Theoretical investigations of an information geometric approach to complexity

Sean Alan Ali\textsuperscript{1} and Carlo Cafaro\textsuperscript{2}

\textsuperscript{1}Albany College of Pharmacy and Health Sciences, 12208 Albany, New York, USA and
\textsuperscript{2}SUNY Polytechnic Institute, 12203 Albany, New York, USA

It is known that statistical model selection as well as identification of dynamical equations from available data are both very challenging tasks. Physical systems behave according to their underlying dynamical equations which, in turn, can be identified from experimental data. Explaining data requires selecting mathematical models that best capture the data regularities. The existence of fundamental links among physical systems, dynamical equations, experimental data and statistical modeling motivate us to present in this article our theoretical modeling scheme which combines information geometry and inductive inference methods to provide a probabilistic description of complex systems in the presence of limited information. Special focus is devoted to describe the role of our entropic information geometric complexity measure. In particular, we provide several illustrative examples wherein our modeling scheme is used to infer macroscopic predictions when only partial knowledge of the microscopic nature of a given system is available. Finally, limitations, possible improvements, and future investigations are discussed.

PACS numbers: Chaos (05.45.-a), Complexity (89.70.Eg), Entropy (89.70.Cf), Probability Theory (02.50.Cw), Riemannian Geometry (02.40.Ky).
I. INTRODUCTION

The inherent relationships among dynamics, modeling and complexity is indeed a remarkable occurrence in the physical sciences [1]. In actual experiments, information associated with the state of a physical system is measured and collected at various points in space and time. In order to obtain an understanding of the physics underlying the behavior of the system, the dynamical equations governing the evolution of the system must be reconstructed from the data. Indeed, the deduction of dynamical laws from empirical data is a fundamental aspect of science [2–4]. In the recent work [5], it was shown that the deduction of the dynamical equations of a system from empirical data is NP hard and computationally intractable. Moreover, this result is valid for both classical and quantum systems, independent of the amount of data that is collected. This seems to suggest that various intimately related issues, such as the determination of dynamical equations that best approximates data, or comparing families of dynamical models to data are generally intractable.

It is known that analysis of a system’s data can, in some cases, enable identification of classes of regularities in the behavior of relevant variable(s) of the actual system. It is widely accepted that classes exhibiting either complete regularity (i.e. perfectly ordered) or nonexistent regularity (i.e. maximally random) would be classified as entirely non-complex in the sense that such systems possess no structure [6–8]. By contrast, structured systems that admit correlations among the constituents of the system can be very complex. Indeed, correlation and structure are not entirely independent of randomness.

Well defined, useful measures of complexity are generally introduced in scenarios that take into consideration the complete sequence of events that lead to the emergence of the system whose complexity is being quantified [9]. For such complexity measures, only those states which are reached through a difficult sequence of intermediate states is deemed complex. For example, the notion of pattern is important in quantifying the complexity of a noisy quantum channel [10]. The logical and thermodynamic depths also play the role of complexity measures. The thermodynamic depth was proposed by Lloyd and Pagels and represents the amount of entropy produced during the evolution of a state of a system [11]. The logical depth was proposed by Bennett and represents the run time needed for a universal Turing machine to execute the minimal program that reproduces (for example) a system’s configuration [12].

Since the sequence of intermediate states leading to the final state of a system is of primary importance when defining a proper measure of complexity, application of simple thermodynamic criteria to the states being compared are generally inadequate. For instance, thermodynamic potentials adequately serve to measure a system’s capacity for irreversible change, but do not agree with accepted notions of complexity [13]. Specifically, the thermodynamic entropy
is a monotone functional of temperature in which high (low) temperature corresponds to high (low) randomness, respectively. However, since there exist multiple functions which vanish in the regimes of extreme order and disorder, it is evident that this property alone does not adequately constrain a useful statistical complexity measure \[8\]. By statistical complexity measure we mean a quantity that measures the average quantity of memory required to statistically reproduce a given configuration. Despite these facts however, it is undeniable that thermodynamics plays a critical role in the characterization and understanding of the complexity in reversible, dissipative systems \[12\].

The difficulty encountered in the construction of a viable theory from a given set of data can be approximately identified with the notion of cripticity. On the other hand, the difficulty associated with extracting predictions from a theory can be viewed as a loose interpretation of the concept of logical depth. Both cripticity and logical depth are deeply related to complexity. The ability to extract predictions from a theory can be quite difficult, but is more so in composite systems involving interactions among subsystems. The introduction of such interactions lead to the so-called fluctuation growth which may give rise to nonlinear and chaotic dynamics. Such phenomena are common and may occur in natural as well as artificial, complex dynamical systems \[14\]. An issue of fundamental concern in the physics of complex systems is model reduction. Model reduction in this context refers to the identification of low-dimensional models that capture the gross features of the original high-dimensional system \[15\]. It is often true that consideration of the dynamics of a system alone may not be sufficient to render reliable predictions. In such cases, entropic considerations must be accounted for as well \[16\].

In this article, we make use of the so-called Entropic Dynamics (ED) \[17\], a theoretical framework built on both Maximum relative Entropy (MrE) methods \[18\] and information geometric techniques \[19\]. We emphasize that ED is formally similar to other generally covariant theories: the dynamics is reversible, the trajectories are geodesics, the system supplies its own notion of intrinsic time, the motion can be derived from a variational principle of the form of Jacobi’s action principle rather than the more familiar principle of Hamilton \[20, 21\]. In brief, the canonical Hamiltonian formulation of ED is an example of an information constrained dynamics where the information-constraints play the role of generators of evolution. The ED approach has been applied to the derivation of Newton’s dynamics \[22\] and various aspects of quantum theory \[23\]. For more details on the ED, we refer to \[17\].

Inspired by the ED approach to physics and motivated by the fundamental links among physical systems, dynamical equations, experimental data and statistical modeling, we present our theoretical information geometric scheme used to model the dynamics of systems (of arbitrary nature) that are described by probability distributions. We focus attention on the role of the our information geometric complexity and entropy measures in characterizing dynamical
systems described by probability distributions. In particular, the relationship among the information geometric entropy, the coarse-grained Boltzmann, von Neumann and the Kolmogorov-Sinai (KS) dynamical entropies in the appropriate regimes will be explored in our examples. The layout of this article is as follows: in Section II, we review the MrE formalism used to update probabilities with both information and data constraints. In Section III, we present a brief introduction to Information Geometry (IG). In Section IV, we introduce the information-geometric indicators of complexity for our theoretical model, namely the information geometric entropy ($S_M$) and the information geometric complexity ($C_M$). In Section V, we present ten applications where our theoretical model is used to study the dynamical complexity of statistical models corresponding to the systems being investigated. Concluding remarks are presented in Section VI.

II. THE MAXIMUM RELATIVE ENTROPY METHOD

The MrE method represents an efficient scheme for the updating of a family of prior probability distribution functions when new information becomes available in the form of constraints on the family of allowed posteriors. The utility and versatility of the MrE method rests in the possibility of updating a family of prior probability distribution functions in presence of both data and constraints of the expected value type. This characteristic was initially presented in a formal manner in [18] wherein it was demonstrated that Bayes updating can be viewed as a special case of the MrE method. An analysis of the practical utility of this powerful feature of the MrE method in real world applications was presented in [24]. We direct the reader to [25] for a recent application of the MrE method to the physics of ferromagnetic materials. In what follows, we present the scheme for updating probabilities in the presence of both expected value and data constraints.

In the remainder of this article, the quantities $x$ and $\theta$ represent the microstate and macrostate of the system, respectively. Furthermore, the microstates $x$ are elements of the microspace $\mathcal{X}$, while the macrostates $\theta$ are elements of the parameter space $\mathcal{D}_\theta$. We utilize the MrE method to update a prior probability distribution into a posterior probability distribution. In particular, we seek to render inferences on some quantity $\theta \in \mathcal{D}_\theta$ given: i) the prior information about quantity $\theta$; ii) the known functional relationship among variables $x \in \mathcal{X}$ and $\theta \in \mathcal{D}_\theta$; and iii) the observed values of the quantity $x \in \mathcal{X}$. The search space for the posterior probability distribution occurs within the product space $\mathcal{X} \times \mathcal{D}_\theta$, while the joint distribution is specified by $P(x, \theta)$. The transition from $P_{\text{old}}(\theta)$ to $P_{\text{new}}(\theta)$ is
stipulated by,

\[ P_{\text{new}}(\theta) \overset{\text{def}}{=} \int dx P_{\text{new}}(x, \theta). \] (1)

The joint probability \( P_{\text{new}}(x, \theta) \) serves to maximize the relative entropy \( S[P|P_{\text{old}}] \),

\[ S[P|P_{\text{old}}] \overset{\text{def}}{=} -\int dx d\theta P(x, \theta) \log \left[ \frac{P(x, \theta)}{P_{\text{old}}(x, \theta)} \right], \] (2)

subject to the known information constraints. Observe that \( P_{\text{old}}(x, \theta) \),

\[ P_{\text{old}}(x, \theta) = P_{\text{old}}(x|\theta) P_{\text{old}}(\theta), \] (3)

is known as the joint prior, while \( P_{\text{old}}(\theta) \) and \( P_{\text{old}}(x|\theta) \) represent the Bayesian prior and likelihood, respectively. It is worth noting that both the joint prior and the standard Bayesian prior both serve to encode prior information concerning the quantity \( \theta \in D_\theta \).

It should be noted at this juncture that the likelihood will be taken as prior information due to its representation as the a priori established relation between \( \theta \in D_\theta \) and \( x \in \mathcal{X} \). The relevant information constraints are enumerated as follows: first, we have the normalization constraint,

\[ \int dx d\theta P(x, \theta) = 1. \] (4)

Second, we have the information constraint associated with some function \( f(\theta) \) specified by,

\[ \int dx d\theta f(\theta) P(x, \theta) \overset{\text{def}}{=} (f(\theta)) = F. \] (5)

Finally, we are left to consider the observed data represented by \( x' \). In the context of the MrE framework, knowledge of this information naturally leads to infinitely many constraints,

\[ \int d\theta P(x, \theta) = P(x) = \delta(x - x'), \] (6)

for any \( x \in \mathcal{X} \) where \( \delta \) denotes the Dirac delta function. By means of the Lagrange multiplier technique, we proceed to maximize the logarithmic relative entropy in Eq. (2) relative to the constraints in Eqs. (4), (5), and (6). We stipulate that the variation of the entropy \( S[P|P_{\text{old}}] \) with respect to \( P \) equals zero identically,

\[ \delta \left\{ S[P|P_{\text{old}}] + \alpha \left[ \int dx d\theta P(x, \theta) - 1 \right] + \beta \left[ \int dx d\theta f(\theta) P(x, \theta) - F \right] + \int dx \gamma(x) \left[ \int d\theta P(x, \theta) - \delta(x - x') \right] \right\} = 0. \] (7)
It is determined that Eq. (7) can be reduced to,

\[ \int dx d\theta \left[ -\log P(x, \theta) - 1 + \log P_{\text{old}}(x, \theta) + \alpha + \beta f(\theta) + \gamma(x) \right] \delta P(x, \theta) = 0, \tag{8} \]

for any \( \delta P(x, \theta) \). Hence, from Eq. (8) we obtain

\[ P_{\text{new}}(x, \theta) = P_{\text{old}}(x, \theta) \exp \left[ -1 + \alpha + \beta f(\theta) + \gamma(x) \right], \tag{9} \]

where the Lagrange multipliers \( \alpha, \beta, \) and \( \gamma(x) \) can be determined via substitution of Eq. (9) into Eqs. (4), (5) and (6). After some algebraic manipulation, we are able to obtain

\[ P_{\text{new}}(x, \theta) = \frac{\exp \left[ \beta f(\theta) \right] P_{\text{old}}(x', \theta) \delta(x - x')}{\int d\theta \exp \left[ \beta f(\theta) \right] P_{\text{old}}(x, \theta)}. \tag{10} \]

Finally, upon marginalizing \( P_{\text{new}}(x, \theta) \) relative to the variable \( x \), we deduce the desired updated prior probability distribution function

\[ P_{\text{new}}(\theta) \overset{\text{def}}{=} \int dx P_{\text{new}}(x, \theta) = \frac{\exp \left[ \beta f(\theta) \right] P_{\text{old}}(x', \theta) \int d\theta \exp \left[ \beta f(\theta) \right] P_{\text{old}}(x, \theta)}{\int d\theta \exp \left[ \beta f(\theta) \right] P_{\text{old}}(x, \theta)}. \tag{11} \]

For convenience we define

\[ \Delta(x', \beta) \overset{\text{def}}{=} \int d\theta \exp \left[ \beta f(\theta) \right] P_{\text{old}}(x', \theta), \tag{12} \]

in terms of which Eq. (11) becomes

\[ P_{\text{new}}(\theta) = P_{\text{old}}(\theta) P_{\text{old}}(x'|\theta) \frac{\exp \left[ \beta f(\theta) \right]}{\Delta(x', \beta)}. \tag{13} \]

Observe that in scenarios lacking constraints of the form of expected values, the parameter \( \beta = 0 \) leading Eq. (13) to reduce to the known Bayes updating relation

\[ P_{\text{new}}(\theta) = \frac{P_{\text{old}}(\theta) P_{\text{old}}(x'|\theta)}{P_{\text{old}}(x')}. \tag{14} \]

In an effort to be complete, we raise attention to the fact that Eq. (14) may be obtained via combination of Bayes theorem,

\[ P_{\text{old}}(\theta|x) = \frac{P_{\text{old}}(\theta) P_{\text{old}}(x|\theta)}{P_{\text{old}}(x)}, \tag{15} \]

with Bayes rule,

\[ P_{\text{new}}(\theta) = P_{\text{old}}(\theta|x'). \tag{16} \]
III. INFORMATION GEOMETRY

In [26], the so-called Information Geometric Approach to Chaos (IGAC) was presented. The IGAC makes use of the ED formalism to characterize and quantify the complexity of geodesic information flows on statistical manifolds underlying the entropic dynamics of physical systems described in terms of probability distributions. The IGAC can be viewed as the information geometric counterpart of conventional geometrodynamical approaches [27, 28], with the usual classical configuration space being replaced with a suitable statistical manifold. In particular, the IGAC represents an information geometric extension of the Jacobi geometrodynamics [29].

A. The Fisher-Rao Information Metric

In this subsection we describe some properties of the statistical manifold \( \mathcal{M} \). An \( n \)-dimensional \( \mathcal{C}^\infty \) differentiable manifold \( \mathcal{M} \) can be viewed as a set of points \( p \) that admits coordinate charts \( U_M \) such that the following two conditions are satisfied: 1) each element \( c \in U_M \) is a one-to-one mapping from \( M \) to some open subset of \( \mathbb{R}^n \); 2) given any one-to-one mapping \( \phi \) from \( M \) to \( \mathbb{R}^n \) for all \( c \in U_M \), the mapping \( \phi \in U_M \leftrightarrow \phi \circ c^{-1} \) is a \( \mathcal{C}^\infty \) diffeomorphism. In particular, the set of probability distributions \( p(x|\theta) \) forms an \( n \)-dimensional statistical manifold \( \mathcal{M} \)

\[
\mathcal{M} \overset{\text{def}}{=} \{ p(x|\theta) : \theta = (\theta^1, \ldots, \theta^n) \in \mathcal{D}_\theta \},
\]

(17)

The points \( p \) belonging to the space \( \mathcal{M} \) are parameterized by \( n \) real-valued variables \( (\theta^1, \ldots, \theta^n) \). Hence, the parameters \( \theta \) serve as coordinates for the point \( p \). The set \( \mathcal{D}_\theta \) is the whole parameter space available to the system and is a subset of \( \mathbb{R}^n \)

\[
\mathcal{D}_\theta \overset{\text{def}}{=} \bigotimes_{k=1}^n \mathcal{I}_{\theta^k} = (\mathcal{I}_{\theta^1} \otimes \mathcal{I}_{\theta^2} \cdots \otimes \mathcal{I}_{\theta^n}) \subseteq \mathbb{R}^n,
\]

(18)

where \( \mathcal{I}_{\theta^k} \) is a subset of \( \mathbb{R} \) and represents the full range of allowed values of the macrostates \( \theta \).

At this point it is important to emphasize that there exists an infinite number of Riemannian metrics on the manifold \( \mathcal{M} \). For this reason, a necessary and fundamental assumption within the IG framework is the choice of the Fisher-Rao information metric as that which underlies the Riemannian geometry of \( \mathcal{M} \) [19, 30, 31], namely

\[
g_{\mu\nu}(\theta) \overset{\text{def}}{=} \int dx p(x|\theta) \partial_\mu \log p(x|\theta) \partial_\nu \log p(x|\theta) = 4 \int dx \partial_\mu \sqrt{p(x|\theta)} \partial_\nu \sqrt{p(x|\theta)} = - \left( \frac{\partial^2 S(\theta', \theta)}{\partial \theta^\nu \partial \theta'^\mu} \right)_{\theta'=\theta},
\]

(19)

with \( \mu, \nu = 1, \ldots, n \) for an \( n \)-dimensional manifold, \( \partial_\mu \overset{\text{def}}{=} \frac{\partial}{\partial \theta^\mu} \) and \( S(\theta', \theta) \) representing the logarithmic relative
The information metric $g_{\mu\nu}(\theta)$ is a positive-definite, symmetric Riemannian metric that defines a measure of distinguishability among macrostates on $\mathcal{M}$. It satisfies the two following properties: 1) invariance under (invertible) transformations of elements of the microspace $\mathcal{X}$; 2) covariance under reparametrization of the elements of the macrospace $\mathcal{D}_\theta$. The invariance of $g_{\mu\nu}(\theta)$ under reparametrization of the microspace $\mathcal{X}$ implies that

$$\begin{split}
\mathbb{R}^l \subseteq \mathcal{X} \ni x \mapsto y \overset{\text{def}}{=} f(x) \in \mathcal{Y} \subseteq \mathbb{R}^l \implies p(x|\theta) \mapsto p'(y|\theta) = \left[ \frac{1}{\frac{\partial f}{\partial x}} p(x|\theta) \right]_{x = f^{-1}(y)}. \quad (21)
\end{split}$$

The covariance under reparametrization of elements of the parameter space $\mathcal{D}_\theta$ (homeomorphic to $\mathcal{M}$) implies that

$$\begin{split}
\mathcal{D}_\theta \ni \theta \mapsto \theta' \overset{\text{def}}{=} f(\theta) \in \mathcal{D}_{\theta'} \implies g_{\mu\nu}(\theta) \mapsto g'_{\mu\nu}(\theta') = \left[ \frac{\partial \theta^\alpha}{\partial \theta'^\nu} \frac{\partial \theta'^\beta}{\partial \theta^\nu} g_{\alpha\beta}(\theta) \right]_{\theta = f^{-1}(\theta')}, \quad (22)
\end{split}$$

where $g_{\mu\nu}(\theta)$ is given in Eq. (19) and

$$g'_{\mu\nu}(\theta') = \int dx p'(x|\theta') \partial'_\mu \log p'(x|\theta') \partial'_\nu \log p'(x|\theta'), \quad (23)$$

with $\partial'_{\mu} = \frac{\partial}{\partial \theta'^{\mu}}$ and $p'(x|\theta') = p(x|\theta = f^{-1}(\theta')).$

It is known from IG [19] that there is a one-to-one relation between elements of $\mathcal{M}$ and $\mathcal{D}_\theta$. In particular, the statistical manifold $\mathcal{M}$ is homeomorphic to the parameter space $\mathcal{D}_\theta$. This implies the existence of a continuous, bijective map $h_{\mathcal{M},\mathcal{D}_\theta}$,

$$h_{\mathcal{M},\mathcal{D}_\theta} : \mathcal{M} \ni p(x|\theta) \mapsto \theta \in \mathcal{D}_\theta,$$

where $h_{\mathcal{M},\mathcal{D}_\theta}^{-1}(\theta) = p(x|\theta)$.

### B. Entropic Motion

Given a statistical manifold $\mathcal{M}$ with metric $g_{\mu\nu}(\theta)$, the ED program is concerned with the following issue [17]: given the known initial $\theta_{\text{initial}}$ and final $\theta_{\text{final}}$ states of a system, what is the expected trajectory of the system as it evolves? In answering this question, the ED framework implicitly assumes that a trajectory exists and furthermore, the trajectory follows from a principle of inference, namely the MrE method [18]. These assumptions imply that large
changes arise as a consequence of a continuous succession of small incremental changes. By considering only small changes in going from $\theta_{\text{initial}}$ to $\theta_{\text{final}} = \theta_{\text{initial}} + \Delta \theta$, the distance $\Delta l$ between such states is given by,

$$\Delta l^2 \overset{\text{def}}{=} g_{\mu \nu}(\theta) \Delta \theta^\mu \Delta \theta^\nu. \quad (25)$$

In what follows, we primarily follow the work of Caticha \[17\]. In going from $\theta_{\text{initial}}$ to $\theta_{\text{final}}$, the system must necessarily sample a midway point, namely a state $\theta$ that is equidistant between $\theta_{\text{initial}}$ and $\theta_{\text{final}}$. We remark that there is no special significance to this chosen halfway state. We could equally well have argued that in evolving from $\theta_{\text{initial}}$ to $\theta_{\text{final}}$, the system must initially sample a state that is doubly distant from $\theta_{\text{final}}$ as it is from $\theta_{\text{initial}}$. More generally, it is necessary only for the system to sample some intermediate state $\theta_\xi$ whereby having already covered a distance $dl$ from $\theta_{\text{initial}}$, a distance $\xi dl$ alone remains to be traversed in order to arrive at $\theta_{\text{final}}$. Midway states correspond to $\xi = 1$, one-third of the way states correspond to $\xi = 2$, and so on. Each value of $\xi$ provides a different criterion by which the trajectory may be selected. If there exist a multiplicity of ways by which a trajectory is determined, consistency demands all the different approaches should agree. As such, the selected trajectory must necessarily be independent of the choice of $\xi$.

In line with the above reasoning, the aforementioned issue of concern to the ED program can be effectively re-framed as follows: if a system in initial state $p(x|\theta_{\text{initial}})$ is subject to new information constraints, then the system will evolve to one of the neighboring states in the family $p(x|\theta_\xi)$. The relevant question becomes: how do we select the proper $p(x|\theta_\xi)$? Such a reformulation of the ED problem is naturally of the type that can be analyzed by means of the MrE method. In particular, recall that the MrE method is designed for the specific purpose of processing information. It enables the transition from an initial family of models described by a prior probability distribution, to a new family of models described by a posterior distribution when the available information is represented by specification of the set of distributions from which the posterior must be selected. Typically, this set of distributions is constrained by the known moments of some relevant set of variables. This is not necessary however, nor do the information-constraints need not be linear functionals. In the ED framework, constraints are defined geometrically.

When using the MrE method, it is important to specify which entropy is to be maximized. The selection of a distribution $p(x|\theta)$ requires maximization of the entropy functional of form

$$S[p|q] \overset{\text{def}}{=} -\int dx p(x|\theta) \log \left[ \frac{p(x|\theta)}{q(x)} \right]. \quad (26)$$

Equation (26) serves to define the entropy of $p(x|\theta)$ relative to the prior $q(x)$. It is obvious that the selected posterior distribution coincides with the prior distribution in the absence of new information-constraints. Since the distribution
that maximizes $S[p|q]$ when new information-constraints are lacking is $p \propto q$, we must take $q(x)$ to represent the prior. Assuming the system is known to be in the initial state $p(x|\theta_{\text{initial}})$ and we do not have any information that the system has changed, then we have no reason to assume any change has occurred. The prior $q(x)$ should therefore be chosen such that maximization of $S[p|q]$ subject to no new information-constraints leads naturally to the posterior $q(x) = p(x|\theta_{\text{initial}})$. By contrast, if the system is known to be in the initial state $p(x|\theta_{\text{initial}})$ and we then obtain new information that the system has evolved to one of the neighboring states within the family $p(x|\theta_\xi)$, then the correct selection of the posterior probability distribution is obtained by maximizing the entropy,

$$S[\theta|\theta_{\text{initial}}] \overset{\text{def}}{=} - \int dx p(x|\theta) \log \left( \frac{p(x|\theta)}{p(x|\theta_{\text{initial}})} \right),$$

subject to the constraint $\theta = \theta_\xi$.

In order to facilitate analysis, we assume the system evolves from $\theta_{\text{initial}}$ to $\theta_{\text{final}} = \theta_{\text{initial}} + \Delta \theta$. Moreover, we denote by $\theta_\xi = \theta_{\text{initial}} + d\theta$ with $\xi \in \mathbb{R}_0^+$ an arbitrary intermediate state that is infinitesimally close to $\theta_{\text{initial}}$. Hence, the distance $d(\theta_{\text{initial}}, \theta_{\text{final}})$ is given by $d\theta_{\text{initial} \rightarrow \text{final}} = g_{\mu\nu}(\theta) \Delta \theta^\mu \Delta \theta^\nu$, while the distance between $\theta_{\text{initial}}$ to and $\theta_\xi$ is given by,

$$d\theta_{\text{initial} \rightarrow \xi} = g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu.$$  

(28)

The distance between $\theta_\xi$ and $\theta_{\text{final}}$ can be written as,

$$d\theta_{\xi \rightarrow \text{final}} = g_{\mu\nu}(\theta) (\Delta \theta^\mu - d\theta^\mu) (\Delta \theta^\nu - d\theta^\nu).$$

(29)

The MrE maximization problem then essentially reduces to that of maximization of the functional

$$S[\theta_{\text{initial}} + d\theta|\theta_{\text{initial}}] = -\frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu = -\frac{1}{2} d\theta_{\text{initial} \rightarrow \xi}^2$$

under variations of $d\theta$ subject to the geometric constraint,

$$\xi d\theta_{\text{initial} \rightarrow \xi} = d\theta_{\xi \rightarrow \text{final}}.$$  

(30)

(31)

It necessarily follows that

$$\delta \left[ -\frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu - \gamma (\xi^2 d\theta_{\text{initial} \rightarrow \xi} - d\theta_{\xi \rightarrow \text{final}}^2) \right] = 0,$$

(32)

where $\gamma$ denotes a Lagrangian multiplier. Substituting Eqs. (28) and (29) into Eq. (32), we find

$$\{ [1 + 2\gamma (\xi^2 - 1)] d\theta_\mu + 2\gamma \Delta \theta_\mu \} \delta (d\theta^\mu) = 0.$$  

(33)
Since Eq. (33) must be satisfied for any \( \delta (d\theta^\mu) \), it must be true that \( \left\{ 1 + 2\gamma (\xi^2 - 1) \right\} d\theta^\mu + 2\gamma \Delta \theta^\mu = 0 \), that is,

\[
d\theta^\mu = \chi \Delta \theta^\mu, \tag{34}
\]

where \( \chi = \chi (\xi, \gamma) \) is defined as,

\[\chi (\xi, \gamma) \overset{\text{def}}{=} \frac{1}{(1 - \xi^2) - \frac{1}{2\gamma}}. \tag{35}\]

In order to determine the value of the Lagrange multiplier \( \gamma \), we recognize that the geometric constraint in Eq. (31) can be recast as, \( \xi^2 dl^2_{\text{initial} \rightarrow \xi} - dl^2_{\xi \rightarrow \text{final}} = 0 \). Then, by using Eqs. (28), (29) and (34), we are able to obtain

\[
\left[ \xi^2 \chi^2 - (1 - \chi)^2 \right] g_{\mu\nu}(\theta) \Delta \theta^\mu \Delta \theta^\nu = 0, \text{ such that } \xi^2 \chi^2 - (1 - \chi)^2 = 0. \tag{36}\]

Combination of Eqs. (35) and (36), leads to

\[\chi (\xi) \overset{\text{def}}{=} \frac{1}{1 + \xi} \text{ and, } \gamma (\xi) \overset{\text{def}}{=} - \frac{1}{2\xi (1 + \xi)}. \tag{37}\]

In this manner we were able to determine

\[dl^2_{\text{initial} \rightarrow \xi} \overset{\text{def}}{=} \frac{1}{(1 + \xi)^2} \Delta \theta^2, \tag{38}\]

and

\[dl^2_{\xi \rightarrow \text{final}} \overset{\text{def}}{=} \frac{\xi^2}{(1 + \xi)^2} \Delta \theta^2. \tag{39}\]

From Eqs. (38) and (39), it is true that

\[dl_{\text{initial} \rightarrow \xi} + dl_{\xi \rightarrow \text{final}} = \frac{1}{1 + \xi} \Delta \theta + \frac{\xi}{1 + \xi} \Delta \theta = \Delta \theta. \tag{40}\]

However, since \( dl^2_{\text{initial} \rightarrow \text{final}} \overset{\text{def}}{=} g_{\mu\nu}(\theta) \Delta \theta^\mu \Delta \theta^\nu = \Delta \theta^2 \), we have

\[dl_{\text{initial} \rightarrow \text{final}} = \Delta \theta. \tag{41}\]

By combining Eqs. (40) and (41), we are able to show that \( dl_{\text{initial} \rightarrow \text{final}} = dl_{\text{initial} \rightarrow \xi} + dl_{\xi \rightarrow \text{final}} \). Thus, given

\[\Delta \theta \overset{\text{def}}{=} d\theta + (\Delta \theta - d\theta), \tag{42}\]

we have determined that,

\[\| \Delta \theta \| = \| d\theta \| + \| \Delta \theta - d\theta \|, \tag{43}\]
where \( \|\Delta \theta\| \defeq \sqrt{dl_{\text{initial}\to\text{final}}^2} \), \( \|d\theta\| \defeq \sqrt{dl_{\text{initial}\to\xi}^2} \) and, \( \|\Delta \theta - d\theta\| \defeq \sqrt{dl_{\xi\to\text{final}}^2} \). In view of Eq. (42), we conclude that Eq. (43) is satisfied provided that \(d\theta\) and \(\Delta \theta - d\theta\) are collinear. That is to say, the triangle defined by the triple \(\theta_{\text{initial}}, \theta_{\xi}, \theta_{\text{final}}\) will degenerate into a straight line. This result is sufficient to determine a short segment of the trajectory since all intermediate states lie on the straight line whose endpoints are \(\theta_{\text{initial}}\) and \(\theta_{\text{final}}\). The generalization beyond short trajectories is as follows: if any three neighboring points along a curve lie along a straight line segment, then the curve in question is necessarily a geodesic. We emphasize that this result is independent of the arbitrarily chosen \(\xi\).

A geodesic on an \(n\)-dimensional statistical manifold \(\mathcal{M}\) represents the maximum probability path a complex dynamical system explores in its evolution from \(\theta_{\text{initial}}\) to \(\theta_{\text{final}}\). Each point that lies on the geodesic curve is parametrized by the variables \(\theta\) defining the macrostates of the system. Each component \(\theta^\kappa\) with \(\kappa = 1, \ldots, n\) is a solution of the geodesic equation \[17, 33, 34\],

\[
\frac{d^2\theta^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu} \frac{d\theta^\mu}{d\tau} \frac{d\theta^\nu}{d\tau} = 0.
\]

Furthermore, each macrostate \(\theta\) is in a one-to-one correspondence with the probability distribution \(p(x|\theta)\) representing the maximally probable description of the system being considered. This is a distribution of the microstates \(x\). In summary, the solution to the main ED problem is as follows \[17\]: the expected trajectory between known initial and final states is the geodesic curve that connects them. It is worth noting however, that this result alone is insufficient to conclude that the actual trajectory coincides with the expected trajectory. An affirmative result to this issue depends on whether the information encoded in the initial state is sufficient for such prediction. For further discussion on this particular issue, we refer to Ref. \[35\].

C. Volumes in Curved Statistical Manifolds

Upon establishing a quantitative notion of distinguishability among probability distributions in terms of distances assigned relative to the Fisher-Rao information metric \(g_{\mu\nu}(\theta)\), we may construct the Riemannian volume element \(dV_{\mathcal{M}}\) to be used as the natural measure in the space of distributions. To this end, we consider an \(n\)-dimensional volume of the statistical manifold \(\mathcal{M}\) of distributions \(p(x|\theta)\) labelled by parameters \(\theta^\kappa\) with \(\kappa = 1, \ldots, n\). The parameters \(\theta^\kappa\) represent coordinates for the point \(p\). Furthermore, we consider very small regions of the manifold \(\mathcal{M}\) wherein we employ Cartesian coordinates where the metric assumes the form of the identity matrix \(\delta_{ab}\) since curved spaces are locally flat. In locally Cartesian coordinates \(\varphi\), the volume element is given by the product \(dV_{\mathcal{M}} \defeq \prod_{k=1}^{n} d\varphi^k\), which in
terms of the old coordinates $\theta^\kappa$ reads,

$$dV_M \overset{\text{def}}{=} \left| \frac{\partial \varphi}{\partial \theta} \right| d\theta^1 d\theta^2 \ldots d\theta^n. \quad (45)$$

The next clear task is the evaluation of the Jacobian $\left| \frac{\partial \varphi}{\partial \theta} \right|$ associated with the transformation that takes the metric $g_{\mu\nu}$ into its flat (i.e. Euclidean) form $\delta_{ab}$. We define the new coordinates as $\varphi^a \overset{\text{def}}{=} \Phi^a (\theta^1, ..., \theta^n)$ where $\Phi$ denotes a coordinate transformation map $\Phi : \theta \rightarrow \varphi$. Therefore, a small change $d\theta$ corresponds to a small change $d\varphi$ according to

$$d\varphi^a \overset{\text{def}}{=} X^a_{\mu} d\theta^\mu$$

and the Jacobian is given by the determinant of the matrix $X^a_{\mu}$, $\left| \frac{\partial \varphi}{\partial \theta} \right| \overset{\text{def}}{=} |\det (X^a_{\mu})|$. The distance between two neighboring points is the same whether it is computed in either the old or new coordinates, $d^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu = \delta_{ab} d\varphi^a d\varphi^b$. Hence, the old and the new metrics are related via $g_{\mu\nu} (\theta) = \delta_{ab} X^a_{\mu} X^b_{\nu}$.

We now consider statistical manifolds with Fisher-Rao information metric tensor $g_{\mu\nu} (\theta)$ with $\theta \overset{\text{def}}{=} (\theta^1, ..., \theta^n)$ whose determinant can be written as,

$$\det [g_{\mu\nu} (\theta)] = g (\theta^1, ..., \theta^n) = \prod_{\kappa=1}^{n} g_{\kappa} (\theta^\kappa). \quad (47)$$

From Eq. (47), we assume that the determinant $g (\theta)$ can be factorized in a product of $n$-functions $g_{\kappa} (\theta^\kappa)$ with $1 \leq \kappa \leq n$, where each function depends, at worst, on a single variable $\theta^\kappa$. If there is no dependence on the variable $\theta^\kappa$, we simply assign $g_{\kappa} (\theta^\kappa) = 1$. Note that both uncorrelated and correlated Gaussian statistical manifolds satisfy this set of assumptions. If Eq. (47) is not satisfied however, it is only required that $\sqrt{g (\theta)}$ to be integrable over the $n$-dimensional space $D_\theta$ defined in Eq. (13). Thus, upon taking the determinant of $g_{\mu\nu} (\theta)$, we obtain $g (\theta) \overset{\text{def}}{=} \det [g_{\mu\nu} (\theta)] = [\det (X^a_{\mu})]^2$ and therefore $|\det (X^a_{\mu})| = \eta_{(M, g)} (\theta^1, ..., \theta^n)$, where $\eta_{(M, g)} (\theta^1, ..., \theta^n)$ is the so-called Fisher density and equals the square root of the determinant of the metric tensor $g_{\mu\nu} (\theta)$.

$$\eta_{(M, g)} (\theta^1, ..., \theta^n) \overset{\text{def}}{=} \sqrt{g (\theta)}. \quad (48)$$

Observe that $\sqrt{g (\theta)} d^n \theta$ is a scalar quantity and is consequently invariant under orientation preserving (i.e. with positive Jacobian) general coordinate transformations $\theta \rightarrow \theta'$. The square root of the determinant $g (\theta)$ of the metric tensor $g_{\mu\nu} (\theta)$ and the flat infinitesimal volume element $d^n \theta$ transform as,

$$\sqrt{g (\theta)} \overset{\theta \rightarrow \theta'}{\rightarrow} |\frac{\partial \theta'}{\partial \theta}| \sqrt{g (\theta')}, \quad d^n \theta \overset{\theta \rightarrow \theta'}{\rightarrow} |\frac{\partial \theta}{\partial \theta'}| d^n \theta', \quad (49)$$
respectively. Hence, it follows that,
\[ \sqrt{g(\theta')}d^n\theta' \rightarrow \sqrt{g(\theta')}d^n\theta'. \]
(50)

For further details on these issues, we refer the reader to Caticha’s 2012 tutorial [36].

We have finally succeeded in obtaining all the elements necessary to express the Riemannian volume element completely in terms of the coordinates \( \theta \) and the known metric \( g_{\mu\nu}(\theta) \), namely
\[ dV_M = \sqrt{g(\theta')}d^n\theta'. \]
(51)

By use of Fubini’s theorem, the volume of any extended region on the manifold can be expressed as follows,
\[ V_M(s) \overset{\text{def}}{=} \int_{D(\text{geodesic})} d\theta = \int_{s_0}^{s_0+s} \int \sqrt{g(\theta', \ldots, \theta^n)}d\theta^n, \]
(53)

where \( s \in \mathbb{R} \). We remark that any permutation of the order of integration may be considered in Eq. (51) leaving the final result unchanged. The integration space \( D(\text{geodesic}) \) in Eq. (51) is defined as
\[ D(\text{geodesic}) \overset{\text{def}}{=} \{ \theta^\kappa(\alpha) : \theta^\kappa(s_0) \leq \theta^\kappa \leq \theta^\kappa(s_0+s) \}, \]
(52)
where \( \kappa = 1, \ldots, n \) and \( s_0 \leq \alpha \leq s_0+s \) such that \( \theta^\kappa = \theta^\kappa(\alpha) \) satisfies Eq. (44). The integration space \( D(\text{geodesic}) \) is an \( n \)-dimensional subspace of the whole (permitted) parameter space \( D_\theta \). The elements of \( D(\text{geodesic}) \) are the \( n \)-dimensional macrostates \( \theta \) whose components \( \theta^\kappa \) are bounded by specified limits of integration \( \theta^\kappa(s_0) \) and \( \theta^\kappa(s_0+s) \).

The limits of integration are obtained via integration of the \( n \)-dimensional set of geodesic equations. Now, by use of Eqs. (47) and (51), we obtain
\[ \int \sqrt{g(\theta', \ldots, \theta^n)}d\theta^n = \prod_{\kappa=1}^{n} \int_{s_0}^{s_0+s} \sqrt{g_{\kappa\kappa}(\theta^\kappa(\alpha))}d\theta^\kappa d\alpha, \]
(53)

where in Eq. (53) we have made use of the following equivalence,
\[ \prod_{\kappa=1}^{n} \int_{\theta^\kappa(s_0)}^{\theta^\kappa(s_0+s)} \sqrt{g_{\kappa\kappa}(\theta^\kappa)}d\theta^\kappa = n \int_{s_0}^{s_0+s} \sqrt{g_{\kappa\kappa}(\theta^\kappa(\alpha))}d\theta^\kappa d\alpha. \]
(54)

The extended volume defined in Eq. (53) depends formally on both \( s_0 \) (the selected initial instant from which we start computing a relevant hyper-volume in \( D(\text{geodesic}) \)) and \( s \) (the measure of the set of instances over which we observe the growth or change of the hyper-volume). This procedure does not present a problem since we are free to perform the additional two steps: 1) Take the limit for \( s_0 \) approaching 0. Indeed, we may also integrate in \( ds_0 \) with \( 0 \leq s_0 \leq \epsilon \), where \( \epsilon \) denotes the measure of the set of initial instances. 2) Take the limit for \( s \) approaching infinity (this is valid since we are primarily interested in the asymptotic behavior of the IGE). One may be concerned about the potential impact of selecting a different set of initial conditions. Such issues however already enter at the level of the definition of the functional forms for the geodesic trajectories \( \theta^\kappa(\alpha) \) used to characterize the extended volume \( V_M(s) \) over which one integrates. For this reason, these issues do not affect the formal definition of the IGE.
IV. THE INFORMATION GEOMETRIC COMPLEXITY

Within the IGAC framework, we are interested in a probabilistic description of the evolution of a given system in terms of its corresponding probability distribution on $\mathcal{M}$ which is homeomorphic to $\mathcal{D}_\theta$. For the sake of argument, consider the evolution of a system from $s_{\text{initial}}$ to $s_{\text{final}}$. In the context of the present probabilistic description of the MrE method [18], analysis of this evolution is equivalent to studying the maximally probable path leading from $\theta (s_{\text{initial}})$ to $\theta (s_{\text{final}})$. In order to quantify the complexity of such path, we propose the so-called information geometric entropy (IGE) $S_{\mathcal{M}} (\tau)$ as a good quantifier of complexity [37, 38].

Within the context of our theoretical modeling scheme, the average dynamical statistical volume $C_{\mathcal{M}} (\tau)$ [which we choose to name the information geometric complexity (IGC)] is defined as [37],

$$C_{\mathcal{M}} (\tau) \overset{\text{def}}{=} \frac{1}{\tau} \int_0^\tau ds V_{\mathcal{M}} (s) . \tag{55}$$

The IGC defined in Eq. (55) represents the volume of the effective parameter space explored by the system at affine time $\tau$. Alternatively, $C_{\mathcal{M}} (\tau)$ may be interpreted as the temporal evolution of the system’s uncertainty volume $C_{\mathcal{M}} (0)$ after an affine temporal duration $\tau$ has elapsed. Its faithful geometric visualization may be highly non trivial, especially in high-dimensional spaces.

The IGE, an indicator of temporal complexity of geodesic information flows, is defined in terms of the IGC as follows,

$$S_{\mathcal{M}} (\tau) \overset{\text{def}}{=} \log [C_{\mathcal{M}} (\tau)] . \tag{56}$$

The original idea underlying the formulation of the IGE was to provide a quantitative means by which to encode dynamical information residing within the hyper-volume $\mathcal{D}_\theta^{(\text{geodesic})}$.

An analogue of our IGE can be found in the work of Myung, Balasubramanian and Pitt [2–4] as well as the work of Rodriguez [39]. In both cases, the authors introduce a quantity that serve to quantify intrinsic complexity, which in both cases is comprised of two contributions. These two contributions are related to the notions of Bayesian complexity penalty and minimum description length. In both cases, each of the two contributions to intrinsic complexity are deemed to be inherent properties of the statistical model describing the system under investigation since each of the two contributions are independent of data. The first contribution is defined as the product of the number of free parameters in the model (up to a constant multiplicative factor) with the natural logarithm of the data sample size (also up to a constant multiplicative factor). The second contribution to the intrinsic complexity is defined as the logarithm of the integral of the Riemannian volume element of a suitable parameter manifold of the model.
Riemannian volume element is in turn specified in terms of the Fisher density on the parameter manifold. With regard to this latter integral, the authors state: we will always cut off the ranges of the parameters to ensure that volumes are finite. These ranges should be considered as part of the functional form of the model. By construction, this volume is independent of the parametrization. This second contribution to the geometric complexity appearing in is reminiscent of our IGE both in terms of its formal construction and its invariance under reparametrization of the statistical model. In the case of our IGE however, there are two noteworthy differences. Firstly, the dependent lower and upper limits of integration (of the extended volume appearing in the definition of the IGE) defines elements of . The functional dependence of these limits in turn depend upon the nature of the geodesic equations underlying the statistical model being considered. Indeed, the elements of are solutions of the geodesic equations of the system.

This system of geodesic equations is integrated with suitable boundary conditions prior to the computation of the hyper-volume . Secondly, our IGE represents an affine temporal average of the -fold integral of the Fisher density over maximum probability trajectories (geodesics) and serves as a measure of the number of the accessible macrostates in the statistical configuration manifold after a finite affine temporal increment . The affine temporal average has been introduced in order to average out the possibly very complex fine details of the entropic dynamical description of the system on .

In what follows, we discuss the connection between the IGE and the Kolmogorov-Sinai dynamical entropy. The notion of entropy is introduced, in both classical and quantum physics, to quantify the missing information about a system’s coarse-grained state . In the case of classical systems, it is convenient to partition the phase space into fine-grained cells of uniform volume , labelled by an index . In the absence of knowledge of which cell the system occupies, one assigns probabilities to each cell. In the limit of infinitesimal cells, and with the same state of knowledge as in the previous coarse-grained case, one instead makes use of the phase-space density . Then, the asymptotic expression for the information required to characterize a particular coarse-grained trajectory up to time is given by the Shannon information entropy (measured in bits),

\[
S_{\text{classical}}^{(\text{chaotic})} = -\int dX \rho(X) \log_2 \left[ \rho(X) \Delta v \right] = -\sum_j p_j \log_2 p_j \approx h_{\text{KS}} t, \tag{57}
\]

where represents the phase-space density and is the probability of the corresponding coarse-grained trajectory. The quantity is the KS dynamical entropy or metric entropy ( is actually an entropy rate, i.e. an entropy per unit time), and represents the rate of information increase. The quantity represents the missing information about which coarse-grained cell the system occupies. According to the Alekseev-Brudno theorem, the information (the quantity is formally known as the Kolmogorov algorithmic complexity) associated
with a segment of a trajectory of length $|t|$ is asymptotically equal to

$$h_{KS} = \lim_{|t| \to \infty} \frac{I(t)}{|t|}. \quad (58)$$

Stated in an alternative manner, the Alekseev-Brudno theorem implies that: the KS entropy measures the algorithmic complexity of classical trajectories [44]. Within the IGAC, the information geometric analogue $h_{KS}^{M}$ of the KS dynamical entropy $h_{KS}$ takes the form

$$h_{KS}^{M} = \lim_{\tau \to \infty} \left\{ \lim_{\Delta \tau \to 0} \frac{S_{M}(\tau + \Delta \tau) - S_{M}(\tau)}{\Delta \tau} \right\}. \quad (59)$$

V. APPLICATIONS

In the following, we outline ten selected applications concerning the complexity characterization of geodesic paths on curved statistical manifolds within the IGAC framework. Work featuring some of the early conceptual developments of the IGAC framework can be found in [45–47]. A more updated overview of the IGAC appear in [38, 48, 49]. The initial series of applications presented in this Section represent systems of arbitrary nature. Such systems are not only instructive, but also serve as building blocks used to construct the more sophisticated, physically motivated models appearing later in the section.

A. Uncorrelated Gaussian Statistical Model

In [37, 50], we apply the IGAC to study the dynamics of an uncorrelated Gaussian statistical model specified by the probability distribution

$$p(x|\theta) = \prod_{k=1}^{l} p(x_k|\mu_k, \sigma_k), \quad \text{with} \quad p(x_k|\mu_k, \sigma_k) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi \sigma_k^2}} \exp \left[ -\frac{(x_k - \mu_k)^2}{2\sigma_k^2} \right], \quad (60)$$

where $x = (x_1, \ldots, x_l)$ and $\theta = (\mu_1, \ldots, \mu_l, \sigma_1, \ldots, \sigma_l)$. The Gaussian distribution in Eq. (60) has $l$ degrees of freedom, each one described by two pieces of relevant information, its mean expected value $\mathbb{E}(x_k) = \langle x_k \rangle = \mu_k$ and its variance $\mathbb{E}(x_k - \mu_k)^2 = \Delta x_k = \sqrt{\langle (x_k - \langle x_k \rangle)^2 \rangle} = \sigma_k$ (Gaussian statistical macrostates). The line element $ds^2 = g_{\alpha\beta}(\theta) d\theta^\alpha d\theta^\beta$ ($\alpha, \beta = 1, \ldots, 2l$) of the Fisher-Rao information metric $g_{\alpha\beta}(\theta)$ on $\mathcal{M}$ is found to be [50],

$$ds^2 = \sum_{k=1}^{l} \left( \frac{1}{\sigma_k^2} d\mu_k^2 + \frac{2}{\sigma_k^2} d\sigma_k^2 \right). \quad (61)$$

This leads to consider a statistical model on a non-maximally symmetric $2l$-dimensional statistical manifold $\mathcal{M}$. The IGE $S_{M}(\tau)$ increases linearly in affine time and is moreover, proportional to the number of degrees of freedom of the
The asymptotic linear growth of the IGE may be viewed as an information-geometric analogue of the von Neumann entropy growth introduced by Zurek-Paz, a quantum feature of chaos. The parameter $\lambda_M \in \mathbb{R}$ serves to characterize the family of probability distributions on $\mathcal{M}$.

At this juncture, in anticipation of the correlated nature of the following applications, we remark that in the presence of correlated constraints among the microstates of a system, the product rule in Eq. (60) assumes a generalized form, while the metric tensor in Eq. (19) will no longer contain identically vanishing off-diagonal elements. Under such scenarios the generalized version of the product rule in Eq. (60) takes the form

$$p_{\text{total}} (x_1, \ldots, x_l) = \prod_{j=1}^{l} p_j (x_j) \rightarrow p_{\text{total}} (x_1, \ldots, x_l) \neq \prod_{j=1}^{l} p_j (x_j),$$

with

$$p'_{\text{total}} (x_1, \ldots, x_l) = p_l (x_l | x_1, \ldots, x_{l-1}) p_{l-1} (x_{l-1} | x_1, \ldots, x_{l-2}) \cdots p_2 (x_2 | x_1) p_1 (x_1).$$

On the one hand, correlations among the microvariables of a system may be introduced via information-constraints of the form $x_j = f_j (x_1, \ldots, x_{j-1}), \forall j = 2, \ldots, l$. In such a case

$$p'_{\text{total}} (x_1, \ldots, x_l) = \delta (x_l - f_l (x_1, \ldots, x_{l-1})) \delta (x_{l-1} - f_{l-1} (x_1, \ldots, x_{l-2})) \cdots \delta (x_2 - f_2 (x_1)) p_1 (x_1),$$

where the $j$-th probability distribution is given by

$$p_j (x_j) = \int \cdots \int dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_l p'_{\text{total}} (x_1, \ldots, x_l).$$

On the other hand, correlations among the microvariables of a system may also be introduced by means of the so-called correlation coefficients $\rho_{ij}$,\[52\],

$$\rho_{ij} = \rho (x_i, x_j) \overset{\text{def}}{=} \frac{(x_i x_j) - (x_i) (x_j)}{\sigma_i \sigma_j},$$

with $\rho_{ij} \in (-1, 1)$ and $i, j = 1, \ldots, l$. The probability distribution describing a $2l$-dimensional Gaussian model with non-vanishing correlations is given by

$$p(x|\theta) = \frac{1}{(2\pi)^l \det C(\theta)^{1/2}} \exp \left[-\frac{1}{2} (x - m)^T \cdot C^{-1}(\theta) \cdot (x - m) \right] \neq \prod_{j=1}^{l} \left(2\pi \sigma_j^2\right)^{-1/2} \exp \left[-\frac{(x_j - \mu_j)^2}{2\sigma_j^2}\right],$$

where $x = (x_1, \ldots, x_l)$, $m = (\mu_1, \ldots, \mu_l)$ and $C(\theta)$ is the $(2l \times 2l)$-dimensional (non-singular) covariance matrix. In the following subsections we consider several statistical models with correlated microstates.
B. Correlated Bivariate Gaussian Statistical Model

In this application we consider a correlated bivariate Gaussian model \[53\]. The ratio between the IGC in presence and absence of micro-correlations is explicitly computed, leading to an intriguing though not yet deeply understood connection with the phenomenon of geometric frustration \[54\]. Specifically, we study in \[53\] a 2D Gaussian model specified by the probability distribution

\[
p(x_1, x_2|\mu, \sigma) = \frac{\exp\left\{ -\frac{1}{2\sigma^2(1-\rho^2)} \left[ (x_1 - \mu)^2 - 2\rho(x_1 - \mu)(x_2 - \mu) + (x_2 - \mu)^2 \right] \right\}}{2\pi\sigma^2\sqrt{1-\rho^2}}, \tag{69}
\]

where the correlation coefficients \(\rho_{ij}\) are defined in \[67\]. The line element of the Fisher-Rao information metric associated with \(p(x_1, x_2|\mu, \sigma)\) is given by

\[
ds^2 = \frac{1}{\sigma^2(1-\rho^2)} d\mu^2 + \frac{4}{\sigma^2} d\sigma^2. \tag{70}
\]

The asymptotic expression of the IGC in this case is found to be

\[
C(\tau) \underset{\tau \to \infty}{\approx} \left( \frac{4\sqrt{2}}{\sigma_0 A_1} \right) \frac{\sqrt{1+\rho}}{\tau} \text{ with } \rho \in (-1, 1), \tag{71}
\]

where \(A_1 \in \mathbb{R}\) is an integration constant, \(\sigma_0 = \sigma(\tau)|_{\tau=0} \in \mathbb{R}\) and \(\sigma(\tau)\) satisfies the geodesic equation \[44\]. We may compare the ratio of the asymptotic expression of the ICGs in the presence and absence of correlations, yielding the IGC ratio

\[
R_{\text{bivariate}}^{\text{strong}}(\rho) \overset{\text{def}}{=} \frac{C_M(\tau)}{C_M(\tau)|_{\rho=0}} = \sqrt{1+\rho}, \tag{72}
\]

where \textit{strong} represents the case in which the underlying microstates of the system are maximally connected. The ratio \(R_{\text{bivariate}}^{\text{strong}}(\rho)\) results in an increasing \textit{monotone} function of \(\rho\). From Eq. \(72\), it is evident that for anti-correlated variables, an increase in one variable gives rise to a corresponding decrease in the remaining variable; this result implies that variables become more distant and therefore more distinguishable relative to the Fisher-Rao information metric. By contrast, for positively correlated variables, an increase or decrease in one variable always predicts a similarly directed change in the remaining variable. In this scenario, variables do not become more distant and consequently, are not more distinguishable relative to the Fisher-Rao metric. This result seems to suggest that when anti-correlations are present, evolution on the statistical manifold induced by the system reduces in complexity.
C. Correlated Trivariate Gaussian Statistical Model

In [53], we study the IG of a trivariate Gaussian statistical model where the multivariate normal joint distributions for \( n \) real-valued microstates \( x_1, \ldots, x_n \) is given by

\[
p(x|\theta) = \frac{1}{\sqrt{(2\pi)^n \det C}} \exp \left[ -\frac{1}{2} (x - \mu)^T C^{-1} (x - \mu) \right],
\]

where \( C \) denotes the \( n \times n \) symmetric, positive definite covariance matrix with entries \( c_{ij} = \text{E}(x_i x_j) - \text{E}(x_i)\text{E}(x_j) \), \( i, j = 1, \ldots, n \). It is assumed that the mean and variance of each of the three microstates are (i.e. \( \mu_x = \mu_y = \mu_z = \mu \) and \( \sigma_x = \sigma_y = \sigma_z = \sigma \)). Furthermore, each model has different correlational structure between microstates. In this section we consider a Gaussian statistical model in Eq. (73) for the case \( n = 3 \). The covariance matrices corresponding to these cases are given by [55],

\[
C_1 = \sigma^2 \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \sigma^2 \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & 0 \\ 0 & \rho & 1 \end{pmatrix}, \quad C_3 = \sigma^2 \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.
\]

(74)

1. Case 1

First, we consider the trivariate Gaussian statistical model corresponding to the case \( C = C_1 \). The line element of the Fisher-Rao information metric corresponding to this choice of covariance matrix \( C_1 \) is given by,

\[
d s^2 = 3 + \frac{\rho}{1 + \rho} 2 \mu^2 + \frac{6}{\sigma^2} d \sigma^2.
\]

(75)

The asymptotic expression of the IGC in this case is given by

\[
C_M(\tau) \xrightarrow{\tau \to \infty} \left( \frac{6\sqrt{6}}{\sigma_0 A_1} \right) \sqrt{\frac{1 + \rho}{3 + \rho}} \frac{1}{\tau} \quad \text{with} \quad \rho \in (-1, 1),
\]

(76)

where \( A_1 \in \mathbb{R} \) is an integration constant, \( \sigma_0 = \sigma(\tau)|_{\tau=0} \in \mathbb{R} \) and \( \sigma(\tau) \) satisfies the geodesic equation (44). Comparing Eq. (76) in the presence and absence of correlations yields the IGC ratio

\[
R_{\text{weak trivariate}}^{\text{weak}}(\rho) \overset{\text{def}}{=} \frac{C_M(\tau)}{C_M(\tau)|_{\rho=0}} = \sqrt{3} \sqrt{\frac{1 + \rho}{3 + \rho}}.
\]

(77)

where weak represents the case in which the underlying microstates of the system are minimally connected. Observe that \( R_{\text{bivariate}}^{\text{weak}}(\rho) \) is an increasing monotone function of the argument \( \rho \in (-1, 1) \).
2. Case 2

In the second case, we consider the trivariate Gaussian statistical model corresponding to Eq. (73) with the choice $C = C_2$. For this choice of covariance matrix, the condition $C > 0$ constrains the correlation coefficient to the range $\rho \in (-\sqrt{2}, \sqrt{2})$. The Fisher-Rao information metric line element associated with this model is given by

$$ds^2 = \frac{3 - 4\rho}{(1 - 2\rho^2)\sigma^2}d\mu^2 + \frac{6}{\sigma^2}d\sigma^2. \quad (78)$$

The asymptotic behavior of the IGC is found to be

$$C_M(\tau) \overset{\tau \to \infty}{\approx} \left(\frac{6\sqrt{6}}{\sigma_0 A_1}\right) \sqrt{\frac{1 - 2\rho^2}{3 - 4\rho}} \frac{1}{\tau}. \quad (79)$$

Then, by means of comparison of Eq. (79) in the presence and absence of correlations yield the IGC ratio

$$R_{\text{trivariate}}^{\text{mildly weak}}(\rho) \overset{\text{def}}{=} \frac{C_M(\tau)}{C_M(\tau)|_{\rho = 0}} = \sqrt{3} \sqrt{\frac{1 - 2\rho^2}{3 - 4\rho}}, \quad (80)$$

where mildly weak represents the case in which the underlying microstates of the system are neither minimally nor maximally connected. The ratio $R_{\text{trivariate}}^{\text{mildly weak}}(\rho)$ is a function of the argument $\rho \in (-\sqrt{2}, \sqrt{2})$ and it attains the maximal value of $\sqrt{3}$ at $\rho = \frac{1}{2}$, while in the extrema of the interval $(-\sqrt{2}, \sqrt{2})$ it tends to zero.

3. Case 3

As our final case for this example, we consider the trivariate Gaussian statistical model of Eq. (73) when $C = C_3$. In this case, the condition $C > 0$ requires that the correlation coefficient assume values in the range $\rho \in (-\frac{1}{2}, 1)$. The Fisher-Rao information metric line element associated with this model is given by

$$ds^2 = \frac{3}{(1 + 2\rho)\sigma^2}d\mu^2 + \frac{6}{\sigma^2}d\sigma^2. \quad (81)$$

The asymptotic behavior of the IGC reduces to

$$C_M(\tau) \overset{\tau \to \infty}{\approx} \left(\frac{12}{\sigma_0 A_1}\right) \sqrt{\frac{1 + 2\rho}{\tau}}, \quad (82)$$

where $A_1 \in \mathbb{R}$ is an integration constant, $\sigma_0 = \sigma(\tau)|_{\tau = 0} \in \mathbb{R}$ and $\sigma(\tau)$ satisfies the geodesic equation (14). The comparison of Eq. (82) in the presence and absence of correlations yield the IGC ratio

$$R_{\text{trivariate}}^{\text{strong}}(\rho) \overset{\text{def}}{=} \frac{C_M(\tau)}{C_M(\tau)|_{\rho = 0}} = \sqrt{1 + 2\rho}, \quad (83)$$
where *strong* represents a maximally connected lattice underlying the trivariate microstates of the system. It is obvious that the ratio $R_{\text{strong trivariate}}^{\text{strong trivariate}}(\rho)$ is an increasing monotonic function of the argument $\rho \in (-\frac{1}{2}, 1)$. Observe that the growth of $R_{\text{mildly weak trivariate}}^{\text{mildly weak trivariate}}(\rho)$ terminates at the critical value of $\rho_{\text{peak}} = \frac{1}{2}$ where $R_{\text{mildly weak trivariate}}^{\text{mildly weak trivariate}}(\rho_{\text{peak}}) = R_{\text{strong trivariate}}^{\text{strong trivariate}}(\rho_{\text{peak}})$. Interestingly, these conclusions are quite similar to those presented for the bivariate case. There is however, a key-feature of the IGC worth emphasizing when transitioning from the two-dimensional to the three-dimensional manifolds associated with cases exhibiting maximal connectedness among the microvariables of the system. In particular, the effects of negative and positive correlations are both *amplified* relative to the respective scenarios lacking correlations, such that

$$
\frac{R_{\text{strong trivariate}}^{\text{strong trivariate}}(\rho)}{R_{\text{bivariate}}^{\text{bivariate}}(\rho)} = \sqrt{1 + 2\rho} \quad (84)
$$

where $\rho \in (-\frac{1}{2}, 1)$. The above results enables us to conclude that the implementation of entropic inferences on higher-dimensional manifolds in the presence of anti-correlations [i.e. $\rho \in (-\frac{1}{2}, 0)$] is less complex than that on lower-dimensional manifolds as is evident form Eq. (84). The converse is true in the presence of positive-correlations [i.e. $\rho \in (0, 1)$].

### D. Complexity Reduction Arising from Microcorrelations

In [56], we consider a three-dimensional Gaussian model specified by the probability distribution

$$
p_{\text{correlated}}(x, y|\mu_x, \mu_y, \sigma) = \frac{\exp \left\{-\frac{1}{2\sigma^2(1-\rho^2)} \left[ (x - \mu_x)^2 - 2\rho(x - \mu_x)(y - \mu_y) + (y - \mu_y)^2 \right] \right\}}{2\pi\sigma^2\sqrt{1-\rho^2}} \quad (85)
$$

where $\sigma \in (0, \infty)$, $\mu_x$ and $\mu_y \in (-\infty, \infty)$ and $\rho \in (0, +1)$, from which the Fisher-Rao information metric line element

$$
ds_{\text{correlated}}^2 = \frac{1}{\sigma^2} \left[ \frac{1}{1-\rho^2} \left( d\mu_x^2 + d\mu_y^2 + 2\rho d\mu_x d\mu_y \right) + 4d^2\sigma \right] \quad (86)
$$

is obtained. The asymptotic expression of the IGC is given by

$$
C_{\text{correlated}}(\tau, \rho) \approx \frac{\sigma^2}{2\sigma_0 A^2} \frac{4(4-\rho^2)}{(2-2\rho^2)^2} \frac{1}{\tau} \quad (87)
$$

where $\sigma_0 = \sigma|_{\tau=0}$, $A \eqdef \frac{A_1^2 + A_2^2 - \rho A_1 A_2}{4(1-\rho^2)}$ and without loss of generality, $A_1 = -A_2 = a \in \mathbb{R}$. Upon comparison of the asymptotic expressions of the IGCs in presence and absence of microcorrelations, where $C_{\text{uncorrelated}}(\tau, 0) = C_{\text{correlated}}(\tau, \rho \to 0)$, we obtain

$$
C_{\text{correlated}}(\tau, \rho) \approx \frac{\sigma^2}{2\sigma_0 A^2} \frac{4(4-\rho^2)}{(2-2\rho^2)^2} \frac{1}{\tau} \quad (88)
$$
where \(0 \leq \mathcal{F}(\rho) \leq 1\) is defined as,

\[
\mathcal{F}(\rho) \overset{\text{def}}{=} \frac{1}{2\pi} \sqrt{\frac{4(4-\rho^2)}{(2-\rho^2)^2}} \left( \frac{2+\rho}{4(1-\rho^2)} \right)^{-\frac{1}{2}}.
\]

(89)

The quantity \(\mathcal{F}(\rho)\) is a monotonic decreasing function for any value of the correlation coefficient \(\rho\) in the open interval (0, +1). In essence, it represents an asymptotic power law decay of the IGC at a rate determined by \(\rho\). The result of this analysis, captured in \([57, 58]\), suggest that systems containing microcorrelations experience an asymptotic compression of the explored statistical microstates at a faster rate than in the absence of microcorrelations. This finding represents an explicit connection between the behavior of the experimentally observable macroscopic quantities of a statistical system on the information encoded in the correlational structure underlying the system’s microscopic degrees of freedom.

E. Complexity Reduction Arising from Macrocorrelations

In \([57, 58]\), we consider a 4\(l\)-dimensional Gaussian statistical model specified by the probability distribution

\[
p (x|\theta) = \prod_{k=1}^{2l} p (x_k|\mu_k, \sigma_k), \quad p (x_k|\mu_k, \sigma_k) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp \left[ -\frac{(x_k - \mu_k)^2}{2\sigma_k^2} \right],
\]

(90)

with \(x \equiv (x_1, ..., x_{2l})\) and \(\theta \equiv (\mu_1, ..., \mu_{2l}, \sigma_1, ..., \sigma_{2l})\), from which the information line element

\[
ds^2 = \sum_{j=1}^{2l} \frac{1}{\sigma_j^2} (d\mu_j^2 + 2d\sigma_j^2),
\]

(91)

is obtained. By subjecting the statistical microstates \(x_k\) to a set of \(2l\) embedding constraints,

\[
\sigma_{2j} = \sigma_{2j-1} \text{ and, } \mu_{2j} = \mu_{2j} (\mu_{2j-1}, \sigma_{2j-1}) \text{ with } j = 1, ..., l,
\]

(92)

the probability distribution \(p (x|\theta)\) reduces to the \(2l\)-dimensional embedded Gaussian statistical model

\[
p_{\text{embedded}} (x|\theta) = \prod_{j=1}^{l} p (x_{2j-1}, x_{2j}|\mu_{2j-1}, \sigma_{2j-1}),
\]

(93)

with \(x = (x_1, ..., x_{2l})\) and \(\theta = (\mu_1, \mu_3, ..., \mu_{2l-1}; \sigma_1, \sigma_3, ..., \sigma_{2l-1})\) where \(p (x_{2j-1}, x_{2j}|\mu_{2j-1}, \sigma_{2j-1})\) is defined as

\[
p (x_{2j-1}, x_{2j}|\mu_{2j-1}, \sigma_{2j-1}) \overset{\text{def}}{=} \frac{1}{2\pi\sigma_{2j-1}} \exp \left[ -\frac{(x_{2j-1} - \mu_{2j-1})^2 + (x_{2j} - \mu_{2j} (\mu_{2j-1}, \sigma_{2j-1}))^2}{2\sigma_{2j-1}^2} \right],
\]

(94)

and \(j = 1, ..., l\). The model in Eq. \([93]\) leads to the information line element

\[
ds^2_{\text{embedded}} = \sum_{j=1}^{l} \frac{1}{\sigma_{2j-1}^2} (d\mu_{2j-1}^2 + 2d\mu_{2j-1}d\sigma_{2j-1} + 2d\sigma_{2j-1}^2),
\]

(95)
with the coefficients $\rho_{2j-1}$ defined as

$$\rho_{2j-1} \equiv \frac{\frac{\partial \mu_{2j}}{\partial \mu_{2j-1}} \frac{\partial \sigma_{2j}}{\partial \sigma_{2j-1}}}{\left[1 + \left(\frac{\partial \mu_{2j}}{\partial \mu_{2j-1}}\right)^2\right]^{\frac{1}{2}} \left[2 + \frac{1}{2} \left(\frac{\partial \mu_{2j}}{\partial \sigma_{2j}} + \frac{\partial \sigma_{2j}}{\partial \mu_{2j-1}}\right)^2\right]^{\frac{1}{2}}}, \quad (96)$$

where the explicit expressions of such coefficients depend on the functional parametric form given to the embedding constraints in Eq. (92). From Eq. (96) it follows that the coefficients $\rho_{2j-1}$ are non-zero if and only if $\mu_{2j-1}$ depends on both $\mu_{2j}$ and $\sigma_{2j-1}$. Therefore, we conclude that the emergence of non-vanishing off-diagonal terms in Eq. (95) arise due to the presence of a correlation among the statistical variables on the larger manifold and are therefore characterized by the macroscopic correlation coefficients $\rho_{2j-1}$. Motivated by these considerations, we will name the coefficients $\rho_{2j-1}$ macroscopic correlation coefficients. The IGE was determined to have the form \[ S_M(\tau; l, \lambda_k, \rho_k) \approx \log \left[ A_1 (r_k) + A_2 (\rho_k, \lambda_k) \right], \quad (97) \]

provided $\rho_k = \rho_s \forall k$ and $s = 1, \ldots, l$, with

$$A_1 (\rho_k) \equiv \frac{2\rho_k \sqrt{2 - \rho_k^2}}{1 + \sqrt{\Delta (\rho_k)}}, \quad A_2 (\rho_k, \lambda_k) \equiv \frac{\sqrt{\Delta (\rho_k) (2 - \rho_k^2)} \log \left[ \Sigma (\rho_k, \lambda_k, \alpha\pm) \right]}{\rho_k \lambda_k}, \quad \text{and} \quad \alpha \pm (\rho_k) \equiv \frac{1}{2} \left(3 \pm \sqrt{\Delta (\rho_k)}\right). \quad (98)$$

The quantity

$$\Sigma (\rho_k, \lambda_k, \alpha\pm) \equiv \frac{\Xi_k}{4 \lambda_k} \frac{1 + \sqrt{\Delta (\rho_k)}}{\lambda_k} \left(\frac{2 \alpha_- (\rho_k)}{\alpha_+ (\rho_k)}\right) > 0, \quad \forall \rho \in [0, 1) \quad (99)$$

is a strictly positive function of its arguments, where $\Xi_k$ and $\lambda_k$ are real, positive constants of integration and

$$\Delta (\rho_k) \equiv 1 + 4 \rho_k^2. \quad (100)$$

It is evident from Eq. (97) that the IGE is characterized by a power law decay, whereby the power is specified by the cardinality of the microscopic degrees of freedom associated with correlated macroscopic information. Furthermore, the IGE attains a maximal value quantified by the set $\{\rho_k\}$. The relevance of these finding is twofold: first, it provides a compact description of the effect of microscopic information on (experimentally observable) macroscopic variables; second, it provides quantitative evidence that the information geometric complexity of a system decreases in the presence of correlational structures.
F. Suppression of Classical Chaos from Quantum-like constraints: The Uncorrelated Case

Building upon the results obtained in [59–61], we investigate in [62] a 3Du (three-dimensional and uncorrelated) Gaussian statistical model specified by the probability distribution
\[ p_{3Du}(x, y \mid \mu_x, \sigma_x, \sigma_y) \overset{\text{def}}{=} \frac{1}{2\pi \sigma_x \sigma_y} \exp \left[ -\frac{1}{2\sigma_x^2} (x - \mu_x)^2 - \frac{1}{2\sigma_y^2} y^2 \right], \quad (101) \]
whose Fisher-Rao information metric line element is given by
\[ ds_{3Du}^2 = \frac{1}{\sigma_x^2} (d\mu_x^2 + 2d\sigma_x^2) + \frac{2}{\sigma_y^2} d\sigma_y^2. \quad (102) \]

We then compare our analysis to that of a 2Du (two-dimensional and uncorrelated) Gaussian statistical model obtained from the higher-dimensional model \( p_{3Du}(x, y \mid \mu_x, \sigma_x, \sigma_y) \) via the introduction of a macroscopic information constraint
\[ \sigma_x \sigma_y = \Sigma^2, \quad \Sigma^2 \in \mathbb{R}_0^+ \quad (103) \]
that resembles the quantum mechanical canonical minimum uncertainty relation, which leads to the 2Du statistical model
\[ p_{2Du}(x, y \mid \mu_x, \sigma) \overset{\text{def}}{=} \frac{1}{2\pi \Sigma^2} \exp \left[ -\frac{1}{2\sigma^2} (x - \mu_x)^2 - \frac{\sigma^2}{2\Sigma^4} y^2 \right], \quad (104) \]
where \( x \) denotes the position of a particle and \( y \) its conjugate momentum. The line element of the Fisher-Rao information metric associated with \( p_{2Du}(x, y \mid \mu_x, \sigma) \) is given by,
\[ ds_{2Du}^2 = \frac{1}{\sigma^2} (d\mu_x^2 + 4d\sigma^2). \quad (105) \]

It was determined that for the 3Du model, the IGE takes the form
\[ S_{M}^{(3Du)}(\tau) \overset{\tau \to \infty}{\approx} \lambda_+^\prime \tau, \quad (106) \]
where \( \lambda_+^\prime \in \mathbb{R}_+^+ \). In the 2Du case, it was found that
\[ S_{M}^{(2Du)}(\tau) \overset{\tau \to \infty}{\approx} \lambda_+ \tau, \quad (107) \]
where \( \lambda_+ = \frac{\lambda_+^\prime}{\sqrt{2}} \in \mathbb{R}_+^+ \). By comparing \( S_{M}^{(3Du)} \) with \( S_{M}^{(2Du)} \), one observes
\[ S_{M}^{(2Du)}(\tau) \overset{\tau \to \infty}{\approx} \left( \frac{\lambda_+}{\lambda_+^\prime} \right) \cdot S_{M}^{(3Du)}(\tau) \quad \text{with} \quad \lambda_+ / \lambda_+^\prime = \frac{1}{\sqrt{2}} < 1. \quad (108) \]
In 

\[ G \text{. Suppression of Classical Chaos from Quantum-like constraints: The Correlated Case} \]

we study a correlated 3Dc (three-dimensional and correlated) Gaussian statistical model with uncorrelated microstates specified by the probability distribution

\[
p_{3Dc}(x, y|\mu_x, \sigma_x, \sigma_y; \rho) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2(1-\rho^2)} \left( \frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho(x - \mu_x)y}{\sigma_x \sigma_y} \right) \right\}
\]

(109)

whose line element of the Fisher-Rao information metric associated with \( p_{3Dc}(x, y|\mu_x, \sigma_x, \sigma_y) \) is given by

\[
ds_{3Dc}^2 = \frac{1}{1-\rho^2} \frac{d\mu_x^2}{\sigma_x^2} + \frac{2-\rho^2}{1-\rho^2} \frac{d\sigma_x^2}{\sigma_x^2} + \frac{2-\rho^2}{1-\rho^2} \frac{d\sigma_y^2}{\sigma_y^2} - \frac{2\rho^2}{1-\rho^2} \frac{d\sigma_x d\sigma_y}{\sigma_x \sigma_y}.
\]

(110)

We then compare our analysis to that of a 2Dc (two-dimensional and correlated) Gaussian statistical model obtained from the higher-dimensional model \( p_{3Dc}(x, y|\mu_x, \sigma_x, \sigma_y) \) via introduction of a covariance constraint

\[
\sigma_{xy} = \rho \sigma_x \sigma_y,
\]

(111)

where the parameter \( \rho \) is the correlation coefficient between \( x \) and \( y \) and assumes values within the ranges \(-1 \leq \rho \leq 1\).

Applying the macroscopic constraint in Eq. (103) to the covariance constraint in Eq. (111) yields the combined constraint

\[
\sigma_{xy} = \rho \Sigma^2 \text{ with } \sigma_x \sigma_y = \Sigma^2 \in \mathbb{R}_0^+
\]

(112)

which when applied to \( p_{3Dc}(x, y|\mu_x, \sigma_x, \sigma_y; \rho) \) leads to

\[
p_{2Dc}(x, y|\mu_x, \sigma_x; \rho) \overset{\text{def}}{=} \frac{1}{2\pi \rho \Sigma^2 \sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2(1-\rho^2)} \left( \frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{\rho \Sigma^2}{\Sigma^2} \frac{y^2}{\Sigma^2} - \frac{2\rho^2 \Sigma^2}{\sigma_x \sigma_y} \right) \right\}. \]

(113)

The line element of the Fisher-Rao information metric associated with \( p_{2Dc}(x, y|\mu_x, \sigma_x) \) is given by

\[
ds_{2Dc}^2 = \frac{1}{\sigma^2 (1-\rho^2)} d\mu_x^2 + \frac{4}{\sigma^2 (1-\rho^2)} d\sigma_x^2.
\]

(114)

It was determined that for the correlated 2Dc model, the IGE takes the form

\[
S_M^{(2Dc)}(\tau) \overset{\tau \to \infty}{\approx} \sigma_0 \lambda_+ \tau,
\]

(115)

where the correlated IGC is given by

\[
C_M^{(2Dc)}(\tau) \overset{\tau \to \infty}{\approx} \frac{1}{(1-\rho^2)} C_M^{(2Du)}(\tau) \text{ with } C_M^{(2Du)}(\tau) \overset{\tau \to \infty}{\approx} \left( \frac{\mu_0 + 2\sigma_0}{\sigma_0^2 \lambda_+} \right) \frac{\exp(\sigma_0 \lambda_+ \tau)}{\tau},
\]

(116)

where \( \lambda_+ \in \mathbb{R}_+^*, \sigma_0 \overset{\text{def}}{=} \sigma(\tau = 0) \) and \( \mu_0 \overset{\text{def}}{=} \mu(\tau = 0) \). By comparing \( S_M^{(2Du)} \) with \( S_M^{(2Dc)} \), one observes

\[
S_M^{(2Dc)}(\tau) \overset{\tau \to \infty}{\approx} S_M^{(2Du)}.
\]

(117)
The IGE does not change asymptotically for either the correlated or uncorrelated $2D$ models considered above. Equation (116) quantitatively demonstrates that the IGC $C_M^{(2D,c)}$ diverges as the correlation coefficient - introduced via the constraint in Eq. (111) - approaches unity. As expected, the two cases are identical for $\rho = 0$.

H. Random Frequency Macroscopic Anisotropic Inverted Harmonic Oscillators

Building upon the results obtained in [22], we present in [64, 65] an information geometric analogue of the Zurek-Paz quantum chaos criterion in the classical reversible limit. This analogy is illustrated by applying our modeling scheme to a set of $l$-uncoupled, three-dimensional anisotropic, inverted harmonic oscillators (IHOs) characterized by a Ohmic distributed frequency spectrum. In this application, we consider a manifold whose metric line element is given by

$$ds^2 = [1 - \Phi(\theta)] \delta_{ab} d\theta^a d\theta^b \text{ with } \Phi(\theta) = \sum_{k=1}^{l} u_k(\theta^k),$$

(118)

where $\delta_{ab}$ is the identity matrix of dimension $l$ and

$$u_k(\theta^k) = -\frac{1}{2} \omega_k(\theta^k)^2, \text{ with } \theta^k = \theta^k(s).$$

(119)

Upon making a suitable change of the affine parameter featured in the geodesic equations from $s$ to $\tau$ in such a manner that $ds^2 = 2 (1 - \Phi)^2 d\tau^2$, we can obtain a simplified form for these differential equations describing a set of macroscopic inverted harmonic oscillators (IHOs). Since the $l$-Newtonian equations of motion for each IHO is given by

$$\frac{d^2 \theta^j}{d\tau^2} - \omega_j^2 \theta^j = 0, \forall j = 1, \ldots, l,$$

(120)

the asymptotic behavior of such macrostates on manifold $\mathcal{M}^{(l)}_{\text{IHO}}$ is determined to be

$$\theta^j(\tau) \xrightarrow{\tau \to \infty} \Xi_j e^{\omega_j \tau}, \text{ with } \Xi_j \in \mathbb{R}, \forall j = 1, \ldots, l.$$  

(121)

Thus, after some analysis, the IGC and IGE were found to have the forms

$$C_{\mathcal{M}^{(l)}_{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l) \xrightarrow{\tau \to \infty} \frac{1}{l} \frac{1}{2\pi} \Xi^2 \left( \frac{\xi^2 \Omega^2}{2} \right)^{\frac{1}{2}} \exp\left( \frac{\xi \Omega \tau}{\tau} \right),$$

(122)

and

$$S_{\mathcal{M}^{(l)}_{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l) \xrightarrow{\tau \to \infty} \Omega \tau,$$

(123)
respectively, where
\[ \Omega = \sum_{i=1}^{l} \omega_i, \quad \Xi_j \in \mathbb{R}, \forall j = 1, \ldots, l \] (124)
and in Eq. (122) we assumed that \( \Xi_i = \Xi_j = \Xi \forall i, j = 1, \ldots, l \) with \( \xi \) being a positive real constant that depends on the specific nature of the system being considered \[64, 65\]. Note that it was further assumed that \( l \to \infty \) such that the spectrum of frequencies transitions into a continuum with linearly distributed spectrum (Ohmic frequency spectrum),
\[ \theta_{\text{Ohmic}}(\omega) = \frac{2\omega}{\Omega_{\text{cut-off}}} \],
where \( \int_{0}^{\Omega_{\text{cut-off}}} \theta_{\text{Ohmic}}(\omega) d\omega = 1 \) and \( \Omega_{\text{cut-off}} = \gamma \Omega \) with \( \gamma \in \mathbb{R} \). (125)

Equation (123) displays an asymptotic, linear IGE growth for the generalized set of inverted harmonic oscillators and serves to extend the result of Zurek-Paz to an arbitrary ensemble of anisotropic inverted harmonic oscillators \[66\] within a classical IG setting. This example may be viewed as the IG counterpart of the Zurek-Paz model used to investigate the effects of decoherence in quantum chaos. In their work, Zurek and Paz considered a single unstable harmonic oscillator characterized by a potential
\[ V(x) = -\frac{\Omega^2 x^2}{2} \] (126)
coupled to an external environment. Note that the quantity \( \Omega \) in Eq. (126) represents the Lyapunov exponent. In the reversible classical limit \[67\], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent according to,
\[ S_{\text{quantum}}^{(\text{chaotic})}(\tau) \xrightarrow{\tau \to \infty} \Omega \tau. \] (127)

Equation (128) is effectively the classical IG analog of Eq. (127).

I. Regular and Chaotic Quantum Spin Chains

In \[68, 69\], we proposed an IG characterization of integrable and chaotic energy level statistics of a quantum anti-ferromagnetic Ising spin chain. In this example, we encode relevant information about the spin-chain in a suitable composite probability distribution taking account of both the quantum spin chain and the configuration of the external magnetic field in which the spins are immersed. Specifically, for the integrable case an anti-ferromagnetic Ising chain is immersed in a transverse, homogeneous magnetic field
\[ \vec{B}_{\text{transverse}} = B_\perp \hat{B}_\perp, \] (128)
which has only one component $B_\perp$, where the level spacing distribution of its spectrum is Poisson distributed. It is known from information theory that the Exponential distribution is identified as the maximum entropy distribution if only one piece of information (in this case the expectation value of the transverse magnetic field) is known. For this reason, we consider a Poisson distribution

$$p^{(\text{Poisson})}_A(x_A|\mu_A) = \frac{1}{\mu_A} \exp \left( - \frac{x_A}{\mu_A} \right),$$

(129)
coupled to an Exponential bath

$$p^{(\text{Exponential})}_B(x_B|\mu_B) = \frac{1}{\mu_B} \exp \left( - \frac{x_B}{\mu_B} \right),$$

(130)
which together, gives rise to the composite statistical model

$$P^{(\text{integrable})}(x_A, x_B|\mu_A, \mu_B) = \frac{1}{\mu_A \mu_B} \exp \left[ - \left( \frac{x_A}{\mu_A} + \frac{x_B}{\mu_B} \right) \right],$$

(131)
inducing manifold $\mathcal{M}^{(\text{integrable})}$. The microstate $x_A$ represents the spacing of the energy levels while the macrostate $\mu_A$ is the average level spacing; the microstate $x_B = |\mathbf{\mu} \cdot \mathbf{B}| = |-\mu_B \cos \varphi|$ is identified with the intensity of the magnetic field, $\varphi$ is the tilt angle and the macrostate $\mu_B$ is the average transverse magnetic field intensity. In the transverse case, $\varphi = 0$ and therefore $x_B = B \equiv B_\perp$. This model represents our best guess as justified by the observation that the magnitude of the magnetic field is a relevant quantity in this scenario. The components of the transverse magnetic field are varied during the transition from integrable to chaotic regimes. In the integrable regime, the magnetic field intensity is set to the well-defined value $\langle x_B \rangle = \mu_B$.

In the chaotic case, an antiferromagnetic Ising chain is immersed in a tilted, homogeneous magnetic field

$$\mathbf{B}_\text{tilted} = B_\perp \mathbf{\hat{B}}_\perp + B_\parallel \mathbf{\hat{B}}_\parallel,$$

(132)
comprised of two components $B_\perp$ and $B_\parallel$, with the level spacing distribution of its spectrum given by the Wigner-Dyson distribution of Poisson form. It is known from information theory that the Gaussian distribution is identified as the maximum entropy distribution when only two pieces of information are known (in this case the expectation value and the variance). For these reasons, we consider a Wigner-Dyson distribution

$$p^{(\text{Wigner-Dyson})}_A(x'_A|\mu'_A) = \frac{\pi}{2\mu'_A^2} \exp \left( - \frac{\pi x'_A^2}{4\mu'_A^2} \right),$$

(133)
coupled to a Gaussian bath

$$p^{(\text{Gaussian})}_B(x'_B|\mu'_B, \sigma'_B) = \frac{1}{\sqrt{2\pi} \sigma'_B} \exp \left( - \frac{(x'_B - \mu'_B)^2}{2\sigma'_B^2} \right),$$

(134)
which together, gives rise to the composite statistical model

\[
P^{(\text{chaotic})}(x'_A, x'_B | \mu'_A, \mu'_B, \sigma'_B) = \pi \left( \frac{2 \pi \sigma'_B^2}{2 \mu'_A^2} \right)^{-\frac{1}{2}} x'_A \exp \left[ - \left( \frac{\pi x'_A^2}{4 \mu'_A^2} + \frac{(x'_B - \mu'_B)^2}{2 \sigma'_B^2} \right) \right],
\]

inducing manifold \( \mathcal{M}^{(\text{chaotic})} \). Note that \( B_\perp \) and \( B_\parallel \) are transverse and longitudinal magnetic field intensities, respectively. The microstate \( x'_B \) is identified with the intensity of the tilted magnetic field, while the macrostate \( \mu'_B \) is the average intensity of the magnetic energy arising from the interaction of the tilted magnetic field with the magnetic moment of the spin \( \frac{1}{2} \) particle and \( \sigma'_B \) is its covariance. During the transition from the integrable to chaotic regimes, the magnetic field is being experimentally varied. Specifically, the magnetic field is being tilted while its two components \( B_\perp \) and \( B_\parallel \) are simultaneously being varied. Our best guess based upon knowledge of the experimental mechanism that drives the transitions between the two regimes is that the the microstate \( \mu B \cos \varphi \) is Gaussian-distributed during this change. In the chaotic regime, the magnetic field intensity is set to a well-defined value \( \langle x'_B \rangle = \mu'_B \) with covariance \( \sigma'_B = \sqrt{\langle (x'_B - \langle x'_B \rangle)^2 \rangle} \).

The line element \( ds^2_{\text{integrable}} \) of the Fisher-Rao information metric on \( \mathcal{M}^{(\text{integrable})} \) is given by

\[
ds^2_{\text{integrable}} = ds^2_{\text{Poisson}} + ds^2_{\text{Exponential}} = \frac{1}{\mu'_A^2} d\mu'_A^2 + \frac{1}{\mu'_B^2} d\mu'_B^2.
\]

Applying the IGAC to the line element in Eq. (136) leads to polynomial growth in \( C_{\mathcal{M}}^{(\text{integrable})} \) and logarithmic IGE growth \[68, 69\], according to

\[
C_{\mathcal{M}}^{(\text{integrable})}(\tau) \xrightarrow{\tau \to \infty} \exp(c'_{\text{IG}}) \tau^{c_{\text{IG}}}, \quad S_{\mathcal{M}}^{(\text{integrable})}(\tau) \xrightarrow{\tau \to \infty} c_{\text{IG}} \log \tau + c'_{\text{IG}}.
\]

The quantity \( c_{\text{IG}} \) is a constant that is proportional to the number of Exponential probability distributions within the composite distribution utilized in the computation of the IGE; \( c'_{\text{IG}} \) is a constant that depends on the values assumed by the statistical macrostates \( \mu_A \) and \( \mu_B \). Equations \[137\] may be interpreted as the IG analogues of the computational complexity and the entanglement entropy defined in standard quantum information theory, respectively.

The Fisher-Rao information metric line element \( ds^2_{\text{chaotic}} \) on \( \mathcal{M}^{(\text{chaotic})} \) is given by

\[
ds^2_{\text{chaotic}} = ds^2_{\text{Wigner-Dyson}} + ds^2_{\text{Gaussian}} = \frac{4}{\mu'_A^2} d\mu'_A^2 + \frac{1}{\sigma'_B^2} d\mu'_B^2 + \frac{2}{\sigma'_B^2} d\sigma'_B^2.
\]

Applying the IGAC machinery to the line element in Eq. (138), we obtain exponential growth for \( C_{\mathcal{M}}^{(\text{chaotic})} \) and linear IGE growth \[68, 69\],

\[
C_{\mathcal{M}}^{(\text{chaotic})}(\tau) \xrightarrow{\tau \to \infty} q_{\text{IG}} \exp (K_{\text{IG}} \tau), \quad S_{\mathcal{M}}^{(\text{chaotic})}(\tau) \xrightarrow{\tau \to \infty} K_{\text{IG}} \tau.
\]
respectively. The constant $q_{IG}$ acts to encode information concerning the initial conditions of the macrostates parametrizing $\mathcal{M}^{(\text{chaotic})}$. The constant $K_{IG}$ given by,

$$K_{IG} \tau \to \infty \approx \frac{dS_M(\tau)}{d\tau} \quad (140)$$

is the model parameter of the chaotic system and depends on the temporal evolution of the macrostates of the system. As in the integrable case, equations (139) may be interpreted as the IG analogues of the computational complexity and the entanglement entropy defined in standard quantum information theory, respectively.

### J. Scattering Induced Quantum Entanglement

Building upon the results obtained in [70], we implement in [71, 72] a hybrid approach (standard quantum theory combined with IG techniques) to the modeling of scattering induced quantum entanglement [73, 74]. In particular, we performed an IG analysis of two identical, distinguishable Continuous Variable Quantum Systems (CVQS) with Gaussian distributed continuous degrees of freedom (i.e. spinless, non-relativistic point particles of mass $m$, each represented by minimum uncertainty Gaussian wave packets where both particles are initially located far from each other, a linear distance $R_o$, and each being characterized before collision by initial average momentum $\langle p_1\rangle_o = p_o$ and $\langle p_2\rangle_o = -p_o$, respectively, with equal momentum dispersion $\sigma_o$) that are prepared independently. The two wave packets interact via a scattering process mediated by an interaction (s-wave scattering) potential $V(x)$ and separate again. Note that by CVQS we refer to quantum mechanical systems on which one can, in principle, perform measurements of certain observables whose eigenvalue spectrum is continuous. For such a system, a complete set of commuting observables is furnished by the momentum operators of each particle [75, 76]. The interaction potential $V(x)$ is isotropic and is active over a short range $L$ such that $V(x) = V$ for $0 \leq x \leq L$ and $V(x) \approx 0$ for $x > L$ where $V$ denotes the height (for $V > 0$: repulsive potential) or depth (for $V < 0$: attractive potential) of the potential.

We investigate the quantum entanglement quantified in terms of a scalar quantity, the purity $P \overset{\text{def}}{=} \text{Tr} (\rho_A^2)$ of the two-particle state function describing the system generated by such a scattering event. We note that $\text{Tr}$ denotes the standard quantum-mechanical trace operation on the operator $\rho_A^2$. In the case being considered, $\rho_A \overset{\text{def}}{=} \text{Tr}_B (\rho_{AB})$ is the reduced density matrix that describes particle $A$ and $\rho_{AB}$ represents the two-particle density matrix associated with the post-collisional two-particle wave function. Note that $\text{Tr}_B$ denotes the partial trace over the particle $B$. The operation of computing $\text{Tr}_B$ is usually referred to as tracing-out system $B$. Briefly speaking, the reduced density operator $\text{Tr}_B (\rho_{AB})$ is the correct tool to use when analyzing physical properties that belong solely to $A$ [77]. For
pure bipartite states, the smaller the value of $\mathcal{P}$, the higher the entanglement. Thus, the loss of purity furnishes an indicator of the degree of entanglement, where a disentangled product state corresponds to $\mathcal{P} = 1$.

1. The Pre and Post Collision Scenarios

The normalized, separable, two-particle Gaussian wave function representing the situation prior to collision is prescribed by \[ \psi^{(\text{Pre})}(k_1, k_2) = \psi_1(k_1) \otimes \psi_2(k_2), \] with \[ \psi_{1(2)}(k_{1(2)}) = a(k_{1(2)}, \langle k_{1(2)} \rangle_o; \sigma_{k_0}) e^{i(k_{1(2)} - \langle k_{1(2)} \rangle_o)q_{1(2)}}, \] (141)

where

\begin{equation}
    a(k_{1(2)}, \langle k_{1(2)} \rangle_o; \sigma_{k_0}) \overset{\text{def}}{=} \left( \frac{1}{2\pi\sigma_{k_0}^2} \right)^{1/4} \exp \left[ -\frac{(k_{1(2)} - \langle k_{1(2)} \rangle_o)^2}{4\sigma_{k_0}^2} \right],
\end{equation}

(142)

and $k_{1(2)} = \frac{p_{1(2)}}{\hbar} \in (-\infty, +\infty)$, $\langle k_{1(2)} \rangle_o = \frac{\langle p_{1(2)} \rangle}{\hbar} = \pm k_o$, $\sigma_{k_0} = \frac{\sigma_k}{\hbar}$, $q_{1(2)} = \mp \frac{1}{2} R_o$, and $\hbar$ is the reduced Planck constant. The pre-collision probability density $|\psi^{(\text{Pre})}(k_1, k_2)|^2$ takes the form

\begin{equation}
    p_{\text{QM}}^{(\text{Pre})} \equiv |\psi^{(\text{Pre})}(k_1, k_2, t)|^2 = \frac{1}{2\pi\sigma_{k_0}^2} \exp \left[ -\frac{(k_1 - \langle k_1 \rangle_o)^2 + (k_2 - \langle k_2 \rangle_o)^2}{2\sigma_{k_0}^2} \right].
\end{equation}

(143)

After collision, the wave function for the two-particle system in the long time limit takes the form

\begin{equation}
    \psi^{(\text{Post})}(k_1, k_2, t) = (N)^{-1/2} \left[ \psi_1(k_1) \psi_2(k_2) e^{-i\hbar(k_1^2 + k_2^2)t/(2m)} + \varepsilon \psi_{\text{scat}}(k_1, k_2, t) \right],
\end{equation}

(144)

where $N$ and $\varepsilon$ are normalization constants and the single-particle wave function $\psi_{1(2)}(k_{1(2)})$ is specified via Eq. (141). The quantity $\psi^{(\text{Post})}(k_1, k_2, t)$ can be rewritten as

\begin{equation}
    \psi^{(\text{Post})}(k_1, k_2, t) = (N)^{-1/2} \left( \frac{1}{2\pi\sigma_{k_0}^2} \right)^{1/2} \exp \left[ -\frac{K^2 + 4(k - k_o)^2}{8\sigma_{k_0}^2} \right] \times [1 + \varrho(k)] e^{-i(k - k_o)R_o - i\hbar k^2 t/(2M) - i\hbar k^2 t/(2\mu)},
\end{equation}

(145)

where we adopt the one-dimensional center of mass and relative coordinates, whose conjugate momenta are defined as $K \overset{\text{def}}{=} k_1 + k_2 \in (-\infty, +\infty)$, $k \overset{\text{def}}{=} \frac{1}{\hbar} (k_1 - k_2) \in (-\infty, +\infty)$, $M \overset{\text{def}}{=} 2m$ is the total mass, $\mu \overset{\text{def}}{=} \frac{m}{2}$ is the reduced mass, and $\varrho(k)$ is given by

\begin{equation}
    \varrho(k) \overset{\text{def}}{=} \frac{4i}{\sigma_{k_0}^2} \frac{k^2 f(k)}{\sigma_{k_0}^2},
\end{equation}

(146)

Here, $f(k)$ is a function that depends on the specific interaction and initial conditions of the system.
where \( f(k) \) is the s-wave scattering amplitude due to the s-wave scattering phase shift \( \theta(k) \) and \( i = \sqrt{-1} \) is the standard imaginary unit. The post-collision probability density \( |\psi^{(Post)}(k_1, k_2, t)|^2 \) takes the form

\[
P_{QM}^{(Post)} \equiv |\psi^{(Post)}(k_1, k_2, t)|^2 \approx \frac{1}{N} \exp \left\{ -\frac{1}{2(1-\rho_{QM})} \left[ \frac{(k_1-k_2)^2}{\sigma_{k_0}^2} - 2\rho_{QM} \frac{(k_1-k_2)(k_2+k_0)}{\sigma_{k_0}^2} + \frac{(k_2+k_0)^2}{\sigma_{k_o}^2} \right] \right\},
\]

where the constant \( N \) in Eq. (147) has been determined so as to normalize the integral. The quantity \( \rho_{QM} \) appearing in (147) is defined as

\[
\rho_{QM} \equiv \left( \frac{8}{k_0^2 + \sigma_{k_0}^2} \right) R_o a_s \ll 1,
\]

where the parameter \( a_s \) has the dimension of length and is defined as the s-wave scattering length \[80\], which comes from

\[
f(k_0) = \frac{e^{i\theta_o} \sin \theta_o}{k_0} \approx \frac{\theta(k_0)}{k_0} + \mathcal{O}(\theta^2),
\]

where

\[
\tan \theta_o = \frac{k_0 \tan(k_o L) - k_r \tan(k_o L)}{k_r + k_r \tan(k_o L) \tan(k_r L)} \approx \frac{\theta(k_o)}{k_o} \approx -\frac{\rho(k_o L)^3}{3}
\]

denotes the s-wave scattering phase shift, under the assumption of low energy s-wave scattering with \( k_o L = p_o L/\hbar \ll 1 \). From our IG analysis,

\[
k_r \overset{\text{def}}{=} \sqrt{1 - \rho k_o}, \quad 0 < x < L
\]

where

\[
k_r = \frac{\sqrt{2\mu (E-V)}}{\hbar}, \quad 0 < x < L, \text{ and }
\]

\[
k_o = \frac{\sqrt{2\mu^*}}{\hbar}, \quad x > L.
\]

We remark that the quantity \( \rho_{QM} \) in Eq. (148) can be related to the two-particle squeezing parameters found in \[81\] for example.

2. Information Geometric Modeling of Quantum Entanglement

Our first conjecture is that the quantum entanglement produced by a head-on collision between two Gaussian wave packets are macroscopic manifestations emerging from underlying microscopic statistical structures. For this reason,
we model the pre and post-collisional scenarios as limiting cases of uncorrelated and correlated Gaussian statistical models

\[ p_{\text{QM}}^{(\text{Pre})} \approx p_{\text{IG}}^{(\text{uncorr})}(x, y|\mu_x, \mu_y, \sigma) = \frac{1}{2\pi\sigma^2} \exp \left[ -\frac{(x - \mu_x)^2 + (y - \mu_y)^2}{2\sigma^2} \right], \quad (154) \]

and

\[ p_{\text{QM}}^{(\text{Post})} \approx p_{\text{IG}}^{(\text{corr})}(x, y|\mu_x, \mu_y, \sigma; \rho) = \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ \frac{(x - \mu_x)^2 + (y - \mu_y)^2}{\sigma^2} \right] \right\} \frac{2\pi\sigma^2}{\sqrt{1 - \rho^2}}, \quad (155) \]

respectively, with the following identifications: \( x \to k_1, y \to k_2; \langle x \rangle = \mu_x \to \mu_{k_1} \overset{\text{def}}{=} +k_o, \langle y \rangle = \mu_y \to \mu_{k_2} \overset{\text{def}}{=} -k_o \) and \( \sigma \to \sigma_o \). The correlation coefficient \( \rho \) is defined in Eq. (67). In the present example, the correlation coefficient \( \rho \) is considered to have compact support over the line segment \([0, 1]\), and is identified with the quantum entanglement strength \( \rho_{\text{QM}} \), such that \( \rho = \rho_{\text{QM}} \). In order to furnish an IG interpretation of quantum entanglement characterized by the purity

\[ \mathcal{P} = \iint \iint \psi(k_1, k_2, t) \psi(k_3, k_4, t) \psi^*(k_1, k_4, t) \psi^*(k_3, k_2, t) \, dk_1 dk_2 dk_3 dk_4, \quad (156) \]

we employ our second conjecture whereby we use IG to model the scattering interaction by patching together two charts, each belonging to a different Gaussian statistical manifolds, one without correlation (pre collision) and the other with correlation (post collision). The two models can be represented by means of \( p_{\text{IG}}^{(\text{uncorr})}(x, y|\mu_x, \mu_y, \sigma) \) and \( p_{\text{IG}}^{(\text{corr})}(x, y|\mu_x, \mu_y, \sigma; \rho) \) with associated statistical manifolds \( \mathcal{M}^{(\text{uncorr})} \) and \( \mathcal{M}^{(\text{corr})} \), respectively. The two charts belonging to the correlated and uncorrelated Gaussian statistical manifolds are patched together by joining the sets of geodesic curves associated with each manifold at the junction \( \tau = 0 \). In particular, the set of geodesic curves defined when \( \tau < 0 \) (pre collision) for the uncorrelated model is joined to the set defined when \( \tau \geq 0 \) (post collision) for the correlated model. As a consequence of our second conjecture, we are able to uncover an interesting quantitative connection between the correlation coefficient \( \rho \) and the scattering potential \( V(x) \) on the one hand, and purity \( \mathcal{P} \) and the IGC \( \mathcal{C} \) on the other. Full details of the implementation of our second conjecture can be found in [71, 72].

By direct computation of the integral in Eq. (156), we obtain

\[ \mathcal{P} \approx 1 - \frac{2(2k_o^2 + \sigma_o^2) R_o \sqrt{\Sigma}}{\sqrt{\pi}}, \quad (157) \]

where \( \Sigma = 4\pi |f(k_o)|^2 \) is the scattering cross-section. Next we seek to determine \( \Sigma \) by obtaining an expression for \( f(k_o) \). Under the assumption that the two particles are well separated both initially (before collision) and finally (after collision), and further assuming that the colliding Gaussian wave packets are very narrow in momentum space
(\sigma_{k_o} \ll 1) such that the phase shift can be treated as a constant \( \theta(k_o) \), we deduce that the scattering potential is given by

\[
V = \rho E = \frac{\hbar^2 k_o^2}{2\mu} = \frac{\rho_o^2}{2\mu}.
\]  

(158)

With the potential determined, we obtain the scattering phase shift

\[
\tan \theta_o \approx -\frac{\rho (k_o L)^3}{3} = -\frac{2\mu V k_o L^3}{3\hbar^2} = -\frac{2\mu V p_o L^3}{3\hbar^3},
\]  

(159)

which is in perfect agreement with [82] (and not found in [78]) where standard Schrödinger’s quantum dynamics was employed. This result is significant because it allows us to state that our conjecture is also physically motivated. As the scattering potential has been determined, so too can the scattering amplitude be obtained. To this end, we write

\[
f(k_o) = e^{i\theta_o} \sin \theta_o \approx \frac{\theta_o}{k_o} \approx -a_s
\]  

(160)

for low energy s-wave scattering, \( k_o L = p_o L/\hbar \ll 1 \). Then the squared modulus of Eq. (160), by means of Eq. (159), reads

\[
|f(k_o)|^2 \approx \frac{\theta_o^2}{k_o^2} = \frac{\rho^2 k_o^4 L^6}{9} = \frac{4\mu^2 V^2 L^6}{9\hbar^4} \approx \alpha_s^2.
\]  

(161)

Thus, we finally obtain the scattering cross section:

\[
\Sigma = 4\pi |f(k_o)|^2 \approx \frac{4\pi \rho^2 k_o^4 L^6}{9} = \frac{16\pi \mu^2 V^2 L^6}{9\hbar^4} \approx 4\pi \alpha_s^2.
\]  

(162)

Having found the scattering cross-section, we can recast the purity of the post-collisional two-particle wave function and the correlation coefficient as

\[
P \approx 1 - \frac{4\rho k_o^3 (2 k_o^2 + \sigma_{k_o}^2) R_o L^3}{3} = 1 - \frac{8\mu V (2 k_o^2 + \sigma_{k_o}^2) R_o L^3}{3\hbar^2},
\]  

(163)

and

\[
\rho = \frac{V}{E} = \frac{2\mu V}{\hbar^2 k_o^2} \approx \frac{3\sqrt{\Sigma}}{2\sqrt{\pi k_o^2 L^3}} \approx \frac{3\alpha_s}{k_o^2 L^2},
\]  

(164)

respectively. By use of our first conjecture \( \rho = \rho_{QM} \), with \( \rho_{QM} \overset{\text{def}}{=} \sqrt{8 (2 k_o^2 + \sigma_{k_o}^2) R_o a_s} \), together with Eq. (162) and Eq. (164), we are able to determine the scattering potential density

\[
\frac{V}{L^3} = \frac{4\hbar^2 k_o^4 (2 k_o^2 + \sigma_{k_o}^2) R_o}{3\mu},
\]  

(165)

as well as the scattering length,

\[
a_s \approx 2\mu V L^3/3\hbar^2.
\]  

(166)
This result for the scattering length agrees with equation (41) of [82]. Equation (163) demonstrates that the purity $P$ can be expressed in terms of physical quantities such as the scattering potential height $V$ and range $L$ together with the initial quantities $k_o$, $\sigma_o$ and $R_o$. This result constitutes the second significant finding obtained within our hybrid approach which explains how the entanglement strength is controlled by the interaction potential height $V$ and the incident kinetic energy $E$ of the two-particle system. The role played by $\rho$ in the quantities $P$ and $V$ seem to suggest that the physical information about quantum entanglement is encoded in the covariance term $\text{Cov}(k_1, k_2) \overset{\text{def}}{=} \langle k_1 k_2 \rangle - \langle k_1 \rangle \langle k_2 \rangle$ appearing in the definition of the correlation coefficient $\rho$.

3. Information Geometric Complexity of Entangled Gaussian Wave-Packets

The line element of the Fisher-Rao information metric on the statistical manifold $M^{(corr)}$ induced by $p_{IG}^{(corr)}(x, y|\mu_x, \mu_y, \sigma; r)$ is given by

$$ds^2_{M^{(corr)}} = \frac{1}{\sigma^2} \left( \frac{1}{1 - \rho^2} d\mu_x^2 + \frac{1}{1 - \rho^2} d\mu_y^2 - \frac{2\rho}{1 - \rho^2} d\mu_x d\mu_y + 4d\sigma^2 \right).$$

(167)

Note that $M^{(uncorr)} = \{ p_{IG}^{(uncorr)}(x, y|\mu_x, \mu_y, \sigma) \big| p_{IG}^{(uncorr)}(x, y|\mu_x, \mu_y, \sigma) \geq 0 \}$, and $ds^2_{M^{(uncorr)}} = ds^2_{M^{(corr)}} (\rho \to 0)$. The IGC is determined to be

$$C_{M^{(corr)}} (\tau; \rho) = \frac{8}{\lambda_M} \sqrt{\frac{1 - \rho}{1 + \rho}} \left[ -\frac{3}{4} \lambda_M + \frac{1}{4} \sinh \left( \frac{\lambda_M \tau}{2} \right) + \tanh \left( \frac{1}{2} \lambda_M \tau \right) \right].$$

(168)

Similarly, $C^{(uncorr)} = C^{(corr)} (\rho \to 0)$ represents the IGC on $M^{(uncorr)}$.

The technical details that will be omitted in what follows may be found in [71, 72]. By direct computation, the post-collision IGE is found to be

$$S_{M^{(corr)}} (\tau; \rho) = \lambda_M \tau - \log (\lambda_M \tau) + \frac{1}{2} \log \left( \frac{1 - \rho}{1 + \rho} \right).$$

(169)

For uncorrelated Gaussian statistical models, the IGE is given by $S_{M^{(uncorr)}} (\tau; 0) = S_{M^{(corr)}} (\tau; \rho \to 0)$. For the specific case being considered in this work, the IG analogue of the KS-entropy is determined to be

$$h_{KS}^M \approx 2A_o = \lambda_M.$$

(170)

The KS-entropy is related to the coarse-grained Boltzmann entropy according to [83],

$$S_B(t) = h_{KS} t.$$  

(171)
By means of Eq. (170) we observe that the IGE is related to the KS-entropy in a similar manner as the coarse-grained Boltzmann entropy in Eq. (171), as seen in the following:

\[ S_{\mathcal{M}}(\text{uncorr}) (\tau; 0) \approx h_{\mathcal{M}}^{\text{KS}} \tau. \]  

(172)

By comparing the asymptotic expressions of the IGCs in the presence and absence of correlations, respectively, we obtain the IGC ratio

\[ \frac{C_{\mathcal{M}}(\text{corr}) [D_{\theta}^{(\text{geodesic})} (\tau; \rho)]}{C_{\mathcal{M}}(\text{uncorr}) [D_{\theta}^{(\text{geodesic})} (\tau; 0)]} = \sqrt{\frac{1 - \rho}{1 + \rho}}. \]  

(173)

From Eq. (169) we are also able to determine,

\[ S_{\mathcal{M}}(\text{corr}) (\tau; \rho) - S_{\mathcal{M}}(\text{uncorr}) (\tau; 0) = \frac{1}{2} \log \left( \frac{1 - \rho}{1 + \rho} \right). \]  

(174)

From Eqs. (168) and (174) we find that both the IGC and IGE decrease in presence of correlations. Specifically, the former decreases by the factor \( \sqrt{\frac{1 - \rho}{1 + \rho}} < 1 \) for \( \rho > 0 \) whereas the latter decreases by \( \frac{1}{2} \log \left( \frac{1 - \rho}{1 + \rho} \right) < 0 \) for \( \rho > 0 \). Furthermore, inspection of Eq. (173) confirms that an increase in the correlational structure among the macrovariables of a system implies a reduction in the complexity of the corresponding geodesic information flows on the underlying statistical manifold [37, 57, 58] of said system. Stated otherwise, drawing macroscopic predictions is easier in the presence of correlations than in their absence.

From Eq. (168) we deduce

\[ \rho = \frac{V}{\mathcal{E}} = \frac{\Delta C^2}{C_{\text{total}}^2}, \]  

(175)

with

\[ \Delta C^2 \overset{\text{def}}{=} \left[ C^{(\text{uncorr})} \right]^2 - \left[ C^{(\text{corr})} \right]^2, \]  

and

\[ C_{\text{total}}^2 \overset{\text{def}}{=} \left[ C^{(\text{uncorr})} \right]^2 + \left[ C^{(\text{corr})} \right]^2. \]  

(176)

By combining Eqs. (168) and (168) it follows that

\[ \mathcal{P} \approx 1 - \eta_c \cdot \frac{(\Delta C)^2}{C_{\text{total}}} \]  

where \( \eta_c \overset{\text{def}}{=} \frac{8}{3} k_0^2 \left( 2k_0^2 + \sigma^2_{k_0} \right) R_{\alpha} L^3. \)  

(177)

A new quantitative relation between quantum entanglement and IGC was uncovered in Eq. (177). From this relation it is evident that our system becomes perfectly pure (i.e. \( \mathcal{P} = 1 \)) for vanishing complexity. On the other hand, as the complexity increases from zero, the mixedness of the system increases, causing the purity to decrease from unity. The appearance of correlation terms leads to the compression of the correlated IGC by the fraction \( \sqrt{\frac{1 - \rho}{1 + \rho}} \). From this result we conclude that the IGC decreases when the quantum wave-packets comprising our system becomes entangled.
By means of the hybrid approach employed in this application, we obtained results which coincide with those in [78] (mathematical support to our conjecture) in addition to those in [82] (physical support to our conjecture). While these results are not new, they nonetheless motivate the utility of our theoretical modeling scheme in the study of entanglement.

VI. CONCLUDING REMARKS

In this article, we presented a theoretical modeling scheme that combines IG techniques with inductive inference methods. This modeling scheme allows to describe the macroscopic behavior of complex systems in terms of the microscopic degrees of freedom of the system. After the MrE and IG formalisms were reviewed, particular emphasis was placed on our information geometric measures of complexity, namely the IGC and the IGE. Ten illustrative examples of this modeling scheme were presented.

- Application 1: For our first example we investigated a Gaussian statistical model describing an arbitrary system with \( l \) uncorrelated degrees of freedom. In this case it was determined that the IGE of this statistical model exhibits linear growth characteristics. This asymptotic linear growth of the IGE may be considered the IG analogue of the von Neumann entropy growth introduced by Zurek and Paz.

Our second and third applications focused on bivariate and trivariate Gaussian statistical models, respectively, each admitting correlations among the microstates of each system.

- Application 2: In the bivariate case, the ratio of the IGCs in the presence and absence of correlations was found to be a monotonic increasing function of the correlation parameter \( \rho \) over the range of values \((-1, 1)\). From this result we conclude that entropic inferences on two Gaussian distributed microstates is implemented in a less (more) efficient manner when the two microstates are negatively (positively) correlated than in the absence of correlations.

- Application 3: The trivariate scenario admits three cases, with each case corresponding to the three possible choices of the covariance matrix. We termed these three cases \textit{weak}, \textit{mildly weak} and \textit{strong}, with each designation corresponding to a minimal, intermediate (i.e. neither minimal nor maximal) and maximal correlations among the microstates of the system respectively. In the weak case, the IGC ratio exhibits a monotonic behavior in the correlation parameter \( \rho \in (-1, 1) \). For the mildly weak case, the IGC ratio exhibits a non-monotonic
behavior in $\rho \in \left(-\sqrt{2}, \sqrt{2}\right)$, assuming a maxima at $\rho = \frac{1}{2}$ and vanishing at $\rho = \sqrt{2}$. For the strong case, the IGC ratio exhibits a monotonic behavior in $\rho \in \left(-\frac{1}{2}, 1\right)$. In the latter case, contrary to the mildly weak case, the IGC ratio cannot be zero at the extrema of the range. This behavior is similar to the geometric frustration phenomena that occurs in the presence of loops. Finally, comparison of the IGC ratios for the maximally correlated trivariate and bivariate cases leads to conclude that carrying out entropic inferences on a higher-dimensional manifold in the presence of anti-correlations is less complex than on a lower-dimensional manifold. The converse is true in the presence of positive-correlations.

- **Application 4:** In our forth example, we sought the understanding of the asymptotic temporal behavior of the IGC of a three-dimensional microcorrelated Gaussian statistical model. In this scenario, it was observed that the presence of microcorrelations result in an asymptotic power law decay of the IGC. This decay leads in effect, to the emergence of an asymptotic information geometric compression of the statistical macrostates explored by the system during its evolution, at a faster rate compared to the case of vanishing correlations.

- **Application 5:** Our fifth example examined the effect upon a statistical model due to the introduction of embedding constraints on the macrostates of the system. In presence of such constraints, it is observed that the introduction of embedding constraints lead to the emergence of an asymptotic compression of the statistical macrostates explored by the system as it evolves, with this compression occurring at a faster rate than that observed in absence of embedding constraints. Although arising through radically different mechanisms, the results of Application 4 and 5 both provide quantitative evidence that the information geometric complexity of a statistical systems decreases in presence of correlational structures of either macroscopic or microscopic nature.

The sixth and seventh examples focused respectively, on the dimensional reduction arising from quantum-like constraints being imposed on the macrostates of both 3-dimensional uncorrelated ($3Du$) and 3-dimensional correlated ($3Dc$) Gaussian statistical systems.

- **Application 6:** In this application, it is found that when a quantum-like constraint reminiscent of Heisenberg’s minimum uncertainty relation is imposed on the macrostates of a $3Du$ Gaussian model, the IGE of the corresponding $2Du$ system is attenuated relative to the $3Du$ case.

- **Application 7:** In this example, it is found that the introduction of quantum-like constraints on the macrostates of a $3Dc$ Gaussian model results in a corresponding $2Dc$ model that exhibits no asymptotic change in the
IGE relative to the $2Du$ case. This comparison between the uncorrelated and correlated $2D$ Gaussian models demonstrate that further constraining the $2Du$ with a quantum-like constraint does not result in further global softening of complexity. By contrast, the IGC of the correlated $2Dc$ model diverges as the correlation coefficient approaches unity.

- **Application 8:** In our eighth example, we studied the IG model of an ensemble of random frequency, macroscopic, inverted harmonic oscillators. Our analysis led to the result that the IGE of the model grew linearly in affine time. This may be viewed as an IG analogue of the Zurek-Paz quantum chaos criterion of asymptotic linear entropy growth, and effectively extends the result of Zurek-Paz to an arbitrary set of anisotropic inverted harmonic oscillators in the classical information-geometric setting.

- **Application 9:** For our ninth application, we investigated the statistical manifolds induced by classical probability distributions commonly used in the study of regular and chaotic quantum energy level statistics. As a result of our analysis, it was determined that the IGE associated with regular and chaotic spin chains displayed asymptotic logarithmic and linear growth characteristics in affine time, respectively. These results may be seen as IG analogues of the regular and chaotic entanglement entropies arising in quantum energy level statistics, since the asymptotic behavior of these latter entropies are known to grow logarithmically and linearly in time, respectively.

- **Application 10:** Our tenth and final example made use of our modeling scheme to describe the scattering-induced quantum entanglement between two Gaussian minimum uncertainty wave-packets. It was found that the correlation coefficient $\rho$ can be viewed as the ratio of the potential to kinetic energy of the system. This result constitutes an explicit connection between correlations and physical observables (the macrostate $p_o$ in this case). When $\rho \neq 0$ the wave-packets experience the effect of a repulsive potential; the magnitude of the wave vectors (momenta) decreases relative to their corresponding uncorrelated value. Relative to the uncorrelated case, it was determined that for scenario in which $\rho > 0$, the IGC is compressed by the factor $\sqrt{\frac{1-\rho}{1+\rho}} < 1$, whereas the IGE decreases by the amount $\frac{1}{2}\log \left( \frac{1-\rho}{1+\rho} \right) < 0$. Finally, it was observed that the uncorrelated IGE is related to the KS-entropy in the same functional manner as the coarse-grained Boltzmann entropy is related to the KS-entropy.

At this juncture, it is worth noting that although the first three applications introduced in this article analyze Gaussian models of arbitrary nature, we emphasize that an understanding of such systems is nevertheless quite useful...
since they facilitate the construction of more physically motivated scenarios such as our IG models of anisotropic, inverted harmonic oscillators and scattering induced quantum entanglement presented in examples eight and ten, respectively. These two examples lead to a connection between our IG modeling scheme and actual physical systems by means of the former reproducing the known results of linear von Neumann entropy growth attributed to Zurek-Paz in the case of example eight, and to the results found in Mishima for the scattering phase shift and scattering length in the case of example ten. In example nine, we utilized our theoretical modeling scheme to describe integrable (by means of a composite Poisson and Exponential model) and chaotic (by means of a composite Wigner-Dyson and Gaussian model) energy level statistics associated with a quantum antiferromagnetic Ising spin chain leading to an IG result that is coincident with the known entanglement entropy of each scenario. Specifically, the IGE exhibits asymptotic logarithmic growth in the former case, while the IGE exhibits asymptotic linear growth in the latter case. These results are consistent with the known results that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their entanglement entropies, respectively. Finally, building upon example four and five, we find that in example six, Eq. quantitatively shows that the IGE is attenuated when approaching the 2D case from the 3D case via the introduction of the macroscopic constraint in Eq. that is reminiscent of Heisenberg’s minimum uncertainty relation. In the same vein of our work in , a recent investigation concludes that quantum mechanics can reduce the statistical complexity of classical models. It is our hope that, based on the above discussion, the connection between our theoretical modeling scheme and physics is made more transparent.

In conclusion, we would like to outline four possible lines of research for future investigations:

- Describe the role of thermodynamics within the IGAC. For instance, thermodynamics is known to play a prominent role in the entropic analysis of chaotic dynamics.

- Better understand the role of thermodynamics as a possible bridge among different complexity measures.

- Further develop our understanding of how the IGE relates to the Kolmogorov-Sinai dynamical entropy, the coarse-grained Boltzmann entropy and the von Neumann entropy depending on the nature of the system being modeled; and finally,

- Explore in a more substantial manner the relationship between our IGE and the intrinsic complexity appearing in the work of Rodriguez as well as in the works by Myung, Balasubramanian and Pitt.

We are aware of several issues that remain unsolved within the IGAC and further development of the framework.
remains to be done. Nevertheless, we are gratified that our theoretical modeling scheme is gaining attention within the community. Indeed, there appears to be an increasing number of scientists who are actively engaged in research that either makes use of, or is related to, the theoretical framework described in the present article [55, 87–107).

Acknowledgments

The authors are grateful to Domenico Felice for helpful discussions in the early version of this manuscript. Finally, C. C. acknowledges the hospitality of the United States Air Force Research Laboratory in Rome (New York) where part of his contribution to this work was completed.

[1] J. P. Crutchfield and B. S. McNamara, *Equations of motions from a data series*, Complex Systems 1, 417 (1987).
[2] I. J. Myung, V. Balasubramanian, and M. A. Pitt, *Counting probability distributions: differential geometry and model selection*, Proc. Natl Acad. Sci. 97, 11170 (2000).
[3] V. Balasubramanian, *A geometric formulation of Occam’s razor for inference of parametric distributions*, arXiv:adap-org/9601001 (1996).
[4] V. Balasubramanian, *Statistical inference, Occam’s razor and statistical mechanics on the space of probability distributions*, Neural Computation 9, 268 (1997).
[5] T. S. Cubitt, J. Eisert, and M. W. Wolf, *Extracting dynamical equations from experimental data is NP hard*, Phys. Rev. Lett. 108, 120503 (2012).
[6] M. Gell-Mann, *What is complexity?*, Complexity 1, 1 (1995).
[7] D. P. Feldman and J. P. Crutchfield, *Measures of complexity: why?*, Phys. Lett. A238, 244 (1998).
[8] J. P. Crutchfield and K. Young, *Inferring statistical complexity*, Phys. Rev. Lett. 63, 105 (1989).
[9] R. Landauer, *A simple measure of complexity*, Nature 336, 306 (1988).
[10] R. Romano and P. van Loock, *Quantum control of noisy channels*, arXiv:quant-ph/0811.3014 (2008).
[11] S. Lloyd and H. Pagels, *Complexity as thermodynamic depth*, Ann. Phys. 188, 186 (1988).
[12] C. H. Bennett, *On the nature and origin of complexity in discrete, homogeneous, locally-interacting systems*, Foundations of Physics 16, 585 (1986).
[13] C. H. Bennett, *How to define complexity in physics and why*, in Complexity, Entropy, and the Physics of Information: Proceedings of the Santa Fe Institute Workshop, ed. by W. H. Zurek (1989).
[14] S. Lloyd, *Core-halo instability in dynamical systems*, arXiv:nlin.CD/1302.3199 (2013).
[15] J. Sun, E. M. Bollt and T. Nishikawa, *Judging model reduction of complex systems*, Phys. Rev. E83, 046125 (2011).
E. T. Jaynes, *Macroscopic prediction*, in Complex Systems and Operational Approaches in Neurobiology, Physics, and Computers, ed. by H. Haken, Springer, Berlin (1985).

A. Caticha, *Entropic dynamics*, AIP Conf. Proc. 617, 302 (2002).

A. Caticha and A. Giffin, *Updating probabilities*, AIP Conf. Proc. 872, 31 (2006).

S. Amari and H. Nagaoka, *Methods of information geometry*, Oxford University Press (2000).

L. Casetti, M. Pettini and E. G. D. Cohen, *Geometric approach to Hamiltonian dynamics and statistical mechanics*, Phys. Rep. 337, 237 (2000).

A. Caticha and C. Cafaro, *From information geometry to Newtonian dynamics*, AIP Conf. Proc. 954, 165 (2007).

A. Caticha, *Entropic dynamics, time and quantum theory*, J. Phys. A: Math. Theor. 44, 225303 (2011).

A. Giffin and A. Caticha, *Updating probabilities with data and moments*, AIP Conf. Proc. 954, 74 (2007).

A. Giffin, C. Cafaro, and S. A. Ali, *Application of the maximum relative entropy method to the physics of ferromagnetic materials*, Physica A455, 11 (2016).

C. Cafaro, *The information geometry of chaos*, PhD Thesis, State University of New York at Albany, Albany-NY, USA (2008).

L. Casetti, C. Clementi, and M. Pettini, *Riemannian theory of Hamiltonian chaos and Lyapunov exponents*, Phys. Rev. E54, 5969 (1996).

M. Di Bari and P. Cipriani, *Geometry and chaos on Riemann and Finsler manifolds*, Planet. Space Sci. 46, 1543 (1998).

C. G. J. Jacobi, *Vorlesungen uber dynamiik*, Reimer, Berlin (1866).

R.A. Fisher, *Theory of statistical estimation*, Proc. Cambridge Philos. Soc. 122, 700 (1925).

C.R. Rao, *Information and accuracy attainable in the estimation of statistical parameters*, Bull. Calcutta Math. Soc. 37, 81 (1945).

A. Caticha, *Lectures on probability, entropy and statistical physics*, in the 28-th International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering, Brazil (2008).

F. De Felice and J. S. Clarke, *Relativity on curved manifolds*, Cambridge University Press (1990).

M. P. do Carmo, *Riemannian geometry*, Birkhauser, Boston (1992).

C. Cafaro and S. A. Ali, *Maximum caliber inference and the stochastic Ising model*, Phys. Rev. E94, 052145 (2016).

A. Caticha, *Entropic inference and the foundations of physics*, USP Press, Sao Paulo, Brazil (2012).

C. Cafaro and S. A. Ali, *Jacobi fields on statistical manifolds of negative curvature*, Physica D234, 70 (2007).

C. Cafaro, A. Giffin, S. A. Ali and D.-H. Kim, *Reexamination of an information geometric construction of entropic indicators of complexity*, Appl. Math. Comput. 217, 2944 (2010).

C. C. Rodriguez, *The volume of bitnets*, AIP Conf. Proc. 735, 555 (2004).
[40] C. M. Caves and R. Schack, *Unpredictability, information, and chaos*, Complexity 3, 46-57 (1997); A. J. Scott, T. A. Brun, C. M. Caves, and R. Schack, *Hypersensitivity and chaos signatures in the quantum baker’s map*, J. Phys. A39, 13405 (2006).

[41] V. M. Alekseev and M. V. Yakobson, *Symbolic dynamics and hyperbolic dynamic systems*, Phys. Reports 75, 287 (1981).

[42] A. N. Kolmogorov, *Three approaches to the quantitative definition of information*, Probl. Inf. Transm. (USSR) 1, 4 (1965); A. N. Kolmogorov, *Logical basis for information theory and probability theory*, IEEE Trans. Inf. Theory, IT14, 662 (1968); T. M. Cover and J. A. Thomas, *Elements of information theory*, John Wiley and Sons, Inc. (2006).

[43] Y. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Mathematics Survey 32, 55 (1977).

[44] F. Benatti, *Classical and quantum entropies: dynamics and Information*, in *Entropy*, Princeton University Press (2003); T. M. Cover and J. A. Thomas, *Elements of information theory*, John Wiley and Sons, Inc. (2006).

[45] C. Cafaro, S. A. Ali, and A. Giffin, *An application of reversible entropic dynamics on curved statistical manifolds*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. 872, 243 (2006).

[46] C. Cafaro, *Information geometry and chaos on negatively curved statistical manifolds*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. 954, 175 (2007).

[47] C. Cafaro, *Recent theoretical progress on an information geometrodynamical approach to chaos*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. 1073, 16 (2008).

[48] S. A. Ali, C. Cafaro, A. Giffin, and D.-H. Kim, *Complexity characterization in a probabilistic approach to dynamical systems through information geometry and inductive inference*, Physica Scripta 85, 025009 (2012).

[49] C. Cafaro, *Information geometric complexity of entropic motion on curved statistical manifolds*, Proceedings of the 12th Joint European Thermodynamics Conference, JETC 2013, Eds. M. Pilotelli and G.P. Beretta (ISBN 978-88-89252-22-2, Snoopy, Brescia, Italy, 2013), pp. 110-118.

[50] C. Cafaro, *Information-geometric indicators of chaos in Gaussian models on statistical manifolds of negative Ricci curvature*, Int. J. Theor. Phys. 47, 2924 (2008).

[51] W. H. Zurek, S. Habib and J. P. Paz, *Coherent states via decoherence*, Phys. Rev. Lett. 70, 1187 (1993).

[52] Y. A. Rozanov, *Probability theory: a concise course*, Dover Publications, New York (1977).

[53] D. Felice, C. Cafaro, and S. Mancini, *Information geometric complexity of a trivariate Gaussian statistical model*, Entropy 16, 2944 (2014).

[54] J. F. Sadoc and R. Mosseri, *Geometrical frustration*, Cambridge University Press (2006).

[55] D. Felice, S. Mancini and M. Pettini, *Quantifying networks complexity from information geometry viewpoint*, J. Math. Phys. 55, 043505 (2014).

[56] S. A. Ali, C. Cafaro, D.-H. Kim and S. Mancini, *The effect of microscopic correlations on the information geometric complexity of Gaussian statistical models*, Physica A389, 3117 (2010).
[57] C. Cafaro and S. Mancini, *On the complexity of statistical models admitting correlations*, Phys. Scr. **82**, 035007 (2010).

[58] C. Cafaro and S. Mancini, *Quantifying the complexity of geodesic paths on curved statistical manifolds through information geometric entropies and Jacobi fields*, Physica **D240**, 607 (2011).

[59] C. Cafaro, A. Giffin, C. Lupo, and S. Mancini, *Insights into the softening of chaotic statistical models by quantum considerations*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. **1443**, 366 (2012).

[60] S. A. Ali, C. Cafaro, A. Giffin, C. Lupo, and S. Mancini, *On a differential geometric viewpoint of Jaynes’ MaxEnt method and its quantum extension*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. **1443**, 120 (2012).

[61] A. Giffin, S. A. Ali, and C. Cafaro, *Local softening of chaotic statistical models with quantum consideration*, in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, AIP Conf. Proc. **1553**, 238 (2013).

[62] C. Cafaro, A. Giffin, C. Lupo and S. Mancini, *Softening the complexity of entropic motion on curved statistical manifolds*, Open Syst. & Inf. Dyn. **19**, 1250001 (2012).

[63] A. Giffin, S. A. Ali, and C. Cafaro, *Local softening of information geometric indicators of chaos in statistical modeling in the presence of quantum-like considerations*, Entropy **15**, 4622 (2013).

[64] C. Cafaro, *Works on an information geometrodynamical approach to chaos*, Chaos, Solitons & Fractals **41**, 886 (2009).

[65] C. Cafaro and S. A. Ali, *Geometrodynamics of information on curved statistical manifolds and its applications to chaos*, EJTP **5**, 139 (2008).

[66] W. H. Zurek and J. P. Paz, *Decoherence, chaos, and the second law*, Phys. Rev. Lett. **72**, 2508 (1994); *Quantum chaos: a decoherent definition*, Physica **D83**, 300 (1995).

[67] W. H. Zurek, *Preferred states, predictability, classicality and environment-induced decoherence*, Prog. Theor. Phys. **89**, 281 (1993).

[68] C. Cafaro, *Information geometry, inference methods and chaotic energy levels statistics*, Mod. Phys. Lett. **B22**, 1879 (2008).

[69] C. Cafaro and S. A. Ali, *Can chaotic quantum energy levels statistics be characterized using information geometry and inference methods?*, Physica **A387**, 6876 (2008).

[70] D.-H. Kim, S. A. Ali, C. Cafaro, and S. Mancini, *An information geometric analysis of entangled continuous variable quantum systems*, Journal of Physics: Conference Series **306**, 012063 (2011).

[71] D.-H. Kim, S. A. Ali, C. Cafaro and S. Mancini, *Information geometric modeling of scattering induced quantum entanglement*, Phys. Lett. **A375**, 2868 (2011).

[72] D.-H. Kim, S. A. Ali, C. Cafaro and S. Mancini, *Information geometry of quantum entangled Gaussian wave-packets*, Physica **A391**, 4517 (2012).
[73] A. Einstein, B. Podolsky, N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Phys. Rev. **47**, 777 (1935).

[74] E. Schrödinger, *Die gegenwartige situation in der quantenmechanik*, Naturwissenschaften **23**: pp. 807-812; 823-828; 844-849 (1935).

[75] N. L. Harshman and G. Hutton, *Entanglement generation in the scattering of one-dimensional particles*, Phys. Rev. **A77**, 042310 (2008).

[76] N. L. Harshman and P. Singh, *Entanglement mechanisms in one-dimensional potential scattering*, J. Phys. **A**: Math. Theor. **41**, 155304 (2008).

[77] P. Kaye, R. Laflamme, and M. Mosca, *An Introduction to Quantum Computing*, Oxford University Press (2007).

[78] J. Wang, C. K. Law and M.-C. Chu, *Loss of purity by wave-packet scattering at low energies*, Phys. Rev. **A73**, 034302 (2006).

[79] C. K. Law, *Entanglement production in colliding wave packets*, Phys. Rev. **A70**, 062311 (2004).

[80] L. D. Landau and E. M. Lifshitz, *Quantum mechanics: Non-relativistic theory*, Butterworth-Heinemann (1981).

[81] A. Serafini and G. Adesso, *Standard forms and entanglement engineering of multimode Gaussian states under local operations*, J. Phys. **A40**, 8041 (2007). V. M. Alekseev and M. V. Yakobson, *Symbolic dynamics and hyperbolic dynamic systems*, Phys. Reports **75**, 287 (1981).

[82] K. Mishima, M. Hayashi and S. H. Lin, *Entanglement in scattering processes*, Phys. Lett. **A333**, 371 (2004).

[83] K. Ropotenko, *Kolmogorov-Sinai entropy and black holes*, Class. Quant. Grav. **25**; 195005 (2008).

[84] T. Prosen, M. Znidaric, *Is the efficiency of classical simulations of quantum dynamics related to integrability?*, Phys. Rev. **E75**, 015202 (2007); T. Prosen, I. Pizorn, *Operator space entanglement entropy in transverse Ising chain*, Phys. Rev. **A76**, 032316 (2007).

[85] M. Gu, K. Wiesner, E. Rieper and V. Vedral, *Quantum mechanics can reduce the complexity of classical models*, Nature Commun. **3**, 1 (2012).

[86] C. Beck and F. Schlogl, *Thermodynamic Analysis of Chaotic Systems: An Introduction*, Cambridge University Press (1995).

[87] L. Peng, H. Sun and G. Xu, *Information geometric characterization of the complexity of fractional Brownian motion*, J. Math. Phys. **53**, 123305 (2012).

[88] L. Peng, H. Sun, D. Sun and J. Yi, *The geometric structures and instability of entropic dynamical models*, Adv. Math. **227**, 459 (2011).

[89] O. Semarak and P. Sukova, *Free motion around black holes with discs or rings: between integrability and chaos-I*, Monthly Notices of the Royal Astronomical Society **404**, 545 (2010).

[90] C. Li, H. Sun and S. Zhang, *Characterization of the complexity of an ED model via information geometry*, Eur. Phys. J.
[91] L. Cao, D. Li, E. Zhang, Z. Zhang and H. Sun, *A statistical cohomogeneity one metric on the upper plane with constant negative curvature*, Adv. Math. Phys., Vol. 2014 (2014), Article ID 832683, 6 pages.

[92] S. M. Abtahi, S. H. Sadati and H. Salarieh, *Ricci-based chaos analysis for roto-translatory motion of a Kelvin-type gyrostat satellite*, Journal of Multi-Body Dynamics **228**, 34 (2014).

[93] J. Mikes and E. Stepanova, *A five-dimensional Riemannian manifold with an irreducible SO (3)-structure as a model of abstract statistical manifold*, Annals of Global Analysis and Geometry **45**, 111 (2014).

[94] S. Weis, *Continuity of the maximum-entropy inference*, Commun. Math. Phys. **330**, 1263 (2014).

[95] C. Li, L. Peng and H. Sun, *Entropic dynamical models with unstable Jacobi fields*, Rom. Journ. Phys. **60**, 1249 (2015).

[96] M. Itoh and H. Satoh, *Geometry of Fisher information metric and the barycenter map*, Entropy **17**, 1814 (2015).

[97] R. Franzosi, D. Felice, S. Mancini and M. Pettini, *A geometric entropy detecting the Erdős-Rényi phase transition*, Eur. Phys. Lett. **111**, 20001 (2015).

[98] A. C. R. Martins, *Opinion particles: Classical physics and opinion dynamics*, Phys. Lett. **A379**, 89 (2015).

[99] S. A. Muhammad, E. Zhang and H. Sun, *Jacobi fields on the manifold of Freund*, Italian Journal of Pure and Applied Mathematics **34**, 181 (2015).

[100] D. Felice and S. Mancini, *Gaussian network’s dynamics reflected into geometric entropy*, Entropy **17**, 5660 (2015).

[101] C. Wen-Haw, *A review of geometric mean of positive definite matrices*, British Journal of Mathematics and Computer Science **5**, 1 (2015).

[102] S. Weis, A. Knauf, N. Ay and M.-J. Zhao, *Maximizing the divergence from a hierarchical model of quantum states*, Open Syst. & Inf. Dyn. **22**, 1550006 (2015).

[103] S. Weis, *Maximum-entropy inference and inverse continuity of the numerical range*, Reports on Mathematical Physics **77**, 251 (2016).

[104] D. S. Shalymov and A. L. Fradkov, *Dynamics of non-stationary processes that follow the maximum of the Rényi entropy principle*, Proc. R. Soc. **A472**, 20150324 (2016).

[105] I. S. Gomez and M. Portesi, *Ergodic statistical models: entropic dynamics and chaos*, AIP Conf. Proc. **1853**, Art. ID:100001 (2017).

[106] I. S. Gomez, *Notions of the ergodic hierarchy for curved statistical manifolds*, Physica **A484**, 117 (2017).

[107] G. Henry and D. Rodriguez, *On the instability of two entropic dynamical models*, Chaos, Solitons & Fractals **91**, 604 (2016).