QUADRATIC RESPONSE OF RANDOM AND DETERMINISTIC DYNAMICAL SYSTEMS.

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Abstract. We consider the linear and quadratic higher order terms associated to the response of the statistical properties of a dynamical system to suitable small perturbations. These terms are related to the first and second derivative of the stationary measure with respect to the changes in the system itself, expressing how the statistical properties of the system varies under the perturbation.

We show a general framework in which one can obtain rigorous convergence and formulas for these two terms. The framework is flexible enough to be applied both to deterministic and random systems. We give examples of such an application computing linear and quadratic response for Arnold maps with additive noise and deterministic expanding maps.

The statistical properties of the long time behavior of the evolution of dynamical system are strongly related to the properties of its invariant or stationary measures. It is important both in the theory and in the applications to understand quantitatively how the invariant measures of interest change when a given system is perturbed in some way. In the case where the invariant measure changes smoothly with the perturbation, the Linear and Quadratic Response express the first and second order leading terms describing the change in the invariant measure with respect to the perturbation, hence this concept is related to the first and second derivative representing how the invariant measure change. The paper gives a general approach for the understanding of these concepts in families of Markov operators with suitable properties, which hold for natural perturbations of transfer operators associated to deterministic and random systems. We show quite general assumptions under which Linear and Quadratic Response hold in these systems and explicit formulas to compute it. We show applications both to deterministic and random systems, providing a unified approach to these cases. As far as we know, formulas for quadratic response terms in the random case are shown in this paper for the first time.

1. Introduction

Linear Response in the dynamical systems context. Random and deterministic dynamical systems are often used as models of physical or social complex systems. In many cases it is natural to describe some aspects of the evolution of a system having many components at different time and size scales as a random input
while other components evolve deterministically. For random dynamical systems, like in deterministic ones, the invariant or stationary measures play a central role in the understanding of the statistical properties of the system. It is then natural to study the robustness of those invariant measures to perturbations of the system, whether in its deterministic or random part. Another motivation comes when the system of interest is submitted to a certain change or perturbation (an external forcing e.g.): it is useful to understand and predict the direction and the intensity of change of the invariant measures of the system, as it provides information on the direction and the intensity of change of its statistical properties after the perturbation.

When such a change is smooth, we say that the system exhibits Linear Response, and this can be described by a suitable derivative. More precisely, but still informally, let \((S_t)_{t \geq 0}\) be a one parameter family of dynamical systems obtained by perturbing an initial system \(S_0\), and let \(h_t\) be the invariant measure of interest of the systems \(S_t\).

The linear response of \(S_0\) under the given perturbation is defined by the limit

\[
R := \lim_{t \to 0} \frac{h_t - h_0}{t}
\]

where the meaning of this convergence can vary from system to system. In some systems and for a given perturbation, one may get \(L^1\)-convergence for this limit; in other systems or for other perturbations one may get convergence in weaker or stronger topologies. The linear response to the perturbation hence represents the first order term of the response of a system to a perturbation and when it holds, a linear response formula can be written:

\[
h_t = h_0 + Rt + o(t)
\]

(1)

which holds in some weaker or stronger sense.

For deterministic dynamical systems, Linear Response, as well as higher-order formulae have been obtained first by Ruelle, in the uniformly hyperbolic case \([28, 29]\). Nowadays these results have been extended to many other situations where one has some hyperbolicity and enough smoothness for the system and its perturbations. On the other hand there are many examples of deterministic systems whose statistical properties do not behave smoothly under quite natural perturbations. We refer to the survey \([5]\) for an extended discussion of the literature about linear response for deterministic systems. Since in our paper we mainly consider the response in the random case, in the next paragraphs we give more details on the literature for random systems, about which no surveys have been written until now.

**Linear Response for random dynamical systems.** In the physical literature, often borrowing the point of view of statistical mechanics, linear response formulae for several kinds of stochastic systems and for several aspects of their statistical behavior have been proposed and applied in various contexts (see \([13]\) and \([4]\) for

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1Typically this is done by modeling the evolution of the system at a small scale as a random perturbation of the large scale dynamics or, in the presence of different time scales (fast-slow systems) one can model the evolution of the fast component as a random perturbation of the slow one. Sometimes random dynamical system appear as a model for an “infinite dimensional limit” of deterministic dynamical systems having many interacting components (for an example related to linear response see \([34]\)).
general surveys), notably in climate science where several applications and estimation methods have been proposed, often in relation with the understanding of the nature of tipping points in the climate evolution (see the introduction of [12] or [24], [23], [25], [26]). The mathematical literature about Linear Response for random dynamical systems is smaller and more recent. In the next paragraphs, we try to give a quick overview of the possible approaches and existing results. As usual, there are mostly two perspectives to study random systems: the annealed and quenched point of view. Although we will focus on the former in this paper, we give a brief account of the latter at the end of this paragraph.

Consider a random dynamical system, in which the future dynamics depend on the initial condition and on some evolution laws containing random parameters. In the annealed case, the focus is on the average behavior of the evolution of the system with respect to these random parameters. In this perspective, when the randomness is strong enough, statistical stability and linear response to perturbations could be expected or considered easier to establish compared to the deterministic case. However it is worth remarking that even in this situation there are examples of non-smooth statistical stability under natural perturbations. The skew products given in [8] can be seen as random i.i.d rotations with a polynomial speed of mixing, having only Hölder statistical stability under small perturbations, even for very smooth observables. Regarding positive results, examples of linear response for small random perturbations of deterministic systems appear in [21] and [11]. In the paper [35], the smoothness of the invariant measure response under suitable perturbations is proved for a class of random diffeomorphisms, but no explicit formula is given for the derivatives. An application to the smoothness of the rotation number of Arnold circle maps with additive noise is presented. In the paper [20], these findings are extended outside the diffeomorphism case and applied to an idealized model of El Niño-Southern Oscillation. Linear response results for suitable classes of random systems were proved in [12] where the technical framework was adapted to stochastic differential equations and in [3], where the authors consider random compositions of expanding or non-uniformly expanding maps. In the paper [10], like in the present paper in Section 5, general discrete time systems with additive noise are considered, i.e. systems where the dynamics map a point deterministically to another point and then some random perturbation is added independently at each iteration according to a certain noise distribution kernel. The response of the stationary measure to perturbations of the deterministic part of the system or to perturbations of the shape of the noise is considered and explicit formulas for the response are given, with convergence in different stronger or weaker spaces according to the kind of perturbation considered. It is notable that in the case of additive noise (like in the case considered by [12]), no strong assumptions on the deterministic part of the dynamics are necessary, and in particular no hyperbolicity assumptions are required. In some sense, in this approach the regularizing effect of the noise on suitable functional spaces plays the role of the Lasota-Yorke-Doebelin-Fortet inequalities, as commonly used in many other functional analytic approaches to the study of the statistical properties of systems.

Another possible perspective on the response of statistical properties to perturbations in random systems concerns quenched result, i.e when one looks at a fixed realization of the random parameters instead of averaging over all possible values of the random parameters. In this approach, one considers random products of
maps instead of iterations of a single system, and the dynamically relevant objects become the Oseleets-Lyapunov spectrum of some transfer operator cocycle. One then studies the response of an appropriate equivariant family of measure to the perturbations. Contrary to the annealed case, one does not expect some regularization effect to come into play. The interest of this perspective was highlighted in the climate literature (notably [6]), but so far the mathematical results in this direction are very sparse: see [31, Chapter 5].

**Linear request, optimal response, numerical methods.** An important problem related to linear response is the control of the statistical properties of a system: how can one perturb the system, in order to modify its statistical properties in a prescribed way? How can one do it optimally? What is the best action to be taken in a possible set of allowed small perturbations in order to achieve a wanted small modification of the statistical behavior of the system? Understanding of this problem has a potentially great importance in the applications of Linear Response, as it is related to questions about optimal strategies in order to influence the behavior of a system. This problem was considered from a mathematical point of view for deterministic systems in [9] and [17]. Similar problems in the case of extended systems were considered in [19]. For random systems with additive noise the problem was briefly considered in [10]. In [1] the problem was considered for systems described by finite states Markov chains. Rigorous numerical approaches for the computation of the linear response are available to some extent, both for deterministic and random systems (see [2, 27]). We note that in principle, the quadratic response can provide important information in these optimization problems, as it can be of help in establishing convexity properties in the response of the statistical properties of a given family of systems under perturbation.

**Quadratic response and the present paper.** In the random case like in the deterministic case, a fruitful strategy to study the stability of a system relies on the remark that the stationary or invariant measures of interest are fixed points of the transfer operator associated to the system we consider; thus, linear response statements or quantitative stability results can be proved by first proving perturbation theorems for suitable operators, as done in [12, 17, 30, 8, 16, 21, 10].

In this paper we adopt this point of view, proving two general theorems about linear response of fixed points of Markov operators to perturbations. Those statements are tailored for operators which naturally appear as transfer operators of random or deterministic dynamical systems.

Once the first order (the linear part) of the response of a system to a perturbation is understood, it is natural to study further orders. The second order of the response may then be related to the second derivative and to other natural questions, as convexity aspects of the response of the system under perturbation, or the stability of the first order response. Hence, if the Linear Response $R$, represents the first order term of the response (see [11]), the Quadratic Response $Q$ will represent the second order term of this response, analogous to the second derivatives in usual Taylor’s expansion:

\[(2) \quad h_t = h_0 + Rt + \frac{1}{2} Qt^2 + o(t^2).\]

The first appearance of higher-order response formulae was in [28]. In the existing literature, there are two other approaches, closer to ours in spirit, to obtain higher-order regularity and explicit formulae for the derivatives of the invariant
measure. It is certainly interesting to discuss the differences and similarities between those papers and ours.

The framework of [11], a far-reaching generalization of [16], allows to obtain high smoothness, as well as explicit formulae at any order, for the whole discrete spectrum of a family of operators, if a uniform spectral gap is present and suitable Taylor expansions are satisfied. This approach is usually referred to as weak spectral perturbation theory, in contrast to classical perturbation theory (à la Kato, [15]). It is applicable for a wide class of deterministic and random perturbations of uniformly hyperbolic systems. However, we notice that the type of random perturbations considered in this paper are limited to zero-noise limit, and that for the very general type of random walk studied there (see [11, p.6]), only Lipschitz continuity results are formulated.

In [30], a construction to get high differentiability and explicit formulae for higher derivatives of perturbations of fixed points of operators in Banach spaces is presented and an application to linear response in uniformly expanding systems is shown. Among its strengths, this approach allows to obtain results both in deterministic and random situations, especially in the quenched case (see [31, Chapter 5]); it may be applied to non-linear operators; although this is not obvious at first sight, it may also be used to study the regularity w.r.t parameters of the whole discrete spectrum [32]. Let us remark that it may also apply in systems where no spectral gap is present. In our opinion, the main weakness of this approach is in the very heavy notation one has to digest in order to implement it.

In this paper, we focus on the first and second term of the Taylor's development of the response using statements which are somewhat simpler than the ones presented in [30] or [11], but lighter notation-wise. Our approach is flexible enough to be applied both to random and deterministic perturbations of a given system. Furthermore, it only requires the unperturbed system to exhibit convergence to equilibrium and a well-defined resolvent, (see Assumption LR2 and LR3), in contrast to the exponential mixing requirement for the whole family of operators in [11]. The existence of a quadratic response and related formulae for zero noise limits of deterministic systems was known (it can be obtained via [11], see also [21]), but those results in the case where the noise is an intrinsic part of the model, as far as we know, are new.

**Plan of the paper and main results.** In Sections 2 and 3 we prove two abstract theorems, giving a framework of general assumptions on the system and its associated transfer operator in which the expansion (2) can be obtained. We also show explicit formulas for $R$ and $Q$ ($R$ and $Q$ will belong to suitable normed vector spaces of measures or distributions). One of the required assumptions is the existence of certain resolvent for the unperturbed transfer operators associated to our systems. In Section 4 we show how this existence can be deduced by suitable regularization properties of the transfer operators we consider (Lasota Yorke inequalities on suitable measure spaces or the regularization brought by the effect of noise e.g.). Our framework is flexible enough to apply both to random and deterministic systems and in Sections 5 and 6 we give examples of such applications.

2. First derivative, linear response

In this section we show a general result for the linear response of fixed points of Markov operators under suitable perturbations. The result is made to be applied
to transfer operators of dynamical systems and suitable perturbations. Let $X$ be a compact metric space. Let us consider the space of signed Borel measures on $X$, $BS(X)$. In the following we consider three normed vectors spaces of signed Borel measures on $X$. The spaces $(B_{ss}, \| \cdot \|_{ss}) \subseteq (B_{s}, \| \cdot \|_{s}) \subseteq (B_{w}, \| \cdot \|_{w}) \subseteq BS(X)$ with norms satisfying

$$\| \cdot \|_{w} \leq \| \cdot \|_{s} \leq \| \cdot \|_{ss}.$$ 

We remark that, a priori, these spaces can be taken equal. Their precise choice depends on the type of system and perturbation under study.

We will assume that the linear form $\mu \to \mu(X)$ is continuous on $B_{i}$, for $i \in \{ss, s, w\}$. Since we will consider Markov operators acting on these spaces, the following (closed) spaces $V_{ss} \subseteq V_{s} \subseteq V_{w}$ of zero average measures defined as:

$$V_{i} := \{ \mu \in B_{i} | \mu(X) = 0 \}$$

where $i \in \{ss, s, w\}$, will play an important role. If $A, B$ are two normed vector spaces and $T : A \to B$ we denote the mixed norm $\| T \|_{A \to B}$ as

$$\| T \|_{A \to B} := \sup_{f \in A, \| f \|_{A} \leq 1} \| T f \|_{B}.$$ 

Suppose hence we have a one parameter family of Markov operators $L_{\delta}$. The following theorem is similar to the linear response theorem for regularizing transfer operators used in [10], the present statement is adapted to a general application on both deterministic and random systems.

**Theorem 1 (Linear Response).** Suppose that the family of bounded Markov operators $L_{\delta} : B_{i} \to B_{i}$, where $i \in \{ss, s, w\}$ satisfy the following:

(LR1) (regularity bounds) for each $\delta \in [0, \delta)$ there is $h_{\delta} \in B_{ss}$, a probability measure such that $L_{\delta} h_{\delta} = h_{\delta}$. Furthermore, there is $M \geq 0$ such that for each $\delta \in [0, \delta)$

$$\| h_{\delta} \|_{ss} \leq M.$$ 

(LR2) (convergence to equilibrium for the unperturbed operator) There is a sequence $a_{n} \to 0$ such that for each $g \in V_{s}$

$$\| L_{0}^{n} g \|_{s} \leq a_{n} \| g \|_{ss};$$ 

(LR3) (resolvent of the unperturbed operator) $(Id - L_{0})^{-1} := \sum_{n=0}^{\infty} L_{0}^{n}$ is a bounded operator $V_{w} \to V_{w}$.

(LR4) (small perturbation and derivative operator) There is $K \geq 0$ such that

$$\| L_{0} - L_{\delta} \|_{B_{s} \to B_{w}} \leq K \delta,$$

and

$$\| L_{0} - L_{\delta} \|_{B_{ss} \to B_{s}} \leq K \delta.$$ 

There is $\dot{L} h_{0} \in V_{w}$ such that

$$\lim_{\delta \to 0} \left\| \frac{(L_{\delta} - L_{0})}{\delta} h_{0} - \dot{L} h_{0} \right\|_{w} = 0.$$ 

Then we have the following Linear Response formula

$$\lim_{\delta \to 0} \left\| \frac{h_{\delta} - h_{0}}{\delta} - (Id - L_{0})^{-1} \dot{L} h_{0} \right\|_{w} = 0.$$

\footnote{A Markov operator is a linear operator preserving positive measures and such that for each positive measure $\mu$, it holds $[L(\mu)](X) = \mu(X)$.}
Remark 2. The choice for the three spaces depends on the system and the perturbation considered. We remark that the space where the response is defined is the same as the one where the derivative operator is defined. Concrete examples will be shown in Sections 2 and 6.

Remark 3. The convergence to equilibrium assumption at Item (LR2) is required only for the unperturbed operator $L_0$. It is sometimes not trivial to prove, but is somehow expected in systems having some sort of indecomposability and chaotic behavior (topological mixing, expansion, hyperbolicity or noise e.g.). In [10] there are several examples of verification of this condition by different methods in systems with additive noise.

Remark 4. The regularity bounds asked in Assumption (LR1) are easily verified in systems satisfying some regularization properties, like a Lasota Yorke inequality or the effect of noise (see Section 5).

Remark 5. The assumption (LR3) on the existence of the resolvent may be harder to verify. This is asked only for the unperturbed transfer operator, allowing a large class of perturbations. This technical point is quite useful and different from the kind of assumptions required in other previous approaches (e.g. [11]). In many systems, it will result from the presence of a spectral gap (compactness or quasi-compactness of the transfer operator acting on $B_v$). In Section 4 we will prove this assumption in the case of regularizing operators, which include systems with additive noise.

Remark 6. As remarked in the introduction, a family of operators might fail to have linear response, sometimes because of lack of hyperbolicity, sometimes because of the non smoothness of the kind of perturbation which is considered along the family (see e.g. [5]). In particular this is related to the type of convergence of the derivative operator

$$
\hat{L} f = \lim_{\delta \to 0} \frac{L_\delta - L_0}{\delta} f.
$$

In deterministic systems and related transfer operators, if the system is perturbed by moving its critical values or discontinuities, this will result in a bad perturbation of the associated transfer operators, and the limit defining $\hat{L}$ will not converge, unless we consider very coarse topologies in which the resolvent operator might not be a bounded operator.

We are ready to prove the main general statement.

Proof of Theorem 1. Let us first prove that under the assumptions the system has strong statistical stability in $B_s$, that is

$$
\lim_{\delta \to 0} \| h_\delta - h_0 \|_s = 0.
$$

Let us consider for any given $\delta$ a probability measure $h_\delta$ such that $L_\delta h_\delta = h_\delta$. Thus

$$
\| h_\delta - h_0 \|_s \leq \| L_\delta^N h_\delta - L_0^N h_0 \|_s \\
\leq \| L_\delta^N h_\delta - L_0^N h_\delta \|_s + \| L_0^N h_\delta - L_0^N h_0 \|_s.
$$

Since $h_\delta, h_0$ are probability measures, $h_\delta - h_0 \in V_{ss}$ and by (LR1), $\| h_\delta - h_0 \|_{ss} \leq 2M$ then we have

$$
\| h_\delta - h_0 \|_s \leq \| L_\delta^N h_\delta - L_0^N h_\delta \|_s + Q(N)
$$
with $Q(N) = 2a_nM \to 0$, not depending on $\delta$, because of the assumption (LR2).

Next we rewrite the operator sum $L^N_0 - L^N_\delta$ telescopically

$$(L^N_0 - L^N_\delta) = \sum_{k=1}^N L^{N-k}_0 (L_\delta - L_0) L^{k-1}_\delta$$

so that

$$(L^N_\delta - L^N_0) h_\delta = \sum_{k=1}^N L^{N-k}_0 (L_\delta - L_0) L^{k-1}_\delta h_\delta$$

$$= \sum_{k=1}^N L^{N-k}_0 (L_\delta - L_0) h_\delta.$$}

The assumption that $\|h_\delta\|_{ss} \leq M$, together with the small perturbation assumption (LR4) implies that $\|(L_\delta - L_0) h_\delta\|_s \leq \delta KM$ as $\delta \to 0$. Thus

$$(7) \quad \|h_\delta - h_0\|_s \leq Q(N) + N M_2 [\delta KM]$$

where $M_2 = \max(1, \|L_0\|_{B_w \to B_w}^{N/2})$. Choosing first $N$ big enough to let $Q(N)$ be close to 0 and then $\delta$ small enough we can make $\|h_\delta - h_0\|_s$ as small as wanted, proving the stability in $B_w$.

Let us now consider $(Id - L_0)^{-1}$ as a continuous operator $V_w \to V_w$. Remark that since $L h_0 \in V_w$, the resolvent can be computed at $L h_0$. Now we are ready to prove the main statement. By using that $h_0$ and $h_\delta$ are fixed points of their respective operators we obtain that

$$(Id - L_0) \frac{h_\delta - h_0}{\delta} = \frac{1}{\delta} (L_\delta - L_0) h_\delta.$$}

By applying the resolvent to both sides

$$(Id - L_0)^{-1}(Id - L_0) \frac{h_\delta - h_0}{\delta} = (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} h_\delta$$

$$= (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} h_0 + (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (h_\delta - h_0)$$

we obtain that the left hand side is equal to $\frac{1}{\delta} (h_\delta - h_0)$. Moreover, with respect to right hand side we observe that, applying assumption (LR4) eventually, as $\delta \to 0$

$$\left\| (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (h_\delta - h_0) \right\|_w \leq (Id - L_0)^{-1} \|_{V_w \to V_w} K \| h_\delta - h_0 \|_s$$

which goes to zero thanks to (7). Thus considering the limit $\delta \to 0$ we are left with

$$\lim_{\delta \to 0} \frac{h_\delta - h_0}{\delta} = (Id - L_0)^{-1} \dot{L} h_0.$$

converging in the $\| \cdot \|_w$ norm, which proves our claim. \qed

3. The second derivative

In this section we show how the previous approach can give us information on the second derivative and the second order term of the response to a perturbation.

Consider a further space $(B_{ww}, \| \|_{ww})$ such that $(B_w, \| \|_w) \subseteq (B_{ww}, \| \|_{ww}) \subseteq BS(X)$ and

$$\| \|_{ww} \leq \| \|_w.$
on which the linear form $\mu \rightarrow \mu(X)$ is continuous. Let us also consider the space of zero average measures in $B_{ww}$

$$V_{ww} := \{ \mu \in B_{ww} | \mu(X) = 0 \}.$$ 

We now prove an abstract response result for the second derivative.

**Theorem 7** (Quadratic term in the response). Let $(L_\delta)_{\delta \in [0,\delta]} : B_i \rightarrow B_i$, $i \in \{ss, ..., ww\}$ be a family of Markov operators as in the previous theorem. Assume furthermore that:

(QR1) The derivative operator $\dot{L}$ admits a bounded extension $\dot{L} : B_w \rightarrow V_{ww}$, such that

$$\left\| \frac{1}{\delta} (L_\delta - L_0) - \dot{L} \right\|_{w \rightarrow ww} \xrightarrow{\delta \rightarrow 0} 0.$$ 

(QR2) There exists a "second derivative operator" at $h_0$, i.e. $\ddot{L}h_0 \in V_{ww}$ such that

$$\left\| \frac{(L_\delta - L_0)h_0 - \delta \ddot{L}h_0}{\delta^2} - \ddot{L}h_0 \right\|_{ww} \xrightarrow{\delta \rightarrow 0} 0.$$ 

(QR3) The resolvent operator $(Id - L_0)^{-1}$ admits a bounded extension as an operator $V_{ww} \rightarrow V_{ww}$.

Then one has the following: the map $\delta \in [0,\delta] \mapsto h_\delta \in B_{ss}$ has an order two Taylor’s expansion at $\delta = 0$, with

$$\left\| \frac{h_\delta - h_0 - \delta(Id - L_0)^{-1}\ddot{L}h_0}{\delta^2} - (Id - L_0)^{-1} \left[ \ddot{L}h_0 + \ddot{L}(Id - L_0)^{-1}\ddot{L}h_0 \right] \right\|_{ww} \xrightarrow{\delta \rightarrow 0} 0.$$ 

**Remark 8.** We require the first derivative operator (see (8)) to be defined not only at the stationary measure, but on the whole space $B_w$ with convergence in the $ww$ topology, while for the second derivative operator (see (9)) we need it to be defined only at $h_0$. We also remark that the Quadratic response converges in the same norm in which the second derivative operator converges.

**Proof.** We write, for $\delta \neq 0$,

$$(Id - L_0) \frac{h_\delta - h_0 - \delta(Id - L_0)^{-1}\ddot{L}h_0}{\delta^2} = \frac{1}{\delta^2} \left[ (Id - L_0)(h_\delta - h_0) - \delta \ddot{L}h_0 \right]$$

$$= \frac{1}{\delta^2} \left[ (L_\delta - L_0)h_\delta - \delta \ddot{L}h_0 \right]$$

$$= \frac{1}{\delta^2} (L_\delta - L_0)(h_\delta - h_0) + \frac{1}{\delta^2} \left[ (L_\delta - L_0)h_0 - \delta \ddot{L}h_0 \right].$$ 

By assumption (QR2), in (11) the second term of the right-hand term,

$$\frac{1}{\delta^2} \left[ (L_\delta - L_0)h_0 - \delta \ddot{L}h_0 \right] \xrightarrow{\delta \rightarrow 0} \dddot{L}h_0$$
in the $V_{ww}$-norm.

The first in (11) can be rewritten as
\[
\frac{(L_\delta - L_0)}{\delta} h_\delta - h_0 = \left( \frac{(L_\delta - L_0)}{\delta} - \hat{L} \right) \left( \frac{h_\delta - h_0}{\delta} \right) + \hat{L} \left( \frac{(h_\delta - h_0)}{\delta} \right),
\]
(12)
where $\hat{h} := \lim_{\delta \to 0} \frac{h_\delta - h_0}{\delta} \in V_w$ (it is well-defined by Theorem 1). By uniform convergence of $\frac{(L_\delta - L_0)}{\delta}$ towards the derivative operator $\hat{L}$ in (8), the first summand in the right hand term of (12) converges in $V_{ww}$ as $\delta \to 0$ to 0.

For the second summand in (12) we write
\[
\left\| \hat{L} \left( \frac{h_\delta - h_0}{\delta} \right) - \hat{L} (I - L_0)^{-1} \hat{L} h_0 \right\|_{V_{ww}} \leq \left\| \hat{L} \right\|_{V_{ww} \to V_{ww}} \left\| \frac{h_\delta - h_0}{\delta} - (I - L_0)^{-1} \hat{L} h_0 \right\|_w
\]
which goes to 0 as $\delta \to 0$ thanks to Theorem 1. Thus, we have that in the $B_{ww}$-norm
\[
(I - L_0) h_\delta - h_0 - \frac{\delta (I - L_0)^{-1} \hat{L} h_0}{\delta^2} \to \hat{L} h_0 + \hat{L} (I - L_0)^{-1} \hat{L} h_0.
\]

We conclude by applying the resolvent $(I - L_0)^{-1}$, well defined on $V_w$.

4. Existence of the resolvent for $L_0$ and regularization.

In this section we show how the presence of some regularization and compactness allows to show that the resolvent operator $(I - L_0)^{-1}$ is well defined and continuous on the space of zero average measures. The following statement, is a version of a classical tool to obtain spectral gap in systems satisfying a Lasota Yorke inequality. It allows one to estimate the contraction rate of zero average measures, and imply spectral gap when applied to Markov operators. Let us consider a transfer operator $L_0$ acting on two normed vector spaces of complex or signed measures $(B_s, \| \cdot \|_s)$, $(B_w, \| \cdot \|_w)$, $B_s \subseteq B_w$ with $\| \cdot \|_s \geq \| \cdot \|_w$. We furthermore assume that $\mu \mapsto \mu(X)$ is continuous in the $\| \cdot \|_s$ and $\| \cdot \|_w$ topologies, and let $V_i := \{ \mu \in B_i ; \mu(X) = 0 \}$, $i \in \{w,s\}$.

**Theorem 9.** Suppose:

1. (Lasota Yorke inequality). For each $g \in B_s$, $\| L_0^n g \|_s \leq A \lambda_1^n \| g \|_s + B \| g \|_w$;

2. (Mixing) for each $g \in V_s$, it holds $\lim_{n \to \infty} \| L_0^n g \|_w = 0$;

3. (Compact inclusion) The image of the closed unit ball in $B_s$ under $L_0$ is relatively compact in $B_w$.

Under these assumptions, we have

1. $L_0$ admits a unique fixed point in $h \in B_s$, satisfying $h(X) = 1$.

2. There are $C > 0, \rho < 1$ such that for all $f \in V_s$ and $m$ large enough,

\[
\| L_0^m f \|_s \leq C \rho^m \| f \|_s.
\]
Proof. First we prove the existence of the fixed point. By Hennion theorem \[14\], the essential spectral radius of $L_0$ on $B_s$ is $\leq \lambda_1$. Hence for any $r \in (\lambda_1, 1)$, there are only isolated eigenvalues of finite multiplicity in the annulus $r \leq |z| \leq 1$. Let $\theta \notin 2\pi \mathbb{Z}$, such that $e^{i\theta}$ is an eigenvalue of $L_0$. Let $f \in B_s$ be an associated eigenfunction. Since $L_0$ is Markov, we have
\[
\int_X f dm = \int_X L_0 f dm = \int_X e^{i\theta} f dm,
\]
hence $\int_X f dm = 0$. But then by the mixing assumption, we have $\|f\|_w = \|L_0^n f\|_w \to 0$ as $n \to \infty$, which is a contradiction.

We conclude that 1 is the only eigenvalue on the unit circle. Furthermore, it is simple, since if there are $h_1, h_2$ satisfying $L_0 h_1 = h_1$, $L_0 h_2 = h_2$ and normalized so that $\int h_1 dm = \int h_2 dm = 1$, then $h_1 - h_2 \in V_s$ and hence by the mixing assumption $\|h_1 - h_2\|_w = 0$.

Now we turn to the estimate on the rate of mixing. What follows is essentially an adaptation of Hennion’s proof\[1] on the bound on the essential spectral radius. Let $S_s$ be the closed unit ball in $V_s$. By the compactness assumption, for each $\epsilon > 0$, there is an $\epsilon$-net $G_\epsilon$ for $L_0(S_s)$ in $||.||_w$, which implies that for any $f \in S_s$ we may find $g \in L_0(S_s) \cap G_\epsilon$ such that $\|L_0 f - g\|_w \leq \epsilon$. By the Lasota-Yorke inequality we then have, for any $m \in \mathbb{N}$,
\[
\|L_0^m f\|_s = \|L_0^{m-1} L_0 f\|_s \leq \|L_0^{m-1} g\|_s + \|L_0^{m-1} (L_0 f - g)\|_s \leq \|L_0^{m-1} g\|_s + 2A\lambda_1^{m-1} \|L_0\| + B\epsilon.
\]
We then write $m - 1 = n + k$, so that
\[
\|L_0^{m-1} g\|_s = \|L_0^n L_0^k g\|_s \leq A\lambda_1^n \|L_0^k g\|_s + B\|L_0^k g\|_w \leq A^2 \lambda_1^n + AB\lambda_1^n \|g\|_w + B\|L_0^k g\|_w,
\]
from which we get
\[
\|L_0^m g\|_s \leq A^2 \lambda_1^{n+k} + AB\lambda_1^n \|g\|_w + B\|L_0^k g\|_w + 2A\lambda_1^{m-1} \|L_0\| + B\epsilon.
\]
We fix $\epsilon$ small enough to get $B\epsilon < 1/4$. By the mixing assumption, we may take $k$ large enough to get $\sup_{g \in G_\epsilon} \|L_0^k g\|_w \leq 1/4B$. We may then choose $n$ large enough to obtain
\[
A^2 \lambda_1^{n+k} + AB\lambda_1^n \|g\|_w + 2A\lambda_1^{m-1} \|L_0\| \leq 1/4,
\]
from which follows $\|L_0^m f\|_s \leq 3/4$. Hence the result. $\square$

By this result, the existence of the resolvent follows easily.

**Corollary 10.** Under the assumptions of Theorem 9, the resolvent $(Id - L_0)^{-1} : V_s \to V_s$ is defined and continuous.

**Proof.** Let $f \in V_s$ then, by definition, $(Id - L_0)^{-1} f = \sum_0^\infty L_0^i f$. By the Markov assumption $L_0^i f \in V_s$ for $i \geq 1$. Since \[13\] holds and $\sum_0^\infty C\rho^n < \infty$, the sum $\sum_0^\infty L_0^i f$ converges in $V_s$ with respect to the $\|\cdot\|_w$ norm, and $\|(Id - L_0)^{-1}\|_{V_s \to V_s} \leq \sum_0^\infty C\rho^n$. $\square$

**Remark 11.** In the case where $B_s = B_w = B$, Theorem 9 still apply. In this case, we get the result that an operator that is power-bounded on $B$, compact and mixing has a unique (up to normalization) fixed point, and has exponential mixing.

\[3\]We are grateful to the anonymous referee for communicating this argument to us.
An important case in which the assumptions of Theorem 9 are satisfied is systems with additive noise, for which the transfer operator often satisfies a regularization assumption: Let $B_s \subset B_w$ be two Banach spaces with $\|\cdot\|_w \leq \|\cdot\|_s$. We say $L_0$ is regularizing from $B_w$ to $B_s$ if $L_0 : B_w \to B_s$ is continuous, i.e. if there is $B > 0$ such that the inequality

$$\|L_0 f\|_s \leq B \|f\|_w$$

is satisfied. If moreover a weak boundedness assumption is verified on $B_w$, that is if there is some $C > 0$ such that

$$\sup_n \|L_0^n f\|_w =: C\|f\|_w$$

for all $n$ and $f \in B_w$, then we have the Lasota Yorke inequality

$$\|L_0^n g\|_s \leq CB\|g\|_w$$

holding for each $n$. If the compactness assumption (2) in Theorem 9 is satisfied, this easily implies (by Hennion theorem [14]) that on the strong space, the operator $L_0$ only has discrete spectrum.

**Corollary 12.** If $L_0$ is regularizing from $B_w$ to $B_s$, and if Assumptions (2) and (3) in Theorem 9 holds, then the resolvent $(Id - L_0)^{-1}$ is defined and continuous also on $V_w$. Furthermore, let $B_{ww} \supseteq B_w$ as at beginning of Section 3. Suppose $L_0$ is regularizing from $B_{ww}$ to $B_w$, i.e. $L_0 : B_{ww} \to B_w$ is continuous, then $(Id - L_0)^{-1}$ is defined and continuous on $V_{ww}$ too.

**Proof.** Let $f \in V_w$. Since $(Id - L_0)^{-1} f = f + \sum_1^{\infty} L_0^n f$, we get

$$\|(Id - L_0)^{-1} f\|_w \leq \|f\|_w + \|L_0\|_{w \to s} \sum_{i=0}^{\infty} \|L_0^i\|_{s \to s} \|f\|_w < \infty$$

by Corollary 10. This shows that $(Id - L_0)^{-1} = Id + \sum_1^{\infty} L_0^i$ is a continuous operator $V_w \to V_w$. In case $L_0 : V_{ww} \to V_w$ is continuous we can repeat the same proof with $V_{ww}$ and $V_w$ in the place of $V_w$ and $V_s$, obtaining that is a continuous operator $V_{ww} \to V_{ww}$. \square

### 5. Linear and Quadratic Response in Systems with Additive Noise

In this section, we consider a non-singular map $T$, defined on the circle $\mathbb{S}^1$, perturbed by composition with a $C^3$ diffeomorphism near identity $D_\delta$ (in a sense explained precisely in (25) and (28)), and an additive noise with smooth kernel $\rho_\xi$. For example, one may take a Gaussian kernel

$$\rho_\xi = \frac{e^{-x^2/2\xi^2}}{\sqrt{2\pi\xi}}.$$  

In other words we consider a random dynamical system, corresponding to the stochastic process $(X_n)_{n \in \mathbb{N}}$ defined by

$$X_{n+1} = D_\delta \circ T(X_n) + \Omega_n \mod 1$$

where $(\Omega_n)_{n \in \mathbb{N}}$ are i.i.d random variables generated by the kernel $\rho$.

To this system we associate the annealed transfer operator defined by

$$L_\delta := \rho_\xi \ast L_{D_\delta \circ T}$$
(see section 5.1 for the proper definition of the convolution * in this context) where $L_{D_\delta \circ T} = L_{D_\delta} \circ L_T$ is the transfer operator (the pushforward map) associated to the deterministic map $D_\delta \circ T$ (see [33], Section 5 for more details about transfer operators associated to this kind of systems).

Our goal in this section is to show how this family of systems exhibits linear and quadratic response, as $\delta \to 0$ by applying Theorems 1 and 7: in the following, we thus verify that the family of transfer operators $(L_\delta)_{\delta \in [-\epsilon, \epsilon]}$ satisfy the assumptions of these two Theorems.

5.1. Convolution with Gaussian kernel and regularization inequalities on the circle. In this section we show the regularization properties of the convolution product of a Gaussian kernel and a finite order distribution on the circle.

Let $\rho_\xi = \frac{1}{\sqrt{2\pi \xi}} e^{-x^2/2\xi^2}$ be the Gaussian kernel. It is not a priori obvious how one can define the convolution product of the Gaussian kernel and a probability density on the circle, as the former is not one-periodic.

To that effect we start by recalling the definition of the periodization of a Schwartz function $\rho$:

**Definition 13.** If $\rho : \mathbb{R} \to \mathbb{R}$ is a Schwartz function, we define its periodization $\tilde{\rho} : \mathbb{S}^1 \to \mathbb{R}$ by

$$\tilde{\rho}(x) = \sum_{k \in \mathbb{Z}} \rho(x + k)$$

It is clear that the latter series converges uniformly on any bounded interval. As the same holds for its derivatives, it follows that $\tilde{\rho}$ defines a $C^\infty$ function on the circle. Thus, for $f \in L^1(\mathbb{S}^1)$, we can define $\rho * f$ as follows:

**Definition 14.** Let $f \in L^1(\mathbb{S}^1)$ and $\rho \in \mathcal{S}(\mathbb{R})$. We define the convolution $\rho * f$ by the formula

$$\rho * f(x) := \int_{\mathbb{S}^1} \tilde{\rho}(x - y) f(y) dy$$

The next proposition is an obvious consequence of the definition. We omit the proof.

**Proposition 15.** Let $\rho$ and $f$ be as before. The convolution $\rho * f$ has the following properties:

1. $\rho * f : \mathbb{S}^1 \to \mathbb{R}$ is $C^\infty$, with $(\rho * f)^{(k)} = \rho^{(k)} * f$ for any $k \in \mathbb{N}$.
2. One has the following regularization inequality: for each $k \in \mathbb{N}$,

$$\|\rho * f\|_{C^k} \leq \|\rho\|_{C^k} \|f\|_{L^1}.$$  

**Remark 16.** We notice that the same construction could be carried out for $\rho \in C^\infty(\mathbb{S}^1)$, or for $\rho \in C^\infty(\mathbb{R})$ decaying sufficiently fast (e.g. so that $|\rho^{(j)}(x+k)| \leq 1/k^2$ for all $j \in \mathbb{N}$ and $x$.)

---

4Recall that $\rho : \mathbb{R} \to \mathbb{R}$ is a Schwartz function if it is a $C^\infty$ function that satisfies, for any $(n,m) \in \mathbb{N}^2$,

$$|x|^n \rho^{(m)}(x) \to 0 \text{ as } |x| \to \infty.$$  

The set of Schwartz functions is traditionally denoted by $\mathcal{S}(\mathbb{R})$. 

Now, we extend the previous definition of convolution to the case of a distribution on the circle, by duality. We introduce the space of distributions on the circle \( \mathcal{D}'(S^1) \), to write the

**Definition 17.** Let \( f \in \mathcal{D}'(S^1) \), and \( \rho \) be a Schwartz function. Then \( \rho * f \) is the distribution on the circle defined for all \( \phi \in C^\infty(S^1) \) by

\[
\langle \rho * f, \phi \rangle = \langle f, \bar{\rho} * \phi \rangle,
\]

where \( \bar{\rho} \) is defined by \( \bar{\rho}(x) := \rho(-x) \) for \( x \in S^1 \).

We also define the space of distributions of order \( N \), \( \mathcal{D}_N(S^1) := \{ f \in \mathcal{D}'(S^1), \|f\|_{\mathcal{D}_N} < \infty \} \), where

\[
\|f\|_{\mathcal{D}_N} := \sup_{|\phi|_{C^N} \leq 1} |\langle f, \phi \rangle|.
\]

and

\[
W^{-N,1}(S^1) := \{ f \in \mathcal{D}'(S^1), \exists F \in L^1(S^1), F^{(N)} = f \text{ in the sense of distributions} \}.
\]

Now assume that \( f \in W^{-N,1}(S^1) \). Then one has the following result:

**Proposition 18.** Let \( f \in W^{-N,1}(S^1) \) and \( \rho \) be a Schwartz function. Then \( f \) induces a distribution of order \( N \), and one has that: \( \rho * f \) is a \( C^\infty \), one-periodic function, and for any \( k \in \mathbb{N} \),

\[
\|\rho * f\|_{C^k} \leq C\|\rho\|_{C^{N+k}}\|f\|_{\mathcal{D}_N}.
\]

**Proof.** First, consider \( F \in L^1(S^1) \) such that \( F^{(N)} = f \) in the sense of distributions. Then one may write that for any \( \phi \in C^\infty(S^1) \),

\[
\langle \rho * f, \phi \rangle = \langle f, \bar{\rho} * \phi \rangle = (-1)^N \langle F, (\bar{\rho})^{(N)} * \phi \rangle = \langle \rho^{(N)} * F, \phi \rangle,
\]

where we used \( (\bar{\rho})^{(N)} = (-1)^N \rho^{(N)} \). Thus, the distribution \( \rho * f \) coincides with the smooth function \( \rho^{(N)} * F \) in the sense of distributions: thus it is a smooth function itself. For the second part, notice that one has, for any \( x \in S^1 \), that

\[
\rho * f(x) = \langle \delta_x, \rho * f \rangle
\]

where \( \delta_x \) is the Dirac mass at \( x \in S^1 \). Consider now \( (\chi_{n,x})_{n \geq 0} \) a mollifier, i.e. a sequence of non-negative, smooth functions with integral one such that for \( g \in C^\infty(S^1) \), \( \langle \delta_x, g \rangle = \lim_{n \to \infty} \langle \chi_{n,x}, g \rangle \).

In particular, we notice that for any \( x \in S^1 \)

\[
\langle \chi_{n,x}, \rho * f \rangle = \int_{S^1} \chi_{n,x}(y) \rho * f(y) dy = \langle \rho * f, \chi_{n,x} \rangle = \langle f, \bar{\rho} * \chi_{n,x} \rangle
\]

and thus

\[
|\langle \chi_{n,x}, \rho * f \rangle| \leq \|f\|_{\mathcal{D}_N} \|\bar{\rho} * \chi_{n,x}\|_{C^N} \leq \|f\|_{\mathcal{D}_N} \|\rho\|_{C^N}\|\chi_{n,x}\|_{L^1}.
\]

Taking the limit \( n \to +\infty \) gives the result for \( k = 0 \). One obtains the general case by replacing \( \rho \) by \( \rho^{(k)} \) in the previous computation. \( \Box \)

The previous discussion allows to give a precise meaning to the annealed transfer operator \( L_\delta \) [18], and to its derivative operators (see Definition 22).
5.2. Small perturbations in the family of transfer operators. In this section, we establish the "small perturbations" assumptions (LR4) and (QR2) of Theorems 11 and 17.
We start by establishing that the perturbed transfer operator $L_\delta$ is close to $L_0$ in the $\|\cdot\|_{L^1(S^1)}$ norm, under the assumption that $D_\delta = Id + o(x \to 0)$ in the $C^0$-topology, i.e if $\sup_{x \in S^1} d(D_\delta(x), x) \to 0$. This is in fact the consequence of the more general, following result:

**Proposition 19.** Let $(T_\delta)_{\delta \in [0, \delta]}$ be a family of continuous maps of the circle, such that $d_{C^0}(T_\delta, T_0) \to 0$ and consider their associated transfer operators $L_{T_\delta}$. Then

$$\|L_{T_0} - L_{T_\delta}\|_{L^1(S^1)} \to 0,$$

**Proof.** First we consider functions $f \in L^1(S^1)$ and $g \in C^1(S^1)$. One has, by duality properties of the transfer operator

$$\langle (L_{T_0} - L_{T_\delta}) f, g \rangle = |\langle f, (g \circ T_\delta - g \circ T_0) \rangle|$$

$$\leq \|f\|_{L^1} \sup_{x \in S^1} |g(T_\delta(x)) - g(T_0(x))|$$

$$\leq \|f\|_{L^1} \|g\|_{C^1} d_{C^0}(T_\delta, T_0)$$

hence the result. \qed

We can then apply Proposition 19 to the family $(D_\delta)_{\delta \in [0, \delta]}$, with $D_0 = Id$.

Now we establish the derivative operator, or Taylor's expansion of order one for the family of operators $(L_{D_\delta})_{\delta \in [0, \delta]}$. Assume that there exists $S \in C^2(S^1, \mathbb{R})$, such that in the $C^0(S^1)$-topology, $D_\delta = Id + \delta S + o(\delta)$, i.e

$$(25) \quad \frac{1}{|\delta|} \|D_\delta - Id - \delta S\|_{C^0} \to 0$$

Note that as $S$ is a bounded function from $S^1$ to $\mathbb{R}$, the product $f.S$ is well-defined for any $f \in L^1(S^1)$; thus, we define $R = \left[ \frac{dL_{D_\delta}}{d\delta} \right]_{\delta = 0} : L^1(S^1) \to W^{-1,1}(S^1) \subset D_1(S^1)$ by

$$(26) \quad Rf := -(f.S)'$$

When $W^{-1,1}(S^1)$ is endowed with the $\|\cdot\|_{D_1}$-topology, $R$ is a bounded operator. Beware that this is not the derivative operator mentioned in Theorem 11 (which will be associated to $\rho \ast L_{D_\delta} \circ L_T$ at $\delta = 0$: see Theorem 24), but the derivative operator associated to the family of diffeomorphisms $D_\delta$.

**Proposition 20.** Let $(L_{D_\delta})_{\delta \in [0, \delta]}$ be the family of transfer operators associated to $(D_\delta)_{\delta \in [0, \delta]}$. Then one has

$$(27) \quad \left\| \frac{L_{D_\delta} - L_{D_0}}{\delta} - R \right\|_{L^1(D_2)} \to 0$$
Proof. Let $f \in L^\infty(\mathbb{S}^1)$ and $g \in C^2(\mathbb{S}^1)$. One may write
\[
\left| \left( \frac{L_{D_{\delta}} - L_{D_{\delta}}}{{\delta}} f - Rf, g \right) \right| = \left| \frac{1}{\delta} (f, g \circ D_{\delta} - g - \delta g') \right| = \left| \frac{1}{\delta} (f, g \circ D_{\delta} - g - (D_{\delta} - Id)g') \right| \leq \frac{1}{\delta} \int_{\mathbb{S}^1} |f| \|g \circ D_{\delta} - g - (D_{\delta} - Id)g'\|_\infty + \|D_{\delta} - Id - \delta S\|_\infty g' \].
\]

One has, by the mean value theorem
\[
|g(D_{\delta}(x)) - g(x) - (D_{\delta}(x) - x)g'(x)| \leq \int_{x}^{D_{\delta}(x)} |g'(t) - g'(x)| dt \leq C|\delta||g|_{C^{2}o_{\delta} \to 0}\]

Together with the Taylor’s expansion of $D_{\delta}$, this yields
\[
\left| \left( \frac{L_{D_{\delta}} - L_{D_{\delta}}}{{\delta}} f - Rf, g \right) \right| \leq C\|f\|_{L^1} \|g\|_{C^{2}o_{\delta} \to 0}\]

establishing (27).

Finally we show that a second order Taylor’s expansion is satisfied. Assume that there are $S_1$, $S_2 \in C^3(\mathbb{S}^1, \mathbb{R})$ such that in the $C^0$-topology, $D_{\delta}$ satisfies
\[
D_{\delta} = Id + \delta S_1 + \frac{\delta^2}{2} S_2 + o(\delta^2)
\]
i.e.
\[
\frac{1}{\delta^2} \|D_{\delta} - Id - \delta S_1 - \frac{\delta^2}{2} S_2\|_{C^0} \xrightarrow[\delta \to 0]{} 0
\]
and let us define the second derivative $Q : L^1(\mathbb{S}^1) \to W^{-2,1}(\mathbb{S}^1) \subset D_2(\mathbb{S}^1)$ by
\[
Qf := (f S_1)^{''} - (f S_2)^{'}.
\]
When $W^{-2,1}(\mathbb{S}^1)$ is endowed with the topology induced by the $\|\cdot\|_{D_2}$ norm, $Q$ is a bounded operator, and one has the following result.

**Proposition 21.** Let $(L_{D_{\delta}})_{\delta \in [0, \delta]}$ be the family of transfer operator associated to $(D_{\delta})_{\delta \in [0, \delta]}$. It satisfies
\[
\left( L_{D_{\delta}} - L_0 - \frac{\delta R}{2} Qf, g \right) = \frac{1}{\delta^2} \int_{\mathbb{S}^1} f(x)(g(D_{\delta}(x)) - g(x) - \delta Sg'(x) - \frac{\delta^2}{2}(S_1^2 g'' + S_2 g'))(x)dx.
\]

Proof. Let $f \in L^1(\mathbb{S}^1)$ and $g \in C^3(\mathbb{S}^1)$. Then one may write
\[
\left( L_{D_{\delta}} - L_0 - \frac{\delta R}{2} f, g \right) = \frac{1}{\delta^2} \int_{\mathbb{S}^1} f(x)(g(D_{\delta}(x)) - g(x) - \delta Sg'(x) - \frac{\delta^2}{2}(S_1^2 g'' + S_2 g'))(x)dx.
\]

Now, notice that one has:
\[
g \circ D_{\delta} - g - \delta Sg' - \frac{\delta^2}{2}(S_1^2 g'' + S_2 g')
\]
\[
= g \circ D_{\delta} - g - (D_{\delta} - Id)g' - \frac{1}{2}(D_{\delta} - Id)^2 g'' + (D_{\delta} - Id - \delta S_1 - \frac{\delta^2}{2} S_2)g' + \frac{1}{2}((D_{\delta} - Id)^2 - \delta^2 S_1^2) g''.
\]

$(I)$ \hspace{1cm} $(II)$
It follows from Taylor’s integral formula at order 3 and the Taylor’s expansion \( (28) \) that
\[
|(I)| \leq \|g\|_{C^3} \alpha(\delta^3)
\]
where the \( o(\delta^3) \) is uniform in \( x \). It also follows from \( (28) \) that \( (D_{\delta} - Id)^2 = \delta^2 S_1 + o(\delta^2) \), (where once again, \( o(\delta^2) \) is uniform in \( x \)) and thus
\[
|(II)| \leq \|g\|_{C^2} \alpha(\delta^2).
\]
Finally, one obtains
\[
\left| \frac{L_{D_{\delta}} - L_0 - \delta R}{\delta^2} f - Qf, g \right| \leq \|f\|_{L^1} \|g\|_{C^3} \alpha_0 \delta \to 0(1)
\]
hence the result. \( \square \)

We now consider the derivative operators of the system with additive noise, defining \( \dot{L} \) and \( \dot{\tilde{L}} \).

**Definition 22.** Let \((L_\delta)_{\delta \in [-\epsilon, \epsilon]}\) be the family of transfer operators associated with \((17)\). The derivative operators \( \dot{L} : C^k(\mathbb{S}^1) \to C^k(\mathbb{S}^1) \) and \( \dot{\tilde{L}} : C^k(\mathbb{S}^1) \to C^k(\mathbb{S}^1) \) are defined by:
\[
\dot{L} := \rho_\xi \ast R \circ L_T
\]
\[
\dot{\tilde{L}} := \rho_\xi \ast Q \circ L_T.
\]

**Remark 23.** The convolution in \((34)\) should be understood in the sense of Definition \((17)\). Notice that the regularization effect of the Gaussian noise allow us to define the derivative operators \( \dot{L} : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1) \), and even from \( L^1(\mathbb{S}^1) \) to \( C^\infty(\mathbb{S}^1) \) (see Proposition \((18)\) and \((22)\)).

**Theorem 24.** Let \((L_\delta)_{\delta \in [0, \delta]}\) be the family of transfer operators associated to systems of the kind described in \((17)\) and perturbations satisfying \((28)\). Then for any \( k \in \mathbb{N} \), the derivative operators \( \dot{L} : C^{k+1}(\mathbb{S}^1) \to C^k(\mathbb{S}^1) \) and \( \dot{\tilde{L}} : C^{k+1}(\mathbb{S}^1) \to C^{k-1}(\mathbb{S}^1) \) satisfy the following estimates:
\[
\|L_0 - L_\delta\|_{C^{k+1} \to C^k} \leq C \delta;
\]
\[
\left| \frac{L_\delta - L_0}{\delta} - \dot{\tilde{L}} \right|_{C^{k} \to C^{k-1}} \to 0;
\]
\[
\left| \frac{L_\delta - L_0 - \dot{\tilde{L}}}{\delta^2} \right|_{C^{k} \to C^{k-2}} \to 0.
\]

**Proof.** Recall that by \((18)\), one has \( L_\delta = \rho_\xi \ast L_{D_\delta} \). Let \( \phi \in C^{k+1}(\mathbb{S}^1) \). Applying Proposition \((19)\) and \((20)\) yields
\[
\|(L_0 - L_\delta)\phi\|_{C^k} \leq C \delta \|\phi\|_{C^k},
\]
\[
\|LT\phi\|_{L^1} \leq C \delta \|\phi\|_{C^{k+1}}.
\]
Take \( \phi \in C^k(\mathbb{S}^1) \). By Proposition \((20)\) and \((22)\) one has
\[
\left| \frac{L_\delta - L_0}{\delta} \phi - \dot{\tilde{L}} \phi \right|_{C^{k-1}} \leq C \|\rho\|_{C^k} \left| \frac{L_{D_\delta} - Id}{\delta} LT\phi - RL_T\phi \right|_{D_2} \leq C \|\rho\|_{C^k} \|LT\phi\|_{L^1} \left| \frac{L_{D_\delta} - Id}{\delta} - R \right|_{L^1 \to D_2}.
\]
Similarly combining Proposition 21 and (22), one obtains
\[
\left\| \frac{L_\delta - L_0 - \delta L}{\delta^2} \phi - \frac{1}{2} L_0 \phi \right\|_{C^{k-2}} \leq C\|\rho\|_{C^{k-1}}\|L_T\phi\|_{L^1} \left\| \frac{L_{D_2} - Id - \delta R}{\delta^2} - \frac{1}{2} Q \right\|_{L^1 \to D_3}
\]
Hence the result.

5.3. Convergence to equilibrium and regularization for the unperturbed transfer operator. In this subsection, we show that the unperturbed transfer operator \( L_0 := \rho_\xi * L_T \) satisfies the rest of the assumptions of Theorems 1 and 7 with the nested sequence of Banach spaces \( C^{k+1}(S^1) \hookrightarrow C^k(S^1) \hookrightarrow C^{k-1}(S^1) \hookrightarrow C^{k-2}(S^1) \), via the result of section 4, namely the convergence to equilibrium on \( C^k(S^1) \), weak boundedness for the sequence \( \{L_0^n\}_{n \in \mathbb{N}} \) and regularizing from \( L^1(S^1) \) to \( C^k(S^1) \) for any \( k \in \mathbb{N} \). Notice that here we use crucially the weak contraction property of the (deterministic) transfer operator \( L_T \) on \( L^1(S^1) \) as a preliminary to the subtler regularization properties.

Lemma 25. Let \( k \geq 0 \).

1. The unperturbed transfer operator \( L_0 \) is regularizing from \( L^1(S^1) \) to \( C^k(S^1) \) and from \( C^k(S^1) \) to \( C^{k+1}(S^1) \) for any \( k \in \mathbb{N} \).

2. The unperturbed transfer operator \( L_0 \) has convergence to equilibrium on \( C^{k+1}(S^1) \), i.e. there is \( a_n \to 0 \) such that for any \( g \in C^{k+1}(S^1) \), such that \( \int_{S^1} g dm = 0 \), then
\[
\|L_0^n g\|_{C^k} \leq a_n \|g\|_{C^{k+1}}.
\]

3. The sequence \( \{L_0^n\}_{n \in \mathbb{N}} \) is bounded, i.e. there exists \( M' > 0 \) such that
\[
\|L_0^n \phi\|_{C^k} \leq M' \|\phi\|_{C^k}
\]
for all \( \phi \in C^k(S^1) \).

Proof. The regularization property from \( C^k(S^1) \) to \( C^{k+1}(S^1) \) is a straightforward consequence of the regularization inequalities (22) for \( N = 0 \). For the regularization property from \( L^1(S^1) \) to \( C^k(S^1) \), one may see that any \( f \in L^1(S^1) \) admits an anti derivative (in the sense of distributions) \( F \in W^{1,1}(S^1) \), so that in fact \( f \in W^{-1,1}(S^1) \). Thus it follows that Proposition 13 applies, so that \( L_0 f \in C^\infty(S^1) \) and, since the injection \( L^1 \to D_0 \) is continuous:
\[
\|L_0 f\|_{C^k} \leq \|\rho_\xi\|_{C^k} \|L_T f\|_{D_0} \leq \|\rho_\xi\|_{C^k} \|L_T f\|_{L^1} \leq \|\rho_\xi\|_{C^k} \|f\|_{L^1}.
\]
Hence \( L_0 : L^1(S^1) \to C^k(S^1) \) is continuous for any \( k \in \mathbb{N} \). For the second item, one may remark that \( L_0 \) has a strictly positive kernel. There is \( l > 0 \) such that the kernel \( k(x,y) \) of \( L_0 \) satisfies \( k(x,y) \geq l \) and thus (see note 6 of [22] or the proof of [13] Corollary 5.7.1) we have that for any \( g \in V_{k+1}^1(S^1) \), \( \|L_0^n g\|_{L^1} \leq (1 - l)^n ||g||_{L^1} \). Thus by the regularization property, one gets
\[
\|L_0^n g\|_{C^k} \leq C \|L_0^{-1} g\|_{L^1} \leq C(1 - l)^n \|g\|_{L^1} \leq C(1 - l)^n \|g\|_{C^{k+1}}
\]
which proves the claim. For the last item, we once again use the regularization property from \( L^1(S^1) \) to \( C^k(S^1) \), as such. First, we start by remarking that for any \( f \in L^1(S^1) \), the convolution product defined in (19) has the following property: the function \( \rho_\xi \ast f \in L^1(S^1) \), and
\[
\|\rho_\xi \ast f\|_{L^1(S^1)} \leq \|\rho_\xi\|_{L^1(\mathbb{R})} \|f\|_{L^1(S^1)} = \|f\|_{L^1(S^1)}
\]
since \( \rho \) is a probability kernel. Hence, one has, for any \( \phi \in C^k(S^1) \subset L^1(S^1) \),

\[
\|L_0\phi\|_{L^1} \leq \|L_T\phi\|_{L^1} \leq \|\phi\|_{L^1}
\]

which gives, by an immediate induction, \( \|L^n_0\phi\|_{L^1} \leq \|\phi\|_{L^1} \) for any \( N \in \mathbb{N} \) and any \( \phi \in C^k(S^1) \). Thus

\[
\|L^n_0\phi\|_{C^k} \leq \|L_0\|_{L^1 \rightarrow C^k} \|L^{n-1}_0\phi\|_{L^1} \leq \|L_0\|\|\phi\|_{L^1} \leq \|L_0\|_{L^1 \rightarrow C^k} \|\phi\|_{C^k}
\]

\[ \square \]

**Remark 26.** The previous Lemma also applies to each of the perturbed operators \( L_\delta, \delta \in [0, \delta] \).

We may summarize the conclusions of Section 5 in the following way:

**Theorem 27.** Let \( T : S^1 \to S^1 \) be a non-singular map and \((D_\delta)_{\delta \in [0, \delta]}\) a family of diffeomorphisms of the circle, satisfying (24) and (25).

We consider the random dynamical system (17) generated by

\[
T_\delta(\omega, x) = D_\delta \circ T(x) + X_\xi(\omega) \mod 1
\]

where \( X_\xi \) is a centered Gaussian random variable with variance \( \xi^2 \), and the associated (annealed) transfer operator \((L_\delta)_{\delta \in [0, \delta]}\) defined by (13). Then Theorem 7 and 8 apply for the sequence of spaces \( C^{k+1}(S^1) \), \( C^k(S^1) \), \( C^{k-1}(S^1) \) and \( C^{k-2}(S^1) \). i.e linear and quadratic response hold for the stationary measure when \( \delta \to 0 \).

**Proof.** By Lemma 25 and the remark right after, such a system satisfies the assumptions of Theorem 9 for (say) \( B_w = C^k(S^1) \) and \( B_s = C^{k+1}(S^1) \). Hence we get the existence and boundedness of the stationary densities (Assumption LR1) \((h_\delta)_{\delta \in [0, \delta]}\), as well as the good definition of the resolvent operator \( R(1, L_0) \) on the spaces \( C^k(S^1) \) (Assumptions LR3, QR3).

The convergence to equilibrium property for the unperturbed operator (Assumption LR2) is directly established in Lemma 25.

The regularity properties for the family of transfer operators (Assumptions LR4, QRT1, QRT2) are established in Theorem 24 Section 5.2 for the nested sequence of spaces \( C^{k+1}(S^1) \subset C^k(S^1) \subset C^{k-1}(S^1) \subset C^{k-2}(S^1) \).

\[ \square \]

**Remark 28.** One may adapt the estimates in the proof of Theorem 24 so that assumptions of Theorems 1 and 7 hold for the spaces \( B_{ss} = B_s = B_w = B_{ww} = C^k(S^1) \). Notice that in this case, the assumptions of Theorem 9 are still satisfied (see Remark 11).

**5.4. Application: Arnold maps with Gaussian noise.** In this subsection we present an example to which the previous approach apply: the Arnold standard map of the circle, perturbed with Gaussian noise.

More precisely, one takes \( D_\delta := \text{Id} + \delta \) to be the rotation of angle \( \delta \), and \( T \) to be the standard Arnold circle map

\[
T(x) := x + a + \epsilon \sin(2\pi x) \mod 1
\]

with \( \epsilon > 0 \): in particular, it does not matter to us whether \( T \) is a diffeomorphism \((\epsilon < 1)\) or not \((\epsilon > 1)\). Then the random dynamical system induced by this data and a sequence of i.i.d Gaussian random variable \((\Omega_n)_{n \geq 0}\),

\[
X_{n+1} = D_\delta \circ T(X_n) + \Omega_n
\]
satisfies the assumptions of Section 5 for the sequence of spaces $C^{k+1}(\mathbb{S}^1) \subset C^k(\mathbb{S}^1) \subset C^{k-1}(\mathbb{S}^1) \subset C^{k-2}(\mathbb{S}^1)$; in particular linear response holds if we see the density of the stationary measure $h_\delta \in C^{k-1}(\mathbb{S}^1)$ and quadratic response holds if we consider $h_\delta \in C^{k-2}(\mathbb{S}^1)$.

It is also possible to proceed as in [20] (Proposition 17) and write the (almost surely constant) rotation number of this random dynamical system as the integral of some well-chosen observable against its stationary measure, and thus deduce its regularity w.r.t. the ”driving frequency” a (Corollary 18).

6. Linear and Quadratic Response for Expanding Maps

In this section we consider smooth expanding maps on the circle and show they have linear and quadratic response with respect to smooth perturbations. We also provide explicit formulas for the response.

To get the linear response we will consider maps $T : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying the following assumptions

1. $T \in C^4$,
2. $|T'(x)| \geq \alpha^{-1} > 1 \forall x$.

For the quadratic response we will consider $T \in C^5$. We consider a family of perturbations of $T := T_0$ of the kind $T_\delta := D_\delta \circ T$ with $D_\delta = Id + o_\delta \to 0(1)$ in a suitable topology.

In the following subsection we show these systems satisfy the assumptions of Theorems [11] and [7].

6.1. Resolvent for Expanding Maps. In this section we show the existence and continuity properties of the resolvent, needed to apply our response statements to deterministic expanding maps, via Section 4.

More precisely, we show that the transfer operators associated to expanding maps satisfy regularization (here Lasota-Yorke) inequalities (see Assumption 1 of Theorem [9] when acting on suitable Sobolev spaces.

Lemma 29. A $C^{k+1}$ expanding map on $\mathbb{S}^1$ satisfies a Lasota-Yorke inequality on $W^{k,1}(\mathbb{S}^1)$: there is $\alpha < 1$, $A_k$, $B_k \geq 0$ such that

$$
\begin{align*}
\|L^n f\|_{W^{k-1,1}} &\leq A_k \|f\|_{W^{k-1,1}}, \\
\|L^n f\|_{W^{k,1}} &\leq \alpha^k \|f\|_{W^{k,1}} + B_k \|f\|_{W^{k-1,1}}.
\end{align*}
$$

Proof. We will proceed by induction on $k \geq 1$. The Lasota-Yorke inequality for $k = 1$ is well-known, and we refer to, e.g. [7] Section 4.1 for details.

Assume that the Lasota-Yorke inequality is established on $W^{k-1,1}(\mathbb{S}^1)$ for the transfer operator of a $C^k$ expanding map, and consider a $C^{k+1}$ expanding map together with its associated transfer operator. Let $f \in W^{k,1}(\mathbb{S}^1)$. We want to evaluate

$$
\|Lf\|_{W^{k,1}} = \|(Lf)'\|_{W^{k-1,1}} + \|Lf\|_{L^1}.
$$

By the well-known formula, we have that $(Lf)' = L \left( \frac{f'}{T'} \right) + L \left( \frac{T''}{(T')^2} f \right)$, thus we may write, using our induction hypothesis

$$
\|(Lf)'\|_{W^{k-1,1}} \leq \alpha^{k-1} \|f'\|_{W^{k-1,1}} + C_k \|f\|_{W^{k-1,1}}.
$$
For our purposes, we only need precise information on the term carrying the highest derivative of $f$: by Leibniz formula, one has
\[
\left( \frac{f'}{T'} \right)^{(k-1)} = \sum_{\ell=0}^{k-1} \left( \frac{1}{T'} \right)^{k-1-\ell} f^{(\ell+1)} = \frac{f^{(k)}}{T'} + \ldots
\]
so that one gets
\[
\left\| \frac{f'}{T'} \right\|_{W^{k-1,1}} \leq \left\| \frac{f^{(k)}}{T'} \right\|_{\infty} \\|f^{(k)}\|_{L^1} + C_k'\|f\|_{W^{k-1,1}}, \text{ whence}
\]
(44)\[
\|Lf\|_{W^{k,1}} \leq \alpha^k\|f\|_{W^{k,1}} + C_k\|f\|_{W^{k-1,1}}.
\]
We may now iterate this inequality; after $n$ steps we obtain
\[
\|L^n f\|_{W^{k,1}} \leq \alpha^{nk}\|f\|_{W^{k,1}} + C_k \sum_{i=0}^{n-1} \alpha^{k(n-1-i)}\|L^i f\|_{W^{k-1,1}}
\]
and the wanted result follows by power-boundedness of $L^i$ on $W^{k-1,1}$, with
\[B_k := \frac{A_k C_k}{1 - \alpha^k} \square \]

From this last result, we classically deduce the following: for any $k \geq 1$, the transfer operator $L_T$ of a $C^{k+1}$ expanding map $T$ is quasi-compact on $W^{k,1}(\mathbb{S}^1)$. Furthermore, by topological transitivity, $1$ is the only eigenvalue on the unit circle. It is simple and the associated (normalized) eigenfunction, $h$, is the invariant density of the system. The rest of the spectrum is contained in disk of radius strictly smaller than one. In particular, we may write $L = \Pi + R$ with $\Pi : W^{k,1}(\mathbb{S}^1) \to W^{k,1}(\mathbb{S}^1)$ the spectral projector, defined by $\Pi(\phi) := \int_{\mathbb{S}^1} \phi dm$ and $R$ satisfying $R\Pi = \Pi R = 0$ and $\|R^n\phi\|_{W^{k,1}} \leq C\rho^n\|\phi\|_{W^{k,1}}$. Thus we have the following result.

**Proposition 30.** For each $g \in V_k := \{g \in W^{k,1}(\mathbb{S}^1) \text{ s.t. } \int_{\mathbb{S}^1} g \, dm = 0\} = \ker(\Pi)$, it holds
\[
\|L^n g\|_{W^{k,1}} \leq C\rho^n\|g\|_{W^{k,1}}.
\]
In particular, the resolvent $R(1,L) := (\text{Id} - L)^{-1} = \sum_{i=0}^{\infty} L^i$ is a well-defined and bounded operator on $V_k$.

6.2. Small perturbations of expanding maps. In this section, we specify the type of perturbations we consider in the deterministic case, and establish that they satisfy the relative continuity, and Taylor’s expansions assumptions (LR4), (QR2) for the spaces $W^{4,1}(\mathbb{S}^1), W^{3,1}(\mathbb{S}^1), W^{2,1}(\mathbb{S}^1), W^{1,1}(\mathbb{S}^1)$. We will focus on the case of a fixed, $C^4$ expanding map of the circle $T : \mathbb{S}^1 \to \mathbb{S}^1$, perturbed by left composition with a family of diffeomorphisms $(D_\delta)_{\delta \in [-\epsilon,\epsilon]}$.

More precisely, let $D_\delta : \mathbb{S}^1 \to \mathbb{S}^1$ be a diffeomorphism, with
\[
(45) \quad D_\delta = \text{Id} + \delta S
\]
and $S \in C^{k+1}(\mathbb{S}^1, \mathbb{R})$. For such a diffeomorphism, one has the following result.

**Lemma 31.** Assume $k = 0$, i.e. $S \in C^1(\mathbb{S}^1)$. Then in the $C^0(\mathbb{S}^1)$-topology
\[
(46) \quad \left\| \frac{1}{\delta} (D_\delta^{-1} - \text{Id}) + S \right\|_{C^0(\mathbb{S}^1)} \xrightarrow{\delta \to 0} 0
\]
which we sum up in
\[
D_\delta^{-1} = \text{Id} - \delta S + o(\delta).
\]
Proof. Let \( x \in \mathbb{S}^1 \), and \( y = D_\delta(x) \). Then
\[
|D_\delta^{-1}(y) - y + \delta S(y)| = |x - D_\delta(x) + \delta S(D_\delta(x))| = \delta |S(D_\delta(x)) - S(x)|
\leq \delta \|S'\|_\infty d(D_\delta(x), x)
\leq \delta^2 \|S'\|_\infty \|S\|_\infty
\]
hence the result. \(\square\)

Let \( k \in \mathbb{N} \). Our starting point is the remark that for any map \( g \in C^k(\mathbb{S}^1) \), the operator \( M_g \) defined by \( M_g(f) := g \circ f \) is bounded on \( W^{k,1}(\mathbb{S}^1) \): this is an easy consequence of Leibniz formula. In turns, this implies the following proposition.

**Proposition 33.** The transfer operator \( L_{D_\delta} \) associated to a \( C^{k+1} \)-diffeomorphism \( D_\delta \) is bounded on \( W^{k,1}(\mathbb{S}^1) \).

We introduce the notation
\[
J_\delta := \frac{1}{1 + \delta S' \circ D_\delta^{-1}}
\]
for the weight of \( L_{D_\delta} \). We prove the proposition by induction on \( k \in \mathbb{N} \).

**Proof.** For \( k = 0 \), the claim is simply that \( L_{D_\delta} \) is bounded on \( L^1(\mathbb{S}^1) \), which is well-known.

Let us assume that \( L_{D_\delta} : W^{k-1,1}(\mathbb{S}^1) \to \mathbb{S}^1 \) is a bounded operator whenever \( D_\delta = Id + \delta S \) with \( S \in C^k(\mathbb{S}^1) \). If \( S \in C^{k+1}(\mathbb{S}^1) \) we write, for \( f \in W^{k,1}(\mathbb{S}^1) \) that
\[
\|L_{D_\delta}f\|_{W^{k,1}} = \|(L_{D_\delta}f)'\|_{W^{k-1,1}} + \|L_{D_\delta}f\|_{L^1}.
\]
We can thus write that \((L_{D_\delta}f)' = J_\delta' f \circ D_\delta^{-1} + J_\delta^2 f' \circ D_\delta^{-1} \). As
\[
J_\delta' = \frac{\delta S'' \circ D_\delta^{-1}}{(1 + \delta S' \circ D_\delta^{-1})^2} J_\delta,
\]
one has by using the remark above the statement of Proposition 33
\[
\|(L_{D_\delta}f)'\|_{W^{k-1,1}} \leq C_\delta \|L_{D_\delta}f\|_{W^{k-1,1}} + C_\delta' \|L_{D_\delta}f\|_{W^{k-1,1}}.
\]
The result follow by induction hypothesis. \(\square\)

We now turn to continuity estimates for the map \( \delta \mapsto L_{D_\delta} \in L(W^{k,1}(\mathbb{S}^1), W^{k-1,1}(\mathbb{S}^1)) \).

Our first step is the following lemma.

**Lemma 34.** Let \( f \in W^{1,1}(\mathbb{S}^1) \), and let \( H : \mathbb{S}^1 \to \mathbb{S}^1 \) be an orientation-preserving homeomorphism. Then
\[
\|f \circ H - f\|_{L^1} \leq \|H^{-1} - Id\|_\infty \|f'\|_{L^1}.
\]

**Proof.** This is a straightforward consequence of Fubini-Tonelli theorem. Lifting everything to \( \mathbb{R} \), we consider a monotone, increasing lift of \( H \) that we still denote \( H \). Since \( f \) is \( W^{1,1} \), it is the integral of its derivative and one has
\[
\int_0^1 (f \circ H(x) - f(x))dx = \int_0^1 \int_0^1 1_{[x,H(x)]} f'(t)dt dx = \int_0^1 f'(t) \int_0^1 1_{[H^{-1}(t), t]}(x) dx dt.
\]
The result follows by taking absolute values in the previous equality. □

**Proposition 35.** For an orientation preserving, $C^{k+1}$ diffeomorphism $D_\delta = \text{Id} + \delta S$, one has

\begin{equation}
\|L_{D_\delta} - \text{Id}\|_{W^{k,1}\to W^{k-1,1}} \leq C\delta. \tag{50}
\end{equation}

**Proof.** We prove the proposition by induction on $k \in \mathbb{N}$. For $k = 1$, let $f \in W^{1,1}(\mathbb{S}^1)$ and write $L_{D_\delta}f - f = (J_\delta - 1)f \circ D_\delta^{-1} + f \circ D_\delta^{-1} - f$, so that

\begin{align*}
\|L_{D_\delta}f - f\|_{L^1} &\leq \delta S' \circ D_\delta^{-1} L_{D_\delta}f \|_{L^1} + \|f \circ D_\delta^{-1} - f\|_{L^1} \\
&\leq \delta \|S'\|_{L^\infty} \|f\|_{L^1} + \delta \|S\|_{L^\infty} \|f'\|_{L^1} \\
&\leq \|S\|_{C^1} \|f\|_{W^{1,1}},
\end{align*}

as in Proposition 33. By using the previous lemma and the remark above the statement of Proposition 33, one has $\|L_{D_\delta}f - f\|_{W^{1,1}} = \|(L_{D_\delta}f - f')\|_{W^{k-1,1}} + \|L_{D_\delta}f - f\|_{L^1}$, with

\begin{align*}
(L_{D_\delta}f - f') &= J_\delta f \circ D_\delta^{-1} + J_\delta f' \circ D_\delta^{-1} - f' \\
&= J_\delta f \circ D_\delta^{-1} + J_\delta L_{D_\delta}(f') - f' + (J_\delta - 1)f'.
\end{align*}

In view of (17), (50) and remark 32, $(J_\delta - 1)/\delta J_\delta$ and $J_\delta/fJ_\delta$ are bounded in $\|\cdot\|_{C^1}$-norm when $\delta \to 0$. By induction hypothesis, $\|L_{D_\delta}(f') - f'\|_{W^{k-1,1}} \leq C\delta \|f'\|_{W^{k,1}}$. Thus,

\begin{align*}
\|(L_{D_\delta}f - f')\|_{W^{k-1,1}} &\leq C\delta \|L_{D_\delta}f\|_{W^{k-1,1}} + C\delta \|f'\|_{W^{k,1}} + C\delta \|L_{D_\delta}f'\|_{W^{k-1,1}},
\end{align*}

and the conclusion follows from the case $k = 1$ and Proposition 33. □

We now turn to differentiability estimates, i.e. we establish first-order Taylor’s expansion for the map $\delta \mapsto L_{D_\delta} \in L(W^{k,1}(\mathbb{S}^1), W^{k-1,1}(\mathbb{S}^1))$. In this endeavor, the first step is to define the derivative operator $R : W^{k,1}(\mathbb{S}^1) \to W^{k-1,1}(\mathbb{S}^1)$, by

\begin{equation}
R(f) := -(fS)' \tag{51}
\end{equation}

Then we have the following proposition.

**Proposition 36.** Consider an orientation-preserving, $C^{k+1}$ diffeomorphism $D_\delta : \mathbb{S}^1 \to \mathbb{S}^1$ as in (13).

Let us consider the associated transfer operator $L_{D_\delta} : W^{k,1}(\mathbb{S}^1) \to W^{k,1}(\mathbb{S}^1)$. Then

\begin{equation}
\lim_{\delta \to 0} \left\| \frac{L_{D_\delta}f - f}{\delta} + (fS)' \right\|_{W^{k-1,1}} = 0 \tag{52}
\end{equation}

for each $f \in W^{k,1}(\mathbb{S}^1)$.

**Proof.** We start by writing, for $f \in W^{k,1}(\mathbb{S}^1),

\begin{align*}
L_{D_\delta}f - f - \delta R(f) &= J_\delta f \circ D_\delta^{-1} - f + \delta(fS)' \\
&= (J_\delta - 1)f \circ D_\delta^{-1} + \delta fS' + f \circ D_\delta^{-1} - f + (\delta S)f'.
\end{align*}

As $(I) = -\delta (L_{D_\delta}(fS') - fS')$, by Proposition 33 we have the bound

\[\|(I)\|_{W^{k-1,1}} \leq C\delta^2 \|f\|_{W^{k,1}}\]
for all $k \in \mathbb{N}$, which obviously implies the wanted result. For (II) we proceed by induction on $k \in \mathbb{N}$. First we consider, for $f \in W^{1,1}(\mathbb{S}^1)$,

$$
\frac{1}{\delta}(f(D_{\delta}^{-1}(x)) - f(x) + \delta Sf'(x)) = \frac{1}{\delta}\int_{x}^{D_{\delta}^{-1}(x)} (f'(t) - f'(x))dt + \frac{1}{\delta}(D_{\delta}^{-1}(x) - x + \delta S(x))f'(x)
$$

$$
= \frac{D_{\delta}^{-1}(x) - x}{\delta} \int_{x}^{D_{\delta}^{-1}(x)} (f'(t) - f'(x))dt + o(1)f'(x)
$$

$$
\to 0,
$$

for a.e $x \in \mathbb{S}^1$ by Lebesgue’s differentiation theorem. Note that the $o(1)$ comes from (46), and should be understood here as a $C^0$ function of $x$ that goes to 0 with $\delta$ uniformly in $x$.

Furthermore, by (46), for $\delta$ small enough,

$$
\left|\frac{D_{\delta}^{-1}(x) - x}{\delta}\right| \leq \frac{3}{2}|S'(x)|
$$

$$
\left|\frac{D_{\delta}^{-1}(x) - x - S'(x)}{\delta}\right| \leq \frac{3}{4}
$$

uniformly in $x$. Similarly, by Lebesgue’s differentiation theorem, for $\delta$ small enough and a.e $x \in \mathbb{S}^1$,

$$
\frac{1}{D_{\delta}^{-1}(x) - x} \int_{x}^{D_{\delta}^{-1}(x)} (f'(t) - f'(x))dt \leq \frac{1}{2}
$$

Hence the quantity considered is bounded almost everywhere by the $L^1$ function $\frac{3}{4}(|S| + |f'|)$, which is independent of $\delta$.

Hence, Lebesgue’s dominated convergence theorem apply, and one has

$$
\frac{1}{\delta}\int_{\mathbb{S}^1} |f \circ D_{\delta}^{-1} - f + \delta Sf'|dm \to 0,
$$

i.e $\|(II)\|_{L^1} = o(\delta)$.

Let us assume now that if $D_{\delta}$ is a $C^{k+1}$ diffeomorphism, $\|(II)\|_{W^{k-1,1}} = o(\delta)$ holds for $f \in W^{k,1}(\mathbb{S}^1)$.

If $D_{\delta}$ is $C^{k+2}$, let $f \in W^{k+1,1}(\mathbb{S}^1)$. We write as usual $\|(II)\|_{W^{k,1}} = \|(II)\|_{W^{k-1,1}} + \|(II)\|_{L^1}$, where

$$
(II)' = (f \circ D_{\delta}^{-1})' - f' + (\delta Sf')' = L_{D_{\delta}}(f') - f' - \delta R(f').
$$

Hence, by induction hypothesis, $\|(II)\|_{W^{k-1,1}} = o(\delta)$ and the conclusion follows from the case $k = 1$.

We are left to verify that the transfer operator $L_{D_{\delta}}$ has a second order Taylor’s expansion at $\delta = 0$. In that perspective, one needs to obtain more precise information on the family of diffeomorphisms $(D_{\delta})_{\delta \in [0,\delta]}$.

**Lemma 37.** Let $S \in C^2(\mathbb{S}^1)$ and $D_{\delta} = Id + \delta S$ be a $C^2$ diffeomorphism. Then

$$
(53) \quad \left\| \frac{D_{\delta}^{-1} - Id + \delta S}{\delta^2} - S.S' \right\|_{C^0(\mathbb{S}^1)} \to 0,
$$

which we sum up in $D_{\delta}^{-1} = Id - \delta S + \delta^2 S.S' + o(\delta^2)$. 

Remark 38. The term $o(\delta^2)$ should be understood as a $C^0$ function that goes to 0 with $\delta$, uniformly in $x$.

Although we do not prove it here, in the general case $S \in C^{k+2}(\mathbb{S}^1)$, the Taylor expansion holds in $C^k(\mathbb{S}^1)$ topology, which means the $o(\delta)$ term should be understood as a $C^k(\mathbb{S}^1)$ function that goes to 0 in $C^k$ norm as $\delta \to 0$.

The proof is similar to the proof of (46).

Next, we introduce a second derivative operator, $Q : W^{k,1}(\mathbb{S}^1) \to W^{k-2}(\mathbb{S}^1)$, defined by

$$Qf := (f.S^2)'' .$$

We are now in a position to formulate the following result:

Proposition 39. Let $k \geq 2$ and $D_\delta : \mathbb{S}^1 \to \mathbb{S}^1$ be a $C^{k+1}$ diffeomorphism (as in (45)), and let $L_{D_\delta} : W^{k+1}(\mathbb{S}^1) \to W^{k+1}(\mathbb{S}^1)$ be its transfer operator, and $Q, R$ be the derivatives operator (51), (54). One has

$$L_{D_\delta}f - f - \delta R(f) - 2^{-1}\delta^2 Q(f) = J_\delta f \circ D_\delta^{-1} - f + \delta(fS)' - 2^{-1}\delta^2(fS^2)''(I) + (II)$$

with

$$(I) := (J_\delta - 1) f \circ D_\delta^{-1} + \delta fS' - \delta^2(fS'S + f((S')^2 + SS''))$$

$$(II) := f \circ D_\delta^{-1} - f + (\delta S - \delta^2 SS')f' - 2^{-1}\delta^2 S^2 f'' .$$

In view of (47) and (51), it is possible to rewrite $(I) = -\delta (L_{D_\delta}(fS') - fS' - \delta R(fS'))$.

Taking into account Proposition 50 we get $\|L_{D_\delta}\|_{W^{k-2,1}} = o(\delta^2)$ as $\delta \to 0$.

For the term $(II)$, we proceed by induction on $k \geq 2$.

For $k = 2$, we start by evaluating $(II)$ at $x \in \mathbb{S}^1$. Using mean value theorem one gets

$$\int_x^{D_\delta^{-1}(x)} (f'(t) - f'(x))dt + (D_\delta^{-1}(x) - x + \delta S(x) - \delta^2 S(x)S'(x)) f' - 2^{-1}\delta^2 S^2(x) f''(x)$$

$$= \int_x^{D_\delta^{-1}(x)} \int_x^t (f''(s) - f''(x))dsdt + (D_\delta^{-1}(x) - x + \delta S(x) - \delta^2 S(x)S'(x)) f'(x) f''(x)$$

$$+ 2^{-1} ((D_\delta^{-1}(x) - x)^2 - \delta^2 S^2(x)) f''(x).$$

For a.e $x \in \mathbb{S}^1$, this last quantity is $o(\delta^2)$. Indeed, by virtue of (46) one has $(D_\delta^{-1} - Id)^2 = \delta^2 S^2 + o(\delta^2)$, hence

$$\frac{1}{\delta^2} \int_x^{D_\delta^{-1}(x)} \int_x^t (f''(s) - f''(x))dsdt = \frac{(D_\delta^{-1}(x) - x)^2}{\delta^2} \frac{1}{(D_\delta^{-1}(x) - x)^2} \int_x^{D_\delta^{-1}(x)} \int_x^t (f''(s) - f''(x))dsdt,$$

by Lebesgue's differentiation theorem, and

$$2^{-1} ((D_\delta^{-1}(x) - x)^2 - \delta^2 S^2(x)) f''(x) = o(\delta^2) f''(x).$$
Similarly, by (53)
\[(D^{-1}_\delta(x) - x + \delta S(x) - \delta^2 S(x)S'(x)) f'(x) = o(\delta^2 f'(x))\]
By virtue of (56) and (53), one has
\[
\frac{1}{\delta^2} \left| 2^{-1} \left(D^{-1}_\delta(x) - x \right)^2 - \delta^2 S^2(x) \right| f''(x) \leq \frac{3}{4} |f''(x)|
\]
\[
\frac{1}{\delta^2} \left| (D^{-1}_\delta(x) - x + \delta S(x) - \delta^2 S(x)S'(x)) f'(x) \right| \leq \frac{3}{4} |f'(x)|,
\]
uniformly in \(x \in S^1\) for \(\delta\) small enough. Finally, for \(\delta\) small enough, the right-hand side of (56) is bounded almost everywhere by \(\frac{3}{4} S^2(x)\).

From what precedes, if \(\delta\) is small enough, (II) is bounded a.e by the \(L^1\) function \(\frac{3}{4} (S^2 + |f'| + |f''|)\), independent of \(\delta\).

Thus, by Lebesgue’s dominated convergence theorem, \(\|(II)\|_{L^1} = o(\delta^2)\).

Assuming that the property holds at \(k \in \mathbb{N}\), we now take \(D_\delta\) to be a \(C^{k+2}\) diffeomorphism, and let \(f \in W^{k+1,1}(S^1)\). We have
\[
\|(II)\|_{W^{k-1,1}} = \|(II)'\|_{W^{k-2,1}} + \|(II)\|_{L^1},
\]
and computing yields
\[
(II)' = LD_\delta(f') - f' - \delta R(f') - 2^{-1} \delta^2 Q(f').
\]
Thus, by induction hypothesis, \(\|(II)\|_{W^{k-2,1}} = o(\delta^2)\), and the conclusion follows from the case \(k = 2\).

6.3. Application: Explicit perturbation of the doubling map. An example of system to which the previous discussion apply is the following perturbation of the doubling map \((T_\delta)_{\delta \in [0,\bar{\delta}]}\) defined by
\[(57)\]
\[T_\delta := 2x + \delta \sin(4\pi x) \mod 1\]
which falls under the setup described in Section 6.2 with \(T_0(x) := 2x \mod 1\) and \(D_\delta(x) := x + \delta \sin(2\pi x) \mod 1\), and the spaces \(B_{ss} = W^{4,1}(S^1) \subset B_s = W^{3,1}(S^1) \subset B_w = W^{2,1}(S^1) \subset W^{1,1}(S^1) = B_w\).

Indeed, the system satisfies uniform Lasota-Yorke estimates (for \(\delta_0\) small enough) by Lemma 29 and Proposition 30; in particular Theorem 9 applies. Furthermore, this example satisfies the regularity requirements of Section 6.2 so that Propositions 35, 36 and 39 apply.

This implies that the assumptions of Theorems 1 and 7 are satisfied, so for this family of systems, linear response holds if one considers the invariant density \(h_\delta\) as a \(W^{2,1}(S^1)\) function, and quadratic response holds if one considers the invariant density \(h_\delta\) as a \(W^{1,1}(S^1)\) function.

This example is certainly well-known, and may be obtained by other methods ([21] for linear response or [11, 30] for higher-order response). However, the nice feature of this example is the possibility to compute everything: here we have
\[
L_0 f(x) = \frac{1}{2} \left[ f \left( \frac{x}{2} \right) + f \left( \frac{1+x}{2} \right) \right]
\]
\[
\hat{L} h_0(x) = R[L_0 h_0](x) = -2\pi \cos(2\pi x)
\]
\[
\hat{L} h_0(x) = Q[L_0 h_0](x) = 8\pi^2 \cos(4\pi x)
\]
with $h_0 = 1$ the invariant density of the unperturbed system. Notice that for $f(x) = \cos(2\pi x)$ we have
\[ L_0 f(x) = \frac{1}{2} \left[ \cos(\pi x) + \cos(\pi x + \pi) \right] = 0. \]

Hence, applying (4) yields
\[
\frac{d}{d\delta} h_\delta(x) \big|_{\delta = 0} = \sum_{n=0}^{\infty} L_0^n \dot{h}_0 = \sum_{n=0}^{\infty} L_0^n (-2\pi \cos(2\pi x))
\]
\[
= -2\pi \cos(2\pi x) - 2\pi \sum_{n \geq 1} L_0^n (\cos(2\pi x))
\]
\[
= -2\pi \cos(2\pi x)
\]

Similarly, we may compute the quadratic term with (10):
\[
\frac{d^2}{d\delta^2} h_\delta(x) \big|_{\delta = 0} = (R(1, L_0)\dot{L})^2 h_0 + R(1, L_0)Q h_0 = Qh_0 + L_0 Qh_0 = 8\pi^2(\cos(4\pi x) + \cos(2\pi x))
\]
so that one may obtain the following explicit, order two Taylor’s expansion for $h_\delta$:
\[
(58) \quad h_\delta(x) = 1 - 2\pi \delta \cos(2\pi x) + 4\pi^2 \delta^2 (\cos(4\pi x) + \cos(2\pi x)) + o(\delta^2).
\]

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