A TOPOLOGY FOR LIMITS OF MARKOV CHAINS

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Abstract. In the investigation of limits of Markov chains, the presence of states which become instantaneous states in the limit may prevent the convergence of the chain in the Skorohod topology. We present in this article a weaker topology adapted to handle this situation. We use this topology to derive the limit of random walks among random traps and sticky zero-range processes.

1. Introduction

Some Markov chains can be approximated by Markovian dynamics evolving in a contracted state space. This is the case of certain zero-range models, whose dynamics can be approximated by the one of a random walk, [2, 14], and of some polymer models whose evolution can be approximated by a two-state Markov chain [7, 8, 6, 13]. We proposed in [1, 5] a formal definition of this phenomenon. In analogy to statistical mechanic models, we named these processes metastable Markov chains, and we developed tools to prove the convergence (of the projection on a contracted state space) of these chains to Markovian dynamics. The erratic behavior of the projected chain in very short time intervals precludes convergence in the Skorohod topology. We introduce in this article a topology in which the convergence takes place. This topology might be useful in other contexts. For example, in the investigation of limits of Markov chains when some states become instantaneous states in the limit, that is, states whose jump rates become infinite. To explain the topological problem created by the asymptotic instantaneous states and to present the main results of the article, we examine in this introduction sticky zero-range processes and random walks among random traps.

1. Zero-range processes.

Fix a finite set \( S_L = \{1, \ldots, L\} \), and denote by \( E_{L,N} \), \( N \geq 1 \), the configurations obtained by distributing \( N \) particles on \( S_L \):

\[
E_{L,N} := \{\eta \in \mathbb{N}^{S_L} : \sum_{x \in S_L} \eta_x = N\}.
\]

Consider an irreducible, continuous-time random walk \( \{Z(t) : t \geq 0\} \) on \( S_L \) which jumps from \( x \) to \( y \) at a rate \( r(x,y) \) which is reversible with respect to the uniform measure, \( r(x,y) = r(y,x), x, y \in S_L \). Fix \( \alpha > 1 \), and let \( g : \mathbb{N} \to \mathbb{R} \) be given by

\[
g(0) = 0, \quad g(1) = 1, \quad \text{and} \quad g(n) = \left(\frac{n}{n-1}\right)^\alpha, \quad n \geq 2,
\]

so that \( \prod_{i=1}^n g(i) = n^\alpha, \quad n \geq 1 \).

Denote by \( \eta^N(t) : t \geq 0 \) the zero-range process on \( S_L \) in which a particle jumps from a site \( x \), occupied by \( k \) particles, to a site \( y \) at rate \( g(k)r(x,y) \). The
generator of this Markov chain \( \eta(t) = \eta^N(t) \), represented by \( L_N \), acts on functions

\[
(L_N F)(\eta) = \sum_{x \in \sigma_L} g(\eta_x) r(x, y) \{ F(\sigma^{+y} \eta) - F(\eta) \},
\]

where \( \sigma^{+y} \eta \) is the configuration obtained from \( \eta \) by moving a particle from \( x \) to \( y \):

\[
(\sigma^{+y} \eta)_z = \begin{cases} 
\eta_x - 1 & \text{for } z = x \\
\eta_y + 1 & \text{for } z = y \\
\eta_z & \text{otherwise}.
\end{cases}
\]

Fix a sequence \( \{ \ell_N : N \geq 1 \} \) such that \( 1 \ll \ell_N \ll N \), where \( a_N \ll b_N \) means that \( a_N/b_N \to 0 \). For \( x \) in \( S_L \), let

\[
\mathcal{E}_N^x := \{ \eta \in E_{L,N} : \eta_x \geq N - \ell_N \}.
\]

Since \( \ell_N/N \to 0 \), on each set \( \mathcal{E}_N^x \) the proportion of particles at \( x \in S_L \), \( \eta_x/N \), is almost one. Assume that \( N \) is large enough so that \( \mathcal{E}_N^x \cap \mathcal{E}_N^y = \emptyset \) for \( x \neq y \), and consider the partition

\[
E_{L,N} = \mathcal{E}_N^1 \cup \cdots \cup \mathcal{E}_N^L \cup \Delta_N,
\]

where \( \Delta_N \) is the set of configurations which do not belong to the set \( \mathcal{E}_N = \bigcup_{1 \leq x \leq N} \mathcal{E}_N^x \).

Denote by \( \pi_N \) the stationary measure of the zero-range dynamics \( \eta(t) \). We proved in \([2]\) that the measure \( \pi_N \) is concentrated on the set \( \mathcal{E}_N^x \):

\[
\lim_{N \to \infty} \pi_N(\mathcal{E}_N^x) = \frac{1}{L}.
\]

For \( N > L \), let the projection \( \Psi_N : E_{L,N} \to \{1, \ldots, L\} \cup \{N\} \) be defined by

\[
\Psi_N(\eta) = \sum_{x=1}^L x \mathbf{1} \{ \eta \in \mathcal{E}_N^x \} + N \mathbf{1} \{ \eta \in \Delta_N \},
\]

and let \( X_N(t) = \Psi_N(\eta(t)), X_N(t) = X_N(t\theta_N) \), where \( \theta_N = N^{1+\alpha} \). By analogy with statistical mechanics models, we call \( \Psi_N(\eta) \) the order parameter. Note that \( X_N(t) \) is not a Markov chain.

A typical trajectory of \( X_N(t) \) is represented in Figure 1. The intervals \( I_1, I_2, \ldots \) correspond to the sojourns of the rescaled process \( \eta(t\theta_N) \) in a set \( \mathcal{E}_N^x \), while the time intervals \( R_1, R_2, \ldots \) correspond to the excursions in the set \( \Delta_N \). When the process \( \eta(t) \) reaches \( \Delta_N \) from \( \mathcal{E}_N^x \), a strong drift drives it back to \( \mathcal{E}_N^x \). With a very small probability it crosses \( \Delta_N \) and reaches a new set \( \mathcal{E}_N^y, y \neq x \), before hitting \( \mathcal{E}_N^x \) again, and with a probability close to 1 it returns to \( \mathcal{E}_N^x \). In the latter case, \( \eta(t) \) remains close to the boundary between \( \mathcal{E}_N^x \) and \( \Delta_N \) for an interval of time, very short compared to the time it stays in the set \( \mathcal{E}_N^x \), and crosses this boundary a certain number of times until it is absorbed again in the set \( \mathcal{E}_N^x \). This explains the erratic behavior of \( X_N(t) \) in the time intervals \( R_j \).

We examined in \([2, 5]\) the limit behavior of the projected process \( X_N(t) \). The short excursions of \( \eta^N(t\theta_N) \) in \( \Delta_N \) which correspond to short visits of \( X_N(t) \) to \( N \) prevent the convergence of \( X_N(t) \) in the Skorohod topology to a \( S_L \)-valued process. The same phenomenon occurs in random walks among random traps.
2. Random walk among random traps. Denote by $\mathbb{T}_N^d$ the discrete $d$-dimensional torus with $N^d$ points, $d \geq 2$. We denote the sites of $\mathbb{T}_N^d$ by the letters $x$, $y$, $z$.

Note that the letters $x$, $y$ are used in this article to represent three different objects. In the zero-range model, we use $x$, $y$ to index the sets $E_N^x$. In the random walk model, $x$, $y$ stand for the sites of the torus $\mathbb{T}_N^d$, and in Sections 2, 3, 4, $x$, $y$ denote trajectories in paths spaces such as $D([0,T],S_L)$.

Consider a sequence $\{W_j : j \geq 1\}$ of summable, non-increasing, positive numbers:

\[ W_j \geq W_{j+1} > 0, \quad j \geq 1, \quad \sum_{j \geq 1} W_j < \infty. \]

For each $N \geq 1$, denote by $\Psi_N : \mathbb{T}_N^d \rightarrow \{1, \ldots, V_N\}$, $V_N = |\mathbb{T}_N^d| = N^d$, a random uniform enumeration of the sites of $\mathbb{T}_N^d$ and let $x_j^N = \Psi_N^{-1}(j)$. Let

\[ W_x^N = W_{\Psi_N(x)}, \quad x \in \mathbb{T}_N^d, \]

and let $\{\eta^N(t) : t \geq 0\}$ be the random walk on $\mathbb{T}_N^d$ which waits a mean $W_x^N$ exponential time at site $x$. The generator $L_N$ of this random walk is given by

\[ (L_N f)(x) = \frac{1}{2d W_x^N} \sum_{y \sim x} [f(y) - f(x)], \]

where the summation is carried over all nearest-neighbor sites $y$ of $x$.

Denote by $\pi_N$ the stationary state of $\eta^N(t)$, $\pi_N(x) = Z_N^{-1} W_x^N$, where $Z_N$ is the normalizing constant $Z_N = \sum_{1 \leq j \leq V_N} W_j$. Note that the stationary state is concentrated on the first sites $x_1^N, \ldots, x_{L}^N$. More precisely, for every $\epsilon > 0$, there exists $L \geq 1$ such that for all $N \geq L$, \( \pi_N(\{x_1^N, \ldots, x_L^N\}) \geq 1 - \epsilon \).

To describe the asymptotic behavior of the Markov chain $\eta^N(t)$, since the geometry of the graph $\mathbb{T}_N^d$ is lost as sites are reshuffled for each $N$, we will examine the chain $X_N(t) = \Psi_N(\eta_N(t))$ instead of the chain $\eta^N(t)$.

Denote by $D(\mathbb{R}_+, \mathbb{T}_N^d)$ the set of trajectories $\mathbf{r} : \mathbb{R}_+ \rightarrow \mathbb{T}_N^d$ which are right-continuous and have left-limits. Denote by $\mathbf{P}_x$, $x \in \mathbb{T}_N^d$, the probability measure on $D(\mathbb{R}_+, \mathbb{T}_N^d)$ induced by the Markov chain $\eta^N(t)$ starting from $x$. Expectation with respect to $\mathbf{P}_x$ is denoted by $\mathbf{E}_x$. 
Denote by \( B(x, \ell), x \in \mathbb{T}_N^d, \ell \geq 1 \), the ball centered at \( x \) and of radius \( \ell \) in the graph \( \mathbb{T}_N^d \). For a subset \( A \) of \( \mathbb{T}_N^d \), denote by \( H_A \) (resp. \( H_A^+ \)) the hitting time of (resp. the return time to) the set \( A \):

\[
H_A = \inf \{ t > 0 : \eta^N(t) \in A \}, \quad H_A^+ = \inf \{ t > 0 : \eta^N(t) \in A \text{ and } \exists 0 < s < t; \eta^N(s) \neq \eta^N(0) \}.
\]

Let

\[
\ell_N = \frac{N}{(\log N)^{1/4}} \quad \text{if } d = 2, \quad \ell_N = \sqrt{N} \quad \text{if } d \geq 3,
\]

and denote by \( v_N \) the escape probability from a ball of radius \( \ell_N \):

\[
v_N = P_x [H_{B(x, \ell_N)} < H_x^+] . \tag{1.1}
\]

Denote by \( v_d, d \geq 3 \), the probability that a nearest-neighbor, symmetric random walk on \( \mathbb{Z}^d \) never returns to its initial state, and let

\[
\theta_N = \frac{2}{\pi} \log N \quad \text{if } d = 2, \quad \theta_N = \frac{1}{v_d} \quad \text{if } d \geq 3.
\]

By [15, Theorem 1.6.6], \( \lim_{N \to \infty} \theta_N v_N = 1 \).

We investigated in [11, 12] the asymptotic behavior of the rescaled process

\[
\mathbb{X}_N(t) = X_N(t \theta_N) . \tag{1.2}
\]

Actually, we examined this problem for a large class of random walks evolving on random graphs with i.i.d. random weights \( W_j \) in the basin of attraction of an \( \alpha \)-stable distribution, \( 0 < \alpha < 1 \).

That the convergence of the rescaled process \( \mathbb{X}_N(t) \) can not occur in the Skorohod topology is easy to understand. Consider the two-dimensional case, for instance. In the time scale \( \theta_N \), a typical trajectory which starts from a site \( j \) has the shape illustrated in Figure 2 with the difference that the number of sites visited in time intervals \( R_j \) is much greater than the one depicted in Figure 2.

![Figure 2. A typical trajectory of a random walk among random traps in \( \mathbb{T}_N^d \).](image)

The time intervals \( I_1, I_2, \ldots \) correspond to the holding times at \( x_j^N \) of the rescaled process \( \eta^N(\theta_N) \), while the time intervals \( R_1, R_2, \ldots \) correspond to the excursions in the ball \( B(x_j^N, \ell_N) \) between two visits to \( x_j^N \). In the time scale \( \theta_N \), the sojourn times at \( x_j^N \) have length of order \( \theta_N^{-1} \), while we proved in [11] that the length of the time intervals \( R_j \) are of an order \( \gamma_N \) which is much smaller than \( \theta_N^{-1} \), \( \gamma_N \ll \theta_N^{-1} \). This
property follows from the fact that with a very large probability, excluding $x_j^N$, all sites in the ball $B(x_j^N, \epsilon_N)$ are shallow traps, i.e., belong to the set $\{x_1^N, \ldots, x_{M_N}^N\}$ for some sequence $M_N$ which increases to $\infty$ fast enough.

The process $\eta^N(t \theta_N)$ reaches the boundary of the ball $B(x_j^N, \epsilon_N)$ after $v_N^{-1} \sim \theta_N$ attempts. Summing the lengths of the time intervals $I_j$ up to the hitting time of the set $B(x_j^N, \epsilon_N)^c$, we obtain a total length of order 1. Actually, the total length is a mean $\theta_N$ geometric sum of i.i.d. mean $W_j \theta_N^{-1}$ exponential random variables and is therefore distributed according to a mean $W_j$ exponential random variable, which foretells a Markovian limit. In contrast, the sum of the lengths of the time intervals $R_j$ is negligible, being of order $\gamma_N \theta_N \ll 1$.

As in the zero-range model, the short excursions in the ball $B(x_j^N, \epsilon_N)$ prevent the convergence in the Skorohod topology of the trajectory presented in Figure 2 to the constant trajectory equal to $j$.

3. Trace, last visit and soft topology. There are three ways to overcome the topological obstruction caused by the short excursions, illustrated in Figures 1 and 2 by the time intervals $R_j$, $j \geq 1$. The first one, proposed in [1, 5], consists in removing the time intervals $R_j$ by considering the trace of the process $X_N(t)$ on the set $A_N$, where $A_N$ represents the set $S_L = \{1, \ldots, L\}$ in the zero-range model, and the set of deep traps $\{1, \ldots, M_N\}$ in the second model.

Denote by $X_N^T(t)$ the trace of $X_N(t)$ on the set $A_N$. In the random walk model, since the sites in the ball $B(x_j^N, \epsilon_N)$, with the exception of $x_j^N$, belong to $A_N$, the trace on the set $A_N$ of the trajectory presented in Figure 2 is the constant trajectory equal to $j$. Therefore, by considering the trace process $X_N^T(t)$, one removes the short excursions among the shallow traps, and one is left with the problem of proving that the trace process $X_N^T(t)$ converges in the Skorohod topology to some Markov chain. The same ideas apply to the zero-range model. In fact, this strategy has been adopted in [2, 14] to describe the asymptotic evolution of the condensate in sticky zero-range dynamics.

A second way to overcome the difficulty created by the short time intervals $R_j$ is to consider the process which records the last visit to the set $A_N$. Denote by $X_N^V(t)$ the process which at time $t$ is equal to last site in $A_N$ visited by the process $X_N(t)$ before time $t$. More precisely, let

$$\sigma_N(t) := \sup \{s \leq t : X_N(s) \in A_N\},$$

with the convention that $\sigma_N(t) = 0$ if the set $\{s \leq t : X_N(s) \in A_N\}$ is empty. Assume that $X_N(0)$ belongs to $A_N$ and define $X_N^V(t)$ by

$$X_N^V(t) = \begin{cases} X_N(\sigma_N(t)) & \text{if } X_N(\sigma_N(t)) \in A_N, \\ X_N(\sigma_N(t)) - & \text{if } X_N(\sigma_N(t)) \notin A_N. \end{cases}$$

We refer to $X_N^V(t)$ as the last visit process. In Figure 2 assuming that all sites in $B(x_j^N, \epsilon_N)$, except $x_j^N$, are shallow traps, the last visit trajectory is constant equal to $j$, the short excursions among the shallow traps in the time intervals $R_i$ being replaced by trajectories which are constant equal to $j$. As for the trace process, after this surgical intervention on the paths, one is left to prove that the last visit process $X_N^V(t)$ converges in the Skorohod topology to some Markovian dynamics.

Proposition 4.3 in [1] asserts that if the trace on a set $A_N$ of an $\Omega_N$-valued chain $X_N(t)$ converges in the Skorohod topology to a Markov process $X(t)$, and if the time spent by $X_N(t)$ on the complement $\Omega_N \setminus A_N$ is negligible, then the process
which records the last visit to the set \( A_N \) also converges in the Skorohod topology to the Markov process \( X(t) \).

The last visit process \( X_N(t) \) has the advantage with respect to the trace process that it does not translate in time the original trajectory. More precisely, if \( X_N(t) \) belongs to \( A_N \) then \( X_N(t) = X_N(t) \), while this may be false for the trace process \( X_N^T(t) \) because by removing short time intervals, the value of the trace process at time \( s \) corresponds to the value of the original process at some later time \( s' \geq s \): \( X_N(s) = X_N(s') \) for some \( s' \geq s \).

A third way to overcome the problem created by the short time intervals \( R_j \) is to define a topology, weaker than the Skorohod topology, which disregards the behavior of the trajectory in short time intervals. A first attempt in this direction has been made in [12], where we introduced a metric in the space of functions \( \eta : [0, T] \to \mathbb{R} \) which induces the topology of the convergence in measure. We proved in [12] that the rescaled process \( X_N(t) \) introduced in (1.2) converges in this metric to the \( K \)-process.

In this article, we introduce another topology in which we can prove the convergence to a Markov chain of the two models introduced above and of all the other dynamics in which a metastable behavior has been identified [2, 11, 3, 12, 7, 8, 5, 13].

This topology has two advantages with respect to the one introduced in [12]. On the one hand, it is defined on the space of paths which are soft right-continuous and have soft left-limits, a much smaller space than the one which appears in [12]. Actually, this set of paths, denoted by \( E([0, T], S) \) in the next section, is precisely the set of trajectories which supports the paths of Markov chains which have instantaneous states [9, Chapter II.7], [10, Section 9.2]. On the other hand, this topology is a natural generalization of the Skorohod topology and is connected to the Skorohod topology through the last visit process (cf. Theorem 4.1).

The main contributions of this article are the introduction of the soft topology, defined by the metric \( d \) in [28], and two results. The first one, Theorem 4.1, establishes that a sequence of probability measures \( P_n \) defined on \( E([0, T], S_0) \) converges in the soft topology to a probability measure \( P \) if and only if for every \( m \geq 1 \), the sequence of probability measures \( P_n \circ \mathcal{R}_m^{-1} \) defined on \( D([0, T], S_m) \) converges in the Skorohod topology to the probability measure \( P \circ \mathcal{R}_m^{-1} \). Here \( S_0 \) stands for the one point compactification of \( \mathbb{N} \), \( S_m \) for the set \( \{1, \ldots, m\} \), and \( \mathcal{R}_m \) for the path which records the last site in \( S_m \) visited by the trajectory \( x \in E([0, T], S_0) \).

The second result, Theorem 4.2, presents sufficient conditions on a sequence of probability measures \( P_n \), defined on \( E([0, T], S_0) \) and converging to a probability measure \( P \) in the soft topology, for the limit \( P \) to be concentrated on a subspace of \( D([0, T], S_0) \), the space of càdlàg trajectories.

4. Scaling limit of metastable Markov chains. In view of these results, to prove that a sequence of Markov chains converges in the soft topology to a Markov chain evolving in a contracted state space, we may proceed as follows. We first introduce a general framework.

Consider a sequence of countable state spaces \( E_N, N \geq 1 \), and a sequence of \( E_N \)-valued continuous-time Markov chains \( \eta^N(t) \). Denote by \( P_\eta, \eta \in E_N \), the probability measure on the path space \( D(\mathbb{R}_+, E_N) \) induced by the Markov chain \( \eta^N(t) \) starting from \( \eta \). Expectation with respect to \( P_\eta \) is denoted by \( \mathbb{E}_\eta \).

Let \( \mathcal{E}_x^L, \mathcal{E}_x^L, L \geq 2 \), be a finite number of disjoint subsets of \( E_N \): \( \mathcal{E}_x^L \cap \mathcal{E}_y^L = \emptyset, x \neq y \). The sets \( \mathcal{E}_x^L \) have to be interpreted as wells of the Markov chains \( \eta^N(t) \).
Let \( S_L = \{1, \ldots, L\} \), \( \mathcal{E}_N = \cup_{x \in S_L} \mathcal{E}_N x \) and \( \Delta_N = E_N \setminus \mathcal{E}_N \) so that
\[
\{\mathcal{E}_N^1, \ldots, \mathcal{E}_N^L, \Delta_N\} \text{ forms a partition of } E_N. \tag{1.3}
\]
We assumed here that the number of wells, \( L \), does not depend on \( N \), but the same analysis can be carried through if it depends on \( N \), as in the case of random walks among random traps.

Denote by \( \eta^N(t) \) the trace of the process \( \eta^N(t) \) on the set \( \mathcal{E}_N \), and by \( \eta^V(t) = (\mathfrak{P} \in \eta^N(t)) \) the process which records the last site visited by \( \eta^N(t) \) in the set \( \mathcal{E}_N \), as defined in (2.5). Denote by \( \Psi = \Psi_N : \mathcal{E}_N \mapsto S_L \cup \{N\} \), the projection given by
\[
\Psi(\eta) = \sum_{x=1}^{L} x \{\eta \in \mathcal{E}_N^x\} + N \{\eta \in \Delta_N\}.
\]

We call \( \Psi(\eta) \) the order parameter. Let \( \{X_N(t) : t \geq 0\} \) (resp. \( X^N_N(t) \), \( X^V_N(t) \)) be the stochastic process on \( S_L \cup \{N\} \) (resp. \( S_L \)) defined by \( X_N(t) = \Psi(\eta^N(t)) \) (resp. \( X^N_N(t) = \Psi(\eta^N(t)) \), \( X^V_N(t) = \Psi(\eta^V(t)) \)). Besides trivial cases, \( X_N(t) \) is not Markovian. Note that \( X^N_N(t) \) is the trace of \( X_N(t) \) on the set \( S_L \) and that \( X^V_N(t) \) is the process which records the last site in \( S_L \) visited by \( X_N(t) \).

**Definition 1.1.** Let \( \nu_N \) be a sequence of probability measures on \( \mathcal{E}_N \) such that \( \nu_N \circ \Psi^{-1} \) converges to a probability measure \( \nu \) on \( S_L \). The sequence of Markov chains \( \{\eta^N(t) : t \geq 0\} \) is said to be a metastable sequence of Markov chains for the partition (1.3) and the initial state \( \nu_N \) if there exist

(a) An increasing sequence \( \theta_N, \theta_N \gg 1 \),
(b) A \( S_L \)-valued Markov chain \( X(t) \) whose distribution we denote by \( \mathbb{P}_x, x \in S_L \), such that the measure \( \mathbb{P}_{\nu_N} \circ X^{-1} \cdot X_N(t) = X_N(t \theta_N) = \Psi(\eta^N(t \theta_N)) \), converges in the soft topology to \( \mathbb{P}_\nu = \sum_{x \in S_L} \nu(x) \mathbb{P}_x \).

To prove that a sequence of Markov chain is metastable one may proceed as follows:

**Step 1:** Prove the convergence in the Skorohod topology of \( X^N_N(t) = X^N_N(\theta_N t) \) to a Markov chain \( X(t) \).

**Step 2:** Prove that the time spent by the chain \( \eta^N(t) \) on the set \( \Delta_N \) in the time scale \( \theta_N \) is negligible.

**Step 3:** Apply Theorem 5.1, which asserts that under the two previous conditions the process \( X_N(t) \) converges in the soft topology to \( X(t) \).

This article is organized as follows. For a metric space \( M \), denote by \( D([0, T], M) \), \( T > 0 \), the space of right-continuous functions \( x : [0, T] \rightarrow M \) with left-limits. We introduce in (2.8) a metric \( d \) in a subspace of \( D([0, T], S_0) \), where \( S_0 \) is the one-point compactification of \( \mathbb{N} \). The completion of this subspace with respect to the metric \( d \) turns out to be the space of soft right-continuous trajectories with soft left-limits. This space, denoted by \( E([0, T], S_0) \), is introduced in Section 2. We derive in this section several properties of the metric \( d \) and we prove in Proposition 2.12 that the space \( E([0, T], S_0) \) endowed with the metric \( d \) is complete and separable. In Section 3 we introduce a subspace \( E^*([0, T], S_0) \) of the space \( E([0, T], S_0) \) and we present in Lemma 3.4 sufficient conditions for the limit trajectory \( x \) in the soft topology of a sequence \( x_n \) in \( E([0, T], S_0) \) to belong to \( E^*([0, T], S_0) \). In Section 4 we state and prove the main results of the article, mentioned above, and in Section 5 we present some applications of these results. We prove the convergence of the order
parameter to a Markov chain in the case of the random walk among random traps presented above and in the case of the condensate in sticky zero-range dynamics.

2. The space \( E([0,T], S_\delta) \)

Let \( S_m = \{1, \ldots, m\} \), \( m \geq 1 \), and let \( S_\delta \) be the one-point compactification of \( S = \mathbb{N} \): \( S_\delta = S \cup \{\emptyset\} \), \( \delta = \infty \), where the metric in \( S_\delta \) is given by \( d(k,j) = |k^{-1} - j^{-1}| \).

Fix \( T > 0 \). Denote by \( \Lambda \) the set of increasing and continuous functions \( \lambda : [0,T] \to [0,T] \) such that \( \lambda(0) = 0, \lambda(T) = T \). For \( \lambda \in \Lambda \), let

\[
\|\lambda\|_\circ = \sup_{0 \leq s < t \leq T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|
\]

Denote by \( d_S \) the Skorohod metric on \( D([0,T], S_m), m \geq 1 \), defined by

\[
d_S(x,y) = \inf_{\lambda \in \Lambda} \max \left\{ \|x - y\|_\circ, \|\lambda\|_\circ \right\},
\]

where \( y = y \circ \lambda \) and \( \|x - y\|_\circ = \sup_{0 \leq t \leq T} d(x(t), y(t)) \).

**Definition 2.1.** A measurable function \( x : [0,T] \to S_\delta \) is said to have a soft left-limit at \( t \in (0,T] \) if one of the following two alternatives holds

(a) The trajectory \( x \) has a left-limit at \( t \), denoted by \( x(t-) \);

(b) The set of cluster points of \( x(s), s \uparrow t \), is a pair formed by \( \emptyset \) and a point in \( S \), denoted by \( x(t\ominus) \).

A soft right-limit at \( t \in [0,T) \) is defined analogously. In this case, the right-limit, when it exists, is denoted by \( x(t+) \), and the cluster point of the sequence \( x(s), s \downarrow t \), which belongs to \( S \) when the second alternative is in force is denoted by \( x(t\oplus) \).

More concisely, a trajectory \( x \) has a soft left-limit at \( t \in (0,T] \) if and only if there exists \( n \in S \) such that for all \( m \geq 1 \), there exists \( \delta > 0 \) for which \( x(s) \in \{n\} \cup S_m^c \) for all \( t - \delta < s < t \). Note the similitude of this definition with the notion of quasiconvergence in [10].

The second alternative in Definition 2.1 asserts that there exist \( n \in S \) and two increasing sequences \( t_j, t_j' \uparrow t \) such that \( \lim_j x(t_j) = n, \lim_j x(t_j') = \emptyset \). Moreover, if \( x(t_j') \) converges for some sequence \( t_j' \uparrow t \), \( \lim_j x(t_j') \in \{n, \emptyset\} \).

We call \( x(t\ominus) \) the finite soft left-limit of \( x \) at \( t \). Whenever we refer to \( x(t-) \) it means that \( x \) has a left-limit at \( t \). Similarly, when we refer to \( x(t\oplus) \), it is understood that \( x \) has not a left-limit at \( t \), but that the alternative (b) of the previous definition is in force. An analogous convention is adopted for \( x(t+) \) and \( x(t\oplus) \).

**Remark 2.2.** Since \( S_\delta \) is a compact set, to prove that \( x \) has a soft right-limit at \( t \) we only have to show uniqueness of limit points in \( S \) (assuming they exist). In other words, we have to prove that if \( t_j \) and \( t_j' \) are sequences decreasing to \( t \) and if \( x(t_j), x(t_j') \) converge to \( m \in S, n \in S \), respectively, then \( m = n \).

**Definition 2.3.** A trajectory \( x : [0,T] \to S_\delta \) which has a soft right-limit at \( t \) is said to be soft right-continuous at \( t \) if one of the following three alternatives holds

(a) \( x(t+) \) exists and is equal to \( \emptyset \);

(b) \( x(t+) \) exists, belongs to \( S \), and \( x(t+) = x(t) \);

(c) \( x(t\oplus) \) exists and \( x(t\oplus) = x(t) \).

A trajectory \( x : [0,T] \to S_\delta \) which is soft right-continuous at every point \( t \in [0,T] \) is said to be soft right-continuous.
A trajectory \( x \) is soft right-continuous at \( t \) if and only if there exists \( n \in S \) such that for all \( m \geq 1 \), there exists \( \delta > 0 \) for which \( x(s) \in \{n\} \cup S_{m}^{c} \) for all \( t \leq s < t + \delta \).

Note that if \( x \) is soft right-continuous at \( t \) and if \( x(t+) = \emptyset \), then \( x(t+) \) may be different from \( x(t) \). In contrast, if \( x \) is soft right-continuous at \( t \) and if \( x(t) = \emptyset \), then \( x(t+) = \emptyset = x(t) \).

Clearly, if \( x \) is soft right-continuous at \( t \), for every \( m \geq 1 \), there exists \( \epsilon > 0 \) such that for all \( t \leq s < t + \epsilon \),

\[ x(s) = x(t) \text{ or } x(s) \geq m . \tag{2.1} \]

Similarly, if \( x \) has a soft left-limit at \( t \), there exists \( n \in S \) with the following property. For every \( m \geq 1 \), there exists \( \epsilon > 0 \) such that for all \( t - \epsilon < s < t \),

\[ x(s) = n \text{ or } x(s) \geq m . \tag{2.2} \]

**Definition 2.4.** Let \( E([0, T], S_{\emptyset}) \) be the space of soft right-continuous trajectories \( x : [0, T] \to S_{\emptyset} \) with soft left-limits.

Note that the space \( E([0, T], S_{\emptyset}) \) corresponds to the space of sample functions of Markov chains with instantaneous states [9, Chapter II.7], [10, Section 9.2].

Fix a trajectory \( x \) in \( E([0, T], S_{\emptyset}) \) such that \( x(r) = \emptyset \) for some \( r \in [0, T) \). Since \( x \) is soft right-continuous, by Definition 2.3,

\[ x(r+) \text{ exists and } x(r+) = \emptyset . \tag{2.3} \]

Recall that \( S_{m} = \{1, \ldots, m\} \), \( m \geq 1 \). For a trajectory \( x \) in \( E([0, T], S_{\emptyset}) \), \( t \in [0, T] \), let

\[ \sigma_{m}^{x}(t) := \sup\{s \leq t : x(s) \in S_{m} \} . \tag{2.4} \]

If the set \( \{s \leq t : x(s) \in S_{m} \} \) is empty, we set \( \sigma_{m}^{x}(t) = 0 \), but this convention does not play any role below and we could have defined \( \sigma_{m}^{x}(t) \) in another way. When there is no ambiguity and it is clear to which trajectory we refer to, we denote \( \sigma_{m}^{x}(t) \) by \( \sigma_{m}(t) \).

\[ \sigma_{1}(t) \quad \sigma_{2}(t) \quad \sigma_{5}(t) \quad t \]

\[ 1 \quad 5 \]

**Figure 3.** The values of \( \sigma_{j}(t) \) and \( (R_{j}x)(t) \) for a trajectory \( x : [0, T] \to \mathbb{N} \). In this example \( \sigma_{2}(t) = \sigma_{3}(t) = \sigma_{4}(t) \), \( \sigma_{5}(t) = \sigma_{0}(t) \), and \( (R_{2}x)(t) = (R_{3}x)(t) = (R_{4}x)(t) = 2 \), \( (R_{5}x)(t) = (R_{0}x)(t) = 5 \).
Fix $t \in (0,T]$ and $m \geq 1$. Suppose that $\sigma_m(t) > 0$ and that $x(\sigma_m(t)) \not\in S_m$, so that $x(s) \not\in S_m$ for $\sigma_m(t) \leq s < t$. By Proposition 2.2, there exist $n \in S$ and $\epsilon > 0$ such that for each $s \in (\sigma_m(t) - \epsilon, \sigma_m(t))$ either $x(s) = n$ or $x(s) > m$. By definition of $\sigma_m(t)$ we must have $n \in S_m$. Moreover, $x(\sigma_m(t)') = n$ if $x$ has a left-limit at $\sigma_m(t)$, and $x(\sigma_m(t)\ominus) = n$ if not.

Let $\mathcal{R}_m x$ be the trajectory which records the last site visited in $S_m$: $(\mathcal{R}_m x)(t) = 1$ if $x(s) \not\in S_m$ for $0 \leq s \leq t$, and

$$(\mathcal{R}_m x)(t) = \begin{cases} x(\sigma_m(t)) & \text{if } x(\sigma_m(t)) \in S_m, \\ x(\sigma_m(t)') & \text{if } x(\sigma_m(t)) \not\in S_m \text{ and } x(\sigma_m(t)') \text{ exists,} \\ x(\sigma_m(t)\ominus) & \text{otherwise,} \end{cases} \tag{2.5}$$

if there exists $0 \leq s \leq t$ such that $x(s) \in S_m$. Figure 3 illustrates the definition of $\sigma_m(t)$ and $\mathcal{R}_m x$ for some trajectory $x$.

Note that $(\mathcal{R}_m x)(0) = x(0)$ if $x(0) \in S_m$ and $(\mathcal{R}_m x)(0) = 1$ if $x(0) \not\in S_m$. The convention that $(\mathcal{R}_m x)(t) = 1$ if $x(s) \not\in S_m$ for $0 \leq s \leq t$ corresponds to the assumption that the trajectory $x$ is defined for $t < 0$ and that $x(t) = 1$ in this time interval.

Consider a trajectory $x$ in $D([0,T[, S_0)$, $m \geq 1$ and $t \in (0,T]$. Assume that $x(t) \not\in S_m$ and that there exists $0 \leq s \leq t$ such that $x(s) \in S_m$. Since $x$ is right-continuous, $\sigma_m(t) > 0$ and $x(\sigma_m(t)) = x(\sigma_m(t)') \not\in S_m$. Hence, since $x$ has left-limits, under the above conditions,

$$(\mathcal{R}_m x)(t) = x(\sigma_m(t)') \tag{2.6}.$$

Note that we may have $\sigma_m(t) = t$ in this example.

**Assertion A.** Fix a trajectory $x$ in $E([0,T[, S_0)$. For each $m \geq 1$, $\mathcal{R}_m x$ is a trajectory in $D([0,T[, S_m)$.

**Proof.** Fix $m \geq 1$. We first prove the right continuity of $\mathcal{R}_m x$. Fix $t \in [0,T)$. By Proposition 2.1, there exists $\delta > 0$ such that for all $t \leq s \leq t + \delta$, either $x(s) = x(t)$ or $x(s) > m$. Suppose that $x(t)$ belongs to $S_m$. In this case, $(\mathcal{R}_m x)(s) = x(t) = (\mathcal{R}_m x)(t)$ for $t \leq s < t + \delta$. On the other hand, if $x(t) \not\in S_m$, $x(s) \not\in S_m$ for $t \leq s < t + \delta$ so that $\sigma_m(s) = \sigma_m(t)$ in this interval. Therefore, in view of Proposition 2.5, $(\mathcal{R}_m x)(s) = (\mathcal{R}_m x)(t)$ for $t \leq s \leq t + \delta$. This proves that $\mathcal{R}_m x$ is right-continuous.

We turn to the proof of the existence of a left limit at $t \in [0,T]$. If $x(t^-)$ exists and belongs to $S_m$, $(\mathcal{R}_m x)(s) = x(t^-)$ for all $s < t$ close enough to $t$. If $x(t^-)$ exists and does not belong to $S_m$, $\sigma_m(s)$ is constant in an open interval $(t - \delta, t)$, which implies that $(\mathcal{R}_m x)(s)$ is constant in the same interval. Finally, suppose that $x(t\ominus)$ exists. In view of Proposition 2.2, there exists $\delta > 0$ such that for all $t - \delta < s < t$, either $x(s) > m$ or $x(s) = x(t\ominus)$. If $x(t\ominus) \leq m$, $(\mathcal{R}_m x)(s) = x(t\ominus)$ in some interval $(t - \delta', t)$, $\delta' > 0$. If $x(t\ominus) > m$, then $\sigma_m(s)$ is constant in the interval $(t - \delta, t)$, so that $\mathcal{R}_m x$ is constant in the same interval. This concludes the proof of the assertion. \[\Box\]

The next example shows that the trajectories $\mathcal{R}_m x$, $m \geq 1$, do not characterize the trajectory $x$. 

Example 2.5. Fix $0 < s < t < T$ and a sequence $\{t_j : j \geq 1\}$ such that $t_1 < T$, $t_j \downarrow t$. Consider the trajectories $x, y \in \mathbb{E}((0, T], S_\delta)$ given by

$$x = 1\{(0, s)\} + 01\{(s, t)\} + \sum_{j \geq 2} j1\{(t_j, t_j-1)\} + 1\{(t_1, T)\},$$

$$y = 1\{(0, t)\} + \sum_{j \geq 2} j1\{(t_j, t_j-1)\} + 1\{(t_1, T)\}.$$

It is clear that $R_m x = R_m y$ for all $m \geq 1$.

For a trajectory $x \in \mathbb{E}([0, T], S_\delta)$, let $\sigma^x_{\infty}(t)$ be the time of the last visit to $S$:

$$\sigma^x_{\infty}(t) := \sup\{s \leq t : x(s) \in S\},$$

with the convention that $\sigma^x_{\infty}(t) = 0$ if $x(s) = \emptyset$ for $0 \leq s \leq t$. As before, when there is no ambiguity and it is clear to which trajectory we refer to, we denote $\sigma^x_{\infty}(t)$ by $\sigma_{\infty}(t)$.

Let $R_\infty x$ be the trajectory which records the last site visited in $S$: $(R_\infty x)(t) = 1$ if $x(s) = \emptyset$ for all $0 \leq s \leq t$, and

$$(\mathcal{R}_\infty x)(t) = \begin{cases} x(\sigma_{\infty}(t)) & \text{if } x(\sigma_{\infty}(t)) \in S, \\ x(\sigma_{\infty}(t)-) & \text{if } x(\sigma_{\infty}(t)) \notin S \text{ and if } x(\sigma_{\infty}(t)) \text{ exists,} \\ x(\sigma_{\infty}(t)\ominus) & \text{otherwise,} \end{cases}$$

if there exists $0 \leq s \leq t$ such that $x(s) \in S$. As for the operator $R_m$, the convention that $(R_\infty x)(0) = 1$ if $x(0) = \emptyset$ corresponds in assuming that the trajectory is defined for $t < 0$ and that $x(t) = 1$ for $t < 0$. Note that $(R_\infty x)(0) \in S$ and that $(R_\infty x)(0) = x(0)$ if and only if $x(0) \in S$. Note also that in Example 2.5 $y = R_\infty x$ and that $y$ is not right-continuous at $t$ but soft right-continuous.

Consider a trajectory $x$ in $D([0, T], S_\delta)$ and $t \in (0, T]$. Assume that $x(t) \notin S$ and that there exists $0 \leq s \leq t$ such that $x(s) \in S$. Since $x$ is right-continuous, $\sigma_{\infty}(t) > 0$ and $x(\sigma_{\infty}(t)) = x(\sigma_{\infty}(t)+) \notin S$. Hence, since $x$ has left-limits, under the above conditions,

$$(\mathcal{R}_\infty x)(t) = x(\sigma_{\infty}(t)-).$$

(2.7)

Definition 2.6. Denote by $E((0, T], S_\delta)$ the set of trajectories in $\mathbb{E}((0, T], S_\delta)$ such that $x(0) \in S$ and which fulfill the following condition. If $x(t) = \emptyset$ for some $t \in (0, T]$, then $\sigma_{\infty}(t) > 0$ and $x(\sigma_{\infty}(t)) = x(\sigma_{\infty}(t)-) = \emptyset$.

In words, a trajectory $x$ belongs to $E((0, T], S_\delta)$ if it possesses the following property. If $x(t) = \emptyset$ for some $t \in (0, T]$, then $x$ has visited $S$ before time $t$, $\sigma_{\infty}(t) > 0$, and at the time of the last visit to $S$ before $t$, that is, at time $\sigma_{\infty}(t)$, $x$ is equal to $\emptyset$ and its left-limit is also equal to $\emptyset$: $x(\sigma_{\infty}(t)) = x(\sigma_{\infty}(t)-) = \emptyset$. Note that the trajectory $x$ of Example 2.5 does not belong to $E((0, T], S_\delta)$ because $x(\sigma_{\infty}(t)-) = 1$.

Lemma 2.7. For every $x \in \mathbb{E}((0, T], S_\delta)$, the trajectory $\mathcal{R}_\infty x$ belongs to the space $E((0, T], S_\delta)$.

Proof. Fix a trajectory $x \in \mathbb{E}((0, T], S_\delta)$. By definition $(\mathcal{R}_\infty x)(0) \in S$. We first show that $\mathcal{R}_\infty x$ belongs to $\mathbb{E}((0, T], S_\delta)$.

We claim that $\mathcal{R}_\infty x$ has a left-limit at $t \in (0, T]$ if $x$ has one. Suppose first that $x(t-) = \emptyset$. If there exists $\delta > 0$ such that $x(s) = \emptyset$ for $s \in (t-\delta, t)$, then $\sigma_{\infty}$ is constant in this interval. By definition, $\mathcal{R}_\infty x$ is constant in the same interval and
has therefore a left-limit at \( t \). On the other hand, if there exists a sequence \( t_j \uparrow t \) such that \( x(t_j) \in S, \sigma_\infty(s) \geq t_1 \) for \( t_1 \leq s < t \). As \( x(t--) = d \), for every \( m \geq 1 \), there exists \( \delta > 0 \) such that \( x(s) \geq m \) for \( t - \delta < s < t \). Therefore \( (\mathcal{R}_\infty x)(s) \geq m \) for \( t^*_n \leq s < t \), where \( t^*_n \) is the smallest element of the sequence \( t_j \) which is greater than \( t - \delta \). This proves that \( (\mathcal{R}_\infty x)(t--) \) exists and is equal to \( d \). Suppose now that \( x(t-) \in S \). In this case \( x(s) = x(t-) \in S \) for \( s \) in some interval \( (t- \delta, t) \). In particular, \( (\mathcal{R}_\infty x)(s) = (x(s) = x(t-)) \in \) the same interval, which proves the claim. The trajectory \( x \) of Example 2.5 shows that the left-limits of \( x \) and \( \mathcal{R}_\infty x \) at some point \( t \) may be different.

Suppose now that \( x(t\infty) \) exists and is equal to \( n \in S \). By definition there exists a sequence \( t_j \uparrow t \) such that \( x(t_j) \rightarrow n \), which means that \( x(t_j) = n \) for \( j \) sufficiently large. By definition, \( (\mathcal{R}_\infty x)(t_j) = n \) for the same indices. Fix \( m > n \). By \[ (2.2) \], there exists \( \delta > 0 \) such that \( x(s) = n \) or \( x(s) \geq m \) for all \( t - \delta < s < t \). Hence, if we denote again by \( t^*_j \) the smallest element of the sequence \( t_j \) which is greater than \( t - \delta \), for \( t^*_j < s < t \), \( (\mathcal{R}_\infty x)(s) = n \) or \( (\mathcal{R}_\infty x)(s) \geq m \). This proves that \( \mathcal{R}_\infty x \) has a soft left-limit at \( t \).

The trajectory \( \mathcal{R}_\infty x \) is soft right-continuous. Fix \( t \in [0, T) \) and assume that \( x(t) = d \). If \( x(s) = d \) in some interval \( (t , t + \epsilon) \), \( \sigma_\infty, \mathcal{R}_\infty, \sigma_\infty, \mathcal{R}_\infty \) are constant on the interval \( [t, t + \epsilon) \); while if there exists a sequence \( t_j \downarrow t \) such that \( x(t_j) \in S \) for all \( j \), \( (\mathcal{R}_\infty x)(t) = d \). In both cases, \( \mathcal{R}_\infty x \) is soft right-continuous at \( t \).

Suppose now that \( x(t) \in S \) so that \( (\mathcal{R}_\infty x)(t) = x(t) \in S \). By soft right-continuity of \( x \) at \( t \), for a fixed \( m \geq 1 \), there exists \( \delta > 0 \) such that \( x(s) \in \{ x(t) \} \cup S^*_\infty \) for all \( t < s < t + \delta \). By definition of \( \mathcal{R}_\infty x \) the same property holds for \( \mathcal{R}_\infty x \), which proves its soft right-continuity.

We conclude the proof of the lemma showing that \( \mathcal{R}_\infty x \) belongs to \( E([0, T], S_\delta) \). Fix \( t \in [0, T] \) such that \( (\mathcal{R}_\infty x)(t) = \sigma_\infty \). Denote by \( \sigma_\infty(t), \hat{\sigma}_\infty(t) \) the time of the last visit to \( S \) before time \( t \) of the trajectory \( x, \mathcal{R}_\infty x \), respectively. Clearly \( x(t) = \sigma_\infty \), otherwise \( (\mathcal{R}_\infty x)(t) = x(t) \in S \). We also have that \( \sigma_\infty(t) > 0 \) because if \( \sigma_\infty(t) = 0 \), \( (\mathcal{R}_\infty x)(t) = 1 \) by definition if \( x(s) = \sigma_\infty \) for \( 0 \leq s \leq t \), and \( (\mathcal{R}_\infty x)(t) = x(0) \in S \) if \( x(s) = d \) for \( 0 \leq s < t \). It follows from the definition of \( \mathcal{R}_\infty x \) and from the identity \( (\mathcal{R}_\infty x)(t) = \sigma_\infty \) that \( x(\sigma_\infty(t)) = x(\sigma_\infty(t-)) = d \).

Since \( x(s) = d \) for \( \sigma_\infty(s) < s \leq t \), and since \( x(\sigma_\infty(t)) = x(\sigma_\infty(t-)) = d \), we have that \( (\mathcal{R}_\infty x)(s) = d \), \( \sigma_\infty(s) \leq s \leq t \). \( \hat{\sigma}_\infty(t) = \sigma_\infty(t) > 0 \), \( (\mathcal{R}_\infty x)(\hat{\sigma}_\infty(t-)) = d = (\mathcal{R}_\infty x)(\hat{\sigma}_\infty(t)) \).

Assertion B. Let \( x \) be a trajectory in \( E([0, T], S_\delta) \). Then, \( \mathcal{R}_\infty x = x \).

The proof of this assertion is elementary. It follows from this claim and from Lemma 2.7 that \( \mathcal{R}_\infty : E([0, T], S_\delta) \rightarrow E([0, T], S_\delta) \) is a projection. The next assertion shows that \( \mathcal{R}_m x \) converges pointwisely to \( x \) if \( x \) belongs to \( E([0, T], S_\delta) \).

Assertion C. Fix a trajectory \( x \) in \( E([0, T], S_\delta) \). Then, \( (\mathcal{R}_m x) \) converges pointwisely and \( \lim_m (\mathcal{R}_m x) = \mathcal{R}_\infty x \).

Proof. It is clear from the definition of \( \mathcal{R}_m x \) that \( \mathcal{R}_m x \leq \mathcal{R}_{m+1} x \). In particular, the pointwise limit always exists. Fix \( 0 \leq t \leq T \) and suppose initially that \( x(t) \in S \). In this case, for \( m > x(t), (\mathcal{R}_m x)(t) \geq (\mathcal{R}_\infty x)(t) \).

Suppose from now on that \( x(t) = \sigma_\infty \). If \( x(s) = \sigma_\infty \) for \( 0 \leq s < t \), \( (\mathcal{R}_m x)(t) = 1 = (\mathcal{R}_{\sigma_\infty} x)(t) \) for all \( m \geq 1 \), while if \( x(0) \in S \) and \( x(s) = \sigma_\infty \) for \( 0 < s < t \), \( (\mathcal{R}_m x)(t) = x(0) = (\mathcal{R}_\infty x)(t) \) for all \( m \geq x(0) \). We may therefore assume that there exists \( 0 < s < t \) such that \( x(s) \in S \) so that \( \sigma_\infty(t) = \sigma_\infty^r(t) > 0 \).
If \( x(\sigma_\infty(t)) \in S \), for \( m > x(\sigma_\infty(t)) \) we have that \((\mathcal{R}_m x)(t) = (\mathcal{R}_\infty x)(t)\), while if \( x(\sigma_\infty(t)) = \emptyset \) and if \( x(\sigma_\infty(t)) \) exists, \((\mathcal{R}_{\infty} x)(t) = x(\sigma_\infty(t)) \). Finally, suppose that \( x(\sigma_\infty(t)) \) does not exist. Then, by definition, \((\mathcal{R}_{\infty} x)(t) = x(\sigma_\infty(t)) \) and for \( m > x(\sigma_\infty(t)) \) \((\mathcal{R}_m x)(t) = x(\sigma_\infty(t)) \). This proves the assertion.

The next statement follows from Assertions [3] and [C]

**Assertion D.** Fix two trajectories \( x, y \) in \( E([0, T], S_\emptyset) \). If \( \mathcal{R}_m x = \mathcal{R}_m y \) for all \( m \) large enough, then \( x = y \).

For two trajectories \( x, y \in E([0, T], S_\emptyset) \), let

\[
d(x, y) = \sum_{m \geq 1} \frac{1}{2^m} d_m(x, y), \quad \text{where } d_m(x, y) = d_S(\mathcal{R}_m x, \mathcal{R}_m y). \tag{2.8}
\]

**Example 2.5.** shows that \( d \) is not a metric in \( E([0, T], S_\emptyset) \), but the next assertion states that it is a metric in \( E([0, T], S_\emptyset) \).

**Assertion E.** The map \( d \) is a metric in \( E([0, T], S_\emptyset) \).

**Proof.** It is clear that \( d \) is finite, non-negative and symmetric, and that \( d \) satisfies the triangular inequality. Suppose that \( d(x, y) = 0 \). Then, \( \mathcal{R}_m x = \mathcal{R}_m y \) for all \( m \geq 1 \), and, by Assertion [D], \( x = y \).

**Example 2.8.** Fix \( t_0 < T \) and let \( x_n \in D([0, T], S_\emptyset) \) be the sequence given by

\[
x_n = 1\{[0, t_0]\} + n 1\{[t_0, t_0 + n^{-1}]\} + 1\{[t_0 + n^{-1}, T]\}.
\]

While this sequence does not converges in the Skorohod topology, it converges to the constant trajectory equal to 1 in the metric \( d \). In contrast and as required, for \( \ell, \in \mathbb{N}, \ell \neq 1 \), the sequence

\[
y_n = 1\{[0, t_0]\} + \ell 1\{[t_0, t_0 + n^{-1}]\} + 1\{[t_0 + n^{-1}, T]\}
\]

does not converge in the metric \( d \).

The undesirable aspect of the metric \( d \) is that the sequence

\[
z_n = 1\{[0, t_0]\} + n 1\{[t_0, T]\}
\]
also converges to the constant trajectory equal to 1. To exclude such cases, we shall introduce in the next section a subset of trajectories in \( E([0, T], S_\emptyset) \) which spend only a negligible amount of time in \( \emptyset \) and we shall introduce compactness conditions which ensure that the limit points of a sequence of trajectories belongs to this set. These compactness conditions will exclude sequences as \( z_n \), which spend uniformly a non-negligible amount of time in a set \( S_m^c \) for some \( m \).

We conclude this section proving in Proposition 2.12 below that the path space \( E([0, T], S_\emptyset) \) endowed with the metric \( d \) is complete and separable. Recall that we denote by \( \Lambda \) the set of increasing and continuous functions \( \lambda : [0, T] \to [0, T] \) such that \( \lambda(0) = 0 \), \( \lambda(T) = T \).

**Assertion F.** Let \( x \) be a trajectory in \( D([0, T], S_{m+1}) \) and fix \( \lambda \in \Lambda \). Then, \( \mathcal{R}_m (x \circ \lambda) = (\mathcal{R}_m x) \circ \lambda \). The same identity holds for a trajectory \( x \) in \( D([0, T], S_\emptyset) \).
Proof. Since \( x \in D([0,T], S_{m+1}) \), there exist \( k \geq 1 \), \( 0 = t_0 < t_1 < \cdots < t_k = T \), and \( \ell_0, \ldots, \ell_k \in S_{m+1} \) such that \( \ell_i \neq \ell_{i+1}, 0 \leq i \leq k-2 \), and
\[
x(t) = \sum_{i=0}^{k-1} \ell_i 1\{[t_i, t_{i+1})\}(t) + \ell_k 1\{t = t_k\}.
\] (2.9)

Note that \( \ell_{k-1} \) may be equal to \( \ell_k \) in which case \( x \) is left continuous at \( T \). It is easy to obtain from this formula explicit expressions for \( \mathcal{R}_m(x \circ \lambda) \) and for \( (\mathcal{R}_m x) \circ \lambda \) and to check that they are equal.

Consider now a trajectory \( x \) in \( D([0,T], S_0) \). Fix \( \lambda \in \Lambda \) and \( m \in S \). Recall that we denote by \( \sigma_m^\lambda(t) \) the last visit to \( S_m \) before time \( t \) for the trajectory \( y \).

Fix \( t \in [0,T] \) and suppose that \( x(s) \notin S_m \) for \( 0 \leq s \leq \lambda(t) \). In this case, \( x(\lambda(s)) \notin S_m \) for \( 0 \leq s \leq \lambda(t) \) and \( (\mathcal{R}_m(x\lambda))(t) = 1 = (\mathcal{R}_m x)(\lambda(t)) \).

If \( x(\lambda(t)) \in S_m \), \( (\mathcal{R}_m(x\lambda))(t) = (x(\lambda)(t) = (\mathcal{R}_m x)(\lambda(t)) \). It remains to consider the case in which \( x(s) \in S_m \) for some \( 0 \leq s \leq \lambda(t) \) and \( x(\lambda(t)) \notin S_m \). By (2.6), and since \( y(\lambda^{-1}(s) - ) = y(\lambda^{-1}(s)-) \),
\[
(\mathcal{R}_m(x\lambda))(t) = (x(\lambda)(\sigma_m^x(\lambda(t))) = (x(\lambda)(\lambda^{-1}(\sigma_m^\lambda(t))) = \lambda(t)
\]
which proves the claim. \( \square \)

**Lemma 2.9.** The map \( \mathcal{R}_m : D([0,T], S_{m+1}) \to D([0,T], S_m) \) is continuous for the Skorohod topology.

**Proof.** Let \( x_n \) be a sequence of trajectories in \( D([0,T], S_{m+1}) \) which converges in the Skorohod topology to \( x \). We will prove that the sequence of trajectories \( \mathcal{R}_m x_n \) in \( D([0,T], S_m) \) converges in the Skorohod topology to \( \mathcal{R}_m x \).

Recall the notation introduced in the beginning of this section. Fix \( \epsilon < [m(m+1)]^{-1} \). Since \( x_n \) converges to \( x \), there exists \( n_0 \) such that for all \( n \geq n_0 \)
\[
\max \left\{ \| x_n - x \lambda \|_\infty, \| \lambda \|^o \right\} < \epsilon,
\]
for some \( \lambda \in \Lambda \). Since we chose \( \epsilon < [m(m+1)]^{-1} \), we must have that \( x_n = x \lambda \) so that \( \mathcal{R}_m x_n = \mathcal{R}_m (x \lambda) \). Since by Assertion F \( \mathcal{R}_m (x \lambda) = (\mathcal{R}_m x) \circ \lambda \), we conclude that
\[
d_S(\mathcal{R}_m x, \mathcal{R}_m x_n) \leq \max \left\{ \| (\mathcal{R}_m x_n) - (\mathcal{R}_m x) \circ \lambda \|_\infty, \| \lambda \|^o \right\} < \epsilon,
\]
which proves the lemma. \( \square \)

Fix \( 1 \leq k < m \), a trajectory \( y \in D([0,T], S_m) \), and \( t \in (0,T) \). Let \( x = \mathcal{R}_k y \). If \( y(t) \leq k \), then \( x(t) = y(t) \), while \( x(t) \neq y(t) \) if \( y(t) \) because in this latter case \( x(t) \leq k < y(t) \). Hence, if \( \mathcal{R}_k y(t) \neq y(t) \), \( y(t) \) is necessarily greater than \( k \).

**Assertion G.** Let \( y \) be a trajectory in \( D([0,T], S_m) \), \( m \geq 2 \), and let \( x = \mathcal{R}_{m-1} y \). Suppose that \( x \) is discontinuous at \( t \in (0,T) \). Then, \( y(t) = x(t) \) and \( y \) is discontinuous at \( t \).

**Proof.** We first show that \( y(t) = x(t) \) if \( x \) is discontinuous at \( t \in (0,T) \). We proceed by contradiction. Fix \( t \in (0,T) \) and suppose that \( y(t) \neq x(t) \). By the remark made just before the assertion, \( y(t) = m \). We want to show that \( x \) is continuous at \( t \). Since \( y \) belongs to \( D([0,T], S_m) \), \( y \) can be represented as in (2.9). By definition of \( \mathcal{R}_{m-1} \), the only points where \( x \) can be discontinuous are the points \( t_i, 1 \leq i \leq k \).
If \( t = t_i \) and \( x(t_i) \neq y(t_i) \), then \( y(t_i) = m \), \( y(t_{i-1}) \in S_{m-1} \) (because \( y(t_{i-1}) \in S_m \) and \( y(t_{i-1}) \neq y(t_i) \)). \( t \) is left-continuous at \( x(t_i) \).

We now prove the second claim of the assertion. Fix \( t \in (0,T] \) and suppose that \( x \) is discontinuous at \( t \). By the first part of the claim, \( y(t) = x(t) \in S_{m-1} \). By definition of \( \mathcal{R}_{m-1} \), \( y(t-) = x(t-) \) or \( y(t-) = m \). In the first case \( y \) is discontinuous at \( t \) because so is \( x \). In the second case \( y \) is also discontinuous at \( t \) because \( y(t) \in S_{m-1} \).

\[ \square \]

**Figure 4.** The figure illustrates how the trajectory \( y_{m+1} \) is obtained from the trajectory \( y_m \) in Lemma 2.10. The value of \( y_{m+1} \) is set to be \( m+1 \) in some intervals \([s,t]\) strictly contained in a constancy interval of \( y_m \): \( y_{m+1}(r) = m+1 \) for \( a < s < t \leq b \) and \( y_m \) is constant in \([a,b]\). In particular, either \( y_{m+1}(t) = y_m(t) \) or \( y_{m+1}(t) = m+1 \). The points \( t = 0 \) and \( t = T \) are special. For example, an interval \([0,r]\) can be lifted to \( m+1 \) if \( y_m \) is constant in \([0,a]\), \( r \leq a \), and if \( y_m = 1 \) on \([0,a]\). Note that \( \mathcal{R}_m y_{m+1} = y_m \).

**Lemma 2.10.** Let \( y_m \in D([0,T],S_m) \), \( m \geq 1 \), be a sequence of trajectories such that \( \mathcal{R}_m y_{m+1} = y_m \) for all \( m \geq 1 \). Then, there exists a trajectory \( y \) in \( E([0,T],S_0) \) such that \( \mathcal{R}_m y = y_m \) for all \( m \geq 1 \).

**Proof.** Since \( \mathcal{R}_m x \leq x \), the sequence \( y_m \) is increasing and has therefore a pointwise limit, denoted by \( y \). Figure 4 illustrates how the trajectory \( y_{m+1} \) is obtained from \( y_m \). The precise mechanism is presented below in the proof.

Suppose that \( y(t) = n \in S \) for some \( t \in [0,T] \). In this case \( y_m(t) = n \) for all \( m \geq n \). Indeed, if \( y_m(t) \neq n \) for some \( m_0 > n \), then for all \( m \geq m_0 \), either \( y_m(t) = y_{m_0}(t) \) or \( y_m(t) = m > n \), which contradicts the fact that \( \lim_{m} y_m(t) = y(t) = n \).

There exists \( 1 \leq m_0 \leq \infty \) such that \( y_m(0) = 1 \) for \( m < m_0 \) and \( y_m(0) = m_0 \) for \( m \geq m_0 \). For any trajectory \( x \), by our convention in the definition of \( \mathcal{R}_m \),

\[
(\mathcal{R}_m x)(0) = \begin{cases} x(0) & \text{if } x(0) \leq m, \\ 1 & \text{if } x(0) > m. \end{cases}
\]
Therefore \( y_m(0) = (\mathfrak{R}_m y_{m+1})(0) \) satisfies the relation

\[
y_m(0) = \begin{cases} 
  y_{m+1}(0) & \text{if } y_{m+1}(0) \leq m, \\
  1 & \text{if } y_{m+1}(0) = m + 1.
\end{cases}
\] (2.10)

Let \( m_0 = \min\{ j \geq 1 : y_j(0) \neq 1 \} \). Assume that \( m_0 < \infty \), otherwise there is nothing to be proven. By (2.10) for \( m = m_0 - 1 \), \( y_{m_0}(0) = m_0 \), and by definition of \( m_0 \), \( y_k(0) = 1 \) for \( k < m_0 \). By (2.10) for \( m = m_0 \), \( y_{m_0+1}(0) = y_{m_0}(0) = m_0 \). Repeating this argument, we conclude that \( y_k(0) = m_0 \) for all \( k \geq m_0 \), as claimed.

The trajectory \( y \) has a soft left-limit at each point \( t \in (0, T] \). Fix \( t \in (0, T] \) and suppose that there exists an increasing sequence \( t_j \) converging to \( t \) and such that \( y(t_j) \to n \in S \). For \( j \) large enough \( y(t_j) = n \). We assume, without loss of generality, that this holds for all \( j \): \( y(t_j) = n \) for all \( j \geq 1 \). By the penultimate paragraph, \( y_m(t_j) = n \) for all \( m \geq n \) and \( j \geq 1 \). This proves that \( y_m(t^-) = n \) for all \( m \geq n \). In particular, by Remark 2.2, \( y \) has a soft left-limit at \( t \).

It is not difficult to construct an example of a sequence \( y_m \) for which \( y \) has a soft left-limit at \( t \in (0, T] \), but not a left-limit, i.e., a sequence \( y_m \) for which there exist increasing sequences \( t_j, t_j' \) converging to \( t \) and such that \( y(t_j) \to n \in S \), \( y(t_j') \to \partial \).

The trajectory \( y \) is soft right-continuous. Fix \( t \in (0, T] \) and suppose that there exists a decreasing sequence \( t_j \) converging to \( t \) and such that \( y(t_j) \to n \in S \). The argument presented above shows that \( y_m(t) = n \) for all \( m \geq n \), which proves, in view of Remark 2.2, that \( y \) has a soft right-limit at \( t \) equal to \( n \). Since \( y_m(t) = n \) for all \( m \geq n \), \( y(t) = n \), which proves that \( y \) is soft right-continuous at \( t \).

Fix \( t \in (0, T] \) and assume that there exists \( m \) for which \( y_m \) is discontinuous at \( t \).

By Assertion \( \square \) \( y_{m+1}(t) = y_m(t) \) and \( y_{m+1} \) is discontinuous at \( t \). Repeating this argument, we conclude that \( y_{m}(t) = y_{m}(t) \) for all \( n \geq m \) so that \( y(t) = y_m(t) \in S \).

The trajectory \( y \) belongs to \( E([0, T], \mathcal{S}_k) \). We proved above that \( y(0) \in S \).

Assume that \( y(t) = \partial \) for some \( t \in (0, T] \). By the previous paragraph, \( t \) is a continuity point of \( y_m \) for every \( m \). Denote by \( [\ell_m, r_m) \) the largest interval which contains \( t \) and in which \( y_m \) is constant. \( \ell_m \) is a non-decreasing sequence bounded above by \( t \). Denote by \( \ell \) its limit. It is clear that \( \ell = \sigma^R_y(t) \), that \( y(\ell) = \partial \) and that \( y(\ell^-) = \partial \). We claim that \( \ell > 0 \). By construction, there exists \( m_0 \) such that \( y_m(0) = 1 \) for \( m < m_0 \) and \( y_m(0) = m_0 \) for \( m \geq m_0 \). As \( y(t) = \partial \), there exists \( m_1 \) such that \( y_m(t) > m_0 \) for \( m \geq m_1 \). In particular, for \( m \geq m_1, \ell_m > 0 \), which proves that \( \ell > 0 \) and that \( y \) belongs to \( E([0, T], \mathcal{S}_k) \).

It remains to show that \( \mathfrak{R}_m y = y_m \) for all \( m \geq 1 \). Fix \( m \geq 1 \) and \( t \in (0, T] \).

If \( t \) is a point of discontinuity of \( y_m \), by Assertion \( \square \) \( y_n(t) = y_m(t) \) for all \( n \geq m \) so that \( y(t) = y_m(t) \in S_m \) and \( \mathfrak{R}_m y(t) = y_m(t) \). If \( t \) is a continuity point of \( y_m \), as above, let \( [\ell_m, r_m) \) the largest constancy interval of \( y_m \) which contains \( t \).

If \( \ell_m > 0 \), \( \ell_m \) is a discontinuity point of \( y_m \) so that \( y(\ell_m) = y_m(\ell_m) \in S_m \). By definition of the sequence \( y_k \), for \( k > m \) and \( \ell_m \leq s \leq t \), \( y_k(s) = y_m(s) = y_m(\ell_m) \) or \( y_k(s) > m \). Hence, for \( \ell_m \leq s \leq t \), \( y(s) = y_m(\ell_m) \) or \( y(s) > m \), so that \( \mathfrak{R}_m y(t) = y_m(t) \). If \( \ell_m = 0 \) and \( y_m(0) \neq 1 \), the same argument holds since the sequence \( y_k(0) \), \( k \geq m \), is constant by the assertion above (2.10). If \( \ell_m = 0 \) and \( y_m(0) = 1 \), the argument can be adapted even if the sequence \( y_k(0) \) may not be constant. By the assertion above (2.10), for \( k \geq m \) and \( 0 \leq s \leq t \), \( y_k(s) = y_m(s) = 1 \) or \( y_k(s) > m \). Hence, for \( 0 \leq s \leq t \), \( y(s) = 1 \) or \( y(s) > m \). If \( y(s) > m \) for all \( 0 \leq s \leq t \), by our convention in the definition of \( \mathfrak{R}_m \), \( (\mathfrak{R}_m y)(t) = 1 = y_m(t) \). If
there exists $0 \leq s \leq t$ such that $y(s) = 1$ we also have that $(\mathcal{R}_m y)(t) = 1 = y_m(t)$. This concludes the proof of the lemma.

Remark 2.11. Let $y \in E([0, T], S_0)$, $y_m(t)$, $m \geq 1$, be the trajectories appearing in the statement of the previous lemma. It follows from Assertion G that if $y_m(t)$ is discontinuous at $t \in (0, T]$, the sequence $\{y_\ell(t) : \ell \geq m\}$ is constant and $y(t) = y_m(t)$.

Fix $m \geq 1$. For $\ell \geq m$, since $y_\ell$ belongs to $D([0, T], S_\ell)$, the set $I_\ell = \{t \in [0, T] : y_\ell(t) = m\}$ is the union of intervals $[s_k^\ell, t_k^\ell)$, $1 \leq k \leq n_\ell$. The last interval may be closed, all the other ones are closed on the left and open on the right. The intervals are disjoint, $t_k^\ell < s_k^{\ell+1}$, $1 \leq k < n_\ell$, and nondegenerate, $s_k^\ell < t_k^\ell$, except the last one which can be reduced to a point.

The sequence $I_\ell$ is decreasing, $I_{\ell+1} \subset I_\ell$, and a left end-point $s_k^\ell$ of $I_\ell$ belongs to $\cap_{\ell \geq I_\ell} I_\ell$: $s_k^\ell = s_j^\ell$ for some $1 \leq j \leq n_{\ell+1}$. In particular, the number of intervals may only increase, $n_\ell \leq n_{\ell+1}$. The set $\{t \in [0, T] : y(t) = m\}$ is equal to the limit of the sets $I_\ell$.

\begin{equation}
\{t \in [0, T] : y(t) = m\} = \bigcap_{\ell \geq m} \{t \in [0, T] : y_\ell(t) = m\}.
\end{equation}

Proposition 2.12. The space $E([0, T], S_0)$ endowed with the metric $d(x, y)$ is complete and separable.

Proof. Consider a Cauchy sequence $\{x_n : n \geq 1\}$ in $E([0, T], S_0)$ for the metric $d$. By definition of $d$, for each $m \geq 1$, $\mathcal{R}_m x_n$ is a Cauchy sequence in $D([0, T], S_m)$ for the metric $d_S$. Since this space is complete, there exists $y_m \in D([0, T], S_m)$ such that $\mathcal{R}_m x_n \to y_m$ as $n \to \infty$. By Lemma 2.9 there exists $y \in E([0, T], S_0)$ such that $\mathcal{R}_m y = y_m$ for all $m \geq 1$. Therefore, $\mathcal{R}_m x_n \to y = \mathcal{R}_m y$, which implies that $x_n$ converges to $y$ in $E([0, T], S_0)$. This proves completeness.

The separability of $E([0, T], S_0)$ follows from the separability of each set $D([0, T], S_m)$. Since the set $D([0, T], S_m)$, $m \geq 1$, endowed with the metric $d_S$ is separable, for each $m \geq 1$ there exists a sequence of trajectories $x_{m,n}$, $n \geq 1$, which is dense in $D([0, T], S_m)$ for the metric $d_S$. We claim that the countable set of trajectories $x_{m,n}$, $n \geq 1$, $m \geq 1$ is dense.

Fix a trajectory $x \in E([0, T], S_0)$ and $\epsilon > 0$. Take $m \geq 1$ such that $2^{-m} < \epsilon$ and $x_{m,n} \in D([0, T], S_m)$ such that $d_S(x_{m,n}, \mathcal{R}_m x) < \min\{\epsilon, |m(m-1)|^{-1}\}$. There exists $\lambda \in \Lambda$ such that

\[\max\{||x_{m,n} - (\mathcal{R}_m x) \circ \lambda||_\infty, ||\lambda||^\circ\} < \min\{\epsilon, |m(m-1)|^{-1}\} .\]

Since $||x_{m,n} - (\mathcal{R}_m x) \circ \lambda||_\infty < |m(m-1)|^{-1}$, $x_{m,n} = (\mathcal{R}_m x) \circ \lambda$. Hence, by Assertion G for $\ell \leq m$, $\mathcal{R}_\ell x_{m,n} = \mathcal{R}_\ell((\mathcal{R}_m x) \circ \lambda) = (\mathcal{R}_\ell x) \circ \lambda$. In particular,

\[d_S(\mathcal{R}_\ell x_{m,n}, \mathcal{R}_\ell x) \leq ||\lambda||^\circ < \epsilon .\]

Putting together the previous estimates, as $d_S(x, y) \leq 1$ for any pair of trajectories in $D([0, T], S_\ell)$, we obtain that

\[\sum_{\ell \geq 1} \frac{1}{2^\ell} d_S(\mathcal{R}_\ell x_{m,n}, \mathcal{R}_\ell x) \leq \sum_{\ell = 1}^m \frac{1}{2^\ell} d_S(\mathcal{R}_\ell x_{m,n}, \mathcal{R}_\ell x) + \epsilon \leq 2\epsilon .\]

This concludes the proof of the proposition. \[\square\]
By extension, we call soft topology the topology in \( E([0, T], S_\delta) \) induced by the metric \( d \). A sequence of trajectories \( x_n \in E([0, T], S_\delta) \) which converges converges in the soft topology is said to s-converges. Denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra of subsets of \( E([0, T], S_\delta) \) spanned by the open sets of the soft topology.

**Assertion H.** The subspaces \( D([0, T], S_m), \ m \geq 1 \) of \( E([0, T], S_m) \) are closed for the soft topology.

**Proof.** Consider a sequence \( x_n \in D([0, T], S_m) \) s-converging to \( x \). For all \( \ell \geq 1 \), \( \mathcal{R}_\ell x_n \) converges in the Skorohod topology to \( \mathcal{R}_\ell x \). Since \( x_n \) belongs to \( D([0, T], S_m) \), \( \mathcal{R}_\ell x_n = \mathcal{R}_m x_n \) for \( \ell \geq m \), so that \( x = \lim_\ell \mathcal{R}_\ell x = \lim_\ell \lim_\ell \mathcal{R}_m x_n = \lim_\ell \mathcal{R}_m x_n = \mathcal{R}_m x \in D([0, T], S_m) \). \( \square \)

3. The space \( D^*([0, T], S_\delta) \)

Denote by \( D^*([0, T], S_\delta) \) the subset of all trajectories in \( D([0, T], S_\delta) \) which spend no time at \( \emptyset \) and which are continuous at time \( T \):

\[
D^*([0, T], S_\delta) = \left\{ x \in D([0, T], S_\delta) : \Lambda_T(x) = 0, \ x(T) = x(T) \right\},
\]

where

\[
\Lambda_T(x) = \int_0^T 1\{x(s) = \emptyset\} \, ds.
\]

Since a trajectory \( x \) in \( D^*([0, T], S_\delta) \) spends no time at \( \emptyset \), \( \sigma_{\infty}^x(t) = t \) for all \( t \in [0, T] \). In particular, by definition of the map \( \mathcal{R}_\infty \), for \( x \in D^*([0, T], S_\delta) \)

\[
(\mathcal{R}_\infty x)(t) = \begin{cases} x(t) & \text{if } x(t) \in S, \\ x(t-) & \text{if } x(t) = \emptyset. \end{cases} \tag{3.1}
\]

Therefore, \( (\mathcal{R}_\infty x)(t) \neq x(t) \) only if \( x(t) = \emptyset \neq x(t-) \) and \( (\mathcal{R}_\infty x)(T) = x(T) \).

**Assertion I.** The map \( \mathcal{R}_\infty : D^*([0, T], S_\delta) \to E([0, T], S_\delta) \) is one-to-one.

**Proof.** Fix two trajectories \( x, y \in D^*([0, T], S_\delta) \) and suppose that \( \mathcal{R}_\infty x = \mathcal{R}_\infty y \). Let \( A = \{ t \in [0, T] : x(t) = \emptyset \text{ or } y(t) = \emptyset \} \). By (3.1), \( x(t) = y(t) \) for \( t \notin A \). Hence, since the set \( A \) has measure zero and since \( x \) and \( y \) are right continuous, \( x(t) = y(t) \) for \( t \in [0, T] \). On the other hand, as we have seen just below (3.1), \( x(T) = (\mathcal{R}_\infty x)(T) = (\mathcal{R}_\infty y)(T) = y(T) \). \( \square \)

**Definition 3.1.** Denote by \( E^*([0, T], S_\delta) \) the range of the map \( \mathcal{R}_\infty : D^*([0, T], S_\delta) \to E([0, T], S_\delta) \).

Fix \( x \in E([0, T], S_\delta) \). By Assertions \([\Box] \) and \([\Box] \), \( x = \lim_m \mathcal{R}_m x \). In particular \( x : [0, T] \to \mathbb{R} \) is Borel measurable and \( \Lambda_T(x) \) is well defined.

**Lemma 3.2.** A trajectory \( y \) in \( E([0, T], S_\delta) \) belongs to \( E^*([0, T], S_\delta) \) if and only if

(a) \( y \) has left and right-limits at every point;
(b) \( \text{if } y(t+) = \emptyset \text{ for some } t \in [0, T], \text{ then } y(t) = y(t-) \);
(c) \( y \) is continuous at \( T \);
(d) \( \Lambda_T(y) = 0 \).

**Proof.** Fix a trajectory \( y \) in \( E^*([0, T], S_\delta) \). Let \( x \in D^*([0, T], S_\delta) \) such that \( y = \mathcal{R}_\infty x \). It follows from (3.1) that \( y(t+) = x(t+) \), \( y(t-) = x(t-) \), which proves (a). Assume that \( y(t+) = \emptyset \) for some \( t \in [0, T] \). As we just have seen, \( x(t+) = \emptyset \). Since \( x \) is right continuous, \( x(t) = \emptyset \). Thus, by (3.1), \( y(t) = x(t-) \). By the first
part of the proof, \( x(t-) = y(t-) \), so that \( y(t) = y(t-) \), which proves (b). To verify (c), recall from (3.1) that \( y(T) = x(T) \) and from the first part of the proof that \( y(T-) = x(T-) \). Since \( x \) belongs to \( D^*(\mathbb{R}, \mathbb{R}) \), \( x(T) = x(T-) \) so that \( y(T) = y(T-) \). Finally, since \( y(t) \in S \) whenever \( x(t) \in S \), \( x = \emptyset \) if \( y(t) = \emptyset \), and \( \Lambda_T(y(t)) \leq \Lambda_T(x) \).

Conversely, let \( y \) be a trajectory in \( E([0, T], \mathbb{R}) \) which fulfills conditions (a)–(d). Let \( x \) be the trajectory defined by \( x(t) = y(t+) \), \( 0 < t < T \), \( x(T) = y(T) \). We claim that \( x \in D^*(\mathbb{R}, \mathbb{R}) \). By definition, \( x \) is right continuous and has left limits, and \( x(t+) = y(t+) \), \( x(t-) = y(t-) \). Therefore, \( x \in D([0, T], \mathbb{R}) \), and, by assumption (c), \( x(T) = x(T-) \).

By definition of \( x \),

\[
\Lambda_T(x) = \int_{0}^{T} 1\{y(s+) = \emptyset\} \, ds.
\]

Fix \( t \in [0, T) \) such that \( y(t+) = \emptyset \). Then, either \( y(t) = \emptyset \) or, by assumption (b), \( y(t-) = y(t) \in S \). The first set of points has Lebesgue measure zero because \( \Lambda_T(y(t)) = 0 \) by assumption (d). The second set is at most countable because \( y \) is constant on an interval \( [t-\delta, t) \) if \( y(t-) = y(t) \in S \). This proves that \( \Lambda_T(x) = 0 \).

It remains to show that \( \mathfrak{R}_{\infty}x = y \). Suppose that \( x(t) \in S \). By the definition (3.1) of \( \mathfrak{R}_{\infty} \), \( \mathfrak{R}_{\infty}(x)(t) = x(t) = y(t-) \). Since \( y \) is soft right-continuous and since \( y \) has a right-limit which belongs to \( S \), \( y(t+) = y(t) \), so that \( \mathfrak{R}_{\infty}(x)(t) = y(t) \). Suppose now that \( x(t) = \emptyset \), so that \( y(t+) = \emptyset \). By definition (3.1) of \( \mathfrak{R}_{\infty} \), \( \mathfrak{R}_{\infty}(x)(t) = x(t-) = y(t-) \). Since \( y(t+) = \emptyset \), by assumption (b), \( y(t-) = y(t) \) so that \( \mathfrak{R}_{\infty}(x)(t) = y(t) \).

The set \( E^*([0, T], \mathbb{R}) \) is clearly not closed for the soft topology. Lemma 3.3 shows that it belongs to \( \mathcal{B} \), and Lemma 3.4 provides sufficient conditions for the limit \( x \) of a converging sequence \( x_n \) in \( E^*([0, T], \mathbb{R}) \) to belong to \( E^*([0, T], \mathbb{R}) \).

For a trajectory \( x \in D([0, T], \mathbb{R}) \) and \( 1 \leq j \leq m \), denote by \( \mathfrak{H}_j = \mathfrak{H}_j(x) \) the number of visits of \( x \) to \( j \) in the time interval \( [0, T] \), and denote by \( T_{j,1}, \ldots, T_{j,\mathfrak{H}_j} \) the holding times at \( j \). Hence, if the trajectory \( x \) is given by

\[
\begin{align*}
x(t) &= \sum_{i=0}^{k-1} \ell_i 1\{[t_i, t_{i+1})\}(t) + \ell_k 1\{[t_k, T]\},
\end{align*}
\]

where \( 0 = t_0 < t_1 < \cdots < t_k = T \), and \( \ell_i \neq \ell_{i+1}, 0 \leq i \leq k - 1 \), and if we denote by \( I_j \) the set \( \{i \in \{0, \ldots, k\} : \ell_i = j\} \), we have that \( \mathfrak{H}_j(x) = |I_j| \). Moreover, if \( \mathfrak{H}_j \geq 1 \) and if \( I_j = \{i_1, \ldots, i_{\mathfrak{H}_j}\} \), where \( i_a < i_{a+1} \) for \( 1 \leq a < \mathfrak{H}_j \),

\[
T_{j,1} = t(i_1 + 1) - t(i_1), \ldots, T_{j,\mathfrak{H}_j} = t(i_{\mathfrak{H}_j} + 1) - t(i_{\mathfrak{H}_j}).
\]

In this formula, to avoid small indices we represented \( t_{i_a} \) by \( t(i_a) \). By convention, \( T_{j,\ell} = 0 \) for \( \ell > \mathfrak{H}_j \).

**Assertion J.** Let \( x \) be a trajectory in \( E^*([0, T], \mathbb{R}) \). Then, for all \( \ell \geq 1 \), \( \mathfrak{R}_\ell x \) is continuous at \( T \).

**Proof.** Fix \( \ell \geq 1 \). By Lemma 3.2, \( x \) is continuous at \( T \). Suppose that \( x(T) \in S \). In this case, \( x \) is constant in an interval \( (T - \delta, T) \), \( \delta > 0 \), and so is \( \mathfrak{R}_\ell x \).

Suppose that \( x(T) = \emptyset \). Fix \( m > \ell \). There exists \( \delta > 0 \) such that \( x(t) \geq m \) on \( (T - \delta, T] \). Hence, \( \sigma_\ell \) is constant in this interval and so is \( \mathfrak{R}_\ell(x) \). \( \square \)
Assertion K. The functionals $\mathfrak{N}_k, 1 \leq k \leq m$, are continuous with respect to the Skorohod topology in $D([0, T], S_m)$, and the sets \{x : T_{j, \ell} \geq a\}, a > 0$, are closed.

Proof. Fix $1 \leq k \leq m$, and let \{x_n : n \geq 1\} be a sequence in $D([0, T], S_m)$ which converges to a trajectory $x$ in the Skorohod topology. Fix $\epsilon < [m(m - 1)]^{-1}$. Since $x_n$ converges to $x$, there exists $n$ sufficiently large and $\lambda_n \in \Lambda$ such that

$$\|x_n - x\lambda_n\|_\infty < \epsilon.$$ 

Since $\epsilon < [m(m - 1)]^{-1}$ we have that $x_n = x\lambda_n$ so that $\mathfrak{N}_k(x\lambda_n) = \mathfrak{N}_k(x_n)$. Since $\mathfrak{N}_k(x\lambda_n) = \mathfrak{N}_k(x)$, we conclude that the sequence $\mathfrak{N}_k(x_n)$ is eventually constant and converges to $\mathfrak{N}_k(x)$.

To prove the assertion, we consider a sequence $\mathfrak{N}_k, 1 \leq k \leq m$, are continuous with respect to the Skorohod topology in some trajectory $x$. Suppose that $T_{j, \ell}(x_n) \geq a$ for all $n \geq 1$ and fix $0 < \epsilon < [m(m - 1)]^{-1}$. There exists $\lambda_n \in \Lambda$ such that $\|x_n - x\lambda_n\|_\infty < \epsilon, \|\lambda_n\|^0 < \epsilon$ for all $n$ large enough. As in the first part of the proof, we deduce from this estimate that $x_n = x\lambda_n$ so that $\mathfrak{N}_k(x_n) = \mathfrak{N}_k(x\lambda_n) = \mathfrak{N}_k(x)$ and $T_{j, \ell}(x_n) = T_{j, \ell}(x\lambda_n)$ for $n$ large enough. Since $T_{j, \ell}(x_n) \geq a, \ell \leq \mathfrak{N}_k(x_n) = \mathfrak{N}_k(x)$. Denote by $[s, t)$ the time interval of the $\ell$-th visit to $j$ for the trajectory $x$, so that $T_{j, \ell}(x\lambda_n) = \lambda_n^{-1}(t) - \lambda_n^{-1}(s)$. Since $T_{j, \ell}(x_n) = T_{j, \ell}(x\lambda_n)$ and $T_{j, \ell}(x_n) \geq a, \lambda_n^{-1}(t) - \lambda_n^{-1}(s) \geq a$. However, as $\|\lambda_n\|^0 < \epsilon, e^{-\epsilon}(t - s) \leq \lambda_n^{-1}(t) - \lambda_n^{-1}(s) \leq e^{\epsilon}(t - s)$. Therefore, $T_{j, \ell}(x) = t - s \geq e^{-\epsilon}\lambda_n^{-1}(t) - \lambda_n^{-1}(s) \geq e^{\epsilon}\lambda_n^{-1}(t - s)$, which proves the assertion.

By expressing all conditions of Lemma 3.2 in terms of the trajectories $\mathfrak{N}_k x$, we show in the next lemma that the set $E^*(\{0, T]\), S_\lambda)$ belongs to the Borel $\sigma$-algebra $B$.

Lemma 3.3. The set $E^*(\{0, T]\), S_\lambda)$ belongs to the Borel $\sigma$-algebra $B$.

Proof. To keep notation simple, denote the set $E(\{0, T]\), S_\lambda)$ by $E_T$. Let

$$\Omega_1 = \{x \in E_T : \Lambda_T(x) = 0\}, \quad \Omega_2 = \{x \in E_T : x(T) = x(T^-)\},$$

$$\Omega_3 = \{x \in E_T : x \text{ has left and right-limits at every point}\},$$

$$\Omega_4 = \{x \in E_T : x(t) = x(t^-) \text{ if } x(t+) = \emptyset \text{ for some } t \in (0, T]\}.$$ 

In view of Lemma 3.2, $E^*(\{0, T]\), S_\lambda) = \cap_{1 \leq j \leq 4}\Omega_j$. To prove the lemma we have to show that the latter set belongs to $B$.

We first show that $\Omega_1$ belongs to $B$. Let $F_j : E_T \rightarrow [0, T], j \geq 1$, be given by

$$F_j(x) = \int_0^T 1\{x(s) = j\} ds.$$ 

Since $\mathfrak{N}_k x$ increases to $x$, $F_j(x) = \inf_{\ell \geq j} F_{j, \ell}(x)$, where $F_{j, \ell}(x) = \int_{[0, T]} 1\{\mathfrak{N}_k x(s) = j\} ds$. Since each function $F_{j, \ell}$ is continuous for the soft topology, the function $F_j$ is $B$-measurable and so is $\Lambda_T = T - \sum_{j \geq 1} F_j$. This proves that $\Omega_1$ belongs to $B$.

We turn to $\Omega_2$. We show separately that $\Omega_2^{1, 2} = \{x \in E_T : x(T) = x(T^-) = \emptyset\}$ and $\Omega_2^{2, 2} = \{x \in E_T : x(T) = x(T^-) \in S\}$ belong to $B$. By definition of $E_T$, we may rewrite $\Omega_2^{2, 2}$ as $\{x \in E_T : x(T) = \emptyset\}$. Since $\mathfrak{N}_k x$ increases to $x$, $\Omega_2^{2, 1} = \bigcap_{m \geq 1} \Omega_2^{1, k \geq m} = \{x \in E_T : (\mathfrak{N}_k x)(T) = \ell\}$. For each $\ell \geq 1$, the set $\{x \in E_T : (\mathfrak{N}_k x)(T) = \ell\}$ is closed, which shows that $\Omega_2^{2, 1}$ belongs to $B$.

On the other hand, we claim that $\Omega_2^{2, 2} = \bigcup_{k \geq 1} \bigcap_{\ell \geq m} \Omega_2^{k, m, \ell}$, where $\Omega_2^{k, m, \ell} = \{x \in E_T : (\mathfrak{N}_k x)(t) = m, T - (1/k) \leq t \leq T\}$. Fix $x \in \Omega_2^{2, 2}$ and set $m = x(T)$. 


Since \( x(T^-) = m \), there exists \( k \geq 1 \) such that \( x(t) = m \) for \( T - (1/k) \leq t \leq T \). By definition of \( \mathcal{R}_t \), for all \( \ell \geq m \), \( \mathcal{R}_t(x)(t) = m \) for \( T - (1/k) \leq t \leq T \). Thus, \( \Omega^{k,m,\ell}_2 = \cup_{k \geq 1} \cup_{m \geq 1} \cap_{\ell \geq m} \Omega^{k,m,\ell}_2 \). Conversely, if \( x \) belongs to \( \cup_{k \geq 1} \cup_{m \geq 1} \cap_{\ell \geq m} \Omega^{k,m,\ell}_2 \), there exists \( k \geq 1 \) and \( m \geq 1 \) such that \( \mathcal{R}_t(x)(t) = m \) for \( T - (1/k) \leq t \leq T \) for all \( \ell \geq m \). Since \( \mathcal{R}_t(x) \) increases pointwise to \( x \), the same property holds for \( x \), which proves that \( \Omega^{k,m,\ell}_2 = \cup_{k \geq 1} \cup_{m \geq 1} \cap_{\ell \geq m} \Omega^{k,m,\ell}_2 \). As the sets \( \Omega^{k,m,\ell}_2 \) are closed, the set \( \Omega^{2,2}_2 \) belongs to \( \mathcal{B} \).

We claim that \( \Omega_3 = \cap_{m \geq 1} \cup_{\ell \geq 1} \cap_{t \geq m} \{ x \in E_T : \mathcal{R}_m(x) \leq k \} \). Denote the right hand side of the equality by \( \Omega_3' \) and fix \( x \in \Omega_3' \). We will show that the trajectory \( x \) has left and right limits at all points \( t \in [0, T] \). Since \( x \) belongs to \( E([0, T], S_\delta) \), it is enough to exclude the possibility that \( x \) has a finite soft limit at some point \( t \in [0, T] \). Fix \( m \geq 1 \) and recall Remark 2.11. As \( x \) belongs to \( \Omega_3' \), there exists \( k \geq 1 \) such that \( \mathcal{R}_m(x) \leq k \) for all \( \ell \geq m \). Since the sequence \( \mathcal{R}_m(x) \) increases with \( \ell \), it is constant for \( \ell \) large enough. Denote by \( [s^1_1, t^1_1), \ldots, [s^N_N, t^N_N) \) the \( N = \mathcal{R}_m(x) \) time-intervals in which \( \mathcal{R}_t(x) \) visits \( m \). If \( t^N_N = T \), the last time interval may be closed at \( T \). By Assertion \( \text{C} \), \( s^l_{i+1} = s^l_i \), \( 1 \leq i \leq N \), and \( t^l_{i+1} \leq t^l_i \). Since \( \mathcal{R}_t(x) \) converges pointwise to \( \mathcal{R}_\infty x \), the set \( \{ s \in [0, T] : x(s) = m \} \) is the union of \( N \) disjoint intervals, some of which can be reduced to a point. In particular, \( m \) can not be the finite soft limit of \( x \) at some point \( t \in [0, T] \). Since this holds for every \( m \), \( x \) does not have a left or a right finite soft limit at some \( t \in [0, T] \).

Conversely, fix a trajectory \( x \) which does not belong to \( \Omega_3' \). In this case, there exists \( m \geq 1 \) such that \( \mathcal{R}_m(x) \) increases to \( \infty \). By Assertion \( \text{C} \), the set \( A_m = \{ t \in [0, T] : x(t) = m \neq x(t^-) \} \) is countably infinite because it contains all the left end-points of the time intervals \( [s, t) \) in which \( \mathcal{R}_t(x) \) is constant equal to \( m \). Let \( t \) be an accumulation point of \( A_m \) and assume, without loss of generality, that there exists \( t_j \uparrow t \). Then, \( x(t_j) = m \) and there exist \( s_j \uparrow \infty \) such that \( x(s_j) \neq m \) for all \( j \). This proves that \( x \) has not a left limit at \( t \), proving that \( \Omega_3 = \Omega_3' \). Since the sets \( \{ x \in E_T : \mathcal{R}_m(x) \leq k \} \) are closed, \( \Omega_3 \) belongs to \( \mathcal{B} \).

Finally, consider the set \( \Omega_4 \). By definition of \( E_T = \{ x(t^-) = x(t) = \emptyset \} \) if \( x(t^-) = \emptyset \). The set \( \Omega_4 \) may therefore be rewritten as \( \{ x \in E_T : x(t^-) = x(t) = \emptyset, x(t) \in S_\delta \) for some \( t \in [0, T] \}. \) Let \( \Omega_4' = \cap_{m \geq 1} \cup_{\ell \geq 1} \cap_{t \geq m} \{ x \in E_T : T_{m,j}(x) \geq \{ 1/k \) for all \( 1 \leq j \leq \mathcal{R}_m(x) \}, \) \( \Omega_13 = \{ x(t) = m \} \) for \( x \in E_T \), \( \mathcal{B} \). We claim that \( \Omega_{13} \cap \Omega_4 = \Omega_13 \cap \Omega_4 \).

Consider a trajectory \( x \in \Omega_{13} \cap \Omega_4 \), \( m \geq 1 \), and recall the notation introduced when we proved that \( \Omega_3 \) belongs to \( \mathcal{B} \). For \( \ell \geq m \), \( [s^1_1, t^1_1), \ldots, [s^N_N, t^N_N) \) represent the \( N = \mathcal{R}_m(x) \) time-intervals in which \( \mathcal{R}_t(x) \) visits \( m \). Since \( T_{m,j}(x) \geq \{ 1/k \) for all \( 1 \leq j \leq \mathcal{R}_m(x) \}, \) \( x \) is equal to \( m \) in \( N \) intervals \( [s_j, t_j) \) of length at least \( 1/k \). Since \( s^l_i = s^l_{i+1} \), taking the limit \( \ell \uparrow \infty \) we obtain that there is no \( t \in [0, T] \) such that \( x(t+) = \emptyset \), \( x(t) = m \). Since this holds for every \( m \), \( x \) belongs to \( \Omega_{13} \cap \Omega_4 \).

Reciprocally, consider a trajectory \( x \) in \( \Omega_{13} \cap \Omega_4 \). By definition, there exist \( m \geq 1 \), a sequence \( \ell_k \) and an interval \( [s^{t_k}_{i}, t^{t_k}_{i}) \), \( 1 \leq i \leq \mathcal{R}_m(x) \), such that \( s^{t_k}_{i} - s^{t_k}_{i-1} \leq 1/k \). The number of intervals, \( \mathcal{R}_m(x), \) is eventually constant because \( x \) belongs to \( \Omega_3 \). There exists, in particular, an index \( i \) such that \( \lim_{k} (t^{t_k}_{i} - s^{t_k}_{i}) = 0 \). Let \( t = s^{t_k}_{i} \), a sequence which is constant in view of Assertion \( \text{C} \). Since \( t^{t_k}_{i} \downarrow t \), \( x(t+) \) exists and is equal to \( \emptyset \), \( x(t) = m \) and \( x(t-) \neq m \), which proves that \( x \) belongs to \( \Omega_{13} \cap \Omega_4 \). This shows that \( \Omega_{13} \cap \Omega_4 = \Omega_{13} \cap \Omega_4 \). Finally, since
\{x \in E_T : T_{m,j} (\mathcal{R}_kx) \geq (1/k) \text{ for all } 1 \leq j \leq \mathcal{N}_m(\mathcal{R}_kx)\} \text{ is a closed set, } \Omega_1 \cap \Omega_4 \text{ belongs to } \mathcal{B}, \text{ which concludes the proof of the lemma.} \]

Pushing further the arguments used in the proof of the previous lemma, we obtain in the next lemma sufficient conditions, all expressed only in terms of the trajectories \(\mathcal{R}_ix_n\), for the limit \(x\) of a sequence \(x_n\) in \(E^*(\[0,T],S_0)\) to belong to \(E^*(\[0,T],S_0)\).

**Lemma 3.4.** Let \(\{x_n : n \geq 1\}\) be a sequence in \(E^*(\[0,T],S_0)\) which converges to \(x \in E([0,T],S_0)\) in the metric \(d\). Assume that

(a) \[
\lim_{m \to \infty} \sup_{\ell \geq 1} \sup_{n \geq 1} \int_0^T 1\{\mathcal{R}_\ell x_n(s) \geq m\} \, ds = 0;
\]

(b) For each \(m \geq 1\), there exists \(k_m \in \mathbb{N}\) such that \(\mathcal{N}_m(\mathcal{R}_\ell x_n) \leq k_m\) for all \(\ell \geq m\) and \(n \geq 1\);

(c) For each \(m \geq 1\), there exists \(\epsilon_m > 0\) such that \(T_{m,k}(\mathcal{R}_\ell x_n) \geq \epsilon_m\) for all \(1 \leq k \leq \mathcal{N}_m(\mathcal{R}_\ell x_n), \ell \geq m\) and \(n \geq 1\);

Then, \(x\) belongs to \(E^*(\[0,T],S_0)\).

**Proof.** We need to prove that the trajectory \(x\) fulfills conditions (a)–(d) of Lemma 3.2. We first claim that \(\Lambda_T(x) = 0\). Fix \(\epsilon > 0\). By assumption (a), there exists \(m \geq 1\) such that

\[
\int_0^T 1\{\mathcal{R}_\ell x_n(s) \geq m\} \, ds \leq \epsilon
\]

for all \(n \geq 1, \ell \geq 1\). Fix \(\ell \geq m\). The sequence \(\mathcal{R}_\ell x_n\) converges almost everywhere to \(\mathcal{R}_\ell x\) because it converges in the Skorohod topology. Hence, since \(\mathcal{R}_\ell x_n\) takes values in a discrete set, by Fatou’s lemma,

\[
\int_0^T 1\{(\mathcal{R}_\ell x)(s) \geq m\} \, ds \leq \liminf_{n \to \infty} \int_0^T 1\{(\mathcal{R}_\ell x_n)(s) \geq m\} \, ds \leq \epsilon.
\]

Since \(\mathcal{R}_\ell x\) converges pointwisely to \(x\), by the dominated convergence theorem,

\[
\int_0^T 1\{x(s) \geq m\} \, ds \leq \epsilon,
\]

so that \(\Lambda_T(x) \leq \epsilon\).

We now show that \(x\) has left and right limits. Since \(x\) belongs to \(E([0,T],S_0)\) to prove this claim it is enough to exclude the possibility that \(x\) has a finite soft limit at some point \(t \in [0,T]\). Fix \(m \geq 1\). By assumptions (b) and (c) of this lemma, there exist \(k_m \geq 1\) and \(\epsilon_m > 0\) such that \(\mathcal{N}_m(\mathcal{R}_\ell x_n) \leq k_m\) and \(T_{m,k}(\mathcal{R}_\ell x_n) \geq \epsilon_m\) for all \(1 \leq k \leq \mathcal{N}_m(\mathcal{R}_\ell x_n), \ell \geq m, n \geq 1\). Since \(\mathcal{R}_\ell x_n\) converges in the Skorohod topology to \(\mathcal{R}_\ell x\), by Assertion [K], \(\mathcal{N}_m(\mathcal{R}_\ell x) \leq k_m\) and \(T_{m,k}(\mathcal{R}_\ell x) \geq \epsilon_m\) for all \(1 \leq k \leq \mathcal{N}_m(\mathcal{R}_\ell x), \ell \geq m\). As the sequence \(\mathcal{N}_m(\mathcal{R}_\ell x)\) increases with \(\ell\), it is constant for \(\ell\) large enough. Denote by \([s_1^\ell,t_1^\ell], \ldots, [s_N^\ell,t_N^\ell]\) the \(N = \mathcal{N}_m(\mathcal{R}_\ell x)\) time-intervals in which \(\mathcal{R}_\ell x\) visits \(m\). Since \(T_{m,k}(\mathcal{R}_\ell x) \geq \epsilon_m\) for all \(k, \ell \geq m\), \(t_i^\ell \geq s_i^\ell + \epsilon_m\). By Assertion [G] \(s_i^{i+1} = s_i^\ell, 1 \leq i \leq N,\) and \(t_i^{i+1} \leq t_i^\ell\). Since \(\mathcal{R}_\ell x\) converges pointwisely to \(\mathcal{R}_\ell x = x\), the set \(\{s \in [0,T] : x(s) = m\}\) is the union of \(N\) disjoint intervals of length greater or equal to \(\epsilon_m\), which are closed at the left boundary and open or closed at the right boundary. In particular, \(m\) cannot be the finite soft limit of \(x\) at some point \(t \in [0,T]\). Since this holds for every \(m, x\) does not have a
left or a right finite soft limit at some time \( t \in [0, T] \). This proves condition (a) of Lemma 3.2.

We turn to condition (b) of Lemma 3.2. Suppose that \( x(t+) = \partial \) for some \( t \in [0, T] \). If \( x(t) = \partial \), since \( x \in E([0, T], S_b) \) and \( \Lambda_T(x) = 0 \), \( \sigma_\partial(t) = t \) and, by definition of the set \( E([0, T], S_b) \), \( x(t-) = x(t) \). If \( x(t) = m \in S \), since \( x(t+) = \partial \), \( t \) is the right endpoint of an interval \([s_i, t_i]\) obtained as the limit of the intervals \([s'_i, t'_i]\) introduced in the previous paragraph. Since the interval is not degenerate, \( x(t-) = m = x(t) \), which proves condition (b) of Lemma 3.2.

We finally prove condition (c) of Lemma 3.2. Suppose that \( x(T) = k \in S \). In this case, since the set \( \{ s \in [0, T] : x(s) = k \} \) is the union of a finite number of disjoint intervals of positive length, \( x \) is continuous at \( T \). Suppose now that \( x(T) = \partial \). By Assertion 1, \( (\mathcal{R}_\ell x_n)(T) = (\mathcal{R}_\ell x_n)(T-) \) for all \( \ell \geq 1 \), \( n \geq 1 \). Since \( \mathcal{R}_\ell x_n \) converges to \( \mathcal{R}_\ell x \) in the Skorohod topology, the continuity at \( T \) is inherited by \( \mathcal{R}_\ell x \). Denote by \([a_\ell, T]\) the constancy interval of \( \mathcal{R}_\ell x \) and fix \( m \geq 1 \). Since \( x(T) = \partial \) and since \( (\mathcal{R}_\ell x)(T) \) converges to \( x(T) \), there exists \( \ell_0 \geq 1 \) such that for all \( \ell \geq \ell_0 \), \( (\mathcal{R}_\ell x)(T) \geq m \). By definition of \( a_\ell \) and since \( x \geq \mathcal{R}_\ell x \), for all \( a_\ell \leq t \leq T \), \( x(t) = (\mathcal{R}_\ell x)(t) = (\mathcal{R}_\ell x)(T) \geq m \). This proves that \( x(T-) = \partial = x(T) \). Condition (c) of Lemma 3.2 is therefore in force, which concludes the proof of the lemma.

Corollary 3.5. Let \( x \) be a trajectory in \( E([0, T], S_b) \) which satisfies conditions (b) and (c) of the previous lemma and such that \( \Lambda_T(x) = 0 \), \( (\mathcal{R}_\ell x)(T) = (\mathcal{R}_\ell x)(T-) \) for all \( \ell \geq 1 \). Then, \( x \) belongs to \( E^*(\Omega, [0, T], S_b) \).

Proof. By the proof of Lemma 3.4, \( x \) satisfies conditions (a)–(c) of Lemma 3.2. Since condition (d) of this assertion holds by assumption, the corollary is proved.

4. Weak Convergence of Probability Measures.

We examine in this section the weak convergence of probability measures on \( E([0, T], S_b) \). Fix \( m \geq 1 \) and consider a sequence \( x_n \) in \( D([0, T], S_m) \) converging to \( x \) in the Skorohod topology. Then, \( x_n \) converges to \( x \) in \( E([0, T], S_b) \). Indeed,

\[
d(x_n, x) = \sum_{\ell \geq 1} \frac{1}{2^\ell} d_S(\mathcal{R}_\ell x_n, \mathcal{R}_\ell x)
\]

\[
= \frac{1}{2^m} d_S(x_n, x) + \sum_{\ell = 1}^m \frac{1}{2^\ell} d_S(\mathcal{R}_\ell x_n, \mathcal{R}_\ell x).
\]

By hypothesis and by Lemma 2.9, this sum vanishes as \( n \uparrow \infty \).

Let \( F : E([0, T], S_b) \rightarrow \mathbb{R} \) be a continuous function for the soft topology. Then, its restriction to \( D([0, T], S_m) \), \( m \geq 1 \), is continuous for the Skorohod topology. Indeed, consider a sequence \( x_n \) converging in \( D([0, T], S_m) \) to \( x \). By the previous paragraph, \( x_n \) converges to \( x \) in the soft topology of \( E([0, T], S_b) \). Since \( F \) is continuous in this topology, \( F(x_n) \) converges to \( F(x) \).

Theorem 4.1. A sequence of probability measures \( P_n \) on \( E([0, T], S_b) \) converges weakly in the soft topology to a measure \( P \) if and only if for each \( m \geq 1 \) the sequence of probability measures \( P_n \circ \mathcal{R}_m^{-1} \) defined on \( D([0, T], S_m) \) converges weakly to \( P \circ \mathcal{R}_m^{-1} \) with respect to the Skorohod topology.

Proof. Suppose that the sequence \( P_n \) converges weakly to \( P \) and fix \( m \geq 1 \). Since \( \mathcal{R}_m : E([0, T], S_b) \rightarrow D([0, T], S_m) \) is continuous for the soft topology, \( P_n \circ \mathcal{R}_m^{-1} \) converges weakly to \( P \circ \mathcal{R}_m^{-1} \).
Conversely, suppose that $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$ for every $m \geq 1$. Fix a bounded, uniformly continuous function $F : E([0, T], S_B) \to \mathbb{R}$ and $\epsilon > 0$. Since $F$ is uniformly continuous, there exists $\delta > 0$ such that $|F(y) - F(x)| \leq \epsilon$ if $d(x, y) \leq \delta$. Let $m \geq 1$ such that $2^{-(m-1)} < \delta$. Since $d(x, \mathfrak{R}m) \leq 2^{-(m-1)} < \delta$, the difference $E_{P_n}[F(x)] - E_{P_n}[F(\mathfrak{R}m)]$ is absolutely bounded by $\epsilon$, uniformly in $n$. A similar estimate holds for $P$ replacing $P_n$.

We have shown right before the statement of the theorem that $F : D([0, T], S_m) \to \mathbb{R}$ is continuous for the Skorohod topology. As $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$ in the Skorohod topology, and since $F$ is bounded and continuous, there exists $n_0$ such that for all $n \geq n_0$, $|E_{P_n}[F(\mathfrak{R}m)] - E_P[F(\mathfrak{R}m)]| \leq \epsilon$. Putting together the previous estimates we conclude that for all $n \geq n_0$,

$$|E_{P_n}[F(x)] - E_P[F(x)]| \leq 3\epsilon,$$

which concludes the proof of the theorem.

**Theorem 4.2.** Let $\{P_n : n \geq 1\}$ be a sequence of probability measures on the path space $E^*([0, T], S_B)$ which converges weakly to a measure $P$ in $E([0, T], S_B)$ endowed with the soft topology. Assume that

(a) $$\lim_{m \to \infty} \limsup_{n \to \infty} \limsup_{t \to \infty} E_{P_n} \left[ \int_0^T \mathbf{1}\{\mathfrak{R}_m(s) \geq m\} \, ds \right] = 0;$$

(b) For each $m \geq 1$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} P_n \left[ \mathfrak{R}_m(\mathfrak{R}_m) \geq k \right] = 0;$$

(c) For each $m \geq 1$,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \limsup_{t \to \infty} P_n \left[ \bigcup_{k=1}^{\mathfrak{R}_m(\mathfrak{R}_m)} \mathcal{T}_{m,k}(\mathfrak{R}_m) < \epsilon \right] = 0.$$

Then, $P$ is concentrated on $E^*([0, T], S_B)$.

**Proof.** It is not difficult to show that for each $m \leq \ell$ the map $y \to \int_0^T \mathbf{1}\{y(s) \geq m\} \, ds$ is continuous in $D([0, T], S)$, and therefore, the map $y \to \int_0^T \mathbf{1}\{\mathfrak{R}_m(y)(s) \geq m\} \, ds$ is bounded and continuous in $E([0, T], S_B)$. By assumption (a), given $\epsilon > 0$, there exists $m_0$ such that for all $m \geq m_0$,

$$\limsup_{t \to \infty} E_P \left[ \int_0^T \mathbf{1}\{\mathfrak{R}_m(s) \geq m\} \, ds \right] \leq \epsilon.$$

Since $\mathfrak{R}_m$ increases pointwisely to $\mathfrak{R}_m$, by the monotone convergence theorem,

$$E_P[\lambda_T(x)] \leq E_P \left[ \int_0^T \mathbf{1}\{x(s) \geq m\} \, ds \right] \leq \epsilon.$$

Letting $\epsilon \downarrow 0$, we conclude that $E_P[\lambda_T(x)] = 0$, i.e., that

$$P[\lambda_T(x) = 0] = 1. \quad (4.1)$$

By Assertion [K] the functionals $\mathfrak{R}_m$, $m \geq 1$, are continuous for the Skorohod topology. The sets $\{x \in D([0, T], S_T) : \mathfrak{R}_m(x) \geq k\} = \{x \in D([0, T], S_T) : \mathfrak{R}_m(x) \leq k\}$
Since this equation holds for every $m \geq 1$,} 
\[
\lim_{k \to \infty} \lim_{\ell \to \infty} P \left[ \mathcal{N}_m(\mathcal{R}_\ell x) \geq k \right] = 0 . \tag{4.2}
\]

As $\mathcal{N}_m(\mathcal{R}_\ell x)$ is a non-decreasing sequence in $\ell$, the set $\{\mathcal{N}_m(\mathcal{R}_\ell x) \geq k\}$ is contained in $\{\mathcal{N}_m(\mathcal{R}_{\ell+1} x) \geq k\}$. Thus, for every $m \geq 1$, 
\[
P \left[ \bigcap_{k \geq 1} \bigcup_{\ell \geq m} \{\mathcal{N}_m(\mathcal{R}_\ell x) \geq k\} \right] = \lim_{k \to \infty} \lim_{\ell \to \infty} P \left[ \mathcal{N}_m(\mathcal{R}_\ell x) \geq k \right] = 0 ,
\]

where the last equality follows from (4.2). Since this identity holds for every $m \geq 1$, 
\[
P \left[ \bigcap_{m \geq 1} \bigcup_{k \geq 1} \bigcap_{\ell \geq m} \{\mathcal{N}_m(\mathcal{R}_\ell x) \leq k\} \right] = 1 . \tag{4.3}
\]

A straightforward modification of the proof of Assertion $K$ shows that for every $\ell \geq m$, the set $\bigcap_{k=1}^{\mathcal{N}_m(\mathcal{R}_\ell x)} \{T_{m,k}(\mathcal{R}_\ell x) < \epsilon\}$ is closed in $D([0,T],S_\ell)$. Therefore, by assumption (c), 
\[
\lim_{\epsilon \to 0} \lim_{\ell \to \infty} P \left[ \bigcup_{k=1}^{\mathcal{N}_m(\mathcal{R}_\ell x)} \{T_{m,k}(\mathcal{R}_\ell x) < \epsilon\} \right] = 0 .
\]

Since the duration of the visits to a point $m$ may only decrease as $\ell$ increases, 
\[
\bigcup_{k=1}^{\mathcal{N}_m(\mathcal{R}_\ell x)} \{T_{m,k}(\mathcal{R}_\ell x) < \epsilon\} \subset \bigcup_{k=1}^{\mathcal{N}_m(\mathcal{R}_{\ell+1} x)} \{T_{m,k}(\mathcal{R}_{\ell+1} x) < \epsilon\} .
\]

In particular, by the previous displayed equation, 
\[
P \left[ \bigcap_{j \geq 1} \bigcup_{\ell \geq m} \bigcup_{k=1}^{\mathcal{N}_m(\mathcal{R}_\ell x)} \{T_{m,k}(\mathcal{R}_\ell x) < \frac{1}{j}\} \right] = 0 .
\]

Since this equation holds for every $m \geq 1$, we conclude that 
\[
P \left[ \bigcap_{m \geq 1} \bigcup_{j \geq 1} \bigcup_{\ell \geq m} \bigcup_{k=1}^{\mathcal{N}_m(\mathcal{R}_\ell x)} \{T_{m,k}(\mathcal{R}_\ell x) \geq \frac{1}{j}\} \right] = 1 . \tag{4.4}
\]

Since the measure $P_n$ is concentrated on $E^*([0,T],S_\ell)$, by Assertion $J$, for every $\ell$, 
\[
P_n \left( (\mathcal{R}_\ell x)(T) = (\mathcal{R}_\ell x)(T^-) \right) = 1 .
\]

As the set $\{x \in D([0,T],S_\ell) : x(T) = x(T^-)\}$ is closed for the Skorohod topology, by Theorem 4.1, for every $\ell \geq 1$, 
\[
P \left[ (\mathcal{R}_\ell x)(T) = (\mathcal{R}_\ell x)(T^-) \right] = 1 ,
\]

so that 
\[
P \left[ \bigcap_{\ell \geq 1} \{ (\mathcal{R}_\ell x)(T) = (\mathcal{R}_\ell x)(T^-) \} \right] = 1 . \tag{4.5}
\]

Denote by $A$ the intersection of the events with full measure appearing in (4.1), (4.3), (4.4), (4.5). By Corollary 3.5, any trajectory in $A$ belongs to $E^*([0,T],S_\ell)$. This proves the theorem. $\square$

In view of condition (b), to prove condition (c) of Theorem 4.2, it is enough to show that for each $k, m \geq 1$, 
\[
\lim_{\epsilon \to 0} \lim_{\ell \to \infty} \lim_{n \to \infty} P_n \left[ T_{m,k}(\mathcal{R}_\ell x) < \epsilon \right] = 0 . \tag{4.6}
\]
5. Applications

In this section, we apply Theorems 4.1 and 4.2 to prove the convergence in the soft topology of the order parameter $X_N(t)$ to a Markov chain $X(t)$ in the two models presented in the introduction. We examine first the case of a finite number of wells. Recall the set-up introduced in Subsection 1.4.

**Theorem 5.1.** Let $\nu_N$ be a sequence of probability measures on $E_N$ such that $\nu_N \circ \Psi^{-1}$ converges to a probability measure $\nu$ on $S_L$, and let $\theta_N$ be a sequence of real positive numbers. Assume that the sequence of probability measures $P_{\nu_N} \circ (X^T_N)^{-1}$, defined on $D([0, T], S_L)$, converges in the Skorohod topology to a measure $P_\nu$ which corresponds to a $S_L$-valued continuous-time Markov chain $X(t)$ starting from $\nu$. Suppose, furthermore, that in the time scale $\theta_N$ the original process $\eta_N(t)$ spends a negligible amount of time in $\Delta_N$:

$$\lim_{N \to \infty} \mathbb{E}_{\nu_N} \left[ \int_0^T 1\{\eta(s \theta_N) \in \Delta_N\} \, ds \right] = 0.$$  

Then, the sequence of probability measures $P_{\nu_N} \circ X_N^{-1}$ converges in the soft topology to $P_\nu$.

**Proof.** Since the sequence of probability measures $P_{\nu_N} \circ (X^T_N)^{-1}$ converges in the Skorohod topology to the measure $P_\nu$, and since in the time scale $\theta_N$ the time spent by the chain $\eta_N(t)$ in the set $\Delta_N$ is negligible, by Proposition 4.3 in [1], the sequence of measures $P_{\nu_N} \circ (X^T_N)^{-1}$ also converges in the Skorohod topology to $P_\nu$.

Denote by $P_N$ the probability measure on $D([0, T], S_L \cup \{N\})$ induced by the process $X_N(t) = \Psi(\eta(t \theta_N))$ starting from $\nu_N$. Since, by Assertion [II], $D([0, T], S_L \cup \{N\})$, $D([0, T], S_L)$ are closed subsets of $E([0, T], S)$, we may extend $P_N$ to $E([0, T], S_N)$. Note that $P_{\nu_N} \circ (X^T_N)^{-1} = P_N \circ R_L^{-1}$. By the previous paragraph, $P_N \circ R_L^{-1} \to P_\nu$ in the Skorohod topology. Since $P_N \circ R_m^{-1} = P_N \circ R_L^{-1}$ for $L \leq m \leq N$, $P_N \circ R_m^{-1} \to P_\nu = P_\nu \circ R_m^{-1}$ for $m \geq L$ in the Skorohod topology. On the other hand, for $1 \leq m < L$, by Lemma [2.9], $P_N \circ R_m^{-1} \to P_\nu \circ R_m^{-1}$ in the Skorohod topology.

In conclusion, $P_N \circ R_m^{-1} \to P_\nu \circ R_m^{-1}$ in the Skorohod topology for all $m \geq 1$. Therefore, by Theorem 4.1, $P_N$ converges to $P_\nu$ in the soft topology, a probability measure concentrated on the closed subset $D([0, T], S_L)$.

We may apply Theorem 5.1 to the zero-range processes presented in the introduction. Fix $x \in S_L$ and a sequence of configurations $\eta^N$ in $E_N$. We proved in [2, 14] that starting from $\Psi_N(\eta^N)$ the process $X_N^T(t)$ in the time scale $\theta_N = N^{1+\alpha}$ converges in the Skorohod topology to a Markov chain $X(t)$ and that the time spent by $\eta^N(t)$ in the set $\Delta_N$ in the time scale $N^{1+\alpha}$ is negligible. Therefore, by Theorem 5.1, the rescaled order parameter $X_N(t)$ converges in the soft topology to $X(t)$.

Consider now the random walk among the random traps on $T_N^d$. It follows from the results proved in [11, 12] that the trace of $X_N(t)$ on the set $\{1, \ldots, L_N\}$, denoted by $X^N_N(t)$, where $L_N$ is a sequence which increases slowly to $\infty$, converges in the Skorohod topology to a $K$-process $X(t)$, and that the time spent by $\eta^N(t)$ in the set $\Delta_N = \{L_N + 1, \ldots, V_N\}$ in the time scale $\theta_N = N^{-1}$ is negligible. By Proposition 4.3 in [1], the process which records the last visit of $X_N(t)$ to $\{1, \ldots, L_N\}$, denoted by $X^N_N(t)$, also converges in the Skorohod topology to $X(t)$.

Fix $j \in \mathbb{N}$ and denote by $P_N$ the probability measure on $D([0, T], S_{V_N})$ induced by the process $X_N(t) = \Psi(\eta(t \theta_N))$ starting from $j$. By the previous paragraph,
$P_N \circ \mathcal{R}^{-1}_{L_N} \to P$ in the Skorohod topology, where $P$ is the probability measure on $D([0,T], S_\delta)$ which corresponds to a $K$-process starting from $j$. By Lemma 2.9 $P_N \circ \mathcal{R}^{-1}_m \to P \circ \mathcal{R}^{-1}_m$ in the Skorohod topology for every $m \geq 1$. Therefore, there exists a probability measure $P$ on the path space $E([0,T], S_\delta)$, concentrated on the subset $D([0,T], S_\delta)$, such that $P_N \circ \mathcal{R}^{-1}_m \to P \circ \mathcal{R}^{-1}_m$ in the Skorohod topology for all $m \geq 1$. By Theorem 4.1 $P_N$ converges to $P$ in the soft topology.

We conclude this section showing that the conditions (a)–(c) of Theorem 4.2 follow from the convergence of the order parameter to a Markov process and from the fact that asymptotically the process spends a negligible amount of time on $\Delta_N$.

1. Random walks among traps. Consider the random walk among traps $\eta(t) = \eta^N(t)$ defined in the introduction, and recall that we denoted by $\pi_N$ the stationary state. Fix $T > 0$ and denote by $Q_k^N$, $k \geq 1$, the probability measure on $D([0,T], S_\delta)$ induced by the random walk $\mathbb{X}_N(t) = \Psi_N(\eta(\theta_N t))$ starting from $k$. Note that time has been speeded-up by $\theta_N = v_N^{-1}$, where $v_N$, defined in (1.1), is the probability to escape from a ball of radius $\ell_N$, and that the measure $Q_k^N$ is concentrated on the set $D([0,T], S_{\delta_N})$.

It is clear from this last observation that $\Lambda_T(x) = 0$, $Q_k^N$–almost surely. On the other hand, if we denote by $\tau_j$, $j \geq 1$, the holding times of the trajectory $x(t)$, $x(t)$ is discontinuous at $T$ if and only if $\tau_1 + \cdots + \tau_j = T$ for some $j$. Since, $Q_k^N[\tau_1 + \cdots + \tau_j = T] = 0$ for each $j \geq 1$, $Q_k^N$ is concentrated on the set $D^*(([0,T], S_\delta))$.

Denote by $P_N$ the probability measure on $E([0,T], S_\delta)$ defined by $P_N = Q_k^N \circ \mathcal{R}_\infty^{-1}$. By the last observation, $P_N$ is concentrated on $E^*([0,T], S_\delta)$. We claim that the sequence $P_N$ fulfills all the assumptions of Theorem 4.2. We start with assumption (a). Since $\mathcal{R}_k x \leq x$, it is enough to show that

$$\lim_{m \to \infty} \limsup_{N \to \infty} E_{P_N} \left[ \int_0^T \mathbf{1}\{x(s) \geq m\} \, ds \right] = 0 .$$  \hspace{1cm} (5.1)

By definition of $P_N$,

$$E_{P_N} \left[ \int_0^T \mathbf{1}\{x(s) \geq m\} \, ds \right] = E_{Q_k^N} \left[ \int_0^T \mathbf{1}\{x(s) \geq m\} \, ds \right] \leq \frac{1}{\pi_N(k)} \sum_{j \geq 1} \pi_N(j) E_{Q_j^N} \left[ \int_0^T \mathbf{1}\{x(s) \geq m\} \, ds \right].$$

Since $\pi_N$ is the stationary state, the previous sum is equal to $T \pi_N\{S_m\}$, where, we recall, $S_m = \{1, \ldots, m\}$. As, for every $k \geq 1$, $\lim_{m \to \infty} \limsup_{N \to \infty} \frac{\pi_N\{S_m\}}{\pi_N(k)} = 0$, condition (5.1) is in force.

We first prove Conditions (b) and (c) of Theorem 4.2 under the assumption that $\theta := \sup_{N \geq 1} \theta_N$ is finite. This is the case of the random walk on a torus $T_N^d$ in dimension $d \geq 3$.

Since $\mathcal{R}_m(\mathcal{R}_\ell x) \leq \mathcal{R}_m(x)$, $\ell \geq 1$, to prove condition (b) of Theorem 4.2 it is enough to show that for each $m \geq 1$,

$$\lim_{j \to \infty} \limsup_{N \to \infty} P_N \left[ \mathcal{R}_m(x) \geq j \right] = 0 .$$  \hspace{1cm} (5.2)
The above probability is equal to $\mathbb{Q}_k^N[\mathcal{R}_m(x) \geq j]$. Denote by $\tau^m_i$, $i \geq 1$, the holding times at $m$. This is a sequence of i.i.d. mean $\theta_N^{-1} W_m$ exponential random variables. Since $\{\mathcal{R}_m(x) \geq j\} \subseteq \{\tau^m_1 + \cdots + \tau^m_j \leq T\}$, the previous probability is bounded by $\mathbb{Q}_k^N[\tau^m_1 + \cdots + \tau^m_j \leq T] \leq P[\tau_1 + \cdots + \tau_j \leq T]$, where $T_i$, $i \geq 1$, is a sequence of i.i.d. mean $\theta^{-1} W_m$ exponential random variables. This expression vanishes as $j \uparrow \infty$, which proves (5.2).

In view of (4.6) and since $T_{m,j}(\mathcal{R}x)$, $j \geq 1$, are identically distributed, to prove condition (c) of Theorem 4.2 we need to show that for each $m \geq 1$,

$$
\lim_{\epsilon \to 0} \lim_{N \to \infty} \limsup_{\ell \to \infty} \limsup_{N \to \infty} P_N \left[ T_{m,1}(\mathcal{R}x) < \epsilon \right] = 0 .
$$

Since $T_{m,1}(\mathcal{R}x) \geq T_{m,1}(x)$, $\ell \geq m \geq 1$, to prove condition (c) of Theorem 4.2 we just have to show that for each $m \geq 1$,

$$
\lim_{\epsilon \to 0} \limsup_{N \to \infty} P_N \left[ T_{m,1}(x) < \epsilon \right] = 0 .
$$

With the notation introduced in the previous paragraph, the probability above is equal to $\mathbb{Q}_k^N[\tau^N_i < \epsilon]$. As $\tau^N_i$ is a mean $\theta_N^{-1} W_m$ exponential random variable and as $\theta_N \leq \theta$, the previous probability is less than or equal to $P[\tau < \epsilon]$, where $\tau$ is a mean $\theta^{-1} W_m$ exponential random variable. This proves condition (c) of Theorem 4.2 in the case where $\sup_N \theta_N < \infty$.

We conclude the analysis proving conditions (b) and (c) of Theorem 4.2 without the assumption that $\sup_N \theta_N < \infty$. Recall that we denote by $A_N$ the set of the first $M_N$ deepest traps, $A_N = \{x_1^N, \ldots, x_{M_N}^N\}$. Choose a sequence $M_N$ so that $M_N^2 \ell_N^d \ll N^d$. In this case with a probability converging to one the balls $B(x_i^N, \ell_N)$, $1 \leq i \leq M_N$, are disjoints. Let $U_1^N$ be the time of the first visit to $A_N$, $U_1^N = \inf\{t \geq 0 : \eta(t) \in A_N\}$, and define recursively the sequence of stopping times $U_j^N$, $j \geq 1$, by

$$
U_{j+1}^N = \inf \left\{ t \geq U_j^N : \eta(t) \in A_N, \exists U_j^N \leq s \leq t \text{ s.t. } \eta(s) \notin B_N \right\},
$$

where $B_N = \bigcup_{i=1}^{M_N} B(x_i^N, \ell_N)$. Hence, the sequence $U_j^N$ represents the successive visits to the deepest traps after escaping from these traps. We refer to the time interval $[U_j^N, U_{j+1}^N]$ as the $j$-th excursion.

For $m \geq 1$, let $e_1(m) = \min\{j \geq 1 : \eta(U_j^N) = x_m^N\}$ be the first excursion to the trap $x_m^N$. Define recursively $e_i(m)$, $i \geq 1$, by

$$
e_{i+1}(m) = \min\{j > e_i(m) : \eta(U_j^N) = x_m^N\} .
$$

Note that we may have $e_{i+1}(m) = e_i(m) + 1$, as the process may escape from the trap $x_m^N$ and then return to it before visiting any other deep trap. We refer to $[U_{e_i(m)}^N, U_{e_i(m)+1}^N]$ as the $i$-th excursion to $x_m^N$.

Let $G_i^N$, $i \geq 1$, be the number of visits to $x_m^N$ during the $i$-th excursion to $x_m^N$, in other words, $G_i^N$ is the number of visits to $x_m^N$ in the time interval $[U_{e_i(m)}^N, U_{e_i(m)+1}^N]$. The random variables $G_i^N$, $i \geq 1$, are i.i.d. and have a mean $\theta_N$ geometric distribution. Let $T_{i,p}$, $1 \leq p \leq G_i^N$, be the $p$-th holding time at $x_m^N$ in the time interval $[U_{e_i(m)}^N, U_{e_i(m)+1}^N]$. The random variables $T_{i,p}$ are i.i.d. and have a mean $W_m/\theta_N$ exponential distribution.

Fix $N$ large enough for $M_N \geq \ell$ so that $\mathcal{R}_m(\mathcal{R}x) \leq \mathcal{R}_m(\mathcal{R}_{M_N}x)$. In this case, since for the trajectory $\mathcal{R}_{M_N}x$ all visits of $\eta(t)$ to $x_m^N$ in the time interval
It follows from the conclusion of the last paragraph that \( \sum_{1 \leq p \leq G^N_i} T_{i,p}^N \), \( i \geq 1 \), forms a sequence of i.i.d. mean \( W_m \) exponential random variables. This proves condition (b) of Theorem 4.2.

In view of (4.6) and since \( T_{m,j}(\mathfrak{R}_m(x)) \), \( j \geq 1 \), are identically distributed, to prove condition (c) of Theorem 4.2 we need to show that for each \( m \geq 1 \),

\[
\lim_{\epsilon \to 0} \lim\sup_{N \to \infty} \sup_{t \in \Delta_N} P_N \left[ T_{m,1}(\mathfrak{R}_m(x)) < \epsilon \right] = 0.
\]

Since \( T_{m,1}(\mathfrak{R}_m(x)) \geq T_{m,1}(\mathfrak{R}_{M_N}(x)) \), it is in fact enough to show that for each \( m \geq 1 \),

\[
\lim_{\epsilon \to 0} \lim\sup_{N \to \infty} P_N \left[ T_{m,1}(\mathfrak{R}_{M_N}(x)) < \epsilon \right] = 0.
\]

This probability is equal to \( Q^N_{\eta^N} \left[ T_{m,1}(\mathfrak{R}_{M_N}(x)) < \epsilon \right] \) and \( T_{m,1}(\mathfrak{R}_{M_N}(x)) \geq \sum_{p=1}^{G^N_i} T_{i,p}^N \), a mean \( W_m \) exponential random variable. Therefore,

\[
P_N \left[ T_{m,1}(\mathfrak{R}_{M_N}(x)) < \epsilon \right] \leq P[T < \epsilon],
\]

where \( T \) is a mean \( W_m \) exponential random variable, which proves condition (c) of Theorem 4.2.

2. Zero-range processes. Consider the zero-range process \( \eta(t) = \eta^N(t) \) introduced in Section 1. Denote by \( \mathbf{P}_\eta, \eta \in E_{L,N} \), the probability measure on the path space \( D(\mathbb{R}_+, E_{L,N}) \) induced by the Markov chain \( \eta(t) \) starting from \( \eta \). Expectation with respect to \( \mathbf{P}_\eta \) is denoted by \( \mathbf{E}_\eta \).

Fix \( T > 0, 1 \leq x \leq L \), and a sequence \( \{ \eta^N : N \geq 1 \} \) of configurations in \( E_N^x \), \( \eta^N \in E_N^x \). Denote by \( Q^N \) the probability measure on \( D([0,T], S_0) \) induced by the process \( \mathbf{X}_N(t) = \mathbf{X}_N(\eta(N^{1+\alpha} t)) \) starting from \( \eta^N \). Note that time has been speeded-up by \( N^{1+\alpha} \) and that the measure \( Q^N \) is concentrated on the set \( D([0,T], S_0) \).

It is clear from this last observation that \( \Lambda_T(x) = 0, Q^N \)-almost surely. On the other hand, if we denote by \( \tau_j, \tau_j^N, j \geq 1 \), the holding times of the processes \( \mathbf{X}_N(t), \eta(N^{1+\alpha} t) \), respectively, \( \mathbf{X}_N(t) \) is discontinuous at \( T \) if and only if \( \tau_1 + \cdots + \tau_j = T \) for some \( j \). Since \( \tau_1 + \cdots + \tau_j = \tau_1\eta^N + \cdots + \tau_k\eta^N \) for some \( k \geq j \) and since \( \mathbf{P}_\eta \{ \tau_1\eta^N + \cdots + \tau_k\eta^N = T \} = 0 \) for all \( k \geq 1 \), we have that \( Q^N \{ \tau_1 + \cdots + \tau_j = T \} = 0 \) for each \( j \geq 1 \). Therefore, \( Q^N \) is concentrated on the set \( D^*([0,T], S_0) \).

Denote by \( P_N \) the probability measure on \( E([0,T], S_0) \) defined by \( P_N = Q^N \circ \mathfrak{R}_N^{-1} \). By the last observation, \( P_N \) is concentrated on \( E^*([0,T], S_0) \). We claim that the sequence \( P_N \) fulfills all the assumptions of Theorem 4.2. We start with assumption (a). As in the previous example, it is enough to show that (5.1) holds. By definition of \( P_N \), for \( N \geq m \geq L \),

\[
E_{P_N} \left[ \int_0^T 1 \{ x(s) \geq m \} \, ds \right] = E_{Q^N} \left[ \int_0^T 1 \{ x(s) \geq m \} \, ds \right] = E_{\eta^N} \left[ \int_0^T 1 \{ \eta(s N^{\alpha+1}) \in \Delta_N \} \, ds \right].
\]

Therefore, (5.1) follows from assertion (M3) of [2] Theorem 2.4.
We turn to condition (b) of Theorem 4.2. As in the example of random walks among traps, it is enough to prove (5.2). Let $E_N = \mathcal{E}_N^1 \cup \cdots \cup \mathcal{E}_N^L$. Denote by $T_j$, $j \geq 1$, the holding times between successive visits to the metastable sets: $T_1 = \inf\{t > 0 : \eta(t) \in \mathcal{E}_N\},$

$$T_{j+1} = \inf\{t > 0 : \eta(T_j + t) \in \mathcal{E}_N \setminus \mathcal{E}_N^j\}, \quad T_j = T_1 + \cdots + T_j, \quad j \geq 1.$$

where $y_j = \Psi(\eta(T_j))$. Denote by $T^\epsilon_j$, $j \geq 1$, the same sequence for the trace process $\eta^T(t), T^\epsilon_1 = \inf\{t > 0 : \eta^T(t) \in \mathcal{E}_N\}$.

For $1 \leq k \leq L$, let $e_1(k) = \min\{j \geq 1 : \eta(T_j) \in \mathcal{E}_N^k\}$ be the first visit to the metastable set $\mathcal{E}_N^k$. Define recursively $e_i(k)$, $i \geq 1$, by

$$e_{i+1}(k) = \min\{j > e_i(k) : \eta(T_j) \in \mathcal{E}_N^k\}.$$

It is clear that $T^\epsilon_j \leq T_j$, $j \geq 1$, and that $\{\mathcal{M}_k(\mathcal{X}_N) \geq j\} \subset \{T_{e_1(k)} + \cdots + T_{e_j(k)} \leq T\} \subset \{T^\epsilon_{e_1(k)} + \cdots + T^\epsilon_{e_j(k)} \leq T\}$. Since the sequence $T^\epsilon_{e_1(k)}$ represents the holding times at $k$ for the process $\mathcal{X}_N^k(t) = \Psi_N(\eta^T(N^{1+\alpha}t))$, and since the process $\mathcal{X}_N^k(t)$ converges in the Skorohod topology to a Markov process on $\{1, \ldots, L\}$,

$$\limsup_{N \to \infty} P_{\eta_N}[T^\epsilon_{e_1(k)} + \cdots + T^\epsilon_{e_j(k)} \leq T] \leq P[S_1 + \cdots + S_j \leq T],$$

where $S_i, i \geq 1$, is a sequence of non-degenerate i.i.d. exponential random variables. As $j \to \infty$, this expression vanishes, which proves (5.2).

It remains to prove assertion (c) of Theorem 4.2. As argued in the previous example, it is enough to show that (5.3) holds for every $m \geq 1$. With the notation introduced above, it means that we have to show that

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} P_{\eta_N}[T^\epsilon_{e_1(m)} < \epsilon] = 0.$$

Since $T^\epsilon_{e_1(m)} \leq T^\epsilon_{e_1(m)}$, it is enough to prove the previous assertion with $T^\epsilon_{e_1(m)}$ replacing $T^\epsilon_{e_1(m)}$. This follows from the convergence of $T^\epsilon_{e_1(m)}$ to a non-degenerate exponential distribution.

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