Consequences of the noncompactness of the Lorentz group

Hans - Jürgen Schmidt

Universität Potsdam, Institut für Mathematik
Projektgruppe Kosmologie
D-14415 POTSDAM, PF 601553, Am Neuen Palais 10, Germany

Abstract

The following four statements have been proven decades ago already, but they continue to induce a strange feeling:

- All curvature invariants of a gravitational wave vanish - in spite of the fact that it represents a nonflat spacetime.
- The eigennullframe components of the curvature tensor (the Cartan "scalars") do not represent curvature scalars.
- The Euclidean topology in the Minkowski spacetime does not possess a basis composed of Lorentz–invariant neighbourhoods.
- There are points in the de Sitter spacetime which cannot be joined to each other by any geodesic.

We explain that our feeling is influenced by the compactness of the rotation group; the strangeness disappears if we fully acknowledge the noncompactness of the Lorentz group.

Output: Imaginary coordinate rotations from Euclidean to Lorentzian signature are very dangerous.

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1 Introduction

A topological space $X$ is compact iff each open cover contains a finite sub-cover. Equivalently one can say: $X$ is compact iff each sequence in $X$ possesses a converging subsequence. $SO(n)$, the $n$–dimensional rotation group, is compact, whereas $SO(n - 1, 1)$, the corresponding Lorentz group, fails to be compact. Nevertheless, one can simply switch from the Euclidean space $E^n$ to the Minkowski spacetime $M^n$ by replacing $x^n \rightarrow it$.

It is the aim of the present essay to show those points where the loss of compactness connected with this replacement has nontrivial consequences.

2 Gravitational waves

Let
\begin{equation}
    ds^2 = 2 du dv - a^2(u) dw^2 - b^2(u) dz^2
\end{equation}
with positive smooth functions $a$ and $b$. It represents a gravitational wave iff
\begin{equation}
    a \cdot \frac{d^2b}{du^2} + b \cdot \frac{d^2a}{du^2} = 0
\end{equation}
cf. e.g. the review [1]. Metric (2.1) represents a flat spacetime iff both $a$ and $b$ are linear functions.

Let $I$ be any curvature invariant of order $k$, i.e.,
\begin{equation}
    I = I(g_{ij}, R_{ijlm}, \ldots, R_{ijlm;i_{1} \ldots i_{k}})
\end{equation}
is a scalar which depends continuously on all its arguments; the domain of dependence is requested to contain the flat space, and $I(g_{ij}, 0, \ldots, 0) \equiv 0$.

It holds, cf. [2, 3]: For gravitational waves of type (2.1), $I$ identically vanishes. Moreover, one can prove that statement for all metrics (2.1) without requiring (2.2). The proof by calculating the components of the curvature tensor is possible but quite technical.

A very short and geometrical proof goes as follows: Apply a Lorentz boost in the $u - v$–plane, i.e., $u \rightarrow \lambda \cdot u$ and $v \rightarrow \lambda^{-1} \cdot v$ for any $\lambda > 0$. Then $a(u)$ is replaced by $a(\lambda \cdot u)$ and $b(u)$ by $b(\lambda \cdot u)$. In the limit $\lambda \rightarrow 0$, metric (2.1) has a unique limit: constant functions $a$ and $b$. It is the flat spacetime,
and so, \( I = 0 \) there. On the other hand, for all positive values \( \lambda \), \( I \) carries the same value. By continuity this value equals zero. q.e.d.

Why all trials failed to generalize the idea of this proof to the positive definite case? Because we need a sequence of Lorentz boosts which does not possess any accumulation points within \( SO(3, 1) \). Such a sequence does not exist in \( SO(4) \) because of compactness.

### 3 Cartan scalars

A variant [4] of the Newman–Penrose formalism uses projections to an eigen-nullframe of the curvature tensor to classify gravitational waves. The corresponding Cartan ”scalars” have different boost weights, and they represent curvature invariants for vanishing boost weight only. The non–vanishing Cartan ”scalars” for metric (2.1) have either non–vanishing boost weight or a discontinuity at flat spacetime. So, the GHP-formalism [4], cf. also [5], does not yield a contradiction to the statement in sect. 2.

Why do not exist analogous ”scalars” with different ”rotation weight” in the Euclidean signature case? Analysing the construction one can see: The boost weights appear because there is a nontrivial vector space isomorphic to a closed subgroup of \( SO(3, 1) \). For \( SO(4) \), however, it holds: Every closed subgroup is compact; in order that it is isomorphic to a vector space it is necessary that it is the trivial one–point space.

More geometrically this looks as follows, cf. [6]: Let \( v \in E^4 \) be a vector and \( g \in SO(4) \) such that \( g(v) \uparrow \downarrow v \), then it holds: \( g(v) = v \). In Minkowski space–time \( M^4 \), however, there exist vectors \( v \in M^4 \) and boosts \( h \in SO(3, 1) \) with \( h(v) \uparrow \downarrow v \) and \( h(v) \neq v \).

### 4 Lorentz–invariant neighbourhoods

There is no doubt that the Euclidean topology \( \tau \) is the adequate topology of the Euclidean space \( E^n \). However, controversies appear if one asks whether \( \tau \) is best suited for the Minkowski spacetime \( M^n \).

The most radical path in answering this question can be found in refs. [7, 8, 9]; it leads to a topology different from \( \tau \) which fails to be a normal one.
Here, we only want to find out in which sense one can say that $\tau$ is better adapted to $E^n$ than to $M^n$. From a first view they appear on an equal footing: Both $SO(n)$ and $SO(n - 1, 1)$ represent subgroups of the homeomorphism group of $\tau$.

The difference appears as follows: For $E^n$, the usual $\epsilon-$spheres form a neighbourhood–basis composed of $SO(n)$-invariant open sets. Moreover, each of these neighbourhoods has a compact closure. Let $U$ be any open neighbourhood with compact closure around the origin in $E^n$. For every $g \in SO(n)$, $U(g)$ is the set $U$ after rotation by $g$. Of course, $U(g)$ is also an open neighbourhood with compact closure around the origin in $E^n$. Let us define

$$V = \bigcup\{U(g) | g \in SO(n)\}$$

and

$$W = \bigcap\{U(g) | g \in SO(n)\}$$

It holds: Both $V$ and $W$ represent $SO(n)$-invariant neighbourhoods of the origin with compact closure. Analysing the proofs one can see: ”$V$ is a neighbourhood of the origin” and ”$W$ has compact closure” are trivial statements, whereas ”$W$ is a neighbourhood of the origin” and ”$V$ has compact closure” essentially need the compactness of $SO(n)$. For $M^n$, however, all these properties fail.

First: No point of $M^n$ possesses a neighbourhood–basis composed of $SO(n - 1, 1)$-invariant open sets.

Second: No $SO(n - 1, 1)$-invariant neighbourhood has a compact closure.

Third: Let $U$ be any open neighbourhood with compact closure around the origin in $M^n$. For every $g \in SO(n - 1, 1)$, $U(g)$ is also an open neighbourhood with compact closure around the origin in $M^n$. However, neither

$$V = \bigcup\{U(g) | g \in SO(n - 1, 1)\}$$

nor

$$W = \bigcap\{U(g) | g \in SO(n - 1, 1)\}$$

represent neighbourhoods of the origin with compact closure.
5 Geodesics

Now we analyze a statement (known already to de Sitter himself, cf. [2, 10]):
Inspite of the fact that the de Sitter spacetime is connected and geodetically complete, there are points in it which cannot be joined to each other by any geodesic.

Let us recall: For Riemannian spaces it holds: If the space is connected and geodetically complete, then each pair of points can be connected by a geodesic.

The proof for Riemannian spaces \( V_n \) goes as follows: Take one of its points as \( x \) and define \( M_x \subset V_n \) to be that set of points which can be reached from \( x \) by a geodesic. One can show that \( M_x \) is non-empty, open and closed. This implies \( M_x = V_n \).

But where does the corresponding proof fail when we try to generalize it to the de Sitter spacetime?

Let us recall: A geodetic \( \epsilon \)–ball is the exponentiated form of a rotation–invariant neighbourhood of the corresponding tangent space. For Riemannian spaces these geodetic \( \epsilon \)–balls form a neighbourhood basis - and just this is needed in the proof.

But where does it fail in detail? \( M_x \) ”non–empty, open and closed” would again imply \( M_x = V_n \). \( M_x \) ”non–empty” is trivially satisfied by \( x \in M_x \). So we can fail by proving ”open” or by proving ”closed”. It turns out, cf. [10], that \( M_x \) is neither open nor closed, and both properties fail by the lack of a neighbourhood basis consisting of geodetic \( \epsilon \)–balls.

So, if compared with sct. 4, we can see that it is again the noncompactness of the Lorentz group which produces the peculiarities.

6 Conclusion

A finite set in set theory, a bounded set in geometry, and a compact set in topology: these are corresponding fundamental notions.

What have we learned from the above analysis on compactness? Let us concentrate on the first point (sct. 2): The fact that non–isometric space-times exist which cannot be distinguished by curvature invariants is neither
connected with the fact that one of them is flat nor with the vanishing of the
curvature invariants, but, as we have seen, with the appearance of a Lorentz
boost which has a limit not belonging to $SO(3, 1)$ but producing a regular
metric there. So we have found the very recipe to construct several classes
of such spacetimes. Let us present one of them [6]:

For a positive $C^\infty$–function $a(u)$ let

$$ds^2 = \frac{1}{z^2}[2 \, du \, dv - a^2(u) \, dy^2 - dz^2]$$

In the region $z > 0$, $ds^2$ represents the anti-de Sitter space–time if and only
if $a(u)$ is linear in $u$. Now, let $d^2a/du^2 < 0$ and

$$\phi := \frac{1}{\sqrt{\kappa}} \int \left( -\frac{1}{a} \, \frac{d^2a}{du^2} \right)^{1/2} \, du$$

Then $\Box\phi = \phi,^i \phi^i = 0$ and $R_{ij} - \frac{\Lambda}{2} \, g_{ij} = \Lambda g_{ij} + \kappa T_{ij}$ with $\Lambda = -3$
and $T_{ij} = \phi,^i \phi,^j$. So $(ds^2, \phi)$ represents a solution of Einstein’s equation
with negative cosmological term $\Lambda$ and a minimally coupled massless scalar
field $\phi$. Let $I$ be a curvature invariant of order $k$. Then for the metric
$ds^2$, $I$ does not depend on the function $a(u)$. So $I$ takes the same value
both for linear and non–linear functions $a(u)$. This seems to be the first
example that non–isometric space–times with non–vanishing curvature scalar
cannot be distinguished by curvature invariants. And having the recipe the
construction of other classes is straightforwardly done.

The fact that the representation theory of the rotation groups $SO(n)$ and
the Lorentz groups $SO(n - 1, 1)$ is quite different is so well–known that we
did not repeat it here - we only want to mention that it is the compactness
of the first one which produces the difference.

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