Quaternionic Mass Matrices
and
CP Symmetry

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Abstract

A viable formulation of gauge theory with extra generations in terms of quaternionic fields is presented. For the theory to be acceptable, the number of generations should be equal to or greater than 4. The quark-lepton mass matrices are generalized into quaternionic matrices. It is concluded that explicit CP violation automatically disappears in both strong- and weak-interaction sectors.
1. Introduction

We still have no conclusive argument on the origin of quark-lepton masses, flavor mixings and CP violation. To understand the origin of these observables seems to be almost equivalent to understanding the origin of quark-lepton mass matrices.

Even if the mass matrices were assumed to be real the phenomenon of flavor mixing would be still possible. The presence of the flavor mixing, of course, demands \( N_g \geq 2 \), with \( N_g \) being the number of generations. In realistic complex mass matrices, the phenomenon of CP violation becomes possible, but only when \( N_g \geq 3 \) as was first discussed by Kobayashi-Maskawa [1].

Then it may be a natural question to ask whether there exists a physical observable or a theoretical framework which necessitates the presence of higher generations, i.e. \( N_g \geq 4 \). A trivial example of such an observable, one may encounter, may be the deviation from unitarity of the \( 3 \times 3 \) mixing matrix, i.e.

\[
\sum_{i=1}^{3} |V_{ai}|^2 < 1
\]

with \( V_{ai} \) being the elements of the matrix. So far there is no experimental hint suggesting higher generations, including such deviation. We, thus, would like to rely on theoretical considerations and search for some theoretical framework, in which \( N_g = 4 \) is a critical number. Apparently, as far as we work in the framework of complex mass matrices the critical value of \( N_g \) is 3 and any essential change from the Kobayashi-Maskawa scheme will not be expected.

We, therefore, try to generalize the complex structure, i.e., we try to formulate the theory in terms of quaternionic spinors, which are obtained by combining ordinary complex spinors in pairs; 2 generations correspond to 1 quaternionic spinor for each type of quarks and leptons. The mass matrices for quarks and leptons can be naturally quaternionic, without contradicting with Lorentz and gauge invariances. The main purpose of the present paper is to investigate the consequences of such quaternionic mass matrices. More concretely we will investigate the following issues, which are closely related with the quaternionic property; (i) At which \( N_g \) does weak CP violation start to occur? (ii) Is a new insight into the
strong CP problem obtained? (iii) Are there any restrictions on quark(lepton) masses and/or flavor mixings as the result of the quaternionic property?

We will see that, in spite of the generalization from the complex to the quaternionic matrices, explicit CP violation actually disappears in the both of strong (assuming $\theta_{QCD} = 0$) and weak interaction sectors. Thus the quaternionic approach necessarily leads to the scenario of spontaneous CP violation. We, however, would like to emphasize the fact that in our approach the “real” mass matrices are not put by hand, but the absence of explicit CP violation is automatically guaranteed as the consequence of the quaternionic property, i.e. guaranteed for arbitrary quaternionic mass matrices to start with. For such quaternionic approach to make sense $N_g$ must be even. Since we already know $N_g \geq 3$, this inevitably means $N_g \geq 4$. Correspondingly, the mass matrices should be regarded as general $(N_g/2) \times (N_g/2)$ quaternionic matrices.

2. A formulation via 6-dimensional theory

Let $H = a_1 + a_2 i + a_3 j + a_4 k$ be an arbitrary quaternion, with imaginary units $i, j$ and $k$ ($a_1$ to $a_4$:real). $H$ can be uniquely decomposed into 2 complex numbers, according to $H = (a_1 + a_2 i) + (a_3 + a_4 i) j$. In this way an arbitrary quaternionic spinor reduces to 2 complex spinors.

To formulate a consistent theory with quaternionic fermions, so that it reduces to a viable theory with ordinary complex spinors in pairs, is a non-trivial task. In particular, we immediately encounter the problem of whether the imaginary unit “$i$” appearing in the momentum operator, $p_\mu = -i \partial_\mu$ in ordinary theories, should be regarded as to commute with quaternions or not. If that “$i$” is identified with $i$ in the quaternionic imaginary units, it will not commute with quaternions, especially not with $j$, leading to $[p_\mu, j] \neq 0$. It, however, means that the momentum operator is generation dependent (multiplying $j$ is equivalent to a unitary rotation among different generations), which is not acceptable for us.

A safe way to construct a viable theory, avoiding such problem, is to start
from a 6-dimensional Yang-Mills theory which, after (naive) dimensional reduction, naturally reduces to a consistent 4-dimensional theory with even number of generations. The reason why D=6 has some relevance is that the 6-dimensional Lorentz group $SO(1, 5)$ is equivalent to $SL(2, \mathbb{H})$ where $\mathbb{H}$ stands for the quaternion $[2]$, just as $SO(1, 3)$ is equivalent to $SL(2, \mathbb{C})$. A D=6 Weyl fermion can be represented as a 2-component quaternionic spinor. $SL(2, \mathbb{H})$ is also equivalent to $SU^*(4)$. A D=6 Weyl fermion, therefore, can be equivalently represented as a 4-component complex spinor, and decomposes into a pair of D=4 Weyl fermions. Apparently, after naive dimensional reduction, for instance, where all massive modes are neglected, the theory automatically reduces to a viable renormalizable D=4 theory. Though utilizing the D=6 theory is quite helpful, once we know the way to construct a viable theory we actually can formulate a theory with quaternionic fermions in D=4, from the beginning.

The linkage between the theory with quaternionic spinors and the theory with ordinary complex spinors can be made through the following one-to-one correspondence between quaternionic units and $2 \times 2$ complex matrices, which relates the representations in $SL(2, \mathbb{H})$ and those in $SU^*(4)$

$$
1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \leftrightarrow i\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j \leftrightarrow i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \leftrightarrow i\sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(1)

The assignment above is a bit different from that of Ref.[2]. Now regarding the imaginary unit "$i$" appearing in the momentum operator as just ordinary $i$, not as a matrix, the commutativity between $i$ and $j$ is trivial:

$$
[i, j] = [i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}] = 0.
$$

(2)

One note is in order on the meaning of the “special” in $SL(2, \mathbb{H})$. Usually “special” means that the determinant of the transformation matrices should be 1. In the present case, however, this condition overconstrains the matrices, since this condition leaves only $2 \times 2 \times 4 - 4 = 12$ real degree of freedom, while the dimension of $SO(1, 5)$ is 15. Thus we have to modify the meaning of the determinant as
follows for an arbitrary quaternionic matrix $M$:

$$
\text{det} M \equiv \exp(\text{Tr} \ln M), \quad (3)
$$

where the operator $\text{Tr}$ should be understood for an arbitrary $M'$ as

$$
\text{Tr} M' \equiv \text{Re}(\text{tr} M'), \quad (4)
$$

where $\text{tr}$ means ordinary trace, while $\text{Re}$ implies to take only the real part. Let us note that, in the matrix representation of quaternion by the use of Eq.(1), such defined $\text{Tr}$ just corresponds to ordinary trace for the matrix. This specific trace, taking the real part, plays an important role when we consider CP symmetry of the theory, leading to the absence of explicit CP violation. Now that the condition of $\text{det} = 1$ reduces only one real degree of freedom, as expected.

The 6-dimensional gamma matrices are given as

$$
\Gamma^M = \begin{pmatrix} 0 & \gamma^M \\ \tilde{\gamma}^M & 0 \end{pmatrix}, \quad (M = 0 - 5), \quad (5)
$$

where

$$
\gamma^M = (\gamma^0, \gamma^i), \quad \tilde{\gamma}^M = (\gamma^0, -\gamma^i), \quad (6)
$$

with the $2 \times 2$ quaternionic matrices being given as

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}. \quad (7)
$$

The representation for the gamma matrices given above is in the chiral basis, i.e.,

$$
\Gamma^7 \equiv -\Gamma^0 \Gamma^1 \cdots \Gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (8)
$$

where a 6-dimensional Weyl spinor is represented as a 2-componet quaternionic spinor, say, the upper half, reducing to a pair of 2-component complex spinors after dimensional reduction. Combining Weyl spinors of different chiralities, $\Psi_R$ and $\Psi_L$, we get a 4-component full spinor $\Psi$, $\Psi = \Psi_R + \Psi_L$.

Let us study the Lorentz invariant free lagrangian for a 6-dimensional fermion,

$$
L = Tr(\bar{\Psi} \Gamma^M i \partial_M \Psi) + Tr(m \bar{\Psi}_R \Psi_L + m^* \bar{\Psi}_L \Psi_R), \quad (9)
$$
where \( m \) represents a quaternionic mass. We now perform the naive dimensional reduction of this theory into 4-dimensional world. First we note that a quaternionic fermion \( \Psi \) can be uniquely decomposed into two fermions as,

\[
\Psi = \Psi_1 + \Psi_2 j, \tag{10}
\]

where each of \( \Psi_1 \) and \( \Psi_2 \) is made of 1 and \( i \) alone. Under the naive dimensional reduction only the 4-dimensional part of \( \partial_M \) and therefore \( \Gamma^M \), which does not contain \( j \) nor \( k \), survives. Thus the resultant lagrangian reads as

\[
L = \overline{\Psi}_1 \gamma^\mu i \partial_\mu \Psi_1 + \overline{\Psi}_2 \gamma^\mu i \partial_\mu \Psi_2 + ( \overline{\Psi}_{1R} \overline{\Psi}_{2R} ) \begin{pmatrix} m_1 & -m_2 \\ m_2^* & m_1^* \end{pmatrix} \begin{pmatrix} \Psi_{1L} \\ \Psi_{2L} \end{pmatrix} + h.c., \tag{11}
\]

where \( \Psi_{1,2} \) are ordinary complex spinors, and \( \gamma^\mu \) is an ordinary 4-dimensional gamma matrices; both are obtained from the quaternionic counterparts, \( \Psi_{1,2} \) and \( \Gamma^\mu \) with the quaternionic \( i \) being replaced by ordinary imaginary unit \( i \). The 2 × 2 mass matrix, expressed in terms of two complex numbers \( m_1 \) and \( m_2 \), is just the (transpose of) matrix representation of the quaternion \( m \). This mass matrix, though it leads to a flavor mixing, has an unacceptable consequence, i.e. degeneracy of the mass eigenvalues, \( |m| = \sqrt{|m_1|^2 + |m_2|^2} \). This degeneracy may be understood as the reflection of the global \( SL(1, \mathbb{H}) \) symmetry, independent of Lorentz transformation \( u \),

\[
\Psi \rightarrow \Psi' = \Psi u, \tag{12}
\]

with a unimodular quaternion \( u, |u| = 1 \). The 4-dimensional spinors \( (\Psi_1, \Psi_2) \) behave as a doublet under \( SU(2) \), which is isomorphic to the \( SL(1, \mathbb{H}) \). The kinetic term is trivially invariant under this transformation, while the mass \( m \) can be modified into \( |m| \). We can show that this degeneracy arises for arbitrary even number of generations.

### 3. A possible alternative formulation

We now have to search for an alternative quaternion-like closed algebra (divi-
sion algebra) in order to describe fermions. The almost unique choice is to modify
the Lie algebra of $SU(2)$, corresponding to the imaginary units of quaternion, into
the Lie algebra of $SU(1,1)$. Namely, now the “imaginary” units $j$ and $k$, corre-
sponding to raising and lowering operators of $SU(1,1)$, should be accompanied
by ordinary $i$:

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
   i \leftrightarrow i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
   j \leftrightarrow \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
   k \leftrightarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (13)$$

Accordingly, the gamma matrices are also modified and the Clifford algebra,
$\Gamma^M \Gamma^N + \Gamma^N \Gamma^M = 2g^{MN}I_4$ tells us $g_{MN} = diag(1,-1,-1,-1,1,1)$, namely
that the extra dimensions of the 6-dimensional space-time are time-like. So
the ordinary conjugate $\Psi$ should be replaced by $\Psi^\dagger \Gamma^0 \Gamma^4 \Gamma^5$, to ensure the full
6-dimensional Lorentz invariance. This conjugate, however, causes an indefinite
metric, i.e.,
$$\bar{\psi}_1 \gamma^\mu i\partial_\mu \psi_1 - \bar{\psi}_2 \gamma^\mu i\partial_\mu \psi_2,$$
after the dimensional reduction.

Since only 4-dimensional Lorentz invariance is what we really have to demand
(the Lorentz invariance of the full space-time is spoiled under the compactifica-
tion, anyway), we will just adopt ordinary Pauli conjugate, $\Psi^\dagger = \Psi^\dagger \Gamma^0$. Thus
we have exactly the same form of the free lagrangian as the one for the original
quaternions, Eq.(9), though the fermions, gamma matrices and $m_1$ should be un-
derstood to be written in terms of the modified $j$ and $k$. We also just follow the
procedure of the dimensional reduction done above. The only difference in the
resultant lagrangian for 4-dimensional fermions is that the mass matrix is now of
the form,

$$\begin{pmatrix} m_1 & m_2 \\ m_2^* & m_1^* \end{pmatrix}. \quad (14)$$

We can easily check that the mass eigenvalues are now non-degenerate, $|m_1| \pm |m_2|$, and the mass matrix deserves to the description of the real world.

4. CP symmetry and the flavor mixing

So far we have discussed the free lagrangian for a quaternionic fermion, i.e.
for two generations. We can immediately generalize the lagrangian so that it
can contain arbitrary even number of generations, $N_g$, with $N_g/2$ quaternionic
spinors, \( \Psi_a(a = 1 \text{ to } N_g/2) \):

\[
L = Tr(\bar{\Psi}_a \Gamma^M i \partial_M \Psi_a) + Tr(m_{ab} \bar{\Psi}_{aR} \Psi_{bL} + m_{ba}^* \bar{\Psi}_{aL} \Psi_{bR}).
\] (15)

We will study what does the \((N_g/2) \times (N_g/2)\) matrix \( m_{ab} \) imply concerning flavor mixing and, in particular, CP symmetry, which is of our main concern. For this purpose, the quaternionic mass matrix \( m_{ab} \) should be replaced by a \( N_g \times N_g \) complex matrix, obtained by substituting the \( 2 \times 2 \) matrix form for each quaternionic element according to the rule Eq.(13), after the dimensional reduction.

(i) Flavor mixing

Concerning the flavor mixing among various generations and its relation with mass eigenvalues, we may have, in principle, some new constraint, as the quaternionic property restricts the form of the mass matrices to some extent, as is seen in Eq.(14). We, however, have not found any physically observable constrict. To see the situation, let us investigate the simplest case of two generations. In this case, the mass matrix of the form of Eq.(14) can be assigned for both of up-type and down-type quarks:

\[
m_U = \begin{pmatrix} m_{u1} & m_{u2} \\
m_{u2}^* & m_{u1}^* \end{pmatrix}, \quad m_D = \begin{pmatrix} m_{d1} & m_{d2} \\
m_{d2}^* & m_{d1}^* \end{pmatrix}.
\] (16)

The mass eigenvalues are non-degenerate as was seen above, i.e., \( m_{u,c} = |m_{u1}| \mp |m_{u2}| \), and \( m_{d,s} = |m_{d1}| \mp |m_{d2}| \). The Cabbibo mixing angle turns out to be given as

\[
\theta_C = \frac{1}{2} \arg \left( \frac{m_{u1} m_{u2}^*}{m_{d1} m_{d2}} \right).
\] (17)

This type of analysis can be easily generalized to higher generation cases.

(ii) Strong CP problem

One of the interesting solutions to the strong CP problem is to assume that the lagrangian does preserve CP symmetry, i.e. no explicit CP violation, and therefore that \( \theta_{QCD} = 0 \). Then it becomes a non-trivial problem to assure the absence of the contribution from the flavor dynamics QFD, \( \theta_{QFD} \), which is expected from the spontaneous CP violation in the QFD sector (\( \theta = \theta_{QCD} + \theta_{QFD} \)).
Nelson and Barr\cite{3} have proposed a contrived mechanism to guarantee $\theta_{QFD} = 0$. It is interesting to note that our quaternionic mass matrices, though they contain various phases, automatically guarantee $\theta_{QFD} = 0$. The proof is quite simple. Namely, in terms of mass matrices for up-type and down-type quarks, $m_U$ and $m_D$,

$$
\theta_{QFD} = \arg(det m_U) + \arg(det m_D),
$$

and each term identically vanishes:

$$
\arg(det m_U) = \arg[\exp(Tr \ln m_U)] = \arg[\exp(Re \tr \ln M_U)] = 0,
$$

etc., where $M_U$ is original $N_g/2 \times N_g/2$ quaternionic mass matrix for up-type quarks. The presence of the operation $Re$ was essential to get this result. We may explicitly check this relation for the two generation case, by taking the determinant of Eq.(14).

(iii) Weak CP

To extract re-phasing invariant measure of weak CP violation, if there is any, it is useful in general to analyze the quantities $Im Tr(P_1(H_D)P_2(H_U) \cdots)$ where $P_i$ denote arbitrary monomials of hermitian matrices $H_U$ and $H_D$, defined as $H_U \equiv m_U^\dagger m_U$, $H_D \equiv m_D^\dagger m_D$\cite{4}. In our case we again get no explicit CP violation, since we do not have any imaginary part for the arbitrary monomials:

$$
Im Tr(P_1(H_D)P_2(H_U) \cdots) = Im(Re[\tr(P_1(H_D)P_2(H_U) \cdots)]) = 0,
$$

with $H_{U,D} \equiv M_{U,D}^\dagger M_{U,D}$. Thus our theory is quite different from the Kobayashi-Maskawa theory, and inevitably necessitates the mechanism of spontaneous CP violation\cite{5}.

The fact that there is no explicit CP violation turns out to be a natural consequence of the quaternionic property of the mass matrices. Let us note the following fact (similar to the “reality” of the $SU(2)$ representations):

$$
kH^{**}k = H,\quad (21)
$$

where $H^{**}$ corresponds to ordinary complex conjugation $i \to -i$, not quaternionic conjugation. From this we learn that for an arbitrary quaternionic matrix $M$, 

(VMV†)** = VMV†, with a unitary transformation $V = \frac{1+i}{\sqrt{2}} I$ ($I$: a unit matrix). This means that we can move to a basis by the unitary transformation, where all mass matrices are real.

While the fermion mass matrices, which result from Yukawa couplings, show characteristic features, as discussed above, other interactions can be incorporated into the theory just as in ordinary gauge theories. This is essentially because the other interactions are “generation blind”. One thing we should care about is the scalar potential, since we have to device a potential with spontaneous CP violation. We, however, may just utilize the existing potential in Ref. [5], replacing the complex scalar fields in the Higgs doublets, by the corresponding quaternionic complex scalars defined over 1 and $i$.

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