A note on shortest circuit cover of 3-edge colorable cubic signed graphs

Ronggui Xu\textsuperscript{a}, Jiaao Li\textsuperscript{b} Xinmin Hou\textsuperscript{a,c}
\textsuperscript{a}School of Mathematical Sciences
University of Science and Technology of China, Hefei, Anhui 230026, China.
\textsuperscript{b}School of Mathematical Sciences
Nankai University, Tianjin 300071, China
\textsuperscript{c}CAS Key Laboratory of Wu Wen-Tsun Mathematics
University of Science and Technology of China, Hefei, Anhui 230026, China.

Abstract

A sign-circuit cover $\mathcal{F}$ of a signed graph $(G, \sigma)$ is a family of sign-circuits which covers all edges of $(G, \sigma)$. The shortest sign-circuit cover problem was initiated by Mácajová, Raspaud, Rollová, and Škoviera (JGT 2016) and received many attentions in recent years. In this paper, we show that every flow-admissible 3-edge colorable cubic signed graph $(G, \sigma)$ has a sign-circuit cover with length at most $\frac{20}{9}|E(G)|$.

1 Introduction

In this paper, graphs may have parallel edges and loops. A circuit is a connected 2-regular graph. A graph is even if every vertex has even degree, and an Eulerian graph is a connected even graph. A circuit cover $\mathcal{C}$ of a graph is a family of circuits which covers all edges of $G$. We call $\mathcal{C}$ a circuit $k$-cover of $G$ if $\mathcal{C}$ covers every edge of $G$ exactly $k$ times. The length of a circuit cover $\mathcal{C}$ is defined as $\ell(\mathcal{C}) = \sum_{C \in \mathcal{C}} |E(C)|$. Determining the shortest length of a circuit cover of a graph $G$ (denoted by $scc(G) = \min \{ \ell(\mathcal{C}) : \mathcal{C}$ is a circuit cover$\}$) is a classic optimization problem initiated by Itai, Lipton, Papadimitriou, and Rodeh [8]. Thomassen [9] showed that it is NP-complete to determine whether a bridgeless graph has a circuit cover with length at most $k$ for a given integer $k$. A well-known conjecture, the Shortest Circuit Cover Conjecture, was proposed by Alon and Tarsi [1] as follows.

Conjecture 1.1 (Shortest Circuit Cover Conjecture). For any 2-edge-connected graph $G$, $scc(G) \leq \frac{7}{5}|E(G)|$. 

\*The work was supported by NNSF of China (No. 12071453) and Anhui Initiative in Quantum Information Technologies (AHY150200) and the National Key R and D Program of China(2020YFA0713100).
The upper bound is achieved by the Petersen graph. Jamshyi and Tarsi [10] proved that Conjecture 1.1 implies the well-known Cycle Double Cover Conjecture proposed by Seymour [11] and Szekeres [12]. The best known general result about Conjecture 1.1 is obtained by Bermond, Jackson, Jaeger [13] and Alon, Tarsi [1], independently.

**Theorem 1.2** (Bermond, Jackson and Jaeger [13], Alon and Tarsi [1]). Let $G$ be a 2-edge-connected graph. Then $scc(G) \leq \frac{5}{3}|E(G)|$.

Several improvements of this upper bound for cubic graphs $G$ have been made in literature. Specifically, Jackson [19] showed that $scc(G) \leq \frac{64}{39}|E(G)|$ and Fan [16] later showed that $scc(G) \leq \frac{44}{27}|E(G)|$ and Hou and Zhang [4] proved that $scc(G) \leq \frac{34}{21}|E(G)|$ if $G$ has girth at least 7 and $scc(G) \leq \frac{8}{5}|E(G)|$ if all 5-circuits of $G$ are disjoint. Recently, Lukotka [14] showed that $scc(G) \leq \frac{212}{135}|E(G)|$ for all 2-edge-connected cubic graphs $G$.

A signed graph $(G, \sigma)$ is a graph $G$ associated with a mapping $\sigma : E(G) \to \{+1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. A signed graph $G$ is called positive if $G$ contains even number of negative edges and otherwise called negative. In a signed graph, a circuit with an even number of negative edges is called a balanced circuit, and otherwise we call it an unbalanced circuit. A barbell is a signed graph consisting of two unbalanced circuits joined by a (possibly trivial) path, intersecting with the circuits only at ends. If the path in a barbell is trivial, the barbell is called a short barbell; otherwise, it is a long barbell. A balanced circuit or a barbell is called a sign-circuit of a signed graph. A sign-circuit cover $\mathcal{F}$ of a signed graph is a family of sign-circuits which covers all edges of $(G, \sigma)$. In fact, it is well-known that a signed graph has a sign-circuit cover if and only if each edge lies in a sign-circuit, which is equivalent to the fact that the signed graph admits a nowhere-zero integer flow, so-called, flow-admissible. (Readers may refer to [2] for details). The shortest length of a sign-circuit cover of a signed graph $(G, \sigma)$ is also denoted by $scc(G)$. The shortest sign-circuit cover problem was initiated by Mácajová, Raspaud, Rollová, and Škoviera [5] and received many attentions in recent years. It is a major open problem for the optimal upper bound of shortest sign-circuit cover in signed graphs.

**Problem 1.3.** What is the optimal constant $c$ such that $scc(G) \leq c \cdot |E(G)|$ for every flow-admissible signed graph $(G, \sigma)$?

As remarked in [5], the signed Petersen graph $(P, \sigma)$ whose negative edges induce a circuit of length five has $scc(P) = \frac{5}{3}|E(P)|$, which indicates $c \geq \frac{5}{3}$. We list some of known results related to Problem 1.3.

1. $c \leq 11$ by Mácajová, Raspaud, Rollová, and Škoviera [5].
2. $c \leq \frac{14}{3}$ by Lu, Cheng, Luo, and Zhang [15].
3. $c \leq \frac{25}{6}$ by Chen and Fan [17].
For any flow-admissible 2-edge-connected cubic signed graph \((G, \sigma)\), Wu and Ye \cite{wu2011} obtained a better upper bound that \(scc(G) \leq \frac{26}{9} |E(G)|\). In this article, we focus on the shortest sign-circuit cover of 3-edge colorable cubic signed graphs and prove the following theorem.

**Theorem 1.4.** Every flow-admissible 3-edge colorable cubic signed graph \((G, \sigma)\) has a sign-circuit cover with length at most \(\frac{20}{9} |E(G)|\).

An equivalent version of the Four-Color Theorem states that every 2-edge-connected cubic planar graph is 3-edge colorable. So we have the following corollary.

**Corollary 1.5.** Every flow-admissible 2-edge-connected cubic planar signed graph \((G, \sigma)\) has a sign-circuit cover with length at most \(\frac{20}{9} |E(G)|\).

Now we introduce more notation and terminologies used in the following sections. Let \(G\) be a graph and \(T \subseteq V(G)\) with \(|T| \equiv 0 \pmod{2}\). A \(T\)-join \(J\) of \(G\) with respect to \(T\) is a subset of edges of \(G\) such that \(d_J(v) \equiv 1 \pmod{2}\) if and only if \(v \in T\), where \(d_J(v)\) denotes the degree of \(v\) in the edge-induced subgraph \(G[J]\). A \(T\)-join is minimum if it has minimum number of edges among all \(T\)-joins. Let \(G'\) be the graph obtained from a graph \(G\) by deleting all the bridges of \(G\). Then the components of \(G'\) are called the bridgeless-blocks of \(G\). By the definition, a bridgeless-block is either a single vertex or a maximal 2-edge-connected subgraph of \(G\). For a vertex subset \(U \subseteq V(G)\), \(\delta_G(U)\) denotes the set of edges with one end in \(U\) and the other in \(V(G) \setminus U\). Let \(u, v\) be two vertices in \(V(G)\). A \((u, v)\)-path is a path connecting \(u\) and \(v\). Let \(C = v_1 \ldots v_r v_1\) be a circuit where \(v_1, v_2, \ldots, v_r\) appear in clockwise on \(C\). A segment of \(C\) is the path \(v_i v_{i+1} \ldots v_{j-1} v_j\) (where the sum of the index is under modulo \(r\)) contained in \(C\) and is denoted by \(v_i C v_j\). A connected graph \(H\) is called a cycle-tree \cite{wu2011} if it has no vertices of degree 1 and all circuits of \(H\) are edge-disjoint. In a signed graph, switching a vertex \(u\) means reversing the signs of all edges incident with \(u\). Two signed graphs are equivalent if one can be obtained from the other by a sequence of switching operations, and a signed graph is balanced if and only if it is equivalent to an ordinary graph. The set of negative edges of \((G, \sigma)\) is denoted by \(E_N(G, \sigma)\).

The rest of the article is organized as follows. Some basic lemmas about signed graph and \(T\)-join and a crucial lemma which deals with a special case in our proof are given in Section 2. Then we are able to complete the proof of Theorem 1.4 in Section 3 and we will finish with some discussions and remark.

## 2 Some Lemmas

The following lemma due to Bouchet \cite{bouchet1985} characterized connected flow-admissible signed graphs.
Lemma 2.1 (Bouchet [2]). A connected signed graph \((G, \sigma)\) is flow-admissible if and only if it is not equivalent to a signed graph with exactly one negative edge and it has no bridge \(e\) such that \((G - e, \sigma|_{G - e})\) has a balanced component.

Lemma 2.2 (Li, Li, Luo, Zhang and Zhang [7]). Let \(T\) be a spanning tree of a signed graph \(G\). For every \(e \in E(T)\), let \(C_e\) be the unique circuit contained in \(T + e\). If the circuit \(C_e\) is balanced for every \(e \in E(T)\), then \(G\) is balanced.

Wu and Ye [21] gave a lemma to control the size of a \(T\)-join.

Lemma 2.3 (Wu and Ye [21]). Let \(G\) be a 2-edge-connected graph and \(T\) be subset of vertices with \(|T|\) even. Then \(G\) has a \(T\)-join of size at most \(\frac{1}{2} |E(G)|\).

The following two results gave upper bounds of \(scc(G)\) with \(G\) under some constrains.

Lemma 2.4 (Chen, Fan [3] and Kaiser, Lukať, Mácajová, Rollová [18]). Let \((G, \sigma)\) be a signed graph and suppose that each bridgeless-block of \(G\) is Eulerian.

(a) (Corollary 1.5 in [3]) If \((G, \sigma)\) is flow-admissible, then \(scc(G) \leq \frac{3}{2} |E(G)|\).

(b) (Corollary 2.6 in [18]) If the union of all the bridgeless-block of \(G\), denoted by \(H\), is positive, then there exists a family of sign-circuits \(F\) covers \(H\) with length at most \(\frac{4}{3} |E(G)|\).

A combination of the above two lemmas leads to the following.

Lemma 2.5. Let \(F\) be a 2-factor of a 2-edge-connected cubic sign graph \((G, \sigma)\). If \(F\) contains even number of negative edges, then there exists a family of sign-circuits \(F\) covers \(F\) with length at most \(\frac{10}{9} |E(G)|\).

Proof. We may assume that the 2-factor \(F\) consists of circuits \(C_1, C_2, \ldots, C_t\). Denote by \(G^*\) the graph obtained from \(G\) by contracting each circuit \(C_i\) of \(F\) to a single vertex \(c_i\).

Since \(F\) contains even number of negative edges, the number of unbalanced circuits in \(F\) is even. Without loss of generality, we may assume that \(C = \{C_1, C_2, \ldots, C_{2t}\}\) is the set of unbalanced circuits of \(F\). Let \(T = \{c_1, c_2, \ldots, c_{2t}\}\) and \(J\) be a minimum \(T\)-join of \(G^*\) with respect to \(T\). Since \(G\) is 2-edge-connected and \(G^*\) is obtained from \(G\) by contracting edges, \(G^*\) is 2-edge-connected as well. By Lemma 2.3 we have \(|J| \leq \frac{1}{2} |E(G^*)| = \frac{1}{2} |E(G)|\). Consider the edge set \(F \cup J\) in \(G\), and we view it as an edge-induced subgraph of \(G\). By the definition of \(T\)-join, we can apply Lemma 2.4(b) to \(F \cup J\), i.e., there exists a family of sign-circuits \(F\) covers \(F\) with length

\[
\ell(F) \leq \frac{4}{3} |E(F \cup J)|
\]

\[
= \frac{4}{3} (|E(F)| + |E(J)|)
\]

\[
\leq \frac{4}{3} \left( \frac{2}{3} |E(G)| + \frac{1}{6} |E(G)| \right)
\]

\[
= \frac{10}{9} |E(G)|.
\]

This proves the lemma. \(\square\)
The proof of the following lemma is inspired by the proof of Lemma 3.7 in [7] on flows of 3-edge colorable cubic signed graphs.

**Lemma 2.6.** Let $C$ be an unbalanced circuit of a cubic signed graph $(G, \sigma)$. If $(G, \sigma)$ is flow-admissible and $G - E(C)$ is balanced, then $(G, \sigma)$ has a family $\mathcal{F}$ of sign-circuits such that

1. $E(C)$ is covered by $\mathcal{F}$, and
2. the length of $\mathcal{F}$ satisfies $\ell(\mathcal{F}) \leq \frac{8}{9}|E(G)| + |E(C)|$.

**Proof.** Let $G' = G - E(C)$. Since $G'$ is balanced, with some switching operations, we may assume that all edges in $E(G')$ are positive and thus $E_N(G, \sigma) \subseteq E(C)$.

Let $M$ be a component of $G'$. The circuit $C$ was divided by the vertices of $M$ into pairwise edge-disjoint paths (called segments) whose end-vertices lie in $M$ and all inner vertices lie in $C$. An end-vertex of a segment is called an attachment of $M$. A segment is called positive (negative, resp.) if it contains an even (odd, resp.) number of negative edges. Note that $M \cup S$ is unbalanced (balanced, resp.) if and only if the segment $S$ is negative (positive, resp.). Since $M \cup C$ is unbalanced, the number of negative segments determined by $M$ is odd.

**Case 1.** There exists a component $M$ of $G'$ that determines more than one negative segments.

Then in this case $M$ determines at least three negative segments and so $|E(C)| \geq 3$. Let $u_1Cu_3, u_2Cu_5, u_3Cu_7$ be three consecutive negative segments (in clockwise order) where $u_i$ and $v_i$ are attachments for $i = 1, 2, 3$. Then $v_1Cu_2, v_2Cu_3, v_3Cu_4$, where each of them contains even number of negative edges. This implies that $C$ can be partitioned into three pieces: $u_1Cu_2, u_2Cu_3$, and $u_3Cu_4$ all contain odd number of negative edges. Note that $u_i$ and $v_i$ are not adjacent in $C$ for distinct $i, j \in \{1, 2, 3\}$ since $G$ is cubic. Let $P_1$ be a $(u_1, u_2)$-path in $M$. Since $M$ is connected, there is a path $P_2$ from $u_3$ to $P_1$ such that $|V(P_2) \cap V(P_1)| = 1$. Let $v$ be the only common vertex in $P_1$ and $P_2$. Then $C, P_1, P_2$ form a signed graph $H_1$ as illustrated in Figure 1.

Note that $|E(H_1)| \leq \frac{8}{9}|E(G)| + 2$ since $G$ is cubic and there are exactly four vertices of degree 3 in $H_1$. Divide $P_1$ into two pieces in $M$: $u_1P_{11}v$ and $vP_{12}u_2$, denote by $B_1 = u_1P_{11}v \cup vP_{12}u_2 \cup u_3Cu_4, B_2 = u_3P_2v \cup vP_{11}u_1 \cup u_1Cu_3, B_3 = u_2P_{12}v \cup vP_{23}u_3 \cup u_3Cu_2$. Note that each of $u_2Cu_1, u_1Cu_3, u_3Cu_2$ contains even number of negative edges. So $B_1, B_2, B_3$ are all balanced circuits and $\mathcal{F}_1 = \{B_1, B_2\}, \mathcal{F}_2 = \{B_2, B_3\}, \mathcal{F}_3 = \{B_3, B_1\}$ are all sign-circuit covers of $H_1$, which are also sign-circuit covers of $C$ since $E(C) \subseteq E(H_1)$. Note that
Figure 1: $H_1 = C \cup P_{11} \cup P_{12} \cup P_2$, negative segments are dashed.

$F = \{F_1, F_2, F_3\}$ covers edge edge of $H_1$ exactly 4 times. So we have

$$\min \{\ell(F_1), \ell(F_2), \ell(F_3)\} \leq \frac{1}{3} (\ell(F_1) + \ell(F_2) + \ell(F_3))$$

$$= \frac{4}{3} |E(H_1)|$$

$$\leq \frac{4}{3} \left( \frac{2}{3} |E(G)| + 2 \right)$$

$$< \frac{8}{9} |E(G)| + |E(C)|.$$

**Case 2.** Each component of $G'$ determines exactly one negative segment.

Let $\mathcal{M}$ denote the set of all components of $G'$. For each component $M$, denote by $S_M = uCv$ the negative segment determined by $M$ where $u$ and $v$ are two attachments of $M$ on $C$. Denote by $S'_M = vCu$ the cosegment of $S_M$, which is the complement of $S_M$ on $C$. Then $E(S_M) \neq \emptyset$ and $S'_M = E(C) - E(S_M)$. We have the following two conclusions:

**Claim 1** (see Claim 3.7.2 in [7]):

$$\cap_{M \in \mathcal{M}} E(S_M) = \emptyset,$$

or equivalently, $\cup_{M \in \mathcal{M}} E(S'_M) = C$ and $|\mathcal{M}| \geq 2$.

Let $S = \{S'_1, S'_2, ..., S'_t\}$ be a minimal cosegment cover of $C$. We have

**Claim 2** (see Claim 3.7.3 in [7]). For any edge $e \in E(C)$, $e$ is contained in at most two cosegments.

**Proof Sketches of Claims 1 and 2:** For the sake of completeness, we present the proof sketches of this two claims here. Suppose to the contrary $\cap_{M \in \mathcal{M}} E(S_M) \neq \emptyset$. 


\( \emptyset \) and \( e^* \in \cap_{M \in \mathcal{M}} E(S_M) \). Then there is a spanning tree \( T \) of \( G - e^* \) containing the path \( P^* = C - e^* \). Let \( e = uv \in E(G) - e^* - E(T) \). Denote the unique circuit contained in \( T + e \) by \( C_e \).

If \( E(C_e) \cap E(P^*) = \emptyset \), then \( C_e \) contains no negative edges and thus is balanced. Otherwise since \( T \) contains all the edges in \( C - e^* \), \( E(C_e) \cap E(C) \) is a path \( P \) on \( C \). Let \( u' \) and \( v' \) be the two end-vertices of \( P \) in clockwise order on \( C \). Then \( C_e - V(P) + u', v' \) is also a path and thus it is contained in some component \( M \in \mathcal{M} \). This implies that \( u' \) and \( v' \) are two attachments of \( M \) on \( C \). Since \( e^* \) belongs to the only negative segment of \( C \) determined by \( M \), \( u'Cv' \) is the union of some positive segments of \( C \) determined by \( M \). Therefore \( C_e \) has an even number of negative edges and thus is balanced. By Lemma 2.2, \( G - e^* \) is balanced, contradicting Lemma 2.1. This proves \( \cap_{M \in \mathcal{M}} E(S_M) = \emptyset \). Since \( E(S_M) = E(C) - E(S_M) \) and \( \cap_{M \in \mathcal{M}} E(S_M) = \emptyset \), we have \( \cup_{M \in \mathcal{M}} E(S'_M) = C \). Since \( E(S_M) \neq \emptyset \) and \( \cap_{M \in \mathcal{M}} E(S_M) = \emptyset \), we have \( |\mathcal{M}| \geq 2 \). This completes the proof of the claim 1.

Suppose to the contrary that there exists an edge \( e = uv \) that belongs to three cosegments \( L_1, L_2, L_3 \) of \( S \). Denote \( L_i = u_iCv_i \) for each \( i = 1, 2, 3 \). Without loss of generality, we may assume that \( u_2 \) belongs to \( u_1Cu \). Then \( v_2 \) doesn’t belong to \( u_1Cu_3 \) (see Figure 2). Note that \( v_3 \) belongs to \( u_1Cu_3 \). If \( u_3 \) belongs to \( u_1Cu_3 \), then both \( v_3 \) and \( u_3 \) belongs to \( u_1Cv_1 \) and thus \( L_1 \cup L_3 = C \) (see Figure 2-(a)), contradicting the minimality of \( S \). If \( u_3 \) doesn’t belong to \( u_1Cu_1 \), then \( u_3 \) belongs to \( vCv_2 \). Since \( L_3 \) contains \( uv \), \( v_3 \) belongs to \( vCv_2 \). Thus both \( v_3 \) and \( u_3 \) belongs to \( u_2Cv_2 \). Therefore \( L_2 \cup L_3 = C \) (see Figure 2-(b)), also contradicting the minimality of \( S \). This completes the proof of the claim 2.

For each \( i = 1, \ldots, t \), denote by \( S'_i = x_iCy_i \) and let \( P_i \) be a path in \( M_i \) connecting \( x_i \) and \( y_i \). Then \( C_i = S'_i \cup P_i \) is a balanced Eulerian subgraph. By Claims 1 and 2 we may assume that the vertices \( x_1, y_t, x_2, y_1, \ldots, x_t, y_{t-1}, x_1 \) appear on \( C \) in clockwise order. Then \( C_i \cap C_j \neq \emptyset \) if and only if \( |j - i| \equiv 1 \)
Figure 3: Minimal cosegment cover of $C$ with $t = 5$.

Let $H_2 = C \cup P_1 \cup P_2 \cup \ldots \cup P_t$ and $B_i = x_iCy_i \cup y_iP_ix_i$ for $i = 1, 2, \ldots, t$. Note that $B_i$ is a balanced circuit and so $F = \{B_1, B_2, \ldots, B_t\}$ is a sign-circuit cover of $C$. Obviously, $|E(H_2)| \leq \frac{2}{3}|E(G)| + t$ since $G$ is cubic and there are exactly $2t$ vertices of degree 3 in $H_2$. By Claim 2, $F$ covers the edges in $C$ at most twice and edges in $P_1 \cup \ldots \cup P_t$ exactly once. Let $W$ be the set of edges covered by $F$ twice. Then $W \subset E(C)$. Since $G$ is cubic, $x_i \neq y_j$ for all $i, j \in \{1, 2, \ldots, t\}$. So we have $|W| \leq |E(C)| - t$. Therefore,

$$t(F) = |E(B_1)| + |E(B_2)| + \ldots + |E(B_t)| \leq |E(H_2)| + |E(C)| - t \leq \frac{2}{3}|E(G)| + t + |E(C)| - t \leq \frac{8}{9}|E(G)| + |E(C)|.$$ 

This completes the proof. \qed

3 Proof of Theorem 1.4

**Proof of Theorem 1.4.** Let $f$ be a 3-edge coloring of connected cubic graph $G$. Let $R, B, Y$ be the three color classes of $f$. Recall that $E_N(G, \sigma)$ is the set of negative edges in $(G, \sigma)$. Without loss of generality, we may assume $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod{2}$. Denote by $M_1, M_2$ the 2-factor induced by $M_1 \cup M_2$ for each pair $M_1, M_2 \in \{R, B, Y\}$. Since $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod{2}$, $RB$ has an even number of unbalanced components. By Lemma 2.5, there exists a family of sign-circuits $F_1$ covers $RB$ with length at most $\frac{10}{9}|E(G)|$. 

8
Case 1. RB contains an unbalanced circuit.

First, assume that $|Y \cap E_N(G, \sigma)|$ has the same parity with $|R \cap E_N(G, \sigma)|$. Then $RY$ has an even number of unbalanced circuits. By Lemma 2.5 we can find a family of sign-circuits $\mathcal{F}_2$ covers $RY$ with length at most $\frac{10}{9}|E(G)|$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a sign-circuit cover of $G$ with length

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) \leq \frac{10}{9}|E(G)| + \frac{10}{9}|E(G)| = \frac{20}{9}|E(G)|.$$

Then, assume instead that $|Y \cap E_N(G, \sigma)|$ has different parity with $|R \cap E_N(G, \sigma)|$. Let $C$ be an unbalanced circuit in RB. Now we swap the colors $R$ and $B$ on $C$, i.e. reset $R' = R \Delta E(C)$ and $B' = B \Delta E(C)$ respectively. Note that the operation will change the parity of $|R \cap E_N(G, \sigma)|$ and $|B \cap E_N(G, \sigma)|$. This implies that $|Y \cap E_N(G, \sigma)| \equiv |R' \cap E_N(G, \sigma)| \equiv |B' \cap E_N(G, \sigma)| \pmod{2}$ now. So, similar as the previous paragraph, we apply Lemma 2.5 to find a family of sign-circuits $\mathcal{F}_2'$ covers $R'Y$ with length at most $\frac{10}{9}|E(G)|$. Notice that $\mathcal{F}_1$ covers $RB = R'B'$ with length at most $\frac{10}{9}|E(G)|$. Hence $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2'$ is a sign-circuit cover of $G$ with length at most $\frac{20}{9}|E(G)|$.

Case 2. RB contains no unbalanced circuit.

In this case, each circuit $C_i$ of RB is a balanced circuit. Let $\mathcal{F}_1 = \{C_i | C_i$ is a balanced circuit of RB$\}$. Then $\mathcal{F}_1$ is a family of sign-circuits covering RB with length $\ell(\mathcal{F}_1) = E(RB) = \frac{4}{3}|E(G)|$.

Subcase 2.1: The number of unbalanced circuits in $RY$ or $BY$ is even.

By Lemma 2.5 we have a family of sign-circuits $\mathcal{F}_2$ which covers $RY$ or $BY$ with length at most $\frac{10}{9}|E(G)|$. Therefore $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a sign-circuit cover of $G$ with length $\ell(\mathcal{F}) \leq \frac{16}{9}|E(G)|$.

Subcase 2.2: The number of unbalanced circuits in $RY$ or $BY$ is equal to one.

Without loss of generality, assume that $RY$ has exactly one unbalanced component, say, $C_1$. Let $C = \{C_1, ..., C_m\}$ be the set of components of $RY$, where each $C_i$ ($i \geq 2$) is balanced. Let $\mathcal{F}_2 = \{C_i : i \geq 2\}$. Then $\mathcal{F}_2$ is a family of sign-circuits covering $RY - E(C_1)$ with length $\frac{4}{3}|E(G)| - |E(C_1)|$. We consider the following two cases in order to cover $C_1$.

Assume first that $G$ contains an unbalanced circuit $C'$ with $E(C') \cap E(C_1) = \emptyset$. Since $G$ is cubic and connected, there is a long barbell $Q$ in $G$ with $P$ as the path connecting $C_1$ and $C'$ with $|E(Q)| \leq \frac{2}{3}|E(G)| + 1$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup Q$ is a sign-circuit cover of $G$ with length

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(Q)$$

$$\leq \frac{2}{3}|E(G)| + \frac{2}{3}|E(Q)| - |E(C_1)| + \frac{2}{3}|E(G)| + 1$$

$$< 2|E(G)|.$$

Then assume instead that $G$ contains no unbalanced circuit $C'$ with $E(C') \cap E(C_1) = \emptyset$. In this case, $G - E(C_1)$ is balanced. By Lemma 2.6 there exists a
family $\mathcal{F}_3$ of sign-circuits covering $E(C_1)$ with length at most $\frac{8}{9}|E(G)| + |E(C_1)|$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a sign-circuit cover of $G$ with length
\[
\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(\mathcal{F}_3)
\leq \frac{2}{3}|E(G)| + \frac{2}{3}|E(G)| - |E(C_1)| + \frac{8}{9}|E(G)| + |E(C_1)|
\leq \frac{20}{9}|E(G)|.
\]

**Subcase 2.3:** The number of unbalanced circuits in each of $RY$, $BY$ is odd and is at least 3.

Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the set of components of $RY$. Denote by $G^*$ the graph obtained from $G$ by contracting each circuit $C_i$ of $RY$ to a single vertex $u_i$, where $i = 1, 2, \ldots, m$. Note that $G^*$ is connected, and so let $T^*$ be a spanning tree of $G^*$. Then $T^* \cup RY$ is a cycle-tree in $G$, denoted by $H$, containing at least 3 unbalanced circuits. Let $B'$ be the set of bridges of $H$ such that $b_i \in B'$ if and only if $(H - b_i, \sigma|_{H - b_i})$ has a balanced component $H_i$. $B = \emptyset$ if no such bridge exists in $H$. Let $H' = H - (B' \cup \cup_{i=1, \ldots, m}|B'|E(H_i))$. Note that $H'$ contains all the unbalanced circuit of $\mathcal{C}$. By Lemmas 2.1 and 2.4(a), $H'$ is flow-admissible and has a sign-circuit cover $\mathcal{F}_2$ with length at most $\frac{3}{2}|E(H')|$. Since the circuits in $H_i$ are all balanced circuits, we can cover them with length at most $\frac{3}{2}|E(H_i)| - |E(H')|$. Thus we have a sign-circuit cover $\mathcal{F}_3$ of $RY$ with length
\[
\ell(\mathcal{F}_3) = \ell(\mathcal{F}_2) + (|E(H)| - |E(H')|)
\leq \frac{3}{2}|E(H')| + |E(H)| - |E(H')|
\leq \frac{3}{2}|E(H)| < \frac{3}{2}|E(G)|.
\]

Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_3$ is a sign-circuit cover of $G$ with length
\[
\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_3)
\leq \frac{2}{3}|E(G)| + \frac{3}{2}|E(G)|
\leq \frac{13}{6}|E(G)| < \frac{20}{9}|E(G)|.
\]

This completes the proof. \qed

**Remark.** The upper bound of $scc(G)$ in Theorem 1.4 seems not to be tight. We realized that the 3-edge colorable cubic signed graph $(G, \sigma)$ as illustrated in Figure 4 has a sign-circuit cover with length $\frac{13}{6}|E(G)|$. The problem to determine the optimal upper bound for the shortest sign-circuit cover of 3-edge colorable cubic signed graph remains open.

**Data Availability:** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
Figure 4: $(G, \sigma)$ with a shortest circuit cover with length $\frac{13}{9} |E(G)|$.

References

[1] Alon, N and Tarsi, M. Covering multigraphs by simple circuits. SIAM Journal on Algebraic Discrete Methods, 6(3):345–350, 1985.

[2] Bouchet, André. Nowhere-zero integral flows on a bidirected graph. Journal of Combinatorial Theory, Series B, 34(3):279–292, 1983.

[3] Chen, Jing and Fan, Genghua. Circuit k-covers of signed graphs. Discrete Applied Mathematics, 294:41–54, 2021.

[4] Hou, Xinmin and Zhang, Cun-Quan. A note on shortest cycle covers of cubic graphs. Journal of Graph Theory, 71(2):123–127, 2012.

[5] Mácajová, Edita and Raspaud, André and Rollová, Edita and Škoviera, Martin. Circuit covers of signed graphs. Journal of Graph Theory, 81(2):120–133, 2016.

[6] Mácajová, Edita and Raspaud, André and Tarsi, Michael and Zhu, Xuding. Short cycle covers of graphs and nowhere-zero flows. Journal of Graph Theory, 68(4):340–348, 2011.

[7] Li, Liangchen and Li, Chong and Luo, Rong and Zhang, Cun-Quan and Zhang, Hailiang. Flows of 3-edge colorable cubic signed graphs. preprint.

[8] Itai, Alon and Lipton, Richard J and Papadimitriou, Christos H and Rodeh, Michael. Covering graphs by simple circuits. SIAM Journal on Computing, 10(4):746–750, 1981.

[9] Thomassen, Carsten. On the complexity of finding a minimum cycle cover of a graph. SIAM Journal on Computing, 26(3):675–677, 1997.
[10] Jamshy, Ury and Tarsi, Michael. Short cycle covers and the cycle double cover conjecture. Journal of Combinatorial Theory, Series B, 56(2):197–204, 1992.

[11] Seymour, Paul D. Sums of circuits. Graph theory and related topics, 1:341–355, 1979.

[12] Szekeres, George. Polyhedral decompositions of cubic graphs. Bulletin of the Australian Mathematical Society, 8(3):367–387, 1973.

[13] Bermond, Jean Claude and Jackson, Bill and Jaeger, François. Shortest coverings of graphs with cycles. Journal of Combinatorial Theory, Series B, 35(3):297–308, 1983.

[14] Lukotka, Robert. Short cycle covers of cubic graphs and intersecting 5-circuits. SIAM Journal on Discrete Mathematics, 34(1):188–211, 2020.

[15] Lu, You and Cheng, Jian and Luo, Rong and Zhang, Cun-Quan. Shortest circuit covers of signed graphs. Journal of Combinatorial Theory, Series B, 134:164–178, 2019.

[16] Fan, Genghua. Short cycle covers of cubic graphs. Journal of Graph Theory, 18(2):131–141, 1994.

[17] Chen, Jing and Fan, Genghua. Short signed circuit covers of signed graphs. Discrete Applied Mathematics, 235:51–58, 2018.

[18] Kaiser, Tomáš and Lukot’ka, Robert and Máčajová, Edita and Rollová, Edita. Shorter signed circuit covers of graphs. Journal of Graph Theory, 92(1):39–56, 2019.

[19] Jackson, Bill. Shortest circuit covers of cubic graphs. Journal of Combinatorial Theory, Series B, 60(2):299–307, 1994.

[20] Kaiser, Tomáš and Král’, Daniel and Lidický, Bernard and Nejedlý, Pavel and Šámal, Robert. Short cycle covers of graphs with minimum degree three. SIAM Journal on Discrete Mathematics, 24(1):330–355 2010.

[21] Wú, Yezhou and Ye, Dong. Minimum T-joins and signed-circuit covering. SIAM Journal on Discrete Mathematics, 34(2):1192–1204, 2020.

[22] Wú, Yezhou and Ye, Dong. Circuit covers of cubic signed graphs. Journal of Graph Theory, 89(1):40–54, 2018.