Kinematic Methods of Designing Free Form Shells

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Abstract. The geometrical shell model is formed in light of the set requirements expressed through surface parameters. The shell is modelled using the kinematic method according to which the shell is formed as a continuous one-parameter set of curves. The authors offer a kinematic method based on the use of second-order curves with a variable eccentricity as a form-making element. Additional guiding ruled surfaces are used to control the designed surface form. The authors made a software application enabling to plot a second-order curve specified by a random set of five coplanar points and tangents.

1. Introduction
Frame and grid shell coatings are widely used in the modern architectural practice. Famous architects P. Nervi, O. Niemeyer, B. Fuller, N. Foster, etc. used free form shells in their art (figure 1). An open grid dome was used in the Reichstag reconstruction project in Berlin (figure 2). Shells have become particularly widespread in architecture and construction over the last two decades due to the implementation of computer-aided design methods and the appearance of new construction materials and technologies [1].

Grid shells demonstrate new approaches to form making. On the one hand, they can be used in extreme Hi-Tech architectural solutions. On the other hand, grid shells enable the creation of forms similar to natural ones. Such environmental style is topical in modern architecture. Various methods of form making simulating the natural relief and the living organism structure are considered in [2,3].

The design of new architectural forms is always a rational compromise of a free search and technological limits. All other conditions being equal, the preference is given to the simplest lines and
surfaces [4,5]. The problem of designing a shell, which frame is formed by conic section segments, is of a particular practical interest [6-9]. For, example, the building in Cincinnati is decoratively completed by elliptical arches (figure 3).

2. Topicality
Currently, the researches in the sphere of the architectural shell design have been noticeably shifted towards using of the computational geometry means and methods [10]. Nevertheless, at the formation of smooth surfaces resting on a closed contour the known kinematic methods can be more preferable than computational algorithms [11-13]. Surface design methods using second-order curves as form-making elements [14,15] are particularly important in the architectural design practice.

3. Problem Statement and Solving Procedure
We have a flat or a spatial closed contour $ABCD$ consisting of straight lines or second-order curves. We have to design a continuous smooth (without fractures) shell resting on this contour.

To solve the problem it is proposed to use a kinematic method, according to which curve $m$, called the generator, slides in guides $AD$ and $BC$ smoothly changing its shape from link $AB$ to link $CD$.

4. “Direct Wedge” Shell
A direct wedge is formed by the movement of a straight line generator along guides $l_0, e_0$. The straight line generator is parallel to plane $zy$ (figure 4). Let us prove that if guiding curve $e_0$ is an ellipse, the section of the direct wedge with a plane parallel to the ellipse plane will also be an ellipse.

Let us mark points $1_0$ and $2_0$ at the intersection of a random horizontal plane $z=h$ with generators $A_0M_0$ and $B_0N_0$. From the consideration of the two pairs of triangles $AKA_0$, $1H1_0$ and $AA_0M_0$, $BB_0N_0$ it follows that $AA_0/AA’=BB_0/BB’$, i.e. a range of points $A’, B’,..$ forms ellipse $e’$ obtained by an orthographic compression of guiding ellipse $e_0$ to plane $xz$. Whereas $1_0A’$ and $2_0B’$, the range of points $1_0, 2_0,…$ obtained in the wedge section by the horizontal plane $z=h$ forms an ellipse congruent to ellipse $e’$, which was to be proved.
5. Elliptical Dome on a Quadrangular Contour

Let us assume that we need to design a smooth shell on a rectangular foundation $TUVW$ passing through semi-ellipse $g_0$. The shell is formed by a parallel movement of generator line $e$ along guide $g_0$. Curve $e$ sliding along $g_0$ changes its shape from semi-ellipse $e_0$ to straight line $TW$ or $UV$ (figure 5).

The surface consists of two generator families. The first family consists of a set of semi-ellipses $e || zy$ resting on sides $TU$ and $VW$ of the foundation. In order to find the generators of the second family, let us consider the surface section with a random plane parallel to $zx$. Let us show that such plane crosses a grid of curves $e$ in points $1, 2, ...$ lying on ellipse $k$. In fact, projections $e_{03}$, $e_{3}^{'}, ...$ of curves $e_0$, $e'$ on the parallel plane to $zy$ form a pencil of conic sections with two pairs of coinciding base points. In this pencil random conic $e_3'$ is in a orthogonal and perspective correspondence with projection $e_{03}$ of ellipse $e_0$, therefore, corresponding straight lines $M_{03}$, $N_{03}$, ..., intersect in one and the same point $H$ on the kinship axis passing through the double base points of the pencil. Curves $e_0$, $e'$ and their projections $e_{03}$, $e_{3}^'$ are congruent, therefore, points $A_0$, $B_0$, $H$ lie on one straight line $l_0$, and straight lines $M_{03}A_0$, $N_{03}B_0$, ..., are the direct wedge generators with guides $g_0$ and $l_0$. However, in cl. 3.1 it has been shown that an ellipse is formed in the wedge surface section with a plane parallel to the plane of guiding ellipse $g_0$. Consequently, curve $k$ bearing a range of points $1, 2, ...$, is an ellipse. A set of ellipses parallel to plane $xz$ comprises the second family of lines of the surface frame. The surface is formed by the two families of elliptic arcs, therefore, it can be called an elliptical dome. The dome smoothly mates with the foundation walls. The dome section with plane $z=0$ falls into two pairs of parallel straight lines: $x=\pm a$, $y=\pm b$, which corresponds to the shape of a rectangular foundation. In vertical plane sections $x=a$, $y=b$ ($a\leq a_0$, $b\leq b_0$) we obtain two families of ellipses with a variable eccentricity. In a small neighborhood of the angular point the dome surface is close to the second-order conical surface [16].

![Figure 5. Dome on a rectangular contour.](image5)

![Figure 6. Dome integrated into a pyramid.](image6)
The random projective transformation preserves the order of algebraic curves. Let us perform the projective transformation, at which the improper (ideal) point of crossing the foundation’s vertical edges is transformed into proper point S, and the foundation remains rectangular. Such transformation preserves a smooth mating of the dome with the foundation walls, which turn into inclined from vertical ones. At the same time, the elliptical dome touches the side planes of the pyramid (figure 6).

The planes of the frame lines form two plane pencils. The planes of the first pencil pass through top S of the pyramid and improper crossing point G∞ of the foundations’ opposite sides TU and VW. The planes of the second pencil pass through S and E∞=UV∩TW. Making the random projective transformation, we obtain an elliptical dome integrated into an irregular quadrangular pyramid. The planes of the family of frame lines ei pass through top S and point E=TW∩UV forming a pencil of planes ε with axis ES. The planes of the family of frame lines gj pass through points S and G=TU∩VW, forming pencil γ with axis GS (figure 7).

![Figure 7. Dome on a flat contour.](image)

![Figure 8. Free form convex-concave shell.](image)

Let us consider an algorithm of building a dome integrated into pyramid TUVWS and passing though this point M. The planes of pencils ε and γ incident to M cross the pyramid by triangles 12S and 34S. Let us draw frame lines (second-order curves) e and g passing through M resting on the foundation sides in the pairs of points 12, 34 and touching the sides of triangles 12S and 34S. In random plane Σ of pencil ε we find frame line e, passing though point M=γ∩Σ and touching the sides of the triangle obtained in the section of the pyramid with plane Σ. A set of planes of pencil ε induces a one-parameter family of second-order curves ei, which form the frame of an elliptical dome integrated into this pyramid. The second family gj of the dome generators is formed analogously.

A random generator is drawn with a Computer Conicograph special software application as a second-order curve passing through three points and touching two straight lines [17,18]. For example, curve e passes through points M, 1, 2 and touches sides S-1, S-2 of triangle S12.

### 6. Free Form Shell

Let us design a shell strained on spatial contour ABCD formed by the arches of second-order curves (figure 8). Let us use a second-order curve as a generator. As opposed to the known geometric modeling methods, in order to control the shell form we specify two guiding conoids η and ψ contacting with the designed surface by guiding links AD and BC. Let us show that additional guide l chosen from technological considerations fully defines the generator shape in any section y=ut parallel to xz. In fact, random section y=const corresponds to two points on side links AD, BC, two tangents in these points (lines of intersection of connoids η, ψ with plane y=const) and a point on additional guide l. Five geometrical elements (points and tangents) fully define the single second-order curve in plane
$y=u$. Changing parameter $u$, we obtain a one-parameter set of second-order curves, which form a continuous shell frame strained on the set contour $ABCD$.

7. Practical Applications

In the design practice it is often necessary to solve the problem of building a smooth compound curve passing through the set points. Smooth curves are widely used in architecture and construction in the design of diffusers and air conduits. A prior use of second-order curves to solve the form making problems is explained by a good knowledge of these curves and their building simplicity. As opposed to algebraic splines, arches of second-order curves have a simple analytical description. A convex curve consisting of arches of second-order curves has a second-order smoothness (a continuous curvature change along the compound curve) sufficient for most practical applications.

7.1. Diffuser

It is known that diffusers with a curved generator are better than those with a straight line generator by technical and economic indices [19]. The diffuser profile is calculated by the numerical methods of gas dynamics. As a result of the calculation we obtain a discrete range of points $1, \ldots, 7$ defining its curved generator. The generator’s characteristic points specify the directions of flow lines $t_2, t_5, t_7$ (figure 9). It is necessary to design a smooth typical bypass passing through these points and having these tangents. Points $1, 3$ and tangents $t_1, t_2$ define the arch of the second-order curve (first link of the bypass). Points $3, 4, 5$ with the tangents specified in points $3, 5$ define the second link of the bypass. Points $5, 6, 7$ with tangents $t_5, t_7$ define the third link. Thus, we obtained a smooth curved generator of the diffuser consisting of three sections of second-order curves with common tangents in the junction points [20].

7.2. Air intake

At the design of air intakes it is necessary to enable a smooth transfer from quadrangular inlet $ABCD$ to circular outlet section $r$ (figure 10). This problem can be solved by the Liming method. The method is based on using conic section segments meeting the conditions of matching in several points. A two-fold symmetrical smooth curve consisting of units of second-order curves is formed in a random horizontal section of the air intake. The section shape depends on the position of the control points on the vertically-aligned flow lines. It results in a smooth transfer from a rectangular to a circular cross-section [10].

Figure 9. Diffuser.  Figure 10. Air intake.

8. Conclusion

The authors offered a kinematic method of geometrical modeling of free form shells characterized by the use of the guiding ruled surface to control the shell form. According to the offered method, the shell is formed as a one-parameter set of variable eccentricity conic section segments.
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