Wavelet estimation of operator fractional Brownian motions *†‡

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Abstract

Operator fractional Brownian motion (OFBM) is the natural vector-valued extension of the univariate fractional Brownian motion. Instead of a scalar parameter, the law of an OFBM scales according to a Hurst matrix that affects every component of the process. Despite the theoretical relevance of OFBMs, the difficulties associated with the estimation of the generally non-diagonal Hurst parameter have effectively prevented its use in applications. This paper develops the wavelet analysis of OFBMs, as well as a new estimator for the Hurst matrix of bivariate OFBMs. Our approach relies on an original change of perspective: instead of considering the entry-wise behavior of the wavelet spectrum as a function of the (wavelet) scales, it draws upon the evolution along scales of the eigenstructure of the wavelet spectrum. This is shown to yield consistent and asymptotically normal estimators of the Hurst index in each coordinate, and of the coordinate system itself under assumptions. A simulation study is included to demonstrate the good performance of the estimators under finite sample sizes.

1 Introduction

The analysis and inference for univariate self-similar processes now comprises a voluminous and well-established literature. To name just a few references, Mandelbrot and Van Ness (1968), Fox and Taqqu (1986), Samorodnitsky and Taqqu (1994), Wornell and Oppenheim (1992), Flandrin (1992), Robinson (1995), Veitch and Abry (1999), Doukhan et al. (2003).

On one hand, the multivariate framework evokes several applications, such as in long range dependent time series (Marinucci and Robinson (2000), Davidson and de Jong (2000), Chung (2002), Dolado and Marmol (2004), Davidson and Hashimzade (2008)), Kechagias and Pipiras (2015) and queueing systems (Konstantopoulos and Lin (1996), Majewski (2003, 2005), Delgado (2007)). On the other hand, it brings forward a richer probabilistic perspective stemming from the so-named theory of operator self-similarity (Laha and Rohatgi (1981), Hudson and Mason (1982), Maejima and Mason (1994), Cohen et al (2010)). We say that a stochastic process

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\( \{X(t)\}_{t \in \mathbb{R}} = \{(X_1(t), \ldots, X_n(t))^*\}_{t \in \mathbb{R}} \) is operator self-similar (o.s.s.) when its law scales according to a matrix (Hurst) exponent \( H \), i.e.,
\[
\{X(ct)\}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{e^{H}X(t)\}_{t \in \mathbb{R}}, \quad c > 0,
\]
where \( e^H = \sum_{k=0}^{\infty} \log^k(c)H^k/k! \). The study of operator self-similarity is related to that of operator stable measures (e.g., Jurek and Mason (1993) and Meerschaert and Scheffler (2001)), and of operator scaling random fields (e.g., Biermé et al. (2007), Baek et al. (2014)).

A core parametric class in operator self-similarity theory is that of operator fractional Brownian motions (OFBMs), namely, proper Gaussian, o.s.s., stationary increment stochastic processes, which naturally generalize the univariate fractional Brownian motions (FBMs). Under a mild condition on the eigenvalues of the exponent \( H \) (see (2.9)), Didier and Pipiras (2011) showed that any OFBM \( B_H \) admits a harmonizable representation
\[
\{B_H(t)\}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \left\{ \int_{\mathbb{R}} e^{itx} \frac{1}{ix} (x^{-D}A + x^{-D}A^*)\tilde{B}(dx) \right\}_{t \in \mathbb{R}}
\]
for some complex-valued matrix \( A \). In (1.2), \( x_\pm = \max\{\pm x, 0\} \),
\[
D = H - \frac{1}{2} I,
\]
and \( \tilde{B}(dx) \) is a complex-valued random measure such that \( \tilde{B}(-dx) = \overline{\tilde{B}(dx)} \), \( E\tilde{B}(dx)\tilde{B}(dx)^* = dx \), where \( * \) represents Hermitian transposition. Expression (1.2) shows that OFBMs are characterized by the matrices \( H \) and \( A \).

Let \( H = PJ_HP^{-1}, P \in GL(n, \mathbb{C}) \), be the Jordan form of the Hurst parameter in (1.2) (see Section 2 for matrix notation). If \( H \) is diagonal with real eigenvalues, then we can assume that \( P \) takes the form of a scalar matrix \( P = pI \), where \( p \in \mathbb{R} \) and \( I \) is the \( n \times n \) identity matrix, and that \( J_H = \text{diag}(h_1, \ldots, h_n) \). In this case, (1.1) can be broken down into simultaneous entry-wise expressions
\[
\{X(ct)\}_{t \in \mathbb{R}} \overset{\mathcal{L}}{=} \{(c_1X_1(t), \ldots, c_nX_n(t))^*\}_{t \in \mathbb{R}}, \quad c > 0.
\]
Relation (1.4) is henceforth called \textit{entry-wise scaling}. In particular, under (1.4) an OFBM is a vector of correlated FBM entries (Amblard et al. (2012), Coeurjolly et al. (2013)). Several estimators have been developed by building upon the univariate, entry-wise scaling laws, e.g., the Fourier-based multivariate local Whittle (e.g., Shimotsu (2007), Nielsen (2011)) and the multivariate wavelet regression (Wendt et al. (2009), Amblard and Coeurjolly (2011), Achard and Gannaz (2014)). However, if \( H \) is non-diagonal, i.e., if \( P \) is not a scalar matrix, then the relation (1.1) mixes together the several entries of \( X \). The estimation problem under a non-scalar \( P \) turns out to be rather intricate and calls for the construction of methods that are multivariate from their inception. To this day, this issue has remained to a great extent open in the literature.

Although the emergence of o.s.s. processes in applications is rightly expected – e.g., as functional weak limits of multivariate time series –, there is no specific reason to believe a priori that scaling laws must occur predominantly entry-wise and \textit{exactly} along the canonical axes. Indeed, this is palpably not true in several applications such as fractional blind-source separation (see Didier et al. (2015)) and fractional co-integration (see Robinson (2008) for a bivariate local Whittle estimator). Both cases are subsumed under the framework of a multivariate mixed fractional time series. Let \( X = \{X_t\}_{t \in \mathbb{Z}} \) be an unobserved signal whose spectral density \( f_X(x) \) satisfies
\[
f_X(x)_{ij} \sim c_{ij}|x|^{\beta_{ij}}, \quad x \to 0^+, \quad -1 < \beta_{ij} < 1, \quad c_{ij} \in \mathbb{C}, \quad i, j = 1, \ldots, n.
\]
The observed signal is

\[ Y = \{ Y_t \}_{t \in \mathbb{Z}} = \{ PX_t \}_{t \in \mathbb{Z}}, \]  

(1.6)

for a non-scalar, mixing matrix \( P \in GL(n, \mathbb{R}) \). If \( X_t = B_H(t) - B_H(t - 1) \), i.e., an operator fractional Gaussian noise, then \( H = P \text{diag}(d_1 + 1/2, \ldots, d_n + 1/2)P^{-1} \) and \( \beta_{ij} = d_i + d_j \). In blind source separation, the entries of \( X \) are uncorrelated, whereas in cointegration they are typically correlated.

From a mathematical standpoint, the mixing of scaling laws can be illustrated by means of the expression for the spectral density \( f_X \) of an OFBM with Hurst parameter

\[ H = P \text{diag}(h_1, h_2)P^{-1}, \quad 0 < h_1 < h_2 < 1, \quad P \in GL(2, \mathbb{R}). \]  

(1.7)

For \( M := P^{-1}AA^*(P^*)^{-1} \) and \( x > 0 \), the spectral density takes the form \( \left(f_X(x)_{ij}\right) = x^{-D}AA^*x^{-D^*} \), where

\[
\begin{align*}
    f_X(x)_{11} &= p_{11}^2 m_{11}x^{-2d_1} + p_{11}p_{12} (m_{12} + m_{12}) x^{-(d_1 + d_2)} + p_{12}^2 m_{22} x^{-2d_2}, \\
    f_X(x)_{21} &= p_{21} p_{11} m_{11} x^{-2d_1} + (p_{22} p_{11} m_{12} + p_{21} p_{12} m_{12}) x^{-(d_1 + d_2)} + p_{22} p_{12} m_{22} x^{-2d_2}, \\
    f_X(x)_{22} &= p_{21}^2 m_{11} x^{-2d_1} + p_{22} p_{21} (m_{12} + m_{12}) x^{-(d_1 + d_2)} + p_{22}^2 m_{22} x^{-2d_2} \quad (1.8)
\end{align*}
\]

(see (4.21) for the analogous expression in the wavelet domain). The univariate-inspired approach of setting up a Fourier-domain log-regression – e.g., Whittle-type estimators – has to cope with the double-sided challenge of mixed power laws. On one hand, under mild assumptions on the amplitude coefficients, the strongest power law \( x^{-2d_2} \) always prevails around the origin of the spectrum. On the other hand, and paradoxically, even if the estimation of \( d_2 \) is the target, the magnitude of the amplitude coefficients themselves can arbitrarily bias the estimate over finite samples by masking the power laws involved.

In this work, we carry out the wavelet analysis of OFBMs, which is of interest by itself. Moreover, we build upon such analysis to propose a novel wavelet-based estimation method for bivariate (and potentially multivariate) OFBMs. The method yields the Hurst eigenvalues of \( H \) and, under assumptions, also its eigenvectors. Its essential ingredient, and the main theme of this paper, is an original change of perspective: instead of considering the entry-wise behavior of the wavelet spectrum as a function of wavelet scales, it draws upon the evolution along scales of the eigenstructure of the wavelet spectrum. This way, one avoids much of the difficulty associated with inference in the presence of mixed power laws, as we now explain.

For a wavelet function \( \psi \in L^2(\mathbb{R}) \) with a number \( N_\psi \) of vanishing moments (see (2.6)), the (vector) wavelet transforms of OFBMs are naturally defined as

\[
\mathbb{R}^n \ni D(2^j, k) = \int_\mathbb{R} 2^{-j/2} \psi(2^{-j}t - k)B_H(t) dt, \quad j \in \mathbb{N} \cup \{0\}, \quad k \in \mathbb{Z}, \]  

(1.9)

provided the integral in (1.9) exists in an appropriate sense. The wavelet-domain process \( \{D(2^j, k)\}_{k \in \mathbb{Z}} \) is stationary in \( k \) and o.s.s. in \( j \) (Proposition 3.1). Moreover, whereas the original stochastic process \( B_H(t) \) displayed fractional memory, the covariance between (multivariate) wavelet coefficients decays as a function of \( |2^j k - 2^j k'| \) according to an inverse fractional power controlled by \( N_\psi \) (Proposition 3.2). The wavelet spectrum at scale \( j \) is the positive definite matrix

\[
ED(2^j, k)D(2^j, k)^* = ED(2^j, 0)D(2^j, 0)^* =: EW(2^j),
\]
and its natural estimator, the sample wavelet transform, is the matrix statistic

$$W(2^j) = \frac{1}{K_j} \sum_{k=1}^{K_j} D(2^j, k) D(2^j, k)^*, \quad K_j = \frac{\nu}{2^j}, \quad j = j_1, \ldots, j_m,$$

(1.10)

for a total of $\nu$ (data) points. Under the bivariate framework (1.7), the univariate-like entry-wise scaling approach would consist of exploiting the behavior of each component $W(2^j)_{i_1, i_2}$, $i_1, i_2 = 1, 2$, of the sample wavelet transform $W(2^j)$ as functions of the scales $2^j$. Apart from an amplitude effect, the entries are then controlled by the largest Hurst eigenvalue $h_2$. Figure 1, top panels, illustrates the fact that this precludes the estimation of $h_1$.

The proposed estimators of the Hurst eigenvalues $h_1$ and $h_2$ are

$$\hat{h}_1(2^j) = \frac{\log \lambda_1(2^j)}{2 \log(2^j)}, \quad \hat{h}_2(2^j) = \frac{\log \lambda_2(2^j)}{2 \log(2^j)},$$

(1.11)

where $\lambda_1(2^j) \leq \lambda_2(2^j)$ are the eigenvalues of the positive definite symmetric matrix $W(2^j)$ (see Definition 4.1 for the precise assumptions). However, as usual with operator self-similarity, also the finite sample expressions for $\lambda_1(2^j)$ and $\lambda_2(2^j)$ involve a mixture of distinct power laws $2^{jh_1}$, $2^{(h_1+h_2)}$, $2^{2h_2}$ ($h_1 < h_2$). For this reason, one must take the limit at coarse scales, namely, the scale itself must go to infinity. It is a remarkable fact that the power law $2^{2h_1}$ ends up prevailing in the expression for $\lambda_1(2^j)$ (see Figure 1, bottom panels, and the striking contrast with the top panels; see Section 4.1 for a mathematically motivated, intuitive discussion). The convergence of (1.11) in turn allows for the convergence of associated sequences of eigenvectors, under the assumption that $P$ is orthogonal. Moreover, simulation studies show that the estimation procedure is accurate and computationally fast. The asymptotics are mathematically developed in two stages. In the first, the wavelet scales (octaves) are held fixed and the asymptotic distribution of the sample wavelet transform is obtained (Proposition 3.3 and Theorem 3.1). In the second, one takes the limit with respect to the scales themselves; however, the latter must go to infinity slower than the sample size, a feature our estimator shares with Fourier or wavelet-based semiparametric estimators in general (e.g., Robinson (1995a), Moulines et al. (2007b, 2007a, 2008)).

Our results are related to the literature on the estimation of operator stable laws via eigenvalues and eigenvectors of sample quadratic forms (see Meerschaert and Scheffler (1999, 2003)). In this context, one encounters the same problem with the prevalence of some stronger power law (i.e., the tail exponent) in most directions. In Becker-Kern and Pap (2008), a similar philosophy is applied in the time domain to produce one of the very few available estimators for authentic, mixed scaling o.s.s. processes of dimension up to 4. However, the asymptotics provided are restricted to consistency. In our work, the wavelet transform is the main tool for ensuring the consistency and asymptotic normality of the proposed estimators.

The paper is organized as follows. Section 2 contains the notation, assumptions and basic concepts. Section 3 is dedicated to the wavelet analysis of $n$-dimensional OFBMs, as well as the asymptotics of the wavelet transform for fixed scales. In Section 4, the estimation method for the Hurst exponent of bivariate OFBMs is laid out in full detail and its asymptotics are established at coarse scales. Section 5 displays finite sample computational studies, including one of the performance of the estimators under blind source separation and cointegrated instances. The research contained in this paper leads to a number of interesting open questions, which are mentioned in Section 6. The appendix contains several auxiliary mathematical results.
2 Notation and assumptions

All through the paper, the dimension of OFBMs is denoted by $n \geq 2$.

We shall use throughout the paper the following notation for finite-dimensional operators (matrices). All with respect to the field $\mathbb{R}$, $M(n)$ or $M(n, \mathbb{R})$ is the vector space of all $n \times n$ matrices (endomorphisms), $GL(n)$ or $GL(n, \mathbb{R})$ is the general linear group (invertible matrices, or automorphisms), $O(n)$ is the orthogonal group of matrices $O$ such that $OO^* = I = O^*O$ (i.e., the adjoint operator is the inverse), and $S(n, \mathbb{R})$ is the space of symmetric matrices. A block-diagonal matrix with main diagonal blocks $\mathcal{P}_1, \ldots, \mathcal{P}_n$ or $m$ times repeated diagonal block $\mathcal{P}$ is represented by

$$\text{diag}(\mathcal{P}_1, \ldots, \mathcal{P}_n), \quad \text{diag}_m(\mathcal{P}),$$

respectively. The symbol $\| \cdot \|$ represents a generic matrix or vector norm. The $l_p$ entry-wise norm of the matrix $A$ is denoted by

$$\|A\|_{l_p} = \| (a_{i_1, i_2})_{i_1=1, \ldots, m} \|_{l_p} = \left( \sum_{i_1=1}^{m} \sum_{i_2=1}^{n} |a_{i_1, i_2}|^p \right)^{1/p}.$$  

(2.2)

$\Re(M)$ and $\Im(M)$ denote the real and imaginary parts, respectively, of a generic matrix $M \in M(n, \mathbb{R})$. 

Figure 1: Entry-wise vs eigenvalue-based estimation. From one synthetic realization of OFBM, the top row displays (black solid lines with $\ast$), from left to right, $\log_2 W(2^j)_{1,1}$ vs $j$, $\log_2 W(2^j)_{1,2}$ vs $j$ and $\log_2 W(2^j)_{2,2}$ vs. $j$, on which the asymptotic behavior $j2h_2$ is superimposed (red dashed line with ‘o’). All auto- and cross-components are then driven by the largest Hurst eigenvalue $h_2$, which precludes the estimation of the smallest eigenvalue $h_1$. The bottom row displays (black solid lines with $\ast$), from left to right, $\log_2 \lambda_1(2^j)$ vs $j$ and $\log_2 \lambda_2(2^j)$ vs $j$, with their respective asymptotic trends $j2h_1$ and $j2h_2$ superimposed (red dashed line with ‘o’). This shows that both Hurst eigenvalues $h_1$ and $h_2$ can be estimated (see Section 5.1 for details on the numerical simulations).
The functions
\[ \pi_{i_1}(v), \pi_{i_1,i_2}(M), \quad v \in \mathbb{R}^n, \quad M \in M(n, \mathbb{R}), \] (2.3)
denote, respectively, the \(i_1\)-th projection (entry) of the vector \(v\) and the \((i_1,i_2)\)-th projection (entry) of the matrix \(M\). For notational simplicity, we write \(\pi_{i_1,i_1} = \pi_{i_1}\). For \(S = (s_{i_1,i_2})_{i_1,i_2=1,...,n} \in \mathcal{S}(n, \mathbb{R})\), we define the operator
\[ \text{vec}_S(S) = (s_{11}, \ldots, s_{nn}, s_{12}, \ldots, s_{1n}, s_{23}, \ldots, s_{2n}, \ldots, s_{n-1,n})^* \] (2.4)
In other words, \(\text{vec}_S(S)\) vectorizes the lower triangular entries of the symmetric matrix \(S\).

When establishing bounds, \(C\) is used to denote a positive constant whose value can change from one inequality to another. For a sequence of random vectors \(\{X_l,Y_l\}_{l \in \mathbb{N}}\), \(P(Y_l = 0) = 0\), we write
\[ X_l \overset{P}{\sim} Y_l \] (2.5)
to mean that \(X_l/Y_l \overset{P}{\to} 1, \ l \to \infty\). Note that this does not imply that \(\{X_l,Y_l\}_{l \in \mathbb{N}}\) converges in probability. Relations of the type (2.5) will often appear in the proofs of results found in Section 4.

All through the paper, we will make the following assumptions on the underlying wavelet basis. For this reason, such assumptions will be omitted from the statements.

**Assumption (W1):** \(\psi \in \mathbb{R}\) is a wavelet function, namely,
\[ \int_{\mathbb{R}} \psi(t)^2 dt = 1, \quad \int_{\mathbb{R}} t^q \psi(t) dt = 0, \quad q = 0, 1, \ldots, N_\psi - 1, \quad N_\psi \geq 2. \] (2.6)

**Assumption (W2):** \(\text{supp}(\psi)\) is a compact interval. (2.7)

**Assumption (W3):** there is \(\alpha > 1\) such that
\[ \sup_{x \in \mathbb{R}} |\hat{\psi}(x)|(1 + |x|)^\alpha < \infty. \] (2.8)

Under (2.6), (2.7) and (2.8), \(\psi\) is continuous, \(\hat{\psi}(x)\) is everywhere differentiable and its first \(N_\psi - 1\) derivatives are zero at \(x = 0\) (see Mallat (1999), Theorem 6.1 and the proof of Theorem 7.4).

**Example 2.1** If \(\psi\) is a Daubechies wavelet with \(N_\psi\) vanishing moments, \(\text{supp}(\psi) = [0, 2N_\psi - 1]\) (see Mallat (1999), Proposition 7.4).

Under the framework of the harmonizable representation (1.2), throughout the paper we will make the following assumptions on the OFBMs \(B_H = \{B_H(t)\}_{t \in \mathbb{R}}\).

**Assumption (OFBM1):** the eigenvalues \(h_k\) of the matrix exponent \(H\) satisfy
\[ 0 < \Re(h_k) < 1, \quad k = 1, \ldots, n. \] (2.9)

**Assumption (OFBM2):** \(\Re(\AA^*)\) has full rank. (2.10)

The condition (2.9) generalizes the familiar constraint \(0 < H < 1\) on the Hurst parameter of a FBM. As shown in Didier and Pipiras (2011), it ensures the existence of the harmonizable
representation (1.2). In turn, recall that a stochastic process is called proper when, except at \( t = 0 \), its one-dimensional distributions are full dimensional. The condition (2.10) is sufficient (though not necessary) for the integral on the right-hand side of (1.2) to be a proper stochastic process and hence to define an OFBM.

The next two assumptions will appear in some of the results.

**Assumption (OFBM3):**

\[ \Im(AA^*) = 0. \]  
(2.11)

**Assumption (OFBM4):** \( B_H = \{ B_H(t) \}_{t \in \mathbb{R}} \) is a bivariate OFBM with scaling matrix

\[ H = PJ_H P^{-1} = P \text{diag}(h_1, h_2) P^{-1}, \quad h_1 < h_2, \quad P \in GL(2, \mathbb{R}). \]  
(2.12)

The condition (2.11) is equivalent to time reversibility, namely, \( \{ B_H(-t) \} \overset{c}{=} \{ B_H(t) \} \). In turn, the latter is equivalent to the existence of a closed form expression for the covariance function, i.e.,

\[ \text{EB}_H(s)B_H(t)^* = \frac{1}{2} \{ |t|H \Sigma |t|^H + |s|^H \Sigma |s|^H - |t - s|^H \Sigma |t - s|^H \}, \]  
(2.13)

where \( \Sigma := \text{EB}_H(1)B_H(1)^* \) (Didier and Pipiras (2011), Proposition 5.2). Time reversibility is used in some of the results in Section 3 and in all Section 4. The bivariate framework (2.12) is only used towards results on the limit at coarse scales (Section 4), due to the availability of convenient formulas for eigenvalues and eigenvectors.

### 3 Wavelet analysis

In this section, we carry out the wavelet analysis of \( n \)-dimensional OFBMs. All proofs can be found in the appendix.

#### 3.1 Basic properties

The wavelet transform (1.9) is itself a vector-valued random field in the scale and shift parameters \( j \) and \( k \), respectively. It will be convenient to make the change-of-variables \( z = 2^{-j}t - k \), and reexpress

\[ D(2^j, k) = 2^{j/2} \int_{\mathbb{R}} \psi(z) B_H(2^j z + 2^j k) dz. \]  
(3.1)

As in the univariate case, the wavelet coefficients of OFBMs exhibit a number of good properties. The next proposition describes such properties as well as the general form of the wavelet spectrum (variance).

**Proposition 3.1** Under the assumptions (OFBM1)–(2), let \( \{ D(2^j, k) \}_{j \in \mathbb{N}, k \in \mathbb{Z}} \) be as in (3.1). Then,

(P1) the wavelet transform (1.9) is well-defined in the mean square sense, and

\[ ED(a, k) = 0; \]  
(3.2)

(P2) (stationarity for a fixed scale)

\[ \{ D(2^j, k + h) \}_{k \in \mathbb{Z}} \overset{c}{=} \{ D(2^j, k) \}_{k \in \mathbb{Z}}, \quad h \in \mathbb{Z}; \]  
(3.3)
(P3) (operator self-similarity over different scales)

\[ \{D(2^j, k)\}_{k \in \mathbb{Z}} \subseteq \{2^j H D(0, k)\}_{k \in \mathbb{Z}}; \quad (3.4) \]

(P4) the wavelet spectrum \( EW(2^j, k) \equiv EW(2^j) \) is given by

\[ EW(2^j) = 2^j \int \left( x_+^{-D} A A^* x_+^{-D} + x_+^{-D} A A^* x_+^{-D} \right) \frac{\hat{\psi}(2^j x)^2}{x^2} \, dx \quad (3.5) \]

(P5) the wavelet spectrum satisfies the operator scaling relation

\[ EW(2^j) = 2^j H \, EW(0) 2^{jH^*} = 2^j H \left\{ \int \int R(t) R(t') \, EB_H(t) B_H(t') \, dt \, dt' \right\} 2^{jH^*}, \quad (3.6) \]

\( j \in \mathbb{N} \). In particular, under (2.11),

\[ EW(2^j) = -\frac{1}{2} \, 2^j H \left\{ \int \int R(t) R(t') \, |t - t'|^H \Sigma |t - t'|^H \, dt \, dt' \right\} 2^{jH^*}; \quad (3.7) \]

(P6) analogously to (P5),

\[ W(2^j) = 2^j H \left( \frac{1}{K_j} \sum_{k=1}^{K_j} D(0, k) D(0, k)^* \right) 2^{jH^*}; \quad (3.8) \]

(P7) the wavelet spectrum has full rank, namely,

\[ \det EW(2^j) \neq 0, \quad j \in \mathbb{N}. \quad (3.9) \]

Fix some \( 0 < \delta < 1 \), and consider the range of wavelet parameters \( j, j', k, k' \) such that

\[ \frac{1 - \delta}{\max\{2^j, 2^{j'}\}} \leq \frac{1 - \delta}{\text{length}(\text{supp}(\psi))}. \quad (3.10) \]

If the parameters \((j, k)\) and \((j', k')\) of two wavelet coefficients satisfy (3.10), then we can interpret that they are “far apart” in the parameter space. The next result provides a notion of decay of the covariance between wavelet coefficients under (3.10). The proof is similar to that for the univariate case, but we provide it (in the appendix) for the reader’s convenience.

**Proposition 3.2** Under the assumptions (OFBM1)–(3) and (3.10), the covariance between wavelet coefficients (3.1) satisfies the relation

\[ ED(2^j, k) D(2^{j'}, k')^* \]

\[ = -\frac{1}{2} \, 2^{j + j'}(1/2 + N_v) \left\{ |2^j k - 2^{j'} k'|^H \left( O^{N_v}_{1,1} \right)_{i_1,i_2=1,...,n} \right\}; \quad (3.11) \]

where \( O^{N_v}_{1,1} \) is an entry-wise bounded symmetric-matrix-valued function that depends only on \( N_v \). As a consequence,

\[ \| ED(2^j, k) D(2^{j'}, k')^* \| \]

\[ \leq C(N_v) \, 2^{(j + j')(1/2 + N_v)} \frac{1}{|2^j k - 2^{j'} k'|^{2N_v - 2 \max_{h \in \text{arg}(H)} \mathbb{R}(h)} \, \log^\delta |2^j k - 2^{j'} k'|, \quad (3.12) \]

where \( \delta \) is the dimension of the largest Jordan block in the spectrum of \( H \).
3.2 Asymptotics for sample wavelet transforms: fixed scales

As typical in the asymptotic study of averages, we begin by investigating the asymptotic covariance structure of sample wavelet transforms $W(2^j)$.

For FBM, the asymptotic covariance between wavelet transforms $W(2^j), W(2^j') \in \mathbb{R}$ is not available in closed form since it depends on the wavelet function, which is itself not available in closed form (c.f. Bardet (2002), Proposition II.3). Operator self-similarity adds a layer of intricacy, since in general exact entry-wise scaling relations are not present.

For notational simplicity, let

$$X = (X_1, \ldots, X_n) = (d_1(2^j, k), \ldots, d_n(2^j, k)), \quad Y = (Y_1, \ldots, Y_n) = (d_1(2^{j'}, k'), \ldots, d_n(2^{j'}, k')).$$

The bivariate case serves to illustrate the computation of covariances.

**Example 3.1** For a zero mean, Gaussian random vector $Z \in \mathbb{R}^m$, the Isserlis’ theorem (e.g., Vignat (2012)) yields

$$E(Z_1 \ldots Z_{2k}) = \sum \prod E(Z_iZ_j), \quad E(Z_1 \ldots Z_{2k+1}) = 0, \quad k = 0, \ldots, \lfloor m/2 \rfloor. \quad (3.14)$$

The notation $\sum \prod$ stands for adding over all possible $k$-fold products of pairs $E(Z_iZ_j)$, where the indices partition the set $1, \ldots, 2k$. So, let $X$ and $Y$ be as in (3.13) with $n = 2$. Then,

$$\text{Cov} \left( \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_1X_2 \end{pmatrix}, \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_1Y_2 \end{pmatrix} \right) = \begin{pmatrix} 2E(X_1Y_1)^2 & 2E(X_1Y_2)^2 & 2E(X_1Y_1)E(X_1Y_2) \\ 2E(X_2Y_1)^2 & 2E(X_2Y_2)^2 & 2E(X_2Y_1)E(X_2Y_2) \\ 2E(X_1Y_1)E(X_2Y_1) & 2E(X_1Y_2)E(X_2Y_2) & c_{33} \end{pmatrix}, \quad (3.15)$$

where $c_{33} = E(X_1Y_1)E(X_2Y_2) + E(X_1Y_2)E(X_2Y_1)$.

Expression (3.15) shows that the asymptotic behavior of the second moments of $W(2^j)$ involves several cross products. A notationally economical way of tackling this difficulty is by resorting to Kronecker products. For instance, in the bivariate case,

$$M(4, \mathbb{R}) \ni EXY^* \otimes EYX^* = \begin{pmatrix} E(X_1Y_1) & E(X_1Y_2) \\ E(X_2Y_1) & E(X_2Y_2) \end{pmatrix} \otimes \begin{pmatrix} E(X_1Y_1) & E(X_1Y_2) \\ E(X_2Y_1) & E(X_2Y_2) \end{pmatrix}$$

contains all the 9 terms (two-fold products of cross moments), as well as a few repeated ones, needed to express (3.15). In view of (3.14), this fact extends to general dimension $n$ by means of the relations

$$\text{Cov}(X_{i_1}X_{i_2}, Y_{j_1}Y_{j_2}) = E(X_{i_1}Y_{j_1})E(X_{i_2}Y_{j_2}) + E(X_{i_1}Y_{j_2})E(X_{i_2}Y_{j_1}), \quad i_1, i_2, j_1, j_2 = 1, \ldots, n. \quad (3.16)$$

The next proposition provides an expression that encompasses the asymptotic fourth moments of the wavelet coefficients.

**Proposition 3.3** Let $B_H = \{B_H(t)\}_{t \in \mathbb{R}}$ be an OFBM under the assumptions (OFBM1) – (3). As $\nu \to \infty$, for every pair of octaves $j, j'$,
Theorem 3.1

The definition of a wavelet only requires $N_{\psi} \geq 1$, but $N_{\psi} \geq 2$ (see (2.6)) is needed for the convergence in Proposition 3.3.

The next theorem establishes the asymptotics for the vectorized sample wavelet transforms $(\text{vec}_S W(2^j))_{j=j_1,\ldots,j_m}$ (see (2.4)) at a fixed set of scales.

Theorem 3.1

Let $B_H = \{B_H(t)\}_{t \in \mathbb{R}}$ be an OFBM under the assumptions (OFBM1)–(3). Let $F \in S(\frac{n(n+1)}{2}, \mathbb{R})$ be the asymptotic covariance matrix described in Proposition 3.3. Then,

$$
\left(\sqrt{K_j} (\text{vec}_S W(2^j) - \text{vec}_S EW(2^j))\right)_{j=j_1,\ldots,j_m} \xrightarrow{d} \mathcal{N}_{\frac{n(n+1)}{2}}(0, F),
$$

as $\nu \to \infty$, where $j_1 < \ldots < j_m$.

4 A wavelet-based estimator for bivariate OFBMs

In this section, we switch to the bivariate framework (2.12), i.e., $n = 2$. As a consequence of the mixing of scaling laws which results from a non-diagonal $P$, the estimators (1.11) are generally biased at finite scales. However, in this section we draw upon explicit expressions for eigenvalues to establish the consistency and asymptotic normality of the estimators (1.11) as the wavelet scale grows according to a factor $a(\nu) \to \infty$, as $\nu \to \infty$, where $a(\nu) \in \mathbb{N}$. We are also able to show the consistency and asymptotic normality, in a sense to be defined, of a sequence of eigenvectors associated with the smallest eigenvalue.

The next definition describes the proposed estimators for the Hurst eigenvalues $h_1 < h_2$.

Definition 4.1

Let $B_H = \{B_H(t)\}_{t \in \mathbb{R}}$ be an OFBM under the assumptions (OFBM1)–(4). For $a(\nu) \in \mathbb{N}$, let $W(a(\nu)2^j)$ be the associated (symmetric) sample wavelet spectrum at scale $a(\nu)2^j$, and let

$$
\lambda_1(a(\nu)2^j) \leq \lambda_2(a(\nu)2^j)
$$

be its eigenvalues. The wavelet estimators at scale $a(\nu)2^j$ of the eigenvalues $h_1 < h_2$ are defined, respectively, as in expression (1.11) (with $a(\nu)2^j$ in place of $2^j$).
By analogy to (4.1) and (1.11), we denote the eigenvalues and normalized eigenvalues of \( EW(a(\nu)2^j) \), respectively, by
\[
\lambda_1^E(a(\nu)2^j) \leq \lambda_2^E(a(\nu)2^j), \quad \lambda_1^E(a(\nu)2^j) = \frac{\lambda_1^E(a(\nu)2^j)}{2 \log(a(\nu)2^j)}, \quad \lambda_2^E(a(\nu)2^j) = \frac{\lambda_2^E(a(\nu)2^j)}{2 \log(a(\nu)2^j)}.
\]

(4.2)

### 4.1 Motivation

In order to develop intuition on the proposed method, we address the following questions.

(Q.1) since the usual analysis of scaling behavior suggests that the fastest growth \( a(\nu)^{2h_2} \) should prevail, why is it so that \( \lambda_1(a(\nu)2^j) \sim Ca(\nu)^{2h_1} \)?

(Q.2) why isn’t the scaling law for \( \lambda_1(a(\nu)2^j) \) in general exact, namely, some fixed power law of \( a(\nu) \)?

It is instructive to reason in terms of expected values. We know that the operator scaling property (3.6) of the wavelet transform yields
\[
EW(a(\nu)2^j) = P \text{diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^{-1} EW(2^j)(P^*)^{-1} \text{diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^*.
\]

(4.3)

For the sake of clarity, we will focus on the particular case \( EW(2^j) = P \text{diag}(w_1, w_2) P^* \), for \( w_1 = w_1(2^j) > 0, w_2 = w_2(2^j) > 0 \). This corresponds to the choice \( A = P \) in (1.2). The wavelet scaling relation (4.3) becomes
\[
EW(a(\nu)2^j) = P \text{diag}(w_1 a(\nu)^{2h_1}, w_2 a(\nu)^{2h_2}) P^*.
\]

(4.4)

The expression (4.4) is quite informative. When the matrix \( P \) is orthogonal, (4.4) is the spectral decomposition of \( EW(a(\nu)2^j) \). Thus, (Q.1) is straightforward, and in regard to (Q.2), the scaling law for \( \lambda_1^E(a(\nu)2^j) \) is, indeed, exact. But even when the matrix \( P \) is not orthogonal, (4.4) implies that every entry in \( EW(a(\nu)2^j) \) is a linear combination, with fixed coefficients, of the power laws \( a(\nu)^{2h_1}, a(\nu)^{2h_2} \). Now let \( u_0 = P^{-1} e_1 \), where \( e_1 \) is the first Euclidean vector. Then,
\[
\lambda_1^E(a(\nu)2^j) = \inf_{u \in S^1} u^* EW(a(\nu)2^j) u \leq u_0^* EW(a(\nu)2^j) u_0 = w_1 a(\nu)^{2h_1}.
\]

(4.5)

Since \( \lambda_1^E(a(\nu)2^j) \) is itself a linear combination of the entries of \( EW(a(\nu)2^j) \), then in general it blows up according to a power law which is no faster than \( a(\nu)^{j h_1} \). Therefore, \( \lambda_1^E(a(\nu)2^j) \sim Ca(\nu)^{2h_1} \). This provides an answer to (Q.1).

Furthermore, and more explicitly,
\[
\lambda_1^E(a(\nu)2^j) = \frac{1}{2} \left\{ EW(a(\nu)2^j)_{11} + EW(a(\nu)2^j)_{22} - \sqrt{\Delta(a(\nu)2^j)} \right\}
\]

(4.6)

where \( \Delta(a(\nu)2^j) = (EW(a(\nu)2^j)_{11} - EW(a(\nu)2^j)_{22})^2 + 4(EW(a(\nu)2^j)_{12})^2 \), and
\[
EW(a(\nu)2^j) = \left( EW(a(\nu)2^j)_{l_1,l_2} \right)_{l_1,l_2=1,2}
\]

\[
= \begin{pmatrix}
    p_{11}^2 w_1 a(\nu)^{2h_1} + p_{12}^2 w_2 a(\nu)^{2h_2} & p_{11} p_{21} w_1 a(\nu)^{2h_1} + p_{12} p_{22} w_2 a(\nu)^{2h_2} \\
    p_{11} p_{21} w_1 a(\nu)^{2h_1} + p_{12} p_{22} w_2 a(\nu)^{2h_2} & p_{21}^2 w_1 a(\nu)^{2h_1} + p_{22}^2 w_2 a(\nu)^{2h_2}
\end{pmatrix}
\]
Further calculations based on a second order Taylor expansion of the function $f(x) = \sqrt{x}$ around 1 yield

$$
\lambda^E_1(a(\nu)2^j) = \frac{2 \det EW(2^j)}{(p_{11}^2 + p_{21}^2)w_{11}a(\nu)^{2(h_1-h_2)} + (p_{12}^2 + p_{22}^2)w_{22}} a(\nu)^{2h_1} \\
\{ \frac{1}{2} + \frac{1}{2} [(p_{11}^2 + p_{21}^2)w_{11}a(\nu)^{2(h_1-h_2)} + (p_{12}^2 + p_{22}^2)w_{22}]^2 a(\nu)^{2(h_2-h_1)} \} + o\left(\frac{1}{a(\nu)^{2(h_2-h_1)}}\right) 
$$

(see Lemma B.1 and expression (B.7)). Unsurprisingly, (4.7) is not an exact scaling law; this answers question (Q.2) under (4.4). However, it is in accordance with (4.5) and confirms our intuition by disclosing that the quality of the approximation $\lambda^E_1(a(\nu)2^j) \sim Ca(\nu)^{2h_1}$ increases with the difference $h_2 - h_1$.

**Remark 4.1** The expression (4.7) can be easily generalized beyond the assumption (4.4).

**Remark 4.2** Expression (4.5) further shows that not only the eigendirection associated with the smallest eigenvalue $\lambda^E_1(a(\nu)2^j)$ has “slow” increase of order $a(\nu)^{2h_1}$, namely, the direction $u_0 = P^{-1}e_1$ always displays the same asymptotic behavior. See Meerschaert and Scheffler (2003) for a related discussion under the framework of operator-stable measures.

### 4.2 The weak limit of eigenvalues

As illustrated in Section 4.1, this paper’s estimation undertaking draws upon the behavior at coarse scales of the sample wavelet spectra (variances)

$$
W(a(\nu)2^j) = \frac{1}{K_{a,j}} \sum_{k=1}^{K_{a,j}} D(a(\nu)2^j, k)D(a(\nu)2^j, k)^*, \quad K_{a,j} = \frac{\nu}{a(\nu)2^j}, 
$$

(4.8)

where $a(\nu)$ is a positive integer sequence such that

$$
\frac{1}{a(\nu)} + \frac{a(\nu)^3}{\nu} \to 0, \quad \nu \to \infty. 
$$

(4.9)

**Remark 4.3** A sequence satisfying (4.9) and commonly found in the practice of wavelet estimation is $a(\nu) = \lceil \log_2(\nu) \rceil$. This is so because doubling the sample size allows for shifting up by 2 the (largest) wavelet scale used.

We will make use of some basic relations for bivariate symmetric matrices. Recall that for some matrix

$$
S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, 
$$

(4.10)

the eigenvalues can be expressed in closed form as

$$
\lambda_1 = \frac{(a + c) - \sqrt{\Delta}}{2} \leq \lambda_2 = \frac{(a + c) + \sqrt{\Delta}}{2}, \quad \Delta = (a + c)^2 - 4(ac - b^2). 
$$

(4.11)

As a consequence, if $\det(S) \neq 0$,

$$
\lambda_1 = \frac{1}{2}(a + c) \left( 1 - \sqrt{1 - \frac{4(ac - b^2)}{(a + c)^2}} \right) = \frac{1}{2} \frac{4(ac - b^2)}{a + c} \frac{1 - \sqrt{1 - \frac{4(ac - b^2)}{(a + c)^2}}}{4(ac - b^2)/(a + c)^2}, 
$$

(4.12)
\[ \lambda_2 = \frac{1}{2} (a + c) \left( 1 + \sqrt{1 - \frac{4(ac - b^2)}{(a + c)^2}} \right). \] (4.13)

Moreover, assuming \( b \neq 0 \), for an eigenvector \( v = (v_1, v_2)^* \in \mathbb{R}^2 \) associated with an eigenvalue \( \lambda \), the relation \( S(v_1, v_2)^* = \lambda (v_1, v_2)^* \) yields

\[ v_2 = \frac{\lambda - a}{b} v_1. \] (4.14)

In all developments below, in view of the operator self-similarity property (3.8) we will consider the matrix statistics

\[ \hat{B}_a(2^j) := P^{-1}W_a(2^j)(P^*)^{-1}, \quad j = j_1 < \ldots < j_m, \] (4.15)

where \( W_a(2^j) = \frac{1}{K_{a,j}} \sum_{k=0}^{K_{a,j}-1} D(2^j, k)D(2^j, k)^* \) (compare with (1.10) and (4.8)). Each \( \hat{B}_a(2^j) \) is only a pseudo-estimator of

\[ B(2^j) := P^{-1}EW(2^j)(P^*)^{-1}, \quad P \in GL(2, \mathbb{R}), \] (4.16)

because it depends on the unknown parameter \( P \). It will be convenient to describe the matrices entry-wise as

\[ \hat{B}_a(2^j) = \left( \hat{b}_{i_1,i_2}(2^j) \right)_{i_1,i_2=1,2}, \quad B(2^j) = \left( b_{i_1,i_2}(2^j) \right)_{i_1,i_2=1,2}. \] (4.17)

The asymptotic distribution of the statistics (4.15) is described in the following lemma.

**Lemma 4.1** For \( m \in \mathbb{N} \), let \( j_1 < \ldots < j_m \) be a set of fixed scales \( j \). Let \( \Pi = \left( \pi_{i_1,i_2} \right)_{i_1,i_2=1,2} = P^{-1} \), and let \( \hat{B}_a(2^j) \), \( B(2^j) \) be as in (4.15), (4.16), respectively. Under the assumptions (OFBM1) – (4) and (4.9),

\[ \sqrt{K_{a,j}}(\text{vec}_S \hat{B}_a(2^j) - \text{vec}_SB(2^j)) \xrightarrow{d} N(0, \Sigma_B(j_1, \ldots, j_m)), \quad \nu \to \infty, \] (4.18)

In (4.18),

\[ \Sigma_B(j_1, \ldots, j_m) = \text{diag}_m(\mathcal{P})F\text{diag}_m(\mathcal{P})^* \] (4.19)

where \( F \) is the asymptotic covariance matrix in (3.18), \( \text{diag}_m(\mathcal{P}) \) is as defined in (2.1), and

\[ \mathcal{P} = \begin{pmatrix} \pi_{11}^2 & 2\pi_{11}\pi_{12} & \pi_{12}^2 \\ \pi_{11}\pi_{21} & \pi_{11}\pi_{22} + \pi_{12}\pi_{21} & \pi_{12}\pi_{22} \\ \pi_{21}^2 & 2\pi_{21}\pi_{22} & \pi_{22}^2 \end{pmatrix}. \]

**PROOF:** For any \( j \), a brief calculation shows that \( \text{vec}_S(\Pi W_a(2^j)\Pi^*) = \mathcal{P}\text{vec}_SW_a(2^j) \). Likewise, \( \text{vec}_S(\Pi EW(2^j)\Pi^*) = \mathcal{P}\text{vec}_SEW(2^j) \). Therefore, we can recast the left-hand side of (4.18) as

\[ \text{diag}_m(\mathcal{P}) \left( \sqrt{K_{a,j}}(\text{vec}_SW_a(2^j) - \text{vec}_SEW(2^j)) \right)_{j=j_1,\ldots,j_m}. \]

The weak limit in (4.18) is now a consequence of Theorem 3.1. \( \square \)

The next lemma contains expressions for the wavelet spectrum and its sample counterpart. The notation \( (\cdot)_{j=j_1,\ldots,j_m} \) designates a vector of \( 2 \times 2 \) matrices.
Lemma 4.2 For \( m \in \mathbb{N} \), let \( j_1 < \ldots < j_m \) be a set of fixed scales \( j \). Let \( \widehat{B}_a(2^j), B(2^j) \) be as in (4.15), (4.16), respectively, and \( \widehat{b}_{i_1,i_2}(2^j), b_{i_1,i_2}(2^j) \) be as in (4.17). For \( a(\nu) > 0 \), under the assumptions (OFBM1) – (4), we can express

\[
EW(a(\nu)2^j) = \left( \begin{array}{c}
\frac{a_{j,a}(\nu)}{b_{j,a}(\nu)} \\
\frac{b_{j,a}(\nu)}{c_{j,a}(\nu)}
\end{array} \right),
\]

where
\[
a_{j,a}(\nu) = p_{11}^2 b_{11}(2^j) a(\nu)^{2h_1} + 2p_{12}p_{11} b_{12}(2^j) a(\nu)^{h_1+h_2} + p_{12}^2 b_{22}(2^j) a(\nu)^{2h_2},
\]
\[
b_{j,a}(\nu) = p_{11} p_{21} b_{11}(2^j) a(\nu)^{2h_1} + (p_{12}p_{21} + p_{11}p_{22}) b_{12}(2^j) a(\nu)^{h_1+h_2} + p_{12}p_{22} b_{22}(2^j) a(\nu)^{2h_2},
\]
\[
c_{j,a}(\nu) = p_{21}^2 b_{11}(2^j) a(\nu)^{2h_1} + 2p_{22}p_{21} b_{12}(2^j) a(\nu)^{h_1+h_2} + p_{22}^2 b_{22}(2^j) a(\nu)^{2h_2}.
\]

Likewise,
\[
\left(W(a(\nu)2^j)\right)_{j = j_1, \ldots, j_m} \overset{d}{=} \left( \begin{array}{c}
\hat{a}_{j,a}(\nu) \\
\hat{b}_{j,a}(\nu) \\
\hat{c}_{j,a}(\nu)
\end{array} \right)_{j = j_1, \ldots, j_m},
\]

where
\[
\hat{a}_{j,a}(\nu) = p_{11}^2 \hat{b}_{11}(2^j) a(\nu)^{2h_1} + 2p_{12}p_{11} \hat{b}_{12}(2^j) a(\nu)^{h_1+h_2} + p_{12}^2 \hat{b}_{22}(2^j) a(\nu)^{2h_2},
\]
\[
\hat{b}_{j,a}(\nu) = p_{11} p_{21} \hat{b}_{11}(2^j) a(\nu)^{2h_1} + (p_{12}p_{21} + p_{11}p_{22}) \hat{b}_{12}(2^j) a(\nu)^{h_1+h_2} + p_{12}p_{22} \hat{b}_{22}(2^j) a(\nu)^{2h_2},
\]
\[
\hat{c}_{j,a}(\nu) = p_{21}^2 \hat{b}_{11}(2^j) a(\nu)^{2h_1} + 2p_{22}p_{21} \hat{b}_{12}(2^j) a(\nu)^{h_1+h_2} + p_{22}^2 \hat{b}_{22}(2^j) a(\nu)^{2h_2}.
\]

Proof: The operator scaling properties (3.6) and (3.8) yield
\[
EW(a(\nu)2^j) = P \text{diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) B(2^j) \text{ diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^*,
\]
\[
\left(W(a(\nu)2^j)\right)_{j = j_1, \ldots, j_m} \overset{d}{=} \left( P \text{ diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^{-1} W(2^j) (P^*)^{-1} \text{ diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^* \right)_{j = j_1, \ldots, j_m}
\]
\[
\overset{=}{=} \left( P \text{ diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) \widehat{B}_a(2^j) \text{ diag}(a(\nu)^{h_1}, a(\nu)^{h_2}) P^* \right)_{j = j_1, \ldots, j_m},
\]
whence (4.20) and (4.21) follow. \( \Box \)

The next theorem establishes the consistency and asymptotic normality of the estimators described in Definition 4.1. Intuitively, the theorem states that
\[
h_1(a(\nu)2^j) \approx h_1, \quad h_2(a(\nu)2^j) \approx h_2 \text{ with convergence rate } 2 \log(a(\nu)2^j) \sqrt{K_{a,j}}.
\]

Theorem 4.1 For \( m \in \mathbb{N} \), let \( j_1 < \ldots < j_m \) be a set of fixed scales \( j \). Let \( h_1, h_2, h_1^E, h_2^E \) be as in (1.11) and (4.2). Under the assumptions (OFBM1) – (4) and (4.9), as \( \nu \to \infty \),

(i)
\[
\left( \hat{h}_1(a(\nu)2^j), \hat{h}_2(a(\nu)2^j) \right) \overset{D}{\to} (h_1, h_2), \quad j = j_1, \ldots, j_m;
\]

(ii)
\[
\left( \frac{2 \log(a(\nu)2^j) \sqrt{K_{a,j}} [ \hat{h}_1(a(\nu)2^j) - h_1^E(a(\nu)2^j)]}{2 \log(a(\nu)2^j) \sqrt{K_{a,j}} [ \hat{h}_2(a(\nu)2^j) - h_2^E(a(\nu)2^j)]} \right)_{j = j_1, \ldots, j_m}
\]
\[
\overset{D}{\to} N(0, \Sigma_{h_1,h_2}(j_1, \ldots, j_m)).
\]
In (4.23),
\[ \Sigma_{h_1,h_2}(j_1,\ldots,j_m) = Q \Sigma_B(j_1,\ldots,j_m) Q^*, \]
where \( \Sigma_B(j_1,\ldots,j_m) \) is the covariance matrix in (4.19), and \( Q = (Q_{jj'})_{j,j'=j_1,\ldots,j_m} \) is a block matrix whose blocks have dimension 2 \times 3 and satisfy
\[ Q_{jj} = \left( \begin{array}{ccc} \frac{b_{22}(2j)}{\text{det } B(2j)} & - \frac{2h_{12}(2j)}{\text{det } B(2j)} & \left( \frac{b_{11}(2j) b_{22}(2j)}{\text{det } B(2j)} - 1 \right) \frac{1}{b_{22}(2j)} \\ 0 & 0 & -1 \\ \end{array} \right) \quad \text{and } Q_{j,j'} = 0, \text{ if } j \neq j'. \]

**Proof:** Fix an arbitrary \( j \). For notational simplicity we write \( B = B(2) = \left( b_{ij} \right)_{i,j=1,2} \). We will also drop the subscripts \( j, \nu \) in the expressions (4.20) and (4.21). Recall that the asymptotic distribution for \( \left( \hat{B}_a(2) \right)_{j=j_1,\ldots,j_m} \) is given by (4.18).

The statement (i) is a consequence of Lemma B.1 and Theorem 3.1. To show statement (ii), since \( 2 \log(a(\nu)^2) [h_1(a(\nu)^2) - h_1^E(a(\nu)^2)] = \log \lambda_1(a(\nu)^2) - \log \lambda_1(a(\nu)^2) \), \( i = 1, 2 \), we can use a Taylor expansion to rewrite the left-hand side of (4.23) as
\[ \left( \begin{array}{c} 
\sqrt{K_{a,j}} \left( \frac{\lambda_1(a(\nu)^2)}{\lambda_1^E(a(\nu)^2)} - 1 \right) \\
\sqrt{K_{a,j}} \left( \frac{\lambda_2(a(\nu)^2)}{\lambda_2^E(a(\nu)^2)} - 1 \right) 
\end{array} \right)_{j} + \left( \begin{array}{c} 
\sqrt{K_{a,j}} \frac{\lambda_1(a(\nu)^2)}{\lambda_1^E(a(\nu)^2)} - 1 \\
\sqrt{K_{a,j}} \frac{\lambda_2(a(\nu)^2)}{\lambda_2^E(a(\nu)^2)} - 1 
\end{array} \right)_{j}, \]
where \( j = j_1,\ldots,j_m \), and the residual function \( o_1(\cdot) \) does not depend on \( j \). The asymptotics will be written out for just one general term indexed \( j \), but the conclusions apply to the whole vector comprising the terms associated with \( j = j_1,\ldots,j_m \). Since all meaningful limits will boil down to a function of the variables that appear on the right-hand side of (4.18), they depend on the omitted, fixed octave \( j \).

We work only with the linear term in (4.25); after establishing its asymptotic normality, the usual argument shows that the residual term in (4.25) converges in probability to zero. In regard to \( \lambda_1 \), rewrite
\[ \sqrt{K_{a,j}} \left( \frac{\lambda_1(a(\nu)^2)}{\lambda_1^E(a(\nu)^2)} - 1 \right) = \sqrt{K_{a,j}} \left( \frac{\hat{\alpha} + \hat{\gamma}}{\alpha + c} - 1 \right) \left\{ \frac{1 - \sqrt{1 - 4(\hat{\alpha} \hat{\gamma} - \hat{b}^2) / (\hat{\alpha} + \hat{\gamma})^2}}{1 - \sqrt{1 - 4(ac - b^2) / (a + c)^2}} \right\} + \sqrt{K_{a,j}} \left\{ \frac{1 - \sqrt{1 - 4(\hat{\alpha} \hat{\gamma} - \hat{b}^2) / (\hat{\alpha} + \hat{\gamma})^2}}{1 - \sqrt{1 - 4(ac - b^2) / (a + c)^2}} - 1 \right\}, \]
(4.26)

If we show that the vector
\[ \sqrt{K_{a,j}} \left( \frac{\hat{\alpha} + \hat{\gamma}}{\alpha + c} - 1, 1 - \sqrt{1 - 4(\hat{\alpha} \hat{\gamma} - \hat{b}^2) / (\hat{\alpha} + \hat{\gamma})^2} \right)_{j=j_1,\ldots,j_m} \]
converges to a zero mean, \( 2m \)-variate Gaussian distribution, then by Slutsky’s theorem the asymptotic distribution of the expression (4.26) will be the same as that of
\[ \sqrt{K_{a,j}} \left( \frac{\hat{\alpha} + \hat{\gamma}}{\alpha + c} - 1 \right) + \sqrt{K_{a,j}} \left\{ \frac{1 - \sqrt{1 - 4(\hat{\alpha} \hat{\gamma} - \hat{b}^2) / (\hat{\alpha} + \hat{\gamma})^2}}{1 - \sqrt{1 - 4(ac - b^2) / (a + c)^2}} - 1 \right\}. \]
(4.27)

We first investigate the left entry of the vector (4.27). Based on the expression
\[ a + c = a(\nu)^2 h_2 \left\{ (p_{11}^2 + p_{21}^2) b_{11} a(\nu)^2 b_{11} + 2(p_{11} p_{12} + p_{21} p_{22}) b_{12} a(\nu)^h_1 + (p_{11}^2 + p_{22}^2) b_{22} \right\}, \]
(4.29)
and its sample counterpart \( \hat{a} + \hat{c} \) – namely, with the terms \( \hat{b}_{1,i_2} \) in place of \( b_{i_1,i_2} \), we have

\[
\sqrt{K_{a,j}} \frac{1}{a + c} \left\{ (\hat{a} + \hat{c}) - (a + c) \right\}
\]

\[
= \frac{1}{o(a(\nu)) + (p_{12}^2 + p_{22}^2)b_{22}} \left\{ (p_{11}^2 + p_{21}^2)\sqrt{K_{a,j}} (\hat{b}_{11} - b_{11})a(\nu)^{2(h_1-h_2)} + (p_{11}p_{12} + p_{21}p_{22})\sqrt{K_{a,j}} (\hat{b}_{12} - b_{12})a(\nu)^{h_1-h_2} + (p_{12}^2 + p_{22}^2)\sqrt{K_{a,j}} (\hat{b}_{22} - b_{22}) \right\} 
\]

\[
P \sim \sqrt{K_{a,j}} \frac{b_{22} - b_{22}}{b_{22}}. \quad (4.30)
\]

By contrast, tackling the right entry of the vector (4.27) is not as straightforward and will require resorting to the mean value theorem twice. By Lemma B.1,

\[
\left( \frac{-1}{4(ac-b^2)/(a+c)^2} \left( \frac{1}{1 - \sqrt{1 - 4(ac-b^2)/(a+c)^2}} \right) \right) 
\]

\[
\cdot \sqrt{K_{a,j}} \left\{ \sqrt{1 - 4(\hat{a}c - \hat{b})^2}/(\hat{a} + \hat{c})^2} - \sqrt{1 - 4(ac-b^2)/(a+c)^2}\right\}
\]

\[
P \sim (-1) \left( \frac{4 \det {kW(2j)}}{[p_{12}^2 + p_{22}^2]b_{22}^2 a(\nu)^{2(h_2-h_1)}} \right)^{-1} \left( \frac{1}{2} \right)^{-1} 
\]

\[
\sqrt{K_{a,j}} \left\{ \sqrt{1 - 4(\hat{a}c - \hat{b})^2}/(\hat{a} + \hat{c})^2} - \sqrt{1 - 4(ac-b^2)/(a+c)^2}\right\}. \quad (4.31)
\]

Let \( f_1(x) = \sqrt{x} \). Since \( 4(\hat{a}c - \hat{b})^2/(\hat{a} + \hat{c})^2 \xrightarrow{P} 0 \), \( 4(ac-b^2)/(a+c)^2 \rightarrow 0 \), then with probability going to 1 we can apply the mean value theorem to \( f_1(\cdot) \) to write

\[
\sqrt{K_{a,j}} \left\{ \sqrt{1 - 4(\hat{a}c - \hat{b})^2}/(\hat{a} + \hat{c})^2} - \sqrt{1 - 4(ac-b^2)/(a+c)^2}\right\}
\]

\[
= \frac{1}{2} \eta^{-1/2} \left( 1 - \frac{4(\hat{a}c - \hat{b})^2}{(\hat{a} + \hat{c})^2}, 1 - \frac{4(ac-b^2)}{(a+c)^2} \right) \sqrt{K_{a,j}} \left\{ \frac{4(ac-b^2)}{(a+c)^2} - \frac{4(\hat{a}c - \hat{b})^2}{(\hat{a} + \hat{c})^2}\right\}, \quad (4.32)
\]

where

\[
\eta \left( 1 - \frac{4(\hat{a}c - \hat{b})^2}{(\hat{a} + \hat{c})^2}, 1 - \frac{4(ac-b^2)}{(a+c)^2} \right) \xrightarrow{P} 1.
\]

Moreover,

\[
-\frac{1}{2} \sqrt{K_{a,j}} \left\{ \frac{4(\hat{a}c - \hat{b})^2}{(\hat{a} + \hat{c})^2} - \frac{4(ac-b^2)}{(a+c)^2}\right\}
\]

\[
= -2\sqrt{K_{a,j}} \left\{ \frac{(\hat{a}c - \hat{b})^2 - (ac-b^2)}{(\hat{a} + \hat{c})^2} \right\} - 2(ac-b^2)\sqrt{K_{a,j}} \left\{ \frac{1}{(\hat{a} + \hat{c})^2} - \frac{1}{(a+c)^2}\right\}. \quad (4.33)
\]

The second term on the right-hand side of (4.33) can be recast as

\[
-2\sqrt{K_{a,j}} \left\{ \frac{(\hat{a}c - \hat{b})^2 - (ac-b^2)}{(\hat{a} + \hat{c})^2} \right\} (1).
\]

So, let \( f_2(x) = x^2 \). Again we use the mean value theorem with \( f_2(\cdot) \) to write

\[
\sqrt{K_{a,j}} \left\{ (\hat{a} + \hat{c})^2 - (a + c)^2 \right\} = 2\eta(\hat{a} + \hat{c}, a + c) \sqrt{K_{a,j}} \left\{ (\hat{a} + \hat{c}) - (a + c) \right\}, \quad (4.34)
\]
where \( \eta(\tilde{a} + \tilde{c}, a + c) \in [\min\{\tilde{a} + \tilde{c}, a + c\}, \max\{\tilde{a} + \tilde{c}, a + c\}] \). However, with probability going to 1,

\[
\eta(\tilde{a} + \tilde{c}, a + c) \leq (p_{12}^2 + p_{22}^2)[\min\{b_{22}, \hat{b}_{22}\}, \max\{b_{22}, \hat{b}_{22}\}]
\]

where, in turn, \([\min\{b_{22}, \hat{b}_{22}\}, \max\{b_{22}, \hat{b}_{22}\}] \xrightarrow{P} b_{22} \) by (4.18). Furthermore,

\[
\sqrt{K_{a,j}}((\tilde{a} + \tilde{c}) - (a + c)) \leq a(\nu)^{2h_2}(p_{12}^2 + p_{22}^2)\sqrt{K_{a,j}}(\hat{b}_{22} - b_{22}).
\]

By taking determinants on (4.20) and (4.21),

\[
a c - b^2 = a(\nu)^{2(h_1 + h_2)} \det E W(2^j), \quad \tilde{a} c - \tilde{b}^2 \leq a(\nu)^{2(h_1 + h_2)} \det W_a(2^j). \tag{4.35}
\]

Therefore, by (4.35) the second term on the right-hand side of (4.33) is asymptotically equivalent in probability to

\[
\frac{4}{\eta(\tilde{a} + \tilde{c})} a(\nu)^{2(h_1 + h_2)} \det E W(2^j) \left( \frac{p_{12}^2 + p_{22}^2}{b_{22}^2 a(\nu)^{2h_2}} \right)^2 \frac{(a(\nu)^{2h_2})^2 (p_{12}^2 + p_{22}^2)^2 b_{22} \sqrt{K_{a,j}}(\hat{b}_{22} - b_{22})}{\det E W(2^j)} \xrightarrow{P} \frac{2}{\eta(\tilde{a} + \tilde{c})} \sqrt{K_{a,j}}(\tilde{a}^2 - \tilde{b}^2) \right) \tag{4.36}
\]

On the other hand, again by (4.35) the first term on the right-hand side of (4.33) is asymptotically equivalent in probability to

\[
-\frac{2}{\eta(\tilde{a} + \tilde{c})} \sqrt{K_{a,j}}(\tilde{a}^2 - \tilde{b}^2) \right) \tag{4.37}
\]

By combining (4.30), (4.31), (4.32), (4.36) and (4.37) for \( j = j_1, \ldots, j_m \), we obtain the asymptotic Gaussianity of (4.27), where the second entry of the vector is asymptotically equivalent in probability to

\[
\left( \frac{2 \det E W(2^j)}{[(p_{12}^2 + p_{22}^2)^{2b_{22}}]^{a(\nu)^{2(h_2-h_1)}}} \right)^{-1} \left\{ \frac{1}{\eta(\tilde{a} + \tilde{c})} \left[ \frac{-2 \det E W(2^j) \sqrt{K_{a,j}} \det W_a(2^j) - \det E W(2^j)}{\det E W(2^j)} \right] \right\} \tag{4.38}
\]

\[
\xrightarrow{P} \sqrt{K_{a,j}} \frac{\det W_a(2^j) - \det E W(2^j)}{\det E W(2^j)} - 2 \sqrt{K_{a,j}} \frac{\hat{b}_{22} - b_{22}}{b_{22}} \right) \tag{4.39}
\]

Therefore, the sum of the expressions (4.30) and (4.39) is asymptotically equivalent to

\[
\sqrt{K_{a,j}} \frac{\det \hat{B}_a(2^j) - \det B(2^j)}{\det B(2^j)} - \sqrt{K_{a,j}} \frac{\hat{b}_{22} - b_{22}}{b_{22}}. \tag{4.40}
\]
Let \( f_3(x, y, z) = xz - y^2 \). By a first order Taylor expansion,

\[
\det \hat{B}_a(2^j) - \det B(2^j) = \nabla f_3(b_{11}, b_{12}, b_{22}) \left( \text{vec}_S \hat{B}_a(2^j) - \text{vec}_S B(2^j) \right) + o \left( \text{vec}_S \hat{B}_a(2^j) - \text{vec}_S B(2^j) \right).
\]

Reintroducing \( j \), the expression (4.40) is asymptotically equivalent in probability to

\[
\frac{b_{22}(2^j)}{\det B(2^j)} \sqrt{K_{a,j} \left( \hat{b}_{11}(2^j) - b_{11}(2^j) \right)} - 2 \frac{b_{12}(2^j)}{\det B(2^j)} \sqrt{K_{a,j} \left( \hat{b}_{12}(2^j) - b_{12}(2^j) \right)}
\]

\[
+ \left( \frac{b_{11}(2^j) b_{22}(2^j)}{\det B(2^j)} - 1 \right) \sqrt{K_{a,j} \left( \frac{\hat{b}_{22}(2^j) - b_{22}(2^j)}{b_{22}(2^j)} \right)}.
\]

(4.41)

The asymptotic distribution for \( \sqrt{K_{a,j} \left\{ \frac{\lambda_2(a(\nu)2^j)}{\lambda_5^2(a(\nu)2^j)} - 1 \right\}} \) can be similarly established. Analogously, it suffices to look at

\[
\sqrt{K_{a,j} \left\{ \frac{\tilde{a} + \tilde{c}}{a + c} - 1 \right\}} + \sqrt{K_{a,j} \left\{ \frac{1 + \frac{1}{\sqrt{1 - 4(\tilde{a} \tilde{c} - \hat{b}^2) / (\tilde{a} + \tilde{c})^2}}}{1 + \frac{1}{\sqrt{1 - 4(ac - b^2) / (a + c)^2}} - 1} \right\}}
\]

(4.42)

In view of (4.28) and the ensuing calculations, it remains to develop the second term in the sum (4.42). Recast the latter as

\[
\left( 1 + \frac{1}{\sqrt{1 - 4(ac - b^2) / (a + c)^2}} \right)^{-1} \left\{ \sqrt{1 - 4(\tilde{a} \tilde{c} - \hat{b}^2) / (\tilde{a} + \tilde{c})^2} - \sqrt{1 - 4(ac - b^2) / (a + c)^2} \right\}.
\]

(4.43)

By Lemma B.1, \( \left( 1 + \frac{1}{\sqrt{1 - 4(ac - b^2) / (a + c)^2}} \right)^{-1} \rightarrow \frac{1}{2} \), whereas the term between braces in (4.43) behaves asymptotically in probability like the term between braces in (4.38). As a consequence, (4.43) goes in probability to zero and (4.30) thus implies that

\[
\sqrt{K_{a,j} \left\{ \frac{\lambda_2(a(\nu)2^j)}{\lambda_5^2(a(\nu)2^j)} - 1 \right\}} \sim \sqrt{K_{a,j} \left( \frac{\tilde{b}_{22}(2^j) - b_{22}(2^j)}{b_{22}(2^j)} \right)}.
\]

(4.44)

The expressions (4.41) and (4.44) for \( j = j_1 < \ldots < j_m \) imply the weak limit (4.23) with asymptotic covariance matrix (4.24). \( \square \)

### 4.3 The weak limit of unit eigenvectors

For a given scale \( j \in \mathbb{N} \) and \( a(\nu) > 0 \), consider the relation (4.14) for the eigenvector entries

\[
(v_1(a(\nu)2^j), v_2(a(\nu)2^j))^* \in \mathbb{R}^2
\]

(4.45)

associated with the smallest eigenvalue \( \lambda_5^2 := \lambda_5^2(a(\nu)2^j) \) of the (symmetric) wavelet variance matrix \( S := EW(a(\nu)2^j) \), as well as their sample counterparts \( (\hat{v}_1(a(\nu)2^j), \hat{v}_2(a(\nu)2^j))^* \), \( \lambda_1 := \lambda_1(a(\nu)2^j) \) and \( S := W(a(\nu)j) \). The ratios \( (v_2/v_1)(a(\nu)2^j), (\hat{v}_2/\hat{v}_1)(a(\nu)2^j) \) represent the tangents of the angles that determine the associated eigenspaces. At coarse wavelet scales \( a(\nu)2^j \), one expects \( (\hat{v}_2/\hat{v}_1)(a(\nu)2^j) \) to be a consistent estimator of the angle determined by the entries of \( P \) associated with \( \lambda_1 \) when \( P \in O(2) \), i.e., \( \theta = -p_{12}/p_{22} \). This motivates the next definition.
Definition 4.2 Let \( j \in \mathbb{N} \) and \( a(\nu) > 0 \). Under the assumptions in Definition 4.1, we define the estimator of \( \theta \) at scale \( a(\nu)2^j \) as

\[
\hat{\theta}(a(\nu)2^j) = \left( \frac{\hat{\nu}_2}{\nu} \right)(a(\nu)2^j) = \frac{\lambda_1(a(\nu)2^j) - \tilde{a}_{j,a(\nu)}}{b_{j,a(\nu)}}. \tag{4.46}
\]

Indeed, the consistency and asymptotic normality of the estimator (4.46) are precisely stated and shown in Theorem 4.2 below. The limits themselves do not depend on the orthogonality of \( P \). Note that the parametric assumption (4.47) below rules out diagonal wavelet spectra, the latter being associated with entry-wise scaling OFBMs (as in (1.4)). For further comments on the assumptions of Theorem 4.2, see Remark 4.5.

Theorem 4.2 Let \( j \in \mathbb{N} \), and let \( \lambda_1, \lambda_1^E \) be as in (4.1) and (4.2). Suppose that the assumptions (OFBM1)–(4) and (4.9) hold, as well as the parametric assumption \( p_{22} \neq 0 \).

(i) If

\[
either p_{12} \neq 0 \quad \text{or} \quad p_{12} = 0 \quad \text{and} \quad b_{12} \neq 0, \tag{4.47}
\]

then

\[
\hat{\theta}(a(\nu)2^j) \xrightarrow{P} \theta = -\frac{p_{12}}{p_{22}}, \quad \nu \to \infty; \tag{4.48}
\]

(ii) if

\[
p_{12} \neq 0 \quad \text{and} \quad b_{12} \neq 0, \tag{4.49}
\]

then,

\[
a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \left\{ \left( \frac{\lambda_1 - \tilde{a}}{b} \right) - \left( \frac{\lambda_1^E - a}{b} \right) \right\} \overset{d}{\to} N(0, \sigma^2_{b}), \quad \nu \to \infty. \tag{4.50}
\]

In (4.50), \( \sigma^2_{b} = \mathcal{R}^* \Sigma_B(2^j) \mathcal{R} \), where \( \Sigma_B(2^j) \) is the \( 3 \times 3 \) block, associated with \( j \), on the main diagonal of \( \Sigma_B(j_1, \ldots, j_m) \) (see (4.18)), and \( \mathcal{R}^* = \frac{\det P}{b_{22}p_{22}} \begin{pmatrix} 0, & -1, & b_{12} \end{pmatrix} \).

\[
\frac{v_2}{v_1} = \frac{a}{b} \left( \frac{\lambda_1^E}{a} - 1 \right) \sim -\frac{p_{12}}{p_{22}}, \quad \nu \to \infty, \tag{4.51}
\]

by Lemma B.1.

Now assume that \( p_{12} \neq 0 \). The property (3.9) ensures that \( b_{22} \neq 0 \), whence \( b = b_{j,a(\nu)} \neq 0 \) in expression (4.20). The same applies to \( a = a_{j,a(\nu)} \). Therefore, \( v_1 \neq 0 \) in (4.14), and

\[
by \text{Lemma B.1.}

In either case in (4.47), the relations above can be simply rewritten for \( \hat{S} = W(a(\nu)2^j) \) with eigenvector \( \hat{v} \). Because of (4.18), \( \hat{B}_a(2^j) \xrightarrow{P} B(2^j) \) and thus the expression \( \hat{v}_2/\hat{v}_1 \) is well-defined with probability increasing to 1, as \( \nu \to \infty \). Again by Lemma B.1, \( \hat{v}_2/\hat{v}_1 \xrightarrow{P} -\frac{p_{12}}{p_{22}} \).

To show (ii), reexpress

\[
a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \left[ \left( \frac{\lambda_1 - \tilde{a}}{b} \right) - \left( \frac{\lambda_1^E - a}{b} \right) \right]
\]

\[
= a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \left[ \frac{\lambda_1}{b} - \frac{\lambda_1^E}{b} \right] + a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \left\{ \frac{a}{b} - \frac{\tilde{a}}{b} \right\}. \tag{4.52}
\]
Up to the rate factor \( a(\nu)^{h_2-h_1} \), the first term on the right-hand side of (4.52) can be further developed into

\[
\frac{\lambda_1^E}{b} \sqrt{K_{a,j}} \left\{ \frac{\lambda_1}{\lambda_1^E} - 1 \right\} + (-1) \frac{\lambda_1^E}{b} \sqrt{K_{a,j}} \left\{ \frac{\hat{b} - b}{b} \right\}. \tag{4.53}
\]

In view of Lemma B.1,

\[
\frac{\lambda_1^E}{b} \sim \frac{1}{P_{12P22}b_{22}a(\nu)^{2h_2}} a(\nu)^{2h_1} \det E W(2^i) = \frac{1}{a(\nu)^{2(h_2-h_1)}} \frac{\det E W(2^i)}{p_{12P22}b_{22}^2(p_{12}^2 + p_{22}^2)},
\]

\[
\frac{b}{P} \sim \frac{p_{12P22}b_{22}a(\nu)^{2h_2}}{p_{12P22}b_{22}a(\nu)^{2h_2}} P \to 1.
\]

Therefore, by premultiplying by the rate factor \( a(\nu)^{2(h_2-h_1)} \) and applying Theorem 4.1,

\[
a(\nu)^{2(h_2-h_1)} \frac{\lambda_1^E}{b} \sqrt{K_{a,j}} \left\{ \frac{\lambda_1}{\lambda_1^E} - 1 \right\} \cdot \frac{\det E W(2^i)}{p_{12P22}b_{22}^2(p_{12}^2 + p_{22}^2)} \pi_1(\Lambda_j), \tag{4.54}
\]

where \( \pi_1(\Lambda_j) \) is a projection of the random vector \( \Lambda_j \) obtained in the limit (4.23). Moreover,

\[
\sqrt{K_{a,j}} \left\{ \frac{\hat{b} - b}{b} \right\} \sim \frac{1}{p_{12P22}b_{22}a(\nu)^{2h_2}} p_{12P22} \sqrt{K_{a,j}} \left\{ \hat{b}_{22} - b_{22} \right\} a(\nu)^{2h_2} d \to N(0, \sigma^2(b_{22}(2^j))) \frac{\det E W(2^i)}{b_{22}(2^j)},
\]

where \( \sigma^2(b_{22}(2^j)) \) comes from the matrix \( \Sigma_B(j_1, \ldots, j_m) \) in (4.18). Therefore, after premultiplying by the rate factor \( a(\nu)^{2(h_2-h_1)} \), we obtain

\[
a(\nu)^{2(h_2-h_1)}(-1) \frac{\lambda_1^E}{b} \sqrt{K_{a,j}} \left\{ \frac{\hat{b} - b}{b} \right\} \cdot \frac{\det E W(2^i)}{p_{12P22}b_{22}^2(p_{12}^2 + p_{22}^2)} \frac{N(0, \sigma^2(b_{22}))}{b_{22}(2^j)}. \tag{4.55}
\]

In (4.55), \( \det E W(2^i) \neq 0 \), thus implying that \( b_{22}^{-1} \) is well-defined, where \( b_{22} \) comes from \( B = P^{-1} E W(2^j)(P^{-1})^\ast \). Expressions (4.54) and (4.55) reveal that the rate of convergence of each term in (4.53) is \( a(\nu)^{2(h_2-h_1)} \sqrt{K_{a,j}} \), i.e., faster than \( a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \). As a consequence, for the first term on the right-hand side of (4.52),

\[
a(\nu)^{h_2-h_1} \sqrt{K_{a,j}} \left\{ \frac{\lambda_1}{b} - \frac{\lambda_1^E}{b} \right\} = \frac{1}{a(\nu)^{h_2-h_1}} a(\nu)^{2(h_2-h_1)} \sqrt{K_{a,j}} \left\{ \frac{\lambda_1}{b} - \frac{\lambda_1^E}{b} \right\} P \to 0.
\]

We now turn to the second term on the right-hand side of (4.52). Up to a negative sign, we can recast it as

\[
\frac{\tilde{\alpha}}{b} \sqrt{K_{a,j}} \left( \frac{\tilde{a} - a}{\tilde{\alpha}} \right) - \frac{a}{b} \sqrt{K_{a,j}} \left( \frac{\hat{b} - b}{b} \right). \tag{4.56}
\]

For notational simplicity, we will denote the “slow-growing” bits of \( a \) and \( b \) in (4.20) as

\[
\alpha = p_{11}^2 b_{11} a(\nu)^{2h_1} + 2p_{12} p_{11} b_{12} a(\nu)^{h_1+h_2}, \quad \beta = p_{11} p_{21} b_{11} a(\nu)^{2h_1} + (p_{12} p_{21} + p_{11} p_{22}) b_{12} a(\nu)^{h_1+h_2},
\]

where their sample counterparts \( \tilde{\alpha}, \tilde{\beta} \) display \( \hat{b}_{11}, \hat{b}_{12} \) in place of \( b_{11}, b_{12} \). Note that, by (4.49), \( \tilde{\alpha} \sim p_{12}^2 p_{11} b_{12} a(\nu)^{h_1+h_2} \), \( \alpha \sim 2p_{12} p_{11} b_{12} a(\nu)^{h_1+h_2}, \beta \sim (p_{12} p_{21} + p_{11} p_{22}) b_{12} a(\nu)^{h_1+h_2} \). Under the new notation, expression (4.56) can be expanded into

\[
\left( \frac{\tilde{\alpha} + p_{12}^2 \hat{b}_{22} a(\nu)^{2h_2}}{\tilde{\beta} + p_{12} p_{22} \hat{b}_{22} a(\nu)^{2h_2}} \right) \sqrt{K_{a,j}} \left( \frac{\tilde{\alpha} - a}{\tilde{\alpha}} + \frac{p_{12}^2 (\hat{b}_{22} - b_{22}) a(\nu)^{2h_2}}{\tilde{\alpha} + p_{12}^2 \hat{b}_{22} a(\nu)^{2h_2}} \right).
\]
We will break up (4.57) into two terms and premultiply them by the rate factor \(a(\nu)^{h_2-h_1}\). We will see that this way we eventually arrive at a meaningful stochastic limit. Indeed, after premultiplication by \(a(\nu)^{h_2-h_1}\) the first term becomes

\[
\frac{(\alpha a(\nu)^{-2h_2} + \frac{1}{2}\hat{\nu}^2 b_{22})}{\hat{\beta} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \sqrt{K_{a,j}} \left( \frac{(\hat{\alpha} - \alpha) a(\nu)^{-(h_1+h_2)} + \frac{1}{2} p_{12}^{2} b_{22}}{\hat{\alpha} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \right) 
\]

\[
= \frac{(\hat{\alpha} a(\nu)^{-2h_2} + \frac{1}{2}\hat{\nu}^2 b_{22})}{\hat{\beta} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \left( \frac{p_{11}^{2} \sqrt{K_{a,j} (b_{11} - b_{11})} a(\nu)^{h_1-h_2} + 2p_{12} p_{11} \sqrt{K_{a,j} (b_{12} - b_{12})}}{\hat{\alpha} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \right) 
\]

The second term in (4.60) is asymptotically equivalent in probability to

\[
\frac{\frac{1}{2} p_{12}^{2} b_{22}}{p_{12} p_{22} b_{22}} \sqrt{K_{a,j}} (b_{22} - b_{22}).
\]

Note that \(\text{det } P \neq 0\), since \(P \in GL(2, \mathbb{R})\), and \(p_{22} \neq 0\), by assumption. So, the remaining term in (4.57) is

\[
\left[ \frac{(\hat{\alpha} a(\nu)^{-2h_2} + \hat{\nu}^2 b_{22})}{\hat{\beta} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \right] \frac{p_{12}^{2}}{(\alpha a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22})} a(\nu)^{h_2-h_1}.
\]

Since \(\sqrt{K_{a,j} (b_{22} - b_{22})} \xrightarrow{d} N(0, \sigma^2(b_{22}))\), it suffices to look at the term between brackets. The latter can be reexpressed as

\[
\frac{(\hat{\alpha} a(\nu)^{-2h_2} + \hat{\nu}^2 b_{22})}{\hat{\beta} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} \left( \frac{\frac{1}{2} p_{12}^{2} b_{22}}{\hat{\alpha} a(\nu)^{-2h_2} + \frac{1}{2} p_{12}^{2} b_{22}} - \frac{1}{p_{12} p_{22} b_{22}} \right) 
\]

When premultiplied by the rate factor \(a(\nu)^{h_2-h_1}\), the second term in (4.60) is asymptotically equivalent in probability to

\[
a(\nu)^{h_2-h_1} \frac{p_{12}^{2} b_{22}}{p_{12} p_{22} b_{22}} \left( \frac{1}{\hat{\alpha} p_{12}^{2} b_{22} + \frac{1}{2} b_{22}} - \frac{1}{(\beta/p_{12} p_{22}) a(\nu)^{-2h_2} + \frac{1}{2} b_{22}} \right)
\]
= a(\nu^{h_2-h_1})p_{12} \frac{(\beta/p_{12}p_{22})a(\nu)^{-2h_2} - (\alpha/p_{12}^2)a(\nu)^{-2h_2}}{p_{12}^2 ((\alpha/p_{12}^2)a(\nu)^{-2h_2} + \hat{b}_{22})} \frac{\hat{b}_{12}}{p_{12}p_{22}} = \frac{p_{12}}{p_{22}^2} \left\{ \frac{(p_{12}p_{21} + p_{11}p_{22})\hat{b}_{12}}{p_{12}p_{22}} - 2p_{12}p_{11}\hat{b}_{12} \right\} \frac{p_{12}}{p_{12}^2} - \frac{b_{12}}{b_{22}^2} \frac{1}{p_{22}^2} \det P. \quad (4.61)

When premultiplied by the rate factor \(a(\nu)^{h_2-h_1}\), the first term in (4.60) can be reexpressed as

\[
\begin{align*}
a(\nu)^{h_2-h_1}
\frac{p_{12}^2}{\alpha a(\nu)^{-2h_2} + \hat{b}_{22}^2} \left[ \frac{(\alpha - \alpha) a(\nu)^{-2h_2} + p_{12}^2(\hat{b}_{22} - b_{22})}{\beta a(\nu)^{-2h_2} + p_{12}p_{22}\hat{b}_{22}} \right] \\
+ (\alpha a(\nu)^{-2h_2} + p_{12}^2\hat{b}_{22}) \left[ \frac{1}{\beta a(\nu)^{-2h_2} + p_{12}p_{22}\hat{b}_{22}} - \frac{1}{\beta a(\nu)^{-2h_2} + p_{12}p_{22}\hat{b}_{22}} \right].
\end{align*}
\]

On one hand,

\[
\begin{align*}
a(\nu)^{h_2-h_1}
\frac{p_{12}^2}{\alpha a(\nu)^{-2h_2} + \hat{b}_{22}^2} \left[ \frac{(\alpha - \alpha) a(\nu)^{-2h_2} + p_{12}^2(\hat{b}_{22} - b_{22})}{\beta a(\nu)^{-2h_2} + p_{12}p_{22}\hat{b}_{22}} \right] \\
\frac{p_{12}}{b_{22}} \frac{2p_{12}p_{11}(\hat{b}_{12} - b_{12}) + p_{12}^2\sqrt{K_{a,j}}}{} \frac{(\hat{b}_{22} - b_{22})a(\nu)^{h_2-h_1}}{\sqrt{K_{a,j}}} \to 0, \quad (4.62)
\end{align*}
\]

where the limit follows from (4.9). On the other hand,

\[
\begin{align*}
\frac{p_{12}^2}{\alpha a(\nu)^{-2h_2} + \hat{b}_{22}^2} \left[ \frac{(\alpha - \alpha) a(\nu)^{-2h_2} + p_{12}^2(\hat{b}_{22} - b_{22})}{\beta a(\nu)^{-2h_2} + p_{12}p_{22}\hat{b}_{22}} \right] \\
\frac{p_{12}}{b_{22}} \frac{2p_{12}p_{11}(\hat{b}_{12} - b_{12}) + p_{12}^2\sqrt{K_{a,j}}}{} \frac{(\hat{b}_{22} - b_{22})a(\nu)^{h_2-h_1}}{\sqrt{K_{a,j}}} \to 0, \quad (4.63)
\end{align*}
\]

where, again, we use (4.9). By piecing together (4.61), (4.62) and (4.63), we conclude that the term (4.59) converges in law to

\[
\frac{b_{12}}{b_{22}^2 \frac{1}{p_{22}^2}} \det(P) N(0, \sigma^2(b_{22})). \quad (4.64)
\]

By adding together the weak limits (4.58) and (4.64) (and switching the sign), (4.50) follows. \(\Box\)

**Remark 4.4** From a different but mathematically equivalent perspective, statement (4.48) implies that \(EW(a(\nu)^{2j})\) and \(W(a(\nu)^{2j})\) have sequences of unit eigenvectors associated with \(\lambda_j^E(a(\nu)^{2j})\) and \(\lambda_j(a(\nu)^{2j})\), respectively, which converge (deterministically and in probability, respectively) to the limiting unit vector

\[
\left( \frac{|p_{22}|}{\sqrt{p_{12}^2 + p_{22}^2}}, -\frac{p_{12} \text{sign}(p_{22})}{\sqrt{p_{12}^2 + p_{22}^2}} \right). \quad (4.65)
\]

If \(P \in O(2)\), then eigenvectors of \(H\) in two orthogonal directions are consistently estimated by the eigenvectors of \(W(a(\nu)^{2j})\). In view of (4.65), there is also a unit eigenvector associated with \(\lambda_2(\nu)\) that converges to \((p_{21}, -p_{11})^*\), since the latter is orthogonal to \((p_{22}, -p_{12})^*\). It is easy to see that the latter pair of vectors generates the same (eigen)spaces as \((p_{11}, p_{21})^*, (p_{12}, p_{22})^*\).
Remark 4.5 In regard to the assumptions of Theorem 4.2, when $p_{22} = 0$, it is clear that (4.46) cannot be consistent. By a similar proof to that for Theorem 4.2, asymptotic normality (with different variances) for \( \left\{ \left( \frac{\lambda_1 - \hat{\lambda}}{b} \right) - \left( \frac{\lambda_1^0 - \hat{\lambda}}{b} \right) \right\} \) can also be established when $p_{12} = 0$ or $b_{12} = 0$. In the former case, for instance, the convergence rate is $a(\nu)^{h_2 - h_1} \sqrt{K_{a,j}}$ or $\sqrt{K_{a,j}}$, respectively, depending on whether $b_{12} \neq 0$ or both $b_{12} = 0$ and $p_{21} \neq 0$.

Remark 4.6 In Theorem 4.2, the convergence rate is the non-standard $a(\nu)^{h_2 - h_1} \sqrt{K_{a,j}} \sim a(\nu)^{h_2 - h_1 - 1/2} \nu^{p/2}$, which depends on the parameters to be estimated $h_1$, $h_2$. In practice, since $a(\nu)$ is much slower than $\nu$ (see condition (4.9)), then its effect may not be noticeable (see Section 5 below).

Remark 4.7 Substituting $\lambda_2(a(\nu)^{2i})$ for $\lambda_1(a(\nu)^{2i})$ in (4.46) does not provide information on $p_{11}$ or $p_{21}$. By Lemma B.1 under mild assumptions on the parameters,

\[
\frac{a}{b} \left( \frac{\lambda_2(a(\nu)^{2i})}{a} - 1 \right) \frac{p_{12}}{p_{22}} \left( \frac{p_{12}^2 + p_{22}^2}{p_{12}^2} - 1 \right).
\]

5 Simulation studies

In this section, we carry out several computational studies of the performance of the proposed estimators. The synthetic OFBMs used were generated by the authors using the Hermir Toolbox, devised in Helgason et al. (2011b, 2011a) and available at \texttt{www.hermir.org}. For the description of the studies, we drop the asymptotic scaling factor $a(\nu)$ used in Theorems 4.1 and 4.2 and only speak of shifting scales $j \in \mathbb{N}$.

The simulation sample path size was chosen large on purpose: $N = 2^{16}$. This is so because the main computational goal is to provide a compelling illustration of the estimators’ ability to capture both Hurst eigenvalues, whereas state-of-the-art approaches would only hit upon the largest one. In biomedical applications, typical recordings can be much shorter (of the order $N = 2^{10}$; see, for instance, Ivanov (1999)). Nevertheless, sample paths of size $N = 2^{16}$ are, indeed, encountered in Internet traffic analysis (see Abry et al. (2002)) and hydrodynamic turbulence (see Frisch (1995)); see also Section 6.

5.1 Entry-wise vs eigenvalue-based estimation

A simulation experiment serves to illustrate the difficulties associated with the naive use of entry-wise scaling laws under operator self-similarity, as well as to contrast the performance of the latter approach with that of the estimator (1.11).

We compute the matrix $W(2^j)$ based on a single sample path of size $N = 2^{16}$ from a synthetic OFBM with matrix parameters $P = \begin{pmatrix} 0.98 & 0.57 \\ 0.20 & 0.82 \end{pmatrix}$, $J_H = \text{diag}(0.25, 0.85)$. In the spirit of the entry-wise approach, the wavelet-based estimation of the scalar Hurst parameters relies on performing a linear regression on a $\log_2 W(2^j)$ vs $j = \log_2 2^j$ diagram, motivated by the log-transformed scalar version of (3.6), i.e., $EW(2^j) = C(H)2^{j2H}$ for some $C(H) > 0$. This is shown for each auto- and cross- wavelet components of the bivariate OFBM in Figure 1. The top row displays, in order, plots for $\log_2 W(2^j)_{1,1}$ vs $j = \log_2 2^j$, $\log_2 W(2^j)_{1,2}$ vs $j = \log_2 2^j$ and $\log_2 W(2^j)_{2,2}$ vs $j = \log_2 2^j$. The asymptotic behavior $2h_2$ is superimposed on each of these plots. In view of the expression (4.21), it is unsurprising that at coarse scales all auto- and cross-components end up driven by the largest Hurst eigenvalue $h_2$. In other words, the conspicuous prevalence of the latter precludes the estimation of the smallest Hurst eigenvalue $h_1$. 

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Figure 1, bottom row, displays, in order, plots for $\log_2 \lambda_1(2^j)$ vs $j = \log_2 2^j$ and $\log_2 \lambda_2(2^j)$ vs $j = \log_2 2^j$, as well as the superimposed asymptotic behaviors $j2h_1$ and $j2h_2$, respectively. This conveys a striking demonstration that the eigenvalue-based procedure permits the accurate estimation of both parameters $h_1$ and $h_2$.

5.2 Estimation performance and asymptotic normality

To study the finite-sample effectiveness of the normal approximation described in Theorem 4.1, we numerically synthesize $10^4$ bivariate OFBM sample paths. For each path, the estimates $W(2^j)$, $\lambda_1(2^j)$, $\lambda_2(2^j)$ and $\hat{\gamma}_2(2^j)/\hat{\gamma}_1(2^j)$ are computed. Averaging over realizations yields estimates $E\lambda_1(2^j)$, $E\lambda_2(2^j)$ and $E_{p12}(2^j)/p_{22}(2^j)$ of the ensemble average $E\lambda_1(2^j)$, $E\lambda_2(2^j)$ and $E_{p12}(2^j)/p_{22}(2^j)$, together with estimates of the variances $\hat{\text{Var}}\lambda_1(2^j)$, $\hat{\text{Var}}\lambda_2(2^j)$, $\hat{\text{Var}}\beta(2^j)$.

Figures 2, 3 and 4 correspond to OFBM with parameters $J_H = \text{diag}(0.25, 0.85)$ and path sizes $N = 2^{16}$ for the three cases, but with different mixing matrices of the general form $P = \begin{pmatrix} 1/\sqrt{1 + \gamma^2} & \beta/\sqrt{1 + \beta^2} \\ \gamma/\sqrt{1 + \gamma^2} & 1/\sqrt{1 + \beta^2} \end{pmatrix}$. The different examples are representative of the general case $\beta = 0.7$, $\gamma = 0.2$ (Figure 2), of the case where $P \in O(2)$, $\beta = -\gamma$ and $\beta/\sqrt{1 + \beta^2} = \sin \pi/6$ (Figure 3), and of the case often referred to as co-integration $\gamma = 0$ and $\beta = 0.2$ (Figure 4). In Figures 2, 3, and 4, the comparison of the estimates (black solid lines with ‘o’) with the theoretical values (red dashed lines) reveals the excellent performance of the proposed estimators in the three cases (top row). The qq-plots (bottom row) also show that no deviation from a $N(0, 1)$ distribution can be observed within $\pm 2$ standard deviations for $\log_2 \lambda_1(2^j)$ and $\log_2 \lambda_1(2^j)$, and within $\pm 2$ standard deviations for $p_{12}/p_{22}$, a fairly impressive empirical result.

The simulations indicate that, beyond asymptotics, Theorem 4.1 for $\log_2 \lambda_1(2^j)$ and $\log_2 \lambda_1(2^j)$, and Theorem 4.2 for $-p_{12}/p_{22}$ provide effective normal approximations to the finite-sample estimator distributions. This is of great importance in practice, as it points to the usefulness of the estimators for the analysis of real world data sets.

5.3 The case $P \in O(2)$

In the two examples described above, $P \in O(2)$ in the first one, but not in the second. The simulations show that whether or not $P \in O(2)$ does not impact the estimation performance of $\log_2 \lambda_1(2^j)/2j$, $\log_2 \lambda_2(2^j)/2j$ and $p_{12}(2^j)/p_{22}(2^j)$. However, assuming $P \in O(2)$ additionally allows for the full identification of $P$, and thus of $H$, as discussed in Remark 4.4 and illustrated in Figure 5.

5.4 Beyond the bivariate setting

It is natural to ask whether in higher dimension the eigenvalues of the wavelet spectrum are good estimators of the eigenvalues of $H$. Simulation studies provide evidence that this is, indeed, the case.

Figure 6 shows eigenvalue-based estimation at work for $n = 4$, with $\log_2 \lambda_p(2^j)/2j$, for $p = 1, 2, 3, 4$, averaged over $2,000$ realizations of an OFBM with parameters

$$P = \begin{pmatrix} 0.90 & -0.22 & -0.30 & -0.22 \\ 0.43 & 0.45 & 0.63 & 0.46 \\ 0 & -0.85 & 0.40 & 0.30 \\ 0 & 0 & -0.59 & 0.81 \end{pmatrix}$$
and \( J_H = \text{diag}(0.20, 0.40, 0.70, 0.90) \), \( N = 2^{16} \).

The computational results provide a clear indication that the smallest and largest eigenvalues of \( W(2^j) \) are still good estimators of the smallest and largest entries of \( J_H \). Furthermore, there is evidence that the method does produce reasonable estimates of all the intermediate-valued entries of \( J_H \) based on the corresponding intermediate eigenvalues of \( W(2^j) \).

In regard to eigenvector estimation under the assumption \( P \in O(4) \), unreported numerical simulations show that the performance of wavelet spectrum eigenvectors looks promising. The multivariate setting \( n > 2 \) will be further explored in future work.

6 Perspectives and open issues

OFBMs constitute the natural multivariate extension of the univariate FBM, allowing a different Hurst eigenvalue in each coordinate, in an arbitrary coordinate system. When the mixing matrix \( P \) is non-diagonal, the problem of estimating \( H \) becomes distinctively multivariate and not easily amenable to approaches inspired in the univariate context. In this work, we propose a change of perspective from univariate-like, entry-wise scaling relations to the eigenstructure of the wavelet spectrum \( W(2^j) \) across scales. In the bivariate setting, this methodology is mathematically shown to yield consistent and asymptotically normal estimates of the Hurst indices as well as of the eigenspace angle parameter \(-p_{12}/p_{22}\). The matrix Hurst parameter \( H \) can then be fully identified when \( P \in O(2) \). Numerical simulations reveal that the asymptotic results can be accurately used in practice, with finite-size data. The research contained in this paper has lead to five open issues, currently under investigation: (i) the quantitative assessment of the performance of the estimators as a function of sample size; how much data is demanded by the difficult problem of estimating
operator-scaling systems, especially in high dimension?; (ii) is there an advantage to using multiple scales in a regression, as in univariate wavelet-based estimation?; (iii) mathematical extensions to higher dimensional multivariate OFBMs; (iv) the estimation of non-orthogonal coordinate systems; (v) applications in real data. In the near future, a MATLAB toolbox for the estimators proposed in this paper will be made publicly available.

A Asymptotic normality at fixed scales: proofs

A.1 Wavelet analysis

Proof of Proposition 3.1 We first show (P1). Since the covariance function $EB_{H}(s)B_{H}(t)^{*}$ is continuous, by Cramér and Leadbetter (1967), p. 86, it suffices to show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \|EB_{H}(s)B_{H}(t)^{*}\|_{1}\|\psi(s)\|\|\psi(t)\| \, ds \, dt < \infty.$$  \hfill (A.1)

In fact,

$$\|EB_{H}(s)B_{H}(t)^{*}\|_{1} = \sum_{i_{1}=1}^{n} \sum_{i_{2}=1}^{n} |EB_{H}(s)_{i_{1}}B_{H}(t)_{i_{2}}| \leq \left( \sum_{i_{1}=1}^{n} \sqrt{EB_{H}(s)_{i_{1}}} \right) \left( \sum_{i_{2}=1}^{n} \sqrt{EB_{H}(t)_{i_{2}}} \right).$$  \hfill (A.2)

However, for $t \in \mathbb{R}$,

$$\max_{i_{1},i_{2}=1,...,n} |EB_{H}(t)_{i_{1}}B_{H}(t)_{i_{2}}| = \|EB_{H}(t)B_{H}(t)^{*}\|_{l_{\infty}} = \|t^{H}\sum t^{H^{*}}\|_{l_{\infty}} \leq C|t|^{2\max \Re \text{eg}(H)}.$$
Therefore, (A.2) is bounded by $C|t|^{\max \Re \sigma(H)|s|^{\max \Re \sigma(H)}}$. By conditions (2.7) and (2.8), (A.1) holds. The fact that $E\|B_H(t)\|_{l_1} < \infty$ and the assumption (2.7) yield
\[
E\int_{\mathbb{R}} |\psi(z)||B_H(2jz + 2jk)||_{l_1}dz \leq C\int_{\mathbb{R}} |\psi(z)||B_H(z + k)||_{l_1}dz
\leq CE\|B_H(1)\| \int_{\mathbb{R}} |\psi(s - k)||s^H||_{l_1}ds < \infty,
\]
where we used the change-of-variables $s = z + k$. By Fubini, this yields (3.2).

The properties (P2), (P3), (P5) and (P6) can be established by arguments similar to those for the univariate case. The property (P4) is a consequence of the harmonizable representation (1.2), and also of the conditions (2.6), (2.7) and (2.8); the latter ensure that the integrand in (3.5) is well-behaved in $\mathbb{R}$. We now show (P7). Since $\psi \in \mathbb{R}$, then $\tilde{\psi}(-2jx) = \tilde{\psi}(2jx)$. Thus, by a change-of-variables $y = -x$ over the integration domain $x < 0$ we can rewrite (3.5) as
\[
EW(2j) = 2j \int_{0}^{\infty} x^{-D/2}2\Re(AA^*)x^{-D/2} \frac{\tilde{\psi}(2jx)^2}{a^2}dx.
\]
Since $\text{supp} \tilde{\psi}(x)$ has positive Lebesgue measure, then (2.10) yields $v^*EW(2j)v > 0, v \in \mathbb{C}^n \setminus \{0\}$. □

A.2 The asymptotic covariance matrix

All through this section, we assume that $a_j, a_{j'} \in \mathbb{N}$. 

---

Figure 4: Estimation performance and asymptotic normality: OFBM with $\gamma = 0$ and $\beta = 0.2$. Top row: black ‘o’ in solid lines show $\hat{E}_{\hat{\theta}} \pm \sqrt{\text{Var}\hat{\theta}/n}$ for $\hat{\theta} = \log_2 \lambda_1(2j)/2j, \log_2 \lambda_2(2j)/2j$ and $p_{12}(2j)/p_{22}(2j)$ (target parameters: $h_1$ (left plots), $h_2$ (center plots), $p_{12}/p_{22}$ (right plots), respectively), red dashed lines correspond to theoretical values. Bottom row, corresponding q-q-plots (against $\mathcal{N}(0,1)$ distributions) for $j = 10$. 
We start by reexpressing the range of indices of $\sin \frac{\pi}{2}$. Estimation performance when

Figure 5: Estimation performance when $P \in O(2)$: OFBM with $\gamma = -\beta$ and $\beta/\sqrt{1 + \beta^2} = \sin \pi/6$. Estimates $\hat{p}_{k,l}(2^j)$, $k,l \in \{1,2\}^2$ (black solid lines with ‘o’) of the entries of $P$, $p_{k,l}$ (red dashed line), from the eigenvectors of the $W(2^j)$. When $P \in O(2)$, both $P$ and $J_H$ can be estimated, and the matrix exponent $H$ is fully identified.

Let $\{\phi_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers. We are interested in calculating the limit

$$\lim_{\nu \to \infty} \frac{1}{\nu} \sum_{k=1}^{K} \sum_{k'=1}^{K'} \phi_{a_j k - a_{j'} k'}.$$  \hfill (A.3)

We start by reexpressing the range of indices of $\phi$, i.e.,

$$R := \{a_j N - a_{j'} N\}.$$  \hfill (A.4)

**Lemma A.1**

$$R = \gcd(a_j, a_{j'}) \mathbb{Z}. \hfill (A.5)$$

**Proof:** We begin by showing that $\gcd(a_j, a_{j'}) \mathbb{Z} \subseteq R$. Let $x \in \gcd(a_j, a_{j'}) \mathbb{Z}$, whence we can write $x = \gcd(a_j, a_{j'}) z$ for some $z \in \mathbb{Z}$. If $z = 0$, just pick $k = a_{j'}$, $k' = a_j$, whence $x = a_j k - a_{j'} k'$, $k, k' \in \mathbb{N}$. Now assume that $z \neq 0$. By Bézout’s lemma (Jones and Jones (1998), pp. 7–11), there are $k_0, k'_0 \in \mathbb{Z}$ such that

$$a_j k_0 + a_{j'} (-k'_0) = \gcd(a_j, a_{j'}). \hfill (A.6)$$

If $z > 0$, choose a large enough $m_+ \in \mathbb{N}$ such that

$$k_0 + (m_+ a_{j'})/\gcd(a_j, a_{j'}) > 0, \quad (-k'_0) - (m_+ a_{j'})/\gcd(a_j, a_{j'}) < 0.$$  \hfill (A.7)

By setting

$$k := z(k_0 + m_+ a_{j'}/\gcd(a_j, a_{j'})), \quad k' := z(k'_0 + m_+ a_j/\gcd(a_j, a_{j'})),$$  \hfill (A.8)

then $a_j k - a_{j'} k' = \gcd(a_j, a_{j'}) z$, $k, k' \in \mathbb{N}$, as claimed. Otherwise, if $z < 0$, choose a negative integer $m_-$ such that

$$k_0 + (m_- a_{j'})/\gcd(a_j, a_{j'}) < 0, \quad -k'_0 - (m_- a_j)/\gcd(a_j, a_{j'}) > 0.$$  \hfill (A.9)

By setting $k = z(k_0 + m_- a_{j'}/\gcd(a_j, a_{j'})) > 0$ and $k' = z(k'_0 + m_- a_j/\gcd(a_j, a_{j'})) > 0$, then we arrive at the same conclusion.

For the converse, namely, $\gcd(a_j, a_{j'}) \mathbb{Z} \supseteq R$, let $x \in R$. Write $x = a_j k - a_{j'} k'$, where $k, k' \in \mathbb{N}$, and assume without loss of generality that $x \neq 0$. Then, $x = \gcd(a_j, a_{j'}) \left( \frac{a_j}{\gcd(a_j, a_{j'})} k - \right.$
Figure 6: **Estimation performance and asymptotic normality:** OFBM in dimension 4. Top row: black solid lines with ‘o’ show $\hat{E}_{\hat{\theta}} \pm \frac{\sqrt{\text{Var}(\hat{\theta})}}{n}$ for $\hat{\theta} = \log_2 \lambda_{1}^{2j}/2j$, $\log_2 \lambda_{2}^{2j}/2j$, $\log_2 \lambda_{3}^{2j}/2j$ and $\log_2 \lambda_{4}^{2j}/2j$ (target parameters: $h_1$, $h_2$, $h_3$, $h_4$, respectively, with the plots in the same order), red dashed lines correspond to theoretical values. Bottom row, corresponding qq-plots (against $N(0,1)$ distributions) for $j = 10$.

$\frac{a_{j'}}{\gcd(a_j, a_{j'})} k'$, where the second term on the right-hand side is an integer. □

The next lemma provides an estimate of the number of terms in the summation (A.3). Consider the range $k = 1, \ldots, a_{j'} \nu, k' = 1, \ldots, a_j \nu, \nu \in \mathbb{N}$. To every $r \in R$, we can associate the affine level curve

$$k'(k) = \frac{a_j}{a_{j'}} k - \frac{r}{a_{j'}}, \quad k \in \mathbb{R},$$

(A.10)

that contains all the solution pairs $(k, k') \in \mathbb{N}^2$. So, let

$$k_r(\nu) = \min\{k \in \{a_{j'} \nu + 1, \ldots, a_{j'} (\nu + 1)\} : (k, k'(k)) \text{ satisfies (A.10)}\}.$$

**Lemma A.2** Let $a_j, a_{j'}, \nu \in \mathbb{N}$ and $r = z \gcd(a_j, a_{j'}) \in R$. Let $k'(\cdot)$ be the function defined in (A.10). Then,

(i) $k_1(\nu)$ is well-defined and

$$\left\{ \left( (k_1(\nu) + \frac{a_{j'}}{\gcd(a_j, a_{j'})} m, k'(k_1(\nu) + \frac{a_{j'}}{\gcd(a_j, a_{j'})} m) \right) : m = 1, \ldots, \gcd(a_j, a_{j'}) \right\}$$

is the set of pairs $(k, k') \in \mathbb{Z}^2$ lying in the affine level curve associated with $r$ and such that $a_{j'} \nu < k \leq a_{j'} (\nu + 1)$, $a_j \nu < k' \leq a_j (\nu + 1)$;

(ii) the number $\xi_r(\nu)$ of pairs $(k, k') \in \mathbb{Z}^2$, $1 \leq k \leq a_{j'} \nu$, $1 \leq k' \leq a_j \nu$, in the affine level curve (A.10) associated with $r$ satisfies $\lim_{\nu \to \infty} \frac{\xi_r(\nu)}{\nu \gcd(a_j, a_{j'})} = 1$.  

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PROOF: We first show statement (i). By Bézout’s lemma, at least one solution pair \((k^*, k'(k^*)) \in \mathbb{Z}^2\) for (A.10) exists and the full set of solutions is
\[
\left\{ \left( k^* + \frac{a_{j'}}{\gcd(a_j, a_{j'})} z, k'(k^*) + \frac{a_{j'}}{\gcd(a_j, a_{j'})} z \right) : z \in \mathbb{Z} \right\}.
\]

Since \(a_{j'}/\gcd(a_j, a_{j'}) \leq a_{j'}\), then there exists a solution \((k_0, k'(k_0))\) such that \(a_{j'} \nu < k_0 \leq a_{j'}(\nu + 1)\). Therefore, \(k(\nu) > a_{j'} \nu\) is well-defined and \(k(\nu) - a_{j'}/\gcd(a_j, a_{j'}) \leq a_{j'} \nu\). Let \(m = 1, \ldots, \gcd(a_j, a_{j'})\). Then,
\[
k(\nu) + \frac{a_{j'}}{\gcd(a_j, a_{j'})} (m - 1) \leq a_{j'}(\nu + 1) < k(\nu) + \frac{a_{j'}}{\gcd(a_j, a_{j'})} \gcd(a_j, a_{j'}).
\]

In regard to statement (ii), for \(r = \gcd(a_j, a_{j'}) \) (i) shows that the total number \(\xi_r(\nu)\) of pairs in (A.10) over \(1 \leq k \leq a_{j'} \nu\), \(1 \leq k' \leq a_{j'} \nu\) is \(\gcd(a_j, a_{j'}) \nu\). For general \(r \in \gcd(a_j, a_{j'})\mathbb{Z}\), \(0 \leq \xi_r(\nu) \leq \xi_{\gcd(a_j, a_{j'})}(\nu)\) and \(\xi_r(\nu) \sim \nu \gcd(a_j, a_{j'})\), \(\nu \to \infty\), due to a fixed shift in the linear coefficient of the function \(k'(k)\).

Lemma A.3 Let \(\{\phi_r\} \in \mathbb{R}\) be a sequence such that \(\sum_{z=-\infty}^{\infty} |\phi_{z \gcd(a_j, a_{j'})}| < \infty\). Then,
\[
\frac{1}{\nu} \sum_{k=1}^{\nu} \sum_{k'=1}^{\nu} \phi_{a_j k - a_{j'} k'} \to \gcd(a_j, a_{j'}) \sum_{z=-\infty}^{\infty} \phi_{z \gcd(a_j, a_{j'})}, \quad \nu \to \infty.
\] (A.11)

PROOF: By Lemma A.2, \(\nu^{-1} \sum_{k=1}^{\nu} \sum_{k'=1}^{\nu} \phi_{a_j k - a_{j'} k'} = \gcd(a_j, a_{j'})^{-1} \sum_{r \in \mathbb{R}} \xi_r(\nu) \phi_r\). Then, expression (A.11) is a consequence of dominated convergence by noting that for every \(r \in \mathbb{R}\), \(\nu^{-1} \xi_r(\nu) \to 1\) as \(\nu \to \infty\).

Proof of Proposition 3.3 Statement (ii) is a direct consequence of (i), so we only prove the latter. It suffices to consider the subsequence \(\nu = 2^j 2^l \nu_s\), \(\nu_s \to \infty\). Then, \(K_j = 2^j \nu^s\), \(K_{j'} = 2^l \nu_s\), and \(\sqrt{K_j} \sqrt{K_{j'}} K_j^{-1} K_{j'}^{-1} = 2^{-(j+l)/2} \nu_s^{-1}\). By the same reasoning leading to the expression (3.7),
\[
\text{Cov}(D(2^j, k), D(2^j, k')) = -\frac{1}{2} \int_\mathbb{R} \int_\mathbb{R} \psi(t) \psi(t') |2^j t - 2^j t' + 2^j k - 2^j k'|^\Sigma |2^j t - 2^j t' + 2^j k - 2^j k'| H^* dt dt'
\]
\[
= \Phi_{2^j k - 2^j k'}.
\]

Let \(\Xi_{2^j k - 2^j k'} = \Phi_{2^j k - 2^j k'} \otimes \Phi_{2^j k - 2^j k'}\). By Lemma A.1, the range of indices spanned by \(2^j k - 2^j k'\) is \(\mathbb{Z} \gcd(2^j, 2^j)\). Thus, we would like to show that
\[
\sum_{z=-\infty}^{\infty} \left\| \Xi_{z \gcd(2^j, 2^j)} \right\| < \infty.
\] (A.12)

However, \(\left\| \Xi_{z \gcd(2^j, 2^j)} \right\|_1 = \left\| \text{vec}(\Phi_{z \gcd(2^j, 2^j)}) \text{vec}(\Phi_{z \gcd(2^j, 2^j)})^* \right\|_1 \leq \left\| \text{vec}(\Phi_{z \gcd(2^j, 2^j)}) \right\|_1^2\), which is a summable sequence by Proposition 3.2 (with \(N_\psi \geq 2\); see (2.6)). This shows (A.12). Then, the expression (3.17) is a consequence of Lemma A.3. □
A.3 Decay of the covariance and asymptotic normality of wavelet coefficients

**Proof of Proposition 3.2:** In view of (2.7), we can assume without loss of generality that

\[ \text{supp}(\psi) = [0, K], \quad K > 0. \]  

(A.13)

By (2.6), (2.11) and (A.13), the wavelet covariance can be reexpressed as

\[
ED(2^j, k)D(2^{j'}, k') = (2^{j+j'})^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(t)\psi(t')EB_H(2^j t + 2^j k)B_H(2^{j'} t' + 2^{j'} k')dt\,dt'
\]

\[
= \frac{(2^{j+j'})^{1/2}}{2} \int_0^K \int_0^K \psi(t)\psi(t')\{ |2^j t + 2^j k|^H \Sigma |2^j t + 2^j k|^H^* + |2^{j'} t' + 2^{j'} k'|^H \Sigma |2^{j'} t' + 2^{j'} k'|^H^* - |(2^j t - 2^{j'} t') + (2^j k - 2^{j'} k')|^H \Sigma |(2^j t - 2^{j'} t') + (2^j k - 2^{j'} k')|^H \} dt\,dt'
\]

\[
= -\frac{(2^{j+j'})^{1/2}}{2} |2^j k - 2^{j'} k'|^H \left\{ \int_0^K \int_0^K \psi(t)\psi(t') \left| 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right|^H \Sigma \left| 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right|^H^* dt\,dt' \right\} |2^j k - 2^{j'} k'|^H^*
\]  

(A.14)

Let \( f(x) = x^H \Sigma x^{H^*} = (f_{i_1,i_2}(x))_{i_1,i_2=1,...,n} \). Within the radius defined by (3.10), the integrand in (A.14) can be reexpressed analytically as

\[
\left( 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^H \Sigma \left( 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^{H^*} = \sum_{r=0}^{\infty} \left( \frac{f_{i_1,i_2}(1)}{r!} \left( \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^r \right)_{i_1,i_2=1,...,n}.
\]

Thus, again (2.6) yields

\[
\int_0^K \int_0^K \psi(t)\psi(t') \left( 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^H \Sigma \left( 1 + \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^{H^*} dt\,dt'
\]

\[
= \sum_{r=2N_\psi}^{\infty} \left( \frac{f_{i_1,i_2}(1)}{r!} \int_0^K \int_0^K \psi(t)\psi(t') \left( \frac{2^j t - 2^{j'} t'}{2^j k - 2^{j'} k'} \right)^r dt\,dt' \right)_{i_1,i_2=1,...,n}.
\]  

(A.15)

We now look at each summation term in (A.15). Up to the matrix constant \( (f_{i_1,i_2}(1))/(2N_\psi) \), the term associated with the index \( r = 2N_\psi \) is

\[
\frac{1}{(2^j k - 2^{j'} k')^{2N_\psi}} \int_0^K \int_0^K \psi(t)\psi(t') (2^j t - 2^{j'} t')^{2N_\psi} dt\,dt'
\]

\[
= \sum_{\nu=0}^{2N_\psi} \left( 2N_\psi \right) (2^j)^\nu (-2^{j'})^{2N_\psi - \nu} \frac{1}{(2^j k - 2^{j'} k')^{2N_\psi}} \int_0^K \int_0^K \psi(t)\psi(t') t^\nu t'^{2N_\psi - \nu} dt\,dt'
\]

\[
= \frac{2N_\psi}{N_\psi} (-1)^{N_\psi} (2^j)^{2j'}^{N_\psi} \frac{1}{(2^j k - 2^{j'} k')^{2N_\psi}} \int_0^K \int_0^K \psi(t)\psi(t') t^{N_\psi} t'^{N_\psi} dt\,dt'.
\]  

(A.16)

By induction, the double integral in (A.16) associated with the index \( r = 2N_\psi + l, l \in \mathbb{N} \), can be written as

\[
\frac{1}{(2^j k - 2^{j'} k')^{2N_\psi}} \left\{ \sum_{\nu=N_\psi+1}^{N_\psi+l} \left( 2N_\psi + l \right) (2^j)^\nu (-2^{j'})^{(2N_\psi+l) - \nu} \frac{1}{(2^j k - 2^{j'} k')^l} \int_0^K \int_0^K \psi(t)\psi(t') t^l t'^{(2N_\psi+l) - \nu} dt\,dt' \right\}
\]
Therefore, (A.16) is, indeed, summable, and by (A.14) and (A.17) the expression (3.11) holds.

We would like to establish the limiting distribution of the statistic

\[ T_\nu = \sum_{j=j_1}^{j_m} \frac{\alpha_j}{2^j} \text{vec} G(2^j) = Y^*DY, \]

where

\[ Y = (d_1(2^{j_1}), d_2(2^{j_1}), \ldots, d_1(2^{j_1}, K_{j_1} - 1), d_2(2^{j_1}, K_{j_1} - 1), \ldots) \]

To establish the absolute convergence of the expansion (A.15), note that in (A.16) the matrix norms \( \|f^{(2N_\psi + L)}(t)\| \) are uniformly bounded, that for large enough \( L \in \mathbb{N} \), \( \sum_{l=L}^{\infty} \frac{1}{(2N_\psi + l)!} \sum_{l=0}^{t} \frac{2N_\psi + l}{(2N_\psi + l)!} < \infty \) and that \( \int_0^K \int_0^K \psi(t)\psi(t')t^{r+N_\psi} t' N_\psi + l - r dt dt' \leq Cl^{-1}K^{2N_\psi + l + 2} \).

Therefore, (A.16) is, indeed, summable, and by (A.14) and (A.17) the expression (3.11) holds.

In regard to (3.12), note that for \( c > 1 \) we can construct the bound for the matrix exponential

\[ \|e^H\| \leq C \|P\|\|P^{-1}\|\|c^H\|_{11} \leq C' c^{\max_{k \in \text{eig}(H)} R(k)} \log^k(c). \]

Now set \( c = |2^j k - 2^j k'| \) under the restriction (3.10). □

**Proof of Theorem 3.1** The argument is reminiscent of those in Bardet (2000), pp. 510-513, Bardet (2002), pp. 997, and Istas and Lang (1997), Lemma 2. For notational simplicity, we will restrict ourselves to the bivariate context \( (n=2) \). The argument for general \( n \) can be worked out based on a simple adaptation.

The proof is by means of the Cramér-Wold device. Form the vector of wavelet coefficients

\[ Y = (d_1(2^{j_1}, 0), d_2(2^{j_1}, 0), \ldots, d_1(2^{j_1}, K_{j_1} - 1), d_2(2^{j_1}, K_{j_1} - 1), \ldots) \]

where \( \Upsilon(\nu) := 2 \sum_{j=j_1}^{j_m} K_j \). Notice that \( m, j_1, \ldots, j_m \) are fixed, but each \( K_j \) goes to infinity with \( \nu \). Let

\[ \alpha = (\alpha_{j_1}, \ldots, \alpha_{j_m}) \in \mathbb{R}^{3m}, \quad (A.18) \]

where

\[ \alpha_j = (\alpha_{j,1}, \alpha_{j,2}, \alpha_{j,12})^* \in \mathbb{R}^3, \quad j = j_1, \ldots, j_m. \]

Now form the block-diagonal matrix

\[ D = \text{diag}\left( \frac{1}{K_{j_1}} \sqrt{\frac{1}{2^{j_1}}} \Omega_{j_1}, \ldots, \frac{1}{K_{j_1}} \sqrt{\frac{1}{2^{j_1}}} \Omega_{j_1}, \ldots, \frac{1}{K_{j_m}} \sqrt{\frac{1}{2^{j_m}}} \Omega_{j_m}, \ldots, \frac{1}{K_{j_m}} \sqrt{\frac{1}{2^{j_m}}} \Omega_{j_m} \right), \quad (A.19) \]

where

\[ \Omega_j = \begin{pmatrix} \alpha_{j,1} & \alpha_{j,12}/2 & \alpha_{j,2}/2 \\ \alpha_{j,12}/2 & \alpha_{j,2} & \alpha_{j,1} \end{pmatrix}, \quad j = j_1, \ldots, j_m. \]

We would like to establish the limiting distribution of the statistic

\[ T_\nu = \sum_{j=j_1}^{j_m} \frac{\alpha_j}{2^j} \text{vec} G(2^j) = Y^*DY, \]

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where it suffices to consider \( \alpha \) in (A.18) such that
\[
\alpha^* \Sigma(H, AA^*) \alpha > 0 \tag{A.20}
\]
(see Brockwell and Davis (1991), pp. 211 and 214). The matrix \( \Sigma(H, AA^*) \) in (A.20) is obtained from Proposition 3.3, and can be written in block form as
\[
\Sigma(H, AA^*) = (G_{jj})_{j,j'=j_1,...,j_m}, \tag{A.21}
\]
corresponding to block entries of the vector \( \alpha = (\alpha_{j_1}, \ldots, \alpha_{j_m})^*. \) Let \( \Gamma = \text{Cov}(Y, Y) \) and consider the spectral decomposition \( \Gamma^{1/2} D \Gamma^{1/2} = O \Lambda O^* \), where \( \Lambda \) is diagonal with real, and not necessarily positive, eigenvalues \( \lambda \) and \( O \) is an orthogonal matrix. Now let \( Z \sim N(0, I_{\nu}) \), where \( I \) is an identity matrix. Then,
\[
T_\nu \overset{d}{=} Z^* \Gamma^{1/2} D \Gamma^{1/2} Z = Z^* O \Lambda O^* Z = Z^* \Lambda Z = \sum_{i=1}^{T(\nu)} \lambda_i \nu Z_i^2, \tag{A.22}
\]
Assume for the moment that
\[
\max_{i=1,...,T(\nu)} |\lambda_i(\nu)| = o\left(\frac{1}{\nu^{1/2}}\right). \tag{A.23}
\]
By (A.20) and Proposition 3.3,
\[
|\nu \text{Var}(T_\nu) = \sum_{j=j_1}^{j_m} \sum_{j'=j_1}^{j_m} \alpha_j^* \left\{ \sqrt{\nu} \frac{\nu}{2} \text{Cov}(vec_s W(2j), vec_s W(2j')) \right\} \alpha_{j'} \right| \rightarrow \sum_{j=j_1}^{j_m} \sum_{j'=j_1}^{j_m} \alpha_j^* G_{jj'} \alpha_{j'} > 0.
\]
Therefore, there exists a constant \( C > 0 \) such that, for large enough \( \nu \), \( \nu \text{Var}(T_\nu) \geq C > 0 \). In view of condition (A.22),
\[
\frac{\max_{i=1,...,n} |\lambda_i(\nu)|}{\sqrt{\text{Var}(T_\nu)}} \leq C' \nu^{1/2} \max_{i=1,...,T(\nu)} |\lambda_i(\nu)| \rightarrow 0, \quad \nu \rightarrow \infty.
\]
The claim (3.18) is now a consequence of Lemma B.2.

So, we need to show (A.22). The first step is to establish the bound
\[
\sup_{u \in S^{T(\nu)-1}} |u^* \Gamma^{1/2} D \Gamma^{1/2} u| \leq C \max_{j=j_1,...,j_m} \frac{1}{K_j} \|\Omega_j\|_1 \sup_{u \in S^{T(\nu)-1}} u^* \Gamma u. \tag{A.23}
\]
Let \( u \in S^{T(\nu)-1} \) and let \( v = u \Gamma^{1/2} \). We can break up the vector \( v \) into two-dimensional subvectors \( v_\nu \), to reexpress
\[
v = (v_{j_1,1}, \ldots, v_{j_1,K_1}; \ldots; v_{j_m,1}, \ldots, v_{j_m,K_m})^*.
\]
Based on the block-diagonal structure of \( D \) expressed in (A.19),
\[
|u^* \Gamma^{1/2} D \Gamma^{1/2} u| = |v^* D v| = \left| \sum_{j=j_1}^{j_m} \sum_{l=1}^{K_j} v_{j,l}^* \frac{\Omega_j}{K_j \sqrt{2j}} v_{j,l} \right| \leq C \sum_{j=j_1}^{j_m} \sum_{l=1}^{K_j} \frac{1}{K_j \sqrt{2j}} \|\Omega_j\|_1 \|v_{j,l}\|^2
\]
\[
\leq C \left( \max_{j=j_1,...,j_m} \frac{1}{K_j \sqrt{2j}} \|\Omega_j\|_1 \right) \sum_{j=j_1}^{j_m} \sum_{l=1}^{K_j} \|v_{j,l}\|^2 = C \left( \max_{j=j_1,...,j_m} \frac{1}{K_j \sqrt{2j}} \|\Omega_j\|_1 \right) u^* \Gamma u, \tag{A.24}
\]
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where the constant $C$ comes from a change of matrix norms (see (2.2)) and only depends on the fixed dimension $n = 2$. By taking $\sup_{u \in S(\nu)} - 1$ on both sides of (A.24), we arrive at (A.23).

The second step towards showing (A.22) consists of analyzing the asymptotic behavior of the right-hand side of (A.23), as $\nu \to \infty$. Note that

$$\max_{j=j_{1},...,j_{m}} \frac{1}{K_{j}} \| \Omega_{j} \|_{l_{1}} \sim C \frac{1}{\nu}, \quad \nu \to \infty.$$  

(A.25)

In view of (A.25), for establishing (A.22) it suffices to show that $\sup_{u \in S(\nu)} u^{T}u$ is bounded. So, rewrite $Y = \{Y_{i}\}_{i=1,...,Y(\nu)}$. Since

$$\sup_{u \in S(\nu)} u^{T}u \leq \max_{i_{1}=1,...,Y(\nu)} \sum_{i_{2}=1}^{Y(\nu)} |\text{Cov}(Y_{i_{1}}, Y_{i_{2}})|$$  

(A.26)

(see Lemma 1 in Bardet (2000), p.509), we only need to show that the right-hand side of (A.26) is bounded. Fix $0 < \delta < 1$ and without loss of generality assume that $\text{length}(K) = 1$. We turn back to wavelet and dimensionality parameters (indices) to reexpress, and then bound, the right-hand side of (A.26) as

$$\max_{j=j_{1},...,j_{m}} \max_{k=1,...,K_{j}} \max_{i=1,...,2} \sum_{K' \geq j_{1}}^{j_{m}} \sum_{K' \geq k_{1}}^{K_{j}} \sum_{K' \geq k'_{1}}^{K_{j}'} |\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))|$$

$$\leq 2m \max_{j,j'=j_{1},...,j_{m}} \max_{k=1,...,K_{j}} \max_{i,i'=1,2} \sum_{k'=1}^{K_{j}'} |\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))|$$

$$= 2m \max_{j,j'=j_{1},...,j_{m}} \max_{k=1,...,K_{j}} \max_{i,i'=1,2} \sum_{k'=1}^{K_{j}'} 1 \left\{ \frac{\max\{2^{j}, 2^{j'}\}}{|2^{j} - 2^{j'}|} \right\} \leq 1 - \delta \right\} |\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))|.$$  

(A.27)

One can interpret the bound in (A.27) as dividing up the covariance terms into those that, parameter-wise, are either distant apart or close together (see (3.10)).

We now set out to develop a bound for the first summation term in (A.27). By (3.3),

$$|\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))| \leq \max_{i_{1},i_{2}=1,2} \max_{j_{1}',j_{2}'} \max_{j_{1}'} \sqrt{\text{Var} d_{i_{1}}(2^{j}, 0)} \sqrt{\text{Var} d_{i_{2}}(2^{j'}, 0)} \leq C,$$

i.e., the covariance terms have a common bound. Moreover, in regard to the associated indicators in (A.27), when $j' \geq j$, $\#\{k' : |2^{j-j'} k - k'_{1}| < (1 - \delta)^{-1}\} \leq 2(1 - \delta)^{-1} + 1$. Alternatively, when $j' < j$, $\#\{k' : |2^{j-j'} k - k'_{1}| < 2^{j'}(1 - \delta)^{-1}\} \leq 2^{j'-j'}(1 - \delta)^{-1} + 1$. Therefore, the first summation term in (A.27) comprises finitely many terms and is bounded by a constant, irrespective of $\nu$.

To bound the second summation term in (A.27), since $|\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))| \leq \|\text{Cov}(D(2^{j}, k), D(2^{j'}, k'))\|_{l_{1}}$, the bound (3.12) implies that

$$\sum_{K_{j}' \geq 1} 1 \left\{ \frac{\max\{2^{j}, 2^{j'}\}}{|2^{j} - 2^{j'}|} \right\} \leq 1 - \delta \right\} |\text{Cov}(d_{i}(2^{j}, k), d_{i'}(2^{j'}, k'))|$$

$$\leq C.$$
\[ \leq C \sum_{k' = 1}^{K_j} \frac{1}{\max_{|2j' - k'| \leq 1 - \delta}} \frac{1}{2^{2j' k' - 2N_0 - 2}} \log^\delta |2^j k - 2^j k'| \leq C \sum_{z \neq 0} \frac{1}{|z| 2N_0 - 2} \log^\delta |z| < \infty, \]

where \(2N_0 - 2 > 1\) by (2.6) and \(C\) does not depend on \(k, k'\). Consequently, (A.27) is bounded, and so is (A.26), as we wished to show. This establishes (A.22), and as a result, also (3.18). \(\square\)

**B Asymptotic normality at coarse scales: auxiliary results**

**Lemma B.1** Fix \(j \in \mathbb{N}\). Under (OBFM1)–(4) and (4.9), let \(EW(a(\nu)2^j)\) and \(W(a(\nu)2^j)\) be as in Lemma 4.2. Then, the following limits hold, as \(\nu \to \infty\):

\[
\frac{4(ac - b^2)}{a + c} \sim \frac{4 \det EW(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2h_1}, \quad \frac{4(ac - b^2)}{a + c} \sim \frac{4 \det EW(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2(h_2 - h_1)} \quad (B.1)
\]

\[
\frac{4(ac - b^2)}{a + c} \sim \frac{4 \det W(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2h_1}, \quad \frac{4(ac - b^2)}{a + c} \sim \frac{4 \det W(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2(h_2 - h_1)} \quad (B.2)
\]

\[
\lambda_1^E = 4 \bigg(1 - \frac{1 - \sqrt{1 - \frac{4(ac - b^2)}{(a + c)^2}}}{4(ac - b^2)/(a + c)^2}\bigg) \sim \frac{4 \det EW(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2h_1}, \quad \lambda_1 \sim \frac{4 \det W(2^j)}{(p_{12}^2 + p_{22}^2)b_{22}(2^j)} a(\nu)^{2h_1}, \quad (B.3)
\]

\[
\lambda_2^E \sim (p_{12}^2 + p_{22}^2)b_{22}(2^j)a(\nu)^{2h_2}, \quad \lambda_2 \sim (p_{12}^2 + p_{22}^2)b_{22}(2^j)a(\nu)^{2h_2}. \quad (B.4)
\]

**Proof:** In regard to \(\lambda_1\) and \(\lambda_1^E\), consider the relation (4.12), where \(a, b\) and \(c\) are expressed in (4.10). By Lemma 4.2,

\[
\frac{4(ac - b^2)}{a + c} = \frac{4a(\nu)^{2(h_1 + h_2)} \det EW(2^j)}{(p_{12}^2 + p_{22}^2)b_{11}a(\nu)^{2h_1} + 2(p_{12}p_{11} + p_{22}p_{21})b_{12}a(\nu)^{h_1 + h_2} + (p_{12} + p_{22})b_{22}a(\nu)^{2h_2}}, \quad (B.7)
\]

where the determinant is non-trivial due to (3.9). Again from Lemma 4.2 and by applying Theorem 3.1, an analogous expression holds for \(4(\hat{a}c - \tilde{b}^2)/(\hat{a} + \tilde{c})\). Then, the expressions (B.1), (B.2), (B.3), (B.4), (B.5) follow.

An analogous reasoning applied to \(\lambda_2\) and \(\lambda_2^E\) leads to (B.6), since the relation (4.13) shows that \(\lambda_2^E \sim a + c, \lambda_2 \sim \hat{a} + \tilde{c}. \quad \square\)

The next lemma, stated without proof, is used in the proof of Theorem 3.1. It is a simple extension of Lemma 2 in Istas and Lang (1997).
Lemma B.2  Let \( \{ W_{j,n} \}_{j=1,...,n}, n \in \mathbb{N} \), be an array of i.i.d. random variables such that \( EW_{j,k} = 0 \) and \( EW^2_{j,k} < \infty \). Let \( \{ \lambda_{j,n} \}_{j=1,...,n}, n \in \mathbb{N} \), be an associated array of constants \( \lambda_{j,n} \in \mathbb{R} \). Define the statistic \( V_n = \sum_{j=1}^n \lambda_{j,n} W_{j,n} \) and its variance \( \sigma^2_n = \text{Var}(V_n) \). If \( \max_{j=1,...,n} |\lambda_{j,n}| = o(\sigma_n) \), then \( \frac{V_n}{\sigma_n} \overset{d}{\to} N(0,1) \).

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