SD-effect for circular plates of lamina cribrosa and optic nerve

Galina V Pavilaynen¹ and Dmitry V Franus²

¹ Saint-Petersburg State University, 7/9 Universitetskaya Emb., St-Petersburg, 199034, Russia
² Noncommercial organization "Foundation for promotion of mathematical education and researches in the field of exact sciences "UniChance", 6 Doroga na Turukhtannyye Ostrova, Saint-Petersburg, 198096, Russia
E-mail: g.pavilaynen@spbu.ru, franusdv@gmail.com

Abstract. Asymptotic equations for stresses are presented and used for numerical modeling and graphical representation of elastoplastic properties of circular SD-plates. A numerical solution of the plate bend is obtained after calculating the system of fifth-order differential equations. Euler difference method and software package COMSOL 5.6 are applied to solve the problem for biological tissue.

1. Introduction
Modern problems of biomechanics, particularly the problems of modeling vision correction in ophthalmology, are associated with studying the stress-strain state of shells and plates made of complex materials [1]. The bending of circular plate is a nonlinear problem in physical properties and essentially in geometry. It is necessary to model large and inelastic deformations on the one hand and to set complex loading systems on the other [2]. Due to difficulties in an analytical solution to such problems, modeling using finite-element (FE) methods is the best way to study them.

This paper considers the bending of a significantly plastic anisotropic plate under hydrostatic loading, simulating the lamina cribrosa when intraocular pressure increases. Figure 1 shows eye nerve structure. The biological tissue of nerve fibers has different elastic properties under tension and compression. Such materials are called plastically anisotropic or materials with SD-effect.

The elastic characteristics of the nerve tissue are deficient, and its strength is negligible. Determining these characteristics by experiments is a rather difficult task. Medical data of biological SD-tissue is used together with the classical Hill’s theory of plasticity and various mathematical models with transversal isotropy parameter $A$ and the plastic anisotropy parameter (SD-parameter) $\beta$.

R. von Mises, R. Hill, L. H. Donnell developed the classical theory of elastic and elastoplastic bending. S.P. Timoshenko introduced elasticity theory for thin isotropic plates. V.V. Sokolovsky added plasticity to elasticity theory [3]. Another group of scholars solved bending problems of SD-plates [2, 4, 5].
Figure 1. Schematic anatomy of optic nerve (right) and microscopic picture of lamina cribrosa (left).

2. Mathematical model

This paper considers the problem of elastic-plastic bending of a round freely supported SD-plate, which has transverse anisotropy properties and is evenly loaded with pressure $p$ on the upper surface.

Figure 2 shows the central cross-section of a bent circular plate with following parameters:

- $h$ is the half-thickness of the plate,
- $x_1$, $x_2$ are the radiiuses of the plastic regions on the bottom and the top of the plate’s surfaces respectively,
- $a_1$, $a_2$ are the depths of the plastic zones on the bottom and on the top of the plate respectively.

Figure 2. Elastic-plastic bending of a circular plate from SD material

The plastic regions are shaded. In the case presented in Figure 2 the neutral surface does not coincide with the geometrically average surface. A solid line is a neutral surface, a dashed line is a geometrically mid-plane.

Point O in Figure 2 represents the beginning of the coordinate system, which is the plate’s center on the neutral surface.

In the article [5], authors made the mathematical model for the SD-plate more complicated, suggesting a new criterion of plasticity:
Here $\sigma_r$, $\sigma_\theta$ are the stresses in the plate’s plane, $k$ is a function of the yield stresses, $\sigma$ is hydrostatic stress. The average stress $\sigma$ is equal to:

$$\sigma = \frac{\sigma_r + \sigma_\theta}{3} \quad (2)$$

In (1), scholars use the transversal isotropy parameter $A$, which varies from 1 to 2. The parameter $\beta$ characterizes the plastic anisotropy property (SD effect) and can be found by the following relation:

$$A = 2 - \frac{(\sigma_{pz} + \sigma_{cz})^2 \sigma_p^2 \sigma_c^2}{(\sigma_p + \sigma_c)^2 \sigma_{pz} \sigma_{cz}} \quad (3)$$

where $\sigma_p$ is the yield point for uniaxial tension in the plane of the plate, $\sigma_{pz}$ the yield point for uniaxial tension in a direction perpendicular to the plate’s plane, $\sigma_c$ is the yield point by uniaxial compression in the plane of the plate, $\sigma_{cz}$ is the yield point for uniaxial compression in a direction perpendicular to the plate’s plane. For uniaxial stretching (the equation on the left) and uniaxial compression (the equation on the right), criterion (1) is equal to:

$$k = \sigma_p + \frac{1}{3} \sigma_p \beta, \quad k = \sigma_c - \frac{1}{3} \sigma_c \beta. \quad (4)$$

Thus, the relationship between $\beta$, $\sigma_p$ and $\sigma_c$ is:

$$\frac{\sigma_c}{\sigma_p} = \frac{3 + \beta}{3 - \beta} \quad (5)$$

or

$$\beta = \frac{3(\sigma_c - 1)}{\sigma_p + 1} \quad (6)$$

In the case of a biaxial stress state, the criterions for stretching and compression can be written accordingly:

$$\bar{k} = \sigma_{pz} \sqrt{2 - A} + 2 \frac{1}{3} \sigma_{pz} \beta, \quad \bar{k} = \sigma_{cz} \sqrt{2 - A} - 2 \frac{1}{3} \sigma_{cz} \beta, \quad (7)$$

The transversal isotropy parameter $A$ is a function of four yield points $A = f(\sigma_c, \sigma_p, \sigma_{pz}, \sigma_{cz})$. The SD-parameter $\beta$ is a function of two yield points $\beta = g(\sigma_c, \sigma_p)$ and doesn’t depend on the properties of the plate in the direction of the axis $OZ$. The relationship between parameters $A$ and $\beta$ can be written as follows:

$$\frac{\sqrt{2 - A} + 2 \frac{1}{3} \beta}{\sqrt{2 - A} - 2 \frac{1}{3} \beta} = \frac{\sigma_{cz}}{\sigma_{pz}} \quad (8)$$

Only three yield points are independent. The relationship between the yield points is following:

$$\frac{1}{\sigma_{cz}} = \frac{1}{\sigma_{pz}} + 2 \frac{1}{\sigma_c} - \frac{1}{\sigma_p} \quad (9)$$
The bending theory is based on the plane stress state model. The deformation of the transverse shear is ignored. The stress in the direction perpendicular to the plane of the plate is assumed to be zero.

If the $\sigma$, $\sigma_c$, $\sigma_{pz}$, $\sigma_{cz}$ are known, then the values of $\beta$ and $A$ can be calculated by equations (3), (6). Let’s assume that $\sigma_p \leq \sigma_c$, then from equation (5) follows that $\beta \geq 0$, and from equation (3) that $A \leq 2$.

Bending metal plates have $\beta \leq 1$, but it should be adopted for biological tissues, for example, elastin’s $\beta$ varies from 0.5 to 3.

From the point of view of evaluating the stressed state of the plate, the most critical place is its center. Therefore, the stresses in the plastic regions near the centers of the top and the bottom surfaces of the plate should be considered.

In the center of the plate $\sigma_\theta = \sigma_r$ and the asymptotic equations together with stresses equations take the form [6]:

$$
\frac{\sigma_+}{\sigma_p} = \frac{\beta + 3}{3\sqrt{(2 - A)}} \left(1 - \frac{2\beta}{3\sqrt{(2 - A)}}\right)
$$

(10)

$$
\frac{\sigma_-}{\sigma_p} = -\frac{\beta + 3}{3\sqrt{(2 - A)}} \left(1 + \frac{2\beta}{3\sqrt{(2 - A)}}\right)
$$

(11)

3. Numerical modeling

It becomes possible to estimate the influence of the parameters $A$ and $\beta$ on the plate’s stresses without solving the enormous problem of elastoplastic equilibrium of the plate [6].

We want to point out that parameters $A$ and $\beta$ are not independent. The relationship between them is introduced in equation (8).

Table 1 shows the results of $\beta$ calculations.

| $\sigma_c/\sigma_p$ | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 2.0 | 3.0 | 4.0 |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta$             | 0   | 0.14| 0.27| 0.39| 0.5 | 0.6 | 1.0 | 1.25| 1.8 |

Referencing the Figure 3 (right), the yield curves for metals (1-6) considerably differ from the yield curves for biomaterials (7-9). Mentioned differences in yield curves affect the plate’s bending and significantly change the nature of the occurrence and development of plastic deformations. The transversal isotropy parameter for biomaterials $A$ is less than zero.

4. Results of finite element modeling

Research shows that plasticity area in the plate’s compression zone is substantially smaller than those in the tension zone. We assume that the yield point during compression is more significant than that under tension. We use the COMSOL 5.6 software package to calculate the bending. Sizes of plasticity zones are calculated depending on the pressure.

Based on the results, the magnitude of the plasticity ”spot” and the depth of plasticity areas depend on compression or tension (Figure 4).
5. Finite-element modeling of collagen plate

Lamina cribrosa is formed by a multilayered network (beams) of collagen fibers, which comprise the nerve fibers that run through pores formed by the network.

Firstly, we model a solid plate made entirely out of collagen. Due to different behavior biological systems, it is necessary to change boundary conditions for materials made out of biological tissues [7]. From a mechanical perspective, it is helpful to recognize two components of IOP-induced deformation of the lamina cribrosa (Figure 5). One component is the effect of IOP on the anterior laminar surface, which deforms the lamina posteriorly (middle). Another component is the effect of IOP on the sclera, which causes an expansion of the canal (right). The deformations are transmitted to the lamina through its insertion into the canal wall, resulting in a lamina that pulls ”taut”, thereby displacing anteriorly. As IOP increases, both components act simultaneously [8].

Two different boundary conditions are analyzed:

- round freely supported collagen plate;
- fixed constraint surface on the edge of the collagen plate.

Figure 3. The dependence (1) of the anisotropy parameter $A$ on the values of the yield points in different directions (left) and qualitative difference in yield curves for metals and biomaterials (right).

Figure 4. The plasticity ”spot” at the top (left) and bottom (right) of the plate for $P=36$ MPa.
FE model of plate made of collagen is build as circular plate to research SD-effect with equations (1), (8), (10), and (11), where $A = 1.41$, $\beta = 1.8$.

Pressure $p$ on the upper surface is taken 15 kPa, which corresponds to intraocular injection or high tension due to eye rubbing [9].

Modeling results show (Figure 6 and Figure 7) that the deformations do not start in the center of the plate but along the edge in case of fixed constraint boundary conditions and vice versa for round freely supported plate. The situation is fundamentally different from metals, described in previous section. The top surface of the plate has a plasticity area on edge. Plasticity "spot" appears on the bottom side later for fixed constraint and earlier for round freely supported plate.

Stresses in the center of the plate on top and bottom surfaces are very similar by value and behavior; on the contrary, stresses on the edge is highly depended on boundary conditions, which leads to more profound research of the problem of boundary conditions.
6. FE modeling of lamina cribrosa bending
Based on real microscopic picture (see fig.1 (left)) 2D geometrical model is built with 3 different isotropic materials (see fig.8) corresponding to:

- nerve bundles (orange color);
- soft shell, walls of blood vessels, and septa (blue color), which separates nerve bundles;
- central retinal artery and vein and small capillary (red color).

We build the model in plate interface (branch of Structural Mechanics). It allows to use of thin flat structures of a cross-section of the optic nerve, being loaded with intraocular pressure (IOP) $P_0$ in a direction out of the plane. In this calculation $P_0 = 80$ mmHg, which corresponds to IOP level during creation of corneal flap (step of vision correction), intraocular injection, or eyes rubbing [9, 10]. The reason to model such high pressure is to investigate influence of such a level of IOP on the stress-strain state in the area of central retinal blood vessels.

Figure 8 shows that geometry of the model is very difficult with no pattern or simple models. The outer edge of the plate is pinned for boundary conditions. So, zero displacement of that edge in any direction are prescribed, but rotations are free.

7. Results of finite element modeling
Figure 8 (on right) shows displacement of the cross-section of the optic nerve in a perpendicular direction.

Figure 9 shows stress on the surface of the cross-section of the optic nerve.

8. Conclusions
In the current paper, we used numerical modeling and graphical representation of the elastoplastic properties of circular transversely isotropic and plastic anisotropic plates. This method allowed us to show the solution to the problem of optimizing the selection of the parameters of transversal isotropy and plastic anisotropy under the condition of minimum stresses for surface stress functions.

We constructed asymptotic equations for its calculation using the yield criterion, taking into account the transversal isotropy and the SD effect for the elastoplastic bending of a circular plate. Obtained equations are universal and estimate the influence of the parameters
Figure 9. Midsurface von Mises stress of the cross-section of the optic nerve (left) and in the central zone around central retinal blood vessels (right)

of transversal isotropy and SD effect on the stress-strain state of any material satisfying the described conditions. Asymptotic equations allow a rapid evaluation of a plate’s stress state without heavy calculations, which is vital in engineering practice.

Collagen plate had shown plasticity areas for high level of IOP and that stress level in center of top and bottom surfaces of the plate doesn’t depend on boundary conditions. Plasticity spot appears only on the bottom surface of the plate, while appearing next to the edge of the plate on the top surface. This is very different from metals. Irreversible deformations appear in these areas and can lead to different eye diseases, including glaucoma.

FE modeling of the optic nerve shows that areas next to the central artery and vein are highly stressed, leading to a reduction in the size of the vein and artery’s cross-section. This means that high IOP harms blood circulation and can lead to eye diseases.

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