Abstract. We consider dynamic boundary conditions involving non-local operators. Our analysis includes a detailed description of such operators together with their relations with random times and random (additive) functionals. We provide some new characterizations for the boundary behaviour of the Brownian motion based on the interplay between non-local operators and boundary value problems. Our main focus is on Feller-Wentzell diffusions. We first consider the instructive case of the real line, then we extend our results on star graphs with trapping points or repulsive vertices.

Keywords. Non-local operators, dynamic boundary conditions, star graphs, Brownian motions

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1. INTRODUCTION

Let us consider the non-local boundary value problem (NLBVP)

\[
\begin{aligned}
\dot{u} &= Au & t > 0, \ x \in E \\
\dot{v} &= Tu, \ \mathcal{D}_t^\psi v = \dot{v} \ast \Pi^\psi & t > 0, \ x \in \partial E \\
\mathcal{D}_t^\psi v + \Phi(B)u &= 0 & t > 0, \ x \in \partial E \\
u(0, x) &= f(x) & x \in \overline{E}
\end{aligned}
\]  

(1)

for a good initial datum \(f\) and some operators \(A, B\) for which \(D(\Phi(B)) \subset D(B) \subset D(A)\). The symbol \(\mathcal{D}_t^\psi\) denotes a convolution-type operator to be characterized below. Suppose that \(E \subset \mathbb{R}^d\) is a smooth domain and denote by \(T\) the trace operator properly defined with respect to a finite Borel measure on \(\partial E\). In general, the operator \(\Phi(B)\) can be associated with a boundary motion. The reader can consult [17] for a discussion in case \(\Phi(B) = B\) is local. Consider the case in which \((B, D(B))\) generates a Markov process on \(\partial E\) and \(\Phi(B)\) is a non-local operator defined via the Bernstein symbol \(\Phi\). In case \(\partial E\) is a compact manifold without boundary, the probabilistic reading of the non-local Cauchy problem (NLCP)

\[
\begin{aligned}
\mathcal{D}_t^\psi u &= \Phi(B)u, \quad t > 0, \ x \in \partial E \\
u(0, x) &= f(x), \quad x \in \partial E
\end{aligned}
\]  

(2)

can be given in terms of a time-changed Markov process where the time change is obtained via subordinate semigroup associated with \(\Phi(B)\) and an independent non Markov random time associated with \(\mathcal{D}_t^\psi\). Roughly speaking the operator \(\Phi(B)\) introduces jumps whereas the operator \(\mathcal{D}_t^\psi\) introduces delays. In turn, the non-local boundary condition in (1) introduces a behaviour near the boundary with a very tricky characterization. In essence, for the process on \(E\), we have jumps near \(\partial E\) and delays on \(\partial E\). Depending on \(d\), a process on \(E\) may have interesting properties. For example a Brownian motion on \(E \subset \mathbb{R}\) can hit its starting point infinitely many times. As \(d > 1\), there are many results on the visited points of a Brownian motion. For \(d = 2\) every point is almost surely not visited and despite of this, there exists a random set of points visited infinitely often. In both cases \((d = 1\) and \(d = 2\)) such sets are big and uncountable. For the process associated with (1) we focus on the set of boundary points with their hitting times. Although the problem (1) for \(d > 1\) is interesting, the case \(E = [0, \infty)\) turns out to be very instructive. The boundary point \(\{0\}\) has zero Lebesgue measure and represents a repulsive point as prescribed by \(\Phi(B)\). On the other hand the operator \(\mathcal{D}_t^\psi\) introduces random holding times on the boundary if \(\Phi(B) = B\) or alternatively on a random point in the interior, after the jump, in case \(\Phi(B)\) is non-local.

The intuition and the first rigorous results on boundary conditions associated with jumps away from the boundary have been given in [24; 29] in case \(E = [0, \infty)\). However, the connection with non-local operators has a recent development. The NLCP (2) on compact domains has been investigated by many researcher in the last decades. In case of Caputo-Džrbashjan derivative in [41] the authors introduced the Fractional Cauchy Problem (FCP) on bounded domains and previously in [39] the space-time fractional diffusion equation on the real line has been investigated. The general problem has been studied in [33] and [52]. Concerning the NLBVP, in the literature there are clear references on dynamic boundary conditions. The non-local dynamic boundary condition as in (1) has been introduced, to the best of our knowledge, in [18] for bounded domains and in [16; 17] for the real line. We refer to the book [45] for the discussion on the visited points of the Brownian motion.

In the present work we first provide rigorous results for the problem (1) on the real line, then we use such results in order to investigate the associated motions on networks. Here we extend the results in [4] and consider the problem to find a solution together with the probabilistic characterization to

\[
\begin{aligned}
\eta \mathcal{D}_t^\psi u(t, x) + \sum_{e \in \mathcal{E}} \rho_e \mathcal{D}_t^\psi_{\mathcal{E}, e} u(t, x) &= 0 & t > 0, \ x \in \mathcal{G} \\
u(0, x) &= f(x) & x \in \mathcal{G}
\end{aligned}
\]

where \(\mathcal{G}\) is a metric graph. We respectively focus on the half-line \(E = [0, \infty)\) and the metric graph obtained from the collection of half-lines

\[E = \bigsqcup [0, \infty)\].\]
We study the non-local dynamic boundary condition

\[ \mathcal{D}_t^\Psi v + \Phi(B)u = 0 \]

where

\[ \Phi(B)u = -D_x^\Phi u - u, \quad \text{on } (0, \infty) \times \{0\} \]

for the problem on the positive half-line and

\[ \Phi(B)u = -\sum_{e \in \mathcal{E}} \rho_e D_x^\Phi u_e, \quad \text{on } (0, \infty) \times \{v\} \]

for the problem on \( \mathcal{G} \), the star graph.

The non-local operators appearing above are given by the Caputo-Džrbašjan (type) derivative \( \mathcal{D}_t^\Psi \) and the Marchaud (type) derivative \( D^\Phi_x \), where \( \Psi \) and \( \Phi \) are Bernstein symbols characterizing the operators. The probabilistic representation of the solution is written in terms of a Brownian motion reflected at zero, we show that the spatial condition controls an additive part acting on the reflecting Brownian motion whereas the time condition introduces a time change acting on the local time (at zero) of the reflecting Brownian motion. The additive part pushes away from zero the process and the time change forces the process to stop for a random amount of time. Due to the right-continuity, the stop occurs immediately after the jump. Furthermore, the process stops for an independent amount of time and it jumps for an independent distance from the origin.

In Section 2 we introduce some basic facts and notations about subordinators and inverses. We discuss the relations between the operators we deal with and the associated processes. In particular, the left and right Marchaud (type) derivatives \( D^\Phi_x^- \) and \( D^\Phi_x^+ \) can be respectively regarded as the generator and its adjoint for a subordinator with symbol \( \Phi \). From this point of view, our analysis completely agrees with the boundary conditions introduced by Feller in [24]. Then, we discuss the heat equation on the half-line with non-local boundary conditions in Section 3. We provide the main result (Theorem 3.1) which generalizes the work [17] for the time operator and the work [24] for the space operator. We also provide a discussion on the probabilistic representation of the solution by extending the results given in [16, 17] and the probabilistic representation given by Itô and McKean in [30, 29]. In Section 4 we generalize our results on metric graphs. In Section 5 we discuss on interpretations, applications and extensions of our results.

Among the motivations driving our investigation into these models, lies a number of potential applications, from financial models to models of traffic for example. Specifically, dynamic non-local conditions prove invaluable for constructing a model of sticky expectations, wherein investors adjust their beliefs at a deliberately gradual pace. Meanwhile, spatial boundary conditions characterized by abrupt data jumps find relevance, for instance, in addressing structural breaks within financial systems. To further expand the scope of our research, we have extended our results to star graphs, offering a framework for networks with distinct behaviours. For example, traffic models with different holding times for given nodes.

In order to streamline the notation as much as possible we write

\[ \dot{u} = \frac{\partial u}{\partial t}, \quad u' = \frac{\partial u}{\partial x}, \quad u'' = \frac{\partial^2 u}{\partial x^2}. \]

Moreover, we write

\[ W^{1,p}(0, \infty) = \{ u \in L^p(0, \infty) : u' \in L^p(0, \infty) \} \]

and

\[ W_0^{1,p}(0, \infty) = \{ u \in W^{1,p}(0, \infty) : u(0) = 0 \} \]

with \( p \in [1, \infty] \) for the Sobolev spaces. For the reader’s convenience we list below some symbols:

- \( H^\Phi, H^\Psi \) are independent subordinators with symbols \( \Phi, \Psi \);
- \( L^\Phi, L^\Psi \) are their inverse processes;
- \( \gamma(X) \) is the local time \( \gamma \) of the process \( X \). If no otherwise specified \( \gamma = \gamma(B^+) \);
- \( B^+ \) is a reflected Brownian motion;
- \( B^* = B^+ + L \circ \gamma \) is driven by (space) non-local boundary conditions;
- \( B^* = B^* \circ S^{-1} \) is driven by (space/time) non-local boundary conditions.
2. Preliminaries and notations

We recall some basic facts and introduce some notations.

2.1. Subordinators and symbols. Let \( H^\Phi = \{H^\Phi_t, \ t \geq 0\} \) be a subordinator (see [3] for details). Then, \( H^\Phi \) can be characterized by the Laplace exponent \( \Phi \), that is,

\[
E_0[\exp(-\lambda H^\Phi_t)] = \exp(-t\Phi(\lambda)), \quad \lambda \geq 0.
\]

We denote by \( E_x \) the expected value with respect to \( P_x \) where \( x \) is the starting point. Since \( \Phi(\lambda) \) is the Laplace exponent of a subordinator, it is uniquely characterized by the pair of non-negative real numbers \( (\kappa, d) \) and by the Lévy measure \( \Pi^\Phi \) on \((0, \infty)\) such that \( \int_0^\infty (1 + z) \Pi^\Phi(dz) < \infty \). For the symbol \( \Phi \), the following Lévy-Khintchine representation holds ([51, Theorem 3.2])

\[
\Phi(\lambda) = \kappa + d\lambda + \int_0^\infty (1 - e^{-\lambda z}) \Pi^\Phi(dz), \quad \lambda > 0
\]

where the killing rate \( \kappa \) and the drift coefficient \( d \) are given by

\[
\kappa = \lim_{\lambda \to 0} \Phi(\lambda), \quad d = \lim_{\lambda \to \infty} \frac{\Phi(\lambda)}{\lambda}.
\]

The symbol \( \Phi \) is a Bernstein function (non-negative, non-decreasing and continuous, see for example [51]) uniquely associated with \( H^\Phi \) ([51, Theorem 5.2]). For the reader’s convenience we also recall that

\[
\Phi(\lambda) = d + \int_0^\infty e^{-\lambda z} \pi^\Phi(z)dz, \quad \pi^\Phi(z) = \kappa + \Pi^\Phi(z, \infty)
\]

where \( \pi^\Phi \) is the so called tail of the Lévy measure \( \Pi^\Phi \). In this paper we only consider symbols for which

\[
\kappa = 0, \quad d = 0.
\]

We also define the process \( L^\Phi = \{L^\Phi_t, \ t \geq 0\} \), with \( L^\Phi_0 = 0 \), as the inverse of \( H^\Phi \), that is

\[
L^\Phi_t = \inf\{s > 0 : H^\Phi_s > t\}, \quad t > 0.
\]

Activity: Let \( \Pi^\Phi \) be the Lévy measure of the subordinator \( H^\Phi \). We recall that \( H^\Phi \) has finite (respectively infinite) activity if \( \Pi^\Phi(0, \infty) < \infty \) (respectively \( \Pi^\Phi(0, \infty) = \infty \)). The associated Lévy measure plays a role in the definition of non-singular (respectively singular) non-local operator we will deal with.

In Section 3.4 we consider subordinators with finite activity and discuss the spacial case of stochastic resetting. Further on, we mainly focus only on strictly increasing subordinators with infinite activity and zero drift (see [31, Theorem 2.13]). Thus, the inverse process \( L^\Phi \) turns out to be a continuous process. In particular, \( H^\Phi \) may have jumps and the inverse \( L^\Phi \) has non decreasing paths. Notice that, an inverse process can be regarded as an exit time for \( H^\Phi \). By definition, we also have

\[
P_0(H^\Phi_t < s) = P_0(L^\Phi_s > t), \quad s, t > 0.
\]

Let us introduce \( h, l \) for which

\[
P_0(H^\Phi_t \in I) = \int_I h(t, x)dx, \quad P_0(L^\Phi_t \in I) = \int_I l(t, x)dx,
\]

for a given set \( I \subset (0, \infty) \). From (3), we have that

\[
\int_0^\infty e^{-\lambda x}h(t, x)dx = e^{-t\Phi(\lambda)}, \quad \lambda > 0
\]

and from [43, formula (3.13)], we get

\[
\int_0^\infty e^{-\lambda t}l(t, x)dt = \frac{\Phi(\lambda)}{\lambda}e^{-x\Phi(\lambda)}, \quad \lambda > 0.
\]

By using the initial value theorem for the Laplace transform, we observe that, from (7),

\[
h(t, 0) = \lim_{\lambda \to \infty} \lambda e^{-t\Phi(\lambda)}
\]

only in case \( \forall t, h(t, \cdot) \) is bounded. For example, \( h(t, 0) < \infty \). The limit above depends on \( \Phi(\lambda) \) and the time variable \( t \). This fact is known as time dependent property (see Definition 23.1 in [31]). Thus, the fact that \( h(t, x) = 0 \) as \( x \leq 0 \) plays a special role in our discussion. Indeed, this condition has non trivial consequences and in general, for some \( \Phi \) and some \( t > 0 \),

\[
h(t, \cdot) \not\in W^1_0(0, \infty).
\]
The gamma subordinator with \( \Phi(\lambda) = \ln(1 + \lambda) \) is an example (see [15]).

Further on we will also introduce the processes \( H^\Psi = \{H^\Psi_t, t \geq 0\} \), independent of \( H^\Phi \), and the inverse \( L^\Psi = \{L^\Psi_t, t \geq 0\} \) of \( H^\Psi \) with symbol

\[
\Psi(\lambda) := \int_0^\infty (1 - e^{-\lambda z})\Pi^\Psi(dz), \quad \lambda > 0.
\]

We always assume that \( H^\Phi \) and \( H^\Psi \) are independent subordinators.

2.2. Marchaud (type) operators. For a continuous (causal) function \( u \) on \( \mathbb{R} \) extended with zero on the negative part of the real line, that is \( u(x) = 0 \) if \( x \leq 0 \), we define the right Marchaud (type) derivative

\[
D^\Phi_{x+} u(x) = \int_0^\infty (u(x) - u(x - y))\Pi^\Phi(dy)
\]

and the left Marchaud (type) derivative

\[
D^\Phi_{x-} u(x) = \int_0^\infty (u(x) - u(x + y))\Pi^\Phi(dy).
\]

If \( \Pi^\Phi \) is the Lévy measure associated to a stable subordinator, formulas (10) and (11) respectively coincide with the right and the left Marchaud derivatives, usually denoted by \( D^\Phi_{x+} \) and \( D^\Phi_{x-} \) respectively. The reader can consult the famous book [50, formula (5.57) and (5.58)] or the paper [25, section 6.1] for a nice recent discussion. The operators \( D^\Phi_{x+} \) and \( D^\Phi_{x-} \) can be defined on different spaces depending on \( \Pi^\Phi \). For a general definition, given the symbol \( \Phi \), we consider \( u \) bounded and locally Lipschitz continuous, then

\[
|D^\Phi_{x \pm} u(x)| \leq \int_0^1 |u(x) - u(x \mp y)|\Pi^\Phi(dy) + \int_1^\infty |u(x) - u(x \mp y)|\Pi^\Phi(dy)
\leq K \int_0^1 y\Pi^\Phi(dy) + 2||u||_\infty \int_1^\infty \Pi^\Phi(dy)
\leq (K + 2||u||_\infty) \int_1^\infty (1 \land y)\Pi^\Phi(dy) < \infty.
\]

Indeed, the Lipschitz property for a positive constant \( K > 0 \) holds in the first integral and the boundedness of \( u \) holds in the second integral. Since \( \int (1 \land z)\Pi^\Phi(dz) < \infty \), then the last inequality directly emerges.

We provide the following two examples in case of explicit representations for \( \Pi^\Phi \):

- The case of stable subordinator with \( \Phi(\lambda) = \lambda^\alpha \) for \( \alpha \in (0, 1) \) and \( \Pi^\alpha(dy) = \frac{\alpha}{(1 - \alpha)\Gamma(1 - \alpha)} \frac{dy}{y^{\alpha+1}} \). The operators (10) and (11) are therefore defined for locally \( \gamma \)-Hölder continuous functions with \( \gamma > \alpha \) (it can be easily checked by calculation);

- The case of gamma subordinator with \( \Phi(\lambda) = a \ln(1 + \lambda/b) \) for \( a, b > 0 \) and \( \Pi^\Phi(dy) = a \frac{e^{-by}}{y} dy \). This case may be a bit demanding. This is due to the time dependent continuity of \( h(t, x) \) at \( x = 0 \). The operator (10) is defined for \( \gamma \)-Hölder continuous functions with \( \gamma > 0 \) (see [15] for details).

We also underline the following fact. The operator \( D^\Phi_{x+} \) can be obtained by the Phillips’ representation ([48]) in the set of functions extended with zero on the negative part of the real line. Indeed, for the shift semigroup \( S_y u(x) = u(x - y) \) we have the representation

\[
D^\Phi_{x+} u(x) = \int_0^\infty (u(x) - S_y u(x))\Pi^\Phi(dy)
\]

for which

\[
\int_0^\infty e^{-\lambda x}D^\Phi_{x+} u(x)dx = \Phi(\lambda) \int_0^\infty e^{-\lambda x} u(x)dx
\]

where we used (4) with \( \kappa = 0 \) and \( d = 0 \) in order to obtain \( \Phi(\lambda) \). The representation given by Phillips well agrees with a Riemann-Liouville (type) operator described in Section 8 and denoted by \( D^\Phi_{(0, x)} \).

We will investigate the role of \( D^\Phi_{x+} u \) and \( D^\Phi_{x-} u \) in the governing equations of \( H^\Phi \). We notice that if \( u \in W^{1, \infty}(0, \infty) \), then the Marchaud (type) derivatives (10) and (11) are well defined almost everywhere. Indeed, with the first inequality of (12) at hand, we use the fact that \( u \) is essentially bounded and locally Lipschitz almost everywhere (its derivative is bounded).
2.3. Caputo-Džrbašjan (type) operators. Let $M > 0$ and $w \geq 0$. Let $\mathcal{M}_w$ be the set of (piecewise) continuous function on $[0, \infty)$ of exponential order $w$ such that $|u(t)| \leq Me^{wt}$. Let $u \in \mathcal{M}_0 \cap C[0, \infty)$ with $u' \in \mathcal{M}_0$. Then we define the Caputo-Džrbašjan (type) derivative as the convolution
\[\mathcal{D}_x^\Phi u(x) := \int_0^x u'(y) \Pi_\Phi (x - y)dy\] (14)
whose Laplace transform writes
\[\int_0^\infty e^{-\lambda x} \mathcal{D}_x^\Phi u(x)dx = \Phi(\lambda) \tilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0), \quad \lambda > 0\] (15)
($\tilde{u}$ denotes the Laplace transform of $u$). For explicit representations of the operator $\mathcal{D}_x^\Phi$ see also the recent works [33; 52; 13].

**Proposition 2.1.** We have that
\[||\mathcal{D}_x^\Phi u||_p \leq ||u'||_p \left( \lim_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda} \right), \quad p \geq 1.\] (16)

**Proof.** First we observe that the Laplace transform can be considered in order to obtain
\[\lim_{\lambda \to 0} \int_0^\infty e^{-\lambda x} ||\mathcal{D}_x^\Phi u||_p dx = ||\mathcal{D}_x^\Phi u||_p^p.\]
However, from the Young’s inequality, we have
\[||\mathcal{D}_x^\Phi u||_p \leq ||u'||_p \left( \int_0^\infty \Pi_\Phi (y)dy \right)\]
where only the last integral can be obtained via Laplace transform. Indeed,
\[\int_0^\infty \Pi_\Phi (y)dy = \lim_{\lambda \to 0} \int_0^\infty e^{-\lambda y} \Pi_\Phi (y)dy\]
= [from (5) with $\kappa = d = 0$]
\[= \lim_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda}, \quad \lambda > 0.\]
\[\square\]

We remark that $\lim_{\lambda \to 0} \Phi(\lambda)/\lambda$ is finite only in some cases (see [8]). If $\lim_{\lambda \to 0} \Phi(\lambda)/\lambda$ is finite, then (16) gives a clear information about the cases of measurable or essentially bounded functions. In particular, we have that for $u' \in L^1(0, \infty)$, if
\[\lim_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda} < \infty,\]
we can apply (16) and the derivative exists almost everywhere. By considering the well-known relation
\[\mathcal{D}_x^\Phi u(x) = \mathcal{D}_x^\Phi (0, x)(u(x) - u(0))\]
(the definition of $\mathcal{D}_x^\Phi (0, x)$ has been postponed in Section 8) we have equivalence between derivatives as $u(0) = 0$.

We can check that, in this case, (15) coincides with (13). This entails a connection with causal functions, that is $\mathcal{D}_x^\Phi u$ is well-defined for $u \in W^{1,\infty}(0, \infty)$.

A further characterization can be given by considering the class of functions for which
\[\exists M > 0 : |u'(y)| \leq M \frac{\Lambda^\Phi (dy)}{dy}\] (17)
where
\[\Lambda^\Phi (dy) = \int_0^\infty P_0(H_t^\Phi \in dy)dt\]
is the potential measure for the subordinator $H^\Phi$ with symbol $\Phi$. Since $\Lambda^\Phi$ and $\Pi_\Phi$ are associated Sonine kernels for which
\[\int_0^x \Pi_\Phi (x - y)\Lambda^\Phi (dy) = 1\]
and
\[|\mathcal{D}_x^\Phi u(x)| \leq M \int_0^x \Pi_\Phi (x - y)\Lambda^\Phi (dy),\] (18)
we get that $|\mathcal{D}_{x}^{\alpha}u(x)|$ is uniformly bounded on $(0, \infty)$.

The well-known Caputo-Dzhrbashyan derivative is associated with $\Phi(\lambda) = \lambda^{\alpha}$, $\alpha \in (0, 1)$. It has been introduced independently in [10; 11; 12] and [19; 20].

2.4. Subordinators and governing equations. We are now ready to discuss the connection between operators and subordinators. The governing equations of $H_0^\Phi$ and $L_0^\Phi$ have been already studied in the literature (see for example [33; 52] for results with special attention only about the fundamental solutions). Based on the previous discussion, for the reader’s convenience we provide a clear statement for such equations.

Excluding $\Phi$ associated with the time dependent continuity, we consider the problems

$$
\begin{align*}
&\begin{cases}
\frac{\partial}{\partial t} h_f(t, x) = -\mathcal{D}_{x}^{\alpha} h_f(t, x) & t > 0, \ x \in (0, \infty), \\
h_f(t, x) = 0 & t > 0, \ x = 0 \\
h_f(0, x) = f(x), \ f \in W_{0,1}^{1,1}(0, \infty) \ x \in (0, \infty)
\end{cases} \quad (19)
\end{align*}
$$

and

$$
\begin{align*}
&\begin{cases}
\mathcal{D}_{x}^{\alpha} l_f(t, x) = -\frac{\partial}{\partial x} l_f(t, x) & t > 0, \ x \in (0, \infty), \\
l_f(t, x) = 0 & t > 0, \ x = 0, \\
l_f(0, x) = f(x), \ f \in L^{p}(0, \infty) \ x \in (0, \infty).
\end{cases} \quad (20)
\end{align*}
$$

Proposition 2.2. We have that

$$
C((0, \infty), W_{0,1}^{1,1}(0, \infty)) \ni h_f(t, x) = \int_{0}^{x} f(x - y) h(t, y) dy
$$
with probabilistic representation

$$
h_f(t, x) = E_{0}[f(x - H_{t}^{\Phi})1_{(t < L_{t}^{\Phi})}]
$$

is the unique solution to (19).

Proof. From the Laplace transforms technique we simply obtain that $h_f$ is a solution. Moreover, the solution is continuous, then by inverting the Laplace transform we get a unique solution. Since $h_f$ is a convolution and $h$ is a probability density we get $|h_f(t, \cdot)|_1 \leq \|f\|_1$ and $|h_f(t, \cdot)|_1 \leq \|f\|_1$ for every $t > 0$. Thus, $h_f(t, \cdot) \in W_{0,1}^{1,1}(0, \infty)$.

Proposition 2.3. We have that

$$
C(W^{1,\infty}(0, \infty), (0, \infty)) \ni l_f(t, x) = \int_{0}^{x} f(x - y) l(t, y) dy
$$
with probabilistic representation

$$
l_f(t, x) = E_{0}[f(x - L_{t}^{\Phi})1_{(t < H_{t}^{\Phi})}]
$$

is the unique solution to (20).

Proof. The proof follows the same arguments as in the previous proof.

Concerning the limit (9), it worth mentioning the following facts for the density $h$ and the function $h_f$. As $\lambda \to \infty$ we get $\lambda h_f(t, \lambda) \to h_f(t, 0)$ in the following cases: i) For $h(t, \cdot)$ bounded for every $t > 0$, we have that

$$
\lambda h_f(t, \lambda) = f(\lambda)e^{-\Phi(\lambda)} \to \left(\lim_{\lambda \to \infty} f(\lambda)\right) \cdot h(t, 0)
$$

with $h(t, 0) < \infty$, $\forall t > 0$. Since $f \in W_{0}^{1,1}(0, \infty)$, then

$$
\left(\lim_{\lambda \to \infty} f(\lambda)\right) \cdot h(t, 0) \leq \left(\lim_{\lambda \to \infty} \frac{1}{\lambda}\right) \|f\|_{\infty} h(t, 0) = 0.
$$

That is $h_f(t, 0) = 0$ for any $h(t, \cdot)$ bounded; ii) On the other hand, for a bounded $f$,

$$
\lambda h_f(t, \lambda) = \lambda f(\lambda)e^{-\Phi(\lambda)} \to f(0) \cdot \left(\lim_{\lambda \to \infty} e^{-\Phi(\lambda)}\right) = 0, \ \forall t \geq 0
$$

where $f(0) < \infty$ and $\Phi(\lambda) \to \infty$ as $\lambda \to \infty$. That is $h_f(t, 0) = 0$ for any symbol $\Phi$.

We also notice that $l_f(t, \cdot) \in L^{p}(0, \infty)$, $\forall t > 0$. Indeed, $\forall t \geq 0$ we simply have that

$$
\|l_f(t, \cdot)\|_{p} \leq \|f\|_{p}\|l(t, \cdot)\|_{1} = \|f\|_{p}
$$
Now we provide the following result which relates the Marchaud (type) operators previously introduced.

**Theorem 2.1.** For \( 0 < x < y \) and \( t > 0 \), we have that
\[
P_x(H^\Phi_t \in dy) = P_0(x + H^\Phi_t \in dy) = h(t, y - x)dy
\]
where the density \( h \) satisfies
\[
\dot{h} = -D^\phi_{y -}h = -D^\phi_xh. \tag{21}
\]

**Proof.** First we focus on \( \dot{h} = -D^\phi_xh \). For \( \lambda > 0 \), consider the Laplace transform \( \tilde{h}(t, x, \lambda) \) of the density \( h(t, y - x) \) given by
\[
\tilde{h}(t, x, \lambda) = E_x[e^{-\lambda H^\phi_t}] = E_0[e^{-\lambda x - \lambda H^\phi_t}] = e^{-\lambda x - \Phi(\lambda)}, \quad x > 0, \ t > 0.
\]
From (71) and (70) below in Section 8, we have
\[
-D^\phi_x \tilde{h}(t, x, \lambda) = \int_x^\infty (-\lambda e^{-\lambda y - t\Phi(\lambda)}) \Pi^\phi(y - x) \, dy
\]
\[
= -\lambda e^{-t\Phi(\lambda)} \int_x^\infty e^{-\lambda y} \Pi^\phi(y - x) \, dy
\]
\[
= -\lambda e^{-\lambda x - t\Phi(\lambda)} \int_0^\infty e^{-\lambda z} \Pi^\phi(z) \, dz
\]
\[
= -\Phi(\lambda) \tilde{h}(t, x, \lambda)
\]
where in the last step we have used (5). Thus, for \( \lambda > 0 \), the function \( \tilde{h} \) solves the equation
\[
\frac{\partial \tilde{h}}{\partial t} = -D^\phi_x \tilde{h}, \quad \tilde{h}(0, x) = e^{-\lambda x}, \quad x > 0. \tag{22}
\]
We now focus on \( \dot{h} = -D^\phi_yh \) for \( y > x > 0 \), \( t > 0 \) for which we have that
\[
D_1^\phi h(t, y - x)1_{(y > x)} = -\int_0^\infty \left( \tilde{h}(t, y - s - x)1_{(x, \infty)}(y - s) - h(t, y - x)1_{(x, \infty)}(y) \right) \Pi^\phi(ds)
\]
and therefore, the Laplace transform is given by
\[
\int_0^\infty e^{-\lambda y} D_1^\phi h(t, y - x) \, dy = -\int_0^\infty \left( e^{-\lambda(s + x) - t\Phi(\lambda)} - e^{-\lambda x - t\Phi(\lambda)} \right) \Pi^\phi(ds)
\]
\[
= -e^{-\lambda x - t\Phi(\lambda)} \int_0^\infty \left( e^{-\lambda s} - 1 \right) \Pi^\phi(ds)
\]
\[
= \Phi(\lambda) e^{-\lambda x - t\Phi(\lambda)}.
\]
Thus, we get that
\[
\frac{\partial \tilde{h}}{\partial t} = -\Phi(\lambda) \tilde{h} = \int_0^\infty e^{-\lambda y} (-D^\phi_yh1_{(y > x)}) \, dy
\]
which is the claimed result. \( \square \)

3. **Motions on half-lines**

### 3.1. Non-local Boundary Value Problems

In this section we discuss the connection between subordinators and non-local boundary conditions. According with [42] we introduce the regenerative set
\[
M = \{(t, \omega) \subset \mathbb{R} \times \Omega : t = H^\Phi_z(\omega) \text{ for some } z \geq 0\}
\]
where \( H^\Phi_t = H^\Phi_0(\omega) \) is a subordinator with symbol \( \Phi \). This definition does not agree with the definition given for example in [40] and others but it allows a nice formulation of our problem. Moreover, we obtain a random closed set such that the right-hand portion of \( M \) is independent (and identically distributed) from the left-hand portion of \( M \) where the portions can be regarded as determined by a stopping time. The random set \( M \) is a set of renewal points for the inverse \( L^\Phi \). In particular, for a given path of the subordinator, the set \( M(\omega) = \{t : (t, \omega) \in M\} \) is a countable union of the intervals \([H^\Phi_{z-}, H^\Phi_z] \). For any \( t \geq 0 \) we define the last time of regeneration before \( t \),
\[
r_t := \sup\{s \leq t : s \in M(\omega)\}
\]
and the next time of regeneration after \( t \),
\[
R_t := \inf\{ s > t : s \in \mathcal{M}(\omega) \}
\]
for which \( r_t \leq t \leq R_t \). As noted in [42] and in [3, Section 2] we have that a.s.
\[
\Phi(t^-) := \Phi_{t+} := \Phi \circ (L_t^\gamma -) \quad \text{and} \quad R_t = H_{t+}^\Phi := H^\Phi \circ L_t^\Phi.
\]
We also introduce the age process
\[
A_t := t - r_t
\]
and the remaining lifetime
\[
L_t := R_t - t.
\]
Under a semi-Markov approach, for a given \( t \), the total amount of time the particle stops in its location will be given by \( A_t + L_t \). A particle that is already in its location according with \( A_t \) will move again according with \( L_t \).

We now introduce the main object we deal with, that is the process \( B^\bullet = \{ B_t^\bullet, t \geq 0 \} \) written as
\[
B_t^\bullet = B_t^+ + L_{\gamma_t},
\]
where \( L_{\gamma_t} := L \circ \gamma_t \) and \( \gamma_t = \gamma_t(B^+) \) is the local time at zero of the reflecting Brownian motion \( B^+ = \{ B_t^+, t \geq 0 \} \) on \([0, \infty)\). The process \( B^\bullet \) has been first introduced in [29] in order to provide a probabilistic representation for a class of problems involving general boundary conditions. In this regard, a current reading can be given in terms of Marchaud (type) operators. The probabilistic reading of (23) says that \( B^\bullet \) is a reflecting Brownian motion jumping away from the origin, with a length of the jump given by the subordinator \( H^\Phi \) running with the clock \( L_{\gamma_t}^\Phi \). Since \( \Pi^\Phi(0, \infty) = \infty \), then in each time interval the number of jumps of \( H^\Phi \) are infinite. A detailed analysis of the sample paths has be given in [29, section 12]. We provide a discussion below.

We remark that \( L^\Phi \circ H_t^\Phi = t \) almost surely, whereas the study of composition \( H^\Phi \circ L_t^\Phi \) is a bit demanding. We recall that ([2, Proposition 2, section III.2])
\[
P_0(r_t^- < t = R_t) = 0.
\]
If \( d = 0 \) in (4), then \( P_0(R_t > t) = 1 \) for every \( t > 0 \) ([2, Theorem 4, section III.2]). Thus, for a zero drift subordinator, the passage above any given level is almost surely realized by a jump. Notice that \( L_{\gamma_t} \) is constant as \( B^+ > 0 \).

Now we focus on the problem with non-local dynamic boundary conditions and provide the probabilistic representation of the solution.

Let us introduce \( D_0 = D_1 \cap D_2 \) where
\[
D_1 = \left\{ \varphi : \forall x, \varphi(\cdot, x) \in W^{1,\infty}(0, \infty) \text{ and } \lim_{x \downarrow 0} \nabla^\Phi_x \varphi(t, x) \text{ exists} \right\},
\]
\[
D_2 = \left\{ \varphi : \forall t, \varphi(t, \cdot) \in W^{1,\infty}(0, \infty) \text{ and } \lim_{x \downarrow 0} \nabla^\Phi_x \varphi(t, x) \text{ exists} \right\}.
\]
Let us define the random time
\[
S_t = t + H^\Psi(\eta L_{\gamma_t}^\Phi), \quad L_{\gamma_t}^\Phi := L^\Phi \circ \gamma_t
\]
and its inverse
\[
S_t^{-1} = \inf\{ s > 0 : S_s > t \}.
\]

**Theorem 3.1.** The solution \( v \in C([0, \infty), (0, \infty)) \cap D_0 \) to the problem
\[
\begin{aligned}
\dot{v}(t, x) &= v''(t, x) \quad t > 0, \ x \in (0, \infty), \\
\eta \nabla^\Phi_x v(t, x) &= -\nabla^\Phi_x v(t, x), \ \eta \geq 0 \quad t > 0, \ x = 0, \\
v(0, x) &= f(x), \ f \in C[0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty)
\end{aligned}
\]
has the probabilistic representation
\[
v(t, x) = \mathbb{E}_x \left[ f(B^\bullet \circ S_t^{-1}) \right], \quad (24)
\]
We postpone the proof in Section 6.

The lifetime of the process \( B^\bullet \circ S^{-1} \) (which basically moves on a given path of a reflecting Brownian motion) is infinite, that is \( R_0 \mathbf{1} = \infty \) and \( \forall x, v(\cdot, x) \not\in L^1(0, \infty) \) for a bounded initial datum \( f \). However, there is no hope to find \( v(\cdot, x) \in L^1(0, \infty) \) by also asking for \( f \in L^1(0, \infty) \) for which we only have

\[
R_0^D f(x) = 2 \int_0^\infty (y \wedge x) f(y) dy < \infty
\]

in the potential \( R_\lambda f = R_\lambda^D f + \bar{R}_\lambda f \). The last formula can be obtained from (58). Then we ask for \( f \in L^\infty \) in order to obtain \( v(\cdot, x) \in L^\infty(0, \infty), \forall x \).

The previous result is concerned with the following particular cases:

\( \eta = 0 \) Non-local space conditions. The problem takes the form

\[
\begin{align*}
\dot{v}(t, x) &= v''(t, x) & t > 0, x \in (0, \infty), \\
D_{x-}^\Phi v(t, x) &= 0 & t > 0, x = 0, \\
v(0, x) &= f(x), & f \in C[0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty).
\end{align*}
\]

From the probabilistic point view, \( S_t^{-1} \) becomes the inverse to \( S_t = t \) and therefore, the solution \( v \) has the representation

\[
v(t, x) = \mathbb{E}_x \left[ f(B_t^\bullet) \right].
\]

We observe that

\[
-D_{x-}^\Phi v(t, x)|_{x=0} = \lim_{x \to 0} \int_0^\infty (v(t, x + y) - v(t, x)) \Pi^\Phi(dy)
\]

\[
= \int_0^\infty (v(t, y) - v(t, 0)) \Pi^\Phi(dy)
\]

(25)

which is the boundary condition introduced by W. Feller in [24] and studied by K. Itô and P. McKean in [29, section 12].

We now discuss the special case of the \( \alpha \)-stable subordinator \( H^\alpha \), characterized by the symbol

\[
\Phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-\lambda y}) \frac{dy}{y^{\alpha + 1}}, \quad \alpha \in (0, 1).
\]

Then, we write \( D_{x-}^\alpha \) in place of \( D_{x-}^\Phi \) which coincides with the well known Marchaud left derivative. Let us consider the following properties for the Marchaud derivatives

\[
D_{x+}^\alpha u(x) \to u(x) \quad \alpha \downarrow 0,
\]

(26)

\[
D_{x+}^\alpha u(x) \to u'(x) \quad \alpha \uparrow 1.
\]

(27)

Such continuity properties hold for the Riemann-Liouville derivatives (see for example [32, (2.2.5)]), we can adapt such results and obtain (26) and (27) just considering \( u \) continuously differentiable with \( u'(x) \) that vanishes at infinity as \( |x|^\alpha \mathbb{1}_{-\epsilon} \), \( \epsilon > 0 \) (see [50, Section 5.4]). The following analogue formulas hold true

\[
D_{x-}^\alpha u(x) \to u(x) \quad \alpha \downarrow 0,
\]

(28)

\[
D_{x-}^\alpha u(x) \to -u'(x) \quad \alpha \uparrow 1.
\]

(29)

Indeed, formula (72) below which still holds for \( D_{x+}^\Phi u \in L^\infty(0, \infty) \), together with (26) and (27) say that (28) and (29) hold in some sense. Thus, roughly speaking, as \( \alpha \downarrow 0 \) we get

\[
\begin{align*}
\dot{v}(t, x) &= v''(t, x) & t > 0, x \in (0, \infty), \\
v(t, x) &= 0 & t > 0, x = 0, \\
v(0, x) &= f(x), & f \in C(0, \infty) \cap L^\infty(0, \infty) \quad x \in (0, \infty).
\end{align*}
\]

(30)

and for \( \alpha \uparrow 1 \), we get

\[
\begin{align*}
\dot{v}(t, x) &= v''(t, x) & t > 0, x \in (0, \infty), \\
v'(t, x) &= 0 & t > 0, x = 0, \\
v(0, x) &= f(x), & f \in C[0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty).
\end{align*}
\]

(31)
It is well-known that almost surely, for $\alpha = 0$ the subordinator dies immediately whereas for $\alpha = 1$, it is the elementary subordinator $t$ ([3, Section 3.1.1]). The process (23) for $\alpha \downarrow 0$ becomes a killed Brownian motion as expected for the solution of (30). For $\alpha \uparrow 1$, the subordinator becomes $H_t^\alpha = t$ and the process (23) is a reflected Brownian motion that does not jump at the boundary point. It reflects and never dies. Indeed, it coincides with a reflected Brownian motion on $[0, \infty)$ as expected for the solution of (31).

**$\Phi = Id$** Non-local dynamic conditions. For the problem

$$\begin{cases}
\dot{v}(t, x) = v''(t, x) & t > 0, x \in (0, \infty), \\
\eta \mathcal{D}_t^\Phi v(t, x) = v'(t, x), & \eta \geq 0, t > 0, x = 0, \\
v(0, x) = f(x), & f \in C[0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty)
\end{cases}$$

the solution $v$ takes the following probabilistic representation

$$v(t, x) = E_x \left[ f(B^\Phi \circ \tilde{V}^{-1}) \right]$$

where $\tilde{V}_t = t + H^\Phi \circ (\eta \gamma_t)$. This results well agrees with [17, Theorem 3.1]. Here we have $H^\Phi_t = t$ a.s. implies that $L_t \equiv 0$ a.s. and $B^\Phi = B^\pi$. Moreover, $S_t$ equals in law $\tilde{V}_t = t + H^\Phi \circ (\eta \gamma_t)$.

**$\Psi = Id$** Dynamic condition and non-local space conditions. We have

$$\begin{cases}
\dot{v}(t, x) = v''(t, x) & t > 0, x \in (0, \infty), \\
\eta \dot{v}(t, x) = -D_{x^-}^\Phi v(t, x), & \eta \geq 0, t > 0, x = 0, \\
v(0, x) = f(x), & f \in C[0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty).
\end{cases}$$

The boundary condition can be written as $\Delta v(t, x)|_{x=0} = -D_{x^-}^\Phi v(t, x)|_{x=0}$ by means of which we can underline the sticky nature of the motion. Indeed, $\mathcal{D}_t^\Psi v = \partial_t v$ in Theorem 3.1 and $v$ satisfies the heat equation on $[0, \infty)$. The solution has the probabilistic representation

$$v(t, x) = E_x \left[ f(B^\Psi \circ \tilde{T}^{-1}) \right]$$

where $B^\pi_t$ is defined in (23) and $T_t = t + \eta L_{\gamma_t}^\Phi$ must be considered in place of $S_t$. Indeed, $H^\Psi_t = t$ almost surely.

**Remark 3.1. (Non-local reflection on bounded domains)** We only underline the main problem in dealing with $B^\Phi \circ S^{-1}$ in a bounded domain. Since we have jumps as the process approaches the boundary, then we have to control the jumping measure so that the process can not jump outside. As far as we know, there are no results in this direction under Lévy jumps. The authors are working on this case. The case of compactly supported measures (for the jumps) leads to non-local non-singular operators seems to be treatable. Concerning $\Phi = Id$, we have a clear picture of the problem given in [17] for a smooth domain $\Omega \subset \mathbb{R}^d$.

**3.2. Sample paths description.** We recap briefly the dynamics of the processes introduced in Section 3. The process $B^\pi$ defined in (23) is a reflected Brownian motion that jumps from 0 inside $(0, \infty)$ in a random point given by the jumps of $H^\Phi$. This process turns out to be right-continuous since $H^\Phi \circ L^\Phi_{\gamma_t}$ is the composition of the right-continuous subordinator $H^\Phi$ with its inverse (and continuous process) $L^\Phi_{\gamma_t}$.

The Sticky behavior is controlled by the non-local dynamic condition ($\eta > 0$) and realized via time-

![Figure 1](image_url)

**Figure 1.** A possible path for $B^\pi$. The case $\eta = 0$. The process is pushed away from the boundary point, it never hits $x = 0$ and it jumps randomly in $(0, \infty)$ according with $L_{\gamma_t}$.
As $B^+$ hits the origin, then $\gamma_t$ increases in a such a way that the time-change runs slowly and slows down the process. The boundary condition $\eta \dot{v}(t, 0) = -D_\Phi^x v(t, x)|_{x=0}$ leads to (33) which is the case of a (reflected) sticky process. Due to right-continuity of $H^\Phi$, determining an independent jump for $B^\bullet$, the time change $T^{-1}$ slows down $B^\bullet$ immediately after the jump. Then, an independent (exponential) holding time determines the sticky behavior at that state. The same arguments apply for the boundary condition $\eta D^\Phi \dot{v}(t, x)|_{x=0} = -D_\Phi^x v(t, x)|_{x=0}$ leading to (24). The holding time after the jump is no more exponential, it may have heavy tailed distribution. Figure 3 shows and example of these paths, a plateau occurs after a jump. We only underline once again that the plateaus are given by $H^\Psi$ whereas the jumps are given by $H^\Phi$ and they are both independent from $B^+$ which is the base process in the characterization we have proposed.

3.3. Holding and renewal times. Let us consider the construction given in [17] and [18]. We respectively denote by $\{e_i, i \in \mathbb{N}\}$ and $\bar{e}_i := H^\Psi_{e_i}, i \in \mathbb{N}\}$ the sequences of holding times (at the boundary) for the reflecting sticky Brownian motion $X$ on $[0, \infty)$ and the partially delayed reflecting Brownian motion $\bar{X}$ on $[0, \infty)$. We say that $\bar{X}$ is partially delayed in the sense that a delay effect only occurs on the boundary (the process behaves like a Brownian motion away from the boundary point $x = 0$). Indeed, $\bar{X}$ equals in law $B^+ \circ V^{-1}_t = B^+ \circ (t + H^\Psi \circ \eta\gamma_t)^{-1}$ where $\gamma_t = \gamma_t(B^+)$ and can be related with $B^\bullet$ as described in (32). The process $X$ equals in law $B^+ \circ V_t^{-1}$ where $V_t$ corresponds to the case $\Psi = Id$. In particular, $X$ is a reflected sticky Brownian motion which can be considered for the problem

$$
\begin{cases}
\dot{v}(t, x) = v''(t, x) & t > 0, x \in (0, \infty), \\
\eta \dot{v}(t, x) = v'(t, x) & t > 0, x = 0, \\
v(0, x) = f(x), & f \in C(0, \infty) \cap L^\infty(0, \infty) \quad x \in [0, \infty),
\end{cases}
$$

for $\eta \geq 0$. That is,

$$
v(t, x) = E_x[f(X_t)] = E_x[f(B^+ \circ V^{-1}_t)],
$$

where $V_t = t + \eta\gamma_t$. It turns out that $\bar{e}_i, i \in \mathbb{N}$ are i.i.d. random variables for which ([17, Section 4.3])

$$
P_0(\bar{e}_i > t | \bar{X}_{\bar{e}_i} > 0) = E_0 \left[ \exp \left( -(1/\eta)L^\Phi_t \right) \right], \quad \eta > 0
$$
and
\[ E_0[H^\psi_x | B^+] = E_0[\epsilon_1 | B^+] E_0[H^\psi_{\epsilon_1}] = E_0[\epsilon_1 | B^+] \lim_{\lambda \to 0} \frac{\Psi(\lambda)}{\lambda} \]
gives the mean holding time on the boundary point \( x = 0 \) for the process \( B^+ \circ \bar{V}_t^{-1} \).

Let us consider \( r_t := \sup\{ s \geq 0 : \gamma^+_s \leq t \} \) such that \( \gamma^+ \circ r_t = t \). For a given \( t, r_t \) is a stopping time for \( B^+ \). As in [27], we introduce the boundary process \( \bar{X}_t := B^+ \circ r_t \). That is the boundary trace process of \( B^+ \). The set of times \( J = \{ t \geq 0 : \bar{X}_t \neq \bar{X}_t \} \) gives the times in which \( \bar{X}_t \) has a jump. The local time \( \gamma^+_t \) is constant only for the excursions of \( B^+ \) on \((0, \infty)\), a jump for \( \bar{X}_t \) can be realized in \((r_t-, r_t)\) in case of non-empty set. Here \( \bar{X}_t \) is a pure jump process. We write \( J_t = J \cap (0, t] \) and recall that a.s. \( J \) is a dense countable subset of \((0, \infty)\). This can be associated with the countable jumps of a Cauchy process. The processes \( X \) and \( \bar{X} \) move along the path of \( B^+ \) for which the zero set \( \{ 0 \leq t < \infty : B^+_t = 0 \} \) has no isolated points. However, \( X'_{\gamma^+_0} = 0 \) and \( \bar{X}_{\gamma^+_0} = 0 \). Assume \( T^+ \in J_t \), then \( B^+_{T^+} = 0 \). Moreover,
\[ \sum_{j \in J_t} e_j \quad \text{and} \quad \sum_{j \in J_t} \bar{e}_j \]
respectively give the time the processes \( X \) and \( \bar{X} \) spend on the boundary point \( \{0\} \) up to time \( t \).

We now define \( B^*_t := B^* \circ S_t^{-1}, \ t \geq 0 \) and recall that also \( B^* \) can be identified as a process moving along the path of \( B^+ \). The new clock \( S_t^{-1} \) is associated to a non-local dynamic boundary condition and it introduces a sequence of holding times for \( B^* \). Since \( B^* \) jumps according with the jumps of \( H^\psi \), then once it reaches the boundary point \( \{0\} \), it starts afresh after a jump. However, \( B^* \) maintains its Markovian nature (see [29, Section 13]). This does not hold for \( B^* \).

Let us denote by \( \{ e_i^*, i \in \mathbb{N} \} \) the sequence of holding times for \( B^* \).

**Theorem 3.2.** The holding times \( e_i^* \) are i.i.d. random variables for which
\[ P_s(e_i^* > t | B^*_e^* \neq s) = E_0[\exp(-(1/\eta)L^\psi_t)], \ \eta > 0, \ s \neq 0. \] (35)
In particular, \( e_1^* \overset{d}{=} \bar{e}_1 \).

**Proof.** The process \( L^\psi_{\gamma_t^*} := L^\psi \circ \gamma_t \) can be regarded as the local time of \( B^* \) at zero [29, section 14] and
\[ B^*_{t} = L^\psi \circ \gamma_t \circ (t + H^\psi \circ \eta_{\gamma_t^*})^{-1} \] where \( \gamma_t^* = \gamma_t(B^*) \).

The right-hand side of (35) says that \( e_i^* \) is an holding time at the point \( s \in (0, \infty) \) and the left-hand side of (35) can be obtained by following the same arguments as in [17, Lemma 6]. We say that \( s \) is a sticky point for \( B^* \). For \( \Psi = I_d \), we get
\[ P_s(e_i^* > t | B^*_e^* \neq s) = P_s(e_i > t | B^*_e \neq s) = \exp(-(1/\eta)t), \ \eta > 0, \ s \neq 0 \]
and we say that \( s \in (0, \infty) \) is a sticky point for \( B^* \). Thus, the processes \( B^* \) and \( B^* \) near the origin jump to a point \( s \neq 0 \) and stop there according to the distribution of the associated holding time: \( e_i \) are (i.i.d.) exponential holding times with parameter \((1/\eta)\) and \( e_i^* \) equal in law \( H^\psi_{e_i} \).

We stress the fact that \( s \in B^* \circ J \) and \( B^* \circ J \) can be regarded as the set of regenerative points for \( B^* \), that is we have
\[ E_x[f(B_{t+j}^*) | B^*_j] = E_{B^*_j}[f(B^*_j)], \ t > 0. \] (36)

Let us discuss on this. The jumps of \( B^* \) are given by \( L_{\gamma_t} := L \circ \gamma_t \) as previously described. It may have jumps only as \( B^+ \) approaches \( x = 0 \) and then \( \gamma_t \) increases. Moreover, we know that \( P_0(L_t > 0) = 1 \) for every \( t > 0 \) and therefore the process \( B^* \) leaves the origin only through jumps. Moreover, the (length of the) jumps are independent from the base process \( B^+ \). The jumps determine the sticky points. Let us introduce the regenerative sets
\[ T_y = \{(t, \omega) \subset \mathbb{R} \times \Omega : y = L_{\gamma_t}(\omega)\}, \ y \in (0, \infty) \]
and
\[ S = \{(y, \omega) \subset \mathbb{R} \times \Omega : y = L_{\gamma_z}(\omega) \text{ for some } z \geq 0\}. \]
The random set \( \{ T_y, y \in S \} \) is not empty and includes the set \( J \). In particular, the set
\[
\{ L_\gamma \circ \tau \} : \tau \in J
\]
gives the set of *sticky points* for \( B^* \). Furthermore, \( \{ \tau \in J \} \) is a Markov set of points for \( B^* \) by means of which we may recover the associated Markov set of points for \( B^* \) by shifting w.r.t. holding times \( \{ e_i^{< -} : i \in N \} \) and for which (36) holds true. The process \( B^* \) starts afresh at its Markov points. We observe that the set of sticky points has zero Lebesgue measure whereas, the set of holding times is a set of positive Lebesgue measure (which obviously does not hold for the set \( \{ t : B^+_t = 0 \} \)).

In conclusion, the solution to the NLBVP in Theorem 3.1 has a probabilistic representation given by \( B^* \) which can be constructed via time-change \( S \) of a decomposition in terms of \( B^+ \) and \( L_\gamma \). The process moves along the path of the reflected Brownian motion. The time-change gives the amplitude of the plateaus, it is governed by the non-local operator in space. Reflected Brownian motion, jumps and plateaus are independent.

### 3.4. Stochastic resetting

In this section we underline that the time reverse process of \( B^* \) produces paths which can be associated with Brownian motions under stochastic resetting. In particular, we focus on the case where the subordinator \( H^\Phi \) does not have infinite activity, i.e., the case where \( \Pi^\Phi(0, \infty) < \infty \). This case, (which is not the case of infinite activity with \( \eta = 0 \) in Theorem 3.1) has the characteristic that the \( B^* \) is not forced to stop after the jump away from zero: it may simply reflect continuously or jump like a compound Poisson process with measure \( \Pi^\Phi \).

Let us consider the Figure 1 given by the path of \( B^*_t \) with \( 0 \leq t \leq T \) for a given finite time \( T > 0 \). We introduce the process \( \mathfrak{B} = \{ \mathfrak{B}_t \}_{0 \leq t \leq T} \) defined as
\[
\mathfrak{B}_t := B^*_{T-t}, \quad 0 \leq t \leq T
\]
where \( \mathfrak{B}_0 = B^*_T \) and \( \mathfrak{B}_T = B^*_0 = x > 0 \). This produces the paths in Figure 5. Let \( J = \{ J_t \}_{0 \leq t \leq T} \) give the random jumps of \( B^* \) from zero. For example \( J = \{ t : B^*_{t-} \neq B^*_{t+} \} \). Recall that \( \mathfrak{B} \) behaves like \( B^* \) on \((0, \infty)\). Moreover, in both cases the excursions on \((0, \infty)\) are Markovian, that is \( B^* \) is Markov ([29, Section 13]) as well as \( \mathfrak{B} \). We focus on the pictures of Figure 5. The associated process \( B^* \) (started at \( B^*_0 = x > 0 \)) behaves like a Brownian motion on \((0, \infty)\) and
\[
\inf\{ t : B^*_t = 0 \} =: \tau_0
\]
has the distribution \( P_x(\tau_0 > t) = Q^D_t 1(x) \) where \( Q^D_t \) has been introduced in (53). The zero set of a Brownian motion is a Cantor set. However, because of the jumps away from zero, \( \tau_0 \) and in general, the sequence \( \{ \tau_i \}_{i \in \mathbb{N}} \) of (zero) hitting times are well identified. For the process \( B^* \) we can say that \( \tau_i \sim \tau_j \) \( \forall i \) (the times are i.i.d.). The process \( \mathfrak{B} \) (started at \( \mathfrak{B}_0 = B^*_T \)) behaves like the Brownian motion on \((0, \infty)\) until a resetting occurs. As suggested by the right picture in Figure 5, the resetting of \( \mathfrak{B} \) can be associated with the set \( J \) of the jumps of \( B^* \). Since the resetting level is given by the jumps of \( B^* \) with each visit of the boundary point \( \{ 0 \} \), we have that
\[
P(B^* \text{ has a jump away from zero at time } t \mid B^*_0 = s) = P_s(\tau_0 > t), \quad s \in B^* \circ J
\]
coinsides with
\[
P(\text{ the resetting time } \mathfrak{I} \text{ of } \mathfrak{B} \text{ to zero } > t \mid \mathfrak{B}_0 = s) = P_s(\tau_0 > t), \quad s \in B^* \circ J
\]
(37) where \( \mathfrak{I} \) is the resetting time random variable taking values in \( T - J \). In particular, we recall that
\[
P(x > \tau_0) = P_0(H_x > t) = P_0(x < L_t), \quad x \in (0, \infty)
\]
where $H_x$ is a stable subordinator of order $\alpha = 1/2$ and $H_x \overset{d}{=} \inf\{s : B_s = x\}$. Notice that we only have equivalence in law for the hitting times above, the underlying paths of the associated Brownian motions are reasonably different.

4. Motions on graphs

Brownian motions on metric graphs describe the movement of particles across interconnected nodes. We consider non-local conditions on nodes, which is the novel here, by first studying a fundamental case with a single node, that is a star graph. The star graph is a special metric graph where there is a finite collection of sets, isomorphic to the positive half-line, such that all the origin points coincide at the unique point which is the vertex. Since each edge is isomorphic to $(0, \infty)$, it is natural to extend Feller’s theory on boundary conditions [24] and the related probabilistic solutions [29]. The pathwise construction of the Brownian motion on the star graph and the analysis of conditions at the vertex are developed in [35; 36]. The topic has been thoroughly covered also in the thesis [54]. The construction for a general metric graph is treated in [34]. Recently, the sticky Brownian motion, even with non-local dynamic conditions, has been studied in [4].

4.1. Star graph and Brownian motions. Let us begin by introducing the concept of a star graph within the theory of metric graphs. Let $E$ be a countable finite set and $E$ be a family of copies of the positive half-line

$$E := \bigcup_{e \in E} [0, \infty).$$

For simplicity, we denote each point in $E$ by a couple $(j, x)$, where $j \in E$ and $x$ is the Euclidean distance from the origin. As in [47], we introduce the following equivalence relation

$$(j, x) \sim (i, y) \iff \begin{cases} j = i & \text{and} & x = y \\ x = y = 0, & \text{for any } j, i. \end{cases}$$

Then, the star graph is the quotient space $\mathcal{G} := E/\sim$. The unique vertex $v = (\cdot, 0)$ belongs to all the edges. We also have that $\mathcal{G}$ is a metric space with the distance

$$d((j, x), (i, y)) := |x - y|1_{j=i} + (x + y)1_{j \neq i}.$$  

Another possible representation for the star graph $\mathcal{G}$, with star vertex $v$, is the following

$$\mathcal{G} = \{v\} \cup \bigcup_{e \in E} (\{e\} \times (0, \infty)).$$

Now, we define the Brownian motion on the star graph $\mathcal{G}$. With $\epsilon_e$, we indicate

$$\epsilon_e(x) = \epsilon_e(j, x) = \begin{cases} 1 & \text{if } j = e \\ 0 & \text{otherwise} \end{cases}.$$  

**Definition 1.** (Brownian motion on a star graph) Let $B = (U, B^+)$ be defined on $\mathcal{G}$, such that

$$\mu_U := \sum_{e \in E} \frac{1}{|E|} \epsilon_e.$$  

The process $\mathcal{B}$ satisfies:

- $B^+$ is a reflected Brownian motion;
- If $B_0 = v$, then for $t > 0$, the distribution of $U_t$ is given by $\mu_U$.
For $B_0 = (l, x)$, for $x > 0$, then $U_t = l$ if $t \leq T^U$ and if $t > T^U$ the distribution of $U$ is equal to $\mu_U$ and independent of $B^+$, where

$$T^U := \inf\{t > 0 : B_t = v\}.$$ 

**Remark 4.1.** If in Definition 1, we consider $X = (\Theta, B^+)$ and $\Theta$ is not a uniform random variable on the external edges, for example if $\Theta \sim \mu$ such that $\mu := \sum_{e \in E} \rho_e \epsilon_e$ and $\sum_{e \in E} \rho_e = 1$, then we are dealing with a Walsh Brownian motion on the star graph [53].

**Remark 4.2.** Under the hypothesis of equiangular configuration between edges, the well-known properties of Brownian motion are satisfied by $B$. For example, we have

$$E_v[B_t] = \sum_{e \in E} 1_e = 0,$$

since $1_e$ are unit vectors that sum zero. When we deal with stochastic processes on graphs, it is not uncommon to require that the vector sum of the components be zero. An example is the connection between spider martingales and martingales, see [56, Exercise 17.3].

### 4.2. Functions on star graphs

Contrary to Brownian motion on the real line, where we can visualize a particle free to move, on the star graph, the presence of the vertex introduces necessarily boundary conditions. First of all, we must grasp the concepts of functions, continuity, and differentiability on the graphs.

Let us introduce the space functions that we need. Since $(G, d)$ is a metric space, a function $f : G \to \mathbb{C}$ is continuous if the standard definition of continuous functions on metric spaces holds. In addition, we observe that $f$ can be considered as

$$f = \bigoplus_{e \in E} f_e,$$

where $f_e(\cdot) = f(e, \cdot) : (0, \infty) \to \mathbb{C}$ is the projection of $f$ on the edge $e$. Then for all $k \in \mathbb{N}$ we define recursively

$$C(G) := \left\{ f = \bigoplus_{e \in E} f_e \in \bigoplus_{e \in E} C(0, \infty) \text{ and } f_e(0) = f_l(0), \forall e, l \in E \right\},$$

$$C^k(G) := \left\{ f = \bigoplus_{e \in E} f_e \in \bigoplus_{e \in E} C^k(0, \infty) \text{ and } f_e^{(k)}(0) = f_l^{(k)}(0), \forall e, l \in E : \right\},$$

$$f^{(h)} := \bigoplus_{e \in E} f_e^{(h)} \in C(G), 1 \leq h \leq k,$$

with $f_e^{(h)}$ we denote the $h-$derivative of $f_e$. In this way the functions are continuous up to the vertex.

Similarly to what we observed for functions, measures can also be regarded as direct sums of measures on edges. Consequently, we can, for instance, define Lebesgue spaces. By following [46, Definition 3.15], we
denote by \( L^p(\mathcal{G}) \) the space of measurable functions \( f : \mathcal{G} \to \mathbb{C} \) such that
\[
||f||_{L^p(\mathcal{G})} := \left( \sum_{e \in \mathcal{G}} \int_0^\infty |f(e, x)|^p \, dx \right)^{\frac{1}{p}} < \infty \quad p \in [1, \infty)
\]
\[
||f||_{L^\infty(\mathcal{G})} := \inf \{ c \in \mathbb{R} : |f(e, x)| \leq c \text{ for a.e. } x \in (0, \infty), \text{ and for all } e \in \mathcal{E} \}.
\]
Now, the Sobolev space is, for \( p \in [1, \infty) \),
\[
W^{k,p}(\mathcal{G}) := \left\{ f \in L^p(\mathcal{G}), f^{(h)} \in C(\mathcal{G}) \text{ for } 1 \leq h \leq k - 1, \right. \\
and \left. f^{(j)} \in L^p(\mathcal{G}) \text{ for all } 0 \leq j \leq k \right\}.
\]
Let us turn our attention to the concept of derivatives. In our exploration of the heat equation on graphs, we direct our focus toward the Laplacian. Let \( C_0(\mathcal{G}) \) be the space of the continuous functions \( f \) such that, for every \( e \in \mathcal{E} \), \( f_e \) vanishes at infinity. The Laplacian is the operator \( \Delta : C^2_0(\mathcal{G}) \to C_0(\mathcal{G}) \) written as
\[
\Delta u(x) = \frac{d^2}{dx^2} u(l, x)
\]
where \( x = (l, x) \in \mathcal{G} \) and \( x \neq v \). For the star vertex \( v = (e, 0) \), with \( e \in \mathcal{E} \)
\[
\Delta f(v) := \lim_{\xi \to 0} f''_e(\xi),
\]
for which the limit exists and it is a function in \( C_0(\mathcal{G}) \). Representing the vertex as \( v = (e, 0) \), when computing the derivative, it is crucial to keep track of the direction in which we are moving. For this reason, we take the limit along the projection onto the edge \( e \). In the case of the Laplacian, we stipulate that it is a continuous operator across the entire graph, ensuring that the limit \((39)\) holds the same value for every edge \( e \in \mathcal{E} \).
Similarly, we can define other derivatives, as we have done for the second one, and require them to be continuous outside the vertex; in such instances, the limits, like \((39)\), depend on the edge \( e \). For example, for \( u \in C^1(\mathcal{G} \setminus \{v\}) \)
\[
u'(x) := \frac{d}{dx} u(l, x)
\]
with \( x = (l, x) \). But, in this case, we are not considering that at the vertex, the derivatives are all equal.

4.3. Stochastic processes on the edges. In Definition 1, our focus was on Brownian motion, examining it across the entire graph. Presently, we aim to comprehend its behavior on each individual edge. Formally, we define the process \( X_t^e \), on the edge \( e \in \mathcal{E} \) as
\[
X_t^e = f(B_t),
\]
where \( f : \mathcal{G} \to \mathbb{R} \) is
\[
f(B_t) := \begin{cases} B_t^+ & \text{if } U_t = e \text{ and } U_t \neq 0, \\
0 & \text{if } U_t \neq e. \end{cases}
\]
Based on the definition, it is evident that \( X^e \) behaves as a reflecting Brownian motion up to the point where \( U = e \). However, when the Brownian motion \( B \) shifts between edges on the graph, the \( X^e \) process remains stuck at zero until \( B \) occurs again on the same edge \( e \). As illustrated in Figure 7, we observe that the plateaus (intervals of consistency) of \( X^e \) are a result of excursions of \( B \) to different edges. This observation leads us to anticipate a non-Markovian dynamic. We observe that this case is very close to the case \( \eta > 0 \) and \( \Phi = \text{Id} \) already studied (see Figure 2). In both cases, we deal with processes that remains stuck at zero: for \( X^e \), the slowdown is caused by the excursions of \( B \) on the other edges, while for \( B^+ \circ V^{-1} \) the slowdown is due the non-local dynamic boundary conditions.

**Theorem 4.1.** The process \( X^e \) is non-Markov.

**Proof.** We recall that \( B \) is a Brownian motion, \( \tau_0 := \inf\{ t > 0 : B_t = 0 \} \) and, as in [14, Section 2],
\[
\beta_t := \inf\{ s \geq t : B_s = 0 \}.
\]
From the Markov property of \( B \), we have
\[
P(\beta_t - t > s \mid B_t = x) = P(\beta_0 > s \mid B_0 = x)
\]
\[
= P(\tau_0 > s \mid B_0 = x),
\]
and, by adapting [5, Formula 2.0.2], we know
\[
P(\tau_0 > s | B_0 = x) = \frac{|x|}{\sqrt{4\pi s^3}} e^{-\frac{x^2}{4s}}.
\]

Then, we obtain
\[
P(\beta_t - t > s) = \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{4\pi s^2}} e^{-\frac{x^2}{4s}} P_0(B_t \in dx)
\]
\[
= \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{4\pi s^2}} \frac{1}{\sqrt{4\pi t^2}} e^{-\frac{x^2}{4t^2}} dx
\]
\[
= 2 \int_{0}^{\infty} \frac{x}{\sqrt{4\pi s^2}} \frac{1}{\sqrt{4\pi t^2}} e^{-\frac{x^2}{4t^2}} dx
\]
\[
= \frac{1}{\pi} \frac{1}{\sqrt{s t + s}}.
\]

We move now on the process on the metric graph and we define \( T^e_t := \inf \{ s > t : U_s = e \} \). Trivially, we have
\[
P(T^e_t - t = 0 | U_t = e) = 1.
\]

We recall that the Brownian motion \( B, B_0 = 0 \), with probability one has infinitely many zeros in every time interval \((0, \varepsilon), \varepsilon > 0\). Then, for the process on the graph \( \mathcal{B} \), we have
\[
P(T^e_0 = 0 | \mathcal{B}_0 = v) = 1, \quad \forall e \in \mathcal{E}.
\]

Since the Brownian motion visits infinitely many times zero and \( \forall \varepsilon > 0 \), then
\[
P(T^e_0 > \varepsilon | \mathcal{B}_0 = v) = \lim_{n \to \infty} \left( 1 - \frac{1}{|\mathcal{E}|} \right)^n = 0.
\]

From (40), we have seen that immediately the Brownian motion on the graph visits all edges, then, by using the strong Markov property for \( \mathcal{B} \), we have
\[
P(T^e_t - t > s | U_t \neq e) = P(\beta_t - t > s),
\]
which we calculated before and it was greater than zero. Now we observe that
\[
0 = P(T^e_t - t > s | \mathcal{B}_t = v) \neq P(T^e_t - t > s | U_t \neq e) > 0,
\]
which, again from the strong Markov property of \( \mathcal{B} \) and the definition of \( X^e_t \), becomes
\[
0 = P(T^e_0 > s | \mathcal{B}_0 = v) = P(T^e_0 > s | X^e_0 = 0) \neq P(T^e_t - t > s | U_t \neq e)
\]
\[
= P(T^e_t - t > s | X^e_t = 0) > 0,
\]
so \( X^e \) cannot be Markovian. \( \square \)
4.4. **Fundamental solution of the heat equation.** In this section, we analyze the heat equation on the star graph and observe how only with the choice of uniform weights on the edges we achieve the expected inclusion in operator’s domain.

Let $P_t^\mathcal{X} f(x) = E_x[f(X_t)]$ be the semigroup of the Walsh Brownian motion $\mathcal{X}$, defined in Remark 4.1, for $f \in C(\mathcal{G})$ and $x = (l, x)$. From [1, Formula (2.1) and (2.2)], we have

$$P_t^\mathcal{X} f(x) = \sum_{e \in \mathcal{E}} \rho_e (P_t^+ f_e(x) - P_t^D (f_l - f_e)(x)),$$

where $P_t^+$ is the semigroup of a reflected Brownian motion on the half-line and $P_t^D$ is the semigroup of the Brownian motion on the half-line killed in zero. Then, the infinitesimal generator of $\mathcal{X}$ is $A = \Delta$, with $\Delta$ the laplacian on the star graph defined in (38), and the domain is

$$D(A) = \left\{ \varphi, \Delta \varphi \in C_0(\mathcal{G}) : \sum_{e \in \mathcal{E}} \rho_e \frac{\partial}{\partial x} \varphi(e, x) \bigg|_{x=0} = 0 \right\},$$

so that the Kirchoff’s condition on the vertex appears.

**Theorem 4.2.** The density of the Walsh Brownian motion on $\mathcal{G}$ is not a function in $D(A)$.

**Proof.** For the point $x = (l, x)$, the density of the Walsh Brownian motion is given by

$$g(t, v, x) := P(\mathcal{X}_t \in dx | \mathcal{X}_0 = v) = \rho_l P(B_t^+ \in dx | B_0^+ = 0) = 2\rho_l g(t, x),$$

where $g$ is the Gaussian kernel. Therefore, we show that the heat equation holds

$$\frac{\partial}{\partial t} g(t, v, x) = 2\rho_l \frac{\partial}{\partial t} g(t, x) = 2\rho_l \frac{\partial^2}{\partial x^2} g(t, x) = \Delta g(t, v, x),$$

but if we want continuity at the vertex

$$\lim_{x \downarrow 0} g(t, v, x) = \lim_{x \to 0} 2\rho_l g(t, x)$$

then it is not continuous and $g(t, v, x) \notin C(\mathcal{G})$. So the density of the Walsh Brownian motion does not belong to $D(A)$.

On the contrary, when we consider the Brownian motion $\mathcal{B}$, defined in Definition 1, the limit (42) does not depend on the edge, since the distribution of edge selection is uniform. For the other derivatives, we provide

$$\lim_{x \downarrow 0} \frac{d}{dx} g(t, v, x) = \lim_{x \to 0} \frac{\partial}{\partial x} g(t, v, (l, x)) = \lim_{x \to 0} \frac{2}{|E|} (-\Delta g(t, x)) = 0$$

and

$$\Delta g(t, v, x) = \frac{2}{|E|} (-g(t, x) + x^2 g(t, x)) \to -\frac{2}{|E|} g(t, 0) \quad x \downarrow 0$$

which does not depend on the edge $l$. We conclude that $g(t, v, x) \in D(A)$ if and only if we deal with a Brownian motion, such that the distribution of the edge selection is uniform.

4.5. **Non-local operators on the star graph.** Here we introduce the non-local operators on the star graph $\mathcal{G}$ as a natural extension of the results in Section 2.2 and Section 8.1.

For $u \in W^{1,\infty}(\mathcal{G})$, we define the left Marchaud-type derivative on $x = (e, x) \in \mathcal{G} \setminus \{v\}$

$$D^\mathcal{G}_{x^-} u(g) = \int_0^\infty (u(e, x) - u(e, x + y)) \Pi^\mathcal{G}(dy).$$

In this way, the function $u$ is $W^{1,\infty}$ on each edge and we use the analogue of (12) to see that (43) is well defined.

As for the Marchaud (type) operator, we exploit the same idea for Riemann-Liouville (type) operators, which are

$$D^\mathcal{G}_{x^-} u(x) := -\frac{d}{dx} \int_x^\infty u(e, y) \Pi^\mathcal{G}(y - x) dy$$

and

$$D^\mathcal{G}_{x^+} u(g) := \frac{d}{dx} \int_0^x u(e, y) \Pi^\mathcal{G}(x - y) dy$$

respectively defined for function $u$ such that

$$u(e, \cdot) \Pi^\mathcal{G}(\cdot - x) \in L^1(x, \infty), \quad \text{and} \quad u(e, \cdot) \Pi^\mathcal{G}(x - \cdot) \in L^1(0, x) \quad \forall x \in (0, \infty), e \in \mathcal{E}.$$
Regarding the star vertex $v$, careful consideration of the definitions is essential. We represent $v$ as $(e,0)$ across all edges, necessitating specification of the direction in which we are progressing. For example, for (43), we have for $u \in W^{1,\infty}(G)$

$$D^{\Phi}_{e-} u(v) = \lim_{\xi \to 0} D^{\Phi}_{\xi-} u(e,\xi) = \lim_{\xi \to 0} D^{\Phi}_{\xi-} u_e(\xi),$$

(46)

with $D^{\Phi}_{\xi-} u_e(\xi)$ is the edgewise Marchaud-type derivative (11) on $\xi$.

4.6. **Non-local conditions on star graphs.** As previously anticipated, the fact that each edge is isomorphic to the positive half-line leads us to think that Feller’s theorem is easily extendable to the star graph $G$. This is true, see for example [54, Theorem 20.26]. In general, the integral boundary condition is typically interpreted as the integral over a measure on the graph $G$. However, we prefer to focus on Lévy measures and provide an interpretation involving non-local operators.

**Definition 2.** Let $X^\ast = (\Theta, B^\ast)$ be defined on $G$, with weights $\rho_e \geq 0$ for all $e \in E$ such that

$$\sum_{e \in E} \rho_e = 1 \text{ and } \mu := \sum_{e \in E} \rho_e e_e.$$ 

The process $X^\ast$ satisfies:

- $B^\ast$ is the process defined in (23);

- If $X^\ast_0 = v$, then for $t > 0$, the distribution of $\Theta_t$ is given by $\mu$;

- For $X^\ast_0 = (l,x)$, for $x > 0$, then $\Theta_t = l$ if $t \leq T$ and if $t > T$ the distribution of $\Theta$ is equal to $\mu$ and independent of $B^\ast$, where

$$T := \inf\{ t > 0 : X^\ast_t = v \}.$$ 

Since $B^\ast$ is a Markov process, the same property is inherited by $X^\ast$. $X^\ast$ starts its paths in a point $x$, it behaves like a Walsh Brownian motion on a star graph and, when it reaches the vertex $v$, it jumps (as the last jumps of $H^\Phi$) in an edge (by choosing it trough the distribution of $\Theta$). On each edge the paths are right-continuous, since the same is true for $B^\ast$.

**Remark 4.3.** This definition could be also obtained from the Walsh Brownian motion on the star graph, as in [54, Section 21].

**Remark 4.4.** The second component of $X^\ast$ on $G \setminus \{v\}$ and $B^\ast$ on $(0, +\infty)$ have the same excursions. Indeed, from the definition of $X^\ast$, we have that outside from the node $v$ the process coincides with $B^\ast$.

We have seen that, even for simple Brownian motion on $G$, it is necessary to impose Kirchhoff conditions at the vertex. Therefore, it makes sense to search solutions in the space

$$D := \{ \varphi(\cdot, x) \in C(0,\infty) \forall x \in G; \varphi(t, \cdot), \Delta \varphi(t, \cdot) \in C_0(G), \varphi'(t, \cdot) \in C(G \setminus \{v\}) \forall t > 0 \}.$$ 

For the non-local Kirchhoff condition, we need $D_H \subset D$ defined as

$$D_H := \{ \varphi \in D : \varphi(t, \cdot) \in W^{1,\infty}(G), D^{\Phi}_{x-} \varphi_e(t, x) \mid_{x=0} \text{ exists} \forall e \in E, \forall t > 0 \},$$

where the Marchaud (type) derivative at the vertex $v$ is rewritten trough the Marchaud (type) derivatives on the projections $\varphi_e$, as in (46).

**Theorem 4.3.** The probabilistic representation of the solution $u \in D_H$

$$\begin{cases}
 u(t,x) = \Delta u(t,x) & t > 0, x \in G \setminus \{v\} \\
 \sum_{e \in E} \rho_e D^{\Phi}_{x-} u_e(t,x) \mid_{x=0} = 0 & t > 0 \\
 u(0,x) = f(x) & x \in G
\end{cases}$$

(47)

with $f \in C(G) \cap L^\infty(G)$ and $x = (l,x)$ is

$$u(t,x) = E_x[f(X^\ast_t)].$$

We postpone the proof in Section 7.

**Remark 4.5.** In this remark, we delve into how our process can be viewed as a generalization of the Walsh Brownian motion on the star graph, by taking advantage of the properties of fractional calculus. Let $f$ be a function on $G$ such that $f \in W^{1,1}(G)$ and $\Phi(\lambda) = \lambda^\alpha$ with $\alpha \in (0,1)$. Then, the associated Lévy measure, from (4), is denoted by $\Pi^\alpha$ and $D^{\alpha}_{x-} f(x)$ is the Marchaud derivative, with $x = (e,x) \in G$. We see that

$$D^{\alpha}_{x-} f(x) := \int_0^\infty (f(e,x) - f(e,x+y)) \Pi^\alpha(dy)$$

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\[
- \int_0^\infty \frac{d}{dz} f(e, x + z) \Pi^\alpha(dy)
= - \int_0^\infty \frac{d}{dz} f(e, x) \Pi^\alpha((z, \infty))dz
= [\text{by using (5)}]
= - \int_x^\infty \frac{d}{dz} f(e, z) \Pi^\alpha(z - x)dz.
\]

We observe that
\[
\int_0^\infty e^{-\lambda z} \Pi^\alpha(z)dz = \frac{\lambda_\alpha}{\lambda} \to 1, \quad \alpha \uparrow 1,
\]
then, at the limit \(\alpha \uparrow 1\), \(\Pi^\alpha\) behaves like a delta distribution. This leads us to
\[
-\int_x^\infty \frac{d}{dz} f(e, z) \Pi^\alpha(z - x)dz \to -f'(x), \quad \alpha \uparrow 1.
\]

We can apply the same argument to the vertex \(v = (e, 0)\) and we have, for \(\alpha \uparrow 1\),
\[
D^\Phi_{v-} f(v) \to -f'_e(v),
\]
where
\[
f'_e(v) = \lim_{\xi \to 0} \frac{d}{d\xi} f(e, \xi).
\]

Hence, the boundary conditions in Theorem 4.3 correspond to the boundary condition of a Walsh Brownian motion when \(\alpha\) tends to 1 in the case of an \(\alpha\)-stable subordinator. We anticipate that this result holds for the paths of the process as well. The process \(B^\alpha\) defined in (23) approaches a reflecting Brownian motion as \(\alpha\) tends to 1 due to \(H^\alpha \to t\). Consequently, we have \(X^\alpha = (\Theta, B^\alpha)\), where in Definition 2, \(B^\alpha\) is replaced by \(B^+\), resulting in a Walsh Brownian motion.

We observed that, also in the case of the star graph, Marchaud (type) derivatives lead us to a process that jumps as soon as it reaches the vertex. The difference compared to the positive half-line is that now the jump can land on other edges based on the \(\Theta\) law.

Presently, we want to check if the slowdowns caused by Caputo-Džrbašjan (type) derivatives still hold. The non-local sticky Brownian motion on the star graph has been introduced in [4, Theorem 22], where at the vertex there are the conditions
\[
c \mathcal{D}^\Phi_t u(t, v) = \sum_{e \in \mathcal{E}} \rho_e u'_e(t, 0),
\]
with \(b + c = 1\) and \(\mathcal{D}^\Phi\) the Caputo-Džrbašjan (type) derivative introduced in (14). The aim is to extend this result in the presence of jumps, just as we did for the positive half-line. To achieve this, we introduce the conditions
\[
\varrho, \dot{\varrho} \in C(0, \infty) \text{ and } \dot{\varrho}(s) \Pi^\Phi(t - s) \in L^1(0, t), \ t > s > 0 \quad (48)
\]
for a given function \(\varrho\) and the space
\[
D_L := \{ \varphi \in C((0, \infty) \times \mathcal{G}) \text{ with } \varrho = \varphi|_{x=v} \text{ s.t. (48) are satisfied} \}.
\]

We also define the random times
\[
\mathcal{S}_t = t + H^\Phi \circ (\eta L^\Phi \circ \ell_t)
\]
where \(\ell\) is the local time at the vertex of a Walsh Brownian motion on \(\mathcal{G}\) with weights \(\{\rho_e\}\), and \(\mathcal{S}^{-1}\) is the right inverse of \(\mathcal{S}\).

We know that \(\ell\) coincides with the local time at zero \(\gamma\) of the reflected Brownian motion \(B^+\) (see [4, Corollary 8]).

**Theorem 4.4.** The probabilistic representation of the solution \(u \in D_H \cap D_L\) of the problem
\[
\begin{aligned}
\left\{
\begin{array}{ll}
\dot{u}(t, x) = \Delta u(t, x) & \quad t > 0, x \in \mathcal{G} \setminus \{v\} \\
\eta \mathcal{D}^\Phi_t u(t, v) + \sum_{e \in \mathcal{E}} \rho_e \mathcal{D}^\Phi_{x-e} u_e(t, x)|_{x=0} = 0 & \quad t > 0 \\
u(0, x) = f(x) & \quad x \in \mathcal{G}
\end{array}
\right.
\end{aligned}
\]
with \( f \in C(G) \cap L^\infty(G), \eta \geq 0 \) and \( x = (l, x) \) is
\[
 u(t, x) = E_x[\lambda^*(\mathcal{S}^{-1}_t)].
\]

**Proof.** The proof of this theorem follows that of Theorem 3.1, where the key observation is that once the edge is specified, the dynamics on \((0, \infty)\) remain the same. Let us examine the principal steps.

We write the \( \lambda \)--potential, for \( \lambda > 0 \),
\[
\mathcal{R}_\lambda f(x) = \int_0^\infty e^{-\lambda t} u(t, x) dt = \int_0^\infty e^{-\lambda t} E_x[\lambda^*(\mathcal{S}^{-1}_t)] dt.
\]

Since \( x = (l, x) \), before hitting the vertex the motion is a killed Brownian motion on the edge \( e \). Then, with the same arguments that led us to write (53), and once a vertex is reached, we choose the edge as in (65), we have, for \( \lambda > 0 \),
\[
\mathcal{R}_\lambda f(x) = \mathcal{R}_\lambda^D f(x) + e^{-x \sqrt{\lambda}} \sum_{e \in E} \rho_e E_{(e, 0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B^* \circ \mathcal{S}_t^{-1}) dt \right],
\]

where \( \mathcal{S}^{-1} \) is the same random time introduced in Theorem 3.1, this is possible since the local time \( \ell \) at the vertex coincides with \( \gamma \).

As previously noted, the heat equation holds, and we only need to verify the conditions at the vertex. As mentioned earlier, once the edge is specified, the \( \lambda \)--potential to be computed remains that of the process \( B^* \circ \mathcal{S}^{-1} \) on \((0, \infty)\), which is already determined in (62) through the sum of \( I_1^\circ + I_2^\circ \). Then, we have
\[
E_{(e, 0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B^* \circ \mathcal{S}_t^{-1}) dt \right] = \frac{1}{\eta \Psi(\lambda) + \Phi(\sqrt{\lambda})} \left( \int_0^\infty \mathcal{R}_\lambda^D f(e, y) \Phi(\lambda) dy + \frac{\eta}{\lambda} \Psi(\lambda) f(e, 0) \right),
\]

and the \( \lambda \)--potential at the vertex, by using (49), is
\[
\mathcal{R}_\lambda f(v) = \sum_{e \in E} \rho_e \left( \frac{1}{\eta \Psi(\lambda) + \Phi(\sqrt{\lambda})} \left( \int_0^\infty \mathcal{R}_\lambda^D f(e, y) \Phi(\lambda) dy + \frac{\eta}{\lambda} \Psi(\lambda) f(e, 0) \right) \right).
\]

We now verify the Laplace transforms for the conditions at the vertex. For the Caputo-Džrbašjan (type) derivative, as in (60), we get
\[
\int_0^\infty e^{-\lambda t} \eta \mathcal{D}_t^\Phi u(t, v) dt = \eta \Psi(\lambda) \mathcal{R}_\lambda f(v) - \eta \frac{\Psi(\lambda)}{\lambda} f(v), \quad \lambda > 0,
\]

where we emphasize that, in exploring solutions within space \( D_H \), the function at the vertex is continuous. For the Marchaud (type) derivative, from (66), we have
\[
\sum_{e \in E} \rho_e \mathcal{D}_x^\Phi \mathcal{R}_\lambda f(e, x) |_{x=0} = -\sum_{e \in E} \rho_e \int_0^\infty \mathcal{R}_\lambda^D f(e, y) \Phi(\lambda) dy + \Phi(\sqrt{\lambda}) \mathcal{R}_\lambda f(v). \]

By combining (50) and (51), the vertex conditions applied to the \( \lambda \)--potential become
\[
\eta \Psi(\lambda) \mathcal{R}_\lambda f(v) - \eta \frac{\Psi(\lambda)}{\lambda} f(v) - \sum_{e \in E} \rho_e \int_0^\infty \mathcal{R}_\lambda^D f(e, y) \Phi(\lambda) dy + \Phi(\sqrt{\lambda}) \mathcal{R}_\lambda f(v) = 0
\]

which are satisfied by
\[
\mathcal{R}_\lambda f(v) = \sum_{e \in E} \rho_e \left( \frac{1}{\eta \Psi(\lambda) + \Phi(\sqrt{\lambda})} \left( \int_0^\infty \mathcal{R}_\lambda^D f(e, y) \Phi(\lambda) dy + \frac{\eta}{\lambda} \Psi(\lambda) f(e, 0) \right) \right),
\]

that is the \( \lambda \)--potential at the vertex of the process \( \lambda^* \circ \mathcal{S}^{-1} \). This concludes that \( u(t, x) = E_x[\lambda^*(\mathcal{S}^{-1}_t)] \) is a solution to the problem under consideration. \( \square \)

In case \( |E| = 1 \), we have a description of the paths in terms of \( B^* \circ \mathcal{S}^{-1} \) on the half-line, as shown in Figure 3. We now turn to the general scenario involving the graph \( G \). According to Definition 2, the process \( \lambda^* \) is a reflecting Brownian motion on \( G \) which jumps away from the vertex according with the jump of \( B^* \) and the angular process \( \Theta \). Once near the vertex, the probability of choosing a specific edge is determined by the weights \( \{\rho_e = P(\Theta(t) = e)\}_{e \in E} \). The jumps of \( \lambda^* \) well accords with the jumps of \( B^* \). This dynamic is prescribed according with Theorem 4.3, that is case \( \eta = 0 \). We underline that \( \mathcal{T} \), the time at which the process \( \lambda^* \) hits the vertex, plays the same role as \( \tau_0 \) in the case of the process \( B^* \). For \( \eta > 0 \), as presented in Theorem 4.4, the time change \( \mathcal{S}^{-1} \) introduces the same type of delay as in \( B^* \circ \mathcal{S}^{-1} \). Specifically, the process \( \lambda^* \circ \mathcal{S}^{-1} \) behaves like a Brownian motion on the graph that, upon reaching a vertex, jumps to a new point in \( G \) (that is, some edge of \( G \)), it stops there for a random amount of time according with \( L^\Psi \), the inverse of \( H^\Psi \). Then, it starts afresh.
5. Applications

5.1. Stochastic resetting. In Section 3.4, we have highlighted a potential connection between NLBVPs and stochastic resetting, a concept widely applied in statistical mechanics, particularly in the context of search problems. We focus on a Brownian particle that intermittently returns to a fixed point (zero in our case), as introduced in [22; 23]. The resets occur at random intervals governed by a Poisson process, meaning that the time between successive resets follows an exponential distribution. In this framework, there is a constant probability \( r \) of resetting at any given moment, independent of the time elapsed since the previous reset.

Stochastic resetting proves especially advantageous in search problems due to a key feature: while standard Brownian motion leads to an infinite mean first passage time (MFPT), the introduction of resetting ensures that the MFPT becomes finite [22].

The process we describe through NLBVPs is closely related to a Brownian motion that resets to zero and subsequently reflects on the positive half-line. This connection with NLBVPs allows us to explore search problems by analyzing repelling boundaries, which induce jumps in the process. These boundaries provide deeper insights into the system’s dynamics, offering a richer structure for understanding how resetting and boundary conditions influence behavior.

The NLBVPs together with the time reversal and the translation of the associated motions give a very large family of processes describing stochastic resetting and stochastic delay.

5.2. Traffic flow models. We can mention general models not only restricted to road traffic. By setting the jumps according to the length of the edge we can move from metric graphs to graphs in which the particles move along the vertices. Moreover, the time non-local boundary condition implies a random holding times on the vertices to be charged to very clear and common situations. For example in road traffic, the holding time on a given edge can be given by a broken traffic light. If you consider data traffic, then the random holding time on a given edge can be associated with some failure in between the corresponding servers.

5.3. Financial Applications. Figure 3 gives an example of possible applications. We may consider motions different from Brownian motions, then our model may entail different properties. Concerning the applications in case we have no jumps (only time non-local operator on the boundary), an example is given in [6] where the authors deal with a model of sticky expectations in which investors update their beliefs too slowly. Our results in this regards give a control for the slow update depending on \( \alpha \in (0, 1) \). A further example is given by the bank interest rates (or the Vasicek model for instance) which are termed sticky if they react slowly to changes in the corresponding market rates or in the policy rate ([26]). Moreover, in case we have only jumps (only space non-local operator on the boundary) can be given in terms of structural breaks. The process jumps according with the jumps of \( B_t \) (as in Figure 1).

5.4. Delayed and rushed reflection. In the same spirit of the results in [9] we are able to give a classification for the reflection near the boundary. Consider (34) and write

\[
c := \lim_{\lambda \downarrow 0} \frac{\Psi(\lambda)}{\Psi(\lambda)}
\]

assuming the limit exists (\( c = \infty \) is a clear case). The process \( \tilde{X} \) may have (in terms of mean holding time on the boundary):

- **B1)** delayed boundary behavior if \( c < 1 \);
- **B2)** rushed boundary behavior if \( c > 1 \);
- **B3)** base (reflected) boundary behavior if \( c = 1 \).

A special case of subordinator including either delayed or rushed effect is the gamma subordinator for which, for \( a, b > 0 \),

\[
\Psi(\lambda) = a \ln(1 + \lambda/b) \quad \text{and} \quad \lim_{\lambda \downarrow 0} \frac{\Psi(\lambda)}{\lambda} = \frac{a}{b}
\]

and \( c < 1, c > 1 \) or \( c = 1 \) depending on the ratio \( a/b \). On the other hand, by considering \( \Psi = Id \), we have that \( B^* \) jumps away from the boundary according with the last jump of \( H^\Phi \). Due to nature of the subordinator, the process \( B^* \) never hits the boundary, it reflects instantaneously before hitting the boundary. In this case, the time that the process spends on the boundary can be only given (eventually) by the starting point as a point of the boundary, case in which the process is pushed immediately away once again for the nature of the subordinator. We recall that our focus is only on strictly increasing subordinators.

Our analysis introduces the following further characterization:

- **R1)** delayed/rushed (continuous) slow reflection;
R2) jumping (discontinuous) reflection;
R3) (continuous) reflection.

The delayed reflection can be related with a reflection on the boundary (like an insulating boundary for example) whereas the rushed reflection seems to be related with a reflection near the boundary (like an inflating boundary for example). These characterizations are therefore given according with the comparison between the total (mean) amounts of time the processes \( B^+ \) and \( \tilde{X} \) would spend on the boundary. By base boundary behavior we mean the behavior of the reflecting process \( B \) between the total (mean) amounts of time the processes \( B \) inflating boundary for example). These characterizations are therefore given according with the comparison example) whereas the rushed reflection seems to be related with a reflection near the boundary (like an insulating boundary for example). The delayed reflection can be related with a reflection on the boundary (like an insulating boundary for example).

Concerning \( R_3 \) we easily make the connection with \( B_3 \), that is the case of Neumann diffusion (according with the definition given in [9]). \( R_2 \) turns out to be associated with the symbol \( \Phi \) of the boundary condition.

The behaviour of a "regular process" on an irregular domain (trap boundary for example) can be captured by an "irregular process" on a regular domain (smooth boundary for example).

5.5. **Irregular domains.** The behaviors \( B_1-B_2-B_3 \) can be associated with a macroscopic description of motions on irregular domains. Indeed, irregular domains in a microscopic point of view need a geometric characterization in order to be completely described. By irregular domain here we mean domains with irregular boundaries, for example the Koch domain still is regular (it is non trap for the Brownian motion, see [7]).

Our message here is the following: Consider \( E \subset \mathbb{R}^d \) with \( d > 1 \), then

The behaviour of a "regular process" on an irregular domain (trap boundary for example) can be captured by an "irregular process" on a regular domain (smooth boundary for example).

In case \( E = [0, \infty) \) we are considering the trapping point \{0\}. Despite of the apparently simple situation, it basically represents a very general setting.

### 6. Proof of Theorem 3.1

As a solution to the heat equation, the function \( v \) can be written as

\[
v(t, x) = a(t, x) + \int_0^t b(t - s, x) v(s, 0) ds
\]

with Laplace transform of both side

\[
\hat{v}(\lambda, x) = \hat{a}(\lambda, x) + \hat{b}(\lambda, x) \hat{v}(\lambda, 0), \quad \lambda > 0
\]

for some suitable (sufficiently regular and integrable) functions \( a, b \). Simple arguments say that \( \hat{a}(\lambda, 0) = 0 \) and \( \hat{b}(\lambda, 0) = 1 \). Moreover, it must be \( v(0, x) = a(0, x) \) and \( \hat{v}'' = \lambda \hat{v} - f \). In particular, it turns out that \( a, b \) are given by

\[
v(t, x) = Q^D_t f(x) + \int_0^t \frac{x}{\tau} g(\tau, x) v(t - \tau, 0) d\tau
\]

where

\[
Q^D_t f(x) = \int_0^\infty (g(t, x - y) - g(t, x + y)) f(y) dy
\]

is the Dirichlet semigroup and \( g(t, z) = e^{-z^2/4t}/\sqrt{4\pi t} \) for which we recall the well-known transforms

\[
\int_0^\infty e^{-\lambda t} g(t, x) dt = \frac{1}{2} e^{-x\sqrt{\lambda}}, \quad \lambda > 0
\]

and

\[
\int_0^\infty e^{-\lambda t} \frac{x}{\tau} g(t, x) dt = e^{-x\sqrt{\lambda}}, \quad \lambda > 0.
\]

In particular, if \( a \) is the Dirichlet semigroup, then \( b \) must be obtained from the unique (continuous and bounded) solution to \( \hat{b}'' = \lambda \hat{b} \) with \( \hat{b}(\lambda, 0) = 1 \). Thus, \( v(t, 0) \) in (53) completely identifies the boundary condition and vice versa.

Denote by \( \tau^* \) the first time \( B^\bullet \) hits the boundary point \( 0 \) and observe that \( \mathbf{P}_x(B^\bullet_0 = 0) = 1, \forall x > 0 \).

From the probability viewpoint, first we notice that

\[
R_1(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(B^\bullet_t) dt \right]
\]
where $\lambda$ is the $\lambda$-potential associated with the process $B^\bullet$ whereas the second part of $R_2$ writes

$$E_x\left[\int_{\tau^*}^\infty e^{-\lambda t} f(B^\bullet_t) dt\right] = E_x\left[\int_{S_0*}^\infty e^{-\lambda S_t} f(B^\bullet_t) dS_t\right] = -\frac{1}{\lambda} E_x\left[\int_{S_0*}^\infty f(B^\bullet_t) d\lambda e^{-\lambda S_t}\right].$$

We can invoke the Markov property of $B^\bullet$ in order to confirm the analytic discussion on (52). We will provide conclusive arguments on Section 3.3. Now we focus on the probabilistic representation of the solution $\nu$.

Let us write

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} \nu(t, x) dt, \quad \lambda > 0. \quad (56)$$

Our aim is to prove that

$$R_\lambda f(x) = E_x\left[\int_0^\infty e^{-\lambda t} f(B^\bullet_t) dt\right], \quad \lambda > 0 \quad (57)$$

is the $\lambda$-potential associated with the process $B^\bullet \circ S^{-1}$.

Under the representation (53), formula (56) takes the form

$$R_\lambda f(x) = R^D_\lambda f(x) + \tilde{R}_\lambda f(x)$$

where $R^D_\lambda f$ and $\tilde{R}_\lambda f$ are defined below. From (54) and (55) we respectively write

$$R^D_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q^D_\lambda f(x) dt$$

$$= \frac{1}{2} \int_0^\infty \left(\frac{e^{-|x-y|\sqrt{\lambda}}}{\sqrt{\lambda}} - \frac{e^{-(x+y)\sqrt{\lambda}}}{\sqrt{\lambda}}\right) f(y) dy \quad (58)$$

$$= \frac{1}{2} \int_0^\infty \frac{e^{(x-y)\sqrt{\lambda}}}{\sqrt{\lambda}} f(y) dy - \frac{1}{2} \int_0^\infty \frac{e^{-(x+y)\sqrt{\lambda}}}{\sqrt{\lambda}} f(y) dy +$$

$$- \frac{1}{2} \int_0^x \frac{e^{(x-y)\sqrt{\lambda}}}{\sqrt{\lambda}} - \frac{e^{-(x-y)\sqrt{\lambda}}}{\sqrt{\lambda}} f(y) dy, \quad \lambda > 0$$

and

$$\tilde{R}_\lambda f(x) = e^{-x\sqrt{\lambda}} R_\lambda f(0), \quad \lambda > 0.$$

We can easily check that

$$(R_\lambda f(x))'' = \lambda R_\lambda f(x) - f(x) = \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} \nu(t, x) dt, \quad \lambda > 0, x > 0 \quad (59)$$

and $\nu$ solves the heat equation. We also notice that $R_\lambda f$ is continuous at $x = 0$ and $R_\lambda f(0) = \tilde{R}_\lambda f(0)$. The characterization of $R_\lambda f(0)$ and therefore $\nu(t, 0)$ gives (53). Thus, we focus on the boundary condition for which, first we observe that

$$\int_0^\infty e^{-\lambda t} D_t^\psi \nu(t, x) \Big|_{x=0} dt = \int_0^\infty e^{-\lambda t} D_t^\psi \nu(t, x) \Big|_{x=0}, \quad \lambda > 0$$

and, thanks to (15), we have

$$\int_0^\infty e^{-\lambda t} \eta D_t^\psi \nu(t, x) \Big|_{x=0} dt = \eta \Psi(\lambda) R_\lambda f(0) - \eta \frac{\Psi(\lambda)}{\lambda} f(0), \quad \lambda > 0. \quad (60)$$
A key point in our proof is the linearity of $D^\Phi_{x-}R_{\lambda}f(x)|_{x=0}$ for which
\[
D^\Phi_{x-}R_{\lambda}f(x)|_{x=0} = (D^\Phi_{x-}R^D_{\lambda}f(x) + D^\Phi_{x-}R_{\lambda}f(x))|_{x=0}
\]
as we can check from the definition (11). We use the fact that
\[
\lim_{x \downarrow 0} \int_0^\infty \left( R^D_{\lambda}f(x) - R^D_{\lambda}f(x+y) \right) \Pi^\Phi(dy) = - \int_0^\infty R^D_{\lambda}f(y) \Pi^\Phi(dy)
\]
and
\[
\lim_{x \downarrow 0} \int_0^\infty \left( R_{\lambda}f(x) - R_{\lambda}f(x+y) \right) \Pi^\Phi(dy) = \lim_{x \downarrow 0} \int_0^\infty \left( e^{-x\sqrt{\lambda}} - e^{-(x+y)\sqrt{\lambda}} \right) \Pi^\Phi(dy) R_{\lambda}f(0) = \Phi(\sqrt{\lambda}) R_{\lambda}f(0).
\]
Notice that, for $|f(x)| \leq K$,
\[
\int_0^\infty R^D_{\lambda}f(y) \Pi^\Phi(dy) \leq \frac{K}{\lambda} \int_0^\infty (1 - e^{-\sqrt{\lambda}y}) \Pi^\Phi(dy) = \frac{K}{\lambda} \Phi(\sqrt{\lambda}), \quad \lambda > 0.
\]
We therefore obtain
\[
-D^\Phi_{x-}R_{\lambda}f(x)|_{x=0} = \int_0^\infty R^D_{\lambda}f(y) \Pi^\Phi(dy) - \Phi(\sqrt{\lambda}) R_{\lambda}f(0).
\]
This together with (60) leads to
\[
R_{\lambda}f(0) = \frac{1}{\eta \Psi(\lambda) + \Phi(\sqrt{\lambda})} \left( \frac{\eta}{\lambda} \Psi f(0) + \int_0^\infty R^D_{\lambda}f(y) \Pi^\Phi(dy) \right), \quad \lambda > 0
\]
which has been obtained under the representation (53).

Now we study the probabilistic representation of $R_{\lambda}f(0)$. Assume that (57) holds true. Recall that $S_t = t + H_\Phi \circ (\eta L_{\gamma_t})$ is strictly increasing. The inverse $S^{-1}$ may have plateaus. Observe that $R_{\lambda}f$ can be written in terms of the path integral
\[
R_{\lambda}f(0) = \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda f(B^t) \circ S_{t^{-1}}(t)} dt \right]
\]
where
\[
I_1^\Psi = \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda \Psi f(B^t)} dt \right], \quad \lambda > 0
\]
and
\[
I_2^\Psi = f(0) \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda \Psi (f(L^t_{\gamma_t}) + f(B^t_{\gamma_t}) dS_t \circ H_\Phi)} dt \right], \quad \lambda > 0.
\]
The computation of $I_1^\Psi$ follows from the computation of $e_1$ in [29, page 215], recalling that
\[
\mathbb{E}_0[e^{-\lambda H^\Phi_\gamma}] = e^{-\eta t \Psi(\lambda)}, \quad \lambda > 0.
\]
Indeed, $H_\Phi$ is independent from $B^t$ (and therefore, from $L^\Phi$) and we get
\[
I_1^\Psi = \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda \Psi f(B^t)} dt \right] = \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda \Psi} \lambda t \Psi f(B^t) dt \right], \quad \lambda > 0
\]
which has been evaluated in [29, page 215]. For completeness, we prove this result in a more general setting, see Remark 7.1. We have that
\[
I_1^\Psi = \frac{1}{\eta \Psi(\lambda) + \Phi(\sqrt{\lambda})} \int_0^\infty R^D_{\lambda}f(y) \Pi^\Phi(dy), \quad \lambda > 0.
\]
For the path integral $I_2^\Psi$, we get that
\[
I_2^\Psi = f(0) \mathbb{E}_0 \left[ \int_0^\infty e^{-\lambda \Psi (f(L^t_{\gamma_t}) + f(B^t_{\gamma_t}) dH^\Phi_{\gamma_t})} dt \right]
\]
and
guarantees uniqueness of the solution in terms of positivity (Widder’s theorem [55]) and exponential growth. Since

\[ υ \]

which coincides with (61). Thus, we conclude that (57) holds true. In particular, $υ$ is a continuous function with Laplace transform $X$.

By collecting all pieces together, we obtain

\[
\begin{aligned}
E_0 \left[ \int_0^\infty e^{-\lambda t} e^{-\eta \Psi(\lambda)L_{\gamma t}^\Phi} dL_{\gamma t}^\Phi \right] &= - \frac{1}{\eta \Psi(\lambda)} E_0 \left[ \int_0^\infty e^{-\lambda t} e^{-\eta \Psi(\lambda)L_{\gamma t}^\Phi} dt \right] \\
&= - \frac{1}{\eta \Psi(\lambda)} E_0 \left[ 1 + \lambda \int_0^\infty e^{-\lambda t} e^{-\eta \Psi(\lambda)L_{\gamma t}^\Phi} dt \right] \\
&= \frac{1}{\eta \Psi(\lambda)} - \frac{\lambda}{\eta \Psi(\lambda)} E_0 \left[ \int_0^\infty e^{-\lambda t} e^{-\eta \Psi(\lambda)L_{\gamma t}^\Phi} dt \right] \\
&= \frac{1}{\eta \Psi(\lambda)} - \Phi(\sqrt{\lambda}) \\
&= \frac{1}{\eta \Psi(\lambda)} + \Phi(\sqrt{\lambda}).
\end{aligned}
\]

We arrive at

\[
I_2^\psi = \frac{1}{\eta \Psi(\lambda)} + \frac{\eta}{\lambda} \Psi(\lambda) f(0), \quad \lambda > 0.
\]

By collecting all pieces together, we obtain

\[
I_1^\psi + I_2^\psi = \frac{1}{\eta \Psi(\lambda)} + \Phi(\sqrt{\lambda}) \left( \int_0^\infty R_\lambda^D f(y) \Pi^D(dy) + \frac{\eta}{\lambda} \Psi(\lambda) f(0) \right)
\]

which coincides with (61). Thus, we conclude that (57) holds true. In particular, $v$ is a continuous function with Laplace transform $R_\lambda f$, then $v$ is unique. There exists only one continuous inverse to $R_\lambda f$.

We also stress the fact that we deal with positive functions on unbounded domain. Standard arguments guarantees uniqueness of the solution in terms of positivity (Widder’s theorem [55]) and exponential growth ([21, section 2.3.3]).

7. Proof of Theorem 4.3

Let us introduce the resolvent, for $\lambda > 0$,

\[
R_\lambda f(x) = \int_0^\infty e^{-\lambda t} u(t, x) dt = \int_0^\infty e^{-\lambda t} E_x[f(\lambda t^*)]dt.
\]

Since $\lambda^*$ starts afresh with a jump after $T$, by proceeding as [29, Formula 2, Section 15], we obtain

\[
R_\lambda f(x) = E_x \left[ \int_0^T e^{-\lambda t} f(\lambda t^*) dt \right] + E_x \left[ e^{-\lambda T} \right] E_x \left[ \int_0^\infty e^{-\lambda t} f(\lambda t^*) dt \right],
\]

where now the role of the origin on the positive half-line is taken by the vertex $v$. A key observation is that by expressing the process $\lambda^*$ as $(\Theta, B^*)$, then the first-hitting time of the vertex is equivalent to the
first-hitting time of \(B^*\) at zero. For the first part, in the same way of (58), since the process \(B^*\) before hitting zero is a killed Brownian motion \(B^D\), associated to a Dirichlet boundary condition, we get

\[
E_x \left[ \int_0^T e^{-\lambda t} f(X_t^*) dt \right] = E_{(l,x)} \left[ \int_0^{\tau_0} e^{-\lambda t} f(l, B^*_t) dt \right]
\]

\[
= E_{(l,x)} \left[ \int_0^\infty e^{-\lambda t} f(l, B^*_t) dt \right]
\]

\[
= \int_0^\infty e^{-\lambda t} \int_0^\infty (g(t, x - y) - g(t, x + y)) f(l, y) dy dt
\]

\[
=: R^D f(l, x),
\]

where \(g(t, z) = e^{-z^2/4t} / \sqrt{4\pi t}\). By using (54) we conclude that

\[
R^D f(l, x) = E_{(l,x)} \left[ \int_0^T e^{-\lambda t} f(l, B^*_t) dt \right] = \frac{1}{2} \int_0^\infty \left( \frac{e^{-(z+y)\sqrt{\lambda}}}{\sqrt{\lambda}} - \frac{e^{-(z-y)\sqrt{\lambda}}}{\sqrt{\lambda}} \right) f(l, y) dy
\]

\[
= \frac{1}{2} \int_0^\infty \frac{e^{-(z+y)\sqrt{\lambda}}}{\sqrt{\lambda}} f(l, y) dy - \frac{1}{2} \int_0^\infty \frac{e^{-(z-y)\sqrt{\lambda}}}{\sqrt{\lambda}} f(l, y) dy + \frac{1}{2} \int_0^x \left( \frac{e^{-(z+y)\sqrt{\lambda}}}{\sqrt{\lambda}} - \frac{e^{-(z-y)\sqrt{\lambda}}}{\sqrt{\lambda}} \right) f(l, y) dy, \quad \lambda > 0. \tag{64}
\]

For the second part of (63), since \(\Theta \sim \mu\), we have

\[
E_x \left[ e^{-\lambda T} \right] E_v \left[ \int_0^\infty e^{-\lambda t} f(X^*_t) dt \right] = E_{(l,x)} \left[ e^{-\lambda T} \sum_{e \in \mathcal{E}} \rho_e E_{e(0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B^*_t) dt \right] \right]
\]

\[
= E_{(l,x)} \left[ e^{-\lambda T} \sum_{e \in \mathcal{E}} \rho_e R_{\lambda f}(e, 0) \right].
\]

We note that \(E_{(l,x)}[e^{-\lambda T}] = E_x[e^{-\lambda \tau_0}] = e^{-x\sqrt{\lambda}}\) for all \(e \in \mathcal{E}\), since, once the edge is fixed, it remains a simple Brownian excursion. Then, we get

\[
E_x \left[ e^{-\lambda T} \right] E_v \left[ \int_0^\infty e^{-\lambda t} f(X^*_t) dt \right] = e^{-x\sqrt{\lambda}} \sum_{e \in \mathcal{E}} \rho_e E_{e(0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B^*_t) dt \right]. \tag{65}
\]

By combining (64) and (65), we see that, for the resolvent (63), the following holds

\[
\lambda R_{\lambda f}(l, x) - f(l, x) = \frac{\partial^2}{\partial x^2} R_{\lambda f}(l, x),
\]

then the heat equation

\[
\begin{cases}
\dot{u}(t, x) = \Delta u(t, x) & t > 0, x \in \mathcal{G} \setminus \{v\} \\
v(0, x) = f(x) & x \in \mathcal{G}
\end{cases}
\]

is satisfied. Now, let us move on the boundary conditions. When the process is in \(x = v\), the resolvent simply reduces to

\[
R_{\lambda f}(v) = \sum_{e \in \mathcal{E}} \rho_e E_{e(0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B^*_t) dt \right].
\]

The following part is a generalization of [29, Formula 6, Section 15], treated for other boundary conditions in [54, Section 21.11].

In the definition of (1) we have the process \(H^\Phi L^\Phi\) that is a right-continuous process with jumps (given the way the subordinator \(H^\Phi\) is constructed). By enumerating the jumps of \(H^\Phi\) with \(l_1, l_2, ...\), we decompose \((0, \infty)\) in two sets:

\[
\mathcal{J}^c := \{ t \geq 0 : H^\Phi L^\Phi = t \}
\]

\[
\mathcal{J} := \bigcup_{n \geq 1} \mathcal{J}_n = \bigcup_{n \geq 1} [l_n^-, l_n^+),
\]

\[
\mathcal{J}^c := \{ t \geq 0 : H^\Phi L^\Phi = t \}
\]

\[
\mathcal{J} := \bigcup_{n \geq 1} \mathcal{J}_n = \bigcup_{n \geq 1} [l_n^-, l_n^+),
\]

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where we have to include the point \( l_n^- \) because \( H^\Phi L^\Phi \) is right-continuous. Hence, for all \( e \in \mathcal{E} \), the process \( \lambda^\bullet \) turns out to be

\[
(e, \lambda^\bullet) = \begin{cases} 
(e, B^+) & \gamma_t \in \mathcal{J}^c \\
(e, l_n^+ - \gamma_t + B^+) & \gamma_t \in \mathcal{J}_n.
\end{cases}
\]

indeed if \( \gamma_t \in \mathcal{J}_n \) we have \( H^\Phi L^\Phi \gamma = l_n^- \). We observe that \( \gamma_t \in [l_n^-, l_n^+] \) if and only if \( t \in [\gamma_t^{-1}(l_n^-), \gamma_t^{-1}(l_n^+)) \), in fact \( \gamma_t \) is a continuous process and its right inverse \( \gamma_t^{-1} \) is a right-continuous process with jumps, then to make sure that the point \( l_n^- \) is included we have to introduce

\[
\gamma_t^{-1}(t) := \inf\{s \geq 0 : \gamma_s \geq t\},
\]

which is left-continuous. We prefer writing \( \gamma_t^{-1}(t) \) over \( \gamma_t^{-1} \), to avoid confusion with right and left points of \( l_n \). Then, for the resolvent of the process, we have

\[
E_{(e,0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = E_{(e,0)} \left[ \int_{\mathcal{J} \cup \mathcal{J}^c} e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = E_{(e,0)} \left[ \int_{\mathcal{J}^c} e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] + E_{(e,0)} \left[ \int_{\mathcal{J}} e^{-\lambda t} f(e, B_t^\bullet) \, dt \right],
\]

but we are dealing only with pure jump subordinators (with drift zero), then \( \mathcal{J}^c \) has zero measure. For the other part, we get

\[
E_{(e,0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = \sum_{n \geq 1} E_{(e,0)} \left[ \int_{\gamma_t^{-1}(l_n^+)}^{\gamma_t^{-1}(l_n^-)} e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = \sum_{n \geq 1} E_{(e,0)} \left[ \int_0^{\gamma_t^{-1}(l_n^+)-\gamma_t^{-1}(l_n^-)} e^{-\lambda s+\gamma_t^{-1}(l_n^-)} f(e, B_s^\bullet+\gamma_t^{-1}(l_n^-)) \, ds \right].
\]

Prior to commencing integration, it is necessary to note certain aspects concerning the local time \( \gamma \) and its left inverse \( \gamma_t^{-1} \). Since \( \gamma \) is continuous, we have \( \gamma_t^{-1}(l_n^-) = l_n^- \). For the left inverse, serving as the upper bound within the integral, we know that (see [49, Proposition 1.3, Chapter X]) \( \gamma_t^{-1}(l_n^+)-\gamma_t^{-1}(l_n^-) = \gamma_t^{-1}(l_n^-) - \theta \gamma_t^{-1}(l_n^-) \), where \( l_n = l_n^+ - l_n^- \) and \( \theta \) is the shift operator. We also recall the additive functional property for the local time: \( \gamma_t+\gamma_t^{-1}(l_n^-) = \gamma_s \circ \theta_t \gamma_t^{-1}(l_n^-) + \gamma_t^{-1}(l_n^-) = \gamma_s \circ \theta_t \gamma_t^{-1}(l_n^-) + l_n^- \). Then, we rewrite

\[
E_{(e,0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = \sum_{n \geq 1} E_{(e,0)} \left[ \int_0^{\gamma_t^{-1}(l_n^+)-\gamma_t^{-1}(l_n^-)} e^{-\lambda s+\gamma_t^{-1}(l_n^-)} f(e, B_s^\bullet+\gamma_t^{-1}(l_n^-)) \, ds \right] = \sum_{n \geq 1} E_{(e,0)} \left[ \left( \int_0^{\gamma_t^{-1}(l_n^-)} e^{-\lambda s} f(e, l_n^- + B_s^+ - \gamma_t^-) \, ds \right) \circ \theta_t \gamma_t^{-1}(l_n^-) \right] \mathcal{F}_t^{+} \theta_t \gamma_t^{-1}(l_n^-) )
\]

where \( \mathcal{F}_t^{+} \) is the natural filtration of \( B^+ \). For more details on the strong Markov property on the star graph and the characterization of the filtration, see [54, Section 21]. By using the strong Markov property of \( B^+ \) with respect to the stopping time \( \gamma_t^{-1}(l_n^-) \) and the fact that \( \gamma_t^{-1}(l_n^-) = \gamma_t^{-1} \) a.s., we obtain

\[
E_{(e,0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B_t^\bullet) \, dt \right] = \sum_{n \geq 1} E_{(e,0)} \left[ e^{-\lambda \gamma_t^{-1}(l_n^-)} E_{(e,0)} \left[ \int_0^{\gamma_t^{-1}(l_n^-)} e^{-\lambda s} f(e, l_n^- + B_s^+ - \gamma_t^-) \, ds \right] \right]
\]

but, the process \( (e, l_n^- - \gamma_t + B^+) \), since \( t \leq \gamma_t^{-1}(l_n^-) \), behaves like the Brownian motion \( \{(e, B_t), t \leq \tau_0\} \) started at \( (e, l_n^-) \), with \( \tau_0 \) hitting time at zero for \( B \). Then, we have

\[
E_{(e,0)} \left[ \int_0^\infty e^{-\lambda t} f(e, B_t^\bullet) \, dt \right]
\]
\[
= \sum_{n \geq 1} E(e, 0) \left[ e^{\frac{-\lambda}{\tau_n}} E(e, l_n) \left[ \int_0^{\tau_0} e^{-\lambda t} f(e, B_t) dt \right] \right]
\]
\[
= \sum_{n \geq 1} E(e, 0) \left[ e^{\frac{-\sqrt{\lambda}}{\tau_n}} R^D_\lambda f(e, l_n) \right]
\]
\[
= \sum_{n \geq 1} E(e, 0) \left[ e^{\frac{-\sqrt{\lambda} \Phi}{\tau_n}} R^D_\lambda f(e, l_n) \right]
\]
\[
= E(e, 0) \left[ \int_{(0, \infty) \times (0, \infty)} e^{-\sqrt{\lambda} \Phi} R^D_\lambda f(e, l) N(dt \times dl) \right],
\]
where \(N(dt \times dl)\) is the random measure associated to \(H^\Phi\). From [28, Example II.4.1] we know that \(N(dt \times dl) = dt \Pi^\Phi (dl)\), then we get
\[
E(e, 0) \left[ \int_{(0, \infty) \times (0, \infty)} e^{-\sqrt{\lambda} \Phi} R^D_\lambda f(e, l) N(dt \times dl) \right]
\]
\[
\lim_{\varepsilon \to 0} E(e, 0) \left[ \int_{(0, \infty) \times (0, \infty)} e^{-\sqrt{\lambda} \Phi} R^D_\lambda f(e, l) N(dt \times dl) \right]
\]
\[
= \int_0^{\infty} \int_0^{\infty} e^{-\sqrt{\lambda} \Phi} R^D_\lambda f(e, l) dt \Pi^\Phi (dl)
\]
\[
= \int_0^{\infty} R^D_\lambda f(e, l) \Pi^\Phi (dl)
\]
\[
= \frac{\Phi(\sqrt{\lambda})}{\sqrt{\lambda}}.
\]
Hence, the resolvent in \(v\) is
\[
R^D_\lambda f(v) = \sum_{e \in \mathcal{E}} \rho_e E(e, 0) \left[ \int_0^{\infty} e^{-\lambda t} f(e, B_t^v) dt \right] = \sum_{e \in \mathcal{E}} \rho_e \int_0^{\infty} R^D_\lambda f(e, l) \Pi^\Phi (dl) \frac{\Phi(\sqrt{\lambda})}{\Phi(\sqrt{\lambda})}.
\]
Now we show that the vertex conditions have the same Laplace transform. For the Marchaud (type) derivative, fixed an \(l \in \mathcal{E}\), we see that
\[
\sum_{e \in \mathcal{E}} \rho_e D^\Phi_{x, e} R^D_\lambda f(e, x) \big|_{x=0} = \left( \sum_{e \in \mathcal{E}} \rho_e D^\Phi_{x, e} R^D_\lambda f(e, x) \big|_{x=0} + D^\Phi_{x, e} f(e, x) \big|_{x=0} \sum_{e \in \mathcal{E}} \rho_e R^D_\lambda f(e, 0) \right).
\]
In particular, for the killed part
\[
\sum_{e \in \mathcal{E}} \rho_e \lim_{x \downarrow 0} \int_0^{\infty} (R^D_\lambda f(e, x) - R^D_\lambda f(e, x + y)) \Pi^\Phi (dy) = \sum_{e \in \mathcal{E}} \rho_e \int_0^{\infty} (R^D_\lambda f(e, x) - R^D_\lambda f(e, y)) \Pi^\Phi (dy)
\]
\[
= - \sum_{e \in \mathcal{E}} \rho_e \int_0^{\infty} R^D_\lambda f(e, y) \Pi^\Phi (dy)
\]
and
\[
\lim_{x \downarrow 0} \int_0^{\infty} \left( e^{-x \sqrt{\lambda}} - e^{-(x+y) \sqrt{\lambda}} \right) \Pi^\Phi (dy) \sum_{e \in \mathcal{E}} \rho_e R^D_\lambda f(e, 0) = \Phi(\sqrt{\lambda}) \sum_{e \in \mathcal{E}} \rho_e R^D_\lambda f(e, 0).
\]
We therefore obtain that
\[
0 = - \sum_{e \in \mathcal{E}} \rho_e D^\Phi_{x, e} R^D_\lambda f(e, x) \big|_{x=0} = \sum_{e \in \mathcal{E}} \rho_e \int_0^{\infty} R^D_\lambda f(e, y) \Pi^\Phi (dy) - \Phi(\sqrt{\lambda}) \sum_{e \in \mathcal{E}} \rho_e R^D_\lambda f(e, 0),
\]
\[
(66)
\]
it is satisfied by
\[
R^D_\lambda f(v) = \sum_{e \in \mathcal{E}} \rho_e R^D_\lambda f(e, 0) = \sum_{e \in \mathcal{E}} \rho_e \int_0^{\infty} R^D_\lambda f(e, l) \Pi^\Phi (dl) \frac{\Phi(\sqrt{\lambda})}{\Phi(\sqrt{\lambda})}, \quad \lambda > 0.
\]
\[
(67)
\]
Then, we provide that \(u(t, x) = E_x[f(\mathcal{X}^v_t)]\) solves the boundary conditions
\[
\sum_{e \in \mathcal{E}} \rho_e D^\Phi_{x, e} f(e, x) \big|_{x=0} = 0.
\]
and since we have already seen that it solves the heat equation, it is the probabilistic solution of (47).
Remark 7.1. In the case $|\mathcal{E}| = 1$, we obtain
\[ E_0 \left[ \int_0^\infty e^{-\lambda t} f(B_t^\ast)dt \right] = \int_0^\infty R^D_\lambda f(y) \Pi^{\Phi}(dy) / \Phi(\sqrt{\lambda}). \]

We use this result for the computation of $I^\Phi_1$ in Theorem 3.1, see Section 6.

8. Appendix

8.1. Some auxiliary results. Observe that $AC(I)$ coincides with $W^{1,1}(I)$ only if $I \subset (0, \infty)$ is bounded. We recall that $AC$ denotes the set of absolutely continuous functions. For the interval $I$, $W^{1,\infty}(I)$ coincides with the space of Lipschitz continuous functions.

In order to define the Riemann-Liouville (type) derivatives on the positive real line, we first consider a closed interval $\bar{I} \subset (0, \infty)$ and $u \in AC(\bar{I})$. Then we extend the result on $\mathbb{R}^+$ as in [32, page 79]. We now introduce the Riemann-Liouville (type) derivatives
\[ D_{(x,\infty)}^{\Phi} u(x) := -\frac{d}{dx} \int_x^\infty u(y)\Pi^{\Phi}(y-x)dy \]  
(68) and
\[ D_{(0,x)}^{\Phi} u(x) := \frac{d}{dx} \int_0^x u(y)\Pi^{\Phi}(x-y)dy \]  
(69)
respectively defined for function $u$ such that
\[ u(\cdot)\Pi^{\Phi}(\cdot - x) \in L^1(x, \infty), \quad \text{and} \quad u(\cdot)\Pi^{\Phi}(x - \cdot) \in L^1(0, x) \quad \forall x. \]

Let us focus on (68). We observe that
\[ D_{(x,\infty)}^{\Phi} u(x) = -\frac{d}{dx} \int_x^\infty u(y)\Pi^{\Phi}(y-x)dy \]
\[ = \lim_{b \to \infty} -\frac{d}{dx} \int_x^b u(y)\Pi^{\Phi}(y-x)dy \]
\[ = \lim_{b \to \infty} -\frac{d}{dx} \int_0^{b-x} u(z+x)\Pi^{\Phi}(z)dz \]
\[ = \lim_{b \to \infty} u(b)\Pi^{\Phi}(b-x) - \int_0^{b-x} u'(z+x)\Pi^{\Phi}(z)dz \]
\[ = \lim_{b \to \infty} u(b)\Pi^{\Phi}(b-x) - \int_0^\infty u'(y)\Pi^{\Phi}(y-x)dy \]
where, from the final value theorem for the Laplace transform in (5), we have
\[ 0 = \Phi(0) = \lim_{b \to \infty} \Pi^{\Phi}(b). \]

Assuming that the growth of $u$ is asymptotically bounded, then
\[ D_{(x,\infty)}^{\Phi} u(x) = -\int_x^\infty u'(y)\Pi^{\Phi}(y-x)dy. \]  
(70)
The right-hand side in (70) can be regarded as a Caputo-Džrbašjan (type) derivative. A further relevant fact for (68) is the equivalence with (11). If (70) holds, then
\[ D_{x-}^{\Phi} u(x) = D_{(x,\infty)}^{\Phi} u(x). \]  
(71)
Indeed, using first (70) and then the second formula in (5), we have
\[ D_{(x,\infty)}^{\Phi} u(x) = -\int_x^\infty u'(y)\Pi^{\Phi}(y-x)dy \]
\[ = -\int_0^\infty \frac{d}{dy} u(x+y)\Pi^{\Phi}(y)dy \]
\[ = -\int_0^\infty \frac{d}{dy} u(x+y)\Pi^{\Phi}(y, \infty)dy \]
\[ = -\int_0^\infty \int_0^z \frac{d}{dy} u(x+y)dy\Pi^{\Phi}(dz) \]
\[ = -\int_0^\infty (u(x+z) - u(x))\Pi^{\Phi}(dz) \]
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= D^\Phi_{x-} u(x).

Concerning (69), the analogous result for $D^\Phi_{x+}$ and $D^\Phi_{(0,x)}$ can be proved, a clue is due to their Laplace transforms.

For the operators (11) and (10) we introduce the following integration by parts formula.

**Theorem 8.1.** If $u, v \in W^{1,1}_{0}(0, \infty)$ and $D^\Phi_{x-} v(x), D^\Phi_{x+} u(x) \in L^1(0, \infty)$, then

$$
\int_0^\infty u(x) (D^\Phi_{x-} v(x)) \, dx = \int_0^\infty (D^\Phi_{x+} u(x)) \, v(x) \, dx.
$$

(72)

**Proof.** First of all, from Hölder’s inequality, we observe that

$$
|u| \leq ||u||_\infty |D^\Phi_{x-} v|_1 < \infty
$$

because $D^\Phi_{x-} v \in L^1(0, \infty)$ and $W^{1,1}_{0}(0, \infty)$ embeds into $L^\infty(0, \infty)$ (see [38, section 11.2]). Similarly for $D^\Phi_{x+}$. By Fubini’s theorem, from the definition (11),

$$
\int_0^\infty u(x) (D^\Phi_{x-} v(x)) \, dx = \int_0^\infty \int_0^\infty u(x)[v(x) - v(x + y)] \Pi^\Phi(dy) \, dx
$$

$$
= \int_0^\infty \int_0^\infty [u(x) - u(x - y) + u(x - y)][v(x) - v(x + y)] \Pi^\Phi(dy) \, dx.
$$

By using (10),

$$
\int_0^\infty \int_0^\infty [u(x) - u(x - y)] v(x) \Pi^\Phi(dy) \, dx = \int_0^\infty (D^\Phi_{x+} u(x)) \, v(x) \, dx,
$$

and

$$
\int_0^\infty u(x) (D^\Phi_{x-} v(x)) \, dx = \int_0^\infty (D^\Phi_{x+} u(x)) \, v(x) \, dx + F(u, v)
$$

where

$$
F(u, v) := \int_0^\infty \int_0^\infty (u(x) - u(x - y))[v(x) - v(x + y)] \Pi^\Phi(dy) \, dx
$$

$$
= \int_0^\infty \int_0^\infty (u(x) - u(x - y)) v(x) \, dx \Pi^\Phi(dy) - \int_0^\infty \int_0^\infty u(x) v(x + y) \Pi^\Phi(dy) \, dx
$$

$$
= \int_0^\infty \int_0^\infty u(x) v(x + y) \, dx \Pi^\Phi(dy) - \int_0^\infty \int_0^\infty u(x) v(x + y) \Pi^\Phi(dy) \, dx
$$

$$
= \int_0^\infty \int_0^\infty u(x) v(x + y) \Pi^\Phi(dy) \, dx - \int_0^\infty \int_0^\infty u(x) v(x + y) \Pi^\Phi(dy) \, dx = 0,
$$

hence (72) holds.

$\square$

**Remark 8.1.** The same result for left and right Marchaud derivatives $D^\alpha_{x-}$ and $D^\alpha_{x+}$, when $x \in \mathbb{R}$ is presented in [50, (6.27)] and in [37, Exercise 1.8.2].

Let us write

$$
\tilde{h}_f(x) = \int_0^\infty h_f(t, x) \, dt \quad \text{and} \quad \tilde{l}_f(x) = \int_0^\infty l_f(t, x) \, dt.
$$

(73)

We can immediately check that the Abel (type) equation $f(x) = D^\Phi_{x+} \tilde{h}_f(x)$ gives the elliptic problem associated with (19). On the other hand, the elliptic problem associated with (20) exists only if $\lim_{\lambda \downarrow 0} \Phi(\lambda) / \lambda < \infty$, this gives a clear meaning to (16). Such a result is not surprising, indeed by considering $f = 1$,

$$
\tilde{l}_1(x) = \int_0^\infty E_0[1_{t < H^\Phi_x}] \, dt = E_0[H^\Phi_x] = x \lim_{\lambda \downarrow 0} \frac{\Phi(\lambda)}{\lambda}.
$$

In case the elliptic problem exists, it takes the form

$$
f(x) = (\lim_{\lambda \downarrow 0} \frac{\lambda}{\Phi(\lambda)}) \frac{\partial}{\partial x} \tilde{l}_f(x).
$$

Moving to the elliptic problems associated with (21) we notice that the solution to

$$
-D^\Phi_{x-} w_-(x) = -f(x) + \lambda w_-(x), \quad x > 0, \lambda > 0
$$

(74)

is given by

$$
w_-(x) = E_0 \left[ \int_0^\infty e^{-\lambda t} f(x + H^\Phi_x) \, dt \right].
$$

(75)
Thus, the solution to (74) is given by (75). Similarly, we observe that the solution to

Indeed, by using (21), we have

where

be represented as

I

the following notation, for a set

be regarded as a non-Markovian time-changed process, being the random time

L

composition

H

On the compositions involving subordinators and their inverses. We now briefly discuss the composition \( H_{\Phi} \circ L_{\Phi} \) of a subordinator with an inverse to an independent subordinator. The process can be regarded as a non-Markovian time-changed process, being the random time \( L_{\Phi} \) non-Markovian. We use the following notation, for a set \( I \subset (0, \infty) \),

\[
P_0(H_{\Phi}^I \in I) = \int_I h_{\Phi}(t, x)dx, \quad P_0(L_{\Phi}^I \in I) = \int_I t_{\Phi}(t, x)dx,
\]

and

\[
P_0(H_{\Phi} \circ L_{\Phi}^I \in I) = \int_I w(t, x)dx.
\]

We have that

\[
D_t^{\Phi} w(t, x) = -D_{x+}^{\Phi} w(t, x), \quad t > 0, \ x > 0
\]

where \( D_t^{\Phi} \) is defined in (14) and \( D_{x+}^{\Phi} \) is defined in (10). For the reader’s convenience we give the following sketch of proof. By applying the \( \lambda \)-time Laplace transform in both sides of (77), from (15), we obtain

\[
\Phi(\lambda) \tilde{w}(\lambda, x) - \frac{\Phi(\lambda)}{\lambda} \delta(x) = -D_x^{\Phi} \tilde{w}(\lambda, x), \quad \lambda > 0, \ t > 0.
\]
From (13), by taking into account the $\xi$–space Laplace transform, we have

$$\Phi(\lambda)\tilde{w}(\lambda, \xi) - \frac{\Phi(\lambda)}{\lambda} = -\Psi(\xi)\tilde{w}(\lambda, \xi), \quad \lambda > 0, \xi > 0$$

which leads to

$$\tilde{w}(\lambda, \xi) = \frac{\Phi(\lambda)}{\lambda (\Phi(\lambda) + \Psi(\xi))}, \quad \lambda > 0, \xi > 0.$$  

The inverse of $\tilde{w}(\lambda, \xi)$ gives the density function of $H^\Psi \circ L_1^\Phi$ which is written as

$$w(t, x) = \int_0^\infty \Psi(s, x)l^\Phi(t, s)ds, \quad t > 0, x > 0$$

for the independence between $H^\Psi$ and $L_1^\Phi$.

Concerning the special case of dependent processes, $L_1^\Phi$ and $H^\Phi$, we focus on the case of stable subordinators.

**Proposition 8.1.** Let $H^\alpha$ be an $\alpha$–stable subordinator and $L_1^\alpha$ its inverse. We have:

(i) For $z > t > 0$

$$P_0(H^\alpha \circ L_1^\alpha \in dz) = \frac{\sin(\pi \alpha)}{\pi} \left( \frac{t}{z-t} \right)^\alpha dz.$$  

(ii) For $0 < y < t$

$$P_0(H^\alpha \circ (L_1^\alpha -) \in dy) = \frac{\sin(\pi \alpha)}{\pi} \left( \frac{y}{t-y} \right)^\alpha dy.$$  

**Proof.** (i) From [2, Proposition 2, section III.2] we have that, for each $t \geq 0$ and $0 \leq y \leq t < z$,

$$P_0(H^\alpha \circ (L_1^\alpha -) \in dy, H^\alpha \circ L_1^\alpha \in dz) = U(dy)\Pi^\alpha(dz - y),$$

(78)

where $U(dy)$ is the potential measure for which

$$\int_0^\infty e^{-\lambda y}U(dy) = \frac{1}{\lambda^\alpha}, \quad \lambda > 0.$$  

It turns out that $\Pi^\alpha$ and $U$ are written as

$$\Pi^\alpha(dz) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dz}{z^{\alpha+1}} \quad \text{and} \quad U(dy) = \frac{1}{\Gamma(\alpha)} \frac{dy}{y^{\alpha-1}}.$$  

Hence, by integrating (78),

$$P_0(H^\alpha \circ L_1^\alpha \in dz) = \int_0^t P_0(H^\alpha \circ (L_1^\alpha -) \in dy, H^\alpha L_1^\alpha \in dz)$$

$$= \frac{\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^t \frac{y^{\alpha-1}}{(z-y)^{\alpha+1}}dy dz$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{1}{z} \left( \frac{t}{z-t} \right)^\alpha dz.$$  

(ii) From [3, Lemma 1.10], we have

$$P_0(H^\alpha \circ (L_1^\alpha -) \in dy) = \Pi^\alpha(t-y)U(dy).$$

We obtain

$$P_0(H^\alpha \circ (L_1^\alpha -) \in dy) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(t-y)^\alpha \Gamma(\alpha)} y^{\alpha-1}dy,$$

which is the claim. □

We observe that, for $t = 1$, $H^\alpha \circ (L_1^\alpha -)$ has the so-called generalized arcsine law

$$P_0(H^\alpha \circ (L_1^\alpha -) \in dy) = \frac{\sin \alpha \pi}{\pi} y^{\alpha-1}(1-y)^{-\alpha}dy$$

and for $\alpha = \frac{1}{2}$ we get the well known arcsine law for the standard Brownian motion.
Figure 8. A simulation of the $\alpha$–stable subordinator if $\alpha = 0.5$, based on [44, Section 5.2].

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