Solomon-Tukachinsky’s vs. Welschinger’s
Open Gromov-Witten Invariants of Symplectic Sixfolds

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Abstract

Our previous paper describes a geometric translation of the construction of open Gromov-Witten invariants by J. Solomon and S. Tukachinsky from a perspective of $A_{\infty}$-algebras of differential forms. We now use this geometric perspective to show that these invariants reduce to Welschinger’s open Gromov-Witten invariants in dimension 6, inline with their and G. Tian’s expectations. As an immediate corollary, we obtain a translation of Solomon-Tukachinsky’s open WDVV equations into relations for Welschinger’s invariants.

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1 Introduction

Suppose $(X, \omega)$ is a compact symplectic sixfold, $Y \subset X$ is a compact Lagrangian submanifold, and $\mathcal{O}_s = (\mathfrak{O}, \mathfrak{s})$ is an OSpin-structure on $Y$, i.e. a pair consisting of an orientation $\mathfrak{O}$ on $Y$ and a Spin-structure $\mathfrak{s}$ on the oriented manifold $(Y, \mathfrak{O})$. If

$$\omega(\beta) > 0, \quad \mu_Y(\beta) \geq 0 \quad \implies \quad \mu_Y(\beta) > 0$$

(1.1)

for every $\beta \in H_2(X, Y; \mathbb{Z})$ representable by a map from $(\mathbb{D}^2, S^1)$, every non-constant $J$-holomorphic map from $(\mathbb{D}^2, S^1)$ to $(X, Y)$ has a positive Maslov index for a generic $\omega$-compatible almost complex structure $J$; the same happens along a generic path of almost complex structures.

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Let $R$ be a commutative ring with unity 1. If the homomorphism
\[ \iota_Y : H_1(Y; R) \longrightarrow H_1(X; R) \] (1.2)
induced by the inclusion $\iota_Y : Y \longrightarrow X$ is injective, the boundaries of maps from $(\mathbb{D}^2, S^1)$ to $(X, Y)$ are homologically trivial in $Y$ and thus have well-defined linking numbers with values in $R$. Under this injectivity assumption and the positivity condition (1.1), Welschinger [17] defines open Gromov-Witten invariants
\[ \langle \cdot, \ldots, \cdot \rangle_{\beta, k}^\omega : \bigoplus_{l=0}^{\infty} H^{2*}(X, Y; R)^{\otimes l} \longrightarrow \mathbb{R}, \quad \beta \in H_2(X, Y; \mathbb{Z}) - \{0\}, \ k \in \mathbb{Z}^+, \] (1.3)
enumerating $J$-holomorphic multi-disks of total degree $\beta$ weighted by $R$-valued linking numbers of their boundaries; see Section 4.1 in [17] and (3.3). In “real settings” (when $Y$ is a topological component of the fixed locus of anti-symplectic involution on $X$), these invariants encode the real Gromov-Witten invariants defined in [15, 16, 11]; see Remark 3.3. A special case of the setting of [17] is when $Y$ is an $R$-homology $S^3$.

Based on $A_\infty$-algebra considerations, K. Fukaya [4] defines a count $\langle \cdot, \ldots, \cdot \rangle_{\beta, 0}^\omega \in \mathbb{Q}$ of $J$-holomorphic degree $\beta$ disks in $X$ with boundary in $Y$ under the assumption that $(X, \omega)$ is a Calabi-Yau threefold and the Maslov index
\[ \mu_\omega : H_2(X, Y; \mathbb{Z}) \longrightarrow \mathbb{Z} \] (1.4)
of $(X, Y)$ vanishes; this count may in general depend on the $\omega$-compatible almost complex structure $J$ on $X$. Motivated by [4], J. Solomon and S. Tukachinsky [12] use a bounding chain of differential forms to define counts
\[ \langle \cdot, \ldots, \cdot \rangle_{\beta, k}^{\omega, \partial} : \bigoplus_{l=0}^{\infty} H^{2*}(X, Y; \mathbb{R})^{\otimes l} \longrightarrow \mathbb{R}, \quad \beta \in H_2(X, Y; \mathbb{Z}), \ k \in \mathbb{Z}^+, \] (1.5)
of $J$-holomorphic disks in symplectic manifolds $X$ of arbitrary dimension $2n$ and show that bounding chains that are equivalent in a suitable sense define the same counts. They also prove that bounding chains exist and any two relevant bounding chains are equivalent if $n$ is odd and $Y$ is an $R$-homology sphere. The resulting open Gromov-Witten invariants (1.5) of $(X, Y)$ depend on the choice of a (relative) OSpin-structure $\partial$ on $Y$, but are independent of all other auxiliary choices (such as $J$).

Well before [12], G. Tian expressed a belief that the construction of [17] is a geometric realization of the algebraic considerations behind the construction of [4]. The same sentiment is expressed in [12] Sec 1.2.7. The present paper uses the geometric interpretation of the construction of [12] described in [2], which is applicable over any commutative ring $R$ with unity under the positivity condition (1.1) in the case of symplectic sixfolds, to confirm G. Tian’s and Solomon-Tukachinsky’s expectations; see Theorem 1.1 below. Proposition 3.2, a kind of open divisor relation which trades real codimension 1 bordered insertions at boundary marked points for linking numbers of their boundaries with the boundaries of the disks, provides a transition from bounding chains to linking numbers. As an immediate consequence of Theorem 1.1 the basic structural properties and the WDVV-type relations for open Gromov-Witten invariants obtained in [13] yield similar properties and relations for Welschinger’s invariants (1.3); see Corollaries 1.2 and 1.3.

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The positivity condition (1.1) ensures that virtual techniques are not necessary for the purposes of this paper. However, the reasoning extends to general symplectic sixfolds satisfying the injectivity condition (1.2) via the setup of Appendix A in [2] if $R \supseteq \mathbb{Q}$. This setup is compatible with standard virtual class approaches, such as in [9, 6, 8, 10]. As we only need evaluation maps from the $(J, \nu)$-spaces to be pseudocycles, a full virtual cycle construction and gluing across all strata of the $(J, \nu)$-spaces are not necessary.

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1.1 Comparison theorem

Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n$, $Y \subset X$ be a compact Lagrangian submanifold, and $\mathfrak{o}_s \equiv (\alpha, s)$ be a relative $OSpin$-structure on $Y$. Let $\alpha \equiv (\beta, K, L)$ be a tuple consisting of $\beta \in H_2(X, Y; \mathbb{Z})$, a generic finite subset $K$ of $Y$, and a generic set $L$ of pseudocycles $\gamma_1, \ldots, \gamma_l$ to $X-Y$ representing Poincare duals of some cohomology classes $\gamma_1, \ldots, \gamma_l$ on $(X, Y)$. For a generic $\omega$-compatible almost complex structure $J$ on $X$, a bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega, \alpha}(Y)}$ on $(\alpha, J)$ in the geometric perspective of [2] is a tuple of bordered pseudocycles with certain properties specified in the dim $X = 6$ case by Definition 2.5 in the present paper. Such a tuple determines a pseudocycle $\mathfrak{b}_\alpha$ into $Y$; see (2.3). It has a well-defined degree, and we set

$$
\langle \gamma_1, \ldots, \gamma_l \rangle_{\beta, |K|+1}^{\omega, \mathfrak{o}_s} \equiv \langle L \rangle_{\beta, K}^{\omega, \mathfrak{o}_s} \equiv \deg \mathfrak{b}_\alpha.
$$

This degree may depend on the choices of $K$, $L$, $J$, and $(b_{\alpha'})_{\alpha' \in C_{\omega, \alpha}(Y)}$. Bounding chains differing by a pseudo-isotopy of [2] Dfn. 2.2 determine the same degrees (1.6); see [2] Sec. 2.2]. This guarantees that the numbers in (1.6) depend only on $\omega$, $\mathfrak{o}_s$, $\beta$, $|K|$, and $\gamma_1, \ldots, \gamma_l$ if $n$ is odd and $Y$ is an $R$-homology sphere. However, the injectivity of (1.2) does not guarantee the existence of a pseudo-isotopy between a pair of bounding chains associated even with the same $K$, $L$, and $J$.

Nevertheless, we establish the following.

**Theorem 1.1** Suppose $R$ is a commutative ring with unity, $(X, \omega)$ is a compact symplectic sixfold, $Y \subset X$ is a compact Lagrangian submanifold so that the positivity condition (1.1) holds and the homomorphism (1.2) is injective, and $\mathfrak{o}_s$ is a relative $OSpin$-structure on $Y$. Let $\beta \in H_2(X, Y; \mathbb{Z})$, $K$ be a generic finite subset of $Y$, $L \equiv \{\Gamma_1, \ldots, \Gamma_l\}$ be a generic set of even-dimensional pseudocycles to $X-Y$, $\alpha \equiv (\beta, K, L)$, and $J$ be a generic $\omega$-compatible almost complex structure on $X$.

(W1) There exists a bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega, \alpha}(Y)}$ on $(\alpha, J)$.

(W2) If $\gamma_1, \ldots, \gamma_l \in H^{2*}(X, Y; R)$ are the Poincare duals of $\Gamma_1, \ldots, \Gamma_l$, then

$$
\langle \gamma_1, \ldots, \gamma_l \rangle_{\beta, |K|+1}^{\omega, \mathfrak{o}_s} = (-1)^{|K|+1} \langle L \rangle_{\beta, K}^{\omega, \mathfrak{o}_s}
$$

for any bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega, \alpha}(Y)}$ on $(\alpha, J)$.

For each $\alpha' \in C_{\omega, \alpha}(Y)$ satisfying the dimension condition in Definition 2.5(BC4) the right-hand side of (2.6) consists of the boundaries of some maps from $(\mathbb{D}^2, S^1)$ to $(X, Y)$; see Proposition 3.1. By the injectivity of (1.2), we can thus choose a bordered pseudocycle $b_{\alpha'}$ to $Y$ satisfying (2.6). This implies that a bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega, \alpha}(Y)}$ on $(\alpha, J)$ can be constructed by induction on the partially ordered set $C_{\omega, \alpha}(Y)$ and establishes Theorem 1.1(W1).
A key ingredient in the proof of Proposition 3.1 is the Open Divisor Relation of Proposition 3.2, which replaces real codimension 1 bordered insertions at boundary marked points with linking numbers. This relation is also combined with Proposition 3.1 to obtain the identification of the disk counts (1.6) with Welschinger’s invariants (1.3) stated in Theorem 1.1(W2). This identification in turn implies that the numbers (1.6) depend only on \( \omega, \os, \beta, |K|, \) and \( \gamma_1, \ldots, \gamma_l \).

We denote by

\[
q_Y : H_2(X; \mathbb{Z}) \to H_2(X, Y; \mathbb{Z})
\]

the natural homomorphism. A bounding chain \( (\alpha)_{|a| \leq \omega} \) as in Definition 2.5 can also be used to define a count of \( J \)-holomorphic degree \( \beta \) disks through \( |K| \) points in \( Y \) if \( k \neq |K| \neq 0 \) or \( \beta \not\in \text{Im}(q_Y : H_2(X; \mathbb{Z}) \to H_2(X, Y; \mathbb{Z})) \);

\[
q_Y : H_2(X; \mathbb{Z}) \to H_2(X, Y; \mathbb{Z})
\]

see (2.9). The definition of the invariants (1.3) in [17] immediately extends to counts of multi-disks with \( k 
eq 0 \) points in \( Y \) if \( \beta \) satisfies the second condition in (1.8). The proof of Theorem 1.1 can be slightly modified to cover this case. It can also be readily extended to the open invariants with insertions from \( H_2(X, Y; \mathbb{R}) \), which are defined in [17].

It is immediate from (3.4) that Welschinger’s open invariants are symmetric linear functionals that satisfy an open divisor relation:

\[
\left\langle \gamma, \gamma_1, \ldots, \gamma_l \right\rangle_{\beta, k}^{\omega, s} = \left\langle \gamma, \beta \right\rangle \left\langle \gamma_1, \ldots, \gamma_l \right\rangle_{\beta, k}^{\omega, s} \quad \forall \gamma \in H^2(X, Y; \mathbb{R}).
\]

Combining Theorem 1.1 above with the last three statements of Theorem 2.9 in [2], which in turn are analogues of Proposition 2.1 in [3] and Corollary 1.5 in [13], we obtain below additional properties of Welschinger’s open invariants.

Suppose \( Y \) is connected. The kernel of the homomorphism

\[
H_2(X - Y; R) \to H_2(X; R)
\]

is then generated by the homology class \([S(N_Y Y)]_{X - Y}\) of a unit sphere \( S(N_Y Y) \) in the fiber of \( N_Y \) over any \( y \in Y \). We orient \( S(N_Y Y) \) as in [3 Sec 2.5] and denote the image of \([S(N_Y Y)]_{X - Y}\) under the Lefschetz Duality isomorphism

\[
PD_{X,Y} : H_2(X - Y; R) \isom H^4(X, Y; R)
\]

by \( \eta_{X,Y} \). For \( B \in H_2(X; \mathbb{Z}) \), let

\[
\left\langle \gamma, \ldots, \gamma \right\rangle_{B}^{\omega, s} : \bigoplus_{l=0}^{\infty} H^*(X; R)^{\otimes l} \to R
\]

be the standard GW-invariants of \( (X, \omega) \). We denote by \([Y]_X\) the homology class on \( X \) determined by \( Y \).

**Corollary 1.2** Let \((X, \omega, Y), \os, \beta, k, \) and \( \gamma_1, \ldots, \gamma_l \) be as in Theorem 1.1. If the pair \((k, \beta)\) satisfies (1.8) and \( Y \) is connected, Welschinger’s open invariants (1.3) satisfy the following properties.

(WGW1) \( \left\langle \gamma_X Y, \gamma_1, \ldots, \gamma_l \right\rangle_{\beta, k}^{\omega, s} = -\left\langle \gamma_1, \ldots, \gamma_l \right\rangle_{\beta, k+1}^{\omega, s} \).
be the evaluation morphism at the $i$-th marked point. If in addition $\Gamma$, and $\gamma$ is a two-dimensional pseudocycle to $X - Y$ bounding a pseudocycle $b$ transverse to $Y$, we define

$$\text{lk}_{\omega}(\gamma) \equiv |b \times_{fb} \iota_Y|^{\pm};$$

see Section 2.1 for the sign conventions for fiber products. This linking number of $\gamma$ and $Y$ with the orientation determined by the relative OSpin-structure $\omega$ does not depend on the choice of $b$. We set $\text{lk}_{\omega}(\gamma) = 0$ if $\gamma$ is not a two-dimensional pseudocycle.

For $l \in \mathbb{Z}_{\geq 0}$, $B \in H_2(X; \mathbb{Z})$, and an $\omega$-tame almost complex structure $J$, we denote by $\mathcal{M}_{0,\{0\} \cup [l]}(B; J)$ the moduli space of stable $J$-holomorphic degree $B$ maps with marked points indexed by the set $\{0\} \cup [l]$. It carries a canonical orientation. For each $i \in \{0\} \cup [l]$, let

$$\text{ev}_i: \mathcal{M}_{0,\{0\} \cup [l]}(B; J) \longrightarrow X$$

be the evaluation morphism at the $i$-th marked point. If in addition $\Gamma_1, \ldots, \Gamma_l$ are maps to $X$, let

$$\mathcal{M}_{0,\{0\} \cup [l]}(B; J) \times_{fb} \text{ev}_i (\{i, \Gamma_i\} \subseteq [l]) = \mathcal{M}_{0,\{0\} \cup [l]}(B; J)_{(\text{ev}_1, \ldots, \text{ev}_l) \times \Gamma_1 \times \ldots \times \Gamma_l ((\text{dom } \Gamma_1) \times \ldots \times (\text{dom } \Gamma_l))).$$
If $J$ is generic and $\Gamma_1, \ldots, \Gamma_l$ are pseudocycles in general position, then

$$f^C_{B,(\Gamma_i)_{i\in[l]}} = \left( ev_0 : 2\mathfrak{m}^C_{0,\omega} (B; J) \rightarrow \mathfrak{m}_0 (B; J) \right) \rightarrow X $$

is a pseudocycle of dimension

$$\dim f^C_{B,(\Gamma_i)_{i\in[l]}} = \mu_\omega(q_Y(B)) - \sum_{i=1}^l (\text{codim } \Gamma; -2) + 2$$

transverse to $Y$.

Since $\dim Y = 3$, the cohomology long exact sequence for the pair $(X,Y)$ implies that the restriction homomorphism

$$ H^p(X,Y; R) \rightarrow H^p(X; R) \hspace{1cm} (1.10) $$

is surjective for $p = 4, 6$. Since $R$ is a field and the homomorphism $(1.9)$ is trivial, $(1.10)$ is also surjective for $p = 2$. Let

$$ \gamma_1^\ast \equiv 1 \in H^0(X; R) \quad \text{and} \quad \gamma_2^\ast, \ldots, \gamma_N^\ast \in H^{2*}(X,Y; R) $$

be such that $\gamma_1^\ast, \gamma_2^\ast |_X, \ldots, \gamma_N^\ast |_X$ is a basis for $H^{2*}(X; R)$, $(g_{ij})_{i,j}$ be the $N \times N$-matrix given by

$$ g_{ij} = \langle \gamma_i^\ast, \gamma_j^\ast; [X] \rangle $$

and $(g^{ij})_{i,j}$ be its inverse. Let $\Gamma^\ast_1 = \text{id}_X$ and $\Gamma^\ast_2, \ldots, \Gamma^\ast_N$ be pseudocycles to $X-Y$ representing the Poincare duals of $\gamma_2^\ast, \ldots, \gamma_N^\ast$.

For the purpose of WDVV-type equations for the invariants $(1.3)$, we extend the signed counts $(3.4)$ to the pairs $(k, \beta)$ not satisfying $(1.5)$, i.e. $k = 0$ and $\beta \in H_2 (X, Y; \mathbb{Z})$ is in the image of the homomorphism $q_Y$ in $(1.7)$, as follows. Let $\gamma_1, \ldots, \gamma_l$ be elements of $\{1\} \cup H^{2*}(X,Y; R)$. If $[Y]^X \neq 0$, we define

$$ \langle \gamma_1, \ldots, \gamma_l \rangle^\omega_{\beta,0} = 0. $$

Suppose next that $[Y]^X = 0$. Let $\Gamma_1, \ldots, \Gamma_l$ be generic pseudocycles to $X$ so that $\Gamma_i = \text{id}_X$ if $\gamma_i = 1$ and $\Gamma_i$ is a pseudocycle into $X-Y$ representing the Poincare dual of $\gamma_i$ otherwise. For $B \in H_2(X; \mathbb{Z})$, let $(\lambda^j_{B,(\gamma_i)_{i\in[l]}})_{j\in[N]} \in R^N$ be such that

$$ [f^C_{B,(\Gamma_i)_{i\in[l]}}] = \sum_{j=1}^N \lambda^j_{B,(\gamma_i)_{i\in[l]}} \text{PD} X (\gamma_j^\ast |_X) \in H_*(X; R); $$

the tuple $(\lambda^j_{B,(\gamma_i)_{i\in[l]}})_{j\in[N]}$ depends only on $B$, $\gamma_1, \ldots, \gamma_l$, and $\gamma_2^\ast, \ldots, \gamma_N^\ast$. Define

$$ \langle \gamma_1, \ldots, \gamma_l \rangle^\omega_{\beta,0} = \text{RHS of (3.4)} + \sum_{B \in q_\beta} (-1)^{w_2(B)} \text{lk}_B \left( f^C_{B,(\Gamma_i)_{i\in[l]}} - \sum_{j=1}^N \lambda^j_{B,(\gamma_i)_{i\in[l]}} \Gamma_j^\ast \right) $$

in this case. This number depends on the span of the elements $\gamma_2^\ast, \ldots, \gamma_N^\ast$ in $H^{2*}(X,Y; R)$, but not on the choice of pseudocycles $\Gamma_1, \ldots, \Gamma_l$ and $\Gamma_2^\ast, \ldots, \Gamma_N^\ast$ representing the Poincare duals of $\gamma_1, \ldots, \gamma_l$ and $\gamma_2^\ast, \ldots, \gamma_N^\ast$, respectively. For example,

$$ \langle \gamma_1, \gamma_2 \rangle^\omega_{0,0} = \text{lk}_\beta \left( \Gamma_1 \cap \Gamma_2 - \sum_{j=1}^N \lambda^j_{\gamma_1 \gamma_2, \gamma_j^\ast} \right), \quad \text{where} \quad \gamma_1 \gamma_2 \equiv \sum_{j=1}^N \lambda^j_{\gamma_1 \gamma_2, \gamma_j^\ast} |_X \in H^*(X; \mathbb{Q}). $$
Let \( \gamma \equiv (\gamma_1, \ldots, \gamma_l) \) be a tuple of elements of \( \{1\} \cup H^{2*}(X,Y;R) \). For \( I \subset \{1,2,\ldots,l\} \), we denote by \( \gamma_I \) the \(|I|\)-tuple consisting of the entries of \( \gamma \) indexed by \( I \). If in addition \( \beta \in H_2(X,Y;\mathbb{Z}) \), define

\[
k_\beta(\gamma_I) \equiv \frac{1}{2} \left( \mu_\omega(\beta) - \sum_{i \in I} (\deg \gamma_i - 2) \right), \quad \langle \gamma_I \rangle_{\beta,k_\beta(\gamma_I)}^{\omega,0} = \begin{cases} \langle \gamma_I \rangle_{\beta,k_\beta(\gamma_I)}^{\omega,0} & \text{if } k_\beta(\gamma_I) \geq 0; \\ 0, \quad & \text{otherwise.} \end{cases}
\]

For \( i, j = 1,2,\ldots,l \), we define

\[
\mathcal{P}(l) = \{(I,J) : \{1,2,\ldots,l\} = I \cup J, \ 1 \in I\}, \\
\mathcal{P}_i(l) = \{(I,J) \in \mathcal{P}(l) : i \in I\}, \\
\mathcal{P}_{ij}(l) = \{(I,J) \in \mathcal{P}(l) : j \in J\}, \\
\mathcal{P}_{i,j}(l) = \mathcal{P}_i(l) \cap \mathcal{P}_j(l).
\]

For \( \beta \in H_2(X,Y;\mathbb{Z}) \), let

\[
\mathcal{P}_C(\beta) = \{ (\beta', B) \in H_2(X,Y;\mathbb{Z}) \oplus H_2(X;\mathbb{Z}) : \beta' + q_Y(B) = \beta \}, \\
\mathcal{P}_R(\beta) = \{ (\beta_1, \beta_2) \in H_2(X,Y;\mathbb{Z}) \oplus H_2(X,Y;\mathbb{Z}) : \beta_1 + \beta_2 = \beta \}.
\]

Combining Theorem 1.1 above with Theorem 2.10 in [2], we obtain relations between Welschinger’s open invariants [1.3] as stated below.

**Corollary 1.3** Let \( R \) be a field and \( (X,\omega,Y) \), \( \omega_5 \), \( \beta \), and \( \gamma \equiv (\gamma_1, \ldots, \gamma_l) \) be as in Theorem 1.1 with

\[
k \equiv \frac{1}{2} \left( \mu_\omega(\beta) - \sum_{i=1}^l (\deg \mu_i - 2) \right) - 1 \geq 0.
\]

Suppose in addition that the homomorphism 1.9 is trivial.

(R\textsuperscript{WDVV1}) If \( l \geq 2 \) and \( k \geq 1 \), then

\[
\sum_{(\beta',B) \in \mathcal{P}_C(\beta)} \sum_{i,j \in [N]} \langle \gamma_I | X, \gamma_\ast_i | X \rangle_B g^{ij} \langle \gamma_j, \gamma_J \rangle_{\beta'}^{\omega,05} = \sum_{(\beta_1,\beta_2) \in \mathcal{P}_R(\beta)} \left( \begin{array}{c} k-1 \\ k_\beta_1(\gamma_I) \end{array} \right) \langle \gamma_I \rangle_{\beta_1}^{\omega,05} \langle \gamma_J \rangle_{\beta_2}^{\omega,05} - \sum_{(\beta_1,\beta_2) \in \mathcal{P}_R(\beta)} \left( \begin{array}{c} k-1 \\ k_\beta_1(\gamma_I) - 1 \end{array} \right) \langle \gamma_I \rangle_{\beta_1}^{\omega,05} \langle \gamma_J \rangle_{\beta_2}^{\omega,05}.
\]

(R\textsuperscript{WDVV2}) If \( l \geq 3 \), then

\[
\sum_{(\beta',B) \in \mathcal{P}_C(\beta)} \sum_{i,j \in [N]} \langle \gamma_I | X, \gamma_\ast_i | X \rangle_B g^{ij} \langle \gamma_j, \gamma_J \rangle_{\beta'}^{\omega,05} = \sum_{(\beta_1,\beta_2) \in \mathcal{P}_R(\beta)} \left( \begin{array}{c} k \\ k_\beta_1(\gamma_I) \end{array} \right) \langle \gamma_I \rangle_{\beta_1}^{\omega,05} \langle \gamma_J \rangle_{\beta_2}^{\omega,05} - \sum_{(\beta_1,\beta_2) \in \mathcal{P}_R(\beta)} \left( \begin{array}{c} k \\ k_\beta_1(\gamma_I) \end{array} \right) \langle \gamma_I \rangle_{\beta_1}^{\omega,05} \langle \gamma_J \rangle_{\beta_2}^{\omega,05}.
\]

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2 Preliminaries

Section 2.1 recalls the orientation conventions for fiber products and some of their properties from [2, Sec. 5.1]. The combinatorial objects needed for the geometric presentation of the open invariants of [12] in symplectic sixfolds is gathered in Section 2.2. We describe the relevant moduli spaces of stable disk maps and specify their orientations in Section 2.3. Section 2.4 specializes the geometric definition of bounding chain from [2] to symplectic sixfolds and uses it to define counts $J$-holomorphic disks.

2.1 Fiber products

We say a short exact sequence of oriented vector spaces

\[ 0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \]

is orientation-compatible if for an oriented basis \( (v'_1, \ldots, v'_m) \) of \( V' \), an oriented basis \( (v''_1, \ldots, v''_n) \) of \( V'' \), and a splitting \( j : V'' \longrightarrow V, (v'_1, \ldots, v'_m, j(v''_1), \ldots, j(v''_n)) \) is an oriented basis of \( V \). We say it has sign \( (-1)^{c} \) if it becomes orientation-compatible after twisting the orientation of \( V \) by \( (-1)^{c} \). We use the analogous terminology for short exact sequences of Fredholm operators with respect to orientations of their determinants; see [19, Section 2].

Let \( M \) be an oriented manifold with boundary \( \partial M \). We orient the normal bundle \( N \) to \( \partial M \) by the outer normal direction and orient \( \partial M \) so that the short exact sequence

\[ 0 \longrightarrow T_p \partial M \longrightarrow T_p M \longrightarrow N \longrightarrow 0 \]

is orientation-compatible at each point \( p \in \partial M \). We refer to this orientation of \( \partial M \) as the boundary orientation.

We orient \( M \times M \) by the usual product orientation and the diagonal \( \Delta_M \subset M \times M \) by the diffeomorphism

\[ M \longrightarrow \Delta_M, \quad p \longrightarrow (p, p). \]

We orient the normal bundle \( N \Delta_M \) of \( \Delta_M \) so that the short exact sequence

\[ 0 \longrightarrow T_{(p,p)} \Delta_M \longrightarrow T_{(p,p)} (M \times M) \longrightarrow N \Delta_M |_{(p,p)} \longrightarrow 0 \]

is orientation-compatible for each point \( p \in M \). Thus, the isomorphism

\[ N \Delta_M |_{(p,p)} \longrightarrow T_p M, \quad [v, w] \longrightarrow w - v, \]

respects the orientations.

For maps \( f : M \longrightarrow X \) and \( g : \Gamma \longrightarrow X \), we denote by

\[ f \times_{\Gamma} g \equiv M_f \times_g \Gamma \equiv \{(p, q) \in M \times \Gamma : f(p) = g(q)\} \]

their fiber product. If \( M, \Gamma, \) and \( X \) are oriented manifolds \((M, \Gamma \) possibly with boundary) and \( f, f |_{\partial M} \) are transverse to \( g, g |_{\partial \Gamma} \), we orient \( M_f \times_g \Gamma \) so that the short exact sequence

\[ 0 \longrightarrow T_{(p,q)} (M_f \times_g \Gamma) \longrightarrow T_{(p,q)} (M \times \Gamma) \xrightarrow{[d \eta f, d \eta g]} N \Delta_X |_{(f(p), g(q))} \longrightarrow 0 \]

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is orientation-compatible for every \((p, q) \in M_f \times_g \Gamma\). The exact sequence

\[
0 \longrightarrow T_{(p, q)}(M_f \times_g \Gamma) \longrightarrow T_{(p, q)}(M \times \Gamma) \xrightarrow{d_{g \circ f} - d_f} T_{f(p)}X \longrightarrow 0
\]

is then orientation-compatible as well. We refer to this orientation of \(M_f \times_g \Gamma\) as the fiber product orientation.

**Lemma 2.1** With the assumptions as above,

\[
\hat{\partial}(M_f \times_g \Gamma) = (-1)^{\dim X} (-1)^{\dim \Gamma} (\hat{\partial}M_f \times_g \Gamma \sqcup M_f \times_g \Gamma).
\]

For a diffeomorphism \(\sigma: M \longrightarrow M\) between oriented manifolds, we define \(\text{sgn } \sigma = 1\) if \(\sigma\) is everywhere orientation-preserving and \(\text{sgn } \sigma = -1\) if \(\sigma\) is everywhere orientation-reversing; this notion is also well-defined if \(M\) is orientable and \(\sigma\) preserves each connected component of \(M\).

**Lemma 2.2** Suppose \(\sigma_M, \sigma_\Gamma, \sigma_X\) are diffeomorphisms of \(M, \Gamma, X\), respectively, with well-defined signs. If the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\sigma_M \downarrow & & \sigma_X \\
M & \xrightarrow{f} & X
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma & \xleftarrow{g} & \Gamma \\
\sigma_\Gamma \downarrow & & \sigma_\Gamma \\
\Gamma & \xleftarrow{g} & \Gamma
\end{array}
\]

commutes, then the sign of the diffeomorphism

\[
(M_f \times_g \Gamma) \longrightarrow (\sigma_M(p), \sigma_\Gamma(q)),
\]

is \((\text{sgn } \sigma_M)(\text{sgn } \sigma_\Gamma)(\text{sgn } \sigma_X)\).

Let \(M, \Gamma, X\) and \(f, g\) be as above Lemma 2.1. Suppose in addition that \(e: M \longrightarrow Y\) and \(h: C \longrightarrow Y\). Let \(e': M_f \times_g \Gamma \longrightarrow Y\) be the map induced by \(e\); see Figure 1. There is then a natural bijection

\[
(M_f \times_g \Gamma)_{e' \times h} C \approx M_{(f,e) \times h} (\Gamma \times C) \quad (2.1)
\]

If \(C, Y\) are oriented manifolds and all relevant maps are transverse, then both sides of this bijection inherit fiber product orientations. For any map \(h: M \longrightarrow Z\) between manifolds, let

\[
\text{codim } h = \dim Z - \dim M.
\]

**Lemma 2.3** The diffeomorphism (2.1) has sign \((-1)^{(\dim X)(\text{codim } h)}\) with respect to the fiber product orientations on the two sides.

### 2.2 Combinatorial notation

Let \((X, \omega)\) be a compact symplectic sixfold, \(Y \subset X\) be a compact Lagrangian submanifold,

\[
H_2^\omega(X, Y) = \{ \beta \in H_2(X, Y; \mathbb{Z}) : \omega(\beta) > 0 \text{ or } \beta = 0 \},
\]

and \(\mathcal{J}_\omega\) be the space of \(\omega\)-compatible almost complex structures on \(X\). We denote by \(\text{PC}(X - Y)\) the collection of pseudocycles to \(X - Y\) with coefficients in \(R\) as defined in [18, Sec 1], by \(\text{FPC}(X - Y)\)
the collection of finite subsets of $PC(X-Y)$, and by $FP_t(Y)$ the collection of finite subsets of $Y$. Let
\[ C_\omega(Y) = \{ (\beta, K, L) : \beta \in H_2^\omega(X, Y), K \in FP_t(Y), L \in FP(X-Y), (\beta, K, L) \neq (0, \emptyset, \emptyset) \}. \]
This collection has a natural partial order:
\[ (\beta', K', L') \leq (\beta, K, L) \quad \text{if} \quad \beta - \beta' \in H_2^\omega(X, Y), \quad K' \subset K, \quad \text{and} \quad L' \subset L. \]
The elements $(0, K, L)$ of $C_\omega(Y)$ with $|K| + |L| = 1$ are minimal with respect to this partial order. For each element $\alpha = (\beta, K, L)$ of $C_\omega(Y)$, we define
\[ \beta(\alpha) = \beta, \quad K(\alpha) = K, \quad L(\alpha) = L, \]
\[ \dim(\alpha) = \mu_Y^\omega(\beta) - 2|K| - \sum_{\Gamma \in L}(\text{codim} \Gamma - 2), \quad C_{\omega \alpha}(Y) = \{ \alpha' \in C_\omega(Y) : \alpha' < \alpha \}. \]
For $\alpha \in C_\omega(Y)$, let
\[ D_\omega(\alpha) = \left\{ \left( \beta_*, k_*, L_*, (\alpha_i)_{i \in [k_*]} \right) : \beta_* \in H_2^\omega(X, Y), k_* \in \mathbb{Z}^\geq 0, L_* \subset L(\alpha), (\beta_*, k_*, L_*) \neq (0, 1, \emptyset), \alpha_i \in C_\omega(Y) \forall i \in [k_*], \sum_{i=1}^{k_*} \beta(\alpha_i) = \beta(\alpha), \bigcup_{i=1}^{k_*} K(\alpha_i) = K(\alpha), L_* \cup \bigcup_{i=1}^{k_*} L_i(\alpha) = L(\alpha) \right\}. \]
Since $\alpha_i < \alpha$ for every
\[ \eta = (\beta_*, k_*, L_*, (\alpha_i)_{i \in [k_*]}) \equiv (\beta_*, k_*, L_*, (\beta_i, K_i, L_i)_{i \in [k_*]}) \in D_\omega(\alpha) \tag{2.2} \]
and every $i \in [k_*]$, $k_* = 0$ if $\alpha$ is a minimal element of $C_\omega(Y)$. Thus,
\[ D_\omega(0, \{ pt \}, \emptyset) = \emptyset \quad \forall \; pt \in Y \quad \text{and} \quad D_\omega(0, \emptyset, \{ \Gamma \}) = \{ (0, 0, \{ \Gamma \}, \emptyset) \} \quad \forall \Gamma \in PC(X-Y). \]
For $\eta \in D_\omega(\alpha)$ as in $(2.2)$ and $i \in [k_*]$, we define
\[ \beta_*(\eta) = \beta_*, \quad k_*(\eta) = k_*, \quad L_*(\eta) = L_*, \]
\[ \beta_i(\eta) = \beta_i, \quad K_i(\eta) = K_i, \quad L_i(\eta) = L_i, \quad \alpha_i(\eta) = \alpha_i = (\beta_i, K_i, L_i). \]
2.3 Moduli spaces

We denote by $\mathbb{D}^2 \subset \mathbb{C}$ the unit disk with the induced complex structure, by $\mathbb{D}^2 \cup \mathbb{D}^2$ the union of two disks joined at a pair of boundary points, and by $S^1 \subset \mathbb{D}^2$ and $S^1 \cup S^1 \subset \mathbb{D}^2 \cup \mathbb{D}^2$ the respective boundaries. We orient the boundaries counterclockwise; thus, starting from a smooth point $x_0$ of $S^1 \cup S^1$, we proceed counterclockwise to the node nd, then circle the second copy of $S^1$ counterclockwise back to nd, and return to $x_0$ counterclockwise from nd. We call smooth points $x_0, x_1, \ldots, x_k$ on $S^1$ or $S^1 \cup S^1$ ordered by position if they are traversed counterclockwise.

Let $k, l \in \mathbb{Z}_{\geq 0}$ with $k + 2l \geq 3$. We denote by $\mathcal{M}^{\text{uo}}_{k,l}$ the moduli space of $k$ distinct boundary marked points $x_1, \ldots, x_k$ and $l$ distinct interior marked points $z_1, \ldots, z_l$ on the unit disk $\mathbb{D}$ (the superscript $\text{uo}$ refers to the boundary marked points being unordered with respect to their position on $S^1 \subset \mathbb{D}^2$ relative to the order on $[k]$). We orient $\mathcal{M}^{\text{uo}}_{1,1}$ as a plus point. The space $\mathcal{M}^{\text{uo}}_{3,0}$ consists of two points, $\mathcal{C}^+_0$ with the three boundary points ordered by position and $\mathcal{C}^-_{3,0}$ with the three boundary points not ordered by position. We orient $\mathcal{C}^+_0$ as a plus point and $\mathcal{C}^-_{3,0}$ as a minus point. We identify $\mathcal{M}^{\text{uo}}_{0,2}$ with the interval $(0, 1)$ by taking $z_1 = 0$ and $z_2 \in (0, 1)$ and orient it by the negative orientation of $(0, 1)$.

We orient other $\mathcal{M}^{\text{uo}}_{k,l}$ inductively. If $k \geq 1$, we orient $\mathcal{M}^{\text{uo}}_{k,l}$ so that the short exact sequence

$$0 \rightarrow T_{x_k}S^1 \rightarrow TM^{\text{uo}}_{k,l} \xrightarrow{df_k^R} TM^{\text{uo}}_{k-1,l} \rightarrow 0 \quad (2.3)$$

induced by the forgetful morphism $f_k^R$ dropping $x_k$ has sign $(-1)^k$ with respect to the counterclockwise orientation of $S^1$. Thus,

$$TM^{\text{uo}}_{k,l} \approx TM^{\text{uo}}_{k-1,l} \oplus T_{x_k}S^1.$$ 

If $l \geq 1$, we orient $\mathcal{M}^{\text{uo}}_{k,l}$ so that the short exact sequence

$$0 \rightarrow T_{z_l} \mathbb{D} \rightarrow TM^{\text{uo}}_{k,l} \xrightarrow{df_l^C} TM^{\text{uo}}_{k,l-1} \rightarrow 0 \quad (2.4)$$

induced by the forgetful morphism $f_l^C$ dropping $z_l$ is orientation-compatible with respect to the complex orientation of $\mathbb{D}$. By a direct check, the orientations of $\mathcal{M}^{\text{uo}}_{1,2}$ induced from $\mathcal{M}^{\text{uo}}_{0,2}$ via $(2.3)$ and from $\mathcal{M}^{\text{uo}}_{1,1}$ via $(2.4)$ are the same, and the orientations of $\mathcal{M}^{\text{uo}}_{3,1}$ induced from $\mathcal{M}^{\text{uo}}_{3,0}$ via $(2.3)$ and from $\mathcal{M}^{\text{uo}}_{3,0}$ via $(2.4)$ are also the same. Since the fibers of $f_l^C$ are even-dimensional, it follows that the orientation on $\mathcal{M}^{\text{uo}}_{k,l}$ above is well-defined.

Let $(X, \omega)$ be a symplectic manifold, $Y \subset X$ be a Lagrangian submanifold, $\beta \in H^2(X, \mathbb{R})$, and $J \in \mathcal{J}_\omega$. For a finite ordered set $K$ and a finite set $L$, we denote by $\mathcal{M}^{\text{uo}, \star}_{K,L}(\beta; J)$ the moduli space of stable $J$-holomorphic degree $\beta$ maps from $(\mathbb{D}^2, S^1)$ and $(\mathbb{D}^2 \cup \mathbb{D}^2, S^1 \cup S^1)$ to $(X, Y)$ with boundary and interior marked points indexed by $K$ and $L$, respectively. Let

$$\mathcal{M}^{\text{uo}, \star}_{K,L}(\beta; J) \subset \mathcal{M}^{\text{uo}, \star}_{K,L}(\beta; J)$$

be the subspace of maps from $(\mathbb{D}^2, S^1)$. If $K = [k]$ for $k \in \mathbb{Z}_{\geq 0}$ (resp. $L = [l]$ for $l \in \mathbb{Z}_{\geq 0}$), we write $k$ for $K$ (resp. $l$ for $L$) in the subscripts of these moduli spaces. For

$$[u] \equiv \left[ u: (\mathbb{D}, S^1) \rightarrow (X, Y), (x_i)_{i \in [k]}, (z_i)_{i \in [l]} \right] \in \mathcal{M}^{\text{uo}, \star}_{K,L}(\beta; J), \quad (2.5)$$
let
\[ D_{f,u} : \Gamma(u^*TX, u|_{S^1}^*TY) \longrightarrow \Gamma(T^d S^{0,1} \otimes \mathbb{C} u^*(TX, J)) \]
be the linearization of the \((\bar{\partial}_J)\)-operator on the space of maps from \((\mathbb{D}, S^1)\) to \((X, Y)\). By Proposition 8.1.1 in [5], a relative OSpin-structure \(\mathfrak{os}\) on \(Y\) determines an orientation on \(\det(D_{f,u})\).

We orient \(\mathcal{M}^{\mu}_{k,l}(\beta; J)\) by requiring the short exact sequence
\[ 0 \longrightarrow \ker D_{f,u} \longrightarrow T_u\mathcal{M}^{\mu}_{k,l}(\beta; J) \xrightarrow{\text{id}} T_{\bar{f}(u)}\mathcal{M}^{\mu}_{k,l} \longrightarrow 0 \]
to be orientation-compatible, where \(\bar{f}\) is the forgetful morphism dropping the map part of \(u\). This orientation extends over \(\mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J)\). If \(K\) is a finite ordered set and \(L\) is a finite set, we orient \(\mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J)\) from \(\mathcal{M}^{\mu,\bullet}_{|K|,|L|}(\beta; J)\) by identifying \(K\) with \([|K|]\) as ordered sets and \(L\) with \([|L|]\) as sets.

Remark 2.4. The above paragraph endows \(\mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J)\) with an orientation under the assumption that \(|K|+2|L| \geq 3\). If \(|K|+2|L| < 3\), one first stabilizes the domain of \(u\) by adding one or two interior marked points, then orients the tangent space of the resulting map as above, and finally drops the added marked points using the canonical complex orientation of \(\mathbb{D}\); see the proof of Corollary 1.8 in [7].

For \(i \in K\) and \(i \in L\), let
\[ \text{ev}_{ib_i} : \mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J) \longrightarrow Y \quad \text{and} \quad \text{ev}_i : \mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J) \longrightarrow X \]
be the evaluation morphisms at the \(i\)-th boundary marked point and the \(i\)-th interior marked point, respectively. If \(M \subset \mathcal{M}^{\mu,\bullet}_{K,L}(\beta; J)\), we denote the restrictions of \(\text{ev}_{ib_i}\) and \(\text{ev}_i\) to \(M\) also by \(\text{ev}_{ib_i}\) and \(\text{ev}_i\). If in addition \(m, m' \in \mathbb{Z}_{\geq 0}\),
\[ (b_s : Z_{bs} \longrightarrow Y)_{s \in [m]} \quad \text{and} \quad (\Gamma_i : Z_{\Gamma_i} \longrightarrow X)_{s \in [m']} \]
are tuples of maps and \(i_1, \ldots, i_m \in [k]\) and \(j_1, \ldots, j_{m'} \in L\) are distinct elements, let
\[ M \times_{fb} ((i_s, b_s)_{s \in [m]} ; (j_s, \Gamma_s)_{s \in [m']}) \]
be their fiber product with \(M\). If \(M\) is an oriented manifold and \(b_s\) and \(\Gamma_s\) are smooth maps from oriented manifolds satisfying the appropriate transversality conditions, then we orient this space as in Section 2.1. For \(i \in [k]\) with \(i \neq i_s\) for any \(s \in [m]\) (resp. \(i \in L\) with \(i \neq j_s\) for any \(s \in [m']\)), we define
\[ \text{ev}_{ib_i} \ (\text{resp.} \ \text{ev}_i) : M \times_{fb} ((i_s, b_s)_{s \in [m]} ; (j_s, \Gamma_s)_{s \in [m']}) \longrightarrow Y \ (\text{resp.} \ X) \]
to be the composition of the evaluation map \(\text{ev}_{ib_i}\) (resp. \(\text{ev}_i\)) defined above with the projection to the first component.

### 2.4 Open Gromov-Witten invariants

In the remainder of this paper, we use the terms (bordered) \(\mathbb{Z}_2\)-pseudocycle and pseudocycle to mean the respective pseudocycles in the usual sense taken with \(R\)-coefficients; see the last part of Section 3 in [3] for precise definitions. We recall that every \(R\)-homology class in a manifold can
be represented by a pseudocycle in this sense, which is unique up to equivalence; see Theorem 1.1 in [13].

Let \((X, \omega)\) be a symplectic sixfold and \(Y \subset X\) be a Lagrangian submanifold. For a point \(pt \in Y\), we denote its inclusion into \(Y\) also by \(pt\). For \(\beta \in H^2_\Sigma (X, Y)\), \(k \in \mathbb{Z}^\geq 0\), a finite set \(L\), and \(J \in \mathcal{J}_\omega\), let

\[
\mathcal{M}^\star_{k,L} (\beta; J) \subset \mathcal{M}^\star_{k,L} (\beta; J)
\]

be the subspace of maps with the boundary marked points ordered by position. If in addition \(\eta \in \mathcal{D}_\omega (\alpha)\) for some \(\alpha \in \mathcal{C}_\omega (Y)\), define

\[
\mathcal{M}^\bullet_{\eta, J} = \mathcal{M}^\star_{k, \eta, L} (\beta_\bullet (\eta); J), \quad \mathcal{M}^+_{\eta, J} = \mathcal{M}^\star_{k, \eta, 1, L} (\beta_\bullet (\eta); J).
\]

**Definition 2.5.** Let \(R\), \((X, \omega)\), \(Y\), \(\mathfrak{os}\), and \(\alpha = (\beta, K, L)\) be as in Theorem 1.1. A bounding chain on \((\alpha, J)\) is a collection \((b_{\alpha'})_{\alpha' \in \mathfrak{C}_{\omega, \alpha} (Y)}\) of bordered pseudocycles into \(Y\) such that

1. \(\dim b_{\alpha'} = \dim (\alpha') + 2\) for all \(\alpha' \in \mathfrak{C}_{\omega, \alpha} (Y)\);
2. \(b_{\alpha'} = \emptyset\) unless \(\alpha' = (0, \{pt\}, \emptyset)\) for some \(pt \in K\) or \(\dim (\alpha') = 0\);
3. \(b_{(0, \{pt\}, \emptyset)} = pt\) for all \(pt \in K\);
4. for all \(\alpha' \in \mathfrak{C}_{\omega, \alpha} (Y)\) such that \(\dim (\alpha') = 0\),

\[
\partial b_{\alpha'} = \left( \text{evb}_1 : \bigcup_{\eta \in \mathcal{D}_\omega (\alpha')} (-1)^{k_\bullet (\eta)} \mathcal{M}^+_{\eta, J} \times \mathcal{F} ((i + 1, b_{\alpha_i (\eta)}) \in [k_\bullet (\eta)]; (i, \Gamma_i) \in L_\bullet (\eta)) \to Y \right).
\]

Since the dimension of every pseudocycle \(\Gamma_i \in L\) is even, the oriented morphism

\[
b_{\alpha'} = \left( \text{evb}_1 : (-1)^{k_\bullet (\eta)} \mathcal{M}^+_{\eta, J} \times \mathcal{F} ((i + 1, b_{\alpha_i (\eta)}) \in [k_\bullet (\eta)]; (i, \Gamma_i) \in L_\bullet (\eta)) \to Y \right)
\]

in (2.6) does not depend on the choice of identification of \(L_\bullet (\eta)\) with \([L_\bullet (\eta)]\). By Lemma 3.1 in [2], the map

\[
b_{\alpha'} = \bigcup_{\eta \in \mathcal{D}_\omega (\alpha')} b_{\alpha'}
\]

with orientation induced by the relative OSpin-structure \(\mathfrak{os}\) on \(Y\) is a pseudocycle for every \(\alpha' \in \mathfrak{C}_{\omega, \alpha} (Y)\). If in addition \(\dim (\alpha) = 2\), then \(b_{\alpha}\) is a pseudocycle of codimension 0. Its degree determines a count of \(J\)-holomorphic disks in \((X, Y)\) through \(|K|+1\) points in \(Y\) as in (1.5).

A bounding chain \((b_{\alpha'})_{\alpha' \in \mathfrak{C}_{\omega, \alpha} (Y)}\) as in Definition 2.5 can also be used to define the counts (1.5) of \(J\)-holomorphic disks in the following way. We denote the signed cardinality of a finite set \(S\) of signed points by \(|S|^\pm\). If \(S\) is not a finite set of signed points, we set \(|S|^\pm = 0\). If \(\eta \in \mathcal{D}_\omega (\alpha)\), let

\[
s^\bullet (\eta) = \begin{cases} \frac{1}{k_\bullet (\eta)} - \frac{1}{2}, & \text{if } k_\bullet (\eta) \neq 0, \\ 1, & \text{if } k_\bullet (\eta) = 0. \end{cases}
\]

Define

\[
\langle L \rangle^\omega, \mathfrak{os}_{\beta; K} \equiv \sum_{\eta \in \mathcal{D}_\omega (\alpha)} (-1)^{k_\bullet (\eta)} s^\bullet (\eta) \left| \mathcal{M}^\bullet_{\eta, J} \times \mathcal{F} ((i, b_{\alpha_i (\eta)}) \in [k_\bullet (\eta)]; (i, \Gamma_i) \in L_\bullet (\eta)) \right|^\pm + \frac{1}{2} \sum_{p \in K} \langle L \rangle^\omega, \mathfrak{os}_{\beta; K \setminus \{p\}}.
\]
This number vanishes unless \( \dim(\alpha) = 0 \). Unlike (1.6), (2.9) provides a definition of the counts (1.5) with \( k = 0 \). By Theorem 2.7(2) in [2],
\[
\langle L \rangle^{\omega, \partial}_\beta; K \text{ in (2.9)} = \langle L \rangle^{\omega, \partial}_\beta; K - \{p_t\} \text{ in (1.6)}
\]
for any \( p_t \in K \) if \( \langle L \rangle^{\beta, K - \{p_t\}} \) does not depend on \( p_t \in K \).

3 Proof of Theorem 1.1

For the remainder of the paper, we take \((X, \omega, Y)\) and \(\mathcal{OS}\) as in Theorem 1.1. Let \( \gamma_1, \gamma_2 \) be smooth maps from oriented closed one-manifolds into the oriented closed three-manifold \( Y \) with disjoint images. If \( \gamma_1 = \partial b_1 \) and \( \gamma_2 = \partial b_2 \) for some bordered pseudocycles \( b_1 \) and \( b_2 \) into \( Y \) so that \( b_1 \) is transverse to \( \gamma_2 \) and \( b_2 \) is transverse to \( \gamma_1 \), we define
\[
\text{lk}(\gamma_1, \gamma_2) \equiv \left| [b_1 \times b_2]^{\pm} \right| = -[\gamma_1 \times b_2]^{\pm} = [b_2 \times b_1]^{\pm} = -[\gamma_2 \times b_1]^{\pm}:
\]
(3.1)
the first and last equalities above hold by Lemma 2.1 while the middle one follows from Lemma 2.2.

The sign of a point \((p, q)\) of \( b_1 \times b_2 \) is the sign of the isomorphism
\[
T_p \text{dom}(b_1) \oplus T_q \text{dom}(\gamma_2) \longrightarrow T_{b_1(p)} Y = T_{\gamma_2(q)} Y, \quad (v, w) \longrightarrow d_p b_1(v) + d_q \gamma_2(w).
\]
The linking number (3.1) of the one-cycles \( \gamma_1 \) and \( \gamma_2 \) that bound in \( Y \) does not depend on the choice of \( b_1, b_2 \). In this section, we take linking numbers of the boundaries \( \partial u \) of \( J \)-holomorphic maps \( u \) from \((D^2, S^1)\) to \((X, Y)\). By the injectivity of (1.2), these boundaries also bound in \( Y \) and thus have well-defined linking numbers.

3.1 Bounding chains and Welschinger’s invariants

For \( \beta_1, \ldots, \beta_m \in H^2_\partial(X, Y) \), we denote by \( \mathcal{M}^{\mu_0}_{K, \beta, L}(\beta_1, \ldots, \beta_m; J) \) the moduli space of unions of \( m \) \( J \)-holomorphic disks in classes \( \beta_1, \ldots, \beta_m \) with \( L \)-labeled interior marked points and \( K \)-labeled boundary marked points between the \( m \) disks. In contrast to Section 2.4 in [17], we do not order the disks or orient this moduli space. Let
\[
\mathcal{M}^{\mu_0 \circ, \partial}_{K, \beta, L}(\beta_1, \ldots, \beta_m; J) \subset \mathcal{M}^{\mu_0}_{K, \beta, L}(\beta_1, \ldots, \beta_m; J)
\]
(3.2)
be the dense open subset of the multi-disks whose \( m \) components have pairwise disjoint boundaries in \( Y \). We extend the definitions of the evaluations maps \( e_{b_i} \) and \( e_{v_i} \) and of the associated fiber product \( \times_{b_i} \) of Section 2.3 to the moduli spaces in (3.2).

Let \( \alpha \equiv (\beta, K, L) \) and \( J \) be as in Theorem 1.1 and \( p_1, \ldots, p_k \) be an ordering of the elements of \( K \). For any element \( \alpha' \equiv (\beta', K', L') \) of \( C_{\omega, \alpha}(Y) \cup \{\alpha\} \), we endow \( K' \subset K \) with the order induced from \( K \). We define the spaces of (constrained) single \( \alpha' \)-disks and \( \alpha' \)-multi-disks by
\[
\text{SD}(\alpha') \equiv \mathcal{M}^{\mu_0}_{K', \beta', L'}(\beta'; J) \times_{b_i} ((i, p_i) \in K'; (i, \Gamma_i) \in L') \quad \text{and}
\]
\[
\text{MD}(\alpha') \equiv \bigcup_{m=1}^{\infty} \bigcup_{\beta_1, \ldots, \beta_m \in H^2_\partial(X, Y)} \mathcal{M}^{\mu_0 \circ}_{K', \beta, L'}(\beta_1, \ldots, \beta_m; J) \times_{b_i} ((i, p_i) \in K'; (i, \Gamma_i) \in L').
\]

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There exists a bounding chain $pW_{\text{elschinger's definition of the open GW-invariant (1.3)}}$ in [17, Sect. 4.1] is equivalent to

and denote by $ST$ be the forgetful morphisms dropping the boundary marked point with index $K$ marked points indexed by the set $K$.

For such an element $u$ of $MD(\alpha')$, we write $u_r \in u$ to indicate that $u_r$ is a component of the multi-disk $u$. Let $\hat{u} : S^1 \sqcup \ldots \sqcup S^1 \to Y$ be the boundary of the components of $u$ with the orientation induced by the complex orientation on the unit disk. If $\dim(\alpha') = 0$ and $u_r \in u$, we denote by $\text{sgn}(u_r)$ the sign of $u_r$ as an element of the fiber product in (3.3) and set

$$\text{sgn}(u) = \prod_{u_r \in u} \text{sgn}(u_r);$$

this sign does not depend on the order on $K$. If $\dim(\alpha') \neq 0$, we define $\text{sgn}(u) = 0$.

For $u \in MD(\alpha')$ as in (3.3), we denote by $K_u$ the complete graph with vertices $u_1, \ldots, u_m$. We call a tree $T \subset K_u$, i.e. a connected subgraph without loops, spanning if $T$ contains all vertices of $K_u$ and denote by $ST(u)$ the set of all spanning trees $T \subset K_u$. Let $\text{lk}(u; T) = \prod_{\text{edge } e \in T} \text{lk}(\hat{u_r}, \hat{u_s}) \forall T \in ST(u)$ and $\text{lk}(u) = \sum_{T \in ST(u)} \text{lk}(u; T)$.

Welschinger’s definition of the open GW-invariant (1.3) in [17, Sect. 4.1] is equivalent to

$$\left\langle \Gamma_1, \ldots, \Gamma_l \right\rangle_{\beta, |K|}^{\omega, GS} = \sum_{u \in MD(\alpha)} \text{sgn}(u) \text{lk}(u).$$

(3.4)

The first statement of the next proposition establishes [W1]. We deduce [W2] from the second statement of this proposition and Proposition 3.2. The two propositions are proved in Sections 3.2 and 3.3.

**Proposition 3.1.** Let $\alpha \in C_\omega(Y)$ and $J$ be as in Theorem 1.1.

(1) There exists a bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega;\alpha}(Y)}$ on $(\alpha, J)$.

(2) For every such bounding chain $(b_{\alpha'})_{\alpha' \in C_{\omega;\alpha}(Y)}$ and $\alpha' \in C_{\omega;\alpha}(Y)$ with $\dim(\alpha') = 0$, the associated closed pseudocyclus (2.8) satisfies

$$\hat{\partial} b_{\alpha'} = b_{\hat{\partial} \alpha'} = (-1)^{|K(\alpha')}| \sum_{u \in MD(\alpha')} \text{sgn}(u) \text{lk}(u) \hat{\partial} u.$$  

(3.5)

If $k \in \mathbb{Z}^+$ and $K \subset [k]$, let

$$\hat{f}_k : \mathcal{M}_{k, 1}^{\text{lo}, *}(\beta; J) \to \mathcal{M}_{k-1, 1}^{\text{lo}, *}(\beta; J) \quad \text{and} \quad \hat{f}_{k, K} : \mathcal{M}_{k, 1}^{\text{lo}, *}(\beta; J) \to \mathcal{M}_{[k]-K, 1}^{\text{lo}, *}(\beta; J)$$

be the forgetful morphisms dropping the boundary marked point with index $k$ and the boundary marked points indexed by the set $K$, respectively.

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Proposition 3.2 (Open Divisor Relation) Suppose \( K' \subset K \subset [k] \), \( L \subset [l] \), and \((b_i)_{i \in K}\) and \((\Gamma_i)_{i \in L}\) are tuples of bordered pseudocycles into \(Y\) and \(X\), respectively. If the codimension of \(b_i\) is 1 for every \(i \in K'\) and

\[ K' \subset \{k', \ldots, k\} \subset K \]

for \( k' \in [k] \), then there exists a dense open subset \( \mathcal{M}^{*}_{[k]-K',t} \) of the target of the induced forgetful morphism

\[
\phi_{k,K'} \colon \mathcal{M}^{\text{no,}*}_{k,t}(\beta; J) \times_{\text{BD}} (i, b_i)_{i \in K}; (i, \Gamma_i)_{i \in L} \to \mathcal{M}^{\text{no,}*}_{[k]-K',t}(\beta; J) \times_{\text{BD}} ((i, b_i)_{i \in K-K'}; (i, \Gamma_i)_{i \in L}) \tag{3.6}
\]

so that \( \phi_{k,K'} \) restricts to a covering map over each connected component \( \mathcal{M} \) of \( \mathcal{M}^{*}_{[k]-K',t} \). If in addition the codimensions of all \( b_i \) are odd and the codimensions of \( \Gamma_i \) are even, then

\[
\deg (\phi_{k,K'}|_{\mathcal{M}}) = (-1)^{|K'|} \prod_{i \in K'} \text{lk}(\partial u_i, \partial b_i)
\]

for any \( u \in \mathcal{M} \).

Remark 3.3. Suppose that \( \phi \) is an anti-symplectic involution on \((X, \omega)\), i.e. \( \phi^2 = \text{id}_X \) and \( \phi^* \omega = -\omega \), \( Y \subset X \) is a topological component of the fixed locus \( X^\phi \) of \( \phi \), \( \phi^* J = -J \), and for every \( i \in [l] \) there exists a diffeomorphism \( \phi_{\Gamma_i} \) of \( \Gamma_i \) such that

\[
\phi \circ \Gamma_i = \Gamma_i \circ \phi_{\Gamma_i} \quad \text{and} \quad \text{sgn} \phi_{\Gamma_i} = -(-1)^{(\dim \Gamma_i)/2}.
\]

Let \( H_{2,\phi}(X, Y) \) be the quotient of \( H_2(X, Y; \mathbb{Z}) \) by the image of the endomorphism \( \{\text{id} + \phi_*\} \). For \( B \in H_{2,\phi}(X, Y) \), let

\[
\text{SD}(B, K, L) = \bigcup_{\alpha' \in \mathcal{C}_\omega(Y), \beta(\alpha') \in B} \text{SD}(\alpha'), \quad \text{MD}(B, K, L) = \bigcup_{\alpha' \in \mathcal{C}_\omega(Y), \beta(\alpha') \in B} \text{MD}(\alpha').
\]

If \( \mathfrak{c} \) denotes the complex conjugation on \( \mathbb{D}^2 \), the replacement of \( u_r \in u \) as in (3.3) with

\[
u' \equiv \left( \left\{ \phi \circ u_r \circ \mathfrak{c}, (x_i)_{i \in K'}, (\mathfrak{c}(z_i))_{i \in L_r}, (i, p_i)_{i \in K}, (i, \Gamma_i)_{i \in L_r} \right\} \in \mathcal{M}^{\text{no,}*}_{K_r, L_r}(-\phi_*(\beta_r); J) \times_{\text{BD}} ((i, p_i)_{i \in K}; (i, \Gamma_i)_{i \in L_r}) \right.
\]

preserves \( \text{MD}(B, K, L) \). If \( u' \in \text{MD}(B, K, L) \) is the resulting element, \( \text{sgn}(u') = \text{sgn}(u) \) by [11, Prop. 5.1]; see also [2, Lem. B.7]. However, \( \partial u \), and \( \partial u' \) are the same circles with the opposite orientations. If precisely one edge of a tree \( T \in \text{ST}(u) \) contains \( u_r \in u \) as a vertex, this implies that \( \text{lk}(u; T) = -\text{lk}(u'; T') \), where \( T' \) is the tree obtained from \( T \) by replacing the vertex \( u_r \) with \( u'_r \). It follows that the collection

\[
\left\{ (u, T) : u \in \text{MD}(B, K, L) - \text{SD}(B, K, L), T \in \text{ST}(u) \right\}
\]

can be split into pairs of elements with opposite values of \( \text{sgn}(u) \text{lk}(u; T) \). Thus,

\[
\sum_{\beta \in B} \langle \Gamma_1, \ldots, \Gamma_l \rangle_{\beta, [K]}^{\omega, \text{reg}} = \sum_{u \in \text{MD}(B, K, L)} \sum_{T \in \text{ST}(u)} \text{sgn}(u) \text{lk}(u; T) = \sum_{u \in \text{SD}(B, K, L)} \text{sgn}(u).
\]

The right-hand side above is the GW-invariant of \((X, \omega, \phi)\) enumerating degree B rational J-holomorphic curves through the specified constraints as defined in [3]. This invariant is a reinterpretation of the invariants defined in [13, 16, 11]; see also [2, (B.12)].
3.2 Main argument

We continue with the setting of Theorem 1.1 and Proposition 3.1. Let $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y) \cup \{\alpha\}$. For $\eta \in \mathcal{D}_\omega(\alpha')$, define

$$K^*_\alpha(\eta) = \{ i \in [k_\alpha(\eta)] : \alpha_i(\eta) \neq (0, \{ pt \}, \emptyset) \forall pt \in K \},$$

$$K^{pt}_\alpha(\eta) = \{ pt \in K : (0, \{ pt \}, \emptyset) = \alpha_i(\eta) \text{ for some } i \in [k_\alpha(\eta)] \},$$

$$\alpha^pt_i(\eta) \equiv (\beta_i(\eta), K^pt_i(\eta), L_i(\eta)) \in \mathcal{C}_{\omega;\alpha}(Y) \cup \{ \alpha \},$$

$$\mathcal{M}^{\alpha \omega, +}_{\eta, J} = \mathcal{M}^{\alpha \omega, \neq}_{[k_\alpha(\eta) + 1], L_i(\eta)}(\beta_i(\eta); J).$$

For $\eta, \eta' \in \mathcal{D}_\omega(\alpha')$, define $\eta \sim \eta'$ if

$$\left( \beta_i(\eta), k_i(\eta), L_i(\eta) \right) = \left( \beta_i(\eta'), k_i(\eta'), L_i(\eta') \right) \text{ and } \left( \alpha_i(\eta) \right)_{i \in [k_\alpha(\eta)]} \text{ is a permutation of } \left( \alpha_i(\eta') \right)_{i \in [k_\alpha(\eta')]}.$$

Denote by $[\eta]$ the equivalence class of $\eta$. With $bb_\eta$ as in (2.7), let

$$bb_{[\eta]} = \bigsqcup_{\eta' \in [\eta]} bb_{\eta'}.$$

We define

$$\text{DMD}(\alpha') \equiv \{ (u, u_\ast, T) : u \in \text{MD}(\alpha'), u_\ast \in u, T \in \text{ST}(u) \},$$

$$\overline{\text{DMD}}(\alpha') \equiv \{ (\eta, u_\ast, (\tilde{u}_i)_{i \in K^*_\alpha(\eta)}) : \eta \in \mathcal{D}_\omega(\alpha'), u_\ast \in \text{SD}(\alpha^pt_i(\eta)), \tilde{u}_i \in \text{DMD}(\alpha_i(\eta)) \forall i \in K^*_\alpha(\eta) \}. \quad (3.7)$$

We define elements $(\eta, u_\ast, (\tilde{u}_i)_{i \in K^*_\alpha(\eta)})$ and $(\eta', u_\ast', (\tilde{u}_i')_{i \in K^*_\alpha(\eta')})$ of the last space to be equivalent if $u_\ast = u_\ast'$, $k_i(\eta) = k_i(\eta')$, and there exists a permutation $\sigma$ of $[k_\alpha(\eta)]$ such that

$$\alpha_i(\eta) = \alpha_{\sigma(i)}(\eta') \forall i \in [k_\alpha(\eta)], \quad \sigma(K^*_\alpha(\eta)) \subset K^*_\alpha(\eta'), \quad \text{and} \quad \tilde{u}_i = \tilde{u}'_{\sigma(i)} \forall i \in K^*_\alpha(\eta).$$

We denote by $\overline{\text{DMD}}(\alpha')$ the quotient of the space in (3.7) by this equivalence relation.

Let $(u, u_\ast, T) \in \text{DMD}(\alpha')$. For each $u_i \in u$, let

$$\beta_i(u_r) \in H^2(X, Y), \quad L_i(u_r) \subset L(\alpha'), \quad \text{and} \quad K(u_i) \subset K(\alpha')$$

be the degree of $u_r$, the interior marked points carried by $u_i$, and the boundary marked points carried by $u_i$, respectively. We denote by $\text{Br}(u_\ast; T)$ the set of branches of $T$ at $u_\ast$, i.e. the trees $T_i$ obtained by removing the vertex $u_\ast$ from the graph $T$. For each $i \in \text{Br}(u_\ast; T)$, we denote by $u_i'$ the set of all vertices of $T_i$ and by $u_i' \in u_i'$ the vertex connected by an edge of $T$ to $u_\ast$. Define

$$\beta_i = \sum_{u_i' \in u_i'} \beta_i(u_r), \quad K_i = \bigsqcup_{u_i' \in u_i'} K_i(u_r), \quad \alpha_i = (\beta_i, K_i, L_i) \in \mathcal{C}_{\omega;\alpha}(Y).$$

Let $\alpha^pt = (0, \{ pt \}, \emptyset)$ for each $pt \in K(u_\ast)$ and

$$k_\ast = |K(u_\ast)| + |\text{Br}(u_\ast; T)|.$$

Identifying $K(u_\ast) \cup \text{Br}(u_\ast; T)$ with $[k_\ast]$, we obtain an element

$$\left( \eta \equiv (\beta(u_\ast), k_\ast, L(u_\ast), (\alpha_i)_{i \in [k_\ast]}), u_\ast, (u_i', u_i'; T_i)_{i \in \text{Br}(u_\ast; T)} \right) \in \overline{\text{DMD}}(\alpha').$$
The induced element of $\overline{\text{DMD}}(\alpha')$ does not depend on the choice of this identification. In this way, we obtain a natural bijection

$$\text{DMD}(\alpha') \rightarrow \overline{\text{DMD}}(\alpha').$$ (3.8)

For $k \in \mathbb{Z}_{>0}$, we denote by $S_k$ the $k$-th symmetric group. For $\sigma \in S_{k}(\eta)$, define

$$\iota_{\eta,\sigma}: \mathfrak{m}^+_{\eta,\sigma} \rightarrow \mathfrak{m}_{\eta,\sigma}^{\text{uo}+},$$

$$\iota_{\eta,\sigma}(u; x_1, x_2, \ldots, x_{k(\eta)+1}, (z_i)_{i \in L_{k}(\eta)}) = (u; x_1, x_1+\sigma(1), \ldots, x_1+\sigma(k(\eta)), (z_i)_{i \in L_{k}(\eta)}).$$

This map is an open embedding and

$$\mathfrak{m}_{\eta,\sigma}^{\text{uo}+} = \bigsqcup_{\sigma \in S_{k}(\eta)} (\text{sgn } \sigma)(\text{Im } \iota_{\eta,\sigma}^+) .$$

If $\eta \sim \eta'$ are such that $\alpha_i(\eta) = \alpha_i(\eta')$ for all $i \in [k(\eta)]$, then

$$(-1)\kappa(\eta) \cdot \mathfrak{b}_{i, \eta} \approx (\text{evb}_1: (\text{sgn } \sigma)(\text{Im } \iota_{\eta,\sigma}^+) \times \mathfrak{b}_{i, \alpha_i(\eta)}; (i, \Gamma_i)_{i \in L_{k}(\eta)} \rightarrow Y)$$

by Lemma [2.2]. Therefore,

$$\mathfrak{b}_{i, \eta} \approx (-1)\kappa(\eta) (\text{evb}_1: \mathfrak{m}_{\eta,\sigma}^{\text{uo}+} \times \mathfrak{b}_{i, \alpha_i(\eta)}; (i, \Gamma_i)_{i \in L_{k}(\eta)} \rightarrow Y).$$ (3.9)

**Proof of (W2).** We establish this statement with $K$ replaced by $K - \{p_1\}$ under the assumption that $\dim(\alpha) = 0$.

Let $\alpha'$ and $\eta$ be as above with $1 \notin K(\alpha')$. With $p_1 \equiv (0, \{p_1\}, \emptyset)$,

$$\mathfrak{b}_{[\eta]} \times \mathfrak{b}_{p_1} = (-1)\kappa(\eta) (\text{evb}_1: \mathfrak{m}_{\eta,\sigma}^{\text{uo}+} \times \mathfrak{b}_{\alpha_i(\eta)}; (i, \Gamma_i)_{i \in L_{k}(\eta)} \rightarrow Y).$$

Suppose in addition $\dim(\alpha') = 2$. By the above identity, Proposition [3.2] with $K' = K(\alpha')$, and (3.10) with $\alpha'$ replaced by $\alpha_i(\eta)$,

$$|\mathfrak{b}_{i, \eta} \times \mathfrak{b}_{i, \eta} | \approx (-1)\kappa(\eta) \sum \left( \text{sgn } (u_i) \prod_{\iota \in K_{\alpha_i}(\eta)} \text{lk}(\eta_i, \eta_{\iota}) \right).$$

Since the dimension of $Y$ is odd,

$$-\deg \mathfrak{b}_{\alpha} = |\mathfrak{b}_{\alpha} \times \mathfrak{b}_{\alpha} | \approx \sum_{[\eta] \in \overline{\text{D}}(\alpha')} |\mathfrak{b}_{[\eta]} \times \mathfrak{b}_{[\eta]} | \approx \sum_{[\eta] \in \overline{\text{D}}(\alpha')} (-1)^{\kappa(\eta)} \sum \text{sgn } (u_i) \text{lk}(u_i; T_i) \text{lk}(u_i, u_i).$$

Summing up (3.10) over the equivalence classes $[\eta]$ of $\eta$ in $\overline{\text{D}}(\alpha')$ and using the bijectivity of $[\eta]$ in $\overline{\text{D}}(\alpha')$ and using the bijection of $[\eta]$ in $\overline{\text{D}}(\alpha')$ and using the bijection of the map (3.8), we thus obtain

$$-\deg \mathfrak{b}_{\alpha} = (-1)^{\kappa(\alpha')} \sum \text{sgn } (u_i) \text{lk}(u_i; T_i) = (-1)^{\kappa(\alpha')} \sum \text{sgn } (u_i) \text{lk}(u_i).$$
Taking $\alpha' = (\beta, K - \{p_1\}, L)$ above and using (1.6) and (3.4), we obtain
\[
\langle L \rangle_{\beta,K-\{p_1\}}^\omega = (-1)^{|K|} \langle \Gamma_1, \ldots, \Gamma_L \rangle_{\beta,|K|}^\omega
\]
and establish the claim.

Proof of Proposition 3.1: We prove both statements by induction on the set $C_\omega(Y)$ with respect to the partial order $< \in$ defined in Section 2.2. It is sufficient to consider the elements $\alpha' \in C_\omega(Y)$ with $\dim(\alpha') = 0$ only.

Suppose $\alpha \in C_\omega(Y)$ with $\dim(\alpha) = 0$ and $(b_\alpha')_{\alpha' \in C_\omega(Y)}$ is a collection of bordered pseudocycles into $Y$ satisfying the conditions of Definition 2.5 as well as the second equality in (3.5) if $\dim(\alpha') = 0$. By (3.9), Proposition 3.2 with $K' = K^*_{\eta}$, and (3.5) with $\alpha'$ replaced by $\alpha(\eta)$,
\[
bb_{[\eta]} = (-1)^{|K^*_{\eta}(\eta)|} \bigcup_{u \in SD(\alpha^*_{\eta}(\eta))} \left( \sum_{i \in K^*_{\eta}(\eta)} \text{sgn}(u_i) \prod_{i \in K^*_{\eta}(\eta)} \text{lk}(\partial u_i, \partial b_{\alpha(\eta)}) \right) \partial u_i.
\]

Summing up (3.11) over the equivalence classes $[\eta]$ of $\eta$ in $D_\omega(\alpha)$ and using the bijectivity of the map (3.8), we obtain
\[
bb_{\alpha} \equiv \bigcup_{[\eta] \in D_\omega(\alpha)/\sim} \langle \eta \rangle (3.11) = (-1)^{|K(\alpha)|} \bigcup_{u \in MD(\alpha), u \in T \in ST(u)} \text{sgn}(u) \prod_{i \in K^*_{\eta}(\eta)} \text{lk}(\partial u_i, \partial u_i) \partial u_i.
\]

This establishes the second equality in (3.5) with $\alpha'$ replaced by $\alpha$. Along with the injectivity of (1.2), it implies that $bb_{\alpha}$ bounds in $Y$. Thus, we can choose a bordered pseudocycle $b_\alpha$ into $Y$ satisfying the first equality in (3.5) with $\alpha'$ replaced by $\alpha$.

3.3 Open divisor relation

We deduce Proposition 3.2 from the following lemma, which confirms the $K' = \{k\}$ case of this proposition.

Lemma 3.4 Let $\beta, k, l, K, L, (b_i)_{i \in K}$, and $(\Gamma_i)_{i \in L}$ be as in Proposition 3.2. If $k \in K$ and the codimension of $b_k$ is 1, then there exists a dense open subset $M_{k-1,l}$ of the target of the induced forgetful morphism
\[
f_k^b : M_{k,l}^\text{iso} \times \mathfrak{B}(J) \times \mathfrak{F}(i, b_i)_{i \in K}; (i, \Gamma_i)_{i \in L}) \longrightarrow M_{k-1,l}^\text{iso} \times \mathfrak{B}(J) \times \mathfrak{F}(i, b_i)_{i \in K - \{k\}}; (i, \Gamma_i)_{i \in L})
\]
so that (3.12) restricts to a covering map over each connected component $M$ of $M_{k-1,l}$. If in addition the codimensions of all $b_i$ are odd and the codimensions of $\Gamma_i$ are even, then the degree of this restriction is $-\text{lk}(\partial u, \partial b)$ for any $u \in M$.

Proof. We denote the right-hand side of (3.12) by $M$ and define
\[
K' = K - \{k\}, \quad M_{k,l}^\text{iso} = M_{k,l}^\text{iso} \times \mathfrak{B}(J) \times \mathfrak{F}(i, b_i)_{i \in K'}; (i, \Gamma_i)_{i \in L}).
\]
Figure 2: Commutative squares of exact sequences for the proof of Lemma 3.4

By Lemma 2.3,
\[
\text{LHS of } (3.12) = (-1)^{|K|^{-1}}(\tilde{M}_{\text{evb}} \times b_k(\text{dom } b_k)).
\] (3.13)

If the pseudocycles \(b_i\) with \(i \in K\) have odd codimensions and the pseudocycles \(\Gamma_i\) have even codimensions, then
\[
\dim(\tilde{M}_{\text{evb}} \times b_k(\text{dom } b_k)) = k - |K| \mod 2.
\] (3.14)

Let \(M' \subset M\) be the image of the elements of the left-hand side of (3.12), which meet the boundary of any of the pseudocycles \(b_i\) and \(\Gamma_i\) or the pairwise intersection of any pair of these pseudocycles. The dense open subset \(\mathfrak{M}_{k-1,F}^*\) of \(M-M'\) consisting of the maps \(u\) from \(\mathbb{D}^2\) with \(\hat{\epsilon}u\) transverse to \(b_k\).

We compute the sign of \(f_k^b\) at a preimage \((\hat{u}, q_k)\) of \(u\) in the fiber product space in (3.13) under (3.12). Denote the \(k\)-th boundary marked point of \(\hat{u}\) by \(x_k\) and the image of \(u\) in \(Y^{K'} \times X^I\) by \(y\). All rows and the right column in the first diagram of Figure 2 are orientation-compatible. The short exact
sequence
\[ 0 \rightarrow T_{x_k} S^1 \rightarrow T_{\tilde{u}'} \mathcal{M}_{k,l}^\text{uo}(\beta; J) \rightarrow T_{u'} \mathcal{M}_{k-1,l}^\text{uo}(\beta; J) \rightarrow 0, \]

where \( \tilde{u}' \) and \( u' \) are the projections of \( \tilde{u} \) and \( u \), respectively, to the corresponding disk moduli spaces, has sign \((-1)^{k-1}\). Along with Lemma 6.3 in [1], this implies that the middle and left columns in the first diagram also have signs \((-1)^{k-1}\). Thus, the middle column in the second diagram has sign \((-1)^{k-1}\) as well. The middle row and the side columns in this diagram are orientation-compatible. The sign of the top row is the sign of \( (x_k, q_k) \) in the fiber product \((\tilde{u}) \times_{\mathcal{A} \mathcal{B}_k} b_k\). Along with Lemma 6.3 in [1] and (3.14), this implies that the sign of the bottom row is \((-1)^{|K'|-1}\) times the sign of \( (x_k, q_k) \) in the fiber product \((\tilde{u}) \times_{\mathcal{A} \mathcal{B}_k} b_k\).

Combining the last conclusion with (3.13), we obtain
\[
\sum_{(\tilde{u},q_k) \in \{f_k^\text{uo}\}^{-1}(u)} \text{sgn}(d_{(\tilde{u},q_k)} f_k^\text{uo}) = \left| (\tilde{u}) \times_{\mathcal{A} \mathcal{B}_k} b_k \right|^{\pm}.
\]

Along with (3.1), this establishes the degree claim. \(\square\)

**Proof of Proposition 3.2.** The first claim follows immediately from the first claim of Lemma 3.4. By Lemma 2.2, a reordering of the pseudocycles \( b_i \)'s with \( i = k', \ldots, k \) does not change the oriented space on the left-hand side of (3.6). We can thus assume that
\[ K' = \{k-\lfloor K'\rfloor+1, \ldots, k\} \subset [k]. \]

The second claim then follows from the second claim of Lemma 3.4 by induction. \(\square\)

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