Rigidity theorems for $K$- and $H$-cohomology and other functors

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Abstract

Suslin [Su] proved that for an extension $K/k$ of algebraically closed fields the induced maps $K_m(k)[n] \to K_m(K)[n]$ and $K_m(k)/n \to K_m(K)/n$ for the higher algebraic $K$-groups are isomorphisms, where $A[n]$ is the subgroup of $n$-torsion in an abelian group $A$, and $A/n = A/NA$, by definition. In this paper we generalize this to other functors and other field extensions.

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0 Introduction

The results in this paper hold in particular for the following functors on separated noetherian $k$-schemes for a field $k$, which are contravariant for flat morphisms.

Example 0.1 (1) $V(Y) = H^i(Y_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(j))$ (étale cohomology), where $i, j \in \mathbb{Z}$, $n \in \mathbb{N}$ is invertible in $k$, and $\mathbb{Z}/n\mathbb{Z}(j) = \mu_n^{\otimes j}$, for the sheaf $\mu_n$ of $n$-th roots of unity on $V$.

(2) $V(Y) = K_m(V)$ (algebraic $K$-theory) for $m \in \mathbb{N}_0$.

(3) $V(Y) = CH^r(Y,s)$ (Bloch’s higher Chow groups) for $r, s \in \mathbb{N}_0$, which contains the classical Chow groups $CH^r(Y)$ as the special case $CH^r(Y,0)$.

(4) $V(Y) = H^i(Y, \mathcal{K}_m)$ ($\mathcal{K}$-cohomology) for $i, m \in \mathbb{N}_0$, the $i$-th Zariski cohomology of the Zariski sheaf associated to the presheaf $U \mapsto K_m(U)$.  
(5) \( V(Y) = H^i(Y, \mathcal{H}(\mathbb{Z}/n\mathbb{Z}(k))) \) (\( \mathcal{H} \)-cohomology), the \( i \)-th cohomology of the Zariski sheaf on \( Y \) associated to the presheaf \( U \mapsto H^j(U_{et}, \mathbb{Z}/n\mathbb{Z}(k)) \), for \( i \in \mathbb{N}_0 \) and \( j, k \in \mathbb{Z} \).

(6) \( V(Y) = H^i(Y, \mathcal{CH}(s)) \), where \( \mathcal{CH}(s) \) is the Zariski sheaf on \( Y \) associated to the presheaf \( U \mapsto \mathcal{CH}(U, s) \).

On the other hand, we consider field extensions \( L^0 \subset L \) over \( k \) and \( n \in \mathbb{N} \) such that one of the following conditions hold

**Conditions 0.2**

(i) Both fields are algebraically closed, or
(ii) \( L = \mathbb{R} \) and \( L^0 \) is dense and algebraically closed in \( \mathbb{R} \), or
(iii) \( L \) is a complete discrete valuation field, \( L^0 \) is algebraically closed in \( L \) and dense in \( L \) for the valuation topology, and \( n \) is invertible in \( L \).

An application we have in mind for (iii) is the situation where \( K \) is a global field (i.e., a number field or a function field in one variable over a finite field), and where \( L = K_v \) is the completion of \( K \) with respect to a discrete valuation \( v \) of \( K \), and \( L^0 \) is the Henselization \( K_{(v)}(v) \) of \( K \) at \( v \). An example for (ii) is the situation where \( L \) is the completion of \( K \) at a real place \( v \) (hence isomorphic to the field \( \mathbb{R} \) of real numbers), and \( L^0 \) is the associated real closure of \( K \) in \( L \), the algebraic elements in the extension \( L/K \). An example for (i) is the situation where \( L \) is the completion of \( K \) at a complex place (hence isomorphic to \( \mathbb{C} \)), and \( L^0 \) is the algebraic closure of \( K \) in \( L \).

In particular we show the following.

**Theorem 0.3** If \( V \) is one of the functors (1) - (5) in Example 0.1 and the field extension \( L^0 \subset L \) is one of the field extensions (i) - (iii) in Condition 0.2, and \( X \) is a smooth projective variety over \( L^0 \), then, with \( n \) as in the respective cases, the restriction maps

\[
V(X)[n] \xrightarrow{\sim} V(X_L)[n] \quad \text{and} \quad V(X)/n \xrightarrow{\sim} V(X_L)/n
\]

are isomorphisms, where \( X_L = X \times_{L^0} L \), \( A[n] = \ker(A \xrightarrow{n} A) \), and \( A/n = \coker(A \xrightarrow{n} A) \).

**Remark 0.4**

(a) This is well-known for (1), where case (i) is well-known and implies (ii) and (iii) by Galois descent.

(b) For (2), case (i), and \( Y = \text{Spec}(L^0) \) this is the result of Suslin quoted in the beginning.

(c) For (4) and case (i) this was proved by F. Lecomte [Lec].

(d) The cases (ii) and (iii) of (b) and (c) do not follow - because these theories do not have Galois descent.

However, with our methods explained below, we can treat more cases.
Example 0.5 Let $K$ be a global field, let $v$ be a non-archimedean place.

(a) For case (3) and $r = s$ we consider the Chow groups $H^r(X, K_r) = CH^r(X)$, i.e., the classical Chow groups of a smooth projective variety $X$ over a field. In case (iii) we get isomorphisms, for a global field $K$, and an integer $n$ invertible in $K$, and a non-archimedean place $v$ of $K$:

$$CH^r(X_{K(v)})[n] \xrightarrow{\cong} CH^r(X_{K_v})[n] \quad \text{and} \quad CH^r(X_{K(v)}/n) \xrightarrow{\cong} CH^r(X_{K_v})/n.$$ 

(b) Let $X$ be a smooth projective curve over $K$ and consider the “residue map” for $n$ invertible in $K$

$$H^i(K(X), \mathbb{Z}/n\mathbb{Z}(j)) \xrightarrow{\alpha} \bigoplus_{x \in |X|} H^{i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(j)),$$

where $|X|$ is the set of closed points of $X$. Then one has $\ker(\alpha) = H^0(X, \mathcal{H}_n^i(j))$, and by the above this group for $X_{K(v)}$ is isomorphic to the one for $X_{K_v}$. The same holds for $\text{coker}(\alpha) = H^1(X, \mathcal{H}_n^i(j))$.

(c) For a field $F$ one has a functorial isomorphism $K^M_m(F) \cong CH^m(F, m)$ between the Milnor $K$-group and the written Bloch higher Chow group, by work of Nesterenko and Suslin [NeSu], see also Totaro [To]. Hence one gets rigidity for Milnor $K$-theory mod $n$ and the $n$-torsion of Milnor $K$-theory, for all three cases (i), (ii) and (iii).

The above results are implied by some more general theorems. We introduce a notion of rigid functors (see Definition 1.1) and sufficiently rigid functors (see Definition 3.1), and prove that the above rigidity result hold for such functors, justifying their names in retrospective.

1 Rigid Functors

Let $S$ be a category of schemes, and let $S^{flat}$ be the category just endowed with the flat morphisms.

**Definition 1.1** A contravariant functor $V$ on $S^{flat}$ with values in the category $Ab$ of abelian groups is called rigid, if it satisfies the following properties, provided the occurring schemes and morphisms are in $S$.

(a) For any flat finite morphism $\pi : X \to Y$ there is a transfer morphism $\pi_* : V(X) \to V(Y)$, such that for another flat finite morphism $\rho : Y \to Z$ one has $(\rho \pi)_* = \rho_* \pi_*$. 

(b) For every cartesian diagram of schemes

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\pi' \downarrow & & \downarrow \pi \\
Y' & \xrightarrow{f} & Y
\end{array}
$$
with \( \pi \) finite and flat, one has \( f^* \pi_* = \pi'_* f'^* : V(X) \to V(Y') \).

(c) If \( X = X_1 \amalg X_2 \), then the immersions \( \pi_i : X_i \hookrightarrow X \) \((i = 1, 2)\) induce an isomorphism

\[
(\pi_1^*, \pi_2^*) : V(X) \xrightarrow{\sim} V(X_1) \oplus V(X_2)
\]

with inverse \( (\pi_1)_* + (\pi_2)_* \).

(d) If \( X_m = X \times_{\mathbb{Z}} \text{Spec} (\mathbb{Z}[T]/(T^m)) \) is the \( m \)-fold thickening of \( X \), then for the morphism \( \pi : X_m \to X \) one has \( \pi^* \pi_* = \text{multiplication by } m \).

(e) If \( \mathbb{A}^1_X = X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \) is the affine line over \( X \), then the projection \( p : \mathbb{A}^1_X \to X \) induces an isomorphism \( p^* : V(X) \xrightarrow{\sim} V(\mathbb{A}^1_X) \).

(f) Let \( i \mapsto X_i \) be a filtered projective system of schemes such that the transition morphisms \( X_i \to X_j \) are affine, and let \( X = \lim X_i \). Then the canonical map

\[
\lim V(X_i) \longrightarrow V(X)
\]

is an isomorphism.

**Theorem 1.2** Let \( K^0 \subset K \) be a field extension as in Condition 0.2 and let \( V \) be a rigid functor on the category of all noetherian \( K^0 \)-schemes, such that the value groups \( V(Y) \) are torsion groups.

(a) If both fields are algebraically closed, or if \( K = \mathbb{R} \) and \( K^0 \) is algebraically closed in \( \mathbb{R} \), then the restriction map

\[
V(K^0) \longrightarrow V(K)
\]

is an isomorphism (where we write \( V(L) \) for \( V(\text{Spec} L) \) if \( L \) is a field).

(b) Let \( K \) be a complete discrete valuation field, and let \( K^0 \) be a dense subfield of \( K \) which is algebraically closed in \( K \). Then the restriction map

\[
V(K^0) \longrightarrow V(K)
\]

is an isomorphism if \( \text{char}(K) = 0 \). If \( \text{char}(K) = p > 0 \), then it is an isomorphism if \( K \) is separable over \( K^0 \), and the value groups of \( V \) do not have any \( p \)-torsion.

**Proof** For the surjectivity it suffices to prove

**Claim 1:** If \( F \) is a function field over \( K^0 \), contained in \( K \), then

\[
\text{Im}(V(F) \to V(K)) \subseteq \text{Im}(V(K^0) \to V(K))
\]

In fact, by the limit property 1.1 (f), \( V(K) \) is generated by the images of the maps \( V(F) \to V(K) \) for all such fields \( F \).

We prove Claim 1 by induction on \( f = \text{deg.tr.}(F/K^0) \), the degree of transcendence of \( F \) over \( K^0 \). If \( d = 0 \), then necessarily \( F = K^0 \), since \( K^0 \) is algebraically closed in \( K \), and
the claim is trivially true. If \( d > 0 \), then by Noether normalization and separability of \( F \) over \( K^0 \), there exists a function field \( F_1 \) with \( \text{deg.tr.}(F_1/K^0) = d - 1 \) and a smooth, geometrically irreducible curve \( C_1 \) over \( F_1 \) such that \( F = F_1(C_1) \), the function field of \( C_1 \). Let \( \tilde{F}_1 \) be the algebraic closure of \( F_1 \) in \( K \) and let \( \tilde{C}_1 = C_1 \times_{F_1} \tilde{F}_1 \), and \( \tilde{F} = \tilde{F}_1(\tilde{C}_1) \). Then it suffices to show

**Claim 2:** \( \text{Im}(V(\tilde{F}) \to V(K)) \subseteq \text{Im}(V(\tilde{F}_1) \to V(K)) \).

In fact, by 1.1 (f) every \( \alpha \in \text{Im}(V(\tilde{F}_1) \to V(K)) \) lies in \( \text{Im}(V(F_2) \to V(K)) \) for a function field \( F_2 \) over \( K \) with \( \text{deg.tr.}(F_2/K^0) = d-1 \), so by induction \( \alpha \) lies in \( \text{Im}(V(K^0) \to V(K)) \).

In other words, it suffices to prove Claim 1 for a field \( L \subseteq K \) in place of \( K^0 \), which is dense and algebraically closed in \( K \) but for which \( K \) is not necessarily separable over \( L \), and for a function field \( F \) with \( \text{deg.tr.}(F/L) = 1 \) which is separable over \( L \).

Let \( \alpha \in \text{Im}(V(F) \to V(K)) \). By 1.1 (f) there is a smooth, geometrically irreducible curve \( C \) over \( L \) with function field \( L(C) = F' \) such that \( \alpha \in \text{Im}(V(C) \to V(K)) \). Let \( \text{Div}(C) \) be the group of divisors on \( C \), i.e., the free abelian group on the closed points \( x \in C \). Consider the bilinear pairing

\[
\text{Div}(C) \times V(C) \longrightarrow V(L)
\]

defined by sending \((x, \beta)\) to \((\pi_x)_{\ast} \varphi_x^\ast(\beta)\), where \( \varphi_x : \text{Spec} \, \kappa(x) \to C \) and \( \pi_x : \text{Spec} \, \kappa(x) \to \text{Spec} \, L \) are the canonical morphisms. Denote by \( \overline{C} \) the regular proper model of \( C \) and set \( C_\infty = \overline{C} \setminus C \). Let \( f \) be a meromorphic function on \( \overline{C} \) which is defined and equal to one. Now one has a commutative diagram of pairings

\[
\begin{array}{ccc}
\text{Div}(C) & \times & V(C) \\
\cup & & \downarrow j^* \\
\text{Div}(C') & \times & V(C') \\
\pi^* & \uparrow & \downarrow \pi_* \\
\text{Div}(\mathbb{A}_L^1) & \times & V(\mathbb{A}_L^1) \\
\end{array}
\]

where \( j^* \) is induced by the open immersion \( j : C' \hookrightarrow C \). Indeed, recall that for a closed point \( x \in \mathbb{A}_L^1 \) one has

\[
\pi^*(x) = \sum_{\pi(y) = x} e(y/x) \cdot x,
\]

where \( e(y/x) = \text{length}(\mathcal{O}_{C,y} \otimes \kappa(x)) \) is the ramification index of \( y \) over \( x \) (the tensor product is over \( \mathcal{O}_{\mathbb{A}_L^1,x} \)). Consider the following cartesian diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{\varphi'_x} & C' \\
\pi' \downarrow & & \downarrow \pi \\
\text{Spec} \, \kappa(x) & \xrightarrow{\varphi_x} & \mathbb{A}_L^1
\end{array}
\]

Then \( \varphi_x^* \pi_* = (\pi')_*(\varphi'_x)^* \) by 1.1 (b), and it suffices to show that

\[
(\pi')_* = \sum_{\pi(y) = x} e(y/x)(\pi' \alpha_y)^* \alpha^*_y,
\]

5
as a closed subscheme (Note that
\( \text{Jacobian}_J \) where
\( C \) is geometrically connected group variety over \( L \) where the right hand side is the group of
the complete regular model of the smooth curve
\( C \). Now\( \sum \) with the corresponding Cartier divisor and hence with the Weil divisor
\( [\psi, J, x] \) where
\( \psi \) necessarily reduced, if\( \text{char} K > 0 \) and \( [CC]. \)) In fact, one has (1.2.1) can most easily be seen by the fact that
\( \psi((f), \beta) \in V(L) \) coincides with \( \psi'(0 - \infty, \pi_*(\beta)) \), which is zero in view of the homotopy invariance 1.1 (e).

We have proved that the pairing factors through\( \text{Pic}(\mathcal{C}, C_\infty) \otimes V(C) \), where
\( \text{Pic}(\mathcal{C}, C_\infty) = \text{Div}(C)/\{f \in L(C)\times | f = 1 \text{ on } C_\infty\} \)
is the divisor class group of modulus \( C_\infty \), where the closed subscheme \( C_\infty \subset \mathcal{C} \) is identified with the corresponding Cartier divisor and hence with the Weil divisor \( \sum_y \).

Let\( \text{Pic}^0(\mathcal{C}, C_\infty) \) be the subgroup defined by the divisors of degree zero, where \( \text{deg} x = [K(x) : L] \). Then we have a canonical isomorphism
\( \text{Pic}^0(\mathcal{C}, C_\infty) = J_{C_\infty}(\mathcal{C})(L) \),
where the right hand side is the group of \( L \)-rational points of the Rosenlicht generalized Jacobian \( J_{C_\infty}(\mathcal{C}) \) (cf. [Ro], [Se] Chap. V and [CC]), which is a commutative, smooth, geometrically connected group variety over \( L \). Now
(1.2.1)
\( J_{C_\infty}(\mathcal{C}) \times_L K = J_{C_\infty,K}(\mathcal{C}_K) = J_{C_\infty}(\mathcal{C}_K) \),
where \( C_\infty, K \subset \mathcal{C} \times_L \mathcal{K} \subset \mathcal{C} \times_L \mathcal{K} \) as a closed subscheme, \( \mathcal{C}_K \) is the complete regular model of the smooth curve \( C_K = C \times_K K \) and \( \mathcal{C}_\infty = C_\infty \mathcal{C}_\infty \mathcal{C}_K \) as a closed subscheme (Note that \( \mathcal{C}_K \) is not necessarily regular and that \( C_\infty,K \) is not necessarily reduced, if\( \text{char} K > 0 \) and \( K/L \) is not separable). In the language of schemes the equalities (1.2.1) can most easily be seen by the fact that \( J_D(X) \), for a geometrically integral curve \( X \) over a field \( k \) and a Cartier divisor \( D \subset X \) such that \( X - D \) is smooth, represents the Picard functor \( \text{Pic}_{X_D/k} \), where \( X_D \) is the curve obtained by contracting \( D \) to a point, i.e., \( X_D \) is the scheme theoretic amalgamated sum \( X \amalg_{k} \text{Spec} k \) (cf. [Se] p. 85 and [CC]). In fact, one has \( (\mathcal{C}_K)_{C_\infty,K} = (\mathcal{C}_K)_{C_\infty} \), since the diagram
\[
\begin{array}{ccc}
\mathcal{C}_\infty & \xrightarrow{\sim} & \mathcal{C}_K \\
\downarrow & & \downarrow \\
C_{\infty,K} & \xrightarrow{\sim} & \mathcal{C}_K
\end{array}
\]
is cartesian and cocartesian.

If the pairing
\( \psi_K : \text{Div}(C_K) \times V(C_K) \longrightarrow V(K) \)
is defined for \( C_K \) over \( K \) in the same way as for \( C \) over \( L \) above, then by the same argument as there, this pairing factors through \( \text{Pic}(\mathcal{C}_K, (\mathcal{C}_\infty)_{\text{red}}) \otimes V(C_K) \) and therefore also through \( \text{Pic}(\mathcal{C}_K, \mathcal{C}_\infty) \otimes V(C_K) \), where
\( \text{Pic}(\mathcal{C}_K, \mathcal{C}_\infty) = \text{Div}(\mathcal{C}_K)/\{f \in K(C_K)^\times | f \equiv 1 \text{mod } \mathcal{C}_\infty\} \)
is the divisor class group of modulus $\tilde{C}_\infty$ ($f \equiv 1 \mod \tilde{C}_\infty$ meaning that $f$ lies in $\Gamma(U, \mathcal{O}_U)$ for an open neighbourhood of $(\tilde{C}_\infty)_{\text{red}}$ and has image 1 in $\Gamma(\tilde{C}_\infty, \mathcal{O}_{\tilde{C}_\infty})$). Let $p : C_K \to C$ be the projection induced by $p : \text{Spec } K \to \text{Spec } L$. Then by 1.1 (b) we have a commutative diagram of pairings

$$
\psi_K : \text{Div}(C_K) \times V(C_K) \to V(K)
$$

$$
\psi : \text{Div}(C) \times V(C) \to V(L),
$$

where the left hand map is the pull-back of Cartier divisors (which sends $x \in | C |$ to the unique $x' \in | C_K |$ with $p(x') = x$; note that $\kappa(x) \otimes_L K$ is a field, since $\kappa(x)$ is separable over $L$ and $L$ is separably closed in $K$).

For the following we may assume that $V$ is annihilated by a natural number $n$, since it suffices to prove Theorem 1.2 for all subfunctors $V[n]$ for such $n$. Then we get an induced diagram

$$
\begin{align*}
\text{Pic}(\tilde{C}_K, \tilde{C}_\infty)/n &\otimes V(C_K) \to V(K) \\
\text{Pic}(\tilde{C}, \tilde{C}_\infty)/n &\otimes V(C) \to V(L),
\end{align*}
(1.2.2)
$$

To prove Claim 1 in this situation it then suffices to show

**Claim 3:** $p^* : \text{Pic}^0(\tilde{C}, \tilde{C}_\infty)/n \to \text{Pic}^0(\tilde{C}_K, \tilde{C}_\infty)/n$ is surjective.

In fact, consider an element $\beta \in V(C)$ mapping to our element $\alpha \in \text{Im}(V(C) \to V(K))$. Then $\alpha = \psi_K(y_0, p^*(\beta))$, where $y_0$ is the $K$-rational point of $C_K$ corresponding to the generic point $\text{Spec } K \to \text{Spec } F \to C$. If $K$ is algebraically closed, then $L$ is algebraically closed as well (since $K$ is algebraically closed in $K$). Hence there exists an $L$-rational point $x_0$ of $C$. The same holds for the other two cases by Lemma 1.3 below, since $C$ has the $K$-rational point $y_0$. Then $y_0 - p^*(x_0)$ lies in $\text{Div}^0(C_K)$, and by Claim 3 there exists an element $z \in \text{Div}(C)$ with $\psi_K(y_0 - p^*(x_0), p^*(\beta)) = \psi_K(p^*(z), p^*(\beta))$. Thus $\alpha = \psi_K(y_0, p^*(\beta)) = p^*\psi(x_0 + z, \beta)$ lies in the image of $p^* : V(L) \to V(K)$ as wanted.

To prove Claim 3, note that the considered map can be identified with the natural map

$$
J_{C_\infty}(C)(L)/n \to J_{C_\infty}(C)(K)/n,
$$

since there are identifications $J_{C_\infty}(C)(K) = (J_{C_\infty}(C) \times_L K)(K) = J_{\tilde{C}_\infty}(\tilde{C}_K)(K) = \text{Pic}(\tilde{C}_K, \tilde{C}_\infty)$, functorially in the field $K$, and since $J_D(X)$ represents $\text{Pic}_{X_{\text{ét}/k}}$ as mentioned above.

If $K$ is algebraically closed, then $L$ is algebraically closed as well, as remarked above, and Suslin’s proof of [Su] Prop. 2.3 and Cor. 2.3.3 applies to $V$. In the notations above, we have $\tilde{C}_K = \tilde{C}_K$ and $C_{\infty,K} = \tilde{C}_\infty$, and Claim 3 follows from the fact that $\text{Pic}^0(\tilde{C}_K, \tilde{C}_\infty)/n$ is zero for every $N$, since $\tilde{C}_\infty$ is reduced, so that $J_{\tilde{C}_\infty}(\tilde{C}_K)$ is a semi-abelian variety.

For the other two cases $nJ_{C_\infty}(C)(K)$ is open in $J_{C_\infty}(C)(K)$ for the strong topology on this set, i.e., the topology coming from the topology of $K$), since the morphism $n : J_{C_\infty}(C) \to J_{C_\infty}(C)$ is étale, $n$ being invertible in $L$. Therefore the claim follows from

**Lemma 1.3** Let $K$ be $\mathbb{R}$, $\mathbb{C}$, or a complete discrete valuation field, let $L$ be a dense subfield, and let $X$ be a scheme of finite type over $L$. If $K$ is separable over $L$ or if $X$ is smooth over $L$, then $X(L)$ is dense in $X(K)$ for the strong topology.
In fact, for \( x \in J_{C_\infty}(C)(K) \), \( x + nJ_{C_\infty}(C)(K) \) is open, hence by Lemma 1.3 there is \( y \in J_{C_\infty}(C)(L) \) contained in \( x + nJ_{C_\infty}(C)(K) \).

Lemma 1.3 is proved in [KaSa], Lemma 4, where the first case is explicitly stated and reduced to the second case in the proof, and where the reader may also find a definition of the strong topology (called the usual topology there).

It remains to prove the injectivity of the map \( V(K^0) \to V(K) \). Since \( K = \lim A_i \) for smooth \( K^0 \)-algebras \( A_i \), by 1.1 (f) it suffices to show that

\[
q_i^* : V(\text{Spec } K^0) \to V(X_i)
\]

is injective for every \( q_i : X_i = \text{Spec } A_i \to K^0 \). But every \( X_i \) has a \( K \)-rational point, hence a \( K^0 \)-rational point \( s_i : \text{Spec } K^0 \to X_i \) by lemma 1.3. Since \( s_i^* q_i^* = (q_i s_i)^* = id, q_i^* \) must be injective.

**Corollary 1.4** It \( K, K^0 \) and \( V \) are as in Theorem 1.2, then for every scheme \( X \) of finite type over \( K^0 \) the restriction map

\[
V(X) \to V(X_K)
\]

is an isomorphism.

Indeed, the functor \( V_X \) with \( V_X(Y) = V(X \times_{K^0} Y) \) is again a rigid functor on the category of all noetherian \( K^0 \)-schemes.

**Remark 1.5** Let \( R \) be an excellent discrete valuation ring, let \( \hat{R} \) be its completion, and denote by \( k \) and \( K \) the fraction fields of \( R \) and \( \hat{R} \), respectively. Then by definition ([EGA IV](2),7.8.2), \( K \) is separable over \( k \). Hence, if \( K^0 \) is the algebraic (= separable) closure of \( k \) in \( K \), then the assumptions of theorem 1.1 are fulfilled for \( K \) and \( K^0 \). Note that \( K^0 \) is the fraction field of the Henselization \( \hat{R} \) of \( R \) ([EGA IV](4),18.9.3). In particular, if \( K \) is a global field and \( v \) is a place of \( K \), then theorem 1.1 holds for the pair \( (K_v,K_v) \), where \( K_v \) is the completion of \( K \) at \( v \) and \( K_v \) is the algebraic closure of \( K \) in \( K_v \). In fact, if \( v \) is non-archimedean, then the corresponding valuation ring is excellent ([EGA IV](2),7.8.3(ii),(iii). If \( K_v \) is a separable closure of \( K \) and \( w \) is a place of \( K_v \) extending \( v \), then \( K_w \) can also be replaced by the isomorphic decomposition field \( K_w \) which is the classical Henselian field associated to \( K \) and \( w \).

## 2 Examples of rigid functors

The first two examples of rigid functors are given by étale cohomology and algebraic \( K \)-theory.

**Proposition 2.1** Let \( S \) be a category of quasi-compact schemes, and let \( \mathcal{F} \) be an étale torsion sheaf on \( S \) whose torsion is invertible in \( \mathcal{S} \). Assume that \( \mathcal{F} |_{X} = f^* \mathcal{F} |_{Y} \) for every morphism \( f : X \to Y \) in \( S \). (e.g., for any natural number \( n \) we may take \( S = \text{Sch}^{sp}/\mathbb{Z}[1/n] \), the category of quasi-compact schemes on which \( n \) is invertible, and \( \mathcal{F} = \).
\( \mathbb{Z}/n(j) = \mu_n^{\otimes j} \), the \( j \)-th Tate twist of the constant sheaf \( \mathbb{Z}/n \), cf. [Mi], p. 163). Then for any integer \( i \geq 0 \) the functor
\[
V(Y) = H^i(Y, \mathcal{F})
\]
given by the \( i \)-th étale cohomology with coefficients in \( \mathcal{F} \) is a rigid functor on \( \mathcal{S} \).

**Proof** This is well-known: The contravariance of \( V \) is induced by the adjunction maps \( \text{adj}_f : \mathcal{F} \to f_*f^*\mathcal{F} \) for morphisms \( f : X \to Y \) (cf. [Mi] III 1.6(c)), viz., \( f^* \) is the composition
\[
f^* : H^i(Y, \mathcal{F}) \xrightarrow{\text{adj}_f} H^i(Y, f_*f^*\mathcal{F}) \xrightarrow{\text{can}} H^i(X, f^*\mathcal{F}) = H^i(X, \mathcal{F}) \,.
\]
For the limit property 1.1(f) cf. [Mi] III 1.16, and for the homotopy property 1.1 (e), which is related to smooth base change, cf. [Mi] VI 4.15. The transfer \( \pi_* \) for a finite flat morphism \( \pi : X \to Y \) is defined as follows. By [SGA 4] XVII 6.2.3 there is a canonical trace morphism
\[
\text{Tr}_{\pi} : \pi_*\pi^*\mathcal{F} \to \mathcal{F}
\]
for every abelian sheaf \( \mathcal{F} \) on \( Y \) (note that \( \pi_* = \pi! \) for a finite morphism). Then \( \pi_* \) is defined as the composition
\[
\pi_* : H^i(X, \pi^*\mathcal{F}) \xrightarrow{\text{can}} H^i(Y, \pi_*\pi^*\mathcal{F}) \xrightarrow{\text{Tr}_{\pi}} H^i(Y, \mathcal{F})
\]
in which the first map is an isomorphism since \( \pi \) is finite (so that \( G \mapsto \pi_* G \) is exact, cf. [Mi] II 3.6). The functoriality (\( \rho\pi^* \)) \( = \rho_* \pi_* \) in 1.1 (a) then follows from the transitivity of the trace morphism ([SGA 4] XVII 6.2.3 (Var 3)). For property 1.1 (d) we first note that the composition
\[
\mathcal{F} \xrightarrow{\text{adj}} \pi_*\pi^*\mathcal{F} \xrightarrow{\text{Tr}_{\pi}} \mathcal{F}
\]
is multiplication by \( d \) for every finite flat morphism \( \pi : X \to Y \) of (constant) rank \( d \) (loc.cit. (Var 4)). Hence \( \pi_*\pi^* \) is multiplication by \( d \) for such \( \pi \). Now let \( \pi : X_d = X[T]/(T^{d+1}) \to X \) be as in 1.1 (d), and let \( s : X \to X_d \) be the obvious “zero” section (\( T \mapsto 0 \)) of \( \pi \). Then \( s^* \) is an isomorphism by the topological invariance of étale cohomology ([SGA 4] VIII 1.2), hence \( \pi^* \) is an isomorphism as well. Thus the equality \( \pi^*\pi_*\pi^* = d\pi^* \) implies that \( \pi^*\pi_* = d \) as wanted. If
\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\pi' \downarrow & & \pi \downarrow \\
Y' & \xrightarrow{f} & Y
\end{array}
\]
is a cartesian diagram, with \( \pi \) (and hence \( \pi' \)) finite and flat, then the base change property \( f^*\pi_* = \pi'_*f'^* \) of 1.1 (b) follows from the commutativity of the diagram
\[
\begin{array}{c}
\pi_*f'_*f'^*\pi^*G \xrightarrow{\text{adj}_f} \pi_*\pi^*G \xrightarrow{\text{Tr}_{\pi}} G \\
\downarrow \quad \downarrow \quad \downarrow \text{adj} \\
f_*\pi'_*\pi'^*f^*G \xrightarrow{f_*\Phi} f_*f^*\pi_*\pi^*G \xrightarrow{f_*f^*\text{Tr}_{\pi}} f_*f^*G \\
\downarrow \quad \downarrow \quad \downarrow \\
f_*\pi'_*\pi'^*f^*G \xrightarrow{f_*\text{Tr}_{\pi'}} f_*f^*G
\end{array}
\]
for any torsion sheaf \( \mathcal{G} \) on \( Y \), where \( \Phi : f^* \pi_* \to \pi'_* f'^* \) is the base change morphism. Here (3) is commutative by loc. cit. (Var.2), (2) commutes by the functoriality in sheaves of \( \text{ad} f \), and (1) commutes by the definition of the base change morphism (cf., e.g., [Mi] p. 223). Note that we get an induced commutative diagram

\[
\begin{array}{ccc}
H^i(Y, \pi_* \pi^* \mathcal{G}) & \cong & H^i(Y, f^* \pi'_* \pi'^* \mathcal{G}) \\
\downarrow \text{can} & & \downarrow \text{can} \\
H^i(X, \pi^* \mathcal{G}) & \xrightarrow{\text{ad} f^*} & H^i(X, f^* \pi'^* \mathcal{G}) \\
\downarrow \text{can} & & \downarrow \text{can} \\
H^i(X', \pi'^* \mathcal{G}) & \xrightarrow{\text{ad} f'^*} & H^i(Y', f^* \mathcal{G}) \\
\end{array}
\]

Finally, property 1.1 (c) is a straightforward consequence of [SGA 4] 6.2.3.1: it implies that for \( X = X_1 \amalg X_2 \) with open and closed immersions \( \alpha_i : X_i \hookrightarrow X \) \((i = 1, 2)\) one has \((\alpha_i)^* (\alpha_j)_* \cong \delta_{ij} \text{id}_{X_i}\).

**Proposition 2.2** For every integer \( m \geq 0 \), the functor

\[ V(X) = K_m(X) \]

given by the \( m \)-th algebraic \( K \)-group is a rigid functor on the category \( \mathcal{S} \) of all noetherian schemes, except that the homotopy axiom 1.1 (c) possibly only holds for a regular base \( X \).

**Proof** This follows from the results of Quillen in [Qui]: By [Qui] §7.2, \( X \mapsto K_m(X) \) is a contravariant functor, and the limitproperty 1.1(f) is proved in loc. cit. §7, 2.2. The transfer map for a finite flat morphism \( \pi : X \to Y \) is defined as follows. Recall that \( K_m(X) = K_m(P(X)) \), the \( m \)-th \( K \)-group of the exact category \( P(X) \) of locally free coherent \( \mathcal{O}_X \)-modules. Then \( \pi_* : K_m(X) \to K_m(Y) \) is induced by the exact functor \( \pi_* : P(X) \to P(Y) \) sending an \( \mathcal{O}_X \)-module \( \mathcal{P} \) in \( P(X) \) to the \( \mathcal{O}_Y \)-module \( \pi_* \mathcal{P} \) in \( P(Y) \) (cf. loc. cit. §7). The functoriality \( (\rho \pi)_* = \rho_* \pi_* \) is immediate. For 1.1 (b) recall that the pull-back \( f^* : K_m(Y) \to K_m(Y') \) for a morphism \( f : Y' \to Y \) is induced by the exact functor \( f^* : P(Y) \to P(Y') \) sending \( \mathcal{Q} \) in \( P(Y) \) to \( f^* \mathcal{Q} = \mathcal{O}_{Y'} \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \mathcal{Q} \) (coherent pull-back) in \( P(Y') \). If now

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow \pi' & & \downarrow \pi \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

is a cartesian diagram, with \( \pi \) finite and flat, then the base change morphism \( f^* \pi_* \to \pi'_* f'^* \) is an isomorphism of exact functors from \( P(X) \) to \( P(Y') \); therefore \( f^* \pi_* = \pi'_* f'^* \) on the level of \( K \)-groups ([Qui] §1 Prop. 2). The additivity property 1.1 (c) follows from [Qui] §1, (8) and the fact that for \( X = X_1 \amalg X_2 \) the immersions \( \alpha_i : X_i \hookrightarrow X \) induce an equivalence of exact categories \( (\alpha^*_1, \alpha^*_2) : P(X) \to P(X_1) \times P(X_2) \) with \( (\alpha_i)^*(\alpha_j)_* \to \delta_{ij} \text{id} : P(X_j) \to P(X_i) \). Property 1.1 (d) follows from the more general fact that for any flat finite morphism \( \pi : X \to Y \) for which \( \pi_* \mathcal{O}_X \cong \mathcal{O}_Y^d \) as an \( \mathcal{O}_Y \)-module one has a functorial isomorphism

\[
\pi_* \pi^* P \xrightarrow{\sim} P \otimes_{\mathcal{O}_Y} \pi_* \mathcal{O}_X \xrightarrow{\sim} P^d
\]
for \( P \) in \( P(Y) \). This implies that \( \pi_*\pi^* \) is the multiplication by \( d \) on \( K_m(Y) \) by \([\text{Qui}]\) §3 Corollary 1. Finally, the homotopy property 1.1(e) for \( V(X) = K_m(X) \) is proved in \([\text{Qui}]\) §7, Proposition 4.1, for a regular base \( X \).

This covers case (3) in the introduction, while the following implies case (3) in the introduction.

**Proposition 2.3** The functors \( X \mapsto CH^m(X,n) \) of higher Chow groups are rigid functors. In particular, this holds for the classical Chow groups \( CH^r(X) = CH^r(X,0) \).

**Proof** All properties in Definition 1.1 hold, see for example \([\text{Lev}]\), section 2.1.

### 3 Sufficiently rigid functors

In view of the applications we have in mind, we note that the full strength of the axioms in 1.1 was not needed in the proof of theorem 1.2. Consider the following weakening.

**Definition 3.1** Let \( S = Sch^{noeth}/k \) be the category of noetherian \( k \)-schemes, for a field \( k \).

A contravariant functor \( V : S \mapsto Ab \) is called sufficiently rigid, if it satisfies the following properties.

(a) Call a morphism \( \pi : X \to Y \) in \( S \) admissible, if it is a finite and flat morphism of smooth \( L \)-schemes, for a field extension \( L \) of \( k \). Then for any admissible morphism \( \pi : X \to Y \) there is a transfer map \( \pi_* : V(X) \to V(Y) \), such that \((g\pi)_* = g_*\pi_* \) for another admissible morphism \( g : Y \to Z \).

(b) For every cartesian diagram of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

lying in \( S \), with \( \pi \) and \( \pi' \) admissible, one has \( f^*\pi_* = \pi'_*f'^* : V(X) \to V(Y') \).

(c) Let \( \pi : C \to D \) be a finite flat morphism of smooth curves over an extension \( L \) of \( k \). For \( y \in D \) and \( x \in C \) with \( \pi(x) = y \) consider the commutative (in general not cartesian!) diagram

\[
\begin{array}{ccc}
Spec(\kappa(x)) & \xrightarrow{\varphi_x} & C \\
\downarrow{\pi_x} & & \downarrow{\pi} \\
Spec(\kappa(y)) & \xrightarrow{\varphi_y} & D
\end{array}
\]

(3.1.1)
where \( \varphi_x, \varphi_y \), and \( \pi_x \) are the canonical morphisms, and for every smooth \( L \)-scheme \( X \) denote by

\[
(3.1.2) \quad \begin{array}{ccc}
X \times_L \kappa(x) & \xrightarrow{\varphi_x} & X \times_L C \\
\pi_x & \downarrow & \phantom{\downarrow} \\
X \times_L \kappa(y) & \xrightarrow{\varphi_y} & X \times_L D
\end{array}
\]
also the diagram obtained by base change with \( X \) over \( L \). Then

\[
\varphi_y^* \pi_* = \sum_{\pi(x) = y} e(x/y) (\varphi_x)^* : V(X \times_L C) \to V(X \times_L \kappa(y)),
\]
where \( e(x/y) \) is the ramification index of \( x \) over \( y \), and the same equality holds after base change with an open subscheme \( U \) of \( X \times L D \).

(d) If \( X \) is a smooth \( L \)-scheme, for an extension \( L \) of \( k \), then the projection \( p : \mathbb{A}^1_X \to X \) induces an isomorphism \( p^* : V(X) \to V(\mathbb{A}^1_X) \).

(e) Let \( i \mapsto X_i \) be a filtered projective system of schemes in \( S \) such that the transition morphisms \( X_i \to X_j \) are affine, and assume that \( X = \lim X_i \) is in \( S \). Then the canonical map

\[
\lim V(X_i) \to V(X)
\]
is an isomorphism.

Then we have

**Theorem 3.2** Let \( K^0 \subset K \) be fields, and let \( V : \text{Sch}^{\text{noeth}}/K^0 \to \text{Ab} \) be a sufficiently rigid functor. Then for every smooth \( K^0 \)-scheme \( X \) the restriction map

\[
V(X) \to V(X_K)
\]
is an isomorphism, provided we are in one of the following two situations:

(i) \( V \) has values in torsion abelian groups, and the fields \( K^0 \) and \( K \) are algebraically closed, or \( K = \mathbb{R} \) and \( K^0 \) is algebraically closed in \( \mathbb{R} \), or \( K \) is a complete discrete valuation field of characteristic 0, and \( K^0 \) is dense and algebraically closed in \( K \).

(ii) \( K \) is a complete discrete valuation field of characteristic \( p > 0 \), \( K^0 \) is dense and algebraically and \( K \), and \( V \) has values in torsion abelian groups without \( p \)-torsion.

**Proof** The proof of theorem 1.2 applies to the functor \( V_X \) with \( V_X(Y) = V(X \times_{K^0} Y) \). In fact, by directly using 1.6 (c), applied to \( X \times_{K^0} L \xrightarrow{id \times X \pi} X \times_{K^0} \mathbb{A}^1_L \), we need only consider transfer maps for admissible morphisms, and we only need the homotopy axiom 1.1 (e) for \( \mathbb{A}^1_{X \times_{K^0} L} = X \times_{K^0} \mathbb{A}^1_L \to X \times_{K^0} L \) with \( L \) and \( \pi : C \to \mathbb{A}^1_L \) as in the proof of theorem 1.2. Note that \( X \times_{K^0} Y = (X \times_{K^0} L) \times_L Y \) is a smooth \( L \)-scheme for a smooth \( L \)-scheme \( Y, L \) an extension of \( K^0 \).

Of course, here we only needed 3.1 (c) for the diagrams (3.1.1), and not for their localizations. This additional property will be needed in section 2. Therefore we record the following, not completely obvious fact.
Lemma 3.3 Let \( k \) be a field. A rigid functor \( V : \text{Sch}^{\text{noeth}}/k \rightarrow \text{Ab} \) is also a sufficiently rigid functor.

**Proof** We only have to show that 3.1 (c) holds for \( V \). With the notations of 3.1 (c) we have a commutative diagram

\[
\begin{array}{ccc}
\Pi \times L \kappa(x) & \xrightarrow{(\alpha_x)} & \Pi \times L (\kappa(x)[T]/(T^e(x/y))) \\
\downarrow \Pi \pi_x & & \downarrow \Pi \pi_x \\
\Pi \times L \kappa(y) & \xrightarrow{\gamma_x} & \Pi \times L \kappa(y)
\end{array}
\]

such that the square is cartesian and \( \varphi'^* \alpha_x = \varphi_x \). By assumption, the triangle can be rewritten and factored as

\[
\begin{array}{ccc}
\Pi \times L \kappa(x) & \xrightarrow{(\alpha_x)} & \Pi \times L (\kappa(x)[T]/(T^e(x/y))) \\
\downarrow id & & \downarrow \Pi \pi_x \\
\Pi \times L \kappa(x) & \xrightarrow{(\gamma_x)} & \Pi \times L \kappa(y)
\end{array}
\]

By forming the base change of the diagrams with an open subscheme \( U \) of \( X \times L D \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\Pi \times L \kappa(x) & \xrightarrow{(\alpha_x)} & \Pi \times L (\kappa(x)[T]/(T^e(x/y)+1)) \\
\downarrow \Pi \pi_x & & \downarrow \Pi \pi_x \\
\Pi \times L \kappa(x) & \xrightarrow{(\gamma_x)} & \Pi \times L \kappa(y)
\end{array}
\]

with cartesian square, and we want to show that

\[
\varphi'^* \pi_x = \sum_{\pi(x)=y} e(x/y)(\pi_x)^* \varphi'^* x : V(U') \rightarrow V(Y),
\]

where \( \varphi_x = \varphi'_x \alpha_x \). Since \( \varphi'^* \pi_x = (\Pi \pi_x)^*(\gamma_x)^*(\varphi'^*_y) \) by 1.1 (b), it suffices to show that

\[
(\gamma_x)^* = e(x/y)\alpha_x^*;
\]

by 1.1 (c). Since \( \gamma_x \alpha_x = \text{id}_{Y_x} \), we have \( \alpha_x^* \gamma_x^* = \text{id} \), and \( \gamma_x^* \) is injective. Hence it suffices to show the equality

\[
\gamma_x^* (\gamma_x)^* = e(x/y)\gamma_x^* \alpha_x^* = e(x/y)\text{id},
\]

which holds by 1.1 (d).
4 The treatment of associated Zariski sheaves

The following considerations will be useful for treating $\mathcal{K}$- and $\mathcal{H}$-cohomologies.

Let $\mathcal{S}$ be a category of schemes such that for a morphism $f : X \to Y$ in $\mathcal{S}$ and an open immersion $j : U \hookrightarrow Y$ all morphisms of the cartesian diagram

\[
\begin{array}{ccc}
U' & \xrightarrow{j'} & X \\
\downarrow f' & & \downarrow f \\
U & \xrightarrow{j} & Y
\end{array}
\]

lie in $\mathcal{S}$. Thus $\mathcal{S}$ becomes a site, if we endow it with the Zariski topology. Let $V$ be a contravariant functor on $\mathcal{S}$ with values in the category $\text{Ab}$ of abelian groups, i.e., a presheaf (of abelian groups) on $\mathcal{S}$, and let $V$ be the associated Zariski sheaf on $\mathcal{S}$. If $V_X$ and $V_X$ denote the restrictions of $V$ and $V$ to the small Zariski site $X_{\text{Zar}}$ for a scheme $X$ in $\mathcal{S}$ (which consists of all open immersions $(U \hookrightarrow X)$, then $V_X = a V_X$, where $a = a_X$ is the functor mapping a presheaf on $X_{\text{Zar}}$ to its associated sheaf (on $X_{\text{Zar}}$).

For a scheme $X$ in $\mathcal{S}$, let $H^i(X, V)$ be the $i$-th Zariski cohomology of $V$ on $X$. This is equivalently, the $i$-th derived functor of the functor $\mathcal{F} \mapsto H^0(X, \mathcal{F}) = \mathcal{F}(X)$, from sheaves on $\mathcal{S}$ to $\text{Ab}$, and also equal to $H^i(X, V_X)$, the $i$-th cohomology of $V_X$ on $X_{\text{Zar}}$ (cf. [Mi] III 1.5 (b), 1.10, and 3.1 (c)). It is clear from the first description that $X \mapsto H^i(X, V)$ is a contravariant functor from $\mathcal{S}$ to $\text{Ab}$. In the second description, this functoriality can be described as follows. Since $V$ is a sheaf on $\mathcal{S}$, one gets a canonical morphism

$$\alpha_f : V_Y \to f_* V_*$$

for every morphism $f : X \to Y$ in $\mathcal{S}$, defined via the maps

$$V_Y(U) = V(U) \xrightarrow{(f)^*} V(U') = V_X(U') = f_* V_*(U)$$

for each diagram (4.1.1). Then the wanted pull-back is the composition

\[
(4.1.2) \quad H^i(Y, V_Y) \xrightarrow{\alpha_f} H^i(Y, f_* V_*) \xrightarrow{\text{can}} H^i(X, V_X).
\]

We have the following result.

**Theorem 4.1** Let $\mathcal{S} = \text{Sch}^{\text{noeth}}/k$ be the category of noetherian $k$-schemes, for a field $k$, and let $V$ be a sufficiently rigid functor on $\mathcal{S}$. Assume that the following properties hold for rings $A$ which are localizations of (the coordinate rings of) smooth affine $L$-schemes $Y$, where $L$ is an extension field of $k$.

(i) If $A$ is semi-local, then the natural map $V(\text{Spec } A) \to H^0(\text{Spec } A, V)$ is an isomorphism, and $H^i(\text{Spec } A, V) = 0$ for $i > 0$.

(ii) If $A$ is local, then the restriction map $H^0(\text{Spec } A, V) \to H(\text{Spec } A[T], V)$ is an isomorphism, and $H^i(\text{Spec } A[T], V) = 0$ for $i > 0$. 14
Then for every $\nu \geq 0$ the functor

$$Y \mapsto H^\nu(Y, \mathcal{V})$$

is a sufficiently rigid functor on $\mathcal{S}$.

**Proof** For a scheme $Y$ denote by $P(X_{\text{Zar}})$ and $S(X_{\text{Zar}})$ the category of abelian presheaves and sheaves on the small Zariski site $X_{\text{Zar}}$, respectively, let $i : S(X_{\text{Zar}}) \to P(X_{\text{Zar}})$ be the inclusion, and let $a = a_X : P(X_{\text{Zar}}) \to S(X_{\text{Zar}})$ be its left adjoint, mapping a presheaf $P$ to its associated sheaf. For a morphism $f : X \to Y$ let $f_P : P(Y_{\text{Zar}}) \to P(Y_{\text{Zar}})$ be the direct image functor (defined by $(f_P)(U) = P(f^{-1}(U)))$, and let $f^P : P(Y_{\text{Zar}}) \to P(X_{\text{Zar}})$ be its left adjoint.

We first show the limit property 3.1 (e) for the functors $H^\nu(\cdot, \mathcal{V})$. Let $(X_i)_{i \in I}$ be a filtered projective system of schemes in $\mathcal{S}$ with affine transition maps. Assume that $X = \lim_i X_i$ is in $\mathcal{S}$, and let $u_i : X_i \to X$ be the canonical morphism. Then the morphism

$$\lim_i u_i^! \mathcal{V}_{X_i} \to \mathcal{V}_X$$

is an isomorphism. In fact, the stalk at a point $x \in X$ with images $x_i$ in $X_i$ is the map

$$\lim_i V(\mathcal{O}_{X_i, x_i}) \to V(\mathcal{O}_{X, x}) ,$$

which is an isomorphism by 3.1 (e) for $\mathcal{V}$, since $\mathcal{O}_{X, x} = \lim_i \mathcal{O}_{X_i, x_i}$.

It now follows that the map

$$\lim_i H^\nu(X_i, \mathcal{V}_{X_i}) \to H^\nu(X, \mathcal{V}_X)$$

is an isomorphism since $X_{\text{Zar}}$ is the limit of the sites $(X_i)_{\text{Zar}}$ (cf. [SGA 4] VII 5.7 for the case of étale sites; the case of Zariski sites is much simpler and follows from [EGA IV] 3, 8.6.3 and 8.10.5 (vi), cf. the proof of [SGA 4] VII 5.6).

Now let $\pi : X \to Y$ be an admissible finite and flat morphism in $\mathcal{S}$. The transfer morphisms for $\pi$ and all base changes with open immersions $U \hookrightarrow Y$ define a morphism

$$\pi_P \mathcal{V}_X \to \mathcal{V}_Y .$$

We get an induced diagram

$$\begin{array}{ccc}
\pi_* \mathcal{V}_X & \xrightarrow{\phi} & \pi_* \mathcal{V}_Y \\
\downarrow & & \downarrow \\
\pi_* \mathcal{V}_X & = & \pi_* \mathcal{V}_Y ,
\end{array}$$

where the morphism $\phi$ is the canonical map, which for instance arises from the adjunction map $id \to i a$ and the equality $\pi_* = \pi_P i$. We claim that $\phi$ is an isomorphism, Let $y \in Y$. Then the stalk of $\phi$ at $y$ is the canonical map

$$\begin{array}{ccc}
\lim_U V(U \times_Y X) & \xrightarrow{\ell} & \lim_U H^0(U \times_Y X, \mathcal{V}) \\
\downarrow & & \downarrow \\
V(\text{Spec } \mathcal{O}_{Y, y} \times_Y X) & \xrightarrow{\phi_y} & H^0(\text{Spec } \mathcal{O}_{Y, y} \times_Y X, \mathcal{V}) ,
\end{array}$$

15
where \( U \) runs through all open neighborhoods of \( y \) in \( Y \). The vertical maps are isomorphisms since 3.1 (e) holds for \( V \) and \( H^0(-, V) \). Since \( \pi \) is finite, \( \text{Spec} \mathcal{O}_{Y,y} \times_Y X = \text{Spec} A \) for a semi-local ring \( A \), which is a localization of \( X \). Since \( \pi \) is admissible, \( X \) is smooth over an extension \( L \) of \( k \). Hence \( \phi_y \) is an isomorphism by our assumption (i) on \( V \). As a consequence, we have a canonical morphism

\[
\text{Tr}_\pi : \pi_* \mathcal{V}_X \to \mathcal{V}_Y.
\]

Next we claim that \( R^i \pi_* \mathcal{V}_X = 0 \) for \( i > 0 \). In fact, the stalk at \( y \in Y \) of \( R^i \pi_* \mathcal{V}_X \) is

\[
\lim_{\to U} H^i(U \times_Y X, V),
\]

where the limit is over all open neighborhoods of \( y \) in \( Y \). This equals \( H^i(\text{Spec} \mathcal{O}_{Y,y} \times_Y X, V) \) by the limit property 3.1 (e) for \( H^i(-, V) \), and this is zero for \( i > 0 \) by the assumption (i) on \( V \) and the same arguments as before.

As a consequence, the canonical map \( H^\nu(Y, \pi_* \mathcal{V}_X) \to H^\nu(X, \mathcal{V}_X) \) is an isomorphism for all \( \nu \geq 0 \), and we define the transfer maps for \( \pi \) as the compositions

\[
\pi_* : H^\nu(X, \mathcal{V}_X) \xrightarrow{\text{Tr}_\pi} H^\nu(Y, \mathcal{V}_Y),
\]

for all \( \nu \geq 0 \). That \( (\varphi \pi)_* = \varphi_* \pi_* \), for an admissible morphism \( \varphi : Y \to Z \), is a straightforward consequence of the fact that \( \text{Tr}_{\varphi \pi} \) coincides with the composition

\[
\varphi_* \pi_* \mathcal{V}_X \xrightarrow{\varphi_* \text{Tr}_\pi} \varphi_* \mathcal{V}_Y \xrightarrow{\text{Tr}_\varphi} \mathcal{V}_Z,
\]

which easily follows from the definitions and the observation that \( a_{\varphi \pi} \pi_* \mathcal{V}_X \to \varphi_* a_{\varphi \pi} \mathcal{V}_X \) is an isomorphism.

Now we show 3.1 (b). Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\pi' \downarrow & & \downarrow \pi \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

be a cartesian diagram in \( S \), with \( \pi \) and \( \pi' \) admissible. Then the diagram

\[
\begin{array}{ccc}
\pi_* \mathcal{V}_X & \xrightarrow{\pi_* f'_*} & \pi_* f'_* \mathcal{V}_X' \\
\downarrow \text{Tr}_\pi & & \downarrow \text{Tr}_{f'} \\
\mathcal{V}_Y & \xrightarrow{\alpha_f} & f_* \mathcal{V}_Y'
\end{array}
\]

is commutative. In fact, by passing to the stalks at \( y \in Y \) we get the diagram

\[
\begin{array}{ccc}
V(\text{Spec} \mathcal{O}_{Y,y} \times_Y X) & \xrightarrow{f_*} & V(\text{Spec} \mathcal{O}_{Y,y} \times_Y X') \\
\pi_* \downarrow & & \downarrow \pi'_* \\
V(\text{Spec} \mathcal{O}_{Y,y}) & \xrightarrow{f_*} & V(\text{Spec} \mathcal{O}_{Y,y} \times_Y Y')
\end{array}
\]

which is commutative by 3.1 (b) for \( V \). The morphism \( \pi : \text{Spec} \mathcal{O}_{Y,y} \times_Y X \to \text{Spec} \mathcal{O}_{Y,y} \) is only the localization of an admissible morphism, but the map \( \pi_* \) can be defined via
passing to the limit over the maps \( \pi_s : V(U \times_Y X) \to V(U) \) for \( U \) running through the open neighborhoods of \( y \) in \( Y \). The analogous statement holds for \( \pi'_s \) and the equality \( f^* \pi_s = \pi'_s f^* \) then follows from the corresponding equalities for the \( U \).

The equality \( f^* \pi_s = \pi'_s f^* \) for the functors \( H^\nu(\cdot, \mathcal{V}) \) now follows from the commutative diagram

\[
\begin{array}{ccc}
H^\nu(X, \mathcal{V}_X) & \xrightarrow{\alpha_f} & H^\nu(X', \mathcal{V}_{X'}) \\
\downarrow\text{can} & & \downarrow\text{can} \\
H^\nu(Y, \pi_* \mathcal{V}_X) & \xrightarrow{\pi_* \alpha_f} & H^\nu(Y, \pi'_* \mathcal{V}_{X'}) \\
\downarrow\text{can} & & \downarrow\text{can} \\
H^\nu(Y, \mathcal{V}_Y) & \xrightarrow{\alpha_f} & H^\nu(Y, \mathcal{V}_{Y'}) \\
\end{array}
\]

The proof of 3.1 (c) for the \( H^\nu(\cdot, \mathcal{V}) \) is similar. By similar arguments as above, it suffices to show that the diagram

\[
\pi_* \mathcal{V}_{X \times C} \xrightarrow{\oplus \pi_* \varphi_x} \mathcal{V}_{X \times \kappa(x)} = \oplus \pi((\varphi_y)_*) \mathcal{V}_{X \times \kappa(x)}
\]

commutes, where the notations are as in 3.1 (c). But by taking the stalks at \( t \in X \times \kappa(y) \) we obtain the diagram

\[
V(Spec \ (R \times_D C)) \xrightarrow{\oplus \varphi_y} V(Spec \ (R \times_D \kappa(x)))
\]

\[
\pi_* \quad \Sigma e(x/y)(\varphi_y)_* \quad \Sigma e(x/y)(\varphi_y)_*
\]

with \( R = O_{X \times D, t} \), which is commutative by 1.6 (c) for \( V \) (for \( \pi_* \) and the \( (\pi)_* \) the same remarks as before apply).

Finally, we prove the homotopy property 3.1 (d) for the functors \( H^\nu(\cdot, \mathcal{V}) \). Let \( Y \) be smooth over an extension \( L \) of \( k \), and let \( p : \mathbb{A}^1_Y \to Y \) be the affine line over \( Y \). Then

\[
\alpha_p : \mathcal{V}_Y \to p_* \mathcal{V}_{\mathbb{A}^1_Y}
\]

is an isomorphism, and \( R^i p_* \mathcal{V}_{\mathbb{A}^1_Y} = 0 \) for \( i > 0 \), by assumption (ii) on \( V \). In fact, for \( y \in Y \), the stalk of \( \alpha_f \) at \( y \) is the pull-back map

\[
H^0(Spec \ O_{Y,y}, \mathcal{V}) \to H^0(A^1_{Spec \ O_{Y,y}}, \mathcal{V}),
\]

and the stalk \( (R^i p_* \mathcal{V})_y \) is isomorphic to \( H^i(A^1_{Spec \ O_{Y,y}}, \mathcal{V}) \) for all \( i \geq 0 \). As a consequence, we obtain the bijectivity of the maps in the composition

\[
p^* : H^\nu(Y, \mathcal{V}_Y) \xrightarrow{\text{can}} H^\nu(Y, p_* \mathcal{V}_{\mathbb{A}^1_Y}) \xrightarrow{\text{can}} H^\nu(A^1_Y, \mathcal{V}_{\mathbb{A}^1_Y}),
\]

for every \( \nu \geq 0 \). q.e.d.
Remark 4.2. (a) It seems unlikely that one can define natural transfer maps 
\[ \pi_* : H^\nu(X, \mathcal{V}) \to H^\nu(Y, \mathcal{V}) \]
for arbitrary finite and flat maps \( \pi : X \to Y \), even if one has transfer maps \( \pi_* : V(X) \to V(Y) \) for all such \( \pi \).

(b) Fix an extension \( L \) of \( k \), and let \( \pi : \mathbb{A}^1_Y \to Y \) be the affine line over a smooth \( L \)-
scheme \( Y \). Given the limit property 1.6 (e) for \( V \) (and hence for the \( H^\nu(-, \mathcal{V}) \)) the
following statements are equivalent.

1. \( \pi^* : H^\nu(U, \mathcal{V}) \to H^\nu(\mathbb{A}^1_U, \mathcal{V}) \) is an isomorphism for all \( \nu \geq 0 \) and all open subschemes \( U \) of \( Y \).

2. The morphism \( \mathcal{V}_Y \to Rp_\mathcal{V}_Y \) is a quasi-isomorphism.

3. Property 4.1 (ii) holds for all local rings \( \mathcal{O}_{Y,y} \) of \( Y \).

Theorem 4.3. Let \( k \) be any field. The functor
\[ Y \mapsto H^\nu(Y, \mathcal{K}_m) \]
where \( \mathcal{K}_m \) is the Zariski sheaf associated to the presheaf \( U \mapsto K_m(U) \) given by the \( m \)-
th algebraic \( K \)-groups, is a sufficiently rigid functor on the category of all noetherian \( k \)-
schemes.

Proof. By Theorem 4.1 we have to show the properties 4.1 (i) and (ii). The first property
is a direct consequence of results of Quillen. In fact, let \( Y \) be a noetherian scheme. In
[Qui] §7.5.4 Quillen constructed a spectral sequence
\[ E_1^{p,q}(Y) = \bigoplus_{x \in Y^{(p)}} K_{p-q}(\kappa(x)) \Rightarrow K'_{p-q}(Y) \]
which is contravariant for flat morphisms. Here \( K'_m(Y) \) is the \( m \)-th \( K \)-group of the
category \( M(Y) \) of coherent \( \mathcal{O}_Y \)-modules, but for a regular noetherian scheme \( Y \), this

group is canonically isomorphic to \( K_m(Y) \) ([Qui] §7, (1.1)), functorially for flat pull-backs
(both functorialities are induced by the exact functor mapping an \( \mathcal{O}_Y \)-module \( \mathcal{M} \) to its
coherent pull-back \( f^*\mathcal{M} = \mathcal{O}_{Y'} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M} \) for a flat morphism \( f : Y' \to Y \)). If \( Y \)
is smooth over a field \( L \), or any localization of such a scheme, then Quillen constructed a
canonical isomorphism
\[ E_2^{p,q}(Y) = H^p(Y, K_{-q}) \quad (p, q \in \mathbb{Z}), \]
compatible with flat pull-backs, by showing that for any ring \( A \) obtained by localizing (an
open affine subscheme of) \( Y \) in finitely many points, the edge morphism
\[ K_{-m}(A) \to E_2^{0,m}(\text{Spec } A) \]
is an isomorphism for all \( m \in \mathbb{Z} \) and
\[ E_2^{p,q}(\text{Spec } A) = 0 \quad \text{for } p \neq 0. \]
(loc. cit. 5.6, 5.8 and 5.10). This proves 4.1 (i) for such $A$ (by definition, the composition $K_{-m}(\text{Spec } A) \to E^{0,m}_{2}(\text{Spec } A) = H^0(\text{Spec } A, K_{-m})$ is the canonical map).

In view of (4.3.1) and Remark 4.2 (b), property 4.1 (ii) follows from results of Gillet. For it follows from [Gi] Thm. 8.3 that for the affine line $p : \mathbb{A}^1_Y \to Y$ over a smooth $L$-scheme $Y$, the pull-back

\begin{equation}
(4.3.2) \quad p^* : E^p,q_2(Y) \to E^p,q_2(\mathbb{A}^1_Y)
\end{equation}

is an isomorphism for all $p, q \in \mathbb{Z}$. In fact, without restriction, $Y$ is equidimensional of dimension $m$, and then (4.3.2) is the map $p^* : CH_{m-p,m+g}(Y) \to CH_{m+1-p,m+1+g}(\mathbb{A}^1_Y)$ of loc. cit., which is an isomorphism.

The following observation will be useful in the next section.

**Lemma 4.4** Let $V$ be a sufficiently rigid functor on the category $\text{Sch}^{\text{smooth}}/k$, where $k$ is a perfect field. Then properties 4.2 (i) and (ii) hold if and only if they hold for rings $A$ which are localizations of smooth $k$-schemes.

**Proof** (cf. [Qui ] §7, Proof of thm. 5.11) Let $L$ be an extension of $k, Y = \text{Spec } R$ a smooth affine $L$-scheme, and $A$ a semi-local ring obtained by localizing $R$ in a finite set of primes $S$. Then there exists a subfield $L'$ of $L$ finitely generated over $k$, a smooth $L'$-Algebra $R'$ and a finite subset $S'$ of $\text{Spec } R'$ such that $R = L \otimes_{L'} R'$ and such that the primes in $S$ are the base extensions of the primes in $S'$. This follows by applying [EGA IV] 8.8.2, 8.10.5, 8.7.3 and 17.7.8 to the family $L_i$ ($i \in I$) of subfields of $L$ which are finitely generated over $k$. Let $L_j$ ($j \in J$) be the subfamily of those fields which contain $L'$, and for each $j \in J$ let $S_j$ be the set of primes in $R_j = L_j \otimes_{L'} R'$ obtained by tensoring the primes in $S'$ with $L_j$, and let $A_j$ be the localization of $R_j$ in $S_j$. Then $A = \lim_{\to j} A_j$, so by the limit property 1.6 (e) for $V$ and the $H^\nu(-, V)$, we see that it suffices to show 4.2 (i) and (ii) for the localization $A$ of a smooth $L$-scheme $Y$, where $L$ is finitely generated over $k$. But since $k$ is perfect, every such $L$ is the function field of a smooth $k$-scheme; hence $A$ is the localization of a smooth $k$-scheme, and the claim follows.

## 5 Poincaré duality theories and $\mathcal{H}$-cohomology

To show the properties 4.1 (i) and (ii) for $\mathcal{H}$-cohomology, where $\mathcal{H}$ is the Zariski sheaf associated to étale cohomology, we will consider more generally the case of a twisted Poincaré duality theory (with supports) as introduced by Bloch and Ogus [BO] (1.3). It encodes the usual properties of a cohomology theory with supports for algebraic schemes over a field $k$, with an associated (Borel-Moore type) homology theory. We recall the axioms, since we need to consider them more closely.

**Definition 5.1** Let $k$ be a field, and let $\mathcal{S}$ be a category of algebraic $k$-schemes such that $Y \in \text{Ob}(\mathcal{S})$ if $X \in \text{Ob}(\mathcal{S})$ and $Y \subset X$ is locally closed.

(1) Let $\mathcal{S}^\ast$ be the category whose objects are the closed immersions $Y \subset X$ in $\mathcal{S}$ and whose morphisms are cartesian squares

\[
\begin{array}{ccc}
Y' & \subset & X' \\
\downarrow & & \downarrow \\
Y & \subset & X 
\end{array}
\]
A twisted cohomology theory (with supports) is a sequence (indexed by \( j \in \mathbb{Z} \)) of contravariant functors

\[
S \rightarrow (\text{graded abelian groups})
\]

\[(Y \subset X) \mapsto \oplus_i H^i_Y(X,j)\]

satisfying the following properties

(a) (long exact cohomology sequence) For \( Z \subset Y \subset X \), there is a long exact sequence

\[
\ldots \rightarrow H^i_Z(X,j) \rightarrow H^i_Y(X,j) \rightarrow H^i_{Y-Z}(X \setminus Z,j) \rightarrow H^{i+1}_Z(X,j) \rightarrow \ldots,
\]

functorial with respect to morphisms

\[
\begin{array}{ccc}
Z' & \subset & Y' \subset X' \\
\downarrow & & \downarrow \\
Z & \subset & Y \subset X
\end{array}
\]

in the obvious way.

(b) (excision) If \( Z \subset X \in \text{Ob} \mathcal{V}^* \) and if \( U \subset X \) is open in \( X \) and contains \( Z \), then the map \( H^i_Z(X,j) \rightarrow H^i_Z(U,j) \) is an isomorphism.

(2) Let \( S_* \) be the category with \( \text{Ob}(S_*) = \text{Ob}S \) but whose arrows are only the proper morphisms in \( S \). A twisted homology theory is a sequence (indexed by \( b \in \mathbb{Z} \)) of covariant functors

\[
\varphi_* : \rightarrow (\text{graded abelian groups})
\]

\[
X \mapsto \oplus_a H_a(X,b)
\]

such that the following properties hold.

(c) If \( \alpha : X' \rightarrow X \) is étale, there is a map

\[
\alpha^* : H_a(X,b) \rightarrow H_a(X',b)
\]

such that \((\alpha \circ \alpha')^* = \alpha^* \alpha^* \) for \( \alpha' : X'' \rightarrow X' \) étale.

(d) If the diagram below on the left is cartesian, with proper \( f \) and \( g \), and étale \( \alpha \) and \( \beta \), then the diagram on the right commutes.

\[
\begin{array}{ccc}
X' & \xrightarrow{\beta} & X \\
| & \downarrow{\alpha} & | \\
Y' & \xrightarrow{\alpha} & Y
\end{array}
\quad
\begin{array}{ccc}
H_a(X,b) & \xrightarrow{\beta^*} & H_a(X',b) \\
| & \downarrow{\alpha^*} & | \\
H_a(Y,b) & \xrightarrow{\alpha^*} & H_a(Y',b)
\end{array}
\]

(e) If \( i : Z \hookrightarrow X \) is a closed immersion in \( \mathcal{V} \), with open complement \( \alpha : V \hookrightarrow X \), then there are long exact sequences

\[
\ldots \rightarrow H_a(Z,b) \xrightarrow{i^*} H_a(X,b) \xrightarrow{\alpha^*} H_a(U,b) \rightarrow H_{a-1}(Z,b) \rightarrow \ldots
\]

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which satisfy the following compatibilities (NB only the commutativity of the squares (1) and (2) below is a new statement):

(f) If \( f : X' \to X \) is a proper morphism, restricting to \( f' : Z' \to Z \) for a closed subscheme \( Z' \subseteq X \) then the diagram

\[
\begin{array}{cccccc}
\cdots & H_a(Z', b) & \longrightarrow & H_a(X', b) & \longrightarrow & H_a(U', b) & \longrightarrow & H_a(Z', b) & \longrightarrow & \cdots \\
| & f' & | & f_* & | & f_* & & f_* & | & \alpha^* \\
\cdots & H_a(Z, b) & \longrightarrow & H_a(X, b) & \longrightarrow & H_a(U, b) & \longrightarrow & H_a(Z, b) & \longrightarrow & \cdots \\
\end{array}
\]

commutes, where \( \alpha : f^{-1}(U) = X' - f^{-1}(Z) \to X' - Z' = U' \) is the open immersion.

(g) If \( \alpha : X' \to X \) is étale, then the diagram

\[
\begin{array}{cccccc}
\cdots & H_a(Z', b) & \longrightarrow & H_a(X', b) & \longrightarrow & H_a(U', b) & \longrightarrow & H_a(Z', b) & \longrightarrow & \cdots \\
| & \alpha^* & | & \alpha^* & | & \alpha^* & & \alpha^* & | & \alpha^* \\
\cdots & H_a(Z, b) & \longrightarrow & H_a(X, b) & \longrightarrow & H_a(U, b) & \longrightarrow & H_a(Z, b) & \longrightarrow & \cdots \\
\end{array}
\]

commutes, where \( Z' = f^{-1}(Z) \) and \( U' = f^{-1}(U) \).

(3) A Poincaré duality theory is given by a cohomology and homology theory as above, together with the following structures.

(h) (cap product) For \( Y \subset X \in \text{Ob } \varphi^* \) there is a pairing

\[ H_a(X, b) \otimes H^i_Y(X, j) \to H_{a-i}(Y, b - j) \]

compatible with étale pull-backs, in the obvious way.

(i) (projection formula) For a cartesian diagram below on the left, with proper \( f \), the diagram on the right commutes.

\[
\begin{array}{ccccccc}
Y' & \longrightarrow & X' & \longrightarrow & H_a(X', b) & \otimes & H^i_Y(X', j) & \longrightarrow & H_{a-i}(Y', b - j) \\
\downarrow & & \downarrow f & & f_* & & f_* & & f_* \\
Y & \longrightarrow & X & \longrightarrow & H_a(X, b) & \otimes & H^i_Y(X, j) & \longrightarrow & H_{a-i}(Y, b - j) \\
\end{array}
\]

(j) (fundamental class) If \( X \in \text{Ob } \varphi \) is irreducible and of dimension \( d \), then there is a canonical element \( \eta_X \in H_{2d}(X, d) \). If \( \alpha : X' \to X \) is étale, then \( \alpha^* \eta_X = \eta'_X \).

(k) (Poincaré duality) If \( X \in \text{Ob } \varphi \) is smooth of pure dimension \( d \), and \( Y \subset X \) is a closed subscheme, then cap-product with \( \eta_X \) induces an isomorphism

\[ \eta_X \cap : H^i_Y(X, j) \xrightarrow{\sim} H_{2d-i}(Y, d - j) \]

(l) (Principal triviality) Let \( i : W \hookrightarrow X \) be a smooth principal divisor in the smooth scheme \( X \). Then \( i_* \eta_W = 0 \).

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Write $H^i(X,j)$ for $H^i_X(X,j)$. By results of Bloch and Ogus we then have:

**Proposition 5.2** Let $k$ be a perfect field, and let

\[(Z \subset X) \mapsto H^*_Z(X,*) , \quad X \mapsto H_*(X,*)\]

be a twisted Poincaré duality theory on the category $\text{Sch}^{alg}/k$ of all algebraic $k$-schemes. Let $H^i(j)$ be the Zariski sheaf associated to the presheaf $U \mapsto H^i(U,j)$. Then for all $i,j \in \mathbb{Z}$ the functor $V : (\text{Sch}^{alg}/k)^0 \to \text{Ab}$ defined by

\[V(Y) = H^i(Y,j)\]

satisfies property 4.1 (i) for semi-local regular rings $A$ which are localizations of smooth $k$-schemes.

Since $V$ is a priori only defined on $k$-schemes of finite type, this statement has to be interpreted in the following way: the functors $V$ and $H^\nu(-,V)$ are defined on $\text{Spec} A$ by taking the limit of the value groups at all opens $U \subset X$ containing $\text{Spec} A$ (or, equivalently, the maximal ideals of $A$).

**Proof of 5.2** For any algebraic $k$-scheme $X$ denote by $X_{(p)}$ the set of points $x \in X$ of dimension $p$, and for $x \in X$ put

\[H_a(x,b) = \lim_{U \subset Z \text{ open}} H_a(U,b) , \]

where $Z = \{x\}$ is the Zariski closure of $x$. Then Bloch and Ogus construct a homological spectral sequence

\[(5.2.1) \quad E^1_{p,q} = E^1_{p,q}(X,b) = \bigoplus_{x \in X_{(p)}} H_{p+q}(x,b) \Rightarrow H_{p,q}(X,b)\]

as follows ([BO] (3.7)). Let $Z_p = Z_p(X)$ be the set of all closed subsets $Z \subset X$ of dimension $\leq p$, ordered by inclusion, and put

\[H_a(Z_p(X),b) = \lim_{Z \in Z_p} H_a(Z,b) . \]

Then by 4.1 (e) one gets exact sequences

\[(5.2.2) \quad \ldots \to H_a(Z_{p-1},b) \xrightarrow{i} H_a(Z_p,b) \xrightarrow{j} \bigoplus_{x \in X_{(p)}} H_a(x,b) \xrightarrow{k} H_{a-1}(Z_{p-1},b) \to \ldots . \]

The method of exact couples now gives the desired spectral sequence. Note that by definition the differential $d^1_{p,q}$ is the composition

\[d^1_{p,q} : E^1_{p,q} \xrightarrow{k} H_{p+q-1}(Z_{p-1},b) \xrightarrow{j} E^1_{p-1,q} . \]

Moreover, if $\dim X = d$, one has a complex

\[(5.2.3) \quad 0 \to H_a(X,b) \xrightarrow{\epsilon} E^1_{d,a-d}(X,b) \xrightarrow{d^1} E^1_{d-1,a-d}(X,b) \xrightarrow{d^1} \ldots , \]

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in which $\varepsilon$ is the edge morphism. By 5.1 (g) the sequences (5.2.3) for all opens $U \subseteq X$ form a complex of Zariski presheaves. Let $\mathcal{H}_a(b)$ and $\mathcal{E}^1_{p,q}(b)$ be the Zariski sheaves associated to $U \mapsto H_a(U, b)$ and $U \mapsto E^1_{p,q}(U, b) = \oplus_{x \in U} H_{p+q}(x, b)$, respectively, so that we get a complex of Zariski sheaves

$$0 \to \mathcal{H}_a(b) \to \mathcal{E}^1_{d,a-d}(b) \to \mathcal{E}^1_{d-a,a-d}(b) \to \ldots.$$  

(5.2.4)

It is clear from the definition that $U \mapsto E^1_{p,q}(U, b)$ is already a sheaf, and is flabby.

Now let $X$ be smooth. Then Bloch and Ogus [BO] show that for any finite set $S \subset X$ which is contained in an affine open, the sequence (5.2.3) becomes exact after passing to the limit over all opens $V \subset X$ containing $S$. In fact, this is equivalent to the statement that all maps $i : H_a(Z_{p-1}(X), b) \to H_a(Z_p(X), b)$ vanish after passing to the limit over such $U$, and this is shown in [BO], section 4 and 5 (in the claim on p.191 loc. cit. only the case of a one-element set $S$ is stated, but the proof works more generally, since the trick of Quillen quoted loc. cit. is valid for a finite $S$ as above). In particular, (5.2.4) is an exact sequence of Zariski sheaves, hence a resolution of $\mathcal{H}_a(b)$ by the complex with flabby components

$$R_{a-d}(b) = \mathcal{E}^1_{d-a-d}(b).$$  

(5.2.5)

As a consequence, we get a canonical isomorphism

$$H^\nu(U, \mathcal{H}_a(b)) = E^2_{d-\nu,a-d}(U, b)$$  

(5.2.6)

for every open $U$ in $X$ and $\nu \geq 0$, if $X$ is smooth. Moreover, if $A$ is a semi-local ring obtained by localizing $X$, then $H^\nu(Spec A, \mathcal{H}_a(b)) = 0$ for all $\nu > 0$, and the map

$$H_a(Spec A, b) \to E^2_{a-d-a}(Spec A, b) = H^0(Spec A, \mathcal{H}_a(b))$$

is an isomorphism. Since the presheaves $U \mapsto H_a(U, b)$ and $U \mapsto H^{2d-a}(U, d - b)$ are isomorphic by 4.1 (h), (j) and (k), the proposition follows.

For the treatment of the homotopy property 4.1 (ii) for $\mathcal{H}$-cohomology we need the following extended version of a Poincaré duality theory.

**Definition 5.3** Let $k$ be a field. A twisted Poincaré duality theory

$$(Z \subset X) \mapsto H^*_Z(X, \ast), \ X \mapsto H_* (X, \ast)$$

on $Sch^{alg}/k$ is called an extended Poincaré duality theory, if for every flat morphism $f : X' \to X$ which is equidimensional of dimension $m$ (i.e., whose fibres are either empty or equidimensional of dimension $m$), cf. [EGA IV], 13.3) there are functorially associated maps

$$f^* : H_a(X, b) \to H_{a+2m}(X', b + m),$$

agreeing with the pull-back maps in 5.1 (c) for étale $f$ and $m = 0$, such that the following further properties hold.

(m) If $X$ and $X'$ are irreducible, then $f^*\eta_X = \eta_{X'}$.  

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(n) If $Z \subset X$ is a closed subscheme, and $Z' = Z \times_X X' \subset X'$, then the following diagram commutes.

$$
\begin{array}{ccc}
H_{a+2m}(X', b + m) & \otimes & H_{a+2m}(Z', b - j + m) \\
| f^* & & | f^* \\
H_a(X, b) & \otimes & H_a(Z, b - j)
\end{array}
$$

(o) If $Z$ is closed in $X$ and $U$ is the open complement, then one has a commutative diagram

$$
\begin{array}{ccc}
H_{a+2m}(Z', b+m) & \longrightarrow & H_{a+2m}(Z', b+m) \\
| f^* & & | f^* \\
H_a(Z, b) & \longrightarrow & H_a(U, b)
\end{array}
$$

where $Z' = Z \times_X X'$ and $U' = f^{-1}(U)$.

We shall give some examples below. First we note

**Proposition 5.4** Let $k$ be a perfect field, and let $(Z \subset X) \mapsto H^i(Z, *)$, $X \mapsto H_*(X, *)$ be an extended Poincaré duality theory on $\text{Sch}^{\text{alg}}/k$. Assume that the following “homotopy invariance” holds:

(p) For every smooth $k$-scheme $X$ the maps

$$p^*: H^i(X, j) \longrightarrow H^i(\mathbb{A}^1_X, j)$$

induced by the projection $p : \mathbb{A}^1_X \to X$ are isomorphisms for all $i, j \in \mathbb{Z}$.

Then for all $i, j \in \mathbb{Z}$ the contravariant functor $V$ on $\text{Sch}^{\text{alg}}/k$ given by

$$V(Y) = H^i(Y, j)$$

satisfies property 4.1 (ii) for local rings $A$ which are local rings $\mathcal{O}_{Y,y}$ of smooth $k$-schemes $Y$.

**Proof** (The values on $\text{Spec} A$ are defined as in Proposition 5.2). Let $f : X' \to X$ be a flat morphism in $\text{Sch}^{\text{alg}}/k$ which is equidimensional of dimension $m$.

We first note that by 5.3 (o) the flat pull-backs induce maps between the sequences (5.2.2) for $X$ and $X'$, with appropriate shift of degrees (for $Z \subset X$ of dimension $p$ the preimage $f^{-1}(Z) = Z \times_X X'$ is of dimension $p + m$). This induces a map of spectral sequences from (5.2.1) (with the indicated shift of degrees)

$$
\begin{array}{ccc}
E^1_{p+m, q+m}(X', b + m) & \Longrightarrow & H_{p+q+z}(X', b + m) \\
| f^* & & | f^* \\
E^1_{p, q}(X, b) & \Longrightarrow & H_{p+q}(X, b)
\end{array}
$$

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In particular, there are natural pull-back maps

\[(5.4.2) \quad f^*: E^2_{p,q}(X, b) \to E^2_{p+m,q+m}(X', b + m)\]

On the other hand, if \(X\) and \(X'\) are irreducible of dimensions \(d\) and \(d'\), respectively, (so that \(m = d' - d\)), then by 3.3 (m) and (n) the diagram

\[(5.4.3) \quad \begin{array}{ccc}
H^i(X', j) & \xrightarrow{\eta(X')} & H_{2d'-i}(X', d' - j) \\
\downarrow f^* & & \downarrow f^* \\
H^i(X, j) & \xrightarrow{\eta(X)} & H_{2d-i}(X, d - j)
\end{array}\]

is commutative. Now let \(X\) and \(X'\) be smooth. Then the horizontal maps in the above diagram are isomorphisms by Poincaré duality 3.1 (k). The same is true for open subschemes; hence the pull-back map \(f^*: H^\nu(X, H^i(j)) \to H^\nu(X', H^i(j))\) can be identified with a pull-back map

\[(5.4.4) \quad f^*: H^\nu(X, H_{2d-i}(d - j)) \to H^\nu(X', H_{2d'-i}(d' - j))\]

which is defined in a way analogous to (4.2.2), using the compatibility of flat pull-backs with open immersions. Here \(H_a(b)\) is a Zariski sheaf on \(X\) or \(X'\) associated to the presheaf \(U \mapsto H^a(U, b)\).

Furthermore, it follows from (5.4.1) that, via the isomorphisms (5.2.6) for \(X\) and \(X'\), the pull-back map (5.4.4) can be identified with the pull-back map (5.4.2) for \((p, q, b, m) = (d - \nu, d - i, d - j, d' - d)\). Thus, in view of Remark 4.2 (b), the Proposition follows from part (ii) of the Lemma below, applied to smooth \(k\)-schemes \(X\).

**Lemma 5.5** In the situation of Proposition 5.4, let \(X\) be any algebraic \(k\)-scheme (not necessarily smooth), and let \(p: A^1_X \to X\) be the affine line over \(X\). Then the following holds.

(i) The flat pull-back maps \(p^*: H_a(X, b) \to H_{a+2}(A^1_X, b + 1)\) are isomorphisms for all \(a, b \in \mathbb{Z}\).

(ii) The pull-back maps \(p^*: E^2_{p,q}(X, b) \to E^2_{p+1,q+1}(A^1_X, b + 1)\) are isomorphisms for all \(p, q, b \in \mathbb{Z}\).

**Proof** We proceed by induction on \(\text{dim}(X)\), and we may and will consider only reduced schemes (since \(H_a(X, b) = H_a(X_{\text{red}}, b)\) by 4.1 (e)). For \(\text{dim}(X) = 0\) we then may assume that \(X = \text{Spec} K\) for a finite extension \(K\) of \(k\), necessarily separable since \(k\) is perfect. Then (5.4.1) gives a commutative diagram with exact top row

\[(5.5.1) \quad \begin{array}{cccccc}
0 & \xrightarrow{} & E^2_{0,a}(A^1_K, b + 1) & \xrightarrow{} & H_{a+2}(A^1_K, b + 1) & \xrightarrow{} & E^2_{1,a-1}(A^1_K, b + 1) & \xrightarrow{} & 0 \\
& & \uparrow p^* & & \uparrow p^* & & \uparrow p^* & \\
& & H_a(\text{Spec}(K), b) & \xrightarrow{\sim} & E^2_{0,a}(\text{Spec}(K), b)
\end{array}\]
in which the middle vertical map is an isomorphism by the assumption 5.4 (p) and Poincaré duality (cf. (5.4.3)). On the other hand, the right hand vertical map is injective: it can be identified with

\[ p^* : H^0(\text{Spec} \ (K), \mathcal{H}^{-a}(-b)) \to H^0(\mathbb{A}^1_K, \mathcal{H}^{-a}(-b)) \]

and any \( K \)-rational point \( s : \text{Spec} \ K \to \mathbb{A}^1_K \) gives a left inverse \( s^* \) of \( p^* \). Putting this together, we deduce that both vertical maps are isomorphisms and that \( E^2_{p,q}(\mathbb{A}^1_K, b + 1) = 0 \). This shows (i) and (ii) in this case, since \( E^2_{p,q}(\text{Spec} \ (K), b) = 0 \) for \( p \neq 0 \) and \( E^2_{p,q}(\mathbb{A}^1_K, b) = 0 \) for \( p \neq 0 \) or 1.

If \( X \) has positive dimension (and is reduced), then there is a dense open subscheme \( U \subseteq X \) which is smooth (because \( k \) is perfect). Since (i) holds for \( U \) by Poincaré duality and assumption, and for \( Z = X - U \) by induction hypothesis, it holds for \( X \) by 5.3 (o) and the five-lemma.

For (ii) we observe the following: if \( Z \hookrightarrow X \) is a closed subscheme and \( U = X - Z \hookrightarrow X \) is the open complement, then one has an exact sequence of complexes

\[
(5.5.2) \quad 0 \to E^1_{q,Z}(Z, b) \xrightarrow{i_*} E^1_{q,U}(X, b) \xrightarrow{j^*} E^1_{q,U}(U, b) \to 0.
\]

Here \( j^* \) comes from (5.4.1) for \( f = j \), and \( i_* \) comes from the contravariance of the spectral sequence (5.2.1) for the proper morphism \( i \), which is an immediate consequence of 3.1 (f). The exactness of the sequence follows from the easily verified fact that in degree \( p \) the sequence is given by

\[
0 \to \bigoplus_{x\in Z(p)} H^{p+q}(\kappa(x), b) \xrightarrow{i_*} \bigoplus_{x\in X(p)} H^{p+q}(\kappa(x), b) \xrightarrow{j^*} \bigoplus_{x\in U(p)} H^{p+q}(\kappa(x), b) \to 0,
\]

where \( i_* \) and \( j^* \) are the obvious inclusion and projection, respectively. There is a corresponding exact sequence for the triple \( \mathbb{A}^1_Z \hookrightarrow \mathbb{A}^1_X \hookleftarrow \mathbb{A}^1_U \), and both exact sequences are connected by the pull-back maps for the projections \( p_X : \mathbb{A}^1_X \to X, p_U \) and \( p_Z \), in a commutative map. Passing to the cohomology, we obtain a commutative diagram with long exact rows

\[
\begin{align*}
\ldots E^{2}_{p+1,q+1}(\mathbb{A}^1_Z, b+1) & \xrightarrow{p^*_Z} E^{2}_{p+1,q+1}(\mathbb{A}^1_X, b+1) & \xrightarrow{p^*_X} E^{2}_{p+1,q+1}(\mathbb{A}^1_U, b+1) & \xrightarrow{p^*_U} E^{2}_{p+1,q+1}(\mathbb{A}^1_Z, b+1) .. \\
E^{2}_{p,q,Z}(Z, b) & \xrightarrow{p^*_Z} E^{2}_{p,q,U}(U, b) & \xrightarrow{p^*_U} E^{2}_{p,q,Z}(Z, b) & \xrightarrow{p^*_Z} ..
\end{align*}
\]

By induction hypothesis we may assume that all \( p^*_Z \) are isomorphisms for \( \dim Z < \dim X \). Hence it suffices to show that the \( p^*_U \) become isomorphisms after passing to the limit over all dense opens \( U \subseteq X \). Moreover, we may restrict to the case that we consider the limit over all open neighbourhoods of a fixed generic point \( \eta \). If \( K = \kappa(\eta) \) is the function field of the corresponding connected component of \( X \), then we obtain formally the same diagram as (5.5.1), by putting \( H_a(K, b) = \lim H_a(U, b), H_a(\mathbb{A}^1_K, b) = \lim H_a(\mathbb{A}^1_U, b) \) etc. By reasoning in a completely similar way as in the case of a finite extension \( \mathcal{K} \) of \( k \), we deduce that the pull-back map

\[
p^* : E^2_{p,q}(K, b) \to E^2_{p+1,q+1}(\mathbb{A}^1_K, b + 1)
\]

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is an isomorphism as wanted (the map is injective since \( p^* : E^2_{p,q}(U,b) \to E^2_{p+1,q+1}(\mathbb{A}_U^1, b+1) \) is injective for each smooth open \( U \subseteq X \)).

**Remark 5.6** The proof of 5.5 follows very much Gillet’s proof of the corresponding statement for the Chow groups \( CH_{r,s}(X) \) ([Gi] Thm. 8.3), except that the proof for \( X = \text{Spec} \,(K) \) via (5.5.1) is more direct than the recursion to the projective bundle theorem in [Gi] Lemma 8.4.

We want to apply the above to the following example.

**Proposition 5.7** Let \( k \) be a field, and fix \( n \in \mathbb{N} \) invertible in \( k \) The following functors form an extended Poincaré duality theory on \( \text{Sch}^{ad}/k \), and the properties 4.1 (i) and (ii) hold for them (In particular, by Poincaré duality, the homotopy invariance 5.4 (p) holds for them):

\[
\begin{align*}
H^j_{Z}(X,j) & = H^j_{Z}(X,\mathbb{Z}/n(j)) & \text{(étale cohomology with supports)}, \\
H_a(X,b) & = H_a(X,\mathbb{Z}/n(b)) := H^{-a}(X,a_X!\mathbb{Z}/n(-b)) & \text{(étale homology)}
\end{align*}
\]

Here \( \mathbb{Z}/n(j) = \mu_{n^j} \) as in 2.1, \( a_X : X \to \text{Spec} \,(k) \) is the structural morphism, and \( a_X^! : D^b_c(\text{Spec}(k),\mathbb{Z}/n) \to D^b_c(X,\mathbb{Z}/n) \) (= derived category of bounded complexes of étale \( \mathbb{Z}/n \)-sheaves on \( X \) with constructible cohomology) is the right adjoint of \( R(a_X)_! \) constructed in Grothendieck-Verdier duality [SGA 4] XVII, 3.

**Proof** That étale cohomology and homology from a twisted Poincaré duality theory, follows from the results in [SGA 4], cf. the sketch in [BO] 2.1. We now describe flat pullbacks in the homology, for a flat morphism \( f : X' \to X \) which is equidimensional of dimension \( m \) (cf. the discussion for an algebraically closed field \( k \) in Laumon’s article [DV] VIII, 5). By [SGA 4] XVIII 2.9, there is a canonical trace morphism

\[
(5.7.1) \quad Tr_f : R^{2m}f_*f^*\mathcal{F}(m) \to \mathcal{F},
\]

for any étale \( \mathbb{Z}/n \)-sheaf \( \mathcal{F} \) on \( X \), coinciding with the trace morphism used in 2.1 for finite \( f \) (in which case \( m = 0 \) and \( f^! = f_* \)). Since \( R^if_*\mathcal{F} = 0 \) for \( i > 2m \), this trace morphism can be regarded as a morphism

\[
Tr_f : Rf_!f^*\mathcal{F}(m)[2m] \to \mathcal{F}
\]

in \( D^b_c(X,\mathbb{Z}/n) \). This can be extended to arbitrary complexes \( \mathcal{L} \) in \( D^b_c(X,\mathbb{Z}/n) \) as follows. If let \( \mathcal{F} = \mathbb{Z}/n \) and tensor with \( \mathcal{L} \), then by the “projection formula” isomorphism

\[
(5.7.2) \quad \gamma_f : Rf_!\mathcal{K} \otimes_{\mathbb{Z}/n} f^*\mathcal{L} \sim Rf_!(\mathcal{K} \otimes_{\mathbb{Z}/n} f^*\mathcal{L})
\]

([SGA 4] XVII 5.2.9), we obtain a trace morphism

\[
(5.7.3) \quad Tr_f : Rf_!f^*\mathcal{L}(m)[zm] \to \mathcal{L}
\]

(cf. [SGA 4] XVIII 2.13.2). By adjunction between \( Rf_! \) and \( f^! \), we now get a morphism

\[
t_f : f^*\mathcal{L}(m)[2m] \to f^!\mathcal{L}.
\]
induces the wanted pull-back maps

$f^*: H^{-a}(X, a^1_X\mathbb{Z}/n(-b)) \rightarrow H^{-a}(X', f^*a^1_X\mathbb{Z}/n(-b))$,

where the first arrow is the restriction morphism which exists for any complex of sheaves $\mathcal{K}$ (by the composition $H^*(X, \mathcal{K}) \rightarrow H^*(X, Rf_*f^*\mathcal{K}) = H^*(X', f^*\mathcal{K})$)

If $f$ is étale, then $f^!$ is identified with $f^*$ via $t_f$, and the pull-back by definition is the one used for étale morphisms in homology (cf. [BO] 2.1). The functoriality of flat pull-backs is a direct consequence of the “transitivity” of the trace maps (3.7.1) (cf. [SGA 4] XVIII 2.9, (Var 3)). This in turn implies 5.3 (m), because $\eta_X$ is the image of $1 \in \mathbb{Z}/n = H_0(Spec(k), \mathbb{Z}/n)$ under

$a^*_X : H_0(Spec(k), \mathbb{Z}/n) \rightarrow H_{2d}(X, \mathbb{Z}/n(d))$

if $X$ is irreducible of dimension $d$ (cf. [SGA 4], [cycle], 2.3).

For 5.3 (n) we recall that the cap product is induced by a pairing $a^1_X\mathbb{Z}/n(r) \otimes a^1_X\mathbb{Z}/n(s) \rightarrow a^1_X\mathbb{Z}/n(r + s)$ (cf. [BO] 2.1) which is a special case of the following, more general one. For any morphism $g : X \rightarrow Y$ in $Sch^{alg}/k$ and any $\mathbb{Z}/n$-sheaves $\mathcal{F}, \mathcal{G}$ on $Y$ (in fact, any objects $\mathcal{F}, \mathcal{G}$ in $D^b(Y, \mathbb{Z}/n)$), one has a pairing

$\varphi_g : g^!\mathcal{F} \otimes_L g^*\mathcal{G} \rightarrow g^!(\mathcal{F} \otimes_L \mathcal{G})$

which by adjunction corresponds to the horizontal morphism making the diagram

\[
\begin{array}{ccc}
R g_!(g^!\mathcal{F} \otimes_L g^*\mathcal{G}) & \rightarrow & \mathcal{F} \otimes_L \mathcal{G} \\
\gamma_g \downarrow & & \downarrow Ad_g \otimes id \\
R g_!g^!\mathcal{F} \otimes_L \mathcal{G}
\end{array}
\]

commutative. Here the vertical isomorphism is the projection formula isomorphism (5.7.2), and $Ad_g : Rg_!g^!\mathcal{F} \rightarrow \mathcal{F}$ is the adjunction map.

Now let $f : X' \rightarrow X$ be a flat morphism which is equidimensional of dimension $m$, and put $g' = gf : X' \rightarrow Y$. Then we claim that the diagram

\[
(5.7.4) \quad f^!g^!\mathcal{F} \otimes_L f^*g^*\mathcal{G} \quad \begin{array}{ccc} t_f \otimes id \end{array} \quad \rightarrow \quad \begin{array}{ccc} \varphi_{g'} \end{array} \quad g^!\mathcal{F} \otimes_L g^*\mathcal{G} \quad \begin{array}{ccc} t_f \end{array} \quad f^*g^!\mathcal{F}\{m\} \otimes_L f^*g^*\mathcal{G} \quad \begin{array}{ccc} f^!\varphi_{g'} \end{array} \quad f^*g^!\mathcal{F}\{m\} \otimes_L f^*g^*\mathcal{G}\{m\}
\]

commutes, where we put $\mathcal{H}\{m\} = \mathcal{H}(m)[2m]$ for a complex $\mathcal{H}$. By adjunction, this amounts to the commutating of the following two diagrams (where we have written $f_i$ for
$Rf_1$, etc.)

\[
g_! (g^! F \otimes_L g^* G) \xrightarrow{g_! \varphi_g} g_! g^! (F \otimes G) \xrightarrow{Ad_g} F \otimes G
\]

(1)

\[
g_! (f^* g^! F \{m\} \otimes g^* G) \xrightarrow{g_!^{\gamma g}} g_! f^* g_!^! F \{m\} \otimes g^* G
\]

(4)

\[
g_! f_! f^* (g_!^! F \otimes g^* G) \{m\}
\]

(5)

Here the diagrams (1) and (3) commute by the definitions of \( \varphi \) and \( t \), respectively, and the identification \( g^! = f^! g_! \) is just the one for which the diagrams (2) commute (these morphisms being defined as the adjoints of \( g_!^i \), \( f_! \) and \( g_! \)). The commutativity of (4) is easily checked (cf. also [SGA 4] XVII 5.2.4), and (5) commutes by our definition of (5.7.3), together with an obvious ”associativity” for \( \varphi_f \). The remaining squares commute by functoriality.

Now let \( Y = \text{Spec} \ (k) \). Then by definition the map product for \( X \) is the composition

\[
H^{-a}(X, a_1 X \mathbb{Z}/n(-b)) \otimes H^a_Z(X, a_1 X \mathbb{Z}/n(j)) \xrightarrow{H_Z^{-a}(X, a_1 X \mathbb{Z}/n(j - b))} H^a_Z(X, a_1 X \mathbb{Z}/n(j - b))
\]

induced by the usual cup product (with supports) and \( \varphi_{ax} \), together with the identification \( H_Z(X, a_1 X F) = H(Z, R^i a_1 X F) = H(Z, a_1 X F) \) for \( i : Z \hookrightarrow X \). Thus 3.3 (n) follows from the commutativity of (5.7.4).

Next we consider 5.3 (o). Let \( i : Z \hookrightarrow X \) be a closed immersion with open complement \( j : U = X - Z \hookrightarrow X \). For a flat morphism \( f : X' \to X \), which is equidimensional of
dimension \( m \), consider the cartesian squares

\[
\begin{array}{ccc}
Z' & \rightarrow & X' \\
\downarrow f_Z & & \downarrow f \\
Z & \rightarrow & X \\
\end{array}
\]

The relative sequence 3.1 (e) for the triple \((Z, X, U)\) is obtained by taking the cohomology on \( X \) of the canonical exact triangle

\[
i_*R^1a_X^1\mathbb{Z}/n(-b) \xrightarrow{\text{Ad}} a_X^1\mathbb{Z}/n(-b) \xrightarrow{a} Rj_*j^*a_X^1\mathbb{Z}/n(-b) \rightarrow ,
\]

and identifying \( Ri^1a_X = a_Z^1 \) and \( j^*a_X = a_U^1 \) (Here we used that \( Ri_* = i_* \), and have written \( Ri^1 \) since \( i^! \) may be misinterpreted as the functor “ sections with support in \( Z \)” whose derivative \( Ri^1 \) is). Hence 5.3 (o). follows from the fact that one has natural identifications of exact triangles

\[
i_*R^1a_X^1\mathcal{F} \rightarrow a_X^1\mathcal{F} \rightarrow Rj_*j^*a_X^1\mathcal{F} \rightarrow
\]

\[
i_*R^1a_X^1\mathcal{F} \rightarrow f^!a_X^1\mathcal{F} \rightarrow Rj_*f^!j^*a_X^1\mathcal{F} \rightarrow
\]

\[
i_*f_Z^!R^1a_X^1\mathcal{F} \rightarrow f^!a_X^1\mathcal{F} \rightarrow Rj_*f^!j^*a_X^1\mathcal{F} \rightarrow
\]

for any complex of sheaves \( \mathcal{F} \) on \( \text{Spec}(k) \), where \( \beta' \) and \( \beta'' \) are obtained from the ”adjoint base change isomorphism” [SGA 4] XVIII 3.1.12.3. Note that by definition the diagram

\[
i_*R^1f^!G \xrightarrow{\text{Ad}} f^!G \]

\[
i_*f_Z^!R^1G \xrightarrow{\beta} f^!i_*R^1G
\]

commutes, similarly for \( \beta'' \) (where \( j^! = j^* \) and \( j'' = j^{**} \)).

Finally we show that the homotopy invariance 5.5 (i) holds for étale homology (this implies property 5.5 (ii) as well, as is clear from the proof of 5.5). Since \( p : A_X^1 \rightarrow X \) is acyclic (cf. [Mi] VI 4.20), the restriction map

\[
H^{-a}(X, a_X^1\mathbb{Z}/n(-b)) \rightarrow H^{-a}(A_X^1, p^*a_X^1\mathbb{Z}/n(-b))
\]

is an isomorphism for all \( a, b, \in \mathbb{Z} \). We conclude by recalling that

\[
t_p : p^*\mathcal{L}(1)[2] \rightarrow p^!\mathcal{L}
\]

is an isomorphism for any \( \mathcal{L} \) in \( D^b_c(X, \mathbb{Z}/n) \) (for this it suffices that \( p \) is smooth of relative dimension 1, cf. [SGA 4] XVIII proof of 3.2.5). We can now collect the fruits of our efforts.
Theorem 5.8 Let $k$ be a field, and let $n$ be a natural number invertible in $k$. For $i, j \in \mathbb{Z}$ let $\mathcal{H}^i(j)$ be the Zariski sheaf on the category $\text{Sch}^{\text{noeth}}/k$ of all noetherian $k$-schemes associated to the presheaf

$$U \mapsto H^i(U, \mathbb{Z}/n(j))$$

given by étale cohomology. Then for all $\nu, i, j \in \mathbb{Z}$ the functor

$$X \mapsto H^\nu(X, \mathcal{H}^i(j))$$

is a sufficiently rigid functor on $\text{Sch}^{\text{noeth}}/k$.

Proof By Proposition 2.1 and Lemma 3.3, étale cohomology is a sufficiently rigid functor on $\text{Sch}^{\text{noeth}}/k$. In view of Theorem 4.1, we then have to show 4.1 (i) and (ii) for the $\mathcal{H}^i(j)$. Since it suffices to consider the bigger category $\text{Sch}^{\text{noeth}}/k_0$, where $k_0$ is the prime field, we may assume that $k$ is perfect, and by Lemma 4.4 we may restrict our attention to algebraic $k$-schemes. By Proposition 5.7, $X \mapsto H^\nu(X, \mathbb{Z}/n(\cdot))$ is part of an extended Poincaré duality theory with homotopy invariance. Therefore the claim follows from Propositions 5.2 and 5.4.

By Theorem 5.8, we obtain case (5) of Theorem 0.3 in the introduction.

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