Koszul Duality for Semidirect Products and Generalized Takiff Algebras

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Abstract We obtain Koszul-type dualities for categories of graded modules over a graded associative algebra which can be realized as the semidirect product of a bialgebra coinciding with its degree zero part and a graded module algebra for the latter. In particular, this applies to graded representations of the universal enveloping algebra of the Takiff Lie algebra (or the truncated current algebra) and its (super)analogues, and also to semidirect products of quantum groups with braided symmetric and exterior module algebras in case the latter are flat deformations of classical ones.

Keywords Module algebras · Semi-direct products · Graded algebras · Koszul duality

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1 Introduction

Koszul property, as defined in [17], plays an important role in modern representation and structural theory of graded associative algebras. It typically occurs in the following setting. Let \( A \) be a \( \mathbb{Z} \)-graded associative algebra over a field \( k \) whose non-zero homogeneous components appear only in non-negative degrees and are finite dimensional and whose

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degree zero part $A_0$ is semisimple. Consider the category of locally finite dimensional graded $A$-modules, which can be regarded as a non-semisimple deformation of the semisimple category of finite dimensional $A_0$-modules. Koszulity of $A$ is then formulated via the requirement that simple graded $A$-modules have so-called linear projective resolutions. One consequence of Koszulity is a derived equivalence between the bounded derived category of finitely generated graded $A$-modules and a similar category for the quadratic dual of $A$ (see [2]). This classical Koszul duality has numerous generalizations and extensions (see [12, 14, 16] and references therein).

However, it often happens that one needs to consider graded modules over a graded algebra whose degree zero part is not semisimple. A typical example is the current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ of a finite dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ which is intimately connected to, in particular, the quantum affine algebra corresponding to $\mathfrak{g}$. The universal enveloping algebra of $\mathfrak{g}[t]$ is naturally $\mathbb{Z}$-graded, with the degree zero part being isomorphic to the enveloping algebra of $\mathfrak{g}$. The latter is very far from being semisimple. However, the category of finite dimensional $\mathfrak{g}$-modules is semisimple. This allows one to associate a Koszul algebra with the full subcategory of the category of finitely generated graded locally finite dimensional $\mathfrak{g}[t]$-modules whose objects are annihilated by the Lie ideal $\mathfrak{g} \otimes t^2 \mathbb{C}[t]$ of $\mathfrak{g}[t]$ (cf. [6, 7]). Alternatively, such modules can be regarded as modules over the Takiff Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$ ([18]) which is naturally isomorphic to $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$. The interest in that category stems from the observation that $q = 1$ limits of celebrated Kirillov-Reshetikhin modules over quantum affine algebras ([13]) and also of certain generalizations of Kirillov-Reshetikhin modules known as minimal affinizations ([5]) are its objects, provided that $\mathfrak{g}$ is of a classical type. Once one has a Koszul algebra, its quadratic dual is also Koszul, and the natural question is whether one can find a Lie-theoretic background for a representation theory of that quadratic dual.

In the present paper we answer that question and establish the most general framework in which such a question can be answered. Namely, it turns out that for the Takiff Lie algebra and its generalizations, the “Koszul graded dual” is, morally, a certain Lie superalgebra which is isomorphic to our initial Lie algebra as a $\mathfrak{g}$-module. This is a special case of the following setup. Suppose that we have a $\mathbb{Z}$-graded algebra $A$ such that $A_0$ is a bialgebra and $A$ is a semidirect product of $A_0$ with a $\mathbb{Z}$-graded right $A_0$-module algebra $H$. We consider a category of $\mathbb{Z}$-graded left $A$-modules whose graded pieces are in a suitable “underlying” category $\mathcal{C}$ of $A_0$-modules (for example, a semisimple category of finite dimensional $A_0$-modules if it is available). Assuming that $H$ is Koszul, we establish (see Theorem 4.5 and its Corollary) a Koszul-type duality between that category and a category of $\mathbb{Z}$-graded left modules with graded pieces in $\mathcal{C}$ over the algebra $A^{\otimes}$, which is the semidirect product of the quadratic dual of $H$ with the bialgebra $A^{\text{cop}}$ which coincides with $A$ as an algebra and has the opposite comultiplication. The case of the Takiff algebra described above corresponds to taking $A_0$ to be the enveloping algebra of $\mathfrak{g}$, $H$ to be the symmetric algebra of the adjoint representation of $\mathfrak{g}$, and $\mathcal{C}$ to be the category of finite dimensional $\mathfrak{g}$-modules.

The paper is organized as follows. In Section 2 we collected basic generalities on graded algebras and categories of graded modules. Section 3 contains results pertaining to module algebras and semidirect products that are needed for our construction. In Section 4 we establish Koszul duality in the setting of semidirect products of bialgebras with their module algebras. In particular, it contains the main results of the paper (Theorem 4.5 and Corollary 4.6). Finally, Section 5 provides examples illustrating applications of our main result.