A GENERALIZATION OF NEUMANN'S QUESTION

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Abstract. Let $G$ be a group, $m \geq 2$ and $n \geq 1$. We say that $G$ is an $T(m, n)$-group if for every $m$ subsets $X_1, X_2, \ldots, X_m$ of $G$ of cardinality $n$, there exists $i \neq j$ and $x_i \in X_i, x_j \in X_j$ such that $x_i x_j = x_j x_i$. In this paper, we give some examples of finite and infinite non-abelian $T(m, n)$-groups and we discuss finiteness and commutativity of such groups. We also show solvability length of a solvable $T(m, n)$-group is bounded in terms of $m$ and $n$.

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1. Introduction

Let $m, n$ be positive integers or infinity (denoted $\infty$) and $\mathcal{X}$ be a class of groups. We say that a group $G$ satisfies the condition $\mathcal{X}(m, n)$ ($G$ is an $\mathcal{X}(m, n)$-group, or $G \in \mathcal{X}(m, n)$), if for every two subsets $M$ and $N$ of cardinalities $m$ and $n$, respectively, there exist $x \in M$ and $y \in N$ such that $\langle x, y \rangle \in \mathcal{X}$. Bernhard H. Neumann in 2000 [9], put forward the question: Let $G$ be a finite group of order $|G|$ and assume that however a set $M$ of $m$ elements and a set $N$ of $n$ elements of the group is chosen, at least one element of $M$ commutes with at least one element of $N$, that is $G$ is an $C(m, n)$-group, where $C$ is the class of abelian groups. What relations between $|G|, m$ and $n$ guarantee that $G$ is abelian? Even though the latter question was posed for finite groups, the property introduced therein can be considered for all groups.

Following Neumann's question, authors in [1], showed that infinite groups satisfying the condition $\mathcal{C}(m, n)$ for some $m$ and $n$ are abelian. They obtained an upper bound in terms of $m$ and $n$ for the solvability length of a solvable group $G$. Also the third author in [13] studied the $N(m, n)$-groups, where $N$ is the class of nilpotent groups. Considering the analogous question for rings, Bell and Zarrin in [3] studied the $C(m, n)$-rings and they showed that all infinite $C(m, n)$-rings (like infinite $C(m, n)$-groups) are commutative and proved several commutativity results.

As a substantial generalization of $C(m, n)$-rings, Bell and Zarrin in [4] studied the $T(m, n)$-rings. Let $m \geq 2$ and $n \geq 1$. A ring $R$ (or a semigroup) is said to be a $T(m, n)$-ring (or is an $T(m, n)$-semigroup), if for every $m$-subsets $A_1, A_2, \ldots, A_m$ of $R$, there exists $i \neq j$ and $x_i \in A_i, x_j \in A_j$ such that $x_i x_j = x_j x_i$. They showed that torsion-free $T(m, n)$-rings are commutative. Note that unlike $C(m, n)$-rings there are the vast classes of infinite noncommutative $T(m, n)$-rings. Also they discussed finiteness and commutativity of such rings.

In this paper, considering the analogous definition for groups, we prove some results for $T(m, n)$-groups and present some examples of such groups and give several
commutativity theorems. Note that infinite $T(m, n)$-groups, unlike infinite $C(m, n)$-groups, need not be commutative. For instance, we can see $A_5 \times A$ is an infinite non-Abelian $T(22, 1)$-group, where $A_5$ is the alternating group of degree 5 and $A$ is an arbitrary infinite abelian group. However, certain infinite $T(m, n)$-groups can be shown to be commutative. Clearly, every finite group is an $T(m, n)$-group, for some $m$ and $n$. It is not necessary that every group is an $T(m, n)$-group, for some $m \geq 2, n \geq 1$. For example, if $F$ be a free group, then it is not difficult to see that $F$ is not an $T(m, n)$-group, for every $m \geq 2, n \geq 1$.

It is easy to see that every $C(m, n)$-group is an $C(\max\{m, n\}, \max\{m, n\})$-group and every $C(\max\{m, n\}, \min\{m, n\})$-group is an $T(2, \max\{m, n\})$-group. Therefore every $C(m, n)$-group is an $T(2, r)$-group for some $r$. Thus a next step might be to consider $T(3, r)$-groups. We show solvability length of a solvable $T(3, n)$-group is bounded above in terms of $n$. Also we give a solvability criterion for $T(m, n)$-groups in terms of $m$ and $n$.

Finally, in view of $T(m, n)$-groups and $\mathcal{X}(m, n)$-groups, we can give a substantial generalization of $\mathcal{X}(m, n)$-groups. Let $m, n$ be positive integers or infinity (denoted $\infty$) and $\mathcal{X}$ be a class of groups. We say that a group $G$ satisfies the condition $G \mathcal{X}(m, n)$ (or $G \in G \mathcal{X}(m, n)$), if for every $m, n$-subsets $A_1, A_2, \ldots, A_m$ of $G$, there exists $j \neq j$ and $x_i \in A_i, x_j \in A_j$ such that $\langle x_i, x_j \rangle \in \mathcal{X}$. Also a set $\{A_1, A_2, \ldots, A_m\}$ of $n$-subsets of a group $G$ is called $(m, n)$-obstruction if it prevents $G$ from being a $G \mathcal{X}(m, n)$-group. Therefore, with this definition, $T(m, n)$-groups are exactly $G \mathcal{C}(m, n)$-groups. Obviously, every $\mathcal{X}(m, n)$-group is an $G \mathcal{X}(m + n, 1)$-group.

2. Some properties of $T(m, n)$-groups

Here, we use the usual notation, for example $A_n, S_n, SL_n(q), PSL_n(q)$ and $Sz(q)$, respectively, denote the alternating group on $n$ letters, the symmetric group on $n$ letters, the special linear group of degree $n$ over the finite field of size $q$, the projective special linear group of degree $n$ over the finite field of size $q$ and the Suzuki group over the field of size $q$.

At first, we give some properties of $T(m, n)$-groups and then give some examples of such groups. If $2 \leq m_1 \leq m_2, n_1 \leq n_2$, then every $T(m_1, n_1)$-group is a $T(m_2, n_2)$-group. Therefore every $T(m, n)$-group is an $T(\max\{m, n\}, \max\{m, n\})$-group.

Lemma 2.1. If $G$ is an $T(m, n)$-group, $H \leq G$ and $N \leq G$, then two groups $H$ and $\frac{G}{H}$ are $T(m, n)$-group.

Lemma 2.2. Let $m + n \leq 4$, then $G$ is an $T(m, n)$-group if and only if it is abelian.

Proof. It is enough to consider only the groups that belongs to $T(3, 1)$ and $T(2, 2)$. If $G$ be a non-Abelian $T(3, 1)$-group, then there exist elements $x$ and $y$ of $G$, such that $[x, y] \neq 1$. Therefore $A_1 = \{x\}, A_2 = \{y\}, A_3 = \{xy\}$ is a $(3, 1)$-obstruction of $G$, a contrary.

If $G$ is a non-Abelian $T(2, 2)$-group and $[x, y] \neq 1$. Then we can see that $A_1 = \{x, y\}, A_2 = \{xy, yx\}$ is a $(2, 2)$-obstruction of $G$, a contrary.

We note that the bound 4 in the above Lemma is the best possible. As $D_8$ and $Q_8$ are $T(4, 1)$-groups.
Proposition 2.3. Assume that a finite group $G$ is not $\mathcal{T}(m,n)$-group, for two positive integers $m, n$. Then $mn \leq \vert G \vert - \vert Z(G) \vert$. Moreover, if $mn = \vert G \vert - \vert Z(G) \vert$, then for every $a \in G \setminus Z(G)$, $\vert a \vert \leq n + \vert Z(G) \vert$.

Proof. As $G$ is not an $\mathcal{T}(m,n)$-group, then there exists $(m, n)$-obstruction for $G$ like $\{A_1, A_2, \ldots, A_m\}$. It follows that $A_i \cap A_j = \emptyset$ for every $i \neq j$ and $Z(G) \cap A_i = \emptyset$. Therefore $\bigcup_{i=1}^{m} A_i \subseteq G \setminus Z(G)$ and so $mn \leq \vert G \vert - \vert Z(G) \vert$.

Now if $mn = \vert G \vert - \vert Z(G) \vert$, then we can see that, for every noncentral element $a \in G$, there exists $1 \leq i \leq m$ such that $C_{G}(a) \setminus Z(G) \subseteq A_i$. Thus $\vert C_{G}(a) \vert \leq n + \vert Z(G) \vert$ and so $\vert a \vert \leq n + \vert Z(G) \vert$. □

Corollary 2.4. $S_3$ is an $\mathcal{T}(2,3)$ and $\mathcal{T}(3,2)$-group. Also the dihedral group of order $2n$, $D_{2n}$ is an $\mathcal{T}(2,n)$. If $n$ is even integer, then $D_{2n}$ is an $\mathcal{T}(2,n - 1)$ but not an $\mathcal{T}(2,n - 2)$-group. For this, if we put $D_{2n} = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle$, $A_1 = \langle a \rangle \setminus Z(D_{2n})$ and $A_2 = \bigcup_{i=1}^{n-1} C_{G}(ba^i) \setminus Z(D_{2n})$. Then we can see that $\{A_1, A_2\}$ is a $(2, n - 2)$-obstruction. Moreover it is easy to see that, if $n$ is even integer, then $D_{2n}$ is an $\mathcal{T}(\frac{n}{2} + 2, 1)$-group. Also, if $n$ is odd integer, then $D_{2n}$ is an $\mathcal{T}(n + 2, 1)$-group. Finally, every finite $p$-group of order $p^n$ is an $\mathcal{T}(p^{n-1}, p)$ and also an $\mathcal{T}(p, p^{n-1})$-group.

Example 2.5. Let $G = S_4$. It is not difficult to see that, according to the centralizers of $G$, $G$ is an $\mathcal{T}(14,1)$, $\mathcal{T}(11,2)$, $\mathcal{T}(6,3)$, $\mathcal{T}(4,5)$ and $\mathcal{T}(3,7)$-group.

Lemma 2.6. Every finite group $G$ is an $\mathcal{T}(m, \lceil \frac{|G|}{m} \rceil)$ and so is an $\mathcal{T}(m, \lceil \frac{|G|}{2} \rceil)$, for every $m \geq 2$.

Proof. The result follows from Proposition 2.3. □

Remark 2.7. Assume that $G_1$ is an $\mathcal{T}(m_1, n_1)$-group and $G_2$ is an $\mathcal{T}(m_2, n_2)$-group. Then the group $G_1 \times G_2$ need not to be an $\mathcal{T}(m,n)$-group, where $m = \max \{m_1, m_2\}$ and $n = \max \{n_1, n_2\}$. For example, $S_3$ is an $\mathcal{T}(3,2)$-group but $S_3 \times S_3$ is not an $\mathcal{T}(3,2)$-group (note that $S_3 \times S_3$ is an $\mathcal{T}(7,3)$-group). In particular, the group $G_1 \times G_2$ need not to be even an $\mathcal{T}(m_1m_2, n_1n_2)$-group. For example, it is easy to see that the quaternion group $Q_8$ is an $\mathcal{T}(4,1)$-group, but $Q_8 \times S_3$ is not an $\mathcal{T}(12,2)$-group (in fact, $Q_8 \times S_3$ is an $\mathcal{T}(13,2)$-group). For, if we consider the subsets of $Q_8 \times S_3$ as follows:

$A_1 = \{(i, (1, 2)), (-i, (1, 2))\}$, $A_2 = \{(i, (1, 3)), (-i, (1, 3))\}$,

$A_3 = \{(i, (2, 3)), (-i, (2, 3))\}$, $A_4 = \{(i, (1, 2, 3)), (-i, (1, 2, 3))\}$,

$A_5 = \{(j, (1, 2)), (-j, (1, 2))\}$, $A_6 = \{(j, (1, 3)), (-j, (1, 3))\}$,

$A_7 = \{(j, (2, 3)), (-j, (2, 3))\}$, $A_8 = \{(j, (1, 2, 3)), (-j, (1, 2, 3))\}$,

$A_9 = \{(k, (1, 2)), (-k, (1, 2))\}$, $A_{10} = \{(k, (1, 3)), (-k, (1, 3))\}$,

$A_{11} = \{(k, (2, 3)), (-k, (2, 3))\}$, $A_{12} = \{(k, (1, 2, 3)), (-k, (1, 2, 3))\}$.

Then it is easy to see that the subsets $\{A_1, A_2, \ldots, A_{12}\}$ is a $(12,2)$-obstruction for the group $Q_8 \times S_3$. 

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Lemma 2.8. Let $G_1$ be an $T(m,1)$-group, $G_2$ be an abelian group, then $G_1 \times G_2$ is an $T(m,1)$-group.

Remark 2.9. Clearly every finite group is an $T(m,n)$-group for some $m, n$. But it is not true that every infinite group is an $T(m,n)$-group. For instance, every group which contain a free subgroup, is not an $T(m,n)$-group, for every $m \geq 2, n \geq 1$. Moreover, it is well-known that every free group is a residually finite group (even though the converse in not necessarily true). But there exist some residually finite groups that are not again an $T(m,n)$-group, for every $m \geq 2, n \geq 1$. For example, the group $SL_2(\mathbb{Z})$ is a residually finite group. The subgroup of $SL_2(\mathbb{Z})$ generated by matrices $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a free group of rank 2. So $SL_2(\mathbb{Z})$ is not an $T(m,n)$-group, for every $m \geq 2$ and $n \geq 1$.

For any nonempty set $X$, $|X|$ denotes the cardinality of $X$. Let $A$ be a subset of a group $G$. Then a subset $X$ of $A$ is a set of pairwise non-commuting elements if $xy \neq yx$ for any two distinct elements $x$ and $y$ in $X$. If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements $Y$ in $A$, then the cardinality of $X$ (if it exists) is denoted by $w(A)$ and is called the clique number of $A$ (for more information concerning the clique number of groups, see for example [12] and [2]).

Lemma 2.10. Let $G$ be not an $T(m,n)$-group and $\{A_1, A_2, \ldots, A_m\}$ be a $(m,n)$-obstruction for $G$. Then

$$m + \max\{w(A_i) \mid 1 \leq i \leq m\} \leq w(G).$$

Proof. Clearly.

Lemma 2.11. Let $G$ be an $T(m,n)$-group. Then $w(G) < mn$ and $G$ is center-by-finite.

Proof. We show that for any set $X$ of pairwise non-commuting elements of $G$, we have $|X| < mn$. Suppose that $|X| \geq mn$, then we can take $m n$-subsets of $X$ that is a $(m,n)$-obstruction for $G$. It is a contradiction. By the famous theorem of B. H. Neumann [8], since every set of non-commuting elements of $T(m,n)$-group $G$ is finite, therefore it is center-by-finite.

Now we show that for $T(m,n)$-groups with $|Z(G)| \geq n$, we get even $w(G) < m$. In fact, we have

Proposition 2.12. If $G$ is an $T(m,n)$-group, then $|Z(G)| < n$ or $w(G) < m$.

Proof. Let $G$ be an $T(m,n)$-group and $|Z(G)| \geq n$. We may assume $Z_1 \subseteq Z(G)$ and $|Z_1| = n$. Now if $w(G) \geq m$ and $\{x_1, x_2, \ldots, x_m\}$ be a pairwise non-commuting set of $G$, then $\{x_1Z_1, x_2Z_1, \ldots, x_mZ_1\}$ is a $(m,n)$-obstruction for $G$, which is a contradiction.

It is easy to see that a group $G$ is an $T(m,1)$-group, if and only if $w(G) < m$.

Corollary 2.13. Assume that $G$ is a nilpotent finite $T(m,n)$-group and $p$ is a prime divisor of $|G|$ such that $n \leq p$. Then $G$ is an $T(m,1)$-group. In particular, every nilpotent $T(m,2)$-group is an $T(m,1)$-group.

If $G$ is a non-Abelian group, then $G$ is not an $T(3,z)$, which $z = |Z(G)|$, since $w(G) \geq 3$.

If $G$ is an $T(w(G),2)$-group, then $Z(G) = 1$. 

\[ \]
Corollary 2.14. Let $G$ be a non-Abelian $T(m, n)$-group with at least $m$ pairwise non-commuting elements, then $G$ is a finite group.

Lemma 2.15. Let $G$ be a non-Abelian $T(2, n)$ or $T(3, n)$-group and $N$ be a normal subgroup of $G$ such that $G/N$ is non-Abelian. Then $|N| < n$.

Proof. Suppose that $N = \{a_1, a_2, \ldots, a_t\}$ and $t \geq n$. It is enough to prove the theorem for non-Abelian $T(3, n)$-groups. We chose elements $x, y$ in $G \setminus N$, and we consider three subsets of $G$, as follows:

$$A_1 = \{xa_1, xa_2, \ldots, xa_t\}, \quad A_2 = \{ya_1, ya_2, \ldots, ya_t\}$$

and

$$A_3 = \{xya_1, xya_2, \ldots, xya_t\}.$$ 

Now as $G$ is an $T(3, n)$-group, we can follow that $[x, y] \in N$, that is $G/N$ is abelian, which is a contradiction.

Theorem 2.16. Let $G$ be a non-Abelian group and its clique number is finite. Then there exist a natural number $m$ such that $G$ is an $T(m, n)$-group for all $n \in \mathbb{N}$.

Proof. As the clique number of $G$ is finite, so according to the famous Theorem of B. H. Neumann, $G$ is center-by-finite. So we put $|G : Z(G)| = m$. We claim that for every $n \in \mathbb{N}$, $G$ is an $T(m, n)$-group. There exists $m - 1$ elements $g_1, g_2, \ldots, g_{m-1}$ in $G$, such that $Z(G), g_1Z(G), g_2Z(G), \ldots, g_{m-1}Z(G)$ are distinct cosets of $Z(G)$ in $G$ and

$$G = Z(G) \bigcup_{j=1}^{m-1} (g_jZ(G)).$$

Let $\{A_1, A_2, \ldots, A_m\}$ be an $(m, n)$-obstruction of $G$. Now as every $g_iZ(G)$ is abelian, therefore if $g_iZ(G) \cap A_r \neq \emptyset$ for some $1 \leq i \leq m - 1$ and $1 \leq r \leq m$, then $g_iZ(G) \cap A_j = \emptyset$ for every $j \neq r$. On the other hand $Z(G) \cap A_i = \emptyset$ for every $1 \leq i \leq m$, thus $A_i \subseteq \bigcup_{j=1}^{m-1} (g_jZ(G))$ for every $1 \leq i \leq m$. From this one can follow that there exist two subsets like $A_r$ and $A_s$ such that $A_r \cup A_s \subseteq g_jZ(G)$ for some $1 \leq j \leq m - 1$, a contradiction. Therefore $G$ is an $T(m, n)$-group, for all $n \in \mathbb{N}$.

Remark 2.17. In the above Theorem, the finiteness of clique number is necessary. For example, it should be borne in mind that infinite $p$-groups can easily have trivial center. The group $G = C_p \wr C_p$, the regular wreath product $C_p$ by $C_p$, is an infinite centerless $p$-group, while $C_p$ is a cyclic group of order $p$ and $C_p^\infty$ is a quasi-cyclic (or Prüfer) group. So $|G : Z(G)|$ is infinite and so $w(G)$ is infinite and $G$ is not an $T(m, n)$-group.

B. H. Neumann showed that if every set of non-commuting elements of group $G$ is finite, then $G$ is center-by-finite. Moreover, it is not difficult to see that every center-by-finite group has finite clique number. Here, by using the above Theorem, we will obtain the following result.

Corollary 2.18. If $G$ be a group and $|G : Z(G)| = m$. Then $w(G) < m$.

Proof. As $|G : Z(G)| = m$ and $G$ is an $T(m, 1)$-group, if and only if $w(G) < m$, the result follows by Theorem 2.16.

Corollary 2.19. Every infinite $T(m, n)$-group with $m \leq 3$, is an abelian group.
Proof. If $G$ is non-Abelian group, then there exists $x, y$ such that $xy \neq yx$. So \( \{x, y, xy\} \) is a subset of pairwise non-commuting elements of $G$. Therefore by Proposition 2.12 \(|Z(G)| < n\) and by Lemma 2.11 $G$ is center-by-finite and so $G$ is a finite group, a contradiction. \( \square \)

**Theorem 2.20.** Let $G$ be a finite $T(m, n)$-group, $m \leq 4$, $n > 1$ and \( (p, |G|) = 1 \), for every prime number $p \leq n$. Then $G$ is abelian.

**Proof.** It is enough to prove the theorem for the case $m = 4$. Suppose, a contrary, that $G$ is a non-Abelian $T(4, n)$-group. Then there exists elements $x$ and $y$ in $G$, such that $xy \neq yx$. Now we consider four subsets of $G$ as follows:

\[
A_1 = \{x, x^2, \ldots, x^n\}, \quad A_2 = \{y, y^2, \ldots, y^n\},
\]

\[
A_3 = \{xy, (xy)^2, \ldots, (xy)^n\} \quad \text{and} \quad A_4 = \{xy, xy^2, \ldots, xy^{n-1}, x^2 y\}.
\]

Then it is not difficult to see that \( \{A_1, A_2, A_3, A_4\} \) is a \((4, n)\)-obstruction for $G$, a contradiction (note that if \( a \in G \) and \( (i, |a|) = 1 \), then \( C_G(a^i) = C_G(a) \)). \( \square \)

As a corollary, for $p > 2$ every finite $p$-group, $G \in T(4, p-1)$ is abelian.

Note that the group $D_8$ is a non-Abelian $T(4, 1)$-group. This example suggests that it may be necessary to restrict ourselves to $T(m, n)$-groups with $n > 1$ in the above Theorem.

**Proposition 2.21.** Assume that $G$ is a non-Abelian group. Then

1. If there exist a positive integer $n$ such that \((p, |G|) = 1\), for every prime number $p \leq n$. Then $w(G) \geq n + 2$.
2. If $p$ is the smallest prime divisor of $|G|$. Then $w(G) \geq p + 1$. Moreover, if $G$ is a finite $p$-group and $G \in T(m, p-1)$, then $p + 1 \leq w(G) < m$.

**Proof.** (1) Since $G$ is non-Abelian group, there exists elements $x, y$ in $G$, such that $xy \neq yx$. Now $X = \{x, y, xy, xy^2, \ldots, xy^n\}$ is a set of pairwise non-commuting elements of $G$ of cardinality $n + 2$.

(2) For every prime number $q \leq p - 1$, \((q, |G|) = 1\), then by part (1), $w(G) \geq (p - 1) + 2$. Thus $w(G) \geq p + 1$. \( \square \)

**Lemma 2.22.** Let $G$ be a finite $T(m, n)$-group where $Z(G) \neq 1$ and $p$ be smallest prime divisor of $|G|$. Then

1. $p \leq \max\{m - 2, n - 1\}$.
2. If $G$ is a nilpotent, then $p \leq \max\{m - 2, \sqrt[n]{n} - 1\}$, where $t$ is the number of prime divisors of order $G$, \(|\pi(G)|\).

**Proof.** (1) As $G$ is a $T(m, n)$-group, by Proposition 2.12 $p \leq |Z(G)| < n$ or $p + 1 \leq w(G) < m$. Therefore $p \leq n - 1$ or $p \leq m - 2$ and so $p \leq \max\{m - 2, n - 1\}$.

(2) In this case it is enough to note that the set of prime divisors of the center of $G$ is equal to $\pi(G)$, so $p^t \leq \prod_{i=1}^t p_i \leq |Z(G)| < n$. Thus $p \leq \sqrt[n]{n} - 1$ or $p \leq m - 2$. \( \square \)

**Corollary 2.23.** (1) If $G$ is a finite $p$-group and $G \in T(p, p)$, then $G$ is abelian.

(2) Every finite non-Abelian nilpotent $T(m, n)$-group with $3 \leq n \leq 6$ and $m \leq 3$ is a $p$-group.

(3) If $G$ is a finite nilpotent $T(4, n)$-group with $n \leq 3$ and odd order, then $G$ is an abelian group.

(4) Every finite nilpotent $T(4, n)$-group with $n \leq 3$, is an abelian-by-2-group.
Theorem 2.24. Let $G$ be a non-Abelian nilpotent $T(3,n)$-group. Then

$$|\pi(G)| \leq \log_3(n + 2).$$

Proof. Use induction on $|\pi(G)|$, the case $3 \leq n \leq 6$ being clear by Case (2) of Corollary 2.23. Assume that $n \geq 7$ and the result holds for $|\pi(G)| - 1$. Since $G$ is finite non-Abelian nilpotent, then there exist a Sylow subgroup $P$ of $G$, such that $\frac{G}{P}$ is non-Abelian and $\frac{G}{P} \in T(3,n-t)$-group, for every $2t \leq n$. So $|\pi(\frac{G}{P})| \leq \log_3(n - t + 2)$, therefore $|\pi(G)| - 1 \leq \log_3(n - t + 2) < \log_3(n + 2)$ and hence $|\pi(G)| \leq \log_3(n + 2)$, as wanted. \hfill $\square$

Remark 2.25. By argument similar to the one in the proof of Theorem 2.24, we can follow that if $G$ is a non-Abelian nilpotent $T(4,n)$-group with odd order, then $|\pi(G)| \leq \log_4(n + 6)$ (in this case note that if $4 \leq n \leq 9$, then $G$ is a $p$-group, for some prime number $p$).

3. On solvable $T(m,n)$-groups

In this section we investigate solvable $T(m,n)$-groups. At first we obtain the derived length of a solvable $T(m,n)$-group in terms $n$, for $m = 3$ or $4$ and then give a solvability criterion for $T(m,n)$-groups in terms $m$ and $n$. To prove our results it is necessary to establish a technical lemma.

Lemma 3.1. Let $G$ be an $T(m,n)$-group for some integers $m \geq 2$, $n > 1$ and $N$ be a proper non-trivial normal subgroup of $G$, then $\frac{G}{N}$ is an $T(m,n-t)$-group, where $n \geq 2t$.

Proof. Suppose that $G$ is an $T(m,n)$-group and $N \triangleleft G$, but $\frac{G}{N}$ is not an $T(m,n-t)$-group. We can take $m$ subsets $X_i = \{x_{i1}N, x_{i2}N, \ldots, x_{in_i}N\}$, $1 \leq i \leq m$ of $\frac{G}{N}$ of cardinality $n - t$, such that for every $1 \leq i, j \leq m$ and $1 \leq k, l \leq n - t$, $[x_{ik}, x_{jl}]$ is not belongs to $N$. Let $a$ be a nontrivial element of $N$, then we can obtain $m$ $n$-subsets $Y_i = \{ax_{i1}, ax_{i2}, \ldots, ax_{in_i}, x_{i1}, x_{i2}, \ldots, x_{it}\}$ of $G$, for some $2t \leq n$. Thus $\{Y_1, Y_2, \ldots, Y_m\}$ is a $(m,n)$-obstruction for $G$, a contrary. \hfill $\square$

Corollary 3.2. Let $G$ be a non-simple $T(wG), 2)$-group. Then for proper non-trivial normal subgroup $N$ of $G$, $\frac{G}{N}$ is an $T(wG), 1)$-group and so $w(\frac{G}{N}) < w(G)$.

Theorem 3.3. Let $G$ be a solvable $T(3,n)$-group (or $T(4,n)$-group and the order of $G$ is odd). Then the derived length of $G$, $d$ is at most $\log_3(2n)$.

Proof. We argue by induction on $d$. The case $d = 1$ being obvious. Assume that $d \geq 2$ and so, by Lemma 3.1 the group $\frac{G}{O_{d-1}(G)}$, has solvability length $d - 1$, is an $T(3,n-t)$-group, where $2t \leq n$. Therefore $d - 1 \leq \log_3(2(n - t)) < \log_3(2(n))$. Thus $d - 1 < \log_3(2n)$, so $d \leq \log_3(2n)$, as wanted. (We note that the bound $\log_3(2n)$ is the best possible, as $S_3$ is an $T(3,2)$-group and $d(S_3) = 2 = \log_3(4)$.)

Now if $G$ is $T(4,n)$-group and the order of $G$ is odd, then by argument similar, the result follows (for proof it is enough to note that $T(4,1)$-group of odd order is abelian). \hfill $\square$

Note that the group $D_8$ is a solvable $T(4,1)$-group with solvability length 2, but $2 \not\leq \log_3(2)$. This example suggests that it may be necessary to restrict ourselves to groups with odd order in the above Theorem.

If $G$ is a finite group, then for each prime divisor $p$ of $|G|$, we denote by $v_p(G)$ the number of Sylow $p$-subgroups of $G$. 
Lemma 3.4. Let $G$ be a finite $T(m, n)$-group and $p$ be a prime number dividing $|G|$ such that every two distinct Sylow $p$-subgroups of $G$ have trivial intersection, then $v_p(G) \leq mn - 1$.

Proof. Since $G$ is an $T(m, n)$-group, we have $w(G) < mn$. Now Lemma 3 of [6] completes the proof.

Now we obtain a solvability criteria for $T(m, n)$-groups in terms of $m$ and $n$.

Theorem 3.5. Let $G$ be an $T(m, n)$-group. Then we have

(i) If $mn \leq 21$, then $G$ (not necessarily finite) is solvable and this estimate is sharp.
(ii) If $mn \leq 60$ and $G$ is non-solvable finite group, then $G = A_5$ (in fact, $A_5$ is the only non-solvable $T(m, n)$-group, which $mn \leq 60$).

Proof. (i) Since $G$ is an $T(m, n)$-group, we have $w(G) < mn$. Then by Theorem 1.2 of [12], $G$ is solvable. This estimate is sharp, because $A_5$ is an $T(22, 1)$-group.
(ii) We know that the alternating group $A_5$ has five Sylow 2-subgroups of order 4, ten Sylow 3-subgroups of order 3 and six Sylow 5-subgroups of order 5, that their intersections are trivial. From this we can follow that $A_5$ is an $T(22, 1)$, $T(22, 2)$, $T(17, 3)$ and $T(14, 4)$-group. For uniqueness, suppose, on the contrary, that there exists a non-Abelian finite simple group not isomorphic to $A_5$ and of least possible order which is an $T(m, n)$-group, which $mn \leq 60$. Then by Proposition 3 of [5], Lemma 3.4 and by argument similar to the one in the proof of Theorem 1.3 of [1], gives a contradiction in each case.

Corollary 3.6. Every arbitrary $T(21, i)$-group $G$ with $i \leq z$ is solvable, where $z = |Z(G)|$.

In the follow we show that the influence of the value of $n$ for the solvability of $T(m, n)$-groups is more important than the value of $m$.

Corollary 3.7. Let $G$ be an $T(m, n)$-group. Each of the following conditions implies that $G$ is solvable.

(a) $n = 2$ and $m \leq 21$. (b) $n = 3$ and $m \leq 16$. (c) $n = 4$ and $m \leq 13$. (d) $n = 5$ and $m \leq 8$. (e) $n = 6$ and $m \leq 8$. (f) $n = 7$ and $m \leq 7$. (g) $n = 8$ and $m \leq 7$.

Proof. Suppose, on the contrary, that $G$ is a non-solvable group. By Theorem 3.5, $G \cong A_5$. This is a contradiction, since $A_5$ is an $T(22, 2)$-group but is not an $T(21, 2)$-group.

By similar argument of Case (a) the rest Cases would be proved, since $A_5 \notin T(17, 3) \bigcap T(14, 4) \bigcap T(9, 5) \bigcap T(9, 6) \bigcap T(8, 7) \bigcap T(8, 8)$ but $A_5 \notin T(16, 3) \bigcup T(13, 4) \bigcup T(8, 6) \bigcup T(8, 7) \bigcup T(7, 8)$.

\[ \square \]

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