Notes on the Cauchy problem for the self-adjoint and non-self-adjoint Schrödinger equations with polynomially growing potentials

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Abstract

The Cauchy problem is studied for the self-adjoint and non-self-adjoint Schrödinger equations. We first prove the existence and uniqueness of solutions in the weighted Sobolev spaces. Secondly we prove that if potentials are depending continuously and differentiably on a parameter, so are the solutions, respectively. The non-self-adjoint Schrödinger equations that we study are those used in the theory of continuous quantum measurements. The results on the existence and uniqueness of solutions in the weighted Sobolev spaces will play a crucial role in the

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proof for the convergence of the Feynman path integrals in the theories of quantum mechanics and continuous quantum measurements.

**Keywords** Schrödinger equation; non-self-adjoint equation; dependence on a parameter; quantum mechanics; quantum measurement

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## 1 Introduction

Let $T > 0$ be an arbitrary constant. We will study the self-adjoint and non-self-adjoint Schrödinger equations

$$i\hbar \frac{\partial u}{\partial t}(t) = \tilde{H}(t)u(t) \equiv \left[H(t) - i\hbar K(t)\right]u(t)$$

$$:= \left[\frac{1}{2m} \sum_{j=1}^{d} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - qA_j(t, x)\right)^2 + qV(t, x) - i\hbar K(t)\right]u(t), \quad (1.1)$$

where $t \in [0, T], x = (x_1, \ldots, x_d) \in \mathbb{R}^d, (V(t, x), A(t, x)) = (V, A_1, A_2, \ldots, A_d) \in \mathbb{R}^{d+1}$ are electromagnetic potentials, $\hbar$ the Planck constant, $m > 0$ the mass of a particle and $q \in \mathbb{R}$ its charge. In addition, $K(t)$ is the pseudo-differential operator with a real-valued double symbol $k(t, (x + x')/2, \xi)$ defined by

$$K\left(t, \frac{X + X'}{2}, D_x\right) f = \int \int e^{i(x-y)\cdot\xi} k\left(t, \frac{x + y}{2}, \xi\right) f(y)dyd\xi \quad (1.2)$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, where $d\xi = (2\pi)^{-d}d\xi$ and $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^d$. The non-self-adjoint Schrödinger equations that we study in the present paper are those used in the theory of continuous quantum measurements. See §4.2 in [13] and §5.1.3 in [14]. Accordingly, we assume

$$(K(t)f, f) \geq -C \|f\|^2 \quad (1.3)$$
in $[0, T]$ for $f \in \mathcal{S}(\mathbb{R}^d)$ with a constant $C \geq 0$, where we denote by $L^2 = L^2(\mathbb{R}^d)$ the space of all square integrable functions on $\mathbb{R}^d$ with inner product $(f, g) := \int f(x)g(x)^*dx$ for the complex conjugate $g^*$ of $g$ and norm $\|f\|$. For the sake of simplicity we will suppose $\hbar = 1$ and $q = 1$ hereafter.

In the present paper we consider the potentials $(V, A)$ satisfying

$$|\partial^\alpha_x V(t, x)| \leq C_\alpha < x >, \ |\alpha| \geq 1,$$

$$\sum_{j=1}^d |\partial^\alpha_x A_j(t, x)| \leq C_\alpha, \ |\alpha| \geq 1 \quad (1.4)$$

with constants $C_\alpha \geq 0$ or

$$C_0 < x >^{2(M+1)} - C_1 \leq V(t, x) \leq C_2 < x >^{2(M+1)} \quad (1.5)$$

with constants $M > 0, C_0 > 0, C_1 \geq 0$ and $C_2 > 0$ in $[0, T] \times \mathbb{R}^d$, where $|x| = (\sum_{j=1}^d x_j^2)^{1/2}$, $< x > = (1 + |x|^2)^{1/2}$, $\partial_{x_j} = \partial / \partial x_j$, $|\alpha| = \sum_{j=1}^d \alpha_j$, and $\partial^\alpha_x = \partial^\alpha_{x_1} \cdots \partial^\alpha_{x_d}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$. As well known, if $a$ is a constant greater than 2, the uniqueness of solutions to (1.1) with $H(t) = -\sum_{j=1}^d \partial^2_{x_j} - |x|^a$ and $K(t) = 0$ does not hold (cf. pp. 157-159 in [2], Theorem X.2 in [16]). Therefore the assumptions (1.4) and (1.5) are not so restrictive.

For a constant $M \geq 0$ let us introduce the weighted Sobolev spaces

$$B^a_M(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d); \|f\|_{a, M} := \|f\| + \sum_{|\alpha| \leq 2a} \|\partial^\alpha_x f\| + \| < \cdot >^{2a(M+1)} f\| < \infty \} \quad (1.6)$$

for $a = 1, 2, \ldots$. We denote the dual space of $B^a_M$ ($a = 1, 2, \ldots$) by $B^{-a}_M$ and the $L^2$ space by $B^0_M$.

The first aim in the present paper is to prove that for any $u_0 \in B^a_M$ ($a = 0, \pm 1, \pm 2, \ldots$) there exists the unique solution $u(t) \in \mathcal{E}_t^a([0, T]; B^a_M) \cap \mathcal{E}_t([0, T];$
$B_{M}^{-1}$) with $u_0$ at $t = 0$ to (1.1), where $\mathcal{E}_{j}([0, T]; B_{M}^{\alpha})$ ($j = 0, 1, \ldots$) denotes the space of all $B_{M}^{\alpha}$-valued, $j$-times continuously differentiable functions on $[0, T]$. This result will play a crucial role in the proof of the convergence of the Feynman path integrals for (1.1) in [9] and [10] as in the proofs of the theorems in [6] and [8].

The second aim in the present paper is to prove that if potentials are depending continuously and differentiably on a parameter, so are the solutions to (1.1) in $\mathcal{E}_{0}^{a}([0, T]; B_{M}^{\alpha})$ ($a = 0, \pm 1, \pm 2, \cdots$), respectively. Such results have been well known in the theory of ordinary differential equations as the fundamental results.

In the present paper the results stated above to (1.1) will be extended to multi-particle systems. For simplicity we will consider 4-particle systems

\[
i \frac{\partial u}{\partial t}(t) = \tilde{H}(t)u(t) := \left[ \sum_{k=1}^{4} \left\{ \frac{1}{2m_k} \sum_{j=1}^{d} \left( \frac{1}{i} \frac{\partial}{\partial x_j} - A^{(k)}_j(t, x^{(k)}) \right)^2 + V_k(t, x^{(k)}) - iK_k(t) \right\} + \sum_{1 \leq i < j \leq 4} W_{ij}(t, x^{(i)} - x^{(j)}) \right] u(t)
\]

\[
\equiv \left[ \sum_{k=1}^{4} \left\{ H_k(t) - iK_k(t) \right\} + \sum_{1 \leq i < j \leq 4} W_{ij}(t, x^{(i)} - x^{(j)}) \right] u(t), \quad (1.7)
\]

where $x^{(k)} \in \mathbb{R}^{d}$ ($k = 1, 2, 3, 4$) and $K_k(t) = K_k(t, (X^{(k)} + X'^{(k)})/2, D_{x^{(k)}})$. Let’s consider the self-adjoint equations, i.e. $K(t) = 0$. When the Hamiltonian $H(t) = \tilde{H}(t)$ is independent of $t \in [0, T]$, the existence and uniqueness of solutions in the $L^2$ space to (1.1) and (1.7) are equivalent to the self-adjointness of $H(t) = H$ (cf. §8.4 in [13]). The self-adjointness of $H$ in $L^2$ has almost been settled now (cf. [3, 12, 16]). If $H(t)$ is not independent of $t \in [0, T]$, the problem is not simple. In [18] Yajima has proved the existence and uniqueness of
solutions to (1.1) in $B_0^a$ ($a = 0, \pm 1, \pm 2, \ldots$) under the assumptions

$$|\partial_x^n V(t, x)| \leq C_\alpha, \; |\alpha| \geq 2,$$
$$\sum_{j=1}^{d} (|\partial_x^n A_j(t, x)| + |\partial_x^n \partial_t A_j(t, x)|) \leq C_\alpha, \; |\alpha| \geq 1,$$
$$\sum_{1 \leq j < k \leq d} |\partial_x^n B_{jk}(t, x)| \leq C_\alpha < x >^{-1(1+\delta_\alpha)}, \; |\alpha| \geq 1$$

with constants $\delta_\alpha > 0$ and $C_\alpha \geq 0$ by the theory of Fourier integral operators, where $B_{jk} = \partial A_k/\partial x_j - \partial A_j/\partial x_k$. In [5] the first author has proved the existence and uniqueness of solutions in $B_0^a$ ($a = 0, \pm 1, \pm 2, \ldots$) under the assumptions (1.4) by the energy method. Recently, Yajima in [19, 20] has proved by the semi-group method the existence and uniqueness of solutions in the $L^2$ space to (1.1) and (1.7) with singular potentials.

We consider the self-adjoint equations (1.1) and (1.7) again. When the Hamiltonian $H(t)$ is independent of $t \in [0, T]$, it follows from Theorems VIII. 21 and VIII. 25 in [15] that if potentials are depending continuously on a parameter, so are the solutions in the $L^2$ space. If $H(t)$ is not independent of $t \in [0, T]$, the problem is not simple again. In [7] the first author has proved by the energy method under the assumptions (1.4) that if potentials are depending continuously and differentiably on a parameter, so are the solutions to (1.1) in $\mathcal{E}_0^0([0, T]; B_0^a)$, respectively.

As for the non-self-adjoint Schrödinger equations, there are many papers on the spectral analysis (cf. [4], [17]). The authors don’t know the results related to our results.

Therefore, our aims in the present paper are to generalize the results for the self-adjoint equations (1.1) with potentials (1.4) to the non-self-adjoint equations (1.1) and (1.7) with potentials (1.4) or (1.5).
We will prove our results by the energy method as in [5] and [7]. The crucial point in the proofs of our results for (1.1) is to introduce a family of bounded operators \( \{ \tilde{H}_\epsilon(t) \}_{0 < \epsilon \leq 1} \) on \( B_a^M \) \((a = 0, \pm 1, \pm 2, \ldots)\) satisfying Proposition 4.2 in the present paper by (4.1) as an approximation of \( \tilde{H}(t) \). Then, using the assumption (1.3), we can complete the proofs as in [5] and [7]. In the same way the crucial point in the proofs of our results for (1.7) is to introduce \( \{ \tilde{H}_\epsilon(t) \}_{0 < \epsilon \leq 1} \) by (5.31) as an approximation of \( \tilde{H}(t) \). As in the proofs for (1.1) we can complete the proofs.

The plan of the present paper is as follows. In §2 we will state all theorems. §3 is devoted to preparing for the proofs of the theorems for (1.1). In §4 we will prove all theorems for (1.1). In §5 we will prove all theorems for (1.7).

2 Theorems

In the present paper we often use symbols \( C, C_a, C_{\alpha}, C_{\alpha\beta}, \) and \( \delta \) to write down constants, though these values are different in general.

**Assumption 2.1.** We assume (1.3), (1.4) and

\[
|k^{(\alpha)}_{(\beta)}(t, x, \xi)| \leq C_{\alpha\beta}(1 + |x| + |\xi|), \quad |\alpha + \beta| \geq 1
\]

in \([0, T] \times \mathbb{R}^d\), where \( k^{(\alpha)}_{(\beta)}(t, x, \xi) = (i)^{-|\beta|} \partial_\xi^\beta \partial_x^\alpha k(t, x, \xi) \).

**Assumption 2.2.** Let \( M > 0 \) be a constant. We assume (1.3), (1.5) and

\[
|k_{(\beta)}(t, x, \xi)| \leq C_{\beta} \quad x > M + 1, \quad |\beta| \geq 1,
|k^{(\alpha)}_{(\beta)}(t, x, \xi)| \leq C_{\alpha\beta}, \quad |\alpha| \geq 1, |\beta| \geq 0.
\]

Suppose for all \( \alpha \) and \( l = 0, 1 \) that \( \partial_x^\alpha \partial_t^l V(t, x) \) and \( \partial_x^\alpha \partial_t^l A_j(t, x) \) \((j = 1, 2, \ldots, d)\) are continuous in \([0, T] \times \mathbb{R}^d\) and assume the following. We have

\[
|\partial_x^\alpha V(t, x)| \leq C_{\alpha} \quad x > 2(M + 1), |\alpha| \geq 1,
\]

(2.3)
\[ |\partial_x^2 \partial_t V(t, x)| \leq C_\alpha < x >^{2(M+1)} \]  \hspace{1cm} (2.4)

for all \( \alpha \),

\[ |A_j(t, x)| \leq C < x >^{M+1-\delta} \]  \hspace{1cm} (2.5)

with a constant \( \delta > 0 \),

\[ |\partial_x^\alpha A_j(t, x)| \leq C_\alpha < x >^{M+1}, |\alpha| \geq 1 \]  \hspace{1cm} (2.6)

and

\[ |\partial_x^\alpha \partial_t A_j(t, x)| \leq C_\alpha < x >^{M+1} \]  \hspace{1cm} (2.7)

for all \( \alpha \).

**Theorem 2.1.** (1) Suppose Assumption 2.1. Then, for any \( u_0 \in B_{a_0}^0 \) \((a = 0, \pm 1, \pm 2, \ldots)\) there exists the unique solution \( u(t) \in \mathcal{E}_1^0([0, T]; B_{a_0}^0) \cap \mathcal{E}_1^1([0, T]; B_{a_0}^{-1}) \) with \( u(0) = u_0 \) to (1.1). This solution \( u(t) \) satisfies

\[ \|u(t)\|_{a,0} \leq C_a \|u_0\|_{a,0} \hspace{0.5cm} (0 \leq t \leq T). \]  \hspace{1cm} (2.8)

(2) Suppose Assumption 2.2. Then we have the same assertions as in (1) where \( B_{a_0}^0 \) is replaced with \( B_{a_M}^0 \).

**Remark 2.1.** Let \( a(t) \) be a continuous function on \([0, T]\) such that \( a(0) = 0 \) and \( a(t) > 0 \) \((0 < t \leq T)\). Since \( V := a(t)|x|^4 + |x|^2 \) does not satisfy either (1.4) or (1.5) for any \( M > 0 \), Theorem 2.1 cannot be applied to (1.1) with \( \widetilde{H}(t) := (1/2m) \sum_{j=1}^d (-i \partial_{x_j})^2 + a(t)|x|^4 + |x|^2 \). Theorems 1.2 and 1.4 in [19] cannot be applied either, because the self-adjoint operators \( \widetilde{H}(t) \) \((0 \leq t \leq T)\) don’t have a common domain in \( L^2(\mathbb{R}^d) \).

Next, let us consider the equations (1.1) depending on a parameter \( \rho \in \mathcal{O} \), where \( \mathcal{O} \) is an open set in \( \mathbb{R} \).
Theorem 2.2. We suppose that \( \partial_x^\alpha V(t, x; \rho), \partial_x^\alpha A_j(t, x; \rho) \) \((j = 1, 2, \ldots, d)\) and \( k^{(\alpha)}(t, x, \xi; \rho) \) are continuous in \([0, T] \times \mathbb{R}^{2d} \times \mathcal{O}\) for all \( \alpha \) and \( \beta \). (1) We assume that \((V(t, x; \rho), A(t, x; \rho))\) and \( k(t, x, \xi; \rho) \) satisfy Assumption 2.1 for all \( \rho \in \mathcal{O} \) and have the uniform estimates \((1.3), (1.4)\) with respect to \( \rho \in \mathcal{O} \). Let \( u_0 \in B_0^a \((a = 0, \pm 1, \pm 2, \ldots)\) be independent of \( \rho \) and \( u(t; \rho) \) the solutions to \((1.1)\) with \( u(0; \rho) = u_0 \) determined in Theorem 2.1. Then, the mapping : \( \mathcal{O} \ni \rho \rightarrow u(t; \rho) \in E_0^t[0, T]; B_0^a \) is continuous, where the norm in \( E_0^t[0, T]; B_0^a \) is \( \max_{0 \leq t \leq T} \| f(t) \|_{a,0} \). (2) We assume that \((V(t, x; \rho), A(t, x; \rho))\) and \( k(t, x, \xi; \rho) \) satisfy Assumption 2.2 for all \( \rho \in \mathcal{O} \) and have the uniform estimates \((2.2) - (2.7)\) with respect to \( \rho \in \mathcal{O} \). Then we have the same assertions as in (1) where \( B_0^a \) is replaced with \( B_{M}^a \).

We set
\[
h(t, x, \xi) := \frac{1}{2m} |\xi - A(t, x)|^2 + V(t, x). \tag{2.9}
\]
Then by \((1.1)\) and \((1.2)\) we have
\[
H(t)f = H\left( t, \frac{X + X'}{2}, D_x \right) f
\]
for \( f \in S(\mathbb{R}^d) \).

Theorem 2.3. We suppose for \( l = 0, 1 \) that \( \partial^l_{\rho} \partial_x^\alpha V(t, x; \rho), \partial^l_{\rho} \partial_x^\alpha A_j(t, x; \rho) \) \((j = 1, 2, \ldots, d)\) and \( \partial^l_{\rho} k^{(\alpha)}(t, x, \xi; \rho) \) are continuous in \([0, T] \times \mathbb{R}^{2d} \times \mathcal{O}\) for all \( \alpha \) and \( \beta \). (1) Besides the assumptions of (1) in Theorem 2.2 we assume
\[
\sup_{\rho \in \mathcal{O}} |\partial^l_{\rho} \partial_x^\alpha V(t, x; \rho)| \leq C_{\alpha} < x >^2, \tag{2.10}
\]
\[
\sup_{\rho \in \mathcal{O}} |\partial^l_{\rho} \partial_x^\alpha A_j(t, x; \rho)| \leq C_{\alpha} < x >, \tag{2.11}
\]
\[
\sup_{\rho \in \mathcal{O}} |\partial^l_{\rho} k^{(\alpha)}(t, x, \xi; \rho)| \leq C_{\alpha \beta}(1 + |x|^2 + |\xi|^2) \tag{2.12}
\]
in \([0, T] \times \mathbb{R}^d\) for all \(\alpha\) and \(\beta\). Let \(u_0 \in B^{a+1}_0\) \((a = 0, \pm 1, \pm 2, \ldots)\) be independent of \(\rho\) and \(u(t; \rho)\) the solutions to (1.1) with \(u(0) = u_0\). Then, the mapping \(O \ni \rho \rightarrow u(t; \rho) \in \mathcal{E}_t^0([0, T]; B^a_0)\) is continuously differentiable with respect to \(\rho\), we have

\[
\sup_{\rho \in O} \|\partial_{\rho} u(t; \rho)\|_{a,0} \leq C_a \|u_0\|_{a+1,0} \quad (0 \leq t \leq T) \tag{2.13}
\]

and \(\partial_{\rho} u(t; \rho)\) is the solution to

\[
\frac{\partial}{\partial t} w(t; \rho) = \tilde{H}(t; \rho) w(t; \rho) + \frac{\partial \tilde{H}(t; \rho)}{\partial \rho} u(t; \rho) \tag{2.14}
\]

with \(w(0) = 0\). Here, \(\partial_{\rho} \tilde{H}(t; \rho)\) denotes the pseudo-differential operator with the double symbol \(\partial_{\rho} \tilde{h}(t, x + x', \xi; \rho)\), where \(\tilde{h}(t, x, \xi; \rho) = h(t, x, \xi; \rho) - ik(t, x, \xi; \rho)\). (2) Besides the assumptions of (2) in Theorem 2.2 we assume

\[
\sup_{\rho \in O} |\partial_{\rho} \partial_x^\alpha V(t, x; \rho)| \leq C_\alpha < x >^{2(M+1)}, \tag{2.15}
\]

\[
\sup_{\rho \in O} |\partial_{\rho} \partial_x^\alpha A_j(t, x; \rho)| \leq C_\alpha < x >^{M+1}, \tag{2.16}
\]

\[
\sup_{\rho \in O} |\partial_{\rho} k^{(\alpha)}(t, x, \xi; \rho)| \leq C_{\alpha\beta}(< x >^{2(M+1)} + < \xi >^2) \tag{2.17}
\]

in \([0, T] \times \mathbb{R}^d\) for all \(\alpha\) and \(\beta\). Then we have the same assertions as in (1) where \(B^a_0\) is replaced with \(B^a_M\).

Now, we consider the 4-particle systems (1.7).

**Assumption 2.3.** (1) Each \((V_k(t, x), A_k(t, x))\) and \(K_k(t)\) \((k = 1, 2)\) satisfies Assumption 2.2 with \(M = M_k > 0\). (2) Each \((V_k, A_k)\) and \(K_k(t)\) \((k = 3, 4)\) satisfies Assumption 2.1. (3) For \(M_0 := \min(M_1, M_2)\) \(W_{12}\) satisfies

\[
|W_{12}(t, x)| \leq C < x >^{2(M_0+1)-\delta} \tag{2.18}
\]

with a constant \(\delta > 0\) and

\[
|\partial_x^\alpha W_{12}(t, x)| \leq C_\alpha < x >^{2(M_0+1)}, \quad |\alpha| \geq 1. \tag{2.19}
\]
(4) Each $W_{ij}(t, x)$ except $W_{12}$ satisfies
\[ |\partial_x^\alpha W_{ij}(t, x)| \leq C_\alpha < x >, \quad |\alpha| \geq 1. \] (2.20)

We introduce the weighted Sobolev spaces $B^a(\mathbb{R}^{4d}) := \{ f \in L^2(\mathbb{R}^{4d}); \| f \|_a := \| f \| + \sum_{|\alpha| \leq 2a} \| \partial_x^\alpha f \| + \sum_{k=1}^4 \| x^{(k)} >^{2a(M_k+1)} f \| < \infty \}$ for all $a = 1, 2, \ldots$ with $M_3 = M_4 = 0$. We denote the dual space of $B^a$ by $B^{-a}$ and $L^2$ by $B^0$.

**Theorem 2.4.** Under Assumption 2.3 for any $u_0 \in B^a(\mathbb{R}^{4d})$ ($a = 0, \pm 1, \pm 2, \ldots$) there exists the unique solution $u(t) \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$ with $u(0) = u_0$ to (1.7). This solution $u(t)$ satisfies
\[ \| u(t) \|_a \leq C_a \| u_0 \|_a \quad (0 \leq t \leq T). \] (2.21)

We will consider the 4-particle systems (1.7) depending on a parameter $\rho \in \mathcal{O}$.

**Theorem 2.5.** We suppose that $\partial_x^\alpha V_k(t, x; \rho)$, $\partial_x^\alpha A_j^{(k)}(t, x; \rho)$ ($k = 1, 2, 3, 4$, $j = 1, 2, \ldots, d$), $k_i^{(\alpha)}(t, x, \xi; \rho)$ ($l = 1, 2, 3, 4$) and $\partial_x^\alpha W_{ij}(t, x; \rho)$ ($1 \leq i < j \leq 4$) are continuous in $[0, T] \times \mathbb{R}^{2d} \times \mathcal{O}$ for all $\alpha$ and $\beta$. In addition, we assume that $(V_k(t, x; \rho), A^{(k)}(t, x; \rho))$ and $K_k(t; \rho)$ ($k = 1, 2, 3, 4$) and $W_{ij}(t, x; \rho)$ ($1 \leq i < j \leq 4$) satisfy Assumption 2.3 for all $\rho \in \mathcal{O}$ and have the uniform estimates with respect to $\rho \in \mathcal{O}$ for all inequalities stated in Assumption 2.3.

Let $u_0 \in B^a$ ($a = 0, \pm 1, \pm 2, \ldots$) be independent of $\rho$ and $u(t; \rho)$ the solutions to (1.7) with $u(0; \rho) = u_0$ determined in Theorem 2.4. Then, the mapping $\mathcal{O} \ni \rho \rightarrow u(t; \rho) \in \mathcal{E}_t^0([0, T]; B^a)$ is continuous.

**Theorem 2.6.** Besides the assumptions of Theorem 2.5 we suppose for all $\alpha$ and $\beta$ that all functions $\partial_\rho \partial_x^\alpha V_k(t, x; \rho)$, $\partial_\rho \partial_x^\alpha A_j^{(k)}(t, x; \rho)$, $\partial_\rho k_i^{(\alpha)}(t, x, \xi; \rho)$ and $\partial_\rho \partial_x^\alpha W_{ij}(t, x; \rho)$ are continuous in $[0, T] \times \mathbb{R}^{2d} \times \mathcal{O}$. In addition, we assume
with $M = M_k$ for $(V_k, A^{(k)})$ and $K_k(t)$ ($k = 1, 2$), \(2.10\) - \(2.12\) for $(V_k, A^{(k)})$ and $K_k(t)$ ($k = 3, 4$),

\[
\sup_{\rho \in \Omega} |\partial_\rho \partial_\alpha \partial_x W_{12}(t, x; \rho)| \leq C_\alpha < x >^{2(M_0 + 1)}
\]

(2.22)

for all $\alpha$ and

\[
\sup_{\rho \in \Omega} |\partial_\rho \partial_\alpha \partial_x W_{ij}(t, x; \rho)| \leq C_\alpha < x >^{2}, \ (i, j) \neq (1, 2)
\]

(2.23)

for all $\alpha$.

Let $u_0 \in B^{a+1}$ ($a = 0, \pm 1, \pm 2, \ldots$) be independent of $\rho$ and $u(t; \rho)$ the solutions to \(1.7\) with $u(0; \rho) = u_0$. Then we have the same assertion as in Theorem 2.3.

### 3 Preliminaries

Let $h(t, x, \xi)$ be the function defined by \(2.9\).

**Lemma 3.1.** Assume \(1.5\) and \(2.5\). Then there exist constants $C_0^* > 0$ and $C_1^* \geq 0$ such that

\[
C_0^*( < \xi >^2 + < x >^{2(M + 1)}) - C_1^* \leq h(t, x, \xi) \leq C_0^{*-1}( < \xi >^2 + < x >^{2(M + 1)})
\]

(3.1)

in $[0, T] \times \mathbb{R}^{2d}$.

**Proof.** From \(2.9\) we have $h(t, x, \xi) \leq (|\xi|^2 + |A(t, x)|^2)/m + V(t, x)$ and hence by \(1.3\) and \(2.5\)

\[
h(t, x, \xi) \leq C(< \xi >^2 + < x >^{2(M + 1)})
\]

in $[0, T] \times \mathbb{R}^{2d}$ with a constant $C \geq 0$. 

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We may assume $0 < \delta \leq M + 1$ in (2.5). Take $p > 1$ and $q > 1$ so that
\[
\frac{1}{p} = \frac{1}{2} \left(1 - \frac{1}{2} \cdot \frac{\delta}{M+1}\right), \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
Then we have
\[
p(M + 1 - \delta) = 2(M + 1) \cdot \frac{1 - \frac{\delta}{M+1}}{1 - \frac{1}{2} \cdot \frac{\delta}{M+1}} \equiv 2(M + 1)\delta_1,
\]
\[
q = \frac{2}{1 + \frac{1}{2} \cdot \frac{\delta}{M+1}} \equiv 2\delta_2
\]
with $0 < \delta_j < 1$ ($j = 1, 2$). Hence, Young’s inequality and (2.5) show
\[
|A(t, x)| \cdot |\xi| \leq \frac{1}{p} |A|^p + \frac{1}{q} |\xi|^q \leq \frac{1}{p} < x >^p(M+1-\delta) + \frac{1}{q} |\xi|^q
\]
\[
= \frac{1}{p} < x >^{2(M+1)\delta_1} + \frac{1}{q} |\xi|^{2\delta_2}.
\]
Applying this, (1.5) and (2.5) to (2.9), we have
\[
h(t, x, \xi) \geq \frac{1}{2m}(|\xi|^2 - 2|A| \cdot |\xi|) + V
\]
\[
\geq C_0(\xi >^2 < x >^{2(M+1)\delta_1} - < \xi >^{2\delta_2} + < x >^{2(M+1)} - C_1
\]
with constants $C_0 > 0$ and $C_1 \geq 0$. Therefore, we obtain (3.1).

We fix $C_0^*$ and $C_1^*$ in Lemma 3.1 hereafter. We set
\[
h_s(t, x, \xi) = h(t, x, \xi) + \frac{i}{2m} \nabla \cdot A(t, x),
\]
where $\nabla \cdot A(t, x) = \sum_{j=1}^d \partial_{x_j} A_j(t, x)$. Since the real part $\text{Re} h_s(t, x, \xi)$ of $h_s(t, x, \xi)$ is equal to $h(t, x, \xi)$, we can determine
\[
p_{\mu}(t, x, \xi) := \frac{1}{\mu + h_s(t, x, \xi)}
\]
for $\mu \geq C_1^*$ under the assumptions of Lemma 3.1. We denote by $H_s(t, X, D_x)f$ the pseudo-differential operator
\[
\int e^{ix \cdot \xi} h_s(t, x, \xi) d\xi \int e^{-iy \cdot \xi} f(y) dy
\]
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for \( f \in S(\mathbb{R}^d) \) with the symbol \( h_s(t,x,\xi) \). As is well known (cf. Theorem 2.5 in Chapter 2 of [11]), \( H_s(t,X,D_x) = H(t) \) holds, where \( H(t) \) is the operator defined by (1.1).

**Lemma 3.2.** Assume (1.5), (2.3) and (2.5) - (2.6). Then we have

\[
[\mu + H(t)] P_\mu(t,X,D_x) = I + R_\mu(t,X,D_x),
\]

\[
|\Gamma_{\mu \beta}^{(\alpha)}(t,x,\xi)| \leq C_{\alpha \beta} (\mu - C_1^*)^{-1/2}
\]

in \([0,T] \times \mathbb{R}^{2d}\) for \( \mu \geq C_0^*/2 + C_1^* \) with constants \( C_{\alpha \beta} \) independent of \( \mu \).

**Proof.** Let \( \mu \geq C_1^* \). By Lemma 3.1 and (3.2) we have

\[
C_0^* (<\xi>^2 + <x>^{2(M+1)}) + \mu - C_1^* \leq \mu + \text{Re} \ h_s(t,x,\xi).
\]

Since \( H(t) = H_s(t,X,D_x) \), from (2.13) in [5] we have

\[
r_\mu(t,x,\xi) = \sum_{|\alpha|=1} \int_0^1 d\theta \text{Os} - \int\int e^{-iy\cdot\eta} h_s^{(\alpha)}(t,x,\xi+\theta\eta)p_{\mu(\alpha)}(t,x+y,\xi)dyd\eta
\]

\[
= \sum_{|\alpha|=1} \int_0^1 d\theta \text{Os} - \int\int e^{-iy\cdot\eta} <y>^{-2l_0} <D_{\eta}>^{2l_0} <\eta>^{-2l_1} <D_{y}>^{2l_1}
\]

\[
\cdot h_s^{(\alpha)}(t,x,\xi+\theta\eta)p_{\mu(\alpha)}(t,x+y,\xi)dyd\eta
\]

for large integers \( l_0 \) and \( l_1 \), where \( <D_{\eta}>^2 = 1 - \sum_{j=1}^d \partial_{\eta j}^2 \).

Now, using (2.3) and (2.5) - (2.6), from (3.2) we have

\[
|\partial_x h_s(t,x+y,\xi)| \leq C \left(<\xi><x+y>^{M+1} + <x+y>^{2(M+1)}\right)
\]

\[
\leq C' \left(<\xi>^2 + <x+y>^{2(M+1)}\right).
\]

In the same way we can prove

\[
|h_s^{(\alpha)}(t,x+y,\xi)| \leq C \left(<\xi>^2 + <x+y>^{2(M+1)}\right)
\]

(3.8)
for all $\alpha$ and $|\beta| \geq 1$, and
\[ |h_{(\alpha)}(\beta, t, x, \xi + \theta \eta)| \leq C \left( \frac{\xi > + x > M + 1}{\xi > + x > 2(M + 1)} \right) < \eta > \quad (3.9) \]
for $|\alpha| \geq 1$ and all $\beta$. We also note
\[
\frac{1}{\xi > + x > 2(M + 1)} \leq \frac{1}{(\sqrt{2} < \eta >)^{2(M + 1)}} \leq \frac{1}{\xi > + x > 2(M + 1)}.
\]

Apply (3.6) and (3.8) - (3.9) to (3.7). Then, taking integers $l_0$ and $l_1$ so that
\[
2l_0 - 2(M + 1) > d \quad \text{and} \quad 2l_1 - 1 > d,
\]
we have
\[
|r_\mu(t, x, \xi)| \leq C \int \int < y >^{-2l_0} < \eta >^{-2l_1} < \eta > < y >^{2(M + 1)} dy \, d\eta
\]
\[
\times \frac{\Theta^{1/2}}{\mu - C_1^* + C_0^* \Theta} \leq C' \max_{1 \leq \theta} \frac{\theta^{1/2}}{\mu - C_1^* + C_0^* \theta} \quad (3.10)
\]
with constants $C$ and $C'$ independent of $\mu \geq C_1^*$, where $\Theta = < \xi >^2 + < x >^{2(M + 1)}$. Applying (2.9) in [5] with $\kappa = 1$ and $\tau = 2$ to (3.10), we have
\[
|r_\mu(t, x, \xi)| \leq C_0 (\mu - C_1^*)^{-1/2}
\]
for $\mu \geq C_0^*/2 + C_1^*$. In the same way we can prove (3.5) from (3.7) - (3.9). \qed

**Proposition 3.3.** Under the assumptions of Lemma 3.2 there exist a constant $\mu \geq C_0^*/2 + C_1^*$ and a function $w(t, x, \xi)$ in $[0, T] \times \mathbb{R}^{2d}$ satisfying
\[
|w_{(\alpha)}(\beta, t, x, \xi)| \leq C_{\alpha \beta} \left( \frac{\xi > + x > 2(M + 1)}{\xi > + x > 2(M + 1)} \right)^{-1} \quad (3.11)
\]
for all $\alpha, \beta$ and
\[
W(t, X, D_x) = (\mu + H(t))^{-1}. \quad (3.12)
\]

**Proof.** Let $\mu \geq C_0^*/2 + C_1^*$. From (3.3), (3.6) and (3.8) - (3.9) we see
\[
|p_{(\alpha)}(\beta, t, x, \xi)| \leq C_{\alpha \beta} \left( \frac{\xi > + x > 2(M + 1)}{\xi > + x > 2(M + 1)} \right)^{-1}
\]
for all $\alpha$ and $\beta$. Hence we can complete the proof of Proposition 3.3 as in the proof of (2.16) of [5] by using Lemma 3.2.

We take a constant $\mu \geq C_0^*/2 + C_1^*$ stated in Proposition 3.3 and fix it hereafter throughout §3 and §4. Set

$$\lambda(t, x, \xi) := \mu + h_s(t, x, \xi).$$

Then from (3.13) we have

$$\Lambda(t, X, D_x) = \mu + H_s(t, X, D_x) = \mu + H(t).$$

We take a $\chi \in \mathcal{S}(\mathbb{R}^d)$ such that $\chi(0) = 1$ and set

$$\chi_\epsilon(t, x, \xi) := \chi(\epsilon(\mu + h(t, x, \xi)))$$

for constants $0 < \epsilon \leq 1$. We note that $h(t, x, \xi)$ defined by (2.9) is real-valued.

The following is crucial in the present paper.

**Lemma 3.4.** Under the assumptions of Lemma 3.2 there exist functions $\omega_\epsilon(t, x, \xi) (0 < \epsilon \leq 1)$ in $[0, T] \times \mathbb{R}^{2d}$ satisfying

$$\sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |\omega_\epsilon^{(\alpha)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty$$

for all $\alpha, \beta$ and

$$\Omega_\epsilon(t, X, D_x) = \left[ X_\epsilon(t, X, D_x), \Lambda(t, X, D_x) \right],$$

where $[\cdot, \cdot]$ denotes the commutator of operators.

**Proof.** Apply Theorem 3.1 in Chapter 2 of [11] to the right-hand side of (3.17).
Then we have

$$
\omega(\epsilon; t, x, \xi) = \sum_{|\alpha| = 1} \left\{ \chi^{(\alpha)}_\epsilon(t, x, \xi) \lambda^{(\alpha)}(t, x, \xi) \right\} - \lambda^{(\alpha)}(t, x, \xi) \chi^{(\alpha)}_\epsilon(t, x, \xi) \\
+ 2 \sum_{|\gamma| = 2} \frac{1}{\gamma!} \int_0^1 (1 - \theta) d\theta \mathcal{O}_s - \int \int e^{-i\eta \cdot \theta} \left\{ \chi^{(\gamma)}_\epsilon(t, x, \xi + \theta \eta) \lambda^{(\gamma)}(t, x + y, \xi) - \lambda^{(\gamma)}(t, x, \xi + \theta \eta) \chi^{(\gamma)}_\epsilon(t, x + y, \xi) \right\} dy \, d\eta \equiv I_1 + I_2. \quad (3.18)
$$

By (3.2), (3.13) and (3.15) we can write

$$
I_1(t, x, \xi) = \epsilon \chi'((\mu + h)) \sum_{|\alpha| = 1} \left\{ h^{(\alpha)} h_{s(\alpha)} - h_{s(\alpha)} h^{(\alpha)} \right\} \\
= \epsilon \chi'((\mu + h(t, x, \xi))) \sum_{|\alpha| = 1} \frac{i}{2m^2} (\xi - A(t, x))(-i\partial_x)^2 \nabla \cdot A(t, x). \quad (3.19)
$$

Hence, using \(\epsilon \chi'((\mu + h)) = (\mu + h)^{-1} \{ \epsilon(\mu + h) \chi'((\mu + h)) \} \) and Lemma 3.1, from (2.5) - (2.6) we can prove \(\sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_1| < \infty\). In the same way from (3.19) we can prove

$$
\sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_1^{(\alpha)}(\epsilon; t, x, \xi)| \leq C_{\alpha \beta} < \infty \quad (3.20)
$$

for all \(\alpha\) and \(\beta\).

Next we will consider \(I_2\). Let \(|\gamma| = 2\). Since from (3.15) we have

$$
\partial_{\xi_j} \chi^{(\alpha)}_\epsilon(t, x, \xi) = \frac{1}{m} \epsilon \chi'((\mu + h)) (\xi_j - A_j)
$$

and

$$
\epsilon(\xi_j - A_j) \partial_{\xi_k} \chi'((\mu + h)) = \frac{1}{m} \epsilon^2 (\xi_j - A_j) (\xi_k - A_k) \chi''((\mu + h)),
$$

as in the proof of (3.20) we can easily prove

$$
\sup_{0 < \epsilon \leq 1} |\chi^{(\alpha + \gamma)}_\epsilon(t, x, \xi)| \leq C_{\alpha \beta} \left( < \xi >^2 + < x >^{2(M+1)} \right)^{-1} \quad (3.21)
$$

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for all $\alpha$ and $\beta$. In the same way we can also prove

$$
\sup_{0<\epsilon\leq1} |\chi_{(\alpha)_{(\beta+\gamma)}}(t,x,\xi)| \leq C_{\alpha\beta} < \infty
$$

(3.22)

for all $\alpha$ and $\beta$. We also note from (3.13) that each of $\lambda(\gamma) = h_{s}(\gamma)$ for $|\gamma| = 2$ is equal to 0 or 1/m. Hence, applying (3.8) and (3.21) - (3.22) to $I_{2\epsilon}$ in (3.18), as in the proof of (3.10) we have $\sup_{0<\epsilon\leq1} \sup_{t,x,\xi} \left| I_{2\epsilon} \right| < \infty$. In the same way we can prove

$$
\sup_{0<\epsilon\leq1} \sup_{t,x,\xi} \left| I_{2\epsilon}^{(\alpha)}(t,x,\xi) \right| \leq C_{\alpha\beta} < \infty
$$

for all $\alpha$ and $\beta$, which completes the proof of Lemma 3.4 together with (3.20).

Let

$$
\lambda_{M}(x,\xi) := \mu^{'} + \frac{1}{2m}|\xi|^{2} + <x>^{2(M+1)},
$$

(3.23)

which is equal to $\lambda(t,x,\xi)$ defined by (3.13) with $V = <x>^{2(M+1)}$ and $A = 0$. Let $B_{M}^{a}$ be the weighted Sobolev spaces introduced in §1.

**Proposition 3.5.** (1) There exist a constant $\mu^{'} \geq 0$ and a function $w_{M}(x,\xi)$ in $[0,T] \times \mathbb{R}^{2d}$ satisfying (3.11) for all $\alpha, \beta$ and

$$
W_{M}(X,D_{x}) = \Lambda_{M}(X,D_{x})^{-1}.
$$

(3.24)

(2) We take a $\mu^{'} \geq 0$ satisfying (1). Let $f$ be in the dual space $S^{\prime}(\mathbb{R}^{d})$ of $S(\mathbb{R}^{d})$. Then, $B_{M}^{a} \ni f \ (a = 0, \pm1, \pm2, \ldots)$ is equivalent to $(\Lambda_{M})^{a}f \in L^{2}$.

**Proof.** The assertion (1) follows from Proposition 3.3. The assertion (2) follows from Lemma 2.4 of [5] with $s = a, a = 2(M+1)$ and $b = 2$.

We take a constant $\mu^{'} \geq 0$ stated in Proposition 3.5 and fix it hereafter throughout §3 and §4. We can easily see from (3.1), (3.8) and (3.9) that under
the assumptions of Lemma 3.2 we have
\[ |h_s^{(\alpha)}(t, x, \xi)| \leq C_{\alpha\beta} \left( <\xi>^2 + <x>^{2(M+1)} \right) \] (3.25)
in \([0, T] \times \mathbb{R}^{2d}\) for all \(\alpha\) and \(\beta\).

4 Proofs of Theorems 2.1 - 2.3

Let \(\lambda(t, x, \xi)\) and \(\chi_\epsilon(t, x, \xi)\) \((0 < \epsilon \leq 1)\) be the functions defined by (3.13) and (3.15), respectively. We define an approximation of \(\tilde{H}(t)\) by the product of operators
\[ \tilde{H}_\epsilon(t) := X_\epsilon(t, X, D_x) \dagger \tilde{H}(t) X_\epsilon(t, X, D_x), \] (4.1)
where \(X_\epsilon(t, X, D_x)\) denotes the formally adjoint operator of \(X_\epsilon(t, X, D_x)\).

**Lemma 4.1.** Under Assumption 2.2 there exist functions \(q_\epsilon(t, x, \xi)\) \((0 < \epsilon \leq 1)\) satisfying
\[ \sup_{0<\epsilon\leq1} \sup_{t,x,\xi} |q^{(\alpha)}_\epsilon(t, x, \xi)| \leq C_{\alpha\beta} < \infty \] (4.2)
for all \(\alpha, \beta\) and
\[ Q_\epsilon(t, X, D_x) = \left[ \Lambda(t, X, D_x), \tilde{H}_\epsilon(t) \right] \Lambda(t, X, D_x)^{-1} \]
\[ + i \frac{\partial \Lambda}{\partial t}(t, X, D_x) \Lambda(t, X, D_x)^{-1}. \] (4.3)

**Proof.** We first note
\[ \left[ \Lambda(t, X, D_x), \tilde{H}_\epsilon(t) \right] = \left[ \Lambda(t), X_\epsilon(t) \dagger \right] \tilde{H}(t) X_\epsilon(t) + X_\epsilon(t) \dagger \left[ \Lambda(t), H(t) \right] X_\epsilon(t) \]
\[ - i X_\epsilon(t) \dagger \left[ \Lambda(t), K(t) \right] X_\epsilon(t) + X_\epsilon(t) \dagger \tilde{H}(t) \left[ \Lambda(t), X_\epsilon(t) \right]. \]
Since $\Lambda(t) = \mu + H(t)$ from (3.14), we have $[\Lambda(t), H(t)] = 0$ and $\Lambda(t)^\dagger = \Lambda(t)$.

Hence

$$[\Lambda(t), \tilde{H}_\epsilon(t)] = -[\Lambda(t), X_\epsilon(t)]^\dagger \tilde{H}(t)X_\epsilon(t) + X_\epsilon(t)^\dagger \tilde{H}(t)[\Lambda(t), X_\epsilon(t)]$$

$$- iX_\epsilon(t)^\dagger [\Lambda(t), K(t)] X_\epsilon(t).$$

As in the proof of Lemma 3.4, we can prove from Assumption 2.2 and Proposition 3.3 that there exist $q_\epsilon'(t, x, \xi) (0 < \epsilon \leq 1)$ satisfying (4.2) and

$$Q_\epsilon'(t, X, D_x) = X_\epsilon(t)^\dagger [\Lambda(t), K(t)] X_\epsilon(t)\Lambda(t)^{-1}.$$}

From (2.2) we have

$$|k^{(\alpha)}_{(\beta)}(t, x, \xi)| \leq C_{\alpha\beta}(<\xi>^2 + <x>^{2(M+1)})$$

(4.4)

for all $\alpha$ and $\beta$. Consequently, using Proposition 3.3 and Lemma 3.4, we see together with (3.25) that there exist functions $q_\epsilon''(t, x, \xi) (0 < \epsilon \leq 1)$ satisfying (4.2) and

$$Q_\epsilon''(t, X, D_x) = [\Lambda(t, X, D_x), \tilde{H}_\epsilon(t)]\Lambda(t, X, D_x)^{-1}.$$}

It is easy to study the second term on the right-hand side of (4.3) by Proposition 3.3. Thus our proof is complete.

**Proposition 4.2.** Under Assumption 2.2 there exist functions $q_{a\epsilon}(t, x, \xi) (a = 0, \pm 1, \pm 2, \ldots, 0 < \epsilon \leq 1)$ satisfying (4.2) and

$$Q_{a\epsilon}(t, X, D_x) = \left[i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^a\right] \Lambda(t)^{-a}.$$}

(4.5)

**Proof.** For $a = 0$ the assertion is clear. For $a = 1$ the assertion follows from Lemma 4.1. Consider the case $a = 2$. We note

$$[P, QR] = [P, Q]R + Q[P, R].$$

(4.6)
and thereby
\[
\begin{align*}
\left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^2 \right] \Lambda(t)^{-2} &= \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} \\
+ \Lambda(t) \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-2} &= Q_\epsilon(t) + \Lambda(t)Q_\epsilon(t)\Lambda(t)^{-1}.
\end{align*}
\]
Hence it follows from Lemma 4.1 and Proposition 3.3 that the assertion holds.

We consider the case \( a = 3 \). From (4.6) we have
\[
\begin{align*}
\left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^3 \right] \Lambda(t)^{-3} &= \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} \\
+ \Lambda(t) \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^2 \right] \Lambda(t)^{-2} \Lambda(t)^{-1} &= Q_\epsilon(t) + \Lambda(t)Q_\epsilon(t)\Lambda(t)^{-1}.
\end{align*}
\]
Consequently, using the results for \( a = 1 \) and \( 2 \), we see that the assertion holds.

In the same way we can prove the assertion for \( a = 0, 1, 2, \ldots \) by induction.

Next we consider the case \( a = -1, -2, \ldots \). From (4.6) we have
\[
\begin{align*}
0 &= \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-1}\Lambda(t) \right] = \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) \\
+ \Lambda(t)^{-1} \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t) \right],
\end{align*}
\]
which shows
\[
\begin{align*}
\left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) &= -\Lambda(t)^{-1} \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} \Lambda(t) \\
&= -\Lambda(t)^{-1}Q_\epsilon(t)\Lambda(t).
\end{align*}
\]
Hence the assertion for \( a = -1 \) holds. We consider the case \( a = -2 \). From
we have
\[
\left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-2} \right] \Lambda(t)^2 = \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t)
\]
\[
+ \Lambda(t)^{-1} \left\{ \left[ i \frac{\partial}{\partial t} - \tilde{H}_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) \right\} \Lambda(t)
\]
\[
= Q_{-1\epsilon}(t) + \Lambda(t)^{-1} Q_{-1\epsilon}(t) \Lambda(t),
\]
which shows the assertion. In the same way we can prove the assertion for \(a = -1, -2, \ldots\) by induction. Thus, our proof is complete.

We consider the equation
\[
i \frac{\partial u}{\partial t}(t) = \tilde{H}_\epsilon(t) u(t) + f(t). \tag{4.7}
\]

**Proposition 4.3.** Suppose Assumption 2.2. Let \(u_0 \in B_M^a (a = 0, \pm 1, \pm 2, \ldots)\) and \(f(t) \in E^0([0, T]; B_M^a).\) Then, there exist solutions \(u_\epsilon(t) \in E^1([0, T]; B_M^a) \ (0 < \epsilon \leq 1)\) with \(u_\epsilon(0) = u_0\) to (4.7) satisfying
\[
\sup_{0 < \epsilon \leq 1} \|u_\epsilon(t)\|_{a, M} \leq C\alpha \left( \|u_0\|_{a, M} + \int_0^t \|f(\theta)\|_{a, M} d\theta \right). \tag{4.8}
\]

**Proof.** Applying Theorem 2.5 in Chapter 2 of [11] to (4.1), we see that each of \(\tilde{H}_\epsilon(t) \ (0 < \epsilon \leq 1)\) is written as the pseudo-differential operator with a symbol \(p_\epsilon(t, x, \xi)\) satisfying
\[
\sup_{t, x, \xi} |p_\epsilon^{(\alpha)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty
\]
for all \(\alpha\) and \(\beta,\) where \(C_{\alpha \beta}\) may depend on \(0 < \epsilon \leq 1.\) Consequently, it follows from Lemma 2.5 of [9] with \(s = a, a = 2(M + 1)\) and \(b = 2\) that we have
\[
\sup_{0 \leq t \leq T} \|\tilde{H}_\epsilon(t) f\|_{a, M} \leq C_{a\epsilon} \|f\|_{a, M}
\]
for $a = 0, \pm 1, \pm 2, \ldots$ with constants $C_{\alpha \epsilon} \geq 0$ dependent on $0 < \epsilon \leq 1$. Hence, noting that the equation (4.7) is equivalent to

$$iu(t) = iu_0 + \int_0^t \{ \tilde{H}_\epsilon(\theta)u(\theta) + f(\theta) \} \, d\theta,$$

we can find a solution $u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B_{M^a})$ by the successive iteration for each $0 < \epsilon \leq 1$. From (4.5) and (4.7) we have

$$i \frac{\partial}{\partial t} \Lambda(t)^a u_\epsilon(t) = \tilde{H}_\epsilon(t) \Lambda(t)^a u_\epsilon(t) + Q_{\alpha \epsilon}(t) \Lambda(t)^a u_\epsilon(t) + \Lambda(t)^a f(t). \quad (4.9)$$

Applying the Calderón-Vaillancourt theorem (cf. p.224 in [11]), from (3.25), Propositions 3.3 and 3.5 we have $\Lambda(t)^a u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; L^2)$ because of $u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B_{M^a})$. Noting (4.1) and that $H(t)$ is symmetric on $L^2$, from (4.9) we have

$$\frac{d}{dt} \| \Lambda(t)^a u_\epsilon(t) \|^2 = 2 \text{Re} \left( \frac{\partial}{\partial t} \Lambda(t)^a u_\epsilon(t), \Lambda(t)^a u_\epsilon(t) \right)$$

$$= -2 \left( K(t) X_\epsilon(t) \Lambda(t)^a u_\epsilon(t), X_\epsilon(t) \Lambda(t)^a u_\epsilon(t) \right)$$

$$- 2 \text{Re} \left( i Q_{\alpha \epsilon}(t) \Lambda(t)^a u_\epsilon(t), \Lambda(t)^a u_\epsilon(t) \right) - 2 \text{Re} \left( i \Lambda(t)^a f(t), \Lambda(t)^a u_\epsilon(t) \right).$$

Hence, using (1.3), Proposition 4.2 and the Calderón-Vaillancourt theorem, we have

$$\frac{d}{dt} \| \Lambda(t)^a u_\epsilon(t) \|^2 \leq 2 C_a \left( \| \Lambda(t)^a u_\epsilon(t) \|^2 + \| \Lambda(t)^a f(t) \| \cdot \| \Lambda(t)^a u_\epsilon(t) \| \right) \quad (4.10)$$

with a constant $C_a$ independent of $0 < \epsilon \leq 1$.

For a moment take a constant $\eta > 0$ and set $v(t) := \left( \| \Lambda(t)^a u_\epsilon(t) \|^2 + \eta \right)^{1/2}$, which is a positive, continuously differentiable function with respect to $t$. From (4.10) we have

$$\frac{d}{dt} v(t)^2 \leq 2 C_a \left( v(t)^2 + \| \Lambda(t)^a f(t) \| v(t) \right)$$
and so \( v'(t) \leq C_a \left( v(t) + \| \Lambda(t)^a f(t) \| \right) \). Hence we see
\[
v(t) \leq e^{C_a t} v(0) + C_a \int_0^t e^{C_a (t-\theta)} \| \Lambda(\theta)^a f(\theta) \| d\theta.
\]
Letting \( \eta \) to 0, we get
\[
\| \Lambda(t)^a u_\epsilon(t) \| \leq e^{C_a t} \| \Lambda(0)^a u_0 \| + C_a \int_0^t e^{C_a (t-\theta)} \| \Lambda(\theta)^a f(\theta) \| d\theta. \tag{4.11}
\]
Therefore, noting (3.25), Propositions 3.3 and 3.5, we can prove (4.8) with another constant \( C_a \geq 0 \).

The following has been proved in Lemma 3.1 of [7].

**Lemma 4.4.** Let \( a = 0, \pm 1, \pm 2, \ldots \). Then the embedding map from \( B_M^{a+1} \) into \( B_M^a \) is compact.

Now, we will prove Theorem 2.1 under Assumption 2.2. Our proof is similar to that of Theorem in [5].

**1st step.** Throughout 1st step we suppose \( u_0 \in B_M^{a+1} \) and \( f(t) \in \mathcal{E}_t^0([0, T]; B_M^{a+1}) \). Let \( u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B_M^{a+1}) \) \( (0 < \epsilon \leq 1) \) be the solutions to (4.7) with \( u(0) = u_0 \), found in Proposition 4.3. We see from (3.25), (4.4), Propositions 3.5 and 4.3 that the family \( \{ u_\epsilon(t) \}_{0 < \epsilon \leq 1} \) is bounded in \( \mathcal{E}_t^0([0, T]; B_M^{a+1}) \) and equi-continuous in \( \mathcal{E}_t^0([0, T]; B_M^a) \) because
\[
i \{ u_\epsilon(t) - u_\epsilon(t') \} = \int_{t'}^t \tilde{H}_\epsilon(\theta) u_\epsilon(\theta) d\theta + \int_{t}^{t'} f(\theta) d\theta
\]
and
\[
\sup_{0 < \epsilon \leq 1, 0 \leq t \leq T} \| \tilde{H}_\epsilon(t) u_\epsilon(t) \|_{a, M} \leq C_a \sup_{0 < \epsilon \leq 1, 0 \leq t \leq T} \| u_\epsilon(t) \|_{a+1, M} \leq C_a' \| u_0 \|_{a+1, M}.
\]
Consequently, it follows from Lemma 4.4 that we can apply the Ascoli-Arzelà theorem to \( \{u_\epsilon(t)\}_{0 < \epsilon \leq 1} \) in \( \mathcal{E}_t^0([0, T]; B_M^a) \). Then, there exist a sequence \( \{\epsilon_j\}_{j=1}^\infty \) tending to zero and a function \( u(t) \in \mathcal{E}_t^0([0, T]; B_M^a) \) such that

\[
\lim_{j \to \infty} u_{\epsilon_j}(t) = u(t) \text{ in } \mathcal{E}_t^0([0, T]; B_M^a).
\]  

(4.12)

Since from (4.7) we have

\[
u_{\epsilon_j}(t) = u_0 - i \int_0^t \tilde{H}_{\epsilon_j}(\theta)u_{\epsilon_j}(\theta)d\theta - i \int_0^t f(\theta)d\theta
\]

\[
= u_0 - i \int_0^t \tilde{H}_{\epsilon_j}(\theta)u(\theta)d\theta - i \int_0^t \tilde{H}_{\epsilon_j}(\theta)\{u_{\epsilon_j}(\theta) - u(\theta)\}d\theta - i \int_0^t f(\theta)d\theta,
\]

as in the proof of (3.14) in [5] from (3.25) and (4.4) we have

\[
u(t) = u_0 - i \int_0^t \tilde{H}(\theta)u(\theta)d\theta - i \int_0^t f(\theta)d\theta
\]

in \( \mathcal{E}_t^0([0, T]; B_{M}^{a-1}) \) by using Lemma 2.2 in [5]. Therefore we see that \( u(t) \) belongs to \( \mathcal{E}_t^0([0, T]; B_M^a) \cap \mathcal{E}_t^1([0, T]; B_{M}^{a-1}) \) and satisfies

\[
i \frac{\partial u}{\partial t}(t) = \tilde{H}(t)u(t) + f(t)
\]  

(4.13)

with \( u(0) = u_0 \). From (4.8) and (4.12) we also have

\[
\|u(t)\|_{a,M} \leq C_a \left( \|u_0\|_{a,M} + \int_0^t \|f(\theta)\|_{a,M}d\theta \right).
\]  

(4.14)

**2nd step.** In this step we will prove the uniqueness of solutions to (1.1) in \( \mathcal{E}_t^0([0, T]; B_M^a) \cap \mathcal{E}_t^1([0, T]; B_{M}^{a-1}) \) for any \( a = 0, \pm 1, \pm 2, \ldots \).

Let \( u(t) \in \mathcal{E}_t^0([0, T]; B_M^a) \cap \mathcal{E}_t^1([0, T]; B_{M}^{a-1}) \) be a solution to (1.1), i.e.

\[
i \frac{\partial u}{\partial t}(t) = \tilde{H}(t)u(t)
\]

with \( u(0) = 0 \). We may assume \( a \leq 0 \) because of \( B_{M}^{a+1} \subset B_{M}^a \). Let \( g(t) \in \mathcal{E}_t^0([0, T]; B_{M}^{a+2}) \) be an arbitrary function and consider the backward Cauchy problem

\[
i \frac{\partial v}{\partial t}(t) = \{H(t) + iK(t)\}v(t) + g(t)
\]
with \( v(T) = 0 \). Since \( (1.3) \) is assumed, as in the proof of the 1st step we can get a solution \( v(t) \in E^0_t([0, T]; B_M^{-a+1}) \cap E^1_t([0, T]; B_M^{-a}) \). Then we have

\[
0 = \int_0^T \left( i \frac{\partial u}{\partial t}(t) - \tilde{H}(t)u(t), v(t) \right) dt \\
= \int_0^T \left( u(t), i \frac{\partial v}{\partial t}(t) - \{ H(t) + iK(t) \} v(t) \right) dt = \int_0^T (u(t), g(t)) dt,
\]

which shows \( u(t) = 0 \).

**3rd step.** Let \( u_0 \in B_M^a \). We take \( \{ u_{0j} \}_{j=1}^\infty \) in \( B_M^{a+1} \) such that \( \lim_{j \to \infty} u_{0j} = u_0 \) in \( B_M^a \). Let \( u_j(t) \in E^0_t([0, T]; B_M^a) \cap E^1_t([0, T]; B_M^{a-1}) \) be the solution to \( (1.1) \) with \( u(0) = u_{0j} \), uniquely determined in the above 2 steps. Since \( u_j(t) - u_k(t) \in E^0_t([0, T]; B_M^a) \cap E^1_t([0, T]; B_M^{a-1}) \) is the solution to \( (1.1) \) with \( u(0) = u_{0j} - u_{0k} \in B_M^{a+1} \), from \( (4.14) \) we have

\[
\| u_j(t) - u_k(t) \|_{a,M} \leq C_a \| u_{0j} - u_{0k} \|_{a,M}.
\]

Consequently, there exists a \( u(t) \in E^0_t([0, T]; B_M^a) \) such that \( \lim_{j \to \infty} u_j(t) = u(t) \) in \( E^0_t([0, T]; B_M^a) \). Since

\[
u_j(t) = u_{0j} - i \int_0^t \tilde{H}(\theta)u_j(\theta) d\theta,
\]

\( u(t) \) belongs to \( E^0_t([0, T]; B_M^a) \cap E^1_t([0, T]; B_M^{a-1}) \) and satisfies \( (1.1) \) with \( u(0) = u_0 \). We can also prove \( (2.8) \) because \( \| u_j(t) \|_{a,M} \leq C_a \| u_{0j} \|_{a,M} \) holds from \( (4.14) \). Thus we have completed the proof of Theorem 2.1 under Assumption 2.2.

We will prove Theorem 2.1 under Assumption 2.1. We define \( \Lambda(X, D_x) \) by \( (3.14) \) where \( h_s(x, \xi) \) is replaced with \( |x|^2 + |\xi|^2 \). We also define \( \chi_s(x, \xi) \) by \( (3.15) \) where \( h(x, \xi) \) is replaced with \( |x|^2 + |\xi|^2 \). Then it is easy to show the same assertions as in Lemma 3.4. Noting Proposition 3.5, we can also prove the same assertions as in Lemma 4.1 and Propositions 4.2 - 4.3 where \( M = 0 \).
Consequently, we can prove Theorem 2.1 under Assumption 2.1 as in the proof of that under Assumption 2.2. Thus, our proof of Theorem 2.1 is complete.

Next, we will prove (2) of Theorem 2.2. The proof of (1) of Theorem 2.2 can be given in the same way. Our proof below is similar to that of Theorem 4.1 in [7]. For simplicity we write \( \|f\|_{a,M} \) as \( \|f\|_a \) hereafter in this section.

Let \( u(t; \rho) (\rho \in \mathcal{O}) \) be the solutions to (1.1) with \( u(0; \rho) = u_0 \in B^a_M (a = 0, \pm 1, \pm 2, \ldots) \). Then, following the proof of Theorem 2.1, under the assumptions of Theorem 2.2 we have

\[
\sup_{\rho \in \mathcal{O}} \|u(t; \rho)\|_a \leq C_a \|u_0\|_a \quad (0 \leq t \leq T). \tag{4.16}
\]

We first assume \( u_0 \in B^{a+1}_M \). Then from (4.16) we have

\[
\sup_{\rho \in \mathcal{O}} \|u(t; \rho)\|_{a+1} \leq C_{a+1} \|u_0\|_{a+1}
\]

and hence as in the 1st step of the proof of Theorem 2.1

\[
\|u(t; \rho) - u(t'; \rho)\|_a \leq C'_a |t - t'| \|u_0\|_{a+1}
\]

with a constant \( C'_a \) independent of \( \rho \). Consequently, we see that the family \( \{u(t; \rho)\}_{\rho \in \mathcal{O}} \) is bounded in \( E^0_t([0, T]; B^{a+1}_M) \) and equi-continuous in \( E^0_t([0, T]; B^a_M) \).

Let \( \rho_j \to \rho \) in \( \mathcal{O} \) as \( j \to \infty \). Noting Lemma 4.4, we can apply the Ascoli-Arzelà theorem to \( \{u(t; \rho_j)\}_{j=1}^\infty \) in \( E^0_t([0, T]; B^{a+1}_M) \). Then, there exist a subsequence \( \{j_k\}_{k=1}^\infty \) and a function \( v(t) \in E^0_t([0, T]; B^{a+1}_M) \) such that \( \lim_{k \to \infty} u(t; \rho_{j_k}) = v(t) \) in \( E^0_t([0, T]; B^{a+1}_M) \). As in the proof of (4.13) we see that \( v(t) \) belongs to \( E^1_t([0, T]; B^{a-1}_M) \) and satisfies (1.1) with \( u(0) = u_0 \). The uniqueness of solutions to (1.1) gives \( v(t) = u(t; \rho) \), which shows \( \lim_{k \to \infty} u(t; \rho_{j_k}) = u(t; \rho) \). Using the uniqueness of solutions to (1.1) again, we have

\[
\lim_{j \to \infty} u(t; \rho_j) = u(t; \rho) \text{ in } E^0_t([0, T]; B^a_M).
\]
Therefore we see that the mapping \( \mathcal{O} \ni \rho \mapsto u(t; \rho) \in \mathcal{E}^0_t([0, T]; B^a_M) \) is continuous.

Now let \( u_0 \in B^a_M \) and \( u(t; \rho) \ (\rho \in \mathcal{O}) \) the solutions to (1.1) with \( u(0) = u_0 \). We take \( \{u_{0k}\}_{k=1}^{\infty} \) in \( B^{a+1}_M \) such that \( \lim_{k \to \infty} u_{0k} = u_0 \) in \( B^a_M \) and let \( u_k(t; \rho) \in \mathcal{E}^0_t([0, T]; B^a_M) \cap \mathcal{E}_l^1([0, T]; B^{a-1}_M) \) be the solutions to (1.1) with \( u(0) = u_{0k} \). Then, from (4.16) we have

\[
\sup_{\rho} \max_t \|u_k(t; \rho) - u(t; \rho)\|_a \leq C_a \|u_{0k} - u_0\|_a,
\]

which shows that \( u(t; \rho) \) is continuous in \( \mathcal{E}^0_t([0, T]; B^a_M) \) with respect to \( \rho \in \mathcal{O} \) because so is \( u_k(t; \rho) \). Thus our proof of Theorem 2.2 is complete.

In the end of this section we will prove (2) of Theorem 2.3. The proof of (1) is given in the same way. Our proof below is similar to that of Theorem 2.3 in [7].

Let \( u_0 \in B^a_M \) \((a = 0, \pm 1, \pm 2, \ldots)\) and \( f(t) \in \mathcal{E}^0_t([0, T]; B^a_M) \). Then, we see that there exists the unique solution \( u(t) \in \mathcal{E}^0_t([0, T]; B^a_M) \cap \mathcal{E}_l^1([0, T]; B^{a-1}_M) \) to (4.13) with \( u(0) = u_0 \), which satisfies

\[
\|u(t)\|_a \leq C_a \left( \|u_0\|_a + \int_0^t \|f(\theta)\|_a d\theta \right). \tag{4.17}
\]

Its proof can be completed by using (4.14) as in the 3rd step of the proof of Theorem 2.1.

Let \( u_0 \in B^{a+1}_M \) and \( u(t; \rho) \in \mathcal{E}^0_t([0, T]; B^{a+1}_M) \cap \mathcal{E}_l^1([0, T]; B^a_M) \) \((\rho \in \mathcal{O})\) the solutions to (1.1) with \( u(0) = u_0 \). Let \( \rho \in \mathcal{O} \) be fixed and \( \tau \neq 0 \) small constants such that \( \rho + \tau \in \mathcal{O} \). We set

\[
w_\tau(t; \rho) := \frac{u(t; \rho + \tau) - u(t; \rho)}{\tau}, \tag{4.18}
\]

which belong to \( \mathcal{E}^0_t([0, T]; B^a_M) \cap \mathcal{E}_l^1([0, T]; B^{a-1}_M) \). Then we have \( w_\tau(0; \rho) = 0 \)
and from (1.11)
\[ i \frac{\partial}{\partial t} w_\tau(t; \rho) = \tilde{H}(t; \rho) w_\tau(t; \rho) + \int_0^1 \frac{\partial \tilde{H}}{\partial \rho}(t; \rho + \theta \tau) \, d\theta \, u(t; \rho + \tau). \quad (4.19) \]

Hence, noting (2.15) - (2.17), from (4.16) and (4.17) we get
\[
\| w_\tau(t; \rho) \|_a \leq C_a \int_0^t \int_0^1 \left\| \frac{\partial \tilde{H}}{\partial \rho}(t'; \rho + \theta \tau) u(t'; \rho + \tau) \right\|_a \, d\theta \, dt' \leq C'_a \| u_0 \|_{a+1}.
\]

Consequently,
\[
\sup_{\tau} \| w_\tau(t; \rho) \|_a \leq C_a \| u_0 \|_{a+1} \quad (4.20)
\]
with another constant $C_a$ independent of $\rho \in \mathcal{O}$.

We first assume $u_0 \in B^{a+2}_M$. From (4.20) we have
\[
\sup_{\tau} \| w_\tau(t; \rho) \|_{a+1} \leq C_{a+1} \| u_0 \|_{a+2}.
\]

Thereby from (4.16) and (4.19) we have
\[
\sup_{\tau} \| w_\tau(t; \rho) - w_\tau(t'; \rho) \|_a \leq C'_a |t - t'| \| u_0 \|_{a+2}
\]

as in the 1st step of the proof of Theorem 2.1. Hence we can apply the Ascoli-Arzelà theorem to $\{ w_\tau(t; \rho) \}_{\tau}$ in $\mathcal{E}_t^0([0, T]; B^a_M)$. In addition, we can use the uniqueness of solutions to (2.14) or (4.13). Then, using Theorem 2.2, as in the 3rd step of the proof of Theorem 2.1 we can prove from (4.19) that there exists a function $w(t; \rho) \in \mathcal{E}_t^0([0, T]; B^a_M) \cap \mathcal{E}_t^1([0, T]; B^{a-1}_M)$ satisfying (2.14) with $w(0) = 0$ and
\[
\lim_{\tau \to 0} w_\tau(t; \rho) = w(t; \rho) \text{ in } \mathcal{E}_t^0([0, T]; B^a_M). \quad (4.21)
\]

Now let $u_0 \in B^{a+1}_M$. Let $u(t; \rho)$ be the solution to (1.1) with $u(0) = u_0$ and define $w_\tau(t; \rho)$ by (4.18). We take $\{ u_{0k} \}_{k=1}^\infty \in B^{a+2}_M$ such that $\lim_{k \to \infty} u_{0k} = u_0$. [28]
in $B_{M}^{a+1}$. Let $u_k(t; \rho) \in \mathcal{E}_{t}^{0}([0, T]; B_{M}^{a+1}) \cap \mathcal{E}_{t}^{1}([0, T]; B_{M}^{a})$ be the solution to (1.1) with $u(0) = u_{0k}$. We define $w_{k\tau}$ by (4.18) with $u = u_k$ and $w_{\tau}$ by (4.21) with $w_{\tau} = w_{k\tau}$. From (4.19) we have

$$i \frac{\partial}{\partial t} \left\{ w_{k\tau}(t; \rho) - w_{\tau}(t; \rho) \right\} = \tilde{H}(t; \rho) \left\{ w_{k\tau}(t; \rho) - w_{\tau}(t; \rho) \right\}$$

$$+ \int_{0}^{1} \frac{\partial \tilde{H}(t; \rho + \theta \tau)}{\partial \rho} d\theta \left\{ u_k(t; \rho + \tau) - u(t; \rho + \tau) \right\}$$

and $w_{k\tau} - w_{\tau} \in \mathcal{E}_{t}^{0}([0, T]; B_{M}^{a}) \cap \mathcal{E}_{t}^{1}([0, T]; B_{M}^{a-1})$. Hence, as in the proof of (4.20) we have

$$\sup_{\tau} \| w_{k\tau}(t; \rho) - w_{\tau}(t; \rho) \|_{a} \leq C_{a} \| u_{0k} - u_{0} \|_{a+1}$$

(4.22)

with the constant $C_{a}$ in (4.20). As noted in the first part of this proof, there exists the solution $w(t; \rho) \in \mathcal{E}_{t}^{0}([0, T]; B_{M}^{a}) \cap \mathcal{E}_{t}^{1}([0, T]; B_{M}^{a-1})$ to (2.14) with $w(0) = 0$ because of $\partial_{\rho} \tilde{H}(t; \rho) u(t; \rho) \in \mathcal{E}_{t}^{0}([0, T]; B_{M}^{a})$. Since both of $w_{k}$ and $w$ are the solutions to (2.14), as in the proof of (4.22) we have

$$\| w_{k}(t; \rho) - w(t; \rho) \|_{a} \leq C_{a} \| u_{0k} - u_{0} \|_{a+1}.$$  

(4.23)

Consequently, we have

$$\| w_{\tau}(t; \rho) - w(t; \rho) \|_{a} \leq \| w_{\tau} - w_{k\tau} \|_{a} + \| w_{k\tau} - w_{k} \|_{a} + \| w_{k} - w \|_{a}$$

$$\leq 2C_{a} \| u_{0k} - u_{0} \|_{a+1} + \| w_{k\tau} - w_{k} \|_{a}.$$  

Hence we see from (4.21) that we get $\lim_{\tau \to 0} \max_{t} \| w_{\tau} - w \|_{a} \leq 2C_{a} \| u_{0k} - u_{0} \|_{a+1}$, which shows

$$\lim_{\tau \to 0} \max_{0 \leq t \leq T} \| w_{\tau}(t; \rho) - w(t; \rho) \|_{a} = 0.$$  

(4.24)

We also have (2.13) from (4.20) and (4.24).
In the end of this proof we will prove that \(w(t; \rho) = \partial_{\rho} u(t; \rho)\) for \(u_0 \in B^{a+1}_M\) is continuous in \(E^0_t([0, T]; B^{a}_M)\) with respect to \(\rho \in \mathcal{O}\). We first assume \(u_0 \in B^{a+2}_M\). Then we have (4.16) where \(a\) is replaced with \(a+2\). Since \(w(t; \rho)\) is the solution to (2.14) with \(w(0) = 0\), we see from (4.17) as in the proof of (4.21) that the family \(\{w(t; \rho)\}_{\rho \in \mathcal{O}}\) is bounded in \(E^0_t([0, T]; B^{a+1}_M)\) and equi-continuous in \(E^0_t([0, T]; B^{a}_M)\). Hence, noting that \(u(t; \rho)\) is continuous in \(E^0_t([0, T]; B^{a+2}_M)\) with respect to \(\rho\), we see that so is \(w(t; \rho)\) in \(E^0_t([0, T]; B^{a}_M)\) as in the proof of Theorem 2.2. Now let \(u_0 \in B^{a+1}_M\). We take \(\{u_{0k}\}_{k=1}^\infty\) in \(B^{a+2}_M\) such that \(\lim_{k \to \infty} u_{0k} = u_0\) in \(B^{a+1}_M\) and write as \(w_k(t; \rho)\) the solutions to (2.14) with \(u(t; \rho) = u_k(t; \rho)\) and \(w(0) = 0\). Then we have (4.23), which shows that \(w(t; \rho)\) is continuous with respect to \(\rho \in \mathcal{O}\) in \(E^0_t([0, T]; B^{a}_M)\) because so is \(w_k(t; \rho)\). Therefore, our proof of Theorem 2.3 is complete.

5 Proofs of Theorems 2.4 - 2.6

In this section we will study the 4-particle systems (1.7). Let \((x, \xi) \in \mathbb{R}^{2d}\) and write

\[
h_k(t, x, \xi) := \frac{1}{2m_k} |\xi - A^{(k)}(t, x)|^2 + V_k(t, x) \quad (k = 1, 2, 3, 4) \quad (5.1)
\]

and

\[
l_k(x, \xi) := \frac{1}{2m_k} |\xi|^2 + <x>^2 \quad (k = 3, 4). \quad (5.2)
\]

We set

\[
\hat{h}(t, z, \zeta) := \sum_{k=1}^2 h_k(t, x^{(k)}, \xi^{(k)}) + W_{12}(t, x^{(1)} - x^{(2)}) + \sum_{k=3}^4 l_k(x^{(k)}, \xi^{(k)}) \quad (5.3)
\]

and write

\[
\hat{H}(t) := \hat{H} \left( t, \frac{Z + Z'}{2}, D_z \right), \quad (5.4)
\]
where \( z = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \) and \( \zeta = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}) \) in \( \mathbb{R}^{4d} \). We also set

\[
\widehat{h}_s(t, z, \zeta) := \widehat{h}(t, z, \zeta) + i \sum_{k=1}^{2} \frac{1}{2m_k} \nabla \cdot A^{(k)}(t, x^{(k)})
\]  

(5.5)

and

\[
p_{\mu}(t, z, \zeta) := \frac{1}{\mu + \widehat{h}_s(t, z, \zeta)}
\]

for large \( \mu \) as in (3.2) and (3.3), respectively.

**Lemma 5.1.** Assume (1.5), (2.3) and (2.5) - (2.6) for \((V_k, A^{(k)}) \) \( (k = 1, 2) \) with \( M = M_k \) and (2.18) - (2.19) for \( W_{12} \). Then, there exist a constant \( \mu^* \geq 0 \) and functions \( r_{\mu}(t, z, \zeta) \) \( (\mu \geq \mu^*) \) such that

\[
\mu^* + \text{Re} \, \widehat{h}_s(t, z, \zeta) \geq C_0^* (\zeta >^2 + \Phi(z)^2),
\]

(5.6)

\[
\left[ \mu + \widehat{H}(t) \right] P_{\mu}(t, Z, D_Z) = I + R_{\mu}(t, Z, D_Z),
\]

(5.7)

\[
\left| r_{\mu}^{(\alpha)}(t, z, \zeta) \right| \leq C_{\alpha\beta} \mu^{-1/2}
\]

(5.8)

in \([0, T] \times \mathbb{R}^{8d} \) for all \( \alpha, \beta \) and \( \mu \geq \mu^* \) with constants \( C_0^* > 0 \) and \( C_{\alpha\beta} \) independent of \( \mu \), where

\[
\Phi(z) = \sum_{k=1}^{2} < x^{(k)} >^{M_k+1} + \sum_{k=3}^{4} < x^{(k)} > .
\]

(5.9)

**Proof.** As in the proof of (3.6) we see

\[
\text{Re} \, \widehat{h}_s(t, z, \zeta) = \widehat{h}(t, z, \zeta) \geq C_0 (\zeta >^2 + \Phi(z)^2) - |W_{12}(t, x^{(1)} - x^{(2)})| - C_1
\]

with constants \( C_0 > 0 \) and \( C_1 \geq 0 \). Hence, using the assumption (2.18), we can determine constants \( \mu^* \geq 0 \) and \( C_0^* > 0 \) satisfying (5.6). Then, using (5.6), as in the proof of (3.7) for \( \mu \geq \mu^* \) we have

\[
r_{\mu}(t, z, \zeta) = \sum_{|\alpha| = 1} \int_{0}^{1} d\theta \, O_s - \int \int e^{-iy \cdot \eta} < y >^{-2l_0} < D_\eta >^{2l_0} < \eta >^{-2l_1} < D_\eta >^{2l_1} \cdot \widehat{h}_s^{(\alpha)}(t, z, \zeta + \theta \eta) P_{\mu(\alpha)}(t, z + y, \xi) dy d\eta
\]

(5.10)
for large integers $l_0$ and $l_1$. In addition, as in the proofs of (3.8) - (3.9) we can show
\[ |\hat{h}_{s, (\beta)}^{(\alpha)}(t, z + y, \zeta)| \leq C_{\alpha \beta} \left( \langle \zeta \rangle^2 + \Phi(z + y)^2 \right) \tag{5.11} \]
for all $\alpha$ and $|\beta| \geq 1$, and
\[ |\hat{h}_{s, (\beta)}^{(\alpha)}(t, z + \theta \eta)| \leq C_{\alpha \beta} \left( \langle \zeta \rangle + \Phi(z) \right) \eta \tag{5.12} \]
for $|\alpha| \geq 1$ and all $\beta$. Therefore, we can complete the proof of Lemma 5.1 from (5.10) - (5.12) as in the proof of Lemma 3.2.

We can easily see from (5.11) and (5.12) as in the proof of (3.25) that under the assumptions of Lemma 5.1 we have
\[ |\hat{h}_{s, (\beta)}^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha \beta} \left( \langle \zeta \rangle^2 + \Phi(z)^2 \right)^{-1} \tag{5.13} \]
for all $\alpha$ and $\beta$.

**Proposition 5.2.** Under the assumptions of Lemma 5.1 there exist a constant $\mu \geq \mu^*$ and a function $w(t, z, \zeta)$ satisfying
\[ |w_{(\beta)}^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha \beta} \left( \langle \zeta \rangle^2 + \Phi(z)^2 \right)^{-1} \tag{5.14} \]
for all $\alpha, \beta$ and
\[ W(t, Z, D_z) = \left( \mu + \tilde{H}(t) \right)^{-1}. \tag{5.15} \]

**Proof.** If $\mu \geq \mu^*$, from (5.6) and (5.11) - (5.12) we see
\[ |p_{\mu, (\beta)}^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha \beta} \left( \langle \zeta \rangle^2 + \Phi(z)^2 \right)^{-1} \]
for all $\alpha$ and $\beta$ as in the proof of Proposition 3.3. Hence, using Lemma 5.1, we can prove Proposition 5.2 as in the proof of Proposition 3.3. \qed
We take a $\mu$ stated in Proposition 5.2 and fix it hereafter. We set
\[ \lambda(t, z, \zeta) := \mu + \hat{h}_s(t, z, \zeta) \]  
(5.16)
as in (3.13). Then, from (5.1) - (5.5) we have
\[ \Lambda(t) = \Lambda(t, Z, D_z) = \mu + \hat{H}(t) \]
(5.17)
where $H_k(t)$ are the operators defined by (1.7) and $L_k(t)$ the pseudo-differential operators with the symbols $l_k(x^{(k)}, \xi^{(k)})$ defined by (5.2). Using the real-valued function $\hat{h}(t, z, \zeta)$ defined by (5.3), we determine
\[ \chi_\epsilon(t, z, \zeta) = \chi(\epsilon(\mu + \hat{h}(t, z, \zeta))) \]
(5.18)
with constants $0 < \epsilon \leq 1$ and $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$ as in (3.15).

Lemmas 5.3 and 5.4 below are crucial in this section.

**Lemma 5.3.** Under the assumptions of Lemma 5.1 there exist functions $\omega_\epsilon(t, z, \zeta)$ $(0 < \epsilon \leq 1)$ in $[0, T] \times \mathbb{R}^d$ satisfying
\[ \sup_{0 < \epsilon \leq 1} \sup_{t, z, \zeta} |\omega^{(\alpha)}_\epsilon(t, z, \zeta)| \leq C_{\alpha\beta} < \infty \]
(5.19)
for all $\alpha, \beta$ and
\[ \Omega_\epsilon(t, Z, D_z) = \left[ X_\epsilon(t, Z, D_z), \Lambda(t, Z, D_z) \right]. \]
(5.20)

**Proof.** As in the proof of (3.18) we see
\[ \omega_\epsilon(t, z, \zeta) = \sum_{|\alpha| = 1} \left\{ \chi^{(\alpha)}_\epsilon(t, z, \zeta) \lambda_\epsilon(t, z, \zeta) - \lambda_\epsilon(t, z, \zeta) \chi^{(\alpha)}_\epsilon(t, z, \zeta) \right\} \]
\[ + 2 \sum_{|\gamma| = 2} \frac{1}{\gamma !} \int_0^1 (1 - \theta) d\theta \left( \int e^{-iy\eta \cdot \eta} \left\{ \chi^{(\gamma)}_\epsilon(t, z, \zeta + \theta\eta) \lambda^{(\gamma)}_\epsilon(t, z + y, \zeta) - \lambda^{(\gamma)}_\epsilon(t, z, \zeta + \theta\eta) \chi^{(\gamma)}_\epsilon(t, z + y, \xi) \right\} dyd\eta \right) \equiv I_{1\epsilon} + I_{2\epsilon}. \]
(5.21)
From (5.5), (5.16) and (5.18) we can write

\[
I_1(t, z, \zeta) = \epsilon \chi'(\epsilon(\mu + \hat{h})) \sum_{|\alpha|=1} \left\{ \hat{h}_{s}^{(\alpha)}(t, z, \zeta) (t, z, \zeta) \sum_{k=1}^{2} \frac{1}{2m_k} (-i \partial_z)^\alpha \nabla \cdot A^{(k)}(t, x^{(k)}, \zeta^{(k)}). \right\}
\]

(5.22)

From (5.6) we have

\[
(\mu + \hat{h}(t, z, \zeta))^{-1} \leq C_{0} (\zeta >^{2} + \Phi(z)^{2})^{-1}
\]

(5.23)

because of \( \hat{h} = \text{Re} \hat{h}_{s} \). Hence, together with (2.6) and (5.12) we can prove \( \sup_{t, z, \zeta} |I_{1\epsilon}| < \infty \) as in the proof of (3.20). In the same way we can prove

\[
\sup_{0<\epsilon \leq 1} \sup_{t, z, \zeta} |I^{(\alpha)}_{1\epsilon}(t, z, \zeta)| \leq C_{\alpha \beta} < \infty
\]

(5.24)

for all \( \alpha \) and \( \beta \).

Let \( |\gamma| = 2 \). Then, from (5.5) and (5.11) - (5.12) we have the similar inequalities

\[
\sup_{0<\epsilon \leq 1} |\chi_{\epsilon}^{(\alpha+\gamma)}(t, z, \zeta)| \leq C_{\alpha \beta} (\zeta >^{2} + \Phi(z)^{2})^{-1}
\]

and

\[
\sup_{0<\epsilon \leq 1} |\chi_{\epsilon}^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha \beta} < \infty
\]

to (3.21) and (3.22) for all \( \alpha \) and \( \beta \), respectively. Consequently, noting that \( \lambda^{(\gamma)}(t, z, \zeta) = \hat{h}_{s}^{(\gamma)}(t, z, \zeta) \) are constants, from (5.21) we can prove

\[
\sup_{0<\epsilon \leq 1} \sup_{t, z, \zeta} |I^{(\alpha)}_{2\epsilon}(t, z, \zeta)| \leq C_{\alpha \beta} < \infty
\]

for all \( \alpha \) and \( \beta \) as in the proof of Lemma 3.4, which completes the proof together with (5.21) and (5.24).
Let $\tilde{H}(t)$ be the operator defined by (1.7).

**Lemma 5.4.** Besides the assumptions of Lemma 5.1 we suppose that each $(V_k, A^{(k)})$ $(k = 3, 4)$ satisfies (1.4) and each $W_{ij}(t, x)$ $(1 \leq i < j \leq 4)$ except $W_{12}$ satisfies (2.20). In addition, we suppose that $k_l(t, x, \xi)$ $(l = 1, 2)$ and $k_l(t, x, \xi)$ $(l = 3, 4)$ satisfy (2.2) with $M = M_l$ and (2.1), respectively. Then, there exists a function $\tilde{q}(t, z, \zeta)$ satisfying

$$\sup_{t, z, \zeta} |q^{(\alpha)}_{(\beta)}(t, z, \zeta)| \leq C_{\alpha\beta} < \infty$$

(5.25)

for all $\alpha, \beta$ and

$$\tilde{Q}(t, Z, D_z) = \left[ \Lambda(t), \tilde{H}(t) \right] \Lambda(t)^{-1}.$$  

(5.26)

**Proof.** We write $\tilde{H}(t)$ as

$$\tilde{H}(t) = \sum_{k=1}^{4} \tilde{H}_k(t) + W_{12}(t) + \sum ' W_{ij}(t),$$

(5.27)

where $\tilde{H}_k(t) = H_k(t) - iK_k(t)$. Then from (5.17) we see

$$[\tilde{H}(t), \Lambda(t)] = [(\tilde{H}_1 + \tilde{H}_2 + W_{12}) + \tilde{H}_3 + \tilde{H}_4 + \sum ' W_{ij} \cdot (H_1 + H_2 + W_{12})$$

$$+ L_3 + L_4] = (-i[K_1, H_1] - i[K_2, H_2] - i[K_1 + K_2, W_{12}] + [\tilde{H}_3, L_3] + [\tilde{H}_4, L_4])$$

$$+ \left[ \sum ' W_{ij}, H_1 + H_2 + L_3 + L_4 \right] \equiv I_1(t) + I_2(t).$$

(5.28)

As in the proof of Lemma 4.1, we can prove that $I_1(t)\Lambda(t)^{-1}$ is written as the pseudo-differential operator with a symbol satisfying (5.25).

We can easily see that $m_1 [W_{13}(t), H_1(t)]$ is written as the pseudo-differential operator with the symbol

$$\tilde{q}_1(t, z, \zeta) = i \frac{\partial W_{13}}{\partial x}(t, x^{(1)} - x^{(3)}) \cdot \xi^{(1)} + \frac{1}{2} \Delta_x W_{13}(t, x^{(1)} - x^{(3)})$$

$$- i A^{(1)}(t, x^{(1)}) \cdot \frac{\partial W_{13}}{\partial x}(t, x^{(1)} - x^{(3)}).$$

(5.29)
Hence from the assumptions we have
\[
|\tilde{q}_1(t, z, \zeta)| \leq C_1 \left( <\xi^{(1)}_1>^2 + <x^{(1)}_1 - x^{(3)}_1>^2 + <x^{(1)}_1 - x^{(3)}_1>^{M_1+1} \right) \\
\leq C_2 \left( <\xi^{(1)}_1>^2 + <x^{(1)}_1>^{M_1+2} + <x^{(1)}_1>^{2(M_1+1)} + <x^{(3)}_1>^2 \right) \\
\leq C_3 \left( <\xi^{(1)}_1>^2 + <x^{(1)}_1>^{2(M_1+1)} + <x^{(3)}_1>^2 \right).
\]
In the same way we have
\[
|\tilde{q}_1^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha \beta} \left( <\zeta>^2 + \Phi(z)^2 \right) \quad (5.30)
\]
for all $\alpha$ and $\beta$. Consequently, by Proposition 5.2 we see that $[W_{13}(t), H_1(t)] \Lambda(t)^{-1}$ is written as the pseudo-differential operator with a symbol satisfying (5.25).

In the same way we can complete the proof of Proposition 5.4.

Lemma 5.5. Under Assumption 2.3 there exist functions $q_{\epsilon}(t, z, \zeta)$ ($0 < \epsilon \leq 1$) satisfying (5.19) and

\[
Q_{\epsilon}(t, Z, D_z) = \left[ \Lambda(t, Z, D_z), \tilde{H}_{\epsilon}(t) \right] \Lambda(t, Z, D_z)^{-1} \\
+ i \frac{\partial \Lambda}{\partial t}(t, Z, D_z) \Lambda(t, Z, D_z)^{-1}. \quad (5.32)
\]

Proof. From (5.31) and $\Lambda(t)^\dagger = \Lambda(t)$ we have

\[
\left[ \Lambda(t), \tilde{H}_{\epsilon}(t) \right] = - \left[ \Lambda(t), X_{\epsilon}(t) \right]^\dagger \tilde{H}(t) X_{\epsilon}(t) \\
+ X_{\epsilon}(t)^\dagger \left[ \Lambda(t), \tilde{H}(t) \right] X_{\epsilon}(t) + X_{\epsilon}(t)^\dagger \tilde{H}(t) \left[ \Lambda(t), X_{\epsilon}(t) \right]
\]
as in the proof of Lemma 4.1. Apply Proposition 5.2 and Lemmas 5.3 - 5.4 to the above. In addition, apply Proposition 5.2 to $(i \partial \Lambda(t)/\partial t) \Lambda(t)^{-1}$. Then, we can prove Lemma 5.5 as in the proof of Lemma 4.1. □
Using Lemma 5.5, we can prove the following as in the proof of Proposition 4.2.

**Proposition 5.6.** Under Assumption 2.3 there exist functions $q_{ae}(t, z, \zeta) (a = 0, \pm 1, \pm 2, \ldots, 0 < \epsilon \leq 1)$ satisfying (5.19) and

$$Q_{ae}(t, Z, D_z) = \left[i \frac{\partial}{\partial t} - \tilde{H}_t(t), \Lambda(t)^{a}\right] \Lambda(t)^{-a}. \tag{5.33}$$

Let $B^{ta}(\mathbb{R}^{4d}) (a = 0, \pm 1, \pm 2, \ldots)$ be the weighted Sobolev spaces introduced in §2. Then we see that the embedding map from $B^{ta+1}$ into $B^{ta}$ is compact and that the similar result to Proposition 3.5 follows from Proposition 5.2. Therefore, using Proposition 5.6, we can prove Theorems 2.4 - 2.6 as in the proofs of Theorems 2.1 - 2.3, respectively.

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