ON THE BRAID MONODROMY GROUP OF A POLYNOMIAL IN ONE VARIABLE

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Abstract. It is proved that the braid monodromy group $\Gamma_{P(z)} \subset Br_n$ of a polynomial $P(z) \in \mathbb{C}[z]$, $\deg P(z) = n$, is the braid group $Br_n$ if the polynomial $P(z)$ has $n-1$ distinct critical values.

0. Introduction

Let $K \subset \Pi$ be a subset of the complex plane $\Pi = \mathbb{C}$ consisting of $n$ distinct points. Denote by $Br_n = Br_n[\Pi, K]$ the braid group of the plane $\Pi$ with $n$ strings and $\sigma : Br_n \to S_n$ the natural epimorphism to the symmetric group $S_n$.

Let $P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 \in \mathbb{C}[z]$ be a polynomial in one variable, $\deg P(z) = n > 1$. The polynomial $P(z)$ defines a finite morphism $p_{P(z)} : \Pi := \mathbb{C} \to \mathbb{C}$ of degree $n$ given by $w = P(z)$. Denote by $C_{P(z)} = \{z \in \Pi \mid P'(z) = 0\}$ the set of critical points of $P(z)$ and $B_{P(z)} = p_{P(z)}(C) \subset \mathbb{C}$ the branch locus of morphism $p_{P(z)}$, so that $p_{P(z)} : \Pi \setminus p^{-1}(B_{P(z)}) \to \mathbb{C} \setminus B_{P(z)}$ is an unramified finite cover of degree $n$.

Let us choose a point $w_0 \in \mathbb{C} \setminus B_{P(z)}$ and consider a loop $\lambda : [0, 1] \to \mathbb{C} \setminus B_{P(z)}$ that starts and ends at the point $w_0$. The inverse image $p_{P(z)}^{-1}(\lambda) \subset \Pi \setminus p_{P(z)}^{-1}(B_{P(z)})$ of the loop $\lambda$ is the $n$ paths $\{b_1(t), \ldots, b_n(t)\}$, $b_j(t) \neq b_{j_2}(t)$ for $j_1 \neq j_2$, that start and end at the points lying in $K = p_{P(z)}^{-1}(w_0) = \{q_1, \ldots, q_n\}$ which define a geometric braid $b(\lambda)$. If $\lambda_1$ and $\lambda_2$ are homotopic loops, then it is easy to see that the braids $b(\lambda_1)$ and $b(\lambda_2)$ are isotopic. Therefore the lift of loops defines a braid monodromy homomorphism $\beta_{P(z)} : \pi_1(\mathbb{C} \setminus B_{P(z)}, w_0) \to Br_n = Br_n[\Pi, K]$. The image $\text{Im}\beta_{P(z)} := \Gamma_{P(z)} \subset Br_n$ is called the braid monodromy group of polynomial $P(z)$. The composition $\mu_{P(z)} = \sigma \circ \beta_{P(z)} : \pi_1(\mathbb{C} \setminus B_{P(z)}, w_0) \to S_n$ is called the monodromy homomorphism of $P(z)$ and its image $\text{Im}\mu := G_{P(z)} \subset S_n$ is called the monodromy group of $P(z)$.

Obviously, a necessary condition for $\beta_{P(z)}$ to be an epimorphism is the equality $G_{P(z)} = S_n$. Note that this condition is not always met. For example, if $P(z) = z^n$ then $G_{P(z)}$ is a cyclic group of order $n$, and it is easy to see that if $P(z) = z^4 - z^2$ then $G_{P(z)} = D_4 \subset S_4$ is the dihedral group of order 8. Therefore, the following question is of interest, which subgroups of $Br_n$ can be realized as the braid monodromy groups of polynomials.

The aim of this paper is to prove the following
Theorem 1. If $B_{P(z)}$ consists of $n - 1$ distinct points, then $\Gamma_{P(z)} = Br_n$.

The proof of Theorem 1 is given in Section 2 and in Section 1, we remind several definitions and well-known results related to the theory of braid groups (see details, for example, in [1]) and introduce notations which are used in the proof of Theorem 1.

1. The braid groups

1.1. As an abstract group, the group $Br_n$ has the following presentation. It is generated by elements $\{a_1, \ldots, a_{n-1}\}$ being subject to the relations

\[
\begin{align*}
 a_1a_ja_j^{-1}a_j &= a_ja_1a_ja_1, & 1 \leq j \leq n-1, \\
 a_ja_k &= a_ka_j, & |j-k| \geq 2
\end{align*}
\]

(such generators of $Br_n$ are called standard).

Below in this section, we remind the well-known realization of the braid group $Br_n$ as a geometric braid group.

1.2. Let $S$ be either the complex plane $\mathbb{C}$ or a disk $D_r(z_0) = \{z \in \mathbb{C} | \|z - z_0\| < r\}$ in $\mathbb{C}$.

Let us fix a set $K = \{q_1, \ldots, q_n\} \subset S$ consisting of $n$ distinct points. The elements $b$ of the geometric braid group $Br_n[S,K]$ are $n$ pairwise nonintersecting paths

\[
\{(b_j(t), t) \in S \times \mathbb{R} | t \in [0, 1]\}, \quad j = 1, \ldots, n,
\]

(considered up to continuous isotopy) the start points of which are $(b_j(0), 0) = (q_j, 0)$ and the end points of which belong to the set $K \times \{1\}$.

A braid $b$ defines a permutation $\sigma(b) \in S_n$ acting on the set $\{1, \ldots, n\}$, $\sigma(b)(j) = k$ if $b_j(1) = q_k$.

The product of braids

\[
b_1 = \{(b_{1,1}(t), \ldots, b_{1,n}(t), t)\} \text{ and } b_2 = \{(b_{2,1}(t), \ldots, b_{2,n}(t), t)\},
\]

is the braid $b = b_1b_2 = \{(b_{1,1}(t), \ldots, b_{n,n}(t), t)\}$, where

\[
b_j(t) = \begin{cases} 
 b_{1,j}(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\
 b_{2,\sigma(b_1)(j)}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1, 
\end{cases} \quad j = 1, \ldots, n.
\]

Let $l_j \subset S$ be a smooth path connecting the point $q_j$ with the point $q_{j+1}$ such that $l_j \cap K = \{q_j, q_{j+1}\}$ and let $U \subset S$ be a sufficiently small neighborhood of the path $l_j$ such that $K \cap U = \{q_j, q_{j+1}\}$ and there is a diffeomorphism $\psi : D_{1+\varepsilon}(0) = \{z \in \mathbb{C} | |z| < 1 + \varepsilon\} \to U$ preserving the "complex" orientation and such that $\psi([-1, 1]) = l_j$, where $[-1, 1] = \{z \in \mathbb{C} | \text{Re } z \in [-1, 1], \text{Im } z = 0\}$, and $\psi(1) = q_j, \psi(-1) = q_{j+1}$. The braid $b(t) = \{(b_{1,1}(t), \ldots, b_{n,n}(t), t)\}$, where

\[
b_k(t) = \begin{cases} 
 q_k, & \text{for } k \neq j, j + 1, \\
 \psi(e^{\pi i t}) & \text{for } k = j, \\
 \psi(e^{\pi i (t+1)}), & \text{for } k = j + 1,
\end{cases}
\]
is called a half-twist associated with the path \( l_j \). (Below, we will assume that \( \psi : D_{1+\varepsilon}(0) \to U \) is a bi-holomorphic isomorphism.)

Let us choose \( n-1 \) smooth paths \( l_j \subset S, j = 1, \ldots, n-1 \), that start at the points \( q_j \) and end at \( q_{j+1} \), and such that the path \( l = l_1 \cup \cdots \cup l_{n-1} \) is a simple path without self-intersections (such collection of the paths \( l_j \) is called a frame of the group \( \text{Br}_n[S, K] \)). Then the set \( \{a_1, \ldots, a_{n-1}\} \in \text{Br}_n[S, K] \) of the half-twists associated with the paths \( l_j \) defines an isomorphism between \( \text{Br}_n[S, K] \) and the abstract braid group \( \text{Br}_n \).

1.3. Let \( n = 2 \). The abstract braid group \( \text{Br}_2 \) is generated by the standard generator \( a_1 \).

Consider a geometric braid \( b = \{(b_1(t), t), (b_2(t), t)\} \). Let \( q_2 - q_1 = |q_2 - q_1|e^{\pi i} \), then there is a continuous function \( \text{arg} \gamma : [0,1] \to \mathbb{R} \) such that \( \text{arg} \gamma(0) = \varphi \) and \( q_2(t) - q_1(t) = |b_2(t) - b_1(t)|e^{\text{arg} \gamma(t)} \).

The number

\[
N_b = \frac{1}{\pi} (\text{arc} \gamma(1) - \text{arc} \gamma(0)) \in \mathbb{Z}
\]

is called the number of half-twists of \( b \). Obviously, if braids \( b_1 \) and \( b_2 \) are isotopic, then \( N_{b_1} = N_{b_2} \). In addition, we have \( N_{b_1b_2} = N_{b_1} + N_{b_2} \). Therefore the map \( N : \text{Br}_2[S, K] \to \mathbb{Z} \) sending braids \( b \) to \( N_b \) is a homomorphism of groups.

**Lemma 1.** Let \( l \subset S \) be a frame of \( \text{Br}_2[S, K] \) and \( b \) a half-twist associated with the path \( l \). Then \( N_b = 1 \) and, in particular, \( b \) is a standard generator of \( \text{Br}_2[S, K] \).

**Proof.** The bi-holomorphic map \( \psi : D_{1+\varepsilon}(0) \to U \), involved in the definition of the half-twist \( b \), defines a holomorphic function \( w(z) = \psi(z) - \psi(-z) \) on \( D_{1+\varepsilon}(0) \) such that \( w(z) = -w(-z) \). Denote by \( \text{arg} w(z) \) a continuous branch along the path \( \gamma = \{e^{\pi i t}\}_{t \in [0,2]} \) of the function \( \text{Arg} \ w(z) \). Obviously, \( N_b = \frac{1}{\pi}(\text{arg} w(e^{\pi i 1}) - \text{arg} w(e^{\pi i 0})) \).

We have

\[
\text{arg} w(e^{\pi i 2}) - \text{arg} w(e^{\pi i 1}) = \text{arg} w(e^{\pi i 1}) - \text{arg} w(e^{\pi i 0}),
\]

since \( w(z) = -w(-z) \). It is easy to see that \( w(z) = 0 \) only if \( z = 0 \) and the order of zero of the function \( w(z) \) at \( z = 0 \) is equal to 1. Therefore, by the argument principle,

\[
\text{arg} w(e^{\pi i 2}) - \text{arg} w(e^{\pi i 0}) = (\text{arg} w(e^{\pi i 2}) - \text{arg} w(e^{\pi i 1})) + (\text{arg} w(e^{\pi i 1}) - \text{arg} w(e^{\pi i 0})) = 2\pi
\]

and hence \( \text{arg} w(e^{\pi i 1}) - \text{arg} w(e^{\pi i 0}) = \pi \).

\[\square\]

2. Proof of Theorem

2.1. Consider a polynomial \( P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 \). First of all, note that after the coordinate change \( \tilde{w} = w - c_0 \) in \( \mathbb{C} \), the morphism \( p_{P(z)} : \Pi \to \mathbb{C} \) is given by \( \tilde{w} = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z \). So, without loss of generality, we can assume that \( c_0 = 0 \).

Denote by \( \mathcal{P} \simeq \mathbb{C}^{n-1} \) the affine space of polynomials \( P(z) \), \( \deg P(z) = n \), of the form \( z^n + c_{n-1}z^{n-1} + \cdots + c_1 z \) and let \( \mathcal{C} \subset \mathcal{P} \) be the set of polynomials \( P(z) \) such that \( B_{P(z)} \) consists of \( n-1 \) distinct points.
Let us show that the set $\mathcal{C}$ is everywhere dense Zariski open subvariety in $\mathcal{P}$ and, in particular, $\mathcal{C}$ is a connected set.

Consider in $\mathcal{P} \times \Pi \simeq \mathbb{C}^n$ the smooth hypersurface $\mathcal{P}'$ given by

$$nz^{n-1} + (n-1)c_{n-1}z^{n-2} + \cdots + c_1 = 0.$$ 

Obviously, the restriction $h : \mathcal{P}' \to \mathcal{P}$ of the projection $pr : \mathcal{P} \times \Pi \to \mathcal{P}$ to $\mathcal{P}'$ is a finite morphism of degree $n-1$. Note that the morphism $h$ can be considered as the composition of two dominant morphisms $h_1$ and $h_2$, where $h_1 : \mathcal{P}' \to \mathcal{P} \times \mathbb{C}$ is the restriction to $\mathcal{P}'$ of morphism $id \times p : \mathcal{P} \times \Pi \to \mathcal{P} \times \mathbb{C}$ given by

$$(c_{n-1}, \ldots, c_1, z) \mapsto (c_{n-1}, \ldots, c_1, z^n + c_{n-1}z^{n-1} + \cdots + c_1z)$$

and $h_2 : \mathcal{V} = h_1(\mathcal{P}') \to \mathcal{P}$ is the restriction to $\mathcal{V}$ of the projection $pr : \mathcal{P} \times \mathbb{C} \to \mathcal{P}$.

It is easy to check that for the polynomial $P_0 = z^n - nz \in \mathcal{P}$ the preimage $h_2^{-1}(P_0)$ consists of $n-1$ distinct points. Therefore $\deg h_2 \geq n-1$ and hence, $\deg h_2 = n-1$, and $h_1 : \mathcal{P}' \to \mathcal{V}$ is a bi-rational morphism such that $h_1 : \mathcal{P}' \setminus h_1^{-1}(\text{Sing } \mathcal{V}) \to \mathcal{V} \setminus \text{Sing } \mathcal{V}$ is a bi-regular morphism, where $\text{Sing } \mathcal{V}$ is the set of singular points of $\mathcal{V}$. Consequently, $\mathcal{C} = \mathcal{P} \setminus (\mathcal{D} \cup h_2(\text{Sing } \mathcal{V}))$ is an everywhere dense Zariski open subvariety (here $\mathcal{D}$ is the hypersurface in $\mathcal{P}$ given by $\Delta_{n-1} = 0$, where $\Delta_{n-1}$ is the discriminant of the polynomial $nz^{n-1} + (n-1)c_{n-1}z^{n-2} + \cdots + c_1$).

2.2. Let us show that it suffices to prove Theorem 1 for a particular polynomial $P_0(z) \in \mathcal{C}$. Indeed, let $P_1(z)$ and $P_2(z) \in \mathcal{C}$ be two very closed to each other polynomials. Then the sets $B_{P_1(z)}$ and $B_{P_2(z)}$ are very closed to each other, and the sets $p^{-1}_{P_1(z)}(w_0)$ and $p^{-1}_{P_2(z)}(w_0)$ are also very closed to each other for a point $w_0 \in \Pi \setminus (B_{P_1(z)} \cup B_{P_2(z)})$. Therefore we can identify the groups $\text{Br}_n(\Pi, p^{-1}_{P_1(z)}(w_0))$ and $\text{Br}_n(\Pi, p^{-1}_{P_2(z)}(w_0))$ and identify the groups $\pi_1(C \setminus B_{P_1(z)}, w_0)$ and $\pi_1(C \setminus B_{P_2(z)}, w_0)$ which are isomorphic to the free group $\mathbb{F}_{n-1}$ generated by elements represented by simple loops $\lambda_j$ around the very close points $w_{1,j} \in B_{P_1(z)}$ and $w_{2,j} \in B_{P_2(z)}$, $j = 1, \ldots, n-1$. These identifications imply that the groups $\Gamma_{P_1(z)}$ and $\Gamma_{P_2(z)}$ are isomorphic, since the geometric braids $p^{-1}_{P_1(z)}(\lambda_j)$ and $p^{-1}_{P_2(z)}(\lambda_j)$ are also very closed to each other for each $j = 1, \ldots, n-1$. Therefore the isomorphisms of the groups $\Gamma_{P(z)}$ for all $P(z) \in \mathcal{C}$ follows from connectedness of $\mathcal{C}$.

2.3. Consider the polynomial $P_0(z) = z^n - nz$. We have

$$C_{P_0(z)} = \{e^{2\pi j(i-1)/n-1}\}_{j=1}^{n-1} \text{ and } B_{P_0(z)} = \{w_j = (1-n)e^{2\pi j(i-1)/n-1}\}_{j=1}^{n-1}.$$ 

Let $w_0 = 0$ and

$$K = p^{-1}_{P_0(z)}(0) = \{q_1 = 0, q_2 = \sqrt[n-1]{n}e^{2\pi(2-2)i/n-1}, \ldots, q_n = \sqrt[n-1]{n}e^{2\pi(n-2)i/n-1}\}.$$ 

Denote by $D_r(w_1) = \{w \in \mathbb{C} \mid |w - w_1| < r\}$ the disk of radius $r > 0$ with center at $w_1$.

The set $p^{-1}_{P_0(z)}(w_1)$ consists of $n-1$ points $z_1 = 1 - n, z_2, \ldots, z_{n-1}$ and there is $\varepsilon_0$ such that
In addition, we can assume that there is a bi-holomorphic map $\varphi_1 : D_{\sqrt{\varepsilon_0}} = \{ \tilde{z} \in \mathbb{C} \mid |\tilde{z}| \leq \sqrt{\varepsilon_0} \} \rightarrow W_1$ such that $p_{P_0(x)}|W_1 \circ \varphi_1 : D_{\sqrt{\varepsilon_0}} \simeq W_1 \rightarrow \mathcal{D}_{\varepsilon_0}(w_1)$ is given by $w = w_1 + \tilde{z}^2$.

Let us choose $\varepsilon < \varepsilon_0$ such that $\varphi_1(D_{\sqrt{\varepsilon}}) \subset \{ z \in \Pi \mid |z - \frac{n - \sqrt{n}}{2}| \leq \frac{n - \sqrt{n}}{2} \}$ and put $V_j = W_j \cap p^{-1}_{P_0(x)}(D_{\varepsilon}(w_1))$, where $D_{\varepsilon}(w_1)$ is the closure of $D_{\varepsilon}(w_1)$ in $\mathcal{D}_{\varepsilon_0}(w_1)$.

Consider a loop $\lambda_1$ around the point $w_1$ that begins and ends at the point $w_0$, $\lambda_1(t) = \begin{cases} 
\lambda_{1,1}(t) = 3(1 - n + \varepsilon)t & \text{for } t \in [0, 1/3], \\
\lambda_{1,2}(t) = 1 - n + \varepsilon e^{2\pi(3t-1)i/3} & \text{for } t \in [1/3, 2/3], \\
\lambda_{1,3}(t) = -3(1 - n + \varepsilon)(t - 1) & \text{for } t \in [2/3, 1]. 
\end{cases}$

**Proposition 1.** The geometric braid $b_1 = p^{-1}_{P_0(x)}(\lambda_1) = \{ b_{1,1}, \ldots, b_{1,n} \}$ is isotopic to a half-twist associated with the path $\Lambda_1 = \{ z = x + iy \in \Pi \mid x \in [0, \frac{n-\sqrt{n}}{2}], y = 0 \}$.

**Proof.** Each strand $b_{1,j}$ of the braid $b_1$ is the union of three paths, $b_{1,j} = \bigcup_{l=1}^{3} b_{1,j,l}$, where $b_{1,j,l} = \{(b_{1,j}(t), t) \mid p_{P_0(x)}(b_{1,j}(t)) = \lambda_1(t), t \in [(l-1)/3, l/3]\}$, $l = 1, 2, 3$.

Note that $b_{1,j,1}(t) = b_{1,j,3}(1 - t)$ for each $j > 2$ and $t \in [0, 1/3]$.

To describe the functions $b_{1,j}(t)$, $j = 1, 2$, consider the graph of the function $\tau = P_0(x)$, $x \in [0, \frac{n-\sqrt{n}}{2}]$, depicted in Fig. 2. The parts of this graph from $(0, 0)$ to $A$ and from $(\frac{n-\sqrt{n}}{2}, 0)$ to $B$ define two functions $x = f_1(\tau)$ and $x = f_2(\tau)$, $\tau \in [1 - n + \varepsilon, 0]$.

![Graph of function $\tau = x^n - nx$, $x \in [0, \frac{n-\sqrt{n}}{2}]$.](Fig.1)
Then
\[
b_{1,1}(t) = \begin{cases} 
  b_{1,1,1}(t) = f_1(3(1 - n + \varepsilon)t) & \text{if } t \in [0, 1/3], \\
  b_{1,1,2}(t) = \varphi_1(\sqrt{\varepsilon}e^{\pi(3t-1)n}) & \text{if } t \in [1/3, 2/3], \\
  b_{1,1,3}(t) = f_2(3(1 - n + \varepsilon)(1 - t)) & \text{if } t \in [2/3, 1]
\end{cases}
\]
and
\[
b_{1,2}(t) = \begin{cases} 
  b_{1,2,1}(t) = f_2(3(1 - n + \varepsilon)t) & \text{if } t \in [0, 1/3], \\
  b_{1,2,2}(t) = \varphi_1(\sqrt{\varepsilon}e^{3\pi t}) & \text{if } t \in [1/3, 2/3], \\
  b_{1,2,3}(t) = f_1(3(1 - n + \varepsilon)(1 - t)) & \text{if } t \in [2/3, 1].
\end{cases}
\]

Denote by \( U_1 = b_{1,1,1}(t) \cup b_{1,2,1} \cup V_1 \) and \( U_j = b_{1,j+1,1} \cup V_j \) for \( j = 2, \ldots, n - 1 \). It is easy to see that \( U_j \cap U_{j+1} = \emptyset \) if \( j \neq j_2 \).

Consider the map \( h : [1/3, 2/3] \times [0, 1] \to \mathbb{D}(w_1) \) given by the rule
\[
h(t, \tau) = 1 - n + \varepsilon(e^{2\pi(3t-1)n/3}(1 - \tau) + \tau).
\]
The map \( h \) defines the continuous family of braids \( \beta_\tau = \{(\beta_{1,\tau}(t), t), \ldots, (\beta_{n,\tau}(t), t)\} \), where \( \beta_{j,\tau}(t) \equiv b_{1,j}(t) \) if \( j = 1, 2 \), \( \beta_{j,\tau}(t) \equiv b_{1,j}(t) \) if \( t \notin (1/3, 2/3) \), and \( \beta_{j,\tau}(t) = \phi_{j-1}(h(t, \tau)) \) for \( \tau \in [1/3, 2/3] \) and \( j > 2 \). The family of braids \( \beta_\tau \) defines an isotopy in \( \Pi \times \mathbb{R} \) between the braids \( b_1 = \beta_0 \) and \( \beta_1 \), since \( U_j \cap U_{j+1} = \emptyset \) if \( j \neq j_2 \) and \( \beta_{j,\tau}(t) \in U_{j-1} \) for \( j > 2 \) and \( (t, \tau) \in [0, 1] \times [0, 1] \).

Note that \( \beta_{j,1}(t) = \beta_{j,1}(1 - t) \) for \( j > 2 \) and \( t \in [0, 1/2] \). Therefore the family of braids \( \widetilde{\beta}_\tau = \{(\widetilde{\beta}_{1,\tau}(t), t), \ldots, (\widetilde{\beta}_{n,\tau}(t), t)\}, \tau \in [0, 1] \), where \( \widetilde{\beta}_{1,\tau}(t) \equiv \beta_{j,1}(t) \) for \( j = 1, 2 \) and
\[
\widetilde{\beta}_{j,\tau}(t) = \begin{cases} 
  \beta_{j,\tau}(t) & \text{if } 2t \leq 1 - \tau \text{ or } 2t \geq 1 + \tau, \\
  \beta_{j,\tau}(1 - \tau) & \text{if } 1 - \tau \leq 2t \leq 1 + \tau
\end{cases}
\]
for \( j > 2 \).

The family \( \widetilde{\beta}_\tau \) defines an isotopy between the braids \( \beta_1 = \beta_0 \) and \( \beta_1 \). Therefore the braid \( b_1 \) is isotopic to the braid \( \beta_1 \equiv \{(b_{1,1}(t), t), (b_{1,2}(t), t), (b_{3}(t), \ldots, (q_n, t)\} \).

The points \( q_2, \ldots, q_n \) belong to the circle \( \partial \mathbb{D}(n \sqrt{3}) \) (0). Therefore if a positive \( \delta \ll 1 \), then \( q_j \notin D_{\frac{n-1}{2}}(\frac{\sqrt{3}n}{2}) \) for \( j > 2 \) and hence, "forgetting" about the strands \( (q_3, t), \ldots, (q_n, t) \) of the braid \( \beta_1 \), we obtain the braid \( \beta'_1 = \{(b_{1,1}(t), t), (b_{1,2}(t), t)\} \in \text{Br}_2[D_{\frac{n-1}{2}}(\frac{\sqrt{3}n}{2}), \{q_1, q_2\}] \). Applying Lemma \ref{lemm}, we obtain that the geometric braid \( \beta'_1 \) is isotopic in \( D_{\frac{n-1}{2}}(\frac{\sqrt{3}n}{2}) \times \mathbb{R} \) to the half twist associated with the path
\[
\Lambda_1 = \{t + (1 - t) \sqrt[4]{3} \}_{t \in [0, 1]} \subset D_{\frac{n-1}{2}}(\frac{\sqrt[4]{3}n}{2})
\]
connecting the points \( q_1 \) and \( q_2 \), since \( N_{\beta'_1} = 1 \). Therefore the braid \( \beta_1 \) (and hence, \( b_1 \)) is also isotopic to the half twist \( \beta_1 \) associated with the path \( \Lambda_1 \). □
2.4. The fundamental group $\pi_1(\mathbb{C} \setminus B_{P_0(z)}, 0)$ is generated by elements represented by the loops $\lambda_j = e^{\frac{2\pi(i-1)}{n-1}}\lambda_1$ and the subgroup $\Gamma_{P_0(z)} \subset Br_n[\Pi, K]$ is generated by the braids $b_j = p_{P_0(z)}^{-1}(\lambda_j)$, $j = 1, \ldots, n - 1$. Note that if we take new coordinates $\tilde{w} = e^{\frac{2\pi(i-1)}{n-1}}w$ in $\mathbb{C}$ and $\tilde{z} = e^{\frac{2\pi(i-1)}{n-1}}z$ in $\Pi$, then in the new coordinates, the morphism $p_{P_0(z)} : \Pi \to \mathbb{C}$ is given by the same formula $\tilde{w} = P_0(\tilde{z})$. Therefore the braids $b_j = e^{\frac{2\pi(i-1)}{n-1}}b_1$ are isotopic to the half-twists $\tilde{b}_j$ associated with the paths $\Lambda_j = e^{\frac{2\pi(i-1)}{n-1}}\Lambda_1$.

2.5. Let us choose a frame $\{l_1, \ldots, l_{n-1}\}$ of $(\Pi, K)$ as follows: $l_1 = \Lambda_1$ and

$$l_j = \left\{ e^{\frac{2\pi(j-1)(i-1)}{n-1}}\sqrt[n]{i} \right\}_{i \in [0, 1]}, \quad j = 2, \ldots, n - 1,$$

are the circular arcs of the circle $\partial D_n = \mathbb{C} \setminus \Pi(0)$ between the points $q_j$ and $q_{j+1}$. The half-twists $a_j$, associated with the paths $l_j$, generate the group $Br_n[\Pi, K]$.

It is easy to see (if to “straighten” the frame (see Fig. 2))

![Braid Diagram](image)

**Fig. 2**

that $\tilde{b}_1 = a_1$ and $\tilde{b}_j = (a_{j-1} \ldots a_1)^{-1}a_j(a_{j-1} \ldots a_1)$ for $j = 2, \ldots, n - 1$ as elements of $Br_n[\Pi, K]$. Therefore the elements $b_j$, $j = 1, \ldots, n - 1$, also generate the group $Br_n[\Pi, K]$.

### References

[1] J. Birman: *Braids, links and mapping class groups*, Annals of Mathematics Studies, Princeton University Press, 1974, ix+230 pp.

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