Singular Quantum Mechanical Viewpoint of Localized Gravity in Brane-World Scenario

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Abstract

The graviton localized on the 3-brane is examined in Randall-Sundrum brane-world scenario from the viewpoint of one-dimensional singular quantum mechanics. For the Randall-Sundrum single brane scenario the one-parameter family of the fixed-energy amplitude is explicitly computed where the free parameter $\xi$ parametrizes the various boundary conditions at the brane. The general criterion for the localized graviton to be massless is derived when $\xi$ is arbitrary but non-zero. When $\xi = 0$, the massless graviton is obtained via a coupling constant renormalization. For the two branes picture the fixed-energy amplitude is in general dependent on the two free parameters. The numerical test indicates that there is no massless graviton in this picture. For the positive-tension brane, however, the localized graviton becomes massless when the distance between branes are infinitely large, which is essentially identical to the single brane picture. For the negative-tension brane there is no massless graviton regardless of the distance between branes and choice of boundary conditions.
I. INTRODUCTION

The first Randall-Sundrum(RS1) brane-world scenario [1] was designed to solve the gauge hierarchy problem which is one of the longstanding puzzle in physics. To examine this problem they have introduced two branes located at the boundary of the compactified fifth dimension. The second Randall-Sundrum(RS2) scenario [2] is followed from RS1 by remoting one of the brane to infinity. The most remarkable feature of RS2 scenario is that it leads to a massless graviton localized on the 3-brane at the linearized fluctuation level [3,4]. In this paper we will explore the localized RS graviton problem at RS1 and RS2 from the viewpoint of the singular quantum mechanics.

In addition to its good features on hierarchy and localized graviton problem, RS picture supports a non-static cosmological solution [5–7] which leads to the conventional Friedmann equation if one introduces the bulk and brane cosmological constants and imposes a particular fine-tuning condition between them. Furthermore, RS scenario is also applied to the cosmological constant hierarchy [8,9] and black hole physics [10–12].

The bulk spacetime of RS scenario is two copies of $AdS_5$ glued in a $Z_2$-symmetric way along a boundary which is interpreted as the 3-brane world-volume. It is explicitly seen by examining the line elements;

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2d\phi^2 \quad \text{(RS1)}$$
$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad \text{(RS2)}$$

where $|\phi| \leq \pi$ and $|y| < \infty$. The parameter $r_c$ is a radius of the compactified fifth dimension.

The analogy of RS scenario to $AdS/CFT$ [13] enables us to explore the finite temperature effect in RS brane-world scenario by extending $AdS_5$ bulk spacetime to Schwarzschild-$AdS_5$ [14,15].

Inserting the small fluctuation equations

$$ds^2 = \left(e^{-2kr_c|\phi|}\eta_{\mu\nu} + h_{\mu\nu}(x,\phi)\right)dx^\mu dx^\nu + r_c^2d\phi^2 \quad \text{(RS1)}$$
$$ds^2 = \left(e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(y)\right)dx^\mu dx^\nu + dy^2 \quad \text{(RS2)}$$
to 5d Einstein equation one can derive a gravitational fluctuation equation

\[ \hat{H}_{RS} \hat{\psi}(z) = \frac{m^2}{2} \hat{\psi}(z) \]  
(1.3)

\[ \hat{H}_{RS} = -\frac{1}{2} \partial_z^2 + V_i(z) \]

where \( i = 1, 2 \) represents \( i^{th} \) RS scenario. For each RS scenario the potential becomes

\[ V_1(z) = \frac{15k^2}{8(k|z| + 1)^2} - \frac{3}{2} k [\delta(z) - \delta(z - z_0)] \]  
(1.4)

\[ V_2(z) = \frac{15k^2}{8(k|z| + 1)^2} - \frac{3}{2} k \delta(z) \]

where \( z_0 = (e^{kr_c \pi} - 1)/k \). The function \( \hat{\psi}(z) \) is related to the linearized gravitational field \( h \) as follows

\[ h(x, \phi) = e^{-\frac{1}{2} kr_c |\phi|} \hat{\psi} e^{ipx} \]  
(RS1)

\[ h(x, y) = e^{-\frac{1}{2} k|y|} \hat{\psi} e^{ipx} \]  
(RS2)

where \( m^2 = -p^2 \) and

\[ z = \epsilon(\phi) \frac{e^{kr_c |\phi|} - 1}{k} \]  
(RS1)

\[ z = \epsilon(y) \frac{e^{k|y|} - 1}{k} \]  
(RS2)

Since all components are same, Lorentz indices \( \mu \) and \( \nu \) are suppressed in Eq.(1.5).

When deriving the linearized fluctuation equation (1.3), we have used the RS gauge choice

\[ h_{\nu,\mu} = h_{\mu} = 0, \quad h_{\nu 5} = h_{\mu 5} = 0 \]  
(1.7)

for each RS scenario. This gauge choice, however, generally generates a non-trivial bending effect on the brane [3]. The bending effect usually makes the linearized fluctuation equation (1.3) to be non-homogeneous form, i.e. \((\hat{H}_{RS} - m^2/2)\hat{\psi} \neq 0\). Thus, inclusion of the bending effect makes the story to be more complicated. In this paper we will not consider the bending effect for simplicity.
The linearized fluctuation equation (1.3) looks like usual Schrödinger equation. From the purely mathematical point of view the Hamiltonian operator $\hat{H}_{RS}$ in Eq.(1.3) is a singular operator due to the singular $\delta$-function potential in $V_i$. In the path-integral framework \[16,17\] the 1d $\delta$-function potential was treated by Schulman about one and half decades ago as follows \[18\].

Let us consider 1d Hamiltonian

$$H = H_V + v\delta(x)$$ \hspace{1cm} (1.8)

where

$$H_V = \frac{p^2}{2} + V(x).$$ \hspace{1cm} (1.9)

It is well-known that the Euclidean propagator $G[x_1, x_2; t]$ for $H$ obeys the following integral equation

$$G[x_1, x_2; t] = G_V[x_1, x_2 : t] - v \int_0^t ds \int dx G_V[x_1, x; t - s]\delta(x)G[x, x_2; s]$$ \hspace{1cm} (1.10)

where $G_V[x_1, x_2 : t]$ is an Euclidean propagator for $H_V$. The Euclidean propagator $G[x_1, x_2; t]$ is related to the usual Feynman propagator(or Kernel) $K[x_1, x_2; t]$ as follows;

$$K[x_1, x_2; t] = G[x_1, x_2; it].$$ \hspace{1cm} (1.11)

Taking a Laplace transform

$$\hat{f} \equiv \mathcal{L}f(t) \equiv \int_0^\infty dt e^{-Et}f(t)$$ \hspace{1cm} (1.12)

to both sides of Eq.(1.10) yields

$$\hat{G}[x_1, x_2; E] = \hat{G}_V[x_1, x_2; E] - v\hat{G}_V[x_1, 0; E]\hat{G}[0, x_2; E]$$ \hspace{1cm} (1.13)

which supports a solution

$$\hat{G}[0, x_2; E] = \frac{\hat{G}_V[0, x_2; E]}{1 + v\hat{G}_V[0, 0, E]}.$$ \hspace{1cm} (1.14)
Inserting Eq.(1.14) into Eq.(1.13) again completes Schulman’s procedure;

\[
\hat{G}[x_1, x_2; E] = \hat{G}_V[x_1, x_2; E] - \frac{\hat{G}_V[x_1, 0; E]\hat{G}_V[0, x_2; E]}{\frac{1}{v} + \hat{G}_V[0, 0; E]}.
\] (1.15)

The usual energy-dependent Green’s function $\hat{K}[x_1, x_2; E]$ which is a Fourier transform of $K[x_1, x_2; t]\theta(t)$, where $\theta(t)$ is a step function, is also evaluated from the corresponding fixed-energy amplitude $\hat{G}[x_1, x_2; E]$ by a relation

\[
\hat{K}[x_1, x_2; E] = -i\hat{G}[x_1, x_2; -E]
\] (1.16)

where $-E$ in $\hat{G}$ is a usual Euclidean nature. Of course, one can compute the Feynman propagator by taking an inverse Laplace transform to $\hat{G}[x_1, x_2; E]$ and using a relation (1.11).

Extension of Schulman’s procedure to higher dimensional cases is not straightforward due to the infinity arising from origin. In these cases we have to modify Eq.(1.15) appropriately to escape the ultraviolet divergence [19]. Especially, 2$d$ case is very interesting because a lot of non-trivial effects are involved in 2$d$ $\delta$-function potential such as scale anomaly and dimensional transmutation. In Ref. [20] Jackiw explored the 2$d$ $\delta$-function potential system by making use of the physically-oriented coupling constant renormalization and the mathematically-oriented self-adjoint extension [21,22]. He also derived the relation of the renormalized coupling constant to the self-adjoint extension parameter. His result is generalized within a path-integral or Green’s function formalism in Ref. [19,23–26].

The purpose of this paper is to examine the property of the localized gravity in RS1 and RS2 scenario by treating Eq.(1.3) as a Schrödinger equation. In this paper we adopt $AdS$/CFT setting, i.e. single copy of $AdS_5$ spacetime with a singular brane on the boundary. The $AdS$/CFT setting generates non-trivial constraints. For RS1 and RS2 it generates 1$d$ box($0 \leq \phi \leq \pi$) and half-line($0 \leq y < \infty$) constraints respectively. These constraints makes the fixed-energy amplitude for $\hat{H}_{RS}$ to be crucially dependent on the boundary conditions(BCs). The combination of these constraints with a singular $\delta$-function potential makes the situation to be complicated. The fascinating fact is that Dirichlet BC requires a coupling constant renormalization to lead a non-trivial fixed-energy amplitude although our case is one-dimensional singular quantum mechanics.
The paper is organized as follows. In section 2 we will consider the free particle case with an half-line constraint and δ-function potential at the boundary as a toy model of RS2 case. In this section we will show how BCs play important roles in this simple singular quantum mechanics. Also we will show why coupling constant renormalization is necessary to lead a non-trivial modification in the fixed-energy amplitude at Dirichlet BC. In section 3 we will compute the fixed-energy amplitude for RS2 which depends on a free parameter ξ where ξ = 0 and ξ = 1 correspond respectively to pure Dirichlet and pure Neumann BC cases. We will derive in this section the general criterion in the parameter space for the localized graviton on the 3-brane to be massless. We will also show that the massless graviton at ξ = 0 is followed via a coupling constant renormalization. In section 4 we will consider the free particle case with an 1d-box constraint and δ-function potentials at the both boundaries as a toy model of RS1. The final expression is dependent on the two free parameters ξ₁ and ξ₂ which parametrize the BCs arising at both boundaries of 1d box. In section 5 we will compute the fixed-energy amplitude for RS1 which depends on two free parameters ξ₁ and ξ₂. We will show in this section that there is no localized massless graviton on both branes. For positive-tension brane, however, the massless graviton can appear when the width of 1d box is infinity, which is essentially identical to RS2. For the negative-tension brane our numerical calculation indicates there is no localized massless graviton regardless of the size of 1d box. In final section a brief conclusion is given.

II. TOY MODEL 1: FREE PARTICLE ON A HALF-LINE WITH δ-FUNCTION POTENTIAL

In this section as a toy model of RS2 we will examine Green’s function for the free particle system defined on a half-line(x ≥ 0) with δ-function potential whose Hamiltonian is

\[ \hat{H} = \hat{H}_0^> - v\delta(x) \]  \hspace{1cm} (2.1)

where \( \hat{H}_0^> \) is a free particle Hamiltonian with the half-line constraint, \textit{i.e.}
\[ \hat{H}_0^\gamma = -\frac{1}{2} \partial_x^2 \quad (x \geq 0). \] (2.2)

Of course the main problem in this model is how to compute the fixed-energy amplitude for \( \hat{H}_0^\gamma \). Once this is completed, one can derive a fixed-energy amplitude for \( \hat{H} \) by employing the Schulman procedure described in the previous section.

We start with a fixed-energy amplitude \( \hat{G}_F[x, y; E] \) for free particle without any constraint

\[ \hat{G}_F[x, y; E] = e^{-\sqrt{2E}|x-y|} \sqrt{2E}. \] (2.3)

Then, the fixed-energy amplitude for \( \hat{H}_0^\gamma \) can be computed as follows from \( \hat{G}_F[x, y; E] \).

First, we have to note that the fixed-energy amplitude for \( \hat{H}_0^\gamma \) is dependent on BC at \( x = 0 \) arising due to the half-line constraint. The usual Dirichlet or Neumann BCs at \( x = 0 \) are properly incorporated into the path-integral formalism using \( \delta - \) and \( \delta' - \)function potentials\[ \begin{align*}
\hat{G}^D_F[a, b; E] &= \hat{G}_F[a, b; E] - \frac{\hat{G}_F[a, 0; E]\hat{G}_F[0, b; E]}{G_F[0^+, 0; E]} \\
\hat{G}^N_F[a, b; E] &= \hat{G}_F[a, b; E] - \frac{\hat{G}_F[a, 0; E]\hat{G}_F[0, b; E]}{G_F,ab[0^+, 0; E]},
\end{align*} \] (2.4)

where the superscripts \( D \) and \( N \) stand for Dirichlet and Neumann respectively. The explicit calculation shows

\[ \begin{align*}
\hat{G}^D_F[a, b; E] &= e^{-\sqrt{2E}|x-y|} - e^{-\sqrt{2E}(|a|+|b|)} \\
\hat{G}^N_F[a, b; E] &= e^{-\sqrt{2E}|x-y|} + \epsilon(a)\epsilon(b)e^{-\sqrt{2E}(|a|+|b|)}. \end{align*} \] (2.5)

One can show easily \( \hat{G}^D_F \) and \( \hat{G}^N_F \) satisfy the following BCs;

\[ \begin{align*}
\hat{G}^D_F[a, 0; E] &= \hat{G}^D_F[0, b; E] = 0 \\
\hat{G}^N_F[a,b, 0; E] &= \hat{G}^N_F[0, b; E] = 0.
\end{align*} \] (2.6)

Then, the general fixed-energy amplitude for \( \hat{H}_0^\gamma \) can be obtained by linearly combining \( \hat{G}^D_F \) and \( \hat{G}^N_F \);
\[ \hat{G}_F^\xi[a, b; E] = \xi \hat{G}_F^N[a, b; E] + (1 - \xi) \hat{G}_F^D[a, b; E] \]  \hspace{1cm} (2.7)

where \(0 \leq \xi \leq 1\) is a real parameter parametrizing the BCs at the origin. Of course \(\xi = 0\) and \(\xi = 1\) represent the pure Dirichlet and pure Neumann BCs respectively. Another interesting case is \(\xi = 1/2\), in which the contribution of Neumann and Dirichlet have an equal weighting factors. Since the fixed-energy amplitude \(\hat{G}_F^\xi\) is in general expressed in terms of eigenvalues \(E_n\) and eigenfunctions \(\phi_n\) of \(\hat{H}_0^\xi\) as follows:

\[ \hat{G}_F^\xi[a, b; E] = \sum_n \frac{\phi_n(a)\phi_n^*(b)}{E - E_n}, \]  \hspace{1cm} (2.8)

the \(\xi = 1/2\) case should correspond to the free particle case without any constraint at the origin.

Following Schulman procedure one can calculate the fixed-energy amplitude \(\hat{G}_F^\xi\) for \(\hat{H}\) from \(\hat{G}_F^\xi\) as follows:

\[ \Delta \hat{G}_F^\xi[a, b; E] \equiv \hat{G}_F^\xi[a, b; E] - \hat{G}_F^\xi[a, b; E] = \frac{4\xi^2}{\sqrt{2E_v} - 2\xi} \frac{e^{-\sqrt{2E(|a|+|b|)}}}{\sqrt{2E}}. \]  \hspace{1cm} (2.9)

At \(\xi = 1\) and \(\xi = 1/2\) the fixed-energy amplitudes are simply reduced to

\[ \hat{G}_F^{\xi=1}[a, b; E] = \frac{e^{-\sqrt{2E}|a-b|}}{\sqrt{2E}} + \frac{\sqrt{2E_v}}{\sqrt{2E_v} - 2} \frac{e^{-\sqrt{2E}(|a|+|b|)}}{\sqrt{2E}}, \]  \hspace{1cm} (2.10)
\[ \hat{G}_F^{\xi=1/2}[a, b; E] = \frac{e^{-\sqrt{2E}|a-b|}}{\sqrt{2E}} + \frac{e^{-\sqrt{2E}(|a|+|b|)}}{\sqrt{2E} \left(\sqrt{2E_v} - 1\right)}, \]

and the corresponding bound state energies \(B(\xi)\) arising due to the \(\delta\)-function potential are

\[ B(\xi = 1) = -2v^2 \]  \hspace{1cm} (2.11)
\[ B(\xi = 1/2) = - \frac{v^2}{2}. \]

Finally, let us consider \(\xi = 0\) case. In this case Eq.(2.9) shows that the modification term \(\Delta \hat{G}_F^{\xi=0}\) vanishes. This means the \(\delta\)-function potential in Eq.(2.1) does not play any important role. In fact this is obvious if we consider the fact that at \(\xi = 0\) the Hamiltonian \(\hat{H}_0^\geq\) describes the free particle system plus \(\lim_{\alpha \to \infty} \alpha \delta(x)\) which makes the half-line constraint. Thus, the \(\delta\)-function potential in eq.(2.1) is absorbed to \(\hat{H}_0^\geq\).
Even in this case, however, one can derive a non-trivial fixed-energy amplitude under the assumption that $v$ is infinite bare coupling constant by adopting the coupling constant renormalization. To show this explicitly we re-express the modification term $\Delta \hat{G}^{\xi=0}$ as follows;

$$\Delta \hat{G}^{\xi=0}[a, b; E] = \lim_{\epsilon \to 0^+} \frac{G^{\xi=0}_F[a, \epsilon; E] G^{\xi=0}_F[\epsilon, b; E]}{\frac{1}{v} - G^{\xi=0}_F[\epsilon, E]}.$$  \hspace{1cm} (2.12)

Expanding the denominator and numerator separately one can conclude

$$\Delta \hat{G}^{\xi=0}[a, b; E] = \frac{2}{\sqrt{2E} - v^{\text{ren}}} e^{-\sqrt{2E}(|a|+|b|)}$$ \hspace{1cm} (2.13)

where the renormalized coupling constant $v^{\text{ren}}$ is defined as

$$v^{\text{ren}} = \frac{1}{2\epsilon^2} \left( 2\epsilon - \frac{1}{v} \right).$$ \hspace{1cm} (2.14)

It is easy to show that $v^{\text{ren}}$ has a same dimension with the bare coupling constant $v$. Following the philosophy of renormalization we regard $v^{\text{ren}}$ as a finite quantity. Combining Eq.(2.3) and Eq.(2.13) we get finally

$$\hat{G}^{\xi=0}[a, b; E] = \frac{e^{-\sqrt{2E}|a-b|}}{\sqrt{2E}} + \frac{\sqrt{2E}}{v^{\text{ren}}} + 1 \frac{e^{-\sqrt{2E}(|a|+|b|)}}{\sqrt{2E} - 1}$$ \hspace{1cm} (2.15)

whose bound state energy is $B(\xi = 0) = -(v^{\text{ren}})^2/2$.

In the next section we will apply the analysis in this toy model to the RS2 scenario.

**III. FIXED-ENERGY AMPLITUDE FOR RS2**

Recently, one of the present authors computed the fixed-energy amplitude for RS2 at Ref. [27] which will be reviewed in this section briefly. Furthermore we will derive the general condition in the parameter space for the appearance of the localized massless graviton.

The Hamiltonian for RS2 can be read from Eq.(1.3) and (1.4) easily;

$$\hat{H}_{\text{RS2}} = \hat{H}_0 - v\delta(z)$$ \hspace{1cm} (3.1)

$$\hat{H}_0 = -\frac{1}{2} \partial^2 + \frac{g}{(|z| + c)^2}.$$
Of course, we can obtain the exact RS2 Hamiltonian by letting \( g = 15/8 \), \( c = 1/k \equiv R \) and \( v = 3k/2 \), where \( R \) is the radius of \( AdS_5 \). In this section, however, we do not require them from the beginning. In other words we will compute the fixed-energy amplitude for arbitrary \( g, c, \) and \( v \) when the half-line constraint \( (z \geq 0) \) is imposed. This will give us the general condition for the localized graviton on the brane to be massless.

The half-line constraint makes \( \hat{H}_0 \) in Eq.(3.1) to be a following simple form;

\[
\hat{H}_0 = -\frac{1}{2} \partial_x^2 + \frac{g}{x^2}
\]

where \( x = z + c \). Thus our half-line constraint \( z \geq 0 \) is changed into \( x \geq c \). If \( c = 0 \), the Euclidean propagator \( G_{0>}[a, b; t] \) and the corresponding fixed-energy amplitude \( \hat{G}_{0>}[a, b; E] \) for Hamiltonian (3.2) are given at Ref. [17];

\[
G_{0>}[a, b; t] = \frac{\sqrt{ab}}{t} e^{-\frac{a^2 + b^2}{4t}} I_\gamma \left( \frac{ab}{t} \right) (3.3)
\]

\[
\hat{G}_{0>}[a, b; E] = 2\sqrt{ab} I_\gamma \left( \sqrt{\frac{E}{2}}[(a + b) - |a - b|] \right) K_\gamma \left( \sqrt{\frac{E}{2}}[(a + b) + |a - b|] \right)
\]

where \( I_\gamma(z) \) and \( K_\gamma(z) \) are the usual modified Bessel functions, and \( \gamma = \sqrt{1 + 8g/2} \).

The main problem for the computation of the fixed energy amplitude for \( \hat{H}_0 \) in Eq.(3.2) is how to adopt an asymmetric constraint \( x \geq c \) in terms of \( x \). However, this is already explained at the previous section by introducing an infinite energy barrier. In the asymmetric barrier the fixed-energy amplitude will be dependent on the BC at \( x = c \). Hence, the final form will be one parameter family type

\[
\hat{G}_0^\xi[a, b; E] = \xi \hat{G}_0^N[a, b; E] + (1 - \xi) \hat{G}_0^D[a, b; E]
\]

(3.4)

where \( \hat{G}_0^N \) and \( \hat{G}_0^D \) are the fixed-energy amplitudes which obey the Neumann and Dirichlet BCs respectively at \( x = c \). Of course, \( \hat{G}_0^N \) and \( \hat{G}_0^D \) can be calculated from \( \hat{G}_{0>}[a, b; E] \) introducing \( \delta \)- and \( \delta' \)-functions as we did in the previous section;

\[
\hat{G}_0^D[a, b; E] = \hat{G}_{0>}[a, b; E] - \frac{\hat{G}_{0>}[a, c; E] \hat{G}_{0>}[c, b; E]}{\hat{G}_{0>}[c^+, c; E]}
\]

(3.5)

\[
\hat{G}_0^N[a, b; E] = \hat{G}_{0>}[a, b; E] - \frac{\hat{G}_{0>}[a, c; E] \hat{G}_{0>}[c, b; E]}{\hat{G}_{0>}[c^+, c; E]}
\]
Inserting Eq. (3.3) into Eq. (3.5) one can derive the explicit forms of $\hat{G}_D^0$ and $\hat{G}_N^0$:

\[
\hat{G}_D^0[a, b; E] = \hat{G}_{>0}[a, b; E] - 2\sqrt{ab} \frac{I_\gamma(\sqrt{2Ec})}{K_\gamma(\sqrt{2Ec})} K_\gamma(\sqrt{2Ea}) K_\gamma(\sqrt{2Eb}) \tag{3.6}
\]

\[
\hat{G}_N^0[a, b; E] = \hat{G}_{>0}[a, b; E] + 2\sqrt{ab} \frac{f_I(c, E)}{f_K(c, E)} K_\gamma(\sqrt{2Ea}) K_\gamma(\sqrt{2Eb})
\]

where

\[
f_K(x, E) = \frac{\gamma - \frac{1}{2}}{\sqrt{2Ex}} K_{\gamma-1}(\sqrt{2Ex}) \tag{3.7}
\]

\[
f_I(x, E) = I_{\gamma-1}(\sqrt{2Ex}) - \frac{\gamma - \frac{1}{2}}{\sqrt{2Ex}} I_\gamma(\sqrt{2Ex}).
\]

The useful relation which will be used frequently is

\[
f_K(x, E) I_\gamma(\sqrt{2Ex}) + f_I(x, E) K_\gamma(\sqrt{2Ex}) = \frac{1}{\sqrt{2Ex}}. \tag{3.8}
\]

Following Schulman procedure it is straightforward to derive a fixed-energy amplitude for $\hat{H}_{RS2}$:

\[
\hat{G}_{RS2}[a, b; E] = \hat{G}_0^c[a, b; E] + \frac{\hat{G}_0^c[a, c; E] \hat{G}_0^c[c, b; E]}{\frac{1}{v} - \hat{G}_0^c[c^+, c; E]} \tag{3.9}
\]

where $\hat{G}_0^c$ is given in Eq. (3.4). The most convenient form of $\hat{G}_{RS2}$ is

\[
\hat{G}_{RS2}[a, b; E] = \hat{G}_0^c[a, b; E] + \frac{\sqrt{ab} K_\gamma(\sqrt{2Ea}) K_\gamma(\sqrt{2Eb})}{K_\gamma^2(\sqrt{2Ec})} \times \left[ \left( \frac{\gamma - \frac{1}{2}}{2\xi cv} - 1 \right) + \frac{\sqrt{2E} K_{\gamma-1}(\sqrt{2Ec})}{2\xi v K_\gamma(\sqrt{2Ec})} \right]^{-1}. \tag{3.10}
\]

Since $\hat{G}_0^D$ satisfies the usual Dirichlet BC, i.e. $\hat{G}_0^D[a, c; E] = \hat{G}_0^D[c, b; E] = 0$, the fixed-energy amplitude on the brane is simply reduced to

\[
\hat{G}_{RS2}[c, c; E] = \left[ \left( \frac{\gamma - \frac{1}{2}}{2\xi c} - v \right) + \frac{\sqrt{2E} K_{\gamma-1}(\sqrt{2Ec})}{2\xi K_\gamma(\sqrt{2Ec})} \right]^{-1}. \tag{3.11}
\]

If $(\gamma - 1/2)/(2\xi c) - v = 0$, $\hat{G}_{RS2}[c, c; E]$ becomes

\[
\hat{G}_{RS2}[c, c; E] = \Delta_0 + \Delta_{KK} \tag{3.12}
\]

where
\[ \Delta_0 = \frac{2\xi(\gamma - 1)}{cE} \quad (3.13) \]
\[ \Delta_{KK} = \frac{2\xi K_{\gamma-2}(\sqrt{2Ec})}{\sqrt{2E} K_{\gamma-1}(\sqrt{2Ec})}. \]

Of course, \( \Delta_0 \) and \( \Delta_{KK} \) represent the zero mode and the higher Kaluza-Klein excitations respectively. This means that the condition for the localized graviton to be massless is

\[ \frac{\gamma - \frac{1}{2}}{2\xi c} - \nu = 0. \quad (3.14) \]

At the RS limit \( \gamma = 2, \ c = R \) and \( \nu = 3/2R \) this condition really holds at \( \xi = 1/2 \). But for other values of \( \xi \) except \( \xi = 0 \) one can also obtain the massless graviton by changing \( c, \ g, \) and \( \nu \) appropriately to obey Eq.(3.14).

At the pure Dirichlet BC case(\( \xi = 0 \)) Eq.(3.14) cannot hold unless \( \nu = \infty \). Thus in this case we can get the massless graviton via a coupling constant renormalization as we did in the previous section.

To show explicitly we have to re-write Eq.(3.9) by introducing a positive infinitesimal parameter \( \epsilon \);

\[ \hat{G}^{\xi=0}_{RS2}[a, b; E] = \hat{G}^D[a, b; E] + \lim_{\epsilon \rightarrow 0^+} \frac{1}{\nabla} \frac{1}{v} - \hat{G}^D[c, c + \epsilon; E]. \quad (3.15) \]

Explicit calculation shows that even in this case one can derive a massless graviton when \( v^{ren} = -3/(2R) \), where \( v^{ren} \) is a renormalized coupling constant defined as

\[ v^{ren} = \frac{1}{2\epsilon^2} \left( \frac{1}{v} - 2\epsilon \right). \quad (3.16) \]

The coupling constant renormalization procedure in RS2 is in detail explained in Ref. [27]. In this case the fixed-energy amplitude and the corresponding gravitational potential is exactly same with that of the original RS result when \( v^{ren} = -3/(2R) \). This result may provide us the compromise of the massless graviton with a small cosmological constant [27].
IV. TOY MODEL 2: FREE PARTICLE IN A BOX WITH $\delta$-FUNCTION POTENTIALS

In this section as a toy model of RS1 we will examine Green’s function for the free particle system in a 1d box ($0 \leq x \leq L$) with $\delta$-function potentials at both end points.

The Hamiltonian for this system is

$$\hat{H}_\delta^{Box} = \hat{H}_0^{Box} - v_1 \delta(x) + v_2 \delta(x - L) \quad (4.1)$$

where

$$\hat{H}_0^{Box} = -\frac{1}{2} \partial_x^2 \quad (0 \leq x \leq L). \quad (4.2)$$

The main problem in this toy model is of course to derive a fixed-energy amplitude $\hat{G}_0^{Box}$ for $\hat{H}_0^{Box}$. Once $\hat{G}_0^{Box}$ is obtained, the fixed-energy amplitude $\hat{G}_\delta^{Box}$ for the total Hamiltonian $\hat{H}_\delta^{Box}$ is straightforwardly obtained by performing the Schulman procedure twice;

$$\hat{G}_\delta^{Box}[a, b; E] = \hat{G}_0^{Box}[a, b; E] - \frac{\hat{G}_0^{Box}[a, L; E]\hat{G}_0^{Box}[L, b; E]}{\frac{1}{v_2} + \hat{G}_0^{Box}[L, L^-; E]} \quad (4.3)$$

where

$$\hat{G}_0^{Box}[a, b; E] = \hat{G}_0^{Box}[a, b; E] + \frac{\hat{G}_0^{Box}[a, 0; E]\hat{G}_0^{Box}[0, b; E]}{\frac{1}{v_2} - \hat{G}_0^{Box}[0^+, 0; E]}. \quad (4.4)$$

The fixed-energy amplitude $\hat{G}_0^{Box}$ for $\hat{H}_0^{Box}$ is also obtained directly from that for free particle on half-line, i.e. $\hat{G}_F^{\xi_1}[a, b; E]$ in Eq.(2.7). Of course, the parameter $\xi_1$ represents the type of BC at $x = 0$. Then, the fixed-energy amplitude $\hat{G}_0^{Box}$ can be computed by introducing an infinite barrier at $x = L$ to the half-line constraint system. As we commented in section 2 the infinite barrier is introduced by $\delta$- and $\delta'$-functions potential at $x = L$ with assumption that the coupling constant is infinite. Thus the final form of $\hat{G}_0^{Box}$ is dependent on the two parameters as follows;

$$\hat{G}_0^{Box}[a, b; E] = \xi_2 \hat{G}_0^{\xi_1,N}[a, b; E] + (1 - \xi_2) \hat{G}_0^{\xi_1,D}[a, b; E] \quad (4.5)$$

where
\[ \hat{G}_{0}^{\xi_1,D}[a, b; E] = \hat{G}_{F}^{\xi_1}[a, b; E] - \frac{\hat{G}_{F}^{\xi_1}[a, L; E] \hat{G}_{F}^{\xi_1}[L, b; E]}{\hat{G}_{F}^{\xi_1}[L, L^-; E]} \]  
(4.6)

\[ \hat{G}_{0}^{\xi_1,N}[a, b; E] = \hat{G}_{F}^{\xi_1}[a, b; E] - \frac{\hat{G}_{F,a}^{\xi_1}[a, L; E] \hat{G}_{F,a}^{\xi_1}[L, b; E]}{\hat{G}_{F,a}^{\xi_1}[L, L^-; E]} . \]

Of course the parameter \( \xi_2 \) in Eq.(4.5) parametrizes the various BCs at \( x = L \).

Explicit calculation shows

\[ \hat{G}_{0}^{\xi_1,D}[a, b; E] = \frac{1}{\sqrt{2E}[\xi_1 \cosh \sqrt{2EL} + (1 - \xi_1) \sinh \sqrt{2EL}]} \times \left[ \xi_1 \left\{ \sinh \sqrt{2E}(L - |a - b|) - \sinh \sqrt{2E}((a + b) - L) \right\} + (1 - \xi_1) \left\{ \cosh \sqrt{2E}(L - |a - b|) - \cosh \sqrt{2E}((a + b) - L) \right\} \right] \]  
(4.7)

\[ \hat{G}_{0}^{\xi_1,N}[a, b; E] = \frac{1}{\sqrt{2E}[(1 - \xi_1) \cosh \sqrt{2EL} + xi \sinh \sqrt{2EL}]} \times \left[ \xi_1 \left\{ \cosh \sqrt{2E}(L - |a - b|) + \cosh \sqrt{2E}((a + b) - L) \right\} + (1 - \xi_1) \left\{ \sinh \sqrt{2E}(L - |a - b|) + \sinh \sqrt{2E}((a + b) - L) \right\} \right]. \]

Inserting eq.(4.7) into Eq.(4.3) we get

\[ \hat{G}_{0}^{Box}[a, b; E] = \frac{1}{\sqrt{2E}} \left[ (\mu(\xi_1, \xi_2) + \mu(1 - \xi_1, 1 - \xi_2)) \cosh \sqrt{2E}(L - |a - b|) + (\mu(\xi_1, \xi_2) - \mu(1 - \xi_1, 1 - \xi_2)) \cosh \sqrt{2E}((a + b) - L) + (\nu(\xi_1, \xi_2) + \nu(1 - \xi_1, 1 - \xi_2)) \sinh \sqrt{2E}(L - |a - b|) + (\nu(\xi_1, \xi_2) - \nu(1 - \xi_1, 1 - \xi_2)) \sinh \sqrt{2E}((a + b) - L) \right] \]  
(4.8)

where

\[ \mu(z, w) = \frac{zw}{(1 - z) \cosh \sqrt{2EL} + z \sinh \sqrt{2EL}} \]  
(4.9)

\[ \nu(z, w) = \frac{(1 - z)w}{(1 - z) \cosh \sqrt{2EL} + z \sinh \sqrt{2EL}} . \]

It is interesting to note the following special cases;

\[ \hat{G}_{0}^{DD}[a, b; E] = \frac{\cosh \sqrt{2E}(L - |a - b|) - \cosh \sqrt{2E}((a + b) - L)}{\sqrt{2E} \sinh \sqrt{2EL}} \]  
(4.10)

\[ \hat{G}_{0}^{NN}[a, b; E] = \frac{\cosh \sqrt{2E}(L - |a - b|) + \cosh \sqrt{2E}((a + b) - L)}{\sqrt{2E} \sinh \sqrt{2EL}} . \]
where the superscript $DD$ (or $NN$) stands for Dirichlet-Dirichlet (or Neumann-Neumann) BCs at $x = 0$ and $x = L$. Similar results to Eq. (1.10) are found at Ref. [28]. Bound state energy spectrum is obtained from poles of $\hat{G}^{DD}_0$ and $\hat{G}^{NN}_0$ which indicates $B^{DD}_n = B^{NN}_n = n^2 \pi^2 / 2L^2$ where $n$ is integer. Another interesting case is $\xi_1 = \xi_2 = 1/2$ case where $\hat{G}^{Box}_0$ is simply reduced to the free particle case without any constraint, i.e. $e^{-\sqrt{2E}|a-b|}/\sqrt{2E}$.

Inserting Eq. (1.8) into Eq. (4.4) and subsequently Eq. (4.3) we get the final form of $\hat{G}^{Box}_\delta$ for the Hamiltonian $\hat{H}^{Box}_\delta$. Since final expression is too long, we do not describe it explicitly in this paper. Instead we will consider two special cases.

The first case we will consider is $\xi_1 = \xi_2 = 1/2$. In this case Eq. (4.3) and Eq. (4.4) yield

$$
\hat{G}^{Box}_{\delta, \xi_1=\xi_2=1/2}[a, b; E] = \frac{e^{-\sqrt{2E}|a-b|}}{\sqrt{2E}} + \left(\frac{\sqrt{2E}}{v_1} - 1\right)^{-1} e^{-\sqrt{2E}(a+b)} - \left[\left(\frac{\sqrt{2E}}{v_1} - 1\right)\left(\frac{\sqrt{2E}}{v_2} + 1\right) + e^{-2\sqrt{2EL}}\right]^{-1} \times \left[\left(\frac{\sqrt{2E}}{v_1} - 1\right)\frac{e^{\sqrt{2E}(a+b)}}{\sqrt{2E}} + \frac{e^{\sqrt{2E}(a-b)}}{\sqrt{2E}} + \frac{e^{-\sqrt{2E}(a-b)}}{\sqrt{2E}} + \frac{\left(\frac{\sqrt{2E}}{v_1} - 1\right)^{-1} e^{-\sqrt{2E}(a+b)}}{\sqrt{2E}}\right].
$$  \hspace{1cm} (4.11)

The second case we will consider is $\xi_1 = \xi_2 = 0$. In this case $\hat{G}^{Box}_\delta$ is $\hat{G}^{DD}_0$ in Eq. (1.10).

As expected $\hat{G}^{DD}_0$ satisfies the usual Dirichlet-Dirichlet BCs;

$$
\hat{G}^{DD}_0[0, b; E] = \hat{G}^{DD}_0[a, 0; E] = \hat{G}^{DD}_0[L, b; E] = \hat{G}^{DD}_0[a, L; E] = 0.
$$  \hspace{1cm} (4.12)

If, therefore, $v_1$ and $v_2$ are finite, we arrive at a conclusion $\hat{G}^{Box}_\delta[a, b; E] = \hat{G}^{Box}_0[a, b; E]$.

If however, $v_1$ and $v_2$ are infinite and unphysical bare quantities, one can arrive at different conclusion via the coupling constant renormalization as we have seen in section 2 and 3. To adopt the coupling constant renormalization we introduce the infinitesimal positive constant $\epsilon$ as follows;

$$
\hat{G}^{Box}_0[0, 0; E] \rightarrow \hat{G}^{Box}_0[\epsilon^-, \epsilon, E] = 2\epsilon - 2\sqrt{2E} \coth \sqrt{2E} L \epsilon^2 + O(\epsilon^3)  \hspace{1cm} (4.13)
$$

$$
\hat{G}^{Box}_0[L, L; E] \rightarrow \hat{G}^{Box}_0[L - \epsilon^-, L - \epsilon, E] = 2\epsilon - 2\sqrt{2E} \coth \sqrt{2E} L \epsilon^2 + O(\epsilon^3)
$$

$$
\hat{G}^{Box}_0[0, L; E] \rightarrow \hat{G}^{Box}_0[\epsilon, L - \epsilon, E] = \frac{2\sqrt{2E}}{\sinh \sqrt{2E} L} \epsilon^2 + O(\epsilon^3)
$$

\begin{align*}
&\hat{G}^{Box}_0[L, 0; E] \rightarrow \hat{G}^{Box}_0[\epsilon^-, L - \epsilon, E] = \frac{2\sqrt{2E}}{\sinh \sqrt{2E} L} \epsilon^2 + O(\epsilon^3)
\end{align*}
\[
\hat{G}_0^{\Box}[L, 0; E] \rightarrow \hat{G}_0^{\Box}[L - \epsilon, \epsilon, E] = \frac{2\sqrt{2E}}{\sinh \sqrt{2EL}} \epsilon^2 + O(\epsilon^3)
\]

\[
\hat{G}_0^{\Box}[a, 0; E] \rightarrow \hat{G}_0^{\Box}[a, \epsilon, E] = \frac{2\sinh \sqrt{2E}(L - a)}{\sinh \sqrt{2EL}} \epsilon + O(\epsilon^3)
\]

\[
\hat{G}_0^{\Box}[a, L; E] \rightarrow \hat{G}_0^{\Box}[a, L - \epsilon, E] = \frac{2\sinh \sqrt{2Ea}}{\sinh \sqrt{2EL}} \epsilon + O(\epsilon^3)
\]

\[
\hat{G}_0^{\Box}[0, b; E] \rightarrow \hat{G}_0^{\Box}[\epsilon, b; E] = \frac{2\sinh \sqrt{2Eb}}{\sinh \sqrt{2EL}} \epsilon + O(\epsilon^3).
\]

Inserting Eq.(4.13) into Eq.(4.14) and Eq.(4.3), and defining the renormalized constants

\[
v_1^{\text{ren}} = \frac{1}{2\epsilon^2} \left( \frac{1}{v_1} - 2\epsilon \right)
\]

\[
v_2^{\text{ren}} = \frac{1}{2\epsilon^2} \left( \frac{1}{v_2} + 2\epsilon \right),
\]

one can arrive at the following long expression after tedious calculation;

\[
\hat{G}_0^{\Box}[a, b; E] = \hat{G}_0^{DD}[a, b; E]
\]

\[
+2(v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})^{-1} \frac{\sinh \sqrt{2E}(L - a) \sinh \sqrt{2E}(L - b)}{\sinh^2 \sqrt{2EL}}
\]

\[
- 2 \left[ (v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})(v_2^{\text{ren}} - \sqrt{2E} \coth \sqrt{2EL}) + \frac{2E}{\sinh^2 \sqrt{2EL}} \right]^{-1}
\]

\[
\times \left[ (v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL}) \frac{\sinh \sqrt{2Ea} \sinh \sqrt{2Eb}}{\sinh^2 \sqrt{2EL}}
\]

\[
+ \sqrt{2E} \frac{\sinh \sqrt{2Ea} \sinh \sqrt{2E}(L - b) + \sinh \sqrt{2E}(L - a) \sinh \sqrt{2Eb}}{\sinh^3 \sqrt{2EL}}
\]

\[
+ 2E(v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})^{-1} \frac{\sinh \sqrt{2E}(L - a) \sinh \sqrt{2E}(L - b)}{\sinh^4 \sqrt{2EL}} \right].
\]

Inserting Eq.(4.13) into Eq.(4.14) and Eq.(4.3), one can arrive at the following long expression after tedious calculation;

\[
\hat{G}_0^{\Box}[\delta, \xi_1 = \xi_2 = 0][a, b; E] = \hat{G}_0^{DD}[a, b; E]
\]

\[
+2(v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})^{-1} \frac{\sinh \sqrt{2E}(L - a) \sinh \sqrt{2E}(L - b)}{\sinh^2 \sqrt{2EL}}
\]

\[
- 2 \left[ (v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})(v_2^{\text{ren}} - \sqrt{2E} \coth \sqrt{2EL}) + \frac{2E}{\sinh^2 \sqrt{2EL}} \right]^{-1}
\]

\[
\times \left[ (v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL}) \frac{\sinh \sqrt{2Ea} \sinh \sqrt{2Eb}}{\sinh^2 \sqrt{2EL}}
\]

\[
+ \sqrt{2E} \frac{\sinh \sqrt{2Ea} \sinh \sqrt{2E}(L - b) + \sinh \sqrt{2E}(L - a) \sinh \sqrt{2Eb}}{\sinh^3 \sqrt{2EL}}
\]

\[
+ 2E(v_1^{\text{ren}} + \sqrt{2E} \coth \sqrt{2EL})^{-1} \frac{\sinh \sqrt{2E}(L - a) \sinh \sqrt{2E}(L - b)}{\sinh^4 \sqrt{2EL}} \right].
\]

In the next section we will apply the analysis in this toy model to the RS1 scenario.

**V. FIXED-ENERGY AMPLITUDE FOR RS1**

In this section we will examine the fixed-energy amplitude for RS1 whose linear gravitational fluctuation is given in Eq.(1.3) and Eq.(1.4). The Hamiltonian for RS1 can be read from these equations easily.
\[ H_{RS1} = \hat{H}_0 - v_1 \delta(z) + v_2 \delta(z - z_0) \] (5.1)

\[ \hat{H}_0 = -\frac{1}{2} \partial_z^2 + \frac{g}{(|z| + c)^2}. \]

Of course, the exact RS1 Hamiltonian can be obtained by letting \( g = 15/8, c = 1/k \equiv R, \)
\( v_1 = v_2 = 3k/2 \) and \( z_0 = (e^{k|x_0|} - 1)/k. \) As we did in section 3, however, we will try to
examine the fixed energy amplitude for arbitrary parameter as much as possible.

Next we impose \( z \) is non-negative. This means we use the single copy of \( AdS_5 \) as a bulk
spacetime. In this sense we have a same setting with that of \( AdS/CFT. \) In this setting
Hamiltonian \( \hat{H}_0 \) in Eq.(5.1) becomes

\[ \hat{H}_0 = -\frac{1}{2} \partial_x^2 + \frac{g}{x^2} \quad (c \leq x \leq L) \] (5.2)

where \( L = c + z_0. \)

Of course, the main problem is to compute the fixed-energy amplitude \( \hat{G}_0[a, b; E] \) for \( \hat{H}_0 \)
in Eq.(5.2). From \( \hat{G}_0[a, b; E] \) it is simple to derive the fixed energy amplitude for \( \hat{H}_{RS1} \) by
applying the Schulman procedure twice;

\[ \hat{G}_{RS1}[a, b; E] = \hat{G}_0[a, b; E] - \frac{\hat{G}_0[a, L; E] \hat{G}_0[L; b; E]}{1 - \hat{G}_0[L, L^-; E]} \] (5.3)

where

\[ \hat{G}_0[a, b; E] = \hat{G}_0[a, b; E] + \frac{\hat{G}_0[a, c; E] \hat{G}_0[c; b; E]}{1 - \hat{G}_0[c^+, c; E]} \]. (5.4)

The fixed-energy amplitude for \( \hat{H}_0 \) is also straightforwardly obtained from \( \hat{G}_0^{\xi_1} \) in Eq.(5.4)
by introducing an infinite barrier at \( x = L \) again. Then, the amplitude is dependent on the
two parameters \( \xi_1 \) and \( \xi_2 \) which represent the various BCs at \( x = 0 \) and \( x = L \) respectively;

\[ \hat{G}_0[a, b; E] \equiv \hat{G}_0^{\xi_1, \xi_2}[a, b; E] \] (5.5)

\[ = \xi_2 \hat{G}_0^{\xi_1, N}[a, b; E] + (1 - \xi_2) \hat{G}_0^{\xi_1, D}[a, b; E] \]

where
\begin{align}
\hat{G}^{\xi_1,D}_0[a, b; E] &= \hat{G}^{\xi_1}_0[a, b; E] - \frac{\hat{G}^{\xi_1}_0[a, L; E] \hat{G}^{\xi_1}_0[L, b; E]}{G^{\xi_1}_0[L, L^{-}; E]} \\
\hat{G}^{\xi_1,N}_0[a, b; E] &= \hat{G}^{\xi_1}_0[a, b; E] - \frac{\hat{G}^{\xi_1}_0[a, L; E] \hat{G}^{\xi_1}_0[L, b; E]}{G^{\xi_1}_{0,ab}[L, L^{-}; E]}.
\end{align}

Explicit calculation shows

\begin{align}
\hat{G}^{\xi_1}_0[a, b; E] &= 2\sqrt{ab} \left[ I_\gamma(\sqrt{2E_{\min}}(a, b)) K_\gamma(\sqrt{2E_{\max}}(a, b)) ight.
onumber \\
&\quad + g_1(\xi_1, E) K_\gamma(\sqrt{2Ea}) K_\gamma(\sqrt{2Eb}) 
\end{align}

\begin{align}
\hat{G}^{\xi_1,D}_0[a, b; E] &= \hat{G}^{\xi_1}_0[a, b; E] - 2\sqrt{ab} g_D(\xi_1, E) [I_\gamma(\sqrt{2Ea}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Ea})] 
&\quad \times [I_\gamma(\sqrt{2Eb}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Eb})] \\
\hat{G}^{\xi_1,N}_0[a, b; E] &= \hat{G}^{\xi_1}_0[a, b; E] - 2\sqrt{ab} g_N(\xi_1, E) [I_\gamma(\sqrt{2Ea}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Ea})] 
&\quad \times [I_\gamma(\sqrt{2Eb}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Eb})]
\end{align}

where

\begin{align}
g_1(\xi_1, E) &= \xi_1 \frac{f_1(c, E)}{f_K(c, E)} - (1 - \xi_1) \frac{I_\gamma(\sqrt{2Ec})}{K_\gamma(\sqrt{2Ec})} \\
g_D(\xi_1, E) &= \left( g_1(\xi_1, E) + \frac{I_\gamma(\sqrt{2EL})}{K_\gamma(\sqrt{2EL})} \right)^{-1} \\
g_N(\xi_1, E) &= \left( g_1(\xi_1, E) - \frac{f_1(L, E)}{f_K(L, E)} \right)^{-1}
\end{align}

and, \( f_K \) and \( f_L \) are defined at Eq. (3.7).

Inserting Eq. (5.7) into Eq. (5.5) one can obtain the fixed-energy amplitude for Hamiltonian (5.2);

\begin{align}
\hat{G}^{\xi_1,\xi_2}_0[a, b; E] &= \hat{G}^{\xi_1}_0[a, b; E] - 2\sqrt{ab} g_{Box}(\xi_1, \xi_2, E) [I_\gamma(\sqrt{2Ea}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Ea})] 
&\quad \times [I_\gamma(\sqrt{2Eb}) + g_1(\xi_1, E) K_\gamma(\sqrt{2Eb})]
\end{align}

where

\begin{align}
g_{Box}(\xi_1, \xi_2, E) &= \xi_2 g_N(\xi_1, E) + (1 - \xi_2) g_D(\xi_1, E).
\end{align}
Thus inserting Eq. (5.9) into Eq. (5.4) and subsequently Eq. (5.3) we can derive the fixed-energy amplitude $\hat{G}_{RS1}[a, b; E]$. The expression is too long to describe it here. So, we rely on the numerical computation to check the occurrence of the localized massless graviton.

First, we will check the possibility for the appearance of the massless graviton at the brane located in $x = c$. Fig. 1 shows $m^2 \hat{G}^{\xi_1, \xi_2}[c, c; m^2/2]$ when $\xi_1 = \xi_2 = 1/2$ and $R = 1$. Of course we have taken RS limit, $i.e.$ $c = 1, \gamma = 2, \text{and } v_1 = v_2 = 1.5$. In order for the massless graviton to appear on the brane we need a pole in $\hat{G}^{\xi_1, \xi_2}[c, c; m^2/2]$ at $m^2 = 0$. This means the numerical value of $m^2 \hat{G}^{\xi_1, \xi_2}[c, c; m^2/2]$ should be non-zero and finite at $m^2 \to 0$. Fig. 1 indicates that the zero mass graviton appears only when $L$ is infinitely large. In fact, this limit is effectively RS2 scenario.

Numerical calculation shows that there is no massless graviton on the brane located at $x = L$ regardless of $L$ if one chooses $\xi_1 = \xi_2 = 1/2$. One may conjecture that the condition (3.14) for the appearance of the massless graviton in RS2 may be modified to

$$\gamma - \frac{1}{2} \frac{2\xi_2 L}{v_2} = 0 \quad (5.11)$$

for the appearance of the massless graviton on negative-tension brane. Fig. 2 shows $m^2 \hat{G}^{\xi_1, \xi_2}[L, L; m^2/2]$ where $\xi_1 = 1/2$ and $\xi_2$ is determined from Eq. (5.11). Fig. 2 shows again that there is no massless graviton. Although we have not tested all kinds of possibility, our numerical results strongly suggest that there is no room for the appearance of the massless graviton in negative-tension brane regardless of $\xi_1, \xi_2, \text{and } L$.

Of course, one can derive a fixed-energy amplitude for $\xi_1 = \xi_2 = 0$ case via the coupling constant renormalization in principle. However, long expression for $\hat{G}_{RS1}$ seems to make the calculation too tedious. So, we do not describe the result of this case in this paper.

VI. CONCLUSION

In this paper we have examined the localized gravity on the brane in RS brane-world scenario from the singular quantum mechanics. Choosing a single copy of $AdS_5$ as a bulk
spacetime we have shown that the fixed-energy amplitude for RS1 and RS2 are non-trivially
dependent on the BCs.

As a result the fixed-energy amplitude for RS2 is dependent on the free parameter ξ, which parametrize the BC at y = 0. Computing the fixed-energy amplitude explicitly one can derive the general criterion (3.14) for the appearance of the localized massless graviton on the brane when ξ is arbitrary but non-zero. When ξ = 0, the massless graviton is obtained via the coupling constant renormalization.

In RS1 scenario the final expression of the fixed-energy amplitude is dependent on the
two free parameters ξ₁ and ξ₂, which parametrize the various BCs at the end-points of 1d box. The appearance of the massless graviton is numerically tested by examining the pole at m² = 0. For the positive-tension brane our numerical test indicates that there is no massless graviton if the length of 1d box is finite. However, the infinite length of 1d box makes the graviton localized on the positive-tension brane to be massless, which is effectively identical to the RS2 scenario. For the negative-tension brane our numerical test shows that there is no massless graviton regardless of the length of 1d box and choice of BCs.

We can consider the various extension for this paper. Firstly, one may include the
ing the bending effect of the brane in the computation. In this case, however, the final expression of the linearized fluctuation does not seem to be like Schrödinger equation. Thus, we think the method used in Ref. [3] is more convenient than the technique of singular quantum mechanics to treat the bending effect. One can extend the method presented in this paper to the higher-dimensional RS scenario [29,30]. If one can find a singular brane solution in the higher-dimensional case, one can apply the self-adjoint extension or a coupling constant renormalization to treat the higher-dimensional δ-function potential. Of course, it is very interesting if we can find a singular solution in six dimension because two-dimensional δ-function potential has various non-trivial properties such as scale anomaly and dimensional transmutation [21]. One may extend the present paper to the moving brane picture [31]. But it is unclear for us whether or not the path-integral solution is in this case analytically obtainable.
We think the most interesting problem is to understand the reason why there is no massless graviton in RS1 scenario. This means that the gauge hierarchy problem is not compatible with the massless graviton problem. Thus, it seems to be important to compromise these two distinct phenomena.
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FIGURES

FIG. 1. $m$-dependence of $m^2 G_{RS_1}^{1/2,1/2}[c,c;m^2/2]$. The finite but non-zero at $L = \infty$ indicates that the graviton localized on the positive-tension brane is massless.

FIG. 2. $m$-dependence of $m^2 G_{RS_1}^{1/2,\xi^2}[L,L;m^2/2]$. This figure indicates that there is no localized massless graviton on the negative-tension brane.
