Tuning the primary resonance of vibrating beam micro-gyroscopes based on piezoelectric actuation and multiple nonlinearities

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Abstract

In this paper, the nonlinear dynamic characteristics of statically piezoelectric actuated vibrating beam micro-gyroscopes are studied. The comprehensive nonlinear model including curvature, inertia and electrostatic force nonlinearities is considered. In the research of electrostatic micro-gyroscopes, it’s a novel way to tune the primary resonance by piezoelectric actuation and multiple nonlinearities. The multiple scales method and numerical continuation technique are used to characterize the frequency-amplitude and force-amplitude responses of the micro-gyroscopes. The effect of varying the size-dependent, fringing field, statically piezoelectric voltage and nonlinear curvature and inertia on the dynamic response of the micro-gyroscope is investigated in detail. The frequency-response results show that small vibrations produce a symmetrical frequency response curve in sense direction while the system actually has a significant softening characteristic in drive direction. The nonlinear multi-value problem effectively reduces in sense direction under the size-dependent effect, which plays an important role in the design of detection instruments for micro-gyroscopes. Choosing a positive piezoelectric actuation voltage will obtain a higher sensitivity. Increasing the curvature nonlinearity and reducing the inertial nonlinearity of the gyroscopic system will help the micro-gyroscope obtain better sensitivity, and may eliminate multi-valued responses as much as possible.

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**Key words:** Multiple nonlinearities; micro-gyroscopes; size-dependent; fringing field; statically piezoelectric actuation.

1. Introduction

In recent years, the investigation and development of micro-inertial sensors based on MEMS technology has been received extensive attention of many researchers [1]. Micro-inertial sensors are widely used in various fields of engineering such as automotive, air vehicles, aerospace, robotics, military systems and consumer electronics [2]. They are more attractive and highly applied due to their many notable advantages on low manufacturing cost, high durability, small size, and low-power [3, 4]. As a key component of micro-inertial sensor, vibrating beam micro-gyroscope has been proposed in tracking orientation, guiding direction, and controlling path. A great deal of design models and performance investigations of vibrating gyroscopes including different types, such as forks, beams, and shells, have been published in several studies [5-9]. The main objective of these vibrating micro-gyroscopes is to measure angular rate or angle via detecting Coriolis force of the system.

The actuation and transduction principles of vibrating beam gyroscopes commonly used in previous literatures are electrostatic and piezoelectric. Based on a simply supported piezoelectric beam gyroscope, Yang and Fang [10] analyzed the parameter performance of the system sensitivity. By considering the cantilever beam gyroscope, Li et al. [11] found the optimal combination of geometric parameters for the double-resonant condition with maximum sensitivity. By analyzing the flexural-torsional vibrations of a piezoelectric beam gyroscope, Bhadbhade et al. [12] studied its gyroscopic effect due to the angular speed of the system by using an assumed mode expansion method. Based on a cantilever beam attached to a proof mass at its free end, Esmaeili et al. [13] used Hamilton principle derive a general 6-dof frequency equation of an electrostatic micro-gyroscope without taking the effect of rotary inertia into account. They utilized a linear approximation method to denote electrostatic forcing and studied the effect of mass and acceleration of the vibratory gyroscope. An electrostatic micro-gyroscope with AC voltage to excite the primary vibration and
Coriolis force caused by rotational speed to generate the secondary vibration has been developed by Ghommen et al. [14]. They proposed a closed-form solution to investigate static deflection and linearized amplitude-frequency response in both drive and sense directions of the electrostatic micro-gyroscopes. Using the same model, Nayfeh et al. [15] presented a novel differential frequency-domain method to measure rotational speed of the electrostatic micro-gyroscopes. Lajimi et al. [16] derived an improved mathematical model of a rigid-body electrostatic micro-gyroscopes to correct the existent simplified model of a beam-mass micro-gyroscopes which examined by Rasekh and Khadem [17].

A lot of researchers have introduced the electrostatic force nonlinearity on the nonlinear response of the vibrating micro-gyroscopes due to its important impact. Based on the multiple scales method, Ghommem et al. [18] analyzed the nonlinear dynamics of the reduced-order model of the vibrating beam micro-gyroscopes. By using differential quadrature and finite difference methods, Ghommem and Abdelkefi [19] studied the primary resonance of the nonlinear vibrating beam micro-gyroscopes for varying grain sizes of nanocrystalline materials and several types of electric actuation conditions. Based on the assumed mode method, the nonlinear dynamics such as primary resonance and mechanical-thermal noise of eccentric micro-gyroscopes are investigated by Lajimi et al. [20-22]. However, few literatures specifically examined the effect of nonlinear curvature and inertia on the dynamic vibrations of the vibrating beam micro-gyroscopes. By considering nonlinear curvature and inertia of the micro/nano gyroscopes, Mojahedi et al. [2, 3, 23-25] mainly studied the static deflection and pull-in instability analysis, but did not investigate the multiple nonlinearities on nonlinear dynamics of the vibrating beam micro-gyroscopes.

Previous studies have shown that the incorporation of the size-dependent effect in the micro-structure system has a significant influence on the dynamic response [26]. Ghayesh et al. [27] developed a mathematical model of the vibrating beam gyroscope and presented that the size-dependent has a notable influence about the dynamic response by using the modified couple stress theory. Meanwhile, the fringing field effect on the pull-in instability is investigated by Ghommem and Abdelkefi [4].
However, they did not analyze the fringing field influence on the dynamic characteristic of electrostatic micro-gyroscopes. Therefore, the overall impact of the size-dependent and the fringing field on the nonlinear dynamic response should be studied for the electrostatic micro-gyroscope.

Axial excitation will be introduced by using piezoelectric layers bond to the four surfaces of the cantilever beam. By applying synchronous periodic voltages to the upper and lower piezoelectric layers, the tensile mechanical stress or compressive stress will be generated along the axial of the beam [28, 29]. For a lot of beam models, such as axially moving beams, accelerating beams, periodic structures and conveying fluid pipes, Yang et al. [30-37] investigated the stability in parametric resonance and nonlinear dynamics. By using this method to produce axial excitation of a micro-beam, Azizi et al. [38-43] investigated many dynamic responses such as stability, bifurcation, and primary resonance of the piezoelectrically actuated beam under different boundary conditions. However, the beams which they studied were all stationary, without rotational speed. By using a thin layer of piezoelectric film (PZF) encircling the circumferential surface of the ring, Liang et al. [44] found that the attached PZF can enhance the sensitivity of the MEMS ring gyroscope.

A vibrating beam micro-gyroscope model with considering multiple nonlinearities, the size-dependent effect and the fringing field effect is developed in this work. The partial differential governing equations with axial excitation of the vibrating micro-gyroscope are derived via the extended Hamilton’s principle in both drive and sense directions. Then the Galerkin technique is used to truncate the partial differential governing equations to ordinary differential equations [45]. The frequency/force amplitude response curves of the vibrating beam micro-gyroscope are studied by utilizing the multi-scale method and numerical continuation technique. The effect of the size-dependent, fringing field, nonlinear curvature and inertia, and piezoelectric DC voltage on nonlinear behaviors of the micro-gyroscope is investigated. It is found that nonlinear multi-value problem is obviously reduced in sense direction due to the effect of size-dependent. A higher sensitivity can be obtained by choosing a positive piezoelectric actuation DC voltage to the four piezoelectric layers which surrounded
the cantilever micro-beam. The influence of nonlinear curvature and inertial on the dynamic response of the vibrating beam micro-gyroscope is also investigated in detail.

2. Modeling

![Diagram of a vibrating beam micro-gyroscope]

**Fig. 1.** A vibrating beam micro-gyroscope under piezoelectric DC actuation.

The micro-cantilever beam is attached to a tip mass $M$ at its free end and surrounded piezoelectric layers on its four surfaces under a base rotation $\Omega$ along $x$ axis as shown in Fig. 1. The deformation of the piezoelectric micro-beam is described by means of two transverse displacements $v(x, t)$ and $w(x, t)$ respectively along sense and drive directions. The same $V_{DC1}$ voltages are applied to two fixed electrodes in sense and drive directions. Furthermore, the tip mass is driven by a $V_{AC\cos(\omega)}$ voltage only in drive direction, which produces a secondary vibration in sense direction due to the angular speed. Especially, the fixed $V_{DC2}$ voltages are applied to the four piezoelectric layers, which will produce axial excitation on the micro-beam due to the piezoelectric effect. By considering the inertia nonlinearity [46, 47], the total kinetic energy can be obtained as follows [14, 15]:

$$
T = \frac{1}{2} \int_0^L (m + M \delta(x - L)) \left[ \ddot{w}^2 + \ddot{v}^2 + \Omega^2 (w'^2 + v'^2) + 2\Omega (w'^{\delta} - w'^{v}) \right] dx + \\
\frac{1}{2} \int_0^L (m + M \delta(x - L)) \left[ -\frac{1}{2} \frac{\partial}{\partial t} (w'^2 + v'^2) \right] dx + LJ_\Omega^2 + \\
\frac{1}{2} \int_0^L \left[ j (\ddot{w}^2 + \ddot{v}^2) + j \Omega^2 (v'^2 + w'^2) + 2j\Omega (w'^{\delta} + w'^{v}) \right] dx,
$$

(1)
where \( \dot{t} \) and \( \dot{x} \) represent the derivatives of \( t \) and \( x \), respectively. \( \delta \) denotes the Dirac delta function. The beam and piezoelectric layer are made of thickness \( h_b \), \( h_p \) and width \( w_b \), \( w_p \) and same length \( L \), respectively. The mass per unit length \( m \) and mass moment of inertia \( j \) can be written as

\[
m = w_b \rho_b h_b^3 + 4w_p \rho_p h_p^3,
\]

\[
j = \rho_b I_b + \rho_p I_p,
\]

and \( \rho \) is the mass density, \( I \) is cross sectional second moments of area

\[
I_b = \frac{1}{12} w_b h_b^3,
\]

\[
I_p = 2 \left[ \frac{1}{12} w_p h_p^3 + w_p h_p \left( \frac{h_p}{2} + \frac{h_b}{2} \right)^2 + \frac{1}{12} h_p w_p^3 \right].
\]

By considering the effects of the size-dependent [27] and the fringing field [4], the total potential energy can be expressed by using the bending energy \( U_{bending} \), which considers the curvature nonlinearity [46, 47], the electrical energy \( U_{electrical} \) and axial strain energy \( U_{axial} \) [38] as follows:

\[
U = U_{bending} + U_{electrical} + U_{axial}
\]

\[
= \frac{EI + \mu Al^2}{2} \int_0^L \left[ \left( \frac{v''}{2} + \frac{1}{2} v'' + \frac{1}{2} v' w' w'' \right)^2 + \left( \frac{w''}{2} + \frac{1}{2} w'' + \frac{1}{2} v' v'' w'' \right)^2 \right] dx
\]

\[
+ w_b \varepsilon_3 V_{DC2} \int_0^L \left( v'^2 + w'^2 \right) dx - \frac{\varepsilon_0 A_v V_{DC1}^2 \delta(x-L)}{d_v} \left( 1 + \beta_F \frac{d_v - v}{h_M} \right) - \frac{\varepsilon_0 A_w [V_{DC1} + V_{AC} \cos(\omega t)]^2 \delta(x-L)}{d_v} \left( 1 + \beta_F \frac{d_v - w}{h_M} \right),
\]

where

\[
EI = E_b I_b + E_p I_p,
\]

and \( \mu \) is the shear modulus, \( l \) is the length-scale parameter, \( \varepsilon_0 \) is the dielectric constant, \( b_M \) and \( h_M \) are the width and thickness of the tip mass, \( e_{31} \) is the piezoelectric constant, \( \beta_F \) is the fringing field parameter, \( (A_w, A_v) \) are the areas and \( (d_w, d_v) \) are initial gap distances of the drive and sense capacitors, respectively.

Applying the Hamilton principle to the total kinetic and potential energy, we can obtain the flexural-flexural differential equations of motion in both sense and drive directions as follows:
\begin{align*}
(m + M \delta(x - L))(\dddot{w} + 2\Omega \ddot{w} - \Omega^2 w) &- J \dddot{w} + jJ \Omega^2 w - 2w_0^2 \epsilon_0 \epsilon_1 V_{DC2} v'' + \\
(EI + \mu A L^2) &\left(v^{(4)} + v^{(3)}w^{(3)} + v''w''w^{(2)} + 3v'w''w'' + v''w''w'' + v''w''w'' + 4v''v''v''\right) + \\
&\frac{m + M \delta(x - L)}{2} \left[ v' \int_0^x \left( \dddot{w} + \ddot{v} + \dddot{w} \right) \, dx + v'' \int_0^x \left( \dddot{w} + \dddot{w} + \dddot{w} \right) \, dx \right]

= \delta(x - L) \frac{\epsilon_0 \epsilon_1 V_{DC1}^2}{2(d_v - v)^2} \left( 1 + \beta_f \frac{d_v - v}{h_M} \right),
\end{align*}

(6)

\begin{align*}
(m + M \delta(x - L))(\dddot{w} + 2\Omega \ddot{w} - \Omega^2 w) &- J \dddot{w} + jJ \Omega^2 w - 2w_0^2 \epsilon_0 \epsilon_1 V_{DC2} w'' + \\
(EI + \mu A L^2) &\left(w^{(4)} + w^{(3)}w^{(3)} + w^{(1)}v^{(3)}w^{(1)} + 3w'v'w''w' + w''w''w'' + w''w''w'' + 4w''v''v''\right) + \\
&\frac{m + M \delta(x - L)}{2} \left[ w' \int_0^x \left( \dddot{w} + \ddot{v} + \dddot{w} \right) \, dx + w'' \int_0^x \left( \dddot{w} + \dddot{w} + \dddot{w} \right) \, dx \right]

= \delta(x - L) \frac{\epsilon_0 \epsilon_1 V_{DC1} + V_{AC} \cos(\omega t)^2}{2(d_w - w)^2} \left( 1 + \beta_f \frac{d_w - w}{b_M} \right),
\end{align*}

(7)

By introducing the following non-dimensional quantities

\begin{align*}
x' &= \frac{x}{L}, & v' &= \frac{v}{d_v}, & w' &= \frac{w}{d_w}, & t' &= \frac{\tau t}{L}, & J &= \frac{j}{mL^2}, & M_v &= \frac{M}{mL}, & \tau &= \sqrt{\frac{EI}{mL^2}}, \\
\Omega' &= \frac{\Omega}{\tau}, & d &= \frac{d_v}{d_w}, & k &= \frac{k}{d_v} / L, & V_{DC1}' &= \frac{V_{DC1}}{V_0}, & V_{DC2}' &= \frac{V_{DC2}}{V_0}, & V_{AC}' &= \frac{V_{AC}}{V_0}, & b'_M &= \frac{b_M}{d_w},
\end{align*}

(8)

\begin{align*}
h_M' &= \frac{h_M}{d_v}, & \eta &= \frac{\mu A L^2}{EI}, & \alpha_v &= \frac{\epsilon_0 \epsilon_1 L V_0^2}{2 E I d_v}, & \alpha_w &= \frac{\epsilon_0 \epsilon_1 L V_0^2}{2 E I d_w}, & F_p &= \frac{2w_0^2 \epsilon_0 \epsilon_1 L V_0}{EI},
\end{align*}

where $V_0$ is the unitary voltage.

Using the non-dimensional quantities, then one may obtain the equations of motion governing the transverse vibrations in its nondimensional form

\begin{align*}
(1 + M \delta(x - 1))(\dddot{w} + 2\Omega \ddot{w} - \Omega^2 w) &- \frac{J \dddot{w}}{d_v} + \frac{J \Omega^2 w}{d_v} - \frac{F_p V_{DC2} v''}{d_v} + \\
(1 + \eta) &\left[v^{(4)} + \frac{k^2}{d_v^2} \left(v^{(3)}w^{(3)} + v''w''w^{(2)} + 3v'w''w'' + v''w''w'' + v''w''w'' + 4v''v''v''\right)\right] + \\
&\frac{m(x)k^2}{2} \left[ v' \int_0^x \left( \dddot{w} + \ddot{v} + \dddot{w} \right) \, dx + v'' \int_0^x \left( \dddot{w} + \dddot{w} + \dddot{w} \right) \, dx \right]

= \delta(x - 1) \frac{\epsilon_0 \epsilon_1 V_{DC1}^2}{(1 - v)^2} \left( 1 + \beta_f \frac{1 - v}{h_M} \right),
\end{align*}

(9)
(1 + M_1 \delta(x-1))(d^2 w/dx^2 + 2 \Omega_1 \delta^2 \Omega^2 w) - Jd \delta^3 w'' - J \Omega^2 \delta w'' - F_p V_{DC1} w'' +
(1 + \eta) \left[ w''^4 + k^2 \left( \frac{w''^3}{2} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} + w'' \frac{w'''}{3} \right) \right]
+ \frac{m(x) k^2}{2} \left[ w' \int_0^x \left( \phi^2 + \phi' \frac{\phi'^2}{d^2} + \frac{\phi''}{d^2} \right) dx + w'' \int_0^x \left( \phi^2 + \phi' \frac{\phi'^2}{d^2} + \frac{\phi''}{d^2} \right) dx \right]
= \delta(x-1) \frac{\alpha_m (V_{DC1} + V_{AC} \cos(\omega t))^2}{(1-w)^2} (1 + \beta_F \frac{1-w}{b_M}).

where

\begin{equation}
m(x) = 1 + M_1 \delta(x-1).
\end{equation}

In Eqs. (9) and (10), \eta represents the length-scale parameter of size-dependent effect and \beta_F denotes the parameter of fringing field effect; the same DC voltages \( V_{DC2} \) are applied to the four piezoelectric layers; \( k \) denotes the coefficient of nonlinear curvature and inertia and the nonlinearities in curvature and inertia are not considered as \( k=0 \).

3. Reduced-order model of the micro-gyroscope

The solutions of the gyroscopic system can be decomposed into static and dynamic components, \( v(x,t)=v_s(x)+v_d(x,t) \) and \( w(x,t)=w_s(x)+w_d(x,t) \), where \( v_s(x) \), \( v_d(x,t) \), \( w_s(x) \) and \( w_d(x,t) \) represent the static and dynamics components in sense and drive directions, respectively. By expanding the right hand side of Eqs. (9) and (10) up to three orders of Tailor series and substituting the static and dynamic components into the equations, and dropping the dynamic terms yields the static equations. The static equilibrium equations are completely symmetric and hence \( v_s(x)=w_s(x) \). The effect of nonlinear curvature and inertia on the static deformation is very small, so it can be neglected. The static equations are substituted into the decomposed equations to result in the dynamics equations.

The approximate solutions of the dynamics responses \( v_d(x,t) \) and \( w_d(x,t) \) are supposed as follows [48]

\begin{equation}
\begin{align*}
v_d(x,t) &= \sum_{r=1}^{M} \phi_r(x) q_r(t), \\
w_d(x,t) &= \sum_{r=1}^{N} \phi_r(x) p_r(t),
\end{align*}
\end{equation}

where \( \phi_r(x) \) is the \( r \)th mode shape of a cantilever beam, \( q_r(t) \) and \( p_r(t) \) are the generalized
coordinates of sense and drive directions, respectively.

Furthermore, by using the single-mode approximation [16, 21, 22] and setting $d_c=d_w$ and $A_c=A_w$, the parameter $d$ equals to 1 and $\alpha_c=\alpha_w$, so the 2DOF nonlinear ordinary differential equations are expressed as

$$\dot{\phi} + \kappa_3 (\phi + q^2 \dot{\phi} + \dot{q} p) = 0,$$

$$\dot{\phi}_w + \kappa_4 (\phi_w - q^2 \dot{\phi}_w + \dot{q}_w p) = 0,$$

where

$$\kappa_3 = \int_0^1 \phi^2 dx + M, \phi(1) - J \int_0^1 \phi \phi'' dx,$$

$$\kappa_4 = 2 \Omega \int_0^1 \phi^2 dx + 2 \Omega M, \phi(1),$$

$$\kappa_5 = \int_0^1 \phi \phi'' dx + \int_0^1 \phi \phi''' dx + \int_0^1 \phi \phi''^2 dx,$$

$$\kappa_6 = \alpha_{\phi} V_{DC1}^2 \left( \frac{2}{(1 - \nu_s (w_y))^2} + \frac{\beta_f}{h_m (1 - \nu_s (w_y))^2} \right) \phi(1)^2,$$

$$\kappa_7 = \alpha_{\phi} V_{DC1}^2 \left( \frac{3}{(1 - \nu_s (w_y))^3} + \frac{\beta_f}{h_m (1 - \nu_s (w_y))^3} \right) \phi(1)^3,$$

$$\kappa_8 = \alpha_{\phi} V_{DC1}^2 \left( \frac{4}{(1 - \nu_s (w_y))^5} + \frac{\beta_f}{h_m (1 - \nu_s (w_y))^5} \right) \phi(1)^4,$$

$$\kappa_9 = 2 \alpha_{\phi} V_{DC1} \left( \frac{1}{(1 - \nu_s (w_y))^2} + \frac{\beta_f}{b_m (1 - \nu_s (w_y))^2} \right) \phi(1),$$

$$\kappa_{10} = 2 \alpha_{\phi} V_{DC1} \left( \frac{2}{(1 - \nu_s (w_y))^2} + \frac{\beta_f}{b_m (1 - \nu_s (w_y))^2} \right) \phi(1)^2.$$

### 4. Application of multi-scale method for nonlinear micro-gyroscopes

The multi-scale method has been widely applied in the investigation of the nonlinear micro-gyroscopes. The nonlinear gyroscopic Eqs. (13) and (14) are studied by using
the procedure of the multi-scale method. Then, the solutions of Eqs. (13) and (14) are expanded in the following form

\[ q = \varepsilon q_1(T_0, T_1, T_2) + \varepsilon^2 q_2(T_0, T_1, T_2) + \varepsilon^3 q_3(T_0, T_1, T_2) + \ldots, \]

\[ p = \varepsilon p_1(T_0, T_1, T_2) + \varepsilon^2 p_2(T_0, T_1, T_2) + \varepsilon^3 p_3(T_0, T_1, T_2) + \ldots \]

(16)

where \( \varepsilon \) is a small parameter, and \( T_0 = t, T_1 = \varepsilon t \) and \( T_2 = \varepsilon^2 t \) are different time scales.

Damping \( c \), angular speed \( \Omega \) and voltage \( V_{AC} \) are scaled with \( \varepsilon^2 c, \varepsilon^2 \Omega \) and \( \varepsilon^3 V_{AC} \) since they are relatively weak. The time derivatives are expressed as

\[ \frac{\partial}{\partial t} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \]

\[ \frac{\partial}{\partial t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \]

(17)

where \( D_0 = \partial \varepsilon T_0, D_1 = \partial \varepsilon T_1, D_2 = \partial \varepsilon T_2 \).

Substituting Eqs. (16)-(17) into Eqs. (13) and (14) and equating the coefficient of different orders of \( \varepsilon \) yields

**Order** (\( \varepsilon \))

\[ \kappa_i D_0^2 q_i + (\kappa_3 - \kappa_6) q_i = 0, \]

\[ \kappa_i D_0^2 p_i + (\kappa_3 - \kappa_6) p_i = 0. \]

(18)

**Order** (\( \varepsilon^2 \))

\[ \kappa_i D_0^2 q_2 + (\kappa_3 - \kappa_6) q_2 = -2\kappa_i D_0 D_1 q_1 + \kappa_i q_1^2, \]

\[ \kappa_i D_0^2 p_2 + (\kappa_3 - \kappa_6) p_2 = -2\kappa_i D_0 D_1 p_1 + \kappa_i p_1^2, \]

(19)

**Order** (\( \varepsilon^3 \))

\[ \kappa_i D_0^3 q_3 + (\kappa_3 - \kappa_6) q_3 = -c D_0 q_1 - 2\kappa_i D_0 D_1 q_2 - \kappa_i D_0^2 q_i - 2\kappa_i D_0 D_2 q_1 + \kappa_i D_0 p_1 + \]

\[ 2\kappa_i q_2 q_2 - (\kappa_4 - \kappa_3) q_1^2 - \kappa_i q_1^2 q_1^2 - \kappa_5 (q_1^2 q_1^2 D_0 + q_1^2 q_1^2 D_0) + \]

\[ q_1 (D_0 q_1)^2 + q_1 (D_0 p_1)^2 + q_1 p_1 D_0^2 p_1, \]

\[ \kappa_i D_0^3 p_3 + (\kappa_3 - \kappa_6) p_3 = -c D_0 p_1 - 2\kappa_i D_0 D_1 p_2 - \kappa_i D_0^2 p_1 - 2\kappa_i D_0 D_2 p_1 - \kappa_i D_0 q_1 + \]

\[ 2\kappa_i p_2 p_2 - (\kappa_4 - \kappa_3) p_1^2 - \kappa_5 (p_1^2 D_0^2 p_1 + \]

\[ p_1 (D_0 q_1)^2 + p_1 (D_0 p_1)^2 + q_1 p_1 D_0^2 q_1) + \kappa_i V_{AC} \cos(\omega_1). \]

(20)

The general solutions of Eq. (18) are supposed as

\[ q_i(T_0, T_1, T_2) = A_q(T_1, T_2)e^{i\omega_q T_0} + A_q(T_1, T_2)e^{-i\omega_q T_0}, \]

\[ p_i(T_0, T_1, T_2) = A_p(T_1, T_2)e^{i\omega_p T_0} + A_p(T_1, T_2)e^{-i\omega_p T_0}. \]

(21)

where \( \omega_p \) and \( \omega_q \) are natural frequencies corresponding to the drive and sense directions, respectively.
Substituting Eq. (21) into Eq. (19), one can obtain
\[
\begin{align*}
\kappa_i D_i^2 q_2 + (\kappa_3 - \kappa_6) q_2 &= -2i\kappa_i \omega_q D_i A_q(T_1, T_2) e^{i\omega_q T_i} + 2i\kappa_i \omega_q D_i \bar{A}_q(T_1, T_2) e^{-i\omega_q T_i} \\
+ \kappa_7 A^2_q(T_1, T_2) e^{2i\omega_q T_i} + \kappa_7 \bar{A}^2_q(T_1, T_2) e^{-2i\omega_q T_i} + 2\kappa_7 A_q \bar{A}_q, \\
\kappa_7 D^2 p_2 + (\kappa_3 - \kappa_6) p_2 &= -2i\kappa_i \omega_p D_i A_p(T_1, T_2) e^{i\omega_p T_i} + 2i\kappa_i \omega_p D_i \bar{A}_p(T_1, T_2) e^{-i\omega_p T_i} \\
+ \kappa_7 A^2_p(T_1, T_2) e^{2i\omega_p T_i} + \kappa_7 \bar{A}^2_p(T_1, T_2) e^{-2i\omega_p T_i} + 2\kappa_7 A_p \bar{A}_p. 
\end{align*}
\] (22)

In order to eliminate the secular terms in the preceding equations, the solvability conditions of the terms are set to zero. Inspection of the results one may find \(A_q(T_1, T_2)\) and \(A_p(T_1, T_2)\) and their complex conjugates to be function of only \(T_2\):
\[
\begin{align*}
q_i(T_0, T_2) &= A_q(T_2) e^{i\omega_q T_i} + \bar{A}_q(T_2) e^{-i\omega_q T_i}, \\
p_i(T_0, T_2) &= A_p(T_2) e^{i\omega_p T_i} + \bar{A}_p(T_2) e^{-i\omega_p T_i}. 
\end{align*}
\] (23)

Considering Eq. (23), the solutions of (22) are obtained in the following form
\[
\begin{align*}
q_2(T_0, T_2) &= G_1 A^2_q(T_1, T_2) e^{2i\omega_q T_i} + G_1 \bar{A}^2_q(T_1, T_2) e^{-2i\omega_q T_i} + 2G_2 A_q \bar{A}_q, \\
p_2(T_0, T_2) &= H_1 A^2_p(T_1, T_2) e^{2i\omega_p T_i} + H_1 \bar{A}^2_p(T_1, T_2) e^{-2i\omega_p T_i} + 2H_2 A_p \bar{A}_p. 
\end{align*}
\] (24)

where
\[
\begin{align*}
G_1 &= -\frac{\kappa_i}{4\kappa_i \omega_q^2 - \kappa_3 + \kappa_6}, \\
H_1 &= -\frac{\kappa_i}{4\kappa_i \omega_p^2 - \kappa_3 + \kappa_6}, \\
G_2 = H_2 &= \frac{\kappa_i}{\kappa_3 - \kappa_6}. 
\end{align*}
\] (25)

To express the primary and internal resonances of the gyroscopic system, two detuning parameters are introduced and defined as:
\[
\begin{align*}
\omega &= \omega_p + \varepsilon^2 \sigma_1, \\
\omega_q &= \omega_p + \varepsilon^2 \sigma_2, 
\end{align*}
\] (26)

where \(\sigma_1\) and \(\sigma_2\) denote the nearness of the excitation frequency \(\omega\) to the drive natural frequency \(\omega_p\) and two modal frequencies \((\omega_p, \omega_q)\) in drive and sense directions, respectively.

Substituting Eqs. (23)-(26) into Eq. (20), one can obtain
\begin{align*}
D_0^2 q_3 + \omega_p^2 q_3 &= \{(2\kappa_f G_1 + 2\kappa_s \omega_p^2 + 2\kappa_f G_2 - 3\kappa_2 + 3\kappa_8) \tilde{A} q^2 - 2i\kappa_q \omega_p A_q \} e^{i\omega T_0} + cc + nst \\
-ic \omega_p A_q - 2\kappa_4 A_q A_p \tilde{A} + i \kappa_q \omega_p A_q e^{i\omega T_0} + cc + nst
\end{align*}

\begin{align*}
D_0^2 p_3 + \omega_p^2 p_3 &= \{(2\kappa_f H_1 + 2\kappa_s \omega_p^2 + 2\kappa_f H_2 - 3\kappa_2 + 3\kappa_8) \tilde{A} p^2 - 2i\kappa_4 \omega_p A_p \} e^{i\omega T_0} \\
-ic \omega_p A_p - (2\kappa_3 \omega_p^2 \tilde{A} - i \kappa_2 \omega_p A_q - \kappa_4 A_p A_q e^{i\omega T_0} \\
-2\kappa_4 A_q A_p \tilde{A} + \kappa_9 \gamma \tilde{A} + \kappa_9 V_{AC} e^{i\omega T_0} + cc + nst
\end{align*}

(27)

where

\[
\omega_p^2 = \frac{\kappa_2 - \kappa_6}{\kappa_1},
\]

(28)

and ‘cc’ denotes the conjugate of the proceeding terms, ‘nst’ denotes non-secular terms.

For both the slow-varying complex amplitudes \( A_q \) and \( A_p \), are expressed in polar form

\[
A_q(T_2) = \frac{1}{2} a_q(T_2) e^{i\gamma_q(T_2)}, \quad A_p(T_2) = \frac{1}{2} a_p(T_2) e^{i\gamma_p(T_2)},
\]

(29)

where \( a_q(T_2), a_p(T_2) \) and \( \gamma_q(T_2), \gamma_p(T_2) \) are real amplitudes and phases of the responses, respectively.

By equating the coefficient of \( e^{i\omega T_0} \), one can obtain two equations about slow-varying complex amplitudes and set these two equations to zero. Furthermore, substituting Eq. (29) into these two equations, and separating real and imaginary parts, the slow-varying amplitudes and phases are computed as

\[
k_1 \omega_q a_q \cos(\gamma_1) (\delta \delta - \delta \delta + \sigma_2 - \sigma_1) - k_1 \omega_p \sin(\gamma_1) \delta \delta - \frac{1}{2} c \omega_p a_q \sin(\gamma_1) + \frac{1}{4} k_4 a_q a_p^2 \cos(\gamma_1)
\]

- \frac{1}{8} (2\kappa_f G_1 + 2\kappa_s \omega_p^2 + 4\kappa_f G_2 - 3\kappa_2 + 3\kappa_8) a_q^3 \cos(\gamma_1) = 0,

\[
k_1 \omega_p a_q \sin(\gamma_1) (\delta \delta - \delta \delta + \sigma_2 - \sigma_1) + k_1 \omega_p \cos(\gamma_1) \delta \delta + \frac{1}{2} c \omega_p a_q \cos(\gamma_1) + \frac{1}{4} k_4 a_q a_p^2 \sin(\gamma_1)
\]

- \frac{1}{2} k_2 \omega_p a_p - \frac{1}{8} (2\kappa_f H_1 + 2\kappa_s \omega_p^2 + 4\kappa_f H_2 - 3\kappa_2 + 3\kappa_8) a_q^3 \sin(\gamma_1) = 0,

\[
k_1 \omega_q a_p (\delta \delta - \sigma_1) - \frac{1}{8} (2\kappa_f H_1 + 2\kappa_s \omega_p^2 + 4\kappa_f H_2 - 3\kappa_2 + 3\k_8) a_q^3 - \frac{1}{2} k_2 \omega_p a_q \sin(\gamma_1)
\]

+ \frac{1}{4} k_4 a_p a_q^2 - \frac{1}{4} k_2 \omega_p a_p a_q^2 \cos(2\gamma_1) + \frac{1}{8} k_2 a_p a_q^2 \cos(2\gamma_1) = k_9 V_{AC} \cos(\gamma_2),

\[
k_1 \omega_p a_p + \frac{1}{2} k_2 \omega_p a_p \cos(\gamma_1) - \frac{1}{4} k_2 \omega_p a_p a_q^2 \sin(2\gamma_1) + \frac{1}{8} k_2 a_p a_q^2 \sin(2\gamma_1) + c \omega_p a_q
\]

= k_9 V_{AC} \sin(\gamma_2),

(30)
where
\[ \gamma_1 = \gamma_q - \gamma_p + \sigma_1 T_2, \]
\[ \gamma_2 = \sigma_1 T_2 - \gamma_p. \]

Substituting Eq. (29) into Eqs. (21) and (24), and using the results into Eq. (16), one may obtain the following second-order approximate analytical solutions
\[ q(t) = \varepsilon_0 a_p \cos(\omega t + \gamma_1 - \gamma_2) + \varepsilon^2 a_q^2 (\cos(2\omega t + 2\gamma_1 - 2\gamma_2) + G_1 + G_2) + \ldots, \]
\[ p(t) = \varepsilon_0 a_p \cos(\omega t - \gamma_2) + \varepsilon^2 a_p^2 (\cos(2\omega t - 2\gamma_2) + H_1 + H_2) + \ldots, \]

where \( G_1, G_2, H_1 \) and \( H_2 \) are obtained by Eq. (25) and \( a_q, a_p, \gamma_1 \) and \( \gamma_2 \) are given by Eq. (30).

In the calculation of this paper, by considering the geometry and material properties of the micro-gyroscopes as follows [4, 14, 27]
\[ L = 400 \times 10^{-6} \text{ m}; \quad E_b = 160 \times 10^9 \text{ N / m}^2; \quad E_p = 76.6 \times 10^9 \text{ N / m}^2; \]
\[ \mu = 79.92 \times 10^9 \text{ N / m}^2; \quad l = 0.9509 \times 10^{-6} \text{ m}; \quad \rho_b = 2300 \text{ kg / m}^3; \]
\[ \rho_p = 7500 \text{ kg / m}^3; \quad w_b = w_p = h_b = 2.8 \times 10^{-6} \text{ m}; \quad h_p = 0.01 \times 10^{-6} \text{ m}; \]
\[ A = w_b \times h_b \text{ m}^2; \quad A_e = A_w = 392 \times 10^{-12} \text{ m}^2; \quad d_e = d_w = 2 \times 10^{-6} \text{ m}; \]
\[ h_M = 4d_v \text{ m}; \quad b_M = 4d_w \text{ m}; \quad \varepsilon_0 = 8.854 \times 10^{-12} \text{ F / m}; \quad e_{31} = -9.29; \]
\[ m = \rho_b \times w_b \times h_b + 4 \rho_p \times w_p \times h_p \text{ kg / m}; \quad M_e = m \times L \text{ kg}; \quad \beta_f = 0.65. \]

5. Effects of the size-dependent and the fringing field

In this section, the effects of the size-dependent and the fringing field on the frequency-amplitude and force-amplitude responses of the vibrating beam micro-gyroscope are investigated. The curves of both drive and sense directions are plotted in Figs. 2-3 for the case of fixed parameters \( k=0 \) (regardless of nonlinear curvature and inertia), \( c=0.065, V_{DC1}=4, V_{DC2}=0, V_{AC}=0.1, \) \& \( \Omega=20 \). The solid lines are stable solutions and dotted lines are unstable solutions. Frequency-response and force-response curves are studied for four types of the gyroscopic system: both without size-dependent (\( \eta=0 \)) and fringing field (\( \beta_f=0 \)); with size-dependent (\( \eta=0.6913 \)) and without fringing field (\( \beta_f=0 \)); without size-dependent (\( \eta=0 \)) and with fringing field (\( \beta_f=0.65 \)); both with size-dependent (\( \eta=0.6913 \)) and fringing field (\( \beta_f=0.65 \)).

As can be seen from Fig. 2, only one external excitation force is given to drive direction, the system will cause vibrations in sense direction since the gyroscopic effect
produced by Coriolis force. The bifurcation points of the stable solution and unstable solution in Fig. 2 indicate that the system has reached the limit points. It can be seen that the frequency of the system is the same value in drive and sense directions as the bifurcation occurs. Fig. 2a shows that the point where the maximum amplitude appears in drive direction is the above bifurcation point. Fig. 2b illustrates that the point where the maximum amplitude appears in sense direction is when the small parameter $\sigma_1$ is equal to 0, that is, the external excitation frequency is equal to the frequency in drive direction. The nonlinear response in drive direction always exhibits a softening characteristic, while the nonlinear response in sense direction describes a multi-valued effect as the excitation frequency less than drive natural frequency. When only the size-dependent is considered, it can be seen that the amplitude in drive direction is reduced as shown in Fig. 2a. Although the amplitude in sense direction does not change much, the size-dependent effectively reduces the nonlinear multi-value problem which plays an important role in gyro detection system. It was further found that, only considering the size-dependent, the softening characteristics of the system are weakened. The size-dependent has little effect on the sensitivity of the system dynamic response as depicted in Fig. 2b. It can also be seen from Fig. 2 that when only the fringing field effect is considered, the maximum amplitude in drive direction increases. Although the amplitude in sense direction does not change much, it increases the nonlinear multi-value problem as shown in Fig. 2b. It was further found that, only considering the fringing field, the soft characteristics of the system in drive direction are enhanced as shown in Fig. 2a. Therefore, if the effects of the size-dependent and the fringing field are ignored, the micro-gyroscope measurement will be inaccurate.
Fig. 2. The frequency-amplitude curves of the vibrating beam micro-gyroscope for the effects of the size-dependent and the fringing field with $V_{dc}=0.1$. (a) drive amplitude (b) sense amplitude

Under the effects of the size-dependent and the fringing field, Fig. 3 describes the force-amplitude response curves in both drive and sense directions. It is known from Fig. 2 that the multi-valued nonlinear response of the system occurs when the small parameter $\sigma_1$ is less than 0, so $\sigma_1 = -0.1$ is taken. It can be seen from Fig. 3 that the trend...
curves of four types parameters of force amplitudes in both drive and sense directions are the similar, except that the values in sense direction are much smaller than the values in drive direction. As the amplitude of the externally excited voltage increases, the maximum amplitude of the gyroscopic system gradually increases. Within a certain interval (such as [0.1-0.2]), the system appears a multi-valued response, including two stable solutions and an unstable solution. Continuing to increase the amplitude of the external excitation voltage, the system reaches a single value again, and the larger amplitude of the external excitation, the higher the sensitivity.

It can be seen from Fig. 3 that when only the size-dependent is considered, the sensitivity of the original system is better as the amplitude of the external excitation voltage is small; increasing the external excitation amplitude can increase the system's multi-value response interval; if continues to increase external excitation voltage, the maximum amplitude is higher than original system. It can also be seen from Fig. 3 that when only the effect of the fringing field is considered, the sensitivity of the system is better than original system when the amplitude of the external excitation voltage is small; increasing the external excitation amplitude, the multi-valued response interval of the system decreases; continue to increase the magnitude, the maximum amplitude is lower than original system. When considering the effects of the size-dependent and the fringing field, the sensitivity of original system is better when the amplitude of the external excitation voltage is small; increasing the external excitation amplitude increases the system's multi-value response interval; maximum amplitudes of two directions are higher than original system as continuing to increase the external excitation voltage. Therefore, the size-dependent effect can significantly affect the force-amplitude curve of the system. Therefore, the conclusion of Fig. 2 can also be proved, if the effects of the size-dependent and the fringing field are ignored, the gyroscope measurement will be inaccurate.
Fig. 3. The force-amplitude curves of the vibrating beam micro-gyroscope for the effects of the size-dependent and the fringing field with $\sigma_1=0.1$. (a) drive amplitude (b) sense amplitude

6. Effect of five values of DC2 voltage

In this section, the effect of the DC2 voltage on the frequency-amplitude and force-amplitude responses of the vibrating beam micro-gyroscope is investigated. The curves of drive and sense directions are plotted in Figs. 4-5 for the case of fixed parameters
$k=0$ (regardless of nonlinearities in curvature and inertia), $c=0.065$, $V_{DC1}=4$, $\eta=0.6819$, $\beta_f=0.65$, $V_{AC}=0.1$, and $\Omega=20$. Frequency-amplitude and force-amplitude curves are studied for five different cases: $V_{DC2}=0$, $V_{DC2}=-1$, $V_{DC2}=-0.5$, $V_{DC2}=0.5$, and $V_{DC2}=1$.

Under the effect of different DC voltages applied on four piezoelectric layers of the micro-beam, Fig. 4 shows the amplitude-frequency response curves in both drive and sense directions. The nonlinear responses in drive direction always exhibits softening characteristics as depicted in Fig. 4a. As the DC voltage goes from positive to negative (from big to small) $V_{DC2} = 1$, $V_{DC2} = 0.5$, $V_{DC2} = 0$, $V_{DC2} = -0.5$ and $V_{DC2} = -1$, the softening characteristic is more and more obvious, and the maximum amplitude increases first and then decreases, hence, there is always one DC voltage where the maximum amplitude in drive direction reaches a maximum value. However, the nonlinear responses in sense direction present multi-valued effect as shown in Fig. 4b. As the DC voltage $V_{DC2}$ decreases from positive to negative, the maximum amplitude point becomes smaller and smaller, and the multi-valued effect becomes more and more obvious. To obtain a higher sensitivity in sense direction, it is better to choose a positive $V_{DC2}$ voltage.

![Diagram](attachment:diagram.png)
Fig. 4. The frequency-amplitude curves of the vibrating beam micro-gyroscope for five DC2 voltages with $V_{ac}=0.1$. (a) drive amplitude (b) sense amplitude

Under the effect of different piezoelectric $V_{DC2}$ voltages on the vibrating beam micro-gyroscope, Fig. 5 shows the force-amplitude curves in drive and sense directions. From Fig. 4 we know that the multi-valued nonlinear response of the system occurs when the small parameter $\sigma_1$ is less than 0, so take $\sigma_1 = -0.1$. As can be seen from Fig. 5, the trend curves of five types parameters of force amplitudes in drive and sense directions are similar, except that the values in sense direction is much smaller than the values in drive direction. With increase of the amplitude of the externally excited voltage, the maximum amplitude of the system gradually increases. Within a certain interval, the system has a multi-valued response, including two stable solutions and one unstable solution.

As can be seen from Fig. 5, as the DC voltage $V_{DC2}$ goes from positive to negative, the multi-value response area of the system decreases. The higher branch represents a stable solution, and the vibrating gyroscope can work at a higher external excitation amplitude.
Fig. 5. The force-amplitude curves of the vibrating beam micro-gyroscope for five DC2 voltages with $\sigma_1=-0.1$. (a) drive amplitude (b) sense amplitude

7. Effect of nonlinear curvature and inertial

In this section, the effect of nonlinear curvature and inertial on frequency-amplitude and force-amplitude responses of the vibrating beam micro-gyroscope is studied. The curves of both drive and sense directions are plotted in Figs. 6-7 for the case of fixed
parameters $c=0.065$, $V_{DC1}=4$, $V_{DC2}=0$, $\eta=0.6819$, $\beta_f=0.65$, and $\Omega=20$. The curvature nonlinearity and inertial nonlinearity share same coefficient $k$ and their values are selected as $k=0$, $k=0.005$, $k=0.05$, and $k=0.5$. Frequency-amplitude and force-amplitude curves are considered for three different cases: with curvature ($\kappa_4\neq0$) and inertia nonlinearities ($\kappa_5\neq0$); with curvature ($\kappa_4\neq0$) and without inertia nonlinearities ($\kappa_5=0$); without curvature ($\kappa_4=0$) and with inertia nonlinearities ($\kappa_5\neq0$).

From Fig. 6a, it can be seen that when the coefficient value $k=0$, $k=0.005$ and $k=0.05$, the nonlinear effect of the system is not change, that is to say, when the value of $k$ is small, the nonlinear curvature and inertia can be ignored. As the nonlinear coefficient $k=0.5$ is taken, when considering the case of $\kappa_4\neq0$ and $\kappa_5=0$, the softening characteristics of the responses in drive direction are weakened compared to the original system; As $\kappa_4=0$ and $\kappa_5\neq0$, the softening characteristics are enhanced; As $\kappa_4\neq0$ and $\kappa_5\neq0$, the softening characteristics of the responses in drive direction are also increased compared to the original system. These results show that curvature nonlinearity makes the system softer and inertial nonlinearity makes the system harder. Under the influence of the same coefficient $k$, the effect of curvature nonlinearity is stronger than that of inertial nonlinearity. In addition, we find that the maximum amplitude of the system in drive direction is constant under different nonlinearities.

In Fig. 6b, as coefficient values of the nonlinearity are $k=0$, $k=0.005$ and $k=0.05$, it can also be found that the nonlinear effect of the system in sense direction is the same. When taking the nonlinear coefficient $k=0.5$, the maximum amplitude in sense direction of the case of $\kappa_4\neq0$ and $\kappa_5=0$ is larger than that cases of $\kappa_4=0$, $\kappa_5\neq0$ and $\kappa_4\neq0$, $\kappa_5\neq0$. One may also find the multi-valued response interval is the smallest for the case of $\kappa_4\neq0$ and $\kappa_5=0$. Therefore, increasing the curvature nonlinearity and reducing the inertial nonlinearity of the system will help the system obtain better sensitivity and eliminate multi-valued responses as much as possible.
Fig. 6. The frequency-amplitude curves of the vibrating beam micro-gyrooscope for the effect of nonlinear curvature and inertial with $V_{AC}=0.1$ (a) drive amplitude (b) sense amplitude

Under the influence of different coefficient values of nonlinearities in curvature and inertia, Fig. 7 shows the force-amplitude response curves in both drive and sense directions. As can be seen from Fig. 7, the trend curves of force-amplitude in drive and sense directions are different, which is different from the force-amplitude curves
studied earlier. From Fig. 7a, it can be seen that when the nonlinearities in curvature and inertia coefficient $k=0.5$ is taken and in the case of $\kappa_4 \neq 0$ and $\kappa_5 = 0$, the system's multi-value response interval increases with the increase of the external excitation amplitude; continue to increase the external excitation amplitude, the maximum amplitude in drive direction is higher than original system. When in the cases of $\kappa_4 \neq 0$, $\kappa_5 \neq 0$ and $\kappa_4 = 0$, $\kappa_5 \neq 0$, the multi-valued response interval of the system reduces with the increase of the external excitation amplitude; continue to increase the external excitation amplitude, the maximum amplitude in drive direction is lower than original system. From Fig. 7b in sense direction, when taking the nonlinear coefficient $k=0.5$, the multi-value interval of the system is the biggest in the case of $\kappa_4 \neq 0$ and $\kappa_5 = 0$. As the cases of $\kappa_4 = 0$, $\kappa_5 \neq 0$ and $\kappa_4 \neq 0$, $\kappa_5 \neq 0$, the multi-value interval of the system is smaller than original system. However, as the amplitude of the externally excited voltage increases, the maximum amplitude of original system in sense direction is bigger than other nonlinear systems.

![Graph](image)

(a) Drive amplitude
Fig. 7. The force-amplitude curves of the vibrating beam micro-gyroscope for the effect of nonlinear curvature and inertial with $\sigma_1=-0.1$ (a) drive amplitude (b) sense amplitude

8. Conclusions

In this research, by considering the effect of statically parametric excitation on the model of the vibrating beam micro-gyroscope, the nonlinear dynamics with nonlinearities in curvature, inertia and electrostatic force is studied. The multi-scale method and numerical continuation technique are used to investigate the nonlinear responses of the gyroscopic system. This paper studies the effect of the size-dependent, fringing field, and static axial excitation on the frequency-amplitude and force-amplitude responses of the nonlinear system. For such complex nonlinear micro-gyroscopes, the main resonance response of the system is investigated under the primary and internal resonance conditions. The following conclusions are drawn:

(1) Small vibrations produce a symmetrical frequency response curve in sense direction while the system actually has a significant softening characteristic in drive direction. When only the size-dependent effect is considered, the amplitude in drive direction decreases, and the nonlinear multi-value problem is effectively reduced in sense direction, which plays an important role in the gyroscope detection system. If the effect of the size-dependent and the fringing field is ignored, the micro-gyroscope
measurement will be inaccurate.

(2) As the piezoelectric DC voltage decreases from positive to negative, the softening characteristic becomes more and more obvious in drive direction, there is always one DC voltage where the amplitude reaches a maximum value. However, the nonlinear responses in sense direction present multi-valued effect. If we want to obtain a higher sensitivity, it is better to choose a positive $V_{DC2}$ voltage. The higher branch represents a stable solution, and the vibratory gyroscope can work at higher external excitation amplitude.

(3) The nonlinear responses of the system are constant when the value of nonlinear coefficient $k$ is small and its influence can be ignored. When the coefficient $k$ of nonlinearities in curvature and inertia is big (e.g. $k=0.5$), the curvature nonlinearity makes the system softer and the inertial nonlinearity makes the system harder. Under the influence of the same coefficient, the effect of curvature nonlinearity is stronger than that of inertial nonlinearity. Increasing the curvature nonlinearity and reducing the inertial nonlinearity of the system help the system obtain better sensitivity, and can eliminate multi-valued responses as much as possible.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Data availability statements
The authors declare that the data supporting the findings of this study are available within the article.
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