Bigraded Betti numbers and Generalized Persistence Diagrams

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Abstract

Commutative diagrams of vector spaces and linear maps over $\mathbb{Z}^2$ are objects of interest in topological data analysis (TDA) where this type of diagrams are called 2-parameter persistence modules. Given that quiver representation theory tells us that such diagrams are of wild type, studying informative invariants of a 2-parameter persistence module $M$ is of central importance in TDA. One of such invariants is the generalized rank invariant, recently introduced by Kim and Mémoli. Via the Möbius inversion of the generalized rank invariant of $M$, we obtain a collection of connected subsets $I \subseteq \mathbb{Z}^2$ with signed multiplicities. This collection generalizes the well known notion of persistence barcode of a persistence module over $\mathbb{R}$ from TDA. In this paper we show that the bigraded Betti numbers of $M$, a classical algebraic invariant of $M$, are obtained by counting the corner points of these subsets $I$s. Along the way, we verify that an invariant of 2-parameter persistence modules called the interval decomposable approximation (introduced by Asashiba et al.) also encodes the bigraded Betti numbers in a similar fashion. We also show that the aforementioned results are optimal in the sense that they cannot be extended to $d$-parameter persistence modules for $d \geq 3$.

\textbf{keywords.} Multiparameter persistence, Multigraded Betti numbers, Quiver representations, Möbius inversion, Persistent homology, Persistence diagram

1 Introduction

Multiparameter persistent homology. Theoretical foundations of persistent homology, one of the main protagonists in topological data analysis (TDA), have been rapidly developed in the

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last two decades, allowing a large number of applications. Persistent homology is obtained by applying the homology functor to an \(\mathbb{R}\) (or \(\mathbb{Z}\)) -indexed increasing family of topological spaces \([12, 28]\). This parametrized family of topological spaces, for example, often arises as either a sublevel set filtration of a real-valued map on a topological space, or the Vietories-Rips simplicial filtration of a metric space.

With more complex input data, we obtain \(\mathbb{R}^d\)-indexed increasing families \((d > 1)\) of topological spaces, e.g. a sublevel set filtration of a topological space that is filtered by multiple real-valued functions, or a Vietories-Rips-sublevel simplicial filtration of a metric space equipped with a map \([12, 16]\). By applying the homology functor (with coefficients in a fixed field \(k\)) to such a multiparameter filtration, we obtain a \(d\)-parameter persistence module \(\mathbb{R}^d\) (or \(\mathbb{Z}^d\)) \(\rightarrow\) \(\text{vec}\), a functor from the poset \(\mathbb{R}^d\) (or \(\mathbb{Z}^d\)) to the category \(\text{vec}\) of finite dimensional vector spaces and linear maps over the field \(k\). In contrast to the case of \(d = 1\), there is no discrete and complete invariant for \(\mathbb{R}^d\) (or \(\mathbb{Z}^d\)) \(\rightarrow\) \(\text{vec}\) for \(d > 1\) \([16]\). In quiver representation theory, functors \(\mathbb{R}^d\) (or \(\mathbb{Z}^d\)) \(\rightarrow\) \(\text{vec}\) \((d > 1)\) are of wild type, implying that there is no simple invariant which completely encodes the isomorphism type of \(\mathbb{R}^d\) (or \(\mathbb{Z}^d\)) \(\rightarrow\) \(\text{vec}\) \([23, 32]\). Nevertheless, there have been many studies on the invariants of \(d\)-parameter persistence modules, e.g. \([16, 18, 33, 39, 47, 53, 55]\).

Special attention has been placed on the case of \(d = 2\) \([1, 2, 8, 9, 20, 26, 31, 42]\) in part because 2-parameter filtrations arise in the study of interlevel set persistence \([8, 14]\), in the study of point cloud data with non-uniform density \([4, 11, 15, 16]\), or in applications in material science and chemical engineering \([31, 35, 36]\). The software RIVET \([42]\) can efficiently compute and visualize the dimension function (a.k.a. the Hilbert function), the fibered barcode, and the bigraded Betti numbers of a 2-parameter persistence module.

**Multigraded Betti numbers.** Multigraded Betti numbers encode important information about the algebraic structure of a multigraded module over the polynomial ring in \(n\) variables \([30, 48]\). For multiparameter persistence modules that arise from data, multigraded Betti numbers provide insight about the coarse-scale topological features of the data (cf. \([11]\)). For 2-parameter persistence modules, the multigraded Betti numbers are also called the bigraded Betti numbers. RIVET \([42]\) represents the bigraded Betti numbers of a 2-parameter persistence module as a collection of colored dots in the plane. More interestingly, RIVET employs the bigraded Betti numbers to implement an interactive visualization of the fibered barcode. Recently, Lesnick-Wright \([43]\) and Kerber-Rolle \([37]\) developed efficient algorithms for computing minimal presentations and the bigraded Betti numbers of 2-parameter persistence modules.

**Persistence diagram and its generalizations.** In most applications of 1-parameter persistent homology, the notion of persistence diagram \([29, 40]\) (or equivalently barcode \([17]\); cf. Definition 2.3) plays a central role. The persistence diagram of any \(M : \mathbb{R} \rightarrow \text{vec}\) is not only a visualizable topological summary of \(M\), but also a stable and complete invariant of \(M\) \([21]\). In contrast, as mentioned before, there is no simple complete invariant for \(d\)-parameter persistence modules when \(d > 1\).

Patel introduced the notion of generalized persistence diagram for constructible functors \(\mathbb{R} \rightarrow \mathcal{C}\), in which \(\mathcal{C}\) satisfies certain properties \([51]\). Construction of the generalized persistence diagram is based on the observation that the persistence diagram of \(M : \mathbb{R} \rightarrow \text{vec}\) \([29]\) is an instance of the Möbius inversion of the rank invariant \([16]\) of \(M\). McCleary and Patel showed
that the generalized persistence diagram is stable when $\mathcal{C}$ is a skeletally small abelian category [45]. Kim and Mémoli further extended Patel’s generalized persistence diagram to the setting of functors $\mathcal{P} \to \mathcal{C}$ in which $\mathcal{P}$ is an essentially finite poset such as a finite $d$-dimensional grid [38]. The generalized persistence diagram of $\mathcal{P} \to \mathcal{C}$ is defined as the Möbius inversion of the generalized rank invariant of $\mathcal{P} \to \mathcal{C}$. The generalized persistence diagram is not only a complete invariant of interval decomposable persistence modules $\mathcal{P} \to \text{vec}$ (Theorem 2.20), but is also well-defined regardless of the interval decomposability. The generalized rank invariant of $\mathbb{R}^d$ (or $\mathbb{Z}^d$) $\to \text{vec}$ is proven to be stable with respect to a certain generalization of the erosion distance [51] and the interleaving distance [41] (see the latest version of the arXiv preprint of [38]).

**Our contributions.** Assume that a given $M: \mathbb{Z}^2 \to \text{vec}$ is finitely generated. We establish a combinatorial formula for extracting the bigraded Betti numbers of $M$ from the generalized persistence diagram of $M$ (Theorem 3.5). More interestingly, the formula we found is a generalization of a well-known formula for extracting the bigraded Betti numbers from interval decomposable persistence modules (Theorem 2.9).

Namely, for any finitely generated interval decomposable $M: \mathbb{Z}^2 \to \text{vec}$, there is a visually intuitive way to find the bigraded Betti numbers of $M$ from the indecomposable summands of $M$. An example of this process is shown in Fig. 1 (A)-(C). For any finitely generated $N: \mathbb{Z}^2 \to \text{vec}$, which may not be interval decomposable, we utilize a similar process to find the bigraded Betti numbers of $N$ from the (Int-)generalized persistence diagram of $N$. This process is shown in Fig. 1 (A’)-(C’). In a sense, Theorem 3.5 thus reinforces the viewpoint that the (Int-)generalized persistence diagram is a proxy for the barcode (Definition 2.3) of persistence modules [2, 38].

One implication of Theorem 3.5 is that all invariants of 2-parameter persistence modules that are computed by the software RIVET [42] are encoded by the generalized persistence diagram. In other words, we obtain the following hierarchy of invariants for any finitely generated $M: \mathbb{Z}^2 \to \text{vec}$, where invariant A is placed above invariant B if invariant B can be recovered from invariant A:

```
Generalized persistence diagram
     /\                  /\  \\
/     \                /     \  \\
Fibered barcode        Bigraded Betti numbers
     |                  |
     \                 /  \
       Hilbert function
```

We remark that the generalized persistence diagram is equivalent to the generalized rank invariant (Definitions 2.17 and 2.18). Also, the fibered barcode is equivalent to the (standard) rank invariant [16]. Hence, in the diagram above, generalized persistence diagram and fibered barcode can be replaced by generalized rank invariant and rank invariant, respectively.

In the course of establishing Theorem 3.5, we verify that the interval decomposable approximation and multirank invariant of 2-parameter persistence modules (introduced by Asashiba et al. [2] and Thomas [54], respectively) also encode the bigraded Betti numbers (Remark 2.19 and Corollary A.10).

**Remark 1.1.** It should not be construed that Theorem 3.5 provides a practically efficient way to compute the bigraded Betti numbers. Rather, we hope that the aforementioned efficient
Figure 1: (A) A $\mathbb{Z}^2$-indexed persistence module $M$ whose support is contained in a $3 \times 4$ grid. (B) $M$ is interval decomposable, and the barcode of $M$ consists of the two blue intervals of $\mathbb{Z}^2$ (Definitions 2.2 and 2.3). (C) Expand each of the blue intervals from (B) to intervals in $\mathbb{R}^2$ as follows: Each point $p = (p_1, p_2)$ in the two intervals is expanded to the unit square $[p_1, p_1 + 1) \times [p_2, p_2 + 1) \subset \mathbb{R}^2$. Black dots, red stars and blue squares indicate three different corner types of the expanded intervals (see Fig. 2). The bigraded Betti numbers of $M$ can be read from these corner types; for each $p \in \mathbb{Z}^2$, $\beta_j(M)(p)$ is equal to the number of black dots, red stars, and blue squares at $p$ when $j = 0, 1, 2$, respectively. (A') Another $\mathbb{Z}^2$-indexed persistence module $N$ whose support is contained in a $3 \times 4$ grid. $N$ is not interval decomposable. (B') The Int-generalized persistence diagram of $N$ (Definition 2.18) is shown, where the multiplicity of the red interval is -1 and the multiplicity of each blue interval is 1. (C') is similarly interpreted as in (C), where corner points of the red interval negatively contribute to the counting of the bigraded Betti numbers. More details are provided in Example 3.6.
algorithms to compute the bigraded Betti numbers could be useful for approximating the generalized persistence diagram.

We also show that for \( d \geq 3 \), the generalized persistence diagram does not determine the multigraded Betti numbers of \( d \)-parameter persistence modules (Theorem 4.1). Lastly, we show that, in general, the generalized persistence diagram and the multirank invariant do not determine each other (Examples A.11 and A.12).

Other related work. McCleary and Patel utilized the Möbius inversion formula for establishing a functorial pipeline to summarize simplicial filtrations over finite lattices into persistence diagrams [46]. Botnan et al. introduced notions of signed barcode and rank decomposition for encoding the rank invariant of multiparameter persistence modules as a linear combination of rank invariants of indicator modules [10]. In their paper, Möbius inversion was utilized for computing the rank decomposition, characterizing the generalized persistence diagram in terms of rank decompositions. Asashiba et al. provided a criterion for determining whether or not a given multiparameter persistence module is interval decomposable without having to explicitly compute indecomposable decompositions [1]. Dey and Xin proposed an efficient algorithm for decomposing multiparameter persistence modules and introduced a notion of persistent graded Betti numbers, a refined version of the graded Betti numbers [27]. Dey et al. reduced the problem of computing the generalized rank invariant of a given 2-parameter persistence module to computing the indecomposable decompositions of zigzag persistence modules [24]. Blanchette et al. developed a theoretical framework for building new invariants of a persistence module over a poset using homological algebra [6].

Organization. In Section 2, we review the notions of persistence modules, multigraded Betti numbers, and generalized persistence diagrams. In Section 3, we show that the bigraded Betti numbers can be recovered from the generalized persistence diagram. In Section 5, we discuss open questions. In the appendix, we prove that (a) the multirank invariant (introduced in [54]) also determines the bigraded Betti numbers, and that (b) in general, the generalized persistence diagram and the multirank invariant do not determine each other.

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2 Preliminaries

In Section 2.1, we review the notions of persistence modules and interval decomposability. In Section 2.2, we recall the notion of multigraded Betti numbers (an invariant of multiparameter persistence modules). In Section 2.3, we review the Möbius inversion formula in combinatorics. In Section 2.4, we review the notions of generalized rank invariant and generalized persistence
diagram. In Section 2.5, we provide a formula of the (Int-)generalized persistence diagram in a certain setting, which will be useful in the next section.

2.1 Persistence modules and their interval decomposability

Let \( \mathbb{P} \) be a poset. We regard \( \mathbb{P} \) as the category that has points of \( \mathbb{P} \) as its objects and for \( p, q \in \mathbb{P} \) there is a unique morphism \( p \to q \) if and only if \( p \leq q \) in \( \mathbb{P} \). For \( d \in \mathbb{N} \), let \( \mathbb{R}^d \) and subsets of \( \mathbb{R}^d \) (such as \( \mathbb{Z}^d \)) be given the partial order defined by \( (a_1, a_2, \ldots, a_d) \leq (b_1, b_2, \ldots, b_d) \) if and only if \( a_i \leq b_i \) for \( i = 1, 2, \ldots, d \).

Every vector space in this paper is over some fixed field \( k \). Let \( \text{vec} \) denote the category of finite dimensional vector spaces and linear maps over \( k \).

A \( \mathbb{P} \)-indexed persistence module, or simply a \( \mathbb{P} \)-module, refers to a functor \( M : \mathbb{P} \to \text{vec} \). In other words, to each \( p \in \mathbb{P} \), a vector space \( M(p) \) is associated, and to each pair \( p \leq q \) in \( \mathbb{P} \), a linear map \( \varphi_p(p, q) : M(p) \to M(q) \) is associated. Importantly, whenever \( p \leq q \leq r \) in \( \mathbb{P} \), it is required that \( \varphi_{p, r} = \varphi_{q, r} \circ \varphi_{p, q} \). When \( \mathbb{P} = \mathbb{R}^d \) or \( \mathbb{Z}^d \), \( M \) is also called a \( d \)-parameter persistence module.

Consider a zigzag poset of \( n \) points,

\begin{align*}
\bullet_1 & \leftrightarrow \bullet_2 \leftrightarrow \ldots \bullet_{n-1} \leftrightarrow \bullet_n
\end{align*}

where \( \leftrightarrow \) stands for either \( \leq \) or \( \geq \). A functor from a zigzag poset (of \( n \) points) to \( \text{vec} \) is called a zigzag module (of length \( n \)) [13].

A morphism between \( \mathbb{P} \)-modules \( M \) and \( N \) is a natural transformation \( f : M \to N \) between \( M \) and \( N \). That is, \( f \) is a collection \( \{ f_p : M(p) \to N(p) \}_{p \in \mathbb{P}} \) of linear maps such that for every pair \( p \leq q \) in \( \mathbb{P} \), the following diagram commutes:

\[
\begin{array}{c}
M(p) \xrightarrow{\varphi_{M(p, q)}} M(q) \\
\downarrow f_p \\
N(p) \xrightarrow{\varphi_{N(p, q)}} N(q).
\end{array}
\]

The kernel of \( f \), denoted by \( \ker(f) : \mathbb{P} \to \text{vec} \), is defined as follows: For \( p \in \mathbb{P} \), \( \ker(f)(p) := \ker(f_p) \subseteq M(p) \). For \( p \leq q \) in \( \mathbb{P} \), \( \varphi_{\ker(f)}(p, q) \) is the restriction of \( \varphi_{M(p, q)} \) to \( \ker(f_p) \). Two \( \mathbb{P} \)-modules \( M \) and \( N \) are (naturally) isomorphic, denoted by \( M \cong N \), if there exists a natural transformation \( \{ f_p \}_{p \in \mathbb{P}} \) from \( M \) to \( N \) where each \( f_p \) is an isomorphism.

The direct sum \( M \oplus N \) of \( M, N : \mathbb{P} \to \text{vec} \) is the \( \mathbb{P} \)-module where \( (M \oplus N)(p) = M(p) \oplus N(p) \) for \( p \in \mathbb{P} \) and \( \varphi_{M \oplus N}(p, q) = \varphi_M(p, q) \oplus \varphi_N(p, q) \) for \( p \leq q \) in \( \mathbb{P} \). A nonzero \( \mathbb{P} \)-module \( M \) is indecomposable if whenever \( M = M_1 \oplus M_2 \) for some \( \mathbb{P} \)-modules \( M_1 \) and \( M_2 \), either \( M_1 = 0 \) or \( M_2 = 0 \).

**Theorem 2.1** (Krull-Remak-Schmidt-Azumaya [3]). Any \( \mathbb{P} \)-module \( M \) has a direct sum decomposition \( M \cong \bigoplus_{i} M_i \) where each \( M_i \) is indecomposable. Such a decomposition is unique up to isomorphism and reordering of the summands.

In what follows, we review the notion of interval decomposability.
Definition 2.2. Let \( \mathbb{P} \) be a poset. An interval of \( \mathbb{P} \) is a subset \( I \subseteq \mathbb{P} \) such that: (i) \( I \) is nonempty. (ii) If \( p, q \in I \) and \( p \leq r \leq q \), then \( r \in I \). (iii) \( I \) is connected, i.e. for any \( p, q \in I \), there is a sequence \( p = p_0, p_1, \ldots, p_\ell = q \) of elements of \( I \) with either \( p_i \leq p_{i+1} \) or \( p_{i+1} \leq p_i \) for each \( i \in [0, \ell - 1] \).\(^{1}\) By \( \text{Int}(\mathbb{P}) \), we denote the set of all intervals of \( \mathbb{P} \).

For example, any interval of a zigzag poset in (1) is a set of consecutive points in \( \{ \bullet_1, \bullet_2, \ldots, \bullet_n \} \).

For an interval \( I \) of a poset \( \mathbb{P} \), the interval module \( V_I : \mathbb{P} \to \text{vec} \) is defined as

\[
V_I(p) = \begin{cases} k & \text{if } p \in I \\ 0 & \text{otherwise}, \end{cases} \quad \varphi_{V_I}(p, q) = \begin{cases} \text{id}_k & \text{if } p, q \in I, \ p \leq q \\ 0 & \text{otherwise}. \end{cases}
\]

It is well-known that any interval module is indecomposable [8, Proposition 2.1].

Definition 2.3. A \( \mathbb{P} \)-module \( M \) is said to be interval decomposable if there exists a multiset \( \text{barc}(M) \) of intervals of \( \mathbb{P} \) such that \( M \cong \bigoplus_{I \in \text{barc}(M)} V_I \). We call \( \text{barc}(M) \) the barcode of \( M \).

Theorem 2.4 ([3, 22, 32]). For \( d = 1 \), any \( M : \mathbb{R}^d \) (or \( \mathbb{Z}^d \)) \to \text{vec} \) is interval decomposable and thus admits a (unique) barcode. However, for \( d \geq 2 \), \( M \) may not be interval decomposable. Lastly, any zigzag module is interval decomposable and thus admits a (unique) barcode.

The following notation is useful in the rest of the paper.

Notation 2.5. Assume that a \( \mathbb{P} \)-module \( M \) is isomorphic to the direct sum \( \bigoplus_{i \in \mathcal{I}} M_i \) for some indexing set \( \mathcal{I} \) where each \( M_i \) is indecomposable. For \( I \in \text{Int}(\mathbb{P}) \), we define \( \text{mult}(I, M) \) as the cardinality of the set \( \{ i \in \mathcal{I} : M_i \cong V_I \} \). In words, \( \text{mult}(I, M) \) is the number of those summands \( M_i \) which are isomorphic to the interval module \( V_I \).

2.2 Multigraded Betti numbers

In this section we review the notion of multigraded Betti numbers [30].

Fix any \( p \in \mathbb{Z}^d \). Then, the upper set \( p^1 := \{ x \in \mathbb{Z}^d : p \leq x \} \) determines an interval of \( \mathbb{Z}^d \). An \( \mathbb{Z}^d \)-module \( F \) is free if there exists \( p_1, p_2, \ldots, p_n \) in \( \mathbb{Z}^d \) such that \( F \cong \bigoplus_{i=1}^{n} V_{p_i^1} \).

Let \( M \) be an \( \mathbb{Z}^d \)-module. An element \( v \in M(p) \) for some \( p \in \mathbb{Z}^d \) is called a homogeneous element of \( M \). Assume that \( M \) is finitely generated, i.e. there exist \( p_1, \ldots, p_n \in \mathbb{Z}^d \) and \( v_i \in M(p_i) \) for \( i = 1, \ldots, n \) such that for any \( p \in \mathbb{Z}^d \) and for any nonzero \( v \in M(p) \), there exist \( c_i \in \mathbb{K} \) for \( i = 1, \ldots, n \) with

\[
v = \sum_{i=1}^{n} c_i \cdot \varphi_M(p_i, p)(v_i).
\]

The collection \( \{ v_1, \ldots, v_n \} \) is called a (homogeneous) generating set for \( M \).

Let us assume that \( \{ v_1, \ldots, v_n \} \) is a minimal homogeneous generating set for \( M \), i.e. there is no homogeneous generating set for \( M \) that includes fewer than \( n \) elements. Let \( F_0 := \bigoplus_{i=1}^{n} V_{p_i^1} \).

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\(^{1}\)This definition of interval is not the standard definition of interval used in order theory but it is often used in the literature concerned with persistence modules over posets; e.g. [7]. In order theory language, \( I \) is a nonempty convex connected subset of \( \mathbb{P} \).
For \( i = 1, \ldots, n \), let \( 1_{p_i} \in V_{\mu_i}(p_i) \). Then, the set \( \{1_{p_1}, \ldots, 1_{p_n}\} \) generates \( F_0 \) and the morphism \( \eta_0 : F_0 \to M \) defined by \( \eta_0(1_{p_i}) = v_i \) for \( i = 1, \ldots, n \) is surjective. Let \( K_0 := \ker(\eta_0) \subseteq F_0 \) and let \( \iota_0 : K_0 \hookrightarrow F_0 \) be the inclusion map. Iterate this process using \( K_0 \) in place of \( M \).

Namely, identify a minimal homogeneous generating set \( \{v_1', \ldots, v_m'\} \) for \( K_0 \) where \( v_j' \in (K_0)_{p_j} \) for some \( p_1', \ldots, p_m' \in \mathbb{Z}^d \) and consider the free module \( F_1 := \bigoplus_{j=1}^m V_{p_j} \) and the surjection \( \eta_1 : F_1 \to K_0 \). Then we have the map \( \iota_0 \circ \eta_1 : F_1 \to F_0 \). By repeating this process, we obtain a \textbf{minimal free resolution of} \( M \):

\[
\cdots \xrightarrow{I_1 \circ \eta_2} F_1 \xrightarrow{\iota_0 \circ \eta_1} F_0 \xrightarrow{\eta_0} M \to 0.
\]

This resolution is unique up to isomorphism \cite[Theorem 1.6]{30}. Hilbert’s Syzygy Theorem guarantees that \( F_j = 0 \) for \( j > d \) \cite{34}.

\textbf{Definition 2.6.} For \( j = 0, \ldots, d \), the \( j^{\text{th}} \) \textbf{multigraded Betti number} \( \beta_j(M) : \mathbb{Z}^d \to \text{vec} \) of \( M \) is defined by mapping each \( p \in \mathbb{Z}^d \) to

\[
\beta_j(M)(p) := \text{mult} \left( p^j, F_j \right) \quad \text{(Notation 2.5)}.
\]

When \( d = 2 \), the multigraded Betti numbers are also called the \textbf{bigraded Betti numbers}.

We will see that for an interval decomposable \( \mathbb{Z}^2 \)-module \( M \), its bigraded Betti numbers can be extracted from \( \text{barc}(M) \). To this end, we will make use of a certain regions that arise by “blowing-up” intervals from \( \text{barc}(M) \):

\textbf{Definition 2.7.} Given any \( I \in \text{Int}(\mathbb{Z}^2) \), the subset of \( \mathbb{R}^2 \)

\[
I^+ := \bigcup_{(p_1, p_2) \in I} [p_1, p_1 + 1) \times [p_2, p_2 + 1)
\]

will be referred to as the region corresponding to \( I \) in \( \mathbb{R}^2 \).

The following remarks are well-known; e.g. \cite[Remarks 2.4 and 3.10]{11}.

\textbf{Remark 2.8.} \quad (i) For any finitely generated \( M, N : \mathbb{Z}^d \to \text{vec} \), we have \( \beta_j(M \oplus N) = \beta_j(M) + \beta_j(N) \) for \( j = 0, \ldots, d \).

(ii) Let \( I \in \text{Int}(\mathbb{Z}^2) \). For the interval module \( V_I : \mathbb{Z}^2 \to \text{vec} \), the \( j^{\text{th}} \) bigraded Betti number \( \beta_j(V_I)(p) \) is equal to 1 if \( p \) is a \( j^{\text{th}} \) type corner point of \( I^+ \) and is equal to 0 otherwise; see Fig. 2.

Remark 2.8 directly implies:

\textbf{Theorem 2.9.} \quad Given any finitely generated interval decomposable module \( M : \mathbb{Z}^2 \to \text{vec} \), the bigraded Betti numbers of \( M \) can be extracted from \( \text{barc}(M) \). More specifically, the bigraded Betti numbers of \( M \) can be extracted from the corner points of the elements in the multiset

\[
\{ I^+ \subseteq \mathbb{R}^2 : I \in \text{barc}(M) \}.
\]

In Theorem 3.5, we remove the assumption that \( M \) be interval decomposable and generalize Theorem 2.9 to the setting of \textit{any} finitely generated \( \mathbb{Z}^2 \)-modules.

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\textsuperscript{2}We remark that \( K_0 \) is finitely generated by the following two facts. (1) A submodule of any finitely generated module over a Noetherian ring is finitely generated. (2) A finitely generated \( \mathbb{Z}^d \)-module can be viewed as a module over the polynomial ring in \( d \) variables, which is Noetherian; this viewpoint can be found in \cite{16,48} for example.
2.3 The Möbius inversion formula in combinatorics

In this section, we briefly review the Möbius inversion formula, a fundamental concept in combinatorics [5, 52].

A poset \( A \) is said to be **locally finite** if for all \( p, q \in A \) with \( p \leq q \), the set \( [p, q] := \{ r \in A : p \leq r \leq q \} \) is finite. Let \( A \) be a locally finite poset. The Möbius function \( \mu_A : A \times A \to \mathbb{Z} \) of \( A \) is defined recursively as

\[
\mu_A(p, q) = \begin{cases} 
1, & p = q, \\
-\sum_{p \leq r < q} \mu_A(p, r), & p < q, \\
0, & \text{otherwise}. 
\end{cases}
\] (3)

For \( q_0 \in A \), consider the principal ideal \( q_0^\downarrow := \{ q \in A : q \leq q_0 \} \). Note that if we assume that \( q_0^\downarrow \) is finite for all \( q \in A \), then \( A \) must be locally finite. To see this, note that, for any \( p, q \in A \) with \( p \leq q \), the set \( [p, q] \) is a subset of the finite set \( q_0^\downarrow \).

**Theorem 2.10** (Möbius Inversion formula). Assume that \( q_0^\downarrow \) is finite for all \( q \in A \). Let \( k \) be a field. For any pair of functions \( f, g : A \to k \),

\[
g(q) = \sum_{r \leq q} f(r) \text{ for all } q \in A
\]

if and only if

\[
f(q) = \sum_{r \leq q} g(r) \cdot \mu_A(r, q) \text{ for all } q \in A.
\]

The function \( f \) is called the **Möbius inversion** of \( g \).

One interpretation of the Möbius inversion formula is that of a discrete analogue of the derivative of a real-valued map in elementary calculus, as explained in the following example:

\[3\text{More precisely, the codomain of } \mu_A \text{ is the multiple of 1 in a specified base ring.}\]
Example 2.11. Let \([m] = \{0, 1, \ldots, m\}\) with the usual order. Then,

\[
\mu_{[m]}(a, b) = \begin{cases} 
1, & a = b, \\
-1, & a = b - 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Hence, for any function \(g : [m] \to \mathbb{R}\), its M"obius inversion \(f : [m] \to \mathbb{R}\) is given by \(f(a) = g(a) - g(a - 1)\) for \(a \neq 0\) and \(f(0) = g(0)\). Hence, at each point \(a \neq 0\), \(f(a)\) captures the rate of change of \(g\) around that point.

2.4 Generalized rank invariant and generalized persistence diagrams

In this section we review the notions of generalized rank invariant and generalized persistence diagram [38, 51].

Throughout this subsection, let \(\mathbb{P}\) denote a finite connected poset (Definition 2.2 (iii)).

Consider any \(\mathbb{P}\)-module \(M\). Then \(M\) admits a limit and a colimit of \(M\): \(\lim M = (L, (\pi_p : L \to M(p))_{p \in \mathbb{P}})\) and \(\text{colimit } M = (C, (\iota_p : M(p) \to C)_{p \in \mathbb{P}})\); see the appendix for a review of the definitions of limits and colimits (Definitions A.4 and A.6). This implies that, for every \(p \leq q\) in \(\mathbb{P}\),

\[
M(p \leq q) \circ \pi_p = \pi_q \quad \text{and} \quad \iota_q \circ M(p \leq q) = \iota_p.
\]

Since \(\mathbb{P}\) is connected, these equalities imply that \(\iota_p \circ \pi_p = \iota_q \circ \pi_q : L \to C\) for any \(p, q \in \mathbb{P}\). In words, the composition \(\iota_p \circ \pi_p\) is independent of \(p\). The canonical limit-to-colimit map \(\psi_M : \lim M \to \text{colimit } M\) is therefore defined to be the linear map \(\iota_p \circ \pi_p\) where \(p\) is any point in \(\mathbb{P}\).

Definition 2.12 ([38]). The rank of \(M : \mathbb{P} \to \text{vec}\) is defined as the rank of the canonical limit-to-colimit map \(\psi_M : \lim M \to \text{colimit } M\).

The rank of \(M : \mathbb{P} \to \text{vec}\) counts the multiplicity of the fully supported interval module \(V_p\) in a direct sum decomposition of \(M\) into indecomposable modules:

Theorem 2.13 ([19, Lemma 3.1]). For any \(M : \mathbb{P} \to \text{vec}\), the rank of \(M\) is equal to \(\text{mult}(\mathbb{P}, M)\).

Let \(p, q \in \mathbb{P}\). We say that \(p\) covers \(q\) and write \(q < p\) if \(q < p\) and there is no \(r \in \mathbb{P}\) such that \(q < r < p\).

A subposet \(I \subseteq \mathbb{P}\) is said to be path-connected in \(\mathbb{P}\) if for any \(p \neq q\) in \(I\), there exists a sequence \(p = p_0, p_1, \ldots, p_n = q\) in \(I\) such that either \(p_i < p_{i+1}\) or \(p_{i+1} < p_i\) in \(\mathbb{P}\) for \(i = 0, \ldots, n - 1\). For example, the set \(\{0, 2\}\) is a connected (Definition 2.2 (iii)) subposet of \(\{0, 1, 2\}\) equipped with the usual order, but is not path-connected in \(\{0, 1, 2\}\).

By \(\text{Con}(\mathbb{P})\) we denote the poset of all path-connected subposets of \(\mathbb{P}\) that is ordered by inclusions. We remark that, since \(\mathbb{P}\) is finite, \(\text{Con}(\mathbb{P})\) is finite. For example, assume that \(\mathbb{P}\) is the zigzag poset \(\{1 < 2 > 3\}\). Then, \(\text{Con}(\mathbb{P})\) consists of the six elements: \(\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\), and \(\{1, 2, 3\}\). All of these are also intervals of \(\{1 < 2 > 3\}\), i.e. \(\text{Int}(\mathbb{P}) = \text{Con}(\mathbb{P})\) (Definition 2.2). In general, \(\text{Int}(\mathbb{P})\) is a subposet of \(\text{Con}(\mathbb{P})\).
**Definition 2.14.** The **generalized rank invariant** of $M : \mathbb{P} \to \text{vec}$ is the function

$$
rk(M) : \text{Con}(\mathbb{P}) \to \mathbb{Z}_{\geq 0}
$$

which maps $I \in \text{Con}(\mathbb{P})$ to the rank of the restriction $M|_I$ of $M$.

In fact, in order to define the generalized rank invariant, $\mathbb{P}$ does not need to be finite [38, Section 3]. However, for this work, it suffices to consider the case when $\mathbb{P}$ is finite.

**Remark 2.15.** Let $I \in \text{Con}(\mathbb{P})$. For any $\mathbb{P}$-module $M$, the following hold:

(i) By Theorem 2.13, if there exists $p \in I$ such that $M(p) = 0$, then $rk(M)(I) = 0$.

(ii) Let $I, J \in \text{Con}(\mathbb{P})$ with $J \supseteq I$. Then $rk(M)(J) \leq rk(M)(I)$, i.e. $rk(M)$ is order-reversing. This is because the canonical limit-to-colimit map $\lim M|_I \to \lim M|_J$ is a factor of the canonical limit-to-colimit map $\lim M|_I \to \lim M|_J$ [38, Proposition 3.7]. This monotonicity implies that if $rk(M)(I) = 0$, then $rk(M)(J) = 0$.

The following is a corollary of Theorem 2.13.

**Proposition 2.16 ([38, Proposition 3.17]).** Let $M : \mathbb{P} \to \text{vec}$ be interval decomposable. Then for any $I \in \text{Con}(\mathbb{P})$,

$$
rk(M)(I) = \sum_{J \supseteq I} \text{mult}(J, M).
$$

In words, $rk(M)(I)$ equals the total multiplicity of intervals $J$ in $\text{barc}(M)$ that contain $I$.

For any poset $\mathbb{A}$, let $\mathbb{A}^{\text{op}}$ denote the **opposite poset** of $\mathbb{A}$, i.e. $p \leq q$ in $\mathbb{A}$ if and only if $q \leq p$ in $\mathbb{A}^{\text{op}}$. By virtue of Theorem 2.10 we have:

**Definition 2.17.** Let $\mathbb{P}$ be a finite connected poset. The **generalized persistence diagram** of $M : \mathbb{P} \to \text{vec}$ is the unique function $dgm(M) : \text{Con}(\mathbb{P}) \to \mathbb{Z}$ that satisfies, for any $I \in \text{Con}(\mathbb{P})$,

$$
rk(M)(I) = \sum_{J \supseteq I} dgm(M)(J). \tag{4}
$$

In other words, $dgm(M)$ is the Möbius inversion of $rk(M)$ over $\text{Con}^{\text{op}}(\mathbb{P})$. That is, for $I \in \text{Con}(\mathbb{P})$,

$$
dgm(M)(I) := \sum_{J \supseteq I} \mu_{\text{Con}^{\text{op}}(\mathbb{P})}(J, I) \cdot rk(M)(J). \tag{5}
$$

The function $\mu_{\text{Con}^{\text{op}}(\mathbb{P})}$ has been precisely computed in [38, Section 3].

Next, we restrict the domain of $rk(M)$ and $dgm(M)$ to the collection $\text{Int}(\mathbb{P})$ of all intervals of $\mathbb{P}$. For $M : \mathbb{P} \to \text{vec}$, let $rk_0(M)$ denote the restriction of $rk(M) : \text{Con}(\mathbb{P}) \to \mathbb{Z}_{\geq 0}$ to $\text{Int}(\mathbb{P})$. We consider the Möbius inversion of $rk_0(M)$ over the poset $\text{Int}^{\text{op}}(\mathbb{P})$. Again by virtue of Theorem 2.10 we have:
Definition 2.18. Let \( P \) be a finite connected poset. The \textbf{Int-generalized persistence diagram} of \( M : P \to \text{vec} \) is the unique function \( \text{dgm}_I(M) : \text{Int}(P) \to \mathbb{Z} \) that satisfies, for any \( I \in \text{Int}(P) \),

\[
\text{rk}_I(M)(I) = \sum_{J \supseteq I} \text{dgm}_I(M)(J).
\]

In other words, by Theorem 2.10, \( \text{dgm}_I(M) \) is the Möbius inversion of \( \text{rk}_I(M) \) over \( \text{Int}^{\text{op}}(P) \), i.e. for \( I \in \text{Int}(P) \),

\[
\text{dgm}_I(M)(I) := \sum_{J \supseteq I} \mu_{\text{Int}^{\text{op}}(P)}(J,I) \cdot \text{rk}_I(M)(J).
\] (6)

Recall from Example 2.11 that, for \( m \in \mathbb{Z}_{\geq 0} \), \([m]\) is defined as the set \( \{0 < 1 < \cdots < m\} \).

Remark 2.19. In Definition 2.18, let \( P \) be the finite product poset \([m] \times [n]\) for any \( m, n \in \mathbb{N} \cup \{0\} \). Then, \( \text{dgm}_1(M) \) is equivalent to the \textit{interval decomposable approximation} \( \delta_{\text{tot}}(M) \) given in [2]; this is a direct corollary of Theorem 2.13. The Möbius function \( \mu_{\text{Int}^{\text{op}}([m] \times [n])} \) has been precisely computed in [2], which leads to Theorem 2.22 below.

Although we do not require \( M : P \to \text{vec} \) to be interval decomposable in order to define \( \text{dgm}(M) \) or \( \text{dgm}_I(M) \), these two diagrams generalize the notion of barcode (Definition 2.3):

Theorem 2.20. Let \( M : P \to \text{vec} \) be interval decomposable. Then we have:

\[
\text{dgm}(M)(I) = \begin{cases} 
\text{mult}(I,M) & I \in \text{Int}(P) \\
0 & I \in \text{Con}(P) \setminus \text{Int}(P), \text{ and} 
\end{cases}
\] (7)

\[
\text{dgm}_I(M)(I) = \text{mult}(I,M) \text{ for all } I \in \text{Int}(P).
\] (8)

The equality given in Equation (7) was first proved in [38, Theorem 3.14], but we include a proof here for completeness.

Proof. By Proposition 2.16, we have that \( \text{rk}(M)(I) = \sum_{J \supseteq I} \text{mult}(I,M) \). By the uniqueness of \( \text{dgm}(M) \) in Definition 2.18, \( \text{dgm}(M)(I) = \text{mult}(I,M) \) for all \( I \in \text{Int}(P) \) and \( \text{dgm}(M)(I) = 0 \) for \( I \in \text{Con}(P) \setminus \text{Int}(P) \). By a similar argument, we have that \( \text{dgm}_I(M)(I) = \text{mult}(I,M) \) for all \( I \in \text{Int}(P) \).

In the restricted case when \( P = [m] \times [n] \), the equality in Equation (8) has been also independently proved in [2, Theorem 5.10].

Theorem 2.20 implies that both \( \text{dgm}(M) \) and \( \text{dgm}_I(M) \) are able to completely determine the isomorphism type of an interval decomposable persistence module \( M \) (which also implies that each of \( \text{rk}(M) \) and \( \text{rk}_I(M) \) is strong enough to determine the isomorphism type of \( M \)). However, in general, the generalized persistence diagram \( \text{dgm}(M) \) is more discriminative than the \textbf{Int}-generalized persistence diagram \( \text{dgm}_I(M) \); see Example A.2 in the appendix.

In Section 3, the case when \( P \) is a zigzag poset of length 3 will be useful.
Example 2.21 ([38, Section 3.2.2]). Assume that $\mathcal{P}$ is any zigzag poset of length 3, i.e. $\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \bullet_3$ where $\leftrightarrow$ stands for either $\leq$ or $\geq$. Then, $dgm(M)$ is computed as follows:

$$dgm(M)(\{\bullet_1\}) = \text{rk}(M)(\{\bullet_1\}) - \text{rk}(M)(\{\bullet_1, \bullet_2\}),$$
$$dgm(M)(\{\bullet_2\}) = \text{rk}(M)(\{\bullet_2\}) - \text{rk}(M)(\{\bullet_1, \bullet_2\}) - \text{rk}(M)(\{\bullet_2, \bullet_3\}) + \text{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}),$$
$$dgm(M)(\{\bullet_3\}) = \text{rk}(M)(\{\bullet_3\}) - \text{rk}(M)(\{\bullet_2, \bullet_3\}),$$
$$dgm(M)(\{\bullet_1, \bullet_2\}) = \text{rk}(M)(\{\bullet_1, \bullet_2\}) - \text{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}),$$
$$dgm(M)(\{\bullet_2, \bullet_3\}) = \text{rk}(M)(\{\bullet_2, \bullet_3\}) - \text{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}).$$

Since $M$ is a zigzag module, it is interval decomposable (Theorem 2.4). Thus, we have $dgm(M)(I) = \text{mult}(I, M)$ for $I \in \text{Con}(\mathcal{P})$, the multiplicity of $I$ in $\text{barc}(M)$. Since $\text{Con}(\mathcal{P}) = \text{Int}(\mathcal{P})$, each $dgm(M)$ above can be replaced by $dgm_{I}(M)$.

2.5 \textbf{Int-Generalized persistence diagram of an }([m] \times [n])\text{-module.}

In this section we review a formula of the Int-generalized persistence diagram of an $([m] \times [n])$-module for any fixed integers $m, n \geq 0$.

Let us consider the poset $\text{Int}([m] \times [n])$. Then, given any two distinct $I, J \in \text{Int}([m] \times [n])$, we say that $J$ covers $I$ if $J \supseteq I$ and there is no interval $K$ such that $J \supsetneq K \supsetneq I$. For $I \in \text{Int}([m] \times [n])$, let us define $\text{cov}(I)$ as the collection of all $J \in \text{Int}([m] \times [n])$ that cover $I$. Given any nonempty $S \subseteq \text{Int}([m] \times [n])$, by $\bigvee S$, we denote the smallest interval $J$ that contains all $I \in S$.

The following theorem is established by invoking Remark 2.19 and finding an explicit formula for the Möbius function $\mu_{\text{Intop}(\mathcal{P})}$ that appears in Equation (6) with $\mathcal{P} = [m] \times [n]$.

\textbf{Theorem 2.22} ([2, Theorem 5.3]). \textit{For any }([m] \times [n])\text{-module }M,\textit{ }

$$dgm_{I}(M)(J) = \text{rk}_{I}(M)(J) + \sum_{S \subseteq \text{cov}(I), S \neq \emptyset} (-1)^{|S|} \text{rk}_{I}(M)(\bigvee S). \quad (9)$$

\textbf{Example 2.23.} Let $I \in \text{Int}([3] \times [2])$ depicted as in Fig. 3 (A). Note that $\text{cov}(I) = \{J_1, J_2, J_3\}$ where $J_1, J_2$ and $J_3$ are depicted as in Fig. 3 (B). For any $([3] \times [2])$-module $M$, we have:

$$dgm_{I}(M)(J) = \text{rk}_{I}(M)(J) - \sum_{i=1}^{3} \text{rk}_{I}(M)(J_i) + \sum_{i \neq j} \text{rk}_{I}(M)(\bigvee \{J_i, J_j\}) - \text{rk}_{I}(M)(\bigvee \{J_1, J_2, J_3\}).$$
where $\mathcal{V}(\mathcal{V}_2) = J_1 \cup J_2 \cup \{0, 2\}$, $\mathcal{V}(\mathcal{V}_3) = J_1 \cup J_3$, $\mathcal{V}(\mathcal{V}_4) = J_2 \cup J_3$, and $\mathcal{V}(\mathcal{V}_5) = J_1 \cup J_2 \cup J_3 \cup \{0, 2\}$.

The following remark will be useful in the next section.

**Remark 2.24.** Let $M$ be an $(\mathcal{M} \times \mathcal{N})$-module and let $I \in \textbf{Int}(\mathcal{M} \times \mathcal{N})$. By Remark 2.15 (ii) and Equation (9), if $\text{rk}(M)(I) = 0$, then $dgm(M)(I) = 0$.

### 3 Extracting the bigraded Betti numbers from the generalized persistence diagram

In this section we aim at establishing Theorem 3.5, as a generalization of Theorem 2.9.

Let $M$ be a finitely generated $\mathcal{Z}^2$-module.\(^4\) We may assume that $M(p) = 0$ for $p \not\in (0,0)$. Then, all algebraic information of $M$ can be recovered from the restricted module $M' := M(\mathcal{M} \times \mathcal{N})$ for some large enough positive integers $m$ and $n$. We will show that the generalized persistence diagram of $M'$ determines the bigraded Betti numbers of $M$.

**Definition 3.1.** A given $\mathcal{Z}^2$-module $M$ is said to be **encoded** by $M' : \mathcal{M} \times \mathcal{N} \to \textbf{vec}$ if the following hold:

- If $p \in \mathcal{Z}^2$ is not greater than equal to $(0,0)$, then $M(p) = 0$.
- For $(0,0) \leq p$ in $\mathcal{Z}^2$, we have that $M(p) = M'(q)$ where $q$ is the maximal element of $\mathcal{M} \times \mathcal{N}$ such that $q \leq p$ (we write $q = \lfloor p \rfloor_{m,n}$ in this case).
- For $(0,0) \leq p_1 \leq p_2$ in $\mathcal{Z}^2$, the map $\varphi_M(p_1, p_2)$ is equal to $\varphi_M((p_1, p_2))_m$.

The $\mathcal{Z}^2$-module $M$ described above is clearly finitely generated and its restriction $M'_{\mathcal{M} \times \mathcal{N}}$ coincides with $M'$. The following proposition is the key to obtain Theorem 3.5. Let $e_1 := (1,0)$ and $e_2 := (0,1)$ in $\mathcal{Z}^2$.

**Proposition 3.2.** Assume that a $\mathcal{Z}^2$-module $M$ is encoded by $M' : \mathcal{M} \times \mathcal{N} \to \textbf{vec}$. Then, $dgm(M')$ determines the bigraded Betti numbers of $M$ via the following formulas: For $p \in [m+1] \times [n+1]$, we have $\beta_j(M)(p) = 0$, $j = 0, 1, 2$. For $p \in [m+1] \times [n+1]$, we have:

\[
\beta_j(M)(p) = \left\{ \begin{array}{ll}
\sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p}
\end{array} \right. \quad \text{for } j = 0
\]

\[
\beta_j(M)(p) = \left\{ \begin{array}{ll}
\sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p}
\end{array} \right. \quad \text{for } j = 1
\]

\[
\beta_j(M)(p) = \left\{ \begin{array}{ll}
\sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p} \sum_{J \not\in p}
\end{array} \right. \quad \text{for } j = 2
\]

\(^4\)Main results in this section (which are Proposition 3.2 and Theorem 3.5) also hold for finitely presented $\mathcal{R}^2$-modules upto rescaling parameters [16, 42].
where each sum is taken over \( J \in \text{Con}([m] \times [n]) \).\(^5\) Moreover, each \( \text{dgm}(M') \) above can be replaced by \( \text{dgm}(M') \) where each sum is taken over \( J \in \text{Int}([m] \times [n]) \).

One direct consequence of this proposition is that if \( p \in [m + 1] \times [n + 1] \) is outside of \([m] \times [n]\), then \( \beta_0(M)(p) = 0 \): no \( J \in \text{Con}([m] \times [n]) \) can include \( p \) and thus the sum \( \sum_{J \ni p} \text{dgm}(M')(J) \) is zero.

We defer the proof of Proposition 3.2 to the end of this section.

**Remark 3.3.** In Proposition 3.2, the equation for \( \beta_1(M) \) with respect to \( \text{dgm}(M') \) can be further simplified by removing the fourth, sixth, and seventh sums, i.e.

\[
\beta_1(M) = \sum_{J \ni p - e_1} \text{dgm}(M')(J) + \sum_{J \ni p - e_2} \text{dgm}(M')(J) + \sum_{J \ni p - e_1, p - e_2} \text{dgm}(M')(J) + \sum_{J \ni p} \text{dgm}(M')(J).
\]

This is because the connected sets \( J \) (Definition 2.2 (iii)) over which the fourth, sixth, and seventh sums are taken cannot be intervals of \([m] \times [n]\). Similarly, if \( \text{dgm}(M')(J) = 0 \) for all non-intervals \( J \in \text{Con}([m] \times [n]) \), then the fourth, sixth, and seventh sums can be eliminated in the equation for \( \beta_1(M) \).\(^6\)

By virtue of Remark 3.3, Proposition 3.2 admits a simple pictorial interpretation which generalizes Remark 2.8 (ii) and Theorem 2.9. To state this interpretation, we introduce the following notation.

**Notation 3.4.** Given any \( I \in \text{Con}(\mathbb{Z}^2) \), let \( I^+ \subset \mathbb{R}^2 \) be the corresponding region (cf. equation (2)). Then \( I^+ \) admits the 3 types of corner points depicted in Figure 4. For \( j = 0, 1, 2 \), we define functions \( \tau_j(I^+) : \mathbb{Z}^2 \to \{0, 1, 2\} \) as follows: for \( j = 0, 2 \), let \( \tau_j(I^+)(p) := 1 \) if \( p \) is a \( j^{th} \) type corner point of \( I^+ \), and 0 otherwise. For \( j = 1 \), let

\[
\tau_1(I^+)(p) := \begin{cases} 
2, & \text{if } p \text{ is a } 1^{st}\text{-type corner point of } I^+ \text{ with multiplicity 2} \\
1, & \text{if } p \text{ is a } 1^{st}\text{-type corner point of } I^+ \text{ with multiplicity 1} \\
0, & \text{otherwise}.
\end{cases}
\]

Our main theorem below says that the bigraded Betti numbers of a given \( \mathbb{Z}^2 \)-module \( M \) encoded by an \((|m| \times |n|)\)-module \( M' \) can be read off from the corner points of the elements in either of

\[\{I^+ \subset \mathbb{R}^2 : \text{dgm}(M')(I) \neq 0\} \text{ and } \{I^+ \subset \mathbb{R}^2 : \text{dgm}(M')(I) \neq 0\}.
\]

**Theorem 3.5.** Assume that a \( \mathbb{Z}^2 \)-module \( M \) is encoded by \( M' : [m] \times [n] \to \text{vec} \). Then, for every \( j = 0, 1, 2 \) and for every \( p \in \mathbb{Z}^2 \), we have

\[
\beta_j(M)(p) = \sum_{I \in \text{Con}([m] \times [n])} \text{dgm}(M')(I) \times \tau_j(I^+)(p).
\]

Also we have:

\[
\beta_j(M)(p) = \sum_{I \in \text{Int}([m] \times [n])} \text{dgm}(M')(I) \times \tau_j(I^+)(p).
\]

\(^5\)For example, when \( j = 0 \), the sum is taken over every \( J \in \text{Con}([m] \times [n]) \) that contains \( p \) and does not contain \( p - e_1 \) and \( p - e_2 \).

\(^6\)We remark that, in general, there can exist a non-interval \( J \in \text{Con}([m] \times [n]) \) where \( \text{dgm}(M')(J) \neq 0 \); see Example A.2 in the appendix.
Figure 4: The three different types of corner points in $I^+ \subset \mathbb{R}^2$ and $J^+ \subset \mathbb{R}^2$. Note that two different 1st type corner points of $J$ are located at $p$. See Definition A.1 for a rigorous description of each of the three types of corner points.

Notice that, by Theorem 2.20, the theorem above is a generalization of Theorem 2.9. We prove Theorem 3.5 at the end of this section.

Example 3.6. Recall that $[3] \times [2] = \{0,1,2,3\} \times \{0,1,2\} \subset \mathbb{Z}^2$ and assume that a $\mathbb{Z}^2$-module $N$ is encoded by the module $N' : [3] \times [2] \to \text{vec}$ depicted in Fig. 1 (A'). Then, Fig. 1 (B') and (C') are explained as follows:

(i) If $J \in \text{Int}([2] \times [3])$ contains any point $p \in [3] \times [2]$ such that $N'(p) = 0$, then $\text{dgm}_i(N')(J) = 0$ by Remarks 2.15 (i) and 2.24.

(ii) Consider $K := \{(0,1),(1,1),(1,0)\} \in \text{Int}([3] \times [2])$ that is depicted in Fig. 5. We claim that for all $J \supseteq K$, $\text{dgm}_i(N')(J) = 0$: By Remarks 2.15 (ii) and 2.24, it suffices to show that
Figure 5: $I_1$, $I_2$, $I_3$, and $I_4$ are the intervals corresponding to Fig. 1 (B').

Figure 6: For $N : \mathbb{Z}^2 \to \text{vec}$ in Example 3.6, a black dot at $p$ indicates $\beta_0(N)(p) = 1$, a red star at $p$ indicates $\beta_1(N)(p) = 1$, and a blue square at $p$ indicates $\beta_2(N)(p) = 1$. For all other $j$ and $p$, $\beta_j(N)(p) = 0$. 
Proposition 3.2. Let \( M \) be any finitely generated \( 2 \)-parameter persistence module. The proofs of Proposition 3.2 and Theorem 3.5 will be used in the proof of Lemma 3.7 below will be used in the proof of \( \lim N'(K) \) is trivial: Note that \( \lim N'|_K \equiv (L, \pi_p)_{p \in K} \), where

\[
L = \{(v_1, v_2, v_3) \in N'(0, 1) \oplus N'(1, 1) \oplus N'(1, 0) : \\
\varphi_{N'}((0, 1), (1, 1))(v_1) = v_2 = \varphi_{N'}((1, 0), (1, 1))(v_3)\}
\]

and \( \pi_p : L \to N'(p) \) are the canonical projections for \( p \in K \). Then, we have:

\[
L = \{(x_1, (x_2, x_3), x_4) \in k \oplus (k^2) \oplus k : x_1 = x_2, \ x_3 = 0, \ x_2 = x_3 = x_4\}
= \{(0, 0, 0, 0)\}.
\]

(iii) We claim that \( \text{dgm}_i(N')(I_1) = \text{dgm}_i(N')(I_2) = 1 \). Fix any \( i \in \{1, 2\} \). By invoking Theorem 2.13, one can check that \( \ker_i(N')(I_1) = \ker_i(N'|_I) = 1 \). Let us observe that any interval \( I \supseteq I_i \) must contain either \( K \) or a point \( p \in [3] \times [2] \) such that \( N'(p) = 0 \). Hence, by Remarks 2.15 (i) and (ii), we have that \( \ker_i(N')(J) = 0 \). Therefore, by Theorem 2.22, we have:

\[
\text{dgm}_i(N')(I_j) = \ker_i(N')(I_j) + \sum_{S \subseteq \operatorname{cov}(I_j)} (-1)^{|S|} \ker_i(N') \{ \bigvee S \}
= 1 + \sum_{S \subseteq \operatorname{cov}(I_j)} (-1)^{|S|} \cdot 0 = 1.
\]

Similarly, one can compute \( \text{dgm}_1(N')(I_3) = 1, \text{dgm}_1(N')(I_4) = -1, \text{dgm}_1(N')(L) = 0 \) for any \( L \in \operatorname{Int([3] \times [2])} \) that has not been considered so far.

Proofs of Proposition 3.2 and Theorem 3.5. Lemma 3.7 below will be used in the proof of Proposition 3.2. Let \( M \) be any finitely generated \( \mathbb{Z}^2 \)-module. For any \( p \in \mathbb{Z}^2 \), consider the subposet \( \{p-e_1 \leq p \geq p-e_2\} \) where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). The restriction of \( M \) to \( \{p-e_1 \leq p \geq p-e_2\} \) is a zigzag module and thus admits a barcode (Theorem 2.4). Let \( n_p \) be the multiplicity of \( \{p\} \) in the barcode of \( M \) and \( m_p \) be the multiplicity of \( \{p\} \) in the barcode of \( M \).

Lemma 3.7 ([49, 50]). Given any finitely generated \( \mathbb{Z}^2 \)-module \( M \), for every \( p \in \mathbb{Z}^2 \), we have:

\[
\beta_j(M)(p) = \begin{cases} 
    n_p & \text{if } j = 0 \\
    n_p - \dim(M(p)) + \dim(M(p-e_1)) + \dim(M(p-e_2)) - \dim(M(p-e_1-e_2)) + m_{p-e_1-e_2} & \text{if } j = 1 \\
    m_{p-e_1-e_2} & \text{if } j = 2.
\end{cases}
\]

A combinatorial proof of this lemma can be found in [49, Corollary 2.3]. This lemma can be also proved by utilizing machinery from commutative algebra as follows (see [30, Section 2.4.3] for details): A finitely generated 2-parameter persistence module \( M \) can equivalently be considered as an \( \mathbb{N}^2 \) graded module over \( k[x_1, x_2] \). The bigraded Betti numbers of \( M \) can be
defined using tensor products, after which Lemma 3.7 follows by tensoring $M$ with the Koszul complex on $x_1$ and $x_2$.

For $p \in \mathbb{Z}^2$, we consider the subposets $\uparrow p := \{p + e_1 > p + e_2\}$ and $\downarrow p := \{p - e_1 < p > p - e_2\}$ of $\mathbb{Z}^2$. Let $\mathbf{M}(\mathbb{Z}^2)$ be the collection of all multisets of subsets of $\mathbb{Z}^2$. Lemma 3.7 implies:

**Corollary 3.8.** Let $M$ be any finitely generated $\mathbb{Z}^2$-module. Then, the two maps $\mathbb{Z}^2 \rightarrow \mathbf{M}(\mathbb{Z}^2)$ given by

$$p \mapsto \text{barc}(M_{\uparrow p}) \quad \text{and} \quad p \mapsto \text{barc}(M_{\downarrow p})$$

determine the bigraded Betti numbers of $M$.

**Proof of Proposition 3.2.** We consider the case $j = 1$, as the other cases are similar. By Lemma 3.7,

$$\beta_1(M)(p) = n_p - \dim(M(p)) + \dim(M(p-e_1)) + \dim(M(p-e_2)) - \dim(M(p-e_1-e_2)) + m_{p-e_1-e_2}. \quad (14)$$

Let $p \in \mathbb{Z}^2$ where $p \notin [m] \times [n]$. Then, we claim that $\beta_1(M)(p) = 0$. This fact can be shown by checking that $0 = n_p = m_{p-e_1-e_2}$ and

$$0 = - \dim(M(p)) + \dim(M(p-e_1)) + \dim(M(p-e_2)) - \dim(M(p-e_1-e_2)).$$

Next, let $p \in [m] \times [n]$. We will now find a formula for each term in the right-hand side (RHS) of Equation (14) in terms of the generalized rank invariant of $M|_{[m] \times [n]} = M'$. Notice that for every $q \in [m] \times [n],

$$\dim(M(q)) = \text{rk}(M')(|q|). \quad (15)$$

Next, consider $M|_{[p-e_1 \leq p \leq p-e_2]}$, which is a zigzag module and thus it is interval decomposable (Theorem 2.4). Recall that $n_p$ is the multiplicity of $\{p\}$ in the barcode of $M|_{[p-e_1 \leq p \leq p-e_2]}$. From Example 2.21, we know that:

$$n_p = \text{rk}(M')(\{p\}) - \text{rk}(M')(\{p-e_1 \leq p\}) - \text{rk}(M')(\{p \geq p-e_2\}) + \text{rk}(M')(\{p-e_1 \leq p \geq p-e_2\}). \quad (16)$$

Similarly, we have:

$$m_{p-e_1-e_2} = \text{rk}(M')(\{p-e_1-e_2\}) - \text{rk}(M')(\{p-e_1-e_2 \leq p-e_1\}) - \text{rk}(M')(\{p-e_1-e_2 \leq p-e_2\}) + \text{rk}(M')(\{p-e_1 \geq p-e_1-e_2 \leq p-e_2\}). \quad (17)$$

Combining equations (14), (15), (16), (17) yields:

$$\beta_1(M)(p) = - \text{rk}(M')(\{p-e_1 \leq p\}) - \text{rk}(M')(\{p \geq p-e_2\}) + \text{rk}(M')(\{p-e_1 \leq p \geq p-e_2\}) + \text{rk}(M')(\{p-e_1 \leq p \leq p-e_2\}) - \text{rk}(M')(\{p-e_1-e_2 \leq p-e_2\}) + \text{rk}(M')(\{p-e_1 \geq p-e_1-e_2 \leq p-e_2\}).$$

Since $p \in [m] \times [n]$, by invoking Equation (4), we obtain:

$$\beta_1(M)(p) = - \sum_{J \ni p-e_1} dgm(M')(J) - \sum_{J \ni p-e_2} dgm(M')(J) + \sum_{J \ni p-e_1, p-e_2} dgm(M')(J)
+ \sum_{J \ni p-e_2} dgm(M')(J) - \sum_{J \ni p-e_1-e_2, p-e_1} dgm(M')(J)
- \sum_{J \ni p-e_1-e_2, p-e_2} dgm(M')(J) \quad (18)$$

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Let $J \in \text{Con}([m] \times [n])$. The multiplicity of $\text{dgm}(M')(J)$ in the RHS of Equation (18) is fully determined by the intersection of $J$ and the four-point set $\{p - e_1 - e_2, p - e_1, p - e_2, p\}$. For example, if $p - e_1, p - e_1 - e_2 \in J$ and $p - e_2 \notin J$, then $\text{dgm}(M')(J)$ occurs only in the fourth and sixth sums, and has an overall multiplicity of zero in the RHS of Equation (18). For another example, if $p - e_1 \in J$ and $p - e_1 - e_2, p - e_2 \notin J$, then $\text{dgm}(M')(J)$ occurs only in the fourth summand, which yields the first sum of the RHS in Equation (19) below. Considering all possible $2^4$ combinations of the intersection of $J$ and the four-point set $\{p - e_1 - e_2, p - e_1, p - e_2, p\}$ yields

$$\beta_1(M)(p) = \sum_{J \ni p - e_1} \sum_{J \ni p - e_2} \sum_{J \ni p - e_1 - e_2} \sum_{J \ni p} \text{dgm}(M')(J).$$

as claimed.

\[\text{Proof of Theorem 3.5.}\] We only prove equation (12) with $j = 1$, as the other cases are similar. By Remark 3.3,

$$\beta_1(M)(p) = \sum_{J \ni p - e_1} \sum_{J \ni p - e_2} \sum_{J \ni p - e_1 - e_2} \sum_{J \ni p} \text{dgm}(M')(J),$$

where each sum is taken over $J \in \text{Int}(\mathbb{Z}^2)$.

Furthermore, for $I \in \text{Int}(\mathbb{Z}^2)$, observe that $\tau_1(I^+) = 1$ if and only if one of the following is true about $I$: (i) $p - e_1 \in I$ and $p - e_1 - e_2, p - e_2, p \notin I$, (ii) $p - e_2 \in I$ and $p - e_1 - e_2, p - e_1, p \notin I$, (iii) $p - e_1 - e_2, p - e_1, p - e_2 \in I$ and $p \notin I$, or (iv) $p - e_1, p - e_2, p \notin I$ and $p - e_1 - e_2 \notin I$. Otherwise, $\tau_1(I^+)(p) = 0$. These four cases (i),(ii),(iii), and (iv) correspond to the four sums on the RHS of Equation (20) in order, and also correspond to the 1st corner types (i),(ii),(iii), (iv) given in Figure 4. Therefore:

$$\beta_1(M)(p) = \sum_{I \in \text{Int}([m] \times [n])} \text{dgm}(M')(I) \times \tau_1(I^+)(p).$$

\[\square\]

4 Imp possibility of extending Theorem 3.5 to $d$-parameter persistence modules for $d \geq 3$

In this section, we show that Theorem 3.5 cannot be extended to $\mathbb{Z}^d$-modules for $d \geq 3$. This impossibility claim directly follows from:

\[\text{Theorem 4.1.}\] For $d \geq 3$, the generalized persistence diagram does not determine the multigraded Betti numbers of $\mathbb{Z}^d$-modules. In particular, there exists a pair of $\mathbb{Z}^d$-modules that have the same generalized persistence diagram, but different multigraded Betti numbers.
To prove this theorem, we find a pair of $\mathbb{Z}^3$-modules that have the same generalized persistence diagram, but different multigraded Betti numbers. Since any $\mathbb{Z}^d$-module $M$ can be trivially extended to the $\mathbb{Z}^{d+1}$-module $M \times 0$, the existence of such a pair proves the claim for arbitrary $d \geq 3$.

**Proof of Theorem 4.1.** Let $\pi_1, \pi_2 : k^2 \to k$ be the canonical projections onto the first and the second coordinates of $k^2$ respectively. Let $M$ be the $\mathbb{Z}^3$-module described as follows: Over the subset $\{(x, y, z) \in \mathbb{Z}^3 : 0 \leq x, y, z \leq 1\} \subset \mathbb{Z}^3$, $M$ is given by

\[
\begin{array}{c}
(0,0,0) \\
(0,0,1) \\
(0,1,0) \\
(1,0,0) \\
(1,1,0) \\
(1,1,1) \\
\end{array}
\]

and $M(p) = 0$ for $p \notin \{(x, y, z) \in \mathbb{Z}^3 : 0 \leq x, y, z \leq 1\}$. In the poset $\mathbb{Z}^3$, consider the intervals

$I_0 := \{(0,0,0)\}, \quad I_1 := I_0 \cup \{(1,0,0)\}, \quad I_2 := I_0 \cup \{(0,1,0)\}, \quad I_3 := I_0 \cup \{(0,0,1)\}.$

Combine the interval modules $V_{I_i}$ for $i = 0, 1, 2, 3$ and $M$ into the modules

$N_1 := M \oplus V_{I_0}$ and $N_2 := V_{I_1} \oplus V_{I_2} \oplus V_{I_3}$.

Since $N_2$ is interval decomposable, by Theorem 2.20, both $\text{dgm}(N_2)$ and $\text{dgm}_1(N_2)$ amount to the barcode of $N_2$. We observe that $\text{rk}(N_1) = \text{rk}(N_2)$, which implies not only $\text{rk}_1(N_1) = \text{rk}_1(N_2)$, but also

$\text{dgm}(N_1) = \text{dgm}(N_2)$ and $\text{dgm}_1(N_1) = \text{dgm}_1(N_2)$.

Now we prove that the multigraded Betti numbers of $N_1$ and $N_2$ do not coincide. Since multigraded Betti numbers are additive (Remark 2.8 (i)), we have that

$\beta_3(N_1)(1,1,1) = \beta_3(M)(1,1,1) + \beta_3(V_{I_0})(1,1,1) \geq \beta_3(V_{I_0})(1,1,1) = 1$,

as seen in Figure 7. However, since $\beta_3(V_{I_1})(1,1,1) = \beta_3(V_{I_2})(1,1,1) = \beta_3(V_{I_3})(1,1,1) = 0$ (also shown in Figure 7), we have that $\beta_3(N_2)(1,1,1) = 0$, completing the proof. \qed

## 5 Conclusions

The formula in Theorem 3.5 for computing the bigraded Betti numbers reinforces the fact that the (Int-)generalized persistence diagram and the *interval decomposable approximation* by Asashiba et al. (Remark 2.19) are a proxy for the “barcode” of $M$ in a novel way. Some open questions follow.

(i) Note that when $M$ is a finitely generated $\mathbb{Z}^2$-module, $\text{dgm}(M)$ can recover $\text{dgm}_1(M)$ by construction while $\text{dgm}_1(M)$ may not be able to recover $\text{dgm}(M)$. However, if $M$ is interval decomposable, then both $\text{dgm}(M)$ and $\text{dgm}_1(M)$ are equivalent to the barcode of $M$ by Theorem 2.20. Are there other settings in which $\text{dgm}_1(M)$ can recover $\text{dgm}(M)$?
(ii) How can we utilize Theorem 3.5 alongside efficient algorithms for computing the bi-
graded Betti numbers \([43, 37]\), as a means to estimate or calculate the generalized per-
sistence diagram of a 2-parameter persistence module (cf. Remark 1.1)?

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A Appendix

Types of corner points. We define the 0th, 1st and 2nd type corner points that are depicted in Fig. 4. Given any \( A \subset \mathbb{R}^2 \), let \( \mathbb{1}_A : \mathbb{R}^2 \to \{0, 1\} \) be the indicator function of \( A \), i.e. \( \mathbb{1}_A(p) = 1 \) if \( p \in A \) and zero otherwise.

Definition A.1. Let \( I \in \text{Con}(\mathbb{Z}^2) \) and let \( I^+ := \bigcup_{(p_1, p_2) \in I} [p_1, p_1 + 1) \times [p_2, p_2 + 1) \subset \mathbb{R}^2 \). Fix \( p \in \mathbb{R}^2 \). This \( p \) is a 0-th type corner point of \( I^+ \) if

\[
\mathbb{1}_{I^+}(p) = 1, \quad \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) = \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) = \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 0.
\]

The point \( p \) is a 1-st type corner point with multiplicity \( k \) \((k = 1, 2)\) of \( I^+ \) if one of the following two conditions holds:

(i) \((k = 1)\) Either the following is evaluated to be -1

\[
\mathbb{1}_{I^+}(p) - \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) - \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) + \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 1,
\]

(cf. (i)-(iv) in the panel corresponding to the 1-st type in Figure 4), or the following holds:

\[
\mathbb{1}_{I^+}(p) = \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) \neq \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon))
\]

(cf. (v) and (vi) in the panel corresponding to the 1-st type in Figure 4).

(ii) \((k = 2)\) The formula given in (21) is evaluated to be -2 (cf. the point \( p \) in Figure 4).

The point \( p \) is a 2-nd type corner point of \( I^+ \) if

\[
\mathbb{1}_{I^+}(p) - \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) = \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) = 0, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 1.
\]

Definition A.1 is closely related to the differential of an interval introduced in [25].

The generalized persistence diagram is more discriminative than the Int-generalized persistence diagram. We provide a pair of persistence modules that are distinguished by their generalized rank invariants (and hence by their generalized persistence diagrams) but have the same Int-generalized rank invariant (and hence the same Int-generalized persistence diagram).

Example A.2. Let \( M, N : [2]^2 \to \text{vec} \) and \( J \in \text{Con}([2]^2) \) be defined as in Figure 8. Then, \( \text{rk}(M)(J) = 1 \) whereas \( \text{rk}(N)(J) = 0 \); this directly follows from Theorem 2.13. Note also that for all \( I \supseteq J \) in \( \text{Con}([2]^2) \), we have that \( \text{rk}(M)(I) = \text{rk}(N)(I) = 0 \). This implies, by equations (3), (5), (6), that \( \text{dgm}(M)(J) = 1 \neq 0 = \text{dgm}(N)(J) \).

Invoking Theorem 2.13 again, one can check that \( \text{rk}(M)(I) = \text{rk}(N)(I) \) for all \( I \in \text{Int}([2]^2) \), i.e. \( \text{rk}_0(M) = \text{rk}_0(N) \) and thus \( \text{dgm}_0(M) = \text{dgm}_0(N) \).
Figure 8: An illustration for Example A.2 illustrating that, in general, \( \text{dgm}(M) \) is a stronger invariant than \( \text{dgm}_1(M) \).

Limits and colimits. We recall the notions of limit and colimit [44, Chapter V]. In what follows, \( I \) stands for a small category, i.e. \( I \) has a set of objects and a set of morphisms. Let \( C \) be any category.

**Definition A.3** (Cone). Let \( F : I \to C \) be a functor. A cone over \( F \) is a pair \( (L, (\pi_x)_{x \in \text{ob}(I)}) \) consisting of an object \( L \) in \( C \) and a collection \( (\pi_x)_{x \in \text{ob}(I)} \) of morphisms \( \pi_x : L \to F(x) \) that commute with the arrows in the diagram of \( F \), i.e. if \( g : x \to y \) is a morphism in \( I \), then \( \pi_y = F(g) \circ \pi_x \) in \( C \). Equivalently, the diagram below commutes:

\[
\begin{array}{ccc}
F(x) & \overset{F(g)}{\longrightarrow} & F(y) \\
\pi_x & \downarrow & \pi_y \\
L & & \\
\end{array}
\]

A limit of \( F : I \to C \) is a terminal object in the collection of all cones over \( F \):

**Definition A.4** (Limit). Let \( F : I \to C \) be a functor. A limit of \( F \) is a cone over \( F \), denoted by \( \left( \lim \, F, (\pi_x)_{x \in \text{ob}(I)} \right) \) or simply \( \lim \, F \), with the following terminal property: If there is another cone \( \left( L', (\pi'_x)_{x \in \text{ob}(I)} \right) \) of \( F \), then there is a unique morphism \( u : L' \to \lim \, F \) such that \( \pi'_x = \pi_x \circ u \) for all \( x \in \text{ob}(I) \).

It is possible that a functor does not have a limit at all. However, if a functor does have a limit then the terminal property of the limit guarantees its uniqueness up to isomorphism. For
this reason, we sometimes refer to a limit as the limit of a functor. When \( I \) is a finite category and \( \mathcal{C} = \text{vec} \), any functor \( F : I \to \text{vec} \) admits a limit in \( \text{vec} \).

Cocones and colimits are defined in a dual manner:

**Definition A.5 (Cocone).** Let \( F : I \to \mathcal{C} \) be a functor. A cocone over \( F \) is a pair \((C, (i_x)_{x \in \text{ob}(I)})\) consisting of an object \( C \) in \( \mathcal{C} \) and a collection \((i_x)_{x \in \text{ob}(I)}\) of morphisms \( i_x : F(x) \to C \) that commute with the arrows in the diagram of \( F \), i.e. if \( g : x \to y \) is a morphism in \( I \), then \( i_y = i_y \circ F(g) \) in \( \mathcal{C} \), i.e. the diagram below commutes.

\[
\begin{array}{ccc}
F(x) & \xrightarrow{F(g)} & F(y) \\
\downarrow{i_x} & & \downarrow{i_y} \\
C & & C
\end{array}
\]

A colimit of a functor \( F : I \to \mathcal{C} \) is an initial object in the collection of cocones over \( F \):

**Definition A.6 (Colimit).** Let \( F : I \to \mathcal{C} \) be a functor. A colimit of \( F \) is a cocone, denoted by \( \left( \lim_{\longrightarrow} F, (i_x)_{x \in \text{ob}(I)} \right) \) or simply \( \lim_{\longrightarrow} F \), with the following initial property: If there is another cocone \((C', (i'_x)_{x \in \text{ob}(I)})\) of \( F \), then there is a unique morphism \( u : \lim_{\longrightarrow} F \to C' \) such that \( i'_x = u \circ i_x \) for all \( x \in \text{ob}(I) \).

It is possible that a functor does not have a colimit at all. However, if a functor does have a colimit then the initial property of the colimit guarantees its uniqueness up to isomorphism. For this reason, we sometimes refer to a colimit as the colimit of a functor. When \( I \) is a finite category and \( \mathcal{C} = \text{vec} \), any functor \( F : I \to \text{vec} \) admits a colimit in \( \text{vec} \).

**Multirank invariant.** We review the notion of the multirank invariant for a persistence module [54], which is a natural generalization of the rank invariant and differs from the generalized rank invariant. Then, we demonstrate that the multirank invariant of a zigzag module \( M \) with a length of 3 completely determines the isomorphism type of \( M \). This fact, along with Corollary 3.8, implies that the multirank invariant of any \( \mathbb{Z}^2 \)-module \( N \) determines the bigraded Betti numbers of \( N \).

Let \( P \) be a poset. For a \( P \)-module \( M \) and any \( s, t \in P \), let us define the map

\[
M(s) \to M(t) = \begin{cases} 
\varphi_M(s, t), & \text{if } s \leq t \\
0, & \text{otherwise.}
\end{cases}
\]  

(22)

**Definition A.7.** For a \( P \)-module \( M \) and finite subsets \( S, T \subset P \), the multirank from \( S \) to \( T \) is defined as

\[
\text{multirk}_M(S, T) := \text{rank} \left( \bigoplus_{s \in S} M(s) \to \bigoplus_{t \in T} M(t) \right),
\]

where the map \( \bigoplus_{s \in S} M(s) \to \bigoplus_{t \in T} M(t) \) is canonically defined by the map given in Equation (22) for each pair of \( s \in S \) and \( t \in T \). The multirank invariant of \( M \) is the map that sends every pair of finite subsets \( S, T \subset P \) to \( \text{multirk}_M(S, T) \).

A list of useful properties of the multirank invariant follows.
**Remark A.8.** Let $M$ be any $\mathbb{P}$-module.

(i) The multirank invariant subsumes the rank invariant: If $s, t \in \mathbb{P}$ with $s \leq t$, then for $S = \{s\}$ and $T = \{t\}$, the rank of $\varphi_M(s, t)$ coincides with $\text{multirk}_M(S, T)$.

(ii) For any $S, T \subseteq \mathbb{P}$ such that there is no pair $(s, t) \in S \times T$ with $s \leq t$ in $\mathbb{P}$, we have that $\text{multirk}_M(S, T) = 0$.

(iii) For any $S \subseteq \mathbb{P}$ and any disjoint pair $T_1, T_2 \subseteq \mathbb{P}$,

$$\text{multirk}_M(S, T_1 \cup T_2) = \text{multirk}_M(S, T_1) + \text{multirk}_M(S, T_2).$$

(iv) For any $T \subseteq S \subseteq \mathbb{P}$,

$$\text{multirk}_M(S, T) = \sum_{t \in T} \dim M_t.$$ 

Let $N$ be another $\mathbb{P}$-module.

(v) For any $S, T \subseteq \mathbb{P}$,

$$\text{multirk}_{M \oplus N}(S, T) = \text{multirk}_M(S, T) + \text{multirk}_N(S, T).$$

(vi) If $\dim(M_p) = \dim(N_p)$ for each $p \in \mathbb{P}$, then item (iv) implies that, whenever $T \subseteq S \subseteq \mathbb{P}$, $\text{multirk}_M(S, T) = \text{multirk}_N(S, T)$.

(vii) By Items (iii) and (vi), if $\dim(M_p) = \dim(N_p)$ for every $p \in \mathbb{P}$, then the multirank invariants of $M$ and $N$ are identical if and only if for every disjoint pair of subsets $S, T \subseteq \mathbb{P}$, $\text{multirk}_M(S, T) = \text{multirk}_N(S, T)$.

The multirank invariant is a complete invariant for zigzag modules of length 3:

**Proposition A.9.** Let $\mathbb{P}$ be any zigzag poset of length 3, and let $M, N$ be any $\mathbb{P}$-modules. If $M$ and $N$ have the same multirank invariant, then $M$ and $N$ are isomorphic.

**Proof.** Assume that $\mathbb{P} = \{\bullet_1 < \bullet_2 < \bullet_3\}$. In this case, the rank invariant of $M$ uniquely determines the isomorphism type of $M$. Since the rank invariant is subsumed by the multirank invariant (Remark A.8 (i)), the claim follows.

Next, assume that $\mathbb{P} = \{\bullet_1 > \bullet_2 < \bullet_3\}$. Since every $\mathbb{P}$-module $M$ is interval decomposable (Theorem 2.4), it suffices to show that the barcode of $M$ can be extracted from the multirank invariant of $M$. The multirank of the interval modules $V_I : \mathbb{P} \rightarrow \text{vec}$ for different pairs of $S, T \subseteq \mathbb{P}$ and for different intervals $I$ are given in the table below.
For example, the entry 1 in the second row and fifth column indicates that for $I = \{\bullet_1, \bullet_2\}$, $\text{multirk}_V(I) = (\{\bullet_1\}, \{\bullet_1\}) = 1$. By Remark A.8 (v), the table above allows us to extract the barcode of $M$ as follows:

(i) From Rows 6 and 7, the multiplicity of $\{\bullet_2, \bullet_3\}$ in $\text{barc}(M)$ equals

$$
\text{multirk}_M(\{\bullet_2\}, \{\bullet_1, \bullet_3\}) - \text{multirk}_M(\{\bullet_2, \{\bullet_1\})
$$

(ii) From Rows 5 and 7, the multiplicity of $\{\bullet_1, \bullet_2\}$ in $\text{barc}(M)$ equals

$$
\text{multirk}_M(\{\bullet_2\}, \{\bullet_1, \bullet_3\}) - \text{multirk}_M(\{\bullet_2, \{\bullet_3\})
$$

(iii) From Rows 5, 6, and 7, the multiplicity of $\{\bullet_1, \bullet_2, \bullet_3\}$ in $\text{barc}(M)$ equals

$$
\text{multirk}_M(\{\bullet_2\}, \{\bullet_3\}) + \text{multirk}_M(\{\bullet_2\}, \{\bullet_1\}) - \text{multirk}_M(\{\bullet_2, \{\bullet_1, \bullet_3\})
$$

(iv) The multiplicity of $\{\bullet_1\}$ in $\text{barc}(M)$ is equal to $\text{multirk}_M(\{\bullet_1\}, \{\bullet_1\})$ (cf. Row 2) minus the sum of the multiplicities of $\{\bullet_1, \bullet_2\}$ and $\{\bullet_1, \bullet_2, \bullet_3\}$ found in the previous two items. The multiplicities of $\{\bullet_2\}$ and $\{\bullet_3\}$ in $\text{barc}(M)$ can be computed in a similar way.

Similarly, when assuming $P = \{\bullet_1 < \bullet_2 > \bullet_3\}$, the barcode of any $P$-module $N$ can be extracted from the multirank invariant of $N$. 

By Corollary 3.8 and Proposition A.9, we have:

**Corollary A.10.** The multirank invariant of a finitely generated $\mathbb{Z}^2$-module $M$ determines the bigraded Betti numbers of $M$.

Since the bigraded Betti numbers do not determine the standard rank invariant of a 2-parameter persistence module [42, Section 1.6], the fact that the multirank invariant subsumes the standard rank invariant (Remark A.8 (i)) implies that the converse of the previous corollary does not hold.
Comparison between the multirank invariant and the generalized rank invariant. We show that neither the generalized rank invariant nor the multirank invariant is a strictly stronger invariant than the other. We do this by providing two pairs of persistence modules: The first pair is distinguishable by their multirank invariants but not by their generalized rank invariants. The second pair exhibits the opposite scenario.

In what follows, let $\mathbb{P} := \{a, b, c, d\}$ be the poset equipped with the partial order $\leq := \{(b, a), (c, a), (d, a)\} \subset \mathbb{P} \times \mathbb{P}$. The Hasse diagram of $\mathbb{P}$ is given below.

Example A.11. A $\mathbb{P}$-module $M$ is given by

$$
\begin{array}{cccc}
i_1 & k^2 & i_1 + i_2 \\
n & n & n & n
\end{array}
$$

where $i_1, i_2 : k \rightarrow k^2$ are the canonical inclusions into the first factor and the second factor of $k^2$, respectively. Consider the interval modules $V_{(a)}$, $V_{(a,b)}$, $V_{(a,c)}$, $V_{(a,d)}$ over $\mathbb{P}$.

It is not hard to verify that the generalized rank invariants of the $\mathbb{P}$-modules $N_1 := M \oplus V_{(a)}$ and $N_2 := V_{(a,b)} \oplus V_{(a,c)} \oplus V_{(a,d)}$ are identical. However, for $S := \{b, c, d\}$ and $T := \{a\}$, we have that

$$\text{multirk}_{N_1}(S, T) = 2 \neq 3 = \text{multirk}_{N_2}(S, T).$$

Example A.12. Consider the $\mathbb{P}$-modules $L_1 := M \oplus V_{\mathbb{P}}$ and $L_2 := V_{(a,b,c)} \oplus V_{(a,c,d)} \oplus V_{(a,b,d)}$. By Theorem 2.13, we have that

$$\text{rk}_{L_1}(\mathbb{P}) = 1 \neq 0 = \text{rk}_{L_2}(\mathbb{P}).$$

On the other hand, we claim that for any pair $S, T \subset \mathbb{P}$,

$$\text{multirk}_{L_1}(S, T) = \text{multirk}_{L_2}(S, T). \quad (23)$$

Since $\dim((L_1)_p) = \dim((L_2)_p)$ for each $p \in \mathbb{P}$, Remark A.8 (vii) implies that one needs to check Equation (23) only for disjoint nonempty sets $S, T \subset \mathbb{P}$. Let $S, T \subset \mathbb{P}$ be nonempty disjoint sets. Note that, unless $a$ belongs to $T$, the both sides of Equation (23) are zero. Therefore, it is only necessary to verify Equation (23) in the case where $S \subset \{b, c, d\}$ and $T = \{a\}$. This verification is straightforward.