LEMOINE POINT AND THE INScribed CONIC

LILIANA GABRIELA GHEORGHE

Abstract. The center of an inscribed conic which have a given perspector is the complement of its isotomic conjugate. We provide a synthetic proof, based on fine proprieties of Lemoine point.

1. Introduction

When a conic touches the sides of a triangle, the lines \(AA', BB', CC'\) joining the vertices with the tangency points meet at a point \(P\), the perspector. This is nothing but a manifestation of Brianchon’s theorem, and this concurrency can be validated by a tandem-use of Carnot and Ceva’s theorems.

There exist one (and only one) inscribed conic, whose perspector coincide with its center: the Steiner conic, the i-conic centered at \(G\). In any other case, the center of the inscribed conic is distinct from its perspector. An elementary geometric construction of the former from the latter is quite simple (see Figure 2 for "a proof without words"). By contrast, to (geometrically) obtain the perspector of a conic whose center is known, is not an easy task: it requires a careful use Brianchon’s theorem.

Keywords: Lemoine point, inscribed conic, perspector, isotomic conjugate, complement.

2010 Mathematics Subject Classification: 51A05, 51A30, 51M15.

Figure 1. In any triangle, there are infinitely many inscribed conics: for each point \(P\) there exists a unique conic that tangents the triangle’s sides precisely at the feet of cevians through \(P\), conic’s perspector. What about its center?
Figure 2. **Left** The line that join the midpoint of the chord of a conic, with its pole, pass through the center of the conic. **Right** If $A', B', C'$ are the tangency points of an inscribed conic into $\triangle ABC$, then the lines that join the vertices $A, B, C$, to the midpoints of $\triangle A'B'C'$ sides, meet at the center of the conic.

Figure 3. I) If $D$ is a point and $A_0, B_0, C_0$ the feet of cevians through $D$, the isotomic conjugate of $D$, here noted by $Iso$, is the intersection of cevians through the reflections of $A_0, B_0, C_0$ into the midpoints of the sides. II) The complement of $Iso$ is the point $\text{AIso}$ such that $G\text{AIso} = -1/2GIso$. III) The center of an inscribed conic is the complement of the isotomic conjugate of its perspector.

In this paper, we give a synthetic proof for the fact that the center of a conic is the complement of its isotomic conjugate (see Figure 3). This simple and rigid relation between the two, make them even: one can be straightforwardly obtained from the other.

We unveil the relation between the perspector and the center of a conic, by connecting them to Lemoine point and the orthic conic. The elementary path we adopt allows to give a fully elementary proof of the Grinberg-Yiu theorems, listed as Theorem 2.1 and Theorem 2.4 in [MM1].

**Main results.** We give a geometric proof for the following facts.
Theorem 1. The orthic conic is centered at Lemoine point.

Theorem 2. Lemoine point is the complement of the isotomic conjugate of the orthocenter.

Theorem 3. The center of an inscribed conic is the complement of the isotomic conjugate of its perspector.

Related work. Inscribed conics, in their classic setting appear in almost any classic book on conics; see [AZ], [C], [G], [GSO]. Inscribed conics in Poncelet pairs, are in [RG], [GRK], [G1]. Inscribed conic in trilinear coordinates, profoundly related both to isotomic or isogonal transform of lines are in the recent paper [A]. A sophisticated study of the relation between perspector, orthocenter and the center of a conic, that uses a mix of projective and affine techniques is in [MM1] and [MM2].

2. Under-exploited proprieties of Lemoine point

Lemoine geometry is nowadays a consolidate chapter in triangle’s geometry and the point itself is still of interest (see [L], [M] or the more recent [P] and the references therein). Curiously enough, the connection between geometric proprieties of Lemoine point and inscribed conics or hexagons is scarcely exploit. Here, we use several proprieties of the Lemoine point, that reconnect it to its natural environment: hexagons with parallel sides and Lemoine circle

For the convenience of the reader, we begin by recalling some useful observations on Lemoine point, all of them classic.

Lemma 1. In any triangle, the lines that join the vertices to the midpoints of its orthic meets at Lemoine’s point.

Proof. Let $A_m, B_m, and C_m$ the midpoints of the orthic triangle (see Figure 4). The sides of the orthic are anti-parallel to the sides of $\triangle ABC$. Since the locus of the midpoints of the anti-parallel at a triangle side is the symmedian that corresponds to the referred side, the lines $AA_m, BB_m, C_m$, are therefore $\triangle ABC$’s symmedians, meeting at Lemoine point. 

The next result is also classic.

Lemma 2. Let $BC$ be a chord of a conic centered in $O$ and $M$ be its midpoint. If the tangents to the conic at $B$ and $C$ meet at a point $A$, then $AM$ pass through the center of the conic.

Proof. Refer to Figure 2. Perform an affine transform that maps the conic into a circle. Affine transform preserves the midpoint. Therefore, the original claim reduces to the following (familiar) statement: if the tangents at the endpoints of a chord of a circle $B'C'$ meet at $A'$, then the line $A'M'$ that join $A'$ to the midpoint of $B'C'$ pass through the circle’s center.

Finally, Lemoine point and the altitudes are interlinked by a curious geometric relation; refer to Figure 5.

Lemma 3. In any triangle, the segment that join the midpoint of one side, to the midpoint of the corresponding altitude passes through Lemoine point.

For a proof refer to [H].
3. Proofs of the main results

3.1. Proof of Theorem 1.

*Proof.* The orthic conic tangents triangle’s sides at the feet of the altitudes. By Lemma 1, the lines that join the triangle’s vertices, to the midpoints of the orthic sides, meet at Lemoine point. On the other hand, by Lemma 2 the same point is (also) the center of the orthic conic, ending the proof. \(\square\)

3.2. Proof of Theorem 2.

*Proof.* Refer to Figure 5. Let \(H_m\) be the isotomic conjugate of the orthocenter \(H\). We shall to prove that

\[
\overrightarrow{KG} = \frac{1}{2}\overrightarrow{GH_m},
\]

where \(G\) is the barycenter. Let \(A'_m, B'_m, C'_m\) be the midpoints of the altitudes; by Lemma 3, the lines \(A_mA'_m, B_mB'_m\) and \(C_mC'_m\) meet at Lemoine point. Let \(A'_h\) the reflection of \(A_h\), in \(A_m\), the midpoint of \(BC\); similarly define \(B'_h, C'_h\). Then the lines \(AA'_h, BB'_h, CC'_h\) intercept in \(H_m\), the isotomic conjugate of the orthocenter \(H\).

Denote (temporarily) by \(G'\) the intersection of \(AA_m\) and \(A'_mA'_h\).

Note that

\[
\triangle A_m G' A'_m \sim \triangle AG' A'_h
\]

since by construction the angle in the common vertex \(G'\) is the same, and \(A_mA'_m\) is a mid-base in \(\triangle AA_hA'_h\); hence

\[
(1) \quad \overrightarrow{G'A_m} = \frac{1}{2}\overrightarrow{GA}, \quad \overrightarrow{A_mA'_m} = \frac{1}{2}\overrightarrow{A'A_h},
\]
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Figure 5. I) The segments $A_mA'_m, B_mB'_m, C_mC'_m$ joining the midpoints of an altitude with the midpoint of the corresponding side, concur at Lemoine point. II) $A'_h, B'_h, C'_h$ are the reflections of the feet of the altitudes $A_h, B_h, C_h$ into the midpoints; $H_m$, the isotomic conjugate of the orthocenter is the intersection of $AA'_h, BB'_h, CC'_h$. II) $K$ shown as the complement of $H_m$, the isotomic conjugate of the orthocenter.

thus, $G' = G$.

This proves that triangles $\triangle KGA_m \sim \triangle H_mA$, since they are inversely homothetic, of ratio $-\frac{1}{2}$ with respect to homothety center $G$. Hence

$$\overrightarrow{GK} = -\frac{1}{2}\overrightarrow{GH}_m,$$

showing that the point $K$ itself is the complement of $H_m$, the isotomic conjugate of $H$.

Now let us consider the "Lemoine hexagon" the hexagon $[A_1A_2B_1B_2C_1C_2]$ whose diagonals are the three congruent antiparallels at triangle’s sides (see Figure 6). It is inscribed into Lemoine circle, centered at $K$, and its vertices are the intersection of $\triangle ABC$’s sides, with the antiparalels through $K$ at its sides.

**Lemma 4.** The hexagon $[A_1A_2B_1B_2C_1C_2]$ is inscribed into Lemoine’s second circle and circumscribes the orthic conic of $\triangle ABC$.

**Proof.** Due to central symmetry, the opposite sides of $[A_1A_2B_1B_2C_1C_2]$ are parallel and congruent. Since the side $A_1A_2$ is tangent to the orthic conic, which is concentric with Lemoine circle, so is the side $B_2C_1$. Hence $[A_1A_2B_1B_2C_1C_2]$ circumscribes the orthic conic and tangents it at the feet of the altitudes of $\triangle ABC$. 

At this point, we learned two facts on orthic conic:

- it is centered at Lemoine point $K$;
- $K$ is (also) the complement of the isotomic conjugate of $H$. 
Figure 6. I) Lemoine circle (blue) is centered at the Lemoine point, $K$. The three diameters $A_1B_2, A_2C_1, B_1C_2$ are antiparallel with triangle’s sides. II) Hexagon $[A_1A_2B_1B_2C_1C_2]$ is inscribed into Lemoine circle and circumscribes the orthic conic of $\triangle ABC$.

Figure 7. $[A_1A_2B_1B_2C_1C_2], \triangle ABC$, and $\triangle A'B'C'$ share the same inscribed conic, centered at $\Omega$ (purple ellipse). The diagonals meet at the center of the conic $\Omega$. By an affine transform, the circumscribed conic of $[A_1A_2B_1B_2C_1C_2]$, (blue circumellipse) can be mapped into a circle, hence the point $\Omega$, is mapped into Lemoine point.

Now, we shall finally prove that (and why) the center of (any) conic with given perspector is the complement the isotomic conjugate of the perspector.

**Lemma 5.** There exists an affine transform that maps an inscribed ellipse $\gamma$, of perspector $P$, into the orthic conic.

**Proof.** In fact, if $\Omega$ is $\gamma$’s center, then reflection of $\triangle ABC$ in $\Omega$ led to a $\triangle A'B'C'$ which share with the former the same i-conic (see Figure 7). Due to central symmetry, the sides of these two triangles mutually intercept at six points, vertices of a hexagon with pairs of opposite parallel and congruent sides. This hexagon also shares with the two reflected triangles the
i-conic. Since this hexagon have opposite parallel sides, by Pascal theorem this hexagon has a circumellipse, \( \Gamma \); the central symmetry ensures that \( \Gamma \) and \( \gamma \) are concentric.

There exists an affine transform mapping hexagon’s \( [A_1A_2B_1B_2C_1C_2] \) circumellipse \( \Gamma \) into a circle, \( \tilde{\Gamma} \) and preserves the center of the conic. This maps the perspector \( P \) of the conic \( \gamma \), into the perspector of the i-conic \( \tilde{\gamma} \) of \( \triangle \tilde{A}\tilde{B}\tilde{C} \) the image of the referred triangle. Thus, by Lemma 4, the circle \( \tilde{\Gamma} \) is precisely \( \triangle \tilde{A}\tilde{B}\tilde{C} \)’s Lemoine circle, centered at its Lemoine’s point \( \tilde{K} \). The proof finishes once reminded that the perspector of the i-conic centered at Lemoine point \( \tilde{K} \) is the orthocenter. \( \square \)

**Proof of Theorem 3.**

*Proof.* Affine transforms preserves colinearity, as well as the proportions between points located on the same line. Therefore, affine transforms preserves the barycenter \( G \), isotomic conjugacy and complementarity, ending the proof. \( \square \)

As a corollary, a classic fact on Steiner conic.

**Corollary 1.** *There exist one (and only one) inscribed conic whose perspector coincides with its center: this is Steiner conic, centered at \( G \).*

*Proof.* The map that associates to a point \( P \), the complement of its isotomic conjugate has a unique fixed point: triangle’s barycenter. \( \square \)

Finally, note that Lemma 1, 2 and 3 led to a straight-forward synthetic proof for both Theorem 2.1 and Theorem 2.4, in [MM1], which are the ingredients of Grindberg-Yiu theorem (Theorem 2.7 in [MM1]).

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DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE PERNAMBUCO
RECIFE, (PE) BRASIL
E-mail address:
liliana@dmat.ufpe.br