Lambda \subset \mathbb{Z}^d \) such that both \( \Lambda \) and \( \mathbb{Z}^d \setminus \Lambda \) are connected. Its internal vertex-boundary, the set of vertices in \( \Lambda \) adjacent to a vertex outside \( \Lambda \), is denoted \( \partial \Lambda \). Given a proper \( q \)-coloring \( f \), we say that

- a vertex \( v \) is in the \((A, B)\)-pattern if either \( v \) is even and \( f(v) \in A \), or \( v \) is odd and \( f(v) \in B \).

We also say that a set of vertices is in the \((A, B)\)-pattern if all its elements are such.

Date: August 13, 2018.

Research of both authors was supported by the Israel Science Foundation grant 861/15 and the European Research Council starting grant 678520 (LocalOrder). Research of Y.S. was additionally supported by the Adams Fellowship Program of the Israel Academy of Sciences and Humanities.
Theorem 1.1. There exists $C \geq 1$ such that for any number of colors $q \geq 3$ and any dimension $d \geq Cq^{10} \log^3 q$, the following holds. Let $(A, B)$ be a dominant pattern. Let $\Lambda \subset \mathbb{Z}^d$ be a domain. Let $\mathbb{P}_{\Lambda, (A, B)}$ be the uniform measure on proper $q$-colorings $f$ of $\Lambda$ satisfying that $\partial_f \Lambda$ is in the $(A, B)$-pattern. Then

$$\mathbb{P}_{\Lambda, (A, B)}(v \text{ is not in the } (A, B)\text{-pattern}) \leq e^{-\frac{d}{q^2(q+\log^2 q)}}, \quad v \in \Lambda.$$  

Theorem 1.2. Let $q \geq 3$ and suppose that the dimension $d$ satisfies (1). There exists a distinct extremal maximal-entropy Gibbs state $\mu_{(A, B)}$ for each dominant pattern $(A, B)$. Moreover, for any sequence of domains $\Lambda_n$ increasing to $\mathbb{Z}^d$, the measures $\mathbb{P}_{\Lambda_n, (A, B)}$ converge weakly to $\mu_{(A, B)}$ as $n \to \infty$. In particular, $\mu_{(A, B)}$ is invariant to automorphisms of $\mathbb{Z}^d$ preserving the two sublattices.
yield information on the mixing properties of $\mu_{(A,B)}$, implying that it is strongly mixing with an exponential rate (see Lemma 8.7).

Theorem 1.2 shows that there are at least $\binom{q}{q/2}$ extremal Gibbs states of maximal entropy when $q$ is even and $\binom{q}{\lfloor q/2 \rfloor}$ such Gibbs states when $q$ is odd. Our third result shows that these exhaust all possibilities.

**Theorem 1.3.** Let $q \geq 3$ and suppose that the dimension $d$ satisfies $1$. Then any maximal-entropy Gibbs state is a mixture of the measures $\{\mu_{(A,B)}\}$.

The main results are not valid in low dimensions due to the uniqueness results discussed above. Nonetheless, a version of the results applies in any dimension $d \geq 2$ provided the underlying graph is suitably modified. Precisely, the results remain true when $Z^d$ is replaced by a graph of the form $Z^{d_1} \times T^{d_2}_{2m}$, $m \geq 1$ integer, provided $d_1 \geq 2$ and $d = d_1 + d_2$ satisfies $1$, where $T^{d_2}_{2m}$ is the cycle graph on $2m$ vertices. The graph $Z^{d_1} \times T^{d_2}_{2m}$ may be viewed as a subset of $Z^d$ in which the last $d_2$ coordinates are restricted to take value in $\{0,1,\ldots,2m-1\}$ and are endowed with periodic boundary conditions. In this sense, it is only the local structure of $Z^d$ which matters to the results. To keep the discussion focused, we present the proofs of the results only in the $Z^d$ case. The extension to graphs of the above form requires virtually no modification to the arguments beyond obvious notational changes.

### 1.1. General spin systems

The methods introduced in this paper allow a vast generalization: In the companion paper [51], we extend the ideas from the proper $q$-coloring setting to general discrete spin systems. Under suitable conditions, the results characterize the set of Gibbs states of maximal pressure of such systems, showing that a typical sample from such a Gibbs state mainly follows an $(A,B)$ pattern for suitable sets $A,B$. We briefly describe here the main results of [51]. An introduction aimed at a physics audience appears in [50].

The spin systems considered are described by a finite spin space $S$, a collection $((\lambda_i))_{i \in S}$ of positive numbers called the single-site activities, and a collection $((\lambda_{i,j}))_{i,j \in S}$ of non-negative numbers called the pair interactions. The pair interactions are symmetric, i.e., $\lambda_{i,j} = \lambda_{j,i}$ for all $i, j \in S$, and at least one is positive. The probability of a configuration $f : \Lambda \rightarrow S$ is proportional to

$$\prod_{v \in \Lambda} \lambda_{f(v)} \prod_{\{u,v\} \in E(\Lambda)} \lambda_{f(u),f(v)}^\Lambda = \prod_{\{u,v\} \in E(\Lambda)} \lambda_{f(u),f(v)},$$

where $E(\Lambda)$ is the set of edges of $Z^d$ whose two endpoints belong to $\Lambda$. Classical models obtained as special cases include the Ising, Potts, hard-core, Widom–Rowlinson and clock models.

The $q$-state antiferromagnetic Potts model at temperature $T$ is obtained when $S = \{q\}$ and $\lambda_{i,j} = 1_{\{i\neq j\}} + e^{-\gamma} 1_{\{i=j\}}$. The $((\lambda_i))$ encode external magnetic fields. In the zero-temperature limit $\lambda_{i,j} = 1_{\{i\neq j\}}$, and taking also $\lambda_i = 1$ one recovers the proper $q$-coloring model.

The emergent long-range order will involve spins interacting with the maximal pair interaction weight. In this setting, a pattern is thus defined as a pair $(A,B)$ of subsets of $S$ such that $\lambda_{a,b} = \max_{i,j \in S} \lambda_{i,j}$ for all $a \in A$ and $b \in B$.

The single-site activities then play a role in singling out dominant patterns, defined as patterns maximizing $(\sum_{a \in A} \lambda_a)(\sum_{b \in B} \lambda_b)$ among all patterns. These definitions extend the ones used for proper $q$-colorings.

Two patterns $(A,B)$ and $(A',B')$ are called equivalent if there is a bijection $\varphi : S \rightarrow S$ such that $\varphi(A) = A'$, $\varphi(B) = B'$, $\lambda_{\varphi(i)} = \lambda_i$, $\lambda_{\varphi(i)\varphi(j)} = \lambda_{i,j}$ for all $i, j \in S$.

The results of the companion paper [51] apply to spin systems in which all dominant patterns are equivalent.
As for proper colorings, here too we wish to avoid degenerate situations, and thus restrict attention to (periodic) maximal-pressure Gibbs states (which are the analogues of maximal-entropy Gibbs states in this more general setting).

**Theorem 1.4** ([51]). For each spin system as above (fixing $S$, $(\lambda_i)$ and $(\lambda_{i,j})$) in which all dominant patterns are equivalent there exists $d_0$ such that in any dimension $d \geq d_0$, the set of extremal maximal-pressure Gibbs states is in bijection with the set of dominant patterns. Moreover, typical samples from the Gibbs measure $\mu_{(A,B)}$ corresponding to the dominant pattern $(A,B)$ have a strong tendency to follow the $(A,B)$-pattern in the sense that $\mu_{(A,B)}(f(u) \in A, f(v) \in B) \geq 1 - \epsilon(d)$, for even $u \in \mathbb{Z}^d$ and odd $v \in \mathbb{Z}^d$, where $\epsilon(d) \to 0$ as $d \to \infty$.

A quantitative estimate for $d_0$ in terms of $(\lambda_i)$ and $(\lambda_{i,j})$ is possible, encapsulating conditions of ‘low-temperature’ and ‘significant weight difference between dominant and non-dominant patterns’, as described in [51]. These imply, for instance, that the results obtained for the proper $q$-coloring model extend to the low-temperature regime of the antiferromagnetic $q$-state Potts model, with the temperature even allowed to grow with $d$ at a power-law rate. Also described in [51] are properties of the Gibbs state $\mu_{(A,B)}$ which is in correspondence with the dominant pattern $(A,B)$, among which are: quantitative bounds for $\epsilon(d)$, invariance under automorphisms preserving the two sublattices and convergence of finite-volume measures with $(A,B)$ boundary conditions to $\mu_{(A,B)}$.

As for the $q$-coloring model, a version of Theorem 1.4 remains valid on $\mathbb{Z}^{d_1} \times \mathbb{T}_{2n}^{d_2}$ provided $d_1 \geq 2$ and $d = d_1 + d_2$ is at least the threshold $d_0$ of the theorem.

### 1.2. Discussion and background.

Long-range ordering results of the type obtained here are ubiquitous in statistical physics. Starting from the classical result of Peierls [47] that the Ising model orders at low temperature, such results have been obtained for a wide range of models. In the example of the Ising model, where the state space is $S = \{+, -\}$, the probability distribution biases against different values being placed at adjacent vertices. In the limit of zero temperature, this bias becomes absolute and the only allowed configurations in a domain are the fully + or fully − configurations. The result of Peierls may thus be viewed as saying that the zero-temperature ordering persists to the low-temperature regime, an idea which received systematic treatment starting with the work of Pirogov and Sinai [52, 53] (see Friedli–Velenik [16, Chapter 7] for a pedagogical introduction). In contrast, the proper $q$-coloring model studied here is already a zero-temperature model (for the antiferromagnetic $q$-state Potts model), with the difficulty in its analysis stemming from the fact that it has residual entropy — configurations are sampled uniformly from a set whose cardinality is exponential in the volume. As such, any long-range order present in the model is entropically-driven and its rigorous justification requires new tools.

The question of understanding the type of emergent long-range order, or its absence, in the antiferromagnetic $q$-state Potts model, including proper $q$-colorings, has received significant attention. In the physics literature, to our knowledge, the problem was first considered by Berker–Kadanoff [3] who suggested in 1980 that a phase with algebraically decaying correlations may occur at low temperatures (including zero temperature) with fixed $q$ when $d$ is large. This prediction was challenged by numerical simulations and an $\varepsilon$-expansion argument of Banavar–Grest–Jasnow [2] who predicted a Broken-Sublattice-Symmetry (BSS) phase at low temperatures for the 3 and 4-state models in three dimensions. The BSS phase is exactly of the type proved to occur here, with a global tendency towards a pure $(A,B)$-ordering for a dominant pattern $(A,B)$. Kotecký [37] in 1985 argued for the existence of the BSS phase at low temperature when $q = 3$ and $d$ is large by analyzing the model on a decorated lattice. This prediction became known as *Kotecký’s conjecture*. While our concern here is with the zero-temperature case, we briefly mention that the behavior of the antiferromagnetic Potts model at intermediate temperature regimes is also unclear. The interested reader is directed to the paper of Rahman–Rush–Swendsen [55], where the 3-state model in three dimensions is considered, conflicting predictions regarding Permutationally-Symmetric-Sublattice
RIGIDITY OF PROPER COLORINGS OF $\mathbb{Z}^d$

(PSS) and Rotationally-Symmetric (RS) phases are surveyed and the controversy between them is addressed. We are not aware of mathematically rigorous results on such intermediate-temperature regimes. We also mention that irregularities in a lattice (i.e., having different sublattice densities) often promote the formation of order. This may be used, for instance, to find for each $q$ a planar lattice on which the proper $q$-coloring model is ordered [30]. However, irregularities also modify the nature of the resulting phase, leading to long-range order in which a single spin value appears on most of the lower-density sublattice [38], or to partially ordered states [54].

In the mathematically rigorous literature, Kotecký’s conjecture remained open for 25 years until it was finally answered by the first author [48] and by Galvin–Randall–Sorkin [23] (following closely-related papers by Galvin–Randall [21] and Galvin–Kahn [22]). Its extension to the low-temperature Potts model was resolved by Feldheim and the second author [12]. The results of [23,48] correspond to the $q = 3$ case of Theorem 1.1 and to the existence of 6 extremal maximal-entropy Gibbs states which results from it (the fact that the measures have maximal entropy is shown in [23, Section 5]), while the convergence result in Theorem 1.2 and the characterization result given in Theorem 1.3 are new also for this case. Periodic boundary conditions were considered in [13,24] and in [48] for the corresponding height function (also on tori with non-equal side lengths). Following Kotecký, it is quite natural to predict that multiple maximal-entropy Gibbs states exist for proper $q$-colorings with any $q \geq 3$ provided the dimension is sufficiently large as a function of $q$. Related questions and conjectures have been made by several authors:

- Salas–Sokal [56] write in 1997 that any lattice $G$ should admit a value $q_c(G)$ such that the antiferromagnetic $q$-state Potts model on $G$ is disordered at all $q > q_c(G)$ and all temperatures, has a critical point at zero temperature when $q = q_c(G)$, and often (though not always) has a phase transition at non-zero temperature for any $q < q_c(G)$;
- Kotecký–Sokal–Swart [38] Section 1.4, (3) ask to prove the existence of an entropy-driven phase transition on $\mathbb{Z}^d$ for suitable pairs of $(q,d)$ and suggest that this holds for $q < q_c(\mathbb{Z}^d)$ for some function $q_c(\mathbb{Z}^d)$, possibly satisfying $q_c(\mathbb{Z}^d) \approx 2d$.
- Engbers–Galvin [10] Section 6.3 write that it would be of great interest to prove long-range order for weighted graph homomorphisms on $\mathbb{Z}^d$ (including proper $q$-colorings) and deduce information on the Gibbs states of the model.
- Galvin–Kahn–Randall–Sorkin [23] Conjecture 1.3 conjecture that, for any $q > 3$, there are multiple maximal-entropy Gibbs states for proper $q$-colorings of $\mathbb{Z}^d$ when $d$ is sufficiently large.
- Feldheim and the authors ask in [13] Section 8 and [12] Section 1.3 to show long-range order of the BSS type (with $\lceil q/2 \rceil$ colors predominant on one sublattice and the remaining $\lfloor q/2 \rfloor$ colors on the other sublattice) for each $q$ when $d$ is sufficiently large.

Our work resolves the prediction by exhibiting long-range order for all $q$ when $d$ is sufficiently large, and further allows for a quantitative power-law dependence between $q$ and $d$ (the companion paper [51] addresses more general models including weighted graph homomorphisms). Compared with the aforementioned uniqueness of Gibbs states results which hold when $q > Cd$, we see that a power-law dependence is best possible though the precise power is yet to be determined.

The previously addressed case of $q = 3$ colors has a special additional structure as proper 3-colorings of $\mathbb{Z}^d$ admit a height function representation. This special structure manifests in a natural cyclic order on the 6 dominant patterns and is essential to the analysis presented in [48] and [23]. Already the extension to low temperatures in [12] is quite significant as the global height representation is lost, but the analysis there still relies on the height function existing locally, away from the rare places where the coloring is not proper. As nothing of this structure remains when the number of colors increases beyond 3, the previously used methods are insufficient for the analysis of proper $q$-colorings with any $q \geq 4$. Specific new challenges arising include the difficulty in identifying ordered regions (which, if any, dominant pattern does a vertex follow?), the many more ways in which the proper coloring can order and transition between the different orders (the
complicated ‘adjacency structure’ of the numerous dominant patterns), and the more significant role played by disordered regions (where vertices do not follow any pattern) and sub-optimally ordered regions (where vertices follow a non-dominant pattern).

A common ingredient in the proofs of long-range order for \( q = 3 \) colors in \( \mathbb{Z}^d \), as well as for the hard-core model, is the use of sophisticated contour methods. The underlying idea is similar to the argument of Peierls – identify regions of ‘excitations’ away from the ordered state, show that any specific excitation is unlikely and use a union bound to show that the probability that there exists an excitation is small. However, the idea in this form fails for the models mentioned above, as the probability of specific excitations is not sufficiently small to allow the use of the union bound. As a remedy, one is led to a coarse-graining technique where several different excitations are grouped together according to a common ‘approximation’, the probability of each approximation is shown to be small, the number of approximations is shown to be small (compared with the number of excitations) and a union bound over approximations is then applied to show that the probability that there exists an excitation is small. The notion of approximation which turns out to be fruitful takes advantage of the following geometric property of the excitation regions in the 3-coloring and hard-core models – these regions have all their internal vertex boundary on a single sublattice of \( \mathbb{Z}^d \). Regions with this property have been termed ‘odd cutsets’ in [48]. The idea to approximate several such regions according to a common approximation can be traced back to the works of Korshunov and Sapozhenko \([35,36,57,59]\) in the context of general bipartite graphs, with further developments and applications to statistical physics questions on \( \mathbb{Z}^d \) made by Galvin \([18,20,21]\), Galvin–Kahn \([22]\), Galvin–Kahn–Randall–Sorkin \([23]\), Galvin–Randall \([24]\), Galvin–Tetali \([26,27]\), Feldheim–Spinka \([12,14]\), Peled \([48]\) and Peled–Samotij \([49]\). This core idea is also used and further developed in this work.

In a parallel development, entropy methods have been identified as a powerful tool to analyze models of graph homomorphisms. Pioneered by Kahn–Lawrentz \([34]\) in 1999 and Kahn \([32,33]\) in 2001, the ideas were further developed by Galvin–Tetali \([25]\) (see also Lubetzky–Zhao \([42]\), Galvin \([19]\), Madiman–Tetali \([43]\) and Engbers–Galvin \([10,11]\). The basic method applies to graph homomorphisms from a finite bipartite regular (or bi-regular) graph \( G \) to a general finite graph \( H \). Relying on Shearer’s inequality \([7]\), it implies that most such graph homomorphisms are locally ordered at most vertices, in the sense that the neighborhood of all but \( \epsilon(\Delta(G)) \) fraction of the vertices follow some dominant pattern (as in Section 1.1), where \( \Delta(G) \) is the degree of \( G \) and \( \epsilon(\Delta) \) is a function satisfying \( \epsilon(\Delta) \to 0 \) as \( \Delta \to \infty \). This suffices to estimate rather accurately the exponential growth rate of the number of graph homomorphisms, up to an error term which decreases as the degree of \( G \) grows (for proper colorings of \( \mathbb{Z}^d \) the obtained error decays as \( C(q)/d \) as \( d \to \infty \). Our results imply improved error bounds, see Section 8.3. Generalizations from graph homomorphisms to discrete spin systems of the type considered in Section 1.1 are possible \([19,25]\). The method does not generally imply global ordering in typical graph homomorphisms, as it allows for different regions to be ordered according to different dominant patterns. Nonetheless, it was discovered in \([10]\) that global ordering follows on hypercube graphs – discrete tori with vertex set \( \{0,1,\ldots,2m-1\}^d \) which are considered with \( m \geq 1 \) fixed and \( d \to \infty \) – due to the interplay between their isoperimetric properties and the smallness of the function \( \epsilon \) above. One may further allow \( m \) to grow slowly with \( d \) but this approach does not extend to the \( \mathbb{Z}^d \) lattice \([10]\) Section 6.3.

The main technical novelty introduced in this paper is a non-trivial synthesis of the contour and entropy methods discussed above. Our approach begins by identifying ordered and disordered regions in a given coloring, where vertices are classified according to the coloring of their local neighborhoods. The abundance of possible local colorings gives rise to a complicated classification where regions ordered according to one dominant pattern may overlap with those of another and where many types of disordered behavior may arise. The contours separating the different regions are then approximated with a similar, albeit more involved, technique to that used in the \( q = 3 \) case. It then remains to prove that any given picture of approximated contours is unlikely, in order
to deduce long-range order via a union bound. This is resolved here by use of the entropy method extended in the following two manners: (i) The method is applied to a partial set of colorings, restricted by various pieces of information known from the contour picture and taken into account by the entropy estimates to produce a sufficiently tight bound. (ii) The method is applied to colorings defined on bounded subsets of $\mathbb{Z}^d$, specifically on the disordered regions and on the interfaces between ordered regions. This is in contrast with previous applications of the method where it was applied to the full set of colorings (or graph homomorphisms), which were themselves defined on a regular graph. New difficulties thus arise in integrating the externally available information with the entropy estimates and in carefully tracking and cancelling the boundary terms arising from the irregularity of the bounded subsets. A detailed overview of the method is given in Section 2.

We end the discussion with several questions for future research.

1. Determine for all pairs $(q,d)$ whether there is a unique maximal-entropy Gibbs state. Is the dependence on $q$ monotone in the sense that there is a $q_c(d)$ with multiplicity holding if and only if $q < q_c(d)$? Does $q_c(d)$ tend to a positive limit as $d \to \infty$? As mentioned above, uniqueness is known when $q > 3.53d$ [29] (even without the entropy restriction).

The $d$-regular tree case was studied by Brightwell–Winkler [3] who noted that frozen Gibbs states exist whenever $q \leq d$, and by Jonasson [31] who proved uniqueness whenever $q \geq d+1$ and $d$ is large. We note regarding frozen Gibbs states on $\mathbb{Z}^d$ the following construction explained to us by Nishant Chandgotia: If $q = d + 1$ there is a frozen Gibbs state supported on $f : \mathbb{Z}^d \to \{0, 1, \ldots, q - 1\}$ defined by $f(x_1, \ldots, x_d) := \sum_{i=1}^{d} ix_i \pmod{q}$. If $q < d + 1$ one has frozen Gibbs states which behave as the previous construction on $(q - 1)$-dimensional sections.

2. Prove an analogous result to Theorem 1.1 for free and periodic boundary conditions. Our methods should be relevant also for these cases, with the periodic case with even side length possibly being a direct extension (see [13, Section 8] for a prediction regarding 3-colorings of tori with odd side length), and the free case seeming more difficult as issues regarding excitations (deviations from the long-range order) touching the boundary of the domain must be dealt with carefully. Of course, the characterization of Gibbs states given in Theorem 1.3 does not depend on the choice of boundary conditions.

3. As discussed, our results apply also in low dimensions provided the underlying lattice is enhanced to $\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} / \mathbb{Z}^{2m}$, $m \geq 1$ integer, $d_1 \geq 2$ and $d = d_1 + d_2$ satisfying [1]. Another natural enhancement used in low-dimensional lattices, e.g., in the context of percolation [61], is the spread-out lattice. In our context, this corresponds to $\mathbb{Z}^d$ with additional edges connecting every two vertices of different parity whose graph distance in $\mathbb{Z}^d$ is at most some fixed threshold $M$. We expect our results to hold also with this enhancement provided $d \geq 2$ and $M$ is sufficiently large as a function of $q$ (raising $d$ should only assist the long-range order).

1.3. Organization. The rest of the paper is organized as follows. In Section 2, we provide an overview of the proof. In Section 3, definitions and preliminary results which will be needed throughout the paper are given. In Section 4, we give the main steps of the proof of Theorem 1.1 including the definitions of breakup and approximation and the statements of several propositions which are then used to deduce Theorem 1.1. In Section 5, we prove the propositions about the breakup (existence of non-trivial breakup, almost-sure absence of infinite breakups, bounds on the probability of breakups). In Section 6, we prove Lemma 4.7 which provides a general bound on the probability of an event and which is used in the proofs in Section 5.3 and Section 5.4. In Section 7, we prove Proposition 4.9 about the exists of a small family of approximations. Finally, in Section 8, we prove results about the infinite-volume Gibbs states, namely, Theorem 1.2 and Theorem 1.3.

1.4. Acknowledgments. We thank Raimundo Briceño, Nishant Chandgotia, Ohad Feldheim and Wojciech Samotij for early discussions on proper colorings and other graph homomorphisms. We are grateful to Christian Borgs for valuable advice on the way to present the material of this paper.
and its companion \[51\]. We thank Michael Aizenman, Jeff Khan, Eyal Lubetzky, Dana Randall, Prasad Tetali and Peter Winkler for useful discussions and encouragement.

2. OVERVIEW OF PROOF

In this section we give a high-level view of the proof of Theorem 1.1.

We recall that \((A, B)\) is a dominant pattern if \(A, B \subset [q]\) and \(\{|A|, |B|\} = \{\lfloor \frac{q}{2} \rfloor, \lceil \frac{q}{2} \rceil\}\). Throughout this section, we fix a domain \(\Lambda \subset \mathbb{Z}^d\) and a dominant pattern

\[ P_0 = (A_0, B_0) \quad \text{such that} \quad |A_0| = \lfloor \frac{q}{2} \rfloor, \quad |B_0| = \lceil \frac{q}{2} \rceil. \tag{5} \]

We think of \(P_0\) as the boundary pattern so that we will later consider a coloring chosen from \(\mathbb{P}_{\Lambda, P_0}\).

We use \(\partial U\) to denote the edge-boundary of a set \(U \subset \mathbb{Z}^d\), and \(N(U)\) to denote its neighborhood (vertices adjacent to some vertex in \(U\)). We also denote \(U^+ := U \cup N(U)\), \(\partial_* U := U \cap N(U^c)\), \(\partial_\ast U := N(U) \setminus U\) and \(\partial_{\ast*} U := \partial_* U \cup \partial_\ast U\). We say that \(U\) is an even (odd) set if \(\partial_{\ast*} U\) is contained in the even (odd) sublattice of \(\mathbb{Z}^d\). An even (odd) set \(U\) is called regular if both it and its complement contain no isolated vertices. See Section 3 for more notation and definitions.

2.1. A toy scenario. To gain intuition, let us analyze the entropic loss in the toy scenario in which the \(P_0\)-pattern is disturbed by a single droplet of a different dominant pattern \(P = (A, B)\); see Figure 1. More precisely, let \(U \subset \mathbb{Z}^d\) be such that \(U^+ \subset \Lambda\) and let \(n(U)\) be the number of proper colorings of \(\Lambda\), for which \(U^+\) is in the \(P\)-pattern and \((\Lambda \setminus U)^+\) is in the \(P_0\)-pattern. A straightforward computation yields that, when \(q\) is even,

\[ \frac{n(U)}{n(\emptyset)} \leq \left( \frac{q - 2}{q} \right)^{|\partial_{\ast*} U|}, \]

with equality if and only if \(|A_0 \Delta A| = 2\). When \(q\) is odd, a straightforward (though somewhat more involved) computation yields that

\[ \frac{n(U)}{n(\emptyset)} \leq \left( \frac{q - 1}{q + 1} \right)^{\frac{1}{2d} |\partial U|}, \]

with equality if and only if either \(U\) is an odd set and \(A_0 \subset A\) or \(U\) is an even set and \(B_0 \subset B\). This example shows a difference in behavior between the even and odd \(q\) cases, with the odd case more difficult due to the lower entropic cost of creating interfaces between \(P_0\)- and \(P\)-ordered regions. It is the odd \(q\) case that motivates many of our definitions and ideas, including the idea that such interfaces should be even or odd, according to the relative size of \(A_0\) and \(A\).

2.2. Identification of ordered and disordered regions. Given a proper \(q\)-coloring \(f\) of \(\mathbb{Z}^d\), we wish first to identify regions where \(f\) follows, in a suitable sense, a dominant pattern. A first idea is that the decision regarding a vertex \(v\) will be made based on the values that \(f\) takes on the neighbors of \(v\). Indeed, the color that \(v\) takes cannot itself be sufficient as it has only \(q\) options whereas there are many more dominant patterns, but the colors of the neighbors turn out to suit the job. A second idea, motivated by the toy scenario described earlier and also by questions of
approximation of contours which will be soon described, is that each region will be a (regular) even or odd set. More precisely, the region associated with a dominant pattern \((A, B)\) is an even set if \(|A| \leq |B|\) and an odd set if \(|A| > |B|\) (thus odd sets appear only if \(q\) is odd). Let us now describe the regions precisely. Let \(\mathcal{P}\) be the set of all dominant patterns. For each \(P = (A, B) \in \mathcal{P}\), define the terms

\[
P\text{-even} = \begin{cases} 
\text{even} & |A| \leq |B| \\
\text{odd} & |A| > |B| 
\end{cases}
\quad \text{and similarly} \quad P\text{-odd} = \begin{cases} 
\text{odd} & |A| \leq |B| \\
\text{even} & |A| > |B| 
\end{cases}.
\]

Thus, for instance, if \(|A| \leq |B|\) then even vertices (having even sum of coordinates) are \(P\)-even and odd vertices are \(P\)-odd. The region associated to \(P\) is denoted \(Z_P(f)\) and defined by

\[
Z_P = Z_P(f) := \{v \in \mathbb{Z}^d : v \text{ is } P\text{-odd}, \ N(v) \text{ is in the } P\text{-pattern}\}^+.
\]

For technical reasons, only \(P\)-odd vertices whose neighbors are in the \(P\)-pattern are included in \(Z_P\), and then \(Z_P\) is taken to be the smallest \(P\)-even set containing them. Note that a \(P\)-odd vertex in \(Z_P\) is not itself required to be in the \(P\)-pattern, whereas a \(P\)-even vertex in \(Z_P\) is necessarily in the \(P\)-pattern, but need not have its neighbors in the \(P\)-pattern. In addition, there may be \(P\)-even vertices which are not in \(Z_P\) although their neighbors are in the \(P\)-pattern. These somewhat undesirable consequences of our definition are allowed in order to ensure that \(Z_P\) is a regular \(P\)-even set, which will be important in the proof.

Having defined the regions \((Z_P)\), let us examine more closely their interrelations. It is possible for a vertex \(v\) to belong to two (or more) of the \(Z_P\) and also possible that it lies outside all of the \(Z_P\). These possibilities are captured by the following definitions:

\[
Z_{\text{overlap}} := \bigcup_{P \neq Q} (Z_P \cap Z_Q) \quad \text{and} \quad Z_{\text{bad}} := \bigcap_P (Z_P)^c
\]

(see Figure 2). Regions of this type, along with the boundaries of \(Z_P\), are regions where the coloring \(f\) does not achieve its maximal entropy per vertex, in a way which is quantified later. It will be our task to prove that such regions are not numerous and this will lead to a proof of Theorem 1.1.

To this end, we define

\[
Z_* := \bigcup_P \partial_5 Z_P \cup Z_{\text{overlap}} \cup Z_{\text{bad}}.
\]

The region \(Z_*\) plays a similar role in our analysis as the contours used in arguments of the Peierls or Pirogov-Sinai type.

2.3. Breakups. With Theorem 1.1 in mind, let \(f\) be sampled from \(\mathbb{P}_{\Lambda, P_0}\) and fix a vertex \(v \in \Lambda\). It is convenient to extend \(f\) to a coloring of \(\mathbb{Z}^d\) by coloring vertices of \(\Lambda^c\) independently and uniformly from \(A_0\) or \(B_0\) according to their parity (so that they are in the \(P_0\)-pattern). The collection \((Z_P)\) then identifies ordered and disordered regions in \(f\). Our goal is to show that \(v\) is typically in the \(P_0\)-pattern. One checks that \(Z_P \setminus Z_{\text{overlap}}\) is in the \(P\)-pattern, and therefore it suffices to show that, with high probability, \(Z_{P_0}\) is the unique set among \((Z_P)\) to which \(v\) belongs. This, in turn, follows by showing that there is a path from \(v\) to infinity avoiding \(Z_*\). If no such path exists, there needs to be a connected component of \(Z_*^+\) which disconnects \(v\) from infinity. Our focus is then on these connected components and this motivates the following notion of a breakup seen from \(v\), which encodes the partial information from \((Z_P)\) relevant to these components.

The breakup seen from \(v\) is a collection \((X_P)_{P \in \mathcal{P}}\) of subsets of \(\mathbb{Z}^d\), from which one defines \(X_*\) in the same manner as \(Z_*\) is defined from \((Z_P)\), with the following properties: The \((X_P)\) coincide with the \((Z_P)\) in the neighborhood of \(X_*\) in the sense that \(X_P \cap X_*^+ = Z_P \cap X_*^+\) for each \(P\), and \(X_*^+\) is composed of the connected components of \(Z_*^+\) which disconnect \(v\) from infinity. The definition implies that each \(X_P\) is a regular \(P\)-even set, a property important for the approximations described in the section below. The choice to consider connected components of \(Z_*^+\) rather than just connected components of \(Z_*^+\) implies that near \(X_*\) (in its 5-neighborhood)
there are no additional violations of the pure dominant pattern coloring. This will be convenient in the proof (though the specific number 5 is not important and could just as well be taken larger).

For technical reasons, we define a slightly more general notion of breakup (which is not associated to a vertex \( v \)), as it streamlines some of the proofs, but we will not refer to this extension in the overview here. See Section 4.2 for the definition.

2.4. Approximations. Our method of proof is, in essence, an involved variant of the Peierls argument. As such, it consists of two parts: the first is to obtain a bound on the probability of a given breakup (this is discussed in the subsequent section), and the second is to conclude the unlikeliness of any breakup. As can be seen from the toy scenario considered above, the “entropic loss per edge” on the interfaces between different \( X_P \) is positive, but small. This leads to a bound on the probability of a given breakup which, however, does not allow to conclude with the standard union bound – indeed, the number of possible breakups is too large in comparison. Instead, we employ a delicate coarse-graining scheme, termed here approximation, of the possible breakups according to their rough features.

The crucial property of the breakup which allows its approximation is that each \( X_P \) is either regular even or regular odd. Let us briefly discuss the theory of such sets: The number of odd sets \( U \subset \mathbb{Z}^d \) which are connected, have connected complement, contain the origin and have \( |\partial U| = L \) boundary plaquettes grows as \( 2^{\frac{\log 2}{2} L} \) for \( L \) large \([4]\), with \( 2^{-2d} \leq \varepsilon_d \leq \frac{C\log^{\frac{3}{2}d}}{\sqrt{d}} \). This contrasts with the same count when the set is not required to be odd, which grows faster, roughly as...
The different growth rates are indicative of a deeper structural difference. Typical odd sets of the above type have a macroscopic shape (e.g., an axis-parallel box) from which they deviate on the microscopic scale, while sets of the above type without the parity restriction should scale to integrated super-Brownian excursion \([11,60]\). The distinction between these very different behaviors is akin to the breathing transition undergone by random surfaces \([15\text{ Section 7.3}]\). This phenomenon has been exploited in previous works \([12,22,48]\) to provide a natural coarse-graining scheme for odd sets, grouping them according to their macroscopic shape, and noting that this shape has significantly less entropy in high dimensions (of order at most \((\log d)^{3/2}L\)) than the odd sets themselves. We proceed in the same manner here, extending the previous schemes from a single \(X_P\) to breakups.

It is natural to approximate breakups by applying the coarse-graining technique separately to each \(X_P\). This can indeed be done, but due to the amount of dominant phases it leads to a version of Theorem \([1,39]\) which requires the dimension \(d\) to be larger than an exponential function of \(q\), rather than the stated power-law dependence \([1]\). Instead, we use a more sophisticated scheme which takes into account the interplay between the different \(X_P\) sets themselves. We proceed in the same manner here, extending the previous schemes from a single \(X_P\) to breakups.

An approximation of a breakup \(X = (X_P)_P\) is a collection \(\mathcal{A} = ((A_P)_P \in \mathcal{P}, A^*, A^{**})\) of subsets of \(\mathbb{Z}^d\), with the following properties: \(A_P \subset X_P \subset A_P \cup A^{**}\) for all \(P\), so that \(A_P\) represents the region known to be in the corresponding set of the breakup and \(A^{**}\) indicates the (joint) region which is unknown to belong to some \(X_P\). On a subset \(A^* \subset A^{**}\) of the unknown region, one has more information, namely, every vertex in \(A^*\) has many neighbors in \(\bigcup_P A_P\). Additional properties ensure that the unknown region is not large and that it is only present near \(X_\ast\). Thus, an approximation provides enough information to recover the breakup everywhere but near \(X_\ast\). See Section 4.5 for a precise definition.

The advantage of approximations is that, as we will show, one may find a relatively small family \(\mathcal{A}\) of them (much smaller than the number of breakups) with the property that every breakup seen from \(v\) is approximated by some element in \(\mathcal{A}\).

### 2.5. Repair transformation.

We proceed to explain, for a given \(X = (X_P)_P\), how to bound the probability that \(X\) is the breakup seen from \(v\), when \(f\) is sampled from \(P_{A_P}\). In the full proof the arguments need to be adapted to the case that only an approximation of \(X\) is given rather than \(X\) itself, but this adaptation is not the essence of the argument so our focus in the overview is on the case that \(X\) is given.

Let \(\Omega\) be the set of proper colorings for which \(X\) is the breakup seen from \(v\). To establish the desired bound on \(P_{A_P}(\Omega)\), we apply the following one-to-many operation to every coloring \(f \in \Omega\): (i) Delete the colors at all vertices of \(X_\ast\). (ii) For each connected component \(D\) of \(X_P \setminus X_\ast\), apply a permutation \(\phi\) taking \(P\) to \(P_\phi\) to the colors of \(f\) on \(D\), and also, if \(P = (A, B)\) is such that \(|A| > |B|\), then shift the configuration in \(D\) by a single lattice site in the \((1,0,\ldots,0)\) direction (such a shift was first used by Dobrushin for the hard-core model \([9]\)). (iii) Fill colors following \(P_\phi\) in all remaining vertices.

Noting that the resulting configuration is always a proper coloring, and that no entropy is lost in step (ii), it remains to show that the entropy gain in step (iii) is much larger than the entropy loss in step (i). The gain in step (iii) is either \(\log \lfloor \frac{q}{2} \rfloor\) or \(\log \lceil \frac{q}{2} \rceil\) per vertex according to its parity, making the entropy gain an easily computable quantity. The main challenge is thus to bound the loss in step (i), and the method used for its resolution is described next.

### 2.6. Upper bounds on entropy loss.

We make use of the following extension of the subadditivity of entropy (see Section 3.5 for basic definitions and properties), first used in a similar context by Kahn \([32]\), followed by Galvin–Tetali \([25]\).
Lemma 2.1 (Shearer’s inequality \[7\]). Let \(Z_1, \ldots, Z_n\) be discrete random variables. Let \(I\) be a collection of subsets of \(\{1, \ldots, n\}\) such that \(|\{I \in I : i \in I\}| \geq k\) for every \(i\). Then

\[
\text{Ent}(Z_1, \ldots, Z_n) \leq \frac{1}{k} \sum_{I \in I} \text{Ent}((Z_{i})_{i \in I}).
\]

Recall that \(X\) is fixed and that \(\Omega\) is the set of proper colorings for which \(X\) is the breakup seen from \(v\). Let \(f\) be sampled from \(P_{A,P_0}\) conditioned on \(f \in \Omega\). Let \(F\) be the configuration coinciding with \(f\) on \(X_s\) and equaling a fixed symbol \(\ast\) on \(X^c_s\). Applying Shearer’s inequality to \((F_v)_{v \in \text{Even}}\) with \(I = \{N(v)\}_{v \in \text{Odd}}\), yields

\[
\text{Ent}(f_{X_v}) = \text{Ent}(F) \leq \frac{1}{2} \sum_v \left[ \frac{\text{Ent}(F(N(v)))}{2d} + \frac{\text{Ent}(F(N(v)) | F|_{\text{Even}})}{2d} + \frac{\text{Ent}(F(v) | F(N(v)))}{2d} \right].
\]

Averaging this with the inequality obtained by reversing the roles of odd and even yields that

\[
\text{Ent}(f_{X_v}) = \text{Ent}(F) \leq \frac{1}{2} \sum_v \left[ \frac{\text{Ent}(F(N(v)))}{2d} + \frac{\text{Ent}(F(N(v)) | F|_{\text{Even}})}{2d} + \frac{\text{Ent}(F(v) | F(N(v)))}{2d} \right]. \tag{9}
\]

Of course, the terms corresponding to vertices \(v\) at distance 2 or more from \(X_s\) equal zero as \(F\) is deterministic in their neighborhood. The boundary terms corresponding to vertices \(v\) in \(\partial_B X_s\) need to be handled with careful bookkeeping, which we do not elaborate on here. The advantage of this bound is that it is local, with each term involving only the values of \(F\) on a vertex and its neighbors. Each term admits the simple bounds \(I \leq \frac{q \log(\frac{q}{2} \frac{q}{2})}{2d} + \text{II} \leq (\frac{q}{2} \frac{q}{2}) 2d\), which only take into account the fact that \(f\) is a proper coloring, i.e., that \(F(v) \notin F(N(v))\). The main contribution in these bounds comes from the possibility that \((F_v), F|_{N(v)}\) is approximately uniformly distributed in \(A \times B^{2d}\) for some dominant pattern \((A, B)\). To obtain stronger bounds, we use additional information implied by the knowledge that \(f \in \Omega\).

Let us illustrate this idea through examples. Consider a vertex \(v \in X_{\text{overlap}}\); which for concreteness, we assume to be even. By definition, there exist distinct dominant patterns \(P = (A, B)\) and \(Q = (A', B')\) such that \(v \in X_P \cap X_Q\). Suppose first that \(v\) is both \(P\)-even and \(Q\)-even (so that \(|A| = |A'| = \frac{q}{2}\)). Recalling \(7\), one may check that \(v\) is both in the \(P\)-phase and in the \(Q\)-phase, so that \(f(v) \in A \cap A'\). In particular, \(f(v), f|_{N(v)}\) belongs to \((A \cap A') (\{q\} \setminus (A \cap A')) 2d\). Hence

\[
\text{II} \leq \log(|A \cap A'| \cdot (q - |A \cap A'|)) \leq \log((\frac{q}{2} 1) + 1) \leq \log(\frac{q}{2} - 1).
\]

The improvement obtained over the simple bound \(\log(\frac{q}{2} \frac{q}{2})\) is significant, as these quantities are to be compared with the entropy gain per vertex described in Section 2.6, which is either \(\log(\frac{q}{2})\) or \(\log(\frac{q}{2})\) according to the parity of the vertex. Next, suppose instead that \(v\) is \(P\)-odd and \(Q\)-odd (so that \(q\) is necessarily odd, \(|A| = \frac{q}{2}\) and \(|A'| = \frac{q}{2}\)). Using \(7\), one may again check that \(v\) is in the \(P\)-phase and that \(N(v)\) in the \(Q\)-phase, so that \(f(v) \in A\) and \(f(N(v)) \in B\). Hence

\[
\text{II} \leq \log(|A| \cdot |B'|) = 2 \log(\frac{q}{2}) = \log(\frac{q}{2} \frac{q}{2} - \frac{q}{2}) \leq \log(\frac{q}{2} - 1).
\]

The case when \(v\) is both \(P\)-odd and \(Q\)-odd is similar.

To handle vertices in \(X_{\text{bad}}\) or in \(\partial_{\ast} X_P\) requires more work. For vertices \(v \in X_{\text{bad}}\), the deterministic information implied by \(f \in \Omega\) does not suffice to obtain a good bound. Indeed, the conditional entropies in II are averages over entropies on events of the form \(\{f(N(v)) = A\}\) with \(A \subset [q]\), and only in certain cases do we have good control on these entropies (for instance, when \(|A| \notin \{\frac{q}{2}, \frac{q}{2}\}\)). This is overcome by controlling the probabilities of such events. For vertices \(v \in \partial_{\ast} X_P\), the problem is of a different nature. Intuitively, the loss of entropy is not tied to the vertex-boundary \(\partial_{\ast} X_P\), but rather to the edge-boundary \(\partial X_P\). Indeed, given an edge \((u, v) \in \partial X_P\) (i.e., \([u, v] \in \partial X_P \text{ with } u \in X_P \text{ and } v \notin X_P\)) with \(u\) even (say), \(7\) implies that \(f(u) \in A\) and \(f(N(v)) \notin A\). After II is rewritten as a sum over neighbors \(w\) of \(v\), the summand corresponding to \(w = u\) can be effectively bounded in this case.
These ideas are explained in more detail in Section 4.6 where we introduce four notions on which our entropy bounds are based: vertices having unbalanced neighborhoods, non-dominant vertices, restricted edges and vertices having a unique pattern.

3. Preliminaries

3.1. Notation. Let $G = (V, E)$ be a graph. For vertices $u, v \in V$ such that $\{u, v\} \in E$, we say that $u$ and $v$ are adjacent and write $u \sim v$. We denote the graph-distance between $u$ and $v$ by $\text{dist}(u, v)$. For two non-empty sets $U, W \subset V$, we denote by $\text{dist}(U, W)$ the minimum graph-distance between a vertex in $U$ and a vertex in $W$. For a subset $U \subset V$, denote by $N(U)$ the neighbors of $U$, i.e., vertices in $V$ adjacent to some vertex in $U$, and define for $t > 0$,

$$N_t(U) := \{v \in V : |N(v) \cap U| \geq t\}.$$ 

In particular, $N_1(U) = N(U)$. Denote the external boundary and the internal boundary of $U$ by

$$\partial U := N(U) \setminus U \quad \text{and} \quad \partial_0 U := \partial U^c,$$

respectively. Denote also

$$\partial_\bullet U := \partial U \cup \partial_0 U \quad \text{and} \quad U^+ := U \cup \partial_0 U.$$ 

For a positive integer $r$, we denote

$$U^{+r} := \{v \in V : \text{dist}(v, U) \leq r\}.$$ 

The set of edges between two sets $U$ and $W$ is denoted by

$$\partial(U, W) := \{\{u, w\} : u \in U, w \in W\}.$$ 

The edge-boundary of $U$ is denoted by $\partial U := \partial(U, U^c)$. We also define the set of out-directed boundary edges of $U$ to be

$$\tilde{\partial}U := \{(u, v) : u \in U, v \in U^c, u \sim v\}.$$ 

We write $\tilde{\partial}U := \tilde{\partial}(U^c)$ for the in-directed boundary edges of $U$. We also use the shorthands $u^+ := \{u\}^+$, $\partial u := \partial\{u\}$ and $\tilde{\partial}u := \tilde{\partial}\{u\}$. The diameter of $U$, denoted by $\text{diam } U$, is the maximum graph-distance between two vertices in $U$, where we follow the convention that the diameter of the empty set is $-\infty$. For a positive integer $r$, we denote by $G^{\otimes r}$ the graph on $V$ in which two vertices are adjacent if their distance in $G$ is at most $r$.

We consider the graph $\mathbb{Z}^d$ with nearest-neighbor adjacency, i.e., the edge set $E(\mathbb{Z}^d)$ is the set of $\{u, v\}$ such that $u$ and $v$ differ by one in exactly one coordinate. A vertex of $\mathbb{Z}^d$ is called even (odd) if it is at even (odd) graph-distance from the origin. We denote the set of even and odd vertices of $\mathbb{Z}^d$ by $\text{Even}$ and $\text{Odd}$, respectively.

Policy on constants: In the rest of the paper, we employ the following policy on constants. We write $C, c, C', c'$ for positive absolute constants, whose values may change from line to line. Specifically, the values of $C, C'$ may increase and the values of $c, c'$ may decrease from line to line.

3.2. Odd sets and regular odd sets. We say that a set $U \subset \mathbb{Z}^d$ is odd (even) if its internal boundary consists solely of odd (even) vertices, i.e., $U$ is odd if and only if $\partial_\bullet U \subset \text{Odd}$ and it is even if and only if $\partial_\bullet U \subset \text{Even}$. We say that an odd or even set $U$ is regular if both it and its complement contain no isolated vertices. Observe that $U$ is odd if and only if $(\text{Even} \cap U)^+ \subset U$ and that $U$ is regular odd if and only if $U = (\text{Even} \cap U)^+$ and $U^c = (\text{Odd} \cap U^c)^+$.

An important property of odd sets is that the size of their edge-boundary cannot be too small. The following is by now rather well-known (see, e.g., [14], Corollary 1.3).

Lemma 3.1. Let $A \subset \mathbb{Z}^d$ be finite and odd. If $A$ contains an even vertex then $|\partial A| \geq 2d(2d - 1)$. 
3.3. Co-connected sets. In this section, we fix an arbitrary connected graph \( G = (V, E) \). A set \( U \subset V \) is called co-connected if its complement \( V \setminus U \) is connected. For a set \( U \subset V \) and a vertex \( v \in V \), we define the co-connected closure of \( U \) with respect to \( v \) to be the complement of the connected component of \( V \setminus U \) containing \( v \), where it is understood that this results in \( V \) when \( v \in U \). We say that a set \( U' \subset V \) is a co-connected closure of a set \( U \subset V \) if it is its co-connected closure with respect to some \( v \in V \). Evidently, every co-connected closure of a set \( U \) is co-connected and contains \( U \). The following simple lemma summarizes some basic properties of the co-connected closure (see [12, Lemma 2.5] for a proof).

**Lemma 3.2.** Let \( A, B \subset V \) be disjoint and let \( A' \) be a co-connected closure of \( A \). Then

(a) \( \partial A' \subset \partial A \).
(b) \( \partial (B \setminus A') \subset \partial B \).
(c) If \( B \) is co-connected then \( B \setminus A' \) is also co-connected.
(d) If \( B \) is connected then either \( B \subset A' \) or \( B \cap A' = \emptyset \).

The following lemma, taken from [13, Proposition 3.1] and based on ideas of Timár [62], establishes the connectivity of the boundary of subsets of \( \mathbb{Z}^d \) which are both connected and co-connected.

**Lemma 3.3.** Let \( A \subset \mathbb{Z}^d \) be connected and co-connected. Then \( \partial \circ A \) is connected.

As noted in the introduction, our results and proofs continue to apply when the lattice \( \mathbb{Z}^d \) is replaced by \( \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \) with \( T_{2m} \) the cycle graph on \( 2m \) vertices. This extension requires the following corollary of Lemma 3.3.

**Corollary 3.4.** Let \( A \subset \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \), \( d_1 \geq 1 \), be finite, connected and co-connected. Then \( \partial \circ \circ A \) is connected.

**Proof.** Let \( d = d_1 + d_2 \) and identify the vertex set of \( \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \) as the subset of \( \mathbb{Z}^d \) in which the last \( d_2 \) coordinates are restricted to take value in \( \{0, 1, \ldots, 2m - 1\} \). For \( v \in \mathbb{Z}^d \) let \( P(v) \) be the vertex in \( \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \) obtained from \( v \) by performing modulo \( 2m \) in the last \( d_2 \) coordinates. Define a set \( \bar{A} \subset \mathbb{Z}^d \) from \( A \) by ‘unwrapping’ the torus dimensions. Precisely, \( v \in \bar{A} \) if and only if \( P(v) \in A \).

Let us check that \( \bar{A} \) is co-connected: as \( A \) is finite, there exists \( v \in \mathbb{Z}^d \) such that any vertex agreeing with \( v \) on the first \( d_1 \) coordinates lies outside \( \bar{A} \). Let \( w \in \mathbb{Z}^d \setminus \bar{A} \). As \( w \) is arbitrary, co-connectedness of \( \bar{A} \) is implied by the existence of a path in \( \mathbb{Z}^d \setminus \bar{A} \) joining \( w \) to \( v \). To this end note that, as \( A \) is co-connected, there is a path in \( \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \setminus A \) joining \( P(w) \) with \( P(v) \). Thus there is a ‘lift’ of this path to \( \mathbb{Z}^d \setminus \bar{A} \) which joins \( w \) with a vertex \( \bar{v} \) having \( P(\bar{v}) = P(v) \). Lastly, this path may be continued in \( \mathbb{Z}^d \setminus \bar{A} \) to connect \( \bar{v} \) with \( v \), by the definition of \( v \).

Let \( \bar{A}_0 \) be a connected component of \( \bar{A} \). Then \( \bar{A}_0 \) is connected and co-connected in \( \mathbb{Z}^d \) and thus Lemma 3.3 implies that \( \partial \circ \circ \bar{A}_0 \) is connected. This then implies that \( \partial \circ \circ A \) is connected in \( \mathbb{Z}^{d_1} \times T_{2m}^{d_2} \) as one may check that \( P(\partial \circ \circ \bar{A}_0) = \partial \circ \circ A \). \( \square \)

3.4. Graph properties. In this section, we gather some elementary combinatorial facts about graphs. Here, we fix an arbitrary graph \( G = (V, E) \) of maximum degree \( \Delta \).

**Lemma 3.5.** Let \( U \subset V \) be finite and let \( t > 0 \). Then

\[
|N_t(U)| \leq \frac{\Delta}{t} |U|.
\]

**Proof.** This follows from a simple double counting argument.

\[
t |N_t(U)| \leq \sum_{v \in N_t(U)} |N(v) \cap U| = \sum_{u \in U} \sum_{v \in N_t(U)} \mathbb{1}_{N(u)}(v) = \sum_{u \in U} |N(u) \cap N_t(U)| \leq \Delta |U|.
\] \( \square \)

The next lemma follows from a classical result of Lovász [40, Corollary 2] about fractional vertex covers, applied to a weight function assigning a weight of \( \frac{1}{t} \) to each vertex of \( S \).
Equality holds in (11) if and only if $Z$ and $g$ gives the following two useful properties:

4.1. Notation. Lemmas and propositions are given in subsequent sections. Throughout this section, we fix a domain $\Lambda \subset \mathbb{Z}^d$ and a dominant pattern $P_0 = (A_0, B_0)$ satisfying $|A_0| \leq |B_0|$ as in \[5\]. Recall that $\mathbb{P}_{\Lambda, P_0}$ is supported on proper colorings satisfying that $\partial_v \Lambda$ is in the $P_0$-pattern. In proving statements for this finite-volume measure, it will be technically convenient to work in an infinite-volume setting as follows. Sample $f$ from $\mathbb{P}_{\Lambda, P_0}$ and extend it to a proper coloring of $\mathbb{Z}^d$ by requiring that

\[ \{f(v)\}_{v \in \Lambda^c} \text{ are independent random variables} \]

and

\[ f(v) \text{ is uniformly distributed in } A_0 \quad \text{for all even } v \notin \Lambda, \]

\[ f(v) \text{ is uniformly distributed in } B_0 \quad \text{for all odd } v \notin \Lambda. \]

Equality holds in (11) if and only if $Z$ and $g$ gives the following two useful properties:

4.1. Notation. Lemmas and propositions are given in subsequent sections.

Lemma 3.6. Let $S \subset V$ be finite and let $t \geq 1$. Then there exists a set $T \subset S$ of size $|T| \leq \frac{1+\log \Delta}{t} |S|$ such that $N_t(S) \subset N(T)$.

The following standard lemma gives a bound on the number of connected subsets of a graph.

Lemma 3.7 (\[4\], Chapter 45). The number of connected subsets of $V$ of size $k + 1$ which contain the origin is at most $(\varepsilon(\Delta - 1))^k$.

3.5. Entropy. In this section, we give a brief background on entropy (see, e.g., \[44\] for a more thorough discussion). Let $Z$ be a discrete random variable and denote its support by $\text{supp } Z$. The Shannon entropy of $Z$ is

\[ \text{Ent}(Z) := -\sum_z \mathbb{P}(Z = z) \log \mathbb{P}(Z = z), \]

where we use the convention that such sums are always over the support of the random variable in question. Given another discrete random variable $Y$, the conditional entropy of $Z$ given $Y$ is

\[ \text{Ent}(Z \mid Y) := \mathbb{E}[\text{Ent}(Z \mid Y = y)] = -\sum_y \mathbb{P}(Y = y) \sum_z \mathbb{P}(Z = z \mid Y = y) \log \mathbb{P}(Z = z \mid Y = y). \]

This gives rise to the following chain rule:

\[ \text{Ent}(Y, Z) = \text{Ent}(Y) + \text{Ent}(Z \mid Y), \]

where $\text{Ent}(Y, Z)$ is shorthand for the entropy of $(Y, Z)$. A simple application of Jensen’s inequality gives the following two useful properties:

\[ \text{Ent}(Z) \leq \log |\text{supp } Z| \]

and

\[ \text{Ent}(Z \mid Y) \leq \text{Ent}(Z \mid \phi(Y)) \quad \text{for any function } \phi. \]

Equality holds in (11) if and only if $Z$ is a uniform random variable. Together with the chain rule, (12) implies that entropy is subadditive. That is, if $Z_1, \ldots, Z_n$ are discrete random variables, then

\[ \text{Ent}(Z_1, \ldots, Z_n) \leq \text{Ent}(Z_1) + \cdots + \text{Ent}(Z_n). \]

As discussed in the overview, Shearer’s inequality (Lemma 2.1) is an extension of this inequality.

4. Main steps of proof

In this section, we give the main steps of the proof of Theorem 1.1, providing definitions, stating lemmas and propositions, and concluding Theorem 1.1 from them. The proofs of the technical lemmas and propositions are given in subsequent sections.

4.1. Notation. Throughout this section, we fix a domain $\Lambda \subset \mathbb{Z}^d$ and a dominant pattern $P_0 = (A_0, B_0)$ satisfying $|A_0| \leq |B_0|$ as in \[5\]. Recall that $\mathbb{P}_{\Lambda, P_0}$ is supported on proper colorings satisfying that $\partial_v \Lambda$ is in the $P_0$-pattern. In proving statements for this finite-volume measure, it will be technically convenient to work in an infinite-volume setting as follows. Sample $f$ from $\mathbb{P}_{\Lambda, P_0}$ and extend it to a proper coloring of $\mathbb{Z}^d$ by requiring that

\[ \{f(v)\}_{v \in \Lambda^c} \text{ are independent random variables} \]

and

\[ f(v) \text{ is uniformly distributed in } A_0 \quad \text{for all even } v \notin \Lambda, \]

\[ f(v) \text{ is uniformly distributed in } B_0 \quad \text{for all odd } v \notin \Lambda. \]

With a slight abuse of notation, we continue to denote the distribution of the random coloring $f$ obtained as such by $\mathbb{P}_{\Lambda, P_0}$.

Denote the set of dominant patterns by $\mathcal{P}$. Let $\mathcal{P}_0$ be the set of dominant patterns $P = (A, B)$ having $|A| \leq |B|$ and set $\mathcal{P}_1 := \mathcal{P} \setminus \mathcal{P}_0$. Note that $\mathcal{P}_1$ is empty when $q$ is even and that $|\mathcal{P}_0| = |\mathcal{P}_1|$. 

when \( q \) is odd. The difference between dominant patterns in \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) plays an important role. For this reason, it will be convenient to use a notation which distinguishes the two. For \( P = (A, B) \in \mathcal{P} \), denote
\[
(P_{\text{bdry}}, P_{\text{int}}) := \begin{cases} (A, B) & \text{if } P \in \mathcal{P}_0, \\ (B, A) & \text{if } P \in \mathcal{P}_1, \end{cases}
\]
so that, for any \( P \in \mathcal{P} \),
\[
|P_{\text{bdry}}| = \left\lceil \frac{q}{2} \right\rceil \quad \text{and} \quad |P_{\text{int}}| = \left\lfloor \frac{q}{2} \right\rfloor.
\]
Recall also the convention (6). With this terminology, for any \( P \in \mathcal{P} \) and \( v \in \mathbb{Z}^d \),
\[
\begin{align*}
v \in \text{the } P\text{-pattern} & \iff f(v) \in P_{\text{bdry}} \quad \text{when } v \text{ is } P\text{-even}, \\
v \in \text{the } P\text{-pattern} & \iff f(v) \in P_{\text{int}} \quad \text{when } v \text{ is } P\text{-odd}.
\end{align*}
\]
Note that \( P_{0}\text{-even} \) is even and \( P_{0}\text{-odd} \) is odd. We denote by \( \text{Even}_P \) and \( \text{Odd}_P \) the set of \( P\text{-even} \) and \( P\text{-odd} \) vertices of \( \mathbb{Z}^d \), respectively.

4.2. The breakup — definition and existence. Recall the definition of \( Z_P(f) \) from (7). As we explained in Section 2.2, \( Z_P(f) \) indicates the regions that are ordered according to the \( P\)-pattern. As we then explained in Section 2.3, in order to bound the probability that a given vertex \( v \) is not in the \( P_{0}\text{-pattern} \), we aim to define a “breakup seen from \( v \)” (in a similar way that one may distinguish a single contour surrounding \( v \) in the Ising model). It will be convenient for us to define this in a somewhat abstract manner, but the reader may have in mind that it is just a modified version of the \( Z_P(f) \).

The geometric structure of the breakup is captured by the following notion of an atlas. An atlas is a collection \( X = (X_P)_{P \in \mathcal{P}} \) of subsets of \( \mathbb{Z}^d \) such that, for every \( P \),
\[
X_P \text{ is a regular } P\text{-even set.}
\]
For an atlas \( X \), we define
\[
X_\text{bad} := \bigcap_P (X_P)^c, \quad X_{\text{overlap}} := \bigcup_{P \neq Q} (X_P \cap X_Q), \quad X_* := \bigcup_P \partial_{\text{bdry}} X_P \cup X_\text{bad} \cup X_{\text{overlap}}.
\]
Let \( f \) be a proper coloring of \( \mathbb{Z}^d \). An atlas \( X \) is a breakup (with respect to \( f \)) if
\[
\Lambda^c \subset X_{f_0}
\]
and the following condition holds for every dominant pattern \( P \) and every vertex \( v \):
\[
\text{If } v \in X_*^{+5} \text{ is } P\text{-odd then } v \in X_P \iff N(v) \text{ is in the } P\text{-pattern.}
\]
It is instructive to observe that \( (Z_P(f))_P \) is a breakup with respect to \( f \). The above property (22) is formulated via the values of \( f \) on the neighbors of a vertex \( v \). It is convenient to note its implication on the value of \( f \) at \( v \) itself. Suppose that \( X \) is a breakup and let \( P \) be a dominant pattern. Then, by (19), (20) and (22),
\[
\begin{align*}
f(v) & \in P_{\text{bdry}} \quad \text{for any } P\text{-even } v \in X_*^{+5} \cap X_P, \\
f(v) & \in P_{\text{int}} \quad \text{for any } P\text{-odd } v \in X_*^{+5} \cap X_P \setminus X_{\text{overlap}}.
\end{align*}
\]
Thus, \( P\text{-even} \) vertices in \( X_*^{+5} \cap X_P \) are always in the \( P\)-pattern, while in regions of \( X_*^{+5} \cap X_P \) which do not overlap with any other \( X_P \), all vertices are in the \( P\)-pattern. This property of \( (X_P)_P \) is analogous to that of \( (Z_P(f))_P \), except that here we do not have information on vertices of \( X_P \) that are not near \( X_* \). Observe also that, by (22) and (23),
\[
\begin{align*}
f(N(v)) & \not\in P_{\text{bdry}} \quad \text{for any } P\text{-odd } v \in X_\text{bad}, \\
f(u) & \in P_{\text{bdry}} \quad \text{and } f(N(v)) \not\in P_{\text{bdry}} \quad \text{for any } (u, v) \in \partial X_P.
\end{align*}
\]
See Figure 2 for an illustration of a breakup.

The following lemma, whose proof is given in Section 5.1, shows that whenever there is a violation of the boundary pattern, there exists a breakup that “captures” that violation. We say that $X$ is non-trivial if $X_*$ is non-empty and that it is finite if $X_*$ is finite. We also say that $X$ is seen from a set $V$ if every finite connected component of $X^+_{v}$ disconnects some vertex $v \in V$ from infinity.

Let $Z^{+5}(f,V)$ denote the union of connected components of $Z_*^+(f)$ that are infinite or disconnect some vertex in $V$ from infinity.

**Lemma 4.1.** Let $f$ be a proper coloring of $\mathbb{Z}^d$ such that $\text{int}(\Lambda)^c$ is in the $P_0$-pattern and let $V \subseteq \Lambda$. Then there exists a breakup $X$ such that $X^+_{v} = Z^+_{v}(f,V)$.

In particular, $X$ is seen from $V$, it is non-trivial if $V^+_{v}$ is not in the $P_0$-pattern or intersects $Z_*^+(f)$, and any vertex in $V^+_{v}$ is in not in the $P_0$-pattern is not in $X_{P_0} \setminus X_{\text{overlap}}$.

### 4.3. Unlikeliness of breakups.

Now that we have a definition of breakup and we know that any violation of the boundary pattern creates a non-trivial breakup, it remains to show that breakups are unlikely.

The main part of the proof consists of obtaining a quantitative bound on the probability of a large breakup. Nevertheless, formally one also needs to rule out the existence of an infinite breakup. As this does not require a quantitative bound, it is actually rather simple to do so. The following lemma is proved in Section 5.2.

**Lemma 4.2.** $P_{\Lambda,P_0}$-almost surely, every breakup seen from a finite set is finite.

We now discuss the quantitative bound on finite breakups. To this end, denote by $X$ the collection of atlases which have a positive probability of being a breakup and, for integers $L, M, N \geq 0$, denote

$$X_{L,M,N} := \left\{ X \in X : \left| \bigcup_P \partial X_P \right| = L, \ |X_{\text{overlap}}| = M, \ |X_{\text{bad}}| = N \right\}.$$ 

**Proposition 4.3.** For any finite $V \subset \mathbb{Z}^d$ and any integers $L, M, N \geq 0$, we have

$$P_{\Lambda,P_0}(\text{there exists a breakup in } X_{L,M,N} \text{ seen from } V) \leq 2^{|V|} \cdot \exp \left( -\frac{c}{q(q+\log d)} \left( \frac{L}{q} + \frac{M}{q} + \frac{N}{q^2} \right) \right).$$

This is the main technical proposition of this paper. An overview of the tools to prove the proposition is given in the rest of Section 4 with the detailed proofs appearing in Section 5, Section 6 and Section 7.

Based on the above lemmas, it is now a simple matter to deduce Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that $v$ is not in the $P_0$-pattern. Lemma 4.1 implies the existence of a non-trivial breakup $X$ seen from $v$. By Lemma 4.2, we may assume that $X$ is finite so that $X \in X_{L,M,N}$ for some $L, M, N \geq 0$. Since $X$ is also non-trivial, some set in $\{X_P, X_P^c\}_P$ is both non-empty and not $\mathbb{Z}^d$. Recalling (20) and applying Lemma 3.1 (or its analogue for even sets) to any such set shows that $L \geq d^2$. Therefore, by Proposition 4.3,

$$P_{\Lambda,P_0}(v \text{ is not in the } P_0\text{-pattern}) \leq 2 \sum_{L \geq d^2, M, N \geq 0} \exp \left( -\frac{c}{q(q+\log d)} \left( \frac{L}{q} + \frac{M}{q} + \frac{N}{q^2} \right) \right).$$

Using (11), the desired inequality follows (perhaps with a larger constant $C$ in (11)).

### 4.4. Unlikeliness of specific breakups.

In light of the bound in Proposition 4.3, it is natural to first prove that a specific atlas is unlikely to be a breakup. Precisely, we would like to show the following.

**Proposition 4.4.** For any $X \in X_{L,M,N}$, we have

$$P_{\Lambda,P_0}(X \text{ is a breakup}) \leq \exp \left( -\frac{c}{q} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right) \right).$$
Proposition 4.6. For any approximation \( b \) of having a breakup which is approximated by a given approximation comes a more complicated task. The following proposition provides a similar bound on the probability of this event.

Let \( \Omega \) be the event that there exists a breakup in \( X \) such that any \( A \) may or may not belong to \( X \). The latter collection exceeds the reciprocal of the bound of Proposition 4.4. Overcoming this obstacle forms the second main technical part of our paper and requires an analysis of the structure of atlases. We discuss this in the following section.

4.5. Approximations. As mentioned above, the standard union bound does not allow to upgrade the bound of Proposition 4.4 to that of Proposition 4.3. Instead, we employ a delicate coarse-graining scheme of the possible breakups according to their rough features.

Let \( A = ((A_p)_{p \in P}, A^*, A^{**}) \) be a collection of subsets of \( \mathbb{Z}^d \) such that each \( A_p \) is \( P \)-even and \( A^* \subset A^{**} \). For notational convenience, we write \( Q \simeq P \) if \( Q, P \in P_i \) for some \( i \in \{0, 1\} \). We say that \( A \) is an approximation of an atlas \( X \in \mathcal{X}_{L,M,N} \) if the following conditions hold for all \( P \):

1. \( A_P \subset X_P \subset A_P \cup (\text{Odd}_P \cap A^*) \cup (\text{Even}_P \cap A^{**}) \).
2. \( \text{Odd}_P \cap A^* \subset N_d(\bigcup_{Q \simeq P} A_Q) \).
3. \( |A^{**}| \leq C_L \log d \sqrt{d} \).
4. \( A^{**} \subset \bigcup_Q (\partial_Q X_Q)^{+3} \).

In light of (A1), we think of \( A_P \) as the region known to belong to \( X_P \), of \( A^{**} \) as a region which may or may not belong to \( X_P \), and of \( A_P \cup A^{**} \) as a region known to contain \( X_P \). The other three properties ensure together that there is a “small amount” of unknown information. Specifically, (A2) guarantees that near any unknown vertex (with the correct parity) there are many known vertices, whereas (A3) and (A4) control the number and location of the unknown vertices.

The following proposition shows that one may find a small family which contains an approximation of every atlas seen from a given set.

**Proposition 4.5.** For any integers \( L, M, N \geq 0 \) and any finite set \( V \subset \mathbb{Z}^d \), there exists a family \( A \) of approximations of size

\[
|A| \leq 2^{|V|} \cdot \exp \left( C q L \left( \frac{\log d}{d} \right)^{3/2} + C(M + N) \frac{\log^2 d}{d} \right)
\]

such that any \( X \in \mathcal{X}_{L,M,N} \) seen from \( V \) is approximated by some element in \( A \).

Of course, working with approximations, finding a suitable modification of Proposition 4.4 becomes a more complicated task. The following proposition provides a similar bound on the probability of having a breakup which is approximated by a given approximation.

**Proposition 4.6.** For any approximation \( A \) and any integers \( L, M, N \geq 0 \), we have

\[
\Pr_{A,P_b}(A \text{ approximates some breakup in } \mathcal{X}_{L,M,N}) \leq \exp \left( -\frac{c}{q^2(q+\log d)} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right) \right).
\]

We are now ready to complete the proof of Proposition 4.3.

**Proof of Proposition 4.3.** Let \( A \) be a family of approximations as guaranteed by Proposition 4.5. Let \( \Omega \) be the event that there exists a breakup in \( \mathcal{X}_{L,M,N} \) seen from \( V \) and let \( \Omega(A) \) be the event that there exists a breakup in \( \mathcal{X}_{L,M,N} \) seen from \( V \) and approximated by \( A \). Then, by Proposition 4.5 and Proposition 4.6

\[
\Pr(\Omega) \leq \sum_{A \in A} \Pr(\Omega(A)) \leq 2^{|V|} \cdot \exp \left( C q L \left( \frac{\log d}{d} \right)^{3/2} + C(M + N) \frac{\log^2 d}{d} - \frac{c}{q^2(q+\log d)} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right) \right).
\]

The proposition now follows using (4).
The proofs of Lemma 4.1, Lemma 4.2, Proposition 4.4, and Proposition 4.6 are given in Section 5. Proposition 4.5 is proven in Section 7.

4.6. Bounding the probability of breakups and approximations. Here we explain the main ideas behind the proofs of Proposition 4.4 and Proposition 4.6. As explained in the proof overview, the idea of the proof is to apply a one-to-many transformation in which the colors in $X_*$ are first erased, the sets $X_F$ are then shifted and the colors there are permuted, and then the empty sites are colored afresh with coloring in the $P_0$-pattern. The main technical difficulty in this proof is to bound the loss of entropy in the first step, i.e., to bound the entropy of $f|_{X_*}$. After applying Shearer’s inequality to obtain (28), we wish to use information implies by having $X$ as a breakup in order to improve the naive bound on $I + II$ in (29), as discussed in the overview.

The type of additional information we shall use in order to improve the above bounds is based on four notions — vertices having unbalanced neighborhoods, non-dominant vertices, restricted edges and vertices having a unique pattern — all of which we now define. These notions are somewhat abstract (and not directly related to the breakup) in order to allow sufficient flexibility for the proof of both propositions.

Let $f : \mathbb{Z}^d \rightarrow [q]$ be a proper coloring and let $\Omega$ be a collection of proper colorings of $\mathbb{Z}^d$. The four notions implicitly depend on $f$ and $\Omega$. Let $v \in \mathbb{Z}^d$ be a vertex and let $u$ be adjacent to $v$. Recall that $(v, u) \in \partial v$ is the directed edge from $v$ to $u$. We say that

- $v$ is non-dominant (in $f$) if
  \[ |f(N(v))| \notin \left\{ \left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil \right\} \]

Thus, a vertex is non-dominant if the set of colors which appear on its neighbors does not determine a dominant pattern.

- $(v, u)$ is restricted (in $(f, \Omega)$) if
  \[ \{ g(u) : g \in \Omega, g(N(v)) = f(N(v)) \} \cup \{ g(v) : g \in \Omega, g(N(v)) = f(N(v)) \} \neq [q]. \]

Observe that $(v, u)$ is restricted if and only if

either \[ \{ g(u) : g \in \Omega, g(N(v)) = f(N(v)) \} \neq f(N(v)), \] or \[ \{ g(v) : g \in \Omega, g(N(v)) = f(N(v)) \} \neq f(N(v))^c. \]

Thus, roughly speaking, $(v, u)$ is restricted if upon inspection of the set of values which appears on the neighbors of $v$, one is guaranteed that either $u$ or $v$ cannot take all possible values which they should typically take, i.e., either $u$ cannot take some value in $f(N(v))$, or $v$ cannot take some value in $f(N(v))^c$. Note that (30) actually implies that all edges in $\partial v$ are restricted as it does not involve $u$.

- $v$ has an unbalanced neighborhood (in $f$) if
  \[ |\{ u \in N(v) : f(u) = i \}| \leq \frac{d}{q} \quad \text{for some } i \in f(N(v)). \]

As $f(N(v))$ increases, the set of values which $v$ may take, namely $f(N(v))^c$, is reduced, resulting in a trade-off in the entropy contribution at $v$ quantified by the term $II$ above. In order to have high entropy, if some neighbor of $v$ takes a value $i$, many other neighbors of $v$ should take advantage of this as well. The neighborhood of $v$ is therefore deemed unbalanced if some value is taken by few (but at least one) neighbors of $v$.

- $v$ has a unique pattern (in $\Omega$) if there exists $A \subset [q]$ such that, for every $g \in \Omega$, either
  \[ g(N(v)) = A \] or $v$ is non-dominant in $g$ or all edges in $\partial v$ are restricted in $(g, \Omega)$.

We may more appropriately term this notion as a unique high-entropy pattern or unique unrestricted pattern, the reason being that there is at most one choice for $g(N(v))$ which does not lead to a reduction of entropy at $v$ by making $v$ non-dominant or causing all edges in $\partial v$ to be restricted.
Given $f$ and $\Omega$ as above and given $S \subset \mathbb{Z}^d$, let $S^f_{\text{unbal}}$ be the set of vertices in $S$ which have unbalanced neighborhoods in $f$, let $S^f_{\text{nondom}}$ be the set of vertices in $S$ which are non-dominant in $f$, let $S^f_{\text{rest}}$ be the set of directed edges $(v, u)$ with $v \in S$ which are restricted in $(f, \Omega)$ and let $S^f_{\text{uniq}}$ be the set of vertices in $S$ which have a unique pattern in $\Omega$.

The following lemma, which is proved in Section 6, provides a general upper bound for the probability of an event.

**Lemma 4.7.** Let $S \subset \mathbb{Z}^d$ be finite and let $\{S_P\}_{P \in \mathcal{P}}$ be a partition of $S^c$ such that $\partial_0 S_P \subset S$ for all $P$. Suppose that $S \cup S_P$ contains $N^c$. Let $\Omega$ be an event on which $(\partial_0 S_P)^c$ is in the $P$-pattern for every $P$ and denote

$$k(\Omega) := \min_{f \in \Omega} |S^f_{\text{unbal}}| + \frac{1}{q} |S^f_{\text{nondom}}| + \frac{1}{d} |S^f_{\text{rest}}|.$$

Then

$$\mathbb{P}_{\Lambda, P_0}(\Omega) \leq \exp \left[ -\frac{k(\Omega)}{128q} + \frac{3}{q} |S \setminus S^f_{\text{uniq}}| + e^{-dq/65q^2} |S| \right].$$

We conclude with a short outline as to how Lemma 4.7 is used to prove Proposition 4.4. To this end, we take $S$ to be $X_\ast$, $S_P$ to be $X_P \setminus X_\ast$ and $\Omega$ to be the event that $X$ is a breakup. Unfortunately, concluding Proposition 4.4 from Lemma 4.7 is not straightforward, as the latter gives an insufficient bound on the probability of $\Omega$. The difficulty here is that, while $k(\Omega)$ is always large in comparison to $L$ and $M$, it is not necessarily large in comparison to $N$. Indeed, as we will show (see Lemma 5.3), every edge in $\partial X_P$ is necessarily restricted in $f$ and every edge incident to $X_\text{overlap}$ is either restricted in $f$ or incident to a non-dominant vertex in $f$, so that

$$\frac{1}{q} |S^f_{\text{nondom}}| + \frac{1}{d} |S^f_{\text{rest}}| \geq \frac{L}{2q} + \frac{M}{4q}.$$ 

Unfortunately, $X_{\text{bad}}$ need not contain enough restricted edges (or non-dominant vertices or vertices having unbalanced neighborhood) – the main reason being that, when $q$ is even, $X_{\text{bad}}$ may contain even vertices $v$ for which $|f(N(v))| = \frac{q}{2}$, and when $q$ is odd, $X_{\text{bad}}$ may contain vertices $v$ for which $|f(N(v))| = \lceil \frac{q}{2} \rceil$ (that is, there is no analogue of (25) for $P$-even vertices). Instead, to obtain a good bound, we shall apply Lemma 4.7 to subevents $\Omega' \subset \Omega$ on which we have additional information about the configuration on the set $X_{\text{bad}}$. For suitably chosen subevents (see Lemma 5.3), the number of restricted edges in $X_{\text{bad}}$ (and non-dominant vertices and vertices having unbalanced neighborhood) increases enough to ensure that

$$k(\Omega') \geq \frac{L}{3q} + \frac{M}{6q} + \frac{N}{18q^2}.$$ 

As the entropy of this additional information is negligible with our assumptions (see Lemma 5.5), this will allow us to conclude Proposition 4.4 by taking a union bound over the subevents $\Omega'$. This is carried out in detail in Section 5.3.

5. Breakups

In this section, we prove Lemma 4.1 about the existence of a non-trivial breakup, we prove Lemma 4.2 about the absence of infinite breakups, we prove Proposition 4.4 about the probability of a given breakup, and we prove Proposition 4.6 about the probability of an approximation.

5.1. Constructing the breakup. Here we prove Lemma 4.1. As we have mentioned, the collection $Z = (Z_P(f))_P$ defined in (7) is always a breakup. The main difficulty is therefore to ensure that the breakup is seen from a given set. For this, we require the following lemma which allows to “close holes”. The proof is accompanied by Figure 3.

**Lemma 5.1.** Let $V, W \subset \mathbb{Z}^d$ and let $B$ be the union of connected components of $W$ that are infinite or disconnect some vertex in $V$ from infinity. Let $A$ be a connected component of $B^c$. Then $\partial_0 A$ is contained in a connected component of $(W^c)^+$. 

Lemma 3.3 implies that \( S \) is connected and since \( \partial_0S \) is connected, we see that \( \partial_0S \) is contained in a connected component \( D \) of \( W \).

Let \( S \) be the connected component of \( a \) in \((W^c)^+\) and note that \( \partial_0S \subset W \). Let \( \tilde{S} \) be the co-connected closure of \( S \) with respect to \( a' \). Since \( a' \notin S \) by assumption, we have \( a' \notin \tilde{S} \). Since Lemma 3.3 implies that \( \partial_0\tilde{S} \) is connected and since \( \partial_0S \subset \partial_0\tilde{S} \subset W \), we see that \( \partial_0\tilde{S} \) is contained in a connected component \( D \) of \( W \).

Since \( \partial_0A \subset \partial_0B \subset \partial_0W \), the connected components \( D_a \) and \( D_{a'} \) of \( a \) and \( a' \) in \( W \) are contained in \( B \). Since any path between \( a \) and \( a' \) must intersect \( \partial_0\tilde{S} \setminus \{a,a'\} \) and since there is a path in \( B^c \cup \{a,a'\} \) between \( a \) and \( a' \), it follows that \( \partial_0\tilde{S} \notin B \). In particular, \( D \notin B \) so that \( D \neq D_a, D_{a'} \).

Hence, \( D \) is disjoint from both \( D_a \) and \( D_{a'} \).

We now show that \( D \subset B \), which leads to a contradiction, and thus concludes the proof. If \( D \) is infinite then this follows from the definition of \( B \). Otherwise, \( \partial_0\tilde{S} \subset D \) is finite, so that either \( \tilde{S} \) or \( \tilde{S}^c \) is finite. Thus, \( \partial_0\tilde{S} \) disconnects either \( a \) or \( a' \) from infinity. Therefore, \( D \) disconnects either \( D_a \) or \( D_{a'} \) from infinity. In particular, \( D \) disconnects some vertex in \( V \) from infinity, so that \( D \subset B \) by the definition of \( B \).

The next lemma shows that an atlas can be “localized” into an atlas which is seen from \( V \).

**Lemma 5.2.** Let \( \Lambda \) be a domain, let \( V \subset \Lambda \), let \( P_0 \) be a dominant pattern and let \( Y \) be an atlas such that \( \Lambda^c \subset Y_{P_0} \). Then there exists an atlas \( X \) which is seen from \( V \) and satisfies that

\[
X^+_{\ast} \cap X_P = X^+_{\ast} \cap Y_P \quad \text{for every dominant pattern } P.
\]

Moreover, \( \Lambda^c \subset X_{P_0} \) and \( X^+_{\ast} \) is the union of connected components of \( Y^+_{\ast} \) that are infinite or disconnect some vertex in \( V \) from infinity.

**Proof.** Let \( B \) be the union of connected components of \( Y^+_{\ast} \) that are infinite or disconnect some vertex in \( V \) from infinity. Let \( \mathcal{A} \) be the set of connected components of \( B^c \). We claim that

for every \( A \in \mathcal{A} \), there exists a unique dominant pattern \( P_A \) such that \( A^+_{\ast} \setminus A \subset Y_{P_A} \setminus Y_{\ast} \).

Indeed, it follows from the definition of \( Y_{\ast} \) that for every \( a \in A^+_{\ast} \setminus A \subset Y^c_{\ast} \), there exists a unique dominant pattern \( P_a \) such that \( a \in Y_{P_a} \). Since Lemma 3.3 applied with \( W := Y^+_{\ast} \) yields that \( \partial_0A \) is contained in a connected component of \((W^c)^+ \subset (Y^+_{\ast})^c \), we see that \( P_a = P_{a'} \) for all \( a, a' \in \partial_0A \). The claim follows. Note also that, since \( \Lambda^c \subset Y_{P_0} \), we have \( P_A = P_0 \) for all \( A \in \mathcal{A} \) such that \( A \not\subset \Lambda \).

We now define \( X = (X_P)_P \) by

\[
X_P := (Y_P \cap B) \cup \bigcup \{A \in \mathcal{A} : P_A = P\}, \quad P \in \mathcal{P}.
\]
Let us show that $X$ satisfies the conclusion of the lemma. Note first that $X_P \cap B = Y_P \cap B$ and $X_\ast \subseteq B$, so that $X_\ast = Y_\ast \cap B$ and $X_{\ast +5} = B$. It easily follows that $X$ is an atlas satisfying $\{1\}$. Let us check that $X$ is seen from $V$. Indeed, every finite connected component of $X_{\ast +5} = B$ is by definition a connected component of $Y_{\ast +5}$ that disconnects some vertex in $V$ from infinity. Finally, $\Lambda^c \subseteq X_P$, since $\Lambda^c \subseteq Y_P$ and $P_A = P_0$ for all $A \in A$ such that $A \not\subseteq \Lambda$.

**Proof of Lemma 4.2.** Recall the definition of $Z_P(f)$ from [1]. It is straightforward to check that $Z = (Z_P(f))_P$ is an atlas and that, for any $P$-odd vertex $v$, we have $v \in Z_P(f)$ if and only if $N(v)$ is in the $P$-pattern. Thus, (31) implies (22). Moreover, using the assumption that $\text{int}(\Lambda)^c$ is in the $P_0$-pattern, it is easy to check that $\Lambda^c \subseteq X_P$. Thus, the lemma follows from Lemma 5.2. □

5.2. **No infinite breakups.** Here we prove Lemma 4.2. As mentioned above, our main argument (namely, Proposition 4.4 and Proposition 4.6) is concerned only with finite breakups. However, it is easy to rule out the existence of an infinite breakup in a random coloring. In doing so, there are two possibilities to have in mind: either there exists an infinite pattern of $Z_{\ast +5}$ or infinitely many finite components surrounding a vertex.

**Proof of Lemma 4.2.** By (15) and (16), for any $u \not\subseteq \Lambda^+$ and $P \neq P_0$ for which $u$ is $P$-even, 

$$\mathbb{P}(u \text{ is in the } P\text{-pattern } | (f(v))_{v \neq u}) \leq \frac{\left\lfloor \frac{q-1}{q} \right\rfloor}{q} \leq \frac{q-1}{q+1}.$$ 

Say that $u$ is in a double pattern if $u$ is $P$-odd and $N(u)$ is in the $P$-pattern for some $P \neq P_0$. Then 

$$\mathbb{P}(u \text{ is in a double pattern } | (f(v))_{v \neq N(u)}) \leq 2^d \left(\frac{q-1}{q+1}\right)^{2d+5}.$$ 

Note that if a vertex $u \in \mathbb{Z}^d \setminus \Lambda^+$ belongs to $Z_\ast$, then some vertex in $u^+$ is in a double pattern.

We wish to show that, almost surely, every breakup seen from $V$ is finite. For $v \in \mathbb{Z}^d$, let $E_v$ be the event that $v$ is in an infinite connected component of $Z_{\ast +5}$. Let $E'_v$ be the event that $v$ is disconnected from infinity by infinitely many connected components of $Z_{\ast +5}$. It suffices to show that $\mathbb{P}(E_v) = \mathbb{P}(E'_v) = 0$ for any $v \in \mathbb{Z}^d$. Let us show that $\mathbb{P}(E'_v) = 0$; the proof that $\mathbb{P}(E_v) = 0$ is very similar. On the complement the event $E'_v$ for any $m$, there exists a set $B \subset \mathbb{Z}^d \setminus \Lambda^+$ of size at least $m$ such that $B_{\ast +5}$ is connected and disconnects $v$ from infinity and such that for every vertex $v \in B$ there exists a vertex in $u^+$ which is in a double pattern. In particular, for any $m$, there exists a path $\gamma$ in $(\mathbb{Z}^d \setminus \Lambda^+)_{\ast +50}$ of length $n \geq m$ such that $\{\gamma_i^+\}_{i=0}^n$ are pairwise disjoint, $\text{dist}(v, \gamma_0) \leq Cn$ and all vertices $\{\gamma_i\}_{i=0}^n$ are in a double pattern. Since $\mathbb{P}(\gamma) \leq 2^m \left(\frac{q^2}{q+1}\right)^{2d+5}$ for any such fixed $\gamma$, and since the number of simple paths $\gamma$ in $(\mathbb{Z}^d)_{\ast +50}$ of length $n$ with $\text{dist}(v, \gamma_0) \leq Cn$ is at most $d^{Cn}$, the lemma follows using (11). □

5.3. **The probability of a given breakup.** In this section, we prove Proposition 4.4. Let $\Omega$ be the set of proper colorings $f$ having $X$ as a breakup. In order to bound the probability of $\Omega$, we aim to apply Lemma 4.7 with

$$S := X_\ast \quad \text{and} \quad S_P := X_P \setminus X_\ast.$$ 

The definition of $X_\ast$ implies that $\{S^+_P\}_P$ are pairwise disjoint so that, in particular, $\{S_P\}_P$ is a partition of $S'$. By [21], $S \cup S_P$ contains $\Lambda^c$. By [23], [24] and [20], $S^+_P \cap S_{\ast +2}$ is in the $P$-pattern on the event $\Omega$. Thus, the assumptions of Lemma 4.7 are satisfied.

The following lemma guarantees that there are many restricted edges in $(f, \Omega)$. Recall the definitions of $S^f_\text{ubal}$, $S^f_\text{nondom}$, $S^\Omega f_{\text{rest}}$ and $S^\Omega f_{\text{uniq}}$ from Section 4.3.

**Lemma 5.3.** For any $f \in \Omega$, we have

$$|S^\Omega f_{\text{rest}}| \geq L \quad \text{and} \quad \frac{1}{d}|S^\Omega f_{\text{rest}}| + 2|S^f_\text{nondom}| \geq M.$$
Proof. Fix \( f \in \Omega \) and write \( S_{\text{rest}}^f \) for \( S_{\text{rest}}^{\Omega,f} \) and \( S_{\text{nondom}}^f \) for \( S_{\text{nondom}}^{\Omega,f} \).

To show that \( |S_{\text{rest}}| \geq L \), it suffices to show that
\[
\partial X_P \subset S_{\text{rest}} \quad \text{for any } P.
\] (32)

To this end, let \((v, u) \in \partial X_P\). Then \( g(u) \in P_{\text{bdry}} \) and \( g(N(v)) \not\in P_{\text{bdry}} \) for any \( g \in \Omega \) by \[26\], from which it follows that \((v, u)\) is restricted by \[29\].

We now show that \( \frac{1}{2} |S_{\text{rest}}| + 2|S_{\text{nondom}}| \geq M \). Letting \( S_{\text{nondom}}^* \) denote the set of edges having an endpoint in \( S_{\text{nondom}} \) and noting that \( 2|S_{\text{nondom}}| \geq \frac{1}{2} |S_{\text{nondom}}^*| \), we see that it suffices to show that
\[
(v, u) \in S_{\text{rest}} \quad \text{or} \quad u \in S_{\text{nondom}} \quad \text{or} \quad v \in S_{\text{nondom}} \quad \text{for any } u \in X_{\text{overlap}} \text{ and } v \sim u.
\]

Let \( u \in X_{\text{overlap}} \) and let \( P \neq Q \) be such that \( u \in X_P \cap X_Q \). If \( v \not\in X_P \cap X_Q \) then \((v, u)\) is restricted by \[32\]. Otherwise, \( v \in X_P \cap X_Q \). If \( P \simeq Q \), letting \( w \in \{u, v\} \) be \( P\)-odd, we have \( w^+ \subset X_P \cap X_Q \) by \[20\]. Thus, \( g(N(w)) \subset P_{\text{bdry}} \cap Q_{\text{bdry}} \) for any \( g \in \Omega \) by \[23\], and it follows that \( w \in S_{\text{nondom}}^* \) for any \( P \neq Q \) and we may assume without loss of generality that \( v \) is \( P\)-even and \( u \) is \( Q\)-even, in which case \( g(v) \in P_{\text{bdry}} \) and \( g(u) \in Q_{\text{bdry}} \) for any \( g \in \Omega \) by \[23\], so that it follows from \[28\] that \((v, u)\) is restricted (note that \( P \neq Q \) can only occur when \( q \) is odd).

As explained in the outline, applying Lemma \[4.7\] directly for \( \Omega \) does not produce the bound stated in Proposition \[4.4\]. This bound will instead follow by applying Lemma \[4.7\] to subevents of \( \Omega \) on which we have additional information about the configuration on the set \( X_{\text{bad}} \) and then summing the resulting bounds. To explain the reason for this and to motivate the definitions below, we note that, although \[25\] prohibits the possibility that the neighborhood \( N(v) \) of a \( P\)-odd vertex \( v \in X_{\text{bad}} \) is in the \( P\)-pattern, this is possible for a \( P\)-even vertex. That is, when \( q \) is even, it cannot happen that \( |f(N(v))| = \frac{q}{2} \) for an odd vertex, but it may happen that \( |f(N(v))| = \frac{q}{2} \) for an even vertex, and when \( q \) is odd, it cannot happen that \( |f(N(v))| = \left\lfloor \frac{q}{2} \right\rfloor \), but it may happen that \( |f(N(v))| = \left\lceil \frac{q}{2} \right\rceil \). A vertex for which the latter occurs is problematic as it does not immediately reduce the entropy of the configuration (since it may also have a balanced neighborhood and no or few restricted edges incident to it). However, if many (perhaps even all or almost all) of the vertices in \( X_{\text{bad}} \) are of this type, then by recording a small subset of these vertices, we may ensure that most of them become restricted (more precisely, either an unbalanced neighborhood, non-dominant vertex, or many incident restricted edges). We now describe the structure of this additional information.

For \( f \in \Omega \) and a dominant pattern \( P \), define
\[
U_P(f) := \{ u \in X_{\text{bad}} : u \text{ is } P\text{-even, } f(N(u)) = P_{\text{int}} \}.
\] (33)

Note that the sets \( \{U_P(f)\}_P \) are pairwise disjoint. Note also that \( u \in U_P(f) \) implies that \( u^+ \) is in the \( P\)-pattern and that \( N(u) \) is not in the \( Q\)-pattern for any \( Q \neq P \). In particular,
\[
f(U_P(f)) \subset P_{\text{bdry}}.
\] (34)

The collection \( \langle U_P(f) \rangle_P \) contains the relevant information on \( f \) beyond that which is given by the breakup \( X \). However, it contains more information than is necessary and this comes at a large enumeration cost. Instead, we only wish to specify a certain approximation of this information. Given a collection \( V = \langle V_P \rangle_P \) of subsets of \( \mathbb{Z}^d \), let \( \Omega(V) \) denote the set of \( f \in \Omega \) satisfying that, for every dominant pattern \( P \),
\[
V_P \subset U_P(f) \quad \text{and} \quad N_{d/3q} \left( \bigcup_{Q \neq P} U_Q(f) \right) \subset N \left( \bigcup_{Q \neq P} V_Q \right).
\] (35)

Thus, \( V \) is a kind of approximation of \( \langle U_P(f) \rangle_P \). With this definition at hand, there are now two goals. The first is to show that the additional information given by \( V \) is enough to improve the bound given in Lemma \[5.3\]. The second is to show that the cost of enumerating \( V \) is not too large.
Lemma 5.4. For any \( V \) and any \( f \in \Omega(V) \), we have
\[
|S_{unbal}^f| + \frac{1}{q}|S_{nondom}^f| + \frac{1}{q}|S_{rest}^{\Omega(V),f}| \geq \frac{L}{d^d} + \frac{M}{d^{d+1}} + \frac{N}{d^{d^2}}.
\]

Proof. We fix \( V \) and \( f \in \Omega(V) \) and suppress these in the notation of \( S_{unbal}^f, S_{nondom}^f, S_{rest}^{\Omega(V),f} \) and \( U_P(f) \). It suffices to show that
\[
|S_{unbal}| + \frac{1}{q}|S_{nondom}| + \frac{1}{q}|S_{rest}| \geq \frac{N}{d^{d^2}}.
\]
as the lemma then follows by averaging this bound with the one given by Lemma 5.3. Let \( S_{rest}^* \) denote the set of vertices which are incident to at least \( d/3q \) edges in \( S_{rest} \), i.e.,
\[
S_{rest}^* := \left\{ v : |(\tilde{\partial} v \cup \tilde{\partial} v) \cap S_{rest}| \geq \frac{d}{3q} \right\}.
\]
Note that \( |S_{rest}| \geq \frac{d}{3q}|S_{rest}^*| \) so that it suffices to show that
\[
X_{bad} \subset S_{rest}^* \cup S_{unbal} \cup N_{d/3q}(S_{nondom}).
\]

Let us first show that
\[
X_{bad} \setminus U \subset S_{nondom}, \quad \text{where } U := \bigcup_P U_P.
\]

To this end, let \( u \in X_{bad} \setminus U \) and note that, by the definition of a non-dominant vertex, we must show that \( |f(N(u))| \notin \{|\frac{q}{2}, \lfloor \frac{q}{2} \rfloor \} \). Let us consider separately the cases of even and odd \( q \). Assume first that \( q \) is even. Assume now that \( q \) is odd. Note that \( |f(N(u))| \notin \{|\frac{q}{2}, \lfloor \frac{q}{2} \rfloor \} \) by (25) if \( u \) is odd and that \( |f(N(u))| \notin \{|\frac{q}{2}, \lfloor \frac{q}{2} \rfloor \} \) by (33) if \( u \) is even. Assume now that \( q \) is odd. Note that \( |f(N(u))| \notin \{|\frac{q}{2}, \lfloor \frac{q}{2} \rfloor \} \) by (25) and that \( |f(N(u))| \notin \{|\frac{q}{2}, \lfloor \frac{q}{2} \rfloor \} \) by (33).

Next, we show that
\[
\bigcap_P N_{d/3q}(U \setminus U_P) \subset S_{rest}^*.
\]
To see this, let \( u \in \bigcap_P N_{d/3q}(U \setminus U_P) \) and note that, by (35), \( u \in N(V_P) \) for some \( P \). Since \( u \in N_{d/3q}(U \setminus U_P) \), another application of (35) yields that \( u \in N(Q) \) for some \( Q \neq P \). Since \( V_P \subset U_P \) and \( V_Q \subset U_Q \) by (35), it follows from (33) that \( g(u) \in P_{\text{int}} \cap Q_{\text{int}} \) for any \( g \in \Omega(V) \). Since \( u \in N_{d/3q}(U) \), in order to show that \( u \in S_{rest}^* \), it suffices to show that if \( v \in N(u) \cap U_T \) for some \( T \), then \((v, u)\) is restricted. Indeed, this follows since \( f(N(v)) = T_{\text{int}} \) by (33), which implies that \((v, u)\) is restricted by (29).

Finally, towards showing (36), let \( u \in X_{bad} \) and assume that
\[
u \notin S_{rest}^* \cup N_{d/3q}(S_{nondom}).
\]
We show that \( u \in S_{unbal} \). By (37), we have \( u \notin N_{d/3q}(X_{bad} \setminus U) \) so that \( u \in N_{2d-d/3q}(U_P \setminus X_P) \). Since \( X_{bad} \cap N_{d/3q}(U) \subset S_{rest}^* \) by (32), it follows that \( u \in N_{2d-2d/3q}(U) \). Hence, by (38), we have that \( u \in N_{2d-d/3q}(U_P) \) for some \( P \). In particular, \( |N(u) \cap f^{-1}(P_{\text{bdry}})| \geq 2d-2d/q \) by (34). Since \( f(N(u)) \notin P_{\text{bdry}} \) by (25) (note that \( u \) is \( P \)-odd as it is adjacent to \( U_P \)), it follows that \( u \in S_{unbal} \).

Lemma 5.5. There exists a family \( \mathcal{V} \) satisfying that
\[
|\mathcal{V}| \leq \exp \left( \frac{CN(q+\log d) \log d}{d} \right) \quad \text{and} \quad \Omega \subset \bigcup_{V \in \mathcal{V}} \Omega(V).
\]

Proof. Let \( \mathcal{V} \) be the collection of all \( \{V_P\}_P \) such that \( \{V_P\}_P \) are disjoint subsets of \( X_{bad} \) having \( \sum_P |V_P| \leq 3rN \), where \( r := 3q(1 + \log 2d)/d \). Let us check that \( \mathcal{V} \) satisfies the requirements of the lemma. Since \( |X_{bad}| = N \), we have
\[
|\mathcal{V}| \leq \sum_{k=0}^{3rN} \binom{N}{k} \cdot (2q)^{3rN} \leq \left( \frac{e^{2q}}{3r} \right)^{3rN} \leq e^{CN(q+\log d)(\log d)/d}.
\]
Fix $f \in \Omega$. We must find a collection $(V_P)_P \in \mathcal{V}$ for which (35) holds. We write $U_P$ for $U_P(f)$, and we denote $U_I := \bigcup_{P \in I} U_P$ for $I \subset \mathcal{P}$ and $U := U_P$. Define a bipartite graph $G$ with vertex set $(\mathbb{Z}^d \times \{0,1\}) \cup U$ as follows. For each $v \in \mathbb{Z}^d$, let $I_v$ be a minimal set of dominant patterns for which $|N(v) \cap U_{I_v}| \geq \frac{1}{4}|N(v) \cap U|$, and place an edge between $(v, i) \in \mathbb{Z}^d \times \{0,1\}$ and $u \in U$ if and only if $v \sim u$ and $1(u \in U_{I_v}) = i$. Note that $G$ has maximum degree at most $2d$.

By Lemma 5.6 applied to $G$ with $d = 9q$, we obtain a set $W \subset U$ of size $|W| \leq 3rN$ such that

$$v \in N_d/9q(U_I) \implies v \in N(W \cap U_I) \quad \text{for any } v \in \mathbb{Z}^d \text{ and } I \in \{I_v, \mathcal{P} \setminus I_v\}.$$ 

Set $V_P := W \cap U_P$ for all $P$ and note that $W = \bigcup_P V_P$. Towards showing (35), let $P \in \mathcal{P}$ and $v \in N_d/3q(U \setminus U_P)$. Suppose first that $P \notin I_v$. Then

$$v \in N_d/3q(U) \subset N_d/9q(U_{I_v}) \subset N(W \cap U_{I_v}) \subset N(W \setminus U_P) = N(W \setminus V_P).$$

Suppose next that $P \in I_v$. By the minimality of $I_v$, either $I_v = \{P\}$ or $|N(v) \cap U_{I_v}| < \frac{2}{3}|N(v) \cap U|$. In either case, we have $|N(v) \cap U_{P \setminus I_v}| \geq d/9q$ so that

$$v \in N_d/9q(U_{P \setminus I_v}) \subset N(W \cap U_{P \setminus I_v}) \subset N(W \setminus U_P) = N(W \setminus V_P).$$

Lemma 5.6. $S \setminus X_{bad} \subset S_{uniq}$.

Proof. Let $v \in S \setminus X_{bad}$ and note that there exists $P$ such that $v \in U_P$. Assume first that $v$ is $P$-even. Then, by (23), $g(v) \in P_{bdry}$ for all $g \in \Omega$, so that if $g(N(v)) \neq P_{int}$ then either $|g(N(v))| \notin \left\{\left\lfloor \frac{2}{3}\right\rfloor, \left\lceil \frac{2}{3}\right\rceil\right\}$ or all edges in $\partial v$ are restricted in $g$ by (30). Hence, $v$ has a unique pattern. Assume next that $v$ is $P$-odd. Then $v^+ \subset X_P$ by (20) so that, by (23), $g(N(v)) \subset P_{bdry}$ for all $g \in \Omega$. Thus, either $g(N(v)) = P_{bdry}$ or $|g(N(v))| < \frac{2}{3}$. In particular, $v$ has a unique pattern. □

Proof of Proposition 4.4. Note that $|S| \leq 2L + M + N$. Thus, Lemma 4.7 and Lemma 5.6 imply that, for any $V$,

$$\mathbb{P}(\Omega(V)) \leq \exp\left(-\frac{1}{32q}\left(L \frac{d}{3d} + M \frac{q \log d}{6q} + \frac{N}{15q^2}\right) + 2N + e^{-d/65q^2}(2L + M + N)\right).$$

Therefore, by Lemma 5.5 and (1),

$$\mathbb{P}(\Omega) \leq \exp\left(CNq(q + \log d)\frac{\log d}{d} + e^{-cd/q^2}(2L + M + N) - \frac{\xi}{q}\left(L \frac{d}{3d} + M \frac{q \log d}{6q} + \frac{N}{15q^2}\right)\right) \leq e^{-\frac{\xi}{q}\left(L \frac{d}{3d} + M \frac{q \log d}{6q} + \frac{N}{15q^2}\right)}.$$ □

5.4. The probability of an approximated breakup. In this section, we prove Proposition 4.6.

Fix integers $L, M, N \geq 0$ and an approximation $A$. Denote

$$A_{bad} := \bigcap_P (A_P \cup A^{**})^c, \quad A_{overlap} := \bigcup_P (A_P \cap A_Q), \quad U := A^{**} \cup A_{bad} \cup A_{overlap}.$$

Further define

$$S_P := \text{int}(A_P \setminus U) \quad \text{and} \quad S := \bigcap_P (S_P)^c.$$

Note that $U^+ \subset S$, that $\{S_P\}_P$ is a partition of $S^c$ and that $\{S_P^+\}_P$ are pairwise disjoint. Let $X$ be an atlas which is approximated by $A$. Note that, by (A1)

$$A_{bad} \subset X_{bad} \subset X_{bad} \cup A^{**}, \quad A_{overlap} \subset X_{overlap} \subset A_{overlap} \cup A^{**}, \quad U = A^{**} \cup A_{bad} \cup A_{overlap}.$$

Claim 5.7.

$$S = X_s \cup (A^{**})^+.$$

Proof. Let us first show that $S \subset X_s \cup (A^{**})^+$. Let $v \in S$ and note that $v \notin \text{int}(A_P \setminus U)$ for all $P$. Thus, for any $P$, there exists $u \in v^+$ such that $u \notin A_P$ or $u \in U$. If the latter occurs for some $P$, then $u \notin U \subset A^{**} \cup X$ and we are done. Otherwise, for every $P$, there exists $u \in v^+$ such that $u \notin A_P$. That is, $u \in \bigcap_P \text{int}(A_P)^c$. Suppose that $u \notin X_s$ so that $u \in \text{int}(X_P)$ for some $P$. By (A1) $u \in \text{int}(A_P \cup A^{**})$. Since $u \notin \text{int}(A_P)$, it must be that $u \in (A^{**})^+$. 


Let us now show that $X_s \cup (A^*)^+ \subset S$. Since $A^* \subset U$ and $U^+ \subset S$, we see that $(A^*)^+ \subset S$. Similarly, $X_{bad} \cup X_{overlap} \subset U \subset S$. It remains to show that $\bigcup P \partial_{\infty} X_P \subset S$. Let $v \in \partial_{\infty} X_P$ for some $P$ and suppose towards a contradiction that $v \in S_Q$ for some $Q$. Then (A1) implies that $v \in \text{int}(X_Q \setminus X_{overlap})$, which clearly contradicts the fact that $v \in \partial_{\infty} X_P$.

Thus, using (A4) we see that $S \subset X^+$. Recall that $A_P \subset X_P$ by (A1) and note that $\partial_{\infty} S_P \subset \partial_{\infty} S \cap X_P \setminus X_{overlap}$. Thus, (23) and (24) imply that, for any coloring $f$ having $X$ as a breakup,

$$\text{Even}_P \cap A_P \cap S^+ \text{ and } \partial_{\infty} S_P \text{ are in the } P\text{-pattern.}$$

Finally, by (21), (A1) and the fact that $X_s \subset S$, we have that $S \cup S_{P_0}$ contains $(A^c)^+$. We have thus established that the assumptions of Lemma 4.7 are satisfied for the sets $(S,(S_P)^+)$ and the event $\Omega$ that $A$ approximates some breakup in $X_{L,M,N}$.

**Lemma 5.8.** Every vertex in $S \setminus U$ has a unique pattern. That is, $S \setminus U \subset S^{\Omega}_{\text{uniq}}$.

**Proof.** The proof is essentially the same as that of Lemma 5.6. Let $v \in S \setminus U$ and note that $v \notin A^* \cup A_{bad}$ so that $v \in A_P$ for some $P$. Assume first that $v$ is $P$-even. Then, by (39), we have $g(v) \in P_{\text{dry}}$ for all $g \in \Omega$. Thus, by (30), if $g(N(v)) \neq P_{\text{int}}$ and $|g(N(v))| \in \{\lfloor \frac{d}{2}\rfloor,\lceil \frac{d}{2}\rceil\}$, then all edges in $\partial v$ are restricted in $g$. Hence, $v$ has a unique pattern. Assume next that $v$ is $P$-odd. Then, since $A_P$ is $P$-even, $v^+ \subset A_P$ so that $g(N(v)) \in P_{\text{dry}}$ for all $g \in \Omega$ by (39). Thus, either $g(N(v)) = P_{\text{dry}}$ or $|g(N(v))| < \lfloor \frac{d}{2}\rfloor$. In particular, $v$ has a unique pattern. \hfill \Box

The proof of Proposition 4.6 is based on the idea that one of two situations can occur: either there are enough restricted edges so that one may directly apply Lemma 4.7 to obtain the desired bound, or there are not many possible breakups so that one may apply Proposition 4.4 together with a union bound. At the heart of this approach lies the fact that an unknown vertex is either non-dominant or adjacent to many restricted edges or there is a unique way to reconstruct the $X_P$ in its neighborhood. We now make this precise.

Denote

$$S^{\Omega,f,1/2}_{\text{rest}} := \{ v : |\partial v \cap S^{\Omega,f}_{\text{rest}}| \geq \frac{d}{2} \}.$$ 

For an atlas $X$, let $\Omega_X$ denote the event that $X$ is a breakup. With a slight abuse of notation, denote

$$S^{\Omega,X,1/2}_{\text{rest}} := \bigcap_{f \in \Omega_X} S^{\Omega,f,1/2}_{\text{rest}} \quad \text{and} \quad S^{X}_{\text{nondom}} := \bigcap_{f \in \Omega_X} S^f_{\text{nondom}}.$$ 

**Lemma 5.9.** Let $X$ be an atlas which is approximated by $A$, let $P$ be a dominant pattern and let $v \in A^*$ be a $P$-odd vertex. Then

either $v \in S^{\Omega,X,1/2}_{\text{rest}} \cup S^X_{\text{nondom}}$ or $v \in X_P \iff v \in N_{d/2}(A_P)$.

**Proof.** Fix a dominant pattern $P$ and a $P$-odd vertex $v \in A^*$. Recall that $v^+ \subset (A^*)^+ \subset S \subset X^+$. Denote $I := \{ Q \simeq P : v \in X_Q \}$. We first show that

$$|I| > 1 \implies v \in S^X_{\text{nondom}}.$$ 

Indeed, if $Q,T \in I$ are distinct, then $f(N(v)) \subset Q_{\text{bdry}} \cap T_{\text{bdry}}$ for any $f \in \Omega_X$ by (20) and (23), and it follows that $v$ is a non-dominant vertex in $f$.

Next, we show that

for every $Q \simeq P$ and $u \in N(v) \cap A_Q$, $Q \notin I \implies (v,u) \in S^{\Omega,f}_{\text{rest}}$ for all $f \in \Omega_X$. To this end, let $Q \simeq P$, $u \in N(v) \cap A_Q$ and $f \in \Omega_X$, and note that $g(u) \in Q_{\text{bdry}}$ for all $g \in \Omega$ by (39). If $Q \notin I$ then $f(N(v)) \not\subset Q_{\text{bdry}}$ by (26) so that $(v,u)$ is restricted by (29).

Suppose now that $v \notin S^{\Omega,X,1/2}_{\text{rest}} \cup S^X_{\text{nondom}}$. Note that $v \in N_{d}(\bigcup_{Q \simeq P} A_Q)$ by (A2). It therefore follows from what we have just shown that $I = \{ Q \} = \{ T \simeq P : v \in N_{d/2}(A_T) \}$ for some $Q \simeq P$. In particular, $v \in X_P$ if and only if $P = Q$ if and only if $v \in N_{d/2}(A_P)$.
Using (1) and Lemma 5.8, we may apply Lemma 4.7 to obtain
\[ A \approx B \] and then apply Proposition 4.4. Formally, let
\[ \text{Indeed, (40), (41) and (44) imply that} \]
\[ |B| \]
\[ \text{By (20) and (A1), we have} \]
\[ \text{Using Proposition 4.4, when} \]
\[ \text{Note that} \]
\[ \text{We bound separately the probabilities of} \Omega' \text{ and} \Omega \setminus \Omega'. \text{ Let us begin with} \Omega'. \text{ Note that} \]
\[ \text{By (A3) and Claim 5.7,} \]
\[ \text{Let} \]
\[ \text{To show (44), it suffices to show that, for any} \]
\[ \text{As this mapping is injective, we use Proposition 4.4 when} a \text{ is chosen sufficiently small, this will then yield that} \]
\[ \text{We now bound the probability of} \Omega \setminus \Omega'. \text{ To do this, as explained above, we recover the breakup and then apply Proposition 4.4. Formally, let} \mathcal{B} \text{ be the collection of atlases} X \in \mathcal{X}_{L,M,N} \text{ which are approximated by} A \text{ and have} \]
\[ \text{Note that} \Omega \setminus \Omega' \subset \bigcup_{X \in \mathcal{B}} \Omega_X. \text{ We shall show that} \]
\[ \text{Using Proposition 4.4 when} a \text{ is chosen sufficiently small, this will then yield that} \]
\[ \text{Toward establishing (42), we show that the mapping} \]
\[ \text{is injective on} \mathcal{B}, \text{ where} \]
\[ \text{By (20) and (A1), we have} \]
\[ \text{Thus, to determine} X_P, \text{ we only need to know the set} \text{Odd} \cap X \cap A^* \text{. In other words, we only need to know for each vertex} v \in \text{Odd} \cap A^*, \text{ whether it belongs to} X_P \text{ or not. If} v \in S_{\text{rest}}^{\Omega \setminus \Omega'} \cup S_{\text{nondom}}^X \text{ then this is given by} I_X(v), \text{ and otherwise, Lemma 5.9 implies that this is determined by the approximation.} \]
\[ \text{Let} \mathcal{R} \text{ be the image of the mapping in (43) as} X \text{ ranges over} \mathcal{B}. \text{ As this mapping is injective, we have} |\mathcal{B}| = |\mathcal{R}|. \text{ The bound (42) will then easily follow once we show that} \]
\[ \text{Indeed, (40), (41) and (44) imply that} \]
\[ |\mathcal{B}| = |\mathcal{R}| \leq \left( \frac{|S|}{q(q+\log d)} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right) \right) \cdot (2^q) \left( \frac{q}{q(q+\log d)} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right) \right) \leq e^{\frac{C_4}{q} \left( \frac{L}{d} + \frac{M}{q} + \frac{N}{q^2} \right)}. \]
To see this, let \( f : \mathbb{Z}^d \to [q] \) be such that \( X \) is a breakup of \( f \), and set \( I := f(N(v)) \). By (22), for \( P \in \mathcal{P}_i \), we have \( v \in X_P \) if and only if \( I \subset P_{\text{bdry}} \). For \( P \in \mathcal{P} \setminus \mathcal{P}_i \), we clearly have \( P \notin I_X(v) \), since \( v \) is \( P \)-even.

6. Repair transformation and Shearer’s inequality

In this section, we prove the following generalization of Lemma 4.7.

**Lemma 6.1.** Let \( S \subset \mathbb{Z}^d \) be finite and let \( \{S_P\}_{P \in \mathcal{P}} \) be a partition of \( S^e \) such that \( \partial \bullet S_P \subset \partial_{\bullet} S \) for all \( P \). Suppose that \( S \cup S_P^c \) contains \( (\Lambda^c)^+ \). Let \( \Omega \) be the event which is determined by the values of \( f \) on \( S^+ \). Let \( \mathbb{P} \) be the set of all \( \Omega \)'s for which \( \Omega \) is in the \( \mathcal{P} \)-pattern for all \( P \). Then

\[
\mathbb{P}_P(\Omega) \leq \exp \left[ -\frac{1}{128q} \mathbb{E} \left( |S^f_{\text{unbal}}| + \frac{1}{q} |S^f_{\text{nondom}}| + \frac{1}{d} |S^F_{\text{rest}}| \right) \right],
\]

where the expectation is taken with respect to a random function \( f \) chosen from \( \mathbb{P}_\Lambda.P_0(\cdot | \Omega) \).

Let us show how this lemma yields Lemma 4.7.

**Proof of Lemma 4.7.** Let \( E \) be the event that \( f|_{S^+} = \phi|_{S^+} \) for some \( \phi \in \Omega \) and let \( \Omega' \) be the event that \( E \) occurs and \( (\partial \bullet S_P)^+ \) is in the \( \mathcal{P} \)-pattern for all \( P \). Note that \( E \) is determined by \( f|_{S^+} \), \( \Omega \subset \Omega' \), \( k(\Omega) = k(\Omega') \) and \( S^\Omega_{\text{unbal}} = S^\Omega'_{\text{unbal}} \). Thus, Lemma 4.7 follows from Lemma 6.1.

The proof is based on a general upper bound on the total weight of configurations in an event, given in Proposition 6.2 below. For a set \( U \subset \mathbb{Z}^d \), we denote \( U^{\text{even}} := \text{Even} \cap U \) and \( U^{\text{odd}} := \text{Odd} \cap U \). For two sets \( U, V \subset \mathbb{Z}^d \), we denote

\[
\partial^{\text{even}}(U, V) := \partial(U^{\text{even}}, V^{\text{odd}}) \quad \text{and} \quad \partial^{\text{odd}}(U, V) := \partial(U^{\text{odd}}, V^{\text{even}}),
\]

so that \( \partial(U, V) = \partial^{\text{even}}(U, V) \cup \partial^{\text{odd}}(U, V) \). We also write \( \partial^{\text{even}}U := \partial^{\text{even}}(U, U^c) \) and \( \partial^{\text{odd}}U := \partial^{\text{odd}}(U^c, U) \), and for a dominant phase \( P \), we use the notation \( \partial^{P,\text{even}} \) and \( \partial^{P,\text{odd}} \) with the meanings inferred from the notions of \( P \)-even and \( P \)-odd. Recall the notions of unbalanced neighborhood, non-dominant vertex, restricted edge and unique pattern defined in Section 1.3. Note that, although those notions were defined for proper colorings \( f \) of \( \mathbb{Z}^d \), they are well-defined for any \( v \in S \) when \( f \) is a proper coloring of \( S^+ \).

**Proposition 6.2.** Let \( S \subset \mathbb{Z}^d \) be finite and let \( \{S_P\}_{P \in \mathcal{P}} \) be a partition of \( S^e \). Let \( F \) be a set of proper colorings of \( S^+ \) satisfying that \( S^+ \cap (\partial \bullet S_P)^+ \) is in the \( \mathcal{P} \)-pattern for every \( P \). Sample \( f \in F \) uniformly at random. Then

\[
|F| \leq \left( \left( \frac{q}{2} \right)^2 \right)^{|S^+|} \cdot \exp \left[ -\frac{1}{128q} \mathbb{E} \left( |S^f_{\text{unbal}}| + \frac{1}{q} |S^f_{\text{nondom}}| + \frac{1}{d} |S^F_{\text{rest}}| \right) \right]
\cdot \prod \left( \frac{\mathbb{P}_{\Lambda_\Lambda}(\cdot | \Omega)}{\mathbb{P}_{\Lambda_\Lambda}(\cdot | \Lambda^c)} \right)^{\frac{1}{2}}.
\]

Before proving the above proposition, let us show it implies Lemma 4.7.

**Proof of Lemma 6.7.** Note that \( \Omega \) is measurable with respect to the values of \( f \) on \( S^{++} \). Let \( \tilde{\Lambda} \) be a finite subset of \( \mathbb{Z}^d \) that contains \( \Lambda \cup S^{++} \). Let \( \bar{\Omega} \) be the support of the marginal of \( \mathbb{P}_{\Lambda_\Lambda} \) on \([q]^{\tilde{\Lambda}}\). We henceforth view \( \Omega \) as a subset of \( \bar{\Omega} \), and consider all complements to be within \( \tilde{\Lambda} \), e.g., \( S^c = \tilde{\Lambda} \setminus S \). Denote

\[
\Omega_0 := \{ f|_{S^+} : f \in \Omega \} \subset [q]^{\tilde{\Lambda}\setminus S^+} \quad \text{and} \quad \Omega_1 := \{ f|_{S^+} : f \in \Omega \} \subset [q]^{S^+}.
\]

Let \( T : \Omega_0 \to P(\Omega) \) be a map to be defined shortly which satisfies \( T(f) \cap T(f') = \emptyset \) for distinct \( f, f' \in \Omega_0 \). Recalling (15) and (16), we note that

\[
\mathbb{P}_{\Lambda_\Lambda}(\Omega) = \frac{\Omega}{|\Omega|} \leq \frac{|\Omega_0| \cdot |\Omega_1|}{\sum_{f \in \Omega_0} |T(f)|} \leq \frac{|\Omega_1|}{\min_{f \in \Omega_0} |T(f)|}.
\]
Before defining \( T \), let us bound \( |\Omega| \). To this end, we aim to apply Proposition 6.2 with \( \mathcal{F} = \Omega \). Observe that, since \((\partial \bullet S_P)^+\) is in the \( P \)-pattern on \( \Omega \) and since \( E \) is determined by \( f|_{S^+} \), the collection \( \mathcal{F} \) satisfies the assumption of the proposition and, moreover, \( \mathbb{P}_{A,P_0}(f|_{S^+} \in \cdot \mid \Omega) \) is the uniform distribution on \( \mathcal{F} \). For \( i \in \{0,1\} \), denote \( S_i := \bigcup_{P \in \mathcal{P}_i} S_P \setminus S^+ \), where \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) were defined in Section 4.1. Then, by Proposition 6.2

\[
|\Omega_1| \leq \left( \left\lfloor \frac{q}{2} \right\rfloor \right)^2 |S^+| \cdot e^{-\frac{1}{4q^2} \left( |S^+| + \frac{1}{q} |S^+_\text{random}| + \frac{1}{2} |S^+_\text{rest}| + \frac{1}{2} |S^+ - S^+_\text{random}| + e^{-\frac{d}{5q^2}} |S| \right)} \cdot \left( \left\lfloor \frac{q}{2} \right\rfloor / \left\lceil \frac{q}{2} \right\rceil \right)^{\frac{1}{2}} (|\text{even}(S^+, S_0)| - |\text{odd}(S^+, S_0)| - |\text{even}(S^+, S_1)| - |\text{odd}(S^+, S_1)|).
\]

Thus, the lemma will follow if we find such a map \( T \) which satisfies

\[
\min_{f \in \Omega_0} |T(f)| \geq \left( \left\lfloor \frac{q}{2} \right\rfloor \right)^2 |S^+| \cdot \left( \left\lfloor \frac{q}{2} \right\rfloor / \left\lceil \frac{q}{2} \right\rceil \right)^{\frac{1}{2}} (|\text{even}(S^+, S_0)| - |\text{odd}(S^+, S_0)| - |\text{even}(S^+, S_1)| - |\text{odd}(S^+, S_1)|). \tag{45}
\]

We now turn to the definition of \( T \). Fix a unit vector \( e \in \mathbb{Z}^d \). For \( u \in \mathbb{Z}^d \), we denote \( u^+ := u + e \) and \( u^- := u - e \). For a set \( U \subset \mathbb{Z}^d \), we also write \( U^+ := \{ u^+ : u \in U \} \) and \( U^- := \{ u^- : u \in U \} \). For each \( P \in \mathcal{P} \), let \( \psi_P \) be a permutation of \([q]\) taking \( P_0 \) if \( P \in \mathcal{P}_0 \) or to \((B_0, A_0)\) otherwise. Let \( \mathcal{H} \) be the set of all functions \( h : S_* \mapsto [q] \) which are in the \( P_0 \)-pattern, where \( S_* := (S_0 \cup S_1^\downarrow)^c \).

For \( f \in \Omega_0 \) and \( h \in \mathcal{H} \), define \( \phi_{f,h} \) by

\[
\phi_{f,h}(v) :=
\begin{cases}
\psi_P(f(v)) & \text{if } v \in S_P \setminus S^+ \text{ for } P \in \mathcal{P}_0 \\
\psi_P(f(v^+)) & \text{if } v \in (S_P \setminus S^+)^\downarrow \text{ for } P \in \mathcal{P}_1 \\
h(v) & \text{if } v \in S_*
\end{cases}
\]

Note that \( \phi_{f,h} \) is well-defined, since the assumption that \( \partial \bullet S_P \subset \partial \bullet S \) for all \( P \) implies that

\[
\text{dist}(S_P \setminus S^+, S_Q \setminus S^+) \geq 3 \quad \text{for distinct } P \text{ and } Q \tag{46}
\]

so that, in particular, \( \{S_0, S_1^\downarrow, S_*\} \) is a partition of \( \mathbb{Z}^d \).

Let us check that \( \phi := \phi_{f,h} \) is a proper coloring. In light of [10], it suffices to show that \( \partial \bullet S_0, \partial \bullet S_1^\downarrow \) and \( S_* \) are in the \( P_0 \)-pattern in \( \phi \). It is immediate from the definition that \( S_* \) is in the \( P_0 \)-pattern in \( \phi \). If \( w \in \partial \bullet S_0 \) then \( w \in \partial \bullet (S_P \setminus S^+) \subset (\partial \bullet S_P)^+ \) for some \( P \in \mathcal{P}_0 \). By the assumption of the lemma, \( w \) is in the \( P \)-pattern in \( f \), and thus, by the definition of \( \psi_P \), \( w \) is in the \( P_0 \)-pattern in \( \psi_P \circ f \) and hence also in \( \phi \). Similarly, if \( w \in \partial \bullet S_1^\downarrow \) then \( w \in \partial \bullet (S_P \setminus S^+) \subset (\partial \bullet S_P)^+ \) for some
$P \in \mathcal{P}_1$, so that $w^+ \in$ is the $P$-pattern in $f$, and thus, $w^+$ is in the $(B_0, A_0)$-pattern in $\psi_P \circ f$ so that $w$ is in the $P_0$-pattern in $\phi$.

Let us check that $\phi \in \Omega$. By (14), (15) and (16), we must check that $(\Lambda^c)^+$ is in the $P_0$-pattern in $\phi$. Let $v \in (\Lambda^c)^+$ and recall that, by assumption, $(\Lambda^c)^+ \subset S \cup S_{P_0}$. Since $S \subset S_\ast$ and $S_\ast$ is in the $P_0$-pattern, we may assume that $v \in (S \cup S_{P_0}) \setminus S_\ast \subset S_{P_0} \setminus S^+$, in which case, $\phi(v) = f(v)$ and it is clear that $v$ is in the $P_0$-pattern.

Finally, define
\[ T(f) := \{ \phi_{f,h} : h \in \mathcal{H} \}. \]
To see that the desired property that $T(f) \cap T(f') = \emptyset$ for distinct $f, f' \in \Omega_0$ holds, we now show that the mapping $(f, h) \mapsto \phi_{f,h}$ is injective on $\Omega_0 \times \mathcal{H}$. To this end, we show how to recover $(f, h)$ from a given $g$ in the image of this mapping. It is straightforward to check that
\[ f(v) = \begin{cases} \psi_P^{-1}(g(v)) & \text{if } v \in S_P \setminus S^+ \text{ for } P \in \mathcal{P}_0, \\ \psi_P^{-1}(g(v^\dagger)) & \text{if } v \in S_P \setminus S^+ \text{ for } P \in \mathcal{P}_1 \end{cases} \quad \text{and} \quad h(v) = g(v) \text{ for } v \in S_\ast. \]

It remains to check that (45) holds. By injectivity, we have
\[ |T(f)| = |\mathcal{H}| \quad \text{for all } f \in \Omega_0. \]
Since the definition of $\mathcal{H}$ immediately implies that
\[ |\mathcal{H}| = \left[ \frac{q}{2} \right]^{S_\ast^\text{even}} \cdot \left[ \frac{q}{2} \right]^{S_\ast^\text{odd}}, \]
concluding (45) is essentially just a computation. To see this, using the fact (which we prove below) that, for any finite set $U \subset \mathbb{Z}^d$,
\[ |U^\text{even}| - |U^\text{odd}| = \frac{1}{2^d} (|\partial^\text{even} U| - |\partial^\text{odd} U|), \quad (47) \]
and writing $|S_\ast^\text{even}| = \frac{1}{2} |S_\ast| + |S_\ast^\text{even}| - |S_\ast^\text{odd}|$, and similarly for $|S_\ast^\text{odd}|$, we have
\[ |\mathcal{H}| = (\left[ \frac{q}{2} \right] / \left[ \frac{q}{2} \right]) \frac{1}{2} |S_\ast| \cdot (\left[ \frac{q}{2} \right] / \left[ \frac{q}{2} \right]) \frac{1}{2} (|S_\ast^\text{even}| - |S_\ast^\text{odd}|). \]
Noting that $|S_\ast| = |S^+|$, it thus suffices to show that
\[ |\partial^\text{even} S_\ast| = |\partial^\text{even}(S^+, S_0)| + |\partial^\text{odd}(S^+, S_1)|, \]
\[ |\partial^\text{odd} S_\ast| = |\partial^\text{odd}(S^+, S_0)| + |\partial^\text{even}(S^+, S_1)|. \]

Since $\partial S_\ast = \partial(S^+, S_0) \cup \partial((S^+, S_0^\dagger), S_1^\dagger)$, this easily follows.

It remains to prove (47). To see this, first observe that $u \mapsto u^\dagger$ is a bijection between $U^\text{even} \cap U^\dagger$ and $U^\text{odd} \cap U^\dagger$, so that
\[ |U^\text{even}| - |U^\text{odd}| = |(U \setminus U^\dagger)^\text{even}| - |(U \setminus U^\dagger)^\text{odd}|. \]
As this equality holds for any direction $\dagger$, summing it up over the $2d$ possible choices yields (47). \qed

The proof of the Proposition 6.2 relies on two lemmas. The first lemma, whose proof is based on Shearer’s inequality, provides a bound on the total weight of a collection of configurations $\mu$, which is conveniently factorized into “local terms” involving the values of $f$ on a vertex and its neighbors.

**Lemma 6.3.** Let $S \subset \mathbb{Z}^d$ be finite and even and let $\{ S_u \}_{u \in \partial S}$ be a collection of subsets of $[q]$. Let $\mathcal{F} \subset[q]^S$ be a set of proper colorings such that $f(u) \in S_u$ for every $f \in \mathcal{F}$ and $u \in \partial S$. Let $f$ be an element of $\mathcal{F}$ chosen uniformly at random. For each odd vertex $v \in S$, let $X_v$ be a random variable which is measurable with respect to $\mathcal{F}|_{N(v)}$. Then
\[ \log |\mathcal{F}| \leq \sum_{v \in S^\text{odd}} \left[ \frac{1}{2d} \text{Ent}(X_v) + \frac{1}{2d} \mathbb{E} \log |\Psi_v| + \mathbb{E} \log |I_v| \right] + \frac{1}{2d} \sum_{u \in \partial S} |\partial u \cap \partial S| \log |S_u|, \]
where $\Psi_v$ and $I_v$ are the supports of $f|_{N(v)}$ and $f(v)$ given $X_v$, respectively.
Lemma 6.4. \[\text{indices or to the condition that }\]

\[\text{restricted edges (in the sense of the definition in Section 4.3) correspond to either semi-restricted}
\]

\[\text{is disjoint from} \]

\[J_k \]

\[\text{with suitable bounds on}
\]

\[f\]

\[\text{the type of}
\]

\[\text{such that}
\]

\[\text{entropy of}
\]

\[\text{proceed to define this information, which we call the}
\]

\[\text{type}
\]

\[\text{we aim to expose just enough information to allow determining the occurrence of these. We now}
\]

\[\text{as little information as possible, which still suffices to obtain good bounds on}
\]

\[f\]

\[\text{Note that, by the assumption on}
\]

\[\text{variables}\]

\[\Ent\]

\[\text{Since}
\]

\[\Ent\]

\[\text{Besides factorizing the bound on}
\]

\[\Ent\]

\[\text{f}\]

\[\text{we make use of (10)-(13) throughout the proof. We begin by writing}
\]

\[\Ent(f) = \Ent(f_{\text{even}}) + \Ent(f_{\text{odd}} | f_{\text{even}}).
\]

\[\text{By the sub-additivity of entropy, we have}
\]

\[\Ent(f_{\text{odd}} | f_{\text{even}}) \leq \sum_{v \in S_{\text{odd}}} \Ent(f(v) | f|_{N(v)}).
\]

\[\text{We use Shearer's inequality to bound }\]

\[\Ent(f_{\text{even}}). \text{ Namely, Lemma 2.1 applied with the random}
\]

\[\text{variables } (Z_i) = (f(v))_{v \in S_{\text{odd}}}, \text{ the collection } I = \{N(v)\}_{v \in S_{\text{odd}}} \cup \{N(v) \cap S\}_{v \in \partial S} \text{ and } k = 2d, \text{ yields}
\]

\[\Ent(f_{\text{even}}) \leq \frac{1}{2d} \sum_{v \in S_{\text{odd}}} \Ent(f|_{N(v)}) + \frac{1}{2d} \sum_{v \in \partial S} \Ent(f|_{N(v) \cap S}).
\]

\[\text{Note that, by the assumption on } F,
\]

\[\sum_{v \in \partial S} \Ent(f|_{N(v)} \cap S) \leq \sum_{v \in \partial S} \sum_{w \in N(v) \cap S} \Ent(f(u)) = \sum_{u \in \partial S} |\partial u \cap \partial S| \cdot \Ent(f(u)).
\]

\[\text{Thus, the lemma will follow once we show that}
\]

\[\frac{1}{2d} \cdot \Ent(f|_{N(v)}) + \Ent(f(v) | f|_{N(v)}) \leq \frac{1}{2d} \Ent(X_v) + \frac{1}{2d} E \log |\Psi_v| + E \log |I_v|.
\]

\[\text{Indeed,}
\]

\[\Ent(f|_{N(v)}) \leq \Ent(X_v) + \Ent(f|_{N(v)} | X_v) \leq \Ent(X_v) + E \log |\Psi_v|,
\]

\[\text{and}
\]

\[\Ent(f(v) | f|_{N(v)}) \leq \Ent(f(v) | X_v) \leq E \log |I_v|. \]

\[\square
\]

\[\text{Besides factorizing the bound on } |F| \text{ over the odd vertices in } S, \text{ Lemma 6.3 allows to expose some}
\]

\[\text{information about } f|_{N(v)} \text{ which can then be used to bound } |\Psi_v| \cdot |I_v|^{2d}. \text{ One could theoretically}
\]

\[\text{expose } f|_{N(v)} \text{ completely (i.e., by taking } X_v \text{ to equal } f|_{N(v)} \text{ above), but this would increase the}
\]

\[\text{entropy of } X_v, \text{ making it harder to bound } \Ent(X_v) \text{ effectively. One would therefore like to expose as}
\]

\[\text{little information as possible, which still suffices to obtain good bounds on } |\Psi_v| : |I_v|^{2d}.
\]

\[\text{Recalling the notions of unbalanced neighborhood and restricted edge introduced in Section 4.3, we aim to expose just enough information to allow determining the occurrence of these. We now}
\]

\[\text{proceed to define this information, which we call the type of } f|_{N(v)} \text{. Given a function } \psi: [2d] \rightarrow [q], \text{ which is later identified with } f|_{N(v)} \text{, let }\]

\[\psi_{\text{unbal}} \text{ be the indicator of whether there exists } i \in \psi([2d]) \text{ such that } |\psi^{-1}(i)| \leq d/q. \text{ The type of } \psi \text{ is then defined to be } (\psi([2d]), \psi_{\text{unbal}}).
\]

\[\text{In the proof of Proposition 6.2, we will use Lemma 6.3 with the random variable } X_v \text{ taken to be the type of } f|_{N(v)} \text{. To make use of the inequality given in Lemma 6.3, we will need to accompany it with suitable bounds on } |\Psi| : |I|^{2d}, \text{ where } \Psi \text{ is a collection of functions of type } (J, z) \text{ and } I \subset [q] \text{ is disjoint from } J. \text{ The next lemma provides such bounds. For } \Psi \text{ consisting of functions of type}
\]

\[\text{we say that } j \in [2d] \text{ is a semi-restricted index in } \Psi \text{ if } \{\psi(j) : \psi \in \Psi\} \neq J. \text{ We note that}
\]

\[\text{restricted edges (in the sense of the definition in Section 4.3) correspond to either semi-restricted indices or to the condition that } I \cup J \neq [q].
\]

\[\text{Lemma 6.4. Let } \Psi \text{ be a collection of functions of type } (J, z) \text{ and let } I \subset J^c. \text{ Suppose that there are } k \text{ semi-restricted indices in } \Psi. \text{ Then}
\]

\[|\Psi| : |I|^{2d} \leq \left(\left\lceil \frac{d}{2} \right\rceil \left\lceil \frac{d}{2} \right\rceil \right)^{2d} \cdot \begin{cases} e^{-k/q} & \text{if } |J| \notin \left\{\left\lfloor \frac{d}{2}\right\rfloor, \left\lceil \frac{d}{2}\right\rceil\right\}, \\ e^{-d/4q} & \text{if } I \cup J \neq [q] \text{ or } z = 1 \end{cases}
\]
Let us explain the terms in the above bound. Observe that $|\Psi| \cdot |I|^{2d}$ is the number of proper colorings $\varphi$ of $K_{2d, 2d}$ whose restriction to the left side of $K_{2d, 2d}$ belongs to $\Psi$ and whose restriction to the right side belongs to $I^{[2d]}$. The first term, $(|\varphi| \cdot |I|^{2d})^{2d}$, comes from considering those $\varphi$ whose left and right sides takes values in $A$ and $B$, respectively, for some dominant phase $(A, B)$. In the second term, the first case reflects the reduction in the number of choices for $\varphi$ on the left side caused by the existence of semi-restricted indicies. The second case corresponds to a non-dominant vertex. Finally, the third case corresponds to either an unbalanced neighborhood or a partial restriction on the values of $\varphi$ on the right side.

**Proof.** For the first inequality in the lemma, we note that, by definition, $\{\psi(j) : \psi \in \Psi\} \subseteq J$ for any $j \in [2d]$ which is semi-restricted in $\Psi$. Thus,

$$|\Psi| \leq \prod_{j \in [2d]} |\{\psi(j) : \psi \in \Psi\}| \leq |J|^{2d} \cdot (1 - \frac{k}{\eta})^k \leq |J|^{2d} \cdot e^{-k/q},$$

so that

$$|\Psi| \cdot |I|^{2d} \leq (|J| \cdot |J_c|)^{2d} \cdot e^{-k/q} \leq (|\varphi| \cdot |I|^{2d})^{2d} \cdot e^{-k/q}.$$ 

For the second inequality in the lemma, observe that if $|J| \notin \{\frac{1}{2}, \frac{3}{2}\}$, then

$$|\Psi|^{1/2d} \cdot |I| \leq |J| \cdot |J_c| \leq (\frac{q}{2} - 1) \cdot (\frac{q}{2} + 1) \leq \frac{q}{2} \cdot \frac{q}{2} \cdot e^{-4/q^2}.$$ 

For the third inequality in the lemma, suppose first that $I \cup J \neq [q]$ and note that

$$|\Psi|^{1/2d} \cdot |I| \leq |J| \cdot |I| \leq \frac{q}{2} \cdot (\frac{q}{2} - 1) \leq \frac{q}{2} \cdot \frac{q}{2} \cdot e^{-1/q}.$$ 

Suppose now that $z = 1$ and note that, by a Chernoff bound and (1),

$$|\Psi| \cdot |J|^{-2d} \leq |J| \cdot \mathbb{P}(\text{Bin}(2d, \frac{1}{q^2}) \leq \frac{q}{2} \leq \frac{2d}{q-1} \leq e^{-d/4q},$$

so that

$$|\Psi| \cdot |I|^{2d} \leq |\Psi| \cdot |J_c|^{2d} \leq (|\varphi| \cdot |J|^{2d})^{2d} \cdot |\Psi| \cdot |J|^{-2d} \leq (|\varphi| \cdot |I|^{2d})^{2d} \cdot e^{-d/4q}. \quad \square$$

Let us now give the proof of the main proposition.

**Proof of Proposition 6.2.** We prove something slightly stronger than the inequality stated in the lemma. Namely, we show that

$$|F| \leq (|\varphi| \cdot |J|^{2d})^{2d} \cdot e^{-\frac{d}{20q^2} \cdot \left(\sum_{v \in \text{int}(S)} |S_v^{\text{odd}}| \cdot e^{-d/20q^2} \right)} \leq e^{-d/20q^2} \cdot \mathbb{E} \left[ |\Psi| \cdot |J|^{2d} \right],$$

where

$$S_v^{\text{odd}} := (S_v^{\text{odd}})^{\text{odd}}, \quad S_v^{\text{adj}} = (S_v^{\text{adj}})^{\text{odd}} \quad \text{and} \quad S_v^{f, \text{odd}} = \text{the set of restricted edges } (v, u) \text{ with } v \in \text{int}(S),$$

and

$$\lambda_P := \begin{cases} \frac{q}{2} & \text{if } P \in P_0 \\ \frac{3}{2} & \text{if } P \in P_1 \end{cases}.$$ 

Indeed, the lemma then follows by taking the geometric average of the above bound and its symmetric version in which the roles of odd and even are exchanged.

In proving (48), instead of working directly with $S$, it is convenient to work with its even expansion, defined as

$$S' := S' \cup (\partial_0 S)^{\text{even}} = S^+ \setminus (\partial_0 S)^{\text{odd}}.$$

Note that $S \subset S' \subset S^+$ and $S^{\text{odd}} = (S')^{\text{odd}}$. Let $F'$ be the set of functions $f' \in [q]^{S'}$ such that $f'|_S \in F$ and for which $S_P$ is in the $P$-pattern for every $P$. Observe that if one samples an element $f' \in F'$ uniformly at random, then $f'|_S$ has the same distribution as $f$, and the random variables
$\{f'(u)\}_{u \in S'}$ are independent and uniformly distributed on $A$, where $P = (A, B)$ is the unique dominant pattern such that $u \in S_P$. It follows that

$$|F'| = |F| \cdot \prod_P (\lambda_P)^{|S' \cap S_P|}.$$  

Thus, noting that $|S' \cap S_P| = |\partiald(S, S_P)| + |\partiald(S^c, S_P)|$, we see that (48) is equivalent to

$$|F'| \leq \left(\left(\frac{d}{2}\right)^{\frac{d}{2}}\right)^{|\partiald|} \cdot e^{-\frac{2d}{\sqrt{dq}} + \frac{1}{2} \cdot |\partiald_{\text{unbal}}| + \frac{1}{2} \cdot |\partiald_{\text{nonnull}}|} \cdot e^{-d/(\sqrt{dq^2} + \frac{1}{2} \cdot |\partiald_{\text{rest}}|)} \cdot \prod_P (\lambda_P)^{\frac{|\partiald(S, S_P)| + |\partiald(S^c, S_P)|}{2}} \cdot \prod_P (\lambda_P)^{|\partiald(S, S_P)| + |\partiald(S^c, S_P)|}.$$  

We now aim to apply Lemma 6.3 with $S'$ and $F'$. For $u \in \partial_u S'$, define

$$S_u := \begin{cases} A & \text{if } u \in \partial_u S \text{ and } u \in S_{(A, B)} \cap \{A : (A, B) \in P : u \in N(S_{(A, B)}) \} \text{ if } u \in \partial_u S \end{cases}.$$  

Note that, by the assumption on $F$ and by the definition of $F'$, we have $\delta(u) \in S_u$ for all $\delta \in F'$ and all $u \in \partial_u S'$. For an odd vertex $v \in S$, define

$$X_v := \begin{cases} the \text{ type of } f'|_{N(v)} & if \quad v \in \text{int}(S) \quad \text{ and } \quad \begin{cases} 0 & if \quad v \in \partial_u S \end{cases} \end{cases}.$$  

Then, by Lemma 6.3

$$\log |F'| \leq \sum_{v \in \partialodd} \left[ \frac{1}{2d} \text{Ent}(X_v) + \frac{1}{2d} \mathbb{E} \log |\Psi_v| + \mathbb{E} \log |I_v| \right] + \frac{1}{2d} \sum_{u \in \partial_u S'} |\partialu \cap \partial S'| \log |S_u|,$$

where $\Psi_v$ and $I_v$ are the supports of $f'|_{N(v)}$ and $f'(v)$ given $X_v$, respectively. We stress that the probabilities above are with respect to $f'$, but we also remind that $f'|_S$ equals $f$ in distribution so that these probabilities are the same when taken with respect to $f$.

We first show that

$$\sum_{u \in \partial_u S'} |\partialu \cap \partial S'| \log |S_u| \leq \sum_{P \in \mathcal{P}_0} (|\partialeven(S, S_P)| + |\partialodd(S^c, S_P)|) \cdot \log \lambda_P.$$  

Since $\{\partialu \cap \partial S'\}_{u \in \partial_u S'}$ and $\{\partialeven(S, S_P), \partialodd(S^c, S_P)\}_P$ are two partitions of $\partial S'$, it suffices to show an inequality for each edge separately, namely, that $|S_u| \leq \lambda_P$ for any $u \in \partial_u S'$ and $P = (A, B)$ such that $\partialu \cap (\partialeven(S, S_P) \cup \partialodd(S^c, S_P)) \neq \emptyset$. To this end, suppose that $\{u, w\} \in \partialeven(S, S_P) \cup \partialodd(S^c, S_P)$ for some $u$ and note that $u$ is even. If $u \notin S$ then $u \in S_P$ so that $S_u = A$ and $|S_u| = \lambda_P$. If $u \in S$ then $w \in S_P$ so that $S_u \subset A$ and $|S_u| \leq \lambda_P$.

Thus, to obtain (49), it suffices to show that, for any $v \in Sodd$,

$$\frac{\text{Ent}(X_v) + \mathbb{E} \log (|\Psi_v| \cdot |I_v|)^{2d}}{2d} \leq \log (\left(\frac{d}{2d}\right)^{\frac{d}{2d}}) + \left\{ \begin{array}{ll} -\frac{2d}{\sqrt{dq}} \cdot p_v + e^{-d/\sqrt{dq^2}} + \frac{1}{2} \cdot |\Psi_v| \cdot |I_v|^{2d} & \text{if } v \in \text{int}(S) \\ 0 & \text{if } v \in \partial_u S' \end{array} \right.,$$  

where

$$p_v := \mathbb{P}(v \in S^f_{\text{unbal}}) + \frac{1}{q} \cdot \mathbb{P}(v \in S^f_{\text{nonnull}}) + \frac{1}{2} \cdot \mathbb{E} |\partialu \cap \partial S^f_{\text{rest}}|.$$  

Suppose first that $v \in \partial_u S$. By the assumption on $F$ and by definition of $F'$, we have that $\Psi_v \subset A^{N(v)}$ and $I_v \subset B$, where $P = (A, B)$ is a dominant pattern such that $v \in S_P^+$. Thus,

$$\frac{\text{Ent}(X_v) + \mathbb{E} \log (|\Psi_v| \cdot |I_v|)^{2d}}{2d} = \frac{1}{2d} \log (|\Psi_v| \cdot |I_v|^{2d}) \leq \mathbb{E} \log (|\Psi_v| \cdot |I_v|^{2d}) \leq \log (|\Psi_v| \cdot |I_v|^{2d}) = \log (\left(\frac{d}{2d}\right)^{\frac{d}{2d}}).$$  

Suppose now that $v \in \text{int}(S)$. The desired inequality in (50) will follow if we show that

$$\frac{1}{2d} \mathbb{E} \log (|\Psi_v| \cdot |I_v|^{2d}) \leq \log (\left(\frac{d}{2d}\right)^{\frac{d}{2d}}) = \frac{1}{32q} \cdot p_v$$  

(51)
Thus, it suffices to show that, for any \( v \) inequality easily follows.

There exists some \( J \) for which \( \mathbf{Ent}_v \) tight-totally-separates \( x \). Hence, \( \mathbf{Ent}(X_v) \leq \log |\mathbb{P}(X_v)| \leq \log 2^{q+1} \leq 2q \).

Thus, it suffices to show that, for any \( v \in S_{\text{uniq}}^f \),

\[
\mathbf{Ent}(X_v) \leq \frac{d}{32q} \cdot p_v + e^{-d/65q^2}.
\]

Fix \( v \in S_{\text{uniq}}^f \) and denote \( p := p_v \). When \( p \geq 1/2q \), the above bound follows from the trivial bound on \( \mathbf{Ent}(X_v) \) using \([1]\). Thus, we may assume that \( p < 1/2q \). By the definition of unique pattern, there exists some \( J \) for which \( X_v \neq (J, 0) \) implies that \( v \in S_{\text{nondom}}^f \) or \( \partial v \in S_{\text{rest}}^f \). In particular, \( \mathbb{P}(X_v \neq (J, 0)) \leq pq < 1/2q \). Hence,

\[
\mathbf{Ent}(X_v) \leq pq \log \frac{2^{q+1}}{pq} + (1 - pq) \log \frac{1}{1-pq} \leq 2pq \log \frac{2^{q+1}}{pq}.
\]

Thus,

\[
\mathbf{Ent}(X_v) - \frac{d}{32q} \cdot p \leq 2pq \log \left( \frac{2^{q+1} \cdot e^{-q/64q^2}}{pq} \right) \leq 2^{q+1} \cdot e^{-d/65q^2} \leq e^{-d/65q^2},
\]

where we used the fact that \( x \log(a/x) \leq a/e \) for \( 0 < x < 1 \) and \([1]\). \hfill \( \square \)

7. Approximations

In this section, we prove Proposition \([4.5]\). That is, we show that there exists a small family of approximations which contains an approximation of every atlas in \( \mathcal{X}_{L,M,N} \) that is seen from a given set.

Say that a set \( W \) separates an atlas \( X \) if every edge in \( \bigcup_p \partial X_p \) has an endpoint in \( W \), and that it tightly-separates \( X \) if also \( W \subset \bigcup_p (\partial_{\circ p} X_p)^2 \). The first step is to construct a small family of small sets which contains a separating set of every atlas in \( \mathcal{X}_{L,M,N} \) that is seen from a given set.

**Lemma 7.1.** There exists \( C > 0 \) such that for any integers \( d \geq 2 \) and \( L, M, N \geq 0 \) and any finite set \( V \subset \mathbb{Z}^d \), there exists a family \( \mathcal{W} \) of subsets of \( \mathbb{Z}^d \), each of size at most \( CL(\log d)/\sqrt{d} \), such that

\[
|\mathcal{W}| \leq 2^{|V|} \cdot \exp \left( \frac{C(M+N)\log d}{d^{3/2}} + \frac{C(\log d)^2}{d^{3/2}} \right)
\]

and any atlas \( X \in \mathcal{X}_{L,M,N} \) seen from \( V \) is tightly-separated by some set in \( \mathcal{W} \).
Most of the arguments in this section are applied separately to the two collections \((X_P)_{P \in P_0}\) and \((X_P)_{P \in P_1}\), which consist of even and odd sets, respectively. The desired approximation defined in Section 4.5 is then constructed by combining the two independent pieces. For simplicity of writing, we fix the parity of the sets we work with here to be odd, even sets being completely analogous.

The definition of an atlas does not require any relation between \(X_P\) for different \(P\). In particular, the set of \(P\) for which a given vertex belongs to \(X_P\) could be any subset of the dominant phases. Since there are doubly-exponentially many such subsets, this would not lead to the correct dependency on \(q\). In light of this, we require an additional property of atlases, satisfied by any breakup, namely, \([44]\). In order to keep this section as independent as possible, we introduce some abstract definitions.

Let \(S = (S_i)_i\) be a collection of regular odd sets (we do not explicitly specify the index set as it has no significance in what follows). A rule is a family \(Q\) of subsets of indices. We say that a rule \(Q\) has rank at most \(q\) if \(|Q| \leq 2^q\). We say that \(S\) is an odd \(Q\)-collection if it obeys the rule \(Q\) in the following sense:

\[\{i : v \in S_i\} \in Q\quad \text{for any even vertex } v.\]

We say that a set \(W\) separates \(S\) if every edge in \(\bigcup_i \partial S_i\) has an endpoint in \(W\). An approximation of \(S\) is a collection \(A = ((A_i)_i, A_*)\) such that \(A_i \subset S_i \subset A_i \cup A_*\) and \(A_i\) is odd for all \(i\) and such that \(\text{Even} \cap A_* \subset N_d(\bigcup_i A_i)\). We say that \(A\) is controlled by \(W\) if \(|A_i| \leq C|W|\) and \(A_* \subset W^+\).

**Lemma 7.2.** There exists \(C > 0\) such that for any integers \(d \geq 2\) and \(q \geq 1\), any rule \(Q\) of rank at most \(q\) and any finite set \(W \subset \mathbb{Z}^d\), there exists a family \(A\) of approximations, each of which is controlled by \(W\), such that

\[|A| \leq \exp \left( \frac{Cq|W| \log d}{d} \right)\]

and any odd \(Q\)-collection which is separated by \(W\) is approximated by some element in \(A\).

Lemma 7.1 and Lemma 7.2 are proved in Sections 7.1 and 7.2 below.

**Proof of Proposition 4.5.** Applying Lemma 7.1 we obtain a family \(W\) of subsets of \(\mathbb{Z}^d\), each of size at most \(r := CL(\log d)/\sqrt{d}\), such that every \(X \in \mathcal{X}_{L,M,N}\) seen from \(V\) is tightly-separated by some set in \(W\). By \([20]\) and \([44]\), there exists a rule \(Q\) of rank at most \(q\) such that \((X_P)_{P \in P_1}\) is an odd \(Q\)-collection for any \(X \in \mathcal{X}\). Now, for each \(W \in \mathcal{W}\), we apply Lemma 7.2 to obtain a family \(A^1_W\) of approximations, each of which is controlled by \(W\), such that \(|A^1_W| \leq \exp(Crqd^{-1} \log d)\) and satisfying that any odd \(Q\)-collection which is separated by \(W\) is approximated by some element in \(A^1_W\). Reversing the roles of even and odd, we also obtain a family \(A^0_W\) in a similar manner. Finally, define \(A := \bigcup_{W \in \mathcal{W}} \bigcup_{A^0 \in A^0_W, A^1 \in A^1_W} \phi(A^0, A^1)\), where \(\phi(A^0, A^1) := ((A^0_P)_{P \in P_0} \cup (A^1_P)_{P \in P_1}, (\text{Odd} \cap A^0) \cup (\text{Even} \cap A^1), A^0 \cup A^1)\).

It is straightforward to verify that \(A\) satisfies the requirements of the lemma. \(\square\)

### 7.1. Constructing separating sets

This section is devoted to the proof of Lemma 7.1. That is, we construct a small family of sets, each of size at most \(CL(\log d)/\sqrt{d}\), which contains a tightly-separating set of every atlas \(X \in \mathcal{X}_{L,M,N}\) seen from \(V\). We begin by showing that for every collection \(S = (S_i)_i\) of regular odd sets, there exists a small set \(U\) such that \(N(U)\) tightly-separates \(S\). For such a collection, denote \(\partial S := \bigcup_i \partial S_i\) and \(\partial^+ S := \bigcup_i \partial^+ S_i\).

**Lemma 7.3.** Let \(S = (S_i)_i\) be a collection of regular odd sets. Then there exists \(U \subset (\partial^+ S)^+\) of size at most \(|\partial S| \cdot Cd^{-3/2} \log d\) such that \(N(U)\) separates \(S\).

The proof of Lemma 7.3 is given at the end of the section. Before proving Lemma 7.1, we require another lemma.

**Lemma 7.4.** For any \(n \geq 1\), the number of sets \(U \subset \mathbb{Z}^d\) of size at most \(n\) such that \(U^{+10}\) is connected and disconnects the origin from infinity is at most \(\exp(Cn \log d)\).
Corollary 7.6. Let $S$ be an odd set. Then $S^{\text{rev}}$ separates $S$.

Proof. Assume without loss of generality that $u$ is odd. Since $S$ is odd, we have $u \in S$ and $v \notin S$. Similarly, if $u + e \in S$ then $v + e \in S$. Thus, either $\{u, u + e\} \in \partial S$ or $\{v, v + e\} \in \partial S$.

For a set $S$, denote the revealed vertices in $S$ by
$$S^{\text{rev}} := \{v \in \mathbb{Z}^d : |\partial v \cap \partial S| \geq d\}.$$ That is, a vertex is revealed if it sees the boundary in at least half of the $2d$ directions. The following is an immediate corollary of Lemma 7.5.
Proof of Lemma 7.3. Let \( S = (S_i)_i \) be a collection of regular odd sets and denote \( L := |\partial S| \), \( \partial_s S := \bigcup_i \partial_s S_i \) and \( \partial_o S := \bigcup_i \partial_o S_i \). Note that a set separates \( S \) if and only if it separates \( S_i \) for all \( i \). Note also that \( \partial S_i = \partial S_i^c \) implies that \( S_i^{rev} = (S_i^c)^{rev} \). Thus, in light of Corollary 7.6 and by even-odd symmetry, it suffices to show that there exists a set \( U \subset N(\bigcup_i \partial_o S_i) \) such that \( \bigcup_i (S_i \cap S_i^{rev}) \subset N(U) \) and \( |U| \leq CLd^{-3/2} \log d \).

Denote \( s := \sqrt{d} \) and \( t := d/6 \), and define
\[
A := \{ v \text{ even : } |\partial v \cap \partial S| \geq s \} \quad \text{and} \quad A_i := \{ u \text{ odd : } |\partial u \cap \partial S_i| \geq 2d - s \},
\]
and observe that, by Lemma 3.5,
\[
|A| \leq \frac{L}{s} \quad \text{and} \quad \bigcup_i A_i \leq \frac{L}{2d - s}.
\]

For an odd vertex \( w \) and a vertex \( v \sim w \), denote
\[
M(w) := |\{ z \sim w : I(w, z) \neq \emptyset \}|, \quad M(w, v) := |\{ z \sim w : I(w, z) \not\subset I(w, v) \}|,
\]
where
\[
I(w, z) := \{ i : w \in A_i, z \in S_i \}.
\]

Denote
\[
T := \{ v \text{ even : } \exists v \sim w \in M(w, v) < \frac{1}{2} M(w) \}, \quad T' := \{ w \text{ odd : } 1 \leq M(w) \leq 2s \}.
\]

We claim that
\[
|T| \leq 2s \cdot \left| \bigcup_i A_i \right| \quad \text{and} \quad |T'| \leq \left| \bigcup_i A_i \right|.
\]

The second inequality is straightforward since \( M(w) \geq 1 \) implies that \( w \in \bigcup_i A_i \). Let us show the first inequality. Observe that \( T = \bigcup_w T(w) \), where the union is over odd \( w \) and
\[
T(w) := \{ v \sim w : M(w, v) < \frac{1}{2} M(w) \}.
\]

Then
\[
\frac{1}{2} M(w) \cdot |T(w)| < \left| \left\{ (v, z) \in N(w)^2 : \emptyset \neq I(w, z) \subset I(w, v) \right\} \right| \leq sM(w).
\]

Since \( T(w) \neq \emptyset \) implies \( M(w) \geq 1 \), it follows that \( |T(w)| \leq 2s \). Since \( T(w) \neq \emptyset \) also implies that \( w \in \bigcup_i A_i \), the desired inequality follows.

We now use Lemma 3.6 with \( A \) to obtain a set \( B \subset A \subset \partial_o S \) such that
\[
|B| \leq \frac{4 \log d}{t} |A| \quad \text{and} \quad N_t(A) \subset N(B).
\]

Applying the same lemma again, we obtain a set \( B' \subset T \subset N(\partial_o S) \) such that
\[
|B'| \leq \frac{4 \log d}{t} |T| \quad \text{and} \quad N_t(T) \subset N(B').
\]

We also define
\[
B'' := \bigcup_i (S_i \cap N_t(A_i \cap T')).
\]

By Lemma 3.5 and the definition of \( T' \), we have
\[
|B''| \leq \frac{2s}{t} |T'|.
\]

Finally, we define \( U := B \cup B' \cup B'' \). Clearly, \( U \subset N(\partial_o S) \) and
\[
|U| \leq \frac{4L \log d}{t} \left( \frac{1}{s} + \frac{2s}{2d - s} \right) + \frac{2sL}{t(2d - s)} \leq \frac{CL \log d}{d^{3/2}}.
\]

It remains to show that \( S_i \cap S_i^{rev} \subset N(U) \) for all \( i \). Towards showing this, let \( u \in S_i \cap S_i^{rev} = \partial_o S_i \cap N_d(\partial_o S_i) \) for some \( i \). Since \( S_i \) is regular, there exists a vertex \( z \in N(u) \cap S_i \). Let \( F \) denote
the set of pairs \((v, w)\) such that \((u, v, w, z)\) is a four-cycle and \(v \in \partial S_i\), and note that \(|F| \geq d - 1\). Define
\[
G^0 := \{(v, w) \in F : v \in A\}, \quad G^1 := \{(v, w) \in F : v \in T\}, \quad G^2 := \{(v, w) \in F : w \in A_i \cap T'\}.
\]
It suffices to show that \(F = G^0 \cup G^1 \cup G^2\), since then, either \(|G^0| \geq |F|/3 \geq t\) in which case \(u \in N_t(A) \subset N(B) \subset N(U)\), or \(|G^1| \geq t\) in which case \(u \in N_t(T) \subset N(B') \subset N(U)\), or \(|G^2| \geq t\) in which case \(z \in N_t(A_i \cap T')\) so that \(z \in B'\) and \(u \in N(B') \subset N(U)\).

Towards showing this, let \((v, w) \in F\) and note that \(w \in S_i\). By Lemma 7.5 \(v \in A\) or \(w \in A_i\). In the former case, \((v, w) \in G^0\), so we may assume that \(v \notin A\) and \(w \in A_i\). Thus, if \(w \in T'\) then \((v, w) \in G^2\) so that we may also assume that \(w \notin T'\). Since \(w \in A_i \setminus T'\), we have \(M(w) > 2s\). Thus, to obtain that \(v \in T\) and hence that \((v, w) \in G^1\), it suffices to show that \(M(w, v) \leq s\). Since \(v \notin A\), this will follow if we show that \(|\partial v \cap \partial S| \geq M(w, v) - 1\). For this, it is enough to show that if \((v, w, x, y)\) is a four-cycle such that \(I(w, x) \notin I(w, v)\), then \(\{v, y\} \in \partial S\). Indeed, this statement is straightforward, since \(j \in I(w, x) \setminus I(w, v)\) implies that \(x \in S_j\) (so that \(y \in S_j\)) and \(\{v \notin S_j\)  

7.2. Constructing approximations. The proof of Lemma 7.2 is split into two parts. We first show that every separating set gives rise to a small family of weak approximations. A weak approximation of a collection \(S = (S_i)_i\) is a collection \(A = ((A_i)_i, A_s)\) such that \(A_i \subset S_i \subset A_i \cup A_s\) for all \(i\). As before, we say that \(A\) is controlled by \(W\) if \(|A_i| \leq C|W|\) and \(A_s \subset W^+\).

**Lemma 7.7.** For any integers \(d \geq 2\) and \(q \geq 1\), any rule \(Q\) of rank at most \(q\) and any finite set \(W \subset \mathbb{Z}^d\), there exists a family \(A\) of weak approximations, each controlled by \(W\), such that
\[
|A| \leq 4^{W/q/d}
\]
and any odd \(Q\)-collection which is separated by \(W\) is weakly approximated by some \(A \in \mathcal{A}\).

The second step is to upgrade a weak approximation to a small family of approximations which covers at least the same set of \(Q\)-collections.

**Lemma 7.8.** There exists \(C > 0\) such that for any integers \(d \geq 2\) and \(q \geq 1\), any rule \(Q\) of rank at most \(q\) and any weak approximation \(A\) controlled by some \(W\), there exists a family \(A\) of approximations, each of which is also controlled by \(W\), such that
\[
|A| \leq \exp(Cq|A_i| \log d / d)
\]
and any odd \(Q\)-collection which is weakly approximated by \(A\) is approximated by some element in \(\mathcal{A}\).

Note that Lemma 7.2 follows immediately from Lemma 7.7 and Lemma 7.8. We now prove these two lemmas.

**Proof of Lemma 7.7.** Let \(Q\) be a rule of rank at most \(q\) and let \(W \subset \mathbb{Z}^d\) be finite. Consider the set \(X := \mathbb{Z}^d \setminus W\). Say that a connected component of \(X\) is small if its size is at most \(d\), and that it is large otherwise.

Let \(S = (S_i)_i\) be a collection of regular odd sets which is tightly-separated by \(W\), and observe that, for each \(i\), every connected component \(T\) of \(X\) is entirely contained in either \(S_i\) or \(S_i^c\). Define
\[
A_i := \bigcup \{T \text{ large component of } X : i \in I(T)\}, \quad \text{where } I(T) := \{i : T \subset S_i\}.
\]
Note that \(A_i\) is contained in \(S_i\) and that if \(I(T) = \emptyset\) then \(T \subset (\bigcup_i S_i)^c\). Let \(Y\) be the union of all the small components of \(X\) and define \(A_s := Y \cup W\). Clearly, \(A = A(S) := ((A_i)_i, A_s)\) is a weak approximation of \(S\).

Next, we bound the size of \(A_s\). For this we require a simple consequence of a well-known isoperimetric inequality (see, e.g., [12 Corollary 2.3]), namely,
\[
|\partial T| \geq d \cdot \min\{d, |T|\} \quad \text{for any finite } T \subset \mathbb{Z}^d.
\]
Since any small component $T$ of $X$ has $|T| \leq \lvert \partial T \rvert / d$ and $\partial T \subset \partial W$, we obtain

$$|Y| \leq \frac{\lvert \partial W \rvert}{d} \leq \frac{2d|W|}{d} \leq 2|W|.$$

Thus, $|A_*| = |Y \cup W| \leq 3|W|$.

Let us now show that $A$ is controlled by $W$. For this, it remains only to show that $A_* \subset W^+$. It suffices to show that $Y \subset N(W)$. To this end, let $v \in Y$ and note that $v^+ \not\subset Y$ by the definition of small component. Since $\partial v Y \subset W$, we see that $v \in \partial W$.

Now, denote by $\mathcal{A}$ the collection of weak approximations $A(S)$ constructed above for all odd $Q$-collections $S$ which are separated by $W$. To conclude the proof, it remains to bound $|A_*|$. Let $\ell$ be the number of large components of $X$. Since every large component $T$ must contain an even vertex (it is a connected set of size at least 2), and since every $S \in Q$ is a $Q$-collection, the set $I(T)$ defined above always belongs to $Q$. Hence, as $Q$ has rank at most $q$, we have $|A_*| \leq |Q|^{\ell} \leq 2^{q \ell}$.

Since any large component $T$ of $X$ has $|\partial T| \geq d^2$ and $\partial T \subset \partial W$, we obtain $\ell \leq |\partial W|/d^2 \leq |W|/d$ so that $|A_*| \leq 4^{Wq/d}$, as required.

**Proof of Lemma 7.3.** Let $Q$ be a rule of rank at most $q$ and let $A = ((A_i)_i, A_*)$ be a weak approximation. Let us first show that we may assume that

$$\text{Odd} \cap N(A_*) \subset \bigcup_i A_i \cup A_*.$$  \hspace{1cm} (53)

Define $A'_* := A_* \setminus N(U)$, where $U := \text{Odd} \cap (\bigcup_i A_i \cup A_*)^c$. Note that $\text{Odd} \cap A'_* = \text{Odd} \cap A_*$ and $\text{Odd} \cap N(A'_*) \subset \bigcup_i A_i \cup A'_*$. Thus, we may replace $A$ with $A' = ((A_i)_i, A'_*)$ once we show that any odd $Q$-collection $S$ which is weakly approximated by $A$ is also weakly approximated by $A'$. For this, it suffices to show that $S_i \subset A_i \cup A'_*$ for any $i$. Let $v \in S_i \setminus A_i \subset A_*$. If $v$ is odd then clearly $v \in A'_*$. If $v$ is even, then we must show that $v \notin N(U)$. Indeed, since $U \subset S_i^c$ and since $S_i$ is odd, this easily follows.

For a set $W \subset \text{Even} \cap A_*$, define

$$W_o := \text{Even} \cap N_d(A_* \setminus W^+).$$

Observe that $W^+$ and $W_o$ are disjoint. Here one should think of $W$ as recording the location of a subset of even vertices in $A_* \cap \bigcup_i S_i$. We shall see that if this subset is chosen suitably then $W^+ \subset \bigcup_i S_i$ and $W_o \subset \bigcap_i S_i^c$.

Let $W$ denote the family of such sets $W$ having size at most $m/d$, where $m := |A_*|$. We say that a collection $(W_i)_i$ is a $Q$-partition of $W$ if $W = \bigcup_i W_i$ and $\{i : v \in W_i\} \subset Q$ for all $v \in W$. Define

$$\mathcal{A} := \{((A_i \cup W_i^+)_i, A_* \setminus W_o) : W \in W, (W_i)_i \text{ is a } Q\text{-partition of } W\}.$$  \hspace{1cm} (53)

Let us show that $\mathcal{A}$ satisfies the requirements of the lemma. To this end, we first bound the size of $\mathcal{A}$. We have

$$|\mathcal{A}| \leq \sum_{k=0}^{\lfloor m/d \rfloor} \binom{m}{k} \leq (m + 1)(ed)^{m/d} \leq d^{Cm/d}.$$

Hence,

$$|A_*| \leq |\mathcal{A}| \cdot |Q|^{m/d} \leq 2^{(q+C\log d)m/d}.$$

Next, let us show that, for any $B = ((B_i)_i, B_*) \in \mathcal{A}$, we have $\text{Even} \cap B_* \subset N_d(\bigcup_i B_i)$. To this end, let $W \in \mathcal{W}$ be such that $\bigcup_i B_i = \bigcup_i A_i \cup W^+$ and $B_* = A_* \setminus W_o$. Let $v \in \text{Even} \cap B_*$. Note that, by (53), $N(v) \setminus A_* \subset \bigcup_i A_i$. Thus, it suffices to show that $v \in N_d(W^+ \cup (A^*)^c)$. This in turn follows from $v \notin W_o$.

It remains to show that any odd $Q$-collection $S$ which is weakly approximated by $A$ is approximated by some element in $\mathcal{A}$. Let $S$ be such a collection. Let $W$ be a maximal subset of $\text{Even} \cap A_* \cap \bigcup_i S_i$ among those satisfying $d|W| \leq |A_* \cap W^+|$, and note that $W \in \mathcal{W}$. Now define $B_i := A_i \cup W_i^+$ and $B_* := A_* \setminus W_o$, where $W_i := W \cap S_i$. To show that $B := ((B_i)_i, B_*) \in \mathcal{A}$, we
must show that \((W_i)_i\) is a \(Q\)-partition of \(W\). Indeed, since \(W\) is a set of even vertices, this follows from the fact that \(S\) is a \(Q\)-collection.

To conclude that \(B\) approximates \(S\), we must show that \(B_i \subset S_i \subset B_i \cup B_*\) for all \(i\) and that \(B_* \subset (\partial_0 S)^+\). Since \(A_i \subset S_i\) and \(W_i \subset \text{Even}\cap S_i\), and since \(S_i\) is odd, it follows that \(B_i \subset S_i\). Let \(v \in S_i \setminus B_i\) and note that \(v \in A_i\) since \(A_i \subset B_i\). If \(v\) is even, then \(v \in \text{Odd} \cap A_* = \text{Odd} \cap B_*\). Suppose that \(v\) is even. To obtain that \(v \in B_*\), it remains to show that \(v \not\in N_d(A_* \setminus W^+)\). Indeed, by the maximality of \(W\), and since \(v \in A_* \setminus W\), we have

\[
d[W \cup \{v\}] > |A_* \cap (W \cup \{v\})| = |A_* \cap W^+| + |A_* \cap v^+ \setminus W^+| \geq d[W] + |A_* \cap v^+ \setminus W^+|,
\]

so that \(|A_* \cap v^+ \setminus W^+| < d\). Finally, \(B_* \subset (\partial_0 S)^+\) follows from \(B_* \subset A_*\) and the fact that \(S\) is weakly approximated by \(A\).

8. Infinite-volume Gibbs states

In this section, we prove Theorem 1.2 and Theorem 1.3. The former is about the existence of a Gibbs state for each dominant pattern and the properties of this measure. The latter is about the characterization of all maximal-entropy Gibbs states. The first is proven in Section 8.2 and the second in Section 8.3. We assume throughout this section that \(q \geq 3\) and that \(d\) satisfies (1).

Let us first provide a formal definition of a Gibbs state (for uniform proper \(q\)-colorings). A probability measure \(\mu\) on \([q]^Z\) (with the natural product \(\sigma\)-algebra) is a Gibbs state if it is supported on proper \(q\)-colorings of \(Z^d\) and a random coloring \(f\) sampled according to \(\mu\) has the property that, for any finite \(\Lambda \subset Z^d\), conditioned on the restriction \(f|_{\Lambda^+}\), the restriction \(f|_{\Lambda^-}\) is almost surely uniformly distributed on the set of proper \(q\)-colorings of \(\Lambda^+\) that agree with \(f\) on \(\partial_0 \Lambda\).

For a distribution \(\mu\) on \([q]^Z\), we denote by \(\mu|_U\) the marginal distribution of \(\mu\) on \([q]^U\). Given two discrete distributions \(\mu\) and \(\lambda\) on a common space, we denote the total-variation distance between \(\mu\) and \(\lambda\) by \(d_{TV}(\mu, \lambda) := \max_A |\mu(A) - \lambda(A)|\) where the maximum is over all events \(A\). Recall that a domain is a finite, non-empty, connected and co-connected subset of \(Z^d\).

8.1. Large violations. For the proofs of Theorem 1.2 and Theorem 1.3 we require two extensions of Theorem 1.1 to larger violations of the boundary pattern rather than just single-site violations. Recall the definitions of \(Z_P(f)\) and \(Z_*(f)\) from (7) and (8) and the definition of \(Z_+^5(f, V)\) from before Lemma 4.1.

**Proposition 8.1.** Let \(\Lambda\) be a domain and let \(V \subset Z^d\) be finite. Then, for any \(k \geq 1\),

\[
\mathbb{P}_{A, P_0}\left(|Z_* (f) \cap Z_+^5 (f, V)| \geq k\right) \leq 2^{|V|} \cdot e^{-\frac{k}{q^3 (q+\log d)^d}}.
\]

**Proof.** Let \(\Omega_{L, M, N}\) denote the event that there exists a breakup in \(X_{L, M, N}\) seen from \(V\). Let us show that \(|Z_* \cap Z_+^5 (V)| \geq k\) implies the occurrence of \(\Omega_{L, M, N}\) for some \(L, M, N \geq 0\) satisfying that \(L/2 + M + N \geq k\).

Lemma 4.1 implies the existence of a breakup \(X\) such that \(X_+^5 = Z_+^5 (V)\). Note that this implies that \(X_* = Z_* \cap Z_+^5 (V)\) so that \(|X_*| \geq k\). Since every vertex in \(\bigcup P \partial_0 X_P\) is an endpoint of an edge in \(\bigcup P \partial_0 X_P\), and since every edge has only two endpoints, we see that \(X \in X_{L, M, N}\) implies that \(L/2 + M + N \geq k\). Note also that Lemma 3.1 implies that \(X_{L, M, N} = \emptyset\) when \(L < d^2\). Therefore, by Proposition 4.3,

\[
\mathbb{P}\left(|Z_* \cap Z_+^5 (V)| \geq k\right) \leq 2^{|V|} \sum_{L \geq d^2, M, N \geq 0 \atop L/2 + M + N \geq k} \exp \left(-\frac{c}{q^3 (q + \log d)} \left(\frac{L}{d} + \frac{M}{q} + \frac{N}{q^2}\right)\right).
\]

Using (1), the desired inequality follows.

Using (1), the desired inequality follows.
V \subset \mathbb{Z}^d$, define \( \mathcal{B}_P(f,V) \) to be the union of the \((\mathbb{Z}^d)^{\otimes 2}\)-connected components of \( \bar{Z}_P(f)^c \) that intersect \( V \). For a set \( U \subset \mathbb{Z}^d \), define \( \text{diam}^* U := 2m + \text{diam} U_1 + \cdots + \text{diam} U_m \), where \( \{U_i\}_{i=1}^m \) are the \((\mathbb{Z}^d)^{\otimes 2}\)-connected components of \( U \).

**Proposition 8.2.** Let \( \Lambda \) be a domain and let \( V \subset \mathbb{Z}^d \) be finite. Then, for any \( k \geq 1 \),

\[
\mathbb{P}_{\Lambda,P}(\text{diam}^* \mathcal{B}_P(f,V) \geq k) \leq 2^{|V|} e^{-\frac{cdk}{q(q+\log d)}}. \tag{54}
\]

For the proof, we require the following adaptation of Lemma 2.4 in \[12\]. For \( A \subset \mathbb{Z}^d \), denote

\[
A_{\text{iso}} := \{ v \in A : N(v) \cap A = \emptyset \}. \tag{55}
\]

**Lemma 8.3.** Let \( A \subset \mathbb{Z}^d \) be finite, odd and \((\mathbb{Z}^d)^{\otimes 2}\)-connected. Then

\[
|\partial A| + |\partial (A_{\text{iso}}^+)| \geq \frac{1}{2}(d-1)^2(2 + \text{diam } A).
\]

**Proof of Proposition 8.2.** We denote \( \mathcal{B} := \mathcal{B}_P(f,V) \) and omit \( f \) from notation. Let \( \Omega_{L,M,N} \) denote the event that there exists a breakup in \( X_{L,M,N} \) seen from \( V \). Let us show that \( \text{diam}^* \mathcal{B} \geq k \) implies the occurrence of \( \Omega_{L,M,N} \) for some \( L,M,N \geq 0 \) satisfying that \( L + 2dM \geq \text{cd}^2k \).

Note that \( \partial_0 \mathcal{B} \subset \partial_0 \bar{Z}_P \) so that, in particular, \( \mathcal{B} \) is an odd set. Also note that \( \mathcal{B}^{+2} \setminus B \subset \bar{Z}_P \) and that \( \mathcal{B}_{\text{iso}} \subset Z_P \setminus Z_P \). We claim that \( \partial (\mathcal{B} \setminus \mathcal{B}_{\text{iso}}) \subset \partial Z_P \) and \( \mathcal{B}^{+\text{iso}} \subset \Omega_{\text{overlap}} \), so that, in particular, \( \partial \mathcal{B} \subset Z_\Lambda \). To see the former, let \( (u,v) \in \partial (\mathcal{B} \setminus \mathcal{B}_{\text{iso}}) \) and \( w \in N(\{u,v\}) \cap \mathcal{B} \), so that \( v \in \mathcal{B} \) and \( w \notin \mathcal{B} \). It follows that \( u \notin Z_P \) and hence that \( \{u,v\} \in \partial Z_P \). To see the latter, let \( u \in \mathcal{B}^{+\text{iso}} \) and \( w \in u^+ \cap \mathcal{B}_{\text{iso}} \), so that \( f(u) \in P_{\text{bdry}}(\mathcal{B}_{\text{iso}}) \) such that \( P_{\text{bdry}}(\mathcal{B}_{\text{iso}}) \setminus \{f(w)\} \). It follows that \( w \in \mathcal{B}^{+\text{iso}} \subset \Omega_{\text{overlap}} \).

**Lemma 4.3** implies the existence of a breakup \( X \) such that \( X_{5}^{+5} = Z_5^{+5}(V) \). Note that this implies that \( X_x = Z_x \cap Z_x^{+5}(V) \). Since \( \partial_0 \mathcal{B} \subset Z_x \) and since every \((\mathbb{Z}^d)^{\otimes 2}\)-connected component of \( \mathcal{B} \) intersects \( V \), it follows that \( \partial_0 \mathcal{B} \subset Z_x^{+5}(V) \) and hence that \( \partial_0 \mathcal{B} \subset X_x \). In particular, by \[22\], \( X_Q \) and \( Z_Q \) coincide near the boundary of \( B \) for all \( Q \), so that \( \partial (\mathcal{B} \setminus \mathcal{B}_{\text{iso}}) \subset \partial X_P \) and \( \mathcal{B}^{+\text{iso}} \subset X_{\text{overlap}} \).

Let \( L,M,N \geq 0 \) be such that \( X \in X_{L,M,N} \). Applying \textit{Lemma 8.3} to each \((\mathbb{Z}^d)^{\otimes 2}\)-connected component of \( \mathcal{B} \) yields that \( |\partial \mathcal{B}| + |\partial (\mathcal{B}^{+\text{iso}})| \geq \text{cd}^2k \). Since \( |\partial (\mathcal{B}^{+\text{iso}})| \leq 2d|\mathcal{B}^{+\text{iso}}| \), we conclude that \( L + 2dM \geq \text{cd}^2k \). Therefore, by \textit{Proposition 4.3}

\[
\mathbb{P}(\text{diam}^* \mathcal{B} \geq k) \leq 2^{|V|} \sum_{L,M,N \geq 0 \atop L+2dM \geq \text{cd}^2k} \exp \left( -\frac{c}{q(q+\log d)} \left( \frac{L}{q} + \frac{M}{q} + \frac{N}{q^2} \right) \right).
\]

Using \[1\], the desired inequality follows.

For the proof of \textit{Theorem 1.2}, we also require a corollary of \textit{Proposition 8.2} for violations of the boundary pattern in a pair of proper colorings. Given two proper colorings \( f \) and \( f' \) of \( \mathbb{Z}^d \), define \( \mathcal{B}_P(f,f',u) \) to be the \((\mathbb{Z}^d)^{\otimes 2}\)-connected component of \( u \) in \((\bar{Z}_P(f) \cap \bar{Z}_P(f'))^c\).

**Corollary 8.4.** Let \( \Lambda \) and \( \Lambda' \) be two domains and \( f \sim \mathbb{P}_{\Lambda,P} \) and \( f' \sim \mathbb{P}_{\Lambda',P} \) be independent. Then

\[
\mathbb{P}(\text{diam} \mathcal{B}_P(f,f',u) \geq r) \leq e^{-\frac{crd}{q(q+\log d)}} \text{ for any } r \geq 1 \text{ and } u \in \mathbb{Z}^d.
\]

For the proof of \textit{Corollary 8.4}, we require the following simple adaptation of \[12\] \textit{Lemma 6.9}. 

Lemma 8.5. Let \( U, V \subset \mathbb{Z}^d \) be finite and assume that \( U \cup V \) is \((\mathbb{Z}^d)^{\otimes 2}\)-connected. Then for any \( u, v \in U \cup V \) there exists a path \( p \) from \( u \) to \( v \) of length at most \( \text{diam}^* U_p + \text{diam}^* V_p \), where \( U_p \) and \( V_p \) are the union of \((\mathbb{Z}^d)^{\otimes 2}\)-connected components of \( U \) and \( V \) which intersect \( p \).

Proof. Let \( W \) be the collection of \((\mathbb{Z}^d)^{\otimes 2}\)-connected components of \( U \) and \( V \). Consider the graph \( G \) on vertex set \( W \) in which \( W, W' \in W \) are adjacent if and only if \( \text{dist}(W, W') \leq 2 \). Note that \( G \) is connected. Consider a simple path \( q = (W_1, \ldots, W_k) \) in \( G \), where \( u \in W_1 \) and \( v \in W_k \). For each \( 1 \leq i \leq k - 1 \), let \( u_i \in W_i \) and \( v_i \in W_{i+1} \) be such that \( \text{dist}(u_i, v_i) \leq 2 \). Let \( p \) be a path from \( u \) to \( v \) constructed by connecting \( v_{i-1} \) to \( u_i \) by a shortest-path for every \( 1 \leq i \leq k \) (where we set \( v_0 := u \) and \( u_k := v \)), and \( u_i \) to \( v_i \) by at most one other vertex for every \( 1 \leq i \leq k - 1 \). Then the length of \( p \) is at most \( \sum_{i=1}^k (\text{diam} W_i + 2) \). On the other hand, \( \text{diam}^* U_p + \text{diam}^* V_p \geq \sum_{i=1}^k (\text{diam} W_i + 2) \), and the lemma follows.

Proof of Corollary 8.4. Denote \( B := B_p(f, f', u) \) and suppose that \( \text{diam} B \geq r \). Let \( v \in B \) be such that \( \text{dist}(u, v) \geq r/2 \). Note that \( B \) is contained in \( B(f, \mathbb{Z}^d) \cup B(f', \mathbb{Z}^d) \). By Lemma 8.5 applied to \( B \cap B(f, \mathbb{Z}^d) \) and \( B \cap B(f', \mathbb{Z}^d) \), there exists a path \( p \) from \( u \) to \( v \) of length \( s \leq \text{diam} B(f, p) + \text{diam}^* B(f', p) \). In particular, \( t := \max\{\text{diam}^* B(f, p), \text{diam}^* B(f', p)\} \) is at least \( s/2 \). Thus, by a union bound on the choices of \( p \) and on \( t \), Proposition 8.2 and (1),

\[
\mathbb{P}(\text{diam} B \geq r) \leq \sum_{t=\lceil r/4 \rceil} 2(2d)^{2t} e^{-\frac{4t}{q^2(2d)}} \leq e^{-\frac{\frac{4r}{q^2(2d)}}{q^2(2d)}}. 
\]

\[\square\]

8.2. The \( P \)-pattern Gibbs state. In this section, we fix a dominant pattern \( P \) and prove that \( \mathbb{P}_{\Lambda, P} \) converges as \( \Lambda \uparrow \mathbb{Z}^d \) to an infinite-volume Gibbs state \( \mu_P \) that satisfies a mixing property called quite weak Bernoulli with exponential rate (abbreviated as QWBE; see, e.g., [3]), which, in particular, implies that \( \mu_P \) is ergodic, strongly mixing and extremal. This is the content of the following two lemmas.

Lemma 8.6. Let \( \Lambda \) and \( \Lambda' \) be two domains. Let \( r \geq 1 \) and let \( U \) be a domain such that \( U^{+r} \subset \Lambda \cap \Lambda' \). Then

\[
d_{\text{TV}}(\mathbb{P}_{\Lambda, P}|U, \mathbb{P}_{\Lambda', P}|U) \leq |U| \cdot e^{-\frac{c_{\text{dr}}}{q^2(2d)}}. 
\]

Lemma 8.7. Let \( \Lambda \) be a domain, let \( V \subset \Lambda \) be a domain, let \( r \geq 1 \) and let \( U \subset \Lambda \) be such that \( U^{+2r} \subset V \). Then

\[
d_{\text{TV}}(\mathbb{P}_{\Lambda, P}|U \cup (\Lambda \setminus V), \mathbb{P}_{\Lambda, P}|U \times \mathbb{P}_{\Lambda, P}|\Lambda \setminus V) \leq |U| \cdot e^{-\frac{c_{\text{dr}}}{q^2(2d)}}. 
\]

Lemma 8.6 easily implies that the finite-volume \( P \)-pattern measures converge to an infinite-volume Gibbs state \( \mu_P \). Indeed, if \( \Lambda_n \) is a sequence of domains increasing to \( \mathbb{Z}^d \), then for any domain \( U \), \( \text{dist}(U, \Lambda_n^c) \to \infty \) as \( n \to \infty \), so that Lemma 8.6 implies that the sequence of measures \((\mathbb{P}_{\Lambda_n, P}|U)_{n=1}^\infty \) is a Cauchy sequence with respect to the total-variation metric, and therefore, converges. This establishes the convergence of \( \mathbb{P}_{\Lambda_n, P} \) as \( n \to \infty \) towards an infinite-volume measure \( \mu_P \) and it is standard that such a limit is a Gibbs state. Since this holds for any such sequence \( \Lambda_n \), it follows that \( \mu_P \) is invariant to all automorphisms preserving the two sublattices. Lemma 8.7 then shows that \( \mu_P \) is QWBE. It is also not hard to see that Lemma 8.7 implies that \( \mu_P \) is ergodic, strongly mixing and extremal (within the set of all Gibbs states). As this is fairly standard and straightforward, we do not elaborate further on this (see [28]). Noting that [3] implies that different \( P \) yield different measures \( \mu_P \), Theorem 1.2 will follow once we show that \( \mu_P \) is of maximal entropy. We postpone this part to Section 8.3 (see Proposition 8.11).

The proofs of Lemma 8.6 and Lemma 8.7 make use of the following fact which exploits the domain Markov property of the model. We say that a collection \( S \) of proper subsets of \( \mathbb{Z}^d \) is a boundary semi-lattice if for any \( S_1, S_2 \in S \) there exists \( S \in S \) such that \( S_1 \cup S_2 \subset S \) and \( \partial S \subset \partial S_1 \cup \partial S_2 \). Two boundary semi-lattices which we require are \( S(U, V) := \{ S \subset \mathbb{Z}^d : U \subset S \subset V \} \).
and $S(f, P) := \{ S \subseteq \mathbb{Z}^d : \partial_x S \text{ is in the } P\text{-pattern with respect to } f \}$. The latter has the property that if $S$ is any boundary semi-lattice, then $S \cap S(f, P)$ is also a boundary semi-lattice.

**Lemma 8.8.** Let $\Lambda, \Lambda' \subseteq \mathbb{Z}^d$ be finite and let $U \subset V \subset \Lambda \cap \Lambda'$ be non-empty. Let $f \sim \mathbb{P}_{\Lambda, P}$ and $f' \sim \mathbb{P}_{\Lambda', P}$ be independent.

(a) $d_{TV}(\mathbb{P}_{\Lambda, P}|_U, \mathbb{P}_{\Lambda', P}|_U) \leq \mathbb{P}(S(U, V) \cap S(f, P) \cap S(f', P) = \emptyset)$.

(b) Assume that $U$ is connected, $V$ is co-connected and $\mathbb{P}(S(U, V) \cap S(f, P) \neq \emptyset) > 0$. Then, conditioned on $\{ S(U, V) \cap S(f, P) \neq \emptyset \}$, the distribution of $f|_U$ is a convex combination of the measures $\{ \mathbb{P}_S|_U \}_{S \in S_{\text{dom}}(U, V)}$, where $S_{\text{dom}}(U, V)$ is the collection of domains in $S(U, V)$.

**Proof.** We shall prove both items together. To this end, let $f''$ be either $f$ or $f'$, and denote $S := S(U, V) \cap S(f, P) \cap S(f'', P)$. Since $S$ is a finite boundary semi-lattice, it has a unique maximal element $S$ (if $S = \emptyset$, we set $S := \emptyset$). Let $S \neq \emptyset$ be such that $\mathbb{P}(S = S) > 0$. Observe that the event $\{ S = S \}$ is determined by $f|_{(S^c)^+}$ and $f''|_{(S^c)^+}$. Therefore, by the domain Markov property, conditioned on $\{ S = S \}$, $f|_S$ and $f''|_S$ are distributed according to $\mathbb{P}_S|_S$. In particular, conditioned on $\{ S \neq \emptyset \}$, the distribution of both $f|_U$ and $f''|_U$ is $\sum_{S} \mathbb{P}(S = S | S \neq \emptyset) \mathbb{P}_S|_S$, from which the first item follows. Moreover, if $U$ is connected and $V$ is co-connected, then $S$ is always a domain, since Lemma 3.2(a) and Lemma 3.4(d) imply that the co-connected closure of $S$ (with respect to infinity) belongs to $S$ for any $S \in S$. Hence, the second item also follows. \hfill $\square$

We are now ready to prove Lemma 8.6 and Lemma 8.7.

**Proof of Lemma 8.6.** Denote $S := U \cup \bigcup_{u \in \partial_x S} B_p(f, f', u)^+$ and observe that, by definition, $\partial_x S$ is in the $P$-pattern with respect to both $f$ and $f'$. Let $\mathcal{E}$ be the event that $S$ intersects $(U^r)^c$, so that $S \subset U^r$ on the complement of $\mathcal{E}$. Then, by Lemma 8.8 and Corollary 8.4

$$d_{TV}(\mathbb{P}_{\Lambda, P}|_U, \mathbb{P}_{\Lambda', P}|_U) \leq \mathbb{P}(\mathcal{E}) \leq \sum_{u \in U} \mathbb{P}(\text{diam } B_p(f, f', u) \geq r) \leq |U| \cdot e^{-\frac{cdr}{q^3(4q \log d)}}.$$ \hfill $\square$

**Proof of Lemma 8.7.** We begin with a simple observation. Let $X$ and $Y$ be discrete random variables and let $\mu_{X|Y}$ denote the conditional (random) distribution of $X$ given $Y$. Then

$$d_{TV}(\mu_{(X,Y)}, \mu_X \times \mu_Y) \leq \mathbb{E}[d_{TV}(\mu_{X|Y}, \mu_X)],$$

where we write $\mu_Z$ for the distribution of a random variable $Z$. Indeed, this follows immediately from the fact that

$$\text{Cov} \{ \{ X \in A \}, \{ Y \in B \} \} = \mathbb{E}[(\mu_{X|Y}(A) - \mu_X(A))\mathbb{1}_{\{ Y \in B \}}].$$

Let $\mu$ be the conditional (random) distribution of $f|_U$ given $f|_{V^c}$. Let $\mathcal{E}'$ be the event that there exists a set $S$ such that $U^r \subset S \subset V$ and such that $\partial_x S$ is in the $P$-pattern. By Lemma 8.8, conditioned on $\mathcal{E}'$, $\mu$ is a convex combination of measures $\mathbb{P}_S|_U$, where $S$ is a domain containing $U^r$. For any such $S$, by Lemma 8.6, we have

$$d_{TV}(\mathbb{P}_S|_U, \mathbb{P}_{\Lambda, P}|_U) \leq |U| \cdot e^{-\frac{cdr}{q^3(4q \log d)}}.$$ Let $\mathcal{E}$ be the event that $B_p(f, u)^+$ intersects $V^c$ for some $u \in U^r$, and note that $\mathcal{E}^c \subset \mathcal{E}'$. Hence,

$$\mathbb{E}[d_{TV}(\mu, \mathbb{P}_{\Lambda, P}|_U)] \leq |U| \cdot e^{-\frac{cdr}{q^3(4q \log d)}} + \mathbb{E}[\mu(\mathcal{E})].$$

By Proposition 8.2

$$\mathbb{E}[\mu(\mathcal{E})] = \mathbb{P}(\mathcal{E}) \leq |U^r| \cdot e^{-\frac{cdr}{q^3(4q \log d)}} \leq |U| \cdot (Cd)^r \cdot e^{-\frac{cdr}{q^3(4q \log d)}} \leq |U| \cdot e^{-\frac{cdr}{q^3(4q \log d)}}.$$ Thus, $\mathbb{E}[d_{TV}(\mu, \mathbb{P}_{\Lambda, P}|_U)] \leq |U| \cdot e^{-\frac{cdr}{q^3(4q \log d)}}$, and the lemma follows from the above observation. \hfill $\square$
8.3. The maximal-entropy Gibbs states. The purpose of this section is to characterize all maximal-entropy Gibbs states. Let us begin by defining the relevant notions. Let \( \mu \) be a probability measure on \( [q]^d \). Given a transformation \( T: \mathbb{Z}^d \to \mathbb{Z}^d \), we say that \( \mu \) is \( T \)-invariant if \( \mu(T^{-1} A) = \mu(A) \) for any measurable event \( A \). We say that \( \mu \) is periodic if it is \( \Gamma \)-invariant for a (full-dimensional) lattice \( \Gamma \) of translations of \( \mathbb{Z}^d \). Observe that every periodic measure \( \mu \) is \((N\mathbb{Z}^d)\)-invariant for some positive integer \( N \).

To define the notion of a maximal-entropy Gibbs state, we first require some other definitions. Let \( \Omega^\text{free}_\Lambda \) be the set of proper colorings of \( \Lambda \). The topological entropy of proper colorings is the exponential rate of growth of the number of proper colorings, i.e.,

\[
    h_{\text{top}} := \lim_{n \to \infty} \frac{\log |\Omega^\text{free}_n|}{n^d}.
\]

The above limit exists by subadditivity. Note also that \( \frac{1}{2} \log(\frac{q}{2} \left\lfloor \frac{q}{2} \right\rfloor) \) is a trivial lower bound on \( h_{\text{top}} \). Let \( \mu \) be a periodic measure which is supported on proper \( q \)-colorings of \( \mathbb{Z}^d \). The measure-theoretic entropy (also known as Kolmogorov–Sinai entropy) of \( \mu \) is

\[
    h(\mu) := \lim_{n \to \infty} \frac{\text{Ent}(\mu|\Lambda_n)}{|\Lambda_n|}, \quad \text{where } \Lambda_n := \{0, 1, \ldots, n\}^d,
\]

which exists by subadditivity (see [28, Theorem 5.12]). Using [11], one easily checks that \( h(\mu) \leq h_{\text{top}} \). The variational principle tells us that equality is achieved by some \( \mu \). Such a \( \mu \) is said to be of maximal entropy. A theorem of Lanford–Ruelle (see, e.g., [46]) tells us that every measure of maximal entropy is also a Gibbs state (so that there is some redundancy when speaking about a maximal-entropy Gibbs state). We stress that a measure of maximal entropy is, by definition, always assumed to be periodic.

Before proceeding with the proof of Theorem 1.3 let us give a simple consequence of our results to the enumeration of proper colorings. Using the sub-additivity of entropy [13], it is straightforward to see that Theorem 1.1 together with the fact that \( \mu_P \) is of maximal entropy, implies that, when (1) holds, the topological entropy is bounded by

\[
    h_{\text{top}} \leq \frac{1}{2} \log(\frac{q}{2} \left\lfloor \frac{q}{2} \right\rfloor) + e^{-\frac{cd}{q^3(q + \log d)}}.
\]

Galvin–Tetali [25] showed a weaker bound of this form (where the exponential correction term is replaced by a term of order \( \frac{1}{d} \)) on any bipartite regular graph (in which case their bound is of the correct order). Using either bound, we see that \( h_{\text{top}} \to \frac{1}{2} \log(\frac{q}{2} \left\lfloor \frac{q}{2} \right\rfloor) \) as \( d \to \infty \). An analogue of this for isotropic subshifts was shown by Meyerovitch–Pavlov [45].

Let us come back to the proof of Theorem 1.3. We wish to show that the \( P \)-pattern Gibbs states are the only extremal maximal-entropy measures. Our technique is inspired by the work of Gallavotti and Miracle-Solé [17] on the translation-invariant Gibbs states of the low-temperature Ising model. In order to allow ourselves to appeal directly to Proposition 8.1 in the proof (instead of repeating similar arguments), we first show that proper colorings with periodic boundary conditions may be extended to \( P \)-pattern boundary conditions.

A proper coloring \( f \) of \( \{-n, \ldots, n\}^{d-1} \) is symmetric if \( f(x_1, \ldots, x_{d-1}) = f(|x_1|, \ldots, |x_{d-1}|) \) for all \( x \in \{-n, \ldots, n\}^{d-1} \). A proper coloring of \( U \subset \mathbb{Z}^{d-1} \) is \( n \)-periodic if \( f(x) \) depends only on \( (x_1 \mod n, \ldots, x_{d-1} \mod n) \) for \( x \in U \). A proper coloring of \( \{-kn, \ldots, kn\}^{d-1} \) is \( n \)-symmetric if it is \( 2n \)-periodic and its restriction to \( \{-n, \ldots, n\}^{d-1} \) is symmetric. A proper coloring of \( \Lambda_{2kn} \) is \( n \)-symmetric if its restriction to any of the \( 2d \) faces is \( n \)-symmetric (after an appropriate translation). Finally, a proper coloring of \( U \) has \( (a, b) \)-boundary conditions if the even vertices in \( \partial_a U \) take the value \( a \) and the odd ones take \( b \).

**Lemma 8.9.** Any \( n \)-symmetric proper coloring \( f \) of \( \Lambda_{2kn} \) can be extended to a proper coloring of \( (\Lambda_{2kn})^{+dn} \) having \( (a, b) \)-boundary conditions, where \( a := f(0, \ldots, 0) \) and \( b := f(1, 0, \ldots, 0) \).
Proof. Let \( K_q \) be the complete graph on \([q]\). Say that two paths \( p = (p_m)_{m \geq 0} \) and \( q = (q_m)_{m \geq 0} \) in \( K_q \) are adjacent if \( p_m \neq q_m \) for all \( m \geq 0 \). Denote \( \Lambda := \Lambda_{2kn} \). Let \( (p^u)_{u \in \partial \Lambda} \) be a family of paths such that \( p^u \) and \( p^v \) are adjacent whenever \( u \sim v \) and such that \( p^u_0 = f(u) \) for every \( u \in \partial \Lambda \). Observe that every \( x \in \mathbb{Z}^d \) has a unique \( u(x) \in \Lambda \) closest to \( x \) and that \( \text{dist}(u(x), u(y)) \leq \text{dist}(x, y) \). In particular, if \( x \sim y \) then either \( \text{dist}(u(x), x) = \text{dist}(u(y), y) \) and \( u(x) \sim u(y) \) or \( \text{dist}(u(x), x) = \text{dist}(u(y), y) + 1 \) and \( u(x) = u(y) \). Hence, defining \( g: \mathbb{Z}^d \to [q] \) by

\[
g(x) := \begin{cases} f(x) & \text{if } x \in \Lambda \\ p_{\text{dist}(u(x),x)} & \text{if } x \notin \Lambda \end{cases}
\]

we have that \( g(x) \neq g(y) \) whenever \( x \sim y \). Thus, \( g \) is a proper coloring of \( \mathbb{Z}^d \) which extends \( f \). To conclude, it suffices to show the existence of such a family of paths \( (p^u)_{u \in \partial \Lambda} \) which also satisfies that

for every \( u \in \partial \Lambda \) there exists \( 0 \leq m \leq dn \) such that \( (p^u_m, p^u_{m+1}, \ldots) = (a, b, a, b, \ldots) \). (56)

Indeed, the lemma will then follow as \( g|_{\Lambda^{4+dn}} \) has \((a,b)\)-boundary conditions. To construct such a family, we first define \( p^u \) for \( u \in \Lambda' := \Lambda_n \cap \partial \Lambda \) by

\[
p^u := (f(u), f(Su), f(S^2u), \ldots, f(S^{\ell_u}u), b, a, b, a, \ldots),
\]

where \( S: \Lambda' \setminus \{0\} \to \partial \Lambda \) is the lexicographical successor operator defined by \( Su := u - e_j \), where \( j := \min\{i \geq 1 : u_i > 0\} \), and \( \ell_u := \min\{\ell \geq 0 : S^\ell u = 0\} \). Note that \( \ell_u = |u| := \sum_{1 \leq i \leq d} |u_i| \leq dn \) so that \( (56) \) holds for all \( u \in \Lambda' \). For \( u \in \partial \Lambda \setminus \Lambda' \), define \( p^u := p^{(\ell_1|\ldots|\ell_d)} \), where \( \ell_i \) is uniquely determined by writing \( u_i = 2k_i + r_i \) for \( k_i \in \{0, 1, \ldots, k\} \) and \( r_i \in \{-n+1, \ldots, n\} \). It is easy to see that \( p^u \) and \( p^v \) are adjacent whenever \( u, v \in \partial \Lambda \) are adjacent. It remains to check that \( p^u_0 = f(0) \) for all \( u \in \Lambda_0 \). For \( u \in \Lambda_0 \) this follows from the definition, and in general, this holds since the fact that \( f \) is n-symmetric implies that \( f(u) \) depends only on \((|r_1|, \ldots, |r_d|)\), where \( r_i \) is defined as before.

Recall the definition of \( Z_*(f) \) from \([8]\).

**Lemma 8.10.** Assume that \([11]\) holds and suppose that \( f \) is sampled according to some (periodic) measure of maximal entropy. Then \( Z_*(f) \) almost surely has no infinite \((\mathbb{Z}^d)^{\otimes 2}\)-connected component.

**Proof.** Let \( \mu \) be a measure of maximal entropy and let \( f \) be sampled according to \( \mu \). Denote the lattice of \( \mu \)-preserving translations by \( \Gamma \). We call the elements of \( Z_* \) interface vertices. For a vertex \( u \), let \( E_u \) be the event that \( u \) belongs to an infinite \((\mathbb{Z}^d)^{\otimes 2}\)-path of interface vertices. Since \( \mu \) is \( \Gamma \)-periodic, \( \mu(E_u) \) depends only on the \( \Gamma \)-equivalence class \([u]\) of \( u \). Assume towards a contradiction that \( \mu(E_u) > \delta \) for some \( u \) and \( \delta > 0 \). By ergodic decomposition, we may assume that \( \mu \) is ergodic with respect to the \( \Gamma \)-action. Then by the ergodic theorem, the density of the set of vertices \( v \in [u] \) for which \( E_v \) occurs is \( \mu(E_u) \) almost surely. In particular, \( \mu(E_n) \to 1 \) as \( n \to \infty \), where \( E_n \) is the event that at least a \( \delta \)-proportion of vertices in \( \Lambda_n \) are connected to \( (\partial \Lambda_n)^{+4} \) by a \((\mathbb{Z}^d)^{\otimes 2}\)-path of interface vertices in \( \Lambda_n \setminus (\partial \Lambda_n)^{+4} \). Note that the event that a vertex \( v \) is an interface vertex is measurable with respect to the values of \( f \) on \( v^{+3} \), and thus, \( E_n \) is measurable with respect to the values of \( f \) on \( \Lambda_n \) so that we may regard it as a collection of proper colorings of \( \Lambda_n \).

Denote by \( \Omega_{\Lambda}^{\tau,B} \) the set of proper colorings of \( \Lambda \) that agree with \( \tau \) on \( B \). Denote also \( \Omega_{\Lambda}^{\tau} := \Omega_{\Lambda}^{\tau,\partial \Lambda} \).

Then, using \([10] \otimes [13]\),

\[
\text{Ent}(f|_{\Lambda_n}) \leq \text{Ent}(f|_{\partial \Lambda_n}) + \text{Ent}(E_n) + \mu(E_n) \cdot \log |\Omega_{\Lambda_n}^{\text{free}}| + \max_{\tau \in [q]} \log |\Omega_{\Lambda_n}^{\tau} \cap E_n|.
\]

In particular, there exists a fixed (deterministic) boundary condition \( \tau \in [q]^{\mathbb{Z}^d} \) such that

\[
\frac{\log |\Omega_{\Lambda_n}^{\tau} \cap E_n|}{|\Lambda_n|} \to h(\mu) \quad \text{as } n \to \infty.
\]
(a) Boundary conditions on a box of side-length $n$ are reflected to obtain boundary conditions on a box of side-length $2n$ (more precisely, on the union of the boundaries of the $2^d$ boxes of side-length $n$).

(b) Many translated copies of $\Lambda_{2n}$ with a suitably chosen boundary condition are placed within a slightly larger box $U_{n,k}$. Connected components of $Z_\ast$ are depicted, with shaded regions representing components that intersect $B_{n,k}$. Proposition 8.1 implies that the total area covered by the latter is typically not large when $f$ is sampled from a measure of maximal entropy.

**Figure 5.** Excluding the possibility of infinite components of $Z_\ast$.

Using the assumption that $\mu$ has maximal entropy, we shall show that this is impossible.

The first step is to magnify the effect at a given scale $n$ by replicating it many times. Namely, we take the model in domain $\Lambda_n$ with $\tau$ boundary conditions, and duplicate it to obtain a model in domain $\Lambda_{2kn}$, with each of the $(2k)^d$ shifted copies of the smaller box $\Lambda_n$ having the same boundary conditions (up to reflections). Indeed, by reflecting $\tau$ along the sides of the box $\Lambda_n$ some $2k - 1$ number of times in each coordinate direction, we get boundary conditions $\tau_{n,k}$ defined on $B_{n,k} := n\{0, 1, \ldots, 2k - 1\}^d + \partial \Lambda_n$. Let $E_{n,k}$ denote the event that at least a $\delta$-proportion of vertices in $\Lambda_{2nk}$ are connected to $B_{n,k} + 4n$ by a $(\mathbb{Z}^d \otimes 2)$-path of interface vertices in $\Lambda_{2nk} \setminus (\partial \Lambda_{2nk})^+$. With a slight abuse of notation, we regard $E_{n,k}$ below as a collection of proper colorings of either $\Lambda_{2kn}$ or $U_{n,k} := \{-dn, \ldots, 2kn + dn\}^d$, according to the context. Then

$$\log |\Omega_{\Lambda_{2kn}}^{\tau_{n,k}, B_{n,k} \cap E_{n,k}}| \geq (2k)^d \cdot \log |\Omega_{\Lambda_n}^{\tau_n \cap E_n}| = h(\mu) - o(1) \quad \text{as } n \to \infty.$$ 

By Lemma 8.9 each proper coloring of $\Lambda_{2kn}$ having $\tau_{n,k}$ boundary conditions can be extended to a proper coloring of $U_{n,k}$ having $(a, b)$-boundary conditions for some $a \neq b$ depending only on $\tau_{n,k}$. Thus, letting $P$ be a dominant pattern extending $\{(a), \{b\}\}$ and letting $\Omega_P^\Lambda$ be the set of proper colorings of $\Lambda$ for which $\partial \Lambda$ is in the $P$-pattern, we have

$$\mathbb{P}_{U_{n,k}, P}(E_{n,k}) \cdot |\Omega_P^{U_{n,k}}| = |\Omega_P^{U_{n,k} \cap E_{n,k}}| \geq |\Omega_{\Lambda_{2kn}}^{\tau_{n,k}, B_{n,k} \cap E_{n,k}}|.$$ 

On the other hand,

$$\log |\Omega_{U_{n,k}}^P| - \log |\Omega_{\Lambda_{2kn}}^{\text{free}}| \leq \log |\Omega_{U_{n,k} \setminus \Lambda_{2kn}}^{\text{free}}| \leq |U_{n,k} \setminus \Lambda_{2kn}| \log q \leq C_{d, q} n^{d(d-1)}$$

so that

$$h(\mu) \leq \frac{\log |\Omega_{\Lambda_{2kn}}^{\text{free}}|}{|\Lambda_{2kn}|} + C_{d, q} \frac{\log \mathbb{P}_{U_{n,k}, P}(E_{n,k})}{|\Lambda_{2kn}|} + o(1) \quad \text{as } n \to \infty.$$
Thus, since $\mu$ has maximal entropy, we will arrive at a contradiction if
\[
\limsup_{k \to \infty} \limsup_{n \to \infty} \log \frac{\mathbb{P}_{U_{n,k}, \mathcal{P}}(\mathcal{E}_{n,k})}{|A_{2kn}|} < 0.
\]
This follows from Proposition 8.1 as it implies that
\[
\mathbb{P}_{U_{n,k}, \mathcal{P}}(\mathcal{E}_{n,k}) \leq 2^{C_{\mathcal{P}} d^d n^d - 1} \cdot e^{-c_{\mathcal{P}} \delta kn^d}.
\]

**Proposition 8.11.** Assume that \( \mathbb{P}_1 \) holds. Then every (periodic) measure of maximal entropy is a mixture of the $\mathcal{P}$-pattern Gibbs states.

**Proof.** Let $f$ be sampled according to a Gibbs state $\mu$ under which $Z_*(f)$ almost surely has no infinite $(\mathbb{Z}^d)^{\otimes 2}$-connected components. In light of Lemma 8.10, it suffices to show that such a measure $\mu$ is a mixture of the $\mathcal{P}$-pattern Gibbs states.

Let $U \subseteq \mathbb{Z}^d$ be finite and connected. Let us show that, almost surely, there exists a dominant pattern $\mathcal{P}$ and a finite set $V$ containing $U$ such that $(\partial_\mathcal{P} V)^+$ is in the $\mathcal{P}$-pattern. Indeed, if we let $W$ denote the $(\mathbb{Z}^d)^{\otimes 2}$-connected component of $U \cup Z_*$ containing $U$, then $W$ is almost surely finite. Thus, if $V$ denotes the co-connected closure of $W^+$ with respect to infinity, then $V$ is finite, connected, co-connected and contains $U$. Since $\partial_\mathcal{P} W$ is connected by Lemma 3.3 and is contained in $\partial_\mathcal{P} W^+ = W^{+2} \setminus W$, which is disjoint from $Z_*$, it follows from the definition of $Z_*$ that $(\partial_\mathcal{P} V)^+$ is in the $\mathcal{P}$-pattern for some $\mathcal{P}$.

Now consider the boxes $U_n := \{-n, \ldots, n\}^d$ and let $P_n$ and $V_n$ be as above. For a dominant pattern $\mathcal{P}$, let $\mathcal{E}_\mathcal{P}$ be the event that $\{ n : P_n = \mathcal{P} \}$ is infinite. As there are finitely many dominant patterns, $\bigcup_{\mathcal{P}} \mathcal{E}_\mathcal{P}$ occurs almost surely. By a similar argument as in the proof of Lemma 8.8, and using the fact that the finite-volume $\mathcal{P}$-pattern measures converge, it follows that $\mu(\cdot \mid \mathcal{E}_\mathcal{P})$ is precisely the $\mathcal{P}$-pattern Gibbs state $\mu_\mathcal{P}$. Thus, the events $\{\mathcal{E}_\mathcal{P}\}_\mathcal{P}$ are disjoint and $\mu$ is the mixture $\sum_\mathcal{P} \mu(\mathcal{E}_\mathcal{P}) \mu_\mathcal{P}$. \qed

**References**

[1] PN Balister and B Bollobás, *Counting regions with bounded surface area*, Communications in mathematical physics 273 (2007), no. 2, 305–315.

[2] Jayanth R. Banavar, Gary S. Grest, and David Jasnow, *Ordering and phase transitions in antiferromagnetic Potts models*, Physical Review Letters 45 (1980), no. 17, 1424–1428.

[3] AN Berker and Leo P Kadanoff, *Ground-state entropy and algebraic order at low temperatures*, Communications in mathematical physics 273 (2007), no. 2, 305–315.

[4] Béla Bollobás, *The art of mathematics: Coffee time in memphis*, Cambridge University Press, 2006.

[5] Graham R Brightwell and Peter Winkler, *Random colorings of a Cayley tree*, Contemporary combinatorics 10 (2002), 247–276.

[6] Robert M Burton and Jeffrey E Steif, *Quite weak Bernoulli with exponential rate and percolation for random fields*, Stochastic processes and their applications 98 (1995), no. 1, 35–55.

[7] Fan RK Chung, Ronald L Graham, Peter Frankl, and James B Shearer, *Some intersection theorems for ordered sets and graphs*, Journal of Combinatorial Theory, Series A 43 (1986), no. 1, 23–37.

[8] Roland L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theor. Probab. Appl. 13 (1968), 197–224.

[9] Roland L’vovich Dobrushin, *The problem of uniqueness of a Gibbsian random field and the problem of phase transitions*, Functional analysis and its applications 2 (1968), no. 4, 302–312.

[10] John Engbers and David Galvin, *H-coloring tori*, Journal of Combinatorial Theory, Series B 102 (2012), no. 5, 1110–1133.

[11] John Engbers and David Galvin, *H-colouring bipartite graphs*, Journal of Combinatorial Theory, Series B 102 (2012), no. 3, 726–742.

[12] Ohad Feldheim and Yinon Spinka, *Long-range order in the 3-state antiferromagnetic Potts model in high dimensions*, arXiv preprint arXiv:1511.07877, to appear in Journal of the European Mathematical Society (2015).

[13] Ohad N. Feldheim and Ron Peled, *Rigidity of 3-colorings of the discrete torus*, arXiv preprint arXiv:1309.2340, to appear in Annals de l’Institut Henri Poincaré (B) (2013).

[14] Ohad Noy Feldheim and Yinon Spinka, *The growth constant of odd cutsets in high dimensions*, Combinatorics, Probability and Computing (2017), 1–20.
RUDIGITY OF PROPER COLORINGS OF $\mathbb{Z}^d$

[47] Rudolf Peierls, *On Ising’s model of ferromagnetism*, Mathematical proceedings of the cambridge philosophical society, 1936, pp. 477–481.

[48] Ron Peled, *High-dimensional Lipschitz functions are typically flat*, The Annals of Probability 45 (2017), no. 3, 1351–1447.

[49] Ron Peled and Wojciech Samotij, *Odd cutsets and the hard-core model on $\mathbb{Z}^d$*, Annales de l’institut henri poincaré, probabilités et statistiques, 2014, pp. 975–998.

[50] Ron Peled and Yinon Spinka, *A condition for long-range order in discrete spin systems with application to the antiferromagnetic Potts model*, arXiv preprint arXiv:1712.03699 (2017).

[51] Ron Peled and Yinon Spinka, *Long-range order in discrete spin systems*, in preparation (2018).

[52] Sergey Anatol’evich Pirogov and Ya G Sinai, *Phase diagrams of classical lattice systems*, Theoretical and Mathematical Physics 25 (1975), no. 3, 1185–1192.

[53] Sergey Anatol’evich Pirogov and Ya G Sinai, *Phase diagrams of classical lattice systems continuation*, Theoretical and Mathematical Physics 26 (1976), no. 1, 39–49.

[54] MP Qin, QN Chen, ZY Xie, J Chen, JF Yu, HH Zhao, B Normand, and T Xiang, *Partial long-range order in antiferromagnetic Potts models*, Physical Review B 90 (2014), no. 14, 144424.

[55] Shafiqur Rahman, Eric Rush, and Robert H Swendsen, *Intermediate-temperature ordering in a three-state antiferromagnetic Potts model*, Physical Review B 58 (1998), no. 14, 9125.

[56] Jesús Salas and Alan D Sokal, *Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem*, Journal of Statistical Physics 86 (1997), no. 3, 551–579.

[57] Aleksandr Antonovich Sapozhenko, *The number of antichains in ranked partially ordered sets*, Diskretnaya Matematika 1 (1989), no. 1, 74–93.

[58] Alexander. A. Sapozhenko, *On the number of connected subsets with given cardinality of the boundary in bipartite graphs*, Metody Diskretnogo Analiza (Russian) 45 (1987).

[59] Alexander A. Sapozhenko, *On the number of antichains in multilevelled ranked posets*, Discrete Math. Appl. 1 (1991), no. 2, 149–170.

[60] Gordon Slade, *Lattice trees, percolation and super-Brownian motion*, Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten, Basel (1999).

[61] Gordon Slade, *The lace expansion and its applications: Ecole d’été de probabilités de Saint-Flour xxxiv-2004*, Springer, 2006.

[62] Ádám Timár, *Boundary-connectivity via graph theory*, Proceedings of the American Mathematical Society 141 (2013), no. 2, 475–480.

[63] Eric Vigoda, *Improved bounds for sampling colorings*, Journal of Mathematical Physics 41 (2000), no. 3, 1555–1569.

RUDIGITY OF PROPER COLORINGS OF $\mathbb{Z}^d$

Ron Peled
Tel Aviv University, School of Mathematical Sciences, Tel Aviv, 69978, Israel.
E-mail address: peledron@post.tau.ac.il
URL: http://www.math.tau.ac.il/~peledron

Yinon Spinka
Tel Aviv University, School of Mathematical Sciences, Tel Aviv, 69978, Israel.
E-mail address: yinonspi@post.tau.ac.il
URL: http://www.math.tau.ac.il/~yinonspi