Bethe ansatz replica derivation of the GOE Tracy–Widom distribution in one-dimensional directed polymers with free endpoints

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Abstract. The distribution function of the free energy fluctuations in one-dimensional directed polymers with free boundary conditions is derived by mapping the replicated problem to the $N$-particle quantum boson system with attractive interactions. It is shown that in the thermodynamic limit this function is described by the universal Tracy–Widom distribution of the Gaussian orthogonal ensemble.

Keywords: rigorous results in statistical mechanics, thermodynamic Bethe ansatz, disordered systems (theory)
1. Introduction

Directed polymers in quenched random potentials have been the subject of intense investigation during the past two decades (see, e.g., [1]–[6]). In the one-dimensional case we deal with an elastic string directed along the $\tau$-axis within an interval $[0, t]$. Randomness enters the problem through a disorder potential $V[\phi(\tau), \tau]$, which competes against the elastic energy. The problem is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^t d\tau \left\{ \frac{1}{2} [\partial_\tau \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\},$$

where the disorder potential $V[\phi, \tau]$ is Gaussian distributed with a zero mean, $\overline{V(\phi, \tau)} = 0$, and the $\delta$-correlations

$$\overline{V(\phi, \tau)V(\phi', \tau')} = u \delta(\tau - \tau') \delta(\phi - \phi').$$

Here, the parameter $u$ describes the strength of the disorder. Note that such a system is equivalent to the problem of the Kardar–Parisi–Zhang (KPZ) equation describing the growth in time of an interface in the presence of noise [7].

In what follows we consider the problem in which the polymer is fixed at the origin, $\phi(0) = 0$, and is free at $\tau = t$. In other words, for a given realization of the random potential $V$ the partition function of the considered system is

$$Z = \int_{-\infty}^{+\infty} dx Z(x) = \exp\{-\beta F\},$$

where

$$Z(x) = \int_{\phi(0) = 0}^{\phi(t) = x} D\phi(\tau)e^{-\beta H[\phi]}$$

is the partition function of the system with the fixed boundary conditions $\phi(0) = 0$ and $\phi(t) = x$ and where $F$ is the total free energy. Besides the usual extensive part $f_0 t$ (where $f_0$ is the linear free energy density), the total free energy $F$ of such a system is known to contain the disorder dependent fluctuating contribution $\tilde{F}$. In the limit of large $t$ the

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typical value of the free energy fluctuations scales with $t$ as $\tilde{F} \propto t^{1/3}$ (see, e.g., [3]–[6]). In other words, the total free energy of the system can be represented as

$$ F = f_0 t + c t^{1/3} f, $$

where $c$ is a non-universal parameter which depends on the temperature and the strength of the disorder and $f$ is a random quantity which in the thermodynamic limit $t \to \infty$ is described by a non-trivial universal distribution function $P(f)$. Note that, according to equations (3)–(5), the trivial self-averaging contribution $f_0 t$ to the free energy can be eliminated by a simple redefinition of the partition function as

$$ Z = \exp\{ -\beta f_0 t \} \tilde{Z}, $$

so that

$$ \tilde{Z} = \exp\{ -\lambda f \}, $$

where

$$ \lambda = \beta c t^{1/3}. $$

For the similar problem with zero boundary conditions, $\phi(0) = \phi(t) = 0$, the corresponding distribution function was shown to be described by the Gaussian unitary ensemble (GUE) Tracy–Widom distribution [8]–[11]. In the course of this proof a rather efficient Bethe ansatz replica technique was developed [10, 11]. In particular, in terms of this technique the corresponding multi-point free energy distribution functions were derived [12]. Recently, the free energy distribution function for the directed polymer problem with a free endpoint, equations (1)–(4), has been obtained [13]. It was shown that the function $P(f)$ is the Gaussian orthogonal ensemble (GOE) Tracy–Widom distribution. In this paper I would like to present a sufficiently simple alternative way of deriving the same result which does not require the rather complicated technique of the Fredholm Pfaffian described in [13].

Let us introduce the function

$$ W(f) \equiv \int_f^\infty df' P(f'), $$

which gives the probability that the random free energy is larger that a given value $f$. It will be shown that in the thermodynamic limit, $t \to \infty$, this function is equal to the Fredholm determinant

$$ W(f) = \det(1 - \hat{K}_{-f}) \equiv F_1(-f) $$

with the kernel

$$ K_{-f}(\omega, \omega') = \text{Ai}(\omega + \omega' - f) \quad (\omega, \omega' > 0), $$

which is the GOE Tracy–Widom distribution [14, 15]. Explicitly,

$$ F_1(s) = \exp\left[ -\frac{1}{2} \int_s^{+\infty} d\xi (\xi - s)q^2(\xi) - \frac{1}{2} \int_s^{+\infty} d\xi q(\xi) \right], $$

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where \( q(\xi) \) is the solution of the Painlevé II differential equation, \( q''(\xi) = \xi q(\xi) + 2q^3(\xi) \), with the boundary condition \( q(\xi \to +\infty) = \text{Ai}(\xi) \).

It should be noted that this paper is rather technical. The main message of this work is not the final result itself (which is well known anyway) but the presentation of the general method and new technical tricks used in the derivation. Section 2 is devoted to the standard reformulation of the considered problem in terms of a one-dimensional \( N \)-particle system of quantum bosons with attractive \( \delta \)-interactions [6]. Here, it is shown that the calculation of the free energy probability distribution function, equation (9), reduces to a summation over the whole spectrum of eigenstates of this \( N \)-particle problem. This summation is performed in section 3, where in the thermodynamic limit, \( t \to \infty \), the result, equations (10) and (11), is derived. The concluding remarks as well as the key points of the calculations are listed in section 4.

2. Mapping to quantum bosons

In terms of the partition function \( \tilde{Z} \), equation (7), the function \( W(f) \), equation (9), can be defined as follows:

\[
W(f) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda N f) \overline{Z^N},
\]  

(13)

where \( \overline{\cdots} \) denotes averaging over the quenched disorder. Indeed, substituting here equation (7), we have

\[
W(f) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \int_{-\infty}^{+\infty} df' P(f') \exp\{\lambda N (f - f')\}
\]

\[
= \lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} df' P(f') \exp[-\exp\{\lambda (f - f')\}]
\]

\[
= \int_{-\infty}^{+\infty} df' P(f') \theta(f - f'),
\]

(14)

which coincides with the definition (9).

Later on we will see that the integration over \( x \) in the definition of the partition function, equation (3), requires proper regularization at both limits \( \pm \infty \). For that reason it is convenient to represent it in the form of two contributions

\[
Z = \int_{-\infty}^{0} dx \, Z(x) + \int_{0}^{+\infty} dx \, Z(x) \equiv Z(-) + Z(+).
\]

(15)

Thus, taking into account the definition (6), we get

\[
W(f) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp\{\lambda N f + \beta N f_0 t\} (Z(-) + Z(+))^N
\]

\[
= \lim_{\lambda \to \infty} \sum_{K,L=0}^{\infty} \frac{(-1)^{K+L}}{K! L!} \exp\{\lambda (K + L) f + \beta (K + L) f_0 t\} \overline{Z^K Z^L}
\]

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\[
= \lim_{\lambda \to \infty} \sum_{K,L=0}^{\infty} \frac{(-1)^{K+L}}{K! L!} \exp\{\lambda (K + L) f + \beta (K + L) f_0 t\}
\times \int_{-\infty}^{0} dx_1 \cdots dx_K \int_{0}^{+\infty} dy_1 \cdots dy_L \Psi(x_1, \ldots, x_K, y_L, \ldots, y_1; t),
\]  
(16)

where

\[
\Psi(x_1, \ldots, x_N; t) \equiv \frac{Z(x_1)}{Z(x_2)} \cdots \frac{Z(x_N)}{Z}.
\]  
(17)

Using the relations (1), (2) and (4), after simple Gaussian averaging we obtain

\[
\Psi(x_1, \ldots, x_N; t) = N \prod_{a=1}^{N} \left[ \int_{\phi_a(0)=0}^{\phi_a(t)=x_a} D\phi_a(\tau) \right] \exp\left(-\beta H_N[\phi_1, \phi_2, \ldots, \phi_N]\right),
\]  
(18)

where

\[
H_N[\phi_1, \phi_2, \ldots, \phi_N] = \frac{1}{2} \int_0^t d\tau \left( \sum_{a=1}^{N} [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a \neq b}^{N} \delta[\phi_a(\tau) - \phi_b(\tau)] \right).
\]  
(19)

The propagator \(\Psi(x; t)\), equation (18), describes \(N\) trajectories \(\phi_a(\tau)\), all starting at zero \((\phi_a(0) = 0)\) and coming to \(N\) different points \(\{x_1, \ldots, x_N\}\) at \(\tau = t\). One can easily show that \(\Psi(x; t)\) can be obtained as the solution of the linear differential equation

\[
\beta \partial_t \Psi(x; t) = \frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 \Psi(x; t) + \frac{1}{2} \kappa \sum_{a \neq b}^{N} \delta(x_a - x_b) \Psi(x; t)
\]  
(20)

with the initial condition

\[
\Psi(x; 0) = \Pi_{a=1}^{N} \delta(x_a)
\]  
(21)

and the interaction parameter \(\kappa = \beta^3 u\). One can easily see that equation (20) is the imaginary-time Schrödinger equation

\[
- \beta \partial_t \Psi(x; t) = \hat{H} \Psi(x; t)
\]  
(22)

with the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 - \frac{1}{2} \kappa \sum_{a \neq b}^{N} \delta(x_a - x_b)
\]  
(23)

which describes \(N\) Bose particles interacting via the attractive two-body potential \(-\kappa \delta(x)\). A generic eigenstate of such a system is characterized by \(N\) momenta \(\{q_a\}(a = 1, \ldots, N)\) which are split into \(M\) \((1 \leq M \leq N)\) ‘clusters’ described by continuous real momenta \(q_a(\alpha = 1, \ldots, M)\) and having \(n_a\) discrete imaginary ‘components’ (for details see [10], [16]–[20]),

\[
q_a = q_\alpha^r = q_\alpha - \frac{i \kappa}{2} (n_\alpha + 1 - 2r) \quad (r = 1, \ldots, n_\alpha)
\]  
(24)
with the constraint
\[ \sum_{\alpha=1}^{M} n_{\alpha} = N. \]  
(25)

A generic solution \( \Psi(x, t) \) of the Schrödinger equation (20) with the initial conditions (21) can be represented in the form of a linear combination of the eigenfunctions \( \Psi_{q}^{(M)}(x) \) as
\[
\Psi(x_{1}, \ldots, x_{N}; t) = \sum_{M=1}^{N} \frac{1}{M!} \left[ \int D^{(M)}(q, n) \right] \\
\times |C_{M}(q, n)|^{2} \Psi_{q}^{(M)}(x) \Psi_{q}^{(M)*}(0) \exp\{ -E_{M}(q)t \},
\]  
(26)

where we have introduced the notation
\[
\int D^{(M)}(q, n) \equiv \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dq_{\alpha}}{2\pi} \sum_{n_{\alpha}=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) \right]
\]  
(27)

and \( \delta(k, m) \) is the Kronecker symbol; note that the presence of this Kronecker symbol in the above equation allows the summations over \( n_{\alpha}s \) to be extended to infinity. Here, the (non-normalized) eigenfunctions are \[10, 20\]
\[
\Psi_{q}^{(M)}(x) = \sum_{P} \prod_{a<b} \left[ 1 + i\kappa \text{sgn}(x_{a} - x_{b}) \right] \exp \left[ i \sum_{a=1}^{N} q_{Pa} x_{a} \right],
\]  
(28)

where the summation goes over \( N! \) permutations \( P \) of \( N \) momenta \( q_{a} \), equation (24), over \( N \) particles \( x_{a} \), the normalization factor
\[
|C_{M}(q, n)|^{2} = \frac{\kappa^{N}}{N! \prod_{\alpha} (\kappa n_{\alpha})} \prod_{\alpha<\beta} \frac{|q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} - n_{\beta})|^{2}}{|q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} + n_{\beta})|^{2}}
\]  
(29)

and the eigenvalues
\[
E_{M}(q) = \frac{1}{2\beta} \sum_{a=1}^{N} q_{a}^{2} = \frac{1}{2\beta} \sum_{a=1}^{M} n_{a} q_{a}^{2} - \frac{\kappa^{2}}{24\beta} \sum_{a=1}^{M} (n_{a}^{3} - n_{a}).
\]  
(30)

Note that the eigenfunctions, equation (28), are symmetric with respect to permutations of all their arguments \( x_{1}, \ldots, x_{N} \) and
\[
\Psi_{q}^{(M)}(0) = N!.
\]  
(31)

In this way the problem of the calculation of the free energy probability distribution function, equation (16), reduces to a summation over all the spectrum of eigenstates of the \( N \)-particle bosonic problem, which is parametrized by the set of both the continuous, \( \{q_{1}, \ldots, q_{M}\} \), and the discrete, \( \{n_{1}, \ldots, n_{M}\} \); \( (M = 1, \ldots, N); (N = 1, \ldots, \infty) \), degrees of freedom.
3. The free energy probability distribution function

Substituting equations (26)–(31) into equation (16) (defining \( f_0 = (1/24)\beta^4u^2 \), the factor \( f_0 \) drops out of the further calculations), we get

\[
W(f) = 1 + \lim_{\lambda \to \infty} \sum_{K+L \geq 1}^{\infty} (-1)^{K+L} e^{\lambda(K+L)f} \\
	imes \sum_{M=1}^{K+L} \frac{1}{M!} \prod_{a=1}^{M} \left[ \sum_{n_a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_{\alpha}}{2\pi\kappa n_{\alpha}} q_{n_{\alpha}}^{2n_{\alpha}^2 + (\epsilon/2\beta)n_{\alpha}q_{n_{\alpha}}^2} \right] \\
\times \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) |\tilde{C}_M(q,n)|^2 I_{K,L}(q,n), \quad (32)
\]

where

\[
|\tilde{C}_M(q,n)|^2 = \prod_{\alpha<\beta}^{M} \left| q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} - n_{\beta}) \right|^2 \\
(33)
\]

and

\[
I_{K,L}(q,n) = \sum_{P(K,L)} \sum_{P(K)} \sum_{P(L)} \prod_{a=1}^{K} \prod_{b=1}^{L} \left[ \frac{q_{p_a}^{(K)} - q_{p_b}^{(K)} - i\kappa}{q_{p_a}^{(K)} - q_{p_b}^{(K)}} \right] \\
\times \prod_{a<b}^{K} \left[ \frac{q_{p_a}^{(K)} - q_{p_b}^{(K)} - i\kappa}{q_{p_a}^{(K)} - q_{p_b}^{(K)}} \right] \times \prod_{c<d}^{L} \left[ \frac{q_{p_c}^{(L)} - q_{p_d}^{(L)} + i\kappa}{q_{p_c}^{(L)} - q_{p_d}^{(L)}} \right] \\
\times \int_{-\infty<x_1\leq \cdots \leq x_K\leq 0} dx_1 \cdots dx_K \exp \left[ i \sum_{a=1}^{K} (q_{p_a}^{(K)} - i\epsilon)x_a \right] \\
\times \int_{0\leq y_L\leq \cdots \leq y_1\leq +\infty} dy_L \cdots dy_1 \exp \left[ i \sum_{c=1}^{L} (q_{p_c}^{(L)} + i\epsilon)y_c \right]. \quad (34)
\]

Here, the summation over all permutations \( P \) of \((K + L)\) momenta \{q_1, \ldots, q_{K+L}\} over \( K \) ‘negative’ particles \{x_1, \ldots, x_K\} and \( L \) ‘positive’ particles \{y_L, \ldots, y_1\} is divided into three parts: the permutations \( P(K) \) of \( K \) momenta (taken at random out of the total list \{q_1, \ldots, q_{K+L}\}) over \( K \) ‘negative’ particles, the permutations \( P(L) \) of the remaining \( L \) momenta over \( L \) ‘positive’ particles, and finally the permutations \( P(K,L) \) (or the exchange) of the momenta between the group ‘\( K \)’ and the group ‘\( L \)’. Note also that the integrations over both \( x_a \)s and \( y_s \)s in equation (34) require proper regularization at \(-\infty\) and \(+\infty\) correspondingly. This is carried out in the standard way by introducing a supplementary parameter \( \epsilon \) which will be set to zero in the final results. The result of the integrations
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can be represented as follows:

\[ I_{K,L}(q,n) = i^{(K+L)} \sum_{p,(K,L)} \prod_{a=1}^{K} \prod_{c=1}^{L} \left[ \frac{q_{p_a}^{(K)} - q_{p_c}^{(L)} - \imath \kappa}{q_{p_a}^{(K)} - q_{p_c}^{(L)}} \right] \]

\[ \times \sum_{p,(K)} \frac{1}{q_{p_1}^{(-)}(q_{p_1}^{(-)} + q_{p_2}^{(-)}) \cdots (q_{p_1}^{(-)} + \cdots + q_{p_K}^{(-)})} \prod_{a,b}^{K} \left[ \frac{q_{p_a}^{(-)} - q_{p_b}^{(-)} - \imath \kappa}{q_{p_a}^{(-)} - q_{p_b}^{(-)}} \right] \]

\[ \times \sum_{p,(L)} \frac{(-1)^L}{q_{p_1}^{(+)}(q_{p_1}^{(+)} + q_{p_2}^{(+)}) \cdots (q_{p_1}^{(+)} + \cdots + q_{p_L}^{(+)})} \prod_{c,d}^{L} \left[ \frac{q_{p_c}^{(+) - q_{p_d}^{(+)}} + \imath \kappa}{q_{p_c}^{(+) - q_{p_d}^{(+)}}} \right], \]

(35)

where

\[ q_{a}^{(\pm)} \equiv q_{a} \pm \imath \epsilon. \]

(36)

Using the ‘magic’ Bethe ansatz combinatorial identity [13],

\[ \sum_{p} q_{p_1}(q_{p_1} + q_{p_2}) \cdots (q_{p_1} + \cdots + q_{p_N}) \prod_{a,b}^{N} \left[ \frac{q_{p_a} - q_{p_b} - \imath \kappa}{q_{p_a} - q_{p_b}} \right] = \frac{1}{\prod_{a=1}^{N} q_{a}} \prod_{a < b}^{N} \left[ \frac{q_{a} + q_{b} + \imath \kappa}{q_{a} + q_{b}} \right] \]

(37)

(where the summation goes over all permutations \( P \) of \( N \) momenta \( \{q_1, \ldots, q_N\} \)), we get

\[ I_{K,L}(q,n) = i^{(K+L)} \sum_{p,(K,L)} \prod_{a=1}^{K} \prod_{c=1}^{L} \left[ \frac{q_{p_a}^{(K)} - q_{p_c}^{(L)} - \imath \kappa}{q_{p_a}^{(K)} - q_{p_c}^{(L)}} \right] \]

\[ \times \frac{1}{\prod_{a=1}^{K} q_{p_a}^{(-)}(q_{p_a}^{(-)} + q_{p_b}^{(-)}) \cdots (q_{p_a}^{(-)} + \cdots + q_{p_K}^{(-)})} \prod_{a,b}^{K} \left[ \frac{q_{p_a}^{(-)} + q_{p_b}^{(-)} + \imath \kappa}{q_{p_a}^{(-)} + q_{p_b}^{(-)}} \right] \]

\[ \times \frac{(-1)^L}{\prod_{c=1}^{L} q_{p_c}^{(+)}(q_{p_c}^{(+)} + q_{p_d}^{(+)}) \cdots (q_{p_c}^{(+)} + \cdots + q_{p_L}^{(+)})} \prod_{c,d}^{L} \left[ \frac{q_{p_c}^{(+)} + q_{p_d}^{(+)} - \imath \kappa}{q_{p_c}^{(+)} + q_{p_d}^{(+)}} \right]. \]

(38)

Further simplification comes from one important property of the Bethe ansatz wavefunction, equation (28). It has such a structure that for ordered particle positions (e.g. \( x_1 < x_2 < \cdots < x_N \)) in the summation over permutations the momenta \( q_{a} \) belonging to the same cluster also remain ordered. In other words, if we consider the momenta, equation (24), of a cluster \( \alpha \), \( \{q_{a}^{\alpha}, q_{b}^{\alpha}, \ldots, q_{n}^{\alpha}\} \), belonging correspondingly to the particles \( \{x_{1} < x_{2} < \cdots < x_{n}\} \), the permutation of any two momenta \( q_{a}^{\alpha} \) and \( q_{b}^{\alpha} \) of this ordered set gives zero contribution. Thus, in order to perform the summation over the permutations \( P^{(K,L)} \) in equation (38) it is sufficient to split the momenta of each cluster into two parts, \( \{q_{1}^{0}, \ldots, q_{m}^{0} \parallel q_{m+1}^{0}, \ldots, q_{n}^{0}\} \), where \( m = 0, 1, \ldots, n \) and where the momenta \( q_{1}^{0}, \ldots, q_{m}^{0} \) belong to the particles of the sector ‘\( K \)’, while the momenta \( q_{m+1}^{0}, \ldots, q_{n}^{0} \) belong to the particles of the sector ‘\( L \)’. 

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Let us introduce the numbering of the momenta of the sector ‘L’ in the reversed order,

\[ q_{m_a}^\alpha \rightarrow q_{1}^{s_\alpha}, \]
\[ q_{m_a-1}^\alpha \rightarrow q_{2}^{s_\alpha}, \]
\[ \ldots \]
\[ q_{m_a+1}^\alpha \rightarrow q_{s_\alpha}^{s_\alpha}, \]

where \( m_a + s_\alpha = n_\alpha \) and (s.f. equation (24))

\[ q_{r}^{s_\alpha} = q_{\alpha} + \frac{\alpha}{2}(n_\alpha + 1 - 2r) = q_{\alpha} + \frac{\alpha}{2}(m_\alpha + s_\alpha + 1 - 2r). \]

By definition, the integer parameters \( \{m_\alpha\} \) and \( \{s_\alpha\} \) fulfil the global constraints

\[ \sum_{\alpha=1}^{M} m_\alpha = K, \]
\[ \sum_{\alpha=1}^{M} s_\alpha = L. \]

In this way the summation over permutations \( \mathcal{P}(K,L) \) in equation (38) is changed by the summations over the integer parameters \( \{m_\alpha\} \) and \( \{s_\alpha\} \),

\[ \sum_{\mathcal{P}(K,L)} (\ldots) \rightarrow \prod_{\alpha=1}^{M} \left[ \sum_{m_\alpha+s_\alpha \geq 1} \delta(m_\alpha + s_\alpha, n_\alpha) \right] \delta\left(\sum_{\alpha=1}^{M} m_\alpha, K\right) \delta\left(\sum_{\alpha=1}^{M} s_\alpha, L\right) (\ldots), \]

which allows us to lift the summations over \( K, L \) and \( \{n_\alpha\} \) in equation (32). In terms of the parameters \( \{m_\alpha\} \) and \( \{s_\alpha\} \) the product factors in equation (38) are expressed as follows:

\[ \prod_{a=1}^{K} q_{p_a(K)}^{'(\alpha)} = \prod_{a=1}^{M} \prod_{r=1}^{M} q_{r}^{m_\alpha(-)}, \]
\[ \prod_{a=1}^{L} q_{p_a(L)}^{'(\alpha)} = \prod_{a=1}^{M} \prod_{r=1}^{M} q_{r}^{s_\alpha(+)}, \]
\[ \prod_{a<b} \left[ \frac{q_{p_a(K)}^{'(\alpha)}}{q_{p_b(K)}^{'(\alpha)}} + \frac{i\kappa}{2} \right] = \prod_{a=1}^{M} \prod_{1 \leq r < r'} q_{r}^{m_\alpha(-)} + q_{r'}^{m_\alpha(-)} + i\kappa, \]
\[ \times \prod_{1 \leq a < b} \prod_{1 \leq r < r'} \left[ q_{r}^{m_\alpha(-)} + q_{r'}^{m_\alpha(-)} + i\kappa \right] \]
\[ \prod_{a<b} \left[ \frac{q_{p_a(L)}^{'(\alpha)}}{q_{p_b(L)}^{'(\alpha)}} + \frac{i\kappa}{2} \right] = \prod_{a=1}^{M} \prod_{1 \leq r < r'} q_{r}^{s_\alpha(+) + \alpha r(+) - \alpha r'} + \kappa, \]
\[ \times \prod_{1 \leq a < b} \prod_{1 \leq r < r'} \left[ q_{r}^{s_\alpha(+) + \alpha r(+) - \alpha r'} + \kappa \right] \]
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\[ K \prod_{a=1}^{L} \prod_{c=1}^{K} \left[ \frac{q_{pc}(\kappa) - q_{pc}(\kappa) - i\kappa}{q_{pc}(\kappa) - q_{pc}(\kappa)} \right] = \prod_{1 \leq \alpha < \beta} \left\{ \prod_{r=1}^{s} \prod_{r'=1}^{s} \left[ \frac{q_{r}^{\alpha} - q_{r'}^{\beta} - i\kappa}{q_{r}^{\alpha} - q_{r'}^{\beta}} \right] \right\} \]

\[ \times \prod_{\alpha=1}^{M} \prod_{\alpha'=1}^{M} \left[ \frac{q_{\alpha}^{\alpha} - q_{\alpha'}^{\alpha} - i\kappa}{q_{\alpha}^{\alpha} - q_{\alpha'}^{\alpha}} \right] \]

\[ \times \prod_{r=1}^{M} \prod_{s} q_{s}^{m_{s}} \left[ \frac{q_{\alpha}^{\alpha} - q_{s}^{\alpha} - i\kappa}{q_{\alpha}^{\alpha} - q_{s}^{\alpha}} \right]. \] (48)

Substituting equations (43)–(48) into equation (38), and then substituting the resulting expression into equation (32), we obtain

\[ W(f) = \lim_{\lambda \to \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^{M} M!}{M!} \prod_{\alpha=1}^{M} \sum_{m_{\alpha} + s_{\alpha} \geq 1} (-1)^{m_{\alpha} + s_{\alpha} - 1} \int_{-\infty}^{+\infty} \frac{\mathcal{G}(q_{\alpha}, m_{\alpha}, s_{\alpha})}{2\pi\kappa(m_{\alpha} + s_{\alpha})} \times \exp \left( -\frac{1}{2\beta} \left( m_{\alpha} + s_{\alpha} \right) q_{\alpha}^{2} + \frac{\kappa^{2}}{24\beta} \left( m_{\alpha} + s_{\alpha} \right)^{3} + \lambda(m_{\alpha} + s_{\alpha})f \right) \right\} \]

\[ \times \prod_{1 \leq \alpha < \beta} \mathcal{G}_{\alpha\beta}(q_{\alpha}, m_{\alpha}, s_{\alpha}) \}, \] (49)

where

\[ |\tilde{C}_{M}(q, m + s)|^{2} = \prod_{\alpha < \beta} \left[ q_{\alpha} - q_{\beta} - (i\kappa/2)(m_{\alpha} + s_{\alpha} - m_{\beta} - s_{\beta}) \right]^{2}, \] (50)

\[ \mathcal{G} = \frac{(-1)^{s_{\alpha}}(-i\kappa)^{m_{\alpha} + s_{\alpha}}}{\prod_{r=1}^{m_{\alpha}} q_{r}^{\alpha}(-) \prod_{r'=1}^{s_{\alpha}} q_{r'}^{\alpha}(-)} \prod_{r < r'}^{m_{\alpha}} q_{r}^{\alpha}(-) + q_{r'}^{\alpha}(-) + i\kappa \prod_{r < r'}^{s_{\alpha}} \left[ q_{r}^{\alpha}(-) + q_{r'}^{\alpha}(-) \right] \prod_{r < r'}^{s_{\alpha}} \left[ q_{r}^{\alpha}(-) + q_{r'}^{\alpha}(-) \right] \]

\[ \times \prod_{r=1}^{m_{\alpha}} \prod_{s=1}^{s_{\alpha}} \left[ q_{r}^{\alpha}(-) - q_{r}^{\alpha}(-) - i\kappa \right] \] (51)

and

\[ \mathcal{G}_{\alpha\beta} = \prod_{r=1}^{m_{\alpha}} \prod_{r'=1}^{m_{\beta}} \left[ q_{r}^{\beta}(-) + q_{r'}^{\beta}(-) + i\kappa \right] \prod_{r=1}^{s_{\alpha}} \prod_{s=1}^{s_{\beta}} \left[ q_{r}^{\alpha}(-) + q_{r}^{\alpha}(-) - i\kappa \right] \]

\[ \times \prod_{r=1}^{m_{\alpha}} \prod_{s=1}^{s_{\beta}} \left[ q_{r}^{\alpha}(-) - q_{r}^{\alpha}(-) - i\kappa \right] \times \prod_{r=1}^{s_{\alpha}} \prod_{s=1}^{m_{\beta}} \left[ q_{r}^{\alpha}(-) - q_{r}^{\alpha}(-) - i\kappa \right]. \] (52)

The product factors in equation (51) can be easily expressed in terms of the Gamma functions

\[ \prod_{r=1}^{m_{\alpha}} q_{r}^{\alpha}(-) = \prod_{r=1}^{m_{\alpha}} \left[ q_{r}^{\alpha}(-) - \frac{i\kappa}{2} (m_{\alpha} + s_{\alpha} + 1) + i\kappa r \right] \]

\[ = (i\kappa)^{m_{\alpha}} \Gamma \left( 1/2 - (s_{\alpha} - m_{\alpha})/2 - i\kappa q_{\alpha}(-)/\kappa \right) / \Gamma \left( 1/2 - (s_{\alpha} + m_{\alpha})/2 - i\kappa q_{\alpha}(-)/\kappa \right). \] (53)
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\[ \prod_{r=1}^{s} q_r^{\ast(+) - m} = \prod_{r=1}^{s} \left[ q_r^{(+) - m} + \frac{i\kappa}{2} (m + s + 1) - ikr \right] \]

\[ = (-i\kappa)^{s} \frac{\Gamma(1/2 - (m - s)/2 + iq_{a}^{(+)}/\kappa)}{\Gamma(1/2 - (m + s)/2 + iq_{a}^{(+)}/\kappa)} \]  

\[ \prod_{r<r'}^{m} \left[ \frac{q_r^{(-)} + q_{r'}^{(-)} + i\kappa}{q_r^{(-)} + q_{r'}^{(-)} - i\kappa} \right] = 2^{-(m - s - 1)} \frac{\Gamma((s - m)/2 - i\kappa^{(-)}/\kappa)}{\Gamma((s - m)/2 + i\kappa^{(+)/\kappa})} \times \frac{\Gamma(1 - (m + s)/2 - i\kappa^{(-)/\kappa})}{\Gamma(1 - (m + s)/2 + i\kappa^{(+)/\kappa})} \]

\[ \prod_{r=1}^{m} \prod_{r'=1}^{s} \left[ \frac{q_r^{(-)} - q_{r'}^{(-) - m}}{q_r^{(-)} - q_{r'}^{(-) - m}} \right] = \frac{\Gamma(1 + m + s) \Gamma(1 + s)}{\Gamma(1 + m) \Gamma(1 + s)} \]  

Substituting the above expressions into equation (51) and using the standard relations for the Gamma functions,

\[ \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \]  
\[ \Gamma(1 + z) = z \Gamma(z) \]  
\[ \Gamma \left( \frac{1}{2} + z \right) = \frac{\sqrt{\pi} \Gamma(1 + 2z)}{2^{2z} \Gamma(1 + z)} \]

for the factor \( \mathcal{G} \), equation (51), we get

\[ \mathcal{G}(q_{a}, m_{a}, s_{a}) = \frac{\Gamma(s_{a} + (2i/\kappa)q_{a}^{(-)}) \Gamma(m_{a} - (2i/\kappa)q_{a}^{(+)})(1 + m_{a} + s_{a})}{2^{(m_{a} + s_{a})} \Gamma(m_{a} + s_{a} + (2i/\kappa)q_{a}^{(-)})(1 + m_{a} + s_{a})} \]

Similar calculations for the factor \( \mathcal{G}_{\alpha\beta}(q, m, s) \) yield the following expression:

\[ \mathcal{G}_{\alpha\beta}(q, m, s) \]

\[ = \frac{\Gamma[1 + \frac{m_{a} + m_{\beta} - s_{a} - s_{\beta}}{2} - \frac{1}{\kappa}(q_{a}^{(-)} + q_{\beta}^{(-)})][1 - \frac{m_{a} + m_{\beta} + s_{a} + s_{\beta}}{2} - \frac{1}{\kappa}(q_{a}^{(-)} + q_{\beta}^{(-)})]}{\Gamma[1 - \frac{m_{a} - m_{\beta} - s_{a} + s_{\beta}}{2} - \frac{1}{\kappa}(q_{a}^{(-)} + q_{\beta}^{(-)})][1 + \frac{m_{a} - m_{\beta} + s_{a} - s_{\beta}}{2} - \frac{1}{\kappa}(q_{a}^{(-)} + q_{\beta}^{(-)})]} \times \]

\[ \frac{\Gamma[1 + \frac{m_{a} + m_{\beta} - s_{a} - s_{\beta}}{2} + \frac{1}{\kappa}(q_{a}^{(+)} + q_{\beta}^{(+)})][1 - \frac{m_{a} + m_{\beta} + s_{a} + s_{\beta}}{2} + \frac{1}{\kappa}(q_{a}^{(+)} + q_{\beta}^{(+)})]}{\Gamma[1 - \frac{m_{a} - m_{\beta} - s_{a} + s_{\beta}}{2} + \frac{1}{\kappa}(q_{a}^{(+)} + q_{\beta}^{(+)})][1 + \frac{m_{a} - m_{\beta} + s_{a} - s_{\beta}}{2} + \frac{1}{\kappa}(q_{a}^{(+)} + q_{\beta}^{(+)})]} \times \]

\[ \frac{\Gamma[1 + \frac{m_{a} + m_{\beta} - s_{a} + s_{\beta}}{2} + \frac{1}{\kappa}(q_{a} - q_{\beta})][1 - \frac{m_{a} + m_{\beta} + s_{a} - s_{\beta}}{2} + \frac{1}{\kappa}(q_{a} - q_{\beta})]}{\Gamma[1 - \frac{m_{a} - m_{\beta} - s_{a} + s_{\beta}}{2} + \frac{1}{\kappa}(q_{a} - q_{\beta})][1 + \frac{m_{a} - m_{\beta} + s_{a} - s_{\beta}}{2} + \frac{1}{\kappa}(q_{a} - q_{\beta})]} \]
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\[
\times \frac{\Gamma[1 + \frac{m_\alpha + m_\beta + s_\alpha + s_\beta}{2} - \frac{i}{\kappa}(q_\alpha - q_\beta)] \Gamma[1 + \frac{m_\alpha - m_\beta - s_\alpha + s_\beta}{2} - \frac{i}{\kappa}(q_\alpha - q_\beta)]}{\Gamma[1 + \frac{m_\alpha + m_\beta - s_\alpha + s_\beta}{2} - \frac{i}{\kappa}(q_\alpha - q_\beta)] \Gamma[1 + \frac{m_\alpha - m_\beta + s_\alpha + s_\beta}{2} - \frac{i}{\kappa}(q_\alpha - q_\beta)]}.
\]

(62)

Redefining

\[
q_\alpha = \frac{\kappa}{2\lambda} p_\alpha,
\]

with

\[
\lambda = \frac{1}{2} \left( \frac{\kappa^2 t}{\beta} \right)^{1/3} = \frac{1}{2} (\beta u^2 t)^{1/3},
\]

the normalization factor \(|\hat{C}_M(\mathbf{q}, \mathbf{m} + \mathbf{s})|^2\), equation (50), can be represented as follows:

\[
|\hat{C}_M(\mathbf{q}, \mathbf{m} + \mathbf{s})|^2 = \prod_{a \leq \beta} \frac{\lambda(m_\alpha + s_\alpha) - \lambda(m_\beta + s_\beta) - ip_\alpha + ip_\beta|^2}{\lambda(m_\alpha + s_\alpha) + \lambda(m_\beta + s_\beta) - ip_\alpha + ip_\beta|^2}
\]

\[
= \prod_{a=1}^{M} \left[ 2\lambda(m_\alpha + s_\alpha) \right] \times \det \left[ \frac{1}{\lambda(m_\alpha + s_\alpha) - ip_\alpha + \lambda(m_\beta + s_\beta) + ip_\beta} \right]_{a,\beta = 1, \ldots, M},
\]

(65)

where we have used the Cauchy double alternate identity

\[
\prod_{a < \beta}(a_\alpha - a_\beta)(b_\alpha - b_\beta) \prod_{a, \beta = 1}^{M}(a_\alpha - b_\beta) = (-1)^{M(M-1)/2} \det \left[ \frac{1}{a_\alpha - b_\beta} \right]_{a,\beta = 1, \ldots, M},
\]

(66)

with \(a_\alpha = p_\alpha - i\lambda(m_\alpha + s_\alpha)\) and \(b_\alpha = p_\alpha + i\lambda(m_\beta + s_\beta)\).

After rescaling, equation (63), for the exponential factor in equation (49) we find

\[
-\frac{t}{2\beta}(m_\alpha + s_\alpha)q_\alpha^2 + \frac{\kappa^2}{24\beta}(m_\alpha + s_\alpha)^3 + \lambda(m_\alpha + s_\alpha) f
\]

\[
= - \lambda(m_\alpha + s_\alpha)p_\alpha^2 + \frac{1}{3} \lambda^3(m_\alpha + s_\alpha)^3 + \lambda(m_\alpha + s_\alpha) f.
\]

(67)

The cubic exponential term can be linearized using the Airy function relation

\[
\exp \left[ \frac{1}{3} \lambda^3(m_\alpha + s_\alpha)^3 \right] = \int_{-\infty}^{+\infty} dy_\alpha \Ai(y_\alpha) \exp \left[ \lambda(m_\alpha + s_\alpha) y_\alpha \right].
\]

(68)

Substituting equations (68), (67) and (65) into equation (49), and redefining \(y_\alpha \rightarrow y_\alpha + p_\alpha^2 - f\), we get

\[
W(f) = \lim_{\lambda \rightarrow \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \left[ \int_{-\infty}^{+\infty} dy_\alpha \Ai(y_\alpha + p_\alpha^2 - f) \right] \right. \times \sum_{m_\alpha + s_\alpha \geq 1} (-1)^{m_\alpha + s_\alpha - 1} \exp \{ \lambda(m_\alpha + s_\alpha) y_\alpha \} G \left( \frac{p_\alpha}{\lambda}, m_\alpha, s_\alpha \right) \]

\[
\left. \times \det \hat{K} [(\lambda m_\alpha, \lambda s_\alpha, p_\alpha); (\lambda m_\beta, \lambda s_\beta, p_\beta)]_{\alpha,\beta = 1, \ldots, M} \times \prod_{1 < \alpha < \beta} G_{\alpha\beta} \left( \frac{p_\lambda}{\lambda}, m_\alpha, s_\alpha \right) \right\}.
\]

(69)

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Figure 1. The contours of integration in the complex plane used for summing the series: (a) the original contour $C$; (b) the deformed contour $C'$.

where

$$\hat{K}[(\lambda m, \lambda s, p); (\lambda m', \lambda s', p')] = \frac{1}{\lambda m + \lambda s - ip + \lambda m' + \lambda s' + ip'}.$$  \hfill (70)

The crucial point of the further calculations is the procedure of taking the thermodynamic limit $\lambda \to \infty$. In this limit the summations over $\{m_\alpha\}$ and $\{s_\alpha\}$ are performed according to the following algorithm. Let us consider the example of a sum of the general type

$$R(y, p) = \lim_{\lambda \to \infty} \prod_{\alpha=1}^{M} \left[ \sum_{n_\alpha=1}^{\infty} (-1)^{n_\alpha-1} \exp\{\lambda n_\alpha y_\alpha\} \right] \Phi\left( \frac{p}{\lambda}, p, \lambda n, n \right),$$  \hfill (71)

where $\Phi$ is a function which depends on the factors $\lambda n_\alpha$, $p_\alpha/\lambda$ as well as on the parameters $n_\alpha$ and $p_\alpha$ (which do not contain $\lambda$). The summations in the above example can be represented in terms of the integrals in the complex plane,

$$R(y, p) = \lim_{\lambda \to \infty} \prod_{\alpha=1}^{M} \left[ \frac{1}{2\pi i} \int_{C} \frac{dz_\alpha}{\sin(\pi z_\alpha)} \exp\{\lambda z_\alpha y_\alpha\} \right] \Phi\left( \frac{p}{\lambda}, p, z, z \right),$$  \hfill (72)

where the integration goes over the contour $C$ shown in figure 1(a). Shifting the contour to the position $C'$ shown in figure 1(b) (assuming that there is no contribution from $\infty$), and redefining $z \to z/\lambda$, in the limit $\lambda \to \infty$ we get

$$R(y, p) = \prod_{\alpha=1}^{M} \left[ \frac{1}{2\pi i} \int_{C'} \frac{dz_\alpha}{z_\alpha} \exp\{\lambda z_\alpha y_\alpha\} \right] \lim_{\lambda \to \infty} \Phi\left( \frac{p}{\lambda}, p, z, z \lambda \right),$$  \hfill (73)

where the parameters $y_\alpha$, $p_\alpha$ and $z_\alpha$ remain finite in the limit $\lambda \to \infty$.

To perform the summations over $m_\alpha$ and $s_\alpha$ in equation (69) it is convenient to represent it in the following way:

$$V_\alpha(f_1, f_2) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} dy_\alpha dp_\alpha \frac{2\pi}{2\pi} \text{Ai}(y_\alpha + p_\alpha^2 - f) \right] S_M(p, y),$$  \hfill (74)

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where

\[ S_M(p, y) = \lim_{\lambda \to \infty} \prod_{a=1}^{M} \left[ \sum_{m_a+s_a \geq 1}^{\infty} (-1)^{m_a+s_a} \exp\{\lambda m_a y_a + \lambda s_a y_a\} \right] \]

\times \prod_{a=1}^{M} \left[ \mathcal{G} \left( \frac{p_a}{\lambda}, m_a, s_a \right) \right] \prod_{1 \leq \alpha < \beta} \left[ \mathcal{G}_{\alpha\beta} \left( \frac{P}{\lambda}, m, s \right) \right] \text{det} \hat{K} \left[ (\lambda m_a, \lambda s_a, p_a); (\lambda m_\beta, \lambda s_\beta, p_\beta) \right]. \quad (75)

The summations over \( m_a \) and \( s_a \) in the above expression can be represented as follows:

\[ \sum_{m_a+s_a \geq 1}^{\infty} (-1)^{m_a+s_a} = \sum_{m_a=1}^{\infty} (-1)^{m_a-1} \delta(s_a, 0) \]

\[ + \sum_{s_a=1}^{\infty} (-1)^{s_a-1} \delta(m_a, 0) - \sum_{m_a=1}^{\infty} (-1)^{m_a-1} \sum_{s_a=1}^{\infty} (-1)^{s_a-1}. \quad (76) \]

Thus in the integral representation, equations (71)–(73), for the function \( S_M(p, y) \), equation (75), we get

\[ S_M(p, y) = \prod_{a=1}^{M} \left[ \int_{C'} \frac{dz_1 adz_2 a}{(2\pi i)^2} \left( \frac{2\pi i}{z_1 a} \delta(z_2 a) + \frac{2\pi i}{z_2 a} \delta(z_1 a) - \frac{1}{z_1 a z_2 a} \right) \exp\{z_1 a y_a + z_2 a y_a\} \right] \]

\[ = \text{det} \hat{K} \left[ (z_1 a, z_2 a, p_a); (z_1 \beta, z_2 \beta, p_\beta) \right] \]

\[ \times \lim_{\lambda \to \infty} \left\{ \prod_{a=1}^{M} \mathcal{G} \left( \frac{p_a}{\lambda}, \frac{z_1 a}{\lambda}, \frac{z_2 a}{\lambda} \lambda \right) \right\} \prod_{1 \leq \alpha < \beta} \left[ \mathcal{G}_{\alpha\beta} \left( \frac{P}{\lambda}, \frac{z_1}{\lambda}, \frac{z_2}{\lambda} \right) \right]. \quad (77) \]

Taking into account the Gamma function properties, \( \Gamma(z)|_{z \to 0} = 1/z \) and \( \Gamma(1+z)|_{z \to 0} = 1 \), for the factors \( \mathcal{G} \), equation (61), and \( \mathcal{G}_{\alpha\beta} \), equation (62), we obtain

\[ \lim_{\lambda \to \infty} \mathcal{G} \left( \frac{p_a}{\lambda}, \frac{z_1 a}{\lambda}, \frac{z_2 a}{\lambda} \lambda \right) = \frac{(z_1 a + z_2 a + ip_a^{(-)})(z_1 a + z_2 a - ip_a^{(+))}}{(z_2 a + ip_a^{(-)})(z_1 a - ip_a^{(+)}))} \quad (78) \]

and

\[ \lim_{\lambda \to \infty} \mathcal{G} \left( \frac{p}{\lambda}, \frac{z_1}{\lambda}, \frac{z_2}{\lambda} \right) = 1. \quad (79) \]

Thus, in the limit \( \lambda \to \infty \) the expression for the probability distribution function, equation (69), takes the form of the Fredholm determinant

\[ W(f) = 1 + \frac{\sum_{M=1}^{\infty} (-1)^M}{M!} \prod_{a=1}^{M} \int_{-\infty}^{\infty} \frac{dy_a dp_a}{2\pi} \text{Ai}(y_a + p_a^2 - f) \]

\[ \times \int_{C'} \frac{dz_1 a dz_2 a}{(2\pi i)^2} \left( \frac{2\pi i}{z_1 a} \delta(z_2 a) + \frac{2\pi i}{z_2 a} \delta(z_1 a) - \frac{1}{z_1 a z_2 a} \right) \]

\[ \times \left( 1 + \frac{z_1 a}{z_2 a + ip_a^{(-)}} \right) \left( 1 + \frac{z_2 a}{z_1 a - ip_a^{(+)}\right). \]

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\[ \times \exp\{z_1 y_1 + z_2 y_2\} \det \left[ \frac{1}{z_1 + z_2 - ip_1 + z_1 + z_2 + ip_1} \right]_{(\alpha, \beta) = 1, 2, \ldots, M} \]

\[ = \det \left[ 1 - \hat{B} \right], \quad (80) \]

with the kernel

\[ \hat{B} [(z_1, \ z_2, \ p); (z_1', \ z_2', \ p')] = \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \ Ai(y + p^2 - f) \left( \frac{2\pi i}{z_1} \delta(z_2) + \frac{2\pi i}{z_2} \delta(z_1) - \frac{1}{z_1 z_2} \right) \]

\[ \times \left( 1 + \frac{z_1}{z_2 + ip(-)} \right) \left( 1 + \frac{z_2}{z_1 - ip(+)} \right) \exp\{z_1 y + z_2 y\} \]

\[ \times \frac{1}{z_1 + z_2 - ip + z_1' + z_2' + ip'}, \quad (81) \]

In the exponential representation of this determinant we get

\[ W(f) = \exp \left[ -\sum_{M=1}^{\infty} \frac{1}{M} \Tr \hat{B}^M \right], \quad (82) \]

where

\[ \Tr \hat{B}^M = \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \ Ai(y_\alpha + p_\alpha^2 - f) \right. \]

\[ \times \int \int_{C'} \frac{dz_1 \alpha dz_2 \alpha}{(2\pi)^2} \left( \frac{2\pi i}{z_1 \alpha} \delta(z_2 \alpha) + \frac{2\pi i}{z_2 \alpha} \delta(z_1 \alpha) - \frac{1}{z_1 \alpha z_2 \alpha} \right) \]

\[ \left. \times \left( 1 + \frac{z_1 \alpha}{z_2 \alpha + ip(-)} \right) \left( 1 + \frac{z_2 \alpha}{z_1 \alpha - ip(+)} \right) \right] \exp\{z_1 \alpha y + z_2 \alpha y\} \right] \prod_{\alpha=1}^{M} \left[ \frac{1}{z_1 \alpha + z_2 \alpha - ip_\alpha + z_1 \alpha + z_2 \alpha + ip_\alpha + 1} \right]. \quad (83) \]

Here, by definition, it is assumed that \( z_{i_{M+1}} \equiv z_{i_1} (i = 1, 2) \) and \( p_{M+1} \equiv p_1 \). Substituting

\[ \frac{1}{z_1 + z_2 - ip + z_1 + z_2 + ip + 1} \]

\[ = \int_{0}^{\infty} d\omega_\alpha \exp \left[ -(z_1 + z_2 - ip + z_1 + z_2 + ip + 1) \omega_\alpha \right] \quad (84) \]

into equation (83), we obtain

\[ \Tr \hat{B}^M = \int_{0}^{\infty} d\omega_1 \cdots d\omega_M \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dy dp}{2\pi} \ Ai(y + p^2 + \omega_\alpha + \omega_{\alpha-1} - f) \right. \]

\[ \times \left. \exp\{ip(\omega_\alpha - \omega_{\alpha-1})\} S(p, y) \right], \quad (85) \]
where, by definition, $\omega_0 \equiv \omega_M$, and

$$S(p, y) = \int \int_{C'} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \left( \frac{2\pi i}{z_1} \delta(z_2) + \frac{2\pi i}{z_2} \delta(z_1) - \frac{1}{z_1 z_2} \right) \left( 1 + \frac{z_1}{z_2 + ip(-)} \right) \times \left( 1 + \frac{z_2}{z_1 - ip(+)} \right) \exp\{z_1 y + z_2 y\}. \tag{86}$$

Simple calculations yield

$$S(p, y) = 1 - \frac{1}{(2\pi i)^2} \int \int_{C'} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \left( 1 + \frac{z_1}{z_2 + i(p - i\epsilon)} \right) \times \left( 1 + \frac{z_2}{z_1 - i(p + i\epsilon)} \right) \exp\{(z_1 + z_2)y\} \tag{87}$$

Taking the limit $\epsilon \to 0$ we find

$$S(p, y) = \delta(y) \delta(p). \tag{88}$$

Substituting this result into equation (85) we obtain

$$\text{Tr} \hat{B}^M = \int_0^\infty d\omega_1 \cdots d\omega_M \prod_{a=1}^M [\text{Ai}(\omega_a + \omega_{a-1} - f)]. \tag{89}$$

In other words, the free energy distribution function of our problem is given by the Fredholm determinant

$$W(f) = \det \left[ 1 - \hat{B}_{-f} \right] \tag{90}$$

with the kernel

$$B_{-f}(\omega; \omega') = \text{Ai}(\omega + \omega' - f) \quad (\omega, \omega' > 0) \tag{91}$$

which is the GOE Tracy–Widom distribution [14, 15].

4. Conclusions

In this paper we have presented a sufficiently simple derivation of the GOE Tracy–Widom distribution function for the free energy fluctuations in random directed polymers with free boundary conditions. The main message of this somewhat technical work is not the final result itself (which is not new anyway), but the demonstration of the efficiency of the general method and new technical tricks used in the derivation. By mapping the original problem to the $N$-particle quantum boson system with attractive interactions the derivation is made in the framework of the integer replica series summations and the Bethe ansatz formalism for the quantum boson system.

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The key technical tricks of the presented calculations include the following points. First of all, to make the integration over particle coordinates of the Bethe ansatz propagator well defined one has to introduce proper regularization at \( \pm \infty \), which requires formal splitting of the partition function into two parts: one in the positive particle coordinate sector (up to \(+\infty\)) and another one in the negative particle coordinate sector (down to \(-\infty\)), equations (15) and (16). Next is the ‘magic’ Bethe ansatz combinatorial identity, equation (37), which allows us to perform the summation over the momentum permutations and ‘disentangle’ the sophisticated products contained in the Bethe ansatz propagator. One more trick is the reformulation of the summation over permutations of the momenta between the positive and the negative particle position sectors in terms of the series summations, equation (43), which allows us to represent the probability distribution function in terms of the problem of the series summations, equation (49). Finally, the crucial point of the considered derivation is the procedure of the series summations in the thermodynamic limit \( t \to \infty \). In this limit, due to the integral representation of the series, equations (71)–(73), one obtains dramatic simplification of some factors, equations (78) and (79), in the expression for the probability distribution function, which allows us to represent it in the form of the Fredholm determinant, equation (80).

Hopefully, the experience gained from the presented calculations will help in solving more serious long standing problems of this scope, such as the joint statistical properties of the free energy fluctuations at different times.

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