ON THE VOLUME CONJECTURE FOR HYPERBOLIC KNOTS

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1. Introduction

In [1], R.M. Kashaev introduced certain invariants of oriented links motivated by his study of quantum dilogarithm functions. Since the classical dilogarithm functions are related to the hyperbolic volumes, he naturally expected that, for hyperbolic knots, the asymptotic behaviors of his invariants determine their volumes, which is in fact confirmed for a few hyperbolic knots by himself in [2]. Some other examples are now given in [5], where the asymptotic behaviors of the invariants suggestively determine not only the volumes but also the Chern-Simons invariants.

Later, in [4], H. Murakami and J. Murakami have shown that Kashaev’s invariant coincides with certain special value of the colored Jones function, and then reformulated Kashaev’s conjecture, that is, the asymptotic behavior of the colored Jones function determines the simplicial volume of a knot, which is now called the volume conjecture of knots. For torus knots, known as typical non-hyperbolic knots, this conjecture is recently confirmed by Kashaev and O. Tirkkonen in [3].

The purpose of this article is to give a rough, and so not yet complete unfortunately, proof of Kashaev’s conjecture, that is, the volume conjecture for hyperbolic knots. In fact, for a hyperbolic knot in $S^3$, the quantum factorials in its Kashaev’s invariant naturally correspond to the tetrahedra in an ideal triangulation of its complement, which establish a surprising correspondence between the stationary phase equations for the invariant and the hyperbolicity equations for the triangulation. Then, the asymptotic behavior of the invariant is determined by the promised solution, which is nothing but the volume of the knot complement because the quantum factorials asymptotically goes to the dilogarithm functions.

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2. Ideal triangulations

Throughout of this article, M denotes the complement of a hyperbolic knot $K$ in $S^3$. In this section, associated with a diagram of $K$, we construct an ideal triangulation of $M$ from which the hyperbolicity equations for $M$ follows quite nicely. We suppose $K$ does not hit the two poles $\pm \infty$ of $S^3$ and denote $M$ with $\pm \infty$ removed by $\hat{M}$.

Let $D$ be a diagram of $K$ in $S^2$ with $n$ crossings and $R_0, \ldots, R_{n+1}$ its faces. Then, $D$ together with its dual graph gives a decomposition $\mathcal{D}$ of $S^2$ into $n$ octagons $Q_1, \ldots, Q_n$, where each $Q_\nu$ is further divided into four quadrangles as shown in Figure 1. We put $Q_\nu = \{ \mu | Q_\nu \cap R_\mu \neq \emptyset \}$ and $R_\mu = \{ \nu | Q_\nu \cap R_\mu \neq \emptyset \}$, where we can suppose $|Q_\nu| = 4$ for any $\nu$, otherwise we can reduce the number of crossings of $D$.

Due to D. Thurston [6], $\hat{M}$ decomposes into ideal octahedra $P_1, \ldots, P_n$, where each $P_\nu$ corresponds to $Q_\nu$ and further decomposes into four ideal tetrahedra as shown in Figure 2. Thus, we have an ideal triangulation $\hat{\mathcal{S}}$ of $\hat{M}$ which associates $Q_\nu \cap R_\mu$ of $\mathcal{D}$ with an ideal tetrahedron $S_{\nu \mu}$. Notice that four tetrahedra corresponding to $Q_\nu$ share an edge $\hat{E}_\nu$ while all the tetrahedra corresponding to $R_\mu$ also share an edge $\hat{F}_\mu$. As usual, we put a hyperbolic structure on $S_{\nu \mu}$ by assigning a complex number $z_{\nu \mu}$ to a pair of opposite edges of $S_{\nu \mu}$ corresponding to $\hat{E}_\nu$ and $\hat{F}_\mu$.

To make $\hat{\mathcal{S}}$ an ideal triangulation of $M$, we have to specify a non-singular point on $D$, where we meet an overpass $A$ followed by an undercrossing $X$ in one side and an underpass $B$ followed by an overcrossing $Y$ in the other side. Without loss of generality, we can suppose $Q_1 \cap Q_n = \{ 0, n + 1 \}$ and that $A, X, Y, B$ are covered by $\cup_{\nu=1}^a Q_\nu, Q_x, Q_y, \cup_{\nu=b}^n Q_\nu$ respectively, where $a < x \leq y < b$, otherwise we can reduce $n$, the number of crossings in $D$. For simplicity, we put $B = \{ 1, \ldots, a, b, \ldots, n \}$, $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_{n+1}$ and $\mathcal{Q} = \cup_{\nu \in B} Q_\nu$. Then, we can further suppose $B \cap \mathcal{R} = \{ 1, n \}$ and $Q_a \cap Q_x \cap Q_y \cap Q_b = \emptyset$, and so $x \neq y$ in particular, otherwise we can reduce $n$ again or change the specified point to $D \cap Q_a \cap Q_x$ and so on, which should stop sometime because $K$ can not be a satellite of an elementary torus link.

Now, we contract the ideal bigon $P_1 \cap P_n$ bounded by $L = \{ \pm \infty \} \cup \hat{F}_0 \cup \hat{F}_{n+1}$, which makes the ideal tetrahedra touching the bigon degenerate, and come up with an ideal triangulation $\mathcal{S}$ of $M$. We choose a cusp cross section $T$ of $M$ so that $T$ does not touch the bigon, and denote by $\mathcal{T}$ and $\mathcal{S}$ the triangulations of $T$ induced by $\hat{\mathcal{S}}$ and $\mathcal{S}$ respectively. Then, $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}, \mathcal{S})$ are related as follows.

If $\nu \in \{ 1, n \}$, then $P_\nu$ intersects the bigon in two triangles and so is truncated by $T$ as shown in Figure 4a. Thus, no tetrahedron in $P_\nu$ survives in $\mathcal{S}$ as shown in Figure 4b. In particular, $S_{\nu \emptyset}$ and $S_{\nu (n+1)}$ degenerate into an edge in $\mathcal{S}$. Note that $Q_\nu \setminus \mathcal{Q} = \emptyset$. 

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)
If \( \nu \in \mathcal{B} \setminus \{1, n\} \), then two edges of \( P_\nu \) are identified with an edge in the bigon, and so is truncated by \( T \) as shown in Figure 5a. Thus, no tetrahedron in \( P_\nu \) survives in \( S \) as shown in Figure 5b, and \( Q_\nu \setminus Q = \emptyset \) again.

If \( \nu \in \{x, y\} \cap \mathcal{R} \), then \( P_\nu \) intersects the bigon in two edges, and so is truncated by \( T \) as shown in Figure 6a. Thus, one tetrahedron, \( S_{\nu c} \) say, in \( P_\nu \) survives in \( S \) as shown in Figure 6b. Note that \( Q_\nu \setminus Q = \{c\} \).

If \( \nu \in \{x, y\} \setminus \mathcal{R} \), then \( P_\nu \) intersects the bigon in one edge, and so is truncated by \( T \) as shown in Figure 7a. Thus, two tetrahedra, \( S_{\nu e}, S_{\nu f} \) say, in \( P_\nu \) survive in \( S \). Note that \( Q_\nu \setminus Q = \{e, f\} \).

If \( \nu \in \mathcal{R} \setminus \{1, x, y, n\} \), then \( P_\nu \) intersects the bigon in one edge, and so is truncated by \( T \) as shown in Figure 8a. Thus, three tetrahedra other than \( S_{\nu g} \), where \( g \) is 0 or \( n + 1 \), in \( P_\nu \) survive in \( S \) as shown in Figure 8b. Note that \( Q_\nu \setminus Q = Q_\nu \setminus \{g\} \).

Otherwise, that is, if \( \nu \not\in \mathcal{B} \cup \{x, y\} \cup \mathcal{R} \), \( P_\nu \) does not touch the bigon, and so no degeneration occurs as shown in Figure 9. Note that \( Q_\nu \setminus Q = Q_\nu \).

Such observations enable us to write down \( T \) and so the hyperbolicity equations for \( M \) explicitly. In what follows, we illustrate how to do this through an example depicted in Figure 10.

Let \( N(K), N(L), N(\pm \infty) \) be the regular neighborhoods of \( K, L, \pm \infty \) in \( S^3 \), and \( \mathcal{K}, \mathcal{L}, \mathcal{D}_\pm \) the decompositions of \( \partial N(K), \partial N(L), \partial N(\pm \infty) \) induced by \( \mathcal{S} \) respectively. Then, it is not difficult to write down \( \mathcal{K} \) which associates an edge of \( D \) with either an annulus or a pinched annulus, divided into four triangles anyway, according as the edge is alternating or not.
On the other hand, \( \mathcal{D}_+ \) is \( \mathcal{D} \) with the overpasses of \( \mathcal{D} \) contracted, while \( \mathcal{D}_- \) is the mirror image of \( \mathcal{D} \) with the underpasses of \( \mathcal{D} \) contracted, and \( \hat{\mathcal{L}} \) is then given by gluing \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) along \( \hat{F}_0 \) and \( \hat{F}_{n+1} \).

In \( S^3 \) with \( N(K) \cup N(L) \) removed, the bigon becomes a properly embedded annulus and so \( \hat{T} \) is obtained by gluing \( \hat{K} \) and \( \hat{L} \) along the annulus as shown in Figure 13. Then, by contracting a number of edges, the dotted ones in Figure 13, in \( \hat{T} \) as we have observed above, we obtain \( T \) which naturally decomposes into two triangulations \( \mathcal{K} \) and \( \mathcal{L} \) of annuli corresponding to \( \hat{K} \) and \( \hat{L} \). Note that \( \mathcal{K} \cap \mathcal{L} \) represents two meridians of \( K \) each of which consists of two edges.

To describe the hyperbolicity equations for \( M \), we consider a diagram \( G \) with two trivalent vertices which is obtained from \( D \) by removing the simple arc between \( X \) and \( Y \). Let \( m \) denote the number of crossings of \( G \), and \( \mathcal{M}_0 \cup \cdots \cup \mathcal{M}_{m+1} \) a partition of \( \{1, \ldots, n\} \) so that \( \cup_{\mu \in \mathcal{M}_\lambda} R_\mu \) form a face of \( G \) other than \( R_0 \cup R_{n+1} \), where we can suppose \( \mathcal{Q}_a \cap \mathcal{Q}_x \subset \mathcal{M}_0 \), \( \mathcal{Q}_b \cap \mathcal{Q}_y \subset \mathcal{M}_{m+1} \). Let \( \mathcal{E} \) denote the sets of edges of \( G \) and \( \mathcal{F} \) the set of edges of \( G \) lying in \( \partial R_0 \cup \partial R_{n+1} \).

If \( \nu \notin \mathcal{B} \cup \{x, y\} \), \( \hat{E}_\nu \) survives alone as an edge \( E_\nu \) in \( S \), which connects two vertices in \( \mathcal{K} \setminus \mathcal{L} \), and the edge relation around \( E_\nu \) reads
\[
\prod_{\mu \in \mathcal{Q}_\nu \setminus \mathcal{Q}} z_{\nu\mu} = 1.
\]

On the other hand, if \( \lambda \notin \{0, m+1\} \), the edges in \( \{\hat{F}_\mu | \mu \in \mathcal{M}_\lambda \} \) reduce to an edge \( F_\lambda \) in \( S \), which connects two vertices in \( \mathcal{L} \setminus \mathcal{K} \), and the edge relation around \( F_\lambda \) reads
\[
\prod_{\mu \in \mathcal{M}_\lambda} \prod_{\nu \in \mathcal{R}_\mu \setminus \mathcal{B}} z_{\nu\mu} = 1.
\]

If \( \nu \in \{x, y\} \) and \( \lambda \in \{0, m+1\} \), \( \hat{E}_\nu \), together with some other edges in \( \hat{S} \), survives as an edge \( E_\nu \) in \( S \), connecting two vertices in \( \mathcal{K} \setminus \mathcal{L} \) and \( \mathcal{K} \cap \mathcal{L} \), while the edges in \( \{\hat{F}_\mu | \mu \in \mathcal{M}_\lambda \} \), together with some other edges in \( \hat{S} \) too, reduce to an edge \( F_\lambda \) in \( S \), connecting two vertices in \( \mathcal{L} \setminus \mathcal{K} \) and \( \mathcal{K} \cap \mathcal{L} \). Thus, the edge relations around \( E_x, E_y \) should be read from \( \mathcal{K} \setminus \mathcal{L} \), and those around \( F_0, F_{m+1} \) may be substituted with the cusp conditions along the two annuli lying on the borders of \( \mathcal{K} \) and \( \mathcal{L} \), which read
\[
\prod_{\mu \in \mathcal{M}_0} \prod_{\nu \in \mathcal{R}_\mu \setminus \mathcal{B}} z_{\nu\mu} = \prod_{\mu \in \mathcal{Q}_x \setminus \mathcal{Q}} z_{x\mu}, \quad \prod_{\mu \in \mathcal{M}_{m+1}} \prod_{\nu \in \mathcal{R}_\mu \setminus \mathcal{B}} z_{\nu\mu} = \prod_{\mu \in \mathcal{Q}_y \setminus \mathcal{Q}} z_{y\mu}.
\]
Note that the set of solutions to these equations coincides with the set of functions from $E$ to $C$ which take 1 on $F$. In fact, for such $z$, $z_{\nu\mu}$ is given by $z(\varphi_{\nu\mu})/z(\psi_{\nu\mu})$, where $\varphi_{\nu\mu}, \psi_{\nu\mu} \in E$ touch $Q_\nu \cap R_\mu$ as shown in Figure 15. We here put $\epsilon(\nu, \mu) = 1$ or $-1$ according as $\psi_{\nu\mu}$ is over $\varphi_{\nu\mu}$ in $D$ or not.

Figure 15

Now, any other equation can be read from $K \setminus L$. Let $\hat{A}_\varphi$ be an annulus, which may be pinched, in $\hat{K}$ corresponding to $\varphi \in E$ and $A_\varphi$ denote $\hat{A}_\varphi$ in $K$. Furthermore, $U$ denotes the set of vertices in $K$ corresponding to $E_\nu$’s and $V$ denotes the set of vertices in $K$ which does not lie in $\partial A_\varphi$ for any $\varphi \in E$. If $\varphi \in F$, then $A_\varphi$ is a circle or a pinched annulus in $K$ such that $A_\varphi \cap V = \emptyset$, and so no equation arises. We then suppose $\varphi \in E \setminus F$ is surrounded by $\alpha, \beta, \gamma, \delta \in E$ as shown in Figure 16, and put

$$
\epsilon(\varphi) = \begin{cases} 
1 & \text{if } \varphi \text{ is over } \alpha \cup \beta \text{ and } \gamma \cup \delta \text{ in } D, \\
-1 & \text{if } \varphi \text{ is under } \alpha \cup \beta \text{ and } \gamma \cup \delta \text{ in } D, \\
0 & \text{otherwise.}
\end{cases}
$$

Figure 16

If $\epsilon(\varphi) = 1$, then $A_\varphi$ is a pinched annulus with $A_\varphi \cap V \neq \emptyset$ as shown in Figure 17, and so the edge relation around $A_\varphi \cap V$ reads

$$
\frac{1 - z(\alpha)/z(\varphi)}{1 - z(\beta)/z(\varphi)} \cdot \frac{1 - z(\delta)/z(\varphi)}{1 - z(\gamma)/z(\varphi)} = 1,
$$

where $z(\varepsilon)/z(\varphi)$ is deleted if $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ is empty.

Figure 17

If $\epsilon(\varphi) = -1$, then $A_\varphi$ is a pinched annulus with $A_\varphi \cap V \neq \emptyset$ too as shown in Figure 18, and so the edge relation around $A_\varphi \cap V$ reads

$$
\frac{1 - z(\varphi)/z(\beta)}{1 - z(\varphi)/z(\alpha)} \cdot \frac{1 - z(\varphi)/z(\gamma)}{1 - z(\varphi)/z(\delta)} = 1,
$$

where $z(\varphi)/z(\varepsilon)$ is deleted if $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ is empty.

Figure 18

If $\epsilon(\varphi) = 0$, we can suppose $\varphi$ is over $\alpha \cup \beta$ and under $\gamma \cup \delta$ in $D$. Then, $A_\varphi$ is an annulus in $K$ as shown in Figure 19, and the cusp condition along $A_\varphi$ reads

$$
\frac{1 - z(\alpha)/z(\varphi)}{1 - z(\beta)/z(\varphi)} = \frac{1 - z(\varphi)/z(\gamma)}{1 - z(\varphi)/z(\delta)},
$$

where $z(\varepsilon)/z(\varphi)$ is deleted if $\varepsilon \in \{\alpha, \beta\}$ is empty and $z(\varphi)/z(\varepsilon)$ is deleted if $\varepsilon \in \{\gamma, \delta\}$ is empty. The edge relations around the vertices not in $U \cup V$ may be substituted with
these cusp conditions, and so we obtain a complete system of hyperbolicity equations for $M$ which has $2m + 3$ unknowns and $2m + 3$ equations corresponding to $\mathcal{E} \setminus \mathcal{F}$.

To be more precise, we shall introduce a potential function for these equations by using Euler’s dilogarithm function

$$\text{Li}_2(\omega) = - \int_0^\infty \frac{\log(1 - w)}{w} dw.$$  

For $z : \mathcal{E} \to \mathbb{C}$, we put

$$V(z) = \sum_{\nu \notin B} \sum_{\mu \in \mathbb{Q}, \nu} V_{\nu \mu}(z) - 2\pi \sqrt{-1} \sum_{\varphi \in \mathcal{E} \setminus \mathcal{F}} \epsilon(\varphi) \log z(\varphi),$$

where

$$V_{\nu \mu}(z) = \epsilon(\nu, \mu) \cdot \{\text{Li}_2(z(\varphi_{\nu \mu})^{\epsilon(\nu, \mu)}/z(\psi_{\nu \mu})^{\epsilon(\nu, \mu)}) - \pi^2/6\}.$$  

Then, a simple calculation shows

$$\text{Im} V(z) = \sum_{\nu \notin B} \sum_{\mu \in \mathbb{Q}, \nu} D(z(\varphi_{\nu \mu})/z(\psi_{\nu \mu})) + \sum_{\varphi \in \mathcal{E} \setminus \mathcal{F}} \log |z(\varphi)| \cdot \text{Im} z(\varphi) \frac{\partial V(z)}{\partial z(\varphi)},$$

where

$$D(\omega) = \text{Im} \text{Li}_2(\omega) + \log |\omega| \arg(1 - \omega).$$  

It is well-known that the hyperbolic volume of $S_{\nu \mu}$ is given by $D(z(\varphi_{\nu \mu})/z(\psi_{\nu \mu}))$, and so our observations can be summarized as follows.

**Proposition 1.** The hyperbolicity equations for $M$ associated to $D$ are given by

$$\{\partial V_0(z)/\partial z(\varphi) = 0 \mid \varphi \in \mathcal{E} \setminus \mathcal{F}\},$$

where $V_0(z)$ is a branch of $V(z)$, and the volume $\text{vol}(M)$ of $M$ is then given by

$$\text{vol}(M) = \text{Im} V_0(z_0),$$

where $z_0$ is the promised solution to the hyperbolicity equations.

### 3. Kashaev’s invariants

Let $N$ be a positive integer, which will be sent to $\infty$, and $\mathcal{N} = \{0, 1, \ldots, N - 1\}$. For $h \in \mathbb{Z}$, we denote by $[h] \in \mathcal{N}$ the residue modulo $N$. In this section, we compute Kashaev’s invariant $\langle K \rangle_N$ of $K$, which is nothing but the $N$-colored Jones function of $K$ evaluated at $q = \exp 2\pi \sqrt{-1}/N$ due to [4], and detect its asymptotic behavior. Put

$$R_{ijkl}^{ij} = \frac{Nq^{-1-(k-j)(i-l+1)}\theta_{ijkl}}{(q)[i-j](q)[j-l](q)[l-k-1](q)[k-i]}, \quad \bar{R}_{ijkl}^{ij} = \frac{Nq^{1+(i-l)(k-j+1)}\theta_{ijkl}}{(q)[i-j](q)[j-l](q)[l-k-1](q)[k-i]}$$
Lemma 2. If \( (\omega)[h] = (1 - \omega)(1 - \omega^2) \ldots (1 - \omega^{|h|}) \) and
\[
\theta_{ijkl} = \begin{cases} 
1 & \text{if } [i - j] + [j - l] + [l - k - 1] + [k - i] = N - 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Furthermore, we denote \( k \in [i, j] \) if \([i - k] + [k - i] = [i - j] \).

In what follows, we suppose \( D \) is a closed braid, where \( A, X, Y, B \) are arranged in this order along its orientation, and put \( D \) in \( \mathbb{R}^2 \) by removing the specified point. Then, we can suppose \( |Q_\nu| = 4, B \cap \{x, y\} = \emptyset \) and \( B \cap \mathcal{R} = \{1, n\} \) as before, otherwise we can reduce the number of strings of \( D \). Of course, we can suppose \( Q_a \cap Q_x \cap Q_y \cap Q_b = \emptyset \).

Let \( \mathcal{X} \) be the set of maxima of \( D \) where \( D \) is oriented clockwise and \( \mathcal{Y} \) the set of maxima of \( D \) where \( D \) is oriented counter-clockwise. In what follows, such maxima and minima are regarded as vertices of \( D \) together with crossings, and the set of the edges of \( D \) is then denoted by \( \hat{\mathcal{E}} \). Furthermore, as shown in Figure 20, \( \alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu \) denote the four edges in \( \hat{\mathcal{E}} \) incident to the crossing with sign \( \epsilon(\nu) \) in \( Q_\nu \) while \( \alpha_\xi, \delta_\xi \) and \( \alpha_\eta, \delta_\eta \) denote the edges in \( \hat{\mathcal{E}} \) incident to \( \xi \in \mathcal{X} \) and \( \eta \in \mathcal{Y} \) respectively.

![Figure 20](image)

A state of \( D \) is a function \( \sigma : \hat{\mathcal{E}} \rightarrow \mathcal{N} \) which assigns 0 to the non-compact edges of \( D \). For such \( \sigma \), we put
\[
\langle D|\sigma \rangle_\nu = \begin{cases} 
R_{\sigma(\alpha_\nu)\sigma(\beta_\nu)}^{\sigma(\gamma_\nu)\sigma(\delta_\nu)} & \text{if } \epsilon(\nu) = +1, \\
R_{\sigma(\beta_\nu)\sigma(\alpha_\nu)}^{\sigma(\gamma_\nu)\sigma(\delta_\nu)} & \text{if } \epsilon(\nu) = -1.
\end{cases}
\]

for \( \nu \in \{1, \ldots, n\} \). Then, Kashaev’s invariant \( \langle K \rangle_N \) of \( K \) is given by
\[
\langle K \rangle_N = \prod_{\nu=1}^n q^{(\nu)/2} \sum_{\sigma \in \mathcal{Z}} \langle D|\sigma \rangle,
\]
where \( \mathcal{Z} \) denotes the set of states of \( D \) and
\[
\langle D|\sigma \rangle = \prod_{\nu=1}^n \langle D|\sigma \rangle_\nu \prod_{\xi \in \mathcal{X}} \{ -q^{1/2} \delta_{\sigma(\alpha_\xi)+1,\sigma(\delta_\xi)} \} \prod_{\eta \in \mathcal{Y}} \{ -q^{-1/2} \delta_{\sigma(\alpha_\eta)+1,\sigma(\delta_\eta)} \}.
\]

However, a lot of states do not contribute to \( \langle K \rangle_N \) at all. To describe them, in what follows, we denote
\[
\{ [\epsilon(\nu)\sigma(\alpha_\nu - \beta_\nu)], [\epsilon(\nu)\sigma(\beta_\nu - \delta_\nu)], [\epsilon(\nu)\sigma(\gamma_\nu - \alpha_\nu)], [\epsilon(\nu)\sigma(\delta_\nu - \gamma_\nu) - 1] \}
\]
by \( \{ \sigma(\nu, \mu) \mid \mu \in Q_\nu \} \) for \( \nu \in \{1, \ldots, n\} \), where \( \sigma(\varphi - \psi) \) stands for \( \sigma(\varphi) - \sigma(\psi) \).

**Lemma 2.** If \( \langle D|\sigma \rangle \neq 0 \),
\[
\sum_{\mu \in Q_\nu} \sigma(\nu, \mu) = \sum_{\nu \in \mathcal{R}_{\mu}} \sigma(\nu, \mu) = N - 1.
\]

Furthermore, \( \sigma(\nu, 0) = 0 \) for \( \nu \in \mathcal{R}_0 \) and \( \sigma(\nu, n + 1) = 0 \) for \( \nu \in \mathcal{R}_{n+1} \).
Proof. By definition, we have
\[
\sum_{\mu \in \mathcal{Q}_\nu} \sigma(\nu, \mu) = N - 1, \quad \sum_{\nu = 1}^{n} \sum_{\mu \in \mathcal{Q}_\nu} \sigma(\nu, \mu) = \sum_{\mu = 0}^{n+1} \sum_{\nu \in \mathcal{R}_\mu} \sigma(\nu, \mu)
\]
and so
\[
\sum_{\mu = 0}^{n+1} \sum_{\nu \in \mathcal{R}_\mu} \sigma(\nu, \mu) = n(N - 1).
\]
On the other hand, we can observe
\[
\sum_{\nu \in \mathcal{R}_\mu} \sigma(\nu, \mu) \geq N - 1
\]
unless \(\mu \in \{0, n + 1\}\), and so Lemma 2 follows immediately. \(\square\)

Lemma 2 definitely reduces the definition of \(\langle K \rangle_N\) but not enough. We can further reduce certain \(q\)-factorials in \(\langle K \rangle_N\) by using

Lemma 3. We have
\[
\sum_{i \in [k,j]} q^{-i} \bar{R}_{kl}^{ij} = \delta_{j,k} q^{1-l}, \quad \sum_{j \in [i,l]} q^{-j} R_{kl}^{ij} = \delta_{i,l} q^{1-k},
\]
\[
\sum_{k \in [l,i]} q^k \bar{R}_{kl}^{ij} = \delta_{i+1,l} q^j, \quad \sum_{l \in [j,k]} q^l R_{kl}^{ij} = \delta_{j,k+1} q^i
\]
and
\[
\sum_{i \in [k,j]} q^{-i} \bar{R}_{kl}^{ij} = \frac{-Nq^{-1-k}}{(\bar{q})_{[j-l]}(q)_{[l-k-1]}}, \quad \sum_{j \in [i,l]} q^{-j} R_{kl}^{ij} = \frac{-Nq^{1-l}}{(q)_{[l-k-1]}(\bar{q})_{[k-i]}},
\]
\[
\sum_{k \in [j,i]} q^k \bar{R}_{kl}^{ij} = \frac{-Nq^{-1+i}}{(q)_{[i-j]}(\bar{q})_{[j-l]}}, \quad \sum_{l \in [j,k]} q^l R_{kl}^{ij} = \frac{-Nq^{1+j}}{(\bar{q})_{[i-j]}(q)_{[k-i]}},
\]

Proof. We shall prove
\[
\sum_{i \in [k,j]} q^{-i} \bar{R}_{kl}^{ij} = \delta_{j,k} q^{1-l}, \quad \sum_{i \in [k,j]} q^{-i} R_{kl}^{ij} = \frac{-Nq^{-1-k}}{(\bar{q})_{[j-l]}(q)_{[l-k-1]}}.
\]
By Lemma A.1 of [4], we have
\[
\sum_{k \in [i,j]} \frac{q^{-k(i-j)}}{(q)_{[i-k]}(\bar{q})_{[k-j]}} = \delta_{i,j}, \quad \sum_{k \in [i,j]} \frac{q^{-k(i-j+1)}}{(q)_{[i-k]}(\bar{q})_{[k-j]}} = (-1)^{i-j} q^{-(i+j)(i-j+1)/2},
\]
and so
\[
\sum_{i \in [k,j]} q^{-i} \bar{R}_{kl}^{ij} = \sum_{i \in [k,j]} \frac{q^{-i(j-k)}}{(q)_{[i-j]}(\bar{q})_{[k-i]}} \frac{Nq^{1+l(j-k-1)}}{(q)_{[l-k-1]}(\bar{q})_{[j-l]}} = \delta_{j,k} q^{1-l},
\]
Thus, a simple state can be considered as a function from 
\[ \langle \sigma \rangle \text{ where } \langle \sigma \rangle \text{ and so we come up with the following definition. A state } \sigma \text{ of } D \text{ is said to be simple if } \sigma(\varphi) = 0 \text{ for } \varphi \subset D \setminus G \text{ and } \]

\[
\prod_{\nu=1}^{a} \delta_{\sigma(\alpha_{\nu}), \sigma(\delta_{\nu})} \prod_{\nu=b}^{n} \delta_{\sigma(\beta_{\nu}), \sigma(\gamma_{\nu})+\epsilon(\nu)} \prod_{\nu \in B} q^{-\epsilon(\nu)}
\]

\[
\times \frac{N^2}{(q^\epsilon(x))[\epsilon(x)\sigma(\delta_{x})^2-\gamma_{x}][q^\epsilon(y)][\epsilon(y)\sigma(\alpha_{y})^2-\beta_{y}]} \prod_{\nu \notin \{x,y\}} \frac{q^{-\epsilon(\nu)}}{(q^{-\epsilon(\nu)})[\epsilon(\nu)\sigma(\beta_{\nu})+\delta_{\nu}]} \]

and so we come up with the following definition. A state } \sigma \text{ of } D \text{ is said to be simple if } \sigma(\varphi) = 0 \text{ for } \varphi \subset D \setminus G \text{ and } \]

\[
\prod_{\nu=1}^{a} \delta_{\sigma(\alpha_{\nu}), \sigma(\delta_{\nu})} \prod_{\nu=b}^{n} \delta_{\sigma(\beta_{\nu}), \sigma(\gamma_{\nu})+\epsilon(\nu)} \prod_{\nu \in B} q^{-\epsilon(\nu)}
\]

\[
\times \frac{N^2}{(q^\epsilon(x))[\epsilon(x)\sigma(\delta_{x})^2-\gamma_{x}][q^\epsilon(y)][\epsilon(y)\sigma(\alpha_{y})^2-\beta_{y}]} \prod_{\nu \notin \{x,y\}} \frac{q^{-\epsilon(\nu)}}{(q^{-\epsilon(\nu)})[\epsilon(\nu)\sigma(\beta_{\nu})+\delta_{\nu}]} \]

for a simple state } \sigma \text{. Furthermore, by using } \]

\[
(q)[h] = \pm (-1)^h q^{h(h+1)/2}(\bar{q})[h],
\]

we have

\[
R_{ij}^{kl} = \frac{\pm N q^{i-k-1} \theta_{ijkl}}{(q)[i-j][q][j-l][q][l-k-1][q][k-i]}, \quad \bar{R}_{ij}^{kl} = \frac{\pm N q^{j-l+1} \theta_{ijkl}}{(q)[i-j][q][j-l][q][l-k-1][q][k-i]}
\]

and

\[
\langle D|\sigma \rangle = \frac{-N q^{\sigma(\alpha_{\nu})-\gamma_{\nu}-\epsilon(\nu)}}{\prod_{\mu \in Q_{\nu} \setminus Q(q^\epsilon(\nu)) \sigma(\nu, \mu)}}.
\]

Consequently, } \langle K \rangle \text{ can be rewritten as } \]

\[
(-N)^{n-a-b}(q^{1/2})[\chi]-[\gamma]^n \prod_{\nu=1}^{n} q^{-\epsilon(\nu)/2} \sum_{\nu \notin B} \prod_{\mu \in Q_{\nu} \setminus Q(q^\epsilon(\nu)) \sigma(\nu, \mu)} \]

where } Z_0 \text{ denotes the set of simple states of } D \text{. It should be noted that the } q \text{-factorials above curiously corresponds to the tetrahedra in } S.\]
From now onward, we suppose $N$ is sufficiently large so that
\[
\frac{1}{(q^{\pm 1})^{|h|}} \sim \exp \frac{N \{\pm \text{Li}_2(q^{|h|}) \mp \pi^2/6\}}{2\pi \sqrt{-1}}.
\]
Then, due to Kashaev [2], $\langle K \rangle_N$ can be approximated by the integral
\[
\int \cdots \int \exp \frac{N V(z)}{2\pi \sqrt{-1}} \prod_{\varphi \in \mathcal{E} \setminus \mathcal{F}} dz(\varphi),
\]
where $\{z(\varphi) | \varphi \in \mathcal{E} \setminus \mathcal{F}\}$ correspond to $\{q^\sigma(\varphi) | \varphi \in \mathcal{E} \setminus \mathcal{F}\}$ in $\langle K \rangle_N$. The maximal contributions to this integral comes from the solutions to the stationary phase equations for the branches of $V(z)$ which of course contain the solution $z_0$ to
\[
\{\partial V_0(z)/\partial z(\varphi) = 0 | \varphi \in \mathcal{E} \setminus \mathcal{F}\},
\]
the hyperbolicity equations before. Let $z_1$ be such a solution other than $z_0$ which is derived from the stationary phase equations for a branch $V_1(z)$ of $V(z)$. Then, the following assumption is likely to be true, see p. 367 in [7], but the author has no proof.

**Assumption.** $\text{Im} V_1(z_1) < \text{Im} V_0(z_0)$.

If this is true, the integral is simply approximated by
\[
\exp \frac{N V_0(z_0)}{2\pi \sqrt{-1}}
\]
by the saddle point method, where we have to use another assumption.

**Assumption.** There is a deformation of the contour to apply the saddle point method.

Then, the absolute value of the invariant exponentially grows like
\[
e^{\frac{N}{2\pi} \text{Im} V_0(z_0)} = e^{\frac{N}{2\pi} \text{vol}(M)},
\]
by Proposition 1, which proves the volume conjecture for hyperbolic knots.

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