Severi varieties on blow-ups of the symmetric square of an elliptic curve

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Abstract
We prove that certain Severi varieties of nodal curves of positive genus on general blow-ups of the twofold symmetric product of a general elliptic curve are nonempty and smooth of the expected dimension. This result, besides its intrinsic value, is an important preliminary step for the proof of nonemptiness of Severi varieties on general Enriques surfaces.

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degenerations, elliptic ruled surfaces, nodal curves, Severi varieties

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1 | INTRODUCTION

Let $S$ be a smooth, projective complex surface and $\xi \in \text{Num}(S)$. Let

$$p_a(\xi) = \frac{1}{2} \xi \cdot (\xi + K_S) + 1$$

be the arithmetic genus of $\xi$. If $L$ is a line bundle or divisor on $S$ with class $\xi$ in $\text{Num}(S)$, we set $p_a(L) = p_a(\xi)$. We denote by $V^\xi(S)$ the locus in the Hilbert scheme of $S$ parameterizing the curves $C$ on $S$ such that the class of $O_C(C)$ in $\text{Num}(S)$
coincides with \( \xi \). Assume that \( L \) is a line bundle or divisor on \( S \) with class \( \xi \) in \( \text{Num}(S) \) and that \( p_a(L) \geq 0 \). For any integer \( \delta \) satisfying \( 0 \leq \delta \leq p_a(\xi) \), we denote by \( V_{\delta}^S(S) \) the Severi variety parameterizing irreducible \( \delta \)-nodal curves contained in \( V^S(S) \). This is a possibly empty locally closed set in \( V^S(S) \).

Let \( V \) be an irreducible component of \( V^S(S) \) and, for any \( \delta \) such that \( 0 \leq \delta \leq p_a(\xi) \), let \( V_{\delta} \) be an irreducible component of \( V \cap V_{\delta}^S(S) \). It is well known that

\[
\dim(V_{\delta}) \geq \dim(V) - \delta,
\]

where the right-hand side is called the expected dimension of \( V_{\delta} \). Moreover, if the equality holds in \( (1.1) \), then, for all \( 0 \leq \delta' \leq \delta \), the closure of the intersection of \( V_{\delta'}^S(S) \) with \( V \) contains \( V_{\delta} \), and any of its components whose closure contains \( V_{\delta} \) has the expected dimension \( \dim(V) - \delta' \) (see [25, Theorem 6.3]).

Severi varieties were introduced by Severi in Anhang F of [26], where he proved that all Severi varieties of irreducible \( \delta \)-nodal curves of degree \( d \) in \( \mathbb{P}^2 \) are nonempty and smooth of the expected dimension. Severi also claimed irreducibility of such varieties, but his proof contains a gap. The irreducibility was proved by Harris [17] more than 60 years later.

Severi varieties on other surfaces have received much attention in recent years, especially in connection with enumerative formulas computing their degrees.

Nonemptiness is known to hold for all Severi varieties associated to big and nef classes on Del Pezzo surfaces (as well as rational surfaces under certain assumptions) by [16, Theorems 3-4] and for Hirzebruch surfaces, a result implicitly contained in [27, section 3]. In both cases of Del Pezzo and Hirzebruch surfaces, all Severi varieties are smooth of the expected dimensions, cf., for example, [27, Lemma 2.9] or [7, p. 45]. Moreover, all Severi varieties of Hirzebruch surfaces are irreducible [31], and Severi varieties parameterizing rational curves on Del Pezzo surfaces of degrees \( \geq 2 \) are irreducible as well [30]; the same holds true for general Del Pezzo surfaces of degree 1, except for the Severi variety parameterizing rational curves in the anticanonical class [29, Corollary 6.4].

On a general primitively polarized K3 surface \((S, \xi)\), all Severi varieties \( V_{n\xi}^S(S) \), where \( 0 \leq \delta \leq p_a(n\xi) \), are nonempty by a result of Mumford [23] if \( n = 1 \) and of Chen [5] for all \( n \); moreover, all components are always smooth of the expected dimension \( p_a(n\xi) - \delta \) [7, 28]. The irreducibility question (for \( \delta < p_a(n\xi) \)) has been the object of much attention, see [1, 8, 13, 18, 19], and was recently solved in the case \( n = 1 \) for all \( \delta \leq p_a(\xi) - 4 \) in the preprint [2].

Similarly, on a general primitively polarized abelian surface \((S, \xi)\), all Severi varieties \( V_{n\xi}^S(S) \), where \( 0 \leq \delta \leq p_a(n\xi) - 2 \), are nonempty (by [21] if \( n = 1 \) and [20] in general) and smooth of the expected dimension \( p_a(n\xi) - \delta \) [22]. Irreducibility does not hold: The various irreducible components in the case \( n = 1 \) have been described by Zahariuc [32]. Very little is known on other surfaces, where problems such as nonemptiness, smoothness, dimension, and irreducibility are regarded as very hard. In particular, Severi varieties may have unexpected behavior: Examples are given in [6] of surfaces of general type with reducible Severi varieties, and also with components of dimension different from the expected one.

In this paper, we consider the case of blow-ups of a particular type of ruled surface over an elliptic curve.

Let \( E \) be a general smooth irreducible projective curve of genus 1 and set \( R := \text{Sym}^2(E) \). Let \( \bar{R} \) be the blow-up of \( R \) at any finite set of general points. Our main result in this paper shows that Severi varieties of a large class of line bundles on \( \bar{R} \) are well behaved:

**Theorem 1.1.** In the above setting, let \( L \) be a line bundle on \( \bar{R} \) verifying condition (\( \ast \)) (cf. Definition 2.1) and let \( \xi \) be the class of \( L \) in \( \text{Num}(\bar{R}) \). Let \( \delta \) be an integer satisfying \( 0 \leq \delta < p_a(L) \). Then, \( V_{\delta}^\xi(\bar{R}) \) is nonempty and smooth with all components of the expected dimension \(-L \cdot K_{\bar{R}} + p_a(L) - \delta - 1\).

The statement about smoothness and dimension follows from standard arguments of deformation theory, once nonemptiness has been proved, cf. Proposition 2.2 below. Moreover, we remark that, by what we said above, it suffices to prove Theorem 1.1 for the maximal number of nodes, that is, \( \delta = p_a(L) - 1 \). This will follow from Proposition 2.3 below, which treats the special case in which the blown-up points are in a special position. In [10], we will make use of Proposition 2.3 in order to prove nonemptiness of Severi varieties on Enriques surfaces. The question of smoothness and dimension of Severi varieties on Enriques surfaces has been treated in [9].

The irreducibility question for \( V_{\delta}^\xi(\bar{R}) \) is not treated in this paper; thus, we pose the following:

**Question 1.** Are the varieties \( V_{\delta}^\xi(\bar{R}) \) from Theorem 1.1 irreducible?
The paper is organized as follows. In Section 2, we recall some preliminaries concerning twofold symmetric products of elliptic curves. Section 3 is devoted to recalling a degeneration of the symmetric product of a general elliptic curve studied in [11]. In Section 4, we construct certain families of curves on some blow-ups of the projective plane that turn out to be useful in the proof of Proposition 2.3, which is proved by degeneration in Section 5.

## 2 THE TWOFOLD SYMMETRIC PRODUCT OF AN ELLIPTIC CURVE

Let $E$ be a smooth irreducible projective elliptic curve. Denote by $\oplus$ (and $\ominus$) the group operation on $E$ and by $e_0$ the neutral element. Let $R := \text{Sym}^2(E)$ and $\pi : R \to E$ be the (Albanese) morphism sending $x + y$ to $x \oplus y$. We denote the fiber of $\pi$ over a point $e \in E$ by

$$f_e := \pi^{-1}(e) = \{x + y \in \text{Sym}^2(E) \mid x \oplus y = e \text{ (equivalently, } x + y \sim e + e_0)\},$$

(where $\sim$ denotes linear equivalence), which is the $\mathbb{P}^1$ defined by the linear system $|e + e_0|$. We denote the algebraic equivalence class of the fibers by $\mathfrak{f}_e$. For each $e \in E$, we define the curve $s_e$ (called $D_e$ in [4]) as the image of the section $E \to R$ of the Albanese morphism mapping $x$ to $e + (x \ominus e)$. We let $\mathfrak{s}$ denote the algebraic equivalence class of these sections, which are the ones with minimal self-intersection, namely, 1, cf. [4]. One has

$$K_R \sim -2s_e + f_e.$$ 

Let $y_1, \ldots, y_n \in R$ be distinct points and let $\tilde{R} := \text{Bl}_{y_1, \ldots, y_n}(R) \to R$ denote the blow-up of $R$ at $y_1, \ldots, y_n$, with exceptional divisors $e_i$ over $y_i$. We denote the strict transforms of $\mathfrak{s}$ and $\mathfrak{f}$ on $\tilde{R}$ by the same symbols.

**Definition 2.1.** A line bundle or Cartier divisor $L$ on $\tilde{R}$ is said to verify condition $(\ast)$ if it is of the form $L \equiv \alpha \mathfrak{s} + \beta \mathfrak{f} - \sum_{i=1}^n \gamma_i e_i$ (where $\equiv$ denotes numerical or, equivalently, algebraic equivalence), with $\alpha, \beta, \gamma_1, \ldots, \gamma_n$ as integers such that:

(i) $\alpha \geq 1, \beta \geq 0$;
(ii) $\alpha \geq \gamma_i$ for $i = 1, \ldots, n$;
(iii) $\alpha + \beta \geq \sum_{i=1}^n \gamma_i$;
(iv) $\alpha + 2\beta \geq \sum_{i=1}^n \gamma_i + 4$.

Condition (ii) is satisfied if $L$ is nef. Condition (iv) is equivalent to $-L \cdot K_{\tilde{R}} \geq 4$.

The statement about smoothness and dimension in Theorem 1.1 follows from the following more general result, well known to experts:

**Proposition 2.2.** Let $S$ be a smooth projective complex surface and $\xi \in \text{Num}(S)$ such that $-\xi \cdot K_S > 0$. Let $\delta$ be an integer satisfying $0 \leq \delta \leq p_a(\xi)$.

If $V^{\xi}_\delta(S)$ is nonempty, it is smooth and every component has the expected dimension $-\xi \cdot K_S + p_a(\xi) - \delta - 1$.

**Proof.** Let $X$ be any curve in $V^{\xi}_\delta(S)$ and let $V^{\xi}(S)$ be the Hilbert scheme defined in the introduction. Since

$$\text{deg}(N'_{X/S}) = \xi^2 = \xi \cdot (\xi + K_S) - \xi \cdot K_S = 2p_a(\xi) - 2 - \xi \cdot K_S > 2p_a(\xi) - 2,$$

the normal bundle $N'_{X/S}$ is nonspecial, whence $V^{\xi}(S)$ is smooth at $[X]$ of dimension $h^0(N'_{X/S}) = -\xi \cdot K_S + p_a(\xi) - 1$ (cf., e.g., [24, section 4.3]).

Let $\varphi : \tilde{X} \to S$ be the composition of the normalization $\tilde{X} \to X$ with the inclusion $X \subset S$ and consider the normal sheaf $N'_{\varphi}$ defined by the short exact sequence

$$0 \to \tau_{\tilde{X}} \to \varphi^*T_S \to N'_{\varphi} \to 0.$$
The tangent space to \( V^\xi_\delta(S) \) at \([X]\) is isomorphic to \( H^0(\overline{X}, \mathcal{N}_\phi) \), and \( \mathcal{N}_\phi \) is a line bundle, as \( X \) is nodal (cf., e.g., [24, section 3.4.3] or [14]). Let \( g \) be the geometric genus of \( X \). Since \( \deg \mathcal{N}_\phi = -X \cdot K_S + 2g - 2 > 2g - 2 \) by the above sequence, the line bundle \( \mathcal{N}_\phi \) is nonspecial, and
\[
H^0(\mathcal{N}_\phi) = -\xi \cdot K_S + g - 1 = -\xi \cdot K_S + p_d(\xi) - \delta - 1 = \dim(V^\xi(S)) - \delta,
\]
which is the expected dimension of \( V^\xi_\delta(S) \). Thus, \( V^\xi_\delta(S) \) is smooth at \([X]\) and of the expected dimension.

By what we said in the introduction, it suffices to prove Theorem 1.1 for the highest possible \( \delta \), that is, for \( \delta = p_a(L) - 1 \), in which case, the Severi variety in question parameterizes nodal curves of geometric genus 1. We will prove the theorem by specializing the points \( y_1, \ldots, y_n \) as we now explain.

Let \( \eta \) be any of the three nonzero 2-torsion points of \( E \). The map \( E \to R \) defined by mapping \( e \) to \( e + (e \oplus \eta) \) realizes \( E \) as an unramified double cover of its image curve
\[
T = \{ e + (e \oplus \eta) \mid e \in E \}.
\]
We have
\[
T \sim -K_R + \mathfrak{f}_\eta - \mathfrak{f}_{e_0} \sim 2\mathfrak{g}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta,
\]
by [4, (2.10)]. In particular,
\[
T \sim -K_R \quad \text{and} \quad 2T \sim -2K_R.
\]
We denote the strict transform of \( T \) on \( \overline{R} \) by the same symbol. Suppose that \( y_1, \ldots, y_n \in T \) are general points. Then, by (2.1)–(2.2) we have
\[
T \sim 2\mathfrak{g}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - e_1 - \cdots - e_n \sim -K_{\overline{R}}, \quad 2T \sim -2K_{\overline{R}}
\]
on \( \overline{R} \). In particular,
\[
T \equiv -K_{\overline{R}} \equiv 2\mathfrak{g} - \mathfrak{f} - e_1 - \cdots - e_n.
\]

As remarked in the introduction, Theorem 1.1 is a consequence of the following result, which we will prove in Section 5.

**Proposition 2.3.** Let \( E \) be a general irreducible smooth projective curve of genus 1. Let \( y_1, \ldots, y_n \in T \) be general points, with \( T \) on \( R = \text{Sym}^2(E) \) as defined above. Let \( L \) be a line bundle on \( \overline{R} = \text{Bl}_{y_1, \ldots, y_n}(R) \) verifying condition (\( \star \)) with class \( \xi \) in \( \text{Num}(\overline{R}) \). Then, \( V^\xi_{p_a(L) - 1}(\overline{R}) \) is nonempty and smooth, of the expected dimension \( L \cdot T = -L \cdot K_{\overline{R}} \).

## 3 A DEGENERATION OF THE TWOFOLD SYMMETRIC PRODUCT OF A GENERAL ELLIPTIC CURVE

Let \( E \) be a smooth irreducible projective elliptic curve. We recall a degeneration of \( R = \text{Sym}^2(E) \) from [11], to which we refer the reader for details.

Let \( \lambda : \mathcal{X} \to D \) be a flat projective family of curves over the unit disk \( D \), with \( \mathcal{X} \) smooth, such that the fiber \( X_0 \) over \( 0 \in D \) is an irreducible rational 1-nodal curve and all other fibers \( X_t \), \( t \in D \setminus \{0\} \), are smooth irreducible elliptic curves. Let \( p : \mathcal{Y} \to D \) be the relative 2-symmetric product. Then, for \( t \neq 0 \), the fiber \( Y_t = p^{-1}(t) \simeq \text{Sym}^2(X_t) \) is smooth, whereas the special fiber \( Y_0 = p^{-1}(0) = \text{Sym}^2(X_0) \) is irreducible, but singular. The singular locus of \( Y_0 \) consists of the curve
\[
X_P := \{ x + P \mid x \in X_0 \},
\]
where \( P \) is the node of \( X_0 \).
Let \( \nu : P^1 \simeq X_0 \to X_0 \) be the normalization, with \( P_1 \) and \( P_2 \) the preimages of \( P \) (with the notation of \([11, p. 328]\), this is the case \( g = 1 \) with \( P = P_1 \)). Then, \( \nu \) induces a birational morphism

\[
\text{Sym}^2(\nu) : P^2 \simeq \text{Sym}^2(X_0) \to \text{Sym}^2(\tilde{X}_0) = Y_0.
\]

Under the isomorphism on the left, the diagonal in \( \text{Sym}^2(\tilde{X}_0) \) corresponds to a smooth conic \( \Gamma \) in \( P^2 \) that is mapped by \( \text{Sym}^2(\nu) \) to the diagonal \( \Delta_0 \) of \( Y_0 \) and, for each \( x \in \tilde{X}_0 \), the curve

\[
\{ x + Q \mid Q \in \tilde{X}_0 \} \subset \text{Sym}^2(\tilde{X}_0)
\]
corresponds to the line in \( P^2 \) tangent to \( \Gamma \) at the point corresponding to \( 2x \).

The threefold \( \mathcal{Y} \) has local equations at the point \( 2P \in Y_0 \) given by

\[
\begin{align*}
z_5^2 - z_3z_4 = 0, \quad z_1z_5 + z_2z_3 = 0, \quad z_1z_4 + z_2z_5 = 0, \quad z_1z_2 + z_5 + t = 0,
\end{align*}
\]

with \((z_1, \ldots, z_5, t) \in \mathbb{C}^5 \times \mathbb{D}\) (cf. \([11, p. 329]\)). In particular, \( \mathcal{Y} \) is singular only at the point \( 2P \), corresponding to the origin \( 0 \). Its special fiber \( Y_0 \) is locally reducible at \( 0 = 2P \), where it consists of three irreducible components \( S^1 \cup S^2 \cup S^3 \) (named \( S'_1 \) in \([11]\)), where \( S^1 \) is the \( z_2z_4 \)-plane, \( S^2 \) is the \( z_1z_3 \)-plane, and \( S^3 \) has equations \( z_3 = z_1^2, \quad z_4 = z_2^2, \quad z_5 = -z_1z_2 \), meeting as in \([11, \text{fig. 1}]\). The singular locus \( X_P = \text{Sing}(Y_0) \) of \( Y_0 \) has a node at the origin, \( Y_0 \) has double normal crossing singularities along \( X_P \setminus 2P \) and the intersection curves \( C^1 = S^1 \cap S^2 \) and \( C^2 = S^2 \cap S^3 \) (named \( C_i^j \) in \([11]\)) are the two branches of the curve \( X_P \) at \( 0 = 2P \). Finally, in these local coordinates, the diagonal \( \Delta_0 \) of \( Y_0 \) \((\Delta_0 = \Delta^1_{0,1} \cup \Delta^1_{0,2} \text{ in } [11, \text{fig. 1}] \) consists of the \( z_2, z_1 \)-axes and it has a node at the point \( 2P \).

Let \( \mu : \tilde{Y} \to \mathcal{Y} \) be the blow-up at the point \( 2P \in \text{Sym}^2(X_0) = Y_0 \subset \mathcal{Y} \) and denote the exceptional divisor by \( E \) (called \( E_1 \) in \([11]\)). Then, \( \mathcal{Y} \simeq \mathbb{F}_1 \) and \( \tilde{Y} \) is smooth (see \([11, p. 330]\)). All fibers over \( t \neq 0 \) are unchanged. The special fiber \( \tilde{Y}_0 \) of \( \tilde{Y} \to \mathbb{D} \) is the union of \( E \) and of an irreducible surface \( \tilde{S} \), which is the strict transform of \( Y_0 \). We have

\[
\tilde{S} \cap E = s_0 + e_1 + e_2,
\]

where \( e_1 \) and \( e_2 \) are two fibers of \( E \simeq \mathbb{F}_1 \) and \( s_0 \) (called \( \eta \) in \([11, \text{fig. 2}]\)) is the section satisfying \( s_0^2 = -1 \). The surface \( \tilde{S} \) is singular, with double normal crossings singularities along the proper transform \( \tilde{X}_P \) of the curve \( X_P \). The proper transform on \( \tilde{Y} \) of the diagonal of \( \mathcal{Y} \) intersects \( \tilde{Y}_0 \) along

\[
\tilde{\Delta}_0 + s_0,
\]

where \( \tilde{\Delta}_0 \) is the proper transform of the diagonal \( \Delta_0 \) on \( Y_0 \).

To normalize \( \tilde{S} \), one unfolds along \( \tilde{X}_P \). The resulting surface \( W \) is smooth. Denote the normalization map by \( \sigma : W \to \tilde{S} \). The preimage of \( \tilde{X}_P \) is a pair of curves, which we call \( \tilde{X}_{P_1} \) and \( \tilde{X}_{P_2} \). Denoting the inverse images on \( W \) of the curves \( e_1, e_2, s_0 \) on \( S \) by the same symbols, the intersection configuration between the curves \( e_1, e_2, s_0, \tilde{X}_{P_1}, \tilde{X}_{P_2} \) on \( W \) looks as follows:

Under the map \( \sigma \), the two curves \( \tilde{X}_{P_1} \) and \( \tilde{X}_{P_2} \) are identified: We denote the identification morphism by \( \omega : \tilde{X}_{P_1} \simeq \tilde{X}_{P_2} \). Under this morphism, the intersection points of the above configuration are mapped as follows:
Definition 3.1. We say that a curve $C \subset W$ is $\omega$-compatible if $C$ contains neither $\tilde{X}_{P_1}$ nor $\tilde{X}_{P_2}$ and $\omega$ maps the 0-dimensional intersection scheme of $C$ with $\tilde{X}_{P_1}$ to the intersection scheme of $C$ with $\tilde{X}_{P_2}$.

If the curve $C$ is $\omega$-compatible, then $\sigma(C)$ is a Cartier divisor on $\tilde{S}$. Conversely, any curve on $\tilde{S}$ that is a Cartier divisor and does not contain the singular curve of $\tilde{S}$ is the image by $\sigma$ of an $\omega$-compatible curve on $W$.

One sees that the curves $s_0, e_1, e_2$ are $(-1)$-curves on $W$ (see [11, pp. 331–332]). Contracting them, we obtain a morphism $\phi : W \to \mathbb{P}^2 \simeq \text{Sym}^2(\tilde{X}_0)$ such that

$$\phi(s_0) = P_1 + P_2, \quad \phi(e_1) = 2P_1, \quad \phi(e_2) = 2P_2$$

and

$$\phi(\tilde{X}_{P_i}) = X_{P_i} := \{P_i + Q \mid Q \in \tilde{X}_0\},$$

fitting in a commutative diagram

(see [11, p. 332]). This is shown in the next picture:

Remark 3.2. The morphism $\omega : \tilde{X}_{P_1} \to \tilde{X}_{P_2}$ is geometrically interpreted in the following way (see [11]). Via $\phi$ the curves $\tilde{X}_{P_1}$ and $\tilde{X}_{P_2}$ map isomorphically to the two lines on the plane $\mathbb{P}^2$ in red in Figure 1 joining the point $P_1 + P_2$ with the points $2P_1$ and $2P_2$, respectively. In $\mathbb{P}^2$, we have the conic $\Gamma$ (mapped by $\text{Sym}^2(\nu)$ to the diagonal $\Delta_0$ of $Y_0$), which is tangent to these lines at the points $2P_1$ and $2P_2$. The map $\omega$ associates two points if and only if their images in the plane lie on a tangent line to $\Gamma$. (The two points $P_1 + Q$ and $P_2 + Q$ of $\mathbb{P}^2$ lie on the tangent line to $\Gamma$ at the point $2Q$ and are the

\[ W \]

\[ \phi \]

\[ \mathbb{P}^2 \]
intersection points of this tangent line with the two lines joining $2P_1$ with $P_1 + P_2$ and $2P_2$ with $P_1 + P_2$, namely, $\phi(\overline{X}_{P_1})$ and $\phi(\overline{X}_{P_2}).$)

The Picard group of $W$ is generated by $h, e_1, e_2, s_0$, where $h$ is the pullback by $\phi$ of a line. In particular,

$$\overline{X}_{P_i} \sim h - s_0 - e_i, \quad i = 1, 2.$$  \hspace{1cm} (3.2)

One has

$$-K_W = 3h - e_1 - e_2 - s_0.$$

Let us look at what happens to the classes of $\mathfrak{s}$ and $\mathfrak{f}$ under the degeneration of $R$ to $\overline{Y}_0$. This is described in [11, section 2] together with the more general description of the degeneration of line bundles on $R$ under the degeneration of $R$ to $\overline{Y}_0$, which we are now going to recall.

Let $h'$ be an $\omega$-compatible member of $|h|$ on $W$ (cf. Definition 3.1), not containing any of $e_1, e_2, s_0$. There is a one-dimensional irreducible family of such curves whose general member is the strict transform on $W$ of a general tangent line to the conic $\Gamma$ of $\mathbb{P}^2$ mapped to the diagonal of $Y_0$ by $\text{Sym}^2(\nu)$ (cf. Remark 3.2). Since $h \cdot e_1 = h \cdot e_2 = h \cdot s_0 = 0$, we have $\sigma(h') \cap E = \emptyset$, so that $\sigma(h') \subset \overline{S}$ determines a Cartier divisor on $\overline{Y}_0$. The class of $\mathfrak{s}$ on $R$ degenerates to the class of $\sigma(h')$.

The class $h - s_0$ on $W$ satisfies $(h - s_0) \cdot s_0 = 1$ and $(h - s_0) \cdot e_i = (h - s_0) \cdot \overline{X}_{P_i} = 0, i = 1, 2$. Thus, the general member $F$ of the pencil $|h - s_0|$ is $\omega$-compatible and $\sigma(F)$ intersects $E$ in one point along $s_0$. Therefore, the union of $\sigma(F)$ with the fiber of $E$ over the intersection point on $s_0$ is a Cartier divisor on $\overline{Y}_0$, which turns out to be the limit of $\mathfrak{f}$.

Let $C \equiv a\mathfrak{s} + b\mathfrak{f}$ on $R$, with $a, b > 0$, and let $C_0$ be its limit on $\overline{Y}_0$. Assume that it neither contains any of $e_1, e_2, s_0$ nor the double curve of $\overline{S}$. We may write $C_0 = C_\mathfrak{S} \cup C_\mathfrak{E}$ with $C_\mathfrak{S} \subset \overline{S}$ and $C_\mathfrak{E} \subset E$. Then, $C_\mathfrak{S} \cap C_\mathfrak{E} \subset s_0$ and $C_\mathfrak{E}$ is a union of fibers of $E$. We have $C_\mathfrak{S} = \sigma(C_W)$, with $C_W$ a $\omega$-compatible curve satisfying

$$C_W \sim ah + b(h - s_0) = (a + b)h - bs_0.$$  

This is because the transform of the limit of $\mathfrak{s}$ is numerically equivalent to $h$ on $W$ and the transform of the limit of $\mathfrak{f}$ is equivalent to $(h - s_0)$. This means that $\phi(C_W) \subset \mathbb{P}^2$ is a plane curve of degree $a + b$ with a point of multiplicity $b$ at $P_1 + P_2$, with intersection points with $X_{P_1}$ and $X_{P_2}$ satisfying the suitable “gluing conditions” given by $\omega$.

Conversely, we have the following:

**Lemma 3.3.** Let $a, b \geq 0$ and $C_W \equiv |(a + b)h - bs_0|$ be an $\omega$-compatible curve not containing any of $e_1, e_2, s_0$ and intersecting $s_0$ in distinct points. Let $C_\mathfrak{E}$ denote the union of fibers on $E \subset \mathfrak{F}$ such that $C_\mathfrak{E} \cap s_0 = C_W \cap s_0$. Then, $\sigma(C_W) \cup C_\mathfrak{E}$ is the flat limit of a curve algebraically equivalent to $a\mathfrak{s} + b\mathfrak{f}$.

**Proof.** Since $C_W \cdot \overline{X}_{P_i} = a$, the locus of $\omega$-compatible curves in $|C_W|$ has dimension

$$\dim |C_W| - a = \frac{1}{2}(a^2 + 3a + 2ab + 2b) - a = \frac{1}{2}(a^2 + a + 2ab + 2b),$$

which equals the dimension of the Hilbert scheme of curves algebraically equivalent to $a\mathfrak{s} + b\mathfrak{f}$. The result follows from the discussion prior to the lemma. \hfill $\square$

Let us now go back to the degeneration $\mathcal{X} \to \mathcal{D}$ of a general elliptic curve $E$ to a rational nodal curve $X_0$ we considered at the beginning of this section. This can be viewed as a degeneration of the group $E$ to $\mathbb{C}^*$, where (keeping the notation introduced at the beginning of this section) $\mathbb{C}^* = \mathbb{P}^1 \setminus \{P_1, P_2\}$. Since $\mathbb{C}^*$ has a unique nontrivial point of order 2, that is, $-1$, we see that in the degeneration $\mathcal{X} \to \mathcal{D}$ one of the three nontrivial points of order 2 of the general fiber degenerates to $-1$, so it is fixed by the monodromy of the family $\mathcal{X} \to \mathcal{D}$. (The other two nontrivial points of order 2 must degenerate to the node of $X_0$.) This implies that we have a divisor $T$ on $\overline{Y}$ such that the fiber $T_t$ for $t \neq 0$ is a curve $T \subset R$ like the one we considered in Section 2. Since $T \equiv 2\mathfrak{s} - \mathfrak{f}$, the proper transform $T_W$ on $W$ of the limit of the curve $T$ is such that

$$T_W \sim 2h - (h - s_0) \sim h + s_0.$$
\( T_W = h_0 + e_1 + e_2 + s_0, \)

where \( h_0 \) is the strict transform by \( \phi \) of the line in \( \mathbb{P}^2 \) through \( 2P_1 \) and \( 2P_2 \).

Since \( T_W \) contains \( s_0 \), the divisor \( T \) contains \( E \). By subtracting \( E \) from \( T \), the resulting irreducible effective divisor \( T - E \) intersects the central fiber in a curve that consists of two components: one component on \( \tilde{S} \), which is \( \sigma(h_0) \), and another component sitting on \( E \) that is the pull-back on \( E \approx F_1 \) of the unique line of \( \mathbb{P}^2 \) passing through the two points in which \( h_0 \) intersects \( e_1 \) and \( e_2 \). However, what will be important for us in what follows is that \( \sigma(h_0) \) is in the limit of \( T \).

4 A USEFUL FAMILY OF RATIONAL CURVES ON SOME BLOW-UPS OF THE PLANE

In this section, we prove some results on certain line bundles on some blow-ups of the surface \( W \) introduced in the previous section. They will be useful in the proof of Proposition 2.3 in Section 5. We go on keeping the notation and convention we introduced in the previous section.

Let \( y_1, \ldots, y_n \in h_0 \) be general points. Choose sections of \( p : \tilde{Y} \to D \) passing through \( \sigma(y_1), \ldots, \sigma(y_n) \in \sigma(h_0) \) and through general points \( y_1', \ldots, y_n' \in T_1 \) on a general fiber \( Y_1 \). Blowing up \( \tilde{Y} \) along these sections, we obtain a smooth threefold \( Y' \) with a morphism \( p' : Y' \to D \) with general fiber the blow-up of \( Y_1 = \text{Sym}^2(X_1) \) at \( n \) general points of \( T_1 \) and special fiber \( Y' : = S' \cup E \), where \( S' = \text{Bl}_{\sigma(y_1), \ldots, \sigma(y_n)}(\tilde{S}) \), and there is a normalization morphism \( \sigma' : W' \to S' \), where \( W' = \text{Bl}_{y_1, \ldots, y_n}(W) \). Let \( e_i' \) denote the exceptional divisor in \( W' \) over \( y_i \), for \( i = 1, \ldots, n \). We denote the strict transforms of \( e_1, e_2, s_0, \tilde{X}_{P_1}, \tilde{X}_{P_2}, h_0 \) on \( W' \) by the same symbols. Note that (3.2) still holds; furthermore,

\[
\begin{align*}
\sigma(y) &\sim h - e_1 - e_2 - e_{y_1} - \cdots - e_{y_n} \\
- K_{W'} &\sim 3h - e_1 - e_2 - s_0 - e_{y_1} - \cdots - e_{y_n} \sim h_0 + e_1 + e_2 + \tilde{X}_{P_1} + \tilde{X}_{P_2} + s_0.
\end{align*}
\]

Moreover, the pull-back on \( W' \) of the limit of \( T \) on \( S' \) contains \( h_0 \). We next fix a general point \( x_1 \in \tilde{X}_{P_1} \) and set \( x_2 = \omega(x_2) \in \tilde{X}_{P_2} \). The following picture summarizes the situation:

![Diagram of a threefold with labeled components and points](image)

We introduce the following notation. For a line bundle \( \mathcal{M} \) on \( W' \), we denote by \( V_{\mathcal{M}} \) the locus of curves \( C \) in \( |\mathcal{M}| \) on \( W' \) such that

1. \( C \) is irreducible and rational,
2. \( C \) intersects \( \tilde{X}_{P_i} \) only at \( x_i, i = 1, 2 \), and it is unibranch there.
We denote by $V_{\mathcal{A}_4}^*$ the open sublocus of $V_{\mathcal{A}_4}$ of curves $C$ with the further properties that

1. $C$ intersects $s_0$ transversely,
2. $C$ is smooth at $x_i$, $i = 1, 2$,
3. $C$ is nodal.

**Lemma 4.1.** Assume $V_{\mathcal{A}_4} \neq \emptyset$.

(i) If $\mathcal{M} \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) \geq 1$, then for each component $V$ of $V_{\mathcal{A}_4}$ one has

$$\dim(V) = -K_{W'} \cdot \mathcal{M} - 1 - \mathcal{M} \cdot \bar{X}_{P_1} - \mathcal{M} \cdot \bar{X}_{P_2} = \mathcal{M} \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) - 1.$$  

(ii) If $\mathcal{M} \cdot (h + s_0 - e_{y_1} - \cdots - e_{y_n}) \geq 4$, then $V_{\mathcal{A}_4}^* \neq \emptyset$.

**Proof.** The result follows from [3, section 2], as outlined in [12, Theorem (1.4)].

**Proposition 4.2.** Let $\alpha, \beta, \gamma_1, \ldots, \gamma_n$ be nonnegative integers verifying conditions (i)–(iv) in Definition 2.1. Set $\mathcal{M} = (\alpha + \beta)h - \beta s_0 - \sum \gamma_i e_{y_i}$. Then, $V_{\mathcal{A}_4}^*$ is nonempty with all components of dimension $\alpha + 2\beta - \sum \gamma_i - 1$.

In the proof of Proposition 4.2, we will need the following:

**Lemma 4.3.** Given three lines $\ell_1, \ell_2, \ell_3$ in the plane $\mathbb{P}$ not passing through the same point, we set $y_{ij} = \ell_i \cap \ell_j$ for $1 \leq i < j \leq 3$. Fix integers $d > m \geq 0$, $n \geq 0$, $m_1, \ldots, m_n \geq 1$, such that

$$d \geq \sum_{i=1}^n m_i \quad \text{and} \quad d \geq m + m_i, \quad i = 1, \ldots, n.$$  

Then, there is a reduced and irreducible rational curve $\gamma$ in $\mathbb{P}$ of degree $d$ with the following properties:

1. $\gamma$ has a point of multiplicity $m$ at $y_{12}$,
2. the pull-back on the normalization of $\gamma$ of the $g_1^{d-m}$ cut out by the lines through $y_{12}$ has two total ramification points mapping to generic points $x_1 \in \ell_1$ and $x_2 \in \ell_2$, respectively (in particular, different from $y_{12}, y_{13}, y_{23}$),
3. $\gamma$ has $n$ points of multiplicities $m_1, \ldots, m_n$ that are pairwise distinct points on $\ell_3$ (in particular, different from $y_{13}$ and $y_{23}$).

**Proof.** Set $\delta = d - m$. The assertion is trivial if $\delta = 1$. So we assume $\delta \geq 2$. Consider a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $\delta$ with two points of total ramification, that is, a $g_\delta^1$ with no base points. Fix a general effective divisor $D$ of degree $m$ on $\mathbb{P}^1$, so that $g_\delta^1 + D$ is a $g_d^1$. Fix $n$ general points $P_1, \ldots, P_n$ of $\mathbb{P}^1$, and consider the fibers

$$f^{-1}(P_i) = P_{i1} + \cdots + P_{i\delta}, \quad i = 1, \ldots, n.$$
Consider then the divisor
\[ E = \sum_{i=1}^{n} \sum_{j=1}^{m_i} P_{ij} + F, \]
where \( F \) is a general effective divisor of degree \( d = \sum_{i=1}^{n} m_i \) on \( \mathbb{P}^1 \). The divisor \( E \) has no common point with the general divisor of \( g_1^1 + D \). Hence, \( E \) and \( g_1^1 + D \) span a \( g_2^2 \) with no base points. Moreover, this \( g_2^2 \) is birational. Indeed, if \( g_2^2 \) were composed with a \( g_1^1 \), then, by the generality of the divisor \( D \), the \( g_1^1 \) would have base points, a contradiction.

Let \( y \) be the image of \( \mathbb{P}^1 \) via the \( g_2^2 \). This is a rational plane curve of degree \( d \), with a point of multiplicity \( m \) at a point \( y_{12} \). Moreover, there are two lines \( \ell_1, \ell_2 \) passing through \( y_{12} \), that each intersects \( y \) at one point apart from \( y_{12} \), call it \( x_1 \) and \( x_2 \), respectively, where the \( g_1^1 \) has total ramification. Finally, there is a third line \( \ell_3 \) that pulls back to \( \mathbb{P}^1 \) to the divisor \( E \). By the choices we made, this line does not pass through \( x_1 \) and \( x_2 \) and the divisors \( \sum_{i=1}^{n} m_i \) are contracted by the \( g_2^2 \) to \( n \) distinct points on \( \ell_3 \) that have multiplicities \( m_1, \ldots, m_n \). The genercity of \( x_1, x_2 \) can be achieved by acting with projective transformations of the plane fixing the lines \( \ell_1, \ell_2, \ell_3 \), which keep the points of multiplicities \( m_1, \ldots, m_n \) pairwise distinct.

**Proof of Proposition 4.2.** Let \( d = \alpha + \beta \), \( m_i = \gamma_i \), and \( m = \beta \). Consider the plane \( \mathbb{P} \) containing the curve \( y \) constructed in Lemma 4.3. Let us blow up the points \( y_{12}, y_{13}, y_{23} \) and the \( n \) points of multiplicities \( m_1, \ldots, m_n \) on \( y \) along \( \ell_3 \).

We will consider the family \( W' \) of the surfaces \( W' \) as above where the points \( y_1, \ldots, y_n \) are no longer general but simply pairwise distinct. We call \( b > 0 \) the dimension of the parameter space of this family. There is a line bundle \( M \) on \( W \) that restricts on each member of \( W \) to the line bundle \( M \) as in the statement of the proposition. Accordingly, we can consider the families \( V_M \) and \( V^*_M \) of all varieties \( V_M \) and \( V^*_M \) as before.

The blow-up at the beginning of the proof can be interpreted as a member \( W'_0 \) of \( W' \) with \( \omega(x_1) = x_2 \), since there is a unique irreducible conic \( \Gamma \) tangent to the lines \( \ell_1, \ell_2 \) at \( y_{13} \) and \( y_{23} \), respectively, and tangent also to the line joining the two points \( x_1 \) and \( x_2 \) (cf. Remark 3.2). We denote by \( M_0 \) the restriction of \( M \) to \( W'_0 \).

Lemma 4.3 implies that \( V_{M_0} \) is nonempty, which by Lemma 4.1 implies in turn that \( V^*_M \) is nonempty, with all components of the expected dimension \( \alpha + 2\beta - \sum \gamma_i - 1 \). This yields that \( V^*_M \) is nonempty. Then, its dimension is at least the expected dimension, which is \( \alpha + 2\beta - \sum \gamma_i - 1 + b \) that is strictly larger than the dimension of \( V^*_M \). This implies that for \( W' \) general in \( W \), the variety \( V^*_M \) is nonempty of the expected dimension \( \alpha + 2\beta - \sum \gamma_i - 1 \), as wanted.

**5 PROOF OF PROPOSITION 2.3**

Let us go back to \( R \) and \( \bar{R} = B_{y_1, \ldots, y_n}(R) \), where \( y_1, \ldots, y_n \) are general points on \( T \), with exceptional divisors \( e_i \) over \( y_i \), for \( i = 1, \ldots, n \).

**Proof of Proposition 2.3.** As we already noted, condition (iv) in Definition 2.1 of (⋆) is equivalent to \( -K_{\bar{R}} \cdot L \geq 4. \) Hence, as remarked for Theorem 1.1 in the introduction, the statements about dimension and smoothness follow from Proposition 2.2 once nonemptiness is proved. So it remains to prove nonemptiness.

We prove the result by degeneration of \( \bar{R} \) to \( Y' \), as indicated in Section 4, from which we keep the notation.

On the surface \( W' \), consider
\[ L_0 \sim \alpha h + \beta(h-s_0) - \sum \gamma_i e_i. \]

Denote by \( \{L_0\}_{W'} \subset |L_0| \) the sublocus of \( \omega \)-compatible curves.

**Claim 5.1.** \( \dim(\{L_0\}_{W'}) = \dim(|L_0|) - \alpha = \frac{1}{2} (\alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2) + \beta(\alpha + 1). \)

**Proof of Claim.** By Proposition 4.2, the linear system \( |L_0| \) contains an irreducible curve. Since \( h^1(\mathcal{O}_{W'}) = 0 \), it therefore follows that
\[ h^1(-L_0) = h^1(L_0 + K_{W'}) = 0. \] (5.1)
Set \( A := h_0 + e_1 + e_2 + \bar{X}_{p_1} + \bar{X}_{p_2} + s_0 \). Then, \( A \) is a reduced cycle of rational curves, thus of arithmetic genus 1, and it is anticanonical by (4.1). Since \( L_0 \cdot A \geq 4 \) by condition (iv) of (*) it follows in particular that \( |L_0 + A| \) contains a reduced, connected member. It therefore follows that

\[
h^1(-L_0 - A) = h^1(L_0) = 0 \quad \text{and} \quad h^0(-L_0 - A) = h^2(L_0) = 0.
\]

In particular, using Riemann–Roch on \( W' \), one computes that

\[
\dim(|L_0|) = \frac{1}{2} \left( \alpha^2 + 3\alpha - \sum \gamma_i - \sum \gamma_i^2 \right) + \beta(\alpha + 1),
\]

thus proving the right-hand equality of the claim.

We have left to prove that \( |L_0|_{W'} \) has codimension \( \alpha \) in \( |L_0| \).

To this end, let \( Z_1 \in \text{Sym}^\alpha(\bar{X}_{p_1}) \) be general, and set \( Z_2 := \omega(Z_1) \in \text{Sym}^\alpha(\bar{X}_{p_2}) \). From the two restriction sequences,

\[
0 \longrightarrow L_0 + K_{W'} \longrightarrow L_0 \longrightarrow L_0|_A \longrightarrow 0
\]

and

\[
0 \longrightarrow L_0 + K_{W'} \longrightarrow L_0 \otimes J_{Z_1 \cup Z_2} \longrightarrow L_0|_A(-Z_1 - Z_2) \longrightarrow 0,
\]

together with (5.1), we see that

\[
\text{codim} \left( |L_0 \otimes J_{Z_1 \cup Z_2}|, |L_0| \right) = \text{codim} \left( |L_0|_A(-Z_1 - Z_2)|, |L_0| \right).
\]

A standard computation involving restriction sequences to the various components of \( A \) shows that the latter codimension is \( 2\alpha \). Therefore,

\[
\dim \left( |L_0 \otimes J_{Z_1 \cup Z_2}| \right) = \dim(|L_0|) - 2\alpha.
\]

(This equality can also be obtained applying \([12, \text{Theorem (1.4.0)}]\).) Varying \( Z_1 \in \text{Sym}^\alpha(\bar{X}_{p_1}) \), we obtain the whole of \( |L_0|_{W'} \). Thus,

\[
\dim(|L_0|_{W'}) = \dim \left( |L_0 \otimes J_{Z_1 \cup Z_2}| \right) + \dim(\text{Sym}^\alpha(\bar{X}_{p_1})) = \dim(|L_0|) - \alpha,
\]

finishing the proof of the claim. \( \square \)

Denote by \( \{L_0\} \) the locus of curves on \( Y' = S' \cup \mathcal{E} \) of the form \( \sigma'(C) \cup C_\mathcal{E} \), where \( C \) is an element of \( |L_0|_{W'} \) and \( C_\mathcal{E} \) is the union of fibers on \( \mathcal{E} \simeq F_1 \) such that \( C_\mathcal{E} \cap S' = \sigma'(C) \cap s_0 \). Then, there is a one-to-one correspondence between \( |L_0|_{W'} \) and \( \{L_0\} \). Thus, by the last claim,

\[
\dim(|L_0|) = \frac{1}{2} \left( \alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2 \right) + \beta(\alpha + 1).
\]

(5.3)

Note that all members of \( \{L_0\} \) are Cartier divisors on \( Y' \). Moreover, by Lemma 3.3, the closure of the locus \( \{L_0\} \) is (a component of) the limit of the algebraic system \( \{L\} \) on \( \bar{R} \) of curves of class \( \alpha \delta + \beta \mathfrak{f} - \sum \gamma_i \epsilon_i \). Since the anticanonical divisor on \( \bar{R} \) is effective, we have \( h^2(L) = h^0(K_{\bar{R}} - L) = 0 \), whence Riemann–Roch yields

\[
\dim[L] = \dim |L| + 1 = \chi(L) + h^1(L) = \frac{1}{2} L \cdot (L - K_{\bar{R}}) + h^1(L)
\]

\[
= \frac{1}{2} \left( \alpha^2 + \alpha - \sum \gamma_i - \sum \gamma_i^2 \right) + \beta(\alpha + 1) + h^1(L).
\]
By (5.3) and semicontinuity, we must have $h^1(L) = 0$ and
\[ \dim\{L_0\} = \dim\{L\}. \] (5.4)

Let $x_1$ and $x_2$ be as in Section 4 and pick a general $C$ in a component of $V^e_{L_0}$ in $W'$ (which is nonempty by Proposition 4.2). Then, $C$ intersects $s_0$ transversely at $L_0 \cdot s_0 = \beta$ distinct points. Denote as above by $C_\epsilon$ the union of the $\beta$ fibers on $\epsilon$ such that $C_\epsilon \cap s_0 = \sigma'(C) \cap s_0$. Then, $\sigma'(C) \cup C_\epsilon$ is a member of $\{L_0\}$, with an $\alpha$-tacnode at $\sigma'(x_1) = \sigma'(x_2)$, and nodal otherwise, stably equivalent to $\sigma'(C)$. Varying $x_1$, we obtain by Proposition 4.2 a family $\mathcal{C}$ of dimension $\alpha + 2\beta - \sum \gamma_i = -L \cdot K_\epsilon$ of such curves, and this is the expected dimension of $V^e_{p_a(L)-1}(\tilde{R})$.

Let $\delta_0$ be the number of nodes of $C$, which equals the number of singular points of $\sigma'(C)$ on the smooth locus of $Y'$. Then,
\[ \delta_0 = p_a(L_0) = \frac{1}{2}(\alpha^2 - 3\alpha + \sum \gamma_i - \sum \gamma_i^2) + \beta(\alpha - 1) + 1. \]

Grant for the moment the following:

Claim 5.2. The family of curves in $\{L_0\}$ passing through the $\delta_0$ nodes of $\sigma'(C)$ and having an $(\alpha - 1)$-tacnode at $\sigma'(x_1) = \sigma'(x_2)$ has codimension $\delta_0 + \alpha - 1$ in $\{L_0\}$, that is, it has dimension $\alpha + 2\beta - \sum \gamma_i = \dim \mathcal{C}$.

Arguing as in [15, Theorem 3.3, Corollary 3.12, and proof of Theorem 1.1], we may deform $Y'$ to $\tilde{R}$ deforming the $\alpha$-tacnode of $\sigma'(C)$ to $\alpha - 1$ nodes, while preserving the $\delta_0$ nodes and smoothing the nodes $\sigma'(C) \cap C_\epsilon$. Thus, $\sigma'(C) \cup C_\epsilon$ deforms to a curve algebraically equivalent to $L$ with $\delta$ nodes, where
\[ \delta = \delta_0 + \alpha - 1 = \frac{1}{2}(\alpha^2 - \alpha + \sum \gamma_i - \sum \gamma_i^2) + \beta(\alpha - 1). \]

One computes
\[ p_a(L) = \frac{1}{2}(L^2 + L \cdot K_\epsilon) + 1 = \delta + 1. \]

This shows that $C$ has geometric genus 1, as wanted.

We have left to prove the claim.

Proof of Claim 5.2. Let $F$ be the family of curves in $\{L_0\}_{W'}$ passing through the $\delta_0$ nodes of $C$ and being tangent to $X_{P_i}$ at $x_i$ with order $\alpha - 1$, for $i = 1, 2$. The statement of the claim is equivalent to $\dim F = \alpha + 2\beta - \sum \gamma_i$.

Denoting by $N$ the scheme of the $\delta_0$ nodes of $C$ and by $Z_i = (\alpha - 1)x_i$ the subscheme on $X_i$, whence on $W'$, we have that $F$ is the locus of $\omega$-compatible curves in $|L_0 \otimes J_{N \cup Z_1 \cup Z_2}|$, which has codimension 1, as $L_0 \cdot X_i = \alpha$. Thus,
\[ \dim F = \dim |L_0 \otimes J_{N \cup Z_1 \cup Z_2}| - 1. \]

To compute this, let $q : W'' \rightarrow W'$ denote the blow-up of $W'$ at $N$ and denote the total exceptional divisor by $E$. Denote the inverse images of $Z_i$ by the same names. Then,
\[ \dim F = \dim \left( |(q^*L_0 - E) \otimes J_{Z_1 \cup Z_2}| \right) - 1. \] (5.5)

Let $\tilde{C}$ be the strict transform of $C$ on $W''$, which is a smooth rational curve. Then, $\tilde{C} \sim q^*L_0 - 2E$. We therefore have a short exact sequence
\[ 0 \rightarrow \mathcal{O}_{W''}(E) \rightarrow (q^*L_0 - E) \otimes J_{Z_1 \cup Z_2} \rightarrow \mathcal{O}_{\tilde{C}}(q^*L_0 - (-(\alpha - 1)(x_1 + x_2))) \rightarrow 0, \]
whence, from (5.5), we have
\[
\dim(F) = h^0((q^*L_0 - E) \otimes J_{Z_1 \cup Z_2}) - 2
\]
\[
= h^0(\mathcal{O}_C(q^*L_0 - E)((\alpha - 1)(x_1 + x_2))) - 1
\]
\[
= \deg(\mathcal{O}_C(q^*L_0 - E)((\alpha - 1)(x_1 + x_2)))
\]
\[
= (q^*L_0 - 2E)(q^*L_0 - E) - 2(\alpha - 1)
\]
\[
= L_0^2 - 2\delta_0 - 2\alpha + 2
\]
\[
= \alpha + 2\beta - \sum \gamma_i,
\]
as desired.

The proof of Proposition 2.3 is now complete.

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ENDNOTES
1From a deformation-theoretic point of view, the claim asserts the smoothness of the equisingular deformation locus of $\sigma'(C) \cup C\epsilon$, which is a dense open subset of $C\epsilon$, cf. [15, Lemma 3.4].
2The setting in [15] is slightly different, as the central fiber in the degeneration is a transversal union of two smooth surfaces, whereas $S'$ in the present setting is singular. Moreover, the central fiber in the degeneration in [15] is regular, whence linear and algebraic equivalence coincide, which is not the case on $Y'$. However, the reasoning in [15] is local, so the proof goes through in the present setting as well. The two hypotheses (1) and (2) in [15, Theorem 3.3] correspond, respectively, to (5.4) and the statement in Claim 5.2.

REFERENCES
[1] E. Ballico, On the irreducibility of the Severi variety of nodal curves in a smooth surface, Arch. Math. (Basel) 113 (2019), no. 5, 483–487.
[2] A. Bruno and M. Lelli Chiesa, Irreducibility of Severi varieties on K3 surfaces, preprint arxiv:2112.09398v1.
[3] L. Caporaso and J. Harris, Counting plane curves of any genus, Invent. Math. 131 (1998), 345–392.
[4] F. Catanese and C. Ciliberto, Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$, J. Algebraic Geom. 2 (1993), 389–411.
[5] X. Chen, Rational curves on K3 surfaces, J. Algebraic Geom. 8 (1999), 245–278.
[6] L. Chiantini and C. Ciliberto, On the Severi varieties of surfaces in $\mathbb{P}^3$, J. Alg. Geom. 8 (1999), 67–83.
[7] L. Chiantini and E. Sernesi, Nodal curves on surfaces of general type, Math. Ann. 307 (1997), 41–56.
[8] C. Ciliberto and Th. Dedieu, On the irreducibility of Severi varieties on K3 surfaces, Proc. Amer. Math. Soc. 147 (2019), 4233–4244.
[9] C. Ciliberto, T. Dedieu, C. Galati, and A. L. Knutsen, A note on Severi varieties of nodal curves on Enriques surfaces, in “Birational Geometry and Moduli Spaces,” Springer INdAM Ser. 39 (2018), 29–36.
[10] C. Ciliberto, T. Dedieu, C. Galati, and A. L. Knutsen, Nonemptiness of Severi varieties on Enriques surfaces, arXiv:2109.10735.
[11] C. Ciliberto and A. Kouvidakis, On the symmetric product of a curve with general moduli, Geom. Dedicata 78 (1999), 327–343.
[12] Th. Dedieu, Geometry of logarithmic Severi varieties at a general point, https://hal.archives-ouvertes.fr/hal-02913705.
[13] Th. Dedieu, Comment on: on the irreducibility of the Severi variety of nodal curves in a smooth surface, by E. Ballico, Arch. Math. (Basel) 114 (2020), no. 2, 171–174.
[14] Th. Dedieu and E. Sernesi, Equigeneric and equisingular families of curves on surfaces, Publ. Mat. 61 (2017), 175–212.
[15] C. Galati and A. L. Knutsen, On the existence of curves with $A_1$-singularities on K3 surfaces, Math. Res. Lett. 21 (2014), 1069–1109.
[16] G.-M. Greuel, C. Lossen, and E. Shustin, Geometry of families of nodal curves on the blown-up projective plane, Trans. Amer. Math. Soc. 350 (1998), 251–274.
[17] J. Harris, On the Severi problem, Invent. Math. 84 (1986), 445–461.
[18] Th. Keilen, *Irreducibility of equisingular families of curves*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3485–3512.
[19] M. Kemeny, *The universal Severi variety of rational curves on K3 surfaces*, Bull. Lond. Math. Soc. **45** (2013), no. 1, 159–174.
[20] A. L. Knutsen and M. Lelli-Chiesa, *Genus two curves on abelian surfaces*, Ann. Sci. Éc. Norm. Supér. (4) **55** (2022), no. 4, 905–918.
[21] A. L. Knutsen, M. Lelli-Chiesa, and G. Mongardi, *Severi Varieties and Brill-Noether theory of curves on abelian surfaces*, J. Reine Angew. Math. **749** (2019), 161–200.
[22] H. Lange and E. Sernesi, *Severi varieties and branch curves of abelian surfaces of type (1, 3)*, Internat. J. Math. **13** (2002), 227–244.
[23] S. Mori and S. Mukai, *The uniruledness of the moduli space of curves of genus II*, Algebraic geometry (Tokyo/Kyoto, 1982), pp. 334–353, Lecture Notes in Math., 1016, Springer, Berlin, 1983.
[24] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der mathematischen Wissenschaften 334. Springer, Berlin, Heidelberg, New York, 2006.
[25] E. Sernesi, *A smoothing criterion for families of curves*, preprint February 2009, http://www.mat.uniroma3.it/users/sernesi/papers.html.
[26] F. Severi, *Vorlesungen über Algebraische Geometrie*, reprinted, 1968; 1st. ed., Johnson Pub., Leipzig, 1921.
[27] A. Tannenbaum, *Families of algebraic curves with nodes*, Compositio Math. **41** (1980), 107–126.
[28] A. Tannenbaum, *Families of curves with nodes on K3 surfaces*, Math. Annalen **260** (1982), 239–253.
[29] D. Testa, *The Severi problem for rational curves on del Pezzo surfaces*, Ph.D. Thesis, Massachusetts Institute of Technology, 2005, ProQuest LLC.
[30] D. Testa, *The irreducibility of the moduli spaces of rational curves on del Pezzo surfaces*, J. Algebraic Geometry **18** (2009), 37–61.
[31] I. Tyomkin, *On Severi varieties on Hirzebruch surfaces*, Int. Math. Res. Not. IMRN (2007), no. 23, Art. ID rnm109, 31 pp.
[32] A. Zahariuc, *The Severi problem for abelian surfaces in the primitive case*, J. Math. Pures Appl. (9) **158** (2022), 320–349.

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