THE VERTEX IDEAL OF A LATTICE

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ABSTRACT. We introduce a monomial ideal whose standard monomials encode the vertices of all fibers of a lattice. We study the minimal generators, the radical, the associated primes and the primary decomposition of this ideal, as well as its relation to initial ideals of lattice ideals.

1. Introduction

The main purpose of this paper is to introduce and study a monomial ideal, the vertex ideal, associated to a sublattice of $\mathbb{Z}^n$. We relate algebraic properties of this ideal to combinatorial properties of the lattice.

Definition 1.1. Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$ with $\dim(\mathcal{L}) = m$. For $u \in \mathbb{N}^n$ we define $P_u := \text{conv}\{v \in \mathbb{N}^n : u - v \in \mathcal{L}\}$ to be the fiber of $u$ with respect to $\mathcal{L}$. Clearly, if $v \in P_u$ then $P_u = P_v$.

Each fiber $P_u$ is a rational polyhedron, by Theorem 16.1 in [9], and hence has finitely many vertices $\text{Vert}(P_u)$. We start with the observation (Proposition 2.1) that the union of all $\text{Vert}(P_u), u \in \mathbb{N}^n$ forms an order ideal of $\mathbb{N}^n$. We call the monomial ideal which is the complement of this order ideal the vertex ideal of $\mathcal{L}$, and denote it by $V_\mathcal{L}$.

One motivation for studying vertex ideals comes from the theory of integer programming. Suppose $A \in \mathbb{N}^{d \times n}$ is a matrix of rank $d$ with no zero columns. Let $\mathbb{N}A$ be the submonoid of $\mathbb{N}^d$ consisting of nonnegative integer combinations of the columns of $A := [a_1 \ldots a_n]$. Integer programming is concerned with minimizing a fixed linear form $c \cdot x$, where $c \in \mathbb{R}^n$, over $\{u \in \mathbb{N}^n : Au = b\}$ for a fixed $b \in \mathbb{N}A$. Note that if we let $\mathcal{L} = \ker(A) \cap \mathbb{Z}^n$, then for $v \in \mathbb{N}^n$, the fiber of $v$ is a polytope $\text{conv}\{u \in \mathbb{N} : Au = Av\}$ (in this case we denote the fiber of $v$ by $P_b$ where $b = Av$). Hence studying the vertex ideal $V_\mathcal{L}$ in this context gives information about the vertices of all integer programming polytopes as $b$ varies in $\mathbb{N}A$. Commutative algebra and computational algebraic geometry enter this picture through the connection between integer programming and Gröbner bases and initial ideals of the toric ideal of $A$ (see [10], [11] and [14]).
A second motivation comes from the recent work of Saito, Sturmfels and Takayama [8] on hypergeometric differential equations. One observation these authors make is that the set of all generic $A$-hypergeometric series solutions to a GKZ $A$-hypergeometric system is indexed by the top-dimensional standard pairs of $V_L$ where $L = \ker(A) \cap \mathbb{Z}^n$ (pp. 129-131 in [8]). This leads us to studying the standard pairs (and hence the associated primes) of $V_L$.

In Section 2 of this paper we start by giving a naive algorithm to construct $V_L$ in Theorem 2.3. This first algorithm needs all initial ideals of the associated lattice ideal $I_L$, and therefore it is highly inefficient for large problems. We remedy this by giving an improved algorithm to construct a generating set for $V_L$, using the Graver basis elements of $I_L$. This second algorithm depends on a characterization of $V_L$ which is derived from only the geometric properties of the lattice. We also describe the radical of $V_L$ as the Stanley-Reisner ideal of a matroid complex.

In Section 3 the second motivation we cited above for studying $V_L$ leads us to investigate the associated primes of $V_L$. First we give a characterization of $\text{Ass}(V_L)$ and compute the irreducible primary decomposition of $V_L$ in terms of a family of polytopes using similar methods to those found in [11]. This allows us to give some necessary conditions for a prime being an embedded associated prime of $V_L$ when $\text{dim}(L) = 2$. In particular, we show that the irrelevant maximal ideal $\langle x_1, \ldots, x_n \rangle$ cannot be associated to $V_L$ in this case. This result fails when $\text{dim}(L) \geq 3$, and we give a counterexample. This seemingly harmless counterexample turns out to be a very interesting one for our first motivation, integer programming. It provides a counterexample to a conjecture about the complexity of codimension three integer programs. More precisely, it gives a counterexample to Conjecture 6.1 in [11] which hypothesized that every cone in the Gröbner fan of a codimension three toric ideal has at most four facets.

In Section 4 we define another monomial ideal, $P_L$, closely related to $V_L$. We show that the product ideal $P_L$ has the same radical as $V_L$. In two interesting special cases, we prove that $P_L$ (which is easier to compute) is equal to $V_L$. The first case is when $L$ comes from a unimodular matrix $A$. The second case is when $L \subseteq \mathbb{Z}^2$ and $\text{dim}(L) = 2$. This implies that for any two dimensional lattice, we have $\text{Top}(P_L) = \text{Top}(V_L)$. 
2. The Minimal Generators and the Radical of the Vertex Ideal

The first goal of this section is to come up with useful characterizations of $V_L$ which we use for devising a relatively efficient algorithm. We then give a combinatorial description of the radical of the vertex ideal. We first show the existence of the vertex ideal.

Proposition 2.1. Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$, and let $P_u$ be a fiber of $\mathcal{L}$. For any vertex $v$ of $P_u$, if $v_i > 0$, then $v - e_i$ is a vertex of $P_{u-e_i}$ where $e_i$ is the $i$-th unit vector. In other words, there exists a monomial ideal $V_L$ in $S = k[x_1, \ldots, x_n]$ where $x^v \notin V_L$ if and only if $v \in \text{Vert}(P_u)$ for a fiber $P_u$ of $\mathcal{L}$.

Proof. If $v - e_i$ is not a vertex of $P_{u-e_i}$, then it is in the convex hull of vertices $v_1', v_2', \ldots, v_k'$ of $P_{u-e_i}$. But then $v$ would be in the convex hull of $v_1' + e_i, v_2' + e_i, \ldots, v_k' + e_i$. This contradiction proves the first statement, and hence implies that the union of all $\text{Vert}(P_u), u \in \mathbb{N}^n$ forms an order ideal of $\mathbb{N}^n$. This is equivalent to the second statement.

We now give a first algorithm to compute $V_L$. To do this, we first associate a binomial ideal to $L$.

Definition 2.2. The lattice ideal $I_L$ is defined by

$$I_L = \langle x^u - x^v : u, v \in \mathbb{N}^n, u - v \in \mathcal{L} \rangle.$$  

Lattice ideals have been widely studied, see for example [3], [6], [7]. In this context we are interested in the initial ideals of $I_L$. For a weight vector $\omega \in \mathbb{R}^n$ such that $\omega \cdot u > 0$ for every non-zero vector $u \in \mathbb{N}^n \cap \mathcal{L}$, we let $\text{in}_\omega(I_L)$ be the ideal $\langle \text{in}_\omega(f) : f \in I_L \rangle$ where $\text{in}_\omega(f)$ is the sum of all terms of $f$ with maximum $\omega$-value. If the initial ideal $\text{in}_\omega(I_L)$ is a monomial ideal we call $\omega$ a generic weight vector. Our assumption on $\omega$ ensures that each fiber $P_u$ has a bounded face which minimizes the linear functional $\omega \cdot x$. Then the genericity of $\omega$ is equivalent to the condition that each such bounded face is a vertex $v$ of $P_u$.

Theorem 2.3. The vertex ideal $V_L$ is equal to $\bigcap_\omega \text{in}_\omega(I_L)$ where $\omega$ is a generic weight vector.

Proof. Since for any two lattice points $u, v \in P_u$ we have $u - v \in \mathcal{L}$, a monomial is a standard monomial of $\text{in}_\omega(I_L)$ if and only if its exponent vector minimizes the linear functional $\omega \cdot x$ in $P_u$ [13]. Hence the monomial $x^v$ is a standard monomial of $\bigcap_\omega \text{in}_\omega(I_L)$ if and only if $v$ is the minimizer of $\omega \cdot u$ for $u \in P_v$ for some generic weight vector. But these are precisely the vertices of the fibers of $\mathcal{L}$.  \qed
Using this theorem we have a first algorithm for computing $V_L$: compute all initial monomial ideals of $I_L$ and take their intersection. We note that this is a finite algorithm, as any ideal in $S$ has only a finite number of different initial ideals. The list of all initial ideals of $I_L$ can be computed with the software TiGERS [5]. This first algorithm is not, however, completely satisfactory, as the number of initial ideals can be much larger than the subset needed to define the intersection. In order to illustrate this point we use the following example, where the number of initial ideals depends exponentially on the data of the lattice.

Example 2.4. Let $I_L$ be the ideal generated by the $2 \times 2$ minors of a generic $2 \times n$ matrix $X = (x_{ij})$. This is a prime lattice ideal which is the defining ideal of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ into $\mathbb{P}^{2n-1}$. Proposition 5.4 of [10] shows that with respect to the reverse lexicographic term order $x_1 \prec \cdots \prec x_n \prec x_2 \prec \cdots \prec x_{2n}$, these $2 \times 2$ minors form a reduced Gröbner basis. By permuting columns of $X$, and using the corresponding reverse lexicographic term order, one gets $n!$ distinct initial ideals. This shows that $I_L$ has at least $n!$ initial ideals. In Remark 2.13 we will see that the vertex ideal can be constructed as the intersection of only $n^2 \frac{2^{n-1}}{n}!$ initial ideals. As $\frac{n^2 \frac{2^{n-1}}{n}!}{n!} \to 0$ as $n \to \infty$, a vanishingly small proportion of the initial ideals are needed to construct $V_L$ in this family.

Below we give a more efficient description of the minimal generators of the vertex ideal. For this, we need to define the Graver basis of $L$.

Definition 2.5. Suppose $L \subseteq \mathbb{Z}^n$ and let $R_\rho$ be the orthant defined by the sign pattern $\rho \in \{+, -\}^n$. Then $L \cap R_\rho$ is a finitely generated monoid with a unique minimal generating set $H_\rho$, its Hilbert basis (Theorem 16.4 in [9]). The Graver basis $Gr_L$ of $L$ (or $I_L$) is defined to be the union of all such $H_\rho$.

Lemma 2.6. Suppose $\sum_i c_i (\alpha_i - \beta_i) = 0$, where $c_i > 0$, and $\alpha_i - \beta_i \in Gr_L$ with $\alpha_i, \beta_i \in \mathbb{N}^n$ and $\text{supp}(\alpha_i) \cap \text{supp}(\beta_i) = \emptyset$. Then $x^v = \text{lcm}_i (x^{\alpha_i})$ is in $V_L$.

Proof. Suppose $x^v$ is not in $V_L$. This means that $v$ is a vertex of $P_v$, so there is some $\omega \in \mathbb{R}^n$ such that $\omega \cdot v > \omega \cdot u$ for all lattice points $u \in P_v \setminus \{v\}$. But now

$$\omega \cdot v > \omega \cdot (v - (\alpha_i - \beta_i))$$

for each $i$

because $\alpha_i \leq v$ means that $v' = (v - \alpha_i + \beta_i) \in \mathbb{N}^n$, and thus $v'$ is a lattice point in $P_u \setminus \{v\}$. This implies
\[ \sum_i \omega \cdot (c_i v) > \sum_i \omega \cdot (c_i v - c_i (\alpha_i - \beta_i)) \]
\[ = \sum_i \omega \cdot (c_i v) - \omega \cdot \sum_i c_i (\alpha_i - \beta_i) \]
\[ = \sum_i \omega \cdot (c_i v) \]

This contradiction shows that \( x^u \) is in \( V_\mathcal{L} \). \qedhere

**Corollary 2.7.** The minimal generators of \( V_\mathcal{L} \) are of the form as in Lemma 2.6.

**Proof.** Let \( x^u \) be a minimal generator of \( V_\mathcal{L} \). Hence \( u \) is not a vertex of its fiber, and therefore it is a convex combination \( \sum \lambda_i v_i \) of some vertices \( v_i \) of \( P_u \), where \( 0 \leq \lambda_i \leq 1 \) and \( \sum \lambda_i = 1 \). Since \( u - v_i \) is in \( \mathcal{L} \), we have \( u - v_i = \sum j c_{ij} (\alpha_{ij} - \beta_{ij}) \) where \( \alpha_{ij} - \beta_{ij} \) are Graver basis elements with \( \alpha_{ij} \leq u \) and \( \beta_{ij} \leq v_i \), and \( c_{ij} \in \mathbb{Z}_{\geq 0} \). Now clearly \( \sum \lambda_i (u - v_i) = 0 \), and thus \( \sum i,j \lambda_i c_{ij} (\alpha_{ij} - \beta_{ij}) = 0 \). By Lemma 2.6, \( x^u = \text{lcm}_{ij} (x^{\alpha_{ij}}) \) is in \( V_\mathcal{L} \). But \( x^u \) divides \( x^u \), and \( x^u \) is a minimal generator, so \( x^u = x^u \). \qedhere

Corollary 2.7 implies that the minimal generators of \( V_\mathcal{L} \) can be computed by identifying all positive linear dependencies among Graver basis elements of \( \mathcal{L} \). In fact only the minimal positive dependencies, known as theme positive circuits, are needed. We summarize this as follows.

**Theorem 2.8.** Let \( \text{Gr}_\mathcal{L} = \{ \alpha_i - \beta_i \} \) be an ordered Graver basis of \( \mathcal{L} \), so that \( \alpha - \beta \in \text{Gr}_\mathcal{L} \) implies \( \beta - \alpha \in \text{Gr}_\mathcal{L} \). If \( \tau \) is the support of a positive circuit \( \sum \in \tau c_i (\alpha_i - \beta_i) = 0 \) we define \( x^{m_\tau} \) to be \( \text{lcm}_{j \in \tau} x^{\alpha_j} \). Then
\[ V_\mathcal{L} = \langle x^{m_\tau} | \tau \text{ is the support of a positive circuit of } \text{Gr}_\mathcal{L} \rangle. \]

**Proof.** If \( \tau \) is the support of a positive circuit of \( \text{Gr}_\mathcal{L} \), Lemma 2.6 implies that \( x^{m_\tau} \) is in \( V_\mathcal{L} \). And Corollary 2.7 says that every minimal generator of \( V_\mathcal{L} \) is of this form. \qedhere

Theorem 2.8 gives our second, more efficient, algorithm to compute \( V_\mathcal{L} \): after computing the Graver basis \( \text{Gr}_\mathcal{L} \), identify each positive circuit \( \tau \) of \( \text{Gr}_\mathcal{L} \) and compute \( x^{m_\tau} = \text{lcm}_{j \in \tau} x^{\alpha_j} \).

We observe that not all vectors of \( \text{Gr}_\mathcal{L} \) are necessary. When \( \mathcal{L} \cap \mathbb{N}^n = \{ 0 \} \), it suffices to replace \( \text{Gr}_\mathcal{L} \) by the ordered universal Gröbner basis of \( \mathcal{L} \). See [10, Chapter 7] for information on computing the universal Gröbner basis.
The next result in this section describes the radical of $V_L$. Let $B \in \mathbb{Z}^{n \times m}$ be a matrix whose columns form a basis for the $m$-dimensional lattice $L$. We will denote the rows of $B$ by $b_1, \ldots, b_n$. Now if $\omega$ is a generic cost vector, the vector $\omega B$ is contained in the relative interior of a set $C$ of $m$-dimensional simplicial cones with generators from $\{b_1, \ldots, b_n\}$. We define $\Delta_\omega$ to be the simplicial complex generated by the complementary indices of the generators of the cones in $C$. By its definition, $\Delta_\omega$ is an $(n - m)$-dimensional pure simplicial complex on $\{1, \ldots, n\}$. We also note that this simplicial complex is the regular triangulation of $A$ with respect to $\omega$ when $L = \ker(A) \cap \mathbb{Z}^n$ (see Chapter 8 in [10]). Extending the connection between Stanley-Reisner ideals of regular triangulations of $A$ and the radicals of the initial ideals of $I_L$, we get the following proposition (Corollary 2.9 in [3], see also Section 7 in [13]). Recall that the Stanley-Reisner ideal of a simplicial complex is the ideal generated by the minimal non-faces of the complex.

**Proposition 2.9.** The radical of $\text{in}_\omega(I_L)$ is the Stanley-Reisner ideal of the simplicial complex $\Delta_\omega$.

Now we are ready to prove the following theorem:

**Theorem 2.10.** The radical of $V_L$ is $\bigcap_{\sigma} \langle x_i : i \in \sigma \rangle$ where the intersection is over all linearly independent subsets of $\{b_1, \ldots, b_n\}$ of size $m$.

**Proof.**

$$\text{rad}(V_L) = \text{rad}(\bigcap_{\omega \text{ generic}} \text{in}_\omega(I_L))$$

$$= \bigcap_{\omega \text{ generic}} \text{rad}(\text{in}_\omega(I_L)) = \bigcap_{\Delta_\omega} I_{\Delta_\omega}$$

$$= \bigcap_{\Delta_\omega} \bigcap_{\tau \in \Delta_\omega} \langle x_i : i \notin \tau \rangle$$

$$= \bigcap_{\sigma : \dim(\sigma) = m} \langle x_i : i \in \sigma \rangle$$

where $I_{\Delta_\omega}$ is the Stanley-Reisner ideal of $\Delta_\omega$. We have the first equality on the second line because taking the radical commutes with intersections, while the second equality follows from Proposition 2.9. The third line is a standard result on Stanley-Reisner ideals, and the last line follows because the complement of the indices of the generators of any full dimensional simplicial cone $\{b_{i_1}, \ldots, b_{i_m}\}$ is involved in some $\Delta_\omega$. 

This result can be interpreted using the notion of a matroid complex.
Definition 2.11. The matroid complex \( \Delta(M) \) of a matroid \( M \) is the simplicial complex where the simplices are the independent sets of \( M \).

If \( L \subset \mathbb{Z}^n \) is a lattice of dimension \( m \) generated by the columns of a matrix \( B \in \mathbb{Z}^{n \times m} \), then the complements of bases (i.e. linearly independent subsets of rows of size \( m \)) of \( B \) form the maximal independent sets of a matroid \( M(L) \). Hence the matroid complex \( \Delta(M(L)) \) is the simplicial complex whose maximal simplices are the union of the maximal simplices occurring in \( \Delta_\omega \) for all generic \( \omega \). Note that when \( L = \ker(A) \cap \mathbb{Z}^n \) for a matrix \( A \), then \( M(L) \) is the matroid of all linearly independent subsets of the columns of \( A \), and \( \Delta(M(L)) \) is the simplicial complex whose maximal simplices are the union of all maximal simplices appearing in the regular triangulations of \( A \). We now get the following corollary.

Corollary 2.12. The Stanley-Reisner ideal of \( \Delta(M(L)) \) is the radical of \( V_L \).

Proof. The Stanley-Reisner ideal of \( \Delta(M(L)) \) is

\[
I_{\Delta(M(L))} = \bigcap_{\tau \in \Delta(M(L))} \langle x_i : i \notin \tau \rangle.
\]

The above intersection can be taken over all \( \tau \) where \( \tau \) is a maximal face. Then since \( \tau \in \Delta(M(L)) \) if and only if \( \{b_i : i \notin \tau\} \) forms a basis of \( B \) where \( B \) is matrix whose columns are a basis for \( L \), Theorem 2.10 implies that \( I_{\Delta(M(L))} = V_L \).

Remark 2.13. We can now prove the last claim in Example 2.4. The vertex ideal \( V_L \) of the \( 2 \times 2 \) minors of a generic \( 2 \times n \) matrix is a radical ideal, as all the initial ideals are radical, because the corresponding configuration is unimodular. Hence we can use the intersection formula in the proof above. The maximal faces over which we need to take the intersection are determined by maximal independent sets of the collection \( \{e_i \oplus e_j : i = 1, 2, \text{ and } 1 \leq j \leq n\} \). These are in bijection with the distinct spanning trees of the complete bipartite graph \( K_{2,n} \). There are \( n2^{n-1} \) such spanning trees, as exactly one vertex in the \( n \)-block is connected to both vertices in the 2-block.

Finally, we observe that the Hilbert series of \( V_L \) gives us information about the number of vertices of the fibers \( P_u \) of \( L \).

Proposition 2.14. The Hilbert series \( H(S/V_L; z_1, \ldots, z_n) \) of \( S/V_L \) is

\[
\sum_u z^u,
\]

where the sum is taken over all vertices \( u \) of all fibers \( P_u \). When
$\mathcal{L} = \ker(A) \cap \mathbb{Z}^n$ for an integer matrix $A = [a_1, \ldots, a_n]$ then

$$H(S/V_{\mathcal{L}}; z_1t^{a_1}, \ldots, z_n t^{a_n}) = \sum_{b \in \mathbb{N}_A} \bigg( \sum_{u \in \text{Vert}(P_b)} z^u \bigg) \cdot t^b, \text{ and,}$$

$$H(S/V_{\mathcal{L}}; t) = \sum_{b \in \mathbb{N}_A} |\text{Vert}(P_b)| \cdot t^b.$$

We can derive information about the fibers $P_u$ from the Hilbert function for $V_{\mathcal{L}}$. An example is given in the following proposition.

**Proposition 2.15.** If $\mathcal{L} = \ker(A) \cap \mathbb{Z}^n$ for a $1 \times n$ matrix $A = [a_1, \ldots, a_n]$ where $a_i \in \mathbb{N}$, then the number of vertices of a fiber $P_u$ is eventually periodic, with period dividing $\text{lcm}(a_i)$.

**Proof.** The Hilbert series $H(S/V_{\mathcal{L}}; t)$ can be written in the form

$$\frac{p(t)}{\prod_{i=1}^n (1 - t^{a_i})},$$

for some polynomial $p(t)$. This means that the Hilbert function of $S/V_{\mathcal{L}}$ at $b$, which counts the number of vertices of $P_u$ when $Au = b$, eventually agrees with a quasi-polynomial evaluated at $b$. As there is an upper bound, given by the number of initial ideals of $I_{\mathcal{L}}$, on the number of vertices of any $P_u$, this polynomial part of the quasi-polynomial must be a constant. As the period of the quasi-polynomial divides $\text{lcm}(a_i)$, the result follows.

We observe that a more constructive proof of this proposition can also be given using the notion of atomic fibers, defined in [1].

### 3. Associated Primes and Standard Pairs of $V_{\mathcal{L}}$

With the relation between initial ideals and $V_{\mathcal{L}}$ given in Theorem 2.3, it is natural to ask which properties of the initial ideals of a lattice ideal pass to $V_{\mathcal{L}}$. For example, these initial ideals possess the rare property that their associated primes come in saturated chains [3]. Although we do not determine if this property holds for the vertex ideal, this section provides some tools for approaching this question. Furthermore, while investigating the associated primes of $V_{\mathcal{L}}$, we construct a lattice which provides a counterexample to a conjecture about codimension three toric ideals.

Since $V_{\mathcal{L}}$ is a monomial ideal, all of its associated primes are monomial primes of the form $\mathcal{P}_\sigma = \langle x_i : i \notin \sigma \rangle$ where $\sigma \subseteq [n] := \{1, \ldots, n\}$. 
Lemma 3.1. The set of associated primes $\text{Ass}(V_L)$ of $V_L$ is contained in $\bigcup_\omega \text{Ass}(in_\omega(I_L))$, the union of the associated primes of all initial ideals of $I_L$. Furthermore, the set of minimal primes of $V_L$ is precisely the union of the minimal primes of all initial ideals of $I_L$.

Proof. Using Theorem 2.3, the first statement follows from the fact that if two ideals $I$ and $J$ have minimal primary decompositions $\cap_i P_i$ and $\cap_j P'_j$, then $\left(\cap_i P_i\right) \cap \left(\cap_j P'_j\right)$ is a (not necessarily minimal) primary decomposition of $I \cap J$. Minimal primes of a intersection of monomials ideals are always contained in the union of the minomial primes of the ideals. The fact that this containment is an equality in this case follows from the fact, used in Theorem 2.10, that all minimal primes of all initial ideals have the same dimension. 

Example 3.2. The associated primes of $V_L$ can be strictly contained in $\bigcup_\omega \text{Ass}(in_\omega(I_L))$. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $L = \ker(A) \cap \mathbb{Z}^n$. For this lattice, $V_L = \langle abc, a^2b, a^3c, b^3c^2 \rangle$, which has primary decomposition $\langle a^3, ab, b^3 \rangle \cap \langle a^2, ac, c^2 \rangle \cap \langle b, c \rangle$, so the associated primes of $V_L$ are $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$. For $\omega = (100, 10, 1)$ in $\omega(I_L) = \langle a^2, ab, ac, b^3 \rangle$. This has primary decomposition $\langle a, b^3 \rangle \cap \langle a^2, b, c \rangle$, so we have $\langle a, b, c \rangle \in \bigcup_\omega \text{Ass}(in_\omega(I_L))$.

Corollary 3.5 of [3] gives bounds on the dimensions and codimensions of initial ideals of $I_L$. Combined with Lemma 3.1 we get the following fact about the dimension and codimension of the associated primes of $V_L$.

Proposition 3.3. The dimension of an associated prime of $V_L$ for a lattice of dimension $m$ is at least $\max(0, n - (2^m - 1))$ and the codimension is at most $\min(n, 2^m - 1)$.

For our purposes it is more convenient to study the associated primes of $V_L$ via its standard pairs [12]. For a vector $u \in \mathbb{N}^n$ we denote by $\text{supp}(u)$ the set $\{i : u_i \neq 0\}$.

Definition 3.4. An admissible pair of a monomial ideal $M$ is a pair $(x^u, \tau)$ with $\tau \subseteq [n]$ such that $\text{supp}(u) \cap \tau = \emptyset$, and $x^{u+v} \notin M$ for all $v$ with $\text{supp}(v) \subseteq \tau$. We place a partial order on the set of admissible pairs of $M$ by declaring $(x^u, \tau) \prec (x^{\nu}, \sigma)$ if $x^\nu | x^u$ and $\text{supp}(u - v) \cup \tau \subseteq \sigma$. The maximal elements of the set of admissible pairs with respect to this order are called standard pairs.

In the rest of the paper we use a polyhedral characterization of the standard pairs of $V_L$ following the results and terminology in [3] and [4]. We start with a characterization which follows from the definition of standard pairs.
Proposition 3.5. The pair \((x^u, \tau)\) is a standard pair of \(V_\mathcal{L}\) if and only if \(u\) is a vertex of \(P_u\), \(\text{supp}(u) \cap \tau = \emptyset\), \(u + v\) is a vertex of \(P_{u+v}\) for all \(v\) with \(\text{supp}(v) \subseteq \tau\), and for all \(i \notin \tau\) there is some \(v'\) with support in \(\tau \cup \{i\}\) such that \(u + v'\) is not a vertex of \(P_{u+v'}\).

As in the previous section, let \(B \in \mathbb{Z}^{n \times m}\) such that the columns of \(B\) form a lattice basis for \(\mathcal{L}\). Given \(u \in \mathbb{N}^n\), we can define the polyhedron \(Q_u := \{x \in \mathbb{R}^m : Bx \leq u\}\). The lattice points in \(Q_u\) and the lattice points in \(P_u\) are in bijection by the correspondence \(z \in Q_u \cap \mathbb{Z}^m \longleftrightarrow u - Bz \in P_u\). The origin of \(\mathbb{Z}^m\) is in \(Q_u\) for all \(u \in \mathbb{N}^n\) and corresponds to \(u \in P_u\). We let \(R_u\) be the convex hull of the lattice points in \(Q_u\). Note that \(R_u\) is affinely isomorphic to \(P_u\). For a subset \(\tau \subseteq [n]\) we denote by \(\bar{\tau}\) the complement of \(\tau\), so \(\bar{\tau} = [n] \setminus \tau\). With this convention we define \(Q_u^\tau\) to be the polyhedron \(\{x \in \mathbb{R}^m : B^\tau x \leq u^\tau\}\) where the inequalities defining \(Q_u\) corresponding to \(\tau\) are omitted. \(R_u^\tau\) denotes the convex hull of the lattice points in \(Q_u^\tau\). We now reformulate the characterization of standard monomials and standard pairs of \(V_\mathcal{L}\).

Theorem 3.6. The monomial \(x^u\) is a standard monomial of \(V_\mathcal{L}\) if and only if the origin is a vertex of \(R_u\). Moreover, a pair \((x^u, \tau)\) is a standard pair of \(V_\mathcal{L}\) if and only if the origin is a vertex of \(R_u^\tau\) and it is not a vertex of \(R_u^\tau\setminus i\) for any \(i \in \bar{\tau}\).

Proof. The first statement follows from Theorem 2.3 and the fact that the origin is a vertex of \(R_u\) if and only if \(u\) is a vertex of \(P_u\). For the second claim we use Proposition 3.5. The statement that \(u\) is a vertex of \(P_u\), and \(u + v\) is a vertex of \(P_{u+v}\) for all \(v\) with \(\text{supp}(v) \subseteq \tau\) is equivalent to the statement that the origin is a vertex of \(R_u\) and it remains a vertex of \(R_{u+v}\) for all such \(v\). Since \(\text{supp}(v) \subseteq \tau\), this is the same thing as the origin being the vertex of \(R_u^\tau\). Similarly, if for all \(i \notin \tau\) there exists a \(v'\) with \(\text{supp}(v') \subseteq \tau \cup \{i\}\) such that the origin fails to be a vertex of \(P_{u+v'}\), then the origin is also not a vertex of \(R_{u+v'}^\tau\), and hence not a vertex of \(R_u^\tau\setminus i\), and vice versa.

The characterization of the standard pairs in the above theorem also gives rise to a description of the irredundant irreducible primary decomposition of \(V_\mathcal{L}\). This is very similar to the description of the irredundant irreducible primary decompositions of \(in_\omega(J_\mathcal{L})\) given in Section 4 of [3]. In order to give this characterization we make the following definition.

Definition 3.7. We call the polyhedron \(Q_u\) critical if the origin is a vertex of \(R_u\), but not a vertex of \(R_{u+e_i}\) for any \(i = 1, \ldots, k\).
Theorem 3.8. The ideal $V_L$ has the irreducible primary decomposition

$$V_L = \bigcap_{Q_u^\tau} \langle x_i^{u_i+1} : i \in \tau \rangle$$

where the intersection is taken over all critical $Q_u^\tau$.

Proof. The proof of Lemma 3.3 in [12] implies that

$$V_L = \bigcap_{(x^u, \tau)} \langle x_i^{u_i+1} : i \in \tau \rangle$$

where the intersection is taken over all standard pairs $(x^u, \tau)$ such that $x^u x_i \in V_L$ for all $i \in \tau$. By Theorem 3.6 these standard monomials are in bijection with critical $Q_u^\tau$. \qed

When we have a two-dimensional saturated lattice $L = \ker(A) \cap \mathbb{Z}^n$, the codimension of $V_L$ is two, and so Proposition 3.3 implies that if an embedded prime $P_\tau$ of $V_L$ exists, the codimension of $P_\tau$ must be three, which means $|\tau| = n - 3$. Our next task is to show that in this case cone$\{a_i : i \in \tau\}$ cannot be a face of cone$\{a_i : i = 1, \ldots, n\}$ where $a_i$ is the $i$-th column of the matrix $A$. The result is a consequence of the following lemma.

Lemma 3.9. Let $Q \subset \mathbb{R}^2$ be a polygon defined by $n$ facet-defining inequalities $b_i \cdot x \leq u_i$, and let $R$ be the convex hull of the lattice points in $Q$. Let $v$ be a vertex of $R$. Then there exists a facet $j$ of $Q$ such that $v$ is a vertex of the convex hull, $R_j$, of the lattice points in $Q_j := \{x \in \mathbb{R}^2 : b_i \cdot x \leq u_i, i \neq j\}$.

Proof. Suppose not. Clearly we can assume that $R$ is two-dimensional and that $v$ is the origin. Let $v_1$ and $v_2$ be the two vertices of $R$ which are the neighboring vertices of the origin, in the clockwise and counterclockwise directions respectively. We define the pointed cone $K$ generated by $v_1$ and $v_2$, and $-K$, the opposite cone generated by $(-v_1)$ and $(-v_2)$. These constructions are illustrated in Figure 1. We first claim that each edge of $Q$ has to intersect $-K$. Suppose there is an edge $e$, lying on the hyperplane $b_k \cdot x \leq u_k$, which does not intersect $-K$. Then the convex region $S := \{x \in \mathbb{R}^2 : b_i \cdot x \leq u_i, i \neq k, b_k \cdot x \geq u_k\}$ does not intersect $-K$ as well. This is true because if $v \in S \cap -K$ there is a point $w$ on the line segment joining $v$ to the origin lying on $e$, and $w$ would then be in $-K$. Since the origin is not a vertex of conv$(Q_k \cap \mathbb{Z}^2)$, either 0 is in the interior of an edge of $R_k$ or it is in the interior of $R_k$. In the first case there exists two vertices $y$ and $z$ of $R_k$ such that $y \in S$ and $z \in R \subset K$ with $0 = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$. But then $y \in S \cap -K$, contrary to our assumption. If 0
is an interior point of $R_k$, then there exist three vertices $y_1, y_2$ and $y_3$ of $R_k$ such that \( 0 = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \) with \( 0 < \lambda_1, \lambda_2, \lambda_3 < 1 \) and \( \sum \lambda_1 + \lambda_2 + \lambda_3 = 1 \). Now, either exactly one or exactly two of these vertices are in $S$. In the first case, say $y_1 \in S$ and $y_2, y_3 \in R$, we have $y_1 = -\frac{\lambda_2}{\lambda_1} y_2 - \frac{\lambda_3}{\lambda_1} y_3$ and hence $y_1 \in (-K) \cap S$. In the second case, say $y_1, y_2 \in S$ and $y_3 \in R$, we have $\frac{\lambda_1}{\lambda_1 + \lambda_2} y_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} y_2 = -\frac{\lambda_3}{\lambda_1 + \lambda_2} y_3$, and hence $-\frac{\lambda_2}{\lambda_1 + \lambda_2} y_3 \in (-K) \cap S$. In both cases we get a contradiction to our assumption that edge $e$ does not intersect $-K$. This shows that all edges of $Q$ intersect $-K$.

Because $Q$ contains $v_1$ but not $-v_2$, and $v_2$ but not $-v_1$, some edge of $Q$ must intersect the line segment $[v_1, -v_2]$, and another one the line segment $[v_2, -v_1]$. If we assume that the facets of $Q$ are labeled going clockwise and the edge 1 is the first edge intersecting the facet of $-K$ defined by $(-v_1)$, then edge 1 must be the edge intersecting $[v_2, -v_1]$. And if edge $n$ is the last edge intersecting the facet of $-K$ defined by $(-v_2)$, then edge $n$ must be the edge intersecting $[v_1, -v_2]$. Edge 1 and edge $n$ are the only edges of $Q$ not lying entirely in $-K$, so they need to meet in a common vertex of $Q$. But their endpoints outside $-K$ are on opposite sides of the parallel line segments $[v_1, -v_2]$ and $[v_2, -v_1]$, which makes this impossible.

Remark 3.10. Note that we cannot relax the hypothesis in Lemma 3.9 that $Q$ is a polygon to $Q$ being a possibly unbounded polyhedron. An example of this phenomenon is in Figure 2. If any of the facets of $Q$ are removed, the origin, $O$, ceases to be a vertex of $R$.
Theorem 3.11. Let $I_L$ be a codimension two lattice ideal where $L = \ker(A) \cap \mathbb{Z}^n$ with $L \cap \mathbb{N}^n = \{0\}$. If $P_\tau$ is an embedded prime of $V_L$ then cone${\{a_i : i \in \tau}\}}$ is not a face of cone${\{a_i : i = 1, \ldots, n}\}}$ where $a_i$ is the $i$-th column of the matrix $A$. In particular, the irrelevant maximal ideal $P_\emptyset$ is not associated to $V_L$.

Proof. Let $(x^n, \tau)$ be a standard pair of $V_L$. Suppose that cone${\{a_i : i \in \tau}\}}$ is a face of cone${\{a_i : i = 1, \ldots, n}\}}$. This means that the origin in $\mathbb{R}^2$ is in the convex hull of $\{b_i : i \in \tau\}$, where $b_i$ is the $i$-th row of the $B$ defined after Proposition 3.3. This follows because positive covectors of (the oriented matroid of) $A$ correspond to positive vectors of (the oriented matroid of) $B$ (see [13, Chapter 6]). So $Q_u^\tau$ is a polygon. Theorem 3.6 now implies that the origin in $\mathbb{R}^2$ is a vertex of $R_u^\tau$, but not a vertex of any $R_u^{\tau \setminus i}$ for $i \in \overline{\tau}$. But this is a contradiction to Lemma 3.9.

Remark 3.12. The statement of Lemma 3.9 also cannot be generalized to higher dimensional polytopes. Similarly, the statement of Theorem 3.11 cannot be generalized to higher codimension. In particular, the irrelevant maximal ideal $P_\emptyset$ could be an embedded prime for some $V_L$ as the following example shows.
Example 3.13. Let $A := [15, 247, 248, 345]$. A lattice basis for $\mathcal{L} = \ker(A) \cap \mathbb{Z}^4$ is given by the columns of

$$B := \begin{bmatrix} -4 & -3 & -3 \\ -6 & 9 & -2 \\ 9 & -6 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$ 

If we choose $u = (9, 7, 7, 1)^T$, then $Q_u^{[1,2,3,4]} = Q_u = \{ x \in \mathbb{R}^3 : Bx \leq u \}$ is a tetrahedron. The polytope $R_u^{[1,2,3,4]} = R_u$ has the following six vertices:

$$(0, 0, -3), (0, 0, 0), (1, 0, 1), (0, 1, 1), (3, 3, 1), (23, 23, 31).$$

Now $R_u^{[1,3,4]}$ contains the lattice point $(-1, 0, -1)$ and the origin is in $\text{conv}\{(1, 0, 1), (-1, 0, -1)\}$; $R_u^{[1,2,4]}$ contains the lattice point $(0, -1, -1)$ and the origin is in $\text{conv}\{(0, 1, 1), (0, -1, -1)\}$; $R_u^{[1,2,3]}$ contains the lattice points $(-1, 0, 0)$ and $(0, -1, 0)$, and the origin is in $\text{conv}\{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)\}$, and finally $R_u^{[2,3,4]}$ contains the lattice point $(-1, -1, -1)$ and the origin is in $\text{conv}\{(1, 1, 1), (-1, -1, -1)\}$. This shows the origin is not a vertex in any of these new polytopes. In particular, $(x^u, \emptyset)$ is a standard pair of $V_\mathcal{L}$, and hence the irrelevant ideal is an associated prime of $V_\mathcal{L}$.

The above example also provides a counterexample to a conjecture about the complexity of Gröbner fans of codimension three toric ideals (Conjecture 6.2 in [4]). This conjecture stated that any Gröbner cone of a codimension three toric ideal has at most four facets.

Theorem 3.14. There exists a toric ideal $I_A$ with $\text{codim}(I_A) = 3$ which has a Gröbner cone with five facets.

Proof. Let $A$ be as in the above remark. If we choose $\omega = (111, 0, 342, 1)$ as the cost vector we get the following reduced Gröbner basis:

$$\{a^{23} - d, da^{10} - bc, d^{12}a^4 - b^{16}c, d^{55}a^3 - b^{76}c, d^{16}a^2 - b^{225}, d^{204}a - b^{285}, d^{247} - b^{345}, cd^6a^7 - b^{14}, cd^{20}a - b^{29}, cd^{63} - b^{89}, c^2d^8 - b^{13}a^3, c^4d^5 - b^{11}, c^5d^4 - b^{10}a^{10}, c^6d^2a^3 - b^{9}, c^7a^{16} - b^8, c^7d - b^4a^7, c^8 - b^7a^{17}, bca^{13} - d^2, b^2c^2a^3 - d^3, b^3c^3 - d^4a^7, b^5a^{20} - c^6d^3, b^{12}a^{13} - c^3d^7, b^{15}a^6 - d^{11}, b^{31}ca^2 - d^{23}, b^{44}a^5 - cd^{31}, b^{47}c^2 - d^{35}a^2, b^{66}a - d^{43}, b^{136}c - d^{98}a^2 \}.$$
The corresponding Gröbner cone is given by
\[
\begin{align*}
+345b &\leq -247d \\
-20a &\quad -9b \quad +6c \quad 3d \quad \leq 0 \\
+2a &\quad -136b \quad -c \quad +98d \quad \leq 0 \\
-3a &\quad +76b \quad +c \quad -55d \quad \leq 0 \\
+7a &\quad -3b \quad -3c \quad +4d \quad \leq 0
\end{align*}
\]
which are all facet defining.

This counterexample was found by using TiGERS \cite{tiigers}, an implementation to compute Gröbner fans of toric ideals developed by Birkett Huber and Rekha Thomas. Computer experiments with TiGERS have yielded many other examples of Gröbner cones of codimension 3 toric ideals with five facets, and the following (thus far unique) codimension 3 toric ideal with a Gröbner cone with six facets.

**Example 3.15.** For the matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 8 & 9 & 7 & 10 & 6 & 5 \\
8 & 7 & 4 & 8 & 7 & 2 & 2 \\
5 & 9 & 4 & 2 & 9 & 8 & 3
\end{bmatrix}
\]
the initial ideal of \( I_A \) with respect to the weight vector \((252, 197, 0, 0, 153, 0, 0)\) corresponds to a Gröbner cone with six facets.

4. The Product Ideal

In this section we define the *product ideal* of \( \mathcal{L} \) which is closely related to \( V_\mathcal{L} \), and which is much easier to compute. Although in general the two ideals are not equal, we will look at two special cases where they are: the case when \( \mathcal{L} \) is unimodular, and when \( \mathcal{L} \) is a two-dimensional lattice in \( \mathbb{Z}^2 \). Even in the cases where they are not equal, we will show that the product ideal carries valuable information about \( V_\mathcal{L} \). For instance we will show that the radicals of the two ideals are equal.

**Definition 4.1.** The product ideal \( P_\mathcal{L} \) is the monomial ideal defined by
\[
P_\mathcal{L} = \langle x^u x^v : u - v \in Gr_\mathcal{L} \rangle.
\]

Since each initial ideal in \( \omega (I_\mathcal{L}) \) contains one of \( x^u \) or \( x^v \) whenever \( u - v \in \mathcal{L} \), we have \( P_\mathcal{L} \subseteq V_\mathcal{L} \). This containment could be strict, however, as shown in the following example. Let \( \mathcal{L} = ker(A) \cap \mathbb{Z}^3 \) where \( A = [3 \ 4 \ 5] \). Then \( P_\mathcal{L} = \langle a^2c^{\cdot}, a^2bc^2, a^3bc, a^4b^3, a^5c^3, b^5c^4 \rangle \) is strictly contained in \( V_\mathcal{L} = \langle ab^2c, a^2bc, a^4b^3, a^5c^3, b^5c^4 \rangle \). There are two special cases, though, in which the product ideal and the vertex ideal
are equal. The first case is when $L$ comes from a unimodular matrix. We recall that a $d \times n$ matrix is unimodular if all $d \times d$ minors have the same absolute value.

**Proposition 4.2.** If $L = \ker(A) \cap \mathbb{Z}^n$ where $A$ is a unimodular matrix, then $P_L = V_L$, and $P_L$ coincides with the matroid ideal $I_{\Delta(M(L))}$.

**Proof.** The initial ideals $\in_{\omega}(I_L)$ are all square-free (Corollary 8.9 in [10]) and hence $V_L$ is radical. Therefore, by Corollary 2.12, $V_L = I_{\Delta(M(L))}$. But the minimal generators of $I_{\Delta(M(L))}$ are of the form $\prod_{i \in J} x_i$ for some $J = \{i_1, \ldots, i_k\}$ such that $\{a_{i_1}, \ldots, a_{i_k}\}$ is a circuit of $A$. Now Proposition 8.11 of [10] implies that the Graver basis of $I_L$ is $\{\alpha_i - \beta_i \in L : \text{supp}(\alpha_i - \beta_i) \text{ is the support of a circuit }\}$. Therefore $P_L = \langle \prod_{i \in J} x_i : J = \{i_1, \ldots, i_k\} \text{ is the support of a circuit }\rangle$, and hence $P_L = V_L = I_{\Delta(M(L))}$. \vspace{1em}

**Proposition 4.3.** If $L$ is a two-dimensional lattice in $\mathbb{Z}^2$, then $P_L = V_L$.

**Proof.** Let $S = k[x, y]$ and suppose $x^u y^v \in V_L$, so $(u, v)$ is not a vertex of its fiber $P_{(u,v)}$, but $x^u y^v \notin P_L$. If $(a, b) \in P_{(u,v)}$ where $(0,0) \leq (a, b) \leq (u, v)$, then $x^u y^v \in P_L$ because $x^{u-a} y^{v-b} - 1 \in I_L$ and hence $x^{u-a} y^{v-b} \in P_L$. So no such point in $P_{(u,v)}$ exists. Now there must be a vertex $(a, b)$ of this fiber with $b < v$, because otherwise $(u, v)$ would be a vertex. Let $(a, b)$ be the vertex with $b < v$ such that $(a, b)$ is the maximum with this property. Let $H$ be the line through $(u, v)$ and $(a, b)$, let $H^-$ be the halfspace containing the origin, and let $H^+$ be the other halfspace. If $P_{(u,v)} \subset H^+$, since $(u, v)$ is not a vertex of $P_{(u,v)}$, the line $H$ must contain $(c, d) \in P_{(u,v)}$ such that $0 \leq c \leq u$ and $v < d < 2v - b$. But then $(u - c, v - d) \in L$ and $x^{u-c} y^{d-v} \in I_L$, which implies $x^{u-c} y^{d-v} \in P_L$. This implies $x^u y^v \in P_L$ since $d - v \leq v$. Hence we are reduced to the case that $P_{(u,v)}$ is not contained in $H^+$ and $2u < a$ (so no such $(c, d) \in P_{(u,v)}$). This means that there exists a vertex $(e, f) \in H^- \cap P_{(u,v)}$. Now if $e > u$, by the construction of $(a, b)$ we must have $f < b$. If in addition $e < a$, the existence of a vector of the form $(k, 0) \in L$ for some $k$ means that $(e + kN, f) \in P_{(u,v)}$ for $N \gg 0$ which contradicts $(a, b)$ being a vertex. On the other hand, if $a < e$, $(a, b)$ would not be a vertex of $P_{(u,v)}$. So we conclude that $e < u$. But now we must have $v < f < 2v - b$, where the second inequality follows from the assumption that $(e, f) \in H^-$ and $2u < a$. Since $(e - u, f - v) \in L$, it follows that $x^u y^v \in P_L$, a contradiction, so $P_L = V_L$. \vspace{1em}
Example 4.4. Proposition 4.3 fails when \( \dim(L) \geq 3 \). For instance, let \( L \) be the lattice in \( \mathbb{Z}^3 \) generated by the columns of the matrix

\[
\begin{pmatrix}
1 & 4 & 3 \\
-2 & 0 & 5 \\
-1 & 1 & -9
\end{pmatrix}.
\]

One can verify using Macaulay2 \(^3\) that \( P_L = \langle ab^2c, a^4c, a^5b^2, b^8c^5, abc^{12}, b^3c^{11}, b^{19}c, ab^{21}, a^4b^{19}, ac^{26}, a^3c^{26}, b^2c^{27}, bc^{38}, a^{49}b, c^{103}, b^{103}, a^{103} \rangle \), and it is strictly contained in \( V_L = \langle c^3, ab^2c, a^4b^2, b^{19}c, ab^{21}, a^4b^{19}, a^{49}b, b^{103}, a^{103} \rangle \).

The example at the beginning of this section shows that Proposition 4.3 does not hold even for a two-dimensional lattice \( L \) when \( L \) is in \( \mathbb{Z}^n \) for \( n \geq 3 \). However, we will show that \( P_L \) and \( V_L \) have the same radical, and that for two dimensional lattices they are almost equal.

For \( \sigma \subseteq \{1, \ldots, n\} \), let \( \pi_\sigma : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-|\sigma|} \) be the projection map which eliminates the coordinates indexed by \( \sigma \). We will denote the image of a lattice \( L \) under this map by \( L_\sigma \). It is clear that if \( \dim(L) = \dim(L_\sigma) \) then \( L \) and \( L_\sigma \) are isomorphic lattices. This observation implies the following lemma.

Lemma 4.5. Let \( Gr_L \) and \( Gr_{L_\sigma} \) be the Graver bases of the lattices \( L \) and \( L_\sigma \) where \( \dim(L) = \dim(L_\sigma) \). Then \( Gr_{L_\sigma} \subseteq \pi_\sigma(Gr_L) \).

Proof. If \( \alpha' - \beta' \in Gr_{L_\sigma} \), there is a unique \( \alpha - \beta \in L \) such that \( \pi_\sigma(\alpha - \beta) = \alpha' - \beta' \). If \( \alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2 \), with \( \alpha_i - \beta_i \in L \) for \( i = 1, 2 \), and \( \alpha_i, \beta_i \in \mathbb{N}^n \), then \( \alpha' = \pi_\sigma(\alpha_1) + \pi_\sigma(\alpha_2) \) and \( \beta' = \pi_\sigma(\beta_1) + \pi_\sigma(\beta_2) \), with \( \pi_\sigma(\alpha_i) - \pi_\sigma(\beta_i) \in L_\sigma \) for \( i = 1, 2 \). As this contradicts \( \alpha' - \beta' \in Gr_{L_\sigma} \), we conclude that \( \alpha - \beta \in Gr_L \), so \( \alpha' - \beta' \in \pi_\sigma(Gr_L) \).

The algebraic analogue of the projection map \( \pi_\sigma \) is the localization map \( \hat{\pi}_\sigma : k[x_1, \ldots, x_n] \longrightarrow k[x_i : i \notin \sigma] \) where \( \hat{\pi}_\sigma(x_i) = x_i \) if \( i \notin \sigma \) and \( \hat{\pi}_\sigma(x_i) = 1 \) otherwise. This corresponds to localizing at the monomial prime \( \mathcal{P}_\sigma = \langle x_i : i \notin \sigma \rangle \). We now compare \( Top(P_L) \) with \( Top(V_L) \), where \( Top(M) \) is the intersection of the top-dimensional primary components of the ideal \( M \). When we consider a monomial ideal \( M \) with top-dimensional minimal primes \( \mathcal{P}_{\sigma_1}, \ldots, \mathcal{P}_{\sigma_k} \), we have \( Top(M) = \cap_{i=1}^k \hat{\pi}_{\sigma_i}(M) \).

Proposition 4.6. If \( \dim(L) = \dim(L_\sigma) \) then \( \hat{\pi}_\sigma(V_L) = V_{L_\sigma} \) and \( \hat{\pi}_\sigma(P_L) = P_{L_\sigma} \).

Proof. From Lemma 4.3 we know that \( \pi_\sigma(Gr_L) \subseteq Gr_{L_\sigma} \). Let \( x^a \) be a minimal generator of \( V_L \). By Corollary 2.7 we know that \( x^a = \text{lcm}_i(x^\alpha_i) \) where \( \sum_i c_i(\alpha_i - \beta_i) = 0 \) for \( \alpha_i - \beta_i \in Gr_L \) and \( c_i > 0 \). Now \( \sum_i c_i \pi_\sigma(\alpha_i - \beta_i) \).
Proof. Proposition 4.3 says that \( \beta_i = 0 \). Writing \( \pi_\sigma(\alpha_i - \beta_i) = \sum_j (\alpha_{ij} - \beta_{ij}) \) where \( \alpha_{ij} - \beta_{ij} \in Gr_{L_\sigma} \) and \( \alpha_{ij} \leq \alpha_i \), \( \beta_{ij} \leq \beta_i \) for all \( j \), we see that for \( x^v = \text{lcm}_i (x^{\alpha_{ij}}) \), \( x^v \in V_{L_\sigma} \). Since \( x^v \) divides \( \text{lcm}(\alpha_i) \), it follows that \( \hat{\pi}_\sigma(x^v) \in V_{L_\sigma} \).

For the other inclusion, let \( x^u \) be a minimal generator of \( V_{L_\sigma} \), so \( x^u = \text{lcm}_i (x^{\alpha_i}) \) for \( \sum_i c_i (\alpha_i - \beta_i) = 0 \), where \( \alpha_i - \beta_i \in Gr_{L_\sigma} \) and \( c_i > 0 \). Let \( \alpha'_i - \beta'_i \) be the preimage of \( \alpha_i - \beta_i \) under \( \pi_\sigma \). We still have \( \sum_i c_i (\alpha'_i - \beta'_i) = 0 \), so for \( x^v = \text{lcm}_i (x^{\alpha'_i}) \), \( x^v \in V_L \), and thus \( \hat{\pi}_\sigma(x^v) = x^u \in \hat{\pi}_\sigma(V_L) \).

The second statement of the proposition follows from the definition of the product ideal, and the observation that if \( \pi_\sigma(\alpha - \beta) \notin Gr_{L_\sigma} \) for \( \alpha - \beta \in Gr_L \), we can write \( \pi_\sigma(\alpha - \beta) \) as the sum of \( \alpha_i - \beta_i \in Gr_{L_\sigma} \) so that \( x^{\alpha_i+\beta_i}|x^{\pi_\sigma(\alpha)+\pi_\sigma(\beta)} = \hat{\pi}_\sigma(x^{\alpha+\beta}) \).

\[ \square \]

**Corollary 4.7.** The radical of \( P_L \) and the radical of \( V_L \) coincide. Moreover, \( \text{Top}(P_L) \subseteq \text{Top}(V_L) \).

**Proof.** Theorem 2.10 shows that \( \text{rad}(V_L) \) is an equidimensional ideal. Now an associated prime \( P_\sigma \) of \( V_L \) is a minimal prime if and only if \( L_\sigma \) is a full dimensional lattice in \( \mathbb{Z}^{n-|\sigma|} \). But this is true if and only if there exist \( n_i \) such that \( n_i e_i \in L_\sigma \) for all \( i \notin \sigma \). This happens if and only if \( x_i^{n_i} \in P_{L_\sigma} \) for all \( i \notin \sigma \), which happens exactly whenever \( P_{L_\sigma} = \hat{\pi}_\sigma(P_L) \) is a zero-dimensional ideal, and hence \( P_\sigma \) is a minimal prime of \( P_L \). This shows that \( \text{rad}(V_L) = \text{rad}(P_L) \).

The second statement follows from Proposition 1.6 and the discussion before it, and the fact that \( P_{L_\sigma} \subseteq V_{L_\sigma} \).

\[ \square \]

**Corollary 4.8.** If \( \dim(L) = 2 \) then \( \text{Top}(P_L) = \text{Top}(V_L) \).

**Proof.** Proposition 1.3 says that \( P_{L_\sigma} = V_{L_\sigma} \) when \( P_\sigma \) is a minimal prime of \( V_L \) (and of \( P_L \)). Now Proposition 1.6, Corollary 4.7, and the discussion before them imply the result.

We note that the above corollary fails when \( \dim(L) \geq 3 \). Example 4.3 provides a lattice \( L \in \mathbb{Z}^3 \) of dimension three. Therefore \( \text{Top}(V_L) = V_L \) and \( \text{Top}(P_L) = P_L \), but in that example we saw that \( P_L \neq V_L \).

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