Skorohod equation and BSDE’s with two reflecting barriers

Soufiane Aazizi *

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Abstract

We solve a class of doubly reflected backward stochastic differential equation whose generator depends on the resistance due to reflections, which extend the recent work of Qian and Xu on reflected BSDE with one barrier. We then obtain the existence and uniqueness and the continuous dependence theorem for this reflected BSDE.

Key words : Local Time; Reflected backward stochastic differential equation; Picard iteration; Optional projection; Extended Skorohod problem.

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*Department of Mathematics, Faculty of Sciences Semlalia Cadi Ayyad University, B.P. 2390 Marrakesh, Morocco. aazizi.soufiane@gmail.com
1 Introduction

In this paper we are interested to study the following doubly reflected backward stochastic differential equation (DBSDE for short)

\[
\begin{align*}
    Y_t &= \xi + \int_t^T f(r, Y_r, Z_r, K_r)dr - \int_t^T Z_r dW_r + K_T - K_t \\
    K &= K^l - K^u \\
    \forall t \leq T, \quad L_t \leq Y_t \leq U_t \quad \text{and} \quad \int_0^T (U_r - Y_r) dK^u_r = \int_0^T (Y_r - L_r) dK^l_r = 0
\end{align*}
\]

(1.1)

where \( W \) is a Brownian motion defined on some complete filtered probability space \((\Omega, \mathcal{F}, P)\), a terminal value \( \xi \in \mathcal{F}_T \), a continuous barriers \( L \) and \( U \) which are modeled by a semimartingales, and \( K^l \) and \( K^u \) are nondecreasing processes.

The BSDE was firstly initiated by Bismut [3] and later developed by Pardoux and Peng [14] to prove existence and uniqueness of adapted solution, EL Karoui et al. [8] introduced the notion of reflected BSDEs with lower barrier, in which the component \( Y \) is forced to stay above a given obstacle, Cvitanic, Karatzas and Soner [6], and later Aazizi and Ouknine [1] considered the case where the constraint is imposed on the component \( Z \). In the same frame, the generalization of BSDE with two continuous reflecting barriers is introduced by Cvitanic-Karatzas [5]. Since then, there were many works on the latter kind of BSDEs. A new kind of reflected BSDE has been introduced by Bank and El Karoui [2], by a variation of Skorohod’s obstacle problem, known as variant reflected BSDE, which takes the following forme:

\[
\begin{align*}
    Y_t &= X_T + \int_t^T f(r, Y_r, Z_r, A_r)dr - \int_t^T Z_r dW_r \\
    Y &\leq X \\
    \int_t^T |X_r - Y_r| dA_r = 0
\end{align*}
\]

(1.2)

where \( A \) is an increasing process, with \( A_{0-} = -\infty \). The process \( A \) does not directly act on \( Y \) to push the solution downwards such that \( Y_t \leq X_t \) like in standard BSDE with one barrier, but it acts through the generator \( f \). This work has been generalized by Ma and Wang in [12], to prove that a solution in a small-time duration, under some extra conditions, exists and is unique. Recently in the same framework, Qian and Xu [18] studied the following class of reflected BSDE

\[
\begin{align*}
    Y_t &= \xi_T + \int_t^T f(r, Y_r, Z_r, K_r)dr - \int_t^T Z_r dW_r + K_T - K_t \\
    S_t &\leq Y_t \\
    \int_t^T (S_r - Y_r) dK_r = 0
\end{align*}
\]

(1.3)

Here, \( K \) appears in the driver as a resistance force, if \( f \) is decreasing in \( K \), then we get an extra force from the Lebesgue integral, if \( f \) is increasing in \( K \), then there is a kind of cancelation of the positive force, in general case, they consider this RBSDE as an equation with resistance. They derived an explicit formula of the increasing process \( K \) by using the result on Skorohod equation together with the theory of optional dual projection (see [10]).
In this paper, we extend the approach of [18], to the doubly reflected BSDE (1.1). The paper is organized as follows. In Section 2, we provide and explicit formula of the process $K := K^l - K^u$ using Tanaka formula (see 2.7) and the extended Skorohod map (ESM in short) (see (2.18)) introduced by Ramanan [16]. In Section 3 we study a doubly reflected BSDE with generator depending on $K$, we then prove existence and uniqueness through fixed point theorem. Finally, in Section 4, we obtain the continuous dependence property of the solution.

2 Explicit formula of the process $K$

2.1 General formulation

On a given complete probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, let $W := (W^1, W^2, ..., W^d)$ be a d-dimensional Brownian motion defined on a finite interval $[0, T]$, and denote $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the filtration generated by the Brownian motion and the $P$-null sets. We denote by $\mathcal{P}$ the progressive $\sigma$-field on the product space $[0, T] \times \Omega$.

We consider the following spaces:

- $L^2(\mathcal{F}_t)$ the space of all $\mathcal{F}_t$-measurable real random variable $\phi$ such that $\mathbb{E}|\phi|^2 < \infty$.
- $H^2_d(0, T)$ the space of $\mathbb{R}^d$-valued predictable process $\psi$ such that $\mathbb{E} \left[ \int_0^T |\psi_t|^2 dt \right] < \infty$.
- $\mathcal{S}^2(0, T)$ is the space of all continuous semimartingales over $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0 \leq t \leq T})$.
- $\mathcal{A}^2(0, T)$ the space of all $\mathcal{F}_t$-measurable continuous and increasing process $K$ with $K_0 = 0$ and such that $\mathbb{E}|K_T|^2 < \infty$.
- $D^2_F(0, T)$ the set of $\mathbb{F}$-progressively measurable càdlàg process $\phi$ with $\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty$.

In the sequel of this paper, we denote

(i) The process $K$ to be the difference of the two increasing processes $K^l$ ans $K^u$ such that $K := K^l - K^u$ with $K^l, K^u \in \mathcal{A}^2$.

(ii) The generator $f : [0, T] \times \Omega \times \mathcal{S}^2 \times H^2 \times \mathcal{A}^2 \times \mathcal{A}^2 \to \mathbb{R}$ is globally Lipschitz w.r.t $y, z$ and $k$ such that:

$$|f(r, y, z, k) - f(r, y', z', k')| \leq L_1(|y - y'| + |z - z'|) + L_2|k - k'|.$$  (2.1)
We recall the definition of optional (dual) projection (See Nikeghbali [13] for more details).

**Definition 2.1** Optional projection

Let $X$ be a measurable process either positive or bounded. There exists a unique (up to indistinguishability) optimal process $X\flat$ called optional projection such that:

$$
\mathbb{E}[X_T 1_{T<\infty}/\mathcal{F}_T] = X_T^\flat 1_{T<\infty} \quad \text{a.s.}
$$

(2.2)

for every stopping time $T$.

**Definition 2.2** Dual optional projection

Let $(A_t)$ be an integrable raw increasing process. We call dual optional projection of $A$ the (optional) increasing process $(A_t^\circ)$ defined by:

$$
\mathbb{E} \left[ \int_{\mathbb{R}^+} X_r dA_r^\circ \right] = \mathbb{E} \left[ \int_{\mathbb{R}^+} X_r^\flat dA_r \right],
$$

(2.3)

for any bounded measurable $X$.

By the definitions above, we define the optional projection of $K_l$ (resp. $K_u$) by $K_l^\flat$ (resp. $K_u^\flat$) and its dual optional projection by $K_l^{\circ}$ (resp. $K_u^{\circ}$) which is continuous and increasing. Furthermore, we have $K_l^\flat - K_l^{\circ}$ (resp. $K_u^\flat - K_u^{\circ}$) a continuous martingale.

In the first time, we propose to derive some explicit formulas of the process $K$, when the doubly reflected BSDE has a generator $f$ not depending on $y$, $z$ and $k$ taking the following form

$$
\begin{cases}
Y_t = \xi + \int_0^T f_r dr - \int_0^T Z_r dW_r + K_T - K_t \\
L_t \leq Y_t \leq U_t, \forall t \leq T \\
The Skorohod conditions hold: \int_0^T (U_r - Y_r) dK_r^u = \int_0^T (Y_r - L_r) dK_r^l = 0.
\end{cases}
$$

(2.4)

where $(f_t)_{0 \leq t \leq T}$ is optional, $\mathbb{E} \left( \int_0^T f_r^2 dr \right) < \infty$, and $L, U$ are two continuous semimartingale. Observe that $Y$ could be written as

$$
Y_t = Y_0 - \int_0^t f_r dr + \int_0^t Z_r dW_r - K_t^l + K_t^u.
$$

(2.5)

In this whole paper, we suppose that

$$
\inf_{t \geq 0} (U_t - L_t) > 0 \quad \text{a.s.}
$$

(2.6)

### 2.2 Local time

The main objective in this part is to provide an explicit formula of the two increasing processes $K_u$ and $K_l$ in term of local time associated to $Y$, based on Tanaka formula. According to [LS], we can interpret the increasing processes $K_l$ and $K_u$ in this framework, as a time-reversed local time, in order that $K_l$ (resp. $K_u$) will be called the reflected local time of $Y$ at $L$ (resp. at $U$).
Corollary 2.1 Assume that $L \leq Y \leq U$ are three continuous semimartingales, such that $L$ and $U$ take the form $X = M^X + A^X$ with $X := L, U$ and $(M^X, A^X) \in \mathcal{M}^2 \times BV[0, \infty)$, when $Y$ is solution to (2.5). Then

$$
K_t = - \int_0^t (I_{Y_r = L_r} + I_{Y_r = U_r}) f_r dr - \int_0^t I_{Y_r = U_r} dA_r^U - \int_0^t I_{Y_r = L_r} dA_r^L + L^U_t - L_t^L - Y_t - L_t^Y,
$$

(2.7)

with $L^X$ denote the local time of the continuous semimartingale $X$ at 0.

Proof. From equation (2.5), and the following Tanaka formula

$$
\begin{align*}
(Y_t - L_t)^- & = (Y_0 - L_0)^- - \int_0^t I_{Y_r \leq L_r} d(Y_r - L_r) + L_t^Y - L_t^L \\
(U_t - Y_t)^- & = (U_0 - Y_0)^- - \int_0^t I_{U_r \leq Y_r} d(U_r - Y_r) + L_t^U - L_t^Y
\end{align*}
$$

(2.8)

Together with the fact that $(Y - L)^- = (U - Y)^- = 0$ and increasing property of the local time, we have:

$$
\begin{align*}
L_t^Y - L_t^L & = - \int_0^t I_{Y_r = L_r} f_r d \mathbb{A}_r - \int_0^t I_{Y_r = L_r} dK_r - \int_0^t I_{Y_r = L_r} dA_r^L \\
L_t^U - L_t^Y & = \int_0^t I_{Y_r = U_r} f_r d \mathbb{A}_r - \int_0^t I_{Y_r = U_r} dK_r + \int_0^t I_{Y_r = U_r} dA_r^U
\end{align*}
$$

(2.9)

Then

$$
\begin{align*}
K_t^L & = - \int_0^t I_{Y_r = L_r} f_r d \mathbb{A}_r - \int_0^t I_{Y_r = L_r} dA_r^L - L_t^Y - L_t^L \\
K_t^U & = \int_0^t I_{Y_r = U_r} f_r d \mathbb{A}_r + \int_0^t I_{Y_r = U_r} dA_r^U - L_t^U - L_t^Y
\end{align*}
$$

(2.10)

Since $K = K^L - K^U$, we come at the end of the proof. □

2.3 Extended Skorohod problem

We derive an explicit formula of the process $K$ solution of doubly reflected BSDE (2.5), using Skorohod equation. Throughout this part, $D[0, \infty)$ will denote real-valued càdlàg functions on $[0, \infty)$, $D^-[0, \infty)$ (resp. $D^+[0, \infty)$) will denote càdlàg functions on $[0, \infty)$ taking values in $\mathbb{R} \cup \{-\infty\}$ (reps. in $\mathbb{R} \cup \{\infty\}$) and $BV[0, \infty)$ denotes the subspace of functions with bounded variation on every finite interval. According to Skorohod equation with two time-boundaries, we have the following definition.

Definition 2.3 (Skorohod problem)

Let $\alpha, \beta \in D[0, \infty)$ be such that $\alpha \leq \beta$. Given $x \in D[0, \infty)$ a pair of functions $(y, \eta) \in D[0, \infty) \times BV[0, \infty)$ is said to be a solution of the Skorohod problem on $[\alpha, \beta]$ for $x$ if the following two properties are satisfied:

1. $y_t = x_t + \eta_t \in [\alpha_t, \beta_t], \forall t \geq 0$. 

2. \( \eta(0^-) = 0 \), and \( \eta \) have the decomposition \( \eta := \eta^l - \eta^u \), where \( \eta^l, \eta^u \in I[0, \infty) \),

\[
\int_{0}^{\infty} I_{y_s < \beta_s} d\eta^u_s = \int_{0}^{\infty} I_{y_s > \alpha_s} d\eta^l_s = 0. \tag{2.11}
\]

A more general Skorohod problem is called Extended Skorohod Problem (ESP in short) firstly introduced by Ramanan \[16\] (see Definition 2.2 in \[17\]) which allows a pathwise construction of reflected Brownian motion that is not necessarily semimartingales. More recently, Burdzy et al. have shown in Theorem 5 in \[4\], that for any \( \alpha \in D^-[0, \infty) \) and \( \beta \in D^+[0, \infty) \) such that \( \alpha \leq \beta \), there is a well defined Extended Skorohod Map (ESM in short) \( \Gamma_{\alpha, \beta} \) represented by

\[
\Gamma_{\alpha, \beta}(x_t) = x_t - \Xi_{\alpha, \beta}(x_t).
\tag{2.12}
\]

where \( \Xi_{\alpha, \beta}(x) : D[0, \infty) \rightarrow D[0, \infty) \) is given by

\[
\Xi_{\alpha, \beta}(x_t) = \max \left\{ \left[ (x_0 - \beta_0) + \inf_{0 \leq r \leq t} (x_r - \alpha_r) \right] ; \\
\sup_{0 \leq s \leq t} \left[ (x_s - \beta_s) + \inf_{s \leq r \leq t} (x_r - \alpha_r) \right] \right\}. \tag{2.13}
\]

Note that if \((y, \eta)\) is a solution of the Skorohod problem (SP) on \([\alpha, \beta]\) for \(x\), then it is also a solution of extended Skorohod problem (ESP) on \([\alpha, \beta]\) for \(x\). However, the expression of the process \(\Xi\) is slightly complicate to handle.

In his paper, Slaby \[17\] provided an alternative formula (see (2.16)) for the two sided Skorohod map with time depended boundaries, that is easier to understand and has more interesting properties, especially the Lipschitz property of \(\Gamma_{\alpha, \beta}\). Those results are reminded below.

Let us introduce the following notations:

- Two pairs of times depending on \(x\):

\[
T_{\alpha} := \min \{ t > 0, / \alpha_t - x_t \geq 0 \}, \\
T_{\beta} := \min \{ t > 0, / x_t - \beta_t \geq 0 \}. \tag{2.14}
\]

- Two functions:

\[
H_{\alpha, \beta}(x_t) := \sup_{0 \leq s \leq t} \left\{ (x_s - \beta_s) \wedge \inf_{s \leq r \leq t} (x_r - \alpha_r) \right\}, \\
J_{\alpha, \beta}(x_t) := \inf_{0 \leq s \leq t} \left\{ (x_s - \alpha_s) \vee \sup_{s \leq r \leq t} (x_r - \beta_r) \right\}. \tag{2.15}
\]

**Corollary 2.2** Let \( \alpha \in D^-[0, \infty) \) and \( \beta \in D^+[0, \infty) \) be such that \( \inf_{t \geq 0} (\beta_t - \alpha_t) > 0 \). Then for every \( x \in D[0, \infty) \), we have:

\[
\Xi_{\alpha, \beta}(x_t) = I_{[T_{\beta} < T_{\alpha}]} I_{[T_{\beta} \infty]}(t) H_{\alpha, \beta}(x_t) + I_{[T_{\alpha} < T_{\beta}]} I_{[T_{\alpha} \infty]}(t) J_{\alpha, \beta}(x_t). \tag{2.16}
\]
Theorem 2.1  Lipschitz continuity
Under the same conditions of corollary 2.2, we have for any \( x, x' \in D[0, \infty) \)
\[
\| \Gamma_{\alpha, \beta}(x) - \Gamma_{\alpha, \beta}(x') \| \leq \| x - x' \|,
\]
where \( \| x \| = \sup_{0 \leq t \leq T} |x_t| \).

According to the doubly reflected BSDE (2.4) and the expression (2.5) of \( Y \), denote
\[
\Gamma_{L,U}(x_t) := Y_{T-t} , \quad \Xi_{L,U}(x_t) = K_{T-t} - K_T,
\]
where \( x_t = \xi + \int_{T-t}^T f_r dr - \int_{T-t}^{T_t} Z_r dW_r \).

It follows from (2.12) that \( K_t = \Xi_{L,U}(x_{T-t}) - \Xi_{L,U}(x_T) \) and more explicitly, we have
\[
K_t = I_{(T_U < T_L)} I_{[T_U, \infty)}(t) \left[ H_{L,U}(x_{T-t}) - H_{L,U}(x_T) \right] + I_{(T_L < T_U)} I_{[T_L, \infty)}(t) \left[ J_{L,U}(x_{T-t}) - J_{L,U}(x_T) \right].
\]

Proposition 2.1 The two continuous processes \( K^u \) and \( K^l \) are increasing. Moreover, the
measures \( dK^u \) and \( dK^l \) are carried by the sets \( \{ Y = U \} \) and \( \{ Y = L \} \) respectively.

Proof. We can write the process \( K^u \) as:
\[
K_t^u = \int_{0}^{t} I_{\{Y_r = U_r\}} dK_r.
\]
According to Skorohod condition in (2.4) we have
\[
\int_{0}^{T} (U_r - Y_r) dK_r^u = \int_{0}^{T} (U_r - Y_r) I_{\{Y_r = U_r\}} dK_r = 0,
\]
which means that the support of the measure \( dK^u \) is carried by the set \( \{ Y = U \} \). Similarly, the measure \( dK^l \) associated to the process \( K_t^l = \int_{0}^{t} I_{\{Y_r = L_r\}} dK_r \) is carried by the set \( \{ Y = L \} \).

From other side, if \( T_L < T_U \), then by (2.14) and Skorohod condition in (2.4) is the process \( K^l \) who plays a role in making \( Y \) above the barrier \( L \), and according to the explicit formula (2.18) of the process \( K = K^l - K^u \), we observe that
\[
K_t^l = I_{[T_L, \infty)}(t) \left[ \inf_{0 \leq s \leq T-t} \left( (x_s - L_s) \vee \sup_{s \leq r \leq T-t} (x_r - U_r) \right) \right.
\]
\[
\left. - \inf_{0 \leq s \leq T} \left( (x_s - L_s) \vee \sup_{s \leq r \leq T} (x_r - U_r) \right) \right],
\]
which is an increasing process.

If \( T_U < T_L \), then \( K^u \) plays the role in making \( Y \) below the barrier \( U \) such that :
\[
K_t^u = -I_{[T_U, \infty)}(t) \left[ \sup_{0 \leq s \leq T-t} \left( (x_s - U_s) \wedge \inf_{s \leq r \leq T-t} (x_r - L_r) \right) \right.
\]
\[
\left. - \sup_{0 \leq s \leq T} \left( (x_s - U_s) \wedge \inf_{s \leq r \leq T} (x_r - L_r) \right) \right],
\]
which is increasing. \( \square \)
3 Doubly reflected BSDEs with resistance

In this section, we prove existence and uniqueness of a class of doubly reflected BSDE with resistance, by constructing a Picard iteration. We formulate this class of BSDE as the following.

**Definition 3.1** A solution of BSDE with resistance reflected between lower barrier $L \in S^2$ and upper barrier $U \in S^2$ associated to $(\xi, f)$ is a quadruple $(Y, Z, K^l, K^u) \in \mathcal{D} := S^2 \times H^2_\delta \times \mathcal{A}^2 \times \mathcal{A}^2$ satisfying

(i) $(Y, Z, K^l, K^u)$ solves the following BSDE on $[0, T]$:

$Y_t = \xi + \int_t^T f(r, Y_r, Z_r, K^l_r - K^u_r)dr - \int_t^T Z_r dW_r + K^l_T - K^l_t - (K^u_T - K^u_t).$ \hspace{1cm} (3.1)

(ii) $L_t \leq Y_t \leq U_t$, a.s. a.e \hspace{1cm} \forall t \leq T.

(iii) Skorohod conditions hold:

$\int_0^T (U_r - Y_r) dK^u_r = \int_0^T (Y_r - L_r) dK^l_r = 0,$ \hspace{1cm} a.s. \hspace{1cm} (3.2)

According to (2.18), if $(Y, Z, K^l, K^u)$ is solution to reflected BSDE with resistance in the sense of Definition 3.1, then $K_t$ must be:

$K_t = I_{\{T^U < T_L\}} I_{[T^U, \infty)}(t) \left[ \sup_{0 \leq s \leq T-t} \left\{ (x_s - U_s) \wedge \inf_{0 \leq r \leq T-t} (x_r - L_r) \right\} - \sup_{0 \leq s \leq T} \left\{ (x_s - U_s) \wedge \inf_{0 \leq r \leq T} (x_r - L_r) \right\} \right] + I_{\{T_L < T^U\}} I_{[T_L, \infty)}(t) \left[ \inf_{0 \leq s \leq T} \left\{ (x_s - L_s) \vee \sup_{0 \leq r \leq T-t} (x_r - U_r) \right\} - \inf_{0 \leq s \leq T} \left\{ (x_s - L_s) \vee \sup_{0 \leq r \leq T} (x_r - U_r) \right\} \right],$ \hspace{1cm} (3.3)

where

$x_t = \xi + \int_{T-t}^T f(r, Y_r, Z_r, K^l_r - K^u_r)dr - \int_{T-t}^T Z_r dW_r,$ \hspace{1cm} (3.4)

**3.1 Existence and uniqueness By Picard iteration**

One approach to prove existence of the solution of reflected BSDE’s with two barriers, is to use the solution of Skorohod problem by constructing a Picard-type iterative procedure (See e.g. [7] or [8]) to the reflected BSDE with resistance. Throughout this section, we adapt the new method of [18] to our setting.

The following proposition state the mapping which leads to prove existence and uniqueness of the solution $(Y, Z, K^l, K^u)$. 


Proposition 3.1 Picard iteration

The mapping $\phi : \mathcal{D} := S^2 \times H^2_\mathbb{P} \times \mathcal{A}^2 \times \mathcal{A}^2 \rightarrow \mathcal{D}$ which associates $(Y, Z, K^l, K^u)$ to $\phi(Y, Z, K^l, K^u) = (\tilde{Y}, \tilde{Z}, \tilde{K}^l, \tilde{K}^u)$ is well defined. Moreover, the decomposition of $\tilde{Y}$ is given by:

$$\tilde{Y}_t = \xi + \int_t^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr - \int_t^T \tilde{Z}_r dW_r + \tilde{K}^o_T - \tilde{K}^u_T - (\tilde{K}^l_t - \tilde{K}^u_t). \quad (3.5)$$

**Proof.** To develop this iteration, we suppose that $(Y, Z, K^u, K^l) \in \mathcal{D}$, and $L \leq Y \leq U$, after the first iteration we obtain $(\tilde{Y}, \tilde{Z}, \tilde{K}^u, \tilde{K}^l) \in \mathcal{D}$, and according to (3.3)-(3.4), we define

$$\tilde{K}_t := \tilde{K}^l - \tilde{K}^u = I_{\{(T,U) < T_L\}} I_{[T,V,\infty)}(t) [H_{L,U}(x_{T-}) - H_{L,U}(x_T)] + I_{\{(T,L) < T_U\}} I_{[T,L,\infty)}(t) [J_{L,U}(x_{T-}) - J_{L,U}(x_T)], \quad (3.6)$$

where

$$x_t = \xi + \int_{T-t}^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr - \int_{T-t}^T Z_r dW_r. \quad (3.7)$$

Here $K^l$ and $K^u$ is replaced by the optional projections $K^b$ and $K^u_b$ respectively. To define $\tilde{Y}$, we first consider $\hat{Y}$ such that

$$\hat{Y}_t = \xi + \int_t^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr + \tilde{K}^l_t - \tilde{K}^u_t - (\tilde{K}^l_t - \tilde{K}^u_t) - \int_t^T Z_r dW_r, \quad (3.8)$$

However, $\hat{Y}$ is not necessary adapted, the reason for which we consider its optional projection $\hat{Y}^o := \hat{Y}$ in the sense of Definition 2.1 then

$$\tilde{Y}_t = \mathbb{E} \left\{ \xi + \int_t^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr + \tilde{K}^l_t - \tilde{K}^u_t - (\tilde{K}^l_t - \tilde{K}^u_t) / \mathcal{F}_t \right\}$$

$$= M_t - \int_0^t f(r, Y_r, Z_r, K^b_r - K^u_r)dr - (\tilde{K}^l_t - \tilde{K}^u_t),$$

where $M$ is a continuous martingale given by

$$M_t = \mathbb{E} \left\{ \xi + \int_0^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr + \tilde{K}^l_t - \tilde{K}^u_t / \mathcal{F}_t \right\}$$

$$- (\tilde{K}^l_t - \tilde{K}^u_t) + (\tilde{K}^u_t - \tilde{K}^l_t).$$

By martingale representation theorem, there exist a predictable process $\tilde{Z}$ such that

$$\tilde{Y}_t = \xi + \int_t^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr - \int_t^T \tilde{Z}_r dW_r + \tilde{K}^o_T - \tilde{K}^u_T - (\tilde{K}^l_t - \tilde{K}^u_t). \quad (3.9)$$

From Lipschitz property of $f$, and Proposition 2.1 we have $(\tilde{Y}, \tilde{Z}, \tilde{K}^l, \tilde{K}^u) \in \mathcal{D}$. Moreover, the mapping $Y \rightarrow \tilde{Y}$ preserves the constraint $L \leq \tilde{Y} \leq U$. □
Remark 3.1 The process $\tilde{K}^l$ and $\tilde{K}^u$ increase only on set $\{\tilde{Y} = L\}$ and $\{\tilde{Y} = U\}$ respectively.

We are going now to prove existence of the solution.

Theorem 3.1 Assume that $(Y, Z, K^l, K^u)$ is a fixed point of $\phi$, then $(Y, Z, K^l, K^u)$ is a solution of the reflected BSDE with resistance in the sense of Definition 3.1. Moreover, the processes $K^l$ and $K^u$ are adapted.

Proof. Since we suppose that $(Y, Z, K^l, K^u)$ is a fixed point of $\phi$, then $\phi(Y, Z, K^l, K^u) = (Y, Z, K^l, K^u)$ and according to Proposition 3.1 we have:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr - \int_t^T Z_r dW_r + K^{l}_r - K^{u}_r - (K^{l}_T - K^{u}_T)$$

Then by (3.6), it follows that

$$K^l_t - K^u_t = I_{\{T^u < T^l\}}I_{\{T^u, \infty\}}[H_{L,U}(x_{T-t}) - H_{L,U}(x_T)] + I_{\{T^l < T^u\}}I_{\{T^l, \infty\}}[J_{L,U}(x_{T-t}) - J_{L,U}(x_T)],$$

with

$$x_t = \xi + \int_{T-t}^T f(r, Y_r, Z_r, K^b_r - K^u_r)dr - \int_{T-t}^T Z_r dW_r.$$ (3.12)

By uniqueness of Skorohod equation, we have $K^{l_0} - K^{u_0} = K^l - K^u$, and since $K^{l_0}$ and $K^{l_0}$ are optional, it follows the adaptness of the process $K^l - K^u$, so that $K^l - K^u = K^b - K^u$.

In the following, we prove the main result in this paper, which state uniqueness of the solution in the sense of Theorem 2.3 in Peng and Xu [15].

Theorem 3.2 Assume Lipschitz continuity of $f$. There exists a unique quadriple $(Y, Z, K^l, K^u)$ in the space $\mathcal{D}$, solution to doubly reflected BSDE with resistance in the sense of Definition 3.1. The solution is unique in the following sense: if $(Y', Z', K'^l, K'^u)$ is another solution, then $Y \equiv Y'$, $Z \equiv Z'$, and $K^l - K^u \equiv K'^l - K'^u$, $\forall t \in [0, T]$.

Proof. In order to simplify the notation, we not $\tilde{K}^j = K^j - K'^j$ for $j = l, u$.

Let the space $\mathcal{D} := \mathcal{S}^2 \times H^{2, d} \times \mathcal{A}^2 \times \mathcal{A}^2$ be endowed with the norm:

$$\|(Y, Z, K^l, K^u) - (Y', Z', K'^l, K'^u)\|^2_{\alpha, \beta} := \|Y - Y'||^2_\alpha + \|Z - Z'||^2_\alpha + \beta \|\tilde{K}^l - \tilde{K}^u\|^2_\infty,$$

where $\alpha \geq 0$, $\beta \geq 0$ and

$$\|\tilde{K}^l - \tilde{K}^u\|^2_\infty := \sup_{0 \leq t \leq T} \mathbb{E}|\tilde{K}^l - \tilde{K}^u|^2$$

$$\|Y\|^2_\alpha := \int_0^T e^{\alpha r} \mathbb{E}|Y_r|^2 dr.$$
Let the mapping $\phi$ defined in proposition 3.1 be such that $\phi(Y, Z, K^l, K^u) = (\tilde{Y}, \tilde{Z}, \tilde{K}^l, \tilde{K}^u)$ and $\phi(Y', Z', K'^l, K'^u) = (\tilde{Y}', \tilde{Z}', \tilde{K}'^l, \tilde{K}'^u)$. From now, the proof will be divided into three steps.

**Step 1.** We show that:

$$
\|\tilde{Y} - \tilde{Y}'\|_\alpha^2 + \|\tilde{Z} - \tilde{Z}'\|_\alpha^2 \leq \frac{2L_1}{\gamma_1} (\|Y - Y'\|_\alpha^2 + \|Z - Z'\|_\alpha^2) + \frac{2L_2}{\gamma_2} \|\tilde{K}^b - \tilde{K}'^b\|_\alpha^2.  \tag{3.13}
$$

We applied Itô’s formula to $e^{at}(\tilde{Y}_t - \tilde{Y}'_t)^2$, and taking its expectation we have,

$$
\mathbb{E} e^{at}(\tilde{Y}_t - \tilde{Y}'_t)^2 = -\alpha \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)^2 dr - \mathbb{E} \int_t^T e^{ar}|\tilde{Z}_r - \tilde{Z}'_r|^2 dr
$$

$$
+ 2 \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r^o - \tilde{K}'_r^o)
$$

$$
+ 2 \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)[f(r, Y_r, Z_r, K^b_r) - f(r, Y'_r, Z'_r, K'^b_r)] dr,  \tag{3.14}
$$

where $\tilde{K}^o = \tilde{K}^o - \tilde{K}^u$ and $\tilde{K}^b = \tilde{K}^b - \tilde{K}^u$. Now, since $\tilde{Y}$ and $\tilde{Y}'$ are optional, then by Definition 2.2

$$
\mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r^o - \tilde{K}'_r^o) = \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r - \tilde{K}'_r)
$$

$$
= \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r^l - \tilde{K}_r^u - \tilde{K}_r^o + \tilde{K}_r^o').
$$

Observe that

$$
\mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r^l - \tilde{K}_r^u) = \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - L_r)d\tilde{K}_r^l - \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - L_r)d\tilde{K}_r^u
$$

$$
- \mathbb{E} \int_t^T e^{ar}(\tilde{Y}'_r - L_r)d\tilde{K}_r^l + \mathbb{E} \int_t^T e^{ar}(\tilde{Y}'_r - L_r)d\tilde{K}_r^u
$$

$$
\leq \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - L_r)d\tilde{K}_r^l + \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - L_r)d\tilde{K}_r^u,
$$

and similarly

$$
-\mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - \tilde{Y}'_r)d(\tilde{K}_r^o - \tilde{K}_r^o')
$$

$$
= -\mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - U_r)d\tilde{K}_r^o + \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - U_r)d\tilde{K}_r^o'
$$

$$
+ \mathbb{E} \int_t^T e^{ar}(\tilde{Y}'_r - U_r)d\tilde{K}_r^o - \mathbb{E} \int_t^T e^{ar}(\tilde{Y}'_r - U_r)d\tilde{K}_r^o'
$$

$$
\leq -\mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - U_r)d\tilde{K}_r^o - \mathbb{E} \int_t^T e^{ar}(\tilde{Y}_r - U_r)d\tilde{K}_r^o'.
$$
Since $\tilde{K}^t$ is increasing and $\tilde{Y}$ is the optional projection of $\hat{Y}$, then by Theorem 4.16 in [13], and Remark 3.1

$$
\mathbb{E} \int_t^T e^{\alpha r} (\hat{Y}_r - L_r) d\tilde{K}^t_r = \mathbb{E} \int_t^T e^{\alpha r} (\hat{Y}_r - L_r) d\tilde{K}_r^t = 0.
$$

Similarly, we show that $\mathbb{E} \int_t^T e^{\alpha r} (\hat{Y}_r' - L_r) d\tilde{K}^t_r = \mathbb{E} \int_t^T e^{\alpha r} (\hat{Y}_r' - U_r) d\tilde{K}_r^u = \mathbb{E} \int_t^T e^{\alpha r} (\hat{Y}_r' - U_r) d\tilde{K}_r^u = 0.$

Plugging this in to (3.14), and using the Lipschitz continuity of $f$,

$$
\mathbb{E} \left( e^{\alpha t} (\hat{Y}_t - \hat{Y}_t')^2 \right) \leq -\alpha \int_t^T \mathbb{E} \left( e^{\alpha r} (\hat{Y}_r - \hat{Y}_r')^2 \right) dr - \mathbb{E} \int_t^T e^{\alpha r} |\tilde{Z}_r - \tilde{Z}_r'| dr
$$

$$
+ 2L_1 \int_t^T e^{\alpha r} \mathbb{E} \left( |\hat{Y}_r - \hat{Y}_r'| (|Y_r - Y_r'| + |Z_r - Z_r'|) \right) dr
$$

$$
+ 2L_2 \int_t^T e^{\alpha r} \mathbb{E} \left( |\hat{Y}_r - \hat{Y}_r'| |K_r^\alpha - K_r'^\alpha| \right) dr.
$$

Using the fact that $ab \leq \gamma a^2 + \frac{1}{\gamma} b^2$ for some positive constant $\gamma$:

$$
\mathbb{E} \left( e^{\alpha t} (\hat{Y}_t - \hat{Y}_t')^2 \right) \leq - (\alpha - \gamma_1 L_1 - \gamma_2 L_2) \int_t^T \mathbb{E} \left( e^{\alpha r} (\hat{Y}_r - \hat{Y}_r')^2 \right) dr
$$

$$
+ \frac{2L_1}{\gamma_1} \int_t^T e^{\alpha r} \mathbb{E} \left( |Y_r - Y_r'|^2 + |Z_r - Z_r'|^2 \right) dr
$$

$$
+ \frac{2L_2}{\gamma_2} \int_t^T e^{\alpha r} |K_r^\alpha - K_r'^\alpha|^2 dr - \mathbb{E} \int_t^T e^{\alpha r} |\tilde{Z}_r - \tilde{Z}_r'|^2 dr. \quad (3.15)
$$

The result follows for $\alpha = \gamma_1 L_1 + \gamma_2 L_2$ and $t = 0$.

**Step 2.** We show that

$$
\|\tilde{K}_t - \tilde{K}_t'\|_\infty \leq 45(T L_1^2 + C_b) \left( \|Y - Y'\|_0^2 + \|Z - Z'\|_0^2 \right) + 45T L_2^2 \|K^\alpha - K'^\alpha\|^2_\infty. \quad (3.16)
$$

where $C_b$ is the constant appearing in Burkholder-Davis-Gundy inequality.

From Skorohod equation (2.12), equation (3.8) and the fact that $K_t = \Xi_{L,U}(x_{T-t}) - \Xi_{L,U}(x_T)$, we have:

$$
\tilde{K}_t = - \int_0^t f(r, Y_r, Z_r, K_r^\alpha) dr + \int_0^t Z_r dW_r + \Gamma_{L,U}(x_T) - \Gamma_{L,U}(x_{T-t})
$$

$$
\tilde{K}_t' = - \int_0^t f(r, Y'_r, Z'_r, K_r'^\alpha) dr + \int_0^t Z'_r dW_r + \Gamma_{L,U}(x'_T) - \Gamma_{L,U}(x'_{T-t}),
$$

where

$$
x_t = \xi + \int_{T-t}^T f(r, Y_r, Z_r, K_r^\alpha) dr - \int_{T-t}^T Z_r dW_r,
$$

$$
x'_t = \xi + \int_{T-t}^T f(r, Y'_r, Z'_r, K_r'^\alpha) dr - \int_{T-t}^T Z'_r dW_r.
$$
It follows
\[
|\tilde{K}_t - \tilde{K}'_t| \leq \left| \int_0^t \left[ f(r, Y_r, Z_r, K_r^\psi) - f(r, Y'_r, Z'_r, K'_r^\psi) \right] dr \right| \\
+ |\Gamma_{L,U}(x_T) - \Gamma_{L,U}(x'_T)| + |\Gamma_{L,U}(x_{T-t}) - \Gamma_{L,U}(x'_{T-t})| \\
+ \left| \int_0^t [Z_r - Z'_r] dW_r \right|.
\]

Then
\[
|\tilde{K}_t - \tilde{K}'_t|^2 \leq 3T \int_0^T \left| f(r, Y_r, Z_r, K_r^\psi) - f(r, Y'_r, Z'_r, K'_r^\psi) \right|^2 dr \\
+ 6\| \Gamma_{L,U}(x) - \Gamma_{L,U}(x') \|^2 + 3 \left| \int_0^T [Z_r - Z'_r] dW_r \right|^2.
\] (3.17)

From other side, we have by Theorem 2.1
\[
\| \Gamma_{L,U}(x) - \Gamma_{L,U}(x') \|^2 \leq \left| \sup_{0 \leq t \leq T} |x_t - x'_t| \right|^2 \\
\leq 2T \int_0^T \left| f(r, Y_r, Z_r, K_r^\psi) - f(r, Y'_r, Z'_r, K'_r^\psi) \right|^2 dr \\
+ 2 \left| \int_0^T [Z_r - Z'_r] dW_r \right|^2.
\]

Plugging this in (3.17), and taking expectation in both hand side, we have:
\[
\mathbb{E}[\tilde{K}_t - \tilde{K}'_t] \leq 15T\mathbb{E} \int_0^T \left| f(r, Y_r, Z_r, K_r^\psi) - f(r, Y'_r, Z'_r, K'_r^\psi) \right|^2 dr \\
+ 15\mathbb{E} \left| \int_0^T [Z_r - Z'_r] dW_r \right|^2.
\] (3.18)

Taking the supremum on $t$ over $[0, T]$, using Lipschitz continuity of $f$, Jensen inequality together with Burkholder-Davis-Gundy inequality:
\[
\| \tilde{K} - \tilde{K}' \|_\infty^2 \leq 15T \mathbb{E} \int_0^T \left| f(r, Y_r, Z_r, K_r^\psi) - f(r, Y'_r, Z'_r, K'_r^\psi) \right|^2 dr \\
+ 15\mathbb{E} \left[ \left| \int_0^T [Z_r - Z'_r] dW_r \right|^2 \right] \\
\leq 45(TL_1^2 + C_b) \mathbb{E} \int_0^T (|Y_r - Y'_r|^2 + |Z_r - Z'_r|^2) dr \\
+ 45TL_2^2 \mathbb{E} \int_0^T |K_r^\psi - K'_r^\psi|^2 dr \\
\leq 45(TL_1^2 + C_b) (\| Y - Y' \|_0^2 + \| Z - Z' \|_0^2) + 45TL_2^2 \| K^\psi - K'^\psi \|_\infty^2.
\]
Step 3.
Since we have:
\[ \|K^b - K'^b\|_2^2 \leq \frac{e^{\alpha T} - 1}{\alpha} \|K - K'\|_\infty^2. \]  
(3.19)

Combining (3.13) and (3.16), leads to:
\[ \|\tilde{Y} - \tilde{Y}'\|_2^2 + \|\tilde{Z} - \tilde{Z}'\|_2^2 + \beta \|\tilde{K} - \tilde{K}'\|_\infty^2 \leq \left( \frac{2L_1}{\gamma_1} + 45\beta(TL_1^2 + C_b) \right)(\|Y - Y'\|_0 + \|Z - Z'\|_0) \]
\[ + \left( \frac{2L_2(e^{\alpha T} - 1)}{\alpha \beta \gamma_2} + 45TL_2^2 \right)\beta \|K - K'\|_\infty. \]  
(3.20)

We set \( \gamma_1 = 2L_1^{-1}, \gamma_2 = 2L_2^{-1}, \alpha = 5 \) and we choose \( L_1, L_2 \) small enough such that
\[ L_1 < \sqrt{\frac{1}{2} - \frac{C_b}{45T\beta}}, \quad L_2 < \sqrt{\frac{1}{2} \times \frac{5\beta}{e^{5T} + 225\beta - 1}}. \]

Then
\[ \|\tilde{Y}, \tilde{Z}, \tilde{K}^l, \tilde{K}^u\|_{\alpha, \beta}^2 \leq \frac{1}{2} \|(Y, Z, K^l, K^u) - (Y', Z', K'^l, K'^u)\|_{\alpha, \beta}^2. \]

So, the mapping is a contraction, and there is a fixed point \((Y, Z, K^l, K^u)\), which is the solution. \( \square \)

4 Continuous dependence

Our formulation of doubly reflected BSDE with resistance permits us to derive the following continuous dependence theorem.

**Proposition 4.2** Under the same assumptions in Theorem 3.2, Let \((Y^i, Z^i, K^i, K^{iu})\) with \(i = 1, 2\), be solution of the following DRBSDE:
\[ Y^i_t = \xi^i + \int_t^T f(r, Y^i_r, Z^i_r, K^i_r)dr - \int_t^T Z^i_r dW_r + K^i_T - K^i_T - (K^{iu}_T - K^{iu}_T) \]
with two obstacles \(L\) and \(U\), where \(K^i := K^i - K^{iu}\), in the sense of Definition 3.1. Then we have
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^1_t - Y^2_t|^2 \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| K^{1l}_t - K^{2l}_t - (K^{1u}_t - K^{2u}_t) \right|^2 \right] \]
\[ + \mathbb{E} \left[ \int_0^T |Z^1_r - Z^2_r|^2 dr \right] \leq C \mathbb{E} \left[ |\xi^1 - \xi^2|^2 \right]. \]

The constant \(C\) depends only on \(L_1, L_2\) and \(T\).
**Proof.** We set \( \hat{Y} = Y^1 - Y^2 \), \( \hat{Z} = Z^1 - Z^2 \), \( \hat{K}^l = K^{1l} - K^{2l} \), \( \hat{K}^u = K^{1u} - K^{2u} \), \( \hat{K} = K^1 - K^2 \), \( \hat{\xi} = \xi^1 - \xi^2 \), \( \hat{f}_r = f(r, Y^1, Z^1, K^1) - f(r, Y^2, Z^2, K^2) \), we have

\[
\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f}_r dr - \int_t^T \hat{Z}_r dW_r + \hat{K}^l_T - \hat{K}^l_t - (\hat{K}^u_T - \hat{K}^u_t)
\]  

(4.21)

Apply Itô Formula to \( |\hat{Y}_t|^2 \), then

\[
|\hat{Y}_t|^2 + \int_t^T |\hat{Z}_r|^2 dr = |\hat{\xi}|^2 + 2 \int_t^T \hat{Y}_r \hat{f}_r dr + 2 \int_t^T \hat{Y}_r \hat{K}^l_r - 2 \int_t^T \hat{Y}_r \hat{Z}_r dW_r
\]  

(4.22)

We first observe that

\[
\int_t^T (Y^1_r - L_s) dK^{1l}_r = \int_t^T (Y^2_r - L_r) dK^{2l}_r = 0
\]

\[
\int_t^T (Y^1_r - U_r) dK^{1u}_r = \int_t^T (Y^2_r - U_r) dK^{2u}_r = 0
\]  

(4.23)

Thus

\[
\int_t^T \hat{Y}_r d\hat{K}^l_r = \int_t^T (Y^1_r - L_r) dK^{1l}_r + \int_t^T (L_r - Y^2_r) dK^{1l}_r + \int_t^T (Y^2_r - L_r) dK^{2l}_r + \int_t^T (L_r - Y^1_r) dK^{2l}_r
\]

\[
\leq 0
\]  

(4.24)

And

\[
\int_t^T \hat{Y}_r d\hat{K}^u_r = \int_t^T (Y^1_r - U_r) dK^{1u}_r + \int_t^T (U_r - Y^2_r) dK^{1u}_r + \int_t^T (Y^2_r - U_r) dK^{2u}_r + \int_t^T (U_r - Y^1_r) dK^{2u}_r
\]

\[
\geq 0
\]  

(4.25)

Applying this to the equation (4.22) we obtain:

\[
|\hat{Y}_t|^2 + \int_t^T |\hat{Z}_r|^2 dr \leq |\hat{\xi}|^2 + 2 \int_t^T \hat{Y}_r \hat{f}_r dr - 2 \int_t^T \hat{Y}_r \hat{Z}_r dW_r
\]  

(4.26)
By Lipschitz condition of $f$ we have:

$$|\hat{Y}_t|^2 + \int_t^T |\hat{Z}_r|^2 dr \leq |\hat{\xi}|^2 + 2 \int_t^T \hat{Y}_r (L_1(|\hat{Y}_r| + |\hat{Z}_r|) + L_2|\hat{K}_r|) dr$$

$$- 2 \int_t^T \hat{Y}_r \hat{Z}_r dW_r$$

$$\leq |\hat{\xi}|^2 + (2L_1 + \alpha L_1^2 + \beta) \int_t^T |\hat{Y}_r|^2 dr + \frac{1}{\alpha} \int_t^T |\hat{Z}_r|^2 dr$$

$$+ \frac{L_2^2}{\beta} \int_t^T |\hat{K}_r|^2 dr - 2 \int_t^T \hat{Y}_r \hat{Z}_r dW_r$$

Set $\alpha = 2$ we have

$$E \left[ |\hat{Y}_t|^2 \right] \leq E \left[ |\hat{\xi}|^2 \right] + (2L_1 + 2L_1^2 + \beta) E \int_t^T |\hat{Y}_r|^2 dr + \frac{TL_2^2}{\beta} E \left[ \sup_{0 \leq t \leq T} \hat{K}_r^2 \right]$$

By Gronwall Lemma we have:

$$E \left[ |\hat{Y}_t|^2 \right] \leq e^{2L_1 + 2L_1^2 + \beta} E \left[ |\hat{\xi}|^2 + \frac{TL_2^2}{\beta} \sup_{0 \leq t \leq T} \hat{K}_r^2 \right]$$

it follows that

$$E \left[ |\hat{Y}_t|^2 + \int_t^T |\hat{Z}_r|^2 dr \right] \leq e^{2L_1 + 2L_1^2 + \beta} E \left[ |\hat{\xi}|^2 + \frac{TL_2^2}{\beta} \sup_{0 \leq t \leq T} \hat{K}_r^2 \right]$$

Using Burkholder-Davis-Gundy inequality, we have

$$E \left[ \sup_{0 \leq t \leq T} |\hat{Y}_t|^2 + \int_0^T |\hat{Z}_r|^2 dr \right] \leq e^{2L_1 + 2L_1^2 + \beta} E \left[ |\hat{\xi}|^2 + \frac{TL_2^2}{\beta} \sup_{0 \leq t \leq T} \hat{K}_r^2 \right]$$

(4.27)

From other side, we have by equation (4.21):

$$\hat{K}_t = \hat{Y}_0 - \hat{Y}_t - \int_0^t \hat{f}_r dr + \int_0^t \hat{Z}_r dW_r$$

Using again Burkholder-Davis-Gundy inequality, Jensen inequality and (4.27):

$$\sup_{0 \leq t \leq T} |\hat{K}_t|^2 \leq C \left( \sup_{0 \leq t \leq T} |\hat{Y}_t|^2 + T \int_0^T \left[ L_1^2 (|\hat{Y}_r|^2 + |\hat{Z}_r|^2) + L_2^2 |\hat{K}_r|^2 \right] dr + \sup_{0 \leq t \leq T} \left( \int_0^t |\hat{Z}_r dW_r| \right)^2 \right)$$

Set $\beta = 1$, leads to

$$E \sup_{0 \leq t \leq T} |\hat{K}_t|^2 \leq C e^{2L_1 + 2L_1^2 + \beta (2 + TL_1^2 + T^2 L_1^2)} E[\hat{\xi}^2]$$

$$+ C T L_2^2 (e^{2L_1 + 2L_1^2 + \beta (2 + TL_1^2 + T^2 L_1^2)} + 1) E \sup_{0 \leq t \leq T} |\hat{Y}_t|^2$$

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Choosing $L_1$ and $L_2$ small enough we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} |\hat{K}_t|^2 \leq Ce^{2L_1+2L_2^2+\beta}(2 + TL_1^2 + T^2L_2^2)\mathbb{E}[\xi^2]
\]
Plugging this in (4.27), we get the required result.

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