Resonant Dampers for Parametric Instabilities in Gravitational Wave Detectors

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Advanced gravitational wave interferometric detectors will operate at their design sensitivity with nearly ~1MW of laser power stored in the arm cavities. Such large power may lead to the uncontrolled growth of acoustic modes in the test masses due to the transfer of optical energy to the mechanical modes of the arm cavity mirrors. These parametric instabilities have the potential of significantly compromising the detector performance and control. Here we present the design of “acoustic mode dampers” that use the piezoelectric effect to reduce the coupling of optical to mechanical energy. Experimental measurements carried on an Advanced LIGO-like test mass shown a 10-fold reduction in the amplitude of several mechanical modes, thus suggesting that this technique can greatly mitigate the impact of parametric instabilities in advanced detectors.

I. INTRODUCTION

The network of advanced gravitational wave detectors currently under construction (two LIGO detectors in the USA, the Advanced Virgo detector in Italy, and Kagra in Japan) promises to open the new window of gravitational wave astronomy within this decade.

Sensitivity to gravitational wave strains of order $10^{-24}$ requires high optical power circulating in the arm cavities of these detectors. For instance, up to 750 kW of optical power will be sustained in the steady-state regime inside the Advanced LIGO arm cavities.

It has been experimentally observed that the stored energy in a resonant cavity can leak from the optical modes to the mechanical modes of the cavity mirrors via a 3-mode interaction [4]. Given sufficiently high circulating optical power, and mirror materials with very low mechanical loss as required to avoid thermal noise, the uncontrolled growth of test mass acoustic modes can lead to Parametric Instabilities (PI) [5, 6]. If left unaddressed PI will prevent high power operation, and thus limit the astrophysical output of gravitational wave detectors. While Advanced LIGO will serve as the primary example in this paper, all advanced gravitational wave detectors are susceptible to these instabilities.

Several schemes have been proposed to damp PI [7, 8]. In particular, solutions directly applicable to Advanced LIGO involve active damping of acoustic modes by means of the test mass electro-static drive actuators [9], and thermal tuning of the optical modes using the test mass ring heaters [10]. A significant constraint on any technique is that it must preserve the inherently low mechanical loss of the test mass in the gravitational wave frequency band to maintain a low level of thermal noise.

Here we present a novel method to passively control PI by reducing the Q-factor of the test mass acoustic modes with small resonant dampers. These “acoustic mode dampers” (AMD) dissipate the strain energy of the test mass mode through a resistive element after converting it to electrical energy via the piezoelectric effect (see figure 2).

The resonant nature of the AMD allows it to effectively damp test mass acoustic modes without introducing significant mechanical loss at lower frequencies where thermal noise can limit detector performance. With respect to other proposed solutions, this approach has the advantage of being simple, self-contained, and completely passive. Models indicate that AMDs can provide a broadband reduction in the Q of mechanical modes relevant to PI, and are therefore particularly beneficial in the presence of a large number of unstable modes.

The structure of this paper is as follows. In section II we set the stage by giving a brief overview of parametric instabilities, including equations of particular relevance to evaluating AMD performance. Section III presents a simple 1-dimensional model of the AMD which highlights the principles of AMD operation. This is followed by a description of the detailed finite element model (FEM) used to analyze the AMDs ability to suppress PI when attached to an Advanced LIGO test mass. The FEM predictions are compared with experimental results from a full-scale prototype in section IV. Finally, in section V we discuss an AMD design that will provide Advanced LIGO with protection from instability, without significantly increasing test mass thermal noise.

II. PARAMETRIC INSTABILITIES (PI)

The acousto-optic interactions responsible for parametric instabilities have been extensively studied [11, 12]. They consist of a scattering process and radiation pressure operating together in an optical cavity in a closed-loop manner. The graphical representation of this process is shown in Fig.1.

The e-folding growth time, or “ring-up time”, of an acoustic mode in the presence of a 3-mode interaction is

$$\tau = \frac{2Q_m}{(\omega_m(R-1))} \text{[1]}$$

where $Q_m$ and $\omega_m$ are the Q-factor and angular frequency of the mode, respectively, and $R$ is the parametric gain. When $R > 1$ the amplitude increases exponentially until a saturation point is reached.

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FIG. 1. PI described as a positive feedback process. A steady state cavity field inside the interferometer arm cavity is distorted by scattering off a vibrating mirror surface. Two transverse optical sidebands are created. Both sidebands exert force on the test mass via radiation pressure. When the energy dissipation of the acoustic mode and the rate of work done by the radiation pressure is unbalanced, one of the sidebands excites the exponential growth of the acoustic mode amplitude. The dynamic of this process is commonly described in terms of the parametric gain $R$, with $R > 1$ in the case of instability.

The parametric gain $R$ can be defined as

$$ R = 4\pi^2 P_c \Lambda Q_m \times \frac{\nu_o}{\delta \nu_{hom}} \times \frac{\nu_o}{\delta \nu_{00}} \times \Gamma(\Delta \omega) $$

where $P_c$ is the optical power circulating in the arm cavity, $\nu_o$ is the optical frequency of the light, and $\delta \nu_{00}$ and $\delta \nu_{hom}$ are the cavity linewidths (full width, half maximum) for the fundamental and the higher-order optical mode, respectively. The parameter $\Lambda$ measures the spatial overlap between the acoustic and the higher-order optical modes; $\Gamma$ is representative of the interferometer optical configuration and is a function of the 3-mode interaction tuning $\Delta \omega = \omega_m - 2\pi \Delta \nu_{hom}$, where $\Delta \nu_{hom}$ is the frequency difference between the fundamental and higher-order optical modes. For $\Delta \omega \rightarrow 0$, the parameter $\Gamma$ reaches its maximum (see [14] for a more detailed description).

Unstable acoustic modes with parametric gain up to $R \simeq 100$ may arise in Advanced LIGO in the 10-90 kHz band [5]. To prevent these instabilities, a damping mechanism must be introduced to reduce the Q-factor of all unstable acoustic modes in this frequency band without introducing excess thermal fluctuation in the detection band of 10 Hz to 1 kHz.

III. MODEL OF THE ACOUSTIC MODE DAMPER (AMD)

In order to reduce the Q test mass acoustic modes we designed a resonant AMD which can be attached to the test mass and provide dissipation via the piezo-electric effect.

In this section we first describe the interaction between the AMD and the test mass with a simple 1-D model, then we present a complete Finite Element Model of the entire system.

A. Simplified 1-D Model

The AMD and test-mass system can be described as a pair of coupled oscillators with a large mass ratio. The AMD mass $m$ is attached to the much more massive test mass via piezo electric shear plates, which are modeled as a lossy spring with complex spring constant of magnitude $k$ and loss angle $\eta$.

The test mass acoustic mode for which we would like to estimate the impact of the AMD is simplified in this model to a mass $M$, equal to the modal mass of the acoustic mode, attached to a fixed reference by a lossless spring $K$. The coupled systems is then excited by the radiation pressure force $F$ applied to the TM mode, as shown in Fig. 2.

At frequencies near the resonance of the AMD, the lossy spring produced by the piezoelectric material and resistive load will dissipate the energy of the excited acoustic mode, as seen in Fig. 3.

For this system of coupled oscillators, the amplitude $A(\omega)$ of the acoustic mode driven by force $F$ at angular frequency $\omega$ is

$$ A(\omega) = \frac{F}{M\omega^2} \sqrt{\frac{\epsilon^2 + \eta^2}{(\delta + \mu)^2 + \eta^2(\delta + \mu)^2}} $$

where $\delta = 1 - \omega_0^2/\omega^2$, $\epsilon = 1 - \omega^2/\omega_D^2$, $\omega_0^2 = K/M$, $\omega_D^2 = k/m$, and $\mu = m/M$.

The resulting effective Q-factor is

$$ Q_{eff} = \frac{\max(A(\omega))}{A(\omega = 0)} \simeq \frac{\eta^2 + (1 - \rho)^2}{\eta \mu} $$

where $\rho = \omega_0/\omega_1$, and we assume $\mu \ll 1$.

When the acoustic mode resonance is near that of the AMD, $\eta \gg |1 - \rho|$, the acoustic mode Q is reduced to $Q_{eff} \simeq \eta/\mu$. When the acoustic mode resonance is well above the AMD resonance, $Q_{eff} \simeq \rho/\eta \mu$, and when it is well below the AMD resonance, $Q_{eff} \simeq 1/\eta \mu$, assuming $\eta^2 \ll 1$.

To suppress PIs, the test mass acoustic mode Q-factors only need to be reduced from $\approx 10^7$ to $10^5 - 10^6$. Using...
FIG. 2. Overview of the Acoustic Mode Damper (AMD). The AMD can be described as a small spring-mass system attached to a larger mass $M$. $M$ represents a modal mass of an acoustic test mass mode which vibrates due to an external force $F$ (radiation pressure). Such induced vibration causes deformation of the lossy spring of the damper. The non-zero loss angle of the damping spring (piezoelectric material) results in the lag of strain with respect to stress and thus energy dissipation. As a consequence of this dissipation process the amplitude of the acoustic mode vibration is reduced.

this simple model we can estimate that a 3g AMD with $\eta = 0.1$ on a 10kg test mass, can give $Q_{\text{eff}} \lesssim 10^5$ for resonances with $\frac{1}{3} < \rho < 3$.

However, this model ignores a number of important factors. One of these is the location of the AMD relative to the nodes and anti-nodes of each test mass acoustic mode. Quantitatively speaking, the modal mass $M$ of a given mode should be increased in this model by the ratio of the displacement at the AMD location to that of the mode’s anti-node squared $M' = M(x_{\text{max}}/x_{\text{AMD}})^2$. Thus, an AMD located near a node will have a reduced value of $\mu$, and will provide little damping.

Other important factors include the multiple coupled degrees of freedom of the AMD and the directional nature of the piezoelectric damping material, both of which are covered in the follow section.

B. Finite Element Model (FEM)

The simple 1-D model introduced in the previous section is useful to provide an intuitive understanding of the AMD damping mechanism. However, it is inadequate to represent the details of the interaction between the AMD and the test mass acoustic modes (TMAMs). Each AMD has at least six resonant modes and hundreds of TMAMs are present in the frequency band of interest; a Finite Element Model is required in order to properly reproduce these modes and calculate the mode overlap between them.

A FEM of the Advanced LIGO test mass with two attached AMDs was constructed with the ANSYS program [16]. The AMD model corresponds to the parameters of our prototype AMD (see Table I), and the test mass model parameters are reported in Table [VII]. All dissipation mechanisms in the test mass substrate, coating and bonds were included in the FEM, along with all the losses related to the acoustic mode damper structure (see table [V] for a full list).

A piezoelectric material (PZT), for which energy dissipation can be easily controlled, is ideal for AMD construction. The AMD design modeled here has 2 PZTs sandwiched between a reaction mass and the interferometer test mass (see figure [2]). The PZTs operate in shear,
plate with a resistor. The active stiffness component $c_{55}$ is the shear stiffness due to the shunt. The magnitude of $Im$ becomes a complex quantity with a nonzero imaginary part.

The loss factor $\eta$ is strictly related to the impedance of the sheared piezoelectric material. The total loss factor is a combination of the shunting loss and structural loss of the piezoelectric material. The maximum peak position is located at $40\,\text{kHz}$ which corresponds to the shunt with $10\,\text{k}\Omega$ resistor. The magnitude of $Im$ is the height of the shear plate, respectively. The loss factor $\eta^{pzt}$ is frequency dependent. As such, careful selection of the shunting resistor and PZT dimensions can be used to maximize loss in the band of interest, as shown in figure 3. A more detailed discussion of the piezoelectric loss angle can be found in Appendix A.

We validated the FEM of the AMD by computing its principle resonances, and comparing them with direct measurements performed on a prototype AMD. A total of five principle resonances were identified, with three types of modes: two flag, two anti-flag, and one rotation mode (see figure 4). The loss associated with each structural component is treated separately; $E_m$ is the modulus of the strain energy in the $m$th component and $\tan(\phi_m)$ is its loss factor (see tables I and VI).

The modal Q-factor of each resonant mode of the system was calculated as

$$Q(f_n) = \frac{\sum_mE_m(f_n)\tan(\phi_m)}{\sum_mE_m(f_n)}.$$  

where $f_n$ is the frequency of the $n$th acoustic mode (see figure 3). The loss associated with each structural component is treated separately; $E_m$ is the modulus of the strain energy in the $m$th component and $\tan(\phi_m)$ is its loss factor (see tables I and VI).

The modeled resonant frequencies of 12 TMAMs were then compared to the measured resonance frequencies through harmonic analysis (finite element analysis with an excitation at a fixed frequency) was conducted to estimate the Q-factor of each of the TMAMs. This approach allows us to include frequency dependent variables such as the shear plate stiffness $\Re(c_{55}^{pzt})$, and its loss angle (see Appendix A). Table II shows the good agreement between the output of the model and measurements on the AMD prototype.
TABLE II. Verification of the finite element model for AMD. Five principle resonances (two flag, two anti-flag and one rotational mode) obtained with the model are compared to the measured values. The x, y suffix corresponds to the shear plate polarization direction. The principle resonances of the AMD were measured with a capacitive bridge circuit where one of the matching capacitors was the AMD prototype. The difference between flag and anti-flag modes corresponds to the location of the rotation axis about which reaction mass rocks. For the flag pole the rotation axis is at the bottom surface of the shear plate whereas for the anti-flag mode the rotation axis is at the height of mass center of the reaction mass.

| Mode type FEM [kHz] | Measured [kHz] |
|---------------------|---------------|
| Fy-mode             | 9.96          |
| Fx-mode             | 12.87         |
| R-mode              | 23.28         |
| aFy-mode            | 38.36         |
| aFx-mode            | 50.51         |

(see table III), obtained according to methods described in section IV. The front face displacement of the modes are shown in Fig. 4.

Note that the agreement between calculated and measured Q-values increases with frequency. This may indicate an additional dissipation process which is missing in the model. Nevertheless, a small relative frequency offset $\Delta f$ below 1% in Table III indicates a good agreement of the FEM with measured values.

TABLE III. The Q-factor of the test mass modes without AMDs installed. The 4th and 7th column correspond to the relative frequency noise $\Delta f$ % and Q-factor ratio between calculated and measured value, respectively.

| Mode # | Freq. [Hz] Freq. [Hz] $\Delta f$ % Q-factor |
|--------|------------------|----------|
|        | FEM Measured    |          |
| 1      | 8128.3 8150.9   | 0.3      | 37M 1.9M 19.5 |
| 2      | 10391.1 10418.1 | 0.3      | 63M 14M 4.5  |
| 3      | 12999.1 12984.7 | 0.1      | 29M 15M 1.9  |
| 4      | 15101.4 15047.2 | 0.4      | 56M 17M 3.3  |
| 5      | 15151.0 15539.1 | 2.5      | 55M 16M 3.4  |
| 6      | 19487.0 19544.9 | 0.3      | 30M 7.0M 4.3 |
| 7      | 20113.6 20185.5 | 0.3      | 27M 13M 2.1  |
| 8      | 24824.0 24901.8 | 0.3      | 32M 16M 2.0  |
| 9      | 26504.3 26681.2 | 0.7      | 48M 18M 2.7  |
| 10     | 29767.4 29699.5 | 1.0      | 18M 15M 1.1  |
| 11     | 30912.1 31003.3 | 0.3      | 18M 12M 1.5  |
| 12     | 32664.1 32743.2 | 0.2      | 14M 13M 1.1  |

IV. EXPERIMENTAL RESULTS

Several AMD prototypes were constructed, each consisting of six components (see Table I):

**Reaction Mass**: A 10 g tungsten cylinder which tunes the AMD principle resonances to frequencies above 10 kHz is located on top of the shear plates.

**Shear Plate**: Two piezoelectric shear plates, oriented with perpendicular polarizations are glued to the reaction mass and base with conductive epoxy Tra-Duct2902. The epoxy serves to electrically connect the PZT electrodes to the reaction mass and base.

**Base**: The interface between flat shear plates and curved barrel of the test mass. The top flat surface is gold coated with separate sections to which shear plates are glued. The bottom surface is curved and match to the test mass barrel curvature. The base is made from fused silica and glued to the test mass with nonconductive epoxy.

**Shunting circuit**: One 10 kΩ resistor for each shear plate is glued to the reaction mass with conductive epoxy. The circuit is closed with 100 $\mu$m diameter silver coated copper wire, which is soldered to the resistor and gold coated surface of the base.

To measure Q-factors of the test mass modes, both with and without AMDs attached, a 16 m optical cavity was used, with the test mass forming the end mirror of the cavity. The optical cavity supports a 2 mm diameter Gaussian mode; the resonant beam probes the motion in the center of the test mass face, so that modes with an anti-node in the center of the mass are easily measured. A total of 12 modes were identified (Fig. 4) and measured, with the results shown in Table III.

In order to measure the impact of the AMD on the Q of the TMAMs, several modes of the test mass were excited using electrostatic actuators [17] and their ringdown times observed. This measurement was repeated in...
three configurations: with no AMD, with 2 AMDs, with 2 AMDs which lacked resistive damping (shunt wires cut).

Each acoustic mode was detected in the cavity locking error signal as a peak in the Fourier domain. After being excited, each mode amplitude was recorded as a function of time to estimate the decay time $\tau$. The Q-factor was determined from the decay time $\tau$, according to

$$Q = \pi f_0 \tau.$$  

(8)

where $f_0$ is the resonant frequency in Hz and $\tau$ is the exponential decay time constant.

![FIG. 5. Two acoustic mode dampers (AMDs) attached to the barrel of a suspended aLIGO test mass. The AMDs are on opposite sides of the test mass, $\sim 45^\circ$ down from the midline.](image)

Measurements of the test mass mode Q-factor performed after installing two AMDs clearly show the substantial damping capability of the AMD prototype, as reported in Table III (see also Table VII in Appendix for additional details).

The results indicate that out of 12 modes, 11 are suppressed by at least factor of 10 and in some cases by more than two orders of magnitude. The relatively large discrepancy between model and measurement for mode #2 can be explained by the AMD and TM interaction condition for this particular mode. The FEM predicts that mode #2 will be within 500 Hz of the AMD Fy-mode at 10 kHz, while the measured values give a separation of 800 Hz (mostly due to the AMD resonance being off), reducing the interaction between the AMD and TM mode.

In the off-resonance interactions, which are more common and set the lower limit to AMD performance, the discrepancy is generally less than a factor of a few.

Surprisingly, modes #2 and #6 show a counterintuitive behavior; opening the resistive circuit of the AMDs decreases the TMAM Q-factor. However, since electrical circuit of AMD affects mechanical stiffness of the shear plates it is expected that principle resonances also changed when the circuit is opened. Also, if the AMD resonance is close to the TMAM frequency, equation [4] indicates that the mode Q can be decreased by lowering the AMD loss.

The large Q reduction for modes #3 and #4 is due to the on-resonance interaction between TMAM and AMD, whereas for modes #5 and #12 the large Q reduction is due to the anti-node AMD location on the TMAM.

| Mode # | Freq. [Hz] | Damping factor | Resistive contribution [%] |
|--------|------------|----------------|---------------------------|
| 1      | 8150.9     | 32.2           | 12                        |
| 2      | 10418.1    | 31.8           | -57                       |
| 3      | 12984.7    | 441.2          | 21                        |
| 4      | 15047.2    | 81.0           | 9                         |
| 5      | 15539.1    | $>320$         | $>71$                     |
| 6      | 19544.9    | 15.6           | -2                        |
| 7      | 20185.5    | 13.8           | 41                        |
| 8      | 24901.8    | 5.2            | 4                         |
| 9      | 26681.2    | $>360$         | $>0$                      |
| 10     | 29699.5    | 23.8           | 43                        |
| 11     | 31003.3    | 307.7          | 68                        |
| 12     | 32743.2    | $>260$         | $>4$                      |

V. ANALYSIS OF THE AMD THERMAL NOISE

The AMD is designed to increase the mechanical damping of the test mass acoustic modes above 10 kHz. At the same time, the AMD must introduce minimal additional mechanical loss in the 10-1000 Hz band, where low mechanical loss is required to keep test mass thermal noise small [18]. Thus it is critical to calculate the thermal noise resulting from the AMDs in our overall evaluation of this PI mitigation technique.

We used the FEM described above to calculate the thermal noise resulting from our experimental test of two AMDs attached to a test mass. The AMD thermal noise was calculated numerically at 100 Hz, the most sensitive part of the detection band, using Levin’s approach [19]. The energy dissipation per cycle was computed using Eqn. A15 for a pressure profile corresponding to the Advanced LIGO geometry (laser beam radius $\omega_0 = 5.5$ cm, incident on the front face of the TM). Results are shown in Table V.

The test mass thermal noise level of $5.2 \times 10^{-21}$ m/$\sqrt{\text{Hz}}$ corresponds to the design level for Advanced LIGO, and is dominated by optical coating loss [20, 21]. As Table V shows, while our prototype AMD would contribute significantly to thermal noise it is not orders of magnitude above the more fundamental sources of thermal noise.

Relative to the prototype device, our model points to several design and material improvements that can be
TABLE V. Thermal noise budget for test mass and acoustic mode damper prototype when attached to the test mass. The thermal noise was calculated for the laser beam spot size of $\omega = 5.5$ cm.

| Component          | Thermal noise @ 100 Hz $[10^{-21}m/\sqrt{Hz}]$ |
|--------------------|-----------------------------------------------|
| **Test Mass:**     |                                               |
| Substrate          | 0.8                                           |
| Optical coating    | 5.1                                           |
| Suspension ears    | 0.0                                           |
| Ears bond          | 0.6                                           |
| Total per TM       | **5.2**                                       |
| **AMD:**           |                                               |
| Reaction mass (RM)| 0.3                                           |
| Epoxy (RM-PZT)     | 4.8                                           |
| Shear plate (PZT-X)| **5.5**                                       |
| Shear plate (PZT-Y)| **5.5**                                       |
| Epoxy Base-PZT     | 7.2                                           |
| Base               | 0.0                                           |
| Epoxy Base-TM      | 6.0                                           |
| Total per AMD      | **13**                                        |

* A product of a structural+resistive loss angle. Note that a major contribution to that value comes from the structural loss, see Fig. 3.

made to significantly reduce the thermal noise impact. The major AMD thermal noise contributors are the epoxies used to bond the AMD elements, and structural loss in the piezo shear plates. The former can be improved with lower loss epoxy and thinner bond layers. The latter can be improved with a more suitable choice of piezo material.

Other design elements can also be altered. The mass of the reaction mass can be reduced to lower the thermal noise without significantly affecting the acoustic mode damping performance. Another modification would be to avoid alignment of the piezo shear plate polarization with the laser beam axis, to minimize the contribution of resistive loss to the thermal noise. These and other design optimizations will be explored in a future paper.

VI. CONCLUSION

Acoustic mode dampers represent a simple yet effective approach to damping parametric instabilities. The great advantage of this approach over active damping \cite{9} is that many test mass acoustic modes are effected simultaneously, and no further intervention is necessary. This is likely to be a critical feature in instruments that would otherwise suffer from multiple acoustic modes simultaneously excited by parametric instabilities.

The investigation presented here involved modeling and construction of a prototype AMD, which was shown to effectively damp test mass acoustic modes. The thermal noise associated with this prototype AMD was also computed. Though the prototype AMD does not meet the stringent thermal noise requirements of gravitational wave detectors, several design elements were identified for improvement, making this approach a viable solution to parametric instabilities.

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Appendix A: Piezoelectric material

Any piezoelectric material is strictly characterized by electromechanical properties. The fact that both electrical and mechanical properties are inseparable allows us to represent the dissipation process in a shunted piezoelectric material either as a Jonson heat or mechanical loss. For the purpose of this work we focus on dissipation process in terms of the mechanical loss.

From the stress-charge of the piezoelectric constitutive equation it is straightforward to derive the total induced stress in the shunted piezoelectric material 22. Assuming no external current plugged to the piezoelectric electrode we get

$$\sigma_{ij} = \left( E c_{ij} - i \omega e_{ij} Z_{TOT} A e_{ij} \right) S_{ij}, \quad (A1)$$

where $E$ is the mechanical stiffness matrix under constant electric field, $e$ and $e$ are the piezoelectric stress constant. The upper script $t$ refers to transpose operator 23. Electrode area is represented by matrix $A$, whereas thickness of the piezomaterial between electrodes by matrix $L$. The strain $S$ is a product of the acting stress $\sigma$ on piezo-element and charge accumulation in the piezoelement. Each bracket corresponds to the matrix dimension. It is assumed that piezoelectric element has a brick shape thus matrices $Z_{TOT}$, $A$, and $L$ are diagonal.

The total impedance in Eqn. $A1$ is inverse sum of the piezelement admittance under constant electric field $Y_e$ and the admittance of the external circuit $Y_{SU}$ connected to the piezoelement electrodes, thus

$$Z_{TOT} = (Y_e + Y_{SU})^{-1}. \quad (A2)$$

The admittance of piezoelement $Y_e$ is assumed to be exclusively capacitive. Hence,

$$Y_e = i \omega A c^S L^{-1} = i \omega C^S = i \omega C^T (e^T)^{-1} e^S, \quad (A3)$$

where $C$, $e$ is the capacitance and the dielectric constant matrices under constant strain $S$ and constant stress $T$, respectively. It is convenient to operate with $C^T$ since this quantity can be easily measured at stress free conditions and no shortened piezoelement electrodes.

According to Eqn. $A1$ if the piezoelement is integrated with a nonzero impedance electric circuit, the imaginary part stiffness tensor arises. The imaginary part can be interpreted as a dissipative component of the stiffness tensor. The total shunted stiffness matrix is

$$e^S_{SU} = e^E - i e^S Z_{TOT} A c^E L^{-1}. \quad (A4)$$

Because matrices $A$, $Z_{TOT}$, and $L$ are diagonal and knowing that $AL^{-1} = C^T (e^T)^{-1}$ Eqn. $A4$ can be written as

$$e^S_{SU} = e^E - i e^S Z (e^S)^{-1}, \quad (A5)$$

where $Z$ is the nondimensional impedance

$$Z = Z_{TOT} Y_e. \quad (A6)$$

Note, for zero impedance shunted circuit sets $Z$ to be a unity matrix $Z = I$ whereas for the nonzero impedance shunting $Z$ is a complex quantity. Using indexing Eqn. $A4$ can be written as

$$e^S_{ij} = e^E_{ij} - \frac{e_{ki} e_{kj} \tilde{Z}_{kk}}{e_{kk}}. \quad (A7)$$

where $k = 1, 2, 3$ and corresponds to the electrode position, $e^S$ is the dielectric permittivity of the piezoelectric material.

From Eqn. $A3$ and $A7$ becomes clear that loss can be control with the shunting circuit. Moreover, such loss is in fact a frequency dependent quantity with arbitrary shape of the loss curve dependent on shunting circuit.

For the shear plate with the single pair of electrodes only a stiffness matrix element $e_{55}^{SU}$ is affected by shunting resistor, thus

$$e_{55}^{SU} = e_{55}^E - \frac{e_{15} e_{15} \tilde{Z}_{11}}{e_{11}}, \quad (A8)$$

and the nondimensional complex impedance $\tilde{Z}_{kk}$ has a
single nonunity element

\[ \tilde{Z}_{11} = \frac{i\omega e^{S}_{11}RC_{T}^{T}}{(\omega e^{S}_{11}RC_{T}^{T})^{2} + (c_{11}^{T})^{2}} + \frac{\omega e^{S}_{11}RC_{T}^{T}}{(\omega e^{S}_{11}RC_{T}^{T})^{2} + (\epsilon_{11}^{T})^{2}}. \]  

(A9)

Since the stiffness matrix \( c^{su} \) is a complex quantity we can define the loss factor \( \eta \) as

\[ \eta = \frac{Im(c^{su})}{Re(c^{su})}, \]  

(A10)

and thus

\[ \eta_{kij} = \frac{Im(\tilde{Z}_{kk})\chi_{kij}}{Re(\tilde{Z}_{kk})\chi_{kij} + 1}, \]  

(A11)

where \( \chi_{kij} = e_{ki}e_{kj}(e_{ij}^{E}S_{k}^{S})^{-1} \). Index \( k \) is associated with the electrodes orientation, whereas indices \( i \) and \( j \) correspond to stress-strain directions in the stiffness matrix \( c^{su} \). The shunting loss factor of the shear plate becomes

\[ \eta_{155} = \frac{3(\tilde{Z}_{11})\chi_{155}}{Re(\tilde{Z}_{11})\chi_{155} + 1}. \]  

(A12)

This is the main loss mechanism based on which shear AMD operates. Note that according to Eqn. A8 the material stiffness (the real part of \( c_{55}^{st} \)) is also reduced and should be included in analysis.

1. Total mechanical loss angle

An assumption of the stiffness matrix being real in the absence of shunt is not sufficient for accurate estimation of energy dissipation. It becomes especially important in the thermal noise analysis, see Sec. [V].

For known material loss angle of the nonshunted piezoelement, the total loss factor of piezoelectric material can be defined as

\[ \eta_{tot}^{mat} = \frac{\tan(\phi_{ij}^{mat}) + Im(\tilde{Z}_{ii})\chi_{kij}}{1 + Re(\tilde{Z}_{ii})\chi_{kij}}, \]  

(A13)

where \( \tan(\phi_{ij}^{mat}) \) is the material loss factor matrix of piezoelement. The loss factor \( \tan(\phi_{ij}^{mat}) \) can be easily computed using Eqn. A10

All piezoelectric material has anisotropic structure what implies that strain energy dissipation in such material must depend on the piezoelement geometric shape. It is more convenient to use effective loss angle \( \phi_{eff} \) rather than the loss angle \( \phi \) for each stiffness matrix component. We define the energy dissipation per cycle in the piezoelement

\[ W_{dis} = 2\pi W_{st}\eta_{eff} \]  

(A14)

\[ = \int \text{Re}(S_{i}S_{j}^{*})\text{Re}(e_{ij}^{su})\eta_{ij}^{tot}dV \]  

(A15)

where \( W_{st} \) is the stored strain energy such that \( 2\pi W_{st} = \int \text{Re}(S_{i}S_{j}^{*})dV \), where \( S, T \) are the complex strain and stress, respectively and \( V \) is the volume of the piezoelement.

Since loss factor of the piezoelement is inverse of its Q-factor or equally a ratio of dissipated energy per cycle \( W_{dis} \) to energy stored \( W_{st} \) in the piezoelement, we can defined the effective noise factor \( \eta_{eff} \)

\[ \eta_{eff} = \frac{W_{dis}}{2\pi W_{st}} = \frac{\int \text{Re}(S_{i}S_{j}^{*})\text{Re}(e_{ij}^{su})\eta_{ij}^{tot}dV}{\int \text{Re}(S_{i}S_{j}^{*})dV}, \]  

(A16)

where \( S, T \) are the complex strain and stress, respectively and \( V \) is the volume of the piezoelement. This is a key equation in the finite element analysis which properly estimate contribution of piezoelement in the strain energy dissipation in the test mass.

In our analysis we assumed constant material intrinsic loss factor for all \( c^{E} \) elements equal to \( \tan(\phi_{155}) = 0.014 \) [24], what leads to

\[ \eta_{155}^{tot} = \frac{0.014 + \text{Im}(\tilde{Z}_{11})\chi_{155}}{1 + \text{Re}(\tilde{Z}_{11})\chi_{155}}. \]  

(A17)

Note, the remaining elements of shunting induced loss \( \eta_{kij}^{tot} \) are equal to the material structural loss factor. For the shear plate configuration \( \chi_{155} \) is simply a function of electromechanical coupling coefficient \( k_{15} \) and is equal to \( \chi_{155} = \frac{k_{15}^{2}}{1 - k_{15}^{2}} \). The total loss angle for the \( c_{55}^{st} \) stiffness element is shown in Fig.[2]

Appendix B: Additional Tables
### TABLE VI. Values for the aLIGO end test mass parameters used in this paper.

**Optical parameters:**
- End test mass transmissivity: 5.0 ppm
  - 1 layer each of Ta$_2$O$_5$/SiO$_2$ with thickness of 2.32 µm, and 3.49 µm, respectively

**Mechanical properties of optical coating and ear bond:**

|                         | Ta$_2$O$_5$ | SiO$_2$ | Ear bond |
|-------------------------|-------------|---------|----------|
| Young’s modulus         | 140 GPa     | 70 GPa  | 7.2 GPa  |
| Poisson ratio           | 0.23        | 0.17    | 0.17     |
| Density                 | 8300 kg/m$^3$ | 2201 kg/m$^3$ | 2202 kg/m$^3$ |
| Refractive index        | 2.06539     | 1.45    | -        |
| Loss angle$^a$          | 2.4·10$^{-4}$+$f$·1.8·10$^{-9}$ | 0.4·10$^{-4}$+$f$·1.1·10$^{-9}$ | 0.1 |

**Test mass dimensions:**
- Radius: 0.17 m
- Thickness: 0.2 m
- Flats width: 0.095 m
- Wedge angle: 0.07 deg
- Mass: 40 kg
- Loss angle$^b$f
  - 7.6·10$^{-12}$·$f$ $^0.77$
- Material: fused silica

$^a$ For the purpose of numerical analysis, a multilayer coating was reduced to double layers with a total thickness which corresponds to the sum of all 18/19 layers of Ta$_2$O$_5$/SiO$_2$ for the ETM.

$^b$f-frequency. Note, the difference in loss angles for the substrate, and for the optical coating made from fused silica. Losses were obtained from [25]. Additionally, coating loss angles were revised to the current measured values.

$^c$ Both test mass and suspension ears.

### TABLE VII. Results of the Q-factor measurement with attached AMDs for shunted and non-shunted cases.

| Mode # | Resistive | Open Circuit |
|--------|-----------|--------------|
|        | FEM Measured | FEM Measured |
| 1      | 52k       | 59k          | 70k | 67k |
| 2      | 7.9k      | 440k         | 9.9k | 280k |
| 3      | 23k       | 34k          | 23k | 43k |
| 4      | 420k      | 210k         | 510k | 230k |
| 5      | 2.4M      | <50k         | 2.2M | 170k |
| 6      | 1.9M      | 450k         | 4.8M | 440k |
| 7      | 1.1M      | 940k         | 1.6M | 1.6M |
| 8      | 6.7M      | 3.1M         | 9.5M | 3.2M |
| 9      | 49k       | <50k         | 64k | <50k |
| 10     | 1.9M      | 630k         | 3.3M | 1.1M |
| 11     | 61k       | 39k          | 116k | 120k |
| 12     | 25k       | <50k         | 260k | 52k |