INTEGRABLE COUPLING IN A MODEL FOR
JOSEPHSON TUNNELING BETWEEN
NON-IDENTICAL BCS SYSTEMS

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Abstract
We extend a recent construction for an integrable model describing
Josephson tunneling between identical BCS systems to the case where
the BCS systems have different single particle energy levels. The exact
solution of this generalized model is obtained through the Bethe ansatz.

1 Introduction.
The experimental work of Ralph, Black and Tinkham[1, 2] on the discrete energy
spectrum in small metallic aluminium grains has generated substantial interest
in understanding the nature of superconducting correlations at the nano-scale
level. Their results indicate significant parity effects due to the number of
electrons in the system. For grains with an odd number of electrons, the gap in
the energy spectrum reduces with the size of the system, in contrast to the case
of a grain with an even number of electrons, where a gap larger than the single
electron energy levels persists. In the latter case the gap can be closed by a
strong applied magnetic field. The conclusion drawn from these results is that
pairing interactions are prominent in these nano-scale systems. For a grain with
an odd number of electrons there will always be at least one unpaired electron,
so it is not necessary to break a Cooper pair in order to create an excited state.
For a grain with an even number of electrons, all excited states have a least one

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broken Cooper pair, resulting in a gap in the spectrum. In the presence of a strongly applied magnetic field, it is energetically more favourable for a grain with an even number of electrons to have broken pairs, and hence in this case there are excitations which show no gap in the spectrum.

The physical properties of a small metallic grain are described by the reduced BCS Hamiltonian

\[
H_{BCS} = \sum_{j=1}^{L} \epsilon_j n_j - g \sum_{j,k} c_{j+}^\dagger c_{j-} c_{k+} c_{k-} .
\]  

Above, \( j = 1, \ldots, L \) labels a shell of doubly degenerate single particle energy levels with energies \( \epsilon_j \) and \( n_j \) is the fermion number operator for level \( j \). The operators \( c_{j}\pm, c_{j}\pm^\dagger \) are the annihilation and creation operators for the fermions at level \( j \). The labels \( \pm \) refer to time reversed states.

One of the prominent features of the Hamiltonian (1) is the blocking effect. For any unpaired electron at level \( j \) the action of the pairing interaction is zero since only paired electrons are scattered. This means that the Hilbert space can be decoupled into a product of paired and unpaired electron states in which the action of the Hamiltonian on the subspace for the unpaired electrons is automatically diagonal in the natural basis. In view of the blocking effect, it is convenient to introduce hard-core boson operators

\[
b_j = c_j - c_j^\dagger, \quad b_j^\dagger = c_j^\dagger c_j \}
\]

which satisfy the relations

\[
(b_j^\dagger)^2 = 0, \quad [b_j, b_k^\dagger] = \delta_{jk} (1 - 2b_j^\dagger b_j)
\]

on the subspace excluding single particle states. In this setting the hard-core boson operators realise the \( su(2) \) algebra in the pseudo-spin representation, which will be utilized below.

The original approach of Bardeen, Cooper and Schrieffer to describe the phenomenon of superconductivity was to employ a mean field theory using a variational wavefunction for the ground state which has an undetermined number of electrons. The expectation value for the number operator is then fixed by means of a chemical potential term \( \mu \). One of the predictions of the BCS theory is that the number of Cooper pairs in the ground state of the system is given by the ratio \( \Delta/d \) where \( \Delta \) is the BCS “bulk gap” and \( d \) is the mean level spacing for the single electron eigenstates. For nano-scale systems, this ratio is of the order of unity, in seeming contradiction with the experimental results discussed above. The explanation for this is that the mean-field approach is inappropriate for nano-scale systems due to large superconducting fluctuations.

As an alternative to the BCS mean field approach, one can appeal to the exact solution of the Hamiltonian (1) derived by Richardson and Sherman. It has also been shown by Cambiaggio, Rivas and Saraceno that (1) is integrable in the sense that there exists a set of mutually commutative operators which commute with the Hamiltonian. These features have recently been shown to
be a consequence of the fact that the model can be derived in the context of
the Quantum Inverse Scattering Method (QISM) using a solution of the Yang-
Baxter equation associated with the Lie algebra $su(2)$\cite{8, 9}. One of the aims
of the present work is to extend this approach for application to generalised
models. As a specific example, we will show that a model for strong Josephson
coupling between two BCS systems falls into this class.

Recall first that electron pairing interactions manifest themselves in macro-
scopic systems via three well known phenomena:

- supercurrents
- Meissner effect
- Josephson effect

As noted by von Delft\cite{4}, the notion of a supercurrent in a nano-scale system
is inapplicable because the mean free path of an electron is comparable to the
system size. Likewise, the penetration depth of an applied magnetic field is
comparable to the system size, which prohibits any Meissner effect.

Josephson\cite{11} put forth a proposal for the tunneling of electron pairs between
superconductors separated by an insulating barrier. A theory was derived to
describe weak coupling between two superconductors treated at the mean field
level in the grand-canonical ensemble. A remarkable prediction of the theory
was that it is possible for a direct current to flow across the insulator for the
case of zero applied voltage, whereas a constant voltage across the insulator
produces an alternating current. The essential features of the theory stem from
the phase difference between the superconductors, which is well defined since the
variational wavefunctions for the superconductors have undetermined particle
numbers.

For the case of nano-scale systems, the above predictions are again invalid
due to the finite particle numbers for each system, giving rise to phase uncer-
tainty. However, if we are to consider strong coupling where individual particle
numbers are not conserved, only total particle number, it is appropriate to study
the effective Hamiltonian

$$
H = H_{BCS}(1) + H_{BCS}(2) - \varepsilon J \sum_{j,k} \left( b_j^\dagger(1)b_k(2) + b_j^\dagger(2)b_k(1) \right),
$$

where $\varepsilon J$ is the Josephson coupling energy, for the purpose of investigating the
nature of pair tunneling at the nano-scale level. In a previous work\cite{10} it was
shown that the above Hamiltonian is integrable for $\varepsilon J = g$ for the case when
$H_{BCS}(1), H_{BCS}(2)$ have identical single electron energy levels. Below we will
extend this construction to the case where $H_{BCS}(1), H_{BCS}(2)$ describe non-
identical systems.
2 A universal integrable system.

First we introduce the Lie algebra $su(2)$ with generators $S^+, S^-, S^z$ satisfying the commutation relations

$$[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \quad (3)$$

The Casimir invariant, which commutes with each element of the algebra, has the form

$$C = S^+ S^- + S^- S^+ + 2(S^z)^2.$$ 

Associated with the $su(2)$ algebra there is a solution of the Yang-Baxter equation in $\text{End}V \otimes \text{End}V \otimes su(2)$, where $V$ denotes a two-dimensional vector space. This solution reads\cite{12}

$$R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v)$$

with

$$R(u) = I \otimes I + \frac{\eta}{u} \sum_{m,n}^2 e_m^m \otimes e_n^n, \quad L(u) = I \otimes I + \frac{\eta}{u} \left( e_1^1 \otimes S^z - e_2^2 \otimes S^z + e_2^1 \otimes S^- + e_1^2 \otimes S^+ \right)$$

where $\{ e_m^n \}$ are $2 \times 2$ matrices with 1 in the $(m, n)$ entry and zeroes elsewhere. Above, $I$ is the identity operator and $\eta$ is a scaling parameter for the rapidity variable $u$ which plays an important role in the subsequent analysis. With this solution we construct the transfer matrix

$$t(u) = \text{tr}_0 \left( G_0 L_0 L_0 \ldots L_0 (u - \epsilon_1) \right)$$

(4)

which is an element of the $L$-fold tensor algebra of $su(2)$. Above, $\text{tr}_0$ denotes the trace taken over the auxiliary space and $G = \exp(\alpha \eta \sigma)$ with $\sigma = \text{diag}(1, -1)$. A consequence of the Yang-Baxter equation is that $[t(u), t(v)] = 0$ for all values of the parameters $u$ and $v$, and independent of the representations of $su(2)$ in the tensor algebra. Defining

$$T_j = \lim_{u \to \epsilon_j} \frac{u - \epsilon_j}{\eta^2} t(u)$$

for $j = 1, 2, \ldots, L$, we may write in the quasi-classical limit $T_j = \tau_j + o(\eta)$ and it follows that $[\tau_j, \tau_k] = 0, \forall j, k.$ Explicitly, these operators read

$$\tau_j = 2\alpha S_j^z + \sum_{k \neq j}^L \frac{\theta_{jk}}{\epsilon_j - \epsilon_k}$$

(5)

with $\theta = S^+ \otimes S^- + S^- \otimes S^+ + 2S^z \otimes S^z$. 

4
We define a Hamiltonian through

\[
H = - \frac{1}{\alpha} \sum_j \epsilon_j \tau_j + \frac{1}{4\alpha^3} \sum_{j,k} \tau_j \tau_k + \frac{1}{2\alpha^2} \sum_j \tau_j - \frac{1}{2\alpha} \sum_j C_j \tag{6}
\]

\[
= - \sum_j 2\epsilon_j S_j^z - \frac{1}{\alpha} \sum_{j,k} S_j^- S_k^+ . \tag{7}
\]

The Hamiltonian is universally integrable since it is clear that \([H, \tau_j] = 0\), \(\forall j\) irrespective of the realizations of the \(su(2)\) algebra in the tensor algebra.

Realizing the \(su(2)\) generators through the hard-core bosons; viz

\[
S_j^+ = b_j, \quad S_j^- = b_j^\dagger, \quad S_j^z = \frac{1}{2}(I - n_j) \tag{8}
\]

one obtains (4) (up to a constant) with \(g = 1/\alpha\) as shown by Zhou et al.\(^8\) and von Delft and Poghossian\(^9\).

We now turn to applying (4) for the study of two coupled BCS systems. To accommodate this, it is convenient to first consider three index sets \(P_0, P_1, P_2\) such that individually the BCS Hamiltonians are expressible

\[
H_{BCS}(i) = \sum_{j \in (P_0 \cup P_i)} \epsilon_j n_j - g \sum_{j,k \in (P_0 \cup P_i)} b_k^\dagger b_j .
\]

If the single particle energy \(\epsilon_j\) is common to both systems, then \(j \in P_0\). Hence it is meant to be understood that \(\epsilon_j \neq \epsilon_k \neq \epsilon_l \forall j \in P_1, k \in P_2, l \in P_3\). In the case that \(j \in P_0\), the local \(su(2)\) operators are described by the tensor product of two pseudo-spin realisations acting on the four-dimensional tensor product space. We can now realise (4) in terms of the hard-core boson representation (8)

\[
S_j^+ = b_j(i), \quad S_j^- = b_j^\dagger(i), \quad S_j^z = \frac{1}{2}(I - n_j(i))
\]

for \(j \in P_t, i = 1, 2\) whereas for \(j \in P_0\) we take the tensor product representation

\[
S_j^+ = b_j(1) + b_j(2) \\
S_j^- = b_j^\dagger(1) + b_j^\dagger(2) \\
S_j^z = I - \frac{1}{2}(n_j(1) + n_j(2)) .
\]

Under this representation of (4) we obtain (5) with \(\varepsilon_J = g = 1/\alpha\), establishing integrability at this value of the Josephson coupling energy. For the case when the index sets \(P_1, P_2\) are both empty, i.e., the two BCS systems are identical, this result was previously shown by Links et al.\(^10\).
3 The exact solution.

In addition to proving integrability for \( \varepsilon_J = g \), we can also obtain the exact solution from the Bethe ansatz. Below we will derive the energy eigenvalues for the Hamiltonian (7) in a very general context, which includes those of (3) with \( \varepsilon_J = g \) as a particular case.

For each index \( k \) in the tensor algebra in which the transfer matrix acts, and accordingly in (7), suppose that we represent the \( su(2) \) algebra through the irreducible representation with spin \( s_k \). Thus \( \{ S_k^+, S_k^-, S_k^z \} \) act on a \((2s_k + 1)\)-dimensional space. Employing the standard method of the algebraic Bethe ansatz\[12\] gives that the eigenvalues of the transfer matrix (4) take the form

\[
\Lambda(u) = \exp(\alpha \eta) \prod_{k} \frac{L}{u - \varepsilon_k + \eta s_k} \prod_{j} \frac{M}{u - w_j - \eta} + \exp(-\alpha \eta) \prod_{k} \frac{L}{u - \varepsilon_k - \eta s_k} \prod_{j} \frac{M}{u - w_j + \eta}.
\]

Above, the parameters \( w_j \) are required to satisfy the Bethe ansatz equations

\[
\exp(2\alpha \eta) \prod_{k} \frac{W_l - \varepsilon_k + \eta s_k}{W_l - \varepsilon_k - \eta s_k} = -\prod_{j} \frac{W_l - w_j + \eta}{W_l - w_j - \eta}.
\]

The eigenvalues of the conserved operators (5) are obtained through the appropriate terms in the expansion of the transfer matrix eigenvalues in the parameter \( \eta \). This yields the following result for the eigenvalues \( \lambda_j \) of \( \tau_j \)

\[
\lambda_j = \left(2\alpha + \sum_{k \neq j}^{L} \frac{2s_k}{\varepsilon_j - \varepsilon_k} - \sum_{i}^{M} \frac{2}{\varepsilon_j - v_i}\right)s_j
\]

such that the parameters \( v_j \) satisfy the coupled algebraic equations

\[
2\alpha + \sum_{k}^{L} \frac{2s_k}{v_j - \varepsilon_k} = \sum_{i \neq j}^{M} \frac{2}{v_j - v_i}.
\]

Through (9) we can now determine the energy eigenvalues of (7). It is useful to note the following identities

\[
2\alpha \sum_{j}^{M} v_j + 2 \sum_{j}^{M} \sum_{k}^{L} \frac{v_j s_k}{v_j - \varepsilon_k} = M(M - 1)
\]

\[
\alpha M + \sum_{j}^{M} \sum_{k}^{L} \frac{s_k}{v_j - \varepsilon_k} = 0
\]

\[
\sum_{j}^{M} \sum_{k}^{L} \frac{s_k}{v_j - \varepsilon_k} - \sum_{j}^{M} \sum_{k}^{L} \frac{s_k}{v_j - \varepsilon_k} = M \sum_{k}^{L} s_k.
\]
Employing the above it is deduced that

$$\sum_j \lambda_j = 2\alpha \sum_j s_j - 2\alpha M$$

$$\sum_j \epsilon_j \lambda_j = 2\alpha \sum_j \epsilon_j s_j + \sum_j \sum_{k \neq j} s_j s_k - 2M \sum_k s_k - 2\alpha \sum_j v_j + M(M - 1)$$

which, combined with the eigenvalues $2s_j(s_j + 1)$ for the Casimir invariants $C_j$, yields the energy eigenvalues

$$E = 2 \sum_j v_j - 2 \sum_k s_k \epsilon_k.$$  \hfill (11)

From the above expression we see that the quasi-particle excitation energies are given by twice the Bethe ansatz roots $\{v_j\}$ of \hfill (10).

In order to specialise this result to \hfill (2) at integrable coupling, it is useful to first make the following observation. For $j \in P_0$, in which case the $su(2)$ algebra is realised via the tensor product of two hard-core boson representations, it is well known that the representation space is completely reducible into triplet states and a singlet state. Note however, that for the singlet state the $su(2)$ generators act trivially, and hence this state is blocked from scattering in analogy with the blocking of single particle states discussed in the introduction. Hence the $su(2)$ algebra will only act non-trivially on the triplet states. In specialising \hfill (10,11) to the case of \hfill (2), we need only to set $s_j = 1/2$ for $j \in P_1 \cup P_2$ and $s_j = 1$ for $j \in P_0$.

4 Conclusion

We have displayed the existence of a general class of integrable systems which includes the reduced BCS Hamiltonian and a model for strong Josephson tunneling between two reduced BCS systems. By deriving the models through the QISM we have also determined the exact solution via the Bethe ansatz. A further application of this approach is the computation of form factors and correlation functions\hfill [8, 10].

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