One-Loop Flatness of Membrane Fuzzy Sphere Interaction in Plane-Wave Matrix Model

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Abstract

In the plane-wave matrix model, the background configuration of two membrane fuzzy spheres, one of which rotates around the other one in the $SO(6)$ symmetric space, is allowed as a classical solution. We study the one-loop quantum corrections to this background in the path integral formulation. Firstly, we show that each fuzzy sphere is stable under the quantum correction. Secondly, the effective potential describing the interaction between fuzzy spheres is obtained as a function of $r$, which is the distance between two fuzzy spheres. It is shown that the effective potential is flat and hence the fuzzy spheres do not feel any force. The possibility on the existence of flat directions is discussed.

Keywords : pp-wave, Matrix model, Fuzzy sphere
PACS numbers : 11.25.-w, 11.27.+d, 12.60.Jv
1 Introduction

The plane-wave matrix model \( [1] \) is a microscopic description of the discrete light cone quantized (DLCQ) M-theory in the eleven-dimensional \( pp \)-wave or plane-wave background. The eleven-dimensional plane-wave \( [2] \) is maximally supersymmetric and the limiting case of the eleven-dimensional \( AdS \) type geometries \( [3] \). Its explicit form is given by

\[
\begin{align*}
\text{ds}^2 &= -2dx^+ dx^- - \left( \sum_{i=1}^{3} \left( \frac{\mu}{3} \right)^2 (x^i)^2 + \sum_{a=4}^{9} \left( \frac{\mu}{6} \right)^2 (x^a)^2 \right) (dx^+)^2 + \sum_{I=1}^{9} (dx^I)^2, \\
F_{+123} &= \mu,
\end{align*}
\]

where \( I = (i, a) \). Due to the effect of the ++ component of the metric and the presence of the four-form field strength, the plane-wave matrix model has some \( \mu \) dependent terms, which make the difference between the usual flat space matrix model and the plane-wave one.

The presence of the \( \mu \) dependent terms makes the plane-wave matrix model have some peculiar properties. One of them is that there are various vacuum structures classified by the \( SU(2) \) algebra \( [4] \). The crucial ingredient for vacua is the membrane fuzzy sphere. It preserves the full 16 supersymmetries of the plane-wave matrix model and exists even at finite \( N \) which is the size of matrix.

After the plane-wave matrix model was proposed and its basic aspects were uncovered, there have been lots of investigations in various directions. The structure of vacua has been studied in more detail especially related to the protected multiplet \( [4, 5, 6] \). The possible BPS objects contained in the plane-wave matrix model have been searched \( [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] \). The algebraic and structural study of the model itself and the various BPS objects present in it has been performed \( [4, 8, 9, 18, 19, 20, 21, 22, 23, 24, 25] \). Based on the fact that the low energy description of the M theory is the eleven dimensional supergravity, there also have been supergravity side analysis \( [26, 27, 28] \).

If the M theory is compactified on a circle, then we have ten-dimensional Type IIA string theory. Under the circle compactification, the \( pp \)-wave geometry \( [11] \) becomes the IIA \( pp \)-wave background which is not maximally supersymmetric and has 24 supersymmetries \( [29, 30, 31, 32] \). For the purpose of understanding the plane-wave matrix model as well as the string theory itself, the IIA string theory in the \( pp \)-wave background has been also extensively studied, in parallel with the progress in the study of the plane-wave matrix model \( [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 11] \).

However, despite of quite amount of progress in the study of the plane-wave matrix model, there has been lack of the investigation about the dynamical aspects. However, see
for example \[12\]. In fact, the present status of the plane-wave matrix model enables us to study the dynamics of the model. In this paper, we consider the basic objects of the plane-wave matrix model and study their interaction.

For the interacting objects, we take two membrane fuzzy spheres, both of which are supersymmetric. In the $SO(3)$ symmetric subspace which one may see in the $pp$-wave background \[11\], two fuzzy spheres are taken to be at the origin. In the $SO(6)$ symmetric space, one fuzzy sphere is located at the origin, while the other fuzzy sphere is taken to rotate around the origin with a fixed distance. It should be noted that this configuration is allowed as a classical solution of the equations of motion and furthermore the rotating fuzzy sphere itself is supersymmetric. We will evaluate the one-loop corrections to the configuration and obtain the effective potential. As we will see, the effective potential is flat. This implies that the whole configuration of fuzzy spheres is also supersymmetric. One may argue that the flat potential is natural since each fuzzy sphere configuration is supersymmetric. However, the situation is unconventional from the viewpoint of the flat space matrix model \[13\] and indicates one of intriguing properties of the plane-wave matrix model. Moreover, the flat potential shows us the possibility that the plane-wave matrix model has the flat directions which have not been observed in it.

The organization of this paper is as follows. In the next section, we give the action of the plane-wave matrix model and consider its classical solutions focused on our concern. The expansion of the action around a given arbitrary background is given in section 3. In section 4, we set up the background configuration and consider the fluctuations around it. In section 5, the one-loop stability of each fuzzy sphere is checked for arbitrary size. In section 6, we evaluate the path integration of fluctuations responsible for the interaction between fuzzy spheres. It will be shown that the one-loop effective potential is flat. Thus, the fuzzy spheres do not feel any force. Finally, conclusion and discussion will be given in section 7. We discuss the possibility on the existence of flat directions.

## 2 Plane-wave matrix model and classical solutions

The plane-wave matrix model is basically composed of two parts. One part is the usual matrix model based on eleven-dimensional flat space-time, that is, the flat space matrix model, and another is a set of terms reflecting the structure of the maximally supersymmetric eleven dimensional plane-wave background, Eq. \[11\]. Its action is

$$S_{pp} = S_{\text{flat}} + S_{\mu} ,$$

(2.1)
where each part of the action on the right hand side is given by

\[
S_{\text{flat}} = \int dt \text{Tr} \left( \frac{1}{2R} D_t X^I D_t X^I + \frac{R}{4} ([X^I, X^J])^2 + i \Theta^I D_t \Theta - R \Theta^I \gamma^I [\Theta, X^I] \right),
\]

\[
S_\mu = \int dt \text{Tr} \left( -\frac{1}{2R} \left( \frac{\mu}{3} \right)^2 (X^i)^2 - \frac{1}{2R} \left( \frac{\mu}{6} \right)^2 (X^a)^2 - i \frac{\mu}{3} \epsilon^{ijk} X^i X^j X^k - i \frac{\mu}{4} \Theta^I \gamma^{123} \Theta \right). \tag{2.2}
\]

Here, \( R \) is the radius of circle compactification along \( x^- \) and \( D_t \) is the covariant derivative with the gauge field \( A \),

\[
D_t = \partial_t - i[A, \cdot]. \tag{2.3}
\]

For dealing with the problem in this paper, it is convenient to rescale the gauge field and parameters as

\[
A \to RA, \quad t \to \frac{1}{R} t, \quad \mu \to R\mu. \tag{2.4}
\]

With this rescaling, the radius parameter \( R \) disappears and the actions in Eq. (2.2) become

\[
S_{\text{flat}} = \int dt \text{Tr} \left( \frac{1}{2} D_t X^I D_t X^I + \frac{1}{4} ([X^I, X^J])^2 + i \Theta^I D_t \Theta - \Theta^I \gamma^I [\Theta, X^I] \right),
\]

\[
S_\mu = \int dt \text{Tr} \left( -\frac{1}{2} \left( \frac{\mu}{3} \right)^2 (X^i)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 (X^a)^2 - i \frac{\mu}{3} \epsilon^{ijk} X^i X^j X^k - i \frac{\mu}{4} \Theta^I \gamma^{123} \Theta \right). \tag{2.5}
\]

The possible backgrounds allowed by the plane-wave matrix model are the classical solutions of the equations of motion for the matrix fields. Since the background that we are concerned about is purely bosonic, we concentrate on solutions of the bosonic fields \( X^I \). We would like to note that we will not consider all possible solutions but only those relevant to our interest for the fuzzy sphere interaction. Then, from the rescaled action, (2.5), the bosonic equations of motion are derived as

\[
\ddot{X}^i = -[[X^i, X^I], X^I] - \left( \frac{\mu}{3} \right)^2 X^i - i \mu \epsilon^{ijk} X^j X^k,
\]

\[
\ddot{X}^a = -[[X^a, X^I], X^I] - \left( \frac{\mu}{6} \right)^2 X^a, \tag{2.6}
\]

where the over dot implies the time derivative \( \partial_t \).

Except for the trivial \( X^I = 0 \) solution, the simplest one is the simple harmonic oscillator solution;

\[
X^i_{\text{osc}} = A^i \cos \left( \frac{\mu}{3} t + \phi_i \right) 1_{N \times N}, \quad X^a_{\text{osc}} = A^a \cos \left( \frac{\mu}{6} t + \phi_a \right) 1_{N \times N}, \tag{2.7}
\]

where \( A^I \) and \( \phi_I \) (\( I = (i, a) \)) are the amplitudes and phases of oscillations respectively, and \( 1_{N \times N} \) is the \( N \times N \) unit matrix. This oscillatory solution is special to the plane-wave matrix
model due to the presence of mass terms for $X^I$. It should be noted that, because of the mass terms, the configuration corresponding to the time dependent straight line motion, say $v^I t + c^I$ with non-zero constants $v^I$ and $c^I$, is not possible as a solution of (2.6), that is, a classical background of plane-wave matrix model, contrary to the case of the flat space matrix model. As the generalization of the oscillatory solution, Eq. (2.7), we get the solution of the form of diagonal matrix with each diagonal element having independent amplitude and phase.

As for the non-trivial constant matrix solution, Eq. (2.6) allows the following membrane fuzzy sphere or giant graviton solution:

$$X^i_{\text{sphere}} = \frac{\mu}{3} J^i,$$  \hspace{1cm} (2.8)

where $J^i$ satisfies the $SU(2)$ algebra,

$$[J^i, J^j] = i \epsilon^{ijk} J^k.$$  \hspace{1cm} (2.9)

The reason why this solution is possible is basically because of the fact that the matrix field $X^i$ feels an extra force due to the Myers interaction which may stabilize the oscillatory force. The fuzzy sphere solution $X^i_{\text{sphere}}$ preserves the full 16 dynamical supersymmetries of the plane-wave and hence is 1/2-BPS object. We note that actually there is another fuzzy sphere solution of the form $\frac{\mu}{6} J^i$. However, it has been shown that such solution does not have quantum stability and is thus non-BPS object \[12\].

3  Matrix model expansion around general background

In this section, the plane-wave matrix model is expanded around the general bosonic background, which is supposed to satisfy the classical equations of motion, Eq. (2.6).

We first split the matrix quantities into as follows:

$$X^I = B^I + Y^I, \quad \Theta = F + \Psi,$$  \hspace{1cm} (3.1)

where $B^I$ and $F$ are the classical background fields while $Y^I$ and $\Psi$ are the quantum fluctuations around them. The fermionic background field $F$ is taken to vanish from now on, since we will only consider the purely bosonic background. The quantum fluctuations are the fields subject to the path integration. In taking into account the quantum fluctuations, we should recall that the matrix model itself is a gauge theory. This implies that the gauge fixing condition should be specified before proceed further. In this paper, we take the background field gauge which is usually chosen in the matrix model calculation as

$$D^a_{\mu} A_{\text{qu}}^\mu \equiv D_t A + i [B^I, X^I] = 0.$$  \hspace{1cm} (3.2)
Then the corresponding gauge-fixing \( S_{GF} \) and Faddeev-Popov ghost \( S_{FP} \) terms are given by

\[
S_{GF} + S_{FP} = \int dt \, \text{Tr} \left( -\frac{1}{2} (D^a_{\mu} A^a_{\mu})^2 - \bar{C} \partial_t D_t C + [B^I, \bar{C}][X^I, C] \right).
\] (3.3)

Now by inserting the decomposition of the matrix fields (3.1) into Eqs. (2.5) and (3.3), we get the gauge fixed plane-wave action \( S \equiv S_{pp} + S_{GF} + S_{FP} \) expanded around the background. The resulting acting is read as

\[
S = S_0 + S_2 + S_3 + S_4,
\] (3.4)

where \( S_n \) represents the action of order \( n \) with respect to the quantum fluctuations and, for each \( n \), its expression is

\[
S_0 = \int dt \, \text{Tr} \left[ \frac{1}{2} (\dot{B}^I)^2 - \frac{1}{2} \left( \frac{\mu}{3} \right)^2 (B^i)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 (B^a)^2 + \frac{1}{4} ([B^I, B^J])^2 - \frac{i \epsilon^{ijk} B^i B^j B^k}{3} \right],
\]

\[
S_2 = \int dt \, \text{Tr} \left[ \frac{1}{2} (\dot{Y}^I)^2 - 2i \dot{B}^I [A, Y^I] + \frac{1}{2} ([B^I, Y^J])^2 + [B^I, B^J][Y^I, Y^J] - i \epsilon^{ijk} B^i Y^j Y^k
\]

\[
- \frac{1}{2} \left( \frac{\mu}{3} \right)^2 (Y^i)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 (Y^a)^2 + i \Psi \dot{\Psi} - \Psi \gamma^I [\Psi, B^I] - \frac{i \mu}{4} \Psi \gamma^{123} \Psi
\]

\[
- \frac{1}{2} \dot{A}^2 - \frac{1}{2} ([B^I, A])^2 + \dot{C} \dot{C} + [B^I, \bar{C}][B^I, C] \right],
\]

\[
S_3 = \int dt \, \text{Tr} \left[ -i \dot{Y}^I [A, Y^I] - [A, B^I][A, Y^I] + [B^I, Y^J][Y^I, Y^J] + \Psi [A, \Psi]
\]

\[
- \Psi \gamma^I [\Psi, Y^I] - \frac{i \mu}{3} \epsilon^{ijk} Y^i Y^j Y^k - i \dot{C} [A, C] + [B^I, \bar{C}][Y^I, C] \right],
\]

\[
S_4 = \int dt \, \text{Tr} \left[ -\frac{1}{2} ([A, Y^I])^2 + \frac{1}{4} ([Y^I, Y^J])^2 \right].
\] (3.5)

Some comments are in order for the background gauge choice, Eq. (3.2). One advantage of this gauge choice is that the quadratic part of the action in terms of fluctuations, that is, the quadratic action, is simplified. In more detail, there appears the term \(-\frac{1}{2} ([B^I, Y^I])^2 \) in the expansion of the potential \( \frac{1}{4} ([X^I, X^J])^2 \), which is canceled exactly by the same term with the opposite sign coming from the gauge fixing term of Eq. (3.3) and hence absent in \( S_2 \) of Eq. (3.5). This cancellation has given some benefits in the actual flat space matrix model calculation. This is also the case in the present plane-wave matrix model, except however for the potential of \( Y^i \). As we will see later, \(-\frac{1}{2} ([B^I, Y^I])^2 \) is responsible for completing the \( Y^i \) potential into the nice square form. Thus the same term with the opposite sign from the gauge fixing term remains in the quadratic action. At later stage, the presence of this term will have an important implication in taking into account of the unphysical gauge degrees of freedom which are eventually eliminated by those of ghosts.
4 Fuzzy sphere configuration and fluctuations

We now set up the background configuration for the membrane fuzzy spheres. Since we will study the interaction of two fuzzy spheres, the matrices representing the background have the $2 \times 2$ block diagonal form as

$$B^I = \begin{pmatrix} B^I_{(1)} & 0 \\ 0 & B^I_{(2)} \end{pmatrix},$$

(4.1)

where $B^I_{(s)}$ with $s = 1, 2$ are $N_s \times N_s$ matrices. If we take $B^I$ as $N \times N$ matrices, then $N = N_1 + N_2$.

The two fuzzy spheres are taken to be static in the space where they span, and hence represented by the classical solution, (2.8);

$$B_i^{(s)} = \mu_j J_i^{(s)},$$

(4.2)

where, for each $s$, $J_i^{(s)}$ is in the $N_s$-dimensional irreducible representation of $SU(2)$ and satisfies the $SU(2)$ algebra, Eq. (2.9). In the $SO(6)$ symmetric transverse space, the fuzzy spheres are regarded as point objects, of course, in a sense of ignoring the matrix nature.

We first let the second fuzzy sphere given by the background of $s = 2$ be at the origin in the transverse space and stay there. As for the first fuzzy sphere, it is made to move around the second sphere in the form of circular motion with the radius $r$. Obviously, this configuration is one of the classical solutions of the equations of motion as one can see from Eq. (2.7). Recalling that the transverse space is $SO(6)$ symmetric, all the possible choices of two-dimensional sub-plane where the circular motion takes place are equivalent. Thus, without loss of generality, we can take a certain plane for the circular motion. In this paper, the $x^4$-$x^5$ plane is chosen. Then the configuration in the transverse space is given by

$$B^4_{(1)} = r \cos \left( \frac{\mu}{6} t \right) 1_{N_1 \times N_1}, \quad B^5_{(1)} = r \sin \left( \frac{\mu}{6} t \right) 1_{N_1 \times N_1}.$$

(4.3)

Eqs. (4.2) and (4.3) compose the background configuration about which we are concerned, and all other elements of matrices $B^I$ are set to zero. We would like to note that not only the fuzzy sphere at the origin given by $B^I_{(2)}$ but also the rotating one, $B^I_{(1)}$, is supersymmetric [14]. A schematic view of the background configuration is presented in Fig. 1.

If we evaluate the classical value of the action for this background, it is zero;

$$S_0 = 0.$$

(4.4)

From now on, we are going to compute the one-loop correction to this action, that is, to the background, (4.2) and (4.3), due to the quantum fluctuations via the path integration of the
Figure 1: Schematic view of the configuration for two membrane fuzzy spheres. The plane of the circular motion is $x^4$-$x^5$ plane, and the fuzzy spheres are actually points in this plane.

quadratic action $S_2$, and obtain the one-loop effective action $\Gamma_{\text{eff}}$ or the effective potential $V_{\text{eff}}$ as a function of $r$, the radius of the circular motion.

For the justification of one-loop computation or the semi-classical analysis, it should be made clear that $S_3$ and $S_4$ of Eq. (3.5) can be regarded as perturbations. For this purpose, following [4], we rescale the fluctuations and parameters as

$$A \rightarrow \mu^{-1/2} A, \quad Y^I \rightarrow \mu^{-1/2} Y^I, \quad C \rightarrow \mu^{-1/2} C, \quad \bar{C} \rightarrow \mu^{-1/2} \bar{C},$$

$$r \rightarrow \mu r, \quad t \rightarrow \mu^{-1} t. \quad (4.5)$$

Under this rescaling, the action $S$ in the background (4.2) and (4.3) becomes

$$S = S_2 + \mu^{-3/2} S_3 + \mu^{-3} S_4, \quad (4.6)$$

where $S_2$, $S_3$ and $S_4$ do not have $\mu$ dependence. Now it is obvious that, in the large $\mu$ limit, $S_3$ and $S_4$ can be treated as perturbations and the one-loop computation gives the sensible result.

Based on the structure of (4.1), we now write the quantum fluctuations in the $2 \times 2$ block matrix form as follows.

$$A = \begin{pmatrix} Z_0^0 & \Phi^0 \\ \Phi_0^f & Z_0^2 \end{pmatrix}, \quad Y^I = \begin{pmatrix} Z_I^0 & \Phi^I \\ \Phi_I^f & Z_I^2 \end{pmatrix}, \quad C = \begin{pmatrix} C^1_C \\ C^2 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{C}^1 & \bar{C}^2 \end{pmatrix}. \quad (4.7)$$

Although we denote the block off-diagonal matrices for the ghosts by the same symbols with those of the original ghost matrices, there will be no confusion since $N \times N$ matrices will never appear in what follows. The above form of matrices is convenient, since the block
diagonal and block off-diagonal parts decouple from each other in the quadratic action and thus can be taken into account separately at one-loop level;

\[ S_2 = S_{\text{diag}} + S_{\text{off-diag}}, \quad (4.8) \]

where \( S_{\text{diag}} \) (\( S_{\text{off-diag}} \)) implies the action for the block (off-) diagonal fluctuations. We note that there is no \( \mu \) parameter in \( S_{\text{diag}} \) and \( S_{\text{off-diag}} \) due to the above rescaling (4.5).

5 Stability of fuzzy sphere

In this section, the path integration of \( S_{\text{diag}} \) is performed. The resulting effective action will enable us to check the one-loop stability of the fuzzy sphere configuration, Eqs. (4.2) and (4.3). In fact, the background encoding the circular motion, Eq. (4.3), does not contribute to \( S_{\text{diag}} \) basically because of the fact that it is proportional to the identity matrix. This leads to the situation that the quantum fluctuations around one fuzzy sphere do not interact with those around another fuzzy sphere, and the effective action is just the sum of that for each fuzzy sphere. This can then be stated as

\[ S_{\text{diag}} = \int dt \sum_{s=1}^{2} (L_B^{(s)} + L_F^{(s)} + L_G^{(s)}), \quad (5.1) \]

where three Lagrangians, \( L_B^{(s)} \), \( L_F^{(s)} \), and \( L_G^{(s)} \), are those for the bosonic, fermionic, and ghost fluctuations respectively around the \( s \)-th fuzzy sphere. We note that, at quadratic level, there is no mixing between these three kinds of fluctuations.

5.1 Bosonic fluctuation

We first evaluate the path integral of bosonic fluctuations. The corresponding Lagrangian is given by

\[
L_B^{(s)} = \frac{1}{2} \text{Tr} \left[ -\left( \dot{Z}_i^{(s)} \right)^2 - \frac{1}{32} \left( [J_i^{(s)}, Z_i^{(s)}] \right)^2 + \left( \dot{Z}_i^{(s)} \right)^2 - \frac{1}{32} \left( Z_i^{(s)} + i\epsilon^{ijk} [J_j^{(s)}, Z_k^{(s)}] \right)^2 \\
+ \frac{1}{32} \left( [J_i^{(s)}, Z_i^{(s)}] \right)^2 + \left( \dot{Z}_a^{(s)} \right)^2 - \frac{1}{32} \left( Z_a^{(s)} \right)^2 - \frac{1}{4} \left( [J_i^{(s)}, Z_i^{(s)}] \right)^2 \right], \quad (5.2)
\]

where, as alluded to at the end of section 4, the quadratic term of \( [J_i^{(s)}, Z_i^{(s)}] \) coming from the gauge fixing term appears because the same term with opposite sign has been used for making the complete square form \( (Z_i^{(s)} + i\epsilon^{ijk} [J_j^{(s)}, Z_k^{(s)}])^2 \).

In order to perform the actual path integration, it is useful to diagonalize the fluctuation matrices and obtain the mass spectrum. By the way, since the diagonalization itself has
been already given in [4], we will be brief in its presentation and present only the essential points.

The starting point is the observation that the mass terms are written in terms of the commutators with $J_i^{(s)}$ satisfying Eq. [2,9], the $SU(2)$ algebra. This indicates that we can use the representation theory of $SU(2)$ for the diagonalization. We regard an $N_s \times N_s$ matrix as an $N_s^2$-dimensional reducible representation of $SU(2)$, which decomposes into irreducible spin $j$ representations with the range of $j$ from 0 to $N_s - 1$, that is, $N_s^2 = 1 \oplus 3 \oplus \cdots \oplus (2N_s - 1)$. Based on this decomposition, an $N_s \times N_s$ matrix may be expanded as

$$Z^{(s)} = \sum_{j=0}^{N_s-1} \sum_{m=-j}^{j} z_{(s)jm}Y^{(s)}_{jm},$$

(5.3)

where the $N_s \times N_s$ matrix $Y^{(s)}_{jm}$ is the matrix spherical harmonics transforming in the irreducible spin $j$ representation and $z_{(s)jm}$ is the corresponding spherical mode. We note that $z_{(s)jm}$ satisfies the following reality condition since the fluctuation matrices are Hermitian.

$$z^*_{(s)jm} = (-1)^m z_{(s)j-m}.$$  

(5.4)

The expansion of matrices in the $SU(2)$ language enables us to use the properties of the $SU(2)$ generators, which are given by

$$[J_3^{(s)}, Y^{(s)}_{jm}] = mY^{(s)}_{jm},$$

$$[J^+_i^{(s)}, Y^{(s)}_{jm}] = \sqrt{(j-m)(j+1+m)} Y_{jm+1}^{(s)},$$

$$[J^-_i^{(s)}, Y^{(s)}_{jm}] = \sqrt{(j+m)(j+1-m)} Y_{jm-1}^{(s)},$$

(5.5)

where $J^\pm_i^{(s)} = J_i^{(s)} \pm iJ_2^{(s)}$. For the normalization of the matrix spherical harmonics, we choose

$$\text{Tr}(Y^{(s)*}_{jm}Y^{(s)}_{jm}) = N_s \delta_{j'j}\delta_{m'm}.$$  

(5.6)

Having equipped with the necessary machinery, we now proceed the diagonalization.

By direct application of Eqs. (5.5), (5.5), and (5.6), the gauge field fluctuation $Z_z^{(s)}$ and the fluctuations in the $SO(6)$ directions $Z^a_{(s)}$ are immediately diagonalized. The corresponding spherical modes and their masses are

$$z^0_{(s)jm} = \frac{1}{3} \sqrt{j(j+1)},$$

$$z^a_{(s)jm} = \frac{1}{3} (j + \frac{1}{2}),$$

(5.7)

where $0 \leq j \leq N_s - 1$ and $-j \leq m \leq j$.

As for the fluctuations in the $SO(3)$ directions, we should first consider the potential \( \frac{1}{2\times 3^2}(Z_i^{(s)} + i\epsilon^{ijk}[J_j^{(s)}, Z_k^{(s)}])^2 \) and diagonalize it by solving the eigenvalue problem,

$$Z^{(s)}_i + i\epsilon^{ijk}[J_j^{(s)}, Z_k^{(s)}] = \lambda Z^{(s)}_i.$$  

(5.8)
By defining the combinations of matrices $Z_{(s)}^\pm = Z_{(s)}^1 \pm i Z_{(s)}^2$ and using Eqs. (5.3) and (5.5), it turns out that the eigenvalue $\lambda$ takes the values of $-j$, $j+1$, or 0. Let us now denote the spherical eigenmodes of matrix eigenvectors as $u_{(s)jm}$, $v_{(s)jm}$, and $w_{(s)jm}$ for $\lambda = -j$, $j+1$, and 0, respectively. The eigenmodes satisfy the reality condition in the form of Eq. (5.4). Then the fluctuation matrices may be expressed with respect to these eigenmodes for each eigenvalue.

For $\lambda = -j$, we have the following expressions:

$$Z_{(s)}^+ = -\frac{1}{\sqrt{N_s}} \sqrt{\frac{(j+m)(j+1+m)}{j(2j+1)}} u_{(s)j-1m} Y_{jm+1}^{(s)},$$

$$Z_{(s)}^- = \frac{1}{\sqrt{N_s}} \sqrt{\frac{(j-m)(j+1-m)}{j(2j+1)}} u_{(s)j-1m} Y_{jm-1}^{(s)},$$

$$Z_{(s)}^3 = \frac{1}{\sqrt{N_s}} \sqrt{\frac{(j+m)(j-m)}{j(2j+1)}} u_{(s)j-1m} Y_{jm}^{(s)},$$

(5.9)

where $0 < j < N_s$, $-j < m < j$, and the normalization constant are chosen so that the kinetic term for the mode $u_{(s)jm}$ is of the form $\frac{1}{2} (\dot{u}_{(s)jm})^2$. Here and in what follows, the summations over $j$ and $m$ with specified ranges are implicit.

For $\lambda = j+1$, we have

$$Z_{(s)}^+ = \frac{1}{\sqrt{N_s}} \sqrt{\frac{(j-m)(j+1-m)}{(j+1)(2j+1)}} v_{(s)j+1m} Y_{jm+1}^{(s)},$$

$$Z_{(s)}^- = -\frac{1}{\sqrt{N_s}} \sqrt{\frac{(j+m)(j+1+m)}{(j+1)(2j+1)}} v_{(s)j+1m} Y_{jm-1}^{(s)},$$

$$Z_{(s)}^3 = \frac{1}{\sqrt{N_s}} \sqrt{\frac{(j+1+m)(j+1-m)}{(j+1)(2j+1)}} v_{(s)j+1m} Y_{jm}^{(s)},$$

(5.10)

where $0 \leq j < N_s$, $-j-1 \leq m \leq j+1$, and the normalization constant are chosen in the same way with the case of $u_{(s)jm}$.

Finally, for $\lambda = 0$,

$$Z_{(s)}^i = \frac{1}{\sqrt{N_s}} w_{(s)jm} [J_{(s)}^i, Y_{jm}^{(s)}]$$

(5.11)

with $0 < j < N_s$ and $-j \leq m \leq j$. As pointed out in [4], the modes for $\lambda = 0$ case correspond to the degrees of freedom for the gauge transformation and are thus unphysical. Since the authors of [4] took the physical Weyl gauge, $A = 0$, and worked in the operator formulation, it was not necessary to consider these unphysical modes seriously. However, they should be involved properly in the present context because our gauge choice is the covariant background gauge and we work in the path integral formulation.
We turn to the remaining potential term in the Lagrangian (5.2) which is the square of $[J^i_{(s)}, Z^i_{(s)}]$. Interestingly enough, the matrices expanded in terms of the eigenmodes $u_{(s)jm}$ and $v_{(s)jm}$, Eqs. (5.9) and (5.10), do not give any contribution, since we obtain

$$[J^i_{(s)}, Z^i_{(s)}] = 0,$$

(5.12)

for $\lambda = -j$ and $j + 1$. Only the modes corresponding to $\lambda = 0$ contribute to the potential, which is evaluated as

$$\frac{1}{2} \cdot 3^j j(j + 1)|w_{(s)jm}|^2,$$

(5.13)

for each $j$ and $m$.

Having diagonalized the fluctuations $Z^i_{(s)}$, we get the following list of spherical modes in the $SO(3)$ directions with their masses and the ranges of spin $j$:

$$u_{(s)jm} = \frac{1}{3}(j + 1) 0 \leq j \leq N_s - 2,$$

$$v_{(s)jm} = \frac{1}{3} j 1 \leq j \leq N_s,$$

$$w_{(s)jm} = \frac{1}{3} \sqrt{j(j + 1)} 1 \leq j \leq N_s - 1,$$

(5.14)

where $-j \leq m \leq j$.

With respect to the spherical modes of Eqs. (5.7) and (5.14), the Lagrangian (5.2) is then written in the diagonalized form as

$$L_B^{(s)} = \frac{1}{2} \sum_{j=0}^{N_s-1} \left( -|\dot{z}^0_{(s)jm}|^2 + \frac{1}{3^2} j(j + 1)|z^0_{(s)jm}|^2 + |\dot{z}^a_{(s)jm}|^2 - \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 |z^a_{(s)jm}|^2 \right)
+ \frac{1}{2} \sum_{j=0}^{N_s-1} \left( |\dot{u}_{(s)jm}|^2 - \frac{1}{3^2} j(j + 1)^2 |u_{(s)jm}|^2 \right) + \frac{1}{2} \sum_{j=1}^{N_s} \left( |\dot{v}_{(s)jm}|^2 - \frac{1}{3^2} j^2 |v_{(s)jm}|^2 \right)
+ \frac{1}{2} \sum_{j=1}^{N_s-1} \left( -|\dot{w}_{(s)jm}|^2 + \frac{1}{3^2} j(j + 1)|w_{(s)jm}|^2 \right),$$

(5.15)

where the sum over $m$ with the range $-j \leq m \leq j$ is understood. The Lagrangian is just the sum of various harmonic oscillator Lagrangians, which are non-interacting with each other, and therefore the path integration is now straightforward. As a result, what we obtain is

$$\prod_{j=0}^{N_s-1} \left[ \det \left( \partial^2 + \frac{1}{3^2} j(j + 1) \right) \det \left( \partial^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 \right) \right]^{-j-\frac{1}{2}}
\times \prod_{j'=0}^{N_s-2} \left[ \det \left( \partial^2 + \frac{1}{3^2} (j' + 1)^2 \right) \right]^{-j'-\frac{1}{2}} \prod_{j''=1}^{N_s} \left[ \det \left( \partial^2 + \frac{1}{3^2} j''^2 \right) \right]^{-j''-\frac{1}{2}}
\times \prod_{j'''=1}^{N_s-1} \left[ \det \left( \partial^2 + \frac{1}{3^2} j'''(j'' + 1) \right) \right]^{-j'''-\frac{1}{2}}.$$

(5.16)
5.2 Fermionic fluctuation

We turn to the path integral of fermionic fluctuations. The Lagrangian is written as

\[ L_F^{(s)} = \text{Tr} \left( i\dot{\Psi}^{(s)} \Psi^{(s)} - \frac{1}{3} \Psi^{(s)} \gamma^i [\Psi^{(s)}, J_i^{(s)}] - \frac{i}{4} \Psi^{(s)} \gamma^{123} \Psi^{(s)} \right) , \quad (5.17) \]

It is convenient for our calculation of fermionic part to introduce the \( SU(2) \times SU(4) \) formulation since the preserved symmetry in the plane-wave matrix model is \( SO(3) \times SO(6) \sim SU(2) \times SU(4) \) rather than \( SO(9) \). In this formulation the \( SO(9) \) spinor \( \Psi^{(s)} \) is decomposed as

\[ 16 \rightarrow (2, 4) + (\bar{2}, \bar{4}) \]

where \( A \) implies a fundamental \( SU(4) \) index and \( \alpha \) is a fundamental \( SU(2) \) index. According to this decomposition, we may take the expression of \( \Psi^{(s)} \) as

\[ \Psi^{(s)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \psi^{(s)}_{A\alpha} \\ \epsilon_{\alpha\beta} \psi^{(s)}_{A\beta} \end{array} \right) , \quad (5.19) \]

We also rewrite the \( SO(9) \) gamma matrices \( \gamma^i \)'s in terms of \( SU(2) \) and \( SU(4) \) ones as follows:

\[ \gamma^i = \left( \begin{array}{cc} -\sigma^i \times 1 & 0 \\ 0 & \sigma^i \times 1 \end{array} \right) , \quad \gamma^a = \left( \begin{array}{cc} 0 & 1 \times (\rho^a)^\dagger \\ 1 \times (\rho^a) & 0 \end{array} \right) , \quad (5.20) \]

where the \( \sigma^i \)'s are the standard \( 2 \times 2 \) Pauli matrices and six of \( \rho^a \) are taken to form a basis of \( 4 \times 4 \) anti-symmetric matrices. The original \( SO(9) \) Clifford algebra is satisfied as long as we take normalizations so that the gamma matrices \( \rho^a \) with \( SU(4) \) indices satisfy the algebra

\[ \rho^a (\rho^b)^\dagger + \rho^b (\rho^a)^\dagger = 2 \delta^{ab} . \quad (5.21) \]

By introducing the above \( SU(2) \times SU(4) \) formulation, the Lagrangian for the fermionic fluctuations is rewritten as

\[ L_F^{(s)} = \text{Tr} \left( i\dot{\psi}_{(s)}^{A\alpha} \psi_{(s)A\alpha} + \frac{1}{3} \psi_{(s)}^{A\alpha} (\sigma^i)^{\alpha\beta} [\psi_{(s)A\beta}, J_i^{(s)}] - \frac{i}{4} \psi_{(s)}^{A\alpha} \gamma^{123} \psi_{(s)A\alpha} \right) . \quad (5.22) \]

The diagonalization of the Lagrangian proceeds in the same way as in the previous subsection. In the present case, it is achieved by solving the following eigenvalue problem:

\[ (\sigma^i)^{\alpha\beta} [J_i^{(s)}, \psi_{(s)A\beta}] = \lambda \psi_{(s)A\alpha} . \quad (5.23) \]

We first expand the \( N_s \times N_s \) fermionic matrix \( \psi_{(s)A\alpha} \) in terms of the matrix spherical harmonics as

\[ \psi_{(s)A\alpha} = \sum_{j=0}^{N_s-1} \sum_{m=-j}^{j} \psi_{(s)A\alpha}^{jm} Y^{(s)}_{jm} . \quad (5.24) \]
If we plug this expansion into the above eigenvalue equation and use the SU(2) algebra Eq. (5.5), we see that the eigenvalues are $\lambda = j$ and $\lambda = -j - 1$. Let us now introduce the fermionic spherical eigenmodes $\eta_{jm}^{(s)}$ and $\pi_{jm}^{(s)}$ corresponding to $\lambda = j$ and $-j - 1$ respectively. As for the spinorial structure, $\eta_{jm}^{(s)} \ (\pi_{jm}^{(s)})$ carries the (anti) fundamental SU(4) index. We note that, from now on, we suppress the SU(4) indices.

Then, as the eigenstate for $\lambda = j$, the matrix $\psi_{(s)\alpha}$ has the expansion in terms of the eigenmode $\eta_{jm}$ as

\[
\psi_{(s)+} = \frac{1}{\sqrt{N_s}} \sqrt{\frac{j + 1 + m}{2j + 1}} \eta_{jm}^{j + \frac{1}{2}, m + \frac{1}{2}} (s) Y_{jm}^{(s)},
\]

\[
\psi_{(s)-} = \frac{1}{\sqrt{N_s}} \sqrt{\frac{j - m}{2j + 1}} \eta_{jm}^{j + \frac{1}{2}, m + \frac{1}{2}} (s) Y_{jm}^{(s)},
\]

where $0 \leq j \leq N_s - 1$, $-j - 1 \leq m \leq j$, and the subscripts $\pm$ denote the SU(2) indices measured by $\sigma^3$.

On the other hand, for $\lambda = -j - 1$, we have

\[
\psi_{(s)+} = -\frac{1}{\sqrt{N_s}} \sqrt{\frac{j - m}{2j + 1}} (\pi_{(s)})^{j - \frac{1}{2}, m + \frac{1}{2}} (s) Y_{jm}^{(s)},
\]

\[
\psi_{(s)-} = \frac{1}{\sqrt{N_s}} \sqrt{\frac{j + 1 + m}{2j + 1}} (\pi_{(s)})^{j + \frac{1}{2}, m + \frac{1}{2}} (s) Y_{jm}^{(s)},
\]

where $1 \leq j \leq N_s - 1$ and $-j \leq m \leq j - 1$.

By using the mode-expansions, Eqs. (5.25) and (5.26), and the SU(2) algebra (5.5), the fermionic Lagrangian (5.22) becomes

\[
L_{(s)}^F = \sum_{j=\frac{1}{2}}^{N_s-\frac{3}{2}} \left( i\pi_{(s)jm} \pi_{(s)jm} - \frac{1}{3} \left( j + \frac{3}{4} \right) \pi_{(s)jm} \pi_{(s)jm} \right)
\]

\[
+ \sum_{j=\frac{1}{2}}^{N_s-\frac{3}{2}} \left( i\eta_{(s)jm} \eta_{(s)jm} - \frac{1}{3} \left( j + \frac{1}{4} \right) \eta_{(s)jm} \eta_{(s)jm} \right),
\]

where it should be understood that there is the summation over $m$ with the range $-j \leq m \leq j$. Now, the path integration for this Lagrangian may be evaluated immediately, and
gives
\[
\prod_{j=1}^{N_s-\frac{3}{2}} \left[ \det \left( \partial_t^2 + \frac{1}{3^2} \left( j + \frac{3}{4} \right)^2 \right) \right]^{2(2j+1)}
\]
\[
\times \prod_{j'=1}^{N_s-\frac{3}{2}} \left[ \det \left( \partial_t^2 + \frac{1}{3^2} \left( j' + \frac{1}{4} \right)^2 \right) \right]^{2(2j'+1)}.
\]

(5.28)

5.3 Ghost fluctuation

As the final part of the diagonal fluctuations, we consider the Lagrangian for the ghost fluctuations, which is given by
\[
L^G_{(s)} = \text{Tr} \left( \dot{\hat{C}}_{(s)} \dot{C}_{(s)} + \frac{1}{3^2} [J_{(s)}^i, \dot{\hat{C}}_{(s)}][J_{(s)}^i, \dot{C}_{(s)}] \right).
\]
(5.29)

By using the SU(2) algebra (5.5) and the following expansions in terms of the matrix spherical harmonics
\[
C_{(s)} = \frac{1}{\sqrt{N_{(s)}}} c_{(s)jm} Y_{jm}^{(s)}, \quad \dot{C}_{(s)} = \frac{1}{\sqrt{N_{(s)}}} \dot{c}_{(s)jm} Y_{jm}^{(s)},
\]
(5.30)
with \(0 \leq j \leq N_s - 1\) and \(-j \leq m \leq j\), we may rewrite the above Lagrangian as
\[
L^G_{(s)} = \sum_{j=0}^{N_s-1} \left( \ddot{c}_{(s)jm} \dot{c}_{(s)jm} - \frac{1}{3^2} j(j+1) \ddot{c}_{(s)jm} \dot{c}_{(s)jm} \right),
\]
(5.31)
where the sum over \(m\) is implicit. Then the result of the path integration for this Lagrangian is
\[
\prod_{j=0}^{N_s-1} \left[ \det \left( \partial_t^2 + \frac{1}{3^2} j(j+1) \right) \right]^{2j+1}.
\]
(5.32)

5.4 One-loop stability

In the previous subsections, we have evaluated the path integrals for the block diagonal fluctuations of the action \(S_{\text{diag}}\). Thus, we may now consider the effective action. As can be inferred from Eq. (5.1), it is enough to consider only the effective action of a given fuzzy sphere. If we let \(\Gamma_{\text{eff}}^{(s)}\) be the effective action for the \(s\)-th fuzzy sphere, then it is given by
\[
e^{i\Gamma_{\text{eff}}^{(s)}} = (5.16) \times (5.28) \times (5.32).
\]
(5.33)
The right hand side may be viewed just as the product of determinants of non-interacting quantum mechanical simple harmonic oscillators with various frequencies. In fact, as we will see in the next section, this is also the case for the effective action describing the interaction between two fuzzy spheres. Thus, it is worthwhile to consider a generic situation, which is useful both in the present and the next section.

Let us then consider a situation,

$$e^{i \Gamma_{\text{eff}}} = \prod_n \det^a_n (\partial^2_t - 2ip_n \partial_t + m^2_n),$$

(5.34)

where $p_n$ is included for the later usage. The formal expression of the effective action is read as

$$\Gamma_{\text{eff}} = -i \sum_n a_n \ln \det (\partial^2_t - 2ip_n \partial_t + m^2_n).$$

(5.35)

By using the relation $\ln \det M = \text{Tr} \ln M$ for a given matrix $M$, we may present a prototype calculation of single determinant as

$$\text{Tr} \ln (\partial^2_t - 2ip \partial_t + m^2) = \int dt \langle t | \ln (\partial^2_t - 2ip \partial_t + m^2) | t \rangle$$

$$= \int dt \int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln (-k^2 + 2pk + m^2)$$

$$= i \int dt \sqrt{m^2 + p^2},$$

(5.36)

where the momentum integration has appeared by inserting the momentum space identity, $\int dk |k\rangle\langle k| = 1$, between bra and ket vectors for time, and the final result has been derived by using the formula

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln (-k^2 + 2pk + m^2 - i\epsilon) = i \sqrt{m^2 + p^2}.$$  

(5.37)

If we apply the result of this single determinant calculation to Eq. (5.35), then the effective action is finally obtained as

$$\Gamma_{\text{eff}} = \int dt \sum_n a_n \sqrt{m^2_n + p^2_n}.$$  

(5.38)

We now return to the present case. First of all, we observe that the ghost part (5.32) eliminates two determinant factors of the bosonic result (5.16) which are actually the contributions from the unphysical modes. Thus, what we get from the bosonic and the ghost parts is only the determinant factors from the physical bosonic modes. By using the generic expression, Eq. (5.38), it turns out that their contributions to the effective action are

$$-\frac{1}{32} N_s (8N_s^2 + 1).$$

(5.39)
On the other hand, the contribution from the fermionic part (5.28) is obtained as
\[
\frac{1}{32} N_s (8 N_s^2 + 1). 
\]  
(5.40)

Therefore, there is no net contribution to the effective action and we can conclude that the one loop effective action obtained after integrating out each of the block diagonal fluctuations vanishes;
\[
\Gamma_{\text{eff}}^{(s)} = 0.
\]  
(5.41)

This indicates that each membrane fuzzy sphere has quantum stability at least at one-loop level, by which we mean that the fuzzy sphere does not receive quantum corrections. We note that, if we take \(N_s = 2\) in the above two contributions, the result of the previous path integral computation [12] is recovered.

6 Interaction between fuzzy spheres

We turn to the action \(S_{\text{off-diag}}\) in Eq. (4.8) for the block off-diagonal fluctuations and compute the one-loop effective potential describing the interaction between two fuzzy spheres. The action is given by
\[
S_{\text{off-diag}} = \int dt (L_B + L_F + L_G),
\]  
(6.1)

where \(L_B, L_F\) and \(L_G\) are the Lagrangians for the off-diagonal bosonic, fermionic and ghost fluctuations of Eq. (4.7) respectively and their explicit expressions will be presented in due course. Since, as in the previous section, there is no mixing between different kinds of fluctuations, each Lagrangian can be considered independently.

In calculating the effective action, the prescription given by Kabat and Taylor [44] is usually used. We note however that, at the present situation, it is more helpful to use the expansion in terms of the matrix spherical harmonics as in the previous section.
6.1 Bosonic fluctuation

Let us first consider the bosonic Lagrangian and evaluate its path integral. The Lagrangian is

\[
L^B = \text{Tr} \left\{ \begin{array}{l}
- |\dot{\Phi}^0|^2 + r^2 |\Phi^0|^2 + \frac{1}{32} \Phi^0 \dagger J^i \circ (J^i \circ \Phi^0) \\
+ |\dot{\Phi}^i|^2 - r^2 |\Phi^i|^2 - \frac{1}{32} |\Phi^i + i\epsilon^{ijk} J^j \circ \Phi^k|^2 + \frac{1}{32} |J^i \circ \Phi^i|^2 \\
+ |\Phi^a|^2 - \left( r^2 + \frac{1}{6^2} + \frac{1}{32} \Phi^a \dagger J^i \circ (J^i \circ \Phi^a) \\
- \frac{1}{3} r \left[ \sin \left( \frac{t}{6} \right) (\Phi^a \dagger \Phi^4 - \Phi^4 \dagger \Phi^0) - \cos \left( \frac{t}{6} \right) (\Phi^a \dagger \Phi^5 - \Phi^5 \dagger \Phi^0) \right] \end{array} \right\}
\]

(6.2)

Here, adopting the notation of [4], we have defined

\[
J^i \circ M_{(rs)} \equiv J^i_{(r)} M_{(rs)} - M_{(rs)} J^i_{(s)}
\]

(6.3)

where \( M_{(rs)} \) is the \( N_r \times N_s \) matrix which is a block at \( r \)-th row and \( s \)-th column in the blocked form of a given matrix \( M \). In the present case, \( r \) and \( s \) take values of 1 and 2. For example, if we look at the \( 2 \times 2 \) block matrix form of the gauge field fluctuation \( A \) in Eq. (4.7), then \( A_{(ss)} = Z^0_{(s)} \), \( A_{(12)} = \Phi^0 \), and \( A_{(21)} = \Phi^0 \dagger \).

The matrix fields, \( \Phi^0 \), \( \Phi^4 \), and \( \Phi^5 \) are coupled with each other through the circular motion background. Since the time dependent trigonometric functions may make the formulation annoying, we consider the newly defined matrix variables as

\[
\Phi^r \equiv \cos \left( \frac{t}{6} \right) \Phi^4 + \sin \left( \frac{t}{6} \right) \Phi^5,
\]

\[
\Phi^\theta \equiv - \sin \left( \frac{t}{6} \right) \Phi^4 + \cos \left( \frac{t}{6} \right) \Phi^5,
\]

(6.4)

where \( \Phi^\theta \) may be interpreted as the fluctuation tangential to the circular motion at time \( t \) and \( \Phi^r \) as the normal fluctuation. In terms of these fluctuations, the terms in the Lagrangian (6.2), which are dependent on \( \Phi^4 \) and \( \Phi^5 \), are rewritten as

\[
\text{Tr} \left[ |\dot{\Phi}^r|^2 + |\dot{\Phi}^\theta|^2 - r^2 (|\Phi^r|^2 + |\Phi^\theta|^2) - \frac{1}{32} (\Phi^r \dagger J^i \circ (J^i \circ \Phi^r) + \Phi^\theta \dagger J^i \circ (J^i \circ \Phi^\theta))
+ \frac{1}{3} (\Phi^r \dagger \Phi^\theta - \Phi^\theta \dagger \Phi^r) + \frac{r}{3} (\Phi^4 \dagger \Phi^5 - \Phi^5 \dagger \Phi^4) \right],
\]

(6.5)

where we no longer see the explicit time dependent classical functions.

We are now in a position to consider the diagonalization of the Lagrangian \( L^B \). We note that, compared to the case in the previous section, we are in a somewhat different situation.
The fluctuation matrices are $N_1 \times N_2$ or $N_2 \times N_1$ ones, while those in the last section are square $N_s \times N_s$ matrices. This means that, when we regard an $N_1 \times N_2$ block off-diagonal matrix as an $N_1N_2$-dimensional reducible representation of $SU(2)$, it has the decomposition into irreducible spin $j$ representations with the range $|N_1 - N_2|/2 \leq j \leq (N_1 + N_2)/2 - 1$, that is, $N_1N_2 = \bigoplus_{j=|N_1-N_2|/2}^{(N_1+N_2)/2-1} (2j + 1)$, and may be expanded as

$$L = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \sum_{m=-j}^{j} \phi_{jm} Y_{jm}^{N_1 \times N_2}, \quad (6.6)$$

where $Y_{jm}^{N_1 \times N_2}$ is the $N_1 \times N_2$ matrix spherical harmonics transforming in the irreducible spin $j$ representation and $\phi_{jm}$ is the corresponding spherical mode. The basic operation between $SU(2)$ generators $J_{i}^{(s)}$ and $Y_{jm}^{N_1 \times N_2}$ is given not by the commutator but by the $\circ$ operator $[6.3]$. Thus the algebraic properties for the present situation are given by Eq. (5.5) where the commutator is replaced by the $\circ$ operator.

Having the expansion and algebraic properties, the diagonalization proceeds in exactly the same way as in the previous section. Hence, by noting that one may find a detailed procedure in [4], we will present only the results of diagonalization for the Lagrangian. We observe that, because of the background for the circular motion in $x^4$-$x^5$ plane, the $SO(6)$ symmetry is broken to $SO(4) \times SO(2)$, while the $SO(3)$ symmetry remains intact. This fact naturally leads us to break the bosonic Lagrangian (6.2) into three parts as follows:

$$L^B = L_{SO(3)} + L_{SO(4)} + L_{\text{rot}}, \quad (6.7)$$

where $L_{SO(3)}$ is the Lagrangian for $\Phi^i$, $L_{SO(4)}$ is for $\Phi^{a'}$ with $a' = 5, 6, 7, 8$, and $L_{\text{rot}}$ represents the $SO(2)$ rotational part described by $\Phi^4$, $\Phi^5$, and the gauge fluctuation $\Phi^0$.

We first consider $L_{SO(3)}$ and its path integration. Its diagonalized form is obtained by

$$L_{SO(3)} = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-2} \left[ |\dot{\alpha}_{jm}|^2 - \left( r^2 + \frac{1}{3^2} (j + 1)^2 \right) |\alpha_{jm}|^2 \right] + \sum_{j=\frac{1}{2}|N_1-N_2|+1}^{\frac{1}{2}(N_1+N_2)} \left[ |\dot{\beta}_{jm}|^2 - \left( r^2 + \frac{1}{3^2} j^2 \right) |\beta_{jm}|^2 \right] + \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ - |\omega_{jm}|^2 + \left( r^2 + \frac{1}{3^2} j(j + 1) \right) |\omega_{jm}|^2 \right], \quad (6.8)$$

where the sum of $m$ over the range $-j \leq m \leq j$ is implicit and the spherical modes $\omega_{jm}$ are the degrees of freedom for the gauge transformation, the block off-diagonal counterpart.
of \( w_{jm} \), in the previous section. \( \alpha_{jm} \) and \( \beta_{jm} \) are the block off-diagonal counterparts of \( u_{jm} \) and \( v_{jm} \), respectively. The path integral of this Lagrangian is straightforward and results in

\[
\frac{1}{2}(N_1+N_2)-2 \prod_{j=\frac{1}{2}|N_1-N_2|-1}^{\frac{1}{2}(N_1+N_2)} \left[ \det \left( \partial_t^2 + r^2 + \frac{1}{32}(j+1)^2 \right) \right]^{-(2j+1)} \\
\times \frac{1}{2}(N_1+N_2) \prod_{j'=\frac{1}{2}|N_1-N_2|+1}^{\frac{1}{2}(N_1+N_2)-1} \left[ \det \left( \partial_t^2 + r^2 + \frac{1}{32}j'^2 \right) \right]^{-(2j'+1)} \\
\times \prod_{j''=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)} \left[ \det \left( \partial_t^2 + r^2 + \frac{1}{32}j''(j''+1) \right) \right]^{-(2j''+1)} .
\] (6.9)

As for \( L_{SO(4)} \), we have as its diagonalized form

\[
L_{SO(4)} = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ |\phi'_{jm}|^2 - \left( r^2 + \frac{1}{32} \left( j + \frac{1}{2} \right) \right) |\phi'^*_{jm}|^2 \right] ,
\] (6.10)

where \(-j \leq m \leq j\) and \( a' = 5, 6, 7, 8 \). Its path integration leads us to have

\[
\prod_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ \det \left( \partial_t^2 + r^2 + \frac{1}{32}j^2 \right) \right]^{-(2j+1)} .
\] (6.11)

For the rotational part, the diagonalized Lagrangian is obtained as

\[
L_{\text{rot}} = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ -|\phi'^*_{jm}|^2 + \left( r^2 + \frac{1}{32}j(j+1) \right) |\phi'^{0}_{jm}|^2 \right. \\
+ |\phi'^*_{jm}|^2 + |\phi'^{0}_{jm}|^2 - \left( r^2 + \frac{1}{32}j(j+1) \right) \left( |\phi'^*_{jm}|^2 + |\phi'^{0}_{jm}|^2 \right) \\
+ \frac{1}{3} \left( \phi'^*_{jm} \phi'^{0}_{jm} - \phi'^*_{jm} \phi'^{0}_{jm} \right) + i \frac{r}{3} \left( \phi'^*_{jm} \phi'^{0}_{jm} - \phi'^*_{jm} \phi'^{0}_{jm} \right) \right] ,
\] (6.12)

where \(-j \leq m \leq j\). Since all the coupling coefficients between the modes are time-independent constants, the path integral of this Lagrangian is readily evaluated and gives

\[
\prod_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ \det \left( \partial_t^2 + r^2 + \frac{1}{32}j(j+1) \right) \right] \det \left( \partial_t^2 + r^2 + \frac{1}{32}j^2 \right) \\
\times \det \left( \partial_t^2 + r^2 + \frac{1}{32}(j+1)^2 \right) \right]^{-(2j+1)} .
\] (6.13)
6.2 Fermionic fluctuation

The fermionic Lagrangian of the block off-diagonal action $S_{\text{off-diag}}$, (6.1), is

$$L^F = 2 \text{Tr} \left[ i \chi^\dagger \dot{\chi} - \frac{1}{4} \chi^\dagger \gamma^{123} \chi + \frac{1}{3} \chi^\dagger \gamma^i J^i \chi + r \chi^\dagger \left( \gamma^4 \cos \left( \frac{t}{6} \right) + \gamma^5 \sin \left( \frac{t}{6} \right) \right) \chi \right],$$

(6.14)

As was done in Eq. (5.18), we decompose the fermion $\chi$ into $\chi_A\alpha$ and $\hat{\chi}^A\alpha$ according to $16 \rightarrow (2, 4) + (\bar{2}, \bar{4})$. Then the expression of $\chi$ may be taken as

$$\chi = \frac{1}{\sqrt{2}} \left( \chi_A\alpha \hat{\chi}^A\alpha \right),$$

(6.15)

where $\hat{\chi}^A\alpha = \epsilon_{\alpha\beta} \hat{\chi}^{A\beta}$. It should be noted that, contrary to the block diagonal fermionic matrix $\Psi_{(s)}$ of (5.19), $\chi$ is the two copy of the $SO(9)$ representation $16$ as one may see from Eq. (4.7). This means that $\chi_A\alpha$ and $\hat{\chi}^A\alpha$ should be treated as independent spinors not related in any way. By plugging the decomposition (6.15) into the Lagrangian (6.14), we have

$$L^F = \text{Tr} \left[ i \chi^\dagger A\alpha \dot{\chi}_A\alpha - \frac{1}{4} \chi^\dagger A\alpha \chi_A\alpha - \frac{1}{3} \chi^\dagger A\alpha (\sigma^i)_{\alpha \beta} J^i \chi_B\beta 
+ i \hat{\chi}_A^\dagger \hat{\chi}^A\alpha + \frac{1}{4} \hat{\chi}_A^\dagger \hat{\chi}^A\alpha + \frac{1}{3} \hat{\chi}_A^\dagger (\sigma^i)_{\alpha \beta} J^i \hat{\chi}^B\beta
+ r \chi^\dagger A\alpha \left( \rho_A^4 \cos \left( \frac{t}{6} \right) + \rho_A^5 \sin \left( \frac{t}{6} \right) \right) \hat{\chi}^B\alpha
+ r \hat{\chi}^\dagger A\alpha \left( (\rho_A^4)^\dagger AB \cos \left( \frac{t}{6} \right) + (\rho_A^5)^\dagger AB \sin \left( \frac{t}{6} \right) \right) \chi_B\alpha \right].$$

(6.16)

The Lagrangian has explicit time dependence due to the presence of the circular motion background. In order to hide it, we take the fermionic field $\hat{\chi}_A^\dagger \alpha$ as

$$\hat{\chi}_A^\dagger \alpha \equiv \left( (\rho_A^4)^\dagger AB \cos \left( \frac{t}{6} \right) + (\rho_A^5)^\dagger AB \sin \left( \frac{t}{6} \right) \right) \tilde{\chi}_{Ba} \alpha,$$

(6.17)

where we have introduced a new fermionic field $\tilde{\chi}_{Ba} \alpha$ which is in the $4$ of $SU(4)$. Then, by using the following identities,

$$\left( \rho_A^4 \cos \left( \frac{t}{6} \right) + \rho_A^5 \sin \left( \frac{t}{6} \right) \right) \left( (\rho_A^4)^\dagger \cos \left( \frac{t}{6} \right) + (\rho_A^5)^\dagger \sin \left( \frac{t}{6} \right) \right) = 1,$$

$$\left( \rho_A^4 \cos \left( \frac{t}{6} \right) + \rho_A^5 \sin \left( \frac{t}{6} \right) \right) \left( -(\rho_A^4)^\dagger \sin \left( \frac{t}{6} \right) + (\rho_A^5)^\dagger \cos \left( \frac{t}{6} \right) \right) = \rho_A^i (\rho_A^5)^\dagger,$$

(6.18)
which are proved via the Clifford algebra (5.21), we may show that the Lagrangian (6.16) becomes

\[
L^F = \text{Tr} \left[ i\chi^A A^a \dot{\chi}_A - \frac{1}{4} \chi^A A^a \chi_\alpha - \frac{1}{3} \chi^A (\sigma^i)^A_B J^i \circ \chi_{\alpha B} \right] + i\dot{\chi}^A A^a \chi_\alpha + \frac{1}{4} \dot{\chi}^A A^a \chi_\alpha + \frac{1}{3} \dot{\chi}^A (\sigma^i)^A_B J^i \circ \chi_{\alpha B} + r \left( \chi^A A^a \chi_\alpha + \dot{\chi}^A A^a \chi_\alpha \right) + \frac{i}{6} \chi^A (\rho^A (\rho^5)^A)^B \dot{\chi}_{AB} \right] .
\]

(6.19)

The explicit time dependent classical functions disappear and the term containing \(\rho^4 (\rho^5)^4\) appears, which originates from the kinetic term of \(\dot{\chi}_A^4\) in (6.16).

With the above Lagrangian (6.19), the diagonalization proceeds in the same manner with that for the block diagonal fermionic Lagrangian (5.22), except for some differences pointed out in the previous subsection. In the expansion of \(\chi_\alpha\) and \(\tilde{\chi}_\alpha\ (SU(4)\ indices\ are\ suppressed.)\) in terms of the matrix spherical harmonics like (6.6), let us denote their spherical modes as \((\chi_\alpha)_{jm}\) and \((\tilde{\chi}_\alpha)_{jm}\) respectively. Then the diagonalization results in

\[(\chi_\alpha)_{jm} \rightarrow (\pi_{jm}, \eta_{jm})\ and\ (\tilde{\chi}_\alpha)_{jm} \rightarrow (\tilde{\pi}_{jm}, \tilde{\eta}_{jm}).\]

The modes \(\pi_{jm}\) and \(\tilde{\pi}_{jm}\) have the same mass of \(\frac{1}{3}(j + \frac{3}{2})\) with \(\frac{1}{2}|N_1 - N_2| - \frac{1}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{3}{2}\). For the modes \(\eta_{jm}\) and \(\tilde{\eta}_{jm}\), their mass is \(\frac{1}{2}(j + \frac{3}{2})\) with \(\frac{1}{2}|N_1 - N_2| + \frac{1}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{1}{2}\). All the modes have the same range of \(m\) as \(-j \leq m \leq j\). With respect to these diagonalized spherical modes, the Lagrangian (6.19) is written as

\[
L^F = \sum_{j = \frac{1}{2} |N_1 - N_2| - \frac{1}{2}}^{\frac{1}{2} (N_1 + N_2) - \frac{1}{2}} \left[ i \pi^a_{jm} \dot{\pi}_{jm} + i \tilde{\pi}^a_{jm} \dot{\tilde{\pi}}_{jm} - \frac{1}{3} \left( j + \frac{3}{4} \right) \left( \pi^a_{jm} \pi_{jm} - \tilde{\pi}^a_{jm} \tilde{\pi}_{jm} \right) \right]
\]

\[
+ r \left( \pi^a_{jm} \tilde{\pi}_{jm} + \tilde{\pi}^a_{jm} \pi_{jm} \right) + \frac{i}{6} \pi^a_{jm} \rho^4 (\rho^5)^a \tilde{\pi}_{jm} \right] + \sum_{j = \frac{1}{2} |N_1 - N_2| + \frac{1}{2}}^{\frac{1}{2} (N_1 + N_2) + \frac{1}{2}} \left[ i \eta^a_{jm} \dot{\eta}_{jm} + i \tilde{\eta}^a_{jm} \dot{\tilde{\eta}}_{jm} - \frac{1}{3} \left( j + \frac{3}{4} \right) \left( \eta^a_{jm} \eta_{jm} - \tilde{\eta}^a_{jm} \tilde{\eta}_{jm} \right) \right]
\]

\[
+ r \left( \eta^a_{jm} \tilde{\eta}_{jm} + \tilde{\eta}^a_{jm} \eta_{jm} \right) + \frac{i}{6} \eta^a_{jm} \rho^4 (\rho^5)^a \tilde{\eta}_{jm} \right] ,
\]

(6.20)

where \(-j \leq m \leq j\) and the \(SU(4)\) indices are suppressed.

The product \(\rho^4 (\rho^5)^4\) measures the \(SO(2)\) chirality in the \(x^4-x^5\) plane where the circular motion takes place. Since \((\rho^4 (\rho^5)^4)^2 = -1\), its eigenvalues are \(\pm i\). Each spherical mode may split into modes having definite \(\rho^4 (\rho^5)^4\) eigenvalues as follows:

\[
\pi_{jm} = \pi_{jm} + \pi_{-jm} \quad \eta_{jm} = \eta_{jm} + \eta_{-jm} \quad \tilde{\pi}_{jm} = \tilde{\pi}_{jm} + \tilde{\pi}_{-jm} \quad \tilde{\eta}_{jm} = \tilde{\eta}_{jm} + \tilde{\eta}_{-jm} \quad \rho^4 (\rho^5)^4 = (\pm i).
\]

(6.21)
where the modes on the right hand sides satisfy
\[
\rho^4(\rho^5)^\dagger \pi_{\pm jm} = \pm i\pi_{\pm jm} , \\
\rho^4(\rho^5)^\dagger \eta_{\pm jm} = \pm i\eta_{\pm jm} , \\
\rho^4(\rho^5)^\dagger \tilde{\eta}_{\pm jm} = \pm i\tilde{\eta}_{\pm jm} .
\] (6.22)

One may see that the Lagrangian (6.20) composed of two independent parts. One is for \(\pi_{jm}\) and \(\tilde{\pi}_{jm}\), and the other one for \(\eta_{jm}\) and \(\tilde{\eta}_{jm}\). If we first consider the part for \(\pi_{jm}\) and \(\tilde{\pi}_{jm}\), then, according to the above splitting of modes, we have
\[
\sum_{j=\frac{1}{2}[N_1-N_2]-\frac{1}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{3}{2}} \left[ i\pi_{+jm}^\dagger \hat{\pi}+jm + i\pi_{-jm}^\dagger \hat{\pi}-jm - \frac{1}{3} \left( j + \frac{3}{4} \right) (\pi_{+jm}^\dagger \pi_{+jm} + \pi_{-jm}^\dagger \pi_{-jm}) \\
+ \frac{1}{3} \left( j + \frac{1}{4} \right) \tilde{\pi}_{+jm}^\dagger \tilde{\pi}_{+jm} + i\tilde{\pi}_{-jm}^\dagger \tilde{\pi}_{-jm} + \frac{1}{3} \left( j + \frac{5}{4} \right) \tilde{\pi}_{-jm}^\dagger \tilde{\pi}_{-jm} \\
+ r(\pi_{+jm}^\dagger \tilde{\pi}_{+jm} + \pi_{+jm}^\dagger \pi_{+jm}) + r(\pi_{-jm}^\dagger \tilde{\pi}_{-jm} + \pi_{-jm}^\dagger \pi_{-jm}) \right],
\] (6.23)

where again \(-j \leq m \leq j\). The path integration of this part is straightforward and gives
\[
\prod_{j=\frac{1}{2}[N_1-N_2]-\frac{1}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{3}{2}} \left[ \det \left( \partial_t^2 + \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 - \frac{1}{3^2 \cdot 4^2} \right) \\
\times \det \left( \partial_t^2 - \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 - \frac{1}{3^2 \cdot 4^2} \right) \\
\times \det \left( \partial_t^2 + \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} (j+1)^2 - \frac{1}{3^2 \cdot 4^2} \right) \\
\times \det \left( \partial_t^2 - \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} (j+1)^2 - \frac{1}{3^2 \cdot 4^2} \right) \right]^{2j+1}.
\] (6.24)

The other part for \(\eta_{jm}\) and \(\tilde{\eta}_{jm}\) is obtained as
\[
\sum_{j=\frac{1}{2}[N_1-N_2]+\frac{1}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{1}{2}} \left[ i\eta_{+jm}^\dagger \hat{\eta}+jm + i\eta_{-jm}^\dagger \hat{\eta}-jm - \frac{1}{3} \left( j + \frac{1}{4} \right) (\eta_{+jm}^\dagger \eta_{+jm} + \eta_{-jm}^\dagger \eta_{-jm}) \\
+ \frac{1}{3} \left( j - \frac{1}{4} \right) \tilde{\eta}_{+jm}^\dagger \tilde{\eta}_{+jm} + i\tilde{\eta}_{-jm}^\dagger \tilde{\eta}_{-jm} + \frac{1}{3} \left( j + \frac{3}{4} \right) \tilde{\eta}_{-jm}^\dagger \tilde{\eta}_{-jm} \\
+ r(\eta_{+jm}^\dagger \tilde{\eta}_{+jm} + \eta_{+jm}^\dagger \eta_{+jm}) + r(\eta_{-jm}^\dagger \tilde{\eta}_{-jm} + \eta_{-jm}^\dagger \eta_{-jm}) \right],
\] (6.25)

where the sum over \(m\) for \(-j \leq m \leq j\) is implicit. The path integration of this part results
in
\[
\prod_{j=\frac{1}{2}|N_1-N_2|+\frac{1}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{1}{2}} \left[ \det \left( \partial^2_t + \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2 j^2} - \frac{1}{3^2 \cdot 4^2} \right) \right. \\
\times \det \left( \partial^2_t - \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2 j^2} - \frac{1}{3^2 \cdot 4^2} \right) \\
\times \det \left( \partial^2_t + \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 - \frac{1}{3^2 \cdot 4^2} \right) \\
\left. \times \det \left( \partial^2_t - \frac{i}{6} \partial_t + r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2 - \frac{1}{3^2 \cdot 4^2} \right) \right]^{2j+1}.
\]  
(6.26)

### 6.3 Ghost fluctuation

Finally, we consider the path integration for the ghost part of the action $S_{\text{off-diag}}$ (6.1). The block off-diagonal Lagrangian for the ghosts is

\[
L^G = \text{Tr} \left[ \dot{\bar{C}} \ddot{\bar{C}} + r^2 \bar{C} \bar{C}^\dagger + \frac{1}{3^2} (J^i \circ \bar{C})(J^i \circ \bar{C}^\dagger) \right] \\
+ \text{Tr} \left[ \dot{\bar{C}}^\dagger \ddot{\bar{C}}^\dagger + r^2 \bar{C}^\dagger \bar{C} + \frac{1}{3^2} (J^i \circ \bar{C}^\dagger)(J^i \circ \bar{C}) \right].
\]  
(6.27)

The diagonalization may be carried out by using the same logic in the previous subsections. We expand the ghost fields $\bar{C}$ and $\bar{C}$ in terms of the matrix spherical harmonics according to (6.6), and denote their spherical modes as $c_{jm}$ and $\bar{c}_{jm}$ respectively. Then the diagonalized Lagrangian is obtained as

\[
L^G = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ \dot{c}_{jm} \ddot{c}_{jm} + \dot{\bar{c}}_{jm} \ddot{\bar{c}}_{jm} - \left( r^2 + \frac{1}{3^2} j(j+1) \right) (\bar{c}_{jm}^\dagger c_{jm} + \bar{c}_{jm}^\dagger \bar{c}_{jm} c_{jm}) \right],
\]  
(6.28)

where the sum over $m$ for the range $-j \leq m \leq j$ is implicit.

The path integral for the above diagonalized Lagrangian is then immediately evaluated as follows:

\[
\prod_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \left[ \det \left( \partial^2_t + r^2 + \frac{1}{3^2} j(j+1) \right) \right]^{2(2j+1)}.
\]  
(6.29)

### 6.4 Effective potential

Having evaluated the path integral for each part of the block off-diagonal action $S_{\text{off-diag}}$ (6.1), the one-loop effective action, $\Gamma_{\text{eff}}^{(\text{int})}$, which describes the interaction between two membrane
fuzzy spheres with the classical configuration (4.2) and (4.3), is now given by

\[ e^{\Gamma_{\text{eff}}^{(\text{int})}} = \left[ (6.9) \times (6.11) \times (6.13) \right]_B \times \left[ (6.24) \times (6.26) \right]_F \times \left[ (6.23) \right]_G, \]

where the subscripts on the right hand side denote the bosonic, the fermionic, and the ghost contributions. We see that the ghost contribution, Eq. (6.29), eliminates those of unphysical gauge degrees of freedom present in bosonic contributions, Eqs. (6.9) and (6.13). Thus only the physical degrees of freedom contribute to the effective action, as it should be.

The explicit expression of the effective action is obtained by consulting the generic result presented from Eq. (5.34) to (5.38). The effective potential, \( V_{\text{eff}} \), about which we are concerned in this subsection, is then given by \( \Gamma_{\text{eff}}^{(\text{int})} = - \int dt V_{\text{eff}} \). In expressing the effective potential, it is convenient to write \( V_{\text{eff}} \) as

\[ V_{\text{eff}} = V_{\text{eff}}^B + V_{\text{eff}}^F, \]

where \( V_{\text{eff}}^B \) (\( V_{\text{eff}}^F \)) is the contribution of the physical bosonic (fermionic) degrees of freedom to the effective potential. The expression that we obtain for \( V_{\text{eff}}^B \) is then

\[
V_{\text{eff}}^B = \frac{1}{2} (N_1 + N_2) - 1 \sum_{j = \frac{1}{2} |N_1 - N_2| - \frac{1}{2}}^{\frac{1}{2} (N_1 + N_2) - \frac{1}{2}} (2j + 1) \left[ \sqrt{r^2 + \frac{1}{3^2} (j + 1)^2} + \frac{1}{2} \left( j + \frac{1}{2} \right)^2 \right]
+ \sum_{j = \frac{1}{2} |N_1 - N_2|}^{\frac{1}{2} (N_1 + N_2) - 1} 4(2j + 1) \left[ \sqrt{r^2 + \frac{1}{3^2} (j + 1)^2} + \sqrt{r^2 + \frac{1}{3^2} j^2} \right],
\]

while, for the fermionic contribution \( V_{\text{eff}}^F \), we obtain

\[
V_{\text{eff}}^F = - \sum_{j = \frac{1}{2} |N_1 - N_2| - \frac{1}{2}}^{\frac{1}{2} (N_1 + N_2) - \frac{1}{2}} 2(2j + 1) \left[ \sqrt{r^2 + \frac{1}{3^2} (j + 1)^2} + \sqrt{r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2} \right]
+ \sum_{j = \frac{1}{2} |N_1 - N_2| + \frac{1}{2}}^{\frac{1}{2} (N_1 + N_2) - \frac{1}{2}} 2(2j + 1) \left[ \sqrt{r^2 + \frac{1}{3^2} \left( j + \frac{1}{2} \right)^2} + \sqrt{r^2 + \frac{1}{3^2} j^2} \right].
\]

The above expressions show that \( V_{\text{eff}}^B \) and \( V_{\text{eff}}^F \) have the same structure except for the ranges of \( j \). This leads us to expect a great amount of cancellation. Indeed, what we have found is that they are exactly the same. One way to see the cancellation is to adjust all the ranges of the summation parameter \( j \) to the range \( \frac{1}{2} |N_1 - N_2| \leq j \leq \frac{1}{2} (N_1 + N_2) - 1 \).
Therefore, the one-loop effective potential $V_{\text{eff}}$, (6.31), as a function of the distance $r$ between two fuzzy spheres is just flat potential;

$$V_{\text{eff}}(r) = 0 .$$

(6.34)

7 Conclusion and discussion

We have studied the one-loop quantum corrections to a classical background of the plane-wave matrix model in the framework of the path integration. The background is composed of two supersymmetric membrane fuzzy spheres, (4.2) and (4.3). One fuzzy sphere is located at the origin in the $SO(6)$ symmetric space and the other one rotates around it with the distance $r$.

Firstly, the quantum stability of each fuzzy sphere has been shown. In fact, the stability check already has been done in the operator [4] as well as in the path integral [12] formulation. However, while the size $N_s$ of the fuzzy sphere has been taken arbitrary in the operator formulation, it has been restricted to the minimal one, that is $N_s = 2$, in the path integral formulation. Since the fuzzy sphere in this paper has an arbitrary size, our stability check can be regarded as the full generalization of the previous path integral result.

Secondly, the one-loop effective potential describing the interaction between two fuzzy spheres has been calculated as a function of the distance $r$. Interestingly, the result is that the effective potential $V_{\text{eff}}$ is flat and thus the fuzzy spheres do not feel any force. This implies that the whole configuration of two fuzzy spheres given by Eqs. (4.2) and (4.3) is supersymmetric. Although the flatness of the effective potential is the one-loop result, we expect that the result holds also for higher loops. For the supersymmetric properties of the fuzzy sphere configuration itself, the study of supersymmetry algebra may be more helpful rather than the path integral formulation. It would be interesting to investigate the configuration in this paper through the supersymmetry algebra.

Let us consider the radial distance $r$ between two fuzzy spheres and discuss about its possible interpretation. We first consider the form of the effective action when $r$ is time dependent. Since the circular motion is taken as the background, $r$ is constant in this paper. However, if we slightly deform the circular motion to the elliptic one, $r$ can be made to have time dependence. We can also make the time variation of it, $\dot{r}$, arbitrarily small by controlling the degree of deformation. In this case, it is expected that the fuzzy spheres begin to interact and the effective action $\Gamma_{\text{eff}}^{(\text{int})}$ may be written as

$$\Gamma_{\text{eff}}^{(\text{int})} = \frac{\mu^3 N_1}{2} \int dt \dot{r}^2 + f(\dot{r}, r) + O(\mu^{-3/2}) ,$$

(7.1)
where the kinetic term for $r$ is the value of the classical action $S_0$ with the rescaling (4.3); $f(\dot{r}, r)$ is the would-be one-loop contribution to the effective action with the property $f(0, r) = 0$, and the term of order $\mu^{-3/2}$ implies the higher loop corrections.

If $\dot{r} = 0$, the above effective action vanishes and the supersymmetric situation is recovered. This is reminiscent of the effective action for graviton-graviton scattering in the flat space matrix model [43, 45]. The distance between two gravitons comes from the flat directions which are continuous moduli or supersymmetric vacua making the potential of the flat space matrix model vanish. In the plane-wave matrix model, it is known that there is no continuous moduli and hence we do not have flat directions. However, if we look at the tree level action (4.4) evaluated for the fuzzy sphere configuration, (4.2) and (4.3), we see that it vanishes exactly and does not depend on $r$ which is continuous from 0 to $\infty$. This means that, as long as the fuzzy sphere dynamics is concerned, the radius of the circular motion may be interpreted as the flat direction.

In this paper, the circular motion takes place in the $x^4$-$x^5$ sub-plane of the $SO(6)$ symmetric space. Since the $SO(6)$ symmetric space has two other sub-planes, that is $x^6$-$x^7$ and $x^8$-$x^9$, and we may embed the circular motion in one of those, there may be three flat directions in total, which are radial directions of three sub-planes. However, it is not obvious whether these three directions are connected in a continuous way or not, because of the supersymmetric property of rotating fuzzy sphere; all the points in the supersymmetric moduli are expected to preserve a fixed fraction of supersymmetry. While the fuzzy sphere rotating in only one sub-plane is 1/2-BPS object, it is generically 1/4-BPS when it has angular momenta also in the other sub-planes [14]. If we turn to the $SO(3)$ symmetric space, it seems that we do not have flat directions, since only the fuzzy sphere rotating with fixed radius is supersymmetric [7,13,14].

In view of the gauge/gravity duality, it is interesting to study the same situation in the supergravity side. In the large $N$ limit, the leading order interaction terms obtained from the supergravity side analysis would match with those from the matrix theory analysis. We expect that the supergravity side analysis is helpful and provides clearer understanding about the structure of the effective potential.

The calculation in this paper has been carried out in the Minkowskian time signature. The basic reason why we have not taken the Wick rotation for some convenience in the actual calculation is the periodic nature of the circular motion background (4.3). The naive change of the Minkowskian time to the Euclidean one in the process of calculation breaks the periodicity of the background and the reality of the action. If one wants to study the time dependent background with the Euclidean time signature, he or she should begin with
the Euclidean action at the first setup. In the Euclidean case, the time dependent solutions to the equations of motion are given by hyperbolic functions, which give open paths not periodic ones. In the sense that there is no classical background solution corresponding to open path in the Minkowskian time plane-wave matrix model, it would be interesting to consider the model in the Euclidean time for studying such a background.

In the situation where the one-loop effective potential is flat, it is a natural step to consider the case where the radius $r$ between two fuzzy spheres is time dependent and calculate the form of the interaction. We hope to return to this issue in the near future.

Acknowledgments

One of us, K.Y., would like to thank M. Sakaguchi and D. Tomino for helpful discussions and comments. The work of H.S. was supported by Korea Research Foundation (KRF) Grant KRF-2001-015-DP0082.

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