COHOMOLOGICAL MILNOR FORMULA AND NON-ACYCLICITY CLASSES
FOR CONSTRUCTIBLE SHEAVES

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Abstract. For a separated morphism $f : X \to Y$ of finite type between Noetherian schemes and a constructible sheaf $\mathcal{F}$ on $X$, we construct a cohomological characteristic class supported on the non-locally acyclic locus of $f$ relatively to $\mathcal{F}$. We propose a conjectural formula to relate this class with its classical version and show this formula under certain transversality condition. We prove a pull-back and a proper push-forward properties for this class.

As applications, we prove cohomological analogues of the Milnor formula [Sai17, Invent.Math.2017, Theorem 5.9], and of the conductor formula [Sai21, J.Amer.Math.Soc.2021, Theorem 2.2.3].

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1. Introduction

1.1. Let $k$ be a field of characteristic $p$. Let $\Lambda$ be a Noetherian ring such that $m\Lambda = 0$ for some integer $m$ which is prime to $p$. Let $g : Y \to S$ be a smooth morphism over $k$ of relative dimension $r$ and $f : X \to Y$ a separated morphism of finite type. Let $\mathcal{K}_{X/Y} = Rf^!\Lambda$ and $\mathcal{F}$ a constructible étale sheaf of $\Lambda$-modules on $X$. The (cohomological) characteristic class $C_{X/k}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/k})$ is introduced by Abbes and Saito in [AS07] by using Verdier pairing (cf. [SGA5])$^1$. In order to obtain a relative version, it is necessary to add some transversality conditions. In fact, if $f : X \to Y$ is (universally) locally acyclic relatively to $\mathcal{F}$, the relative (cohomological) characteristic class $C_{X/Y}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/Y})$ is defined in [YZ21, Definition 3.6] (and [LZ22, 2.20] in general). The following theorem reveals a relation between the relative characteristic class and its classical version (if one takes $S = \text{Spec} k$).

Theorem 1.2 (Fibration formula: transversal case, Theorem 6.4). If $f : X \to Y$ is (universally) locally acyclic relatively to $\mathcal{F}$, then we have

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^1_{Y/S}) \cap C_{X/Y}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/S}).$$

\footnotesize
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$^1$See also [KS90] for a transcendental setting.
Here the top Chern class \( c_r(f^*\Omega_{Y/S}^{1,\vee}) \in H^{2r}(X, \Lambda(r)) \) defines a map

\[
H^0(X, \mathcal{K}_{X/Y}) \xrightarrow{c_r(f^*\Omega_{Y/S}^{1,\vee})} H^0(X, \mathcal{K}_{X/S}).
\]

In the case that \( f = \text{id} \), the cohomology sheaves of \( \mathcal{F} \) must be locally constant on \( X \), and the formula (1.2.1) is just the (relative) Gauss-Bonnet-Chern formula, i.e., \( C_{X/S}(\mathcal{F}) = \text{rank}\mathcal{F} \cdot c_r(\Omega_{X/S}^{1,\vee}) \).

1.3. If \( f : X \to Y \) is not locally acyclic relatively to \( \mathcal{F} \), the formula (1.2.1) does not hold in general (even \( C_{X/Y}(\mathcal{F}) \) can be defined under some special conditions). We introduce two cohomological class \( C_{X/Y/S}^{\text{int}}(\mathcal{F}) \) and \( C_{X/Y/S}^{\text{nt}}(\mathcal{F}) \) in Definition 4.5, which measures the difference \( C_{X/S}(\mathcal{F}) - c_r(f^*\Omega_{Y/S}^{1,\vee}) \cap C_{X/Y}(\mathcal{F}) \). More importantly, these two classes are supported on the non-locally acyclic locus of \( f \) relatively to \( \mathcal{F} \). To be precisely, let \( Z \subseteq X \) be a closed subscheme such that \( X \setminus Z \to Y \) is universally locally acyclic relatively to \( \mathcal{F}|_{X\setminus Z} \) and that \( X \to S \) is universally locally acyclic relatively to \( \mathcal{F} \). If \( H^0_Z(X, \mathcal{K}_{X/Y}) = 0 \) and \( H^1_Z(X, \mathcal{K}_{X/Y}) = 0 \), then \( C_{X/Y/S}^{\text{nt}}(\mathcal{F}) \) and \( C_{X/Y/S}^{\text{int}}(\mathcal{F}) \) define the same class in \( H^0_{Z}(X, \mathcal{K}_{X/S}) \). In this case, the relative cohomological characteristic class \( C_{X/Y}(\mathcal{F}) \) is also well-defined (cf. Definition 3.4). Theorem 1.2 implies that the pull-back of \( C_{X/S}(\mathcal{F}) - c_r(f^*\Omega_{Y/S}^{1,\vee}) \cap C_{X/Y}(\mathcal{F}) \) to \( X\setminus Z \) is zero, hence \( C_{X/S}(\mathcal{F}) - c_r(f^*\Omega_{Y/S}^{1,\vee}) \cap C_{X/Y}(\mathcal{F}) \) is a class supported on \( Z \). We conjecture that the following formula should hold.

**Conjecture 1.4** (Fibration formula, Conjecture 4.7 and Conjecture 6.7). Let \( Z \subseteq X \) be a closed subscheme such that \( H^0_{Z}(X, \mathcal{K}_{X/Y}) = 0 \) and \( H^1_{Z}(X, \mathcal{K}_{X/Y}) = 0 \). Then for any \( \mathcal{F} \in D^b_{Z}(X, \Lambda) \) such that \( X \setminus Z \to Y \) is universally locally acyclic relatively to \( \mathcal{F}|_{X\setminus Z} \) and that \( X \to S \) is universally locally acyclic relatively to \( \mathcal{F} \), we have

\[
(1.4.1) \quad C_{X/S}(\mathcal{F}) = c_r(f^*\Omega_{Y/S}^{1,\vee}) \cap C_{X/Y}(\mathcal{F}) + C_{X/Y/S}^{\text{nt}}(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}).
\]

We call \( C_{X/Y/S}^{\text{nt}}(\mathcal{F}) \) the non-acylicity class of \( \mathcal{F} \). If \( Z \) is empty, then the conjecture is Theorem 1.2. For the case that \( f = \text{id} \) and \( S = \text{Spec} k \). Since \( \text{id} : X\setminus Z \to X\setminus Z \) is universally locally acyclic relatively to \( \mathcal{F}|_{X\setminus Z} \), the cohomology sheaves of \( \mathcal{F}|_{X\setminus Z} \) are locally constant on \( X\setminus Z \). Then (1.4.1) is just the Abbes-Saito’s localization formula [AS07, Proposition 5.2.3].

1.5. In the case where \( S = \text{Spec} k \) is an algebraically closed field, \( X \) is projective and smooth, and \( f : X \to Y \) is a good fibration over a projective smooth curve (cf. 6.5), then the formula (1.4.1) follows from the global index formula [Sai17, Theorem 7.13] together with [UYZ20, Proposition 5.3.7]. This hints that there should have a cohomological description of the total dimension of vanishing cycles at isolated non-locally acyclic points. Indeed, we have

**Theorem 1.6** (Theorem 4.19). Let \( Y \) be a smooth curve over a perfect field \( k \) of characteristic \( p > 0 \). Let \( f : X \to Y \) be a separated morphism of finite type and \( x \in |X| \) a closed point. Let \( \mathcal{F} \in D^b_{X}(X, \Lambda) \) such that \( f|_{X\setminus \{x\}} \) is universally locally acyclic relatively to \( \mathcal{F}|_{X\setminus \{x\}} \). Then we have

\[
(1.6.1) \quad C_{X/Y/k}^{\text{nt}}(\mathcal{F}) = -\text{dim}_{\text{tot}} R\Phi_k(\mathcal{F}, f) \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}),
\]

where \( R\Phi(\mathcal{F}, f) \) is the complex of vanishing cycles.

The above theorem gives a cohomological analogue of the Milnor formula proved by Saito in [Sai17, Theorem 5.9], and does not involve the smoothness assumption on \( X \). Its proof is based on a pull-back property of the non-acyclicity class \( C_{X/Y/S}^{\text{nt}}(\mathcal{F}) \) (cf. Proposition 4.17) together with the method in [Abe22]. The construction of non-acyclicity classes in this paper can be applied to motivic categories with six functor formalism. If there is a generalization argument as [Abe22] to
the $p$-adic cohomology theory (cf. [Abe22, 3.2]), we could expect that the construction will lead to a solution of Deligne’s conjecture on the Milnor formula in the mixed case [Del72, Conjecture 1.9].

As an application of Theorem 1.6, we have the following cohomological version of Grothendieck-Ogg-Shafarevich formula:

**Corollary 1.7** (Corollary 4.20). Let $X$ be a smooth and connected curve over a perfect field $k$ of characteristic $p > 0$. Let $F \in D^b_c(X, \Lambda$) and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $F|_{X\setminus Z}$ are locally constant. Then we have

$$C_{X/k}(F) = \text{rank}_F \cdot c_r(\Omega^{1,\vee}_X) - \sum_{x \in Z} a_x(F) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}),$$

where $a_x(F) = \text{rank}_F|_\eta - \text{rank}_F + \text{Sw}_xF$ is the Artin conductor and $\eta$ is the generic point of $X$.

1.8. We also prove a proper push-forward property for non-acyclicity classes in Theorem 5.12. But unlike the pull-back property, we have to use the class $C_{X/Y/S}^{\text{nt,} \delta}$. As an application, we prove a cohomological version of the conductor formula [Sai21, Theorem 2.2.3] (also of Bloch’s conjecture on the conductor formula for constant sheaves [Blo87]).

**Theorem 1.9** (Theorem 5.14). Let $Y$ be a smooth connected curve over a perfect field $k$ of characteristic $p > 0$. Let $\Lambda$ be a Noetherian ring such that $m\Lambda = 0$ for some integer $m$ which is prime to $p$. Let $f : X \to Y$ be a proper morphism over $k$ and $y \in |Y|$ a closed point. Let $F \in D^b_c(X, \Lambda)$. Assume that $f|_{X\setminus Y}$ is universally locally acyclic relatively to $F|_{X\setminus Y}$. Then we have

$$f_* C_{X/Y/k}^{\text{nt,} \delta}(F) = -a_y(Rf_*F) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}).$$

Here the class $C_{X/Y/k}^{\text{nt,} \delta}(F)$ is defined in Definition 4.5.

In [Sai21, Theorem 2.2.3], Saito assumes that $f$ is projective since the global index formula for the characteristic cycle is only known for projective schemes. However using a proper push-forward property for cohomological characteristic class, we could get rid of this assumption in Corollary 1.9. This is one of the advantages by using cohomological argument.

1.10. Let us explain one of the main motivation for proposing Conjecture 1.4. Using ramification theory in [AS07], Abbes and Saito calculate the cohomological characteristic classes for rank 1 sheaf under certain ramification conditions. Later, after successfully defining characteristic cycle, Saito [Sai17, Conjecture 6.8.1] proposes a conjecture to compute the cohomological characteristic classes in terms of the characteristic cycles. Corollary 1.7 confirms the curve case of Saito’s conjecture. In Theorem 6.10, we show that Conjecture 1.4 implies the projective case of Saito’s conjecture by using fibration method.

1.11. This article is organized as follows. In Section 2, we first introduce the definition of transversality conditions, which is crucial for defining $C_{X/Y/S}^{\text{nt}}(F)$. In Section 3, we review the construction of the relative characteristic class and prove that its formation is compatible with base change. In Section 4, we construct the non-acyclicity classes, and formulate the Conjecture 1.4. We prove a pull-back property in Proposition 4.17 and show prove Theorem 1.6. In Section 5, we prove a proper push-forward property for non-acyclicity classes in Theorem 5.12 and deduce the cohomological conductor formula in Theorem 5.14 as an application. In Section 6, we first prove Theorem 1.2 and then explain why we call (1.4.1) the fibration formula.
Acknowledgments. We would like to thank Tomoyuki Abe for helpful discussion. In the previous version of the paper, Theorem 1.6 is stated as a conjecture. After discussing, he suggests us to prove a pullback formula for non-acyclicity classes, which will imply Theorem 1.6 by a similar argument in [Abe22]. We thank Takeshi Saito for suggesting the terminology “non-acyclicity class”. We also thank Haoyu Hu and Fangzhou Jin for comments. This work was partially supported by the National Key R&D Program of China (Grant No.2021YFA1001400), NSFC Grant No.11901008 and NSFC Grant No.12271006.

Notation and Conventions.

(1) Let $S$ be a Noetherian scheme and Sch$_S$ the category of separated schemes of finite type over $S$.

(2) Let $\Lambda$ be a Noetherian ring such that $m\Lambda = 0$ for some integer $m$ invertible on $S$.

(3) For any scheme $X \in$ Sch$_S$, we denote by $D^b_c(X, \Lambda)$ the derived category of bounded complexes of $\Lambda$-modules with constructible cohomology groups on $X$.

(4) For any morphism $f : X \to Y$ in Sch$_S$, we denote by $D^b_c(X/Y, \Lambda) \subseteq D^b_c(X, \Lambda)$ the full subcategory generated by objects $\mathcal{F}$ such that $f$ is universally locally acyclic relatively to $\mathcal{F}$ (also say $\mathcal{F}$ is universally locally acyclic over $Y$). We define $K_0(X/Y, \Lambda)$ to be the Grothendieck group of $D^b_c(X/Y, \Lambda)$.

(5) For any separated morphism $f : X \to Y$ in Sch$_S$, we use the following notation

$$K_{X/Y} = Rf^!\Lambda, \quad D_{X/Y}(-) = R\text{Hom}(-, K_{X/Y}).$$

(6) For $\mathcal{F} \in D^b_c(X, \Lambda)$ and $\mathcal{G} \in D^b_c(Y, \Lambda)$ on $S$-schemes $X$ and $Y$ respectively, $\mathcal{F} \boxtimes^L \mathcal{G}$ denotes $\text{pr}_1^*\mathcal{F} \otimes^L \text{pr}_2^*\mathcal{G}$ on $X \times_S Y$.

(7) To simplify our notation, we omit to write $R$ or $L$ to denote the derived functors unless otherwise stated explicitly or for $R\text{Hom}$.

2. Transversality condition

2.1. Saito studies the transversality condition in [Sai17], which is closely related to universally locally acyclicity. For our purpose, we define the following generalized version. Consider the following cartesian diagram in Sch$_S$

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow p & & \downarrow f \\
W & \rightarrow & T.
\end{array}
$$

Let $\mathcal{F} \in D^b_c(Y, \Lambda)$ and $\mathcal{G} \in D^b_c(T, \Lambda)$. We define a morphism

$$c_{S, f, \mathcal{F}, \mathcal{G}} : i^*\mathcal{F} \otimes^L p^*\delta^!\mathcal{G} \rightarrow i^!(\mathcal{F} \otimes^L f^*\mathcal{G})$$

to be the adjunction of the composition

$$
\begin{align*}
\text{proj-formula} & \quad \simeq \\
\text{base change} & \quad \simeq \\
\simeq & \quad \mathcal{F} \otimes^L f^*\delta^!\mathcal{G}
\end{align*}
$$

We also put $c_{S, f, \mathcal{F}} := c_{S, f, \mathcal{F}, \Lambda} : i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \rightarrow i^!\mathcal{F}$. If $c_{S, f, \mathcal{F}}$ is an isomorphism, then we say the cartesian diagram (2.1.1) is $\mathcal{F}$-transversal (and also say that the morphism $\delta$ is $\mathcal{F}$-transversal). In (2.1.1), if we take $f = \text{id}$, then we obtain Saito’s definition [Sai17, Definition 8.5]. The following proposition generalizes [Sai17, Corollary 8.10].
Proposition 2.2. Consider the cartesian diagram (2.1.1). Assume that $\delta$ is an immersion and that $f : Y \to T$ is universally locally acyclic relatively to $\mathcal{F} \in D^b_c(Y, \Lambda)$. Then for any $\mathcal{G} \in D^b_c(T, \Lambda)$, $c_{\delta, f, \mathcal{F}, \mathcal{G}}$ is an isomorphism. In particular, $\delta$ is $\mathcal{F}$-transversal.

Proof. We may assume that $\delta$ is a closed immersion and put $V = T \setminus W$. Consider the following cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y & \xrightarrow{j} & U \\
\downarrow{p} & \square & \downarrow{f} & \square & \downarrow{f_V} \\
W & \xrightarrow{\delta} & T & \xrightarrow{j} & V.
\end{array}
$$

(2.2.1)

Since $f$ is universally locally acyclic relatively to $\mathcal{F}$, $f$ is also strongly locally acyclic relatively to $\mathcal{F}$. By [Sai17, Proposition 8.9], we obtain an isomorphism $\mathcal{F} \otimes^L f^* \mathcal{G} \cong j_* j'^*(\mathcal{F} \otimes^L f^* \mathcal{G})$. Note that $i_* (i^* \mathcal{F} \otimes^L p^* \delta^! \mathcal{G}) \cong \mathcal{F} \otimes^L i_* p^* \delta^! \mathcal{G} \cong \mathcal{F} \otimes^L f^* \delta^! \mathcal{G}$. Now we consider the following diagram between distinguished triangles:

$$
\begin{array}{ccc}
i_* (\mathcal{F} \otimes^L f^* \mathcal{G}) & \xrightarrow{i_* (c_{\delta, f, \mathcal{F}, \mathcal{G}})} & \mathcal{F} \otimes^L f^* \mathcal{G} & \xrightarrow{j_* j'^*(\mathcal{F} \otimes^L f^* \mathcal{G})} & +1 \\
i_* (i^* \mathcal{F} \otimes^L p^* \delta^! \mathcal{G}) & \cong & \mathcal{F} \otimes^L f^* \delta^! \mathcal{G} & \cong & \mathcal{F} \otimes^L f^* \mathcal{G} & \cong & \mathcal{F} \otimes^L f^* j_* j^* \mathcal{G} & \cong & +1
\end{array}
$$

(2.2.2)

The commutativity of the above diagram can be verified by using the proof of [Sai22, Lemma 1.1.3]. Thus $i_* (c_{\delta, f, \mathcal{F}, \mathcal{G}})$ is an isomorphism. Since $i$ is a closed immersion, the morphism $c_{\delta, f, \mathcal{F}, \mathcal{G}}$ is also an isomorphism.

The following proposition gives a converse of Proposition 2.2, which is not used in this paper.

Proposition 2.3 (Saito, [Sai17, Proposition 8.11]). Let $f : Y \to T$ be a morphism of schemes of finite type over a field $k$ of characteristic $p \neq \ell$ and let $\mathcal{F} \in D^b_c(Y, \Lambda)$. The morphism $f$ is locally acyclic relatively to $\mathcal{F}$ if the following condition is satisfied:

Let $T'$ and $W$ be smooth schemes over a finite extension of $k$, and

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y' & \xrightarrow{b'} & Y \\
\downarrow{p} & \square & \downarrow{f'} & \square & \downarrow{f} \\
W & \xrightarrow{\delta} & T' & \xrightarrow{b} & T
\end{array}
$$

(2.3.1)

be a cartesian diagram of schemes where $b : T' \to T$ is proper and generically finite and $\delta : W \to T'$ is a closed immersion. Then $\delta$ is $b^* \mathcal{F}$-transversal.

Proposition 2.4. Consider a commutative diagram in $\text{Sch}_S$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & \square & \downarrow{g} \\
P & \xrightarrow{\rho} & S
\end{array}
$$

(2.4.1)
Let $\mathcal{F} \in D^b_c(X, \Lambda)$ and $\mathcal{G} \in D^b_c(X, \Lambda)$. Let $i : X \times_Y X \to X \times_S X$ be the base change of the diagonal morphism $\delta : Y \to Y \times_S Y$:

$$
\begin{array}{c}
\xrightarrow{i} X \\
\xrightarrow{f \times f} X \times_S X \\
\xleftarrow{Y} \\
\xleftarrow{\delta} Y \times_S Y.
\end{array}
$$

(2.4.2)

Assume that $\mathcal{K}_{Y/S} = g^! \Lambda$ is locally constant and that $\mathcal{K}_{Y/S} \otimes^L \delta^! \Lambda \simeq \Lambda$ (for example, these two conditions hold if $g$ is smooth). Then we have

1. The morphism $f$ is $\mathcal{K}_{Y/S}$-transversal, i.e., $\mathcal{K}_{X/Y} \otimes^L f^* \mathcal{K}_{Y/S} = f^! \Lambda \otimes^L f^* \mathcal{K}_{Y/S} \simeq f^! \mathcal{K}_{X/S} = \mathcal{K}_{X/S}$ is an isomorphism.
2. The morphism $\delta$ is $\mathcal{K}_{X/S}$-transversal, i.e., $\mathcal{K}_{X/S} \otimes^L f^* \delta^! \Lambda \simeq \mathcal{K}_{X/Y}$ is an isomorphism.
3. If moreover $\mathcal{F} \in D^b_c(X/Y, \Lambda)$ and $\mathcal{G} \in D^b_c(X/Y, \Lambda)$, then $\delta$ is $D \boxtimes_S D_{X/S}(\mathcal{G})$-transversal, i.e.,

$$
\mathcal{F} \boxtimes_S D_{X/Y}(\mathcal{G}) \xrightarrow{(a)} i^* (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{G})) \otimes^L p^* \delta^! \Lambda \to i^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{G}))
$$

is an isomorphism, where $(a)$ is induced by the following composition:

$$
D_{X/S}(\mathcal{G}) \otimes^L f^* \delta^! \Lambda = R\text{Hom}(\mathcal{G}, \mathcal{K}_{X/S}) \otimes^L f^* \delta^! \Lambda \simeq R\text{Hom}(\mathcal{G}, \mathcal{K}_{X/S} \otimes^L f^* \delta^! \Lambda)
$$

(2.4.3)

(2.4.4)

Proof. Since $\mathcal{K}_{Y/S} = g^! \Lambda$ is locally constant, the assertion (1) follows from [Sai17, Proposition 8.6.2]. The second claim follows from (1) and $\delta^! \Lambda \otimes^L \mathcal{K}_{Y/S} \simeq \Lambda$. Now we prove (3). Since $\mathcal{G} \in D^b_c(X/Y, \Lambda)$, we have $D_{X/Y}(\mathcal{G}) \in D^b_c(X/Y, \Lambda)$ by [LZ22, Corollary 2.18]. Since $D_{X/S}(\mathcal{G}) \otimes^L f^* \delta^! \Lambda \simeq D_{X/Y}(\mathcal{G})$ is an isomorphism and $f^* \delta^! \Lambda$ is locally constant, we have $D_{X/S}(\mathcal{G}) \in D^b_c(X/Y, \Lambda)$. By [Ill17, Corollary 2.5], $f \times f : X \times_S Y \to Y \times_S Y$ is universally locally acyclic relatively to $\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{G})$. By Proposition 2.2, $\delta$ is $\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{G})$-transversal.

2.5. Consider the notation in 2.1. If $\delta$ is a closed immersion, we can imitate the construction of [AS07, 5.1] to define a functor $\hat{\delta}^! : D^b_c(Y, \Lambda) \to D^b_c(X, \Lambda)$ such that for any $\mathcal{F} \in D^b_c(Y, \Lambda)$, we have a distinguished triangle

$$
i^* \mathcal{F} \otimes^L p^* \delta^! \Lambda \xrightarrow{\mathcal{C}_{i, f, \mathcal{F}}} i^! \mathcal{F} \to \delta^! \mathcal{F} \to +1
$$

(2.5.1)

For a general $\delta$, one could use $\infty$-enhancement of the étale cohomology theory and define $\hat{\delta}^!$ to be the cofiber of $i^*(-) \otimes^L p^* \delta^! \Lambda \to i^!(-)$. Then $\delta^! : D^b_c(Y, \Lambda) \to D^b_c(X, \Lambda)$ is a functor fitting in the distinguished triangle (2.5.1). In this paper, we essentially only need the case where $\delta$ is a closed immersion.

Note that the canonical morphism $i_! p^* \delta^! \Lambda \simeq f^* \delta^! \Lambda \to f^* \Lambda = \Lambda$ induces a morphism $p^* \delta^! \Lambda \to i^! \Lambda$ by adjunction. Now we have a commutative diagram

$$
\begin{array}{c}
i^* \mathcal{F} \otimes^L i^! \Lambda \xrightarrow{\mathcal{C}_{i, i, \mathcal{F}}} i^! \mathcal{F} \to \delta^! \mathcal{F} \to +1 \\
\xrightarrow{3} \end{array}
$$

(2.5.2)
Lemma 2.6. Consider the cartesian diagram (2.1.1). For any $F \in D^b_c(Y, \Lambda)$, the morphism $\delta$ is $\mathcal{F}$-transversal if and only if the complex $\delta^2\mathcal{F}$ is acyclic on $X$.

Proof. By definition, the morphism $\delta$ is $\mathcal{F}$-transversal if and only if $c_{\delta,f,\mathcal{F}}$ is an isomorphism. By the distinguished triangle (2.5.1), the morphism $c_{\delta,f,\mathcal{F}}$ is an isomorphism if and only if $\delta^2\mathcal{F}$ is acyclic. \qed

3. Cohomological characteristic class

In this section, let us recall the construction of relative cohomological characteristic class. We first summarize the construction of relative cohomological characteristic classes. A correspondence over $S$ is a pair of morphisms $X \xrightarrow{c_1} C \xrightarrow{c_2} Y$ over $S$ (or equivalently, a morphism $c = (c_1, c_2) : C \to X \times_S Y$). For such $c$, we define $P$ to be the pull-back of $C$ along the diagonal map $\delta : X \to X \times_S X$:

\[
P \xleftarrow{c'} \xrightarrow{\delta'} C \xrightarrow{c} X \xrightarrow{\delta} X \times_S X.
\]

For any $F \in D^b_c(X/S, \Lambda)$ and $\mathcal{G} \in D^b_c(X, \Lambda)$, we have canonical isomorphisms

\[
F \boxtimes_S^{L} D_{X/S}(\mathcal{G}) \xrightarrow{(a)} R\text{Hom}(\text{pr}_2^*\mathcal{G}, \text{pr}_1^*\mathcal{F})
\]

\[
c^l (F \boxtimes_S^{L} D_{X/S}(\mathcal{G})) \xrightarrow{\sim} c^l \text{RHom}(\text{pr}_2^*\mathcal{G}, \text{pr}_1^*\mathcal{F}) \xrightarrow{(b)} \text{RHom}(c_2^*\mathcal{G}, c_1^*\mathcal{F}),
\]

where the isomorphism (a) follows from [LZ22, Proposition 2.5](see also[YZ21, Corollary 3.1.5]) and the isomorphism (b) follows from [SGA4, XVIII, 3.1.12.2]. Combining with the canonical map

\[
c^l (F \boxtimes_S^{L} D_{X/S}(\mathcal{F})) \xrightarrow{\text{id} \to \delta_*^s} c^l \delta_*^s (F \otimes^L D_{X/S}(\mathcal{F})) \xrightarrow{\text{evaluation}} c^l \delta_*^s \mathcal{K}_{X/S} \to \delta_*^s \mathcal{K}_{P/S},
\]

we obtain

\[
R\text{Hom}(c_2^*\mathcal{F}, c_1^*\mathcal{F}) \to \delta_*^s \mathcal{K}_{P/S},
\]

\[
\text{Tr} : \text{Hom}(c_2^*\mathcal{F}, c_1^*\mathcal{F}) \to \text{Hom}(\Lambda_P, \delta_*^s \mathcal{K}_{P/S}) = H^0(P, \mathcal{K}_{P/S}).
\]

In particular, taking the special correspondence $C = X$ and $c = \delta$, we have canonical morphisms

\[
R\text{Hom}(\mathcal{F}, \mathcal{F}) \to \mathcal{K}_{X/S},
\]

\[
\text{Tr} : \text{Hom}(\mathcal{F}, \mathcal{F}) \to H^0(X, \mathcal{K}_{X/S}).
\]

The relative (cohomological) characteristic class of $\mathcal{F}$ is defined to be (cf. [YZ21, Definition 3.6] and [LZ22, 2.20])

\[
C_{X/S}(\mathcal{F}) := \text{Tr}(\text{id}_\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}).
\]

The map $\mathcal{F} \mapsto C_{X/S}(\mathcal{F})$ defines a group homomorphism

\[
C_{X/S} : K_0(X/S, \Lambda) \to H^0(X, \mathcal{K}_{X/S}).
\]

Note that if $S$ is the spectrum of a field $k$, then $C_{X/k}(\mathcal{F})$ is the characteristic class introduced by Abbes and Saito (cf. [AS07, Definition 2.1.1]). The following lemma is useful for computing the characteristic classes of locally constant sheaves.
Lemma 3.2. Let $X$ be a connected scheme in $\text{Sch}_S$ and $\mathcal{L}$ a locally constant constructible sheaf of $\Lambda$-modules of constant rank on $X$. Then for any $\mathcal{F} \in D^b_c(X/S, \Lambda)$, we have

$$C_{X/S}(\mathcal{F} \otimes \mathcal{L}) = \text{rank} \mathcal{L} \cdot C_{X/S}(\mathcal{F}).$$

(3.2.1)

In particular, $C_{X/S}(\mathcal{L}) = \text{rank} \mathcal{L} \cdot C_{X/S}(\Lambda)$.

Proof. In the following, we put $\mathcal{L}' = R\text{Hom}(\mathcal{L}, \Lambda) = D_X(\mathcal{L})$. By definition, the class $C_{X/X}(\mathcal{L}) = \text{rank} \mathcal{L} \in H^0(X, K_{X/X}) = \Lambda$ is the composition

$$\Lambda \xrightarrow{\text{ev}} R\text{Hom}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{L} \otimes \mathcal{L}' \xrightarrow{\text{ev}} \Lambda,$$

(3.2.2)

where $\text{ev}$ is the evaluation map. We have a commutative diagram

$$\Lambda \xrightarrow{1\otimes \mathcal{L}} R\text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{L}) \xrightarrow{(3.1.3) \& (3.1.4)} (\mathcal{F} \otimes \mathcal{L}) \otimes \mathcal{L}' \xrightarrow{\text{ev}} K_{X/S}$$

(3.2.3)

$$\Lambda \xrightarrow{1\otimes \mathcal{L}} R\text{Hom}(\mathcal{F}, \mathcal{F}) \xrightarrow{(3.1.3) \& (3.1.4)} \mathcal{F} \otimes \mathcal{L}' \xrightarrow{\text{ev} \circ \text{ev}} K_{X/S} \otimes \Lambda.$$

The composition of the first row is $C_{X/S}(\mathcal{F} \otimes \mathcal{L})$, while the composition of the bottom row is $\text{rank} \mathcal{L} \cdot C_{X/S}(\mathcal{F})$. Thus $C_{X/S}(\mathcal{F} \otimes \mathcal{L}) = \text{rank} \mathcal{L} \cdot C_{X/S}(\mathcal{F})$, which finishes the proof.

In certain cases, we could also define relative characteristic classes without assuming locally acyclic condition. We first note the following lemma.

Lemma 3.3. Let $X \in \text{Sch}_S$ and $Z \subseteq X$ a closed subscheme. Let $U = X \setminus Z$. Assume that $H^0_Z(X, \mathcal{K}_{X/S}) = 0$ and $H^1_Z(X, \mathcal{K}_{X/S}) = 0$. Then we have

$$H^0(X, \mathcal{K}_{X/S}) \simeq H^0(U, \mathcal{K}_{U/S}).$$

(3.3.1)

Proof. The assertion follows from the following long exact sequence

$$H^0_Z(X, \mathcal{K}_{X/S}) \rightarrow H^0(X, \mathcal{K}_{X/S}) \rightarrow H^0(U, \mathcal{K}_{U/S}) \rightarrow H^1_Z(X, \mathcal{K}_{X/S}).$$

(3.3.2)

Definition 3.4. Under the assumptions in Lemma 3.3. Let $\mathcal{F} \in D^b_c(X, \Lambda)$ such that $\mathcal{F}|_U \in D^b_c(U/S, \Lambda)$. We define the relative (cohomological) characteristic class of $\mathcal{F}$ to be

$$C_{X/S}(\mathcal{F}) := C_{U/S}(\mathcal{F}|_U) \text{ in } H^0(X, \mathcal{K}_{X/S}).$$

(3.4.1)

3.5. For any $\mathcal{F} \in D^b_c(X/S, \Lambda)$ and $\mathcal{G} \in D^b_c(X, \Lambda)$, we call elements of $\text{Hom}(c_2^\Lambda \mathcal{G}, c_1^\Lambda \mathcal{F})$ the cohomological correspondences from $\mathcal{G}$ to $\mathcal{F}$. By the isomorphism (3.1.3), a cohomological correspondence $\eta : c_2^\Lambda \mathcal{G} \rightarrow c_1^\Lambda \mathcal{F}$ is equivalent to a map $\Lambda \rightarrow c_1^\Lambda R\text{Hom}(pr_2^\Lambda \mathcal{G}, pr_1^\Lambda \mathcal{F})$. By adjunction, this is also equivalent to a map $c_0 \Lambda \rightarrow R\text{Hom}(pr_2^\Lambda \mathcal{G}, pr_1^\Lambda \mathcal{F})$. In the following, we will identify these three descriptions freely.
3.6. We give an equivalent description of the relative (cohomological) characteristic class. Let $X \in \text{Sch}_S$ and $\delta : X \to X \times_S X$ the diagonal morphism. Let $\mathcal{F} \in D^b_c(X/S, \Lambda)$. The canonical isomorphism

$$R\text{Hom}_{X \times_S X}(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes^L_S D_{X/S}(\mathcal{F}).$$

induces an isomorphism

$$\delta^* R\text{Hom}_{X \times_S X}(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes^L D_{X/S}(\mathcal{F}).$$

Thus the evaluation map $\mathcal{F} \otimes^L D_{X/S}(\mathcal{F}) \to \mathcal{K}_{X/S}$ induces a map

$$(3.6.1) \quad \nu_\mathcal{F} : \delta^* R\text{Hom}_{X \times_S X}(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) \to \mathcal{K}_{X/S}.$$

Let $u : \delta \Lambda \to R\text{Hom}(pr_2^* \mathcal{F}, pr_1^! \mathcal{F})$ be a cohomological correspondence. Then $\text{Tr}(u) \in H^0(X, \mathcal{K}_{X/S})$ equals the cohomological class of the following composition

$$(3.6.2) \quad \Lambda = \delta^* \delta \Lambda \xrightarrow{\delta^* (u)} \delta^* R\text{Hom}_{X \times_S X}(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) \xrightarrow{\nu_\mathcal{F}} \mathcal{K}_{X/S}.$$

3.7. Now we study the pull-back property of the relative (cohomological) characteristic class. We consider a cartesian diagram in $\text{Sch}_S$

$$\begin{array}{ccc}
Y & \xrightarrow{p} & X \\
\downarrow{g} & & \downarrow{f} \\
W & \xleftarrow{h} & S.
\end{array}$$

(3.7.1)

For any $\mathcal{F} \in D^b_c(X/S, \Lambda)$, the object $p^* \mathcal{F}$ on $Y$ is also universally locally acyclic over $W$. Thus we have $p^* \mathcal{F} \in D^b_c(Y/W, \Lambda)$. This shows that we have well-defined maps

$$(3.7.2) \quad p^* : D^b_c(X/S, \Lambda) \to D^b_c(Y/W, \Lambda),$$

$$(3.7.3) \quad p^* : K_0(X/S, \Lambda) \to K_0(Y/W, \Lambda).$$

Note that the canonical maps

$$(3.7.4) \quad \mathcal{K}_{X/S} = f^! \Lambda_S \to f^! h_* h^* \Lambda_S = f^! h_* \Lambda_W \cong p_* g^! \Lambda_W = p_* \mathcal{K}_{Y/W}$$

induces a pull-back morphism

$$(3.7.5) \quad p^* : H^0(X, \mathcal{K}_{X/S}) \to H^0(Y, \mathcal{K}_{Y/W}).$$

**Proposition 3.8.** Given the cartesian diagram (3.7.1), the following diagram commutes

$$\begin{array}{ccc}
K_0(X/S, \Lambda) & \xrightarrow{C_{X/S}} & H^0(X, \mathcal{K}_{X/S}) \\
(3.8.1) \quad p^* \text{(3.7.3)} & & p^* \text{(3.7.5)} \\
K_0(Y/W, \Lambda) & \xrightarrow{C_{Y/W}} & H^0(Y, \mathcal{K}_{Y/S}).
\end{array}$$

**Proof.** We thank Weizhe Zheng for pointing out that Proposition 3.8 follows from [LZ22, Remark 1.9] and the proof of [LZ22, Prop. 2.26] (see also [Abe22, Theorem 1.3]). \(\square\)
3.9. The above result will be used in the proof of Proposition 4.17, which is the key to prove the cohomological analogue of the Milnor formula. For our purpose, we would like to reformulate Proposition 3.8 in another form, which will be convenient for proving Proposition 4.17. Consider the cartesian diagram (3.7.1), it induces the following commutative diagram with cartesian squares (cf. (3.1.1))

\[ \begin{array}{ccc}
E & \rightarrow & C \\
\downarrow & & \downarrow \\
Q & \rightarrow & P \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array} \]

(3.9.1)

where \( e = (e_1, e_2): E \rightarrow Y \times_W Y \) is defined by the pull-back of the correspondence \( C \rightarrow X \times_S X \) along the morphism \( q: Y \times_W Y \rightarrow X \times_S X \).

Let \( \mathcal{F} \in D^b_c(X/S, \Lambda) \) and put \( \mathcal{G} = p^* \mathcal{F} \). Then \( \mathcal{G} \in D^b_c(Y/W, \Lambda) \) by (3.7.2). Now we construct the following three pull-back morphisms:

\[ R\text{Hom}(c_2^* \mathcal{F}, c_1^* \mathcal{F}) \rightarrow q_* R\text{Hom}(e_2^* \mathcal{G}, e_1^* \mathcal{G}) \]
(3.9.2)

\[ c_1^*(\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) \rightarrow q_* e_1^*(\mathcal{G} \boxtimes_W D_{Y/W}(\mathcal{G})) \]
(3.9.3)

\[ \delta_{X*} K_{P/S} \rightarrow q_* \delta_{Y*} K_{Q/W} \]
(3.9.4)

3.10. We first define (3.9.2). Consider the following cartesian diagrams \((i = 0, 1)\)

\[ \begin{array}{ccc}
E & \rightarrow & C \\
\downarrow & & \downarrow \\
Y \times_W Y & \rightarrow & X \times_S X \\
\downarrow & & \downarrow \\
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
W & \rightarrow & S
\end{array} \]

(3.10.1)

By base change, we get a map \( e_1 q^* c_1^* \mathcal{F} \simeq p^* c_1^* c_1^* \mathcal{F} \xrightarrow{\text{adj}} p^* \mathcal{F} = \mathcal{G} \). By adjunction, we get a morphism

\[ q^* c_1^* \mathcal{F} \rightarrow e_1^* \mathcal{G} \]
(3.10.2)

Now the morphism (3.9.2) is defined to be the adjunction of the following composition

\[ q^* R\text{Hom}(e_2^* \mathcal{F}, c_1^* \mathcal{F}) \rightarrow R\text{Hom}(q^* e_2^* \mathcal{F}, q^* c_1^* \mathcal{F}) \simeq R\text{Hom}(e_2^* p^* \mathcal{F}, q^* e_1^* \mathcal{F}) \xrightarrow{(3.10.2)} R\text{Hom}(e_2^* \mathcal{G}, e_1^* \mathcal{G}) \]
(3.10.3)
3.11. For (3.9.3), we consider the following composition

\[
q^*(F \boxtimes_S^L D_{X/S}(F)) = q^*(pr_1^*F \boxtimes^L pr_2^*D_{X/S}(F)) \simeq pr_1^*p^*F \boxtimes^L pr_2^*p^*D_{X/S}(F)
\]

(3.11.1)

\[
\longrightarrow \text{pr}_1^*p^*F \boxtimes^L \text{pr}_2^*R\hom(p^*F, p^*K_{X/S})
\]

(3.7.4)

\[
\text{adjunction} \: q_* \rightarrow \text{pr}_1^* \mathcal{G} \boxtimes^L \text{pr}_2^*D_{Y/W}(G) = G \boxtimes^L_W D_{Y/W}(G).
\]

By adjunction we get a map \( F \boxtimes_S^L D_{X/S}(F) \to q_*(G \boxtimes^L_W D_{Y/W}(G)) \). Then the morphism (3.9.3) is defined to be the following composition

\[
c^!(F \boxtimes_S^L D_{X/S}(F)) \to c^!q_*(G \boxtimes^L_W D_{Y/W}(G)) \xrightarrow{\sim} q'_*c^!(G \boxtimes^L_W D_{Y/W}(G)),
\]

where the last map is the base change isomorphism.

3.12. Finally the morphism (3.9.4) is defined to be the following composition

\[
\delta'^!_{X*}K_{P/S} \simeq \delta'^!_{X*}c'^!p^*K_{X/S} \xrightarrow{\text{adj}} \delta'^!_{X*}c'^!p^*p^*K_{X/S} \xrightarrow{(3.7.4)} \delta'^!_{X*}c'^!p^*K_{Y/W}
\]

\[
\simeq \delta'^!_{X*}c'^!p'^!K_{Y/W} \simeq \delta'^!_{X*}c'^!p'^!K_{Q/W} \simeq q'_*\delta'^!_{Y*}K_{Q/W}.
\]

By construction, we get the following proposition.

**Proposition 3.13.** We consider the commutative diagrams (3.7.1) and (3.9.1). Then there is a commutative diagram

\[
\begin{array}{ccc}
R\hom(c_2^*F, c_1^*F) & \longrightarrow & c^!(F \boxtimes_S^L D_{X/S}(F)) & \longrightarrow & \delta'^!_{X*}K_{P/S} \\
\downarrow (3.9.2) & & \downarrow (3.9.3) & & \downarrow (3.9.4) \\
q'_*R\hom(c_2^*G, c_1^*G) & \longrightarrow & q'_*c^!(G \boxtimes^L_W D_{Y/W}(G)) & \longrightarrow & q'_*\delta'^!_{Y*}K_{Q/W}.
\end{array}
\]

In particular, we get a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(c_2^*F, c_1^*F) & \xrightarrow{\text{Tr}} & H^0(P, K_{P/S}) \\
\downarrow (3.15) & & \downarrow \\
\text{Hom}(c_2^*G, c_1^*G) & \xrightarrow{\text{Tr}} & H^0(Q, K_{Q/W}).
\end{array}
\]

In (3.9.1), if we take \( C = X \) and \( E = Y \) to be the diagonal, then we get Proposition 3.8.

3.14. Now we recall the proper push-forward property of the relative characteristic class. Let \( q : P \to Q \) be a proper morphism in Sch\textsubscript{S}. For any \( F \in D^b_c(P/S, \Lambda) \), then \( q_*F \) is universally locally acyclic over \( S \). Thus we have well-defined maps

\[
q_* : D^b_c(P/S, \Lambda) \to D^b_c(Q/S, \Lambda),
\]

(3.14.1)

\[
q_* : K_0(P/S, \Lambda) \to K_0(Q/S, \Lambda).
\]

(3.14.2)

For the proof of the following proposition, we refer to [LZ22, Corollary 2.22] (see also [YZ21, Corollary 3.3.4]).
Proposition 3.15. If $q: P \to Q$ is a proper morphism in $\text{Sch}_S$, then the following diagram commutes

\[
\begin{array}{c}
K_0(P/S, \Lambda) \xrightarrow{C_{P/S}} H^0(P, K_{P/S}) \\
q_* \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
We know that
\[(4.2.3) \quad K_{X/Y} \xrightarrow{\text{Prop.2.4.(2)}} K_{X/S} \otimes^L f^* \delta^! \Lambda \simeq \delta^*_1(i^* \delta_{0*} K_{X/S} \otimes p^* \delta^! \Lambda),\]
and that \(\delta^*_1 i^* \delta_{0*} K_{X/S} \simeq \delta^*_1 \delta_{1*} K_{X/S} \simeq K_{X/S}\). Hence we rewrite (4.2.2) as the following distinguished triangle
\[(4.2.4) \quad K_{X/Y} \to K_{X/S} \to \delta^*_1 \delta^! \delta_{0*} K_{X/S} \overset{+1}{\to} .\]

4.3. Since \(i^*(F \boxtimes_S D_{X/S}(F)) \otimes p^* \delta^! \Lambda \simeq (F \boxtimes_Y D_{X/S}(F)) \otimes p^* \delta^! \Lambda \simeq F \boxtimes_Y D_{X/Y}(F)\), we have distinguished triangles by (2.5.1):
\begin{align*}
(4.3.1) \quad F \otimes^L D_{X/Y}(F) & \to i^*(F \boxtimes_Y D_{X/S}(F)) \to \delta^! (F \boxtimes_Y D_{X/S}(F)) \overset{+1}{\to} \\
(4.3.2) \quad F \otimes^L D_{X/Y}(F) & \to \delta^*_1 i^!(F \boxtimes_Y D_{X/S}(F)) \to \delta^*_1 \delta^! (F \boxtimes_Y D_{X/S}(F)) \overset{+1}{\to} .
\end{align*}

Since \(X \setminus Z \to Y\) is universally locally acyclic relatively to \(F\), the morphism \(F \otimes^L D_{X/Y}(F) \to \delta^*_1 i^!(F \boxtimes_Y D_{X/S}(F))\) is an isomorphism on \(X \setminus Z\) by Proposition 2.4.3. We deduce that
\[\delta^*_1 \delta^! (F \boxtimes_Y D_{X/S}(F)) = \delta^*_1 \delta^! (R\hom_{X \times_S X}(pr^*_2 F, pr^!_1 F))\]
is supported on \(Z\).

4.4. Recall that the cohomological characteristic class \(C_{X/S}(F)\) is defined by the composition (cf. 3.5 and 3.6.2)
\[(4.4.1) \quad \delta_{0*} \Lambda = \delta_{0!} \Lambda \to F \boxtimes_Y D_{X/S}(F) \to \delta_{0*} K_{X/S} .\]

Now we consider
\[(4.4.2) \quad \delta^*_1 \delta^! \delta_{0*} \Lambda \to \delta^*_1 \delta^! (F \boxtimes_Y D_{X/S}(F)) \to \delta^*_1 \delta^! \delta_{0*} K_{X/S} .\]

By definition of \(\delta^!\) (cf.(2.5.1)), we have morphisms
\[(4.4.3) \quad \Lambda = \delta^*_1 \delta_{1*} \Lambda \xrightarrow{\text{base change}} \delta^*_1 i^! \delta_{0*} \Lambda \xrightarrow{(2.5.1)} \delta^*_1 \delta^! \delta_{0*} \Lambda .\]

Since the complex \(\delta^*_1 \delta^! (F \boxtimes_Y D_{X/S}(F)) \simeq \tau_0 \tau^* \delta^! (F \boxtimes_Y D_{X/S}(F))\) is supported on \(Z\), the composition of (4.4.3) and (4.4.2) defines a cohomological class \(\tau^* \Lambda \to \tau^* \delta^*_1 \delta^! (F \boxtimes_Y D_{X/S}(F)) \to \tau^! (\delta^*_1 \delta^! \delta_{0*} K_{X/S})\), which we denote by
\[C^\int_{X/Y}(F) \in H^0_Z(X, \delta^*_1 \delta^! \delta_{0*} K_{X/S}) .\]

The distinguished triangle (4.2.4) induces an exact sequence
\[(4.4.4) \quad H^0_Z(X, K_{X/Y}) \to H^0_Z(X, K_{X/S}) \to H^0_Z(X, \delta^*_1 \delta^! \delta_{0*} K_{X/S}) \to H^1_Z(X, K_{X/Y}) .\]

If the condition (C2) in 4.1 holds, then the map \(H^0_Z(X, K_{X/S}) \to H^0_Z(X, \delta^*_1 \delta^! \delta_{0*} K_{X/S})\) is an isomorphism. In this case, the class \(C^\int_{X/Y}(F) \in H^0_Z(X, \delta^*_1 \delta^! \delta_{0*} K_{X/S})\) defines an element of \(H^0_Z(X, K_{X/S})\), which is denoted by \(C^\int_{X/Y}(F)\) (or by \(C^\int_{X/Y}(F)\) for short).

**Definition 4.5.** Under the notation and conditions (C1) and (C3) in 4.1. We define the non-acyclicity class of \(F\) to be the class \(C^\int_{X/Y}(F) \in H^0_Z(X, \delta^*_1 \delta^! \delta_{0*} K_{X/S})\). If moreover the condition (C2) holds, we also call \(C^\int_{X/Y}(F) = C^\int_{X/Y}(F) \in H^0_Z(X, K_{X/S})\) the non-acyclicity class of \(F\).
Remark 4.6. In [AS07, Definition 5.2.1], Abbes and Saito define a localized characteristic class for $\mathcal{F} \in D^b(X, \Lambda)$. Their class is supported on the non-locally constant locus of $\mathcal{F}$. However the class $C_{\mathcal{X}, Y}^\text{nt}(\mathcal{F})$ is supported on the locus of non-locally acyclic points of $X \to Y$ relatively to $\mathcal{F}$. If $X = Y$ and $S = \text{Spec} k$, then these two definitions coincide.

Conjecture 4.7 (Fibration formula). Under the notation and conditions (C1)-(C3) in 4.1. Let
\begin{equation}
\delta^! : H^0(X, \mathcal{K}_{X/Y}) \to H^0(X, \mathcal{K}_{X/S}).
\end{equation}
be the homomorphism induced by the first morphism of (4.2.4). Then we have an equality
\begin{equation}
C_{X/S}(\mathcal{F}) = \delta^!(C_{X/Y}(\mathcal{F})) + C_{\mathcal{X}, Y}^\text{nt}(\mathcal{F}) \text{ in } H^0(X, \mathcal{K}_{X/S}).
\end{equation}
For the definition of $C_{X/Y}(\mathcal{F})$, we refer to 3.4.

When $g : Y \to S$ is smooth of relative dimension $r$. By [ILO14, Exposé XVI, Théorème 1.3], we view the top Chern class $c_r(f^*\Omega^1_{Y/S})$ as an element of $H^{2r}(X, \Lambda(r))$. The cup product with this element induces a map
\begin{equation}
H^0(X, \mathcal{K}_{X/Y}) \xrightarrow{c_r(f^*\Omega^1_{Y/S})} H^0(X, \mathcal{K}_{X/S}).
\end{equation}
We will show in Lemma 6.2 that $\delta^! = c_r(f^*\Omega^1_{Y/S})$. Then (4.7.2) can be rewritten as
\begin{equation}
C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^1_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_{\mathcal{X}, Y}^\text{nt}(\mathcal{F}) \text{ in } H^0(X, \mathcal{K}_{X/S}).
\end{equation}

4.8. Abbes-Saito’s formula. In the case where $S = \text{Spec} k$ and $X = Y$ is a smooth scheme over a field $k$. Since $\text{id} : X \setminus Z \to X \setminus Z$ is universally locally acyclic relatively to $\mathcal{F}|_{X\setminus Z}$, the cohomology sheaves of $\mathcal{F}|_{X\setminus Z}$ are locally constant on $X \setminus Z$. In this case, (4.7.2) follows from [AS07, Proposition 5.2.3].

4.9. The case where $Z = \emptyset$ will be verified in Theorem 6.4. Here we verify some other special cases of Conjecture 4.7. Let $U = X \setminus Z$ and $j : U \to X$ the open immersion. In general, we may reduce to the cases where $\mathcal{F}|_U = 0$ and the support of $\mathcal{F}$ is contained in $U$, i.e., $\mathcal{F} = j_!(j^*\mathcal{F})$, such that $U \to Y$ is universally locally acyclic relatively to $j^*\mathcal{F}$.

4.10. Case $\mathcal{F}|_U = 0$. In this case, $C_{X/Y}(\mathcal{F}) = 0$ and the complex $R\text{Hom}_{X \times Y}(\mathcal{F}, pr^*_{2,1}\mathcal{F}, pr^*_{1,1}\mathcal{F})$ supported on $Z \times Y Z$ and the class $C_{\mathcal{X}, Y}^\text{nt}(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{X/S})$ is defined by the composition $\delta^! \circ \delta : \delta_0^*\mathcal{K}_{X/S} \to \delta_0^*\delta^! \delta^\Delta \delta_0^*\mathcal{K}_{X/S}$. Thus $C_{X/S}(\mathcal{F}) = C_{\mathcal{X}, Y}^\text{nt}(\mathcal{F})$ and the assertion follows in this case.

4.11. Case $\mathcal{F} = j_!(\Lambda)$. In this case we have $C_{X/Y}(j_!(\Lambda)) = C_{\mathcal{X}, Y}^\text{nt}(\Lambda) = 0$ and $C_{\mathcal{X}, Y}^\text{nt}(\Lambda) = C_{\mathcal{X}, Y}^\text{nt}(\tau_*\Lambda) - C_{\mathcal{X}, Y}^\text{nt}(\tau_*\Lambda)$. By 4.10, the image $-C_{\mathcal{X}, Y}^\text{nt}(\tau_*\Lambda)$ in $H^0(X, \mathcal{K}_{X/S})$ is
\begin{align*}
-C_{X/S}(\tau_*\Lambda) &= C_{X/S}(j_!(\Lambda)) - C_{X/S}(\Lambda) \\
&\text{Thm. 4.10} C_{X/S}(j_!(\Lambda)) - \delta^!(C_{X/Y}(\Lambda)) \\
&= C_{X/S}(j_!(\Lambda)) - \delta^!(C_{X/Y}(\Lambda)).
\end{align*}
Thus the assertion is proved in this case.

4.12. Case $\mathcal{F} = \Lambda$. It is well-known that the trace for derived categories is non-additive in general [Fer]. But using filtered triangulated categories, one could prove a similar result as [AS07, Lemma 2.1.3]. Then $C_{X/S}(\Lambda) = C_{X/S}(\tau_*\Lambda) + C_{X/S}(j_!(\Lambda))$. Same for $C_{X/Y}(\Lambda)$ and $C_{\mathcal{X}, Y}^\text{nt}(\Lambda)$. Now the result follows from 4.10 and 4.11.
4.13. Since relative characteristic class is compatible with base change by Corollary 4.7.2, Conjecture 4.14.2 implies that the non-acyclicity class should also satisfy a base change property. But we cannot expect it in general for $C_{X/Y/S}^{int}$. For example, the condition (C2) (4.12.2) may not hold after a general base change. In the following, we formulate and prove such a property.

4.14. Let $b : S' \to S$ be a morphism in $\text{Sch}_S$. Let $U = X \setminus Z$. We denote by $X' := X \times_S S'$ the base change of $X$. Similar to define $F', Y', Z', U', f', g', h'$ and $\tau' : Z' \to X'$. Let $b_X$ and $b_Z$ be the base change of $b$ by $X \to S$ and $Z \to S$ respectively. We form the following commutative diagrams

\[
\begin{align*}
Z' & \xrightarrow{b_Z} Z \\
X' & \xrightarrow{b_X} X \\
Y' & \xrightarrow{b_Y} Y \\
S' & \xrightarrow{i} S
\end{align*}
(4.14.1)
\[
\begin{align*}
X' & \xrightarrow{\delta_X} X \\
Y' & \xrightarrow{\delta_Y} Y \\
X \times_S X' & \xrightarrow{\delta} X \times X
\end{align*}
\]

where squares and parallelograms are cartesian diagrams.

4.15. In the following, we always assume that the following condition (C'1) holds:

(C'1) $K_{Y'/S'} = g^* \Lambda$ is locally constant and that $K_{Y'/S'} \otimes^L \delta^* \Lambda \cong \Lambda$.

Then the class $C_{X'/Y'}^{int}(F')$ is well-defined. If moreover the following condition holds:

(C'2) The closed subscheme $Z' \subseteq X'$ satisfies

\[(4.15.1) \quad H^0_{Z'}(X', K_{X'/Y'}) = 0 \text{ and } H^1_{Z'}(X', K_{X'/Y'}) = 0.
\]

Then the class $C_{X'/Y'}^{int}(F')$ is well-defined.

4.16. Applying $i^*(-) \otimes^L p^* \delta^* \Lambda \to i^*(-) \to \delta^*(-) \to 1$ to (4.14.1), we get a commutative diagram

\[
\begin{align*}
f^* \delta^* \Lambda & \to \delta^*_1 (F \boxtimes_S D_{X/Y}(F)) \to K_{X/Y} \\
\Lambda & \to \delta^*_1 i^!(F \boxtimes_S D_{X/S}(F)) \to K_{X/S} \\
\delta^*_1 \delta^\Delta \delta^0 e \Lambda & \to \delta^*_1 \delta^\Delta (F \boxtimes_S D_{X/S}(F)) \to \delta^*_1 \delta^\Delta \delta^0 e K_{X/S}
\end{align*}
(4.16.1)

where vertical sequences are distinguished triangles. We also have a similar diagram after base change. In order to reduce the width of the diagram (4.16.2) below, we put $H_{X/S} = F \boxtimes_S D_{X/S}(F)$,
\[ \mathcal{H}_{X/Y} = \mathcal{F} \boxtimes_Y D_{X/Y}(\mathcal{F}), \mathcal{H}_{X'/S'} = \mathcal{F'} \boxtimes_{S'} D_{X'/S'}(\mathcal{F}') \] and \( \mathcal{H}_{X'/Y'} = \mathcal{F'} \boxtimes_Y D_{X'/Y'}(\mathcal{F}') \) for short. By Proposition 3.13, there is a commutative diagram

\[
\begin{array}{c}
\text{Diagram 4.16.2} \\

\end{array}
\]

where the existence of the dashed lines are due to properties of distinguished triangles. Consider the pull-back morphism \( b_X^*: H^0_Z(X, \delta_1^* \delta^\Delta \delta_{0X} K_{X/S}) \to H^0_Z(X', \delta_1^* \delta^\Delta \delta_{0X} K_{X'/S'}) \) (resp. \( b_X^*: H^0_Z(X, K_{X/S}) \to H^0_Z(X', K_{X'/S'}) \)). The above commutative diagram implies the following result:

**Proposition 4.17.** Under the notation in 4.1 and 4.14, and the condition (C'1) in 4.15, we have

\[
\begin{align*}
\langle 4.17.1 \rangle & \quad b_X^*C^\text{nt,}^\Delta_{X/Y/S}(\mathcal{F}) = C^\text{nt,}^\Delta_{X'/Y'/S'}(\mathcal{F}') \quad \text{in} \quad H^0_Z(X', \delta_1^* \delta^\Delta \delta_{0X} K_{X'/S'}). \\
\langle 4.17.2 \rangle & \quad b_X^*C^\text{nt}_{X/Y/S}(\mathcal{F}) = C^\text{nt}_{X'/Y'/S'}(\mathcal{F}') \quad \text{in} \quad H^0_Z(X', K_{X'/S'}). 
\end{align*}
\]

4.18. Now we apply Proposition 4.17 and Abe’s method [Abe22] to prove a cohomological Milnor formula.

**Theorem 4.19.** Let \( Y \) be a smooth curve over a perfect field \( k \) of characteristic \( p > 0 \). Let \( \Lambda \) be a Noetherian ring such that \( m\Lambda = 0 \) for some integer \( m \) which is prime to \( p \). Let \( f: X \to Y \) be a separated morphism of finite type and \( x \in |X| \) a closed point. Let \( \mathcal{F} \in D^b_c(X, \Lambda) \) such that \( f|_{X\setminus\{x\}} \) is universally locally acyclic relatively to \( \mathcal{F}|_{X\setminus\{x\}} \). Then we have

\[
\langle 4.19.1 \rangle \quad C^\text{nt}_{X/Y/k}(\mathcal{F}) = -\dim \text{tot} R\Phi(\mathcal{F}, f) \quad \text{in} \quad \Lambda = H^0_x(X, K_{X/k}),
\]

where \( R\Phi(\mathcal{F}, f) \) is the complex of vanishing cycles.

The above theorem gives a cohomological analogue of the Milnor formula proved by Saito in [Sai17, Theorem 5.9]. Thanks to the cohomological argument, we don’t need the smoothness assumption on \( X \).

**Proof.** The argument here follows closely [Abe22, Theorem 1.6]. By Proposition 4.21 below, we may assume that \( Y \) is an étale neighborhood of \( 0 \) in \( \mathbb{A}^1 = \text{Spec}(k) \). Let \( \text{pr}'_2: X \times \mathbb{A}^1_{t'} \to \mathbb{A}^1_{t'} \) be the second projection, where \( t' \) denotes the coordinate. Let \( f': X \times \mathbb{A}^1_{t'} \to Y \times \mathbb{A}^1_{t'} \to \mathbb{A}^1_{t'} \times \mathbb{A}^1_{t'} \to \mathbb{A}^1_1 = \mathbb{A}^1 \), where
μ is the multiplication morphism. We choose a non-trivial additive character ψ : F_p → Λ*, and let \( L \) be the Artin-Schreier sheaf on \( \mathbb{A}^1 \) associated with ψ. Let \( L(f^t) \) be zero extension to \( X \times \mathbb{P}_k^1 \) of \((f^t)^*L\). By Lemma [Abe22, Lemma 1.5], there exist a finite surjective morphism \( \mathbb{P}' \to \mathbb{P}_k^1 \) with \( \mathbb{P}' \) integral, and an object \( \tilde{\mathcal{F}} \in D_c^b(X \times \mathbb{P}', \Lambda) \) such that the following hold:

- \( \tilde{\mathcal{F}} \) is an extension of \( pr_{1*}^t\mathcal{F} \otimes L(f^t)|_U \), where \( U = (X \times \mathbb{P}') \times \mathbb{A}_k^1 \times \mathbb{P}_k^1 \) (\( \mathbb{A}_k^1 \times \mathbb{A}_k^1 \)).
- The projection \( \pi' : X \times \mathbb{P}' \to \mathbb{P}' \) is universally locally acyclic relatively to \( \tilde{\mathcal{F}} \).

We fix geometric points \( 0' \) and \( \infty' \) of \( \mathbb{P}' \) over \( 0 \) and \( \infty \) of \( \mathbb{P}_k^1 \). Let \( \eta' \) be a geometric generic point of \( \mathbb{P}_k^1 \). Let \( \Psi_{pr_2} = \Psi_{pr_2, \infty' - \eta'} \) be the notation defined in [Abe21, 4.2]. We have \( \tilde{\mathcal{F}}|_{X \times \infty'} \simeq \Psi_{pr_2} (pr_{1*}^t\mathcal{F} \otimes L(f^t)), \) which is supported on \( x \). We also have \( \tilde{\mathcal{F}}|_{X \times 0'} \simeq \mathcal{F} \). Now we apply Proposition 4.17 to the following diagram

\[
\begin{array}{ccc}
Z := \{x\} \times \mathbb{P}' & \xrightarrow{f \times \text{id}} & Y \times \mathbb{P}' \\
\pi \downarrow & & \downarrow \\
\mathbb{P}_k^1 & \to & \\
\end{array}
\]
\[
(4.19.2)
\]

We have \( Z_0' = \{x\} \times 0' \) and \( Z_{\infty'} = \{x\} \times \infty' \). Consider the following commutative diagram

\[
\begin{array}{ccc}
H^0_{Z_0'}(X \times 0', \mathcal{K}_{X \times 0'/0'}) & \xrightarrow{\text{res}_0} & \Lambda \\
\downarrow \hspace{1cm} \Downarrow \text{res}_0 \\
H^0_{Z}(X \times \mathbb{P}', \mathcal{K}_{X \times \mathbb{P}'/\mathbb{P}'}) & \xrightarrow{\simeq} & \Lambda \\
\downarrow \hspace{1cm} \downarrow \text{res}_x \\
H^0_{Z_{\infty'}}(X \times \infty', \mathcal{K}_{X \times \infty'/\mathbb{P}'}) & \xrightarrow{\simeq} & \Lambda \\
\end{array}
\]
\[
(4.19.3)
\]

Thus we get \( \text{res}_0(C_{X \times \mathbb{P}'/Y \times \mathbb{P}'/\mathbb{P}'}^r(\tilde{\mathcal{F}})) = \text{res}_x(C_{X \times \mathbb{P}'/Y \times \mathbb{P}'/\mathbb{P}'}^r(\tilde{\mathcal{F}})) \). By Proposition 4.17, we have

\[
\begin{align}
\text{res}_0(C_{X \times \mathbb{P}'/Y \times \mathbb{P}'/\mathbb{P}'}^r(\tilde{\mathcal{F}})) &= C_{X \times 0'/Y \times 0'/k}^r(\tilde{\mathcal{F}}|_{X \times 0'}) = C_{X/Y/k}(\mathcal{F}) \\
\text{res}_x(C_{X \times \mathbb{P}'/Y \times \mathbb{P}'/\mathbb{P}'}^r(\tilde{\mathcal{F}})) &= C_{X \times \infty'/Y \times \infty'/k}^r(\tilde{\mathcal{F}}|_{X \times \infty'}) = C_{X/Y/k}(\Psi_{pr_2} (pr_{1*}^t\mathcal{F} \otimes L(f^t)))
\end{align}
\]
\[
(4.19.4, 4.19.5)
\]

Since \( \Psi_{pr_2} (pr_{1*}^t\mathcal{F} \otimes L(f^t)) \) is supported on \( x \), by 4.10 we have

\[
C_{X/Y/k}^r(\Psi_{pr_2} (pr_{1*}^t\mathcal{F} \otimes L(f^t))) = \dim \Psi_{pr_2} (pr_{1*}^t\mathcal{F} \otimes L(f^t)))x = -\dim \text{tot}R\Phi_x(F, f),
\]
where the last equality follows from [Abe21, Proposition 6.5.2]. Indeed, the same conclusion and proof in loc.cit also hold for an isolated non-locally acyclic point \( x \) of \( f \) relatively to \( \mathcal{F} \). Now the result follows.

The following corollary gives a cohomological version of the Grothendieck-Ogg-Shafarevich formula:

**Corollary 4.20.** Let \( X \) be a smooth and connected curve over a perfect field \( k \) of characteristic \( p > 0 \). Let \( \mathcal{F} \in D_c^b(X, \Lambda) \) and \( Z \subseteq X \) be a finite set of closed points such that the cohomology sheaves of \( \mathcal{F}|_{X \setminus Z} \) are locally constant. Then we have

\[
C_{X/k}(\mathcal{F}) = \text{rank} \mathcal{F} \cdot c_r(\Omega_{X/k}^1) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}),
\]
where \( a_x(\mathcal{F}) = \text{rank} \mathcal{F}|_x - \text{rank} \mathcal{F}_x + \text{Sw}_x \mathcal{F} \) is the Artin conductor and \( \eta \) is the generic point of \( X \).
**Proof.** This follows from 4.8 and Theorem 4.19 together with the fact that \( \dim_{\text{tot}} R\Phi_x(F, \text{id}) = a_x(F) \). □

**Proposition 4.21.** Under the notation and conditions (C1) and (C3) in 4.1. Let \( e : Y \to Y_1 \) be an étale morphism in \( \text{Sch}_S \). Consider the diagram

\[
\begin{array}{ccc}
Z \xrightarrow{\tau} X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{e} \\
S & \xrightarrow{g} & Y_1
\end{array}
\]

(4.21.1)

where \( f' = e \circ f \) and \( g' \) is the structure morphism. Assume that the assumption (C1) in 4.1 holds for \( g_1 \). Then we have

\[
C_{X/Y/S}^{\text{nt, } \delta}(F) = C_{X/Y'/S}^{\text{nt, } \delta}(F) \quad \text{in} \quad H^0_Z(X, \delta^*_f \delta^*_e \delta_{g_0} \mathcal{K}_{X/S}).
\]

(4.21.2)

If moreover the condition (C2) in 4.1 holds, then we have

\[
C_{X/Y'/S}^{\text{nt, } \delta}(F) = C_{X/Y'/S}^{\text{nt, } \delta}(F) \quad \text{in} \quad H^0_Z(X, \mathcal{K}_{X/S}).
\]

(4.21.3)

**Proof.** We first show that \( C_{X/Y'/S}^{\text{nt, } \delta}(F) \) is well-defined. Since \( Y \to Y' \) is smooth and \( X \to Z \to Y \) is universally locally acyclic relatively to \( F|_{X \setminus Z} \), hence \( X \to Y' \) is also universally locally acyclic relatively to \( F|_{X \setminus Z} \) by [SGA4h, Th.finitude, Lemme 2.14]. So the assumption (C3) in 4.1 holds for \( X/Y' \). This proves that \( C_{X/Y'/S}^{\text{nt, } \delta}(F) \) is well defined. Now we show (4.21.2). Consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\delta_1} & X \\
\downarrow{\delta_0} & & \downarrow{\delta_0'} \\
X \times Y & \xrightarrow{\eta} & X \times_S X \\
\downarrow{\eta'} & & \downarrow{\eta'} \\
Y & \xrightarrow{\gamma_0} & Y \times_{Y'} Y \\
\downarrow{\gamma_1} & & \downarrow{\gamma_1} \\
Y' \times_S Y' & \xrightarrow{e \times e} & Y' \times_{Y'} Y'
\end{array}
\]

(4.21.4)

where squares and parallelograms are cartesian diagrams. Since \( e : Y \to Y' \) is étale, the diagonal morphism \( \gamma_0 : Y \to Y \times_{Y'} Y \) is an open immersion. So is its base change \( \eta : X \times_Y X \to X \times_Y X \). For \( X/Y/S \), we have a diagram (4.16.1). Similar for \( X/Y'/S \). Since \( \gamma_0 \) is an open immersion and that \( e \times e \) is étale, we have isomorphisms \( f^* \delta^* \Lambda \simeq f^* e^* \delta^* \Lambda \simeq f^* \gamma_0^* \gamma_1^* \Lambda \simeq f^* \delta' \Lambda \) and
\[ \delta_1^*(F \boxtimes^L D_{X/Y}(F)) \approx \delta_1^*(F \boxtimes^L D_{X/Y'}(F)) \]

Similarly, one can check items in the diagram (4.16.1) for \( X/Y/S \) are isomorphic to that for \( X/Y'/S \). This proves (4.21.2).

Now assume the condition (C2) in 4.1 holds. Since \( K_{X/Y} = K_{X/Y'} \), hence (C2) also holds for \( X/Y' \), i.e., \( H^0_Z(X, K_{X/Y'}) = H^1_Z(X, K_{X/Y'}) = 0 \). Thus \( C^\text{alt}_{X/Y'/S}(F) \) is well defined. Now (4.21.3) follows from (4.21.2). \( \square \)

5. Conductor formula

5.1. In this section, we prove the cohomological conductor formula 1.9 as a consequence of the proper push-forward property of non-acyclicity classes. Before that, we summarize the proof of Proposition 3.15 following [SGA5] (see also [YZ21, Corollary 3.3.4]). Note that a categorical interpretation is given in [LZZ22, Corollary 2.22].

5.2. Let \( \pi_1 : X_1 \to S \) and \( \pi_2 : X_2 \to S \) be morphisms in Sch\(_S\). We put \( X := X_1 \times_S X_2 \) and consider the following cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{pr_2} & X_2 \\
\downarrow{pr_1} & & \downarrow{\pi_2} \\
X_1 & \xrightarrow{\pi_1} & S.
\end{array}
\]

Let \( E_i \) and \( F_i \) be objects of \( D_c^b(X_i, \Lambda) \) for \( i = 1, 2 \). We put

\[
F := F_1 \boxtimes^L F_2 := pr_1^*F_1 \otimes^L pr_2^*F_2,
\]

\[
E := E_1 \boxtimes^L E_2 := pr_1^*E_1 \otimes^L pr_2^*E_2,
\]

which are objects of \( D_c^b(X, \Lambda) \).

5.3. For \( i = 1, 2 \), consider the following diagram in Sch\(_S\)

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow{\pi_i} & & \downarrow{q_i} \\
S & & .
\end{array}
\]

We put \( X := X_1 \times_S X_2, Y := Y_1 \times_S Y_2 \) and \( f := f_1 \times_S f_2 : X \to Y \). Let \( M_i \in D_c^b(Y_i, \Lambda) \) for \( i = 1, 2 \). We have a canonical map (cf. [Zh15, Construction 7.4] and [SGA5, Exposé III, (1.7.3)])

\[
f_1^!M_1 \boxtimes^L f_2^!M_2 \to f^!(M_1 \boxtimes^L M_2)
\]

which is adjoint to the composition

\[
f_1(f_1^!M_1 \boxtimes^L f_2^!M_2) \xrightarrow{\approx} f_1 f_1^!M_1 \boxtimes^L f_2 f_2^!M_2 \xrightarrow{\text{adj}[\text{adj}]} M_1 \boxtimes^L M_2
\]

where (a) is the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3]. In particular, we have a canonical morphism

\[
K_{X_1/S} \boxtimes^L K_{X_2/S} \to K_{X/S}.
\]

**Proposition 5.4.** If \( q_i : Y_i \to S \) is universally locally acyclic relatively to \( M_i \) for \( i = 1, 2 \), then the map (5.3.2) is an isomorphism.
Proof. Consider the following diagrams

\[
\begin{array}{c}
X_1 \times_S X_2 \xrightarrow{f_1 \times \id} Y_1 \times_S X_2 \\
\downarrow \quad \downarrow \\
X_1 \times_S Y_2 \xrightarrow{f_1 \times \id} Y_1 \times_S Y_2 \\
\downarrow \quad \downarrow \\
X_1 \xrightarrow{f_1} Y_1 \xrightarrow{q_1} S
\end{array}
\]

We may assume that \(X_2 = Y_2\) and \(f_2 = \id\), i.e., it suffices to show that the canonical map

\[(5.4.1) \quad f_1^! \mathcal{M}_1 \boxtimes_S \mathcal{M}_2 \xrightarrow{\cong} (f_1 \times \id)^!(\mathcal{M}_1 \boxtimes_S \mathcal{M}_2)\]

is an isomorphism. This is exactly the conclusion of [LZ22, Proposition 2.3]. We finish the proof. □

5.5. Consider a cartesian diagram

\[(5.5.1) \quad E \xrightarrow{e} D \xrightarrow{d} C \xleftarrow{c} X\]

of schemes in \(\text{Sch}_S\). Let \(F, G\) and \(H\) be objects of \(D^b_c(X, \Lambda)\) and \(F \otimes G \to H\) any morphism. By the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3] and adjunction, we have

\[e_!(c^! F \boxtimes_X^L d^! G) \xrightarrow{\cong} c_! c^! F \otimes^L d_! d^! G \to F \otimes G \to H.\]

By adjunction, we get a morphism

\[(5.5.2) \quad c_! c^! F \boxtimes_X^L d^! G \to e_! H.\]

Thus we get a pairing

\[(5.5.3) \quad \langle \cdot, \cdot \rangle : H^0(C, c^! F) \otimes H^0(D, d^! G) \to H^0(E, e_! H).\]

5.6. The relative Verdier pairing is defined by applying the map (5.5.3) to relative cohomological correspondences. Consider a cartesian diagram

\[(5.6.1) \quad E \xrightarrow{e} D \xrightarrow{d=(d_1,d_2)} C = (c_1,c_2) X = X_1 \times_S X_2\]

of schemes over \(S\). Let \(F_1 \in D^b_c(X_1, \Lambda)\) and \(F_2 \in D^b_c(X_2, \Lambda)\). Assume that \(\pi_i : X_i \to S\) is universally locally acyclic relatively to \(F_i\) for \(i = 1, 2\).

By the isomorphism (3.1.2), we have

\[(5.6.2) \quad \xrightarrow{\cong} (F_1 \boxtimes_S^L R\text{Hom}(F_2, \pi_2^! \Lambda_S)) \otimes^L (R\text{Hom}(F_1, \pi_1^! \Lambda_S) \boxtimes_S^L F_2) \xrightarrow{\text{evaluation}} \pi_1^! \Lambda_S \boxtimes_S^L \pi_2^! \Lambda_S \xrightarrow{(5.3.4)} \mathcal{K}_{X/S}.)
By (3.1.3), (5.5.2), (5.5.3) and (5.6.2), we get the following pairings
\begin{align}
(5.6.3) & \quad c_1 R\text{Hom}(c_2^* F_2, c_1^* F_1) \otimes_L d_1 R\text{Hom}(d_1^* F_1, d_2^* F_2) \to e_1 \mathcal{K}_{E/S}, \\
(5.6.4) & \quad \langle \cdot, \rangle : \text{Hom}(c_2^* F_2, c_1^* F_1) \otimes \text{Hom}(d_1^* F_1, d_2^* F_2) \to H^0(E, e_1^!(\mathcal{K}_{X/S})) = H^0(E, \mathcal{K}_{E/S}).
\end{align}

The pairing (5.6.4) is called the relative Verdier pairing (cf. [SGA5, Exposé III (4.2.5)]).

5.7. The relative characteristic class can be defined by using relative Verdier pairing. Let $f : X \to S$ be a morphism in $\text{Sch}_S$ and $F \in D^b_c(X, \Lambda)$. We assume that $f$ is universally locally acyclic relatively to $F$. Let $c = (c_1, c_2) : C \to X \times_S X$ be a closed immersion and $u : c_2^* F \to c_1^* F$ be a relative cohomological correspondence on $C$. Then the relative (cohomological) characteristic class $C_{X/S}(u)$ of $u$ equals to the cohomology class $\langle u, 1 \rangle \in H^0_{\operatorname{rel}}(X, \mathcal{K}_{X/S})$ defined by the pairing (5.6.4). In particular, if $C = X$ and $c : C \to X \times_S X$ is the diagonal and if $u : F \to F$ is the identity, then we have
$$C_{X/S}(F) = \langle 1, 1 \rangle \text{ in } H^0(X, \mathcal{K}_{X/S}).$$

5.8. For $i = 1, 2$, let $f_i : X_i \to Y_i$ be a proper morphism in $\text{Sch}_S$. Let $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2$. Let $p_i : X \to X_i$ and $q_i : Y \to Y_i$ be the canonical projections for $i = 1, 2$. Consider a commutative diagram
\begin{equation}
(5.8.1)
X \xleftarrow{c} C \xrightarrow{g} D
\end{equation}
of schemes over $S$. Assume that $c$ is proper. Put $c_i = p_i c$ and $d_i = q_i d$. By [Zh15, Construction 7.17], we have the following push-forward maps for cohomological correspondence (see also [SGA5, Exposé III, (3.7.6)] if $S$ is the spectrum of a field):
\begin{align}
(5.8.2) & \quad f_* : \text{Hom}(c_2^* \mathcal{L}_2, c_1^* \mathcal{L}_1) \to \text{Hom}(d_2^* (f_2 \mathcal{L}_2), d_1^* (f_1 \mathcal{L}_1)), \\
(5.8.3) & \quad f_* : g_* R\text{Hom}(c_2^* \mathcal{L}_2, c_1^* \mathcal{L}_1) \to R\text{Hom}(d_2^* (f_2 \mathcal{L}_2), d_1^* (f_1 \mathcal{L}_1)).
\end{align}

The following theorem follows from [LZ22, Theorem 2.21] (see also [YZ21, Theorem 3.3.2]).

**Theorem 5.9.** For $i = 1, 2$, let $f_i : X_i \to Y_i$ be a proper morphism in $\text{Sch}_S$. Let $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2$. Let $p_i : X \to X_i$ and $q_i : Y \to Y_i$ be the canonical projections for $i = 1, 2$. Consider the following commutative diagram with cartesian horizontal faces
\begin{equation}
(5.9.1)
C' \xleftarrow{c'} C \xrightarrow{g} C'' \\
\text{down arrows to } D', D, D''
\end{equation}
where \( c', c'', d' \) and \( d'' \) are proper morphisms in \( \text{Sch}_S \). Let \( c'_i = p_i c' \), \( c''_i = p_i c'' \), \( d'_i = q_i d' \), \( d''_i = q_i d'' \) for \( i = 1, 2 \). Let \( \mathcal{L}_i \in \text{Der}_X(X, \Lambda) \) and we put \( \mathcal{M}_i = f_{i*} \mathcal{L}_i \) for \( i = 1, 2 \). Assume that \( X_i \to S \) is universally locally acyclic relatively to \( \mathcal{L}_i \) for \( i = 1, 2 \). Then we have the following commutative diagram

\[
\begin{array}{c}
\xymatrix{ f_* c'_* R \text{Hom}(c''_2 \mathcal{L}_2, c''_1 \mathcal{L}_1) \otimes^L f_* c''_* R \text{Hom}(c''_* \mathcal{L}_1, c'_* \mathcal{L}_2) \ar[r]^{(1)} & f_* c_* \mathcal{K}_{C/S} \\
 d'_* R \text{Hom}(d''_* \mathcal{M}_2, d''_* \mathcal{M}_1) \otimes^L d''_* R \text{Hom}(d''_* \mathcal{M}_1, d''_* \mathcal{M}_2) \ar[r]^{(3)} & d_* \mathcal{K}_{D/S} \\
}
\end{array}
\]

where \( (3) \) is given by \((5.6.3)\), \( (1) \) is the composition of \( f_*((5.6.3)) \) with the canonical map \( f_* c'_* \otimes^L f_* c''_* \to f_* (c'_* \otimes c''_*) \), \( (2) \) is induced from \((5.8.3)\), and \( (4) \) is defined by

\[
f_* c_* \mathcal{K}_{C/S} \cong d_* g_* \mathcal{K}_{C/S} = d_* g! g^! \mathcal{K}_{D/S} \xrightarrow{\text{adj}} d_* \mathcal{K}_{D/S}.
\]

If \( S \) is the spectrum of a field, this is proved in [SGA5, Exposé III, Théorème 4.4].

**Proof.** For convenience, we sketch the main step of the proof following [SGA5]. Since \( f_i \) is proper, the morphism \( Y_i \to S \) is also universally locally acyclic relatively to \( \mathcal{M}_i = f_{i*} \mathcal{L}_i \) for \( i = 1, 2 \). Now we can use the same proof of [SGA5, Exposé III, Théorème 4.4]. Put

\[
\begin{align*}
\mathcal{P} &= \mathcal{L}_1 \boxtimes_S R \text{Hom}(\mathcal{L}_2, \mathcal{K}_{X_2/S}), & \mathcal{Q} &= R \text{Hom}(\mathcal{L}_1, \mathcal{K}_{X_1/S}) \boxtimes_S \mathcal{L}_2 \\
\mathcal{E} &= \mathcal{M}_1 \boxtimes_S R \text{Hom}(\mathcal{M}_2, \mathcal{K}_{Y_2/S}), & \mathcal{F} &= R \text{Hom}(\mathcal{M}_1, \mathcal{K}_{Y_1/S}) \boxtimes_S \mathcal{M}_2.
\end{align*}
\]

Then the theorem follows from the following commutative diagram

\[
\begin{array}{c}
\xymatrix{ f_* c'_* d' \mathcal{P} \otimes^L f_* c''_* d'' \mathcal{Q} \ar[r] & f_* c_* c^! (\mathcal{P} \otimes^L \mathcal{Q}) \ar[r] & f_* c_* c^! \mathcal{K}_{X/S} \\
 d'_* d' f_* \mathcal{P} \otimes^L d''_* d'' f_* \mathcal{Q} \ar[r] & d_* d^! (f_* \mathcal{P} \otimes^L f_* \mathcal{Q}) \ar[r] & d_* d^! f_* \mathcal{K}_{X/S} \\
 d'_* d' \mathcal{E} \otimes^L d''_* d'' \mathcal{F} \ar[r] & d_* d^! (\mathcal{E} \otimes^L \mathcal{F}) \ar[r] & d_* d^! \mathcal{K}_{Y/S} 
}
\end{array}
\]

where commutativity can be verified following the same argument of [SGA5, Exposé III, Théorème 4.4], and note that the universally locally acyclicity assumption is used to define the right three horizontal arrows.

\( \square \)

5.10. Now we formulate and prove a proper push-forward property for non-acyclicity classes. Consider the following first commutative diagram in \( \text{Sch}_S \):
induces a morphism \( s \) induces the second one, where squares and parallelograms are cartesian diagrams. Assume that the following conditions hold:

1. \( K_{Y/S} = g^! \Lambda \) is locally constant and that \( K_{Y/S} \otimes^L \delta^! \Lambda = \Lambda \), where \( \delta : Y \to Y \times_S Y \) is the diagonal morphism.
2. Let \( F \in D^b_c(X/S, \Lambda) \) such that \( X \setminus Z \to Y \) is universally locally acyclic relatively to \( F|_{X \setminus Z} \).
3. The morphism \( s : X \to X' \) is a proper morphism.

Under the condition (3), the morphism \( X' \setminus Z' \to Y \) is also universally locally acyclic relatively to \( s_*F|_{X' \setminus Z'} \). Now the non-acyclicity classes

\[
(5.10.2) \quad C_{X'/Y/S}^{\text{int}, \delta}(F) \in H^0_Z(X, \delta^! \delta \Delta \delta_0^* \mathcal{K}_{X/S}) \quad \text{and} \quad C_{X'/Y/S}^{\text{int}, \delta}(s_*F) \in H^0_{Z'}(X', \delta_1^! \delta \Delta \delta_0^* \mathcal{K}_{X'/S})
\]

are well-defined.

5.11. We define proper push-forward maps

\[
(5.11.1) \quad s_* : H^0_Z(X, \mathcal{K}_{X/S}) \to H^0_{Z'}(X', \mathcal{K}_{X'/S}),
\]

\[
(5.11.2) \quad s_* : H^0_Z(X, \delta^! \delta \Delta \delta_0^* \mathcal{K}_{X/S}) \to H^0_{Z'}(X', \delta_1^! \delta \Delta \delta_0^* \mathcal{K}_{X'/S}).
\]

First, we have a morphism

\[
(5.11.3) \quad s_* \mathcal{K}_{X/Y} = s_! \mathcal{K}_{X/Y} \to \mathcal{K}_{X'/Y} = f^! \Lambda,
\]

which is defined to be the adjunction of the following composition

\[
(5.11.4) \quad f'_! s_! f^! \Lambda \simeq f_1 f^! \Lambda \xrightarrow{\text{adj}} \Lambda.
\]

Similarly, there is a canonical morphism

\[
(5.11.5) \quad s_* \mathcal{K}_{X/S} \to \mathcal{K}_{X'/S},
\]

Therefore the composition

\[
(5.11.6) \quad s_{Z^!} s_{\tau^!} \mathcal{K}_{X/S} = s_{\tau^!} \tau_0 \tau_{1*} \mathcal{K}_{X/S} \simeq s_{\tau^!} \tau_{1*} \mathcal{K}_{X/S} \xrightarrow{\text{adj}} s_* \mathcal{K}_{X/S} \xrightarrow{(5.11.5)} \mathcal{K}_{X'/S},
\]

induces a morphism

\[
(5.11.7) \quad s_{Z^!} \mathcal{K}_{X/S} \to \tau_{1*} \mathcal{K}_{X'/S},
\]
which gives the required morphism (5.11.1). Applying $s_*(-)$ to the distinguished triangle (4.2.4), we obtain a commutative diagram

\[
\begin{array}{cccccc}
s_*K_{X/Y} & \xrightarrow{s_*\delta_1^*\delta^\Delta} & s_*\delta_0^*K_{X/S} & \xrightarrow{+1} \\
\downarrow & & \downarrow & \\
K_{X'/Y} & \xrightarrow{s_0^*\delta^\Delta} & K_{X'/S} & \xrightarrow{+1} \\
\end{array}
\]

(5.11.8)

The right vertical dashed arrow induces the required morphism (5.11.2).

**Theorem 5.12.** Under the notation and conditions (C1) and (C3) in 5.10, we have

\[
s_*(C_{X/Y/S}(\mathcal{F})) = C_{X'/Y'/S}(s_*(\mathcal{F})) \text{ in } H^0_Z(X', \delta_1^*\delta^\Delta\delta_0^*K_{X'/S}),
\]

(5.12.1)

where $s_*$ is the morphism (5.11.2). In particular, if $H^0_Z(X', K_{X'/Y'}) = H^1_Z(X', K_{X'/Y'}) = 0$, then we have

\[
s_*(C_{X/Y/S}(\mathcal{F})) = C_{X'/Y'/S}(s_*(\mathcal{F})) \text{ in } H^0_Z(X', K_{X'/S}).
\]

(5.12.2)

Proof. In the following, we put $\mathcal{F}' = s_*(\mathcal{F})$. In order to reduce the width of the diagram (5.12.3) below, we put $\mathcal{H}_{X/S} = \mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F})$, $\mathcal{H}_{X/Y} = \mathcal{F} \boxtimes_Y^L D_{X/Y}(\mathcal{F})$, $\mathcal{H}_{X'/S} = \mathcal{F}' \boxtimes_S^L D_{X'/S}(\mathcal{F}')$ and $\mathcal{H}_{X'/Y'} = \mathcal{F}' \boxtimes_Y^L D_{X'/Y'}(\mathcal{F}')$ for short. By (5.11.8) and Theorem 5.9, there is a commutative diagram (5.12.3)

where the existence of the dashed lines are due to properties of distinguished triangles. This proves the equality (5.12.1).

5.13. Now we apply Theorem 5.12 to prove a cohomological conductor formula.

**Theorem 5.14.** Let $Y$ be a smooth and connected curve over a perfect field $k$ of characteristic $p > 0$. Let $\Lambda$ be a Noetherian ring such that $m\Lambda = 0$ for some integer $m$ which is prime to $p$. Let
We view the cycle class $\mathcal{C}_\tau$ as
inex $\Lambda = H^0_y(Y, \mathcal{K}_{Y/k})$. This gives a cohomological version of the conductor formula [Sai21, Theorem 2.2.3] (also of Bloch’s conjecture on the conductor formula for constant sheaves [Blo87]).

Proof. In the Theorem 5.12, we take $s = f$ and $s' = \text{id}$. Then we get
\begin{equation}
(5.14.2)
f_* C^\text{nt}_X^\tau(Y/k)(\mathcal{F}) = C^\text{nt}_Y(Y/k)(Rf_* \mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}),
\end{equation}
where $f|_{X \setminus f^{-1}(y)}$ is proper and universally locally acyclic relatively to $\mathcal{F}|_{X \setminus f^{-1}(y)}$, the cohomology sheaves of $Rf_* \mathcal{F}$ are locally constant on $Y \setminus \{y\}$. By Theorem 4.19, we have $C^\text{nt}_Y(Y/k)(Rf_* \mathcal{F}) = -\dim \text{tot} R\Phi_y(Rf_* \mathcal{F}, \text{id}) = -a_y(Rf_* \mathcal{F})$. This proves (5.14.1). \hfill $\Box$

6. Fibration formula

In this section, we prove Theorem 1.2. Before that, let us briefly recall the cycle class defined by a regular immersion. Let $\tau : P \hookrightarrow Q$ be a closed immersion in $\text{Sch}_S$, which is a local complete intersection of pure codimension $c$. By [Fuj02, Definition 1.1.2] and [ILO14, Exposé XVI, Définition 2.3.1], there is a cycle class $\mathcal{C}_\tau \in H^2_Z(Q, \Lambda(c))$ which refines the $c$-th Chern class $c_c(N_{P/Q}^\tau) \in H^{2r}(Q, \Lambda(c))$, where $N_{P/Q}$ is the conormal sheaf associated to the closed immersion $\tau$.

We view the cycle class $\mathcal{C}_\tau$ as a morphism $\mathcal{C}_\tau : \Lambda \to \tau^!\Lambda(c)[2c]$. 

**Lemma 6.2.** Consider the notation in 4.1. If $g : Y \to S$ is a smooth morphism, then the two maps (4.7.1) and (4.7.3) are the same, i.e.,
\begin{equation}
(6.2.1) \quad \delta^r = c_r(f^*\Omega^1_{Y/S}^\vee) : H^0(Y, \mathcal{K}_{X/Y}) \to H^0(X, \mathcal{K}_{X/S}).
\end{equation}

Proof. Note that the excess normal sheaf of the following cartesian diagram (cf. (4.2.1))
\begin{equation}
(6.2.2) \quad \begin{array}{c}
X \\
\downarrow \phi \\
Y \to \quad Y \times_S Y
\end{array}
\end{equation}
equals $f^*\Omega^1_{Y/S}$. By [ILO14, Exposé XVI, Proposition 2.3.2], the image of $\mathcal{C}_\delta : \Lambda \to \delta^r\Lambda(r)[2r]$ under the map
\begin{equation}
(6.2.3) \quad H^2_{\wedge}(Y \times_S Y, \Lambda(r)) \xrightarrow{f^*} H^2_{\wedge}(X, \Lambda(r)) = H^2(X, \Lambda(r))
\end{equation}
equals $c_r(f^*\Omega^1_{Y/S}) \in H^{2r}(X, \Lambda(r))$. By definition of (6.2.3), the class $c_r(f^*\Omega^1_{Y/S})$ equals to the composition
\begin{equation}
(6.2.4) \quad \Lambda = f^*\Lambda \xrightarrow{f^*\mathcal{C}_\delta} f^*\delta^r\Lambda(r)[2r] \xrightarrow{b} s^*\Lambda(r)[2r] = \Lambda(r)[2r],
\end{equation}
where $b$ is the canonical morphism induced by the cartesian diagram (6.2.2). For any $\alpha \in H^0(X, \mathcal{K}_{X/Y})$, we write it as $\alpha : \Lambda \to \mathcal{K}_{X/Y}$. Now by definition, $c_r(f^*\Omega^1_{Y/S}) \cap \alpha$ equals to
\begin{equation}
(6.2.5) \quad \alpha \wedge f^*\mathcal{C}_\delta \wedge f^*\delta^r\Lambda(r)[2r] \xrightarrow{\text{id} \otimes b} \mathcal{K}_{X/Y}(r)[2r] \simeq \mathcal{K}_{X/S}.
\end{equation}
\footnote{For the sign problem of the cup-product, we refer to [ILO14, Exposé XVI, 4.5.3].}
Now we calculate $\delta^l(\alpha)$. By definition of (4.2.4), the morphism $\mathcal{K}_{X/Y} \rightarrow \mathcal{K}_{X/S}$ is defined to be

$$\mathcal{K}_{X/Y} = \mathcal{K}_{X/S} \otimes^L f^* \delta^l \Lambda \xrightarrow{\text{id} \otimes \Phi} \mathcal{K}_{X/S} \otimes^L \delta^l_0 \Lambda = \mathcal{K}_{X/S}.$$ (6.2.6)

Hence $\delta^l(\alpha)$ is the composition $\Lambda \xrightarrow{\alpha} \mathcal{K}_{X/Y} \xrightarrow{\text{id} \otimes \Phi} \mathcal{K}_{X/S}$. Compared with (6.2.5), we get $\delta^l(\alpha) = c_r(f^* \Omega^1_{Y/S}) \cap \alpha$, which finishes the proof.

6.3. When $g : Y \rightarrow S$ is a smooth morphism, for any $\mathcal{F} \in D^b_c(X/Y, \Lambda)$, we have $\mathcal{F} \in D^b_c(X/S, \Lambda)$ by [SGA4h, Th.finitude, Lemme 2.14]. In this case, there is a canonical homomorphism

$$K_0(X/Y, \Lambda) \rightarrow K_0(X/S, \Lambda).$$ (6.3.1)

The trace formula (transversal case) Consider the notation and condition (C1) in 4.1. Assume that $X \rightarrow Y$ and $X \rightarrow S$ are universally locally acyclic relatively to $\mathcal{F} \in D^b_c(X, \Lambda)$. Then we have

$$\text{Hom}(\mathcal{F}, \mathcal{F}) \xrightarrow{\text{Tr}} H^0(X, \mathcal{K}_{X/Y})$$ (6.4.1)

$$\text{Hom}(\mathcal{F}, \mathcal{F}) \xrightarrow{\text{Tr}} H^0(X, \mathcal{K}_{X/S}).$$

If $g : Y \rightarrow S$ is smooth, then the following diagram commutes

$$K_0(X/Y, \Lambda) \xrightarrow{C_{X/Y}} H^0(X, \mathcal{K}_{X/Y})$$ (6.3.1)

$$K_0(X/S, \Lambda) \xrightarrow{C_{X/S}} H^0(X, \mathcal{K}_{X/S}).$$

If $X = Y$ is smooth and if $S = \text{Spec} k$, then any $\mathcal{F} \in D^b_c(X/X, \Lambda)$ is smooth and $C_{X/k}(\mathcal{F}) = \text{rank} \mathcal{F} \cdot c_r(\Omega^1_{X/k}).$

**Proof.** For any $\alpha : \mathcal{F} \rightarrow \mathcal{F}$ which is given by a morphism $\Lambda \rightarrow R\text{Hom}(\mathcal{F}, \mathcal{F})$, the trace $C_{X/S}(\alpha) := \text{Tr}(\alpha) \in H^0(X, \mathcal{K}_{X/S})$ is given by the composition (cf.$(3.6.2)$)

$$\Lambda \xrightarrow{\delta^l_0(\alpha)} \delta^l_0(\mathcal{F} \boxtimes^L_{S} D_{X/S} \mathcal{F}) \xrightarrow{\text{tr}} \mathcal{K}_{X/S}.$$ (6.4.3)

The trace $C_{X/Y}(\alpha) := \text{Tr}(\alpha) \in H^0(X, \mathcal{K}_{X/Y})$ is given by the composition

$$\Lambda \xrightarrow{\delta^l(\alpha)} \delta^l(\mathcal{F} \boxtimes^L_{S} D_{X/S} \mathcal{F}) \xrightarrow{\text{tr}} \mathcal{K}_{X/Y}.\ (6.4.4)$$

Applying $i^*(-) \otimes^L p^* \delta^l \Lambda \rightarrow i^l(-)$ to $\mathcal{F} \boxtimes^L_{S} D_{X/S} \mathcal{F} \xrightarrow{\text{tr}} \delta^l \mathcal{K}_{X/S}$, we obtain a commutative diagram

$$\delta^l_0(\mathcal{F} \boxtimes^L_{S} D_{X/S}(\mathcal{F})) \xrightarrow{\text{tr}} \delta^l_1(i^*(\mathcal{F} \boxtimes^L_{S} D_{X/S}(\mathcal{F})) \otimes p^* \delta^l \Lambda) \xrightarrow{\text{tr}} \delta^l_1(i^* \delta^l_0 \mathcal{K}_{X/S} \otimes^L \delta^l \Lambda) \simeq \mathcal{K}_{X/Y}$$

$$\Lambda \xrightarrow{\delta^l_0(\alpha)} \delta^l_0(\mathcal{F} \boxtimes^L_{S} D_{X/S}(\mathcal{F})) \xrightarrow{(6.4.3)} \delta^l_1(i^*(\mathcal{F} \boxtimes^L_{S} D_{X/S}(\mathcal{F}))) \xrightarrow{\text{tr}} \delta^l_1(i^l \delta^l_0 \mathcal{K}_{X/S} \simeq \mathcal{K}_{X/S}}$$

where the middle vertical morphism is an isomorphism by Proposition 2.4.(3). Thus $C_{X/S}(\alpha) = \delta^l(C_{X/Y}(\alpha))$. The commutativity of the diagram (6.4.4) follows from Lemma 6.2 and (6.4.1). This finishes the proof. \qed
6.5. **Good fibration.** Let us explain the reason for calling (4.7.2) a fibration formula. For the definition of singular supports and characteristic cycles, we refer to [Bei16] and [Sai17]. Let $X$ be a smooth scheme purely of dimension $d$ over a perfect field $k$ of characteristic $p$, and let $\mathcal{F} \in D^b_c(X, \Lambda)$. Let $Y$ be a smooth connected and projective curve over $k$. Assume that $f : X \to Y$ is a good fibration with respect to the singular support $SS(\mathcal{F})$ of $\mathcal{F}$, and $Z := \{x_v\}_{v \in \Sigma}$ is the set of isolated characteristic points of $f$ and $\Sigma \subseteq Y$, i.e., the following conditions hold (cf. [UYZ20, Definition 3.4.2])

1. There exist finitely many closed points $x_v (v \in \Sigma)$ of $X$ such that $f$ is $SS(\mathcal{F})$-transversal on $X \setminus Z$ (thus $X \setminus Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X \setminus Z}$).
2. For each $v \in \Sigma$, $f(x_v) = v$. If $x_v \neq x_u$, then $v \neq u$. (Each fiber $X_v$ contains at most one isolated characteristic point)

Since $\dim Z = 0$, the relative cohomological characteristic class $C_{X/Y}(\mathcal{F})$ is well-defined (cf. Definition 3.4). Let $U = X \setminus Z$. Then the image of $C_{X/Y}(\mathcal{F})$ by the map $H^0(X, \mathcal{K}_{X/Y}) \to H^0(U, \mathcal{K}_{U/Y})$ equals $C_{U/Y}(\mathcal{F})$.

6.6. Let $\omega$ be a non-zero rational 1-form on the curve $Y$ such that $\omega$ has neither zeros nor poles on $\Sigma$. Then we have an equality in $H^0(Y, \mathcal{K}_{Y/k})$

$$c_1(\Omega^1_{Y/k}) = - \sum_{v \in \Sigma \setminus \{y\}} \text{ord}_v(\omega) \cdot [v].$$

Since $U \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_U$, by Theorem 6.4, we have an equality in $H^0(U, \mathcal{K}_{U/k})$

$$C_{U/k}(\mathcal{F}|_U) = c_1(f^*\Omega^1_{Y/k}) \cap C_{U/Y}(\mathcal{F}|_U)$$

By Theorem 4.19, we have

$$C^\text{ct}_{X/Y}(\mathcal{F}) = - \sum_{v \in \Sigma} \text{dimtot} R\Phi_{x_v}(\mathcal{F}, f) \cdot [x_v] \text{ in } H^0_Z(X, \mathcal{K}_{X/k}) = \bigoplus_{v \in \Sigma} \Lambda \cdot [x_v].$$

Conjecture 4.7 implies that the following conjecture should hold:

**Conjecture 6.7.** Under the assumptions in 6.5, we have

$$C_{X/k}(\mathcal{F}) = - \sum_{v \in \Sigma \setminus \{y\}} \text{ord}_v(\omega) \cdot C_{X/v}(\mathcal{F}|_{X_v}) - \sum_{v \in \Sigma} \text{dimtot} R\Phi_{x_v}(\mathcal{F}, f) \cdot [x_v].$$

The formula (6.7.1) gives an induction formula for the cohomological characteristic classes. More precisely, it implies that the characteristic class $C_{X/k}(\mathcal{F})$ can be built from characteristic classes $C_{X/v}(\mathcal{F}|_{X_v})$ on schemes $X_v$ of dimension smaller than $X$. Based on this idea, let us explain that Conjecture 6.7 (hence Conjecture 4.7) implies the projective case of Saito’s conjecture [Sai17, Conjecture 6.8.1].

**Conjecture 6.8** (Saito, [Sai17, Conjecture 6.8.1]). Let $X$ be an embeddable scheme of finite type over a perfect field $k$. Let $cc_X : K_0(X, \Lambda) \to CH_0(X)$ be the morphism defined in [Sai17, Definition

\[\footnote{We refer to [UYZ20] for a weak version of Saito’s another conjecture [Sai17, Conjecture 6.8.2].}]

Then we have a commutative diagram
\[
\begin{array}{ccc}
K_0(X, \Lambda) & \xrightarrow{cc_X} & \text{CH}_0(X) \\
\downarrow{c_{X/k}} & & \downarrow{\text{cl}} \\
H^0(X, \mathcal{K}_{X/k}) & & \\
\end{array}
\]
(6.8.1)

where \(\text{cl} : \text{CH}_0(X) \to H^0(X, \mathcal{K}_{X/k})\) is the cycle class map.

If \(X\) is projective and smooth over an algebraically closed field \(k\), then \(H^0(X, \mathcal{K}_{X/k}) \simeq \Lambda\) and Saito’s conjecture follows from the global index formula [Sai17, Theorem 7.13].

We first note the following blow up formula for characteristic classes.

**Lemma 6.9.** Let \(X\) and \(Y\) be smooth projective connected schemes over a perfect field \(k\) and let \(i : Y \to X\) be a closed immersion of codimension \(r \geq 1\). Let \(\pi : \bar{X} \to X\) be the blow up of \(X\) along \(Y\) and \(F \in D^b_c(X, \Lambda)\). Then we have:

\[
\begin{align*}
\pi_* (cc_X(\pi^* F)) &= cc_X(F) + (r - 1) \cdot i_* (cc_Y(i^* F)) \quad \text{in} \ CH_0(X), \\
\pi_* (C_{\bar{X}/k}(\pi^* F)) &= C_{X/k}(F) + (r - 1) \cdot i_* (C_{Y/k}(i^* F)) \quad \text{in} \ H^0(X, \mathcal{K}_{X/k}).
\end{align*}
\]

In this paper, we only need the case where \(r = 2\).

**Proof.** The first equality (6.9.1) follows from [UYZ20, Lemma 3.3.2 and Remark 3.3.3]. We prove the second one. By [ILO14, Exposé XVI, Proposition 2.2.2.1], we have an isomorphism in \(D^b_c(X, \Lambda)\):

\[
R\pi_* \Lambda \simeq \Lambda \oplus \bigoplus_{t=1}^{r-1} (i_\Lambda)(-t)[-2t].
\]

By the projection formula, we have

\[
R\pi_* \Lambda \otimes^L F \simeq R\pi_* (\Lambda \otimes^L \pi^* F) = R\pi_* \pi^* F \quad \text{and} \quad i_\Lambda \otimes^L F \simeq i_* i^* F.
\]

By (6.9.3) and (6.9.4), we get

\[
R\pi_* \pi^* F \simeq F \oplus \bigoplus_{t=1}^{r-1} (i_* i^* F)(-t)[-2t].
\]

Now (6.9.2) follows from Proposition 3.15 and (6.9.5). \(\Box\)

**Theorem 6.10.** Assume Conjecture 6.7 holds. Then Conjecture 6.8 holds for any smooth and projective scheme \(X\) over a perfect field \(k\).

**Proof.** We prove the theorem by induction on the dimension \(d = \dim(X)\). If \(d = 0\), then both \(C_{X/k}\) and \(cc_X\) equal the rank function. If \(d = 1\), the result follows from the Grothendieck-Ogg-Shafarevich formula and Corollary 4.20. Suppose that \(d \geq 1\) and that \(X\) has a good fibration \(f : X \to \mathbb{P}^1\) with respect to \(SS(F)\). We use the notation in 6.5. By [UYZ20, Proposition 5.3.7], the characteristic class \(cc_X(F)\) satisfies the following fibration formula

\[
cc_X(F) = - \sum_{v \in Y \setminus \Sigma} \text{ord}_{v}(\omega) \cdot cc_{X_v}(F|_{X_v}) - \sum_{v \in \Sigma} \dim_{\text{tot}} R\Phi_{X_v}(F, f) \cdot [x_v] \quad \text{in} \ CH_0(X).
\]

By the induction hypothesis and (6.7.1), we get \(\text{cl}(cc_X(F)) = C_{X/k}(F)\). Thus the result holds when \(X\) has a good fibration.
In general, for a finite Galois extension $k'/k$ of degree invertible in $\Lambda$, assume the result holds for $(X_{k'}, \mathcal{F}_{X_{k'}})$. We show the result also holds for $(X, \mathcal{F})$. Indeed, let $\sigma : X_{k'} \to X$ be the projection. Since $\sigma_*\sigma^*\mathcal{F} = \mathcal{F} \otimes \deg(k'/k)$, we get
\[
\deg(k'/k) \cdot \text{cl}(c_{cX}(\mathcal{F})) = \text{cl}(c_{cX}(\sigma_*\sigma^*\mathcal{F}))
\]
(6.10.2)
\[
\overset{(a)}{=} \sigma_* (\text{cl}(c_{cX}(\sigma^*\mathcal{F}))) = \sigma_*(C_{X}(\sigma^*\mathcal{F})) = C_{X}(\sigma_*\sigma^*\mathcal{F}) = \deg(k'/k) \cdot C_{X}(\mathcal{F}),
\]
where (a) follows from [Sai18, Lemma 2]. Since $\deg(k'/k)$ is invertible in $\Lambda$, hence $\text{cl}(c_{cX}(\mathcal{F})) = C_{X}(\mathcal{F})$.

By [UYZ20, Lemma 4.2.7], after taking a finite extension of $k$ if necessary, we may assume there is a blow up $\pi : \tilde{X} \to X$ of $X$ along a closed subscheme of codimension 2 and a good fibration $f : \tilde{X} \to \mathbb{P}^1$. Since $\tilde{X}$ has a good fibration, the result holds for $(\tilde{X}, \pi^*\mathcal{F})$. By Lemma 6.9, the result also holds for $(X, \mathcal{F})$. This finishes the proof.

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