Monte Carlo error analyses of Spearman’s rank test

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Spearman’s rank correlation test is commonly used in astronomy to discern whether a set of two variables are correlated or not. Unlike most other quantities quoted in astronomical literature, the Spearman’s rank correlation coefficient is generally quoted with no attempt to estimate the errors on its value. This is a practice that would not be accepted for those other quantities, as it is often regarded that an estimate of a quantity without an estimate of its associated uncertainties is meaningless. This manuscript describes a number of easily implemented, Monte Carlo based methods to estimate the uncertainty on the Spearman’s rank correlation coefficient, or more precisely to estimate its probability distribution.

1 Introduction

Spearman’s rank correlation test (Spearman, 1904) is commonly used in astronomy to discern whether a pair of two variables are correlated or not. It has an advantage over the Pearson correlation test – which requires a linear relationship between the two variables – as it is non-parametric. This non-parametric nature of the test is useful as it is model independent so long as the relationship is monotonic, i.e. is a steadily increasing or decreasing function. Unlike other quantities in literature, Spearman’s rank correlation coefficient, \( \rho \), is generally quoted with no attempt to estimate the uncertainties on its value; though the data used to calculate the coefficient will almost certainly have some measurement uncertainty, in at least one of the variables.

In the fields of e.g. bio-medicine and clinical medicine, where Spearman’s rank test is regularly used, the method of Monte Carlo bootstrapping (Efron, 1979) is used to estimate the confidence intervals of the coefficient (e.g., Haukoos & Lewis 2005); this involves resampling the data by drawing random entries from the original data set to create multiple resampled data sets of the same size as the original. The method does have limitations, as described by Haukoos & Lewis (2005), primarily that the method assumes that the sample is representative of the overall population which is of particular concern for small sample sizes. However, few other methods are available to estimate the uncertainties.

The principal difference between astronomical and clinical data is that the astronomical data will often, though not always, have associated measurement uncertainties (e.g., flux, magnitude, distance) while clinical data is often count-based (e.g., frequency of success in clinical trials) with no intrinsic

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1Here error analyses of the Spearman test are discussed, though the methods can (and should) also be applied to the Pearson test.
probability distribution on the individual data points. Of course count based astronomical data is commonplace (e.g. population studies) but even in these cases an attempt is often made to approximate the probability distribution of the data by assuming e.g. Poissonian uncertainties. Here three Monte Carlo methods of error analyses of the Spearman’s rank coefficient – the bootstrap/resampling method, the perturbation method and composite method – are described and demonstrated using a sample data set. Importantly, the latter two methods exploit the uncertainties or probability distributions which are generally associated with astronomical data.

2 Method

Given a data set consisting of $N$ data pairs, $X_i$ and $Y_i$, each individual entry is assigned an ascending order rank, $RX_i$ and $RY_i$, and the Spearman’s rank correlation coefficient, $\rho$, for the sample is calculated from the square of the difference of the two ranks for each pair by

$$\rho = 1 - \frac{6 \sum_{i=1}^{N} (RX_i - RY_i)^2}{N(N^2 - 1)}.$$  

A coefficient of 0 corresponds to no correlation between the variables, while a value of +1 or -1 corresponds to a perfect increasing or decreasing monotonic correlation. The significance of the correlation may be calculated in a number of ways, such as a Student’s t-test (Zar, 1972), but here the z-score, $z$, is used since it approximately follows a Gaussian distribution in this case. The z-score is calculated by $z = F(\rho) \sqrt{\frac{N-3}{1.16}}$, where $F(\rho) = \frac{\ln[(1 + \rho)/(1 - \rho)]}{2}$ is the Fisher transformation of the correlation coefficient, which goes to infinity as $\rho$ goes to 1.

The aforementioned bootstrap or resampling method of error analysis involves creating $M$ new data sets, each consisting of $N$ data pairs, $x_i$ and $y_i$, where for statistical significance, $M \gtrsim 1000$. Each of these new pairs is a randomly chosen pair from the original data set, $X_j$ and $Y_j$, where $j$ is the randomly chosen entry, e.g., $x_i = X_j$ and $y_i = Y_j$, such that some of the original pairs may appear more than once in a given data set or not at all. The data points are again assigned a rank, $RX_i$ and $RY_i$, and the Spearman’s rank correlation coefficient and z-score for each of these $M$ new data sets calculated. The distributions of the returned values are used to estimate the probability distribution of the two quantities, by normalising so that the integral is equal to one. In the simplest case of this probability distribution being Gaussian\footnote{For a much fuller discussion of using probability distributions, Gaussian or otherwise, to estimate quantities see Andrae (2010), which also includes discussion on the use of Monte Carlo methods to estimate errors.}, the estimate of the correlation coefficient, $\hat{\rho} = \bar{\rho}$, the average of the calculated values and the estimate of the error is the Gaussian width of the distribution, $\sigma_{\rho}$, i.e., the standard deviation of the calculated values; the z-score is similarly estimated as $\bar{z} \pm \sigma_z$.

The resampling method obviously does not take into account the uncertainties, $\Delta X_i$ and $\Delta Y_i$, likely associated with the pairs. To do so a perturbation method may be used, where one creates multiple new data sets ($M \gtrsim 1000$) consisting of $N$ data pairs, $x_i$ and $y_i$, each of which is perturbed from the original values, $X_i$ and $Y_i$, by adding a random Gaussian\footnote{Here, as is often done, it is assumed that the measurement uncertainties are Gaussian; however, these methods only require that the probability distribution of the measurement uncertainties are known.} number times the uncertainty on that point, e.g., $x_i = X_i + G \times \Delta X_i$, where $G$ is a number, drawn randomly from a Gaussian distribution of width 1 and centered on 0. This is done independently per point (assuming that the uncertainties are independent) so that a different random Gaussian number, $G$, is used to perturb the $X$ and $Y$ values.
The Spearman’s rank correlation coefficient and z-score are then calculated for each of these \( M \) data sets and the distribution of returned values used as an estimate of the probability distribution of the two quantities, as before.

Alternatively, the composite method may be used, combining the traditional resampling method and the above outlined perturbation method. In this case each new data pair, \( x_i \) and \( y_i \), is perturbed from a random entry of the original data set, \( X_j \) and \( Y_j \), where \( j \) is the randomly chosen entry, e.g., \( x_i = X_j + G \times \Delta X_j \). Again, the random Gaussian numbers, \( G \), must be independent, in so far as that is possible given whatever random number generator is being used, but the two variables must have originated from the same source pair. The probability distribution of the two quantities, \( \rho \) and \( z \), are again estimated as above.

### 3 Example

Here I apply the standard method of estimating the Spearman’s rank correlation coefficient (no error) and the different Monte Carlo methods (resampling, perturbation and composite)\(^4\) to real physical data from the X-ray transient, MAXIJ0556-332. During its 2011 outburst, MAXIJ0556-332 was observed by the UVOT instrument on board the \textit{Swift} satellite. When the photon indices of these optical observations were compared to the \( v \)-band magnitudes, an apparent correlation was clear (Figure 1). In fact, the Spearman’s rank correlation coefficient was 0.83, implying a z-score significance of \( z = 8.2 \) (\( N = 53 \)). If one were to take the basic step of applying the resampling method to this (\( M = 1000 \) in this and the following cases), which is not often done, the distributions of \( \rho \) and \( z \) plotted in Figure 2 (black) are obtained. Assuming that these may be described as Gaussian distributions (clearly not the case for the correlation coefficient but a reasonable simplifying assumption nonetheless), their averages and standard deviations are as given in Table 1.

It is clear from Figure 1 that there is significant uncertainties on both the spectral indices and the magnitudes which may reduce the significance of the correlation, and should be taken into account. Applying the perturbation method one finds that, indeed, the distribution and average of the correlation coefficient is reduced, as is the z-score (Figure 2, red). Applying the composite method, one

\(^4\)For these results, the methods were implemented within a C program (available online at \url{https://github.com/PACurran/MCSpearman}) but are easily implemented in many languages or programs.
finds that it returns a similar average to the perturbation method but with a wider distribution of values (Figure 2, green). In fact, the plotted distributions of the composite method clearly demonstrate that what was considered a correlation at the $\approx 8.2\sigma$ level is better described by a significance of $\approx (7.1 \pm 1.0)\sigma$.

### 4 Discussion

Only one test case of these methods is presented so general conclusions should not be drawn without serious caution. For example, the results will be heavily dependent on the number of data points, the distribution of these data points in $x$ and $y$, and, in the cases of the perturbation and composite methods, the size of the uncertainties on the data points. For a discussion regarding the number and distribution of data points see Haukoos & Lewis (2005).

Clearly, taking the data uncertainties into account weakens the correlation between the two variables; as the data uncertainties go to zero, the perturbation method will tend to a delta function at the value returned by the standard method, while the results of the composite method will tend to those of the resampling method.

It is important to understand the difference between the two (non-composite) methods and the distribution they return. The resampling method estimates the uncertainty of the correlation coefficient given the uncertainty of the sample, i.e., the sample being tested is only a sub-sample of the population.
of all possible data pairs and the resampling method estimates the uncertainties associated with the lack of information of all those data pairs. The perturbation method estimates the uncertainty of the correlation coefficient, given only the uncertainties on the data points, i.e., this method assumes the given sample is absolutely representative of the population, or alternatively, the method only estimates the error of the given pairs in the sample, not the population as a whole. In some circumstances, one may only want to estimate the correlation coefficient (and uncertainty) of the given sample in which case the perturbation method should be used. In other circumstances, it is the uncertainty associated with the population, including its unknown entries, which dominates, and the resampling method should be used. In many cases, though only one of these dominate the probability distribution of the correlation coefficient, both should be taken into account via the composite approach.

5 Conclusion

When calculating the Spearman’s rank correlation coefficient of a data set, an estimate of the uncertainty of the coefficient is required, just as it is for most other quantities in astronomy. Furthermore, when the given data set has uncertainties on the individual entries, these uncertainties must be taken into account when estimating the correlation coefficient. Here I have suggested and discussed three Monte Carlo methods of accounting for the uncertainties in the data points which are easily implemented and return significant information regarding the probability distribution of the correlation coefficient.

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