ON THE MAXIMAL REGULARITY FOR A CLASSE OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We propose an approach based on perturbation theory to establish maximal $L^p$-regularity for a class of Volterra integro-differential equations. As the left shift semigroup is involved for such equations, we study maximal regularity on Bergman spaces for autonomous and non-autonomous integro-differential equations. Our method is based on the formulation of the integro-differential equations to a Cauchy problems, infinite dimensional systems theory and some recent results on the perturbation of maximal $L^p$-regularity (see [1, 2]). Applications to heat equations driven by the Dirichlet (or Neumann)-Laplacian are considered.

1. Introduction

In recent years, somewhat more progress on the concept of maximal $L^p$-regularity ($p \in (1, \infty)$) has been made in the evolution equations literature. This property plays an important role in the well-posedness of nonlinear evolution equations, quasilinear ones and non-autonomous evolution ones. Various approaches have been proposed for the concept of maximal $L^p$-regularity, we cite the variational approach e.g. [18], the operator one e.g. [3], [16], and the perturbation one e.g. [1, 2]. For more facts on this property, the reader is invited to consult this non-exhaustive list [9], [8], [17], [16], [3], [14], [21], [22] and references therein.

This paper focuses on proving the maximal $L^p$-regularity for some classes of Volterra integro-differential equations using the recent results based on the perturbation approach developed in [1, 2]. On the one hand, the results displayed throughout this article draw from our recent papers [1, 2], where the above equations are studied with kernels $a(\cdot) = 0$, and, on the other hand, from Bárta [1] where these problems are studied on UMD spaces by using the concept of $R$-sectoriality. One remarkable fact is that in the context of UMD spaces the left shift semigroup on Bergman space enjoys the maximal $L^p$-regularity.
We first consider in Section 3 the following autonomous Volterra integro-differential equation
\[
\begin{aligned}
&z(t) = A z(t) + \int_0^t a(t - s) Fz(s)ds + f(t), \quad t \geq 0, \\
&z(0) = 0,
\end{aligned}
\tag{1.1}
\]
where \(A : D(A) \subset X \rightarrow X\) is the generator of a \(C_0\)-semigroup \(T := (T(t))_{t \geq 0}\) on a Banach space \(X\) and \(F : D(A) \rightarrow X\) a linear operator, and \(a : \mathbb{C} \rightarrow \mathbb{C}\) and \(f : [0, \infty) \rightarrow X\) are measurable functions.

In a suitable product space, the previous problem is reformulated as the following non-homogeneous problem
\[
\begin{aligned}
&\dot{\rho}(t) = A \rho(t) + \zeta(t), \quad t \geq 0, \\
&\rho(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
\]
where \(A\) is a matrix operator (see Section 3). Using a recent perturbation result of maximal \(L^p\)-regularity, we prove, under assumptions, that the operator \(A\) has maximal \(L^p\)-regularity and we give an estimate for the solution of the problem (1.1).

In [20], the author studied maximal \(L^p\)-regularity of type \(C^\alpha\) of (1.1), which differs from the maximal \(L^p\)-regularity of type \(L^p\) presented in this paper. Here we use a direct approach in the treatment of (1.1) without appealing the concept of \(\kappa\)-regular kernels as in the paper [24].

In Section 4, we study the maximal \(L^p\)-regularity for Volterra integro-differential equations with boundary conditions of the form
\[
\begin{aligned}
&\dot{z}(t) = A_m z(t) + \int_0^t a(t - s) Pz(s)ds + f(t), \quad t \in [0, T], \\
&z(0) = 0, \\
&Gz(t) = K z(t), \quad t \in [0, T],
\end{aligned}
\tag{1.2}
\]
where \(A_m : Z \rightarrow X\) is a closed linear operator with \(Z\) is a Banach space that is densely and continuously embedded in the Banach space \(X\). \(G, K : Z \rightarrow U\) are linear operators with \(U\) another Banach space. \(P : Z \rightarrow X\) is a linear operator.

In [1] and [2], the authors studied the problem (1.2) in the case \(a(\cdot) = 0\) using the feedback theory of infinite dimensional linear systems.

In order to prove maximal \(L^p\)-regularity of (1.2), we reformulate the problem as
\[
\begin{aligned}
&\dot{\rho}(t) = A \rho(t) + \zeta(t), \quad t \in [0, T] \\
&\rho(0) = 0,
\end{aligned}
\]
where \(A\) is some matrix operator (see Section 3). Using results from [2], we prove, under some assumptions, that \(A\) has maximal \(L^p\)-regularity and we derive a useful estimate satisfied by the solution of (1.2).

Section 2 is devoted to recall the definition of maximal \(L^p\)-regularity and a useful perturbation result. In Section 3 we study well-posedness and maximal \(L^p\)-regularity of
evolution equations (1.1). In Section 4, we review some useful results on feedback theory of infinite dimensional systems and prove maximal $L^p$-regularity of (1.2) under suitable assumptions.

2. Background on maximal $L^p$-regularity for Cauchy problems

In this section, we collect necessary background on maximal $L^p$-regularity that will be used in this paper. Let $\mathcal{X}$ be a Banach space with norm $\| \cdot \|$, $p \in (1, \infty)$ a real number, and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ a closed linear operator.

**Definition 2.1.** We say that $\mathcal{A}$ has the maximal $L^p$-regularity, and we write $\mathcal{A} \in MR_p(0, T; \mathcal{X})$, if for every $f \in L^p([0, T], \mathcal{X})$ there exists a unique $u \in W^{1,p}([0, T], \mathcal{X}) \cap L^p([0, T], D(\mathcal{A}))$ such that

$$\dot{u}(t) = \mathcal{A}u(t) + f(t) \quad \text{for } t \in [0, T] \quad \text{and } u(0) = 0.$$ 

By ”maximal” we mean that the applications $f$, $\mathcal{A}u$ and $\dot{u}$ have the same regularity. According to the closed graph theorem, if $\mathcal{A}$ has maximal $L^p$-regularity,

$$\|\dot{u}\|_{L^p([0, T], \mathcal{X})} + \|u\|_{L^p([0, T], \mathcal{X})} + \|\mathcal{A}u\|_{L^p([0, T], \mathcal{X})} \leq \kappa \|f\|_{L^p([0, T], \mathcal{X})}$$

for a constant $\kappa := \kappa(p) > 0$ independent of $f$.

It is known that a necessary condition for the maximal $L^p$-regularity is that $\mathcal{A}$ generates an analytic semigroup $\mathcal{T} := (\mathcal{T}(t))_{t \geq 0}$. This condition is also sufficient if $\mathcal{X}$ is a Hilbert space, see De Simon [7]. Moreover, it is shown in [9] that if $\mathcal{A}$ has maximal $L^p$-regularity for one $p \in [1, \infty]$ then $\mathcal{A}$ has maximal $L^q$-regularity for all $q \in ]1, \infty[$.

Next we recall a perturbation result on maximal $L^p$-regularity. To this end, we need the following concept.

**Definition 2.2.** let $\mathcal{A}$ be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on a Banach space $\mathcal{X}$, and let $\mathcal{Y}$ be another Banach space. An operator $\mathcal{C} \in \mathcal{L}(D(\mathcal{A}), \mathcal{Y})$ is called $p$-admissible observation operator for $\mathcal{A}$, if there exist (hence all) $\alpha > 0$ and a constant $\gamma := \gamma(\alpha) > 0$ such that:

$$\int_0^\alpha \| \mathcal{C} \mathcal{T}(t)x \|^p_y dt \leq \gamma^p \| x \|^p,$$

for all $x \in D(\mathcal{A})$. We also say that $(\mathcal{C}, \mathcal{A})$ is $p$-admissible.

The following theorem gives an invariance result of the maximal $L^p$-regularity, see [1].

**Theorem 2.3.** If $\mathcal{P} \in \mathcal{L}(D(\mathcal{A}), \mathcal{X})$ is $l$-admissible for $\mathcal{A}$ for some $l > 1$ and $\mathcal{A}$ has maximal $L^p$-regularity then the operator $\mathcal{A} + \mathcal{P} : D(\mathcal{A}) \to \mathcal{X}$ is so.

3. Maximal regularity for Volterra integro-differential equations

In this section, we study the maximal $L^p$-regularity of the autonomous Volterra integro-differential equation (1.1). First, certain conventions are defined. Let the Banach product
Let us define the left shift semigroup on \( L^q(\mathbb{R}^+, X) \) by
\[
(\mathbb{S}(t)f)(s) = f(t+s), \quad t, s \geq 0.
\]
We select
\[
\Upsilon x := a(\cdot)Fx, \quad x \in D(A),
\]
\[
\mathfrak{A} := \begin{pmatrix} A & \delta_0 \\ \Upsilon & \frac{d}{ds} \end{pmatrix}, \quad D(\mathfrak{A}) = D(\mathbb{A}) \times D(\frac{d}{ds}).
\] (3.1)
Moreover, we consider the function
\[
\zeta : [0, \infty) \to X, \quad \zeta(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad t \in [0, T].
\] (3.2)
Let \( z : [0, \infty) \to X \) satisfies (1.1). According to [10, Section VI.7], the Volterra equation (1.1) can be reformulated as the following non-homogeneous Cauchy problem on \( \mathcal{X} \),
\[
\begin{aligned}
\dot{\varrho}(t) &= \mathfrak{A} \varrho(t) + \zeta(t), & t \in [0, T], \\
\varrho(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
\] (3.3)
where
\[
\varrho(t) = \begin{pmatrix} z(t) \\ g(t, \cdot) \end{pmatrix}, \quad g(t, \cdot) = \mathbb{S}(t)g(0, \cdot) + \int_0^t \mathbb{S}(t-s)\Upsilon z(s)ds, \quad t \geq 0.
\]
In order to study the maximal \( L^p \)-regularity of (1.1), it suffices to study the one of the linear operator \( \mathfrak{A} \). To this end, we will use Theorem 2.3. In fact, we first split the operator \( \mathfrak{A} \) as
\[
\mathfrak{A} = \mathcal{A} + \mathcal{P}
\] (3.4)
where
\[
\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{ds} \end{pmatrix}, \quad D(\mathcal{A}) = D(\mathbb{A}) \times W^{1,q}(\mathbb{R}^+, X),
\]
\[
\mathcal{P} := \begin{pmatrix} 0 & \delta_0 \\ \Upsilon & 0 \end{pmatrix}, \quad D(\mathcal{P}) = D(\mathcal{A}).
\] (3.5)
Clearly, the operator \( \mathcal{A} \) generates the following \( C_0 \)-semigroup on \( \mathcal{X} \),
\[
\mathcal{T}(t) = \begin{pmatrix} \mathbb{T}(t) & 0 \\ 0 & \mathbb{S}(t) \end{pmatrix}, \quad t \geq 0.
\]
Moreover, \( \mathcal{T} \) is not an analytic semigroup on \( \mathcal{X} \) even if we assume that the semigroup \( \mathbb{T} \) is analytic on \( X \). This is due to the fact that the left shift semigroup \( \mathbb{S} := (\mathbb{S}(t))_{t \geq 0} \) is not analytic in \( L^q(\mathbb{R}^+, X) \). This means that \( \mathcal{A} \) has not the maximal \( L^p \)-regularity on the space \( \mathcal{X} \). To overcome this problem, one way is to look for subspaces of \( L^q(\mathbb{R}^+, X) \) in which the shift semigroup \( \mathbb{S} \) is analytic. As shown [4] and [5] a perfect space in which the shift semigroup is analytic is the Begrman space which we define as follows:
Definition 3.1. For $q \in (1, \infty)$, we define the Bergman space of holomorphic $L^q$-integrable functions by:

$$B^q(\Sigma_\theta; X) := \left\{ f : \Sigma_\theta \to X \text{ holomorphic} ; \int_{\Sigma_\theta} \| f(\tau + i\sigma) \|^q_X d\tau d\sigma < \infty \right\}.$$  

where

$$\Sigma_\theta := \{ \lambda \in \mathbb{C} : |\arg(\lambda)| < \theta \}, \quad 0 < \theta \leq \frac{\pi}{2}.$$

This space is also defined by $B^q_\theta, X$. Moreover, if $X = \mathbb{C}$, then we write $B^q_\theta$ instead of $B^q_\theta, \mathbb{C}$.

The space $B^q_\theta, X$, endowed with the following norm

$$\| f \|_{B^q_\theta, X} := \left( \int_{\Sigma_\theta} \| f(\tau + i\sigma) \|^q_X d\tau d\sigma \right)^{\frac{1}{q}},$$

is a Banach space.

The proof of the following result can be found in [5] and [6].

**Proposition 3.2.** The complex derivative $\frac{d}{dz}$ with its natural domain:

$$D \left( \frac{d}{dz} \right) := \{ f \in B^q_\theta, X ; f' \in B^q_\theta, X \}$$

generates an analytic semigroup of translation on $B^q_\theta, X$. Furthermore, if $X$ is an UMD space, then $\frac{d}{dz}$ enjoys the maximal $L^p$-regularity on $B^q_\theta, X$.

This result motivated us to replace the state space $\mathcal{X}$ with the following appropriate space

$$\mathcal{X}^q := X \times B^q_\theta, X, \quad \|(x, f)\|_{\mathcal{X}^q} := \| x \|_X + \| f \|_{B^q_\theta, X}.$$  

According to Proposition 3.2, the following result becomes trivial.

**Lemma 3.3.** Assume that $X$ is an UMD space. If $A$ has the maximal $L^p$-regularity in $X$, then the operator $\mathcal{A}$ defined in (3.5) has the maximal $L^p$-regularity in $\mathcal{X}^q$.

Now if $\frac{d}{dz}$ has the maximal $L^p$-regularity on $B^q_\theta, X$ for some $\theta \in (0, \frac{\pi}{2}]$ then according to Theorem 2.3 and the decomposition (3.4), $\mathcal{A}$ will have the maximal $L^p$-regularity on $\mathcal{X}^q$ as long as one proves that the perturbation $\mathcal{P}$ is $l$-admissible for $\mathcal{A}$ for some $l \in (1, \infty)$.

We have the following technical result.

**Lemma 3.4.** Assume that $a(\cdot) \in B^q_\theta$ for $0 < \theta \leq \frac{\pi}{2}$ and let $F \in \mathcal{L}(D(\mathcal{A}), X)$ be $p$-admissible for $\mathcal{A}$. Then

$$\int_0^\alpha \| \mathcal{P} \mathcal{T}(t)(\frac{\tau}{\alpha}) \|^p dt \leq 2^{p-1} \left( \| a \|^p_{B^q_\theta} \gamma^p \| x \|^p_X + \int_0^\alpha \| f(t) \|^p dt \right) \quad (3.6)$$

$\alpha > 0$ and $(\frac{\tau}{\alpha}) \in D(\mathcal{A})$.  

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Proof. Let $\alpha > 0$ and $(\vec{f}) \in D(\mathcal{A})$, we have

\[
\int_0^\alpha \| \mathcal{P} \mathcal{T}(t)(\vec{f}) \|^p dt \leq 2^{p-1} \left( \int_0^\alpha \| \Upsilon \mathcal{T}(t)x \|^p_{B^q_\theta} dt + \int_0^\alpha \| f(t) \|^p dt \right) \leq 2^{p-1} \left( \| a \|^p_{B^q_\theta} \int_0^\alpha \| F \mathcal{T}(t)x \|^p_{X} dt + \int_0^\alpha \| f(t) \|^p dt \right).
\]

Now the estimate (3.6) immediately follows the $p$-admissible of $F$ for $\mathcal{A}$. \qed

According to Lemma 3.4, To prove the $p$-admissibility of $\mathcal{P}$ for $\mathcal{A}$ for some $p \in (1, \infty)$ it suffices to estimate the $L^p$ norm of $f$ by its norm on the Bergman space $B^{q}_{\theta,X}$. To this end, we need the following lemma inspired from [6, lem.4.3] (here we slightly modify the result proved in [6, lem.4.3] and give a sharp estimate.)

**Lemma 3.5.** Let $s \in (1, 2)$ and $q > 2$. For $p_{s,q} := \frac{q(s-1)}{s} (> 1)$ and $f \in B^{q}_{\theta,X}$ for some $\theta \in (0, \frac{\pi}{2})$, we have

\[
\left( \int_0^R \| f(t) \|^p_{X} dt \right)^{\frac{1}{p_{s,q}}} \leq C_R \| f \|_{B^{q}_{\theta,X}}.
\]

for all $R > 0$ and $C_R > 0$ only depends on $R$ that verifies $C_R \to 0$ as $R \to 0$.

**Proof.** First, let $\theta \in (0, \frac{\pi}{2})$ and let us estimate the value of $\| f(t) \|_{p_{s,q}}$ using the Cauchy formula. The integration path will consist of two circle segments (see the next Figure). Let $\gamma_1(t) := r - acr + are^{it}$ and $\gamma_2(t) := r + acr - are^{it}$, $t \in [-\alpha, \alpha]$ with $c := \cos \alpha$ and $a = \tan(\theta)$ such that $ac < 1$.
\[ f(r) = \frac{1}{2i\pi} \int_{\gamma} \frac{f(z)}{z - r} \, dz \]

\[ = \frac{1}{2i\pi} \left( \int_{-\alpha}^{\alpha} \frac{f(r - \text{arc} + \text{are}^{it})\text{iar}e^{it}}{-\text{arc} + \text{are}^{it}} \, dt + \int_{-\alpha}^{\alpha} \frac{f(r + \text{arc} - \text{are}^{it})\text{iar}e^{it}}{\text{arc} - \text{are}^{it}} \, dt \right) \]

then

\[ \|f(r)\|^{p_{q}} \leq \frac{1}{(2\pi)^{p_{q}}} \left( \int_{-\alpha}^{\alpha} \|f(r - \text{arc} + \text{are}^{it})\| \, dt \right)^{p_{q}} + \left( \int_{-\alpha}^{\alpha} \|f(r + \text{arc} - \text{are}^{it})\| \, dt \right)^{p_{q}} \]

\[ \leq \frac{2^{p_{q} - 1}}{(2\pi(1 - c))^{p_{q}}} \left( \int_{-\alpha}^{\alpha} \|f(r - \text{arc} + \text{are}^{it})\|^{p_{q}} \, dt \right) + \left( \int_{-\alpha}^{\alpha} \|f(r + \text{arc} - \text{are}^{it})\|^{p_{q}} \, dt \right) \]

\[ \leq \frac{(4\alpha)^{p_{q} - 1}}{(2\pi(1 - c))^{p_{q}}} \left( \int_{-\alpha}^{\alpha} \|f(r - \text{arc} + \text{are}^{it})\|^{p_{q}} \, dt \right) + \int_{-\alpha}^{\alpha} \|f(r + \text{arc} - \text{are}^{it})\|^{p_{q}} \, dt \)

Now we shall estimate the first integral in the above inequality, and the second one can be estimated in a similar way. We set \( \psi(t, r) = (r - \text{arc} + \text{ar} \cos t, \text{ar} \sin t) \) then the Jacobian
of $\psi$ satisfies

$$|J_\psi| = \begin{vmatrix} 1 - ac + a \cos t & -ar \sin t \\ ar \sin t & ar \cos t \end{vmatrix} = ar((1 - ac) \cos t + a)$$

$$\geq ar(c(1 - ac) + a)$$

Since $x = r + ar(\cos t - c) < r(1 + a(1 - c))$, we have

$$|J_\psi| \geq ax(c(1 - ac) + a) := c_1 x$$

then

$$\int_0^R \int_{-\alpha}^{\alpha} \|f(\psi(t, r))\|^{p_{s,q}} dx dy = \int_0^R \int_{|J_\psi|} \|f(x + iy)\|^{p_{s,q}} dx dy$$

where $M := \psi((0, R) \times [-\alpha, \alpha])$ is contained in $M' := \{x + iy \in \mathbb{C}; 0 < x < R + \delta \}$ with $\delta := R(1 - c)a$, and $|y| \leq ax \sin \alpha < ax \sin \alpha$. This inclusion and Hölder inequality imply that

$$\int_M \|f(x + iy)\|^{p_{s,q}} dx dy \leq \frac{1}{c_1} \int_{M'} \|f(x + iy)\|^{p_{s,q}} x dx dy$$

$$\leq \frac{1}{c_1} \left( \int_{M'} \|f(x + iy)\|^{p_{s,q,s'}} x^{s'} dx \right)^{\frac{1}{s'}} \left( \int_{M'} x^{-\frac{s}{2}} dx \right)^{\frac{s}{s'}}$$

$$\leq \frac{(2a \sin \alpha)^{\frac{1}{2}}}{c_1} \|f\|_{B_{s,q,s'}^{p_{s,q}}(\mathbb{C}; \mathbb{R}^2)} \left( \int_0^{R + \delta} x^{1-s} dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{ax \sin \alpha}{x}} 2dy dx \right)^{\frac{1}{s'}}$$

$$\leq \frac{(2a \sin \alpha)^{\frac{1}{2}}}{c_1} \|f\|_{B_{s,q,s'}^{p_{s,q}}(\mathbb{C}; \mathbb{R}^2)} \left( \int_0^{R + \delta} x^{1-s} dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{ax \sin \alpha}{x}} 2dy dx \right)^{\frac{1}{s'}}$$

$$= \frac{(2a \sin \alpha)^{\frac{1}{2}}}{a(a + c(1 - ac))} \left( R(1 + (1 - c)a) \right)^{\frac{2-s}{2}} \|f\|_{B_{s,q,s'}^{p_{s,q}}(\mathbb{C}; \mathbb{R}^2)}$$

$$:= \tilde{C} R^{\frac{2-s}{2}} \|f\|_{B_{s,q,s'}^{p_{s,q}}(\mathbb{C}; \mathbb{R}^2)}.$$

where the constant $\tilde{C}$ does not depend on $R$.

Finally we have

$$\left( \int_0^R \|f(t)\|^{p_{s,q}} dt \right)^{\frac{1}{p_{s,q}}} \leq C_R \|f\|_{B_0^{q}}$$

with $C_R \to 0$ as $R \to 0$.

The case of $\theta = \frac{\pi}{2}$ follows easily due to the fact that the space $B_{q,X}^\theta$ is decreasing with respect to $\theta$ with continuous injection. \qed

**Remark 3.6.** Given $q, l > 1$, there always exists $s_{q,l} \in (1, 2)$ such that:

$$1 < p_{s,q,l} := \frac{q(s_{q,l} - 1)}{s_{q,l}} \leq l.$$
In fact, if \( q \in (1, l] \) the assertion is trivial. Now if \( q \in [2l, \infty) \) we have \( \frac{q}{q - l} \in (1, 2) \). Hence all \( s_{q,l} \in (1, \frac{q}{q - l}] \) satisfy the required estimation. Finally, for \( q \in (1, 2l] \) we have \( 2 \leq \frac{q}{q - l} \). Thus all \( s \in (1, 2) \) will satisfy the estimation. The fact that the space of \( l \)-admissible operators is decreasing with respect to the exponent \( l \), the discussion above shows that a sufficient condition to have the required \( p_{s,q,l} \)-admissibility for \( A \) in Theorem 3.7 is in fact the \( l \)-admissibility for some \( l > 1 \).

Now we state the first result of this paper:

**Theorem 3.7.** Let \( X \) be a UMD space and that \( a(\cdot) \in B_0^q \) for some \( q > 2 \) and \( \theta \in (0, \pi/2] \) and \( F \in \mathcal{L}(D(A), X) \) is a \( l_0 \)-admissible observation operator for \( A \) for some \( l_0 \in (1, \infty) \). If both \( A \) and \( \frac{dz}{dz} \) have the maximal \( L^p \)-regularity in \( X \) and \( B_{\theta,X}^q \) respectively, then \( A \) has the maximal \( L^p \)-regularity on \( \mathcal{X}^q \). Moreover, if \( p \in (1, l_0] \) and \( z \) is the solution of the problem (1.1), then there exists \( C_p > 0 \) independent of \( f \in L^p([0, T], X) \) such that

\[
\| \dot{z} \|_{L^p([0, T], X)} + \| A z \|_{L^p([0, T], X)} + \| z \|_{L^p([0, T], X)} \leq C_p \| f \|_{L^p([0, T], X)}.
\]  

(3.7)

**Proof.** The proof uses Theorem 2.3 and the decomposition \( \mathfrak{A} = \mathcal{A} + \mathcal{P} \) given in (3.4). Lemma 3.3 shows that the operator \( \mathcal{A} \) has the maximal \( L^p \)-regularity on \( \mathcal{X}^q \). Let \( s_{q,l_0} \) and \( p_{s,q,l_0} \) as in Remark 3.6. Now by combining Lemma 3.4 and Lemma 3.5, it is clear that the operator \( \mathcal{P} \) is \( p_{s,q,l_0} \)-admissible for \( \mathcal{A} \). Appealing to Theorem 2.3, the operator \( \mathfrak{A} \) also enjoys the maximal \( L^{p_{s,q,l_0}} \)-regularity on \( \mathcal{X}^q \). It is well known that if an operator has maximal \( L^p \)-regularity for some \( p \in (1, \infty) \), then it has maximal \( L^p \)-regularity for all \( p \in (1, \infty) \) (see for instance [9]) and hence \( \mathfrak{A} \) has the maximal \( L^p \)-regularity on \( \mathcal{X}^q \). Thus there is a constant \( C_p > 0 \) such that

\[
\| \dot{\mathfrak{A}} \|_{L^p([0, T], \mathcal{X}^q)} + \| \mathfrak{A} \mathfrak{A} \|_{L^p([0, T], \mathcal{X}^q)} + \| \mathfrak{A} \|_{L^p([0, T], \mathcal{X}^q)} \leq C_p \| f \|_{L^p([0, T], X)}.
\]

Since \( \mathfrak{A} \mathfrak{A} (t) = \left( \frac{A z(t) + g(t, 0)}{\Theta z(t) + \frac{d g(t, \cdot)}{d z}} \right) \) and \( \dot{\mathfrak{A}}(t) = \left( \frac{\dot{z}(t)}{\dot{g}(t, \cdot)} \right) \), we have

\[
\| \dot{z} \|_{L^p([0, T], X)} + \| A z \|_{L^p([0, T], X)} + \| z \|_{L^p([0, T], X)} \\
\leq \| \dot{z} \|_{L^p([0, T], X)} + \| A z + g(\cdot, 0) \|_{L^p([0, T], X)} + \| g(\cdot, 0) \|_{L^p([0, T], X)} + \| z \|_{L^p([0, T], X)} \\
\leq C \| f \|_{L^p([0, T], X)} + \| g(\cdot, 0) \|_{L^p([0, T], X)}.
\]

On the other hand, we have \( \dot{g}(t, \cdot) = \Theta z(t) + \frac{d g(t, \cdot)}{d z} \), hence

\[
g(t, \cdot) = \mathcal{S}(t) g(0, \cdot) + \int_0^t \mathcal{S}(t-s) a(\cdot) F z(s) ds.
\]
Thus
\[ \|g(\cdot, 0)\|_{L^p([0,T], X)}^p = \int_0^T \|g(t, 0)\|_X^p dt \]
\[ = \int_0^T \| \int_0^t a(t-s)Fz(s)\|^p dt \]
\[ \leq T^{p-1} \int_0^T \int_0^T |a(t-s)|^p \|Fz(s)\|^p ds dt \]
\[ = T^{p-1} \int_0^T \int_s^T |a(t-s)|^p \|Fz(s)\|^p dt ds \]
\[ = T^{p-1} \left( \int_0^T |a(t)|^p dt \right) \left( \int_0^T \|Fz(s)\|^p ds \right) \]
\[ \leq T^{p-1} \left( \int_0^T |a(t)|^p dt \right) \int_0^T \|Fz(s)\|^p ds \]
\[ \leq T^{p-1} C_T^p \|a\|_{B^p_\theta}^p \int_0^T \|Fz(s)\|^p ds. \]

where $C_T > 0$ is the constant in Lemma 3.5.

Note that $z(t) \in D(A)$ for almost every $t \in [0, T]$, due to the maximal $L^p$-regularity of $A$. This shows that
\[ \|g(\cdot, 0)\|_{L^p([0,T], X)} \leq T^{\frac{p-1}{p}} C_T \|a\|_{B^p_\theta} \|Fz\|_{L^p([0,T], X)}. \]

It suffices only to estimate $\|Fz\|_{L^p([0,T], X)}$ by $\|f\|_{L^p([0,T], X)}$. In fact, we know that
\[ \dot{z}(t) = A z(t) + g(t, 0) + f(t), \quad t \in [0, T], \]

which gives
\[ Fz(t) = F \int_0^t \mathbb{T}(t-s)g(s, 0)ds + F \int_0^t \mathbb{T}(t-s)f(s)ds, \quad \text{a.e. } t \in [0, T]. \]

Keeping in mind that the space of $l$-admissible operators is decreasing with respect to the exponent $l$ and that $p \leq l_0$ implies that $F$ is $p$-admissible for $A$. Therefore, by using [12 Prop 3.3],
\[ \|Fz\|_{L^p([0,T], X)} \leq \gamma_T \|g(\cdot, 0)\|_{L^p([0,T], X)} + \gamma_T \|f\|_{L^p([0,T], X)} \]
\[ \leq \gamma_T T^{\frac{p-1}{p}} C_T \|a\|_{B^p_\theta} \|Fz\|_{L^p([0,T], X)} + \gamma_T \|f\|_{L^p([0,T], X)}, \]

where $\gamma_T > 0$ is the constant satisfying $\gamma_T \to 0$ as $T \to 0$. Hence, if we choose $T$ such that $\beta_T := \gamma_T T^{\frac{p-1}{p}} C_T \|a\|_{B^p_\theta} < 1$, we obtain
\[ \|Fz\|_{L^p([0,T], X)} \leq \frac{\gamma_T}{1 - \beta_T} \|f\|_{L^p([0,T], X)}. \]
Hence we have
\[
\|\dot{z}\|_{L^p([0,T],X)} + \|Az\|_{L^p([0,T],X)} + \|z\|_{L^p([0,T],X)} \leq C\|f\|_{L^p([0,T],X)}.
\]
\[
\square
\]

**Remark 3.8.** In contrast to Barta’s result and as for Cauchy problem, we have obtained the estimation (3.7). One wonders if the method used in [5] can prove the aforementioned estimate.

**Example 3.9.** In this section we investigate the maximal \(L^p\)-regularity of the following Volterra integro-differential heat equation involving a fractional power of the Laplace operator

\[
\begin{cases}
\dot{u}(t,x) = (\Delta + P)u(t,x) + \int_0^t \beta e^{-\gamma(t-s)}(-\Delta)^\alpha u(s,x)ds + f(t), & t \in [0,T], x \in \Omega \\
u(t,x) = 0, & t \in [0,T], x \in \partial \Omega \\
u(0,x) = 0, & x \in \Omega
\end{cases}
\]

(3.8)

where \(\Omega \in \mathbb{R}^n\) is a bounded Lipschitz domain and \(\alpha \in (0,1/2]\) and \(\beta, \gamma > 0\). It is an example of anomalous equation of diffusion type. Let us first verify that the following Dirichlet-Laplacian operator defined on suitable \(L^r(\Omega)\) by

\[
D(\Delta^D) = W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega),
\]

\[
\Delta^D = \Delta,
\]

for certain domains \(\Omega\) and a range of exponents \(r\), has the maximal \(L^p\)-regularity. For \(n \geq 2\) and \(1 < r < 2\), it is shown in [23] that for an \(\Omega\) satisfying a uniform outer ball condition, the operator \(\Delta^D\) generates a positive, contractive and exponentially stable \(C_0\)-semigroup on \(L^r(\Omega)\) enjoying the maximal \(L^2\)-regularity. Therefore, due to Kalton and Weis’ result [15, Corollary 5.2], the Dirichlet operator \(-\Delta^D\) admits a bounded \(H^\infty(\Sigma_0)\) functional calculus with \(\theta < \pi/2\). The fact that \(L^r(\Omega)\) is of cotype 2, thanks to [19, Theorem 4.2], the fractional power \((\Delta^D)^{1/2}\) is 2-admissible for \(\Delta^D\). Now assume that the unbounded operator \(P\) satisfies the following resolvent estimate \(P : D(\Delta^D) \to L^r(\Omega)\) such that

\[
\|\sqrt{\lambda}PR(\lambda,\Delta^D)\| \leq M
\]

for some constant \(M > 0\) and for all \(\lambda > 0\).

In view of [19, Theorem 4.1], the operator \(P\) is 2-admissible for \(\Delta^D\). Now Theorem 2.2 show that \(\Delta^D + P\) enjoys the maximal \(L^2\)-regularity on \(L^r(\Omega)\). Since both \(P\) and \((\Delta^D)^{1/2}\) are 2-admissible for \(\Delta^D\), we deduce that \((\Delta^D)^{1/2}\) is 2-admissible for \(\Delta^D + P\) (see. [12]) and in virtue of Theorem (3.7) and Remark 3.6 we conclude that the problem (3.8) has maximal \(L^2\)-regularity \(L^r(\Omega)\) and the estimation (3.7) takes place in particular for \(p = 2\).
For a bounded Lipschitz (or convex) domain and \( n \geq 3 \), similar result can now be also obtained for the Neumann-Laplacian defined on \( L^r(\Omega) \) by

\[
D(\Delta^N_r) = \left\{ u \in W^{2,r}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\}.
\]

\( \Delta^N_r = \Delta. \)

Indeed, in virtue of [23, Theorem 6.4] and by proceeding in a very similar way as for Dirichlet boundary conditions we obtain the maximal \( L^p \)-regularity result for the above integro-differential equation with Neumann-Laplacian. For \( \alpha \in (0, \frac{1}{2}) \) the result follows in a similar way since analyticity shows that \( (−\Delta^D_r)^{\alpha} \) is always \( 2 \)-admissible for \( \Delta^D_r \).

4. MAXIMAL REGULARITY FOR BOUNDARY VOLterra INTEGRO-DIFFERENTIAL EQUATIONS

Let \( X, U \) and \( Z \) be Banach spaces such that \( Z \subset X \) continuously and densely. Let \( A_m : D(A_m) := Z \to X \) be a closed linear operator, and let \( F : Z \to X \), \( G, K : Z \to U \) be linear operators.

The object of this section is to investigate the maximal \( L^p \)-regularity of the problem (1.2)

\[
\begin{align*}
\dot{z}(t) &= A_m z(t) + \int_0^t a(t-s) Fz(s)ds + f(t), \quad t \in [0, T], \\
z(0) &= 0, \\
Gz(t) &= Kz(t), \quad t \in [0, T].
\end{align*}
\]

We introduce the linear operator

\[
A := A_m, \quad D(A) = \{ x \in Z : Gx = Kz \}.
\]

and we set \( \mathcal{P} := Fi \) where \( i \) the continuous injection from \( D(A) \) to \( Z \). Then the equation (1.2) is similar to the equation (1.1) with \( \Upsilon := a(\cdot)\mathcal{P} \). Thus, to prove maximal \( L^p \)-regularity of (1.2) it suffices to find conditions for which \( A \) has maximal regularity and \( \Upsilon \) is admissible for \( A \).

Throughout this section we assume that the operator \( G \) is surjective and

\[
A := (A_m)|_{D(A)} \quad \text{with} \quad D(A) := \ker(G),
\]

generates a strongly continuous semigroup \( T := (T(t))_{t \geq 0} \) on \( X \). We denote by \( \rho(A) \) the resolvent set of \( A \), \( R(\lambda, A) = (\lambda - A)^{-1}, \lambda \in \rho(A) \), the resolvent operator of \( A \). We also consider a new norm on \( X \) defines by \( \| x \|_{-1} := \| R(\mu, A)x \| \) for \( x \in X \) and \( \mu \in \rho(A) \) (this norm is independent of the choice of \( \mu \), due to the resolvent equation). We denote by \( X_{-1} \) the completion of \( X \) with respect to the norm \( \| \cdot \|_{-1} \), which a Banach space, called the extrapolation space associated with \( X \) and \( A \). We have the following continuous and dense embedding \( D(A) \subset X \subset X_{-1} \). The semigroup \( T \) is extended to another strongly continuous semigroup \( T_{-1} := (T_{-1}(t))_{t \geq 0} \) on \( X_{-1} \), whose generator \( A_{-1} : X \to X_{-1} \) is the extension of \( A \) to \( X \), see [10, Chapter II].
According to Greiner [11], for each \( \lambda \in \rho(A) \), the restriction of \( G \) to \( \ker(\lambda - A_m) \) is invertible with inverse \( \mathbb{D}_\lambda \) (called the Dirichlet operator) given by
\[
\mathbb{D}_\lambda := (G|_{\ker(\lambda - A_m)})^{-1} \in \mathcal{L}(U, X).
\]
Now, define the operators
\[
B := (\lambda - A - 1)\mathbb{D}_\lambda \in \mathcal{L}(U, X_{-1}), \quad \lambda \in \rho(A),
\]
\[
C := Kj \in \mathcal{L}(D(A), U),
\]
\[
\mathbb{P} := Fj \in \mathcal{L}(D(A), X)
\]
where \( j \) is the continuous injection from \( D(A) \) to \( Z \). Due to the resolvent equation, the operator \( B \) is independent of \( \lambda \).

We also need the operators
\[
\Phi_t u := \int_0^t T_1(t - s)Bu(s)ds, \quad t \geq 0, \quad u \in L^p([0, +\infty), U).
\]
This integral takes its values in \( X_{-1} \). However, by using an integration by parts, for any \( t \geq 0 \) and \( u \in W^{2,p}_{0,t}(U) \), where
\[
W^{2,p}_{0,t}(U) := \{ u \in W^{2,p}([0, t], U) : u(0) = \dot{u}(0) = 0 \},
\]
we have
\[
\Phi_t u = \mathbb{D}_0 u(t) - \int_0^t T(t - s)\mathbb{D}_0 \dot{u}(s)ds \in Z,
\]
where we assumed that \( 0 \in \rho(A) \) (without loss of generality).

Thus the following operator is well defined
\[
(\mathbb{F}u)(t) = K\Phi_t u, \quad u \in W^{2,p}_{0,t}(U), \quad t \geq 0.
\]
Here, we make the following assumption

(\textbf{H}) The triple \((A, B, C)\) is \( p \)-regular, \( p \in (1, \infty) \), with \( I_U : U \to U \) as an admissible feedback. That is the following assertions hold:

1. \( C \) is a \( p \)-admissible observation operator for \( A \) (see Definition 2.2),
2. \( B \) is a \( p \)-admissible control operator for \( A \). This means that there exists \( \tau > 0 \) such that \( \Phi_\tau u \in X \) for any \( u \in L^p([0, +\infty), U) \),
3. For any \( \tau > 0 \), there exists \( \kappa := \kappa(\tau) > 0 \) such that
   \[
   \|\mathbb{F}u\|_{L^p([0, \tau], U)} \leq \kappa \|u\|_{L^p([0, \tau], U)}, \quad \forall u \in W^{2,p}_{0,\tau}(U),
   \]
   (hence we can extend \( \mathbb{F} \) to a bounded operator on \( L^p([0, \tau], U) \) for any \( \tau > 0 \)).
4. The following limit exists in \( U \) for any \( v \in U \):
   \[
   \lim_{h \to 0^+} \frac{1}{h} \int_0^h (\mathbb{F}(1_U, v))(s)ds = 0.
   \]
5. \( 1 \in \rho(\mathbb{F}) \).

We have the following observations about the condition (\textbf{H}).
Remark 4.1. (i) If \( C \) is bounded from \( X \) to \( U \) (hence \( C = K \in \mathcal{L}(X,U) \)), and \( B \) is \( p \)-admissible for \( A \), then the condition (\( H \)) is satisfied. 
(ii) Assume that \( A \) has the maximal \( L^p \)-regularity for \( p \in (1, \infty) \), \( B \) is \( p \)-admissible for \( A \), and \( K = (-A)^{\alpha} \) for \( \alpha \in (0, \frac{1}{p}) \). Then the condition (\( H \)) is verified, see [1].

The first part of the following result was obtained in [13], while the second part is taken from [2].

Theorem 4.2. Let the condition (\( H \)) be satisfied. Then \( Z \subset D(C_\Lambda) \), \( C_\Lambda = K \) on \( Z \) and the operator \( A \) defined by \( 4.1 \) generates a strongly continuous semigroup \( T := (T(t))_{t \geq 0} \) on \( X \) given by

\[
T(t)x_0 = T(t)x_0 + \int_0^t T_{-1}(t-s)BC_\Lambda T(s)x_0ds \quad x_0 \in X, \ t \geq 0. \tag{4.2}
\]

Furthermore, if in addition \( T \) is an analytic semigroup, then it is so for \( T \).

Now we state the following lemma.

Lemma 4.3. If \( (A, B, \mathcal{P}) \) generates a \( p \)-regular linear system, then \( \mathcal{P} \) is a \( p \)-admissible observation for \( A \) whenever condition (\( H \)) is satisfied.

Proof. Condition (\( H \)) asserts that \( A \) is a generator of a \( C_0 \)-semigroup \( T \) given by (4.2). Now, we first remark that \( Z \subset D(\mathcal{P}_\Lambda) \) and \( P = \mathcal{P}_\Lambda \) on \( Z \), where \( \mathcal{P}_\Lambda \) denotes the Yosida extension of \( \mathcal{P} \) w.r.t. \( A \). Let \( x \in D(\mathcal{A}) \) and \( \alpha > 0 \). The facts that \( (A, B, \mathcal{P}) \) is \( p \)-regular and \( C_\Lambda \) is a \( p \)-admissible observation for \( T \) imply respectively that

\[
\int_0^t T_{-1}(t-s)BC_\Lambda T(s)x \in D(\mathcal{P}_\Lambda) \quad \text{a.e. } t \geq 0,
\]

and

\[
\left\| \mathcal{P}_\Lambda \int_0^t T_{-1}(t-s)BC_\Lambda T(s)x \right\|_{L^p([0,\alpha],X)} \leq \beta_\alpha \|x\|, \tag{4.3}
\]

where \( \beta_\alpha > 0 \) is a constant. On the other hand, by (4.2), we have

\[
\mathcal{P}T(t)x = \mathcal{P}_\Lambda T(t)x,
\]

\[
= \mathcal{P}_\Lambda T(t)x + \mathcal{P}_\Lambda \int_0^t T_{-1}(t-s)BC_\Lambda T(s)x.
\]

Hence \( p \)-admissibility of \( \mathcal{P} \) for \( \mathcal{A} \) follows by (4.3) and \( p \)-admissibility of \( \mathcal{P} \) for \( A \). \hfill \Box

The following theorem is a result of maximal \( L^p \)-regularity for the operator \( A \) in the case of bounded perturbation \( K \), see Remark 4.1(i), Theorem 4.2 and [1].

Theorem 4.4. If \( B \) is a \( p \)-admissible control operator for \( A \) and \( K \) is bounded, then if \( A \) has maximal \( L^p \)-regularity, then so has \( \mathcal{A} \).

The following result (see [2, Thm.4]) gives conditions implying the maximal \( L^p \)-regularity for \( \mathcal{A} \) when the state space \( X \) is a UMD space. The result is based on the concept of \( \mathcal{R} \)-boundedness.
Theorem 4.5. Let $X$ be an UMD space, $p, p' \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and the condition $(H)$ be satisfied. Assume that there exists $\omega > \max\{\omega_0(A), \omega_0(\mathbb{A})\}$ such that the sets $\{s^{\frac{1}{p}} R(\omega + is, A_{-1}) B, s \neq 0\}$ and $\{s^{\frac{1}{p'}} CR(\omega + is, A), s \neq 0\}$ are $\mathcal{R}$-bounded. If $A$ has maximal $L^p$-regularity, then $\mathbb{A}$ has the same property.

Let us now prove the maximal $L^p$-regularity for the boundary Volterra integro-differential equations (1.2). We start with the following special case of bounded boundary perturbation $K$.

Theorem 4.6. Let $X$ be a UMD space and $q \in (1, \infty)$. Assume that $a(\cdot) \in B^q_\theta$ for some $\theta \in (0, \frac{\pi}{2}]$, $B$ is an $l_0$-admissible control operator for $A$ for some $l_0 \in (1, \infty)$, $K \in \mathcal{L}(X,U)$ and $(A, B, \mathbb{P})$ generates an $l_0$-regular linear system. If both $A$ and $\frac{d}{dz}$ have maximal $L^p$-regularity on $X$ and $B^q_{\theta,X}$ respectively for some $p \in (1, \infty)$, then $\mathbb{A}$ has maximal $L^p$-regularity on $\mathcal{X}^q$. Moreover, if $p \in (1, l_0]$ then there exists a constant $C_p > 0$ independent of $f \in L^p([0,T], X)$ such that the solution $z$ of (1.2) satisfies

$$
\|\dot{z}\|_{L^p([0,T],X)} + \|A_m z\|_{L^p([0,T],X)} + \|z\|_{L^p([0,T],X)} \leq C_p \|f\|_{L^p([0,T],X)}.
$$

(4.4)

Proof. By Theorem 4.4, $l_0$-admissibility of $B$ and maximal regularity of $A$ imply that $\mathbb{A}$ has maximal $L^p$-regularity. By Lemma 4.3, the $l_0$-regularity of the system generated by $(A, B, \mathbb{P})$ asserts that $\mathcal{P}$ is an $l_0$-admissible observation operator for $\mathbb{A}$. These facts together with maximal $L^p$-regularity of $\frac{d}{dz}$ imply, by Theorem 3.7, that $\mathbb{A}$ has maximal $L^p$-regularity. The estimate follows by the same theorem. This ends the proof.

Now, we suppose that $K$ is unbounded and state the second main theorem of this paper.

Theorem 4.7. Let $X$ be a UMD space and $q \in (1, \infty)$. Assume that $a(\cdot) \in B^q_\theta$ for some $\theta \in (0, \frac{\pi}{2}]$, $(A, B, C)$ and $(A, B, \mathbb{P})$ generate $l_0$-regular linear systems for some $l_0 \in (1, \infty)$ and the identity $I_U$ is an admissible feedback for the system generated by $(A, B, C)$. If both $A$ and $\frac{d}{dz}$ have maximal $L^p$-regularity on $X$ and $B^q_{\theta,X}$ respectively for some $p \in (1, \infty)$ and the sets $\{s^{\frac{1}{p}} R(is, A_{-1}) B, s \neq 0\}$ and $\{s^{\frac{1}{p'}} CR(is, A), s \neq 0\}$ are $\mathcal{R}$-bounded, then $\mathbb{A}$ has maximal $L^p$-regularity on $\mathcal{X}^q$. Moreover, if $p \in (1, l_0]$ then there exists a constant $C_p > 0$ independent of $f \in L^p([0,T], X)$ such that the solution $z$ of (1.2) satisfies

$$
\|\dot{z}\|_{L^p([0,T],X)} + \|A_m z\|_{L^p([0,T],X)} + \|z\|_{L^p([0,T],X)} \leq C_p \|f\|_{L^p([0,T],X)}.
$$

(4.5)

Proof. By Theorem 4.5, the assumptions imply that $\mathbb{A}$ has maximal $L^p$-regularity. By Lemma 4.3, $l_0$-regularity of the system generated by $(A, B, \mathbb{P})$ asserts that $\mathcal{P}$ is an $l_0$-admissible observation operator for $\mathbb{A}$. Gathering these facts with maximal $L^p$-regularity of $\frac{d}{dz}$, we conclude by Theorem 3.7, that $\mathbb{A}$ has maximal $L^p$-regularity. The same theorem justifies the estimate which ends the proof.

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