Reparametrization-Invariant Effective Action
in Field-Antifield Formalism

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Abstract

We introduce classical and quantum antifields in the reparametrization-invariant effective action,
and derive a deformed classical master equation.

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1 Introduction

It is well-known that the basic/standard notion of effective action $\Gamma(\Phi)$ in quantum field theory is not reparametrization-invariant, cf. Sections 2–3. A remedy was proposed by Vilkovisky [1, 2, 3] by using a connection $\Gamma_{\alpha\beta}$ on the field configuration manifold $\mathcal{M}$, cf. Sections 4–5 and Appendix B. In this paper, we amend the reparametrization-invariant construction with antifields, and develop the corresponding field-antifield formalism [4, 5, 6, 7]. We derive a Ward identity (6.8) and a deformed classical master equation (6.10), cf. Section 6 and Appendix A. The resulting approach works in principle for an arbitrary gauge theory. A manifest superfield approach is considered in Appendix C.

In the following we will use DeWitt condensed notation.

2 Legendre Transformation

In quantum field theory, one often performs a Legendre transformation

$$W_c(J) - \Gamma(\Phi) \equiv J_\alpha \Phi^\alpha$$

(2.1)

to change variables $J_\alpha \leftrightarrow \Phi^\alpha$ from sources $J_\alpha$ to classical fields $\Phi^\alpha$. Here $W_c \equiv \frac{\hbar}{i} \ln \mathcal{Z}$ is the generating action for connected diagrams and $\Gamma(\Phi)$ is the effective action. One usually takes $n$ implicit relations

$$J_\alpha = J_\alpha(\Phi) \iff \Phi^\alpha = \Phi^\alpha(J)$$

(2.2)

to be of the form

$$\Phi^\alpha = \left( \frac{\partial}{\partial J_\alpha} W_c \right)^{(2.1)} \iff J_\alpha = -\left( \Gamma \frac{\partial}{\partial \Phi^\alpha} \right).$$

(2.3)

We stress that although the relations (2.3) are the most natural choice of implicit relations (2.2), they are not the only possibility, as we shall see in Section 5.

3 Standard Partition Function and Effective Action

The standard (non-reparametrization-invariant) partition function $\mathcal{Z}(J)$ depends on sources $J_\alpha$

$$e^{\frac{i}{\hbar} W_c(J)} \equiv \mathcal{Z}(J) := \int d\mu \ e^{\frac{i}{\hbar} (W(\varphi) + J_\alpha \varphi^\alpha)}.$$  

(3.1)

The quantum average is

$$\langle F \rangle_J := \frac{1}{\mathcal{Z}(J)} \int d\mu \ e^{\frac{i}{\hbar} (W(\varphi) + J_\alpha \varphi^\alpha)} F, \quad F = F(\varphi).$$

(3.2)

Here $W = W(\varphi)$ is a (gauge-fixed) quantum action, and $\varphi^\alpha$ is the quantum field/integration variable of Grassmann parity $\varepsilon_\alpha$. The level-zero* measure in the path integral (3.1) is

$$d\mu := \rho(d\varphi) = \rho \prod_\alpha d\varphi^\alpha, \quad \rho = \rho(\varphi).$$

(3.3)

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*The multi-level formalism was introduced in Refs. [8, 9, 10, 11, 12] and reviewed in Ref. [13].
The effective action is defined as

\[ e^{\frac{i}{\hbar} \Gamma(\Phi)} := \left. \int d\mu e^{\frac{i}{\hbar} (W(\phi) + J_\alpha (\phi^\alpha - \Phi^\alpha))} \right|_{J=J(\Phi)}, \tag{3.4} \]

where the implicit relations (2.2) are given by the standard Legendre relations (2.3). Standard reasoning yields that

\[ \Phi^\alpha(J) \overset{(2.3)}{=} \left( \frac{\partial}{\partial J_\alpha} W_\varepsilon \right) \overset{(3.1)}{=} \left( \frac{\partial}{\partial J_\alpha} W^{\text{expl}} \right) \overset{(3.5)}{=} \langle \phi^\alpha \rangle_J, \]

while

\[ \left( \frac{\partial}{\partial \phi^\alpha} \right) \overset{(3.4)}{=} -J_\alpha(\Phi), \tag{3.6} \]

and

\[ \left( \frac{\partial}{\partial \phi^\alpha} \right) - \left( \frac{\partial}{\partial \phi^\alpha} \right)^{\text{impl}} = \left( \frac{\partial}{\partial \phi^\alpha} \right)^{\text{expl}} \overset{(3.6) + (3.5)}{=} 0. \tag{3.7} \]

Moreover,

\[ \langle F(\phi) \rangle_J \overset{(3.1) + (3.2)}{=} e^{-\frac{i}{\hbar} W_\varepsilon(J)} F \left( \frac{\hbar}{i} \frac{\partial}{\partial J_\alpha} \right) e^{\frac{i}{\hbar} W_\varepsilon(J)} = F(\Phi(J)) + O(\hbar). \tag{3.8} \]

Here explicit dependence “expl” means dependence that is not via the implicit relations (2.2). Note that in the standard Legendre transformation (2.3), total and explicit differentiations are the same.

4 Logarithmic Map

Let \((M, \nabla)\) be the \(n\)-dimensional field configuration manifold \(M\) endowed with a torsion-free (tangent space) connection \(\nabla\). Let \(M\) have local (position) coordinates \(\phi^\alpha\) with Grassmann parity \(\varepsilon(\phi^\alpha) = \varepsilon_\alpha\), where \(\alpha = 1, \ldots, n\).

Let \(\Phi\) be a fixed base point. Let \(V \subseteq T_\Phi M\) be an sufficiently small open neighborhood of the zero (velocity) vector \(0 \in T_\Phi M\). The exponential map \(\text{Exp}_\Phi: V \subseteq T_\Phi M \to M\) takes a (velocity) vector \(v_\Phi \in V\) and maps it to the unique point \(\phi \in M\) on the manifold that is reached along a geodesic \(\gamma: [t_0, t_1] \to M\), i.e.,

\[ \gamma(t=t_0) = \Phi, \quad (t_1 - t_0) \dot{\gamma}(t=t_0) = v_\Phi, \quad \text{Exp}_\Phi(v_\Phi) := \gamma(t=t_1) = \phi. \tag{4.1} \]

These formulas are invariant under affine reparametrizations \(t \to at+b\) of the geodesic \(\gamma\). The geodesic differential equation reads

\[ 0 = (\nabla_\gamma \dot{\gamma})^\alpha = \dot{\phi}^\beta (\nabla_\beta \dot{\gamma})^\alpha = \dot{\phi}^\beta \left( \partial_\beta \dot{\gamma} + \Gamma^\alpha_\beta \dot{\gamma}^\beta \right) = \dot{\gamma}^\alpha + (-1)^{\varepsilon_\beta} \Gamma^\alpha_\beta \dot{\gamma}^\beta. \tag{4.2} \]

For a point \(\phi\) sufficiently close to the fixed point \(\Phi \in M\) (technically speaking, for points in a so-called normal neighborhood \(\phi \in U(\Phi) \subseteq M\)), there exists a unique geodesic \(\gamma: [t_0, t_1] \to M\) that goes from \(\Phi\) to \(\phi\). One defines the logarithmic map \(\text{Ln}_\Phi : U(\Phi) \to T_\Phi M\) as the inverse of the exponential map, i.e., it has the corresponding initial velocity vector as output,

\[ \gamma(t=t_0) = \Phi, \quad \gamma(t=t_1) = \phi, \quad \text{Ln}_\Phi(\phi) := (t_1 - t_0) \dot{\gamma}(t=t_0). \tag{4.3} \]
Often in the literature, the logarithmic map (i.e., the initial velocity vector) is denoted as
\[ Ln_\Phi(\varphi) = -\sigma^\alpha(\Phi, \varphi) \frac{\partial^\ell}{\partial \Phi^\alpha} , \quad \sigma^\alpha(\Phi, \varphi) = (t_0 - t_1) \dot{\gamma}^\alpha(t = t_0) . \] (4.4)

The coordinate functions \(-\sigma^\alpha(\Phi, \varphi)\) are also known as the **Riemann normal coordinates** based at \(\Phi\). Note that the bi-local coordinate function \(\sigma^\alpha(\Phi, \varphi)\) behaves geometrically as a vector with respect to the point \(\Phi\) and as a scalar with respect to the point \(\varphi\). A short-distance expansion of the logarithmic map reads
\[ \sigma^\alpha(\Phi, \varphi) = (\Phi - \varphi)^\alpha - \frac{(-1)^\varepsilon}{2} \Gamma^\alpha_{\beta\gamma}(\Phi) (\Phi - \varphi)^\gamma (\Phi - \varphi)^\beta + \mathcal{O}((\Phi - \varphi)^3) . \] (4.5)

If the Christoffel symbols \(\Gamma^\alpha_{\beta\gamma} = 0\) vanish identically in the neighborhood \(\mathcal{U}(\Phi)\), then the logarithmic map is simply given by
\[ \sigma^\alpha(\Phi, \varphi) = \Phi^\alpha - \varphi^\alpha \quad \text{if} \quad \Gamma^\alpha_{\beta\gamma} = 0 . \] (4.6)

The logarithmic map satisfies the differential equation
\[ \sigma^\beta(\Phi, \varphi) \left( \nabla^\beta(\Phi) \sigma^\alpha(\Phi, \varphi) \right) = \sigma^\alpha(\Phi, \varphi) . \] (4.7)

### 5 Reparametrization-Invariant Effective Action

The standard effective action (3.4) is not invariant under reparametrizations of the quantum field \(\varphi^\alpha\) and the classical field \(\Phi^\alpha\).

The source \(J_\alpha\) behaves by definition as a co-vector (scalar) under reparametrizations of the point \(\Phi\) (the point \(\varphi\)), respectively. In particular the term \(J_\alpha \Phi^\alpha\) is *not* a scalar under reparametrizations \(\Phi^\alpha \rightarrow \Phi'^\beta = f^\beta(\Phi)\). Since we want to maintain eq. (2.1), it therefore becomes impossible to make both \(W_c\) and \(\Gamma\) reparametrization-invariant quantities simultaneously. We will focus on the latter, *i.e.*, the effective action \(\Gamma\).

A reparametrization-invariant effective action can be achieved by using the logarithmic map [1, 2]
\[ e^{\frac{i}{\hbar} \Gamma(\Phi)} := \int d\mu \exp(i(W(\varphi) - J_\alpha \sigma^\alpha(\Phi, \varphi))) \bigg|_{J = J(\Phi)} . \] (5.1)

The quantum average is
\[ \langle F \rangle := e^{-\frac{i}{\hbar} \Gamma(\Phi)} \int d\mu \exp(i(W(\varphi) - J_\alpha \sigma^\alpha(\Phi, \varphi))) F \bigg|_{J = J(\Phi)} . \] (5.2)

Since we assume the Legendre relation (2.1), the partition function becomes
\[ e^{\frac{i}{\hbar} W_c(J)} \equiv Z(J) := \int d\mu \exp(i(W(\varphi) + J_\alpha(\Phi^\alpha - \sigma^\alpha(\Phi, \varphi))) \bigg|_{\Phi = \Phi(J)} . \] (5.3)

Let us now elaborate on the status of the implicit dependence (2.2).

If one uses the standard Legendre relations (2.3), one gets
\[ J_\alpha(\Phi) \overset{(2.3)}{=} -\left( \frac{\partial}{\partial \Phi^\alpha} \right) \overset{(5.1)}{=} \left( J_\beta(\Phi) \sigma^\beta(\Phi, \varphi) \right) \frac{\partial}{\partial \Phi^\alpha} , \quad \text{(Not in use!)} \] (5.4)
or equivalently,
\[
0 \stackrel{(2.3)}{=} \Phi^\alpha(J) - \left( \frac{\partial}{\partial J_\alpha} W_\gamma \right) \stackrel{(5.3)}{=} \left( \frac{\partial}{\partial J_\alpha} \left( J_\beta \sigma^\beta(\Phi(J), \varphi) \right) \right), \quad \text{(Not in use!)} \quad (5.5)
\]
However, we shall here not use the standard Legendre relations (5.4) and (5.5).

Instead we shall impose reparametrization-invariant implicit conditions
\[
\left( \frac{\partial}{\partial J_\alpha} W_\gamma \right) = \Phi^\alpha(J) \quad \iff \quad -\left( \frac{\partial}{\partial J_\alpha} \Gamma \right) \equiv \langle \sigma^\alpha(\Phi, \varphi) \rangle = 0,
\]
(5.6)
as advocated by Vilkovisky [1, 2, 3]. The main point is that condition (5.6) is covariant (invariant) under reparametrizations of the classical field \( \Phi^\alpha \) (quantum field \( \varphi^\alpha \)), respectively. The condition (5.6) implies that the total and explicit differentiations of the effective action \( \Gamma \) with respect to the classical field \( \Phi^\alpha \) are the same
\[
\left( \frac{\partial}{\partial \Phi^\alpha} \Gamma \right) - \left( \frac{\partial}{\partial \Phi^\beta} \Gamma \right) \equiv \left( \frac{\partial}{\partial \Phi^\gamma} \Gamma \right) \left( \frac{\partial}{\partial \Phi^\delta} \Gamma \right) \stackrel{(5.1)+(5.6)}{=} 0. \quad (5.7)
\]
Note that the condition (5.6) means that the classical field \( \Phi^\alpha \) and the quantum average \( \langle \varphi^\alpha \rangle \) may differ (even at the classical level). In particular, the classical decomposition formula (3.8) may no longer hold.

However, if the Christoffel symbols \( \Gamma^\gamma_{\alpha \beta} = 0 \) vanish identically, then
1. the reparametrization-invariant effective action (5.1) reduces to the standard effective action (3.4);
2. the \( n \) implicit conditions (5.6) reduce to the standard conditions
\[
\Phi^\alpha = \langle \varphi^\alpha \rangle \quad \text{if} \quad \Gamma^\gamma_{\alpha \beta} = 0,
\]
(5.8)

cf. eq. (4.6).

Finally, let us mention that one could in principle perform a change of integration variables
\[
\varphi^\alpha \rightarrow \varphi^\alpha := \sigma^\alpha(\Phi, \varphi)
\]
(5.9)
to bring the path integral (5.1) back to the form (3.4) (and similarly bring the average (5.6) back to eq. (3.5)), with the caveat that the new action \( W'(\Phi, \varphi) = W(\Phi, \sigma(\Phi, \varphi)) \) and measure \( \rho'(\Phi, \varphi) \) would depend on the classical fields \( \Phi \).

6 Antifields

Next we introduce quantum antifields \( \varphi^\alpha_* \) and classical antifields \( \Phi^\alpha_* \) with opposite Grassmann parity \( \varepsilon_\alpha + 1 \) of the corresponding field variables \( \varphi^\alpha \) and \( \Phi^\alpha \), which in turn carry Grassmann parity \( \varepsilon_\alpha \). The antifields are co-vectors under reparametrizations of \( \varphi \) and \( \Phi \), respectively. The reparametrization-invariant effective action is
\[
e^{\frac{i}{\hbar} \Gamma(\Phi, \Phi^*)} := \int d\mu \ e^{\frac{i}{\hbar} (W(\varphi, \varphi^*) - J_\alpha \sigma^\alpha(\Phi, \varphi)) \big|_{J=J(\Phi, \Phi^*)}} \big|_{\varphi^* = \frac{\partial \Phi^*}{\partial \varphi}}.
\]
(6.1)
(The vertical line notation on the right-hand side of eq. (6.1) means that the two formulas to the right of the vertical line should be substituted into the path integral. By definition the substitution \( J = J(\Phi, \Phi^*) \) counts as implicit dependence, while the substitution \( \varphi^* = \frac{\partial \Phi}{\partial \varphi} \) counts as explicit dependence.) The quantum average is

\[
\langle F \rangle := e^{-\frac{i}{\hbar} \Gamma(\Phi, \Phi^*)} \int d\mu e^{\frac{i}{\hbar}(W(\varphi, \varphi^*) - J_\alpha \sigma^\alpha(\Phi, \varphi))} F |_{J = J(\Phi, \Phi^*)}^{\varphi^* = \frac{\partial \Phi}{\partial \varphi}} .
\]

(6.2)

The \( n \) implicit relations \( J_\alpha = J_\alpha(\Phi, \Phi^*) \) come by definition from the \( n \) conditions

\[
\langle \sigma^\alpha(\Phi, \varphi) \rangle = 0 \quad \Leftrightarrow \quad J_\alpha = J_\alpha(\Phi, \Phi^*) ,
\]

cf. condition (5.6). The extended gauge-fixing Fermion \( \tilde{\Psi} \) is assumed to be affine in the classical antifields

\[
\tilde{\Psi}(\Phi^*, \Phi, \varphi) := \Psi(\Phi, \varphi) - \Phi^*_\alpha \sigma^\alpha(\Phi, \varphi) .
\]

(6.4)

The Fermion \( \tilde{\Psi} \) is a scalar under reparametrizations of both \( \varphi \) and \( \Phi \). We stress that the antifield-free part \( \Psi = \Psi(\Phi, \varphi) \) of the gauge-fixing Fermion \( \tilde{\Psi} \) is allowed to depend on the classical fields \( \Phi \), in contrast to the construction in Sections 3 and 5. The condition (6.3) implies that the total and explicit differentiations of the effective action \( \Gamma \) with respect to the classical field \( \Phi^\alpha \) and antifield \( \Phi^*_\alpha \) are the same

\[
\left( \frac{\partial}{\partial \Phi^\alpha} \right) - \left( \frac{\partial_{\text{expl}}}{\partial \Phi^\alpha} \right) \equiv \left( \frac{\partial_{\text{impl}}}{\partial \Phi^\alpha} \right) = 0 , \quad \left( \frac{\partial}{\partial \Phi^*_\alpha} \right) - \left( \frac{\partial_{\text{expl}}}{\partial \Phi^*_\alpha} \right) \equiv \left( \frac{\partial_{\text{impl}}}{\partial \Phi^*_\alpha} \right) = 0 .
\]

(6.5)

The quantum master equation \([4, 5, 6]\) reads

\[
\Delta e^{\frac{i}{\hbar} W} = 0 \quad \Leftrightarrow \quad \frac{1}{2}(W, W) = i\hbar(\Delta W) ,
\]

(6.6)

with the odd Laplacian\(^\dagger\)

\[
\Delta := \frac{(-1)^{\varepsilon_\alpha}}{\rho} \frac{\partial}{\partial \varphi^\alpha} \rho \frac{\partial}{\partial \varphi^*_\alpha} , \quad \rho = \rho(\varphi) .
\]

(6.7)

The Ward identities read

\[
J_\alpha(\Phi, \Phi^*)(\frac{\partial}{\partial \Phi^*_\beta} \Gamma) = 0 ,
\]

(6.8)

and

\[
\left( \frac{\partial}{\partial \Phi^\beta} \right) = -J_\alpha(\Phi, \Phi^*) C^\alpha_\beta , \quad C^\alpha_\beta := -\frac{\partial_{\text{expl}}}{\partial \Phi^*_\alpha}(\tilde{\Psi} \frac{\partial}{\partial \Phi^*_\beta}) ,
\]

(6.9)

see Appendix A. The Zinn-Justin/classical master equation becomes

\[
\frac{1}{2}(\Gamma, \Gamma)_{\text{cl}} = (\Gamma(\frac{\partial}{\partial \Phi^\alpha} (C^{-1})^\alpha_\beta \frac{\partial}{\partial \Phi^*_\beta} \Gamma) = 0 ,
\]

(6.10)

where we have defined a deformed antibracket of classical variables as

\[
(f, g)_{\text{cl}} := (f \frac{\partial}{\partial \Phi^\alpha} (C^{-1})^\alpha_\beta \frac{\partial}{\partial \Phi^*_\beta} g) - (-1)^{\varepsilon_f + 1}\varepsilon_g + 1)(f \leftrightarrow g) .
\]

(6.11)

\(^\dagger\)Here we for simplicity assume that the odd scalar curvature \([14, 15, 16, 17, 18]\) vanishes.
The deformed classical master equation (6.10) is our main result. The antibracket (6.11) may in
general violate the Jacobi identity (even at the classical level), cf. eq. (A.16).

Finally, the change of integration variables (5.9) is now part of a type-2 anticanonical transformation

\[ \varphi^\alpha = \frac{\partial}{\partial \varphi^\alpha} \Psi_2, \quad \varphi^*_\alpha = \frac{\partial}{\partial \varphi^\alpha}, \quad (6.12) \]

with a type-2 Fermionic generator

\[ \Psi_2(\varphi, \varphi^*) = \varphi^\alpha \sigma^\alpha(\Phi, \varphi), \quad (6.13) \]

which depends on the un-primed fields and the primed antifields. The anticanonical transformation
respects the quantum master eq. (6.6) as well.

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A Extended Formalism

In this Appendix A, we promote (for technical rather than fundamental/profound reasons) the quantum fields \( \varphi^\alpha \), the classical fields \( \Phi^\alpha \) and antifields \( \Phi^*_\alpha \) to superfields

\[ \varphi^\alpha(\theta) := \varphi^\alpha + \lambda^\alpha \theta, \quad \Phi^\alpha(\theta) := \Phi^\alpha + \Lambda^\alpha \theta, \quad \Phi^*_\alpha(\theta) := \Phi^*_\alpha - \theta J_\alpha, \quad (A.1) \]

where \( \theta \) is a Fermionic parameter. Note that the superpartners of the classical antifields \( \Phi^*_\alpha \) are (minus) the sources \( J_\alpha \). Our primary aim in this Appendix A is not to create a superfield formalism, but merely a convenient platform to derive the pertinent Ward identities (6.8) and (6.9). (A treatment
from a manifest superfield perspective is developed in the next Appendix C.) Our sign convention for
the Berezin integral is

\[ \int d\theta \, \theta = 1. \quad (A.2) \]

The extended effective action

\[ \Gamma = \Gamma[\Phi(\cdot); \Phi^*(\cdot)] \quad (A.3) \]

depends on \( 4n \) variables \( \Phi^\alpha, \Lambda^\alpha, \Phi^*_\alpha \) and \( J_\alpha \). It is given as a level-one path integral

\[ e^{\frac{i}{\hbar} \Gamma} := \int d\Pi \, e^{\frac{i}{\hbar} \overline{\Gamma}}, \quad (A.4) \]

with level-one path integral measure

\[ d\Pi := \rho[d\varphi][d\varphi^*][d\lambda], \quad \rho = \rho(\varphi), \quad (A.5) \]

and action

\[ A := W + \varphi^*_\alpha \lambda^\alpha - Y, \quad (A.6) \]

where

\[ W = W(\varphi, \varphi^*) \quad (A.7) \]
is the usual quantum master action, and

\[ Y = Y[\Phi^*(\cdot), \Phi(\cdot), \varphi(\cdot)] \]  

\[ \text{(A.8)} \]

is given by

\[ Y := \int d\theta \, \tilde{\Psi}(\Phi^*(\theta), \Phi(\theta), \varphi(\theta)) = J_\alpha \sigma^\alpha(\Phi, \varphi) + \tilde{\Psi} \left( \frac{\partial r}{\partial \varphi^\alpha} \lambda^\alpha + \frac{\partial^r}{\partial \Phi^\alpha} \Lambda^\alpha \right) . \]  

\[ \text{(A.9)} \]

Later in eq. (A.21) we will introduce \( n \) implicit relations \( J_\alpha = J_\alpha(\Phi, \Phi^*, \Lambda) \). In anticipation of this, we will already now begin to distinguish between total and explicit derivative. Note e.g., that

\[ (\tilde{\Psi} \frac{\partial}{\partial \Phi^\alpha}) \overset{(6.4)}{=} (\tilde{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha}) , \quad (\tilde{\Psi} \frac{\partial}{\partial \Phi^\alpha}) \overset{(6.4)}{=} (\tilde{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha}) , \]  

\[ \text{(A.10)} \]

as \( \tilde{\Psi} = \tilde{\Psi}(\Phi^*, \Phi, \varphi) \) does not depend on \( J \).

Extended Ward identity for \( Y \):

\[ (J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} Y) + Y \left( \frac{\partial r}{\partial \varphi^\alpha} \lambda^\alpha + \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} \Lambda^\alpha \right) = 0 . \]  

\[ \text{(A.11)} \]

The extended Ward identity (A.11) can be seen by shifting integration variable \( \theta \to \theta + \theta_0 \) in the formula (A.9) for \( Y \), and collecting terms proportional to the Fermionic constant \( \theta_0 \).

Extended Ward identity for \( \Gamma \):

\[ J_\alpha (\frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} \Gamma) + (\Gamma \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha}) \Lambda^\alpha = 0 . \]  

\[ \text{(A.12)} \]

**Proof of eq. (A.12):**

\[ 0 = \int [d\varphi][d\varphi^*][d\lambda](-1)^{\varepsilon_\alpha} \frac{\partial^r}{\partial \varphi^\alpha} \left( \rho(\frac{\partial^r}{\partial \varphi^\alpha} e^\pi W) e^\pi (\varphi^* \lambda^\alpha - Y) \right) \]

\[ \overset{(6.6)}{=} - \int d\mu \, e^{\pi W} (\frac{\partial^r}{\partial \varphi^\alpha} e^{\pi (\varphi^* \lambda^\alpha - Y)} \frac{\partial^r}{\partial \varphi^\alpha}) \]

\[ = - \int d\mu \, e^{\pi W} + e^{\pi Y} (\frac{\partial^r}{\partial \varphi^\alpha} \lambda^\alpha) \]

\[ \overset{(A.11)}{=} \int d\mu \, e^{\pi W} + e^{\pi Y} (J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} \Gamma^* + e^{\pi Y} \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} \Lambda^\alpha) \]

\[ \overset{(A.14)}{=} J_\alpha (\frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} e^\pi \Gamma^*) + (e^{\pi Y} \frac{\partial^{\text{expl}}}{\partial \Phi^\alpha}) \Lambda^\alpha . \]  

\[ \text{(A.13)} \]

Expansion of the extended Ward identity (A.13) around \( \Lambda = 0 \) to second order in \( \Lambda \):

\[ J_\alpha (\frac{\partial^{\text{expl}}}{\partial \Phi^\alpha} e^{\pi Y} \Bigg|_{\Lambda=0} = 0 , \]  

\[ \text{(A.14)} \]
\[
\frac{i}{\hbar} J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^*_\alpha} \int \frac{d^\mu}{\sqrt{-g}} \left[ \bar{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}(\Psi) \frac{\partial^{\text{expl}}}{\partial \Phi^\gamma}(\Psi) \right] \bigg|_{\Lambda=0} = e^{\imath \hbar \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}} \bigg|_{\Lambda=0} ,
\]

(A.15)

\[
\frac{i}{\hbar} J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^*_\alpha} \int \frac{d^\mu}{\sqrt{-g}} \left[ \bar{\Psi} \Psi \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}(\Psi) \frac{\partial^{\text{expl}}}{\partial \Phi^\gamma}(\Psi) \right] \bigg|_{\Lambda=0} = \left( -1 \right)^{\varepsilon_{\beta} \varepsilon_{\gamma}} (\beta \leftrightarrow \gamma) .
\]

(A.16)

Extended quantum average
\[
\langle F \rangle : = e^{-\frac{i \hbar}{\hbar} \Gamma} \int \frac{d^\mu}{\sqrt{-g}} e^{\frac{i \hbar}{\hbar} A} F .
\]

(A.17)

Expansion of the extended Ward identity (A.12) around \( \Lambda = 0 \) to second order in \( \Lambda \):
\[
J_\alpha \left( \frac{\partial^{\text{expl}}}{\partial \Phi^*_\alpha} \frac{\partial^{\text{expl}}}{\partial \Phi^\beta} \right) \bigg|_{\Lambda=0} = 0 ,
\]

(A.18)

\[
J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^*_\alpha} \left( \bar{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}(\Psi) \frac{\partial^{\text{expl}}}{\partial \Phi^\gamma}(\Psi) \right) \bigg|_{\Lambda=0} = \left( \Gamma \frac{\partial^{\text{expl}}}{\partial \Phi^\beta} \right) \bigg|_{\Lambda=0} ,
\]

(A.19)

\[
J_\alpha \frac{\partial^{\text{expl}}}{\partial \Phi^*_\alpha} \left( \langle \bar{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}(\Psi) \frac{\partial^{\text{expl}}}{\partial \Phi^\gamma}(\Psi) \rangle - \langle \bar{\Psi} \frac{\partial^{\text{expl}}}{\partial \Phi^\beta}(\Psi) \rangle \langle \frac{\partial^{\text{expl}}}{\partial \Phi^\gamma}(\Psi) \rangle \right) \bigg|_{\Lambda=0} = \left( -1 \right)^{\varepsilon_{\beta}}
\]

(A.20)

\[
\langle \sigma^a(\Phi, \varphi) \rangle = 0 \quad \Leftrightarrow \quad J_\alpha = J_\alpha(\Phi, \Phi^*, \Lambda) .
\]

(A.21)

The effective action \( \Gamma = \Gamma(\Phi, \Phi^*) \) from eq. (6.1) can now be defined via the extended effective action (A.4) as
\[
\Gamma(\Phi, \Phi^*) : = \Gamma(\Phi, \Lambda = 0; \Phi^*, J = J(\Phi, \Phi^*, \Lambda = 0)) .
\]

(A.22)

B  Metric and Synge’s World Function

In the main text we assumed that field configuration manifold \( \mathcal{M} \) is equipped with a torsionfree connection \( \nabla \). In this Appendix B we will additionally assume that field configuration manifold \( \mathcal{M} \) is equipped with a (pseudo) Riemannian metric \( g_{\alpha \beta} \), and that \( \nabla \) is the corresponding Levi-Civita connection. We will follow the sign conventions of Ref. [18].


B.1 Metric

Let there be given a (pseudo) Riemannian metric in field configuration manifold $M$, i.e., a covariant symmetric $(0,2)$ tensor field

$$ds^2 = d\varphi^\alpha g_{\alpha\beta} \lor d\varphi^\beta,$$

of Grassmann–parity $\varepsilon(g_{\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta$, and of symmetry

$$g_{\beta\alpha} = -(-1)^{(\varepsilon_\alpha+1)(\varepsilon_\beta+1)}g_{\alpha\beta} .$$

(B.2)

The symmetry (B.2) becomes more transparent if one reorders the Riemannian metric as

$$ds^2 = d\varphi^\beta \lor d\varphi^\alpha \bar{g}_{\alpha\beta} ,$$

(B.3)

where

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta}(-1)^{\varepsilon_\beta} .$$

(B.4)

Then the symmetry (B.2) simply reads

$$\bar{g}_{\beta\alpha} = (-1)^{\varepsilon_\alpha\varepsilon_\beta}\bar{g}_{\alpha\beta} .$$

(B.5)

The Riemannian metric $g_{\alpha\beta}$ is assumed to be non–degenerate, i.e., there exists an inverse contravariant symmetric $(2,0)$ tensor field $g^{\alpha\beta}$ such that

$$g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha .$$

(B.6)

The inverse metric $g^{\alpha\beta}$ has Grassmann–parity $\varepsilon(g^{\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta$, and symmetry

$$g^{\beta\alpha} = (-1)^{\varepsilon_\alpha\varepsilon_\beta}g^{\alpha\beta} .$$

(B.7)

B.2 Levi–Civita Connection

The torsion tensor is just an antisymmetrization of the Christoffel symbol $\Gamma^\gamma_{\beta\alpha}$ with respect to the lower indices,

$$T^\gamma_{\beta\gamma} := \Gamma^\gamma_{\beta\gamma} + (-1)^{(\varepsilon_\beta+1)(\varepsilon_\gamma+1)}(\beta \leftrightarrow \gamma) .$$

(B.8)

In particular, the Christoffel symbol

$$\Gamma^\gamma_{\beta\gamma} = -(-1)^{(\varepsilon_\beta+1)(\varepsilon_\gamma+1)}(\beta \leftrightarrow \gamma)$$

(B.9)

is symmetric with respect to the lower indices when the connection is torsionfree. A connection $\nabla$ is called metric, if it preserves the metric

$$0 = (\nabla_\alpha \bar{g})_{\beta\gamma} = \left( \frac{\partial}{\partial \varphi^\alpha}\bar{g}_{\beta\gamma} \right) - \left( (-1)^{\varepsilon_\alpha\varepsilon_\beta}\delta^\gamma_\alpha \Gamma^\gamma_{\beta\alpha\gamma} + (-1)^{\varepsilon_\beta\varepsilon_\gamma}(\beta \leftrightarrow \gamma) \right) .$$

(B.10)

Here we have lowered the Christoffel symbol with the metric

$$\Gamma^\gamma_{\beta\gamma} := g_{\alpha\beta} \Gamma^\gamma_{\beta\gamma}(-1)^{\varepsilon_\gamma} .$$

(B.11)

\[1\] Vilkovisky [1, 2] assumes that the field configuration manifold $M$ is Bosonic. Our superconventions are related to those of DeWitt [3] via $g^{(bare)}_{\alpha\beta}(-1)^{\varepsilon_\beta} \equiv g^{(bare)}_{\alpha\beta} \equiv (-1)^{\varepsilon_\alpha}g^{(DeWitt)}_{\alpha\beta}$, and $\Gamma^{(bare)}_{\alpha\beta\gamma} \equiv (-1)^{\varepsilon_\alpha}\Gamma^{(DeWitt)}_{\alpha\beta\gamma}$.
The metric condition (B.10) reads in terms of the contravariant inverse metric

\[ 0 = (\nabla_\alpha g)^{\beta\gamma} \equiv \left( \frac{\partial^\ell}{\partial \phi^\alpha} g^{\beta\gamma} \right)_{\alpha\gamma} = \left( \frac{\partial^\ell}{\partial \phi^\alpha} \tilde{g}^{\beta\gamma} \right)_{\alpha\gamma} \]

Here we have introduced a reordered Christoffel symbol

\[ \Gamma_{\alpha \beta}^\gamma := (-1)^{\varepsilon_{\alpha\beta}} \Gamma_{\beta \alpha}^\gamma . \]

The Levi–Civita connection is the unique connection that is both torsionfree \( T = 0 \) and metric (B.10). The Levi–Civita formula for the lowered Christoffel symbol in terms of derivatives of the metric reads

\[ 2\Gamma_{\gamma\alpha\beta} = (-1)^{\varepsilon_{\alpha\gamma}} \left( \frac{\partial^\ell}{\partial \phi^\alpha} \tilde{g}_{\gamma\beta} \right) + (-1)^{\varepsilon_{\alpha\beta} + \varepsilon_{\gamma\beta}} \varepsilon_{\alpha\varepsilon}(\frac{\partial^\ell}{\partial \phi^\beta} \tilde{g}_{\alpha\gamma}) - \left( \frac{\partial^\ell}{\partial \phi^\gamma} \tilde{g}_{\alpha\beta} \right) . \]

### B.3 Synge’s World Functional

Let \( \gamma : [t_0, t_1] \to M \) be a parametrized open curve in the field configuration manifold \( M \). The Synge world functional \( \Sigma[\gamma] \) is an off-shell action functional

\[ \Sigma[\gamma] := \int_\gamma dt \; L(t) , \quad (B.15) \]

with Lagrangian \( L(t) \) given by a normalized squared distance

\[ L(t) := \frac{t_1 - t_0}{2} \lambda(t) , \quad \lambda(t) := \dot{\gamma}^\alpha(t) g_{\alpha\beta}(\gamma(t)) \dot{\gamma}^\beta(t) , \quad (B.16) \]

The Synge world functional \( \Sigma[\gamma] \) is invariant under affine reparametrizations \( t \to at + b \) of the curve \( \gamma \). The corresponding (on-shell) Euler-Lagrange equation is precisely the geodesic eq. (4.2). The momentum \( p_\alpha(t) \) reads

\[ p_\alpha(t) := L(t) \frac{\partial^\ell}{\partial \gamma^\alpha(t)} = \left( t_1 - t_0 \right) \dot{\gamma}^\beta(t) g_{\gamma\beta}(\gamma(t)) \]

Since there is no explicit \( t \)-dependence, the corresponding energy function

\[ h(t) := p_\alpha(t) \dot{\gamma}^\alpha(t) - L(t) = L(t) \]

does not depend on time \( t \) on-shell, cf. Noether’s theorem. In particular, the Lagrangian \( L(t) \) can be pulled outside the action integral \( \Sigma[\gamma] \approx (t_1 - t_0)L(t) \) on-shell. (Here the \( \approx \) symbol means equality modulo the geodesic eq. (4.2). We do not use the \( \approx \) symbol in the main text.)

### B.4 Synge’s World Function

Let there be given two points \( \Phi, \varphi \in M \) that are linked by a unique geodesic \( \gamma : [t_0, t_1] \to M \), so that

\[ \gamma(t=t_0) = \Phi , \quad \gamma(t=t_1) = \varphi . \]

The Synge world function

\[ \sigma(\Phi, \varphi) := \Sigma[\gamma] \]

(B.20)
between the two points \( \Phi \) and \( \varphi \) is defined \cite{22} as the value of the Synge world functional \( \Sigma[\gamma] \) along the geodesic \( \gamma \). In other words, the Synge world function \( \sigma \) is the associated on-shell action function for the off-shell action functional \( \Sigma \). It follows that the Synge world function \( \sigma(\Phi, \varphi) = \sigma(\varphi, \Phi) \) is numerically precisely \( \text{half the square of the geodesic distance from } \Phi \text{ to } \varphi \),

\[
\frac{1}{2} \left[ \text{dist}(\Phi, \varphi) \right]^2 \approx \frac{1}{2} \left[ \int_{t_0}^{t_1} dt \sqrt{\lambda(t)} \right]^2 \approx \frac{(t_1 - t_0)^2}{2} \lambda(t) \approx \Sigma[\gamma] \approx \sigma(\Phi, \varphi). \tag{B.21}
\]

Here in eq. (B.21) it is implicitly understood that the curve \( \gamma \) is the geodesic between \( \Phi \) to \( \varphi \).

### B.5 Logarithmic Map

The first variation of the Synge world function is determined by the end-point momentas

\[
\delta \sigma(\Phi, \varphi) \approx p_\alpha(t=t_1) \delta \varphi^\alpha - p_\alpha(t=t_0) \delta \Phi^\alpha, \tag{B.22}
\]

so that

\[
\sigma_\alpha(\Phi, \varphi) := \sigma(\Phi, \varphi) \frac{\partial}{\partial \Phi^\alpha} \approx -p_\alpha(t=t_0) = (t_0 - t_1) \dot{\gamma}^\beta(t=t_0) g_{\beta\alpha}(\Phi). \tag{B.23}
\]

We next raise the index of eq. (B.23) with the metric

\[
\sigma^\alpha(\Phi, \varphi) := \sigma_\beta(\Phi, \varphi) g^{\beta\alpha}(\Phi) \approx (t_0 - t_1) \dot{\gamma}^\alpha(t=t_0), \tag{B.24}
\]

in agreement with the general definition \((4.4)\) of the logarithmic map. It follows that

\[
\sigma_\alpha(\Phi, \varphi) \sigma^\alpha(\Phi, \varphi) = 2\sigma(\Phi, \varphi). \tag{B.25}
\]

### C Superfield Formalism

In this Appendix C we consider a manifest superfield formalism \cite{19, 20, 21} in the antisymplectic phase space \( \{\varphi^\alpha; \varphi^*_\beta\} \). It is natural to also promote the quantum antifields \( \varphi^*_\alpha \) to superfields

\[
\varphi^*_\alpha(\theta) := \varphi^*_\alpha - \theta \lambda^*_\alpha, \tag{C.1}
\]

with new superpartners \( \lambda^*_\alpha \). In this Appendix C we will not worry about manifest reparametrization-invariance in superfield space. The only requirement we will demand here is that the superfield formalism in components reduces to the previous construction of Appendix A. The superpartners \( \lambda^*_\alpha \) are immediately killed again by modifying the path integral measure \((A.5)\) to also include a delta function

\[
\rho[d\varphi(\cdot)][d\varphi^*(\cdot)] \delta(\lambda(\cdot)) = \rho[d\varphi(\cdot)][d\varphi^*(\cdot)] \delta \left( \int d\theta \varphi^*(\theta) \right). \tag{C.2}
\]

The odd Laplacian \((6.7)\) can be written as

\[
\Delta := \frac{(-1)^{\varepsilon}}{\rho} \int d\theta \frac{\delta^\ell}{\delta \varphi^\alpha(\theta)} \rho \frac{d}{d\theta} \frac{\delta^\ell}{\delta \varphi^*_\alpha(\theta)} = \frac{1}{\rho} \int d\theta \frac{d}{d\theta} \frac{\delta^\ell}{\delta \varphi^\alpha(\theta)} \rho \frac{\delta^\ell}{\delta \varphi^*_\alpha(\theta)}. \tag{C.3}
\]

Here the functional derivatives of the quantum superfields read in components

\[
\frac{\delta^\ell}{\delta \varphi^\alpha(\theta)} = \theta \frac{\partial^\ell}{\partial \varphi^\alpha} - \theta \frac{\partial^\ell}{\partial \lambda^\alpha}, \quad \frac{\delta^\ell}{\delta \varphi^*_\alpha(\theta)} = \frac{\delta^\ell}{\delta \varphi^*_\alpha} + (-1)^{\varepsilon} \theta \frac{\partial^\ell}{\partial \lambda^*_\alpha}, \tag{C.4}
\]

Here the functional derivatives of the quantum superfields read in components.
see also Appendix B in Ref. [20]. To obtain a manifest superfield formulation, the \( \varphi^*_\alpha \lambda^\alpha \)-term in the \( \bar{A} \)-action (A.6) should be replaced

\[
\varphi^*_\alpha \lambda^\alpha \longrightarrow \int d\theta \: \varphi^*_\alpha(\theta) \varphi^\alpha(\theta) = \varphi^*_\alpha \lambda^\alpha - \lambda^*_\alpha \varphi^\alpha \approx \varphi^*_\alpha \lambda^\alpha , \tag{C.5}
\]

which effectively is the same as before, due to the presence of the delta function \( \delta(\lambda^*) \) in the path integral measure (C.2).

Similarly, the quantum master action and density,

\[
W = W(\varphi, \varphi^*) \quad \text{and} \quad \rho = \rho(\varphi) , \tag{C.6}
\]

should strictly speaking be promoted to functionals of superfields,

\[
W = W(\varphi(\cdot), \varphi^*(\cdot)) \quad \text{and} \quad \rho = \rho(\varphi(\cdot)) , \tag{C.7}
\]

respectively. However in practice, this would jeopardize the rôle of the \( \lambda^\alpha \)'s as Lagrange multipliers for the gauge-fixing of the antifields \( \varphi^* \).

If one adds the action term (C.5) to the quantum master action as

\[
\overline{W} := W + \int d\theta \: \varphi^*_\alpha(\theta) \varphi^\alpha(\theta) , \tag{C.8}
\]

if one introduces an odd vector field

\[
V := \int \left[ \frac{d}{d\theta} \varphi^\alpha(\theta) \right] d\theta \: \frac{\delta \ell}{\delta \varphi^\alpha(\theta)} - \int \left[ \frac{d}{d\theta} \varphi^*_\alpha(\theta) \right] d\theta \: \frac{\delta \ell}{\delta \varphi^*_\alpha(\theta)} = \lambda^*_\alpha \frac{\partial \ell}{\partial \varphi^*_\alpha} - (-1)^\varepsilon \lambda^\alpha \frac{\partial \ell}{\partial \varphi^\alpha} , \tag{C.9}
\]

and if one introduces an odd scalar

\[
\nu := - \int d\theta \left[ \frac{d}{d\theta} \varphi^\alpha(\theta) \right] \varphi^*_\alpha(\theta) = \int d\theta \left[ \frac{d}{d\theta} \varphi^\alpha(\theta) \right] \varphi^\alpha(\theta) = \lambda^*_\alpha \lambda^\alpha \approx 0 , \tag{C.10}
\]

then the quantum master equation (6.6) becomes

\[
\left( \Delta + \frac{i}{\hbar^2} V \rho + \frac{\nu}{\hbar^2} \right) e^{\frac{\xi}{2} \overline{W}} = 0 \quad \Leftrightarrow \quad \frac{1}{2}(\overline{W}, \overline{W}) + V[\overline{W}] - \nu = i\hbar(\Delta \overline{W} + V[\ln \rho]) , \tag{C.11}
\]

because

\[
\frac{1}{2}(\overline{W}, \overline{W}) = \frac{1}{2}(W,W) - V[W] - \nu , \quad V[\overline{W}] = V[W] + 2\nu , \quad (\Delta \overline{W}) = (\Delta W) - V[\ln \rho] . \tag{C.12}
\]

Finally let us mention, that if one introduces an odd vector field

\[
U_{\text{expl}} := - \int \left[ \frac{d}{d\theta} \Phi^\alpha(\theta) \right] d\theta \: \frac{\delta \ell_{\text{expl}}}{\delta \Phi^\alpha(\theta)} - \int \left[ \frac{d}{d\theta} \Phi^*_\alpha(\theta) \right] d\theta \: \frac{\delta \ell_{\text{expl}}}{\delta \Phi^*_\alpha(\theta)} = J_{\alpha} \frac{\partial \ell_{\text{expl}}}{\partial \Phi^\alpha} + (-1)^\varepsilon \Delta^\alpha \frac{\partial \ell_{\text{expl}}}{\partial \Phi^\alpha} , \tag{C.13}
\]

then the extended Ward identity (A.12) becomes

\[
U_{\text{expl}}[\Gamma] = 0 . \tag{C.14}
\]
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