Vertex Nomination, Consistent Estimation, and Adversarial Modification

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May 16, 2019

Abstract

Given a pair of graphs $G_1$ and $G_2$ and a vertex set of interest in $G_1$, the vertex nomination problem seeks to find the corresponding vertices of interest in $G_2$ (if they exist) and produce a rank list of the vertices in $G_2$, with the corresponding vertices of interest in $G_2$ concentrating, ideally, at the top of the rank list. In this paper we study the effect of an adversarial contamination model on the performance of a spectral graph embedding-based vertex nomination scheme. In both real and simulated examples, we demonstrate that this vertex nomination scheme performs effectively in the uncontaminated setting; adversarial network contamination adversely impacts the performance of our VN scheme; and network regularization successfully mitigates the impact of the contamination. In addition to furthering the theoretic basis of consistency in vertex nomination, the adversarial noise model posited herein is grounded in theoretical developments that allow us to frame the role of an adversary in terms of maximal vertex nomination consistency classes.

1 Introduction and background

Given graphs $G_1$ and $G_2$ and vertices of interest $V^* \subset V(G_1)$, the aim of the vertex nomination (VN) problem is to rank the vertices of $G_2$ into a nomination list with
the corresponding vertices of interest concentrating at the top of the nomination list. In recent years, a host of VN procedures have been introduced (see, for example, [13, 29, 25, 16, 36, 48]) that have proven to be effective information retrieval tools in both synthetic and real data applications. Moreover, recent work establishing a fundamental statistical framework for VN has led to a novel understanding of the limitations of VN efficacy in evolving network environments [26]. Herein, we consider a general statistical model for adversarial contamination in the context of vertex nomination—here the adversary model can both randomly add or remove edges and/or vertices in the network—and we examine the effect of both this contamination and subsequent data regularization (effectively removing outlier nodes) on VN performance. To motivate our mathematical and statistical results further, we first consider an illustrative real data example in Section 1.1 in which we demonstrate the following: A VN scheme that works effectively; network contamination adversely impacting the performance of our VN scheme; and network regularization successfully mitigating the impact of the contamination. Note that we will provide a more thorough background of the relevant literature after the motivating example in Section 1.2.

1.1 Motivating example

Consider the pair of high school friendship networks in [31]: The first, $G_1$, has 156 nodes, each representing a student, and has two vertices adjacent if the two students made contact with each other at school in a given time period; the second, $G_2$, has 134 vertices, again with each vertex representing a student, and has two vertices adjacent if the two students are friends on Facebook. There are 82 students appearing in both $G_1$ and $G_2$, and we pose the VN problem here as follows: given a student-of-interest in $G_1$, can we nominate the corresponding student (if they exist) in $G_2$. We note here that the vertex nomination approach outlined below easily adapts to the multiple vertices of interest (v.o.i.) scenario (i.e., given students-of-interest in $G_1$, can we nominate the corresponding students, if they exist, in $G_2$)—and we will provide the necessary details for handling both single and multiple v.o.i. below.

In one idealized data setting, all students would appear in both graphs as this would potentially maximize the signal present in the correspondence of labels across graphs. This bears itself out in the following illustrative VN experiment. Consider the following simple VN scheme, which we denote $VN \circ GMM \circ ASE$: Given vertex (or vertices) of interest $v^*$ in $G_1$ and seeded vertices $S \subset V_1 \cap V_2$ (seeds here represent vertices whose identity across networks is known a priori), proceed as follows (see Section 4.1 for full detail):
1. Use Adjacency Spectral Embedding (ASE) [43] to separately embed \( G_1 \) and \( G_2 \) into a common Euclidean space \( \mathbb{R}^d \);
2. Solve the orthogonal Procrustes problem [39] to find an orthogonal transformation aligning the seeded vertices across graphs; use this transformation to align the embeddings of \( G_1 \) and \( G_2 \) in \( \mathbb{R}^d \);
3. Use model-based Gaussian mixture modeling (GMM; e.g., the R package MClust [18]) to simultaneously cluster the vertices of the embedded graphs. If \( u \in V(G_1) \) and \( v \in V(G_2) \) are clustered points in this embedding with respective covariance matrices \( \Sigma_u \) and \( \Sigma_v \) in their components of the GMM, then compute

\[
\Delta(u, v) = \max \left( D_u(u, v), D_v(u, v) \right),
\]

where

\[
D_u(u, v) = \sqrt{(u - v)^T \Sigma_u^{-1} (u - v)} \quad \text{and} \quad D_v(u, v) = \sqrt{(u - v)^T \Sigma_v^{-1} (u - v)}
\]

are the respective Mahalanobis distances from \( u \) to \( v \).
4. In the single v.o.i. setting, rank the candidate v.o.i. in \( G_2 \) by increasing value of \( \Delta(v^*, u) \) (so that the smallest \( \Delta(v^*, u) \) are ranked first). In the multiple v.o.i. setting, we rank the candidate v.o.i. in \( G_2 \) by increasing value of \( \min_{v \in V^*} \Delta(v, u) \).

We can consider running the above procedure in the idealized data setting where we only consider the induced subgraphs of \( G_1 \) and \( G_2 \) containing the 82 common vertices across graphs (call these graphs \( G_1^{(i)} \) and \( G_2^{(i)} \)), and we can also consider running the procedure in the setting where the 52 vertices in \( G_2 \) without matches across graphs are added to \( G_2^{(i)} \) as a form of contamination. These unmatchable vertices can have the effect of obfuscating the correspondence amongst the common vertices across graphs, and thus can diminish VN performance. Indeed, we see this play out in Figure 1.

In Figure 1, we plot the performance of VN\( \circ \)GMM\( \circ \)ASE averaged over \( nMC = 500 \) random seed sets of size \( s = 10 \). In the left figure, the \( x \)-axis shows the ranks in the nomination list and the \( y \)-axis shows the mean (\( \pm 2\)s.e.) number of vertices \( v \in G_1^{(i)} \), when viewed as the lone v.o.i., that had their corresponding vertex of interest ranked in the top \( x \) by VN\( \circ \)GMM\( \circ \)ASE. The right figure shows the same results normalized by chance performance, where we plot

\[
y = \frac{\text{mean } \# \text{ of v.o.i. with corresp. v.o.i. ranked in top } x \text{ by VN}\circ\text{GMM}\circ\text{ASE}}{\text{mean } \# \text{ of v.o.i. with corresp. v.o.i. ranked in top } x \text{ by chance algorithm}}
\]

versus \( x \). The gold line represents performance in the idealized networks \( G_1^{(i)} \) and \( G_2^{(i)} \), and the red line represents performance in the contaminated network pair \((G_1^{(i)}, G_2^2)\). We see that the contamination detrimentally affects the performance of VN\( \circ \)GMM\( \circ \)ASE.
a) Mean # achieving rank ≤ x versus x
b) Chance normalized mean # achieving rank ≤ x versus x

Figure 1: We plot the performance of VN ◦ GMM ◦ ASE averaged over nMC = 500 random seed sets of size s = 10. In the left figure, the x-axis shows the ranks in the nomination list and the y-axis shows the mean (± 2s.e.) number of vertices v ∈ G1(i), when viewed as the v.o.i., that had their corresponding vertex of interest ranked in the top x by VN ◦ GMM ◦ ASE. The right figure shows the same result normalized by chance performance, where we plot (± 2s.e.) y = mean # of v.o.i. with corresp. v.o.i. ranked in top x by VN ◦ GMM ◦ ASE versus x. The gold line represents performance in the idealized network pair (G1(i), G2(i)); the red line for (G1(i), G2); the green line for (G1(i), G2(0.075,0)); the teal line for (G1(i), G2(0.1,0)); the blue line for (G1(i), G2(0.25,0)); and the pink line for (G1(i), G2(0.1,0.1)).

Algorithm 1: Regularization via network trimming

Input: Graph G, x, y ∈ (0, 1), seed set S;

1. Initialize Vt = S
2. Rank the vertices in V(G) \ S by descending degree (ties are broken via averaging over ranks). For each vertex u in V(G) \ S, denote the rank via rk(u);
   for u ∈ V(G) \ S, do
      3. If x < \frac{rk(u)}{|V(G)\setminus S|} ≤ y, add u to Vt;
   end for
4. Output: G(x,y) = G[Vt], the induced subgraph of G on Vt;

ASE at all levels, as for all x, the number of v.o.i. in G1(i) with their corresponding v.o.i. ranked in the top x in the second graph is larger in (G1(i), G2(i)) versus in (G1(i), G2).

How can we mitigate the effect of the contamination in G2? Network regularization is a natural solution, and we here consider as a regularization strategy the network analogue of the classical trimmed mean estimator. To wit, we consider the regularization procedure in Algorithm 1 inspired by the network trimming procedure in [15];
see also the work in [24] for the impact of trimming regularization on random graph concentration.

**Remark 1.** The parameters $x$ and $y$ appearing in Algorithm 1 are unknown a priori, and to data-adaptively choose $x$ and $y$, we sweep over possible values and choose the values of $x$ and $y$ that leads to the maximum network modularity in $G_{2}^{(x,y)}$ when clustering the vertices of $G_{2}^{(x,y)}$ via $GMM \circ ASE$ clustering; i.e., embed $G_{2}^{(x,y)}$ using ASE and cluster the embedding using a model-based GMM procedure. Given a clustering $C$, the modularity is defined as usual via

$$Q(C) = \frac{1}{2|E|} \sum_{i,j} \left[ A_{i,j} - \frac{d_{i}d_{j}}{2|E|} \right] \mathbb{1}\{C_{i} = C_{j}\},$$

where $|E|$ = the number of edges in $G_{2}^{(x,y)}$; $A_{i,j}$ is the $i$, $j$-th element of the adjacency matrix $A$ of $G_{2}^{(x,y)}$; $d_{i}$ is the degree of vertex $i$ in $G_{2}^{(x,y)}$; and $C_{i}$ is the cluster containing vertex $i$ in $C$.

In the left panel of Figure 2, we plot the modularity of the GMM clustering in the trimmed $G_{2}^{(x,y)}$ as a function of $x, y \in \{0, 0.05, 0.1, 0.15, 0.2, 0.25\}$. Note that we average the modularity values over $nMC = 500$ seed sets of size $s = 10$ (the same seed sets as used in Figure 1). The color indicates the value of the modularity, with darker red indicating lower values and lighter yellow–to–white indicating larger values. From the figure, we can see that modularity is maximized when $y = 0$ (i.e., no large degree vertices trimmed) and $x \approx 0.05$–0.1. We note that this trimming process can cut core vertices as well as junk vertices, and core vertices cut from $G_{2}$ can never be recovered via $VN \circ GMM \circ ASE$. This is demonstrated in the right panel of Figure 2, where the horizontal asymptotes for each trimming value indicates the maximum number of core vertices that are recoverable after regularization.

In Figure 1, we see the effect of regularization play out. Indeed, mean $VN \circ GMM \circ ASE$ performance in the regularized setting increases versus in the contaminated setting for $(x, y) = \{(0.075, 0), (0.1, 0), (0.25, 0)\}$, whereas mean regularized performance decreases for $(x, y) = \{(0.1, 0.1)\}$. From the right figure, we observe that mean performance in the $(x, y) = \{(0.075, 0), (0.1, 0), (0.25, 0)\}$ regularized setting is significantly better than chance, while over-regularizing induces worse than chance performance (the pink line in Figure 1 panel b). While over-regularizing can adversely affect performance, this data-adaptive regularization— while not fully recovering the performance of the idealized setting— nonetheless effectively mitigates the impact of the contamination on our $VN \circ GMM \circ ASE$ algorithm.
Figure 2: In the left panel, we plot the modularity of the GMM clustering in the trimmed $G_{2}^{(x,y)}$ as a function of $x, y \in \{0, 0.05, 0.1, 0.15, 0.2, 0.25\}$. Note that we average the modularity values over $nMC = 500$ seed sets of size $s = 10$ (the same seed sets as used in Figure 1). The color indicates the value of the modularity, with darker red indicating lower values and lighter yellow–to–white indicating larger values. In the right panel, we plot the performance of VN $\circ$ GMM $\circ$ ASE averaged over $nMC = 500$ random seed sets of size $s = 10$. In the left figure, the $x$-axis shows the ranks in the nomination list and the $y$-axis shows the mean ($\pm$ 2s.e.) number of vertices $v \in G_{i}^{(i)}$, when viewed as the v.o.i., that had their corresponding vertex of interest ranked in the top $x$ by VN $\circ$ GMM $\circ$ ASE.

1.1.1 The role of seeds

Figure 1 shows performance of VN $\circ$ GMM $\circ$ ASE averaged over 500 randomly chosen seed sets of size 10. While performance, on the whole, increases with proper regularization, the story can vary wildly from seed set to seed set. To demonstrate this, we plot the performance of VN $\circ$ GMM $\circ$ ASE over two particular seed sets (out of the 500 total used in Figure 1) in Figure 3. In the top panels, we plot performance in the setting of “bad” seeds; i.e., those seeds for which the regularization is unable to effectively mitigate the performance loss due to contamination. In the bottom panels, we plot performance in the setting of “good” seeds; i.e., those seeds for which the contamination negatively impacts performance, but subsequent regularization is able to effectively mitigate this performance loss. These two figures (and their respective chance normalizations in the right panels) point to the primacy of seed selection and of understanding what differentiates “good” versus “bad” seeds. While a full exploration of this is beyond the scope of the present text, this is an active area of our work.

1.2 Background

In modern statistics and machine learning, graphs are a common way to take into account the complex relationships between data objects, and graphs have been used
Figure 3: We plot the performance of VN o GMM o ASE for two particular seed sets of size $s = 10$. In the top (resp., bottom) panels we plot the performance with “bad” (resp., “good”) seeds. In the left panels, the $x$-axis shows the ranks in the nomination list and the $y$-axis shows how many vertices $v \in G^{(i)}_1$, when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top $x$ by VN o GMM o ASE. The right panels shows the same result normalized by chance performance, where we plot $y = \frac{\text{mean } \# \text{ of v.o.i. with corresp. v.o.i. ranked in top } x \text{ by VN o GMM o ASE}}{\text{mean } \# \text{ of v.o.i. with corresp. v.o.i. ranked in top } x \text{ by chance algorithm}}$ vs. $x$. The gold line represents performance in the idealized network pair $(G^{(i)}_1, G^{(i)}_2)$; the red line for $(G^{(i)}_1, G_2)$; the green line for $(G^{(i)}_1, G^{(0.075,0)}_2)$; the teal line for $(G^{(i)}_1, G^{(0.1,0)}_2)$; the blue line for $(G^{(i)}_1, G^{(0.25,0)}_2)$; and the pink line for $(G^{(i)}_1, G^{(0.1,0.1)}_2)$.

In applications across the biological (see, for example, [41, 7, 1, 30, 20, 32]) and social sciences (see, for example, [34, 40, 19, 21]). In addition to more traditional statistical inference tasks such as clustering [38, 37, 6, 33], classification [46, 10, 1], and estimation [5, 4, 43], there has been significant work in more network-specific inference tasks such as graph matching [11, 17, 47], and vertex nomination [29, 12, 16].
Loosely speaking, the vertex nomination problem can be stated as follows: given graphs \( G_1 \) and \( G_2 \) and vertices of interest \( V^* \subseteq V(G_1) \), rank the vertices of \( G_2 \) into a nomination list with the corresponding vertices of interest concentrating at the top of the nomination list (see Definition 4 for full detail). While vertex nomination has found applications in a number of different areas, such as social networks in [36] and data associated with human trafficking in [16], there are relatively few results establishing the statistical properties of vertex nomination. In [16], consistency is developed within the stochastic blockmodel random graph framework, where interesting vertices were defined via community membership. In [26], the authors develop the concepts of consistency and Bayes optimality for a very general class of random graph models and a very general definition of what makes the v.o.i. interesting. In this paper, we further develop the ideas in [26], with the aim of developing a theoretical regime in which to ground the notion of adversarial contamination in VN.

There has been significant recent attention towards better understanding the impact of adversarial attacks on machine learning methodologies (see, for example, [23, 8, 35, 14, 50]). Herein, we define an adversarial attack on a machine learning algorithm to be a mechanism that changes the data distribution in order to negatively affect algorithmic performance; see Definition 13. From a practical standpoint, adversarial attacks model the very real problem of having data compromised; if an intelligent agent has access to the data and algorithm, the agent may want to modify the data or the algorithm to give the wrong prediction/inferential conclusion. Although there has been much work on adversarial modeling in machine learning, there has been less theory developed for adversarial attacks from a statistical perspective.

The adversarial framework we consider is similar to the model considered in [8], and it is motivated by the example in the previous section in which the addition of the vertices without correspondences to \( G_2 \) negatively impacted VN performance. Suppose that we are interested in performing vertex nomination on a graph pair, but an adversary randomly adds and deletes some edges and/or vertices in the second graph. For example, suppose we are trying to find influencers on Instagram by vertex matching to Facebook. An influencer that has knowledge of our procedure may attempt to make our algorithm fail in its nominations, perhaps by friending and de-friending people on Facebook. Even if our vertex nomination scheme was working well prior to encountering the adversary, it may not be after modification by the adversary. However, if the adversary adds edges/vertices to a graph with some probability and deletes edges/vertices with another probability, it may be possible to partially recover the structure of the original graph by removing vertices with unusual degree behavior [15]. Such a modification is the graph analogue of the “trimmed mean” estimator [42].
from classical statistics.

Empirically, if we assume the adversary is modifying the data randomly, can we still predict whether our VN scheme will perform well on the regularized graph? From a statistical standpoint, what can we say about the statistical consistency of our original vertex nomination rule? Our motivating example suggests that it may be possible to recover performance after regularization, but theory is needed both to explain why that may be the case and to properly frame the problem. Hence, to answer these questions, we further develop the theory in [26] to situate the notion of adversarial contamination within the idea of maximal consistency classes for a given VN rule (Section 2.1). In this framework, the goal of an adversary is to move a model out of a rule’s consistency class, while regularization enlarges the consistency class to (hopefully) thwart the adversary. While we are unable to rigorously establish this for the VN rule, VN◦GMM◦ASE, considered herein, we demonstrate with real and synthetic data examples that countering such an adversarial attack via network regularization can effectively ameliorate VN performance (Section 4).

**Notation:** Note that the following notation will be used throughout. For a positive integer \( k \), we will let \( G_k \) denote the set of \( k \)-vertex labeled graphs, and we will let \([k] = \{1, 2, 3, \ldots, k\}\).

### 2 Vertex Nomination and Consistency

We will now rigorously define the VN problem and consistency within the VN framework. Combined with the results on consistency classes in Section 2.1, this will allow us to provide a statistical basis for understanding adversarial attacks in VN.

As in our motivating work in [26], we will situate our analysis of the VN problem in the very general framework of nominatable distributions.

**Definition 2.** For a given \( n, m \in \mathbb{Z} > 0 \), the set of **Nominatable Distributions of order** \((n, m)\), denoted \( N_{n,m} \), is the collection of all families of distributions of the following form

\[
F_{\Theta}^{(n,m)} = \{ F_{c,\theta}^{(n,m)} \} \quad \text{s.t.} \quad 0 \leq c \leq \min(n, m) \in \mathbb{Z}, \theta \in \Theta \subset \mathbb{R}^{d(n,m)}
\]

where \( F_{c,\theta}^{(n,m)} \) is a distribution on \( G_n \times G_m \) parameterized by \( \theta \in \Theta \) satisfying:

1. The vertex sets \( V_1 = \{v_1, v_2, \ldots, v_n\} \) and \( V_2 = \{u_1, u_2, \ldots, u_m\} \) satisfy \( v_i = u_i \) for \( 0 < i \leq c \). We refer to \( C = \{v_1, v_2, \ldots, v_c\} = \{u_1, u_2, \ldots, u_c\} \) as the core vertices. These are the vertices that are shared across the two graphs and imbue the model with a natural notion of corresponding vertices.
2. Vertices in $J_1 = V_1 \setminus C$ and $J_2 = V_2 \setminus C$, satisfy $J_1 \cap J_2 = \emptyset$. We refer to $J_1$ and $J_2$ as junk vertices. These are the vertices in each graph that have no corresponding vertex in the other graph.

3. The induced subgraphs $G_1[J_1]$ and $G_2[J_2]$ are conditionally independent given $\theta$. The vertices in $C$ are those that have a corresponding paired vertex in each graph; where corresponding can be defined very generally. Corresponding vertices need not correspond to the same person/user/account, rather corresponding vertices are understood as those that share a desired property across graphs. In particular, we will assume that the vertices of interest in $G_1$ correspond to the vertices of interest in $G_2$. Having access to the vertex labels would then render the VN problem trivial. To model the uncertainty often present in data applications, where the vertex labels (or correspondences) are unknown a priori we adopt the notion of obfuscation functions.

**Definition 3.** Let $(G_1, G_2) \sim F_{\theta, \theta}^{(n,m)} \in \mathcal{N}_{n,m}$, and let $W$ be a set satisfying $W \cap V_i = \emptyset$ for $i = 1, 2$. An obfuscating function $\sigma : V_2 \mapsto W$ is a bijection from $V_2$ to $W$. We refer to $W$ as an obfuscating set, and we let $\mathcal{O}_W$ be the set of all such obfuscation functions.

In this framework, a VN scheme is defined as follows.

**Definition 4.** (VN Scheme) Let $n, m \in \mathbb{Z} > 0$, and for each $g \in \mathcal{G}_m$, $u \in V(g)$, let

$$\mathcal{I}(u; g) = \{w \in V(g) \text{ s.t. } \exists \sigma \text{ an automorphism of } g, \text{ s.t. } \sigma(u) = w\}.$$ 

Let $W$ be an obfuscating set and $\sigma \in \mathcal{O}_W$ be given. For a set $A$, let $\mathcal{T}_A$ denote the set of all total orderings of the elements of $A$. A vertex nomination scheme is a function $\Phi : \mathcal{G}_n \times \sigma(\mathcal{G}_m) \times 2^{V_1} \mapsto \mathcal{T}_W$ satisfying the following consistency property: If for each $u \in V_2$, we define $\text{rank}_{\Phi(g_1, \sigma(g_2), V^*)(\sigma(u))}$ to be the position of $\sigma(u)$ in the total ordering provided by $\Phi(g_1, \sigma(g_2), V^*)$, and we define $\rho_{\Phi} : \mathcal{G}_n \times \mathcal{G}_m \times \mathcal{O}_W \times 2^{V_1} \times 2^{V_2} \mapsto 2^m$ via

$$\rho_{\Phi}(g_1, g_2, \sigma, V^*, S) = \{\text{rank}_{\Phi(g_1, \sigma(g_2), V^*)(\sigma(u))} \text{ s.t. } u \in S\},$$

then we require that for any $g_1 \in \mathcal{G}_n$, $g_2 \in \mathcal{G}_m$, $V^* \subset V_1$, obfuscating functions $\sigma_1, \sigma_2 \in \mathcal{O}_W$ and any $u \in V(g_2)$,

$$\rho_{\Phi}(g_1, g_2, \sigma_1, V^*, \mathcal{I}(u; g_2)) = \rho_{\Phi}(g_1, g_2, \sigma_2, V^*, \mathcal{I}(u; g_2))$$

$$\Leftrightarrow \sigma_2 \circ \sigma_1^{-1}(\mathcal{I}(\Phi(g_1, \sigma_1(g_2), V^*)[k]; \sigma_1(g_2)) = \mathcal{I}(\Phi(g_1, \sigma_2(g_2), V^*)[k]; \sigma_2(g_2))$$

for all $k \in [m]$,

where $\Phi(g_1, \sigma(g_2), V^*)[k]$ denotes the $k$-th element (i.e., the rank-$k$ vertex) in the ordering $\Phi(g_1, \sigma(g_2), V^*)$. We let $\mathcal{V}_{nm}$ denote the set of all such VN schemes.
Remark 5. The consistency criterion, Eq. 1, models the property that a sensibly-defined vertex nomination scheme should view all vertices in a given $I_o(u)$ as being equally “interesting” in $G_2$. These vertices are topologically indistinguishable, and thus are only separated by their labels which have been obfuscated via $o$. Truly obfuscated vertex labels should be independent of the obfuscation function, and the consistency criterion requires that the set of ranks of each set of equivalent vertices (i.e., each $I_o$) are only separated by their labels which have been obfuscated via $o$. For an obfuscating set, let $(g_1, g_2)$ be realized from $(G_1, G_2) \sim F^{(n, m)}_{	heta} \in \mathcal{N}_{n, m}$ with a vertex of interest set $V^* \subset C$. For $k \in [m - 1]$, we define the level-k nomination losses via

$$
\ell_k(1)(\Phi, g_1, g_2, V^*) := \frac{\sum_{v \in V^*} 1 \{ \text{rank}_{\Phi(g_1, o(g_2), V^*)}(o(v)) \geq k + 1 \}}{|V^*|} \quad \text{(recall loss)}
$$

$$
= 1 - \frac{\sum_{v \in V^*} 1 \{ \text{rank}_{\Phi(g_1, o(g_2), V^*)}(o(v)) \leq k \}}{|V^*|}
$$

$$
\ell_k(2)(\Phi, g_1, g_2, V^*) := \frac{\sum_{v \in V^*} 1 \{ \text{rank}_{\Phi(g_1, o(g_2), V^*)}(o(v)) \geq k + 1 \}}{|k|} \quad \text{(precision loss)}
$$

$$
= \frac{|V^*| - \sum_{v \in V^*} 1 \{ \text{rank}_{\Phi(g_1, o(g_2), V^*)}(o(v)) \leq k \}}{|k|}.
$$

The error of a VN scheme is then defined as the expected loss. To wit, we have

$$
L_k(1)(\Phi, V^*) := \mathbb{E}_{(G_1, G_2) \sim F^{(n, m)}_{\theta}}[\ell_k(1)(\Phi, G_1, G_2, V^*)] \quad \text{(recall error)}
$$

$$
= \frac{1}{|V^*|} \sum_{v \in V^*} \mathbb{P}_{(G_1, G_2) \sim F^{(n, m)}_{\theta}}(\text{rank}_{\Phi(G_1, o(G_2), V^*)}(o(v)) \geq k + 1)
$$

$$
L_k(2)(\Phi, V^*) := \mathbb{E}_{(G_1, G_2) \sim F^{(n, m)}_{\theta}}[\ell_k(2)(\Phi, G_1, G_2, V^*)] \quad \text{(precision error)}
$$

$$
= \frac{1}{|k|} \sum_{v \in V^*} \mathbb{P}_{(G_1, G_2) \sim F^{(n, m)}_{\theta}}(\text{rank}_{\Phi(G_1, o(G_2), V^*)}(o(v)) \geq k + 1)
$$

The level-k Bayes optimal scheme is defined as any element

$$
\Phi_{k, V^*} \in \arg\min_{\Phi \in \mathcal{V}_{nm}} L_k(1)(\Phi, V^*) = \arg\min_{\Phi \in \mathcal{V}_{nm}} L_k(2)(\Phi, V^*),
$$
with corresponding errors $L_k^{*(1)}$ and $L_k^{*(2)}$.

In the absence of symmetries amongst the vertices in $V^*$ (i.e., $\mathcal{I}(v,G_2) = \{v\}$ for all $v \in V^*$), the derivation of the Bayes optimal scheme in the present $|V^*| > 1$ setting mimics that of the $|V^*| = 1$ setting presented in [26]. See Appendix A for full detail. Bayes optimal schemes when symmetries exist for the v.o.i., i.e., $|I(v_i, g_2)| > 1$, offer additional complications and, in the case when $|V^*| = 1$ done in [26], little additional insight. Precisely defining the Bayes optimal scheme in the case of symmetries when $|V^*| > 1$ is notationally and technically nontrivial, and is the subject of current research. Consistency in the VN framework is then defined as follows.

**Definition 8.** Let $F = \left( F_{c_n,\theta_n}^{(n,m_n)} \right)_{n=n_0}^{\infty}$ be a sequence of distributions in $\mathcal{N}$. We say that $F$ has nested cores if there exists an $n_1$ such that for all $n_1 \leq n < n'$, if $(G_1,G_2) \sim F_{c_n,\theta_n}^{(n,m_n)}$ and $(G_1',G_2') \sim F_{c_n',\theta_n'}^{(n',m_n')}$, we have, letting $C$ and $C'$ be the core vertices associated with $F_{c_n,\theta_n}^{(n,m_n)}$ and $F_{c_n',\theta_n'}^{(n',m_n')}$ respectively, and denoting the junk vertices $J_1, J_1', J_2, J_2'$ analogously,

i. $V(G_1) = C \cup J_1 \subset V(G_1') = C' \cup J_1'$;

ii. $V(G_2) = C \cup J_2 \subset V(G_2') = C' \cup J_2'$;

iii. $C \subset C'$.

**Definition 9.** Let $F = \left( F_{c_n,\theta_n}^{(n,m_n)} \right)_{n=n_0}^{n=\infty}$ be a sequence of nominatable distributions in $\mathcal{N}$ with nested cores satisfying $\lim_{n \rightarrow \infty} m_n = \infty$. For a given non-decreasing sequence $(k_n)$, we say that a VN rule $\Phi = \left( \Phi_{n,m_n} \right)_{n=n_0}^{n=\infty}$ is

i. level-$(k_n)$ recall consistent for nested $V_n^* \subset C_n$ with respect to $F$ if

$$\lim_{n \rightarrow \infty} \left( L_{k_n}^{*(1)}(\Phi_{n,m_n}, V_n^*) - L_{k_n}^{*(1)}(V_n^*) \right) = 0,$$

for any sequence of obfuscating functions of $V_2$ with $|V_2| = m_n$.

ii. level-$(k_n)$ precision consistent for for nested $V_n^* \subset C_n$ with respect to $F$ if

$$\lim_{n \rightarrow \infty} \left( L_{k_n}^{*(2)}(\Phi_{n,m_n}, V_n^*) - L_{k_n}^{*(2)}(V_n^*) \right) = 0,$$

for any sequence of obfuscating functions of $V_2$ with $|V_2| = m_n$.

We say that a VN rule $\Phi$ is universally level-$(k_n)$ (precision recall) consistent if it is level-$(k_n)$ (precision recall) consistent for all nested-core nominatable sequences $F$. Corollary 19 from [26] proves that universally consistent VN schemes do not exist for any nondecreasing integral sequences $(k_n)$ satisfying $k_n = o(m_n)$ and any $(V_n^*)$ satisfying $|V_n^*| = \Theta(1)$. Beyond the ramifications for practically implementing VN in streaming or evolving network environments considered in [26], this lack of universal consistency is also the
motivating result for our statistical approach to adversarial contamination in VN. Indeed, a simple consequence of the lack of universal consistency is that for any VN rule there are nominatable sequences for which the rule is not consistent. An adversary could then be understood as a probabilistic mechanism designed to transform nominatable sequences for which the rule is consistent into nominatable sequences for which the rule is not consistent.

To develop this reasoning further, we next develop the notion of (maximal) consistency classes in the VN framework.

### 2.1 VN Consistency Classes

We next explore the concept of consistency classes in VN, with an eye towards the development of a statistical adversarial contamination framework for VN. First, let \( \mathcal{N}_V^* \) be the collection of all nested-core nominatable sequences with nested v.o.i. \( V^* = (V_n^* \subset C_n) \). For a given VN rule \( \Phi \), v.o.i. sequence \( V^* \) satisfying \( |V_n^*| = \Theta(1) \), and nondecreasing sequence \( (k_n) \) (satisfying the growth conditions of Lemma 11), the level-(\( (k_n) \)) consistency class of \( \Phi \) is defined to be

\[
\mathcal{C}_{\Phi}^{(k_n)} = \left\{ F \in \mathcal{N}_V^* \text{ s.t. } \Phi \text{ is level-}(k_n) \left( \begin{array}{c}
\text{precision} \\
\text{recall}
\end{array} \right) \text{ consistent for } F \right\}.
\]

The lack of universal consistency ensures that \( \mathcal{C}_{\Phi}^{(k_n)} \neq \mathcal{N}_V^* \) for any rule \( \Phi \).

It is natural to ask if there are a finite number of VN rules \( \{ \Phi_i \} \) such that \( \bigcup_i \mathcal{C}_{\Phi_i}^{(k_n)} = \mathcal{N}_V^* \). An affirmative answer would allow for ensemble methods to practically overcome the lack of universally consistent rules, and hence practically overcome any adversarial attack in the VN framework. We will see in Section 2.1.1 that the answer is, as expected, no, and any partition of \( \mathcal{N}_V^* \) into maximal consistency classes necessarily contains infinite parts; see Lemma 11. As a consequence, ensemble methods cannot recover universal consistency in VN. The insights developed in Section 2.1.1 further motivate the development of adversarial contamination regimes for a given rule \( \Phi \).

The idea behind adversarial contamination is simple in this framework: the adversary contaminates elements \( F \in \mathcal{C}_{\Phi}^{(k_n)} \) transforming them into \( F' \in \mathcal{N}_V^* \setminus \mathcal{C}_{\Phi}^{(k_n)} \).

### 2.1.1 Counting Consistency Classes

How can a practitioner mitigate the impact of a lack of universal consistency? One idea would be to consider ensemble methods, as the practical implications of the lack of universal consistency can be mitigated if universally consistent ensemble schemes exist. In this section, we will formalize the notion of maximal VN consistency classes and
prove that infinitely many maximal consistency classes exist. We begin with defining the notion of maximal consistency classes in the VN-framework.

**Definition 10.** As above, let $\mathcal{N}_{V^*}$ be the collection of all nested-core nominatable sequences with nested v.o.i. $V^* = (V^*_n \subset C_n)$. For a nondecreasing integer sequence $(k_n)$, we say that $\mathcal{C} \in \mathcal{N}_{V^*}$ is a maximal level-$(k_n)$ precision recall consistency class for $V^*$ if the following two conditions hold.

i. There exists a VN rule $\Phi$ that is jointly level-$(k_n)$ precision recall consistent for $V^*$ for each $F \in \mathcal{C}$;

ii. If $F' \notin \mathcal{C}$, then there does not exist a VN rule $\Phi$ that is jointly level-$(k_n)$ precision recall consistent for $V^*$ for each $F \in \mathcal{C} \cup \{F'\}$.

A natural question to ask is whether it is possible to partition $\mathcal{N}_{V^*}$ into a finite number of maximal level-$(k_n)$ consistency classes for a particular sequence $(k_n)^\infty_{n=1}$? Our next result—Lemma 11—shows that for any integer sequence $(k_n)$ satisfying a modest growth condition, any partition of $\mathcal{N}$ into maximal level-$(k_n)$ consistency classes must include at least countably infinite parts, thus erasing the hope that ensemble methods can recover universal consistency and practically mitigate the effect of any VN adversarial attack.

**Lemma 11.** Let $(k_n)$ be a sequence of nondecreasing integers satisfying $k_n = o(n)$, and let $V^*$ be a nested sequence of vertices of interest satisfying $|V^*_n| = \Theta(1)$.

i. Let $\mathcal{N}_{V^*} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{C}_\alpha$ be a partition of $\mathcal{N}_{V^*}$ into maximal level-$(k_n)$ recall consistency classes, then $|\mathcal{A}| = \infty$.

ii. Let $\mathcal{N}_{V^*} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{C}_\alpha$ be a partition of $\mathcal{N}_{V^*}$ into maximal level-$(k_n)$ precision consistency classes. If $k_n = \Theta(1)$, then $|\mathcal{A}| = \infty$.

The proof of this lemma can be found in Appendix B

### 2.1.2 Verification functions

In the presence of an adversarial attack, is it possible to, without additional supervision, verify if a given VN scheme is working on a given $F^{(n,m)}_{c,\theta} \in \mathcal{N}_{n,m}$? In other words, given a nondecreasing integer sequence $(k_n)$, $(g_1, g_2) \in \mathcal{G}_n \times \mathcal{G}_m$, and v.o.i. $V^*_n$, can we consistently estimate the verification function

$$h_{\Phi_n}(g_1, o(g_2), V^*_n) = h_{\Phi_n, o, k_n}(g_1, o(g_2)) = \sum_{v \in V^*_n} \mathbb{I} \{ \text{rank}_{\Phi_n}(g_1, o(g_2), V^*)(o(v)) \leq k_n \}$$
Note that the scaling by $|V^*_n|$ in the recall setting and by $k_n$ in the precision setting do not affect consistent estimation of $h$ given $|V^*_n| = \Theta(1)$ and in the precision setting $k_n = \Theta(1)$. As such, the scaling is omitted.

The internal consistency criterion, Eq. 1 guarantees that

$$h_{\phi_n}(g_1, o_n(g_2), V^*_n) = h_{\phi_n}(g_1, \delta_n(g_2), V^*_n)$$

(2)

for all obfuscation functions $o_n, \delta_n \in \mathcal{D}_n$. Indeed, the v.o.i.'s in $g_2$ are identical (though obfuscated differently) in $o_n(g_2)$ and $\delta_n(g_2)$. If we consider an alternate $(g'_1, g'_2) \sim F'_n \subset \mathcal{F}'$, it could be the case that $g_1 = g'_1$ and $g_2 \approx g'_2$, while $h_{\phi_n}(g_1, o_n(g_2), V^*_n) \neq h_{\phi_n}(g_1, \delta_n(g_2), V^*_n)$ for any $o_n \in \mathcal{D}_n$; indeed, consider letting the v.o.i.'s in $g'_2$ be different from those in $g_2$ (i.e., the behavior of the v.o.i. in $F'_n$ is different from the behavior of the v.o.i. in $F_n$).

Consider the problem of estimating $h_{\phi_n}$ via $\hat{h}_{\phi_n}$. If the estimator is label-agnostic (i.e., there is no information in the obfuscated labeling of $o(g_2)$), then it is sensible to require that for all $g_2 \approx g'_2$, we have that

$$\hat{h}_{\phi_n}(g_1, o_n(g_2), V^*_n) = \hat{h}_{\phi_n}(g_1, \delta_n(g_2), V^*_n)$$

(3)

Contrasting this to Eq. (2), we see that $(\hat{h}_{\phi_n})$ cannot universally consistently estimate $(h_{\phi_n})$, as the sequence of estimators cannot account for the potentially different behaviors of the v.o.i.'s under the umbrella of nominatable distributions. To wit, we have the following lemma.

**Lemma 12.** With notation as above, let $(\hat{h}_{\phi_n})_n$ be any sequence of label-agnostic (i.e., satisfying Eq. 3) estimators of $(h_{\phi_n})_n$. There exists sequences of nested-core nominatable distributions $\mathcal{F} = (F_n)$ and $\mathcal{F}' = (F'_n)$ such that for $n$ sufficiently large, if $(G_1, G_2) \sim F_n$, and $(G'_1, G'_2) \sim F'_n$, then

$$d_{TV}(\mathcal{L}(h_{\phi_n}(G_1, o(G_2), V^*_n)), \mathcal{L}(h_{\phi_n}(G'_1, o(G'_2), V^*_n))) > 0,$$

while $\mathcal{L}(\hat{h}_{\phi_n}(G_1, o(G_2), V^*_n)) = \mathcal{L}(\hat{h}_{\phi_n}(G'_1, o(G'_2), V^*_n))$ (where $d_{TV}$ is the total variation distance).

As a result of the above discussion and Lemma, we are unable to verify, without additional supervision, if an adversary has moved the distribution out of a given VN rule’s consistency class. This points to the primacy of additional supervision, which in the VN framework often comes in the form of a user-in-the-loop. Indeed, we are currently exploring the role/impact a use-in-the-loop in VN—where the user can evaluate the interestingness of the vertices in the top $k$ of the nomination list for a cost $c_k$. This supervision can also be thought of as a form of regularization, designed to increase the consistency class of a given VN rule.
3 Adversarial Vertex Nomination

In order to actively model adversarial attacks in the VN-framework, we formalize the notion of an edge adversary.

**Definition 13.** Let $F$ be a distribution on graphs in $\mathcal{G}_m$, and let $U$ be a random variable independent of $G \sim F$. We say $\mathcal{A} = \{f_\mathcal{A}, V_\mathcal{A}, U, \theta\}$ is an adversary parameterized by $\theta \in \Theta$ if

1. $f_\mathcal{A} : \mathcal{G}_m \times \mathbb{R} \times \Theta \mapsto \mathcal{G}_m$ is a measurable function such that $V(f_\mathcal{A}(G, U, \theta)) = V(G)$, so that $f_\mathcal{A}(G, U, \theta)$ is a $\mathcal{G}_m$-valued random variable.
2. $V_\mathcal{A} : \mathcal{G}_m \times \mathbb{R} \times \Theta \mapsto 2^{[m]}$ is a measurable function that satisfies $V_\mathcal{A}(G, U, \theta) \subset V(G)$, so that $V_\mathcal{A}(G, U, \theta)$ is a (potentially) random subset of $V(G)$.
3. If $K = \{v, w \in V(G) \text{ s.t. } (v, w) \in E(f_\mathcal{A}(G, U, \theta)) \Delta E(g)\}$, (where $\Delta$ represents the symmetric difference) then $K \subset V_\mathcal{A}(G, U, \theta)$. Succinctly put, if an edge is added or removed from $E(G)$, then the vertices adjacent to that edge must be in $V_\mathcal{A}(G, U, \theta)$.

In the above, $U$ represents an independent source of randomness utilized in the adversarial attack.

Note that $f_\mathcal{A}$ is simply a function that adds/deletes edges from a network potentially randomly, and these edges must be incident to the vertices of $V_\mathcal{A}$. To that end, we will refer to $V_\mathcal{A}$ as the vertices contaminated by $\mathcal{A}$.

If we are given a sequence of nominatable distributions $\mathbf{F} = (F_n)_{n=n_0}^\infty$, where $F_n$ is a distribution on $\mathcal{G}_n \times \mathcal{G}_m$, then we will let $f_{\mathcal{A}_n}(F_n)$ denote a sequence of graphs realized from $F_n$, with the second graph $G_2$ contaminated by $f_{\mathcal{A}_n}$; we call a sequence $(f_{\mathcal{A}_n})_{n=n_0}^\infty$ an adversary rule. In the language of VN consistency classes, we posit that an adversary rule aims to contaminate a VN rule $\Phi$ via

$$
\mathbf{F} = (F_n)_{n=n_0}^\infty \in \mathbf{c}_\Phi^{(k_n)} \implies (f_{\mathcal{A}_n}(F_n))_{n=n_0}^\infty \in \mathfrak{N}^* \setminus \mathbf{c}_\Phi^{(k_n)}.
$$

**Remark 14.** Let $G_2 = (V_2, E_2)$ and $G'_2 = (V'_2, E'_2)$. Consider an edge adversary $f_\mathcal{A}$ acting on $G'_2$. By considering $V_2 = V(G'_2) \setminus V_\mathcal{A}$, we can also consider this adversary as a vertex adversary that randomly adds vertices to $G_2$. Vertex addition and deletion can be simultaneously modeled by first considering a mechanism for randomly deleting vertices from $G_2 = (V_2, E_2)$ before using the above approach to add adversarial vertices to the network.

**Remark 15.** In [50], the authors consider direct attacks and influencer attacks in which, given a vertex of interest $v^*$, either $v^* \in V_\mathcal{A}$ or $v^* \notin V_\mathcal{A}$ respectively. However, note
that in [50], the objective is vertex classification, whereas we are not directly classifying vertices. Rather, we are interested in ranking vertices in $G_2$ by interestingness given limited training data in $G_1$. We will typically assume that $v^* \notin V_A$ (i.e. the adversary does not control the vertex of interest), so that we are examining influencer attacks.

### 3.1 A Simple VN Adversarial Contamination Model

Now that we have developed the requisite theory for framing the idea of adversarial contamination in the VN-setting, we will consider a simple model for adversarial contamination in the stochastic blockmodel (SBM) of [22].

**Definition 16.** We say that an $n$-vertex random graph $G$ is an instantiation of a stochastic blockmodel with parameters $(n, K, B, b)$ (written $A \sim \text{SBM}(n, K, B, \pi)$) if

1. The block membership vector $\pi \in \mathbb{R}^K$ satisfies $\pi_i \geq 0$ for all $i \in [K]$, and $\sum_i \pi(i) = 1$;
2. The vertex set $V = V(G)$ is the disjoint union of $K$ blocks $V = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_K$, where each vertex $v \in V$ is independently assigned to a block according to a Multinomial$(1, \pi)$ distribution. If vertex $v$ is assigned to block $i \in [K]$, then the block membership function $b : V \rightarrow [K]$ satisfies $b(v) = i$;
3. The block probability matrix $B \in [0, 1]^{K \times K}$ is such that, for each pair of vertices $\{u, v\} \in \binom{V}{2}$, $\mathbb{1}_{u \sim_G v} \sim \text{Bernoulli}(B_{b(u), b(v)})$, and the collection of indicator random variables $\{\mathbb{1}_{u \sim_G v}\}_{\{u, v\} \in \binom{V}{2}}$ is mutually independent (here $u \sim_G v \iff \{\{u, v\} \in E\}$).

In addition, we will say that a pair of graphs $(G_1, G_2)$ is an instantiation of a $\rho$-correlated SBM$(n, K, B, b)$ (written $(G_1, G_2) \sim \text{SBM}(\rho, n, K, B, \pi)$) if marginally $G_1 \sim \text{SBM}(n, K, B, b)$ and $G_2 \sim \text{SBM}(n, K, B, b)$, and the collection of indicator random variables

$$\left\{\mathbb{1}_{u \sim_{G_1} v}\right\}_{\{u, v\} \in \binom{V}{2}} \bigcup \left\{\mathbb{1}_{u \sim_{G_2} v}\right\}_{\{u, v\} \in \binom{V}{2}}$$

is mutually independent except that for each $\{u, v\} \in \binom{V}{2}$, $\text{Corr}(\mathbb{1}_{u \sim_{G_1} v}, \mathbb{1}_{u \sim_{G_2} v}) = \rho$.

Consider $G$ as an $n$-vertex stochastic blockmodel, with two blocks, $B_1$ and $B_2$, and with $\pi = (1/2, 1/2)$ . The block-probability matrix $B$ is given by

$$B = \begin{pmatrix} p & r \\ r & q \end{pmatrix},$$

with $p \geq q \geq r > 0$. Given $G = g$, we define the following VN adversarial contamination procedure $A = (f_A, V_A, U, \theta)$ acting on $g$ as follows:
1. \( \theta = (c_+, c_-, \pi_+, \pi_-, s_+, s_-) \) is a vector of parameters where \( c_+, c_- \in \mathbb{Z} \) satisfy \( c_+ + c_- \leq n, \pi_+, \pi_- \in (0, 1), \) and \( s_+, s_- \in [0, 1] \);

2. \( U \) is a uniformly distributed random variable independent of \( G \);

3. \( f_A(g, U, \theta) \in \mathcal{G}_n \) is defined as follows:
   i. Initialize \( g_c = g \)
   ii. Independently select vertices from \( V = [n] \) with probability \( \pi_+ \) (call them \( W_+ \)). Then, independently select vertices \( V \setminus W_+ \) with probability \( \pi_- \) (call then \( W_- \)).
   iii. For each vertex pair \( \{v, u\} \in W_+ \times (V \setminus W_-) \),
      i. If \( \{v, u\} \notin E(g_c) \), nothing happens.
      ii. If \( \{v, u\} \notin E(g_c) \), an edge is independently added connecting \( \{v, u\} \) in \( g_c \) with probability \( s_+ \).
   iv. For each vertex pair \( \{v, u\} \in W_- \times (V \setminus W_+) \),
      i. If \( \{v, u\} \notin E(g_c) \), nothing happens.
      ii. If \( \{v, u\} \in E(g_c) \), the edge is independently deleted from \( g_c \) with probability \( s_- \).
   v. Set \( f_A(g, U, \theta) = g_c \in \mathcal{G}_n \).

The auxiliary randomness \( U \) in \( A \) is utilized to make the random vertex selections in ii., the random edge additions in iii., and the random edge deletions in iv.

Notice that this adversarial model gives rise to a new stochastic blockmodel with the edge-probability matrix \( \tilde{B} \) given by

\[
\tilde{B} = \begin{pmatrix}
\tilde{B}_1 & \tilde{B}_1^+ & \tilde{B}_1^- & \tilde{B}_2 & \tilde{B}_2^+ & \tilde{B}_2^-
\end{pmatrix}
\begin{pmatrix}
p & x_1 & x_2 & r & x_3 & x_4 \\
x_1 & x_2 & p & x_3 & x_5 & r \\
x_2 & p & x_2 & x_4 & r & x_6 \\
r & x_3 & x_4 & q & x_7 & x_8 \\
x_3 & x_5 & r & x_7 & x_7 & q \\
x_4 & r & x_6 & x_8 & q & x_8
\end{pmatrix}
\]

where

\[
x_1 = p + s_+(1 - p), \quad x_2 = p(1 - s_-), \quad x_3 = r + s_+(1 - r),
\]
\[
x_4 = (1 - s_-)r, \quad x_5 = r + (2s_+ - s_+)^2(1 - r), \quad x_6 = r(1 - s_-)^2
\]
\[
x_7 = q + s_+(1 - q), \quad x_8 = q(1 - s_-),
\]
and where $\tilde{B}_1^+$ are the vertices in $W_+ \cap B_1$; $\tilde{B}_1^-$ are the vertices in $B_1 \cap W_-$; and $\tilde{B}_1$ are the vertices in $B_1 \setminus (\tilde{B}_1^+ \cup \tilde{B}_1^-)$; with $\tilde{B}_2$ defined analogously. We note here that this adversarial contamination model is similar to the contamination model considered in [8].

Note also that the original block structure is preserved amongst vertices in $\tilde{B}_1 \cup \tilde{B}_2$, and we can view this contamination model as adding vertices randomly to $G[\tilde{B}_1 \cup \tilde{B}_2]$, i.e., the induced subgraph on $\tilde{B}_1 \cup \tilde{B}_2$. When $(G_1, G_2) \sim \text{SBM}(\rho, n, K, B, \pi)$ and this adversarial procedure is applied to $G_2$, we will denote

$$G_1^{(i)} = G_1[\tilde{B}_1 \cup \tilde{B}_2]$$
$$G_2^{(i)} = G_2[\tilde{B}_1 \cup \tilde{B}_2]$$

**Remark 17.** Let $A_n$ be the simple adversarial rule outlined above. A very simple VN rule $\Phi$ and nested core nominatable sequence $F$ for which

$$F = (F_n)_{n=n_0}^{\infty} \in \mathcal{C}_\Phi^{(k_n)} \implies (f_{A_n}(F_n))_{n=n_0}^{\infty} \in \mathcal{N} \setminus \mathcal{C}_\Phi^{(k_n)}.$$ 

proceeds as follows. Consider $F_n = \text{SBM}(\rho, n, K, B, \pi)$ supported on $G_n \times G_n$ where $B$ is as in Eq. 4 with $\pi = (1/2, 1/2)$, $p > q > r$ fixed, and $\rho > 0$ fixed. Suppose that $\Phi_n$ is a VN scheme that runs spectral clustering on the contaminated graph by first selecting the number of communities in a consistent manner (via adjacency spectral clustering for example [27]) and ranking all the vertices in the group with the highest probability of within-group connection (in a fixed but arbitrary order), and then ranks the rest of the vertices in fixed but arbitrary order. Suppose that we consider $k_n = n/2$. It is immediate that $F = (F_n)_{n=n_0}^{\infty} \in \mathcal{C}_\Phi^{(k_n)}$ and that the adversary acting on $G_2$ impacts this consistency. Indeed, if either

1. $p - q < s_-$, or
2. $\frac{p-q}{1-q} < s_+$,

then $\Phi_n$ is no longer consistent with respect to the adversarially contaminated model sequence.

## 4 Experiments

We next explore the effect of our adversarial noise model in a simulated data experiment, and the effect of adversarial contamination and regularization in a real data example derived from Bing entity transition graphs. First, we explain in detail the steps of the VN scheme we will consider in our experiments.
4.1 VN via ASE o GMM

In the contamination model of Section 3.1, we consider the following VN scheme, denoted VN \( \circ \) GMM \( \circ \) ASE. Letting \( v^* \in V(G_1) \) (resp., \( V^* \subset V(G_1) \)) be the vertex (resp., vertices) of interest in \( G_1 \), we seek the corresponding vertex (resp., vertices) of interest in \( V(G_2) \) as follows:

1. Given two graphs, \( G_1 \) and \( G_2 \), we use Adjacency Spectral Embedding (ASE) [43] to separately embed \( G_1 \) and \( G_2 \) into a common Euclidean space \( \mathbb{R}^d \). Given the \( n \times n \) adjacency matrix \( A \) of \( G_1 \), the \( d \)-dimensional ASE of \( G_1 \) is defined as follows.

   **Definition 18 (Adjacency spectral embedding (ASE)).** Given \( d \in \mathbb{Z} > 0 \), the adjacency spectral embedding (ASE) of \( A \) into \( \mathbb{R}^d \) is defined via
   \[
   \hat{X} = U_A S_A^{1/2}
   \]
   where
   \[
   |A| = [U_A |U_A^\perp| [S_A \oplus S_{A}^\perp] [U_A |U_A^\perp]]
   \]
   is the spectral decomposition of \( |A| = (A^T A)^{1/2} \), \( S_A \in \mathbb{R}^{d \times d} \) is the diagonal matrix with the \( d \) largest eigenvalues of \( |A| \) on its diagonal and \( U_A \in \mathbb{R}^{n \times d} \) has columns which are the eigenvectors corresponding to the eigenvalues of \( S_A \).

   Simply stated, the ASE of a graph \( G \) provides Euclidean features for each vertex in \( G \) on which to perform subsequent inference. Combined with recent efforts to prove that the ASE provides consistent estimators of the latent position parameters in random dot product graphs and positive-definite stochastic blockmodels [43, 2], the ASE allows for a host classical inference methodologies to be successfully employed within these random graph frameworks [44, 45, 28]. To choose \( d \) above, we use the machinery of [49, 9] to develop the principled heuristic of estimating \( d \) as the larger of the two elbows of the associated scree plots of the singular values of \( G_1 \) and \( G_2 \).

2. Solve the orthogonal Procrustes problem [39] to find an orthogonal transformation aligning the seeded vertices across graphs. Let \( \hat{X}_S \) (resp., \( \hat{Y}_S \)) be the matrix composed of the rows of ASE\( (G_1) \) (resp., ASE\( (G_2) \)) corresponding to the seeded vertices in \( S \). Letting the SVD of \( \hat{Y}_S^T \hat{X}_S = U \Sigma V^T \), the solution to
   \[
   R = \arg\min_{O} \ s.t. \ O^T O = I \| \hat{X}_S - \hat{Y}_S O \|_F,
   \]
   is given by \( R = UV^T \). Use this transformation to align the embeddings of \( G_1 \) and \( G_2 \) in \( \mathbb{R}^d \), i.e., rotate \( \hat{Y} \) via \( \hat{Y} O \) to align \( \hat{Y} \) to \( \hat{X} \).

3. Motivated by the central limit theorem of [3] for the residual errors between the rows of the ASE and the latent position parameters in random dot product graphs, we use model-based Gaussian mixture modeling (GMM) to simultaneously cluster the vertices of the embedded graphs. Here, we employ the \( \mathbb{R} \) package \texttt{MClust} [18].
4. Rank the candidate matches in $G_2$ according to the following heuristic. If $u \in V(G_1)$ and $v \in V(G_2)$ are clustered points in the Procrustes-aligned embedding of $G_1$ and $G_2$ with respective covariance matrices $\Sigma_u$ and $\Sigma_v$ in their components of the GMM, then compute

$$\Delta(u, v) = \max(D_u(u, v), D_v(u, v)),$$

where

$$D_u(u, v) = \sqrt{(u - v)\Sigma_u^{-1}(u - v)^T}$$

and

$$D_v(u, v) = \sqrt{(u - v)\Sigma_v^{-1}(u - v)^T}$$

are the respective Mahalanobis distances from $u$ to $v$. In the case of a single v.o.i. $v^*$, rank the vertices in $G_2$ then by increasing value of $\Delta(v^*, u)$, i.e., with ties broken in a fixed deterministic fashion, we rank via (where $n_2 = |V(G_2)|$)

$$\Phi_n(g_1, g_2, v^*)[1] \in \arg \min_{u \in V(G_2)} \Delta(v^*, u)$$

$$\Phi_n(g_1, g_2, v^*)[2] \in \arg \min_{u \in V(G_2) \setminus \{\Phi_n[1]\}} \Delta(v^*, u)$$

$$\vdots$$

$$\Phi_n(g_1, g_2, v^*)[n_2 - 1] \in \arg \min_{u \in V(G_2) \setminus \bigcup_{j \leq n_2 - 2} \Phi_n[j]} \Delta(v^*, u)$$

$$\Phi_n(g_1, g_2, v^*)[n_2] \in \arg \min_{u \in C_{v^*} \setminus \bigcup_{j \leq n_2 - 1} \Phi_n[j]} \Delta(v^*, u).$$

In the case of multiple v.o.i. $V^*$, rank the vertices in $G_2$ then by increasing value of $\min_{u \in V^*} \Delta(v, u)$ with ties broken in a fixed deterministic fashion.

### 4.2 Simulation

We consider the model in Section 3.1 with the following parameter choices:

$$n = 200; \quad \pi = (1/2, 1/2); \quad \pi_- = 0.1, \quad \pi_+ = 0.1;$$

$$p = 0.4; \quad q = 0.5; \quad r = 0.3;$$

$$s_+ = 0.8; \quad s_- = 0.8; \quad \rho \in (0.3, 0.5, 0.7).$$

Note that, in the notation of Section 3.1, if $(G_1, G_2) \sim \text{SBM}(\rho, n, K, B, \pi)$, we will consider

$$G_1^{(i)} = G_1[\tilde{B}_1 \cup \tilde{B}_2]$$

$$G_2^{(i)} = G_2[\tilde{B}_1 \cup \tilde{B}_2]$$

$$G_2^{(c)} = G_2 \text{ acted upon by the adversary described in Section 3.1;}$$

$$G_2^{(x,y)} = G_2^{(c)} \text{ trimmed as in Algorithm 1.}$$
In this simulation example, we observe that the adversarial contamination model significantly decreases VN performance and that the trimming regularization mitigates this contamination and recovers much of the lost inferential performance.

In Figure 4 we plot the performance of VN $\circ$ GMM $\circ$ ASE over a number of $(x, y)$ trimming pairs (we note that for all correlation/regularized/contaminated/trimmed combinations, mean performance is significantly better than chance and chance normalized plots are omitted). In the left panel, we plot the modularity of the GMM clustering in the trimmed $G_2^{(x,y)}$ as a function of $x, y \in \{0, 0.05, 0.1, 0.15, 0.2, 0.25\}$. Note that we average the modularity values over $nMC = 50$ randomly selected seed sets of size $s = 10$. The color indicates the value of the modularity, with darker red indicating lower values and lighter yellow–white indicating larger values. We see that modularity is maximized near $(x, y) \approx (0.1, 0)$, and that the model-true trimming values $(x, y) = (0.1, 0.1)$ achieves relatively high modularity as well.

In the right panel, we plot the performance of VN $\circ$ GMM $\circ$ ASE $(\pm 2\text{s.e.})$ in $(G_1, G_2) \sim \text{SBM}(0.7, 200, 2, B, \pi = (1/2, 1/2))$ again averaged over $nMC = 50$ random seed sets of size $s = 10$. The $x$-axis shows the ranks in the nomination list and the $y$-axis shows (on average) how many vertices $v \in G_1^{(i)}$, when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top $x$ by VN $\circ$ GMM $\circ$ ASE. The gold line represents performance in the idealized network pair $(G_1^{(i)}, G_2^{(i)})$; the red line for $(G_1^{(i)}, G_2^{(c)})$; the green line for $(G_1^{(i)}, G_2^{(0.1, 0.1)})$; the blue line for $(G_1^{(i)}, G_2^{(0.1, 0.1, 0.1)})$; and the pink line for $(G_1^{(i)}, G_2^{(0.2, 0.2)})$. 

Figure 4: In the left panel, we plot the modularity of the GMM clustering in the trimmed $G_2^{(x,y)}$ as a function of $x, y \in \{0, 0.05, 0.1, 0.15, 0.2, 0.25\}$. Note that we average the modularity values over $nMC = 50$ randomly selected seed sets of size $s = 10$. The color indicates the value of the modularity, with darker red indicating lower values and lighter yellow–white indicating larger values. In the right panel, we plot the performance of VN $\circ$ GMM $\circ$ ASE $(\pm 2\text{s.e.})$ in $(G_1, G_2) \sim \text{SBM}(0.7, 200, 2, B, \pi = (1/2, 1/2))$ again averaged over $nMC = 50$ random seed sets of size $s = 10$. The $x$-axis shows the ranks in the nomination list and the $y$-axis shows (on average) how many vertices $v \in G_1^{(i)}$, when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top $x$ by VN $\circ$ GMM $\circ$ ASE. The gold line represents performance in the idealized network pair $(G_1^{(i)}, G_2^{(i)})$; the red line for $(G_1^{(i)}, G_2^{(c)})$; the green line for $(G_1^{(i)}, G_2^{(0.1, 0.1)})$; the blue line for $(G_1^{(i)}, G_2^{(0.1, 0.1, 0.1)})$; and the pink line for $(G_1^{(i)}, G_2^{(0.2, 0.2)})$. 

In this simulation example, we observe that the adversarial contamination model significantly decreases VN performance and that the trimming regularization mitigates this contamination and recovers much of the lost inferential performance.
Figure 5: In the right panel (resp., left panel), we plot the performance of VN ◦ GMM ◦ ASE (±2s.e.) in $(G_1, G_2) \sim \text{SBM}(0.3, 200, 2, B, \pi = (1/2, 1/2))$ with $\rho = 0.3$ (resp., $\rho = 0.5$) again averaged over $nMC = 50$ random seed sets of size $s = 10$. The x-axis shows the ranks in the nomination list and the y-axis shows (on average) how many vertices $v \in G_1^{(i)}$, when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top $x$ by VN ◦ GMM ◦ ASE. The gold line represents performance in the idealized network pair $(G_1^{(i)}, G_2^{(c)})$; the red line for $(G_1^{(i)}, G_2^{(c)})$; the green line for $(G_1^{(i)}, G_2^{(0,1.0)})$; the blue line for $(G_1^{(i)}, G_2^{(0,1.0,1)})$; and the pink line for $(G_1^{(i)}, G_2^{(0,2.0,2)})$.

The y-axis shows (on average) how many vertices $v \in G_1^{(i)}$, when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top $x$ by VN ◦ GMM ◦ ASE. The gold line represents performance in the idealized network pair $(G_1^{(i)}, G_2^{(c)})$; the red line for $(G_1^{(i)}, G_2^{(c)})$; the green line for $(G_1^{(i)}, G_2^{(0,1.0)})$; the blue line for $(G_1^{(i)}, G_2^{(0,1.0,1)})$; and the pink line for $(G_1^{(i)}, G_2^{(0,2.0,2)})$. We see here that, as expected, performance loss due to contamination is mitigated by using the true model-based trimming parameters $x = y = 0.1$, and using the modularity maximizing $x = 0.1, y = 0$. If we over-trim, here represented by $x = y = 0.2$, we see a degradation in performance; as expected from the low modularity value in the left panel for $x = y = 0.2$. We again see here the interesting phenomena observed in the motivating high school friendship network example of Section 1.1: modularity and subsequently VN performance tends to emphasize more trimming of the low degree vertices and less trimming of the high degree vertices. This suggests that low-degree contamination is most effective at thwarting the performance on VN ◦ GMM ◦ ASE, perhaps contrary to the intuition that high-degree nodes adversely affect concentration of adjacency matrices [24].

As in our motivating example, trimming can have the effect of removing v.o.i. from $G_2^{(c)}$, and we see this play out in Figure 6. As expected, over-regularizing results in a significant number of v.o.i. being trimmed and significant performance loss as compared to the more moderate choices of regularization. Lastly, exploring the affect
Figure 6: We plot the performance of VN ◦ GMM ◦ ASE (±2s.e.) in \((G_1, G_2) \sim \text{SBM}(0.3, 200, 2, B, \pi = (1/2, 1/2))\) with \(\rho = 0.7\) again averaged over \(nMC = 50\) random seed sets of size \(s = 10\). The \(x\)-axis shows the ranks in the nomination list and the \(y\)-axis shows (on average) how many vertices \(v \in G_1^{(i)}\), when viewed as the v.o.i., had their corresponding vertex of interest ranked in the top \(x\) by VN ◦ GMM ◦ ASE. The gold line represents performance in the idealized network pair \((G_1^{(i)}, G_2^{(i)})\); the red line for \((G_1^{(i)}, G_2^{(c)})\); the green line for \((G_1^{(i)}, G_2^{(0.1,0)})\); the blue line for \((G_1^{(i)}, G_2^{(0.1,0.1)})\); and the pink line for \((G_1^{(i)}, G_2^{(0.2,0.2)})\).

of \(\rho\) on VN ◦ GMM ◦ ASE performance, we repeat the above experiment with \(\rho = 0.5\), and \(\rho = 0.3\). Results are plotted in Figure 5. As expected, the trends observed in Figure 4 hold here as well, with an across the board performance decrease as \(\rho\) decreases.

4.3 Microsoft Bing Entity Graph Transitions

In the next example, we consider a multigraph derived from one month of aggregate Bing entity graph transitions. The multigraph represents entity transitions, and each weighted edge-type of the multigraph represents aggregated signal that capture a transition rate between two entities while browsing. There are multiple ways that a transition between those entities could be made, so we count each aggregated signal separately using the different edge-types in the multigraph: one edge-type represents transitions that were made via a suggestion interface; the other edge-type represents transitions that we made independent of any suggestion interface. As such, one type will have a constrained set of transition probabilities (it can realistically only connect to a subset of the vertices in the graph), while the other will be more “unlimited” in that it may connect to any other entity in the entire graph.

The resulting graphs are symmetric, weighted and loop-free, with \(G_1^{(i)}\) containing 13535 vertices and 519389 edges, \(G_2^{(i)}\) containing 13535 vertices and 595047 edges, and
Figure 7: Considering 2 Monte Carlo replicates of $s = 100$ randomly chosen seeds, we run VN $\circ$ GMM $\circ$ ASE on $(G_1^{(i)}, G_2^{(i)})$ (yellow line), on $(G_1^{(i)}, G_2^{(c)})$ (red line), on $(G_1^{(i)}, G_2^{(0.1,0.1)})$ (pink line); on $(G_1^{(i)}, G_2^{(0.0,0.5)})$ (green line); and on $(G_1^{(i)}, G_2^{(0.5,0.0)})$ (blue line). In panel a), we consider each vertex in $G_1^{(i)}$ as the v.o.i., and we plot the number of vertices amongst these v.o.i. (x-axis) that had their corresponding v.o.i. in $G_2$ ranked in the top $x$. The right panel shows the same result normalized by chance performance, where we plot $y = \frac{\text{mean # of v.o.i. with corresp. v.o.i. ranked in top $x$ by VN$\circ$GMME$\circ$ASE}}{\text{mean # of v.o.i. with corresp. v.o.i. ranked in top $x$ by chance algorithm}}$ vs. $x$.

Considering two randomly chosen sets of $s = 100$ seeds, we run VN $\circ$ GMM $\circ$ ASE on $(G_1^{(i)}, G_2^{(i)})$ (yellow line in Figure 7), on $(G_1^{(i)}, G_2^{(c)})$ (red line), on $(G_1^{(i)}, G_2^{(0.1,0.1)})$ (pink line); on $(G_1^{(i)}, G_2^{(0.0,0.5)})$ (green line); and on $(G_1^{(i)}, G_2^{(0.5,0.0)})$ (blue line). As in the simulations and motivating data example, we see the general trend of contamination adversely affecting performance and regularization ameliorating the effect of the contamination. Here, the regularized graph $G_2^{(0.1,0.1)}$ has 36808 vertices, and as expected, absolute performance (the left panel in Figure 7) in the clean case is better than in the regularized setting. From the right panel, we observe however, that the relative improvement over chance achieved in the regularized setting exceeds that in the clean setting, and we observe that VN $\circ$ GMM $\circ$ ASE performance is worse than chance in the contaminated and over-regularized network settings. While regularization has not recovered the performance in the idealized setting, the improvement induced via regularization is dramatic versus the contaminated setting. We also note that the modularity levels for automating the choice of $(x,y)$ in this example are relatively stable.
Figure 8: Considering 2 Monte Carlo replicates of $s = 100$ randomly chosen seeds (same seed sets as in Figure 7), we run VN o GMM o ASE on $(G_1^{(i)}, G_2^{(i)})$ (yellow line), on $(G_1^{(i)}, G_2^{(c)})$ (red line), on $(G_1^{(i)}, G_2^{(0,1,0.1)})$ (pink line); on $(G_1^{(i)}, G_2^{(0,1,0.1)})$ (green line); and on $(G_1^{(i)}, G_2^{(0,5,0)})$ (blue line). In panel a), we consider each vertex in $G_1^{(i)}$ as the v.o.i., and we plot the number of vertices amongst these v.o.i. (x-axis) that had their corresponding v.o.i. in $G_2$ ranked in the top $x$.

to the trimming value, with the clustered $G_2^{(c)}$ achieving $Q = 0.52$, the clustered $G_2^{(c)}$ achieving $Q = 0.52$, the clustered $G_2^{(0,0.5)}$ achieving $Q = 0.57$, the clustered $G_2^{(0,5,0)}$ achieving $Q = 0.52$, and the clustered $G_2^{(0,1,0.1)}$ achieving $Q = 0.53$. Indeed, in this data example the graphs do not cluster particularly well under any trimming conditions, and a more modest trimming scheme is more effective for the subsequent VN inference task.

In Figure 8, we again consider the performance of VN o GMM o ASE with the same $nMC = 2$ randomly chose 100 vertex seed sets and various levels of regularization, here plotting over an extended $x$-axis. In pink we plot VN o GMM o ASE run on $(G_1^{(i)}, G_2^{(0,1,0.1)})$; in blue on $(G_1^{(i)}, G_2^{(0,5,0)})$; in green on $(G_1^{(i)}, G_2^{(0,3,0.3)})$; and in red on $(G_1^{(i)}, G_2^{(0,0.5)})$. This figure demonstrates another dramatic side effect of over-regularization: v.o.i. that are trimmed for $G_2^{(c)}$ can never be recovered by VN o GMM o ASE. This is represented by the horizontal asymptotes in Figure 8.

5 Discussion

Our motivating question is two-fold: What effect does adversarial contamination have on the performance of vertex nomination; and can (statistically) successful vertex nomination be retained in the presence or absence of unmatchable vertices? Herein, we have
demonstrated both theoretically and empirically that an adversary can cause our VN scheme to fail (i.e., nominate the wrong vertices). Empirically, we have also demonstrated that regularization can be effective for mitigating the effect of the contamination model posited herein. Establishing the theoretical effect of regularization on VN is an open problem, and the subject of our present research.

In [26], the authors showed that there can be no universally consistent vertex nomination scheme assuming only one vertex of interest. In this paper, we have seen that with a suitable definition of a maximal consistency class and (possibly) multiple vertices of interest, there are infinitely many such consistency classes, which implies that ensemble methods cannot recover consistency and/or thwart an arbitrary adversary. This allows us to formulate our model of adversarial contamination in terms of consistency classes; indeed, an adversary for a particular VN rule aims to move the distribution out of the rule’s consistency class. A natural next question to consider would be what effect regularization has on a VN rule’s consistency class. Ideally, regularization enlarges the consistency class of a VN rule thereby making the adversary’s job (i.e., moving the model out of the consistency class) more difficult. The interplay between the adversary and regularization in VN is central to this story, although we are only at the infancy of understanding it.

Our proposed definition of an adversary is suited to a general random graph setting, and it provides a simple surrogate in which to study the effect of contamination in real data examples. From our simulation study and real data examples we have seen that a particular VN rule (VN ◦ GMM ◦ ASE) succeeds before adversarial contamination, fails after contamination, and succeeds after graph regularization. We are currently exploring the effect of contamination on a broader class of VN rules, and considering other models for adversarial contamination and subsequent regularization. Finally, while we have partially answered in the negative our question about whether consistency can be retained in the general adversarial setting, another valid consideration is whether there are adversarial models for which the adversary does not affect consistency. While we believe even simple manipulation on the edges of $G_2$ can affect consistency, it may be possible to derive bounds and phase transitions on the number of edges (or vertices) that an adversary would need to modify to change the result. Mathematically, this is akin to finding limits on the size of $|V_A|$ in our definition of an adversary.

**Acknowledgments**

This material is based on research sponsored by the Air Force Research Laboratory and DARPA under agreement number FA8750-18-2-0066. This work is also supported in part by the D3M program of the Defense Advanced Research Projects Agency. The
A Construction of the Bayes optimal scheme

Given the notation of Section 2, we now develop the Bayes optimal nomination scheme. Let $n, m$ be fixed and let $V^* \subset V_1 \cap V_2$ be fixed. Let $W$ be an obfuscating set and $\varnothing \in \mathcal{D}_W$. Define $\mathcal{G}_n^a \times \mathcal{G}_m^a$ to be the set of graphs

$$\mathcal{G}_n^a \times \mathcal{G}_m^a := \{(g_1, g_2) \in \mathcal{G}_n \times \mathcal{G}_m \text{ s.t. } I(v; g_2) = \{v\} \text{ for all } v \in V^*\}$$

For each $(g_1, g_2) \in \mathcal{G}_n^a \times \mathcal{G}_m^a$ define

$$(g_1, [\varnothing(g_2)]) = \{(g_1, \tilde{g}_2) \in \mathcal{G}_n^a \times \mathcal{G}_m^a : \varnothing(\tilde{g}_2) \simeq \varnothing(g_2)\}$$

where $\simeq$ denotes graph isomorphism. For each $w \in W$ and $u \in V_2$, we also define the following restriction

$$(g_1, [\varnothing(g_2)])_{w=\varnothing(u)} = \{(g_1, g_2) \in \mathcal{G}_n^a \times \mathcal{G}_m^a : \varnothing(g_2) = \sigma(\varnothing(g_2)), \sigma \text{ an isomorphism, } \sigma(w) = \varnothing(u)\}$$

and for $S \subset V_2$, define

$$(g_1, [\varnothing(g_2)])_{w=\varnothing(S)} = \bigcup_{u \in S} (g_1, [\varnothing(g_2)])_{w=\varnothing(u)}.$$

Choose graphs

$$g = \left\{ \left( g_1^{(i)}, g_2^{(i)} \right) \right\}_{i=1}^{h}$$

so that the sets

$$\left\{ \left( g_1^{(i)}, [\varnothing(g_2^{(i)})] \right) \right\}_{i=1}^{h}$$

partition $\mathcal{G}_n^a \times \mathcal{G}_m^a$. To ease notation, we will denote this partition via $\mathcal{P}_n^{g_m}$. For $F_{c, \theta}^{(n, m)} \in \mathcal{N}$ supported on $\mathcal{G}_n^a \times \mathcal{G}_m^a$, we will next define a Bayes optimal scheme $\Phi^*$
(optimal under both loss functions simultaneously for all $k \in [m - 1]$). For each $i \in [h]$, set (where ties are broken in a fixed but arbitrary manner)

$$
\Phi^*(g_1^{(i)}, o(g_2^{(i)}), V^*)[1] \in \arg \max_{u \in W} \mathbb{P}_{F_{e, \theta}^{(n,m)}} \left( (g_1^{(i)}, [o(g_2^{(i)})])_{u \in o(V^*)} \mid (g_1^{(i)}, [o(g_2^{(i)})]) \right)
$$

$$
\Phi^*(g_1^{(i)}, o(g_2^{(i)}), V^*)[2] \in \arg \max_{u \in W \setminus \{\Phi^*[1]\}} \mathbb{P}_{F_{e, \theta}^{(n,m)}} \left( (g_1^{(i)}, [o(g_2^{(i)})])_{u \in o(V^*)} \mid (g_1^{(i)}, [o(g_2^{(i)})]) \right)
$$

$$
\vdots
$$

$$
\Phi^*(g_1^{(i)}, o(g_2^{(i)}), V^*)[m] \in \arg \max_{u \in W \setminus \bigcup_{j < m} \{\Phi^*[j]\}} \mathbb{P}_{F_{e, \theta}^{(n,m)}} \left( (g_1^{(i)}, [o(g_2^{(i)})])_{u \in o(V^*)} \mid (g_1^{(i)}, [o(g_2^{(i)})]) \right).
$$

For each element

$$
(g_1, g_2) \in (g_1^{(i)}, [o(g_2^{(i)})]) \setminus \{(g_1^{(i)}, g_2^{(i)})\},
$$

choose an isomorphism $\sigma$ such that $o(g_2) = \sigma(o(g_2^{(i)}))$, and define

$$
\Phi^*(g_1, o(g_2), V^*) = \sigma(\Phi^*(g_1^{(i)}, o(g_2^{(i)}), V^*)).
$$

For each $i \in [h]$, $j \in [m]$, $v \in V^*$, $\Phi \in \mathcal{V}_{nm}$, define

$$
U_{i, g}^{j,v} = \left\{ (g_1, g_2) \in (g_1^{(i)}, [o(g_2^{(i)})]) \mid \text{s.t. rank}_{\Phi(g_1, o(g_2), V^*)}(o(v)) = j \right\}
$$

$$
= \left\{ (g_1, g_2) \in (g_1^{(i)}, [o(g_2^{(i)})]) \mid \text{s.t. } \Phi(g_1, o(g_2), V^*)[j] = o(v) \right\}
$$

$$
= \left\{ (g_1, g_2) \in (g_1^{(i)}, [o(g_2^{(i)})]) \mid \exists \text{ iso. } \sigma \text{ s.t. } \sigma(o(g_2^{(i)})) = o(g_2) \text{ and } \sigma \left( \Phi(g_1^{(i)}, o(g_2^{(i)}), V^*)[j] \right) = o(v) \right\}
$$

$$
= \left( g_1^{(i)}, [o(g_2^{(i)})] \right)_{\sigma(g_2^{(i)}, V^*)[j] = o(v)}.
$$

Lastly, for $(g_1, g_2) \in G_n^a \times G_m^a$, define $p_{\Phi} \in [0, 1]^m$ via

$$
p_{\Phi}^{(i)}[g_1, o(g_2), V^*][j] = p_{\Phi}^{(i)}[j] : = \sum_{v \in V^*} \mathbb{P}_{F_{e, \theta}^{(n,m)}} \left[ U_{i, g}^{j,v} \mid (g_1^{(i)}, [o(g_2^{(i)})]) \right]
$$

$$
= \mathbb{P}_{F_{e, \theta}^{(n,m)}} \left[ (g_1, [o(g_2)])_{\Phi(g_1, o(g_2), V^*)[j] \in o(V^*)} \mid (g_1^{(i)}, [o(g_2^{(i)})]) \right]
$$

Note that, by definition, $p_{\Phi}$ majorizes $p_{\Phi}^{(i)}$.

To show that $\Phi^*$ is Bayes optimal for $L_k^{(1)}$ (the proof for $L_k^{(2)}$ being completely
analogous), we have that for $k \leq m - 1$,

$$L_k^{(1)}(\Phi, V^*) = 1 - \frac{1}{|V^*|} \sum_{v \in V^*} \mathbb{P}_{\mathcal{F}^{(n,m)}_{c,\theta}}(\text{rank}_{\Phi(G_1, \sigma(G_2), V^*)}(\sigma(v)) \leq k)$$

$$= 1 - \frac{1}{|V^*|} \sum_{v \in V^*} \sum_{j \leq k} \mathbb{P}_{\mathcal{F}^{(n,m)}_{c,\theta}}(\text{rank}_{\Phi(G_1, \sigma(G_2), V^*)}(\sigma(v)) = j)$$

$$= 1 - \frac{1}{|V^*|} \sum_{P_g} \sum_{j \leq k} \sum_{v \in V^*} \mathbb{P}_{\mathcal{F}^{(n,m)}_{c,\theta}} \left[U_{i,g}^j \left(\phi(1) g_1^{(i)}, \phi(2) g_2^{(i)}\right)\right] \mathbb{P}_{\mathcal{F}^{(n,m)}_{c,\theta}} \left[\left(\phi(1) g_1^{(i)}, \phi(2) g_2^{(i)}\right)\right]$$

$$\geq 1 - \frac{1}{|V^*|} \sum_{P_g} \sum_{j \leq k} \mathbb{P}_{\mathcal{F}^{(n,m)}_{c,\theta}}(\phi(1) g_1^{(i)}, \phi(2) g_2^{(i)})$$

$$= L_k^{(1)}(\Phi^*, V^*),$$

as desired.

B Proof of Lemma 11

We first note that the growth condition on $|V^*_n|$ and on $k_n$ in the precision case ensures that the result for precision and recall consistency follow from each other, and so we will focus our attention on recall consistency. The analogous result for precision follows mutatis mutandis.

Consider the following network construction for a network of size $n$. Let $\xi_n = \max(k_n, |V^*_n|)$. For a fixed $p \in (0, 1)$, let $B_1, \ldots, B_{\left\lceil \frac{n}{\xi_n^3} \right\rceil}$ be i.i.d. ER($\xi_n, p$) random graphs. Let $H_n$ be a complete graph on $n - \xi_n \left\lceil \frac{n}{\xi_n^3} \right\rceil$ vertices. Label the vertices

- of $B_i$ with $\{1, 2, 3, \cdots, \xi_n\}$;
- of $B_2$ with $\{\xi_n + 1, \xi_n + 2, \xi_n + 3, \cdots, 2\xi_n\}$;
- of $B_{i-1}$ with $\{(i - 2)\xi_n + 1, (i - 2)\xi_n + 2, (i - 2)\xi_n + 3, \cdots, (i - 1)\xi_n\}$;
- of $B_1$ with $\{(i - 1)\xi_n + 1, (i - 1)\xi_n + 2, (i - 1)\xi_n + 3, \cdots, i\xi_n\}$;
- of $B_{i+1}$ with $\{i\xi_n + 1, i\xi_n + 2, i\xi_n + 3, \cdots, (i + 1)\xi_n\}$;
- of $B_{\left\lceil \frac{n}{\xi_n^3} \right\rceil}$ with $\left\{\left\lfloor \frac{n/3}{\xi_n} \right\rfloor - 1\right\rfloor \xi_n + 1, \left\lfloor \frac{n/3}{\xi_n} \right\rfloor - 1\right\rfloor \xi_n + 2, \cdots, \left\lceil \frac{n/3}{\xi_n} \right\rceil \xi_n$;
- of $H_n$ with $\left\{\left\lceil \frac{n/3}{\xi_n} \right\rceil \xi_n + 1, \left\lceil \frac{n/3}{\xi_n} \right\rceil \xi_n + 2, \cdots, n\right\}$.
For each \( \ell \in \left[ \left\lfloor \frac{n/3}{\xi_n} \right\rfloor \right] \) and each vertex \( v \) in \( V(B_\ell) \), independent of all other edges in the network, select \( \ell \) vertices uniformly at random from \( H_n \), i.e., from
\[
\left\{ \left\lfloor \frac{n/3}{\xi_n} \right\rfloor \xi_n + 1, \left\lfloor \frac{n/3}{\xi_n} \right\rfloor \xi_n + 2, \ldots, n \right\}.
\]
Denote this set of \( \ell \) vertices via \( V_v, \ell \) and place an edge between \( v \) and each vertex in \( V_v, \ell \). Let \( \mathcal{H}_{n,i} \) be the collection of all graphs possible under the above construction, and let \( F_{n,i} \) be the distribution on \( \mathcal{H}_{n,i} \) outlined above.

With \( c = n \), the correspondence the identity, and (where \( |V_n^*| = \nu_n \)) \( V_n^* = \{v_i\}_{i=1}^{\nu_n} = \{u_i\}_{i=1}^{\nu_n} = [\nu_n] \), define the collection of nominatable distributions
\[
\{\bar{F}_{n,i}\}_{i=1}^{\left\lfloor \frac{n/3}{\xi_n} \right\rfloor}
\]
via \( \bar{F}_{n,i} = F_{n,1} \times F_{n,i} \) (where “\( \times \)” denotes the usual product measure).

Suppose a VN rule \( \Phi = (\Phi_n)_{n=n_0}^\infty \) is level-(\( k_n \)) recall consistent for \( F_i = (\bar{F}_{n,i})_{n=n_0}^\infty \).
Then, by definition
\[
\lim_{n \to \infty} L_{k_n}^{(1)}(\Phi_n, V^*) - L_{k_n}^{*,(1)}(V^*, \bar{F}_{n,i}) = 0.
\]
However, note that here
\[
L_{k_n}^{*,(1)}(V^*, \bar{F}_{n,i}) \leq 1 - \frac{k_n}{\xi_n}.
\]
Indeed, for a given \( \bar{F}_{n,i} \), consider the following VN scheme \( \Psi_n \). First identify the vertices of \( H_n \); this is possible as \( H_n \) is a complete subgraph of order \( \geq 2n/3 \), and each \( B_i \) is of order \( o(n) \) with vertices of degree at most \( \left\lfloor \frac{n/3}{\xi_n} \right\rfloor \leq n/3 \). Each \( B_\ell \) can then be recovered and identified by computing the number of edges between \( H_n \) and each vertex \( v \in V \setminus V(H_n) \); in particular \( B_i \) can be identified as the set of vertices in \( V \setminus V(H_n) \) with \( i \) edges to \( V(H_n) \). Let \( \psi_n \) then rank the vertices in \( B_i \) (in arbitrary order) at the top of its nomination list. It is immediate then that
\[
L_{k_n}^{(1)}(\Psi_n, V^*) = 1 - \frac{k_n}{\xi_n}.
\]
By the distributional symmetry of the v.o.i., we have that for \( v \in V^* \),
\[
\mathbb{P}_{\bar{F}_{n,i}}(\text{rank}_{\Phi_n(G_1, \sigma(G_2), V^*)}(\sigma(v)) \geq k_n + 1) = L_{k_n}^{(1)}(\Phi_n, V^*). \]
For any \( \epsilon > 0 \) and sufficiently large \( n \), consistency ensures that
\[
\mathbb{P}_{\bar{F}_{n,i}}(\text{rank}_{\Phi_n(G_1, \sigma(G_2), V^*)}(\sigma(v)) \geq k_n + 1) \leq \epsilon + \left( 1 - \frac{k_n}{\xi_n} \right).
\]
The internal consistency criterion in the definition of VN schemes (Eq. 1), then implies that
\[
\mathbb{P}_{\bar{F}_{n,i}}(\text{rank}_{\Phi_n(G_1, \sigma(G_2), V^*)}(\sigma(v)) \geq k_n + 1) \leq \epsilon + \left( 1 - \frac{k_n}{\xi_n} \right) \tag{8}
\]
for each $v \in \{1, 2, \cdots, \xi_n\}$. Now, suppose that $\Phi$ is also level-$k_n$ recall consistent for $F_j$ for $j \neq i$. By similar logic, we must have that

$$
\mathbb{P}_{\tilde{F}_{n,j}}(\text{rank}_{\Phi_n(G_1, \sigma(G_2), V^*_n)}(\sigma(v)) \geq k_n + 1) \leq \epsilon + \left(1 - \frac{k_n}{\xi_n}\right)
$$

(9)

for each $v \in \{1, 2, \cdots, \xi_n\}$ for sufficiently large $n$.

Let $\sigma_{i\leftrightarrow j}$ be the permutation on $\{1, \ldots, n\}$ defined as

$$
\sigma(\ell) := \begin{cases} 
(i - 1)\xi_n + \ell & \ell \in \{1, 2, \ldots, \xi_n\} \\
\ell - (j - 1)\xi_n & \ell \in \{(j - 1)\xi_n + 1, \ldots, j\xi_n\} \\
(j - i)\xi_n + \ell & \ell \in \{(i - 1)\xi_n + 1, \ldots, i\xi_n\} \\
\ell & \text{otherwise.}
\end{cases}
$$

Now, for each $v \in [\xi_n]$, define the sets

$$
E_{n,i}^v := \{(g_1, g_2) \in \mathcal{H}_{n,1} \times \mathcal{H}_{n,i} : \text{rank}_{\Phi_n(g_1, \sigma(g_2), V^*_n)}(\sigma(v)) \leq k_n\}
$$

$$
B_{n,i,j}^v := \{(g_1, g_2) \in \mathcal{H}_{n,1} \times \mathcal{H}_{n,i} : \text{rank}_{\Phi_n(g_1, \sigma(g_2), V^*_n)}(\sigma([j - 1]\xi_n + v)) \leq k_n\}
$$

$$
E_{n,j}^v := \{(g_1, g_2) \in \mathcal{H}_{n,1} \times \mathcal{H}_{n,j} : \text{rank}_{\Phi_n(g_1, \sigma(g_2), V^*_n)}(\sigma(v)) \leq k_n\}
$$

By consistency with respect to $\tilde{F}_{n,i}$ and $\tilde{F}_{n,j}$, i.e., by Eqs. 8–9, we have that for any $\epsilon > 0$, there exists $\tilde{n}$ such that for $n \geq \tilde{n}$, we have

$$
\mathbb{P}_{\tilde{F}_{n,i}}(E_{n,i}^v) \geq \frac{k_n}{\xi_n} - \epsilon;
$$

$$
\mathbb{P}_{\tilde{F}_{n,j}}(E_{n,j}^v) \geq \frac{k_n}{\xi_n} - \epsilon.
$$

(10)

(11)

As $(G_1, G_2) \sim \tilde{F}_{n,i} \Leftrightarrow (G_1, \sigma(G_2)) \sim \tilde{F}_{n,j}$, the internal consistency criterion (Eq. 1) of a VN scheme then implies that

$$
\mathbb{P}_{\tilde{F}_{n,j}}(E_{n,j}^v) = \mathbb{P}_{\tilde{F}_{n,i}}(B_{n,j}^v) \geq \frac{k_n}{\xi_n} - \epsilon.
$$

Now, for each $v \in [\xi_n]$ and $h \in [k_n]$ define

$$
\alpha_{v,h} = \mathbb{P}_{\tilde{F}_{n,i}}\left[ (g_1, g_2) \in \mathcal{H}_{n,1} \times \mathcal{H}_{n,i} : \text{rank}_{\Phi_n(g_1, \sigma(g_2), V^*_n)}(\sigma(v)) = h \right]
$$

$$
\beta_{v,h} = \mathbb{P}_{\tilde{F}_{n,i}}\left[ (g_1, g_2) \in \mathcal{H}_{n,1} \times \mathcal{H}_{n,i} : \text{rank}_{\Phi_n(g_1, \sigma(g_2), V^*_n)}(\sigma([j - 1]\xi_n + v)) = h \right].
$$

By Eq. 10, we have that $\sum_{h=1}^{k_n} \alpha_{v,h} \geq \frac{k_n}{\xi_n} - \epsilon$, and by Eq. 11, we have that $\sum_{h=1}^{k_n} \beta_{v,h} \geq$
\[ k_n - \epsilon. \] Noting that for each \( h \in [k_n] \)
\[
1 \geq \mathbb{P}_{\tilde{F}_{n,i}} \left[ \left( \bigcup_{v \in \xi_n} R_{i,v,h} \right) \cup \left( \bigcup_{v \in \xi_n} S_{i,v,h} \right) \right]
= \sum_{v \in \xi_n} \mathbb{P}_{\tilde{F}_{n,i}} \left( R_{i,v,h} \right) + \mathbb{P}_{\tilde{F}_{n,i}} \left( S_{i,v,h} \right)
= \xi_n \alpha_{v,h} + \xi_n \beta_{v,h},
\]
and hence
\[
\beta_{v,h} \leq \frac{1}{\xi_n} - \alpha_{v,h}.
\]
Plugging this into Eq. 11 then yields
\[
\frac{k_n}{\xi_n} - \epsilon \leq \mathbb{P}_{\tilde{F}_{n,i}} \left( B_{n,i,j}^v \right)
= \sum_{h=1}^{k_n} \beta_{v,h}
\leq \frac{k_n}{\xi_n} - \sum_{h=1}^{k_n} \alpha_{v,h}
\leq \epsilon.
\]
As \( \epsilon \) was chosen arbitrarily, and \( \frac{k_n}{\xi_n} \) is bounded away from 0 by assumption, we reach
our desired contradiction, and \( \Phi \) cannot be consistent with respect to both \( F_i \) and \( F_j \).
As \( i,j \in \lfloor \frac{n_0/3}{\xi_{n_0}} \rfloor \) were arbitrary, we see that there must be at least countably many
consistency classes (since there are at least \( \lfloor \frac{n_0/3}{\xi_{n_0}} \rfloor \) and we can let \( n_0 \) tend to infinity).

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