Nonlinear neutral inclusions: assemblages of coated ellipsoids

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1. Summary

The problem of determining nonlinear neutral inclusions in (electrical or thermal) conductivity is considered. Neutral inclusions, inserted in a matrix containing a uniform applied electric field, do not disturb the field outside the inclusions. The well-known Hashin-coated sphere construction is an example of a neutral inclusion. In this paper, we consider the problem of constructing neutral inclusions from nonlinear materials. In particular, we discuss assemblages of coated ellipsoids. The proposed construction is neutral for a given applied field.

2. Introduction

A neutral inclusion, when inserted in a matrix containing a uniform applied electric field, does not disturb the outside field. Mansfield was the first to observe that reinforced holes, ‘neutral holes’, could be cut out of a uniformly stressed plate without disturbing the surrounding stress field in the plate [1].

The well-known Hashin-coated sphere construction [2] is an example of a neutral-coated inclusion for the conductivity problem. In Hashin & Shtrikman [3], an exact expression for the effective conductivity of the coated sphere assemblage was found, which coincides with the Maxwell [4] approximate formula. Thus, the approximate formula is realizable and was shown to be an attainable bound for the effective conductivity of a composite, given the volume fractions of the two materials. This construction was extended to coated confocal ellipsoids [5]. Ellipsoids as neutral inclusions have been also studied [6].

Spheres and ellipsoids are not the only possible shapes for neutral inclusions; indeed, in Milton & Serkov [7], other shapes of neutral inclusions are constructed.

The existence of neutral inclusions was also found in the case of materials with imperfect interfaces, for which the potential (or displacement) field has discontinuities across these interfaces. For these materials neutral inclusions have been studied for the
conductivity problem [8,9], for highly conducting interfaces [10,11], for the elasticity problem [12–14] and for nonlinear materials [15]. General neutral inclusions have been constructed by solving a free boundary problem (for example [16–18]).

For other references related to neutral inclusions in composites, see also Milton [19] and Mei & Vernescu [20] and the references therein.

We consider here nonlinear materials for which the constitutive law relating the current $J$ to the electric field $\nabla u$ is described by a nonlinear constitutive model of the form

$$J = \sigma_1|\nabla u|^{p-2} \nabla u,$$

where $u$ is the potential, and $\sigma_1|\nabla u|^{p-2}$ is a nonlinear conductivity. This constitutive model is used to describe the nonlinear behaviour of several materials, including nonlinear dielectrics [21–27], and is also used to model thermorheological and electrorheological fluids [28–30], viscous flows in glaciology [31] and also in plasticity problems [32–36].

In this paper, we show that even for nonlinear materials, one can construct neutral inclusions by a suitable coating with a linear material. In particular, we show that a coated ellipsoid with core of phase 1 (nonlinear material) surrounded by a coating of phase 2 (linear material) can be constructed as a neutral inclusion. In Jiménez et al. [37], we showed that coated spheres with nonlinear core and linear coating can be constructed as neutral inclusions.

Because the equations for conductivity are local equations, one could continue to add similarly aligned coated ellipsoids of various sizes without disturbing the prescribed uniform applied field surrounding the inclusions. In fact, one can fill the entire space (aside from a set of measure zero) with assemblages of these aligned coated ellipsoids by adding coated ellipsoids of various sizes ranging to the infinitesimal and it is assumed that they do not overlap the boundary of the unit cell of periodicity. The ellipsoids can be of any size, but the volume fraction $\theta_1$ (see equation (3.3)) of nonlinear material is the same for all ellipsoids. While adding the coated ellipsoids, the flux of current and electrical potential at the boundary of the unit cell remains unaltered. Therefore, the effective conductivity does not change.

This paper is structured as follows: §3 provides the statement of the problem and the main result for an assemblage of coated ellipsoids and §4 provides the proofs of the statements in §3. Conclusions are given in §5.

3. Assemblage of coated ellipsoids: statement of the problem

We need to introduce ellipsoidal coordinates $\rho$, $\mu$ and $\nu$, which are defined implicitly as the solution of the set of equations [38,39]

$$\begin{align*}
\frac{x_1^2}{\gamma_1^2} + \rho + \frac{x_2^2}{\gamma_2^2} + \frac{x_3^2}{\gamma_3^2} = 1 : \text{confocal ellipsoids} \\
\frac{x_1^2}{\gamma_1^2} + \frac{x_2^2}{\gamma_2^2} + \frac{x_3^2}{\gamma_3^2} = 1 : \text{hyperboloids of one sheet} \\
\frac{x_1^2}{\gamma_1^2} + \frac{x_2^2}{\gamma_2^2} + \frac{x_3^2}{\gamma_3^2} = 1 : \text{hyperboloids of two sheets}
\end{align*}$$

subjected to the restrictions

$$\rho > -\gamma_1^2 > \mu > -\gamma_2^2 > \nu > -\gamma_3^2,$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are fixed positive constants that determine the coordinate system, all confocal with the ellipsoid

$$\frac{x_1^2}{\gamma_1^2} + \frac{x_2^2}{\gamma_2^2} + \frac{x_3^2}{\gamma_3^2} = 1.$$

One surface of each of the three families passes through each point in space, and the three surfaces are orthogonal. The equations can be solved explicitly for the Cartesian coordinates in terms of the ellipsoidal coordinates. For all permutations $j$, $k$, $l$ of 1, 2, 3, we have

$$x_j^2 = \frac{(\gamma_j^2 + \rho)(\gamma_j^2 + \mu)(\gamma_j^2 + \nu)}{(\gamma_j^2 - \gamma_k^2)(\gamma_j^2 - \gamma_l^2)}.$$

(3.1)
The coordinate $\rho$ plays the role that the radius plays in spherical coordinates. Our prototype ellipsoid is defined by the region $\rho < \rho_e$ with a nonlinear core $0 < \rho < \rho_c$ and a linear coating $\rho_c < \rho < \rho_e$ and is given by
\[
\frac{x_1^2}{c_1^2 + \rho} + \frac{x_2^2}{c_2^2 + \rho} + \frac{x_3^2}{c_3^2 + \rho} = 1,
\]
where $c_i^2 + \rho > 0$, $c_i \in \mathbb{R}$, $i = 1, 2, 3$. Within the ellipsoid, the conductivity depends only on the coordinate $\rho$.

We introduce the lengths
\[
l_{ci} = \sqrt{c_i^2 + \rho_c}, \quad l_{ei} = \sqrt{c_i^2 + \rho_e}, \quad j = 1, 2, 3, \tag{3.2}
\]
which represent the semi-axis lengths of the core and exterior surfaces of the coated ellipsoid, the volume fraction
\[
\theta_1 = \frac{l_{c1}l_{c2}l_{c3}}{l_{e1}l_{e2}l_{e3}}, \tag{3.3}
\]
occupied by phase 1 (nonlinear material in the core), and $\theta_2 = 1 - \theta_1$, the volume fraction occupied by phase 2 (linear material in the coating).

The coated ellipsoid is embedded in a medium with isotropic conductivity tensor $\sigma^*_1 \mathbf{I}$, where the value of $\sigma^*_1$ needs to be chosen so that the conductivity equations have a solution with the uniform field aligned in the $x_1$-direction in the region exterior to the ellipsoid. Once this is done, it follows by the usual argument that $\sigma^*_1$ represents the effective conductivity in the $x_1$-direction of the assemblage of aligned ellipsoids, each identical within a scale factor to the given prototype. We apply a linear electric field $\mathbf{E} \cdot \mathbf{x} = E x_1$ at infinity (where for simplicity $\mathbf{E} = E \mathbf{e}^1$, with $\mathbf{e}^1 = (1, 0, 0)$ and $\mathbf{x} = (x_1, x_2, x_3)$).

Thus, the problem of finding a neutral inclusion reduces to finding the electric potential $u$ that solves
\[
\begin{align*}
\nabla \cdot (\sigma_1 |\nabla u|^{p-2}\nabla u) &= 0 \quad \text{in the core} \\
\nabla \cdot (\sigma_2 \nabla u) &= 0 \quad \text{in the coating},
\end{align*}
\]
where the material conductivities are $\sigma_1 |\nabla u|^{p-2}$ in the core, and $\sigma_2$ in the coating, with $\infty > \sigma_1 > \sigma_2 > 0$, and satisfies continuity conditions of the electric potential and of the normal component of the current at the interfaces.

4. Assemblage of coated ellipsoids: results

Inside the coated ellipsoid, we ask that
\[
\begin{align*}
\sigma_1 \Delta_p u &= 0 \quad \text{for } 0 < \rho < \rho_c \\
\sigma_2 \Delta u &= 0 \quad \text{for } \rho_c < \rho < \rho_e,
\end{align*}
\]
where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ represents the $p$-Laplacian ($p > 1$), $\sigma_1$ and $\sigma_2$ are positive, together with the usual continuity conditions of the electric potential and of the normal component of the current across the interfaces:
\[
\begin{align*}
u \text{ continuous across } \rho = \rho_c \tag{4.2}
\end{align*}
\]
and
\[
\begin{align*}
u = Ex_1 \quad \text{at } \rho = \rho_e \tag{4.3}
\end{align*}
\]
and
\[
\begin{align*}
\sigma_1 |\nabla u|^{p-2} \nabla u \cdot \mathbf{n} &= \sigma_2 \nabla u \cdot \mathbf{n}, \quad \text{across } \rho = \rho_c, \tag{4.4}
\end{align*}
\]
and
\[
\begin{align*}
\sigma_2 \nabla u \cdot \mathbf{n} &= \sigma_1^* \nabla u \cdot \mathbf{n}, \quad \text{across } \rho = \rho_e. \tag{4.5}
\end{align*}
\]

We look for a solution $u$ of (4.1) of the form
\[
\begin{align*}
u(x) &= \begin{cases} A_1 x_1 & \text{for } 0 < \rho < \rho_c, \\
\varphi(\rho) x_1 & \text{for } \rho_c \leq \rho \leq \rho_e. \end{cases} \tag{4.6}
\end{align*}
\]
Because (4.6) satisfies (4.1), it is left to determine $A_1$ and $\varphi(\rho)$, so that $u$ satisfies the conditions (4.2)–(4.5) at the interfaces.
Written in ellipsoidal coordinates, the conductivity equation in the coating (4.1) becomes

\[ 0 = \Delta u = \frac{4g(\rho)}{(\rho - \mu)(\rho - v)} \frac{\partial}{\partial \rho} \left[ g(\rho) \frac{\partial u}{\partial \rho} \right] + \frac{4g(\mu)}{(\mu - \rho)(\mu - v)} \frac{\partial}{\partial \mu} \left[ g(\mu) \frac{\partial u}{\partial \mu} \right] + \frac{4g(v)}{(v - \rho)(v - \mu)} \frac{\partial}{\partial v} \left[ g(v) \frac{\partial u}{\partial v} \right], \] \tag{4.7}

where

\[ g(t) = \sqrt{(c_1^2 + t)(c_2^2 + t)(c_3^2 + t)}. \] \tag{4.8}

**Remark 4.1.** Observe that \( \theta_1 = g(\rho_c)/g(\rho_e) \).

Using (4.6) and (4.7), and the fact that \( \Delta \chi_1 = 0 \), we obtain the following second-order differential equation for \( \phi(\rho) \)

\[ 0 = \frac{d^2 \phi(\rho)}{d \rho^2} + \left[ \frac{1}{g(\rho)} \frac{d g(\rho)}{d \rho} + \frac{1}{(\rho + \rho)} \right] \frac{d \phi(\rho)}{d \rho}. \] \tag{4.9}

Solving (4.9), we obtain

\[ \phi(\rho) = A_2 + B_2 \int_{\rho_c}^{\rho} \frac{1}{(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho)^{3/2} (\rho_2^2 + \rho_3^2 + \rho)^{1/2}} \, d\rho. \] \tag{4.10}

In what follows, we explain how the unknowns \( A_1, A_2 \) and \( B_2 \) and \( \sigma_1^* \) are determined from (4.2), (4.3), (4.4) and (4.5). First, we look at the conditions \( u \) must satisfy when \( \rho = \rho_c \). From (4.2), we have that

\[ A_1 = A_2 + B_2 \int_{\rho_c}^{\rho} \frac{1}{(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho)^{3/2} (\rho_2^2 + \rho_3^2 + \rho)^{1/2}} \, d\rho = A_2, \] \tag{4.11}

and from (4.4) and (4.11), we obtain

\[ B_2 = \frac{A_1 g(\rho_c)(\sigma_1^* |A_1|^{\rho^2 - \sigma_2})}{2 \sigma_2}. \] \tag{4.12}

We now look at the conditions that \( u \) must satisfy on the outer interface \( \rho = \rho_e \). From (4.3) and (4.11), we have

\[ E = A_1 + B_2 \int_{\rho_c}^{\rho_e} \frac{1}{(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho)^{3/2} (\rho_2^2 + \rho_3^2 + \rho)^{1/2}} \, d\rho, \] \tag{4.13}

and from (4.5), we obtain

\[ B_2 = \frac{E g(\rho_c)(\sigma_1^* - \sigma_2)}{2 \sigma_2}. \] \tag{4.14}

We now introduce the depolarization factors

\[ d_{ij} = d_j(l_1, l_2, l_3), \quad d_{ij} = d_j(l_1, l_2, l_3), \quad j = 1, 2, 3, \] \tag{4.15}

where

\[ d_j(l_1, l_2, l_3) = \frac{l_1 l_2 l_3}{2} \int_0^\infty \frac{dy}{(l_1^2 + y)(l_2^2 + y)(l_3^2 + y)}. \] \tag{4.16}

is the depolarization factor in the direction \( j = 1, 2, 3 \) of an ellipsoid with semi-axis lengths \( l_1, l_2, l_3 \). The depolarization factors always sum to unity (see [19])

\[ d_1 + d_2 + d_3 = 1. \] \tag{4.17}

In addition, observe that \( d_j(l_1, \lambda l_2, \lambda l_3) = \lambda d_j(l_1, l_2, l_3) \) for \( \lambda > 0 \), which means that the depolarization factors are independent of scale.
In terms of these depolarization factors, we have
\[
\int_{\rho_{e}}^{\rho} \frac{d\rho}{(c_{1}^{2} + \rho)^{3/2}(c_{2}^{2} + \rho)^{1/2}(c_{3}^{2} + \rho)^{1/2}} = \frac{2d_{c1}}{g(\rho_{c})} - \frac{2d_{e1}}{g(\rho_{e})}.
\]
Rearranging (4.14), we have
\[
E = \frac{2B_{2}\sigma_{2}}{g(\rho_{c})(\sigma_{1}^{*} - \sigma_{2})}.
\]
Using (4.18) and (4.12) in (4.13), we obtain
\[
\sigma_{1}^{*} = \sigma_{2} + \frac{\sigma_{2}\theta_{1}(\sigma_{2}^{1}\|^{\rho-2} - \sigma_{2})}{\sigma_{2} + (\sigma_{1}^{1}\|^{\rho-2} - \sigma_{2})[d_{c1} - \theta_{1}d_{c1}]}.
\]
From (4.13), we have
\[
\sigma_{1}^{*} = \sigma_{2} + \frac{\sigma_{2}\theta_{1}(\sigma_{2}^{1}\|^{\rho-2} - \sigma_{2})}{\sigma_{2} + (\sigma_{1}^{1}\|^{\rho-2} - \sigma_{2})[d_{c1} - \theta_{1}d_{c1}]}.
\]
where \(K = d_{c1} - \theta_{1}d_{c1} > 0\) is independent of scale.

Using (4.20) in (4.12), we obtain the following identity
\[
\sigma_{1}^{*} = \sigma_{2} + \frac{\sigma_{2}\theta_{1}(\sigma_{2}^{1}\|^{\rho-2} - \sigma_{2})}{\sigma_{2} + (\sigma_{1}^{1}\|^{\rho-2} - \sigma_{2})[d_{c1} - \theta_{1}d_{c1}]}.
\]
At this point, we consider the function
\[
f(x) = \sigma_{1}(E - Kx)^{\rho-2}(E - Kx) - \sigma_{2}(E - Kx) - \sigma_{2}x.
\]
Taking the derivative of the \(f(x)\), we have
\[
f'(x) = -K\sigma_{1}(\rho - 1)(E - Kx)^{\rho-2} + \sigma_{2}(K - 1).
\]
Note that the first term of \(f'(x)\) is negative, and the second term is also negative because \(K < 1\). To see this, note that by (4.17) and the fact that \(K > 0\),
\[
K < K + (d_{c2} - \theta_{1}d_{c2}) + (d_{c3} - \theta_{1}d_{c3})
\]
\[
= (d_{c1} + d_{c2} + d_{c3} - \theta_{1}d_{c1} + d_{c2} + d_{c3})
\]
\[
= 1 - \theta_{1} = \theta_{2} < 1.
\]
Therefore, \(f(x)\) is a decreasing function. If \(E - Kx < 0\), we have
\[
f(x) = -\sigma_{1}(Kx - E)^{\rho-1} - \sigma_{2}(E - Kx) - \sigma_{2}x,
\]
and here
\[
f'(x) = -K\sigma_{1}(\rho - 1)(E - Kx)^{\rho-2} + \sigma_{2}(K - 1)
\]
is negative for all \(x\), so the function \(f(x)\) is also decreasing in this case.

Observe that as \(x\) approaches \(\infty\), the function \(f(x)\) approaches \(-\infty\) and as \(x\) approaches \(-\infty\), the function \(f(x)\) approaches \(\infty\). Therefore, we conclude that the equation \(f(x) = 0\) has a unique solution \(x_{0}\).

Moreover, observe that the coefficients of \(f(x)\) depend only on \(\sigma_{1}, \sigma_{2}, E, K, p, \) thus
\[
x_{0} = \frac{2B_{2}}{g(\rho_{c})} = C(\sigma_{1}, \sigma_{2}, E, K, p).
\]

Consequently, from (4.23) and (4.20) we obtain that \(A_{1} = E - Kx_{0}\), which together with (4.19) gives
\[
\sigma_{1}^{*} = \sigma_{2} + \frac{\sigma_{2}\theta_{1}(\sigma_{2}^{1}\|^{\rho-2} - \sigma_{2})}{\sigma_{2} + (\sigma_{1}^{1}\|^{\rho-2} - \sigma_{2})[d_{c1} - \theta_{1}d_{c1}]}.
\]
Here, we would like to emphasize that (4.24) shows that \(\sigma_{1}^{*}\) is independent of scale. In an analogous way, the conductivities in the \(x_{2}\)- and \(x_{3}\)-directions are obtained and given by similar expressions, also independent of scale.
Remark 4.2. If \( p = 2 \), (4.22) becomes
\[
 f(x) = E(\sigma_1 - \sigma_2) - x(K\sigma_1 - \sigma_2) + \sigma_2, \tag{4.25}
\]
which has a unique root \( x_0 = E(\sigma_1 - \sigma_2)/(K\sigma_1 - \sigma_2) + \sigma_2 \). In this case, \( \sigma^*_1 \) (see (4.24)) becomes
\[
 \sigma^*_1 = \sigma_2 + \frac{\sigma_2\theta_1(\sigma_1 - \sigma_2)}{\sigma_1 + (\sigma_1 - \sigma_2)\rho_1\rho_2}, \tag{4.26}
\]
The conductivities in the \( x_2 \)- and \( x_3 \)-directions are obtained in the same manner and have similar expressions (same results as in §7.8 in [19]).

Remark 4.3. If \( c_1 = c_2 = c_3 = c \), we have a sphere. In this case, (3.2) becomes
\[
 l_{ij} = r_e = \sqrt{c^2 + \rho_c} \tag{4.27}
\]
and
\[
 l_{ij} = r_e = \sqrt{c^2 + \rho_c}, \quad j = 1, 2, 3,
\]
where \( r_c \) is the radius of the core of the sphere and \( r_e \) the radius of the entire sphere (core and coating). Here, the volume fraction (3.3) becomes
\[
 \theta_1 = \frac{l_{11}l_{22}l_{33}}{l_{11}l_{22}l_{33}} = \frac{r_c^3}{r_e^3} \quad \text{and} \quad \theta_2 = 1 - \theta_1. \tag{4.28}
\]
The depolarization factors (4.15) are all equal and their value is \( \frac{1}{3} \), which implies that the integral in (4.13) becomes
\[
 \int_{r_c}^{r_e} \frac{d\rho}{(c^2 + \rho)^{3/2}} = \frac{2}{3} r_c^3 \theta_2.
\]
Therefore, we have \( \sigma^*_1 = \sigma^*_2 = \sigma^*_3 = \sigma^* \), where
\[
 \sigma^* = \sigma_2 + \frac{3\sigma_2\theta_1(\sigma_1 - (1/3)\theta_2 x_0)^{p-2} - \sigma_2}{3\sigma_2 + \theta_2(\sigma_1 - (1/3)\theta_2 x_0)^{p-2} - \sigma_2}, \tag{4.29}
\]
with \( x_0 \) being the unique and scale-independent solution of
\[
 f(x) = \sigma_1[E - \frac{1}{3}\theta_2 x]^{p-2}(E - \frac{1}{3}\theta_2 x) - \sigma_2(E - \frac{1}{3}\theta_2 x) - \sigma_2 x. \tag{4.30}
\]
In this way, we recovered the results presented in Jiménez et al. [37]. If \( p = 2 \), we have
\[
 \sigma^* = \sigma_2 + \frac{3\sigma_2\theta_1(\sigma_1 - \sigma_2)}{3\sigma_2 + \theta_2(\sigma_1 - \sigma_2)},
\]
which is the Hashin–Shtrikman formula.

Remark 4.4. Observe that the effective conductivity (4.24) depends on the value of the applied electric field \( E \) unlike in the linear case (4.26).

5. Conclusion

In this paper, we showed that neutral inclusions of coated ellipsoids can be constructed from nonlinear materials in the context of (electrical or thermal) conductivity for a given applied field.

Author contributions. S.J.B. and B.V. worked together to solve the problem described and helped draft the manuscript. S.J.B. and B.V. gave final approval for publication.

Conflict of interests. We have no competing interests.

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