Local quantum uncertainty of SU(2) invariant states

Esfandyar Faizi *, Bahram Ahansaz †
Physics Department, Azarbaijan shahid madani university, Iran.

July 16, 2014

Abstract
In this paper, we investigated quantum correlation in SU(2) invariant quantum spin systems by local quantum uncertainty (LQU). These states are invariant under global rotations of both subsystems and in real physical systems, such states arise from thermal equilibrium states of isotropic spin systems. We derive an analytic expression for the LQU of $(2j + 1) \otimes 2$ and $(2j + 1) \otimes 3$ quantum spin systems with SU(2) symmetry.

Keywords: Local quantum uncertainty, SU(2) invariant states

1 Introduction

The quantum correlation for a quantum state contains entanglement and the other type of nonclassical correlations [1, 2]. Thus, quantum correlations become the subject of intense studies in the last two decades [3]. Among the various researches, it is of great significance to measure the quantum correlations quantitatively. There are much attention put on the quantification of bipartite quantum correlations, including quantum discord [4], geometric discord [5, 6], quantum deficit [7], measurement-induced disturbance [8], etc, and each of these measures are useful in particular physical contexts. Recently, LQU for bipartite quantum systems, as another measure for quantum correlations, has been proposed by Girolami [9]. The LQU quantifies the uncertainty in a quantum state due to measurement of a local observable. Nevertheless, such quantifier has strong reasons to be considered as a faithful measure of quantumness in quantum states. But due to inherent optimization, finding closed formula is a difficult problem for most of the correlations measures. For instance, there is no analytical formulae of quantum discord even for two-qubit quantum systems and in bipartite systems with higher dimension, the results are known for only some certain states [10]. It is possible to derive closed formula geometric discord and also for quantum discord for Werner and Isotropic classes of states due to their highly inherent symmetry in the structures. However, LQU has derived for a large class of arbitrary-dimensional bipartite quantum states [11].

*E-mail:efaizi@azaruniv.edu
†E-mail:b.ahansaz@azaruniv.edu
More recently, it has been studied that mixed states being invariant under certain joint symmetry operations of the bipartite system. We consider a slightly more general class of states, rotationally symmetric states, also known as the $SU(2)$-invariant states. These states are invariant under the global rotations of both subsystems and their density matrices arise from thermal equilibrium states of low-dimensional spin systems with a rotationally invariant Hamiltonian by tracing out all degrees of freedom except those two spins. $SU(2)$-invariant density matrices of two spins $S_1$ and $S_2$ are defined to be invariant under $U_1 \otimes U_2$, and we have $(U_1 \otimes U_2)\rho(U_1^\dagger \otimes U_2^\dagger) = \rho$, where $U_a = \exp(i\vec{\eta}_a \cdot \vec{S}_a)$, $a = 1, 2$ are the usual rotation operator representation of $SU(2)$ with real parameter $\vec{\eta}$ and $\hbar = 1$. For $SU(2)$ invariant quantum spin systems, negativity is shown to be necessary and sufficient for separability \[12, 13\], and the relative entropy of entanglement has been analytically calculated \[14\]. Furthermore, the entanglement of formation (EoF), I-concurrence, I-tangle and convex-roof-extended negativity of the $SU(2)$-invariant states of a spin-$j$ and spin-$1/2$ \[15\] have been analytically calculated by using the approach in \[16\]. Recently, Quantum discord for $SU(2)$-invariant states composed of spin-$j$ and spin-$1/2$ systems has been analytically calculated in \[17\]. Also, the one-way deficit has been calculated for $SU(2)$-invariant states and show that the one-way deficit is equal to the quantum discord for half-integer $j$, and is larger than the quantum discord for integer $j$ \[18\]. Abundance of $SU(2)$ invariant states in real physical systems, make them a good candidate for utilization in quantum computing protocols.

In the present work the behavior of LQU of $SU(2)$-invariant states is studied and the paper consist of the following sections: In section 1 we review the definition of LQU for bipartite systems and in section 2 we show the definition of $SU(2)$ invariant states. Finally, in section 3 we have calculated the LQU of a spin-$1/2$ particle and arbitrary spin-$j$ particle or of a spin-$1$ particle and a spin-$j$ particle with $SU(2)$ symmetry.

## 2 Local quantum uncertainty

Classically, it is possible to measure any two observables with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation gives the statistical nature of errors in these kinds of measurement. Measurement of single observable can also help to detect uncertainty of a quantum observable. For a quantum state $\rho$, an observable is called quantum certain, if the error in measurement of the observable is due to only the ignorance about the classical mixing in $\rho$. A good quantifier of this uncertainty to an observable is the skew information, defined by Wigner and Yanase \[19\] as

$$I(\rho, K) := \frac{1}{2} Tr[\sqrt{\rho}, K^A]^2$$

For a bipartite quantum state $\rho_{AB}$, Girolami et.al. \[9\] introduced the concept of local quantum uncertainty (LQU) defined as

$$U_A := \min_{K^A} I(\rho_{AB}, K^A)$$

The minimization is performed over all local maximally informative observable (or non-degenerate) $K^A = K_A \otimes I$. This quantity quantifies the minimum amount of uncertainty in a quantum state.
The closed form of the LQU for $2 \otimes d$ quantum systems as introduced by Girolami [9] is

$$U_A := 1 - \lambda_{\text{max}}(W)$$

(3)

In which $\lambda_{\text{max}}$ is the maximum eigenvalue of the $3 \times 3$ matrix $W$ with the elements $W_{ij} = Tr \sqrt{\rho}(\sigma_i \otimes I)\sqrt{\rho}(\sigma_j \otimes I)$ and $\sigma_i, i = 1, 2, 3$ is the Pauli matrices.

3 SU(2)-invariant states

Considering two particles, one with spin $j_1$ and the other $j_2$, and the states that are symmetric under global rotations. The symmetry group is $R = \{D^{j_1}(R) \otimes D^{j_2}(R)\}$, where $D^j(R) = e^{x(-i\theta \vec{J}.\vec{n})}$ and $J$ is the angular-momentum vector. The twirling operator can be shown

$$P_R(\rho) = \int d\mu(R) D^{j_1}(R) \otimes D^{j_2}(R) \rho D^{j_1}(R)^\dagger \otimes D^{j_2}(R)^\dagger$$

(4)

where, $\mu(R)$ is the group-invariant measure for the rotation group. The states that are invariant under the twirling operation are those that are convex combinations of the states associated with the projectors onto the irreducible subspaces of total angular momentum. These projectors are

$$\Pi_J = \sum_{m=-J}^{m=J} |J, m\rangle \langle J, m|$$

(5)

Thus the $R$-invariant states are

$$\rho(P) = \sum_{J=\lfloor j_1 - j_2 \rfloor}^{J=j_1+j_2} \frac{p_J}{2J+1} \Pi_J$$

(6)

where $p_J = Tr(\rho(P)\Pi_J) \geq 0$ and $\sum p_J = 1$ [20].

4 Main Discussion

4.1 spin-$j$ and spin-1/2

An SU(2)-invariant state $\rho$ of a bipartite system that composed of a spin-$j$ and a spin-1/2 subsystems are parameterized by a single parameter which will be denoted by $P$ throughout this paper and can be written in total spin basis as [21]

$$\rho^{AB} = \frac{p}{2j} \sum_{m=-j+1/2}^{m=-j-1/2} |j-1/2, m\rangle \langle j-1/2, m| + \frac{1-p}{2(j+1)} \sum_{m=-j-1/2}^{m=-j+1/2} |j+1/2, m\rangle \langle j+1/2, m|$$

(7)

The eigenstates of total angular momentum are well known, given by the Clebsch-Gordon coefficients and can be written as

$$|j \pm 1/2, m\rangle = a_\pm |j, m-1/2\rangle \otimes |1/2, 1/2\rangle + b_\pm |j, m+1/2\rangle \otimes |1/2, -1/2\rangle$$

(8)
where $a_\pm = \pm \sqrt{\frac{j+1/2 \pm m}{2j+1}}$ and $b_\pm = \sqrt{\frac{j+1/2 \pm m}{2j+1}}$. By substitution the Clebsch-Gordon coefficients in eq.4, we can write the density matrix in product basis as

$$
\rho^{AB} = \sum_{m=-j}^{j} |m\rangle \langle m| \otimes (u|1/2\rangle \langle 1/2| + v|-1/2\rangle \langle -1/2|) + \\
\sum_{m=-j}^{j} w(|m\rangle \langle m + 1| \otimes |1/2\rangle \langle -1/2| + |m+1\rangle \langle m| \otimes |-1/2\rangle \langle 1/2|)
$$

where

$$u = \frac{P(j-m)}{(2j+1)} + \frac{1-P}{2(j+1)} \left(\frac{j+m+1}{2j+1}\right)$$
$$v = \frac{P(j+m)}{(2j+1)} + \frac{1-P}{2(j+1)} \left(\frac{j+m+1}{2j+1}\right)$$
$$w = -\frac{\sqrt{(j-m)(j+m+1)}}{2j+1}(\frac{P}{2j} - \frac{1-P}{2j+1})
$$

It is straightforward to calculate the $\sqrt{\rho^{AB}}$ and to evaluate the LQU of $\rho^{AB}$, let $K^B = (I_A \otimes K_B)$ denote a local observable, with $K_B = \vec{n}.\vec{\sigma}$ and ($|\vec{n}| = 1$), where $\sigma_i$, $i= (1, 2, 3)$ represent the Pauli matrices, which are the generators of $SU(2)$. Since making the measurements on spin-$j$ subsystem will make the minimization procedure even harder, thus local observables are made on the spin-1/2 subsystem or B subsystems. After calculations, we find that LQU do not depend on the measurement parameters, therefore, our calculation do not require any minimization over the projective measurements.

Then for integer and half-integer $j$, we have

$$U_A = \frac{8j(j+1)(\sqrt{\frac{P}{2j}} - \sqrt{\frac{1-P}{2(j+1)}})^2}{3(2j+1)}
$$

(11)

Fig.1 and Fig.2 shows the LQU of spin-$j$ and spin-1/2 system for some half-integer and integer $j$, respectively. It is important to note that as $P= j/(2j+1)$ the LQU is exactly zero and as we can see, for high dimensional systems (large $j$), the LQU becomes symmetric around the point $P=1/2$ where LQU vanishes. Also, as we can see in Fig.1, when $j=1/2$, the state becomes the $2 \otimes 2$ Werner state, and the LQU is equal to 1 at $P=1$.

4.2 spin-$j$ and spin-1

An $SU(2)$-invariant state $\rho$ of a bipartite system that is composed of a spin-$j$ and a spin-1 subsystems are parameterized by parameters denoted by $P$ and $Q$ is \[21\]

$$
\rho^{AB} = \frac{P}{2j-1} \sum_{m=-j+1}^{j-1} |j-1, m\rangle \langle j-1, m| + \frac{Q}{2j+1} \sum_{m=-j}^{j} |j, m\rangle \langle j, m| + \\
\frac{1-P-Q}{2j+3} \sum_{m=-j-1}^{j+1} |j+1, m\rangle \langle j+1, m|
$$

(12)
The eigenstates of total angular momentum are well known, given by the Clebsch-Gordon coefficients and can be written as

\[ |j + 1, m\rangle = a_1 |m - 1\rangle \otimes |1\rangle + a_2 |m\rangle \otimes |0\rangle + a_3 |m + 1\rangle \otimes |-1\rangle \]

\[ |j, m\rangle = b_1 |m - 1\rangle \otimes |1\rangle + b_2 |m\rangle \otimes |0\rangle + b_3 |m + 1\rangle \otimes |-1\rangle \]

\[ |j - 1, m\rangle = c_1 |m - 1\rangle \otimes |1\rangle + c_2 |m\rangle \otimes |0\rangle + c_3 |m + 1\rangle \otimes |-1\rangle \]

where

\[ a_1 = \sqrt{\frac{(j+m)(j+m+1)}{(2j+1)(2j+2)}}, \quad a_2 = \sqrt{\frac{(j-m+1)(j+m+1)}{(2j+1)(2j+2)}}, \quad a_3 = \sqrt{\frac{(j-m)(j-m+1)}{(2j+1)(2j+2)}}, \]

\[ b_1 = -\sqrt{\frac{(j-m)(j-m+1)}{2j(j+1)}}, \quad b_2 = \frac{m}{\sqrt{j(j+1)}}, \quad b_3 = \sqrt{\frac{(j-m)(j+m+1)}{2j(j+1)}} \]

\[ c_1 = \sqrt{\frac{(j-m)(j-m+1)}{2j(2j+1)}}, \quad c_2 = -\sqrt{\frac{(j-m)(j+m+1)}{2j(j+1)}}, \quad c_3 = \sqrt{\frac{(j+m)(j+m+1)}{2j(2j+1)}}. \]

By using the Clebsch-Gordon coefficients, density matrix in product basis can be written as

\[ \rho^{AB} = \sum_{m=-j}^{j} |m\rangle \langle m| \otimes (v_1 |1\rangle \langle 1| + v_2 |0\rangle \langle 0| + v_3 |-1\rangle \langle -1|) + \]

\[ \left( \sum_{m=-j}^{j} u_1(|m\rangle \langle m-1| \otimes |0\rangle \langle 1| + |m-1\rangle \langle m| \otimes |1\rangle \langle 0|) \right) \]

\[ \left( \sum_{m=-j}^{j} u_2(|m\rangle \langle m+1| \otimes |0\rangle \langle -1| + |m+1\rangle \langle m| \otimes |1\rangle \langle 0|) \right) \]

\[ \left( \sum_{m=-j}^{j} u_3(|m-1\rangle \langle m+1| \otimes |1\rangle \langle -1| + |m+1\rangle \langle m-1| \otimes |1\rangle \langle 1|) \right) \]

where

\[ v_1 = \sqrt{\frac{P}{2j-1}} c_1^2 + \sqrt{\frac{Q}{2j+1}} b_1^2 + \sqrt{\frac{1-P-Q}{2j+3}} a_1^2 \]

\[ v_2 = \sqrt{\frac{P}{2j-1}} c_2^2 + \sqrt{\frac{Q}{2j+1}} b_2^2 + \sqrt{\frac{1-P-Q}{2j+3}} a_2^2 \]

\[ v_3 = \sqrt{\frac{P}{2j-1}} c_3^2 + \sqrt{\frac{Q}{2j+1}} b_3^2 + \sqrt{\frac{1-P-Q}{2j+3}} a_3^2 \]

\[ u_1 = \sqrt{\frac{P}{2j-1}} c_1 c_2 + \sqrt{\frac{Q}{2j+1}} b_1 b_2 + \sqrt{\frac{1-P-Q}{2j+3}} a_1 a_2 \]

\[ u_2 = \sqrt{\frac{P}{2j-1}} c_3 c_2 + \sqrt{\frac{Q}{2j+1}} b_3 b_2 + \sqrt{\frac{1-P-Q}{2j+3}} a_3 a_2 \]

\[ u_3 = \sqrt{\frac{P}{2j-1}} c_1 c_3 + \sqrt{\frac{Q}{2j+1}} b_1 b_3 + \sqrt{\frac{1-P-Q}{2j+3}} a_1 a_3 \]

Now, let \( K^B = (I_A \otimes K_B) \) denote a local observable, with \( K_B = \vec{n} \cdot \vec{X} \) and (\(|\vec{n}| = 1\)), where \( \lambda_i \), with \( i = (1, 2, 3, 4, 5, 6, 7, 8) \) represent the generators of \( SU(3) \). Since the
LQU definition requires a minimization over the local observable, then we need to optimize the LQU value over the local observable. The procedure of our approach is straightforward, we used the method of Lagrange multipliers finding the local maxima and minima of a function subject to equality constraint. Firstly, we find the optimum value of the LQU is achieved when \((n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) = (0, 0, \pm 1/2, 0, 0, 0, 0, \pm \sqrt{3}/2)\). This shows that if the local measurements are performed by them, then the related LQU are obtained as: 

\[ u_A = \sum_{m=-j}^{m=j} (u_1^2 + u_2^2 + 4u_3^2) \]

The other optimum value of LQU is achieved when \((n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) = (0, 0, \mp \sqrt{3}/2, 0, 0, 0, 0, \pm 1/2)\), yielding the LQU as: 

\[ u_A = 3 \sum_{m=-j}^{m=j} u_1^2 + u_2^2 \]

Now, we have;

\[ U_A = \min(\sum_{m=-j}^{m=j} (u_1^2 + u_2^2 + 4u_3^2), 3 \sum_{m=-j}^{m=j} (u_1^2 + u_2^2)) \]  \hspace{1cm} (17)

In Fig.3, Fig.4, Fig.5 and Fig.6 we show the LQU of spin-\(j\) and spin-1 system that is a function of \(P\) and \(Q\) for some half-integer and integer \(j\).

## 5 Conclusions

We have analytically calculated the LQU of \(SU(2)\) invariant states consisting of a spin-1/2 particle and arbitrary spin-\(j\) particle or a spin-1 particle and a spin-\(j\) particle with measurement on the spin-1/2 subsystem and derived an analytic expression for the LQU on these spin states. However, we suggest that the LQU for other states (or states being invariant under other transformation groups) can be studied in the future investigation.
References

[1] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin and W. K. Wootters, Phys. Rev. A 59, 1070 (1999).

[2] L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001).

[3] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. 84, 1655 (2012).

[4] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).

[5] Dakic, Vedral, and Brukner, Phys. Rev. Lett. 105, 190502 (2010).

[6] B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. A 86, 012312 (2012).

[7] A. K. Rajagopal and R.W. Rendell, Phys. Rev. A 66, 022104 (2002).

[8] S. Luo, Phys. Rev. A 77, 022301 (2008).

[9] D. Girolami, T. Tufarelli and G. Adesso, Phys. Rev. Lett. 110, 240402 (2013).

[10] S. Vinjanampathy, A. R. P. Rau, J. Phys. A 45, 095303 (2012).

[11] Shuhao Wang, Hui Li, Xian Lu and Gui-Lu Long arXiv:1307.0576 [quant-ph].

[12] J. Schliemann, Phys. Rev. A 68, 012309 (2003).

[13] J. Schliemann, Phys. Rev. A 72, 012307 (2005).

[14] Z. Wang and Z. X. Wang, Phys. Lett. A 372, 7033 (2008).

[15] K. K. Manne, C. M. Caves, Quantum. Inf. Comp. 8, 0295 (2008).

[16] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A 64, 062307 (2001).

[17] B. Cakmak and Z. Gedik, J. Phys. A: Math. Theor. 46 (2013) 465302.

[18] Yao-Kun Wang, Teng Ma, Shao-Ming Fei, and Zhi-Xi Wang, arXiv:1307.3468 [quant-ph].

[19] E. P. Wigner, M. M. Yanase, Proc. Natl. Acad. Sci. U.S.A. 49, 910-918 (1963).

[20] Kiran K. Manne and Carlton M. Caves, arXiv:0506151 [quant-ph].

[21] F. F. Fanchini, L. K. Castelano, M. F. Cornelio, and M. C. de Oliviera, New J. Phys. 14, 013027 (2012).
Fig. 1. LQU of the bipartite state composed of a spin-$j$ and a spin-$1/2$ vs. $P$ for $j = 1/2$ ($d = 2$), $j = 5/2$ ($d = 6$) and $j = 21/2$ ($d = 22$).
Figure 2:

Fig. 2. LQU of the bipartite state composed of a spin-$j$ and a spin-$1/2$ vs. $P$ for $j = 1$ $(d = 3)$, $j = 2$ $(d = 5)$ and $j = 100$ $(d = 201)$. 
Fig. 3. LQU of the bipartite state composed of a spin-$j$ and a spin-1 vs. $Q$ with fixed $P=0.5$ for $j = 1$ (d = 3), $j = 3/2$ (d = 4), $j = 2$ (d = 5) and $j = 5/2$ (d = 6).
Fig. 4. LQU of the bipartite state composed of a spin-$j$ and a spin-1 vs. $Q$ with fixed $P=0.3$ for $j = 1$ ($d = 3$), $j = 3/2$ ($d = 4$), $j = 2$ ($d = 5$) and $j = 5/2$ ($d = 6$).
Fig. 5. LQU of the bipartite state composed of a spin-$j$ and a spin-1 vs. $P$ with fixed $Q=0.5$ for $j = 1$ ($d = 3$), $j = 3/2$ ($d = 4$), $j = 2$ ($d = 5$) and $j = 5/2$ ($d = 6$).
Fig. 6. LQU of the bipartite state composed of a spin-$j$ and a spin-1 vs. $P$ with fixed $Q=0.3$ for $j = 1$ ($d = 3$), $j = 3/2$ ($d = 4$), $j = 2$ ($d = 5$) and $j = 5/2$ ($d = 6$).