ON HNN-EXTENSIONS IN THE CLASS OF GROUPS OF LARGE ODD EXPONENT

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Abstract. A sufficient condition for the existence of HNN-extensions in the class of groups of odd exponent \( n \gg 1 \) is given in the following form. Let \( Q \) be a group of odd exponent \( n > 2^{48} \) and \( \mathcal{S} \) be an HNN-extension of \( Q \). If \( A \in \mathcal{S} \) is a fixed isomorphism. Then the standard HNN-extension \( \mathcal{S} \) of \( Q \) isomorphic to \( G \) isomorphic copy of \( Q \), be a subgroup of \( Q \), \( \rho \) be an isomorphism that is naturally embedded in the quotient \( \mathcal{S}/\mathcal{G}^{n} \), that is, there exists an analog of the HNN-extension \( \mathcal{S} \) of \( Q \) in the class of groups of exponent \( n \).

1. Introduction

In this paper, we will generalize one of technical ideas of article [I02] which seems to be of independent interest and could be useful for future references.

Consider the following construction. Let \( Q \) be a group of exponent \( n \) (that is, elements of \( Q \) satisfy the identity \( x^n = 1 \)), \( Y = \{y_1, \ldots, y_m\} \) be an alphabet and \( m \geq 1 \). Let \( P_{k,1}, P_{k,2} \) be two subgroups of \( Q \), \( k = 1, \ldots, m \), such that \( P_{k,1} \) is isomorphic to \( P_{k,2} \) and

\[ \rho_k : P_{k,1} \to P_{k,2} \]

be a fixed isomorphism. Then the standard HNN-extension \( \mathcal{S} \) of \( Q \) with stable letters \( y_1, \ldots, y_m \) and isomorphisms \( \rho_1, \ldots, \rho_m \) is defined by the following relative presentation

\[ \mathcal{S} = \langle Q, y_1, \ldots, y_m \mid y_k \rho_k^{-1} = \rho_k(p), \ p \in P_{k,1}, \ k = 1, \ldots, m \rangle. \] (1)

One might inquire whether there is a group \( \mathcal{K} \) of exponent \( n \) which contains an isomorphic copy of \( Q \) and some elements \( y_1, \ldots, y_m \) so that \( y_k \rho_k^{-1} = \rho_k(p) \) for all \( p \in P_{k,1}, \ k = 1, \ldots, m \), that is, there is an analog \( \mathcal{K} \) of HNN-extension of \( Q \) with stable letters \( y_1, \ldots, y_m \) and isomorphisms \( \rho_k : P_{k,1} \to P_{k,2}, \ k = 1, \ldots, m \), in the class of groups of exponent \( n \). Clearly, the existence of such a group \( \mathcal{K} \) is equivalent to the natural embedding of \( Q \) into the quotient \( \mathcal{S}/\mathcal{S}^n \).

It is also clear that in general the quotient \( \mathcal{S}/\mathcal{S}^n \) need not contain the natural copy of \( Q \). For example, let \( n \) be prime, \( m = 1 \), let \( P_{1,1} = P_{1,2} \) be a subgroup of order \( n \) and \( \rho_1(p) \neq p \), where \( p \in P_{1,1}, \ p \neq 1 \). Then \( y_1 \rho_1^{-1} = \rho_1(p) \neq p \), whence \( p \in \mathcal{G}^n \). Here is another example. Let \( P_{1,1} = P_{1,2} = Q \), that is, \( \rho_1 \in \text{Aut}Q \) and \( \rho_1^n \neq 1 \) in \( \text{Aut}Q \). Then it is clear that \( Q \cap \mathcal{S}^n \neq \{1\} \).

2000 Mathematics Subject Classification. Primary 20E06, 20F50; Secondary 20F05, 20F06.
Supported in part by NSF grant DMS 00-99612.
The aim of this paper is to give a sufficient condition for the embedding $Q \to \mathcal{S}/\mathcal{S}^n$. To do this, for every element $A \in \mathcal{S}$ we consider the maximal subgroup $\mathcal{F}(A) \subseteq Q$ which is normalized by $A$ (clearly, such an $\mathcal{F}(A) \subseteq Q$ is unique). For example, if $A \in Q$ then $\mathcal{F}(A) = Q$. Let $\tau_A$ denote the automorphism of $\mathcal{F}(A)$ which is induced by conjugation by $A$. Using this notation, it is easy to state and prove (see Sect. 3) the following necessary condition for the embedding $Q \to \mathcal{S}/\mathcal{S}^n$.

**Proposition.** Suppose that $Q$ is a group of exponent $n$, an HNN-extension $\mathcal{S}$ of $Q$ is given by presentation (1) and $Q$ naturally embeds in the quotient $\mathcal{S}/\mathcal{S}^n$. Then for every $A \in \mathcal{S}$ the subgroup $\langle \tau_A, \mathcal{F}(A) \rangle$ of $\text{Hol}\mathcal{F}(A)$ has exponent $n$.

A sufficient condition for the embedding $Q \to \mathcal{S}/\mathcal{S}^n$ given in the following Theorem is our main result and is as close to the necessary condition stated above as we can get (in the statement of Theorem, $[r]$ denotes the integer part of a real number $r$).

**Theorem.** Suppose that $Q$ is a group of odd exponent $n > 2^{48}$ and an HNN-extension $\mathcal{S}$ of $Q$ is given by presentation (1). Furthermore, assume that for every $A \in \mathcal{S}$, which is not conjugate to an element of $Q$, the subgroup $\langle \tau_A, \mathcal{F}(A) \rangle$ of $\text{Hol}\mathcal{F}(A)$ has exponent $n$ and, in addition, equalities $A^{-k}q_0A^k = q_k$, where $q_k \in Q$ and $k = 0, 1, \ldots, 2^{-16}n$, imply that $q_0 \in \mathcal{F}(A)$. Then the group $Q$ naturally embeds in the quotient $\mathcal{S}/\mathcal{S}^n$.

To prove this Theorem, we will make use of the machinery of article [94] (but first we will have to adjust it to the new situation which is similar to [102]). Interestingly, most of the machinery of article [94], created to solve the Burnside problem for even exponents $n \gg 1$, is "recycled" in this paper which, in particular, explains why we use the estimate $n \geq 2^{48}$ of [94].

As examples of application of this Theorem, we will state a couple of immediate corollaries (other applications will be given elsewhere). Note that these examples are similar to classical applications of standard HNN-extensions proposed by Higman, B. Neumann and H. Neumann [HNN49].

**Corollary 1.** Suppose that $G$ is a group of odd exponent $n > 2^{48}$. Then $G$ embeds into a group of exponent $n$ in which every maximal cyclic subgroup has order $n$.

**Corollary 2.** Suppose that $G$ is a group of prime exponent $n > 2^{48}$. Then $G$ embeds into a group of exponent $n$ with $n$ classes of conjugate elements.

Recall that a subgroup $H$ of a group $G$ is called *antinormal* if for every $g \in G$ the inequality $gHG^{-1} \cap H \neq \{1\}$ implies that $g \in H$. Note that earlier Mikhailovskii [M94] established the embedding $Q \to \mathcal{S}/\mathcal{S}^n$, where odd $n \gg 1$, under the following assumptions: $m = 1$, the subgroups $P_{1,1}, P_{1,2}$ are both antinormal in $Q$ and $qP_{1,1}q^{-1} \cap P_{1,2} = \{1\}$ for every $q \in Q$. We also remark that this result follows from our Theorem (because, under these assumptions, the subgroup $\mathcal{F}(A)$ is trivial for every $A \in \mathcal{S}$ which is not conjugate to an element of $Q$).

2. **Proving Theorem**

According to the presentation (1), our basic alphabet is

$$y_Q = \{y_1, \ldots, y_m, Q\}$$
and, from now on (unless stated otherwise), all words will be those in the alphabet $y_{Q}^{\pm 1} = \{ y_{1}, y_{1}^{-1}, \ldots, y_{m}, y_{m}^{-1}, Q \}$, called $y_{Q}$-words. Let

\[ U_{1} = S_{0.1} y_{k_{1,1}}^{\varepsilon_{1,1}} S_{1,1} \cdots y_{k_{\ell_{1,1}}}^{\varepsilon_{\ell_{1,1}}} S_{\ell_{1,1}}, \quad U_{2} = S_{0.2} y_{k_{1,2}}^{\varepsilon_{1,2}} S_{1,2} \cdots y_{k_{\ell_{2,2}}}^{\varepsilon_{\ell_{2,2}}} S_{\ell_{2,2}}, \]

be $y_{Q}$-words, where

\[ \varepsilon_{1,1}, \ldots, \varepsilon_{\ell_{1,1}}, \varepsilon_{1,2}, \ldots, \varepsilon_{\ell_{2,2}} \in \{ \pm 1 \}, \]

\[ k_{1,1}, \ldots, k_{\ell_{1,1}}, k_{1,2}, \ldots, k_{\ell_{2,2}} \in \{ 1, \ldots, m \}, \]

$S_{0.1}, \ldots, S_{\ell_{1,1}}$ are $Q$-syllables of the word $U_{1}$ (that is, maximal subwords of $U_{1}$ all of whose letters are in $Q$; if $S_{0.1}$ is, in fact, missing in $U_{1}$, then we assume that $S_{0.1} = 1 \in Q$ and set $S_{\ell_{1,1}} = 1$ if $S_{\ell_{1,1}}$ is not present in $U_{1}$) and $S_{0.2}, \ldots, S_{\ell_{2,2}}$ are $Q$-syllables of the word $U_{2}$.

We will write $U_{1} = U_{2}$ if $\ell_{1} = \ell_{2}$, $\varepsilon_{j,1} = \varepsilon_{j,2}$, $k_{j,1} = k_{j,2}$ for all $j = 1, \ldots, \ell_{1}$, and $S_{j,1} = S_{j,2}$ in the group $Q$ for all $j = 0, \ldots, \ell_{1}$.

The length $|U_{1}|$ of a word $U_{1}$ is $\ell_{1}$, that is, the number of occurrences of letters $y_{i}^{\pm 1}$ in $U_{1}$, where $y \in Y = \{ y_{1}, \ldots, y_{m} \}$.

If the (images of) words $U_{1}, U_{2}$ are equal in the group $G(0) = G$ given by the presentation (1) then we will write $U_{1} \equiv U_{2}$.

By induction on $i \geq 0$ we will construct groups $G(i)$. Assume that the group $G(i), i \geq 0$, is already constructed as a quotient of the group of $G(0)$ by means of defining relations. Define $X_{i+1}$ to be a maximal set of all $y_{Q}$-words of length $i + 1$ (if any) with respect to the following three properties.

\begin{itemize}
  \item[(AC1)] Every word $A \in X_{i+1}$ begins with $y$ or $y^{-1}$, where $y \in Y = \{ y_{1}, \ldots, y_{m} \}$.
  \item[(AC2)] The image of every word $A \in X_{i+1}$ has infinite order in the group $G(i)$.
  \item[(AC3)] If $A, B$ are distinct elements of $X_{i+1}$ then the image of $A^{n}$ is not conjugate in $G(i)$ to the image of $B^{n}$ or $B^{-n}$.
\end{itemize}

Note that it follows from the analog of Lemma 18.2 in rank $i \geq 0$ that the set $X_{i+1}$ is nonempty.

Similar to [O82], [O89], [I94], we will call a word $A \in X_{i+1}$ a \textit{period of rank} $i + 1$.

Now we define the group $G(i + 1)$ by imposing all relations $A^{n} = 1$, $A \in X_{i+1}$, called \textit{relations of rank} $i + 1$, on the group $G(i)$:

\[ G(i + 1) = \langle G(i) \parallel A^{n} = 1, \ A \in X_{i+1} \rangle. \]

It is clear that

\[ G(i + 1) = \langle G \parallel A^{n} = 1, \ A \in \cup_{j=1}^{i+1} X_{j} \rangle. \]

We also define the limit group $G(\infty)$ by imposing on the free product

\[ Q \ast \langle y_{1} \rangle_{\infty} \ast \cdots \ast \langle y_{m} \rangle_{\infty} \]

of all relations of all ranks $i = 0, 1, 2, \ldots$

\[ G(\infty) = \langle Q, y_{1}, \ldots, y_{m} \parallel y_{k} p y_{k}^{-1} = \rho_{k}(p), \ p \in P_{1,k}, \ k = 1, \ldots, m, \ A^{n} = 1, \ A \in \cup_{j=1}^{\infty} X_{j} \rangle. \] (2)

The main technical result relating to the group $G(\infty)$ is the following. (Observe that Lemma A obviously implies Theorem.)

\begin{lemma}
  Suppose that the hypothesis of Theorem holds. Then the group $G(\infty)$ given by presentation (2) is naturally isomorphic to $G/G^{n}$ and the group $Q$ naturally embeds in $G(\infty)$.
\end{lemma}
We will make use of the machinery of article [94] to prove Lemma A (more applications of this machinery can be found in [96], [97], [00], [02]). In order to do this we will have to make necessary changes in definitions, statements of lemmas of [94] and their proofs.

First of all, diagrams over the group $\mathcal{G}(0) = \mathcal{G}$ given by presentation (1) (or, briefly, over $\mathcal{G}$), called diagrams of rank 0, are defined to be maps that have two types of 2-cells.

A 2-cell $\Pi$ of the first type, called a 0-square, has four edges in its counterclockwise oriented boundary (called contour) $\partial\Pi = e_1e_2e_3e_4$ and

$$\varphi(e_1) = \varphi(e_3)^{-1} = y_k, \quad \varphi(e_2) = g, \quad \varphi(e_4) = \rho_k(g)^{-1},$$

where, as in [94], $\varphi$ is the labelling function, $g \in P_{k,1}$ (perhaps, $g = 1$) and $y_k \in \{y_1, \ldots, y_m\}$. Observe that we use Greek letters with no indices exactly as in [94] (in particular, see table (2.4) in [94]).

A 2-cell $\Pi'$ of the second type, called a 0-circle, has $\ell \geq 2$ edges in its contour $\partial\Pi' = e_1 \ldots e_\ell$ so that $\varphi(e_1), \ldots, \varphi(e_\ell) \in Q$ and the word

$$\varphi(\partial\Pi') = \varphi(e_1) \ldots \varphi(e_\ell)$$

equals 1 in $Q$.

Note that this definition of 2-cells in a diagram of rank 0 is analogous to the corresponding definitions of [02].

Furthermore, we take into account that there are multiple periods of rank $i$ and the symbol $A_i$ will denote one of many periods of rank $i$ (note that the length $|A_i|$ of $A_i$ is now $i$).

In the definition of an $A$-periodic word, it is now assumed that $A$ starts with $y$ or $y^{-1}$, where $y \in \mathcal{Y}$, and $A$ is not conjugate in $\mathcal{G}$ to a power $B^\ell$ with $|B| < |A|$.

In addition to cells of positive rank, we also have (as in [89], [96], [02]) cells of rank 0 (which are now either 0-squares or 0-circles). The equality $r(\Delta) = 0$ now means that all cells in $\Delta$ have rank 0.

If $p = e_1 \ldots e_\ell$, where $e_1, \ldots, e_\ell$ are edges, is a path in a diagram $\Delta$ of rank $i$ (that is, a diagram over the group $\mathcal{G}(i)$) then the $y$-length $|p|$ of $p$ is $|\varphi(p)|$, that is, $|p|$ is the number of all edges of $p$ labelled by $y^{\pm 1}$, where $y \in \mathcal{Y}$. The (strict) length of $p$, that is, the total number of edges of $p$, is $\ell$ and denoted by $\|p\|$.

In the definition (A1)–(A2) of $j$-compatibility (p.13 [94]) we eliminate the part (A2) because $n$ is odd and, similar to [94] in the case when $n$ is odd, it will be proven in a new version of Sect. 19 [94] that there are no $J(A_j)$-involutions and there is no $j$-compatibility of type (A2).

We can also drop the definition of self-compatible cells (p.13 [94]) because they do not exist when $n$ is odd (which is again analogous to [94] in the case when $n$ is odd). Thus all lemmas in [94] whose conclusions deal with self-compatible cells, compatibility of type (A2) actually claim that their assumptions are false (e.g., see lemmas of Sect. 12 [94]). On the other hand, the existence of self-compatible cells in assumptions of lemmas of [94] is now understood as the existence of noncontractible $y$-annuli which we are about to define.

A $y$-annulus, where $y \in \mathcal{Y}$, is defined to be an annular subdiagram $\Gamma$ in a diagram $\Delta$ of rank $i$ such that $\Gamma$ consists of 0-squares $S_1, \ldots, S_k$ so that if

$$\partial S_\ell = f_1, e_1, f_2, e_2, \ell,$$
where \( e_{1,\ell}, e_{2,\ell} \) are \( y \)-edges (that is, labelled by \( y \) or \( y^{-1} \), where \( y \in \mathcal{Y} \)) of \( \partial S_\ell \), 
\( 1 \leq \ell \leq k \), then \( e_{2,\ell} = e^{-1}_{1,\ell+1} \), where the second subscript is \( \text{mod} k \), \( \ell = 1, \ldots, k \). If \( \Gamma \) is contractible into a point in \( \Delta \) then we will call \( \Gamma \) a contractible \( y \)-annulus. Otherwise, \( \Gamma \) is a noncontractible \( y \)-annulus. In general, repeating arguments of [94], we additionally require that \( \Delta \) contain no reducible \( y \)-annuli.

In the definition of a reduced (simply connected or not) diagram \( \Delta \) of rank \( i \) (p.13 [94]), we additionally require that \( \Delta \) contain no reducible \( y \)-annuli.

As in [94], we can always remove reducible pairs and reducible \( y \)-annuli in a diagram \( \Delta \) of rank \( i \) to obtain from \( \Delta \) a reduced diagram \( \Delta' \) of rank \( i \). Note that in general it is not possible to get rid of noncontractible \( y \)-annuli (in non-simply connected diagrams of rank \( i \)).

In the definition of a 0-bond \( E \) between \( p \) and \( q \) (p.15 [94]) we now require that \( E \) consist of several 0-squares \( S_1, \ldots, S_k \) so that if

\[
\partial S_\ell = f_{1,\ell}e_{1,\ell}f_{2,\ell}e_{2,\ell},
\]

where \( e_{1,\ell}, e_{2,\ell} \) are \( y \)-edges of \( \partial S_\ell \), \( 1 \leq \ell \leq k \), then \( e^{-1}_{1,1} \in p \), \( e_{2,\ell} = e^{-1}_{1,\ell+1} \), \( \ell = 1, \ldots, k-1 \), and \( e^{-1}_{2,k} \in q \).

The standard contour of the 0-bond \( E \) between \( p \) and \( q \) is

\[
\partial E = e^{-1}_{1,1}(f^{-1}_{1,1} \cdots f^{-1}_{1,k})e^{-1}_{2,k}(f^{-1}_{2,k} \cdots f^{-1}_{2,1})
\]

and the edges \( e^{-1}_{1,1}, e^{-1}_{2,k} \) are denoted by \( E \wedge p, E \wedge q \), respectively.

In the definition of a simple in rank \( i \) word \( A \) (p.19 [94]), we additionally require that \( |A| > 0 \) and \( A \) start with \( y^{1} \), where \( y \in \mathcal{Y} \). Observe that it follows from Lemma 18.2 (in rank \( i-1 \)) and definitions that a period of rank \( i \) is simple in rank \( i-1 \) (and hence in any rank \( j \leq i-1 \)).

In the definition of a tame diagram of rank \( i \) (p.19 [94]), we make two changes. First, in property (D2), we require that if 0-squares \( S_1, \ldots, S_k \) form a subdiagram \( E \) as in the definition of a 0-bond and \( p = q = \partial \Pi \), where \( \Pi \) is a cell of rank \( j > 0 \) in \( \Delta \), then \( E \) is a 0-bond between \( \partial \Pi \) and \( \partial \Pi \) in \( \Delta \). Second, we add the following property.

(D3) \( \Delta \) contains no contractible \( y \)-annuli.

In the definition of a complete system (p.23 [94]) we require in (E3) that \( e \) be a \( y \)-edge.

In Lemma 4.2, the strict length \( ||s_1||, ||s_2|| \) of \( s_1, s_2 \) is meant.

In the definition of the weight function \( \nu \) (p.28 [94]), we require in (F1) that \( e \) be a \( y \)-edge. In (F2), we allow that \( e \) is not a \( y \)-edge.

In the beginning of the proof of Lemma 6.5, we note that the lemma is obvious if \( \Delta \) contains no cells of positive rank. In general, repeating arguments of [94], we always understand ”cells” as cells of positive rank and keep in mind the existence of cells of rank 0.

In the definition of the height \( h(W) \) of a word \( W \) (p.89 [94]), we additionally set \( h(W) = \frac{1}{2} \) if \( W \neq 1 \) and \( W \) is conjugate in rank \( i \) to a word \( U \) with \( |U| = 0 \) (that is, \( U \in Q \) and \( U \neq 1 \)).

In Lemma 10.2, we allow the extra case when \( h(W) = \frac{1}{2} \).

Here is a new version of Lemma 10.4.
Lemma 10.4. (a) If a word $W$ has finite order $d > 1$ in the group $\mathcal{S}(i)$ then $n$ is divisible by $d$.

(b) Every word $W$ with $|W| \leq i$ has finite order in rank $i$.

Proof. (a) By Lemma 10.2, either $h(W) = \frac{1}{2}$ (and then our claim is immediate from $Q$ being of exponent $n$) or, otherwise, $W$ is conjugate in rank $i$ to a word of the form $A^k F$, where $A$ is a period of rank $j \leq i$, $0 < k < n$ and $F \in \mathcal{F}(A)$. In the latter case, it follows from Lemma 18.5(c) in rank $j - 1 < i$ that $(A^k F)^{j-1} = A^k n$. Therefore, $(A^k F)^{n j-1} A^k n \equiv 1$, whence $W^n \equiv 1$ as required.

(b) By induction, it suffices to show that every word $W$ with $|W| = i$ has finite order in rank $i$ (for $i = 0$ this is obvious). It follows from the definition of periods of rank $i \geq 1$ that if $W$ has infinite order in rank $i - 1$ then $W^n$ is conjugate in rank $i - 1$ to $A^{kn}$, where $A$ is a period of rank $i$. Therefore, $W^n \equiv 1$ as desired.

Lemma 10.4 is proved. \qed

In Lemma 10.8, we drop part (b) of its conclusion (and keep in mind that the term "reducible cell" now means "$y$-annulus"). Note that the height of $\varphi(q_1 t_1)$ in Lemma 10.8 is at least 1 hence noncontractible $y$-annuli in $\Delta_0$, $\Delta_0^r$ are impossible (for otherwise, the height $h(\varphi(q_1 t_1))$ of $\varphi(q_1 t_1)$ would be at most $\frac{1}{2}$).

Lemma 10.9 is no longer needed for no path $q$ is (weakly) $j$-compatible with itself.

In the definition of a $U$-diagram of rank $i$ (p.134 [I94]), we allow in property (U3) that the height $h(\varphi(e))$ of $\varphi(e)$ is $\frac{1}{2}$.

Lemma 12.3 now claims that there are no $U$-diagrams of rank $i$. Recall that this agrees with our convention that if the conclusion states the existence of self-compatible cells or $j$-compatibility of type (A2) then the assumption is false.

The analogs of Lemmas 13.1–16.6 are not needed.

In the hypothesis of Lemma 17.1, we now suppose that one can obtain from $\Delta_0$ an annular reduced diagram of rank $i$ which contains no noncontractible $y$-annuli by means of removal of reducible pairs and reducible $y$-annuli.

According to our convention, in the statement of Lemma 17.2, we replace the phrase "one has to remove a reducible cell to reduce $\Delta_0$" by "one encounters a noncontractible $y$-annulus when reducing $\Delta_0$". In the conclusion of Lemma 17.2 and in its proof, we disregard reducible cells, $\mathcal{F}(A_j)$-involution and consider, instead, noncontractible $y$-annuli and their 0-squares.

The new version of Lemma 17.3 is stated as follows.

Lemma 17.3. Let $\Delta$ be a disk reduced diagram of rank $i$ whose contour is $\partial \Delta = bpeq$, where $\varphi(p)$ and $\varphi(q)^{-1}$ are $A$-periodic words and $A$ is a simple in rank $i$ word with $|A| = i + 1$ (in particular, $A$ is a period of rank $i + 1$). Suppose also that $\Delta$ itself is a contiguity subdiagram between $p$ and $q$ with $\min(|p|, |q|) > L|A|$. Then, in the notation of Lemmas 9.1–9.4, there exists a rigid subdiagram of the form $\Delta(m_1, m_2)$ in $\Delta (= \Delta(1, k))$ such that

$$r(\Delta(m_1, m_2)) = 0$$

and the following analogs of inequalities (17.25)–(17.26) (p.222 [I94]) hold.

$$|q(m_2, m_1)| > |q(k, 1)| - 4.4|A| > (L - 4.4)|A|,$$

$$|p(m_1, m_2)| > |p(1, k)| - 4.4|A| > (L - 4.4)|A|,$$

$$|x_{m_1}| = |y_{m_2}| = 0.$$
Proof. To prove this new version of Lemma 17.3, we repeat the argument of the beginning of the proof of Lemma 17.3 [94]. As there, making use of Lemma 17.2, we prove Lemma 17.3.1. After that, arguing as in the proof of Lemma 17.3.2, it is easy to show, using Lemma 12.3, that

\[ r(\Delta(m_1, m_2)) = 0, \]

as required. \qed

When proving the analog of Lemma 18.2, we pick a word \( B = B(a_1, a_2) \) in the alphabet \( \{a_1, a_2\} \) of length \( i + 1 \) so that \( B \) has the same properties as those in [94] and, in addition, first and last letters of \( B \) are distinct (the existence of such a word easily follows from Lemma 1.7 [94]). Next, consider a word \( B(a_1, a_2, q) \) in the alphabet \( \{a_1, a_2, q\} \) which is obtained from \( B(a_1, a_2) \) by plugging in an element \( q \in Q, q \neq 1 \), between each pair of consecutive letters of the word \( B(a_1, a_2) \). Then we replace each occurrence of the letter \( a_1 \) in \( B(a_1, a_2, q) \) by \( y, y \in Y \), and each occurrence of the letter \( a_2 \) in \( B(a_1, a_2, q) \) by \( y^{-1} \). Clearly, we have a word \( B(y, y^{-1}, q) \) with \( |B(y, y^{-1}, q)| = i + 1 \). Now, in view of Lemmas 10.2, 10.4, we can repeat the arguments of the proof of Lemma 18.2 without any changes.

Let \( A \) be a period of rank \( i + 1 \). As in Sect. 1, by

\[ \mathcal{F}(A) \]

denote a maximal subgroup of \( Q \subseteq \mathcal{G}(0) \) with respect to the property that \( A \) in rank 0 normalizes this subgroup \( \mathcal{F}(A) \). Observe that \( Q \) naturally embeds in \( \mathcal{G}(i) \) by Lemma 6.2 and so we can also consider \( \mathcal{F}(A) \) as a subgroup of \( \mathcal{G}(i) \).

Here is a new version of Lemma 18.3.

**Lemma 18.3.** Suppose that \( A \) is a period of rank \( i + 1 \). Furthermore, let \( \Delta \) be a disk reduced diagram of rank \( i \) such that \( \partial \Delta = bpcq \), where \( \varphi(p), \varphi(q)^{-1} \) are \( A \)-periodic words with \( \min(|p|, |q|) > \frac{1}{2} \beta n |A| \), and \( \Delta \) itself be a contiguity subdiagram between sections \( p \) and \( q \). Then there exists a 0-bond \( E \) in \( \Delta \) with the standard contour \( \partial E = bpcq E \), where \( p_E = E \land p, q_E = E \land q, (p_E)_-, (q_E)_+ \) are phase vertices of \( p, q \), respectively, such that

\[ \varphi(b_E) \in \mathcal{F}(A), \quad A^n \varphi(b_E) A^{-n} \cong \varphi(b_E) \]

and for every integer \( k \) one has

\[ (A^k \varphi(b_E))^n \cong A^{kn}. \]

**Proof.** Lemma 17.3 enables us to assume that

\[ r(\Delta) = 0, \quad \min(|p|, |q|) > (\frac{1}{2} \beta n - 5)|A|, \quad |b| = |c| = 0. \]

In particular, there are \( |p| \) 0-bonds between \( p \) and \( q \) in \( \Delta \). Let \( E \) be a 0-bond between sections \( p \) and \( q \) and

\[ \partial E = bpcq E \]

be the standard contour of \( E \), where \( p_E = E \land p, q_E = E \land q \). It is clear that \( \text{div}((p_E)_-, (q_E)_+) \) does not depend on \( E \). Suppose that

\[ \text{div}((p_E)_-, (q_E)_+) \neq 0. \tag{3} \]

A-periodically extending \( p \) or \( q^{-1} \) on the left as in the beginning of the proof of Lemma 17.3 (see Fig. 17.4(a)–(b) in [94]), we will get a diagram \( \Delta' \) with
Let \( \partial \Delta' = b'p'c'q' \) such that both \( \varphi(p') \) and \( \varphi(q')^{-1} \) begin with a cyclic permutation \( \tilde{A} \) of \( A \) such that \( \tilde{A} \) starts with \( y^{-1} \), where \( y \in \mathcal{Y} \).

As in the proof of Lemma 17.1, we can easily get, making use of (3), that the annular diagram \( \Delta'_0 \) (obtained from \( \Delta' \) as in Lemma 17.1) contains no \( y \)-annuli, in particular, \( \Delta'_0 \) is already reduced. Therefore, Lemma 17.1 applies to \( \Delta' \) and yields that \( \varphi(b') \neq 1 \). It follows from (3) that \( |b'| > 0 \) and so \( \tilde{A} \) is not cyclically reduced in rank \( i \). This contradiction proves that (3) is false and \( \text{div}(\langle p_E \rangle_-, \langle q_E \rangle_+ ) = 0 \).

Without loss of generality, we may assume that words \( \varphi(p), \varphi(q)^{-1} \) start with \( A \) and the word \( A \) starts with \( y^{-1} \), where \( y \in \mathcal{Y} \).

Using the notation of Lemma 9.1, let \( E_1, \ldots, E_{|p|} \) be all (consecutive along \( p \)) 0-bonds between \( p \) and \( q \) with standard contours

\[ \partial E_\ell = b_q p c q, \]

where \( p_\ell = E_\ell \wedge p, q_\ell = E_\ell \wedge q \) and \( 1 \leq \ell \leq |p| \).

Next, consider words

\[ V_0 = \varphi(b_1), V_1 = \varphi(b_{1|A|+1}), \ldots, V_t = \varphi(b_{t|A|+1}), \ldots, V_{|2^{-16}n|+1} = \varphi(b_{t(|2^{-16}n|+1)|A|+1}). \]

Recall that \( |p| > (\frac{1}{2} \beta n - 5)|A| > (2^{-16}n + 2)|A| \).

It is clear that

\[ A^{-1}V_tA = V_{t+1} \]

and \( V_t, V_{t+1} \) are words in \( Q \), where \( t = 0, \ldots, 2^{-16}n \).

By definitions, \( A \) is simple in rank \( i \) and so \( A \) is not conjugate in rank \( i \) to an element of \( Q \). Therefore, it follows from Theorem’s hypothesis that the word \( V_0 = \varphi(b_1) \) belongs to the subgroup \( \mathcal{F}(A) \subseteq Q \). Since the group \( \langle \tau_A, \mathcal{F}(A) \rangle \) has exponent \( n \), it follows that

\[ A^n \phi(b_1) A^{-n} = \tau_n^A (\varphi(b_1)) = \varphi(b_1) \]

and

\[ (A^k \phi(b_1))^n = (A^k V_0)^n = A^k V_0 A^{-k} A^{2k} V_0 A^{-2k} \ldots A^{k^n} V_0 A^{-k^n} A^{k^n} = \tau_A^k V_0 \tau_A^{-k} \ldots \tau_A^{k^n} V_0 \tau_A^{-k^n} A^{k^n} = (\tau_A^k V_0)^n \tau_A^{-n} A^{k^n} = A^{kn}, \]

as required. Lemma 18.3 is proven.

\( \square \)

Lemma 18.4 is not needed.

**Lemma 18.5.** Let \( A \) be a period of rank \( i + 1 \) and \( \mathcal{F}(A) \) be the maximal subgroup of \( Q \subset \mathcal{S}(0) \) with respect to the property that \( A \) normalizes \( \mathcal{F}(A) \) in rank \( 0 \). Then the following are true.

(a) The subgroup \( \mathcal{F}(A) \) is defined uniquely.

(b) Suppose \( \Delta \) is a disk reduced diagram of rank \( i \) with \( \partial \Delta = bpcq \), where \( \varphi(p) \), \( \varphi(q)^{-1} \) are \( A \)-periodic words with \( \min(|p|, |q|) > \frac{1}{2} \beta n |A| \), such that \( \Delta \) itself is a contiguity subdiagram between \( p \) and \( q \). Then there is a 0-bond \( E \) between \( p \) and \( q \) with the standard contour \( \partial E = b_q p c q_e \), where \( p_E = E \wedge p, q_E = E \wedge q \), such that \( (p_E)_-, (q_E)_+ \) are phase vertices of \( p, q \), respectively, and \( \varphi (b_E) \in \mathcal{F}(A) \).

(c) \( A^k \) centralizes the subgroup \( \mathcal{F}(A) \) and if \( F \in \mathcal{F}(A) \), \( k \) is an integer then

\[ (A^k F)^n = A^{kn}. \]

(d) The subgroup \( \langle \mathcal{F}(A), A \rangle \) of \( \mathcal{S}(i) \) has the property that a word \( X \) belongs to \( \langle \mathcal{F}(A), A \rangle \) if and only if there is an integer \( m \neq 0 \) such that \( X A^m X^{-1} = A^m \).
Proof. (a) This is obvious from definitions.
(b) This follows from Lemma 18.3.
(c) These claims can be proved as similar claims of Lemma 18.3.
(d) By part (c), it suffices to show that an equality

$$X A^m X^{-1} = A^m,$$

where \( m \neq 0 \), implies that \( X \in \langle A, \mathcal{F}(A) \rangle \). Arguing exactly as in the proof of part (c) of Lemma 18.5 [I02], we can show that

$$X \in \langle A \rangle \phi(b) \langle A \rangle \subseteq G(i),$$

where \( \phi(b) \in \mathcal{F}(A) \) by Lemma 18.3. Thus \( X \in \mathcal{F}(A) \) and Lemma 18.5 is proven.

Let us state a new version of Lemma 19.1.

**Lemma 19.1.** There is no disk diagram \( \Delta \) of rank \( i \) such that \( \partial \Delta = bpcq \), where \( p, q \) are \( A \)-periodic sections with \( |p|, |q| > \frac{1}{2} \beta n |A| \), \( A \) is a period of rank \( i + 1 \), and \( \Delta \) itself is a contiguity subdiagram between \( p \) and \( q \).

**Proof.** Arguing on the contrary, we assume the existence of such a diagram \( \Delta \) and, replacing the coefficient \( N \) (\( N = 484 \) as defined in (17.1) on p.212 [I94]) by \( \frac{1}{2} \beta n \), repeat the proof of Lemma 19.1 [I94] up to getting equality (19.23) (p.290 [I94]) which now reads

$$\varphi(d) A^n \varphi(d)^{-1} \equiv A^{-n}. \quad (4)$$

**Lemma 19.1.3.** The equality (4) is impossible.

**Proof.** Arguing on the contrary, we note that it follows from (4) that

$$\varphi(d)^2 A^n \varphi(d)^{-2} \equiv A^n. \quad (5)$$

Recall that \( |A| = i + 1 \) and, by Lemma 19.1.1, \( \varphi(d) \) is conjugate in rank \( i \) to a word \( W \) with

$$|W| < (0.5 + \xi)|A| < |A|.$$

Hence, by Lemma 10.4(b), \( \varphi(d) \) has finite order in rank \( i \) and, by Lemma 10.4(a),

$$\varphi(d)^n \equiv 1.$$

This means that \( \varphi(d) \in \langle \varphi(d)^2 \rangle \) in rank \( i \) and so equalities (4) and (5) imply that

$$A^{2n} \equiv 1.$$

A contradiction to the definition of a period of rank \( i + 1 \) completes proofs of Lemmas 19.1.3 and 19.1. \( \square \)

Analogues of Lemmas 19.2–19.6 are no longer needed.
The statements and proofs of Lemmas 20.1–20.2 are retained.

Having made all necessary changes, we can now turn to the group \( G(\infty) \) given by presentation (2).

It follows from Lemma 10.4(b) that every word \( W \) has finite order in rank \( i \geq |W| \). Then, by Lemma 10.4(a), \( W^n \equiv 1 \) provided that \( i \geq |W| \). Thus, the group \( G(\infty) \) has exponent \( n \). Now it is clear that the group \( G(\infty) \) is naturally isomorphic to the quotient \( G/\mathcal{F}^n \).

Suppose that \( W \) is a word with \( |W| = 0 \), that is, \( W \in Q \). Let \( W = 1 \) in the group \( G(\infty) \). Then there is an \( i \) such that \( W \equiv 1 \). Consider a reduced diagram \( \Delta \) of
rank \(i\) for this equality. Since \(|\partial \Delta| = 0\), it follows from Lemma 6.2 that \(r(\Delta) = 0\), that is, \(\varphi(\partial \Delta) = 1\) in \(Q\). Thus \(Q\) naturally embeds in \(\mathcal{G}(\infty)\).

The proofs of Lemma A and Theorem are complete. \(\square\)

3. PROOFS OF PROPOSITION AND COROLLARIES

First we prove Proposition. Arguing on the contrary, assume that for some \(A \in \mathcal{G}\) the subgroup \((\tau_A, A)\) of \(\text{Hol}\mathcal{F}(A)\) has no exponent \(n\). Then either \(\tau^n_A \neq 1\) in \(\text{Aut}\mathcal{F}(A)\) or \(\tau^n_A = 1\) and there is a \(q \in Q\) so that \((\tau^n_A q)^n \neq 1\) in \(\text{Hol}\mathcal{F}(A)\) for some \(k\).

In the first case, there are \(q_1, q_2 \in Q\) such that \(q_1 \neq q_2\) and \(A^n q_1 A^{-n} = q_2\). Hence, in view of \(A^n \in \mathcal{G}\), we have that \(q_1^{-1} q_2 \in \mathcal{G}^n\), contrary to \(\mathcal{G}^n \cap Q = \{1\}\).

Now suppose that \(\tau^n_A = 1\) and \((\tau^n_A q)^n \neq 1\) in \(\text{Hol}\mathcal{F}(A)\) for some \(q \in Q\) and \(k\). Then

\[(A^k q)^n = A^k q A^{-k} A^{2k} q A^{-2k} \ldots A^{kn} q A^{-kn} A^{kn} = (\tau^n_A)^k \tau^n_A q A^{-k} \ldots \tau^n_A q A^{-kn} A^{kn} = (\tau^n_A q)^n \tau^n_A A^{kn}.
\]

Then it follows from \(\tau^n_A \neq 1\) and \((\tau^n_A q)^n \neq 1\) that \((A^k q)^n = q' A^{nk}\), where \(q' \in Q\), \(q' \neq 1\). Hence, \(q' \in \mathcal{G}^n\), contrary to \(\mathcal{G}^n \cap Q = \{1\}\). Proposition is proved. \(\square\)

Let us prove Corollary 1. Let \(G\) be a group of odd exponent \(n > 2^{48}\). By the standard induction argument (see similar applications of HNN-extensions in \(\text{HNN49}\) or \(\text{LS77}\)), it suffices to show that if \(g \in G\) then \(G\) embeds in a group \(H\) such that \(H\) has exponent \(n\) and \(H\) contains an element \(z\) of order \(n\) with \(g \in \langle z \rangle\).

To do this, consider a cyclic group \(\langle x \rangle_n\) of order \(n\) generated by \(x\) and let

\[Q = G \ast_n \langle x \rangle_n = G \ast \langle x \rangle_n / (G \ast \langle x \rangle_n)^n\]

be the free Burnside \(n\)-product of \(G\) and \(\langle x \rangle_n\) (which is the quotient of the free product \(G \ast \langle x \rangle_n\) by \((G \ast \langle x \rangle_n)^n\)). Recall that, by Lemma 34.10 \(\text{OS81}\), if \(n\) is odd and \(n \gg 1\) (say, \(n > 10^{10}\)) then the factors of a free Burnside \(n\)-product are antinormal in the product.

Let \(x^k\) have the same order as that of \(g\) and consider the following HNN-extension of \(Q\)

\[\mathcal{G} = \langle Q, y \mid yy^{-1} = x^k \rangle.
\]

Assume that \(A \in \mathcal{G}\), \(A\) is not conjugate in \(\mathcal{G}\) to an element of \(Q\), and \(A^{-1} q_0 A = q_1, A^{-1} q_1 A = q_2 \in \mathcal{G}\), where \(q_0, q_1, q_2 \in Q\). Then it is not difficult to see that for every \(\ell > 0\) it is true that \(A^{-\ell} q_0 A^\ell \in Q\) and, more specifically, \(A^{-\ell} q_0 A^\ell = q_\ell^{-1} q_0 q_\ell\), where \(q_\ell A \in Q\) is the result of deletion in \(A\) of all occurrences of \(y^{\ell+1}\) and \(x\)-syllables (we assume here that \(A\) is a reduced in \(\mathcal{G}\) word, that is, \(|A|\) is minimal). Therefore, \(q_0 \in \mathcal{F}(A)\), \(\tau^n_A = 1\) and

\[(\tau_A^{-\ell} q_0)^n = \tau_A^{-\ell} q_0 \tau_A^{-2\ell} q_0 \tau_A^{-3\ell} \ldots \tau_A^{-n\ell} q_0 \tau_A^{-n\ell} = (q_\ell^{-1} q_0 q_\ell)^n = (q_A^{-\ell} q_0 q_\ell)^n = (q_A^{-\ell} q_0 q_\ell)^n \tau_A^{-n\ell} = 1.
\]

Since \(q_0\) is an arbitrary element of \(\mathcal{F}(A)\), this implies that the group \(\langle \tau_A, \mathcal{F}(A) \rangle\) has exponent \(n\) and so Theorem applies to \(\mathcal{G}\). Hence, \(Q\) naturally embeds in \(\mathcal{G}/\mathcal{G}^n\) and \((y^{-1} x y)^k = g \in \mathcal{G}/\mathcal{G}^n\), as required. Corollary 1 is proved. \(\square\)

The proof of Corollary 2 is similar. Let \(Q\) be a group of prime exponent \(n > 2^{48}\). As before, by induction, it suffices to show that if \(\langle p_1 \rangle, \langle p_2 \rangle\) are two nontrivial cyclic
subgroups of $Q$, which are not conjugate in $Q$, and

$$\mathcal{G} = \langle Q, y \mid yp_1y^{-1} = p_2 \rangle$$

then $Q$ embeds in $\mathcal{G}/\mathcal{G}^n$.

Let $A \in \mathcal{G}$, $A$ not conjugate in $\mathcal{G}$ to an element of $Q$, and $A^{-1}q_0A = q_1$, $A^{-1}q_1A = q_2$ in $\mathcal{G}$, where $q_0, q_1, q_2 \in Q$. Without loss of generality, we can suppose that $A$ is a word of minimal length (that is, $A = B$ in $\mathcal{G}$ implies $|A| \leq |B|$). Then, using the assumption that subgroups $\langle p_1 \rangle$, $\langle p_2 \rangle$ are not conjugate in $Q$ and the remark that an equality of the form $qp_kq^{-1} = p^l \neq 1$ in $Q$ implies $p^k = p^l$, we can obtain from the equalities $A^{-1}q_0A = q_1$, $A^{-1}q_1A = q_2$ that $q_0 = q_1$. Therefore, $q_0 \in \mathcal{F}(A)$, the automorphism $\tau_A$ is trivial and the group $\langle \tau_A, \mathcal{F}(A) \rangle = \langle \mathcal{F}(A) \rangle$ has exponent $n$. By Theorem, $Q$ naturally embeds in $\mathcal{G}/\mathcal{G}^n$ and Corollary 2 is proved. □

Acknowledgements. The author is grateful to the referee for finding a mistake in the proof of Corollary 1 and bringing K.V. Mikhajlovskii’s paper [M94] to author’s attention. The author also wishes to thank A.Yu. Ol’shanskii for his kind comments on results of this article and for pointing out that some of them could also be proved using the techniques developed in his joint with M.V. Sapir paper [OS02] (see [OS02] for details).

References

[HNN49] G. Higman, B.H. Neumann and H. Neumann, Embedding theorems for groups, J. London Math. Soc. 24(1949), 247–254.

[I94] S.V. Ivanov, The free Burnside groups of sufficiently large exponents, Internat. J. Algebra and Comp. 4(1994), 1–308.

[I00] S.V. Ivanov, On finitely presented groups given by periodic relators, J. Group Theory 3(2000), 95–99.

[I02] S.V. Ivanov, Weakly finitely presented infinite periodic groups, Contemporary Math. 296(2002), 139–154.

[IO96] S.V. Ivanov and A.Yu. Ol’shanskii, Hyperbolic groups and their quotients of bounded exponents, Trans. Amer. Math. Soc. 348(1996), 2091–2138.

[IO97] S.V. Ivanov and A.Yu. Ol’shanskii, On finite and locally finite subgroups of free Burnside groups of large even exponents, J. Algebra 195(1997), 241–284.

[LS77] R.C. Lyndon and P.E. Schupp, Combinatorial group theory, Springer-Verlag, 1977.

[M94] K.V. Mikhajlovskii, Some generalizations of HNN-extensions in the periodic case, # 1063-B94, VINITI, Moscow, 1994 (this is kept in Depot of VINITI, Moscow, and is available upon request) 59pp. (in Russian)

[O82] A.Yu. Ol’shanskii, On the Novikov-Adian theorem, Mat. Sbornik 118(1982), 203–235.

[O89] A.Yu. Ol’shanskii, Geometry of Defining Relations in Groups, Nauka, Moscow, 1989; English translation in Math. and Its Applications (Soviet series) 70 (Kluwer Acad. Publishers, 1991).

[OS02] A.Yu. Ol’shanskii and M.V. Sapir, Non-amenable finitely presented torsion-by-cyclic groups, Publ. IHES 96(2002), to appear.

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