EQUIVARIANT LATTICE GENERATORS AND MARKOV BASES

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Abstract. It has been shown recently that monomial maps in a large class respecting the action of the infinite symmetric group have, up to symmetry, finitely generated kernels. We study the simplest nontrivial family in this class: the maps given by a single monomial. Considering the corresponding lattice map, we explicitly construct an equivariant lattice generating set, whose width depends linearly on the width of the map. This result is sharp and improves dramatically the previously known upper bound on the width as it does not depend on the degree of the map. For a single-monomial map of width two, we also explicitly construct a finite set of binomials generating the toric ideal up to symmetry. Both width and degree of this generating set are sharply bounded by linear functions in the exponents of the monomial.

1. Introduction

Let $K[Y] := K[y_{ij} \mid i, j \in \mathbb{N}, i \neq j]$ and $K[X] := K[x_i \mid i \in \mathbb{N}]$ be polynomial rings with countably many indeterminates over a ring $K$. Consider the monomial map

$$\pi: K[Y] \to K[X], \quad y_{ij} \mapsto x_a^i x_b^j.$$ (1.1)

The infinite symmetric group $S_\infty$ of all bijections $\sigma: \mathbb{N} \to \mathbb{N}$, acts on $X$ and $Y$:

$$\sigma(x_i) = x_{\sigma(i)} \text{ and } \sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}.$$ The ring $K[X]$ is equivariantly Noetherian: every ideal that is closed under the action of $S_\infty$ is finitely generated up to symmetry [AH07, AH09]. Although $K[Y]$ is not $S_\infty$-Noetherian, for a large family of monomial maps $\pi$ (including those considered here) the ideal $\ker(\pi) \subset K[Y]$—an infinite-dimensional toric ideal—is finitely generated up to symmetry [DEKL13]. Any monomial map $\pi$—finite or infinite-dimensional—is closely related to its linearization $A_\pi$: the $\mathbb{Z}$-linear map on exponents

$$A_\pi: \mathbb{Z}^{N \times N} \to \mathbb{Z}^N, \quad A_\pi(e_{ij}) = ae_i + be_j,$$

where $e_{ij}$ and $e_i$ are the standard basis vectors of $\mathbb{Z}^{N \times N}$ and $\mathbb{Z}^N$, respectively. Each vector $v \in \mathbb{Z}^{N \times N}$ translates to a binomial $y^{v^+} - y^{v^-} \in K[Y]$ where $(v_\pm)_i = \max\{\pm v_i, 0\}$ are the positive and negative parts of $v$, respectively. The binomials of this form are exactly those whose terms have greatest common divisor equal to one. All minimal generators of toric ideals are of this form.

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The kernel $L = \ker(A_\pi)$ is an infinite-dimensional (integer) lattice. We call $V \subset \mathbb{Z}^{N \times N}$ an equivariant lattice generating set (or equivariant lattice generators) if the $S_\infty$-orbits of its elements generate $L$. This happens if and only if the $S_\infty$-orbits of $\{y^{v+} - y^{v-} \mid v \in V\}$ generate the extension of $\ker(\pi)$ in the ring of Laurent polynomials $K[Y^\pm]$. An $S_\infty$-generating set of $\ker(\pi)$ is an equivariant Markov basis and also spans $\ker(\pi)K[Y^\pm]$.

Remark 1.1. Ideally, one would like to define an equivariant lattice basis, a generating set whose orbits freely generate the lattice. However, already in the finite-dimensional case this seems hard: Consider the sublattice of $\mathbb{Z}^3$ generated by $S_3$ acting on $(1, -1, 0)$. One would like to call $\{(1, -1, 0)\}$ an equivariant lattice basis, but there is a nontrivial linear relations among the elements of the orbit:

$$(1, 0, -1) + (-1, 1, 0) + (0, -1, 1).$$

Even if there are no relations among elements of the orbit, one has the problem that lattice bases can not be defined by inclusion minimality (the integers 2 and 3 span $\mathbb{Z}$, but no subset does). One remedy (in the finite-dimensional setting) are matroids over rings as defined by Fink and Moci [FM12]. There each subset of the base set is assigned a module, instead of just its rank.

Any infinite-dimensional object in our setting has truncations which are its images in finite-dimensional subspaces.

Definition 1.2. The $n$-th truncation of an object indexed by $\mathbb{N}$ is the sub-object indexed by $[n] \subset \mathbb{N}$. The width of an equivariant object is the minimal $n \in \mathbb{N}$ such that the $n$-th truncation determines it up to action of $S_\infty$. The width of on object which can not be described finitely up to symmetry is infinity.

The “objects” in Definition 1.2 are usually sets of indeterminates or lattice vectors.

Example 1.3. The $n$-th truncations of the sets of indeterminates $X$ and $Y$ are $X_n = \{x_i \in X \mid i \leq n\}$ and $Y_n = \{y_{ij} \in Y \mid i, j \leq n\}$, respectively. Their widths are $\text{width}(X) = 1$, $\text{width}(Y) = \text{width}(\pi) = \text{width}(\text{im}(\pi)) = \text{width}(A_\pi) = 2$.

Remark 1.4. An $n$-th truncation of an $S_\infty$-invariant object is $S_n$-invariant.

We study the following problems.

(L) Find an equivariant generating set of $\ker(A_\pi)$.

(BL) Bound the width of an equivariant generating set of smallest width.

(M) Find an equivariant Markov basis of $\ker(\pi)$.

(BM) Bound the width of an equivariant Markov basis of smallest width.

We are also interested in questions (L), (BL), (M), and (BM) in more general settings than that in (1.1):
(1) For $k \in \mathbb{N}$ let $Y = \{ y_\alpha \mid \alpha \in \mathbb{N}^k, \alpha_j \text{ distinct} \}$ be a set of indeterminates. For some fixed values $a_1, \ldots, a_k \in \mathbb{N}$ one can consider the width $k$ monomial map

$$\pi : K[Y] \to K[X], \quad y_\alpha \mapsto x_{\alpha_1}^{a_1} \cdots x_{\alpha_k}^{a_k}.$$ 

(2) For $k, m, N \in \mathbb{N}$, let

$$Y = \{ y_{i\alpha} \mid i \in [N], \alpha \in \mathbb{N}^k, \alpha_j \text{ distinct} \}$$

$$X = \{ x_{lj} \mid l \in [m], j \in \mathbb{N} \}$$

be sets of indeterminates on which $S_\infty$ acts on the second index. It is known that $K[X]$ is equivariantly Noetherian. One can now consider a monomial map $\pi : K[Y] \to K[X]$ of width up to $k$, that is, each $y_{i\alpha}$ maps to some monomial $x_{lj}$ with $j \in [k]$. This is the most general case for which $\ker(\pi)$ is known to be equivariantly Noetherian [DEKL13].

A bound in [HdC13] answers the question (BL1). If $\pi$ is defined by a single monomial $x_{\alpha_1}^{a_1} \cdots x_{\alpha_k}^{a_k}$, there is an equivariant lattice generating set of $\ker(A_\phi)$ of width $2d - 1$, where $d = a_1 + \cdots + a_k$ is the degree of the image monomial. In Section 2 we improve this bound by an explicit construction of an equivariant lattice generators (Theorem 2.5), thus answering question (L1). The width of our basis is two more than the width $k$ of the map and thus independent of degree $d$ (Corollary 2.6).

One of our tools is an idea from [DEKL13]: The map $\pi$ factors as

$$(1.2) \quad \pi : K[Y] \xrightarrow{\phi} K[Z] \xrightarrow{\psi} K[X],$$

into two maps that are easier to analyze. For instance, in the setting of question (L1), we introduce indeterminates $Z_k = \{ z_{ij}, i = 1, \ldots, k, j \in \mathbb{N} \}$ and define $\phi, \psi$ by linear extension of

$$\phi : y_\alpha \mapsto \prod_{i=1}^{k} z_{i\alpha_i} \quad \text{and} \quad \psi : z_{ij} \mapsto x_{j}^{a_j}.$$ 

The union of a Markov basis for $\ker(\phi)$ and the pullback of a Markov basis of $\text{im}(\phi) \cap \ker(\psi)$ forms a Markov basis for $\ker(\pi)$ and similarly for $\ker(\pi)K[Y^\pm]$.

One could hope to compute an equivariant Markov basis of $\ker(\pi)$ by computing a (usual) Markov basis $M_n$ for some $n$-th truncation $\ker(\pi)_n = \ker(\pi) \cap K[Y_n]$ and check if it $S_{n+l}$ generates $\ker(\pi)_{n+l}$, for sufficiently many $l$. Unfortunately it is unknown how large $l$ needs to be to guarantee stabilization.

The paper is structured as follows. In Section 2 we give an explicit construction of an equivariant lattice generators of $\ker(A_\phi)$ for arbitrary width of the image monomial (in the setting of (L1)). Section 3 briefly outlines our computational experiments with truncated Markov bases. Section 4 proceeds with an explicit combinatorial construction of a Markov basis in the width two case. We conclude in Section 5 with some discussion and open problems.
2. Equivariant lattice generators

To get started, consider question (L1) in width two \((k = 2)\). That is, consider the map \(\pi : K[Y] \to K[X]\), defined by \(y_{ij} \mapsto x_i^a x_j^b\). We look for a generating set of the extension \(\ker(\pi)K[Y^\pm]\). To this end we split \(\pi\) as in (1.2), and think of exponents of monomials in \(K[z_1, z_2]\) as \(2 \times \mathbb{N}\) matrices.

**Proposition 2.1.** For \(\pi\) with width two, \(\ker(\pi)K[Y^\pm]\) is generated up to symmetry by two binomials:

\[
y_{12}y_{34} - y_{14}y_{23} \quad \text{and} \quad y_{21}y_{31}^{a-b} - y_{12}y_{32}^{a-b}.
\]

**Proof.** The basic quadric \(y_{12}y_{34} - y_{14}y_{23}\) suffices to generate \(\ker(\phi)K[Y^\pm]\) up to symmetry. This is the first lemma of algebraic statistics: basic quadrics are a Markov basis for the independence model (see Remark 2.3 and [DSS09, §1.1]). The non-existing diagonal variables pose no problem for us since we only need the Laurent case.

On the exponents of monomials in \(K[z_1, z_2]\), the linearization \(A_\psi\) of \(\psi\) acts by left multiplication with the matrix \([a \ b]\). Since \(a\) and \(b\) are relatively prime, \(\ker(\psi)\) consists of all matrices of the form

\[
\begin{bmatrix}
-b \\
a
\end{bmatrix}
\begin{bmatrix}
n_0 \\
1 \\
\vdots
\end{bmatrix}
\]

with \(n_i \in \mathbb{Z}\). Since \(\ker(\pi)\) is homogeneous, the elements of \(\text{im}(\ker(\phi))\) have row sums equal to zero. Therefore every element in \(\text{im}(\phi) \cap \ker(\psi)\) must satisfy \(\sum_i n_i = 0\). The permutations of

\[
w := \begin{bmatrix}
-b & b & 0 & \cdots \\
a & -a & 0 & \cdots
\end{bmatrix}
\]

form an equivariant lattice generating set for such elements. Consequently \(\text{im}(\phi) \cap \ker(\psi)\) is contained in the lattice generated up to symmetry by \(w\). Now

\[
\phi(y_{21}^{b}y_{31}^{a-b} - y_{12}^{b}y_{32}^{a-b}) = z_{13}^{a-b}(z_{21}^{b}z_{31}^{a} - z_{21}^{b}z_{13}^{a})
\]

which has lattice element \(w\), so \(y_{21}^{b}y_{31}^{a-b} - y_{12}^{b}y_{32}^{a-b}\) is the pullback of the generator of \(\text{im}(\phi) \cap \ker(\psi)\). \(\square\)

**Remark 2.2.** A generating set in \(K[Y]\) for the kernel of a map of width two requires the 3-cycle cubic \(y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}\) (see Propositions 4.1). However in the Laurent ring \(K[Y^\pm]\), this binomial is redundant modulo the basic quadric \(y_{12}y_{34} - y_{14}y_{32}\). The 3-cycle cubic’s exponent is

\[
c = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
\vdots & \ddots & \ddots
\end{bmatrix}
\]
which can be expressed in terms of basic quadrics as

\[
c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ \vdots \end{bmatrix}.
\]

We now generalize Proposition 2.1 to arbitrary \( k \). When necessary, we write \( \phi^{(k)} \) instead of \( \phi \) to emphasize the width of the image monomial but usually the level of generality is clear from the context and we avoid overloading the notation too much. Elements of \( \mathbb{Z}^{nk} \) should be thought of as \( k \)-dimensional tables of infinite size with integer entries. Our setup additionally requires that these tables be zero along their diagonals (defined as entries indexed by \((i_1, \ldots, i_k)\) with any \( i_j = i_l \) for \( j \neq l \)). Let \( e_{i_1 \cdots i_k} \) denote the standard basis elements of \( \mathbb{Z}^{nk} \). Then \( Y \) consists of indeterminates \( y_{i_1 \cdots i_k} = e_{i_1 \cdots i_k} \). The factorization (1.2) gives a map \( \phi^{(k)} \) as follows:

\[
\phi^{(k)} : K[Y^\pm] \to K[Z^\pm], \quad \phi^{(k)}(y_{i_1 \cdots i_k}) = z_{i_1} z_{2i_2} \cdots z_{ki_k}.
\]

**Remark 2.3.** In algebraic statistics, the independence model on \( k \) factors is (the non-negative real part of) the image of the monomial map \( y_{i_1 \cdots i_k} \mapsto z_{i_1} z_{2i_2} \cdots z_{ki_k} \) where \( i_j \in [l_j] \) for some integers \( l_j \) [DSS09]. In algebraic geometry, this map represents the Segre embedding \( \mathbb{P}^{l_1-1} \times \cdots \times \mathbb{P}^{l_k-1} \hookrightarrow \mathbb{P}^{l_1l_2 \cdots l_k - 1} \). The coordinate ring of the Segre embedding is presented by quadrics of the form

\[
y_{i_1 \cdots i_r \cdots i_s \cdots i_k} y_{i_1 \cdots i'_r \cdots i'_s \cdots i_k} - y_{i_1 \cdots i'_r \cdots i_s \cdots i_k} y_{i_1 \cdots i_r \cdots i'_s \cdots i_k},
\]

where \( i_j, i'_j \in [l_j] \). This setup differs from ours because diagonal entries like \( y_{i_1} \) are forbidden for us. In the analysis of contingency tables, this restriction is known as a specific subtable-sum condition, namely the sum over all diagonal entries equals zero [HTY09a, HTY09b]. Subtable-sum models have more complicated Markov bases than just independence models, but their lattice bases are still quadratic.

**Proposition 2.4.** The lattice elements

\[
\text{Quad}^{(k)} := \{ e_{i_1 \cdots i_r \cdots i_s \cdots i_k} + e_{i_1 \cdots i'_r \cdots i'_s \cdots i_k} - e_{i_1 \cdots i'_r \cdots i_s \cdots i_k} - e_{i_1 \cdots i_r \cdots i'_s \cdots i_k}, \quad i_l, i'_l \in [k + 2] \}
\]

are an equivariant lattice generating set of \( \ker(A_{\phi^{(k)}}) \).

The elements of \( \text{Quad}^{(k)} \) are moves which take two elements differing in their indices at exactly two positions and then swap the values in one of those positions.

**Proof of Proposition 2.4.** It is easy to see that \( \text{Quad}^{(k)} \subseteq \ker(\phi^{(k)}) \). To see \( \langle \text{Quad}^{(k)} \rangle \supseteq \ker(\phi^{(k)}) \), we first show that \( \langle \text{Quad}^{(k)} \rangle \) contains all elements of the form

\[
e_{a_1 \cdots a_k} + e_{b_1 \cdots b_k} - e_{a_1 \cdots a_{k-1} b_k} - e_{b_1 \cdots b_{k-1} a_k}
\]
where $a_1, \ldots, a_k, b_k$ are distinct and also $b_1, \ldots, b_k, a_k$ are distinct. Now denote $N = \max\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ and consider the following telescopic sum in $\text{Quad}^{(k)}$:

$$
e_{a_1} \ldots a_k - e_{a_1} \ldots a_{k-1} b_k - (e_{(N+1)} a_2 \ldots a_k - e_{(N+1)} a_2 \ldots a_{k-1} b_k) + e_{(N+1)} a_2 \ldots a_k - e_{(N+1)} a_2 \ldots a_{k-1} b_k - (e_{(N+1)} (N+2) a_3 \ldots a_k - e_{(N+1)} (N+2) a_3 \ldots a_{k-1} b_k)
$$

$$
\vdots
$$

$$
+ e_{(N+1)} (N+k-2) a_{k-1} b_{k-1} - e_{(N+1)} (N+k-1) a_{k-1} b_{k-1} - (e_{(N+1)} (N+k-1) a_{k-1} b_{k-1} - e_{(N+1)} (N+k-1) b_{k-1}) = e_{a_1} a_k - e_{a_1} a_{k-1} b_k - (e_{(N+1)} (N+k-1) a_{k-1} b_{k-1} - e_{(N+1)} (N+k-1) b_{k-1})
$$

Similarly, $e_{b_1} \ldots b_k - e_{b_1} \ldots b_{k-1} a_k - (e_{(N+1)} (N+k-1) b_{k-1} - e_{(N+1)} (N+k-1) a_{k-1} M) \in \langle \text{Quad}^{(k)} \rangle$, where $M$ is some fixed constant larger than any index value appearing in $C$. Summing up these moves shows

$$
C_m - \sum_{I \in \mathbb{N}^{k-1}} c_{I m} e_{I M} \in \langle \text{Quad}^{(k)} \rangle.
$$

Summing over $m$ shows that $C - D \in \langle \text{Quad}^{(k)} \rangle$ where $D := \sum_{I \in \mathbb{N}^k} c_I e_{I_1 \ldots i_{k-1} M}$. Since $\langle \text{Quad}^{(k)} \rangle \subseteq \ker(\phi^{(k)})$, also $D \in \ker(\phi^{(k)})$. All non-zero entries of $D$ have $M$ as their last index entry and dropping it we get an element $D' \in \ker(\phi^{(k-1)})$. In the base case $k = 2$, $\phi^{(k-1)}$ is an isomorphism, so $D'$ and then $D$ are 0 and therefore $C \in \langle \text{Quad}^{(k)} \rangle$. For $k > 2$, we can assume by induction that $\langle \text{Quad}^{(k-1)} \rangle = \ker(\phi^{(k-1)})$, so $D'$ can be decomposed into moves in $\text{Quad}^{(k-1)}$. Since $D'$ doesn’t depend of the the choice of $M$, we can choose $M$ larger than any index value used in this decomposition. Therefore appending $M$ as the $k$-th index value produces a decomposition of $D$ in $\text{Quad}^{(k)}$, which proves that $C \in \langle \text{Quad}^{(k)} \rangle$.

To describe $\ker(\pi) K[Y^\pm]$, we proceed to describe $\ker(\psi)$ and its intersection with $\text{im}(\phi)$, working directly with the respective linearizations $A_\pi, A_\phi$, and $A_\psi$. The linearization of $\psi : z_{ij} \mapsto x_{ij}^a$ acts on lattice elements by left multiplication with the matrix $A_\psi = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$. The kernel of $A_\psi$ is a $(k-1)$-dimensional sublattice of $\mathbb{Z}^k$. Let $B = (b_1, \ldots, b_{k-1})$ be a $k \times (k-1)$ matrix whose columns $b_1, \ldots, b_{k-1}$ are a lattice basis. Any element in $\ker(\psi \circ \phi)$ is homogeneous: the entries of its exponent vector sum to zero. Consequently the columns of any $C \in A_\phi(\ker(A_\psi \circ \phi)) = \text{im}(A_\phi) \cap \ker(A_\psi)$ also sum to zero. With the basis $B$, if $C = B C'$ with $C' \in \mathbb{Z}^{k-1 \times N}$, then the columns of $C'$ sum to zero as well. The lattice of matrices in $\mathbb{Z}^{(k-1) \times N}$ with zero row sums is generated by the matrices with a 1 and $-1$ in any two entries of a particular row, and
zero elsewhere. Therefore $\text{im}(A_\phi) \cap \ker(A_\psi)$ is contained in the lattice generated by the orbits of

$$B_i := [b_i - b_i \ 0 \ \ldots]$$

for $1 \leq i < k$. More specifically $\text{im}(A_\phi) \cap \ker(A_\psi) \subseteq \langle B_1, \ldots, B_{k-1} \rangle \subseteq \ker(A_\psi)$. We show constructively that $B_i \in \text{im}(A_\phi)$, so in fact the orbits of $B_1, \ldots, B_{k-1}$ generate $\text{im}(A_\phi) \cap \ker(A_\psi)$. For each $1 \leq j \leq k$ consider the lattice element

$$f_j := e_{a_1 \ldots a_{j-1}}a_{j+1} \ldots a_k - e_{a_1 \ldots a_{j-1}2a_{j+1} \ldots a_k} \in \mathbb{Z}^{n_k} \text{ with } a_t \geq 3 \text{ arbitrary.}$$

Applying $A_\phi$, all entries cancel except for the two in the $j$-th row, producing the matrix with 1 in the $(j, 1)$ entry and $-1$ in the $(j, 2)$ entry. Any $B_i$ can be expressed as a linear combination of such matrices. In particular if $b_i$ has entries $c_1, \ldots, c_k$ then

$$w_i := c_1f_1 + \cdots + c_kf_k \in A_\phi^{-1}(B_i).$$

This proves the following theorem.

**Theorem 2.5.** Up to symmetry, $\text{Quad}^{(k)} \cup \{w_1, \ldots, w_{k-1}\}$ is an equivariant lattice generating set of $\ker(A_\phi)$, where $k$ is the width of the map $\pi$.

**Corollary 2.6.** The lattice $\ker(A_k)$ has an equivariant lattice generating set consisting of $(k^2 + k - 2)/2$ elements of width $k + 2$.

**Proof.** Up to $S_\infty$-action, each element of $\text{Quad}^{(k)}$ is determined by the two index positions where the swap takes place. So $\text{Quad}^{(k)}$ contributes $\binom{k}{2}$ generators. Additionally we have $w_1, \ldots, w_{k-1}$, which totals $(k^2 + k - 2)/2$. Choosing every $f_j$ with $a_1, \ldots, a_j, \ldots, a_k$ being $3, \ldots, k + 1$ produces the width bound. \hfill \Box

This generating set is often not minimal in size. In fact, we can do away with all of $\text{Quad}^{(k)}$ at the expense of width of the $w_i$.

**Corollary 2.7.** The lattice $\ker(A_k)$ has an equivariant lattice generating set consisting of $k - 1$ elements of width $2k$.

**Proof.** Suppose $b_t$ is a generator of $\ker(A_\phi)$ which is non-zero in the $i$-th coordinate for some $1 \leq i \leq k$. Choose $w_l$ as in Corollary 2.6, except that $f_i$ is replaced by

$$f'_i := e_{a_1 \ldots a'_{t-1}1a'_{t+1} \ldots a'_k} - e_{a_1 \ldots a'_{t-1}2a'_{t+1} \ldots a'_k}$$

which has $a'_1, \ldots, a'_i, \ldots, a'_k$ equal to $k + 2, \ldots, 2k$. Then for any $j \neq i$ consider the lattice element $w_l - \sigma w_l$ where $\sigma \in S_\infty$ is the permutation switching $a'_j$ and $2k + 1$. All terms cancel except for $f'_i - \sigma f'_i$ which (up to permutation) is the element of $\text{Quad}^{(k)}$ which switches the indices at positions $i$ and $j$.

For any generating set $b_1, \ldots, b_{k-1}$ of $\ker(A_\phi)$, by Hall’s marriage theorem we can assign to each $b_t$ a distinct $i_t$ such that the the $i_t$-th coordinate of $b_t$ is non-zero. Then $i_1, \ldots, i_{k-1}$ include all but one of the values from 1 to $k$. Construct each $w_l$ as above so that it generates the elements of $\text{Quad}^{(k)}$ corresponding to all pairs $(i_t, j)$ with $j \neq i_t$. 
Every pair represented in Quad\(^{(k)}\) includes some \(i_l\) so together \(w_1, \ldots, w_{k-1}\) generates all of Quad\(^{(k)}\), Therefore \(w_1, \ldots, w_{k-1}\) is a lattice generating set. \(\square\)

Note that neither the bounds in Corollary 2.6 nor 2.7 are sharp: For example, the kernel of \(y_{ij} \mapsto x_i^2 x_j\) in \(K[Y^\pm]\) is generated by a single binomial of width three: \(y_{12} y_{32} - y_{21} y_{31}\).

Theorem 2.5 settles (L1). For (L2), we would like to be able to extend these techniques to a more general domain ring \(K[Y^\pm]\) and a more general target ring \(K[X^\pm]\). For the latter case, where \(X = \{x_{lj} \mid l \in [m], j \in \mathbb{N}\}\) with \(m > 1\) this is straightforward. Factoring \(\pi\) into \(\pi : K[Y^\pm] \xrightarrow{\phi} K[Z^\pm] \xrightarrow{\psi} K[X^\pm]\), we have the same \(\phi\) as before, with the same kernel. The linearization \(A_\psi\) of \(\psi\) is left multiplication by an \(m \times k\) matrix with non-negative entries. A lattice basis \(\{b_1, \ldots, b_s\}\) for \(\ker(A_\psi)\) can be computed using standard algorithms. Since any binomials in \(\ker(\pi)\) is homogeneous, and every variable in \(Y\) contributes exactly one to each row in \(Z\), the matrices for \(\text{im}(A_\phi) \cap \ker(A_\psi)\) have row sums equal to zero. Therefore \(\text{im}(A_\phi) \cap \ker(A_\psi)\) is again generated by \(B_i := [b_i - b_i 0 \cdots]\).

On the other hand, extending the domain to \(K[Y^\pm]\) with \(Y = \{y_{i\alpha} \mid i \in [N], \alpha \in \mathbb{N}^k, \alpha_j \text{ distinct}\}\) presents obstacles for \(N > 1\). Here the lattice \(Z^\pm\) is represented by \(Nk \times \mathbb{N}\) matrices, with \(k\) rows in the image of each of the \(N\) orbits of \(Y\). Our previous argument breaks down because the matrices corresponding to binomials in \(\phi(\ker(\pi))\) need not have all row sums equal to zero, which was critical to the construction used when \(N = 1\). Binomials in \(\ker(\pi)\) need not be homogeneous, and even homogeneous binomials need not correspond to matrices in \(Z^\pm\) with zero row sums.

### 3. Examples and Tools

During experimental investigations leading to the results in this paper we used several different ways to represent binomials in the various rings. Let us introduce the most useful ones in the setting of width two, that is \(y_{ij} \mapsto x_i^a x_j^b\). The extension to higher width is simple.

A simple way to represent monomials in the various polynomial rings is with a table of its exponents, as used in the previous section. These tables have an infinite number of entries, but only a finite number are non-zero for a given monomial.

The monomials in \(K[Y]\) with \(k = 2\) correspond to \(\mathbb{N} \times \mathbb{N}\) matrices \((a_{ij})_{i,j \in \mathbb{N}}\) where the entry \(a_{ij}\) is the exponent of \(y_{ij}\). All diagonal entries \(a_{ii}\) are zero. The action of \(S_\infty\) simultaneously permutes the rows and columns of the matrix. A binomial \(y^A - y^B \in K[Y]\) is in \(\ker(\phi)\) if the corresponding pair of matrices \(A, B\) have each row sum and each column sum equal. The monomials of \(K[Z]\) correspond to \(2 \times \mathbb{N}\) matrices and the action of \(S_\infty\) permutes the columns. In particular we are interested
in the monomials in the image of $\phi$, but these are easy to identify due to [DEKL13, Proposition 3.1]. They correspond precisely to the matrices whose row sums are both equal to some $d \in \mathbb{N}$, and whose column sums don’t exceed $d$. Finally, the monomials of $K[X]$ can be represented by infinite row vectors. The map $\psi$ corresponds to left multiplication by the $(1 \times 2)$ matrix $[a, b]$.

While thinking about generators we also used the box shape formalism which we explain now. Clearly every monomial in $K[X]$ is specified by the exponents of the variables it contains. An exponent $a_i$ in $x_i^a$ can be represented by a column of height $a_i$ in position $i$ in some diagram. For instance the monomial $x_1^3x_2^2x_3$ displays as

Up to the action of $S_\infty$, the order of columns is irrelevant. Therefore one may choose an arbitrary convention like ordering the columns by size. Extending this formalism we represent monomials in the matching monoid, the image of $K[Y]$ in $K[Z]$, by subdivided columns, according to the following rule: A variable $z_{ik}$ corresponds to a box of height $a_i$ in column $k$. In the width two case, $z_{1k}$ gives a box of height $a$ and $z_{2k}$ a box of height $b$. Since we are using commutative variables, the ordering of boxes in a column plays no role. For example, when $a = 2, b = 1$, the two monomials $z_{11}z_{13}z_{22}$ (the image of $y_{12}y_{32}$) and $z_{12}z_{13}z_{21}$ (the image of $y_{21}y_{31}$) display as

Note that both monomials have the same image in $K[X]$ which is just their outer shape, in this case three columns of height two each. From these displays it is obvious that $\psi(z_{11}z_{13}z_{22}z_{21}) = 0$. Finally, we represent monomials in $K[Y]$ by decorated box shapes, which also record the information of which pairs of boxes (one of height $a$, one of height $b$ in different columns) originated from the same variable. Consider the following two decorated box piles:

This display illustrates that $\phi(y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}) = 0$. Note that the two monomials are in the same $S_\infty$-orbit, which may or may not happen for binomials in a Markov basis (see Example 3.1 below).

Our computational powers in $K[Y]$ are limited (though not zero). Therefore it is advantageous for experiments to approximate equivariant computations with their truncations. To this end, fix a truncation width $n$ and coprime exponents $a > b$. The computation of an equivariant Markov basis can be approximated as follows. Consider the matrix $A_n$ whose columns are the elements of the orbit of $(a, b, 0, \ldots, 0)$ under the
action of $S_n$. The size of the orbit is $n(n - 1)$ and each column is indexed by a pair $(i, j)$ of indices $i \neq j$. The group $S_n$ acts on the columns by the rule $\sigma(i, j) = (\sigma(i), \sigma(j))$. If $n$ is reasonably small, say $n = 6$, then 4ti2 [4ti2] computes a usual Markov basis of this matrix in no time. A simple algorithm reduces the result modulo symmetry: One can check for each element of the usual Markov basis, if it is in the orbit of some other element, by enumerating the orbit. The result of this algorithm is the truncated Markov basis. Computations with these truncated bases have lead us to conjecture the results of this paper.

**Example 3.1.** On a standard notebook, for $n = 6$, $a = 2$, $b = 1$, the 4ti2 computation took .03 seconds. Determining the representation of $S_6$ on $S_{30}$, the permutation group of the columns of $A$, takes about two minutes, and the resulting 270 moves, reduced to the 5 orbits in less than 2 seconds. Here are their decorated box representations:

From this computation one conjectures that $\ker(\pi)$ is generated (up to symmetry) by moves of width and degree at most three. Note however that the computation is not a proof, since there is no a-priori bound on the width. In principle there could be some hidden width seven move that our truncated computation has not found. Theorem 4.7 shows that this is not the case. The result of this computation should be compared to the degree five equivariant Gröbner basis determined in [DEKL13, Example 7.2].

**4. Equivariant Markov Bases (case of width 2)**

In this section we return to the width two map from (1.1) and construct an equivariant Markov basis. Fix exponents $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. This requirement is not much of a restriction. If the gcd is larger, then the following results apply with small changes after dividing by the gcd. The Markov basis in Theorem 4.7 has two contributions: one $\phi$-preimage of each element in the two families in Proposition 4.2 and two elements from the following proposition.

**Proposition 4.1.** The 3-cycle cubic $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$ and the basic quadric $y_{12}y_{34} - y_{14}y_{23}$ are an equivariant Markov basis of $\ker(\phi)$.

The proof of Proposition 4.1 is an adaptation of a standard technique from algebraic statistics. It appeared in [AT05, Section 5] but our version is due to Jan Draisma and Jan-Willem Knopper. We give it for the sake of completeness.
Proof of Proposition 4.1. Representing a variable $y_{ij}$ as a directed edge $i \to j$, monomials in $K[y_{ij}]$ correspond to finite loop-free directed multigraphs on $\mathbb{N}$. For each such graph $G$, let $y^G$ denote corresponding monomial. A binomial $y^G - y^H \in \ker(\phi)$ corresponds to a pair of graphs with the same in-degree and out-degree on each vertex. The proof is by induction on the degree $d$ of the binomial. If $G$ and $H$ share an edge, we can divide by that edge and are done by induction. If they don’t share an edge, then it suffices to find an applicable 3-cycle cubic or basic quadric to either $G$ or $H$ and obtain a new graph $G'$ or $H'$ which shares an edge with $H$ or $G$, respectively.

Without loss of generality, let $(1,2) \in G$ be an edge. Then $H$ has an edge out from 1, which we can assume is $(1,3)$, and an edge $(i,2)$ with $i \neq 1$. If $i \neq 3$, apply the basic cubic to the edges $(1,3)$ and $(i,2)$ to get a graph $H'$ with edges $(1,2)$ and $(i,3)$. Now $G$ and $H'$ share the edge $(1,2)$. If $i = 3$, then $G$ has edges $(3,j)$ and $(k,3)$ with $j \neq 2$ and $k \neq 1$. If $j \neq 1$ then apply the basic quadric to $(3,j)$ and $(1,2)$ to get $G'$ with $(3,2)$ and $(1,j)$, sharing $(3,2)$ with $H$. Similarly, if $k \neq 2$, apply the basic quadric to $(k,3)$ and $(1,2)$. Finally, if $j = 1$ and $k = 2$, then $G$ has a 3-cycle $(1,2)$, $(2,3)$, $(3,1)$. Applying the 3-cycle cubic to reverse the direction produces $G'$ with $(2,1)$, $(3,2)$, $(1,3)$ which has edges in common with $H$.

It remains to find generators for $\im(\phi) \cap \ker(\psi)$. For the remainder of this section, we consider the restriction of $\psi$ to $\im(\phi)$—the matching monoid ring:

$$\im(\phi) = K[z_{1i}z_{2j} \mid i, j \in \mathbb{N}, i \neq j] \subseteq K[Z].$$

Because of the interpretation of its generators as matchings on $[2] \times \mathbb{N}$, the multiplicative monoid of $\im(\phi)$ is called the matching monoid [DEKL13, Section 2].

Proposition 4.2. As an ideal in the matching monoid ring, $\ker(\psi)$ is generated by the $S_\infty$-orbits of the binomials $z^A - z^B$ from the following two finite families:

(i) For each $0 \leq n \leq a - b$,

$$A = \begin{bmatrix} b + n & a & c_{13} & c_{14} & \cdots \\ 0 & a & c_{23} & c_{24} & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} n & b + n & c_{13} & c_{14} & \cdots \\ a & 0 & c_{23} & c_{24} & \cdots \end{bmatrix}$$

where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3} c_{2j} = n$.

(ii) For each $1 \leq n \leq b$,

$$A = \begin{bmatrix} b & 0 & a - b + n & 0 & \cdots \\ 0 & a & n & 0 & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b & a - b + n & 0 & \cdots \\ a & 0 & n & 0 & \cdots \end{bmatrix}.$$ 

Additionally, all these binomials are minimal with respect to division in the matching monoid ring.

A generating set of the kernel $\pi = \psi \circ \phi$ consists of preimages of the generators in Proposition 4.2 in $K[Y]$, combined with the generators for $\ker(\phi)$ in Proposition 4.1. The remainder of this section comprises the proof of Proposition 4.2. To deal with divisibility in the matching monoid, recall that its generators are the $(2 \times \mathbb{N})$-matrices
that have exactly one entry 1 in each row, but not in the same column. In fact, more holds: according to [DEKL13, Proposition 3.1], a monomial \( z^A \in K[Z] \) is contained in the matching monoid if and only if there is some \( d \) such that both row sums of \( A \) are equal to \( d \) and all column sums of \( A \) are \( \leq d \) (the matching monoid is normal). Consequently, a monomial is divisible by a generator if we can subtract one in two different columns (reducing the row sum), without violating the new column bound \( d-1 \).

**Proposition 4.3.** As an ideal in the matching monoid ring, \( \ker(\psi) \) is generated up to symmetry by binomials \( z^A - z^B \) with

\[
A - B = \begin{bmatrix} b & -b & 0 & \cdots \\ -a & a & 0 & \cdots \end{bmatrix}.
\]

**Proof.** Let \( z^A - z^B \in \ker(\psi) \). Like in Section 2, the difference \( A - B \) must be of the form

\[
\begin{bmatrix} b \\ -a \end{bmatrix} \begin{bmatrix} n_1 & n_2 & \cdots \end{bmatrix}
\]

where the row vector \( n = [n_1 \ n_2 \ \ldots] \) has entries summing to zero. Such a vector can be expressed as a sum \( n = v_1 + \cdots + v_s \) where each \( v_i \) is in the \( S_\infty \)-orbit of \( [1 \ -1 \ 0 \ \ldots] \). Even more, the decomposition can be chosen *sign-consistently*, that is, each \( v_i \) has 1 in a position \( j \) where \( n_j > 0 \) and has \(-1\) where \( n_j < 0 \).

Consider the sequence \( B = B_0, B_1, \ldots, B_s = A \) of matrices in \( \psi^{-1}(B) \) defined by

\[
B_i = B + \begin{bmatrix} b \\ -a \end{bmatrix} (v_1 + \cdots + v_i).
\]

The sequence is monotonic in each entry, and every column sum is also monotonic. Note that the all row sums of all \( B_i \) are equal to \( d \). Since \( A \) and \( B \) are in the matching monoid, they have non-negative entries and all column sums \( \leq d \). By the monotonicity of the sequence, each \( B_i \) also satisfies these properties and therefore is also in the matching monoid. The proof is complete since \( z^{B_i} - z^{B_{i-1}} \in \ker(\psi) \) for any \( i \), and

\[
B_i - B_{i-1} = \begin{bmatrix} b \\ -a \end{bmatrix} v_i = \sigma_i \begin{bmatrix} b & -b & 0 & \cdots \\ -a & a & 0 & \cdots \end{bmatrix}
\]

for some \( \sigma_i \in S_\infty \).

To prove Proposition 4.2 we need to intersect the matching monoid ring with the equivariant ideal generated by binomials \( z^A - z^B \) with

\[
A - B = \begin{bmatrix} b & -b & 0 & \cdots \\ -a & a & 0 & \cdots \end{bmatrix}.
\]

A general such pair \( A, B \) is of the form

\[
A = \begin{bmatrix} c_{11} + b & c_{12} & c_{13} & c_{14} & \cdots \\ c_{21} & c_{22} + a & c_{23} & c_{24} & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} c_{11} & c_{12} + b & c_{13} & c_{14} & \cdots \\ c_{21} + a & c_{22} & c_{23} & c_{24} & \cdots \end{bmatrix}.
\]
Let $C_j = c_{1j} + c_{2j}$ and $R_i = \sum_{j=1}^{\infty} c_{ij}$ be the column and row sums, respectively, excluding the contributions $a$ and $b$ in the first two columns.

We show that either the pair $(A,B)$ is on the list in Proposition 4.2, or $A$ and $B$ are both divisible (in the matching monoid ring) by a common generator. Let $d = R_1 + b = R_2 + a$ be the degree of $A$ and $B$ which gives a bound on column sums: $C_j \leq d - a$ for $j = 1,2$ and $C_j \leq d$ otherwise. We say that a column is loaded if it achieves its bound. Loaded columns are obstacles to dividing by a common factor, since the degree can’t be decreased without also decreasing the loaded columns by the same amount. $A$ and $B$ have a common factor if there exist positive $c_{1j}$ and $c_{2k}$ such that $j \neq k$ and there are no loaded columns outside of $j$ and $k$.

**Proof of Proposition 4.2.** We distinguish four cases depending on the locations of the (at most two) loaded columns.

**Case 1:** No columns are loaded. We have $d > a$, so $R_1$ and $R_2$ are both positive. The monomials $z^A$ and $z^B$ have a common factor if there are positive $c_{1j}$ and $c_{2k}$ in different columns $j \neq k$, therefore $c_{ij} > 0$ only for one particular column $j$. If $j = 1$, then $C_1 = R_1 + R_2 = 2d - a - b > d - a$ which is a contradiction, and similarly for $j = 2$. Consequently $j \geq 3$ and thus $A, B$ are of the second type for some $1 \leq n < b$.

**Case 2:** Column $j \geq 3$ is loaded. Let $C_j = d$. Since $\sum_j C_j = 2d - a - b$, any other column has $C_k \leq d - a - b$ and is not loaded. Because of the bounds $c_{1j} \leq R_1 = d - b$ and $c_{2j} \leq R_2 = d - a$ and the sum $c_{1j} + c_{2j} = d$, both $c_{1j}$ and $c_{2j}$ are positive. Again, all other values of $c$ must be zero or else $A$ and $B$ have a common factor. Then we have $d = c_{1j} + c_{2j} = b + c_{1j} = a + c_{2j}$ and thus $c_{1j} = a$ and $c_{2j} = b$. Up to symmetry, this is the binomial of type 2 with $n = b$.

**Case 3:** Exactly one of Columns one and two is loaded. Say column one is loaded. In this case no column $j$ can be loaded for $j \geq 3$: If $c_{11} > 0$ then by the divisibility argument $c_{2j} = 0$ for all $j \neq 1$, and similarly if $c_{21} > 0$, then $c_{1j} = 0$ for $j \neq 1$. Thus either $C_j = 0$ for $j > 1$ or one of $R_1$ or $R_2$ is zero. The first case leads to a contradiction as in Case 1. So all positive $c$ values are in one row, which must be the first row since $R_1 > R_2$. This implies $d = a$ and thus that column one is loaded contradicting the assumption. By the same argument, we cannot have column two loaded and column one not loaded.

**Case 4:** Columns one and two are loaded. Either $z^A, z^B$ are divisible by a common generator or we are in one of the following four situations: $c_{11} = c_{12} = 0$; $c_{21} = c_{22} = 0$; $C_1 = 0$; or $C_2 = 0$. However since both column 1 and column 2 are loaded, $C_1 = C_2 = d - a$, so $C_1 = 0$ if and only if $C_2 = 0$, and these cases are subsumed by the other two. If $c_{11} = c_{12} = 0$, then $c_{21} = c_{22} = d - a$. This implies

$$2(d - a) + \sum_{j \geq 3} c_{2j} = R_2 = d - a$$
so $R_2 = 0$. Therefore we need only consider the case $c_{21} = c_{22} = 0$. Here $c_{11} = c_{12} = d - a$ and

$$2(d - a) + \sum_{j \geq 3} c_{1j} = R_1 = d - b.$$ 

Therefore $\sum_{j \geq 3} c_{1j} = a - b - (d - a) = d - a$ and $\sum_{j \geq 3} c_{2j} = R_2 = d - a$. With $n = d - a$ this yields the binomials of type 1. 

**Remark 4.4.** We have gone through some of the many bases of an integer lattice (ideal) [DSS09, § 1.3] and consequently one may ask if it is possible to define an $S_\infty$-equivariant Graver basis. Graver bases originated in optimization problems in economy and are now an important tool in the complexity theory of integer programming [DHK13, Part II]. Recall that the Graver basis $G$ of a lattice ideal $I$ is the unique subset of $I$ satisfying two equivalent properties. First, $G$ is the set of all primitive binomials in the ideal, meaning that for any $x^A - x^B \in I$, there is $x^{A'} - x^{B'} \in G$ such that $x^{A'} | x^A$ and $x^{B'} | x^B$. Second, $G$ is the minimal set such that for every binomial $x^A - x^B \in I$, the difference $A - B$ has a sign-consistent decomposition using $G$, meaning that there is a sequence of exponents $B = B_0, B_1, \ldots, B_s = A$ which is monotonic in each entry and every $x^{B_{i+1}} - x^{B_i}$ is a monomial times an element of $G$. In a general monoid ring such as the matching monoid ring, these two properties are not equivalent. The set of generators in Proposition 4.2 satisfies the sign-consistent decomposition property when considered as a subring of $K[Z]$ as demonstrated in the proof of Proposition 4.3. However it fails the primitiveness condition. A counterexample with $(a, b) = (2, 1)$ is the binomial $z^A - z^B$ with

$$A = \begin{bmatrix} 3 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 2 & 0 & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & \cdots \\ 2 & 0 & 2 & 0 & 0 & \cdots \end{bmatrix}.$$ 

In a monoid ring, the primitiveness condition is the more natural property to use in the definition of a Graver basis. The sign-consistent decomposition property may not be meaningful if the ring is not a subset of a polynomial ring. More convincingly, the set of primitive binomials forms a universal Gröbner basis, a very important conclusion in the classical case. Although our generating set does not contain all primitive binomials, its $S_\infty$-orbits form a universal Gröbner basis nonetheless.

**Proposition 4.5.** The $S_\infty$-orbits of the generators in Proposition 4.2 form a universal Gröbner basis of $\ker(\psi)$ as an ideal in the matching monoid ring.

**Proof.** Fix any monomial $z^B$ in the matching monoid ring and a monomial order $\leq$. Let $z^A$ be the standard monomial in the equivalence class of $z^B$ (that the normal form of $z^B$). From the proof of Proposition 4.3, we have a path $B = B_0, B_1, \ldots, B_s = A$ which is monotonic in each entry and such that each $z^{B_{i+1}} - z^{B_i}$ is a monomial multiple of an element in $S_\infty G$ where $G$ is the generating set in Proposition 4.2.

Suppose this path is not strictly decreasing in the monomial order, so there is some $z^{B_{i+1}} > z^{B_i}$. Let $C = A + B_i - B_{i+1}$. Because of the monotonicity of the sequence,
the entries of $C$ are between $A$ and $B$ so $z^C$ is in the matching monoid, and $z^A > z^C$. This contradicts the assumption that $z^A$ is a standard monomial. Therefore $S_\infty G$ is a Gröbner basis for this order.

**Remark 4.6.** In the theory of equivariant Gröbner bases, only monomial orders that respect the monoid action are considered. However the set $S_\infty G$ is a Gröbner basis for any monomial order.

To get a generating set for $\ker(\pi)$ we combine the results of Propositions 4.1 and 4.2. In particular, for each generator $g$ of $\ker(\psi)$, we find a representative of $\phi^{-1}(g) \subset K[Y]$, and then combine the resulting list with the two generators of $\ker(\phi)$. Interestingly, each generator of $\ker(\psi)$ has a unique binomial preimage in $K[Y]$.

**Theorem 4.7.** In the setup of (1.1) with coprime $a > b$, the following binomials form a Markov basis of $\ker(\pi)$.

(i) $y_{12}y_{34} - y_{14}y_{23}$;
(ii) $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$;
(iii) for each $0 \leq n \leq a - b$,
$$y_{12}^{b+n} \prod_{j \geq 3} y_{j2}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n} \prod_{j \geq 3} y_{j1}^{c_{1j}} y_{1j}^{c_{2j}}$$
where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3} c_{2j} = n$;
(iv) for each $1 \leq n \leq b$,
$$y_{12}^{b-n} y_{13}^{a-b+n} - y_{21}^{b-n} y_{23}^{a-b+n} y_{31}^{a}.$$  

The maximum degree of binomials above is $\max(a + b, 2a - b)$ and

$$\text{width}(\ker(\pi)) = \max(4, a - b + 2).$$

**Proof.** The only open items, the upper bound on the degree and the width formula, are easily checked: first is achieved by generators of type (iii) or (iv), second – by the basic quadric (i) or a generator of type (iii).

To see the sharpness we show that for $n = 0$, the two monomials of the binomial in (iii) are the only two elements in their multidegree. This multidegree is

$$d = (ab, ab, a, a, \ldots, a, 0, \ldots)$$

where there are $a - b$ entries equal to $a$. Let $m \in k[Y]$ be any monomial of multidegree $d$. The total degree of $m$ equals $a$ since $2ab + (a - b)a = a(a + b)$. Because of the $a$ entries in $d$, $m$ must divisible by $y_{13} y_{j4} \cdots y_{a_j} a_j$, where each $j_i$ is either one or two. Now since the first two entries of $d$ both equal $ab$, the only possibility is that all $j_i$ are equal. Consequently the only two monomials of multidegree $d$ are two monomials in the type (iii) binomial for $n = 0$ and whenever there are only two monomials of a given multidegree, their difference appears in every Markov basis. □
Remark 4.8. As in Proposition 4.2 the list of binomials of the third type in Theorem 4.7 is finite up to $S_\infty$-action. In particular, we need a representative for each partition of the pair $(a - b - n, n)$ into a sum of pairs of nonnegative numbers such that in no pair both entries are zero.

Example 4.9. Reading line-wise from left to right, the decorated box shapes of moves in Example 3.1 are of types (iii) (with $n = 0$), (i), (ii), (iii) (with $n = 1$), and (iv).

Remark 4.10. The maximal degree of the generators in Theorem 4.7 matches the degrees in Table 1 of [HdC13]. However, we stop short of proving that our generating set is an equivariant Gröbner basis and we doubt that there needs to exist a term order for which it is one. According to our experiments in truncations, we expect the degrees in Gröbner bases to exceed those in Theorem 4.7. For instance in width five for $a = 2, b = 1$, the Markov complexity in Theorem 4.7 is three, while among many thousand random weight orders we have not found one with complexity smaller than five. In fact, we don’t even know if kernels of the form considered here always admit finite equivariant Gröbner bases.

5. Conclusion

The main result of [DEKL13] teaches us that the situation we consider is much better behaved than in other settings where equivariant Noetherianity may be expected, but does not hold. Once the parameters of the single monomial map are fixed (e.g., the exponents $a, b$ in the case of width 2), there is a finite equivariant Markov basis. That is, the description of a Markov basis for a truncated problem does not depend on the width of truncation, which can be viewed as another parameter “tending to infinity”.

One striking example of a family of toric maps where Noetherianity of equivariant kernels fails is summarized by De Loera and Onn’s no hope theorem [DO06]: Markov bases for three-way contingency tables become arbitrarily complicated as the sizes of the tables tend to infinity.

Our work settles problems (L1) and (M1) in width two via explicit constructions. The sharp width bounds that follow (i.e., the answers to (BL1) and (BM1) in width two), are a linear function in the width of the monomial map and a linear function in its exponents, respectively. The more general questions (L2) and (M2) remain open, and already the combinatorics of (M1) in width larger than two appears to be a lot more complicated than that in the case of width two.

Nonetheless, the very modest bounds on the width in the settled cases shall fuel optimism as to practical computation of up-to-symmetry generators of kernels of equivariant toric maps in applications.
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