Some topics pertaining to algebras of linear operators

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Abstract

Here we consider finite-dimensional vector spaces and linear operators on them. Some of the basic ingredients will be motivated by \( \mathcal{A} \)-algebras of linear transformations on inner product spaces, as in [Arv, Dou]. In particular, let us note that in the theory of \( C^* \) and Von Neumann algebras, the “double commutant” of a \( \mathcal{A} \)-algebra is a kind of basic closure of the algebra.

A special case concerns algebras of operators generated by representations of finite groups [Bur, Cur2, Ser2]. The use of characters is included in each of these three books, and is generally avoided in the present article.

Part of the point of view about the \( p \)-adic absolute value and \( p \)-adic numbers that we follow here is that they can be interesting in connection with theory of computation and related settings just because of their nice properties and simplicity, aside from more traditional number-theoretic uses, even if those are also very interesting.

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1 Preliminaries

As usual, the real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$, respectively.

1.1 Finite groups

Normally groups in this article will be finite.

A finite group is a finite set $G$ with a choice of identity element $e$ in $G$ and a binary operation $G \times G \to G$, $(x, y) \mapsto xy$, such that $xe = ex = x$ for all $x \in G$, every element $x$ of $G$ has an inverse $x^{-1}$, which satisfies

$$xx^{-1} = x^{-1}x = e,$$

and the group operation is associative, so that

$$x(yz) = (xy)z$$

for all $x$, $y$, and $z$ in $G$.

Two elements $x$, $y$ of $G$ are said to commute if

$$xy = yx.$$

The group $G$ is said to be commutative, or abelian, if every pair of elements in $G$ commutes.

A subset $H$ of $G$ is a subgroup if $e \in H$, and if $x^{-1} \in H$ and $xy \in H$ whenever $x, y \in H$.

If $G$ and $L$ are groups and $f : G \to L$ is a mapping between them, then $f$ is said to be a homomorphism if

$$f(xy) = f(x)f(y)$$

for all $x$, $y$ in $G$. Note that $f$ automatically maps the identity element of $G$ to the identity element of $L$ in this case, and that

$$f(x^{-1}) = f(x)^{-1}$$

for all $x$ in $G$. An isomorphism between two groups is a homomorphism that is one-to-one and maps the first group onto the second one. In other words,
the homomorphism should be invertible as a mapping, and one can check
that the inverse mapping is a homomorphism too.

If \( f : G \to L \) is a homomorphism between groups, then the kernel of \( f \)
is the subset \( K \) of \( G \) consisting of the elements \( x \) of \( G \) such that \( f(x) \) is the
identity element of \( L \). It is easy to see that \( K \) is a subgroup of \( G \).

A subgroup \( N \) of \( G \) is said to be normal if \( xyx^{-1} \in N \) whenever \( y \in N \) and \( x \in G \). The kernel of a homomorphism between groups is clearly a
normal subgroup. Conversely, if \( N \) is a normal subgroup of \( G \), then \( N \) is the
kernel of a homomorphism. This follows from the standard quotient subgroup
construction.

### 1.2 Vector spaces

We make the convention that a vector space means a vector space defined
over some field \( k \) which has positive finite dimension (unless the contrary is
indicated). If \( V \) and \( W \) are vector spaces with the same scalar field \( k \), then we
let \( \mathcal{L}(V, W) \) denote the set of linear mappings from \( V \) to \( W \). Thus \( \mathcal{L}(V, W) \)
is also a vector space over \( k \), using ordinary addition of linear operators and
multiplication of them by scalars. The dimension of \( \mathcal{L}(V, W) \) is equal to the
product of the dimensions of \( V \) and \( W \).

For \( V = W \) we simply write \( \mathcal{L}(V) \) for \( \mathcal{L}(V, V) \). For each \( L_1, L_2 \in \mathcal{L}(V) \),
the composition \( L_1 \circ L_2 \), \( (L_1 \circ L_2)(v) = L_1(L_2(v)) \) for all \( v \in V \), is also a
linear mapping on \( V \), and hence an element of \( \mathcal{L}(V) \). We sometimes write
this simply as \( L_1 L_2 \). This defines a product operation on \( \mathcal{L}(V) \), which is
not commutative in general, but which does satisfy the associative law and
distributive laws relative to addition and scalar multiplication. The identity
mapping \( I \) or \( I_V \) on \( V \), which takes every element of \( V \) to itself, is linear and
is the identity element of \( \mathcal{L}(V) \) with respect to this product.

Let us write \( GL(V) \) for the set of linear transformations on \( G \) which are
invertible. (If a linear mapping is invertible simply as a mapping, then the
inverse mapping is automatically linear as well.) Thus \( GL(V) \) is a group
which is infinite.

Suppose that \( v_1, \ldots, v_n \) is a basis for the vector space \( V \) with scalar field
\( k \). If \( T \) is any linear transformation on \( V \), then there is a unique \( n \times n \) matrix
\((t_{j,k})\) with entries in \( k \) such that

\[
T(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} t_{j,k} \alpha_k v_j
\]

(1.6)
for all $\alpha_1, \ldots, \alpha_n$ in $k$. Conversely, if $(t_{j,k})$ is an $n \times n$ matrix with entries in $k$, then (1.4) defines a linear transformation $T$ on $V$.

As usual, if $S$ and $T$ are linear transformations on $V$ corresponding to $n \times n$ matrices $(s_{j,k})$ and $(t_{l,m})$ (with respect to the basis $v_1, \ldots, v_n$), then $S + T$ corresponds to the sum of the matrices, given by $(s_{j,k} + t_{j,k})$. If $\gamma$ is any scalar, then the matrix of $\gamma T$ is $(\gamma t_{j,k})$. The composition $S \circ T$ has matrix $(u_{j,m})$ given by the classical matrix product of $(s_{j,k})$ and $(t_{l,m})$, namely,

$$u_{j,m} = \sum_{k=1}^{n} s_{j,k} t_{k,m}.$$  

(1.7)

The matrix associated to the identity transformation $I$ is $(\delta_{j,k})$, where $\delta_{j,k} = 1$ when $j = k$ and $\delta_{j,k} = 0$ when $j \neq k$. It is easy to check directly that the matrix product of $(\delta_{j,k})$ with another matrix (in either order) is always equal to the other matrix.

### 1.3 Representations of finite groups

Let $G$ be a group and let $V$ be a vector space. A representation of $G$ on $V$ is a mapping $\rho$ from $G$ to invertible linear transformations on $V$ such that

$$\rho_{xy} = \rho_x \circ \rho_y$$  

(1.8)

for all $x$ and $y$ in $G$. Here we use $\rho_x$ to denote the invertible linear transformation on $V$ associated to $x$ in $G$, so that we may write $\rho_x(v)$ for the image of a vector $v \in V$ under $\rho_x$. As a result of (1.8), we have that

$$\rho_e = I,$$  

(1.9)

where $I$ denotes the identity transformation on $V$, and

$$\rho_{x^{-1}} = (\rho_x)^{-1}$$  

(1.10)

for all $x$ in $G$.

In other words, a representation of $G$ on $V$ is a homomorphism from $G$ into $GL(V)$. The dimension of $V$ is called the degree of the representation.

Basic examples of representations are the left regular representation and right regular representation over a field $k$, defined as follows. We take $V$ to
be the vector space of functions on $G$ with values in $k$. For the left regular representation, we define $L_x : V \rightarrow V$ for each $x$ in $G$ by

$$L_x(f)(z) = f(x^{-1}z)$$

(1.11)

for each function $f(z)$ in $V$. For the right regular representation, we define $R_x : V \rightarrow V$ for each $x$ in $G$ by

$$R_x(f)(z) = f(z x)$$

(1.12)

for each function $f(z)$ in $V$. Thus if $x$ and $y$ are elements of $G$, then

$$L_x \circ L_y = L_y \circ L_x$$

(1.17)

and

$$R_x \circ R_y = R_y \circ R_x$$

(1.18)

for all $x$ and $y$ in $G$.

Observe that

$$L_x \circ R_y = R_y \circ L_x$$

(1.17)

for all $x$ and $y$ in $G$.

More generally, suppose that we have a homomorphism from the group $G$ to the group of permutations on a nonempty finite set $E$. That is, suppose that for each $x$ in $G$ we have a permutation $\pi_x$ on $E$, i.e., a one-to-one mapping from $E$ onto $E$, such that

$$\pi_x \circ \pi_y = \pi_{xy}$$

(1.18)
As usual, this implies that $\pi_e$ is the identity mapping on $E$, and that $\pi_{x^{-1}}$ is the inverse mapping of $\pi_x$ on $E$. Let $V$ be the vector space of $k$-valued functions on $E$. Then we get a representation of $G$ on $V$ by associating to each $x$ in $G$ the linear mapping $\Pi_x : V \to V$ defined by

$$\Pi_x(f)(a) = f(\pi_{x^{-1}}(a))$$

for every function $f(a)$ in $V$. This is called the permutation representation corresponding to the homomorphism $x \mapsto \pi_x$ from $G$ to permutations on $E$. It is indeed a representation, because for each $x$ and $y$ in $G$ and each function $f(a)$ in $V$ we have that

$$(\Pi_x \circ \Pi_y)(f)(a) = \Pi_x(\Pi_y(f))(a) = (\Pi_y(f))(\pi_{x^{-1}}(a))$$

$$= f(\pi_{y^{-1}}(\pi_{x^{-1}}(a))) = f(\pi_{(xy)^{-1}}(a)).$$

Alternatively, for each $b \in E$ one can define $\psi_b(a)$ to be the function on $E$ defined by

$$\psi_b(a) = 1 \text{ when } a = b, \quad \psi_b(a) = 0 \text{ when } a \neq b.$$ 

Then the collection of functions $\psi_b$ for $b \in E$ is a basis for $V$, and

$$\Pi_x(\psi_b) = \psi_{\pi_x(b)}$$

for all $x$ in $G$ and $b$ in $E$.

Suppose that $V_1$ and $V_2$ are vector spaces over the same field $k$, and that $T$ is a linear isomorphism from $V_1$ onto $V_2$. Assume also that $\rho^1$ and $\rho^2$ are representations of a group $G$ on $V_1$ and $V_2$, respectively. If

$$T \circ \rho^1_x = \rho^2_x \circ T$$

for all $x$ in $G$, then we say that $T$ determines an isomorphism between the representations $\rho^1$ and $\rho^2$. We may also say that $\rho^1$ and $\rho^2$ are isomorphic group representations, without referring to $T$ specifically. Of course isomorphic representations have equal degrees, but the converse is not true in general.

For example, let $V_1 = V_2$ be the vector space of $k$-valued functions on $G$, and define $T$ on $V_1 = V_2$ by $T(f)(a) = f(a^{-1})$. This is a one-to-one linear mapping from the space of $k$-valued functions on $G$ onto itself, and

$$T \circ R_x = L_x \circ T$$
for all $x$ in $G$. For if $f(a)$ is a function on $G$, then

$$
(T \circ R_x)(f)(a) = T(R_x(f))(a) = R_x(f)(a^{-1})
= f(a^{-1}x) = T(f)(x^{-1}a)
= L_x(T(f))(a) = (L_x \circ T)(f)(a).
$$

Therefore the left and right regular representations of $G$ are isomorphic to each other, in either the real or complex case.

Now suppose that $G$ is a group and $\rho$ is a representation of $G$ on a vector space $V$ over the field $k$, and that $v_1, \ldots, v_n$ is a basis of $V$. For each $x$ in $G$ we can associate to $\rho_x$ an $n \times n$ invertible matrix with entries in $k$ using this basis, as in Subsection 1.2. We denote this matrix by $M_x$. The composition rule (1.26) can be rewritten as

$$
M_{xy} = M_x M_y,
$$

where the matrix product is used on the right side of the equation.

A different choice of basis for $V$ will lead to a different mapping $x \mapsto N_x$ from $G$ to invertible $n \times n$ matrices. However, the two mappings $x \mapsto M_x$, $x \mapsto N_x$ will be similar, in the sense that there is an invertible $n \times n$ matrix $S$ with entries in $k$ such that

$$
N_x = S M_x S^{-1}
$$

for all $x$ in $G$. That is, $S$ is the matrix corresponding to the change of basis.

On the other hand, we can start with a mapping $x \mapsto M_x$ from $G$ into invertible $n \times n$ matrices with entries in $k$ which satisfies (1.26), and convert it into a representation on an $n$-dimensional vector space over $k$ by reversing the process. That is, one chooses a basis for the vector space, and then converts $n \times n$ matrices into linear transformations on the vector space using the basis to get the representation. One might as well take the vector space to be $k^n$, the space of $n$-tuples with coordinates in $k$ (using coordinatewise addition and scalar multiplication), and the basis $v_1, \ldots, v_n$ to be the standard basis, where $v_i$ has $i$th coordinate 1 and all others 0. In this way one gets the usual correspondence between $n \times n$ matrices and linear transformations on $k^n$.

If one has two representations of $G$ on vector spaces $V_1$, $V_2$ with the same scalar field $k$, then these two representations are isomorphic if and only if the associated mappings from $G$ to invertible matrices as above, using any choices of bases on $V_1$ and $V_2$, are similar, with the similarity matrix $S$ having entries in $k$. 

8
1.4 Reducibility

Let $G$ be a finite group, $V$ a vector space over a field $k$, and $\rho$ a representation of $G$ on $V$. Suppose that there is a vector subspace $W$ of $V$ such that

\[ \rho_x(W) \subseteq W \]

for all $x$ in $G$. This is equivalent to saying that

\[ \rho_x(W) = W \]

for all $x$ in $G$, since one could apply (1.28) also to $\rho_x^{-1} = (\rho_x)^{-1}$. We say that $W$ is invariant or stable under the representation $\rho$.

Recall that a subspace $Z$ of $V$ is said to be a complement of a subspace $W$ if

\[ W \cap Z = \{0\}, \quad W + Z = V. \tag{1.30} \]

Here $W + Z$ denotes the span of $W$ and $Z$, which is the subspace of $V$ consisting of vectors of the form $w + z$, $w \in W$, $z \in Z$. The conditions (1.30) are equivalent to saying that

\[ \text{every vector } v \in V \text{ can be written in a} \]
\[ \text{unique way as } w + z, \quad w \in W, \quad z \in Z. \tag{1.31} \]

Complementary subspaces always exist, because a basis for a vector subspace of a vector space can be enlarged to a basis of the whole vector space.

If $W$, $Z$ are complementary subspace of a vector space $V$, then we get a linear mapping $P$ on $V$ which is the projection of $V$ onto $W$ along $Z$ and which is defined by

\[ P(w + z) = w \quad \text{for all } w \in W, \quad z \in Z. \tag{1.32} \]

Thus $I - P$ is the projection of $V$ onto $Z$ along $W$, where $I$ denotes the identity transformation on $V$.

Note that $P^2 = P$, when $P$ is a projection. Conversely, if $P$ is a linear operator on $V$ such that $P^2 = P$, then $P$ is the projection of $V$ onto the subspace of $V$ which is the image of $P$ along the subspace of $V$ which is the kernel of $P$. 
1.5 Reducibility, continued

Again let $G$ be a finite group, $V$ a vector space over a field $k$, $\rho$ a representation of $G$ on $V$, and $W$ a subspace of $V$ which is invariant under $\rho$. In this subsection we assume that either

\[(1.33) \quad k \text{ has characteristic 0}\]

or

\[(1.34) \quad k \text{ has positive characteristic and the number of elements of } G \text{ is not divisible by the characteristic of } k.\]

Let us show that there is a subspace $Z$ of $V$ such that $Z$ is a complement of $W$ and $Z$ is also invariant under the representation $\rho$ of $G$ on $V$. To do this, we start with any complement $Z_0$ of $W$ in $V$, and we let $P_0 : V \to V$ be the projection of $V$ onto $W$ along $Z_0$. Thus $P_0$ maps $V$ to $W$, and $P_0(w) = w$ for all $w \in W$.

Let $m$ denote the number of elements of $G$. Define a linear mapping $P : V \to V$ by

\[(1.35) \quad P = \frac{1}{m} \sum_{x \in G} \rho_x \circ P_0 \circ (\rho_x)^{-1}.\]

Our assumption on $k$ implies that $1/m$ makes sense as an element of $k$, i.e., as the multiplicative inverse of a sum of $m$ 1's in $k$, where 1 refers to the multiplicative identity element of $k$. This expression defines a linear mapping on $V$, because $P_0$ and the $\rho_x$'s are. We actually have that $P$ maps $V$ to $W$, because $P_0$ maps $V$ to $W$, and because the $\rho_x$'s map $W$ to $W$, by hypothesis. If $w \in W$, then $(\rho_x)^{-1}(w) \in W$ for all $x$ in $G$, and then $P_0((\rho_x)^{-1}(w)) = (\rho_x)^{-1}(w)$. Thus we conclude that

\[(1.36) \quad P(w) = w \quad \text{for all } w \in W,\]

by the definition of $P$.

The definition of $P$ also implies that

\[(1.37) \quad \rho_y \circ P \circ (\rho_y)^{-1} = P\]

for all $y$ in $G$. Indeed,

\[(1.38) \quad \rho_y \circ P \circ (\rho_y)^{-1} = \frac{1}{m} \sum_{x \in G} \rho_y \circ \rho_x \circ P_0 \circ (\rho_x)^{-1} \circ (\rho_y)^{-1}\]
This would work as well for other linear transformations instead of $P_0$ as the initial input, but a subtlety is that in general one can get the zero operator after taking the sum. The remarks in the preceding paragraph ensure that this does not happen here, except in the degenerate situation where $W = \{0\}.$

Because $P(V) \subseteq W$ and $P(w) = w$ for all $w \in W$, $P$ is the projection of $V$ onto $W$ along some subspace $Z$ of $V$. Specifically, one should take $Z$ to be the kernel of $P$. It is easy to see that $W \cap Z = \{0\}$, since $P(w) = w$ for all $w \in W$. On the other hand, if $v$ is any element of $V$, then we can write $v$ as $P(v) + (v - P(v))$. We have that $P(v) \in W$, and that

\begin{equation}
P(v - P(v)) = P(v) - P(P(v)) = P(v) - P(v) = 0,
\end{equation}

where the second equality uses the fact that $P(v) \in W$. Thus $v - P(v)$ lies in $Z$, the kernel of $P$. This shows that $W$ and $Z$ satisfy (1.30), so that $Z$ is a complement of $W$ in $V$. The invariance of $Z$ under the representation $\rho$ follows from (1.37).

Thus the representation $\rho$ of $G$ on $V$ is isomorphic to the direct sum of the representations of $G$ on $W$ and $Z$ that are the restrictions of $\rho$ to $W$ and $Z$.

There can be smaller invariant subspaces within these invariant subspaces, so that one can repeat the process. Before addressing this, let us introduce some terminology. We say that subspaces $W_1, W_2, \ldots, W_h$ of $V$ form an independent system if $W_j \neq \{0\}$ for each $j$ and if $w_j \in W_j$, $1 \leq j \leq h$, and

\begin{equation}
\sum_{j=1}^{h} w_j = 0
\end{equation}

imply
\begin{equation}
w_j = 0, \quad j = 1, 2, \ldots, h.
\end{equation}

If, in addition, $\text{span}(W_1, \ldots, W_h) = V$, then every vector $v$ in $V$ can be written in a unique way as $\sum_{j=1}^{h} u_j$ with $u_j \in W_j$ for each $j$.

**Definition 1.42** Let $G$ be a finite group, $U$ be a vector space, and $\sigma$ be a representation of $G$ on $U$. We say that $\sigma$ is irreducible if there are no vector subspaces of $U$ which are invariant under $\sigma$ except for $\{0\}$ and $U$ itself.
Lemma 1.43 Suppose that $G$ is a finite group, $V$ is a vector space over a field $k$ which satisfies $(1.33)$ or $(1.34)$, and $\rho$ is a representation of $G$ on $V$. Then there is an independent system of subspaces $W_1, \ldots, W_h$ of $V$ such that $\text{span}(W_1, \ldots, W_h) = V$, each $W_j$ is invariant under $\rho$, and the restriction of $\rho$ to each $W_j$ is an irreducible representation of $G$.

In other words, the representation $\rho$ of $G$ on $V$ is isomorphic to a direct sum of irreducible representations of $G$ (which are restrictions of $\rho$ to subspaces of $V$). It is not hard to prove the lemma, by repeatedly finding invariant complements for invariant subspaces, until one reaches subspaces to which the restriction of $\rho$ is irreducible. More precisely, suppose that one has an independent system of subspaces of $V$ whose span is $V$ and for which each subspace in the system is invariant under $\rho$. Initially this system could consist of $V$ by itself. If the restriction of $\rho$ to each subspace in the system is irreducible, then we have what we want for the lemma. Otherwise, for each subspace where the restriction of $\rho$ is not irreducible, there is a pair of nontrivial complementary subspaces in that subspace which are invariant under $\rho$. We can use these complementary subspaces in place of the subspace in which they were found, and this leads to a new independent system of invariant subspaces of $V$ whose span is all of $V$. (It is not hard to check that the new collection of subspaces of $V$ still forms an independent system.) By repeating this process, we can get irreducibility, as in the lemma.

Remark 1.44 Exercise 16.7 on p136 of [Ser2] discusses examples where representations over fields of positive characteristic (dividing the order of the group) do not “come from” representations over fields of characteristic 0.

1.6 Positive elements and symmetric fields

Let $k_0$ be a field of characteristic 0. We say that a subset $A$ of $k_0$ is a set of positive elements if (i) 0 does not lie in $A$, (ii) $x + y$ and $xy$ lie in $A$ whenever $x, y \in A$, and (iii) $w^2$ lies in $A$ whenever $w$ is a nonzero element of $k_0$. Observe that the multiplicative identity element 1 in $k_0$ lies in $A$ by (iii), and that $-1$ does not lie in $A$, since otherwise one could use (ii) to get that 0 is in $A$. Any sum of 1’s lies in $A$, so that $k_0$ must have characteristic 0 in order for a set $A$ of positive elements to exist. If $x$ is an element of $A$, then $1/x$ lies in $A$, because $x \neq 0$ by (i), $1/x^2 \in A$ by (iii), and hence $1/x = x(1/x^2)$ is in $A$ by (ii).
A field $k$ of characteristic 0 is said to be a symmetric field if the following conditions are satisfied. First, we ask that $k$ be equipped with an automorphism called conjugation which is an involution, i.e., the conjugate of the conjugate of an element $x$ of $k$ is equal to $x$. The conjugate of $x \in k$ is denoted $\overline{x}$, and we write $k_0$ for the subfield of $k$ consisting of elements $x$ such that $\overline{x} = x$. The second condition is that $k_0$ be equipped with a set $A$ of positive elements. (Thus $A$ and the conjugation automorphism are part of the data for a symmetric field.) The last condition is that $x \overline{x} \in A$ for all nonzero elements $x$ of $k$.

It may be that the conjugation automorphism is equal to the identity, so that $k = k_0$. In this case $k$ is a symmetric field if it is equipped with a set of positive elements.

Of course any subfield of the field of real numbers has a natural set of positive elements, namely the elements that are positive in the usual sense. A basic class of symmetric fields are subfields of the complex numbers which are invariant under complex conjugation, using complex conjugation as the conjugation automorphism on the field. In this case the subfield of elements fixed by the conjugation is the subfield of real numbers in the field, and for the set of positive elements we again use the elements of the field which are positive real numbers in the usual sense.

### 1.7 Inner product spaces

Let $k$ be a symmetric field, and suppose that $V$ is a vector space over $k$. A function $\langle v, w \rangle$ on $V \times V$ with values in $k$ is an inner product on $V$ if it satisfies the following three conditions. First, for each $w \in V$, the function

$$v \mapsto \langle v, w \rangle$$

is linear. Second,

$$\langle w, v \rangle = \overline{\langle v, w \rangle}$$

for all $v, w \in V$, where $\overline{x}$ denotes the conjugate of an element $x$ of $k$, as in Subsection 1.6. Third, if $v$ is a nonzero vector in $V$, then $\langle v, v \rangle$ lies in the set of positive elements associated to $k$ as in Subsection 1.6, and is nonzero in particular. Of course the second condition implies that $\langle v, v \rangle$ lies in the subfield $k_0$ of $k$ consisting of $x$ such that $\overline{x} = x$.

A vector space equipped with an inner product is called an inner product space. It is easy to see that there are plenty of inner products on a vector space over a symmetric field, by writing them down explicitly using a basis.
Suppose that \( \langle \cdot, \cdot \rangle \) is an inner product on the vector space \( V \) over the symmetric field \( k \). Two vectors \( u, w \in V \) are said to be \textit{orthogonal} if
\[ \langle u, w \rangle = 0. \] (1.47)
A collection of vectors \( v_1, \ldots, v_m \) in \( V \) is said to be \textit{orthogonal} if \( v_j \) and \( v_l \) are orthogonal when \( j \neq l \). It is easy to see that any orthogonal collection of nonzero vectors in \( V \) is linearly independent. A collection \( Z_1, \ldots, Z_r \) of vector subspaces of \( V \) is said to be \textit{orthogonal} if any vectors in \( Z_j \) and \( Z_p \), \( j \neq p, 1 \leq j, p \leq r \), are orthogonal.

Assume that \( u_1, \ldots, u_m \) are nonzero orthogonal vectors in \( V \), and let \( U \) denote their span. For each \( w \) in \( U \), we have that
\[ w = \sum_{j=1}^{m} \frac{\langle w, u_j \rangle}{\langle u_j, u_j \rangle} u_j. \] (1.48)
In other words, \( w \) is some linear combination of the \( u_j \)'s, and then the inner product and the assumption of orthogonality can be used to determine the coefficients of the \( u_j \)'s, as in the preceding formula. Define a linear operator \( P \) on \( V \) by
\[ P(v) = \sum_{j=1}^{m} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j. \] (1.49)
Then \( P(v) \) lies in \( U \) for all \( v \) in \( V \), \( P(w) = w \) for all \( w \) in \( U \), and
\[ \langle P(v), w \rangle = \langle v, w \rangle \quad \text{for all } v \in V \text{ and } w \in U. \] (1.50)
This last condition is equivalent to saying that \( v - P(v) \) is orthogonal to every element of \( U \).

\textbf{Remark 1.51} The operator \( P : V \to V \) is characterized by the requirements that \( P(v) \) lie in \( U \) and \( v - P(v) \) be orthogonal to every element of \( U \) for all \( v \) in \( V \). Specifically, if \( v \) is a vector in \( V \) and \( z_1 \) and \( z_2 \) are two elements of \( U \) such that \( v - z_1 \) and \( v - z_2 \) are orthogonal to all elements of \( U \), then \( z_1 = z_2 \). This is because \( z_1 - z_2 \) lies in \( U \), and \( z_1 - z_2 = (z_1 - v) + (v - z_2) \) is orthogonal to all elements of \( U \), so that \( z_1 - z_2 \) is orthogonal to itself. As a consequence, \( P \) does not depend on the choice of orthogonal basis for \( U \).

\textbf{Lemma 1.52} If \( v_1, \ldots, v_m \) are linearly independent vectors in \( V \), then there are nonzero orthogonal vectors \( u_1, \ldots, u_m \) in \( V \) such that the span of \( v_1, \ldots, v_m \) is equal to the span of \( u_1, \ldots, u_m \).
To prove this, one can argue by induction. The $m = 1$ case is trivial, and so we suppose that the statement is true for some $m$ and try to establish it for $m + 1$. Let $v_1, \ldots, v_{m+1}$ be a set of $m + 1$ linearly-independent vectors in $V$. By the induction hypothesis, there are nonzero orthogonal vectors $u_1, \ldots, u_m$ such that the span of $v_1, \ldots, v_m$ is equal to the span of $u_1, \ldots, u_m$. Let $P$ be the projection onto the span of $u_1, \ldots, u_m$ defined above, and set $u_{m+1} = v_{m+1} - P(v_{m+1})$. Then $u_{m+1} \neq 0$, because $v_{m+1}$ does not lie in the span of $v_1, \ldots, v_m$, by linear independence. From the properties of $P$ we know that $u_{m+1}$ is orthogonal to $u_1, \ldots, u_m$, and hence $u_1, \ldots, u_{m+1}$ is a collection of nonzero orthogonal vectors. It is not hard to check that the span of $u_1, \ldots, u_{m+1}$ is equal to the span of $v_1, \ldots, v_{m+1}$, using the corresponding statement for $m$ and the definition of $u_{m+1}$. This completes the proof of the lemma.

**Corollary 1.53** Every nonzero subspace of $V$ admits an orthogonal basis.

(One could say that the subspace $\{0\}$ has the empty orthogonal basis.)

Corollary 1.53 follows from Lemma 1.52 by starting with any basis for the subspace.

**Corollary 1.54** If $U$ is a vector subspace of $V$, then there is a unique linear operator $P = P_U$ on $V$ (the orthogonal projection of $V$ onto $U$) such that $P(v)$ lies in $U$ for all $v$ in $V$, $v - P(v)$ is orthogonal to all elements of $U$ for any $v$ in $V$, and $P(w) = w$ for all $w$ in $U$.

This follows from the earlier discussion, since we know from the previous corollary that $U$ has an orthogonal basis.

If $U$ is a vector subspace of $V$, then the orthogonal complement $U^\perp$ of $U$ in $V$ is the vector subspace of $V$ defined by

$$U^\perp = \{ v \in V : \langle v, w \rangle = 0 \text{ for all } w \in U \}. \tag{1.55}$$

Notice that

$$U \cap U^\perp = \{0\}. \tag{1.56}$$

We can reformulate the characterizing properties of the orthogonal projection $P_U$ of $V$ onto $U$ as saying that for each vector $v$ in $V$, $P_U(v)$ lies in $U$ and $v - P_U(v)$ lies in $U^\perp$. Thus $V$ is the span of $U$ and $U^\perp$. Also, $U^\perp$ is exactly the same as the kernel of $P_U$.  

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Let us check that
\[(U^\bot)^\bot = U.\] 
for any subspace \(U\) of \(V\). The inclusion \(U \subseteq (U^\bot)^\bot\) is a simple consequence of the definition, and so it is enough to show that \((U^\bot)^\bot \subseteq U\). Let \(w \in (U^\bot)^\bot\) be given. We know that \(P_U(w) \in U \subseteq (U^\bot)^\bot\), and hence \(w - P_U(w) \in (U^\bot)^\bot\) as well. On the other hand, \(w - P_U(w)\) lies in \(U^\bot\). Hence \(w - P_U(w)\) lies in both \(U^\bot\) and \((U^\bot)^\bot\), and is therefore 0. This shows that \(w = P_U(w)\) is contained in \(U\), as desired.

Notice that
\[P_{U^\bot} = I - P_U,\] 
since \(I - P_U\) satisfies the properties that characterize \(P_{U^\bot}\). Also, if \(U_1, U_2\) are orthogonal subspaces of \(V\), then \(U_1 + U_2 = V\) if and only if \(U_2 = U_1^\bot\), which is equivalent to \(U_1 = U_2^\bot\).

Suppose that \(T\) is a linear operator on \(V\). The adjoint of \(T\) is the unique linear operator \(T^*\) on \(V\) such that
\[\langle T(v), w \rangle = \langle v, T^*(w) \rangle\] 
for all \(v, w \in V\). It is not hard to describe the matrix of \(T^*\) with respect to an orthogonal matrix in terms of the matrix of \(T\) with respect to the same basis.

The adjoint of the identity operator \(I\) on \(V\) is itself. If \(S\) and \(T\) are linear operators on \(V\), then
\[(T^*)^* = T,\] 
\[(S + T)^* = S^* + T^*,\]
and
\[(S \circ T)^* = T^* \circ S^*.\]
If \(a\) is an element of the scalar field \(k\), then
\[(aT)^* = \overline{a}T^*.\]

A linear operator \(S\) on \(V\) is said to be self-adjoint if \(S^* = S\). Sometimes one says instead that \(S\) is symmetric. As a basic class of examples, if \(U\) is a subspace of \(V\) and \(P_U : V \to V\) is the orthogonal projection of \(V\) onto \(U\), then \(P_U\) is self-adjoint. More precisely, if \(v_1\) and \(v_2\) are arbitrary vectors in \(V\), then
\[\langle P_U(v_1), v_2 \rangle = \langle P_U(v_1), P_U(v_2) \rangle = \langle v_1, P_U(v_2) \rangle,\] 

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since $P_U(v) \in U$ and $v - P_U(v) \in U^\perp$ for $v \in V$.

A linear operator $A$ on $V$ is said to be antiself-adjoint, or antisymmetric, if $A^* = -A$. Any linear operator $T$ on $V$ can be written as the sum of a self-adjoint operator $S$ and an antiself-adjoint operator $A$ by taking $S = (T + T^*)/2$ and $A = (T - T^*)/2$.

Suppose that $R$ is a linear operator on $V$ such that

$$\langle R(v), R(w) \rangle = \langle v, w \rangle \quad \text{(1.65)}$$

for all $v, w \in V$. This is equivalent to saying that $R$ is invertible and

$$R^{-1} = R^* \quad \text{(1.66)}$$

One often says that $R$ is an an orthogonal transformation or a unitary transformation.

The notion of adjoints can be extended to mappings between two inner product spaces. More precisely, let $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ be two inner product spaces, with the same symmetric field $k$ of scalars, and let $T$ be a linear mapping from $V_1$ to $V_2$. The adjoint of $T$ is the unique linear mapping $T^* : V_2 \rightarrow V_1$ such that

$$\langle T(u), w \rangle_2 = \langle u, T^*(w) \rangle_1 \quad \text{(1.67)}$$

for all $u \in V_1$ and $w \in V_2$. The adjoint can be described easily in terms of orthogonal bases and matrices again, although now one would use an orthogonal basis in each of $V_1$ and $V_2$.

As before, $(T^*)^* = T$, $(R + T)^* = R^* + T^*$, and $(aT)^* = \pi T^*$ for any two linear operators $R, T : V_1 \rightarrow V_2$ and any scalar $a \in k$. If $(V_3, \langle \cdot, \cdot \rangle_3)$ is another inner product space with the same scalar field $k$, and if $S : V_2 \rightarrow V_3$ is linear, so that $S \circ T : V_1 \rightarrow V_3$ is defined, then $(S \circ T)^* = T^* \circ S^*$, as linear mappings from $V_1$ to $V_3$.

Note that a linear mapping $T : V_1 \rightarrow V_2$ preserves the inner products on $V_1$ and $V_2$, in the sense that

$$\langle T(u), T(v) \rangle_2 = \langle u, v \rangle_1 \quad \text{(1.68)}$$

for all $u, v \in V_1$, if and only if $T^* \circ T$ is the identity operator on $V_1$. In this case $T$ is one-to-one, but it may not map $V_1$ onto $V_2$. If it does, so that $T$ is invertible, then we can simply say that $T$ preserves the inner products if $T^* = T^{-1}$.
1.8 Inner products and representations

Let $G$ be a finite group, let $V$ be a vector space over a symmetric field $k$, and let $\rho$ be a representation of $G$ on $V$. If $\langle \cdot, \cdot \rangle$ is an inner product on $V$, then $\langle \cdot, \cdot \rangle$ is said to be *invariant under the representation* $\rho$, or simply $\rho$-invariant, if every $\rho_x : V \to V$, $x$ in $G$, preserves the inner product, i.e., if

\begin{equation}
\langle \rho_x(v), \rho_x(w) \rangle = \langle v, w \rangle
\end{equation}

for all $x$ in $G$ and $v, w$ in $V$.

If $\langle \cdot, \cdot \rangle_0$ is any inner product on $V$, then we can obtain an invariant inner product $\langle \cdot, \cdot \rangle$ from it by setting

\begin{equation}
\langle v, w \rangle = \sum_{y \in G} \langle \rho_y(v), \rho_y(w) \rangle_0.
\end{equation}

It is easy to check that this does define an inner product on $V$ which is invariant under the representation $\rho$. Notice that the positivity condition for $\langle \cdot, \cdot \rangle$ is implied by the positivity condition for $\langle \cdot, \cdot \rangle_0$, which prevents $\langle \cdot, \cdot \rangle$ from reducing to 0 in particular.

In some situations it is easy to write down an invariant inner product for a representation directly. In the case of the left and right regular representations for a group $G$ over the symmetric field $k$, as in Subsection 1.3, one can use the inner product

\begin{equation}
\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x) \overline{f_2(x)}.
\end{equation}

More generally, for a permutation representation of $G$ relative to a nonempty finite set $E$, as in Subsection 1.3, one can use the inner product

\begin{equation}
\langle f_1, f_2 \rangle = \sum_{a \in E} f_1(a) \overline{f_2(a)}.
\end{equation}

This inner product is invariant under the permutation representation, because the permutations simply rearrange the terms in the sums without affecting the sum as a whole.

Let $\langle \cdot, \cdot \rangle$ be any inner product on $V$ which is invariant under the representation $\rho$. Suppose that $W$ is a subspace of $V$ which is invariant under $\rho$, so that

\begin{equation}
\rho_x(W) = W
\end{equation}

for all $x$ in $G$.
for all $x$ in $G$. Let $W^\perp$ be the orthogonal complement of $W$ in $V$ with respect to this inner product $\langle \cdot , \cdot \rangle$, as in (1.35). Then

\begin{equation}
\rho_x(W^\perp) = W^\perp
\end{equation}

for all $x$ in $G$, since the inner product is invariant under $\rho$. This gives another approach to finding an invariant complement to an invariant subspace, as in Subsection 1.5.

One can repeat this to get invariant subspaces on which $\rho$ restricts to be irreducible, as in Subsection 1.5. The next lemma is the analogue of Lemma 1.43 in this situation.

**Lemma 1.75** Suppose that $G$ is a finite group, $V$ is a vector space over a symmetric field $k$, and $\rho$ is a representation of $G$ on $V$. Assume also that $\langle \cdot , \cdot \rangle$ is an inner product on $V$ which is invariant under $\rho$. Then there are orthogonal nonzero subspaces $W_1, \ldots, W_h$ of $V$ such that $\text{span}(W_1, \ldots, W_h) = V$, each $W_j$ is invariant under $\rho$, and the restriction of $\rho$ to each $W_j$ is an irreducible representation of $G$.

2 \hspace{1em} **Algebras of linear operators**

2.1 \hspace{1em} **Basic notions**

Let $V$ be a vector space over a field $k$, and recall that $\mathcal{L}(V)$ denotes the collection of linear operators on $V$.

**Definition 2.1** A subset $\mathcal{A}$ of $\mathcal{L}(V)$ is said to be an algebra of linear operators on $V$ if $\mathcal{A}$ is a vector subspace of $\mathcal{L}(V)$ which contains the identity transformation on $V$ and which contains $S \circ T$ whenever $S, T \in \mathcal{A}$.

Thus, for example, $\mathcal{L}(V)$ is an algebra of linear operators on $V$, as is the set of scalar multiples of the identity transformation on $V$.

**Definition 2.2** If $\mathcal{A}$ is a subset of $\mathcal{L}(V)$, then the commutant of $\mathcal{A}$ is defined to be the set of operators $T \in \mathcal{L}(V)$ such that $T \circ S = S \circ T$ for all $S \in \mathcal{A}$. The commutant of $\mathcal{A}$ is denoted $\mathcal{A}'$. 

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For example, $\mathcal{L}(V)'$ is the set of scalar multiples of the identity, and the commutant of the set of scalar multiples of the identity is $\mathcal{L}(V)$.

If $\mathcal{A}$ is any subset of $\mathcal{L}(V)$, then the algebra of operators on $V$ generated by $\mathcal{A}$ is the algebra consisting of linear combinations of the identity operator and finite products of elements of $\mathcal{A}$. It is easy to see that the commutant of $\mathcal{A}$ is the same as the commutant of the algebra of operators on $V$ generated by $\mathcal{A}$. The commutant of any subset $\mathcal{A}$ of $\mathcal{L}(V)$ is an algebra of operators on $V$.

The double commutant of a set $\mathcal{A} \subseteq \mathcal{L}(V)$ is simply the commutant of the commutant of $\mathcal{A}$, $(\mathcal{A}')'$. For simplicity we also write this as $\mathcal{A}''$. By definition, we have that
\begin{equation}
\mathcal{A} \subseteq \mathcal{A}''.
\end{equation}
We shall be interested in conditions which imply that $\mathcal{A} = \mathcal{A}''$. Of course it is necessary that $\mathcal{A}$ be an algebra of operators on $V$ for this to hold.

Let us note the following.

**Lemma 2.4** If $\mathcal{A}$ is any set of operators on $V$, then $\mathcal{A}''' = \mathcal{A}'$ (where $\mathcal{A}''' = (\mathcal{A}'')'$ denotes the triple commutant of $\mathcal{A}$).

As in (2.3), $\mathcal{A}' \subseteq \mathcal{A}'''$. Conversely, if $T$ lies in $\mathcal{A}'''$, then $T$ commutes with all elements of $\mathcal{A}''$, and hence $T$ commutes with all elements of $\mathcal{A}$, because $\mathcal{A} \subseteq \mathcal{A}''$. Thus $T \in \mathcal{A}'$.

The next general fact about algebras of operators will also be useful.

**Proposition 2.5** Let $V$ be a vector space over the field $k$, and let $\mathcal{A}$ be an algebra of operators on $V$. Suppose that $T$ lies in $\mathcal{A}$, and that $T$ is invertible as a linear operator on $V$. Then $T^{-1}$ also lies in $\mathcal{A}$.

If $T$ is any linear operator on $V$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial with coefficients in $k$, then we write $P(T)$ for the linear operator $\sum_{j=0}^{n} a_j T^j$. As usual, $z^0$ is interpreted here as being 1, and $T^0$ is interpreted as being the identity operator on $V$.

Because $\mathcal{L}(V)$ has finite dimension as a vector space over $V$, there is a nontrivial polynomial $P$ with coefficients in $k$ such that $P(T) = 0$. In other words, the operators $T^j$, $j = 0, \ldots, n$, cannot be linearly independent in $\mathcal{L}(V)$ when $n$ is large enough (e.g., if $n$ is equal to the square of the dimension of $V$), and a linear relation among them leads to a polynomial $P$ such that $P(T) = 0$. 

If $T$ is invertible, then we may assume that the constant term in $P$ is nonzero. That is, if the constant term were 0, then we can write $P(z)$ as $z \, P_1(z)$, where $P_1(z)$ is another polynomial (with degree 1 less than the degree of $P$). In this case we have that $T \, P_1(T) = 0$, and hence $P_1(T) = 0$, since $T$ is invertible. By repeating this process as necessary, we can reduce to the case where the constant term is not 0.

Thus we assume that $P(0) \neq 0$. Now, $P(z) - P(0)$ is a polynomial whose constant term vanishes, and so we may write it as $z \, Q_1(z)$, where $Q_1(z)$ is a polynomial of degree 1 less than that of $P$. We obtain that $T \, Q_1(T) = P(0) \, I$, which is the same as saying that $T^{-1} = P(0)^{-1} \, Q_1(T)$. Thus $T^{-1}$ can be expressed as a polynomial in $T$, and the proposition follows.

For the original polynomial $P(z)$ one can in fact take $P(z) = \det(T-zI)$, because $P(T) = 0$ in this event by the Cayley–Hamilton theorem. The constant term $P(0) = \det T$ is already nonzero when $T$ is invertible.

2.2 Nice algebras of operators

Let $V$ be a vector space over a field $k$ again.

**Definition 2.6** Let $\mathcal{A}$ be an algebra of operators on $V$. A vector subspace $W$ of $V$ is invariant under $\mathcal{A}$ if $T(W) \subseteq W$ for all $T$ in $\mathcal{A}$.

**Definition 2.7** An algebra $\mathcal{A}$ of operators on $V$ is said to be a nice algebra of operators if every vector subspace $W$ of $V$ which is invariant under $\mathcal{A}$ is also invariant under $\mathcal{A}''$.

Note that every vector subspace $W$ of $V$ which is invariant under $\mathcal{A}''$ is automatically invariant under $\mathcal{A}$, since $\mathcal{A} \subseteq \mathcal{A}''$. Thus one can rephrase Definition 2.7 as saying that an algebra of operators $\mathcal{A}$ on $V$ is a nice algebra of operators if $\mathcal{A}$ and $\mathcal{A}''$ have the same invariant subspaces.

**Lemma 2.8** Let $S$ and $T$ be linear operators on $V$, and suppose that $S \circ T = T \circ S$. If $W$ denotes the image of $S$ and $Z$ denotes the kernel of $S$, then $T(W) \subseteq W$ and $T(Z) \subseteq Z$.

This is easy to check.

**Lemma 2.9** Let $\mathcal{A}$ be an algebra of linear operators on $V$. If $S$ lies in the commutant $\mathcal{A}'$ of $\mathcal{A}$, then the image and kernel of $S$ are invariant subspaces of the double commutant $\mathcal{A}''$.
Thus follows from Lemma 2.8.

Thus, to show that an algebra is nice, one can try to show that all of its invariant subspaces come from linear operators in the commutant in this way.

**Lemma 2.10** Let \( \mathcal{A} \) be an algebra of operators on \( V \). Suppose that \( U_1, U_1 \) are two vector subspaces of \( V \) which are complementary, so that \( U_1 \cap U_2 = \{0\} \) and \( U_1 + U_2 = V \). Let \( P : V \to V \) denote the projection of \( V \) onto \( U_1 \) along \( U_2 \), i.e., \( P(u_1 + u_2) = u_1 \) for all \( u_1 \in U_1 \) and \( u_2 \in U_2 \). Then \( P \) lies in \( \mathcal{A}' \) if and only if \( U_1 \) and \( U_2 \) are invariant subspaces of \( \mathcal{A} \). In this case \( U_1 \) and \( U_2 \) are also invariant subspaces of the double commutant \( \mathcal{A}'' \).

Assume first that \( U_1 \) and \( U_2 \) are invariant subspaces of \( \mathcal{A} \). Thus if \( T \) lies in \( \mathcal{A} \), \( u_1 \) lies in \( U_1 \), and \( u_2 \) lies in \( U_2 \), then \( T(u_1) \) is also an element of \( U_1 \) and \( T(u_2) \) is an element of \( U_2 \), and hence

\[
(2.11) \quad P(T(u_1 + u_2)) = P(T(u_1) + T(u_2)) = T(u_1) = T(P(u_1 + u_2)).
\]

This shows that \( T \) commutes with \( P \). We conclude that \( P \) lies in \( \mathcal{A}' \), since this applies to any \( T \) in \( \mathcal{A} \).

Conversely, if \( P \) lies in \( \mathcal{A}' \), then \( U_1 \) and \( U_2 \) are invariant subspaces for \( \mathcal{A}'' \), and hence for \( \mathcal{A} \), because of Lemma 2.9. More precisely, this uses the fact that \( U_1 \) is the image of \( P \) and \( U_2 \) is the kernel of \( P \), by construction.

### 2.3 \( \ast \)-Algebras

In this subsection we assume that \( V \) is a vector space over a symmetric field \( k \) and that \( V \) is equipped with an inner product \( \langle \cdot, \cdot \rangle \).

**Definition 2.12** A subset \( A \) of \( \mathcal{L}(V) \) is said to be a \( \ast \)-algebra of linear operators on \( V \) if it is an algebra and if the adjoint \( T^\ast \) of an operator \( T \) lies in \( A \) whenever \( T \) does.

The algebra \( \mathcal{L}(V) \) and the set of scalar multiples of the identity are \( \ast \)-algebras.

**Lemma 2.13** If \( A \) is a \( \ast \)-algebra of operators on \( V \), then the commutant \( A' \) of \( A \) is a \( \ast \)-algebra of operators on \( V \) as well.

This is easy to check.
Lemma 2.14 If \( A \) is a \( \ast \)-algebra of linear operators on \( V \), and if \( W \) is a vector subspace of \( V \) which is invariant under \( A \), then \( W^\perp \) is also invariant under \( A \).

The hypothesis that \( W \) be invariant under \( A \) can be rewritten as

\[
\langle T(w), u \rangle = 0
\]

for all \( w \in W \), \( u \in W^\perp \), and \( T \in A \). This implies in turn that

\[
\langle w, T^*(u) \rangle = 0
\]

for all \( w \in W \), \( u \in W^\perp \), and \( T \in A \). In other words, \( T^*(W^\perp) \subseteq W^\perp \) for all \( T \in A \). Because \( A \) is assumed to be a \( \ast \)-algebra, this is equivalent to saying that \( S(W) \subseteq W \) for all \( S \in A \), as desired.

Corollary 2.17 If \( A \) is a \( \ast \)-algebra of operators on \( V \), then \( A \) is a nice algebra of operators on \( V \).

This follows easily from Lemmas 2.14 and 2.10.

2.4 Algebras from group representations

Let \( G \) be a finite group, \( V \) be a vector space over a field \( k \), and \( \rho \) a representation of \( G \) on \( V \). Consider the set \( A \) of linear operators on \( V \) defined by

\[
A = \text{span}\{\rho_x : x \in G\}.
\]

Here “span” means the ordinary linear span inside the vector space \( L(V) \). Thus

\[
\text{the dimension of } A \text{ as a vector space is less than or equal to the number of elements in } G.
\]

On the other hand,

\[
A \text{ is an algebra of linear operators on } V.
\]

This is easy to check, using the fact that \( \rho \) is a representation of \( G \) on \( V \) (so that \( \rho_x \circ \rho_y = \rho_{xy} \) for all \( x \) and \( y \) in \( G \), and hence \( \rho_x \circ \rho_y \) lies in \( A \) for all \( x \) and \( y \) in \( G \)).
Lemma 2.21 Under the conditions just described, a vector subspace \( W \) of \( V \) is invariant under \( A \) if and only if it is invariant under the representation \( \rho \) (as defined in Subsection 1.4).

This is easy to verify.

Lemma 2.22 Under the conditions above, if \( k \) has characteristic 0, or if \( k \) has positive characteristic and the number of elements of \( G \) is not divisible by the characteristic of \( k \), then \( A \) is a nice algebra of operators.

This uses the fact that each subspace of \( V \) which is invariant under the representation \( \rho \) has an invariant complement, as in Subsection 1.3, and Lemma 2.10.

Suppose now that \( k \) is a symmetric field, \( \langle \cdot , \cdot \rangle \) is an inner product on \( V \), and that \( \langle \cdot , \cdot \rangle \) is invariant under the representation \( \rho \), as in Subsection 1.8. This is the same as saying that

\[
(\rho_x)^{-1} = (\rho_x)^* \tag{2.23}
\]

for each \( x \) in \( G \), as in \( (1.66) \) in Subsection 1.7. We can rewrite (2.23) as

\[
(\rho_x)^* = \rho_{x^{-1}}. \tag{2.24}
\]

It follows easily that \( A \) is a \( * \)-algebra in this case, since the adjoint of each of the \( \rho_x \)'s lies in \( A \), by (2.24).

In other words,

\[
\text{the correspondence } T \mapsto T^* \text{ has the effect of} \tag{2.25}
\]

“linearizing” the correspondence \( x \mapsto x^{-1} \) on \( G \).

Note that because the inner product is assumed to be invariant under \( \rho \), one gets immediately that the orthogonal complement of an invariant subspace is also invariant, as in Subsection 1.8. This also follows from the general result for \( * \)-algebras in Lemma 2.14.

2.5 Regular representations

Let \( G \) be a finite group and \( k \) be a field. Define \( V \) to be the vector space of \( k \)-valued functions on \( G \), and consider the left regular representation \( L_x, x \in G \),
and right regular representation \( R_{x}, \ x \in G, \) of \( G \) on \( V \), as in Subsection \[1.3\]. Let \( \mathcal{A}_{L} \) and \( \mathcal{A}_{R} \) denote the algebras of operators on \( V \) generated by the linear operators in the left and right regular representations, respectively. We would like to show that

\[
\mathcal{A}'_{L} = \mathcal{A}_{R} \quad \text{and} \quad \mathcal{A}'_{R} = \mathcal{A}_{L}.
\]

(2.26)

Once we do this, it follows that \( \mathcal{A}''_{L} = \mathcal{A}_{L} \) and \( \mathcal{A}''_{R} = \mathcal{A}_{R} \).

The inclusions \( \mathcal{A}_{R} \subseteq \mathcal{A}'_{L} \) and \( \mathcal{A}_{L} \subseteq \mathcal{A}'_{R} \) follow from (1.17) in Subsection \[1.3\]. Conversely, suppose that \( S \) is an operator on \( V \) that lies in \( \mathcal{A}'_{L} \). This condition is equivalent to

\[
S(x) = S(L(x)\phi_e) = L(x)S(\phi_e)
\]

(2.27)

for all \( x \) in \( G \). We would like to show that \( S \) can be written as a linear combination of the \( R_{y} \)'s for \( y \) in \( G \). For each \( w \) in \( G \), let \( \phi_w(z) \) be the function on \( G \) defined by \( \phi_w(z) = 1 \) when \( z = w \) and \( \phi_w(z) = 0 \) when \( z \neq w \), as in Subsection \[1.3\]. Thus the \( \phi_w \)'s form a basis for \( V \), and \( L(x)\phi_w = \phi_{xw} \) for all \( x, w \) in \( G \) (which was mentioned before in (1.16)). Using (2.27) we obtain that

\[
S(\phi_x) = S(L(x)\phi_e) = L(x)S(\phi_e)
\]

(2.28)

for all \( x \) in \( G \). Of course \( S(\phi_e) \) is a function on \( G \), which we can write as

\[
S(\phi_e) = \sum_{w \in G} c_w \phi_w,
\]

(2.29)

where in fact \( c_w = S(\phi_e)(w) \) for all \( w \) in \( G \). This and (2.28) permit us to write \( S(\phi_x) \) as

\[
S(\phi_x) = \sum_{w \in G} c_w \phi_{xw} = \sum_{w \in G} c_w R_{w^{-1}}(\phi_x),
\]

(2.30)

since \( R_w(\phi_x) = \phi_{xw^{-1}} \), as in (1.16). Thus

\[
S = \sum_{w \in G} c_w R_{w^{-1}},
\]

(2.31)

which is what we wanted. In the same way one can show that if an operator commutes with \( R_{y} \) for all \( y \) in \( G \), then the operator is a linear combination of \( L_{u} \)'s. Therefore \( \mathcal{A}'_{L} \subseteq \mathcal{A}_{R} \) and \( \mathcal{A}'_{R} \subseteq \mathcal{A}_{L} \). This completes the proof of (2.26).
2.6 Expanding an algebra

Let $V$ be a vector space over a field $k$, and let $\mathcal{A}$ be an algebra of linear operators on $V$. Fix a positive integer $n$, and let $V^n$ denote the vector space of functions $f$ from $\{1, 2, \ldots, n\}$ to $V$. In effect, $V^n$ is a direct sum of $n$ copies of $V$, and this is a convenient way to express it.

If $T$ is a linear operator on $V$, then we can associate to it a linear operator $\hat{T}$ on $V^n$ by

$$\hat{T}(f)(j) = T(f(j))$$

for each $f \in V^n$ and $j = 1, 2, \ldots, n$. In other words, $\hat{T}$ is the same as $T$ in the $V$ directions in $V^n$, and does nothing in the other directions. If one thinks of $V^n$ as a direct sum of $n$ copies of $V$, then $\hat{T}$ acts in the same way as $T$ on each of these copies, with no mixing between the copies.

It is not hard to verify that the correspondence $T \mapsto \hat{T}$ from $\mathcal{L}(V)$ to $\mathcal{L}(V^n)$ is linear and cooperates with compositions, so that $S \circ T = \hat{S} \circ \hat{T}$ for all $S, T \in \mathcal{L}(V)$. Also, if $T$ is the identity operator on $V$, then $\hat{T}$ is the identity operator on $V^n$. Set

$$\mathcal{A}_n = \{\hat{T} : T \in \mathcal{A}\}.$$ 

From the preceding remarks it follows that $\mathcal{A}_n$ is an algebra of operators on $V^n$, since $\mathcal{A}$ is an algebra of operators on $V$.

**Proposition 2.34** $(\mathcal{A}')_n = (\mathcal{A}_n)'$.

Here $(\mathcal{A}_n)' = \{\hat{T} : T \in \mathcal{A}_n\}$, in analogy with $\mathcal{A}_n$, while $(\mathcal{A}_n)''$ denotes the double commutant of $\mathcal{A}_n$ as an algebra of operators in $\mathcal{L}(V^n)$. Thus the proposition states that one gets the same result whether one first takes the double commutant of $\mathcal{A}$, and then uses the correspondence $T \mapsto \hat{T}$ to pass to $\mathcal{L}(V^n)$, or one first uses this correspondence to pass to $\mathcal{L}(V^n)$, and then takes the double commutant there.

To prove the proposition, let us look first at what $(\mathcal{A}_n)'$ is. For $\ell = 1, 2, \ldots, n$, let $P_\ell$ denote the natural projection of $V^n$ onto the $\ell$th copy of $V$ inside $V^n$, defined by

$$P_\ell(f)(j) = \begin{cases} f(j) & \text{when } j = \ell, \\ 0 & \text{when } j \neq \ell. \end{cases}$$
Thus \( P_h \circ P_\ell = 0 \) when \( h \neq \ell \), and

\[
\sum_{\ell=1}^{n} P_\ell = \text{identity transformation on } V^n.
\]

(2.36)

For any \( T \in \mathcal{L}(V) \),

\[
\hat{T} \circ P_\ell = P_\ell \circ \hat{T},
\]

by the way that \( \hat{T} \) is defined. Now let \( M \) be any linear transformation on \( V^n \), and set \( M_{h,\ell} = P_h \circ M \circ P_\ell \). To say that \( M \) lies in \( (\mathcal{A}_n)' \) means that

\[
M \circ \hat{T} = \hat{T} \circ M
\]

(2.38)

for all \( T \in \mathcal{A} \). Hence \( M \in (\mathcal{A}_n)' \) if and only if

\[
M_{h,\ell} \circ \hat{T} = \hat{T} \circ M_{h,\ell}
\]

(2.39)

for all \( T \in \mathcal{A} \) and \( h, \ell = 1, 2, \ldots, n \), because of the preceding observations.

For \( \ell = 1, 2, \ldots, n \), let \( \theta_\ell : V \to V^n \) denote the linear mapping defined by

\[
\theta_\ell(v)(j) =\begin{cases} v & \text{when } j = \ell \\ 0 & \text{when } j \neq \ell. \end{cases}
\]

(2.40)

In other words, \( \theta_\ell \) is the obvious identification between \( V \) and the \( \ell \)th copy of \( V \) inside \( V^n \). For each linear transformation \( M \) on \( V^n \) and each \( h, \ell = 1, 2, \ldots, n \), let \( \tilde{M}_{h,\ell} \) be the linear transformation on \( V \) such that

\[
M_{h,\ell} \circ \theta_\ell = \theta_h \circ \tilde{M}_{h,\ell}.
\]

(2.41)

By construction, \( \tilde{M}_{h,\ell} \) essentially corresponds to a linear mapping from the \( \ell \)th copy of \( V \) in \( V^n \) to the \( h \)th copy of \( V \) in \( V^n \), and with \( \tilde{M}_{h,\ell} \) we rewrite this as a mapping on \( V \) in the obvious way.

We also have that

\[
\hat{T} \circ \theta_\ell = \theta_\ell \circ T
\]

(2.42)

for all \( T \in \mathcal{L}(V) \) and \( \ell = 1, 2, \ldots, n \), again just by the way that everything is defined here. The bottom line is that \( \theta_\ell \) is equivalent to

\[
M_{h,\ell} \circ \theta_\ell = \theta_h \circ \tilde{M}_{h,\ell}.
\]

(2.43)

for all \( T \in \mathcal{A} \) and \( h, \ell = 1, 2, \ldots, n \). To summarize, \( M \in \mathcal{L}(V^n) \) lies in \( (\mathcal{A}_n)' \) if and only if the “blocks” \( \tilde{M}_{h,\ell} \), \( 1 \leq h, \ell \leq n \), in \( \mathcal{L}(V) \) all lie in \( \mathcal{A}' \).
Now suppose that $S \in \mathcal{L}(V)$ lies in $\mathcal{A}''$. Thus $S$ commutes with all linear operators on $V$ in $\mathcal{A}'$. Using this, it is not hard to show that

$$M \circ \hat{S} = \hat{S} \circ M$$

for all $M \in \mathcal{L}(V)$ which lie in $(\mathcal{A}_n)'$. To be more precise, (2.44) holds if and only if

$$M_{h,\ell} \circ \hat{S} = \hat{S} \circ M_{h,\ell}$$

for all $h, \ell = 1, 2, \ldots, n$, as before. This is in turn equivalent to

$$\tilde{M}_{h,\ell} \circ S = S \circ \tilde{M}_{h,\ell}$$

for $h, \ell = 1, 2, \ldots, n$, as operators now on $V$ instead of $V^n$. This condition holds for $S \in \mathcal{A}''$ because $\tilde{M}_{h,\ell}$ lies in $\mathcal{A}'$ for $h, \ell = 1, 2, \ldots, n$.

From (2.44) we obtain the inclusion

$$(\mathcal{A}'')_n \subseteq (\mathcal{A}_n)''.\tag{2.47}$$

Next we wish to show that

$$(\mathcal{A}_n)'' \subseteq (\mathcal{A}'')_n.\tag{2.48}$$

Let $M \in \mathcal{L}(V^n)$ be any element of $(\mathcal{A}_n)''$. The first point is that each of the $n$ copies of $V$ inside $V^n$ are invariant under $M$, which is the same as saying that

$$M_{h,\ell} = 0 \quad \text{when } h \neq \ell.\tag{2.49}$$

This follows from the fact that each $P_\ell$ lies in $(\mathcal{A}_n)'$, so that $M$ commutes with each $P_\ell$.

If $\pi$ is any permutation on $\{1, 2, \ldots, n\}$, consider the linear transformation on $V^n$ which takes $f(j)$ in $V^n$ to $f(\pi(j))$. This transformation commutes with $\tilde{T}$ on $V^n$ for any $T \in \mathcal{L}(V)$, and thus this transformation lies in $(\mathcal{A}_n)'$.

As a result, $M$ in $(\mathcal{A}_n)''$ commutes with all of these linear transformations on $\mathcal{L}(V^n)$ obtained from permutations on $\{1, 2, \ldots, n\}$. Using this it is not hard to verify that $M = \tilde{R}$ for some linear operator $R$ on $V$.

The remaining observation is that $R$ lies in $\mathcal{A}''$. Indeed, if $S$ is any linear operator on $V$ which lies in $\mathcal{A}'$, then it is easy to see that $\hat{S}$ lies in $(\mathcal{A}_n)'$. This implies that $M$ commutes with $\hat{S}$, and hence $R$ commutes with $S$. Therefore $R \in \mathcal{A}''$, since this applies to any $S$ in $\mathcal{A}'$. In other words, $M = \tilde{R}$ lies in $(\mathcal{A}'')_n$, as desired. This completes the proof of Proposition 2.34.
Suppose now that $k$ is a symmetric field and that $V$ is equipped with an inner product $\langle \cdot, \cdot \rangle$. We define an inner product $\langle \cdot, \cdot \rangle_n$ on $V^n$ by

\[
\langle f, h \rangle_n = \sum_{j=1}^{n} \langle f(j), h(j) \rangle
\]

for all $f, h \in V^n$. In this way $V^n$ is an orthogonal direct sum of $n$ copies of $V$, all with the same inner product as on $V$.

**Lemma 2.51** For each linear operator $T$ on $V$, $(\hat{T})^* = (\hat{T})^*$ (where $T^*$ is defined in terms of the inner product on $V$, while $(\hat{T})^*$ is defined using the inner product on $V^n$).

This is easy to check.

**Corollary 2.52** Under the conditions above, if $A$ is a $\ast$-algebra of operators on $V$, then $A_n$ is a $\ast$-algebra of operators on $V^n$, and hence is nice.

Now assume that $k$ is a field, $V$ is a vector space over $k$, and $A$ is the algebra of operators on $V$ generated by a representation $\rho$ of a finite group $G$ on $V$. We can associate to $\rho$ a representation $\rho^n$ of $G$ on $V^n$ in a simple way, namely by setting

\[
\rho^n_x = \hat{\rho}_x
\]

for each $x$ in $G$.

**Lemma 2.54** Under the conditions just described, $A_n$ is the algebra of operators generated by the representation $\rho^n$ of $G$ on $V^n$.

This follows easily from the definitions.

**Corollary 2.55** If we also assume that either $k$ has characteristic 0, or $k$ has positive characteristic and the number of elements of the group $G$ is not divisible by the characteristic of $k$, then $A_n$ is nice.

This uses Lemma 2.22 in Subsection 2.4.
2.7 Very nice algebras of operators

Definition 2.56 Let $k$ be a field and $V$ a vector space over $k$. An algebra of operators $\mathcal{A}$ on $V$ is said to be very nice if $\mathcal{A}_n \subseteq \mathcal{L}(V^n)$ (defined in Subsection 2.4) is a nice algebra of operators for each positive integer $n$.

The $n = 1$ case corresponds to nice algebras of operators. From Corollaries 2.52 and 2.55 we know that $\mathcal{A}$ is very nice if either it is a $\ast$-algebra, assuming that $k$ is a symmetric field and that $V$ is equipped with an inner product, or if $\mathcal{A}$ is generated by a representation of a finite group on $V$ and either the characteristic of $k$ is 0 or the number of elements of $G$ is not divisible by the characteristic of $k$ when it is positive.

Theorem 2.57 Let $k$ be a field, $V$ a vector space over $k$, and $\mathcal{A}$ an algebra of operators on $V$. If $\mathcal{A}$ is very nice, then $\mathcal{A}'' = \mathcal{A}$.

This is a relative of the “double commutant theorem”, and for the proof (including some of the preliminary steps in the previous subsections) we are essentially following the argument indicated on p118 of [Dou]. As in [Dou], this argument applies to “Von Neumann algebras” (with suitable adjustments).

Let $k$, $V$, and $\mathcal{A}$ be as in the statement of the theorem. As in (2.3) in Subsection 2.1, $\mathcal{A} \subseteq \mathcal{A}''$ holds automatically, and so we need only show that $\mathcal{A}'' \subseteq \mathcal{A}$.

Let us first check that for each $v \in V$ and $S \in \mathcal{A}''$ there is a $T \in \mathcal{A}$ such that $S(v) = T(v)$, assuming only that $\mathcal{A}$ is nice. Fix $v \in V$, and set

\[ W = \{T(v) : T \in \mathcal{A}\}. \]

(2.58)

It is easy to see that $W$ is a vector subspace of $V$ which is invariant under $\mathcal{A}$, because $\mathcal{A}$ is an algebra of operators. If $\mathcal{A}$ is nice, then $W$ is also invariant under $\mathcal{A}''$. Of course $v \in W$, since $I \in \mathcal{A}$, and therefore $S(v) \in W$ for any $S \in \mathcal{A}''$, because $W$ is invariant under $\mathcal{A}''$. This shows that for each $S \in \mathcal{A}''$ there is a $T \in \mathcal{A}$ such that $T(v) = S(v)$, by the definition of $W$.

Now let $n$ be a positive integer, and let $V^n$, $\mathcal{A}_n$, etc., be as in Subsection 2.3. The preceding observation can be applied to $V^n$ and $\mathcal{A}_n$ (assuming that $\mathcal{A}$ is very nice, so that $\mathcal{A}_n$ is nice) to obtain that for each $f \in V^n$ and each $M \in (\mathcal{A}_n)''$ there is an $N \in \mathcal{A}_n$ such that $N(f) = M(f)$.

From Proposition 2.34 we know that $(\mathcal{A}_n)'' = (\mathcal{A}'')_n$. Thus the previous assertion can be rephrased as saying that for each $f \in V^n$ and each $M \in$
(\mathcal{A}^\prime\prime)_n \) there is an \( N \in \mathcal{A}_n \) such that \( N(f) = M(f) \). This can be rephrased again as saying that for each \( f \in V^n \) and each \( S \in \mathcal{A}^\prime\prime \) there is a \( T \in \mathcal{A} \) such that
\[
(2.59) \quad \hat{T}(f) = \hat{S}(f).
\]

This is in turn equivalent to the following statement. If \( v_1, v_2, \ldots, v_n \) are elements of \( V \), and if \( S \in \mathcal{A}^\prime\prime \), then there is a \( T \in \mathcal{A} \) such that \( T(v_j) = S(v_j) \) for \( j = 1, 2, \ldots, n \).

If \( n \) is the dimension of \( V \), then we can choose \( v_1, v_2, \ldots, v_n \) to be a basis of \( V \). In this case the preceding statement reduces to saying that if \( S \in \mathcal{A}^\prime\prime \), then there is a \( T \in \mathcal{A} \) such that \( T = S \). This implies that \( \mathcal{A}^\prime\prime \subseteq \mathcal{A} \), and hence \( \mathcal{A}^\prime\prime = \mathcal{A} \), since the other inclusion is automatic. This completes the proof of the theorem.

### 2.8 Some other approaches

Let \( k \) be a field, \( V \) a vector space over \( k \), and \( \mathcal{A} \) an algebra of operators on \( V \). Since \( \mathcal{A} \subseteq \mathcal{A}^\prime\prime \) automatically, another way to try to show that \( \mathcal{A} = \mathcal{A}^\prime\prime \) is to show that the dimension of \( \mathcal{A} \) is equal to the dimension of \( \mathcal{A}^\prime\prime \), as vector spaces over \( k \). Of course this is easier to do if \( \mathcal{A}^\prime\prime \) is known and reasonably simple.

If \( \mathcal{A}' \) consists of only scalar multiples of the identity operator on \( V \), then \( \mathcal{A}^\prime\prime = \mathcal{L}(V) \), and the dimension of \( \mathcal{A}^\prime\prime \) is equal to the square of the dimension of \( V \). In this case we can just say that \( \mathcal{A} = \mathcal{L}(V) \) if \( \mathcal{A} \) has dimension equal to the square of the dimension of \( V \). Let us give a couple of other criteria for \( \mathcal{A} \) to be equal to \( \mathcal{L}(V) \).

**Lemma 2.60** Let \( \mathcal{A} \) be an algebra of linear operators on \( V \) which satisfies the following two conditions: (a) for any vectors \( v, w \) in \( V \) with \( v \neq 0 \) there is a linear operator \( T \) in \( \mathcal{A} \) such that \( T(v) = w \); (b) there is a nonzero linear operator on \( V \) with rank 1 which lies in \( \mathcal{A} \). Then \( \mathcal{A} = \mathcal{L}(V) \).

Condition (a) in the hypothesis is equivalent to asking that \( \mathcal{A} \) be irreducible in the sense of Definition \[1.4].

To show that \( \mathcal{A} = \mathcal{L}(V) \), it is enough to show that every linear operator on \( V \) with rank 1 lies in \( \mathcal{A} \). By assumption, there is a nonzero operator \( R \) of rank 1 which lies in \( \mathcal{A} \), and we can write \( R \) as \( R(u) = f(u) z \), where \( z \) is a nonzero element of \( V \), and \( f \) is a nonzero linear mapping from \( V \) to \( k \), i.e., a nonzero linear functional on \( V \). If \( w \) is any other nonzero vector
in \(v\), then there is an operator \(T\) in \(\mathcal{A}\) such that \(T(z) = w\), and hence \((T \circ R)(u) = f(u) T(z) = f(u) w\) and \(T \circ R\) lies in \(\mathcal{A}\).

Let \(h\) be any nonzero linear functional on \(V\). We would like to show that there is a linear operator \(S\) in \(\mathcal{A}\) such that \(h = f \circ S\). If we can do that, then \((T \circ R \circ S)(u) = h(u) w\) and \(T \circ R \circ S\) lies in \(\mathcal{A}\), and we get that any linear operator on \(V\) of rank 1 lies in \(\mathcal{A}\).

Consider the set

\[(2.61) \quad Z = \{ y \in V : f \circ S(y) = 0 \text{ for all } S \in \mathcal{A} \} .\]

The irreducibility assumption for \(\mathcal{A}\) implies that \(Z\) contains only the zero vector in \(V\). On the other hand, the set of linear functionals on \(V\) of the form \(f \circ S\) is a vector subspace of the vector space of all linear functionals on \(V\). The set \(Z\) is exactly the intersection of the kernels of these linear functionals, and if they did not span the whole space of linear functionals on \(V\), then \(Z\) would contain a nonzero vector. This is a basic result of linear algebra. Since \(Z\) does contain only the zero vector, we obtain that the set of linear functionals on \(V\) of the form \(f \circ S\), \(S \in \mathcal{A}\), is equal to the set of all linear functionals on \(V\), as desired. This completes the proof of Lemma 2.60.

**Lemma 2.62** Let \(\mathcal{A}\) be an algebra of linear operators on \(V\) such that for any pair \(v_1, v_2\) of linearly independent vectors in \(V\) and any arbitrary pair of vectors \(w_1, w_2\) in \(V\) there is a \(T\) in \(\mathcal{A}\) such that \(T(v_1) = w_1, T(v_2) = w_2\). Then \(\mathcal{A} = \mathcal{L}(V)\).

We may as well assume that \(V\) has dimension at least 2, since the conclusion of the lemma is automatic when the dimension of \(V\) is 1. The hypothesis of the lemma is also vacuous in this case. Because of Lemma 2.60, it suffices to show that \(\mathcal{A}\) contains a nonzero operator of rank 1.

Let \(R\) be any nonzero operator in \(\mathcal{A}\). If \(R\) has rank 1, then we are finished, and so we assume that \(R\) has rank at least 2. This means that there are vectors \(u_1, u_2\) in \(V\) such that \(R(u_1), R(u_2)\) are linearly independent.

By assumption, there is an operator \(T\) in \(\mathcal{A}\) such that \(T(R(u_1)) = 0\) and \(T(R(u_2)) \neq 0\). Thus \(T \circ R\) is nonzero and has rank smaller than the rank of \(R\), because the dimension of the image has been reduced, or, equivalently, the dimension of the kernel has been increased. We also have that \(T \circ R\) is in \(\mathcal{A}\), since \(R\) and \(T\) are elements of \(\mathcal{A}\). By repeating this process, as needed, we can get a nonzero element of \(\mathcal{A}\) of rank 1.
Lemma 2.63 Suppose that $A$ is an algebra of linear operators on $V$ such that $A'$ consists of only the scalar multiples of the identity and $A_n \subseteq \mathcal{L}(V^n)$ (defined in Subsection 2.4) is a nice algebra of operators when $n = 2$. Then $A = \mathcal{L}(V) = A''$.

The assumptions in this lemma imply the hypothesis of Lemma 2.62.

3 Vector spaces with definite scalar product

Let $k$ be a field and let $V$ be a vector space. Suppose that $(v, w)$ is a definite scalar product on $V$, by which we mean the following: (a) $(v, w)$ is a bilinear form on $V$, i.e., it is a function from $V \times V$ into $k$ such that $v \mapsto (v, w)$ is linear for each $w$ in $V$, and $w \mapsto (v, w)$ is linear for each $v$ in $V$; (b) $(v, w)$ is symmetric in $v$ and $w$, so that $(v, w) = (w, v)$ for all $v$ and $w$ in $V$; and (c) $(v, v) = 0$ if and only if $v = 0$. This last can be a somewhat complicated condition, which does not necessarily entail positivity. See [Ser3], for instance. For finite fields any homogeneous polynomial of degree 2 in at least 3 variables has a nontrivial solution, as on p338 of [Cas] and Corollary 2 on p6 of [Ser3].

Two vectors $v$, $w$ in $V$ are said to be orthogonal if $(v, w) = 0$. A collection of vectors $v_1, \ldots, v_k$ is said to be orthogonal if $(v_j, v_l) = 0$ for all $j, l$ such that $1 \leq j, l \leq k$ and $j \neq l$. As usual, nonzero orthogonal vectors are linearly independent.

If $W$ is a linear subspace of $V$, then we define the orthogonal complement $U^\perp$ of $U$ by

$$U^\perp = \{v \in V : (v, u) = 0 \text{ for all } u \in U\}. \quad (3.1)$$

Thus $U^\perp$ is also a linear subspace of $V$, and

$$U \cap U^\perp = \{0\}, \quad (3.2)$$

since our scalar product is assumed to be definite.

Suppose that $u_1, \ldots, u_n$ is a collection of nonzero orthogonal vectors in $V$, and let $U$ denote the span of $u_1, \ldots, u_n$. Define a linear operator $P$ on $V$ by

$$P(v) = \sum_{j=1}^n \frac{(v, u_j)}{(u_j, u_j)} u_j. \quad (3.3)$$
We have that  
\[ P(u) = u \quad \text{when} \quad u \in U, \quad P(v) = 0 \quad \text{when} \quad v \in U^\perp, \]
and
\[
P(w) \in U, \quad w - P(w) \in U^\perp \quad \text{for all} \quad w \in V.
\]

For any \( w \) in \( V \), there is at most one vector \( y \) in \( V \) such that \( y \in U \) and \( w - y \in U^\perp \), because if \( y' \) is another such vector, then \( y - y' \in U \cap U^\perp \). Hence \( P \) is characterized by (3.4), and depends only on \( U \), and not the choice of \( u_1, \ldots, u_n \). We say that \( P \) is the **orthogonal projection** of \( V \) onto \( U \).

**Lemma 3.5** If \( x_1, \ldots, x_n \) are linearly independent vectors in \( V \), then there are nonzero orthogonal vectors \( u_1, \ldots, u_n \) in \( V \) such that \( \text{span}(x_1, \ldots, x_n) = \text{span}(u_1, \ldots, u_n) \).

As usual, this can be proved using induction on \( n \). The orthogonalization of the last vector is obtained with the help of an orthogonal projection onto the span of the previous vectors, where the orthogonal projection just mentioned is derived from the induction hypothesis.

**Corollary 3.6** If \( U \) is any linear subspace of \( V \), then there is an orthogonal projection of \( V \) onto \( U \). In particular, \( V \) is spanned by \( U \) and \( U^\perp \), and \((U^\perp)^\perp = U\).

Now let \( T \) be a linear operator on \( V \). The **transpose** of \( T \) is the linear operator \( T^t \) on \( V \) such that
\[
(T(v), w) = (v, T^t(w))
\]
for all \( v, w \) in \( V \). It is not difficult to establish the existence of such an operator \( T^t \), and uniqueness is an easy consequence of (3.7).

Note that
\[
(\alpha S + \beta T) = \alpha S^t + \beta T^t
\]
and
\[
(S \circ T)^t = T^t \circ S^t
\]
for any scalars \( \alpha, \beta \) and linear operators \( S, T \) on \( V \). The transpose of the identity operator \( I \) on \( V \) is itself, and
\[
(T^t)^t = T
\]
for any linear operator \( T \) on \( V \).
Definition 3.11 Let $\mathcal{A}$ be an algebra of linear operators on $V$, as in Definition 2.7. We say that $\mathcal{A}$ is a $t$-algebra if $T^t$ lies in $\mathcal{A}$ whenever $T$ does.

Of course the algebra $\mathcal{L}(V)$ of all operators on $V$ is a $t$-algebra, as is the algebra consisting only of scalar multiples of the identity.

Lemma 3.12 If $\mathcal{A}$ is a $t$-algebra of linear operators on $V$, then the commutant $\mathcal{A}'$ is also a $t$-algebra of linear operators on $V$.

This is a simple exercise.

Lemma 3.13 Let $\mathcal{A}$ be an algebra of operators on $V$, and let $U$ be a linear subspace of $V$ which is invariant under $\mathcal{A}$. If $\mathcal{A}$ is a $t$-algebra, then $U^\perp$ is invariant under $\mathcal{A}$, and the orthogonal projection $P_U$ of $V$ onto $U$ lies in $\mathcal{A}'$.

The statement that $U$ is invariant under $\mathcal{A}$ is equivalent to

\begin{equation}
(T(u), v) = 0
\end{equation}

for all $u \in U$, $v \in U^\perp$, and $T \in \mathcal{A}$. This implies that

\begin{equation}
(u, T^t(v)) = 0
\end{equation}

for all $u \in U$, $v \in U^\perp$, and $T \in \mathcal{A}$, so that $U^\perp$ is invariant under $T^t$ for all $T$ in $\mathcal{A}$. If $\mathcal{A}$ is a $t$-algebra, then it follows that $U^\perp$ is invariant under $\mathcal{A}$. The information that $U$ and $U^\perp$ are both invariant under $\mathcal{A}$ implies that $P_U \in \mathcal{A}'$, as in Lemma 2.10.

As a result, if $\mathcal{A}$ is a $t$-algebra of operators on $V$, then $\mathcal{A}$ is a nice algebra of operators, in the sense of Definition 2.7.

Let $n$ be a positive integer, and define $V^n$, $\mathcal{A}_n$ as in Subsection 2.6. Fix nonzero elements $\lambda_1, \ldots, \lambda_n$ in $k$, and consider

\begin{equation}
(f, h)_n = \sum_{j=1}^{n} \lambda_j (f(j), h(j)), \quad f, h \in V^n.
\end{equation}

In general, this will not be definite, but if it is, then it is easy to see that $\mathcal{A}_n$ is a nice algebra of operators on $V^n$, because it is a $t$-algebra with respect to $(f, h)_n$, as in Corollary 2.52.
4 Irreducibility

**Definition 4.1** Let $V$ be a vector space over a field $k$. An algebra of operators $\mathcal{A}$ on $V$ is said to be irreducible if there are no vector subspaces $W$ of $V$ which are invariant under $\mathcal{A}$ (Definition 2.6) except for $W = \{0\}$ and $W = V$.

**Lemma 4.2** An algebra of operators $\mathcal{A}$ on a vector space $V$ over a field $k$ is irreducible if and only if for every pair of vectors $v, w$ in $V$ with $v \neq 0$ there is an operator $T$ in $\mathcal{A}$ such that $T(v) = w$.

The “if” part is a simple consequence of the definition. For the “only if” part, let $v$ be a nonzero vector in $V$, and consider the set of vectors in $V$ of the form $T(v)$, $T \in \mathcal{A}$. This set of vectors is a vector subspace of $V$ which is invariant under $\mathcal{A}$, because $\mathcal{A}$ is an algebra of operators, and it is nonzero because the identity operator lies in $\mathcal{A}$, so that $v$ lies in the subspace. Thus irreducibility of $\mathcal{A}$ implies that this subspace is all of $V$, which means exactly that for every vector $w$ in $V$ there is an operator $T$ in $\mathcal{A}$ such that $T(v) = w$.

**Lemma 4.3** If $V$ is a vector space, $G$ is a finite group, $\rho$ is a representation of $G$ on $V$, and $\mathcal{A}$ is the algebra of operators on $V$ generated by $\rho_x$ for $x$ in $G$ as in Subsection 2.4, then $\rho$ is an irreducible representation of $G$ if and only if $\mathcal{A}$ is irreducible.

This follows from Lemma 2.21 in Subsection 2.4.

**Proposition 4.4** Let $V$ be a vector space over a field $k$, and let $\mathcal{A}$ be an algebra of operators on $V$. If $\mathcal{A}$ is irreducible, then for each $T$ in the commutant $\mathcal{A}'$, either $T = 0$ or $T$ is invertible as an operator on $V$. In the latter event $T^{-1}$ also lies in $\mathcal{A}'$.

If $T$ lies in $\mathcal{A}'$, then the kernel and image of $T$ are invariant subspaces for $\mathcal{A}$, as in Lemma 2.9. From this it is easy to see that either $T$ is the zero operator or $T$ is invertible when $\mathcal{A}$ is irreducible. Proposition 2.5 implies that $T^{-1}$ lies in $\mathcal{A}'$ when $T$ is invertible, since $\mathcal{A}'$ is an algebra of operators.

**Proposition 4.5** Let $V$ be a vector space over an algebraically closed field $k$. If $\mathcal{A}$ is an irreducible algebra of operators on $V$, then the commutant $\mathcal{A}'$ is equal to the set of multiples of the identity operator by scalars (elements of $k$).
The commutant of any algebra always contains the scalar multiples of the identity, since the identity operator commutes with all other linear mappings, and so it suffices to show that every linear transformation in $\mathcal{A}'$ is a multiple of the identity. This is in essence the same as one-half of the classical Schur’s lemma, which can be proved as follows.

Let $T$ be any linear operator in $\mathcal{A}'$. Because $k$ is algebraically closed, there is at least one nontrivial eigenvalue $\lambda \in k$ of $T$. In other words, there is at least one $\lambda$ in $k$ for which the eigenspace

$$E(T, \lambda) = \{ v \in V : T(v) = \lambda v \} \tag{4.6}$$

contains nonzero vectors. This is because $\lambda$ is an eigenvalue of $T$ if and only if $\det(T - \lambda I) = 0$, and $\det(T - \lambda I)$ is a polynomial in $\lambda$ (of degree equal to the dimension of $V$, which is positive).

If $E(T, \lambda) = V$, then $T = \lambda I$, which is what we want. Thus we suppose that $E(T, \lambda)$ is not all of $V$. It is not hard to check that if $S$ is a linear operator on $V$ which commutes with $T$, then

$$S(E(T, \lambda)) \subseteq E(T, \lambda). \tag{4.7}$$

In particular, this holds for all $S \in \mathcal{A}$, since $T \in \mathcal{A}'$ by hypothesis. Hence $E(T, \lambda)$ is an invariant subspace of $\mathcal{A}$, in contradiction to the assumption that $\mathcal{A}$ is irreducible. We conclude that $E(T, \lambda) = V$.

An alternative approach is to say that if $T$ lies in $\mathcal{A}'$, then $T$ generates a field extension of $k$ of finite degree (since $\mathcal{A}' \subseteq \mathcal{L}(V)$ has finite dimension over $k$), and that this field extension should be equal to $k$ when $k$ is algebraically closed.

Now let us consider the case of a vector space $V$ over the real numbers $\mathbb{R}$. Suppose as before that $\mathcal{A}$ is an algebra of operators on $V$ which is irreducible. One can start with Proposition 4.4 and apply a famous result of Frobenius (as in [Alb]) to conclude that $\mathcal{A}'$ either consists of real multiples of the identity, or that $\mathcal{A}'$ is isomorphic to the complex numbers or to the quaternions, as algebras over the real numbers. Here we shall describe a different (also classical) analysis, under the additional assumption that $V$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ and that $\mathcal{A}$ is a $\ast$-algebra. A broader discussion can be found in [Cur1, Har], which also allows for nonassociative algebras.

**Lemma 4.8** If $T \in \mathcal{A}'$, then $T = T_1 + T_2$, where $T_1^* = T_1$, $T_2^* = -T_2$, and $T_1, T_2 \in \mathcal{A}'$. 

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As usual, one can take \( T_1 = (T + T^*)/2 \) and \( (T - T^*)/2 \) to get \( T_1, T_2 \) so that \( T = T_1 + T_2, T_1^* = T_1, \) and \( T_2^* = -T_2 \). Because \( \mathcal{A}' \) is a \(*\)-algebra, we also have that \( T_1, T_2 \in \mathcal{A}' \) when \( T \in \mathcal{A}_1 \).

**Lemma 4.9** If \( T \in \mathcal{A}' \) is symmetric, \( T^* = T \), then \( T = \alpha I \) for some real number \( \alpha \).

For this we use the fact that \( T \) is diagonalizable, since it is symmetric. We really only need that \( T \) has an eigenvalue, as in the setting of Proposition 4.3. The eigenspace is an invariant subspace for \( \mathcal{A} \), for the same reason as before, and hence the eigenspace is all of \( V \), by the irreducibility of \( V \). This says exactly that \( T \) is a scalar multiple of the identity.

If \( R \) and \( T \) are elements of \( \mathcal{A}' \), then \( (R^* T + T^* R)/2 \) is an element of \( \mathcal{A}' \), since \( \mathcal{A}' \) is a \(*\)-algebra, and it is also symmetric as an operator on \( V \). Hence it is a real number times the identity operator, by Lemma 4.9. Let \( (R, T) \) denote this real number, so that

\[
(4.10) \quad (R^* T + T^* R)/2 = (R, T) I.
\]

It is easy to see that \( (R, T) \) is linear in \( R \) and in \( T \), and symmetric in \( R \) and \( T \).

If \( R = T \), then we can rewrite (4.10) as

\[
(4.11) \quad R^* R = (R, R) I.
\]

If \( v \) is any vector in \( V \), then

\[
(4.12) \quad \langle R(v), R(v) \rangle = \langle (R^* R)(v), v \rangle = (R, R) \langle v, v \rangle.
\]

This implies that \( (R, R) \) is always nonnegative, and that it is equal to 0 exactly when \( R = 0 \). Thus \( (R, T) \) defines an inner product on \( \mathcal{A}' \). If \( (R, R) > 0 \), we also get that \( R \) is invertible.

Next, if \( R_1, R_2 \in \mathcal{A}' \), then

\[
(4.13) \quad (R_1 R_2, R_1 R_2) = (R_1, R_1) (R_2, R_2).
\]

This is because

\[
(4.14) \quad (R_1 R_2)^* (R_1 R_2) = R_2^* R_1^* R_1 R_2 = (R_1, R_1) R_2^* R_2 = (R_1, R_1) (R_2, R_2) I.
\]

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Now let us focus somewhat on antisymmetric operators in $\mathcal{A}'$. If $R \in \mathcal{A}'$ is antisymmetric, then
\begin{equation}
R^2 = -(R, R) I. \tag{4.15}
\end{equation}
If $R_1, R_2 \in \mathcal{A}'$ are antisymmetric and $(R_1, R_2) = 0$, then $R_1^* R_2 + R_2^* R_1 = 0$, which reduces to
\begin{equation}
R_1 R_2 = -R_2 R_1. \tag{4.16}
\end{equation}
In this case we also have that
\begin{equation}
(R_1 R_2)^* = R_2^* R_1^* = R_2 R_1 = -R_1 R_2, \tag{4.17}
\end{equation}
so that $R_1 R_2$ is also antisymmetric.

**Lemma 4.18** Suppose that $R_1$ and $R_2$ are antisymmetric elements of $\mathcal{A}'$ such that $(R_1, R_2) = 0$. Then
\begin{equation}
R_1 (R_1 R_2) = -(R_1 R_2) R_1, \quad R_2 (R_1 R_2) = -(R_1 R_2) R_2, \tag{4.19}
\end{equation}
these elements of $\mathcal{A}'$ are antisymmetric, and
\begin{equation}
(R_1, R_1 R_2) = (R_2, R_1 R_2) = 0. \tag{4.20}
\end{equation}

This is not hard to check.

Assume for the moment that $T_1$ and $T_2$ are nonzero antisymmetric elements of $\mathcal{A}'$ such that $(T_1, T_2) = 0$, and that the set of antisymmetric elements of $\mathcal{A}'$ is not spanned by $T_1$, $T_2$, and $T_1 T_2$. Then there is another nonzero antisymmetric element $U$ of $\mathcal{A}'$ such that
\begin{equation}
(U, T_1) = (U, T_2) = (U, T_1 T_2) = 0. \tag{4.21}
\end{equation}
We know from the observations above that $T_1 T_2$ is antisymmetric, and hence $(U, T_1 T_2) = 0$ implies that
\begin{equation}
U (T_1 T_2) = -(T_1 T_2) U, \tag{4.22}
\end{equation}
as in (4.10). Similarly,
\begin{equation}
U T_1 = -T_1 U \quad \text{and} \quad U T_2 = -T_2 U, \tag{4.23}
\end{equation}
since $(U, T_1) = (U, T_2) = 0$. These two equations imply that
\begin{align}
U (T_1 T_2) &= (U T_1) T_2 = -(T_1 U) T_2 \\
&= -T_1 (U T_2) = -T_1 (-T_2 U) = (T_1 T_2) U. \tag{4.24}
\end{align}
This together with (4.22) lead to

\[ (4.25) \quad U(T_1 T_2) = 0. \]

This is a contradiction, because \( T_1, T_2, \) and \( U \) are assumed to be nonzero elements of \( \mathcal{A}' \), and hence are invertible, so that the product cannot be 0.

The conclusion of this is that if \( T_1 \) and \( T_2 \) are nonzero antisymmetric elements of \( \mathcal{A}' \), then the set of antisymmetric elements of \( \mathcal{A}' \) is spanned by \( T_1, T_2, \) and \( T_3 \).

Thus we obtain three possibilities for \( \mathcal{A}' \). The first is that there are no nonzero antisymmetric elements of \( \mathcal{A}' \). In other words, all elements of \( \mathcal{A}' \) are symmetric, because of Lemma 4.8. From Lemma 4.9 it follows that \( \mathcal{A}' \) consists exactly of multiples of the identity operator by real numbers.

The second possibility is that there is a nonzero antisymmetric element \( J \) of \( \mathcal{A}' \), and that all other antisymmetric elements of \( \mathcal{A}' \) are multiples of \( J \) by real numbers.

The third possibility is that \( \mathcal{A}' \) contains nonzero antisymmetric elements \( T_1, T_2 \) such that \( (T_1, T_2) = 0 \). In this case \( T_1 T_2 \) is also a nonzero antisymmetric element of \( \mathcal{A}' \), \( T_1 T_2 \) is orthogonal to \( T_1 \) and \( T_2 \) relative to the inner product \( (\cdot, \cdot) \) on \( \mathcal{A}' \), and all antisymmetric elements of \( \mathcal{A}' \) are linear combinations of \( T_1, T_2, \) and \( T_1 T_2 \).

These are the only possibilities for \( \mathcal{A}' \). In other words, the dimension of the real vector space of antisymmetric elements of \( \mathcal{A}' \) is either 0, 1, or greater than or equal to 2, and these cases correspond exactly to the possibilities just described. In particular, if the vector space of antisymmetric elements of \( \mathcal{A}' \) has dimension greater than or equal to 2, then the dimension is in fact equal to 3.

In the first situation, where \( \mathcal{A}' \) consists of only multiples of the identity, it follows from Theorem 2.57 that \( \mathcal{A} \) is equal to the algebra \( \mathcal{L}(V) \) of all linear transformations on \( V \). Note that if \( \mathcal{A} = \mathcal{L}(V) \), then \( \mathcal{A}' \) is equal to the set of multiples of the identity operator by real numbers.

In the second situation, there is a nonzero antisymmetric element \( J \) of \( \mathcal{A}' \) such that all other antisymmetric elements of \( \mathcal{A}' \) are equal to real numbers times \( J \). We may as well assume that \( (J, J) = 1 \), since we can multiply \( J \) by a positive real number to obtain this. From (4.15) we obtain that

\[ (4.26) \quad J^2 = -I. \]

We also have that

\[ (4.27) \quad \langle J(v), J(v) \rangle = \langle v, v \rangle \]
for all \( v \in V \), as in (1.12). (Notice that any two of the conditions \( J^* = -J \), (4.24), and (4.27) implies the third.)

In general, if \( J \) is a linear operator on a real vector space that satisfies \( J^2 = -I \), then \( J \) defines a complex structure on the vector space. That is, one can use \( J \) as a definition for multiplication by \( i \) on the vector space (and this choice satisfies the requirements of a complex vector space). A real linear operator on the vector space that commutes with \( J \) is the same as a complex-linear operator on the vector space, with respect to this complex structure. If the vector space is equipped with a (real) inner product, then the additional condition of antisymmetry, or equivalently (4.27), is a compatibility property of the complex structure with the inner product.

Here we have such an operator \( J \), and \( \mathcal{A}' \) is spanned by the identity operator \( I \) and \( J \). By Theorem 2.57, \( \mathcal{A} = \mathcal{A}'' \) is the set of linear operators on \( V \) which commute with \( J \), i.e., which are complex-linear with respect to the complex structure defined by \( J \).

Conversely, one could start with a linear operator \( J \) on \( V \) such that (4.26) holds, and then define \( \mathcal{A} \) to be the set of linear operators on \( V \) which are complex-linear with respect to the complex structure defined by \( J \). It is easy to see that \( \mathcal{A} \) is then an algebra. If \( J \) is compatible with the inner product \( \langle \cdot, \cdot \rangle \), in the sense of (4.27) (or, equivalently, by being antisymmetric), then one can check that \( \mathcal{A} \) is a \( * \)-algebra. Also, \( \mathcal{A}' \) is equal to the span of \( J \) and the identity operator in this case.

The third situation is analogous to the second one, except for having a “quaternionic structure” on \( V \) rather than a complex structure, corresponding to the larger family of operators in \( \mathcal{A}' \). Let us briefly review some aspects of the quaternions.

Just as one might think of a complex number as being a number of the form \( a + bi \), where \( i^2 = -1 \) and \( i \) commutes with all real numbers, one can think of a quaternion as being something which can be expressed in a unique way as

\[
(4.28) \quad a + b i + c j + d k,
\]

where \( a, b, c, \) and \( d \) are real numbers, and \( i, j, \) and \( k \) are special quaternions analogous to \( i \) in the complex numbers. Specifically, \( i, j, \) and \( k \) commute with real numbers and satisfy

\[
(4.29) \quad i^2 = j^2 = k^2 = -1, \quad ij = -ji = k.
\]
As a result of the latter equations, one also has

\[ ik = -ki, \quad jk = -kj. \tag{4.30} \]

If \( x = a + bi + cj + dk \) is a quaternion, then its *conjugate* \( \overline{x} \) is defined by

\[ \overline{x} = a - bi - cj - dk, \tag{4.31} \]

in analogy with complex conjugation. As in the setting of complex numbers, one can check that

\[ x \overline{x} = \overline{x} x = a^2 + b^2 + c^2 + d^2, \tag{4.32} \]

where the right side is always a real number. One defines the “norm” or “modulus” \( |x| \) of \( x \) to be \( \sqrt{x \overline{x}} \). This is the same as the usual Euclidean norm of \( (a, b, c, d) \in \mathbb{R}^4 \). Clearly the conjugate of the conjugate of \( x \) is \( x \) again, and one can verify that

\[ \overline{(xy)} = \overline{y} \overline{x} \tag{4.33} \]

for all quaternions \( x \) and \( y \). As a result,

\[ |xy|^2 = (xy)(\overline{xy}) = xy \overline{y} \overline{x} = |x|^2 |y|^2 \tag{4.34} \]

for all quaternions \( x, y \), i.e., \( |xy| = |x| |y| \).

In the third situation above, \( \mathcal{A}' \) is isomorphic to the quaternions in a natural sense. One can think of the identity operator in \( \mathcal{A}' \) as corresponding to the real number 1 in the quaternions, and in general the real multiples of the identity operator in \( \mathcal{A}' \) correspond to the real numbers inside the quaternions. The antisymmetric operators in \( \mathcal{A}' \) correspond to the “imaginary” quaternions, which are the quaternions of the form \( bi + cj + dk \). There are not necessarily specific counterparts for \( i, j, \) and \( k \) in \( \mathcal{A}' \), but one can use any antisymmetric operators \( R_1, R_2, \) and \( R_3 \) such that \( R_3 = R_1 R_2, \ (R_1, R_1) = (R_2, R_2) = 1, \) and \( (R_1, R_2) = 0 \) (as in the earlier discussion).

This leads to a one-to-one correspondence between \( \mathcal{A}' \) and the quaternions that preserves sums and products. Similarly, the conjugation operation \( x \mapsto \overline{x} \) on the quaternions matches with the adjoint operation \( T \mapsto T^* \) on \( \mathcal{A}' \). The norm on \( \mathcal{A}' \) coming from the inner product \( \langle \cdot, \cdot \rangle \) matches with the norm on quaternions.

In this manner the operators in \( \mathcal{A}' \) define a quaternionic structure on \( V \). That is, a complex structure on a real vector space gives a way to have
complex numbers operate on vectors in the vector space, and now we have a way for the quaternions to operate on a vector space, in a way that extends the usual scalar multiplication by real numbers. This quaternionic structure is also compatible with the inner product on $V$, in the sense that a quaternion with norm 1 corresponds to an operator on $V$ which preserves the inner product there. This uses (4.11).

Theorem 2.57 says that $A = A''$, so that $A$ consists exactly of the real-linear operators on $V$ that commute with the operators that come from the quaternionic structure (i.e., the operators in $A'$ in this case). In other words, $A$ consists exactly of the operators which are “quaternionic-linear” on $V$ with respect to this quaternionic structure coming from $A'$, analogous to the second situation and complex-linear operators with respect to the complex structure on $V$ that arose there.

5 Division algebras of operators

Definition 5.1 Let $\mathcal{B}$ be an algebra of operators on a vector space $V$ over a field $k$. Suppose that for each $T$ in $\mathcal{B}$, either $T = 0$ or $T$ is an invertible operator on $V$ (so that $T^{-1}$ lies in $\mathcal{B}$, by Proposition 2.3). In this case $\mathcal{B}$ is said to be a division algebra of operators.

For the rest of this section we assume that $k$ is a field, $V$ is a vector space over $k$, and $\mathcal{B}$ is a division algebra of operators on $V$.

If $v_1, \ldots, v_m$ are vectors in $V$, then we say that they are $\mathcal{B}$-independent if

\begin{equation}
T_1(v_1) + \cdots + T_m(v_m) = 0
\end{equation}

implies that

\begin{equation}
T_1 = \cdots = T_m = 0
\end{equation}

for all $T_1, \ldots, T_m$ in $\mathcal{B}$. The $\mathcal{B}$-span of $v_1, \ldots, v_m$ is defined to be the subspace of $V$ consisting of vectors of the form

\begin{equation}
T_1(v_1) + \cdots + T_m(v_m),
\end{equation}

where $T_1, \ldots, T_m$ lie in $\mathcal{B}$. Thus $\mathcal{B}$-independence of $v_1, \ldots, v_m$ is equivalent to saying that if a vector can be represented as (5.4) with $T_1, \ldots, T_m \in \mathcal{B}$, then this representation is unique.

Note that a single nonzero vector in $V$ is $\mathcal{B}$-independent. This uses the assumption that an operator in $\mathcal{B}$ is either 0 or invertible.
Lemma 5.5 Assume that $v_1, \ldots, v_m$ are $\mathcal{B}$-independent vectors in $V$. If $w$ is a vector in $V$ that does not lie in the $\mathcal{B}$-span of $v_1, \ldots, v_m$, then the vectors $v_1, \ldots, v_m, w$ are $\mathcal{B}$-independent.

Indeed, suppose that $T_1, \ldots, T_m, U$ are operators in $\mathcal{B}$ such that

\begin{equation}
T_1(v_1) + \cdots + T_m(v_m) + U(w) = 0. \tag{5.6}
\end{equation}

We would like to show that $T_1 = \cdots = T_m = U = 0$. If $U = 0$, then this reduces to the $\mathcal{B}$-independence of $v_1, \ldots, v_m$. If $U \neq 0$, then $U$ is invertible, by our assumptions on $\mathcal{B}$, and we can rewrite (5.6) as

\begin{equation}
w = -(U^{-1} \circ T_1)(v_1) - \cdots - (U^{-1} \circ T_m)(v_m). \tag{5.7}
\end{equation}

This implies that $w$ is in the $\mathcal{B}$-span of $v_1, \ldots, v_m$, contrary to hypothesis, and Lemma 5.5 follows.

Corollary 5.8 There are vectors $u_1, \ldots, u_\ell$ in $V$ which are $\mathcal{B}$-independent and whose $\mathcal{B}$-span is equal to $V$.

To see this, one can start with a single nonzero vector $u_1$ in $V$. If the $\mathcal{B}$-span of this vector is all of $V$, then we stop. Otherwise, we choose a vector $u_2$ which is not in the $\mathcal{B}$-span of $u_1$. Lemma 5.5 implies that $u_1, u_2$ are $\mathcal{B}$-independent. If the $\mathcal{B}$-span of $u_1, u_2$ is all of $V$, then we stop, and otherwise we choose $u_3$ to be a vector in $V$ not in the $\mathcal{B}$-span of $u_1, u_2$. Proceeding in this manner, one eventually gets a collection of vectors $u_1, \ldots, u_\ell$ in $V$ as in the corollary.

Corollary 5.9 The dimension of $V$ is divisible by the dimension of $\mathcal{B}$ (as vector spaces over $k$).

Specifically, the dimension of $V$ is equal to $\ell$ times the dimension of $\mathcal{B}$, if $\ell$ is as in the previous corollary.

Corollary 5.10 Suppose that $W$ is a nonzero vector subspace of $V$ which is invariant under $\mathcal{B}$. There are vectors $u_1, \ldots, u_\ell$ in $V$ and a positive integer $m$ such that $u_1, \ldots, u_\ell$ are $\mathcal{B}$-independent, the $\mathcal{B}$-span of $u_1, \ldots, u_\ell$ is equal to $V$, $u_j$ lies in $W$ exactly when $j \leq m$, and $W$ is equal to the $\mathcal{B}$-span of the $u_j$’s, $j \leq m$. 

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This can be established in the same manner as the Corollary 5.8. The main difference is that in the beginning of the construction one chooses the $u_j$'s in $W$, for as long as one can. One stops doing this exactly when there are enough $u_j$'s so that their $B$-span is $W$. One then continues as before, with additional $u_j$'s as needed so that the $B$-span of the total collection of vectors is $V$.

**Corollary 5.11** If $W$ is a nonzero vector subspace of $V$ which is invariant under $B$ and which is not equal to $V$, then there is a vector subspace $Z$ of $V$ which is invariant under $B$ and which is a complement of $W$ (so that $W \cap Z = \{0\}$ and $W + Z = V$).

For this one can take $Z$ to be the $B$-span of $u_{m+1}, \ldots, u_\ell$, where $u_1, \ldots, u_\ell$ is as in Corollary 5.10.

Now let us consider $B'$. Let $u_1, \ldots, u_\ell$ be vectors in $V$ which are $B$-independent and whose $B$-span is equal to $V$, as in Corollary 5.8. If $R \in B'$, then $R$ is determined uniquely by $R(u_1), \ldots, R(u_\ell)$. For if $T \in B$, then $R(T(u_j)) = T(R(u_j))$ for each $j$, and this and linearity determine $R$ on all of $V$.

Conversely, if $w_1, \ldots, w_\ell$ are arbitrary vectors in $V$, then there is an $R \in B'$ so that $R(u_j) = w_j$ for each $j$. One can simply define $R$ on $V$ by

$$R(T_1(u_1) + \cdots + T_\ell(u_\ell)) = T_1(w_1) + \cdots + T_\ell(w_\ell),$$

for all $T_1, \ldots, T_\ell \in B$. It is easy to check that this gives a linear operator $R$ on $V$ which lies in $B'$.

**Lemma 5.13** The dimension of $B'$ is equal to the square of the dimension of $V$ divided by the dimension of $B$, as vector spaces over $k$.

This follows from the preceding description of $B'$.

**Proposition 5.14** $B'' = B$.

This can be verified directly, using our characterization of $B'$. Let us mention another argument, in terms of the methods of Section 2. It is enough to show that $B$ is a very nice algebra of operators, by Theorem 2.57 in Subsection 2.7. Using Lemma 2.10 in Subsection 2.2 and Corollary 5.11 one obtains that $B$ is a nice algebra of operators (Definition 2.7). To say that $B$
is very nice means (Definition 2.56) that the expanded algebras \( B_n \), defined in Subsection 2.6, are all nice. By construction, the expanded algebras \( B_n \) are all division algebras of operators, and hence they are nice for the same reason that \( B \) is.

Next we discuss some analogues of the results in Subsection 2.8. Let us make the standing assumption for the rest of the section that

\[
(5.15) \quad \mathcal{A} \text{ is an algebra of operators on } V \text{ such that } \mathcal{A} \subseteq B'.
\]

**Lemma 5.16** If the dimension of \( V \) is equal to the dimension of \( B \), as vector spaces over \( k \), then \( \mathcal{A} = B' \) if and only if \( \mathcal{A} \) is irreducible.

The assumption that the dimension of \( V \) is equal to the dimension of \( B \) implies that \( V \) is equal to the \( B \)-span of any nonzero vector in \( V \). Using this and the reformulation of irreducibility in Lemma 4.2, it is easy to check that \( \mathcal{A} = B' \) when \( \mathcal{A} \) is irreducible. For the converse, notice that \( B' \) is always irreducible, without any restriction on the dimension of \( V \).

**Lemma 5.17** If \( \mathcal{A} \) is irreducible and there is a nonzero operator in \( \mathcal{A} \) whose image is contained in the \( B \)-span of a single vector, then \( \mathcal{A} = B' \).

The assumptions in this lemma are also necessary for \( \mathcal{A} \) to be equal to \( B' \).

For each nonzero vector \( u \) in \( V \), consider the vector space of linear operators on \( V \) which lie in \( B' \) and have image contained in the \( B \)-span of \( u \). We would like to show that this vector space of linear operators is contained in \( \mathcal{A} \). If we can do this for any nonzero \( u \) in \( V \), then it follows that \( \mathcal{A} = B' \).

Fix \( u \neq 0 \) in \( V \). Observe first that there is a nonzero linear operator on \( V \) which lies in \( \mathcal{A} \) and has image contained in the \( B \)-span of \( u \). The hypothesis of the lemma says that this is true for at least one \( u_1 \), and one can get any other \( u \) by composing with an operator in \( \mathcal{A} \) that sends \( u_1 \) to \( u \). Such an operator exists, by irreducibility.

As a vector space over \( k \), the collection of operators in \( B' \) whose image is contained in the \( B \)-span of \( u \) has dimension equal to the dimension of \( V \). This can be derived from the earlier description of \( B' \). Thus it is enough to show that the collection of operators in \( \mathcal{A} \) with image contained in the \( B \)-span of \( u \) has dimension which is at least the dimension of \( v \). Let \( R \) be a nonzero operator in \( \mathcal{A} \) whose image is contained in, and hence equal to, the \( B \)-span of \( u \), and let \( z \) be a vector in \( V \) such that \( R(z) = u \). For each vector
there is a linear operator $T$ in $\mathcal{A}$ such that $T(v) = z$. For this $T$ we have that $R \circ T$ lies in $\mathcal{A}$, the image of $R \circ T$ is contained in the $\mathcal{B}$-span of $u$, and $(R \circ T)(v) = u$. Because we can do this for each $v$ in $V$, it is not hard to see that the dimension of the collection of linear operators in $\mathcal{A}$ with image contained in the $\mathcal{B}$-span of $u$ has dimension at least the dimension of $V$, as desired.

**Lemma 5.18** Suppose that $\mathcal{A}$ is irreducible, and that for every four vectors $v_1, v_2, w_1, w_2$ in $V$ such that $v_1, v_2$ are $\mathcal{B}$-independent there is a $T$ in $\mathcal{A}$ which satisfies $T(v_1) = w_1, T(v_2) = w_2$. Then $\mathcal{A} = \mathcal{B}'$.

The assumption of irreducibility of $\mathcal{A}$ follows from the second condition when the dimension of $V$ is larger than the dimension of $\mathcal{B}$. When the two dimensions are equal, the second condition is vacuous, and one could just as well apply Lemma 5.14.

To prove Lemma 5.18, it suffices to show that there is a nonzero operator in $\mathcal{A}$ whose image is contained in the $\mathcal{B}$-span of a single vector, by Lemma 5.17. Let $R$ be any nonzero operator in $\mathcal{A}$. If the image of $R$ is not contained in the $\mathcal{B}$-span of a single vector, then there exist $x_1, x_2$ in $V$ such that $R(x_1), R(x_2)$ are $\mathcal{B}$-independent. By hypothesis, there is an operator $T$ in $\mathcal{A}$ such that $T(R(x_1)) \neq 0$ and $T(R(x_2)) = 0$. Thus $T \circ R$ lies in $\mathcal{A}$, is nonzero, and the dimension of its image strictly less than that of $R$. If the image of $T \circ R$ is contained in the $\mathcal{B}$-span of a single vector, then we are finished, and otherwise we can repeat this process until such an operator is obtained. This proves the lemma.

**Lemma 5.19** Suppose that $\mathcal{A}' = \mathcal{B}$ and $\mathcal{A}_n \subseteq \mathcal{L}(V^n)$ (defined in Subsection 2.4) is a nice algebra of operators when $n = 2$. Then $\mathcal{A} = \mathcal{B}' = \mathcal{A}'$.

The hypotheses of this lemma imply those of Lemma 5.18.

6 Group representations and their algebras

Throughout this section, $G$ will be a finite group, and $k$ will be a field.

**Proposition 6.1** Let $V_1$ and $V_2$ be vector spaces over $k$, and let $\rho^1$, $\rho^2$ be representations of $G$ on $V_1$, $V_2$. Suppose that $T : V_1 \to V_2$ is a linear mapping which intertwines the representations $\rho^1$, $\rho^2$, in the sense that

$$T \circ \rho^1_x = \rho^2_x \circ T$$

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for all $x$ in $G$. If the representations $\rho^1$, $\rho^2$ are irreducible, then either $T$ is 0, or $T$ is a one-to-one mapping from $V_1$ onto $V_2$, and the representations $\rho^1$, $\rho^2$ are isomorphic to each other.

This is one half of Schur’s lemma. (The other half came up in Section 3.)

Let $V_1$, $V_2$, etc., be as in the statement above. Consider the kernel of $T$, which is a vector subspace of $V_1$. From the intertwining property it is easy to see that the kernel of $T$ is invariant under the representation $\rho^1$. The irreducibility of $\rho^1$ then implies that the kernel is either all of $V_1$ or the subspace consisting of only the zero vector. In the first case $T = 0$, and we are finished. In the second case we get that $T$ is one-to-one.

Now consider the image of $T$ in $V_2$. This is a vector subspace that is invariant under $\rho^2$, because of the intertwining property. The irreducibility of $\rho^2$ implies that the image of $T$ is either the subspace of $V_2$ consisting of only the zero vector or all of $V_2$. This is the same as saying that $T$ is either equal to 0 or it maps $V_1$ onto $V_2$. Combining this with the conclusion of the preceding paragraph, we obtain that $T$ is either equal to 0, or that it is one-to-one and maps $V_1$ onto $V_2$. This proves the proposition.

For the rest of this section, let us assume that

\begin{equation}
\tag{6.3}
\text{either the characteristic of } k \text{ is 0, or it is positive and does not divide the number of elements of the group } G.
\end{equation}

Let $V$ be a vector space over $k$, and let $\rho$ be a representation of $G$ on $V$. As in Lemma 1.43 in Subsection 1.3, there is an independent system of subspaces $W_1, \ldots, W_h$ of $V$ such that the span of the $W_j$’s is equal to $V$, each $W_j$ is invariant under $\rho$, and the restriction of $\rho$ to each $W_j$ is irreducible. Let us make the following additional assumption:

\begin{equation}
\tag{6.4}
\text{for each } j, l \text{ such that } 1 \leq j, l \leq h \text{ and } j \neq l, \text{ the restriction of } \rho \text{ to } W_j \text{ is not isomorphic to the restriction of } \rho \text{ to } W_l, \text{ as a representation of } G.
\end{equation}

**Notation 6.5** Suppose that $\mathcal{A}$ is an algebra of operators on a vector space $V$. If $U$ is a vector subspace of $V$ which is invariant under $\mathcal{A}$ (Definition 2.6), then we let $\mathcal{A}(U)$ denote the set of operators on $U$ which are the restrictions to $U$ of the operators in $\mathcal{A}$. It is easy to see that $\mathcal{A}(U)$ is an algebra of operators on $U$. The commutant and double commutant of $\mathcal{A}(U)$ in $\mathcal{L}(U)$ are denoted $\mathcal{A}(U)'$ and $\mathcal{A}(U)''$. 

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In our present circumstances, we let $\mathcal{A}$ denote the algebra of operators on $V$ generated by $\rho$ as in Subsection 2.4. A subspace $U$ of $V$ is invariant under $\mathcal{A}$ if and only if it is invariant under $\rho$, and in this case $\mathcal{A}(U)$ is the same as the algebra generated by the restrictions of the $\rho_x$’s to $U$, $x$ in $G$.

**Theorem 6.6** Under the preceding conditions, $\mathcal{A}$ consists of the linear operators $T$ on $V$ such that $T(W_j) \subseteq W_j$ and the restriction of $T$ to $W_j$ lies in $\mathcal{A}(W_j)$ for each $j$, $1 \leq j \leq h$.

It is clear from the definitions that each $T \in \mathcal{A}$ satisfies the properties described in the theorem. The key point is that in the converse, the restrictions of $T$ to the various $W_j$’s can be chosen independently of each other. To prove this, we use Theorem 2.57.

Because $W_1, \ldots, W_h$ is an independent system of subspaces of $V$ which spans $V$, there is for each $j = 1, \ldots, h$ a natural linear operator $P_j$ on $V$ which projects $V$ onto $W_j$. Specifically, $P_j$ is the linear operator which satisfies $P_j(u) = u$ when $u \in W_j$ and $P_j(z) = 0$ when $z \in W_l$, $l \neq j$. Thus $P_j \circ P_l = 0$ when $j \neq l$, and $\sum_{j=1}^h P_j = I$, the identity operator on $V$. We also have that each $P_j$ lies in $\mathcal{A}'$, because the $W_j$’s are invariant subspaces.

**Lemma 6.7** If $S \in \mathcal{A}'$ and $1 \leq j, l \leq h$, $j \neq l$, then $P_l \circ S \circ P_j = 0$.

Let $S$, $j$, and $l$ be given as in the lemma, and set $S_{j,l} = P_l \circ S \circ P_j$. Note that $S_{j,l}$ lies in $\mathcal{A}'$, since $S$, $P_j$, and $P_l$ do. Thus

$$S_{j,l} \circ \rho_x = \rho_x \circ S_{j,l} \quad (6.8)$$

for all $x$ in $G$ (and this is equivalent to the statement that $S_{j,l} \in \mathcal{A}'$, since $\mathcal{A}$ is generated by the $\rho_x$’s).

Let $R_{j,l}$ be the linear mapping from $W_j$ to $W_l$ which is the restriction of $S_{j,l}$ to $W_j$. Thus $R_{j,l}$ is essentially the same as $S_{j,l}$, except that it is formally viewed as a mapping from $W_j$ to $W_l$ rather than as an operator on $V$. On $W_j$ and $W_l$ we have representations of $G$, namely, the restrictions of $\rho$ to $W_j$ and $W_l$. By hypothesis, these representations are irreducible and not isomorphically equivalent. The mapping $R_{j,l}$ intertwines these representations, by (6.8). Therefore, $R_{j,l} = 0$, by Proposition 6.1. This is equivalent to $S_{j,l} = 0$, and the lemma follows.

**Lemma 6.9** A linear operator $S$ on $V$ lies in $\mathcal{A}'$ if and only if $S(W_j) \subseteq W_j$ and the restriction of $S$ to $W_j$ lies in $\mathcal{A}(W_j)'$ for each $j$, $1 \leq j \leq h$. 49
If $S \in \mathcal{A}'$, then $S(W_j) \subseteq W_j$ for each $j$ by Lemma 6.7. To say that $S \in \mathcal{A}'$ means that $Z \circ S = S \circ Z$ for all $Z \in \mathcal{A}$, or

\begin{equation}
(Z \circ S)(v) = (S \circ Z)(v)
\end{equation}

for all $Z \in \mathcal{A}$ and all vectors $v$ in $V$. Because $V$ is spanned by $W_1, \ldots, W_h$, (6.10) holds for all $v$ in $V$ if and only if it holds for all $v$ in $W_j$ for each $j = 1, \ldots, h$. If $v \in W_j$, then $S(v) \in W_j$ and $Z(v) \in W_j$ for all $Z \in \mathcal{A}$. Let $S_j$ and $Z_j$ denote the restrictions of $S$ and $Z$ to $W_j$. Then we get that (6.10) holds for all $v$ in $V$ and all $Z$ in $\mathcal{A}$ if and only if

\begin{equation}
(Z_j \circ S_j)(v) = (S_j \circ Z_j)(v)
\end{equation}

for all $v$ in $W_j$, all $Z$ in $\mathcal{A}$, and all $j = 1, \ldots, h$. This is the same as saying that for every $j$ the restriction of $S$ to $W_j$ commutes with the restriction of every element of $\mathcal{A}$ to $W_j$, or, equivalently, with every element of $\mathcal{A}(W_j)$. This completes the proof of the lemma.

**Lemma 6.12** A linear operator $T$ on $V$ lies in $\mathcal{A}''$ if and only if $T(W_j) \subseteq W_j$ and the restriction of $T$ to $W_j$ lies in $\mathcal{A}(W_j)''$ for each $j$.

We know that each $W_j$ is an invariant subspace of $T \in \mathcal{A}''$ because each $P_j$ lies in $\mathcal{A}'$. This is essentially the same as applying Lemma 2.4 in Subsection 2.2. One can finish the argument in essentially the same way as for Lemma 6.3.

Theorem 6.6 can be derived from Lemma 6.12. Alternatively, it is enough to show that the operators $T$ on $V$ which satisfy the conditions in Theorem 6.6 should lie in $\mathcal{A}$, and for this it is enough to show that they commute with every element of $\mathcal{A}'$ (so that they then lie in $\mathcal{A}''$, which is equal to $\mathcal{A}$ by the Theorem 2.57 in Subsection 2.7). This is an easy consequence of Lemma 6.9.

### 7 Basic facts about decompositions

Throughout this section $G$ is a finite group, and $k$ is a field which is assumed to have characteristic 0 or positive characteristic which does not divide the number of elements in $G$.

Suppose that $\rho$ is a representation of $G$ on a vector space $V$ over $k$. As in Lemma 1.43 in Subsection 1.3, there is an independent system of subspaces
$W_1, \ldots, W_h$ of $V$ such that $V = \text{span}(W_1, \ldots, W_h)$, each $W_j$ is invariant under $\rho$, and the restriction of $\rho$ to each $W_j$ is irreducible. We do not ask that the restrictions of $\rho$ to the $W_j$’s be isomorphically distinct.

**Lemma 7.1** In addition to the preceding conditions, assume that $U$ is a nonzero vector subspace of $V$ which is invariant under $\rho$ and for which the restriction of $\rho$ to $U$ is irreducible. Let $I$ denote the set of integers $j$, $1 \leq j \leq h$, such that the restriction of $\rho$ to $W_j$ is isomorphic to the restriction of $\rho$ to $U$. Then $I$ is not empty, and

$$U \subseteq \text{span}\{W_j : j \in I\}. \tag{7.2}$$

For each $j = 1, \ldots, h$, let $P_j : V \to V$ be the projection onto $W_j$ that comes naturally from the independent system of subspaces $W_1, \ldots, W_h$, i.e., $P_j(u) = u$ when $u \in W_j$ and $P_j(z) = 0$ when $z \in W_l, l \neq j$. Thus $\sum_{j=1}^h P_j = I$ and each $P_j$ commutes with the operators $\rho_x, x$ in $G$, from the representation, because the $W_j$’s are invariant under the representation.

Define $T_j : U \to W_j$ for $j = 1, \ldots, h$ to be the restriction of $P_j$ to $U$, now viewed as a mapping into $W_j$. Notice that $T_j$ intertwines the representations on $U, W_j$ which are the restrictions of $\rho$ to $U, W_j$, since $P_j$ commutes with the operators $\rho_x, x$ in $G$. Because the restrictions of $\rho$ to $U, W_j$ are irreducible, we have that each $T_j$ is either equal to 0 or is a one-to-one linear mapping onto $W_j$, by Proposition [6.1].

If $J$ denotes the set of $j$’s such that $T_j \neq 0$, then

$$U \subseteq \text{span}\{W_j : j \in J\}, \tag{7.3}$$

simply because $P_j$ is equal to 0 on $U$ for $j \notin J$. In particular, $J$ is not equal to the empty set, since $U \neq \{0\}$. We also have that $J \subseteq I$, since $T_j : U \to W_j$ defines an isomorphism between the restriction of $\rho$ to $U$ and the restriction of $\rho$ to $W_j$ when $j \in J$. This proves Lemma [7.1].

Now suppose that $Y_1, \ldots, Y_m$ are nonzero subspaces of $V$ with properties analogous to those of $W_1, \ldots, W_h$, namely, $Y_1, \ldots, Y_m$ is an independent system of subspaces of $V$ such that $V = \text{span}(Y_1, \ldots, Y_m)$, each $Y_j$ is invariant under $\rho$, and the restriction of $\rho$ to each $Y_j$ is irreducible. In general it is not true that the $Y_j$’s have to be the same subspaces of $V$ as the $W_l$’s, after a permutation of the indices. For instance, if $\rho$ is the trivial representation, so that each $\rho_x$ is equal to the identity mapping on $V$, then any subspace of $V$ is invariant under $\rho$, and any 1-dimensional subspace has the property
that the restriction of $\rho$ to it is irreducible. If the dimension of $V$ is strictly larger than 1, then there are numerous ways in which to decompose $V$ into 1-dimensional subspaces.

However, it is true that the two decompositions of $V$ have to be nearly the same in some respects. Let $J$ be a nonempty subset of $\{1, \ldots, h\}$ such that the restriction of $\rho$ to $W_j$ is isomorphic to the restriction of $\rho$ to $W_l$ when $j, l \in J$, and such that the restriction of $\rho$ to $W_j$ is not isomorphic to the restriction of $\rho$ to $W_l$ when $j \in J$ and $l \not\in J$. Let $L$ be the set of $l \in \{1, \ldots, m\}$ such that the restriction of $\rho$ to $W_j$ is isomorphic to the restriction of $\rho$ to $Y_l$ when $j \in J$. Then

(7.4) \[ \text{span}\{W_j : j \in J\} = \text{span}\{Y_l : l \in L\}. \]

This is because $Y_l \subseteq \text{span}\{W_j : j \in J\}$ when $l \in L$, by Lemma 7.1, and similarly $W_j \subseteq \text{span}\{Y_l : l \in L\}$ when $j \in J$.

Because of the isomorphisms, the $W_j$'s for $j \in J$ and the $Y_l$'s for $l \in L$ all have the same dimension. We also have that

(7.5) \[ \dim \text{span}\{W_j : j \in J\} = \sum_{j \in J} \dim W_j \]

and

(7.6) \[ \dim \text{span}\{Y_l : l \in L\} = \sum_{l \in L} \dim Y_l, \]

since the $W_j$'s and $Y_l$'s form independent systems of subspaces. Using (7.4), we obtain that $J$ and $L$ have the same number of elements.

We can do this for all subsets $J$ of $\{1, \ldots, h\}$ with the properties described above, and these subsets exhaust all of $\{1, \ldots, h\}$. The corresponding sets $L \subseteq \{1, \ldots, m\}$ exhaust all of $\{1, \ldots, m\}$ as well, because of Lemma 7.1. To summarize, the same irreducible representations of $G$ occur as restrictions of $\rho$ to the $W_j$'s as occur as restrictions of $\rho$ to the $Y_l$'s, up to isomorphic equivalence, and they occur the same number of times. In particular, we get that $h = m$. Although the various subspaces do not have to match up exactly, we do have (7.4).

Now let $Z$ be a vector space over $k$, and $\sigma$ be an irreducible representation of $G$ on $Z$. Let $F(G)$ denote the vector space of $k$-valued functions on $G$.

**Lemma 7.7** Suppose that $\lambda$ is a nonzero linear functional on $Z$, i.e., a nonzero linear mapping from $Z$ into $k$. For each $v$ in $Z$, consider the function $f_v(y)$ on $G$ defined by

(7.8) \[ f_v(y) = \lambda(\sigma_{y^{-1}}(v)). \]
Define $U \subseteq F(G)$ by
\begin{equation}
U = \{ f_v(y) : v \in Z \}.
\end{equation}

Then $U$ is a vector subspace of $F(G)$ which is invariant under the left regular representation (Subsection 1.3), and the mapping $v \mapsto f_v$ is a one-to-one linear mapping from $Z$ onto $U$ which intertwines the representations $\sigma$ on $Z$ and the restriction of the left regular representation to $U$. In particular, these two representations are isomorphic.

Clearly $v \mapsto f_v$ is a linear mapping from $Z$ onto $U$, so that $U$ is a vector subspace of $F(G)$. Let us check that the kernel of this mapping is the zero subspace of $Z$.

Suppose that $v$ is a vector in $Z$ such that $f_v(y)$ is the zero function on $G$. This is the same as saying that $\lambda(\sigma_y^{-1}(v)) = 0$ for all $y$ in $G$. Let $W$ be the subspace of $Z$ which is spanned by $\sigma_y(v)$, $y \in G$. Then $W$ is invariant under the representation $\sigma$, because of the way that it is defined. If $v \neq 0$, then $W$ is not the zero subspace, and indeed $v$ lies in $W$ since $\sigma_e(v) = v$. The irreducibility of $\sigma$ then implies that $W = Z$. On the other hand, $\lambda$ is equal to 0 on $W$, and we are assuming that $\lambda$ is a nonzero linear functional on $Z$. Thus we conclude that $v = 0$. This shows that the kernel of $v \mapsto f_v$ is the zero subspace of $Z$, and hence that this mapping is one-to-one.

Let $L_x$, $x \in G$, denote the left regular representation of $G$, as in Subsection 1.3. For each $x$ in $G$ and $v$ in $Z$ we have that
\begin{equation}
L_x(f_v)(y) = f_v(x^{-1}y) = \lambda(\sigma_y^{-1}(v)) = \lambda(\sigma_y^{-1}(\sigma_x(v))) = f_{\sigma_x(v)}(y).
\end{equation}

This says exactly that the mapping $v \mapsto f_v$ intertwines the representations $\sigma$ on $Z$ and the left regular representation on $F(G)$. In particular, $U$ is invariant under the left regular representation. Lemma 7.7 follows easily from these observations.

**Remark 7.11** Lemma 7.7 and its proof work for any field $k$, without any assumption on the characteristic of $k$.

**Proposition 7.12** Suppose that for each $x$ in $G$ an element $a_x$ of the field $k$ is chosen, where $a_x \neq 0$ for at least one $x$. Then there is a vector space $Y$ over $k$ and an irreducible representation $\tau$ on $Y$ such that the linear operator
\begin{equation}
\sum_{x \in G} a_x \tau_x
\end{equation}
on $Y$ is not the zero operator.

Suppose first that we do not worry about having $\tau$ be an irreducible representation. Then we can use the left regular representation on the space of $k$-valued functions on $G$. The operator in question is

\[
\sum_{x \in G} a_x L_x.
\] (7.14)

Let $\phi_e(z)$ denote the function on $G$ defined by $\phi_e(z) = 1$ when $z = e$ and $\phi_e(z) = 0$ when $z \neq e$. Then

\[
\sum_{x \in G} a_x L_x(\phi_e)(w) = a_w
\] (7.15)

for all $w$ in $G$. The assumption that the $a_x$'s are not all 0 says exactly that this is not the zero function on $G$, and hence that (7.14) is not the zero operator on the space of functions on $G$.

From Lemma 1.43 in Subsection 1.5 we know that the vector space of $k$-valued functions on $G$ is spanned by subspaces which are invariant under the left regular representation, and to which the restriction of the left regular representation is irreducible. It follows easily that the restriction of the operator (7.14) to at least one of these subspaces is nonzero, since it is not the zero operator on the whole space of functions. To get Proposition 7.12, we take $\tau$ to be the restriction of the left regular representation to such a subspace.

Now let $V$ be a vector space over $k$, and let $\rho$ be a representation of $G$ on $V$ of the following special type (which is unique up to isomorphism). We assume that $V$ has an independent system of subspaces $W_1, \ldots, W_m$ such that $V = \text{span}(W_1, \ldots, W_m)$, each $W_j$ is invariant under $\rho$, the restriction of $\rho$ to each $W_j$ is an irreducible representation of $G$, the restriction of $\rho$ to $W_j$ is not isomorphic to the restriction of $\rho$ to $W_l$ when $j \neq l$, and every irreducible representation of $G$ on a vector space over $k$ is isomorphic to the restriction of $\rho$ to some $W_j$, $1 \leq j \leq m$. In other words, this representation is isomorphic to a direct sum of irreducible representations, in which every irreducible representation of $G$ on a vector space over $k$ appears exactly once, up to isomorphism. One can also think in terms of starting with the left regular representation of $G$ (over $k$) and passing to a suitable invariant subspace, to avoid repetitions of irreducible representations (up to isomorphism).

Let $A$ denote the set of linear operators on $V$ associated to the representation $\rho$, as in Subsection 2.4.
Lemma 7.16 A linear operator $T \in \mathcal{L}(V)$ lies in $\mathcal{A}$ if and only if it can be written as

\begin{equation}
T = \sum_{x \in G} a_x \rho_x,
\end{equation}

where each $a_x$ lies in $k$. Each operator $T \in \mathcal{A}$ can be written as (7.17) in a unique way. In particular, the dimension of $\mathcal{A}$ as a vector space is equal to the number of elements of $G$.

The first statement is a rephrasal of the definition of $\mathcal{A}$ given at the beginning of Subsection 2.4. The uniqueness of the expression for $T$ in (7.17) follows from Proposition 7.12, since all irreducible representations of $G$ occur as restrictions of $\rho$ to a subspace of $V$. The last assertion in the lemma is an immediate consequence of the first two.

Because the restrictions of $\rho$ to the $W_j$'s are isomorphically distinct, Theorem 6.6 leads to rather precise information about how $\mathcal{A}$ looks in this situation.

If the field $k$ is algebraically closed, such as the field $\mathbb{C}$ of complex numbers, then the algebra of operators generated by an irreducible representation is the algebra of all linear operators on the corresponding vector space. For this we use Proposition 4.5 to say the commutant of the algebra consists of only the scalar multiples of the identity, and then Theorem 2.57 in Subsection 2.7. It follows that the number of elements of $G$ is equal to the sum of the squares of the degrees of the isomorphically-distinct irreducible representations of $G$ in this case. Note that there is always an irreducible representation of degree 1, namely the representation of $G$ on a vector space of dimension 1 in which each $x \in G$ is associated to the identity operator.

Let us look now at the center of $\mathcal{A}$, which is the set of operators in $\mathcal{A}$ which commute with all other elements of $\mathcal{A}$.

Lemma 7.18 An operator $T = \sum_{x \in G} a_x \rho_x$ in $\mathcal{A}$ lies in the center of $\mathcal{A}$ if and only if $a_x = a_y$ whenever $x$ and $y$ are conjugate inside the group $G$, i.e., whenever there is a $w \in G$ such that $y = wxw^{-1}$.

Indeed, $T$ commutes with all elements of $\mathcal{A}$ if and only if $T$ commutes with $\rho_z$ for all $z$ in the group. This is the same as saying that

\begin{equation}
\rho_z^{-1} \left( \sum_{x \in G} a_x \rho_x \right) \rho_z = \sum_{x \in G} a_x x
\end{equation}
for all $z$ in $G$. This can be rewritten as

\begin{equation}
\sum_{x \in G} a_x \rho_{z^{-1}xz} = \sum_{x \in G} a_x \rho_x,
\end{equation}

or as

\begin{equation}
\sum_{x \in G} a_{zzz^{-1}} \rho_x = \sum_{x \in G} a_x \rho_x.
\end{equation}

In other words, the mapping from $x$ to $zxz^{-1}$ permutes the elements of the group, and in the last step we made a change of variables in the sum using this permutation. We conclude that $T$ lies in the center of $\mathcal{A}$ if and only if $a_{zzz^{-1}} = a_z$ for all $x$ and $z$ in the group. This proves the lemma.

If we write $x \sim y$ when $x$ and $y$ are conjugate elements of the group, then this defines an equivalence relation on the group, as is well-known and easy to verify. The equivalence classes for this relation are called conjugacy classes. The lemma can be rephrased as saying that $T = \sum_{x \in G} a_x \rho_x$ lies in the center of $\mathcal{A}$ if and only if the coefficients $a_x$ are constant on the conjugacy classes of $G$.

Note that the center of $\mathcal{A}$ is automatically a subalgebra, and a vector subspace in particular.

**Corollary 7.22** The dimension of the center of $\mathcal{A}$ is equal to the number of conjugacy classes in the group.

This is an easy consequence of the previous remarks.

If the field $k$ is algebraically closed, then it follows that the number of conjugacy classes in the group is equal to the number of isomorphically-distinct irreducible representations of the group. This uses Proposition 4.5.

### 8 p-adic numbers

Let $\mathbb{Q}$ denote the field of rational numbers, and fix a prime number $p$.

The $p$-adic absolute value $| \cdot |_p$ on $\mathbb{Q}$ is defined as follows. If $x = \frac{a}{b}p^k$, where $a$, $b$, and $k$ are integers, with $a, b \neq 0$ and neither $a$ nor $b$ divisible by $p$, then we set

\begin{equation}
|x|_p = p^{-k}.
\end{equation}

If $x = 0$, then we set $|x|_p = 0$. 

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The $p$-adic absolute value $|\cdot|_p$ enjoys many of the same features as the classical absolute value $|\cdot|$, defined by $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. In particular,

(8.2) \[ |x + y|_p \leq |x|_p + |y|_p \]

and

(8.3) \[ |xy|_p = |x|_p |y|_p \]

for all $x, y \in \mathbb{Q}$. In fact, instead of (8.2) there is a stronger “ultrametric” version of the triangle inequality, which states that

(8.4) \[ |x + y|_p \leq \max(|x|_p, |y|_p) \]

whose analogue for the standard absolute value $|\cdot|$ is not true. It is not difficult to verify (8.3) and (8.4), just from the definitions.

Just as for the standard absolute value, we can define a $p$-adic distance function $d_p(x, y)$ on $\mathbb{Q}$ using $|\cdot|_p$, by

(8.5) \[ d_p(x, y) = |x - y|_p. \]

This satisfies the usual requirements for a metric (as in [Rud]), which is to say that $d_p(x, y)$ is a nonnegative real number which is equal to 0 if and only if $x = y$, $d_p(x, y) = d_p(y, x)$, and $d_p(x, y)$ satisfies the triangle inequality. In this case we have the stronger condition

(8.6) \[ d_p(x, z) \leq \max(d_p(x, y), d_p(y, z)) \]

for all $x, y, z \in \mathbb{Q}$, because of (8.4). Thus one says that $d_p(x, y)$ is an ultrametric, referring to this stronger version of the triangle inequality.

Recall that if $(M, D(u, v))$ is any metric space, then a sequence $\{u_j\}_{j=1}^\infty$ of points in $M$ is said to converge to a point $u$ in $M$ if for every positive real number $\epsilon$ there is a positive integer $L$ such that

(8.7) \[ D(u_j, u) < \epsilon \quad \text{for all } j \geq L. \]

Similarly, a sequence $\{v_j\}_{j=1}^\infty$ in $M$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an $L > 0$ so that

(8.8) \[ D(v_j, v_k) < \epsilon \quad \text{for all } j, k \geq L. \]
It is easy to see that every convergent sequence is also a Cauchy sequence. It is not true in general that every Cauchy sequence converges to some point in the metric space, but this is true in some situations.

A metric space \((M, D(u, v))\) is said to be \textit{complete} if every Cauchy sequence in \(M\) also converges in \(M\). A classical example is the real line \(\mathbb{R}\), equipped with the standard metric \(|x - y|\), as in [Rud]. The set \(\mathbb{Q}\) of rational numbers equipped with the standard metric \(|x - y|\) is not complete, because a sequence in \(\mathbb{Q}\) which converges as a sequence in \(\mathbb{R}\) is always a Cauchy sequence, whether or not the limit lies in \(\mathbb{Q}\).

By a \textit{completion} of a metric space \((M, D(u, v))\) we mean a metric space \((M_1, D_1(u, v))\) together with an embedding \(\theta_1 : M \to M_1\) with the following three properties: (a) the embedding is isometric, in the sense that

\[
D_1(\theta_1(u), \theta_1(v)) = D(u, v) \quad \text{for all } u, v \in M;
\]

(b) the image of \(M\) in \(M_1\) under \(\theta_1\) is a dense subset of \(M_1\), so that

\[
\text{for every } w \in M_1 \text{ and } \epsilon > 0 \text{ there is } \quad a \text{ } u \in M \text{ such that } D_1(w, \theta_1(u)) < \epsilon;
\]

and (c) the metric space \((M_1, D_1(u, v))\) is itself complete. A completion of a metric space always exists, as in Problem 24 on p82 and Problem 24 on p170 of [Rud]. If \((M_1, D_1(u, v)), \theta_1 : M \to M_1\) and \((M_2, D_2(z, w)), \theta_2 : M \to M_2\) are two completions of the same metric space \((M, D(x, y))\), then there is an isometry \(\phi\) from \(M_1\) onto \(M_2\) such that \(\theta_2 = \phi \circ \theta_1\). Indeed, one can define \(\phi\) on the dense subset \(\theta_1(M)\) of \(M_1\) through this equation, and then extend \(\phi\) to all of \(M_1\) using uniform continuity and completeness (as in Problem 13 on p99 of [Rud]).

The standard embedding of \(\mathbb{Q}\) in \(\mathbb{R}\) provides a completion of \(\mathbb{Q}\) equipped with the standard metric \(|x - y|\). What about \(\mathbb{Q}\) equipped with the \(p\)-adic metric \(d_p(x, y)\)? Let us begin with a representation for rational numbers connected to the \(p\)-adic geometry.

**Lemma 8.11** Let \(x\) be a nonzero rational number, and let \(l\) be the nonzero integer such that \(|x|_p = p^{-l}\). There is a sequence \(\{\alpha_j\}_{j=1}^\infty\) of nonnegative integers such that \(\alpha_l \neq 0\), \(\alpha_j \leq p - 1\) for all \(j\), and the series

\[
\sum_{j=l}^{\infty} \alpha_j p^j
\]

converges to \(x\) in \(\mathbb{Q}\) with respect to the \(p\)-adic metric.
In other words, if
\[ s_n = \sum_{j=l}^{n} \alpha_j p^j \]
for \( n \geq l \), then \( d_p(s_n, x) \to 0 \) as \( n \to \infty \). In fact,
\[ d_p(s_n, x) \leq p^{-n-1} \quad \text{for each} \quad n \geq l. \] (8.14)

We can use (8.14) to choose the \( \alpha_j \)'s one after the other. Let us start with \( \alpha_l \). Since \( |x|_p = p^{-l} \), we can write \( x \) as \( (a/b) p^l \), where \( a \) and \( b \) are nonzero integers which are not divisible by \( p \). We choose \( \alpha_l, 1 \leq \alpha_l \leq p-1 \), so that \( a \equiv \alpha_l b \) modulo \( p \). This ensures that \( d_p(s_l, x) \leq p^{-l-1} \), where \( s_l = \alpha_l p^l \).

Now suppose that \( \alpha_j \) has been chosen for \( j = l, l+1, \ldots, n \) for some integer \( n \), in such a way that (8.14) holds, and let us choose \( \alpha_{n+1} \). Since \( d_p(s_n, x) \leq p^{-n-1} \), we can write \( x - s_n \) as \( (a'/b') p^{n+1} \), where \( a' \) and \( b' \) are integers, and \( b' \) is not zero and not divisible by \( p \). We choose \( \alpha_{n+1}, 0 \leq \alpha_{n+1} \leq p-1 \), so that \( a' \equiv \alpha_{n+1} b' \) modulo \( p \). This ensures that \( d_p(s_{n+1}, x) \leq p^{-n-2} \), with \( s_{n+1} = s_n + \alpha_{n+1} p^{n+1} \), as in (8.13).

This gives a sequence \( \{\alpha_j\}_{j=l}^{\infty} \) as required in Lemma 8.11. We shall explain another way to get such a sequence in a moment, using the next lemma.

**Lemma 8.15** Let \( l \) be an integer, and let \( \{\beta_j\}_{j=l}^{\infty} \) be a sequence of arbitrary nonnegative integers. There is a sequence \( \{\alpha_j\}_{j=l}^{\infty} \) of nonnegative integers such that \( \alpha_j \leq p-1 \) for all \( j \) and
\[ \sum_{j=l}^{n} \beta_j p^j - \sum_{j=l}^{n} \alpha_j p^j \]
can be written as \( c_n p^{n+1} \) with \( c_n \) a nonnegative integer for all integers \( n \geq l \).

This can be derived in much the same manner as before. We choose \( \alpha_l \) so that it is equal to \( \beta_l \) modulo \( p \). Suppose that \( \alpha_j, l \leq j \leq n \), have the properties described in the lemma, and let us choose \( \alpha_{n+1} \). By assumption,
\[ c_n = p^{-n-1} \left( \sum_{j=l}^{n} \beta_j p^j - \sum_{j=l}^{n} \alpha_j p^j \right) \]
is a nonnegative integer, and we choose \( \alpha_{n+1}, 0 \leq \alpha_{n+1} \leq p-1 \), so that it is equal to \( c_n + \beta_{n+1} \) modulo \( p \). This implies that
\[ c_{n+1} = p^{-n-2} \left( \sum_{j=l}^{n+1} \beta_j p^j - \sum_{j=l}^{n+1} \alpha_j p^j \right) \]
is also a nonnegative integer, as desired.

In the context of Lemma 8.15, if \( \sum_{j=l}^{\infty} \beta_j p^j \) converges to a rational number in the \( p \)-adic metric, then \( \sum_{j=l}^{\infty} \alpha_j p^j \) converges to the same rational number in the \( p \)-adic metric, since the difference between the partial sums tends to 0 in the \( p \)-adic absolute value.

Let us return to the setting of Lemma 8.11. Let \( x \) be a nonzero rational number, and assume first that \( x \) is a positive integer. Then we can write \( x \) as

\[
(8.19) \quad x = \sum_{j=0}^{n} \alpha_j p^j,
\]

where \( n \) is a nonnegative integer and each \( \alpha_j \) is a nonnegative integer such that \( 0 \leq \alpha_j \leq p - 1 \). One can begin by choosing \( n \) as large as possible so that \( p^n \leq x \), and then take \( \alpha_n \) so that \( 0 \leq x - \alpha_n p^n < p^n \). Afterwards the remaining part \( x - \alpha_n p^n \) can be treated in a similar manner, and so on. Alternatively, \( \alpha_0 \) is uniquely determined by the requirement that \( x \equiv \alpha_0 \) modulo \( p \), \( \alpha_1 \) is determined by \( x - \alpha_0 \equiv \alpha_1 p \) modulo \( p^2 \), etc.

More generally, if \( x \) is a positive integer times \( p^l \) for some \( l \in \mathbb{Z} \), then there is a finite expansion for \( x \) as in Lemma 8.11. Of course we can write \( x \) in this way where the integer factor is not divisible by \( p \), so that the leading term in the expansion is nonzero.

What about negative numbers? Consider \( x = -1 \), for instance. We can write \( -1 \) as \( (p - 1)/(1 - p) \), and this leads to the expansion

\[
(8.20) \quad -1 = \sum_{j=0}^{\infty} (p - 1) p^j.
\]

This is a kind of geometric series, which would not converge in the classical sense, because \( p > 1 \). However, this series does converge to \(-1\) with respect to the \( p \)-adic metric. Indeed, by the usual formula,

\[
(8.21) \quad \sum_{j=0}^{n} (p - 1) p^j = (p - 1) \sum_{j=0}^{n} p^j = (p - 1) \frac{1 - p^{n+1}}{1 - p} = -1 - p^{n+1},
\]

so that

\[
(8.22) d_p \left( \sum_{j=0}^{n} (p - 1) p^j, -1 \right) = |(-1 - p^{n+1}) - 1|_p = | - p^{n+1} |_p = p^{-n-1}.
\]

More generally, suppose that \( x = -1/b \), where \( b \) is a positive integer which is not divisible by \( p \). Let \( c \) be a positive integer such that \( c \leq p - 1 \)}
and \(-bc \equiv 1 \mod p\). Thus \(x = -1/b = c/(1-b_1 p)\), where \(b_1\) is a positive integer. As in the preceding situation, we get that
\[
x = \sum_{j=0}^{\infty} c b_1^j p^j,
\]
with convergence in the \(p\)-adic metric, because
\[
\sum_{j=0}^{n} c b_1^j p^j = c \sum_{j=0}^{n} (b_1 p)^j = c \frac{1-(b_1 p)^{n+1}}{1-b_1 p} = x (1-(b_1 p)^{n+1}).
\]

This expansion does not quite fit the conditions in Lemma 8.11, because the coefficients \(c b_1^j\), which are positive integers, are not bounded by \(p-1\) in general. However, one can use Lemma 8.15 to convert this expansion into one that does satisfy the additional restriction on the coefficients.

Before proceeding, let us note the following.

**Lemma 8.25** Let \(l, l'\) be integers, and let \(\{\beta_j\}_{j=l}^{\infty}\) and \(\{\beta'_j\}_{j=l'}^{\infty}\) be sequences of nonnegative integers. Define a sequence \(\{\gamma_i\}_{i=l+l'}^{\infty}\) by
\[
\gamma_i = \sum \{\beta_j \beta'_k : i = j+k, j \geq l, k \geq l'\}
\]
(i.e., \(\{\gamma_i\}_{i=l+l'}^{\infty}\) is the “Cauchy product” of \(\{\beta_j\}_{j=l}^{\infty}\) and \(\{\beta'_j\}_{j=l'}^{\infty}\)). Then, for each integer \(n \geq 0\),
\[
\sum_{j=l}^{l+n} \beta_j p^j \left( \sum_{k=l'}^{l'+n} \beta'_k p^k \right) - \sum_{i=l+l'}^{l+l'+n} \gamma_i p^i
\]
is a nonnegative integer which is divisible by \(p^{l+l'+n+1}\).

To see this, one might prefer to think about the case where \(l = l' = 0\), to which one can easily reduce. The main point is that every term occurring in the sum of the \(\gamma_i\)'s in (8.27) also occurs when one expands the product of the two sums in (8.27). There are additional terms in the product of the sums, which are all of the form \(c p^m\), where \(c\) is a nonnegative integer and \(m \geq l + l' + n + 1\).

**Corollary 8.28** In the situation of Lemma 8.22, suppose that the series \(\sum_{j=l}^{\infty} \beta_j p^j\) and \(\sum_{k=l'}^{\infty} \beta'_k p^k\) converge to rational numbers \(u, u'\), respectively, in the \(p\)-adic metric. Then the series \(\sum_{i=l+l'}^{\infty} \gamma_i p^i\) converges to the product \(uu'\) in the \(p\)-adic metric.
This is an easy consequence of the lemma. Note that this result does not hold for infinite series in the classical sense without some additional hypotheses. See [Rud] for more information.

To finish our alternate discussion of Lemma 8.11, suppose that \( x \) is a nonzero rational number. For special cases we have given expansions for \( x \) in terms of sums of products of nonnegative integers with powers of \( p \). One can get arbitrary \( x \) by taking products, using Corollary 8.28 to ensure that this leads to the correct limits. One also employs Lemma 8.15 to obtain an expansion in which the coefficients are bounded by \( p - 1 \).

Let us record some more lemmas.

**Lemma 8.29** Suppose that \( l \) is an integer, \( \{ \beta_j \}_{j=l}^\infty \) is a sequence of integers (of arbitrary sign), and that the sum \( \sum_{j=l}^\infty \beta_j p^j \) converges in \( \mathbb{Q} \) in the \( p \)-adic metric. Then
\[
\left| \sum_{j=l}^\infty \beta_j p^j \right|_p \leq p^{-l}.
\]

This is not hard to verify, from the definitions. Of course
\[
\left| \sum_{j=m}^n \beta_j p^j \right|_p \leq p^{-m}
\]
for all integers \( m \) and \( n \) such that \( l \leq m \leq n \).

**Lemma 8.32** Suppose that \( l \) is an integer, \( \{ \beta_j \}_{j=l}^\infty \) is a sequence of integers (of arbitrary sign), and that the sum \( \sum_{j=l}^\infty \beta_j p^j \) converges in \( \mathbb{Q} \) in the \( p \)-adic metric. If \( \beta_l \not\equiv 0 \pmod{p} \), then
\[
\left| \sum_{j=l}^\infty \beta_j p^j \right|_p = p^{-l}.
\]

Again, this is not hard to verify. By assumption, the leading term in the sum has \( p \)-adic absolute value \( p^{-l} \), while the rest has \( p \)-adic absolute value \( \leq p^{-l-1} \).

**Lemma 8.34** The expansion described in Lemma 8.11 is unique. In other words, suppose that \( x \) is a nonzero rational number, \( l, l' \) are integers, and \( \{ \alpha_j \}_{j=l}^\infty, \{ \alpha'_j \}_{j=l'}^\infty \) are sequences of nonnegative integers such that \( \alpha_l \neq 0, \alpha'_l \neq 0, 0 \leq \alpha_j \leq p - 1 \) for each \( j \geq l \), \( 0 \leq \alpha'_j \leq p - 1 \) for each \( j \geq l' \), and the sums \( \sum_{j=l}^\infty \alpha_j p^j, \sum_{j=l'}^\infty \alpha'_j p^j \) both converge in \( \mathbb{Q} \) to \( x \) in the \( p \)-adic metric. Then \( l = l' \) and \( \alpha_j = \alpha'_j \) for all \( j \geq l \).
Notice first that $l = l'$ under these conditions, since $|x| = p^{-l}$ and $|x| = p^{-l'}$, as in Lemma 8.32. Next, $\alpha_l = \alpha'_l$, because otherwise the difference between the two series has $p$-adic absolute value $p^{-l}$. One can subtract the leading terms and repeat the argument to get that $\alpha_j = \alpha'_j$ for all $j$, as desired.

**Lemma 8.35** Suppose that $l$ is an integer and that $\{\alpha_j\}_{j=l}^\infty$ is a sequence of nonnegative integers such that $\alpha_j \leq p - 1$ for all $j$. If $\sum_{j=l}^\infty \alpha_j p^j$ converges in $\mathbb{Q}$ to 0 in the $p$-adic metric, then $\alpha_j = 0$ for all $j$.

This is easy to check.

Let us now define $\mathbb{Q}_p$, the set of $p$-adic numbers, to consist of the formal sums of the form

$$\sum_{j=l}^\infty \alpha_j p^j,$$

where $l$ is an integer, and each $\alpha_j$ is an integer such that $0 \leq \alpha_j \leq p - 1$ for each $j \geq l$. If

$$\sum_{j=l'}^\infty \alpha'_j p^j$$

is another such formal sum, with $l' < l$, say, then the two sums are viewed as representing the same element of $\mathbb{Q}_p$ if $\alpha'_j = 0$ when $l' \leq j < l$ and $\alpha'_j = \alpha_j$ when $j \geq l$.

If a formal sum (8.36) actually converges to a rational number $x$ with respect to the $p$-adic metric, then let us identify that element of $\mathbb{Q}_p$, with $x \in \mathbb{Q}$. The preceding lemmas imply that no more than one element of $\mathbb{Q}_p$ is identified in this manner with a single rational number $x$, and Lemma 8.11 says that every rational number is included in $\mathbb{Q}_p$ through this recipe. Thus we can think of $\mathbb{Q}$ as a subset of $\mathbb{Q}_p$.

The $p$-adic absolute value $| \cdot |_p$ can be extended to $\mathbb{Q}_p$ by setting

$$\left| \sum_{j=l}^\infty \alpha_j p^j \right|_p = p^{-l}$$

when $\alpha_l \neq 0$ and $0 \leq \alpha_j \leq p - 1$, $\alpha_j \in \mathbb{Z}$ for all $j$ (as usual). The zero element of $\mathbb{Q}_p$, which corresponds to the series with all coefficients equal to 0, and to the rational number 0 under the identification discussed in the previous paragraph, has $p$-adic absolute value equal to the number 0. For the
elements of $\mathbb{Q}_p$, which correspond to rational numbers, this definition of the $p$-adic absolute value agrees with the original one, as in the earlier lemmas.

Next, let us extend the $p$-adic distance function $d_p(\cdot, \cdot)$ to $\mathbb{Q}_p$. Suppose that

$$
\sum_{j=l}^{\infty} \alpha_j p^j, \quad \sum_{j=l'}^{\infty} \alpha'_j p^j
$$

are two elements of $\mathbb{Q}_p$, where, as usual, $l$ and $l'$ are integers, $\alpha_j$ is an integer such that $0 \leq \alpha_j \leq p - 1$ for all $j \geq l$, and $\alpha'_j$ is an integer such that $0 \leq \alpha'_j \leq p - 1$ for all $j \geq l'$. If either of these two sums is the zero element of $\mathbb{Q}_p$, so that all of the coefficients are 0, then the $p$-adic distance between the two sums is defined to be the $p$-adic absolute value of the other sum. If they are both the zero element of $\mathbb{Q}_p$, then the $p$-adic distance is equal to 0. Assume now that both sums correspond to nonzero elements of $\mathbb{Q}_p$. In this event we ask that $\alpha_l \neq 0$ and $\alpha'_{l'} \neq 0$, which can always be arranged by dropping initial terms with coefficient 0, if necessary. If $l \neq l'$, then the $p$-adic distance between the two sums is defined to be $p^{-\min(l, l')}$. Suppose instead that $l = l'$. If $\alpha_j = \alpha'_j$ for all $j \geq l$, then the two sums are the same, and the $p$-adic distance between them is defined to be 0. Otherwise, let $j_0$ be the smallest integer $\geq l$ such that $\alpha_{j_0} \neq \alpha'_{j_0}$. In this case the distance between the two sums is defined to be $p^{-j_0}$. It is not hard to check that this definition of the $p$-adic distance agrees with the earlier one when both sums correspond to rational numbers.

By definition, the $p$-adic distance on $\mathbb{Q}_p$ is nonnegative and equal to 0 exactly when the two elements of $\mathbb{Q}_p$ are the same. The distance is also symmetric in the two elements of $\mathbb{Q}_p$. As before, the $p$-adic distance satisfies the ultrametric condition

$$
d_p(w_1, w_3) \leq \max(d_p(w_1, w_2), d_p(w_2, w_3))
$$

for all $w_1$, $w_2$, and $w_3$ in $\mathbb{Q}_p$. This is not difficult to verify.

In this way $\mathbb{Q}_p$ becomes a metric space. It is easy to see that $\mathbb{Q}$ defines a dense subset of $\mathbb{Q}_p$, since sums (8.36) in which all but finitely many coefficients are 0 correspond to rational numbers, and arbitrary sums can be approximated by these. One can also show that $\mathbb{Q}_p$ is complete as a metric space, so that it is indeed a completion of $\mathbb{Q}$.

Like the real line with its standard metric, $\mathbb{Q}_p$ enjoys the property that closed and bounded subsets of it are compact. For this it is enough to know that closed balls around the origin are compact. Fix an integer $l$, and consider
the set of \( w \) in \( \mathbb{Q}_p \) such that \( |w|_p \leq p^{-t} \). This can be described exactly as the set of sums

\[
\sum_{j=l}^{\infty} \alpha_j p^j,
\]

where each \( \alpha_j \) is an integer such that \( 0 \leq \alpha_j \leq p - 1 \). Now all of the sums have the same starting point (at \( l \)), but it is important to allow the initial coefficients to be 0, or even all of the coefficients to be 0, to get the right subset of \( \mathbb{Q}_p \).

Topologically, this set is equivalent to a Cantor set (with \( p \) pieces at each stage). It is convenient to look at the topology in terms of sequences in the set, and convergence of sequences. Namely, a sequence \( \{w_t\}_{t=1}^{\infty} \) of elements of this set converges to another element \( w \) of this set exactly if, for each \( j \geq l \), the \( j \)th coefficient in the sum associated to the \( w_t \)'s is equal to the \( j \)th coefficient of the sum associated to \( w \) for all sufficiently large \( t \). (How large \( t \) should be is permitted to depend on \( j \).) The statement that the set is compact means that for any sequence \( \{w_t\}_{t=1}^{\infty} \) of elements of the set there is a subsequence that converges to an element of the set. This can be established through classical arguments.

Suppose that \( l \) is an integer, and that \( \{\beta_j\}_{j=l}^{\infty} \) is a sequence of nonnegative integers, without the restriction \( \beta_j \leq p - 1 \). Consider the formal sum

\[
\sum_{j=l}^{\infty} \beta_j p^j.
\]

We can view this as still giving rise to an element of \( \mathbb{Q}_p \), using Lemma \ref{lemma:8.15} to convert this to a sum \( \sum_{j=l}^{\infty} \alpha_j p^j \) where each \( \alpha_j \) is a nonnegative integer such that \( \alpha_j \leq p - 1 \).

This remark permits us to define operations of addition and multiplication on \( \mathbb{Q}_p \). Specifically, one first adds or multiplies two sums in the obvious manner, to get a sum in the form (8.42). One then gets an element of \( \mathbb{Q}_p \) as described in the previous paragraph.

For elements of \( \mathbb{Q}_p \) that correspond to rational numbers, these operations of addition and multiplication give the same result as the usual ones. The elements of \( \mathbb{Q}_p \), corresponding to the rational numbers 0 and 1 are additive and multiplicative identity elements for all of \( \mathbb{Q}_p \). One also has the usual commutative, associative, and distributive laws on \( \mathbb{Q}_p \). Furthermore, these operations define continuous mappings from \( \mathbb{Q}_p \times \mathbb{Q}_p \) into \( \mathbb{Q}_p \), with respect to the \( p \)-adic metric.
As before, \(-1\) can be written as \(\sum_{j=0}^{\infty} (p - 1) p^j\). Multiplication by \(-1\) gives additive inverses in \(\mathbb{Q}_p\), just as in \(\mathbb{Q}\). If \(w\) is a nonzero element of \(\mathbb{Q}_p\), then \(w\) has a multiplicative inverse in \(\mathbb{Q}_p\). To see this, it is convenient to find a representation for \(-1/w\), from which one can get \(1/w\) by multiplication by \(-1\). Assume that \(w\) is given as \(\sum_{j=0}^{\infty} \alpha_j p^j\), where \(l\) is an integer, \(\alpha_j\) is a nonnegative integer such that \(\alpha_j \leq p - 1\) for all \(j\), and \(\alpha_l \neq 0\). Let \(c\) be an integer such that \(1 \leq c \leq p - 1\) and \(c \alpha_l \equiv -1\) modulo \(p\), and let \(\beta\) be the nonnegative integer such that \(c\alpha_l + 1 = \beta p\). Consider the expression

\[
\sum_{j=0}^{\infty} c p^{-l} \left( \beta p + \sum_{i=1}^{\infty} c \alpha_{l+i} p^i \right)^j.
\]

(8.43)

It is easy to expand this out to get a sum of the form (8.42), and hence an element of \(\mathbb{Q}_p\). One can think of (8.43) as corresponding to

\[
\frac{-1}{w} = \frac{cp^{-l}}{1 - (cw p^{-l} - 1)} = \frac{cp^{-l}}{1 - (\beta p + \sum_{i=1}^{\infty} c \alpha_{l+i} p^i)}.
\]

(8.44)

This works, since the product of

\[
\sum_{j=0}^{\infty} \left( \beta p + \sum_{i=1}^{\infty} c \alpha_{l+i} p^i \right)^j
\]

(8.45)

and

\[
(\beta p - 1) + \sum_{i=1}^{\infty} c \alpha_{l+i} p^i = \sum_{i=0}^{\infty} c \alpha_{l+i} p^i
\]

(8.46)

is equal to 1.

Thus \(\mathbb{Q}_p\) is a field. With subtraction defined, it makes sense to say that \(d_p(w, z) = |w - z|_p\) on \(\mathbb{Q}_p\). It can be somewhat more convenient to write this as \(d_p(w, w + u) = |u|_p\) for \(w, u\) in \(\mathbb{Q}_p\). Notice too that

\[
|w + z|_p \leq \max(|w|_p, |z|_p)
\]

(8.47)

and

\[
|wz|_p = |w|_p |z|_p
\]

(8.48)

for \(w, z\) in \(\mathbb{Q}_p\).
9 Absolute values on fields

Let $k$ be a field. A function $|\cdot|_*$ on $k$ is called an absolute value function on $k$, or a choice of absolute values, if $|x|_*$ is a nonnegative real number for all $x$ in $k$, $|x|_* = 0$ if and only if $x = 0$, and

\[(9.1) \quad |xy|_* = |x|_* |y|_*\]

and

\[(9.2) \quad |x + y|_* \leq |x|_* + |y|_*\]

for all $x, y$ in $k$.

Notice that (9.1) yields

\[(9.3) \quad |1|_* = 1,\]

where the 1 on the left side is the multiplicative identity element in $k$, and the 1 on the right side is the real number 1. If $x$ is a nonzero element of $k$, then we obtain that

\[(9.4) \quad |x^{-1}|_* = |x|_*^{-1},\]

because $1 = |1|_* = |xx^{-1}|_* = |x|_* |x^{-1}|_*$. Also,

\[(9.5) \quad |\ - 1|_* = 1,\]

since $ |- 1|^2_* = |(-1)^2|_* = |1|_* = 1$. As a result, $| - x|_* = |x|_*$ for all $x$ in $k$.

For example, the usual absolute values for real numbers defines an absolute value function on $\mathbb{R}$ and $\mathbb{Q}$. The usual modulus of a complex number defines an absolute value function on $\mathbb{C}$.

An absolute value function $|\cdot|_*$ on $k$ is said to be non-Archimedean, or ultrametric, if it satisfies the stronger condition

\[(9.6) \quad |x + y|_* \leq \max(|x|_* , |y|_*)\]

for all $x, y$ in $k$, instead of (9.2). The $p$-adic absolute values on $\mathbb{Q}$ or on $\mathbb{Q}_p$ have this property.

On any field one can define the trivial absolute value function, which is equal to 0 at the zero element of the field and to 1 at all nonzero elements in the field. Note that this is an ultrametric absolute value function. If $k$ is a finite field, then it is not hard to see that the only absolute value function on $k$ is the trivial one.
If \( k_0 \) is a field, let \( k_0(t) \) denote the field of rational functions in one variable \( t \) over \( k_0 \). Thus every element of \( k_0(t) \) can be written as \( P(t)/Q(t) \), where \( P(t) \) and \( Q(t) \) are polynomials in \( t \) with coefficients in \( k_0 \), and \( Q(t) \) has at least one nonzero coefficient. Two such representations

\[
P_1(t)/Q_1(t), \ P_2(t)/Q_2(t)
\]

are considered to define the same element of \( k_0(t) \) when

\[
P_1(t)Q_2(t) = P_2(t)Q_1(t).
\]

If \( P(t) \) is the zero polynomial, so that all of its coefficients are 0, then \( P(t)/Q(t) \) is the zero rational function in \( k_0(t) \) for any \( Q(t) \) which is not the zero polynomial.

Fix a real number \( A \) with \( A > 1 \). We can define a nontrivial absolute value function on \( k_0(t) \) by setting it equal to 0 for the zero rational function, and to \( A^{-l} \) when the rational function can be expressed as \( t^l P_0(t)/Q_0(t) \), where \( P_0(t) \) and \( Q_0(t) \) are polynomials in \( t \) with nonzero constant terms. It is easy to see that this satisfies the conditions described before, including the ultrametric version of the triangle inequality. If we restrict this absolute value function to the subfield of \( k_0(t) \) consisting of constant functions, then we get the trivial absolute value function on \( k_0 \).

Let \( k_0((t)) \) denote the field of formal Laurent expansions of finite order over \( k_0 \) in \( t \). In other words, an element of \( k_0((t)) \) is given by a formal series

\[
\sum_{j=n}^{\infty} a_j t^j,
\]

where \( n \) is an integer and the \( a_j \)'s are arbitrary elements of \( k_0 \). If \( m \) is an integer with \( m \leq n \) and

\[
\sum_{l=m}^{\infty} b_j t^j
\]

is another such series, then the two series are viewed as defining the same element of \( k_0((t)) \) if and only if \( b_j = 0 \) when \( m \leq j < n \) and \( a_j = b_j \) when \( j \geq n \). The series with all coefficients equal to 0 and whatever starting point \( t^n \) correspond to the zero element of \( k_0((t)) \). Elements of \( k_0((t)) \) can be added and multiplied in the usual manner, term by term. The series with constant term equal to 1 and all others equal to 0 is the multiplicative identity element.
in $k_0((t))$. If $R(t)$ is a nonzero element of $k_0((t))$, then $R(t)$ can be written as
\begin{equation}
R(t) = a_n t^n (1 - R_0(t)), \tag{9.11}
\end{equation}
where $n$ is an integer, $a_n$ is a nonzero element of $k_0$, and $R_0(t) \in k_0((t))$ is a given by a sum with only positive powers of $t$. One can check that the standard formula
\begin{equation}
(1 - R_0(t))^{-1} = \sum_{j=0}^{\infty} R_0(t)^j, \tag{9.12}
\end{equation}
with $R_0(t)^0$ interpreted as being 1, makes sense and works in $k_0((t))$. Thus nonzero elements of $k_0((t))$ have multiplicative inverses, so that $k_0((t))$ is a field. There is a natural embedding of $k_0((t))$ in $k_0((t))$, by associating to a rational function over $k_0$ its Laurent expansion around 0. Of course polynomials are contained in $k_0((t))$ as finite sums of multiples of nonnegative powers of $t$, and rational functions can be obtained from this using products and multiplicative inverses.

Let $A$ be a real number with $A > 1$, as before. One can define an absolute value function on $k_0((t))$ by taking the absolute value of the zero element of $k_0((t))$ to be 0, and the absolute value of (9.9) to be $A^{-n}$ when $a_n \neq 0$. It is not difficult to check that this defines an absolute value function with the ultrametric version of the triangle inequality, and that it agrees with the one described earlier for $k_0((t))$ when applied to Laurent series coming from rational functions.

Suppose that $k$ is a field and that $| \cdot |_*$ is an absolute value function on $k$ which satisfies the ultrametric version of the triangle inequality. We shall say that $| \cdot |_*$ is nice if there is a subset $E$ of the set of nonnegative real numbers such that $|x|_* \in E$ for all $x$ in $k$ and $E$ has no limit point in the real line except possibly at 0. This is equivalent to saying that for any real numbers $a, b$ such that $0 < a < b$, the set $E \cap [a, b]$ is finite.

**Lemma 9.13** An ultrametric absolute value function $| \cdot |_*$ on a field $k$ is nice if and only if there is a real number $r$ such that $0 \leq r < 1$ and $|x|_* \leq r$ for all $x \in k$ such that $|x|_* < 1$.

The “only if” part of this statement is immediate from the definition. Conversely, suppose that there is a real number $r$ as in the lemma. If $x, y$ are elements of $k$ such that $|x|_* < |y|_*$, then $|x y^{-1}|_* < 1$, so that $|x y^{-1}|_* \leq r$ and $|x|_* \leq r |y|_*$. One can use this to show that the absolute value function takes values in a set $E$ as above.
Lemma 9.14 An ultrametric absolute value function \(|\cdot|_*\) on a field \(k\) is nice if there is a positive real number \(s < 1\) and a finite collection \(x_1, \ldots, x_m\) of elements of \(k\) such that for each \(y\) in \(k\) with \(|y|_* < 1\) there is an \(x_j, 1 \leq j \leq m\), such that \(|y - x_j|_* \leq s\).

To prove this, suppose that the hypothesis of Lemma 9.14 holds, and let us check that the condition in Lemma 9.13 is satisfied. If \(1 \leq j \leq m\) and \(y \in k\) satisfy \(|y|_* < 1\) and \(|y - x_j|_* \leq s\), then

\[
|x_j|_* \leq \max(|x_j - y|_*, |y|_*) < 1. \tag{9.15}
\]

We may as well assume that \(|x_j|_* < 1\) for all \(j\), since otherwise we can reduce to a smaller collection of \(x_j\)'s with the same property as in the lemma. Take \(r\) to be the maximum of \(s\) and the numbers \(|x_j|_*\), \(1 \leq j \leq m\), so that \(r < 1\).

If \(y\) is an element of \(k\) and \(|y|_* < 1\), then \(|y - x_j|_* \leq s\) for some \(j\), and hence \(|y|_* \leq \max(|y - x_j|_*, |x_j|_*) \leq r\), as desired.

10 Norms on vector spaces

Fix a field \(k\) and an absolute value function \(|\cdot|_*\) on \(k\), and let \(V\) be a vector space over \(k\). A norm on \(V\) with respect to this choice of absolute value function on \(k\) is a real-valued function \(N(\cdot)\) on \(V\) such that the following three properties are satisfied: (a) \(N(v) \geq 0\) for all \(v \in V\), with \(N(v) = 0\) if and only if \(v = 0\); (b) \(N(\alpha v) = |\alpha|_* N(v)\) for all \(\alpha\) in \(k\) and \(v \in V\); (c) \(N(v + w) \leq N(v) + N(w)\) for all \(v, w \in V\).

In this section we make the standing assumption that \(|\cdot|_*\) is an ultrametric absolute value function on \(k\), and we shall restrict our attention to norms \(N\) on vector spaces \(V\) over \(k\) with respect to \(|\cdot|_*\) that are ultrametric norms, in the sense that

\[
N(v + w) \leq \max(N(v), N(w)) \tag{10.1}
\]

for all \(v, w \in V\). Observe that if \(N(\cdot)\) is an ultrametric norm on \(V\), then \(d(v, w) = N(v - w)\) is an ultrametric on \(V\), so that

\[
d(u, w) \leq \max(d(u, v), d(v, w)) \tag{10.2}
\]

for all \(u, v, w \in V\).

One can think of \(k\) as a 1-dimensional vector space over itself, and then the absolute value function \(|\cdot|_*\) defines an ultrametric norm on this vector space.
space. If \( n \) is a positive integer, then \( k^n \), the space of \( n \)-tuples of elements of \( k \), is an \( n \)-dimensional vector space over \( k \), with respect to coordinatewise addition and scalar multiplication. Consider the expression

\[
\max_{1 \leq j \leq n} |x_j|_*
\]

for each \( x = (x_1, \ldots, x_n) \) in \( k^n \). It is easy to check that this defines an ultrametric norm on \( k^n \).

**Lemma 10.4** Let \( V \) be a vector space over \( k \), and let \( N \) be an ultrametric norm on \( V \). Suppose that \( v, w \) are elements of \( V \), and that \( N(v) \neq N(w) \). Then \( N(v + w) = \max(N(v), N(w)) \).

We have that \( N(v + w) \leq \max(N(v), N(w)) \) for all \( v, w \) in \( V \), and so we want to show that the reverse inequality holds when \( N(v) \neq N(w) \). Assume for the sake of definiteness that \( N(v) < N(w) \). Notice that

\[
N(w) = N((v + w) + (-v)) \leq \max(N(v + w), N(-v)) = \max(N(v + w), N(v)).
\]

Because \( N(v) < N(w) \), this implies that \( N(w) \leq N(v + w) \). Hence

\[
\max(N(v), N(w)) \leq N(v + w),
\]

as desired.

**Corollary 10.7** Let \( V \) be a vector space over \( k \), and let \( N \) be an ultrametric norm on \( V \). Suppose that \( m \) is a positive integer, and that \( v_1, \ldots, v_m \) are elements of \( V \) such that \( N(v_j) = N(v_l) \) only when either \( j = l \) or \( v_j = v_l = 0 \). Then

\[
N\left(\sum_{j=1}^{m} v_j\right) = \max_{1 \leq j \leq m} N(v_j).
\]

This can be derived from Lemma 10.4 using induction.

**Lemma 10.9** Suppose that \( V \) is a vector space over \( k \) of dimension \( n \), and that \( N \) is an ultrametric norm on \( V \). Let \( E \) be a subset of the set of nonnegative real numbers such that \( |x|_* \in E \) for all \( x \) in \( k \). If \( v_1, \ldots, v_{n+1} \) are nonzero elements of \( V \), then at least one of the ratios \( N(v_j)/N(v_l) \), \( 1 \leq j < l \leq n+1 \), lies in \( E \).
Let \( v_1, \ldots, v_{n+1} \) be nonzero vectors in \( V \). If the ratios \( N(v_j)/N(v_l) \) do not lie in \( E \) for any \( j, l \) with \( j \neq l \), then one can check that \( v_1, \ldots, v_{n+1} \) are linearly independent in \( V \), using Corollary \[\text{10.7}\]. This contradicts the assumption that \( V \) has dimension \( n \), and the lemma follows.

In analogy with the definition of a nice ultrametric absolute value function in Section \[\text{9}\], let us say that an ultrametric norm \( N \) on a vector space \( V \) over \( k \) is nice if there is a subset \( E_1 \) of the set of nonnegative real numbers such that \( N(v) \) lies in \( E_1 \) for every vector \( v \) in \( V \) and \( E_1 \) has no limit point in the real line except possibly for 0. The latter is equivalent to asking that \( E_1 \cap [a, b] \) be finite for every pair \( a, b \) of positive real numbers.

**Corollary 10.10** Let \( V \) be a vector space over \( k \), and let \( N \) be an ultrametric norm on \( V \). If the absolute value function \( |\cdot|_* \) on \( k \) is nice, then \( N \) is a nice ultrametric norm on \( V \).

More precisely, if \( V \) has dimension \( n \), and if \( E \) is a subset of the set of nonnegative real numbers such that the absolute value function \( |\cdot|_* \) on \( k \) takes values in \( E \), then there are positive real numbers \( a_1, \ldots, a_n \) such that \( N \) takes values in the set \( E_1 = \bigcup_{j=1}^n a_j E \). Here \( a E = \{a s : s \in E\} \). This can be derived from Lemma \[\text{10.9}\]. (Note that the \( a_j \)'s need not be distinct.)

Fix a positive integer \( n \), and let us take our vector space to be \( k^n \). We shall say that a norm \( N \) on \( k^n \) is nondegenerate if there is a positive real number \( c \) such that

\[
(10.11) \quad c \max_{1 \leq j \leq n} |x_j|_* \leq N(x)
\]

for all \( x = (x_1, \ldots, x_n) \) in \( k^n \). This condition is automatically satisfied if \( \{y \in k : |y|_* \leq 1\} \) is a compact subset of \( k \), using the metric \( |u - v|_* \) on \( k \). To see this, observe that it is enough to check that \( N \) has a positive lower bound on the set of \( x \) in \( k^n \) such that \( \max_{1 \leq j \leq n} |x_j|_* = 1 \), because of homogeneity. By assumption, this is a compact subset of \( k^n \) in the product topology, and \( N \) is positive at each element of this set since \( N \) is a norm. One can verify that \( N \) is continuous, and in fact \( N \) is locally constant away from 0 in \( k^n \).

**Remark 10.12** For any norm \( N \) on \( k^n \) there is a positive real number \( C \) so that

\[
(10.13) \quad N(x) \leq C \max_{1 \leq j \leq n} |x_j|_*
\]

for all \( x \) in \( k^n \). This is easy to see, by writing \( x \) as a linear combination of standard basis vectors, and it applies to norms in general, whether or not
they are ultrametric norms. For that matter, the notion of nondegeneracy makes sense for norms in general.

**Lemma 10.14** Suppose that the absolute value function $|\cdot|_*$ on $k$ is nice. Let $N$ be a nondegenerate ultrametric norm on $k^n$, let $W$ be a vector subspace of $k^n$, and let $z$ be an element of $k^n$ which does not lie in $W$. There exists an element $x_0$ of $W$ such that the distance $N(z-x_0)$ is as small as possible.

Notice first that there is a $\delta > 0$ such that $N(z-x) \geq \delta$ for all $x \in W$. This uses the fact that $W$ is a closed subset of $k^n$ with respect to the product topology. For instance, one can describe $W$ as the set of vectors where finitely many linear functions vanish, and these linear functions are continuous. Because $|\cdot|_*$ is nice, $N$ is nice, as in Corollary [10.10], and hence the infimum of $N(z-x)$ over $x$ in $W$ is attained, since it is positive.

**Remark 10.15** If $w \in W$ satisfies $N(w) \leq N(z-x_0)$, then $N(z-(x_0+w)) \leq N(z-x_0)$, so that $N(z-(x_0+w)) = N(z-x_0)$ if $N(z-x_0)$ is as small as possible. Conversely, if $N(z-(x_0+w)) = N(z-x_0)$, then $N(w) \leq N(z-x_0)$.

**Lemma 10.16** Assume that $|\cdot|_*$ is a nice absolute value function on $k$. Let $N$ be a nondegenerate ultrametric norm on $k^n$, let $V_1$ be a vector space over $k$, and let $N_1$ be an ultrametric norm on $V_1$. Suppose that $W$ is a vector subspace of $k^n$, and that $T$ is a linear mapping from $W$ to $V_1$. Assume also that $A$ is a nonnegative real number such that

\begin{equation}
(10.17) \quad N_1(T(v)) \leq A N(v)
\end{equation}

for all $v$ in $W$. Then there is a linear mapping $T_1$ from $k^n$ to $V_1$ such that $T_1(v) = T(v)$ when $v$ lies in $W$, and

\begin{equation}
(10.18) \quad N_1(T_1(v)) \leq A N(v)
\end{equation}

for all $v$ in $k^n$.

To prove this, let $V_1$, $W$, $T$, etc., be given as above. We may as well assume that $W$ is not all of $k^n$, since otherwise there is nothing to do. Let $z$ be an element of $k^n$ which does not lie in $W$, and set $W_1 = \text{span}(W, z)$. We would like to extend $T$ first to $W_1$. 

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Let \( x_0 \) be an element of \( W \) such that \( N(z - x_0) \) is as small as possible, as in Lemma 10.14, and set \( y_0 = T(x_0) \). Define \( T_1 \) on \( W_1 \) by

\[
T_1(v + \alpha z) = T(v) + \alpha y_0
\]

for all \( v \) in \( W \) and \( \alpha \) in \( k \). With this definition \( T_1 \) is clearly linear on \( W_1 \) and agrees with \( T \) on \( W \). We want to check that

\[
N_1(T_1(v + \alpha z)) \leq A N(v + \alpha z)
\]

for all \( v \) in \( W \) and \( \alpha \) in \( k \). This inequality holds when \( \alpha = 0 \) by assumption, and so we may restrict ourselves to \( \alpha \neq 0 \). By the homogeneity of the norms, we are reduced to showing that

\[
N_1(T_1(v + z)) \leq A N(v + z)
\]

for all \( v \) in \( W \), which is the same as

\[
N_1(T(v) + y_0) \leq A N(v + z).
\]

We can rewrite this further as

\[
N_1(T(v + x_0)) \leq A N(v + z),
\]

by the definition of \( y_0 \).

Since

\[
N_1(T(v + x_0)) \leq A N(v + x_0)
\]

by hypothesis (because \( x_0 \) lies in \( W \)), it suffices to show that

\[
N(v + x_0) \leq N(v + z)
\]

for all \( v \) in \( W \). Notice that

\[
N(v + x_0) = N((v + z) + (-z + x_0)) \leq \max(N(v + z), N(-z + x_0)) = \max(N(v + z), N(z - x_0)).
\]

By the manner in which \( x_0 \) was chosen, \( N(v + z) \geq N(z - x_0) \). This yields (10.25).

Thus \( T \) can be extended to a linear mapping \( T_1 \) from the larger subspace \( W_1 \) into \( V_1 \) with the inequality (10.18) holding there. By repeating the process, we can get an extension to all of \( k^n \). This proves Lemma 10.16.
Corollary 10.27 Assume that $|\cdot|$ is a nice absolute value function on $k$. Let $N$ be a nondegenerate ultrametric norm on $k^n$, and let $W$ be a vector subspace of $k^n$. There is a linear mapping $P : k^n \to W$ which is a projection, so that $P(w) = w$ when $w \in W$ and $P(v)$ lies in $W$ for all $v$ in $k^n$, and which satisfies

$$N(P(v)) \leq N(v)$$

for all $v$ in $k^n$.

This follows by defining $P$ first on $W$ to be the identity mapping, and then extending this to a mapping from $k^n$ to $W$ using Lemma 10.16.

Suppose that $W$ and $P$ are as in the corollary, and let $W_0$ denote the kernel of $P$. We may as well assume that $W$ is neither the zero subspace of $k^n$ nor all of $k^n$, so that the same is true of $W_0$. As usual, if $I$ denotes the identity mapping on $k^n$, then $I - P$ is a projection of $k^n$ onto $W_0$, i.e., $(I - P)(v)$ lies in $W_0$ for all $v$ in $k^n$ and $(I - P)(v) = v$ when $v$ lies in $W_0$. Here we also have that

$$N((I - P)(v)) \leq N(v)$$

for all $v$ in $k^n$, because

$$N((I - P)(v)) = N(v - P(v)) \leq \max(N(v), N(P(v))) = N(v).$$

We can go further and say that

$$N(v) = \max(N(P(v)), N((I - P)(v)))$$

for all $v$ in $k^n$. That is,

$$N(v) = N(P(v) + (I - P)(v)) \leq \max(N(P(v)), N((I - P)(v))),$$

while the reverse inequality follows from the ones that have already been derived.

Let us pause a moment for some terminology. Suppose that $V$ is a vector space over $k$ of dimension $n$, and that $x_1, \ldots, x_n$ is a basis for $V$. This determines a dual family $f_1, \ldots, f_n$ of linear functionals on $V$, i.e., each $f_j$ is
a linear mapping from $V$ into $k$ such that $f_j(x_j) = 1$ and $f_j(x_l) = 0$ when $j \neq l$.

Now assume that $V$ is equipped with an ultrametric norm $N$. We say that the basis $x_1, \ldots, x_n$ is normalized if

\[(10.33) \quad N(x_j) \cdot |f_j(v)|_* \leq N(v)\]

for all vectors $v$ in $V$. Note that equality occurs when $v = x_j$.

**Lemma 10.34** Assume that $|\cdot|_*$ is a nice absolute value function on $k$. If $N$ is a nondegenerate ultrametric norm on $k^n$, then there is a normalized basis for $k^n$ with respect to $N$.

When $n = 1$ there is nothing to do. For $n > 1$ one can use induction, with the projection operators discussed above permitting one to go from $n$ to $n + 1$.

**Lemma 10.35** Let $V$ be a vector space over $k$ with dimension $n$, equipped with an ultrametric norm $N$, and let $x_1, \ldots, x_n$ be a normalized basis for $V$, with dual linear functionals $f_1, \ldots, f_n$. Then

\[(10.36) \quad N(v) = \max \{N(x_j) \cdot |f_j(v)|_* : 1 \leq j \leq n\}\]

for all vectors $v$ in $V$.

The normalization condition (10.33) gives one inequality in (10.36). For the opposite inequality, we write

\[(10.37) \quad v = f_1(v) x_1 + \cdots + f_n(v) x_n,\]

so that

\[(10.38) \quad N(v) \leq \max \{N(f_j(v) x_j) : 1 \leq j \leq n\}.\]

**Remark 10.39** If $x_1, \ldots, x_n$ is a normalized basis for $V$, then we can multiply the $x_j$’s by arbitrary nonzero elements of $k$ and get a new normalized basis for $V$. In particular, if $N$ has the feature that it takes values in the set of nonnegative real numbers which occur as values of $|\cdot|_*$, then we can multiply the $x_j$’s by elements of $k$ to get a basis $y_1, \ldots, y_n$ which is normalized and satisfies $N(y_j) = 1$ for each $j$. If $h_1, \ldots, h_n$ is the corresponding dual basis of linear functionals on $V$, then (10.36) becomes

\[(10.40) \quad N(v) = \max \{|h_j(v)|_* : 1 \leq j \leq n\}\]

for all $v$ in $V$. 

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Let $G$ be a finite group, and $\rho$ be a representation of $G$ on $k^n$. Let $N_0$ be any fixed ultrametric norm on $k^n$. Define $N(v)$ on $V$ by
\[
(10.41) \quad N(v) = \max \{ N_0(\rho_a(v)) : a \in G \}.
\]
It is not difficult to check that this defines an ultrametric norm on $k^n$. Indeed, each $N(\rho_a(v))$ defines an ultrametric norm on $k^n$, and passing to the maximum yields an ultrametric norm as well. By construction, this norm has the property that
\[
(10.42) \quad N(\rho_b(v)) = N(v)
\]
for all $b$ in the group $G$ and all vectors $v$ in $V$. If we choose $N_0$ so that it is nondegenerate, such as $N_0(x) = \max_{1 \leq j \leq n} |x_j|_*$, then $N$ is also nondegenerate. Similarly, if we choose $N_0$ so that it takes values in the set of nonnegative real numbers which occur as values of the absolute value function $| \cdot |_*$ on $k$, as with $N_0(x) = \max_{1 \leq j \leq n} |x_j|_*$, then $N$ has the same feature.

11 Operator norms

Let $k$ be a field with an absolute value function $| \cdot |_*$. Fix a positive integer $n$, and consider the vector space $k^n$ of $n$-tuples of elements of $k$. Let $N(\cdot)$ be a norm on $k^n$ with respect to $| \cdot |_*$, which we assume to be nondegenerate, in the sense of (10.11).

If $T$ is a linear operator from $k^n$ to itself, then the operator norm $\| T \|_{op}$ of $T$ with respect to $N$ can be defined by
\[
(11.1) \quad \| T \|_{op} = \sup \{ N(T(x)) : x \in k^n, \ N(x) = 1 \}.
\]
This supremum is finite because of the nondegeneracy condition for $N$. One can reformulate this definition as saying that
\[
(11.2) \quad N(T(x)) \leq \| T \|_{op} N(x)
\]
for all $x$ in $k^n$, and that $\| T \|_{op}$ is the smallest nonnegative real number with this property.

The collection of linear operators from $k^n$ to itself forms a vector space over $k$ in the usual manner, and $\| T \|_{op}$ defines a norm on this vector space, i.e., $\| T \|_{op}$ is a nonnegative real number which is equal to 0 exactly when $T$ is the zero linear operator on $k^n$,
\[
(11.3) \quad \| \alpha T \|_{op} = |\alpha|_* \| T \|_{op}
\]
for all $\alpha$ in $k$ and linear operators $T$ on $k^n$, and

\begin{equation}
\|T_1 + T_2\|_{op} \leq \|T_1\|_{op} + \|T_2\|_{op}
\end{equation}

(11.4)

for all linear operators $T_1, T_2$ on $k^n$. These properties follow from the corresponding features of the norm $N$. Notice that

\begin{equation}
\|I\|_{op} = 1,
\end{equation}

(11.5)

where $I$ denotes the identity operator on $k^n$, and that

\begin{equation}
\|T_1 \circ T_2\|_{op} \leq \|T_1\|_{op} \|T_2\|_{op}
\end{equation}

(11.6)

for any two linear operators $T_1, T_2$. If $|\cdot|$ is an ultrametric absolute value function and $N(\cdot)$ is an ultrametric norm, then the operator norm $\|\cdot\|_{op}$ is also an ultrametric norm, so that

\begin{equation}
\|T_1 + T_2\|_{op} \leq \max(\|T_1\|_{op}, \|T_2\|_{op})
\end{equation}

(11.7)

for all linear operators $T_1, T_2$ on $k^n$.

Suppose that $T, A$ are linear operators on $k^n$, $T$ is invertible, and

\begin{equation}
\|A\|_{op} < \|T^{-1}\|_{op}^{-1}.
\end{equation}

(11.8)

In this case $T + A$ is also an invertible linear operator on $k^n$. To see this, it suffices to show that the kernel of $T + A$ is trivial. Let $x$ be any element of $k^n$. If $y = T(x)$, then

\begin{equation}
N(T^{-1}(y)) \leq \|T^{-1}\|_{op} N(y),
\end{equation}

(11.9)

which is the same as saying that

\begin{equation}
\|T^{-1}\|_{op}^{-1} N(x) \leq N(T(x)).
\end{equation}

(11.10)

We also have that

\begin{equation}
N(T(x)) = N((T + A)(x) - A(x)) \leq N((T + A)(x)) + N(A(x)) \leq N((T + A)(x)) + \|A\|_{op} N(x),
\end{equation}

(11.11)

and hence

\begin{equation}
(\|T^{-1}\|_{op}^{-1} - \|A\|_{op}) N(x) \leq N((T + A)(x)).
\end{equation}

(11.12)
Thus \((T + A)(x) \neq 0\) when \(x \neq 0\), so that \(T + A\) is invertible, and we get the norm estimate

\[
\|(T + A)^{-1}\|_{op} \leq (\|T^{-1}\|_{op}^{-1} - \|A\|_{op})^{-1}.
\]

(11.13)

If \(|·|_s\) is an ultrametric absolute value function and \(N\) is an ultrametric norm, then (11.11) can be replaced with

\[
N(T(x)) \leq \max(N((T + A)(x)), N(A(x))) \\
\leq \max(N((T + A)(x)), \|A\|_{op} N(x)),
\]

so that

\[
\|T^{-1}\|_{op}^{-1} N(x) \leq \max(N((T + A)(x)), \|A\|_{op} N(x))
\]

for all \(x\) in \(k^n\). Because \(\|A\|_{op} < \|T^{-1}\|_{op}^{-1}\), we obtain that

\[
\|T^{-1}\|_{op}^{-1} N(x) \leq N((T + A)(x)).
\]

(11.15)

More precisely, one derives this initially for \(x \neq 0\), and then notes that it holds automatically for \(x = 0\), so that the inequality applies to all \(x\) in \(k^n\). This leads to

\[
\|(T + A)^{-1}\|_{op} \leq \|T^{-1}\|_{op}
\]

in the ultrametric case (under the condition (11.8)).

**Definition 11.18** Let \(\mathcal{B}\) be an algebra of operators on \(k^n\) which is a division algebra (Definition 5.1). We say that \(\mathcal{B}\) is a uniform division algebra if there is a positive real number \(C\) so that

\[
\|T^{-1}\|_{op} \leq C \|T\|_{op}^{-1}
\]

(11.19)

for all nonzero operators \(T\) in \(\mathcal{B}\).

**Remark 11.20** The opposite inequality is automatic, i.e.,

\[
1 \leq \|T^{-1}\|_{op} \|T\|_{op},
\]

(11.21)

since \(T^{-1} \circ T = I\) has norm 1.
This property does not depend on the choice of a nondegenerate norm $N$ on $k^n$, although the choice of $N$ can affect the constant in the definition. If $k$ is locally compact, so that $\{\alpha \in k : |\alpha|_* \leq 1\}$ is a compact set with respect to the metric $|\alpha - \beta|_*$ on $k$, then any algebra of operators on $k^n$ which is a division algebra is in fact a uniform division algebra. Indeed, it suffices to establish (11.19) for the $T$’s such that $\|T\|_{op} = 1$, and this is a compact subset of the vector space of linear operators on $k^n$ under our assumption. It is not hard to use compactness to get a uniform bound for $\|T^{-1}\|_{op}$, as desired.

For the rest of this section, let us assume that $|\cdot|_*$ is an ultrametric absolute value function on $k$, and consider the ultrametric norm

$$N_1(x) = \max_{1 \leq j \leq n} |x_j|_*$$

on $k^n$.

Let $T$ be a linear operator on $k^n$. Thus $T$ can be described by an $n \times n$ matrix $(a_{j,l})$ with entries in $k$, so that

$$(T(x))_j = \sum_{l=1}^{n} a_{j,l} x_l$$

for all $x$ in $k^n$, where $(T(x))_j$ denotes the $j$th component of $T(x)$, just as $x_l$ denotes the $l$th component of $x$. The operator norm $\|T\|_{op}$ of $T$ with respect to $N_1$ can be given by

$$\|T\|_{op} = \max_{1 \leq j,l \leq n} |a_{j,l}|_*.$$

To check that $\|T\|_{op}$ is less than or equal to the right side of the equation, one can use the ultrametric property of $|\cdot|_*$. For the opposite inequality, one can look at $T(x)$ in the special case where $x$ has one component equal to 1 and the rest equal to 0.

When is $T$ an isometry, i.e., when does $T$ satisfy

$$N_1(T(x)) = N_1(x)$$

for all $x$ in $k^n$? This is clearly equivalent to asking that $\|T\|_{op} \leq 1$ and $\|T^{-1}\|_{op} \leq 1$. Let us check that this happens if and only if $\|T\|_{op} \leq 1$ and $|\det T|_* = 1$, where $\det T$ denotes the determinant of $T$ (or, equivalently, of the matrix $(a_{j,l})$). Notice first that if $\|T\|_{op} \leq 1$, then $|\det T|_* \leq 1$. This uses merely the definition of the determinant, together with the fact that
the entries of the matrix associated to $T$ have absolute value less than or equal to 1, and the properties of the absolute value function. Similarly, the determinant of any square submatrix of $T$ has absolute value in $k$ less than or equal to 1. By Cramer’s rule, $T^{-1}$ is given by $(\det T)^{-1}$ times the cofactor transpose of $T$. The matrix entries for the cofactor transpose have absolute value less than or equal to 1 when $\|T\|_{\text{op}} \leq 1$, since they are given in terms of determinants of submatrices of the matrix of $T$. Thus the operator norm of the cofactor transpose is less than or equal to 1 when $\|T\|_{\text{op}} \leq 1$, because of the formula (11.24) applied to the cofactor transpose (instead of $T$). If $|\det T|_* = 1$, then it follows that $\|T^{-1}\|_{\text{op}} \leq 1$. Conversely, if $\|T^{-1}\|_{\text{op}} \leq 1$, then $|\det T^{-1}|_* \leq 1$, and hence $|\det T|_* \geq 1$, since $\det T^{-1} = (\det T)^{-1}$. This implies that $|\det T|_* = 1$ if $\|T\|_{\text{op}} \leq 1$ too.

As in Section 10, one way that isometries on $k^n$ come up is through representations of finite groups on $k^n$. That is, we might start with the norm $N_1$, not necessarily invariant under the group representation, and then obtain an invariant norm using (10.41). The new norm can be converted back into $N_1$ after a linear change of variables on $k^n$, if the absolute value function $|\cdot|_*$ is nice, as in Section 10. In other words, the representation of the finite group can be conjugated by an invertible linear transformation on $k^n$ to get a representation that acts by isometries with respect to $N_1$. Compare with Appendix 1 of Chapter IV in Part II of [Ser1].

In addition to norms for operators on a single vector space, one can consider norms for operators between vector spaces. Suppose that $V_1$, $V_2$ are vector spaces over the field $k$, and that $N_1(\cdot)$, $N_2(\cdot)$ are norms on $V_1$, $V_2$ with respect to the absolute value function $|\cdot|_*$ on $k$. We assume that $N_1$ is nondegenerate, as before, with respect to a linear equivalence between $V_1$ and $k^n$, where $n$ is the dimension of $V_1$. It is easy to check that the condition of nondegeneracy does not depend on the choice of linear isomorphism between $V_1$ and $k^n$.

If $T$ is a linear mapping from $V_1$ to $V_2$, then the operator norm $\|T\|_{\text{op},12}$ of $T$ with respect to $N_1$, $N_2$ on $V_1$, $V_2$ is defined by

$$(11.26) \quad \|T\|_{\text{op},12} = \sup \{ N_2(T(x)) : x \in V_1, N_1(x) = 1 \}. $$

The supremum makes sense because of the assumption of nondegeneracy for $N_1$. As before, the operator norm is characterized by the conditions that

$$(11.27) \quad N_2(T(x)) \leq \|T\|_{\text{op},12} N_1(x) $$

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for all $x$ in $V_1$ and that $\|T\|_{op,12}$ be the smallest nonnegative real number for which this property holds.

The operator norm defines a norm on the vector space of linear mappings from $V_1$ to $V_2$, as one can easily verify. If $|\cdot|_*$ is an ultrametric absolute value function on $k$, and $N_1, N_2$ are ultrametric norms on $V_1, V_2$, then $\|\cdot\|_{op,12}$ is an ultrametric norm on the vector space of linear mappings from $V_1$ to $V_2$.

Suppose that $V_3$ is another vector space over $k$, and that $N_3$ is a norm on $V_3$ with respect to the absolute value function $|\cdot|_*$ on $k$. Assume that the norm $N_2$ on $V_2$ is also nondegenerate in the sense described before. We can define operator norms $\|\cdot\|_{op,13}$ and $\|\cdot\|_{op,23}$ for linear mappings from $V_1$ to $V_3$ and from $V_2$ to $V_3$, respectively, using the norms $N_1, N_2, N_3$ on $V_1, V_2, V_3$. If $T$ is a linear mapping from $V_1$ to $V_2$ and $S$ is a linear mapping from $V_2$ to $V_3$, so that the composition $S \circ T$ defines a linear mapping from $V_1$ to $V_3$, then
\[
\|S \circ T\|_{op,13} \leq \|S\|_{op,23} \|T\|_{op,12}.
\]
(11.28)
This is easy to check.

## 12 Vector spaces of linear mappings

Let $k$ be a field, and let $V, W$ be vector spaces over $k$. Recall that $\mathcal{L}(V,W)$ denotes the vector space of linear mappings from $V$ to $W$. For notational simplicity, we shall write $H$ for $\mathcal{L}(V,W)$ in this section.

Suppose that $\mathcal{A}$ is an algebra of operators on $V$. We can associate to $\mathcal{A}$ an algebra of operators $\mathcal{A}_{H,1}$ on $H$, where a linear operator on $H$ lies in $\mathcal{A}_{H,1}$ if it is of the form $R \mapsto R \circ T$, $R \in H$, where $T$ lies in $\mathcal{A}$. Similarly, if $\mathcal{B}$ is an algebra of linear operators on $W$, then we can associate to it an algebra $\mathcal{B}_{H,2}$ consisting of operators on $H$ of the form $R \mapsto S \circ R$, $R \in H$, where $S$ lies in $\mathcal{B}$. In other words, we are using composition of operators in $H$ with operators on $V$ or on $W$ to define linear operators on $H$.

**Remark 12.1** As a basic case, let $\mathcal{A}$ be $\mathcal{L}(V)$ and let $\mathcal{B}$ be all of $\mathcal{L}(W)$. It is easy to check that every element of $\mathcal{A}_{H,1}$ commutes with every element of $\mathcal{B}_{H,2}$. In fact,
\[
(\mathcal{A}_{H,1})' = \mathcal{B}_{H,2}, \quad (\mathcal{B}_{H,2})' = \mathcal{A}_{H,1},
\]
(12.2)
where the prime refers to the commutant of the algebra as a subalgebra of $\mathcal{L}(H)$. This is not difficult to show.
Remark 12.3 Let $\mathcal{B}$ be any algebra of operators on $W$. The algebra of operators $\mathcal{B}_{H,2}$ on $H$ is essentially the same as “expanding” $\mathcal{B}$ as in Subsection 2.3, with $n$ equal to the dimension of $V$. In this way, Proposition 2.6 can be reformulated as saying that $(\mathcal{B}'_{H,2})'' = (\mathcal{B}_{H,2})''$. The description of $(\mathcal{B}'_{H,2})'$ in the proof of the proposition can be rephrased as saying that $(\mathcal{B}'_{H,2})'$ contains $(\mathcal{B}')_{H,2}$ and $(\mathcal{L}(V))_{H,1}$, and is generated by them.

Remark 12.4 Let $G_1, G_2$ be finite groups, and let $\sigma, \tau$ be representations of $G_1, G_2$ on $V, W$, respectively. We can define representations $\bar{\sigma}, \bar{\tau}$ of $G_1, G_2$ on $H$ by

$$\bar{\sigma}_x(R) = R \circ (\sigma_x)^{-1}, \quad \bar{\tau}_y(R) = \tau_y \circ R,$$

for $x \in G_1, y \in G_2$, and $R \in H$. If $A$ is the algebra of operators on $V$ generated by $\sigma$, then $A_{H,1}$ is the same as the algebra of operators on $H$ generated by $\bar{\sigma}$. Similarly, if $B$ is the algebra of operators on $W$ generated by $\tau$, then $B_{H,2}$ is the same as the algebra of operators on $H$ generated by $\bar{\tau}$.

Remark 12.6 Suppose that $k$ is a symmetric field, and that $V, W$ are equipped with inner products $\langle \cdot, \cdot \rangle_V, \langle \cdot, \cdot \rangle_W$, as in Subsection 1.7. For every linear mapping $R : V \to W$ there is an “adjoint” $R^*$, a linear mapping from $W$ to $V$, characterized by the condition

$$\langle R(v), w \rangle_W = \langle v, R^*(w) \rangle_V,$$

for all $v \in V, w \in W$. Note that if $T$ is a linear operator on $V$, and if $S$ is a linear operator on $W$, then $R \circ T, S \circ R$ are linear mappings from $V$ to $W$, and

$$(R \circ T)^* = T^* \circ R^*, \quad (S \circ R)^* = R^* \circ S^*.$$

Here $T^*$ is the adjoint of $T$ as an operator on $V$, $S^*$ is the adjoint of $S$ as an operator on $W$, and $R^*, (R \circ T)^*$, and $(S \circ R)^*$ are the adjoints of $R, R \circ T,$ and $S \circ R$ as operators from $V$ to $W$. (In general, if one has three inner product spaces, a linear mapping from the first to the second, and a linear mapping from the second to the third, then the adjoint of the composition is equal to the composition of the adjoints, in the opposite order.)

We can define an inner product $\langle \cdot, \cdot \rangle_H$ on $H$ by

$$\langle R_1, R_2 \rangle_H = \text{tr}_V R_2^* R_1 = \text{tr}_W R_1 R_2^*,$$

where $\text{tr}_V T$ denotes the trace of a linear operator $T$ on $V$ and $\text{tr}_W S$ denotes the trace of a linear operator $S$ on $W$. Recall that if $U_1$ is any linear mapping
from $V$ to $W$, and $U_2$ is any linear mapping from $W$ to $V$, so that $U_2 \circ U_1$ is a linear operator on $V$ and $U_1 \circ U_2$ is a linear operator on $W$, then $\text{tr}_V U_2 \circ U_1 = \text{tr}_W U_1 \circ U_2$. This is a standard property of the trace, and it implies the second equality in (12.9). To see that $\langle R_1, R_2 \rangle_H$ satisfies the positivity condition required of an inner product, one can compute $\text{tr}_V R^* \circ R$ using an orthogonal basis of $V$, and reduce to the positivity condition for the inner product on $W$, since

\[
\langle (R^* \circ R)(v), v \rangle_V = \langle R(v), R(v) \rangle_W
\]

for all $v$ in $V$. Of course one could instead use an orthogonal basis for $W$ and reduce to the positivity condition for the inner product on $W$.

Another way to describe the inner product on $H$ is to observe that if $R_1, R_2$ are rank-1 operators given by

\[
R_1(v) = \langle v, v_1 \rangle_V w_1, \quad R_2(v) = \langle v, v_2 \rangle_V w_2,
\]

where $v_1, v_2$ are elements of $V$ and $w_1, w_2$ are elements of $W$, then $R_2^*(w) = \langle w, w_2 \rangle_W v_2$ and

\[
\langle R_1, R_2 \rangle_H = \langle v_2, v_1 \rangle_V \langle w_1, w_2 \rangle_W.
\]

If $v_1, \ldots, v_n$ is an orthogonal basis for $V$, and $w_1, \ldots, w_m$ is an orthogonal basis for $W$, then

\[
\langle \cdot, v_j \rangle_V w_l, \quad 1 \leq j \leq n, \quad 1 \leq l \leq m
\]

is an orthogonal basis for $H$.

If $T$ is a linear operator on $V$, then $R \mapsto R \circ T$ defines a linear operator on $H$. One can verify that the adjoint of this linear operator, with respect to the inner product just defined on $H$, is given by $R \mapsto R \circ T^*$. Similarly, if $S$ is a linear operator on $W$, then $R \mapsto S \circ R$ defines a linear operator on $H$, and the adjoint of this operator is given by $R \mapsto S^* \circ R$.

Thus, if $\mathcal{A}$ is a $\ast$-algebra of operators on $V$, then $\mathcal{A}_{H,1}$ is a $\ast$-algebra of operators on $H$, and if $\mathcal{B}$ is a $\ast$-algebra of operators on $W$, then $\mathcal{B}_{H,2}$ is a $\ast$-algebra of operators on $H$. As in Remark 12.3, this also came up in Subsection 2.6.

Let $\mathcal{A}, \mathcal{B}$ be algebras of operators on $V$, $W$, respectively. The combined algebra of operators $\mathcal{C}$ on $H$ is defined to be the algebra generated by $\mathcal{A}_{H,1}$ and $\mathcal{B}_{H,2}$. Thus the elements of $\mathcal{C}$ are the operators on $H$ which can be written as

\[
A_1 B_1 + A_2 B_2 + \cdots + A_r B_r,
\]

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where each $A_j$ lies in $\mathcal{A}_{H,1}$ and each $B_j$ lies in $\mathcal{B}_{H,2}$. Because the elements of $\mathcal{A}_{H,1}$ and $\mathcal{B}_{H,2}$ commute with each other, one does not need more complicated products in (12.14).

**Remark 12.15** If $\mathcal{A} = \mathcal{L}(V)$ and $\mathcal{B} = \mathcal{L}(W)$, then the combined algebra $\mathcal{C}$ is all of $\mathcal{L}(H)$. This is not hard to check.

**Remark 12.16** Suppose that $k$ is a symmetric field and that $V$ and $W$ are equipped with inner products, as in Remark 12.6. If $\mathcal{A}$ and $\mathcal{B}$ are $\ast$-algebras of operators on $V$ and $W$, then the combined algebra $\mathcal{C}$ is a $\ast$-algebra on $H$ (using the inner product on $H$ defined in Remark 12.6).

**Remark 12.17** Suppose that $G_1, G_2$ are finite groups, and that $\sigma, \tau$ are representations of $G_1, G_2$ on $V, W$, respectively. Consider the product group $G_1 \times G_2$, in which the group operation is defined componentwise, using the group operations on $G_1, G_2$. Let $\rho$ be the representation on $H$ obtained from $\sigma, \tau$ by setting
\[
(12.18) \quad \rho_{(x,y)}(R) = \tau_y \circ R \circ (\sigma_x)^{-1}
\]
for all $(x,y)$ in $G_1 \times G_2$ and $R$ in $H$. If $\mathcal{A}$ is the algebra of operators on $V$ generated by $\sigma$, and $\mathcal{B}$ is the algebra of operators on $W$ generated by $\tau$, then the combined algebra $\mathcal{C}$ is the same as the algebra of operators on $H$ generated by the representation $\rho$ of $G_1 \times G_2$.

**Lemma 12.19** If $\mathcal{A}, \mathcal{B}$ are algebras of operators on $V, W$ with dimensions $r, s$, respectively, as vector spaces over $k$, then the combined algebra $\mathcal{C}$ has dimension $rs$.

To see this, suppose that $A_1, \ldots, A_r$ is a basis for $\mathcal{A}$, and that $B_1, \ldots, B_s$ is a basis for $\mathcal{B}$. Consider the operators
\[
(12.20) \quad R \mapsto B_j \circ R \circ A_l, \quad 1 \leq j \leq s, \quad 1 \leq l \leq r,
\]
on $H$. It is easy to check that the combined algebra $\mathcal{C}$ is spanned by these $rs$ operators, and we would like to show that they are linearly independent.

Let $c_{j,l}, 1 \leq j \leq s, 1 \leq l \leq r$, be an family of elements of $k$ such that the operator
\[
(12.21) \quad R \mapsto \sum_{j=1}^{s} \sum_{l=1}^{r} c_{j,l} B_j \circ R \circ A_l
\]
on $H$ is equal to 0, i.e.,

$$\sum_{j=1}^{s} \sum_{l=1}^{r} c_{j,l} B_{j} \circ R \circ A_{l} = 0$$

(12.22)

for all $R$ in $H$. If $w$ is any element of $W$ and $f$ is any linear mapping from $V$ to $k$, then $R(v) = f(v) w$ is a linear mapping from $V$ to $W$, and (12.22) can be rewritten for this choice of $R$ as

$$\sum_{j=1}^{s} \sum_{l=1}^{r} c_{j,l} B_{j}(w) f(A_{l}(v)) = 0$$

(12.23)

for all $v$ in $V$. Let us rewrite this again as

$$\sum_{j=1}^{s} \left( \sum_{l=1}^{r} c_{j,l} f(A_{l}(v)) \right) B_{j}(w) = 0$$

(12.24)

This holds for all $w$ in $W$, $v$ in $V$, and linear mappings $f$ from $V$ to $k$. For any fixed $v$ and $f$, the linear independence of the $B_{j}$’s as operators on $W$ leads to

$$\sum_{l=1}^{r} c_{j,l} f(A_{l}(v)) = 0$$

(12.25)

for each $j$. The linear independence of the $A_{l}$’s now implies that $c_{j,l} = 0$ for all $j$ and $l$, which is what we wanted.

### 13 Division algebras of operators, 2

Let $k$ be a field, let $V$ be a vector space, and let $\mathcal{A}$ be an algebra of operators on $V$ which is a division algebra. Put

$$\Gamma = \mathcal{A} \cap \mathcal{A}',$$

(13.1)

so that $\Gamma$ is the center of $\mathcal{A}$, i.e., the collection of elements of $\mathcal{A}$ which commute with all other elements of $\mathcal{A}$. Thus $\Gamma$ is an algebra of operators on $V$ which is commutative and a division algebra of operators.

It will be convenient to write $Y$ for $\mathcal{A}$ viewed as a vector space over $k$, for the purpose of considering linear operators on $Y$. Define $\mathcal{A}_{1}$ to be the algebra of linear operators on $Y$ of the form

$$R \mapsto R \circ T, \quad T \in \mathcal{A},$$

(13.2)
and define $\mathcal{A}_2$ to be the algebra of linear operators on $Y$ of the form
\begin{equation}
R \mapsto T \circ R, \quad T \in \mathcal{A}.
\end{equation}
When $T$ lies in $\Gamma$, $R \circ T = T \circ R$ for all $R$ in $Y$, and we write $\Gamma_0$ for the algebra of linear operators on $Y$ of the form
\begin{equation}
R \mapsto R \circ T = T \circ R, \quad T \in \Gamma.
\end{equation}

Lemma 13.5
(a) $\mathcal{A}_1$, $\mathcal{A}_2$ are division algebras of operators.
(b) Every element of $\mathcal{A}_1$ commutes with every element of $\mathcal{A}_2$.
(c) $\mathcal{A}_1$, $\mathcal{A}_2$ are irreducible algebras of operators on $Y$.

These properties are easy to verify, just from the definitions and the assumption that $\mathcal{A}$ is a division algebra of operators. Concerning (c), it can be helpful to rephrase the question as follows: if $R$, $\tilde{R}$ are elements of $Y$ and $R \neq 0$, then there are elements of $\mathcal{A}_1$, $\mathcal{A}_2$ which take $R$ to $\tilde{R}$.

Lemma 13.6
$\Gamma_0 = \mathcal{A}_1 \cap \mathcal{A}_2$.

Indeed, suppose that $S$, $T$ are elements of $\mathcal{A}$ such that the operators
\begin{equation}
R \mapsto R \circ S, \quad R \mapsto T \circ R
\end{equation}
on $Y$ are the same. We can take $R$ to be the identity operator on $V$ to obtain that $S = T$, and then our hypothesis becomes $R \circ T = T \circ R$ for all $R$ in $\mathcal{A}$. This says exactly that $T$ lies in $\Gamma$, as desired.

Lemma 13.8
$(\mathcal{A}_1)' = \mathcal{A}_2$ and $(\mathcal{A}_2)' = \mathcal{A}_1$, where the primes indicate that we take the commutant of the given algebra as an algebra of operators on $Y$.

The fact that $\mathcal{A}_2 \subseteq (\mathcal{A}_1)'$ and $\mathcal{A}_1 \subseteq (\mathcal{A}_2)'$ is the same as part (b) of Lemma 13.5. Conversely, suppose that $\Phi$ is a linear transformation on $Y$ which commutes with every element of $\mathcal{A}_1$. This is the same as saying that
\begin{equation}
\Phi(R \circ S) = \Phi(R) \circ S
\end{equation}
for all $R$, $S$ in $Y$, i.e., in $\mathcal{A}$. Applying this to $R$ equal to the identity operator $I$ on $V$ we get that
\begin{equation}
\Phi(S) = \Phi(I) \circ S
\end{equation}
for all $S$ in $Y$. This says exactly that $\Phi$ lies in $\mathcal{A}_2$, so that $(\mathcal{A}_1)' \subseteq \mathcal{A}_2$. Similarly, $(\mathcal{A}_2)' \subseteq \mathcal{A}_1$, and the lemma follows.

Let $\mathcal{A}_{12}$ be the algebra of operators on $Y$ which is generated by $\mathcal{A}_1$, $\mathcal{A}_2$. 87
Lemma 13.11  \( \Gamma_0 \) is the center of \( A_{12} \).

This is easy to check.

Proposition 13.12  \( A_{12} = \Gamma_0' \).

To prove this we use Lemma 5.18, where now \( Y \) has the role that \( V \) had before, \( A_{12} \) has the role that \( A \) had before, and \( \Gamma_0 \) has the role that \( B \) had before. Of course \( A_{12} \subseteq \Gamma_0' \), which was a standing assumption for Lemma 5.18, mentioned just before the statement of Lemma 5.16. According to 5.18, it suffices to show that \( A_{12} \) is irreducible, and that for any elements \( R_1, R_2, U_1, U_2 \) of \( Y \) such that \( R_1, R_2 \) are \( \Gamma_0 \)-independent there is a \( \Phi \) in \( A_{12} \) such that \( \Phi(R_1) = U_1 \) and \( \Phi(R_2) = U_2 \). (The notion of independence employed here was defined near the beginning of Section 5.) The irreducibility of \( A_{12} \) is a consequence of the irreducibility of either \( A_1 \) or \( A_2 \), and so it remains to verify the second condition.

For the second condition, it is enough to show that if \( R, U_1, U_2 \) are elements of \( Y \) such that \( R \) is not an element of \( \Gamma \subseteq A \), then there is a \( \Psi \) in \( A_{12} \) such that \( \Psi(I) = U_1, \Psi(R) = U_2 \). Here \( I \) denotes the identity operator on \( V \), as an element of \( Y = A \), just as \( \Gamma \) can be viewed as a subspace of \( Y \). In other words, if \( R_1, R_2 \) are elements of \( Y \) which are \( \Gamma_0 \)-independent, then they can be mapped to \( I, R_1^{-1} R_2 \) by an element of \( A_2 \subseteq A_{12} \), and the \( \Gamma_0 \)-independence of \( R_1, R_2 \) says exactly that \( R_1^{-1} R_2 \) does not lie in \( \Gamma \).

In fact it is enough to show that for every \( R, U \) in \( Y \) such that \( R \) is not in \( \Gamma \) there is a \( \Psi \) in \( A_{12} \) which satisfies \( \Psi(I) = 0 \) and \( \Psi(R) = U \). The reason for this is that for any \( U_1 \) in \( Y \) there is an element of \( A_1 \) (or \( A_2 \)) which sends \( I \) to \( U_1 \) (corresponding to composition with \( U_1 \)). To prescribe a value also at \( R \), one can use a \( \Psi \) in \( A_{12} \) as in the new version of the condition.

Thus we let \( R, U \) in \( Y \) be given, with \( R \) not in \( \Gamma \). Because \( R \) is not in \( \Gamma \), there is an \( S \) in \( Y = A \) such that \( R \circ S - S \circ R \neq 0 \). Thus the inverse of \( R \circ S - S \circ R \) exists and lies in \( A \). Let \( \Psi \) be the linear operator on \( Y \) defined by

\[
\Psi(A) = U \circ (R \circ S - S \circ R)^{-1} \circ (A \circ S - S \circ A),
\]

\( A \in Y = A \). It is easy to see that \( \Psi \) lies in \( A_{12} \), i.e., it can be written as a sum of compositions of operators on \( Y \) in \( A_1, A_2 \). Clearly \( \Psi(I) = 0 \) and \( \Psi(R) = U \), as desired. This proves Proposition 13.12.

Corollary 13.14  The dimension of \( A_{12} \) is equal to the square of the dimension of \( A = Y \) divided by the dimension of \( \Gamma \), as vector spaces over \( k \).
Compare with Lemma 5.13, and note that the dimension of \( \Gamma \) is equal to the dimension of \( \Gamma_0 \).

Let us now relate this to the set-up in Section 12. What were the vector spaces \( V, W \) before are now both taken to be \( V \). Thus we write \( H \) for the vector space of linear mappings from \( V \) to itself. The algebra of operators \( A \) on \( V \) is now used for both \( A \) and \( B \) in Section 12. Let \( A_{H,1}, A_{H,2}, \text{ and the combined algebra } C \) be the algebras of operators defined on \( H \) in Section 12 (with \( B = A \)).

We can view \( Y = A \) as a vector subspace of \( H \). As such, it is an invariant subspace for \( A_{H,1}, A_{H,2}, \) and the combined algebra \( C \). As in Notation 5.5, \( A_{H,1}(Y), A_{H,2}(Y), \) and \( C(Y) \) denote the algebras of operators on \( Y \) which occur as restrictions of operators in \( A_{H,1}, A_{H,2}, \) and \( C \) to the invariant subspace \( Y \). It is easy to check that

\[
(13.15) \quad A_1 = A_{H,1}(Y), \quad A_2 = A_{H,2}(Y), \quad A_{12} = C(Y).
\]

Just as for \( A \), we can associate to \( \Gamma \) the algebras of linear operators \( \Gamma_{H,1}, \Gamma_{H,2} \) on \( H \) of the form

\[
(13.16) \quad R \mapsto R \circ T, \quad R \mapsto T \circ R,
\]

respectively, where \( T \) lies in \( \Gamma \). Thus

\[
(13.17) \quad \Gamma_{H,1} \subseteq A_{H,1}, \quad \Gamma_{H,2} \subseteq A_{H,2},
\]

and \( Y \) is an invariant subspace for \( \Gamma_{H,1}, \Gamma_{H,2} \). For \( T \) in \( \Gamma \), the linear operators in (13.16) are different as operators on \( H \) unless \( T \) is a scalar multiple of the identity, but the restrictions of these operators to \( Y \) are the same, since \( \Gamma \) is the center of \( A \). In particular,

\[
(13.18) \quad \Gamma_{H,1}(Y) = \Gamma_{H,2}(Y) = \Gamma_0.
\]

Let \( T_1, \ldots, T_\ell \) be a collection of operators in \( A \) which are \( \Gamma_0 \)-independent and whose \( \Gamma_0 \)-span is all of \( A \), as discussed in Section 5, where we think of \( \Gamma_0 \simeq \Gamma \) as a division algebra of operators on \( A \). In fact, we can view \( A \) as a vector space over the field \( \Gamma \), since \( \Gamma \) is commutative and commutes with all elements of \( A \). In these terms \( T_1, \ldots, T_\ell \) is a basis for \( A \) as the vector space over \( \Gamma \). If \( \gamma_1, \ldots, \gamma_m \) is a basis for \( \Gamma \) as a vector space over \( k \), then the family of products \( \gamma_i \circ T_j, 1 \leq i \leq m, 1 \leq j \leq \ell \), is a basis for \( A \) as a vector space over \( k \).
As in the proof of Lemma 12.19, the operators on $H$ given by

$R \mapsto \gamma_{i_2} \circ T_{j_2} \circ R \circ \gamma_{i_1} \circ T_{j_1}$, \quad $1 \leq i_1, i_2 \leq m$, \quad $1 \leq j_1, j_2 \leq \ell$, \quad (13.19)

form a basis for the combined algebra $\mathcal{C}$. When one restricts to $Y$, as with $\mathcal{C}(Y)$, then it is enough to consider the operators of the form

$R \mapsto \gamma_i \circ T_{j_2} \circ R \circ T_{j_1}$, \quad $1 \leq i \leq m$, \quad $1 \leq j_1, j_2 \leq \ell$, \quad (13.20)

since the $\gamma_i$’s commute with $R \in Y$ and the $T_j$’s. In other words, on $Y$, the span of these operators is the same as the span of the previous ones. This fits with Corollary 13.14, which implies that $m\ell^2$ is equal to the dimension of $\mathcal{C}(Y) = \mathcal{A}_{12}$.

### 14 Some basic situations

#### 14.1 Some algebras of operators

Let $k$ be a field, and let $V$ be a vector space over $k$. Suppose that $V_1, \ldots, V_\ell$ are vector subspaces of $V$ such that

$\{0\} \subseteq V_1 \subseteq \cdots \subseteq V_\ell \subseteq V$, \quad (14.1)

where each inclusion is strict. Consider the algebra of operators $\mathcal{A}$ on $V$ consisting of the linear mappings $T : V \rightarrow V$ such that

$T(V_j) \subseteq V_j$, \quad $1 \leq j \leq \ell$, \quad (14.2)

**Lemma 14.3** Under the conditions above, if $Z$ and $W$ are arbitrary vector subspaces of $V$, then there are linear operators $T_1, T_2$ in $\mathcal{A}$ such that $T_1(V) = W$ and the kernel of $T_2$ is $Z$.

To be more precise, one can choose $T_1$ so that it satisfies

$T_1(V_j) = W \cap V_j$, \quad $1 \leq j \leq \ell$, \quad (14.4)

in addition to $T_1(V) = W$. This is not hard to arrange, using a suitable basis for $V$. Notice that this is not the only possibility, however; for instance, if the dimension of $W$ is less than or equal to the codimension of $V_\ell$ in $V$, then
one can choose $T_1$ so that $T_1$ is equal to 0 on $V_\ell$, but maps a subspace of $V$ whose intersection with $V_\ell$ is trivial onto $W$. As for $T_2$, we do need to have

$$(14.5) \quad \{v \in V_j : T_2(v) = 0\} = Z \cap V_j, \quad 1 \leq j \leq \ell,$$

in order for the kernel of $T_2$ to be equal to $Z$, and again this is not hard to arrange, through suitable bases.

**Proposition 14.6** Under the conditions above, the commutant $\mathcal{A}'$ of $\mathcal{A}$ consists of scalar multiples of the identity operator on $V$.

Of course $\mathcal{A}'$ always contains the scalar multiples of the identity. Conversely, suppose that $S$ lies in $\mathcal{A}'$. Because of Lemmas 14.3 and 2.8, if $U$ is any vector subspace of $V$, then $S(U) \subseteq U$. (For this one might as well even consider only 1-dimensional subspaces $U$ of $V$.) From this it is not hard to see that $S$ must be a scalar multiple of the identity operator.

### 14.2 A construction

Let $k$ be a field, let $V$ be a vector space, and let $V^*$ denote the dual space of $V$, i.e., the vector space of linear mappings from $V$ into $k$. If $T$ is a linear mapping from $V$ to itself, then there is an associated dual linear operator $\tilde{T}$ on $V^*$, which is defined by saying that if $\phi$ is a linear functional on $V$, then $\tilde{T}(\phi)$ is the linear functional given by $\tilde{T}(\phi)(v) = \phi(T(v))$. The transformation from a linear operator $T$ on $V$ to the dual linear operator $\tilde{T}$ is linear in $T$, and satisfies $(\tilde{T}_1 \circ \tilde{T}_2) = \tilde{T}_2 \circ \tilde{T}_1$ for all linear operators $T_1, T_2$ on $V$. The dual of the identity operator on $V$ is equal to the identity operator on $V^*$, and the dual of an invertible operator on $V$ is an invertible operator on $V^*$, with the inverse of the dual being the dual of the inverse. There is a natural identification of $V$ with the second dual space $V^{**}$, and the dual of the dual of an operator $T$ on $V$ coincides with $T$ under this identification.

Assume now that $\mathcal{A}$ is an algebra of linear operators on $V$. We can define $\tilde{\mathcal{A}}$ to be the algebra of linear operators on the dual space $V^*$ consisting of the duals of the linear operators in $\mathcal{A}$.

Let $W$ be the vector space which is the direct sum of $V$ and $V^*$. Thus $W$ consists of ordered pairs $(u, \phi)$ with $u$ in $V$ and $\phi$ in $V^*$, with addition and scalar multiplication of vectors defined coordinatewise. Define $\hat{\mathcal{A}}$ to be the collection of operators on $W$ of the form

$$(14.7) \quad (u, \phi) \mapsto (S(u), \tilde{T}(\phi)),$$
where \( S, T \) lie in \( \mathcal{A} \). It is not hard to see that \( \hat{A} \) is an algebra of operators on \( W \). The subspaces \( V \times \{0\} \) and \( \{0\} \times V^* \) of \( W \) are complementary to each other and invariant under \( \hat{A} \), and, in effect, the restriction of \( \hat{A} \) to these subspaces reduces to \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \), respectively.

Define bilinear forms \( B_1, B_2 \) on \( W \) by

\[
B_1((u, \phi), (v, \psi)) = \psi(u) + \phi(v)
\]

and

\[
B_2((u, \phi), (v, \psi)) = \psi(u) - \phi(v).
\]

Notice that \( B_1 \) is a symmetric bilinear form while \( B_2 \) is antisymmetric, i.e.,

\[
B_1((u, \phi), (v, \psi)) = B_1((v, \psi), (u, \phi)),
\]

\[
B_2((u, \phi), (v, \psi)) = -B_2((v, \psi), (u, \phi)).
\]

Both are nondegenerate, which is to say that for \( i = 1 \) or \( 2 \) and for any nonzero element \((u, \phi)\) of \( W \) there is a \((v, \psi)\) in \( W \) such that

\[
B_i((u, \phi), (v, \psi)) \neq 0.
\]

On the other hand, it is not true that \( B_1 \) is definite, because there are plenty of nonzero elements \((u, \phi)\) of \( W \) which satisfy

\[
B_1((u, \phi), (u, \phi)) = 0.
\]

For \( B_2 \) we have that

\[
B_2((u, \phi), (u, \phi)) = 0
\]

for all \((u, \phi)\) in \( W \), which is in fact equivalent to the antisymmetry of \( B_2 \).

Suppose that \( S, T \) are elements of \( \mathcal{A} \), and that \( R \) is the element of \( \hat{A} \) given by \( R(u, \phi) = (S(u), \bar{T}(\phi)) \), as in (14.7). Observe that

\[
B_1(R(u, \phi), (v, \psi)) = \psi(S(u)) + \phi(T(v)) = B_1((u, \phi), R'(v, \psi)),
\]

where \( R' \) is the element of \( \hat{A} \) defined by

\[
R'(v, \psi) = (T(v), \bar{S}(\psi)).
\]

Similarly,

\[
B_2(R(u, \phi), (v, \psi)) = \psi(S(u)) - \phi(T(v)) = B_2((u, \phi), R'(v, \psi)).
\]
To put it another way, $R^t$ is the “transpose” of $R$ as a linear mapping from $W$ to itself with respect to each of the bilinear forms $B_1$ and $B_2$. Thus we see that $\hat{\mathcal{A}}$ contains the transposes of all of its elements, which is to say that it is a “$t$-algebra” of operators on $W$, in a sense analogous to that of Definition 3.11, considered before in the setting of definite scalar products.

Let us specialize to the case where $\mathcal{A}$ is as in Subsection 14.1, corresponding to the subspaces $V_1, \ldots, V_\ell$ of $V$. For any vector subspace $U$ of $V$, we denote by $U^\perp$ the subspace of $V^*$ consisting of linear functionals $\phi$ on $V$ such that $\phi(u) = 0$ for all $u \in U$. A standard observation from linear algebra is that a vector $v$ in $V$ lies in $U$ if and only if $\phi(v) = 0$ for all $\phi$ in $U^\perp$. From (14.17) we get that
\[
\{0\} \subseteq V_\ell^\perp \subseteq \cdots \subseteq V_1^\perp \subseteq V^*,
\]
with the inclusions again being strict since they were strict before. One can check that a linear operator on $V^*$ lies in $\hat{\mathcal{A}}$ in these circumstances if and only if each subspace $V_j^\perp$, $1 \leq j \leq \ell$, of $V^*$ is invariant under the operator.

One can also verify that the commutant of $\hat{\mathcal{A}}$ in $\mathcal{L}(W)$ consists of linear combinations of the coordinate projections from $W \simeq V \times V^*$ onto $V \times \{0\}$ and $\{0\} \times V^*$.

15 Positive characteristic

Throughout this section, we let $p$ be a prime number, and we let $k$ be a field of characteristic $p$.

15.1 A few basic facts

The most basic example of a field with characteristic $p$ is $\boldsymbol{Z}/p\boldsymbol{Z}$, the field with $p$ elements obtained by taking the integers $\boldsymbol{Z}$ modulo $p$. Any field with characteristic $p$ contains a copy of this field in a canonical way, as the set of elements obtained by taking sums of the multiplicative identity element. In particular, any such field can be viewed as a vector space over $\boldsymbol{Z}/p\boldsymbol{Z}$, and if a field of characteristic $p$ has a finite number of elements, then that number is a power of $p$. A well-known result is that for each positive integer $m$ there is a field of characteristic $p$ with $p^m$ elements, and this field is unique up to isomorphism.
In a field $k$ of characteristic $p$ we have that
\begin{align}
(x + y)^p &= x^p + y^p,
\end{align}
for all $x, y$ in $k$. Indeed,
\begin{align}
(x + y)^p &= \sum_{j=0}^{p} \binom{p}{j} x^j y^{p-j} = x^p + y^p,
\end{align}
where the first step is an instance of the binomial theorem and the second step uses the observation that the binomial coefficient $\binom{p}{j}$ is a positive integer which is divisible by $p$ when $1 \leq j \leq p - 1$, and hence gives 0 in the field $k$.

Notice that
\begin{align}
(x - y)^p &= x^p - y^p
\end{align}
for all $x, y$ in $k$. This follows from (15.1) and the fact that $(-1)^p = -1$ in $k$. More precisely, $(-1)^2 = 1 = -1$ when $p = 2$, while $(-1)^p = -1$ when $p$ is odd because $(-1)^a = 1$ when $a$ is an even integer, such as $p - 1$.

By iterating these identities one obtains
\begin{align}
(x + y)^p &= x^p + y^p
\end{align}
and
\begin{align}
(x - y)^p &= x^p - y^p
\end{align}
for all positive integers $l$ and all $x, y$ in $k$. As a consequence of the latter we obtain the following.

**Lemma 15.6** If $l$ is a positive integer and $x$ is an element of $k$ which satisfies $x^p = 1$, then $x = 1$.

By contrast, there can be nontrivial roots of unity of other orders. If $k$ is finite, with $p^m$ elements for some positive integer $m$, then the set of nonzero elements of $k$ is a group under multiplication of order $p^m - 1$. It is well known that this group is in fact cyclic. The identity
\begin{align}
x^{p^m-1} &= 1
\end{align}
for all nonzero elements $x$ of $k$ can be derived more simply, from the general result that the order of a group is divisible by the order of any subgroup, and hence by the order of any element of the group. Of course $p^m - 1$ is not
divisible by $p$. Since $p^m - 1$ is the number of nonzero elements in $k$ when $k$ has $p^m$ elements, we see that the polynomial $x^{p^m-1} - 1$ completely factors in $k$, i.e., it is the product of the $p^m - 1$ linear polynomials $x - \alpha$, where $\alpha$ runs through the set of nonzero elements of $k$. Alternatively, one can say that the polynomial $x^{p^m} - x = x(x^{p^m-1} - 1)$ completely factors, as the product of the linear polynomials $x - \alpha$, where now $\alpha$ runs through all elements of $k$.

15.2 Representations

Let $V$ be a vector space over $k$, let $G$ be a finite group, and let $\rho$ be a representation of $G$ on $V$. If $p$ divides the order of $G$, then the argument described in Subsection 1.5 for finding invariant complements of invariant subspaces of a representation does not work, and indeed the result is not true.

If $W$ is a subspace of $V$ which is invariant under $\rho$, then one can at least consider the quotient vector space $V/W$, and the representation of $G$ on $V/W$ obtained from $\rho$ in the obvious manner. Recall that $V/W$ is defined using the equivalence relation $\sim$ on $V$ given by

$$v_1 \sim v_2 \text{ if and only if } v_1 - v_2 \in W,$$

by passing to the corresponding equivalence classes. The assumption that $W$ is invariant under $\rho$ implies that this equivalence relation is preserved by $\rho$, so that $\rho$ leads to a representation on the quotient space $V/W$.

Thus if the representation $\rho$ on $V$ is not irreducible, so that there is an invariant subspace $W$ which is neither the zero subspace nor $V$, then we get two representations of smaller positive degree, namely the restriction of $\rho$ to $W$ and the representation on $V/W$ obtained from $\rho$. The sum of the degrees of these two new representations is equal to the degree of the original representation, i.e., the sum of the dimensions of $W$ and $V/W$ is equal to the dimension of $V$. We can repeat the process, of passing to subspaces and quotients, to get a family of irreducible representations, where the sum of the degrees of these representations is equal to the degree of $\rho$ (the dimension of $V$). However, because we do not necessarily have direct sum decompositions at each step, the family of irreducible representations may not contain as much information as the original representation.

Let $F(G)$ denote the vector space of $k$-valued functions on $G$. As in Subsection 1.3, one has the left and right regular representations $L, R,$ of
Let $Z$ be a vector space over $k$, $\sigma$ a representation of $G$ on $Z$, and $\lambda$ a nonzero linear mapping from $Z$ to $k$. For each $v$ in $Z$, define $f_v(y)$ on $G$ by
\begin{equation}
(15.10) \quad f_v(y) = \lambda(\sigma_{y^{-1}}(v)),
\end{equation}
and put
\begin{equation}
(15.11) \quad U = \{f_v(y) : v \in Z\}.
\end{equation}

The mapping
\begin{equation}
(15.12) \quad v \mapsto f_v
\end{equation}
is a nonzero linear mapping from $Z$ onto $U$, and $U$ is a nonzero vector subspace of $F(G)$. This mapping intertwines the representations $\sigma$ on $Z$ and the restriction of the left regular representation to $U$, and $U$ is invariant under the left regular representation. If $\sigma$ is an irreducible representation of $G$, then (15.12) is one-to-one and yields an isomorphism between $\sigma$ and the restriction of the left regular representation to $U$.

This is essentially the same as Lemma 7.7, and the proof is the same as before.

### 15.3 Linear transformations

Let $V$ be a vector space over $k$. Suppose that $R$ and $S$ are two linear transformations on $V$ which commute. Once again we have that
\begin{equation}
(15.13) \quad (R + S)^p = R^p + S^p.
\end{equation}

This can be established in the same manner as before, by expanding $(R + S)^p$ using the binomial theorem, and then observing that the intermediate terms vanish because $k$ has characteristic $p$. We also get
\begin{equation}
(15.14) \quad (R - S)^p = R^p - S^p,
\end{equation}
and, for any positive integer $l$,
\begin{align}
(15.15) \quad (R + S)^p &= R^p + S^p \\
(15.16) \quad (R - S)^p &= R^p - S^p.
\end{align}

Now assume that $T$ is a linear transformation on $V$ such that $T^p = I$ for some positive integer $l$, where $I$ denotes the identity transformation on $V$. Since $T$ and $I$ automatically commute, we get that
\begin{equation}
(15.17) \quad (T - I)^p = T^p - I = 0.
\end{equation}
Thus $T - I$ is nilpotent of order (at most) $p^l$.

If $n$ is a positive integer and $S$ is a linear transformation on $V$ such that
\begin{equation}
(15.18) \quad S^n = I,
\end{equation}
then there is a well-known way to factor $S$ in order to separate the factors of $p$ in $n$ from the rest. We begin by writing $n$ as $p^l m$, where $l$ is a nonnegative integer and $m$ is a positive integer which is not divisible by $p$. Because $p^l$ and $m$ are relatively prime, it is a standard result that there are integers $a$, $b$ such that
\begin{equation}
(15.19) \quad a m + b p^l = 1.
\end{equation}
Thus we can write
\begin{equation}
(15.20) \quad S = S_1 S_2, \quad \text{where} \quad S_1 = S^{a m}, \quad S_2 = S^{b p^l},
\end{equation}
and where it is understood that $S^c$ is interpreted as being the identity transformation when $c = 0$. With these choices we have that
\begin{equation}
(15.21) \quad S_1^p = I, \quad S_2^n = I, \quad \text{and} \quad S_1 S_2 = S_2 S_1.
\end{equation}

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