WELL PRODUCTIVITY INDEX
FOR COMPRESSIBLE FLUIDS AND GASES

EUGENIO AULISA
Texas Tech University, Department of Mathematics and Statistics
Broadway and Boston, Lubbock, TX 79409-1042, USA

LIDIA BLOSHANSKAYA AND AKIF IBRAGIMOV*
SUNY New Paltz, Department of Mathematics, 1 Hawk Dr
New Paltz, NY 12561, USA
and
Texas Tech University, Department of Mathematics and Statistics
Broadway and Boston, Lubbock, TX 79409-1042, USA

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Abstract. We discuss the notion of the well productivity index (PI) for the
generalized Forchheimer flow of fluid through porous media. The PI character-
izes the well capacity with respect to drainage area of the well and in general
is time dependent. In case of the slightly compressible fluid the PI stabilizes in
time to the specific value, determined by the so-called pseudo steady state solu-
tion, [5, 3, 4]. Here we generalize our results from [4] in case of arbitrary order
of the nonlinearity of the flow. In case of the compressible gas flow the math-
ematical model of the PI is studied for the first time. In contrast to slightly
compressible fluid the PI stays “almost” constant for a long period of time,
but then it blows up as time approaches the certain critical value. This value
depends on the initial data (initial reserves) of the reservoir. The “greater”
are the initial reserves, the larger is this critical value. We present numerical
and theoretical results for the time asymptotic of the PI and its stability with
respect to the initial data.

1. Historical remarks and review of the results. The classical equation de-
scribing the fluid flow in porous media is the Darcy’s law, stating the linear relation
between the pressure gradient $\nabla p$ and the velocity $u$. Darcy himself observed in [11]
that the area of applicability of linear relation is very limited. When, for instance,
the fluid has high velocity or in the presence of fractures in the media, the nonlinear
models are necessary to capture the properties of the flow.

One of the widely-used nonlinear models is the Forchheimer equation in the form
$g(|u|)u = -\nabla p$, where $g(s)$ is a polynomial, [7, 20]. Originally Forchheimer in his
work [13] proposed three particular equations to match the experimental data: two
term, three term and power laws, with $g(s)$ being up to second order degree poly-
nomial. To embrace the recent findings on the nonlinearity of the fluid flow (see

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* Corresponding author: Akif Ibragimov.
[18, 24]) and simplify the mathematical handling, it is convenient to consider the generalization of classical Forchheimer equations to the case where \( g(s) \) is the generalized polynomial with non-negative coefficients and, possibly, non-integer powers (see [2]). We call this family of equations the \( g \)-Forchheimer equations. The \( g \)-Forchheimer equation combined with the equation of state and the conservation of mass results in a single degenerate parabolic equation for the pressure \( p(x,t) \) only. Note, that in case of natural reservoirs the inertial and viscous terms in Brinkman-Forchheimer equation (e.g. [23]) have very small impact on engineering parameters such as the productivity index, and these terms are usually neglected, [20, 10].

In this paper we discuss two types of fluids: slightly compressible fluid, characterized by the equation of state \( \rho(p) \sim \exp(\gamma p) \) with very small compressibility constant \( (\gamma \sim 10^{-8}) \), and strongly compressible fluid (in particular, ideal gas) characterized by the equation of state \( \rho(p) \sim p \), [20]. While in general porosity depends on pressure, see [21, 26, 8], we only consider it to be a function of spatial variable \( x \). In case of slightly compressible fluid we studied different properties of \( g \)-Forchheimer equations in [2, 4, 14]. In this paper we extend the results of our work [4] on asymptotic behavior of the pressure function to the case when the degree of the \( g \)-polynomial is arbitrary. In case of ideal gas we discuss both numerical and analytical results for the time dynamics of the solution of the corresponding parabolic equation.

Keeping in mind that the applications of our findings are in geophysics and in particular in reservoir engineering, we restrict out studies to fluid flow in a reservoir with bounded domain \( U \). The reservoir is bounded by the exterior impermeable boundary \( \Gamma_e \) and the interior well-boundary \( \Gamma_i \). \( \Gamma_i \) is subject to various boundary conditions depending on the flow regime. We introduce the capacity type functional to study the asymptotic behavior of the fluid flow in the reservoir with respect to time \( t \). This functional is motivated by the well Productivity Index (PI) and is often used by reservoir engineers to measure well capacity, see [10, 25, 15]. Productivity index is the total amount of fluid per unit pressure drawdown (the difference between reservoir and well pressures) that can be extracted by the well from a reservoir (see [25]) and as such describes the ability of the reservoir to deliver fluids to the well. It is defined as \( J(t) = Q(t)/PDD(t) \), where \( Q(t) \) is the total flux through the boundary \( \Gamma_i \), and the pressure drawdown \( PDD(t) \) is equal to the difference between averages of the pressure in the domain and on the boundary \( \Gamma_i \). Note, that in our previous articles [3, 4, 5] instead of PI we used the notion of the diffusive capacity that has more general physical meaning for other evolutional problems. In current article we use the term PI that is identical to the diffusive capacity and has clear engineering interpretation.

In general the PI is time dependent for both slightly and strongly compressible fluid flows, [10, 25]. However its time dynamics differs greatly in these two cases.

For slightly compressible fluids and specific regimes of production the PI stabilizes in time to a constant value. This value can be determined using the solution of a particular boundary value problem. Namely, the time-invariant PI is associated with the pressure distribution \( p_s(x,t) = -\frac{Q_s}{|U|} t + W(x) \). Here the initial pressure distribution \( W(x) \) is the solution of the specific steady state BVP, \( Q_s \) is a constant flux through the well-boundary, and \( |U| \) is the volume of the reservoir domain. Such pressure is called the pseudo-steady state (PSS) pressure and satisfies the split boundary condition on the well \(-\frac{Q_s}{|U|} t + \varphi(x)\) for some known function \( \varphi(x) \).
For arbitrary initial data one can not expect the PSS pressure distribution and constant PI. However, as it appears from the engineering practice (see [10, 26]) the constant PSS PI serves as an attractor for the transient PI. We proved this fact in [3, 4] under a number of conditions. In this paper we will generalize our previous results. We consider two types of boundary value problems.

In the IBVP-I we impose total flux $Q(t)$ on the well-boundary and consider the trace of the pressure function to be split as $\gamma(t) + \psi(x, t)$, with $\int_{\Gamma} \psi \, ds = 0$. In this case $\gamma(t)$ can be considered as the average of the trace function and $\psi$ as the deviation of the actual trace from its average. Note that we impose conditions only on the function $\psi$, while $\gamma$ is unknown. For $t \rightarrow \infty$ the boundary data $\{\psi(x, t), Q(t)\}$ are assumed to be localized in the neighborhood of the time independent $\{\varphi(x), Q_s\}$.

In the IBVP-II no assumptions on the well-boundary flux are made. Instead, we specify the Dirichlet condition $\gamma(t) + \psi(x, t)$ on the well boundary. In this case both $\gamma$ and $\psi$ are known functions. For $t \rightarrow \infty$ the boundary data $\{\psi(x, t), \gamma(t)\}$ are assumed to be localized in the neighborhood of functions $\{\varphi(x), -\frac{f^*_i}{\alpha}t\}$.

The main result for both IBVP-I and II is that the time dependent PI asymptotically stabilizes in the neighborhood of the PI for the PSS regime associated with the pair $\varphi(x)$ and $Q_s$ (see Theorems 4.11 and 5.4). The corresponding value for the steady state PI can be calculated just by solving an auxiliary Dirichlet time independent BVP.

In [4] we obtained this result under the degree condition, stating that the degree of $g$-polynomial $\deg(g) \leq \frac{4}{n-2}$, where $n$ is the dimension of space. Mathematically this condition arises from the theory of Sobolev spaces and insures the continuous embedding $W^{1,2-a}(U) \subset L^2(U)$. While for $n = 2$ this constraint holds for any degree $\deg(g)$, for $n = 3$ it holds only for $\deg(g) \leq 4$. In this paper we study the asymptotic behavior of the PI without any constraint on $\deg(g)$. In this case the classical Poincaré-Sobolev inequality does not hold, and we use weighted inequality for the mixed term $|\nabla p|^{2-a}|p|^\alpha$, with $\alpha 
eq 2, a < 2$. We obtain estimates for the bounds of the $L^\alpha$-norm for both the difference between transient and PSS solutions, and the difference between transient and PSS solution time derivatives.

In Sec. 6 we discuss the concept of the PI for an ideal gas flow and some results on its time dynamics. In this case the productivity index is defined, [6, 9], as $J_g(t) = Q(t)/\overline{PDD}(t)$, where $\overline{PDD}(t)$ is the difference between the average of $p^2$ in the domain $U$ and on the boundary $\Gamma_i$. In contrast to the case of slightly compressible flow, we show numerically that until certain critical time $T_{crit}$ the transient PI remains almost constant and then as time approaches $T_{crit}$ it blows up. This result obtained on actual field data corresponds with the engineering observations. Time $T_{crit}$ depends on the initial reserves/initial data: the “greater” the initial reserves are the greater $T_{crit}$ is. Similar to our approach in case of slightly compressible fluid, we use the auxiliary pressure $p_0(x, t)$, resulting in time independent PI to investigate the behavior of time dependent PI. The $p_0(x, t)$ is the solution of the equation with positive function on the RHS. This function can be considered as fluid injection inside the reservoir. When gas reserves are considerably larger than pressure drawdown on the well, this source term is negligible, and the PI for $p_0(x, t)$ is almost identical with the general time dependent PI.

The paper is organized as follows.

- In Sec. 2 we discuss the various aspects of nonlinear Forchheimer equations and associated parabolic PDE. We give the definition of the PI, (10), on the solution of this parabolic equation.
• The IBVP-I for total flux and IBVP-II for Dirichlet boundary conditions are introduced in Sec. 3.
• In Sec. 4 we investigate the asymptotic properties of the PI for IBVP-I with given total flux boundary condition. We state the constraints (43) - (47) imposed on the boundary trace of pressure and prove the asymptotic convergence of the transient PI to the constant PI for the PSS solution, Theorem 4.11. In order to prove this result the additional estimates (Lemma 4.1) on the pressure and its gradient and time derivative are obtained in Sec. 4.2 - 4.4.
• In Sec. 5 we investigate the asymptotic properties of the PI for IBVP-II with given Dirichlet boundary data. We state the constraints (121) - (125) imposed on the boundary data and prove the asymptotic convergence of the transient PI to the constant PI for PSS solution, Theorem 5.4.
• In Sec. 6 we introduce the concept of PI (153) for an ideal gas flow, based on the so-called pseudo-pressure function. We show numerically that the gas PI stays the same until it sharply blows up when time approaches the critical value $T_{crit}$ dependent on the parameter characterizing the initial reserves. This fact corresponds well with the field data. We associate this constant PI with the auxiliary pressure in (154) and prove the estimate for the difference between the actual and auxiliary pressure in terms of PI, Sec. 6.1. Finally, in Theorem 6.10 we obtain the stability of the PI with respect to the parameter characterizing the initial gas reserves.

2. Problem statement and preliminary properties. Consider a fluid in a porous medium occupying a bounded domain $U \subset \mathbb{R}^n$. Let $x \in \mathbb{R}^n$, $n \geq 2$, be a spatial variable and $t \in \mathbb{R}$ be a time variable. Let $u(x,t) \in \mathbb{R}^n$ be the fluid velocity and $p(x,t) \in \mathbb{R}$ be the pressure.

We consider a generalized Forchheimer equation

$$g(|u|)u = -\nabla p,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$. In particular we consider function $g$ to be the generalized polynomial with non-negative coefficients. Namely

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \cdots + a_k s^{\alpha_k} = a_0 + \sum_{j=1}^k a_j s^{\alpha_j},$$

with $k \geq 0$, the real exponents satisfy $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$, and the coefficients $a_0, a_1, \ldots, a_k > 0$. The largest exponent $\alpha_k$ is the degree of $g$ and is denoted by $\text{deg}(g)$.

One can notice that the equation (1) with $g(s)$ defined as in (2) includes the linear Darcy’s equation and all classical forms of Forchheimer equations [13] (for details see our previous works [2, 4]).

In case of slightly compressible fluid we consider the case when the Degree Condition in formula (2.14), [4], is not satisfied, namely

$$a = \frac{\alpha_k}{\alpha_k + 1} \leq \frac{4}{2 + n} \quad \iff \quad \alpha_k = \text{deg}(g) > \frac{4}{n - 2}. \quad (3)$$

In this case there is no continuous embedding $W^{1,2-a}(U) \subset L^2(U)$ and corresponding Poincaré inequality does not hold. Clearly, if $n = 3$ condition (3) will hold for the nonlinearities $\alpha_k > 4$. 
From (1) one can obtain the non-linear Darcy equation explicitly solved for the velocity $u$:

$$u = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = \frac{1}{g(G^{-1}(\xi)), \ \xi \geq 0, \ G(s) = sg(s), \ s \geq 0.}$$

Along with (1), which is considered as a momentum equation, the dynamical system is subject to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

where $\rho(x,t) \in \mathbb{R}^+$ is density of the fluid. The fluid is considered to be slightly compressible, subject to the equation of state

$$\frac{\partial \rho}{\partial p} = \gamma \rho \quad \text{or} \quad \rho(p) = \rho_0 e^{\gamma(p-p_0)},$$

where $\gamma \sim 10^{-8}$ is the compressibility constant, see [20].

Combining (1), (6) and (7) we obtain the pressure equation (see [2] for details):

$$\frac{\partial p}{\partial t} = \gamma^{-1} \nabla \cdot (K(|\nabla p|)|\nabla p| + |\nabla p||\nabla p|).$$

Since in natural reservoirs $\gamma \sim 10^{-8}$, in engineering practice the second term is always neglected in comparison to the first one, [25, 10]. We thus arrive to the truncated degenerate parabolic equation

$$\frac{\partial p}{\partial t} = \gamma^{-1} \nabla \cdot (K(|\nabla p|)|\nabla p|).$$

It is not difficult to show that for small $\gamma$ the difference between the solutions of full and truncated equations is also small. By scaling the time variable we will assume further on that $\gamma = 1$.

In case of the flow of ideal gas the equation (1) for $n = 2$ will take the form (see, for example, [22])

$$\alpha u + \beta \rho |u| = -\nabla p,$$

where $\beta$ is the Forchheimer coefficient. As in case of slightly compressible fluid, the above equation can be solved for $u$:

$$u = -\frac{2}{\alpha + \sqrt{\alpha^2 + 4\beta \rho |\nabla p|}} \nabla p.$$

Combining the equation above with the equation of state $\rho(p) = M p$ (without loss of generality we take $M = 1$) and continuity equation (6) we obtain (for the details see Sec. 6)

$$\frac{\partial p}{\partial t} = \nabla \cdot \left( \frac{2p}{\alpha + \sqrt{\alpha^2 + 4\beta p |\nabla p|}} \nabla p \right).$$

We will study equations (8) and (9) in the open domain $U \subset \mathbb{R}^n$ with the $C^2$ boundary $\partial U = \Gamma = \Gamma_e \cup \Gamma_i$. The $\Gamma_e$ is considered to be external impermeable boundary of the reservoir with $u \cdot N = 0$, where $N$ is the outward normal vector to $\Gamma_e$. The $\Gamma_i$ is the internal boundary of the well with the total flux or Dirichlet boundary conditions.
To study the long term dynamics of the $g$-Forchheimer flow in the domain $U$ we introduce special capacity type functional. We call it the Productivity Index (PI) (see [10, 25, 28]) and define as

$$J(t) = \frac{Q(t)}{PDD(t)}. \quad (10)$$

where $Q(t)$ is the total flux through the well-boundary $\Gamma_i$ and $PDD(t)$ is called the pressure drawdown in the domain $U$.

In case of slightly compressible fluid these quantities are defined as

$$Q(t) = \int_{\Gamma_i} u \cdot N \, ds, \quad (11)$$

$$PDD(t) = \frac{1}{|U|} \int_U p(x,t) \, dx - \frac{1}{|\Gamma_i|} \int_{\Gamma_i} p(x,t) \, ds. \quad (12)$$

are the pressure averages in the domain and on the well-boundary correspondingly.

For gas filtration in porous media these quantities are defined as, see [6, 9]:

$$Q(t) = \int_{\Gamma_i} \rho u \cdot N \, ds, \quad (11)$$

$$PDD(t) = \frac{1}{|U|} \int_U p^2 \, dx - \frac{1}{|\Gamma_i|} \int_{\Gamma_i} p^2 \, ds. \quad (12)$$

We end this section with the recollection of some properties of $K(\xi)$ that will be used in this study (see [2, 4]). If $g(\cdot)$ is defined as in (2), then the inverse function $K(\xi)$, $\xi \in [0, \infty)$ in (5) is decreasing and satisfies

$$0 \geq K'(\xi) \geq -a \frac{K(\xi)}{\xi}, \quad (13)$$

$$C_0(1 + \xi)^{-a} \leq K(\xi) \leq C_1(1 + \xi)^{-a} \leq C_1, \quad (14)$$

$$C_2 \xi^{2-a} - 1 \leq K(\xi) \xi^2 \leq C_1 \xi^{2-a}. \quad (15)$$

Moreover the function $K(|y|)y$, $y \in \mathbb{R}^n$, is monotone, i.e. for any two functions $u_1, u_2 \in W^{2,1}(U)$, one has

$$\int_U (K(|\nabla u_1|)\nabla u_1 - K(|\nabla u_2|)\nabla u_2) \cdot \nabla (u_1 - u_2) \, dx$$

$$\geq C_\Phi^{-1} \left( \int_U |\nabla (u_1 - u_2)|^{2-a} \, dx \right)^{\frac{2}{2-a}}, \quad (16)$$

where

$$C_\Phi = C_\Phi(u_1, u_2) = C_1(1 + \max(\|\nabla u_1\|_{L^{2-a}(U)}, \|\nabla u_2\|_{L^{2-a}(U)}))^{a}. \quad (17)$$

Following [2, 4] we define the functional

$$H(\xi) = \int^\xi_0 K(\sqrt{s}) \, ds \quad \text{for} \quad \xi \geq 0. \quad (18)$$

Function $H(\xi)$ can be compared with $K(\xi)$ (see [2]) as follows:

$$K(\xi) \xi^2 \leq H(\xi) \leq 2K(\xi) \xi^2. \quad (19)$$

Combining (19) with (15) there exist constants $C_1$ and $C_2$ such that

$$C_1 \xi^{2-a} - 1 \leq H(\xi) \leq C_2 \xi^{2-a}. \quad (20)$$
3. Two types of boundary conditions and pseudo-steady state solution.

In Sections 3, 4 and 5 we consider the case of slightly compressible fluid.

Let $U \subset \mathbb{R}^n$ be an open set with the $C^2$ boundary $\partial U = \Gamma = \Gamma_e \cup \Gamma_i$, see Fig. 1. The $\Gamma_e$ is considered to be external impermeable boundary of the reservoir and the $\Gamma_i$ is the internal boundary of the reservoir (well surface). With respect to boundary $\Gamma_i$ two different problems will be considered: IBVP-I with imposed total flux (generalized Neumann condition) and IBVP-II with imposed Dirichlet boundary condition.

**Figure 1. Reservoir domain $U$ with boundaries $\Gamma_i$ and $\Gamma_e$.**

**I. IBVP-I for the total flux condition.** The function $p(x,t)$ is a solution of the IBVP-I if it satisfies

$$
\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) \quad \text{in} \quad D = U \times (0, \infty),
$$

$$
- \int_{\Gamma_i} K(|\nabla p|)\nabla p \cdot N \, ds = Q(t) \quad \text{on} \quad \Gamma_i \times (0, \infty),
$$

$$
\frac{\partial p}{\partial N}\bigg|_{\Gamma_e} = 0 \quad \text{on} \quad \Gamma_e \times (0, \infty),
$$

$$
p(x,0) = p_0(x) \quad \text{in} \quad U.
$$

Throughout this paper, we consider solutions with sufficient regularities both in $x$ and $t$ variables such that our calculations can be performed legitimately. Though the results are applicable to to weak solutions with enough regularities up to the boundary, we consider classical solution, namely $p \in C^{2,1}(U \times (0, \infty)) \cap C^{1,1}(U \times [0, \infty)) \cap C^1(\Gamma)$.

Since the IBVP-I (21)-(24) lacks uniqueness due to conditions (22) and (23), following [4] we further restrict the boundary data. In particular, let the trace of $p(x,t)$ on well-boundary $\Gamma_i$ be of the form

$$
p(x,t)|_{\Gamma_i} = \gamma(t) + \psi(x,t),
$$

with

$$
\int_{\Gamma_i} \psi(x,t) \, ds = 0.
$$

Note, that function $\gamma(t)$ is not specified and will be determined by the flux $Q(t)$. Such restriction of the boundary trace ensures the uniqueness of the solution of IBVP (21)-(24) (see Remark 3.2 in [4]).

Transient boundary data $\psi(x,t)$ and $Q(t)$ will be compared with the time-independent boundary data $\varphi(x)$ and $Q_s$. As in [4] we consider functions $\Psi(x,t)$ and $\Phi(x)$ to be defined in the domain $U$ with the traces on $\Gamma_i$ being $\psi(x,t)$ and $\varphi(x)$.
correspondingly. The results are obtained under the constraints on the parameters of difference of boundary data
\[ \Delta Q(t) = Q(t) - Q_s; \quad \Delta \Psi(x,t) = \Psi(x,t) - \Phi(x). \] (27)

II. IBVP-II for Dirichlet boundary condition. The function \( p(x,t) \) is a solution of the IBVP-II if it satisfies (21), (23), (24) with the condition on the well boundary
\[ p(x,t)|_{\Gamma_i} = \psi(x,t) + \gamma(t) \quad \text{on} \quad \Gamma_i \times (0, \infty), \] (28)
for known function \( \psi(x,t) \).

Transient boundary data \( \psi(x,t) \) and \( \gamma(t) \) will be compared with the boundary data \( \phi(x) \) and \( -At \), \( A = Q_s/|U| \), where \( Q_s \) is constant flux on \( \Gamma_i \). We again consider functions \( \Psi(x,t) \) and \( \Phi(x) \) to be the \( W^{1,2}(U) \) extensions of \( \psi(x,t) \) and \( \phi(x) \) on the domain \( U \). The results are obtained under the constraints on the differences
\[ \Delta \Psi(x,t) = \Psi(x,t) - \Phi(x); \quad \Delta \gamma(t) = \gamma(t) + At. \] (29)

For the existence and regularity theory of degenerate parabolic equations, see e.g. [16, 19, 14] and references therein.

3.1. PSS solution. For the time-independent boundary data \( \phi(x) \), \( Q_s \) we define the pseudo steady state (PSS) solution of IBVP for equation (21)
\[ p_s(x,t) = -At + W(x), \quad A = \frac{Q_s}{|U|} \] (30)
for a given function \( \phi(x) \) and a constant total flux \( Q_s \) on the boundary \( \Gamma_i \). Function \( W(x) \) is called the basic profile corresponding to the flux \( Q_s \) and is defined as a solution of BVP
\[ -A = \nabla \cdot (K(|\nabla W|)\nabla W), \] (31)
\[ W(x)|_{\Gamma_i} = \phi(x), \] (32)
\[ \frac{\partial W}{\partial N} \bigg|_{\Gamma_e} = 0. \] (33)

The existence and uniqueness of a solution for the given pair \( \phi(x) \), \( Q_s \) is proved in [3].

It is not difficult to see that in case of the PSS solution when \( p(x,t) = p_s(x,t) \) the functional \( J(t) \) in (10) is time independent and
\[ J(t) = J_{PSS} = \frac{Q_s}{|\Gamma_i|} \int_{\Gamma_i} W(x) \, dx - \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \phi(x) \, ds = \text{const}. \] (34)
We assume that \( \phi(x) \) and \( Q_s \) are such that the denominator of (34) is not equal zero. Obviously PSS solution is the only solution for given \( \phi(x) \), \( Q_s \) which obey both types of boundary conditions. Therefore PSS will serve us a reference solution for comparison for both IBVP-I and II.

3.2. Time convergence of \( J(t) \to J_{PSS} - \text{general case} \). Let \( p(x,t) \) be the transient solution of IBVP-I (21)-(24) with boundary data \( \Psi(x,t), Q(t) \) or IBVP-II (21), (23), (24), (28) with boundary data \( \Psi(x,t) \). Let \( J(t) \) be the corresponding PI defined as in (10). Let \( p_s(x,t) \) be the PSS solution of this IBVP with boundary data \( \Phi(x), Q_s \) and the corresponding constant PI \( J_{PSS} \) defined as in (34).

For slightly compressible fluid flow subjected to Forchheimer equation the aim of this study can be formulated as follows
Theorem 3.1. There is a wide class of transient boundary data such that for both problem IBVP-I and IBVP-II

\[ J(t) - J_{PSS} \to 0 \quad \text{as} \quad t \to \infty. \]

The proof uses the following Lemma which reduces the question of the convergence of \( J(t) \) to the convergence of the gradient of the solution in the appropriate norm and of the total fluxes to a corresponding constant values.

Lemma 3.2. As \( t \to \infty \) the difference between transient and PSS PI's

\[ J(t) - J_{PSS} \to 0 \quad \text{if} \quad \| \nabla (p - p_s) \|_{L^2(I_t)} \to 0 \quad \text{and} \quad \Delta Q(t) \to 0. \]  

Proof. From (10) and (34) it follows

\[
J(t) - J_{PSS} = \frac{Q(t)}{p_U - p_{\Gamma_i}} - \frac{Q_s}{p_s U - p_{\Gamma_i}} = \frac{Q(t)(p_{sU} - p_{s\Gamma_i})}{(p_U - p_{\Gamma_i})(p_{sU} - p_{s\Gamma_i})} - \frac{Q_s(p_U - p_{\Gamma_i})}{(p_{sU} - p_{s\Gamma_i})}
\]

\[
= \frac{Q(t) \cdot \Delta_p(t)}{(p_{sU} - p_{s\Gamma_i})} + \Delta Q(t) \frac{J_{PSS}}{Q_s}. \]  

(36)

Here \( \Delta_p(t) = (p_{sU} - p_{s\Gamma_i}) - (p_U - p_{\Gamma_i}) \) is the difference of pressure drawdowns for transient and PSS regimes.

The pressure drawdown for the PSS solution is time independent, and is not equal zero. Hence \( J(t) - J_{PSS} \to 0 \) if and only if both \( \Delta_p(t) \to 0 \) and \( \Delta Q(t) \to 0 \) as \( t \to \infty \).

The difference in pressure drawdowns \( \Delta_p(t) \) can be written as

\[
\Delta_p(t) = \frac{1}{|I_i|} \int_{I_i} (p - p_s) \, ds - \frac{1}{|I|} \int_I (p - p_s) \, dx
\]

\[
= \frac{1}{|I_i|} \int_{I_i} \left( p - p_s - \frac{1}{|U|} \int_U (p - p_s) \, dx \right) \, ds
\]

\[
- \frac{1}{|I|} \int_I \left( p - p_s - \frac{1}{|U|} \int_U (p - p_s) \, dx \right) \, dx = \frac{1}{|I_i|} \int_{I_i} \tilde{z} \, ds,
\]

where

\[
\tilde{z}(x, t) = p - p_s - \frac{1}{|U|} \int_U (p - p_s) \, dx.
\]

Applying Trace theorem and Poincaré inequality, as \( \int_U \tilde{z} \, dx = 0 \), one has

\[
|\Delta_p(t)|^{2-a} \leq C \int_{I_i} |\tilde{z}|^{2-a} \, ds \leq C \int_U |\nabla \tilde{z}|^{2-a} \, dx = C \int_U |\nabla (p - p_s)|^{2-a} \, dx.
\]

The condition (35) follows.

The results for the solution of IBVP-I will be obtained under the conditions on the deviation of the boundary data (27) in Sec. 4, and for the solution of IBVP-II in terms of deviations (29) in Sec. 5.

4. IBVP-I: Asymptotic convergence of the productivity index. In this section we prove the Theorem 3.1 on asymptotic convergence of the transient PI \( J(t) \) defined on the solution of IBVP-I (21)-(24) with boundary data \( \Psi(x, t), Q(t) \). Let \( p_s(x, t) \) be the PSS solution of this IBVP with boundary data \( \Phi(x), Q_s \) and the corresponding constant PI \( J_{PSS} \).
4.1. Assumptions on boundary data. In order to formulate the assumptions in the Theorem 3.1 we first introduce the following notations.

Suppose $\alpha \geq \frac{n-1}{2-a} > 2$ and let $2 - a < b = \frac{a(2-a)}{2} < \alpha$. We define

$$F_1(t) = \frac{1}{|U|} \int_0^t \Delta Q(\tau) d\tau + \frac{1}{|U|} \int_U \Delta \Phi \, dx - \frac{1}{|U|} \int_U (p(x,0) - p_s(x,0)) \, dx.$$  
(37)

$$F_2(t) = 1 + \|\nabla W\|_{L^{2-a}(U)} + |F_1(t)| + \int_U |\nabla (\Delta \Phi)| \, dx + |\Delta Q(t)|^{\frac{1}{\alpha - 2}}.$$  
(38)

$$A_1(\alpha, t) = \|\nabla W\|_{L^a}^{\alpha - a} + \|\nabla (\Delta \Phi)\|_{L^a}^{\alpha - a} + |\Delta Q(t)| + \int_U \Psi_t \, dx \right|^a$$  

$$+ \left( \int_U |\Psi_t| \, dx \right)^{\alpha (1-a)} + \Delta Q(t).$$  
(39)

$$A_1(\alpha) = \limsup_{t \to \infty} A_1(\alpha, t)^{\frac{\alpha}{\alpha - 2}}.$$  
(40)

$$A_2(t) = \int_U |\nabla W|^{2-a} \, dx + \int_U |\nabla \Psi_t|^2 \, dx + \int_U |\nabla (\Delta \Phi)|^{2-a} \, dx$$  

$$+ \left( \int_U |\Psi_t| \, dx \right)^{2-a} + \int_U |\Psi_t|^2 \, dx + \int_U |\Psi_t|^b \, dx$$  

$$+ |\Delta Q(t)| + |Q'(t)| + |\Delta Q(t)|^b + |Q'(t)|^b + C|F_1(t)|^{2-a}.$$  
(41)

$$A_3(\varepsilon, t) = \int_U |\nabla \Psi_t|^2 \, dx + \frac{1}{4\varepsilon} \int_U |\Psi_{tt}|^2 \, dx + |Q'(t)|^2 + |F_1(t)|^2.$$  
(42)

The certain way the conditions A1-A5 will be formulated is dictated by the fact, that while the main result will be obtained under all the assumptions below, some auxiliary estimates require less restrictive constraints. As a note for each assumption we state the resulting bounds on parameters $F_1$, $F_2$, $A_1$, $A_2$ and $A_3$.

Assumptions 1 (A1). ($A_1(\alpha) \leq C$)

$$\limsup_{t \to \infty} \left( |\Delta Q(t)| + \|\Psi_t\|_{L^b} + \|\nabla \Phi\|_{L^b} + \|\nabla \Phi\|_{L^b} \right) \leq C.$$  
(43)

Assumptions 2 (A2). ($A_1(\alpha) + \limsup_{t \to \infty} (A_2(t) + F_2(t)) \leq C$)

$$\left\{ \begin{array}{l}
\text{Assumptions A1;}
\limsup_{t \to \infty} \left( |Q'(t)| + |F_1(t)| + \|\Psi_t\|_{L^2} \right) \leq C.
\end{array} \right.$$  
(44)

Assumptions 3-\beta (A3-\beta). for any $\beta \geq 2$

$$|\Delta'_\mu(t)| + |\Delta Q(t)| + \int_1^\infty \|\Psi_{tt}\|_{L^0(U)} + \|\nabla \Psi_t\|_{L^0(U)}^{2} \, dt \leq C.$$  
(45)

Assumptions 4 (A4). $\limsup_{t \to \infty} A_3(t) = 0$

$$\left\{ \begin{array}{l}
\text{Assumptions A2;}
\limsup_{t \to \infty} \left( \int_U (|\nabla \Psi_t|^2 + |\Psi_{tt}|^2) \, dx + |Q'(t)|^2 + \Delta Q(t) \right) = 0.
\end{array} \right.$$  
(46)
Assumptions 5 (A5).

\[
\begin{cases}
\text{Assumptions A3 and A4;} \\
\limsup_{t \to \infty} \|\nabla (\Delta \phi)\|_{L^2} = 0.
\end{cases}
\] (47)

Along with the solution of IBVP-I \( p(x, t) \) and the PSS solution \( p_s(x, t) \) of IBVP-I with boundary data \( \Phi(x), Q_s \), we will use the following shifts of the solutions:

\[
q(x, t) = p(x, t) - \frac{1}{|U|} \int_U p(x, t) \, dx - \left( \Psi(x, t) - \frac{1}{|U|} \int_U \Psi(x, t) \, dx \right); 
\] (48)

\[
q_s(x) = p_s(x, t) - \frac{1}{|U|} \int_U p_s(x, t) \, dx - \left( \Phi(x) - \frac{1}{|U|} \int_U \Phi(x) \, dx \right).
\] (49)

It follows that \( q \) satisfies the BVP

\[
q_t(x, t) = p_t(x, t) + \frac{1}{|U|} Q(t) - \left( \Psi_t(x, t) - \frac{1}{|U|} \int_U \Psi_t(x, t) \, dx \right),
\] (50)

\[
q|_{r_i} = \gamma(t) - \frac{1}{|U|} \int_U p(x, 0) \, dx + \frac{1}{|U|} \int_0^t Q(\tau) \, d\tau + \frac{1}{|U|} \int_U \Psi(x, t) \, dx.
\] (51)

Similarly \( q_s \) satisfies the BVP

\[
\frac{\partial q_s}{\partial t}(x, t) = \frac{\partial p_s}{\partial t}(x, t) + A = 0,
\] (52)

\[
q_s|_{r_i} = - \frac{1}{|U|} \int_U (p_s(x, 0) - \Phi(x)) \, dx.
\] (53)

Note that

\[
\int_U q(x, t) \, dx = 0 \quad \text{and} \quad \int_U q_s(x, t) \, dx = 0.
\] (54)

According to Lemma 3.2 we need the convergence \( \Delta Q(t) \to 0 \) and \( \|\nabla (p - p_s)\|_{L^{2-a}} \to 0 \) as \( t \to \infty \). The affinity of \( Q(t) \) and \( Q_s \) will be imposed as a condition on the boundary data (see Assumptions A4). The estimates necessary to prove the convergence of \( \|\nabla (p - p_s)\|_{L^{2-a}} \) are outlined in the following Lemma. The references to the corresponding results further in the paper are given.

**Lemma 4.1.** Let \( q \) and \( q_s \) be defined as in (48) and (49). Then

\[
\|\nabla (p - p_s)\|_{L^{2-a}} \to 0 \quad \text{as} \quad t \to \infty
\]

if

\[
\lim_{t \to \infty} \left( \|\nabla (\Delta \phi)\|_{L^2} + |\Delta Q(t)||F_1(t) + 1| \right) = 0
\] (55)

and

1. \( \int_U |q - q_s|^2 \, dx \leq C \int_U |q - q_s|^a \, dx \leq C \), see Theorem 4.6;
2. \( \int_U |\nabla p|^{2-a} \, dx \leq C \), see Theorem 4.7;
3. \( \int_U |p_t + A|^2 \, dx = \int_U |p_t| \, dx \to 0 \), see Theorem 4.10;
4. \( \int_U |\nabla p_s|^{2-a} \, dx \leq C \), see Lemma 4.1, [4].

(56) (57) (58) (59)
Proof. According to the inequality (5.20) in proof of Theorem 5.6 in [4] we have:

\[
\|\nabla(p - p_s)\|_{L^2}^2 \leq C\Phi \int_U |p_t + A||q - q_s| \, dx + C\Phi \left( \int_U |\nabla^2\Phi|^2 \, dx + \Delta^2\Phi(t) + |F_1(t)\Delta Q(t)| + \|\nabla(\Delta\Phi)\|_{L^2} \right).
\]  

(60)

Here \(C\Phi = C\Phi(p, W)\) is as in (17) and is bounded by virtue of (57) and (59). It is then clear that the RHS of (60) converges to 0 under conditions (55)-(59). \(\Box\)

4.2. Bounds for the solutions. As before let \(q(x, t)\) be the solution of BVP (50), (51) and \(q_s(x, t)\) be the solution of PSS BVP (52), (53). Let \(z = q - q_s\). We will prove that under certain condition \(\int_U |z|^{\alpha} \, dx\) is bounded at time infinity.

In order to obtain the suitable differential inequality we will use the following result (see Lemma 2.3, [14])

**Lemma 4.2.** Let \(U\) be an open bounded domain in \(\mathbb{R}^n\) and \(\alpha \geq \frac{na}{2 - a}\). If \(u|\partial U = 0\), there exists constant \(C\) such that

\[
\int_U K(|u|)|\nabla u|^{2-\alpha}|u|^\alpha \, dx \geq C\int_U |\nabla u|^{2-\alpha}|u|^{\alpha - 2} - C \, dx \\
\geq C \left( \int_U |u|^{\alpha} \, dx \right)^\frac{1}{\alpha_0} - C,
\]

(61)

where \(\gamma_0 = \alpha/(\alpha - a)\).

We first prove the following lemma to estimate the boundary data.

**Lemma 4.3.** Let \(F_1(t)\) be as in (37), \(\Delta Q = Q(t) - Q_s\) and \(\Delta_q = \gamma(t) + At\). Then for \(\alpha \geq \frac{na}{2 - a}\)

\[
|\Delta_q(t) + F_1(t)|^{\alpha - 1}|\Delta Q(t)|
\]

\[
\leq \varepsilon_1 \left( \int_U |z|^{\alpha} \, dx \right)^{\frac{1}{\alpha_0}} + \varepsilon_2 \int_U |\nabla z|^{2-\alpha}|z|^{\alpha - 2} + C \Delta^\frac{\alpha - a}{\alpha}
\]

with \(\gamma_0 = \alpha/(\alpha - a)\).

**Proof.** We have \(z|\Gamma_i = (q - q_s)|\Gamma_i = \Delta_q(t) + F_1(t)\). Then by Trace theorem

\[
((\Delta_q + F_1)|\Gamma_i)^{\alpha - a} = \left( \int_{\Gamma_i} |z| \, ds \right)^{\alpha - a} \leq C \int_{\Gamma_i} |z|^{\alpha - a} \, ds \\
= C \int_{\Gamma_i} |z|^{\frac{2a}{2a - (2-a)}} (2-a) \, ds \leq C \int_U |z|^{\alpha - a} \, dx + C \int_U \left[ \nabla |z|^{\frac{2a}{2a - a}} \right]^{2-a} \, dx \\
= C \int_U |z|^{\alpha - a} \, dx + C \int_U \left[ \nabla |z|^{\frac{2a}{2a - a}} \frac{2a}{2a - a} \right]^{2-a} \, dx \\
= C \int_U |z|^{\alpha - a} \, dx + C \int_U |\nabla z|^{2-a}|z|^{\alpha - 2} \, dx.
\]

Then

\[
|\Delta_q + F_1|^{\alpha - 1} \leq C \left( \int_U |z|^{\alpha - a} \, dx \right)^{\frac{\alpha - 1}{\alpha}} + C \left( \int_U |\nabla z|^{2-a}|z|^{\alpha - 2} \, dx \right)^{\frac{\alpha - 1}{\alpha}} \\
\leq C \left( \int_U |z|^{\alpha} \, dx \right)^{\frac{\alpha - 1}{\alpha}} + C \left( \int_U |\nabla z|^{2-a}|z|^{\alpha - 2} \, dx \right)^{\frac{\alpha - 1}{\alpha}}.
\]

(64)
Applying Young’s inequality with $\varepsilon$ we get

$$|\Delta_\gamma + F_1|^{\alpha-1}|\Delta Q(t)| \leq \varepsilon_1 \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} + \varepsilon_2 \int_U |\nabla z|^{2-\alpha}|z|^{\alpha-2} \, dx$$

$$+ (C_{\varepsilon_1} + C_{\varepsilon_2}) \Delta_q^{\frac{\alpha-a}{2}}$$

which proves the result. \qed

**Lemma 4.4.** Suppose $\alpha \geq \frac{na}{2-a} > 2$ and let $\gamma_0 = \frac{\alpha}{\alpha-a}$. Then for $z = q - q_s$ and $A_1(\alpha, t)$ as in (39)

$$\frac{d}{dt} \int_U |z|^\alpha \, dx \leq -C \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} + C(1 + A_1(\alpha, t)).$$

**Proof.** Subtracting equations (50) and (52) from each other, multiplying on $|z|^{\alpha-1}$ sign($z$) and integrating over $U$ one has

$$\frac{1}{\alpha} \frac{d}{dt} \int_U |z|^\alpha \, dx = - (\alpha - 1) \int_U (K(|\nabla p|)\nabla p - K(|\nabla W|)\nabla W)\nabla \alpha |z|^{\alpha-2} \, dx$$

$$+ \int_{\Gamma_i} (K(|\nabla p|)\nabla p - K(|\nabla W|)\nabla W) \cdot N |z|^{\alpha-1} \, ds + \frac{1}{|U|} \Delta Q(t) \int_U |z|^{\alpha-1} \text{sign}(z) \, dx$$

$$- \int_U \left[ \Psi_t - \frac{1}{|U|} \int_U \Psi_t \, dy \right] |z|^{\alpha-1} \text{sign}(z) \, dx.$$

From boundary conditions (51) and (53) it follows that $z|_{\Gamma_i} = \Delta_\gamma(t) + F_1(t)$, and one has

$$\frac{1}{\alpha} \frac{d}{dt} \int_U |z|^\alpha \, dx = - (\alpha - 1) \int_U (K(|\nabla p|)\nabla p - K(|\nabla W|)\nabla W)\nabla (p - p_s)|z|^{\alpha-2} \, dx$$

$$+ (\alpha - 1) \int_U (K(|\nabla p|)\nabla p - K(|\nabla W|)\nabla W)\nabla (\Delta_\gamma)|z|^{\alpha-2} \, dx$$

$$+ \frac{1}{|U|} \int_U \Delta Q(t) + \int_U \Psi_t \, dx - |U| \Psi_t |z|^{\alpha-1} \text{sign}(z) \, dx + |\Delta_\gamma(t) + F_1(t)|^{\alpha-1} \Delta Q(t).$$

We split the first integral in RHS of (66) in four separate integrals and estimate them one-by-one. Namely,

$$- \int_U (K(|\nabla p|)\nabla p - K(|\nabla W|)\nabla W)\nabla (p - p_s)|z|^{\alpha-2} \, dx = I_p + I_w + I_{pw} + I_{wp},$$

where (notice that $\nabla p_s = \nabla W$)

$$I_p = - \int_U K(|\nabla p|)\nabla p^2 |z|^{\alpha-2} \, dx, \quad I_w = - \int_U K(|\nabla W|)\nabla W^2 |z|^{\alpha-2} \, dx \leq 0,$$

$$I_{pw} = \int_U K(|\nabla p|)(\nabla p \cdot \nabla W)|z|^{\alpha-2} \, dx, \quad I_{wp} = \int_U K(|\nabla W|)(\nabla W \cdot \nabla p)|z|^{\alpha-2} \, dx.$$

Since $\nabla z = \nabla (p - W) - \nabla (\Delta_\gamma)$ and $\frac{\alpha-2}{\alpha} < \frac{\alpha-a}{\alpha} = \frac{1}{\alpha}$, similar to (4.5) in [14] we have

$$I_p \leq -C \int_U (|\nabla p|^{2-a} - 1)|z|^{\alpha-2} \, dx \leq -C \int_U |\nabla z|^{2-a}|z|^{\alpha-2} \, dx.$$
The first integral in (69) can be estimated similar to (4.8)

\[ + C \int_{U} (|\nabla(\Delta \psi)|^{2-a} + |\nabla p_{a}|^{2-a} + 1)|z|^\alpha - 2 \, dx \]

\[ \leq - C \int_{U} |\nabla z|^{2-a} |z|^\alpha - 2 \, dx + \varepsilon_{1} \left( \int_{U} |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} \]

\[ + C_{\varepsilon_{1}} \left[ 1 + \|\nabla W\|^{\alpha-a}_{L^{\infty}(U)} + \|\nabla(\Delta \psi)\|^{\alpha-a}_{L^{q}(U)} \right]. \quad (68) \]

The integral \( I_{pw} \) can be estimated similar to (4.6)-(4.9) in [14]:

\[ I_{pw} \leq \int_{U} K(|\nabla p|)|\nabla p||\nabla W||z|^{\alpha - 2} \, dx \leq \int_{U} |\nabla p|^{1-a} |\nabla W||z|^{\alpha - 2} \, dx \quad (69) \]

\[ \leq \int_{U} |\nabla z|^{1-a} |\nabla W||z|^{\alpha - 2} \, dx + \int_{U} (|\nabla W|^{2-a} + |\nabla(\Delta \psi)|^{1-a} |\nabla W|)|z|^{\alpha - 2} \, dx. \]

The first integral in (69) can be estimated similar to (4.8)

\[ \int_{U} |\nabla z|^{1-a} |\nabla W||z|^{\alpha - 2} \, dx \]

\[ \leq \varepsilon \int_{U} |\nabla z|^{2-a} |z|^{\alpha - 2} \, dx + \varepsilon \left( \int_{U} |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} + C_{\varepsilon} \|\nabla W\|^{\alpha-a}_{L^{\infty}(U)}. \quad (70) \]

The second integral in (69) is estimated similar to (4.7)

\[ \int_{U} |\nabla W|^{2-a} |z|^{\alpha - 2} \, dx \leq \varepsilon \left( \int_{U} |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} + C_{\varepsilon} \|\nabla W\|^{\alpha-a}_{L^{\infty}(U)}. \quad (71) \]

Similar the third integral in (69) can be estimated as

\[ \int_{U} |\nabla(\Delta \psi)|^{1-a} |\nabla W||z|^{\alpha - 2} \, dx \leq \varepsilon \left( \int_{U} |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} \]

\[ + C_{\varepsilon} \left( \int_{U} |\nabla(\Delta \psi)|^{\frac{1-a}{2}} |\nabla W|^{\frac{a}{2}} \, dx \right)^{\frac{2-a}{a}}. \quad (72) \]

Since \( \frac{1}{2-a} + \frac{a}{2-a} = 1 \), from Hölder and Young’s inequalities it follows

\[ \left( \int_{U} |\nabla(\Delta \psi)|^{\frac{1-a}{2}} |\nabla W|^{\frac{a}{2}} \, dx \right)^{\frac{2-a}{a}} \leq C \left( \|\nabla W\|^{\alpha-a}_{L^{\infty}(U)} + \|\nabla(\Delta \psi)\|^{\alpha-a}_{L^{\infty}(U)} \right). \quad (73) \]

Combining the estimates above in (69) we have the estimate for the integral \( I_{pw} \)

\[ I_{pw} \leq \varepsilon \int_{U} |\nabla z|^{2-a} |z|^{\alpha - 2} \, dx + \varepsilon \left( \int_{U} |z|^\alpha \, dx \right)^{\frac{1}{\alpha}} \]

\[ + C \left( \|\nabla W\|^{\alpha-a}_{L^{\infty}(U)} + \|\nabla(\Delta \psi)\|^{\alpha-a}_{L^{\infty}(U)} \right). \quad (74) \]

Finally, for the integral \( I_{wp} \) we have

\[ I_{wp} \leq \int_{U} K(|\nabla W|)|\nabla W||\nabla p||z|^{\alpha - 2} \, dx \leq \int_{U} |\nabla W|^{1-a} |\nabla p||z|^{\alpha - 2} \, dx \quad (75) \]

\[ \leq C \int_{U} |\nabla W|^{1-a} |\nabla z||z|^{\alpha - 2} \, dx + C \int_{U} |\nabla W|^{1-a} |\nabla(W + \Delta \psi)||z|^{\alpha - 2} \, dx. \]
Applying Hölder’s and Young’s inequalities for the first integral in the RHS of (75) we get

\[ \int_U |\nabla W|^{-\alpha} |\nabla z||z|^\alpha - 2 \, dx \]

\[ = \int_U |\nabla W|^{-\alpha} |\nabla z||z|^\alpha - 2 \, dx \]

\[ \leq \left( \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx \right)^{\frac{1}{2} - \frac{\alpha}{2 - \alpha}} \cdot \left( \int_U |\nabla W|^{-2} |z|^\alpha - 2 \, dx \right)^{\frac{1}{2}} \]

\[ \leq \varepsilon \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx + C \varepsilon \int_U |\nabla W|^{-2} |z|^\alpha - 2 \, dx \]

\[ \leq \varepsilon \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx + \varepsilon \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{2}} + C \varepsilon \|\nabla W\|_{L^p(U)}^{\alpha - 2} \|

The second integral in the RHS of (75) can be estimated similar to (72)

\[ \int_U |\nabla W|^{-\alpha} |\nabla (W + \Delta \Phi)||z|^\alpha - 2 \, dx \]

\[ \leq \varepsilon \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{2}} + C \left( \|\nabla W\|_{L^p(U)}^{\alpha - 2} +\|

Then in (75) the integral \( I_{wp} \) can be estimated as

\[ I_{wp} \leq \varepsilon \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx + \varepsilon \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{2}} + C \left( \|\nabla W\|_{L^p(U)}^{\alpha - 2} +\|

Substituting (68), (74) and (76) in (67) we get the estimate for the first integral in the RHS of (66)

\[ - \int_U (K(|\nabla p|) \nabla p - K(|\nabla W|) \nabla W) \nabla (p - p_s)|z|^\alpha - 2 \, dx \]

\[ \leq - (C - \varepsilon) \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx + \varepsilon \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{2}} \]

\[ + C \left( 1 + \|\nabla W\|_{L^p(U)}^{\alpha - 2} +\|\nabla (\Delta \Phi)\|_{L^p(U)}^{\alpha - 2} \right). \]

The second integral in the RHS of (66) can be estimated as

\[ \int_U (K(|\nabla p|) \nabla p - K(|\nabla W|) \nabla W) \nabla (\Delta \Phi)|z|^\alpha - 2 \, dx \]

\[ \leq C \int_U K(|\nabla p|) \nabla p \|\nabla (\Delta \Phi)\||z|^\alpha - 2 \, dx + \int_U K(|\nabla W|) \nabla W \|\nabla (\Delta \Phi)\||z|^\alpha - 2 \, dx. \]

Then similar to the estimate for \( I_{pw} \) and (72) we get

\[ \int_U (K(|\nabla p|) \nabla p - K(|\nabla W|) \nabla W) \nabla (\Delta \Phi)|z|^\alpha - 2 \, dx \leq \varepsilon \int_U |\nabla z|^{-2} |z|^\alpha - 2 \, dx \]

\[ + \varepsilon \left( \int_U |z|^\alpha \, dx \right)^{\frac{1}{2}} + C \left( \|\nabla W\|_{L^p(U)}^{\alpha - 2} +\|\nabla (\Delta \Phi)\|_{L^p(U)}^{\alpha - 2} \right). \]
By Hölder’s and Young’s inequalities, we get
\[
\begin{align*}
\frac{1}{|U|} \int_U \left( |\Delta Q(t)| + \int_U \Psi_t \, dx \right) & \leq C \left( \int_U \left| |\Psi_t| \right|^\frac{1}{(1-a)} \right)^{\frac{a}{a}} + C \left( \int_U \left| |\Delta Q(t)| + \int_U \Psi_t \, dx \right|^\frac{1}{a} \right)^{\frac{a}{(1-a)}} \\
& \leq \varepsilon \left( \int_U \left| |\Psi_t| \right|^\alpha \right)^{\frac{1}{\alpha}} + C \left( \int_U \left| |\Delta Q(t)| + \int_U \Psi_t \, dx \right|^\alpha \right)^{\frac{a}{(1-a)}}.
\end{align*}
\]  

Finally combining (77), (79), (80) and Lemma 4.3 in (66) we get after choosing small enough \( \varepsilon \)
\[
\frac{d}{dt} \int_U |z|^\alpha \, dx \leq -C \int_U |\nabla z|^2 \cdot |z|^{a-2} \, dx + \varepsilon \left( \int_U |z|^\alpha \right)^{\frac{1}{\alpha}} + C(1 + A_1(\alpha, t)).
\]  

Applying (61) to the first term in the RHS of (81) we get
\[
\frac{d}{dt} \int_U |z|^\alpha \, dx \leq -C \left( \left( \int_U |z|^\alpha \right)^{\frac{1}{\alpha}} - 1 \right) + \varepsilon \left( \int_U |z|^\alpha \right)^{\frac{1}{\alpha}} + C(1 + A_1(\alpha, t)).
\]

Selecting sufficiently small \( \varepsilon \) we obtain (65).

In order to prove the estimate on \( \int_U |z|^\alpha \, dx \) we recall the Lemma A.1 from [14].

**Lemma 4.5.** Let \( \phi : [0, \infty) \to [0, \infty) \) be a continuous, strictly increasing function. Suppose \( y(t) \geq 0 \) is a continuous function on \([0, \infty)\) such that
\[
y'(t) \leq -C\phi^{-1}(y(t)) + f(t), \quad t > 0,
\]
where \( f(t) \geq 0 \) for \( t \geq 0 \) is a continuous function. Then
\[
\limsup_{t \to \infty} y(t) \leq \phi \left( \limsup_{t \to \infty} \frac{f(t)}{h(t)} \right).
\]

The following result will follow from Lemma 4.4.

**Theorem 4.6.** For \( \alpha \geq \frac{na}{2 - a} \) and \( A_1(\alpha) \) as in (40)
\[
\limsup_{t \to \infty} \int_U |z|^\alpha \, dx \leq C \left( 1 + A_1(\alpha) \right).
\]  

Consequently, if \( Q(t) \) and \( \Psi(x, t) \) satisfy condition A1 then
\[
\limsup_{t \to \infty} \int_U |z|^\alpha \, dx \leq C.
\]  

**Proof.** The result follows from Lemma 4.5.

4.3. Bounds for the gradient of solutions. We will obtain the bounds for the integral \( \int_U H(|\nabla p|) \, dx \), where \( H(|\nabla p|) \) is defined in (18). This result is necessary for our estimates of the difference in time derivative of fully transient and PSS solutions.

**Theorem 4.7.** Let \( A_2(t) \) be as in (41). For \( \alpha \geq \frac{na}{2 - a} \)
\[
\limsup_{t \to \infty} \int_U H(|\nabla p|) \, dx \leq C \left( 1 + A_1(\alpha) + \limsup_{t \to \infty} A_2(t) \right).
\]
Consequently, if $Q(t)$ and $\Psi(x,t)$ satisfy conditions A2 then
\[
\int_U |\nabla p|^2 \, dx \leq C \left( \int_U H(|\nabla p|) \, dx + 1 \right) \leq C. \tag{85}
\]

Proof. In formula (4.46) in [4] we have for $z = q - q^s$:
\[
\int_U q^s_t \, dx + \frac{d}{dt} \left[ I_1[p](t) + \int_U z^2 \, dx \right] \leq -C I_1[p](t) + C(1 + A_2(t)), \tag{86}
\]
where
\[
I_1[p](t) = \int_U H(|\nabla p|) \, dx - \int_U K(|\nabla W|) \nabla W \nabla p \, dx + \Delta Q(t) \Delta_\gamma(t). \tag{87}
\]

Notice that
\[
\frac{d}{dt} \int_U z^2 \, dx = 2 \int_U z z_t \, dx \geq - \int_U z^2 \, dx - \int_U q^s_t \, dx. \tag{88}
\]

Subtracting (88) from (86) we get
\[
\frac{d}{dt} I_1[p](t) \leq -C I_1[p](t) + \int_U z^2 \, dx + C(1 + A_2(t)). \tag{89}
\]

The term $\int_U z^2 \, dx$ can be estimated using (82) as $\alpha > 2$. Then in (89) we have
\[
\frac{d}{dt} I_1[p](t) \leq -C I_1[p](t) + C \left[ 1 + A_2(t) + A_1(\alpha) \right]. \tag{90}
\]

Applying Gronwall’s inequality one has for $t \geq 0$
\[
I_1[p](t) \leq e^{-c_1 t} \left[ I_1[p](0) + C \int_0^t e^{c_1 \tau} (1 + A_2(\tau) + A_1(\alpha)) \, d\tau \right] \\
\leq C(1 + A_1(\alpha)) + e^{-c_1 t} I_1[p](0) + C \int_0^t e^{-c_1(t-\tau)} A_2(\tau) \, d\tau. \tag{91}
\]

According to formula (2.35) in [14]
\[
\limsup_{t \to \infty} \int_0^t e^{-k(t-\tau)} f(\tau) \, d\tau \leq \limsup_{t \to \infty} k^{-1} f(t). \tag{92}
\]

Then using estimates (92) and (82) in (90) one has
\[
\limsup_{t \to \infty} I_1[p](t) \leq C \left( 1 + A_1(\alpha) + \limsup_{t \to \infty} A_2(t) \right). \tag{93}
\]

Under assumptions A2 the terms $\limsup_{t \to \infty} A_2(t)$ and $A_1(\alpha)$ are bounded.

Thus $\limsup_{t \to \infty} I_1[p](t) \leq C$. Then from formula (87) for $I_1[p]$ it follows:
\[
\int_U H(|\nabla p|) \, dx \leq \int_U K(|\nabla W|) |\nabla W| |\nabla p| \, dx + |\Delta Q(t)\Delta_\gamma(t)| + C. \tag{94}
\]

For the first integral in the RHS of (94) we have
\[
\int_U K(|\nabla W|) |\nabla W| |\nabla p| \, dx \leq \varepsilon_1 \int_U H(|\nabla p|) \, dx + C \|\nabla W\|_{L^{2-n}(U)} + C. \tag{95}
\]

According to Lemma 4.4 in [4] with $F_2(t)$ as in (38) we have
\[
|\Delta_\gamma(t)\Delta Q(t)| \leq \varepsilon_2 \int_U H(|\nabla p|) \, dx + C F_2(t) |\Delta Q(t)|. \tag{96}
\]

Choosing $\varepsilon_1$ and $\varepsilon_2$ to be small enough and using (94) in (93) we obtain (84). Since $F_2(t)$ is uniformly bounded under assumptions A2, the estimate (85) follows. \qed
4.4. Estimate of time derivatives. We will obtain the estimate on the difference of time derivative of fully transient and PSS solutions.

Let $\overline{p}(x, t) = p(x, t) - \Phi(x, t)$ and $\overline{p}_s(x, t) = p_s(x, t) - \Phi(x)$, then

$$\frac{\partial \overline{p}}{\partial t} = L[p] - \Psi_t(x, t), \quad \overline{p}|_{\Gamma_1} = \gamma(t); \quad (97)$$

$$\frac{\partial \overline{p}_s}{\partial t} = -A = L[W], \quad \overline{p}_s|_{\Gamma_1} = -At. \quad (98)$$

**Theorem 4.8.** Suppose that boundary data satisfies Assumptions A3-β. Then for any $\beta \geq 2$

$$\int_U |\overline{p}_t - \overline{p}_{s,t}|^\beta \, dx = \int_U |\overline{p}_t + A|^\beta \, dx \leq C. \quad (99)$$

**Proof.** Let $z_t = p_t + A$ and notice that $\nabla z_t = \nabla p_t$. Subtracting (97) and (98), taking derivative in $t$, multiplying on $|z_t|^\beta - 1 \text{sign}(z_t)$, integrating over $U$ and integrating by parts we get

$$\frac{1}{\beta} \frac{d}{dt} \int_U |z_t|^\beta \, dx = \int_U (L[p] - L[W])|z_t|^\beta - 1 \text{sign}(z_t) \, dx \quad (100)$$

$$= \int_U (\nabla \cdot (\nabla p)|\nabla p|)|z_t|^\beta - 1 \text{sign}(z_t) \, dx - \int_U \Psi_t(z_t)|z_t|^\beta - 1 \text{sign}(z_t) \, dx$$

$$= - (\beta - 1) \int_U (\nabla(|\nabla p|)|\nabla p|){\nabla p}_t \cdot \nabla z_t |z_t|^\beta - 2 \, dx - \int_U \Psi_t(z_t)|z_t|^\beta - 1 \text{sign}(z_t) \, dx$$

$$+ \int_{\Gamma_1} (\nabla(|\nabla p|)|\nabla p|){\nabla p}_t \cdot N|z_t|^\beta - 1 \text{sign}(z_t) \, ds.$$

Since $(p_t + A)|_{\Gamma_1} = \Delta'_\gamma(t)$ the boundary integral is equal to $(\Delta'_\gamma(t))^{\beta - 1}Q'(t)$.

The quantity in the first integral in the RHS of (100) is equal to

$$- (\Delta'_\gamma(t))^{\beta - 2}$$

$$= -K(|\nabla p|)|\nabla p|{\nabla p}_t \cdot \nabla z_t |z_t|^\beta - 2 - K'(|\nabla p|) \frac{(\nabla p_t \cdot \nabla p)(\nabla p \cdot \nabla z_t)}{|\nabla p|}|z_t|^\beta - 2.$$

From (13) it follows that

$$K'(|\nabla p|) \frac{(\nabla p_t \cdot \nabla p)(\nabla p \cdot \nabla z_t)}{|\nabla p|}|z_t|^\beta - 2 \leq aK(|\nabla p|)|\nabla p_t| |\nabla z_t| |z_t|^\beta - 2$$

$$\leq aK(|\nabla p|)|\nabla p_t|^2 |z_t|^\beta - 2 + aK(|\nabla p|)|\nabla z_t| |\nabla z_t| |z_t|^\beta - 2.$$

Then in (100) we have

$$\frac{1}{\beta} \frac{d}{dt} \int_U |z_t|^\beta \, dx \leq -(1 - a)(\beta - 1) \int_U K(|\nabla p|)|\nabla p_t|^2 |z_t|^\beta - 2 \, dx$$

$$+ (1 + a)(\beta - 1) \int_U K(|\nabla p|)|\nabla z_t| |\nabla p_t| |z_t|^\beta - 2 \, dx$$

$$+ \int_U |z_t|^{\beta - 1} |\Psi_t| \, dx + |(\Delta'_\gamma(t))^{\beta - 1}Q'(t)|. \quad (101)$$

To estimate the second term in the RHS of (101) we apply Hölder inequality and Young’s inequality with $\varepsilon = (1 + a)/(1 - a)$. Since $K(|\nabla p|)$ is bounded (see (14))
applying once more H"older inequality with powers $\beta/(\beta - 2)$ and $\beta/2$ we get
\[
\int_U K((\nabla p)|\nabla \Psi_t| |\nabla \pi| \xi_t|^{\beta - 2} \, dx \leq \frac{1 - a}{2(1 + a)} \int_U K((\nabla p)|\nabla \pi|^{2} |\xi_t|^{\beta - 2} \, dx
\]
\[
+ C\|\nabla \Psi_t\|_{L^\theta(U)}^2 \left( \int_U |\xi_t|^{\beta} \, dx \right)^{\beta - 2} \beta^{-2} . \tag{102}
\]

Similar for the third term in the RHS of (101) one has
\[
\int_U |\xi_t|^{\beta - 1} |\Psi_{tt}| \, dx \leq C\|\Psi_{tt}\|_{L^\beta(U)} \left( \int_U |\xi_t|^{\beta} \, dx + 1 \right) . \tag{103}
\]

Then in (101) we have
\[
\frac{1}{\beta} \frac{d}{dt} \int_U |\xi_t|^{\beta} \, dx \leq - C \int_U K((\nabla p)|\nabla \pi|^{2} |\xi_t|^{\beta - 2} \, dx
\]
\[
+ C\|\nabla \Psi_t\|_{L^\theta(U)}^2 \left( \int_U |\xi_t|^{\beta} \, dx + 1 \right)
\]
\[
+ C\|\Psi_{tt}\|_{L^\beta(U)} \left( \int_U |\xi_t|^{\beta} \, dx + 1 \right) + |(\Delta'_\theta(t))^{\beta - 1} Q'(t)| . \tag{104}
\]

Since the first term in the RHS of (104) is negative, it can be neglected. We then have
\[
\frac{d}{dt} \int_U |\xi_t|^{\beta} \, dx \leq f_1(t) \int_U |\xi_t|^{\beta} \, dx + f_2(t), \tag{105}
\]

where
\[
f_1(t) = C\|\Psi_{tt}\|_{L^\beta(U)} + C\|\nabla \Psi_t\|_{L^\beta(U)}^2,
\]
\[
f_2(t) = f_1(t) + |(\Delta'_\theta(t))^{\beta - 1} Q'(t)| .
\]

The result follows from the Gronwall’s inequality under assumptions A3-$\beta$. \qed

In order to prove that $\limsup_{t \to \infty} \int_U \|p_t + A\|^2 \, dx = 0$ we will need the Weighted Poincaré inequality (see Lemma 2.5., [14]):

**Lemma 4.9.** Let $\xi = \xi(x) \geq 0$ and function $u(x)$ be defined on $U$ and is vanishing on the boundary $\partial U$. Assume $\alpha \geq 2$. Given two numbers $\theta$ and $\theta_1$ such that
\[
\theta > \frac{2}{(2 - a)^*} \quad \text{and} \quad \max \left\{ 1, \frac{2n}{n \theta + 2} \right\} \leq \theta_1 < 2 - a,
\]
there is constant $C$ such that
\[
\int_U |u|^\alpha \, dx \leq CM_P(\xi, |u|) \left[ \int_U K(\xi) |\nabla u|^2 |u|^\alpha - 2 \, dx \right]^{\beta/2} , \tag{106}
\]
where
\[
M_P(\xi, |u|) = \left[ 1 + \int_U \xi^{2 - \alpha} + |u|^{\theta_2 \alpha} \, dx \right]^{\frac{2 - \alpha}{\theta_2 \alpha}} , \tag{107}
\]
\[
\theta_2 = \frac{\theta_1(\theta - 1)(2 - a)}{2(2 - a - \theta_1)} > 0.
\]

**Theorem 4.10.** If $Q(t)$ and $\Psi(x,t)$ satisfy assumptions A3-$\beta$ and A4, then
\[
\limsup_{t \to \infty} \int_U |p_t - p_{ss,t}|^2 \, dx = \limsup_{t \to \infty} \int_U |p_t + A|^2 \, dx = 0.
\]
Proof. In (5.15) of [4] we have
\[ \frac{d}{dt} \int_U (\mathbf{p}_t + A)^2 \, dx \leq -C \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx + \varepsilon_2 \int_U |\mathbf{p}_t + A|^2 \, dx + CA_3(\varepsilon_2, t), \]
(108)
where the constant \( C \) depends on \( \int_U H(|\nabla p|) \, dx \) and is bounded under assumptions \( A_2 \).

We will use Weighted Poincaré inequality (106) to estimate \( \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx \) in terms of \( \int_U |\mathbf{p}_t + A|^2 \, dx \). Namely, let \( u = \mathbf{p}_t + A - \Delta'_\gamma(t) \) and \( \xi(x) = |\nabla p| \). Then \( u|_{\Gamma_1} = 0 \) and \( \nabla u = \nabla \mathbf{p}_t \). Then we have
\[ \int_U |\mathbf{p}_t + A|^2 \, dx \leq \frac{3}{2} \int_U |u + \Delta'_\gamma(t)|^2 \, dx \leq \frac{3}{2} \int_U |u|^2 \, dx + \frac{3}{2} \int_U |\Delta'_\gamma(t)|^2 \, dx \]
(109)
\[ \leq CM_P(\xi, |u|) \cdot \left[ \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx \right]^{\frac{1}{2}} + \frac{|U|}{2} |\Delta'_\gamma(t)|^2. \]
Here
\[ M_P(|\nabla p|, |u|) = \left[ 1 + \int_U |\nabla p|^{2-a} + |\mathbf{p}_t + A - \Delta'_\gamma(t)|^{2\alpha_2} \, dx \right]^{\frac{2-a}{2\alpha_2}} \leq C \]
in view of Theorem 4.7 under Assumptions \( A_2 \) and Theorem 4.8 under Assumptions \( A_3-\beta \).

Thus
\[ \int_U |\mathbf{p}_t + A|^2 \, dx \leq C \left[ \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx \right]^{\frac{1}{2}} + \frac{|U|}{2} |\Delta'_\gamma(t)|^2. \]

On the other hand note that \( \int_U q_t \, dx = 0 \) and \( q_t|_{\Gamma_1} = (q_t - q_{s,t})|_{\Gamma_1} = \Delta'_\gamma(t) + F'_s(t) \). Here \( F'_s(t) = \Delta Q(t) + \int_U \Delta \Psi_t \, dx \) (see (37)). Notice that \( \limsup_{t \to \infty} |F'_s(t)| = 0 \) under conditions \( A_4 \). Then by Trace theorem, Poincaré inequality and the fact that \( \int_U q_t \, dx = 0 \) we have
\[ |\Delta'_\gamma(t)|^2 \leq C \left( \int_{\Gamma_1} |q_t| \, ds + C|F'_s(t)| \right)^2 \leq C \left( \int_U |\nabla q_t| \, dx \right)^2 + C|F'_s(t)|^2. \]

Next, as \( \nabla q_t = \nabla \mathbf{p}_t \), applying Hölder inequality we have
\[ \left( \int_U |\nabla q_t| \, dx \right)^2 \leq C \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx \cdot \int_U K(|\nabla p|)^{-1} \, dx. \]
(110)
Notice that from (14) and Hölder inequality it follows
\[ \int_U K(|\nabla p|)^{-1} \, dx \leq C \int_U (1 + |\nabla p|)^a \, dx \leq C + C \int_U |\nabla p|^{2-a} \, dx \leq C, \]
(111)
The last estimate in (111) follows from Theorem 4.7 under conditions \( A_2 \).

Thus
\[ |\Delta'_\gamma(t)|^2 \leq C \int_U K(|\nabla p|)|\nabla \mathbf{p}_t|^2 \, dx + C|F'_s(t)|^2. \]

Then in (109) we have
\[ \int_U |\mathbf{p}_t + A|^2 \, dx \leq C \left( I^+(t) + I(t) \right) + C|F'_s(t)|^2 = C \cdot \varphi \left( I(t) + |F'_s(t)|^2 \right), \]
(112)
where
\[ \varphi(s) = s^\frac{1}{\alpha} + s + |F_1(t)|^2; \quad I(t) = \int_U K(|\nabla p|)|\nabla p_t|^2 \, dx. \] (113)

Due to condition on \(|F_1(t)||
\[ I(t) \geq C \varphi^{-1} \left( \int_U |\nabla_t + A|^2 \, dx \right) - C|F_1(t)|^2. \]

Then due to definition (42) of \(A_3(\varepsilon, t)\) one has in (108)
\[ \frac{d}{dt} \int_U (|\nabla_t + A|^2 \, dx) \leq -C \varphi^{-1} \left( \int_U |\nabla_t + A|^2 \, dx \right) + \varepsilon_2 \int_U |\nabla_t + A|^2 \, dx + CA_3(\varepsilon, t). \] (114)

Let \(y(t) = \int_U |\nabla_t + A|^2 \, dx. \) Due to Theorem (4.8) one has in (114)
\[ y'(t) \leq -C f(y(t)) + CA_3(\varepsilon, t) + C\varepsilon_2, \] (115)

where
\[ f(s) = \varphi^{-1}(s), \quad f(s) > 0 \quad \text{for} \quad s > 0, \quad \text{and} \quad f(0) = 0. \] (116)

Under the Assumptions \(A_3-\beta\) we have \(0 \leq y(t) \leq C. \) According to Lemma 4.5 we will get \(\limsup_{t \to \infty} y(t) \leq C \varphi(\limsup_{t \to \infty} A_3(\varepsilon, t) + \varepsilon_2). \) Under Assumptions \(A_4\) \(\limsup_{t \to \infty} A_3(\varepsilon, t) = 0\) and therefore
\[ \limsup_{t \to \infty} y(t) \leq \varphi(\varepsilon_2). \]

Taking in above \(\varepsilon_2 \to 0\) we complete proof of the Theorem 4.10. \(\square\)

Finally Theorems 4.6, 4.7 and 4.10 and Lemma 4.1, [4], under the conditions A1-A5 allow us to conclude that \(\limsup_{t \to \infty} y(t) \leq 0\). According to Lemma 4.1. we will get \(\limsup_{t \to \infty} y(t) \leq C \varphi(\limsup_{t \to \infty} A_3(\varepsilon, t) + \varepsilon_2). \) Under Assumptions \(A_4\) \(\limsup_{t \to \infty} A_3(\varepsilon, t) = 0\) and therefore
\[ \limsup_{t \to \infty} y(t) \leq \varphi(\varepsilon_2). \]

Taking in above \(\varepsilon_2 \to 0\) we complete proof of the Theorem 4.10. \(\square\)

Finally Theorems 4.6, 4.7 and 4.10 and Lemma 4.1, [4], under the conditions A1-A5 allow us to conclude that \(\limsup_{t \to \infty} y(t) \leq 0\). According to Lemma 4.1. we will get \(\limsup_{t \to \infty} y(t) \leq C \varphi(\limsup_{t \to \infty} A_3(\varepsilon, t) + \varepsilon_2). \) Under Assumptions \(A_4\) \(\limsup_{t \to \infty} A_3(\varepsilon, t) = 0\) and therefore
\[ \limsup_{t \to \infty} y(t) \leq \varphi(\varepsilon_2). \]

Taking in above \(\varepsilon_2 \to 0\) we complete proof of the Theorem 4.10. \(\square\)

5. IBVP-II: Asymptotic convergence of the productivity index. Let \(p(x, t)\) be the solution of IBVP-II (21), (23), (24) with Dirichlet boundary condition (28) with boundary data \(\Psi(x, t)\). Let \(J(t)\) be the corresponding PI. Let \(p_\alpha(x, t) = -At + W(x), W(x)|_{\Gamma} = \varphi(x), \) be the PSS solution of this IBVP defined by the boundary data \(-At, A = Q_s/|U|, \) and \(\Phi(x)\). Let \(J_{PSS}\) be the corresponding PI.

5.1. Assumptions on the boundary data. Let
\[ D(\beta, t) = \|\Psi_t\|_{L^2(U)} + \|\nabla \Psi_t\|_{L^2(U)}^2 \] (118)

for any \(\beta \geq 2. \)

We also will use some notations from [14]. For \(\alpha \geq \frac{na}{2n}, b = \frac{a(2-a)}{2}, r_0 = \frac{n(2-a)}{(2-a)(n+1)-n}, \) let
\[ A(\alpha, t) = \|\nabla \Psi_t\|_{L^2(U)}^{\frac{\alpha-a}{\alpha-n_a}} + \left( \int_{U} |\Psi_t|^{\alpha} \, dx \right)^{\frac{\alpha-a}{\alpha-n_a}}; \] (119)
\[ G_4(t) = \int_{U} |\nabla \Psi_t|^2 \, dx + \left[ \int_{U} |\Psi_t|^{\alpha} \, dx \right]^{\frac{2}{\alpha(1-n_a)}} + \|\Psi_t\|_{L^0} + \int_{U} |\nabla \Psi_t|^2 \, dx \] (120)
The results in this section are obtained under the following conditions on the boundary data

**Assumptions D1.**

\[ \int_{1}^{\infty} D(\beta, t) \, dt < \infty \quad \text{for any } \beta \geq 2. \]  
\[ (121) \]

**Assumptions D2.**

\[ \limsup_{t \to \infty} (A(\alpha, t) + G_4(t)) \leq C; \]  
\[ (122) \]

\[ \limsup_{t \to \infty} \left( D(2, t) + \int_{U} |\Psi_t + A|^2 \, dx \right) = 0. \]  
\[ (123) \]

**Assumptions D3.**

\[ \limsup_{t \to \infty} A(\alpha, t) + \|\nabla \Phi(x)\|_{L^a} \leq C; \]  
\[ (124) \]

\[ \limsup_{t \to \infty} \|\nabla(\Delta \Psi)\|_{L^2(U)} = 0. \]  
\[ (125) \]

According to Lemma 3.2 in order to prove \( J(t) \to J_{PSS} \) as \( t \to 0 \) we need to prove the following estimates:

\[ \Delta_Q(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{see Lemma 5.2}; \]  
\[ (126) \]

\[ \int_{U} |\nabla (p - p_s)|^{2-a} \, dx \to 0 \quad \text{as} \quad t \to \infty \quad \text{see Theorem 5.3}. \]  
\[ (127) \]

**5.2. Estimates on the solution of IBVP-II.** Along with the solutions \( p \) and \( p_s \) we will use the shifts \( \tilde{p}(x, t) = p(x, t) - \Psi(x, t) \) and \( \tilde{p}_s(x, t) = p_s(x, t) - \Phi(x) + At \). These shifts obviously satisfy

\[ \frac{\partial \tilde{p}}{\partial t} = L[p] - \Psi_t(x, t), \quad \tilde{p}|_{\Gamma} = 0; \]  
\[ (128) \]

\[ \frac{\partial \tilde{p}_s}{\partial t} = 0 = L[W] + A, \quad \tilde{p}_s|_{\Gamma} = 0. \]  
\[ (129) \]

**Lemma 5.1.** Suppose boundary data \( \Psi(x, t) \) satisfies assumptions D1, then

\[ \int_{U} |\tilde{p}_t|^\beta \, dx \leq C \quad \text{for any } \beta \geq 2. \]  
\[ (130) \]

**Proof.** Similar to (100) we have

\[ \frac{1}{\beta} \frac{d}{dt} \int_{U} |\tilde{p}_t|^\beta \, dx \leq - (1 - a)(\beta - 1) \int_{U} K(|\nabla p|) |\nabla \tilde{p}_t|^\beta |\tilde{p}_t|^{\beta-2} \, dx \]

\[ + (1 + a)(\beta - 1) \int_{U} K(|\nabla p|) |\nabla \Psi_t||\nabla \tilde{p}_t||\tilde{p}_t|^{\beta-2} \, dx \]

\[ + \int_{U} |\tilde{p}_t|^{\beta-1} |\Psi_t| \, dx. \]  
\[ (131) \]

Similar to (102) - (105) we have

\[ \frac{d}{dt} \int_{U} |\tilde{p}_t|^\beta \, dx \leq D(\beta, t) \int_{U} |\tilde{p}_t|^\beta \, dx + D(\beta, t). \]  
\[ (132) \]

The result follows from the Gronwall’s inequality under assumptions D1.  
\[ \square \]
Further we will need the following result obtained in [14]: under assumptions D2.1 (122) (see [14], Th. 4.5)
\[
\int_U |\nabla p|^{2-\alpha} \, dx \leq C \int_U H(|\nabla p|) \, dx + C. \tag{133}
\]

**Lemma 5.2.** Suppose boundary data \( \Psi(x,t) \) satisfies assumptions D1 and D2. Then
\[
\limsup_{t \to \infty} \left( |\Delta Q(t)| + C \int_U |p_t + A|^2 \, dx \right) = 0 \quad \text{as} \quad t \to \infty. \tag{134}
\]

**Proof.** First, notice that
\[
\text{Lemma 5.2.}\]
\[
\int_U \frac{\partial}{\partial t} p(x,t) \, dx = Q(t) \quad \text{and} \quad \frac{d}{dt} \int_U p_s(x,t) \, dx = Q_s.
\]

Thus
\[
|\Delta Q(t)|^2 = |Q(t) - Q_s|^2 = \left| \int_U p_t \, dx - Q_s \right|^2 = \left| \int_U (p_t + A) \, dx \right|^2 \leq C \int_U |p_t + A|^2 \, dx \leq C \int_U |\tilde{p}_t|^2 \, dx + C \int_U |\Psi_t + A|^2 \, dx. \tag{135}
\]

Assuming \( \int_U |\Psi_t + A|^2 \, dx \to 0 \) as \( t \to \infty \) it is sufficient to prove that \( \int_U |\tilde{p}_t|^2 \, dx \to 0 \) as \( t \to \infty \).

Differentiating both sides of (128) in \( t \), multiplying by \( \tilde{p}_t \) and integrating over \( U \) we have due to zero boundary condition
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{p}_t^2 \, dx = -\int_U (K(|\nabla p|) \nabla p) \nabla \tilde{p}_t \, dx - \int_U \Psi_{tt} \tilde{p}_t \, dx. \tag{136}
\]

Similar to estimate (5.10) in [4] we get since \( \nabla \tilde{p} = \nabla p - \nabla \Psi \)
\[
\frac{d}{dt} \int_U \tilde{p}_t^2 \, dx \leq -C \int_U K(|\nabla p|) |\nabla \tilde{p}_t|^2 \, dx + \varepsilon \left( \int_U |\tilde{p}_t|^2 \, dx \right)^{\frac{2}{2-\alpha}} + C \tilde{D}(2,t). \tag{137}
\]
where
\[
\tilde{D}(2,t) = \|\Psi_{tt}\|_{L^2}^{\frac{2}{2-\alpha}} + \|\nabla \Psi_t\|_{L^2}^2. \tag{138}
\]

Applying the Weighted Poincaré inequality (106) with \( u = \tilde{p}_t, \tilde{p}_t|_{\Gamma} = 0 \) and \( \xi = \nabla p \) with powers \( \alpha = 2 \) and \( \theta = \frac{2}{2-\alpha} \) to the first integral on the RHS of (137) we get:
\[
\frac{d}{dt} \int_U \tilde{p}_t^2 \, dx \leq -C M_P \left( \int_U |\tilde{p}_t|^2 \, dx \right)^{\frac{2}{2-\alpha}} + \varepsilon \left( \int_U |\tilde{p}_t|^2 \, dx \right)^{\frac{2}{2-\alpha}} + C \tilde{D}(2,t).
\]
Here
\[
M_P(|\nabla p|, |\tilde{p}_t|) = \left[ 1 + \int_U |\nabla p|^{2-\alpha} + |\tilde{p}_t|^{2\theta_2} \, dx \right]^{\frac{2-\alpha}{2-\alpha}} \leq C \tag{139}
\]
by virtue of (133) under assumptions D2.1 and Lemma 5.1 under assumptions D1.

Choosing sufficiently small \( \varepsilon \) we get
\[
\limsup_{t \to \infty} \int_U |\tilde{p}_t|^2 \, dx \leq \limsup_{t \to \infty} \tilde{D}(2,t). \tag{140}
\]
Under assumptions D2.2 the RHS of (140) converges to 0 as \( t \to \infty \) and the result of Lemma follows from (135).
Further we will use another result from [14]: under assumptions D3.1 (124) (see [14], Th. 4.3)
\[ \int_U |\tilde{\rho}(x,t)|^2 + |\tilde{\rho}_s(x,t)|^2 \, dx \leq C. \] (141)

**Theorem 5.3.** Suppose the boundary data \( \Psi(x,t) \) and \( \Phi(x) \) satisfies assumptions \( D1 - D3 \). Then
\[ \| \nabla(p - p_s) \|^2_{L^2-\alpha(U)} \to 0 \quad \text{as} \quad t \to \infty. \] (142)

**Proof.** From (16) we get
\[ \left( \int_U |\nabla(p - p_s)|^{2-\alpha} \, dx \right)^{\frac{2}{2-\alpha}} \leq C_\Phi \int_U \left( K(|\nabla p|) \nabla p - K(|\nabla p_s|) \nabla p_s \right) \cdot \nabla(p - p_s) \, dx, \] (143)
where \( C_\Phi = C_\Phi(p, p_s) \) is as in (17). Since \( \nabla(p - p_s) = \nabla(\tilde{\rho} - W) + \nabla(\Delta \psi) \), \( C_\Phi \) is bounded by virtue of (133) and (59) under condition \( \| \nabla(\Delta \psi) \|_{L^2-\alpha(U)} \leq C \).

Since \( \tilde{\rho} - \tilde{\rho}_s |_{\Gamma_i} = 0 \), integration by parts in the RHS of (143) gives
\[ \| \nabla(p - p_s) \|^2_{L^2-\alpha} \leq -C \int_U (p_t - p_{s,t})(\tilde{\rho} - \tilde{\rho}_s) \, dx \\
+ C \int_U \left( K(|\nabla p|) \nabla p - K(|\nabla p_s|) \nabla p_s \right) \cdot \nabla(\Delta \psi) \, dx. \]
Since \( K(\cdot) \) is bounded we have
\[ \int_U \left( K(|\nabla p|) \nabla p - K(|\nabla p_s|) \nabla p_s \right) \nabla(\Delta \psi) \, dx \] (144)
\[ \leq \left( \int_U K(|\nabla p|)|\nabla p|^2 \, dx \right)^{\frac{1}{2}} \left( \int_U K(|\nabla p_s|)|\nabla p_s|^2 \, dx \right)^{\frac{1}{2}} \\
+ \left( \int_U K(|\nabla W|)|\nabla W|^2 \, dx \right)^{\frac{1}{2}} \left( \int_U K(|\nabla p_s|)|\nabla p_s|^2 \, dx \right)^{\frac{1}{2}} \\
\leq \left[ \left( \int_U H(|\nabla p|) \, dx \right)^{\frac{1}{2}} + \left( \int_U |\nabla W|^{2-\alpha} \, dx \right)^{\frac{1}{2}} \right] \| \nabla(\Delta \psi) \|_{L^2(U)} \]
\[ \leq C \| \nabla(\Delta \psi) \|_{L^2(U)}. \]

The last inequality holds in view of (133) and (59).

Thus
\[ \| \nabla(p - p_s) \|^2_{L^2-\alpha} \leq -C \int_U (p_t + A)(\tilde{\rho} - \tilde{\rho}_s) \, dx + C \| \nabla(\Delta \psi) \|_{L^2(U)}. \]

The result follows in view of Lemma 5.2 and estimate (141) under the assumptions D1 - D3.

Lemma 5.2 and Theorem 5.3 combined with Lemma 3.2 prove the main result for the solution of IBVP-II:

**Theorem 5.4.** Suppose the boundary data satisfies assumptions D1 – D3. Then
\[ J(t) - J_{PSS} \to 0 \quad \text{as} \quad t \to \infty. \]
6. **Productivity index concept for the flow of ideal gas.** In this section we discuss the concept of the PI for an ideal gas flow in the porous media for a well-reservoir system. We will repeat some discussion from Sec. 2 and provide more details. The equation of state for an ideal gas takes the form (see [1], [20])

\[ \rho(p) = Mp, \]  

(145)

where \( \rho \) is the density of the fluid and \( M \) is a proportionality constant. Without loss of generality we assume \( M = 1 \). As a momentum equation we consider the second order Forchheimer equation (1) which takes the form (see, for example, [22])

\[ \alpha u + \beta \rho |u| = -\nabla p, \]  

(146)

where \( \beta \) is the Forchheimer coefficient.

As in case of slightly compressible fluid, the above equation can be solved for \( u \) and rewritten in terms of a nonlinear permeability function \( K_2(u) \):

\[ u = -K_2(\rho |\nabla p|)\nabla p = -\frac{2}{\alpha + \sqrt{\alpha^2 + 4\beta \rho |\nabla p|}} \nabla p. \]  

(147)

Combining (145) and (147) together with the continuity equation (6) yields the parabolic equation for pressure

\[ \frac{\partial p}{\partial t} = L[p] \equiv \nabla \cdot (K_2(p |\nabla p|)p \nabla p) = \nabla \cdot \left( \frac{2p}{\alpha + \sqrt{\alpha^2 + 4\beta \rho |\nabla p|}} \nabla p \right). \]  

(148)

As before the domain \( U \) models the reservoir with boundary split in two parts \( \Gamma_i \) and \( \Gamma_e \). \( \Gamma_i \) is the well-boundary while \( \Gamma_e \) is an exterior non-permeable boundary of the reservoir. We impose Dirichlet Pseudo Steady State (PSS) boundary condition on \( \Gamma_i \)

\[ p|_{\Gamma_i} = B - At, \]  

(149)

zero mass flux boundary condition on \( \Gamma_e \)

\[ \rho u \cdot N|_{\Gamma_e} = 0 \]  

(150)

and the initial condition

\[ p(x, 0) = \sqrt{B^2 + \phi_0(x)} \]  

(151)

for \( \phi_0(x) \geq 0 \).

**Remark 6.1.** Constant \( B \) is a positive parameter (generally large) characterizing the initial reserves of gas in the reservoir domain \( U \). The constant \( A \) is associated with the amount of gas extracted at the well-bore \( \Gamma_i \). Thus the quantity \( B - At \) quantifies the gas reserves in the reservoir at the moment \( t \).

For the flow of an ideal gas in porous media, engineers define the Productivity Index as, see [6, 9]:

**Definition 6.2.** Let \( p(x, t) \) be a classical solution of equation (148) satisfying boundary conditions (149) and (150). Let \( Q(t) \) be the total mass flow through the well-bore boundary \( \Gamma_i \)

\[ Q(t) = \int_{\Gamma_i} \rho u \cdot N \, ds = \int_{\Gamma_i} K_2(p |\nabla p|) p \nabla p \cdot N \, ds, \]  

(152)
where $N$ is an outward normal to $\Gamma_i$. Then productivity index for the well-reservoir system is defined by

$$J_g[p] = \frac{Q(t)}{|U| \int_U p^2 \; dx - \frac{1}{|\Gamma_i|} \int_{\Gamma_i} p^2 \; ds}.$$  \hfill (153)

As we saw earlier in case of slightly compressible flow, under the boundary conditions (149) and (150) the value of the PI, defined as in (10) is stabilizing to a constant value (34), which is defined by the PSS solution and depends only on the parameter $A$. Obviously, in case of the gas flow described by (148) this feature is not valid anymore and the PI is not stabilizing in time. Nevertheless the approach developed in previous sections is applicable to the gas flow as well. Namely we will introduce special auxiliary pressure function characterized by the time independent PI. We will then study the relation between the general time dependent PI and this time independent one.

Let $Q_0 = \frac{A}{|U|}$ be a given constant mass-rate of gas production and consider the auxiliary pressure distribution given by

$$p_0(x, t) = \sqrt{(B - At)^2 + 2W(x)},$$  \hfill (154)

where $W(x)$ is the solution of the following BVP

$$- \nabla \cdot (K_2(|\nabla W|)\nabla W) = - \nabla \cdot \left( \frac{2\nabla W}{\alpha + \sqrt{\alpha^2 + 4\beta|\nabla W|}} \right) = A,$$  \hfill (155)

$$W = 0, \quad \text{on} \quad \Gamma_i,$$  \hfill (156)

$$\nabla W \cdot N = 0, \quad \text{on} \quad \Gamma_e.$$  \hfill (157)

Integrating (155) over the domain $U$, using integration by parts and boundary condition (157) yields to the integral flux condition

$$\int_{\Gamma_i} -K_2(|\nabla W|)\nabla W \cdot N \; ds = |U|A = Q_0.$$  \hfill (158)

From (154) we can also derive that

$$p_0\nabla p_0 = \nabla W.$$  \hfill (159)

Combining this property, definition (154) and BVP (155)-(157) it can be easily verified that the auxiliary pressure $p_0$ satisfies the following equations

$$\frac{\partial p_0}{\partial t} - L[p_0] = f_0(x, t) \quad \text{in} \ U,$$  \hfill (160)

$$p_0(x, t) = B - At \quad \text{on} \ \Gamma_i,$$  \hfill (161)

$$p_0\nabla p_0 \cdot N = 0 \quad \text{on} \ \Gamma_e.$$  \hfill (162)

where

$$f_0(x, t) = \frac{\partial p_0}{\partial t} - \nabla \cdot (K_2(p_0)|\nabla p_0|)p_0 \nabla p_0$$  \hfill (163)

$$= \frac{-A(B - At)}{\sqrt{(B - At)^2 + 2W}} - \nabla \cdot (K_2(|\nabla W|)\nabla W)$$

$$= \frac{-A(B - At)}{\sqrt{(B - At)^2 + 2W}} + A$$
Figure 2. Comparison between the time dependent productivity index $J_g[p]$ and the PSS productivity index $J_g[p_0]$ as the time $t$ approaches the critical value $T_{crit} = \frac{B}{A} = 2000$.

From (158) and (159), it also follows the total mass flux condition for $p_0$

$$\int_{\Gamma_i} -K_2(p_0|\nabla p_0|)p_0 \nabla p_0 \cdot N ds = \int_{\Gamma_i} -K_2(|\nabla W|)\nabla W \cdot N ds = Q_0. \quad (164)$$

Using this last result and the explicit representation of $p_0(x,t)$ in formula (153) leads to the following Proposition.

**Proposition 6.3.** The Productivity Index for an ideal gas flow defined on $p_0(x,t)$ is time independent and is given by

$$J_g[p_0] = \frac{Q_0}{\| U \|_\infty} \int_U 2W(x) dx. \quad (165)$$

Numerical computations performed for several basic reservoir geometries, show that if the initial data in the system (147)-(150) is given by $p(x,0) = p_0(x,0)$, then the corresponding productivity indices $J_g[p_0]$ and $J_g[p](t)$ are almost identical for a long time, see Fig. 2, as long as the quantity

$$\frac{B - At}{\sqrt{(B - At)^2 + 2W}} \sim 1, \quad (166)$$

or, equivalently, as long as

$$(B - At)^2 \gg 2\| W \|_\infty \quad \iff \quad t \leq T_{crit} < \frac{B - \sqrt{2\| W \|_\infty}}{A}. \quad (167)$$

In case of compressible flow, we have to fix a critical time

$$T_{crit} = \frac{B}{A},$$
and for \( t > T_{crit} \) the negative boundary data (149) leads to the violation of ellipticity. This behavior is qualitatively justified by the fact that as long as (166) holds, function \( f_0(x,t) \) in the RHS of (160) is negligible and \( p(x,t) \) and \( p_0(x,t) \) behave similarly. On the other hand, as \( (B-At) \to 0 \) then \( f_0(x,t) \) approaches the constant value \( A \), and the two solutions diverge from each other. Then two productivity indices \( J_g[p_0] \) and \( J_g[p](t) \) also diverge from each other.

This phenomenon is observed on the actual field data and has a clear practical explanation. Notice that the denominator in the formula for \( J_g[p_0] \) (165)

\[
\frac{1}{|U|} \int_U 2W(x) \, dx = \frac{2\|W\|_1}{|U|}
\]

(since \( W \geq 0 \) for any \( x \in U \)) is a measure of the pressure drawdown needed to maintain constant production \( Q_0 \). As long as the gas reserves are considerably larger than the pressure drawdown

\[
(B - At)^2 \geq C > 2\|W\|_\infty > \frac{2\|W\|_1}{|U|},
\]

then the distributed source term \( f_0 \) (equivalent to reservoir fluid injection) needed to maintain constant production \( Q_0 \) is negligible. Otherwise when the gas reserves are comparable in magnitude with the pressure drawdown, then a possible way to maintain constant production rate is by resupplying the reservoir by fluid injections.

In the remaining part of this section we will theoretically investigate the difference between the functions \( p(x,t) \) (the actual solution of the problem) and \( p_0(x,t) \) (the solution of the auxiliary problem) depending on the key parameter \( T_{crit} = B/A \). Unfortunately for the Forchheimer case we were not yet able to obtain the appropriate estimates for the differences between \( p_0(x,t) \) and \( p(x,t) \). We will report mathematically rigorous result only for the case of compressible Darcy flow. We will show that for the fixed \( T_0 \) for all times \( t \in [0,T_0] \) when \( p(x,t) > 0 \) on \( U \times [0,T_0] \) the productivity indices \( J_g[p] \) and \( J_g[p_0] \) are becoming closer to each other with the increasing parameter \( B \). The obtained results make us believe this comparison can be proved in the general Forchheimer case as well.

We consider the case of positive solutions. For that assume that our time space domain belongs to the time layer \( 0 \leq t \leq T_0 < T_{crit} \). Let \( D = U \times (0,T_0) \). The following useful inequality follows directly from maximum principle

**Lemma 6.4.** Let \( p(x,t) > 0 \) for all \( 0 \leq t \leq T_0 < T_{crit} \) is a classical solution of the equation (148) with boundary conditions (149) and (150). Then for any \( 0 \leq t \leq T_0 \)

\[
p(x,t) \geq \min\{p(x,0); B - At\}. \tag{169}
\]

**Proof.** Inequality (169) follows from the maximum principle (see, for example, [12]), since the nonlinear equation (148) is uniformly parabolic in the domain \( D \), and the boundary function in (149) is decreasing with time. \( \square \)

### 6.1. Analytical comparison of the solutions for case of the Darcy flow

First the following maximum principle follows from the results in [27].

**Lemma 6.5.** Let \( (x,t) \subset D = U \times (0,T], U \subset \mathbb{R}^n \) and an elliptic operator \( \mathcal{L} \) is defined on \( U \times (0,T] \)

\[
\mathcal{L} = \sum_{i,j} a_{i,j}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,t) \frac{\partial}{\partial x_i} \tag{170}
\]
with \( C^{-1} | \xi |^2 \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq C | \xi |^2 \), and \( \sum_{i=1}^n | b_i | \leq C \).

Let \( \partial U = \Gamma_1 \cup \Gamma_2 \), where for simplicity \( \Gamma_1 \) and \( \Gamma_2 \) are nonintersecting compact sets. Let \( \partial D = U \times \{ 0 \} \cup \partial U \times (0, T] \) be the parabolic boundary of the domain.

Assume \( \bar{u}(x,t) \) and \( u(x,t) > 0 \) in \( D \) are the solution of inequality \((171)\) and equation \((172)\) correspondingly:

\[
\frac{\partial \bar{u}}{\partial t} - \mathcal{L}[\bar{u}^2] > 0, \tag{171}
\]

\[
\frac{\partial u}{\partial t} - \mathcal{L}[u^2] = 0. \tag{172}
\]

Assume also that \( u(x,t) \) and \( \bar{u}(x,t) \) satisfy the homogeneous Neumann conditions on the boundary \( \Gamma_2 \):

\[
\frac{\partial u}{\partial n} \bigg|_{\Gamma_2} = \frac{\partial \bar{u}}{\partial n} \bigg|_{\Gamma_2} = 0. \tag{173}
\]

The following comparison principle holds: if

\[
\bar{u} \geq u \text{ on } U \times \{ 0 \} \cup \Gamma_1 \times (0, T], \tag{174}
\]

then

\[
\bar{u} \geq u \text{ in } D. \tag{175}
\]

We will now obtain some integral comparison results between actual and auxiliary pressures in case of linear Darcy flow. Namely, let the Forchheimer coefficient \( F = 0 \) in \((146)\). Let \( p_0(x,t) \) be the auxiliary pressure given in \((154)\) and \( p(x,t) \) be the classical solution of IBVP \((148)\) with the PSS boundary conditions \((149)-(150)\) and initial data given by \( p(x,0) = p_0(x,0) = \sqrt{B^2 + 2W(x)} \). First, we will prove a useful integral identity for the difference \( p(x,t) - p_0(x,t) \).

**Lemma 6.6.** Suppose \( T < T_{\text{crit}} = B/A \). Then the following identity holds

\[
\int_0^T \int_U (p + p_0)(p - p_0)^2 \, dx \, dt + \frac{1}{4} \int_U \left( \nabla \int_0^T (p^2 - p_0^2) \, dt \right)^2 \, dx = -\int_0^T \int_U \left( f_0(x,t) \int_t^T (p^2 - p_0^2) \, d\tau \right) dx \, dt, \tag{176}
\]

where \( f_0 \) is defined in \((163)\). The identity above can be further rewritten as

\[
\int_0^T \int_U (p + p_0)(p - p_0)^2 \, dx \, dt + \frac{1}{4} \int_U \left( \nabla \int_0^T (p^2 - p_0^2) \, dt \right)^2 \, dx = \int_U \int_0^T \left( p_0^2(x,t) - p^2(x,t) \right) \int_0^t f_0(x,\tau) \, d\tau \, dx \, dt. \tag{177}
\]

**Proof.** For the Darcy case the equations for \( p(x,t) \) and \( p_0(x,t) \) take the form

\[
\frac{\partial p}{\partial t} = \nabla \cdot (p \nabla p) = \frac{1}{2} \Delta (p^2), \tag{178}
\]

\[
\frac{\partial p_0}{\partial t} = \nabla \cdot (p_0 \nabla p_0) + f_0(x,t) = \frac{1}{2} \Delta (p_0^2) + f_0(x,t). \tag{179}
\]

Subtracting the second equation from the first one yields

\[
\frac{\partial}{\partial t} (p - p_0) = \frac{1}{2} \Delta (p^2 - p_0^2) - f_0(x,t).
\]
Following the ideas of Oleinik (see [27]), we multiply both sides by the test function 
\[ \int_t^T (p^2 - p_0^2) \, d\tau, \]
integrate in time from 0 to \( T \) and in space over the domain \( U \). Then it follows
\[
\int_U \int_0^T \left( \frac{\partial}{\partial t} (p - p_0) \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
= \frac{1}{2} \int_U \int_0^T \left( \Delta (p^2 - p_0^2) \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
- \int_U \int_0^T \left( f_0 \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx.
\]
By using integration by part and the fact that 
\[ p(x, 0) - p_0(x, 0) = 0 \] and \( \int_0^T (\cdot) \, dt = 0 \),
the integral on the LHS of (180) can be rewritten as
\[
\int_U \int_0^T \left( \frac{\partial}{\partial t} (p - p_0) \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
= -\int_U \int_0^T (p - p_0) \cdot \frac{\partial}{\partial t} \left( \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
= \int_U \int_0^T (p - p_0) (p^2 - p_0^2) \, dt \, dx = \int_U \int_0^T (p + p_0)(p - p_0)^2 \, dt \, dx.
\]
By using the divergence theorem the first integral in the RHS of (180) can be rewritten as
\[
\frac{1}{2} \int_U \int_0^T \left( \Delta (p^2 - p_0^2) \int_t^T (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
= -\frac{1}{2} \int_U \int_0^T \left( \nabla (p^2 - p_0^2) \cdot \int_t^T \nabla (p^2 - p_0^2) \, d\tau \right) \, dt \, dx
= \frac{1}{4} \int_U \int_0^T \frac{\partial}{\partial t} \left( \int_t^T \nabla (p^2 - p_0^2) \, d\tau \right)^2 \, dt \, dx - \frac{1}{4} \int_U \left[ \int_0^T \nabla (p^2 - p_0^2) \, d\tau \right]^2 \, dx.
\]
Substituting (181) and (182) back in (180) we obtain (176). Finally in order to obtain the alternative identity (177) we use the following integration by part for the right hand side of (176)
\[
-\int_U \int_0^T \left( f_0(x, t) \int_t^T (p^2(x, \tau) - p_0^2(x, \tau)) \, d\tau \right) \, dt \, dx
= -\int_U \left[ \left( \int_0^t f_0(x, \tau) \, d\tau \int_t^T (p^2(x, \tau) - p_0^2(x, \tau)) \, d\tau \right) \bigg|_0^T \right]
- \int_0^T \left( \int_0^t f_0(x, \tau) \, d\tau \right) \left[ -(p^2(x, t) - p_0^2(x, t)) \right] \, dt \, dx
= -\int_U \int_0^T \left( (p^2(x, t) - p_0^2(x, t)) \int_0^t f_0(x, \tau) \, d\tau \right) \, dt \, dx.
\]
From the above Lemma follows

**Proposition 6.7.** Under the conditions of Lemma 6.6 then the following comparison holds

\[
\left[ \int_0^T \int_U (p^2 - p_0^2) \, dx \, dt \right]^2 \leq C \int_U \left[ \int_0^T (p^2 - p_0^2) \, dt \right]^2 \, dx \leq C \int_U \left( \int_0^T f_0(x,t) \, dt \right)^2 \, dx.
\]

(183)

**Proof.** Since \( f_0(x,t) > 0 \) for all \((x,t)\) (see (163)) and according to Lemma 6.5 \( p_0^2(x,t) - p^2(x,t) > 0 \) for all \((x,t)\), in the RHS of (177) we have

\[
\int_U \int_0^T \left( (p_0^2 - p^2) \int_0^T f_0(x,\tau) \, d\tau \right) \, dt \, dx \leq \int_U \int_0^T (p_0^2 - p^2) \int_0^T f_0(x,\tau) \, d\tau \, dt \, dx
\]

\[
= \int_U \int_0^T (p_0^2 - p^2) \, dt \int_0^T f_0(x,t) \, dt \, dx
\]

\[
\leq \varepsilon \int_U \left( \int_0^T (p_0^2 - p^2) \, dt \right)^2 \, dx + C \varepsilon \int_U \left( \int_0^T f_0(x,t) \, dt \right)^2 \, dx. \quad (184)
\]

By Poincaré inequality we have

\[
\left[ \int_0^T \int_U (p^2 - p_0^2) \, dx \, dt \right]^2 \leq C \int_U \left[ \int_0^T (p^2 - p_0^2) \, dt \right]^2 \, dx
\]

\[
\leq C \int_U \left[ \nabla \int_0^T (p^2 - p_0^2) \, dt \right]^2 \, dx. \quad (185)
\]

Finally, estimating the RHS of (177) using (184), neglecting the first term in LHS, and using (185) for the second one, we get:

\[
\left[ \int_0^T \int_U (p^2 - p_0^2) \, dx \, dt \right]^2 \leq \varepsilon \int_U \left( \int_0^T (p_0^2 - p^2) \, dt \right)^2 \, dx + C \varepsilon \int_U \left( \int_0^T f_0(x,t) \, dt \right)^2 \, dx.
\]

Choosing appropriate \( \varepsilon \) we get (183).

\[\square\]

**6.2. Stability of PI with respect to initial and boundary data.** In this section we will show that for the given time interval \( 0 < t \leq T_0 < T_{crit} \) the difference between the PI’s for the actual and the auxiliary problems becomes small as the parameter \( B \) for the initial reserves becomes large. As before we consider only the linear Darcy case with \( F = 0 \) in (146).

Let \( p(x,t) \) be the classical solution of IBVP (148)-(150) with the corresponding productivity index \( J_g[p](t) \) defined by (153). Let \( p_0(x,t) \) be the auxiliary pressure given by (154) with the corresponding productivity index \( J_g[p_0] \) defined by (165). In linear case the original pressure \( p(x,t) \) inherits the following properties of the auxiliary pressure \( p_0(x,t) \) (for the details see [27]): for all \( t, 0 < t \leq T_0 < T_{crit} \)

\[
|\nabla p(x,t)| + \Delta p(x,t) \leq C < \infty; \quad (186)
\]

\[
|p_t(x,t)| \leq C < \infty. \quad (187)
\]

In the following two lemmas we will obtain estimates for \( p - p_0 \) and \( p_t - p_{0,t} \) as \( B \to 0 \). These will be used later to prove Theorem 6.10.
Lemma 6.8. Under the constraints (186) for $B > 1$

\[ 0 \geq p - p_0 \geq -\frac{C}{B^2} \quad \text{and} \quad 0 \geq p^2 - p_0^2 \geq -\frac{C}{B}. \]  

Moreover, if $B \to \infty$

\[ p - p_0 \to 0 \quad \text{and} \quad p^2 - p_0^2 \to 0. \]

Proof. Let $a(x, t) = p + p_0$, $b_i(x, t) = \partial(p + p_0)/\partial x_i$, $i = 1, \ldots, n$ and $c(x, t) = \Delta(p + p_0)$. The direct calculations show that the function $u = p - p_0$ is a solution of

\[ u_t - a \Delta u + \sum_{i=1}^{n} b_i u_{x_i} + cu = -f_0(x, t) \quad \text{in} \quad U \times (0, T_0], \quad u = 0 \quad \text{on} \quad \partial U \times (0, T_0]. \]  

Under the conditions (186) it follows that $|b_i(x, t)| + c(x, t) \leq C$ independently of time $t$. Eq. (190) is parabolic in $u = p - p_0$ and

\[ C_1 B > a(x, t) > C_0 B \]  

for some $C_0, C_1 > 0$. Thus following the standard arguments using the barrier functions (see [17]) we get that

\[ -C \max_{0 < t \leq T_0} f_0(x, t) \leq p - p_0 \leq 0. \]  

Taking $T_0 \leq \frac{1}{2} T_{\text{crit}} = \frac{B}{2A}$, one can get that

\[ f_0 \leq \frac{A}{\sqrt{B^2 + 2W + \frac{1}{16} B^2}} \leq \frac{C}{1 + B^2} \to 0 \quad \text{as} \quad B \to \infty. \]  

Then the result follows from (193) and (192). \[ \square \]

Lemma 6.9. Under the constraints (186) and (187)

\[ p_t - p_{0,t} \to 0 \quad \text{as} \quad B \to \infty. \]  

Proof. Multiplying equations (178) and (179) on $p$ and $p_0$ correspondingly we get

\[ 2pp_t - p\Delta(p^2) = 0, \]
\[ 2p_0p_{0,t} - p_0\Delta(p_0^2) = f_0p_0. \]

Let $u = pp_t = \frac{1}{2}(p^2)_t$ and $u_0 = p_0p_{0,t} = \frac{1}{2}(p_0^2)_t$. Then differentiating the equations above in $t$ we get

\[ 2u_t - p\Delta u = p_t \Delta(p^2), \]
\[ 2u_{0,t} - p_0\Delta u_0 = p_{0,t} \Delta(p_0^2) + 2f_0p_0 + 2f_0p_{0,t}. \]

Since $2p_t = \Delta(p^2)$ and $2p_{0,t} - f_0 = \Delta(p_0^2)$ we can rewrite

\[ 2u_t - p\Delta u = 2p_t^2, \]
\[ 2u_{0,t} - p_0\Delta u_0 = 2p_{0,t}^2 + 2f_0p_0 + f_0p_{0,t}. \]

Subtracting two last equations from each other we get

\[ 2 \frac{\partial}{\partial t} (u - u_0) - p\Delta u + p_0\Delta u_0 = 2(p^2_t - p_{0,t}^2) - F_1(x, t) \]  

(195)
where \( F_1(x,t) = 2f_{0,t}p_0 + f_0p_{0,t} \). Denoting \( z = u - u_0 \) and adding on both sides of (195) the term \( p\Delta u_0 \) we get

\[
2\frac{\partial z}{\partial t} - p\Delta z = 2(p^2 - p_0^2) - (p_0 - p)\Delta u_0 - F_1(x,t).
\]

Further, since \( p_t = u/p \) and \( p_{0,t} = u_0/p_0 \) we have

\[
2\frac{\partial z}{\partial t} - p\Delta z = 2\left(\frac{u}{p} - \frac{u_0}{p_0}\right)(p_t + p_{0,t}) - (p_0 - p)\Delta u_0 - F_1(x,t).
\]

(196)

In the first term in RHS adding and subtracting the term \( u_0p_0 \) in the numerator we get

\[
\frac{u}{p} - \frac{u_0}{p_0} = \frac{p_0(u - u_0)}{pp_0} - \frac{u_0(p - p_0)}{pp_0} = \frac{z}{p} - \frac{(p - p_0)p_{0,t}}{p}.
\]

Thus from (196) we have linear equation for \( z \)

\[
2\frac{\partial z}{\partial t} - p\Delta z = 2\frac{z}{p}C(x,t) + F_2(x,t) - F_1(x,t),
\]

where

\[
C(x,t) = (p_t + p_{0,t}),
\]

\[
F_2(x,t) = (p - p_0)\left(\Delta u_0 - \frac{2p_{0,t}}{p}(p_t + p_{0,t})\right).
\]

Under conditions (186) and (187) and in view of Lemma 6.8 \(|C(x,t)| \leq C < \infty \) and \( F_2(x,t) + F_1(x,t) \rightarrow 0 \) as \( B \rightarrow \infty \). Then \(|z| \rightarrow 0 \) as \( B \rightarrow \infty \). The result follows since

\[
z = u - u_0 = pp_t - p_0p_{0,t} = p_0(p_t - p_{0,t}) - p_t(p_0 - p).
\]

Finally, we will state the main theorem of this section.

**Theorem 6.10.** Under constraints (186) and (187)

\[
J_g[p](t) - J_g[p_0] \rightarrow 0 \quad \text{for } B \rightarrow \infty.
\]

(197)

**Proof.** From the boundary condition \( p|_\Gamma = B - At \) and the fact that \((B - At)^2 = p_0^2(x,t) - 2W(x)\) (see (154)) we get

\[
J_g[p_0] - J_g[p](t) = \frac{Q(t)}{|\mathcal{U}| \int_U p^2 dx} - \frac{1}{|\mathcal{U}| \int_U p^2 dx} + \frac{Q_0}{|\mathcal{U}| \int_U 2W(x) dx}
\]

\[
= \frac{-\int_{\Gamma} \nabla p \cdot N ds}{|\mathcal{U}| \int_U p^2 dx} - \frac{-\int_{\Gamma} p_0 \nabla p_0 \cdot N ds}{|\mathcal{U}| \int_U p_0^2 dx} + \frac{\int_U 2W(x) dx}{|\mathcal{U}| \int_U 2W(x) dx}
\]

\[
= \frac{\frac{d}{dt} \int_{\Gamma} p(x,t) dx}{|\mathcal{U}| \int_U (p^2 - p_0^2) dx + \frac{1}{|\mathcal{U}| \int_U 2W(x) dx}} + \frac{\frac{d}{dt} \int_U p_0(x,t) dx}{|\mathcal{U}| \int_U 2W(x) dx} - \frac{\frac{d}{dt} \int_U f_0(x,t) dx}{|\mathcal{U}| \int_U 2W(x) dx}
\]

Adding and subtracting the term \( \frac{d}{dt} \int_U p_0(x,t) dx \) in the numerator of the first fraction we get

\[
(J_g[p_0] - J_g[p](t)) \cdot |\mathcal{U}| = \frac{\frac{d}{dt} \int_{\Gamma} (p - p_0) dx}{|\mathcal{U}| \int_U (p^2 - p_0^2) dx + 2 \int_U W dx} \frac{\int_U f_0(x,t) dx}{2 |\mathcal{U}| \int_U W dx}.
\]
Finally

\[
(J_g[p] - J_g[p](t)) \cdot |U| = \frac{\int_U f_0(x,t) \, dx}{2 \int_U W(x) \, dx} + \frac{1}{2 \int_U (p^2 - p_0^2) \, dx + 2 \int_U W(x) \, dx} \times \left[ \frac{d}{dt} \int_U (p - p_0) \, dx - \int_U (p^2 - p_0^2) \, dx - \frac{d}{dt} \left( \int_U p_0 \, dx \right) \right]
\]

Since since \( f_0 \to 0 \) as \( B \to \infty \) (see (163)) then

\[
\frac{\int_U f_0(x,t) \, dx}{2 \int_U W(x) \, dx} \to 0 \quad \text{as} \quad B \to \infty.
\]

Then (197) follows from Lemma 6.8 and Lemma 6.9. \qed

7. Conclusions.

- The notion of the productivity index is studied in case of Forchheimer flow of slightly compressible and strongly compressible fluids (in particular, ideal gas). In general case the PI is a time dependent integral functional over the pressure function.

- In case of slightly compressible fluid we consider two types of boundary profile on the well-boundary: the total flux condition and Dirichlet boundary condition. In both cases we prove the convergence of the time dependent PI to a constant value without any constraints on the degree of nonlinearity of Forchheimer polynomial. This generalizes our previous work [4], requiring the estimates on the mixed term \(|\nabla p|^2 \cdot |p|^{\alpha - 2}, \alpha \neq 2\), and resulting in stronger constraints on smoothness of boundary data.

- In case of the ideal gas flow, we study the Dirichlet boundary problem. The quantity \( B - At \) prescribed on the well-boundary specifies the gas reserves at the moment \( t \). We show numerically that the PI stays the same until it sharply blows up when time approaches the critical value \( T_{crit} = B/A \). This fact corresponds with the field observations. We associate the constant PI with the auxiliary pressure and prove the estimate for the difference between actual and auxiliary pressure. The results on stability of the PI with respect to the initial gas reserves are obtained.

REFERENCES

[1] D. G. Aronson, The porous medium equation, Lecture Notes in Mathematics, Springer, 1224 (1986), 1–46.
[2] E. Aulisa, L. Bloshanskaya, L. Hoang and A. Ibragimov, Analysis of generalized Forchheimer flows of compressible fluids in porous media, J. Math. Phys., 50 (2009), 103102, 44pp.
[3] E. Aulisa, L. Bloshanskaya and A. Ibragimov, Long-term dynamics for well productivity index for nonlinear flows in porous media, J. Math. Phys., 52 (2011), 023506, 26pp.
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[4] E. Aulisa, L. Bloshanskaya and A. Ibragimov, *Time asymptotics of non-Darcy flows controlled by total flux on the boundary*, J. Math. Sci., 184 (2012), 399–430.

[5] E. Aulisa, A. Ibragimov, P. Valko and J. R. Walton, *Mathematical framework of the well productivity index for fast Forchheimer (non-Darcy) flows in porous media*, Mathematical Models and Methods in Applied Sciences, 19 (2009), 1241–1275.

[6] K. Aziz, L. Matta, S. Ko and G. S. Brar, *Use of pressure, pressure-squared or pseudo-pressure in the analysis of transient pressure drawdown data from gas wells*, Petroleum Society of Canada, 15 (1976), 9pp.

[7] J. Bear, *Dynamics of Fluids in Porous Media*, Dover Publications Inc., New York, 1988.

[8] M. Bulíček, J. Málek and J. Žabenský, *On generalized Stokes’ and Brinkman’s equations with a pressure- and shear-dependent viscosity and drag coefficient*, Nonlinear Analysis: Real World Applications, 26 (2015), 109–132.

[9] T. Christopher and O. Uche, *Evaluating productivity index in a gas well using regression analysis*, International Journal of Engineering Sciences & Research Technology, 3 (2014), 661–675.

[10] L. P. Dake, *Fundamentals of Reservoir Engineering*, Elsevier, Amsterdam, 1983.

[11] H. Darcy, *Les Fontaines Publiques de la Ville de Dijon*, Dalmont, Paris, 1856.

[12] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, 1993.

[13] P. Forchheimer, *Wasserbewegung durch boden zeit*, Ver. Deut. Ing., 45 (1901), 1782.

[14] L. Hoang, A. Ibragimov, T. Kieu and Z. Sobol, *Stability of solutions to generalized Forchheimer equations of any degree*, Journal of Mathematical Sciences, 210 (2015), 476–544.

[15] A. Ibragimov, D. Khalmanova, P. P. Valko and J. R. Walton, *On a mathematical model of the productivity index of a well from reservoir engineering*, SIAM Journal on Applied Mathematics, 65 (2005), 1952–1980.

[16] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.

[17] E. M. Landis, *Second Order Equations of Elliptic and Parabolic Type*, vol. 171 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1998, Translated from the 1971 Russian original by Tamara Rozhkovskaya, With a preface by Nina Ural’tseva.

[18] D. Li and T. W. Engler, *Literature review on correlations of the non-Darcy coefficient*, SPE, (2001), 8pp.

[19] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes Aux Limites non Linéaires*, Dunod, 1969.

[20] M. Muskat, *The Flow of Homogeneous Fluids Through Porous Media*, International Human Resources Development, 1982.

[21] K. Nakshatrala and K. Rajagopal, *A numerical study of fluids with pressure-dependent viscosity flowing through a rigid porous medium*, International Journal for Numerical Methods in Fluids, 67 (2011), 342–368.

[22] L. E. Payne, J. C. Song and B. Straughan, *Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity*, The Royal Society, 455 (1999), 2173–2190.

[23] L. E. Payne and B. Straughan, *Convergence and continuous dependence for the brinkman-forchheimer equations*, Studies in Applied Mathematics, 102 (1999), 419–439.

[24] C. A. Pereira, H. Kazemi and E. Ozkan, *Combined effect of non-Darcy flow and formation damage on gas well performance of dual-porosity and dual-permeability reservoirs*, SPEJ, 9 (2006), 543–552.

[25] R. Raghavan, *Well Test Analysis*, Prentice Hall, New York, 1993.

[26] S. Srinivasan and K. Rajagopal, *A thermodynamic basis for the derivation of the Darcy, Forchheimer and Brinkman models for flows through porous media and their generalizations*, International Journal of Non-Linear Mechanics, 58 (2014), 162–166.

[27] J. L. Vázquez, *The Porous Medium Equation*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2007, Mathematical theory.
[28] C. Wolfsteiner, L. Durlofsky and K. Aziz, Calculation of well index for nonconventional wells on arbitrary grids, *Computational Geosciences*, 7 (2003), 61–82.

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E-mail address: eugenio.aulisa@ttu.edu
E-mail address: bloshanl@newpaltz.edu
E-mail address: akif.ibragimov@ttu.edu