TOPOLOGY OF THE OCTONIONIC FLAG MANIFOLD

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Abstract. The octonionic flag manifold $Fl(\mathbb{O})$ is the space of all pairs in $\mathbb{O}P^2 \times \mathbb{O}P^2$ (where $\mathbb{O}P^2$ denotes the octonionic projective plane) which satisfy a certain “incidence” relation. It comes equipped with the projections $\pi_1, \pi_2 : Fl(\mathbb{O}) \to \mathbb{O}P^2$, which are $\mathbb{O}P^1$ bundles, as well as with an action of the group $Spin(8)$. The first two results of this paper give Borel type descriptions of the usual, respectively $Spin(8)$-equivariant cohomology of $Fl(\mathbb{O})$ in terms of $\pi_1$ and $\pi_2$ (actually the Euler classes of the tangent spaces to the fibers of $\pi_1$, respectively $\pi_2$, which are rank 8 vector bundles on $Fl(\mathbb{O})$). Then we obtain a Goresky-Kottwitz-MacPherson type description of the ring $H_{Spin(8)}^\ast(Fl(\mathbb{O}))$. Finally, we consider the $Spin(8)$-equivariant $K$-theory ring of $Fl(\mathbb{O})$ and obtain a Goresky-Kottwitz-MacPherson type description of this ring.

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1. Introduction

Let \( \mathbb{O} \) denote the (normed, unital, non-commutative and non-associative) algebra of octonions and let \( \mathbb{O} P^2 \) be the octonionic projective plane (see for instance \([Ba],[Fr],[Mu]\)). This space is an important example in incidence geometry. It turns out that there exists a natural identification between the space of lines in \( \mathbb{O} P^2 \) and \( \mathbb{O} P^2 \) itself. The octonionic flag manifold \( Fl(\mathbb{O}) \) is the space of all pairs \((p, \ell) \in \mathbb{O} P^2 \times \mathbb{O} P^2\), where \( p \) is a point and \( \ell \) a line, such that \( p \) and \( \ell \) are incident (see Definition 2.1.1 below). Both \( \mathbb{O} P^2 \) and \( Fl(\mathbb{O}) \) carry natural structures of differentiable manifolds. More precisely, we have the natural identifications

\[
\mathbb{O} P^2 = \mathbb{F}_4/\text{Spin}(9), \quad Fl(\mathbb{O}) = \mathbb{F}_4/\text{Spin}(8),
\]

where \( \mathbb{F}_4 \) denotes the compact, connected, simply connected Lie group whose Lie algebra is the (compact) real form of the complex simple Lie algebra of type \( \mathbb{F}_4 \). We consider the natural \( \mathbb{O} P^1 \)-bundles \( \pi_1, \pi_2: Fl(\mathbb{O}) \to \mathbb{O} P^2 \) given by

\[
\pi_1(p, \ell) := p \quad \text{and} \quad \pi_2(p, \ell) := \ell.
\]

Let also \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) denote the rank 8 vector bundles on \( Fl(\mathbb{O}) \) given by

\[
\mathcal{E}_1(p, \ell) := T_{(p, \ell)} \pi^{-1}_1(p), \quad \mathcal{E}_2(p, \ell) := T_{(p, \ell)} \pi^{-1}_2(\ell)
\]

for all \((p, \ell) \in Fl(\mathbb{O})\). Our first result concerns the integral cohomology ring of \( Fl(\mathbb{O}) \). Before stating it, we make the following convention, which will be used throughout this paper: if it is not specified, the coefficient ring of a cohomology group is \( \mathbb{R} \).

**Theorem 1.1.** We can orient the bundles \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) in such a way that the ring \( H^*(Fl(\mathbb{O}); \mathbb{Z}) \) is generated by \( \frac{1}{3}(2e(\mathcal{E}_1) + e(\mathcal{E}_2)) \) and \( \frac{1}{3}(e(\mathcal{E}_1) + 2e(\mathcal{E}_2)) \), the ideal of relations being generated by

\[
S_i \left( \frac{1}{3}(2e(\mathcal{E}_1) + e(\mathcal{E}_2)), \frac{1}{3}(-e(\mathcal{E}_1) + e(\mathcal{E}_2)), -\frac{1}{3}(e(\mathcal{E}_1) + 2e(\mathcal{E}_2)) \right) = 0
\]

\( i = 2, 3 \). Here \( e(\mathcal{E}_1) \) and \( e(\mathcal{E}_2) \) are the Euler classes of the bundles \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) and \( S_2 \) and \( S_3 \) denote the second and third elementary symmetric polynomials in three variables.

This theorem will be proved in Section 3.

We also study the topology of \( Fl(\mathbb{O}) \) from the point of view of the action of the group \( M := \text{Spin}(8) \) induced canonically by Equation (1.1). More precisely, we are interested in the equivariant cohomology ring \( H^*_M(Fl(\mathbb{O})) \). We recall that this ring has a natural structure of \( H^*(BM) \)-module, which is defined as follows: we pick a point \( x_0 \in Fl(\mathbb{O}) \) which is fixed by the \( M \) action and consider the map \( P^*: H^*(BM) \to H^*_M(Fl(\mathbb{O})) \) induced by the constant map \( P: Fl(\mathbb{O}) \to \{x_0\} \). We define

\[
\beta.\alpha := P^*(\beta)\alpha,
\]
for all $\beta \in H^*(BM)$ and all $\alpha \in H^*_M(Fl(\mathbb{O}))$. It is worth noting that, since $M$ is a compact Lie group of rank four, $H^*(BM)$ is a polynomial ring with four generators. More precisely, we have

$$H^*(BM) = H^*(BT)^W,$$

where $T \subset M$ is a maximal torus and $W_M$ the Weyl group of the pair $(M, T)$. This gives

(1.3) $$H^*(BM) = \mathbb{R}[u_1, u_2, u_3, u_4],$$

where $u_1$ has degree 4, $u_2$ and $u_3$ have degree 8, and $u_4$ has degree 12 (cf. [Hu, Section 3.7]).

The group $H^*(BM; \mathbb{Z})$ is described in [Kon]; we note that it contains 2-torsioned elements, and this is the reason which prevented us from discussing the $M$-equivariant cohomology with integer coefficients in this paper.

Our first result concerning the ring $H^*_M(Fl(\mathbb{O}))$ is stated below. The vector bundles $E_1$ and $E_2$ are $M$-equivariant and orientable, so we can associate to them the equivariant Euler classes $e_M(E_1)$ and $e_M(E_2)$, which are elements of $H^*_M(Fl(\mathbb{O}))$. We also consider the equivariant Euler classes

$$b_k := e_M(E_k|_{x_0}),$$

$k = 1, 2$, which are elements of $H^*_M(\{x_0\}) = H^*(BM)$.

**Theorem 1.2.** We can orient the bundles $E_1$ and $E_2$ in such a way that, as a $H^*(BM)$-algebra, $H^*_M(Fl(\mathbb{O}))$ is generated by $e_M(E_1)$ and $e_M(E_2)$, the ideal of relations being generated by:

(1.4) $$S_i(2e_M(E_1) + e_M(E_2), -e_M(E_1) + e_M(E_2), -(e_M(E_1) + 2e_M(E_2))) = S_i(2b_1 - b_1 + b_2, -(b_1 + 2b_2)),$$

$i = 2, 3$. As before, $S_2$ and $S_3$ are the elementary symmetric polynomials of degree two, respectively three, in three variables.

We prove this theorem in Section 4.

The second result about $H^*_M(Fl(\mathbb{O}))$ gives a Goresky-Kottwitz-MacPherson type presentation of this ring (cf. [Go-Ko-Ma], where formulas for the equivariant cohomology of certain spaces with actions of tori have been obtained). We will see that the fixed point set of the $M$ action on $Fl(\mathbb{O})$ can be identified with the symmetric group $\Sigma_3$. We set

$$\bar{e}_{12} := b_1, \bar{e}_{23} := b_2, \bar{e}_{13} := b_1 + b_2.$$

The following result will be proved in Section 5.

**Theorem 1.3.** (a) The (restriction) map

$$i^* : H^*_M(Fl(\mathbb{O})) \to H^*_M(\Sigma_3) = \prod_{\sigma \in \Sigma_3} H^*(BM)$$

induced by the inclusion $i : \Sigma_3 = Fl(\mathbb{O})^M \hookrightarrow Fl(\mathbb{O})$ is injective.
(b) The image of $i^*$ consists of all ordered sets $(f_\sigma) \in \prod_{\sigma \in \Sigma_3} H^*(BM)$ such that $f_\sigma - f_{(i,j)\sigma}$ is a multiple of $\bar{e}_{ij}$, for all $\sigma \in \Sigma_3$ and all $i, j$ with $1 \leq i < j \leq 3$. Here $(1, 2)$, $(2, 3)$ and $(1, 3)$ denote the obvious elements (transpositions) of $\Sigma_3$.

The last main result of the paper concerns the $M$-equivariant $K$-theory ring of $Fl(O)$. By the “equivariant $K$-theory” of a $M$-space we always mean the Grothendieck group of all topological $M$-equivariant complex vector bundles over that space, with the multiplication induced by the tensor product (for more details, we refer the reader to [Se]). To describe this ring for $Fl(O)$, we need some information about the (complex) representation ring $R[\Sigma] = \prod_{\sigma \in \Sigma_3} Z[X_1, X_2, X_3, X_4]$.

A proof of this theorem can be found in Section 6.

Remark. The space $Fl(O)$ is a generalized real flag manifolds. By definition, such a manifold is an orbit of the isotropy representation of a Riemannian symmetric space (for more details, see Appendix B). The cohomology ring of the principal orbits of these representations was computed by Hsiang, Palais, and Terng in [Hs-Pa-Te]. An important class of such manifolds consists of those with uniform multiplicity 2, 4 or 8: these are the principal adjoint orbits of compact Lie groups, the quaternionic flag manifold $Fl_n(\mathbb{H})$, and $Fl(O)$. The descriptions given in [Hs-Pa-Te] show that the cohomology ring of each of those spaces is expressed by a Borel type formula, that is, it is isomorphic to the coinvariant ring of a certain Weyl group. The spaces $Fl_n(\mathbb{H})$ and $Fl(O)$ admit natural group actions similar to the action of a maximal torus on an adjoint orbit (e.g., for $Fl(O)$ this group is $Spin(8)$, see above). The equivariant cohomology and equivariant $K$-theory of a principal adjoint orbit with the action of a maximal torus is well understood (see [Kos-Ku]). A natural goal is to decide whether $Fl_n(\mathbb{H})$ and $Fl(O)$ behave like adjoint orbits also in the equivariant setting. Positive answers have been given for $Fl_n(\mathbb{H})$ from the point of view of equivariant cohomology (see [Ma2]) and equivariant $K$-theory (see [Ma-Wi]). In this paper we discuss the remaining space, which is $Fl(O)$. 

Theorem 1.4. The canonical homomorphism

$$K_M(Fl(O)) \to K_M(\Sigma_3) = \prod_{\sigma \in \Sigma_3} R[M] = \prod_{\sigma \in \Sigma_3} Z[X_1, X_2, X_3, X_4]$$

induced by the inclusion $\Sigma_3 = Fl(O)^M \hookrightarrow Fl(O)$ is injective. Its image consists of all $(f_\sigma) \in \prod_{\sigma \in \Sigma_3} Z[X_1, X_2, X_3, X_4]$ such that $f_\sigma - f_{(i,j)\sigma}$ is a multiple of $X_i - X_j$, for all $\sigma \in \Sigma_3$ and all $i, j$ with $1 \leq i < j \leq 3$. Here $(1, 3)$, $(2, 3)$ and $(1, 2)$ have the same meaning as in Theorem 1.3.
2. The octonionic flag manifold

The goal of this section is to define the flag manifold $Fl(O)$ and discuss some of its basic properties. For reader’s convenience we have included an appendix (see Appendix A) where the complex flag manifold $Fl_3(C)$ is discussed in a way appropriate to serve us as a model here.

2.1. $Fl(O)$ via the Jordan algebra $(h_3(O), \circ)$. We first recall that by definition, the space $O$ has a basis consisting of the elements $e_1 = 1, e_2, \ldots, e_8$; these satisfy certain multiplication rules which make $O$ into a non-associative algebra with division (for more details, see [Ba, Section 2]). Let

$$p = x_1 + x_2e_2 + \ldots + x_8e_8$$

be an element of $O$, where $x_1, x_2, \ldots, x_8 \in \mathbb{R}$. We define its real part

$$\text{Re}(p) = x_1,$$

its conjugate

$$\overline{p} = x_1 - x_2e_2 - \ldots - x_8e_8,$$

as well as its norm $|p|$ given by

$$|p|^2 = p \cdot \overline{p} = x_1^2 + x_2^2 + \ldots + x_8^2.$$

Let us consider

$$h_3(O) := \left\{ \begin{pmatrix} x_1 & p & q \\ \overline{p} & x_2 & r \\ q & \overline{r} & x_3 \end{pmatrix} : p, q, r \in O, \ x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

the space of all $3 \times 3$ Hermitian matrices with entries in $O$. This is closed under the usual matrix multiplication.

**Definition 2.1.1.** (a) The octonionic projective plane $O P^2$ is the set of all matrices $a \in h_3(O)$ with

$$a^2 = a \text{ and } \text{tr}(a) = 1.$$

(b) The octonionic flag manifold $Fl(O)$ is the set of all pairs $(a, b) \in O P^2 \times O P^2$ with

$$\text{Re}(\text{tr}(ab)) = 0.$$

In the language of incidence geometry, this condition says that the “point” $a$ and the “line” $b$ are “incident” (see for instance [Fr, Section 7.2]).

We equip $h_3(O)$ with the $\mathbb{R}$-linear product

$$a \circ b := \frac{1}{2}(ab + ba),$$

for all $a, b \in h_3(O)$.

---

1The pair $(h_3(O), \circ)$ is actually a Jordan algebra (cf. [Ba, Fr]).
Definition 2.1.2. The group $F_4$ consists of all $\mathbb{R}$-linear transformations $g$ of $h_3(\mathbb{O})$ such that

$$g.(a \circ b) = (g.a) \circ (g.b),$$

for all $a, b \in h_3(\mathbb{O})$.

The following is a list of properties of the group $F_4$ which will be needed later. The details can be found for instance in [Fr], [Mu], and [Ad].

- The group $F_4$ is a compact, connected, simply connected Lie group whose Lie algebra is the compact real form of the complex simple Lie algebra of type $F_4$.
- For any $a \in h_3(\mathbb{O})$ there exist $g \in F_4$ and $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$x_1 \geq x_2 \geq x_3$$

and

$$g.a = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}.$$  

The numbers $x_1, x_2, x_3$ are uniquely determined by $a$.
- We have

$$\text{tr}(g.a) = \text{tr}(a),$$

for all $g \in F_4$ and all $a \in h_3(\mathbb{O})$.
- We have

$$(2.1) \quad g.I = I,$$

for any $g \in F_4$. Here $I$ denotes the diagonal matrix $\text{Diag}(1, 1, 1)$.
- Denote by $\mathfrak{d} \simeq \mathbb{R}^3$ the space of all diagonal matrices in $h_3(\mathbb{O})$. We have

$$(2.2) \quad \{g \in F_4 : g.x = x \text{ for all } x \in \mathfrak{d}\} \simeq \text{Spin}(8)$$

- The space $\mathbb{O}P^2$ is the $F_4$ orbit of

$$d_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

The stabilizer of $d_1$ is isomorphic to the Lie group $\text{Spin}(9)$. Thus we have the identification

$$\mathbb{O}P^2 = F_4/\text{Spin}(9).$$

We also have the following description of $\text{Fl}(\mathbb{O})$.

Proposition 2.1.3. The (diagonal) action of $F_4$ on $\text{Fl}(\mathbb{O})$ is transitive. If

$$d_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
then the stabilizer of \((d_1, d_3)\) is isomorphic to the group \(\text{Spin}(8)\) given by Equation (2.2).

Thus we have the identification

\[ Fl(\mathfrak{O}) = F_4/\text{Spin}(8). \]

**Proof.** The transitivity of the \(F_4\) action follows from [Fr, Sections 7.2 and 7.6]. The second assertion follows from the fact that \(g \in F_4\) fixes \(\mathfrak{O}\) pointwise if and only if it fixes \(d_1\) and \(d_3\) (by Equation (2.1)). \(\square\)

Let us now consider the maps \(\pi_1, \pi_2 : Fl(\mathfrak{O}) \to \mathbb{O}P^2\), given by

\[ \pi_1(a, b) := a, \quad \pi_2(a, b) := b, \]

for all \((a, b) \in Fl(\mathfrak{O})\). From the previous considerations we deduce that they are both \(F_4\)-equivariant maps.

**Proposition 2.1.4.** The maps \(\pi_1\) and \(\pi_2\) are \(\mathbb{O}P^1\) bundles. Here, in analogy with Definition 2.1.1 (a), \(\mathbb{O}P^1\) (the octonionic projective line) is the space of all idempotent elements of \(\mathfrak{h}_2(\mathfrak{O})\) with trace equal to 1.

**Proof.** We show that \(\pi_1\) is an \(\mathbb{O}P^1\) bundle. Since \(\pi_1\) is \(F_4\)-equivariant, it is sufficient to prove that \(\pi_1^{-1}(d_1) = \mathbb{O}P^1\) (because then, if \(g \in F_4\), then \(\pi_1^{-1}(g.d_1) = g.\mathbb{O}P^1\)). Indeed, the elements of \(\pi_1^{-1}(d_1)\) are of the form \((d_1, a)\), where \(a \in \mathbb{O}P^2\) is such that

\[ \text{tr}(a \cdot d_1) = 0. \]

The last equation and the fact that \(a^2 = a\) implies that

\[ a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_2 & r \\ 0 & \bar{r} & x_3 \end{pmatrix} \]

for \(x_2, x_3 \in \mathbb{R}\) and \(r \in \mathbb{O}\). The set of all such \(a\) with \(a^2 = a\) and \(\text{tr}(a) = 1\) is the subspace \(\mathbb{O}P^1\) of \(\{0\} \times \mathfrak{h}_2(\mathfrak{O})\) (the latter being canonically embedded in \(\mathfrak{h}_2(\mathfrak{O})\)). This finishes the proof. \(\square\)

### 2.2. \(Fl(\mathfrak{O})\) as a real flag manifold.

Let \(\mathfrak{h}_3^0(\mathfrak{O})\) be the space of all elements of \(\mathfrak{h}_3(\mathfrak{O})\) with trace equal to 0. The representation of \(F_4\) on the space \(\mathfrak{h}_3(\mathfrak{O})\) mentioned in the previous subsection leaves \(\mathfrak{h}_3^0\) invariant. Indeed, let us consider the inner product \(\langle \ , \ \rangle\) on \(\mathfrak{h}_3(\mathfrak{O})\) given by

\[ \langle a, b \rangle = \text{Re}(\text{tr}(ab)), \]

for all \(a, b \in \mathfrak{h}_3(\mathfrak{O})\). This product is \(F_4\) invariant, in the sense that

\[ \langle g.a, g.b \rangle = \langle a, b \rangle, \]

for all \(a, b \in \mathfrak{h}_3(\mathfrak{O})\) and \(g \in F_4\) (see [Fr, Section 4.5]). One can easily verify that \(\mathfrak{h}_3^0(\mathfrak{O})\) is the orthogonal complement of \(I\) in \(\mathfrak{h}_3(\mathfrak{O})\). By Equation (2.1), \(F_4\) leaves \(\mathfrak{h}_3^0(\mathfrak{O})\) invariant. The main point of this subsection is that this representation of \(F_4\) on \(\mathfrak{h}_3^0(\mathfrak{O})\) is just the isotropy representation of the (non-compact) Riemannian symmetric space \(E_{6(-26)}/F_4\). Here \(E_{6(-26)}\)
is a certain non-compact real simple Lie group whose Lie algebra $\mathfrak{e}_{6(-26)}$ is a real form of the simple complex Lie algebra of type $E_6$ (see [He, Table V, Section 6, Chapter X]). Appendix C contains more details about this. We extract from there the relevant information, as follows. We have the Cartan decomposition

$$\mathfrak{e}_{6(-26)} = \mathfrak{f}_4 \oplus h_3^0(\mathbb{O})$$

(2.3)

where $\mathfrak{f}_4$ is the Lie algebra of $F_4$ and $h_3^0(\mathbb{O})$ the space of all elements of $h_3(\mathbb{O})$ with trace equal to 0. We denote by $\mathfrak{d}^0$ the space of all elements of $\mathfrak{d}$ with trace equal to 0. It is a maximal abelian subspace of $h_3^0(\mathbb{O})$. Let us also consider the following subspaces of $h_3(\mathbb{O})$:

$$h_{\gamma_1} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & \bar{r} & 0 \end{pmatrix} : r \in \mathbb{O} \right\},$$

$$h_{\gamma_2} := \left\{ \begin{pmatrix} 0 & p & 0 \\ \bar{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : p \in \mathbb{O} \right\},$$

and

$$h_{\gamma_3} := \left\{ \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ \bar{q} & 0 & 0 \end{pmatrix} : q \in \mathbb{O} \right\}.$$

We have the obvious decomposition

$$h_3^0(\mathbb{O}) = \mathfrak{d}^0 + h_{\gamma_1} + h_{\gamma_2} + h_{\gamma_3}.$$

The spaces $h_{\gamma_k}$ are in fact root spaces, in the sense that we have

$$h_{\gamma_k} = \{ a \in h_3(\mathbb{O}) : [x, [x, a]] = \gamma_k(x)^2a \text{ for all } x \in \mathfrak{d}^0 \},$$

(2.4)

$k = 1, 2, 3$. Here the bracket $[,]$ is the usual commutator of matrices and $\gamma_1, \gamma_2, \gamma_3 : \mathfrak{d}^0 \to \mathbb{R}$ are described by

$$\gamma_1(x_1, x_2, x_3) := x_2 - x_3,$$

$$\gamma_2(x_1, x_2, x_3) := x_1 - x_2,$$

$$\gamma_3(x_1, x_2, x_3) := x_1 - x_3,$$

(2.5)

where $(x_1, x_2, x_3)$ stands for Diag$(x_1, x_2, x_3)$, for any $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 + x_2 + x_3 = 0$ (for more details concerning Equation (2.4), see Appendix C). The elements of $\Phi = \{ \pm \gamma_1, \pm \gamma_2, \pm \gamma_3 \}$ are the roots of $E_6(-26)/F_4$ with respect to $\mathfrak{d}^0$. We also consider the subsets

$$\Phi^+ = \{ \gamma_1, \gamma_2, \gamma_3 \} \text{ and } \Phi^- = \{-\gamma_1, -\gamma_2, -\gamma_3\}$$

of $\Phi$. The following proposition concerns the action of $F_4$ on $h_3^0(\mathbb{O})$ mentioned above.

---

\[2\text{This also explains the subscript } -26 \text{ from } \mathfrak{e}_{6(-26)}. \text{ It is the signature of the Killing form of this Lie algebra. This form is negative definite on } \mathfrak{f}_4 \text{ (of dimension 52) and positive definite on } h_3^0(\mathbb{O}) \text{ (of dimension 26).}\]

\[3\text{Strictly speaking, the roots are } \pm \frac{1}{2}(x_2 - x_3), \pm \frac{1}{2}(x_1 - x_2), \text{ and } \pm \frac{1}{2}(x_1 - x_3) \text{ (see the end of Appendix C).}\]
Proposition 2.2.1. Take \( x_0 = \text{Diag}(x_1^0, x_2^0, x_3^0) \in \mathfrak{d}^0 \) such that \( x_1^0, x_2^0, \) and \( x_3^0 \) are any two different. Then the \( F_4 \) stabilizer of \( x_0 \) is the group \( \text{Spin}(8) \) in Proposition 2.1.3. One identifies in this way

\[
\text{Fl}(\mathcal{O}) = F_4.x_0.
\]

Proof. An element \( g \in F_4 \) leaves \( x_0 \) fixed if and only if it leaves the entire \( \mathfrak{d}^0 \) pointwise fixed (see Proposition 2.1.1). By Equation (2.1) this is equivalent to \( g \) leaves \( \mathfrak{d} \) pointwise fixed. By Equation (2.2), this is equivalent to \( g \in \text{Spin}(8) \) \( \square \)

Consequently \( \text{Fl}(\mathcal{O}) \) is a real flag manifold (see Appendix B for more about this general notion). We deduce from here that the root spaces \( \mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}, \) and \( \mathfrak{h}_{\gamma_3} \) are \( \text{Spin}(8) \)-invariant. In fact, the corresponding representations can be described explicitly as follows (see [Ba, p. 179]):

- \( \mathfrak{h}_{\gamma_3} = V_8 \), the standard (matrix) representation of \( \text{SO}(8) \) on \( \mathbb{R}^8 \), composed with the covering map \( \pi : \text{Spin}(8) \to \text{SO}(8) \)
- \( \mathfrak{h}_{\gamma_2} = S_8^- \)
- \( \mathfrak{h}_{\gamma_3} = S_8^+ \)

where \( S_8^\pm \) are the two half-spin representations of \( \text{Spin}(8) \).

The Weyl group of \( E_6(\mathbb{C})/F_4 \) with respect to \( \mathfrak{d}^0 \) is

\[
W := \{ n \in F_4 : n.\mathfrak{d}^0 \subset \mathfrak{d}^0 \}/\text{Spin}(8).
\]

The obvious action of this group on \( \mathfrak{d}^0 \) is faithful. The corresponding group of transformations of \( \mathfrak{d}_0 \) is generated by the reflections of \( \mathfrak{d}^0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \} \) with respect to the lines \( \ker \gamma_1, \ker \gamma_2, \) and \( \ker \gamma_3 \). Thus \( W \) can be identified with the symmetric group \( \Sigma_3 \) which acts on \( \mathfrak{d}^0 \) by permuting the coordinates \( x_1, x_2, x_3 \). Consequently it also acts on \( \Phi \), by

\[
(\sigma \gamma)(x) = \gamma(\sigma^{-1}x),
\]

for all \( \sigma \in \Sigma_3, \gamma \in \Phi \) and \( x \in \mathfrak{d}^0 \).

The tangent space to \( \text{Fl}(\mathcal{O}) \) (regarded as a submanifold of euclidean space \( \mathfrak{h}_3^0(\mathcal{O}) \)) at \( x_0 \) is

\[
T_{x_0}\text{Fl}(\mathcal{O}) = \mathfrak{h}_{\gamma_1} + \mathfrak{h}_{\gamma_2} + \mathfrak{h}_{\gamma_3}.
\]

Consider the vector bundles \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) on \( \text{Fl}(\mathcal{O}) \) given by

\[
\mathcal{E}_k|_{g.x_0} = g.\mathfrak{h}_{\gamma_k},
\]

for any \( g \in F_4, k = 1, 2, 3 \). These are sub-bundles of the tangent bundle of \( \text{Fl}(\mathcal{O}) \). In what follows we will show that \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) defined above are the same as \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) defined by Equation (1.2).

Proposition 2.2.2. The vector bundles \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) defined by Equation (2.7) satisfy

\[
\mathcal{E}_1|_{g.x_0} = T_{g.x_0}\pi_1^{-1}(\pi_1(g.x_0)) \text{ and } \mathcal{E}_2|_{g.x_0} = T_{g.x_0}\pi_2^{-1}(\pi_2(g.x_0))
\]

for all \( g \in F_4 \).
Proof. We prove the first equality. By $F_4$-equivariance, we only need to prove that
\[ h_{\gamma_1} = T_{(d_1, d_3)} \pi_1^{-1}(d_1). \]
Here we have used that $x_0$ corresponds to $(d_1, d_3)$ via the isomorphism (2.6). We saw in the
proof of Proposition 2.1.4 that $\pi_1^{-1}(d_1)$ consists of all
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 - x_3 & r \\
0 & \bar{r} & x_3
\end{pmatrix}
\]
where $x_3 \in \mathbb{R}$ and $r \in \mathcal{O}$ such that $a^2 = a$. This gives
\[
|r|^2 + (x_3 - \frac{1}{2})^2 = \frac{1}{4}.
\]
This is an 8-sphere whose tangent space at $(r, x_3) = (0, 1)$ is described by $x_3 = 0$. The latter
space is just $h_{\gamma_1}$. \[\square\]

Since $Fl(\mathcal{O})$ is a real flag manifold, we deduce from Appendix B (especially Theorem B.2)
that it has a natural cell decomposition
\[ Fl(\mathcal{O}) = \bigsqcup_{\sigma \in \Sigma_3} C_\sigma. \]
For each $\sigma \in \Sigma_3$, the cell $C_\sigma$ is invariant under the action of $Spin(8)$ and we have a $Spin(8)$-
equivariant diffeomorphism
\[ C_\sigma \simeq \bigoplus h_\gamma \]
where the sum runs over all $\gamma \in \Phi^+$ such that $\sigma^{-1}\gamma \in \Phi^-$ (see Corollary B.4). The following
result will play an important role in our development:

Proposition 2.2.3. Each $C_\sigma$ can be identified with $\mathbb{C}^{n(\sigma)}$ for some number $n(\sigma)$. In this
way, the canonical maximal torus $T$ of $Spin(8)$ (cf. e.g. [Ad, Chapter 3] or [Br-tD, Chapter
IV, Theorem 3.9]) acts $\mathbb{C}$-linearly on $C_\sigma$.

Proof. By the decomposition (2.9), it is sufficient to study the action of $T$ on $V_8$, $S_8^+$ and
$S_8^-$. The last two representations of $Spin(8)$ are obtained from the first one by (outer)
automorphisms of $Spin(8)$ (see [Ad, Theorem 5.6]). Since any of these automorphisms leave $T$
invariant, it is sufficient to consider the action of $T$ on $V_8$. Without giving the exact
description of $T$ (see the references above), we recall that if $\pi : Spin(8) \rightarrow SO(8)$ is the
canonical double covering, then the elements of $\pi(T)$ are block diagonal $8 \times 8$ matrices
consisting of four blocks of the form
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
where $\theta \in \mathbb{R}$. If we identify $\mathbb{R}^8 = \mathbb{C}^4$ via
\[
(x_1, x_2, \ldots, x_7, x_8) = (x_1 + ix_2, \ldots, x_7 + ix_8),
\]
then the action of any element of \( T \) is given by four copies of a map of the form
\[
x_1 + ix_2 \mapsto (\cos \theta + i \sin \theta)(x_1 + ix_2)
\]
for all \( x_1 + ix_2 \in \mathbb{C} \). This map is obviously \( \mathbb{C} \)-linear (since the multiplication of complex numbers is commutative).

\[\square\]

Finally, we describe the fixed points of the \( Spin(8) \) action on \( Fl(\mathbb{O}) \).

**Proposition 2.2.4.** The fixed point set of the \( Spin(8) \) action on \( Fl(\mathbb{O}) = F_4. x_0 \) is
\[
Fl(\mathbb{O})^{Spin(8)} = \Sigma_3 x_0.
\]
If \( T \subset Spin(8) \) is the canonical maximal torus, then the fixed points of the \( T \) and \( Spin(8) \) actions on \( Fl(\mathbb{O}) \) are the same.

**Proof.** We start with the following claim.

**Claim 1.** If \( a \in \mathfrak{h}_3^0(\mathbb{O}) \) is fixed by \( T \), then \( a \) is in \( \mathfrak{d}^0 \).

To prove this we decompose
\[
a = a_0 + a_1 + a_2 + a_3,
\]
where \( a_0 \in \mathfrak{d}^0, a_j \in \mathfrak{h}_{\gamma_j}, j = 1, 2, 3 \). Since \( \mathfrak{d}^0, \mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2} \) and \( \mathfrak{h}_{\gamma_3} \) are \( Spin(8) \)-invariant (see above), all four of \( a_0, a_1, a_2, a_3 \) are fixed by \( T \). Assume that \( a \) is not in \( \mathfrak{d}^0 \). Then at least one of \( a_1, a_2, \) and \( a_3 \) is non-zero. Say first that \( a_1 \) is non-zero. We have
\[
\pi(g) \cdot a_1 = a_1,
\]
for all \( g \in T \). Here \( \pi : Spin(8) \to SO(8) \) is the canonical double covering and “.” is the matrix multiplication. The \( SO(8) \) stabilizer of \( a_1 \) is isomorphic to \( SO(7) \). Equation (2.10) says that this stabilizer contains the four dimensional torus \( \pi(T) \) as a subgroup, which contradicts \( \text{rank}(SO(7)) = 3 \). If \( a_2 \) (or \( a_3 \)) is different from 0, the argument we use is similar: the representation of \( Spin(8) \) on \( \mathfrak{h}_{\gamma_2} = S_8^+ \) (respectively \( \mathfrak{h}_{\gamma_3} = S_8^- \)) differs from \( V_8 \) by an (outer) automorphism of \( Spin(8) \).

The claim implies that
\[
Fl(\mathbb{O})^T \subset Fl(\mathbb{O}) \cap \mathfrak{d}^0 = \Sigma_3 x_0.
\]
For the last equality we have used [Fr Section 5 (Hauptachsentransformation von \( I \))] (see also [Mu Section 5, Lemma 1]). On the other hand, Equation (2.2) implies
\[
Fl(\mathbb{O}) \cap \mathfrak{d}^0 \subset Fl(\mathbb{O})^{Spin(8)}.
\]
This finishes the proof. \(\square\)
3. Cohomology of the octonionic flag manifold

Let us consider again the projection maps $\pi_1, \pi_2 : Fl(O) \to \mathbb{O}P^2$ defined by Equation (1.2). We would like to describe $\pi_1$ and $\pi_2$ by using the identification between $Fl(O)$ and the orbit $F_4.x_0$ (see Proposition 2.2.1). To this end, we consider the following two elements of $\mathfrak{o}^0$:

$$d_1^0 = d_1 - \frac{1}{3}I = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad d_3^0 = d_3 - \frac{1}{3}I = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}.$$ 

For each of them, the $F_4$ stabilizer is a copy of $\text{Spin}(9)$ which contains the $F_4$ stabilizer of $x_0$; thus, their orbits are both diffeomorphic to $\mathbb{O}P^2$. The maps $p_1 : F_4.x_0 \to F_4.d_1^0, \quad p_2 : F_4.x_0 \to F_4.d_3^0$
given by $p_1(g.x_0) = g.d_1^0, \quad p_2(g.x_0) = g.d_3^0$ are well defined. Let us consider the following diagram:

$$\begin{array}{ccc}
Fl(O) & \xrightarrow{\pi_1} & \mathbb{O}P^2 \\
\downarrow & & \downarrow \\
F_4.x_0 & \xrightarrow{p_1} & F_4.d_1^0 \\
\end{array}$$

Here, the vertical arrow in the left-hand side is the $F_4$-equivariant diffeomorphism which maps $(d_1, d_3)$ to $x_0$ (see Proposition 2.2.1). The other vertical arrow in the diagram is the diffeomorphism given by

$$x \mapsto x - \frac{1}{3}I,$$

for all $x \in \mathbb{O}P^2$: it is an $F_4$-equivariant diffeomorphism too. The diagram is commutative.

We also have a similar diagram which involves $p_2$ and $\pi_2$. Thus, if we identify $F_4.x_0 = Fl(O), \quad F_4.d_1^0 = \mathbb{O}P^2, \quad F_4.d_3^0 = \mathbb{O}P^2,$
then we have the following result:

**Proposition 3.1.** The maps $p_1, p_2 : Fl(O) \to \mathbb{O}P^2$ defined above are $\text{Spin}(8)$-equivariant $\mathbb{O}P^1$-bundles. The vector bundles $E_1$ and $E_2$ defined by Equation (2.7) satisfy

$$E_1|_{g.x_0} = T_{g.x_0}p_1^{-1}(p_1(g.x_0)) \quad \text{and} \quad E_2|_{g.x_0} = T_{g.x_0}p_2^{-1}(p_2(g.x_0))$$

for all $g \in F_4$.

This proposition is a direct consequence of Propositions 2.1.4 and 2.2.2.

We will use the notation

$$X = Fl(O) = F_4.x_0.$$
Let us consider again the functions $\gamma_1, \gamma_2, \gamma_3 : \mathfrak{d}^0 \to \mathbb{R}$ defined in the previous section (actually the restrictions to $\mathfrak{d}^0$ of the functions given by Equation (2.5)). Then $\pm \gamma_1, \pm \gamma_2, \pm \gamma_3$ are a root system of type $A_2$. We choose the simple root system consisting of $\gamma_1$ and $\gamma_2$; then $\gamma_3 = \gamma_1 + \gamma_2$ is the third positive root.

In what follows we will construct an orientation on each of the bundles $\mathcal{E}_k$, $k = 1, 2, 3$. First, we pick an orientation on $\mathcal{E}_k|_{x_0} = \mathfrak{h}_{\gamma_k}$ (see below). Then, if $g \in F_4$, we choose the orientation on $\mathcal{E}_k|_{g.x_0} = g.\mathfrak{h}_{\gamma_k}$ in such a way that the map $g$ is orientation preserving (note that this definition does not depend on $g$, since the stabilizer group $(F_4)_{x_0} = Spin(8)$ is connected and each of its elements acts on $\mathfrak{h}_{\gamma_k}$ as a linear orthogonal transformation, cf. Section 2). Thus orienting $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ amounts to choosing orientations on $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$ and $\mathfrak{h}_{\gamma_3}$. We proceed as follows: First we take into account that $\gamma_3 = s_2 \gamma_1$, where $s_2$ denotes the element of the Weyl group $W = \Sigma_3$ given by the reflection of $\mathfrak{d}^0$ about $ker \gamma_2$. Thus there exists $n_2 \in F_4$ with $n_2.\mathfrak{d}^0 = \mathfrak{d}^0$ such that $s_2$ is the same as the coset $[n_2] = n_2 Spin(8)$ in $\Sigma_3$. Then we have

$$\gamma_3 = \gamma_1 \circ n_2^{-1}.$$ 

This implies that $n_2$ maps $\mathfrak{h}_{\gamma_1}$ to $\mathfrak{h}_{\gamma_3}$. Similarly, there exists $n_1 \in F_4$ such that

$$\gamma_3 = \gamma_2 \circ n_1^{-1}.$$ 

Thus $n_1^{-1}$ maps $\mathfrak{h}_{\gamma_3}$ to $\mathfrak{h}_{\gamma_2}$. We pick and fix an orientation on $\mathfrak{h}_{\gamma_1}$; the orientations we equip $\mathfrak{h}_{\gamma_2}$ and $\mathfrak{h}_{\gamma_3}$ with are such that the maps $n_1$ and $n_2$ are orientation preserving.

The main goal of this section is to prove Theorem 1.1. We proceed as follows. First, we recall from the previous section that $X$ is a 24-dimensional manifold. It is known (see for instance [Hs-Pa-Te], Section 5)) that the group $H^* (X; \mathbb{Z})$ is a free $\mathbb{Z}$-module such that

$$\dim H^k (X; \mathbb{Z}) = \begin{cases} 
0, & \text{if } k \notin \{0, 8, 16, 24\} \\
2, & \text{if } k \in \{8, 16\} \\
1, & \text{if } k \in \{0, 24\}.
\end{cases}$$

(3.1)

A basis of $H^8 (X; \mathbb{Z})$ can be constructed as follows. By Proposition 3.1, the subspaces $S_1 := p_1^{-1}(d_1^0)$, $S_2 := p_2^{-1}(d_2^0)$ of $Fl(\mathcal{O})$ are diffeomorphic to $\mathcal{O}P^1$, hence to the sphere $S^8$. By Proposition 3.1, the tangent bundle of $S_1$ is just $\mathcal{E}_1|_{S_1}$; thus, the orientation of $\mathcal{E}_1$ chosen above induces an orientation of $S_1$. Similarly we can orient $S_2$. The homology classes $[S_1], [S_2]$ carried by $S_1$ and $S_2$ are a basis of $H_8 (X; \mathbb{Z})$. Then $\beta_1, \beta_2 \in H^8 (X; \mathbb{Z})$ determined by

$$\langle \beta_i, [S_j] \rangle = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{otherwise}
\end{cases}$$

$1 \leq i, j \leq 2$, are a basis of $H^8 (X; \mathbb{Z})$ (here $(, ) : H^8 (X; \mathbb{Z}) \otimes H_8 (X; \mathbb{Z}) \to \mathbb{Z}$ denotes the evaluation pairing).
We take into account that the elements $d_0^0$ and $d_1^3$ of $\mathfrak{d}^0$ satisfy $\gamma_1(d_0^1) = 0$ and $\gamma_2(d_3^0) = 0$. The following equations can be deduced from [Hs-Pa-Te, Proof of Theorem 6.12] (see also [Ma1, Proof of Lemma 3.3]).

\[
\begin{align*}
  e(\mathcal{E}_1) &= 2\beta_1 + \frac{2(\gamma_1, \gamma_2)}{(\gamma_2, \gamma_2)} \beta_2 = 2\beta_1 - \beta_2 \\
  e(\mathcal{E}_2) &= \frac{2(\gamma_2, \gamma_1)}{(\gamma_1, \gamma_1)} \beta_1 + 2\beta_2 = -\beta_1 + 2\beta_2 \\
  e(\mathcal{E}_3) &= e(\mathcal{E}_1) + e(\mathcal{E}_2)
\end{align*}
\]

Thus we have

\[
\beta_1 = \frac{1}{3}(2e(\mathcal{E}_1) + e(\mathcal{E}_2)) \quad \text{and} \quad \beta_2 = \frac{1}{3}(e(\mathcal{E}_1) + 2e(\mathcal{E}_2)).
\]

From Equation (2.7) we deduce that the tangent bundle $TX$ can be written as $TX = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$.

This implies

\[
e(TX) = e(\mathcal{E}_1)e(\mathcal{E}_2)e(\mathcal{E}_3) = e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)).
\]

If $[X]$ is the fundamental homology class of $X$, then

\[
(e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)), [X]) = (e(TX), [X]) = \chi(X) = 6,
\]

where $\chi(X)$ is the Euler-Poincaré characteristic of $X$. We know it is equal to 6 thanks to Equation (3.1). Consequently, the cohomology class

\[
\frac{1}{6}e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2))
\]

is a basis of $H^{24}(X; \mathbb{Z})$ over $\mathbb{Z}$.

Let us now consider separately the root system $\pm \gamma_1, \pm \gamma_2, \pm \gamma_3$. The fundamental weights corresponding to the simple roots $\gamma_1, \gamma_2$ are

\[
\lambda_1 = \frac{1}{3}(2\gamma_1 + \gamma_2), \quad \lambda_2 = \frac{1}{3}(\gamma_1 + 2\gamma_2).
\]

We know that there exists a canonical isomorphism between the ring

\[
\mathbb{Q}[\lambda_1, \lambda_2]/\langle\text{nonconstant symmetric polynomials in } \lambda_1, \lambda_2 - \lambda_1, -\lambda_2\rangle
\]

and $H^*(Fl_3(\mathbb{C}); \mathbb{Q})$. By a theorem of Bernstein, I. M. Gelfand, and S. I. Gelfand [Be-Ge-Ge], a basis of $H^*(Fl_3(\mathbb{C}); \mathbb{Q})$ over $\mathbb{Q}$ is obtained by considering

\[
\frac{1}{6}\gamma_1\gamma_2(\gamma_1 + \gamma_2)
\]

and applying successively the divided difference operators $\Delta_{\gamma_1}$ and $\Delta_{\gamma_2}$. Here, the operator $\Delta_{\gamma}$ corresponding to the root $\gamma \in \{\gamma_1, \gamma_2\}$ is defined by:

\[
\Delta_{\gamma}(f) = \frac{f - f \circ s_{\gamma}}{\gamma}
\]
for any $f \in \mathbb{Q}[\lambda_1, \lambda_2]$ (by $s_\gamma$ we denote the reflection of $\omega^0$ about the line $\ker \gamma$). The Bernstein-Gelfand-Gelfand basis of $H^*(Fl_3(\mathbb{C}); \mathbb{Q})$ mentioned above consists of the cosets of the following polynomials:

$$\frac{1}{3} \gamma_1 (\gamma_1 + \gamma_2), \frac{1}{3} \gamma_2 (\gamma_1 + \gamma_2)$$

$$\lambda_1, \lambda_2$$

$$1.$$

We deduce that

$$(3.4) \quad \lambda_1 \cdot \frac{1}{3} \gamma_1 (\gamma_1 + \gamma_2) = \frac{1}{6} \gamma_1 \gamma_2 (\gamma_1 + \gamma_2) + f$$

where $f \in \mathbb{Q}[\lambda_1, \lambda_2]$ is in the ideal generated by the non-constant symmetric polynomials in $\lambda_1, \lambda_2 - \lambda_1, -\lambda_2$.

We now return to the cohomology of $X$. By [Hs-Pa-Tc, Theorem 6.12] (see also [Ma1, Section 3]), the ring $H^*(X; \mathbb{Q})$ is generated by $\beta_1$ and $\beta_2$, the ideal of relations being generated by the symmetric polynomials in $\beta_1, \beta_2 - \beta_1, -\beta_2$. Equations (3.2) and (3.4) imply that the equality

$$\beta_1 \frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1) + e(\mathcal{E}_2)) = \frac{1}{6} e(\mathcal{E}_1) e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)),$$

holds in $H^*(X; \mathbb{Q})$. The right-hand side of the equation is the fundamental cohomology class of $X$ over $\mathbb{Z}$ (see Equation (3.3)). Since $\beta_1$ is in $H^*(X; \mathbb{Z})$, we deduce that the cohomology class

$$(3.5) \quad \frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1) + e(\mathcal{E}_2))$$

belongs to $H^*(X; \mathbb{Z})$, being the Poincaré dual of $\beta_1$ in $H^*(X; \mathbb{Z})$. Similarly, the class

$$(3.6) \quad \frac{1}{3} e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2))$$

is in $H^*(X; \mathbb{Z})$, being the Poincaré dual of $\beta_2$. Consequently, the classes given by (3.5) and (3.6) are a basis of $H^{16}(X; \mathbb{Z})$.

To complete the proof, it only remains to show that the cohomology classes given by (3.5) and (3.6) can be expressed as polynomials with integer coefficients in $\beta_1$ and $\beta_2$. Indeed, by using (3.2), we can see that

$$\frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1) + e(\mathcal{E}_2)) = \beta_1^2$$

and

$$\frac{1}{3} e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)) = \beta_2^2.$$
Remark. The result stated in Theorem 1.1 is not entirely new: a similar description has been obtained for example in [Yo, Theorem 2.3]. The novelty of Theorem 1.1 is that it gives geometric descriptions of the generators of the cohomology ring.

4. Equivariant cohomology of $Fl(O)$: generators and relations

In this section we will prove Theorem 1.2. Like before, we denote $M := Spin(8)$ and $X := Fl_3(O) = F_4 x_0$.

We start with the following result, which is an immediate consequence of Theorem 1.1 (see also Section 3):

Lemma 4.1. The ring $H^*(X)$ is generated by $e(E_1)$ and $e(E_2)$ with the relations

$$S_i(2e(E_1) + e(E_2), -e(E_1) + e(E_2), -e(E_1) - 2e(E_2)) = 0,$$

$i = 2, 3$. Here $S_i$ denotes the $i$-th fundamental symmetric polynomial in three variables.

We are actually interested here in the equivariant cohomology ring $H^*_M(X)$. The space $X$ is a real flag manifold (see Appendix B) which satisfies the hypotheses of [Ma1, Theorem 1.1]. Namely, the multiplicities of the roots of the symmetric space $E_6(-26)/F_4$ are strictly greater than 1: indeed, for all $k \in \{1, 2, 3\}$ the multiplicity of $\gamma_k$ is, by definition, the dimension of $s_{\gamma_k}$; the latter space is just $h_{\gamma_k}$ (see Appendix C), so all multiplicities are equal to 8. By the result of [Ma1] mentioned above, the action of $M$ on $X$ is equivariantly formal. For more on the notion of equivariant formality, we address the reader to [Gu-Gi-Ka, Section 4, Appendix C] or [Ha-Ho, Section 4.3]. This condition is saying that, as a module over $H^*(BM)$, the space $H^*_M(X)$ is free, of dimension equal to $dim H^*(X)$. The properties stated in the following proposition are consequence of equivariant formality (cf. [Ha-Ho Proposition 4.4]):

Proposition 4.2. The graded ring homomorphism $j^*: H^*_M(X) \to H^*(X)$ induced by the canonical inclusion $j: X \to EM \times_M X$ is surjective. Its kernel is

$$\ker j^* = \langle H^+(BM).H^*_M(X) \rangle,$$

where $H^+(BM)$ denotes the space of all elements of $H^*(BM)$ of strictly positive degree and $\langle H^+(BM).H^*_M(X) \rangle$ is the $\mathbb{R}$-span of all elements of the form $\beta.\alpha$, with $\beta \in H^+(BM)$ and $\alpha \in H^*_M(X)$.

Our first goal is to prove that Equations 1.4 hold true. The elements $b_1, b_2$ of $H^*(BM)$ involved there can actually be expressed as

$$b_k = e_M(h_{\gamma_k}),$$
Let us also define
\[ b_3 := e_M(b_{x_3}). \]

The following notation is standard: if \( \alpha \in H^*_M(X) \) and \( x \in X^M \), then the restriction of \( \alpha \) to \( x \) is
\[ \alpha|_x := i_x^*(\alpha) \]

where \( i_x : \{x\} \to X \) is the inclusion map (note that \( \alpha|_x \in H^*_M(\{x\}) = H^*(BM) \)). The next lemma will be needed later.

**Lemma 4.3.** We have
\[ e_M(E_1) + e_M(E_2) - e_M(E_3) = b_1 + b_2 - b_3. \] (4.1)

**Proof.** For any \( k \in \{1, 2, 3\} \) we have \( j^*(e_M(E_k)) = e(E_k) \) (since, by definition, \( e_M(E_k) \) is the Euler class of a vector bundle over \( EM \times M X \) whose pullback via \( j \) is \( E_k \)). We deduce that
\[ j^*(e_M(E_1) + e_M(E_2) - e_M(E_3)) = e(E_1) + e(E_2) - e(E_3) = 0. \]

From Proposition 4.2 and the fact that \( H^k(X) = \{0\} \) for all \( 1 \leq k \leq 7 \) (see Equation (3.1)) we deduce that
\[ e_M(E_1) + e_M(E_2) - e_M(E_3) \in P^*(H^*(BM)). \]

The composition \( P \circ i_{x_0} \) is the identity function of \( \{x_0\} \). Thus, it is now sufficient to note that
\[ (e_M(E_1) + e_M(E_2) - e_M(E_3)|_{x_0} = e_M(E_1|_{x_0}) + e_M(E_2|_{x_0}) - e_M(E_3|_{x_0}) = b_1 + b_2 - b_3, \]
where we have used Equation (2.7). \( \square \)

The following localization result will also be used here. It will be proved in Section 5. We recall (see Lemma 2.2.4) that the fixed points of the \( M \) action on \( X \) are given by
\[ X^M = \Sigma_3 x_0 \]

**Lemma 4.4.** The restriction map
\[ H^*_M(X) \to H^*_M(X^M) \]

is injective.

The strategy we will use in order to prove Equations (1.4) is to show that both sides are equal when restricted to any fixed point \( \sigma x_0 \), where \( \sigma \in \Sigma_3 \). The next lemma will help us accomplish this strategy.

**Lemma 4.5.** The restrictions of \( e_M(E_1), e_M(E_2), \) and \( e_M(E_3) \) to \( X^M \) are as follows:

| \( \sigma \) | \( 1 \) | \( s_1 \) | \( s_2 \) | \( s_1s_2 \) | \( s_2s_1 \) | \( s_1s_2s_1 \) |
|---|---|---|---|---|---|---|
| \( e_M(E_1)|_{x_0} \) | \( b_1 \) | \( -b_1 \) | \( b_3 \) | \( b_2 \) | \( -b_3 \) | \( -b_2 \) |
| \( e_M(E_2)|_{x_0} \) | \( b_2 \) | \( b_3 \) | \( -b_2 \) | \( -b_3 \) | \( b_1 \) | \( -b_1 \) |
| \( e_M(E_3)|_{x_0} \) | \( b_3 \) | \( b_2 \) | \( b_1 \) | \( -b_1 \) | \( -b_2 \) | \( -b_3 \) |
Proof. We have
\[ e_M(\mathcal{E}_1)|_{s_1x_0} = e_M(\mathcal{E}_1|_{s_1x_0}). \]
By definition, \( \mathcal{E}_1|_{x_0} = \mathfrak{h}_{\gamma_1} \). The points \( x_0 \) and \( s_1x_0 \) are the antipodal points of the eight dimensional sphere \( S_1 = \pi_1^{-1}(e^0_1) \) embedded in \( X \) (see Section 3). By Proposition 3.1, the tangent bundle of \( S_1 \) is just the restriction of \( \mathcal{E}_1 \) to \( S_1 \). The orientation of \( \mathcal{E}_1 \) induces an orientation of the sphere \( S_1 \). The space \( \mathcal{E}_1|_{s_1x_0} \) is the same as \( \mathcal{E}_1|_{x_0} = \mathfrak{h}_{\gamma_1} \), but with the reversed orientation. Consequently,
\[ e_M(\mathcal{E}_1)|_{s_1x_0} = -e_M(\mathfrak{h}_{\gamma_1}) = -b_1. \]

Let us now determine
\[ e_M(\mathcal{E}_1)|_{s_2x_0} = e_M(\mathcal{E}_1|_{s_2x_0}). \]
Like in Section 3, we consider again \( n_2 \in F_4 \) such that \( s_2 \) is the coset \([n_2] = n_2Spin(8)\) in the Weyl group \( W = \Sigma_3 \). By definition, since \( s_2x_0 = n_2.x_0 \), we have
\[ \mathcal{E}_1|_{s_2x_0} = n_2.\mathfrak{h}_{\gamma_1}. \]
Moreover, \( n_2 \) is an orientation preserving map from \( \mathfrak{h}_{\gamma_1} \) to \( \mathcal{E}_1|_{s_2x_0} \). On the other hand, we saw in Section 3 that \( n_2 \) maps \( \mathfrak{h}_{\gamma_1} \) to \( \mathfrak{h}_{\gamma_3} \) by preserving the orientation. We deduce that
\[ e_M(\mathcal{E}_1|_{s_2x_0}) = e_M(\mathfrak{h}_{\gamma_3}) = b_3. \]

We determine now
\[ e_M(\mathcal{E}_1)|_{s_1s_2x_0} = e_M(\mathcal{E}_1|_{s_1s_2x_0}). \]
Take \( n_1 \in F_4 \) such that \( s_1 = [n_1] = n_1Spin(8) \) in \( \Sigma_3 \). We have
\[ s_1s_2x_0 = s_1^{-1}s_2x_0 = n_1^{-1}(n_2.x_0). \]
Thus \( \mathcal{E}_1|_{s_1s_2x_0} \) is obtained from \( \mathfrak{h}_{\gamma_1} \) by applying first \( n_2 \) (and obtaining \( \mathfrak{h}_{\gamma_3} \)), followed by \( n_1^{-1} \) (which gives \( \mathfrak{h}_{\gamma_2} \)). Consequently,
\[ e_M(\mathcal{E}_1|_{s_1s_2x_0}) = e_M(\mathfrak{h}_{\gamma_2}) = b_2. \]

All other restriction formulas can be proved similarly. \( \square \)

The following lemma expresses \( e_M(\mathcal{E}_3) \) in terms of \( e_M(\mathcal{E}_1) \) and \( e_M(\mathcal{E}_2) \).

Lemma 4.6. We have
\[ b_3 = b_2 + b_1 \]
and
\[ e_M(\mathcal{E}_3) = e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2). \]

Proof. We take Equation (4.1) and restrict both sides to \( s_1x_0 \). The left-hand side changes according to Lemma 4.5. The right-hand side doesn’t change. Indeed, for any \( k \in \{1, 2, 3\} \) we have
\[ P^*(b_k)|_{s_1x_0} = i^*_{s_1x_0}(P^*(b_k)) = i^*_{s_1x_0}(P^*(e_M(\mathfrak{h}_{\gamma_k}))) = (P \circ i_{s_1x_0})^*(e_M(\mathfrak{h}_{\gamma_k})), \]
which is the same as the $M$-equivariant Euler class of the pullback of $\gamma_k$ via the map $P \circ i_{s_1x_0} : \{s_1x_0\} \rightarrow \{x_0\}$; this is equal to $b_k$. Equation (4.1) implies

$$-b_1 + b_3 - b_2 = b_1 + b_2 - b_3,$$

which, in turn, implies the desired equations. \qed

We are now ready to prove that Equations (4.4) hold. For each of them we restrict the left-hand side to $x_0, s_1x_0, s_2x_0, \ldots, s_1s_2s_1x_0$ and use Lemmas 4.5 and 4.6 each time we do this, we obtain $S_2(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2)$, respectively $S_3(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2)$. Indeed, let $S$ be one of the (symmetric) polynomials $S_2$ and $S_3$. We have as follows:

$$S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))_{s_1x_0}$$

$$= S(-2b_1 + b_3, b_1 + b_3, b_1 - 2b_3)$$

$$= S(-b_1 + b_2, 2b_1 + b_2, -b_1 - 2b_2)$$

$$= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).$$

$$S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))_{s_2x_0}$$

$$= S(2b_3 - b_2, -b_3 - b_2, -b_3 + 2b_2)$$

$$= S(2b_1 + b_2, -b_1 - 2b_2, -b_1 + b_2)$$

$$= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).$$

$$S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))_{s_3x_0}$$

$$= S(-2b_3 + b_1, b_3 + b_1, b_3 - 2b_1)$$

$$= S(-b_1 - 2b_2, 2b_1 + b_2, -b_1 + b_2)$$

$$= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).$$

$$S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))_{s_4x_0}$$

$$= S(-2b_2 - b_1, b_2 - b_1, b_2 + 2b_1)$$

$$= S(-b_1 - 2b_2, -b_1 + b_2, 2b_1 + 2b_2)$$

$$= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).$$

Our second goal is to show that $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$ generate $H^*_M(X)$ as a $H^*(BM)$-algebra. To this end we first recall that the action of $M$ on $X$ is equivariantly formal. From Equation

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Lemma 4.7. There exists a basis \{\bar{\alpha}_k : k = 0, \ldots, 5\} of \(H^*_M(X)\) as \(H^*(BM)\)-module such that:

(i) if \(k \in \{0, \ldots, 5\}\), then both \(\bar{\alpha}_k\) and \(\alpha_k := j^*(\bar{\alpha}_k) \in H^*(X)\) are homogeneous of degree given by

\[
\deg \bar{\alpha}_k = \begin{cases}
0, & \text{if } k = 0 \\
8, & \text{if } k \in \{1, 2\} \\
16, & \text{if } k \in \{3, 4\} \\
24, & \text{if } k = 5.
\end{cases}
\]

(ii) the set \(\{\alpha_k : k = 0, \ldots, 5\}\) is a basis of \(H^*(X)\) over \(\mathbb{R}\)

(iii) we have

\[
\bar{\alpha}_1 = e_M(\mathcal{E}_1), \: \bar{\alpha}_2 = e_M(\mathcal{E}_2),
\]

and

\[
\alpha_1 = e(\mathcal{E}_1), \: \alpha_2 = e(\mathcal{E}_2).
\]

Proof. We set

\[
\bar{\alpha}_k := \begin{cases}
\bar{\alpha}_k, & \text{if } k \neq 1, 2 \\
e_M(\mathcal{E}_1), & \text{if } k = 1 \\
e_M(\mathcal{E}_2), & \text{if } k = 2.
\end{cases}
\]

We need to show that these are a spanning set of \(H^*_M(X)\) over \(H^*(BM)\). To this end, it is sufficient to show that

\[
\bar{\alpha}_1 = a_{11}e_M(\mathcal{E}_1) + a_{21}e_M(\mathcal{E}_2) + a_{31}
\]

\[
\bar{\alpha}_2 = a_{12}e_M(\mathcal{E}_1) + a_{22}e_M(\mathcal{E}_2) + a_{32}
\]

where \(a_{11}, a_{21}, a_{12}, a_{22}\) are real numbers and \(a_{31}\) and \(a_{32}\) are in \(H^*(BM)\). Indeed, we have

\[
j^*(e_M(\mathcal{E}_1)) = e(\mathcal{E}_1) \quad \text{and} \quad j^*(e_M(\mathcal{E}_2)) = e(\mathcal{E}_2).
\]

The cohomology classes \(e(\mathcal{E}_1)\) and \(e(\mathcal{E}_2)\) are a basis of \(H^8(X)\) (see Section 3). Also \(j^*(\bar{\alpha}_1)\) and \(j^*(\bar{\alpha}_2)\) are a basis of \(H^8(X)\) (because ker \(j^* = \langle H^+(BM)H^*_M(X) \rangle\)). Thus, we can write

\[
j^*(\bar{\alpha}_1) = a_{11}j^*(e_M(\mathcal{E}_1)) + a_{21}j^*(e_M(\mathcal{E}_2))
\]

\[
j^*(\bar{\alpha}_2) = a_{12}j^*(e_M(\mathcal{E}_1)) + a_{22}j^*(e_M(\mathcal{E}_2))
\]
for some numbers $a_{11}, a_{21}, a_{12}, a_{22}$. Consequently, the differences $\bar{\alpha}_1 - a_{11}e_M(E_1) - a_{21}e_M(E_2)$ and $\bar{\alpha}_2 - a_{12}e_M(E_1) - a_{22}e_M(E_2)$ are linear combinations with coefficients in $H^+(BM)$ of $\bar{\alpha}_0, \ldots, \bar{\alpha}_5$. By dimension reasons, they are actually in $H^+(BM)$. This finishes the proof. \[\square\]

Let us now consider the isomorphism of $H^*(BM)$-modules
\[\Phi : H^*_M(X) \rightarrow H^*(X) \otimes H^*(BM)\]
given by $\Phi(\tilde{\alpha}_k) = \alpha_k$, for all $k = 0, \ldots, 5$.

From now on we identify the $H^*(BM)$-algebra $H^*_M(X)$ with $H^*(X) \otimes H^*(BM)$ equipped with the product $\circ$. The latter is defined by the fact that it is $H^*(BM)$-bilinear and it satisfies the condition
\[\alpha_k \circ \alpha_\ell := \Phi(\tilde{\alpha}_k \tilde{\alpha}_\ell),\]
for all $k, \ell \in \{0, \ldots, 5\}$. We stress that
\[(4.2) \quad H^*_M(X) = (H^*(X) \otimes \mathbb{R}[u_1, u_2, u_3, u_4], \circ)\]
as $\mathbb{R}[u_1, u_2, u_3, u_4]$-algebras (see Equation (1.3)). The usual grading of $H^*(X)$ together with $\deg u_1 = 4, \deg u_2 = \deg u_3 = 8, \deg u_4 = 12$, induces a grading on $H^*(X) \otimes H^*(BM)$. The following two properties of the product $\circ$ will be used later. If $a, b \in H^*(X)$ are homogeneous elements, then we have:

(i) $a \circ b$ is a homogeneous element of $H^*(X) \otimes H^*(BM)$ of degree equal to $\deg(a \circ b) = \deg a + \deg b$,

(ii) $a \circ b = ab + (a \text{ linear combination of multiples of } H^+(BM))$.

Point (i) follows from the fact that the map $\Phi$ is degree preserving. To justify point (ii) it is sufficient to take $a = \alpha_k$ and $b = \alpha_\ell$, where $k, \ell \in \{0, \ldots, 5\}$; we use the fact that the following diagram is commutative
\[H^*_M(X) \xrightarrow{\Phi} H^*(X) \otimes H^*(BM) \xrightarrow{j^*} H^*(X)\]
Here the arrow in the right-hand side is the canonical projection.

**Lemma 4.8.** The classes
\[e_1 := e_M(E_1) \text{ and } e_2 := e_M(E_2)\]
generate $H^*_M(X)$ as $H^*(BM)$-algebra. Equivalently, in terms of the identification (4.2), the classes
\[e_1 = e(E_1) \text{ and } e_2 = e(E_2)\]
generate $(H^*(X) \otimes H^*(BM), \circ)$ as $H^*(BM)$-algebra.
Proof. It is sufficient to prove that for any $k \in \{0, \ldots, 5\}$, $\alpha_k$ can be written as a polynomial expression in $e_1$ and $e_2$ with coefficients in $H^*(BM)$, the product being $\circ$. We prove this by induction on $\ell(\sigma)$. The claim is obvious for $k = 0$, as $\alpha_0$ is just a number (element of $H^0(X)$). Let us now make the induction step: take $k \in \{0, \ldots, 5\}$, $k \geq 1$. We know that $e_1$ and $e_2$ generate $H^*(X)$ (see Lemma 4.1). Thus we have

$$\alpha_k = f(e_1, e_2),$$

where $f$ is a polynomial in two variables and the product in the right hand side is the usual (cup) product. Let $f^\circ(e_1, e_2)$ be the element of $H^*(X) \otimes H^*(BM)$ obtained by evaluating $f$ in terms of the product $\circ$. By property (ii) of $\circ$, $\alpha_k - f^\circ(e_1, e_2)$ is a linear combination of terms of the form $\beta.\alpha_\ell$, where $\beta \in H^+(BM)$ and $\ell \in \{0, \ldots, 5\}$ with $\deg \alpha_\ell < \deg \alpha_k$. The last condition implies $\ell < k$: we only need to use the induction hypothesis.

The following lemma will finish the proof of Theorem 1.2.

Lemma 4.9. The ideal of relations in $H^*_M(X)$ with respect to $e_1$ and $e_2$ is generated by Equations (1.4).

Proof. Let us consider the polynomials $g_2, g_3 \in \mathbb{R}[x_1, x_2]$ given by

$$(4.3) \quad g_i = S_i(2x_1 + x_2, -x_1 + x_2, -(x_1 + 2x_2))$$

$i = 2, 3$. We prove that if $f(x_1, x_2) \in H^*(BM) \otimes \mathbb{R}[x_1, x_2]$ such that

$$(4.4) \quad f^\circ(e_1, e_2) = 0,$$

then $f$ is in the ideal generated by the polynomials

$$f_i(x_1, x_2) := g_i(x_1, x_2) - g_i(b_1, b_2),$$

$i = 2, 3$ (here $f^\circ(e_1, e_2)$ is the element of $H^*(X) \otimes H^*(BM)$ obtained by evaluating $f(x_1, x_2)$ on $e_1, e_2$ in the ring $(H^*(X) \otimes H^*(BM), \circ)$). We prove this claim by induction on $\deg f$: throughout this proof, degree will always be considered only with respect to $x_1$ and $x_2$. If $\deg f = 0$ then the claim is obvious. Let us now make the induction step. We consider a non-constant polynomial $f(x_1, x_2)$ as above, satisfying equation (4.4). Let $g(x_1, x_2)$ be the component of $f(x_1, x_2)$ of highest degree (with respect to $x_1, x_2$). From the fact that $f^\circ(e_1, e_2) = 0$ and property (ii) of $\circ$ we deduce that

$$g(e_1, e_2) = 0.$$

By Lemma 4.1, $g(x_1, x_2)$ is a combination with coefficients in $H^*(BM) \otimes \mathbb{R}[x_1, x_2]$ of $g_2(x_1, x_2)$ and $g_3(x_1, x_2)$. We come back to Equation (4.4) and replace $g$ by the expression mentioned above, where we complete each occurrence of $g_i$ to $f_i$ (by adding and subtracting the necessary quantity). The cancellations which we obtain allow us to obtain another condition of type (4.3), this time with a polynomial $f$ of degree strictly smaller than the previous one. Finally, we use the induction hypothesis. \qed
Remark. Another presentation of the ring $H^*_M(X) = H^*_{Spin(8)}(F_4/Spin(8))$ can be deduced from [Ho-Sj, Corollary 5.10], since $F_4$ and $Spin(8)$ have the same rank.

5. Equivariant cohomology of $Fl(\mathbb{O})$: the GKM presentation

We recall that

$$X := Fl(\mathbb{O}) = F_4.x_0,$$

where $x_0 = \text{Diag}(x_0^1, x_0^2, x_0^3)$, with $x_0^1, x_0^2, x_0^3 \in \mathbb{R}$, any two distinct, such that $x_0^1 + x_0^2 + x_0^3 = 0$; this time we also assume that $x_0^1 > x_0^2 > x_0^3$. Like in the introduction, we consider $\bar{e}_{ij} := e_M(b_\gamma) \in H^8(BM)$, where the root $\gamma$ is given by

$$\gamma(x_1, x_2, x_3) = x_i - x_j \quad 1 \leq i < j \leq 3.$$ Since $E_k|_{x_0} = b_{\gamma_k}, k = 1, 2, 3$, we actually have

$$\bar{e}_{12} = b_1, \bar{e}_{23} = b_2, \bar{e}_{13} = b_3.$$ In this section we prove Theorem 1.3. Our main tool is the following result, which can be deduced from the main theorem of [Ha-He-Ho].

**Theorem 5.1.** ([Ha-He-Ho]) Let $M$ be a compact connected topological group and

$$X = \bigsqcup_{k=1}^s C_k$$

a finite CW complex whose open cells $C_k, 1 \leq k \leq s$, satisfy the following properties:

(i) $C_k$ is an even dimensional real vector space equipped with an $M$-linear action which has a unique fixed point, say $p_k$, which is identified with 0.

(ii) We can decompose

$$C_k = \bigoplus_{1 \leq \ell \leq k} C_{k\ell},$$

where $C_{k\ell}$ are vector subspaces (possibly equal to $\{0\}$) of $C_k$; the boundary $\partial_X(C_{k\ell})$ of $C_{k\ell}$ in $X$ consists of only one point, which is fixed by the $M$ action (in the case where $C_{k\ell} = \{0\}$, the fixed point is $p_k$).

(iii) For any $k \in \{1, \ldots, s\}$, the equivariant Euler classes $e_M(C_{k\ell})$, where $1 \leq \ell \leq k$ such that $C_{k\ell} \neq \{0\}$, are relatively prime elements of $H^*(BM)$.

Then the map $i^*: H^*_M(X) \to H^*_M(X^M)$ induced by the inclusion of the $M$-fixed point set $X^M$ into $X$ is injective. Moreover, the image of $i^*$ consists of all

$$(f_k) \in H^*_M(X^M) = \prod_{k=1}^s H^*(BM)$$

such that $f_k - f_{\ell}$ is divisible by $e_M(C_{k\ell})$ for all $1 \leq \ell < k \leq s$ with $C_{k\ell} \neq \{0\}$. 

In the case of $X = Fl(\mathbb{O})$ and $M = Spin(8)$, we use the CW decomposition mentioned in Section 2.2. That is, we choose $C_k = C_{\sigma}$, for $\sigma \in \Sigma_3$. The decomposition (5.1) is the one described by Equation (2.3). We will verify the assumptions (i)-(iii) above. Assumption (i) follows from Proposition 2.2.4 and the fact that $C_{\sigma} \cap \Sigma_3 x_0 = \{ \sigma x_0 \}$. For assumption (ii), we will need the explicit embedding of $C_{\sigma}$ in $X$, as given in Theorem 3.2 (b). That is, we consider the root spaces $\mathfrak{g}_\gamma \subset \mathfrak{e}_6(-26)$, where $\gamma \in \Phi$, as well as the diffeomorphism $\sum \mathfrak{g}_\gamma \rightarrow X$, $x \mapsto \exp(x)(\sigma x_0)$, where the sum in the domain runs over all $\gamma \in \Phi^+$ such that $\sigma^{-1} \gamma \in \Phi^-$. Assumption (ii) follows readily from the following lemma.

**Lemma 5.2.** If $\sigma$ and $\gamma$ are as above, then the boundary of $\exp(\mathfrak{g}_\gamma)(\sigma x_0)$ in $X$ is $\{ s_\gamma \sigma x_0 \}$.

**Proof.** Let us consider again the functions $\gamma_1, \gamma_2, \gamma_3 : \mathfrak{d}^0 \rightarrow \mathbb{R}$ given by Equation (2.3). The set $\Phi = \{ \pm \gamma_1, \pm \gamma_2, \pm \gamma_3 \}$ is a root system of type $A_2$. To any $\gamma \in \{ \gamma_1, \gamma_2, \gamma_3 \}$ we assign the reflection $s_\gamma$ of $\mathfrak{d}^0$ about the line $\ker \gamma$. These three reflections generate a group of transformations of $\mathfrak{d}^0$ which is isomorphic to the symmetric group $\Sigma_3$. More specifically, we have

$$s_{\gamma_1} = (2, 3), \quad s_{\gamma_2} = (1, 2), \quad s_{\gamma_3} = (1, 3).$$

Here, as usual, $(i, j)$ denotes the $i, j$ transposition in $\Sigma_3$. The group $\Sigma_3$ is generated by $s_1 := s_{\gamma_1}$ and $s_2 := s_{\gamma_2}$. We actually have

$$\Sigma_3 = \{ 1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 \}.$$

The following table gives for all $\sigma \in \Sigma_3$ the set of all $\gamma \in \Phi^+$ such that $\sigma^{-1} \gamma \in \Phi^-$.}

| $\sigma$ | $\gamma$ |
|---------|---------|
| $s_1$   | $\gamma_1$ |
| $s_2$   | $\gamma_2$ |
| $s_2 s_1$ | $\gamma_2, \gamma_3$ |
| $s_1 s_2$ | $\gamma_1, \gamma_3$ |
| $s_1 s_2 s_1$ | $\gamma_1, \gamma_2, \gamma_3$ |

Here we have used the formulas: $s_k(\gamma_k) = -\gamma_k$ for $k = 1, 2$, $s_1(\gamma_2) = s_2(\gamma_1) = \gamma_3$, $s_1(\gamma_3) = \gamma_2$, $s_2(\gamma_3) = \gamma_1$.

We will prove the lemma by a case by case verification.

**Case 1.** $(\sigma, \gamma) = (s_1, \gamma_1)$. We need to show that the boundary of $\exp(\mathfrak{g}_{\gamma_1})(s_1 x_0)$ is $x_0$. To this end, we note that $\exp(\mathfrak{g}_{\gamma_1})(s_1 x_0)$ is a Schubert cell (see Appendix B). Thus, by Du-Ko-Va, Section 4, especially Equation (4.10)], its closure consists of the cell itself together with the 0 dimensional cell $\{ x_0 \}$.

**Case 2.** $(\sigma, \gamma) = (s_1 s_2 s_1, \gamma_3) = (s_{\gamma_3}, \gamma_3)$. We now show that the boundary of $\exp(\mathfrak{g}_{\gamma_3})(s_3 x_0)$ is $\{ x_0 \}$. To simplify notations, we set

$$G := E_6(-26), \quad K := F_4, \quad \mathfrak{g} := \mathfrak{e}_6(-26), \quad \mathfrak{k} := \mathfrak{f}_4, \quad \mathfrak{s} := \mathfrak{h}_3^0(\mathbb{O}), \quad \gamma_3 := \gamma.$$

As usual, we denote $M = Spin(8)$. We also denote by $N$ and $A$ the connected Lie subgroups of $G$ of Lie algebras $\mathfrak{g}_{\gamma_1} + \mathfrak{g}_{\gamma_2} + \mathfrak{g}_{\gamma_3}$, respectively $\mathfrak{a}$ (the notations above have been used in the
general case in Appendix B). We will use the rank-one reduction procedure, as described in [He, Chapter IX, Section 2]. Let us denote by $\mathfrak{g}^\gamma$ the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$. Take $h_\gamma \in \mathfrak{a}$ determined by $\langle h_\gamma, h \rangle = \gamma(h)$, for all $h \in \mathfrak{a}$ (here $\langle , \rangle$ is the Killing form of $\mathfrak{g}$). We have the Cartan decomposition

$$\mathfrak{g}^\gamma = \mathfrak{k}^\gamma + \mathfrak{s}^\gamma,$$

where $\mathfrak{k}^\gamma = \mathfrak{k} \cap \mathfrak{g}^\gamma$ and $\mathfrak{s}^\gamma = \mathfrak{s} \cap \mathfrak{g}^\gamma$ (see also Equation (2.3)). The space $\mathbb{R}h_\gamma$ is maximal abelian in $\mathfrak{s}^\gamma$. Let $G^\gamma$, $K^\gamma$, and $A^\gamma$ denote the the connected Lie subgroups of $G$ of Lie algebras $\mathfrak{g}^\gamma$, $\mathfrak{k}^\gamma$, respectively $\mathbb{R}h_\gamma$. Then we have $K^\gamma = K \cap G^\gamma$ and $A^\gamma = A \cap G^\gamma$. Moreover, if $M^\gamma$ denotes the centralizer of $h_\gamma$ in $K^\gamma$, then we have $M^\gamma = M \cap G^\gamma$. The connected Lie subgroup of $G^\gamma$ of Lie algebra $\mathfrak{g}_\gamma$ is $N^\gamma = G^\gamma \cap N$. The Iwasawa decomposition of $G^\gamma$ is

$$G^\gamma = K^\gamma A^\gamma N^\gamma.$$

Without loss of generality we can assume that $x_0 = h_\gamma$: since the last two vectors are in the same Weyl chamber (see Figure 1), their $K$ orbits are $G$-equivariantly diffeomorphic. Consequently, we have $s_{\gamma_3}x_0 = -h_\gamma$. The orbit $X^\gamma := K^\gamma.h_\gamma$ is contained in $X = K.h_\gamma$ (for both orbits, the group action is the Adjoint one). In fact, the inclusion is $G^\gamma$-equivariant. Indeed, the action of $G$ on $X$ is induced by the identification $X = G/MAN$. Consequently, the subgroup $G^\gamma$ of $G$ acts on $X$ and the orbit of the coset of $e$ is $G^\gamma/(MAN \cap G^\gamma) = G^\gamma/(M^\gamma A^\gamma N^\gamma) = X^\gamma$ (here we have used that the map $K \times A \times N \to G$, $(k, a, n) \mapsto kan$, for all $(k, a, n) \in K \times A \times N$ is a diffeomorphism). The Schubert cell decomposition of $X^\gamma$ described in Theorem B.2 (b) is

$$X^\gamma = \exp(\mathfrak{g}^\gamma)(-h_\gamma) \bigsqcup \{h_\gamma\}.$$
Thus, the cell \( \exp(g^\gamma)(-h_\gamma) \) is dense in \( X^\gamma \) (since the latter space is compact). We deduce that the closure of \( \exp(g^\gamma)(-h_\gamma) \) in \( X \) is equal to \( X^\gamma \). This finishes the proof.

The other cases follow immediately from the two above. For instance, to show that the boundary of \( \exp(g_{\gamma_1})(s_1s_2x_0) \) is \( s_1(s_1s_2x_0) = s_2x_0 \) we use Case 2. Indeed, we replace \( x_0 \) by \( s_2x_0 \) and \( s_1, s_2 \) by \( s_2, s_3 \) (reflections about the walls of the Weyl chamber which contains \( s_2x_0 \)).

Assumption (iii) is a direct consequence of the following lemma.

**Lemma 5.3.** (a) Let \( T \subset M \) denote as usual the canonical maximal torus. The equivariant Euler classes \( e_T(h_{\gamma_1}), e_T(h_{\gamma_2}), \) and \( e_T(h_{\gamma_3}) \) are (non-zero and) pairwise relatively prime elements of \( H^*(BT) \).

(b) The equivariant Euler classes \( e_M(h_{\gamma_1}), e_M(h_{\gamma_2}), \) and \( e_M(h_{\gamma_3}) \) are (non-zero and) pairwise relatively prime elements of \( H^*(BM) \).

**Proof.** (a) We recall from Section 2 that, as complex \( M \)-representations we have

\[
(5.3) \quad h_{\gamma_1} \otimes \mathbb{C} = V_8 \otimes \mathbb{C}, \quad h_{\gamma_2} \otimes \mathbb{C} = S_8^+ \otimes \mathbb{C}, \quad h_{\gamma_3} \otimes \mathbb{C} = S_8^- \otimes \mathbb{C},
\]

where \( V_8 \) arises from the rotation action of \( SO(8) \) and \( S_8^\pm \) are the half-spin representations of \( M = \text{Spin}(8) \). We consider the Euler class of

\[
h_{\gamma_k} \otimes \mathbb{C} \cong h_{\gamma_k} \oplus h_{\gamma_k}.
\]

This is

\[
e_T(h_{\gamma_k} \otimes \mathbb{C}) = e_T(h_{\gamma_k})^2,
\]

for \( k = 1, 2, 3 \). Thus it is sufficient to prove that the \( T \)-equivariant Euler classes of \( h_{\gamma_1} \otimes \mathbb{C}, h_{\gamma_2} \otimes \mathbb{C} \) and \( h_{\gamma_3} \otimes \mathbb{C} \) are pairwise relatively prime.

By Equation (5.3), we can identify these classes with the products of the weights of the representations \( V_8 \otimes \mathbb{C}, S_8^+ \otimes \mathbb{C}, \) respectively \( S_8^- \otimes \mathbb{C} \): Indeed, if \( V \) is any of these representations, then we can split \( V = \bigoplus_{i=1}^8 L_i \), where \( L_i \) are 1-dimensional \( T \)-invariant complex vector subspaces. Consequently,

\[
e_T(V) = c_T^V(\bigoplus_{i=1}^8 L_i) = c_1^T(L_1) \cdots c_1^T(L_8),
\]

where \( c_1^T \) and \( c_1^T \) denote the \( T \)-equivariant Chern classes. We know that the 1-dimensional complex representations of \( T \) are labeled by the character group \( \text{Hom}(T, S^1) \), and the map \( \text{Hom}(T, S^1) \to H^2(BT; \mathbb{Z}) \) given by \( L \mapsto c_1^T(L) \) is a group isomorphism (see e.g. [Hu Chapter 20, Section 11]). In turn, \( \text{Hom}(T, S^1) \) is isomorphic to the lattice of integral forms on \( t \). In Section 6 we will consider a basis \( \{\rho_1, \rho_2, \rho_3, \rho_4\} \) of this lattice and we will calculate the weights of \( V_8 \otimes \mathbb{C}, S_8^+ \otimes \mathbb{C}, \) and \( S_8^- \otimes \mathbb{C} \). Namely, if we identify \( H^*(BT; \mathbb{Z}) = \mathbb{Z}[\rho_1, \rho_2, \rho_3, \rho_4], \)
Equations (6.1), (6.2), (6.3), respectively (6.4) give:
\[ e_T(V_8 \otimes \mathbb{C}) = \rho_2^2(\rho_2 - \rho_1)^2(\rho_4 - \rho_3)^2(\rho_4 - \rho_2 + \rho_3)^2 \]
\[ e_T(S_8^+ \otimes \mathbb{C}) = \rho_4^2(\rho_4 - \rho_2)^2(\rho_3 - \rho_1)^2(\rho_3 - \rho_2 + \rho_1)^2 \]
\[ e_T(S_8^- \otimes \mathbb{C}) = \rho_3^2(\rho_3 - \rho_2)^2(\rho_4 - \rho_1)^2(\rho_4 - \rho_2 + \rho_1)^2 \]

We note that the three polynomials above are pairwise relatively prime.

(b) We use the fact that we have a canonical inclusion (of rings) \( H^*(BM) \hookrightarrow H^*(BT) \), which maps \( e_M(\mathfrak{h}_{\gamma_k}) \) to \( e_T(\mathfrak{h}_{\gamma_k}) \), \( k = 1, 2, 3 \). We also use point (a).

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** The result follows from the discussion made throughout this section. Namely, from Theorem 5.1 we deduce that the map \( \iota^*:H^*_M(Fl(O)) \to H^*_M(\Sigma_3x_0) = \prod_{\sigma \in \Sigma_3}H^*(BM) \) is injective. Its image consists of those \( (f_\sigma)_{\sigma \in \Sigma_3} \) with the following property:

(P1) \( f_\sigma - f_{s_\gamma \sigma} \) is divisible by \( e_M(\mathfrak{h}_\gamma) \) for any \( \gamma \in \Phi^+ \) such that \( \sigma^{-1} \gamma \in \Phi^- \).

Condition (P1) is equivalent to:

(P2) \( f_\sigma - f_{s_\gamma \sigma} \) is divisible by \( e_M(\mathfrak{h}_\gamma) \) for any \( \gamma \in \Phi^+ \).

Indeed, (P2) implies (P1). Also (P1) implies (P2): assume that (P1) holds true and take \( \sigma \in \Sigma_3 \) and \( \gamma \in \Phi^+ \) such that \( \sigma^{-1} \gamma \in \Phi^+ \); then we have \( s_\gamma(s_\gamma \sigma) = \sigma \) and also \( (s_\gamma \sigma)^{-1} \gamma = -\sigma^{-1} \gamma \), which is in \( \Phi^- \); thus, by (P1), the difference \( f_{s_\gamma \sigma} - f_\sigma \) is divisible by \( e_M(\mathfrak{h}_\gamma) \).

Finally, we take into account Equation (5.2) and the definition of \( \bar{e}_{ij} \) at the beginning of this section.

6. **Equivariant K-theory of Fl(O)**

The goal of this section is to prove Theorem 1.4. We regard \( X = Fl(O) \) as the homogeneous space \( F_4/Spin(8) \). The following is a description of the roots, weights and the representation ring of \( Spin(8) \), which will be needed in the proof. The details can be found for instance in [Br-tD, Chapter V, Section 6 and Chapter VI, Section 6] or [Ad, Chapter 4]. The Lie algebra of \( Spin(8) \) is the space \( \mathfrak{so}(8) \) of all skew-symmetric \( 8 \times 8 \) matrices whose entries are real numbers. Recall that by \( T \) we have denoted the canonical maximal torus of \( Spin(8) \) (see Proposition 2.2.3). Its Lie algebra, call it \( t \), consists of all matrices of the form

\[
\begin{pmatrix}
0 & \theta_4 & 0 & 0 \\
-\theta_4 & 0 & 0 & 0 \\
0 & \ldots & 0 & \theta_4 \\
0 & \ldots & -\theta_4 & 0
\end{pmatrix},
\]
where \( \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \). For any \( j \in \{1, 2, 3, 4\} \) we denote by \( L^j \) be the linear function on \( t \) which assigns to each matrix of the form above the number \( \theta_j \). The set \( \{L^1, L^2, L^3, L^4\} \) is a basis of the dual space \( t^* \).

- The set of roots of \( Spin(8) \) with respect to \( t \) is
  \[ \Phi_{Spin(8)} = \{\pm L^i \pm L^j : 1 \leq i < j \leq 4\} \].
- A simple root system is
  \[ \Pi = \{L^1 - L^2, L^2 - L^3, L^3 - L^4, L^3 + L^4\} \].
  The corresponding set of positive roots is
  \[ \Phi^+_{Spin(8)} = \{L^i \pm L^j : 1 \leq i < j \leq 4\} \].
- The corresponding fundamental weights are
  \[ \rho_1 = L^1, \rho_2 = L^1 + L^2, \rho_3 = \frac{L^1 + L^2 + L^3 - L^4}{2}, \rho_4 = \frac{L^1 + L^2 + L^3 + L^4}{2} \].
  Since \( Spin(8) \) is simply connected, these weights are a basis of the lattice \( t^*_Z \) of integral forms. We will also use the presentation
  \[ t^*_Z = \oplus_{1 \leq i \leq 5} \mathbb{Z} \omega^i/(2\omega^5 - \omega^1 - \omega^2 - \omega^3 - \omega^4) \],
  where we have denoted as follows:
  \[ \omega^1 := L^1, \omega^2 := L^2, \omega^3 := L^3, \omega^4 := L^4, \omega^5 := \frac{L^1 + L^2 + L^3 + L^4}{2} \].
  As usual, to any integral form \( \lambda \in t^*_Z \) corresponds the character \( e^\lambda \in R[T] \). In this way, if we denote \( y_j = e^{i\omega^j}, 1 \leq j \leq 5 \), we obtain the presentation
  \[ R[T] \simeq \mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4) \].
- The canonical action of the Weyl group \( W_{Spin(8)} = N_{Spin(8)}(T)/T \) on \( t^* \) is faithful. The automorphisms of \( t^* \) induced in this way are those \( \eta \) with the property that for any \( 1 \leq i \leq 4 \), there exists \( 1 \leq j \leq 4 \) such that \( \eta(L^i) = \pm L^j \), the number of “-” signs being even.
- The representation ring of \( Spin(8) \) is \( R[Spin(8)] = \mathbb{Z}[X_1, X_2, X_3, X_4] \) where
  \[ X_1 = S^+_8 \otimes \mathbb{C}, X_2 = S^-_8 \otimes \mathbb{C}, X_3 = V_8 \otimes \mathbb{C} \]
  and \( X_4 \) is the complexified adjoint representation of \( Spin(8) \) (recall that \( S^\pm_8 \) are the half-spin representations of \( Spin(8) \) and \( V_8 \) is induced by the standard representation of \( SO(8) \) on \( \mathbb{R}^8 \) via the covering map \( Spin(8) \to SO(8) \)). Their weights are as follows (see [Ad], Proposition 4.2):
  (i) For \( X_1 \):
  \[ \pm L^1 \pm L^2 \pm L^3 \pm L^4 \]
  where the number of “-” signs is even.
(ii) for $X_2$:
\[
\pm L^1 \pm L^2 \pm L^3 \pm L^4
\]
where the number of “−” signs is odd.

(iii) for $X_3$:
\[
\pm L^1, \pm L^2, \pm L^3, \text{ and } \pm L^4
\]
(iv) for $X_4$: all the roots of $Spin(8)$ relative to $T$.

The (complex) dimension of each weight space is equal to 1. Consequently, the restriction/inclusion map $R[Spin(8)] = R[T]^{\mathbb{W}_{Spin(8)}} \rightarrow R[T]$ is given by
\[
X_1 = y_5 + y_5y_1^{-1}y_2^{-1} + y_5y_1^{-1}y_3^{-1} + y_5y_1^{-1}y_4^{-1} + y_5y_2^{-1}y_3^{-1} + y_5y_2^{-1}y_4^{-1} + y_5y_3^{-1}y_4^{-1}
\]
\[
X_2 = y_5y_1^{-1} + y_5y_2^{-1} + y_5y_3^{-1} + y_5y_4^{-1} + y_5y_1^{-1}y_2^{-1}y_3^{-1} + y_5y_1^{-1}y_2^{-1}y_4^{-1} + y_5y_1^{-1}y_3^{-1}y_4^{-1}
\]
\[
X_3 = y_1 + y_1^{-1} + y_2 + y_2^{-1} + y_3 + y_3^{-1} + y_4 + y_4^{-1}
\]
\[
X_4 = \sum_{1 \leq i < j \leq 4} y_i^{\pm 1}y_j^{\pm 1}.
\]

The group $Spin(8)$ and its subgroup $T$ act on the homogeneous space $X = F_4/\mathbb{W}_{Spin(8)}$ by multiplication of cosets from the left. Recall (see Proposition 2.2.4) that we have
\[(F_4/\mathbb{W}_{Spin(8)})^T = (F_4/\mathbb{W}_{Spin(8)})^{\mathbb{W}_{Spin(8)}}\]
and this set is in bijective correspondence with the symmetric group $\Sigma_3$. We need to say more about this identification (for the details of the following discussion, see [Ad, Chapter 14]). We first note that the maximal torus $T$ of $Spin(8)$ mentioned above is also a maximal torus of $F_4$. Moreover, the Weyl group $\mathbb{W}_{Spin(8)} := N_{Spin(8)}(T)/T$ is a normal subgroup of $W_{F_4} := N_{F_4}(T)/T$. In fact, the latter group is a semidirect product of the former one by a certain copy of the symmetric group $\Sigma_3$, which we describe in what follows. To do this, we first mention that the roots of $F_4$ relative to $T$ are:

long roots : $\pm L^i \pm L^j$, $1 \leq i < j \leq 4$  24 roots of $Spin(8)$,
short roots : $\pm L^1$, $1 \leq i \leq 4$  8 roots,
$\pm L^1 \pm L^2 \pm L^3 \pm L^4$  16 roots.

Among them, the short roots
\[
L^4 = \omega^4, \quad \frac{L^1 + L^2 + L^3 - L^4}{2} = \omega^5 - \omega^4, \quad \frac{L^1 + L^2 + L^3 + L^4}{2} = \omega^5
\]
play a special role: the subgroup of $W_{F_4}$ generated by the reflections of $t$ about their kernels is isomorphic to $\Sigma_3$. Let us denote by $\tilde{\Sigma}_3$ this subgroup of $W_{F_4}$. By [Ad, Theorem 14.2], we have
\[W_{F_4} = \tilde{\Sigma}_3 \ltimes \mathbb{W}_{Spin(8)}.\]
Let us now focus on the $T$-fixed points in $F_4/\text{Spin}(8)$. An easy calculation shows that

$$(F_4/\text{Spin}(8))^T = \{g \in F_4 : gTg^{-1} \subset \text{Spin}(8)\}/\text{Spin}(8).$$

We will prove the following lemma:

**Lemma 6.1.** The map

$$\tilde{\Sigma}_3 = W_{F_4}/W_{\text{Spin}(8)} = \{g \in F_4 : gTg^{-1} = T\}/\{g \in \text{Spin}(8) : gTg^{-1} = T\} \to (F_4/\text{Spin}(8))^T$$

which sends the coset of $g$ to $g\text{Spin}(8)$ for all $g \in F_4$, is bijective.

**Proof.** The map is well defined and injective. Since both the domain and the codomain are in bijective correspondence with $\Sigma_3$, the map is actually bijective. \hfill $\square$

The previous discussion implies that we can make the identification

$$(F_4/\text{Spin}(8))^\text{Spin}(8) = (F_4/\text{Spin}(8))^T = W_{F_4}/W_{\text{Spin}(8)} = \tilde{\Sigma}_3.$$

Let us consider again the inclusion map $\iota : \tilde{\Sigma}_3 \to F_4/\text{Spin}(8)$ and let $\iota^*_T : K_T(F_4/\text{Spin}(8)) \to \prod_{\sigma \in \tilde{\Sigma}_3} R[T]$ be the corresponding ring homomorphism. The first step towards the proof of Theorem 1.4 is made by the following proposition.

**Proposition 6.2.** (a) The $T$-equivariant $K$-theory of $F_4/\text{Spin}(8)$ is a free $R[T]$-module of rank 6.

(b) The map $\iota^*_T : K_T(F_4/\text{Spin}(8)) \to \prod_{\sigma \in \tilde{\Sigma}_3} R[T]$ is injective.

This can be proved in the same way as [Ma-Wi, Proposition 4.1]. We will not give all details here, just the main ideas:

**Proof (outline).** For point (a), we use the CW decomposition given by Equation (2.8). An important ingredient of the proof is the Thom isomorphism theorem for each cell $C_\sigma$, $\sigma \in \tilde{\Sigma}_3$ (cf. [Se]). Indeed, by Proposition 2.2.3, $C_\sigma$ is a complex (open) cell with a $C^*$-linear action of $T$; thus, we have

$$K_T(C_\sigma) = K_T(\text{pt.}) = R[T].$$

We also note that the number of cells $C_\sigma$ is equal to 6. For point (b), we use the localization theorem in equivariant $K$-theory (cf. [Se]). This tells us that the homomorphism $K_T(F_4/\text{Spin}(8)) \otimes_{R[T]} Q[T] \to \prod_{\sigma \in \tilde{\Sigma}_3} Q[T]$ induced by $\iota^*_T$ is an isomorphism (here $Q[T]$ denotes the field of fractions of $R[T]$). This implies that $\iota^*_T$ is injective. \hfill $\square$

The Weyl group $W_{F_4} = N_{F_4}(T)/T$ acts on $F_4/T$ via

$$(6.6) \quad (nT)(gT) = gn^{-1}T,$$

for any $n \in N_{F_4}(T)$ and $g \in F_4$. This action is $T$-equivariant, therefore, by functoriality, it induces an action by ring automorphisms on $K_T(F_4/T)$. We also consider the canonical projection map

$$\pi_T : F_4/T \to F_4/\text{Spin}(8),$$
which is $T$-equivariant, and maps $(F_4/T)^T = W_{F_4}$ onto $\tilde{\Sigma}_3$ (see Lemma 6.1). We denote by $\pi_T^*$ the homomorphism between the $K_T$-rings induced by $\pi_T$. The following result describes $K_T(F_4/Spin(8))$ as a subring of $K_T(F_4/T)$.

**Proposition 6.3.** The map $\pi_T^* : K_T(F_4/Spin(8)) \rightarrow K_T(F_4/T)$ is injective. Its image consists of all $W_{Spin(8)}$-invariant elements of $K_T(F_4/T)$.

This can be proved by using the same arguments as in the proofs of [Ma-Wi, Propositions 4.2 and 4.4]. For the reader’s convenience, we outline the proof.

**Proof (outline).** The injectivity follows from the commutativity of the diagram

$$
\begin{array}{ccc}
K_T(F_4/Spin(8)) & \xrightarrow{\pi_T^*} & K_T(F_4/T) \\
\downarrow i_T^* & & \downarrow i_T^* \\
\prod_{\sigma \in \tilde{\Sigma}_3} R[T] & \xrightarrow{p} & \prod_{w \in W_{F_4}} R[T]
\end{array}
$$

where $p$ is the map induced by the projection $W_{F_4} \rightarrow W_{F_4}/W_{Spin(8)} = \tilde{\Sigma}_3$ (see Lemma 6.1) and $i_T : W_{F_4} = (F_4/T)^T \rightarrow F_4/T$ is the inclusion map. We know that $i_T^*$, $i_T^*$ and $p$ are injective, thus $\pi_T^*$ is injective as well. To prove that the image of $\pi_T^*$ coincides with $K_T(F_4/T)^{W_{Spin(8)}}$, we use the following ring isomorphisms (cf. e.g. [Mc]):

$$
\begin{align*}
K_T(F_4/T) & \simeq R[T] \otimes_{R[F_4]} R[T] \\
K_T(F_4/Spin(8)) & \simeq R[T] \otimes_{R[F_4]} R[Spin(8)] \\
R[Spin(8)] & \simeq R[T]^{W_{Spin(8)}}.
\end{align*}
$$

\[\square\]

We obtain a GKM description of $K_T(F_4/Spin(8))$ as follows: First, from the previous proposition, we have the ring isomorphism

$$
K_T(F_4/Spin(8)) = K_T(F_4/T)^{W_{Spin(8)}}.
$$

Second, $F_4/T$ is a complex complete flag variety: for any such variety, a GKM type description of the equivariant $K$-theory ring corresponding to the action of the maximal torus has been obtain by McLeod [Mc]. For $F_4/T$, this description is made in terms of the roots of $F_4$ (see above). A simple root system is

$$
\frac{L^1 - L^2 - L^3 - L^4}{2}, L^2, -L^2 + L^3, -L^3 + L^4.
$$

We denote by $\Phi_{F_4}^+$ the corresponding set of positive roots. The GKM type presentation mentioned above is as follows. First we note that $(F_4/T)^T = W_{F_4}$ and the homomorphism $K_T(F_4/T) \rightarrow K_T(W_{F_4})$ induced by the inclusion map is injective. By identifying $K_T(F_4/T)$
with the image of this homomorphism, we obtain:
\[ K_T(F_4/T) \cong \{(f_w) \in \prod_{w \in W_{F_4}} R[T] : e^\delta - 1 \text{ divides } f_w - f_{sw} \text{ for all } \delta \in \Phi_{F_4}^+ \}. \]

As usual, by \( s_\delta \) we denote the element of \( W_{F_4} \) induced by the root \( \delta \) (that is, the reflection of \( t \) about \( \ker \delta \)).

Now the ring isomorphism above is \( W_{F_4} \)-equivariant if we let the Weyl group \( W_{F_4} \) act on the space in the right-hand side by
\[ v.(f_w) = (f_{vw^{-1}}) \]
for all \( v, w \in W_{F_4} \). Consequently, \( K_T(F_4/T)^{W_{Spin(8)}} \) can be identified with:
\[ \{(f_w) \in \prod_{w \in W_{F_4}/W_{Spin(8)}} R[T] : e^\delta - 1 \text{ divides } f_w - f_{\bar{w}w} \text{ for all } \delta \in \Phi_{F_4}^+ \text{ such that } s_\delta \notin W_{Spin(8)} \}. \]

Here we have denoted by \( \bar{w} \) the coset of \( w \) in \( W_{F_4}/W_{Spin(8)} \), for any \( w \in W_{F_4} \). We only need to consider roots \( \delta \in \Phi_{F_4}^+ \) such that \( s_\delta \notin W_{Spin(8)} \): if \( s_\delta \) does belong to \( W_{Spin(8)} \), then
\[ \bar{s_\delta \bar{w} = \bar{w}w^{-1}s_\delta \bar{w} = \bar{w}}, \]
for any \( w \in W_{F_4} \) (because \( W_{Spin(8)} \) is a normal subgroup of \( W_{F_4} \)). We now recall that the subgroup \( \tilde{\Sigma}_3 \) of \( W_{F_4} \) is generated by the reflections \( s_{\omega^4}, s_{\omega^5-\omega^4}, \) and \( s_{\omega^5} \) and is isomorphic to the symmetric group \( \Sigma_3 \). In fact, one can see that
\[ s_{\omega^5} = s_{\omega^4}s_{\omega^5-\omega^4}s_{\omega^4}. \]
Thus, we may use the identification of \( \tilde{\Sigma}_3 \) with \( \Sigma_3 \) given by:
\[ s_{\omega^4} = (1,2), \ s_{\omega^5-\omega^4} = (2,3), \ s_{\omega^5} = (1,3). \]

**Remark.** Although this is not relevant for our purposes, we note that the identification above is not arbitrary, as we explain in what follows (cf. [Ad, Chapter 14]). First, an element \( g \in F_4 \) satisfies \( gTg^{-1} \subset T \) if and only if \( gSpin(8)g^{-1} = Spin(8) \). Thus any element of \( W_{F_4} \) induces a group automorphism of \( Spin(8) \). Second, \( \omega^5, \omega^5-\omega^4, \) and \( \omega^4 \) are the highest weights of the \( Spin(8) \)-representations \( X_1, X_2, \) respectively \( X_3 \). It turns out that the automorphisms of \( Spin(8) \) induced by the elements \( s_{\omega^4}, s_{\omega^5-\omega^4}, \) and \( s_{\omega^5} \) of \( W_{F_4} \) are outer and they permute \( X_1, X_2, X_3 \). More precisely, they act on the latter set as the transpositions \( (1,2), (2,3), \) respectively \( (1,3) \).

The following lemma will be needed later.

**Lemma 6.4.** (a) The roots \( \delta \in \Phi_{F_4}^+ \) such that \( \bar{s_\delta} = \bar{s_{\omega^4}} \) are \( L_1, L_2, L_3, \) and \( L_4 \).

(b) The roots \( \delta \in \Phi_{F_4}^+ \) such that \( \bar{s_\delta} = \bar{s_{\omega^5-\omega^4}} \) are
\[ \frac{L_1 + L_2 + L_3 - L_4}{2}, \ \frac{L_1 + L_2 - L_3 + L_4}{2}, \ \frac{L_1 - L_2 + L_3 + L_4}{2}, \ \frac{L_1 - L_2 - L_3 - L_4}{2}. \]
(c) The roots $\delta \in \Phi^+_{F_4}$ such that $\overline{s_5} = \overline{s_{\omega^5}}$ are

\[
\begin{align*}
\frac{L_1 + L_2 + L_3 + L_4}{2}, & \quad \frac{L_1 + L_2 - L_3 - L_4}{2}, \\
\frac{L_1 - L_2 + L_3 - L_4}{2}, & \quad \frac{L_1 - L_2 - L_3 + L_4}{2}.
\end{align*}
\]

(d) The subsets of $\Phi^+_{F_4}$ described at points (a), (b), and (c) are a partition of this set.

**Proof.** We first prove the following claim.

**Claim.** Let $\gamma$ denote one of the roots $\omega^5, \omega^5 - \omega^4$, and $\omega^4$. Take $\eta \in W_{Spin(8)}$. Then we have $\overline{s_\eta(\gamma)} = \overline{s_\gamma}$.

Indeed, we have

$$s_\eta s_\gamma s_\gamma^{-1} = \eta s_\gamma \eta^{-1} s_\gamma^{-1} = \eta(s_\gamma \eta^{-1} s_\gamma^{-1}).$$

This belongs to $W_{Spin(8)}$, since this group is normal in $W_{F_4}$.

We take into account which is the form of a transformation $\eta \in W_{Spin(8)}$ (see the beginning of this section). We also use the fact that the number of roots enumerated at points (a), (b), and (c) is 12: this means that all elements of $\Phi^+_{F_4}$ have been enumerated. \qed

We deduce as follows:

**Proposition 6.5.** The ring homomorphism $i^*_T : K_T(F_4/Spin(8)) \to K_T((F_4/Spin(8))^T)$ is injective and its image consists of all $(f_\sigma) \in \prod_{\sigma \in \Sigma_3} \mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}] / (y_5^2 - y_1 y_2 y_3 y_4)$, with the following properties:

- $f_{(1,2)\sigma} - f_\sigma$ is divisible by $(y_1 - 1)(y_2 - 1)(y_3 - 1)(y_4 - 1)$,
- $f_{(1,3)\sigma} - f_\sigma$ is divisible by $(y_5 y_4 - 1)(y_5 y_3 - 1)(y_5 y_2 - 1)(y_5 y_1 - 1)$,
- $f_{(2,3)\sigma} - f_\sigma$ is divisible by $(y_5 - 1)(y_5 y_1 - 1)(y_5 y_2 - 1)(y_5 y_3 - 1)(y_5 y_4 - 1)$.

The divisibility referred to above is in the ring $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}] / (y_5^2 - y_1 y_2 y_3 y_4)$.

We are now ready to discuss the map $i^*_{Spin(8)} : K_{Spin(8)}(F_4/Spin(8)) \to \prod_{\sigma \in \Sigma_3} R[Spin(8)]$ induced by the inclusion map $i : \Sigma_3 \to F_4/Spin(8)$ (recall that $\Sigma_3$ is the fixed point set of the $Spin(8)$ action on $F_4/Spin(8)$). We consider the diagonal action of $W_{Spin(8)}$ on the product $\prod_{\sigma \in \Sigma_3} R[T]$: we identify $\prod_{\sigma \in \Sigma_3} R[Spin(8)]$ with the fixed point set of this action.

**Proposition 6.6.** The map $i^*_{Spin(8)}$ is injective. Its image is the intersection of the image of $i^*_T$ (see Proposition 6.5) with $\prod_{\sigma \in \Sigma_3} R[Spin(8)]$.

The principles of the proof can be found in [Ma-Wi, Section 5]. In what follows we sketch the main ideas.
Proof (outline). (i) Since the $T$-fixed points of $F_4/\text{Spin}(8)$ are the same as the $\text{Spin}(8)$-fixed points, we can consider the following commutative diagram:

$K_{\text{Spin}(8)}(F_4/\text{Spin}(8)) \xrightarrow{j^*} K_T(F_4/\text{Spin}(8))$

Here, $j^*: K_{\text{Spin}(8)}(F_4/\text{Spin}(8)) \to K_T(F_4/\text{Spin}(8))$ is the map induced by restricting the $\text{Spin}(8)$ action to $T$: this map is injective by [Mc, Theorem 4.4]. Also, $\tilde{\rho}$ consists of six copies of the inclusion map $R[\text{Spin}(8)] = R[T]^{W_{\text{Spin}(8)}} \hookrightarrow R[T]$: thus, $\tilde{\rho}$ is an injective map. By Proposition 6.2 above, $i^*_T$ is injective. From the commutativity of the diagram we deduce that $j^*_T$ is an injective map.

(ii) We identify

$K_T(F_4/\text{Spin}(8)) = R[T] \otimes_{R[F_4]} R[\text{Spin}(8)].$

We let the group $W_{\text{Spin}(8)}$ act on this space by

$w(x_1 \otimes x_2) = (w x_1) \otimes x_2,$

for all $w \in W_{\text{Spin}(8)}$, $x_1 \in R[T]$, and $x_2 \in R[\text{Spin}(8)]$. The result stated in the lemma follows from the commutativity of the diagram above and the following two facts:

- the map $i^*_T$ is $W_{\text{Spin}(8)}$-equivariant.
- the image of $j^*$ is $K_T(F_4/\text{Spin}(8))^{W_{\text{Spin}(8)}}$: more precisely, we can identify $K_{\text{Spin}(8)}(F_4/\text{Spin}(8)) = R[\text{Spin}(8)] \otimes_{R[F_4]} R[\text{Spin}(8)]$ in such a way that $j^*: R[\text{Spin}(8)] \otimes_{R[F_4]} R[\text{Spin}(8)] \to R[T] \otimes_{R[F_4]} R[\text{Spin}(8)]$

is induced by the inclusion $R[\text{Spin}(8)] \hookrightarrow R[T].$

\[\square\]

To obtain the description in Theorem 1.4, it is enough to prove the following three lemmas.

**Lemma 6.7.** If $f \in \mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4)$ is $W_{\text{Spin}(8)}$-invariant, then $f$ is divisible by $(y_1 - 1)(y_2 - 1)(y_3 - 1)(y_4 - 1)$ in $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4)$ if and only if $f$ is divisible by $X_1 - X_2$ in $\mathbb{Z}[X_1, X_2, X_3, X_4]$.

**Proof.** We have

$X_1 - X_2 = y_1^{-1} y_2^{-1} y_3^{-1} y_4^{-1} y_5 (y_1 - 1)(y_2 - 1)(y_3 - 1)(y_4 - 1).$

Thus if $f$ is divisible by $X_1 - X_2$ in $\mathbb{Z}[X_1, X_2, X_3, X_4] = (\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4))^{W_{\text{Spin}(8)}},$

then $f$ is divisible by $(y_1 - 1)(y_2 - 1)(y_3 - 1)(y_4 - 1)$ in $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4).$
Let us now assume that \( f \) is \( W_{Spin(8)} \)-invariant and divisible by \((y_1-1)(y_2-1)(y_3-1)(y_4-1)\). This implies that \( f = (X_1 - X_2)h \) with \( h \in \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4)\). Since \( f \) and \( X_1 - X_2 \) are \( W_{Spin(8)} \)-invariant and since \( \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4) \) is an integral domain, \( h \) is \( W_{Spin(8)} \)-invariant, too. Thus \( f \) is a polynomial in \( X_1, X_2, X_3, X_4 \). Consequently \( f \) is divisible by \( X_1 - X_2 \) in \( \mathbb{Z}[X_1, X_2, X_3, X_4] \).

**Lemma 6.8.** If \( f \in \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4) \) is \( W_{Spin(8)} \)-invariant, then \( f \) is divisible by \((y_5y_4^{-1} - 1)(y_5y_3^{-1} - 1)(y_5y_2^{-1} - 1)(y_5y_1^{-1} - 1)\) in \( \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4) \) if and only if \( f \) is divisible by \( X_1 - X_3 \) in \( \mathbb{Z}[X_1, X_2, X_3, X_4] \).

**Proof.** We have

\[
X_1 - X_3 = y_5^{-1}(y_5y_4^{-1} - 1)(y_5y_3^{-1} - 1)(y_5y_2^{-1} - 1)(y_5y_1^{-1} - 1).
\]

We use the same idea as in the proof of Lemma 6.7. \( \square \)

**Lemma 6.9.** If \( f \in \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4) \) is \( W_{Spin(8)} \)-invariant, then \( f \) is divisible by \((y_5^{-1}y_1^{-1}y_4^{-1} - 1)(y_5y_2^{-1}y_4^{-1} - 1)(y_5y_3^{-1}y_4^{-1} - 1)(y_5y_1^{-1}y_4^{-1} - 1)\) in \( \mathbb{Z}[y_1 \pm 1, y_2 \pm 1, y_3 \pm 1, y_4 \pm 1, y_5 \pm 1]/(y_5^2 - y_1y_2y_3y_4) \) if and only if \( f \) is divisible by \( X_2 - X_3 \) in \( \mathbb{Z}[X_1, X_2, X_3, X_4] \).

**Proof.** We have

\[
X_3 - X_2 = y_3^{-1}(y_5 - 1)(y_5y_4^{-1}y_1^{-1} - 1)(y_5y_2^{-1}y_4^{-1} - 1)(y_5y_1^{-1}y_2^{-1} - 1).
\]

We use the same idea as in the proof of Lemma 6.7. \( \square \)

**Appendix A. The complex flag manifold \( Fl_3(\mathbb{C}) \)**

The goal of this section is to give a description of the flag manifold \( Fl_3(\mathbb{C}) \) which can be naturally extended to the octonionic case (see Section 2). Originally, \( Fl_3(\mathbb{C}) \) is the set of all nested sequences

\[
V_1 \subset V_2 \subset \mathbb{C}^3,
\]

where \( V_1, V_2 \) are complex vector subspaces of \( \mathbb{C}^3 \), \( \dim V_1 = 1 \), \( \dim V_2 = 2 \). Alternatively, let us equip \( \mathbb{C}^3 \) with the standard Hermitian inner product: then \( Fl_3(\mathbb{C}) \) is the set of pairs \((L_1, L_2)\), where \( L_1, L_2 \) are 1-dimensional complex vector subspaces of \( \mathbb{C}^3 \) with \( L_1 \perp L_2 \).

Let us consider the space

\[
\mathfrak{h}_3(\mathbb{C}) = \{ a \in \text{Mat}^{3 \times 3}(\mathbb{C}) : a = a^* \}.
\]

We have the following presentations.

**Proposition A.1.** a) There is a natural identification between the complex projective plane \( \mathbb{CP}^2 \) and the set of all matrices \( a \in \mathfrak{h}_3(\mathbb{C}) \) with \( a^2 = a \) and \( \text{tr}(a) = 1 \).
b) There is a natural identification between the flag manifold $Fl_3(\mathbb{C})$ and the set of all pairs $(a_1, a_2) \in \mathbb{CP}^2 \times \mathbb{CP}^2$ with the property that
\[
\text{Re}(\text{tr}(a_1a_2)) = 0.
\]

The identifications are as follows.

For $\mathbb{CP}^2$. A 1-dimensional complex vector subspace $V$ of $\mathbb{C}^3$ is identified with the element of $\mathfrak{h}_3(\mathbb{C})$ which has eigenvalues $1, 0, 0$ and 1-eigenspace equal to $V$ (the 0-eigenspace is implicitly $V^\perp$). Moreover, an element $a$ of $\mathfrak{h}_3(\mathbb{C})$ has eigenvalues $(1, 0, 0)$ if and only if $a^2 = a$ and $\text{tr}(a) = 1$.

For $Fl_3(\mathbb{C})$. Take $L_1, L_2$ two 1-dimensional complex vector subspaces of $\mathbb{C}^3$ and $a_1, a_2$ the Hermitian matrices with eigenvalues $(1, 0, 0)$ and 1-eigenspaces $L_1$, respectively $L_2$. The point is that $L_1 \perp L_2$ if and only if $\text{Re}(\text{tr}(a_1a_2)) = 0$. Indeed, let us choose an orthonormal basis $v_1, v_2, v_3$ of $\mathbb{C}^3$, where $L_2 = C v_1$. Then we have
\[
\text{tr}(a_1a_2) = \langle a_1a_2(v_1), v_1 \rangle + \langle a_1a_2(v_2), v_2 \rangle + \langle a_1a_2(v_3), v_3 \rangle
\]
\[
= \langle a_1(v_1), v_1 \rangle = \langle a_2^2(v_1), v_1 \rangle = \langle a_1(v_1), a_1^*(v_1) \rangle = \langle a_1(v_1), a_1(v_1) \rangle.
\]
Thus $\text{Re}(\text{tr}(a_1a_2)) = 0$ if and only if $a_1(v_1) = 0$. On the other hand, $L_1$ is perpendicular to $L_2$ if and only if $L_2$ is contained in the 0-eigenspace of $a_1$, that is, $a_1(v_1) = 0$.

Appendix B. Real flag manifolds and their cell decomposition

In this section we will present some general notions and results concerning real flag manifolds. The main reference is [Du-Ko-Va] (the basic material can be found for instance in [He, Chapter IX]).

Let $G$ be a real connected semisimple Lie group and denote by $\mathfrak{g}$ its Lie algebra. Let
\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}
\]
be a Cartan decomposition: this means that the Killing form of $\mathfrak{g}$ is strictly negative definite on $\mathfrak{k}$ and strictly positive definite on $\mathfrak{s}$. The corresponding Cartan involution is $\theta : \mathfrak{g} \to \mathfrak{g}$,
\[
\theta(x + y) = x - y,
\]
for all $x \in \mathfrak{k}$ and $y \in \mathfrak{s}$. We pick a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$ and consider the following root space decomposition:
\[
\text{(B.1)} \quad \mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\gamma \in \Phi} \mathfrak{g}_\gamma.
\]
Here $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\Phi$ the set of roots, that is functions $\gamma : \mathfrak{a} \to \mathbb{R}$ such that the root space
\[
\mathfrak{g}_\gamma := \{ x \in \mathfrak{g} : [h, x] = \gamma(h)x \text{ for all } h \in \mathfrak{a} \}
\]
is non-zero. The set $\Phi$ is a root system in the dual space $\mathfrak{a}^*$. Let us pick a system of simple roots and denote by $\Phi^+$ the corresponding set of positive roots. We set
$$n := \sum_{\gamma \in \Phi^+} g_\gamma,$$
and obtain the Iwasawa decomposition
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus n.$$
If $K, A, N$ are the connected Lie subgroups of $G$ of Lie algebras $\mathfrak{k}, \mathfrak{a},$ respectively $n$, then we have the following Iwasawa decomposition of $G$:
$$G = KAN.$$
Let us also denote by $M$ the centralizer of $a$ in $K$ and by $W$ the Weyl group, which is
$$W = \{k \in K : Ad_G k(a) \subset a\}/M.$$
It turns out that, via the adjoint representation of $G$, the group $K$ leaves $s$ invariant. The orbits of this representation are called real flag manifolds. We need the following result:

**Proposition B.1.** Take $x_0 \in a$ such that $\gamma(x_0) \neq 0$ for all $\gamma \in \Phi$. Then the stabilizer of $x_0$ in $K$ is equal to $M$.

**Proof.** By [Du-Ko-Va, Proposition 1.2] the stabilizer $K_{x_0}$ of $x_0$ satisfies
$$K_{x_0} = MK_{x_0}^0,$$
where $K_{x_0}^0$ denotes the identity component of $K_{x_0}$. The Lie algebra of $K_{x_0}$ is the commutator of $x_0$ in $\mathfrak{k}$. From the root decomposition B.1, this is the same as $m$. Thus we have $K_{x_0}^0 \subset M$ and consequently $K_{x_0} = M$. □

Consequently, we can identify
$$X := \text{Ad}_G(K)x_0 = K/M.$$
From this we can see that there is a canonical embedding of the Weyl group $W$ in $X$.

The natural action of $K$ on $X$ extends to an action of $G$. This arises from the identification
$$X = K/M = KAN/MAN = G/MAN,$$
where we take into account that $MAN$ is a subgroup of $G$. The cell decomposition of $X$ we will describe in the following theorem uses the embedding $W \subset X$ and also the action of $G$ on $X$. The proof can be found in [Du-Ko-Va, Section 3].

**Theorem B.2.** ([Du-Ko-Va]) (a) We have

(B.2) $$X = \bigsqcup_{w \in W} Nw.$$

(b) Fix $w \in W$. The map $\sum g_\gamma \rightarrow Nw, x \mapsto \exp(x)w$ is a diffeomorphism. The sum in the domain runs over all $\gamma \in \Phi^+$ such that $w^{-1} \gamma \in \Phi^-$. 

(c) The decomposition (B.2) makes \( X \) into a CW complex.

The cells \( Nw, w \in W \) are usually referred to as Schubert cells.

Let us now consider the following root space decomposition of \( s \):

\[
    s = a + \sum_{\gamma \in \Phi^+} s_\gamma,
\]

where

\[
    s_\gamma = (g_\gamma + g_{-\gamma}) \cap s = \{x \in s : [h, [h, x]] = \gamma(h)^2 x \text{ for all } h \in a\}.
\]

We can easily see that both \( g_\gamma \) and \( s_\gamma \) are \( M \)-invariant, where \( M \) acts via the Adjoint representation. The following result seems to be known. Since we didn’t find it clearly stated and proved in the literature, we included a proof of it.

**Proposition B.3.** If \( \gamma \in \Phi^+ \), then the map \( \Theta : g_\gamma \to s_\gamma \), given by \( \Theta(x) = x - \theta x \), for all \( x \in g_\gamma \) is an \( M \)-equivariant linear isomorphism.

**Proof.** First, since \( \theta(g_\gamma) = g_{-\gamma} \), \( \Theta \) is well defined, in the sense that it maps \( g_\gamma \) to \( s_\gamma \). The map is also injective: \( x - \theta x = 0 \) and \( x \in g_\gamma \) implies \( x = 0 \). The map is also surjective: if \( y \in s_\gamma \), then we can write it as \( y = y_1 + y_2 \), with \( y_1 \in g_\gamma \) and \( y_2 \in g_{-\gamma} \); since \( y \in s \), we have \( \theta(y) = -y \), which implies \( y_2 = -\theta(y_1) \), thus \( y = y_1 - \theta(y_1) = \Theta(y_1) \). The \( M \)-equivariance of \( \Theta \) follows from the \( M \)-equivariance of \( \theta \). \( \square \)

We now take into account that the map described in Theorem B.2 (b) is \( M \)-equivariant, where \( M \) acts on the domain by the Adjoint representation and on the codomain via the \( G \) action on \( X \). We deduce:

**Corollary B.4.** Fix \( w \in W \). We have an \( M \)-equivariant diffeomorphism between the Schubert cell \( Nw \) and the space \( \sum s_\gamma \), where the sum runs over all \( \gamma \in \Phi^+ \) such that \( w^{-1} \gamma \in \Phi^- \).

### Appendix C. The symmetric space \( E_6(-26)/F_4 \)

In this section we will outline the construction of the (non-compact) symmetric space mentioned in the title. We will try to make more clear several things mentioned in Section 2.2. For instance, we will prove that the root spaces \( s_\gamma \) in the decomposition described by Equation (B.3) are the \( h_\gamma \) described in Section 2.2. This is an important fact, because it allows us to deduce the presentation of \( C_\sigma \) given by Equation (2.9) from Theorem B.2 (b) and Proposition B.3. The main reference of this section is Freudenthal’s article [Fr].

Recall that by Definition 2.7.2, the group of all linear transformations of \( h_3(\mathbb{O}) \) which preserve the product \( \circ \) is \( F_4 \). We define the determinant function on \( h_3(\mathbb{O}) \), by

\[
    \det(a) = \frac{1}{3} \text{tr}(a \circ a \circ a) - \frac{1}{2} \text{tr}(a \circ a) \text{tra} + \frac{1}{6} (\text{tra})^3,
\]

for all \( a \in h_3(\mathbb{O}) \). Let us consider the group of all linear transformations of \( h_3(\mathbb{O}) \) which leave the determinant invariant. It turns out that this group is just \( E_6(-26) \) (see Section 2.2 for the
where \( b \) is a \( 3 \times 3 \) matrix with entries in \( \mathbb{O} \) such that \( b = -b^* \) (that is, \( b \) is skew-Hermitian). Here and everywhere else in this section \([ , ]\) denotes the usual matrix commutator. The corresponding Cartan decomposition of \( e_{6(-26)} \) is described in the following proposition (see [Fr] end of Section 8.1.1):

**Proposition C.1.** If \( c \) is in the Lie algebra \( e_{6(-26)} \), then there exists \( a \in \mathfrak{h}_3^0(\mathbb{O}) \) and \( b \) a \( 3 \times 3 \) skew-Hermitian matrix with entries in \( \mathbb{O} \) such that \( c = \tilde{b} + \hat{a} \). The matrices \( a \) and \( b \) are uniquely determined by \( c \). By \( \hat{a} \) one denotes the transformation of \( \mathfrak{h}_3(\mathbb{O}) \) given by

\[
\hat{a} : \mathfrak{h}_3(\mathbb{O}) \to \mathfrak{h}_3(\mathbb{O}), \quad \hat{a}(y) = a \circ y, \quad \text{for all } y \in \mathfrak{h}_3(\mathbb{O}).
\]

We see from here that \( e_{6(-26)} = \mathfrak{f}_4 + \mathfrak{h}_3^0(\mathbb{O}) \) is a Cartan decomposition, as already mentioned in Section 2.2.

Note that the elements of \( e_{6(-26)} \) are linear endomorphisms of \( \mathfrak{h}_3(\mathbb{O}) \). We denote the Lie bracket by \([ , ]_*\): it is given by the commutator of the endomorphisms. We need the following lemma:

**Lemma C.2.** If \( a, x \in \mathfrak{h}_3^0(\mathbb{O}) \), then

\[
(i) \quad [\hat{x}, \hat{a}]_* = \frac{1}{4} \overline{[x, a]},
\]

\[
(ii) \quad [\hat{x}, [\hat{x}, \hat{a}]_*]_* = \frac{1}{4} \overline{[x, [x, a]]}.
\]

**Proof.** (i) For any \( y \in \mathfrak{h}_3(\mathbb{O}) \) we have

\[
[\hat{x}, \hat{a}]_*(y) = \hat{x}(\hat{a}(y)) - \hat{a}(\hat{x}(y)) = x \circ (a \circ y) - a \circ (x \circ y) = \frac{1}{4} \overline{[[x, a], y]}.
\]

(ii) For any \( y \in \mathfrak{h}_3(\mathbb{O}) \) we have

\[
4[\hat{x}, [\hat{x}, \hat{a}]_*]_*(y) = [\hat{x}, \overline{[x, a]}],_*(y) = \hat{x}(\overline{[x, a]}(y)) - \overline{[x, a]}(\hat{x}(y))
\]

\[
= x \circ (\overline{[[x, a], y]}) - \overline{[[x, a], x \circ y]} = [x, [x, a]] \circ y.
\]

Let us now identify \( \mathfrak{h}_3^0(\mathbb{O}) \) with the subspace \( \{ \hat{x} : x \in \mathfrak{h}_3^0(\mathbb{O}) \} \) of \( e_{6(-26)} \). From Equation (ii) above we deduce that \( \mathfrak{d}^0 \) is a maximal abelian subspace of \( \mathfrak{h}_3^0(\mathbb{O}) \). A vector \( a \in \mathfrak{h}_3^0(\mathbb{O}) \) is a root vector with respect to a root \( \gamma \) if

\[
[\hat{x}, [\hat{x}, \hat{a}]_*]_* = \gamma(\hat{x})^2 \hat{a},
\]

for all \( x \in \mathfrak{d}^0 \). Again from Equation (ii) we deduce the roots of the symmetric space \( E_{6(-26)}/F_4 \) with respect to \( \mathfrak{d}^0 \) are the functions \( \frac{1}{2}(x_2 - x_3) \), \( \frac{1}{2}(x_1 - x_2) \), and \( \frac{1}{2}(x_1 - x_3) \) and
their negatives (see also Equation (2.5)). The corresponding root spaces are the spaces $h_{\gamma_1}$, $h_{\gamma_2}$, and $h_{\gamma_3}$ described in Section 2.2.

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