THE JACOBIAN OF A GRAPH AND GRAPH AUTOMORPHISMS

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Abstract. In the present paper we investigate the faithfulness of certain linear representations of groups of automorphisms of a graph $X$ in the group of symmetries of the Jacobian of $X$. As a consequence we show that if a 3-edge-connected graph $X$ admits a nonabelian semiregular group of automorphisms, then the Jacobian of $X$ cannot be cyclic. In particular, Cayley graphs of degree at least three arising from nonabelian groups have non-cyclic Jacobians. While the size of the Jacobian of $X$ is well-understood – it is equal to the number of spanning trees of $X$ – the combinatorial interpretation of the rank of Jacobian of a graph is unknown. Our paper presents a contribution in this direction.

1. Introduction

The Jacobian of a finite graph is an important algebraic invariant behaving nicely with respect to branched coverings of graphs [2]. It is a certain finite Abelian group associated with the graph; we will introduce its formal definition in Section 3. The notion of the Jacobian of a graph, also known as the Picard group, the critical group, the dollar, or the sandpile group, was independently introduced by many authors (see e.g. [1, 3, 4, 5, 11, 9]). It can be viewed as a discrete version of the Jacobian in the classical theory of Riemann surfaces. The Jacobian also admits natural interpretations in various areas of physics, coding theory, and financial mathematics. The fact that the size of the Jacobian of a connected graph $X$ is equal to the number of spanning trees is perhaps its most interesting property.

The present paper aims to investigate the relation between the symmetries of a graph $X$ and the symmetries of its Jacobian, $\text{Jac}(X)$. The main result is formulated in Theorem 4.3, where we prove that a semiregular group of symmetries of a connected, 3-edge-connected graph $X$ embeds into the automorphism group of $\text{Jac}(X)$. Corollary 4.5 gives a sufficient condition for $\text{Jac}(X)$ to be acyclic, in particular, Jacobians of non-trivial Cayley graphs based on nonabelian groups are not cyclic (Corollary 4.6).

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The main idea of the proof of Theorem 4.3 is to derive two inequalities which are contradictory. One of them is based on properties of the Jacobians with respect to branched coverings of graphs, for details see the paper [2] and Section 4. The other inequality follows from Lemma 2.6 dealing with counting of spanning trees. For technical reasons, we employ an extended model of a graph, developed in Section 2, allowing multiple edges, loops and even semiedges.

In [3, Proposition 4.23] it is proved that for an acyclic 2-connected graph $X$, the group $\text{Aut}(X)$ embeds into the group of symmetries of the homology group $H_1(X)$. On the other hand, there are infinite families of graphs such that $\text{Aut}(X)$ does not embed into $H_1(X)$. These families are completely determined in [7, Theorem 7]. Since $\text{Jac}(X)$ is a quotient of $H_1(X)$, it is reasonable to include requirements on the connectivity of $X$.

2. Graphs and graph coverings

In this paper, we allow graphs to have parallel edges, loops and semiedges. A simple undirected graph is defined in the standard way as a pair $X = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges. In this paper we prefer to define $X$ by means of a triple $(D; \sim, \lambda)$, where $D$ is a set of darts, $\sim$ is an equivalence relation on $D$ and $\lambda \in \text{Sym}(D)$ is an involutory permutation of $D$. An edge $e = \{u, v\}$ gives rise to two darts $uv$ and $vu$, and for every edge $e \in E$ the dart-reversing involution $\lambda$ swaps the two underlying darts of $e$. The equivalence classes of $\sim$ determine the sets of darts $D_v$ incident to the vertex $v$, therefore one can view them as vertices of the graph. The above-mentioned approach includes graphs with parallel edges, loops and possibly with semiedges. Isolated vertices are not allowed in our model.

Formally, a graph $X$ is a triple $(D; \sim, \lambda)$, where $D$ is a finite, non-empty set of darts, $\sim$ is an equivalence on $D$, and $\lambda \in \text{Sym}(D)$ is an involution. The equivalence classes of $\sim$ will be called the vertices of $X$, the set of all vertices will be denoted $V = V(X)$. The orbits of $\lambda$ will be called the edges of $X$. The projection $I: x \mapsto [x]_{\sim}$ from the set of darts onto the set of vertices will be called the incidence function. For simplicity, we often set $\lambda(x) = x^{-1}$, for $x \in D$. An edge $e = \{x, x^{-1}\}$ is a semiedge, if $x = x^{-1}$, the edge $e$ is a loop if $x^{-1} \neq x$, and $I(x^{-1}) = I(x)$. Edges that are neither semiedges, nor loops, will be called ordinary edges. Two ordinary edges $e = \{x, x^{-1}\}, f = \{y, y^{-1}\}$ will be called parallel, or multiple edges if $\{I(x), I(x^{-1})\} = \{I(y), I(y^{-1})\}$. A graph $X$ will be called simple, if every edge of $X$ is ordinary, and $X$ has no parallel edges. The degree of a vertex $v \in X$ is set to be $\deg(v) = |\{y: y \in D, v = I(y)\}|$. A walk $W$ is a sequence $x_1, x_2, \ldots, x_m$ such that for every $i \in \{1, \ldots, m - 1\}$ it holds that $I(x_i) = I(x_{i-1}^{-1})$. Graph $X = (D; \sim, \lambda)$ is connected if for any two darts $x, y \in D$ there exists a walk $W = x_1, x_2, \ldots, x_m$ such that $x = x_1$ and
graphs. Then the following statements are equivalent:

(i) the monodromy group is transitive on each fibre,
(ii) \( \text{CT}(\psi) \) is the centraliser of the monodromy action,
(iii) the group of covering transformations is semiregular both on vertices and on darts, and
(iv) \( |\text{CT}(\psi)| \leq n \leq |\text{Mon}(\psi)| \).

**Theorem 2.2.** Let \( \psi: X_1 \to X_2 \) be an \( n \)-fold covering of connected graphs. Then the following statements are equivalent:

(i) the monodromy group is regular on each fibre,
(ii) \( \text{CT}(\psi) \) is regular on each fibre,
(iii) \( \text{CT}(\psi) \cong \text{Mon}(\psi) \),
(iv) \(|\text{CT}(\psi)| = n = |\text{Mon}(\psi)|\).

A covering satisfying any of the equivalent conditions in Theorem 2.2 is called regular. Note that if \(\text{CT}(\psi) \cong \text{Mon}(\psi)\) is an Abelian group, then \(\text{CT}(\psi) = \text{Mon}(\psi)\).

Example 2.3 (Cayley graphs). Let \(G = \langle S \rangle\) be a group generated by a set \(S = S^{-1}\) of non-trivial elements. Set \(D = G \times S, \lambda(g, s) = (gs, s^{-1})\), and \((g, s) \sim (h, t)\) when \(g = h\). We have defined the Cayley graph \(\text{Cay}(G; S)\) with respect to the generating set \(S\). Let \(\tilde{B}(S) = (\tilde{S}; \tilde{\sim}, \tilde{\lambda})\), where \(\tilde{\sim}\) is the one-class equivalence and \(\tilde{\lambda}(s) = s^{-1}\). Clearly \(\tilde{B}(S)\) is a one-vertex graph. Moreover, the projection \((g, s) \mapsto s\) is a regular covering \(\text{Cay}(G; S) \to \tilde{B}(S)\), with the group of covering transformations isomorphic to \(G\). Since \(S\) contains no trivial element, \(\text{Cay}(G; S)\) has no loops. Further, since the action by left multiplication (on \(G\)) is semiregular, \(\text{Cay}(G; S)\) contains no semiedges. Observe that the degree of every vertex of a Cayley graph \(\text{Cay}(G; S)\) equals to \(|S|\).

We have seen that each Cayley graph is a regular cover over a one-vertex graph, possibly with semiedges. It is natural to ask what is the class of graphs regularly covering one-vertex graphs. To answer the question one has to generalise the definition of a Cayley graph as follows. A Cayley multigraph arising from a group \(G, \text{Cay}(G, M) = (D; \sim, \lambda), \) is given by a multiset \(M = M^{-1}\) of non-trivial elements of \(G\), where \(D = G \times M, \lambda(g, x) = (gx, x^{-1})\), and \((g, x) \sim (h, y)\) if and only if \(g = h\). In particular, there are no loops and no semiedges in \(\text{Cay}(G, M)\); however there may be parallel edges corresponding to the elements of \(M\) of multiplicity greater than one.

The following lemma can be understood as a natural generalisation of the famous Sabidussi theorem [14].

**Lemma 2.4.** Let \(X\) be a graph without loops and semiedges. Then \(X\) regularly covers a one-vertex graph with the group of covering transformations \(G\) if and only if \(X\) is a Cayley multigraph arising from \(G\).

The construction of a Cayley graph as a regular cover over a one-vertex graph is generalised as follows. Let \(Y = (D; \sim, \lambda)\) be a connected graph and \(T \subseteq Y\) be its spanning tree. Let \(G\) be a finite group. A \(T\)-reduced voltage assignment \(\xi: D \to G\) is a mapping satisfying the following properties:

- \(\xi(x^{-1}) = (\xi(x))^{-1}\),
- \(\langle \xi(x), x \in D(Y) \rangle = G\),
- if \(x \in D_T\), then \(\xi(x) = 1\).

The derived graph \(X = Y^\xi = (\tilde{D}, \tilde{\sim}, \tilde{\lambda})\) is defined by setting \(\tilde{D} = G \times D, (g, x) \tilde{\sim} (h, y)\) if and only if \(x \sim y\) and \(g = h\), \(\tilde{\lambda}(g, x) = (g\xi(x), \lambda(x))\).

The following statement well-known, for graphs without semiedges [8, Section 2.1], is proved in [13, Section 6].
Theorem 2.5. Let $X$, $Y$ be connected graphs, $T \subseteq Y$ be a spanning tree, and $\xi: D(Y) \rightarrow G$ be a $T$-reduced voltage assignment. Then the projection $\psi: (g, x) \mapsto x$ is a regular covering $Y^\xi \rightarrow Y$ with the group of covering transformation isomorphic to $G$.

Moreover, every regular covering $X \rightarrow Y$ with the group of covering transformations isomorphic to $G$ can be described by means of a $T$-reduced voltage assignment $\xi: D(Y) \rightarrow G$.

Let $X$ be a connected graph. We denote the number of its spanning trees by $\tau(X)$. The following lemma gives a lower bound for the number of spanning trees for a prime-fold cover over a connected graph. It will be used in the proof of Theorem 4.3.

Lemma 2.6. Let $p$ be a prime and let $X$ be a simple, connected, 2-edge-connected regular $p$-fold cover over a connected graph $Y$. Then $\tau(X) \geq p \cdot \tau(Y)$. If $X$ is 3-edge-connected, then $\tau(X) > p \cdot \tau(Y)$.

Proof. Let $\psi: X \rightarrow Y$ be a $p$-fold covering satisfying the assumptions. By Theorem 2.5 there is an associated $T$-reduced voltage assignment $\xi: D(Y) \rightarrow \mathbb{Z}_p$ such that the natural projection $\varphi: Y^\xi \rightarrow Y$ is equivalent to $\psi$. Hence, we may assume that $X = Y^\xi$ and $\psi = \varphi$. Denote by $F_T = \psi^{-1}(T)$. Clearly, $F_T$ is a spanning forest of $X$, consisting of $p$ isomorphic copies of $T$. If $p > 2$, then since $X$ is simple, there are no semiedges in $X$. Since $X$ is connected, there exists a cycle $C \subseteq Y$ (maybe a loop) which lifts to a cycle $\tilde{C}$ of length $p \cdot |C|$. Since $C$ lifts non-trivially, there exists a co-tree dart $x \in D(C)$ endowed with a non-trivial voltage. Let $e \in E(C) \setminus E(T)$ be the edge that contains $x$. Then $F_T + \psi^{-1}(e)$ is an unicyclic spanning subgraph of $X$. Deleting any edge $g \in f^{-1}(e)$ from $F_T + \psi^{-1}(e)$ we get a spanning tree $\tilde{T}$ of $X$. Removing different edges $g \in f^{-1}(e)$ we obtain a set $S(T, e)$ of $p$ spanning trees of $X$. We claim that for different spanning trees $T_1 \neq T_2$ of $Y$ we have $S(T_1, e_1) \cap S(T_2, e_2) = \emptyset$. If $\tilde{T}$ is a spanning tree constructed above, then it admits a unique decomposition of the edge set into $|V(Y)| - 1$ fibres over edges of $Y$, each of size $p$, and one incomplete fibre of size $p - 1$. By definition, the complete fibres determine edges of a spanning tree $T$ of $Y$, and the incomplete fibre is a subset of $\psi^{-1}(e)$, where $e$ is a co-tree edge with a non-trivial voltage in the $T$-reduced assignment. Since the edge-decomposition of $\tilde{T}$ is unique, the spanning tree $T$ of $Y$ is uniquely determined as well. It follows that the sets $S(T, e)$, where $T$ ranges through the all spanning trees of $Y$ are pairwise disjoint, and we conclude $\tau(X) \geq p \cdot \tau(Y)$.

Let $p = 2$. If there exists a simple cycle in $Y$ which lifts nontrivially, we apply the same argument as above to prove that $\tau(X) \geq 2$. It may happen that there is no cycle in $Y$ which lifts non-trivially. Since $X$ has no semiedges, every semiedge lifts to an ordinary edge. Since $X$ is connected and 2-edge-connected, $Y$ has at least two semiedges $s_1$ and $s_2$ which lift to ordinary edges $e_1$ and $e_2$, respectively. Given
spanning tree $T$ of $Y$, there are two associated spanning trees $F_T + e_1$ and $F_T + e_2$ of $X$. Since all spanning trees of $X$, constructed in this way are pairwise different, we have at least $2 \cdot \tau(Y)$ spanning trees of $X$.

Assume $X$ is 3-edge-connected. If $p > 2$, then given spanning tree $T$, there are at least two co-tree edges $e_1, e_2$ (maybe loops) assigned by a non-trivial voltage. If there was just one such edge, the $n_X$ would contain edge-cuts of size two. For a spanning tree $T$ of $Y$ we construct the sets $S(T,e_i)$, $i = 1, 2$, and $T$ ranges through all spanning trees of $Y$, are pairwise disjoint. Hence, $\tau(X) \geq 2p \tau(Y) > p \tau(Y)$.

If $p = 2$, and there are two co-tree ordinary edges or loops $e_1$ and $e_2$, endowed with a non-trivial voltage, we proceed as above. Otherwise, since $X$ is 3-edge-connected, either there are three semiedges $s_1, s_2$ and $s_3$ in $Y$, or there is a semiedge $s$ and a co-tree ordinary edge (or a loop) $e$ with a non-trivial voltage assignment. Since $X$ has no semiedges, every semiedge of $Y$ lifts to an ordinary edge. In the first case, the sets $S(T,s_i)$, $i = 1, 2, 3$, are singletons. In the second case, $S(T,e)$ is of cardinality two, and $S(T,s)$ contains one tree. Again, the sets are pairwise disjoint. It follows that $\tau(X) \geq 3 \cdot \tau(Y) > 2 \cdot \tau(Y)$. □

The following classical result on the edge connectivity of vertex-transitive graphs was proved independently by Mader and by Watkins.

**Theorem 2.7** (Mader [12], Watkins [15]). The edge-connectivity of a vertex-transitive graph is equal to its valency.

### 3. Jacobian of a Graph

The aim of this section is to introduce the Jacobian of a connected graph $X$ and to summarise some of its properties.

**Definition 3.1.** Let $X = (D; \sim, \lambda)$ be a connected graph. A mapping $\nu: D \rightarrow A$ into an Abelian group $A$ will be called an $A$-flow if the following conditions are satisfied:

- **(FLW):** $\nu(x^{-1}) = -\nu(x)$, for every $x \in D$ and
- **(GEN):** $A$ is generated by $\{\nu(x) \mid x \in D\}$.

If an $A$-flow $\nu$ satisfies

- **(KLV):** $\sum_{x \sim y} \nu(y) = 0$ for every dart $x \in D$ and
- **(KLC):** $\sum_{i=0}^{m-1} \nu(x_i) = 0$ for every oriented cycle $C = (x_0, x_1, \ldots, x_{m-1})$,

it will be called a harmonic flow.

Note that the sets of equations (KLC) and (KLV) are the well-known Kirchhoff’s laws for cycles and for vertices, respectively. In classical graph theory it is usually required that an $A$-flow satisfies (KLV) [6, Chap. 6]. However, for our purposes the aforementioned definition is more appropriate.
Definition 3.2. Given a connected graph $X$, the Jacobian $\text{Jac}(X)$ of $X$ is the maximal Abelian group $A$ such that $X$ admits a harmonic $A$-flow. A harmonic flow $D(X) \to \text{Jac}(X)$ will be called $J$-flow.

The Jacobian $\text{Jac}(X)$ is not maximal just in “numerical sense”. If $A$ is an Abelian group such that there exists a harmonic $A$-flow $D(X) \to A$, then $A$ is a quotient of $\text{Jac}(X)$.

Denote by $\mathcal{A}(S)$ the free Abelian group generated by $S$. Denote by $D^+$ a subset of $D$ containing from each edge $\{x, x^{-1}\}$ exactly one dart. In other words, $D^+$ is a transversal of the set of edges of $X$. Observe that by (FLW) a flow on $X$ is determined by its values on $D^+$. The following lemma gives a formal algebraic definition of a Jacobian.

Lemma 3.3. Let $X = (D; \sim, \lambda)$ be a connected graph. Then $\text{Jac}(X) \cong \mathcal{A}(D^+)/L$, where $L$ is the subgroup generated by the elements $\sum_{x \in C} x$ for every cycle $C$ of $X$ and by the elements $\sum_{y \sim z} y$, for every $z \in D$.

In what follows we list some properties of $\text{Jac}(X)$.

(P1) $|\text{Jac}(X)| = \tau(X)$, in particular, $\text{Jac}(X)$ is a finite Abelian group [1],
(P2) Any Abelian group $A$, in which one can define a harmonic $A$-flow $D(X) \to A$, is a quotient of $\text{Jac}(X)$,
(P3) $\text{Jac}(X)$ is a quotient of the homology group $H_1(X, \mathbb{Z})$,
(P4) The rank of $\text{Jac}(X)$ is at most $\nu(X) - 1$, where $\nu(X)$ is the number of vertices, and
(P5) Finding a canonical decomposition of $\text{Jac}(X)$ into cyclic factors is equivalent to computing the Smith normal form of the matrix of the homogeneous system of equations determined by $L$ over $\mathbb{Z}$ [11].

Let us discuss the properties (P1), (P2) and (P3) which will be used in the next section.

Property (P2) is a consequence of Lemma 3.3. Indeed, if $\nu: D(X) \to A$ is a harmonic $A$-flow, then by (GEN) the group $A$ can be viewed as a quotient of the free Abelian group $\mathcal{A}(D^+)$. Since $\nu$ is harmonic, (KLC) and (KLV) are satisfied.

Property (P3) is based on the following observation: Recall that $H_1(X, \mathbb{Z}) \cong \mathcal{A}(D_T^+)$, where $D_T^+ = D^+ - D(T)$ with respect to a spanning tree $T$ (co-tree darts). Since every tree has pendant darts (darts incident to vertices of degree one), if the values of a harmonic flow on $D_T^+$ are prescribed, then using (KLV) one can extend the flow to the pendant darts in $D^+ \setminus D_T^+$. Repeating the argument, the flow extends to all the darts in $D^+$. Hence, $\text{Jac}(X)$ is a quotient of $H_1(X, \mathbb{Z})$. Each semiedge contributes to the homology group of $X$ by a $\mathbb{Z}_2$-factor and these factors are independent [13]. In particular, the semiedges behave as independent cycles and therefore by (KLC) they do not contribute to $\text{Jac}(X)$. 
Property (P1) is a reformulation of the well-known Kirchhoff matrix-tree theorem. To see this, one has to relate our definition of the Jacobian to the standard one, see [2], using the concept of divisors. A divisor of $X$ is an integer-valued function defined on the set of vertices of $V$. Denote by $\text{Div}(X)$ the set of divisors of $X$. Observe that the sum of two divisors is again a divisor; therefore the divisors of $X$ form a free Abelian group of rank $v(X)$. A divisor $f: V(X) \to \mathbb{Z}$ determines a flow by setting $\nu_f(x) = f(I(x^{-1})) - f(I(x))$. The flow $\nu_f$ satisfies (KLC) for all closed walks of $X$. Vice-versa, any flow $\nu$ in an Abelian group $A$ satisfying (KLC) determines a function $g: V \to A$ such that $\nu = \nu_g$. Note that $g$ is not uniquely determined by $\nu$. To make this correspondence unique, we need to fix a value of $g$ at some vertex (or to introduce any other linear relation on the set of vertices). It follows that the set of flows in a group $A$ satisfying (KLC) is in correspondence with the set of functions $V(X) \to A$ satisfying an extra linear relation. A good question to ask is which functions on vertices correspond to flows satisfying both (KLC) and (KL V). With a little effort one can establish the following correspondence.

**Lemma 3.4.** Let $X = (D; \sim, \lambda)$ be a graph without loops and semiedges and $\Delta$ be its Laplacian. Let $f: V \to A$ be a function. Then a flow $\nu_f: D(X) \to A$ is a harmonic $A$-flow if and only if the function $f$ satisfies $\Delta f = \vec{0}$.

It follows that $\text{Jac}(X) \cong \text{Div}_0 / \Delta(\text{Div})$, where $\text{Div}_0$ is the set of divisors of degree 0. Recall that a divisor $f: V(X) \to A$ is of degree 0 if $\sum_{v \in V(X)} f(v) = 0$. With this correspondence in mind, the proof of (P1) can be found in [2, p. 769]. The equivalence of the various definitions of Jacobians, including the one used throughout the paper can be found in the appendix of [10].

### 4. Representation of $\text{Aut}(X)$ in $\text{Aut}(\text{Jac}(X))$

Assume we have a harmonic $J$-flow $\xi: D \to \text{Jac}(X)$. Then for every $f \in \text{Aut}(X)$ we can define a transformation $\xi \mapsto \xi_f$ by setting $\xi(x) = \xi(f(x))$. Since $f$ takes vertices onto vertices and cycles onto cycles, $\xi_f$ is a $J$-flow. Since $\xi$ generates $\text{Jac}(X)$, every element $a$ of the Jacobian can be written as $a = \sum_{x \in D} c_x \xi(x)$, where $c_x \in \mathbb{Z}$ for each $x \in D$. We define a mapping $\Theta: \text{Aut}(X) \to \text{Aut}(\text{Jac}(X))$ taking $f \mapsto f^*$, $f \in \text{Aut}(X)$, where $f^*$ is given by setting

$$f^*(a) = f^*(\sum_{x \in D} c_x \xi(x)) := \sum_{x \in D} c_x \xi(f(x)).$$

**Lemma 4.1.** With the above notation, $f^*$ is an automorphism of $\text{Jac}(X)$ and $\Theta: \text{Aut}(X) \to \text{Aut}(\text{Jac}(X))$ is a group homomorphism.
Proof. The restriction of $f^*$ onto the generating set $S = \{\xi(x): x \in D\}$ of $\text{Jac}(X)$ permutes the elements of $S$. Moreover, $f \in \text{Aut}(X)$ satisfies the following

- $\xi f(x^{-1}) = \xi((f(x))^{-1}) = -\xi f(x)$, for every $x \in D$,
- the set $D_v := \{x \in D: I(x) = v\}$ of darts based at $v \in V$ is mapped by $f$ onto $D_{f(v)}$, hence

$$f^* \left( \sum_{x \in D_v} \xi(x) \right) = \sum_{f(x) \in D_{f(v)}} \xi(f(x)) = \sum_{z \in D_{f(v)}} \xi(z), \text{ and}$$

- an oriented cycle $C$ is mapped to an oriented cycle $f(C)$, hence

$$f^* \left( \sum_{x \in C} \xi(x) \right) = \sum_{f(x) \in f(C)} \xi(f(x)) = \sum_{z \in f(C)} \xi(z).$$

It follows that a relation in the presentation of $\text{Jac}(X)$ is mapped onto a relation. Consequently, the mapping $\xi(x) \mapsto \xi(f(x))$ extends to a group automorphism.

Now we verify that $\Theta$ is a group homomorphism. Let $f, g \in \text{Aut}(X)$ be automorphisms. Then

$$\Theta(f \circ g)(y) = (f \circ g)^*(y) = \sum_{x \in D} c_x \xi((f \circ g)(x)) = \alpha^* \sum_{x \in D} c_x \xi(g(x))$$

$$= f^*(g^*(y)) = \Theta(f)(\Theta(g)(y)). \quad \square$$

The central problem we are interested in reads as follows: For which subgroups $G \leq \text{Aut}(X)$ is the image $\Theta(G)$ isomorphic to $G$? Equivalently, our aim is to investigate under which conditions the restriction $\Theta|_G$ is a monomorphism. By the first isomorphism theorem, a subgroup $G \leq \text{Aut}(X)$ embeds into $\text{Aut}(\text{Jac}(X))$ if and only if the kernel $\ker(\Theta|_G)$ is trivial. Observe that $f \in \ker(\Theta|_G)$ if $\xi(x) = \xi(f(x))$ for every $x \in D$, i.e., if and only if $\xi$ takes constant values on the dart-orbits of the group $\langle f \rangle$, generated by $f$. An automorphism $f$ will be called $\xi$-invariant if $\xi(x) = \xi(f(x))$, for every dart $x \in D$.

The following lemma deals with properties of $\xi$-invariant automorphisms. For a flow $\xi: D(X) \to A$ the local group $A^\xi \leq G$ is the subgroup of $A$ generated by $\{\xi(C): C \text{ an oriented cycle of } X\}$, where $\xi(C) = \sum_{i=0}^{m-1} \xi(x_i)$, for an oriented cycle $C = (x_0, x_1, \ldots, x_{m-1})$. If $\xi$ is a harmonic flow, then the local group is trivial by (KLC).

**Lemma 4.2.** Let $X$ be a connected graph and let $\xi: D \to A$ be a harmonic flow on $X$. Let $f \in \text{Aut}(X)$ be a $\xi$-invariant automorphism which is semiregular both on darts and vertices. Let $\varphi: X \to X/\langle f \rangle$ be the canonical projection $x \mapsto [x]_f$, for $x \in D(X)$. Then $\xi: D \to A$ defined by $\xi([x]_f) := \xi(x)$ is a flow satisfying (KLV) on $Y := X/\langle f \rangle$ and the local group $A^\xi$ is an epimorphic image of $\langle f \rangle$. 
Proof. Let the order of $f$ be $n$. By definition, $\xi$ is a flow satisfying (KLV). Since the projection $X \to Y$ is a regular covering, the vertices in the fibre over $v \in Y$ can be indexed as $v_0, v_1, v_2, \ldots, v_{n-1}$, where $v_i := f^i(v_0)$. By Theorem 2.2, the action of the covering transformation group $(f)$ coincides with the monodromy action of the group of $v$-based closed walks $\pi(v, Y)$ on fib$_v = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$. More precisely, there is an epimorphism $\Phi: \pi(v, Y) \to (f)$ with ker$(\Phi) = \varphi(\pi(v_0, X))$.

We show that the assignment: $\Psi: \Phi(W) \mapsto \xi(W), W \in \pi(v, Y)$, is an epimorphism from $(f)$ to the local group $A^\xi$.

First we show that $\Psi$ is well-defined. Suppose that $\Phi(W_1) = \Phi(W_2)$.

Then $1 = W_1 W_2^{-1}$ and $\Phi(W_1 W_2^{-1}) = 0$ implying that there are lifts $\tilde{W}_1$ and $\tilde{W}_2$ such that $\tilde{W}_1 \tilde{W}_2^{-1} \in \pi(v_0, X)$. Since $\xi$ is a harmonic flow, we have $\xi(W_1) = 0$, and consequently, $\xi(W_1 W_2^{-1}) = 0$. Thus, $\xi(W_1) = \xi(W_2)$. Hence, $\Psi(\Phi(W_1)) = \xi(W_1) = \xi(W_2) = \Psi(\Phi(W_2))$.

Secondly, we show that it is a homomorphism. Let $g_1, g_2 \in (f)$, where $g_1 = \Phi(W_1)$ and $g_2 = \Phi(W_2)$ for some $W_1, W_2 \in \pi(v, Y)$. Then

$$\Psi(g_1 g_2) = \Psi(\Phi(W_1) \Phi(W_2)) = \xi(W_1 W_2) = \Psi(\Phi(W_1 W_2)) = \tilde{\xi}(W_1) \tilde{\xi}(W_2) = \Psi(\Phi(W_1)) \Psi(\Phi(W_2)) = \Psi(g_1) \Psi(g_2).$$

By definition, $\Psi$ is an epimorphism. \hfill $\square$

Now we are ready to prove the main result of the paper establishing that any semiregular group of automorphisms of a graph acts faithfully on its Jacobian.

**Theorem 4.3.** Let $X$ be simple, connected, 3-edge-connected graph. Let $G \leq \text{Aut}(X)$ be a subgroup acting semiregularly both on darts and on vertices of $X$. Then the subgroup $\Theta(G)$ of $\text{Aut}(\text{Jac}(X))$ is isomorphic to $G$.

**Proof.** Let $\xi$ be a harmonic $J$-flow in $\text{Jac}(X)$. Our aim is to prove that the kernel of the homomorphism $\Theta|_G: G \to \text{Aut}(\text{Jac}(X))$ is trivial. Suppose to the contrary that there exists $f \in G$, $f \neq \text{id}$ such that $f \in \ker(\Theta|_G)$. It follows that $\xi(f(x)) = \xi(x)$ for every dart $x \in D(X)$, in other words $f$ is $\xi$-invariant. Clearly, taking a proper power of $f$, we obtain an element of $\ker(\Theta|_G)$ of prime order $p$. Hence, we may assume that $f$ has order $p$. Denote $\bar{x} = [x]_f$ the orbit of $x \in D(X)$ under $\langle f \rangle$. Since the action of $G$ is semiregular on darts and vertices, the natural projection $x \mapsto [x]_f = \bar{x}$ is a regular cyclic covering $\gamma: X \to Y, Y := X/\langle f \rangle$. By Lemma 4.2, $\gamma$ induces a flow $\bar{\xi}: D(Y) \to \text{Jac}(X)$ on $Y$ satisfying (KLV).

Moreover, the corresponding local group $(\text{Jac}(X)/\langle f \rangle)^\xi$ is cyclic of order dividing the order of $f$, that means it is of order $p$. The flow $\bar{\xi}$ may not satisfy (KLC). However, we can easily understand the “defect” of $\bar{\xi}$. For every oriented simple cycle $\bar{C}$ of $Y = X/\langle f \rangle$ we set $df(\bar{C}) := \sum_{\bar{x} \in D(\bar{C})} \bar{\xi}(\bar{x})$. The total defect of $\bar{\xi}$ is
the subgroup
\[ K := df(\bar{\xi}) := \langle df(\bar{C}) | \bar{C} \in \mathcal{C}(Y) \rangle, \]
where \( \mathcal{C}(Y) \) is the set of oriented cycles in \( Y \). Now we can formulate several claims about \( K \):

- \( K \leq \text{Jac}(X) \),
- the flow \( \xi^* \) in the factor group \( \text{Jac}(X)/K \), defined by \( \xi^*(\bar{x}) := \bar{\xi}(\bar{x})K \), satisfies both Kirchhoff laws, and
- \( K \) is a cyclic group whose order divides \( |f| \).

The first claim holds true, since every generator \( df(\bar{C}) \) of \( K \) belongs to \( \text{Jac}(X) \). For every cycle \( \bar{C} \), we have
\[ \sum_{\bar{x} \in \bar{C}} \xi^*(\bar{x}) = \sum_{\bar{x} \in \bar{C}} \bar{\xi}(\bar{x})K = df(\bar{C})K = K. \] Thus the second claim holds. By definition, \( K \) is a local group with respect to \( \bar{\xi} \). By Lemma 4.2, \( |K| \) divides \( |f| = p \). Since \( \xi^* \) is a harmonic flow in \( \text{Jac}(X)/K \), the group \( \text{Jac}(X)/K \) is an epimorphic image of \( \text{Jac}(Y) \). It follows that
\[ \tau(Y) = \tau(X(\langle f \rangle)) = |\text{Jac}(Y)| \geq |\text{Jac}(X)/K| = |\text{Jac}(X)|/|K| \geq \tau(X)/p. \] Thus, \( \tau(X) \leq p\tau(Y) \). However, Lemma 2.6 gives \( \tau(X) > p\tau(Y) \), a contradiction. \( \square \)

**Example 4.4.** Let \( X \) be a graph with one central vertex and two outer vertices with three parallel edges between the central vertex and each outer vertex. Then \( |\text{Aut}(X)| = 72 \) and \( \text{Jac}(X) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \). It follows that \( \text{Aut}(\text{Jac}(X)) \) is the general linear group \( \text{GL}_2(3) \) of order 48. Hence, \( \text{Aut}(X) \) does not embed into \( \text{Aut}(\text{Jac}(X)) \). The example can be easily generalised to an infinite family. Hence, the assumption of semiregularity of \( G \) in Theorem 4.3 is essential.

**Corollary 4.5.** If a connected, 3-edge-connected graph \( X \) admits a nonabelian semiregular group of automorphisms, then the rank of the Jacobian \( \text{Jac}(X) \) is at least 2.

**Proof.** Assume, to the contrary, that \( \text{Jac}(X) \) is cyclic. Then \( \text{Aut}(\text{Jac}(X)) \) is Abelian. However, by Theorem 4.3 the group \( \text{Aut}(\text{Jac}(X)) \) contains a nonabelian subgroup, a contradiction. \( \square \)

**Corollary 4.6.** Let \( X \) be a Cayley graph arising from a nonabelian group. If the degree of \( X \) is at least three, then \( \text{Jac}(X) \) is not cyclic.

**Proof.** By Theorem 2.7 the graph \( X \) is 3-edge-connected. Now the statement follows from Corollary 4.5. \( \square \)

It is natural to ask whether Theorem 4.3 can be generalised. In particular, the following open problem.

**Problem 1.** Is there a 3-connected graph \( X \) such that \( \text{Aut}(X) \) does not embed into \( \text{Aut}(\text{Jac}(X)) \)?

Corollaries 4.5 and 4.6 suggest that the structure of the automorphism group \( \text{Aut}(X) \) influences the rank of \( \text{Jac}(X) \). In general we have the following general problem.
Problem 2. Which properties of simple connected graphs are related to the rank of its Jacobian?

Computer-aided experiments with small graphs suggest that most graphs have cyclic Jacobians. The following problem is of interest.

Problem 3. Characterise simple 2-connected graphs with acyclic Jacobians.

A sufficient condition for the group $\text{Jac}(X)$ to be cyclic has been obtained in terms of the characteristic polynomial of the Laplacian matrix of $X$ in [11, Lemma 2.13].

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