THE RANK OF $G$-CROSSED BRAIDED EXTENSIONS OF MODULAR TENSOR CATEGORIES

MARCEL BISCHOFF

Abstract. We give a short proof for a well-known formula for the rank of a $G$-crossed braided extension of a modular tensor category.

1. Introduction

$G$-crossed braided extensions of modular tensor categories are an important ingredient for the process of gauging in topological phases of matter in the framework of modular tensor categories [BBCW14]. Let $\mathcal{C}$ be a modular tensor category, which for the purpose of this note is a semisimple, $\mathbb{C}$-linear abelian ribbon category with simple tensor unit, such that the set of isomorphism classes of simple objects $\text{Irr}(\mathcal{C})$ is finite and the braiding is non-degenerate, see e.g. [BK01]. A global symmetry [BBCW14] of $\mathcal{C}$ is a pair $(G, \rho)$ consisting of a finite group $G$ and homomorphism $\rho: G \to \text{Aut}_{\otimes}(\mathcal{C})$, where $\text{Aut}_{\otimes}(\mathcal{C})$ is the group of isomorphism classes of braided autoequivalences of $\mathcal{C}$. Gauging a global symmetry $(G, \rho)$ of $\mathcal{C}$ [CGPW16] is a two step process which eventually produces a new modular tensor category and is given as follows:

1. Construct a $G$-crossed braided extension $\mathcal{F} = \bigoplus_{g \in G} \mathcal{F}_g$ of $\mathcal{C}$.
2. Consider the equivariantization $\mathcal{F}^G$ of $\mathcal{F}$ to obtain a new modular tensor category, which therefore contains $\text{Rep}(G)$ as a symmetric subcategory.

A $G$-crossed braided fusion category is a fusion category $\mathcal{F}$ equipped with the following structure:

- $\mathcal{F}$ is faithfully $G$-graded, i.e. $\mathcal{F} = \bigoplus \mathcal{F}_g$ and $X_g \otimes Y_h \in \mathcal{F}_{gh}$ for every $g, h \in G$, $X_g \in \mathcal{F}_g$, and $Y_h \in \mathcal{F}_h$.
- There is an $G$-action $X \mapsto ^gX = \rho(g)(X)$ given by a monoidal functor $\rho: G \to \text{Aut}_{\otimes}(\mathcal{F})$, where $\text{Aut}_{\otimes}(\mathcal{F})$ is the categorical group of monoidal autoequivalences.
- There is a natural family of isomorphisms $c_{X,Y}: X \otimes Y \to ^gY \otimes X$, $g \in G, X \in \mathcal{F}_g, Y \in \mathcal{F}$ with $\rho(g)(\mathcal{F}_h) \subset \mathcal{F}_{gh^{-1}}$ and $\rho(g)(c_{X,Y}) = c_{X,Y}$. 

Date: July 18, 2018.
Supported in part by NSF Grant DMS-1700192/1821162.
We note that $F_e$ is a braided fusion category and we get a global symmetry $\rho : G \to \text{Aut}^{\text{br}}(F_e)$ given by restriction and truncation of $\rho$. We are only interested in the case where $F_e$ is a modular tensor category. We refer to \cite{DGNO10, Thr10} for more details.

We call a $G$-crossed braided fusion category $F = \bigoplus_{g \in G} F_g$ a $G$-crossed braided extension of a modular tensor category $C$ if $F_e$ is braided equivalent to $C$. The above data of $F$ gives by restriction a global symmetry $(G, \rho)$ on $C \cong F_e$. We note that given a modular tensor category $C$ and a global action $(G, \rho)$ as an input one has to check that certain obstructions have to vanish in order for a $G$-crossed braided to exist, see \cite{ENO10}.

$G$-crossed braided extensions arise naturally in rational conformal field theory. Let $A$ be a (completely) rational conformal net, then the category of representations $\text{Rep}(A)$ is a unitary modular tensor category \cite{KLM01}. Let $G \leq \text{Aut}(A)$ be a finite group of automorphisms of the net $A$. There is a category $G^\text{-Rep}(A)$ of $G$-twisted representations of $A$, which is a $G$-crossed braided extension of the unitary modular tensor category $\text{Rep}(A)$. The fixed point or “orbifold net” $A^G$ is again completely rational and $\text{Rep}(A^G)$ is a gauging of $\text{Rep}(A)$ by $G$, i.e. $\text{Rep}(A^G)$ is braided equivalent to the $G$-equivariantization $(G^\text{-Rep}(A))^G$, see \cite{Müg05} for the original reference and \cite{Bis18} for a review.

Starting with a modular tensor category $C$ we want to compute certain invariants of possible $G$-crossed braided extensions $F = \bigoplus_{g \in G} F_g$ with $F_e \cong C$. The simplest invariant of $F$ is its rank $\text{rk}(F) := |\text{Irr}(F)|$. We note that the global symmetry $(G, \rho)$ associated with a $G$-crossed braided extension $F$ of $C$ equips $\text{Irr}(C)$ with a $G$-action. This $G$-space already contains all the information about the rank of possible $G$-crossed braided extensions associated with $(G, \rho)$. Namely, the following well-know formula, cf. \cite{BBCW14, Eq. (348)} holds.

**Proposition 1.1.** Let $F = \bigoplus_{g \in G} F_g$ be a (faithful) $G$-crossed braided extension of a modular tensor category $C \cong F_e$. Then the rank of $F_g$ is equal to the size of the stabilizer $\text{Irr}(C)^g$, i.e.

$$\text{rk}(F_g) = |\text{Irr}(C)^g|.$$  

In particular, the rank of $F$ is determined via the $G$-action on $\text{Irr}(C)$ by

$$\text{rk}(F) = \sum_{g \in G} |\text{Irr}(C)^g| = |G||\text{Irr}(C)/G|.$$

The goal of this note is to provide a short proof of this statement using modular invariants, which were used in the operator algebra literature, e.g. \cite{BEK99, BEK00}.

2. The rank of module categories and $G$-crossed braided extensions

Let $C$ be a modular tensor category and $A$ a simple non-degenerate algebra object in $C$. Here non-degenerate means that the trace pairing $\Phi_A : A \to \overline{A}$
THE RANK OF G-CROSSED BRAIDED EXTENSIONS OF MTCS

given by
\[ \Phi_A = (\text{ev}_A \circ (\text{id}_A \otimes m) \circ \text{id}_A) \circ (\text{id}_A \otimes (\text{id}_A \otimes \text{coev}_A) \circ m) \circ (\text{coev}_A \otimes \text{id}_A) \]
is invertible, which implies that \( A \) has the structure of a special symmetric Frobenius algebra object, see [KR08, Lemma 2.3].

We denote the category with the opposite braiding by \( \overline{C} \). The full center \( Z(A) \) is a Lagrangian algebra object in \( Z(C) \cong C \boxtimes \overline{C} \) [FFRS06, KR08, DMNO13]. We note that the forgetful functor \( F: C \boxtimes \overline{C} \to C \) maps \( X \boxtimes Y \mapsto X \otimes Y \) and has an adjoint \( I: C \to C \boxtimes \overline{C} \). The image \( I(1) \) has the canonical structure of a Lagrangian algebra in \( C \boxtimes \overline{C} \). If \( X \in C \) we denote by \( \overline{X} \in C \) a dual object. The full center \( Z(A) \) can be realized as the left center of the braided product of \( (A \boxtimes 1) \otimes^+ I(1) \) [FFRS06, KR08]. Given a simple non-degenerate algebra \( A \) there are two functors \( \alpha^\pm \) from \( C \) to the category \( \mathbb{C} \) of \( A \)-bimodules in \( C \). They are given by equipping the image of the free right module functor \( - \otimes A \) with a left action using the braiding or opposite braiding, respectively, see [FFRS06]. The following proposition is well-known to experts.

**Proposition 2.1.** Let \( C \) be a modular tensor category, \( A \) a simple non-degenerate algebra object in \( C \), and \( Z \) the associated modular invariant matrix, i.e. \( Z = (Z_{X,Y})_{X,Y \in \text{Irr}(C)} \) is the square matrix given by
\[ Z_{X,Y} = \dim C \text{Hom}_A C (\alpha^+(X), \alpha^-(Y)). \]
Then

1. The full center \( Z(A) \) of \( A \) as an object in \( C \boxtimes \overline{C} \) is given by
\[ Z(A) \cong \bigoplus_{X,Y \in \text{Irr}(C)} Z_{X,Y} \cdot X \boxtimes \overline{Y}. \]

2. The rank \( \text{rk}(C_A) \) of the category of right \( A \)-modules \( C_A \) is given by the trace of the matrix \( Z \):
\[ \text{rk}(C_A) = \text{tr}(Z) = \sum_{X \in \text{Irr}(C)} Z_{X,X}. \]

**Proof.** The first statement is [FFRS06, Remark 3.7]. The second statement is [BEK99, Corollary 6.1] in the case that \( A \) is a Q-system and \( C \) is a unitary modular tensor category realized as endomorphisms of a type III factor. In the more general setting, it follows from [KR08, Eq. (4.4) and Prop. 4.3]. Namely, \( F(Z(A)) \cong \bigoplus_{X,Y \in \text{Irr}(C)} Z_{X,Y} \cdot X \otimes \overline{Y} \) hence \( \dim \text{Hom}(1, F(Z(A))) = \text{tr}(Z) \) equals the number of isomorphism classes of simple modules in \( C_A \), since \( F(Z(A)) \) is equivalent to \( \bigoplus_{M \in \text{Irr}(C_A)} M \otimes_A M \) and since \( M \otimes_A M \) is a connected algebra by [KR08, Prop. of Prop. 4.10].

For \( \phi \in \text{Aut}^\text{br}(C) \) there is a canonical Lagrangian algebra \( L_\phi = (\text{id} \boxtimes \phi) I(1) \) in \( C \boxtimes \overline{C} \), see e.g. [DMNO13, Sec 3.2-3.3]. The algebra \( L_\phi \) corresponds to the full center of a \( C \)-module category, which we denote by \( \mathcal{C}_\phi \), cf. [KR08, DMNO13]. This module category can be obtained, using e.g. [KR08], by taking any
connected summand $A$ of the algebra $F(L_\phi)$ (which always fulfills $Z(A) = L_\phi$) and considering the $\mathcal{C}$-module category $\mathcal{C}_\phi := \mathcal{C}_A$, see [KR08], and also [ENO10] Lemma 5.1. Since $L_\phi \cong \bigoplus_{X \in \text{Irr}(\mathcal{C})} X \boxtimes \phi(\bar{X})$, i.e. $Z_{X,Y} = \delta_{\phi(X),Y}$, Proposition 2.1 immediately gives the following statement.

**Lemma 2.2.** The rank of $\mathcal{C}_\phi$ equals $|\text{Irr}(\mathcal{C})^\phi|$, i.e. the number of isomorphism classes of simple objects in $\mathcal{C}$ which are fixed by $\phi$.

**Proof of Proposition 1.1.** Let $F = \bigoplus_{g \in G} F_g$ be a $G$-crossed braided extension of a modular tensor category $\mathcal{C}$. By [ENO10, Sec. 5.4] the category $F_g$ is equivalent as a left $\mathcal{C}$-module category to the category $\mathcal{C}_g$ in Lemma 2.2 and the statement follows. □

2.1. The rank of permutation extensions. Let $\mathcal{C}$ be a modular tensor category. Gannon and Jones announced [GJ18] that certain cohomological obstructions vanish and thus that there is always an $S_n$-crossed braided extension of $\mathcal{C} \boxtimes n$ which we denote by $\mathcal{C} \bowtie S_n$, where the action of $S_n$ is given by permutations of objects. More general, for any subgroup $G \leq S_n$ there is a $G$-crossed braided extension $\mathcal{C} \bowtie G$.

For the cyclic subgroup $Z_n \cong \langle (1 \cdot \ldots \cdot n) \rangle \subset S_n$ we obtain

$$\text{rk}(\mathcal{C} \bowtie Z_n) = \text{rk}(\mathcal{C})^n + (n-1) \text{rk}(\mathcal{C})$$

which in the case $n = 2$ is also true by construction for the examples considered in [EMJP18].

In general, for the full symmetric group, we get that $\text{rk}((\mathcal{C} \bowtie S_n)_g)$ depends only on the conjugacy class $C_g$ of $g \in S_n$. Namely, $\text{rk}((\mathcal{C} \bowtie S_n)_g) = \text{rk}(\mathcal{C})^{|a|}$ where $a = (a_1, \ldots, a_n)$ and $a_j$ is the number of $j$-cycles (counting 1-cycles) in the cycle decomposition of $g$ and $|a| = \sum_j a_j$. Thus summing over all elements of the $p(n)$ conjugacy classes we get

$$\text{rk}(\mathcal{C} \bowtie S_n) = \sum_{a = (a_1, \ldots, a_n)} c_a \text{rk}(\mathcal{C})^{|a|}, \quad c_a = \frac{n!}{\prod_j j^{a_j} (a_j)!},$$

where $c_a = |C_g|$ the size of the conjugacy class $C_g$ of an element $g \in S_n$ with cycle type $a = (a_1, \ldots, a_n)$ and the sum runs over all unordered partitions of $\{1, \ldots, n\}$ with $a_j$ the number of partitions of length $j$. Explicitly,

$$\text{rk}(\mathcal{C} \bowtie S_3) = \text{rk}(\mathcal{C})^3 + 3 \text{rk}(\mathcal{C})^2 + 2 \text{rk}(\mathcal{C})$$
$$\text{rk}(\mathcal{C} \bowtie S_4) = \text{rk}(\mathcal{C})^4 + 6 \text{rk}(\mathcal{C})^3 + 11 \text{rk}(\mathcal{C})^2 + 6 \text{rk}(\mathcal{C})$$
$$\text{rk}(\mathcal{C} \bowtie S_5) = \text{rk}(\mathcal{C})^5 + 10 \text{rk}(\mathcal{C})^4 + 35 \text{rk}(\mathcal{C})^3 + 50 \text{rk}(\mathcal{C})^2 + 24 \text{rk}(\mathcal{C}) \\
\cdots$$

**Acknowledgments.** I would like to thank Corey Jones, David Penneys, and Julia Plavnik for discussions.
REFERENCES

[BBCW14] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, *Symmetry, defects, and gauging of topological phases*, arXiv preprint [arXiv:1410.4540] (2014).

[BEK00] J. Böckenhauer, D. E. Evans, and Y. Kawahigashi, *Chiral structure of modular invariants for subfactors*, Comm. Math. Phys. **210** (2000), no. 3, 733–784. MR1777347 (2001k:46097)

[BEK99] *On α-induction, chiral generators and modular invariants for subfactors*, Comm. Math. Phys. **208** (1999), no. 2, 429–487. MR1729094 (2001c:81180)

[Bis18] M. Bischoff, *Conformal net realizability of Tambara-Yamagami categories and generalized metaplectic modular categories*, arXiv preprint [arXiv:1803.04949] (2018).

[BK01] B. Bakalov and A. Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, vol. 21, American Mathematical Society, Providence, RI, 2001. MR1797619

[CGPW16] S. X. Cui, C. Galindo, J. Y. Plavnik, and Z. Wang, *On gauging symmetry of modular categories*, Comm. Math. Phys. **348** (2016), no. 3, 1043–1064. MR3555361

[DGNO10] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, *On braided fusion categories. I*, Selecta Math. (N.S.) **16** (2010), no. 1, 1–119. MR2609644 (2011c:18015)

[DMNO13] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. Reine Angew. Math. **677** (2013), 135–177. MR3039775

[DNO13] A. Davydov, D. Nikshych, and V. Ostrik, *On the structure of the Witt group of braided fusion categories*, Selecta Math. (N.S.) **19** (2013), no. 1, 237–269. MR3022755

[EMJP18] C. Edie-Michell, C. Jones, and J. Plavnik, *Fusion rules for Z/2Z permutation gauging*, arXiv preprint [arXiv:1804.01657] (2018).

[ENO10] P. Etingof, D. Nikshych, and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topol. **1** (2010), no. 3, 209–273. With an appendix by Ehud Meir. MR2677836

[FFRS06] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Correspondences of ribbon categories*, Adv. Math. **199** (2006), no. 1, 192–329. MR2187404 (2007b:18007)

[GJ18] T. Gannon and C. Jones, *Vanishing of categorical obstructions for permutation orbifolds*, arXiv preprint [arXiv:1804.08313] (2018).

[KLM01] Y. Kawahigashi, R. Longo, and M. Müger, *Multi-Interval Subfactors and Modularity of Representations in Conformal Field Theory*, Comm. Math. Phys. **219** (2001), 631–669, available at [arXiv:math/9903104].

[KR08] L. Kong and I. Runkel, *Morita classes of algebras in modular tensor categories*, Adv. Math. **219** (2008), no. 5, 1548–1576. MR2458146 (2009h:18016)

[Müg05] M. Müger, *Conformal Orbifold Theories and Braided Crossed G-Categories*, Comm. Math. Phys. **260** (2005), 727–762.

[Tur10] V. Turan, *Homotopy quantum field theory*, EMS Tracts in Mathematics, vol. 10, European Mathematical Society (EMS), Zürich, 2010. Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier. MR2674592

DEPARTMENT OF MATHEMATICS, MORTON HALL 321, 1 OHIO UNIVERSITY, ATHENS, OH 45701, USA

E-mail address: bischoff@ohio.edu
E-mail address: marcel@localconformal.net