Elastic moduli of vortex lattices within nonlocal London model

P. Miranović and V. G. Kogan

1 Department of Physics, University of Montenegro, P.O. Box 211, Yugoslavia
2 Ames Laboratory and Department of Physics, Ames 50011

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Vortex lattice (VL) elastic response is analyzed within nonlocal London model. The squash modulus turns zero at the field $H\xi$ where VL undergoes square-to-rhombus structural transition. In the field region $H > H\xi$ where the square VL is stable, the rotation modulus turns zero at a field $H\xi$ indicating instability of the square VL with respect to rotations. The shear modulus depends on the shear direction; the dependence is strong in the vicinity of $H\xi$ where the square VL is soft with respect to the shear along [110]. The $H$ dependences of elastic moduli are evaluated for LuNi$_2$B$_2$C.

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The theory of elasticity for vortex lattices (VL) in type-II superconductors is commonly constructed in a manner similar to the standard elastic theory. One introduces the local VL displacement $u$ from equilibrium and expands the elastic energy $E$ in a series of derivatives $u_{i,k} \equiv \partial u_i / \partial x_k$. The similarity, however, ends already at this stage. Elastic energy of an element of a solid does not depend on rigid rotations of this element, so that $E$ depends only on symmetric combinations $u_{ik} = (u_{i,k} + u_{k,i})/2$ which form the strain tensor, whereas the antisymmetric combinations $\omega_{ik} = (u_{i,k} - u_{k,i})/2$ representing rotations are irrelevant. This property can be tracked back to the basic isotropy of the space in which solids are situated.

Vortices, however, are hosted by crystals which at best have the cubic symmetry. In other words, the “space in which vortices live” is never isotropic. The crystal anisotropy affects intervortex interactions in a trivial way through the tensor $m_{ik}$ of “superconducting masses” which determines the anisotropy of the London penetration depth $\lambda$ and the coherence length $\xi$. For all symmetries except cubic (for which $m_{ik}$ reduces to $\delta_{ik}$), rigid rotations of VL cause an energy change even for rotations about the direction of vortex axes (that of the magnetic induction $B$).

Moreover, the current and field distributions around the vortex core are affected by the underlying crystal in a more subtle manner than via the $\lambda$ anisotropy. In full, this influence can be described only by the microscopic theory. In materials with $\lambda / \xi = \kappa \gg 1$ the problem is simplified at distances $r \gg \xi$ relevant for intervortex interactions in fields well below the upper critical field $H_{c2}$. This is done in the framework of the London theory corrected for the nonlocality of the relation between the current and the vector potential. The kernel of this nonlocal relation provides the formal bridge between the Fermi system of electrons in a given crystal and interacting vortices in the superconducting condensate. In general, this kernel depends on the Fermi surface and on the symmetry of the order parameter; for simplicity, we consider here the isotropic order parameter.

The general form of the elastic energy in terms of strains and rotations is $E = (\lambda_{ijkl} u_{ij} u_{kl} + \eta_{ijkl} \omega_{ij} \omega_{kl})/2 + \zeta_{ijkl} u_{ij} \omega_{kl}$. This form is not convenient for counting number of independent coefficients because one has to consider symmetry restrictions upon three 4th rank tensors. A more direct approach is to forgo the splitting of $u_{i,k}$ into symmetric and antisymmetric parts (which no longer simplifies the problem) and to write the energy in terms of $u_{i,k}$:

$$E = \gamma_{iklm} u_{i,k} u_{l,m} / 2.$$  \(1\)

The tensor $\gamma_{iklm}$ is not symmetric relative to $i \leftrightarrow k$ and $l \leftrightarrow m$, but preserves the property of the standard elasticity tensor $\gamma_{ik,m} = \gamma_{lm,ik}$. The symmetries of this tensor and the number of its independent components are de-
determined both by the crystal and by the equilibrium VL structure. In other words, \( \gamma_{ijkl} \) has only the symmetries common to the crystal and the equilibrium VL.

The equilibrium VL in cubic or tetragonal crystals in fields above \( H_0 \) along [001] has a square unit cell with sides along [110] and [1¯10]. We choose the coordinates \( x, y \) in the \( ab \) plane as [100] and [010]. Then, \( x \) may enter the indices of \( \gamma \) only even number of times, and the components obtained from each other by replacements \( x \leftrightarrow y \) are equal:

\[
E = \frac{1}{2} [\gamma_{xxx}(u_{x,x}^2 + u_{y,y}^2) + 2\gamma_{xx}yy u_{x,x}u_{y,y} + \gamma_{xy}y y(u_{x,y}^2 + u_{y,x}^2) + 2\gamma_{yy}xx u_{x,y}u_{y,x}].
\]

(2)

After excluding compressions (\( B \) is constant, \( \text{div} \mathbf{u} = u_{x,x} + u_{y,y} = 0 \)) we are left with three constants:

\[
E = (\gamma_{xxxx} - \gamma_{xx}yy)u_{x,x}^2 + \frac{1}{2} \gamma_{xx}yy u_{x,y}^2 + \gamma_{xy}xy y u_{x,y}u_{y,x} + \frac{1}{2} \gamma_{yy}xx u_{x,y}^2 + \gamma_{yy}xy x u_{y,x}u_{x,x} + 2\gamma_{xy}xx u_{x,y}u_{y,x}.
\]

(3)

How to classify the three constants, to a large extent, is a question of semantics. For our particular problem, a uniform deformation defined as

\[
\mathbf{u} = \mu(x \mathbf{e}_x - y \mathbf{e}_y),
\]

(4)

(\( \mu \) is a small constant and \( \mathbf{e}_x, \mathbf{e}_y \) are unit vectors) is of a special interest because it is this deformation which transforms the square VL above \( H_0 \) into a rhombic one for \( H < H_0 \). Since this is a second order phase transition, the corresponding modulus must vanish at \( H = H_0 \). This deformation was named “squash” \( \Box \). For the displacement \( \Box \), \( u_{x,x} = -u_{y,y} = \mu \), whereas \( u_{x,y} = u_{y,x} = 0 \). Then, the part of the energy related to the squash is \( E_{sq} = \gamma_1 \mu^2/2 \). In other words, \( \gamma_1 = 2(\gamma_{xxxx} - \gamma_{xx}yy) \) can be called the squash modulus and denoted as \( C_{sq} \).

Consider now a uniform shear polarized along \( x \),

\[
\mathbf{u} = \nu y \mathbf{e}_x,
\]

(5)

for which only \( u_{x,y} \neq 0 \). The energy is \( E_x = \gamma_2 u_{x,y}^2/2 = \gamma_2 y^2/2 \). The energy is the same for the \( y \) polarization: \( E_y = E_x = \gamma_2 x^2/2 \). Hence, \( \gamma_2 \) is the shear modulus for either \( x \) or \( y \) polarizations; below we complement the traditional notation with polarization index:

\( \gamma_2 = C_{s66,x} = C_{s66,y} \).

Let now the shear be directed along some direction \( \mathbf{e}_x \), so that the displacement is given by \( \mathbf{u} = \nu y' \mathbf{e}_x \), where

\[
x' = x \cos \theta + y \sin \theta, \quad y' = -x \sin \theta + y \cos \theta
\]

(6)

and \( \theta \) is the angle between \( x' \) and \( x \). In the old coordinates,

\[
\mathbf{u} = \nu (-x \sin \theta + y \cos \theta) (\mathbf{e}_x \cos \theta + \mathbf{e}_y \sin \theta).
\]

(7)

Then \( u_{x,x} = -u_{y,y} = -\nu \sin \theta \cos \theta \), \( u_{x,y} = \nu \cos^2 \theta \), \( u_{y,x} = -\nu \sin^2 \theta \), and

\[
E(\theta) = \frac{\mu^2}{2} [\gamma_2 + (\gamma_1 - 2\gamma_2 - 2\gamma_3) \sin^2 \theta \cos^2 \theta].
\]

(8)

In particular, \( E(\pi/4) = \nu^2(\gamma_1 + 2\gamma_2 - 2\gamma_3)/8 \). Since all \( \gamma_{ij}^2 = \nu^2/4 \), the factor \( \gamma_1 + 2(\gamma_2 - \gamma_3) \) is the modulus for the shear polarized along [110]. Thus, in addition to \( C_{sq} = \gamma_1 \), there are two independent shear moduli in the problem: \( C_{s66,x} = \gamma_2 \) and \( C_{s66,y} = \gamma_1 + 2(\gamma_2 - \gamma_3) \).

Consider now a rigid rotation

\[
\mathbf{u} = \omega(x \mathbf{e}_y - y \mathbf{e}_x),
\]

(9)

for which \( u_{x,y} = -u_{y,x} = \omega \), whereas \( u_{x,x} = 0 \). Then, \( E_\omega = \omega^2(\gamma_2 - \gamma_3) = C_\omega \omega^2/2 \). In other words, the rotational modulus is \( C_\omega = 2(\gamma_2 - \gamma_3) \). Thus, the three independent moduli can also be chosen as

\( C_{sq} = \gamma_1 \), \( C_{s66,x} = \gamma_2 \), and \( C_\omega = 2(\gamma_2 - \gamma_3) \).

(10)

Note that

\[ C_{s66}(\pi/4) = C_{sq} + C_\omega. \]

(11)

The choice of a particular set of elastic moduli is a matter of convenience in a problem at hand.

The above enumeration of the elastic constants pertains only for \( H > H_0 \) (parallel to \( c \)) where both the crystal and the VL have the square symmetry in the \( xy \) plane. In fields under \( H_0 \), the equilibrium VL is rhombic, and the replacement \( x \leftrightarrow y \) is no longer a symmetry operation. We then obtain instead of Eq. (2):

\[
E = \frac{1}{2} (\gamma_{xxxx} u_{x,x}^2 + \gamma_{yyyy} u_{y,y}^2 + 2\gamma_{xx}yy u_{x,x} u_{y,y} + \gamma_{xy}xy y u_{x,y}u_{y,x}) + \gamma_{yy}xy u_{x,y}u_{x,x} + \gamma_{yy}xy u_{y,x}u_{y,y}.
\]

(12)

For an incompressible VL, we obtain four independent constants. We can choose them as the squash \( C_{sq} \), two shears \( C_{s66,x} \neq C_{s66,y} \), and the rotation \( C_\omega \).

We now turn to evaluation of the moduli using the London equations corrected for nonlocality, which are obtained from the BCS theory for superconductors with isotropic gap \( \Box \):

\[
\frac{4\pi}{c} j_i = -\frac{1}{\lambda^2} (m_{ij}^{-1} - \lambda^2 n_{ijlm} k_l k_m) a_j.
\]

(13)

Here, \( m_{ij} \) is the normalized mass tensor (\( \det m_{ij} = 1 \)) and \( \lambda \) is the average penetration depth \( (\lambda^3 = \lambda_1 \lambda_2 \lambda_3, \lambda_1 = \lambda_0 \sqrt{m_1}, \ldots) \). Further, \( a = A + \phi_0 \nabla \theta/2\pi \) and \( \phi_0 \) is the flux quantum. Tensor \( n_{ijlm} \) is defined as.
\[ n_{ijmn} = \frac{3\hbar^2 (v_i v_i v_m)}{4(v^2) \Delta^2 \lambda^2} \gamma(T, \tau), \quad \gamma = \frac{\Delta^2 \sum \beta^{-2} \beta_1^{-3}}{\sum \beta^{-2} \beta_1^{-2}}. \]  

(14)

where \( v \) is the Fermi velocity and the brackets \( \langle \ldots \rangle \) stand for averages over the Fermi surface; \( \beta^2 = \Delta^2 + \hbar^2 \omega^2 \), \( \Delta(T) \) is the uniform BCS gap, \( \hbar \omega = \pi T (2n + 1) \) with an integer \( n \), and sums run over \( \omega > 0 \); \( \beta_1 = \beta + \hbar/2\tau \) with \( \tau \) being the scattering time due to nonmagnetic impurities. In dirty superconductors \( \gamma \sim (\tau \Delta/\hbar)^2 \rightarrow 0 \), i.e., nonlocal effects vanish. In the clean limit \( \gamma \) ranges from 2/3 at \( T = 0 \) to \( \approx 0.3 \) at \( T = T_c \).

The flux quantization \( \nabla \times \mathbf{a} = \mathbf{h} - \phi_0 \mathbf{e}_z \delta(\mathbf{r}) \) combined with Eq. (13) gives the field of a single vortex along \( c \) of cubic or tetragonal materials:

\[ h_z(k) = \frac{\phi_0}{1 + \lambda^2 k^2 + \lambda^4 (n_{xxxyy} k^4 + dk k_2^2 q)^2}, \]  

(15)

where \( d = 2n_{xxxx} - 6n_{xxxx} \). The free energy density is a sum of pairwise interactions of vortices [14]: in the reciprocal space we have

\[ F = \frac{B^2}{8\pi \phi_0} \sum_{\mathbf{q}} s(\mathbf{q}) h_z(\mathbf{q}), \]  

(16)

where \( \mathbf{q} \) form the reciprocal VL. The factor \( s(\mathbf{q}) \) is introduced to deal with the shortcoming of the London model which disregards spatial variations of the order parameter in vortex cores. At low temperatures of our interest, we use \( s(q) = \exp(-q^2 \xi^2/2) \) [14]. By and large, this form of the cutoff is confirmed in neutron scattering experiments from which the form-factor \( |s(q) h_z(q)|^2 \) can be extracted [16]. We consider the typical case of a platelet sample with the \( c \) axis normal to the flat face in fields along \( c \). Then minimization of \( F \) at a given \( B \) yields the equilibrium VL, whereas its elastic properties are evaluated by considering deviations from the minimum.

For high-\( \kappa \) materials, away of the lower critical field, the interaction of vortices extends over several intervortex spacings. This gives rise to the nonlocal elastic response of VL’s [1], for which the elastic moduli, in general, depend on the deformation length scale (on the wave vector \( \mathbf{k} \)). However, even in isotropic case the shear modulus is practically \( \mathbf{k} \) independent. We focus here on the nondispersive modes, i.e., we consider the elastic response to deformations with \( \mathbf{k} = 0 \).

We begin with the equilibrium square VL with basis vectors \( \mathbf{a}_1 = a_0 (\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2} \), \( \mathbf{a}_2 = a_0 (\mathbf{e}_y - \mathbf{e}_x)/\sqrt{2} \). The reciprocal lattice is

\[ q_x = \pi \sqrt{2}(m + n)/a_0, \quad q_y = \pi \sqrt{2}(m - n)/a_0, \]  

(17)

with integers \( m \) and \( n \). For the 2D uniform deformations, the displacement is \( u_i = u_{ij}x_j \) with constant \( u_{ij} \)'s. Since the VL cell area is fixed at a given \( B \), \( u_{xx} + u_{yy} = 0 \) [1]. The deformed cell is given by \( a'_{i1} = a_{i1} + u_{ij}a_{i2} \) with \( \alpha = 1, 2 \). It is readily shown that the reciprocal VL is also homogeneously deformed \( \mathbf{q}' = \mathbf{q} + \mathbf{u}^* \) with

\[ u_i^* = -u_{ij}q_j \]  

(18)

(with the tensor of derivatives being transposed).

We now expand the intervortex interaction \( V(q') = s(q') h_z(q')/\phi_0 \) in powers of \( \mathbf{u}^* \):

\[ V(q') = V(q) + u_i^* \frac{\partial V}{\partial q_i} + \frac{1}{2} u_i^* u_j^* \frac{\partial^2 V}{\partial q_i \partial q_j} + \ldots. \]  

(19)

The term linear in \( u_i^* \) in the sum of \( 14 \) must vanish since \( u_i^* = 0 \) correspond to equilibrium. We then obtain for the elastic energy \( E = F(q') - F(q) \):

\[ E = \frac{B^2}{16\pi} \sum_{\mathbf{q}} \left( \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j \right) u_{ni} u_{nj}. \]  

(20)

For the squash deformation [1] this gives

\[ C_{sq} = \frac{B^2}{8\pi} \sum_{\mathbf{q}} \left( q_i^2 \frac{\partial^2 V}{\partial q_i^2} + q_y^2 \frac{\partial^2 V}{\partial q_y^2} - 2q_x q_y \frac{\partial^2 V}{\partial q_x \partial q_y} \right). \]  

(21)

Similarly, we obtain for the shear [3] and rotation [4]:

\[ C_{66,x} = \frac{B^2}{8\pi} \sum_{\mathbf{q}} \left( q_i^2 \frac{\partial^2 V}{\partial q_i^2} \right), \]  

(22)

\[ C_r = \frac{B^2}{8\pi} \sum_{\mathbf{q}} \left( q_i^2 \frac{\partial^2 V}{\partial q_i^2} + q_y^2 \frac{\partial^2 V}{\partial q_y^2} - 2q_x q_y \frac{\partial^2 V}{\partial q_x \partial q_y} \right). \]  

(23)

These three moduli form the complete set of 2D independent elastic constants for the square VL in cubic or tetragonal crystals in fields \( H > H_\square \) along [001]. If needed, \( C_{66} (\pi/4) \) can be evaluated using Eq. (11).

In a similar manner one can obtain the four moduli for the rhombic VL below \( H_\square \). The moduli can be calculated numerically by evaluating the sums in Eqs. (21) - (23). The results of such a calculation for LuNi2B2C (with parameters given in Ref. [1]) are shown in Fig. 1, where the field dependence of the moduli is shown. It is worth noting that \( C_r \) turns zero at some field \( H_r \) which is about twice as big as \( H_\square \). This implies instability of the square VL with diagonals along \( a \) and \( b \) which is stable for \( H_\square < H < H_r \) with respect to rotations. For fields \( H > H_r \) we have four equilibrium square VL’s of the same energy: the two with diagonals rotated relative to \{100\} counter- and clockwise over an angle \( \varphi \), and two for rotations relative to \{010\}. We have found by the direct numerical energy minimization that \( \varphi = 0 \) at \( H = H_r \) and increases as \( \sqrt{H - H_r} \) for \( (H - H_r) \ll H_r \). For LuNi2B2C we estimate that \( \varphi \) may reach \( \approx 7 \pm 8 \) degrees.
FIG. 1. The shear, rotation, and squash moduli versus field.

FIG. 2. The ratio $C_{66,x}/C_{66}(\pi/4)$ versus field.

As Fig. 1 shows, $C_{66,x}$ increases with field. For $H < H_\square$, $C_{66,x} \neq C_{66,y}$, whereas in the square phase between $H_\square$ and $H_r$ they coincide, as they should by symmetry (the moduli for $B \approx 200$ G are not plotted because the VL in domain differs from that for $200$ G < $B < H_\square$). However, response to the shear other than of $x$ or $y$ polarization, differs from either $C_{66,x}$ or $C_{66,y}$. The difference is maximum for the shear along [110] in the vicinity of the transition field $H_\square$, see Fig. 2.

We thus come to a striking conclusion: weak nonlocal corrections to the London description suffice to make elastic properties of VL’s strongly anisotropic even in the $ab$ plane where cubic or tetragonal materials are macroscopically isotropic.

Softening of the squash mode near $H_\square$ should affect vortex fluctuations. This, however, does not lead to either fluctuations divergence or to a peak in the critical current [7], the question recently discussed in Ref. [17].

The rotation modulus $C_r$ vanishing at $H = H_r$ affects the equilibrium VL structure above $H_r$. Although some indications for the square VL being unstable relative to rotations in sufficiently high fields were seen in decorations of LuNi$_2$B$_2$C [18], experimental confirmation of this instability is still to be established.

The square VL is soft with respect to shear displacement along the square sides (along [110] or [110]). This suggests that the vortex rows parallel to one of these directions can easily slide by each other. If this happens for, e.g., [110] direction, the VL will possess a long range order along [110], but will look disordered if observed along [110]. Such situations have been seen in decorations in fields near $H_\square$ [18]. The anisotropy of the shear moduli can, in principle, be probed experimentally if the critical current is arranged to flow only along [100] or [110] directions.

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