Property Testing of the Boolean and Binary Rank

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Abstract
We present algorithms for testing if a \((0, 1)\)-matrix \(M\) has Boolean/binary rank at most \(d\), or is \(\epsilon\)-far from having Boolean/binary rank at most \(d\) (i.e., at least an \(\epsilon\)-fraction of the entries in \(M\) must be modified so that it has rank at most \(d\)). For the Boolean rank we present a non-adaptive testing algorithm whose query complexity is \(\tilde{O}(d^4/\epsilon^6)\). For the binary rank we present a non-adaptive testing algorithm whose query complexity is \(O(2^{2d}/\epsilon^2)\), and an adaptive testing algorithm whose query complexity is \(O(2^{2d}/\epsilon)\). All algorithms are 1-sided error algorithms that always accept \(M\) if it has Boolean/binary rank at most \(d\), and reject with probability at least 2/3 if \(M\) is \(\epsilon\)-far from having Boolean/binary rank at most \(d\).

Keywords  Property testing · Boolean rank · Binary rank

1 Introduction

The Boolean rank of a \((0, 1)\)-matrix \(M\) of size \(n \times m\) is equal to the minimal \(r\), such that \(M\) can be factorized as a product \(M = X \cdot Y\), where \(X\) is \((0, 1)\)-matrix of size \(n \times r\) and \(Y\) is a \((0, 1)\)-matrix of size \(r \times m\), and all additions and multiplications are Boolean (that is, \(1 + 1 = 1\), \(1 + 0 = 0 + 1 = 1\), \(1 \cdot 1 = 1\)). A similar definition holds for the binary rank, where here the operations are the regular operations over the reals (that is, \(1 + 1 = 2\)).

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These two rank functions have other equivalent definitions: The Boolean (binary) rank is equal to the minimal number of monochromatic rectangles required to cover (partition) all the 1-entries of the matrix. The Boolean (binary) rank is also equal to the minimal number of bipartite cliques needed to cover (partition) all the edges of a bipartite graph whose adjacency matrix is \( M \) (see [1]). Furthermore, the Boolean rank of \( M \) determines exactly the non-deterministic communication complexity of \( M \), and the binary rank of \( M \) gives an approximation up to a polynomial of the deterministic communication complexity of \( M \) (see, for example, [2] for more details).

Given the importance of these two rank functions it is desirable to be able to compute or approximate them efficiently. However, in several works it was shown that computing and even approximating the Boolean or binary rank is NP-hard [3–6]. The strongest inapproximability result [6] shows that it is NP-hard to approximate both ranks to within a factor of \( n^{1-\delta} \) for any given \( \delta > 0 \), and using a stronger complexity assumption they prove a lower bound that is even closer to linear in \( n \).

1.1 Property Testing of the Matrix Rank

In this work we consider a different relaxation of exactly computing the Boolean or binary rank of a matrix, namely, that of property testing [7, 8]. For a parameter \( \epsilon \in [0, 1] \) and an integer \( d \), a matrix \( M \) is said to be \( \epsilon \)-far from having Boolean (binary) rank at most \( d \), if it is necessary to modify more than an \( \epsilon \)-fraction of the entries of \( M \) to obtain a matrix with Boolean (binary) rank at most \( d \). Otherwise, \( M \) is \( \epsilon \)-close to having Boolean (binary) rank at most \( d \).

A property-testing algorithm for the Boolean (binary) rank is given as parameters \( \epsilon \) and \( d \), as well as query access to a matrix \( M \). If \( M \) has Boolean (binary) rank at most \( d \), then the algorithm should accept with probability at least \( 2/3 \), and if \( M \) is \( \epsilon \)-far from having Boolean (binary) rank at most \( d \), then the algorithm should reject with probability at least \( 2/3 \). If the algorithm accepts matrices having rank at most \( d \) with probability 1, then it is a one-sided error testing algorithm. If it selects all its queries in advance, the it is a non-adaptive algorithm. The main complexity measure that we focus on is the query complexity of the testing algorithm.

The real rank If one considers the real rank of matrices, then there are known efficient property testing algorithms. Krauthgamer and Sasson [9] gave a non-adaptive property testing algorithm for the real rank whose query complexity is \( O(d^2/\epsilon^2) \), and Li, Wang and Woodruff [10] showed that by allowing the algorithm to be adaptive, it is possible to reduce the query complexity to \( O(d^2/\epsilon) \). Recently, Balcan et al. [11] gave a non-adaptive testing algorithm for the real rank whose query complexity is \( \tilde{O}(d^2/\epsilon) \).

It should be noted that the aforementioned property testing algorithms for the real rank cannot be simply adapted to the Boolean and binary rank, since they rely heavily on the augmentation property that holds trivially for the real rank (i.e., if the real rank of \((M|x)\) and of \((M|y)\) is \( d \), then the real rank of \((M|x, y)\) is also \( d \), where \((M|x)\) is the matrix \( M \) augmented with a vector \( x \) as the last column). However, the augmentation property does not hold for the Boolean and binary rank (see for example [12]), and thus a different approach is needed.
Implications of results for testing graph properties to our problem. Recalling the formulation of the Boolean and binary rank as properties of bipartite graphs, we observe that these properties can be characterized as being free of a finite collection of induced subgraphs. Alon, Fischer and Newman [13] showed that every such property of bipartite graphs can be tested with a number of queries that is polynomial in $1/\epsilon$, and with no dependence on the size of the graph. By applying their framework to our problems, we obtain algorithms whose query complexity is upper bounded by $(\frac{2^d}{\epsilon})O(2^{2d})$.

The formulation of the Boolean (binary) rank as a covering (partition) problem of edges by complete bicliques is reminiscent of the well-known problem of graph coloring, where the goal is to partition the graph vertices into a small number of independent sets. Graph coloring has been studied in the context of property testing, where the query complexity is polynomial in $1/\epsilon$ and the number, $k$, of colors (see [8, 14, 15]). However, an important difference between the problems is that while $k$-colorability is monotone in terms of the removal of edges, this is not true for the Boolean and binary rank. In particular, this implies that when testing $k$-colorability, the distance of a graph $G$ to the property is the minimal number of 1-entries in the adjacency matrix of $G$ that should be modified to 0, so as to obtain a $k$-colorable graph. On the other hand, for the rank functions we consider, if we want to make the minimal number of modifications in a $(0, 1)$-matrix $M$, so as to obtain a $(0, 1)$-matrix with rank at most $d$, then we might need to modify both 1-entries and 0-entries.

In view of the above, we may consider a generalization of $k$-coloring that is captured by a family of graph partition properties, which were studied in [16], and are a variant of the generalized partition properties studied in [8]. The property of having Boolean rank at most $d$ can be cast as such a partition property with $k = 2^{d+1}$ parts, and the property of having binary rank at most $d$ can be cast as a union of $2^{2d}$ such properties with $k = O(2^d)$ parts.

The testing algorithm analyzed in [16], which is non-adaptive and has one-sided error, has query complexity $O(k^2 \log^2 k/\epsilon^4)$. This implies a testing algorithm with query complexity $O(2^{2d} d^2 / \epsilon^4)$ for the property of having Boolean rank at most $d$, and a testing algorithm with query complexity $O(2^{4d} d^2 / \epsilon^4)$ for the binary rank.

1.2 Our Results

Our first and main result is a property testing algorithm for the Boolean rank that has query complexity polynomial in $d$ and $1/\epsilon$. Recall that the upper bound on the query complexity implied by [16] is exponential in $d$.

**Theorem 1** There exists a one-sided error non-adaptive property testing algorithm for the Boolean rank whose query complexity is $\tilde{O}(d^4/\epsilon^6)$.

The proof of Theorem 1 builds on the framework used in [17], which in turn builds on [18]. Specifically, we introduce the notion of skeletons and beneficial entries for a matrix $M$ in the context of the Boolean rank. These notions allow us to separate the analysis of Algorithm 1 into a purely combinatorial part and a probabilistic part. Part of the challenge in defining these notions in the context of the Boolean
rank, and using them to prove Theorem 1, is the non-monotonicity of the problem described above. More details on the proof structure, as well as the complete proof of Theorem 1, are given in Section 2.

For the binary rank we present testing algorithms whose query complexity is exponential in the rank \( \delta \), but quadratic, or even linear in \( 1/\epsilon \), thus improving on the aforementioned bound that is implied by [16].

**Theorem 2** There exists a one-sided error non-adaptive property testing algorithm for the binary rank whose query complexity is \( O(2^{2\delta}/\epsilon^2) \), and a 1-sided error adaptive property testing algorithm for the binary rank whose query complexity is \( O(2^{2\delta}/\epsilon) \).

The proof of Theorem 2 is given in Section 3. It remains an open problem whether there exists a testing algorithm for the binary rank with query complexity polynomial in \( \delta \) and \( 1/\epsilon \).

## 2 Testing the Boolean Rank

Let \( M \) be a \((0,1)\)-matrix of size \( n \times n \), and let \([n] = \{1, \ldots, n\}\). We say that an entry \((x, y) \in [n] \times [n]\) is a 1-entry of \( M \) if \( M[x, y] = 1 \). For a subset of entries \( U = \{(x_i, y_i)\}_{i=1}^m \), the submatrix of \( M \) induced by \( U \) is the submatrix whose rows are \( \{x_i\}_{i=1}^m \) and whose columns are \( \{y_i\}_{i=1}^m \). The testing algorithm for the Boolean rank is simple:

**Algorithm 1** (Test \( M \) for Boolean rank \( \delta \), given \( \delta \) and \( \epsilon \)).

1. Select uniformly, independently and at random 
   \( m = \Theta\left(\frac{\delta^2}{\epsilon^3} \cdot \log \frac{\delta}{\epsilon}\right) \) entries from \( M \).
2. Let \( U \) be the subset of entries selected and consider the submatrix \( W \) of \( M \) induced by \( U \).
3. If \( W \) has Boolean rank at most \( \delta \), then accept. Otherwise, reject.

The query complexity of Algorithm 1 is clearly \( \tilde{O}\left(\frac{\delta^4}{\epsilon^6}\right) \). As for the running time of the algorithm, we cannot expect it to be efficient since computing the Boolean rank of a \((0,1)\)-matrix is NP-hard. We now proceed to prove Theorem 1, and show that Algorithm 1 is a 1-sided error tester for the Boolean rank.

**Proof Structure** First note that if \( M \) has Boolean rank at most \( \delta \), then so does each of its submatrices, causing Algorithm 1 to accept (with probability 1). Hence, our focus is on proving that if \( M \) is \( \epsilon \)-far from having Boolean rank \( \delta \), then the algorithm rejects with probability at least 2/3. Following [17], which in turn follows [18], we introduce the notion of skeletons and beneficial entries for a matrix \( M \) in the context of the Boolean rank. These notions allow us to separate the analysis of Algorithm 1 into a
purely combinatorial part (which is the main part of the analysis), and a probabilistic part (which is fairly simple).

A skeleton for $M$ is a multiset $S = \{S_1, \ldots, S_d\}$ that contains $d$ subsets of 1-entries of $M$, where all 1-entries in each subset $S_i$ can be in the same monochromatic rectangle. Roughly speaking, an entry $(x, y)$ is beneficial with respect to a skeleton $S = \{S_1, \ldots, S_d\}$, if for each one of the subsets $S_i$, either: (1) $(x, y)$ cannot be added to $S_i$, since it cannot be in the same monochromatic rectangle as the entries already in $S_i$, or (2) adding $(x, y)$ to $S_i$ significantly reduces the number of other entries that can be in the same monochromatic rectangle with the entries of $S_i$ and $(x, y)$.

Observe that if an entry $(x, y)$ cannot be added to any $S_i$ in $S$, then this is evidence that the skeleton $S$ cannot be extended to a cover of all 1-entries of $M$ by $d$ monochromatic rectangles. More generally, allowing also for the second option defined above, beneficial entries make a skeleton more constrained, as we formalize precisely in the next subsection.

We show how, given a matrix $M$, it is possible to define a set $S(M)$ of relatively small skeletons that have certain useful properties. In particular, if $M$ is $\epsilon$-far from having Boolean rank at most $d$, then every skeleton in $S(M)$ has many beneficial entries. We establish this claim by showing how, given a skeleton $S$ in $S(M)$, we can modify $M$ so as to obtain a matrix with Boolean rank at most $d$, where the number of modifications is upper bounded as a function of the number of entries that are beneficial with respect to $S$. On the other hand, we show that if the matrix $M$ has Boolean rank at most $d$, then for every submatrix $W$ of $M$, every subset of entries $U \subseteq W$ contains a skeleton in $S(M)$ with no beneficial entries in $U$. Finally, we prove that if every skeleton in $S(M)$ has many beneficial entries, then with high constant probability over the choice of $U$ in Algorithm 1, for every skeleton $S \in S(M)$ that is contained in $U$, there exists a beneficial entry in $U$ for $S$. We note that the bound on the size of the skeletons in $S(M)$ plays a role in this last proof.

Theorem 1 can then be shown to follow by combining the above.

### 2.1 Central Definitions

Throughout this subsection, the matrix $M$ and the parameters $d$ and $\epsilon$ are fixed.

**Definition 1** (Compatible entries) An entry $(x_1, y_1)$ is compatible with a 1-entry $(x_2, y_2)$ if $M[x_1, y_2] = M[x_2, y_1] = 1$. Otherwise, $(x_1, y_1)$ is incompatible with $(x_2, y_2)$.

An entry $(x, y)$ is compatible with a set $S$ of 1-entries, if $(x, y)$ is compatible with every entry in $S$. Otherwise, $(x, y)$ is incompatible with $S$.

Note that if both entries $(x_1, y_1)$ and $(x_2, y_2)$ are 1-entries, then the compatibility relation is symmetric, but it applies also to pairs of entries $(x_1, y_1)$ and $(x_2, y_2)$ such that $M[x_1, y_1] = 0$ and $M[x_2, y_2] = 1$. When both entries $(x_1, y_1)$ and $(x_2, y_2)$ are 1-entries, compatibility means that these entries can belong to the same monochromatic rectangle.
**Definition 2** (Friendly row/column) A row \( x \) (column \( y \)) is friendly with a set \( S \) of 1-entries, if for every entry \((x', y')\) \( \in S \), it holds that \( M[x, y'] = 1 \) (\( M[x', y] = 1 \)). Otherwise, it is not friendly with \( S \).

Observe that by Definitions 1 and 2, an entry \((x, y)\) is compatible with a set \( S \) of 1-entries if and only if row \( x \) and column \( y \) are both friendly with \( S \). See Fig. 1 for an illustration.

For a set of entries \( S = \{(x_i, y_i)\}_{i=1}^{|S|} \), denote the set of rows of \( S \) by \( R(S) = \{x_i\}_{i=1}^{|S|} \) and the set of columns of \( S \) by \( C(S) = \{y_i\}_{i=1}^{|S|} \).

**Definition 3** (Zeros of a row/column) For a row \( x \), let \( Z(x) \) denote the set of columns \( y \) such that \( M[x, y] = 0 \), and for a column \( y \) let \( Z(y) \) denote the set of rows \( x \) such that \( M[x, y] = 0 \).

We extend the notation \( Z(\cdot) \) to sets of rows/columns. Namely, for a set of rows \( X \), \( Z(X) = \bigcup_{x \in X} Z(x) \), and similarly for a set of columns \( Y \).

**Claim 1** Let \((x, y)\) be an entry such that \( M[x, y] = 0 \) and such that row \( x \) and column \( y \) are both friendly with a set of entries \( S \). Then \( y \in Z(x) \setminus Z(R(S)) \) and \( x \in Z(y) \setminus Z(C(S)) \).

**Proof** Assume, contrary to the claim, that \( y \notin Z(x) \setminus Z(R(S)) \). Since \( M[x, y] = 0 \), then \( y \in Z(x) \), and therefore, \( y \in Z(R(S)) \). Hence, there exists an entry \((x', y')\) \( \in S \) such that \( M[x', y] = 0 \). But this means that column \( y \) is not friendly with \( S \).

An analogous argument shows that \( x \in Z(y) \setminus Z(C(S)) \).

![Fig. 1](image-url) Row \( x \) and column \( y \) are friendly with the subset \( S \) of grey entries, and entry \((x, y)\) is compatible with \( S \). The empty entries can be either 1 or 0.
Fig. 2 An illustration for the definition of a zero-heavy row and influential entries. The entries of $S$ are filled with solid grey. The 0-entries in row $x$ that belong to $Z(x) \setminus Z(R(S))$ are underlined. Consider the 1-entry $(x, y)$ that is filled with vertical lines. Assuming that row $x$ is zero-heavy, then entry $(x, y)$ is influential with respect to $S$. Furthermore, although the 1-entry $(z, w)$, which is filled with horizontal lines, is compatible with $S$, it is not compatible with $S \cup \{(x, y)\}$

**Definition 4** (Zero-heavy row/column) Row $x$ is zero-heavy with respect to a set of entries $S$ if $|Z(x) \setminus Z(R(S))| \geq g(\epsilon, d) \cdot n$, where $g(\epsilon, d) = \epsilon/(4d)$. Otherwise, it is zero-light with respect to $S$. Similarly, column $y$ is zero-heavy with respect to $S$ if $|Z(y) \setminus Z(C(S))| \geq g(\epsilon, d) \cdot n$, and otherwise, it is zero-light.

**Definition 5** (Influential entries) Entry $(x, y)$ is influential with respect to a set $S$ of 1-entries if: (1) $M[x, y] = 1$, (2) $(x, y)$ is compatible with $S$, and (3) either row $x$ or column $y$ is zero-heavy with respect to $S$ (possibly both). Otherwise, $(x, y)$ is non-influential for $S$.

As we will see shortly, only influential entries will be added to a given skeleton. This will allow us to maintain small skeletons and at the same time, when $M$ is $\epsilon$-far from having Boolean rank at most $d$, each skeleton will have many beneficial entries, as defined next. An illustration for Definitions 4 and 5 is given in Fig. 2.

We are now ready to introduce our main definitions of skeletons and beneficial entries.

**Definition 6** (Skeletons and beneficial entries for the Boolean rank) A skeleton for a matrix $M$ is a multiset $S = \{S_1, \ldots, S_d\}$ that includes $d$ subsets of 1-entries of $M$, and is defined inductively as follows:

1. The multiset $S = \{\emptyset, \ldots, \emptyset\}$, which contains the empty set $d$ times, is a skeleton.
2. If $S = \{S_1, \ldots, S_d\}$ is a skeleton and $(x, y)$ is an influential entry with respect to $S_i$ for some $i \in [d]$, then $S' = \{S_1, \ldots, S_{i-1}, S_i \cup \{(x, y)\}, S_{i+1}, \ldots, S_d\}$ is a skeleton.

(Note that there may be more than one way to add $(x, y)$ to the skeleton $S$, and $(x, y)$ can be added to more than one of the subsets $S_i$).

Let $S(M)$ denote the set of all skeletons for $M$. 
A 1-entry \((x, y) \in M\) is beneficial for a skeleton \(S = \{S_1, \ldots, S_d\}\), if for every \(1 \leq i \leq d\), the entry \((x, y)\) is either incompatible or influential with respect to \(S_i\). Otherwise, \((x, y)\) is non-beneficial for \(S\).

Note that by the definition of influential entries, any skeleton \(S \in S(M)\) contains only 1-entries of \(M\), and beneficial entries are always 1-entries.

In the next two subsections we prove that the set of skeletons \(S(M)\), as defined in Definition 6, has certain properties, which are then exploited to prove Theorem 1.

### 2.2 Matrices of Rank \(d\) Have Skeletons with no Beneficial Entries

**Lemma 2** Let \(W\) be a submatrix of \(M\) with Boolean rank at most \(d\). Then for every \(U \subseteq W\), there exists a skeleton \(S = \{S_1, \ldots, S_d\} \in S(M)\), such that \(\bigcup_{i=1}^{d} S_i \subset U\), and there is no beneficial entry in \(U\) for \(S\).

**Proof** First observe that since \(W\) has Boolean rank at most \(d\), there exist \(d\) monochromatic submatrices \(B_1, \ldots, B_d\) of \(W\) that cover all 1-entries of \(W\), and hence all 1-entries of \(U\). We build the skeleton \(S\) in the following iterative manner:

1. We start with the skeleton \(S^1 = \{\emptyset, \ldots, \emptyset\}\).
2. Let \(S^j = \{S^j_i\}_{i=1}^{d}\) be the skeleton at the beginning of the \(j\)'th iteration.
   
   (a) If there exists an index \(i\) and an entry \((x, y) \in B_i \cap U\) that is an influential entry with respect to \(S^j_i\), then we let \(S^{j+1} = \{S^j_1, \ldots, S^j_{i-1}, S^j_i \cup \{(x, y)\}, S^j_{i+1}, \ldots, S^j_d\}\).
   
   (b) If for every \(i\), the subset \(B_i \cap U\) does not contain any influential entry with respect \(S^j_i\), then we stop.

Let \(S = \{S_1, \ldots, S_d\}\) be the final resulting skeleton. It remains to show that there are no beneficial entries in \(U\) for \(S\).

Assume, contrary to this claim, that there is some beneficial 1-entry \((x, y) \in U\) for \(S\). Since the submatrices \(B_1, \ldots, B_d\) cover all 1-entries of \(U\), there must exist an \(i \in [d]\) such that \((x, y) \in B_i\). Therefore, \((x, y)\) is compatible with \(S_i\). Furthermore, \((x, y)\) is not influential with respect to the subset \(S_i\) (otherwise, we would have added it to \(S_i\)). Thus, entry \((x, y)\) cannot be beneficial for \(S\).

### 2.3 Skeletons of matrices far from rank \(d\) have many beneficial entries

In this subsection we show that if the matrix \(M\) is \(\epsilon\)-far from having Boolean rank at most \(d\), then every skeleton has many beneficial entries. To be precise, we prove the contrapositive statement:

**Lemma 3** Let \(S = \{S_1, \ldots, S_d\}\) be a skeleton for \(M\) with at most \(\frac{\epsilon^2}{64} n^2\) beneficial entries. Then \(M\) is \(\epsilon\)-close to Boolean rank \(d\).

In order to prove Lemma 3, we first show how to modify \(M\) in at most \(\epsilon n^2\) entries, and then prove that after this modification the resulting matrix \(M'\) has Boolean rank...
at most \( d \). We note that in all that follows, the reference to beneficial entries, as well as to friendly and zero-light rows/columns is with respect to the given skeleton \( S \) for \( M \) as stated in Lemma 3, that is, before any modifications are performed on \( M \). We start by showing how to modify \( M \) using the following Modification rules:

1. Modify each row/column with at least \( \epsilon n/8 \) beneficial entries to an all-zero row/column. The number of such rows/columns is at most \( \epsilon n/8 \). Otherwise, we get more than \( \epsilon^2 n^2 \) beneficial entries. Therefore, this step accounts for at most \( 2n \cdot \epsilon n/8 = \epsilon n^2/4 \) modifications.

2. Modify to 0’s all beneficial entries in rows/columns with less than \( \epsilon n/8 \) beneficial entries. This accounts for at most \( 2n \cdot \epsilon n/8 = \epsilon n^2/4 \) modifications.

3. Modify a 0-entry \((x, y)\) to a 1 (where \( x \) and \( y \) are a row/column with less than \( \epsilon n/8 \) beneficial entries) if and only if there exists an \( i \in [d] \), such that row \( x \) and column \( y \) are both friendly and zero-light with respect to \( S_i \).

By Claim 1, in this case it holds that \( y \in Z(x) \setminus Z(R(S_i)) \) and \( x \in Z(y) \setminus Z(C(S_i)) \). Thus, the total number of modifications of this type is at most \( 2n \cdot d \cdot g(\epsilon, d)n = \epsilon n^2/2 \), since \( g(\epsilon, d) = \epsilon/(4d) \).

Therefore, the total number of modified entries is upper bounded by:

\[
\epsilon n^2/4 + \epsilon n^2/4 + \epsilon n^2/2 = \epsilon n^2.
\]

The main issue is hence proving that after this modification, the modified matrix \( M' \) has Boolean rank at most \( d \). We first define \( d \) subsets \( B_1, \ldots, B_d \) of 1-entries, such that each 1-entry of the modified matrix \( M' \) is included in one of these subsets:

1. For each \((x, y)\) \( \in \bigcup_{j=1}^{d} S_j \) such that \( M'[x, y] = 1 \): place \((x, y)\) in \( B_i \) for \( i \in [d] \) such that \((x, y) \in S_i \).

2. For each \((x, y) \not\in \bigcup_{j=1}^{d} S_j \) such that \( M'[x, y] = 1 \): place \((x, y)\) in \( B_i \) if both row \( x \) and column \( y \) are both friendly and zero-light (in \( M \)) with respect to \( S_i \).

To verify that such an index \( i \) exists for such an entry \((x, y)\), we consider two cases:

(a) \( M[x, y] = 1 \): Since \( M'[x, y] = 1 \) as well, we know that \((x, y)\) is non-beneficial (since beneficial entries were modified to 0). By the definition of non-beneficial entries, there exists an index \( i \), such that \((x, y)\) is compatible with \( S_i \) and non-influential with respect to \( S_i \). That is, both row \( x \) and column \( y \) are friendly (in \( M \)) with respect to \( S_i \), and are zero-light (in \( M \)) with respect to \( S_i \).

(b) \( M[x, y] = 0 \): Since \( M'[x, y] = 1 \), by Modification rule number 3, such an index \( i \) must exist as well.

It remains to prove that the subsets \( B_1, \ldots, B_d \) induce a cover of \( M' \) by \( d \) monochromatic rectangles. That is, for each subset \( B_i \), every two 1-entries in \( B_i \) are compatible (in \( M' \)). We first prove the next claim, which follows from the modification rules of \( M \) and the definition of these subsets.

**Claim 4** If \( M'[x, y] = 1 \) and \((x, y) \in B_i \), then row \( x \) and column \( y \) are friendly and zero-light in \( M \) with respect to \( S_i \).
Proof Consider the following cases:

- $(x, y) \in \bigcup_{j=1}^{d} S_j$: Thus, $(x, y) \in S_i$ and by the definition of the skeletons this means that $(x, y)$ is compatible with all entries in $S_i$. Hence, row $x$ and column $y$ are friendly with $S_i$. Furthermore, given that $(x, y) \in S_i$, we have that $Z(x) \setminus Z(R(S_i)) = \emptyset$ and $Z(y) \setminus Z(C(S_i)) = \emptyset$, so that row $x$ and column $y$ are zero-light with respect to $S_i$.

- $(x, y) \notin \bigcup_{j=1}^{d} S_j$ and $M[x, y] = 1$: This case corresponds to Case 2a in the definition of the subsets $B_i$, and so row $x$ and column $y$ are friendly and zero-light in $M$ with respect to $S_i$ by the definition.

- $(x, y) \notin \bigcup_{j=1}^{d} S_j$ and $M[x, y] = 0$: This case corresponds to Case 2b in the definition of the subsets $B_i$, and so row $x$ and column $y$ are friendly and zero-light in $M$ with respect to $S_i$ by the definition.

Since every pair $(x, y)$ fits one of the above cases, the claim follows.

The next claim concludes the proof that $M'$ has Boolean rank at most $d$, thus establishing the proof of Lemma 3.

Claim 5 For every $i \in [d]$, every two entries in $B_i$ are compatible in $M'$.

Proof Consider any pair of entries $(x_1, y_1), (x_2, y_2) \in B_i$. By Claim 4, rows $x_1$ and $x_2$ and columns $y_1$ and $y_2$, are friendly and zero-light in $M$ with respect to $S_i$. Furthermore, these rows/columns were not modified by Modification rule number 1.

We now show that $M'[x_1, y_2] = 1$, where a similar proof holds for $(x_2, y_1)$. We consider the following cases:

- $M[x_1, y_2] = 0$: Since rows $x_1$ and $x_2$ and columns $y_1$ and $y_2$ are friendly and zero-light with respect to $S_i$, then by Modification rule number 3 we have $M'[x_1, y_2] = 1$.

- $M[x_1, y_2] = 1$: Since rows $x_1$ and $x_2$ and columns $y_1$ and $y_2$ are friendly and zero-light with respect to $S_i$, then $(x_1, y_2)$ cannot be influential with respect to $S_i$. It remains to show that $(x_1, y_2)$ is compatible with $S_i$, and therefore cannot be beneficial, and thus, was not modified to a 0 by Modification rule number 2.

Let $(x', y') \in S_i$. Since row $x_1$ and column $y_2$ are friendly with respect to $S_i$, then $M'[x', y_2] = 1$ and $M[x_1, y'] = 1$. Therefore, $(x_1, y_2)$ is compatible with $S_i$. Claim 5 follows.

2.4 A Sampling Lemma

Before we state and prove the main lemma of this subsection, we first establish a bound on the size of each of the subsets in a skeleton.

Claim 6 Let $S = \{S_1, \ldots, S_d\}$ be a skeleton for $M$. Then $|S_i| \leq 8d/\epsilon$ for every $i \in [d]$. 
Lemma 7 Let $0 < \alpha < 1$, and suppose that every skeleton in $S(M)$ has at least $\alpha \cdot n^2$ beneficial entries in $M$.

Consider selecting independently and uniformly at random, $m = c \cdot \left( \frac{d^2}{\alpha \varepsilon} \cdot \log \frac{d}{\alpha \varepsilon} \right)$ entries from $M$ for a sufficiently large constant $c$, and denote the subset of selected entries by $U$. Then with probability at least $2/3$, for every skeleton $S = \{S_1, \ldots, S_d\} \in S(M)$ such that $\bigcup_{i=1}^{d} S_i \subset U$, there exists a beneficial entry in $U$ for $S$.

Proof Consider selecting $m$ entries from $M$, independently and uniformly at random, and let $(x_i, y_i)$ be the $i$'th entry selected, so each entry $(x_i, y_i)$ is a random variable. Let $s = 8d/\varepsilon$ and $m = 200 \cdot \frac{d^2}{\alpha \varepsilon} \cdot \ln \frac{d}{\alpha \varepsilon}$.

By Claim 6, for every skeleton $S = \{S_1, \ldots, S_d\} \in S(M)$, we have that $|S_i| \leq s$ for every subset $S_i \in S$. Therefore, $\bigcup_{i=1}^{d} S_i \leq d \cdot s$. Observe that for each subset of entries $T$ of size at most $d \cdot s$, the number of skeletons $\{S_1, \ldots, S_d\}$ such that $\bigcup_{i=1}^{d} S_i = T$ is upper bounded by

$$\left( \sum_{i=0}^{s} \binom{d \cdot s}{i} \right)^d \leq \left( (s+1) \cdot \binom{d \cdot s}{s} \right)^d \leq (s+1)^d \cdot \left( \frac{e \cdot d \cdot s}{s} \right)^{d \cdot s} = (s+1)^d \cdot (e \cdot d)^{d \cdot s}.$$

For each subset of indices $I \subset [m]$, where $|I| \leq d \cdot s$, suppose that we first select entries $\{(x_i, y_i)\}_{i \in I}$, and let $T_I$ be the resulting set of entries. By the premise of the lemma, for each skeleton $S = \{S_1, \ldots, S_d\} \in S(M)$ such that $\bigcup_{i=1}^{d} S_i = T_I$, there are at least $\alpha \cdot n^2$ beneficial entries in $M$.

For our choice of $m$, we have that $m - d \cdot s > m/2$. Therefore, if we now select the remaining entries $\{(x_i, y_i)\}_{i \in [m] \setminus I}$, the probability that we do not obtain any entry that is beneficial for $S$ is at most

$$(1 - \alpha)^m/2 < e^{-\alpha m/2}.$$

By taking a union bound over all subsets $I$ of size at most $d \cdot s$, and all skeletons $S$ such that $\bigcup_{i=1}^{d} S_i = T_I$, we get that the probability that there exists a skeleton $S = \{S_1, \ldots, S_d\} \in S(M)$ such that $\bigcup_{i=1}^{d} S_i \subset U$, and there is no beneficial entry in $U$ for $S$, is upper bounded by

$$m^{d \cdot s} \cdot (s+1)^d \cdot (e \cdot d)^{d \cdot s} \cdot e^{-\alpha m/2} = e^{(d \cdot s \ln m + d \ln(s+1) + d \cdot s \ln(e \cdot d) - \alpha m/2)} \leq e^{-2} \leq \frac{1}{3},$$

where the first inequality holds for our setting of $s$ and $m$. \qed

2.5 Proof of Theorem 1

We can now complete the proof of Theorem 1, which builds on Lemmas 2, 3 and 7.

Proof Theorem 1 If $M$ has Boolean rank at most $d$, then Algorithm 1 always accepts since every submatrix of $M$ has Boolean rank at most $d$. \qed
Assume, therefore, that \( M \) is \( \epsilon \)-far from having Boolean rank at most \( d \).

By Lemma 3, for every skeleton in \( S(M) \) there are at least \( \frac{\epsilon^2 n^2}{64} \) beneficial entries in \( M \). Therefore, by Lemma 7 (applied with \( \alpha = \frac{\epsilon^2}{64} \)), for every skeleton \( S = \{ S_1, \ldots, S_d \} \in S(M) \) such that \( \bigcup_{i=1}^d S_i \subset U \), there exists a beneficial entry \((x, y) \in U \) for \( S \).

But by Lemma 2, if the Boolean rank of \( W \) was at most \( d \), then for every \( U \subseteq W \), there must exist a skeleton \( S = \{ S_1, \ldots, S_d \} \in S(M) \), where \( \bigcup_{i=1}^d S_i \subset U \), with no beneficial entries in \( U \). Hence, the Boolean rank of \( W \) must be larger than \( d \), and thus Algorithm 1 will reject as required.

\( \square \)

3 Testing the Binary Rank

We present simple testing algorithms for the binary rank whose query complexity is exponential in \( d \), but quadratic, or even linear in \( 1/\epsilon \). We first give a non-adaptive algorithm whose query complexity is \( O(2^{2d}/\epsilon^2) \), and then use its analysis to design an adaptive algorithm whose query complexity is \( O(2^{2d}/\epsilon) \). We note that variants of these algorithms are also applicable to the Boolean rank.

3.1 A Non-adaptive Property Testing Algorithm for the Binary Rank

**Algorithm 2** (Test \( M \) for binary rank \( d \), given \( d \) and \( \epsilon \) – non-adaptive version).

1. Select uniformly, independently and at random \( m = 24(2^d + 1)/\epsilon \) entries from \( M \).
2. Let \( U \) be the subset of entries selected and consider the submatrix \( W \) of \( M \) induced by \( U \).
3. If \( W \) has binary rank at most \( d \), then accept. Otherwise, reject.

The query complexity of the algorithm is \( O(2^{2d}/\epsilon^2) \), and it always accepts a matrix \( M \) that has binary rank at most \( d \), as every submatrix of \( M \) has binary rank at most \( d \). Hence, it remains to prove the following lemma:

**Lemma 8** Let \( M \) be a matrix that is \( \epsilon \)-far from having binary rank at most \( d \). Then Algorithm 2 rejects with probability at least \( 2/3 \).

In order to prove Lemma 8, we first establish a couple of claims. The first is a simple claim regarding the number of distinct rows and columns in matrices with rank at most \( d \).

**Claim 9** Let \( W \) be a \((0, 1)\)-matrix of binary (or Boolean) rank at most \( d \). Then every submatrix of \( W \) has at most \( 2^d \) distinct rows and at most \( 2^d \) distinct columns.
Proof If $W$ has binary rank at most $d$, it clearly has Boolean rank at most $d$. Thus, it suffices to prove the claim for the latter case. If $W$ has Boolean rank $d$, then the 1-entries can be covered by $d$ monochromatic rectangles. Any two rows that share a monochromatic rectangle must have 1-entries in the columns that belong to this rectangle. Therefore, there are at most $2^d$ distinct rows in $W$ according to the monochromatic rectangles to which each row can belong. A similar argument holds for the columns.

In order to state our next claim, we introduce a few definitions.

**Definition 7** (Number of Distinct rows/columns) Denote by $N(R(W))$ the number of distinct rows in a submatrix $W$, and by $N(C(W))$ the number of distinct columns in $W$.

**Definition 8** (New row/column) A row index $x \in [n]$ is said to be new with respect to a submatrix $W$ of $M$, if by extending $W$ with $x$ we obtain a row different from all current rows of $W$. That is, if $W$ is the submatrix induced by $(x_1, y_1), \ldots, (x_t, y_t)$, then $(M[x, y_1], \ldots, M[x, y_t]) \neq (M[x_i, y_1], \ldots, M[x_i, y_t])$ for all $i \in [t]$. A new column index $y$ is defined similarly.

**Definition 9** (New corner entry) Let $W$ be a submatrix of $M$ that is induced by entries $(x_1, y_1), \ldots, (x_t, y_t)$, and let $(x, y) \in [n] \times [n]$ be an entry, such that neither $x$ nor $y$ is new for $W$. Then $(x, y)$ is said to be a new corner entry with respect to $W$ if there exist $i, j \in [t]$, such that:

1. $(M[x, y_1], \ldots, M[x, y_t]) = (M[x_i, y_1], \ldots, M[x_i, y_t]),$
2. $(M[x_1, y], \ldots, M[x_t, y]) = (M[x_1, y_j], \ldots, M[x_t, y_j]),$
3. $M[x, y] \neq M[x_i, y_j].$

For an illustration of a new corner entry, see Fig. 3.

![Fig. 3 An illustration for Definition 9 (new corner entry)](image-url)
Claim 10 Let $W$ be a submatrix of $M$ induced by $(x_1, y_1), \ldots, (x_t, y_t)$. If the binary rank of $W$ is at most $d$, and $M$ is $\epsilon$-far from having binary rank at most $d$, then one of the following must hold:

1. The number of row indices $x \in [n]$ that are new with respect to $W$ is greater than $(\epsilon/3)n$;
2. The number of column indices $y \in [n]$ that are new with respect to $W$ is greater than $(\epsilon/3)n$;
3. The number of corner entries $(x, y) \in [n] \times [n]$ that are new with respect to $W$ is greater than $(\epsilon/3)n^2$.

Proof Assume, contrary to the claim, that none of the three statements stated in the claim holds. In such a case, we can modify $M$ as follows, and obtain a matrix $M'$, which we shall show has binary rank at most $d$:

- For each row index $x \in [n]$ that is new with respect to $W$, row $x$ in $M'$ is set to be the all-zero row.
- For each column index $y \in [n]$ that is new with respect to $W$, column $y$ in $M'$ is set to be the all-zero column.
- For each entry $(x, y) \in [n] \times [n]$ that is a new corner entry with respect to $W$: Let $i, j \in [t]$ be such that $(M[x, y_1], \ldots, M[x, y_t]) = (M[x, y_1], \ldots, M[x, y_t])$ and $(M[x_1, y], \ldots, M[x_t, y]) = (M[x_1, y_j], \ldots, M[x_t, y_j])$. Set $M'[x, y] = M[x_i, y_j]$.
- All other entries of $M'$ are as in $M$.

Observe that by the above modification rules, for every entry $(x, y)$ such that neither $x$ nor $y$ is new with respect to $W$, there exist indices $i, j \in [t]$ as specified in the third item above, and it holds that $M'[x, y] = M[x_i, y_j]$.

By the premise of the claim, the number of entries that $M'$ and $M$ differ on, is at most $2 \cdot (\epsilon/3)n - n + (\epsilon/3)n^2 = \epsilon n^2$. As we show next, since $W$ has binary rank at most $d$, so does the resulting matrix $M'$, in contradiction to our assumption that $M$ is $\epsilon$-far from having binary rank at most $d$.

To verify that $M'$ has binary rank at most $d$, consider a partition of the 1-entries of $W$ into $d' \leq d$ monochromatic rectangles $B_1, \ldots, B_d'$. We shall show how, based on this partition, we can define a partition of all 1-entries of $M'$ into $d'$ monochromatic rectangles $B_1', \ldots, B_d'$.

Note that for any 1-entry $(w, z)$ in $M'$, neither the row index $w$ is new with respect to $W$ nor the column index $z$ is a new with respect to $W$ (since otherwise, we would have modified row $w$ and/or column $z$ to the all-zero row, and thus, $M'[w, z] = 0$). Therefore, there exists a row index $i(w)$, and column index $j(w)$ such that:

$$
(M[w, y_1], \ldots, M[w, y_t]) = (M[x_i(w), y_1], \ldots, M[x_i(w), y_t]),
(M[x_1, z], \ldots, M[x_t, z]) = (M[x_1, y_j(z)], \ldots, M[x_t, y_j(z)]). 
$$

where $i()$ is a function that maps row $w$ of $M$ to a row $i(w)$ in $W$ as specified in (1), and if there are several such rows in $W$, then the function $i()$ chooses one arbitrarily. The function $j()$ is defined similarly for the columns. Also observe that if $w = x_s$ for some $s \in [t]$, then $i(w) = s$ and similarly, if $z = y_s$ for some $s \in [t]$. 

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then $j(z) = s$. Furthermore, as stated above, for such a 1-entry $(w, z)$ it holds that $M'[w, z] = M[x_i(w), y_j(w)]$, and therefore, $M[x_i(w), y_j(w)] = 1$.

Now, place $(w, z)$ in $B'_\ell$, where $\ell$ is such that $(x_i(w), y_j(z)) \in B_\ell$. In particular, if $(w, z)$ belongs to $W$ and $(w, z) \in B_\ell$, then $(w, z) \in B'_\ell$.

To verify that $B'_1, \ldots, B'_d$ is a partition of the 1-entries of $M'$ into monochromatic rectangles, consider any pair of 1-entries in $M'$, $(w, z)$ and $(w', z')$, such that $(w, z), (w', z') \in B'_\ell$. We need to show that $(w, z')$ and $(w', z)$ are also 1-entries of $M'$, and that $(w, z'), (w', z) \in B'_\ell$ as well.

Again, since $(w, z')$ and $(w', z)$ do not belong to a row or column that are new with respect to $W$, we have that $M'[w, z'] = M[x_i(w), y_j(z')]$ and $M'[w', z] = M[x_i(w'), y_j(z)]$. But $(x_i(w), y_j(z)) \in B_\ell$ and $(x_i(w'), y_j(z')) \in B_\ell$, and thus, we get that $M[x_i(w), y_j(z')] = 1, M[x_i(w'), y_j(z)] = 1$, so that $(w, z')$ and $(w', z)$ are also 1-entries of $M'$. Furthermore, $(x_i(w), y_j(z')) \in B_\ell$ and $(x_i(w'), y_j(z)) \in B_\ell$ (by the definition of $B_1, \ldots, B_d$), so that $(w, z')$ and $(w', z)$ both belong to $B'_\ell$, as required. See Fig. 4 for an illustration.

We can now prove Lemma 8, thus completing the proof of correctness of Algorithm 2.

\textbf{Proof Lemma 8} For the sake of the analysis, we consider Algorithm 2 as if it proceeds in $m = O(2^d/\epsilon)$ iterations, where it starts with the empty $0 \times 0$ submatrix, $W_0$, and in each iteration it extends the submatrix it has with a row and a column whose indices are selected uniformly, independently, at random from $[n]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{An illustration of the proof of Claim 10. The figure shows the submatrix $W$, as well as the parts of rows $w, w'$ that are identical to rows $x_i(w) = x_i, x_i(w') = x_i'$ in the submatrix $W$, and similarly for columns $z, z'$ that are identical to columns $y_j(z') = y_j, y_j(z') = y_j'$ in $W$. Also shown are entries $(w, z), (w', z'), (w, z'), (w', z)$, where each is filled with the same pattern as the entry it equals to in $W$. If $(w, z)$ and $(w', z')$ belong to $B'_\ell$, then $(x_i, y_j)$ and $(x_i', y_j')$ belong $B_\ell$. This implies that $(x_i, y_j')$ and $(x_i', y_j)$ also belong to $B_\ell$, so that $(w, z')$ and $(w', z)$ belong to $B'_\ell$ as well. For simplicity not all entries are specified.}
\end{figure}
For each \( t \in [m] \), let \( W_{t-1} \) be the submatrix of \( M \) of size \( (t - 1) \times (t - 1) \) that is considered in the beginning of iteration \( t \), and let \( x_t \) and \( y_t \) denote, respectively, the indices of the row and column selected in the \( t \)'th iteration. Therefore, \( W_t \) is the submatrix induced by \( x_1, \ldots, x_t \) and \( y_1, \ldots, y_t \).

We shall show that with probability at least \( 2/3 \), the rank of the final submatrix, \( W_m \), is greater than \( d \), and therefore, the algorithm will reject as required.

We know by Claim 9, that if \( M \) has binary rank at most \( d \), then for any submatrix \( W \) of \( M \) it holds that \( N(R(W)) \leq 2^d \) and \( N(C(W)) \leq 2^d \). Hence, if either \( N(R(W_t)) > 2^d \) or \( N(C(W_t)) > 2^d \), then the rank of \( W_t \) is greater than \( d \), so that the algorithm will certainly reject. It is of course possible that \( W_t \) has binary rank greater than \( d \), although both \( N(R(W_t)) \leq 2^d \) and \( N(C(W_t)) \leq 2^d \), and in this case, the algorithm rejects as well.

Therefore, for any \( t < m \), if the binary rank of \( W_{t-1} \) is at most \( d \), and given that in iteration \( t \), entry \( (x_t, y_t) \) is selected uniformly, independently at random, then by Claim 10, with probability at least \( \epsilon/3 \), either:

- \( x_t \) is new for \( W_{t-1} \), so that \( N(R(W_t)) > N(R(W_{t-1})) \),
- or \( y_t \) is new for \( W_{t-1} \), so that \( N(C(W_t)) > N(C(W_{t-1})) \),
- or \( (x_t, y_t) \) is a new corner entry for \( W_{t-1} \). In this case, let \( x_i \) and \( y_j \) be as defined in Definition 9. Thus, \((M[x_1, y_1], \ldots, M[x_{t-1}, y_1]) = (M[x_1, y_1], \ldots, M[x_{t-1}, y_1])\), and in particular, \( M[x_i, y_1] = M[x_i, y_1] \). However, \( M[x_i, y_1] \neq M[x_i, y_1] \), implying that \( N(R(W_t)) > N(R(W_{t-1})) \) in this case as well. A similar argument shows that \( N(C(W_t)) > N(C(W_{t-1})) \).

To summarize, if the binary rank of \( W_{t-1} \) is at most \( d \), then with probability at least \( \epsilon/3 \), the number of distinct rows or the number of distinct columns, or both, of \( W_t \) increases compared to that of \( W_{t-1} \). We thus, have to bound the probability that after all \( m = \Theta(2^{d}/\epsilon) \) iterations, the number of distinct rows and the number of distinct columns of \( W_m \), are both at most \( 2^d \).

To do so, define for each \( t \in [m] \), a Bernoulli random variable \( \chi_t \), where \( \chi_t = 1 \) if and only if \( N(R(W_t)) > N(R(W_{t-1})) \) or \( N(C(W_t)) > N(C(W_{t-1})) \), or both. While the random variables \( \chi_1, \ldots, \chi_m \) are not independent, we have that for any \( t \in [m] \):

\[
\Pr[\chi_t = 1 | \text{the binary rank of } W_{t-1} \text{ is at most } d] \geq \epsilon/3.
\]

Furthermore, if \( \sum_{t=1}^{m} \chi_t \geq 2^d + (2^d + 1) \), then necessarily \( \max\{N(R(W_m)), N(C(W_m))\} > 2^d \), so that the binary rank of \( W_m \) is greater than \( d \), and the algorithm rejects. Note that it is possible that the binary rank of \( W_m \) is greater than \( d \) although \( \max\{N(R(W_m)), N(C(W_m))\} \leq 2^d \), and it is possible that \( \max\{N(R(W_m)), N(C(W_m))\} > 2^d \) although \( \sum_{t=1}^{m} \chi_t < 2^d + (2^d + 1) \), but in either case the algorithm rejects.

Finally, we show that for \( m = 24 \cdot 2^d/\epsilon \), with probability at least \( 2/3 \), the binary rank of \( W_m \), is greater than \( d \). To this end we define \( m \) independent random variables, \( \tilde{x}_1, \ldots, \tilde{x}_m \), where \( \Pr[\tilde{x}_t = 1] = \epsilon/3 \), so that:

\[
\Pr[\text{the binary rank of } W_m \text{ is at most } d] \leq \Pr[\sum_{t=1}^{m} \tilde{x}_t \leq 2^{d+1}].
\]
By applying a multiplicative Chernoff bound, given the setting of $m = 24 \cdot 2^d / \epsilon$, so that the expected value of $\sum_{t=1}^{m} \tilde{\chi}_t$ is greater than $2 \cdot 2^{d+1}$, we get that:

$$\Pr \left[ \sum_{t=1}^{m} \tilde{\chi}_t \leq 2^{d+1} \right] \leq \exp(- (m/3)(1/2)^2) < 1/3,$$

and the lemma is established.

### 3.2 An Adaptive Property Testing Algorithm for the Binary Rank

We next describe an adaptive algorithm for the binary rank whose query complexity is $O(2^{2d}/\epsilon)$. The idea is simple: we modify Algorithm 2 so that it is “closer” to the analysis described in its proof of correctness. Namely, the modified algorithm works in $m$ iterations, where $m = \Theta(2^d/\epsilon)$ is as set in Algorithm 2. In each iteration it maintains a submatrix $W_{t-1}$ of $M$, and selects a random row index $x_t$ and a random column index $y_t$. It extends $W_{t-1}$ by $x_t$ and $y_t$ only if this increases the number of distinct rows/columns in the submatrix. Therefore, the number of rows/columns of each submatrix $W_t$ never exceeds $2^d$, and thus the total number of queries is bounded by $O(m \cdot 2^d) = O(2^{2d}/\epsilon)$. Specifically, the modified algorithm is as follows:

**Algorithm 3** (Test $M$ for binary rank $d$, given $d$ and $\epsilon$ – adaptive version).

1. Set $W_0$ to be the empty $0 \times 0$ matrix.
2. For $t = 1$ to $m$:
   a. Select, uniformly, independently and at random, an index $x_t \in [n]$ and an index $y_t \in [n]$.
   b. Consider the matrix $W'_{t-1}$ obtained by extending $W_{t-1}$ with $x_t$ and $y_t$. If
      $$\max\{N(R(W'_{t-1})), N(C(W'_{t-1}))\} > \max\{N(R(W_{t-1})), N(C(W_{t-1}))\}$$
      then set $W_t = W'_{t-1}$. Otherwise, $W_t = W_{t-1}$.
   c. If the binary rank of $W_t$ is greater than $d$, then stop and reject.
3. If no step resulted in a rejection then accept.

Similarly to the non-adaptive algorithm (Algorithm 2), if $M$ has binary rank at most $d$, then Algorithm 3 always accepts. On the other hand, the argument given in the proof of Lemma 8 directly implies that if $M$ is $\epsilon$-far from having binary rank at most $d$, then Algorithm 3 rejects with probability at least $2/3$.

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