A Minimal $S_3$-Invariant Extension of the Standard Model

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Abstract. We present a minimal $S_3$-invariant extension of the Standard Model. We find that in leptonic sector, the exact $S_3 \times Z_2$ symmetry, which allows 6 real independent parameters, is consistent with experimental data. With exact $S_3$ symmetry, there are 10 real independent parameters and one independent phase, on which the mixing matrix $V_{\text{CKM}}$ depends. A set of values of these parameters that are consistent with experimental observations is given.

1. Flavour Permutational Symmetry

In Standard Model (SM), analogous fermions in different generations say $u$, $c$ and $t$ (or $d$, $s$ and $b$) have completely identical couplings to all gauge bosons of the SM. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to any permutation of the left and right quark fields. In weak basis, the charged currents $J_\mu$ are invariant under $G_F$ symmetry group ($G_F \sim S_3^L \otimes S_3^R$), if $d$ and $u$-type fields are transformed with the same family group matrix [1], then we have that

$$J_\mu = -i\bar{u}_L \gamma_\mu d_L + h.c. \Rightarrow G_F \sim S_3 \subset S_3^L \otimes S_3^R,$$

where $u_L$ and $d_L$ denote the left and right quarks fields in the current or weak basis. When $\langle 0|\Phi_H|0\rangle \neq 0$, the Yukawa couplings give mass to quarks and leptons, if we assume that the $S_3$ permutational symmetry is not broken, the mass matrix is the democratic mass matrix in which only the $m_t, b \neq 0$ [1]. Also, in this case the mixing matrix $V$ is the unitarity matrix and there is no mixing nor CP-violation. Then, for to obtain the mass terms of the others families the $S_3$ symmetry should be broken [1, 2, 3].

2. The Group $S_3$

$S_3$ is the discrete non-abelian group with the smallest number of elements, contains the six possible permutations of three objects $(f_1, f_2, f_3)$ [4]. The three-dimensional representation is not an irreducible representation of $S_3$. It can be decomposed into the direct sum to two irreducible representations, a singlet $f_S$ and a doublet $f_D$ given as,

$$f_S = \frac{1}{\sqrt{3}}(f_1 + f_2 + f_3), \quad f_D^T = (\frac{1}{\sqrt{2}}(f_1 - f_2), \quad \frac{1}{\sqrt{6}}(f_1 + f_2 - 2f_3)).$$
The most general Higgs potential invariant under $SU(4)$. The Higgs Sector

where each term is given as:

where $\kappa$ and $\eta$ parameters are the following expressions:

In these expressions, doublets carry the indices $I,J = 1,2$, and the singlets carry the index $s$ or 3. Further, the Majorana mass terms for the right handed neutrinos are

where $C$ is the charge conjugation matrix.

4. The Higgs Sector

The most general Higgs potential invariant under $SU(2) \times U(1) \times S_3$ is expressed as [5]:

$$V = \mu_1^2(\overline{H}_1 H_1 + \overline{H}_2 H_2) + \mu_0^2(\overline{H}_3 H_3) + a(\overline{H}_3 H_3)^2 + b(\overline{H}_3 H_3)(\overline{H}_1 H_1 + \overline{H}_2 H_2) +$$

$$c(\overline{H}_1 H_1 + \overline{H}_2 H_2)^2 + d(\overline{H}_1 H_1 + \overline{H}_2 H_2)^2 + g((\overline{H}_1 H_1 + \overline{H}_2 H_2)^2 + (\overline{H}_1 H_1 + \overline{H}_2 H_2)^2) +$$

$$f((\overline{H}_3 H_3)(\overline{H}_1 H_1 + \overline{H}_2 H_2)) + h((\overline{H}_3 H_3)(\overline{H}_1 H_1 + \overline{H}_3 H_3)(\overline{H}_2 H_2 + h.c.),$$
where $H_i$ ($i = 0, 1, 2$) are SU(2) doublets Higgs fields. This extended Higgs sector is symmetric under the $S_3$ symmetry group. In the adapted symmetry basis, the Higgs fields are given as a singlet $H_S$ and a doublet $H_D$. In concordance with (2), these Higgs fields are expressed as:

$$H_{D1} = \frac{1}{\sqrt{2}} (H_1 - H_2), H_{D2} = \frac{1}{\sqrt{6}} (H_1 + H_2 - 2H_3), H_3 = \frac{1}{\sqrt{3}} (H_1 + H_2 + H_3).$$

(9)

Also, we have the relationship

$$\sum_i (H_i^0)^2 = \nu^2 = (246 \text{ GeV})^2.$$

(10)

5. Mass and Mixing Matrices

For obtain the mass matrices, we will assume that all vev’s are real and that $\langle H_{D1} \rangle = \langle H_{D2} \rangle$. Also, they satisfy the constraint (10). Then, the Yukawa interactions (5) yield mass matrices of the general form

$$M = \begin{pmatrix}
\mu_1 + \mu_2 & \mu_2 & \mu_5 \\
\mu_2 & \mu_1 + \mu_2 & \mu_5 \\
\mu_4 & \mu_4 & \mu_3
\end{pmatrix}.$$

(11)

The Majorana masses for $\nu_L$ are obtained from the see-saw mechanism, the corresponding mass matrix is given by $M_\nu = M_{uD} \tilde{M}^{-1}(M_{uD})^T$ with $\tilde{M} = \text{diag}(M_1, M_1, M_2)$. In general, all entries of this mass matrices can be complex. The mass matrices are diagonalized by unitary matrices

$$U_{d(u,e)}^L M_{d(u,e)} U_{d(u,e)} R = \text{diag}(m_{d(u,e)}m_{s(c,\mu)}b_{(t,\tau)}),$$

$$U_{\nu}^T M_\nu U_\nu = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}).$$

(12)

The diagonal masses $m$’s can be complex, and the physical masses are $|m|’$s. The quarks and leptons mixing matrices are defined as:

$$V_{CKM} = U_{uL}^T U_{dL}, \quad V_{MNS} = U_{eL}^T U_\nu.$$

(13)

5.1. The Leptonic Sector

To achieve a further reduction of the number of parameters, in the leptonic sector, we introduce an additional discrete $Z_2$ symmetry with an assignment of the charges in the leptonic sector as is given in Ref. [4]. The $Z_2$ symmetry forbids certain coupling: $Y_{1}^{e} = Y_{3}^{e} = Y_{1}^{\nu} = Y_{5}^{\nu} = 0$. Hence, the leptonic mass matrices are constructed of (11) with $\mu_1 = \mu_5 = 0$ and $\mu_2 = \mu_\nu = 0$ for the charged lepton and neutrino mass matrices, respectively. The unitary matrix $U_{eL}$ is calculated from $U_{eL}^T M_\nu U_{eL} = \text{diag}( |m_e|^2, |m_\mu|^2, |m_\tau|^2)$ and it is expressed as

$$U_{eL} \simeq \begin{pmatrix}
-\frac{y}{\sqrt{2}} (1 + \frac{1}{x}) & \frac{1 - 2}{2} \left(1 - \frac{y^2}{\sqrt{2}} - \frac{y^2}{\sqrt{2}x^2}\right) & \frac{1}{\sqrt{2}} \\
\frac{y}{\sqrt{2}} (1 + \frac{1}{x}) & \frac{1}{2} \left(1 - \frac{y^2}{\sqrt{2}} + \frac{y^2}{\sqrt{2}x^2}\right) & \frac{1}{\sqrt{2}} \\
(1 - \frac{y^2}{\sqrt{2}})e^{i\delta} & \frac{y}{\sqrt{2}z} e^{i\delta} & \frac{y}{\sqrt{2}z^2} e^{i\delta}
\end{pmatrix},$$

(14)

where $x = \mu_2^2 / \mu_\nu^2 \simeq m_\mu / m_\tau, y = \mu_2^2 / \mu_\nu^2 \simeq \sqrt{2}m_e / m_\mu$ and $e^{i\delta} = \frac{\mu_2 \mu_\nu}{|\mu_2 | |\mu_\nu|}$. On other hand, the neutrino mass matrix can be diagonalized as

$$U_\nu^T M_\nu U_\nu = \text{diag}(m_{\nu_1} e^{i\phi_1 - i\phi_\nu}, m_{\nu_2} e^{i\phi_1 + i\phi_\nu}, m_{\nu_3}).$$

(15)
where the phases $\phi_1, \phi_2$ are defined by $m_{\nu_1} \sin \phi_1 = m_{\nu_2} \sin \phi_2 = m_{\nu_3} \sin \phi_3$. Then, the Majorana masses of the left-handed neutrinos take the form

$$M_{\nu} = M_{\nu D} \bar{M}^{-1}(M_{\nu D})^T = \begin{pmatrix}
2(\rho')_2^2 & 0 & 2\rho_2^* \rho'_4 \\
0 & 2(\rho'_2)^2 & 0 \\
2\rho_2^* \rho'_4 & 0 & 2(\rho'_4)^2 + (\rho')_2^2
\end{pmatrix}. \quad (16)$$

In this expression, we consider the rescaled parameters $\rho'_2 = (\mu'_2) M_1, \rho'_3 = (\mu'_3) M_3$. $\rho'_3$ is a real or purely imaginary number. Then, the $U_{\nu}$ matrix takes the form

$$U_{\nu} = \begin{pmatrix}
-\sin \theta_{12} & \cos \theta_{12} e^{i\phi_{12}} & 0 \\
0 & 0 & 1 \\
\cos \theta_{12} e^{-i\phi_{12}} & \sin \theta_{12} & 0
\end{pmatrix}. \quad (17)$$

With that, we can obtain the form of the mixing matrix $V_{MNS}$ of the (13) using the expressions (14) and (17). We have two phases: $\delta$ is a Dirac phase and $\phi_{12}$ is a Majorana phase. We found expressions for the $V_{MNS}$ parameters. In particular, for $\tan^2 \theta_{12}$ and setting $\delta = 0$, we have the following expression:

$$\tan^2 \theta_{12} = \frac{(m_{\nu_2}^2 - m_{\nu_3}^2 \sin^2 \phi_2)^{1/2} - m_{\nu_3}}{(m_{\nu_2}^2 - m_{\nu_3}^2 \sin^2 \phi_2)^{1/2} + m_{\nu_3}} \cos \phi_{12}. \quad (18)$$

In the present model, the experimental restriction $|\Delta m_{23}^2| < |\Delta m_{23}^2|$ implies an inverted neutrino mass spectrum $m_{\nu_3} < m_{\nu_1}, m_{\nu_2}$. The mass $m_{\nu_2}$ assumes its minimal value when $\sin \phi_{12} = 0$, then $m_{\nu_{2,\text{min}}} \simeq \sqrt{\Delta m_{23}^2} / \sin 2\theta_{12}$. With $1.3 \times 10^{-3} \text{eV}^2 \leq \Delta m_{23}^2 \leq 3.0 \times 10^{-3} \text{eV}^2$ and $0.83 \leq \sin 2\theta_{12} \leq 1$, we get $0.036 \text{eV} \leq m_{\nu_{2,\text{min}}} \leq 0.066 \text{eV}$. Comparing with the experimental values, we see that our prediction based on the exact $S_3 \times Z_2$ symmetry in the leptonic sector is consistent with the recent experimental data on neutrino physics.

6. Flavour Changing Neutral Currents (FCNCs)

In the models with more than one SU(2) Higgs doublet, there are tree level FCNC in the Higgs sector. We calculate the flavour changing Yukawa couplings to the neutral Higgs fields, $H_S^0$ and $H_I^0 (I = 1, 2)$, under the assumption $(H_S^0) = (H_I^0) = (H_I^0) \approx 246/\sqrt{2} \text{ GeV} \approx 142/\sqrt{2} \text{ GeV}$. Then, the FCNC Yukawa couplings can be explicitly calculated and the corresponding Lagrangian is of the form:

$$L_{\text{FCNC}} = (\overline{E}_{aL} Y_{ab}^{ES} E_{bR} + \overline{U}_{aL} Y_{ab}^{US} U_{bR} + \overline{D}_{aL} Y_{ab}^{DS} D_{bR}) H_S^0 + \text{h.c.}$$

$$+ (\overline{E}_{aL} Y_{ab}^{E1} E_{bR} + \overline{U}_{aL} Y_{ab}^{U1} U_{bR} + \overline{D}_{aL} Y_{ab}^{D1} D_{bR}) H_I^0 + \text{h.c.}$$

$$+ (\overline{E}_{aL} Y_{ab}^{E2} E_{bR} + \overline{U}_{aL} Y_{ab}^{U2} U_{bR} + \overline{D}_{aL} Y_{ab}^{D2} D_{bR}) H_I^0 + \text{h.c.} \quad (19)$$

Here, the matrices $E, U$ and $D$ stand for the mass eigenstates. All the non diagonal elements in the $Y$‘s are responsible for tree level FCNC processes. For instance, the amplitude for the FCNC process $\mu^- \rightarrow e^+e^-e^-$ is proportional to $(Y^{E1})_{11}(Y^{E1})_{21} \approx 10^{-8}[4]$. Then, we find that its branching ratio is estimated to be

$$B(\mu \rightarrow 3e) \sim 10^{-15}(M_W/M_H)^4 < 10^{-12}, \quad (20)$$

where $M_W$ and $M_H$ are the $W$ boson mass and Higgs boson mass, respectively, and the value $10^{-12}$ is the experimental upper bound. Note that, because of the three Higgs fields, the term $(Im Y)$ contribute to $CP$-violating amplitudes which are not taken into account by the phase of the mixing matrix $V_{\text{CKM}}$. Therefore, the four phases introduced in the mass matrices through $\mu'^{u,d}_{1,3}$ can, in principle, be measured.
7. Conclusions
By introducing three $SU(2)_L$ Higgs doublet fields in the theory, we extended the concept of flavour and generations to the Higgs sector and formulated a minimal $S_3$–invariant extension of the Standard Model. A definite structure of the Yukawa couplings is obtained which permits the calculation of mass and mixing matrices for quarks and leptons in a unified way. A further reduction of free parameters is achieved in the leptonic sector by introducing a $Z_2$ symmetry. The three charged lepton masses, three Majorana masses of the left-handed neutrinos and the three mixing angles are computed in terms of only seven free parameters, in agreement with the experimental observations at this time. We computed the masses and mixing matrices of quarks as well as the CP-violating Kobayashi Maskawa phase in terms of ten free parameters in agreement with the latest experimental values. From these studies we hypothesized that the flavour symmetry which is exact at the Fermi scale is the permutational symmetry $S_3$. The analysis of the FCNC’s in the Higgs sector are not complete, since we gave only one set of consistent parameter values. A more complete study is being made.

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