TIGHT, NOT SEMI–FILLABLE CONTACT CIRCLE BUNDLES

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Abstract. Extending our earlier results, we prove that certain tight contact structures on circle bundles over surfaces are not symplectically semi–fillable, thus confirming a conjecture of Ko Honda.

1. Introduction

Let $Y$ be a closed, oriented three–manifold. A positive, coorientable contact structure on $Y$ is the kernel $\xi = \ker \alpha \subset TY$ of a one–form $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha$ is a positive volume form on $Y$. The pair $(Y, \xi)$ is a contact three–manifold. In this paper we only consider positive, coorientable contact structures, so we call them simply ‘contact structures’. For an introduction to contact structures the reader is referred to [1], Chapter 8 and [7].

There are two kinds of contact structures $\xi$ on $Y$. If there exists an embedded disc $D \subset Y$ tangent to $\xi$ along its boundary, $\xi$ is called overtwisted, otherwise it is said to be tight. The isotopy classification of overtwisted contact structures coincides with their homotopy classification as tangent two–plane fields [4]. Tight contact structures are much more mysterious, and difficult to classify. A contact structure on $Y$ is virtually overtwisted if its pull–back to some finite cover of $Y$ becomes overtwisted, while it is called universally tight if its pull–back to the universal cover of $Y$ is tight.

A contact three–manifold $(Y, \xi)$ is symplectically fillable, or simply fillable, if there exists a compact symplectic four–manifold $(W, \omega)$ such that (i) $\partial W = Y$ as oriented manifolds (here $W$ is oriented by $\omega \wedge \omega$) and (ii) $\omega|_\xi \neq 0$ at every point of $Y$. $(Y, \xi)$ is symplectically semi–fillable if there exists a fillable contact manifold $(N, \eta)$ such that $Y \subset N$ and $\eta|_Y = \xi$. Semi–fillable contact structures are tight [6] [13]. The converse is known to be false by work of Etnyre and Honda, who recently found two examples of tight but not semi–fillable contact three–manifolds [8]. Nevertheless, all such examples known at present are virtually overtwisted, so it is natural to wonder whether every universally tight contact structure is symplectically semi–fillable.

In this paper we study certain virtually overtwisted tight contact structures discovered by Ko Honda. Denote by $Y_{g,n}$ the total space of an oriented $S^1$–bundle over

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\[ \Sigma_g \] with Euler number \( n \). Honda gave a complete classification of the tight contact structures on \( Y_{g,n} \) [14]. The three–manifolds \( Y_{g,n} \) carry infinitely many tight contact structures up to diffeomorphism. The hardest part of the classification involves two virtually overtwisted contact structures \( \xi_0 \) and \( \xi_1 \), which exist only when \( n \geq 2g \). Honda conjectured that \( \xi_0 \) and \( \xi_1 \) are not symplectically semi–fillable [14].

The main theorem of the present paper extends our earlier results regarding these structures [19], establishing Honda’s conjecture:

Theorem 1.1. For \( n \geq 2g > 0 \), the tight contact structures \( \xi_0 \) and \( \xi_1 \) on \( Y_{g,n} \) are not symplectically semi–fillable.

The proof of Theorem 1.1 consists of two steps. In the first step, we derive a contact surgery presentation for \( \xi_0 \) and \( \xi_1 \) in the sense of [3], and we use it to determine the homotopy type of \( \xi_0 \) and \( \xi_1 \) considered as oriented two–plane fields. This is done in Sections 2 and 3.

In the second step, using specific properties of the \( \text{Spin}^c \) structures \( t_{\xi_i} \) on \( Y_{g,n} \) induced by \( \xi_i \) \((i = 0, 1)\) we generalize a result of the first author [17] so it applies to the situation at hand. Using this generalization together with an analytic computation of Nicolaescu’s [20], we are able to determine the possible homotopy types of a semi–fillable contact structure inducing either \( t_{\xi_0} \) or \( t_{\xi_1} \). This is done in Section 4.

Theorem 1.1 follows immediately from the fact that the two sets of homotopy classes determined in the two steps above have empty intersection.

2. Contact surgery presentations for \( \xi_0 \) and \( \xi_1 \)

A smooth knot \( K \) in a contact three–manifold \((Y, \xi)\) which is everywhere tangent to \( \xi \) is called Legendrian. The contact structure \( \xi \) naturally induces a framing of \( K \) called the contact framing.

Let \( \Sigma_g \) be a closed, oriented surface of genus \( g \geq 1 \), and let \( \pi: Y_{g,n} \to \Sigma_g \) denote an oriented circle bundle over \( \Sigma_g \) with Euler number \( n \). Let \( \xi \) be a contact structure on \( Y_{g,n} \) such that a fiber \( f = \pi^{-1}(s) \subset Y_{g,n} \) \((s \in \Sigma_g)\) is Legendrian. We say that \( f \) has twisting number \( k \) if the contact framing of \( f \) is \( k \) with respect to the framing determined by the fibration \( \pi \). A contact structure on \( Y_{g,n} \) is called horizontal if it is isotopic to a contact structure transverse to the fibers of \( \pi \).

Let \( \zeta \) be a horizontal contact structure on \( Y_{g,2g-2} \) such that a fiber \( f \) of the projection \( \pi \) is Legendrian with twisting number \(-1\) (the existence of such a contact structure is well–known, cf. [10], §1.D). Let \( n \geq 2g \), and view the bundle \( Y_{g,n} \to \Sigma_g \) as obtained by performing a \(-\frac{1}{p+1}\)–surgery, where \( p = n - 2g + 1 \), along the fiber \( f \) of \( \pi: Y_{g,2g-2} \to \Sigma_g \) with respect to the trivialization induced by the fibration \( \pi \).

It was observed by Honda ([14], §5) that there are two possible ways of extending \( \zeta \) from the complement of a standard neighborhood of \( f \) to a tight contact structure on \( Y_{g,n} \). This determines the contact structures \( \xi_0 \) and \( \xi_1 \).
The construction of $\xi_0$ and $\xi_1$ can be viewed as a particular case of a more general construction. In fact, given a Legendrian knot $K$ in a contact three–manifold $(Y, \xi)$ and a rational number $r \in \mathbb{Q}$, it is possible to perform a contact $r$–surgery along $K$ to obtain a new contact three–manifold $(Y', \xi')$ [2, 3]. Here $Y'$ is the three–manifold obtained by a smooth $r$–surgery along $K$ with respect to the contact framing, while $\xi'$ is constructed by extending $\xi$ from the complement of a standard neighborhood of $K$ to a tight contact structure on the glued–up solid torus. Such extension exists once $r \neq 0$. In general there are several ways to extend $\xi$, but up to isotopy there is only one if $r = \frac{1}{k}$, $k \in \mathbb{Z}$, and two if $r = \frac{p}{p+1}$ and $p > 1$, as follows from [2], Propositions 3, 4 and 7. When $r = -1$ the corresponding contact surgery coincides with Legendrian surgery [5, 11, 23]. A simple computation using the fact that the fiber $f$ of $Y_{2g-2}$ has twisting number $-1$ with respect to the contact structure $\zeta$ shows that $\{\xi_0, \xi_1\}$ can be defined as the set of contact structures obtainable by contact $\frac{p}{p+1}$–surgery along $f$.

From now on, we shall indicate a contact $r$–surgery along a Legendrian knot $K$ by writing the coefficient $r$ next to it. Consider the result of performing contact $(-1)$–surgery on the Legendrian knot in standard form in Figure 1 (here we are using the notation of [11], see especially Definition 2.1). Since contact $(-1)$–surgery is equivalent to Legendrian surgery, Figure 1 also represents a Stein four–manifold $W$ with boundary $[11]$. As a smooth four–manifold, $W$ is diffeomorphic to the two–disc bundle $D_{2g-2}$ with Euler number $2g - 2$ over a surface of genus $g$. This can be checked by converting the contact surgery coefficient into the corresponding smooth surgery coefficient e.g. via the formulas found in [11] or [12]. Since by construction the boundaries of Stein four–manifolds come equipped with Stein fillable contact structures, we have a Stein fillable contact structure $\zeta(g)$ on $Y_{2g-2}$, which is tight by [6, 13].

**Lemma 2.1.** The contact structure $\zeta(g)$ is horizontal. Moreover, after an isotopy the map $\pi: Y_{2g-2} \to \Sigma_g$ has a fiber with twisting number equal to $-1$.

**Proof.** The existence of a Legendrian knot isotopic to a fiber with twisting number $-1$ is apparent from Figure 1. On the other hand, contact $(-1)$–surgery on a Legendrian knot isotopic to a fiber and having twisting number $\geq 0$ would result in a Stein manifold containing a sphere with self-intersection $\geq -1$, contradicting the adjunction inequality for Stein manifolds [18]. By the classification of tight contact structures on $Y_{2g-2}$ with negative twisting number i.e. such that the twisting number of any closed Legendrian curve isotopic to a fiber is $< 0$ ([13], Theorem 2.11), we conclude that the diagram of Figure 1 represents a horizontal contact structure. □

By [3], Proposition 3, any contact $r$–surgery with $r < 0$ is equivalent to a Legendrian surgery along a Legendrian link. Moreover, the set of Legendrian links which correspond to some contact $r$–surgery is determined via a simple algorithm by the Legendrian knot and the continued fraction expansion of $r$. For example, let $K$ be a Legendrian unknot in the standard contact three–sphere with Thurston–Bennequin
invariant equal to $-1$. Then, a contact $-\frac{p}{p-1}$ surgery ($p > 1$) along $K$ is equivalent to Legendrian surgery along one of the Legendrian links in Figure 2. According to [3], Proposition 7, a contact $\frac{p}{p+1}$ surgery on a Legendrian knot $K$ is equivalent.
to a contact $\frac{1}{2}$–surgery on $K$ followed by a contact $-\frac{p}{p-1}$–surgery on a Legendrian push–off of $K$. By [2], Proposition 9, a contact $\frac{1}{2}$–surgery on a Legendrian knot $K$ can be replaced by two contact $(+1)$–surgeries, one on $K$ and the other on a Legendrian push–off of $K$.

This implies that if we perform a Legendrian $\frac{p}{p+1}$–surgery on a Legendrian fiber of $(Y_{g,2g-2}, \zeta(g))$ with twisting number $-1$, the resulting contact structures will have contact surgery presentations obtained by replacing the “dotted ellipse” in Figure 4 with either Figure 3(a) or 3(b). More precisely, we can define $\xi_0$, respectively $\xi_1$, as the contact structure obtained by using Figure 3(a), respectively Figure 3(b).

![Figure 3](image)

**Figure 3.** Pictures to be pasted in Figure 4 to obtain $\xi_0$ and $\xi_1$

### 3. Homotopy classes of $\xi_0$ and $\xi_1$

**Homotopy theory of oriented two–plane fields on three–manifolds.** Let $\Xi_Y$ denote the space of oriented two–plane fields on the closed, oriented three–manifold $Y$. Since a $Spin^c$ structure on a three–manifold can be interpreted as an equivalence class of nowhere vanishing vector fields [22], by taking the oriented normal, a two–plane field $\xi \in \Xi_Y$ naturally induces a $Spin^c$ structure $t_\xi$, which depends only on the homotopy class $[\xi]$. Therefore there is a map $p: \pi_0(\Xi_Y) \to Spin^c(Y)$ defined as $p(\xi) = t_\xi$. It is not difficult to show that, if $Y$ is connected, there is a non–canonical identification of each fiber $p^{-1}(t_\xi)$ with $\mathbb{Z}/d(t_\xi)\mathbb{Z}$, where $d(t_\xi) \in \mathbb{Z}$ is the divisibility of $c_1(\xi) \in H^2(Y; \mathbb{Z})$, and is zero if $c_1(\xi)$ is a torsion element (see, e.g. [11], Proposition 4.1).

When $c_1(\xi)$ is torsion the two–plane fields inducing the same $Spin^c$ structure $t_\xi$ can be distinguished by a numerical invariant. Suppose that $X$ is a compact 4-manifold with $\partial X = Y$, with $X$ carrying an almost–complex structure $J$ whose complex tangents at the boundary form an oriented two–plane field homotopic to $\xi$ on $Y$. Observe that the fact that $c_1(\xi)$ is torsion implies that $c_1^2(X, J) \in \mathbb{Q}$ makes sense.
Theorem 3.1 ([11]). The rational number
\[ d_3(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)) \in \mathbb{Q} \]
depends only on \([\xi]\), not on the almost–complex four–manifold \((X, J)\). Moreover, two two–plane fields \(\xi_1\) and \(\xi_2\) inducing the same \(\text{Spin}^c\) structure with torsion first Chern class are homotopic if and only if \(d_3(\xi_1) = d_3(\xi_2)\). \(\square\)

In the following we shall refer to the invariant \(d_3\) as the three–dimensional invariant.

Attaching two–handles and homotopy invariants. Recall that contact \((-1)\)–surgery, i.e. Legendrian surgery, can be viewed as the result of attaching a symplectic two–handle [23]. In fact, attaching the two–handle to a contact three–manifold \((Y_1, \xi_1)\) gives rise to a cobordism \(W\) between \(Y_1\) and the three–manifold underlying the contact–three–manifold \((Y_2, \xi_2)\) resulting from the three–dimensional contact surgery. Furthermore, \(W\) carries an almost–complex structure whose complex tangent lines at the boundary coincide with \(\xi_1\) and \(\xi_2\) (see e.g. [9]).

In the case of contact \((+1)\)-surgery, there is still a smooth cobordism \(W\) between \(Y_1\) and \(Y_2\). One can easily check the existence of an almost–complex structure \(J\) on the complement of a ball \(B\) in the interior of \(W\), with \(J\) inducing \(\xi_1\) and \(\xi_2\) as tangent complex lines. We define \(q\) to be the three–dimensional invariant of the two–plane field induced by \(J\) on \(\partial B\). Observe that, although \(J\) may not extend to the whole cobordism, \(J\) induces a \(\text{Spin}^c\) structure \(s\) which does extend – uniquely – to \(W\).

Lemma 3.2. The value of \(q\) is \(\frac{1}{2}\).

Proof. Consider an oriented Legendrian unknot \(K\) in the standard contact three–sphere with Thurston–Bennequin invariant equal to \(-1\) and vanishing rotation number. We view the standard contact three–sphere as the contact boundary of the unit ball \(B_1(0) \subset \mathbb{C}^2\). Attach a smooth two–handle \(H_1\) to \(B_1(0)\) with framing \(+1\) with respect to the contact framing. The result is a smooth four–manifold \(X\) diffeomorphic to \(S^2 \times D^2\). The unique \(\text{Spin}^c\) structure on \(B_1(0)\) extends to a \(\text{Spin}^c\) structure \(s\) on \(X\), restricting to \(H_1\) as the \(\text{Spin}^c\) structure defined above. Denote by \(k\) the value of \(c_1(s)\) on a generator of the second homology group of \(X\).

Let \(K'\) be a Legendrian push–off of \(K\), which we may assume disjoint from \(H_1\), and attach a symplectic two–handle \(H_2\) to \(K'\) realizing Legendrian surgery on \(K'\). The \(\text{Spin}^c\) structure \(s\) extends over \(H_2\), and the value of its first Chern class on the homology generator corresponding to \(K'\) is \(0\), because \(K'\) has vanishing rotation number (see [11], especially the proof of Proposition 2.3). By [2], Proposition 8, the resulting contact three–manifold is just the standard contact three–sphere. Its three–dimensional invariant \(d_3\) is \(-\frac{1}{2}\), but when viewed as the result of the above construction, \(d_3\) can also be expressed as \(\frac{1}{4}(2k^2 - 4) + q\).

We can generalize this argument using Legendrian push–offs \(K_1, K'_1, \ldots, K_n, K'_n\) of \(K\) by performing contact \((+1)\)–surgeries on \(K_1, \ldots, K_n\) and contact \((-1)\)–surgeries
The resulting contact three–manifold is the standard contact three–sphere again. A homological computation as before gives the identity

\[ \frac{1}{4}(n+1)(k^2n-2)+nq=-\frac{1}{2}, \]

which must hold for all \( n \in \mathbb{N} \). This implies that \( k = 0 \) and \( q = \frac{1}{2} \).

**Spin\(^c\) structures on disc and circle bundles.** Let \( D_{g,n} \) be the oriented disc bundle with Euler number \( n \) over a closed oriented surface of genus \( g \). By e.g. fixing a metric on \( D_{g,n} \) one sees that the tangent bundle of \( D_{g,n} \) is isomorphic to the direct sum of the pull–back of \( T\Sigma_g \) and the vertical tangent bundle, which is isomorphic to the pull–back of the real oriented two–plane bundle \( E_{g,n} \to \Sigma_g \) with Euler number \( n \). In short, we have

\[ TD_{g,n} \cong \pi^*(T\Sigma_g \oplus E_{g,n}). \tag{1} \]

This splitting of \( TD_{g,n} \) naturally endows \( D_{g,n} \) with and almost–complex structure which induces a \( Spin\(^c\) \) structure \( s_0 \) on \( D_{g,n} \). The orientation on \( D_{g,n} \) determines an isomorphism \( H^2(D_{g,n};\mathbb{Z}) \cong \mathbb{Z} \), so the set \( Spin\(^c\)(D_{g,n}) = s_0 + H^2(D_{g,n};\mathbb{Z}) \) can be canonically identified with the integers. We denote by \( s_e = s_0 + e \in Spin\(^c\)(D_{g,n}) \) the element corresponding to the integer \( e \in \mathbb{Z} \cong H^2(D_{g,n};\mathbb{Z}) \).

Consider \( Y_{g,n} = \partial D_{g,n} \). We have \( H_1(Y_{g,n};\mathbb{Z}) \cong H^2(Y_{g,n};\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n\mathbb{Z} \), where the summand \( \mathbb{Z}/n\mathbb{Z} \) is generated by the Poincaré dual \( F \) of the class of a fiber of the projection \( \pi: Y_{g,n} \to \Sigma_g \). Each \( Spin\(^c\) \) structure \( s_e \in Spin\(^c\)(D_{g,n}) \) determines by restriction a \( Spin\(^c\) \) structure \( t_e \in Spin\(^c\)(Y_{g,n}) \) with \( t_e = t_0 + eF, \ e \in \mathbb{Z} \). Since \( nF = 0 \), we see that \( t_{e+n} = t_e \) for every \( e \). Therefore, \( t_0, \ldots, t_{n-1} \) is a complete list of torsion \( Spin\(^c\) \) structures on \( Y_{g,n} \), i.e. \( Spin\(^c\) \) structures on \( Y_{g,n} \) with torsion first Chern class. In short, the \( Spin\(^c\) \) structures on \( Y_{g,n} \) which extend to the disc bundle are precisely the torsion ones.

**Homotopy invariants of the contact structures \( \xi_i \).** Let \( W \) be the Stein four–manifold with boundary diffeomorphic to \( D_{g,2g-2} \) as given by Figure 11. Consider the smooth four–dimensional handlebody \( X \) obtained by attaching to \( W \) the two–handles realizing the contact surgeries described in Figure 13(a) or 13(b). Converting the contact framing coefficients into the usual ones, we see that a framed link presentation of \( X \) is obtained by pasting Figure 14(a) in place of the ‘dotted ellipse’ in Figure 11.

By the discussion above on attaching two–handles we know that, corresponding to each of Figure 13(a) and 13(b), there is an almost–complex structure on \( X \) minus two balls lying in the interior of the two–handles realizing the \((+1)\)–surgeries. Moreover, the two almost–complex structures determine the two–plane fields \( \xi_0 \) and \( \xi_1 \) on \( \partial X \) and two \( Spin\(^c\) \) structures \( s_0 \) and \( s_1 \) on \( X \). Observe that, since the rotation number of the Legendrian knot in Figure 11 vanishes, it follows from 11, Theorem 4.12, that \( c_1(W) = 0 \). In the same way, it follows that we can choose an orientation of the \( n-2g \) linking knots with framing \(-3\) in Figure 14(a) so that \( c_1(s_i) \) evaluates as \((-1)^i\) on all the corresponding homology classes. Finally, by the argument given
in the proof of Lemma 3.2, \( c_1(s_i) \) evaluates trivially on the generators of \( H_2(X; \mathbb{Z}) \) determined by the two-handles corresponding to the \((+1)\)-surgeries.

The four-manifold \( X \) is diffeomorphic to \( D_{g,n} \# S^2 \times S^2 \# (n-2g) \mathbb{CP}^2 \). One can see this by performing a sequence of handleslides on the Kirby diagram as shown in Figure 4. In fact, start by sliding over the knot \( K_1 \) in Figure 4(a) the remaining \((n-2g-1)(-3)\)-framed circles. Then, slide \( K_1 \) over \( K_2 \) and finally \( K_2 \) over \( K_3 \), obtaining Figure 4(b). Sliding the long \((2g-2)\)-framed arc over the 2-framed knot and using the 0-framed normal circle to separate the 2-framed circle from the rest of the diagram, we get Figure 4(c). Blowing down the \((-1)\)-circle results in Figure 4(d), and \((n-2g-1)\) further blow downs give Figure 4(e). Following the handle slides of Figure 4 on the homological level we see that \( c_1(s_i) \) evaluates on the generator of the second homology of \( D_{g,n} \) as \((-1)^i(n-2g)\). Moreover, it evaluates as \((-1)^i\) on generators of the \( \mathbb{CP}^2 \) summands, and vanishes when restricted to the \( S^2 \times S^2 \) summand. This immediately implies that the \( Spin^c \) structure \( t_{\xi_i} \) is equal to the restriction of the unique \( Spin^c \) structure \( s_e \in Spin^c(D_{g,n}) \) such that \( c_1(s_e) \) evaluates on the generator of \( H_2(D_{g,n}; \mathbb{Z}) \) as \((-1)^i(n-2g)\). Since the value of \( c_1(s_0) \) on the generator is \( 2 - 2g + n \), \( e \) satisfies the equation:

\[
2 - 2g + n + 2e = (-1)^i(n-2g).
\]

Therefore we get \( e = -1 \) or \( e = 2g - 1 + n \) respectively for \( i = 0 \) or \( i = 1 \). Since \( s_e|_{Y_{g,n}} = t_{e} \), we conclude that

\[
t_{\xi_i} = t_{2i-1}^g
\]

for \( i = 0, 1 \). Observe that this result is consistent with the independent calculation made in [10].

**Lemma 3.3.** The value of the three-dimensional invariant of \( \xi_i \) is

\[
d_3(\xi_i) = \frac{n^2 - 3n + 4g^2}{4n}.
\]

**Proof.** We have \( \chi(X) = n - 4g + 4 \) and \( \sigma(X) = 1 - n + 2g \). From what we know about \( c_1(s_i) \) is easy to deduce that

\[
c_1^2(s_i) = -2 \frac{g(n-2g)}{n}.
\]

In order to compute the three-dimensional invariant we need to take into account the correction term \( q \) for each of the two contact \((+1)\)-surgeries. Using Lemma 3.2 we conclude

\[
d_3(\xi_i) = \frac{1}{4} (c_1^2(s_i) - 2\chi(X) - 3\sigma(X)) + 2 = \frac{n^2 - 3n + 4g^2}{4n}.
\]

\( \square \)
Figure 4. The diffeomorphism between $X$ and $D_{g,n}#S^2 \times S^2 \# (n-2g)\mathbb{CP}^2$

4. The proof of Theorem 4.1

Theorem 4.1. Let $n \geq 2g > 0$, and let $\xi$ be a two-plane field on $Y_{g,n}$ such that $t_\xi \in \{t_{\xi_0}, t_{\xi_1}\}$. If $\xi$ is homotopic to a semi-fillable contact structure, then

$$d_3(\xi) = \frac{n^2 + n + 4g^2}{4n} - 2g - 2.$$
Proof. In the proof of Theorem 2.1 of [17] it is shown that if $Y$ is a closed three–
manifold and $t \in Spin^c(Y)$ is torsion and satisfies:

• all Seiberg-Witten solutions in $t$ are reducible and
• the moduli space of the Seiberg-Witten solutions in $t$ is a smooth manifold
and the corresponding Dirac operators have trivial kernels,

then the expected dimension $d_1$ of the Seiberg-Witten moduli space of solutions
over a potential symplectic semi–filling of $(Y_{g,n}, \xi_i)$ equipped with a cylindrical end
metric and fixed asymptotic limit is equal to $-1 - b_1(Y_{g,n})$.

The moduli space of Seiberg-Witten solutions on $Y_{g,n}$ has been determined in [15]
(see also [21]). These results show that the assumptions listed above hold for the
moduli spaces associated to the $Spin^c$ structures $t_{\xi_i}$. Therefore, the conclusion
$d_1 = -1 - b_1(Y_{g,n})$ holds. This implies that for each $i = 0, 1$, $Y_{g,n}$ carries only
one homotopy type of two–plane field which contains potentially semi–fillable contact structures inducing $t_{\xi_i}$, because $d_1$ is equal to the three–dimensional invariant
plus an expression involving some topological terms and an $\eta$–invariant ([16], For-
numerator(3.1)). In fact, such an expression has been explicitly calculated in [20], in the
formula preceding (3.29), so our proof reduces to translating that formula into our
notations.

In Nicolaescu’s notations the integer $\kappa$ corresponds to $t_{g-1+\kappa}$. This is because his
“base” $Spin^c$ structure is induced by a $Spin$ structure on $Y_{g,n}$ with associated
bundle of spinors $S = \pi^*K^\frac{1}{2}_{\Sigma_g} \oplus \pi^*K^\frac{1}{2}_{\Sigma_g} \to Y_{g,n}$ (see text following Formula (2.6)
in [20]), and $S$ is the restriction of $TD_{g,n} \otimes \pi^*K^\frac{1}{2}_{\Sigma_g} \to D_{g,n}$ to the boundary.

The result we need is obtained by substituting $n$ for $\ell$ and $g$ or $n - g$ in place of $\kappa$
into the formula preceding (3.29) of [20]. (The formula we are using here differs from
Formula (3.29) by the additive term $2g - 1$ because (3.29) computes the dimension
of the whole moduli space rather than the dimension of the moduli space of solutions
with a fixed asymptotic limit, i.e. $d_1$). Explicitly, in our notation we have:

$$-1 - b_1(Y_{g,n}) = d_1 = d_3(\xi) - \frac{1}{2}(2g - 1) - \frac{1}{4}(n - 1) - \frac{\kappa^2}{n} + \kappa$$

where $b_1(Y_{g,n}) = 2g$ and the value of $\kappa$ to be substituted is either $g$ or $n - g$
according to whether $t_{\xi} = t_{\xi_1}$ or $t_{\xi} = t_{\xi_0}$, respectively. In both cases we obtain for
$d_3(\xi)$ the value given in the statement. □

Proof of Theorem 1.1. Let $\xi$ be a two–plane field representing a homotopy class
inducing $t_{\xi}$, which might be represented by a semi–fillable contact structure. Then,
by Theorems 3.3 and 4.1 we have $d_3(\xi_i) - d_3(\xi) = 2g + 1 > 0$. Therefore, the homotopy classes $[\xi_i]$ cannot be represented by semi–fillable contact structures. □

Remarks. (1) For $n < 2g$ the circle bundle $Y_{g,n}$ admits no $Spin^c$ structure for
which the Seiberg-Witten moduli space has the properties required by the proof of
Theorem 4.1.
(2) The assumption \( g > 0 \) in Theorem 4.1 is necessary — \( Y_{0,n} \) is a lens space on which all tight contact structures are Stein fillable. The proof of Theorem 4.1 breaks down since the formula from \([20]\) used in the proof holds only for \( g \geq 1 \).

(3) Notice that for \( n = 2g \) the two contact structures \( \xi_0 \) and \( \xi_1 \) coincide.

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