Granular Described by Attribute Logic Formulas

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Abstract. How to classify and describe granules in a concise and apt way is still an open, interesting and important problem, and also is a fundamental problem in granular computing. In this paper, we attempt to solve this problem. We prove the granular description theorem and the basic grain description theorem based on formal concept analysis. And then we propose an algorithm for finding attribute logic formula to describes the object granular.

1. Introduction
Humans usually describe and recognize objective thing from different levels and different granularity. There is a process of deepening the attribute characteristics of objective things, and hence the knowledge concepts at different levels or at different granularity are obtained. Granular computing is a kind of useful mathematical method for processing complex structure data, the idea of granular computing fits perfectly with the hierarchical and granular thinking mode of “from coarse to fine, from whole to part” in the process of human cognition. The idea of granular computing originated from professor Zadeh. Since Lin summarized relevant studies and introduced the term granular computing in 1997, the thinking and methods of granular computing have appeared in many fields, such as rough set, fuzzy set, evidence theory, cluster analysis, data analysis, machine learning, data mining and knowledge discovery, etc. In recent years, researches on granular computing have been extensive and many meaningful results have been obtained. For example, Yao study on the basic problems and methods of granular computing [1], Lin on the granular structure and representation [4], Pedrycz on granular computing methodology, mathematical framework and information granulation algorithm [5], Zhang on quotient space [6], etc.

However, as one of the basic work of granular reasoning and application, granule description has attracted little attention. This problem deserves to be investigated since it can not only help us to have a better understanding and comprehension of granules, but also shed some light on the unsolved problem “why some concepts are psychologically simple and easy to learn, while others seem difficult” [7]. Motivated by this problem, the main objective of this paper is to propose a granular description method based on formal concept analysis, and then answer the open questions presented in literature [8].

The rest of this paper is organized as follows. Section 2 summarizes some preliminary. Section 3 introduces attribute logic formulas based on formal context and the notion of corresponding valuation. Section 4 proves the granular description theorem and the basic grain description theorem based on formal concept analysis, and proposes an algorithm for finding attribute logic formula to describes the object granular.

2. Preliminary
In this section, we summarize some basic notions and conclusions on concept lattices. For more detail, we refer the reader to Ganter and Wille's works “Formal Concept Analysis” [7].
Definition 1. A formal context is a triplet $K = (G, M, I)$, where $I \subseteq G \times M$ is a binary relation between $G$ and $M$. The elements in $G$ and $M$ are called objects and attributes, respectively. $(g, m) \in I$ or $g I m$ indicates the object $g$ has the attribute $m$.

Definition 2. Let $K = (G, M, I)$ be a formal context. For a set $A \subseteq G$ of objects we define the set of attributes common to all objects in $A$ as

$$\alpha(A) = \{m \in M \mid g I m, \forall g \in A\}.$$  

Correspondingly, for a set $B \subseteq M$ of attributes we define the set of objects that have all attributes in $B$ as

$$\beta(B) = \{g \in G \mid g I m, \forall m \in B\}.$$  

Definition 3. Let $K = (G, M, I)$ be a formal context. For $A \subseteq G$ and $B \subseteq M$, if $\alpha(A) = B$ and $\beta(B) = A$ then a pair $(A, B)$ is called a formal concept of formal context $K$, and $A$ and $B$ are called the extent and the intent of $(A, B)$, respectively. The set of all concepts of formal context $K$ is denoted as $C(K)$.

Definition 4. Let $(A_1, B_1)$ and $(A_2, B_2)$ be two formal concepts of a given formal context $K$. $(A_1, B_1)$ is called a sup-concept of $(A_2, B_2)$ if $A_1 \subseteq A_2$ (or equivalently, $B_1 \supseteq B_2$) which can be denoted by $(A_1, B_1) \leq (A_2, B_2)$.

Obviously, the sub-concept relation $\leq$ is a partial order on the set $C(K)$.

3. Attribute logic formulas and their valuation

This section introduces logic language based on formal context and the notion of corresponding valuation. Logic language enables the formalized representation and interpretation of rules in the process of knowledge discovery. In order to obtain stronger descriptive ability, the attribute logic formulas in this paper are also introduced on the basis of set of atomic formulas as classic logic.

Definition 5. Let $K = (G, M, I)$ be a formal context and $\rightarrow, \land, \rightarrow$ the usual logical connectives. For $a \in M$, $a$ is called an attribute atomic formula of $K$, the $(\rightarrow, \land, \rightarrow)$-type free algebra generated by the all attribute atomic formulas is denoted as $\Gamma(K)$, an element in $\Gamma(K)$ is called an attribute logic formula of formal context $K$ (in this paper, it is called attribute logic formula to distinguish it from the notion of logic formula in classic mathematical logic).

The elements of $\Gamma(K)$ can be obtained in the following way: (1) $M \subseteq \Gamma(K)$; (2) if $\varphi \in \Gamma(K)$ then $\neg \varphi \in \Gamma(K)$; (3) if $\varphi, \psi \in \Gamma(K)$ then $\varphi \land \psi, \varphi \rightarrow \psi \in \Gamma(K)$; (4) $\gamma \in \Gamma(K)$ if and only if $\gamma$ is obtained by the above 3 steps.

It is often abbreviated for need $\varphi \lor \psi = \neg(\neg \varphi \land \neg \psi)$, $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$.

Let $K = (G, M, I)$ be a formal context. Since $\Gamma(K)$ is the $(\rightarrow, \land, \rightarrow)$-type free algebra generated by the attribute atomic formulas set $M$, for each $u \in G$, $v_u^{(0)} : M \rightarrow \{0, 1\}$, $v_u(a) = I(u, a)$ can determines a unique mapping $v_u : \Gamma(K) \rightarrow \{0, 1\}$ such that the restriction $v_u|_{M}$ of $v_u$ on $M$ is just $v_u^{(0)}$, and $v_u$ is called a $K$-valuation of $\Gamma(K)$. For a given attribute atomic formula $\varphi$, if $v_u(\varphi) = 1$ for every $K$-valuation $v_u$ then $\varphi$ is called absolutely true with respect to $K$.

The set of all $K$-valuations of $\Gamma(K)$ is denoted by $\Omega_K$, i.e. $\Omega_K = \{v_u : u \in G\}$. Then there exists a 1-1 correspondence $\pi : G \rightarrow \Omega_K$, $\pi(u) = v_u$ from $G$ to $\Omega_K$. Hence attribute atomic formula $a$ determines a function $\overline{a} : G \rightarrow \{0, 1\}$, $\overline{a}(u) = I(a, u)$ on $G$. If $\varphi = \varphi(a_h, \cdots, a_h)$ is an attribute logic formula consisting of $t$ attribute atomic formulas $a_h, \cdots, a_h$ then $\varphi$ determines a $t$ function $\overline{\varphi}(u) = \overline{\varphi}(a_h(u), \cdots, a_h(u))$, and the way $\overline{\varphi}(\overline{a}_h(u), \cdots, \overline{a}_h(u))$ acts on $\overline{a}_h(u), \cdots, \overline{a}_h(u)$ on $G$ through
acts on $a_i$ in $\Gamma(K)$ for example, let $\phi = \neg a_i \land a_j$ acts on $a_i, a_j$, then $\neg \phi(u) = (1 - a_i(u)) \lor a_j(u)$ acts on $a_i, a_j$. If there is a probability distribution $P$ on $G$, $\neg \phi$ is actually a random variable defined on $G$.

Obviously, an attribute logic formula is absolutely true if and only if $\forall u \in G, \neg \phi(u) = 0$.

In this paper, an attribute $a$ in attribute set $M$ is regarded as an attribute atomic formula. Following the mathematical logical semantic, a mapping $\nu : \Gamma(K) \rightarrow \{0, 1\}$ determines a mapping $\nu : \Gamma(K) \rightarrow \{0, 1\}$, $\nu$ is called a valuation of $\Gamma(K)$, and $\nu(\phi)$ is called the value of logic formula $\phi$.

The set of all valuation is denoted by $\Omega$. Obviously, $\Omega_k \subseteq \Omega$, that is to say the $K$-valuations set $\Omega_k$ is the subset of valuations set $\Omega$.

According to the decision logic method widely used in rough set, if $u \in G$ satisfy attribute $a$, i.e. $I(u, a) = 1$ or $\neg a(u) = 0$, then we denote $u \mapsto a$. In general, if $\neg \phi(u) = 0$, then we called object satisfy attribute logic formula $\phi$, also denote it by $u \mapsto \phi$. For $\phi \in \Gamma(K)$, the set of all objects satisfying $\phi$ is denoted by $h(\phi) = \{u | u \in G, u \mapsto \phi\}$, and it is called the semantic of $\phi$. Obviously, there is $h(\phi) = \{u | u \in G, \neg \phi(u) = 0\}$, sometimes we also write $h(\phi)$ as $\|\phi\|$.

**Proposition 2.** Let $K = (G, M, I)$ be a formal context. Then for $\phi, \psi \in \Gamma(K)$, the following equation holds:

1. $\|\neg \phi\| = G - \|\phi\|$;
2. $\|\phi \lor \psi\| = \|\phi\| \cup \|\psi\|$;
3. $\|\phi \land \psi\| = \|\phi\| \land \|\psi\|$;
4. $\|\phi \rightarrow \psi\| = (G - \|\phi\|) \lor \|\psi\|$.

The proof is easy.

4. Granular description based on formal concept analysis

Granular description is a basic problem in the formal concept analysis theory. The general description of this problem is as follows: Given formal context $K = (G, M, I)$, object granular $X$ and logical connective $\neg, \land, \rightarrow$, we can find an attribute logic formula that is connected (or partially connected) by $\neg, \land, \rightarrow$ to fully depict object granular $X$? That is to say, the objects that satisfy the logical formula are exactly the objects that came from $X$.

The following is a description theorem for general object granules. At first, we prove a lemma.

**Lemma 1.** Suppose that $\Sigma = \{v_{i1}, v_{i2}, \cdots, v_{in}\} \subset \Omega_k$ and $(r_1, r_2, \cdots, r_n)$ is a sequence containing only 0, 1. Then there exists an attribute logic formula $\phi \in \Gamma(K)$ such that: $v_{i1}(\phi) = r_1, i = 1, \cdots, n$.

**Proof.** We prove this lemma by mathematical induction. Let $n = 1$ and denote $A_1 = \{a \in M | v_{a1}(\phi) = 0\}$. If $A_1$ is not an empty set, then we take arbitrarily $a \in A_1$. If $r_1 = 0$ or $r_1 = 1$ then $\phi = a$ or $\phi = \neg a$ respectively. In this case $v_{a1}(\phi) = r_1$ holds. If $A_1$ is an empty set, then we take arbitrarily $a \in M$. If $r_1 = 0$ or $r_1 = 1$ then $\phi = \neg a$ or $\phi = a$ respectively. In this case $v_{a1}(\phi) = r_1$ also holds.

Assume that the conclusion is true as $n = k$. In the following we prove it is also true as $n = k + 1$. Let

$\Sigma = \{v_{i1}, v_{i2}, \cdots, v_{in}\}$ and $(r_1, r_2, \cdots, r_{k+1})$ be a given sequence containing only 0, 1. Denote $A_1 = \{a \in M | v_{a1}(\phi) = 0\}, i = 1, \cdots, k + 1$.

Since functions $v_{i1}, v_{i2}, \cdots, v_{in}$ take only 0 and 1, we know that $A_1, A_2, \cdots, A_{k+1}$ are not equal to each other. Hence there must exists a set $\bar{A}$ that doesn't contain any other set in these $k + 1$ sets.
Otherwise, the contradictory is obtained that a series of infinite sets which are not equal and contained in layers. Without loss of generality, let's say \( A = A_{k+1} \) is the set that doesn't contain any other set in \( A_1, A_2, \cdots, A_{k+1} \). By the induction assumed, there is an attribute logic formula \( \varphi \in \Gamma(K) \) such that

\[
V_{u_i}(\varphi) = r_i, \quad i=1, \cdots, k.
\]  

(2)

Since \( A_i - A' (i=1, \cdots, k) \) is not empty, there are \( a_1, a_2, \cdots, a_k \) such that:

\[
a_i \in M \cap (A_i - A') , \quad i=1, \cdots, k.
\]

(3)

If \( r_{k+1} = 1 \), then we set \( \psi = \varphi \lor (a_1 \land \cdots \land a_k) \). By (3) and (1) we know \( V_{u_i}(a_1 \land \cdots \land a_k) = 0 \). Hence

\[
V_{u_i}(\psi) = V_{u_i}(\varphi) \lor V_{u_i}(a_1 \land \cdots \land a_k) = V_{u_i}(\varphi), \quad i=1, \cdots, k.
\]

(4)

From that \( a_1 \notin A = A_{k+1} \) and (1), we have that \( V_{u_{k+1}}(a_1) = 1 \). Thus

\[
V_{u_{k+1}}(\psi) = V_{u_{k+1}}(\varphi) \lor V_{u_{k+1}}(a_1 \land \cdots \land a_k) = V_{u_{k+1}}(\varphi) \lor 1 = 1 = r_{k+1}.
\]

(5)

By (4) and (5), we obtain

\[
V_{u_i}(\psi) = r_i, \quad i=1, \cdots, k+1.
\]

(6)

If \( r_{k+1} = 0 \), then we set \( \psi = \varphi \land (\neg a_1 \lor \cdots \lor \neg a_k) \). By (3) and (1) we know \( V_{u_i}(\neg a_1 \lor \cdots \lor \neg a_k) = 1 \). Hence

\[
V_{u_i}(\psi) = V_{u_i}(\varphi) \land V_{u_i}(\neg a_1 \lor \cdots \lor \neg a_k) = V_{u_i}(\varphi), \quad i=1, \cdots, k.
\]

(7)

From that \( a_1 \notin A = A_{k+1} \) and (1), we have that \( V_{u_{k+1}}(\neg a_1) = 0 \). Thus

\[
V_{u_{k+1}}(\psi) = V_{u_{k+1}}(\varphi) \land V_{u_{k+1}}(\neg a_1 \lor \cdots \lor \neg a_k) = V_{u_{k+1}}(\varphi) \lor \max(V_{u_{k+1}}(\neg a_1), \cdots, V_{u_{k+1}}(\neg a_k))
\]

\[
= V_{u_{k+1}}(\varphi) \land 0 = 0 = r_{k+1}.
\]

(8)

By (7) and (8), we obtain

\[
V_{u_i}(\psi) = r_i, \quad i=1, \cdots, k+1
\]

(9)

This shows that the conclusion holds as \( n=k+1 \). So we prove the lemma.

**Theorem 1** (Granular description theorem based on formal context analysis). Let \( K=(G,M,I) \) be a formal context and \( X \subseteq G \) an object granular. Then there exists an attribute logic formula \( \varphi \) connected by \( \neg, \land \rightarrow \rightarrow \) such that

\[
h(\varphi) = \{ u | u \in G, \overline{\varphi}(u) = 1 \} = X.
\]

**Proof.** Let \( G = \{ u_1, u_2, \cdots, u_n \} \) and \( X = \{ u_1, u_2, \cdots, u_n \} \subseteq G \). By the object granular \( X \) we can construct the following 0-1 sequence: If \( u_k \in X \) then \( r_k = 1 \) else \( r_k = 0 \). By Lemma 1 there is an attribute logic formula \( \varphi \in \Gamma(K) \) such that \( V_{u_i}(\varphi) = r_i, \quad i=1, \cdots, n \), so \( \overline{\varphi}(u_i) = r_i, \quad i=1, \cdots, n \). It follows that \( h(\varphi) = \{ u | u \in G, \overline{\varphi}(u) = 1 \} = X \).

**Algorithm 1.** Finding attribute logic formula that describes the object granular

**Input:** Formal context \( K=(G,M,I) \) and object granular \( X \).

**Output:** Attribute logic formula \( \varphi \) that describes the object granular \( X \).

Step 1 Construct the 0-1 sequence \( \{ r_1, r_2, \cdots, r_n \} \) corresponding to \( X = \{ u_1, u_2, \cdots, u_n \} \). If \( u_k \in X \), then \( r_k = 1 \) else \( r_k = 0 \).

Step 2 Construct \( A_i = \{ a \in M | V_{u_i}(a) = 0 \}, \quad i=1, \cdots, n \).

Step 3 Choosing \( A_i \in \{ A_1, A_2, \cdots, A_n \} \) such that \( A_i \) is not included in the rest \( n-1 \) set. If \( A_i \) is not a empty set, then we take arbitrarily \( a \in A_i \). If \( r_i = 0 \) or \( r_i = 1 \) then \( \varphi = a \) or \( \varphi = \neg a \) respectively. If \( A_i \) is
an empty set, then we take arbitrarily \( a \in M \). If \( r_i = 0 \) or \( r_i = 1 \) then \( \varphi = \neg a \) or \( \varphi = a \) respectively.

Step 4. Choosing \( A_i \in \{ A_1, A_2, \cdots, A_n \} - \{ A_i \} \) such that \( A_i \) is not included in the rest \( n-2 \) set. Taking \( a_i \in M \cap (A_i - A_i) \), if \( r_i = 1 \) then set \( \varphi_i = \varphi_i \cup a_i \) else set \( \varphi_i = \varphi_i \land \neg a_i \).

Step 5. Choosing \( A_i \in \{ A_1, A_2, \cdots, A_n \} - \{ A_i, A_i \} \) such that \( A_i \) is not included in the rest \( n-3 \) set. Taking \( a_i \in M \cap (A_i - A_i) \), \( a_i \in M \cap (A_i - A_i) \), if \( r_i = 1 \) then set \( \varphi_i = \varphi_i \lor (a_i \land a_i) \) else set \( \varphi_i = \varphi_i \land (\neg a_i \land \neg a_i) \).

Step 6. Repeat step 5 until all sets \( \{ A_1, A_2, \cdots, A_n \} \) are selected. The final attribute logic formula \( \varphi_i \) is the description of object granular \( X \).

**Example 1.** Consider the following formal context.

| \( a \) | \( b \) | \( c \) | \( d \) | \( e \) |
|---|---|---|---|---|
| 1  | 0  | 1  | 1  | 0  |
| 2  | 0  | 0  | 1  | 0  |
| 3  | 1  | 1  | 1  | 0  |
| 4  | 1  | 1  | 0  | 1  |

Applying algorithm 1, for some given object granular \( \{1,3\} \), \( \{3,4\} \) and \( \{1,2,4\} \), their attribute logic formulas are described as \( b \land c \land e \), \( a \land b \) and \( \neg a \lor d \) respectively.

**Definition 8.** Let \( K=(G,M,I) \) be a formal context and \( X \subseteq G \) an object granular. If \( |X|=1 \) then \( X \) is called an atomic granular. If there exists concept \( C \in \mathcal{C}(K) \) such that its extension is just equal to \( X \), then \( X \) is called a basic granular. If \( |X|>1 \) and there is not \( C \in \mathcal{C}(K) \) such that its extension is equal to \( X \), then \( X \) is called a composite granular.

**Theorem 2** (Basic grain description theorem based on formal concept analysis). Let \( K=(G,M,I) \) be a formal context and \( X \subseteq G \) a basic granular. Then there are some attribute atomic formulas \( \{a_i\}(t \in T) \) such that \( X = h(\land_{i \in T} a_i) \).

**Proof.** Since \( X \) is a basic granular, there exists concept \( C \in \mathcal{C}(K) \) such that its extension is just equal to \( X \). Hence we can assume \( C=(X,B) \), where \( B = \{a_i \mid t \in T\} \) is the intension of concept \( C \). Hence if we can take attribute logic formula \( \varphi = \land_{i \in T} a_i \), then

\[
X = g(B) = \{u \in G \mid u I a_i, \forall t \in T\} = \{u \in G \mid \overline{a_i(u)} = 1, \forall t \in T\} = \{u \in G \mid \land_{i \in T} a_i(u) = 1\} = h(\land_{i \in T} a_i).
\]

The converse of theorem 3 is not true, this is due to the conjunction of some atomic formulas of an attribute and all objects satisfying it do not form a concept.

**Example 2.** Consider the following formal context.

| \( a \) | \( b \) | \( c \) | \( d \) |
|---|---|---|---|
| 1  | 1  | 1  | 1  |
| 2  | 0  | 1  | 1  |
| 3  | 0  | 0  | 1  |
| 4  | 0  | 0  | 0  | 1  |
We take atomic formulas \( b, c \), \( h(b \land c) = \{1,2\} \), hence \( \{1,2\}, \{b, c\} \) is not a concept.

This raises an obvious question: what condition does the conjunction of some attribute atomic formulas satisfy such that the conjunction and the all objects that satisfies the conjunctive attribute form a concept. We have the following conclusion.

**Proposition 3.** Let \( K=(G,M,I) \) be a formal context. If a conjunction \( \land_{\mathcal{RT}} a_i \) contains the most attribute atomic formulas, then \( (h(\land_{\mathcal{RT}} a_i), \land_{\mathcal{RT}} a_i) \) is a concept, and \( h(\land_{\mathcal{RT}} a_i) \) is an object granular.

**Proof.** We have only to prove \( f(h(\land_{\mathcal{RT}} a_i)) = \{a_t, t \in T\} \). Since \( \overline{a_i} (u) = 1 \) iff \( u a_i \), we obtain

\[
f(h(\land_{\mathcal{RT}} a_i)) = \{a \in M | \forall u \in \land_{\mathcal{RT}} a_i (u) = 1, u a_i \} = \{a \in M | \forall u \in \land_{\mathcal{RT}} a_i (u) = 1, u a_i \} \supseteq \{a_t, t \in T\}.
\]

If there is an attribute atomic formula \( b \), which is not included in \( \land_{\mathcal{RT}} a_i \), also satisfy: when \( \forall u \in \land_{\mathcal{RT}} a_i (u) = 1 \) \( \overline{b} (u) = 1 \) . Hence \( h(b \land (\land_{\mathcal{RT}} a_i)) = h(\land_{\mathcal{RT}} a_i) \). This contradicts the conjunction \( \land_{\mathcal{RT}} a_i \) contains the most attribute atomic formulas. Therefore we prove \( f(h(\land_{\mathcal{RT}} a_i)) = \{a_t, t \in T\} \).

By theorem 3 and proposition 3, we know that the basic object granules in a formal context \( K=(G,M,I) \) can be described by the conjunction that contains sufficient number of attribute atomic formulas.

### 5. Conclusion

This paper focus on the description of objective granular in formal concept analysis, proves the granular description theorem and the basic grain description theorem, and then we propose an algorithm for finding attribute logic formula to describes the object granular. The further conclusion on granular description will be given in future research.

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