POSITIVE DEFINABILITY PATTERNS

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Abstract. We reformulate Hrushovski’s definability patterns from the setting of first order logic to the setting of positive logic. Given an h-universal theory $T$ we put two structures on the type spaces of models of $T$ in two languages, $L$ and $L_\pi$. It turns out that for sufficiently saturated models, the corresponding h-universal theories $T$ and $T_\pi$ are independent of the model. We show that there is a canonical model $J$ of $T$, and in many interesting cases there is an analogous canonical model $J_\pi$ of $T_\pi$, both of which embed into every type space. We discuss the properties of these canonical models, called cores, and give some concrete examples.

1. Introduction

In [Hru20b], Hrushovski endows the type spaces of a (universal) first order theory $T$ in a language $L$ with a relational structure (in a new language $L$). This structure is meant to capture what he calls “Definability Patterns”, which are a generalization of definability. For instance, in addition to expressing that a type $p$ is definable — that is we have a formula $\alpha$ such that $\alpha(M) = \{a \in M^p : \varphi(x, a) \in p\}$ — the relations of $L$ can also express the situations where we only have $\alpha(M) \subseteq \{a \in M^p : \varphi(x, a) \in p\}$ rather than equality.

Once these $L$-structures are defined, [Hru20b] looks at them in the context of positive logic (see Section 2 for an overview of positive logic), and deduces three remarkable facts:

1. **Common Theory**: All the type spaces share, as positive structures, a common h-universal theory $T$ (see Definition 2.1).

2. **Universality**: Every model of $T$ admits a homomorphism into every type space $S(M)$ for a model $M$ of $T$.1 This implies that $T$ has a canonical compact positively closed universal model (see Proposition 2.21) with a compact automorphism group. This model is called the core of $T$, and denoted $\text{Core}(T)$ or $J$.

3. **Robinson**: Each $S(M)$ admits a weak form of quantifier elimination called strongly Robinson (see Definition 2.35), which provides a relatively simple description of $\text{Core}(T)$.

Later in [Hru20b], Hrushovski shows that the properties of $J$ reflect those of the original theory $T$ in several ways, and proves some remarkable results based on this construction.

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1In particular positively closed models embed into $S(M)$.

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There is an obvious asymmetry in [Hru20b]: while the construction of $\mathcal{T}$ and $\text{Core}(\mathcal{T})$ happens inherently in the context of positive logic, the original theory $\mathcal{T}$ we start with is just a universal first order theory. One might naturally ask what happens if we try to repeat the construction when $\mathcal{T}$ itself operates in the context of positive logic. This is the question this text answers.

We present multiple ways to reformulate Hrushovski’s definability patterns construction in the context of positive logic. In all cases we start with an h-universal theory $\mathcal{T}$, and consider type spaces over (positively closed) models of $\mathcal{T}$. In the appendix we present the well-known technique of positive Morleyzation, which allows us to apply these constructions to classical first order and continuous logic.

In the main section of the paper, Section 3, we present the definability patterns construction for spaces of maximal positive types, which we denote by $\mathcal{S}(\mathcal{M})$. We present two versions of the construction, using two different languages. The first language $\mathcal{L}$ only contains the definability pattern relations, and we show that facts (1)-(3) above always hold for $\mathcal{L}$ (see Corollary 3.20, Theorem 3.23 and Lemma 3.18 respectively). The second language $\mathcal{L}_\pi$ expands $\mathcal{L}$ to also includes functions for the restriction of a type to a smaller tuple of variables. This may seem more natural since we expect homomorphisms of type spaces to respect these restrictions, and indeed $\mathcal{L}_\pi$-homomorphisms are more related to the original theory $\mathcal{T}$ — specifically, they correspond to certain global types in a saturated model of $\mathcal{T}$ (see, Subsection 3.7). If $\mathcal{T}$ is Hausdorff (see Definition 2.31) then $\mathcal{L}_\pi$ adds no expressive power over $\mathcal{L}$ and the cores for $\mathcal{L}$ and $\mathcal{L}_\pi$ coincide (see Theorem 3.31) — in particular, this is the case for Morley-ized first-order and continuous logic theories.

In general, though, the restriction maps are not well behaved and the facts (1)-(3) above need not hold for $\mathcal{L}_\pi$. Even if $\mathcal{T}$ is not Hausdorff, the weaker condition of being thick (see Definition 2.33) is enough for fact (2) to hold for $\mathcal{L}_\pi$ and thus for the core — which we denote by $\text{Core}_\pi(\mathcal{T})$ — to be well defined. Since every bounded theory is thick, $\text{Core}(\mathcal{T})$ is well-defined whenever $\mathcal{T}$ is bounded. Furthermore, if $\mathcal{T}$ is bounded (see Definition 2.18) and $\mathcal{U}$ is a compact positively closed universal model for $\mathcal{T}$ (see Proposition 2.21), then there is a bijection between $\text{Core}(\mathcal{T})$ and $\mathcal{U}$ that preserves the automorphism group (see Theorem 3.29). In particular, this implies that $\text{Core}_\pi(\text{Th}^{\text{hu}}(\mathcal{J}))$ is well defined when $\mathcal{J}$ is itself the core of some other hu theory and furthermore there is a bijection between $\text{Core}_\pi(\text{Th}^{\text{hu}}(\mathcal{J}))$ and $\mathcal{J}$ preversing the automorphism group.

In Subsection 3.9 we provide a few of examples of $\text{Core}(\mathcal{T})$— specifically, we provide an example (Example 3.48) that demonstrates that $\mathcal{L}$ and $\mathcal{L}_\pi$ are not in general equivalent, and that even in thick theories facts (1) and (3) above may not hold for $\mathcal{L}_\pi$.  

\footnote{We do not know if being thick is a necessary condition for fact (2) to hold for $\mathcal{L}_\pi$.} 

\footnote{Or indeed a universal first order theory as in [Hru20b].}
Finally, for completeness, in Section 4 we apply the definability patterns construction to the spaces of all realized positive types (note that a realized positive type need not be maximal), which we will denote by $S^+ (M)$. While this is not the conventional type space in positive logic, the construction also allows us to replicate facts (1)-(3) (see Theorem 4.7, Theorem 4.9, and Proposition 4.11 respectively). However, Core$(T)$ is in many cases (in particular, for all relational $L$) degenerate — see Subsection 4.5. Note that we present no analogue for $\mathcal{L}_\pi$ in this section.

There are some potential applications for this generalization of the definability patterns construction to positive logic. For instance, as a relatively simple application (generalizing [Hru20b, Corollary A.7]), in Corollary 3.47 we prove (using the Core construction, rather than Core$_\pi (T)$):

Corollary. If $M, N$ are positively $\aleph_0$-saturated and $\aleph_0$-homogeneous pc models of the same hu theory $T$, then the Ellis groups of the actions Aut$(M) \sim S (M)$ and Aut$(N) \sim S (N)$ are isomorphic.

Another candidate application is a generalization of the results of [Hru20a]. We present some background and further details.

Let $G$ be a group. A $k$-approximate subgroup is a set $A \subseteq G$ such that $A^{-1} = A, 1 \in A$ and $A \cdot A$ is covered by $k$ left translates of $A$. In [Hru12, Theorem 4.2], Hrushovski proved a theorem which shows that, under certain amenability conditions on $A$ (in particular for finite $A$) $A$ is commensurable to the preimage under a homomorphism of a compact neighborhood of the identity in some Lie group. This result, which later became known as the Lie Model Theorem, specifically applied to the case of a pseudofinite $A$ by Breuillard, Green and Tao in [BGT12] to give a complete classification of finite approximate subgroups.

In [Hru20a] Hrushovski used the construction of Core$(T)$ in order to prove a more general version of the Lie Model Theorem. This new version applies to any approximate subgroup $A$. When $A$ is arbitrary, one must replace the homomorphism in the theorem with a quasi-homomorphism — that is a function $\phi : G_0 \rightarrow H$ such that $\{ \phi (x) \phi (y) \phi (xy)^{-1} \mid x, y \in G_0 \}$ is contained in a compact set.

In [Fan21], Rodriguez Fanlo generalized the Lie Model Theorem to rough approximate subgroups. It is thus natural to wonder whether the improvements in [Hru20a] and [Fan21] can

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4That is, each is covered by finitely many translates of the other.
5To be precise, the homomorphism is not from $G$ itself but rather from a large subgroup of $G$.
6That is, when $A$ is an ultrapower of finite subsets of a sequence of groups.
7When $A$ does satisfy the conditions of the original theorem, this quasi-homomorphism turns out to actually be a homomorphism.
8An example of a rough approximate subgroup is a metric approximate subgroup — a $\delta, k$ metric approximate subgroup of a metric group $(G, d)$ is a subset $A \subseteq G$ such that $A^{-1} = A, 1 \in A$ and every element of $A \cdot A$ is $\delta$ close to an element in one of $k$ left translations of $A$. 
be combined. The strategy followed in [Fan21] first adapts the results of [Hru12] to the case of hyperdefinable sets, which are quotients of type definable sets by type definable equivalence relations. Since positive logic is the natural syntax to describe hyperdefinable sets, the natural way to improve the results in [Fan21] in a manner similar to [Hru20a] starts by adapting the core construction to positive logic.

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2. Positive Logic — Preliminaries

2.1. Basic Definitions. In this section, we fix some first order language $L$.

Definition 2.1. We denote atomic formulas (that is formulas of the form $R(t_1,\ldots,t_n)$ where $R$ is a relation symbol and each $t_i$ is a term) by (at).

We call a formula positive (p) if it is of the form $\exists \psi(\overline{x},\overline{y})$ where $\psi$ is a positive (that is, the only logical connectors it contains are $\lor$ and $\land$) Boolean combination of atomic formulas.

We call a formula primitive positive (pp) if it is of the form $\exists \varphi_i(\overline{x},\overline{y})$ where each $\varphi_i$ is atomic. Note that every positive formula is equivalent to a disjunction of pp formulas.

We call a formula $h$-universal (hu) if it is equivalent to the negation of a positive formula, that is equivalent to $\forall \overline{x} \neg \psi(\overline{x},\overline{y})$ where $\psi$ is a positive Boolean combination of atomic formulas.

We call a formula primitive $h$-universal (pu) if it is equivalent to a negation of a pp formula, that is of the form $\forall \overline{x} \lor \neg \varphi_i(\overline{x},\overline{y})$ where each $\varphi_i$ is atomic. Note that every hu formula is equivalent to a conjunction of pu formulas.

$\text{tp}^p(a/A)$ means $\{\varphi(x) \in \text{tp}(a/A) \mid \varphi \text{ is positive}\}$, and in general when we add a superscript which denotes a class of formulas, we only consider formulas which belong to that class.

Definition 2.2. If there exists a homomorphism $h: M \to N$, we say that $M$ continues into $N$.

Definition 2.3. An $h$-universal theory is a collection of $h$-universal sentences. Such a theory is called irreducible if for some structure $M$, $T = \text{Th}^\text{hu}(M)$.

We will also call $T$ irreducible if its hu deductive closure is irreducible.

Definition 2.4. Let $\Pi$ be a set of hu formulas (where the set of allowed parameters and variables is understood from context). We define

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9The h stands for homomorphism. hu sentences are pulled back by homomorphisms in the same way that universal formulas are inherited by substructures.
\( \Pi^- = \{ \varphi \mid \varphi \text{ is positive, } \Pi \not\vdash \neg \varphi \} \), that is \( \Pi \) is the set of positive formulas whose negation is not implied by \( \Pi \).

We also denote \( \Pi^* = \Pi \cup \Pi^- \).

**Remark 2.5.** If \( \Pi = T \) is an hu theory then given some structure \( M, T = \text{Th}^\text{hu} (M) \) iff \( M \vDash T^* \).

**Remark 2.6.** If \( \Pi \) is closed under implications, for any positive \( \varphi \) we have \( \varphi \in \Pi^- \) iff \( \neg \varphi \not\in \Pi \), and for any hu \( \varphi \) we have \( \varphi \in \Pi \) iff \( \neg \varphi \not\in \Pi^- \). In particular, if \( a \) is a tuple of elements and \( A \) is a set, \( \text{tp}^\text{hu} (a/A)^- = \text{tp}^\Pi (a/A) \).

### 2.2. pc models.

**Definition 2.7.** Let \( h : M \to N \) be a homomorphism. We say that \( h \) is positively closed (pc) if for any pp (equivalently every positive) formula \( \varphi(x) \), \( N \vDash \varphi(h(a)) \) implies \( M \vDash \varphi(a) \). Note that in particular \( h \) must be an embedding, since every atomic formula is pp. We also call such \( h \) an **immersion**, and say that \( M \) **immerses** into \( N \).

If \( A \subseteq M \) is a substructure, we say that it is a pc **substructure** if \( \text{id}_A : A \to M \) is pc.

We say that \( M \) is a pc model of \( T \) (or just pc, when \( T \) is obvious) if every homomorphism \( h : M \to N \) is pc. If \( T \) is a hu (or pu) theory, and \( C \) is a class of models of \( T \), we say that \( C \) is a **universal class** if for any \( M \vDash T \) continues into a model in \( C \).

**Fact 2.8.** ([PY18 Section 2.3]) The class of pc models of an hu theory \( T \) is universal, that is every model of \( T \) continues into a pc model of \( T \).

**Proposition 2.9.** Let \( T \) be an hu theory. Then the following are equivalent:

1. \( T \) is irreducible.
2. If \( \varphi, \psi \) are hu sentences and \( T \vdash \varphi \lor \psi \) then either \( T \vdash \varphi \) or \( T \vdash \psi \).
3. (JCIF10) For every two models \( M_0, M_1 \) of \( T \), there exists a model \( N \) of \( T \) such that both \( M_0 \) and \( M_1 \) continue into \( N \).

**Proof.** (1) \( \Rightarrow \) (2) Since \( \text{Th}^{\text{pu}} (M) \vdash T \vdash \varphi \lor \psi \) then \( M \vDash \varphi \lor \psi \). Without loss of generality \( M \vDash \varphi \Rightarrow \varphi \in \text{Th}^{\text{pu}} \), and thus \( T \vdash \varphi \).

(2) \( \Rightarrow \) (3) Let \( \{ c_a \mid a \in M_0 \}, \{ d_b \mid b \in M_1 \} \) be new constant symbols. Then it is enough to show that \( \Delta_{M_0} \cup \Delta_{M_1} \cup T \) (where we use \( c_a \) in \( \Delta_{M_0} \) and \( d_b \) in \( \Delta_{M_1} \)) is consistent, since then for \( N \vDash \Delta_{M_0} \cup \Delta_{M_1} \cup T \) we have that \( a \to c_a^N \), \( b \to d_b^N \) are homomorphisms.

Assume otherwise. Then there are conjunctions of atomic formulas \( \varphi(c_\overline{x}) \) and \( \psi(d_\overline{y}) \) such that \( M_0 \vDash \varphi(c_\overline{x}) \), \( M_1 \vDash \psi(d_\overline{y}) \) and \( T \cup \{ \varphi(c_\overline{x}), \psi(d_\overline{y}) \} \) is inconsistent. So let \( \overline{x} \) be a variable tuple of the

\[^{10}\text{Joint Continuation Property}\]
same sort as $\overline{a}$ and let $\overline{y}$ be a variable tuple of the same sort as $\overline{b}$. We have

\[
T \vdash \neg \varphi (\overline{a}) \lor \neg \psi (\overline{b}) \Rightarrow T \vdash \forall \overline{x} \forall \overline{y} (\neg \varphi (\overline{x}) \lor \neg \psi (\overline{y})) \Rightarrow
\]

\[
T \vdash (\forall \overline{x} \neg \varphi (\overline{x})) \lor (\forall \overline{y} \neg \psi (\overline{y}))
\]

By assumption without loss of generality we have $T \vdash \forall \overline{x} \neg \varphi (\overline{x})$ thus $M_0 \models \forall \overline{x} \neg \varphi (\overline{x})$ contradicting $M_0 \models \varphi (\overline{a})$.

(3) $\Rightarrow$ (1) Let $N$ be a pc model of $T$, which exists by Fact 2.8.

We want to show that $T \models \text{Th}^\text{pu}(N)$. Assume $\psi = \forall \overline{x} \neg \varphi (\overline{x})$ where $\varphi$ is a quantifier-free pp formula, and $T \not\vdash \psi$. Then there exists $M = T \cup \{ \neg \psi \}$, and by LS we can take $|M| \leq |T|$.

Then there exists $N' = T$ and $h : M \rightarrow N'$, $g : N \rightarrow N'$ homomorphisms. Then since $M = \exists \overline{x} \varphi (\overline{x})$, let $a \in M^\overline{F}$ be such that $M \models \varphi (a)$.

Since $h$ is a homomorphism, $N' \models \varphi (\varphi (a)) \Rightarrow N' \models \exists \overline{x} \varphi (\overline{x})$ thus since $N$ is pc we have $N \models \exists \overline{x} \varphi (\overline{x})$ so $\psi \notin \text{Th}^\text{pu}(N)$ as required. □

Remark 2.10. If $M$ is a pc model of an irreducible $T$ then $M \models T^*$. Indeed let $\varphi$ a positive sentence such that $T \not\vdash \neg \varphi$. Then let $N_0$ a model of $T \cup \{ \varphi \}$ and let $N \models T$ continuing both $N_0$ and $M$. since $N_0 \models \varphi$ then $N \models \varphi$ (since positive sentences are pushed forward by homomorphisms) but $M$ is immersed in $N$ by assumption thus $M \models \varphi$.

Definition 2.11. Assume we have some hu theory $T$, and $\varphi (x), \psi (x)$ are positive formulas.

We say that $\varphi \perp \psi$ if $T \vdash \forall x (\varphi \land \psi)$.

Fact 2.12. ([PY18, Lemma 2]) If $E$ is a pc model of an hu theory $T$ and $\phi (x)$ is positive formula, $a \in E^x \setminus \phi (E)$, then there exists a positive $\psi (x)$ such that $\psi \perp \phi$ and $E \models \psi (a)$.

Definition 2.13. We endow a pc model $M$ with the topology whose basic closed sets are sets definable (over $M$) with positive formulas, and thus the closed sets are those defined by partial positive types.

We call this the positive topology, or the pp topology.

Claim 2.14. Assume $C$ is a universal class. Let $M \models T$ be such that for any $N$ in $C$ and homomorphism $h : M \rightarrow N$ we have that $h$ is pc. Then $M$ is pc.

Proof. Let $h : M \rightarrow N' \models T$ be a homomorphism. Let $f : N' \rightarrow N$ be a homomorphism for $N$ in $C$.

Then if $\varphi (x)$ is pp and $N' \models \varphi (h (a))$ for $a \in M^x$, then $N \models \varphi (f (h (a)))$ thus by assumption $M \models \varphi (a)$. □

Example 2.15. This holds for example for $C$ the class of pc models of $T$ by Fact 2.8.
Proposition 2.16. If $f_i : M \to N_i$ for $i \in \{1, 2\}$ are immersions then there exist $K$ and homomorphisms $h_i : N_i \to K$ such that $h_1 \circ f_1 = h_2 \circ f_2$. Further we can choose $K$ to be an elementary extension of $M$ in which case we can choose $h_i \circ f_i = \text{id}_M$, or of $N_1$ in which case we can choose $h_1 = \text{id}_{N_1}$.

Proof. Since $f_1, f_2$ are immersion, we may assume without loss of generality that they are the identity. Let $\{e_a\}_{a \in M}$, $\{c_a\}_{a \in N_1}$, and $\{d_a\}_{a \in N_2}$ be new constant symbols. We want to show that

$$\Delta_{N_1} \cup \Delta_{N_2} \cup \Delta_M \cup \{c_a = e_a = d_a\}_{a \in M}$$

is consistent.

Let $\varphi_i (\overline{a}, \overline{b}_i) \ (\overline{b}_i \in N_i \setminus M, \overline{a} \in M)$ be positive quantifier free formulas such that $N_i \models \varphi_i (\overline{a}, \overline{b}_i)$.

Then $N_i \models \exists \overline{b} : \varphi_i (\overline{a}, \overline{b})$ thus as $M$ is pc we have for some $\overline{b}_i^* \in M$ that $M \models \varphi_i (\overline{a}, \overline{b}_i^*)$ and further $N_j \models \varphi_i (\overline{a}, \overline{b}_i^*)$.

We conclude that if we set $c_{a'} = d_{a'} = e_{a'} = a'$ for $a' \in M$, $c_{\overline{b}_1^*} = \overline{b}_1^*$ and $d_{\overline{b}_2^*} = \overline{b}_2^*$ (and set $c_{a'}, d_{a'}$ arbitrarily for all other elements) then

$$M \models \{ \varphi_1 (c, c_{\overline{b}_1^*}) \land \varphi_2 (d_{\overline{b}_1^*}, d_{\overline{b}_2^*}) \} \cup \Delta_M \cup \{ c_a = e_a = d_a \}_{a \in M},$$

is as required for an elementary embedding of $M$ (which, by replacing elements of $K$ by the corresponding elements of $M$, we may take to be the identity). Likewise if we define $c^N_a = a$ for all $a \in N_1$ and $d^N_{\overline{b}_2^*} = \overline{b}_2^*$ we see that the homomorphisms are compatible with $\Delta_{N_1}$ and thus by the same argument we can get $N_1 < K$. \hfill \Box

Corollary 2.17. If $p, q$ are partial positive types over $M$ for $M$ pc then there exists $N$ such that both $p$ and $q$ are realized (though they may not be $\text{tp}^p$ of any element) in $N$. We may also find $N$ such that one (but not both) of $p, q$ is equal to $\text{tp}^p (a/M)$ for some tuple $a$ in $N$.

Therefore by induction the same holds for any finite number of types, and by compactness for any number of types (where in any case one of the types can be chosen to be the positive type of an element).

2.3. Bounded Theories.

Definition 2.18. We call an $\text{hu}$ theory pc **bounded** (or just bounded) if there is a cardinal $\kappa$ such that for any pc model $E$ of $T$ we have $|E| \leq \kappa$.

Definition 2.19. A model $V \models T$ is called **positively $\kappa$-saturated** if whenever:

- $A \subseteq V, |A| < \kappa$,
- $\Sigma (x)$ is a set of positive formulas in a variable tuple $x$ over $A$,
- for any finite $\Sigma_0 \subseteq \Sigma$ there is some $a \in V^x$ such that $V \models \bigwedge_{\varphi \in \Sigma_0} \varphi (a)$

Then $\Sigma$ is realized in $V$.

Remark 2.20. Like with usual saturation, if $V$ is positively $\kappa$-saturated and $\text{Th}^{\text{hu}} (A) = \text{Th}^{\text{hu}} (T)$ then there is a homomorphism from $A$ to $T$, by induction.
Proposition 2.21. Every model of $T$ continues into a positively $\kappa$-saturated pc model of $T$ for every $\kappa$.

Further, if $T$ is irreducible and is bounded by $\kappa_0$ then there exists a unique (up to isomorphism) model $U$ of $T^+$ with the following properties:

1. $U$ is pc.
2. If $h : U \to N$ is an embedding into a model $N$ of $T$, then there is a homomorphism $r : N \to U$ such that $r \circ h = \text{Id}_U$. We call such $r$ a retract.
3. Every model of $T$ continues into $U$. In particular, every pc model of $T$ immerses (in particular embeds) into $U$.
4. $\text{End}(U) = \text{Aut}(U)$, and furthermore every homomorphism from $U$ to a pc model of $T$ is an isomorphism.
5. $U$ is homogeneous for positive types (of finite arity). That is if $a,a'$ are finite tuples in $U$ and $\text{tp}^p(a/\emptyset) = \text{tp}^p(a'/\emptyset)$ then there is $\sigma \in \text{Aut}(U)$ such that $\sigma(a) = a'$.
6. $U$ is positively $\kappa$-saturated for every $\kappa$ (that is every positive partial type over $U$ in any number of variables which is finitely satisfiable in $U$ is realized in $U$).
7. $\text{Aut}(U)$ is compact not only in the product topology, but also in the topology generated by the basic closed sets $C_{\varphi,a,b} := \{ g \in \text{Aut}(U) \mid U \models \varphi(a,g(b)) \}$ for a fixed positive formula $\varphi(x,y)$ and tuples $a,b$.

Definition 2.22. For an irreducible bounded hu theory $T$, we call such a $U$ the universal model of $T$ (note that by (4) any $\kappa_0$-saturated pc model of $T$ is isomorphic to $U$).

Proof. [PY18 2.4] proves the existence of a pc positively $\kappa$ saturated model, and in fact such a model continuing any model of $T$.

Assume $T$ is bounded by $\kappa_0$ and has JCP. Let $U$ be a pc $\kappa_0^+$-saturated model. Since $U$ is pc, by assumption $|U| \leq \kappa_0$.

(2) Let $h : U \to M \models T$ be a homomorphism (thus by pc, an immersion). Then $M$ continues into a pc model $E \models T$. Let $g : M \to E$ be a homomorphism. Since $|E|, |U| \leq \kappa_0$, enumerate $E$ as $\bar{e}$. Then $\{ \varphi(\bar{x},a) \mid \varphi \text{ positive}, a \in U, E \models \varphi(\bar{e},g(h(a))) \}$ is realizable in $U$ (it is finitely satisfiable since $g \circ h$ is a homomorphism and $U$ is pc). In other words, there is an immersion $i : E \to U$ such that $i \circ g \circ h = \text{Id}_U$, that is $r = i \circ g$ is a retract.

(3) Assume $M \models T$. Then by JCP there exists $N \models T$ such that both $M,U$ continue into $N$, and in particular $U$ immerses into $N$. Let $h : M \to N$ a homomorphism, and let $g : N \to U$ a homomorphism (for example a retract, like in (2)). Then $g \circ h : M \to U$ is a homomorphism as required.

(4) Assume $h : U \to U$ is an endomorphism, in particular an immersion, in particular an embedding. Assume $h$ is not surjective and take some $a \in U \setminus h(U)$. For any $b \in U$, $U \not\models a = h(b)$,
thus by Fact 2.12 there exists a positive \( \varphi_b(x,y) \) such that \( \varphi \perp (x = y) \) and \( U \models \varphi_b(a, h(b)) \). Since \( h \) is an immersion, for any \( b_0, \ldots, b_{k-1} \) we have

\[
U \models \exists x \bigwedge_{i<k} \varphi_{b_i}(x, h(b_i)) \Rightarrow U \models \exists x \bigwedge_{i<k} \varphi_{b_i}(x, b_i),
\]

thus \( \{ \varphi_b(x, b) \mid b \in U \} \) is finitely satisfiable — so since \( |U| \leq \aleph_0 \) we have that it is realizable in \( U \).

But for any \( b \in U \), \( U \not\models \varphi_b(b, b) \), contradiction. Thus \( h \) is surjective thus an automorphism.

Let \( h: U \to E \) be a homomorphism to an arbitrary pc model of \( T \). Let \( r: E \to U \) a retract, which exists by (2). Then since \( r \circ h = Id_U \), \( r \) is surjective, and since \( E \) is pc \( r \) is an embedding, so \( r \) is as isomorphism and so is \( h = r^{-1} \circ Id_U = r^{-1} \). In particular, if \( V \) is another universal pc model of \( U \), then \( U \cong V \).

(5) Consider the language \( L_{U,U'} \) which contains two constants \( c_a, d_a \) for any \( a \in U \). Denote by \( \Delta_{U,c}^a \) the atomic diagram of \( U \) with the \( c_a \) constants, and by \( \Delta_{U,d}^a \) the atomic diagram of \( U \) with the \( d_a \) constants. Let \( x \) of finite arity, and assume \( a, b \in U^x \) are such that \( \text{tp}^p(a/\emptyset) = \text{tp}^p(b/\emptyset) \).

Then \( T \cup \Delta_{U,c}^a \cup \Delta_{U,d}^a \cup \{c_a = d_a\} \) is finitely satisfiable in \( U \):

Let \( e \in U^y \) and \( \varphi(x, y) \) a positive qf formula such that \( U \models \varphi(b, e) \). Then \( U \models \exists y \varphi(b, y) \) thus \( U \models \exists y \varphi(a, y) \) thus for some \( f \) such that \( U \models \varphi(a, f) \) we get \( U \) satisfies \( \Delta_{U,c}^a \cup \{\varphi(d_b, d_e)\} \) by setting the \( c \)'s to be the elements they represent, \( d_b = a \) and \( d_e = f \).

So there is a model \( M \models T \) together with two homomorphisms \( h_1, h_2: U \to M \) such that \( h_1(a) = h_2(b) \). Taking a retract \( r \) for \( h_2 \) we get \( r \circ h_1 \in \text{End}(U) = \text{Aut}(U) \) and \( r(h_1(a)) = r(h_2(b)) = b \) as required.

(6) Let \( \Sigma(x) \) a positive partial type in a tuple of some length over \( U \) which is finitely satisfiable in \( U \), in particular consistent with \( \Delta_{U}^a \cup T \). Then there exists \( M \models T \) continuing \( U \) (without loss of generality extending \( U \), since \( U \) is pc) and \( a \in M^x \) such that \( M \models \Sigma(a) \). Let \( r \) a retract for the identity embedding. Then for any \( \varphi(x, b) \in \Sigma \), since \( \varphi \) is positive and \( r \) a homomorphism, \( U \models \varphi(r(a), b) \) and \( r(b) = b \) thus \( U \models \varphi(r(a), b) \) that is \( U \models \Sigma(r(a)) \) as required.

(7) The product topology is generated by \( C_{\varphi,a,b} \) where \( b \) is a 1-tuple, so the defined topology is finer, and thus it is sufficient to show that it is compact. Note that the set of \( C_{\varphi,a,b} \) is certainly closed under finite unions (just take \( \varphi = \bigvee_{i<k} \varphi(x_i, y_i) \) and \( a, b \) to be the concatenation of the \( a_i \)'s and \( b_i \)'s). Assume \( \{C_{\varphi,a,b}\}_{i \in I} \) are basic closed sets with the finite intersection property. Let \( \{c_a\}_{a \in U}, \{d_a\}_{a \in U} \) two new sets of constants and let \( \Delta_{U,c}^a, \Delta_{U,d}^a \) the corresponding atomic diagrams. Consider the positive theory

\[
T \cup \Delta_{U,c}^a \cup \Delta_{U,d}^a \cup \{\varphi(c_a, d_b)\}_{i \in I}.
\]

If \( I_0 \subseteq I \) is a finite subset, let \( g \in \bigcap_{i \in I_0} C_{\varphi,a_i,b_i} \). Then setting \( c_{a_i}^g = a, d_{a_i}^g = d(a) \) we have that the enhanced \( U \) is a model of \( T \cup \Delta_{U,c}^a \cup \Delta_{U,d}^a \cup \{\varphi(c_a, d_b)\}_{i \in I_0} \). Thus there is a model of \( V \) of \( T \cup \Delta_{U,c}^a \cup \Delta_{U,d}^a \cup \{\varphi(c_a, d_b)\}_{i \in I_0} \). Defining \( h(a) = c_a^V, f(a) = d_a^V \) we have \( h, f: U \to V \) are
homomorphism from $U$ to $V$ thus immersions. Let $r : V \to U$ a retract for $h$, which exists by (2). Then $g = r \circ f : U \to U$ is an endomorphism thus an automorphism of $U$ (by (4)), and we find that for any $i$, $V \models \varphi_i(h(a_i), f(b_i)) \Rightarrow U \models \varphi_i(r(h(a_i)), r(f(b_i)))$ that us $U \models \varphi_i(a_i, g(b_i))$ thus $g \in \bigcap_{i \ell} C_{\varphi_i, a_i, b_i}$ as required.

\[ \square \]

\textbf{Corollary 2.23.} The universal model is compact in the positive topology (see Definition 2.13), since this is just $|U|^+$-saturation.

Furthermore, every power $U^I$ of $U$ is not only compact in the product topology, but also compact in the finer topology whose basic closed sets are of the form $\{ a \in U^I | U \models \varphi(a(i_0), ..., a(i_{n-1})) \}$ for $\varphi$ positive.

\textbf{Lemma 2.24.} Let $M$ be a structure in a language $L$, and let $A$ be a model of $\text{Th}^{\text{hu}}(M)$.

Assume there is a compact topology on every sort of $M$ such that for any relation symbol $R(x)$, $R(M)$ is closed inside $M^x$ equipped with the product topology (and for a function symbol, its graph is closed). Then $A$ admits a homomorphism into $M$.

In particular if $A$ is pc, it is embeddable in $M$.

\textit{Proof.} Denote $X = M^A$. $X$ is compact in the product topology. For any relation symbol $R(x)$ and $\overline{a} \in R(A)$ (likewise for a function symbol), denote $X_{R, \overline{a}} = \{ f \in X | f(\overline{a}) \in R(M) \}$. The set of homomorphisms from $A$ to $M$ is $\bigcap_{R, \overline{a}} X_{R, \overline{a}}$.

Define the projection $\pi_{R} : X \rightarrow M^x$ as $\pi_{R}(f) = f(\overline{a})$. Since projections from the product topology are continuous and $R(M)$ is closed, $\pi_{R}^{-1}(R(M)) \subseteq X$ is also closed.

Furthermore for any $\{ R_j \}_{j < k}, \{ \overline{a}_j \}_{j < k}$, since $A \models \exists \overline{y}_j \bigwedge_{j < k} R_j(y_j)$ (where if $\overline{a}_j$ intersects $\overline{a}_{j'}$ we use the same variable) we have also

$$M \models \exists \overline{y}_j \bigwedge_{j < k} R_j(y_j)$$

since this is pp and $\text{Th}^{\text{pp}}(A) \subseteq \text{Th}^{\text{pp}}(M)$ (since the converse holds for all hu sentences, in particular pu sentences). We conclude that $X_{R, \overline{a}}$ have the f.i.p.

Thus from compactness a homomorphism exists. \[ \square \]

\textbf{Corollary 2.25.} Under the conditions of Lemma 2.24 $\text{Th}^{\text{hu}}(M)$ is bounded.

\textit{Proof.} If $E$ is a pc model of $T$ then $E$ embeds into $M$ thus $|E| \leq |M|$. \[ \square \]

2.3.1. Finitely bounded theories.

\textbf{Proposition 2.26.} A finite pc model $E$ of an irreducible hu theory $T$ does not have proper pc substructures.

Thus if $T$ is pc bounded, $U$ is its universal model and $|U| < \kappa_0$ then $U$ is the unique pc model of $T$ up to isomorphism.
Proof. Assume $M \subseteq E$ a proper substructure, and take $a \in E \setminus M$. For any $b \in M$, since $E \models \neg a = b$, there exists a pp positive formula $\varphi_b(x,y) \perp x = y$ such that $E \models \varphi_b(a,b)$. So $E \models \exists x \bigwedge_{b \in M} \varphi_b(x,b)$ but for any $b' \in M$, $M \not\models \varphi_b(b,b)$ thus $M \subseteq E$ is not pc.

Proposition 2.27. If $T$ is bounded and $U$ is its universal model, $\neq$ is positively definable in $U$ iff $U$ is finite. For a multisorted $T$, this holds per sort (that is $x \neq y$ is positively definable iff the sort of $x$ is finite).

Proof. Assume $U$ is finite. For any $a \neq b$ in $U$, let $\phi_{a,b}(x,y) \perp x = y$ be a positive formula such that $U \models \phi_{a,b}(a,b)$. Then $(\neq^U) \subseteq (\bigvee_{a,b} \phi_{a,b}^U) \subseteq (\neq^U)^c = \neg (\neq^U)$ thus inequality is positively definable.

On the other hand, assume $\phi$ is positive such that $\phi^U = (\neg^U)$. Define $\Sigma(x) = \{ \phi(x,a) \}_{a \in U} \cup T \cup \Delta_M^U$; and assume that it is consistent. Then it is realized in some $M \models T$ extending $U$. Let $h : M \rightarrow U$ a retract by Proposition 2.21. We find that $\Sigma$ is also realized in $U$, which is impossible — thus $\Sigma$ is inconsistent. But that means that there exist $a_0,...,a_{n-1}$ such that $U \models \exists x : \bigwedge_{i \leq n} \phi(x,a_i) \iff U \models \exists x : \bigwedge_{i \leq n} x \neq a_i$ — that is $U = \{ a_i \}_{i \leq n}$.

For a multisorted language, we only consider elements of a specific sort.

2.4 Types and Classification of Lu Theories.

Proposition 2.28. Let $T$ be an Lu theory and $M$ be a pc model of $T$.

Assume $p(x)$ is a consistent (with $T \cup \Delta_M^{at}$) set of positive formulas in a variable tuple $x$ over $A \subseteq M$. Then $p$ is maximal (among such sets) iff there exists some pc model $N \models M$ of $T$ and some $a \in N$ such that $p = \text{tp}^p(a/A)$.

Remark 2.29. When $M$ is pc, a set of positive formulas $p$ is consistent with $T \cup \Delta_M^{at}$ iff it is realized in some model of $T$ continuing $M$, iff it is finitely satisfiable in $M$ (since such a continuation would necessarily be an immersion, and thus the satisfiability of any finite subset of $p$ would be pulled back to $M$).

Proof. Assume $p$ is maximal. Let $N' = T$, $M \subseteq N'$ and $b \in N'$, such that $N' \models p(b)$. Let $f : N' \rightarrow N$ a homomorphism for $N$ a pc model of $T$ (see Fact 2.12); without loss of generality, $f|_M = \text{id}_M$ (since $f|_M$ is an embedding). Since homomorphisms preserve positive formulas, $N \models p(f(b))$; and since $p$ is maximal, $p = \text{tp}_p(f(b))$.

Conversely, assume $M \models N$ a pc model of $T$ and $a \in N^x$, and let $p = \text{tp}_p(a/A)$. Let $\varphi(x,b)$ be some positive formula for $b \in A^y$. If $\varphi(x,b) \not\models p$ then $N \not\models \varphi(a,b)$ then by Fact 2.12 there exists some pp formula $\psi(x,b)$ such that $\psi \perp \varphi$ and $N \models \psi(a,b) \Rightarrow \psi(x,b) \not\models p$. We conclude $p \cup \varphi(x,b) \cup T \cup \Delta_M^{at}$ is inconsistent, as required.

Remark 2.30. Essentially the same proof works for a general model $M$, if we replace $M \subseteq N$ with a general homomorphism $h : M \rightarrow N$. 
To be more precise, $p(x)$ over $A \subseteq M$ is maximal for a general model $M$ iff there exists $h : M \to N$ an pc model of $T$ and $a \in N^x$ such that $p = h^* (tp(a/h(A)))$.

**Definition 2.31.** An $h$u theory $T$ is called **Hausdorff** if for any two distinct maximal positive types over the empty set $p(x), q(x)$ (in every tuple of variables $x$) there are positive formulas $\psi, \varphi$ such that $\forall x : \varphi \lor \psi$ holds in every pc model of $T$, and $\varphi \notin p, \psi \notin q$.

$T$ is **semi Hausdorff** if type equality is pp definable; that is for any tuple $x$ there exists a partial positive type $p(x,x')$ (where $x'$ is a tuple of the same sort) such that if $a, b \in M^x$ for some pc model $M$ then $tp^p(a/\emptyset) = tp^p (b/\emptyset)$ iff $M \models p(a,b)$.

**Remark 2.32.** Every Hausdorff theory is semi-Hausdorff with the type

$$\{ \varphi(x) \lor \psi(x') \mid \forall x : \varphi(x) \lor \psi(x) \text{ holds in every pc model of } T \}.$$  

The condition of $T$ being Hausdorff is equivalent to saying that the space of maximal positive types over $\emptyset$ is Hausdorff if endowed with the topology generated by the open sets $\{p \mid \varphi(x) \notin p\}$ for $\varphi$ a positive formula.

**Definition 2.33.** Let $T$ an $h$u theory and $M$ a pc model of $T$, $A \subseteq M$, $I$ a linearly ordered set, and $x$ a variable tuple. A sequence $\langle a_i \rangle_{i \in I}$ of tuples $a_i \in M^x$ is called (positively) **indiscernible** over $A$ if for any $i_0 < \ldots < i_{n-1}$ and $j_0 < \ldots < j_{n-1}$ in $I$ we have $tp^p(a_{i_0}, \ldots, a_{i_{n-1}}/A) = tp^p(a_{j_0}, \ldots, a_{j_{n-1}}/A)$.

A theory $T$ is **thick** if for every variable tuple $x$ there is a partial positive type $p$ in the variables $\langle x_i \rangle_{i < \omega}$ such that for any sequence $\langle a_i \rangle_{i < \omega}$ of $x$ tuples in every pc model of $T$, $\langle a_i \rangle_{i < \omega}$ is indiscernible over $\emptyset$ iff $\langle a_i \rangle_{i < \omega} \models p$.

**Remark 2.34.** Every semi-Hausdorff theory is thick with the type

$$\{ tp^p(x_{i_0}, \ldots, x_{i_{n-1}}/\emptyset) = tp^p(x_{j_0}, \ldots, x_{j_{n-1}}/\emptyset) \mid i_0 < \ldots < i_{n-1} < \omega, j_0 < \ldots < j_{n-1} < \omega \}.$$  

In a bounded theory every infinite indiscernible sequence is constant. Indeed assume $\langle a_i \rangle_{i \in I}$ is a non-constant indiscernible sequence in some pc model $M$ of $T$. Then there is some positive $\psi(x,x') \perp x = x'$ such that $M \models \psi(a_{i_1}, a_{i_2})$ for some, thus every, $i_1 < i_2$ in $I$. Thus for every $\kappa$, the partial positive type in variables $\langle x_i \rangle_{i < \kappa}$ defined as $\{ \psi(x_i, x_j) \mid i < j < \kappa \}$ is finitely satisfiable thus realized in some pc model $N$ of $T$ by some $\langle b_i \rangle_{i < \kappa}$. And now we get that for any $i < j < \kappa$, $\psi(b_i, b_j) \Rightarrow b_i \neq b_j$ and thus $|N^x| \geq \kappa \Rightarrow |N| \geq \kappa$.

Since equality is positively definable, every bounded theory is thick.

**Definition 2.35.** Let $T$ be an $h$u theory.

$T$ has positive quantifier elimination if for every positive formula $\varphi(x)$ over $\emptyset$ there is a quantifier free positive formula $\psi(x)$ over $\emptyset$ such that for any pc model $M \models T$, $\varphi(M) = \psi(M)$. 

T is called (positively) Robinson if for every maximal positive type $p$ (consistent with $T$) over $\varnothing$, there exists an positive quantifier free type $q$ over $\varnothing$ such that in every pc model $M \vDash T$, $p(M) = q(M)$.

$T$ is called (positively) strongly Robinson if the same holds for any partial positive type.

**Remark 2.36.** Quantifier elimination implies strongly Robinson which implies Robinson.

If any of these hold for $T$, then the equivalent requirement holds for types and formulas over any set $A \subseteq M$ in every pc model $M$ of $T$. Indeed assume for example that $T$ is strongly Robinson. Let $p(x)$ some partial positive type over $A$, and for any tuple $a$ from $A$ let $p_a = \{ \varphi(x,y) \mid \varphi(x,a) \in p \}$.

Then for any such $p_a$ there is a quantifier free partial type $q_a$ such that $p_a$ and $q_a$ define the same sets in every pc extension $N$ of $M$, and thus $p(N) = \bigcap_a p_a(N,a) = \bigcap_a q_a(N,a)$, and so $p$ is equivalent to $\bigcup_a q_a(x,a)$.

### 2.5. Examples of $\text{hu}$ Theories.

**Lemma 2.37.** Let $M,N$ be structures in a relational language $L$ and let $h : M \to N$ be an injective homomorphism. Assume that for any finite $A \subseteq M$, any finite $B \subseteq N$ such that $h(A) \subseteq B$ and any finite $L_0 \subseteq L$ there exists an $L_0$ homomorphism $h_B : B \to M$ such that $h_B \circ h |_A = \text{id}_A$. Then $h$ is an immersion.

**Proof.** Let $\varphi(\overline{x},\overline{y})$ be a positive quantifier free formula, take some $\overline{a} \in M^{\overline{x}}$, and assume that $N \vDash \exists \overline{y} \varphi(h(\overline{a}),\overline{b})$. Let $\overline{b} \in N^{\overline{y}}$ be such $N \vDash \varphi(h(\overline{a}),\overline{b})$. Let $A$ be the set of elements in $\overline{a}$, and $B$ the set of elements in $h(\overline{a})$ and $\overline{b}$ together, and let $L_0$ be the set of symbols in $\varphi$. Then by assumption there exists an $L_0$ homomorphism $h_B : B \to M$ such $h_B(h(\overline{a})) = \overline{a}$. We get $B \vDash \varphi(h(\overline{a}),\overline{b})$ thus $M \vDash \varphi(h_B(h(\overline{a})),h_B(\overline{b}))$ and $M \vDash \varphi(\overline{a},h_B(\overline{b}))$ thus $M \vDash \exists \overline{y} \varphi(\overline{x},\overline{y})$.

2.5.1. **Disjoint Subsets.** While in normal first order logic a bound on the size of models implies that all models are finite, bounded $\text{hu}$ theories can have arbitrary bounds, as this simple example shows:

**Example 2.38.** Let $\kappa$ be a (possibly finite) cardinal and $L = \{P_i\}_{i \in \kappa}$, where each $P_i$ is unary.

Consider the theory $T = \{ \forall x \rightarrow (P_i(x) \land P_j(x)) \mid i < j < \kappa \}$ (or its $\text{hu}$ deductive closure).

**Proposition 2.39.** The only pc model of $T$ (up to isomorphism) is $M = \kappa$ with $P_i^M = \{i\}$.

**Proof.** $M$ is certainly a model of $T$.

Every homomorphism $h$ from $M$ to a model of $T$ is an embedding — it is injective since if $c = h(i) = h(j)$ then $P_i(c) \land P_j(c)$ thus $i = j$, and if $P_j(h(i))$ then likewise $(P_i \land P_j)(h(i))$ thus $i = j$. From here it is easy to see that $M$ is pc by Lemma 2.37. Since every homomorphism to

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11The name is taken from [BY03, 2.1].
a model of $T$ is an embedding, we may assume $M \leq N$. If $A \subseteq M, B \subseteq N$ are finite and $A \subseteq B$, $h_B : B \to M$ defined as $\left( \bigcup_{i \in \kappa} P_i(B) \times \{i\} \right) \cup \left( B \setminus \bigcup_{i \in \kappa} P_i(B) \times \{0\} \right)$ is a homomorphism, and it is necessarily over $A$.

Let $N$ be a pc model of $T$. We find that

$$h_N = \left( \bigcup_{i \in \kappa} P_i(N) \times \{i\} \right) \cup \left( N \setminus \bigcup_{i \in \kappa} P_i(N) \times \{0\} \right)$$

is a homomorphism to $M$ thus an immersion to $M$. Thus we get that $N \setminus \bigcup_{i \in \kappa} P_i(N) = \emptyset$, and furthermore for any $i$ we get $|P_i(N)| \leq 1$ by injectivity and also $|P_i(N)| \geq 1$ since $M \models \exists x P_i(x)$. Thus $N \cong M$, and in particular $M$ is also pc.

2.5.2. Directed Acyclic Graphs. This example arises naturally when one wonders what pu sentences exists given a single binary relation other than $=$, and it will also be an example of the sense in which positive logic generalizes first order logic.

Example 2.40. Let $L$ be a language consisting of a single binary relation $E$. Let $T$ be the theory of directed acyclic graphs, that is the hu provable closure of $\{ \forall x_0, \ldots, x_n : \neg (x_n E x_0 \land \land \land \land \land x_{i+1}) \}$ $n \in \omega$.

Proposition 2.41. The set of pc models of $T$ is exactly Mod(DLO).

Proof. Assume that $M$ is a pc model of $T$.

- Transitivity: Let $a, b, c \in M$ be such that $aEbEc$. Define $M'$ to have the same universe as $M$, and $E' = E \cup \{(a, c)\}$. If $M'$ was not a model of $T$, then there was some cycle in $M'$. If that cycle does not involve $aEc$ then we have a cycle in $M$, and if it does then by replacing $aEc$ with $aEbEc$ we have again a cycle in $M$. Thus $M' \models T$ and certainly the identity is a homomorphism thus an embedding, so $(a, c) \in E'$.

- Linearity: Assume $a, b \in M$ distinct. If $(M, E \cup \{(a, b)\})$ is not a model of $T$, then there is a (directed) path in $M$ from $b$ to $a$, thus by transitivity $bEa$.

- No ends: Consider the structure $M'$ with universe $M \cup \{\infty, -\infty\}$ (where $\infty, -\infty$ are new elements) and $E' = E \cup \{\infty\} \cup \{-\infty\} \times M$. Since every new element only ever appears on one side of $E$, we did not add any new cycles. Therefore, $M' \models T$, and thus as for any $a \in M$ we have $M' \models \exists x, y : xEaEy$ we must also have $M \models \exists x, y : xEaEy$ thus $a$ is neither a minimum nor a maximum.

- Density: let $a, b \in M$ be such that $aEb$. Let $M'$ be a structure with universe $M \cup \{c\}$ (where $c$ is a new element) and $E' = E \cup \{(a, c), (c, b)\}$. Any cycle involving $c$ in $M'$ must contain $aEcEb$, but then replacing this sequence with $aEb$ we get a cycle in $M$. Therefore the identity is an immersion, and as $M' \models \exists x : aEaEa$ we have $M \models \exists x : aEaEa$.

We conclude that every pc model of $T$ is a model of DLO.

To show the converse, by Claim 2.14 it is enough to show that every homomorphism from a model of DLO to a pc model of $T$ (which is in particular a model of DLO) is an immersion. But
if $h: M \to N$ is a homomorphism for $M, N \models DLO$ then $h$ is an embedding (since $x = y, x < y, y < x$ partition both $M^2$ and $N^2$) thus from model completeness of $DLO$ $h$ is an elementary embedding, so certainly an immersion.

□

Remark 2.42. It is possible to represent the class of models of every first order theory as the class of pc models of an $hu$ theory, in a manner very similar to the one seen here — though without quantifier elimination, the theory would not be quite this simple. See Appendix A.1 for details.

2.5.3. Unit Circle with a Convergent Sequence. Here is an example of a non-Robinson theory (which is also a more interesting bounded theory).

Example 2.43. Let $(r_n)_{n<\omega} \in [0,1]$ be a sequence such that $r_n \to \pi$, and $\frac{r_n\pi}{\pi}$ is irrational for all $n<\omega$. Consider a structure $M$ with universe $S^1$ in the language $L$ consisting of $I_{a,b}$ for $a,b \in [0,1] \setminus \mathbb{Q}$ such that $I^M_{a,b} = \{e^{2\pi it} | a \leq t \leq b\}$, as well as $S$ where $S^M$ consists of the pairs $\{(-1,1)\} \cup \{(e^{ir_n}, e^{2ir_n})\}_{n<\omega}$.

Then in the usual topology on $M$, every relation is closed and $M$ is compact, thus by Lemma 2.24 every model of $\text{Th}^{\text{hu}}(M)$ admits a homomorphism into $M$.

Claim 2.44. $M$ is pc, thus the universal model of $\text{Th}^{\text{hu}}(M)$.

Proof. Assume $f \in \text{End}(M)$, and take some $t \in (0,1)$. Assume $f(e^{2\pi it}) = e^{2\pi is}$ for $t < s \leq 1$, and let $a, b$ be irrational numbers in $(0,1)$ such that $a < t < b < s$. We have $e^{2\pi i} \in I_{a,b}(M)$ but $f(e^{2\pi it}) = e^{2\pi is} \notin I_{a,b}(M)$ which contradicts the choice of $f$. Likewise, if $s < t < 1$ then it cannot be $f(e^{2\pi it}) = e^{2\pi is}$ thus we have that $f$ fixes every element other than 1. But $(-1, f(1)) \in S$ thus $f(1) = 1$, and so $\text{End}(M) = \{id_M\}$. We conclude by Claim 2.14 that $M$ is pc. □

Proposition 2.45. Denote $N = \{e^{2\pi it} | t \in [0,1] \setminus \mathbb{Q}\}$. Then $N$ is pc model of $\text{Th}^{\text{hu}}(M)$.

Proof. Again, the only homomorphism from $N$ to $M$ is the identity, thus to show $N$ is a pc model of $T$ it is enough to show, by Claim 2.14 that the identity is an immersion.

Let $L_0 \subseteq L$ be a finite sub-language. Take some $\{c_i\}_{i<k}$ in $N$ and $\{d_j\}_{j<m}$ in $M \setminus (N \cup \{\pm 1\})$. Let $\varepsilon > 0$ be the minimal distance from a point of $\overline{D}$ or $-1$ to an endpoint of $I_{a,b} \in L_0$ (note that this is indeed positive by choice of $N$). Choose some $r_n$ such that $d(-1, e^{ir_n}) < \varepsilon$, and let $f: \{c_i\}_{i<k} \cup \{d_j\}_{j<m} \cup \{\pm 1\} \to N$ be the function that: fixes $c_i$, sends each $d_j$ to a point in $N$ which is $\varepsilon$-close to it, sends $-1$ to $e^{ir_n}$, and sends 1 to $e^{2ir_n}$. We get that $f$ is an $L_0$ homomorphism and thus by Lemma 2.37 we are done. □

Corollary 2.46. There is a maximal positive type over a pc model of $\text{Th}^{\text{hu}}(M)$ that is not equivalent to any quantifier free type. In particular, by Remark 2.36 $\text{Th}^{\text{hu}}(M)$ is not Robinson.
Proof. Note that \( p = \text{tp}^{af}(1/N) \) is just \( x = x \cup \Delta_N^M \). Thus \( \text{tp}^f(1/N) \) cannot be equivalent to any quantifier free type \( q \) over \( N \): indeed that would imply \( 1 \in q(M) \) thus \( q \subseteq p \). But then \( M = p(M) \subseteq q(M) \) thus \( q(M) = M \) thus every element of \( M \) realizes \( \text{tp}^f(1/N) \) which is maximal, thus the positive type of every element over \( N \) is the same, which is absurd. \( \square \)

2.5.4. Doubled Interval. This will be our first example of a bounded theory with a big automorphism group, and it will prove to be a useful counterexample.

Example 2.47. Let \( L \) be the language \( \{ I_{a,b} \}_{0 \leq a \leq 1, a, b \in \mathbb{Q}} \cup \{ S \} \), where each \( I_{a,b} \) is unary and \( S \) is binary. Consider the structure \( M \) with universe \( [0,1] \times 2 \) where \( I^M_{a,b} = [a, b] \times 2 \) and \( S^M = \{( (r, i), (r, j)) \mid r \in [0,1], i \neq j \} \).

Proposition 2.48. \( M \) is the universal model of \( T = \text{Th}^{hu}(M) \).

Proof. \( M \) is compact in the usual topology (considering \( M \) as the disjoint union of two intervals) and every relation on \( M \) is closed, thus every model of \( \text{Th}^{pu}(M) \) continues into \( M \) by Lemma 2.24 and thus it is enough, by Claim 2.14, to show that \( \text{End}(M) \) consists only of (self) immersions, and thus it is enough to show that every endomorphism of \( M \) is an automorphism.

If \( f : M \to M \) is an endomorphism then for any \( r \in [0,1] \) and \( i \in \{0,1\} \), \( f((r, i)) = (r, j) \) for some \( j \in \{0,1\} \), since if \( r' < r \) then for some rational \( a \in (r', r) \) we have \((r', 0), (r', 1) \notin I_{a,1} \) and \((r, i) \in I_{a,1} \) and likewise for \( r' > r \). Furthermore, \( f((r, 1 - i)) = (r, 1 - j) \) since \((r, 1 - i) S(r, i) \).

Thus every \( f \in \text{End}(M) \) is of the form \( f((r, i)) = \begin{cases} (r, i) & r \in B \text{ for some } B \subseteq [0,1], \text{ and} \\ ((r, 1 - i)) & r \notin B \end{cases} \) every such function is an automorphism (we get also that \( \text{Aut}(M) \cong (\mathbb{Z}/2\mathbb{Z})^{[0,1]} \) as groups). \( \square \)

Let us classify the pc substructures of \( M \) (that is substructures of \( M \) such that the inclusion is an immersion).

Proposition 2.49. The pc substructures of \( M \) are exactly \( \{B \times 2\}_{Q \cap [0,1] \subseteq B \subseteq [0,1]} \).

Proof. Assume \( A \subseteq M \) is pc. Then for any \( q \in \mathbb{Q} \) we have \( M = \exists x I_{q,q}(x) \) thus \( A \subseteq \{ \{ q \} \times 2 \} \neq \emptyset \); furthermore if \((r, i) \in A \cap (\{ r \} \times 2) \) then \( M = \exists x : S((r, i), x) \) thus \((r, 1 - i) \notin A \).

Conversely assume \( A = B \times 2 \subseteq \mathbb{Q} \cap [0,1] \subseteq B \subseteq [0,1] \). Let \( C \subseteq A, D \subseteq M \) be finite sets such that \( C \subseteq D \) and \( L_0 \subseteq L \) finite. By Lemma 2.37 it is enough to find a homomorphism \( h : D \to A \) over \( C \) without loss of generality, \( (D \setminus C) \cap A = \emptyset \) (since we can replace \( C \) with \( D \cap A \)) and \((D \setminus C) = R \times 2 \) for \( R = \{ r_i \}_{i<k} \). Let \( Q_0 \subseteq \mathbb{Q} \) be the set of endpoints of the \( I_{a,b} \)'s in \( L_0 \), and denote \( \varepsilon = d(R, Q_0) > 0 \). For any \( i < k \) choose \( q_i \in (r_i - \varepsilon, r_i + \varepsilon) \cap \mathbb{Q} \). Then for any \( I_{a,b} \in L_0 \), by choice of \( q_i \) we have \( M = I_{a,b}\{(q_i,j)\} \iff I_{a,b}\{(r_i,j)\} \). Furthermore, \( M = S((r_i,j_1),(r_i,j_2)) \iff M = S((q_i,j_1),(q_i,j_2)) \), and \( S^M \cap C \times D = \emptyset \). Thus \( \text{id}_C \cup \{(r_i,j),(q_i,j)\} \mid i < k, j < 2 \) is a homomorphism from \( D \) to \( A \) as required. \( \square \)
Corollary 2.50. The class of pc models of $T$ is exactly the class of structures isomorphic to some $B \times 2$ for $\mathbb{Q} \cap [0, 1] \subseteq B \subseteq [0, 1]$.

Proof. On one hand, every pc model of $T$ embeds into $M$ and is thus isomorphic to a pc substructure of $M$. On the other hand, by the same reasoning as Proposition 2.48, every homomorphism from a pc substructure $A$ of $M$ to $M$ is (considered as a function to $A$) an automorphism of $A$, thus as a function to $M$ it is a composition of immersions thus an immersion. By Claim 2.14 we conclude that every pc substructure of $M$ is a pc model of $T$. □

2.5.5. Inner Product Spaces.

Example 2.51. This example is taken from Ben Yaacov in [BY03, Example 2.39]; for a complete construction see there.

Let $F = \mathbb{R}$ or $\mathbb{C}$, and let $(H, 0, +, \cdot, (-, -))$ be an infinite dimensional Hilbert space over $F$.

Consider the language $L$ consisting of:

- $I_{\lambda_0, \ldots, \lambda_{n-1}, \mu_0, \ldots, \mu_{n-1}, C, N}$ a relation of arity $n$ for any compact $C \subseteq \mathbb{F}$, and for any $\lambda_0, \ldots, \lambda_{n-1}, \mu_0, \ldots, \mu_{n-1} \in \mathbb{F}$ and $N \in \mathbb{R}_{\geq 0}$.
- $E_m$ a relation of arity $2m$ for any $m < \omega$

We make $H$ into an $L$ structure by defining

$$I^H_{\lambda_0, \ldots, \lambda_{n-1}, \mu_0, \ldots, \mu_{k-1}, C, N} := \left\{ (a_0, \ldots, a_{n-1}) \in H^n \middle| \left( \sum_{i<n} \lambda_i a_i \sum_{j<n} \mu_j a_j \right) \in C \land \sum_{j<n} \|a_j\| \leq N \right\}$$

and

$$E^H_m := \left\{ (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \in H^{2m} \middle| \exists \sigma \in \text{Aut}(H) : \sigma (\vec{a}) = \vec{b} \right\}.$$

Then for $T = \text{Th}_{\text{hu}}(H)$ we have that the class of pc models of $T$ is exactly the class of inner product spaces over $F$ (under the same interpretation of $I$, and type equality in the language consisting only of the $I$’s as an interpretation of $E$) — see [BY03, Remark 2.42].

This $T$ is semi-Hausdorff (with $E$ as a definition of type equality) but not Hausdorff — see [BY03, Example 2.41].

Remark 2.52. A standard formulation of Hilbert spaces as the class of models of a theory uses multisorted continuous logic (one sort for each $B_n(0)$).

Like in the first order case, theories in continuous logic can be emulated by hu theories. The result for the standard formulation of Hilbert spaces will be similar, but not identical, to the example given here (the most obvious difference being that this example is single sorted).

See Appendix A.2 for details.
3. Positive Patterns

3.1. Basic Definitions.

Definition 3.1. Let $M$ be pc model of $T$.

We define a multisorted structure $S(M)$ with a sort for each tuple $x$ of variables where

$$S_x(M) = \{ \text{tp}^p(a/M) \mid a \in N^x, M \leq N, N \text{ a pc model of } T \}$$

$$= \{ p(x) \mid p \text{ a maximal set of positive formulas consistent with } \Delta^1_M \cup T \}$$

with relations we will define shortly.

Note that the equality follows from Proposition 2.28.

Definition 3.2. For any choice of positive formulas $\langle \varphi_i(x,y) \rangle_{i < n}$, and $\alpha(y)$ without parameters, define

$$D^{S(M)}_{\varphi_0,\ldots,\varphi_{n-1};\alpha} = \left\{ (p_0,\ldots,p_{n-1}) \mid \forall c \in \alpha(M) : \bigvee_{i \in n} \varphi_i(x_i,c) \in p_i \right\}.$$ 

Let $L$ be the language containing $D^{S(M)}_{\varphi_0,\ldots,\varphi_{n-1};\alpha}$ for all possible choices of positive formulas $\varphi_0,\ldots,\varphi_{n-1},\alpha$. We usually will consider $S(M)$ as an $L$ structure.

Definition 3.3. For any $x',x$ variable tuples such that $x'$ is a subtuple of $x$, let $\pi^{S(M)}_{x,x'}(p(x))$ be $p|_{x'}$. Let $L_\pi$ be $L \cup \{ \pi_{x,x'} \}_{x' \subseteq x}$.

We will sometimes consider $S(M)$ as an $L_\pi$ structure, but only when explicitly stated.

Remark 3.4. Note that if $x_0,\ldots,x_{n-1}$ are tuples and $x_i'$ is a subtuple of $x_i$ for each $i < n$, then for any relation $D' = D_{\varphi_0(x_0,y),\ldots,\varphi_{n-1}(x_{n-1},y);\alpha(y)}$ on the sorts corresponding to $x_0',\ldots,x_{n-1}'$ we have a corresponding $D' = D_{\varphi_0(x_0,y),\ldots,\varphi_{n-1}(x_{n-1},y);\alpha(y)}$ and we have that by definition $D(p_0,\ldots,p_{n-1})$ iff $D'(\pi_{x_0,x_0'}(p_0),\ldots,\pi_{x_{n-1},x_{n-1}'}(p_{n-1}))$.

Proposition 3.5. If $M \leq N$ (which, since we assumed $M$ is pc, is equivalent to the existence of a homomorphism $h : M \rightarrow N$), then the restriction map $r_M : S(N) \rightarrow S(M)$ defined as $\text{id}^*(p) = \{ \varphi(x,a) \in p \mid a \in M \}$ is an $L_\pi$ homomorphism (in particular an $L$ homomorphism).

Proof. First we show that $r_M(p)$ is indeed in $S(M)$. Take some pc extension $N'$ of $N$ and some $d \in N'^x$ such that $p = \text{tp}^p(d/N)$. Then $r_M(p) = \text{tp}^p(d/M) \in S(M)$.

Furthermore $r_M$ is an $L$ homomorphism: if $(p_0,\ldots,p_{n-1}) \in D^{S(N)}_{\varphi_0,\ldots,\varphi_{n-1};\alpha}$ and $a \in \alpha(M)$ then $a \in \alpha(N)$ thus for some $i < n$ we have $\varphi_i(x_i,a) \in p_i \Rightarrow \varphi_i(x_i,a) \in r_M(p_i)$ that is $(r_M(p_0),\ldots,r_M(p_{n-1})) \in D^{S(M)}_{\varphi_0,\ldots,\varphi_{n-1};\alpha}$.

And $r_M$ is also an $L_\pi$ homomorphism, since

$$\pi_{x,x'}(r_M(p)) = \{ \varphi(x',a) \in p \mid a \in M \} = r_M(\pi_{x,x'}(p)).$$

$\square$
Definition 3.6. Define

\[ T = \bigcup_{M \text{ a pc model of } T} \text{Th}^{hu}(S(M)), \]

\[ \mathcal{T}_\pi = \bigcup_{M \text{ a pc model of } T} \text{Th}^{hu}(S(M))_{\mathcal{L}_\pi}. \]

Claim 3.7. If \( N \) is sufficiently positively saturated (see Proposition 2.21) then \( T = \text{Th}^{hu}(S(N)) \) and likewise for \( \mathcal{T}_\pi \) (in particular if \( T \) is bounded and \( U \) is the universal model, \( T = \text{Th}^{hu}(U) \)).

**Proof.** For any \( \varphi \in T \) let \( M_\varphi \) be such that \( \varphi \in \text{Th}^{hu}(M_\varphi) \), and let \( \kappa = \bigcup_{\varphi \in T} |M_\varphi| \). Then if \( N \) is \( \kappa^* \)-positively saturated, for any \( \varphi \) there is an embedding \( M_\varphi \leq N \); and thus there is by the previous remark an \((\mathcal{L}_\pi, \text{even})\) homomorphism \( r_{M_\varphi} : S(N) \to S(M) \); and thus since \( hu \) sentences are pulled back by homomorphisms we have \( S(N) \models \varphi \).

3.2. The Bounded and Hausdorff Cases.

3.2.1. Bounded Theories.

**Proposition 3.8.** Assume \( T \) is bounded and \( U \) is its universal pc model (see Proposition 2.21).

For any variable tuple \( x \), \( a \mapsto \text{tp}^p(a/U) \) defines a natural bijection from \( U^x \) to \( S_x(U) \).

**Definition 3.9.** Denote this bijection by \( \iota : \bigcup_x U^x \to S(U) \).

**Proof.** Certainly \( \{ \text{tp}^p(a/U) \mid a \in U \} \subseteq S_x(U) \). For any \( p \in S_x(U) \) there exists \( N \) a pc model of \( T \) extending \( U \) and \( a \in N \) such that \( p = \text{tp}^p(a/M) \). By Proposition 2.21.4, the embedding from \( U \) to \( N \) must be an isomorphism thus in particular surjective, so we get that in fact \( a \in U \) thus this map is also surjective.

On the other hand, this map is certainly injective, since \( (x = a) \in \text{tp}^p(b/U) \) iff \( a = b \). \( \square \)

**Definition 3.10.** Given \( \sigma \in \text{End}(U) \), denote \( \sigma^t = \iota \circ \sigma \circ \iota^{-1} : S(U) \to S(U) \) (where we extend \( \sigma \) to every \( U^x \) naturally). Given \( h \in \text{End}_\mathcal{L}(S(U)) \) let \( h_s : U \to U \) be \( h_s(a) = \iota^{-1}(h(\iota(a))) \) (that is \( \iota^{-1} \circ h \circ \iota \) restricted to tuples of length 1).

**Proposition 3.11.** \( \sigma \to \sigma^t \) is an isomorphism from \( \text{Aut}(U) \) to \( \text{End}_{\mathcal{L}_\pi}(S(U)) = \text{Aut}_{\mathcal{L}_\pi}(S(U)) \subseteq \text{Aut}_\mathcal{L}(S(U)) \).

**Proof.** Note first that \( \iota(a_0), \ldots, \iota(a_n-1) \in \mathcal{D}_{\varphi_0, \ldots, \varphi_n, \alpha}(S(U)) \) iff \( \forall y : \alpha(y) \rightarrow \bigvee_{i < n} \varphi(a_i, y) \) holds. Thus the preimage under \( \iota \) of any \( \mathcal{L} \) atomic relation on \( S(U) \) is definable in \( U^{12} \). Furthermore if \( x' \) is a subtuple of \( x \), \( a \in U^x \) and \( a' \) is the corresponding subtuple, then \( \pi_{x,x'}(\iota(a)) = \iota(a') \) (since \( x' = a' \in \text{tp}^p(a/U) = \iota(a) \)). In particular, for every automorphism \( \sigma \) of \( U \) (thus every endomorphism, see Proposition 2.21.4) we have \( \iota \circ \sigma \circ \iota^{-1} \in \text{Aut}_{\mathcal{L}_\pi}(S(U)) \).

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12Note that the defining formula is not \( hu \), but it is equivalent to a Boolean combination of \( hu \) formulas.
Furthermore, \((\iota \circ \sigma \circ \iota^{-1}) \circ (\iota \circ \tau \circ \iota^{-1}) = \iota \circ (\sigma \circ \tau) \circ \iota^{-1}\) thus this is a homomorphism, and \(\iota^{-1} \circ (\iota \circ \sigma \circ \iota^{-1}) \circ \iota = \sigma\) so this is injective.

On the other hand, if \(\varphi(x)\) is a positive \(\varnothing\)-definable relation then \(\varphi(a) \iff D_{\varphi_i}(\iota(a))\) for all \(a \in U^x\). Thus if \(h : S(U) \to S(U)\) is an \(L_\pi\) homomorphism, then

\[
U \models \varphi(a) \Rightarrow S(U) \models \varphi(\iota(a)) \Rightarrow
S(U) \models D_{\varphi_i}(h(\iota(a))) \Rightarrow U \models \varphi(\iota^{-1}(h(\iota(a))))
\]

and

\[
\iota^{-1}(h(\iota(a))) = \iota^{-1}(h(tp^P(a/U)))
\]

is the unique \(b \in U^x\) such that \(x = b \in h(tp^P(a/U))\). So if \(x = (x_0, ..., x_{n-1})\), we find

\[
\pi_{x,x_i}(h(\iota(a))) = h(\pi_{x,x_i}(\iota(a))) = h(\iota(a_i))
\]

and thus \((x_i = \iota^{-1}(h(\iota(a_i)))) \in h(tp^P(a/U))\) so

\[
(\iota^{-1} \circ h \circ \iota)(a_0, ..., a_{n-1}) = \left((\iota^{-1} \circ h \circ \iota) a_0, ..., (\iota^{-1} \circ h \circ \iota) a_{n-1}\right).
\]

Therefore we have

\[
U \models \varphi\left((\iota^{-1} \circ h \circ \iota) a_0, ..., (\iota^{-1} \circ h \circ \iota) a_{n-1}\right)
\]

thus if \(\sigma = h_i, \sigma \in \text{End}(U) = \text{Aut}(U)\). Now since as we said \(\iota^{-1} \circ h \circ \iota\) is uniquely determined by \(h_i, \sigma^i = h, and in particular h \in \text{Aut}(S(M))_{L_\pi}. \)

\(\square\)

**Remark 3.12.** It is not true in general that if \(h\) is merely an \(L\) homomorphism we have

\[
(\iota^{-1} \circ h \circ \iota)(a_0, ..., a_{n-1}) = \left((\iota^{-1} \circ h \circ \iota) a_0, ..., (\iota^{-1} \circ h \circ \iota) a_{n-1}\right)
\]

and consequently \(h_i\) is not necessarily an \(L\)-homomorphism. In Example 3.48 we will see an example of an \(L\)-homomorphism which is non-injective when restricted to \(S_1(U)\), and thus \(h_i\) is necessarily not an \(L\)-automorphism thus not an \(L\)-homomorphism.

**Lemma 3.13.** Let \(T\) be a bounded pu theory, \(U\) its universal model. Take some \(a, b \in U\). Then the following are equivalent:

1. \(tp^\#(\iota(a)) \subseteq tp^\#(\iota(b))\).
2. \(tp^P(a) \subseteq tp^P(b)\).
3. \(tp^P(a) = tp^P(b)\).
4. There exists \(\sigma \in \text{Aut}(U)\) such that \(\sigma(a) = b\).
5. There exists \(\sigma \in \text{Aut}(S(U))\) such that \(\sigma(\iota(a)) = \iota(b)\).
6. There exists \(f \in \text{End}(S(U))\) such that \(f(\iota(a)) = \iota(b)\).
Proof. (5) ⇒ (6) ⇒ (1) is obvious.

(1) ⇒ (2) is from the proof of Proposition 3.11.

(2) ⇒ (3) is from maximality of positive types in pc models (Fact 2.12).

(3) ⇒ (4) is from positive homogeneity of $U$ (Proposition 2.21.5).

(4) ⇒ (5) is also from Proposition 3.11. \qed

**Proposition 3.14.** Assume $U = \text{acl}^P(\emptyset)$, where $\text{acl}^P$ is union of all positively defined algebraic sets. Then for any $h \in \text{End}_L(S(U))$, $h_i \in \text{Aut}(U)$ and $h$ is surjective. In particular, this holds for $U$ which is sortwise finite.

Proof. For any variable tuple $x$, write $U^x = \bigcup_{i \in I} \varphi_i(U)$ where each $\varphi_i$ is an algebraic positive formula over $\emptyset$ (each $\varphi_i(U)$ is a conjuction of positive algebraic sets in each individual variable appearing in $x$). Then for any positive formula (without parameters) $\psi(x)$, $\psi(U) = \bigcup_i (\psi \land \varphi_i)(U)$ thus $\neg\psi$ is equivalent in $U$ to the type $\{\neg (\psi \land \varphi_i)\}_{i \in I}$ where $(\psi \land \varphi_i)(U)$ is finite. Denote $n = |x|.$

Let $y$ another variable tuple in the same sorts as $x$. For any $j < n$ and for any $d, d' \in U^{x_j}$ such that $d \neq d'$, choose by Fact 2.12 some positive $\varepsilon^j_i(d_j, y_j) \perp x_j = y_j$ such that $\varepsilon^j_i(d, d') \perp d, d'$. By slight abuse of notation, denote by $\varepsilon^j_i$ also the formula in $x_j, y$ that requires that $\varepsilon^j_i$ holds for $x_j, y_j$.

Fix $a \in U^x$ and denote by $\overline{a}_i$ the tuple $(\iota(a_0), \ldots, \iota(a_{n-1}))$. Then $a \models \neg (\psi \land \varphi_i)$ implies that for any $b \in (\psi \land \varphi_i)(U)$, $a \neq b$ — that is $\overline{a}_i \models D_{\psi, \varphi_i}$. Thus if $a \models \neg\psi$ then $\overline{a}_i = \{D_{\psi, \varphi_i}\}_{i \in I}$.

Fix an atomic relation (or indeed any positive formula without parameters) $\phi(x)$. Then $a \models \phi$ implies

$$\overline{a}_i \models \Sigma^a_i := \{D_{\psi, \varphi_i} \mid \psi \perp \phi, i \in I\}.$$

Note that for any positive formula $\phi$ and any $a \models \phi$, $\overline{a}_i \models \Sigma^a_i$. Furthermore, for any $\psi \perp \phi$, for any $i$ and for any $c \in \psi \land \varphi_i$ we have that $\overline{c}_i \models D_{\psi, \varphi_i}$ (since for any $j \bigvee \varepsilon^j_i(x_j, y_j) \perp x_j = y_j$, but $x_j = c_j \in \iota(c_j)$) thus $\overline{c}_i \not\models \Sigma^a_i$. Since $\neg\psi(U) = \bigcup \varphi_i(U)$, we get that $\Sigma^a_i(S(U)) \subseteq \{\overline{a}_i \mid a \in \psi(U)\}$. We thus find

$$\{\overline{a}_i \mid a \in \phi(U)\} = \bigcup_{a \in \phi(U)} \Sigma^a_i(S(U)).$$

So every positive definable set is sent via $\iota$ to an infinite positive Boolean combination of atomic definable sets in $L$. This means that for such a $U$, if $h : S(U) \to S(U)$ is an $L$ homomorphism then $h_i : U \to U$ is an $L$ homomorphism.

We will now show that $h$ is surjective. Let $\overline{\pi}_0, \ldots, \overline{\pi}_{k-1}$ enumerate $\varphi_i(U^x)$. Then for $y$ a variable tuple of the same sorts as $x$, $S(U) \models D_{x=y, x=y, \ldots, x=y; \varphi_i(y)}(\iota(\overline{\pi}_0), \ldots, \iota(\overline{\pi}_{k-1}))$. 

**Theorem 3.15.** Assume $U = \text{acl}^P(\emptyset)$, where $\text{acl}^P$ is union of all positively defined algebraic sets. Then for any $h \in \text{End}_L(S(U))$, $h_i \in \text{Aut}(U)$ and $h$ is surjective. In particular, this holds for $U$ which is sortwise finite.
and thus we get
\[ S(U) \vDash D_{x=y, x=y, \ldots, x=y, \varphi}(y) (h(\iota(\overline{a}_0)), \ldots, h(\iota(\overline{a}_{k-1}))). \]
So for any \( l < k \), for some \( j < k \), \((x = \overline{a}_j) \in h(\iota(\overline{a}_j)) \Rightarrow h(\iota(\overline{a}_i)) = \iota(\overline{a}_i)\), thus
\[ S_\pi(U) = \bigcup_{i \in I} \{ \iota(\overline{a}) | \overline{a} \in \varphi_i(U) \} \subseteq \operatorname{Im}(h) \]
so \( h \) is surjective.

\[ \square \]

3.2.2. Hausdorff Theories.

**Lemma 3.15.** In a Hausdorff theory \( T \), \( \pi_{x,x'} \) is equivalent in all type spaces to an infinite intersection of atomic binary \( D \)-relations in \( \mathcal{L} \) (where the equivalence does not depend on the model of \( T \) we look at), thus every \( \mathcal{L} \)-homomorphism between type spaces is an \( \mathcal{L}_\pi \) homomorphism.

**Proof.** Let \( \Sigma \) be the set of pairs of positive formulas \((\psi(x', y), \theta(x', y))\) such that every pc model of \( T \) satisfies \( \forall x', y : \psi \lor \theta \).

\( p \in S_x(M), q \in S_{x'}(M) \). Let \( c \in N^x, d \in N^{x'} \) be such that \( p = \operatorname{tp}^P(c/M) \) and \( q = \operatorname{tp}^P(d/M) \) (for \( N, N' \) pc extensions of \( M \)). Let \( c' \) be the subtuple of \( c \) corresponding to \( x' \).

Assume \( \pi_{x,x'}(p) \neq q \); then from maximality of \( \pi_{x,x'}(p) \) for some \( \varphi(x', a) \) we have that \( N \vDash \varphi(c', a), N' \vDash \varphi(d, a) \).

We conclude that \( \operatorname{tp}^P(c'a) \neq \operatorname{tp}^P(da) \) (since both types are maximal) thus from Hausdorff we have some \((\psi, \theta) \in \Sigma \) such that \( \psi(x', a) \notin p, \theta(x, a) \notin q \).

We conclude \( \neg \mathcal{D}_{\psi, \theta}(p, q) \).

On the other hand if there are some \((\psi, \theta) \in \Sigma \) such that \( \neg \mathcal{D}_{\psi, \theta}(p, q) \) then we have some \( a \in M^y \) such that
\[ \psi(x', a) \notin p \iff N \vDash \psi(c', a) \Rightarrow N \vDash \theta(c', a) \]
\[ \theta(x', a) \notin q \Rightarrow N' \vDash \theta(d, a) \]
and thus \( \pi_{x,x'}(p) \neq q \).

So \( \pi_{x,x'}(\xi) \neq \zeta \iff \bigvee_{\psi, \theta} \neg \mathcal{D}_{\psi, \theta}(\xi, \zeta) \) thus \( \pi_{x,x'}(\xi) = \zeta \iff \bigwedge_{\psi, \theta} \mathcal{D}_{\psi, \theta}(\xi, \zeta) \) as required.

\[ \square \]

In fact, this definability of \( \mathcal{L}_\pi \) is pretty close to \( T \) being Hausdorff; indeed:

**Proposition 3.16.** If \( \pi \) is definable (in every sort) by \( D \)-relations then every \( \pi \) is type definable in \( \mathcal{L} \) as in Lemma 3.15. Furthermore, \( S(M) \) is Hausdorff (in every sort) in the topology generated by the basic closed sets \( [\varphi] = \{ p | \varphi \in p \} \).

**Remark 3.17.** Note that when defining \( \mathcal{L}_\pi \) we did not require strict subotypes, thus \( \pi_{x,x} \in \mathcal{L}_\pi \) for any sort \( x \) (where \( \pi_{x,x}^{S(M)} \) is the identity). Therefore \( \mathcal{L}_\pi \) is definable by \( D \)-relations iff \( = \) is.
Lemma 3.18. Let Robinson.

Proof. The first statement follows from Remark 3.4.

Now by assumption for any variable tuple $x$ there is an intersection $\bigcap_{i \in I} D_{\varphi_i, \psi_i; \alpha}(S(M))$ equal to the diagonal in $S_x(M)^2$. Let $p, q \in S_x(M)$ be distinct types. Then for some $i \in I$, $S(M) \models \neg D_{\varphi_i, \psi_i; \alpha}(p, q)$; that is for some $a \in \alpha(M)$ we have $\varphi_i(x, a) \notin p \iff p \models [\varphi_i(x, a)]^c$ and $\psi_i(x, a) \notin q \iff q \models [\psi_i(x, a)]^c$. But for any $r \in S_x(M)$ we have $D_{\varphi_i, \psi_i; \alpha}(r, r)$ thus in particular $\varphi_i(x, a) \in r$ or $\psi_i(x, a) \in r$, that is $[\varphi_i(x, a)]^c \cap [\psi_i(x, a)]^c = \emptyset$ as required. □

3.3. Robinson.

Lemma 3.18. Let $T$ be an irreducible primitive universal theory.

1. In $S(M)$, every atomic formula is equivalent to an atomic relation; that is if $\varphi(\zeta, \xi)$ is a formula consisting of a single relation symbol, it is equivalent to a binary relation symbol in $\zeta, \xi$ — and likewise for formulas with more parameters.
2. Assume $|M| \geq 2$ and $M$ is pc. Then in $S(M)$, every finite conjunction of atomic formulas (none of which involves $=$) is equivalent to an atomic formula.
3. The family of atomic-type-definable subsets of $S(M)$ is closed under projection on all but one coordinate. Formally, if $A \subseteq S(M)^{k+1}$ for $k \geq 1$ is atomic-type-definable, then so is

$$
\pi_1, \ldots, k(A) = \left\{ (p_1, \ldots, p_k) \in S(M)^k \mid \exists p_0 : (p_0, \ldots, p_k) \in A \right\}.
$$

Furthermore the definition of the projection is independent of $M$, that is for any partial atomic type $\Sigma(x_0, \ldots, x_k)$ there exists $\Pi(x_1, \ldots, x_k)$ such that $\exists x_0 \Sigma$ is equivalent to $\Pi$ in every $S(M)$.

If $T$ is Hausdorff, the same holds for $L_\pi$.

4. Every pp formula $\Xi(\mu)$ is equivalent in $S(M)$ to a possibly infinite (but no larger than $|L| = |L_\pi|$) conjunction of atomic formulas (and if $T$ is Hausdorff, the same holds for $L_\pi$).

Proof. (1) Consider a relation symbol $D_{\varphi_0, \ldots, \varphi_{n-1}; \alpha}$. Given a permutation $\sigma$ of $n,$

$$
S(M) \models D_{\varphi_0, \ldots, \varphi_{n-1}; \alpha}(p_0, \ldots, p_{n-1}) \iff S(M) \models D_{\varphi_{\sigma(0)}, \ldots, \varphi_{\sigma(n-1)}; \alpha}(p_{\sigma(0)}, \ldots, p_{\sigma(n-1)}).
$$

Thus when we consider an atomic formula in $\zeta, \xi$, we may assume it is of the form

$$
D_{\varphi_0, \ldots, \varphi_{n-1}, \varphi_0, \ldots, \varphi_{m-1}; \alpha}(\zeta, \ldots, \zeta, \xi, \ldots, \xi)
$$
(with $n$'s and $m$'s in this order). Now since every $p \in S(M)$ is the type of an element, it necessarily satisfies $\varphi_i \lor \varphi_j \in p \iff \varphi_i \in p \lor \varphi_j \in p$. Therefore we find

$$S(M) = D_{\varphi_0,...,\varphi_{m-1},\psi_0,...,\psi_{n-1}:\alpha}(p,...,p,q,...,q) \iff$$

$$\forall a \in \alpha(M): \bigvee_{i<n} (\varphi_i(x,a) \in p) \lor \bigvee_{i<m} (\psi_i(y,a) \in q) \iff$$

$$\forall a \in \alpha(M): \left(\bigvee_{i<n} \varphi_i(x,a) \right) \in p \lor \left(\bigvee_{i<m} \psi_i(y,a) \right) \in q \iff$$

$$S(M) = D_{\psi_0,...,\psi_{n-1}:\alpha}(p,q),$$

as required.

(2) Take some $c_1 \neq c_2 \in M$; then for some pp formula $\varepsilon(x,y)$ such that $\varepsilon \bot x = y, \varepsilon(c_1,c_2)$ holds. Then we claim $D_{\phi_1,...,\phi_n:\alpha} \cap D_{\psi_1,...,\psi_n:\beta}$ is equivalent to $D_{\theta_1,...,\theta_n:\alpha \cap \beta}$ where $\theta_i = (\phi_i(x,1) \land z_1 = z_2) \lor (\psi_i(x,2) \land \varepsilon(z_1,z_2))$ (where $z_1, z_2$ are new parameter variables) and we define $\delta$ likewise.

It is easier to reason about the negation.

$$\neg D_{\phi_1,...,\phi_n:\alpha} \lor \neg D_{\psi_1,...,\psi_n:\beta} \quad \text{holds for } p_1,...,p_n \text{ iff either [there exists a parameter tuple } a \in \alpha(M) \text{ such that } \phi_i(x,a) \notin p_i \text{ for all } i] \text{ or [there exists a parameter tuple } b \in \beta(M) \text{ such that } \psi_i(x,b) \notin p_i \text{ for all } i].$$

So we can choose as parameters $a$, some arbitrary $b$ and $(c_1,c_1)$ in the first case or likewise an arbitrary $a$ and this $b$ and $(c_1,c_2)$ in the second case — to get in both cases that $\neg D_{\theta_1,...,\theta_n:\alpha \cap \beta}$ holds. On the other hand if $\neg D_{\theta_1,...,\theta_n:\alpha \cap \beta}$ holds then either $z_1 = z_2$ in which case we get necessarily $\neg D_{\phi_1,...,\phi_n:\alpha}$ holds or $\varepsilon(z_1,z_2)$ in which case likewise $\neg D_{\psi_1,...,\psi_n:\beta}$ holds.

Note that if $M \models \exists z_1, z_2 \in (z_1, z_2)$ then the same holds for any pc model of $T$ since they all share the same positive theory $T^*$. This means that the same equivalence holds for all type spaces over pc models.

(3) Let $A = \bigcap_{t \in I} D_t \left( S(M)^{k+1} \right)$ be an atomic-type definable subset of $S(M)$.

Note that we can ignore equality for this discussion: If $0 < i < j$ and $A' = \{(p_0,...,p_k) \in A \mid p_i = p_j\}$ then

$$\{(p_1,...,p_k) \mid \exists p_0 : (p_0,...,p_k) \in A' \} =$$

$$\{(p_1,...,p_k) \mid \exists p_0 : (p_0,...,p_k) \in A : p_i = p_j \} =$$

$$\{(p_1,...,p_k) \mid \exists p_0 : (p_0,...,p_k) \in A \cap \{(p_1,...,p_k) \mid p_i = p_j\} ;$$

while if $0 < j$ and $A' = \{(p_0,...,p_k) \in A \mid p_0 = p_j\}$ then

$$\{(p_1,...,p_k) \mid \exists p_0 : (p_0,...,p_k) \in A' \} =$$

$$\{(p_1,...,p_k) \mid (p_j,p_1,...,p_k) \in A\}$$
and by (1) we can replace each $D_i$ with an appropriate $k$-ary relation to type-define $\{(p_1, \ldots, p_k) \mid \exists p_0 : (p_0, \ldots, p_k) \in A'\}$.

To simplify the notation, we will deal with the case $k = 1$. Let then $D_i = D_{\phi_i, \varphi_i; \alpha_i}$ be a binary relation on $S_{x'}(M) \times S_x(M)$ for $x, x'$ some variable tuples $x, x'$ in $L$.

For any $\vec{i} = (i_0, \ldots, i_{m-1})$ in $I$ and any positive, quantifier free formula $\theta(y_0, \ldots, y_{m-1}, z)$ where $y_j$ are the parameter variables of $\phi_{ij}$ define:

$$
D^{\theta; \vec{i}} := D \cup \bigvee_{j \in m} \varphi_{ij}(x, y_j); \theta \land \bigwedge_{j \in m} \alpha_{ij}(y_j).
$$

We are interested in pairs $\vec{i}, \theta$ such that

$$(*) \quad T \vdash \forall x', y, z : \neg \left( \bigwedge_{j \in m} \phi_{ij}(x', y_j) \land \theta(y, z) \right)$$

$$M \models \exists y, z : \theta(y, z)$$

and we claim that $\exists \xi \land D_i(\xi, \mu)$ is equivalent to $\bigwedge_{\theta; \vec{i} \text{ satisfy } (*)} D^{\theta; \vec{i}}(\mu)$.

We find that for any $q \in S(M)$ (of the relevant sort), $q \in \pi_1(A)$ iff the following partial type is consistent:

$$
\Sigma(x') := \{ \phi_i(x', a) \mid i \in I, a \in \alpha_i(M), \varphi_i(x, a) \notin q \} \cup \Delta_{M}^a \cup T
$$

Let us verify this claim.

If $\Sigma$ is consistent then let $N \models T$ be a continuation of $M$ (recall $M$ is pc, so this is necessarily an immersion) and let $c \in N^{x'}$ realizing $\Sigma$. We may assume $N$ is pc, since if $N'$ is a pc continuation of $N$ then the image of $c$ still satisfies $\Sigma$ over $M$ (since $\Sigma$ is a positive type). Then we find that for $p = \text{tp}^p(c/M)$, $(p, q) \in A$ — indeed for any $i \in I$ and for any $a \in \alpha_i(M)$, either $\varphi_i(x, a) \notin q$ or $N \models \phi_i(c, a) \land \Delta_{M}^a$.

On the other hand, if $(p, q) \in A$ for some $p$ then there exists some tuple $c \in N^{x'}$ (for $N \models T$ a pc continuation of $M$) such that $p = \text{tp}^p(c/M)$. Then for any $i \in I$, for any $a \in \alpha_i(M)$ such that $\varphi_i(x, a) \notin q$ we have $D_i(p, q)$ thus $\phi_i(x', a) \in p \Rightarrow N \models \phi_i(c, a)$, so $N \models \Sigma(c)$.

Now from compactness, if $\Sigma$ is inconsistent then we have:

1. Some $\vec{i}$ and $\vec{\alpha}$ such that $a_j \in \alpha_i(M)$, $\varphi_{ij}(x, a_j) \notin q$.
2. Some $e \in M'$.
3. Some quantifier free positive $\theta(\vec{y}, z)$ such that $M \models \theta(\vec{y}, z)$.

Such that

$$
T \vdash \forall x : \neg \left( \bigwedge_{j \in k} \phi_{ij}(x, a_j) \land \theta(\vec{y}, z) \right),
$$
that is we have \( q \models \neg \mathcal{D}^{\theta,\bar{\tau}} (\mu) \) for \( \theta, \bar{\tau} \) satisfying (\( \ast \)), as

\[
T \vdash \forall x' : \neg \left( \bigwedge_{j < m} \phi_{ij} (x', a_j) \land \theta (\bar{\pi}, e) \right) \iff \\
T \vdash \forall x', \bar{y}, z : \neg \left( \bigwedge_{j < m} \phi_{ij} (x', y_j) \land \theta (\bar{\pi}, z) \right).
\]

On the other hand if we have \( q \models \neg \mathcal{D}^{\theta,\bar{\tau}} (\mu) \) then choose \((\bar{\pi}, e) \in \theta \land \bigwedge_{j < m} \alpha_{ij} (y_j)\) (\(M\)) such that \( \forall_{j < k} \varphi_{ij} (x, a_j) \notin q \); then in particular \( T \cup \Delta_M^{\alpha_{ij}} \vdash \forall x' \land \bigwedge_{j < k} \phi_{ij} (x, a_j) \) thus \( \Sigma \) is inconsistent.

Finally, to see that the equivalence is independent of the choice of \( M \), we need only note that since all \( M \) share the same hu theory (and thus the same positive theory), the condition in (\( \ast \)) does not depend on \( M \), thus the atomic type defining \( \pi_1 (A) \) in \( S(M) \) is also independent of \( M \).

If \( T \) is Hausdorff, by Lemma 3.15 every atomic formula in \( L_{\pi} \) is equivalent to a conjunction of atomic formulas in \( L \). Therefore, every conjunction of atomic formulas in \( L_{\pi} \) is also equivalent to a conjunction of atomic formulas in \( L \), thus the projection is also equivalent to a conjunction of atomic formulas in \( L \) — which are also atomic formulas in \( L_{\pi} \), as required.

(4) Follows immediately from (3) by induction on the number of bounded variables in \( \Xi \), together with (1). \( \square \)

### 3.4. Common Theory.

**Theorem 3.19.** Let \( T \) and irreducible hu theory. Assume \( \varphi \) is a pu sentence in \( L \) and \( M, N \) are pc models of \( T \). Then if \( S(M) \models \varphi \), so does \( S(N) \).

Furthermore if \( T \) is semi-Hausdorff then the same holds for \( L_\pi \) sentences.

Since every hu sentence is equivalent to a conjunction of pu sentences, the same holds for hu sentences.

**Proof.** The first part is in fact a special case of the proof of Lemma 3.18, but let us also spell out the case \( k = 0 \).

By Lemma 3.18 \( \neg \varphi \) is equivalent to \( \exists \xi \left( \bigwedge_{i \in I} D_i (\xi) \right) \) (where the equivalence is independent of \( M \)) where \( \xi \) is a single variable in sort \( x \). Denote \( D_i = D_{\varphi_i, \alpha_i} \).

Then \( S(M) \models \neg \varphi \) iff there exists \( p \in S(M) \) such that for any \( i \in I \), and for any \( a \in \alpha_i (M) \), \( \varphi_i (x, a) \in p \); that is iff there exists a pc model \( M' \supseteq M \) of \( T \) and \( c \in M \) such that for any \( i \in I \), \( a \in \alpha_i (M) \), \( M' \models \varphi_i (c, a) \).

This is equivalent to the claim that \( \Sigma_M (x) = \{ \varphi_i (x, a) \mid i \in I, a \in \alpha_i (M) \} \cup T \cup \Delta_M^{\alpha_{ij}} \) is consistent, like in the previous proposition.
Thus $S(M) = \varphi$ iff $\Sigma_M$ is inconsistent; that is iff there exist $i_0, \ldots, i_{k-1} \in I$ and $a_j \in \alpha_{i_j}(M)$; and $e \in M^l$ for some $l$, $\theta(\overline{y}, z)$ positive such that $M \models \theta(\overline{a}, e)$

\[ T \vdash \neg \exists x \left( \bigwedge_{j<k} \varphi_{i_j}(x, a_j) \land \theta(\overline{a}, e) \right) \iff T \vdash \forall x, \overline{y}, z : \neg \left( \bigwedge_{j<k} \varphi_{i_j}(x, y_j) \land \theta(\overline{y}, z) \right) \]

That is iff exist $i_0, \ldots, i_{k-1} \in I$ and positive quantifier free $\theta(\overline{y}, z)$ such that

$M \models \exists \overline{y}, z : \theta(\overline{y}, z) \land \bigwedge_{j<k} \alpha_{i_j}(y_j) \iff \exists \overline{y}, z : \theta(\overline{y}, z) \land \bigwedge_{j<k} \alpha_{i_j}(y_j) \in T^-$

and

$T \vdash \forall x, \overline{y}, z : \neg \left( \bigwedge_{j<k} \varphi_{i_j}(x, y_j) \land \theta(\overline{y}, z) \right) .

But that requirement only depends on $\text{Th}^{bu}(M) = \text{Th}^{bu}(N) = T$ (by Remark 2.10 and Remark 2.5, thus $S(N) \models \varphi$.

For $\mathcal{L}_x$ sentences, consider a pp sentence $\exists \overline{x} \land \bigwedge_{i<cm} D_i(\overline{x}) \land \bigwedge_{j<k} \pi_{x_{i_j}, x'_{i_j}}(\overline{\xi}_j) = \pi_{x'_{i_j}, x'_{i_j}}(\overline{\xi}'_j)$ where $\overline{\xi} = (\overline{\xi}_i)_{i<m}$ and $\overline{\xi}_i$ is from the sort $S_{x_i}$ and $D_i$ denotes $D_{w_0, \ldots, w_{m-1}, \alpha_i}$ (we may assume $D_i$ is a relation of length $m$ by setting $\varphi'_i(x_i)$ be $\varepsilon(x_i, x_i)$ for $\varepsilon \perp x_i = x_i$ for $l$'s that do not appear; since such a $\varphi'_i$ will never be in any type).

For a sort $z$, let $\Sigma_z$ be the positive type defining positive type equality in $z$. Then like in the proof of Lemma 3.18.3 we have an $L$ type that whose consistency is equivalent to an $\mathcal{L}_x$ formula holding. Specifically,

$S(M) = \exists \overline{x} \land \bigwedge_{i<cm} D_i(\overline{x}) \land \bigwedge_{j<k} \pi_{x_{i_j}, x'_{i_j}}(\overline{\xi}_j) = \pi_{x'_{i_j}, x'_{i_j}}(\overline{\xi}'_j)$

iff the following type is consistent (note that we are using the fact that multiple elements of $S(M)$ can always be realized simultaneously, by Proposition 2.16):

$$\left\{ \bigvee_{i<m} \varphi'_i(x_i, a) \mid i < n, a \in \alpha_i(M) \right\} \cup \bigcup_{y \text{ sort } m \in M^0} \bigcup_{j<k} \Sigma_{x'_{j-y}} \left( x_j \sim m, x'_{j} \sim m \right) \cup T \cup \Delta^0_M$$

Which again holds iff there are no $i_0, \ldots, i_{k-1} < n$; $a_j \in \alpha_{i_j}(M)$; some sorts $w_0, \ldots, w_{r-1}$ and $d_f \in M^{w_f}$; some finite subtypes $\Sigma_{x'_{j-w_f}}$ of $\Sigma_{x'_{j-w_f}}$; some $e \in M^2$ for some sort $z$; and $\theta(\overline{y}, \overline{w}, z)$ positive such that $M \models \theta(\overline{a}, \overline{d}, e)$ such that:

$T \vdash \forall x, \overline{y}, \overline{w}, z : \neg \left( \bigwedge_{j<r} \bigwedge_{j<k} \Sigma_{x'_{j-w_f}}(x_j \sim w_f, x'_{j} \sim w_f) \right) \land \bigwedge_{j<k} \bigvee_{i<m} \varphi'_i(x_i, y_i) \land \theta(\overline{y}, \overline{w}, z)$.

Which is once again independent of $M$. 
Note that this pp sentence is actually as general as we want, since composition of projections is equivalent to a projection (that is $\pi_{x,y}((\pi_{y,z}(\zeta)) = \pi_{x,z}(\zeta)$) and every atomic formula of the form $\mathcal{D}(\pi(\xi_0), ..., \pi(\xi_{n-1}))$ is equivalent to one of the form $\mathcal{D}(\xi_0, ..., \xi_{n-1})$ (see Remark 3.4). □

**Corollary 3.20.** $\text{Th}^{\text{hu}}(S(M))$ is independent of $M$.

**Remark 3.21.** The same does not in general hold for $L_\pi$, see Example 3.48 below.

Note that since Corollary 3.20 does not hold in general for $L_\pi$, it cannot be that Lemma 3.18.3 holds in general for $L_\pi$.

**Corollary 3.22.** $T = \text{Th}^{\text{hu}}(S(M))$ for $M$ an arbitrary pc model of $T$.

If $T$ is semi-Hausdorff then $T_\pi = \text{Th}^{\text{hu}}(S(M))_{L_\pi}$

3.5. Universality and Boundedness.

**Theorem 3.23.** Let $M$ be a pc model of $T$.

1. Any model $A$ of $T$ (in particular every $S(N)$) admits a homomorphism into $S(M)$. In particular if $A = E$ is pc, it is embeddable in $S(M)$.

2. If $T$ is thick (see Definition 2.33), in particular if $T$ is semi Hausdorff or if $T$ is bounded (see Remark 2.34), the same holds for $T_\pi$.

**Proof.** (1) Consider the topology on $S(M)$ generated by the basis $[\varphi] = \{p \in S(M) \mid \varphi \notin p\}$ for all positive formulas $\varphi$.

This space is compact, as usual (if $\left\{ [\varphi_i] \right\}_{i < \kappa}$ is a family of basic closed sets with the f.i.p. then $\left\{ \varphi_i \right\}_{i < \kappa}$ is consistent with $\Delta_M^\text{at}$ thus can be realized in an extension, and thus by Fact 2.12 can be realized in a pc model of $T$).

For any $\mathcal{D} = \mathcal{D}_{\varphi_0, ..., \varphi_{n-1}; \alpha}$, we note that

$$\mathcal{D}(S(M)) = \left\{ p_0, ..., p_{n-1} \mid \forall b \in \alpha(M) : \bigvee_{i < n} \varphi_i(x,b) \in p_i \right\} = \bigcap_{b \in \alpha(M)} \bigcup_{i < n} \pi_i^{-1}\left( [\varphi_i(x,b)]^C \right)$$

thus closed.

So from Lemma 2.24 we are done.

(2) This requires a more careful analysis than what Lemma 2.24 provides, since we cannot use the usual product topology. Let $N$ a positively $\kappa$–saturated pc extension of $M$ (which exists by Proposition 2.21 for $\kappa \geq \kappa(\varphi^+)^+$ large enough such that $T_\pi = \text{Th}^{\text{hu}}(S(N))_{L_\pi}$ (see Claim 3.7). It
is enough to show that $S(N)$ is universal, since the restriction from $S(N)$ to $S(M)$ is an $L_\pi$ homomorphism.

By [DK21 Lemma 2.20 and Fact 2.16], for every variable tuple $x$ there is a positive type $p_x$ over $N$ such that for any $x$ tuples $a, b$ in any pc extension of $N$, $tp^p(a/N) = tp^p(b/N)$ iff $p_x(a, b)$ (since type equality holds iff there is a third element that appears in an indiscernible sequence with both, and this is positively definable).

For each $a \in A_x$, (the $x$ sort of $A$, for $x$ some variable tuple) let $x_a$ be a variable of the same sort as $x$. If $x'$ is a subtuple of $x$ we denote by $x'_a$ the respective subtuple of $x_a$.

Consider the partial positive type

$$\Sigma_1(x_a)_{a \in A} = \left\{ \bigvee_{i \in \alpha} \varphi_i(x_{a_i}, a) \right\}_{D_{\psi_0, \ldots, \psi_{n-1}, \alpha(a_0, \ldots, a_{n-1}) \in \alpha(N)}},$$

Clearly, $(b_a)_{a \in A} = \Sigma_1$ for $b_a$ in some pc model of $T$ extending $N$ iff $a \rightarrow.tp(b_a/N)$ defines an $L$ homomorphism.

We also want a type $\Sigma_2(x_a)_{a \in A}$ that will guarantee that $a \rightarrow.tp(b_a/N)$ respects $\pi_{x,x'}$ for each $x$ and subtuple $x'$. Define

$$\Sigma_2(x_a)_{a \in A} = \bigcup \{ p_{x'}(x_{a'}, x_{a''}) \}_{a \in A_x, a' \in A_{x'}, \pi_{x,x'}(a) = a'};$$

then $\Sigma_2$ has the property we desire.

Let us show that $\Sigma_1 \cup \Sigma_2$ is finitely satisfiable with $\Delta^*_N \cup T$. Every finite subtype $\Sigma_0$ comes from a finite number of relations of the form $D(a_0, \ldots, a_{n-1})$ and a finite number of equalities of the form $\pi_{x,x'}(a_i) = a_j$. Since these hold in $A$ and $A \models \text{Th}^\text{int}(S(N))_{L_\pi}$, like in Lemma 2.24 we can choose elements in $S(N)$ satisfying the same requirements. Each of these is satisfied in some pc extension of $N$, and by Proposition 2.16 these types can be satisfied simultaneously in a single pc extension $L$ of $N$. In particular we find that $L \models \exists \pi \Sigma_0$.

Therefore, there exists $(b_a)_{a \in A} = \Sigma_1 \cup \Sigma_2$ in some pc extension of $N$, and $\{(a, tp^p(b_a/N)) | a \in A\}$ defines a homomorphism from $A$ to $S(N)$ as required.

**Corollary 3.24.** $T$ is bounded (by $|S(M)|$ for arbitrary $M$) thus it has a universal pc model (see Proposition 2.21).

If $T$ is semi-Hausdorff or bounded then $T_\pi$ is also bounded.

**Definition 3.25.** Core($T$) is the universal pc model of $T$ in the language $L$.

When $T$ is fixed, we will denote $J = \text{Core}(T)$.

If $T_\pi$ is well defined and pc bounded we will define Core$_\pi(T)$ and $J_\pi$ similarly.

**Theorem 3.26.** Let $T$ be an irreducible primitive universal theory. Let $E$ be a pc model of $T$.

1. Every finite conjunction of atomic formulas in $E$ (none of which involves $\pi$) is equivalent to an atomic formula.
(2) Every atomic formula is equivalent in $E$ to an atomic relation — that is if $\varphi(\zeta, \xi)$ is a formula consisting of a single relation symbol, it is equivalent to a binary relation symbol in $\zeta, \xi$, and likewise for formulas with more parameters — and the equivalence is independent of the model.

(3) Every $pp$ formula $\Xi(\mu)$ is equivalent in $E$ to a possibly infinite (but no larger than $|L| = |L|$) conjunction of atomic formulas.

(4) $J$ is homogeneous for atomic type — if $\text{tp}^{at}(\bar{\pi}) = \text{tp}^{at}(\bar{b})$ for $\bar{\pi}, \bar{b} \in J$ then there is an automorphism of $J$ sending $\bar{\pi}$ to $\bar{b}$.

(5) An atomic type in $T$ is the type of an element of $J$ iff it is maximal, that is there is no atomic type consistent with $T$ that strictly extends it. In particular, if $p \in S(M)$ belongs to some embedding of $J$, the set of formulas represented in $p$ is minimal.

If $T$ is Hausdorff, the same holds for $\mathcal{T}_x$ and $\mathcal{J}_x$.

Proof. Note first that if $h : M \to N$ is an immersion and $\varphi(x), \psi(x)$ are positive formulas over $\emptyset$ that are equivalent in $N$, they are also equivalent in $M$ (since for any $a \in M^\emptyset$, $M \models \varphi(a) \iff N \models \varphi(h(a)) \iff N \models \psi(h(a)) \iff M \models \psi(a)$).

1-3 thus follow immediately from Lemma 3.18 and Theorem 3.23.

Indeed choose an appropriate $pc$ model $M$ of $T$. Every pc model of $T$ is immersed in $S(M)$, thus we are finished. Note that projections of atomic types are not necessarily identical in a pc submodel, thus Proposition 4.11.3 does not translate to every pc model. Note that in (1) the assumption on the existence of a pc model of cardinality $\geq 2$ is unneeded: if not, then the universal model of $T$ is of cardinality 1, and therefore by Proposition 3.8 so is $S(U)$ and thus every pc model of $T$ is of cardinality 1, and the claim is trivial.

4. We already know $J$ is homogeneous for $pp$ types (and positive types) by Proposition 2.215, and by (3) the $pp$ types of elements of $J$ are completely determined by their atomic types.

5. If $P$ is a maximal atomic type, let $A$ be some model and $\bar{\pi}$ some element satisfying $P$; then there exists a homomorphism $f : A \to J$, and thus $J \models P(f(a))$; so from maximality $P$ is the atomic type of $f(a)$.

In the other hand, assume $a \in J$ and let $P$ be the atomic type of $a$. Let $\psi$ be an atomic formula such that $\neg \psi(a)$. Then there exists a $pp$ formula $\phi$ such that $\phi \perp \psi$ (that is $T \vdash \forall x : (\phi \land \psi)$) and $J \models \phi(a)$.

By (3), there exist atomic $\left\{ \Xi_i \right\}_{i \in l}$ such that $\phi(J) = \bigcap_{i \in l} \Xi_i(J)$ (in particular $\Xi_i(a)$ for all $i$, that is $\left\{ \Xi_i \right\}_{i \in l} \subseteq P$). Assume $\left\{ \Xi_i \right\}_{i \in l} \cup \psi$ is consistent with $T$; then exist a model $A$ and some $b \in A$ witnessing that. But for $f : A \to J$ a homomorphism we get $\Xi_i(f(b))$ for all $i$ thus $\phi(f(b))$, but also $\psi(f(b))$, contradiction. Thus for some $\left\{ i_j \right\}_{j \in n} \subseteq I$ we have $T \vdash \bigwedge_{j=1}^{n} \Xi_{i_j} \to \neg \psi$ thus $P \vdash \neg \psi$; so $P$ is maximal. \qed
3.6. The Core in the Bounded and Hausdorff Case.

3.6.1. Bounded $T$.

**Proposition 3.27.** Let $M$ be a pc model of $T$. Then the following are equivalent:

1. $S(M)$ is a pc model of $T$
2. $S(M)$ is isomorphic to $J$
3. Every $L$ endomorphism of $S(M)$ is an automorphism
4. Every $L$ endomorphism of $S(M)$ is an embedding
5. Every $L$ endomorphism of $S(M)$ is surjective.

If the conclusion of Theorem 3.23 holds, the equivalence between 1, 2, 4 and 5 holds for $L_\pi$. If $T$ is Hausdorff, 1 through 5 are also equivalent for $L_\pi$.

The equivalence of 2 and 5 holds per sort, that is to say that every endomorphism of $S(M)$ is onto $S_x(M)$ iff every/any embedding of $J$ into $S(M)$ is onto $S_x(M)$.

**Proof.** (1) $\Rightarrow$ (2) We know that $J$ is embeddable in $S(M)$ by Theorem 3.23 and the fact $J$ is pc, and we know by Proposition 2.21 that every embedding of the universal model into an pc model is an isomorphism.

(2) $\Rightarrow$ (3) Holds for any universal pc model by Proposition 2.21.

(3) $\Rightarrow$ (4), (5) Obvious.

(4) $\Rightarrow$ (1): By Claim 2.14 and Theorem 3.23 it is enough to show every endomorphism is pc. But by 4 in Lemma 3.18 every pp formula is equivalent in $S(M)$ to a conjunction of atomic formulas and is thus pulled back by self embeddings.

(5) $\Rightarrow$ (2) We need only consider a homomorphism $f : S(M) \to J \subseteq S(M)$. By assumption it is onto, and thus the inclusion $J \subseteq S(M)$ is onto. Since $J$ is pc this means that the inclusion is an isomorphism.

Per sort, we consider a homomorphism $f : S(M) \to J \subseteq S(M)$. If every $L$ endomorphism is onto $S_x(M)$ then in particular this holds for $f$ (regardless of which embedding of $J$ we choose) thus $J_x = S_x(M)$, while if $J_x = S_x(M)$ for any embedding then $f \circ \text{id}_J : J \to J$ is an endomorphism thus an automorphism of $J$ and thus in particular onto $J_x$, thus $f$ must be onto $S_x(M)$. □

**Corollary 3.28.** For any bounded theory $T$, $S(U) = J_{\pi}$. If $U = \text{acl}^p(\emptyset)$, we also have $S(U) = J$.

**Proof.** The first part is from Proposition 3.27 (or 3, or 4) and Proposition 3.11.

The second is from Proposition 3.27 and Remark 3.14. □

**Theorem 3.29.** Assume $T$ is an irreducible thick lu theory. Then $\pi \mapsto \text{tp}^p(\pi/\text{Core}_\pi(T))$ is a bijection between $\text{Core}_\pi(T)$ and $\text{Core}_\pi(T_\pi)$ that preserves the automorphism group.

Informally, we can say that applying the core construction a second time results in the same object.
Proof. By the previous corollary and Proposition 3.11.

Example 3.30. In Example 3.48 below we will see a bounded theory $T$ where $S(U) \neq \mathcal{J}$.

3.6.2. Hausdorff $T$.

Theorem 3.31. Assume $T$ is a Hausdorff irreducible $hu$ theory, $M$ a pc model of $T$, and $\mathcal{J} \leq S(M)$ is $\text{Core}(T)$.

Then $\mathcal{J}$ is an $L_\pi$ substructure of $S(M)$, and every $\pi_{x,x'} : J_x \to J_{x'}$ is $L$-type definable over $\emptyset$ in $\mathcal{J}$. Furthermore, $\mathcal{J}$ is universal and pc in $T_\pi$, and is thus $\text{Core}_\pi(T)$.

Proof. Let $r : S(M) \to \mathcal{J}$ be an $L$ retract for the inclusion map, which exists by Fact 2.8. For any $\pi_{x,x'} \in L_\pi$, let $\Sigma_{x,x'} (\xi, \zeta)$ be a partial $L$ type equivalent to $\pi_{x,x'} (\zeta) = \xi$ in $S(M)$ as in Lemma 3.15. For any $p \in S_x(M)$, we find that $\Sigma_{x,x'} (p, \pi_{x,x'} (p))$ therefore $\Sigma_{x,x'} (r (p), r (\pi_{x,x'} (p)))$ (in $\mathcal{J}$, thus also in $S(M)$) thus in $S(M)$ we have $r(\pi_{x,x'} (p)) = \pi_{x,x'} (r (p))$.

In particular, for $p \in J_x$ we find $\pi_{x,x'} (p) = \pi_{x,x'} (r(p)) = r(\pi_{x,x'} (p)) \in J_{x'}$, thus $\mathcal{J}$ is a $L_\pi$ substructure, and in particular $\mathcal{J} \equiv \text{Th}^{hu}(S(M)) = T_\pi$. Furthermore, we also get that $r$ is an $L_\pi$ homomorphism. Thus as $S(M)$ is universal in $T_\pi$ (since Hausdorff implies semi-Hausdorff and by Theorem 3.23, so is $\mathcal{J}$.

To show that $\mathcal{J}$ is pc is it thus enough by Claim 2.14 to show every $L_\pi$ endomorphism of $\mathcal{J}$ is an immersion. Let $h : \mathcal{J} \to \mathcal{J}$ be an $L_\pi$ endomorphism. Then in particular, $h$ is an $L$-endomorphism of $\mathcal{J}$, thus by Fact 2.8 an $L$-automorphism of $\mathcal{J}$. But every symbol in $L_\pi$ which is not in $L$ is a function symbol, so in fact $h$ is a bijective $L_\pi$ homomorphism which preserves every relation symbol in both directions, that is an $L_\pi$-automorphism as required. □

3.7. $L_\pi$ Homomorphisms in Terms of the Original Language.

Lemma 3.32. Let $A$ be a pc model of $T$. For a set $p(x)$ of $hu$ formulas in variables $x$ with parameters from $A$ the following characterizations are equivalent:

1. $p = \text{tp}^{hu}(c/A)$ for some $c \in B^x$ where $B$ is a pc model of $T$ extending $A$.
2. All of the following hold:
   a. If $\theta(a) \in p$ (for $a \in A^n$ for some $n$) then $A \vDash \theta(a)$.
   b. If $\varphi, \psi \notin p$ then $\varphi \land \psi \notin p$.
   c. $\varphi(x,a) \in p$ (for $a \in A^n$) iff there exists an $hu$ formula $\psi(x,y)$ such that $T \vdash \forall x,y: \varphi \land \psi$ and $\psi(x,a) \notin p$.
   d. $p$ is closed under conjunction.
3. $p$ is the minimal set of $hu$ formulas in $x$ over $A$ such that:
   a. $T \subseteq p$.
   b. If $\varphi(x,a)$ is an $hu$ formula over $A$ such that $p \cup \Delta_A \vdash \varphi(x,a)$ (when the variables of $x$ are considered to be new constants).
Definition 3.33. We call such a $p$ a cu\textsuperscript{13} type.

\textbf{Proof.} (1) $\Rightarrow$ (2): (a) follows from the fact that $B$ extends $A$ (note that $\theta$ is h, thus pulled back by homomorphisms).

(b) is obvious for the type of an element.

(c) follows since if $\varphi(x,a) \in p$ we get $B \models \neg \varphi(c,a)$ thus by Fact \textsuperscript{2.12} there exists a pp formula $\neg \psi(x,y)$ such that $\neg \varphi \perp \neg \psi \iff T \models \forall x,y : \neg (\neg \varphi \land \neg \psi)$ and $B \models \neg \psi(c,a) \Rightarrow \psi(x,a) \notin p$. Conversely if $T \models \forall x,y : \varphi(x,y) \lor \psi(x,y)$ and $\psi(x,a) \notin p$ then $B \models \psi(c,a)$ but $B \models \psi(c,a) \lor \varphi(c,a)$ thus $B \models \varphi(c,a) \Rightarrow \varphi(x,a) \in p$.

(d) also obvious for the type of an element.

(2) $\Rightarrow$ (3) Assume $\varphi \in T$. Then we have $T \models \varphi \lor \bot$ and since $A \models \bot$ we have necessarily by (a) and (c) $\bot \notin p \Rightarrow \varphi \in p$.

Assume that $p \lor \Delta^a_A \models \varphi(x,a)$. Then there is some finite conjunction of formulas in $p \lor \Delta^a_A$ implying $\varphi(x,a)$, thus by (d) some formula $\theta(x,a) \in p$ and positive formula $\zeta(a)$ that holds in $A$ such that $\models \forall x,y (\theta(x,y) \land \zeta(y)) \rightarrow \varphi(x,y) \iff \forall x,y \theta(x,y) \rightarrow (\neg \zeta(y) \lor \varphi(x,y))$ (without loss of generality the parameter variables in $\varphi$ and $\psi$ are the same). By (c) there is some $\psi(x,y)$ such that $T \models \forall x,y : \theta \lor \psi$ and $\psi(x,a) \notin p$ — but then certainly $T \models \forall x,y : (\neg \zeta \lor \varphi) \lor \psi$ thus by (c) again we have $\neg \zeta(a) \lor \varphi(a) \notin p$. But since $A \models \neg \zeta(a)$, by (a) we have $\neg \zeta(a) \notin p$ and therefore by (b) we have $\varphi(x,a) \notin p$ as required.

Now have to show that $p^-$ is indeed consistent with $\Delta^a_A(A)$. By (b), $p^-$ is closed under conjunctions, thus it if not consistent with $\Delta^a_A$ then there exist $a \in A^p$ and positive formulas $\psi(x,y), \theta(y)$ (actually $\theta$ is quantifier free) such that:

$- T \models \forall x,y : \neg (\psi \land \theta) = \forall x,y : \neg \psi \lor \neg \theta$.

$- \psi(x,a) \in p^-, A \models \theta(a)$.

Since $A \models \neg \theta(a)$, $\neg \theta(a) \notin p$ by (a) thus by (c) $\neg \psi(x,a) \in p \Rightarrow \psi(x,a) \notin p^-$, contradiction.

Now assume $p' \models p$ is closed under implications and contains $T$, and take some $\varphi(x,a) \in p \setminus p'$. By (c), there exists $\psi(x,y)$ such that $T \models \forall x,y : \varphi \lor \psi$ and $\psi(x,a) \notin p \Rightarrow \psi(x,a) \notin p'$, thus we get $\neg \varphi(x,a), \neg \psi(x,a) \in p^-$ thus $p^-$ is inconsistent with $T$.

(3) $\Rightarrow$ (1) Since $p \cup \Delta^a_A$ is consistent with $T$ and positive, there are $B \models T$ and $c \in B$ realizing it. By Fact \textsuperscript{2.12} and the fact that $p^- \cup \Delta^a_A$ is positive, we may assume without loss of generality that $B$ is pc; and by the fact that $A$ is pc we may assume $A \leq B$.

Then again by the fact $B$ is pc we get that $p^- \models \text{tp}^p(c/A) = \text{tp}^{\text{hu}}(c/A)^- \iff \text{tp}^{\text{hu}}(c/A) \subseteq p$ (note that we use here that $p$ is closed under implication) and by minimality we get $\text{tp}^{\text{hu}}(c/A) = p$ as required. \qed
Remark 3.34. We can immediately see from 1, Remark 2.5 and Proposition 2.28 that $p$ is a cu type iff $p^-$ is a maximal positive type.

**Definition 3.35.** Assume $V$ is a structure and $A \subseteq V$. We call a set $p$ of formulas in $x$ over $V$ *A positively invariant* ($p$-invariant) if for whenever $\varphi(x,c) \in p$ and $c'$ in $V$ satisfies $\text{tp}^p(c/A) = \text{tp}^p(c'/A)$ we also have $\varphi(x,c') \in p$.

Remark 3.36. Let us consider a saturated $pc$ model $V$, $A \subseteq V$ some subsets, and $p \in S(V)$. Like with complete types, if $p$ is finitely satisfiable in $A$ it is also $p$-invariant.

Indeed assume otherwise. In particular there exist $\varphi(x,b) \in p$, $\varphi(x,b') \notin p$ for some $b \equiv_A b'$. We get from maximality that for some $\psi(x,y)$ such that $\psi(x,y) \perp \varphi(x,y)$, $\psi(x,b') \in p$.

Since $p$ is finitely satisfiable there exists $\alpha \in Ax$ such that $\varphi(a,b) \land \psi(a,b')$ hold, and from $b \equiv_A b'$ we get $\psi(a,b)$ holds, which contradicts $\psi \perp \varphi$.

On the other hand, if $p$ was a cu type, it could be finitely satisfiable but not $p$-invariant, as we will see in Example Example 3.48 below.

**Theorem 3.37.** Let $T$ be an irreducible $hu$ theory. Let $V$ be a $\kappa$-saturated pc model of $T$, and let $A, B \subseteq V$ be pc models of $T$ of cardinality $< \kappa$. Let $b$ be an enumeration of $B$ and $y$ a corresponding variable tuple.

Then there is a bijection between $\mathcal{L}_\pi$-homomorphisms $h : S(A) \rightarrow S(B)$ and cu $A$ $p$-invariant types $p(y)$ over $V$ such that $p \cup \text{tp}^p(b/\emptyset)$ is finitely satisfiable in $A$ (in particular $\text{tp}^p(b/\emptyset) \subseteq p^-$).

**Proof.** To see that if $p \cup \text{tp}^p(b/\emptyset)$ is finitely satisfiable in $A$ then $\text{tp}^p(b/\emptyset) \subseteq p^-$, note that if $\varphi(y) \in \text{tp}^p(b/\emptyset)$ then $p \not\models \neg \varphi$ (otherwise $p \cup \{\varphi(y)\}$ would be inconsistent) thus by definition $\varphi \in p^-$. Assume $h$ is given. Define $p = \{\varphi(c,y) \mid \neg \varphi(x,b) \notin h(\text{tp}^p(c/A))\}$. Since $h$ is an $\mathcal{L}_\pi$ homomorphism, this is well defined even if we do not require that $c$ is the exact set of parameters in $\varphi(c,y)$ — if $c'$ is some tuple extending $c$, then

$$\neg \varphi(x,b) \in h(\text{tp}^p(c'/A)) \iff \neg \varphi(x,b) \in \pi_{x',x}(h(\text{tp}^p(c'/A))) = h(\pi_{x',x}(\text{tp}^p(c'/A))) = h(\text{tp}^p(c/A)).$$

To show that $p \cup \text{tp}^p(b/\emptyset)$ is finitely satisfiable, assume $\{\varphi_i(c_i,y)\}_{i<n} \subseteq p$ and $\alpha(y) \in \text{tp}^p(b/\emptyset)$.

Let $y'$ be the finite subtuple of variables that appear in all of $\varphi_i, \alpha$ and $b'$ the corresponding tuple of elements of $b$. Denote $q_i = h(\text{tp}^p(c_i/A))$. Then $B \models \alpha(b')$ and $\neg \varphi(x_i,b') \notin q_i$ for $i < n$.

We get

$$\neg D_{\neg \varphi_0(x_0,y'), \ldots, \neg \varphi_{n-1}(x_{n-1},y') ; \alpha(y')} (q_0, \ldots, q_{n-1}),$$

thus

$$\neg D_{\neg \varphi_0(x_0,y'), \ldots, \neg \varphi_{n-1}(x_{n-1},y') ; \alpha(y')} (\text{tp}^p(c_0/A), \ldots, \text{tp}^p(c_{n-1}/A)).$$
and thus for some $a' \in A^y$ we have:

- $A \models \alpha(a')$.
- $\neg \varphi(x, a') \not\in \text{tp}^p(c_i/A) \Rightarrow V \models \varphi(c_i, a')$ for all $i < n$.

as required.

We will show $p$ is a cu type using Lemma 3.32.2.

- (a) Assume we have $\theta(c)$ an lu formula with parameters $c \in V$. Then

$$\theta(c) \in p \iff \neg \theta(x) \not\in h(\text{tp}^p(c/A)).$$

If $V \models \neg \theta(c)$ then $D_{\neg \theta}(\text{tp}^p(c/A))$ and thus $D_{\neg \theta}(h(\text{tp}^p(c/A)))$ that is

$$\neg \theta \not\in h(\text{tp}^p(c/A)) \Rightarrow \theta(c) \not\in p.$$

- (b) If $\varphi(c, y), \psi(c, y) \not\in p$ we have

$$\varphi(c, y), \psi(c, y) \not\in p \Rightarrow \neg \varphi(x, b), \neg \psi(x, b) \in \text{tp}^p(c/A) \Rightarrow \neg \varphi(x, b) \land \neg \psi(x, b) \in h(\text{tp}^p(c/A)) \Rightarrow (\varphi \lor \psi)(c, y) \not\in p.$$

- (c) Let $\varphi(c, y)$ be an lu formula over $V$. Then $\varphi(c, y) \in p$ iff $\neg \varphi(x, b) \not\in h(\text{tp}^p(c/A))$ iff (by Proposition 2.28 and Fact 2.12) for some positive formula $\neg \psi(x, b)$ such that $\neg \varphi \not\in \neg \psi$ (that is $T \models \forall x, y \varphi \lor \psi$) we have $\neg \psi(x, b) \in h(\text{tp}^p(c/A)) \iff \psi(c, y) \not\in p$.

- (d) If $\varphi(c, y), \psi(c, y) \in p$ (without loss of generality they have the same parameters) then $\neg \varphi(x, b), \neg \psi(x, b) \not\in h(\text{tp}^p(c/A))$ which is a type of an element, thus $\neg (\varphi \land \psi)(x, b) \not\in h(\text{tp}^p(c/A)) \Rightarrow (\varphi \land \psi)(c, y) \in p$ as required.

Finally we note that $p$ is a $p$-invariant by definition.

Conversely assume $p$ is given. Since $p$ is cu, $p^\perp$ is consistent with $\Delta_0^v$ thus we can find $d \models p^\perp$ in some pe $W$ extending $V$ (possibly $V$ itself) — note that since $\text{tp}^p(b/\emptyset)$ is maximal and contained in $\text{tp}^p(d/\emptyset)$, they are equal. For $r \in S(A)$, define $h(r) = \{ \varphi(x, b) \mid \exists c \in V^x c \models r, \neg \varphi(c, y) \not\in p \}$ (note that by $p$-invariance we may replace $\exists c$ with $\forall c$). Since the choice of specific $c$ is unimportant, we get that $h$ respects projections.

We have to show that $h(r) \in S(B)$. For consistency with $\Delta_0^v(B)$, assume $\{ \varphi_i(x, b) \mid i < n \} \in h(r)$ and $B \models \alpha(b)$ for $\alpha$ quantifier free and positive. Choose $c \models r$ in $V$. Then by definition $\neg \varphi_i(c, y) \not\in p \Rightarrow \varphi_i(c, y) \in p^\perp \Rightarrow W \models \varphi_i(c, d)$ for all $i$ as well as $\alpha(d)$. Thus $W \models \exists x \alpha(d) \land \bigwedge \varphi_i(x, d)$ and thus $V \models \exists x \alpha(b) \land \bigwedge \varphi_i(x, b)$ as required. Assume $\varphi(x, b)$ is positive. Then $\varphi \not\in h(r)$ iff $\varphi(c, y) \in p$ iff (by $2c$) for some positive formula $\psi(c, y)$ such that $\psi \not\vdash \varphi$ we have $\neg \psi(c, y) \not\in p \iff \psi(c, y) \in h(r)$ and thus $h(r)$ is maximal.

\footnote{Note that here we use the fact that we can take a longer $c$ if needed.}
Finally we have to show \( h \) is a homomorphism. Assume
\[
S(B) \models \neg D_{\varphi_0(x_0,y')} \ldots \varphi_{n-1}(x_{n-1},y') ; \alpha(y') \ (h(r_0), \ldots, h(r_{n-1})) ,
\]
and choose by saturation \( c_i \in V \) realizing \( r_i \). Then for some \( b' \in \alpha(B) \) (assume without loss of generality \( y' \) is the subtuple of \( y \) corresponding to \( b' \)) we have that \( \varphi_i(x_i, b') \not\in h(r_i) \) for all \( i \).

Thus by definition \( \neg \varphi_i(c_i, y') \in p \) for all \( i \), and therefore since \( p \cup \tp(B/\emptyset) \) is finitely satisfiable in \( A \) we have that for some \( a' \in \alpha(A) \), \( \neg \varphi_i(c_i, a') \iff \varphi_i(x_i, a') \not\in r_i \) holds for all \( i < n \) — so
\[
\neg D_{\varphi_0(x_0,y')} \ldots \varphi_{n-1}(x_{n-1},y') ; \alpha(y') (r_0, \ldots, r_{n-1})
\]
as required.

Note that these operations are indeed inverses:

- If we have \( p \), define \( h \) and define from it \( p' \) we get that
  \[
  \varphi(c, y) \in p' \iff \neg \varphi(x, b) \not\in h(\tp(c/A)) = \{ \varphi(x, b) \mid \forall c' \in V^x c' \models \tp(c/A), \neg \varphi(c', y) \not\in p \} = \{ \varphi(x, b) \mid \neg \varphi(c, y) \not\in p \} \iff \varphi(c, y) \in p.
  \]

- If we have \( h \), define from it \( p \) and from it \( h' \) we find that for \( r \in S_r(A) \), for \( c \in V^x \) realizing \( r \), we have:
  \[
  \varphi(x, b) \in h'(r) = \{ \varphi(x, b) \mid \exists c' \in V^x c' \models r, \neg \varphi(c', y) \not\in p \} \iff \neg \varphi(c, y) \not\in h(\tp(c'/A)) \iff \varphi(x, b) \in h(\tp(c/A)) = h(r).
  \]

### 3.7.1. Bounded \( T \)

In this section we will assume that \( T \) is bounded, which will allow us to replace the type in Theorem [3.37] with a construct which may be easier to understand.

**Proposition 3.38.** Let \( U \) be the universal model of \( T \) and \( B \subseteq U \) a pc model of \( T \).

Then there is a bijection between \( \mathcal{L}_\pi \) homomorphisms from \( S(U) \) to \( S(B) \) and homomorphisms (embeddings) from \( B \) to \( U \).

**Proof.** By Theorem [3.37] an \( \mathcal{L}_\pi \) homomorphism from \( S(U) \) to \( S(B) \) corresponds to a \( U \)-invariant cu type \( p(y) \) over \( U \) such that \( p \cup \tp(B/\emptyset) \) (where \( b \) enumerates \( B \) and \( y \) is a corresponding variable tuple) is finitely satisfiable in \( U \). Since the cu type is over \( U \), being finitely satisfiable in \( U \) is the same as being consistent, and being \( U \)-invariant is an empty requirement. Therefore, the conditions on \( p \) are only that \( p \cup \tp(B/\emptyset) \), which is the same as saying that \( \tp(B/\emptyset) \subseteq p^- \). We can thus rephrase the correspondence as giving for any \( \mathcal{L}_\pi \) homomorphism a maximal positive type over \( U \) extending \( \tp(B/\emptyset) \). Since every such type is \( \tp(b'/U) \) for some \( b' \in U^y \) (since \( U \) is positively \( |U|^\pi \)-saturated), this in turn corresponds to a homomorphism \( h \) from
What happens when we replace \( U \) with a pc \( A \subseteq U? \)

**Remark 3.39.** Assume \( b' \in U^y \) is the tuple whose cu type corresponds to some \( \mathcal{L}_a \)-homomorphism from \( S(A) \) to \( S(B) \). \( \text{tp}^h_u(b'/U) \) is the set of basic open neighborhoods of \( b' \) inside \( U^y \) when we consider the positive topology on \( U^y \) (whose sub-basic closed sets are the pp definable sets over \( U \)). This means that \( \text{tp}^h_u(b'/U) \) is finitely satisfiable in \( A \) iff every basic open neighborhood of \( b' \) intersects \( A^y \), that is iff \( b' \in \overline{A}^y \). Note that \( \overline{A}^y \) (where we take \( \overline{A} \) in the pp topology on \( U \)) is closed in \( U^y \), thus \( A \subseteq \overline{A}^y \).

Since automorphisms of \( U \) are also homeomorphisms, we find that if \( U \) is Hausdorff in the pp topology, \( \overline{A} \) is \( A \) invariant and therefore \( \text{tp}^h_u(b'/U) \) is \( \text{Aut}(U/A) \)-invariant whenever it is finitely satisfiable in \( A \) (since of \( \sigma \in \text{Aut}(U) \) then \( \sigma(\text{tp}^h_u(b'/U)) = \text{tp}^h_u(\sigma^{-1}(b'/U)) \)).

However, for a general \( T \), since \( \text{tp}^h_u(b'/U) \cup \text{tp}^p(b'/\emptyset) \) is finitely satisfiable in \( A \), we can say more. For any subtuple \( b_0 \) of \( b' \) (with \( y_0 \) the corresponding subtuple of \( y \)), for any basic neighborhood \( O = \psi(U,c) \subseteq U^y \) and for any \( \varphi(y_0) \in \text{tp}^p(b'/\emptyset) \) we find that \( O \cap \varphi(A) \neq \emptyset \). Therefore we find that \( b' \in \overline{\pi^{-1}_{y_0}(\varphi(A))} \). By reversing the logic, we find that the property that \( b' \in \overline{\pi^{-1}_{y_0}(\varphi(A))} \) for any subtuple \( b_0 \) of \( b' \) and for any \( \varphi(y_0) \in \text{tp}^p(b'/\emptyset) \) is equivalent to \( \text{tp}^h_u(b'/U) \) being finitely satisfiable in \( A \).

We can name the property we found in the previous remark:

**Definition 3.40.** Assume \( A,B \) are structures in a language \( L \) and \( C \) is a subset of \( B \). Assume that \( B^A \) is equipped with a topology.

We say that \( f : A \rightarrow B \) is a hypo-homomorphism from \( A \) to \( C \) if whenever \( \varphi(x) \) is a conjunction of atomic formulas and \( a \in \varphi(A) \), we have \( f \in \overline{\pi^{-1}_a(\varphi(C))} \subseteq B^A \), where \( \pi_a \) is the projection on the \( a \) coordinates.

**Remark 3.41.** If we replace the requirement of being a conjunction of atomic formulas with being pp, the definition does not change.

**Proof.** One direction is immediate — any \( f \) satisfying the definition for pp formulas satisfies it for conjunctions of atomic formulas.

For the other, assume that \( \varphi(x,y) \) is a conjunction of atomic formulas, define \( \psi(x) = \exists y \varphi \) and assume \( a \in \psi(A) \). Then for some \( a' \in A^y \), \( A \models \varphi(a,a') \). But note that

\[
\pi^{-1}_{a,a'}(\varphi(C)) = \{ f | f(a) \land f(a') \in \varphi(C) \} \subseteq \{ f | f(a) \in \psi(C) \} = \pi^{-1}_a(\psi(C)),
\]

thus we get that \( f \in \overline{\pi^{-1}_{a,a'}(\varphi(C))} \subseteq \overline{\pi^{-1}_a(\psi(C))} \).
Since the closure of a union is the union of the closures, and preprojections respect unions, we can equivalently require that the property holds for any positive combination of atomic formulas, or for positive formulas. \[\square\]

Remark 3.42. The reasoning for the name is as follows: The set of homomorphisms from \(A\) to \(B\) is
\[
\bigcap_{\varphi \in \varphi(A)} \pi^{-1}_a(\varphi(C)),
\]
while the set of hypo-homomorphisms is \(\bigcap_{\varphi \in \varphi(A)} \pi^{-1}_a(\varphi(C))\); that is hypo-homomorphisms are the accumulation points of the (maybe improper) filter \(\mathcal{F} = \{\pi^{-1}_a(\varphi(C))\}\) whose intersection is the set of homomorphisms.

If (the preprojection of) every relation symbol is closed inside \(B^A\), a homomorphism accumulation to \(C\) is the same as a homomorphism to \(C\).

Remark 3.43. We can expand the proof of Lemma 2.24 to conclude that a hypo–homomorphism exists whenever there is a compact topology on \(M^A\). If we do this Lemma 2.24 becomes a special case.

3.8. Type-Definability Patterns.

Definition 3.44. Let \(T\) an irreducible theory. Then \(\text{Core}^{ip}(T)\), or \(\mathcal{J}^{ip}\) if \(T\) is assumed known, is defined to be \(\text{Core}(T^{ip})\) (see Subsection A.3). We likewise define \(\mathcal{L}^{ip}\), \(\mathcal{J}^{ip}_x\) and \(\mathcal{L}^{ip}_x\).

Lemma 3.45. Let \(M\) a positively \(\aleph_0\)-saturated \(pc\) and \(\aleph_0\)-homogeneous model of \(T\) (which is by Theorem A.44 a \(pc\) model of \(T^{ip}\)), and \(A \subseteq S(M)\) an \(\mathcal{L}^{ip}\) substructure.

Let \(i : \text{Aut}(M) \to S(M)^A\) defined to be \(i(\sigma)(a) = \{\varphi(x,\sigma(a)) | \varphi(x,a) \in p\}\). Then \(\overline{i(\text{Aut}(M))} = \text{Hom}_{\mathcal{L}^{ip}}(A,S(M))\) (when the closure it taken with respect to the product topology).

Proof. Note first that by Corollary A.46 the restriction from \(S(M)_{L^{ip}}\) to \(S(M)_L\) is a homeomorphism, so we can refer to \(S(M)\) without specifying the language with no ambiguity. By Corollary A.41 \(\text{Aut}(M) = \text{Aut}(M^{ip})\).

\(f : A \to S(M)\) is a homeomorphism iff for all positive \(\Psi_0(x_0,y), \ldots, \Psi_{k-1}(x_{k-1},y), \Phi(y)\) in \(L^{ip}\) and \(p_0, \ldots, p_{k-1} \in R_{\Psi_0,\ldots,\Psi_{k-1},\Phi}^A\), we have
\[
f(p_0), \ldots, f(p_{k-1}) \in R_{\Psi_0,\ldots,\Psi_{k-1},\Phi}^{S(M)} \iff \\
\forall a \in \Phi(M) : \bigvee_{i \in k} \Psi_i(x_i, a) \in f(p_i)
\]
which is a closed condition; thus \(\text{Hom}_{\mathcal{L}^{ip}}(A,S(M))\) is closed.

Since \(i(\sigma)\) is the restriction of an \(\mathcal{L}^{ip}\) automorphism to \(A\), \(i(\text{Aut}(M)) \subseteq \text{Hom}_{\mathcal{L}^{ip}}(A,S(M))\).

Take some basic open set \(U\) in \(S(M)^A\) intersecting \(\text{Hom}_{\mathcal{L}^{ip}}(A,S(M))\), which is of the form \(\{f | \forall i < k : \varphi_i(x_i, a) \notin f(p_i)\}\) for some fixed positive formulas \(\varphi_i(x_i, a)\) and \(p_i \in A\) (we may assume...
that the parameters in all formulas are identical by ignoring the irrelevant parameters). Let \( f \in U \cap \text{Hom}_{L^p}(A, S(M)) \). Let \( q = tp^p(a/\emptyset) \). Then we know that \( R_{\varphi_0(x_0, y), \ldots, \varphi_{k-1}(x_{k-1}, y), \varphi(y)}(f(p_0), \ldots, f(p_{k-1})) \) does not hold by definition, thus since \( f \) is a homomorphism we also get \( R_{\varphi_0(x_0, y), \ldots, \varphi_{k-1}(x_{k-1}, y), \varphi(y)}(p_0, \ldots, p_{k-1}) \) does not hold; that is there exists \( a' \) such that for all \( i < k \) we have \( \varphi_i(x_i, a') \notin p_i \). Let by homogeneity \( \sigma \in \text{Aut}(M) \) sending \( a' \) to \( a \). Then for all \( i < k \) we have \( \varphi_i(x_i, \sigma(a)) = \varphi_i(x_i, \sigma(a')) \notin i(\sigma)(p_i) \) that is \( i(\sigma) \in U \).

We conclude that \( \overline{i(\text{Aut}(M))} \subseteq \text{Hom}_{L^p}(A, S(M)) \) and that \( \overline{i(\text{Aut}(M))} \subseteq \text{Hom}_{L^p}(A, S(M))^c \) (since every open set that does not intersect \( i(\text{Aut}(M)) \) does not intersect \( \text{Hom}_{L^p}(A, S(M))^c \)), as required.

**Theorem 3.46.** Let \( M \) a positively \( \aleph_0 \)-saturated and \( \aleph_0 \)-homogeneous \( \mathcal{L}^p \) model of \( T \). Then \( \text{Aut}(J^p) \) is the Ellis group (see [KPRLE] Section 1.1) of the action of \( \text{Aut}(M) \) on \( S(M) \).

**Proof.** Let \( e : J^p \to S(M) \) an \( L^p \) immersion, and let \( J = e(J^p) \). Let \( r : S(M) \to J \) an \( L^p \) homomorphism such that \( r_J = \text{id}_J \) (which exists by Fact 2.8.2), where we consider \( r \) as an element of \( \text{Hom}_{L^p}(S(M), S(M)) \).

By Lemma 3.45, the Ellis semigroup \( ES \) of the action is \( \text{Hom}_{L^p}(S(M), S(M)) \). Note that \( r \in ES \) is an idempotent, since \( r \circ r = r_J \circ r = \text{id}_J \circ r = r. \) \( ESr \) is clearly a left ideal in \( ES \), let us show that it is minimal. Assume \( f \circ r \in ESr \). Then \( r \circ f \circ r \in \text{End}(J) = \text{Aut}(J) \), so let \( \sigma = (r \circ f \circ r \circ \text{id}_J)^{-1} \) (in \( \text{Aut}(J) \)) and we find that

\[
\sigma \circ r \circ f \circ r = \sigma \circ r \circ f \circ r \circ r = \sigma \circ r \circ f \circ r \circ \text{id}_J \circ r = r
\]

thus \( r \in ES(f \circ r) \) and thus \( ESr \subseteq ES(f \circ r) \subseteq ESr \).

Then since the Ellis group \( E \) is equal (up to isomorphism) to \( (rESr, \circ) \), so we need only show that \( (rESr, \circ) \) is isomorphic to \( \text{Aut}(J) \). For any \( r \in rESr \), \( r \circ f \circ r \in \text{Aut}(J) \), so let \( \psi : rESr \to \text{Aut}(J) \) be defined as \( \psi(r \circ f \circ r) = r \circ f \circ r \circ \text{id}_J = r \circ f \circ \text{id}_J \). This is a homomorphism, since for \( f, g \in ES \) we have

\[
r \circ f \circ r \circ g \circ r = r \circ f \circ r \circ \text{id}_J \circ r \circ g \circ r
\]

(since \( \text{id}_J \circ r = r \)). It is surjective, since if \( \sigma \in \text{Aut}(J) \) is arbitrary then \( \sigma \circ r \in ES \) and we find that \( r \circ \sigma \circ r \circ \text{id}_J = \text{id}_J \circ \sigma \circ \text{id}_J = \sigma \) (again, since \( r_J = \text{id}_J \)). Finally, if \( r \circ f \circ \text{id}_J = \text{id}_J \) then for any \( p \in S(M) \) we have that \( r(p) \in J \) thus \( r(f(r(p))) = r(f(\text{id}_J(r(p)))) = \text{id}_J(r(p)) = r(p) \), that is \( r \circ f \circ r = r \) is the identity in \( rESr \) and thus \( \psi \) is injective, as required.

**Corollary 3.47.** If \( M, N \) are positively \( \aleph_0 \)-saturated and \( \aleph_0 \)-homogeneous \( \mathcal{L}^p \) models of the same \( \mathcal{L} \) theory \( T \), then the Ellis groups of the actions \( \text{Aut}(M) \sim S(M) \) and \( \text{Aut}(N) \sim S(N) \) are isomorphic.
3.9. Examples. Let us compute $J$ in some specific cases. We will discuss two specific examples: the first, the doubled interval, will demonstrate the necessity of some of the assumptions made in various claims in this section. The other, Hilbert spaces, is a more “real world” example of computing the core, and is an example of the ways in which the core can reflect properties of the original theory.

3.9.1. Doubled Interval.

Example 3.48. Consider the theory in Example 2.47. Consider $Q = (\mathbb{Q} \cap [0, 1]) \times 2$ which as we remarked is a pc submodel of $M$.

Proposition 3.49. $S(M)$ is not pc (in $L$).

Proof. Let $p = \text{tp}^p\left(\left(\frac{1}{\pi}, 0\right) / M\right), q = \text{tp}^p\left(\left(\frac{1}{\pi}, 1\right) / M\right)$. By Corollary 3.20 $\text{Th}^{h_u}(S(M))_{L\pi} = \text{Th}^{h_u}(S(Q))_{L\pi}$ and by Theorem 3.23 $S(M)$ is universal for $\text{Th}^{h_u}(S(M))_{L\pi}$, thus there is a homomorphism $h : S(Q) \to S(M)$. On the other hand the restriction $r_Q : S(M) \to S(Q)$ (see Proposition 3.51) is also a homomorphism. Further, $r_Q$ is not injective, since $r_Q(p) = \text{tp}^p\left(\left(\frac{1}{\pi}, 0\right) / Q\right) = \text{tp}^p\left(\left(\frac{1}{\pi}, 1\right) / Q\right) = r_Q(q)$. Thus $h \circ r_Q : S(M) \to S(M)$ is a non-injective homomorphism. But by Proposition 3.27 if $S(M)$ was pc every endomorphism would have been an automorphism. □

Remark 3.50. Conversely, we get that there is no $L_\pi$ homomorphism from $S(Q)$ to $S(M)$, since every $L_\pi$ endomorphism of $S(M)$ is injective by Proposition 3.11 recalling Proposition 2.48.

Proposition 3.51. $S(M)$ and $S(Q)$ have distinct $L_\pi$ theories. In particular, $\exists \xi : D_{x_1} S_{x_2} (\xi) \land \pi_{x, x_1} (\xi) = \pi_{x, x_2} (\xi) \in \text{Th}^{h_u}(S(M))_{L_\pi} \setminus \text{Th}^{h_u}(S(Q))_{L_\pi}$.

Proof. $S(Q) = \exists \xi : D_{x_1} S_{x_2} (\xi) \land \pi_{x, x_1} (\xi) = \pi_{x, x_2} (\xi)$ (where $\xi$ is a variable from the sort $x = (x_1, x_2)$) — indeed

$$\text{tp}^p\left(\left(\left(\frac{1}{\pi}, 0\right)/\pi, 1\right) / Q\right) = D_{x_1} S_{x_2} (\xi) \land \pi_{x, x_1} (\xi) = \pi_{x, x_2} (\xi)$$

However $S(M) = \exists \xi : D_{x_1} S_{x_2} (\xi) \land \pi_{x, x_1} (\xi) = \pi_{x, x_2} (\xi)$ since $\text{tp}(a_1 a_2 / M) = D_{x_1} S_{x_2} (\xi)$ implies $a_1 S a_2 \Rightarrow a_1 \neq a_2 \Rightarrow \text{tp}(a_1 / M) \neq \text{tp}(a_2 / M)$. □

Corollary 3.52. $T_\pi$ is not strongly Robinson.

Proof. Let $p, q, h, r_Q$ as in the proof of Proposition 3.49 denote $s = h(r_Q(p)) = h(r_Q(q))$ and assume $T_\pi$ is strongly Robinson. Then there is a quantifier free positive type $\Sigma(\zeta_1, \zeta_2)$ in $L_\pi$ such that $S(M)$ thinks $\Sigma$ is equivalent to

$$\exists \xi : D_{x_1} S_{x_2} (\xi) \land \pi_{x, x_1} (\xi) = \zeta_1 \land \pi_{x, x_2} (\xi) = \zeta_2.$$
Since the only subtuple of a tuple of length 1 is itself, the only $\pi_{x_i, x'_i}$ that can appear in $\Sigma$ is the identity, thus $\Sigma$ is effectively an $L$ type. Since $\Sigma(p, q)$ holds, we get $\Sigma(s, s)$ holds thus $S(M) = D_{x_1, x_2} (\xi) \land \pi_{x, x_1} (\xi) = s \land \pi_{x, x_2} (\xi) = s$. But that is impossible, since $S(M) \models \neg \exists \xi : D_{x_1, x_2} (\xi) \land \pi_{x, x_1} (\xi) = \pi_{x, x_2} (\xi)$ by Proposition 3.51.

**Proposition 3.53.** $\text{Core}(T) = S(Q)$.

*Proof.* Take some $h \in \text{End} (S(Q))$. We will restrict ourselves to $S_1 (Q)$, but the argument is the same in every sort.

If $r \in [0, 1] \setminus Q$, $\text{tp}((r, 0)/Q) = \text{tp}((r, 1)/Q)$ as their transposition is an automorphism of $M$ sending one to the other (and fixing $Q$). We find that $h(\text{tp}((r, 0)/Q)) = \text{tp}((r, 0)/Q)$ where $r \in M \setminus Q$, since $D_{I, a, b}(\text{tp}((r, 0)/Q))$ for any $a < r < b$ and the only two elements of $M$ that satisfy this are $(r, 0)$ and $(r, 1)$.

This covers the non-realized types. For realized types, we find that

$$h(\{\text{tp}((q, 0)/Q), \text{tp}((q, 1)/Q)\}) = \{\text{tp}((q, 0)/Q), \text{tp}((q, 1)/Q)\}$$

(whre $q \in Q \cap [0, 1]$) since

$$D_{x=q, x=y; I, q(y)}(\text{tp}((q, 0)/Q), \text{tp}((q, 1)/Q))$$

holds.

So $h$ is surjective thus by Proposition 3.27 we find $S(Q)$ is $\mathcal{J}$. \hfill $\square$

**Proposition 3.54.** *Not all cu types over $M$ which are finitely satisfiable in $Q$ are $Q$ invariant.*

*Proof.* Since $I_{a, b} \subseteq M$ and $S \subseteq M^2$ are closed (in the product topology, in the case of $S$), $M$ is Hausdorff compact and projections are always continuous, we find that every positive $\emptyset$-definable set in $M^n$ is closed.

And since fibers of closed sets are closed, we get that indeed every positively definable set (even over a set) is closed.

Therefore every hu definable set is open, and thus contains an element of the dense set $Q$ — so every hu type is finitely satifiable in $Q$.

However, if we take for example $p = \text{tp}^{hu}(\big(\frac{1}{n}, 0\big)/M)$ we get that it is not $A$ invariant, since it is not invariant under e.g. $\sigma \in \text{Aut}(M/Q)$ which is the trasposition of $\big(\frac{1}{n}, 0\big)$ and $\big(\frac{1}{n}, 1\big)$ — since $\big(x \neq \big(\frac{1}{n}, 1\big)\big) \in p \setminus \sigma(p)$. \hfill $\square$

3.9.2. *Inner Product Spaces.* Let us now compute the core of the theory of Hilbert Spaces, introduces in Example 2.51. We will discuss inner product spaces over $\mathbb{R}$, but complex spaces behave in very much the same way.
Lemma 3.55. Let $\mathcal{H}$ be a Hilbert space.

The following are equivalent for tuples $\overline{a}, \overline{a}' \in \mathcal{H}^n$:

1. There is $\sigma \in \text{Aut}(\mathcal{H})$ such that $\sigma(\overline{a}) = \overline{a}'$
2. $\text{tp}^p(\overline{a} / \mathcal{O}) = \text{tp}^p(\overline{a}' / \mathcal{O})$
3. $\text{tp}^\text{at}(\overline{a} / \mathcal{O}) = \text{tp}^\text{at}(\overline{a}' / \mathcal{O})$
4. For any $i, j < n$ we have $\langle a_i, a_j \rangle = \langle a'_i, a'_j \rangle$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) We need only verify that under the assumptions $f \left( \sum_{i \in \mathbb{N}} \lambda_i a_i \right) = \sum_{i \in \mathbb{N}} \lambda_i a'_i$ is a well defined isomorphism (steming from the bilinearity of the inner product and the fact $\langle x, x \rangle = 0 \iff x = 0$), and that by choosing a suitable basis $f$ can be extended to an automorphism of $\mathcal{H}$.

Definition 3.56. Let $\mathcal{H}$ be an inner product space and $\overline{a} \in \mathcal{H}^n$. Then denote $M(\overline{a}) := (\langle a_i, a_j \rangle)_{i, j < n} \in M_n(\mathbb{R})$.

Corollary 3.57. If $\mathcal{H}_0 \leq \mathcal{H}_1$ are Hilbert spaces and $\overline{a} \in \mathcal{H}_1^n$ satisfies $\overline{a} \perp \mathcal{H}_0$ then $\text{tp}^p(\overline{a} / \mathcal{H}_0)$ depends only on the matrix $M(\overline{a})$.

Proof. Assume $M(\overline{a}) = M(\overline{a}')$ where $\overline{a}, \overline{a}' \perp \overline{b}$. Then for any $\overline{b} \in \mathcal{H}_0^n$ we find that if $i < n, j < m$ then $\langle a_i, b_j \rangle = 0 = \langle a'_i, b_j \rangle$, if $i, i' < n$ then by assumption $\langle a_i, a_{i'} \rangle = \langle a'_i, a'_{i'} \rangle$ and if $j, j' < m$ then obviously $\langle b_j, b_{j'} \rangle = \langle b_j, b_{j'} \rangle$.

Thus by Lemma 3.55 we have $\text{tp}^p(\overline{a}, \overline{b}) = \text{tp}^p(\overline{a}', \overline{b})$.

Proposition 3.58. Assume $\mathcal{H}_0$ is an infinite dimensional Hilbert space. Take some $p_0, \ldots, p_{k-1} \in S(\mathcal{H}_0)$ where $p_i \in S_{x_i}(\mathcal{H}_0)$. Let $\mathcal{H}_1 \supseteq \mathcal{H}_0$ be another Hilbert space which is infinite dimensional over $\mathcal{H}_0$ such that every $p_i$ is realized in $\mathcal{H}_1$ by some $a_i$.

Let $\tau \in \text{Aut}(\mathcal{H}_1)$ be such that $\tau(a_i) \in (\mathcal{H}_0^n)^n$ for all $i$ (for instance let $B$ be an orthonormal basis of $\text{Span}(a_i)_{i \in \mathbb{N}}$ and let $\tau$ sending $B$ to some orthonormal set of the same size in $\mathcal{H}_0^n$ by Lemma 3.55).

Define $q_i = \text{tp}^p(\tau(a_i) / \mathcal{H}_0)$. Then if $\epsilon(\xi_0, \ldots, \xi_{k-1})$ is an atomic formula in $L_\pi$ such that $\epsilon(p_0, \ldots, p_{k-1})$ holds, then also $\epsilon(q_0, \ldots, q_{k-1})$ holds.

Proof. Assume $\epsilon$ is of the form $\pi_{x_i, x'_i}(\xi_i) = \pi_{x_j, x'_j}(\xi_j)$. Then by assumption $\pi_{x_i, x'_i}(p_i) = \pi_{x_j, x'_j}(p_j)$ that is $\text{tp}^p(\pi_{x_i, x'_i}(a_i) / \mathcal{H}_0) = \text{tp}^p(\pi_{x_j, x'_j}(a_j) / \mathcal{H}_0)$ and in particular $\text{tp}(\pi_{x_i, x'_i}(a_i) / \mathcal{O}) = \text{tp}(\pi_{x_j, x'_j}(a_j) / \mathcal{O})$.

This implies $\text{tp}(\pi_{x_i, x'_i}(\tau(a_i)) / \mathcal{O}) = \text{tp}(\pi_{x_j, x'_j}(\tau(a_j)) / \mathcal{O})$. But let $b \in H_0^n$ be an arbitrary tuple; we find that every pair in $\pi_{x_i, x'_i}(\tau(a_i)) \sim b$ satisfies one of the following:

1. It is from $\tau(a_i)$ (in which case by assumption it has the same inner product as the corresponding pair in $\pi_{x_j, x'_j}(\tau(a_j))$).
2. It is in $b$. 
(3) It contains an element of \( \tau (a_i) \) and an element of \( \tilde{b} \), and thus by assumption has inner product 0 (and the same is true for the corresponding pair in \( \pi_{x_i,y}(\tau (a_j)) \sim b \)).

Thus by Lemma 3.55 we have \( \pi_{x_i,y}(q_i) = \pi_{x_j,y}(q_j) \).

Assume \( \epsilon \) is of the form \( D_{\varphi_0,\ldots,\varphi_{k-1};\alpha} \), and take some \( b \in \alpha (H_0) \). Let \( P \) be the projection onto \( H_0 \). Let \( \sigma \in \text{Aut} (H_0) \) be an automorphism sending \( b \) into \( P (a_i) \). Then \( \sigma (b) \in \alpha (H_0) \), thus for some \( i < k \) we have \( \varphi_i (x_i, \sigma (b)) \in p_i \) that is \( \varphi_i (a_i, \sigma (b)) \). However, \( \tau (a_i) \sim b \) and \( a_i \sim \sigma (b) \) satisfy 4 in Lemma 3.55 — since \( a_i = P (a_i) + (I - P) (a_i) \perp \sigma (b) \) and \( \tau (a_i) \perp b \), while both \( \tau \) and \( \sigma \) preserve inner products.

We conclude that \( \varphi_i (\tau (a_i), b) \) holds, that is \( \varphi_i (x_i, b) \in q_i \) as required. \( \square \)

**Corollary 3.59.** If \( \overline{p} \) is an arbitrary tuple of elements in \( S (H_0) \) then there exists a tuple \( \overline{q} \) such that \( \text{tp}^P (\overline{p}) \subseteq \text{tp}^P (\overline{q}) \) and every type in \( q \) is the type of an element perpendicular to \( H_0 \).

**Proof.** Take \( \overline{q} \) from Proposition 3.58 noting that it does not depend on \( \tau \) by Corollary 3.57. If \( \exists \zeta : \Phi (\zeta, \overline{p}) \) is a positive formula (where \( \Phi \) is quantifier free) such that \( \exists \zeta : \Phi (\zeta, \overline{p}) \) holds then let \( \overline{p} \) be such that \( \Phi (\overline{p}, \overline{p}) \) holds.

Then by Proposition 3.58 for some \( \pi \) we have that \( \Phi (\pi, \overline{q}) \) holds thus \( \exists \zeta : \Phi (\zeta, \overline{q}) \) holds as required. \( \square \)

**Remark 3.60.** If \( q \in S_x (H_0) \) is a type of a perpendicular element \( \overline{p} \) then

\[
D_{(x,y) = 0 \wedge \|x\| + \|y\| \leq 2N, (y,y) \leq 0 \wedge \|y\| \leq N} (q)
\]

for all \( i \) and for all \( N \geq \|a\| \), and such a \( D \) only holds for the types of perpendicular elements. Thus the positive type in \( L_\pi \) of a type (or tuple of types) is maximal iff each of these types is perpendicular.

**Corollary 3.61.** \( J_n \cong \{ \text{tp} (\pi / H) \mid \| \pi \| = n, \forall i < n : a_i \perp H \} \leq S_n^\text{max} (H) \), which by Corollary 3.57 and a well known result in the theory of inner product spaces can be indexed as

\[
\{ p_M \mid M \in M_n (R) \text{ symmetric and positive-semi-definite} \}
\]

(where \( \text{tp}^P (\pi) = p_M (\overline{p}) \)).

4. Partial Positive Patterns

We first present the construction of the core where as type spaces we take the space of all realized positive types over \( M \) (not just maximal) where by realized we mean the positive type of some element in an arbitrary continuation \( N \) of \( M \).\textsuperscript{16} As we note at the end of this section, in Subsection 4.5 this construction is less useful than the one in Section 3 which is the main focus of this paper. This section is presented here mainly for completeness.

\textsuperscript{16}That is \( N \) is not necessarily pc.
4.1. Basic Definitions. Let \( L \) be a language, \( T \) an irreducible primitive universal theory.

**Definition 4.1.** Take some \( M \models T^* \). We define for a homomorphism \( h : M \rightarrow N \models T \) and \( a \in N \)

\[
\text{tp}_h^\#(a/M) := \{ \varphi(x,c) \mid c \in M, \varphi \text{ positive}, N \models \varphi(a,h(c)) \}
\]

And then define

\[
S^+(M) = \{ \text{tp}_h^\#(a/M) \mid a \in N, h : M \rightarrow N \models T, h \text{ a homomorphism} \}
\]

(\( S^p(T) = \{ p \mid p \in S(T) \} \)).

Note that if \( h : M \rightarrow N \) is a homomorphism then \( N \models \text{Th}_{hu}(M)^- = T^- \), and in particular \( N \models T \)
iff \( N \models T^* \).

We include a sort for each arity of the type.

**Definition 4.2.** For any choice of positive formulas \( \langle \varphi_i(x,y) \rangle_{i \leq n} \), and \( \alpha(y) \), define

\[
\mathcal{R}_{\varphi_0,\ldots,\varphi_{n-1};\alpha}^S(M) = \left\{ \langle p_0,\ldots,p_{n-1} \rangle \mid \exists c \in M : \bigwedge_{i \leq n} \varphi_i(x,c) \in p_i \right\}
\]

Let \( \mathcal{L} \) be the language with equality and every \( \mathcal{R} \); we consider \( S^+(M) \) as an \( \mathcal{L} \) structure.

**Remark 4.3.** Note that since \( p \in S^+(M) \) is defined over a homomorphism but \( \alpha \) is computed in \( M \), \( \mathcal{R}_{\varphi,\alpha} \) is not equivalent to \( \mathcal{R}_{\varphi \land \alpha} \); indeed consider for example the language \( \{ E \} \) and the theory \( T = \{ \forall x : \neg xEx \} \) (the theory of directed graphs).

Let \( M \) be the empty graph on \( \{a,b\} \). Let \( N \) be \( a \rightarrow c \) and \( h = \text{id}_M \). Then \( \text{tp}_h^\#(c/M) \models \mathcal{R}_{y_1Ex \land Ey_2;y_1Ey_2} \) but not \( \text{tp}_h^\#(c/M) \models \mathcal{R}_{y_1Ex \land Ey_2 \land y_1Ey_2} \).

If however \( M \) is pc then the two are indeed equivalent.

4.2. Common Theory.

**Proposition 4.4.** If \( N \models \text{Th}_{hu}(M) \) then there exists a homomorphism \( g : N \rightarrow M^U \) for some ultrafilter \( \mathcal{U} \).

Furthermore, if \( h : M \rightarrow N \) is a homomorphism and \( M \) is pc, we can choose \( g \) such that \( g \circ h \) is the natural embedding \( e : M \rightarrow M^U \).

**Proof.** Let \( I \) be the set of quantifier free positive sentences in \( L_N \) that hold in \( N \). For any \( i \in I \), let \( I_i \) be the set \( \{ j \mid \text{Th}_{hu}(M) \models j \rightarrow i \} \); then \( \{ I_i \mid i \in I \} \) is a filter thus can be extended to an ultrafilter \( \mathcal{U} \). Make \( M^U \) into a model of \( L_N \) as follows:
Take some \( i = \varphi(\overline{a}) \in I \) (\( \varphi \) quantifier and parameter free, \( a \in N \)). Since \( \forall y \neg \varphi(y) \) is h, \( N \vDash \text{Th}^\text{hu} (M) \) and \( N \vDash \varphi(\overline{a}) \), it cannot be \( M \vDash \forall y \neg \varphi(y) \). Thus there exists \( \overline{a} \in M \) such that \( M \vDash \varphi(\overline{a}) \). Let \( M_i \) be the \( i \)th copy of \( M \) in \( M^I \), and define \( d^{M_i}_\overline{a} = \overline{a} \) and other \( d \)'s arbitrarily. Then for any \( i \in I \), for any \( j \in I_i \), \( M_j \models i \vdash i \) thus by Łoś’s theorem \( M^I \models i \).

For the furthermore, note that if \( i \) is \( \varphi (h(\overline{a}), \overline{b}) \) for \( \overline{a} \in M \) then since \( N \vDash \exists \overline{y} \varphi (h(\overline{a}), \overline{y}) \) we have also \( M \vDash \exists \overline{y} \varphi (\overline{a}, \overline{y}) \) from \( \text{pc} \), thus we can choose \( d^{M_i}_{\overline{a} \overline{y}} = \overline{a} \). This choice guarantees that for \( a \in M, g(h(a)) = d^{M_i}_{\overline{a} \overline{y}} = [(a)_i] = e(a) \). \( \square \)

**Remark 4.5.** The pc assumption is essential for the second part (\( h : M \to N \) being an embedding is also insufficient). Indeed consider for example \( Z \subseteq \mathbb{Q} \) in \( L = (\prec) \); then \( \text{Th}^Z (Z) = \text{Th}^\mathbb{Q} (\mathbb{Q}) \) (in particular \( \text{Th}^\mathbb{Q} (Z) = \text{Th}^\mathbb{Q} (\mathbb{Q}) \)) but in any ultrapower of \( Z \) we find that \( [(1)_i] \) is the successor for \( [(0)_i] \) thus it is impossible to embed \( \mathbb{Q} \) into the ultrapower over \( Z \).

**Lemma 4.6.** 1. Assume \( M, N \vDash T^* \), and assume \( h : M \to N \) is a homomorphism. Define for \( p \in S^+(N), h^* (p) = \{ \varphi(x,c) \mid \varphi(x,h(c)) \in p \} \).

Then \( h^* (p) \in S^+(M) \), and furthermore \( h^* \) is an \( \mathcal{L} \) homomorphism.

2. Let \( \mathcal{U} \) be an ultrafilter. Then there exists an \( \mathcal{L} \) homomorphism \( I : S^+(M) \to S^+(M^\mathcal{U}) \), defined as

\[
I(p) = \{ \varphi(x, [(c)_i]) \mid \{i \mid \varphi(x,c) \in p\} \in \mathcal{U} \}
\]

Furthermore, if we consider the natural embedding \( e : M \to M^\mathcal{U} \), then \( e^* \circ I \) is the identity.

3. If \( N \vDash \text{Th}^\text{hu} (M) \), then there is a homomorphism \( f : S^+(M) \to S^+(N) \).

Furthermore if \( M \) is \( \text{pc} \) model of \( \text{Th}^\text{hu} (M) \) and \( h : M \to N \) is a homomorphism then we can choose \( f \) to be a right inverse for \( h^* \).

**Proof.** 1. Take \( K \vDash T^* \) and \( g : N \to K \vDash T^* \) a homomorphism, and let \( a \in K \) such that \( p = \text{tp}_g^\mathbb{P} (a/N) \). Then for any \( \varphi(x,d) \in L_M \),

\[
\varphi(x,d) \in h^* (p) \iff \varphi(x, h(d)) \in p \iff K \vDash \varphi(a, g(h(d))) \iff \varphi(x, d) \in \text{tp}_{g,h}^\mathbb{P} (a/M)
\]

But \( g \circ h \) is a homomorphism thus \( h^* (p) \in S^+(M) \).

Furthermore assume \( R_{\varphi_0, \ldots , \varphi_{n-1}, \alpha} (p_0, \ldots , p_{n-1}) \) holds. Then for any \( c \in \alpha (N) \), there exists \( i < n \) such that \( \varphi_i(x,c) \notin p_i \).

Take some \( d \in \alpha (M) \); then since \( \alpha \) is positive and \( h \) is a homomorphism, \( h(d) \in \alpha (N) \). Therefore for some \( i < n, \varphi_i(x, h(d)) \notin p_i \Rightarrow \varphi_i(x,d) \notin h^* (p_i) \); thus \( R_{\varphi_0, \ldots , \varphi_{n-1}, \alpha} (h^* (p_0), \ldots , h^* (p_{n-1})) \) and thus \( h^* \) is a homomorphism.

2. Take some \( p \in S^+(M) \). Fix \( h : M \to N \vDash T^*, a \in N \) such that \( p = \text{tp}_h^\mathbb{P} (a/N) \). Then first we claim that \( h^\mathcal{U} ([(c)_i]) = [(h(c)_i)] \) is a well defined homomorphism from \( M^\mathcal{U} \to N^\mathcal{U} \); this is obvious from Łoś if we consider the 2-sorted structure \((M,N,h)\).
We claim \( I(p) = \text{tp}_{\mu}^p \left( [(a)]_{/M^d} \right) \). Indeed
\[
\varphi(x, [(c)]) \in \text{tp}_{\mu}^p \left( [(a)]_{/M^d} \right) \iff \\
N^d = \varphi([(a)], h^d([(c)]) \iff \\
N^d = \varphi([(a)], [(h(c)]) \iff \\
\{ i \mid N = \varphi(a, h(c)) \} \in U \iff \\
\{ i \mid \varphi(x, c) \in p \} \in U,
\]
therefore \( I(p) \in S^+(M^d) \).

We claim this is also a homomorphism — indeed if \( R_{\alpha, \alpha', \ldots, \alpha_{n-1}:\alpha}^S(p_0, \ldots, p_{n-1}) \), and \( [(c_i)] \in \alpha(M^d) \), then \( A = \{ i \mid c_i \in \alpha(M) \} \in U \). For any \( i \in A \), we find that there exists \( j < n \) such that \( \varphi_j(x, c_i) \not\in p_j \). Let \( A_j = \{ i \in A \mid \varphi_j(x, c_i) \not\in p_j \} \). Then \( A = \bigcup_{j<n} A_j \in U \), thus for some \( j < n \), \( A_j \in U \) (since \( U \) is an ultrafilter). We conclude that \( \{ i \mid \varphi_j(x, c_i) \not\in p_j \} \in U \), thus \( R_{\alpha, \alpha', \ldots, \alpha_{n-1}:\alpha}^S(p_0, \ldots, I(p_{n-1})) \) holds.

Let \( p \in S^+(M) \). Then
\[
\varphi(x, c) \in e^* (I(p)) \iff \\
\varphi(x, [(c)]) \in I(p) \iff \\
\{ i \mid \varphi(x, c) \in p \} \in U \iff \\
\varphi(x, c) \in p,
\]
thus \( e^* \circ I = \text{id}_{S^+(M)} \).

3. Let \( g : N \to M^d \) a homomorphism as in Proposition 4.4. Then \( g^* \circ I : S^+(M) \to S^+(N) \) is a homomorphism. Furthermore by Proposition 4.4, if \( M \) is pc and \( h : M \to N \) is a homomorphism then we can choose \( g \) such that \( g \circ h = e : M \to M^d \) thus \( h^* \circ (g^* \circ I) = (g \circ h)^* \circ I = e^* \circ I = \text{id} \). \( \square \)

**Theorem 4.7.** \( \text{Th}^{nu}(S^+(M)) = \text{Th}^{nu}(S^+(N)) \) for all \( M, N \models T^* \).

**Proof.** If \( f : S^+(M) \to S^+(N) \) is a homomorphism, then \( \text{Th}^{nu}(S^+(N)) \subseteq \text{Th}^{nu}(S^+(M)) \); since by Lemma 4.6.3 and Remark 2.5, such an \( f \) exists in both direction, we have equality. \( \square \)

**Definition 4.8.** We denote \( T^* = \text{Th}^{nu}(S^+(M)) \) for \( M \models T^* \) arbitrary.

Since the choice of \( M \) is arbitrary, we can assume \( M \) is pc. In this case we may assume that every homomorphism from \( M \) is the identity (since it must be an embedding).

### 4.3. Universality and Boundedness.

**Theorem 4.9.** Assume \( M \models T^* \). Then any model \( A \) of \( T^* \) admits a homomorphism into \( S^+(M) \). In particular if \( A = E \) is pc, it is embeddable in \( S^+(M) \).

In particular, \( T^* \) is bounded (by \( |S^+(M)| \) for arbitrary \( M \)) thus it has a universal pc model.
Proof. Consider the topology on \( S^+ (M) \) generated by the basis

\[
[\varphi] = \{ p \in S^+ (M) \mid \varphi \in p \}
\]

for all positive formulas \( \varphi \) (note that formally, \( p \) contains only positive formulas).

This space is compact, as usual (if \( \{ [\varphi_i]^C \}_{i < \kappa} \) is a family of basic closed sets with the f.i.p. then \( \{ \neg \varphi_i \}_{i < \kappa} \) is consistent with \( \Delta^+ M \) thus can be realized over a homomorphism).

Furthermore, for any \( R = R_{p_0, \ldots, p_{n-1}; \alpha} \), we have that

\[
R( S^+ (M)) = \left\{ p_0, \ldots, p_{n-1} \in S^+ (M) \mid \forall b \in \alpha (M) : \bigvee_{i < n} \varphi_i (x,b) \notin p_i \right\}
\]

\[
\cap \bigcup_{b \in \alpha (M) ; i < n} [\varphi (x,b)]^C
\]

thus closed.

So from Lemma 2.24 we are done. \( \square \)

**Definition 4.10.** We define \( \text{Core}^+ (T) \) to be the universal pc model of \( T^+ \) in the language \( \mathcal{L} \).

When \( T \) is fixed, we will denote \( \mathcal{J}^+ = \text{Core}(T) \).

### 4.4. Robinson

**Lemma 4.11.** Let \( T \) be an irreducible primitive universal theory. In the following we assume \( M \) is a pc model of \( T \).

1. In \( S^+ (M) \), every atomic formula is equivalent to an atomic relation; that is if \( \varphi (\zeta, \xi) \) is a formula consisting of a single relation symbol, it is equivalent to a binary relation symbol in \( \zeta, \xi \) — and likewise for formulas with more variables.
2. Assume \( |M| \geq 2 \) and \( M \) is pc. Then in \( S^+ (M) \), every finite conjunction of atomic formulas (none of which involves \( = \)) is equivalent to an atomic formula, and the choice of equivalent formula is independent of the model.
3. The family of atomic-type-definable subsets of \( S^+ (M) \) is closed under projection on all but one coordinate (that is if \( A \subseteq S^+ (M)^{k+1} \) for \( k \geq 1 \) is atomic-type-definable, then so is \( \pi_{1, \ldots, k} (A) = \left\{ (p_1, \ldots, p_k) \in S^+ (M)^k \mid \exists \beta : (p_0, \ldots, p_k) \in A \right\} \).

Furthermore the definition of the projection is independent of \( M \), that is for any partial atomic type \( \Sigma (x_0, \ldots, x_k) \) there exists \( \Pi (x_1, \ldots, x_k) \) such that \( \exists x_0 \Sigma \) is equivalent to \( \Pi \) in every \( S^+ (M) \).
4. Every pp formula \( \Xi (\mu) \) is equivalent in \( S^+ (M) \) to a possibly infinite (but no larger than \( |\mathcal{L}| = |L| \)) conjunction of atomic formulas.
Proof. The proof of this proposition is essentially identical to the proof of Lemma 3.18 with the following differences:

1. Here we observe $\varphi_i \land \varphi_j \not\in p \iff \varphi_i \not\in p \lor \varphi_j \not\in p$ (since $p$ is still the type of an element), and conclude

   $$S^+(M) = R_{\varphi_0, \ldots, \varphi_{n-1}, \psi_0, \ldots, \psi_{m-1}; a}(p, \ldots, p, q, \ldots, q) \iff$$

   $$S^+(M) = R_{\land_{icm} \varphi_i, \land_{icm} \psi_i; a}(p, q),$$

3. Here, we define (mirroring the role of $D_\theta$)

   $$\mathcal{R}^{\theta, i} := R_{\land_{j<k} \varphi_{ij}(x,y_j); \land_{j<k} \alpha_{ij}(y_j)}$$

   and are interested in pairs $\vec{t}, \theta$ such that

   $$(*) \, T \vdash \forall x', \vec{y}, z : \neg \left( \land_{j<k} \neg \varphi_{ij}(x', y_j) \land \theta(\vec{y}, z) \right)$$

   $$M \models \exists \vec{y}, z : \theta(\vec{y}, z),$$

   (note that we are missing negations relative to the original proof).

   We define

   $$\Sigma(x') := \{ \neg \varphi_i(x', a) \mid a \in \alpha_i(M), \varphi_i(x, a) \in q \} \cup \Delta_M^{at} \cup T.$$  

   Note that we require $\varphi_i(x, a) \in q$ rather than $\varphi_i(x, a) \not\in q$ like in the proof of Lemma 3.18.  

   This type is not positive, but we do not require that $p$ is maximal so we can take $p = \text{tp}^p(c/M)$ for any $c$ realizing $\Sigma$ in any continuation (which since $M$ is pc any continuation is necessarily an immersion).

   When proving that $q \in \pi_1(A)$ iff $\Sigma$ is inconsistent, we do not need that the extension model also be pc.

\[\square\]

**Theorem 4.12.** Let $T$ be an irreducible primitive universal theory.

1. Assume there is a pc model $M$ of $T$ such that $|M| \geq 2$. Then in every pc model of $T^+$, every finite conjunction of atomic formulas (none of which involves $=$) is equivalent to an atomic formula, and the equivalence is independent of the model.

2. In every pc model of $T^+$, every atomic formula is equivalent to an atomic relation — that is if $\varphi(\zeta, \xi)$ is a formula consisting of a single relation symbol, it is equivalent to a binary relation symbol in $\zeta, \xi,$ and likewise for formulas with more parameters — and the equivalence is independent of the model.

3. Every pp formula $\Xi(\mu)$ is equivalent in every pc model of $T^+$ to a possibly infinite (but no larger than $|L| = |L|$) conjunction of atomic formulas, and the equivalence is independent of the model.
4. \( \mathcal{J} \) is homogeneous for atomic type — if \( \text{tp}^\text{at}(\bar{a}) = \text{tp}^\text{at}(\bar{b}) \) for \( \bar{a}, \bar{b} \in \mathcal{J}^+ \) then there is an automorphism of \( \mathcal{J}^+ \) sending \( \bar{a} \) to \( \bar{b} \).

5. An atomic type in \( \mathcal{T}^+ \) is the type of an element of \( \mathcal{J}^+ \) iff it is maximal, that is there is no atomic type consistent with \( \mathcal{T}^+ \) that strictly extends it. In particular, if \( p \in S(M) \) belongs to some embedding of \( \mathcal{J}^+ \), the set of formulas represented in \( p \) is minimal.

**Proof.** Identical to the proof of Theorem 3.26 but based on Proposition 4.11 and Theorem 4.9. \( \Box \)

4.5. **Shortcomings of this Approach.** While this construction extends the one in [Hru20b] in a natural way, looking at actual examples reveals that the core is, in many cases, degenerate.

Indeed assume that \( L \) is relational 1-sorted language, and consider for \( M \models T^+ \) the extension model \( M' \) with universe \( M \cup \{\infty\} \) and exactly the same relations as \( M \) (that is, \( \infty \) is in no relation other than = to any element of \( M \) nor itself). It is clear that for any \( h : M \to N \models T^+ \) and \( a \in N \), we can extend \( h \) to a homomorphism \( h' : M' \to N \) by \( h' = h \cup \{ (\infty, a) \} \). In particular for \( h = \text{id}_M \) we find that \( \text{Th}^{\text{hu}}(M') \subseteq \text{Th}^{\text{hu}}(M) \subseteq \text{Th}^{\text{hu}}(M') \).

We also find that for any \( p \in S^+_1(M) \), \( \text{tp}^\text{at}(\infty/M) \subseteq p \). And so in particular every formula represented in \( \text{tp}^\text{P}(\infty/M) \) is represented in every element of \( S^+_1(M) \) thus by Theorem 4.12 \( \text{tp}^\text{P}(\infty/M) \) is the unique element of \( \mathcal{J}^+ \). A similar argument shows that the same holds for any sort of \( S^+_1(M) \), though note that the situation may be more complicated in non-relational languages (since the existence of functions implies some more interesting positive sentences).

One also notes that \( S^+_1(M) \) is never Hausdorff (let alone totally disconnected) — since a non-maximal type cannot be separated from a maximal type extending it — thus it is not homeomorphic to the type space of a first order theory.

To anyone familiar with positive logic, this result may be unsurprising, since the class commonly studied in positive logic is the class of \( \text{pc} \) models, rather than the class of all models. We will thus try and repeat the construction, but consider only those positive types that are realized in \( \text{pc} \) models.

**Appendix A. Appendix: Positive Morleyzation**

Here we describe how to get a \( \text{hu} \) theory given a continuous or first order theory, such that the spaces of types accurately represent the original types, and likewise with the models and morphisms. We will also present a way to, given a positive theory, construct a new positive theory in which every \( \emptyset \) type is definable.

Note that these constructions are a special case of [BY03, Theorem 2.23 and Theorem 2.38] and is also mentioned in e.g. [PY18, Section 2.3] (which lists earlier uses of the technique). Nevertheless we will show all steps explicitly for ease of reference.
A.1. **First Order Theories.** For our setting, let $L$ be a language and $T$ a consistent (not necessarily complete) first order theory in that language.

A.1.1. **Basic Definitions.**

**Definition A.1.** We define a new language $L^p$ with the same sorts as $L$ and a new theory $T^p$, as follows:

For any formula $\phi(x)$ in $L$ (where we choose for every formula a variable tuple $x$ which is minimal, though this is not very important), let $R_\varphi(x)$ be a relation symbol on the sorts given by $x$.

We define a new $L^p$-theory $T^p = \neg \exists x \bigwedge_{i<n} R_\varphi(x)$.

Given a model $M$ of $T$, we define a new model $M^p$ of $T^p$ as expected: $M^p$ has the same universe as $M$, and $R^p_\varphi = \varphi(M)$.

The fact that $M^p \models T^p$ is immediate from the definition of $T^p$ and $M \models T$. It is equally obvious that if $M_0, M_1 \models T$ then $h: M^p_0 \rightarrow M^p_1$ is an $L^p$ homomorphism iff the same function is an $L$ elementary embedding.

**Lemma A.2.** Every model of $T^p$ continues into a model of the form $M^p$ for $M \models T$.

**Proof.** Let $N$ be a model of $T^p$. Let $\bar{n}$ be a tuple enumerating $N$, let $\bar{x}$ be a corresponding variable tuple, and consider the type

$$ \Sigma(\bar{x}) = \{ \varphi(x') \mid x' \text{ a subtuple of } \bar{x}; n' \in R_\varphi^N \subseteq N^{x'} \}, $$

where $n'$ is the corresponding subtuple of $\bar{n}$ whenever $x'$ is a subtuple of $\bar{x}$.

By the definition of $T^p$, $\Sigma(x)$ is finitely consistent with $T$, thus by compactness there exists a model $M \models T$ and a tuple $\bar{m} \in M^x$ such that $M \models \Sigma(\bar{m})$.

But by the definition of $M^p$ and $\Sigma$ we get that $\bar{n} \rightarrow \bar{m}$ defines an $L^p$ homomorphism from $N$ to $M$. \hfill \square

**Proposition A.3.** If $M \models T$ then $M^p$ is a pc model of $T^p$.

**Proof.** By Claim 2.14 and Lemma A.2 it suffices to show that if $f: M^p \rightarrow N^p$ is an $L^p$ homomorphism for $N \models T$ then it is pc.
But if $\exists y \land R_{\varphi_i(x,y)}(x,y)$ is pp, $a \in M^x$ then

$$N^p = \exists y \land R_{\varphi_i(x,y)}(f(a),y) \iff$$

$$N = \exists y \land \varphi_i(f(a),y) \iff$$

$$M = \exists y \land \varphi_i(a,y) \iff$$

$$M^p = \exists y \land R_{\varphi_i(x,y)}(a,y)$$

since $f$ is elementary, as required. $\square$

A.1.2. Models of $T^p$ as Models of $T$.

**Proposition A.4.** For every pc model $E$ of $T^p$ there exists an $L$ structure $E_\sim$ with the same universe, such that for every $M \models T$, every $L^p$ homomorphism $h : E \to M^p$ is also an $L$ embedding to $M$.

In particular if $E = N^p$ then for $M = N$ and $h = id_M$ we get that the identity is an $L$ embedding (and thus an $L$ isomorphism, since it is surjective) from $(N^p)$ to $N$.

In other words, $N$ and $(N^p)$ are identical as $L$ structures.

**Proof.** Let $S(x)$ be a relation symbol in $L$; then we define $S^E = R^E_{S(x)}$. Let $F : x \to y$ be a function symbol in $L$; we want to define $F^E$.

Let $M \models T$ and $f : E \to M^p$ an $L^p$ homomorphism, which exists by Lemma A.2. Then for any $a \in E^x$ we have that

$$M \models \exists y : F(f(a)) = y \iff M^p \models \exists y : R_{F(x)=y}(f(a),y).$$

Since $E$ is pc we find $E \models \exists y : R_{F(x)=y}(a,y)$.

Furthermore, assume $b, b' \in E^y$ satisfy $E \models R_{F(x)=y}(a,b), R_{F(x)=y}(a,b')$. Then

$$M^p \models R_{F(x)=y}(f(a),f(b)), R_{F(x)=y}(f(a),f(b')) \iff$$

$$M \models F(f(a)) = f(b) = f(b') \Rightarrow f(b) = f(b') \Rightarrow$$

$$M^p \models f(b) = f(b') \Rightarrow E \models b = b',$$

since every homomorphism from a pc model is injective.

Therefore we can set $F^E = R^E_{F(x)=y}$ and it is a well defined function. And furthermore if $M_1 \models T$ and $h : E \to M^p_1$ is any $L^p$ homomorphism (thus embedding) then we get:
\[
E \models F^E(a) = b \iff \\
E = R_{F(x) = y}(a, b) \iff \\
M^P_1 = R_{F(x) = y}(h(a), h(b)) \iff \\
M_1 \models F(h(a)) = h(b). \\
\]

and

\[
\begin{align*}
E \models a \in S^- & \iff \\
E \models a \in R_{S(x)} & \iff \\
E \models h(a) \in R_{S(x)}^M & \iff \\
E \models h(a) \in S_M & \iff
\end{align*}
\]

Thus \( h : E \rightarrow M \) is an \( L \) embedding.

\[\square\]

**Proposition A.5.** If \( E \) is pc and \( f : E \rightarrow M^P \) is an \( L^P \) homomorphism then \( f : E \rightarrow M \) is an \( L \) elementary embedding; in particular, \( E \models T \).

Furthermore, \( (E)^P \) equals \( E \) for any pc model \( E \) of \( T^P \).

**Proof.** Note first that if for a given \( f : E \rightarrow M^P \) (where \( M \models T \)), and for a given \( L \) formula \( \varphi \), we have \( E \models \varphi(a) \iff M \models \varphi(f(a)) \) then we also have:

\[\ast \] \( E \models R_\varphi(a) \iff M^P \models R_\varphi(f(a)) \iff M \models \varphi(f(a)) \iff \sim E \models \varphi(a). \]

We will show by induction on the complexity of \( \varphi(x) \) that \( E \models \varphi(a) \iff M \models \varphi(f(a)) \).

If \( \varphi(x) \) is an atomic formula, this is just the fact that \( f \) is an \( L \) embedding by Proposition A.4.

Assume \( \varphi, \psi \) satisfy that

\[
\begin{align*}
E \models \varphi(a) & \iff M \models \varphi(f(a)) ; \\
E \models \psi(b) & \iff M \models \psi(f(b)) \\
\end{align*}
\]

Then

\[
\begin{align*}
E \models \varphi(a) \land \psi(b) & \iff \sim E \models \varphi(a), \psi(b) \iff \\
M \models \varphi(f(a)) \land \psi(f(b)) & \iff M \models \varphi(f(a)) \land \psi(f(b))
\end{align*}
\]

And likewise for \( \lnot \varphi \).

Finally if \( E \models \exists y \varphi(a, y) \) then for some \( b \in E^y \) we have \( E \models \varphi(a, b) \) thus \( M \models \varphi(f(a), f(b)) \Rightarrow M \models \exists y \varphi(f(a), y). \)
On the other hand, assume $M \models \exists y : \varphi(f(a), y)$. This is equivalent to

$$M^p \models \exists y : R_{\varphi(x,y)}(f(a), y) \iff E \models \exists y : R_{\varphi(x,y)}(a, y) \iff (\ast)$$

Thus we are done showing that $f$ is an elementary embedding and that $(E)^p = E$ (as this claim is exactly saying that $(\ast)$ holds for all $\varphi$).

Now since $f : E \to M$ is an elementary embedding then in particular $E \equiv M \Rightarrow E \equiv T$. $\square$

A.1.3. Properties of $T^p$.

**Proposition A.6.** $T$ is complete iff $T^p$ is irreducible.

**Proof.** We use Proposition A.5.3. Assume $T$ is complete and $M, N \models T^p$. Let $E_M, E_N$ be pc models of $T^p$ continuing $M, N$ respectively by Proposition 2.21. Then $E_M, E_N \models T$ and thus as $T$ is complete can be jointly elementarily embedded in some sufficiently saturated model $C \models T$.

But now by the definition of the $(\cdot)^p$ construction, the elementary embeddings of $E_M, E_N$ give $L^p$ homomorphisms from $(E_M)^p, (E_N)^p$ to $C^p$ — but by Proposition A.5 this means that $E_M, E_N$ both continue into $C^p$ and thus so do $M, N$ as required.

Assume on the other hand that $T^p$ is irreducible and $M, N \models T$. Then $M^p, N^p \models T^p$ thus by assumption there is some $P \models T^p$ such that $M^p, N^p$ continue into $P$, and by Proposition 2.21 there is a pc model $E \models T^p$ such that $P$ continues into $E$. We get that there are $L^p$ homomorphisms from $M^p, N^p$ to $E$ which by Proposition A.5 and Proposition A.3 give elementary embeddings from $(M^p), (N^p)$ to $E$.

But by Proposition A.4 we have $(M^p) = M, (N^p) = N$ thus $M \equiv E \equiv N$ that is $T$ is complete as required. $\square$

**Corollary A.7.** In pc models of $T^p$, every positive formula is equivalent to an atomic formula and the equivalence is independent of the model, and maximal types over $\emptyset$ in $L^p$ (which are thus determined by their atomic component) are equivalent naturally to complete $L$ types.

Furthermore the correspondence of types is a homeomorphism, and in particular, $S(\emptyset)$ is (sort-wise) totally disconnected (thus Hausdorff).

**Proof.** Quantifier elimination follows from the proof of Proposition A.5. The correspondence between types is given by $tp_{L^p}^P(a/\emptyset) \to tp_L(a/\emptyset)$ for $a \in E^\circ$, where we take $tp_L(a)$ in $E$. $\square$

**Theorem A.8.** If $T$ is a complete first order theory, then $\text{Core}(T^p)$ and $\text{Core}_{s^\ast}(T^p)$ are both well defined, and furthermore $\text{Core}_{s^\ast}(T^p)|_C = \text{Core}(T^p)$ and every symbol in $\text{Core}_{s^\ast}(T^p)$ is $\emptyset$-type definable in $\text{Core}(T^p)$ (in particular, $\text{Aut}(\text{Core}_{s^\ast}(T^p)) = \text{Aut}(\text{Core}(T^p))$).
Proof. By Corollary 3.24 and Theorem 3.31.

Remark A.9. For any $M \models T$ the positive topology on $M^p$ is discrete, since for any $a$, $R_{x 
eq y}(M^p, a)$ is closed thus $(R_{x 
eq y}(M^p, a))^c = \{a\}$ is open.

A.1.4. Relation to First order Definability Patterns. Defining the core for a first order theory by way of its positive Morleyization does not strictly generalize the construction in [Hru20b]. While the fact that [Hru20b] defines the relations to be $R_{\varphi_1, \ldots, \varphi_n; \alpha} = \{(p_1, \ldots, p_n) \mid \forall a \in \alpha(M) \text{ for some } i \varphi_i(x_i, a) \notin p_i\}$, one notes that $R_{\varphi_1, \ldots, \varphi_n; \alpha}$ is the same as $D_{\varphi_1, \ldots, \varphi_n; \alpha}$. The first order construction differs in two more ways. The first is that $T$ is assumed to be an irreducible universal theory rather than a complete theory, and the second is that all formulas are assumed to be quantifier free. In order to translate this construction to the positive framework one has to define $L^p$ to only include relation for arbitrary quantifier free formulas rather than arbitrary formulas. Then one must reformulate the construction in this paper to assume all formulas are quantifier free (though still positive) formulas.

However, for much of [Hru20b] the writer assumes that $T$ is the universal part of a first order theory $\overline{T}$ with quantifier elimination, in which case the existentially closed models of $T$ are in particular models of $\overline{T}$ thus pc models of $\overline{T}^p$, and thus every $L$ formula is equivalent to a quantifier free $L$ formula. We thus get that the assumption that the formulas that appear in $R_{\varphi_1, \ldots, \varphi_n; \alpha}$ are quantifier free is not needed. In this case we get that the core as defined in [Hru20b] is identical to $\text{Core} \left( \overline{T}^p \right)$ (under the obvious translation of the languages).

A.2. Continuous Logic.

A.2.1. Preliminaries. We will start with a brief overview of bounded continuous logic semantics. See e.g. [BYBHU08] for a more comprehensive overview.

We will make some simplifying assumptions in order to simplify notation, but generalizing the construction here is straightforward.

Definition A.10. A (single sorted) signature $L$ in the context of continuous logic consists of the following:

1. A set of predicate symbols $P_i$ together with, for each predicate symbol:
   
   (a) An arity $n(P_i) \in \mathbb{N}$.

2. A range $I_{P_i}$ (which is a closed bounded interval; we will assume $I_{P_i}$ is always $[0, 1]$).
   
   (b) A function $\Delta_{P_i} : (0, \infty) \to (0, \infty)$.

3. A set of function symbols $F_i$ together with, for each:
   
   (a) An arity $n(F_i) \in \mathbb{N}$.

   (b) A function $\Delta_{F_i} : (0, \infty) \to (0, \infty)$.

4. $D_L \in \mathbb{R}_{>0}$ (we will assume $D_L = 1$).
Definition A.11. A structure for a signature $L$ consists of:

1. A complete metric space $(M, d)$ with diameter at most $D_L$.
2. For each predicate symbol $P$, a function $P^M : M^{n(P)} \to I_P$ such that for all $\varepsilon > 0$, if $\langle a_i \rangle_{i \in n(P)}, \langle b_i \rangle_{i \in n(P)} \in M^{n(P)}$ satisfy $d(a_i, b_i) < \Delta_P(\varepsilon)$ for all $i < n(P)$ then
   \[ |P^M(a_i)_{i \in n(P)} - P^M(b_i)_{i \in n(P)}| \leq \varepsilon. \]
3. For each function symbol $F$, a function $F^M : M^{n(F)} \to M$ such that for all $\varepsilon > 0$, if $\langle a_i \rangle_{i \in n(F)}, \langle b_i \rangle_{i \in n(F)} \in M^{n(F)}$ satisfy $d(a_i, b_i) < \Delta_F(\varepsilon)$ for all $i < n(F)$ then
   \[ d\left(F^M\langle a_i \rangle_{i \in n(F)}, F^M\langle b_i \rangle_{i \in n(F)}\right) \leq \varepsilon. \]

If $d$ is a pseudometric or not complete, we say that $M$ is a prestructure. The information given by the language allows us to make the completion (of the induced metric space) of such a prestructure into a full $L$-structure (see [BYBHU08, Section 3, Prestructures]).

Definition A.12. A formula for continuous logic is constructed recursively just as in first order logic, except for the following:

- The equality relation symbol is replaced with the distance $d$.
- Instead of the logical connectives $\lor, \land, \lnot$, we have every uniformly continuous functions from $[0, 1]^k \to [0, 1]$.
  - We will in particular consider $|X - Y|$ and $X \div Y := \max\{X - Y, 0\}$.
- Instead of quantifiers $\exists x, \forall x$ we have $\inf x, \sup x$.

An example formula may be $\sup \inf |d(x, y) - P(F(x), G(y))|$.

Every formula $\varphi(x)$ (where $x$ has arity $k$) gives us a uniformly continuous function $\varphi_M : M^k \to [0, 1]$ for every structure $k$ (where the uniformity is independent of $M$ — see [BYBHU08, Theorem 3.5])

This is similar to the first order case, where every formula gives $\varphi_M : M^k \to \{T, F\}$; we can think of $\varphi_M$ as “distance from the truth”, in a sense.

Definition A.13. An embedding of continuous structures is an isometry $f : (M, d) \to (N, d)$ such that for every predicate symbol $P$ and $a \in M^{n(P)}$ we have $P^M(a) = P^N(f(a))$, and for any function symbol $F$ and $a \in M^{n(F)}$ we have $F^N(f(a)) = f(P^M(a))$.

An elementary embedding is a function $f : M \to N$ such that for any formula $\varphi(x)$ and $a \in M^x$ we have $\varphi^M(a) = \varphi^N(f(a))$. An elementary embedding is an embedding.

Remark. If $\varphi^M(a) = 0 \iff \varphi^N(f(a)) = 0$ then $f$ is an elementary embedding, since for any formula $\varphi$ and $r \in [0, 1]$ we have

$\varphi^M(a) = r \iff |\varphi - r|^M(a) = 0 \iff |\varphi - r|^N(f(a)) = 0 \iff \varphi^N(f(a)) = r.$
Definition A.14. A condition \( E \) is a requirement of the form \( \varphi(x) = 0 \). Naturally, we say that \( M \models E[a] \) iff \( \varphi^M(a) = 0 \). We say that a condition is over \( A \subseteq M \) if the formula has only parameters from \( A \).

We also write \( M \models \Sigma[a] \) when \( \Sigma(x) = \{ E_i(x) \}_{i \in I} \) is a set of conditions to denote that \( M \models E_i[a] \) for all \( i \in I \).

Remark A.15. Note that if \( \varphi, \psi \) are formulas then \( |\varphi(x) - \psi(x)| = 0 \iff \varphi(x) = \psi(x) \) and \( \varphi(x) \vdash \psi(x) = 0 \iff \varphi(x) \leq \psi(x) \); we will use these as shorthands, especially where \( \psi = r \) for \( r \in [0,1] \).

Note furthermore that \( \varphi = 0 \) and \( \psi = 0 \) iff \( \varphi + \psi = 0 \) and in general \( \bigwedge_{i \in I} \varphi_i = 0 \iff \sum_{i \in I} \varphi_i = 0 \).

Likewise \( \prod_{i \in I} \varphi_i = 0 \) iff \( \forall_i \varphi_i = 0 \).

Definition A.16. \( E \) is closed if the formula \( \varphi \) has no free variables.

A theory \( T \) is a set of closed conditions.

A structure \( M \) is a model of a theory \( T \) if \( M \models E \) for all \( E \in T \).

\( E \) is a logical consequence of \( T \) if every model \( M \) of \( T \) satisfies \( M \models E \).

A theory \( T \) is complete if for any formula \( \varphi \) without free variables \( T \) implies the condition \( |\varphi(x) - r| = 0 \) for some \( r \in [0,1] \).

Fact A.17. ([BYBHU08 Theorem 3.7]) If we take a prestructure and turn it into a structure, any conditions fulfilled by the prestructure remain fulfilled, since the infimum and supremum on continuous functions stay the same when we go to the completion.

Definition A.18. Assume \( T \) is complete (this assumption is not strictly necessary).

The space of types of arity \( n \) for a set \( A \subseteq M \) is \( S_n(A) = \{ \text{tp}(a/A) \mid a \in N^n, N \text{ an elementary extension of } M \} \)

where \( \text{tp}(a/A) = \{ \varphi(x) = 0 \mid \varphi \text{ has only parameters from } A \text{ and } \varphi^N(a) = 0 \} \).

It has a natural Hausdorff compact topology where closed sets are \( C_{\Sigma} = \{ p \in S_n(A) \mid \Sigma \subseteq p \} \) where \( \Sigma \) is a set of conditions over \( A \) (see [BYBHU08 Definition 8.4, Lemma 8.5, Proposition 8.6]).

Definition A.19. We say that a model \( M \) is \( \kappa \)-saturated if whenever \( \Sigma(x) \) is a set of conditions over \( A \subseteq M \) such that:

- \( |A| < \kappa \).

- For any finite \( \Sigma_0 \subseteq \Sigma \) and every \( \varepsilon > 0 \), there is some \( a \in M^\varepsilon \) such that \( M \models \Sigma_0[a] \).

Then there is some \( a \in M^\varepsilon \) such that \( M \models \Sigma[a] \).

Claim A.20. If \( M \) is \( \omega \)-saturated, \( \left( \inf_{x} \varphi(x) \right)^M = 0 \) implies that for some \( a \in M^\varepsilon \), \( \varphi^M(a) = 0 \), since \( \Sigma(x) = \{ \varphi \leq \frac{1}{n} \}_{n<\omega} \) is finitely satisfied by assumption and only satisfied when \( \varphi = 0 \) is.
Fact A.21. 1. ([BYBHU08, Theorem 5.12]) If \( T \) is a theory and \( \Sigma(\{x_j : j \in J\}) \) a set of conditions consistent with \( T \) then there exist a model \( M \) if \( T \) and elements \( \{a_j : j \in J\} \subseteq M \) such that \( M \models \Sigma[a_j : j \in J] \) (here \( J \) can be infinite).

2. ([BYBHU08, Proposition 7.10]) If \( M \) is a continuous structure and \( \kappa \) is a cardinal then \( M \) has a \( \kappa \)-saturated elementary extension.

A.2.2. Basic Definitions.

Definition A.22. Let \( L \) be a continuous logic signature and \( T \) a continuous consistent theory in \( L \).

Define the language \( L^p \) consisting of a predicate symbol \( Z_{\varphi}(x) \) for every formula \( \varphi(x) \) (and of the same arity).

Let \( T^p \) be the \( pu \) theory

\[
\neg \exists x : \bigwedge_{i<k} Z_{\varphi_i}(x)
\]

There is no model \( M \models T \) and \( a \in M^x \) such that \( \varphi_i^M(a) = 0 \) for all \( i < k \).

Given an \( L \)-structure \( M \) we define \( M^p \) to be the \( L^p \) structure with the same universe and with \( Z_{\varphi}^M = (\varphi^M)^{-1}(\{0\}) \).

Proposition A.23. 1. \( M \models T \Rightarrow M^p \models T^p \).

2. \( f : M \to N \) is an \( L^p \) homomorphism iff it is an \( L \) elementary embedding.

Proof. (1) For any formula \( \neg \exists x : \bigwedge_{i<k} Z_{\varphi_i}(x) \) in \( T^p \) we have that in particular for any \( a \in M^x \) there is some \( i < k \) such that \( \varphi_i^M(a) \neq 0 \), thus \( M^p \models \neg \bigwedge_{i<k} Z_{\varphi_i}(a) \).

Therefore \( M^p \models T \).

(2) Assume \( f \) is an \( L \) elementary embedding. Then for any formulas \( \varphi(x) \) and \( a \in M^x \) we have

\[
M^p \models Z_{\varphi}(a) \Rightarrow \varphi^M(a) = 0 \Rightarrow \\
\varphi^N(f(a)) = 0 \Rightarrow N^p \models Z_{\varphi}(f(a))
\]

If \( f \) is an \( L^p \) homomorphism then for any formula \( \varphi(x) \) and \( a \in M^x \) we have

\[
M^p \models Z_{\varphi(x)-\varphi^M(a)}(a) \Rightarrow \\
N^p \models Z_{\varphi(x)-\varphi^M(a)}(f(a)) \Rightarrow \\
|\varphi^N(f(a)) - \varphi^M(a)| = |\varphi(x) - \varphi^M(a)|^N(f(a)) = 0 \Rightarrow \\
\varphi^N(f(a)) = \varphi^M(a).
\]

□
A.2.3. Models of $T$ as models of $T^p$.

Lemma A.24. If $N \vDash T^p$ then it continues into a model of $T^p$ of the form $M^p$ for $M \vDash T$; furthermore we may assume $M$ is $\kappa$-saturated for some $\kappa$.

Proof. Using Fact A.21, the proof is identical to Lemma A.2. For the “furthermore” we again use Fact A.21, together with Proposition A.23.2. \hfill \Box

Proposition A.25. If $M \vDash T$ and $M$ is $\omega$-saturated then $M^p$ is a pc model of $T^p$.

Proof. By Claim 2.14 and Lemma A.24 it suffices to show that if $f : M^p \to N^p$ is a homomorphism for $N$ an $\omega$-saturated $T$ model then $f$ is pc.

But if $\exists y \land Z_{\varphi_i(x,y)}$ is pp, $a \in M^x$ then by Proposition A.23

\[
N^p \vDash \exists y \land Z_{\varphi_i(x,y)}(f(a),y) \iff \exists y \in N \land \varphi_i^N(f(a),y) = 0 \iff \exists y \in N \left( \sum_{i < n} \varphi_i(x,y) \right)^N / n \cdot (f(a),y) = 0 \iff \hspace{0.5cm}
\]

\[
\left( \inf_y \left( \sum_{i < n} \varphi_i(x,y) \right) \right)^N / n \cdot (f(a)) = 0 \iff \hspace{0.5cm}
\]

\[
\left( \inf_y \left( \sum_{i < n} \varphi_i(x,y) \right) \right)^M / n \cdot (a) = 0 \iff \hspace{0.5cm}
\]

\[
\exists y \in M \land \varphi_i^M(a,y) = 0 \iff \hspace{0.5cm}
\]

$M^p \vDash \exists y \land Z_{\varphi_i(x,y)}(a,y),$ as required. \hfill \Box

A.2.4. Models of $T^p$ as Models of $T$. For this section, we take $E$ to be a pc model of $T^p$.

Lemma A.26. For any function symbol $F$ and for any $a \in E^n(F)$ there is a unique $b \in E$ such that $E \vDash Z_{d(F(x),y)}(a,b)$; we can thus define $F^E : E^n(F) \to E$ as $F^E(a) = b$.

Furthermore $F^E$ commutes with $L^p$ homomorphisms.

Proof. Existence:

Let $f : E \to M^p$ be an $L^p$ homomorphism as in Lemma A.24.

Then $d^M(F^M(f(a)),F^M(f(a))) = 0$ thus $M^p \vDash Z_{d(F(x),y)}(f(a),F^M(f(a)))$ thus $M^p \vDash \exists y Z_{d(F(x),y)}(f(a),y)$ thus $E \vDash \exists y Z_{d(F(x),y)}(a,y)$ (since $E$ is ps).

Uniqueness:

Assume $b,b'$ both satisfy the condition. Then

\[
M^p \vDash Z_{d(F(x),y)}(f(a),f(b)), Z_{d(F(x),y)}(f(a),f(b')),
\]
that is
\[ d\left(F^M(f(a)), f(b)\right) = d\left(F^M(f(a)), f(b')\right) = 0, \]
thus \( f(b) = f(b') \) since homomorphisms from pc models are injective.

Since we defined \( F^E \) using an LP relation symbol, it is obviously preserved by homomorphisms, and note that \( M^p \models Z_{d(F(x), y)}(a, b) \iff F^M(a) = b. \)

\[ \square \]

**Lemma A.27.** For any \( L \) formula \( \varphi(x), e^E_{\varphi} := \{(a, r) \mid a \in E^x, r \in [0, 1], E \models Z_{\varphi}\} \) is a well defined function.

Furthermore, for and \( M \models T \) and \( f : E \rightarrow M^p \) an LP homomorphism we have \( e^M_{\varphi}(a) = \varphi^M(f(a)) \) for all \( a \in E^x. \)

**Proof.** We need to show that for any \( \varphi \) and any \( a \) there is a unique \( r \in [0, 1] \) such that \( E \models Z_{\varphi}\).

For uniqueness, note that since in every continuous model and for any \( r \neq r' \) we have at most one of

- \( \varphi^M(a) = r \iff |\varphi - r|^M(a) = 0. \)
- \( \varphi^M(a) = r' \iff |\varphi - r|^M(a) = 0. \)

by definition \(-\exists x : Z_{\varphi}\) \( \models T_p. \)

For existence and the “furthermore”, let \( f : E \rightarrow M^p \) be an LP homomorphism for \( M \) a model of \( T \) as in Lemma [A.24]

Then for \( r = \varphi^M(a) \) we have \( M^p \models Z_{\varphi}\) \( f(a) \) thus \( E \models Z_{\varphi}. \)

\[ \square \]

**Proposition A.28.** \( e^E_{\varphi} \) has the following properties (we omit the \( E \) from the notation):

1. \( e_d \) is a metric on \( E \) with diameter \( \leq D_L. \)
2. If \( F \) is a function symbol, \( a, b \in E^{n(F)}, \varepsilon > 0 \) and \( e_d(a_i, b_i) < \Delta_F(\varepsilon) \) for all \( i < n(F) \) then \( d(F^E(a), F^E(b)) \leq \varepsilon. \)

Likewise if \( P \) is a predicate symbol, \( a, b \in E^{n(P)}, \varepsilon > 0 \) and \( e_d(a_i, b_i) < \Delta_P(\varepsilon) \) for all \( i < n(F) \) then \( |e_p(a) - e_p(b)| \leq \varepsilon. \)

3. \( e_{\left(\_\right)} \) respects connectors; that is if \( \rho : [0, 1]^k \rightarrow [0, 1] \) is uniformly continuous and \( \{\varphi_i\}_{i<k} \) are formulas then \( e_{\rho(\varphi_0, \ldots, \varphi_{k-1})} = \rho \circ (e_{\varphi_0}, \ldots, e_{\varphi_{k-1}}). \)
4. \( e_{\left(\_\right)} \) respects continuous quantifiers; that is if \( \varphi(x, y) \) is a formula then \( e_{\inf_{\varphi(x, y)}} = \inf_{\inf_x} e_{\varphi(x, y)} \) and likewise for sup.

Furthermore, the infimum/supremum is always a minimum/maximum (that is the value is attained).

**Proof.** Let us first fix \( f : E \rightarrow M^p \) an LP homomorphism for \( M \models T_{\omega}\)-saturated, as in Lemma [A.24]

(1) Most of the requirements are immediate consequences of Lemma [A.27] for \( f. \)

The only part with noting is that we are using the fact that \( f(a) = f(b) \iff a = b \) (since homomorphisms from a pc model are injective.)
(2) Again, this must hold since it holds in \( M \). We will spell out the case of a function symbol:

\[
\forall i : e_d(a_i, b_i) < \Delta_P(\varepsilon) \Rightarrow \\
\forall i : d^M(f(a_i), f(b_i)) < \Delta_P(\varepsilon) \Rightarrow \\
d^M(F^M(f(a)), F^M(f(b))) \leq \varepsilon \Rightarrow \\
d^M(f(F^E(a)), f(F^E(b))) \leq \varepsilon \Rightarrow \\
e_d(F^E(a), F^E(b)) \leq \varepsilon.
\]

(3) This again holds since the same holds in \( M \).

(4) We will show this for \( \inf \), with \( \sup \) being analogous.

Choose \( b \in E^y \) and define \( r = e_{\inf \varphi(x,y)}(b) \). Then for any \( a \in E^x \)

\[
r = \left( \inf_x \varphi \right)^M_{\varphi} (f(b)) = \inf_{a \in M^x} \varphi^M(a', f(b)) \leq \\
\inf_{a \in M^x} \varphi^M((a), f(b)) = e_\varphi(a, b).
\]

On the other hand, for any \( r' > r \) we have \( r' > \inf_{a \in M^x} \varphi^M(a', f(b)) \) thus for some \( a' \in M^x \) we have \( \varphi^M(a', f(b)) \leq r' \).

So \( \{ \varphi(x, f(b)) \leq r' \}_{1 \leq r' \leq r} \) is finitely satisfiable in \( M \) thus as \( M \) is saturated there is \( a'_0 \in M^x \) such that \( \varphi^M(a'_0, f(b)) \leq r \) — but we saw \( \varphi^M(a'_0, f(b)) \geq \inf_{a \in M^x} \varphi^M(a', f(b)) = r \) thus \( \varphi^M(a'_0, f(b)) = r \).

We get \( M^P \models \exists x : Z_{\varphi^x}(x, f(b)) \) and so from \( \text{pc } E \models \exists x : Z_{\varphi^x}(x, b) \) that is for some \( a \in E^x \) we have \( Z_{\varphi^x}(a, b) \) that is by definition \( e_\varphi(a, b) = r \).

And thus \( r \leq \inf_{a \in E^x} e_\varphi(a, b) \leq r \) as required. \( \Box \)

**Corollary A.29.** The universe of \( E \) together with \( F^E \) and \( P^E := e_P \) is an \( L \)-prestructure we will denote \( \bar{E} \).

For any formula \( \varphi \), \( \varphi^E \equiv e_\varphi \).

Any \( f : E \to M^P \) (or to another \( \text{pc model of } T^P \) ) is also an \( L \) elementary embedding from \( E \to M \) (in particular an isometry). In particular, the completion of \( E \) is a model of \( T \) (see Fact A.17).

The constructions \( (\cdot)^P \) and \( \preceq \) are inverses, when applicable.

**Claim A.30.** If \( E \) is \( \omega_1 \)-positively saturated then \((E, e_d)\) is complete.

**Proof.** Let \( f : E \to M^P \) be an \( L^P \) homomorphism. Let \( (a_n)_{n \in \omega} \) be a Cauchy sequence in \((E, e_d)\).

Since \( f \) is also an isometry by Corollary A.29 \( F^E((a_n))_{n \in \omega} \) is also a Cauchy sequence thus has a limit \( b_\infty \). Define \( r_n = d^M(f(a_n), b_\infty) \).

For any finite \( I_0 \subseteq \omega \), \( M^P \models \exists y : \bigwedge_{n \in I_0} Z_{d=r_n}(f(a_n), y) \) thus \( E \models \exists y : \bigwedge_{n \in I_0} Z_{d=r_n}(a_n, y) \).
By \( \omega_1 \)-saturation, \( \{ Z_{d \cdot r_n} (a_n, y) \}_{n \in \omega} \) is satisfied by some \( a_\infty \in E \); but this means \( c_d (a_n, a_\infty) = r_n \to 0 \) thus \( a_n \to a_\infty \) in \( (E, c_d) \) as required.

\[ \square \]

A.2.5. Properties of \( T^p \).

**Proposition A.31.** In a pc model of \( T^p \), every positive formula is equivalent to an atomic formula.

**Proof.** By Remark [A.15] and Proposition [A.28] 3 and 4, we may replace:

- \( Z_\varphi \land Z_\psi \) by \( Z_{\varphi \land \psi} \).
- \( Z_\varphi \lor Z_\psi \) by \( Z_{\varphi \lor \psi} \).
- \( \exists x : Z_\varphi \) by \( Z_{\varphi} \).

And thus we can proceed by induction on complexity.

**Proposition A.32.** \( T \) is complete iff \( T^p \) is irreducible.

**Proof.** Assume \( T \) is complete. Let \( M \) be an \( \omega \)-saturated model of \( T \), which exists by Fact [A.21]. Then by Proposition [A.25], \( M \) is a pc model of \( T^p \), thus to show \( T^p \) is irreducible it is enough (by Proposition [2.9]) to show that \( \text{Th}^{\text{pu}} (M^p) \subseteq T^p \).

Assume \( \{ \varphi_i (x) \}_{i \in k} \) are \( L \) formulas such that \( \forall x \lor_{i \in k} \neg Z_{\varphi_i} (x) \notin T^p \). Then by definition of \( T^p \) there is a model \( N \) of \( T \) and \( a \in N^x \) such that \( \varphi_i^N (a) = 0 \) for all \( i < k \). This means that for \( \psi = \inf_x \sum_{i \in k} \varphi_i (x) \) we have \( \psi^N = 0 \).

Since \( T \) is complete there exists \( r \in [0, 1] \) such that \( T \models | \psi - r | = 0 \). It cannot be \( r > 0 \) as that would imply (by definition of \( \models \) and \( N \) being a model of \( T \)) that \( \psi^N = 0 \); therefore \( T \models | \psi - 0 | = 0 \) that is \( T \models \psi = 0 \).

In particular, \( \psi^M = 0 \). Since \( M \) is \( \omega \)-saturated, that means by Claim [A.20] that for some \( a \in M^x \) we have

\[
0 = \left( \frac{\sum_{i \in k} \varphi_i (x)}{k} \right)^M (a) = \frac{\sum_{i \in k} \varphi_i^M (a)}{k} \Rightarrow
\forall i < k \varphi_i^M (a) = 0 \Rightarrow \forall i < k a \in Z_{\varphi_i}^{M^p},
\]

thus \( \forall x \lor_{i \in k} \neg Z_{\varphi_i} (x) \notin \text{Th}^{\text{pu}} (M^p) \) as required.

Assume \( T^p \) is irreducible and let \( \psi \) be an \( L \) formula without parameters. Fix \( M \) to be some an \( \omega \)-saturated model of \( T \). We claim that for \( r = \psi^M \), \( T \models | \psi - r | = 0 \).

Indeed let \( N \) be an arbitrary model of \( T \). By Fact [A.21] take \( N_\omega \) to be an \( \omega \)-saturated elementary extension of \( N \). Then \( N^p_+, M^p \) are pc models of \( T^p \), and by irreducibility they continue into the same model \( C \models T^p \). But now \( M^p \models Z_{|\psi - r|} \) thus \( C \models Z_{|\psi - r|} \) thus since \( N_\omega^p \) is pc \( N_\omega \models Z_{|\psi - r|} \) that is \( | \psi - r |^{N_\omega} = 0 \); and since \( N_\omega \) is an elementary extension of \( N \) we find \( | \psi - r |^{N} = 0 \) as required.
Corollary A.33. $(\cdot)^p$ and $\sim$ are natural bijections between the class of $\omega_1$ positively saturated pc models of $T^p$ and the class of $\omega_1$-saturated models of $M$.

Proof. Assume $M = T$ is $\omega_1$-saturated. Then in particular $M$ is $\omega$-saturated thus $M^p$ is a pc model of $T^p$.

Furthermore assume $A \subseteq M$ is at most countable, and $\Sigma(x)$ is a partial positive type (in $L^p$) over $A$ which is finitely satisfiable in $M^p$.

Then by Proposition A.31 every positive formula $\Phi$ in $\Sigma$ is equivalent in $M^p$ to an atomic formula of the form $\varphi(\Phi)(x)$.

Consider the set of conditions $\{\varphi(\Phi) = 0 \mid \Phi \in \Sigma\}$. By assumption it is finitely satisfiable in $M$, thus by assumption there exists $a \in M$ such that

$$\varphi(\Phi)^M(a) = 0 \Rightarrow a \in Z^M_{\varphi(\Phi)} \Rightarrow a \in \Phi(M)$$

for all $\Phi \in \Sigma$; thus $M^p$ is $\omega_1$ positively saturated.

Conversely, assume $E$ is an $\omega_1$ positively saturated pc model of $T^p$. Then $E^p$ is an $L$ structure and a model of $T$ by Claim A.30 and Corollary A.29. Furthermore assume $A \subseteq E$ is at most countable and $\Sigma(x)$ is a set of conditions over $A$ which is finitely satisfiable in $E^p$.

Then since (again by Corollary A.29) $(E^p)^p = E$ we get that $\{Z_\varphi(x) \mid \varphi(x) = 0 \in \Sigma\}$ is finitely satisfiable in $E$, that is exists $a \in E$ such that $a \in Z^E_{\varphi} \iff \varphi^E(a) = 0$ for all $\varphi$ such that $\varphi(x) = 0 \in \Sigma$, that is $E$ is $\omega_1$-saturated. □

Corollary A.34. Since every type (in either $L$ or $L^p$) is realized in an $\omega_1$-saturated extension, $S(M^p)$ is equivalent to the space of continuous types $S(M)$ (via $p \rightarrow \{\varphi = 0 \mid Z_\varphi \in p\}$) and the equivalence is also a homeomorphism (since it takes basic closed sets to basic closed sets).

This means that $T^p$ is Hausdorff, since continuous type spaces are Hausdorff (see [BYBH08, Definition 8.4, Lemma 8.5, Proposition 8.6]).

Theorem A.35. If $T$ is a complete continuous theory, then $\text{Core}(T^p)$ and $\text{Core}_\pi(T^p)$ are both well defined, and furthermore $\text{Core}_\pi(T^p)|_\mathcal{E} = \text{Core}(T^p)$ and every symbol in $\text{Core}_\pi(T^p)$ is $\emptyset$-type definable in $\text{Core}(T^p)$ (in particular, $\text{Aut}(\text{Core}_\pi(T^p)) = \text{Aut}(\text{Core}(T^p))$).

Proof. By Corollary 3.24 and Theorem 3.31. □

Proposition A.36. For every $\omega$-saturated model of $T$, the metric topology is the same as the positive topology (see Definition 2.13) on $M^p$.

Proof. A basic closed set in the metric topology is $B_\varepsilon(a)^c$, that is $Z_{d(x,y)\geq \varepsilon}(M,a)$.
Conversely, when taking a basic closed set in the positive topology, we may without loss of
generality assume it is atomic by Proposition A.31.

Therefore it is \( Z_\varphi(M, \bar{b}) \) which is equal as a set to \( i^{-1}((\varphi^M)^{-1}(\{0\})) \) when \( i(a) = (a, \bar{b}) \in M^{1+[\bar{b}]} \).

But of course both \( i \) and \( \varphi^M \) are continuous, thus \( Z_\varphi(M, \bar{b}) \) is closed. \( \square \)

**Corollary A.37.** \( T^p \) is bounded iff \( T \) has a compact \( \omega \)-saturated model.

**Proof.** If \( T^p \) is bounded, the universal model \( U \) is saturated for every \( \kappa \) thus \( U \) is a \( \omega \)-saturated model of \( T \) which is compact in the positive topology thus in the metric topology.

Conversely, if \( T \) has a compact model \( M, M^p \) is a compact (in the metric topology) pc model of \( T^p \) (in particular \( T^p = \text{Th}^{pu}(M^p) \)), every relation in \( L^p \) is closed in the respective power of \( M^p \) (being \( (\varphi^M)^{-1}(\{0\}) \) for some continuous \( \varphi \)).

Therefore by Lemma 2.24 every pc model of \( T^p \) embeds into \( M^p \) and thus \( T^p \) is \( |M^p| \) bounded. \( \square \)

**Remark A.38.** In continuous logic, maybe more often than in first order logic, one has to consider
type-definable sets or function rather than merely definable ones (since a limit is generally speaking
only type definable). For this reason it is probably more appropriate to consider for continuous
logic the type-core \( \text{Core}^p(T^p) \) as defined in Subsection 3.8.

### A.3. Positive Types

Here we wish to construct, given an hu theory, a new theory in which
every (maximal) positive type over \( \emptyset \) is isolated.

**Definition A.39.** Let \( T \) and hu theory in a language \( L \). Denote by \( L^{hp} \) the language

\[ \{ P_\Sigma(x) \mid x \text{ a variable tuple, } \Sigma(x) \text{ a positive partial type, consistent with } T \} \]

where we take \( L^{hp} \) to be an extension of \( L \) (that is we assume \( P_\Sigma = \Sigma \) is \( \Sigma \) is actually a formula).

Let \( T^{hp} \) be the set of hu implications of

\[ T := T \cup \{ \forall x : P_\Sigma(x) \rightarrow \psi(x) \mid \psi \in \Sigma \} . \]

Note that since \( T \subseteq \overline{T} \), we also have \( T \subseteq T^{hp} \).

By [PY18, Section 3.1], \( \overline{T} \) and \( T^{hp} \) have the same pc models. Note that every model \( M \) of \( T \)
can be extended to a model \( M^{hp} \) of \( \overline{T} \) (thus of \( T^{hp} \)) by setting \( P^M_\Sigma = \Sigma(M) \).

**Lemma A.40.** If \( M \models \overline{T} \), every \( L \) homomorphism from \( M \) to a model \( N \) of \( T \) is also an \( L^{hp} \)
homomorphism to \( N^{hp} \).

**Proof.** If \( a \in P^M_\Sigma \), by assumption for any \( \psi \in \Sigma \) we have \( a \in \psi(M) \) thus \( h(M) \in \psi(N) \). We
conclude that \( N \models \Sigma(h(a)) \) thus \( h(a) \in P^N_\Sigma \). Thus \( h : M \rightarrow N^{hp} \) is an \( L^{hp} \) homomorphism, as required. \( \square \)
Corollary A.41. \textbf{Aut} (M) = \textbf{Aut} (M_{\text{tp}}) for any M \equiv T.

*Proof.* \textbf{Aut} (M_{\text{tp}}) \subseteq \textbf{Aut} (M) is clear, since L \subseteq L_{\text{tp}}. On the other hand if \sigma \in \textbf{Aut} (M), then \sigma, \sigma^{-1} : M_{\text{tp}} \to M are L-homomorphisms from M_{\text{tp}} \equiv T to M \equiv T, and thus by A.40 they are also L_{\text{tp}} homomorphisms from M_{\text{tp}} to itself, and they are still inverses. \hfill \square

Lemma A.42. If M is a pc model of T_{\text{tp}} then M|_{L} is a pc model of T, and further M = (M|_{L})_{\text{tp}}.

*Proof.* Note that as we remarked, M is model of T, and also in particular a model of T. Assume N is an L-structure and N \equiv T, and assume h : M \to N is an L homomorphism. By the remark h is also an L_{\text{tp}} homomorphism thus by assumption on M an L_{\text{tp}} immersion — and therefore M is a pc model of T.

Further since Id_{M} : M \to M|_{L} is an L homomorphism, it is also an L_{\text{tp}} homomorphism from M to (M|_{L})_{\text{tp}}. Thus again by assumption it is an L_{\text{tp}} immersion, in particular an L_{\text{tp}} embedding, that is M = M|_{L} as required. \hfill \square

Corollary A.43. Assume M is a pc model of T_{\text{tp}} and a,b are tuples in M of the same sort. Then they have the same positive L_{\text{tp}} type (over the empty set) iff they have the same positive L type (over the empty set).

Thus if T is Hausdorff, semi-Hausdorff or thick, so is T_{\text{tp}}.

Theorem A.44. The class of pc models of T_{\text{tp}} is exactly the class of models of the form M_{\text{tp}} for M an positively \aleph_{0}-saturated pc model of T.

*Proof.* Let M a pc model of T_{\text{tp}}, which is equal to (M|_{L})_{\text{tp}}. We need to show M|_{L} is positively \aleph_{0}-saturated. Assume a \in M_{y} for y a finite tuple and \Sigma (x,y) is a positive partial L type such that \Sigma (x,a) is finitely satisfiable in M.

Then there is some homomorphism h : M \to N \equiv T and b \in N^{x} such that N \equiv \Sigma (b,h(a)) \Rightarrow N_{\text{tp}} \equiv P_{\Sigma} (b,h(a)) \Rightarrow N_{\text{tp}} \equiv \exists x P_{\Sigma} (x,h(a)) and since by A.40 h is also an L_{\text{tp}} homomorphism to a model of T_{\text{tp}} thus an L_{\text{tp}} immersion, M \equiv \exists x P_{\Sigma} (x,a) as required.

Conversely, assume that M is an positively \aleph_{0}-saturated pc model of L. Let h : M_{\text{tp}} \to N \equiv T_{\text{tp}} an L_{\text{tp}} homomorphism, and assume that for some y tuple a in M and some \Sigma_{0} (x,y) , \ldots , \Sigma_{k-1} (x,y) (we may assume \Sigma_{i} are all in the same variable tuples) we have N \equiv \exists x \bigwedge_{i<k} P_{\Sigma_{i}} (x,h(a)). Let b \in N^{x} such that N \equiv \Lambda_{i<k} P_{\Sigma_{i}} (b,h(a)). Then we also have for \Sigma = \bigcup_{i<k} \Sigma_{i} that N \equiv \Sigma (b,h(a)). Thus for any finite \Gamma \subseteq \Sigma we have N \equiv \exists x : \Gamma (x,h(a)) and since N \equiv T, h is an L-immersion by assumption on M and thus M \equiv \exists x : \Gamma (x,a). Thus \Sigma (x,a) is finitely satisfiable in M then by saturation it is satisfiable in M. Let c \in M^{x} such that M \equiv \Sigma (b,a) and we find M_{\text{tp}} \equiv \bigwedge_{i<k} P_{\Sigma_{i}} (b,a) thus M_{\text{tp}} \equiv \exists x \bigwedge_{i<k} P_{\Sigma_{i}} (x,a), as required. \hfill \square

Corollary A.45. T is irreducible iff T_{\text{tp}} is.
Proof. Assume $T$ is irreducible and $M_0, M_1 \models T^{tp}$. Then they can be continued into pc models $N_0, N_1$ of $T^{tp}$ respectively. Since $N_0, N_1 \models T$ and $T$ is irreducible, there exists a model $N_2 \models T$ and $L$ homomorphisms $h_i : N_i \to N_2$ (for $i \in \{0, 1\}$), which by are also $L^{tp}$ homomorphisms into $N_2^{tp} \models T^{tp}$ as required.

Conversely if $T^{tp}$ is irreducible and $M_0, M_1 \models T^{tp}$ then $M_0^{tp}, M_1^{tp} \models T^{tp}$ thus there exist a model $N \models T^{tp}$ and $L^{tp}$ homomorphisms into $N^{tp}$.

□

Corollary A.46. The function $p \mapsto p|_L$ is a homeomorphism from $S(M)_{L^{tp}}$ to $S(M)_L$.

Proof. This mapping is a bijection since $tp^L(a/M)_{L^{tp}} = tp^L(a/M)_L$ iff for any tuple $b$ in $M$ we have $tp^L(a,b/\emptyset)_{L^{tp}} = tp^L(a,b/\emptyset)_L$ (when $a$ is without loss of generality a tuple in some positively $\aleph_0$-saturated pc model of $M$).

The mapping in clearly continuous, since every basic closed set in the image is also a basic closed set in the domain. On the other hand, assume $\{p \mid \exists y \bigwedge_{i<k} P_{\Sigma_i} (x,y,a) \in p\}$ is a subbasic closed set in the domain, then its image is equal to the closed
\[
\bigcap \left\{ \{p \mid \exists y \bigwedge_{i<k} \Gamma(x,y,a) \in p\} \mid \Gamma \subseteq \bigcup_{i<k} \Sigma_i \text{ finite} \right\}.
\]

Indeed assume $p$ is a maximal positive $L^{tp}$ type, and let $b$ realizing $p$ in some positively $\aleph_0$-saturated pc extension $N$ of $M$. Then by saturation
\[
\exists y \bigwedge_{i<k} P_{\Sigma_i} (x,y,a) \iff N \models \exists y \bigwedge_{i<k} P_{\Sigma_i} (b,y,a) \iff \\
\exists c \in N \text{ s.t. } N \models P_{\Sigma_i} (b,c,a) \iff \exists c \in N \text{ s.t. } b,c,a \models \bigcup_{i<k} \Sigma_i \iff \\
\forall \Gamma \subseteq \bigcup_{i<k} \Sigma_i \text{ finite} : N \models \exists y \bigwedge \Gamma (b,y,a).
\]

□

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