Toric Rigid Spaces

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Abstract
This paper gives a method to construct rigid spaces, which is similar to the method used to construct toric schemes.

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1 Introduction
Toric geometry provides an important way to see many examples and phenomena in algebraic geometry. Toric geometry studies toric schemes, which are a special type of schemes. At least, they are all rational. However they provide intuition for us to study algebraic geometry and they make computation much easier.

Toric schemes correspond to objects like the simplicial complexes studied in algebraic topology. To construct a toric scheme, we need a fan, which consists of (strongly convex rational polyhedral) cones. If a cone is contained in this fan, so are its faces. A strongly convex rational polyhedral cone is, by definition, a cone with an apex at the origin, generated by a finite number of vectors, and satisfies some conditions. Such a cone $\sigma$ provides a finite generates semigroup and this semigroup gives a noetherian (semigroup) algebra $\mathbb{Q}[S_{\sigma}]$ over $\mathbb{Q}$. Therefore each cone provides an affine scheme. Patching these schemes in some way, we get a separated integral scheme over $\mathbb{Q}$. Call it the toric scheme associated to the given fan.

Luckily, the authors find this method can be used to construct rigid spaces, too. Taking the $p$-adic completion of $\mathbb{Q}_p[S_{\sigma}]$, we get an affinoid algebra and then an affinoid rigid space. Patching all these affinoid rigid spaces, we get a rigid space, and call it the toric rigid space associated to the given fan. We find that its natural reduction is exactly the toric scheme over $\mathbb{F}_p$ associated to the same fan.

There are many problems occurring. For example, what is the relation between the toric rigid space that we get and the rigid analytification of the toric scheme defined over $\mathbb{Q}_p$ associated to the same fan? In general, they are not the same, but sometimes they are. How can we understand such a phenomenon? We shall point out that the toric rigid spaces that we construct are more natural than the latter.
Professor S. Bosch and B. Le Stum gave several courses on rigid geometry at Peking university. Thanks to their work, we develop our ideas in this paper.

2 Toric affinoid algebras and toric rigid spaces

2.1 Construction of toric rigid spaces

Let $K$ be a field with a complete non Archimedean absolute value and $\overline{K}$ be its algebraic closure. Assume its residue field $k$ is of characteristic $p$.

Let $N$ be a lattice which is isomorphic to $\mathbb{Z}^n$ for some positive integer $n$ and $\sigma$ be a strongly convex rational polyhedral cone in the vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. A strongly convex rational polyhedral cone is a cone with an apex at the origin, generated by a finite number of vectors; “rational” means that it is generated by vectors in the lattice $N$, and “strong” means that it contains no line through the origin.

Write $V$ for $N_\mathbb{R}$, then the dual cone of $\sigma$ is defined as

$$\sigma^\vee := \{ u \in V^\ast : < u, v > \geq 0, \forall v \in \sigma \}. \quad (1)$$

We denote $M = \text{Hom}(N, \mathbb{Z})$ as its dual lattice with dual pairing denoted by $< , >$.

Set

$$S_\sigma := \sigma^\vee \cap M, \quad (2)$$

which is a semigroup. As a result, $S_\sigma$ is finitely generated.

Given such an $S_\sigma$, we define an affinoid algebra $K\langle S_\sigma \rangle$ associated to it. $K\langle S_\sigma \rangle$ is just the $p$-adic completion of the (semigroup) algebra $K[S_\sigma]$. It’s equivalent to say that

$$K\langle S_\sigma \rangle := K \otimes_{\mathbb{R}} \lim_{\leftarrow} \mathbb{R}/p^n \mathbb{R}[S_\sigma]. \quad (3)$$

Here, $R$ is the integral ring of $K$. It is easy to check that it is a finitely generated commutative affinoid algebra. We call it the toric affinoid algebra associated to $\sigma$.

We denote $\{ \chi^u | u \in S_\sigma \}$ as a basis of the vector space $K\langle S_\sigma \rangle$, and we have

$$\chi^{u_1} \cdot \chi^{u_2} = \chi^{u_1 + u_2}.$$ 

The unit 1 is just $\chi^0$. Generators $\{ u_\lambda \}$ for the semigroup $S$ determine generators $\chi^{u_\lambda}$ for the $K$-algebra $K\langle S \rangle$. Denote $X_\lambda$ for $\chi^{u_\lambda}$ for simple, then

$$K\langle S \rangle = K\langle X_1, X_2, \ldots, X_n \rangle = K\langle \zeta_1, \zeta_2, \ldots, \zeta_n \rangle / I,$$

where $n$ is the number of the generators of $K$-algebra $K\langle S \rangle$, each $\zeta_i$ is an indeterminate element, and $I$ is an ideal of $K\langle \zeta_1, \zeta_2, \ldots, \zeta_n \rangle$.

Set

$$U_\sigma = \text{Sp}(K\langle S_\sigma \rangle) \quad (4)$$

as the corresponding rigid space.

Now we begin to consider a fan $\Delta$ in $N_\mathbb{R}$ and the rigid space defined according to it. A fan $\Delta$ is defined as the collection of “strongly convex rational polyhedral cones” satisfying the following two conditions:

1. Every face of a cone in $\Delta$ is a cone in $\Delta$.
2. The intersection of two cones in $\Delta$ is a face of each other.
When $\tau$ is a face of $\sigma$ in $\Delta$, $S_{\sigma}$ is contained in $S_{\tau}$. We can find a $u \in S_{\sigma}$ such that $\tau = \sigma \cap u^\perp$ and $S_{\tau} = S_{\sigma} + \mathbb{Z}_{\geq 0}(-u)$. Then we have $K(S_{\tau}) = (K(S_{\sigma}))_{\chi_{u^\perp}}$.

So we get an open embedding $i_\sigma^\tau : U_{\tau} \hookrightarrow U_{\sigma}$. We glue all these $U_{\sigma}$ ($\sigma \in \Delta$) by the following way. If $\sigma, \sigma_1 \in \Delta$ and $\sigma \cap \sigma_1 = \tau$, we glue $U_{\sigma}$ and $U_{\sigma_1}$ by the immersions $i_\sigma^\tau$ and $i_{\sigma_1}^\tau$. We denote the rigid space we get by $R(\Delta)$. Write $T(\Delta)$ for the corresponding toric scheme in the same sense as in [4].

Now we come to the following proposition.

**Proposition 1.** $R(\Delta)$ is a rigid space.

We want to show that all toric rigid spaces are separated. We need the following lemma.

**Lemma 1.** If $\sigma$ and $\tau$ are two strongly convex polyhedral cones in $N_\mathbb{R}$, we have $\sigma^\vee + \tau^\vee = (\sigma \cap \tau)^\vee$.

**Proof.** First we prove $\sigma^\vee + \tau^\vee \subseteq (\sigma \cap \tau)^\vee$. If $u \in \sigma^\vee$ and $v \in \tau^\vee$, and $t \in \sigma \cap \tau$, then $u, t > + < v, t \geq 0$ by the definition. So $u + v \in (\sigma \cap \tau)^\vee$.

Then we prove $\sigma^\vee + \tau^\vee \supseteq (\sigma \cap \tau)^\vee$. It is equivalent to show that $(\sigma^\vee + \tau^\vee)^\vee \subseteq ((\sigma \cap \tau)^\vee)^\vee = (\sigma \cap \tau)^\vee$.

From $\sigma^\vee \subseteq \sigma^\vee + \tau^\vee$, we get $\sigma = (\sigma^\vee)^\vee \supseteq (\sigma^\vee + \tau^\vee)^\vee$. And from $\tau^\vee \subseteq \sigma^\vee + \tau^\vee$, we get $\tau = (\tau^\vee)^\vee \supseteq (\sigma^\vee + \tau^\vee)^\vee$. Therefore, $\sigma \cap \tau \supseteq (\sigma^\vee + \tau^\vee)^\vee$.

From the proof of the above lemma, we can find it is also right that

$$(\sigma^\vee + \tau^\vee) \bigcap M = ((\sigma \cap \tau)^\vee) \bigcap M.$$  

(5)

Now we have sufficient preparations to prove the following theorem.

**Theorem 1.** $R(\Delta)$ is separated for a given fan $\Delta$ in $N_\mathbb{R}$.

**Proof.** \{ $U_{\sigma} : \sigma$ is a cone in $\Delta$ \} is an affinoid covering of $R(\Delta)$. We have to prove two facts that $U_{\sigma} \cap U_{\tau}$ is affinoid for $\sigma, \tau$ in $\Delta$, and that $\mathcal{O}(U_{\sigma} \cap U_{\tau})$ is generated by the canonical image of $\mathcal{O}(U_{\sigma})$ and $\mathcal{O}(U_{\tau})$.

Because $U_{\sigma} \cap U_{\tau} = U_{\sigma \cap \tau}$, we have got the first fact.

As $\mathcal{O}(U_{\sigma}) = K(S_{\sigma})$, $\mathcal{O}(U_{\tau}) = K(S_{\tau})$ and $\mathcal{O}(U_{\sigma \cap \tau}) = K(S_{\sigma \cap \tau})$, we only have to prove that $K(S_{\sigma} \cap S_{\tau} \cap S_{\sigma \cap \tau})$ can generate $K(S_{\sigma \cap \tau})$. It is obvious that $S_{\sigma \cap \tau}$ is the sum of $S_{\sigma}$ and $S_{\tau}$ due to the relation (5).

The rigid spaces we get are also integral rigid spaces.

**Theorem 2.** $R(\Delta)$ is integral for a given fan $\Delta$ in $N_\mathbb{R}$.

**Proof.** Since intersect of $U_{\sigma} (\sigma \in \Delta)$ is not empty, we only have to prove that $K(S_{\sigma})$ is integral, which is the next proposition.

**Proposition 2.** If $\sigma$ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$, $K(S_{\sigma})$ is integral.
Proof. Choose a basis $u_1, ..., u_n$ of $M_{\mathbb{R}}$, which lie in $\sigma^v \cap M$. We can define an order of $\sigma^v \cap M$ compatible with the additive operation in $\sigma^v \cap M$ in the following way. Let $u, v \in \sigma^v \cap M$, and write $u = \sum a_i u_i$, $v = \sum b_i u_i$, with $a_i, b_i \in \mathbb{Q}$. We say $u < v$, if and only if there exists an $i$, $1 \leq i \leq n$, such that $a_j = b_j$ for all $1 \leq j < i$ and $a_i < b_i$.

For any nonzero element $f$ in $K(S_\sigma)$, write $f = \sum c_{f,u} x^u$ with $c_{f,u} \in K$. Let $u_f \in \sigma^v \cap M$ be $\min \{ u \in \sigma^v \cap M; |c_{f,u}| \text{ maximal} \}$.

For two nonzero elements $f, g \in K(S_\sigma)$, it is easy to see that $|c_{f,g,u}| < |c_{f,u}| \cdot |c_{g,u}|$ for $u < u_f + u_g$, that $|c_{f,u}| \leq |c_{f,u}| \cdot |c_{g,u}|$ for $u > u_f + u_g$, and that $|c_{f,g,u} + u| = |c_{f,u}| \cdot |c_{g,u}|$. Therefore $fg$ is not zero and $u_{fg} = u_f + u_g$. We have finished the proof.

We write $pt$ for the fan defined by the origin. Then

$$K(S_{pt}) = K(X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}). \quad (6)$$

$U_{pt}$ is a group rigid space (a torus) and has an action on $R(\Delta)$. In fact $K(S_{pt})$ is a Hopf affinoid algebra over $K$. Its comultiplication $m^*$ is defined by $m^*(X_i) = X_i \otimes X_i$, its counit is $s(X_i) = X_i^{-1}$, and its counit is defined by $\varepsilon(X_i) = 1$ for $1 \leq i \leq n$. For any cone $\sigma$, there is a natural map $K(S_\sigma) \rightarrow K(S_{pt})$. It's easy to see that $m^*(K(S_\sigma)) \subseteq K(S_{pt}) \subseteq K(S_\sigma) \otimes K(S_{pt})$, which defines a natural action of $U_{pt}$ on $U_\sigma$. If $\tau$ is a face of $\sigma$, the action of $U_{pt}$ on $U_\tau$ is the same as the restriction of the action of $U_{pt}$ on $U_\sigma$. Therefore, patching all these actions, we get an action of $U_{pt}$ on $R(\Delta)$.

We point out that, since there is a natural affine covering over $R(\Delta)$, we can use it to calculate the Cech cohomology for any given coherent sheaf over $R(\Delta)$.

We can show that when the support of $\Delta$ is the whole $N_{\mathbb{R}}$, $R(\Delta)$ is a proper rigid space (see [4] for its proof).

For a given fan $\Delta$, we can also construct a toric scheme $T(\Delta)$ in the original sense, then its rigid analytification (definition of rigid analytification, see [1]) $RT(\Delta)$ is also a rigid space, but it is not isomorphic to $R(\Delta)$ in general. For example, for a simple cone $\sigma = \Delta$, $RT(\sigma)$ hasn't a finite covering, hence, $R(\sigma)$ is an affinoid space. But the authors don't know whether $R(\Delta)$ and $RT(\Delta)$ are isomorphic when the support of $\Delta$ is the whole $N_{\mathbb{R}}$.

### 2.2 Examples

In this subsection, we give two examples of toric rigid spaces.

**Example 1.** The projective rigid spaces $P_n$ over $\mathbb{Q}_p$.

1. $n = 2$

Assume $\mathbb{N} = \mathbb{Z} \times \mathbb{Z}$ with a basis $\{u_1, u_2\}$. We define three strongly convex rational polyhedral cones $\sigma_1, \sigma_2, \sigma_3$ in $N_{\mathbb{R}}$ in the following way. $\sigma_1$ is the cone generated by $u_1$ and $u_2$, $\sigma_2$ is the cone generated by $-u_1 - u_2$ and $u_2$, and $\sigma_3$ is the cone generated by $-u_1 - u_2$ and $u_1$. Let $\Delta$ be the fan

$$\{ \sigma_1, \sigma_2, \sigma_3, \sigma_1 \cap \sigma_2, \sigma_1 \cap \sigma_3, \sigma_2 \cap \sigma_3, \sigma_1 \cap \sigma_2 \cap \sigma_3 \}.$$

Write $X_i$ for $\chi_{u_i}$ ($i = 1, 2$) for simple. Then $\mathbb{Q}_p(S_{\sigma_1}) = \mathbb{Q}_p(X_1, X_2)$, $\mathbb{Q}_p(S_{\sigma_2}) = \mathbb{Q}_p(X_1^{-1}, X_2^{-1}X_1)$, and $\mathbb{Q}_p(S_{\sigma_3}) = \mathbb{Q}_p(X_2^{-1}, X_1^{-1}X_1)$. $R(\Delta)$ is the two dimensional projective rigid space $\mathbb{P}^2$ over $\mathbb{Q}_p$. $U_{\sigma_1}$ corresponds to $\{ [x_0, x_1, x_2] \in \mathbb{P}_2 : |x_1| \leq |x_0|, |x_2| \leq |x_0| \}$, $U_{\sigma_2}$ corresponds to $\{ [x_0, x_1, x_2] \in \mathbb{P}_2 : |x_0| \leq |x_1|, |x_2| \leq |x_1| \}$, and $U_{\sigma_3}$ corresponds to $\{ [x_0, x_1, x_2] \in \mathbb{P}_2 : |x_0| \leq |x_2|, |x_1| \leq |x_2| \}$. 


2. $n \geq 3$

It is similar to the case $n = 2$. Assume $N = \mathbb{Z}^n$ with a basis $\{u_1, u_2, \ldots, u_n\}$. There are $n + 1$ vectors $v_i (0 \leq i \leq n)$ with $v_i = u_i$ for $1 \leq i \leq n$ and $v_0 = -u_1 - u_2 \cdots - u_n$. For any proper subset $I$ of $\{0, 1, \ldots, n\}$, define

$$
\sigma_I := \begin{cases} 
\text{the cone of origin} & \text{if } I \text{ is empty,} \\
\text{the cone generated by } v_i (i \in I) & \text{if } I \text{ is not empty.}
\end{cases}
$$

(7)

And set

$$
\Delta := \{ \sigma_I | I \subset \{0, 1, \ldots, n\}\}.
$$

Then $R(\Delta)$ is the $n$ dimensional projective rigid space $\mathbb{P}^n$.

$R(\Delta)$ and $RT(\Delta)$ are the same this time.

**Example 2.** Hirzebruch surfaces.

Let $a$ be a fixed positive integer. Assume $N = \mathbb{Z} \times \mathbb{Z}$ with a basis $\{u_1, u_2\}$. We define four strongly convex rational polyhedral cones $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in $N_\mathbb{R}$ in the following way. $\sigma_1$ is the cone generated by $u_1$ and $u_2$, $\sigma_2$ is the cone generated by $u_1$ and $-u_2$, $\sigma_3$ is the cone generated by $-u_1 + au_2$ and $-u_2$, and $\sigma_4$ is the cone generated by $-u_1 + au_2$ and $u_2$. The four corresponding affinoid spaces are $\overline{Q}_p(S_{\sigma_1}) = \overline{Q}_p(X_1, X_2)$, $\overline{Q}_p(S_{\sigma_2}) = \overline{Q}_p(X_1, X_2^{-1})$, $\overline{Q}_p(S_{\sigma_3}) = \overline{Q}_p(X_1^{-1}, X_2^{-a}X_2^{-1})$, $\overline{Q}_p(S_{\sigma_4}) = \overline{Q}_p(X_1^{-1}, X_2^aX_2)$. Let $\Delta$ be the fan

$$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1 \cap \sigma_2, \sigma_1 \cap \sigma_4, \sigma_2 \cap \sigma_3, \sigma_3 \cap \sigma_4, \sigma_1 \cap \sigma_2 \cap \sigma_3 \cap \sigma_4\}.$$

We get a rigid space $R(\Delta)$ and call it a Hirzebruch surface.

Let $N_1 = \mathbb{Z}$ with a basis $\{v\}$. Let $\tau_1 = \{rv : r \geq 0\}$, $\tau_2 = \{rv : r \leq 0\}$, and $\Delta_1$ be the cone $\{\tau_1, \tau_2, \{0\}\}$. Then $R(\Delta_1)$ is just the one dimensional projective space. The linear map $N_\mathbb{R} \rightarrow N_{1, \mathbb{R}}$ defined by $ru_1 + su_2 \mapsto rv$ induces a map from the fan $\Delta_1$ to the fan $\Delta$, which determines a morphism $R(\Delta_1) \rightarrow R(\Delta_1)$. This morphism makes $R(\Delta)$ a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Especially, when $a = 0$, $R(\Delta)$ is the trivial bundle $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1$.

**3 Reductions of toric rigid spaces**

Reduction theory is very important in rigid geometry. In this section, we will study the reductions of toric rigid spaces. At first, let’s recall the reductions of affinoid algebras. For more details of reductions, see [2].

Assume $K$ is a field with a non-Archimedean valuation. Let $A$ be an affinoid algebra over $K$. There is a supremum (semi)-norm over $A$ defined in the following way. For any $f \in A$, define

$$
|f|_{\text{sup}} = \sup_{x \in \text{Max} A} |f(x)|.
$$

(8)

Then define the reduction $\widetilde{A}$ of $A$ to be $\{f \in A : |f|_{\text{sup}} \leq 1\}/\{f \in A : |f|_{\text{sup}} < 1\}$. Reduction of $\text{Sp}(A)$ is just defined to be $\text{Spec}(\widetilde{A})$.

We have the following lemma.

**Lemma 2.** Let $\sigma$ be a cone in $N_{\mathbb{R}}$ and $S_\sigma$ be as in section 2. For $f \in K\langle S \rangle$, write $f = \sum_{u \in \sigma} a_u \chi^u$ with $a_u \in K$, $a_u \rightarrow 0$. Then

$$
|f|_{\text{sup}} = \max_{u \in \sigma} |a_u|.
$$

(9)
Proof. It’s well known that, for any given valuation $| \cdot |$ over an integral affinoid algebra $A$, $|f|_{sup} = \lim_{n \to \infty} |f^n|^{\frac{1}{n^2}}, \forall f \in A$. We use this result to prove our assertion.

We write $|f| = \max_{u \in \sigma} |a_u|$ for $f = \sum_{u \in \sigma} a_u x^u$ in $K(S)$. It is easy to see that it defines a norm over $K(S)$. From the proof of proposition 2, we can show that $|f|^n = |f^n|$. Therefore, we get $|f|_{sup} = \max_{u \in \sigma} |a_u|$. \hfill \Box

Let $k$ be the residue field of $K$, then we have the following corollary.

**Corollary.** The reduction of $K(S_{\sigma})$ is $k[S_{\sigma}]$.

For a separated rigid space $X$, its reduction always depends on a choice of an affinoid covering $\{U_i\}$. Open subsets $U_i$ and their intersections are all affinoid, so they have reductions defined as above. Patching them together, we get a reduction of $X$. It is a scheme over $k$. We call it the reduction of $X$ according to (the affinoid covering) $\{U_i\}$. In general, different affinoid coverings give different reductions.

For a fan $\Delta$ in $N$, we have defined a toric rigid space $R(\Delta)$ in section 2. There is a natural affinoid covering $Sp(K(S_{\sigma})) (\sigma \in \Delta)$ of this rigid space $R(\Delta)$. Each $Sp(K(S_{\sigma}))$ has a reduction $Spec(k[S_{\sigma}])$. If $\sigma$ is a face of $\tau$, the open immersion $Sp(k(S_{\sigma})) \hookrightarrow Sp(k(S_{\tau}))$ induces an open immersion $Spec(k[S_{\sigma}]) \hookrightarrow Spec(k[S_{\tau}])$. Patching all the $Spec(S_{\sigma})$ according to these open immersions, we get a reduction of $R(\Delta)$. It’s easy to see that this reduction is exactly the toric scheme $T(\Delta)_k$ over $k$.

For example, we calculate the reduction of the projective rigid space $P_2$ over $\mathbb{Q}_p$ (example 1 in section 2). It’s patched by three spaces: $Spec(\mathbb{F}_p[x_1, x_2])$, $Spec(\mathbb{F}_p[x_1^{-1}, x_2^{-1}, x_2])$ and $Spec(\mathbb{F}_p[x_1^{-1}, x_2^{-1}, x_1])$. Let $X_1 = T_1/T_0$, $X_2 = T_2/T_0$, these spaces are corresponding to $\{t_1 : t_0 = \xi \neq 0\}$, $\{t_0 : t_1 : t_2 \in \mathbb{P}_2 | t_0 \neq 0\}$, $\{t_0 : t_1 : t_2 \in \mathbb{P}_2 | t_2 \neq 0\}$, $\{t_0 : t_1 : t_2 \in \mathbb{P}_2 | t_1 \neq 0\}$. They form an affine covering of the two dimensional projective spaces $P_2$ over $\mathbb{F}_p$.

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