Park-and-Ride Facility Location Under Nested Logit Demand Function

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Abstract

Park and Ride (P&R) facilities are car parks at which users can transfer to public transportation to reach their final destinations. Commuters can use P&R facilities or choose to travel by car to their destinations, and individual choice behavior is assumed to follow a logit model. The P&R facility location problem locates a fixed number of P&R facilities among potential locations, maximizing the number of users of P&R facilities. The problem is formalized in a nonlinear optimization problem using a nested logit model. Assuming a strong condition of independent irrelevant alternatives, it is a generalized multinomial logit model. This paper solves the nonlinear optimization problem without the unrealistic independency assumption. To solve the nonlinear P&R facility location problem, we develop a rounding heuristic that identifies extremely fast a decent solution to a realistic scale of the problem. The rounding heuristic flips coins to locate facilities, with probabilities that are adaptively updated with the best known integer solution. The adaptive randomized rounding algorithm effectively avoids local optima to solve the P&R facility location problem. We also develop a genetic algorithm and compare its performance with that of the adaptive randomized rounding algorithm.

Keywords: park-and-ride facility location problem, decision-dependent demand, nested logit model, randomized rounding, genetic algorithm, meta-heuristic

1. Introduction

A transit facility is a place providing access to transit services [14]. Among such facilities, this paper focuses on park and ride (P&R) facilities, such as bus, train, and air mobility stations, where people drive and park their cars and transfer to public transportation. Since the construction cost of transit facilities is high and it is difficult to change their locations once they are built, the location and number of new transit facilities must be carefully determined through quantitative and qualitative analyses of related factors such as cost, demand, and service level. In particular, this paper assumes that a large number of new transit facilities are to be built because of the introduction of a new transportation mode. In this paper, the park

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and ride facility location problem (P&R FLP) finds a fixed number of optimal P&R locations maximizing the total transportation demand served by the P&R facilities (e.g., the sum of users or modal share of P&Rs), which we refer to as total utility.

The P&R FLP is a p-hub location problem to locate a fixed number (p) of facilities. (This paper denotes p by N to avoid confusion against probability p.) Goldman [15] first addressed the network hub location problem, and O’Kelly [37] presented the first recognized mathematical formulation for a hub location problem by studying airline passenger networks. Then, Campbell [9] presented a mixed-integer linear programming (MILP) formulation of the p-hub median problem. Based on the p-hub formulation, Aros-Vera et al. [3] proposed a p-hub approach that utilizes a spatial optimization model taking into account origin-destination (O-D) trips, as opposed to closedness to origins, and considers transportation demand to determine the proportion of users patronizing the facility.

Transportation demand is conventionally analyzed with a four-step model that comprises trip generation, trip distribution, mode choice, and route assignment [36]. The first two steps determine O-D trips. Then, given the O-D trips, the modal share of a particular transportation mode (i.e., the probability of choosing that mode) is usually analyzed with a multinomial logit (MNL) model because of this model’s flexibility and computational efficiency [35]. A distinguishing property of the MNL model is the independence of irrelevant alternatives (IIA): the ratio of the probabilities of choosing two alternatives is independent of other alternatives [33]. However, the IIA property is invalid if some alternatives share unobservable attributes [7]. As a result, the probabilities of choosing these alternatives sharing some (unmodeled) attributes are overestimated by the MNL model. The famous red/blue bus problem provides an example of overestimated probability when choosing buses [33].

The nested logit (NL) model was proposed to reflect correlation between alternatives [6, 34]. The NL model is generally represented by a tree structure that groups together alternatives with common attributes. For instance, bus and subway can be grouped as public transportation. If multiple P&R facilities are available (or reachable) to a person, these facilities are grouped in the same nest. The NL model can estimate modal shares better than the MNL model when customers choose a transit option among multiple transit options sharing common attributes.

Along with certain constraints, logit models can be used for the objective function of FLPs. Benati and Hansen [8] used an MNL model to formulate the objective function of a competitive FLP and computed its upper bounds. Aros-Vera et al. [3] utilized another MNL model to calculate decision-dependent demand for P&R facilities, which comprises their objective function. The P&R FLP was linearized as an MILP problem assuming the IIA property for MNL models. Jokar [21] modified the linearized model of Aros-Vera et al. [3] to formulate a facility and hub location problem. López-Ospina et al. [32] estimated customers’ demand with a constrained MNL model and solved an FLP by considering costs and quality of service.

Haase and Müller [17] pointed out limitations of the IIA property, specifically in spatial choice applications, and utilized a mixed MNL model for a free school-choice problem without any assumption of a
particular distribution of random utility. However, using the general random utility model, they simulated students based on population characteristics and solved the problem for the simulated students. Basciftci et al. [5] considered the demand uncertainty affected by facility location and proposed a distributionally robust optimization model that minimizes the worst-case costs.

As López-Ospina et al. [32] stated, the more accurate the estimation of transportation demand, the better the decision on facility locations. Because it is apparent that P&R facilities share common attributes, an NL model can estimate transportation demand more accurately than an MNL model. Thus, this paper proposes a P&R FLP with an objective function maximizing the total utility calculated by the NL model. Then, the results of the P&R FLP are compared to those of the P&R FLP with MNL model and differences between the NL model and the MNL model are discussed.

Figure 1 illustrates an example of a P&R FLP to select 4 vertiports for airport shuttle service based on passenger travel demand forecast by Korea Transport Database [28]. A vertiport is an airport for Urban Air Mobility, which is an emerging air transport mode serving passengers within/between urban area(s). The NL model determines proper locations of two vertiports in Gangnam (South of the River) and the other two in Gangbuk (North of the River) while the MNL model selects only one vertiport in Gangbuk. It is a consensus among transportation researchers in Korea that the NL model is more appropriate than the MNL model on passenger travel data such as Korea Transport Database [28], especially when the independence between alternatives cannot be ignored [41, 23, 38].

The P&R FLP with NL model is a nonlinear optimization problem while the P&R FLP with MNL model can be linearly formulated as an MILP, as in Aros-Vera et al. [3]. To solve the nonlinear optimization problem, we use a rounding heuristic. Since Lin and Vitter [30, 31] gave an elegant rounding technique for linear programming relaxations, there has been a series of developments in the design and analysis of rounding algorithms for FLPs [10, 12, 13, 69]. They solved the linear relaxation of the MILP formulation,
and then rounded the optimal LP solution into an integer solution. A randomized rounding method sets
the optimal fractional solution as a probability distribution of flipping coins to determine the values of the
binary variables that indicate whether to locate the facilities. Generalizing the probability distribution, we
develop three new rounding algorithms for the P&R FLP.

To avoid local optima, there have been adaptive heuristic approaches such as the greedy randomized
adaptive search procedure [25] and the adaptive randomized rounding procedure [11]. Our randomized
rounding procedure moves the generalized probability distribution, which we refer to as relaxed value dis-
tribution, in the adaptive randomized rounding framework. We also implement a genetic algorithm to solve
our P&R FLP and compared its performance to that of the rounding algorithms in terms of optimality and
computational efficiency.

In Section 2, the P&R FLP is proposed for the NL model. Section 3 presents three rounding algorithms
and a genetic algorithm to solve the P&R FLP for the NL model. In Section 4, numerical experiments are
performed to show the validity of the proposed meta-heuristic algorithms and compare their performance.
Also, a sensitivity analysis of correlation between P&Rs is presented. Finally, Section 5 summarizes the
findings of this paper and presents the conclusions.

2. Nested Logit Model of Park-and-Ride Utilization

The NL model is generally represented by a tree structure, in which alternatives with common attributes
are grouped. For instance, bus and subway can be grouped as public transportation. If multiple P&R
facilities are available to (or reachable by) a person, these P&Rs are grouped in the same nest. Figure 2
shows an example of a nesting structure that consists of only private car and P&Rs. The NL model can
estimate modal shares better than can the MNL model when customers choose a transit option among
multiple transit options sharing common attributes.

For simplicity of modelling, this paper assumes that people use a private car or P&R and that no other
transportation mode is available. However, the proposed model can be extended to a general model with multiple transportation modes such as bus and subway. Private cars move directly from origin to destination. In contrast, people who want to use P&R move from their origin to a P&R facility and transfer to public transportation to reach their final destination. All P&Rs share common attributes except location.

As mentioned in Section 1, MNL and NL models are widely used to analyze transportation demand, which is referred to as total utility of P&R in this paper. Both models calculate the total utility using the observed utility of alternatives \( V \), which generally consists of travel time and cost. Holguín et al. [19] proposed a generalized utility of car and P&R as follows:

\[
V_c^j = c_{time}^{TT_c} + c_{cost}^{TC_c} + c_{dist}^{TD_c}
\]

\[
V_p^{ij} = c_{time}^{TT_p} + c_{wait}^{WT} + c_{cost}^{TC_p} + c_{dist}^{TD_p}.
\]

\( V_c^j \) and \( V_p^{ij} \) denote the (observed) utility of using a private car and P&R \( i \) for O-D trip \( j \), and \( c_{time} \), \( c_{cost} \), \( c_{dist} \), and \( c_{wait} \) denote the coefficients for (in-vehicle) travel time, travel cost, travel distance, and waiting (i.e., out-of-vehicle) time, respectively. \( TT_c^j \), \( TC_c^j \), and \( TD_c^j \) denote travel time, cost, and distance of using a private car for O-D trip \( j \), and \( TT_p^{ij} \), \( WT^{ij} \), \( TC_p^{ij} \), and \( TD_p^{ij} \) denote travel time, waiting time, cost, and distance of using P&R \( i \) for O-D trip \( j \), respectively.

The P&R FLP in this paper determines a fixed number (\( N \)) of optimal P&R facilities to be built. Parameter \( N \) is determined with available budget for P&R, by dividing the available budget by the average cost of building a P&R facility. A set of candidate locations (\( P \)) is given, like the illustrative example of vertiports shown in Fig. 1. The \( N \) facilities selected for P&R will be indicated by binary variables \( (x_i \in \{0, 1\} : i \in P) \) satisfying

\[
\sum_{i \in P} x_i = N.
\]

That is, \( x_i = 1 \) if facility \( i \) is selected, and \( x_i = 0 \) otherwise.

The objective function of the P&R FLP maximizes the total utility of P&R (i.e., the total number of P&R users). The number of P&R users for an O-D trip is the product of the total number of travelers of the O-D trip (\( R \)) and the probability of choosing P&Rs for the O-D trip (\( p^P \)). Note that \( R \) is given and multiple P&Rs can be available for an O-D trip. This paper uses the NL model to calculate the probability of choosing P&Rs because the NL model considers correlation between P&Rs. If there is no limit on the number of P&Rs to be built, the optimal combination of P&Rs includes all the candidates \( P \). Due to limited budget, however, only a subset of candidate P&Rs can eventually be built (i.e., \( N < |P| \)), and thus the utility of unselected P&R candidates must be excluded from the probability calculation.

The NL model uses the nesting structure shown in Fig. 2 where all P&Rs are in the same nest. Then, the probability of choosing a P&R for the O-D trip \( j \) is

\[
p_j^P = \frac{e^{\lambda P_j^P}}{e^{V_c^j} + e^{\lambda P_j^P}}, \quad \forall j \in T,
\]

(3)
where \( T \) denotes the set of all O-D trips, \( \lambda \) is a LogSum parameter, and \( \Gamma^p_j \) is the LogSum of the exponent of the P&R utilities for the O-D trip \( j \); i.e.,

\[
\Gamma^p_j = \ln \sum_{i \in P} e^{V^p_{ij}/\lambda} x_i, \quad \forall j \in T.
\]

The probability of choosing a private car for the O-D trip \( j \) is

\[
p^c_j = \frac{e^{V^c_j}}{e^{V^c_j} + e^{\lambda \Gamma^p_j}}, \quad \forall j \in T.
\]

The utility of P&R \( i \) (i.e., \( V^p_i \)) is not included for the calculation of \( \Gamma^p_j \) unless P&R \( i \) is selected by the P&R FLP. Note that \( p^p_i \) is the probability of choosing a P&R for O-D trip \( j \) without specifying the P&R facility. Then, the decision-dependent probability of choosing P&R \( i \) for O-D trip \( j \) is

\[
p^p_{ij} = p^p_i \frac{e^{V^p_{ij}/\lambda} x_i}{\sum_{k \in P} e^{V^p_{ik}/\lambda} x_k} = \frac{e^{\lambda \Gamma^p_j}}{e^{V^c_j} + e^{\lambda \Gamma^p_j}} \frac{e^{V^p_{ij}/\lambda} x_i}{\sum_{k \in P} e^{V^p_{ik}/\lambda} x_k}, \quad \forall i \in P, \forall j \in T.
\]

Thus, \( p^p_{ij} \) is nonzero if and only if \( x_i \) is not zero, and the objective function of the P&R FLP is

\[
\sum_{i \in P} \sum_{j \in T} R_j p^p_{ij} = \sum_{j \in T} R_j p^p_j,
\]

where \( R_j \) denotes the total number of travelers of the O-D trip \( j \).

The P&R FLP with NL model is given in (6)–(10). Equation (7) determines that only \( N \) park-and-ride facilities are to be built.

Maximize \( \sum_{i \in P} \sum_{j \in T} R_j p^p_{ij} \) \hspace{1cm} (6)

subject to

\[
\sum_{i \in P} x_i = N
\]

\[
\Gamma^p_j = \ln \sum_{i \in P} e^{V^p_{ij}/\lambda} x_i, \quad \forall j \in T \hspace{1cm} (8)
\]

\[
p^p_{ij} = \frac{e^{\lambda \Gamma^p_j}}{e^{V^c_j} + e^{\lambda \Gamma^p_j}} \frac{e^{V^p_{ij}/\lambda} x_i}{\sum_{k \in P} e^{V^p_{ik}/\lambda} x_k}, \quad \forall i \in P, \forall j \in T \hspace{1cm} (9)
\]

\[
x_i \in \{0, 1\}, \quad \forall i \in P
\]

This formulation cannot be linearized because of (8)–(9), and thus we develop a randomized rounding procedure in Section 3 to solve the nonlinear optimization problem.

2.1. Relation with multinomial logit model

The MNL model is a special case of our NL model. The LogSum parameter \( \lambda \) of the NL model is bounded by zero and one. When \( \lambda \) is one, P&Rs are uncorrelated (i.e., no nesting structure), and

\[
e^{\lambda \Gamma^p_j} = \sum_{i \in P} e^{V^p_{ij}} x_i.
\]
Then, (4) and (5) are

\[ p_j^c = \frac{e^{V_j}}{\sum_{i \in P} e^{V_i} x_i}, \forall j \in T \]  

(11)

\[ p_j^p = \frac{e^{V_j}}{\sum_{i \in P} e^{V_i} x_i}, \forall i \in P, \forall j \in T. \]  

(12)

Then, the IIA condition between private car and P&R \( i \) for O-D trip \( j \) is

\[ \frac{p_j^c}{p_j^p} = \frac{e^{V_j}}{e^{V_j} x_i}, \forall i \in P, \forall j \in T. \]  

(13)

Note that (13) is valid if and only if P&R \( i \) is determined to be built (i.e., \( x_i = 1 \)). Similarly, the IIA condition between P&R \( i \) and P&R \( k \) for O-D trip \( j \) is

\[ \frac{p_j^p}{p_j^p} = \frac{e^{V_j} x_k}{e^{V_j} x_i}, \forall i, k \in P, \forall j \in T. \]  

(14)

Unlike the P&R FLP with the NL model presented in (6)–(10), the P&R FLP with the MNL model can be formulated as an MILP by using (13)–(14). Among three linearized formulations presented in Haase and Müller [18], this paper adopts the formulation of Aros-Vera et al. [3] because the formulation is quickly solved. The MILP formulation of the P&R FLP with the MNL model is

\[ \text{Maximize } \sum_{i \in P} \sum_{j \in T} R_j p_j^p \]  

(15)

subject to

\[ \sum_{i \in P} x_i = N \]  

(16)

\[ p_j^c + \sum_{i \in P} p_j^p = 1, \forall j \in T \]  

(17)

\[ p_j^p \leq x_i, \forall i \in P, \forall j \in T \]  

(18)

\[ p_j^p \leq \frac{e^{V_j}}{e^{V_j} x_i}, p_j^p, \forall i, k \in P, \forall j \in T \]  

(19)

\[ p_j^p \leq \frac{e^{V_j}}{e^{V_j} p_j^p} + (1 - x_i), \forall i \in P, \forall j \in T \]  

(20)

\[ p_j^p \leq \frac{e^{V_j}}{e^{V_j} p_k^p} + (1 - x_k), \forall i, k \in P, \forall j \in T \]  

(21)

\[ x_i \in \{0, 1\}, \forall i \in P \]  

(22)

\[ p_j^c \geq 0, \forall j \in T \]  

(23)

\[ p_j^p \geq 0, \forall i \in P, \forall j \in T \]  

(24)

Just as the P&R FLP with the NL model, only \( N \) park-and-ride facilities are to be built by (16). Equation (17) constrains that the sum of all probabilities (e.g., \( p_j^c \) and \( p_j^p \)) is equal to one for each O-D trip. Equation (18) forces \( p_j^p = 0 \) if \( x_i = 0 \), as given in (12). The IIA condition of (13) is represented by (19)–(20), and the IIA condition of (14) is represented by (21). Note that these IIA conditions are valid only for selected P&Rs.
3. Meta-heuristic Algorithms

As stated earlier, the P&R FLP with NL model (6)–(10) cannot be linearized, and thus requires a meta-heuristic algorithm that can deal with the nonlinear terms (8)–(9). By contrast, the P&R FLP with MNL model (15)–(24) can be solved using a commercial MILP solver such as Gurobi Optimization [16]. This paper utilizes Randomized Rounding (RR) and the Genetic Algorithm (GA) to solve the P&R FLP with NL model.

3.1. Randomized Rounding.

All combinations of \( N \) facilities that are indicated by \( x_{\text{INT}} \) in the NL model (6)–(10) are feasible solutions to the P&R FLP. Every combination of \( N \) facilities immediately determines all the other variables \( p^p_i, p^p_{ij} \) and allows us to evaluate total utility in (6). We may regard total utility
\[
\text{obj} = \text{obj} (x_{\text{INT}}) = \sum_{i \in \mathcal{P}} \sum_{j \in T} R_j p^p_{ij}
\]
as a function of the combinations \( x_{\text{INT}} \).

A relaxation of the set of the combinations is
\[
\tilde{X} = \left\{ (\tilde{x}_i \in [0,1] : i \in \mathcal{P}) : \sum_{i \in \mathcal{P}} \tilde{x}_i = N \right\}.
\]
We refer to a point \( \tilde{x} \in \tilde{X} \) of the relaxation as relaxed value distribution (or simply distribution). Given a relaxed value distribution \( \tilde{x} \in \tilde{X} \), a trial of our randomized rounding procedure observes a random integer solution \( x_{\text{INT}} = x_{\text{INT}} (\tilde{x}) = (x^p_i \in \{0,1\} : \sum_{i \in \mathcal{P}} x^p_i = N) \). The random variable is denoted by \( X_{\text{INT}} = X_{\text{INT}} (\tilde{x}) \). Performing multiple trials, a randomized rounding procedure selects the combination \( x^* \) of \( N \) facilities with the largest value of \( \text{obj} (x^*) \).

A randomized rounding procedure usually sets the optimal fractional solution as a probability distribution of flipping coins to determine the values of the binary variables that indicate a feasible solution. In this section, we generalize the probability distribution to the relaxed value distribution, and develop a rounding algorithm for the P&R FLP. To avoid local optima, every trial moves the relaxed value distribution in an adaptive manner to the incumbent feasible solution.

3.1.1. Rounding algorithm.

Given relaxed value distribution \( \tilde{x} \), we implement Algorithm 1 to observe a combination \( x_{\text{INT}} \) of \( N \) facilities. Line 1 initializes \( x_{\text{INT}} = 0 \). In Lines 2 and 3 we observe
\[
\tilde{l}_i = \tilde{x}_i + (1 - \tilde{x}_i) \tilde{\pi}_i
\]
with \( \tilde{\pi}_i \in U \{0,1\} \) for the facility candidates \( i \in \mathcal{P} \), and \( x^p_i = 1 \) indicate the \( N \) facilities \( i \) of the largest values of \( \tilde{l}_i \). Note that and \( U \{a,b\} \) denotes a uniform distribution in the interval \( (a,b) \).
Algorithm 1: Rounding Algorithm

Input: a relaxed value distribution $\tilde{x} \in \tilde{X}$

Output: a combination of $N$ facilities $x^{\text{INT}}$

1: (Initialization) $x^{\text{INT}} = 0$
2: Observe $\tilde{l}_i = \tilde{x}_i + (1 - \tilde{x}_i) \tilde{\pi}_i$ with $\tilde{\pi}_i \in U[0,1]$ for $i \in P$
3: Select the $N$ facilities $i \in P$ of the largest $\tilde{l}_i$
4: Let $x^{\text{INT}}_i = 1$ for the $N$ selected facilities
5: Return $x^{\text{INT}}$

Given a relaxed value distribution $\tilde{x}$, Algorithm 1 observes a combination $x^{\text{INT}}$ of $N$ facilities near $\tilde{x}$ in the space of $\tilde{X}$. If a relaxed value $\tilde{x}_i$ equals 1, $\tilde{l}_i(\tilde{x}) = 1$ is always the highest value and the facility $i$ is selected. On the other hand, $x^{\text{INT}}_i$ does not have to be zero for a facility candidate $i$ with $\tilde{x}_i = 0$, giving the facility candidate a positive opportunity to be selected. Since $\tilde{x}_i = 1$ implies $x^{\text{INT}}_i = 1$, and $\sum_{i \in P} \tilde{x}_i = \sum_{i \in P} x^{\text{INT}}_i = N$, the algorithm always observes the same facility location itself when $\tilde{x} = x^{\text{INT}}$ is a feasible combination of $N$ facilities; i.e.,

**Proposition 1.** \( P[ X^{\text{INT}}(x^{\text{INT}}) = x^{\text{INT}}] = 1 \)

Different from usual randomized rounding, the relaxed value distribution is not guaranteed to be the same as the probability distribution (as $\tilde{x}_i = 0$ does not imply $X^{\text{INT}} = 0$). However, the two distributions are highly correlated, as follows:

**Proposition 2.** Given relaxed value distribution $\tilde{x}$,

$$ \tilde{x}_{i'} < \tilde{x}_{i''} \Rightarrow P[ X^{\text{INT}}_i = 1 ] < P[ X^{\text{INT}}_i = 1 ] \text{ for } i', i'' \in P. $$

The observed combination $x^{\text{INT}} = x^{\text{INT}}(\tilde{x})$ of $N$ facilities is immediately followed by calculation of the total utility of the P&R in the objective (6),

$$ \text{obj}(x^{\text{INT}}) = \sum_{i \in P} \sum_{j \in T} R_{ij} p^P_{ij} = \sum_{i \in P} \sum_{j \in T} R_{ij} p^P(x^{\text{INT}}). $$

From (8)-(9) with $x = x^{\text{INT}}, p^P = p^P(x^{\text{INT}})$ is a function of $x^{\text{INT}}$.

### 3.1.2. Moving distribution.

At each trial $r \geq 1$, an adaptive randomized rounding (ARR) procedure moves the relaxed value distribution from the last distribution $\tilde{x}^{(r-1)}$ toward the best known integer solution $x^*$ by exponential smoothing,

$$ \tilde{x}^{(r)} = (1 - \alpha(r)) \tilde{x}^{(r-1)} + \alpha(r) x^*, $$

where $0 < \alpha(r) < 1$. The ARR procedure looks for a better integer solution near the line segment $[\tilde{x}^{(r-1)}, x^*]$ while the distribution $\tilde{x}^{(r)}$ converges to the best known integer solution $x^*$. Since $\tilde{X}$ is a polyhedron, the line segment is contained in $\tilde{X}$.
A randomized rounding procedure usually starts with the fractional optimal solution to the relaxation of the formulation. Since our formulation is nonlinear, it is not easy to calculate the fractional optimal solution. Alternatively, our initial distribution may be

\[ \tilde{x}_i^{(0)} = \tilde{x}_i = \frac{N}{|P|} \text{ for all } i \in P. \]

**Proposition 3.** Let \( \tilde{x}_i = \frac{N}{|P|} \) for all \( i \in P \). Then, the following hold:

(a) The relaxed value distribution \( \tilde{x} \) is the centroid of the convex hull of all the combinations of \( N \) facilities; i.e.,

\[ \tilde{x} = \frac{\sum \{ x^{\text{INT}} \in \{0, 1\}^P : \sum_{i \in P} x_i^{\text{INT}} = N \} \}}{\binom{|P|}{N}}. \]

(b) The relaxed value distribution \( \tilde{x} \) is the probability distribution: i.e.,

\[ P \left[ X_i^{\text{INT}} (\tilde{x}) = 1 \right] = \frac{N}{|P|}. \]

**Proof.** (a) Each facility \( i \in P \) is a member of the same number \( \binom{|P| - 1}{N - 1} \) of combinations of \( N \) facilities, and each component \( x_i \) of the centroid \( x \) has a common value of \( \tilde{x}_i = N/|P| \).

(b) The relaxed value distribution of the centroid \( \tilde{x} \) is the probability distribution: i.e.,

\[ P \left[ X_i^{\text{INT}} (\tilde{x}) = 1 \right] = \frac{\sum_{i \in P} \sum_{x_i^{\text{INT}}} \sum_{i \in P} X_i^{\text{INT}} (\tilde{x})}{|P|} = \frac{\sum_{i \in P} X_i^{\text{INT}} (\tilde{x})}{|P|} = \frac{N}{|P|}. \]

If \( \tilde{x}^{(0)} = \tilde{x} \), the following \( \tilde{x}^{(r)}, r > 0 \) will belong to the convex hull because the convex hull contains the line segment \([x^{(r-1)}, x^*] \). Moving \( \tilde{x}^{(r)} \) in \( \tilde{X} \), our randomized rounding method performs the following 2 step procedure (see Fig. 3):
Step 0. Initialization. With an initial distribution $\tilde{x}(0)$ and an observed $x^* = x^{\text{INT}(0)} = x^{\text{INT}}(\tilde{x}(0))$, Step 0 sets the initial smoothing constant $\alpha(1) = 1/2$ and moves on to Step 1.

Step 1. Adjusting smoothing constant. The first step moves the last distribution $\tilde{x}^{(r-1)}$ to $\tilde{x}^{(r)}$ on the line segment $[\tilde{x}^{(0)}, x^*]$ by exponential smoothing (25), adjusting the smoothing constant $\alpha$, and observes a random combination $x^{\text{INT}(r)}$ of $N$ facilities using distribution $\tilde{x}^{(r)}$.

- Stopping criterion of the whole procedure: If $\alpha(r)$ is less than a certain tolerance (e.g., tolerance = $10^{-5}$ equal to the integrality tolerance of Gurobi Optimization [16]), the whole procedure stops.
- Criterion to go to Step 2: If $\text{obj} \left( x^{\text{INT}(r)} \right) > \text{obj} \left( x^* \right)$, the procedure updates the best known combination $x^* = x^{\text{INT}(r)}$. Assuming that the speed $\alpha(r)$ is appropriate, fix the present smoothing constant $\alpha = \alpha(r)$, set $r \leftarrow r + 1$ and go to Step 2.
- Major decrease of $\alpha$: If $\text{obj} \left( x^{\text{INT}(r)} \right) = \text{obj} \left( x^* \right)$, assume that $\tilde{x}^{(r)}$ is arriving too close at $x^*$ or too fast, and slow down the speed of the move by major decrease of the smoothing constant $\alpha(r + 1) = \alpha(r)/2$. Reset the present distribution back to $\tilde{x}^{(0)}$, set $r \leftarrow r + 1$ and repeat Step 1.
- Minor increase of $\alpha$: If $\text{obj} \left( x^{\text{INT}(r)} \right) < \text{obj} \left( x^* \right)$, assume that $\tilde{x}^{(r)}$ is moving slow and speed up the move a little bit by minor increase of the smoothing constant $\alpha(r + 1) = 2^{\lceil \log_2 \alpha(r) \rceil} + \alpha(r)/2$ such as $\alpha(r) = 1/2 + 1/4$ to $\alpha(r + 1) = 1/2 + 1/4 + 1/8$. Set $r \leftarrow r + 1$ and repeat Step 1 (moving ahead without coming back to $\tilde{x}^{(r+1)} = \tilde{x}^{(0)}$).

Step 2. Fixed smoothing constant. The second step moves $\tilde{x}^{(r)}$ from $\tilde{x}^{(r-1)}$ toward $x^*$ by exponential smoothing (25) on the line segment $[\tilde{x}^{(r-1)}, x^*]$ without changing the smoothing constant $\alpha$ that is adjusted in Step 1. It observes $x^{\text{INT}(r)}$ based on $\tilde{x}^{(r)}$.

- Criterion to go back to Step 1: If $\text{obj} \left( x^{\text{INT}(r)} \right) = \text{obj} \left( x^* \right)$ for a run of say 10 iterations of trials, assume that $\tilde{x}^{(r)}$ is too close to $x^*$ and reset $\tilde{x}^{(r)} = \tilde{x}^{(0)}$ initializing $\alpha(r + 1) = 1/2$. Set $r \leftarrow r + 1$ and go back to Step 1.
- If $\text{obj} \left( x^{\text{INT}(r)} \right) > \text{obj} \left( x^* \right)$, the procedure updates $x^* = x^{\text{INT}(r)}$ with the new best value. Set $r \leftarrow r + 1$ and repeat Step 2 from the present distribution $\tilde{x}^{(r-1)}$ toward the new best known integer solution $x^*$.
- If $\text{obj} \left( x^{\text{INT}(r)} \right) \leq \text{obj} \left( x^* \right)$, set $r \leftarrow r + 1$ and repeat Step 2 going on from the present $\tilde{x}^{(r-1)}$ toward the best known facility location $x^*$.

3.1.3. Advanced randomized rounding.

We generalize the notion of relaxed value distribution, and develop two advanced randomized rounding procedures. While the relaxed value distribution will not be required anymore to satisfy $\sum_{i \in P} \tilde{x}_i = N$, Algorithm 1 works on the generalized distribution $\tilde{x}$ with $0 \leq \tilde{x} \leq 1$ where $\mathbf{0}$ and $\mathbf{1}$ denote the vectors of
all components 0 and 1. The advanced randomized rounding also utilizes the relaxed value distribution $\tilde{x}$ to Algorithm 1, and moves $\tilde{x}$ by exponential smoothing,

$$
\tilde{x}^{(r)} = (1 - \alpha(r)) \tilde{x}^{(r-1)} + \alpha(r)x^*. 
$$

Unlike ARR, the advanced randomized rounding is a single step procedure and determines $\alpha(r)$ with the root-mean-square deviation (RMSD) of $\tilde{x}^{(r)}$ with respect to 0.5; i.e.,

$$
\text{RMSD}(\tilde{x}^{(r)}) = \sqrt{\sum_{i \in P} (\tilde{x}^{(r)}_i - 0.5)^2 / |P|},
$$

which ranges from 0 ($\tilde{x}_i = 0.5$, $\forall i \in P$) to 0.5 ($\tilde{x}_i = 0$ or $1$, $\forall i \in P$). Thus, as $\tilde{x}^{(r)}$ converges to the best known integer solution $x^*$, the RMSD converges to 0.5.

The advanced randomized rounding relies on two contradictory strategies on convergence speed. Accelerating Randomized Rounding (AccRR) moves faster towards $x^*$ as the RMSD increases (i.e., converges to $x^*$). In other words, AccRR spends more time on exploration than exploitation by

$$
\alpha(r) = \frac{1}{1 + e^{-8 \text{RMSD}(\tilde{x}^{(r)}) - 0.5}}. 
$$

On the other hand, Decelerating Randomized Rounding (DecRR) spends more time on exploitation than exploration by

$$
\alpha(r) = \frac{1}{1 + e^{4 \text{RMSD}(\tilde{x}^{(r)})}}.
$$

A pseudo-code of AccRR and DecRR is given in Algorithm 2. The algorithm can escape from a local optimum by resetting $\tilde{x}^{(r)} = 0.5$, and the probability of reset $p^R$ increases linearly from 0 to RMSD as the algorithm continues to reach the local optimum.

For notational convenience, we use the 1-norm $\| \cdot \|_1$ that is the sum of the components of a non-negative vector; i.e., $\|x\|_1 := \sum_{i \in P} x_i$. While $\tilde{x}$ is not required to satisfy $\|\tilde{x}\|_1 = N$, it can be projected to a relaxed value distribution $\tilde{x}'$ satisfying $\|\tilde{x}'\|_1 = N$ so that Algorithm 1 on $\tilde{x}$ may observe $x^\text{INT}(\tilde{x}')$ near $\tilde{x}'$ on $\hat{X}$:

**Theorem 1.** Let $0 \leq \tilde{x} \leq 1$ and let

$$
\tilde{x}' = 1 - \frac{|P| - N}{\|1 - \tilde{x}\|_1} (1 - \tilde{x}). 
$$

Then, Algorithm 1 observes a feasible solution $x^\text{INT}$ with the same probability; i.e.,

$$
P [X^\text{INT} (\tilde{x}) = x^\text{INT}] = P [X^\text{INT} (\tilde{x}') = x^\text{INT}].
$$

**Proof.** Algorithm 1 on $\tilde{x}$ selects the largest $N$ membership values $\tilde{l}_i$ among

$$
\tilde{l}_i = \tilde{x}_i + (1 - \tilde{x}_i) \tilde{\pi}_i
$$

with $\tilde{\pi}_i \in U [0, 1]$ for $i \in P$. It is equivalent to selecting the smallest $N$ values of

$$
1 - \tilde{l}_i = (1 - \tilde{x}_i) (1 - \tilde{\pi}_i) = \left( \frac{\|1 - \tilde{x}\|_1}{|P| - N} \right) (1 - \tilde{x}_i') (1 - \tilde{\pi}_i).
$$
Algorithm 2 Advanced Randomized Rounding

\[ \tilde{x}^{(0)} \leftarrow 0.5 \]

\[ r \leftarrow 1 \]

\[ n_{\text{local}} \leftarrow 0 \]

\[ \text{while} \text{ all termination criteria are not met do} \]

\[ \text{Observe } x_{\text{INT}(r)} \text{ with } \tilde{x}^{(r-1)} \text{ by Algorithm 1} \]

\[ \text{Adjust } \alpha \text{ by (27) or (28)} \]

\[ \text{Evaluate } \text{obj} (x_{\text{INT}(r)}) \]

\[ \text{if } \text{obj} (x_{\text{INT}(r)}) > \text{obj} (x^*) \text{ then} \]

\[ x^* \leftarrow x_{\text{INT}(r)} \]

\[ \text{Move } \tilde{x}^{(r)} \text{ by (25)} \]

\[ n_{\text{local}} \leftarrow 0 \]

\[ \text{else if } \text{obj} (x_{\text{INT}(r)}) = \text{obj} (x^*) \text{ then} \]

\[ n_{\text{local}} \leftarrow n_{\text{local}} + 1 \]

\[ p^R \leftarrow \min (\frac{n_{\text{local}}}{20}, 1) \times \text{RMSD}(\tilde{x}^{(r)}) \]

\[ \text{Reset } \tilde{x}^{(r)} \leftarrow 0.5 \text{ and } n_{\text{local}} \leftarrow 0 \text{ with probability of } p^R \]

\[ \text{else} \]

\[ \text{Move } \tilde{x}^{(r)} \text{ by (25)} \]

\[ \text{end if} \]

\[ r \leftarrow r + 1 \]

\[ \text{end while} \]

\[ \text{Return the best objective value} \]
Since \( \left( \left\| \frac{1 - \tilde{x}_i}{|\mathcal{P}|} \right\|_1 - N \right) \) is a constant factor in (31), selecting the smallest \( N \) values of \( 1 - \tilde{l}_i \) is equivalent to selecting the smallest \( N \) values of \( (1 - \tilde{x}_i') (1 - \tilde{\pi}_i) \), which is equivalent to selecting the largest \( N \) membership values of \( \tilde{l}_i = \tilde{x}_i' + (1 - \tilde{x}_i') \tilde{\pi}_i \). Thus, Algorithm 1 on \( \tilde{x} \) is equivalent to Algorithm 1 on \( \tilde{x}' \), completing the proof of the theorem.

\[ \square \]

3.2. Genetic Algorithm

GA is a population-based meta-heuristic algorithm [20]. It is well-known and widely adopted to solve the FLP [29]. A canonical GA is given in Algorithm 3 [40].

**Algorithm 3** Canonical GA Algorithm

(Initialization) Randomly generate an initial population

while all termination criteria are not met do

(Evaluation) Evaluate fitness values of the population

(Selection) Randomly select chromosomes (i.e., parents) based on fitness

for every two parents do

(Crossover) Mix the parent chromosomes to form a child chromosome

end for

(Mutation) With probability of \( p^m \), randomly select genes for each child and change their values

(Next generation) chromosomes ← children chromosomes

end while

Return the best objective value

3.2.1. 5 Phases

**Phase 1: Initialization.** Each chromosome is a binary string of length \(|\mathcal{P}|\), and \( N \) genes (i.e., elements of a chromosome) are randomly chosen and assigned to 1. Other genes are assigned to 0. Thus, a chromosome is a feasible set of \((x_1, x_2, \ldots, x_{|\mathcal{P}|})\). The population size (pop) is a design parameter.

**Phase 2: Evaluation.** The fitness value of a chromosome is equal to the objective value; i.e.,

\[ \text{Fitness} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{T}} R_{ij} p_{ij}^p, \]

but it can also be determined with the rank of objective value [4].

**Phase 3: Selection.** The selection probability of a chromosome is proportional to its fitness, and the selection process is stochastic sampling with replacement, as in the roulette wheel method [40, 22, 24]. In a canonical GA, parents produce two children; however, in this paper, they produce one child. As a result, \( 2 \times \text{pop} \) parents are selected, and thus most chromosomes are selected multiple times as parents.
Phase 4: Crossover. A canonical GA utilizes 1-point crossover, which randomly selects a crossover point and swaps the fragmented genes between the parents. Some implementations of GA utilize 2-point crossover in which parents’ middle genes between the crossover points are swapped to generate children [1]. However, such a fragment-based crossover does not guarantee the feasibility of children chromosomes because it constrains the number of chosen P&R to be equal to N.

Therefore, this paper proposes a weight-based crossover operator presented in Fig. 4. Each gene of parent chromosomes is randomly weighted between 0 and 1, and genes are added together to form a child chromosome. Then, the N largest genes of the child chromosome are assigned to 1 and the other genes are assigned to 0. This crossover operator guarantees that the child chromosome is inherited only from its parents’ chromosomes and the number of genes with value of 1 remains at N. Thus, the child chromosome is also a feasible solution of the P&R FLP.

![Weight-Based Crossover Operator](image)

Phase 5: Mutation. A canonical GA mutates each gene (i.e., flips a bit from 0 to 1 or vice versa) with a probability of \( p_m \); however, such a mutation operator can make children chromosomes infeasible. Therefore, the mutation operator in this paper exchanges a gene with value of 1 with another gene with value of 0. Each child chromosome is subject to mutation with a probability of \( p_m \), which is a design parameter.

3.2.2. Design Parameter Tuning

Design parameters of GA influence the performance, specifically in terms of optimality and convergence time [2]. A canonical GA has design parameters such as population size, crossover probability, and mutation probability. This paper sets the crossover probability at 1, meaning all children chromosomes are inherited from both parents. However, the other parameters such as population size and mutation probability should be determined by design parameter tuning.

4. Numerical Experiments

Two numerical experiments are conducted to compare the P&R FLP with the MNL and NL models and to evaluate the performance of the proposed meta-heuristic algorithms (i.e., ARR, AccRR, DecRR, and GA). In numerical experiments, the meta-heuristic algorithms are terminated when the time limit is reached. For comparison of algorithms, time at which the optimal solution is found for the first time (\( t^f \)) is recorded.
only if the exact solution is previously known (e.g., by a brute-force search). Table 1 summarizes the two numerical experiments. The numerical experiments were conducted with MATLAB R2020b on a computer with an Intel Core i9-9900T CPU with 16GB RAM.

Table 1: Summary of Numerical Experiment Details

|                      | Experiment 1 | Experiment 2 |
|----------------------|-------------|--------------|
| Number of O-D pairs  | 40          | 1000         |
| Number of P&R candidates | 30        | 100          |
| Number of selected P&R locations | 8        | 35           |
| Brute-Force search   | Yes         | No           |
| Time limit of meta-heuristic algorithms | 5 min     | 60 min       |

4.1. GA Design Parameter Tuning for Numerical Experiments

Prior to the numerical experiments, Monte-Carlo experiments were conducted to tune the population size (pop) and mutation probability ($p^m$). Based on previous experience, pop ranges from 30 to 90 in steps of 10, and $p^m$ ranges from 1% to 5% in steps of 1%. 100 instances of Experiment 1 are solved by GA for all pairs of (pop, $p^m$). The exact optimal solution of instances is found using a brute-force search, and the optimality of a GA solution is the ratio of the GA solution to the brute-force solution. Because the global optimal solution (i.e., the brute-force solution) is known, the computational times at which the GA finds the global optimal solution ($t_f$) are recorded. The true mean (i.e., population mean) of $t_f$ is denoted as $\mu$, and its point estimate is denoted as $\hat{\mu}$. The time limit is set at 20 min based on sample runs.

Figure 5 shows $\hat{\mu}$ and 95% confidence intervals for $\mu$, only for the pairs of (pop, $p^m$) with which the GA found the optimal solution of all instances within the time limit. Figure 6 shows a box plot of $t_f$, also corresponding to (pop, $p^m$) pairs of 100% optimality. Figures 5–6 show that the (70, 0.04) pair has the fastest convergence time on average (i.e., the smallest $\hat{\mu}$), as well as, in the worst case (i.e., the smallest maximum value of $t_f$ excluding outliers). Hence, the parameter pair (70, 0.04) is used for the numerical experiments, but its 95% confidence interval for $\mu$ overlaps with those for other (pop, $p^m$) pairs. For instance, (70, 0.05) can be another good parameter pair for GA.

4.2. Experiment 1

Experiment 1 solves a medium-scale P&R FLP. The number of O-D pairs is 40 and 8 P&R locations are selected out of 30 candidate P&Rs. Thus, the number of feasible solutions of Experiment 1 is $\binom{30}{8} = 5,852,925$, and a brute-force search is computationally tractable for the medium-scale instances. Then, the optimality of the meta-heuristic algorithms is evaluated with the exact optimal solutions. The time limit for the meta-heuristic algorithms is set at 5 min based on sample runs. The LogSum parameter $\lambda$ of the NL model is set to 0.5.
For the numerical experiments, the negative value of travel distance (Euclidean distance) is used as the observed utility for simplicity of calculation. Thus, the longer its travel, the less attractive the alternative. Then, the utility of car and P&R is

\[ V_c^j = -TD_c^j = -\sqrt{(X_{oj}^j - X_{dj}^j)^2 + (Y_{oj}^j - Y_{dj}^j)^2}, \quad \forall j \in T \]

\[ V_p^{ij} = -TD_p^{ij} = -\sqrt{(X_{pi}^i - X_{oj}^j)^2 + (Y_{pi}^i - Y_{oj}^j)^2} - \sqrt{(X_{di}^j - X_{pi}^i)^2 + (Y_{di}^j - Y_{pi}^i)^2}, \quad \forall i \in P, \forall j \in T, \]

where \((X_{oj}^j, Y_{oj}^j)\), \((X_{dj}^j, Y_{dj}^j)\), and \((X_{pi}^i, Y_{pi}^i)\) are the X- and Y-coordinates of the origin \(j\), destination \(j\), and P&R candidate location \(i\), respectively. Note that O-D trip \(j\) comprises \{\((X_{oj}^j, Y_{oj}^j), (X_{dj}^j, Y_{dj}^j)\)\} and the utility of a private car is always greater than or equal to that of all P&Rs. In addition, the number of travelers of every O-D trip is set to one for simplicity; i.e.,

\[ R_j = 1, \quad \forall j \in T. \]

O-D and candidate P&R locations are randomly generated as follows.

- \(X_{oj}^j = U(0, 2), Y_{oj}^j = U(0, 10)\)
- \(X_{dj}^j = U(8, 10), Y_{dj}^j = U(0, 10)\)
- \(X_{pi}^i = U(4, 6), Y_{pi}^i = U(3, 7)\)

Note that X- and Y-coordinates are dimensionless and \(U(a, b)\) denotes a uniform random variable of the interval \((a, b)\). Figure 7 shows an illustrative example of 40 O-D pairs and 30 P&R candidate locations. In
Experiment 1, P&R candidate locations are concentrated near the center of the area, which increases the competitiveness of P&R against private cars.

First, the dependency of optimal P&R locations on demand modelling is analyzed. As stated previously, the MNL model and NL model are based on different assumptions about correlation between P&Rs. As a result, the P&R FLP with both models give not only different values of the total utility of P&R but also different combinations of N P&R locations. Figure 8 compares the probabilities of choosing private car and P&R and the optimal combination of P&R locations for an instance of Experiment 1. Optimally selected N P&R locations are marked with an asterisk.

Because of the IIA assumption of the MNL model, the probability of choosing a private car of the MNL model is smaller than when using the NL model. That is, the total utility of P&R is overestimated by the MNL model. P&R 3, 7, 11, 14, 15, and 16 are selected for both models but two P&Rs are differently selected. The number of different P&Rs between the MNL and NL models is 1.44 on average and 5 at maximum. Therefore, a solution of P&R FLP with the MNL model is suboptimal considering the correlation between P&Rs.

Second, the computational performance of the proposed meta-heuristic algorithms is compared. In Experiment 1, all meta-heuristic algorithms successfully find the exact optimal solutions to all the medium-scale instances. Note that the optimal solution is found with a brute-force search. However, the solution times to exact optimality are different. Table 2 compares the performance of the proposed algorithms, and it is concluded that randomized rounding algorithms (i.e., ARR, AccRR, and DecRR) solves the problem to exact optimality much faster than GA.
Table 2: Comparison of Meta-Heuristic Algorithms in Experiment 1

| Optimality | ARR | AccRR | DecRR | GA |
|------------|-----|-------|-------|----|
| 95% confidence interval for $\mu$ | (0.0840, 0.392) s | (0.111, 0.158) s | (-0.262, 1.19) s | (14.8, 23.1) s |
| Max $t^f$ | 7.6 s | 0.7 s | 36.7 s | 109.0 s |
| Mean iterations until $t^f$ | 4,617 | 2,510 | 9,827 | 33,920 |

4.3. Experiment 2

Experiment 2 solves a P&R FLP on a real-world scale of large instances. GA parameters and $\lambda$ for Experiment 2 are the same as those for Experiment 1, but O-D and candidate P&R locations are not uniformly distributed in Experiment 2. Origins are randomly clustered into groups (e.g., residential areas or neighborhoods) and the number of neighborhoods is a random number between 5 and 10. Each center of neighborhood is randomly located on a radius between 6 and 10 from the center of the region, and origins in a neighborhood are uniformly distributed between $-1$ and 1 around the neighborhood center. Destinations are assumed to be a central business district (CBD). The number of CBD is also a random number between 3 and 5, and CBDs are randomly located on a radius between 1 and 2 from the center of the region. Thus, multiple origins have the same destination and P&Rs must be located between neighborhoods and CBDs. Candidate P&R locations are randomly located on a radius between 5 and 7 from the center of the region. Figure 9 shows an illustrative example of 6 neighborhoods and 4 CBDs.

The number of O-D pairs is 1,000 and 35 P&R locations are selected out of 100 candidate P&Rs.
Then, the number of feasible solutions of Experiment 2 is \( \binom{100}{35} \), which is about 1.87e+20 times larger than that of Experiment 1. As the number of candidate P&R locations increases, search space increases exponentially. Therefore, a brute-force search cannot be completed within a reasonable time, and thus only relative optimality of meta-heuristic algorithms and their convergence speed to the best observed solution are compared. Relative optimality is calculated as the objective value of an algorithm divided by the best objective value for the same instance. Note that the P&R FLP is a maximizing problem. The time limit for the meta-heuristic algorithms is set to 60 min based on the results of Experiment 1.

Table 3 compares the computational performance of the proposed meta-heuristic algorithms. Randomized rounding algorithms (i.e., ARR, AccRR, and DecRR) find a better solution than GA for all instances, and the relative gap of GA solutions is 3.3\% on average. Solutions of the RRs are the same for each instance. So, the solutions are most likely a global or at least local optimal solution. Moreover, the RRs find a decent solution faster than GA on average, as well as in the worst case. Among the RRs, DecRR is the fastest. As indicated by the average number of iterations until the time limit, GA’s population-based search requires more computational time per iteration than do the RRs. In conclusion, DecRR identifies a better solution in a shorter time.

Lastly, the sensitivity of LogSum parameter \( \lambda \) is analyzed. The LogSum parameter \( \lambda \) is estimated using the maximum likelihood estimation [27]. As no a priori knowledge of \( \lambda \) for the P&R FLP is available, the numerical experiments in this study arbitrarily assumed \( \lambda \) to be equal to 0.5. However, \( \lambda \) is an important parameter that indicates the underlying correlation between P&Rs [26]. So, a sensitivity analysis of the effect of \( \lambda \) on the total utility of P&R is conducted.

The same 100 instances of Experiment 2 are solved by DecRR with \( \lambda \) ranging from 0.1 to 1 in steps of 0.1.
Table 3: Comparison of Meta-Heuristic Algorithms in Experiment 2

|                                  | ARR          | AccRR        | DecRR        | GA           |
|----------------------------------|--------------|--------------|--------------|--------------|
| Number of instances with a better solution | 100          | 100          | 100          | 0            |
| Mean relative optimality         | 100%         | 100%         | 100%         | 96.7%        |
| Minimum relative optimality      | 100%         | 100%         | 100%         | 95.7%        |
| 95% confidence interval for $\mu$ | (113, 161) s | (249, 345) s | (66, 93) s   | (1,841, 2,242) s |
| Max $t^f$                        | 598 s        | 1,098 s      | 380 s        | 3,598 s      |
| Mean iterations until time limit | 1,163,329    | 1,165,721    | 1,161,646    | 66,029       |

Because 100 instances are randomly generated, the total utility of P&R varies from instance to instance. To reduce the variability between random instances, the total utility is normalized by that for $\lambda = 1$. Note that the P&R FLP with NL model for $\lambda = 1$ is equal to the P&R FLP with MNL model and the corresponding total utility is at maximum because of the IIA condition of the MNL model.

Figure 10 shows the mean and standard deviation of the normalized total utility. As stated earlier, the normalized total utility equals 1 when $\lambda$ is 1, and decreases quasi-linearly as $\lambda$ decreases. It is noteworthy that the variability between random instances is negligible because of the normalization. The value of $\lambda$ affects not only the total utility, but also the selected P&R. As shown in Fig. 8 the optimal combination of P&Rs of the MNL model is different from that of the NL model. Figure 11 shows the differences in selected P&R with respect to the MNL model (i.e., $\lambda = 1$). It is found that the selected P&R differs by 1 for every 0.2 difference in $\lambda$. Note that the red bars in Fig. 11 are median, which is an integer.
5. Conclusion

With a nested logit model, we define the park and ride facility location problem and develop a rounding algorithm to solve the nonlinear optimization problem. With the algorithm, randomized rounding methods reach an exact optimal solution extremely fast on a medium-scale of 100 random instances with 40 O-D pairs selecting 8 park and ride facilities out of 30 candidate locations. We also developed a genetic algorithm and compared it with the randomized rounding methods on a real-world scale of large instances with 1,000 O-D pairs, selecting 35 park and ride facilities out of 100 candidates. The randomized rounding methods identify better solutions much faster for the large-scale instances.

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Figure 11: Box Plot of Difference in Selected P&R with Respect to MNL Model

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