EVOLUTION OF COMPACT BINARY POPULATIONS IN GLOBULAR CLUSTERS: A BOLTZMANN STUDY. II. INTRODUCING STOCHASTICITY

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ABSTRACT

We continue the exploration that we began in Paper I of using the Boltzmann scheme to study the evolution of compact binary populations of globular clusters, introducing in this paper our method of handling the stochasticity inherent in the dynamical processes of binary formation, destruction, and hardening in globular clusters. We describe these stochastic processes as “Wiener processes,” whereupon the Boltzmann equation becomes a stochastic partial differential equation, the solution of which involves the use of “Itô calculus” (this use being the first, to our knowledge, in this subject), in addition to ordinary calculus. As in Paper I, we focus on the evolution of (1) the number of X-ray binaries $N_{XB}$ in globular clusters and (2) the orbital period distribution of these binaries. We show that, although the details of the fluctuations in the above quantities differ from one “realization” to another of the stochastic processes, the general trends follow those found in the continuous-limit study of Paper I, and the average result over many such realizations is very close to the continuous-limit result. We investigate the dependence of $N_{XB}$ found by these calculations on two essential globular cluster properties, namely, the star-star and star-binary encounter rate parameters $\Gamma$ and $\gamma$, for which we coined the name “Verbunt parameters” in Paper I. We compare our computed results with those from Chandra observations of Galactic globular clusters, showing that the expected scalings of $N_{XB}$ with the Verbunt parameters are in good agreement with those observed. We indicate additional features that can be incorporated into the scheme in the future, as well as how more elaborate problems can be tackled.

Subject headings: binaries: close — globular clusters: general — methods: numerical — scattering — stellar dynamics — X-rays: binaries

1. INTRODUCTION

In this series of papers, we are studying the evolution of compact binary populations of globular clusters with the aid of a Boltzmann scheme, which we introduced in Banerjee & Ghosh (2007, hereafter Paper I). This scheme follows compact binary evolution as a result of both (1) those processes that determine compact binary evolution in isolation (i.e., outside globular clusters), e.g., angular momentum loss by gravitational radiation and magnetic braking, as well as orbital evolution due to mass transfer, and (2) those processes that arise from encounters of compact binaries with the dense stellar background in globular clusters, e.g., collisional hardening (Heggie 1975; Shull 1979; Banerjee & Ghosh 2006), binary formation through tidal capture and exchange processes, and binary destruction (Fabian et al. 1975; Press & Teukolsky 1977; Lee & Ostriker 1986; Di Stefano & Rappaport 1992, 1994; Spitzer 1987; Hut & Bahcall 1983). We treat all of the above processes simultaneously through our Boltzmann scheme, the aim being to see their combined effect on the compact binary population as a whole, and in particular on the evolution of (1) the total number of X-ray binaries as the formation and destruction processes continue to operate and (2) the orbital period distribution of the population. As stressed in Paper I, our scheme is the original Boltzmann one (not the Fokker-Planck reduction of it), which, by definition, is capable of handling both the combined small effects of a large number of frequent, weak, distant encounters and the individual large effects of a small number of rare, strong, close encounters on the same footing. We note here that, although Monte Carlo Fokker-Planck approaches were normally thought to be capable of handling only the former effects, schemes for including the latter have recently been proposed and studied (Fregeau et al. 2003; Fregeau & Rasio 2007).

In Paper I, we studied the problem in the continuous limit, wherein we used continuous representations for both kinds of processes described above: i.e., those of category 1 above, which are inherently continuous, and also those of category 2, which are inherently stochastic. For the latter category, therefore, we used the continuous limit of the above stochastic processes, wherein the probability or cross section of a particular such process happening with a given set of input and output variables was treated as a continuous function of these variables. These cross sections were, of course, those that had been determined from extensive numerical experiments with two-body and three-body encounters performed earlier (Heggie et al. 1996; Portegies Zwart et al. 1997a).

In this paper, we address the next question: namely, how is the inherent stochasticity of the processes of category 2 to be introduced into our scheme, to be handled simultaneously with the inherently continuous nature of those of category 1? As we stressed in Paper I, this step is of great importance, since it is the simultaneous operation of the above continuous and stochastic processes in globular clusters that leads to the observed properties of compact binary populations therein. To this end, we introduce stochasticity into our Boltzmann study in this paper in the following way. For a first look, we consider the rates of the processes of category 2 as randomly fluctuating about the mean rates described in Paper I, while those of the processes of category 1 remain continuous, as before. We model these fluctuations as a “Wiener process” (see Appendix A and references therein), which is the mathematical description of Brownian motion.

With this prescription, the Boltzmann equation governing the evolution of the distribution function $n(a, \lambda)$ of compact binaries in time $t$ and orbital radius $a$ becomes a stochastic partial differential equation (SPDE), instead of the ordinary partial differential equation (OPDE) that it was in the continuous limit. We handle the solution of this SPDE with the aid of techniques developed...
largely during the last 15 years (Kloeden et al. 1994; Gaines 1995; Øksendal 2003). These techniques involve the use of the “Itô calculus” (see Appendix B and references therein), instead of ordinary calculus, for handling the stochastic terms.

Our results show that the full solutions with stochasticity included have fluctuations that vary from one “realization” to another of the stochastic processes, as expected. However, the full results show trends that generally follow those in the continuous limit. Furthermore, the average result over many realizations comes very close to the continuous limit, which shows the importance of the latter limit for understanding mean trends. On the other hand, understanding fluctuations in a typical full run is also very important, as this gives us a first idea of the magnitude of fluctuations that we can expect in the data on X-ray binaries in globular clusters as a result of the stochastic processes, as well as the expected trends in the fluctuations with the essential globular cluster parameters; e.g., the Verbunt parameters introduced in Paper I (also see below).

Comparison of our computed trends in the number of X-ray binaries, $N_{XB}$, in Galactic globular clusters with the Verbunt parameters on the one hand, and with observed trends in recent Chandra data on Galactic globular clusters on the other, shows that our full results are in good agreement with observations. We have thus constructed a straightforward, very inexpensive scheme for following the evolution of compact binary populations in globular clusters, including the essential fluctuating encounter processes that are thought to operate in such clusters, as well as those continuous processes that operate in isolated binaries and so apply here as well. We can also follow the evolution of $N_{XB}$, as well as that of the orbital period distribution of compact binaries in globular clusters. For the latter study, however, proper modeling of stellar evolutionary effects still remains to be done for parts of the parameter space, as is explained in Paper I, and as is discussed in § 5.

In § 2, we briefly review the continuous-limit results of Paper I, in order to put the results of this paper in their proper context. We give only the essentials here, citing figures in Paper I for detailed results. In § 3, we introduce stochasticity explicitly through our prescription, explaining the details of Wiener processes and the Itô calculus in Appendices A and B. We describe our generalization of the Lax-Wendroff scheme, which we introduced in Paper I, to handle the solution of the SPDE that the Boltzmann equation has now become. In § 4, we describe the results of our full calculations including stochasticity, and we compare these with the continuous-limit results of Paper I. In § 4.3, we compare our full results with observations. Finally, in § 5 we discuss our results, putting them in the context of previous studies in the subject, and we indicate some additional physical effects to be included by stages in future versions of our scheme, as well as some future problems to be tackled.

2. BRIEF REVIEW OF THE CONTINUOUS LIMIT

In order to put the stochastic studies of this paper in their proper context, we review in this section the essentials of the continuous-limit studies of Paper I that form this context. In the latter limit, the Boltzmann description works in terms of appropriate mean values of the variables and parameters, which are actually stochastic. Accordingly, the above compact binary distribution function $n(a, t)$ is replaced by its mean value $\pi(a, t)$, and the Boltzmann equation has the form

$$ \frac{\partial \pi(a, t)}{\partial t} = \mathcal{R}(a) - \pi(a, t) \mathcal{D}(a) - \frac{\partial \pi(a, t)}{\partial a} \mathcal{J}(a). $$

Here $\mathcal{R}(a)$ is the mean formation rate (per unit binary radius) of compact binaries of radius $a$, $\mathcal{D}(a)$ is the mean destruction rate per binary, and $\mathcal{J}(a)$ is the mean shrinkage rate $\dot{a}$ of a compact binary of radius $a$, as described in Paper I.

2.1. Mean Rates

The mean shrinkage or “hardening” rate $\dot{a}$ has been given in Figure 2 of Paper I as a function of $a$, describing the situation as the compact binary goes from its widest, pre–X-ray binary (PXB) phase to Roche lobe contact and then continues through the mass-transferring X-ray binary (XB) phase. As shown there, collisional hardening, i.e., that due to encounters between the binary and the stellar background of the globular cluster, dominates at large values of $a$, whereas hardening by gravitational radiation and magnetic braking dominates at small values of $a$. The relative orbit shrinkage rate $\dot{a}/a$ scales roughly as $a$ at large orbital radii, and it passes through a minimum at a critical separation where the gravitational radiation shrinkage rate, which scales as $\dot{a}/a \sim a^{-4}$, takes over. Magnetic braking also contributes at small radii, but Roche lobe contact also occurs at roughly the same point, whereupon the angular momentum transfer associated with mass transfer in the XB phase dominates the rate of change of the orbit, and thus $\dot{a}$ has a very weak dependence on $a$ during this phase. Detailed quantitative expressions for the above rates are given in Paper I, to which we refer the reader, recording here only that the orbit shrinkage rate is given in terms of the angular momentum loss rate by

$$ \frac{\dot{a}}{a} = \frac{2}{J} \left( \frac{\dot{m}_c}{m_c} - 2 \frac{\dot{m}_X}{m_X} \right), $$

where $m_X$ is the mass of the degenerate star in the compact binary and $m_c$ is that of its (low-mass) companion, and the total angular momentum loss rate can be written in terms of its components as

$$ f(a) \equiv \frac{J}{\dot{J}} = f_{GW}(a) + f_{MB}(a) + f_{coll}(a). $$

Here the subscripts “GW,” “MB,” and “coll” stand for gravitational radiation, magnetic braking, and collisional hardening, respectively.

The other essential mean rates are those of compact binary formation and destruction: $\mathcal{R}(a)$ and $\mathcal{D}(a)$, respectively. Consider first the formation of compact binaries with degenerate primaries (white dwarfs or neutron stars) and low-mass companions, in which we are interested in this series of papers, in globular cluster (GC) cores. The two relevant dynamical processes are (1) tidal capture (“tc”) of a degenerate, compact star by an ordinary, low-mass star, and (2) an exchange encounter (“ex1”) between such a compact star and a binary of two ordinary low-mass stars in the GC, wherein the compact star replaces one of the binary members. Accordingly, the total mean rate of formation of compact binaries per unit binary radius, $\mathcal{R}(a)$, consists of the above mean tidal capture rate $r_{tc}(a)$ and mean exchange rate $r_{ex1}(a)$:

$$ \mathcal{R}(a) = r_{tc}(a) + r_{ex1}(a). $$

The above mean rates are shown as functions of $a$ in Figure 3 of Paper I, and detailed mathematical expressions for them are also given in that work, which we do not repeat here. The mean tidal capture rate is nearly constant for $a < 5 R_C$, and it decreases rapidly at larger values of $a$. Extensive discussion of various
issues related to tidal capture and of the current status of our understanding of this process are also given in Paper I, to which we refer the reader. Further discussion of previous studies of tidal capture in this problem are given in §5. The mean exchange (ex1) rate is roughly constant over the range of radii of interest here for the widely adopted radius distribution of primordial binaries, viz., a uniform distribution in $\ln a$, which we adopt throughout our work.

Consider now the destruction of compact binaries with degenerate primaries and low-mass companions, which can occur in the following two ways. First, an encounter with a star that has a relative speed higher than an appropriate critical speed can lead to its dissociation (“dss”). Second, in an exchange encounter (“ex2”) of this binary with a compact star, the latter can replace the low-mass companion in the binary, forming a double compact star binary consisting of two neutron stars, two white dwarfs, or a neutron star and a white dwarf (all with masses of $m_N \approx 1.4 M_\odot$ in our model: see Paper I). This destroys the binary as an X-ray source (as accretion is not possible in such a system) and thus takes it out of the reckoning in our study.\footnote{1 Note that it is essentially impossible for one of the compact stars in such a double compact system to be re-exchanged with an ordinary star in a subsequent exchange encounter, since the average mass of a background GC star (taken to be $m_f = 0.6 M_\odot$ in our work) is much less than the value of $m_N$ given above.}

The total mean destruction rate $D(a)$ per binary is thus the sum of the above mean dissociation and exchange rates:

$$D(a) = r_{\text{dss}}(a) + r_{\text{ex2}}(a).$$

The above mean rates are also shown as functions of $a$ in Figure 3 of Paper I, and detailed mathematical expressions for them are given in that paper, which, once again, we do not repeat here. The mean dissociation rate is negligible below a critical value of the radius, $a_c$, corresponding to the above critical speed, rises extremely sharply for values of $a$ above $a_c$ at first, and eventually scales as $a^2$ for values of $a \gg a_c$. By contrast, the mean exchange (ex2) rate is roughly $\propto a$ and dominates the destruction processes completely at all orbital radii relevant to our study, dissociation becoming important only for very soft binaries ($a > 1000 R_\odot$, say), which are of little interest to us here.

### 2.2. Results

We now summarize the essentials of our continuous-limit results in Paper I. We computed the evolution of the compact binary distribution in this limit, showing the surface $n(a, t)$ explicitly in three dimensions in Figure 5 of Paper I, corresponding to representative GC parameters (rather similar to those of the well-known Galactic cluster 47 Tuc). The surface evolved smoothly, with the compact binary population growing predominantly at shorter radii ($a < 10 R_\odot$, say). We showed that, starting with a small number of binaries at $t = 0$ that followed various distributions, we obtained at times of approximately 1 Gyr or longer a distribution that was independent of the initial distribution and its evolution by displaying slices through the surface at various points along the time axis and the $a$-axis, which were shown in Figures 6 and 7 of Paper I, respectively. The former figure showed that the profile $n(a)$ increased with time, roughly preserving its profile for $t > 1.5$ Gyr, where said profile consisted of a roughly uniform distribution at small orbital radii, $a \leq 6 R_\odot$, say, and a sharp falloff at larger radii. The latter figure showed that the value of $n(a, t)$ at a given value of $a$ increased with time and approached saturation on a timescale of $6-12$ Gyr, the timescale being longer at smaller values of $a$.

For comparison with crucial X-ray observations of Galactic GCs, we computed the total number of XBs, $N_{\text{XB}}$, in a GC at any time, which we obtained by integrating $n(a, t)$ over the range of values of $a$ that are relevant for XBs: viz., $a_{\text{pm}} \leq a \leq a_c$, where $a_{\text{pm}}$ is the value of $a$ corresponding to the period minimum of $P \approx 80$ minutes and $a_c$ is the value of $a$ at the first Roche lobe contact and onset of mass transfer, as explained above:

$$N_{\text{XB}}(t) = \int_{a_{\text{pm}}}^{a_c} n(a, t) \, da.$$  \hspace{1cm} (6)

Taking an evolutionary time of $\approx 8$ Gyr to be representative, we determined the value of $N_{\text{XB}}$ at this point in time and studied its dependence on the Verbunt parameters $\Gamma$ and $\gamma$ that describe the essential dynamical properties of globular clusters in this context, as explained in Paper I, showing our results as the computed $N_{\text{XB}}(\Gamma, \gamma)$ surface in three dimensions in Figure 8 of that paper.

The Verbunt parameters $\Gamma$ and $\gamma$ were introduced in Paper I. Following pioneering suggestions by Verbunt and coauthors (Verbunt & Hut 1987; Verbunt et al. 1989), the crucial importance of these parameters in GC dynamics has recently been lucidly summarized by Verbunt (Verbunt 2003, 2007), and the importance of the parameter $\Gamma$ for scaling between different GCs has been emphasized in a pioneering study of the production of recycled pulsars in GCs by Di Stefano & Rappaport (1992). Briefly, the parameter $\Gamma$ is the two-body stellar encounter rate, which scales with $\rho^2 r_c^2/v_c$, and occurs naturally in the rates of all two-body processes, where the standard GC core variables are the average stellar density $\rho$, the velocity dispersion $v_c$, and the core radius $r_c$. In fact, we defined $\Gamma$ as

$$\Gamma \equiv \frac{\rho^2 r_c^3}{v_c} \propto \rho^{\alpha_c} r_c^2$$

in Paper I. Note that the last scaling in the above equation holds only for virialized cores, in which the scaling $v_c \propto \rho^{1/2} r_c$ can be applied. The second parameter is a measure of the rate of encounters between binaries and single stars in the GC, for which the rate normally used is the encounter rate $\gamma$ of a single binary with the stellar background, with the understanding that the total rate of binary–single star encounters in the cluster will be $\propto n \gamma$. We defined $\gamma$ in Paper I as

$$\gamma \equiv \frac{\rho}{v_c} \propto \rho^{1/2} r_c^{-1},$$

where the last scaling holds, again, only for virialized cores.

We compared the results of our above computations with the systematics of recent observations of X-ray binaries in Galactic globular clusters (Pooley et al. 2003), as displayed in Figures 9 and 10 of Paper I. We showed that the computed total number $N_{\text{XB}}$ of XBs expected in a globular cluster scaled in a characteristic way with the Verbunt parameters. The qualitative nature of this scaling was rather similar to that found in our earlier “toy” model (Banerjee & Ghosh 2006), although the details were different. We found that $N_{\text{XB}}$ scaled with $\Gamma$ (which is a measure of the dynamical formation rate of compact binaries, as above) and, at a given value of $\Gamma$, the value of $N_{\text{XB}}$ decreased with increasing values of $\gamma$ (which is a measure of the rate of destruction of these binaries by dynamical processes) at large values of $\gamma$, as shown in Figure 9 of Paper I. This rough scaling can be expressed as
\(N_{XB} \propto \Gamma g(\gamma)\), where the “universal” function \(g(\gamma)\) of \(\gamma\) (except for a spurious feature at low values of \(\gamma\), which we explained in Paper I) decreased monotonically with increasing values of \(\gamma\), reflecting the increasing strength of dynamical binary destruction processes with increasing values of \(\gamma\).

We further demonstrated that these computed trends with the Verbunt parameters compared very well with the observed trends in the above X-ray data by showing in Figure 10 of Paper I the data. We do so by expressing the above rates in terms, in addition to their mean values studied in Paper I, as independent of those at other values of \(\gamma\). This is appropriate, since the dynamical processes of binary formation and destruction work in GCs and encouraged us to build more “realistic” models by introducing stochastic effects explicitly, to which this paper is devoted.

### 3. INTRODUCING STOCHASTICITY

In order to study the behavior of the inherently stochastic terms in the full Boltzmann equation

\[
\frac{\partial n(a, t)}{\partial t} = R(a, t) - n(a, t)D(a, t) - \frac{\partial n(a, t)}{\partial a}f(a, t),
\]

we must explicitly include stochastic, fluctuating parts in these terms, in addition to their mean values studied in Paper I, as above. We do so by expressing the above rates \(R(a, t)\), \(D(a, t)\), and \(f(a, t)\) as their earlier mean values \(\overline{R}(a)\), \(\overline{D}(a)\), and \(\overline{f}(a)\), augmented by fluctuating components as below:

\[
R(a, t) = \overline{R}(a) + \zeta_{tr}^a + \zeta_{ex1}^a, \quad D(a, t) = \overline{D}(a) + \zeta_{tr}^a + \zeta_{ex2}^a + \zeta_{dss}^a, \quad f(a, t) = \overline{f}(a) + \zeta_{coll}^a.
\]

Here \(\zeta_{tr}^a\) is the random fluctuation rate of events of type \(X\) from their mean rates, and \(X = tr, ex1, ex2, dss, \) and \(coll\) by turn, these notations having been introduced above. In general, \(\zeta_{tr}^a\) is a function of both \(a\) and \(t\), of course.

The crucial question is that of modeling \(\zeta_{tr}^a\) appropriately. In this introductory work, we use the standard normally distributed model

\[
\zeta_{tr}^a = S_X(a)\eta^a,
\]

where \(S_X^2(a)\) is the variance of \(\zeta_{tr}^a\) at a given value of \(a\) and the values of \(\eta^a\) at each value of \(a\) are independent random numbers distributed in a standard normal distribution.

### 3.1. Variance of Stochastic-Process Rates

How do we estimate the variance of a stochastic process of type \(X\) whose mean value is \(\overline{R}_X(a)\)? To answer this question, consider first how it is addressed in Monte Carlo simulations, which have been performed in this area by several authors (see, e.g., Sigurdsson & Phinney 1993; Portegies Zwart et al. 1997b; Fregeau et al. 2003). These works have uniformly used the so-called rejection method for determining whether an event of a given type occurs in a given time interval or not. The method works as follows.

For events of type \(X\), if the mean rate of event occurrence is \(\overline{R}_X\), then the timescale for occurrence of such events is

\[
\Delta t_X = \frac{1}{\overline{R}_X}.
\]

Hence, during a time step \(\Delta t < \Delta t_X\), the quantity \(p_X = \overline{R}_X\Delta t < 1\) is the expected mean number of events during this interval. Values of \(p_X < 1\) can also be interpreted as the probability of occurrence of an event \(X\) within this time step (Portegies Zwart et al. 1997b), and the actual number of such events within \(\Delta t\) will then follow a binomial distribution with the following mean and variance:

\[
\text{mean} = \overline{R}_X(a)\Delta t, \quad \text{variance} = S_X^2(a)\Delta t = \overline{R}_X(a)\Delta t\left[1 - \overline{R}_X(a)\Delta t\right].
\]

Note that the above variance depends on \(a\), since the mean rates depend on \(a\). When several different types of events are considered simultaneously, as in the present problem, we must, of course, choose \(\Delta t\) such that it is shorter than the shortest event-occurrence timescale appearing in the problem. We discuss this point below.

#### 3.1.1. Time Step

The mean rates depend on \(a\) as detailed in Paper I (see Fig. 3 of that paper). The quantity \(R_{ex}(a)\) is a decreasing function of \(a\) and so attains its maximum at \(a = a_{\text{min}}\). All other rates are either constant (e.g., \(ex2\)) or increasing functions of \(a\), so their maximum values can be thought to occur at \(a = a_{\text{max}}\). Accordingly, if we make the following choice for our computational time step \(\Delta t_d\),

\[
\Delta t_d < \min\left\{\frac{1}{R_{dc}(a_{\text{min}})}, \frac{1}{R_{ex1}(a_{\text{max}})}, \frac{1}{R_{ex2}(a_{\text{max}})}, \frac{1}{R_{dss}(a_{\text{max}})}, \frac{1}{\bar{a}_{\text{coll}}(a_{\text{max}})}\right\},
\]

this will ensure that \(\Delta t_d\) is smaller than the shortest of the above event-occurrence timescales.
However, as is well known, this time step must also obey the "Courant condition" (Press et al. 1992) throughout the range of values of $a$ under consideration (i.e., $0.6-60$ $R_c$):

$$\Delta t = \epsilon \frac{\Delta a}{f_{\text{max}}} \quad \epsilon < 1. \quad (15)$$

Here $\Delta a$ is the step size in $a$ and $f_{\text{max}}$ is the largest value of $f(a)$ over the range of values of $a$ under consideration (see above and Paper I). Satisfaction of this condition is essential for the stability (Press et al. 1992) of the solution of equation (9).

To ensure that both of the above conditions are satisfied, we choose the time step $\Delta t$ for solving equation (9) to be

$$\Delta t = \min\{\Delta t_d, \Delta t_c\}. \quad (16)$$

### 3.2. Solution of the Stochastic Boltzmann Equation

The Lax-Wendroff scheme (Press et al. 1992) used by us for finding a numerical solution of the Boltzmann equation in the continuous limit was introduced in §2.6 of Paper I. The stochastic version of this equation, viz., equation (9), can be regarded as the earlier continuous equation with additional stochastic terms, which turns it into a SPDE (see §1). We now discuss our method of solving this SPDE.\(^2\)

It is well known that ordinary calculus cannot be applied to the handling of stochastic terms in SPDEs, since these terms are nondifferentiable in the ordinary sense and the ordinary definition of an integral does not apply to them. Rather, one has to modify the methods of calculus suitably and redefine appropriate integrals. As summarized in Appendix B, one such modified calculus is the Itô calculus, which has been used widely for solving SPDEs in recent years (Øksendal 2003; Kloeden et al. 1994). The corresponding integrals involving the stochastic terms are then called "Itô integrals," which have properties that are appropriately different from those of ordinary integrals, as indicated in Appendix B.

#### 3.2.1. Numerical Method

In solving an SPDE such as equation (9), one integrates the continuous terms in the usual way, but the stochastic terms must be integrated using Itô calculus (Gaines 1995). This means that, in advancing the solution at $t$ by a time step $\Delta t$—which is essentially a Taylor expansion of the solution $n(a, t)$ about $t$—the expansions of the stochastic terms in equation (9) are to be performed using the stochastic Taylor expansion (eq. [B7]), as discussed in Appendix B.

A variety of numerical algorithms have been explored by various authors for numerically solving SPDEs. The particular algorithm that we use is a hybridization of the two-step Lax-Wendroff scheme for the continuous terms, as utilized in Paper I, and the second-order stochastic Taylor expansion according to the "Milstein scheme" for the stochastic terms (Milstein 1974; Gaines 1995), i.e., equation (B13), as explained in Appendix B. In this scheme, there is only one stochastic path to be solved for in our case, viz., that of $n(a, t)$ (corresponding to $X_j$) and the continuous terms, which are the variances in the tidal capture (tc), exchange (ex1, ex2), dissociation (dss) and collisional hardening (coll) rates, being as given above. Note that in each of the two steps in the Lax-Wendroff scheme, the expansion given in equation (B13) needs to be applied, whereupon we arrive at the following discretization scheme\(^3\) for equation (9):

#### Half step:

$$n_{j+1/2}^{N+1} = \frac{1}{2} (n_{j+1}^N + n_j^N) + \left[ R(a_{j+1/2}) - \bar{D}(a_{j+1/2}) \left( \frac{n_{j+1}^N + n_j^N}{2} \right) \right] \Delta t$$

$$+ \left[ (W_{j+1,2tc}^N + W_{j+1,2ex1}^N) - (W_{j+1,2ex2}^N + W_{j+1,2des}^N) \right] \left( \frac{n_{j+1}^N + n_j^N}{2} \right) \Delta t$$

$$+ \left[ \left(W_{j+1,2des}^N \right)^2 - s_{\text{des}}^2(a_{j+1/2}) \right] \left( \frac{n_{j+1}^N + n_j^N}{4} \right)$$

$$+ \left( W_{j+1,2ex2}^N W_{j+1,2des}^N \right) \frac{n_{j+1}^N + n_j^N}{2} \left( \bar{f}(a_{j+1/2}) \Delta t \right) \left( n_{j+1}^N - n_j^N \right)$$

$$- \frac{1}{2\Delta a} \left( n_{j+1/2}^{N+1} - n_{j-1/2}^{N+1} \right) - \frac{W_{j+1,coll}^N}{\Delta a} \left( n_{j+1/2}^{N+1} - n_{j-1/2}^{N+1} \right). \quad (17)$$

Here $W_{j+1,2,x}^N \equiv S(x_j) \eta^N \Delta t$, where $\eta^N$ is the value of a standard normal variate at the $N$th time step.

For any particular run, we compute the values of $W_{j+1,2,x}^N$ for a particular value of $a_j$ ($a_{j+1/2}$) over the $a$ and $t$ intervals of integration and then repeat this process for all values of $a_j$. The standard normal variate quantities $\eta^N$ are generated using the well-known "polar method" (Press et al. 1992). All values of $W_{j+1,2,x}^N$ are stored in a two-dimensional array (i.e., a "Wiener sheet"), which serves as the input for solving equation (17). Because of the fluctuations in the collisional hardening rate (as contained in $\zeta_{\text{coll}}$), it is not impossible that the value of the total hardening rate $f$ might occasionally exceed $f_{\text{max}}$, which would violate the Courant condition, possibly making the solution procedure unstable. To avoid this, we have restricted the variations in the values of $W_{j+1,coll}^N$ and $W_{j+1,2,coll}^N$ such that the amplification factor $\epsilon \equiv \Delta t/\Delta a$ always lies between zero and unity (Press et al. 1992).

### 4. RESULTS

We now present the results obtained from our above computations of the cases that we studied in Paper I in the continuous limit. As before, we study (1) the evolution of the distribution...
1.5 Gyr, the distribution "heals" to a form that is independent of a grows with time predominantly at shorter radii (continuous limit (cf. Fig. 5 of Paper I). In particular, the population evolution fluctuates randomly throughout, but it does show a clear cluster 47 Tuc. As the figure shows, the surface representing the comparison. 

We explicitly demonstrate this below by displaying temporal and saries with various initial distributions and find that by t \(\approx 1 - 1.5 \text{ Gyr} \), the distribution "heals" to a form that is independent of the initial choice of distribution. The fluctuations differ in detail from run to run, of course, as we choose different seeds for ran-don number generation, but the overall nature of the evolution remains the same for all runs. Indeed, the results for different runs seem to represent different variations about a mean surface, which is very close to that in the continuous limit, as given in Paper I. We explicitly demonstrate this below by displaying temporal and radial slices through the above surface \(n(a, t)\) (see Paper I) for different runs and also displaying their averages over a number of runs, which we show to be close to the continuous limit.

To do this, we first show in Figure 2 typical time slices, i.e., the values of \(n(a)\) at fixed values of \(t\) (solid lines), through the surface in Figure 1 for a single run, overplotting the continuous limit from Paper I for comparison. The distribution with fluctuations does indeed follow the continuous-limit distribution generally, the same gross features being visible through fluctuations, particularly that \(n(a)\) is roughly constant for \(a \leq 7 R_\odot\) and falls off sharply at larger radii. The overall nearly self-similar evolution at large times, described in Paper I, can also be vaguely discerned through the fluctuations. We have discussed possible causes of such self-similar evolution in Paper I. Next, in Figure 3, we show radial slices corresponding to the evolution in Figure 1, representing the behavior of \(n(r)\) at a fixed radius \(a\), with the continuous limit from Paper I overplotted. Again, the curves from a single run follow, in a statistical sense, the corresponding continuous limits. In particular, it can be seen that the radial slices corresponding to larger values of \(a\) tend to saturate by about 6 Gyr, while those for smaller values of \(a\) do not show such saturation.

Finally, in Figures 4 and 5, we show the above temporal and radial slices of the average of 12 different runs, with the correspon-ding continuous limits overplotted. These figures clearly demonstrate how the fluctuations average out over many runs, such that the mean result approaches the continuous limit.

4.1. Evolution of the Compact Binary Distribution

In Figure 1, we show a typical evolution of the compact binary population distribution \(n(a, t)\). The GC parameters were chosen, as in Paper I, to be \(\rho = 6.4 \times 10^4 \ M_\odot \ pc^{-3} \), \(r_c = 0.5 \ pc\), and \(v = 11.6 \ km \ s^{-1}\), similar to those of the well-known Galactic cluster 47 Tuc. As the figure shows, the surface representing the evolution fluctuates randomly throughout, but it does show a clear overall evolution that is of the same nature as that in the continu-ous limit (cf. Fig. 5 of Paper I). In particular, the population grows with time predominantly at shorter radii (\(a < 10 R_\odot\)). As before, we start with a small number of primordial compact binaries with various initial distributions and find that by \(t \approx 1 - 1.5 \text{ Gyr} \), the distribution “heals” to a form that is independent of the initial choice of distribution. The fluctuations differ in detail from run to run, of course, as we choose different seeds for ran-dom number generation, but the overall nature of the evolution remains the same for all runs. Indeed, the results for different runs seem to represent different variations about a mean surface, which is very close to that in the continuous limit, as given in Paper I. We explicitly demonstrate this below by displaying temporal and radial slices through the above surface \(n(a, t)\) (see Paper I) for different runs and also displaying their averages over a number of runs, which we show to be close to the continuous limit.

To do this, we first show in Figure 2 typical time slices, i.e., the values of \(n(a)\) at fixed values of \(t\) (solid lines), through the surface in Figure 1 for a single run, overplotting the continuous limit from Paper I for comparison. The distribution with fluctuations does indeed follow the continuous-limit distribution generally, the same gross features being visible through fluctuations, particularly that \(n(a)\) is roughly constant for \(a \leq 7 R_\odot\) and falls off sharply at larger radii. The overall nearly self-similar evolution at large times, described in Paper I, can also be vaguely discerned through the fluctuations. We have discussed possible causes of such self-similar evolution in Paper I. Next, in Figure 3, we show radial slices corresponding to the evolution in Figure 1, representing the behavior of \(n(r)\) at a fixed radius \(a\), with the continuous limit from Paper I overplotted. Again, the curves from a single run follow, in a statistical sense, the corresponding continuous limits. In particular, it can be seen that the radial slices corresponding to larger values of \(a\) tend to saturate by about 6 Gyr, while those for smaller values of \(a\) do not show such saturation.

Finally, in Figures 4 and 5, we show the above temporal and radial slices of the average of 12 different runs, with the correspon-ding continuous limits overplotted. These figures clearly demonstrate how the fluctuations average out over many runs, such that the mean result approaches the continuous limit.

4.2. Number of X-Ray Binaries

The total number of GC X-ray binaries \(N_{XB}\) at a particular time was computed from equation (6), as in Paper I. We determined \(N_{XB}\) for a representative evolution time of \(\sim 8 \text{ Gyr}\) and studied its dependence on the Verbunt parameters \(\Gamma\) and \(\gamma\), so as to relate our computational results to the systematics of recent observations of X-ray binaries in globular clusters (Pooley et al. 2003). For this we computed, as in Paper I, values of \(N_{XB}\) over a rectangular grid in \(\Gamma - \gamma\) space, spanning the ranges \(\gamma = 1 - 10^6\) and \(\Gamma = 10^3 - 10^8\), which encompasses the entire range of Verbunt parameters over which Galactic GCs have been observed (see Paper I). Although the GCs that have actually been observed so

![Fig. 1.](Image1.png)

**Fig. 1.**—Typical example, i.e., one “realization,” of the evolution of the binary distribution function \(n(a, t)\). Globular cluster parameters are chosen to be roughly those of 47 Tuc, as explained in the text (also see Fig. 5 of Paper I).

![Fig. 2.](Image2.png)

**Fig. 2.**—Typical time slices, i.e., \(n(a)\) at specified times, for the evolution shown in Fig. 1 (solid lines). Overplotted are the same time slices in the continuous limit (dashed lines) from Paper I.

![Fig. 3.](Image3.png)

**Fig. 3.**—Typical radial slices, i.e., \(n(r)\) at fixed values of the binary radius, for the evolution shown in Fig. 1. Overplotted are the same radial slices in the continuous limit from Paper I. As in Paper I, we show the evolution beyond 8 Gyr with dashed lines to indicate that such long evolution times may not be applicable to Galactic GCs but are included here to demonstrate the timescales.
far lie along a diagonal patch over this grid, as explained in Paper I, computational results over the whole grid are useful for clarifying the theoretically expected trends.

As explained in Paper I, at a specific grid point \((\Gamma, \gamma)\), the values of \(k_0\) and \(k_{13}\) are evaluated using the definitions of the Verbunt parameters and the virialization condition (see § 3.2 of Paper I for a detailed discussion). Also as before, we take representative values of the primordial stellar binary fraction \((k_0)\) and the compact star fraction \((k_{13})\) to be 10% and 5%, respectively.

Figure 6 shows the resulting \(N_{XB}(\Gamma, \gamma)\) surface. As indicated in Paper I, the overall falloff in this surface for values of \(k_{13} > 3 \times 10^3\) is a signature of the increasing rates of compact binary destruction rates with increasing values of \(k_{13}\), and the above specific value of \(k_{13}\) represents an estimate of the threshold above which destruction rates are very important. Further, the trend in \(N_{XB}\) with \(\Gamma\) is simple: \(N_{XB}\) increases with \(\Gamma\) monotonically, since the dynamical formation rate of compact binaries scales with \(\Gamma\). What we note in Figure 6 is that this surface also shows random fluctuations due to the stochastic processes, but it generally follows the \(N_{XB}\) surface corresponding to the continuous limit, which is overplotted in the same figure. This is similar to what was discussed above for the compact binary distribution, and the point about the mean surface corresponding to the average of many realizations of the stochastic processes being very close to the continuous limit also holds here. We also note that the total fluctuations in \(N_{XB}\) increase with increasing values of \(\Gamma\). However, as will become evident from results discussed below, the relative fluctuations actually decrease with increasing values of \(\Gamma\).

To further clarify the trends and to make comparisons with the results of the “toy” model of Banerjee & Ghosh (2006) and with those of Paper I, we plot the quantity \(\Gamma/N_{XB}\) for a fixed value of \(\gamma\) against \(\gamma\) in Figure 7, displaying the curves for several values of \(\Gamma\) as indicated. As can be seen, the fluctuating \(\Gamma/N_{XB}\) versus \(\gamma\) curves for various values of \(\Gamma\) follow the same mean trend, although the details of the fluctuation are different in different cases. This mean trend is in fact very close to the mean “universal” curve corresponding to the continuous-limit evolution from Paper I, which is overplotted in the figure. Thus, as in the continuous-limit case, the basic scaling of the toy model, viz., \(N_{XB} \propto \Gamma g(\gamma)\), where \(g(\gamma)\) is a “universal” decreasing function (representing the increasing binary destruction rate with increasing values of \(\gamma\), as explained above), does essentially carry over to this detailed model with stochasticity included, suggesting a robust feature of the scaling between different clusters that is expected to be further confirmed by future observations.
X-ray sources are also overplotted (squares line styles for easy comparison. Positions of GCs with significant numbers of responding contours in the continuous-limit case are overplotted, using the same fluctuations are clearly seen to be larger for smaller values of $N_{XB}$ given in parentheses, as in Paper I.

Another feature of Figure 7 is that the relative fluctuations in the curves increase with decreasing values of $\Gamma$. This is consistent with the intuitive notion that, in all phenomena of this nature, the relative fluctuations in $N_{XB}$ are expected to increase at smaller values of $N_{XB}$, which occur at smaller values of $\Gamma$. More formally, this can be seen as follows. From equation (13), it is clear that over an interval $\Delta t$, the relative variance in the number of events of type $X$ is

$$r_X(a) = 1 - \overline{K}_X(a)\Delta t.$$ 

For the range of values of $\Gamma$ and $\gamma$ considered in this work, we found that $\Delta t$ was actually close to $\Delta t$, in most cases, such that $\Delta t \sim \gamma^{-1}$ roughly. Since the formation rates scale as $R_X \sim \Gamma$, we have

$$r_X(a) = 1 - \mathcal{O} \frac{\Gamma}{\gamma}.$$ 

Therefore, for a fixed value of $\gamma$, $r_X(a)$ increases as $\Gamma$ (and hence $N_{XB}$) decreases.

4.3. Comparison with Observations

In § 4.1 and 4.2 we saw that the basic trends of the results, as obtained from the stochastic Boltzmann equation (eq. [9]), are the same as those obtained from the Boltzmann equation in the continuous limit in Paper I. Therefore, as in Paper I, the results from the stochastic Boltzmann equation are consistent with the observational $\gamma$ populations in Galactic GCs. Indeed, since fluctuations are present in the dynamical processes under study here, we should ideally compare theoretical trends including fluctuations with observational results, as we do in this paper, where Figure 6 shows the positions of the observed GCs with significant numbers of X-ray sources in the ($\gamma$, $\Gamma$, $N_{XB}$) coordinates. The observational points do lie near the computed $N_{XB}(\gamma, \Gamma)$ surface. In Figure 7, we compare the $\Gamma/N_{XB} - \gamma$ curves with the positions of the observed points, showing that most points do indeed lie near the curves.

In Figure 8 we plot the computed contours of constant $N_{XB}$ in the plane of Verbunt parameters, similar to what we did in Figure 10 of Paper I, but now with the fluctuations included. The fluctuations are clearly seen to be larger for smaller values of $N_{XB}$, as expected, and as mentioned above. Again, the observed numbers generally agree well with the present contours, which include fluctuations, and these contours do generally follow the continuous-limit contours of Paper I, which are shown overplotted.

5. DISCUSSION

We have described in this paper a scheme for introducing stochasticity into the Boltzmann study of compact binary evolution in globular clusters that we began in Paper I. Our scheme involves the use of stochastic calculus (for the first time in this subject, to the best of our knowledge), whereas previous studies in the subject have normally used Monte Carlo methods of various descriptions, depending on the particular aspect of the problem being studied, for handling stochasticity (see, e.g., Hut et al. 1992; Di Stefano & Rappaport 1994; Fregeau et al. 2003; Fregeau & Rasio 2007). With the aid of this scheme, we have demonstrated that the joint action of inherently stochastic and continuous processes produces evolutionary trends that necessarily contain fluctuations that vary between individual “realizations” of the stochastic processes, as expected. However, these trends do generally follow those found in the continuous-limit approximation of Paper I, and when trends are averaged over more and more realizations, the mean trend comes closer and closer to the continuous-limit one. In this sense, the continuous limit is very useful as an indicator of the expected mean trend. On the other hand, the magnitude of the fluctuations seen in any given realization, particularly in certain parts of parameter space, suggests that one should compare the results of a typical realization to observations in order to get a feel for expected fluctuations in the data from stochastic dynamical processes alone; i.e., apart from those coming from uncertainties in the observational methods of obtaining the data.

The Boltzmann approach in its original form appealed to us because of its ability by definition to handle weak, frequent, distant encounters and strong, rare, close encounters on the same footing. Of course, the approach is of practical use only when probabilities or cross sections of such encounters are known from detailed studies of individual encounters through numerical experiments, as is the case for our current use of this approach. It was generally believed that, since Fokker-Planck methods were normally used for handling only the weak, frequent, distant encounters above, treating their cumulative effect as a diffusion in phase space, this argument would also apply to Monte Carlo Fokker-Planck methods. However, in a novel feature included recently by Fregeau, Rasio, and coauthors (Fregeau et al. 2003; Fregeau & Rasio 2007) in their Monte Carlo method, both of the above types of encounters are handled in the following way.

The dynamical evolution of the cluster is treated by a basically Hénon-type Monte Carlo method, which describes this evolution as a sequence of equilibrium models, subject to regular velocity perturbations that are calculated by the standard Hénon method for representing the average effect of many weak, frequent, distant encounters (see Fregeau et al. [2003] and references therein). In addition, the strong, rare, close encounters are handled by (1) keeping track of the (Monte Carlo–realized) positions of the objects in the cluster and so deciding whether two given objects will undergo a strong, close encounter or not by using a rejection method very similar to that described above in § 3.1, and then (2) treating these encounters first ($a$) through cross sections compiled from analytic fits to numerical scattering experiments (Fregeau et al. 2003), exactly as we have done throughout our approach, and then ($b$) in a more detailed approach, through a direct integration of the strong interaction at hand using standard two- and three-body integrators (Fregeau & Rasio 2007).
A direct comparison of our results with those of the above authors is, for the most part, not possible, since we focused primarily on the formation, destruction, and hardening of a compact binary population in a given GC environment, while Fregeau et al. (2003) and Fregeau & Rasio (2007) focused primarily on the dynamical evolution of the GC environment in the presence of a given primordial binary population. However, there is one feature for which we were able to roughly compare our results with those obtained by these and earlier authors. This is the problem of hardening of primordial binaries in GCs, for which pioneering studies were performed through direct Fokker-Planck integration by Gao et al. (1991), through the Monte Carlo method by Hut et al. (1992), and again recently through the above Monte Carlo method by Fregeau et al. (2003). In an early test run of our scheme, we studied this problem by “turning off” the binary formation and destruction terms in our scheme, thereby studying only the hardening of the primordial binary population through our Boltzmann approach. The results we obtained for the progressive hardening of the binary $a$-distribution profile (from an initial profile that was uniform in $\ln a$, as in all the above references, and in our work) were, indeed, very similar to those given in the above references.

In a pioneering study, Di Stefano & Rappaport (1992, 1994) explored the tidal capture formation and subsequent evolution of compact binaries in GCs, concentrating on recycled millisecond pulsars in the first part of the study (Di Stefano & Rappaport 1992) and on cataclysmic variables (CVs) in the second part (Di Stefano & Rappaport 1994). These authors followed the histories of many neutron stars against a given background representing a GC core (parameters corresponding to 47 Tuc and ω Cen were used as typical examples), employing Monte Carlo methods to generate tidal capture events in this environment. They followed the subsequent orbital evolution of these binaries due to hardening by gravitational radiation and magnetic braking, until Roche lobe contact occurred. In those cases in which such contact occurred through orbit shrinkage before the low-mass companion could reach the giant phase due to its nuclear evolution, these authors did not follow the further evolution of the binary, whereas they did so when the contact occurred due to the evolutionary expansion of the companion.

From the above considerations, Di Stefano & Rappaport (1992, 1994) estimated the expected number of recycled pulsars and CVs in GCs such as 47 Tuc and ω Cen, and they also gave the orbital period distribution of the above binaries at two points, viz., (1) just after tidal capture and orbit circularization and (2) at Roche lobe contact. However, their orbital period distributions cannot be compared directly with those given in this paper (or those in Paper I) for the following reason. In the Monte Carlo method of these authors, tidal capture occurs at different times for different binaries, as does Roche lobe contact. Thus, showing the orbital period distribution at any of the above two points means, in effect, that the period distributions at different times are being mixed. By contrast, we have (in this paper and in Paper I) studied the evolution of the orbital period distribution in time, displaying “snapshots” of the whole distribution at various times, which we called “time slices” above and in Paper I. In our display, for example, at any given time, some binaries are in Roche lobe contact and some are not. Indeed, it seems that the orbital period distributions just after tidal capture, as given by Di Stefano & Rappaport (1992), should be compared with the corresponding $N$-body results given in Portegies Zwart et al. (1997a), and indeed they appear rather similar. We have, of course, pointed out in Paper I, and we stress the point here again, that our orbital period distributions are to be regarded at this stage as intermediate steps in our calculation, rather than as final results to be compared with future data on orbital period distributions of X-ray binaries in GCs, because stellar evolutionary effects on binary evolution have not been included yet in our scheme (also see below). With this inclusion, the aim would be to produce the GC analog of such orbital period distributions as have been computed by Pfahl et al. (2003) for low-mass X-ray binaries (LMXBs) outside GCs.

In addition to the above improvement, we listed in Paper I various other improvements and extensions that are to be implemented in our scheme in the future. For example, the compact binary distribution function above can be regarded as one obtained by integrating the full, multivariate distribution function, which includes other variables, such as the binding energy of the binary in the gravitational potential of the GC—the so-called external binding energy (or, equivalently, the position of the binary within the GC potential well; Hut et al. 1992), over these other variables. It would be most instructive to be able to follow the evolution in these additional variables in a more elaborate future scheme.

Encouraged by the veracity of the continuous limit, as presented in this paper, we plan to conclude our program of the first stage of exploration of our Boltzmann scheme by studying one more problem in the same spirit of demonstration of feasibility as we have followed here and in Paper I. This is the question of compact binary evolution in the environment of an evolving GC. Whereas, in keeping with the tradition of numerous previous studies, we have treated the GC environment in Paper I and here as a fixed (i.e., unchanging in time) stellar background, in reality a GC is believed to undergo considerable evolution following the long, quasi-static, “binary-burning” phase, passing through phases of deep core collapse, (possible) gravothermal oscillations, and so on. In our concluding study, we propose to demonstrate that, at the current level of approximation in our scheme, and in the continuous limit, it is possible to follow the evolution of compact binary populations of GCs through these phases of GC evolution, at the expense of only a modest amount of computing time.

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**APPENDIX A**

**WIENER PROCESSES**

A Wiener process is the formal mathematical description of Brownian motion: a classic example of a stochastic process, wherein a particle (e.g., a grain of pollen) on the surface of water undergoes random motion due to stochastic bombardment of it by water molecules. A standard description of the motion of such a particle is given by the following differential form of a Langevin equation:

$$dX_t = a(t, X_t) dt + \sigma(t, X_t) \zeta_t dt.$$

(A1)
Here $X_t$ is one of the components of the particle velocity at time $t$ and $a(t, X_t)$ is the retarding viscous force. The second term on the right-hand side represents the random molecular force, represented as a product of an intensity factor $\sigma(t, X_t)$ and a random noise factor $\zeta_t$, the latter at each time $t$ being a suitably generated random number.

A standard Wiener process $W(t)$ is often defined as a continuous Gaussian process with independent increments that satisfies the following properties:

$$W(0) = 0, \quad E(W(t)) = 0, \quad \text{Var}(W(t) - W(s)) = t - s$$

(A2)

for all $0 \leq s \leq t$. Here $E$ represents the expectation value and “Var” represents the variance of the indicated stochastic variable. Note that a Wiener process $W_t(\omega)$ can also be thought of as a “pure” Brownian motion with $a = 0$ in equation (A1) (Kloeden et al. 1994), wherein the increments $dW_t(\omega)$ for any sample path $\omega$ represent Gaussian white noise.

Equation (A1) can then be rewritten in terms of the “symbolic differential” (see below) $dW_t(\omega) \equiv \zeta_t(\omega) ds$ of a Wiener process, and its integral form

$$X_t(\omega) = X_0(\omega) + \int_{t_0}^{t} a(s, X_s(\omega)) ds + \int_{t_0}^{t} \sigma(s, X_s(\omega)) dW_s(\omega)$$

(A3)

represents a path integral over the trajectory of the particle for the sample path $X_t(\omega)$, where $\omega$ is a particular trajectory of the Brownian particle.

APPENDIX B

ITO CALCULUS

The problem with the second term on the right-hand side of equation (A3), which represents an integral along a “Wiener path,” is that it is not defined in ordinary calculus, since the function $W_t(\omega)$ is not differentiable in the ordinary sense. Such an integral along a Wiener path has to be redefined suitably to become acceptable mathematically, and the Ito integral is a well-known example of this. The classical limit-of-sum definition of an integral does not hold good for an Ito integral such as

$$X_t(\omega) = \int_{t_0}^{t} f(s, \omega) dW_s(\omega),$$

(B1)

since the corresponding finite sum will be divergent over a Wiener path, as sample paths of a Wiener process do not have bounded variances (see above). However, it can be shown that such a sum is mean-square convergent under very general conditions (Oksendal 2003), due to the well-behaved mean-square properties of Wiener processes. Accordingly, equation (B1) is defined only in the sense of mean-square convergence, with the result that the integral in equation (B1) is a random variable $X_t(\omega)$ with the following properties:

$$E(X_t) = 0, \quad E(X_t^2) = \int_{t_0}^{t} E \left( f(s)^2 \right) ds.$$ 

(B2)

Consider now the well-known “Itô formula” for the transformation of a function $f(X_t)$ of a stochastic variable $X_t$ (Gaines 1995). For simplicity, first assume that $X_t$ follows a stochastic equation of the form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_r) dr + \int_{t_0}^{t} \sigma(X_r) dW_r,$$

(B3)

i.e., the same as equation (A3), but without explicit time dependence in the continuous and stochastic terms. For brevity, we drop the symbol $\omega$, which represents the sample path, in equation (B3) and from here on. Let us divide the entire time span into time steps at $t_1, t_2, \ldots, t_k, \ldots$ of length $h_1, h_2, \ldots, h_k, \ldots$, with a largest step size of $h_{\text{max}}$. Then the values of $X_t$ at times $t_k$ and $t_{k+1}$ are related by

$$X_{k+1} = X_k + \int_{t_k}^{t_{k+1}} a(X_r) dr + \int_{t_k}^{t_{k+1}} \sigma(X_r) dW_r,$$

(B4)

where we write $X_k \equiv X_{t_k}$ and $X_{k+1} \equiv X_{t_{k+1}}$ for brevity. The Itô formula states (Øksendal 2003) that

$$f(X_t) = f(X_k) + \int_{t_k}^{t} \mathcal{L} f(X_r) dr + \int_{t_k}^{t} f'(X_r) \sigma(X_r) dW_r,$$

(B5)

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} f(X_r) \equiv f'(X_r) a(X_r) + \frac{1}{2} f''(X_r) \sigma^2(X_r).$$

(B6)

For explicitly time-dependent continuous and stochastic terms, the Itô formula can be generalized suitably.

---

4 Strictly speaking, the first equation should be written as “$W(0) = 0$, w.p.1,” where “w.p.1” stands for “with probability one,” since we are dealing with random variables here. But we will not go into mathematical rigor here, and instead we refer the reader to Kloeden et al. (1994).
We can use equation (B5) in equation (B4) to expand $a(X_t)$ and $\sigma(X_t)$ around $t_k$:

\[
X_{k+1} = X_k + a(X_k)h_{k+1} + \sigma(X_k)\Delta W_{k+1} + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \mathcal{L}a(X_s) ds \, dt + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} a'(X_s)\sigma(X_s) dW_s \, dt \\
+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \mathcal{L}\sigma(X_s) ds \, dW_s + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \sigma'(X_s)\sigma(X_s) dW_s \, dW_t.
\]  

(B7)

Now, if we discard all terms in equation (B7) that are of $\mathcal{O}(h^\alpha)$ for $\alpha > 1$, we obtain

\[
X_{k+1} = X_k + a(X_k)h_{k+1} + \sigma(X_k)\Delta W_{k+1} + \frac{1}{2} \sigma'(X_k)\sigma(X_k) \left((\Delta W_{k+1})^2 - h_{k+1}\right),
\]

(B8)

which is known as the “Milstein scheme.” This is the stochastic analog of the second-order Taylor expansion of ordinary calculus. The Milstein scheme can be shown to be either strongly or pathwise convergent (Kloeden et al. 1994) to order $h$, in the sense that the solution converges to the actual Brownian path as $h_{\text{max}} \to 0$. If we restrict the expansion up to the $\mathcal{O}(h^{1/2})$ terms (i.e., up to the first three terms in the right-hand side of eq. [B7]), we obtain a slower (~$h^{1/2}$) pathwise convergence, which is known as the “Euler-Maruyama scheme.”

For higher dimensions, with $X_t \in \mathbb{R}^N$ and $W_t \in \mathbb{R}^D$, the second-order stochastic Taylor expansion of $X_t^i$ is given by (see Gaines [1995] and references therein)

\[
X_{k+1}^i = X_k^i + a^i(X_k)h_{k+1} + \sum_{j=1}^{N} \sigma^j(X_k) \Delta W_{k+1}^j + \sum_{p,q=1}^{N} \partial_{X^j}^i \sigma^j(X_k) I_{pq}(k, k+1) + R_i,
\]

(B9)

where

\[
I_{pq}(k, k+1) \equiv \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} dW_p^j \, dW_q^j
\]

(B10)

and $R$ contains all terms of $\mathcal{O}(h^\alpha)$ for $\alpha > 1$. If $D \leq p$, $q$ ($p \neq q$), we obtain, upon integration by parts,

\[
I_{pq}(k, k+1) + I_{qp}(k, k+1) - \Delta W_{k+1}^p \Delta W_{k+1}^q \equiv B_{pq}(k, k+1).
\]

(B11)

If we further define

\[
A_{pq}(k, k+1) \equiv I_{pq}(k, k+1) - I_{qp}(k, k+1),
\]

(B12)

then we can, with the aid of equations (B12) and (B11), express $I_{pq}$ in terms of $A_{pq}$ and $B_{pq}$. Substituting the result in equation (B9), we finally obtain

\[
X_{k+1}^i = X_k^i + a^i(X_k)h + \sum_{p} \sigma^i_p(X_k) \Delta W_{k+1}^p + \frac{1}{2} \sum_{j=1}^{N} \sum_{p=1}^{D} \partial_{X^j}^i \sigma^j_p(X_k) \left((\Delta W_{k+1}^p)^2 - h_{k+1}\right) + \sum_{j=1}^{N} \sum_{0 < p < q < D} \frac{1}{2} \left( \partial_{X^j}^i \sigma^j_p + \partial_{X^j}^i \sigma^j_q \right) (X_k) B_{pq}(k, k+1) + \sum_{j=1}^{N} \sum_{0 < p < q < D} \frac{1}{2} \left( \partial_{X^j}^i \sigma^j_p - \partial_{X^j}^i \sigma^j_q \right) (X_k) A_{pq}(k, k+1) + R_i.
\]

(B13)

If $\forall i, p, q$

\[
\sum_{j=1}^{N} \left( \partial_{X^j}^i \sigma^j_p - \partial_{X^j}^i \sigma^j_q \right) = 0,
\]

(B14)
then the $A_{pq}$ terms drop out of equation (B13). Equation (B14) is called the “commutativity condition” and is usually written as

$$[\sigma_p, \sigma_q] = 0.$$  \hspace{1cm} (B15)

When the above commutativity condition is not satisfied, the quantities $A_{pq}$, known as the “Levy areas,” have to be calculated in order to achieve second-order accuracy.

**REFERENCES**

Banerjee, S., & Ghosh, P. 2006, MNRAS, 373, 1188
---. 2007, ApJ, 670, 1090 (Paper I)
Di Stefano, R., & Rappaport, S. 1992, ApJ, 396, 587
---. 1994, ApJ, 423, 274
Fabian, A. C., Pringle, J. E., & Rees, M. J. 1975, MNRAS, 172, 155
Fregeau, J. M., Gürkan, M. A., Joshi, K. J., & Rasio, F. A. 2003, ApJ, 593, 772
Fregeau, J. M., & Rasio, F. A. 2007, ApJ, 658, 1047
Gaines, J. G. 1995, in Stochastic Partial Differential Equations, ed. A. Etheridge (Cambridge: Cambridge Univ. Press), 55
Gao, B., Goodman, J., Cohn, H., & Murphy, B. 1991, ApJ, 370, 567
Heggie, D. 1975, MNRAS, 173, 729
Heggie, D., Hut, P., & McMillan, S. L. W. 1996, ApJ, 467, 359
Hut, P., & Bahcall, J. N. 1983, ApJ, 268, 319
Hut, P., McMillan, S., & Romani, R. W. 1992, ApJ, 389, 527
Kloeden, P. E., Platen, E., & Schurz, H. 1994, Numerical Solution of SDE through Computer Experiments (Berlin: Springer)
Lee, H. M., & Ostriker, J. P. 1986, ApJ, 310, 176
Milshtein, G. N. 1974, Theor. Prob. Appl., 19, 557
Øksendal, B. 2003, Stochastic Differential Equations (6th ed.; Berlin: Springer)
Pfahl, E. D., Rappaport, S., & Podsiadlowski, P. 2003, ApJ, 597, 1036
Pooley, D., et al. 2003, ApJ, 591, L131
Portegies Zwart, S. F., Hut, P., McMillan, S. L. W., & Verbunt, F. 1997a, A&A, 328, 143
Portegies Zwart, S. F., Hut, P., & Verbunt, F. 1997b, A&A, 328, 130
Press, W. H., & Teukolsky, S. A. 1977, ApJ, 213, 183
Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes in C (2nd ed.; Cambridge: Cambridge Univ. Press)
Shull, J. M. 1979, ApJ, 231, 534
Sigurdsson, S., & Phinney, E. S. 1993, ApJ, 415, 631
Spitzer, L., Jr. 1987, Dynamical Evolution of Globular Clusters (Princeton: Princeton Univ. Press)
Verbunt, F. 2003, in ASP Conf. Ser. 296, New Horizons in Globular Cluster Astronomy, ed. G. Piotto et al. (San Francisco: ASP), 245
---. 2007, in Highlights of Astronomy, Vol. 14, ed. K. A. van der Hucht (Cambridge: Cambridge Univ. Press), 440
Verbunt, F., & Hut, P. 1987, in IAU Symp. 125, The Origin and Evolution of Neutron Stars, ed. D. J. Helfand & J.-H. Huang (Dordrecht: Reidel), 187
Verbunt, F., Lewin, W. G. H., & van Paradijs, J. 1989, MNRAS, 241, 51