Hawking radiation for higher dimensional Einstein-Yang-Mills linear dilaton black holes

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Abstract

Recently, Hawking radiation from a 4–dimensional linear dilaton black hole solution to Einstein-Maxwell-Dilaton (EMD) theory, via a method of computing exactly the semi-classical radiation, has been derived by Clément, et al. Their results show that, whenever solution is available to the massless scalar wave equation, an exact computation of the radiation spectrum leads to the Hawking temperature $T_H$. We apply the same method to derive Hawking radiation spectrum for higher dimensional linear dilaton black holes in the Einstein-Yang-Mills-Dilaton (EYMD) and Einstein-Maxwell-Dilaton (EMD) theories. Our results show that the radiation with high frequencies for these massive black holes reveal some remarkable information about the $T_H$. 
I. INTRODUCTION

Although today there are different methods to compute the Hawking radiation, (see for instance [1, 2, 3, 4, 5, 6]), it still remains of interest to consider alternative derivations. On the other hand, none of them is completely conclusive. Nevertheless, the most direct is Hawking’s original study [1, 2], which computes the Bogoliubov coefficients between in and out states for a realistic collapsing black hole. The most significant remark on this study is that a black hole can emit particles from its event horizon with a temperature proportional to its surface gravity. Another elegant contribution was made to the Hawking radiation by Unruh [7]. He showed that it is possible to obtain the same Hawking temperature when the collapse is replaced by appropriate boundary conditions on the horizon of an eternal black hole. Instead of computing the Bogoliubov coefficients in order to obtain the black hole radiation, one may alternatively compute the reflection and transmission coefficients of an incident wave by the black hole. This method works best if the wave equation can be solved, exactly. From now on, we designate this method with “semi-classical radiation spectrum method” and abbreviate it as SCRSM.

Recently, Clément, et al [8] have studied the SCRSM for a class of non-asymptotically flat charged massive linear dilaton black holes. The metric of the associated linear dilaton black holes is a solution to the Einstein-Maxwell-Dilaton (EMD) theory in 4-dimensions. It is shown that in the high frequency region, the SCRSM for massive black holes yield the same temperature with the surface gravity method. Their result for a massless black hole is in agreement with the fact that a massless object can’t radiate.

In this Letter, we shall follow [8] as a guide and extend the application of SCRSM to N-dimensional linear dilaton black holes, which are the solutions to Einstein-Yang-Mills-Dilaton (EYMD) theory [9]. The spacetimes describing these linear dilaton black holes are non-asymptotically flat. First, we consider the statistical Hawking temperature of the massive linear dilaton black holes, computed by using the surface gravity and discuss their evaporation process. According to the Stefan’s law, we show that only 4-dimensional black holes evaporate in an infinite time, while the others $N \geq 5$ can evaporate in a finite time. In both cases, during the evaporation process, the Hawking temperature remains constant for a given dimension. Besides this, the constant value of the Hawking temperature increases with the dimensionality number. We then apply the SCRSM to the massive linear dilaton
black holes and show that this computation exactly matches with the statistical Hawking temperature in the high frequency region. We explain how the massless extreme black holes do not radiate by making a connection between our work and [8].

In Sec. II we present and solve the EYMD equations and by using the massless Klein-Gordon equation, radiation spectrum is derived. In Sec. III we repeat similar analysis for the EMD theory, intending to compare with the EYMD results. The Letter ends with Conclusion in Sec. IV.

II. EINSTEIN-YANG-MILLS-DILATON (EYMD) BLACK HOLE RADIATION SPECTRUM

Recently, we obtained a static spherically symmetric higher dimensional linear dilaton black hole solution in EYMD theory [9]. The metric ansatz is given by

$$ds^2 = -f(r)\ dt^2 + \frac{dr^2}{f(r)} + h(r)^2\ d\Omega_{N-2}^2$$

where $f(r)$ and $h(r)$ are only functions of $r$ and the spherical line element is

$$d\Omega_{N-2}^2 = d\theta_1^2 + \sum_{i=2}^{N-3} \prod_{j=1}^{i-1} \sin^2 \theta_j \ d\theta_i^2$$

in which $0 \leq \theta_k \leq \pi$ with $k = 1..N-3$, and $0 \leq \theta_{N-2} \leq 2\pi$. Our action is

$$I = -\frac{1}{2}\int_M d^N x\sqrt{g}\left[R-(\nabla\Phi)^2-e^{2\alpha\Phi}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)\right],$$

$$\text{Tr}(\cdot) = \sum_{a=1}^{(N-1)(N-2)/2} (\cdot),$$

in which $R$ is the curvature scalar, $\Phi$ is the dilaton field (with parameter $\alpha$) and $F^{(a)} = F_{\mu\nu}^{(a)} dx^{a} dx^{\nu}$ stands for the Yang-Mills (YM) field. In terms of the gauge potentials $A^{(a)} = A_{\mu}^{(a)} dx^{\mu}$, the YM fields are given by

$$F^{(a)} = dA^{(a)} + \frac{1}{2\sigma}\text{C}^{(a)}_{(b)(c)} A^{(b)} \wedge A^{(c)},$$

in which $\text{C}^{(a)}_{(b)(c)}$ stands for the structure constants of $(N-1)(N-2)/2$ parameter Lie group $G$ and $\sigma$ is a coupling constant. The N-dimensional YM ansatz is chosen from the Wu-Yang ansatz [13]

$$A^{(a)} = \frac{Q}{r^2}(x_i dx_j - x_j dx_i), \quad Q = \text{charge}, \quad r^2 = \sum_{i=1}^{N-1} x_i^2,$$

$$2 \leq j + 1 \leq i \leq N-1, \quad \text{and} \quad 1 \leq a \leq (N-1)(N-2)/2,$
where \( Q \) is the non-zero YM charge. The YM equations

\[
d (e^{2\alpha \Phi(r)}F(a)) + \frac{1}{\sigma} C^{(a)}_{(b)(c)} e^{2\alpha \Phi(r)} A^b \wedge \star F^c = 0,
\]

are satisfied by virtue of (5). The corresponding metric functions by setting \( \alpha = \frac{1}{\sqrt{N-2}} \), are found as

\[
f (r) = \frac{(N-3) r}{(N-2) Q^2} \left[ 1 - \left( \frac{b}{r} \right)^{\frac{N-2}{2}} \right],
\]

\[
h (r) = Q \sqrt{2r}
\]

\[
e^{2\alpha \Phi} = r.
\]

By following the mass definition for the non-asymptotically flat black holes, the so-called quasilocal mass introduced by Brown and York [10], one can see that the horizon \( b \) is related to the mass \( M \), charge \( Q \) and the dimensions \( N \), through

\[
b = \left[ \frac{2^{\frac{N}{2}} M Q^{4-N}}{N-3} \right]^{\frac{2}{N-2}}
\]

For \( b > 0 \), the horizon at \( r = b \) hides the null singularity at \( r = 0 \). On the other hand, in the extreme case \( b = 0 \) metric (1) still exhibits the features of the black holes. Because the central singularity \( r = 0 \) is null and marginally trapped, it prevents outgoing signals to reach external observers. Using the conventional definition of the statistical Hawking temperature [11], one finds

\[
T_H = \frac{1}{4\pi} f^\prime (r_h) = \frac{(N-3)}{8\pi Q^2}.
\]

One can immediately observe that \( T_H \) is constant for an arbitrary dimensions \( N \) and linearly increases with the dimensionality of the spacetime. As we learnt from black body radiation, radiating objects loose mass in which the process is governed by Stefan’s law [8]. Therefore while a black hole radiates, it should also loose mass. According to Stefan’s law, we should calculate the surface area of the black hole (7). The horizon area \( A_H \) is found as

\[
A_H = \frac{4\pi \frac{N-1}{2} Q b}{\Gamma \left( \frac{N-1}{2} \right)}
\]

where \( \Gamma(z) \) stands for the gamma function. After assuming that only neutral quanta, in respect of the Yang Mills charge, are radiated, Stefan’s law admits the following time-dependent horizon solutions
\[ b(t) = \exp \left[ -\frac{\sigma}{(2Q)^{\frac{1}{3}}} (t - t_0) \right], \quad (N = 4) \]

\[ b(t) = [-\mu(t - t_0)]^{\frac{2}{N-4}} \quad , \quad (N \geq 5) \] (11) \]

where

\[ \mu = 2^{-(\frac{12+3}{2})} Q^{-(N+3)} \sigma \frac{(N-4)(N-3)^3 \pi^{\frac{N-2}{2}}}{N-2 \Gamma(\frac{N-1}{2})}, \] (12) \]

and \( \sigma \) is Stefan’s constant. For \( N = 4 \), the Hawking temperature is constant with decreasing mass and the black hole reaches to an extreme black hole state \( b = 0 \) in an infinite time. However, for \( N \geq 5 \), the Hawking temperature has also a constant value for a chosen dimensions \( N \), but the black hole turns out to be the extreme black hole in a finite time according to \( b \sim (t_0 - t)^{\frac{1}{N-4}} \). It is observed that the exponential decay law for \( N = 4 \) dimensions, modifies into a power law decay for \( N > 4 \). Following the \( SCRS \), we now derive a more precise expression for the temperature of the black holes (7). To this end, we should first study the wave scattering in such a spacetime. Contrary to the several black hole cases, here the massless Klein-Gordon equation

\[ \Box \Psi = 0 \] (13) \]

admits an exact solution in the spacetimes (1). The Laplacian operator on the \( N \)-dimensional metric (1) is given by

\[ \Box = \frac{1}{\sqrt{-g}} \partial_A \left( \sqrt{-g} \partial^A \right) \] (14) \]

where \( A \) runs from 1 to \( N \). One may consider a separable solution as

\[ \Psi = R(r) e^{-i\omega t} Y_{l,m_1,..,m_{N-3}}(\theta_1,..,\theta_{N-2}) \] (15) \]

in which \( Y_{l,m_1,..,m_{N-3}}(\theta_1,..,\theta_{N-2}) \) is the spherical harmonics in \( N - 1 \) dimensional space. After substituting harmonic eigenmodes (15) into the wave equation (13) and making a straightforward calculation, one obtains the following radial equation:

\[ \frac{1}{f(r)} \left[ h(r)^{N-2} f(r) \partial_r \right] R(r) + \left[ \frac{\omega^2}{f(r)} - \frac{l(l+1)}{h(r)^2} \right] R(r) = 0. \] (16) \]
After changing the independent variable and the parameters as
\[ y = 1 - \left( \frac{r}{b} \right)^{\frac{N-2}{2}}, \quad \tilde{\omega} = \frac{2Q^2}{N-3} \omega \]
\[ \tilde{\lambda}^2 = \frac{2}{(N-2)(N-3)}l(l+1) \]
one transforms the radial equation (16) into
\[ \partial_y (y(y-1) \partial_y) R(y) + \left( \omega^2 \frac{y-1}{y} - \tilde{\lambda}^2 \right) R(y) = 0. \]

Further, letting
\[ \tilde{\Lambda}^2 = \tilde{\omega}^2 - (\tilde{\lambda}^2 + \frac{1}{4}) \]
we can obtain the general solution of equation (18) in the form
\[ R(y) = C_1 y^{\tilde{\omega}} F \left( \frac{1}{2} + i \left( \tilde{\omega} + \tilde{\Lambda} \right), \frac{1}{2} + i \left( \tilde{\omega} - \tilde{\Lambda} \right), 1 + 2i\tilde{\omega}; y \right) + \\
C_2 y^{-i\tilde{\omega}} F \left( \frac{1}{2} - i \left( \tilde{\omega} + \tilde{\Lambda} \right), \frac{1}{2} - i \left( \tilde{\omega} - \tilde{\Lambda} \right), 1 - 2i\tilde{\omega}; y \right). \]

Thus, solution (20) leads to the general solution of Eq. (18) as
\[ R(\rho) = C_1 \left( \frac{\beta - \rho}{\beta} \right)^{i\tilde{\omega}} F \left( \frac{1}{2} + i \left( \tilde{\omega} + \tilde{\Lambda} \right), \frac{1}{2} + i \left( \tilde{\omega} - \tilde{\Lambda} \right), 1 + 2i\tilde{\omega}; \frac{\beta - \rho}{\beta} \right) + \\
C_2 \left( \frac{\beta - \rho}{\beta} \right)^{-i\tilde{\omega}} F \left( \frac{1}{2} - i \left( \tilde{\omega} + \tilde{\Lambda} \right), \frac{1}{2} - i \left( \tilde{\omega} - \tilde{\Lambda} \right), 1 - 2i\tilde{\omega}; \frac{\beta - \rho}{\beta} \right) \]

in which
\[ \rho = (r)\left( \frac{N-2}{2} \right) \]
\[ \beta = (b)\left( \frac{N-2}{2} \right) \]

One follows the result of the G. Clément et al’s work (see Eq. (19) in [8]), to read the resulting radiation spectrum as
\[ \left( e^{\frac{\tilde{\omega}}{N}} - 1 \right)^{-1} = \frac{\cosh^2 \pi \left( \tilde{\omega} - \tilde{\Lambda} \right)}{\cosh^2 \pi \left( \tilde{\omega} + \tilde{\Lambda} \right) - \cosh^2 \pi \left( \tilde{\omega} - \tilde{\Lambda} \right)} \]

For high frequencies \( \tilde{\Lambda} \simeq \tilde{\omega} = \frac{2Q^2}{N-3} \omega \), and this leads to
\[ e^{\frac{\tilde{\omega}}{N}} = \cosh^2 2\pi \tilde{\omega} \rightarrow T_H = \lim_{\omega \rightarrow \text{large value}} \frac{\omega}{2 \ln (\cosh 2\pi \tilde{\omega})} = \frac{N-3}{8\pi Q^2} \]
which is nothing but the statistical Hawking temperature (9).

On the other hand, for $b = 0$ (i.e. the case of extreme massless black holes), the above analysis for computing the Hawking radiation fails. In [8], it is successfully shown that the wave scattering problem in the extreme linear dilaton black holes in EMD theory reduces to the propagation of eigenmodes of a free Klein-Gordon field in two-dimensional Minkowski spacetime with an effective mass. Conclusively, there is no reflection, so that the extreme linear dilaton black holes can’t radiate, although their surface gravities remain finite. Since setting $b = 0$ reduces metric (1) to a conformal product $M_2 \times S^{N-2}$ of a two dimensional Minkowski spacetime with the $(N - 2)$–sphere of constant radius, the same interpretation is valid for the extreme linear dilaton black holes in EYMD theory. Namely, the massless linear dilaton black holes in EYMD theory also can’t radiate.

III. EINSTEIN-MAXWELL-DILATON (EMD) BLACK HOLE RADIATION SPECTRUM

In order to make a comparison with the foregoing YM results in this section we represent the similar analysis for the EMD case. The $N$–dimensional EMD action is given by [12]

$$S = \int d^N x \sqrt{-g} \left( R - \frac{4}{N-2} (\nabla \Phi)^2 - e^{-\frac{4\alpha}{N-2}} F^2 \right) \quad (25)$$

where $\alpha$ is the dilaton parameter, and $F^2 = F_{\mu\nu} F^{\mu\nu}$ (this form of action was used by Chan et al [12]). The metric and field ansatz are

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + h(r)^2 d\Omega_{N-2}^2 \quad (26)$$

$$F = e^{\frac{4\alpha}{N-2}} \frac{q}{h^{N-2}} dt \wedge dr. \quad (27)$$

In Ref. [12] the general non-asymptotically flat solutions were reported in which by setting $\alpha = N - 3$ one gets the following explicit form

$$f(r) = \frac{4}{\gamma^2} \left( \frac{N-3}{N-2} \right)^2 r \left( 1 - \left( \frac{b}{r} \right)^{\frac{N-2}{2}} \right), \quad (28)$$

$$h(r) = \gamma \sqrt{r}, \quad \gamma = \text{constant}, \quad (29)$$

where the radius of event horizon $r_h = b$, is given by

$$b = \left( \frac{2(N-2) M_{QL}}{(N-3)^2 \gamma^{N-4}} \right)^{\frac{2}{N-2}} \quad (30)$$
and $M_{QL}$ stands for the quasilocal mass of the black hole. One may use the usual definition of the Hawking temperature at the radius of event horizon

$$T_H = \frac{1}{4\pi} f'(r_h) = \frac{(N-3)^2}{2\pi \gamma^2 (N-2)}. \quad (31)$$

Following the same procedure as in the previous section, one can show that, the radiation spectrum of the EMD black holes are same as EYMD case which is given by Eq. (23), provided the definitions of $\tilde{\omega}$ and $\tilde{\lambda}^2$ are changed as

$$\tilde{\lambda}^2 = \frac{l(l+1)}{(N-3)^2}, \text{ and } \tilde{\omega} = \frac{(N-2) \omega \gamma^2}{2(N-3)^2}. \quad (32)$$

With these changes, the high frequency limit of radiation spectrum gives

$$\lim_{\omega \to \text{large value}} e^{i\tilde{\omega}^2} = \cosh^2 2\pi \tilde{\omega} \rightarrow T_H = \lim_{\omega \to \text{large value}} \frac{\omega}{2 \ln (\cosh 2\pi \tilde{\omega})} = \frac{(N-3)^2}{2\pi \gamma^2 (N-2)} \quad (33)$$

which is matched with (31). In Fig. (1) we plot the high frequency limits of the Hawking temperatures in terms of the dimensions of the space-time, for both EMD and EYMD cases. Also, in Fig. (2) we plot the Hawking temperature in terms of the spectrum low frequency $\omega$ for both EMD and EYMD black hole.

IV. CONCLUSION

The semi-classical radiation spectrum method (SCRSM) properly works to compute the Hawking radiation for massive higher-dimensional ($N \geq 5$) linear dilatonic black holes in EYMD and EMD theories. In the high frequency region, the results of SCRSM agree with the temperature ($T_H$) obtained from the surface gravity. The dependence of $T_H$ on the dimensionality $N$ is plotted for both the EYMD and EMD theories to see the differences. From the Stefan’s law it is shown that the $N = 4$ dimensions dilatonic black holes evaporate in an infinite time, while for $N \geq 5$ it takes a finite time. We verify also, once more that a massless object can’t radiate. We do this by studying a massless Klein-Gordon field in our extreme dilatonic black hole background for which $T_H = 0$, whereas the surface gravity is not zero.

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V. FIGURE CAPTIONS

Figure 1: A plot of the high frequency limits of Hawking temperatures $T_H$ versus the dimensions of the space-time. The solid and crossed curves represent the EMD and EYMD cases, respectively.

Figure 2: $T_H$ versus small frequency plot in 5D. Up to a systematic shift, the EMD and EYMD both exhibit similar behavior at small frequencies.
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