ON CERTAIN MAPS DEFINED BY INFINITE SUMS

SYMON SERBENYUK

Abstract. The present article is devoted to some examples of functions whose arguments represented in terms of certain series of the Cantor type.

1. Introduction

Let $Q \equiv (q_k)$ be a fixed sequence of positive integers, $q_k > 1$, $\Theta_k$ be a sequence of the sets $\Theta_k \equiv \{0, 1, \ldots, q_k - 1\}$, and $\varepsilon_k \in \Theta_k$.

Real number expansions of the form

$$\varepsilon_1 / q_1 + \varepsilon_2 / q_1 q_2 + \cdots + \varepsilon_k / q_1 q_2 \cdots q_k + \cdots$$

(1)

for $x \in [0, 1]$, first studied by G. Cantor in [4]. It is easy to see that the last expansion is the $q$-ary expansion

$$\alpha_1 / q + \alpha_2 / q^2 + \cdots + \alpha_n / q^n + \cdots$$

of numbers from $[0, 1]$ whenever the condition $q_k = q$ holds for all positive integers $k$. Here $q$ is a fixed positive integer, $q > 1$, and $\alpha_n \in \{0, 1, \ldots, q - 1\}$.

By $x = \Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots} \cdots$ denote a number $x \in [0, 1]$ represented by series (1). This notation is called the representation of $x$ by (positive) Cantor series (1).

Let us remark that certain numbers from $[0, 1]$ have two different representations by positive Cantor series (1), i.e.,

$$\Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots} \varepsilon_{m-1} \varepsilon_m 000 \cdots = \Delta^Q_{\varepsilon_1 \varepsilon_2 \cdots} \varepsilon_{m-1} |\varepsilon_{m-1}| [q_{m+1-1}] [q_{m+2-1}] \cdots = \sum_{i=1}^{m} \varepsilon_i / q_1 q_2 \cdots q_i.$$

Such numbers are called $Q$-rational. The other numbers in $[0, 1]$ are called $Q$-irrational.

Now a number of researchers introduce and/or study alternating versions (types) of well-known positive expansions. For example, investigations of positive and alternating Lüroth series and Engel series (e.g., see [5, 9, 11, 29]), as well as of $\beta$- and $(-\beta)$-expansions ([8, 13]) are such researches.

Since investigations for the cases of alternating expansions require more complicated techniques, let us consider functions whose arguments defined in terms of

\begin{itemize}
  \item 2010 Mathematics Subject Classification. 26A27, 11B34, 11K55, 39B22.
  \item Key words and phrases. nowhere differentiable function; singular function; expansion of real number; non-monotonic function; Hausdorff dimension.
\end{itemize}
alternating series of the Cantor type. The present investigations are similar with investigations \([20]\) for positive Cantor series but are more complicated.

In \([20]\), the following expansions of real numbers were studied:

\[
\frac{\varepsilon_1}{q_1} + \frac{\varepsilon_2}{q_1q_2} + \frac{\varepsilon_3}{q_1q_2q_3} + \cdots + (-1)^k \frac{\varepsilon_k}{q_1q_2\cdots q_k} + \cdots
\]

(2)

for \(x \in [a_0 - 1, a_0]\), where \(a_0 = \sum_{k=1}^{\infty} \frac{q_k - 1}{q_1q_2\cdots q_k}\). Here \(-Q = (-q_k)\) is a fixed sequence of negative integers \((-q_k) < -1\), \(\Theta_k \equiv \{0, 1, \ldots, q_k - 1\}\), and \(\varepsilon_k \in \Theta_k\).

It is easy to see that the last expansion is the nega-\(q\)-ary expansion

\[
\Delta^{-q}_{\alpha_1q_2\alpha_3\ldots\alpha_k\ldots} \equiv -\frac{\alpha_1}{q} - \frac{\alpha_2}{q^2} - \frac{\alpha_3}{q^3} - \cdots - \frac{(-1)^k\alpha_k}{q^k} - \cdots
\]

(3)

of numbers from \([-\frac{q}{q+1}: \frac{1}{q+1}]\) whenever the condition \(q_k = q\) holds for all positive integers \(k\). Here \(q\) is a fixed positive integer, \(q > 1\), and \(\alpha_n \in \{0, 1, \ldots, q - 1\}\).

By \(x = \Delta^{-Q}_{c_1c_2\ldots c_m}\) denote a number \(x \in [a_0 - 1, a_0]\) represented by series \([2]\). This notation is called the representation of \(x\) by alternating Cantor series \([1]\) or the nega-\(Q\)-representation.

The term “nega” is used in this article, since the alternating Cantor series expansion is a numeral system with a negative base \((-q_k)\).

Some numbers have two different representations by alternating series \([2]\), i.e.,

\[
\Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots} \equiv \Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots} = \Delta^{-Q}_{c_1c_2\ldots c_m}\]

Such numbers are called nega-\(Q\)-rational. The other numbers in \([a_0 - 1, a_0]\) are called nega-\(Q\)-irrational.

Suppose \(c_1, c_2, \ldots, c_m\) is an ordered tuple of integers such that \(c_i \in \{0, 1, \ldots, q_i - 1\}\) for \(i = 1, m\). Then a cylinder \(\Delta^{-Q}_{c_1c_2\ldots c_m}\) of rank \(m\) with base \(c_1c_2\ldots c_m\) is a set of the form

\[
\Delta^{-Q}_{c_1c_2\ldots c_m} \equiv \{x : x = \Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2} \ldots} \}
\]

That is any cylinder \(\Delta^{-Q}_{c_1c_2\ldots c_m}\) is a closed interval of the form:

\[
[\Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots} \Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2} \ldots} \Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2} \ldots} ...
\]

if \(m\) is even,

\[
[\Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots} \Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2} \ldots} \Delta^{-Q}_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2} \ldots} ...
\]

if \(m\) is odd.

Define the shift operator \(\sigma\) of expansion \([2]\) by the rule

\[
\sigma(x) = \sigma(\Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_k\ldots}) = \sum_{k=2}^{\infty} (-1)^k \varepsilon_k \frac{q_1 q_2 \cdots q_k}{q_{k+1}q_{k+2} \cdots} = -q_1 \Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_k\ldots}...
\]

Whence,

\[
\sigma^n(x) = \sigma^n(\Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_k\ldots}) = \sum_{k=n+1}^{\infty} (-1)^{k-n} \varepsilon_k \frac{q_1 q_2 \cdots q_k}{q_{n+1}q_{n+2} \cdots} = (-1)^n q_1 \cdots q_n \Delta^{-Q}_{\varepsilon_1\varepsilon_2\ldots \varepsilon_k\ldots}...
\]

(4)
Therefore,
\[ x = \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{q_1 q_2 \cdots q_i} + \frac{(-1)^n}{q_1 q_2 \cdots q_n} \sigma^n(x). \] (5)

The notion of the shift operator of an alternating Cantor series was studied in detail in the paper [20].

In [14], the following singular function
\[ s(x) = s(\Delta^2_{\alpha_1 \alpha_2 \cdots}) = \beta_{\alpha_1} + \sum_{n=2}^{\infty} \left( \beta_{\alpha_n} \prod_{i=1}^{n-1} q_i \right) = y = \Delta^2_{\alpha_1 \alpha_2 \cdots}, \]
where \( q_0 > 0, q_1 > 0, \) and \( q_0 + q_1 = 1, \) was modeled by Salem. Note that generalizations of the Salem function can be non-differentiable functions or do not have a derivative on a certain set.

Let us consider the following generalizations of the Salem function that are described in the paper [22] as well.

**Example 1** ([16]). Let \((q_n)\) is a fixed sequence of positive integers, \( q_n > 1, \) and \((A_n)\) is a sequence of the sets \( \Theta_n = \{0, 1, \ldots, q_n-1\} \).

Let \( x \in [0, 1] \) be an arbitrary number represented by a positive Cantor series
\[ x = \Delta^2_{\varepsilon_1 \varepsilon_2 \cdots} = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_1 q_2 \cdots q_n}, \]
where \( \varepsilon_n \in \Theta_n. \)

Let \( P = ||p_{i,n}|| \) be a fixed matrix such that \( p_{i,n} \in (-1, 1) \) \( (n = 1, 2, \ldots, \) and \( i = 0, q_n-1), \) \( \sum_{i=0}^{q_n-1} p_{i,n} = 1 \) for an arbitrary \( n \in \mathbb{N}, \) and \( \prod_{n=1}^{\infty} p_{i,n} = 0 \) for any sequence \((i_n)\).

Suppose that elements of the matrix \( P = ||p_{i,n}|| \) can be negative numbers as well but \( \beta_{0,n} = 0, \beta_{i,n} > 0 \) for \( i \neq 0, \) and \( \max_i |p_{i,n}| < 1. \)

Here
\[ \beta_{\varepsilon_k,k} = \begin{cases} 0 & \text{if } \varepsilon_k = 0 \\ \sum_{i=0}^{\varepsilon_k-1} p_{i,k} & \text{if } \varepsilon_k \neq 0. \end{cases} \]

Then the following statement is true.

**Theorem 1** ([16]). Given the matrix \( P \) such that for all \( n \in \mathbb{N} \) the following are true: \( p_{i,n}, p_{\varepsilon_{n-1},n} < 0 \) moreover \( q_n \cdot p_{\varepsilon_{n-1},n} \geq 1 \) or \( q_n \cdot p_{\varepsilon_{n-1},n} \leq 1; \) and the conditions
\[ \lim_{n \to \infty} \prod_{k=1}^{n} q_k p_{0,k} \neq 0, \lim_{n \to \infty} \prod_{k=1}^{n} q_k p_{k-1,k} \neq 0 \]
hold simultaneously. Then the function
\[ F(x) = \beta_{\varepsilon_{1}(x),1} + \sum_{k=2}^{\infty} \left( \beta_{\varepsilon_k(x),k} \prod_{n=1}^{k-1} P_{\varepsilon_n(x),n} \right) \]
is non-differentiable on \([0, 1].\)
Example 2 (17). Let \( P = \|p_{i,n}\| \) be a given matrix such that \( n = 1, 2, \ldots \) and \( i = 0, q_n - 1 \). For this matrix the following system of properties holds:

\[
\begin{align*}
1^\circ. & \forall n \in \mathbb{N} : p_{i,n} \in (-1, 1) \\
2^\circ. & \forall n \in \mathbb{N} : \sum_{i=0}^{q_n-1} p_{i,n} = 1 \\
3^\circ. & \forall (i_n), i_n \in \Theta_n : \prod_{n=1}^{\infty} |p_{i,n,n}| = 0 \\
4^\circ. & \forall i_n \in \Theta_n \setminus \{0\} : 1 > \beta_{i_n,n} = \sum_{i=0}^{\infty} p_{i,n} > \beta_{0,n} = 0.
\end{align*}
\]

Let us consider the following function

\[
\tilde{F}(x) = \beta_{\varepsilon_1(x),1} + \sum_{n=2}^{\infty} \left( \tilde{\beta}_{\varepsilon_n(x),n} \prod_{j=1}^{n-1} \tilde{\beta}_{\varepsilon_j(x),j} \right),
\]

where

\[
\begin{align*}
\tilde{\beta}_{\varepsilon_n(x),n} &= \begin{cases} 
\beta_{\varepsilon_n(x),n} & \text{if } \varepsilon_n \text{ is odd,} \\
\beta_{q_n-1-\varepsilon_n(x),n} & \text{if } \varepsilon_n \text{ is even,}
\end{cases} \\
\tilde{\beta}_{\varepsilon_n(x),n} &= \begin{cases} 
p_{\varepsilon_n(x),n} & \text{if } \varepsilon_n \text{ is odd,} \\
p_{q_n-1-\varepsilon_n(x),n} & \text{if } \varepsilon_n \text{ is even,}
\end{cases} \\
\beta_{\varepsilon_n(x),n} &= \begin{cases} 
0 & \text{if } \varepsilon_n = 0 \\
\sum_{i=0}^{\varepsilon_n-1} p_{i,n} & \text{if } \varepsilon_n \neq 0.
\end{cases}
\end{align*}
\]

Here \( x \) represented by an alternating Cantor series, i.e.,

\[
x = \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n} = \sum_{n=1}^{\infty} \frac{1+\varepsilon_n}{q_1 q_2 \ldots q_n} (-1)^{n+1},
\]

where \( (q_n) \) is a fixed sequence of positive integers, \( q_n > 1 \), and \( (\Theta_n) \) is a sequence of the sets \( \Theta_n = \{0, 1, \ldots, q_n - 1\} \), and \( \varepsilon_n \in \Theta_n \).

Theorem 2. Let \( p_{\varepsilon_n,n}, p_{\varepsilon_{n-1},n} < 0 \) for all \( n \in \mathbb{N} \), \( \varepsilon_n \in \Theta_n \setminus \{0\} \) and conditions

\[
\lim_{n \to \infty} \prod_{k=1}^{n} q_k p_{0,k} \neq 0, \quad \lim_{n \to \infty} \prod_{k=1}^{n} q_k p_{q_n-1,k} \neq 0
\]

hold simultaneously. Then the function \( \tilde{F} \) is non-differentiable on \([0, 1]\).

In the present article, two examples of certain functions with complicated local structure, are constructed and investigated.

Suppose that the condition \( q_n \leq q \) holds for all positive integers \( n \). The first function is following:

\[
f : \ x = \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n} \to \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n} = y.
\]
The function $f$ is interesting, since a some function, whose almost all properties (all properties without the domain of definition; the domains of definition of these functions are different intervals) are identical with properties of the function described in Example 2 can be represented by the following way:

$$F(x) = \hat{F}_{\xi,Q} \circ g \circ f.$$ 

Here by "$\circ$" denote the operation of composition of functions and $g(x) = x - \Delta_-^q_{(q-1)[0]}$. Also, the function $\hat{F}_{\xi,Q}$ is a function of the type:

$$\hat{F}_{\eta,Q}(y) = \hat{\beta}_{\varepsilon_1(y),1} + \sum_{k=2}^{\infty} \left( \hat{\beta}_{\varepsilon_k(y),k} \prod_{j=1}^{k-1} \bar{\beta}_{\varepsilon_j(y),j} \right),$$

where

$$y = \Delta_-^q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n} = \Delta_-^q_{[q_1-1][0][q_3-1][0]...[q_{2k-1}-1][0]...} = \Delta_-^q_{[q_1-1-\varepsilon_1][0][q_3-1-\varepsilon_3]0...[q_{2k-1}-1-\varepsilon_{2k-1}]0...}.$$

Note that the function $\hat{F}_{\eta,Q}$ is a distribution function of a certain random variable $\eta$ whenever elements $p_{n,n}$ of the matrix $P$ (this matrix described in the last-mentioned examples) are non-negative and

$$\bar{\beta}_{\varepsilon_n(x),n} = \begin{cases} p_{\varepsilon_n(x),n} & \text{if } n \text{ is even} \\ p_{q_n-1-\varepsilon_n(x),n} & \text{if } n \text{ is odd}, \end{cases}$$

$$\hat{\beta}_{\varepsilon_n(x),n} = \begin{cases} \beta_{\varepsilon_n(x),n} & \text{if } n \text{ is even} \\ \beta_{q_n-1-\varepsilon_n(x),n} & \text{if } n \text{ is odd}. \end{cases}$$

**Remark 1.** [26] "In the general case, suppose that $(f_n)$ is a finite or infinite sequence of certain functions (the sequence can contain functions with complicated local structure). Let us consider the corresponding composition of the functions

$$f_n \circ \ldots \circ f_2 \circ f_1 = f_{c,\infty}$$

or

$$f_n \circ \ldots \circ f_2 \circ f_1 = f_{c,n}.$$ 

Also, we can take a certain part of the composition, i.e.,

$$f_{n_0+t} \circ \ldots \circ f_{n_0+1} \circ f_{n_0} = f_{c,\overline{n_0,n_0+t}},$$

where $n_0$ is a fixed positive integer (a number from the set $\mathbb{N}$), $t \in \mathbb{Z}_0 = \mathbb{N} \cup \{0\}$, and $n_0 + t \leq n$.

One can use such technique for modeling and studying functions with complicated local structure. Also, one can use new representations of real numbers (numeral systems) of the type

$$x' = \Delta^c_{\xi_1\ldots\xi_n} = \ldots \circ f_n \circ \ldots \circ f_2 \circ f_1(x),$$

$$x' = \Delta^c_{\eta_1\ldots\eta_n} = f_n \circ \ldots \circ f_2 \circ f_1(x)$$

or

$$z' = \Delta^c_{\eta_1\ldots\eta_n+t} = f_{n_0+t} \circ \ldots \circ f_{n_0+1} \circ f_{n_0}(z).$$
in fractal theory, applied mathematics, etc. The next articles of the author of the present article will be devoted to such investigations”.

Remark 2. One can extend the last remark by the following. Really, compositions of functions are useful for modeling functions with complicated local structure. However, for modeling such functions one can use systems of functional equations containing compositions of functions. For example,

\[
f \left( g \circ g \circ \ldots \circ g(x) \right) = a_k + b_k f \left( g \circ g \circ \ldots \circ g(x) \right),
\]

where \( f, g \) are some functions, \( a_k, b_k \in \mathbb{R} \).

For example, in \([27]\), a technique for modeling certain generalizations of the singular Salem function is introduced. That is,

\[
f \left( \sigma_{n_k-1} \circ \sigma_{n_k-2} \circ \ldots \circ \sigma_{n_1}(x) \right) = \beta_{\alpha_{n_k,n_k}} + p_{\alpha_{n_k,n_k}} f \left( \sigma_{n_k} \circ \sigma_{n_k-1} \circ \ldots \circ \sigma_{n_1}(x) \right),
\]

where \( k = 1, 2, \ldots, \), \( \sigma_0(x) = x \), and \( x \) represented in terms of a certain given numeral system, i.e., \( x = \Delta^{\alpha_{n_k,n_k-1}}u \ldots u^{\alpha_{2,1}}u^{\alpha_{1,1}} \) and \( \alpha_n \in \{0, 1, \ldots, m \} \) for all positive integers \( n \). Here \( (\sigma_{n_k}) \) is a sequence of certain functions and \( \beta_{\alpha_{n_k,n_k}}, p_{\alpha_{n_k,n_k}} \) are some real numbers. Note that a given numeral system can be with a finite or infinite, constant or removable (or variable when \( \alpha_i \in A_i \neq A_j \) for some \( i \neq j \) alphabet).

So, these problems introduce the problem on functional equations and systems of functional equations with several variables, on functional equations and systems of functional equations with compositions of functions.

In addition, one can consider expansions of functions and numbers by complicated compositions of functions:

\[
\ldots \circ f_n \left( g_{m_n}^{(n)} \circ \ldots \circ g_1^{(n)}(x) \right) \circ \ldots \circ f_2 \left( g_{m_2}^{(2)} \circ \ldots \circ g_1^{(2)}(x) \right) \circ f_1 \left( g_{m_1}^{(1)} \circ \ldots \circ g_1^{(1)}(x) \right).
\]

Here \( g_{m_n}^{(n)}, f_n \) are certain functions. One can consider partial cases of the last-mentioned complicated composition and the case when \( m_n = \infty \).

The next articles of the author of the present article will be devoted to such investigations.

The second map considered in this article is useful for modeling fractals in space \( \mathbb{R}^2 \). That is, the map

\[
f : x = \Delta^{\alpha_{n_k,n_k-1}}u \ldots u \alpha_{n_1,1} u^{\alpha_{n_2-1}}u \ldots u \alpha_{n-2,1} \ldots u^{\alpha_{n-1,1}} \longrightarrow \Delta^{\alpha_{n_k,n_k-1}}u \ldots u \alpha_{n_1,1} u^{\alpha_{n_2-1}}u \ldots u \alpha_{n-2,1} \ldots u^{\alpha_{n-1,1}},
\]

where \( u \in \{0, 1, \ldots, q-1\} \) is a fixed number, \( \alpha_n \in \{1, 2, \ldots, q-1\} \setminus \{u\} \), and \( 3 < q \) is a fixed positive integer, models a certain fractal in \( \mathbb{R}^2 \).
2. **One function defined in terms of alternating Cantor series**

Let us consider the function

$$f(x) = f\left(\Delta_{\varepsilon_1\varepsilon_2\ldots}\right) = f\left(\sum_{n=1}^{\infty} (-1)^{n+\varepsilon_n}\right) = \sum_{n=1}^{\infty} (-q)^n \Delta_{\varepsilon_1\varepsilon_2\ldots} = y,$$

where \(\varepsilon_n \in \Theta_n\) and the condition \(q_n \leq q\) holds for all positive integers \(n\).

**Lemma 1.** The function \(f\) has the following properties:

1. \(D(f) = [a_0 - 1, a_0]\), where \(D(f)\) is the domain of definition of \(f\) and
   $$a_0 = \sum_{k=1}^{\infty} \frac{q_{2k} - 1}{q_{1}q_{2}\cdots q_{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_{1}q_{2}\cdots q_{2k}}, \quad a_0 - 1 = \sum_{k=1}^{\infty} \frac{q_{2k-1} - 1}{q_{1}q_{2}\cdots q_{2k-1}};$$

2. If \(E(f)\) is the range of values of \(f\), then:
   - \(E(f) = \left[-\frac{q}{q+1}, \frac{1}{q+1}\right]\) whenever the condition \(q_n = q\) holds for all positive integers \(n\),
   - \(E(f) = \left[-\frac{q}{q+1}, \frac{1}{q+1}\right) \setminus C_f\), where \(C_f = C_1 \cup C_2\),
   $$C_1 = \{y : y = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}, \varepsilon_n \notin \{q_n, q_n + 1, \ldots, q - 1\}\text{ for all } n \text{ such that } q_n < q\};$$
   $$C_2 = \{y : y = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n-1}, \varepsilon_n \notin \{q_n, q_n + 1, \ldots, q - 1\}\};$$
3. \(f(x) + f(a_0 - x) = f(a_0) \leq 1\);
4. \(f(\sigma^k(x)) = \sigma^k(f(x))\) for any \(k \in \mathbb{N}\).

**Proof.** The first property follows from the definition of \(f\).

The second property follows from the definition of \(f\) and Theorem 3 (the next theorem).

Let us prove the third property. Since

$$a_0 - x = \sum_{k=1}^{\infty} \frac{q_{2k} - 1 - \varepsilon_{2k}}{q_{1}q_{2}\cdots q_{2k}} = \sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}}{q_{1}q_{2}\cdots q_{2k-1}},$$

we have

$$f(a_0 - x) = \sum_{k=1}^{\infty} \frac{q_{2k} - 1 - \varepsilon_{2k}}{q^{2k}} + \sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}}{q^{2k-1}}.$$

Whence,

$$f(x) + f(a_0 - x) = \sum_{k=1}^{\infty} (-q)^k + \sum_{k=1}^{\infty} \frac{q_{2k} - 1 - \varepsilon_{2k}}{q^{2k}} + \sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}}{q^{2k-1}} = \sum_{k=1}^{\infty} \frac{q_{2k} - 1}{q^{2k}} = f(a_0) \leq 1.$$**

Note that the last inequality is an equality whenever \(y = x\), i.e., when the condition \(q_n = q\) holds for all positive integers \(n\).

Let us prove the fourth property. We have

$$f(\sigma^k(x)) = f\left(\sum_{j=k+1}^{\infty} \frac{(-1)^{j-k} \varepsilon_j}{q_{k+1}q_{k+2}\cdots q_j}\right) = \sum_{j=k+1}^{\infty} \frac{\varepsilon_j}{(-q)^{j-k}} = \sigma^k \left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{(-q)^n}\right) = \sigma^k(f(x)).$$
Theorem 3. The following properties are true:

- The function $f$ is continuous at nega-$Q$-irrational points from $[a_0 - 1, a_0]$.
- The function $f$ is continuous at all nega-$Q$-rational points from $[a_0 - 1, a_0]$ if the condition $q_k = q$ holds for all positive integers $n$.
- If there exist positive integers $n$ such that $q_m < q$, then points of the form

$$\Delta_{Q}^{\Delta_{Q}} - \Delta_{Q}^{\Delta_{Q}} m \in \{q_{m+1} - 1|q_{m+2} - 1|q_{m+3} - 1|q_{m+4} - 1|q_{m+5} - 1|\ldots\}$$

where $m < n$, are points of discontinuity of the function.

Proof. Since for any $x \in [a_0 - 1, a_0]$ the equality

$$x = \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{m} = \bigcap_{m=1}^{\infty} \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{m},$$

is true (see [20]), where $\Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{m}$ is a nega-$Q$-cylinder, let us consider $x, x_0 \in \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{m}$. Here $x$ is an arbitrary number, $x_0$ is a nega-$Q$-irrational number. Then

$$|f(x) - f(x_0)| = \left| \sum_{k=m+1}^{\infty} \frac{\varepsilon_k(f(x)) - \varepsilon_k(f(x_0))}{q^k} \right| \leq \frac{1}{q^m} \left| \sum_{k=m+1}^{\infty} \frac{q_k - 1}{q^k} \right| \leq \sum_{k=m+1}^{\infty} \frac{q - 1}{q^k} = \frac{1}{q^m} \to 0 \text{ as } m \to \infty.$$

So,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

That is, the function $f$ is continuous at nega-$Q$-irrational points.

If $x_0 = \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{n}$ is a nega-$Q$-rational point, then

$$x_0 = x_0^{(1)} = \left\{ \begin{array}{ll} \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{n-1} e_n = q_{n+1} - 1|q_{n+2} - 1|q_{n+3} - 1|q_{n+4} - 1|q_{n+5} - 1|\ldots & \text{if } n \text{ is even} \\ \Delta_{Q}^{\Delta_{Q}} e_{2} \ldots e_{n-1} e_n = q_{n+1} - 1|q_{n+2} - 1|q_{n+3} - 1|q_{n+4} - 1|q_{n+5} - 1|\ldots & \text{if } n \text{ is odd} \end{array} \right.$$ 

Using the technique for the case of nega-$Q$-irrational points, we obtain the following for nega-$Q$-rational points:

$$\lim_{x \to x_0^{(1)}} f(x) = f(x_0^{(1)}) \text{ and } \lim_{x \to x_0^{(2)}} f(x) = f(x_0^{(2)}).$$

Hence

$$\Delta_f = \lim_{x \to x_0^{(1)}} f(x) - \lim_{x \to x_0^{(2)}} f(x) = \frac{1}{q^n} - \frac{1}{q^n} \sum_{k=1}^{\infty} \frac{q_{k+n} - 1}{q^k} \neq 0$$

whenever there exists at least one $q_{k+n} < q$. \hfill \Box

Corollary 1. The set of all points of discontinuity of the function $f$ is:
• the empty set whenever \( q_n = q \) for all \( n \in \mathbb{N} \). 
• a finite set whenever \( q_n \neq q \) for a finite number of \( n \). 
• an infinite set whenever there exists an infinite subsequence \( (n_k) \) of positive integers such that \( q_{n_k} \neq q \).

Remark 3. To reach that the function \( f \) be well-defined on the set of nega-\( Q \)-rational numbers from \([a_0 - 1, a_0]\), we shall not consider the representation

\[
\Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n-1} [\varepsilon_{n-1} - 1] [\varepsilon_{n+1} - 1] [\varepsilon_{n+3} - 1] \cdots}.
\]

Lemma 2. The function \( f \) is a strictly increasing function on the domain.

Proof. Suppose \( x_1 = \Delta_{\alpha_1 \alpha_2 \cdots \alpha_n} \) and \( x_2 = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n} \) such that \( x_1 < x_2 \). Then there exists \( n_0 \) such that \( \alpha_i = \varepsilon_i \) for \( i = 1, n_0 - 1 \) and

\[
\begin{cases}
\alpha_{n_0} < \varepsilon_{n_0} & \text{if } n_0 \text{ is even} \\
\alpha_{n_0} > \varepsilon_{n_0} & \text{if } n_0 \text{ is odd}.
\end{cases}
\]

Whence,

\[
f(x_2) - f(x_1) = \frac{\varepsilon_{n_0} - \alpha_{n_0}}{(-q)^{n_0}} + \sum_{j = n_0 + 1}^{\infty} \frac{\varepsilon_j - \alpha_j}{(-q)^j}.
\]

If \( n_0 \) is even, then

\[
f(x_2) - f(x_1) \geq \frac{\varepsilon_{n_0} - \alpha_{n_0}}{q^{n_0}} + \sum_{k = 1}^{\infty} \frac{q^{n_0 + 2k - 1} - 1}{q^{n_0 + 2k - 1}} \geq \frac{1}{q^{n_0}} - \frac{1}{q^{n_0}} \sum_{k = 1}^{\infty} \frac{q^{n_0 + 2k - 1} - 1}{q^{2k - 1}}
\]

\[
\geq \frac{1}{q^{n_0}} - \frac{1}{q^{n_0}} \sum_{k = 1}^{\infty} \frac{q - 1}{q^{2k - 1}} = \frac{1}{(q + 1)q^{n_0}} > 0,
\]

since \( n_0 \) is even, \( \varepsilon_{n_0} - \alpha_{n_0} \geq 1 \), and \( q_n \leq q \).

If \( n_0 \) is odd, then

\[
f(x_2) - f(x_1) \geq -\frac{\varepsilon_{n_0} - \alpha_{n_0}}{q^{n_0}} - \sum_{k = 1}^{\infty} \frac{q^{n_0 + 2k} - 1}{q^{n_0 + 2k}} \geq \frac{\alpha_{n_0} - \varepsilon_{n_0}}{q^{n_0}} - \frac{1}{q^{n_0}} \sum_{k = 1}^{\infty} \frac{q^{n_0 + 2k - 1} - 1}{q^{2k}}
\]

\[
\geq \frac{1}{q^{n_0}} - \frac{1}{q^{n_0}} \sum_{k = 1}^{\infty} \frac{q - 1}{q^{2k}} = \frac{1}{(q + 1)q^{n_0} - 1} > 0,
\]

since \( n_0 \) is odd, \( \alpha_{n_0} - \varepsilon_{n_0} \geq 1 \), and \( q_n \leq q \). \( \square \)

Theorem 4. For the function \( f \), the following statements are true:

• If the condition \( q_0 = q \) holds for all positive integers \( n \), then \( f'(x_0) = 1 \);
• If there exists an infinite sequence \( (n_k) \) of positive integers such that \( q_{n_k} < q \), then \( f \) is a singular function;
• If there exists a finite sequence \( (n_k) \) of positive integers such that \( q_{n_k} < q \), then \( f \) is a non-differentiable function.
Proof. Since (see \cite{20}) for any nega-$Q$-cylinder $\Delta^{-Q}_{c_{1}c_{2}...c_{m}}$, properties

$$\bigcap_{m=1}^{\infty} \Delta^{-Q}_{c_{1}c_{2}...c_{m}} = \Delta^{-Q}_{c_{1}c_{2}...c_{m}} = x \in [a_{0} - 1, a_{0}],$$

$$\Delta^{-Q}_{c_{1}c_{2}...c_{m}} = \begin{cases} \Delta^{-Q}_{c_{1}c_{2}...c_{m}0(q_{m}+1-1)0(q_{m}+4-1)0(q_{m}+6-1)...} \\
\Delta^{-Q}_{c_{1}c_{2}...c_{m}0(q_{m}+2-1)0(q_{m}+4-1)0(q_{m}+6-1)...} \\
\Delta^{-Q}_{c_{1}c_{2}...c_{m}(q_{m}+1-1)0(q_{m}+3-1)0(q_{m}+5-1)...}
\end{cases}$$

if $m$ is even

and

$$|\Delta^{-Q}_{c_{1}c_{2}...c_{m}}| = \frac{1}{q_{1}q_{2}...q_{m}}$$

hold, we obtain

$$\mu f (\Delta^{-Q}_{c_{1}c_{2}...c_{m}}) = f \left( \sup_{\Delta^{-Q}_{c_{1}c_{2}...c_{m}}} \right) - f \left( \inf_{\Delta^{-Q}_{c_{1}c_{2}...c_{m}}} \right) = \sum_{k=1}^{\infty} \frac{q_{m+k}-1}{q^{m+k}}$$

and

$$f'(x_{0}) = \lim_{n \to \infty} \frac{\mu f (\Delta^{-Q}_{c_{1}c_{2}...c_{m}})}{|\Delta^{-Q}_{c_{1}c_{2}...c_{m}}|} = \lim_{m \to \infty} \left( \frac{q_{1}q_{2}...q_{m}}{q^{m}} \sum_{n=m+1}^{\infty} \frac{q_{n}-1}{q^{n-m}} \right)$$

for $x_{0} \in \Delta^{-Q}_{c_{1}c_{2}...c_{m}}$. Also, since $2 \leq q_{m} \leq q$ holds for all positive integers $m$, we have

$$\frac{1}{q-1} \lim_{m \to \infty} \left( \frac{q_{1}q_{2}...q_{m}}{q^{m}} \right) \leq \lim_{m \to \infty} \left( \frac{q_{1}q_{2}...q_{m}}{q^{m}} \sum_{n=m+1}^{\infty} \frac{q_{n}-1}{q^{n-m}} \right) \leq \lim_{m \to \infty} \left( \frac{q_{1}q_{2}...q_{m}}{q^{m}} \right).$$

This completes the proof. \hfill \Box

Let us consider the following infinite system of functional equations

$$f \left( \sigma^{k-1}(x) \right) = -\frac{\varepsilon_{k}}{q} - \frac{1}{q} f \left( \sigma^{k}(x) \right), \quad (6)$$

where $k = 1, 2, ..., \sigma$ is the shift operator of the nega-$Q$-expansion (here $\sigma^{0}(x) = x$), and $x = \Delta^{-Q}_{c_{1}c_{2}...c_{m}}$.

**Lemma 3.** The function $f$ is the unique solution of infinite system (6) of functional equations in the class of determined and bounded on $[a_{0} - 1, a_{0}]$ functions.

Proof. Really, for an arbitrary $x = \Delta^{-Q}_{c_{1}c_{2}...c_{m}}$ from $[a_{0} - 1, a_{0}]$, we have

$$f(x) = -\frac{\varepsilon_{1}}{q} - \frac{1}{q} f(\sigma(x)) = -\frac{\varepsilon_{1}}{q} - \frac{1}{q} \left( -\frac{\varepsilon_{2}}{q} - \frac{1}{q} f(\sigma^{2}(x)) \right)$$

$$= -\frac{\varepsilon_{1}}{q} + \frac{\varepsilon_{2}}{q^{2}} + \frac{1}{q^{2}} \left( -\frac{\varepsilon_{3}}{q} - \frac{1}{q} f(\sigma^{3}(x)) \right) = \cdots = \sum_{n=1}^{k} \frac{\varepsilon_{n}}{(-q)^{n}} + \frac{1}{(-q)^{k}} f(\sigma^{k}(x)) = \cdots$$

Whence,

$$f(x) = \lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{\varepsilon_{n}}{(-q)^{n}} + \frac{1}{(-q)^{k}} f(\sigma^{k}(x)) \right) = \sum_{k=1}^{\infty} \frac{\varepsilon_{k}}{(-q)^{k}},$$

10 SYMON SERBENYUK
since functions $f, \sigma^k$ (for any $k \in \mathbb{N}$) are determined and bounded on the domains, and also

$$\frac{1}{(-q)^k} \leq \frac{1}{q^k} \to 0 \ (k \to \infty).$$

Let us consider integral properties of $f$. One can use the last lemma, relationships (4) and (5), and definitions of the shift operator, of alternating Cantor series, and of expansion (3).

**Theorem 5.** The Lebesgue integral of the function $f$ can be calculated by the formula

$$\left| \int_{[a_0-1,a_0]} f(x) \, dx \right| = \sum_{k=1}^{\infty} \frac{q_k - 1}{2q^k}.$$

**Proof.** Since cylinders $\Delta_{c_1c_2\ldots c_m}^{-q}$ are left-to-right situated when $m$ is even and are right-to-left situated when $m$ is odd, this property is true for cylinders $\Delta_{c_1c_2\ldots c_m}^{-q}$, and

$$d \left( \sigma^{k-1}(x) \right) = \frac{1}{q_k} d \left( \sigma^k(x) \right),$$

we obtain

$$I = \int_{[a_0-1,a_0]} f(x) \, dx = \lim_{k \to \infty} \left( -\sum_{n=1}^{k} \frac{q_n - 1}{2q^n} + \frac{1}{q} \int_{\inf \Delta_{c_1}^{-q}} f(\sigma^k(x)) \, d(\sigma^k(x)) \right) = -\sum_{n=1}^{k} \frac{q_n - 1}{2q^n},$$

where:

$$I_1 = \sum_{c_1=0}^{q_1-1} \int_{\inf \Delta_{c_1}^{-q}} f(x) \, dx = \sum_{c_1=0}^{q_1-1} \int_{\inf \Delta_{c_1}^{-q}} \left( -\frac{c_1}{q} - \frac{1}{q} f(\sigma(x)) \right) \, dx$$

$$= -\sum_{i=0}^{q_1-1} \frac{i}{qq_1} \int_{\inf \Delta_{c_1}^{-q}} f(\sigma(x)) \, d(\sigma(x)) = \frac{q_1 - 1}{2q} \int_{\inf \Delta_{c_1}^{-q}} f(\sigma(x)) \, d(\sigma(x)),$$

since $|\Delta_{c_1}^{-q}| = \frac{1}{q^1}$.

$$\frac{1}{q} \int_{\inf \Delta_{c_1}^{-q}} f(\sigma(x)) \, d(\sigma(x)) = \frac{1}{q} I_2 = \frac{1}{q} \sum_{c_2=0}^{q_2-1} \int_{\inf \Delta_{c_2}^{-q'}} f(\sigma(x)) \, d(\sigma(x))$$

$$= \frac{1}{q} \sum_{c_2=0}^{q_2-1} \int_{\inf \Delta_{c_2}^{-q'}} \left( -\frac{c_2}{q} - \frac{1}{q} f(\sigma^2(x)) \right) \, d(\sigma(x))$$

$$= \frac{1}{q} \left( -\sum_{i=0}^{q_2-1} \frac{i}{qq_2} \int_{\inf \Delta_{c_2}^{-q'}} f(\sigma(x)) \, d(\sigma^2(x)) \right) = -\frac{q_2 - 1}{2q^2} \int_{\inf \Delta_{c_2}^{-q'}} f(\sigma^2(x)) \, d(\sigma^2(x)),$$
since $|\Delta_{\tau x}Q'| = \frac{1}{q^2}$ and

$$x = \Delta_{\tau x}\mathcal{Q}... = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{q_2q_3...q_k}$$

---

$$\frac{1}{q^{k-1}}\int_{\inf\sigma^k(x)}^{\sup\sigma^k(x)} f(\sigma^k(x))d(\sigma^k(x)) = \frac{1}{q^{k-1}}I_k = \frac{1}{q^{k-1}} \sum_{\ell_k=0}^{q_k-1} \int_{\inf\Delta_{\tau x}^k(\sigma^{k-1})}^{\sup\Delta_{\tau x}^k(\sigma^{k-1})} f(\sigma^{k-1}(x))d(\sigma^{k-1}(x))$$

$$= \frac{1}{q^{k-1}} \sum_{\ell_k=0}^{q_k-1} \int_{\inf\Delta_{\tau x}^k(\sigma^{k-1})}^{\sup\Delta_{\tau x}^k(\sigma^{k-1})} f(\sigma^k(x))d(\sigma^k(x))$$

$$= \frac{1}{q^{k-1}} \left( -\sum_{i=0}^{q_k-1} \frac{i}{qqk} + q_k \int_{\inf\sigma^k(x)}^{\sup\sigma^k(x)} f(\sigma^k(x))d(\sigma^k(x)) \right) = -\frac{q_k-1}{2q^k} + \frac{1}{q^k} \int_{\inf\sigma^k(x)}^{\sup\sigma^k(x)} f(\sigma^k(x))d(\sigma^k(x)),$$

since $|\Delta_{\tau x}Q^{k-1}| = \frac{1}{q_k}$ and

$$x = \Delta_{\tau x}\mathcal{Q}_{k+1}... = \sum_{n=k}^{\infty} \frac{(-1)^{n-k-1}}{q_kq_{k+1}...q_n}$$

This completes the proof. $\square$

3. SOME FRACTALS DEFINED IN TERMS OF CERTAIN MAPS IN $\mathbb{R}^2$

Let us consider the following function

$$h : x = \Delta_{\alpha_1...\alpha_n}... \rightarrow \Delta_{\alpha_1\alpha_2...\alpha_n}...$$

where $u \in \{0, 1, \ldots, q-1\}$ is a fixed number, $\alpha_n \in \Theta = \{1, 2, \ldots, q-1\} \setminus \{u\}$, and $3 < q$ is a fixed positive integers. This function can be represented by the following form.

$$h : x = -\frac{u}{q+1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{(-q)^{\alpha_1+\alpha_2+\ldots+\alpha_n}} \rightarrow \sum_{n=1}^{\infty} \frac{\alpha_n}{(-q)^n} = h(x) = y.$$  

**Theorem 6.** The function $h$ has the following properties:

1. The domain of definition $D(h)$ of the function $h$ is a set having the following properties:
   - $D(h)$ is an uncountable, perfect, and nowhere dense set;
   - the Lebesgue measure of $D(h)$ equals zero;
   - $D(h)$ is a self-similar fractal whose Hausdorff dimension $\alpha_0$ satisfies the equation
     $$\sum_{p \neq u, p \in \{1, 2, \ldots, q-1\}} \left( \frac{1}{q} \right)^{p\alpha_0} = 1.$$
(2) The range of values $E(h)$ of $h$ is a self-similar fractal

$$E(h) = \{ y : y = \Delta^{-q}_{\alpha_1 \alpha_2 \ldots \alpha_n}, \alpha_n \in \Theta \}$$

for which the Hausdorff dimension $\alpha_0$ equals $\log_q|\Theta|$, where $|\cdot|$ is the number of elements of a set.

(3) The function $h$ is well defined and is a bijective mapping on the domain.

(4) The function $h$ is continuous at any point on the domain.

(5) On the domain of definition the function $h$ is:
   - decreasing whenever $u \in \{0, 1\}$ for all $q > 3$;
   - increasing whenever $u \in \{q - 2, q - 1\}$ for all $q > 3$;
   - not monotonic whenever $u \in \{2, 3, \ldots, q - 3\}$ and $q > 4$.

(6) The function $h$ is non-differentiable on the domain.

(7) The following relationship is true for any positive integer $n$:

$$h(\sigma^{\alpha_1 + \alpha_2 + \ldots + \alpha_n}(x)) = \sigma^n(h(x)).$$

Here $\sigma$ is the shift operator.

(8) The function does not preserve the Hausdorff dimension.

Proof. The domain $D(h)$ of the function $h$ is a set whose elements represented in terms of nega-$q$-ary representation \([3]\). Representations of elements of $D(h)$ contain only combinations of digits from a some subset of the following set (it depends on fixed parameters $u$ and $q$):

$$\left\{ 1, u2, uu3, \ldots, u, \ldots, u[u - 1], u \ldots u[u + 1], \ldots, u \ldots u[q - 1] \right\}.$$

That is (see \([23, 24]\)), such set is a self-similar fractal whose Hausdorff dimension $\alpha_0$ satisfies the following equation

$$\sum_{p \neq u, p \in \{1, 2, \ldots, q - 1\}} \left( \frac{1}{q^p} \right)^{\alpha_0} = 1.$$

Hence this set is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure.

The second property follows from the definition of $h$.

Property 8 follows from the first and second properties, since $\alpha_0(D(h)) \neq \alpha_0(E(h))$, where $\alpha_0(\cdot)$ is the Hausdorff dimension of a set.

Let us prove Property 3 and Property 4. The set $D(h)$ does not contain numbers with zeros in own nega-$q$-representations whenever $u > 0$. If $u = 0$, then $D(h)$ does not contain numbers having a period $0(q - 1)]$ or $[q - 1]0$ in own nega-$q$-representations. Whence, $D(h)$ does contain nega-$q$-rational numbers i.e., numbers of the form

$$\Delta^{-q}_{\alpha_1 \alpha_2 \ldots \alpha_n [0(q - 1)[0(q - 1)]} = \Delta^{-q}_{\alpha_1 \alpha_2 \ldots \alpha_n [0(q - 1)[0].$$

That is, any element of $D(h)$ has the unique nega-$q$-representation. Therefore the condition $h(x_1) \neq h(x_2)$ holds for $x_1 \neq x_2$. Let us note that a value $h(x) \in E(h)$ is assigned to an arbitrary $x \in D(h)$ and vice versa.
Let us consider a nega-$q$-cylinder $\Delta_{u_{c_1-1}}^{1-q}...u_{c_{1-1}}u_{c_2-1}...u_{c_{n-1}}$. Since

$$D(h) \ni x = \bigcap_{n=1}^{\infty} \Delta_{u_{c_1-1}}^{1-q}...u_{c_{1-1}}u_{c_2-1}...u_{c_{n-1}},$$

let us consider $x, x_0 \in \left(D(h) \cap \Delta_{u_{c_1-1}}^{1-q}...u_{c_{1-1}}u_{c_2-1}...u_{c_{n-1}}\right)$. Then

$$|h(x) - h(x_0)| = \left| h\left(\Delta_{u_{c_1-1}}^{1-q}...u_{c_{1-1}}u_{c_2-1}...u_{c_{n-1}}u_{\alpha_{n+1}}(x)u_{\alpha_{n+2}}(x)...\right) \right|$$

$$- h\left(\Delta_{u_{c_1-1}}^{1-q}...u_{c_{1-1}}u_{c_2-1}...u_{c_{n-1}}u_{\alpha_{n+1}}(x_0)u_{\alpha_{n+2}}(x_0)...\right)\right|$$

$$\leq \sum_{k=n+1}^{\infty} \frac{\alpha_k(x) - \alpha_k(x_0)}{(-q)^k} \leq \frac{1}{q^n} \to 0 \ (n \to \infty).$$

So, for any $x_0 \in D(h)$, the following holds

$$\lim_{x \to x_0} |h(x) - h(x_0)| = 0.$$

Let us prove Property 5. Consider $x_1, x_2 \in D(h)$, $x_1 \neq x_2$, i.e.,

$$x_1 = \Delta_{u_{c_1-1}}^{1-q}...u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}...\right.$$  

and

$$x_2 = \Delta_{u_{c_1-1}}^{1-q}...u_{\gamma_{k_{0}}-1}u_{\gamma_{k_{0}}-1}u_{\gamma_{k_{0}}-1}u_{\gamma_{k_{0}}-1}...\right.$$  

where $\alpha_i = \gamma_i$ for $i = 1, k_0 - 1$ and $\alpha_{k_0} \neq \gamma_{k_0}$. Whence,

$$y_1 = h(x_1) = \Delta_{u_{c_1-1}}^{1-q}...u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}...\right.$$  

and $y_1 < y_2$ whenever the following system of conditions holds:

$$\left\{ \begin{array}{l} \alpha_{k_0} < \gamma_{k_0} \quad \text{if } k_0 \text{ is even} \\ \alpha_{k_0} > \gamma_{k_0} \quad \text{if } k_0 \text{ is odd.} \end{array} \right. \quad (7)$$

Let us consider $D(h)$ more detail. Suppose $u = 0$. Then

$$x_1 = \Delta_{u_{c_1-1}}^{1-q}...u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}u_{\alpha_{k_{0}}-1}...\right.$$  

$$= \sum_{k=1}^{\infty} \frac{\alpha_k}{(-q)^{\alpha_1+\alpha_2+...+\alpha_k}}.$$
where
\[ x_2 = \sum_{k=1}^{\infty} \frac{\gamma_k}{(-q)^{\gamma_1 + \gamma_2 + \cdots + \gamma_k}}. \]

Suppose that \( \tau_{k_0} := \min\{\alpha_1 + \alpha_2 + \cdots + \alpha_{k_0}, \gamma_1 + \gamma_2 + \cdots + \gamma_{k_0}\} \).

Since, for \( \Delta_{\alpha}^{-q} 0\alpha(0) < \Delta_{\beta}^{-q} 0\beta(0) \), the system of conditions
\[
\begin{align*}
\Delta_{\alpha}^{-q} 0\alpha(0) &< \Delta_{\beta}^{-q} 0\beta(0) \quad \text{if } \min\{\alpha, \beta\} \text{ is even and } \alpha > \beta \\
\Delta_{\alpha}^{-q} 0\alpha(0) &< \Delta_{\beta}^{-q} 0\beta(0) \quad \text{if } \min\{\alpha, \beta\} \text{ is odd and } \alpha < \beta
\end{align*}
\]
is true, we have that \( x_1 < x_2 \) whenever the following system of conditions holds:
\[
\begin{align*}
\alpha_{k_0} > \gamma_{k_0} & \quad \text{if } \tau_{k_0} \text{ is even} \\
\alpha_{k_0} < \gamma_{k_0} & \quad \text{if } \tau_{k_0} \text{ is odd}.
\end{align*}
\]
However, since
\[
\tau_{k_0} = \sum_{j=1}^{k_0} i_j \pmod{2} \equiv \tau_{k_0}' \pmod{2},
\]
where \( i_j, \tau_{k_0}' \in \{0, 1\} \), we obtain \( \tau_{k_0}' \leq k_0' \equiv k_0 \pmod{2} \). Here \( k_0' \in \{0, 1\} \).

So, if \( k_0 \) is even, then \( \tau_{k_0} \) is even and from (11), (12) it follows that \( y_2 > y_1 \) for \( x_1 < x_2 \), i.e., the function \( h \) is decreasing.

By analogy, if \( k_0 \) is odd, then \( \tau_{k_0} \) is even or is odd but \( h \) is a decreasing function.

Let us remark that \( h \) is decreasing for the case when \( u = 1 \) because corresponding considerations for \( u = 0 \) and \( u = 1 \) are identical.

Let \( u \in \{2, 3, \ldots, q - 3\}, q > 4 \). Then \( \Delta_{\alpha}^{-q} u\alpha(0) < \Delta_{\beta}^{-q} u\beta(0) \) whenever one of the following cases holds:
\begin{itemize}
  \item \( \min\{\alpha, \beta\} \) is odd, \( \alpha > \beta \), and \( u > \beta \);
  \item \( \min\{\alpha, \beta\} \) is odd, \( \alpha < \beta \), and \( u < \alpha \);
  \item \( \min\{\alpha, \beta\} \) is even, \( \alpha < \beta \), and \( u > \alpha \);
  \item \( \min\{\alpha, \beta\} \) is even, \( \alpha > \beta \), and \( u < \beta \).
\end{itemize}

Note that exist \( \alpha_n \) such that \( \alpha_n < u \) and also \( \alpha_n > u \) in our case (when \( u \in \{2, 3, \ldots, q - 3\}, q > 4 \)). For example:
\begin{itemize}
  \item if \( u = 2 \), then \( \alpha_n = 1 < 2 \) and \( 2 < \alpha_n \in \{3, 4, \ldots, q - 1\} \);
  \item if \( u = 3 \), then \( \{1, 2\} \ni \alpha_n < 3 \) and \( 3 < \alpha_n \in \{4, 5, \ldots, q - 1\} \);
  \item if \( u = q - 3 \), then \( \{1, 2, \ldots, q - 4\} \ni \alpha_n < u < \alpha_n \in \{q - 2, q - 1\} \).
\end{itemize}
So, \( h \) is not monotonic in this case.
Suppose \( u \in \{ q - 2, q - 1 \} \). Then \( \Delta_{\alpha}^{q-1} u_{\alpha(0)} < \Delta_{\beta}^{q-1} u_{\beta(0)} \) whenever one of the following holds:

- \( \min\{\alpha, \beta\} \) is odd and \( \alpha > \beta \);
- \( \min\{\alpha, \beta\} \) is even and \( \alpha < \beta \).

Hence \( x_1 < x_2 \) whenever

\[
\begin{cases}
\alpha k_0 < \gamma k_0 & \text{if } \tau k_0 \text{ is even} \\
\alpha k_0 > \gamma k_0 & \text{if } \tau k_0 \text{ is odd}.
\end{cases}
\]

Using (7), \( k'_0 \), and \( \tau k_0 \), we obtain that

- if \( k_0 \) is even, then \( \tau k_0 \) is even and \( h \) is an increasing function;
- if \( k_0 \) is odd, then \( h \) is an increasing function for the cases of even and odd \( \tau k_0 \).

Let us prove the \( 6 \)th property. By analogy with similar investigations for positive expansions (see [26]), we obtain the following. Let \( \Delta_{c_1}^{q-1} u_{c_1} u_{c_2} \cdots u_{c_n} u_{c_{n+1}} \cdots \) be an arbitrary cylinder. Then let us consider a sequence \( (x_n) \) of numbers

\[
x_n = \Delta_{c_1}^{q-1} u_{c_1} u_{c_2} \cdots u_{c_n} u_{c_{n+1}} \cdots
\]

and a fixed number \( x_0 = \Delta_{c_1}^{q-1} u_{c_1} u_{c_2} \cdots u_{c_n} u_{c_{n+1}} \cdots \), where \( c \) is a fixed number. Then

\[
\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\alpha_0 - c}{\alpha_n (-q)^{\alpha_n - c}}.
\]

Since conditions \( x_n \to x_0 \) and \( n \to \infty \) are equivalent and \( \frac{\alpha_0 - c}{\alpha_n (-q)^{\alpha_n - c}} \) is a some number, we get

\[
\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{n \to \infty} \left( \frac{\alpha_0 - c}{\alpha_n (-q)^{\alpha_n - c}} \right) = \pm \infty
\]

whenever there exists an infinite number of \( c_n \neq 1 \), because \( c_1 + c_2 + \cdots + c_{n-1} = n - 1 \) and \( c_1 + c_2 + \cdots + c_{n-1} = n - 1 \) when \( c_j = 1, j = 1, n - 1 \).

If all \( c_j = 1 \), where \( j = 1, n - 1 \), then

\[
\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{n \to \infty} \left( \frac{\alpha_0 - c}{\alpha_n (-q)^{\alpha_n - c}} \right).
\]

However, \( \alpha_n, c \) are fixed different numbers. So, the function \( h \) is non-differentiable.

Property 7. It is easy to see that

\[
h(\sigma^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}(x)) = h(\Delta_{\alpha_{n+1}}^{q-1} u_{\alpha_{n+1}} u_{\alpha_{n+2}} \cdots u_{\alpha_{n+3}} \cdots) = \Delta_{\alpha_{n+1}}^{q-1} u_{\alpha_{n+1}} u_{\alpha_{n+2}} \cdots = \sigma^n (h(x)).
\]

\( \square \)
Theorem 7. The Hausdorff dimension of a graph of the function $h$ is equal to 1.

Proof. Let us prove the statement by analogy with the similar proof for positive expansions ([26]; for example, for other functions, such proof is given in [15]).

Suppose that

$$X = \left[ -\frac{q}{q+1}, \frac{1}{q+1} \right] \times \left[ -\frac{q}{q+1}, \frac{1}{q+1} \right]$$

Then the set

$$\{ (x, y) : x = \sum_{m=1}^{\infty} \frac{\alpha_m}{(-q)^m}, \alpha_m \in \Theta_q = \{0, 1, \ldots, q-1\}, y = \sum_{m=1}^{\infty} \frac{\beta_m}{(-q)^m}, \beta_m \in \Theta_q \}$$

is a square with a side length of $q^{-1}$. Then the set

$$\cap_{(\alpha_1, \ldots, \alpha_m)} \cap_{(\beta_1, \ldots, \beta_m)} = \Delta_{\alpha_1, \ldots, \alpha_m} \times \Delta_{\beta_1, \ldots, \beta_m}$$

is a square with a side length of $q^{-m}$. This square is called a square of rank $m$ with the base $(\alpha_1, \beta_1)(\alpha_2, \beta_2)\ldots(\alpha_m, \beta_m)$.

It is known that if $E \subset X$, then the number

$$\alpha^K(E) = \inf \{ \alpha : \hat{H}_\alpha(E) = 0 \} = \sup \{ \alpha : \hat{H}_\alpha(E) = \infty \},$$

where

$$\hat{H}_\alpha(E) = \lim_{\varepsilon \to 0} \inf_{d \leq \varepsilon} K(E, d)d^\alpha$$

and $K(E, d)$ is the minimum number of squares of diameter $d$ required to cover the set $E$, is called the fractal cell entropy dimension of the set $E$. It is easy to see that $\alpha^K(E) \geq \alpha_0(E)$.

From the definition and properties of the function $g$ it follows that the graph of the function belongs to $\tau = [\Theta]$ squares from $q^2$ first-rank squares (here $\tau$ is equal to $(q-1)$ for $u = 0$ and $\tau$ is equal to $(q - 2)$ for $u \neq 0$):

$$\cap_{(i_1, i_1)} = \left[ \Delta^q_{i_1, i_1}, \Delta^q_{i_1, i_1} \right], \quad i_1 \in \Theta_q.$$

The graph of the function $f$ belongs to $\tau^2$ squares from $q^4$ second-rank squares:

$$\cap_{(i_1, i_2)(i_1, i_2)} = \left[ \Delta^q_{i_1, i_1}, \Delta^q_{i_1, i_1} \right], \quad i_1, i_2 \in \Theta_q.$$

The graph $\Gamma_g$ of the function $g$ belongs to $\tau^m$ squares of rank $m$ with sides $q^{\alpha_1 + \alpha_2 + \ldots + \alpha_m}$ and $q^{-m}$. Then

$$\hat{H}_\alpha(\Gamma_g) = \lim_{m \to \infty} \tau^m \left( \frac{q^{-2(\alpha_1 + \alpha_2 + \ldots + \alpha_m)} + q^{-2m}}{q^{-m}} \right)$$

Since $q^{-m(q-1)} \leq q^{-(\alpha_1 + \alpha_2 + \ldots + \alpha_m)} \leq q^{-m}$, we get

$$\hat{H}_\alpha(\Gamma_g) = \lim_{m \to \infty} \tau^m (2 \cdot q^{-2m})^\alpha = \lim_{m \to \infty} \tau^m (2 \cdot q^{-m})^{\frac{\alpha}{m}} = \lim_{m \to \infty} \left( 2^{\frac{\alpha}{m}} \cdot \tau \cdot q^{-m} \cdot q^{m\alpha} \right)$$

$$= \lim_{m \to \infty} \left( 2^{\frac{\alpha}{m}} \left( \frac{\tau}{q^m} \right)^m \right)$$
for $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m$ and
\[
\hat{H}_\alpha(\Gamma_g) = \lim_{m \to \infty} \tau^m \left( q^{-2m(q-1)} + q^{-2m} \right)^{\frac{\alpha}{2}} = \lim_{m \to \infty} \left( \left( \frac{\tau^\pm}{q} \right)^{2m} + \left( q^{1-\tau^\pm} \right)^{2m} \right)^{\frac{\tau}{\alpha_1}}
\]
for $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m(q - 1)$.

It is obvious that if $\left( \frac{\tau}{q^\alpha} \right)^m \to 0$, $\left( \frac{\tau^\pm}{q} \right)^{2m} \to 0$, and $\left( q^{1-\tau^\pm} \right)^{2m} \to 0$ for $\alpha > 1$, and the graph of the function has self-similar properties, then $\alpha^K(\Gamma_g) = \alpha_0(\Gamma_g) = 1$. $\square$

**References**

[1] E. de Amo, M.D. Carrillo and J. Fernández-Sánchez, A Salem generalized function, *Acta Math. Hungar.* 151 (2017), no. 2, 361–378. https://doi.org/10.1007/s10474-017-0690-x

[2] L. Berg and M. Kruppel, De Rham's singular function and related functions, *Z. Anal. Anwendungen.*, 19(2000), no. 1, 227–237.

[3] K. A. Bush, Continuous functions without derivatives, *Amer. Math. Monthly* 59 (1952), No. 4, 222–225.

[4] G. Cantor, Ueber die einfachen Zahlensysteme, *Z. Math. Phys.* 14 (1869), 121–128. (German)

[5] L. Fang, Large and moderate deviation principles for alternating Engel expansions, *Journal of Number Theory* 156 (2015), 263–276. https://doi.org/10.1016/j.jnt.2015.04.008

[6] G. H. Hardy, Weierstrass’s non-differentiable function, *Trans. Amer. Math. Soc.* 17 (1916), 301–325.

[7] J. Gerver, More on the differentiability of the Riemann function, *Amer. J. Math.* 93 (1971), 33–41.

[8] S. Ito, T. Sadahiro, Beta-expansions with negative bases, *Integers* 9 (2009), no. 3, 239–259, https://doi.org/10.1515/INTEG.2009.023

[9] S. Kalpazidou, A. Knopfmacher, J. Knopfmacher. Lüroth-type alternating series representations for real numbers, *Acta Arithmetica* 55 (1990), 311-322. DOI: 10.4064/aa-55-4-311-322

[10] M. Kruppel, De Rham’s singular function, its partial derivatives with respect to the parameter and binary digital sums, *Rostock. Math. Kolloq.* 64 (2009), 57–74.

[11] J. Lüroth, Ueber eine eindeutige Entwicklung von Zahlen in eine unendliche Reihe, *Math. Ann.* 21 (1883), 411–423

[12] Minkowski, H.: Zur Geometrie der Zahlen. In: Minkowski, H. (ed.) Gesammte Abhandlungen, Band 2, pp. 50–51. Druck und Verlag von B. G. Teubner, Leipzig und Berlin (1911)

[13] A. Rényi, Representations for real numbers and their ergodic properties, *Acta. Math. Acad. Sci. Hung.* 8 (1957), 477–493

[14] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.* 53 (1943), 423–439.

[15] S. Serbenyuk, On one class of functions with complicated local structure, *ˇSiauliai Mathematical Seminar* 11 (19) (2016), 75–88.

[16] S. O. Serbenyuk, Functions, that defined by functional equations systems in terms of Cantor series representation of numbers, *Naukovy Zazysky NaUKMA* 165 (2015), 34–40. (Ukrainian), available at https://www.researchgate.net/publication/292606546

[17] S. O. Serbenyuk, Continuous Functions with Complicated Local Structure Defined in Terms of Alternating Cantor Series Representation of Numbers, *Journal of Mathematical Physics, Analysis, Geometry* 13 (2017), No. 1, 57–81.

[18] S. O. Serbenyuk, On one class of functions with complicated local structure that the solutions of infinite systems of functional equations (On one application of infinite systems of functional equations in the functions theory), arXiv:1602.00493v3
[19] S. Serbenyuk, Nega-$\mathbb{Q}$-representation as a generalization of certain alternating representations of real numbers, *Bull. Taras Shevchenko Natl. Univ. Kyiv Math. Mech.* **1** (35) (2016), 32–39. (Ukrainian), available at https://www.researchgate.net/publication/308273000

[20] S. Serbenyuk, Representation of real numbers by the alternating Cantor series, *Integers* **17** (2017), Paper No. A15, 27 pp.

[21] S. Serbenyuk, On one fractal property of the Minkowski function, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **112** (2018), no. 2, 555–559, doi:10.1007/s13398-017-0396-5

[22] S. O. Serbenyuk Non-Differentiable functions defined in terms of classical representations of real numbers, *Zh. Mat. Fiz. Anal. Geom.* **14** (2018), no. 2, 197–213.

[23] S. Serbenyuk, One one class of fractal sets, https://arxiv.org/pdf/1703.05262.pdf

[24] S. Serbenyuk, More on one class of fractals, [arXiv:1706.01546](https://arxiv.org/pdf/1706.01546).v1

[25] S. Serbenyuk, One distribution function on the Moran sets, [arXiv:1809.03085](https://arxiv.org/pdf/1809.03085).v1

[26] S. Serbenyuk, Certain functions defined in terms of Cantor series, [arXiv:1905.12148](https://arxiv.org/pdf/1905.12148).v1

[27] Symon Serbenyuk, On certain functions and related problems, [arXiv:1909.03163](https://arxiv.org/pdf/1909.03163).v3, 6 pp.

[28] Liu Wen, A nowhere differentiable continuous function constructed using Cantor series / Liu Wen // Mathematics Magazine. — December 2001. — 74, 5. — P. 400-402.

[29] Wikipedia contributors, “Engel expansion”, Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/wiki/Engel_expansion (accessed January 26, 2020).

[30] W. Wunderlich, Eine überall stetige und nirgends differenzierbare Funktion, *El. Math.* **7** (1952), 73–79. (German)

45 Shchukina St., Vinnytsia, 21012, Ukraine

E-mail address: simon6@ukr.net