Universality in open system entanglement dynamics

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Received 20 November 2012, in final form 7 January 2013
Published 6 February 2013
Online at stacks.iop.org/JPhysA/46/085301

Abstract
We show that the entanglement evolution of an open quantum system is the same for the vast majority of initial pure states, in the limit of large Hilbert space dimensions. The analytical results are illustrated by a numerical study of the evolution of negativity for many-qubit systems undergoing a local decoherence process. In the limit of many subsystems, the arising mixed state entanglement between one subsystem and the rest grows more robust, whereas for two equally sized partitions it becomes more fragile.

PACS numbers: 03.67.–a, 03.67.Mn, 03.65.Yz

(Some figures may appear in colour only in the online journal)

1. Introduction

The ability to characterize and control quantum systems with an ever increasing number of constituents is an ever more desired and necessary skill in many fields of modern physics [1], materials science [2] and even biology [3]. From a strictly deterministic point of view, however, the rapidly growing generic complexity of such large systems apparently renders this task ineffable. Yet, similarly to thermodynamics, a statistical description often allows for the extraction of robust, generic features which emerge in the limit of large system size, and imply quantitative predictions.

Entanglement, an unmistakable quantum signature, is a prime example of the above, apparent illusiveness of an exhaustive characterization as the system size is scaled up: the structure of many-particle entanglement turns more and more intricate with the exponentially increasing number of possible correlations between different subgroups of particles. Thus, a complete characterization of a large, composite quantum state requires an experimental
overhead that increases exponentially with the number of system constituents. Even worse,
one expects entanglement to get ever more fragile when enlarging the system size: the more
degrees of freedom, the more difficult it becomes to shield quantum coherences, which are
necessary for entanglement, against the detrimental influence of a noisy environment. In
such situations, the strong quantum correlations due to entanglement additionally need to be
distinguished from classical correlations induced by the ambient noise. This is in general
accomplished by high-dimensional optimization procedures on the space of all quantum states
[4, 5], leaving little hope for quantitative predictions on entanglement evolution in large and
noisy systems. On the other hand, the signatures of entanglement in such systems are of
high fundamental and, potentially, practical interest, e.g. when it comes to harnessing the
computational power of quantum algorithms [6] or assessing the role of quantum correlations
in intrinsically noisy biological systems [3, 7]. It is therefore a key issue, and the subject of
this paper, to estimate the characteristic time scales in which entanglement is present in such
adverse situations.

Here we consider the fate of entangled states of large quantum systems, which require
a high-dimensional Hilbert space, in contact with an incoherent environment. We show
that a statistical analysis over generic initial states unveils universal—state independent—
open system entanglement evolution in the limit of large Hilbert space dimensions. In this
thermodynamic limit, an efficient characterization of entanglement dynamics is thus again
possible.

2. Universal entanglement dynamics

Let us start with a composite quantum system in a pure state, characterized by a complex
vector $|\chi\rangle$ in a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cdots \otimes \mathcal{H}_N$ with dimension $d = d_A d_B \cdots d_N$.
The system then undergoes some dynamics generated by its Hamiltonian, and also couples to
uncontrollable degrees of freedom, which define its environment (denoted by the subscript
$E$).

When focusing on the system alone, its state $\rho$ evolves as described by a time-dependent map
$\Lambda_t$,

$$\rho(0) \mapsto \rho(t) = \text{tr}_E[U(t) \rho_{SE}(0) U(t)^\dagger] =: \Lambda_t[\rho(0)]. \quad (1)$$

where $U(t)$ is the unitary time evolution operator for the system and environment (subscript
$SE$). Such maps $\Lambda_t$ are only independent of the system’s initial state $\rho(0)$ if the system and
environment are initially uncorrelated or at most classically correlated [8]. This is the case
for many implementations of quantum information tasks [9], where initial system states are
almost pure, and therefore uncorrelated with the environment in very good approximation.
In what follows, given the above and the fact that our proof requires initial pure states of
the system, we assume the initial state of the system and environment to be of the form
$\rho_{SE}(0) = |\chi\rangle\langle\chi| \otimes \rho_E(0)$. Since, beyond this latter factorization, our ansatz makes no
assumption on the specific form of the Hamiltonian and thus of $U(t)$, and neither of $\rho_E(0)$,
our subsequent results are applicable for arbitrary open system dynamics of the given initial
states, including non-Markovian effects that may arise in the course of the evolution. The
following derivations are solely grounded in few geometrical properties of the set of states,
and properties of entanglement measures thereon. The next steps and properties needed to
prepare our main result are visualized in figure 1.

As a measure for the relative effect of the open system dynamics $\Lambda_t$ onto two states $\rho$ and
$\omega$, we choose the metric distance induced by the trace norm $\| \cdot \|_{\text{tr}}$ on that space:

$$D_\text{tr}(\rho, \omega) = \| \rho - \omega \|_{\text{tr}} = \text{tr} |\rho - \omega| . \quad (2)$$
In general, for an operator $A$ its trace norm is given by $\|A\|_\text{tr} = \text{tr} \sqrt{A^\dagger A}$. The distance of two states measured by $D_\text{tr}$ is also directly related to how well the two states can be distinguished in an experiment. More importantly, it decreases monotonically in the course of open system dynamics, i.e. upon the application of any such $\Lambda_t$ [10],

$$D_\text{tr}[\Lambda_t(\rho), \Lambda_t(\omega)] \leq \eta_{\text{tr}} D_\text{tr}(\rho, \omega) \quad \text{with} \quad \eta_{\text{tr}} \leq 1. \quad (3)$$

Note that, for closed dynamics, i.e. without coupling to the environment, equality holds with $\eta_{\text{tr}} = 1$, as a consequence of the unitarity of quantum mechanics. Open dynamics, however, introduces additional mixing of the system state, such that the state space is effectively contracted. This dynamical effect is captured by the change of $\eta_{\text{tr}}$ with $t$. For example, when coupled to a thermal bath, all initial states converge to thermal equilibrium, and $\eta_{\text{tr}}$ approaches zero asymptotically in time.

Turning to entanglement, we do not focus on a specific entanglement quantifier, but rather only require that it slowly varies on the set of states. More specifically, we demand a ‘strong’ form of continuity, namely Lipschitz continuity [11],

$$|E(\rho) - E(\omega)| \leq \eta_E D_\text{tr}(\rho, \omega) \quad , \quad (4)$$

where $\eta_E$ is called the Lipschitz constant. Examples of such entanglement measures $E(\rho)$ are defined as the minimum distance of the state $\rho$ to the set $S$ of separable states: $E_\text{D}(\rho) = \min_{\sigma \in S} D(\rho, \sigma)$ [5]. All distances $D$ which abide to the dynamic monotonicity condition (3) [12], fulfill the Lipschitz requirement with constant $\eta_E = 1$, by virtue of the triangle inequality. Another example is negativity $N$, a computable entanglement monotone [13], but with a different constant $\eta_N$ (see the appendix).

Under conditions of equations (3) and (4), we can assess how much two initially pure states differ in their entanglement after exposure to the same incoherent dynamics. For states $\rho(t) = \Lambda_t(|\chi\rangle\langle\chi|)$ and $\omega(t) = \Lambda_t(|\psi\rangle\langle\psi|)$ we obtain

$$|E(\rho) - E(\omega)| \leq \eta_E D_\text{tr}[\Lambda_t(|\chi\rangle\langle\chi|), \Lambda_t(|\psi\rangle\langle\psi|)]$$

$$\leq \eta_E \eta_{\text{tr}} D_\text{tr}[|\chi\rangle\langle\chi|, |\psi\rangle\langle\psi|]$$

$$\leq 2\eta_E \eta_{\text{tr}} \| |\chi\rangle - |\psi\rangle \| . \quad (5)$$

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6 So does also the Bures distance, but not the Hilbert–Schmidt distance.
That is, the entanglement quantifier $E$ inherits Lipschitz continuity, with constant $2\eta_E\eta_A$, even with respect to the Euclidean distance $d$ between the two initial state vectors in $\mathcal{H}$. The difference in entanglement of the two mixed final states is essentially bounded by the distance between the initial states. With the knowledge of the entanglement of a single probe state, say of $\rho(t)$, for an initial state $|\chi\rangle$, we are able to predict the entanglement evolution of any other pure initial state $|\psi\rangle$ within an error margin given by (5)—this without the need to evolve the state, and to compute the resulting mixed state’s entanglement.

Rather than studying specific instances of initial states, we aim at a statement about generic pure states in a statistical manner. To avoid any bias, we use random states which uniformly cover the space of pure states. States that share some properties with random states naturally appear, for example, in registers of quantum computers after long sequences of gates [15], and in quantum systems with a chaotic classical counterpart [16]. In addition, for many interesting scenarios of biology and chemistry such as transport phenomena in proteins, the initial state and/or the Hamiltonian are not exactly known, e.g. [17]. In these cases a uniform distribution of initial states in the potential subspace is the best prior.

Let us first note that the set of (pure) state vectors in $\mathbb{S}^{2d-2}$ in $\mathbb{R}^{2d-1}$, where the real and imaginary part of the expansion coefficients into any basis constitute the coordinates, constrained by normalization and ignoring the global phase by choosing the first component real. With the above preparation, we can now infer our central result by employing the well-known Levy’s lemma [18–20]: the probability for a deviation larger than $\epsilon$ of $E[\rho(t)] := E[\Lambda_\rho(|\chi\rangle\langle\chi|)]$ from its mean $\langle E \rangle(t) := \int d\psi E[\Lambda_\psi(|\psi\rangle\langle\psi|)]$ over all initial states, given the uniform initial distribution on $\mathbb{S}^{2d-2}$, exhibits an exponential suppression: (i) in the deviation $\epsilon$, and moreover, (ii) in the system dimension $d$:

$$\Pr(|E[\rho(t)] - \langle E \rangle(t)| > \epsilon) \leq 4 \exp \left(\frac{-C \epsilon^2}{\epsilon^2} \right).$$

The constant $C$ can be chosen $(24 \pi^2)^{-1}$ [14]. This holds for all entanglement quantifiers $E$ that fulfill (4), even when quantifying multipartite entanglement.

As a concrete application of this result, we consider the normalized negativity $\mathcal{N}/\mathcal{N}_{\max}$ of a bipartite system on $\mathcal{H}_A \otimes \mathcal{H}_B$. Negativity is an entanglement monotone defined as $\mathcal{N}(\rho) := (\|I \otimes T(\rho)\|_1 - 1)/2$, where $T$ is the transposition map acting on $\mathcal{H}_B$, and its maximum value is $\mathcal{N}_{\max} = (d_A - 1)/2$ where we assumed $d_A \leq d_B$. Its Lipschitz constant (see appendix) reads $\eta_{\mathcal{N}/\mathcal{N}_{\max}} = d_A/(d_A - 1)$, and the following inequality for a deviation from the mean entanglement holds:

$$\Pr\left(\left|\mathcal{N}[\rho(t)] - \langle \mathcal{N} \rangle(t)\right| > \epsilon\right) \leq 4 \exp \left(\frac{-C (2d_A d_B - 1)(d_A - 1)^2}{4d_A^2 \eta_{\mathcal{N}}^2} \epsilon^2\right).$$

This inequality expresses the concentration effect visualized in figure 2 for ensembles of $N$ qubits (two-level systems), each locally coupled to an environment that destroys only the coherences, i.e. the dynamical map is of the form $\Lambda_{\rho} \otimes \cdots \otimes \Lambda_{\rho}(|\psi\rangle\langle\psi|)$ $^8$. Negativity is here evaluated with respect to the least balanced bipartition of the qubit register, i.e. it quantifies the entanglement of one qubit ($d_A = 2$) with the remaining $N - 1$ qubits ($d_B = 2^{N-1}$), after application of the decoherence dynamics. We uniformly sample pure initial states, and parametrize the dynamics by the probability $p$ for complete decoherence of a single qubit.

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7 The estimate $d_A(|\chi\rangle\langle\chi|, |\psi\rangle\langle\psi|) \leq 2\|\chi - \langle\chi\rangle\|_2$ follows from the fact that $|\psi\rangle$ and $|\chi\rangle$ span at most a two-dimensional space. As such, one obtains $d_A(|\chi\rangle, |\psi\rangle\langle\psi|) = 2\sqrt{1 - \|\psi\rangle\langle\psi\|}$, which can then be related to the Euclidean norm using that $\text{Re}(\langle\psi\rangle\langle\chi|) = 1 - \|\chi\|_2^2$. See [14] for details.

8 The decoherence map for a single qubit reads $\Lambda_{\rho}(\rho) = \sum_{i=1}^{2d-2} K_\rho K_i$, with $K_1 = \sqrt{1 - p}\mathbb{1}$ and $K_2 = \sqrt{p/2}\sigma_z$. 

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Figure 2. Distributions of negativity for the least balanced partition (1 versus $N - 1$ qubits). (Left) Histograms of the distributions of negativity for $N = 3, 5, 7$ qubits, at different snapshots during the evolution. We sample over 100 000 ($N = 3, 5$) or 30 000 ($N = 7$) uniformly distributed pure initial states of $N$ qubits, which are then evolved under local coupling to a decoherence reservoir. The system–environment interaction time $t$ is parametrized by $p = (1 - e^{-\Gamma t})$, with $\Gamma$ being the local decoherence rate. Clearly, the bigger the $N$, the more the negativity distribution concentrates around its mean value, for all times. (Right) For large $N$ the distributions are increasingly well characterized by their mean (bottom) and standard deviation (top). The logarithmically plotted standard deviation (top) confirms the exponential decay with $N$ as shown for several time steps of the evolution. The inset shows the time evolution of the standard deviation for the boxed data points at fixed $N$. While entanglement generally decays with $p$ as plotted for even system sizes $N = 2, \ldots, 14$ (bottom), it grows more robust with increasing $N$ as exemplified for the boxed data points for fixed $p$ in the inset.

Within the Markov approximation $p = 1 - e^{-\Gamma t}$, where $\Gamma$ is the decoherence rate. The log-plot of the standard deviations of the distributions for increasing system size, shown at the top right in figure 2, confirms the exponential concentration around the mean at any point in time during the open system evolution. In addition to the overall concentration effect due to the increase in $N$, there is the influence of the change of $\eta$, with $p$. For example, for $p \to 1$ the whole ensemble of states will become disentangled and thus the distribution will converge to a peak at $N = 0$ even for $p < 1$ and thus lead to an additional concentration effect. It is the general observation for larger $N$, that distributions are more concentrated if their mean is close to one of the extremal values $N = 0$ or $N = 1$, i.e. for $p \approx 1$ and $p \approx 0$, respectively. This effect is visible by the crossing of curves in the top right plot in figure 2. Our numerical results in figure 2 underpin the mixed state entanglement concentration predicted by (6) and (7). Furthermore, the concentration observed in the numerical experiment is even stronger than estimated—the numerically obtained standard deviations for our specific examples are smaller than the values that we would infer from the general (7), with our above estimate of $C$ and the Lipschitz constants. For example, if we assume a normal distribution of negativity, the probability for a deviation from the mean of more than $\epsilon = 3\sigma$ is $\Pr(\ldots) \approx 1 - 0.9973$. With the conservative estimate $\eta = 1$ and fixed $d = 2$, equation (7) provides an upper bound to
the standard deviation $\sigma < 0.613$ for $N = 12$ qubits, which is not tight as can be observed in figure 2 (top, right). As another example, independent of the distribution, equation (7) can be used to obtain bounds on the system size for a given concentration effect. It bounds from above the maximal system size that can still exhibit a given large deviation from the mean, e.g. it predicts that only systems of up to 26 qubits can show a deviation greater than $\epsilon = 0.01$ from the mean entanglement with probability more than 1%. Larger systems show a more concentrated entanglement distribution.

3. Semi-classical limit of generic entanglement dynamics

The numerical study of the negativity decay in figure 2 (left) indicates that the entanglement between one qubit and the $N - 1$ other qubits becomes increasingly robust against local decoherence for large system sizes. This observation is shown in the bottom-right panel of figure 2 and in the inset there in more detail. For every value of $p$, the higher the number of qubits $N$, the higher the negativity in the partition $\{1, N - 1\}$.

This picture is, however, completely different when splitting the system in two equally sized halves. For the partition $\{N/2, N/2\}$, the concentration effect for mixed states also holds as predicted by (6) and (7) for all points during the evolution, as can be seen in figure 3.
(top, left). However, the time evolution of the mean $\langle N \rangle$ is now completely different from the previous case. Now, when as before each qubit is undergoing a local decoherence process, the entanglement decays faster when increasing the number of qubits. Despite of a growing initial entanglement for larger system sizes, after some interaction with the environment (already for $p = 0.05$), the larger the system size $N$, the smaller is the entanglement. This behavior also contrasts the one of a sample of pure states, which grows more entangled for larger $N$ [20]. This example illustrates that it is not possible to straightforwardly conclude from the scaling of the mean entanglement for pure states, $\langle N \rangle$, with the number $N$ of parties [20], how the mean entanglement for a sample of mixed states obtained after some noisy evolution, $\langle N \rangle$, will behave for increasing $N$.

These partition-dependent features are in contrast with the entanglement decay of, for example, GHZ-like states [21]. For those states the entanglement decay rate is upper-bounded by a factor $(1 - p)^N$ for all partitions. Therefore, for both partitions, $\{1, N - 1\}$ and $\{N/2, N/2\}$, for every value of $p$ there will exist a system size $N' > N$ for which the $E_N(p) < E_{N'}(p)$.

We have investigated if a similar decay behavior holds for the decay of the mean negativity for the balanced partition. To this extent we have tested the data against two model functions. A model similar to the decay of GHZ-like states with $\langle N \rangle(p) = \langle N \rangle_0 (1 - \alpha p)^N$ yields a positive $\alpha$ that decreases with $N$. An exponential model for the initial decay as suggested by the inset in figure 3 (right top) with $\langle N \rangle(p) \sim \exp(-\gamma p)$ yields that $\gamma$ grows approximately linearly with $N$ until $N = 14$, for which we could sample data. Both models, however, fit well only for large $N$ and small $p$.

We emphasize again that, independent of the partition, for large $N$ measuring the entanglement of only one mixed test state of the ensemble suffices to predict the mean entanglement of the whole ensemble, $\langle N(\rho(t)) \rangle$, and thus also of the whole ensemble with high probability.

Finally, let us observe that while the numerical observations regarding the robustness of entanglement—as shown in figures 2 and 3—are in contrast to GHZ-like states, they are in full agreement with the intuition on how entanglement should decay for open macroscopic systems. In our daily experience we do not observe entanglement between two macroscopic systems—this is consistent with our observations for equal size partitions, where the larger the system, the faster the entanglement decay. On the other hand, one may expect that within complex macroscopic systems, quantum effects such as entanglement can be present despite the action of the environment—this is compatible with the slower entanglement decay for the partition $\{1, N - 1\}$ with increasing system size. One atom of a macroscopic object is likely to be entangled with nearby atoms within the same object, while the object in itself is not entangled to another object around it. This expected behavior for the entanglement decay of typical (uniformly distributed) quantum states supports their use as a meaningful sample to describe statistical properties of macroscopic quantum systems.

4. Conclusions

The above results determine the open system evolution of entanglement with an error exponentially small in the dimension of the underlying Hilbert space (provided $\eta_E$ does not increase faster than $\sqrt{d}$). Therefore, in high dimensions, it suffices to monitor the entanglement evolution of a single, generic pure state, in order to predict the fate of any other typical state subject to the same dynamics. Qualitatively different entanglement dynamics for different initial states [22] will occur only as singular effects in sufficiently large quantum systems.

Inasmuch as the knowledge of a single state’s entanglement evolution fully determines (in the present, asymptotic sense) the result for arbitrary states, our result is reminiscent of
previously derived entanglement evolution equations [23–25], where the final entanglement of an arbitrary initial pure state was shown to be fully characterized by the entanglement evolution of a maximally entangled state.

The concentration of open system entanglement evolution, as spelled out by (6) and (7), has yet another bearing for the optimization problem routinely encountered [4, 5] when evaluating the entanglement of arbitrary mixed states. For mixed states generated through arbitrary physical dynamics $\Lambda_t$ from a uniform distribution of pure states, mixed state entanglement concentration suggests a reduction of the optimization space: a single representative of the obtained sample generated under a specific physical evolution, selected with convenient properties (symmetries in terms of its pure state decomposition), will suffice to impose exponentially narrow constraints on the optimization for all other states.

Acknowledgments

We enjoyed discussions with Mark Fannes, Alexander Holevo, Carlos Viviescas, and particularly thank Reinhard Werner for discussions on the Lipschitz constant of negativity. We also profited from exchanges with Alejo Sales and Daniel Cavalcanti, and express our gratitude for the hospitality of Berge Englert and the NUS, as well as Luiz Davidovich’s group at the Universidade Federal do Rio de Janeiro, where part of the work was done. DAAD/CAPES is acknowledged for financial support through the PROBRAL program. MT was partially supported by the IMS. FM acknowledges support by the Alexander von Humboldt Foundation, Belgium Interuniversity Attraction Poles Programme P6/02, and by the Netherlands Organization for Scientific Research (NWO) through the Vidi grant 639.072.803.

Appendix. Negativity’s Lipschitz constant

In order to calculate the Lipschitz constant for the negativity of a bipartite system, we use its definition in terms of the trace norm of the partially transposed state [13], $N(\rho) := \frac{1}{2} \| (\mathbb{1} \otimes T) (\rho) \|_1 - 1$, where $T$ represents the transposition acting just on $\mathcal{H}_B$. By means of the reverse triangle inequality, $\| \| x \| - \| y \| \| \leq \| x - y \|$, and the linearity of the partial transpose we arrive at a first estimation for the negativity of states $\rho$ and $\omega$,

$$|N(\rho) - N(\omega)| \leq \frac{1}{2} \| (\mathbb{1} \otimes T) (\rho - \omega) \|_1. \quad (A.1)$$

The remaining norm can be estimated with the operator norm defined as $\| A \|_{op} := \sup_x (\| A x \| / \| x \|)$. By choosing a particular $x$, not necessarily optimal, we obtain a lower bound: $\| (\mathbb{1} \otimes T) \|_{op} \geq \| (\mathbb{1} \otimes T) (\rho - \omega) \|_1 / \| \rho - \omega \|_1$. This leads to the next estimate for negativity

$$|N(\rho) - N(\omega)| \leq \frac{1}{2} \| (\mathbb{1} \otimes T) \|_{op} \| \rho - \omega \|_1. \quad (A.2)$$

The maximization of the partial transposition is obtained with a maximally entangled state, which yields $\| (\mathbb{1} \otimes T) \|_{op} = d_A$, with $d_A$ the dimension of the smallest subsystem of the bipartition. Consequently, we find Lipschitz continuity for negativity,

$$|N(\rho) - N(\omega)| \leq \frac{d_A}{2} \| \rho - \omega \|_1, \quad (A.3)$$

with Lipschitz constant $\eta_N = d_A/2$. For the normalized negativity, i.e. rescaled with respect to its maximal value $N_{max} = (d_A - 1)/2$, we finally obtain $\eta_{N/N_{max}} = d_A/(d_A - 1)$. 

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References

[1] Amico L et al 2008 Rev. Mod. Phys. 80 517
    DiVincenzo D P 2000 Fortschr. Phys. 48 9
    Bennett C H and DiVincenzo D P 2000 Nature 404 247
[2] Redl F X et al 2003 Nature 423 968
[3] Engel G S et al 2007 Nature 446 782
    Briegel H J and Popescu S 2008 arXiv:0806.4552
    Sarovar M et al 2010 Nature Phys. 6 462
[4] Uhlmann A 2000 Phys. Rev. A 62 032307
[5] Vedral V and Plenio M B 1998 Phys. Rev. A 57 1619
    Vedral V et al 1997 Phys. Rev. Lett. 78 2275
[6] Raussendorf R and Briegel H J 2001 Phys. Rev. Lett. 86 5188
    Walther P et al 2005 Nature 434 169
[7] Caruso F et al 2009 J. Chem. Phys. 131 105106
    Cai J, Guerreschi G G and Briegel H J 2010 Phys. Rev. Lett. 104 220502
    Scholak T et al 2011 Phys. Rev. E 83 021912
[8] Shabani A and Lidar D A 2009 Phys. Rev. Lett. 102 100402
[9] Benhelm J et al 2008 Nature Phys. 4 463
    Ansmann M et al 2009 Nature 461 504
    Leibfried D et al 2005 Nature 438 639
    Haffner H et al 2005 Nature 438 643
[10] Ruskai M B 1994 Rev. Math. Phys. 6 1147
[11] Rockafellar R T 1970 Convex Analysis (Princeton, NJ: Princeton University Press)
[12] See, e.g. Oizawa M 2000 Phys. Lett. A 268 158
[13] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
[14] Tiersch M 2009 Benchmarks and statistics of entanglement dynamics PhD Thesis University of Freiburg
    http://www.freidok.uni-freiburg.de/volltexte/6878/
[15] Emerson J et al 2003 Science 302 2098
    Oliveira R, Dahlsten O C O and Plenio M B 2007 Phys. Rev. Lett. 98 130502
[16] García-Mata I et al 2007 Phys. Rev. Lett. 98 120504
    Benenti G 2009 Riv. Nuovo Cimento 32 105
[17] Read E L et al 2008 Biophys. J. 95 847
[18] Ledoux M 2001 The Concentration of Measure Phenomenon (Mathematical Surveys and Monographs) vol 89
    (Providence, RI: American Mathematical Society)
[19] Milman V D and Schechtman G 1986 Asymptotic Theory of Finite Dimensional Normed Spaces (Lecture Notes
    in Mathematics 1200) (Berlin: Springer)
[20] Hayden P, Leung D and Winter A 2006 Commun. Math. Phys. 265 95
[21] Aolita L et al 2009 Phys. Rev. A 79 032322
[22] Yu T and Eberly J H 2009 Science 323 598
[23] Konrad T et al 2008 Nature Phys. 4 99
[24] Tiersch M, de Melo F and Buchleitner A 2008 Phys. Rev. Lett. 101 170502
[25] Gour G 2010 Phys. Rev. Lett. 105 190504