COMPUTING WITH JETS

FEDERICO GALETTO AND NICHOLAS P. IAMMARINO

Abstract. We introduce a Macaulay2 package for working with jet schemes. The main method constructs jets of ideals, polynomial rings and their quotients, ring homomorphisms, affine varieties, and (hyper)graphs. The package also includes additional methods to compute principal components and radicals of jets of monomial ideals.

1. Introduction

Roughly speaking, the scheme of s-jets of a scheme X is the collection of order s Taylor approximations at points of X. More formally, let X be a scheme over a field k. Following [EM09, §2], we call a scheme J_s(X) over k the scheme of s-jets of X, if for every k-algebra A there is a functorial bijection

\[ \text{Hom}(\text{Spec}(A), J_s(X)) \cong \text{Hom}(\text{Spec}(A[t]/(t^{s+1})), X). \]

This means that the A-points of J_s(X) are in bijection with the A[t]/(t^{s+1})-points of X. It follows that J_0(X) \cong X, and J_1(X) is the total tangent scheme of X, in line with the definition of tangent space using dual numbers [Har77, II, Exercise 2.8]. Jet schemes play an important role in the study of singularities, as initially suggested by J. Nash [Nas95], and in connection with other related topics, such as motivic integration and birational geometry [DL01, Mus01, Mus02, EM09].

The existence of jet schemes is proved in detail in [EM09, §2]. We recall an essential step, which is the construction of jets of an affine variety. Let X be an affine variety over k. Consider a closed embedding of X into an affine space A^n over k. Let I = \langle f_1, \ldots, f_r \rangle be the ideal of \( R = k[x_1, \ldots, x_n] \) corresponding to this embedding. For s \in \mathbb{N}, define the polynomial ring

\[ J_s(R) = k[x_{i,j} \mid i = 1, \ldots, n, j = 0, \ldots, s]. \]

For each k = 1, \ldots, n, perform the substitution

\[ x_k \mapsto x_{k,0} + x_{k,1}t + x_{k,2}t^2 + \cdots + x_{k,s}t^s = \sum_{j=0}^{s} x_{k,j}t^j \]

taking elements of R to elements of \( J_s(R)[t] \). Applying this substitution to a generator \( f_i \) of I gives the following decomposition:

\[ f_i \left( \sum_{j=0}^{s} x_{1,j}t^j, \ldots, \sum_{j=0}^{s} x_{n,j}t^j \right) = \sum_{j \geq 0} f_{i,j}t^j, \]

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where the coefficients $f_{i,j}$ are polynomials in $\mathcal{J}_s(R)$. The ideal of $s$-jets of $I = \langle f_1, \ldots, f_r \rangle$ is the ideal of $\mathcal{J}_s(R)$ defined by

$$\mathcal{J}_s(I) = \langle f_{i,j} \mid i = 1, \ldots, r, j = 0, \ldots, s \rangle.$$ 

The scheme of $s$-jets of $X$ is $\text{Spec}(\mathcal{J}_s(R)/\mathcal{J}_s(I))$.

This paper introduces the Jets package\(^1\) for Macaulay2 [GS], streamlining the process of constructing ideals of jets as indicated above. We adopt the following notation: the variables in the polynomial rings containing the equations of jets have the names of the variables of the original equations with the order of the jets appended to them, and the same subscripts. Moreover, the rings containing the equations of jets are constructed incrementally as towers.

Ideals of jets are computed via the jets method applied to objects of type Ideal. In addition, the jets method can also be applied to objects of type QuotientRing, RingMap, and AffineVariety, with the effects one would expect from applying jet functors. For more information, including grading options, we invite the reader to consult the documentation of the package. We showcase some package applications below.

We are grateful to Greg Smith for valuable feedback on an early version of the package.

2. JETS OF MONOMIAL IDEALS

As observed in [GS06], the ideal of jets of a monomial ideal is typically not a monomial ideal.

```plaintext
i1 : needsPackage "Jets";

i2 : R=QQ[x,y,z];

i3 : I=ideal(x*y*z);

o3 : Ideal of R

i4 : J2I=jets(2,I);

o4 : Ideal of QQ[x0, y0, z0][x1, y1, z1][x2, y2, z2]

i5 : netList J2I_*
```

However, by \cite[Theorem 3.1]{GS06}, the radical is always a (squarefree) monomial ideal. In fact, the proof of \cite[Theorem 3.2]{GS06} shows that the radical is generated by the terms of the jet equations constructed as in the introduction. This observation provides an alternative

\(^1\)Available at https://github.com/galettof/Jets.
algorithm for computing radicals of jets of monomial ideals, which can be faster than the
default radical computation in Macaulay2.

```plaintext
i6 : jetsRadical(2,I);

o6 : Ideal of QQ[x0, y0, z0][x1, y1, z1][x2, y2, z2]
```

```plaintext
i7 : netList pack(5,oo_*)
```

+--------+--------+--------+--------+--------+
|y0*z0*x2|x0*z0*y2|x0*y0*z2|z0*x1*y1|y0*x1*z1|
+--------+--------+--------+--------+--------+
|x0*y1*z1|y0*z0*x1|x0*z0*y1|x0*y0*z1|x0*y0*z0|
+--------+--------+--------+--------+--------+

For a monomial hypersurface, [GS06, Theorem 3.2] describes the minimal primes of the
ideal of jets. Moreover, the main theorem in [Yue06] counts the multiplicity of the jet scheme
of a monomial hypersurface along its minimal primes (see also [Yue07b]). We compute the
minimal primes, then use the LocalRings package [SSE] to compute their multiplicities in
the second jet scheme of the example above.

```plaintext
i8 : P=minimalPrimes J2I;

i9 : --flatten ring to use LocalRings package
   (A,f)=flattenRing ring J2I;

i10 : needsPackage "LocalRings";

i11 : --quotient by jets ideal as a module
   M=cokernel gens f J2I;

i12 : --compute the multiplicity of the jets along each component
   mult=for p in P list
          Rp := localRing(A,f p);
          length(M ** Rp)
   );

i13 : netList(pack(4,mingle{P,mult}),HorizontalSpace=>1)
```

+--------------------+---+--------------------+---+
| ideal (z0, y0, x0) | 6 | ideal (z0, y0, z1) | 3 |
+--------------------+---+--------------------+---+
| ideal (z0, y0, y1) | 3 | ideal (z0, x0, z1) | 3 |
+--------------------+---+--------------------+---+
| ideal (z0, x0, x1) | 3 | ideal (z0, z1, z2) | 1 |
+--------------------+---+--------------------+---+
| ideal (y0, x0, y1) | 3 | ideal (y0, x0, x1) | 3 |
+--------------------+---+--------------------+---+
| ideal (y0, y1, y2) | 1 | ideal (x0, x1, x2) | 1 |
+-------------------+---+-------------------+---+
Jets of graphs were introduced in \cite{GHW21}. Starting with a finite, simple graph $G$, one may construct a quadratic squarefree monomial ideal $I(G)$ (known as the edge ideal of the graph) by converting edges to monomials (see for example \cite{VT13}). One may then consider the radical of the ideal of $s$-jets of $I(G)$, which is again a quadratic squarefree monomial ideal. The graph corresponding to this ideal is the graph of $s$-jets of $G$, denoted $J_s(G)$.

Jets of graphs and hypergraphs can be obtained by applying the \texttt{jets} method to objects of type \texttt{Graph} and \texttt{HyperGraph} from the Macaulay2 \texttt{EdgeIdeals} package \cite{FHT, FHVT09} (which is automatically loaded by the \texttt{Jets} package). Consider, for example, the graph in Figure 1.

\begin{center}
\includegraphics[width=0.2\textwidth]{fig1.png}
\end{center}

**Figure 1**

\begin{verbatim}
i1 : needsPackage "Jets";
i2 : R=QQ[a..e];
i3 : G=graph({{a,c},{a,d},{a,e},{b,c},{b,d},{b,e},{c,e}});
i4 : J1G=jets(1,G); netList pack(7,edges J1G)
o5 = |{c1, a0}|{d1, a0}|{e1, a0}|{c1, b0}|{d1, b0}|{e1, b0}|{a1, c0}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{b1, c0}|{e1, c0}|{a0, c0}|{b0, c0}|{a1, d0}|{b1, d0}|{a0, d0}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{b0, d0}|{a1, e0}|{b1, e0}|{c1, e0}|{a0, e0}|{b0, e0}|{c0, e0}|
     +--------+--------+--------+--------+--------+--------+--------+

i6 : J2G=jets(2,G); netList pack(7,edges J2G)
o7 = |{c1, a1}|{d1, a1}|{e1, a1}|{c1, b1}|{d1, b1}|{e1, b1}|{a1, c1}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{b1, c1}|{e1, c1}|{a0, c1}|{b0, c1}|{a1, d1}|{b1, d1}|{a0, d1}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{b0, d1}|{a1, e1}|{b1, e1}|{c1, e1}|{a0, e1}|{b0, e1}|{c0, e1}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{c2, a0}|{d2, a0}|{e2, a0}|{c1, a0}|{d1, a0}|{e1, a0}|{c2, b0}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{d2, b0}|{e2, b0}|{c1, b0}|{d1, b0}|{e1, b0}|{a2, c0}|{b2, c0}|
     +--------+--------+--------+--------+--------+--------+--------+
     |{e2, c0}|{a1, c0}|{b1, c0}|{e1, c0}|{a0, c0}|{b0, c0}|{a2, d0}|
     +--------+--------+--------+--------+--------+--------+--------+
\end{verbatim}
As predicted in [GHW21, Theorem 3.1], all jets have the same chromatic number.

```
i8 : apply({G,J1G,J2G},chromaticNumber)
o8 = {3, 3, 3}
o8 : List

By contrast, jets may not preserve the property of being co-chordal.
```

```
i9 : apply({G,J1G,J2G},x -> isChordal complementGraph x)
o9 = {true, true, false}
o9 : List

Using Fröberg’s Theorem [Fr90], we deduce that although the edge ideal of a graph may have a linear free resolution, the edge ideals of its jets may not have linear resolutions.

Finally, we compare minimal vertex covers of the graph and of its second order jets.
```

```
i10 : vertexCovers G
o10 = {a*b*c, a*b*e, c*d*e}
o10 : List

i11 : netList pack(2,vertexCovers J2G)
```

```
o11 = |
|a2*b2*c2*a1*b1*c1*a0*b0*c0  |
a2*b2*e2*a1*b1*e1*a0*b0*e0  |
|a2*b2*a1*b1*c1*a0*b0*c0*e0  |
a2*b2*a1*b1*e1*a0*b0*c0*e0  |
|c2*d2*e2*c1*d1*e1*c0*d0*e0  |
|a1*b1*c1*a0*b0*c0*d0*e0   |
|a1*b1*e1*a0*b0*c0*d0*e0   |
|c1*d1*e1*a0*b0*c0*d0*e0   |
```

With the exception of the second row, many vertex covers arise as indicated in [GHW21, Proposition 5.2, 5.3].

4. JETS OF DETERMINANTAL VARIETIES

Determinantal varieties are classical geometric objects whose jets have been studied to a certain degree of success [KS05a, KS05b, Yue07a, GJS14, Doc13, Mal21]. For our example, we consider the determinantal varieties $X_r$ of $3 \times 3$ matrices of rank at most $r$, which are defined by the vanishing of minors of size $r + 1$. We illustrate computationally some of the known results about jets.
i1 : needsPackage "Jets";

i2 : R=QQ[x_(1,1)..x_(3,3)];

i3 : G=genericMatrix(R,3,3)
o3 = | x_(1,1) x_(2,1) x_(3,1) |
    | x_(1,2) x_(2,2) x_(3,2) |
    | x_(1,3) x_(2,3) x_(3,3) |

3 3
o3 : Matrix R <--- R

Since \( X_0 \) is a single point, its first jet scheme consists of a single (smooth) point.

i4 : I1=minors(1,G);
o4 : Ideal of R

i5 : JI1=jets(1,I1);
o5 : Ideal of QQ[x0 ..x0 ][x1 ..x1 ]
    1,1 3,3 1,1 3,3

i6 : dim JI1, isPrime JI1
o6 = (0, true)
o6 : Sequence
The jets of \( X_2 \) (the variety of maximal minors) are known to be irreducible (see [KS05a, Theorem 3.1] or [Doc13, Corollary 4.13]).

i7 : I3=minors(3,G);
o7 : Ideal of R

i8 : JI3=jets(1,I3);
o8 : Ideal of QQ[x0 ..x0 ][x1 ..x1 ]
    1,1 3,3 1,1 3,3

i9 : isPrime JI3
o9 = true

As for the case of 2×2 minors, [KS05a, Theorem 5.1], [Yue07a, Theorem 5.1], and [Doc13, Corollary 4.13] all count the number of components; the first two references describe the components further. As expected, the first jet scheme of \( X_1 \) has two components, one of them an affine space.
The other component is the so-called principal component of the jet scheme, i.e., the Zariski closure of the first jets of the smooth locus of $X_1$. To check this, we first establish that the first jet scheme is reduced (i.e. its ideal is radical), then use the \texttt{principalComponent} method with the option \texttt{Saturate=>false} to speed up computations$^2$.

Finally, as observed in [GJS14, Theorem 18], the Hilbert series of the principal component of the first jet scheme of $X_1$ is the square of the Hilbert series of $X_1$.

\texttt{apply(\{P_0,I2\}, X -> hilbertSeries(X,Reduce=>true))}

\texttt{o17} = \{------------------------, -----------
               10 5
               (1 - T) (1 - T)
\}

\texttt{o17} : \texttt{List}

\texttt{i18} : \texttt{numerator (first oo) == (numerator last oo)^2}

\texttt{o18} = \texttt{true}

$^2$We invite the reader to consult the package documentation for more details.
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Department of Mathematics and Statistics, Cleveland State University, Cleveland, OH, 44115-2215, USA
Email address: f.galetto@csuohio.edu
URL: https://math.galetto.org

Department of Mathematics and Statistics, Cleveland State University, Cleveland, OH, 44115-2215, USA
Email address: nickiammarino@gmail.com