VARIABLE LORENTZ ESTIMATE FOR STATIONARY STOKES SYSTEM WITH PARTIALLY BMO COEFFICIENTS

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ABSTRACT. We prove a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients of weak solution pair \((u, P)\) to the Dirichlet problem of stationary Stokes system. It is mainly assumed that the leading coefficients are merely measurable in one spatial variable and have sufficiently small bounded mean oscillation (BMO) seminorm in the other variables, the boundary of underlying domain is Reifenberg flat, and the variable exponents \(p(x)\) satisfy the so-called log-Hölder continuity.

1. Introduction. Let \(\Omega \subset \mathbb{R}^n (n \geq 2)\) be a bounded domain with a rough boundary specified later. We write the unknown velocity vectorial-value functions \(u = (u^1, u^2, \cdots, u^n) : \Omega \rightarrow \mathbb{R}^n\), and the pressure \(P : \Omega \rightarrow \mathbb{R}\). The main purpose of this present article is in minimizing regular requirements on the given datum to study a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients of weak solution pair \((u, P)\) to the Dirichlet problem of stationary Stokes system as follows:

\[
\begin{aligned}
& \sum_{\alpha, \beta} A^{\alpha\beta}_{ij} D_{ij} u + \nabla P = D_{\alpha} f_{\alpha} \quad & \text{in} \ \Omega, \\
& \text{div} \ u = g \quad & \text{in} \ \Omega, \\
& u = 0 \quad & \text{on} \ \partial \Omega.
\end{aligned}
\]

(1)

Throughout this paper, we use the Einstein summation convention on repeated indices. As usual, we suppose that each matrix-value entry of the coefficients \(A^{\alpha\beta} = (A^{\alpha\beta}_{ij})_{i,j} : \Omega \rightarrow \mathbb{R}^{n \times n}\) for every \(\alpha, \beta = 1, 2, \cdots, n\) satisfies the boundedness and the strong ellipticity, which means that there exists a constant \(0 < \Lambda < 1\) with

\[
|A^{\alpha\beta}| \leq \Lambda^{-1}, \quad \sum_{\alpha, \beta = 1}^{n} \xi_{\alpha} A^{\alpha\beta} \xi_{\beta} \geq \Lambda \sum_{\alpha = 1}^{n} |\xi_{\alpha}|^2
\]

(2)

with \(\xi_{\alpha} \in \mathbb{R}^{n^2}\) for \(\alpha = 1, 2, \cdots, n\). The weak solution of (1) is understood in the following usual sense, if for \((u, P) \in W^{1,2}(\Omega)^n \times L^2(\Omega)\) it holds

\[
\int_{\Omega} A^{\alpha\beta} D_{ij} u \cdot D_{ij} \varphi dx + \int_{\Omega} P \cdot \text{div} \ \varphi dx = \int_{\Omega} f_{\alpha} \cdot D_{\alpha} \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega)^n.
\]

(3)

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The solvability and optimal regularity for stationary Stokes system under minimal regular datum are always classical and important problems in the theory of partial differential equations and fluid dynamics. In recent years, there have been a great deal of literatures concerning the interior regularity of Stokes system (cf. [7]) and global regularity of generalized Stokes system in the Lipschitz domain (cf. [17, 18, 20, 21]) or the rough domain with Reifenberg flat boundary (cf. [10, 19, 23]). Particularly, we would like to mention some recent advances concerning the generalized stationary Stokes problems (1) with discontinuous coefficients. Byun and So [10] recently considered the following generalized Stokes system:

\[
\begin{aligned}
\text{div} (A(x) \nabla u) - \nabla P &= \text{div} \mathbf{F} \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and they obtained the global weighted \(L^q\)-estimate for the gradient of weak solution pair \((u, P)\) under some weak regular assumptions that the coefficients have small bounded mean oscillation (BMO) seminorm and the boundary of domain is Reifenberg flat, which implies the fact that \(\mathbf{F} \in L^q_0(\Omega)^{n^2} \Rightarrow \nabla u \in L^q_0(\Omega)^{n^2}, \quad P \in L^q_0(\Omega)\).

Further, Choi and Lee in [15] proved global a priori \(L^q\)-estimate for stationary Stokes systems (1) just replacing \(D_\alpha f_\alpha\) by \(f + D_\alpha f_\alpha\) with BMO coefficients on the Reifenberg domain. On the other hand, Dong and Kim [18] also studied the same stationary Stokes systems with partially BMO coefficients on the bounded Lipschitz domains with small Lipschitz constant, and they derived

\[
\|D_\alpha u\|_{L^{q_1}(\Omega)} + \|P\|_{L^{q_1}(\Omega)} \leq c\left(\|f\|_{L^{q_1}(\Omega)} + \|f_\alpha\|_{L^{q_1}(\Omega)} + \|g\|_{L^{q_1}(\Omega)}\right)
\]

for \(q_1 \geq \frac{nq}{q+n}\). Later, Choi, Dong and Kim in [14, 19] extended it to the global weighted \(L^q\)-estimates for the Dirichlet problem and the conormal derivative problem for the stationary Stokes system with partial regular coefficients, respectively, while \(\Omega\) is a Reifenberg flat domain. Furthermore, Choi and Dong [13] showed that the solution \((Du, P)\) is bounded and its certain linear combinations are continuous if the coefficients are assumed to be merely measurable in one direction and have Dini mean oscillations in the other directions. Inspired by Choi, Dong and Kim’s considerations above, this present article allows the coefficients \(A_{\alpha\beta}\) are merely measurable in one direction and have a sufficiently small Bounded Mean Oscillation (BMO) seminorm in the other directions, such that our estimate is in the frame of Lorentz spaces for the variable power of the gradients of weak solution.

It is well-known fact that there have been a lot of research activities on the Calderón-Zygmund theory for various elliptic and parabolic problems with discontinuous coefficients. Except an earlier technique that used singular integral operators and their commutators, there were mainly three kinds of main different arguments to handle the Calderón-Zygmund theory concerning elliptic and parabolic problems with \(VMO\) or small \(BMO\) discontinuous coefficients. The first one is so-called geometrical approach originally traced from Byun and Wang’s work in [9], which is used to attain global \(L^p\) estimate based on the weak compactness, the boundedness of the Hardy-Littlewood maximal operators and the modified Vitali covering for distributional functions regarding the gradients of solutions. Here, the so-called modified Vitali covering actually refers to the argument as “crawling of ink spots” as in the early papers by Safonov and Krylov [27, 31]. Indeed, this is also a
development from Caffarelli and Peral’s paper in [12] to obtain local $W_{loc}^{1,p}$-estimates for solutions of $p$-Laplacian type elliptic problems with the nonlinearity satisfying the so-called Cordes-Nirenberg condition. Secondly, Kim and Krylov (cf. [25, 26]) gave a unified approach of studying $L^p$ solvability for elliptic and parabolic problems in accordance with the Fefferman-Stein theorem on the sharp functions and the Hardy-Littlewood maximal function theorem for the spatial derivatives of solution. In this present paper, we have to highlight the third technique being called large-$M$-inequality principle originating from Acerbi and Mingione’s work [2, 3], which is directly applied to argue on certain Calderón-Zygmund type covering arguments instead of the classical maximal function operator and other harmonic analysis techniques such as the good-$\lambda$ inequality. Furthermore, we would like to mention that Byun et al very recently have got numerous global Calderón-Zygmund type results to various nonlinear elliptic and parabolic problems over nonsmooth domains by combining large-$M$-inequality principle with their geometrical approach. Regarding our consideration, we particularly point out that Byun, Ok and Wang [8] first attained a global Calderón-Zygmund estimate with a variable exponent of the gradients of weak solution to the Dirichlet problem of linear elliptic system in divergence form with partially BMO coefficients and log-Hölder continuity $p(x)$, which implies that

$$F, D\psi \in L^{p(x)}(\Omega, \mathbb{R}^n) \Rightarrow Du \in L^{p(x)}(\Omega, \mathbb{R}^n).$$

Also, we mention that Tian and Zheng in [32] further generalized the above result to the global Calderón-Zygmund type estimate in Lorentz spaces for a variable power of the gradients of weak solution to the same problem on a rough domain with partial BMO coefficients.

Lorentz space is a two-parameter scale of the Lebesgue space obtained by refining it in the fashion of a second index, and there are a large body of literatures concerning Lorentz regularity for partial differential equations. For examples, Mengesha and Phuc [28] derived the weighted Lorentz estimate for the gradients of weak solution to quasilinear $p$-Laplacian type equations based on the geometric approach. Meanwhile, Baroni [5, 6] obtained interior Lorentz estimate for the gradients of solutions to evolutionary $p$-Laplacian systems and obstacle parabolic $p$-Laplacian with the given obstacle function $D\psi \in L^{(\gamma, q)}$ locally in $\Omega_T$, respectively, by using the large-$M$-inequality principle, which yields the fact that

$$F, D\psi \in L^{(\gamma, q)} \text{ locally in } \Omega_T \Rightarrow Du \in L^{(\gamma, q)} \text{ locally in } \Omega_T$$

with $\gamma > p$ and $q \in (0, \infty]$. Later, Zhang and Zhou [34] extended the above result in [28] to that for general elliptic equations of $p(x)$-Laplacian also using a geometrical argument, Adimurthi and Phuc [4] also proved the global Lorentz and Lorentz-Morrey estimates for quasilinear equations below the natural exponent. Very recently, Tian and Zheng [33] showed a global weighted Lorentz estimate to linear elliptic equations with lower order items under partially BMO coefficients in Reifenberg flat domain. Zhang and Zheng [35, 36] also studied with Hessian Lorentz estimates for fully nonlinear parabolic and elliptic equations with small BMO nonlinearities, and weighted Hessian Lorentz estimates of strong solution for nondivergence linear elliptic equations with partially BMO coefficients, respectively.

Now let us start with related basic notation being useful in the context. The Lorentz space $L(t, q)(U)$ for open subset $U \subset \mathbb{R}^n$ with parameters $1 \leq t < \infty$ and
0 < q < ∞, is the set of all measurable functions \( g : U \rightarrow \mathbb{R} \) requiring
\[
\|g\|_{L^q(U)} := t \int_0^\infty \left( \mu^t\{ |g| > \mu \} \right)^{\frac{1}{q}} \frac{d\mu}{\mu} < \infty;
\]
while the Lorentz space \( L(t, \infty) \) for \( 1 \leq t < \infty \) and \( q = \infty \) is defined by the Marcinkiewicz space \( \mathcal{M}^t(U) \) as usual, which is the set of all measurable functions \( g \) with
\[
\|g\|_{\mathcal{M}^t(U)} := \sup_{\mu > 0} \left( \mu^t\{ |g| > \mu \} \right)^{\frac{1}{t}} < \infty.
\]
The local variant of such spaces is defined as usual way. If \( t = q \), then the Lorentz space \( L(t, t)(U) \) is nothing but a classical Lebesgue space. Indeed, by Fubini’s theorem it yields
\[
\|g\|_{L^t(t, U)} = t \int_0^\infty \mu^t\{ |g| > \mu \} \frac{d\mu}{\mu} = \|g\|_{L(t, t)(U)},
\]
which implies \( L^t(U) = L(t, t)(U) \), see also [5, 6, 28, 35].

A usual assumption on the variable exponent \( p(\cdot) \) is the so-called log-Hölder continuity, which ensures that the Hardy-Littlewood maximal operator is bounded within the framework of generalized Lebesgue space. To this end, we recall the definition that \( p(x) \) is log-Hölder continuous denoted it by \( p(x) \in LH(\Omega) \), if there exist positive constants \( c_0 \) and \( \delta \) such that for all \( x, y \in \Omega \) with \( |x - y| < \delta \) one has
\[
|p(x) - p(y)| \leq \frac{c_0}{-\log(|x - y|)}.
\]
In the context, we assume that \( p(x) : \Omega \rightarrow \mathbb{R} \) is a log-Hölder continuous function and there exist positive constants \( \gamma_1 \) and \( \gamma_2 \) such that
\[
2 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty, \text{ for all } x \in \Omega.
\]
Without loss of generality, let
\[
|p(x) - p(y)| \leq \omega(|x - y|), \text{ for all } x, y \in \Omega,
\]
where \( \omega : [0, \infty) \rightarrow [0, \infty) \) is a modulus of continuity of \( p(x) \) such that \( \omega \) is a nondecreasing continuous function with \( \omega(0) = 0 \) and \( \lim\sup_{r \to 0} \omega(r) \log \left( \frac{1}{r} \right) < \infty \).

With the above assumptions in hand, it is clear that \( p(x) \in LH(\Omega) \) yields that there exists a positive number \( A \) such that
\[
\omega(r) \log \left( \frac{1}{r} \right) \leq A \iff r^{-\omega(r)} \leq e^A \text{ for any } r \in (0, 1).
\]

Before stating our main result, let us recall some basic concepts and related facts. We denote a type point by \( x = (x^1, x') = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n \). Let \( B_r = \{ x \in \mathbb{R}^n : |x| < r \} \), \( B^+_r = B_r \cap \{ x \in \mathbb{R}^n : x^1 > 0 \} \) and \( B'_r = \{ x' \in \mathbb{R}^{n-1} : |x'| < r \} \) with \( B_r(y) = B_r + y, B^+_r(y) = B^+_r + y \) and \( B'_r(y) = B'_r + y' \). Set \( B^+_{r}(\tau, 0') = B_{r}(\tau, 0') \cap \{ x^1 > \tau \} \) for \( 0 \leq \tau < \infty \) and \( 0' \in \mathbb{R}^{n-1} \), and a typical boundary \( \Omega_r(y) = B_r(y) \cap \Omega \). For any \( f \in L^1(U) \) with a bounded measurable subset \( U \subset \mathbb{R}^n \), we denote an average of \( f \) on \( U \) by
\[
f_{U} = \frac{1}{|U|} \int_U f(x)dx.
\]
Now we recall that \( \Omega \) is a Reifenberg flat domain in the following sense.
Assumption 1.1. There exists $R_0 \in (0, +\infty)$ such that for each $x_0 \in \partial \Omega$ and each $0 < r \leq R_0$, there exists a coordinate system depending only on $x_0$ and $r$, such that in the new coordinate system with the origin at $x_0$, we have

$$B_r \cap \{(y^1, y') : y^1 > r/16\} \subset B_r \cap \Omega \subset B_r \cap \{(y^1, y') : y^1 > -r/16\}. \quad (7)$$

We are in a position to assume the regular assumptions on the nonlinearity $A^{\alpha \beta}(x) = A^{\alpha \beta}(x^1, x')$ along with the rough boundary of Reifenberg domain $\Omega$.

Assumption 1.2. Let $0 < \delta < \frac{1}{16}$ be a small constant to be specified later. We say that $(A^{\alpha \beta}, \Omega)$ is $(\delta, R)$-vanishing of codimension 1 if there exists $R \in (0, R_0]$ such that the following conditions hold:

(i) For $x_0 \in \Omega$ and $0 < r \leq \min\{R, \text{dist}(x_0, \partial \Omega)\}$, there exists a coordinate system depending only on $x_0$ and $r$, whose variables are still denoted by $x$, such that in the new coordinate system we have

$$\int_{B_r(x_0)} |A^{\alpha \beta}(x^1, x') - A^{\alpha \beta}(x^1, z')|dz'dx \leq \delta. \quad (8)$$

(ii) For each $x_0 \in \partial \Omega$ and $0 < r < R$, there exists a coordinate system depending only on $x_0$ and $r$, such that in the new coordinate system with the origin at $x_0$ such that (8) holds and

$$B_r \cap \{(y^1, y') : y^1 > \delta r\} \subset B_r \cap \Omega \subset B_r \cap \{(y^1, y') : y^1 > -\delta r\}. \quad (9)$$

The domain with (9) is called $(\delta, R)$-Reifenberg flat domain, which is stronger than Assumption 1.1. Note that the stationary Stokes system has a scaling invariance property, which leads to that a small positive constant $\delta$ is still invariant under such a scaling so that $(\delta, R)$-Reifenberg flat domain is also a non-tangentially accessible domain. We would remark that $\delta > 0$ is determined by a set of parameters including the boundary flatness and $R_0$ in Assumption 1.1, see Remark 2.3 in [19] and Remark 1.5 in [11]. Notice that for a $(\delta, R)$-Reifenberg flat domain we see that the boundary might be very rough between hyperplanes in a sufficiently small region going beyond the boundaries with $C^1$-smooth or the Lipschitz category with a small Lipschitz constant. Moreover, it is obvious fact that this is $A$-type domain, which ensures the following measure density condition (cf. [8]):

$$\sup_{0 < r \leq R_0} \frac{|Q_r(y)|}{|Q_r(y) \cap \Omega|} \leq \sup_{0 < r \leq R_0} \frac{|B_{\sqrt{2}r}(y)|}{|B_r(y) \cap \Omega|} \leq \left(\frac{2\sqrt{2}}{1 - \delta}\right)^n \leq \left(\frac{32\sqrt{2}}{15}\right)^n. \quad (10)$$

Finally, let us summarize our main result of this paper as follows.

Theorem 1.3. Let $(u, P) \in W^{1,2}(\Omega)^n \times L^2(\Omega)$ be weak solution to stationary Stokes systems (1) with $(P)_{\Omega} = 0$ and coefficients $A^{\alpha \beta}(x)$ satisfying (2). Suppose that

$$\left(|f_\alpha| + |g|\right)^{p(x)} \in L(t,q)(\Omega) \quad \text{and} \quad (g)_{\Omega} = 0, \quad t > 1, \quad q \in (0, +\infty],$$

If $p(\cdot) \in LH(\Omega)$ with (4) and (5), and there exists a small constant $\delta_0 = \delta_0(n, \Lambda, \gamma_1, \gamma_2, t, q, R_0, |\Omega|) > 0$ such that for every $\delta \in (0, \delta_0)$, $(A^{\alpha \beta}, \Omega)$ satisfies $(\delta, R_0)$-vanishing of codimension 1 with Assumption 1.2 and Assumption 1.1. Then we have

$$\left\|\left(\left(|Du| + |P|\right)^{p(x)}\right)_{\Omega}\right\|_{L(t,q)(\Omega)} \leq c \left\|\left(|f_\alpha| + |g|\right)^{p(x)}\right\|_{L(t,q)(\Omega)}, \quad (11)$$

where the constant $c$ depends only on $n, \gamma_1, \gamma_2, \Lambda, t, q, R_0, \omega(\cdot)$, and $|\Omega|$ (except in the case $q = \infty$, where $c$ depends on $n, \gamma_1, \gamma_2, \Lambda, t, R_0, \omega(\cdot)$, and $|\Omega|$).
This article is to focus on a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients of weak solutions to the Dirichlet problem of stationary Stokes system (1) with partially regular coefficients on Reifenberg domains. Our problem and its argument are inspired to a great extent by work of Acerbi and Mingione [2, 3] and Baroni [5, 6], and recent work from Byun and So [10]. The point is to make use of the mixed argument of large-M-inequality principle and geometric approach to the Calderón-Zygmund type covering on the super-level set $E(\lambda, \Omega_R)$. Indeed, our main key ingredient is based on our using Calderón-Zygmund type covering, approximate estimate and an iteration argument to obtain an estimate of the measure of the super-level set for the variable power of the gradient of a solution.

The remainder of the paper is organized as follows. In Section 2, we introduce some useful lemmas. In Section 3, we focus on proving our main theorem.

2. Technical tools. In this section we present some useful lemmas, which will play essential roles in proving our main conclusion. From then on, we denote by $C_i(n, \Lambda, \cdots)$ or $c_i(n, \Lambda, \cdots)$ for $i = 1, 2, \cdots$, the universal constant depending only on prescribed quantities and possibly varying from line to line. First, let us do impose nothing regularity on the coefficients $A_{\alpha\beta}(x)$, but a relaxed geometric requirement on the domain $\Omega$.

Definition 2.1. The domain with Babuška-Aziz inequality states that there exists a constant $C_1$ such that for any $u \in L^2(\Omega)$ with $\int_{\Omega} ud\nu = 0$, we have a linear operator $B$ with $Bu \in W^{1,2}_0(\Omega)^n$, div $Bu = u$ in $\Omega$ and

$$\|Du\|_{L^2(\Omega)} \leq C_1 \|u\|_{L^2(\Omega)}.$$  

Nečas had proved $L^p$-version of the Babuška-Aziz inequality for $1 < p < \infty$ on Lipschitz domains. Acosta, Durán and Muschietti [1] extended solvability of the divergence equation on John domains $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ via a constructive approach, see also [16]. If $\Omega$ is a bounded Reifenberg flat domain, then the domain is also a John domain. Therefore, we know that the domain $\Omega$ satisfies Definition 2.1 with a constant $C_1$ depending only on $n, R_0$ and diam$(\Omega)$. More precisely, for a bounded domain we have

Reifenberg flat domains $\subset$ NTA–domains $\subset$ uniform domains $\subset$ John domains, where NTA–domains are briefly written as the non-tangentially accessible domains, see [19, 24].

Let us recall the existence and energy estimate of the weak solution pair to stationary Stokes system (1) defined in a more general domain with the Babuška-Aziz inequality, see Lemma 3.4 in [19].

Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Babuška-Aziz inequality shown by Definition 2.1, and $f, g \in L^2(\Omega)$ with $(g)_{\Omega} = 0$. Then there exists a unique solution $(u, P) \in W^{1,2}_0(\Omega)^n \times L^2(\Omega)$ to stationary Stokes systems (1) with $(P)_{\Omega} = 0$ such that the following standard estimate is valid:

$$\|Du\|_{L^2(\Omega)} + \|P\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)} + c\|g\|_{L^2(\Omega)},$$

where $c = c(n, \Lambda, C_1)$.

Next, we show a reverse Hölder inequality of the gradient of a weak solution for stationary Stokes systems (1).
**Lemma 2.3.** Let \((u, P) \in W^{1,2}_0(\Omega)^n \times L^2(\Omega)\) be the weak solution of stationary Stokes systems \((1)\) under the structural assumption \((2)\). Suppose \((|f_\alpha| + |g|)^{p(x)} \in L^1(\Omega)\) with \((g)_{\Omega} = 0\) for \(p(x) > \gamma_1 > 2\) and \(t > 1\), and the boundary of \(\Omega\) is Reifenberg flat with Assumption 1.1. Then for any \(x_0 \in \Omega\) there exists a small positive constant \(\sigma_0 : 0 < \sigma_0 < \frac{\gamma_1}{2} - 1\) such that, for any \(0 < \sigma \leq \sigma_0\) and \(\Omega_r(x_0) = B_r(x_0) \cap \Omega\) with \(r \in (0, \frac{R_0}{8}]\) we have

\[
\left( \int_{\Omega_r(x_0)} |\nabla u|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} + \left( \int_{\Omega_r(x_0)} |P|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \int_{\Omega_r(x_0)} |\nabla u|^2 \, dx + c \int_{\Omega_r(x_0)} |P|^2 \, dx + c \left( \int_{\Omega_r(x_0)} (|f_\alpha| + |g|)^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}},
\]

where \(c = c(n, \Lambda, C_1, \gamma_1, t) > 0\), \(u, P, f_\alpha,\) and \(g\) are extended by zero from \(\Omega\) to \(\mathbb{R}^n\).

**Proof.** We see that \(f_\alpha, g \in L^{t\gamma_1}(\Omega)\) for \(t\gamma_1 > 2\), since \((|f_\alpha| + |g|)^{p(x)} \in L^1(\Omega)\). Proceeding as in the same way as the proof of Lemma 3.8 in [19], we get (13). \(\square\)

Let \(\sigma_0\) be the same as in Lemma 2.3, and let \(\sigma_2 = \min\{\sigma_0, \frac{\gamma_1}{2} - 1\} > 0\) due to \(\gamma_1 > 2\). For a fixed point \(y \in \Omega\), we take \(r_y < \frac{R}{20}\) with

\[
R \leq \min \left\{ \frac{R_0}{2}, \frac{R_0}{c^*}, 1 \right\} \quad \text{and} \quad \omega(4R) < \gamma_1 \sigma_2,
\]

where \(c^* = c^*(n, \gamma_1, \gamma_2, \nu, \Lambda, \omega(\cdot), |\Omega|) \geq |\Omega| + 1\) determined later. For a fixed point \(x_0 \in \Omega\), we set

\[
\begin{align*}
p^- & := \inf_{\Omega_{2R}(x_0)} p(x), \\
p^+ & := \sup_{\Omega_{2R}(x_0)} p(x), \\
p^-_u & := \inf_{\Omega_{40r_y}(y)} p(x), \\
p^+_u & := \sup_{\Omega_{40r_y}(y)} p(x).
\end{align*}
\]

In the sequel, we give the comparison estimates with a limiting problem, and Lipschitz property of a weak solution for this limiting problem.

**Lemma 2.4.** Let \((u, P) \in W^{1,2}(\Omega)^n \times L^2(\Omega)\) be any weak solution pair to stationary Stokes systems \((1)\) under assumption that \((A^{\alpha\beta}, \Omega)\) satisfies \((\delta, R)\)-vanishing of codimension 1 with Assumption 1.2 and Assumption 1.1. Suppose \(|f_\alpha|^{p(x)}, |g|^{p(x)} \in L^1(\Omega)\) with \(t > 1\) and \((g)_{\Omega} = 0\). If, for any \(0 < \epsilon < 1\) there exists a constant \(\delta = \delta(n, \Lambda, \epsilon, \gamma_1, \gamma_2, t, q, R_0, |\Omega|) \in (0, \frac{1}{20})\) such that

\[
\int_{\Omega_{40r_y}(y)} \left( |\nabla u|^2 + |P|^2 \right) \, dx \leq c \lambda^{r_y^2}, \quad \int_{\Omega_{40r_y}(y)} \left( |f_\alpha|^{2\eta} + |g|^{2\eta} \right) \, dx \leq c \lambda^{r_y^2} \delta^{\frac{s_\eta}{s}}
\]

for \(\eta = 1 + \sigma_2 \leq 1 + \sigma\) with \(\sigma\) as the same of Lemma 2.3. Then there exists

\[(v, P_v), (w, P_w) \in W^{1,2}(\Omega_{10r_y}(y))^n \times L^2(\Omega_{10r_y}(y))\]

such that \((u, P) = (v, P_v) + (w, P_w)\) in \(\Omega_{10r_y}(y)\) with

\[
\int_{\Omega_{10r_y}(y)} \left( |\nabla v|^2 + |P_v|^2 \right) \, dx \leq c^2 \lambda^{r_y^2}
\]

and

\[
\int_{\Omega_{10r_y}(y)} \left( |\nabla w|^2 + |P_w|^2 \right) \, dx \leq c^2 \lambda^{r_y^2} \delta^\frac{s_\eta}{s}
\]

for \(\eta = 1 + \sigma_2 \leq 1 + \sigma\).
where

\[ c = c(n, \Lambda, C_1, \gamma_1, t). \]

**Proof.** Let us recall the proof of Proposition 5.1 in [19]. For the case of \( B_{10r_y}(y) \subset \Omega \) it leads to \( \Omega_{10r_y}(y) = B_{10r_y}(y) \), we may find that \( (v, P_v) \) and \( (w, P_w) \) are the solutions, respectively, to the following boundary problems of

\[
\begin{cases}
D_\alpha(A^{\alpha\beta}(x^1)D_\beta v) + \nabla P_v = D_\alpha(f^\alpha + (\bar{A}^{\alpha\beta}(x^1) - A^{\alpha\beta})D_\beta u) & \text{in } B_{10r_y}(y), \\
\text{div } v = g - (g)_{B_{10r_y}(y)} & \text{in } B_{10r_y}(y), \\
v = 0 & \text{on } \partial B_{10r_y}(y)
\end{cases}
\]

and

\[
\begin{cases}
D_\alpha(A^{\alpha\beta}(x^1)D_\beta w) + \nabla P_w = 0 & \text{in } B_{10r_y}(y), \\
\text{div } w = (g)_{B_{10r_y}(y)} & \text{in } B_{10r_y}(y),
\end{cases}
\]

where

\[
\bar{A}^{\alpha\beta}(x^1) = \int_{B_1(y')} A^{\alpha\beta}(x^1, x') dx'.
\]

For the case of \( B_{10r_y}(y) \cap \partial \Omega \neq \emptyset \), by considering \( \delta \in (0, \frac{y}{20^{r_y}}) \) it deduces \( \Omega_{35r_y}(300\delta r_y, 0') \subset \Omega_{50r_y}(y) \subset \Omega_{300r_y}(y) \). Let us introduce that \( \zeta \) is a smooth cutting-off function on \( \mathbb{R} \) such that \( \zeta(x^1) \equiv 1 \) for \( x^1 \leq 300\delta r_y \), \( \zeta(x^1) \equiv 1 \) for \( x^1 \geq 600\delta r_y \) and \( |\zeta'| \leq \frac{1}{150\delta r_y} \). Therefore, we find that \( (\bar{v}, P_v) \in W^{1,2}(B_{35r_y}^+(300\delta r_y, 0'))^n \times L^2(B_{35r_y}^+(300\delta r_y, 0')) \) with \( (P_v)_{B_{35r_y}^+(300\delta r_y, 0')} = 0 \) is a unique solution of

\[
\begin{cases}
D_\alpha(A^{\alpha\beta}(x^1)D_\beta \bar{v}) + \nabla P_v = D_\alpha(f^\alpha + h^\alpha) & \text{in } B_{35r_y}^+(300\delta r_y, 0'), \\
\text{div } \bar{v} = (\zeta g + u_1\zeta) - (\zeta g + u_1\zeta)_{B_{35r_y}^+(300\delta r_y, 0')} & \text{in } B_{35r_y}^+(300\delta r_y, 0'), \\
\bar{v} = 0 & \text{on } \partial B_{35r_y}^+(300\delta r_y, 0'),
\end{cases}
\]

where

\[
h^\alpha = (\bar{A}^{\alpha\beta}(x^1) - A^{\alpha\beta})D_\beta u + \bar{A}^{\alpha\beta}(x^1)(\zeta - 1)u.
\]

We set \( v = \bar{v} + (1 - \zeta)u \),

\[
P_v = \begin{cases}
P_0 & \text{in } B_{35r_y}^+(300\delta r_y, 0'), \\
P & \text{in } \Omega_{35r_y}(300\delta r_y, 0') \cap \{x^1 < 300\delta r_y\}.
\end{cases}
\]

We now extend \( \bar{v} \) by zero-extension from \( B_{35r_y}^+(300\delta r_y, 0') \) to \( \Omega_{35r_y}(300\delta r_y, 0') \) so that \( \bar{v} \in W^{1,2}(\Omega_{35r_y}(300\delta r_y, 0')) \), then we define \( (v, P_v) \in W^{1,2}(\Omega_{35r_y}(300\delta r_y, 0'))^n \times L^2(\Omega_{35r_y}(300\delta r_y, 0')) \). Let \( (w, P_w) \) be a solution to

\[
\begin{cases}
D_\alpha(A^{\alpha\beta}(x^1)D_\beta w) + \nabla P_w = 0 & \text{in } B_{35r_y}^+(300\delta r_y, 0'), \\
\text{div } w = (\zeta g + u_1\zeta)_{B_{35r_y}^+(300\delta r_y, 0')} & \text{in } B_{35r_y}^+(300\delta r_y, 0'), \\
w = 0 & \text{on } B_{35r_y}(300\delta r_y, 0') \cap \{x^1 = 300\delta r_y\}.
\end{cases}
\]
In what follows, we can proceed as in the proof of Corollary 5.2 in [19] and use (15) to get

\[
\int_{\Omega_{t,q}(y)} (|Dv|^2 + |Pv|^2) \, dx \leq c\delta^\frac{1}{2} \int_{\Omega_{4t,q}(y)} (|\nabla u|^2 + |P|^2) \, dx + c \left( \int_{\Omega_{4t,q}(y)} (|f_\alpha|^2 + |g|^2) \, dx \right)^{\frac{1}{\eta}}
\]

and

\[
\|Dw\|^2_{L^\infty(\Omega_{t,q}(y))} + \|Pw\|^2_{L^\infty(\Omega_{t,q}(y))} \leq c\left( \delta^\frac{1}{2} + 1 \right) \int_{\Omega_{4t,q}(y)} (|\nabla u|^2 + |P|^2) \, dx + c \left( \int_{\Omega_{4t,q}(y)} (|f_\alpha|^2 + |g|^2) \, dx \right)^{\frac{1}{\eta}}
\]

where \( \frac{1}{\eta} = 1 - \frac{1}{\eta} \). We now choose \( \delta > 0 \) small enough such that \( c\left( \delta^\frac{1}{2} + \delta^\frac{1}{2} \right) \leq \epsilon^2 \), which leads to get the desired estimates (16) and (17). □

Let us collect some preliminary results concerning the so-called embedding relations involved the Lorentz spaces, which will be used in the sequel.

**Proposition 1.** Let \( U \) be a bounded measurable subset of \( \mathbb{R}^n \), then the following relations hold:

i) If \( 0 < q \leq \infty \), and \( 1 \leq t_1 < t_2 < \infty \), then \( L(t_2,q)(U) \subset L(t_1,q)(U) \) with the estimate

\[
\|g\|_{L(t_1,q)(U)} \leq c\|U\|^{\frac{1}{t_1} - \frac{1}{t_2}} \|g\|_{L(t_2,q)(U)}. \tag{18}
\]

ii) If \( 1 \leq t < \infty \), and \( 0 < q_1 < q_2 \leq \infty \), then \( L(t,q_1)(U) \subset L(t,q_2)(U) \subset L(t,\infty)(U) \) with the estimate

\[
\|g\|_{L(t,q_2)(U)} \leq c(t,q_1,q_2) \|g\|_{L(t,q_1)(U)}. \tag{19}
\]

iii) If \( |g|^\alpha \in L(t,q)(U) \) for some \( 0 < \alpha < \infty \), then \( g \in L(at,aq)(U) \) with the estimate

\[
\|g\|^\alpha_{L(t,q)(U)} = \|g\|^\alpha_{L(at,aq)(U)}. \tag{20}
\]

The following two lemmas will play important roles in our main proof, which are the variants of classical Hardy’s inequality and a reverse Hölder inequality, respectively, see Lemma 3.4 and 3.5 in [5].

**Lemma 2.5.** Let \( f : [0, +\infty) \to [0, +\infty) \) be a measurable function such that

\[
\int_0^\infty f(\lambda) d\lambda < \infty. \tag{21}
\]

Then for any \( \alpha \geq 1 \) and \( r > 0 \), it holds that

\[
\int_0^\infty \lambda^{\tau} \left( \int_0^\infty f(\mu) d\mu \right)^{\alpha} \frac{d\lambda}{\lambda} \leq \left( \frac{\alpha}{r} \right)^{\alpha} \int_0^\infty \lambda^{\tau} [\lambda f(\lambda)]^{\alpha} \frac{d\lambda}{\lambda}.
\]
Lemma 2.6. Let \( h : [0, +\infty) \to [0, +\infty) \) be a nonincreasing, measurable function. For \( \alpha_1 \leq \alpha_2 \leq \alpha \) and \( r > 0 \), if \( \alpha_2 < \infty \) we have
\[
\left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_2} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_2}} \leq \varepsilon \lambda^r h(\lambda) + \frac{c}{\varepsilon^{\alpha_2/(\alpha_1-1)}} \left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_1} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_1}} \tag{22}
\]
with every \( \varepsilon \in (0, 1] \) and \( \lambda > 0 \). If \( \alpha_2 = \infty \), then it holds that
\[
\sup_{\mu > \lambda} (\mu^r h(\mu)) \leq c \lambda^r h(\lambda) + c \left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_1} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_1}}, \tag{23}
\]
where the constant \( c \) depends only on \( \alpha_1, \alpha_2 \) and \( r \); except in the case \( \alpha_2 = \infty \) with \( c \equiv c(\alpha_1, r) \).

Finally, let us recall the following iteration argument in the main proof, see [22] or Lemma 4.1 in [30].

Lemma 2.7. Let \( \varphi : [r_1, 2r_1] \to [0, \infty) \) be a function such that
\[
\varphi(r_1) \leq \frac{1}{2} \varphi(r_2) + B_0 (r_2 - r_1)^{-\beta} + L, \quad \text{for every } r_1 < r_2 < 2r_1,
\]
where \( B_0, L \geq 0 \) and \( \beta > 0 \). Then
\[
\varphi(r_1) \leq c(\beta) B_0 r_1^{-\beta} + cL.
\]

3. Proof of Theorem 1.3. This section is mainly devoted to proving Theorem 1.3 via a combination of the so-called large-M-inequality principle in [3] and the geometric argument in [8]. For this, we here part it in six steps. In step 1, for given \( \lambda_0 \) in (26), we show the Calderón-Zygmund type covering on the super-level set \( \Omega(\lambda, \Omega_R) \), and establish the decay estimates of \( \Omega_{r_\varepsilon}(y) \). In step 2, we give various comparison estimates with the reference problems so as to get the comparison estimate with the limiting one. In step 3, we employ the so-called “crawling of ink spots” approach to show an essential estimate for the super-level set of the gradients with a variable power of the gradients. In steps 4 and 6, we attain our conclusions in the cases of \( q < \infty \) and \( q = \infty \), respectively, under a priori assumption \( \| (|\nabla u| + |P|)^{p(x)} \|_{L(t,q)(\Omega_{\lambda R})} < \infty \) which will be proved in step 5.

Proof. Note that \( \Omega \) is bounded \((\delta, R_0)\)-Reifenberg domain, then it holds the Babuška-Aziz inequality with \( C_1 \) depending only on \( n, |\Omega| \), also see Remark 3.3 in [19]. In the following we will use Lemma 2.2, Lemma 2.3 and Lemma 2.4 uniformly with \( C_1 = C_1(n, R_0, |\Omega|) \).

For the weak solution pair \( (u, P) \) of original problem and the nonhomogeneous terms \( f_\alpha, g \), by a scaling argument we write
\[
\tilde{u} = \frac{u}{\| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega)}} \quad \tilde{P} = \frac{P}{\| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega)}}, \tag{24}
\]
\[
\tilde{f}_\alpha = \frac{f_\alpha}{\| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega)}} \quad \tilde{g} = \frac{g}{\| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega)}}.
\]

Then, by the assumption \( (|f_\alpha| + |g|)^{p(x)} \in L(t,q)(\Omega) \) we have
\[
\| (|\tilde{f}_\alpha| + |\tilde{g}|)^{p(x)} \|_{L(t,q)(\Omega)} \leq 1. \tag{25}
\]
Hereafter, for the sake of simplicity, we still use \( u, P, f, \) and \( g \) replacing \( \tilde{u}, \tilde{P}, \tilde{f}, \) and \( \tilde{g} \) in the following.

**Step 1.** In this step, we make use of the Calderón-Zygmund type covering on the super-level set \( E(\lambda, \Omega_R) \) to show a decay estimates of \( \Omega_{r_\lambda}(y) \). Let \( u \) be the weak solution of (1). For any \( x_0 \in \Omega \) and \( \Omega_R = \Omega_R(x_0) \), we define the quantity

\[
\lambda_0 := \int_{\Omega_{2R}} \left( |Du| + |P| \right)^{\frac{2p+1}{p}} \, dx + \frac{1}{\delta} \left( \int_{\Omega_{2R}} \left( |f_\alpha| + |g| \right)^{\frac{2p+1}{p} \eta} \, dx + 1 \right)^{\frac{1}{\eta}},
\]

where \( \delta > 0 \) and \( \eta > 1 \) will be specified later. We now introduce the super-level set

\[
E(\lambda, \Omega_R) := \{ x \in \Omega_R, (|Du| + |P|)^{\frac{2p+1}{p}} > \lambda \}
\]

for any \( \lambda > M\lambda_0 \geq 1 \) with \( M = \left( \frac{25600\sqrt{2}}{15} \right)^n \). For \( y \in E(\lambda, \Omega_R) \) and radii \( 0 < r \leq R \), we let

\[
CZ(\Omega_r(y)) := \int_{\Omega_r(y)} \left( |Du| + |P| \right)^{\frac{2p+1}{p}} \, dx + \frac{1}{\delta} \left( \int_{\Omega_r(y)} \left( |f_\alpha| + |g| \right)^{\frac{2p+1}{p} \eta} \, dx + 1 \right)^{\frac{1}{\eta}}.
\]

Note that \( \frac{R}{400} \leq r \leq R \). By simply enlarging the domain of integration we get

\[
CZ(\Omega_r(y)) \leq \frac{|\Omega_{2R}|}{|\Omega_r(y)|} \int_{\Omega_{2R}} \left( |Du| + |P| \right)^{\frac{2p+1}{p}} \, dx
\]

\[
+ \left( \frac{|\Omega_{2R}|}{|\Omega_r(y)|} \right)^{\frac{1}{\eta}} \frac{1}{\delta} \left( \int_{\Omega_{2R}} \left( |f_\alpha| + |g| \right)^{\frac{2p+1}{p} \eta} \, dx \right)^{\frac{1}{\eta}}
\]

\[
\leq \frac{|\Omega_{2R}|}{|\Omega_r(y)|} \left( \int_{\Omega_{2R}} \left( |Du| + |P| \right)^{\frac{2p+1}{p}} \, dx + \frac{1}{\delta} \left( \int_{\Omega_{2R}} \left( |f_\alpha| + |g| \right)^{\frac{2p+1}{p} \eta} \, dx \right) \right)
\]

\[
\leq \frac{|B_{2R}|}{|B_r(y)|} \frac{|B_r(y)|}{|\Omega_r(y)|} \lambda_0
\]

\[
\leq \left( \frac{2R}{r} \right)^n \frac{32\sqrt{2}}{15} \lambda_0
\]

\[
\leq \left( \frac{25600\sqrt{2}}{15} \right)^n \lambda_0 < \lambda,
\]

which means that while \( \frac{R}{400} \leq r \leq R \) one has

\[
CZ(\Omega_r(y)) < \lambda.
\]

On the other hand, by Lebesgue’s differentiation theorem we get that for \( 0 < r < 1 \)

\[
CZ(\Omega_r(y)) > \lambda.
\]

Therefore, by absolute continuity of the integral with respect to the domain we can pick the maximal radius \( r_\lambda \) such that

\[
CZ(\Omega_{r_\lambda}(y)) = \int_{\Omega_{r_\lambda}(y)} \left( |Du| + |P| \right)^{\frac{2p+1}{p}} \, dx + \frac{1}{\delta} \left( \int_{\Omega_{r_\lambda}(y)} \left( |f_\alpha| + |g| \right)^{\frac{2p+1}{p} \eta} \, dx \right) = \lambda
\]

(28)
for each point \(y \in E(\lambda, \Omega_R)\). Moreover, for any \(r \in (r_y, R]\) one has
\[
CZ(\Omega_r(y)) < \lambda.
\] (29)

From (28) we conclude the following alternatives:
\[
\frac{\lambda}{2} \leq \int_{\Omega_r(y)} \left(|Du| + |P|\right)^{\frac{2p(x)}{p-2}} dx \quad \text{or} \quad \left(\frac{\delta \lambda}{2}\right)^{m} \leq \int_{\Omega_r(y)} \left(|f_\alpha| + |g|\right)^{\frac{2p(x)}{p-2}} dx. \tag{30}
\]

First, we suppose that the first case of (30) is valid and we split it as follows:
\[
\begin{align*}
\frac{\lambda}{2} & \leq \int_{\Omega_r(y)} \left(|Du| + |P|\right)^{\frac{2p(x)}{p-2}} dx \\
& \leq \frac{1}{|\Omega_r(y)|} \int_{\Omega_r(y)} \left(E(\Omega_r(y)) \right)^{\frac{2p(x)}{p-2}} dx \\
& \quad + \frac{1}{|\Omega_r(y)|} \int_{\Omega_r(y) \cap E(\Omega_r(y))} \left(E(\Omega_r(y)) \right)^{\frac{2p(x)}{p-2}} dx \\
& \leq \frac{\lambda}{4} + \frac{1}{|\Omega_r(y)|} \int_{\Omega_r(y) \cap E(\Omega_r(y))} \left(E(\Omega_r(y)) \right)^{\frac{2p(x)}{p-2}} dx \\
& \quad + \frac{1}{|\Omega_r(y)|} \int_{\Omega_r(y) \cap E(\Omega_r(y))} \left(E(\Omega_r(y)) \right)^{\frac{2p(x)}{p-2}} dx.
\end{align*}
\]
(31)

where \(\sigma_1 > 0\) is determined later.

Considering that \(\omega(4R) < \gamma_1 \sigma_2\) in (14), which leads to \(\frac{\gamma_1(1 + \sigma_2)}{\gamma_1 + \omega(4R)} - 1 > 0\). Now let us take
\[
0 < \sigma_1 \leq \frac{\gamma_1(1 + \sigma_2)}{\gamma_1 + \omega(4R)} - 1,
\]
which yields the following inequality
\[
\frac{p_y^+}{p^+} \left(1 + \sigma_1\right) = \left(1 + \frac{p^+ - p^-}{p^+} \right) \left(1 + \sigma_1\right) \leq \left(1 + \frac{\omega(4R)}{p^-} \right) \left(1 + \sigma_1\right) \leq \left(1 + \sigma_2\right),
\]
where \(p_y^+ := \sup_{\Omega_{4R}(y)} p(x)\), and \(\omega(\cdot)\) is the modulus of continuity for \(p(x)\). Then we use the reverse Hölder inequality shown in Lemma 2.3 to derive that
\[
\begin{align*}
\left(\int_{\Omega_r(y)} \left(|Du| + |P|\right)^{\frac{2p(x)}{p-2}(1 + \sigma_1)} dx\right)^{\frac{1}{1 + \sigma_1}} & \leq \left(\int_{\Omega_r(y)} \left(|Du| + |P|\right)^{\frac{2p(x)}{p-2}(1 + \sigma_1)} dx + 1\right)^{\frac{1}{1 + \sigma_1}} \\
& \leq c \left(\int_{\Omega_{4R}(y)} \left(|Du|^2 + |P|^2\right) dx\right)^{\frac{p_y^+}{p^+}} + c \left(\int_{\Omega_r(y)} \left(|f_\alpha| + |g|\right)^{\frac{2p(x)}{p-2}(1 + \sigma_2)} dx\right)^{\frac{1}{1 + \sigma_2}} + c.
\end{align*}
\]
Taking into account (29), we have $CZ(\Omega_{Sr_y}(y)) < \lambda$, then by a similar proof of (37) in Step 2, for $\Omega_{Sr_y}(y)$ we can get that

$$\begin{align*}
\int_{\Omega_{Sr_y}(y)} (|Du|^2 + |P|^2) \, dx &\leq c\lambda^{\frac{p}{p'}} \\
\int_{\Omega_{Sr_y}(y)} |f_\alpha|^{2\eta} + |g|^{2\eta} \, dx &\leq c\lambda^{\frac{p}{p'}} \eta \delta \frac{2\eta}{2}. 
\end{align*}$$

Thus, by taking $\eta = 1 + \sigma_2$ we have

$$\left( \int_{\Omega_{r_y}(y)} (|Du| + |P|) \frac{2p(x)}{p'} (1 + \sigma_2) \, dx \right)^{\frac{1}{1 + \sigma_2}} \leq c \left( \lambda + \lambda \delta \frac{2\eta}{2} + 1 \right) \leq c\lambda. \tag{32}$$

Therefore, by combining (30), (31) and (32) we have

$$\frac{\lambda}{4} \leq c \left( \frac{\Omega_{r_y}(y) \cap E(\frac{\lambda}{4}, \Omega_{2R})}{\Omega_{r_y}(y)} \right)^{1 - \frac{1}{1 + \sigma_2}} \lambda,$$

which implies

$$|\Omega_{r_y}(y)| \leq c|\Omega_{r_y}(y) \cap E(\frac{\lambda}{4}, \Omega_{2R})| \tag{33}$$

with the positive constant $c$ depending only on $n, \gamma_2, \gamma_2, \Lambda, t, R_0$, and $|\Omega|$.

For the case of the second estimate in (30), by taking $\zeta = \frac{\delta}{4}$, Fubini’s theorem and a split of the integral we get

$$\left( \frac{\lambda \delta}{2} \right)^\eta \leq \int_{\Omega_{r_y}(y)} (|f_\alpha| + |g|) \frac{2p(x)}{p'} \eta \, dx$$

$$= \frac{\eta}{|\Omega_{r_y}(y)|} \int_0^\infty \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}$$

$$= \frac{\eta}{|\Omega_{r_y}(y)|} \int_0^{\zeta \lambda} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu} + \frac{\eta}{|\Omega_{r_y}(y)|} \int_{\zeta \lambda}^{\infty} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}$$

$$\leq (\zeta \lambda)^\eta + \frac{\eta}{|\Omega_{r_y}(y)|} \int_{\zeta \lambda}^{\infty} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}.$$ 

Let $\delta = 4\zeta$, we derive

$$(\zeta \lambda)^\eta \leq \frac{\eta}{|\Omega_{r_y}(y)|} \int_{\zeta \lambda}^{\infty} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}$$

or

$$|\Omega_{r_y}(y)| \leq \frac{\eta}{(\zeta \lambda)^\eta} \int_{\zeta \lambda}^{\infty} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}. \tag{34}$$

Now we put (33) and (34) together, and get

$$|\Omega_{r_y}(y)| \leq c|\Omega_{r_y}(y) \cap E(\frac{\lambda}{4}, \Omega_{2R})|$$

$$+ \frac{c\eta}{(\zeta \lambda)^\eta} \int_{\zeta \lambda}^{\infty} \mu^n \left\{ x \in \Omega_{r_y}(y) : (|f_\alpha| + |g|) \frac{2p(x)}{p'} > \mu \right\} \frac{d\mu}{\mu}. \tag{35}$$
Step 2. This step is devoted to various comparison estimates with the reference problems and the limiting one. Taking into account (29), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{\Omega_{400r_y}} (|Du| + |P|)^{\frac{2p(x)}{p}} dx \leq c\lambda, \\
\left( \int_{\Omega_{400r_y}} (|f_\alpha| + |g|)^{\frac{2p(x)}{p} + \eta} dx \right)^{\frac{1}{p}} \leq c\lambda\delta.
\end{array} \right.
\end{align*}
\]

(36)

Therefore, it suffices to show that

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \leq c_3 \lambda^{\frac{2}{p' r_y}}, \\
\int_{\Omega_{400r_y}} (|f_\alpha|^2 + |g|^2) dx \leq c_3 \lambda^{\frac{2\eta}{p' r_y}} \delta^{\frac{1}{2} \frac{2}{\eta}}
\end{array} \right.
\end{align*}
\]

(37)

for a constant \( c_3 \geq 1 \). We first claim that

\[
\left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-} \leq c
\]

(38)

with \( c \geq 1 \) a universal constant. In fact, since \( (|f_\alpha| + |g|)^{p(x)} \in L(t, q)(\Omega) \) for \( t > 1 \) and \( 0 < q \leq \infty \), it deduces that

\[
\int_\Omega (|f_\alpha|^2 + |g|^2) dx \leq \int_\Omega (|f_\alpha|^{p(x)} + |g|^{p(x)} + 2) dx \\
\leq c\|(|f_\alpha| + |g|)^{p(x)}\|_{L(t, q)(\Omega)} + 2|\Omega| \leq c\left(1 + |\Omega|\right),
\]

where we used (25). By \( L^2 \)-estimate in Lemma 2.2, it leads to that

\[
\int_\Omega (|Du|^2 + |P|^2) dx \leq c\left(1 + |\Omega|\right).
\]

(39)

By considering \( p_y^+ - p_y^- \leq \omega(800r_y) \) it yields that

\[
\left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-} \\
= \left( \frac{1}{|\Omega_{100r_y}|} \right)^{p_y^+ - p_y^-} \left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-} \\
\leq c \left( \frac{1}{|B_{800r_y}|} \right)^{p_y^+ - p_y^-} \left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-} \\
\leq c \left( \frac{1}{800r_y} \right)^{p_y^+ - p_y^-} \left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-} \\
\leq c \left( \int_{\Omega_{400r_y}} (|Du|^2 + |P|^2) dx \right)^{p_y^+ - p_y^-}.
\]
On the other hand, by using (39) and \( \frac{1}{800 r_y} \geq \frac{1}{\pi} \geq \frac{c}{100} \geq |\Omega| + 1 \) with (14) we find
\[
\left( \int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx \right)^{p_y^+ - p_y^-} \leq \left( \int_\Omega \left( |Du|^2 + |P|^2 \right) dx \right)^{p_y^+ - p_y^-} \leq c(|\Omega| + 1)^{p_y^+ - p_y^-} \leq c \left( \frac{1}{800 r_y} \right)^{\omega(800 r_y)} \leq c,
\]
where we have used the so-called log-Hölder continuity (6) for \( p(x) \) in the last inequality, which yields (38). Recalling \( \gamma_1 \leq p_y^+ \) and (38) with \( \lambda > 1 \), we obtain
\[
\int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx = \left( \int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx \right)^{p_y^+ - p_y^-} \left( \int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx \right)^{p_y^- - p_y^-} \leq c \left( \int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx \right)^{\frac{p_y^+ - p_y^-}{p_y^-}} \leq c \left( \int_{\Omega_{400 r_y}(y)} \left( |Du|^2 + |P|^2 \right) dx \right)^{\frac{p_y^-}{p_y^-}} \leq c \lambda^\frac{p_y^-}{p_y^-}.
\]

Note that \( \left( |f_\alpha| + |g| \right)^{p(x)} \in L(t, q)(\Omega) \) for \( 2 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty \), and \( 2 < 2\eta \leq 2(1 + \sigma) < \gamma_1 t \) with \( \sigma \) as the same of Lemma 2.3, there holds
\[
\int_\Omega \left( |f_\alpha|^{2\eta} + |g|^{2\eta} \right) dx \leq \int_\Omega \left( |f_\alpha|^{2(1+\sigma)} + |g|^{2(1+\sigma)} + 2 \right) dx \leq c \left( 1 + |\Omega| \right).
\]

Similarly, recalling \( \delta \lambda_0 \geq 1 \) and \( \lambda \geq M \lambda_0 \) we find
\[
\int_{\Omega_{400 r_y}(y)} \left( |f_\alpha|^{2\eta} + |g|^{2\eta} \right) dx \leq c \left( \int_{\Omega_{400 r_y}(y)} \left( |f_\alpha| + |g| \right)^{2p(x)} dx \right)^{\frac{p_y^-}{p_y^-}} \leq c \left( \delta \lambda + 1 \right)^{\frac{p_y^-}{p_y^-}} \leq c \left( \delta \lambda + \delta \lambda_0 \right)^{\frac{p_y^-}{p_y^-}} \leq c \lambda^\frac{p_y^-}{p_y^-} \delta_2 \frac{p_y^-}{p_y^-}.
\]

Thus, by Lemma 2.4 we conclude that there exists
\( (v, P_v), (w, P_w) \in W^{1,2}(\Omega_{10 r_y}(y)) \times L^2(\Omega_{10 r_y}(y)) \)
such that \( (u, P) = (v, P_v) + (w, P_w) \) in \( \Omega_{10 r_y}(y) \) with
\[
\int_{\Omega_{10 r_y}(y)} \left( |Du|^2 + |P_u|^2 \right) dx \leq c^2 \lambda^\frac{p_y^-}{p_y^-}
\]
and
\[ \|Dw\|_{L^\infty(\Omega_{5r_i}(y_j))}^2 + \|Pw\|_{L^\infty(\Omega_{5r_i}(y_j))}^2 \leq c_1 \lambda^{\frac{p^-}{p^+}}. \]

**Step 3.** We are here to estimate the super level set \( E(\lambda, \Omega_R) \). For any fixed point \( x \in \Omega \), we select a universal constant \( R \) with \( 0 < R \leq \min \{ \frac{R_0}{2}, \frac{R_0}{|\Omega|+1}, 1 \} \), and there exists a constant \( \delta = \delta(\epsilon) > 0 \) such that Lemma 2.4 holds. For \( c_1 > \) being the same as in the step 2, let us write
\[ c_0 = \max \{ c_1, 1 \}. \] (40)

Let \( A = (8c_0)^{\frac{p^-}{p^+}} \), for any \( x \in E(\lambda, \Omega_R) \) we consider the collection \( \mathcal{B}_\lambda \) of all subset \( \Omega_{r_i}(y_j) \). By "crawling of ink spots" argument, we extract a countable sub-collection \( \{ \Omega_{r_i}(y_j) \} \in \mathcal{B}_\lambda \), such that five times enlarged balls \( \Omega_{5r_i}(y_j) \) cover almost all \( E(\lambda, \Omega_R) \); moreover, the balls \( \{ \Omega_{r_i}(y_j) \}_{i=1}^\infty \) are pointwise disjoint for \( y_j \in E(\lambda, \Omega_R) \) with \( r_i := r_{y_j} \) for any \( i \in \mathbb{N} \). This results in the following relation
\[ \Omega_{r_i}(y_i) \cap \Omega_{r_j}(y_j) = \emptyset, \quad \text{whenever} \ i \neq j, \quad \text{and} \ E(\lambda, \Omega_R) \subset \bigcup_{i \in \mathbb{N}} \Omega_{5r_i}(y_i) \cup N_\lambda \]
with \( |N_\lambda| = 0 \). Let us denote \( p_i^+ = p_{y_j}^+ \), then we have
\[
\begin{align*}
|E(\lambda, \Omega_R)| &= |E((8c_0)^{\frac{p^-}{p^+}} \lambda, \Omega_R)| \\
&= \left| \left\{ x \in \Omega_R : (|Du| + |P|)^{\frac{2(p+)}{p^+}} > (8c_0)^{\frac{p^-}{p^+}} \lambda \right\} \right| \\
&\leq \sum_{i \geq 1} \left| \left\{ x \in \Omega_{5r_i}(y_i) : (|Du| + |P|)^2 \geq 8c_0 \lambda^{\frac{p^-}{p^+}} \right\} \right| \\
&\leq \sum_{i \geq 1} \left| \left\{ x \in \Omega_{5r_i}(y_i) : |Du|^2 + |P|^2 \geq 4c_0 \lambda^{\frac{p^-}{p^+}} \right\} \right| \\
&\leq \sum_{i \geq 1} \left| \left\{ x \in \Omega_{5r_i}(y_i) : |Du|^2 + |P|^2 \geq 4c_1 \lambda^{\frac{p^-}{p^+}} \right\} \right| \\
&\leq \sum_{i \geq 1} \left| \left\{ x \in \Omega_{5r_i}(y_i) : |Dv|^2 + |Pv|^2 \geq c_1 \lambda^{\frac{p^-}{p^+}} \right\} \right| \\
&\quad + \sum_{i \geq 1} \left| \left\{ x \in \Omega_{5r_i}(y_i) : |Dw|^2 + |Pw|^2 \geq c_1 \lambda^{\frac{p^-}{p^+}} \right\} \right| \\
&\leq \sum_{i \geq 1} \frac{1}{c_1 \lambda^{\frac{p^-}{p^+}}} \int_{\Omega_{5r_i}(y_i)} (|Dv|^2 + |Pv|^2) dz \\
&\leq c \sum_{i \geq 1} e^2 |\Omega_{10r_i}(y_i)| \leq ce^2 |B_{10r_i}(y_i)| = ce^2 |B_{r_i}(y_i)| \leq c e^2 \frac{|B_{r_i}(y_i)|}{|\Omega_{r_i}(y_i)|} \Omega_{r_i}(y_i)| \\
&\leq c \sum_{i \geq 1} e^2 \left( \frac{2\sqrt{2}}{1-\delta} \right)^n |\Omega_{r_i}(y_i)| \leq ce^2 \left( \frac{16\sqrt{2}}{15} \right)^n \sum_{i \geq 1} |\Omega_{r_i}(y_i)|, \\
\end{align*}
\]
where we used the weak \((1, 1)\)-type estimate:
\[ \left| \left\{ x \in E : f(x) > \lambda \right\} \right| \leq \frac{1}{\lambda} \int_E f(x) dx. \]
That is,
\[ |E(A\lambda, \Omega_R)| \leq c\epsilon^2 \sum_{i \geq 1} |\Omega_{r_i}(y_i)|. \] (42)

Using “crawling of ink spots” argument again and (35), we conclude that
\[ |E(A\lambda, \Omega_R)| \leq c\epsilon^2 \sum_{i \geq 1} |\Omega_{r_i}(y_i)| \cap E(\frac{\lambda}{4}, \Omega_{2R}) | + c\epsilon^2 \frac{\eta}{(\zeta \lambda)^n} \int_{\zeta \lambda}^\infty \mu^{|\{x \in \Omega_{r_i}(y_i) : (|f_\alpha| + |g|) \frac{2p(x)}{r} > \mu\}|} \frac{d\mu}{\mu}. \] (43)

**Step 4.** This step is devoted to proving our main conclusion under the claim that \( \|(|\nabla u| + |P|)^{p(x)}\|_{L_t^{(t,q)}(\Omega_{2R})} < \infty \) for \( 0 < q < \infty \). Since \( t > 1 \), we multiply the inequality (43) by \( \left(\frac{tp^-}{2}\right)^\frac{q}{4} (A\lambda)^{\frac{4}{2p^-}} \), and integrate the resulting expression with a power \( \frac{q}{4} \) in the measure \( \frac{d\lambda}{\lambda} \) from \( M\lambda_0 \) to \( \infty \), which yields
\[ \leq c\epsilon^2 \left(\frac{tp^-}{2}\right)^\frac{q}{4} \int_{M\lambda_0}^\infty \left(\frac{\lambda^{\frac{4}{2p^-}}}{A\lambda}\right)^\frac{q}{4} \| \{x \in \Omega_R : (|Du| + |P|) \frac{2p(x)}{r} > A\lambda\}| \frac{d\lambda}{\lambda} \] 
\[ + c\epsilon^2 \left(\frac{tp^-}{2}\right)^\frac{q}{4} \int_{0}^\infty \lambda^{\delta(x) - \frac{q}{4}} \left(\int_{\zeta \lambda}^\infty \mu^{|\{x \in \Omega_{2R} : (|f_\alpha| + |g|) \frac{2p(x)}{r} > \mu\}|} \frac{d\mu}{\mu}\right)^\frac{q}{4} \frac{d\lambda}{\lambda} \] 
\[ := c\epsilon^2 (I_1 + I_2), \] (44)
where \( c \) depends on \( n, \gamma_1, \gamma_2, \Lambda, \mu, \nu, t, R_0, |\Omega|, \) and \( \omega(\cdot) \). Thanks to (20) in Proposition 1, we have
\[ \|(|Du| + |P|)^{p(x)}\|_{L_t^{(t,q)}(\Omega_{2R})}^q = \|(|Du| + |P|)\frac{2p(x)}{r} \|_{L_t^{(tp^- \frac{p}{X} - \frac{q}{r})}(\Omega_{2R})}^q \] (45)
By a simple change of variable and (45) it yields
\[ I_1 = c(q) \|(|Du| + |P|)^{p(x)}\|_{L_t^{(t,q)}(\Omega_{2R})}^q. \]

We are now to estimate \( I_2 \), for this we part it in two cases.

**Case 1.** If \( q \geq t \), noticing that (21) is satisfied since \( (|f_\alpha| + |g|) \frac{2p(x)}{r} \in L^q(\Omega_{2R}) \). By making the change of variables \( \bar{\lambda} = \zeta \lambda \) and \( \zeta = \frac{q}{4} \), then we make use of Lemma 2.5 with \( f(\mu) = \mu^{q - 1} \| \{x \in \Omega_{2R} : (|f_\alpha| + |g|) \frac{2p(x)}{r} > \mu\}|, \) \( \alpha = \frac{q}{4} \geq 1, \) \( r = q \left( \frac{p}{X} - \frac{q}{r} \right) > \)
0 and (45) to infer

\[ I_2 = c \frac{tp^-}{2} \int_0^{\infty} \lambda \left( \frac{t - \lambda}{2} \right) \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \mu \right\} d\mu \right) d\lambda \]

\[ \leq c \frac{tp^-}{2} \int_0^{\infty} \lambda \frac{2p^-}{p} \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \lambda \right\} \right)^{\frac{q}{2}} d\lambda \]

\[ = c\left\| (|f_a| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega_{2R})}, \]

where \( c = c(\gamma_1, \gamma_2, q, t). \)

**Case 2.** If \( 0 < q < t, \) we use Lemma 2.6 with \( h(\mu) = \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \mu \right\} \] and \( \alpha_1 = 1 < \frac{q}{t} = \alpha_2 \) and \( \varepsilon = 1. \) This yields

\[ \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \mu \right\} \right)^{\frac{q}{2}} \]

\[ \leq \lambda \frac{2p^-}{p} \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \lambda \right\} \right)^{\frac{q}{2}} \]

\[ + c \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \mu \right\} \frac{d\lambda}{\mu}. \]

After a change variable \( \zeta \lambda \to \lambda, \) (45) and Fubini’s theorem we get

\[ I_2 \leq c \frac{tp^-}{2} \int_0^{\infty} \lambda \left( \frac{t - \lambda}{2} \right) \lambda \frac{2p^-}{p} \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \lambda \right\} \right)^{\frac{q}{2}} d\lambda \]

\[ + c \frac{tp^-}{2} \int_0^{\infty} \lambda \left( \frac{t - \lambda}{2} \right) \left( \int_0^\infty \mu \left\{ x \in \Omega_{2R} : \left( |f_a| + |g| \right) \frac{2\mu(x)}{\mu} > \mu \right\} \right)^{\frac{q}{2}} d\mu \frac{d\lambda}{\mu}. \]

\[ \leq c\left\| (|f_a| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega_{2R})} \]

\[ + c\left\| (|f_a| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega_{2R})}, \]

where \( c = c(\gamma_1, \gamma_2, q, t). \)

We are now in a position to put the estimates of \( I_1 \) and \( I_2 \) into (44), and after simple manipulation, then it follows that for \( t > 1, \)

\[ \left\| (|Du| + |P|)^{p(x)} \right\|_{L(t,q)(\Omega_R)} \]

\[ \leq c \left( \frac{tp^-}{2} \int_0^{\infty} \left( (A\lambda)^{\frac{tp^-}{2}} \left\{ x \in \Omega_R : \left( |Du| + |P| \right) \frac{2\mu(x)}{\mu} > A\lambda \right\} \right) \frac{d(A\lambda)}{A\lambda} \right)^{\frac{1}{q}} \]

\[ + c \left( \frac{tp^-}{2} \int_0^{\infty} \left( (A\lambda)^{\frac{tp^-}{2}} \left\{ x \in \Omega_R : \left( |Du| + |P| \right) \frac{2\mu(x)}{\mu} > A\lambda \right\} \right)^{\frac{q}{2}} d(A\lambda) \right)^{\frac{1}{q}} \]

\[ \leq c \left( \frac{tp^-}{2} \int_0^{\infty} \left( (A\lambda)^{\frac{tp^-}{2}} \left\{ x \in \Omega_R : \left( |Du| + |P| \right) \frac{2\mu(x)}{\mu} > A\lambda \right\} \right)^{\frac{q}{2}} d(A\lambda) \right)^{\frac{1}{q}} \]

\[ + c\frac{2}{t} \left( \left\| (|Du| + |P|)^{p(x)} \right\|_{L(t,q)(\Omega_{2R})} + \left\| (|f_a| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega_{2R})} \right). \]
\[ \leq c_0 \lambda^{-\frac{2}{q}} \| \Omega_2 R \|^\frac{1}{p} + c_0 c \bar{c} \left( \| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} + \| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} \right), \]

where \( \bar{c} = \bar{c}(n, \gamma_1, \gamma_2, \nu, \Lambda, q, t, \omega(\cdot)) \). Now we can choose \( \epsilon > 0 \) small enough that \( c_0 c \bar{c} \leq \frac{1}{2} \). Once the selection of \( \epsilon \) has been made, we can find a corresponding positive constant \( \delta = \delta(n, \Lambda, \epsilon, \gamma_1, \gamma_2, t, q, R_0, |\Omega|) \) such that we deduce

\[ \| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} \leq c_0 \lambda^{-\frac{2}{q}} \| \Omega_2 R \|^\frac{1}{p} + c \left( \| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} + \| (|f_\alpha| + |g|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} \right). \]

(47)

By a standard iteration argument of Lemma 2.7, then we get the desired estimate of (11) in the case \( t > 1 \) and \( 0 < q < \infty \).  

Step 5. This step is devoted to proving the claim of Step 4: \( \| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega_2 R)} < \infty \). To this end, we first refine the estimate of \( (|Du| + |P|)^{p(x)} \) in the scale of Lorentz spaces. Note that the truncated function

\[ \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} k \quad \text{for} \quad x \in \Omega \quad \text{and} \quad k \in \mathbb{N} \cap [M \lambda_0, \infty). \]

By considering \( E_k(\lambda, \Omega_\rho) = \left\{ x \in \Omega_\rho : \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} \leq \lambda \right\} \) in line with (43), we have

\[ E_k(\lambda \Omega_\rho) \leq c \bar{c}^{\frac{2}{p}} E_k \left( \frac{\lambda}{4}, \Omega_2 R \right) + c \bar{c} \left( \frac{\eta}{\lambda} \right)^\eta \int_{\lambda}^\infty \mu^n \left\{ x \in \Omega_2 R : \left( |f_\alpha| + |g| \right)^{\frac{2p(x)}{p - 2}} > \mu \right\} \frac{d\mu}{\mu} \]

for \( k \in \mathbb{N} \cap [M \lambda_0, \infty) \). For \( 0 < k \leq \lambda \) we have \( E_k(\lambda \Omega_R) = \emptyset \), which implies that the above estimate holds trivially. For \( k > \lambda \), it is also valid because \( E_k(\lambda \Omega_R) = E(\lambda \Omega_R) = \left\{ x \in \Omega_\rho : \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} > \lambda \right\} \) and \( E_k(\lambda \Omega_2 R) = E(\lambda \Omega_2 R) \). Working exactly as in the above argument, we get that (47) holds with \( \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} k \) in place of \( \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} \). Now let \( B_0 = 0, L = c \lambda^{-\frac{2}{q}} \| \Omega_2 R \|^\frac{1}{p} + c \left( \| f_\alpha \| + |g| \right)^{\frac{2p(x)}{p - 2}} \|_{L(t,q)(\Omega_2 R)} \) and \( \varphi(\rho) = \left\{ \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} \right\} \|_{L(t,q)(\Omega_\rho)} \). On the basis of

\[ \left\{ \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} \right\|_{L(t,q)(\Omega_2 R)} < \infty, \]

we use the well-known iteration argument of Lemma 2.7 to get

\[ \left\| \left( \frac{|Du| + |P|}{\lambda} \right)^{p(x)} \right\|_{L(t,q)(\Omega_2 R)} \leq c \lambda^{-\frac{2}{q}} \| \Omega_2 R \|^\frac{1}{p} + c \left( \| f_\alpha \| + |g| \right)^{\frac{2p(x)}{p - 2}} \|_{L(t,q)(\Omega_2 R)} \].

In what follows, we use a standard finite covering argument to realize our global estimate. Note that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and let us now take \( x_0 \) as every point in \( \Omega \). Then there exist \( N = N(n, |\Omega|) \in \mathbb{N} \) and \( x_j \in \Omega \) for \( j = 1, 2, \ldots, N \), where we replace the point \( x_0 \) by each \( x_j \), such that

\[ \Omega \subset \bigcup_{j=1}^N \Omega_R(x_j). \]
Then we have
\[
\|(|Du| + |P|)^{p(x)}\|_{L(t,q)(\Omega)} \\
\leq \sum_{j=1}^{N} \|(|Du| + |P|)^{p(x)}\|_{L(t,q)(\Omega_2(x_j))} \\
\leq c \sum_{j=1}^{N} \left( \lambda_0^{\frac{p^+}{p^-}} \|\Omega_R(x_j)\|^\frac{p^+}{p^-} + \|(|f_\alpha| + |g|)^{p(x)}\|_{L(t,q)(\Omega_2(x_j))} \right).
\]

Recalling the definition of \(\lambda_0\), we get
\[
\|(|Du| + |P|)^{p(x)}\|_{L(t,q)(\Omega)} \\
\leq c \sum_{j=1}^{N} \Omega_R(x_j)^\frac{p^+}{p^-} \left( \int_{\Omega_2R(x_j)} (|Du| + |P|)^{\frac{2p(x)}{p^-}} \, dx \right)^{\frac{p^-}{p^+}} \\
\quad + c \sum_{j=1}^{N} \|(|f_\alpha| + |g|)^{p(x)}\|_{L(t,q)(\Omega_2R(x_j))} \\
\leq c \sum_{j=1}^{N} \Omega_R(x_j)^\frac{p^+}{p^-} \left( \int_{\Omega_2R(x_j)} (|Du| + |P|)^{\frac{2p(x)}{p^-}} \, dx \right)^{\frac{p^-}{p^+}} \\
\quad + c \sum_{j=1}^{N} \left( \int_{\Omega_2R(x_j)} (|f_\alpha| + |g|)^{\frac{2p(x)}{p^-}} \, dx \right)^{\frac{p^-}{p^+}} \\
\quad + c N \left( \sum_{j=1}^{N} \left( \int_{\Omega(x_j)} (|f_\alpha| + |g|)^{\frac{2p(x)}{p^-}} \, dx \right) \right)^{\frac{p^-}{p^+}}. \tag{48}
\]

By (14), we notice that
\[
\frac{2p^+}{p^-} = 2 \left( 1 + \frac{p^+ - p^-}{p^-} \right) \leq 2 \left( 1 + \omega(\frac{4R}{\gamma_1}) \right) \leq 2(1 + \sigma),
\]
where \(\sigma\) is the same as in Lemma 2.3. Then, it yield
\[
\int_{\Omega_2R(x_j)} (|Du| + |P|)^{\frac{2p(x)}{p^-}} \, dx \\
\leq \int_{\Omega_2R(x_j)} (|Du| + |P|)^{\frac{2p^+}{p^-}} \, dx + 1 \tag{49}
\]
\[
\leq c \left( \int_{\Omega_16R(x_j)} (|Du|^2 + |P|^2) \, dx + 1 \right)^{\frac{p^+}{p^-}} + c \int_{\Omega_16R(x_j)} (|f_\alpha| + |g|)^{\frac{2p^+}{p^-}} \, dx,
\]
where we employed the reverse Hölder inequality of Lemma 2.3 in the last inequality. Using (12) and Hölder inequality, we obtain that

\[
\left( \int_{\Omega_{16 R}(x_j)} (|Du|^2 + |P|^2) \, dx \right)^{\frac{p^+}{p}} \leq \left( \frac{1}{|\Omega_{16 R}(x_j)|} \right)^{\frac{p^+}{p}} \left( \int_{\Omega} (|Du|^2 + |P|^2) \, dx \right)^{\frac{p^+}{p}}
\]

\[
\leq c \left( \frac{1}{|\Omega_{16 R}(x_j)|} \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^2 \, dx \right)^{\frac{p^+}{p}}
\]

\[
\leq c \left( \frac{1}{|\Omega_{16 R}(x_j)|} \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{2^p(\alpha)} \, dx \right)^{\frac{p^+}{p}}
\]

Combining (48), (49) and (50), we get that

\[
|||((\nabla u) + |P|)^{p(x)}||L(t, q)(\Omega)\leq c \sum_{j=1}^{N} |\Omega_R(x_j)|^{\frac{1}{q}} \left( \frac{1}{|\Omega_{16 R}(x_j)|} \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{2^p(\alpha)} \, dx \right)^{\frac{p^+}{p}} + cN ||(f_\alpha + |g|)^{p(x)}||L(t, q)(\Omega).
\]

Using a standard Hardy’s inequality in the Marcinkiewicz spaces (cf. Lemma 2.3 in [29]) and the reverse Hölder inequality of Lemma 2.6, we obtain

\[
\int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \leq \frac{t(p^{-})^2}{t(p^{-})^2 - 2p^+} |||f_\alpha + |g||||L(t, q)(\Omega)\leq c|\Omega|^{1 - \frac{2p^+}{p(p^+ - 2)}} \left( \sup_{h>0} \left( \frac{t(h^{-})}{t(h^{-}) - 2p^+} \right) \left\{|x \in \Omega : (|f_\alpha| + |g|)^{\frac{2^p(\alpha)}{p^+}} > h \} \right\} \right)^{\frac{p^+}{p}} \leq c|\Omega|^{1 - \frac{2p^+}{p(p^+ - 2)}} \left( \sup_{h>0} \left( \frac{t(h^{-})}{t(h^{-}) - 2p^+} \right) \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{p^+}{p}} \leq c|\Omega|^{1 - \frac{2p^+}{p(p^+ - 2)}} \left( \sup_{h>0} \left( \frac{t(h^{-})}{t(h^{-}) - 2p^+} \right) \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{p^+}{p}} \leq c|\Omega|^{1 - \frac{2p^+}{p(p^+ - 2)}} \left( \sup_{h>0} \left( \frac{t(h^{-})}{t(h^{-}) - 2p^+} \right) \right)^{\frac{p^+}{p}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{p^+}{p}}
\]

Similarly, we also show

\[
\left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{1}{p}} \leq c(\gamma_1, \gamma_2, q, t) |\Omega|^{\frac{1}{q} - \frac{1}{p^+}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{1}{p}} \leq c(\gamma_1, \gamma_2, q, t) |\Omega|^{\frac{1}{q} - \frac{1}{p^+}} \left( \int_{\Omega} \left( |f_\alpha| + |g| \right)^{\frac{2^p(\alpha)}{p^+}} \, dx \right)^{\frac{1}{p}}
\]
For the case of $q < \infty$, from (51) we then infer the following relations

$$\| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega)} \leq C \sum_{j=1}^N \left( \left( \frac{|\Omega|}{|B_{16R}(x_j)|} \right)^{\frac{p^+}{r}} \left( \left\| (|f_\alpha| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega)} + 1 \right) \right)$$

$$\leq C \sum_{j=1}^N \left( \left( \frac{|\Omega|}{|B_{16R}(x_j)|} \right)^{\frac{p^+}{r}} \left( \left\| (|f_\alpha| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega)} + 1 \right) \right)^{\frac{p^+}{2p(x)}}$$

$$\leq C \sum_{j=1}^N \left( \left( \frac{|\Omega|}{|B_{16R}(x_j)|} \right)^{\frac{p^+}{r}} \left( \left\| (|f_\alpha| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega)} + 1 \right) \right)^{\frac{p^+}{2p(x)}}$$

$$\leq C N \left( \left\| (|f_\alpha| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega)} + 1 \right)^{\frac{p^+}{2p(x)}}$$

$$\leq C \left( \left\| (|f_\alpha| + |g|)^{p(x)} \right\|_{L(t,q)(\Omega)} + 1 \right)^{\frac{p^+}{2p(x)}}$$

where we used the uniformly estimate (25). Now let us take $k \to \infty$. By the lower semi-continuity of Lorentz quasi-norm we have

$$\| (|Du| + |P|)^{p(x)} \|_{L(t,q)(\Omega)} \leq C,$$

where $C$ depends only on $n, \gamma_1, \gamma_2, \nu, \Lambda, t, q, R_0, \omega(-)$, and $|\Omega|$. And recalling the definition in (24), we get the desired result (11)

**Step 6.** Finally, for the case of $q = \infty$, we come back to the second inequality in (30) and split it into two parts with a small $\eta > 0$ determined later:

$$\left( \frac{\lambda}{2} \right)^\eta \leq \frac{1}{\delta^\eta} \int_{\Omega_{r\eta}(y)} (|f_\alpha| + |g|)^{\frac{2p(x)}{r}} \delta^\eta \eta dx$$

$$\leq \left( \frac{\lambda}{\delta} \right)^\eta \frac{1}{\delta^\eta |\Omega_{r\eta}(y)|} \int_{\{x \in \Omega_{r\eta}(y) : (|f_\alpha| + |g|)^{\frac{2p(x)}{r}} \eta > \lambda \}} (|f_\alpha| + |g|)^{\frac{2p(x)}{r}} \delta^\eta \eta dx.$$
Similar to the estimate (52), by using Hölder inequality we get

$$
\left(\frac{\lambda}{2}\right)^{\eta} - \left(\frac{t}{\delta}\right)^{\eta} \leq \frac{1}{\delta^{\eta}|\nu_{\Omega}(y)|} \int_{\{x \in \Omega_{\nu}(y) : (|f_x| + |g|)^{\frac{2p(x)}{p-\mu}} > \lambda\}} (|f_x| + |g|)^{\frac{2p(x)}{p-\mu} \eta} dx
$$

$$
\leq \frac{t}{(t-\eta)\delta^{\eta}} \left(\frac{G(\lambda, \Omega_{\nu}(y))}{|\Omega_{\nu}(y)|}\right)^{1-\frac{2}{\eta}} \left(\sup_{\mu > \lambda} \mu^{\eta}\{x \in G(\lambda, \Omega_{\nu}(y)) : (|f_x| + |g|)^{\frac{2p(x)}{p-\mu}} \geq \mu\}\right)^{\frac{\eta}{2}}
$$

$$
\leq \frac{t}{(t-\eta)\delta^{\eta}} \left(\frac{G(\lambda, \Omega_{\nu}(y))}{|\Omega_{\nu}(y)|}\right)^{1-\frac{2}{\eta}} \left(\sup_{\mu > \lambda} \mu^{\eta}|G(\lambda, \Omega_{\nu}(y))|\right)^{\frac{\eta}{2}}
$$

$$
= \frac{t}{(t-\eta)\delta^{\eta}} \left(\frac{1}{\delta^{\eta}} \frac{G(\lambda, \Omega_{\nu}(y))}{|\Omega_{\nu}(y)|} \sup_{\mu > \lambda} \mu^{\eta}|G(\lambda, \Omega_{\nu}(y))|\right)^{\frac{\eta}{2}}.
$$

Now we choose $\epsilon > 0$ appropriately small so as to satisfy

$$
\left(\frac{\lambda}{2}\right)^{\eta} - \left(\frac{t}{\delta}\right)^{\eta} - \frac{t}{t-\eta} \left(\frac{\lambda}{2}\right)^{\eta} = \left(\frac{\lambda}{2}\right)^{\eta} - \left(\frac{t}{\delta}\right)^{\eta} \left(1 + \frac{t}{t-\eta}\right) \geq \left(\frac{\lambda}{4}\right)^{\eta},
$$

and there exists a positive constant $c(t)$ depending only on $t$ such that $\epsilon \leq c(t)\delta$. Therefore, it follows that

$$
|\Omega_{\nu}(y)| \leq \frac{ct}{t-\eta} \frac{|G(\lambda, \Omega_{\nu}(y))|}{(\lambda)^{\eta}} \left(\sup_{\mu > \lambda} \mu^{\eta}|G(\lambda, \Omega_{\nu}(y))|\right)^{\frac{\eta}{2}}
$$

$$
\leq \frac{ct(\lambda)^{-t}}{t-\eta} \left(\frac{G(\lambda, \Omega_{\nu}(y))}{|\Omega_{\nu}(y)|} \sup_{\mu > \lambda} \mu^{\eta}|G(\lambda, \Omega_{\nu}(y))|\right)^{\frac{\eta}{2}}
$$

$$
\leq \frac{ct(\lambda)^{-t}}{t-\eta} \left(\sup_{\mu > \lambda} \mu^{\eta}|G(\lambda, \Omega_{\nu}(y))|\right)^{\frac{\eta}{2}}.
$$

Next we put the estimates of the first and second cases in (30) together into (42), that is, we insert the formulas (33) and (53) into (42) to get

$$
|E(A\lambda, \Omega_{R})| \leq c\epsilon^{2}|E(\frac{\lambda}{4}, \Omega_{2R})| + c\epsilon^{2}(\lambda)^{-t} \sup_{\mu > \lambda} \mu^{\eta}|G(\mu, \Omega_{2R})|,
$$

then we multiply (54) by $(A\lambda)^{\frac{t}{2t-\epsilon}}$, and take the supremum with respect to $\lambda$ over $(M\lambda_0, \infty)$ to show

$$
\sup_{\lambda > M\lambda_0} (A\lambda)^{\frac{t}{2t-\epsilon}} \left\{|x \in \Omega_{R} : (|Dx| + |P|)^{\frac{2p(x)}{p-\mu}} > A\lambda\}\right|\n$$

$$
\leq c\epsilon^{2} A^{\frac{t}{2t-\epsilon}} \left(\sup_{\lambda > M\lambda_0} \lambda^{\frac{t}{2t-\epsilon}} \left\{|x \in \Omega_{2R} : (|Dx| + |P|)^{\frac{2p(x)}{p-\mu}} > \frac{\lambda}{4}\right|\right)
$$

$$
+ \sup_{\lambda > M\lambda_0} \lambda^{\frac{t}{2t-\epsilon} - t} \left(\sup_{\mu > \lambda} \mu^{\eta}|G(\mu, \Omega_{2R})|\right)
$$

$$
\leq c\epsilon^{2} \left(\sup_{\lambda > M\lambda_0} \lambda^{\frac{t}{2t-\epsilon}} \left\{|x \in \Omega_{2R} : (|Dx| + |P|)^{\frac{2p(x)}{p-\mu}} > \frac{\lambda}{4}\right|\right)
$$

$$
+ \sup_{\lambda > M\lambda_0} \left(\sup_{\mu > \lambda} \mu^{\eta}|G(\mu, \Omega_{2R})|\right).$$
Note that
\[
\sup_{\lambda > M_{130}} \sup_{\mu > \lambda} \mu \frac{\epsilon}{\lambda} |G(\mu, \Omega_{2R})| \leq \|(|f_\alpha| + |g|)^{p(x)}\|_{M'(\Omega_{2R})}^t.
\]
If we take \( \epsilon > 0 \) so small that ensure \( c\epsilon^{\frac{t}{2}} \leq \frac{1}{2} \), it follows that
\[
\|(|Du| + |P|)^{p(x)}\|_{M'(\Omega_{2R})} \leq c\epsilon^{\frac{t}{2}} \left( \|(|Du| + |P|)^{p(x)}\|_{M'(\Omega_{2R})} + c(\gamma_1, \gamma_2, q, t) \|(|f_\alpha| + |g|)^{p(x)}\|_{M'(\Omega_{2R})} \right)
+ c\Omega_{2R}^{t\frac{1}{2}} M_{130}^{\frac{t}{2} - 1} \int_{\Omega_{2R}} (|Du| + |P|) \frac{2p(x)}{p(x) + 1} \, dx + \left( \int_{\Omega_{2R}} \left( (|f_\alpha| + |g|)^{\frac{2p(x)}{p(x) + 1}} + 1 \right)^{\frac{n}{p(x) + 1}} \, dx \right)^{\frac{p(x)}{p(x) + 1}}.
\]
In the remainder we use a similar way of the argument in Step 5, and it leads to the desired result for the case \( q = \infty \). \qed

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