Alternative PV Bus Modelling with the Holomorphic Embedding Load Flow Method

I. Wallace, D. Roberts, A. Grothey, and K. I. M. McKinnon — July 4, 2016

Abstract—The Holomorphic Embedding Load Flow Method (HELM) has been suggested as an alternative approach to solve load flow problems. However, the current literature does not provide any HELM models that can accurately handle general power networks containing PV and PQ buses of realistic sizes. The original HELM paper dealt only with PQ buses, while a second paper showed how to include PV buses but suffered from serious accuracy problems. This paper fills this gap by providing several models capable of solving general networks, with computational results for the standard IEEE test cases provided for comparison. In addition this paper also presents a new derivation of the theory behind the method and investigates some of the claims made in the original HELM paper.

Index Terms—Load flow analysis; Power system modelling; Power system simulation; Power engineering computing; Energy management; Decision support systems.

I. INTRODUCTION

Load flow problems consist of solving a set of equations — the bus-power-equilibrium equations (BPEE) — based on the physical laws surrounding a power network so as to obtain a feasible load flow for the network. The physical laws result in the BPEE being nonlinear, and usually having multiple solutions. In addition, many of these solutions would typically correspond to unstable operating points of a real network, so termed low voltage solutions, and so solving the BPEE for the physical solution is not an easy calculation.

Most current methods for solving the BPEE of a network use an iterative approach. These methods all have the same issue — that for specific problems they may not converge to a solution or converge to the “wrong” (i.e., unstable or low voltage) solution. A 2012 paper [1] and a subsequent patent [2] by Trias proposed an alternative to these iterative methods, HELM, which claims to address this issue. The idea was to treat the voltages as holomorphic functions of a complex parameter \( z \) that scales the demands, and to use the simple-to-calculate solution when \( z = 0 \) to determine the desired solution when \( z = 1 \) making use of the powerful theory of holomorphic functions and analytic continuations. There remained some important gaps in the method as described by Trias however, as it did not provide any detail on how to handle PV buses, and so the method was applicable only on networks with exclusively PQ buses (plus one slack bus).

As well, Trias’ papers contain claims about the theoretical properties of the method, namely regarding the method’s ability to always find the “correct” solution, for which only sketchy theoretical justification was given. Subramanian et al [3] suggest an extension of the HELM approach that can be applied to networks that include PV buses, however this approach suffers from numerical problems.

The contribution of this paper is to provide alternative approaches to model PQ buses within the HELM framework that do not suffer from the above numerical problems. Further, we give an investigation of the theoretical underpinnings of the HELM method.

The organisation of this paper is as follows. In the following section, we recap the HELM method for PQ bus networks as proposed in [1], [2]. In Section III we give a derivation of the HELM approach that uses the holomorphic implicit function theorem. This enables us to gain further insight into the theoretical properties of the method. In Section IV we show how the problem changes when considering PV buses, present the current literature and its shortcomings, and then in Section VI provide our own models which are capable of handling general networks. Section VII presents computational results for the test networks.

NOMENCLATURE

Sets

\[ B \] All Buses.
\[ B_{PQ} \] PQ Buses.
\[ B_{PV} \] PV Buses.

Parameters and Variables

\( Y_{ik} \) \((i, k)\) element of bus admittance matrix.
\( V_i \) Complex voltage at bus \( i \).
\( S_i \) Complex power injection at bus \( i \) \((S_i = P_i + jQ_i)\).
\( P_i \) Real power injection at bus \( i \).
\( Q_i \) Reactive power injection at bus \( i \).
\( M_i \) Prescribed voltage magnitude at PV bus \( i \).
\( z \) Complex variable used for holomorphic embedding.

Notation

\( \delta_{i,j} \) \(1\) if \( i = j \), \(0\) otherwise.

II. HELM PQ MODEL

The HELM method was first introduced by Trias in [1]. He begins by considering the BPEE for a PQ-bus network in the general form:

\[
\sum_{k \in B} Y_{ik} V_k = \frac{S_i^*}{V_i^*}, \quad i \in B_{PQ} \tag{1}
\]
where without loss of generality we will set bus 0 to be the slack bus, so that $\mathcal{B} = \mathcal{B}_{PQ} \cup \{0\}$. The voltage at the slack bus is known to be $V_0 = 1$. The remaining $V_i$ are the unknown complex variables. Trias proceeds by setting up a homotopy where the demands are scaled by a complex parameter $z$ and the resulting bus voltages are treated as functions of this complex parameter

$$\sum_{k \in \mathcal{B}} Y_{ik} V_k(z) = \frac{z S_i^*}{V_i^*(z^*)}, \quad i \in \mathcal{B}_{PQ}$$

(2)

here $z = 1$ corresponds to the solution of the BPEE while at $z = 0$ a solution can be easily computed. Note that the solution for (2) will in general not be unique for a given function. For $z = 0$ a unique solution exists under the condition that $V_i(0) \neq 0 \forall i$.

The main claim by Trias is that the voltages $V_i(z)$ implicitly defined by (2) are holomorphic functions at $z = 0$ and can be analytically continued to obtained the “correct” solution to the BPEE at $z = 1$. This is not obvious, due to the use of the complex conjugate in system (2) which is not a holomorphic function.

Trias circumvents this difficulty by embedding (2) in a larger holomorphic system, namely

$$\sum_{k \in \mathcal{B}} Y_{ik} \overline{V}_k(z) = \frac{z S_i^*}{V_i^*(z^*)}, \quad i \in \mathcal{B}_{PQ}$$

(3)

where $\overline{V}_i(z)$ are additional complex variables formally independent of $V_i(z)$. It is easy to check that these equations are indeed holomorphic as functions of the independent complex variables $z, V_i, \overline{V}_i$ for example by checking the Wirtinger derivatives or the Cauchy-Riemann equations. System (3) is a set of polynomial equations (after multiplying through with the denominator in each case) and Trias uses the theory of resultants and Gröbner bases to deduce that all $V_i$ and $\overline{V}_i$ are holomorphic functions everywhere except for a finite set of singularities - all of them branch points - which will not include 0.

If the additional constraint

$$\overline{V}_i(z) = (V_i(z^*))^*, \quad i \in \mathcal{B},$$

(4)

which Trias calls the reflecting condition, holds, system (3) reduces to (2). Trias makes use of the system (3) only to establish that there exist holomorphic solution functions $V_i(z), \overline{V}_i(z)$ and then argues that since we are only interested in those solutions that satisfy the reflecting condition it can be used to eliminate the $\overline{V}_i$. In the remainder of his presentation Trias uses (2) exclusively.

In Section III we will show that the reflecting condition (4) is satisfied automatically under the condition that $V_i(0) \neq 0 \forall i$.

Given that the voltages are holomorphic functions in a neighbourhood of $z = 0$, they, and their reciprocals, can be written as power series expandable about $z = 0$:

$$V_i(z) = \sum_{n=0}^{\infty} V_i[n] z^n, \quad i \in \mathcal{B}$$

(5)

$$\frac{1}{V_i(z)} = W_i(z) = \sum_{n=0}^{\infty} W_i[n] z^n, \quad i \in \mathcal{B}.$$  

(6)

Substituting into (2) one obtains

$$\sum_{k \in \mathcal{B}} Y_{ik} V_k[n] z^n = z S_i^* \sum_{n=0}^{\infty} W_i[n] z^n$$

(7)

From (7), it is now possible to determine the coefficients of the power series up to any desired level. The process is begun by solving for $z = 0$, which (under the condition that $V_i(0) \neq 0$) yields the set of linear equations

$$\sum_{k \in \mathcal{B}} Y_{ik} V_k[0] = 0, \quad i \in \mathcal{B}_{PQ}.$$  

(8)

Note that the sum on the left includes the slack bus $k = 0$ for which $V_0(z)$ is set to 1 for all $z$. At this point we need to impose

**Assumption 1**: The reduced bus admittance matrix $Y'$ obtained from $Y$ by removing the row and column corresponding to the slack bus is non-singular.

This is a standard assumption and will hold for any sensible power system. In particular, in the absence of shunts and phase shifters the assumption is equivalent to requiring the system to be connected.

Under Assumption 1 system (8) has a unique solution. $W_i[0]$ can then be computed using (6)

$$W_i[0] = \frac{1}{V_i[0]}$$

(9)

Having obtained the initial values for $V$ and $W$, an iterative process can be used to determine the remaining values in the power series up to any desired order of $n$ by equating the coefficients of $z^n$ in (7), which yields

$$\sum_{k \in \mathcal{B}} Y_{ik} V_k[n] = S_i^* W_i[n-1], \quad i \in \mathcal{B}_{PQ}, \quad n \geq 1$$

(10)

where $W_i[n-1]$ is calculated using the coefficients of lower orders

$$W_i[n - 1] = - \frac{\sum_{m=0}^{n-2} V_i[n-m-1] W_i[m]}{V_i[0]}$$

(11)

In (8) and (10) the coefficient matrix of the system of equations is constant, and so factorisation of this matrix needs only to be done once and can be used for all iterations.

Having obtained the power series for the voltages up to some desired level, it is now possible to compute the voltages for each bus. However, a direct summation of the power series for $z = 1$ is insufficient, as the radius of convergence of the power series is typically less than 1. Instead, analytic continuation using Padé approximants is used. Padé approximants are a particular type of rational approximations to power series known to have good convergence properties. The $L, M$ Padé approximant is denoted by $[L/M] = P_L(x)/Q_M(x)$, where

1To be precise: The unique seed, or reference, solution at $z = 0$ under the condition $V_i(0) \neq 0$ can be continued into a neighbourhood by the implicit function theorem. This continuation is holomorphic at $z = 0$. 

$$V_i(z) = \sum_{n=0}^{\infty} V_i[n] z^n, \quad i \in \mathcal{B}$$

$$\frac{1}{V_i(z)} = W_i(z) = \sum_{n=0}^{\infty} W_i[n] z^n, \quad i \in \mathcal{B}.$$  

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where

\[ P_L(x) \text{ and } Q_M(x) \text{ are polynomials of degree less than or}
\]
\[ \text{equal to } L \text{ and } M \text{ respectively.} \]

In [1], Trias explains how Stahl’s extremal domain theorem and Stahl’s Padé convergence theorem provide proof that Padé approximants give the maximal analytical continuation. That is, if there is a steady-state solution to the problem, then the diagonal Padé approximants will converge to this answer, while if there is no steady-state solution (voltage collapse), then the Padé approximants will not converge. In fact Stahl’s Theorems [4, Ch 6] asserts that the diagonal Padé approximants (i.e., \( L = M \)) converge to the analytic continuation with the extremal domain of the approximated function. Here extremal domain is understood as the one having a minimal exemption set (in the shape of branch cuts) measured in capacity. The implication is that diagonal Padé approximants (of high enough order) will yield values that satisfy (14) in a neighbourhood of \( z = 0 \).

In this setup, \( f \) is clearly a holomorphic mapping, and the values of \( v \) in the seed solution \( f(0, v, \overline{\nu}) = 0 \) are the solution to (5).

Using (14), the values of \( J \) can be computed as follows

\[
\begin{align*}
\frac{\partial f_i}{\partial \nu_j}(0, v, \overline{\nu}) &= \frac{\nu_j}{V_i}, \quad i, j = 1, \ldots, N \\
\frac{\partial f_i}{\partial \nu_j}(0, v, \overline{\nu}) &= 0, \quad i, j = 1, \ldots, N \\
\frac{\partial f_N+i}{\partial \nu_j}(0, v, \overline{\nu}) &= v_i \nu_{ij}, \quad i, j = 1, \ldots, N \\
\frac{\partial f_N+i}{\partial \nu_j}(0, v, \overline{\nu}) &= 0, \quad i, j = 1, \ldots, N \\
\end{align*}
\]

and so \( J \) can be rewritten as

\[ J = \begin{pmatrix} \nu'Y' & 0 \\ 0 & \nu'(Y')^* \end{pmatrix} \]

where \( \nu' = \text{diag}(\nu_1, \ldots, \nu_N) \) and \( Y' \) represents the admittance matrix without the slack bus row and column. Clearly \( J \) is non-singular iff \( Y' \) is non-singular which is guaranteed by Assumption 1 and therefore \( V(z) \) and \( \overline{V}(z) \) are holomorphic functions of \( z \).

### B. Reflecting Condition Redundancy

Now we will show that, in a neighbourhood of \( z = 0 \), the reflecting condition (4) is redundant. In particular we will show that any solution to (4)—which will automatically satisfy \( V_i(0) \neq 0 \forall i \)—must satisfy the reflecting condition. If we do not use the reflecting condition, then instead of (7) we obtain the following set of equations

\[
\begin{align*}
\sum_{k \in B} Y_{ik} \sum_{n=0}^{\infty} V_k[n]z^n &= zS_i^* \sum_{n=0}^{\infty} W_i[n]z^n \\
\sum_{k \in B} Y_{ik}^* \sum_{n=0}^{\infty} \overline{V}_k[n]z^n &= zS_i \sum_{n=0}^{\infty} W_i[n]z^n \\
\end{align*}
\]

where again \( W_i = 1/V_i \) and likewise \( \overline{W}_i = 1/\overline{V}_i \).

In comparing coefficients of \( z^n \), we get:

\[
\begin{pmatrix} Y' & 0 \\ 0 & (Y')^* \end{pmatrix} \begin{pmatrix} V_i[n] \\ \overline{V}_i[n] \end{pmatrix} = \begin{pmatrix} r_{1,n-1} \\ r_{2,n-1} \end{pmatrix}
\]

with

\[
\begin{pmatrix} r_{1,n-1} \\ r_{2,n-1} \end{pmatrix} = \begin{cases}
-\overline{Y}_0 \\
S_i \overline{W}_i[n] - 1 & n \geq 1
\end{cases}
\]

where \( Y_0 \) is the slack bus column of the admittance matrix.
Now if we take the complex conjugate of the above system, we obtain:

\[
\begin{bmatrix}
(Y')^* & 0 \\
0 & Y'
\end{bmatrix}
\begin{bmatrix}
V^*[n] \\
V[n]
\end{bmatrix}
= \begin{bmatrix}
r_{1,n-1}^* \\
r_{2,n-1}^*
\end{bmatrix}
\]  

(21)

and by rearranging

\[
\begin{bmatrix}
Y' & 0 \\
0 & (Y')^*
\end{bmatrix}
\begin{bmatrix}
V[n] \\
V^*[n]
\end{bmatrix}
= \begin{bmatrix}
r_{2,n-1} \\
r_{1,n-1}^*
\end{bmatrix}
\]  

(22)

From (20) it follows that when \( n = 0 \), \( r_{2,-1} = -Y_0^* = r_{1,-1} \). Thus the right-hand side in (22) is the same as in (19). Under Assumption 1 \( Y' \) is non-singular and therefore the solutions of the two systems must be the same: namely \( V[0] = V'[0]^* \).

We now assume that \( V[n] = V'[n]^* \) — and hence \( W[n] = W'[n]^* \) — is true up to \( n = k + 1 \)

\[
r_{1,k} = S^*W[k] \\
= S^*W[k]^* \\
\]  

from inductive hypothesis (23)

Once again we have that the right hand side in (22) is identical to the one in (19), and so \( V[n] = V'[n]^* \) for \( n \geq 0 \). The reflecting condition holds for all coefficients of the power series of \( V_i(z) \) and \( V_i(z) \) and therefore \( V_i(z) = V_i(z)^* \) holds for the functions themselves. Moreover, if the [L/M] Padé Approximant for \( V(z) \) is given by \( p(z)/q(z) \), then it is straightforward to show that the [L/M] Padé Approximant to \( V'(z) \) is given by \( (p(z)^*)/(q(z)^*) \) and so the reflecting condition holds also for the analytically continued function. From, for example, [6] we know that the analytically continued functions satisfy the polynomial HE equations (14).

IV. INTRODUCING PV BUSES

The HELM method as described in [11] had one major deficiency — it did not describe how to deal with networks that include PV buses. When considering PV buses, as seen in Table II the unknowns in the BPEE are different. At a PQ bus the real and reactive power injections are known and the complex voltage is unknown, whereas at a PV bus the real power injection and the voltage magnitude are known and the reactive power injection and the voltage angle are unknown. As we are solving for different variables in the BPEE, it is necessary to rearrange the equations. Without loss of generality we will consider the systems to be ordered such that the PQ buses are grouped first and then the PV buses come afterwards.

TABLE I

| Type of Bus | Voltage Mag (|V|) | Voltage Angle (δ) | Real Power Injection (P) | Reactive Power Injection (Q) |
|-------------|----------------|-------------------|--------------------------|-----------------------------|
| Slack (VΔ)  | Given          | Given             | Unknown                  | Unknown                     |
| PQ Bus      | Unknown        | Unknown           | Given                    | Given                       |
| PV Bus      | Given          | Unknown           | Given                    | Unknown                     |

In [3], Subramanian et al present an approach to deal with general networks that may include PV buses. For PQ buses, their holomorphic embedding of the BPEE uses the alternative set of equations (12).

For PV buses, the voltage magnitude and real power at the bus are known, but not the reactive power. Thus the authors create equations that use only the real power at a bus. Adding a number’s complex conjugate to itself results in eliminating the imaginary part of that number—using this strategy with equation (1), the authors suggest the following holomorphic embedding to replace (12) for PV buses

\[
M_i^2 \sum_{k \in B} Y_{ik} V_k(z) = z^2 P_i V_i(z) + (1-z) M_i^2 y_i - \]

\[
\sum_{k=0}^N Y_{ik} V_k(z) \]  

(24)

where \( M_i \) is the target voltage magnitude for PV buses. The second equation in the holomorphic embedding, is not explicitly shown in [3], but is required to provide a path for the voltage magnitudes in the PV buses to start at 1 when \( z = 0 \) and finish at \( M_i \) when \( z = 1 \).

Now the reflecting condition (4) substituted into (12) and (24) combine to form the holomorphic embedding of the entire network.

When \( z = 0 \) a solution to the system is simply \( V_i(0) = 1 \), \( i \in B \). The power series coefficients are then determined by the same process as in the original HELM method. However, for PV buses, the term \( V_i(z)^2 \sum_{k=0}^N Y_{ik} V_k(z) \) contains products of three power series. This results in double convolutions, which can have precision limitations and can lead to inaccuracies in the final results. This problem is discussed in [3].

When applied to the IEEE test cases, the Subramanian model in [3] had poor convergence in even the simplest 9-Bus case, which is confirmed by our results in Table III. Here, \( Rs \) refers to the residual when the voltages are substituted into the original BPEE, and \( Δ \) refers to the difference between the model voltage results and the voltage results obtained through MatPower. For more complicated cases the model was unable to provide even approximately correct results, as shown for the 39-Bus case in Table III.

Updating his previous work in [3], Subramanian provides a revised model in his thesis [7]. This revised model no longer has the double convolution issue and is better suited to solving larger models. The model also no longer involves a \( (1-z) \) term, instead it splits the admittance matrix into two parts, creating a diagonal matrix for the shunt effects which allows the remaining transmission elements to have zero row sums. The shunt elements are moved to the right-hand side and are
multiplied by the complex parameter \(z^2\). The holomorphic embedding becomes

\[
\sum_{k \in B} Y_{\text{trans,}k} V_k(z) = \frac{zS_i^*}{V_i^*(z^*)} - zY_{\text{shunt,}i} V_i(z), \quad i \in B_{PQ}
\]

\[
(V_i(z) \sum_{k \in B} Y_{\text{trans,}k} V_k^*(z^*)) + \left(V_i^*(z^*) \sum_{k \in B} Y_{\text{trans,}k} V_k(z) \right) = 2zP_i - 2\Re \left\{Y_{\text{shunt,}i} V_i(z) V_i^*(z^*) \right\}, \quad i \in B_{PV}
\]

\[
\overline{V}_i(z)V_i(z) = 1 + z(M_i - 1), \quad i \in B_{PV}
\]

This has the similar effect of making the seed voltages at every bus equal to 1. This model is different from our general model, which we provide in the next section. By separating the shunt elements from the admittance matrix, it creates an additional term which is dependent on the square of the voltage. In contrast, our general model is capable of creating either a linearly voltage-dependent term or a voltage-independent term. The results of both of Subramanian’s models are given for the IEEE test cases in Section VI.

## V. NEW PV MODELS

In this section we present a general parametrised model for mixed PQ/PV systems that does not have the drawback of having a double convolution. We will show that for models of this general form the \(V\) and \(\overline{V}\) are holomorphic functions in \(z\) and that the reflecting condition (4) remains redundant. Finally we will present specific choices for the parameters of the general model and investigate the numerical behaviour of the resulting methods.

### A. General Model

We suggest the following general model: for PQ buses, the following equations are used in the place of (12):

\[
\overline{V}_i(z) \left( \sum_{k \in B} Y_{ik} V_k(z) + (z-1)a_i \right) + (z-1)b_i = zS_i^* \quad i \in B_{PQ}
\]

\[
V_i(z) \left( \sum_{k \in B} Y_{ik}^* \overline{V}_k(z) + (z-1)a_i^* \right) + (z-1)b_i^* = zS_i \quad i \in B_{PQ}
\]

This is similar to the approach used in both [2] and [3], and is identical if \(a_i = y_i\) and \(b_i = 0\), \(i \in B_{PQ}\). Natural choices for \(a_i\) and \(b_i\) would be \(y_i\) or 0 and these terms would serve a similar purpose as the additional \((1 - z)y_i\) term in (12). \(a_i\) is multiplied by the voltage-dependent term, while \(b_i\) is voltage-independent. For PV buses we use the following equations in the place of (24):

\[
\overline{V}_i(z) \left( \sum_{k \in B} Y_{ik} V_k(z) + (z-1)a_i \right) + V_i(z) \left( \sum_{k \in B} Y_{ik}^* \overline{V}_k(z) + (z-1)a_i^* \right) = (z-1)(b_i + b_i^*) = 2zP_i \quad i \in B_{PV}
\]

\[
\overline{V}_i(z)V_i(z) = L_i^2(z) \quad i \in B_{PV}
\]

where \(L_i(z)\), which is assumed to be real-valued, describes how the target voltage magnitude changes with respect to \(z\). Equation (27) uses the same approach to eliminate the unknown reactive power as shown in [3], except there is one less voltage term in both the left and right hand sides, so our model does not give rise to any double convolutions and their possible accuracy problems. Equation (28) gives us the voltage magnitude at each PV bus, which may depend on the value of \(z\). The form of \(L_i(z)\) is unrestricted except that \(L_i(1)\) must equal \(M_i\).

Equating coefficients of \(z^n, n \geq 1\) in (26) will yield the following equations for \(n = 1\)

\[
\overline{V}_i[0] \sum_{k \in B} Y_{ik} V_k[1] + \overline{V}_i[1] \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) = S_i^* - \overline{V}_i[0]a_i - b_i, \quad i \in B_{PQ}
\]

\[
V_i[0] \sum_{k \in B} Y_{ik}^* \overline{V}_k[1] + V_i[1] \left( \sum_{k \in B} Y_{ik}^* \overline{V}_k[0] - a_i^* \right) = S_i - V_i[0]a_i^* - b_i^*, \quad i \in B_{PQ}
\]

and for \(n \geq 2\)

\[
\overline{V}_i[0] \sum_{k \in B} Y_{ik} V_k[n] + \overline{V}_i[k] \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) = - \sum_{m=1}^{n-1} \overline{V}_i[m] \left( \sum_{k \in B} Y_{ik} V_k[n - m] + \delta_{m,n-1}a_i \right), \quad i \in B_{PQ}
\]

\[
V_i[0] \sum_{k \in B} Y_{ik}^* \overline{V}_k[n] + V_i[k] \left( \sum_{k \in B} Y_{ik}^* \overline{V}_k[0] - a_i^* \right) = - \sum_{m=1}^{n-1} V_i[m] \left( \sum_{k \in B} Y_{ik}^* \overline{V}_k[n - m] + \delta_{m,n-1}a_i^* \right), \quad i \in B_{PQ}
\]
Similarly, equating coefficients of $z^n$ in (27) yields for $n = 1$ and

$$
\nabla_i[0]\sum_{k \in B} Y_{ik} V_k[1] + \nabla_i[1] \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) + \\
V_i[0] \sum_{k \in B} Y_{ik}^* V_k[1] + V_i[1] \left( \sum_{k \in B} Y_{ik}^* V_k[0] - a_i^* \right) = 2P_i - \nabla_i[0] a_i - V_i[0] a_i^* - b_i - b_i^*, \quad i \in B_{PV}
$$

and for $n \geq 2$

$$
\nabla_i[0]\sum_{k \in B} Y_{ik} V_k[n] + \nabla_i[n] \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) + \\
V_i[0] \sum_{k \in B} Y_{ik}^* V_k[n] + V_i[n] \left( \sum_{k \in B} Y_{ik}^* V_k[0] - a_i^* \right) = \\
- \sum_{m=1}^{n-1} \nabla_i[m] \left( \sum_{k \in B} Y_{ik} V_k[n-m] + \delta_{m,n-1} a_i \right) + \\
- \sum_{m=1}^{n-1} V_i[m] \left( \sum_{k \in B} Y_{ik}^* V_k[n-m] + \delta_{m,n-1} a_i^* \right)
$$

Finally, equating coefficients of $z^n$ in (28) gives

$$
\nabla_i[0] V_i[n] + \nabla_i[n] V_i[0] = - \sum_{m=1}^{n-1} \nabla_i[m] V_i[m-n] + L_i^2[n]
$$

Equations (29), (33) can be written in the following simple form, where the matrix on the left is independent of $n$.

$$
\begin{bmatrix}
AP_{Q1} & AP_{Q2} \\
AP_{V1} & AP_{V2} \\
AP_{Q3} & AP_{Q4} \\
AP_{V3} & AP_{V4}
\end{bmatrix}
\begin{bmatrix}
V_i[n] \\
\nabla_i[n]
\end{bmatrix}
= 
\begin{bmatrix}
AP_{Q1,n-1} & AP_{Q2,n-1} \\
AP_{V1,n-1} & AP_{V2,n-1} \\
AP_{Q3,n-1} & AP_{Q4,n-1} \\
AP_{V3,n-1} & AP_{V4,n-1}
\end{bmatrix}
\begin{bmatrix}
r_{PQ1,n-1,i} & r_{PQ2,n-1,i} \\
r_{PV1,n-1,i} & r_{PV2,n-1,i}
\end{bmatrix}
$$

where

$$
AP_{Q1,ij} = \nabla_i[0] Y_{ij} \\
AP_{V1,ij} = \nabla_i[0] Y_{ij} + \delta_{i,j} \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) \\
AP_{Q2,ij} = \delta_{i,j} \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) \\
AP_{V2,ij} = V_i[0] Y_{ij}^* + \delta_{i,j} \left( \sum_{k \in B} Y_{ik} V_k[0] - a_i \right) \\
AP_{Q3,ij} = \delta_{i,j} \left( \sum_{k \in B} Y_{ik}^* V_k[0] - a_i^* \right) \\
AP_{V3,ij} = \delta_{i,j} \nabla_i[0] \\
AP_{Q4,ij} = V_i[0] Y_{ij}^* \\
AP_{V4,ij} = \delta_{i,j} V_i[0]
$$

Values for $V_i[0]$ and $\nabla_i[0]$ can be obtained by solving (26)–(28) at $z = 0$ or equivalently equating the constant terms (coefficient for $z^0$). Unfortunately this leads to a nonlinear system of equations for the $V_i[0], \nabla_i[0]$ and a solution for general $a_i, b_i, L_i$ can not be derived easily. However, for each of the specific models that we consider it is possible to state a simple choice for $V_i[0], \nabla_i[0]$ that furthermore satisfies $V_i[0] = V_i^*[0]$. This choice will be used as the seed, allowing us to calculate all further coefficients of the power series from the recurrence (34).

**B. Holomorphicity and Reflecting Condition**

We now assume that we have a seed solution $(v, \tau)$, with $v_i = V_i[0], \tau_i = \nabla_i[0]$, to (26)–(28) at $z = 0$ and that furthermore this seed satisfies $\tau = v^*$. Proving that this seed can be continued into holomorphic functions $V(z)$ and $\nabla(z)$ of $z$ that solve the general model follows a similar approach as in Section III-A. Converting equations (26)–(28) into functions to replace $f$ in (14), we need to prove that $J$, defined in (15), is non-singular at $(0, v, \tau)$. Examination of $J$ and $A$ from (34) at $(0, v, \tau)$ shows that the two matrices are equivalent at this point. Therefore $A$ being nonsingular is sufficient for $V(z), \nabla(z)$ to be holomorphic using the CIFT. While it is possible to create special networks where this does not hold, especially if no restrictions are placed on the values of $a$ and $b$, for general networks and sensibly chosen values for $a$ and $b$ the condition will hold, as indeed is true for the IEEE models used for the numerical tests in Section VI.

The proof that the reflecting condition (24) is implied follows the same pattern as the proof given in Section III-B. Note that for the PQ-part of (34) we have $r_{PQ2,n-1} = r_{PQ2,n-1}$. In the PV part, $r_{PV1,n-1}$ and $r_{PV2,n-1}$ are both real-valued: Assuming that we have already proven that $\nabla_i[k] = V_i^*[k]$ for all $k < n$, then $r_{PV1,n-1}$ can be written as the sum of
a complex number and its complex conjugate, whereas for \( r_{PV_{2,n-1}} \) we have \( \sum_{m=1}^{n-1} V_m[n-m] \in \mathcal{R} \) since it is equal to its own complex conjugate, and \( L_i(z) \in \mathcal{R} \) by assumption.

With this in mind, taking the complex conjugates of both sides of (34) we obtain

\[
\begin{bmatrix}
A_{PQ_1} & A_{PQ_2} \\
A_{PV_1} & A_{PV_2} \\
A_{PQ_3} & A_{PQ_4} \\
A_{PV_3} & A_{PV_4}
\end{bmatrix}
\begin{bmatrix}
V[n] \\
V[n]^* \\
1
\end{bmatrix}
= \begin{bmatrix}
r_{PQ_{1,n-1}} \\
r_{PV_{1,n-1}} \\
r_{PQ_{3,n-1}} \\
r_{PV_{3,n-1}}
\end{bmatrix}
\tag{35}
\]

which, after noting how \( A_{PQ_1} = A_{PQ_4}, A_{PQ_2} = A_{PQ_3}, A_{PV_1} = A_{PV_2}, \) and \( A_{PV_3} = A_{PV_4} \), we can rearrange and rewrite as

\[
\begin{bmatrix}
A_{PQ_1} & A_{PQ_2} \\
A_{PV_1} & A_{PV_2} \\
A_{PQ_3} & A_{PQ_4} \\
A_{PV_3} & A_{PV_4}
\end{bmatrix}
\begin{bmatrix}
V[n] \\
V[n]^* \\
1
\end{bmatrix}
= \begin{bmatrix}
r_{PQ_{1,n-1}} \\
r_{PV_{1,n-1}} \\
r_{PQ_{3,n-1}} \\
r_{PV_{3,n-1}}
\end{bmatrix}
\tag{36}
\]

Comparing (36) with the original system (34) and assuming as above that \( A \) is non-singular we must have that \( V[n] = V^*[n] \) and \( V^*[n] = V[n] \) for \( n \geq 0 \) as long as the reflecting condition holds for the seed \( V_i(0) = V_i(0), V_i(0) = V_i(0) \). The reflecting condition \( V(z) = V(z)^* \) then follows.

Different choices of \( a_i, b_i, \) and \( L_i(z) \) result in different paths for the bus voltages between \( z = 0 \) and \( z = 1 \). The obvious choices for \( a_i \) and \( b_i \) are to either be 0 or 1, eliminating the terms from the model, or for one to be 0 and the other to be 1, which makes the system of equations trivial at \( z = 0 \).

There is no single obvious choice for \( L_i(z) \), though we have decided to have the voltage magnitudes scale linearly with \( z \) in our models. Table IV provides the values for \( a_i, b_i, \) and \( L_i(z) \) that we have chosen for the four models we present in this paper.

### Table IV

**Outlines of PV Methods**

| Model | \( a_i \) | \( b_i \) | \( L_i(z) \) | \( V_i(0) \) |
|-------|-----------|-----------|-------------|-------------|
| 1     | \( y_i \) | 0         | 0           | 0           |
| 2     | \( \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \) | 0         | \( M_i \)   | \( \lambda_i \) |
| 3     | 0         | 0         | \( \nu_i \) | \( \nu_i \)  |
| 4     | 0         | \( y_i \) | 1           | \( M_i - 1 \) |

\[ r_{i,n-1} = \delta_{n-1} \left( P_i - \text{Re} \left( y_i \right) \right) - \text{Re} \left( \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \left( n-m+\delta_{m,n-1} y_i \right) \right), \quad i \in \mathcal{B} \]

\[ r_{PQ_{2,n-1}} = \delta_{n-1} \left( -Q_i - \text{Im} \left( y_i \right) \right) - \text{Im} \left( \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \left( n-m+\delta_{m,n-1} y_i \right) \right), \quad i \in \mathcal{B}_{PQ} \]

\[ r_{PV_{2,n-1}} = -\sum_{m=1}^{n-1} V_i^*[m] V_i[n-m] + L_i^2[n], \quad i \in \mathcal{B}_{PV} \]

We may calculate the power series of \( V \) and \( \overline{V} \) to any desired degree \( n \). This model does not suffer from the double convolution found in [3], and in Section VII we will show that this model leads to more accurate solutions of the IEEE test cases.

### C. Model 1

The first of our models sets \( a_i = y_i, b_i = 0, \) and \( L_i = 1 + z(M_i - 1) \). With these values, we obtain a model that is very similar to the Subramanian model given in [3], differing only in that the double convolutions are removed. Using the reflecting condition, the holomorphic embedding simplifies to

\[
\text{Re} \left( V_i^* (z^*) \left( \sum_{k \in \mathcal{B}} Y_{ik} V_k(z) - (1-z)y_i \right) \right) = z P_i, \quad i \in \mathcal{B}
\]

\[
\text{Im} \left( V_i^* (z^*) \left( \sum_{k \in \mathcal{B}} Y_{ik} V_k(z) - (1-z)y_i \right) \right) = -z Q_i, \quad i \in \mathcal{B}_{PQ}
\]

\[
V_i^* V_i(z) = (1 + z(M_i - 1))^2, \quad i \in \mathcal{B}_{PV} \tag{37}
\]

The last equation gives the PV buses a voltage magnitude of 1 when at \( z = 0 \), which allows the initial model to have the simple solution of \( V_i(0) = 1, i \in \mathcal{B} \) when \( z = 0 \), while allowing the voltages in the PV buses to reach their required magnitude, \( M_i \), when \( z = 1 \). All the equations are now real, so we split the voltage \( V_i \) into its real and imaginary parts, and so equation (34) becomes

\[
\begin{bmatrix}
A_i \\
A_{PQ_3} \\
A_{PV_3}
\end{bmatrix}
\begin{bmatrix}
\text{Re} \left( V_i[n] \right) \\
\text{Im} \left( V_i[n] \right)
\end{bmatrix}
= \begin{bmatrix}
r_{i,n-1} \\
r_{PQ_{2,n-1}}
\end{bmatrix}
\tag{38}
\]

where the \( A \) matrix is

\[ A_{1,ij} = G_{ij}, \quad i \in \mathcal{B} \]

\[ A_{2,ij} = B_{ij}, \quad i \in \mathcal{B} \]

\[ A_{PQ_{1,ij}} = B_{ij}, \quad i \in \mathcal{B}_{PQ} \]

\[ A_{PQ_{3,ij}} = 2b_{ij}, \quad i \in \mathcal{B}_{PV} \]

\[ A_{PV_{3,ij}} = 0, \quad i \in \mathcal{B}_{PV} \]

and the right-hand side becomes

\[ r_{i,n-1} = \delta_{n-1} \left( P_i - \text{Re} \left( y_i \right) \right) - \text{Re} \left( \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \left( n-m+\delta_{m,n-1} y_i \right) \right), \quad i \in \mathcal{B} \]

\[ r_{PQ_{2,n-1}} = \delta_{n-1} \left( -Q_i - \text{Im} \left( y_i \right) \right) - \text{Im} \left( \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \left( n-m+\delta_{m,n-1} y_i \right) \right), \quad i \in \mathcal{B}_{PQ} \]

\[ r_{PV_{2,n-1}} = -\sum_{m=1}^{n-1} V_i^*[m] V_i[n-m] + L_i^2[n], \quad i \in \mathcal{B}_{PV} \]

The idea behind Model 2 is to keep the voltage magnitudes of PV buses constant regardless of the value of \( z \). This is accomplished by first setting \( a_i = \sum_{k \in \mathcal{B}} Y_{ik} \lambda_k \) and \( b_i = 0 \) in (26) and (27), where all \( \lambda_k \) are constant and \( \lambda_k = \left| V_k \right|, k \in \mathcal{B}_{PV} \). As well, in (28) we set \( L_i(z) = \left| V_i \right| \), which is independent of \( z \). The holomorphic embedding thus becomes:

\[ V_i^* V_i(z) = (1 + z(M_i - 1))^2, \quad i \in \mathcal{B}_{PV} \]
\[
\text{Re} \left\{ V_i^*(z^*) \left( \sum_{k \in B} Y_{ik} V_k(z) - (1-z) \sum_{k \in B} Y_{ik} \lambda_k \right) \right\} = z P_i, \quad i \in B
\]

(1)\[
\text{Im} \left\{ V_i^*(z^*) \left( \sum_{k \in B} Y_{ik} V_k(z) - (1-z) \sum_{k \in B} Y_{ik} \lambda_k \right) \right\} = -z Q_i, \quad i \in B_{PQ}
\]

\[
V_i^*(z^*) V_i(z) = \lambda_i^2, \quad i \in B_{PV}
\]

For Model 2, when \( z = 0 \), \( V_i(0) = \lambda_i \), \( V_i(0) = \lambda_i \) is a valid solution for \( i \in B \).

Here, the \( A \) matrix from equation (38) becomes:

\[
A_{1ij} = G_{ij} \lambda_i, \quad i \in B
\]

\[
A_{2ij} = B_{ij} \lambda_i, \quad i \in B
\]

\[
A_{PQ_{3ij}} = B_{ij} \lambda_i, \quad i \in B_{PQ}
\]

\[
A_{PV_{3ij}} = 2 \delta_{ij} \lambda_i, \quad i \in B_{PV}
\]

\[
A_{PQ_{4ij}} = G_{ij} \lambda_i, \quad i \in B_{PQ}
\]

\[
A_{PV_{4ij}} = 0, \quad i \in B_{PV}
\]

while the right-hand side becomes

\[
r_{i,n-1,i} = \delta_{n,1} \left( P_i - \lambda_i \sum_{k \in B} G_{ik} \lambda_k \right) - \text{Re} \left\{ \sum_{m=1}^{n-1} V_i^*[m] \left( \sum_{k \in B} Y_{ik} V_k[n-m] + \delta_{m,1} \sum_{k \in B} Y_{ik} \lambda_k \right) \right\}, \quad i \in B
\]

\[
r_{PQ_{2,n-1,i}} = \delta_{n,1} \left( -Q_i - \lambda_i \sum_{k \in B} B_{ik} \lambda_k \right) - \text{Im} \left\{ \sum_{m=1}^{n-1} V_i^*[m] \left( \sum_{k \in B} Y_{ik} V_k[n-m] + \delta_{m,1} \sum_{k \in B} Y_{ik} \lambda_k \right) \right\}, \quad i \in B_{PQ}
\]

\[
r_{PV_{2,n-1,i}} = -\sum_{m=1}^{n-1} V_i^*[m] V_i[n-m], \quad i \in B_{PV}
\]

The \( A \) matrix is identical to that in Model 1 save the introduction of \( \lambda \) – the main difference is in the right-hand side. We can once again solve for \( V \) to any desired value of \( n \).

E. Model 3

In Model 3 both \( a_i \) and \( b_i \) are set to 0. This gives a model similar in form to the model used in the HELM method given in (2) —in the absence of PV buses they are identical. The solution at \( z = 0 \) requires solving a simple set of equations. We obtain \( V(0) = \nu \) and \( V(0) = \nu^* \), where \( \nu \) is the solution to

\[
\sum_{k \in B} Y_{ik} \nu_k = 0 \quad i \in B
\]

We also scale the voltage magnitudes linearly with respect to \( z \), so \( L_i(z) = ||\nu_i|| + z(M_i - ||\nu_i||) \). The holomorphic embedding thus becomes:

\[
\text{Re} \left\{ V_i^*(z^*) \sum_{k \in B} Y_{ik} V_k(z) \right\} = z P_i, \quad i \in B
\]

(41)\[
\text{Im} \left\{ V_i^*(z^*) \sum_{k \in B} Y_{ik} V_k(z) \right\} = -z Q_i, \quad i \in B_{PQ}
\]

\[
V_i^*(z^*) V_i(z) = (||\nu_i|| + z(M_i - ||\nu_i||))^2, \quad i \in B_{PV}
\]

Here, the \( A \) matrix from equation (38) becomes:

\[
A_{1ij} = \nu_i G_{ij}, \quad i \in B
\]

\[
A_{2ij} = \nu_i B_{ij}, \quad i \in B
\]

\[
A_{PQ_{3ij}} = \nu_i B_{ij}, \quad i \in B_{PQ}
\]

\[
A_{PV_{3ij}} = 2 \delta_{ij} \nu_i, \quad i \in B_{PV}
\]

\[
A_{PQ_{4ij}} = \nu_i G_{ij}, \quad i \in B_{PQ}
\]

\[
A_{PV_{4ij}} = 0, \quad i \in B_{PV}
\]

while the right-hand side becomes

\[
r_{i,n-1,i} = \delta_{n,1} P_i - \text{Re} \left\{ \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in B} Y_{ik} V_k[n-m] \right\}, \quad i \in B
\]

\[
r_{PQ_{2,n-1,i}} = -\delta_{n,1} Q_i - \text{Im} \left\{ \sum_{m=1}^{n-1} V_i^*[m] \sum_{k \in B} Y_{ik} V_k[n-m] \right\}, \quad i \in B_{PQ}
\]

\[
r_{PV_{2,n-1,i}} = -\sum_{m=1}^{n-1} V_i^*[m] V_i[n-m] + L_i^2[n], \quad i \in B_{PV}
\]

We can once again solve for \( V \) to any desired value of \( n \).

F. Model 4

The holomorphic embedding for Model 4 is the same as Model 1 except that the term involving \((1-z)\) is made voltage independent. This is accomplished by setting \( a_i = 0, b_i = y_i \), and \( L_i = 1 + z(M_i - 1) \). The holomorphic embedding becomes

\[
\text{Re} \left\{ V_i^*(z^*) \sum_{k \in B} Y_{ik} V_k(z) - (1-z)y_i \right\} = z P_i, \quad i \in B
\]

(42)\[
\text{Im} \left\{ V_i^*(z^*) \sum_{k \in B} Y_{ik} V_k(z) - (1-z)y_i \right\} = -z Q_i, \quad i \in B_{PQ}
\]

\[
V_i^*(z^*) V_i(z) = (1 + z(M_i - 1))^2, \quad i \in B_{PV}
\]
Here, the $A$ matrix from equation (38) becomes:

$$A_{ij} = G_{ij} + \delta_{ij} \Re \{y_i\}, \quad i \in B$$

$$A_{2ij} = B_{ij} - \delta_{ij} \Im \{y_i\}, \quad i \in B$$

$$A_{PQ1ij} = B_{ij} - \delta_{ij} \Im \{y_i\}, \quad i \in B_{PQ}$$

$$A_{PV1ij} = 2\delta_{ij}, \quad i \in B_{PV}$$

$$A_{PV4ij} = G_{ij} + \delta_{ij} \Im \{y_i\}, \quad i \in B_{PQ}$$

$$A_{PV4ij} = 0, \quad i \in B_{PV}$$

while the right-hand side becomes:

$$r_{1,n,1,i} = \delta_{n,1} \left(P_i - \Re \{y_i\} \right)$$

$$\Re \left\{ \sum_{m=1}^{n-1} V_i^* m \sum_{k \in B} Y_{ik} V_k [n-m] \right\}, \quad i \in B$$

$$r_{PQ2,n,1,i} = \delta_{n,1} \left(-Q_i - \Im \{y_i\} \right)$$

$$\Im \left\{ \sum_{m=1}^{n-1} V_i^* m \sum_{k \in B} Y_{ik} V_k [n-m] \right\}, \quad i \in B_{PQ}$$

$$r_{PV2,n,1,i} = -\sum_{m=1}^{n-1} V_i^* m [n-m] + E_i^2 [n], \quad i \in B_{PV}$$

and we can once again solve for $V$ to any desired value of $n$.

Having shown how the new models are able to create the required power series for the bus voltages, the next section will show how successful each model was at solving IEEE test cases.

VI. COMPUTATIONAL RESULTS

The models from Section VI were created in Matlab and run on a series of seven standard IEEE test cases (9-, 14-, 30-, 39-, 57-, 118-, and 300-Bus) to obtain power series for the voltages. These power series were then run through the Viskovatov Padé approximant algorithm, as this is also the algorithm used in [1] and [3]. The resulting voltage values were then checked to see how well they solved the initial BPEE [1] as well as how closely they resemble the solution obtained through the power flow function in MatPower 4.1, used here as an example of a traditional method of solving a load flow problem.

In Table V, Max $|R_s|$ refers to the maximum absolute residual obtained from the $(N-1)$ equations when the Padé approximant solutions are substituted back into [1]. Max $|\Delta|$ refers to the maximum absolute difference between the voltages obtained from the models and those obtained from MatPower. In each case the $[15/15]$ Padé approximant is used to determine the bus voltage values. In all cases double precision was used. The times given in the table are the average of 100 runs. All the models, including MatPower, take roughly the same amount of time for the small systems.

From Table V, we can see that the model derived from [3], Subr. 1, converges much more poorly than the rest. For the 9-Bus system, Subramanian’s first model required a $[60/60]$ Padé approximant to achieve a similar level of accuracy to what the other models achieve with the $[15/15]$ Padé approximant. The other models perform relatively equally on all systems up to the 300-Bus system except that Subramanian’s second model has difficulty with the 39-Bus system. All of the models have trouble with the 300-Bus system and fail to provide correct voltages except Model 4, which manages to converge slowly to a proper solution.

Below we will further investigate the behaviour for the 300-bus system. Figures 5 and 6 show the power series coefficients obtained in the 300-Bus network using Models 1 and 4 respectively, while Figures 1 and 2 show the corresponding singularities (zeros of the denominator polynomial) of the $[50/50]$ Padé approximant. The data for the figures was produced using Maple to 100 significant digits.
We can see from Figure 1 that the Bus 8 Padé approximant derived using Model 1 appears to have a set of poles along the real line between $z = 0.1$ and $z = 0.2$, and going off the real axis into the upper and lower half-planes, with yet another set around $z = 0.8$, indicating cuts in the complex voltage function. This is consistent with a small radius of convergence of the corresponding power series as can be seen from its rapidly increasing coefficients (Figure 3). From Table V we see that these singularities adversely affect the rate of convergence of the Padé approximants, and using double precision the model is incapable of providing a sufficiently accurate value at $z = 1$. Further investigation was conducted on the 300-Bus network using Maple, where precision can be set to higher levels than double precision. By setting the precision to 200 digits, much higher order of Padé approximants can be computed to greater accuracy, and in this setup convergence to the correct value at $z = 1$ is obtained.

On the other hand, Figure 2 shows for Model 4 there are no singularities in the disk centred at $z = 0$ of radius 1. The single pole closest to the origin below the real line is a spurious pole and does not appear neighbouring Padé Approximants. Indeed the radius of convergence of the power series is greater than 1 as is indicated by the decreasing nature of its coefficients (Figure 4).

A. Case 39PQ with HELM

It is not possible to test the original HELM model from \cite{1} on examples containing PV buses. Practical examples, including all of the IEEE test cases used by our general models, have PV buses. One workaround to the lack of PQ-only networks is to convert the PV buses in the IEEE test cases into PQ buses. This is accomplished by finding the reactive power output of the PV buses at the power flow solution to the networks at full-load (as obtained by MatPower) and then fixing this value as a parameter at the bus while also now allowing voltage magnitude to vary (as in a PQ bus). The test-case solution should again be a solution to this altered network, and indeed, running these altered IEEE test cases through
MatPower results in the same power flows and voltages as the original cases.

Using the HELM model on these modified results in the same solution as MatPower in all cases except the 39-Bus IEEE test case. In this case, the HELM model converges to a solution at \( z = 1 \) which does indeed solve the BPEE correctly, but which however is different from the solution obtained by MatPower. The MatPower solution has voltages magnitudes close to 1 p.u. whereas the voltages magnitudes in the HELM solution are quite high (1.4–1.8 p.u.)

We have attempted to follow both solutions as the load changes between full load (\( z = 1 \)) and no load (\( z = 0 \)). Figure 5 shows the profile of the voltages at each bus as HELM moves from the initial no-load case to the full-load case. We have attempted to use HELM to trace the MatPower solution back from \( z = 1 \) to \( z = 0 \) but have been unable to do so: there seems to be a singularity around \( z = 0.5 \) which prevents HELM obtaining solutions for real values of \( z \) less than 0.5 due to precision problems. Instead we have followed the MatPower solution back to the no-load case by using a traditional real homotopy on the non-embedded system. The resulting bus voltage profiles as the load factor changes are shown in Figure 6.

By comparing the two figures we see that the HELM solution starts with \( V_i(0) \neq 0, \forall i \), whereas for the MatPower solution for one of the buses (Bus 38) \( V_i(0) = 0 \).

The apparent inconsistencies can be reconciled by looking at the stability of each solution. Using the \( \frac{d\Delta Q}{dV} \) stability criteria, the MatPower solution—which is stable for the original IEEE test case—becomes unstable in the altered, PQ-only network. Indeed, if more reactive power is demanded at any PQ bus except Bus 32, then the voltage magnitude at that bus increases rather than decreases. The higher-voltage HELM solution, by contrast, is stable, giving credence to the claim that HELM will return a stable solution.

![Fig. 5. HELM Solutions going from no-load to full-load](image)

![Fig. 6. Matpower Solutions going from no-load to full-load](image)

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