Characteristic classes in the Chow ring

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Let $G$ be a reductive algebraic group over an algebraically closed field $k$. An algebraic characteristic class of degree $i$ for principal $G$-bundles on schemes is a function $c$ assigning to each principal $G$-bundle $E \to X$ an element $c(E)$ in the Chow group $A^iX$, natural with respect to pullbacks. These classes are analogous to topological characteristic classes (which take values in cohomology), and two natural questions arise. First, for smooth schemes there is a natural map from the Chow ring to cohomology, and we can ask if topological characteristic classes are algebraic. Second, because the notion of algebraic principal bundles on schemes is more restrictive than the notion of topological principal bundles, we can ask if there are algebraic characteristic classes which do not come from topological ones. For example, for rank $n$ vector bundles (corresponding to $G = GL(n)$), the only topological characteristic classes are polynomials in the Chern classes, which are represented by algebraic cycles, but until now it was not known if these were the only algebraic characteristic classes ([V, Problem (2.4)]). In this paper we describe the ring of algebraic characteristic classes and answer these questions.

One subtlety which does not occur in topology is that there are two natural notions of algebraic principal $G$-bundles on schemes, those which are locally trivial in the étale topology and those which are locally trivial in the Zariski topology. (Of course for groups which are special in the sense of [Sem-Chev], all principal bundles are Zariski locally trivial. Tori, $GL(n)$, $SL(n)$, and $Sp(2n)$ are all examples of special groups.) Let $C(G)$ denote the ring of characteristic classes for principal $G$-bundles locally trivial in the

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étale topology, and \( \mathcal{C}_{Zar}(G) \) the analogous ring for bundles locally trivial in the Zariski topology. Since any bundle which is locally trivial in the Zariski topology is locally trivial in the étale topology, there is a natural homomorphism \( \mathcal{C}(G) \to \mathcal{C}_{Zar}(G) \), but the two rings can differ, as we discuss below.

To state our results, we need some notation. Let \( T \) be a maximal torus of \( G \), \( B \) a Borel subgroup containing \( T \), \( U \) the unipotent radical of \( B \), and \( W \) the Weyl group. Let \( \hat{T} \) denote the group of characters of \( T \), \( \hat{\mathcal{T}} \) the symmetric algebra over \( \mathbb{Z} \) of the free abelian group \( \hat{T} \). Let \( S(\hat{T}) \) be the symmetric algebra over \( \mathbb{Z} \) of the free abelian group \( \hat{T} \). If \( \lambda: T \to \mathbb{G}_m \) is a character of \( T \), we extend \( \lambda \) to a character of \( B \) (still denoted \( \lambda \)) by making \( \lambda|_U \) trivial. Let \( k_\lambda \) be the corresponding 1-dimensional representation of \( B \) on the ground field \( k \). If \( E \to X \) is a principal \( G \)-bundle, we can form a line bundle \( L_\lambda = E \times^B k_\lambda \to E/B \). The map \( \lambda \mapsto c_1(L_\lambda) \) is a group homomorphism from \( \hat{T} \) to \( A^1(E/B) \). This extends to a ring homomorphism \( \Phi_E: S(\hat{T}) \to A^*(E/B) \).

**Theorem 1**

(a) Let \( E \to X \) be a Zariski locally trivial principal \( G \)-bundle, and let \( f \in S(\hat{T})^W \). There is a unique element \( c_f(E) \in A^*X \) which pulls back to \( \Phi_E(f) \in A^*(E/B) \). The assignment \( E \mapsto c_f(E) \) is a characteristic class for Zariski locally trivial principal \( G \)-bundles. If \( E \to X \) is not Zariski locally trivial, the same statements hold after tensoring with \( \mathbb{Q} \).

(b) The map \( S(\hat{T})^W \to \mathcal{C}_{Zar}(G) \), \( f \mapsto c_f \), is an isomorphism.

(c) The map \( S(\hat{T})^W \otimes \mathbb{Q} \to \mathcal{C}(G) \otimes \mathbb{Q} \), \( f \mapsto c_f \), is an isomorphism.

The following is an immediate consequence of our result for \( G = GL(n) \).

**Corollary 1** The only algebraic characteristic classes for vector bundles are polynomials in the Chern classes.

In topology the ring of characteristic classes is just \( H^*(BG) \) where \( BG \) is the classifying space of \( G \). Since \( H^*(BG; \mathbb{Q}) = S(\hat{T})^W \otimes \mathbb{Q} \), the theorem says that \( \mathcal{C}(G) \otimes \mathbb{Q} \simeq \mathcal{C}(G)_{Zar} \otimes \mathbb{Q} \simeq H^*(BG, \mathbb{Q}) \). Borel showed that with integer coefficients \( H^*(BG; \mathbb{Z})/Tor \) injects into \( S(\hat{T})^W \) (cf. [1]). This implies the following corollary.

**Corollary 2** All characteristic classes (mod torsion) from topology are algebraic when applied to Zariski locally trivial principal \( G \)-bundles.
Discussion. Vistoli ([V], cf. [A-H]) constructs an injective map from $S(T)^W \otimes \mathbb{Q} \to \mathcal{C}(G) \otimes \mathbb{Q} \simeq \mathcal{C}_{Zar}(G) \otimes \mathbb{Q}$. What is new in Theorem 1(c) is that these are the only characteristic classes with rational coefficients.

Given a representation $V$ of $G$ we can construct characteristic classes which assign to a principal $G$-bundle $E \to X$ a polynomial in the Chern classes of the associated vector bundle $E \times^G V \to X$. The classes Vistoli constructs are $\mathbb{Q}$-linear combinations of these. Combining Vistoli’s result with Theorem 1(c) shows that every element of $\mathcal{C}(G) \otimes \mathbb{Q}$ is of this form.

In general, we do not know if integral polynomials in Chern classes of associated vector bundles generate $\mathcal{C}(G)$ (in part because we do not have a complete description of $\mathcal{C}(G)$ for non-special groups). However, for the classical groups other than $SO(2n)$, $\mathcal{C}_{Zar}(G)$ is generated in this way. For $G = SO(2n)$ the only generator of $\mathcal{C}_{Zar}(G)$ which cannot be so constructed is the Euler class, which was constructed in [E-G] (this paper gives an implicit construction of the Euler class as well). For the exceptional groups, we do not know which characteristic classes come from representations.

In topology, the ring of characteristic classes with arbitrary coefficients can be computed from the ring of characteristic classes with integer coefficients. By contrast, we do not know how to use Theorem 1 to compute the ring $\mathcal{C}_{Zar}(G; R)$ of characteristic classes with values in $A^*(G; R)$, for an arbitrary ring $R$. Note that $\mathcal{C}_{Zar}(G; R)$ need not be isomorphic to $\mathcal{C}_{Zar}(G) \otimes R$. $G = SO(n)$ is an example, since $\mathcal{C}_{Zar}(G; \mathbb{Z}/2\mathbb{Z})$ contains Stiefel-Whitney classes (E-G), which are not in $\mathcal{C}_{Zar}(G) \otimes \mathbb{Z}/2\mathbb{Z}$.

Examples. For $G$ equal to $GL(n)$, $Sp(2n)$, or $SO(2n + 1)$, $\mathcal{C}_{Zar}(G)$ is isomorphic to the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$, where the $t_i$ are constructed from the defining representation of $G$, as follows. For $GL(n)$, we can take $t_i$ to be the $i$-th Chern class of the associated vector bundle. For $Sp(2n)$ and $SO(2n + 1)$, $t_i$ is the $2i$-th Chern class.

For $G = SO(2n)$, $\mathcal{C}_{Zar}(G)$ is also isomorphic to the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$. For $i < n$, $t_i$ is the $2i$-th Chern class of the associated rank $2n$ vector bundle, but $t_n$ is the Euler class; this class satisfies the relation $t_n^2 = (-1)^n c_{2n}$. The Euler class cannot be expressed as a polynomial in the Chern classes of an associated vector bundle. If it could, then it would be defined in $\mathcal{C}(G)$. However, Totaro [12] has shown that in general only some multiple of the Euler class is defined in $\mathcal{C}(G)$. This example shows that the rings $\mathcal{C}(G)$ and $\mathcal{C}_{Zar}(G)$ are not isomorphic in general.

From the examples above, one might suspect that $\mathcal{C}_{Zar}(G)$ is always
a polynomial ring. However, Feshbach [F] shows that for $G = \text{Spin}(10)$, $H^*(BG)/\text{Tors}$ is not a polynomial ring, but is isomorphic to $S(\hat{T})^W$. Since we prove $C_{\text{Zar}}(G) = S(\hat{T})^W$, it follows that $C_{\text{Zar}}(G)$ is not always a polynomial ring. Feshbach also shows that the map $H^*(BG)/\text{Tors} \rightarrow S(\hat{T})^W$ is not surjective for $G$ equal to Spin(11) or Spin(12). This shows that there are characteristic classes for Zariski locally trivial $G$-bundles which cannot be defined in topology.

**Remark on the proof.** We only present the proof of Theorem 1 in the Zariski locally trivial case. The same proof works for general principal bundles after tensoring with $\mathbb{Q}$. The main point is that if $f : Y \rightarrow X$ is a Zariski locally trivial flag bundle then $A^*Y$ is free over $A^*X$. However, for general flag bundles this is not true.

**Notation:** A “scheme” will mean an algebraic scheme over a fixed algebraically closed field $k$. $A^*X$ refers to the operational Chow ring of [Fulton, Chapter 17]. For smooth schemes, $A^*X$ coincides with the usual Chow ring. $A_*X$ refers to the Chow groups of [Fulton, Chapter 2]. The evaluation of a characteristic class $c$ on a principal bundle $E \rightarrow X$ will be denoted $c(E \rightarrow X)$, or simply $c(E)$ when there is no confusion about the base; $c(E \rightarrow X)$ is always an element of $A^*X$.

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**Reduction to the quasi-projective case** We can define the ring of characteristic classes for principal $G$-bundles over quasi-projective schemes as above. Note that this ring might not be the same as the ring of characteristic classes of bundles over all schemes, since there could be characteristic classes over quasi-projective schemes that cannot be defined for all schemes. Conversely, there might be nonzero characteristic classes which are zero when applied to any bundle over a quasiprojective scheme. The following lemma says that neither of these possibilities occurs.

**Lemma 1** Let $G$ be an algebraic group. The ring of characteristic classes of principal $G$-bundles over quasi-projective schemes is the same as the ring of characteristic classes of principal $G$-bundles over arbitrary schemes. The
same statement is true for characteristic classes of Zariski locally trivial principal $G$-bundles.

Recall ([Fulton, Definition 18.3 and Lemma 18.3]) that every scheme $X$ has a Chow envelope $p : X' \to X$, which is a quasi-projective scheme such that for every subvariety $V \subset X$ there is a subvariety $V' \subset X'$ mapping birationally to $V$. The most important property of Chow envelopes is that the pullback $p^* : A^*X \to A^*X'$ is injective. Moreover, $X'$ may be chosen so that there exists an open set $U \subset X$ such that $p$ is an isomorphism over $U$ and the complement $Z = X - U$ has smaller dimension than $X$. For the sake of brevity, we will call such an $X'$ a birational Chow envelope of $X$. In this case, the image of $p^*$ in $A^*X'$ has been described by Kimura ([K]).

Proof: The proof of the lemma is in two steps. We first show that if a characteristic class $c$ for bundles over arbitrary schemes is zero for all bundles over quasi-projective schemes, then it is zero. We then show that a characteristic class $c$ for bundles over quasi-projective schemes is the restriction to such bundles of a characteristic class for bundles over arbitrary schemes.

Step 1: Suppose $E \to X$ is a principal bundle over an arbitrary scheme. Let $p : X' \to X$ be a Chow envelope, and let $E' \to X'$ be the pullback bundle. By hypothesis, $0 = c(E' \to X') = p^*(E \to X)$. Since the pullback $p^* : A^*X \to A^*X'$ is injective, $c(E \to X) = 0$ as desired.

Step 2: It suffices to show the following. Suppose $E \to X$ is a $G$-bundle and $p : X' \to X$ a map with $X'$ quasiprojective. Let $c$ be a characteristic class for bundles over quasi-projective schemes. Then $c(p^*E \to X') \in A^*X'$ is the pullback of a class in $A^*X$.

Suppose the statement is true for all schemes of dimension smaller than $X$. We will first prove the statement assuming that $X' \to X$ is a birational Chow envelope. Let $Z' = p^{-1}Z$, and consider the following diagram:

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X' \\
q \downarrow & & p \downarrow \\
Z & \xrightarrow{i} & X
\end{array}
\]

By [K, Theorem 3.1], to show that $c(p^*E \to X') \in A^*X'$ is the pullback of a class from $A^*X$ is equivalent to showing that $i'^*c(p^*E \to X')$ is the pullback of a class from $A^*Z$. But $i'^*c(p^*E \to X') = c(q^*i^*E \to Z')$, which is a pullback from $A^*Z$ by the inductive hypothesis. This proves the statement.
if $X' \to X$ is a birational Chow envelope. To deduce the statement for arbitrary $X'$, let $\tilde{X} \to X$ be a birational Chow envelope, and consider the Cartesian diagram:

\[
\begin{array}{ccc}
\tilde{X}' & \xrightarrow{q'} & X' \\
\tilde{p} \downarrow & & \downarrow p \\
\tilde{X} & \xrightarrow{q} & X
\end{array}
\]

Let $E'$, $\bar{E}$, and $\bar{E}'$ be the pullback bundles. Then there exists $\alpha \in A^*X$ such that $c(\bar{E}') = \tilde{p}^*q^*\alpha = q'^*p^*\alpha$. But $c(\bar{E}') = q'^*c(E')$. Since $\tilde{X}' \to X'$ is an envelope, $q'^*$ is injective, so $c(E') = p^*\alpha$, as desired. $\triangle$

By the above lemma, to calculate rings of characteristic classes it suffices to do so for bundles over quasi-projective schemes. For the remainder of this paper we will therefore assume that all schemes are quasi-projective.

**Classifying Spaces in Algebraic Geometry**  

The purpose of this section is to describe a construction, due to Totaro, of an algebro-geometric substitute for the classifying spaces of topology. In order to say what we mean by a substitute, it is helpful to recall what classifying spaces are. In topology, given a contractible space $EG$ on which $G$ acts freely, the quotient space $BG$ is called the classifying space for $G$, and the bundle $EG \to BG$ is called the universal bundle. A principal $G$-bundle over any space $X$ is pulled back from the universal bundle by a classifying map $X \to BG$, and homotopic maps induce the same pullback bundle. Hence there is an isomorphism of $H^*BG$ with (cohomology) characteristic classes of principal $G$-bundles.

Because the topological spaces $EG$ and $BG$ are infinite dimensional, it seems unreasonable to expect to find a bundle of schemes which is universal in the category of schemes. The next best thing would be to find a directed system of bundles of schemes $E_n \to B_n$, such that any principal $G$-bundle of schemes pulls back from some $E_n \to B_n$. However, we do not know how to produce this. What Totaro produces instead is a directed system of bundles as above, with the following property: for any principal $G$-bundle $E \to X$, there is a map $X' \to X$ with fibers isomorphic to $\mathbb{A}^m$ such that the pullback bundle $E' \to X'$ is pulled back from one of the bundles in the directed system

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There is a classifying space $BG$, which is a simplicial scheme whose Chow groups can be defined by the work of Bloch. However, computing these Chow groups seems difficult.
by a map $X' \to B_n$. Moreover, in this situation, for fixed $i$, the Chow groups $A^i(B_n)$ are all isomorphic for $n$ sufficiently large. By abuse of notation, we call this group $A^i(BG)$.

The directed system above is Totaro’s algebro-geometric substitute for $BG$. The system possesses the following two properties, analogous to ones familiar from topology. First, there is a map $A^*(BG) \to C(G)$, $\alpha \mapsto c_\alpha$, defined as follows. Because the bundle $X' \to X$ has fibers isomorphic to $A^m$, by [G, Theorem 8.3], the Chow groups $A^*X$ and $A^*X'$ are isomorphic. Thus, given $\alpha \in A^i(BG)$, pullback gives a class, which we denote $c_\alpha(E)$ in $A^iX' \cong A^iX$. The class $c_\alpha(E)$ in $A^iX$ depends only on the class in $A^i(BG)$ and on the bundle $E \to X$, not on the choice of $X' \to X$ or the classifying map $X' \to B_n$. Moreover, the assignment $E \mapsto c_\alpha(E)$ is natural with respect to pullbacks. Hence $c_\alpha$ is a characteristic class. Now arguing as in topology shows the following.

**Theorem 2 (Totaro)** The map $A^*(BG) \to C(G)$ is an isomorphism.

Because of this theorem, we will use the notations $A^*(BG)$ and $C(G)$ interchangeably.

The other nice property that this directed system possesses is simply that by replacing $X$ by $X'$ one can assume that the bundle $E \to X$ is pulled back from some bundle in the directed system. This enables one to reduce some computations to computations in the directed system.

By abuse of notation we will write $EG \to BG$ as schemes, where we really mean we are working in some bundle in the directed system. This should lead to no confusion, since we will never use the topological notion of classifying space.

We give some details of Totaro’s construction, since we will need it to compute the groups $A^*(BG)$. If $V$ is a representation of $G$ let $V^s$ be the set of points $v \in V$ such that $G \cdot v$ is closed in $V$. Let $V^{sf}$ be the set of $v \in V^s$ with trivial stabilizer. $V^{sf}$ is an open (possibly empty) subset of $V$. Since the orbits in $V^{sf}$ are closed in $V$ and have the same dimension as $G$, there is a geometric quotient $V^{sf} \to V^{sf}/G$ (cf. [GIT, Chapter 1, Appendix B]). Furthermore, since the action of $G$ on $V^{sf}$ is free, $V^{sf} \to V^{sf}/G$ is actually a principal $G$-bundle. These bundles form the directed system; the morphisms are just inclusions. Given a representation $V$ such that $V^{sf}$ is non-empty (we will see below that such representations exist), let $W = V \oplus V$. Then $(V^{sf} \oplus V) \cup (V \oplus V^{sf}) \subset W^{sf}$. Hence by replacing $V$ by a direct sum of

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copies of $V$ we may assume that $V - V^{sf}$ has arbitrarily high codimension. Totaro shows that $A^i(V^{sf}/G)$ does not depend on the choice of $V$ as long as the codimension of $V - V^{sf}$ is greater than $i$. His argument uses Bogomolov’s double fibration (cf. the proof of Lemma 2).

The classifying map is constructed as follows. Suppose $E \to X$ is a principal $G$-bundle. Then there exists a bundle $X' \to X$, with fibers $A^m$, such that the total space $E' = E \times_X X'$ is affine. Because $E'$ is affine, there is a representation $V$ of $G$ and a $G$-equivariant embedding $E' \to V$. ($V$ may be taken to be a subspace of the space of regular functions on $E'$.). Since the action of $G$ on $E'$ is closed and free, the map $E' \to V$ has image in $V^{sf}$ (in particular showing there exist $V$ with $V^{sf}$ non-empty). Thus $E' \to X'$ is the pullback of the principal $G$ bundle $V^{sf} \to V^{sf}/G$.

Of particular importance is the case when $G = T$ is a torus. Then $A^*BT = C_{zar}(T)$, since all principal $T$-bundles in algebraic geometry are locally trivial in the Zariski topology [Sem-Chev]. There is a group homomorphism $\hat{T} \to A^1BT$ taking a character of $T$ to the first Chern class of the associated line bundle on $BT$. This extends to a ring homomorphism $S(\hat{T}) \to A^*BT$.

**Lemma 2** The map $S(\hat{T}) \to A^*BT$ is an isomorphism.

Proof: Assume first that $T = \mathbb{G}_m$. In this case, we can take $V$ to be a product of $N$ copies of the 2 dimensional representation acted on by the torus with weights 1 and $-1$. The complement of $(V^{sf})$ has codimension $2N - 1$.

Claim: For $i < 2N - 1$, $A^i(V^{sf}/T) = A^i(\mathbb{P}^{2N-1})$.

Proof of claim: We use Bogomolov’s double fibration argument. Let $W$ be the 2$N$-dimensional representation of $T$ with all weights equal to 1. Then $(W - \{0\})/T = (\mathbb{P}^{2N-1})$, and $(V + (W - \{0\}))/T$ is an affine bundle over $\mathbb{P}^{2N-1}$ so $A^i((V + (W - \{0\}))/T) = A^i(\mathbb{P}^{2N-1})$. On the other hand $A^i((V^{sf} + W)/T) = A^i(V^{sf}/T)$. Thus for $i < 2N - 1$, $A^i(V^{sf}/T) = A^i(\mathbb{P}^{2N-1})$.

Hence $A^*BT = \mathbb{Z}[t]$ where $t$ is a generator of $A^1(\mathbb{P}^n)$. $S(\hat{T})$ is also a polynomial ring in one variable and the above map takes a generator to a generator. This proves the lemma for $T = \mathbb{G}_m$. In the general case, $T$ will be a product of 1-dimensional tori $T_1, \ldots, T_n$, and $S(\hat{T})$ decomposes as the tensor product of the $S(\hat{T}_i)$. On the other hand, $BT$ is the direct product of the $BT_i$ and $A^*BT$ decomposes as the tensor product of the $A^*(BT_i)$. The map $S(\hat{T}) \to A^*BT$ is compatible with these tensor product decompositions and hence is an isomorphism. △
We will also be interested in the case of $B$-bundles, where $B$ is a Borel subgroup of $G$. As is the case with tori, principal $B$-bundles are locally trivial in the Zariski topology, so we do not have to distinguish between $A^*BB$ and $C_{Zar}(B)$. If $E \to X$ is a $B$-bundle, then $E/U \to X$ is a $T$-bundle and $E \to E/U$ is an affine bundle. In particular, characteristic classes of $B$-bundles are uniquely determined by a characteristic class for $T$-bundles. This fact can be summarized in the following useful lemma.

Lemma 3 $A^*BB = A^*BT = S(\hat{T})$. $\triangle$

A bound on characteristic classes of arbitrary bundles Define a map from principal $T$-bundles to Zariski locally trivial principal $G$-bundles by associating to a $T$-bundle $E \to X$ the $G$-bundle $E \times^T G \to X$. The associated bundle is Zariski locally trivial, since the original $T$-bundle is Zariski locally trivial by [Sem-Chev]. Next define a map $C_{Zar}(G) \to A^*BT$, $c \mapsto c_T$ by the formula

$$c_T(E \to X) = c(E \times^T G \to X).$$

The Weyl group $W$ acts on the set of principal $T$-bundles as follows: If $Y \to X$ is a $T$-bundle define $wY \to X$ as having total space $Y$ but with twisted $T$ action given by

$$y \cdot_w t = y(wtw^{-1})$$

(where the action on the right is the original action). The formula $wc(Y \to X) = c(w^{-1}Y \to X)$ defines a $W$ action on $A^*BT$ which is compatible with the $W$ action on the representation ring $S(\hat{T})$.

Lemma 4 The map $C_{Zar}(G) \to A^*BT$, $c \mapsto c_T$ is injective, with image in $A^*BT^W$.

Proof: Suppose $c_T = 0$. Let $p : E \to X$ by any Zariski locally trivial $G$-bundle. Let $q : E/T \to X$ be the associated $G/T$-bundle. Now, $E \to E/T$ is a $T$-bundle. By assumption, $c_T(E \to E/T) = 0$. Thus, by definition, $c(E \times^T G \to E/T) = 0$. But $E \times^T G \simeq q^*E$ as $G$-bundles over $E/T$. Hence $c(q^*E \to E/T) = q^*(c(E \to T)) = 0$. But $q^*$ is injective because $q$ factors into a composition $E/T \to E/B \to X$; the first map is an affine bundle and the second map is a proper locally trivial bundle. Thus $c(E \to X) = 0$. 9
Since $E \to X$ was arbitrary, $c = 0$, so the map is injective. To see that $c_T$ is $W$-invariant notice that the map
\[(w^{-1}Y) \times^T G \to Y \times^T G\]
sending $(y, g) \mapsto (y, w^{-1}g)$ is an isomorphism of $G$-bundles. Hence $wc_T = c_T$.△

**Proof of the theorem**

By Lemma 4 we know that $C_{Zar}(G) \subset S(\hat{T})^W$. It remains to show that any $W$-invariant polynomial $f \in S(\hat{T})$ defines a characteristic class. In particular we must show that if $E \to X$ is a locally trivial $G$-bundle then the class $p = \Phi_E(f)$ in $A^*(E/B)$ is a pullback from $X$.

By [K], because the $G/B$-bundle $E/B \to X$ is locally trivial, showing $\Phi_E(f)$ is a pullback is equivalent to showing that $\pi_1^*p = \pi_2^*p$ in the following diagram.

\[
\begin{array}{ccc}
E/B \times_X E/B & \xrightarrow{\pi_1} & E/B \\
\downarrow \pi_2 & & \downarrow \\
E/B & \to & X
\end{array}
\]

The theorem follows from the following lemma.

**Lemma 5** $\pi_1^*p = \pi_2^*p$.

**Proof:** First note that by [V] (cf. [A-H]), $S(\hat{T})^W \otimes \mathbb{Q} \subset A^*BG \otimes \mathbb{Q} \subset C_{Zar}(G) \otimes \mathbb{Q}$. Thus, there is some constant $n \in \mathbb{Z}$ such that $nf$ is a characteristic class, so $n(\pi_1^*p - \pi_2^*p) = 0$ for any principal bundle $E \to X$ (this bundle need not be locally trivial).

Since the bundle $E \to X$ is pulled back from the universal bundle over $BG$, it suffices to prove the lemma for $X = BG$. In this case, $E/B = BB$ and our fiber square becomes

\[
\begin{array}{ccc}
BB \times_{BG} BB & \xrightarrow{\pi_1} & BB \\
\downarrow \pi_2 & & \downarrow \\
BB & \to & BG
\end{array}
\]

Now, $BB \times_{BG} BB \to BB$ is a fiber bundle with fiber $G/B$. The structure group of this bundle reduces to $B$, so this bundle is Zariski locally trivial. Since $G/B$ has a decomposition into affine cells, Proposition [II] below implies that $A^*(BB \times_{BG} BB) \cong A^*(G/T) \otimes A^*BB$. Since $A^*(G/B)$ and $A^*BB$ are both torsion-free, so is $A^*(BB \times_{BG} BB)$. Thus, $\pi_1^*p - \pi_2^*p = 0$, as desired. △
Proposition 1 Let \( f : Y \to X \) be a smooth proper locally trivial fibration with \( X \) smooth, whose fiber \( F \) has a decomposition into affine cells. Then \( A^*Y \) is (non-canonically) isomorphic to \( A^*X \otimes A^*F \) as an abelian group.

Proof: Since everything is smooth we may identify the Chow cohomology \( A^*Y \) with the Chow homology \( A_*Y \). Let \( U \subset X \) be an open set over which \( f \) is trivial. Since \( F \) has a cellular decomposition, \( A^*(f^{-1}(U)) \simeq A^*U \otimes A^*F \) (cf. [Fulton, Example 1.10.2]). Let \( b_1, \ldots, b_n \) be a basis for \( A_*F_{x_0} \) where \( F_{x_0} \) is the fiber over a point \( x_0 \in U \). Choose representatives \( b_i = \sum a_{ij}[V_{ij}] \), and set \( B_i = \sum a_{ij}[U \times V_{ij}] \in A_*(Y) \). We claim the \( B_1, \ldots, B_n \) form a basis for \( A_*Y \) over \( A_*X \).

To prove the claim argue as follows. Let \( F_x \) be any fiber. Since \( F_x \) is algebraically equivalent to any other fiber, the restrictions of the \( B_i \) to \( F_x \) are algebraically equivalent to their restrictions to \( F_{x_0} \). However, because \( F \) has an affine cellular decomposition, the group of cycles modulo algebraic equivalence is equal to the Chow group. Thus, the classes \( B_i \) restrict to a basis for the Chow group of every fiber.

The proposition now follows from the following algebraic version of the Leray-Hirsch theorem.

Lemma 6 Let \( f : Y \to X \) be a smooth proper locally trivial fibration whose fiber \( F \) has a decomposition into affine cells. Let \( \{B_i\} \in A^*Y \) be a collection of classes that restrict to a basis of the Chow groups of the fibers. Then \( \{B_i\} \) form a basis for \( A_*Y \) over \( A_*X \), i.e. every \( y \in A_*Y \) has a unique expression of the form \( y = \sum B_i \cap f^*x_i \) with \( x_i \in A_*X \) (This makes sense even when \( X \) is singular, so that \( A_*X \) is not a ring.)

Proof: This is an easy generalization of [E-S, Lemma 2.8]. ∆

References

[A-H] M. Atiyah, F. Hirzebruch, Vector bundles and homogeneous spaces in Proc. Symp. Pure Math 3, American Mathematical Society (1961), 7-38.

[B] S. Bloch, Algebraic cycles and higher K-theory, Adv. Math. 61 (1986), 267-304.
[E-G] D. Edidin, W. Graham, *Characteristic classes and quadric bundles*, Duke Math. J., to appear.

[E-S] G. Ellingsrud and S. Strømme, *The Chow ring of a geometric quotient*, Ann. Math. **130** (1989), 159-187.

[F] M. Feshbach, *The image of $H^\ast(BG,\mathbb{Z})$ in $H^\ast(BT,\mathbb{Z})$ for $G$ a compact Lie group with maximal torus $T$*, Topology **20** (1981), 93-95.

[Fulton] W. Fulton, *Intersection Theory*, Ergebnisse, 3. Folge, Band 2, Springer Verlag (1984).

[G] H. Gillet, *Riemann-Roch theorems for higher $K$-theory*, Adv. Math. **40** (1981), 203-289.

[K] S. Kimura, *Fractional intersection and bivariant theory*, Comm. Alg. **20** (1992) 285-302.

[GIT] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd enlarged edition, Springer-Verlag (1994).

[T1] B. Totaro, *The Chow ring of the symmetric group*, preprint 1994.

[T2] B. Totaro, private communication.

[Sem-Chev] *Anneau de Chow et applications*, Seminaire Chevalley, Secrétariat mathématique, Paris (1958).

[V] A. Vistoli, *Characteristic classes of principal bundles in algebraic intersection theory*, Duke Math J. **58**(1989), 299-315.