CONFORMAL BLOCKS AND RATIONAL NORMAL CURVES

NOAH GIANSIRACUSA

Abstract. We prove that the Chow quotient parametrizing configurations of \(n\) points in \(\mathbb{P}^d\) which generically lie on a rational normal curve is isomorphic to \(\overline{\mathcal{M}}_{0,n}\), generalizing the well-known \(d = 1\) result of Kapranov. In particular, \(\overline{\mathcal{M}}_{0,n}\) admits birational morphisms to all the corresponding geometric invariant theory (GIT) quotients. For symmetric linearizations the polarization on each GIT quotient pulls back to a divisor that spans the same extremal ray in the symmetric nef cone of \(\overline{\mathcal{M}}_{0,n}\) as a conformal blocks line bundle. A symmetry in conformal blocks implies a duality of point-configurations that comes from Gale duality and generalizes a result of Goppa in algebraic coding theory. In a suitable sense, \(\overline{\mathcal{M}}_{0,n,2m}\) is fixed pointwise by the Gale transform when \(d = m - 1\) so stable curves correspond to self-associated configurations.

1. Introduction

1.1. Configurations of points. A natural way to compactify the moduli space of \(n\) distinct points on the line, \(\mathcal{M}_{0,n} = ((\mathbb{P}^1)^n \setminus \{\text{diagonals}\}) / \text{Aut}(\mathbb{P}^1)\), is to allow the points to collide by reintroducing the diagonals. We can add more flexibility by considering configurations of points in \(\mathbb{P}^d\). If \(U_{d,n} := \{(p_1, \ldots, p_n) \in (\mathbb{P}^d)^n \mid p_i\) are distinct points of a rational normal curve\} then \(U_{d,n}/SL_{d+1} \cong \mathcal{M}_{0,n}\) for \(d \leq n - 3\). The key is that now we compactify not only by allowing the points to collide but also by degenerating the rational normal curve supporting the points. By taking the topological closure

\[ V_{d,n} := U_{d,n} \subseteq (\mathbb{P}^d)^n \]

with reduced induced structure we get the space of configurations that lie on what we call a quasi-Veronese curve (see Definitions 2.1, 2.2, and Lemma 2.3). The GIT quotients \(V_{d,n}/SL_{d+1} \cong \overline{\mathcal{M}}_{0,n}\). For \(d = 1\) these have been studied extensively, e.g., [DM86, GHP88, Kap93, GIT94, KM96, Hu99, AL02, Has03, Pol95, AS08, Bol10, HMSV10]. For \(d = 2\) they occur as a special case of a construction in [Sim08, GS09]. For \(d \geq 3\) they do not seem to have appeared in the literature previously.

Theorem 1.1. For any \(d \leq n - 3\) there is an isomorphism \(V_{d,n}/SL_{d+1} \cong \overline{\mathcal{M}}_{0,n}\), and for any effective linearization \(L\) on \(V_{d,n}\) there is a birational morphism \(\varphi: \overline{\mathcal{M}}_{0,n} \to V_{d,n}/L_{SL_{d+1}}\) extending \(\mathcal{M}_{0,n} \to U_{d,n}/SL_{d+1}\).
We prove this in §3.2 by adapting ideas from [GG10] to show that \( \overline{M}_{0, n} \) admits a morphism to Chow\((\mathbb{P}^d)^{\nu} \) which is an isomorphism onto its image, \( V_{d,n} \to \text{Ch} \text{SL}_{d+1} \). The statement about GIT quotients then follows from Kapranov’s result that the Chow quotient maps to all GIT quotients. We provide an explicit description of \( \varphi \) in §4.1, compute which F-curves it contracts in §4.3, and show that it factors through Hassett’s space of weighted pointed curves \( \overline{M}_{0,L} \) in Proposition 4.4.

1.2. Conformal blocks. Conformal field theory, first introduced in the physics literature then adopted by the mathematical community, has found applications in many areas. See [TUY89] for background. For our purposes it leads to a family of vector bundles on \( \overline{M}_{g,n} \), each dependent on a choice of Lie algebra and an \( n \)-tuple of dominant integral weights. When \( q = 0 \) these bundles are globally generated [Fak09, Lemma 2.2].

For \( \mathfrak{s}_l \) the weights can be chosen so that the determinant line bundle of the conformal blocks vector bundle spans the same ray in \( N^1(\overline{M}_{0,n}) \) as the pull-back of the GIT polarization on \( (\mathbb{P}^1)^{\nu} / \text{LSL}_2 \) along the Kapranov morphism \( \overline{M}_{0,n} \to (\mathbb{P}^1)^{\nu} / \text{LSL}_2 \) [Fak09, Theorem 4.5].

In [AGSS10] a specific family of conformal blocks line bundles is studied, corresponding to \( \mathfrak{s}_l \) with fundamental dominant weights \( (\omega_k, \ldots, \omega_k), k \in \{2, \ldots, n-2\} \). Let us denote these by \( D_{k}^{\mathfrak{s}_l} \). The spaces \( V_{d,n} \) provide the correct generalization of \( V_{1,n} = (\mathbb{P}^1)^{\nu} \) to extend Fakhruddin’s result about \( \mathfrak{s}_l \) bundles to these \( \mathfrak{s}_l \) bundles, in the following sense. Each GIT quotient \( V_{d,n} / \text{LSL}_{d+1} \) comes with a polarization which by Theorem 1.1 can be pulled-back to a line bundle on \( \overline{M}_{0,n} \). For \( S_n \)-invariant linearization \( L \) denote the GIT polarization by \( \mathcal{L}_d \in \text{Pic}(V_{d,n} / \text{LSL}_{d+1}) \).

**Theorem 1.2.** The line bundle \( \varphi^* \mathcal{L}_d \) spans the same ray in \( N^1(\overline{M}_{0,n}) \) as \( D_{d+1}^{\mathfrak{s}_l} \).

**Corollary 1.3.** For \( d = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \), the line bundles \( \varphi^* \mathcal{L}_d \) span distinct extremal rays of the symmetric nef cone of \( \overline{M}_{0,n} \).

Indeed, in [AGSS10] it was shown that \( D_{2}^{\mathfrak{s}_l}, \ldots, D_{\lfloor \frac{n}{2} \rfloor}^{\mathfrak{s}_l} \) span distinct extremal rays of the symmetric nef cone. They prove this by finding, for each \( d \), a family of \( \rho - 1 \) independent curves contracted by the complete linear system \( |D_{d+1}^{\mathfrak{s}_l}| \), where \( \rho := \dim(N^1(\overline{M}_{0,n})^{S_n}) \). Because of this extremality result, to prove Theorem 1.2 it is enough to show that the morphism \( \varphi \) contracts the same set of independent curves. We do this in §4 by studying a formula of Fakhruddin for the degrees of conformal blocks bundles restricted to F-curves.

1.3. Gale duality. The symmetry of the Dynkin diagram \( \circ \cdots \circ \circ \cdots \circ \circ \) for \( \mathfrak{s}_l \) implies that \( D_{k}^{\mathfrak{s}_l} = D_{n-k}^{\mathfrak{s}_l} \). This observation, which was brought to my attention by Valery Alexeev, together with Theorem 1.2, implies the following:

**Corollary 1.4.** For \( S_n \)-invariant \( L \) there is an isomorphism of normalizations

\[
(V_{d,n} / \text{LSL}_{d+1})^\nu \cong (V_{n-d-2,n} / \text{LSL}_{n-d-1})^\nu.
\]

We believe the morphisms \( \varphi \) in Theorem 1.1 have connected fibers, which would imply that these GIT quotients are normal. Regardless, by restricting to \( \overline{M}_{0,n} \) there is an isomorphism \( U_{d,n} / \text{SL}_{d+1} \cong U_{n-d-2,n} / \text{SL}_{n-d-1} \) which, as was suggested by Brendan Hassett, is induced by the Gale transform [DO88, Corollary III.1]:

\[
(\mathbb{P}^d)^{\nu} / \text{LSL}_{d+1} \cong (\mathbb{P}^{n-d-2})^{\nu} / \text{LSL}_{n-d-1}.
\]
It has been known since the 70s that the Gale transform of a configuration of distinct points lying on a rational normal curve in $\mathbb{P}^d$ lies on a rational normal curve in $\mathbb{P}^{n-d-2}$. This was proven by Goppa in the context of algebraic coding theory to show that the dual of a Goppa code is a Goppa code. See [EP00] for a modern treatment, generalization, and reference to this historical anecdote. Since the Gale transform is involutive, this implies that $U_{d,n}/\text{SL}_{d+1} \cong U_{n-d-2,n}/\text{SL}_{n-d-1}$.

By replacing $U_{d,n}$ with $V_{d,n}$, Corollary 1.4—with the connected fiber assumption—can therefore be viewed as a generalization of Goppa’s result: the Gale transform preserves configurations supported on quasi-Veronese curves, not just rational normal curves. This follows from Goppa’s result when combined with the Dolgachev-Orland GIT form of Gale duality, but it is curious that it also follows immediately, assuming normality of $V_{d,n}$, from a seemingly unrelated symmetry of conformal blocks via the framework of Theorem 1.2.

For $n = 2m$ and $d = m - 1$ one can discuss self-associated configurations, i.e., points of $(\mathbb{P}^{m-1})^{2m}/\text{ChSL}_m$ which are fixed by the Gale transform. By Theorem 1.1 we can identify $\overline{M}_{0,2m}$ with $V_{m-1,2m}/\text{ChSL}_m$ and thus view it as a subvariety of $(\mathbb{P}^{m-1})^{2m}/\text{ChSL}_m$. In §6.4 we show that each stable curve $(C, p_1, \ldots, p_{2m})$ corresponds under this identification to a self-associated configuration.

Acknowledgements. I would like to thank Dan Abramovich, Valery Alexeev, Najmuddin Fakhruddin, Angela Gibney, Danny Gillam, Brendan Hassett, and Dave Swinarski for helpful conversations, and I especially thank Gibney for suggesting this project. Partial support was provided by funds from NSF award DMS-0901278.
bibliography of papers that deal with this space and its various quotients by \( \text{SL}_2 \). In [GS09] it was proven that \( \overline{M}_{0,n} \) admits a birational morphism to each GIT quotient \( \text{RNC}(2,n)/\text{SL}_3 \). For linearizations \( L \) that are trivial on \( \mathcal{H}_2 = \mathbb{P}^5 \) the GIT quotient is obtained by first applying the induced morphism \( L : \text{RNC}(2,n) \to V_{2,n} \) which “forgets” the underlying conic and then taking the usual GIT quotient of the resulting space on which the linearization is ample. Thus a corollary is the existence of the morphisms \( \varphi \) in Theorem 1.1 when \( d = 2 \). In this paper we only study quotients of the spaces \( V_{d,n} \), but it would be interesting to know if Theorem 1.1 generalizes from these quotients to quotients of \( \text{RNC}(d,n) \) as it does in the cases \( d = 1 \) (trivially) and \( d = 2 \) (by [GS09]).

Remark 2.4. Since this paper concerns the locus \( V_{d,n} \) where only points are parametrized, not the curves supporting them, we could have used different compactifications of the space of rational normal curves. For instance, one could take the Kontsevich stable map space \( \overline{M}_{0,n}(\mathbb{P}^d,d) \) then use the product of evaluation maps \( \overline{M}_{0,n}(\mathbb{P}^d,d) \to \mathbb{P}^d \) to define the same locus \( V_{d,n} \subseteq (\mathbb{P}^d)^n \).

3. Chow Quotients

3.1. Background. We briefly recall the definition of a Chow quotient [Kap93, §0.1] (see also [Hu05a, §3]). If an algebraic group \( G \) acts on a projective variety \( X \) then there exists a Zariski dense \( G \)-invariant subset \( U \subseteq X \) for which all the orbit closures \( Gu \subseteq X \), \( u \in U \), have the same dimension \( r \) and homology class \( \delta \in H_{2r}(X,\mathbb{Z}) \). This induces an embedding \( U/G \to \text{Chow}(X,\delta) \) into the Chow variety parametrizing cycles in \( X \) with homology class \( \delta \). By definition the Chow quotient is the closure of the image:

\[
X/G := \overline{U/G} \subseteq \text{Chow}(X,\delta).
\]

If \( G \) is reductive then there is a birational morphism from the Chow quotient to any GIT quotient \( X/LG \) for which there is a stable point [Kap93, Theorem 0.4.3], and by variation of GIT there is a morphism from such quotients to nearby quotients for which there is a semistable point [Tha96, DH98], so the Chow quotient maps to all nonempty GIT quotients.

3.2. Proof of Theorem 1.1. Fix \( n \geq 4 \) and \( 1 \leq d \leq n - 3 \). We first prove the existence of a morphism \( \overline{M}_{0,n} \to V_{d,n}/\text{Ch}\text{SL}_{d+1} \) extending \( \mathcal{M}_{0,n} \to U_{d,n}/\text{SL}_{d+1} \) and then show that it is an isomorphism. The statement about morphisms to the GIT quotients follows from the preceding general remark.

Setup. The inclusion \( V_{d,n} \subseteq (\mathbb{P}^d)^n \) induces a closed embedding

\[
V_{d,n}/\text{Ch}\text{SL}_{d+1} \to (\mathbb{P}^d)^n/\text{Ch}\text{SL}_{d+1}.
\]

These latter Chow quotients were introduced by Kapranov. In [Kap93, Proposition 2.1.7] it was shown that the locus \( U \) of generic points in the definition of the Chow quotient may be taken to be those configurations such that any \( m \) points, \( m \leq d + 1 \), span a \( \mathbb{P}^{m-1} \). It is a classical fact that distinct points on a rational normal curve satisfy this property, so we may take the generic locus to be \( U_{d,n} \) and view \( V_{d,n}/\text{Ch}\text{SL}_{d+1} \) as the closure of \( U_{d,n}/\text{SL}_{d+1} \) inside \( \text{Chow}((\mathbb{P}^d)^n) \). Our goal then is to construct a morphism \( \overline{M}_{0,n} \to \text{Chow}((\mathbb{P}^d)^n) \) extending the embedding \( \mathcal{M}_{0,n} = U_{d,n}/\text{SL}_{d+1} \to \text{Chow}((\mathbb{P}^d)^n) \). The image of such a map is necessarily
contained in $V_{d,n}/\text{CHSL}_{d,n}$. Moreover, there can be at most one such extension since the Chow variety is separated.

**Extending the map.** Let $\Gamma \subseteq \mathcal{M}_{0,n} \times \text{Chow}((\mathbb{P}^d)^n)$ be the graph of the morphism $\mathcal{M}_{0,n} \hookrightarrow \text{Chow}((\mathbb{P}^d)^n)$ and $\Gamma$ its closure in $\mathcal{M}_{0,n} \times \text{Chow}((\mathbb{P}^d)^n)$, with reduced induced structure. It suffices to show that $\pi_1 : \Gamma \to \mathcal{M}_{0,n}$ is an isomorphism, for then $\pi_2 \pi_1^{-1} : \mathcal{M}_{0,n} \to \text{Chow}((\mathbb{P}^d)^n)$ is the desired extension. To show that $\pi_1$ is an isomorphism, we use the fact that it is birational and $\mathcal{M}_{0,n}$ is normal to reduce to showing that it is a finite morphism. And for this, we use the fact that $\Gamma$ is proper to reduce to showing that $\pi_1$ is quasi-finite. In fact, it is enough to show that the fiber over each closed point is finite. Indeed, each fiber has finite cardinality if and only if it is zero-dimensional, and by upper-semicontinuity the set of points with positive-dimensional fibers is closed, so if this set were nonempty then it would contain a closed point. We now proceed to show that the fiber over each closed point is finite—in fact that it has cardinality one.

Let $x \in \mathcal{M}_{0,n}(\mathbb{C})$ and $(x_i, y_i) \in \Gamma_x$, $i = 1, 2$. We want to show $y_1 = y_2$. There are points $(x'_1, y'_1) \in \Gamma$ specializing to $(x_i, y_i)$, and by [EGA60, III.7.1.9] there are DVRs $R_i$ and morphisms $\text{Spec } R_i \to \Gamma$ sending the generic point $\text{Spec } K_i$ to $(x'_i, y'_i)$ and the closed point $\text{Spec } k_i$ to $(x_i, y_i)$. Thus we are reduced to the following situation. Given a stable pointed curve $(C_k, p_1, \ldots, p_n) \in \mathcal{M}_{0,n}$, write it as the special fiber in a family $C_R \to \text{Spec } R$ of stable pointed curves over a DVR such that the generic fiber $C_K \to \text{Spec } K$ is smooth. There is an algebraic cycle in $(\mathbb{P}^d_K)^n$ obtained by applying the $d$th Veronese map to the marked points of $C_K \cong \mathbb{P}^1_k$, and taking the $\text{SL}_{d+1}$-orbit closure of the resulting configuration. This cycle limits to a cycle in $(\mathbb{P}^d_k)^n$, and we must show that this limit cycle is independent of the smoothing of $C_k$. We will do this by explicitly describing this limit cycle.

**The limit cycle.** For $C := (C, p_1, \ldots, p_n) \in \mathcal{M}_{0,n}$, write $C = C_1 \cup \cdots \cup C_r$ as a union of irreducible components, each isomorphic to $\mathbb{P}^1$. For each partition $d = d_1 + \cdots + d_r$ with $d_i \geq 0$ choose auxiliary smooth points $q_1, \ldots, q_d \in C$ so that $C_i$ has $d_i$ points, and consider the line bundle $L := \mathcal{O}_C(q_1 + \cdots + q_d)$. It has vanishing higher cohomology, since by Serre duality $h^1(L) = h^0(\omega_C \otimes L^{-1}) = 0$. So by Riemann-Roch $h^0(L) = \chi(L) = d + 1$. Moreover, it is basepoint free, so it induces a morphism $\phi_L : C \to \mathbb{P}^d$ sending $p_1, \ldots, p_n$ to a configuration of $n$ not necessarily distinct points in $\mathbb{P}^d$: each $C_i$ gets sent to a rational normal curve of degree $d_i$ in the projective space that it spans, and if $d_i = 0$ then $C_i$ gets contracted and all the marked points on it have the same image in $\mathbb{P}^d$. It follows from Lemma 3.1 below (by taking the DVR to be a smoothing of $C$) that the resulting curve is a quasi-Veronese curve so this configuration lies in $\mathcal{V}_{d,n}$. The orbit closure

$$Z_{d_1 \cdots d_r} := \text{SL}_{d+1}(\phi_L(p_1), \ldots, \phi_L(p_n)) \subseteq (\mathbb{P}^d)^n$$

does not depend on the choice of $L$ once the partition of $d$ is specified. Let

$$Z(C) := \bigcup_{d = d_1 + \cdots + d_r} Z_{d_1 \cdots d_r} \subseteq (\mathbb{P}^d)^n$$

with reduced induced subscheme structure. Not every orbit closure $Z_{d_1 \cdots d_r}$ is full-dimensional, but it follows from Lemmas 3.2 and 3.3 below that $Z(C)$ is of pure dimension $(d + 1)^2 - 1$ so that each $Z_{d_1 \cdots d_r}$ must be contained in a full-dimensional
orbit closure appearing in this union. We claim $Z(C) \in \text{Chow}((\mathbb{P}^d)^n)$ is the limit cycle no matter how $C$ is written as a limit of stable curves.

**Lemma 3.1.** Let $(R, m)$ be a DVR with fraction field $K$ and residue field $k$, and let $(C_R, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}(R)$. Then any smooth points $q_1, \ldots, q_d : \text{Spec} k \to C_k$ extend to smooth points $q_1, \ldots, q_d : \text{Spec} R \to C_R$ and every section of the line bundle $L_k := \mathcal{O}_{C_k}(q_1 + \cdots + q_d)$ lifts to a section of $L_R := \mathcal{O}_{C_R}(q_1 + \cdots + q_d)$.

**Proof.** The statement about extending points follows from smoothness, and the fact that sections lift follows from Grauert’s theorem [Har77, Corollary III.12.9] due to the vanishing of higher cohomology mentioned above. 

We can now show that $Z(C)$ is contained in the limit cycle—for any family, not just a smoothing.

**Lemma 3.2.** Let $R$ be a DVR as above, $C_R \in \overline{\mathcal{M}}_{0,n}(R)$, and write $C_K, C_k$ for the general and special fibers, respectively. Then

$$Z(C_k) \subseteq \overline{Z(C_K)}_k$$

where the closure is taken in $(\mathbb{P}^d)^n$.

**Proof.** If $r$ denotes the number of components of $C_k$ then by the definition of $Z(C_k)$ we must show the containment $Z_{d_1 \cdots d_r} \subseteq \overline{Z(C_K)}_k$ for each partition $d = d_1 + \cdots + d_r$.

Since the right-hand side is closed and $\text{SL}_{d+1}$-invariant it is enough to show that $\phi_{L_k}$ sends the marked points of $C_k$ into $\overline{Z(C_K)}_k$, where as above $L_k = \mathcal{O}_{C_k}(q_1 + \cdots + q_d)$ is a line bundle of degree $d$ on $C_k$ determined by this partition and $\phi_{L_k}$ is the induced morphism. By Lemma 3.1 this extends to a line bundle $L_R := \mathcal{O}_R(q_1 + \cdots + q_d)$ on $C_R$ and the sections inducing $\phi_{L_k} : C_k \to \mathbb{P}^d_R$ lift to give a map $\phi_{L_R} : C_R \to \mathbb{P}^d_R$. The restriction of $\phi_{L_R}$ to $C_K$ is induced by $\mathcal{O}_{C_K}(q_1 + \cdots + q_d)$ and sends the marked points of $C_K$ to a point of $Z(C_K)$, so by continuity we are done.

**Comparison of homology classes.** If we show that the containment in Lemma 3.2 is an equality when $C_K$ is smooth then we will have proven that the limit cycle is unique. For this, it is enough to prove the following:

**Lemma 3.3.** With notation as above, there is an inequality of homology classes

$$[Z(C_k)] \geq [\overline{Z(C_K)}_k] \in H_2((d+1)^2-1)((\mathbb{P}^d)^n, \mathbb{Z})$$

when $C_K$ is smooth.

**Proof.** Since $\overline{Z(C_K)}_k$ is a specialization of a point in $\text{Chow}((\mathbb{P}^d)^n)$ corresponding to a generic orbit closure, it must have the same homology class as a generic orbit closure. Therefore, we must show that $[Z(C)] \geq [\overline{Z(C')}_k]$, where $C \in \overline{\mathcal{M}}_{0,n}$ and $C' \in \mathcal{M}_{0,n}$. Now Lemma 3.2 implies that the homology class can only decrease when the curve degenerates, so we can reduce to the case that $C$ is maximally degenerate, i.e., that each component has exactly three special points. Kapranov computed the homology class of a generic orbit closure, so let us now recall this.

By the K"unneth formula, a basis for the $H_2((d+1)^2-1)((\mathbb{P}^d)^n)$ is given by tensor products $[\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]$ where $0 \leq m_i \leq d$ and $\sum_1^n m_i = (d + 1)^2 - 1$. By [Kap93, Proposition 2.1.7],

$$[Z(C')] = \sum_{m_1 + \cdots + m_n = (d+1)^2-1} [\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]$$
so the generic orbit closure has coefficient 1 at each basis element. In other words, if \( L_i \subseteq \mathbb{P}^d \) are generic linear subspaces of codimension \( m_i \) then the intersection number of \( L_1 \times \cdots \times L_n \) with \( Z(C') \) is 1. Our goal is to show that the intersection with \( Z(C) \) is \( \geq 1 \).

The assumption that \( C = (C, p_1, \ldots, p_n) \) is maximally degenerate implies that it has \( n - 2 \) components and that the only partitions \( d = d_1 + \cdots + d_{n-2} \) that yield a full-dimensional cycle \( \mathbb{Z}_{d_1, \ldots, d_{n-2}} \) are those with \( d_i \leq 1 \). For each choice of integers \( 0 \leq m_i \leq d \) with \( \sum_i m_i = (d+1)^2 - 1 \) fix a generic product of linear subspaces \( L_1 \times \cdots \times L_n \) as above. By the construction of \( Z(C) \), therefore, the proof will be complete if we show that there is a degree \( d \) map \( \psi : C \rightarrow \mathbb{P}^d \), sending each component to either a line or a point, such that \( \psi(p_i) \in L_i \) for \( i = 1, \ldots, n \). This follows from the following slightly more general result.

**Lemma 3.4.** Let \( C = (C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n} \) be a maximally degenerate curve. For any integer \( e \leq d \) and generic linear subspaces \( L_i \subseteq \mathbb{P}^d \) such that

\[
\sum_1^n \text{codim}(L_i) = (d+1)(e+1) - 1
\]

there is a unique degree \( e \) map \( \psi : C \rightarrow \mathbb{P}^d \), linear on components, with \( \psi(p_i) \in L_i \).

**Proof.** We use induction on \( n \). Consider a component \( D \subseteq C \) with one node. By the maximal degeneration hypothesis there are two points on \( D \), say \( p_1 \) and \( p_2 \). If \( L_1 \cap L_2 \neq \emptyset \) then contract \( D \) to produce a stable, maximally degenerate curve \( (C', p_2, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n-1} \). By the inductive hypothesis there is a unique map \( \psi' : C' \rightarrow \mathbb{P}^d \) such that \( (\psi'(p_2), \ldots, \psi'(p_n)) \in (L_1 \cap L_2) \times L_3 \times \cdots \times L_n \), since \( \text{codim}(L_1 \cap L_2) = \text{codim}(L_1) + \text{codim}(L_2) \). This map extends uniquely to a map on \( C \) (with degree zero on \( D \)) satisfying the required properties.

So suppose now that \( p_1, p_2 \in D \subseteq C \) as before but that \( L_1 \cap L_2 = \emptyset \). We again consider the maximally degenerate curve \( (C', p_2, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n-1} \) obtained by forgetting \( p_1 \) and stabilizing. By induction there is a unique degree \( e-1 \) map \( \psi' : C' \rightarrow \mathbb{P}^d \) such that \( (\psi'(p_2), \ldots, \psi'(p_n)) \in (L_1 + L_2) \times L_3 \times \cdots \times L_n \). Indeed,

\[
\text{codim}(L_1 + L_2) + \sum_1^n \text{codim}(L_i) = -d - 1 + \sum_1^n \text{codim}(L_i) = (d+1)((e-1)+1) - 1.
\]

To complete the proof we must show that there is a unique way to extend \( \psi' \) to a map \( \psi \) on \( C \), this time with degree 1 on \( D \), such that \( \psi(p_i) \in L_i \), \( i = 1, \ldots, n \).

If we label the node of \( D \) by \( q \), then \( \psi'(p_i) \in L_i \) for \( i = 3, \ldots, n \), and \( \psi'(q) \in L_1 + L_2 \). All that remains is to show there is a unique line in \( \mathbb{P}^d \) containing \( \psi'(q) \) and intersecting both \( L_1 \) and \( L_2 \). Consider the projective space \( \mathbb{P}^m := L_1 + L_2 \subseteq \mathbb{P}^d \) and the projection \( \mathbb{P}^m \setminus \{\psi'(q)\} \rightarrow \mathbb{P}^{m-1} \). By assumption \( L_1 \) and \( L_2 \) are skew, so after projecting to \( \mathbb{P}^{m-1} \) they intersect in a unique point, which means precisely that there is a unique line through \( \psi'(q) \) intersecting both \( L_1 \) and \( L_2 \), as desired. \( \square \)

**The isomorphism.** Since \( \overline{\mathcal{M}}_{0,n} \) is proper, its image in \( \text{Chow}(\mathbb{P}^d)^n \) is closed so at least topologically it coincides with \( \mathbb{V}_{d,n} / \text{ChSL}_{d+1} \). Thus all that remains in the proof of Theorem 1.1 is to show that the morphism \( \overline{\mathcal{M}}_{0,n} \rightarrow \text{Chow}(\mathbb{P}^d)^n \) constructed above is an isomorphism onto its image. For each subset \( I \subseteq \{1, \ldots, n\} \) of size \( d + 3 \) there is a projection map \( \pi_I : (\mathbb{P}^d)^n \rightarrow (\mathbb{P}^d)^{d+3} \). Because \( \pi_I \) is proper,
Proof. There is a factorization of $\mathcal{M}_0,n \to \text{Chow}((\mathbb{P}^d)^n)$ obtained in [Kol96, Theorem 6.8]. Since $\pi_I$ is $\text{SL}_{d+1}$-equivariant this restricts to a map $U_{d,n}/\text{SL}_{d+1} \to U_{d,d+3}/\text{SL}_{d+1}$ and hence a map on the topological closures $V_{d,n}/\text{SL}_{d+1} \to V_{d,d+3}/\text{SL}_{d+1}$ as well. There are also stabilization morphisms $\mathcal{M}_0,n \to \mathcal{M}_{0,d+3}$. This leads to the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}_0,n & \to & V_{d,n}/\text{Ch} \text{SL}_{d+1} \\
\downarrow & & \downarrow \\
\prod_I \mathcal{M}_{0,d+3} & \to & \prod_I V_{d,d+3}/\text{Ch} \text{SL}_{d+1} \\
\end{array}
$$

$$
\begin{array}{ccc}
& & \text{Chow}((\mathbb{P}^d)^n) \\
\downarrow & & \downarrow \\
& & \text{Chow}((\mathbb{P}^d)^{d+3})
\end{array}
$$

Lemma 3.5. The map $\mathcal{M}_{0,d+3} \to V_{d,d+3}/\text{Ch} \text{SL}_{d+1}$ is an isomorphism.

Proof. By Gale duality (see §6), $((\mathbb{P}^d)^{d+3}/\text{Ch} \text{SL}_{d+1}) \cong (\mathbb{P}^1)^{d+3}/\text{Ch} \text{SL}_{2}$ [Kap93, Corollary 2.3.14], and this latter Chow quotient is isomorphic to $\mathcal{M}_{0,d+3}$ [Kap93, Theorem 4.1.8], so we have an isomorphism $\mathcal{M}_{0,d+3} \cong ((\mathbb{P}^d)^{d+3}/\text{Ch} \text{SL}_{d+1})$ which it is easy to see extends the embedding $\mathcal{M}_{0,d+3} \to \text{Chow}((\mathbb{P}^d)^{d+3})$. As discussed earlier, the image must be contained in $V_{d,d+3}/\text{Ch} \text{SL}_{d+1}$, and by uniqueness of the extension this isomorphism must be the same map as in the statement of the lemma. □

Lemma 3.6. For each $4 \leq k \leq n$, the product of stabilization maps $\mathcal{M}_{0,n} \to \prod_I \mathcal{M}_{0,k}$ over all subsets $I \subseteq \{1, \ldots, n\}$ of size $k$ is an isomorphism onto its image.

Proof. There is a factorization $\mathcal{M}_{0,n} \to \prod_I \mathcal{M}_{0,k} \to \prod_I \mathcal{M}_{0,4}$ where the latter product is over all $J \subseteq \{1, \ldots, n\}$ of size 4, so the general result follows immediately from the case $k = 4$. The fact that the product of stabilization maps to $\mathcal{M}_{0,4} \cong \mathbb{P}^1$ is an isomorphism onto its image is proven in [GG10, Theorem 1]: it follows by induction on $n$ using the computability of the stabilization morphisms with the boundary stratification of $\mathcal{M}_{0,n}$. □

Lemmas 3.6 (with $k = d + 3$) and 3.5 imply that $\mathcal{M}_{0,n} \to \prod_I V_{d,d+3}/\text{Ch} \text{SL}_{d+1}$ is an isomorphism onto its image, so by the commutativity of (1) the same is true of $\mathcal{M}_{0,n} \to V_{d,n}/\text{Ch} \text{SL}_{d+1}$. This completes the proof.

4. GIT Quotients

Having proven the existence of the morphisms $\varphi$ in Theorem 1.1, we now focus on deriving some of their properties.

4.1. Explicit description. By [Kap93, Corollary 2.2.6] every cycle parametrized by $((\mathbb{P}^d)^n)/\text{Ch} \text{SL}_{d+1}$ is a sum of closures of full-dimensional orbits with multiplicity one. By the inclusion $V_{d,n}/\text{Ch} \text{SL}_{d+1} \subseteq ((\mathbb{P}^d)^n)/\text{Ch} \text{SL}_{d+1}$ this holds for $V_{d,n}/\text{Ch} \text{SL}_{d+1}$ as well. By [Kap93, Theorem 0.4.3] and its proof, for each linearization $L$ such that the semistable locus is nonempty, at least one of these orbit closures is semistable and all semistable ones are equivalent—in the sense that they determine the same point in the GIT quotient. The Chow-GIT morphism $V_{d,n}/\text{Ch} \text{SL}_{d+1} \to V_{d,n}/\text{Ch} \text{SL}_{d+1}$ is defined by sending each sum of orbit closures to the corresponding semistable orbit class. On the other hand, by our proof of Theorem 1.1 the morphism $\mathcal{M}_{0,n} \to V_{d,n}/\text{Ch} \text{SL}_{d+1}$ sends $C = (C, p_1, \ldots, p_n) \in \mathcal{M}_{0,n}$ to the sum of orbit closures $Z(C)$ obtained by mapping $C$ to a quasi-Veronese curve in all possible ways corresponding to partitions of $d$ among the components of $C$. So for each linearization there is one
partition that leads to a semistable configuration (or multiple partitions leading to GIT-equivalent configurations in the case of strictly semistable points) and \( \varphi \) sends \( C \) to the orbit of this configuration of \( n \) points on a quasi-Veronese curve.

4.2. GIT Stability. Stability for the action of \( \text{SL}_{d+1} \) on \( V_{d,n} \) can be viewed through the inclusion \( V_{d,n} \subseteq (\mathbb{P}^d)^n \) into the space of configurations of \( n \) points in \( \mathbb{P}^d \) where it is worked out in [DH98, Example 3.3.21]. The \( \text{SL}_{d+1} \)-ample cone of fractional linearizations for \( (\mathbb{P}^d)^n \) is \( \mathbb{Q}^n_{\geq 0} \) and we think of a vector \( L = (x_1, \ldots, x_n) \in \mathbb{Q}^n_{\geq 0} \) as assigning a positive rational weight to each point. A configuration is semistable if and only if the total weight lying in any proper linear subspace \( W \subset \mathbb{P}^d \) is at most \( \frac{\dim W + 1}{n} \cdot \sum x_i \). Multiplying \( L \) by a positive constant does not affect stability so one can use the normalization \( \sum x_i = d + 1 \). The semistable locus is then non-empty precisely when \( \max\{x_i\} \leq 1 \) so the space of effective linearizations can be identified with the hypersimplex

\[
\Delta(d + 1, n) := \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid 0 \leq x_i \leq 1, \sum x_i = d + 1\}
\]

Convention 4.1. Any reference to a linearization \( L \) will implicitly mean \( L \in \Delta(d + 1, n) \cap \mathbb{Q}^n_{\geq 0} \) so that \( L \) is ample and scaled as above. In particular, the unique \( S_n \)-invariant linearization is \( L = (\frac{d+1}{n}, \ldots, \frac{d+1}{n}) \).

This polytope is subdivided into closed chambers such that on their interiors the corresponding GIT quotients are constant. The walls for this decomposition are of the form \( \sum_{i \in I} x_i = k \) for \( I \subset \{1, \ldots, n\} \) and \( 1 \leq k \leq d \). Thus for each \( d \in \{1, \ldots, n-3\} \) and \( L \in \Delta(d+1, n) \) there is a morphism \( \varphi : \overline{M}_{0,n} \to V_{d,n} / _L \text{SL}_{d+1} \) and these morphisms do not change as \( L \) varies within a fixed chamber.

4.3. F-curves. The boundary of \( \overline{M}_{0,n} \) is naturally stratified, and the irreducible components of 1-strata are called F-curves. These are isomorphic to \( \overline{M}_{0,4} \cong \mathbb{P}^1 \) and are obtained by attaching maximally degenerate chains of \( \mathbb{P}^1 \)'s to four points on a \( \mathbb{P}^1 \): varying the cross-ratio of the attaching points traces out the F-curve. We call the chains legs of the F-curve and the component with the attaching points the spine. The numerical equivalence class of an F-curve is determined by the partition of \( \{1, \ldots, n\} \) into four subsets indicating which marked points lie on which leg.

By the F-curve corresponding to the partition \( \{1, \ldots, n\} = \sqcup_{j=1}^4 N_j \), we mean the numerical equivalence class of F-curves with points indexed by \( N_j \) on the \( j \)th leg. By a symmetric F-curve we mean the image of an F-curve in \( \overline{M}_{0,n} / S_n \). In this case only the number of marked points on each leg is relevant so we write \( n_j := |N_j| \) and speak of the symmetric F-curve corresponding to \( n = n_1 + n_2 + n_3 + n_4 \). By abuse of language we also refer to an F-curve in \( \overline{M}_{0,n} \) corresponding to \( n = n_1 + n_2 + n_3 + n_4 \), by which we mean any F-curve whose image in \( \overline{M}_{0,n} / S_n \) corresponds to this partition.

Proposition 4.2. Let \( L = (x_1, \ldots, x_n) \in \Delta(d + 1, n) \). Any F-curve corresponding to \( \sqcup_{j=1}^4 N_j \) such that \( \sum_{i \in N_j} x_i = \alpha_j, j = 1, \ldots, 4, \) for some integers \( \alpha_j \geq 0 \) with \( \sum_1^4 \alpha_j = d \) is contracted by \( \varphi : \overline{M}_{0,n} \to V_{d,n} / _L \text{SL}_{d+1} \).

Proof. By the explicit description in §4.1, \( \varphi \) sends a marked curve \((C, p_1, \ldots, p_n) \in \overline{M}_{0,n} \) with irreducible decomposition \( C = C_1 \cup \cdots \cup C_r \) to a semistable configuration of points lying on a quasi-Veronese curve \( V = V_1 \cup \cdots \cup V_r \). This corresponds to
a partition $d = d_1 + \cdots + d_r$ where each $V_i$ is either a point ($d_i = 0$) or a rational normal curve of degree $d_i \geq 1$. Recall from §4.2 that $L = (x_1, \ldots, x_n)$ assigns weight $x_i$ to the marked point $p_i$ and a configuration in $\mathbb{P}^d$ is semistable if and only if there is weight at most 1 at a point, 2 on a line, 3 on a plane, etc.

It follows that when a connected set of components $\cup_{i \in I} C_i$, $I \subseteq \{1, \ldots, r\}$ carries marked points of total weight $> m$ then the corresponding curve $\cup_{i \in I} V_i$ must have degree $\geq m$ since otherwise it would be contained in a $\mathbb{P}^{m-1}$, violating semistability. Since $V$ has degree $d$, the spine will be contracted if the degrees of the legs add up to $d$. By the preceding observation, this occurs if the weight of marked points on the $j$th leg of the F-curve is $> \alpha_j$ for integers $\alpha_j \geq 0$ satisfying $\sum_1^n \alpha_j = d$. In fact, it is enough to require $\sum_{i \in V_j} x_i \geq \alpha_j$ because if the weight on a leg is exactly $\alpha_j$ then the configuration is strictly semistable and the GIT quotient identifies it with a configuration in which the spine has been contracted.

In the case of a symmetric linearization we can say more.

**Proposition 4.3.** For $L = (\frac{d-1}{n}, \ldots, \frac{d-1}{n})$, the morphism $\varphi : \overline{M}_{0,n} \to V_{d,n} \simeq L_{SL_d + 1}$ contracts F-curves corresponding to $n = \sum_1^n n_j$, $n_1 \leq \cdots \leq n_4$, satisfying either of the following conditions, and no others:

- $n_j \geq \frac{n \alpha_j}{d+1}$, $j = 1, \ldots, 4$, for integers $0 \leq \alpha_1 \leq \cdots \leq \alpha_4$ with $\sum_1^4 \alpha_j = d$
- $n_j \leq \frac{n \beta_j}{d+1}$, $j = 1, \ldots, 4$, for integers $1 \leq \beta_1 \leq \cdots \leq \beta_4$ with $\sum_1^4 \beta_j = d + 2$

**Proof.** The contractions described by the first inequality follow from Proposition 4.2 and are equivalent to the spine being given degree zero, since the weight on the $j$th leg in this case is $\frac{d-1}{n} n_j$. The only other way for the F-curve to be contracted is if the spine is given positive degree but varying the cross-ratio of its attaching points no longer yields a 1-dimensional family of orbit closures. Since we can assume each orbit closure is full-dimensional, this occurs precisely when the configuration of points in $\mathbb{P}^d$ is supported on a collection of at most $d + 2$ points, and we claim this occurs precisely when the second inequality in this proposition is satisfied. Since the weight on the $j$th leg is $\frac{d-1}{n} n_j$, the inequality $n_j \leq \frac{n \beta_j}{d+1}$ is equivalent to the $j$th leg having degree $\leq \beta_j - 1$ by the same argument as in the proof of Proposition 4.2. Now each leg of an F-curve is a chain of $\mathbb{P}^1$’s in which each component carries a single marked point except for the last component which carries two marked points. Therefore the image of the leg having degree $\leq \beta_j - 1$ is equivalent to the image carrying $\leq \beta_j$ distinct points. Thus the condition $\sum_1^4 \beta_k = d + 2$ translates to the image of the entire curve carrying $\leq d + 2$ distinct points, as claimed. \hfill \Box

### 4.4. Hassett spaces

Following [Has03], for $A := \{a_1, \ldots, a_n\} \in \mathbb{Q}^n \cap (0,1)^n$ such that $\sum a_i \geq 2$ we denote by $\overline{M}_{0,A}$ the compact moduli space of $n$-pointed nodal rational curves for which $K + a_1p_1 + \cdots + a_np_n$ is ample and such that the $p_i$ are smooth and $\sum_{i \in I} a_i \leq 1$ if $\{p_i\}_{i \in I}$ coincide. Note that setting $A = \{1, \ldots, 1\}$ gives the usual moduli space $\overline{M}_{0,n}$. Note also that $L = (x_1, \ldots, x_n) \in \Delta(d+1,n) \cap \mathbb{Q}^n_{\geq 0}$ is a valid choice of weight data for a Hassett space since $x_i \in \mathbb{Q} \cap (0,1)$ and $d \geq 1$ so $\sum x_i \geq 2$. For any vector of weights $A' = \{a'_1, \ldots, a'_n\}$ with $a'_i \leq a_i$ there is a birational contraction $\overline{M}_{0,A} \to \overline{M}_{0,A'}$ [Has03, Theorem 4.1].

**Proposition 4.4.** The morphism $\varphi : \overline{M}_{0,n} \to V_{d,n} \simeq L_{SL_d + 1}$ factors through $\overline{M}_{0,L}$. 

Proof. By a result of Alexeev (see [Fak09, Lemma 4.6]) it is enough to show that every F-curve contracted by the Hassett morphism $M_{0,n} \to \overline{M}_{0,L}$ is also contracted by $\varphi$. The Hassett morphism contracts precisely those F-curves satisfying
\begin{equation}
\sum_{i \in N_1} x_i + \sum_{i \in N_2} x_i + \sum_{i \in N_3} x_i \leq 1
\end{equation}
where without loss of generality the leg $N_4$ carries the most weight. Indeed, by the ampleness condition in Hassett’s definition of stability a component with one node must have weight $> 1$, so if (2) is satisfied then the first three legs must be contracted and all that remains is the spine and the fourth leg—but then the spine becomes a component with one node and $\leq 1$ weight so it too is contracted. Conversely, if (2) does not hold then either all three of the first legs are contracted but the spine remains since it then carries $> 1$ weight, or at least one of the first three legs remains but then the spine remains as well since it has two nodes so is Hassett-stable regardless of the weight it carries.

The total weight of points is $d + 1$, so (2) is equivalent to $\sum_{i \in N_1} x_i \geq d$, thus by taking $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, 0, d)$ in Proposition 4.2 we see that all the F-curves contracted by Hassett are contracted by $\varphi$ as well. \qed

Note that for $L \in \Delta(d + 1, n)$ the Hassett space $\overline{M}_{0,L}$ parametrizes trees of $\mathbb{P}^1$s with at most $d$ leaves. Indeed, as noted above a component with exactly one node must carry $> 1$ weight for the curve to be stable. But there is $d + 1$ weight total, so there is not enough weight to have $\geq d + 1$ such components.

4.5. Twisted cubics. We conclude this section with an example: the case $d = 3$. By Proposition 4.4 the morphism $\varphi$ factors as $M_{0,n} \to \overline{M}_{0,L} \to V_{3,n} \sslash \mathbb{P}^1$, and the first arrow is well-understood so we focus on the second. Here $\overline{M}_{0,L}$ parametrizes trees with at most three leaves, each leaf with weight $> 1$. Each Hassett-stable curve $C$ gets mapped to a degree 3 quasi-Veronese curve, namely a degeneration of a twisted cubic. If $C$ is smooth then it gets sent to a twisted cubic and $\varphi$ sends the marked points to the configuration of points on this cubic arising from the Veronese map. If $C$ is nodal then any component with weight $> 2$ gets sent to a plane conic in $\mathbb{P}^3$ and any component with weight $> 1$ gets sent to a line. All the marked points on components that are contracted lie at the corresponding singular point of the image curve in $\mathbb{P}^3$. In Figure 1 we show all the types of situations that can occur. The number on a component indicates the degree of its image. Note that in all cases the image curve is non-degenerate, since if it were contained in a $\mathbb{P}^2$ then the configuration of points would be unstable.

5. Conformal Blocks

We refer to the literature for background on conformal blocks, e.g. [TUY89, BL94, Loo95, Bea96]. As is mentioned in the introduction, the main point is that for each Lie algebra $\mathfrak{g}$ and $n$-tuple of dominant weights there is a vector bundle defined on $\overline{M}_{g,n}$. We focus on the conformal blocks bundles $D^k_L$ defined on $\overline{M}_{0,n}$ corresponding to $\mathfrak{sl}_n$ with weights $(\omega_k, \ldots, \omega_k)$. These are line bundles (by [Fak09, §5.2.5] or by the Verlinde formula) and they are the focus of the paper [AGSS10].
5.1. Fakhruddin’s formula. In [Fak09], Fakhruddin computes the Chern classes of the conformal blocks vector bundles on $\overline{M}_{0,n}$. This is then used to find the intersection numbers between their determinants and F-curves. The formula in the case of the line bundles $D_{sl}^k$ is as follows [Fak09, Proposition 5.2]:

**Proposition 5.1.** Let $F$ be an F-curve corresponding to $n = n_1 + \cdots + n_4$. Let $\nu_j = kn_j \pmod{n} \in \{0, 1, \ldots, n-1\}$ and $\nu_M = \max\{\nu_j\}, \nu_m = \min\{\nu_j\}$. Then

\[
D_{sl}^k \cdot F = \begin{cases}
\nu_m, & \text{if } \sum_1^4 \nu_j = 2n \text{ and } \nu_M + \nu_m \leq n \\
n - \nu_M, & \text{if } \sum_1^4 \nu_j = 2n \text{ and } \nu_M + \nu_m \geq n \\
0, & \text{otherwise}
\end{cases}
\]

From this case of Fakhruddin’s formula we deduce the following:

**Proposition 5.2.** The complete linear system $|D_{sl}^k|$ contracts F-curves $n = \sum_1^4 n_j$, $n_1 \leq \cdots \leq n_4$, satisfying either of the following conditions:

- $n_j \geq \frac{n\alpha_j}{k}$, $j = 1, \ldots, 4$, for integers $0 \leq \alpha_1 \leq \cdots \leq \alpha_4$ with $\sum_1^4 \alpha_j = k - 1$
- $n_j \leq \frac{n\beta_j}{k}$, $j = 1, \ldots, 4$, for integers $1 \leq \beta_1 \leq \cdots \leq \beta_4$ with $\sum_1^4 \beta_j = k + 1$

**Proof.** By Fakhruddin’s formula, $D_{sl}^k \cdot F = 0$ if and only if $\sum_1^4 \nu_j \neq 2n$ or $\nu_j = 0$ for some $j$, where $\nu_j = kn_j \pmod{n}$. If there are $\alpha_j \geq 0$ with $kn_j \geq n\alpha_j$, then $\nu_j \leq kn_j - n\alpha_j$. Together with the fact that $\sum_1^4 \alpha_j = k - 1$, we then have

\[
\sum_1^4 \nu_j \leq \sum kn_j - \sum n\alpha_j = kn - n(k - 1) = n \neq 2n
\]
so \( D_{k,n}^F \cdot F = 0 \). Similarly, if there are \( \beta_j \geq 1 \) with \( kn_j \leq n \beta_j \) then either \( \nu_j = 0 \) for some \( j \) or \( \nu_j \geq kn_j - n(\beta_j - 1) \). Then \( \sum_1^4 \beta_j = k + 1 \) implies that

\[
\sum_1^4 \nu_j \geq \sum kn_j - \sum n(\beta_j - 1) = kn - n(k + 1) + 4n = 3n \neq 2n
\]

so again \( D_{k,n}^F \cdot F = 0 \). \( \square \)

For our present purposes we only need to know that \( |D_{k,n}^F| \) contracts the F-curves described by this proposition, but in fact we will see later (Corollary 5.3) that no other F-curves are contracted, so this proposition precisely describes the zeroes of the Fakhruddin formula in the case of \( D_{k,n}^F \).

5.2. Proof of Theorem 1.2. Recall that for symmetric linearization \( L \) we denote the polarization on \( V_{d,n} \parallel L_{SL_{d+1}} \) by \( L_d \). The line bundle \( \varphi^* L_d \) is nef and \( S_n \)-invariant, so it spans a ray in \( \text{SymNef}(\overline{M}_{0,n}) \subseteq N^1(\overline{M}_{0,n})^{SL_n} \). This latter vector space has dimension \( \rho := \left\lfloor \frac{n}{d+1} \right\rfloor - 1 \) [AGSS10, §2.2.1]. Each linearly independent curve contracted by \( \varphi \) forces \( \varphi^* L_d \) to lie on a face of the symmetric nef cone of one higher codimension. Thus if we find \( \rho - 1 \) independent curves that are contracted then \( \varphi^* L_d \) will necessarily span an extremal ray of \( \text{SymNef}(\overline{M}_{0,n}) \) and this set of curves uniquely determines the ray.

The conformal blocks line bundle \( D_{d+1}^{\text{cv}} \) is also nef and \( S_n \)-invariant, and in [AGSS10] a set of \( \rho - 1 \) independent symmetric F-curves that it contracts is described. Thus to show that \( \varphi^* L_d \) and \( D_{d+1}^{\text{cv}} \) span the same ray it is enough to show that \( \varphi \) contracts this same set of F-curves. The set used in [AGSS10] depends on certain divisibility relations between \( d + 1 \) and \( n \), but in each case one can verify that the curves it contains fall into the framework of Proposition 5.2, so that by Proposition 4.3 they are contracted by \( \varphi \).

For example, when \( (d + 1)n \) write \( q := \left\lceil \frac{n}{d+1} \right\rceil \) and let \( F_{a,b,c} \) denote a symmetric F-curve corresponding to \( n = a + b + c + (n - (a + b + c)) \). The family is then given by all curves of the form \( F_{i,1,1} \) or \( F_{i,q,q} \) where \( 1 \leq i \leq \left\lfloor \frac{q}{d} \right\rfloor - 1 \) and \( q \equiv (i + 1) \), with the last \( F_{i,q,q} \) curve removed [AGSS10, §6.2]. We claim the \( F_{i,1,1} \) are contracted by the first inequality in Proposition 4.3, whereas the \( F_{i,q,q} \) are contracted by the second inequality. For this range of \( i \) we have \( i \leq n - i - 2 \) so \( F_{i,1,1} \) corresponds to \( n = \sum_1^3 n_j \) with \( (n_1,n_2,n_3,n_4) = (1,1,i,n - i - 2), n_1 \leq \cdots \leq n_4 \). Setting \( (\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (0,0,\left\lfloor \frac{q}{d} \right\rfloor, \frac{d}{d} - \left\lfloor \frac{q}{d} \right\rfloor) \) we only have to check that \( n_j \geq q \alpha_j \) for \( j = 3, 4 \). For \( j = 3 \) this is obvious: \( n_3 = \frac{q}{d} \). For \( j = 4 \) we have to verify the inequality \( n - i - 2 \geq q(d - \left\lfloor \frac{q}{d} \right\rfloor) \), or equivalently, using that \( n = q(d + 1) \), the inequality \( q \frac{d - 1}{d} \geq i + 2 - q \). Now in general \( \left\lceil \frac{d}{q} \right\rceil \geq i + 2 - q \), but the hypothesis \( q \nmid (i+1) \) implies the stronger inequality \( \left\lceil \frac{d}{q} \right\rceil \geq \frac{i+2-q}{q} \), from which the desired inequality immediately follows. For the \( F_{i,q,q} \) curves we have \( (n_1,n_2,n_3,n_4) = (i,n - i - 2,q,q) \), so we set \( (\beta_1,\beta_2,\beta_3,\beta_4) = (\left\lfloor \frac{q}{d} \right\rfloor, d - \left\lfloor \frac{q}{d} \right\rfloor, 1,1) \) and must check \( n_j \leq q \beta_j \) for \( j = 1,2,3,4 \). This is automatic for \( j = 1,3,4 \), and for \( j = 2 \) we must verify \( n - i - 2q \leq q(d - \left\lfloor \frac{q}{d} \right\rfloor) \).

This latter inequality boils down to \( \frac{q}{d} \leq \frac{q}{q} + 1 \), which holds even without the hypothesis that \( q \nmid (i+1) \).

Thus we have verified that the family of \( \rho - 1 \) curves is contracted by \( \varphi \) when \((d + 1)n \). The remaining cases [AGSS10, Definitions 7.1 and 9.1] are similar. \( \square \)
5.3. Consequences. We immediately derive two results from Theorem 1.2 and the work leading up to it. One is that since $D_{d+1}^{\mathfrak{sl}_n}$ and $\varphi^*L_d$ span the same ray in $N^1(M_{0,n})$ and the former spans an extremal ray in the symmetric nef cone, the latter must do so as well. This is stated formally in Corollary 1.3.

A second consequence is that $D_{d+1}^{\mathfrak{sl}_n}$ and $\varphi^*L_d$ contract the same set of $F$-curves, but Proposition 4.3 completely describes that set of curves, so we see that the list in Proposition 5.2 must be complete as well:

**Corollary 5.3.** The complete linear system $|D_k^{\mathfrak{gl}_n}|$ does not contract any further $F$-curves than those listed in Proposition 5.2.

A third consequence is described in the following section.

6. Gale Duality

6.1. Background. The Gale transform, sometimes called *association*, is a classical construction associating to a configuration of $n$ sufficiently general points in $\mathbb{P}^d$, up to projectivity, a configuration of $n$ general points in $\mathbb{P}^{n-d-2}$, up to projectivity. In [DO88, Corollary III.1] this was extended, using GIT, to partial compactifications of the space of general configurations, yielding an isomorphism of GIT quotients

$$(\mathbb{P}^d)^n // \mathbb{SL}_{d+1} \cong (\mathbb{P}^{n-d-2})^n // \mathbb{SL}_{n-d-1}$$

where $L$ denotes a symmetric linearization. Note that the space of linearizations on the left-hand side is the hypersimplex $\Delta(d+1,n)$, whereas on the right-hand side it is the hypersimplex $\Delta(n-d-1,n)$. However, these two polytopes are naturally isomorphic: the former is the convex hull of all $\binom{n}{d+1}$ vectors in $\{0,1\}^n$ with 1 occurring in $d+1$ entries and the latter is the convex hull of those with 0 occurring in $d+1$ entries. Under this identification it was shown in [Hu05b] that the above isomorphism of GIT quotients applies for arbitrary linearizations, not just symmetric ones. By [Kap93, Corollary 2.3.14] the Gale transform extends to Chow quotients as well.

Perhaps the easiest way to describe the Gale transform is in terms of Grassmannians via the Gelfand-MacPherson correspondence [GM82]. For both Chow and GIT quotients there is an isomorphism

$$(\mathbb{P}^d)^n // \mathbb{SL}_{d+1} \cong \text{Gr}(d+1,n)/(\mathbb{C}^*)^n$$

coming from the fact that both sides may be viewed as double quotients of the same space [Kap93, §2.2]. Indeed, $(\mathbb{C}^*)^n$ acts on the space of matrices $M_{d+1\times n}$ by scaling each column, and $\text{GL}_{d+1}$ acts on the left by multiplication. The quotient by the torus first then $\text{GL}_{d+1}$ is the left-hand side of this isomorphism, and the quotient by $\text{GL}_{d+1}$ then the torus is the right-hand side. In this framework the Gale transform may then be viewed as an isomorphism

$$\text{Gr}(d+1,n)//(\mathbb{C}^*)^n \cong \text{Gr}(n-d-1,n)/(\mathbb{C}^*)^n$$

which is obtained by simply taking orthogonal complements. One sees immediately from this description that the Gale transform is involutive.

6.2. Configurations on curves. As is described in [EP00], Goppa recognized that the Gale transform of a configuration of $n$ distinct points supported on a rational normal curve in $\mathbb{P}^d$ is a configuration supported on a rational normal curve in $\mathbb{P}^{n-d-2}$. He did this in the context of algebraic coding theory to show that the
dual of a Goppa code is a Goppa code. Thus the image under the Gale transform of \( U_{d,n}/\SL d+1 \) is contained in \( U_{n-d-2,n}/\SL n-d-1 \), and for the same reason it sends \( U_{n-d-2,n}/\SL n-d-1 \) to a subset of \( U_{d,n}/\SL d+1 \). But the Gale transform is an involutive isomorphism so the arrows in the top row of the following commutative diagram must be isomorphisms:

\[
\begin{array}{ccc}
U_{d,n}/\SL d+1 & \longrightarrow & U_{n-d-2,n}/\SL n-d-1 \\
\downarrow & & \downarrow \\
V_{d,n}/_L\SL d+1 & \longrightarrow & V_{n-d-2,n}/_L\SL n-d-1 \\
\downarrow & & \downarrow \\
(\mathbb{P}^d)^n/_{L}\SL d+1 & \longrightarrow & (\mathbb{P}^{n-d-2})^n/_{L}\SL n-d-1 \\
\end{array}
\]

By Lemma 2.3 we can view \( V_{d,n} \) as the topological closure of \( U_{n,d} \) so the Gale transform restricts to give maps as in the middle row of this diagram, and for the same reason they must be isomorphisms. Therefore, Goppa’s result extends to quasi-Veronese curves: the Gale transform of a configuration supported on a quasi-Veronese curve lies on a quasi-Veronese curve. Note that in [EP00, Corollary 3.2] it is shown that Goppa’s result extends to non-Gorenstein curves, but our quasi-Veronese curves include examples of non-Gorenstein curves. For instance, a spatial triple point as in Figure 1 is not Gorenstein.

6.3. Conformal blocks symmetry. It is well-known that conformal blocks satisfy \( D_{k}^{\mathfrak{sl}_n} = D_{n-k}^{\mathfrak{sl}_n} \). One way to see this is by the symmetry of the Dynkin diagram

\[
\circ - \circ - \cdots - \circ - \circ
\]

for \( \mathfrak{sl}_n \). From Theorem 1.2 it then follows that if

\[
\begin{align*}
\varphi_1 : \overline{M}_{0,n} & \rightarrow V_{d,n}/_L\SL d+1 \\
\varphi_2 : \overline{M}_{0,n} & \rightarrow V_{n-d-2,n}/_L\SL n-d-1
\end{align*}
\]

then the pulled-back symmetric polarizations \( \varphi_1^*\mathcal{L}_d \) and \( \varphi_2^*\mathcal{L}_{n-d-2} \) span the same ray in \( N^1(\overline{M}_{0,n}) \). Thus \( \varphi_1 \) and \( \varphi_2 \) have the same Stein factorizations up to a finite map, so the normalizations of their images are isomorphic:

\[
(V_{d,n}/_L\SL d+1)^\nu \cong (V_{n-d-2,n}/_L\SL n-d-1)^\nu
\]

This proves Corollary 1.4.

As noted in the introduction, we believe (though have been unable to prove) that the morphisms in Theorem 1.1 have connected fibers. If this is the case, then the above GIT quotients are normal so we get an isomorphism \( V_{d,n}/_L\SL d+1 \cong V_{n-d-2,n}/_L\SL n-d-1 \). Therefore, with this assumption on the fibers we derive the same quasi-Veronese generalization of Goppa’s result about the Gale transform as above.

6.4. Self-association of \( \overline{M}_{0,2m} \). Consider configurations of \( 2m \) points in \( \mathbb{P}^{m-1} \). Here the source and target are the same, so the Gale transform provides an involution

\[
\Gamma \in \text{Aut}(\mathbb{P}^{m-1})^{2m}/_C\SL m
\]

Fixed points of this automorphism are called self-associated configurations. Such configurations have been studied classically. See [DO88, EP00] and the references
therein, as well as [Kap93, Corollary 2.3.10]. It is proven in [Fla98] that any configuration of $2m$ distinct points on a rational normal curve of degree $m - 1$ is self-associated, so $\Gamma$ restricts to the identity on $U_{m-1,2m}/\text{SL}_m \subseteq (\mathbb{P}^{m-1})^{2m}/\text{ChSL}_m$. It therefore is the identity on the topological closure of this locus as well. But the closure is precisely the Chow quotient $V_{m-1,2m}/\text{ChSL}_m$, which by Theorem 1.1 is isomorphic to $\mathcal{M}_{0,2m}$. Thus, under the identification $\mathcal{M}_{0,2m} \cong V_{m-1,2m}/\text{ChSL}_m \hookrightarrow (\mathbb{P}^{m-1})^{2m}/\text{ChSL}_m$ we see that each stable curve $(C,p_1,\ldots,p_{2m})$ corresponds to a self-associated configuration.

References

[AS08] Alexeev, V. and D. Swinarski. “Nef divisors on $\mathcal{M}_{0,n}$.” math.AG/0812.0778
[AGSS10] Arap, M., Gibney, A., Stankewicz, J., and D. Swinarski. “$sl_1$ level 1 conformal blocks divisors on $\mathcal{M}_{0,n}$.” To appear in Int. Math. Res. Notices, math.AG/1009.4664.
[AL02] Avritzer, D. and H. Lange. “The moduli space of hyperelliptic curves and binary forms.” Math. Z. 242 no. 4 (2002), 615–632.
[Bea96] Beauville, A. “Conformal Blocks, Fusion Rules, and the Verlinde formula.” Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, 9 (1996), 75–96.
[BL94] Beauville, A., and Y. Laszlo. “Conformal Blocks and Generalized Theta Functions.” Commun. Math. Phys. 164 (1994), 385–419.
[Bol10] Bolognesi, M. “Forgetful linear systems on the projective space and rational normal curves over $\mathcal{M}_{0,2n}$.” To appear in Bull. Lond. Math. Soc. math.AG/0909.0151.
[DM86] Deligne, P. and G. Mostow. “Monodromy of hypergeometric functions and non-lattice integral monodromy.” Publications of IHES 63 (1986), 5–90.
[DO88] Dolgachev, I. and D. Ortland. “Points sets in projective spaces and theta functions.” Asterisque 165 (1988), 1–210.
[EP00] Eisenbud, D. and S. Popescu. “The projective geometry of the Gale transform.” J. Algebra 230 no. 1 (2000), 127–173.
[Fak09] Fakhruddin, N. “Chern Classes of Conformal Blocks on $\mathcal{M}_{0,n}$.” math.AG/0904.2918
[Fla98] Flamini, F. “Towards an inductive construction of self-associated sets of points.” Le Matematiche LIII (1998), 33–41.
[GLS82] Gelfand, I. and R. MacPherson. “Geometry in Grassmannians and a generalization of the dilogarithm.” Adv. in Math. 44 (1982), 279–312.
[GHP88] Gerritzen, L., Herrlich, F., and M. van der Put. “Stable n-pointed trees of projective lines.” Indag. Math. 50 (1988), 131–163.
[GG10] Giansiracusa, N. and W.D. Gillam. “On Kapranov’s description of $\mathcal{M}_{0,n}$ as a Chow quotient.” Preprint available at author’s website.
[GS09] Giansiracusa, N. and M. Simpson. “Conic Compactifications of $\mathcal{M}_{0,n}$.” Int. Math. Res. Notices Vol. 2010, doi:10.1093/imrn/rnn228 (2010).
[EGA60] Grothendieck, A. and J. Dieudonné, Éléments de géométrie algébrique. Publ. Math. IHES, 1960.
[Har77] Hartshorne, R. Algebraic Geometry. Springer-Verlag GTM 52, 1977.
[Has03] Hassett, B. “Moduli spaces of weighted pointed stable curves.” Adv. Math. 173 no. 2 (2003), 316–352.
[HMSV10] Howard, B., Millson, J., Snowden, A. and R. Vakil. “The ideal of relations for the ring of invariants of n points on the line.” To appear in J. Eur. Math. Soc. math.AG/0909.3230.
[Hu99] Hu, Y. “Moduli spaces of stable polygons and symplectic structures on $\mathcal{M}_{0,n}$.” Compos. Math. 118 (1999), 159–187.
[Hu05a] Hu, Y. “Topological Aspects of Chow Quotients.” *J. Differential Geometry* 69 (2005), 399–440.

[Hu05b] Hu, Y. “Stable configurations of linear subspaces and quotient coherent sheaves.” *Q. J. Pure Appl. Math.* 1 (2005), 127–164.

[KM96] Kapovich, M. and J. Millson. “The symplectic geometry of polygons in Euclidean space.” *J. Differential Geom.* 44 no. 3 (1996), 479–513.

[Kap93] Kapranov, M. “Chow quotients of Grassmannians, I.” *Adv. Sov. Math.* 16 no. 2 (1993), 29–110.

[Kol96] Kollár, J. *Rational curves on algebraic varieties.* Springer, Secaucus, NJ, 1996.

[Loo95] Looijenga, E. “Conformal blocks revisited.” math.AG/0507086.

[GIT94] Mumford, D., Fogarty, J. and F. Kirwan. *Geometric Invariant Theory.* Third Edition. Springer, 1994.

[Pol95] Polito, M. “SL(2, C)-quotients de (P^1)^n.” *C.R. Acad. Sci Paris* 321 Série I (1995), 1577-1582.

[Sim08] Simpson, M. “On Log Canonical Models of the Moduli Space of Stable Pointed Genus Zero Curves.” Ph.D dissertation, Rice University, 2008.

[Tha96] Thaddeus, M. “Geometric invariant theory and flips.” *J. Amer. Math. Soc.* 9 no. 3 (1996), 691-723.

[TUY89] Tsuchiya, A., Ueno, K., and Y. Yamada. “Conformal field theory on universal family of stable curves with gauge symmetries.” *Adv. Stud. Pure Math.* 19 (1989), 459–506.