RESOLUTION OF RICCATI EQUATION BY THE METHOD DECOMPOSITIONAL OF ADOMIAN

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ABSTRACT

In this work, we will use the method of decompositional for Adomian solve the Riccati equation in the form:

\[ u' = a(t) + b(t)u + c(t)u^2 \]  

Keywords:
Adomian decomposition method, Adomian's polynomials, Riccati equation, Development limited.

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1. INTRODUCTION

The Riccati equation it is named in honor of Jacopo Francesco Riccati (1676-1754) and his son Vincenzo Riccati (1707-1775).
In general equation (1) is not solvable by quadratures, if he knows a particular solution up, the Riccati equation (1) reduces to a Bernoulli equation.
And in the 80 G. Adomian proposed a new method to solve differential equations of different types.
This method is to look for the solution in the form of a series, and decompose the non-linear operator in a series of function (polynomials Adomian) [4, 5]
K. Abbaoui and Y. Cherrault, place assumptions on the convergence of series of Adomian to the exact solution [1, 2, 3, 6].
This work mainly concerns the resolution of the Riccati equation by the Adomian method, with application examples.

2. ADOMIAN METHOD

We consider the following problem:

\[ Fu = Lu + Ru + Nu = f(t) \]  

(2)
as N is a nonlinear operator, and L the invertible linear portion of F.
Equation (2) it gives:
\[ u = g(t) - L^{-1}Ru - L^{-1}Nu \]  
(3)

Or
\[ Nu = \sum_{n=0}^{\infty} A_n t^n \]  
(4)

with Ai are called Adomian polynomials.
and the terms of the standard solution defined by:
\[ u_0 = g(t) \]
\[ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n \]  
(5)

with:
\[ u = \sum_{n=0}^{\infty} u_n t^n \]

3. RICCATI DIFFERENTIAL EQUATION

is an ordinary differential equation of first order of the form [7]:
\[ u' = a(t) + b(t)u + c(t)u^2 \]  
(6)
or a, b and c are continuous functions defined on an open interval I of R.
In general there is no solution by quadrature, but if he knows a particular solution, a Riccati equation is reduced by substitution in a Bernoulli equation.

3.1. RESOLUTION KNOWING A PARTICULAR SOLUTION

If it is possible to find a particular solution \( u_p \).
So the general solution is of the form:
\[ u = u_p + y \]  
(7)

By replacing u by \( u_p + y \) in equation (6)
We obtain:
\[ (u_p + y)' = a(t) + b(t)(u_p + y) + c(t)(u_p + y)^2 \]  
(8)

and as \( u_p \) is a particular solution:
\[ u'_p = a(t) + b(t)u_p + c(t)u_p^2 \]

In was:
\[ y' - (b(t) + 2c(t)u_p)y = c(t)y^2 \]  
(9)
is a Bernoulli equation, transformation is used: \( z = 1/y \)
So:
\[ z' + (b(t) + 2c(t)u_p)z = -c(t) \]  
(10)
It is a non-homogeneous linear equation.
We solve the homogeneous equation, then use the method of variation of constants, we find the general solution of the Bernoulli equation.
\[ y = \frac{1}{z} = e^{\int (b(s)+2c(t)u_p)ds} \left( k - \int c(t) e^{\int (b(s)+2c(t)u_p)ds} dt \right)^{-1} \]  
(11)
the more general solution of the Riccati equation given by:
\[ u = u_p + \frac{1}{z} \]
\[ u = u_p + e^{\int (b(s)+2c(t)u_p)ds} \left( k - \int c(t) e^{\int (b(s)+2c(t)u_p)ds} dt \right)^{-1} \]  
(12)
3.2. RESOLUTION BY THE METHOD ADOMIAN

Consider the following problem:

\[ u' = a(t) + b(t)u + c(t)u^2 \]  
\[ u(0) = u_0 \]  

(13)

the Adomian method is used to solve the Riccati equation in the problem (13). We have:

\[ u' = a(t) + b(t)u + c(t)u^2 \]

and we ask:

\[
L = D^1 = \frac{d}{dt} \Rightarrow L^{-1} = \int_0^t ds
\]

(14)

\[ Ru = b(t)u \]

\[ Nu = u^2 \]

such as:

\[ Nu = u^2 = \sum_{n \geq 0} A_n t^n \]

With \( A_t \): polynomials Adomian of the function \( u^2 \)[4,5].

\[
L^{-1} u' = L^{-1} a(t) + L^{-1} b(t)u + L^{-1} c(t)u^2
\]

\[
u(t) - u(0) = L^{-1} a(t) + L^{-1} b(t)u + L^{-1} c(t)u^2
\]

Using the D,L in the neighborhood of 0 of functions \( a, b \) and \( c \):

\[
a(t) = \sum_{n \geq 0} a_n t^n
\]

\[
b(t) = \sum_{n \geq 0} b_n t^n
\]

\[
c(t) = \sum_{n \geq 0} c_n t^n
\]

and we obtain:

\[
\sum_{n \geq 0} u_n t^n = u(0) + L^{-1} \sum_{n \geq 0} a_n t^n + L^{-1} \sum_{n \geq 0} b_n t^n \cdot \sum_{n \geq 0} u_n t^n + L^{-1} \sum_{n \geq 0} c_n t^n \cdot \sum_{n \geq 0} A_n t^n
\]

\[
= u(0) + \sum_{n \geq 0} \frac{a_n}{n+1} t^{n+1} + \sum_{n \geq 0} \frac{1}{n+1} t^{n+1} \left( \sum_{k=0}^n b_k u_{n-k} \right) + \sum_{n \geq 0} \frac{1}{n+1} t^{n+1} \left( \sum_{k=0}^n c_k A_{n-k} \right)
\]

\[
= u(0) + \sum_{n \geq 1} \frac{1}{n} (a_{n-1} + \sum_{k=0}^{n-1} b_k u_{n-1-k} + \sum_{k=0}^{n-1} c_k A_{n-1-k}) t^n
\]

and the solution given by:

\[
u_0 = u(0)
\]

\[
u_n = \left( \frac{1}{n} \right) (a_{n-1} + \sum_{k=0}^{n-1} b_k u_{n-1-k} + \sum_{k=0}^{n-1} c_k A_{n-1-k})
\]  

(15)
4. APPLICATION EXAMPLES

4.1. EXAMPLE1

We consider the following problem (Riccati equation):

\[ u' = -t + (2t - 1)u + (1 - t)u^2 \]  \hspace{1cm} (16)

with the particular solution \( u_p = 1 \)

\section*{4.1.1. DIRECT RESOLUTION}

We have the following equation according to (7), (9) and (10) with the transformation \( z = 1/u \):

\[ z' + z = t - 1 \]  \hspace{1cm} (17)

it is a first-order non-homogeneous equation, and the solution given by:

\[ z(t) = t - 2 + ke^{-t} \]  \hspace{1cm} (18)

and the solution of the problem (16) with the initial condition given by:

\[ u(t) = 1 + \frac{1}{t - 2 + 3e^{-t}} \]  \hspace{1cm} (19)

the D, L of the solution function is: We have:

\[ e^t = \sum_{n=0}^\infty \frac{(-t)^n}{n!} = 1 - t + \frac{1}{2} . t^2 - \frac{1}{6} . t^3 + \frac{1}{24} . t^4 - ... \]

and

\[ t - 2 + 3e^{-t} = 1 - 2t + \frac{3}{2} . t^2 - \frac{1}{2} . t^3 + \frac{1}{8} . t^4 + o(t^4) \]

Dividing the polynomial \( f_1(t) = 1 \) by the polynomial \( f_2(t) = t - 2 + 3e^{-t} \), we obtain:

\[ \frac{1}{t - 2 + 3e^{-t}} = 1 + 2t + \frac{5}{2} . t^2 + \frac{5}{2} . t^3 + \frac{17}{8} . t^4 + o(t^4) \]

So the order of D.L. 4 in the neighborhood of 0 of the solution function \( u(t) \) is:

\[ u(t) = 1 + 2t + \frac{5}{2} . t^2 + \frac{5}{2} . t^3 + \frac{17}{8} . t^4 + o(t^4) \]  \hspace{1cm} (20)

\section*{4.1.2. RESOLUTION BY THE METHOD ADOMIAN}

Consider the Riccati equation:

\[ u' = -t + (2t - 1)u + (1 - t)u^2 \]  \hspace{1cm} (21)

Applying the method of Adomian with \( F = L + R + N \) as:

\[ L = D^1 = \frac{d}{dt} \]

\[ Ru = (2t - 1)u \]

\[ Nu = (1 - t)u^2 \]

We have
\[ L = D^1 = \frac{d}{dt} \Rightarrow L^{-1} = \int_0^t ds \]

Is applied \( L^{-1} \) in equation (21) we obtain:

\[
L^{-1}u' = L^{-1}(-t) + L^{-1}Ru + L^{-1}Nu
\]

\[
\sum_{n=0} u_n t^n = u(0) - \frac{1}{2} t + \sum_{n=2}^{\infty} \frac{2u_{n-2}}{n} t^n - \sum_{n=1}^{\infty} \frac{u_{n-1}}{n} t^n + \sum_{n=1}^{\infty} \frac{A_{n-1}}{n} t^n - \sum_{n=2}^{\infty} \frac{A_{n-2}}{n} t^n
\]

\[
\sum_{n=0} u_n t^n = u(0) - \frac{1}{2} t + \sum_{n=1}^{\infty} \frac{1}{n} (A_{n-1} - u_{n-1}) t^n + \sum_{n=2}^{\infty} \frac{1}{n} (2u_{n-1} - A_{n-1}) t^n
\]

So the coefficients of the series solution Adomian is given by:

\[
\left\{ \begin{array}{l}
u_0 = u(0) = 2 \\
u_1 = A_0 - u_0 \\
u_2 = \frac{1}{2} (A_1 - u_1 + 2u_0 - A_0) - \frac{1}{2} \\
\ldots \\
u_n = \frac{1}{2} (A_{n-1} - u_{n-1} + 2u_{n-2} - u_{n-2}), \forall n \geq 3.
\end{array} \right.
\]

Or \( A_i \) are polynomials of Adomian of \( u^2 \) function [4,5]

\[
\left\{ \begin{array}{l}
A_0 = u_0^2 \\
A_1 = 2u_0 u_1 \\
A_2 = u_1^2 + 2u_0 u_2 \\
A_3 = 2u_1 u_2 + 2u_0 u_3 \\
A_4 = u_2^2 + 2u_1 u_3 + 2u_0 u_4 \\
A_5 = 2u_2 u_3 + 2u_1 u_4 + 2u_0 u_5 \\
\ldots \\
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
u_0 = 2 \\
u_1 = 2 \\
u_2 = \frac{5}{2} \\
u_3 = \frac{5}{2} \\
u_4 = \frac{17}{8} \\
\ldots \\
\end{array} \right.
\]
Finally the solution series is:

\[ u(t) = 2 + 2t + \frac{5}{2}t^2 + \frac{5}{2}t^3 + \frac{17}{8}t^4 + \cdots. \]  

(23)

And we note that the results (19) and (22) are equal.

4.2. EXAMPLE 2

We consider the following problem (Riccati equation):

\[
\begin{cases}
(1 + t^2)u' = u^2 - 1 \\
u(0) = 2
\end{cases}
\]

(24)

with the particular solution: \(u_p = 1\).

4.2.1. DIRECT RESOLUTION

We have the following equation according to (7), (9) and (10) with the transformation \(z = \frac{1}{u^2}\):

\[ z' = \frac{2}{1+t^2}z = \frac{-1}{1+t^2} \]

(25)

it is a first-order non-homogeneous equation, and the solution given by:

\[ z(t) = -\frac{1}{2} + ke^{2\arctan t} \]

(26)

and the solution of the problem (16) with the initial condition given by:

\[ u(t) = 1 + \frac{1}{2^3e^{2\arctan t}} = 1 + \frac{2}{3e^{2\arctan t} - 1} \]

(27)

the D, L of the solution function is we have:

\[ \arctan t = t - \frac{1}{3}t^3 + o(t^4) \]

So:

\[ e^{2\arctan t} = e^{2t - \frac{2}{3}t^3 + o(t^4)} \]

if we pose: \(v = 2t - \frac{2}{3}t^3 + o(t^4)\)

Then:
\[ e^v = 1 + v + \frac{1}{2} v^2 + o(v^2) \]

from where:

\[ e^v = 1 + (2t - \frac{2}{3} t^3) + \frac{1}{2} (2t - \frac{2}{3} t^3)^2 + o(t^4) \]

\[ = 1 - 2t + 2t^2 - \frac{2}{3} t^3 + \frac{4}{3} t^4 + o(t^4) \]

So we have:

\[ 3e^v - 1 = 2 - 6t + 6t^2 - 2t^3 - 4t^4 + o(t^4) \] (28)

Dividing the polynomial \( f_1(t) = 2 \) by the polynomial \( f_2(t) = 3e^{2\arctan t - 1} \), we obtain:

\[ \frac{2}{3e^{2\arctan t - 1}} = 1 + 3t + 6t^2 + 10t^3 + 17t^4 + o(t^4) \]

So the order of \( D, L \) 4 to neighborhood of the origin of the solution function \( u(t) \) is:

\[ u(t) = 2 + 3t + 6t^2 + 10t^3 + 17t^4 + o(t^4) \] (29)

4.2.2. RESOLUTION BY THE METHOD ADOMIAN

Consider the Riccati equation:

\[ u' = \frac{-1}{1+t^2} + \frac{1}{1+t^2} u^2 \] (30)

Applying the method of Adomian with \( F = L + R + N \) as:

\[
\begin{cases}
L = D^1 = \frac{d}{dt} \\
Nu = \frac{1}{1 + t^2} u^2
\end{cases}
\]

We have:

\[ L = \frac{d}{dt} \Rightarrow L^{-1} = \int_0^t ds \]

Applying the operator \( L^{-1} \) in equation (30) we obtain:

\[ L^{-1} u' = L^{-1} \left( \frac{-t}{1 + t^2} \right) + L^{-1} Nu \]
\[\begin{align*}
\Rightarrow u(t) - u(0) &= L^{-1}\left(\frac{-t}{1+t^2}\right) + L^{-1}\left(\frac{1}{1+t^2} u^2(t)\right) \\
\sum_{n \geq 0} u_n t^n &= u(0) + L^{-1}\sum_{n \geq 0} (-1)^{n+1} t^{2n} + L^{-1}\sum_{n \geq 0} (-1)^n t^{2n}. \sum_{n \geq 0} A_n t^n \\
\sum_{n \geq 0} u_n t^n &= u(0) + L^{-1}\sum_{n \geq 0} (-1)^{n+1} t^{2n} + L^{-1}\sum_{n \geq 0} (-1)^n t^{2n}. \left(\sum_{n \geq 0} A_{2n} t^{2n} + \sum_{n \geq 0} A_{2n+1} t^{2n+1}\right) \\
\sum_{n \geq 0} u_n t^n &= u(0) + \frac{1}{2n+1} t^{2n+1} + \frac{1}{2n+1} t^{2n+1}. \left(\sum_{k=0}^n (-1)^k A_{2(n-k)}\right) \\
\sum_{n \geq 0} u_n t^n &= u(0) + \frac{(-1)^{n+1}}{2n+2} t^{2n+2}. \left(\sum_{k=0}^n (-1)^k A_{2(n-k)-1}\right)
\end{align*}\]

So the coefficients of the series solution Adomian is given by:

\[u_0 = u(0) = 2\]
\[u_{2n+1} = \frac{1}{2n+1}((-1)^{n+1} + \sum_{k=0}^n (-1)^k A_{2(n-k)}) , \forall n \geq 0\]
\[u_{2n+1} = \frac{1}{2n+2}(\sum_{k=0}^n (-1)^k A_{2(n-k)-1}) , \forall n \geq 0\]

Or \(A_i\) are polynomials of Adomian of the function \(u^2\) [4,5].

Then:

\[
\begin{align*}
&u_0 = 2 \\
u_1 = (-1)^1 + (-1)^0 A_0 = -1 + 4 = 3 \\
u_2 = \frac{1}{2} A_1 = \frac{1}{2} \cdot 12 = 6 \\
u_3 = \frac{1}{3}((-1)^2 + A_2 + (-1)^1 A_0) = \frac{1}{3}(1 + 33 - 4) = 10 \\
u_4 = \frac{1}{4}(A_3 + (-1) A_1) = \frac{1}{4}(80 - 12) = 17
\end{align*}\]

Finally the solution series is:

\[u(t) = 2 + 3t + 6t^2 + 10t^3 + 17t^4 + \cdots\]  

(32)

And we note that the results (29) and (32) are equal.

### 5. CONCLUSION

Despite generally, there is no Resolution of Riccati equation, but the method of decomposition Adomian always given an approximate solution in the form of a convergent series.
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