Wrapping interactions at strong coupling – the giant magnon

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Abstract

We derive generalized Lüscher formulas for finite size corrections in a theory with a general dispersion relation. For the $AdS_5 \times S^5$ superstring these formulas encode leading wrapping interaction effects. We apply the generalized $\mu$-term formula to calculate finite size corrections to the dispersion relation of the giant magnon at strong coupling. The result exactly agrees with the classical string computation of Arutyunov, Frolov and Zamaklar. The agreement involved a Borel resummation of all even loop-orders of the BES/BHL dressing factor thus providing a strong consistency check for the choice of the dressing factor.

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1 Introduction

One of the most interesting and rapidly developing lines of investigation in recent years has been the study of integrable structures discovered on both sides \([1, 2, 3, 4, 5, 6]\) of the AdS/CFT correspondence \([7]\). The recent developments allow for an interpolation all the way from weak to strong coupling moving us closer to the complete knowledge of the anomalous dimensions of all operators on the \(\mathcal{N} = 4\) SYM gauge theory side, and the energy spectrum of the quantized superstring in \(AdS_5 \times S^5\).

A lot is currently known about the properties of the integrable worldsheet theory of the superstring in \(AdS_5 \times S^5\) on the plane. The full exact S-matrix is now believed to be known. Initially its structure in various subsectors of the theory has been uncovered \([8, 9]\), which culminated with \([10]\) where the \(su(2|2) \times su(2|2) \subset psu(2,2|4)\) symmetry has been exploited to determine the S-matrix up to a scalar function

\[
S(p_1, p_2) = S_0(p_1, p_2) \cdot \left[ \hat{S}_{su(2|2)}(p_1, p_2) \otimes \hat{S}_{su(2|2)}(p_1, p_2) \right]
\]

The function \(S_0(p_1, p_2)\) is related to the the dressing factor \(\sigma^2(p_1, p_2)\) which is 1 at weak coupling up to three loops and whose leading and subleading behaviour at strong coupling has been determined in \([11]\) and \([12]\) respectively.

The task of fixing the dressing factor at all values of the coupling has been concluded in \([13]\), where a specific solution of crossing constraints \([14]\) in \([15]\) was chosen based on arguments of transcendentality \([16, 17, 13]\). A very nontrivial cross-check of this choice was the 4-loop calculation of \([18]\).

The BES/BHL dressing factor satisfies all known constraints both at weak and at strong coupling, yet it is not known how unique is this choice. So it is also interesting to independently test as large part of this solution as possible. A 2-loop test has been performed in the near-flat space limit in \([19]\), while the considerations in \([20]\) on the location of double poles involve the full expression. As a byproduct, the present paper provides a stringent test sensitive to all even loop orders at strong coupling.

Despite our almost complete knowledge of the S matrix of the theory and the knowledge of the energies of states with large R-charge \(J\) (equivalently the anomalous dimensions of long operators) through the asymptotic Bethe ansatz, not much is known for operators for finite \(J\). On the gauge theory side, it is known \([21]\) that at roughly \(J\) loop order wrapping interactions appear, which are not expected to be captured by the asymptotic Bethe
ansatz. As one increases the coupling, and keeps $J$ fixed the problem becomes more and more severe. Recently there appeared some explicit calculations which showed the limitations of the asymptotic Bethe ansatz both at weak [22] and at strong [23, 24, 25] coupling. It would be very interesting to understand these phenomena on some general grounds.

In order to proceed one may try to take here either the gauge theory perspective using the spin chain language or the dual string theory point of view. It seems that in this case it is this dual string formulation which is more fruitful.

In [26] it has been suggested that wrapping interactions correspond to finite size corrections of the integrable worldsheet quantum field theory which arise due to virtual corrections such as a virtual particle going around the circumference of the cylinder. It has been argued that, as it happens in conventional relativistic integrable field theories, these finite size corrections are uniquely determined by the knowledge of the theory at infinite size.

In fact, in relativistic integrable field theories, leading formulas for mass (energy) finite size correction were derived long time ago by Lüscher [27]. The leading correction is the sum of two terms – the F-term

$$\Delta m_F(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-mL \cosh \theta} \cosh \theta \sum_b \left(S_{ab}^b(\theta + i\pi/2) - 1\right)$$  \hspace{1cm} (2)

and the $\mu$-term

$$\Delta m_{\mu}(L) = -\frac{\sqrt{3}}{2} m \sum_{b,c} M_{abc}(-i) \text{ res}_{\theta = 2\pi i/3} S_{ab}^b(\theta) \cdot e^{-\frac{\sqrt{3}}{2}mL}$$  \hspace{1cm} (3)

quoted here for a theory with a single mass scale [28] in 1+1 dimensions. $S_{ab}^b(\theta)$ is the (infinite volume) $S$-matrix element, and $M_{abc} = 1$ if $c$ is a bound state of $a$ and $b$ and zero otherwise. The F-term formula has the interpretation of a virtual particle going around the cylinder and interacting with the physical particle. This process is depicted graphically at the right in fig. 1. The $\mu$-term arises due to the splitting of the particle into a pair of virtual (but on-shell) particles which then recombine. This process is shown on the left in fig. 1.

Unfortunately, we cannot use directly the above formulae since the worldsheet theory is not relativistic. Indeed the elementary excitations (magnons
Figure 1: The diagram to the left (the $\mu$-term) shows a particle splitting in two virtual, on-shell particles, traveling around the cylinder and recombining. The diagram to the right (the F-term) shows a virtual particle going around the circumference of the cylinder.

[29]) obey the dispersion relation

$$E = \sqrt{1 + 8g^2 \sin^2 \frac{P}{2}}$$

(4)

where $g$ is related to the 't Hooft coupling as $g^2 = \frac{\lambda}{8\pi^2}$. In [26] considerations related to the Thermodynamic Bethe Ansatz approach to finite size corrections led to suggestions for the form of exponential terms (magnitudes) of the corrections for the $AdS_5 \times S^5$ worldsheet theory. However a complete formula together with the prefactor was still missing.

The aim of this paper is to provide a diagrammatic derivation for the leading finite size correction to the energy of a magnon and to explicitly evaluate the $\mu$-term in this case at strong coupling. This particular calculation is interesting since there exists a classical string computation for the same quantity [25] with which one can compare.

Finally let us note that the complexity of the dressing phase has prompted several groups to suggest that it can be obtained in some simpler setting by
eliminating other degrees of freedom/higher levels [30, 31, 32, 33, 34]. We hope that the formalism of finite size corrections may be a strong test on the proposed constructions as it is sensitive to all virtual particles of the theory.

The plan of this paper is as follows. In section 2 we will review recent results on finite size corrections both at weak and at strong coupling. In section 3 we will show how the postulated exponential terms reproduce the magnitudes of these corrections. In section 4 we present a diagrammatic derivation of the finite size corrections which generalize (2)-(3) to a theory with a quite general dispersion relation. We then apply the generalized formula for the $\mu$-term to the case of a magnon at strong coupling. We also consider the case of general light-cone gauges ($a$-gauges) for the worldsheet theory. We close the paper with conclusions and some appendices with more technical parts of the calculations.

2 Finite size corrections

As mentioned in the introduction, recently there appeared a number of explicit computations which showed the limitations of the asymptotic Bethe ansatz. In this section we would like to briefly review these results.

At weak coupling, wrapping interactions appear generically at the order $g^{2L}$, although for some operators that order may be higher\(^1\). Thus e.g. for the Konishi operator wrapping interactions appear at four loops. Obtaining explicit predictions seemed therefore to be nearly hopeless. However in [22] it was shown that results from asymptotic Bethe ansatz at four loops, i.e. exactly where we expect additional wrapping contributions, are in conflict with perturbative expectations from BFKL.

At strong coupling, in [23, 24] exponential corrections beyond asymptotic Bethe ansatz for spinning strings in the $su(2)$ and $sl(2)$ sector were determined. The magnitude of these corrections is

$$e^{-\frac{2sJ}{\sqrt{\lambda}}}$$

(5)

Finally, finite size corrections were found for the giant magnon dispersion relation at strong coupling from classical string solutions [25, 35]. They have

\(^1\)This happens when an operator of bigger length is a member of the same supersymmetry multiplet.
the form
\[ \delta E_{\text{string}} = -\frac{\sqrt{\lambda}}{\pi} \cdot \frac{4}{e^2} \cdot \sin^3 \frac{\alpha}{2} \cdot e^{-\frac{2\alpha J}{\sqrt{\lambda} \sin \frac{\alpha}{2}}} \equiv -g \cdot \frac{8\sqrt{2}}{e^2} \cdot \sin^3 \frac{\alpha}{2} \cdot e^{-\frac{1}{\sqrt{2}g \sin \frac{\alpha}{2}} J} \] (6)

where we gave the result both in terms of \( \lambda \) and in terms of \( g \) which we will use. The curious numerical prefactor includes \( e = \exp(1) \), the base of natural logarithm. This result is even more interesting when one compares it to finite size corrections for the magnon computed within the Hubbard model approach [36] which gives [36, 35]

\[ \delta E_{\text{Hubbard}} = -\frac{\pi}{\sqrt{\lambda}} \cdot \frac{2}{\sin \frac{\alpha}{2}} \cdot e^{-\frac{2\alpha J}{\sqrt{\lambda} \sin \frac{\alpha}{2}}} \] (7)

We see that the exponential term is identical in both expressions, while the prefactors differ not only in momentum dependence but also in the scaling with \( \lambda \). The main motivation of the present paper is to understand quantitatively the origin of these expressions and to show that they follow from a worldsheet quantum field theoretical picture of virtual corrections going around the circumference of the (worldsheet) cylinder as advocated in [26].

3 TBA motivated exponential terms

In [26] it was suggested that finite size corrections to energies of string states could be found from a Thermodynamic Bethe Ansatz reasoning following the route applied already with success to relativistic integrable field theories initially for the ground state [37] and later extended to excited states [38]. The TBA framework suggests a natural generalization [26] to the case of the worldsheet superstring theory which is not relativistic due to the nonstandard dispersion relation (4). Other approaches used in the relativistic context, such as NLIE [39, 40] or [41, 42] seem to be more difficult to adapt here.

The basic idea of TBA is to perform a spacetime interchange. In order to find the energy of a (ground) state for the theory on a circle of circumference \( J \), one considers the (euclidean) partition function

\[ Z = \text{tr} e^{-RH} \] (8)

with \( R \to \infty \). The same quantity may be interpreted as the partition function for the theory, with space and time interchanged, on a very big circle of
circumference $R$ at a temperature $1/J$. The advantage is that in this case the Bethe ansatz is exact (since $R \to \infty$) and one can use it to obtain explicit integral equations for the exact energies. This approach was later extended to excited states in [38].

Following these reasonings, [26] suggested that the magnitude of the exponential finite size corrections is

$$e^{-E_{TBA}J}$$

where $E_{TBA} = -ip$ and $p_{TBA} = -iE$. For the dispersion relation (4) this leads to

$$e^{-2J \text{arcsinh} \left( \frac{1}{\sqrt{8g^2}} \sqrt{1+p_{TBA}^2} \right)}$$

This term should be understood as an analog of $e^{-L \cosh \theta}$ in the F-term (2). Generalization of the $\mu$-term is more heuristic. The $\mu$-term comes from a particle with momentum $p$ splitting into on-shell constituents with momenta $p_c$ and $p - p_c$. The exponential term in\(^2\) (3) then can be rewritten as

$$e^{-J \text{Im} p_c}$$

In this paper we will independently derive the complete expressions for the generalized F- and $\mu$-terms including the preexponential prefactors. For the moment let us see how these heuristic expressions (10)-(11) fit the results reviewed in the previous section. Some of these results were already discussed in [26].

At weak coupling, when $g$ is small, the arcsinh behaves like $-\log g + \ldots$, therefore [26]

$$e^{-LE_{TBA}} \sim g^{2L} \ldots$$

exactly as expected for wrapping interactions.

At strong coupling, the argument of arcsinh is small so effectively we get [26]

$$e^{-LE_{TBA}} \sim e^{-\frac{L}{\sqrt{8g}}} \sqrt{1+p_{TBA}^2} \sim e^{-\frac{L}{\sqrt{8g}}} \equiv e^{-\frac{2\pi L}{\sqrt{\lambda}}}$$

which is just the exponential term in the correction for the spinning string in [23, 24].

\(^2\)And its nonzero $p$ generalization.
Finally let us consider (11) at strong coupling. We have to solve the on-shell condition\footnote{In fact the particle with momentum \( p_c \) could be the BPS bound state with energy \( \sqrt{4 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_c}{2}} \). This does not change the subsequent results as long as the particle with momentum \( p - p_c \) is just a magnon.} 
\[
\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_c}{2}} + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p - p_c}{2}}
\]  
(14)
Perturbatively we find 
\[
p_c = p + \frac{2\pi i}{\sqrt{\lambda} \sin \frac{p}{2}}
\]  
(15)
Plugging this back into (11) we obtain 
\[
ed^{-J \cdot \text{Im} p_c} = e^{-\frac{2\pi J}{\sqrt{\lambda} \sin \frac{p}{2}}}
\]  
(16)
which is exactly the exponential term appearing in the finite size corrections to the giant magnon dispersion relation. From the above derivation we see that this term is very generic and depends only on the dispersion relation (4). Thus it is not surprising that both the string result (6) and the Hubbard one (7) have the same exponential term. The aim of this paper is to understand also the prefactor in (6) which should be sensitive not only to the kinematics but also to the details of the dynamics of the worldsheet theory.

## 4 Diagrammatic derivation of the prefactor

In this section we will derive, using an adaptation of the diagrammatic methods of [27, 28], formulas for F- and \( \mu \)-term corrections for a field theory with a generic dispersion relation for elementary excitations. In general we have to make some assumptions about normalizations of states and the form of Green’s functions. These are quite plausible but we do not prove them. In the course of the calculation we will also assume that the same types of diagrams give dominant finite size corrections as in the relativistic case. We thus treat this derivation in a somewhat heuristic manner.

Our derivation follows closely the relativistic diagrammatic derivation as discussed in [28].
Green’s function

We assume that we are dealing with a 2D quantum field theory whose elementary excitations follow a dispersion relation

\[ E^2 = \varepsilon^2(p) \]  

(17)

In the relativistic case \( \varepsilon^2(p) = m^2 + p^2 \) while for the \( AdS_5 \times S^5 \) worldsheet theory \( \varepsilon^2(p) = 1 + 8g^2 \sin^2 \frac{p}{\ell} \). We note the similarity between the two dispersion relations - both of which are of the square-root type.

Let us consider a two point Euclidean Green’s function for the elementary excitations and its Fourier transform:

\[ \langle \phi_a(x)\phi_b(0) \rangle = \delta_{ab} \int \frac{d^2p}{(2\pi)^2} e^{ipx} G_a(p) \]  

(18)

We define the self-energy \( \Sigma \) through

\[ G_a(p)^{-1} = \varepsilon_E^2 + \varepsilon^2(p) - \Sigma \]  

(19)

where \( \varepsilon_E \) is the Euclidean energy. The Green’s function should have a pole along the mass shell manifold \( \varepsilon_E^2 + \varepsilon^2(p) = 0 \). Moreover we fix its residue by analogy with relativistic case

\[ \text{res} G_a(p) = 1 \quad \text{\( \varepsilon_E \) i.e.} \quad \text{res} = \frac{1}{2\varepsilon_E} \]  

(20)

This means that both \( \Sigma \) and its partial derivatives vanish on mass shell. In the following we will also need to know the appropriate residue w.r.t. the spatial momentum which is

\[ \text{res}_{p^1=p_*} G_a(p) = \frac{1}{\varepsilon^2(p_*)'} \]  

(21)

Let us now put the theory on a cylinder of circumference \( L \). Then the self energy will get modified and one can compute the \( L \)-dependent shift of the energy by examining the condition for the pole of the Green’s function

\[ \varepsilon_E^2 + \varepsilon^2(p) - \Sigma_L(p) = 0 \]  

(22)

with \( \varepsilon_E = i(\varepsilon(p) + \delta \varepsilon_L) \). In this way we get the key formula

\[ \delta \varepsilon_L = - \frac{1}{2\varepsilon(p)} \Sigma_L(p) \]  

(23)

We thus have to calculate the shift of the self energy due to finite size effects.
Figure 2: The graphs giving a leading finite size correction to the self energy: a) $I_{abc}$, b) $J_{abc}$, c) $K_{ab}$. The filled circles are the vertex functions $\Gamma$, empty circles represent the 2-point Green’s function. The letter $L$ represents the factor of $e^{-iq^{1}L}$ and the letters in italics label the type of particles.

**Finite size self energy correction $\Sigma_{L}(p)$**

When we put a theory on a cylinder of circumference $L$, the coordinate space cylinder Green’s function can be reconstructed from averaging the infinite volume Green’s function over translations $x \to x + nL$. In momentum space one will get just factors of $e^{inp^{1}L}$ with $n \in \mathbb{Z}$ which should be redistributed over all lines. We will now assume, following [27] in the relativistic case, that the leading finite size correction arises from a graph where only a *single* line has a factor $e^{iq^{1}L} + e^{-iq^{1}L}$. Since we are integrating over the loop variable $q$, we can substitute $e^{iq^{1}L} + e^{-iq^{1}L} \to 2e^{-iq^{1}L}$ under the integral, for that single line. All remaining parts of the graph are computed with infinite $L$ Feynman rules.

Following [28] we get three types of graphs contributing to the self-energy of a particle of type $a$:

$$
\Sigma_{L} = \frac{1}{2} \left( \sum_{bc} I_{abc} + \sum_{bc} J_{abc} + \sum_{b} K_{ab} \right)
$$

(24)
These graphs are shown in figure 2. The corresponding expressions are

\[ I_{abc} = \int \frac{d^2q}{(2\pi)^2} 2e^{-iq^1L}G_b(q)G_c(q + p)\Gamma_{abc}(-p, -q, p + q)\Gamma_{acb}(p, -p - q, q) \]

\[ J_{abc} = \int \frac{d^2q}{(2\pi)^2} 2e^{-iq^1L}G_b(q)\Gamma_{abc}(q, -q, 0)G_c(0)\Gamma_{aac}(-p, p, 0) \]

\[ K_{ab} = \int \frac{d^2q}{(2\pi)^2} 2e^{-iq^1L}G_b(q)\Gamma_{abb}(p, -p, q, -q) \]

(25)

where the \( \Gamma \)'s are the 3- and 4-point vertex functions.

The idea now is to shift the contour of integration over the momentum \( q^1 \) to imaginary values (\( \text{Im } q^1 = \kappa < 0 \) here). Then the contribution of the shifted contour will be exponentially supressed by \( e^{-\kappa L} \), which we will henceforth neglect. However, on the way, we get contributions from the poles of the propagators \( G_i(q) \). This forces the appropriate line to be on-shell i.e. \( q^1 = q_* \) where

\[ q_*^0 + \varepsilon(q_*)^2 = 0 \]

(26)

Let us work this out for the case of the \( \text{AdS}_5 \times S^5 \) superstring theory. Then using \( \varepsilon(p) = \sqrt{1 + 8g^2 \sin^2 p/2} \) we get

\[ q_* = -i2 \text{arcsinh} \frac{\sqrt{1 + q^2}}{\sqrt{8g^2}} \]

(27)

where we denoted \( q_E^0 \) by \( q \). Note that when we interpret the euclidean energy as momentum \( p_{TBA} \), this is exactly \((\pm i)E_{TBA}\) that we obtain from the space-time interchange principle advocated in [26]. Moreover the factor \( e^{-iq^1L} \) becomes

\[ e^{-iq^1L} \equiv e^{-iq_*L} = e^{-L \cdot E_{TBA}(q)} = e^{-L \cdot 2 \text{arcsinh} \frac{\sqrt{1 + q^2}}{\sqrt{8g^2}}} \]

(28)

which coincides with the formula (10) suggested in [26].

Before we proceed let us note that there is a subtlety associated with the graph \( I_{abc} \). There one can pick up two poles – one associated with \( G_b(q) \), the other one associated with \( G_c(q + p) \). We will denote the contribution of the first pole by \( I_{abc}^+ \), and of the second one by \( I_{abc}^- \). It is convenient to shift the integration variable in \( I_{abc} \) as \( q_{old} = q_{new} - p \). Then the value of the momentum at the pole, \( q_* \), will be the same for all graphs \( I_{abc}^\pm \), \( J_{abc} \), and \( K_{ab} \).

The shift of the integration variable \( q_{old} = q_{new} - p \) has another important consequence. Since \( q \) is Euclidean while \( p \) is Minkowskian, the contour of
integration over Euclidean energy got shifted into the complex plane. During this process, one may encounter a pole. The additional contribution of such a pole, evaluated by residues is exactly the $\mu$-term. We will come back to this point at the end of this section.

In addition, following [28], let us change $I_{abc}^{-} \to I_{acb}^{-}$ since we are summing over $b$ and $c$ anyway. At this stage, after taking the residue at (26) and using (21) we are left with

$$\Sigma_L = \frac{1}{2} \cdot \int \frac{dq^0}{2\pi} \frac{i}{\varepsilon^2(q_+)} \cdot e^{-|q_+|L} \cdot \text{Integrand} \quad (29)$$

where the Integrand is given by a sum of terms coming from $I_{abc}^+ + I_{acb}^- + J_{abc} + K_{ab}$:

$$\text{Integrand} = \sum_{bc} \left( \Gamma_{abc}(-p, -q, p + q)G_c(p + q)\Gamma_{acb}(p, -p - q, q) + \Gamma_{acb}(-p, p - q, q)G_c(p - q)\Gamma_{abc}(p, -q, q - p) + \Gamma_{aac}(p, -p, 0)G_c(0)\Gamma_{bbc}(q, -q, 0) + \Gamma_{aab}(p, -p, q, -q) \right) \quad (30)$$

with both $p$ and $q$ being on-shell. Note that in the second line, coming from $I_{acb}^-$, we made the appropriate shift of the momentum $q$. The crucial observation is now that the integrand is just an amputated connected 4-point forward Green’s function between on-shell particles (see e.g. [28]). We thus get

$$\Sigma_L = \int \frac{dq^0}{2\pi} \frac{i}{\varepsilon^2(q_+)} \cdot e^{-|q_+|L} \cdot \sum_b G_{aab}(-p, -q, p, q) \quad (31)$$

In order to obtain the final expression it remains to connect the forward Green’s function with the on-shell S-matrix of the theory.

**Link with the S-matrix**

The 4-point Green’s function appearing in (31) is essentially the S-matrix up to a different normalization convention which we will now discuss.

We define the scalar product between asymptotic states in analogy to the relativistic case as

$$\langle b(q)|a(p) \rangle = \delta_{ab} 2p^0 \cdot 2\pi \delta(p^1 - q^1) \quad (32)$$

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The (forward) S-matrix appears as the scalar product between 2-particle \textit{in} and \textit{out} states, with the normalization (32) factored out:

\[
\langle b'(q') a(p') | b(q) a(p) \rangle_{\text{in}} = 4p^0 q^0 (2\pi)^2 \delta(p'' - p') \delta(q'' - q') S_{ba}^{ab}(q, p) \equiv \delta_{f_i} S_{ba}^{ab}(q, p) \tag{33}
\]

On the other hand the forward Green’s function \(G_{abab}(-p, -q, p, q)\) which is identified with the forward elastic amplitude \(T_{ab}(p, q | p, q)\) is defined through

\[
\langle b'(q') a(p') | b(q) a(p) \rangle_{\text{in}} = \delta_{f_i} + i(2\pi)^2 \delta^{(2)}(p' + q' - p - q) T_{ab}(p, q | p, q) \tag{34}
\]

where \(\delta_{f_i}\) is defined as in (33). In order to express \(T\) through \(S\) we have to calculate the Jacobian between the delta functions appearing in (33) and (34):

\[
\delta^{(2)}(p' + q' - p - q) = \frac{1}{\varepsilon'(p^1) - \varepsilon'(q^1)} \cdot \delta(p'' - p') \delta(q'' - q') \tag{35}
\]

Putting these formulas together we get

\[
G_{abab}(-p, -q, p, q) = -4i \varepsilon(p) \varepsilon(q_*) (\varepsilon'(q_*) - \varepsilon'(p)) \left[ S_{ba}^{ab}(q, p) - 1 \right] \tag{36}
\]

which completes our derivation. For relativistic kinematics, the above formula reduces to the standard one

\[
G_{abab}(-p, -q, p, q) = -4i m_a m_b \sinh(\theta_q - \theta_p) \left[ S_{ba}^{ba}(\theta_q - \theta_p) - 1 \right] \tag{37}
\]

**Final formulas**

Inserting the relation (36) into (31) and (23), we thus arrive at our final formula for the generalized F-term finite size correction to the energy of a particle \(a\) with momentum \(p\):

\[
\delta \varepsilon^F_a = - \int^{\infty}_{-\infty} \frac{dq}{2\pi} \left( 1 - \frac{\varepsilon'(p)}{\varepsilon'__(q_*)} \right) \cdot e^{-iq_* L} \cdot \sum_b \left( S_{ba}^{ba}(q_*, p) - 1 \right) \tag{38}
\]

Here \(q\) is the original euclidean energy which plays the role of momentum \((p_{TBA})\) in the space-time interchanged theory, \(E = \varepsilon(p)\) is the dispersion relation and \(q_*\) is determined by the on-shell condition

\[
q^2 + \varepsilon^2(q_*) = 0 \tag{39}
\]
Let us now proceed to obtain the generalized form of the $\mu$-term contribution. In the course of derivation of the formula (38), we have moved the contour of integration of the euclidean momentum into the complex plane picking up a pole of the Green’s function putting the particle effectively on-shell and thus reducing the original double integral to a single integral over euclidean energy which plays the role of momentum in the space-time interchanged theory. However there was a subtlety that then in the graph $\Gamma_{acb}$ the mass-shell condition was different from the one in the remaining graphs. This could be compensated by a shift of the $q$ contour. In doing so one may encounter additional poles, the residues of which generate the $\mu$-term contribution. Thus we obtain the generalized expression for the $\mu$-term

$$
\delta \varepsilon^\mu_a = -i \left( 1 - \frac{\varepsilon'(p)}{\varepsilon'(\tilde{q}_*)} \right) \cdot e^{-E_{TBA}(\tilde{q}_*)L} \cdot \text{res}_{\tilde{q}^a = \tilde{q}} \sum_b S_{ba}^\mu(q_*, p) \tag{40}
$$

where $\tilde{q}$ is the Euclidean energy of the pole of the S-matrix, while $\tilde{q}_*$ is the corresponding momentum.

Before we apply the above formula to calculate leading finite size corrections to the giant magnon dispersion relation in the following section, let us illustrate how the generalized F-term formula (38) reduces to the ordinary one for the case of relativistic kinematics i.e. when $\varepsilon(p) = \sqrt{1 + p^2}$. It is rather difficult to keep track of all the choices of branches for the rapidity coordinates when performing the Wick rotations and shifts of the contours so we will not attempt to do it here. In addition there are various conventions for the S-matrix (see e.g. [20]). As in the case of the magnon calculation presented in the remaining part of the paper we will justify our choices a-posteriori by the final result. However it would be very interesting to rigorously fix all such ambiguities from first principles. We leave this extension for future work.

In the relativistic case one can reduce the formula (38) to the classical result in a simple manner when one retains a factor of $e^{iq^1 L}$ and not $e^{-iq^1 L}$. Then the on-shell condition is satisfied by

$$q \equiv -iq^0 = \sinh \theta \equiv \cosh \left( \theta + i \frac{\pi}{2} \right) \tag{41}$$
$$q_* \equiv q^1 = i \cosh \theta \equiv \sinh \left( \theta + i \frac{\pi}{2} \right) \tag{42}$$

In the relativistic case $\varepsilon'(p) = p/\varepsilon(p) = \tanh \theta_p$. Thus we obtain after some
manipulations

\[ \delta \varepsilon_{\text{relativistic}}^F = \frac{-1}{\cosh \theta_p} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh (\theta - \theta_p) e^{-L \cosh \theta} \sum_b \left( S_{ba} (\theta + i \frac{\pi}{2} - \theta_p) - 1 \right) \]

which is exactly the formula derived in [28]. When the particle is at rest, \( \theta_p = 0 \), we obtain the classical Luscher formula for the finite size mass shift (2).

5 The giant magnon finite \( J \) corrections

Let us now apply the preceding formalism to compute the leading finite size corrections to the giant magnon. From the discussion in section 3, we see that the exponential term of the classical correction is captured by the \( \mu \)-term. We will now calculate the prefactor of the exponential term using (40). This requires taking the residue at the BPS bound state [38] pole.

Kinematics

In order to apply the generalized formula (40) to the case at hand we have to calculate the various kinematical factors appearing in (40) at the position of the BPS bound state pole.

The energy and momentum of a particle (magnon) in the worldsheet theory of the superstring in \( \text{AdS}_5 \times S^5 \) is encoded in two (complex) variables \( x^\pm \) constrained by the equation

\[ x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i \frac{\sqrt{2}}{g} \]

The above equation defines a torus, so equivalently one may use a parametrization by a single complex parameter (generalized rapidity) in the complex plane being the universal covering space of the torus [14]. We will not, however, use this parametrization in the present paper.

The energy and momentum are then reconstructed from

\[ e^{i p} = \frac{x^+}{x^-}, \quad E = \sqrt{2} g i (x^- - x^+) - 1 \]

(45)
There exist inverse formulas expressing \( x^\pm \) in terms of the momentum, albeit with a branch cut ambiguity:

\[
x^\pm = \frac{1 + \sqrt{1 + 8g^2 \sin^2 \frac{p}{2}} - e^{\pm i \frac{p}{2}}}{\sqrt{8g^2 \sin \frac{p}{2}}} \quad (46)
\]

We will need to find explicitly the \( x^\pm_p \) parameters of the magnon up to \( \mathcal{O}(1/g^2) \) accuracy. We obtain from (46)

\[
x^+_p = e^{i \frac{p}{2}} \left( 1 + \frac{1}{\sqrt{8g^2 \sin \frac{p}{2}}} + \mathcal{O}\left( \frac{1}{g^2} \right) \right)
\]

\[
x^-_p = e^{-i \frac{p}{2}} \left( 1 + \frac{1}{\sqrt{8g^2 \sin \frac{p}{2}}} + \mathcal{O}\left( \frac{1}{g^2} \right) \right) \quad (47)
\]

These expressions indeed satisfy the constraint equation up to \( \mathcal{O}(1/g^2) \) terms. The corresponding BPS bound state is determined by the equation \( x^+_p = x^-_q \). Thus

\[
x^-_q = e^{i \frac{p}{2}} \left( 1 + \frac{1}{\sqrt{8g^2 \sin \frac{p}{2}}} + \mathcal{O}\left( \frac{1}{g^2} \right) \right) \quad (48)
\]

and the corresponding \( x^+_q \) can be found to be\(^4\)

\[
x^+_q = e^{i \frac{p}{2}} \left( 1 + \frac{3}{\sqrt{8g^2 \sin \frac{p}{2}}} + \mathcal{O}\left( \frac{1}{g^2} \right) \right) \quad (49)
\]

It will be useful to obtain explicit formulas for the momentum \( \tilde{q}_* \) corresponding to the BPS pole as a function of \( p \). By definition

\[
e^{i \tilde{q}_*} \equiv \frac{x^+_q}{x^-_q} \sim 1 + \frac{1}{\sqrt{2g^2 \sin \frac{p}{2}}} \quad (50)
\]

so

\[
\tilde{q}_* \sim \frac{-i}{\sqrt{2g \sin \frac{p}{2}}} \quad (51)
\]

Using the above expression we at once get the exponential term of the magnon finite size corrections, justifying from a different point of view the estimate (16) of section 3:

\[
e^{-i \tilde{q}_* L} \sim e^{-\frac{1}{\sqrt{2g \sin \frac{p}{2}}} L} \equiv e^{-\frac{2\pi \epsilon}{\sin \frac{p}{2}}} \quad (52)
\]

\(^4\)Here we pick a solution lying closest to the physical line.
where we identified $L$ with $J$ (we will come back to this point when discussing general uniform light-cone gauges at the end of the paper).

We are now ready to evaluate the final nontrivial missing kinematical factor in (40) namely $\varepsilon'(\tilde{q}_*)$. To this end we have

$$
\varepsilon'(\tilde{q}_*) = \frac{4g^2 \sin \frac{\tilde{q}_*}{2} \cos \frac{\tilde{q}_*}{2}}{\sqrt{1 + 8g^2 \sin^2 \frac{\tilde{q}_*}{2}}} \sim \frac{\sqrt{2}g}{\cos \frac{p}{2}} + O(1) \quad (53)
$$

Combining the above formula with the strong coupling expansion of the derivative of $\varepsilon(p)$ which can be calculated to be

$$
\varepsilon'(p) = \frac{4g^2 \sin \frac{p}{2} \cos \frac{p}{2}}{\sqrt{1 + 8g^2 \sin^2 \frac{p}{2}}} \sim g\sqrt{2} \cos \frac{p}{2} + O\left(\frac{1}{g}\right) \quad (54)
$$

we obtain finally

$$
1 - \frac{\varepsilon'(p)}{\varepsilon'(\tilde{q}_*)} \sim 1 - \cos^2 \frac{p}{2} = \sin^2 \frac{p}{2} \quad (55)
$$

Putting together the above kinematical factors, the resulting formula for the $\mu$-term becomes

$$
\delta \varepsilon^\mu = -i \cdot \sin^2 \frac{p}{2} \cdot e^{- \frac{1}{\sqrt{2}g^2 \sin \frac{\tilde{q}_*}{2}}} \cdot \text{res}_{q_*=\tilde{q}_*} \sum_b S^{ba}(q_*, p) \quad (56)
$$

We will now proceed to evaluate the residue of the forward S matrix at the BPS pole.

The S-matrix contribution

We will concentrate on the finite size corrections to the magnon in the $su(2)$ subsector. The S-matrix has the following form

$$
S(x_q, x_p) = S_0(x_q, x_p) \cdot \left[ \hat{S}(x_q, x_p) \otimes \hat{S}(x_q, x_p) \right] \quad (57)
$$

where $\hat{S}(x_q, x_p)$ is the $su(2|2)$ invariant S-matrix while the scalar factor $S_0(x_q, x_p)$ is expressed in terms of the dressing factor as

$$
S_0(x_q, x_p) = \frac{x_q^- - x_p^+}{x_q^+ - x_p^-} \frac{1 - \frac{1}{x_q^+ x_p^-}}{1 - \frac{1}{x_q^- x_p^+}} \cdot \sigma^2(x_q, x_p) \quad (58)
$$
We will leave the evaluation of the contribution of the dressing factor \( \sigma^2(x_q, x_p) \) to the next section, concentrating now on the remaining matrix structure.

The sum \( \sum_b \left[ \hat{S} \otimes \hat{S} \right]^{ba}_{q_*, p} \) carried out for \( a \) in the \( su(2) \) sector becomes

\[
(2a_1 + a_2 + 2a_6)^2 \tag{59}
\]

where the \( a_i \)'s are the coefficients introduced in [43], which parametrize the \( su(2|2) \) S-matrix. The ones relevant for us are

\[
a_1 = \frac{x_p^- - x_q^+}{x_p^+ - x_q^-} \eta_q \eta_p \tag{60}
\]

\[
a_2 = \frac{\eta_q}{\eta_p} \equiv \frac{x_q^- - x_q^+ (x_q^- - x_q^+)}{x_q^- + x_q^+ - x_q^+ x_p^+} \eta_q \eta_p \tag{61}
\]

\[
a_6 = \frac{x_p^+ - x_p^-}{x_q^- - x_p^-} \frac{\eta_q}{\eta_p} \tag{62}
\]

In the above expressions there are certain phase factors which depend on the choice of basis for multiparticle states. It will be crucial for us to use the choice corresponding to the string frame of [43] which is

\[
\frac{\eta_q}{\eta_q} = e^{i\frac{\pi}{2}} \equiv \sqrt{\frac{x_p^+}{x_p^-}} \tag{63}
\]

\[
\frac{\eta_p}{\eta_p} = e^{-i\frac{\pi}{4}} \equiv \sqrt{\frac{x_q^-}{x_q^+}} \tag{64}
\]

Note that the phase factors are nonlocal – it is this feature that enables one to have an untwisted Yang-Baxter equation. In the spin chain frame corresponding to the original derivation of the \( su(2|2) \) S-matrix in [10] these factors are equal to unity.

We thus have to calculate the residue

\[
\text{res}_{q = q} S_0(q_*, p) \cdot (2a_1 + a_2 + 2a_6)^2 \tag{65}
\]

at the BPS bound state pole. It is most convenient to factor out \( (x_q^- - x_p^+) \) in the denominator and use the de l’Hospital rule to calculate

\[
\lim_{q \to -q} (q - \bar{q}) \cdot \frac{1}{x_q^- - x_p^+} = \frac{1}{x_q^-} \tag{66}
\]
The derivative $x_q^-'$ can be evaluated to

$$
x_q^- = \frac{dx_q^-}{dq} = \frac{i}{4} e^{-i\frac{p}{2}} (1 - e^{ip})^2 
$$

(67)

at the BPS bound state pole. The derivation of this formula is summarized in Appendix A.

One can now insert the formulas (47-49) into the remaining part of the expression to obtain, in the strong coupling limit

$$
\text{res}_{q=q} \sum_b S_{ba}^{\text{BDS}}(q^*, p) = \frac{4\sqrt{2}i}{g \cdot \sin^3 \frac{p}{2}} \cdot e^{ip} \cdot \sigma^2(x_q, x_p) 
$$

(68)

The phase $e^{ip}$ is due to the 'string frame' phase factors. It will be crucial to cancel an analogous phase from the AFS part of the dressing factor. If one would just use the BDS S-matrix for the su(2) sector only, the result would be

$$
\text{res}_{q=q} S_{BDS}(q^*, p) = \frac{\sqrt{2}i}{g \cdot \sin^3 \frac{p}{2}} 
$$

(69)

The full expression for the $\mu$-term finite size correction is

$$
\delta \varepsilon^\mu = \frac{1}{g} \cdot \frac{4\sqrt{2}}{\sin \frac{p}{2}} e^{ip} \cdot \sigma^2(x_q, x_p) \cdot e^{-\frac{1}{\sqrt{2}g \cdot \sin \frac{p}{2}} L} 
$$

(70)

while in the BDS case we get

$$
\delta \varepsilon^{\mu}_{BDS} = \frac{1}{g} \cdot \frac{\sqrt{2}}{\sin \frac{p}{2}} \cdot e^{-\frac{1}{\sqrt{2}g \cdot \sin \frac{p}{2}} L} 
$$

(71)

It is interesting to compare the two expressions with the results for the finite size magnon corrections (6) and Hubbard model calculation (7). In both cases we obtain the correct exponential term which, as we saw, is expected to be very generic. For the prefactors the situation is different. In the case of the magnon, ignoring the dressing phase, we see that both the $g$ and $p$ dependence are different. Thus we may expect crucial effects from the dressing factor which we will evaluate in the following section. On the other hand both the $g$ and $p$ dependence match exactly with the result of Hubbard model. Only the overall numerical coefficient is different. This is not completely unexpected since we considered only the BDS magnon
subsector of the Hubbard model. It would be very interesting to perform a similar computation with the full effective S-matrix and see whether one can get an exact matching. We will not, however, consider this problem in the current paper.

In the following section we will complete the calculation of (70) by evaluating the dressing factor $\sigma^2(x_q, x_p)$ at the position of the BPS bound state pole.

The contribution of the dressing phase

It remains to evaluate the dressing phase factor at the position of the BPS bound state pole. It has the general structure

$$\sigma^2(x_q, x_p) = e^{2i(\chi(x_q^{-}, x_p^{-}) - \chi(x_q^{+}, x_p^{+}) + \chi(x_q^{+}, x_p^{+}))}$$

where $\chi(x, y) = \tilde{\chi}(x, y) - \tilde{\chi}(y, x)$ is antisymmetric and defined through the series expansion [15, 13]

$$\tilde{\chi}(x, y) = \sum_{n=0}^{\infty} \frac{1}{g_{BHL}} \tilde{\chi}^{(n)}(x, y)$$

where

$$g_{BHL} = \frac{g}{\sqrt{2}}$$

and

$$\tilde{\chi}^{(n)}(x, y) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} -c_{r,s}^{(n)} \frac{1}{(r-1)(s-1) x^{r-1} y^{s-1}}$$

with the coefficients $c_{r,s}^{(n)}$ given by the BHL/BES choice

$$c_{r,s}^{(n)} = \frac{(1 - (-1)^{r+s})\zeta(n)}{2(-2\pi)^n \Gamma(n-1)} \frac{\Gamma\left(\frac{1}{2}(s+r+n-3)\right) \Gamma\left(\frac{1}{2}(s-r+n+1)\right)}{\Gamma\left(\frac{1}{2}(s+r-n+3)\right) \Gamma\left(\frac{1}{2}(s-r-n+1)\right)}$$

for $n \geq 2$ and standard expressions for $n = 0, 1$. The term with $n = 0$ is the celebrated AFS phase [11], the function $\chi^{(0)}(x, y)$ can be resummed to

$$\chi^{(0)}(x, y) = -\frac{g}{\sqrt{2}} \left( \frac{1}{y} - \frac{1}{x} \right) \left( 1 - (1 - xy) \log \left( \frac{1}{1 - \frac{1}{xy}} \right) \right)$$

while $n = 1$ is the ‘1-loop’ HL correction [12]. We will first discuss these two terms separately and then calculate the contribution of $\chi^{(n)}$ with $n \geq 2$.

\footnote{Where we took into account that $\chi(x_q^{-}, x_p^{+}) = 0$ at the BPS pole.}
AFS phase contribution

We can now substitute the expressions for $x_q^\pm$ and $x_p^\pm$ into (77) to obtain

$$\sigma^2_{AFS}(x_q, x_p) = -\frac{g^2}{2} \cdot e^{-ip} \cdot \sin^4 \frac{p}{2}$$  \hspace{1cm} (78)

Let us note several salient features of this result. Firstly, there is a factor of $g^2$ which, when inserted into (70), gives the correct $g$ scaling of the magnon result. Secondly, the momentum dependence changes from Hubbard-like (7) to the one of the magnon (6). Thirdly, there is an overall complex phase, which is exactly canceled by the phase factor choice of the ‘string frame’ of [25] in the S-matrix.

At this stage, both the $g$ and $p$ dependence are exactly as in (6). The only difference is an overall numerical coefficient. We will find that the remaining part of the dressing factor will only give a numerical coefficient.

HL phase contribution

The HL part of the dressing factor can be also resummed into a closed form expression in terms of dilogarithm functions. Unfortunately it is quite complicated to use these formulas for the evaluation at the BPS bound state pole as these expressions have various branch cuts. One can fix the cuts by comparison with the result of a direct calculation of the double sum defining the HL phase factor. The latter calculation can in fact be done analytically in the strong coupling limit and we present it in Appendix B. The result for the contribution of the Hernandez-Lopez phase is thus

$$\sigma^2_{HL}(x_q, x_p) = \frac{1}{2}$$ \hspace{1cm} (79)

Putting together (78) and (79) we obtain at this stage

$$\delta\ve_a^\mu = -g \cdot \sqrt{2} \sigma^2_{n\geq 2}(x_q, x_p) \cdot \sin^3 \frac{p}{2} \cdot e^{-\frac{1}{\sqrt{2}} \frac{1}{g \sin \frac{p}{2}}}$$ \hspace{1cm} (80)

while the string result (6) is

$$\delta\ve_a^\mu = -g \cdot \sqrt{2} \frac{8}{e^2} \cdot \sin^3 \frac{p}{2} \cdot e^{-\frac{1}{\sqrt{2}} \frac{1}{g \sin \frac{p}{2}}}$$ \hspace{1cm} (81)

It remains to calculate the contribution of ‘higher-loop’ terms in the dressing factor $\sigma^2_{n\geq 2}(x_q, x_p)$. 
Contribution of $\chi^{(n)}$ with $n \geq 2$

Naively it may seem that $\chi^{(n)}$ with $n \geq 2$ should not contribute at strong coupling, however this turns out not to be the case. At strong coupling $x_q^\pm$ and $x_p^\pm$ are just at the edge of the radius of convergence of the appropriate power series and approach a singularity which compensates the inverse power of the coupling. Indeed, examining the resummed forms of $\tilde{\chi}^{(n)}(x, y)$ for $n < 12$ we see that there is a common structure appearing in the denominator:

$$\tilde{\chi}^{(n)}(x, y) = \frac{\cdots}{g^{n-1}(1-xy)^{n-1}}$$  \hspace{1cm} (82)

Therefore for $(x, y) = (x_q^-, x_p^-)$ and $(x, y) = (x_q^+, x_p^-)$ the large inverse power of the coupling constant is canceled by the second factor. We will thus obtain a nonvanishing result at strong coupling for $\chi^{(n)}(x, y)$ with $x, y$ of the form

$$x = e^{i\frac{p}{2}} \left( 1 + \frac{a}{g_{BHL} \sin \frac{k}{2}} \right)$$ \hspace{1cm} (83)

$$y = e^{-i\frac{p}{2}} \left( 1 + \frac{b}{g_{BHL} \sin \frac{k}{2}} \right)$$ \hspace{1cm} (84)

As shown in Appendix C, it turns out that the odd terms do not contribute $\chi^{(2n+1)}(x, y) = 0$ and the even ones give

$$\chi^{(2n)}(x, y) = i(-1)^n \frac{(2n-2)!}{2^{4n-2} \pi^{2n}} \zeta(2n) \frac{1}{(a + b)^{2n-1}} \equiv \chi^{(2n)}(a + b)$$ \hspace{1cm} (85)

The remaining part of the dressing factor contribution is therefore

$$\sigma_{n \geq 2} = \exp \left( 2i \sum_{n=1}^{\infty} \chi^{(2n)} \left( \frac{1}{2} \right) - \chi^{(2n)}(1) \right)$$ \hspace{1cm} (86)

The above series turns out to be only asymptotic and as it stands is divergent. However it can be resummed using Borel resummation which in this case amounts to performing the following procedure on the sums appearing in (86)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n \frac{1}{(2n-2)!} \cdot (2n-2)! = \sum_{n=1}^{\infty} a_n \frac{1}{(2n-2)!} \cdot \int_0^{\infty} t^{2n-2} e^{-t} dt =$$

$$= \int_0^{\infty} \left( \sum_{n=1}^{\infty} a_n \frac{1}{(2n-2)!} t^{2n-2} \right) \cdot e^{-t} dt$$ \hspace{1cm} (87)
A Borel resummed sum of the asymptotic series is then defined by the last expression:

\[
\sum_{\text{Borel}} \equiv \int_0^\infty \left( \sum_{n=1}^{\infty} \frac{a_n}{(2n-2)!} t^{2n-2} \right) e^{-t} \, dt
\]  \hspace{1cm} (88)

Performing the above procedure on the expression (86) we obtain

\[
\sigma_{n \geq 2}^2 = \exp \left( \int_0^\infty \frac{2 - \frac{1}{t} \sinh \frac{t}{2} - e^{-t}}{t^2} \, dt \right) = \frac{8}{e^2}
\]  \hspace{1cm} (89)

where we checked the last equality numerically using Maple up to 200 digit accuracy. We also proved this result analytically and we give the proof in Appendix D.

We thus obtain finally

\[
\sigma_{n \geq 2}^2 = \frac{8}{e^2}
\]  \hspace{1cm} (90)

which when inserted into (80) reproduces exactly the finite size correction to the magnon dispersion relation at strong coupling (6). Let us note that the agreement involves a contribution from all (even) loop orders in the BHL/BES dressing factor and is thus a very nontrivial test of the proposed expressions together with the Borel resummation procedure. It would be very interesting to obtain the same result directly from the convergent summation/integral formulas in [13, 44, 20].

**General \(a\)-gauges**

Up till now we considered finite size correction to the magnon dispersion relation evaluated in a gauge which is just one member \((a = 0)\) of the family of generalized light cone gauges where the density of

\[
P_+ = J + a(\Delta - J) \equiv J + aE
\]  \hspace{1cm} (91)

is kept fixed on the worldsheet. Changing \(a\), changes the behaviour of elementary degrees of freedom. However such physical properties as the spectrum of energies, after one incorporates the level matching condition, should be independent of \(a\). Despite that, it is also interesting to see if one can also directly understand the \(a\)-dependent properties of the elementary excitations.
In [25] finite size corrections to the magnon dispersion relation were evaluated in an arbitrary $a$-gauge with the result:

$$\delta \varepsilon = -g \frac{8\sqrt{2}}{e^2} \sin^3 \frac{p}{2} \cdot e^{-\frac{1}{\sqrt{2}g \sin \frac{p}{2}}} \cdot e^{-ap \cot \frac{p}{2}} \tag{92}$$

Let us see how this result is reproduced from the formalism of the present paper. The S-matrix in an arbitrary $a$-gauge is related to the one in $a = 0$ gauge, which we considered up to this point, by a simple scalar factor:

$$S_{a \neq 0}(x_q, x_p) = e^{-ia(\varepsilon_q - \varepsilon_p q^*)} \cdot S_{a=0}(x_q, x_p) \tag{93}$$

We will now evaluate this scalar factor. In the strong coupling limit $\varepsilon_p \sim \sqrt{8g^2 \sin \frac{p}{2}}$, while $\varepsilon_q$ can be easily found from

$$\varepsilon_q = \sqrt{2}gi(x_q^- - x_q^+) - 1 \sim -i \cot \frac{p}{2} \tag{94}$$

The momentum $q^*$ is

$$q^* = -i \log \frac{x_q^+}{x_q^-} \sim -i \frac{p}{\sqrt{2}g^2 \sin \frac{p}{2}} \tag{95}$$

We thus see that from the change in the S-matrix we will get an additional factor

$$e^{-ap \cot \frac{p}{2}} \cdot e^{2a} \tag{96}$$

The first factor is exactly the $a$ dependent correction to the magnon dispersion relation. The second factor will cancel with another source of $a$ dependence – namely the fact that the length of the string is no longer equal to $J$ but is rather $L = J + aE$. Therefore the exponential term will get modified to

$$e^{-\frac{1}{\sqrt{2}g \sin \frac{p}{2}}} L \equiv e^{-\frac{1}{\sqrt{2}g \sin \frac{p}{2}}(J+\varepsilon_p)} \sim e^{-\frac{1}{\sqrt{2}g \sin \frac{p}{2}}} J \cdot e^{-2a} \tag{97}$$

Putting the above terms together we recover exactly (92).

### 6 Conclusions

The aim of the present paper was to derive a generalization of Lüscher formulas for finite size corrections in relativistic quantum field theories which
could be applied to the worldsheet integrable theory of the $AdS_5 \times S^5$ superstring. In [26] it was suggested that such corrections, coming from virtual particles going around the worldsheet cylinder are the string counterparts of gauge theoretical wrapping interactions which go beyond the asymptotic Bethe ansatz. The string theory point of view is very fruitful here as such effects and corrections to the Bethe ansatz are inherent to the very structure of quantum field theory of which the worldsheet theory is an example. This is in contrast to the spin chain language where there seems to be no guiding principle for incorporating wrapping interaction effects.

We have shown that generic exponential terms describe well the magnitude of various types of finite size corrections both at weak and at strong coupling. The new result here is the very universal origin of the exponential term in the finite size correction to the magnon dispersion relation which just depends on the infinite volume dispersion relation.

We adapted the diagrammatic arguments of [27, 28] to derive the generalized formulas for the $F$- and $\mu$-terms, recovering in the course of derivation the space-time interchange appearing as the motivation of the exponential terms proposed in [26].

Finally we used the formula for the generalized $\mu$-term to evaluate finite size corrections to the giant magnon dispersion relation derived in [25, 35]. We found that the dressing factor had a crucial contribution and is responsible for the difference in the finite size structure of the Hubbard and string result. Moreover we found that we had to use the S-matrix in the so-called string frame [43] in order to cancel a complex phase from the AFS dressing phase.

We found that all even loop-orders of the BHL/BES dressing factor contribute to the resulting expression, which after Borel resummation exactly reproduces the result of [25] coming from a completely independent classical string calculation.

Finally we considered the same calculation in a general $\alpha$-gauge, again reproducing exactly the result of [25].

It would be very interesting to apply the formalism of the present paper to other cases, as well as to generalize it to multiparticle states. Ultimately, one would like to obtain exact results valid for any $J$ in analogy to similar treatments for certain relativistic integrable field theories.

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A Evaluation of $x_q^-'$

We have to compute the derivative of $x_q^-$ w.r.t. the Euclidean energy $q$. It turns out to be convenient to rewrite both quantities as functions of $p$ and evaluate

$$x_q^- = \frac{dx_q^-}{dp}$$  \hspace{1cm} (98)

At strong coupling we may use formula (48) and get

$$\frac{dx_q^-}{dp} \sim \frac{i}{2} e^{i\frac{p}{2}}$$  \hspace{1cm} (99)

The Euclidean energy $q$ is related to $\varepsilon(q_*)$ through $q = i\varepsilon(q_*)$, and the latter quantity has already been evaluated in terms of $p$ in (94). Taking the derivative

$$\frac{dq}{dp} = -\frac{1}{2\sin^2 \frac{p}{2}}$$  \hspace{1cm} (100)

we obtain finally

$$x_q^- = \frac{dx_q^-}{dq} = -ie^{i\frac{p}{2}} \sin^2 \frac{p}{2} = i\frac{e^{-i\frac{p}{2}}}{4}(1 - e^{ip})^2$$  \hspace{1cm} (101)

B Evaluation of $\sigma_{HL}^2(x_q, x_p)$

In this section we want to derive

$$\sigma_{HL}^2(x_p, x_q) = e^{2i(\chi^{(1)}(x_q^-; x_p^+) - \chi^{(1)}(x_q^+; x_p^-) + \chi^{(1)}(x_q^+; x_p^+))}$$  \hspace{1cm} (102)

analytically. We start with the expression for $c_{r,s}^{(1)}$

$$c_{r,s}^{(1)} = -\frac{(1 - (-1)^{r+s})}{\pi} \frac{(r - 1)(s - 1)}{(s + r - 2)(s - r)}$$  \hspace{1cm} (103)
which we substitute into (75) and introduce a new summation index $2k = s - r - 1$ to obtain

$$\tilde{\chi}^{(1)}(x, y) = \frac{1}{\pi} \sum_{r=2}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2k + 1} \frac{1}{y^{2k+1}} \frac{z^{r-1}}{r + k - 1}$$  \hspace{1cm} (104)$$

where $z = \frac{1}{xy}$.

Firstly let us focus on $\chi^{(1)}(x^+_q, x^+_p)$. At strong coupling we have $x^+_q = x^+_p = e^{ip/2}$ and

$$\chi^{(1)}(x^+_q, x^+_p) = \tilde{\chi}^{(1)}(x^+_q, x^+_p) - \tilde{\chi}^{(1)}(x^+_p, x^+_q) = 0$$ \hspace{1cm} (105)$$

For the two remaining terms $z$ is close to 1 at strong coupling. Indeed we have

$$z = 1 - a + b \frac{g_{BHL} \sin \frac{p}{2}}{2}$$ \hspace{1cm} (106)$$

where $a + b = 1$ for $(x, y) = (x^+_q, x^-_p)$ and $a + b = \frac{1}{2}$ for $(x, y) = (x^-_q, x^+_p)$. Performing the summation over $r$ in (104)

$$\tilde{\chi}^{(1)}(x, y) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k + 1} \frac{1}{y^{2k+1}} \frac{2F_1(1, k + 1; k + 2; z)}{k + 1} z$$ \hspace{1cm} (107)$$

and we can use relations for hypergeometric functions to change argument $z \to 1 - z$ and expand the result around $1 - z = 0$. Leaving only the leading term we have

$$2F_1(1, k + 1; k + 2; z) z \sim -(k + 1) \log(1 - z)$$ \hspace{1cm} (108)$$

Using in addition the relation

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k + 1} = \text{arctanh} \ x$$ \hspace{1cm} (109)$$

we get the final form of the function $\tilde{\chi}^{(1)}(x, y)$

$$\tilde{\chi}^{(1)}(x, y) = \frac{-1}{\pi} \log \left( \frac{a + b}{g_{BHL} \sin \frac{p}{2}} \right) \text{arctanh} \left( \frac{1}{y} \right)$$ \hspace{1cm} (110)$$
which gives after the antisymmetrization
\[ \chi^{(1)}(x, y) = -\frac{1}{\pi} \log\left( \frac{a + b}{g_{BHL} \sin \frac{\pi}{2}} \right) \left( \arctanh\left( \frac{1}{y} \right) - \arctanh\left( \frac{1}{x} \right) \right) \] (111)

It is easy to notice that at the strong coupling we have \( y = \frac{1}{x} \) in the cases which we consider. Using the well-known relation
\[ \arctanh w = \frac{1}{2} \log(1 + w) - \frac{1}{2} \log(1 - w) \] (112)
we derive
\[ \chi^{(1)}(x, y) = -\frac{1}{\pi} \log\left( \frac{a + b}{g_{BHL} \sin \frac{\pi}{2}} \right) \frac{i\pi}{2} \] (113)
and finally
\[ \chi^{(1)}(x_q^-, x_p^-) - \chi^{(1)}(x_q^+, x_p^-) = \frac{i}{2} \log(2) \] (114)
This gives the HL part of the dressing factor contribution at the BPS pole
\[ \sigma^2_{HL} = e^{2i\frac{\log(2)}{2}} = \frac{1}{2} \] (115)

C Evaluation of \( \chi^{(n)}(x, y) \) for \( n \geq 2 \)

In this section we want to derive \( \chi^{(n)}(x, y) \) for \( n \geq 2 \) at strong coupling. Let \( x, y \) be defined by the relations (57) and (58)
\[ x = e^{ip}(1 + \frac{a}{g_{BHL} \sin \frac{\pi}{2}}) \]
\[ y = e^{-ip}(1 + \frac{b}{g_{BHL} \sin \frac{\pi}{2}}) \]

Let us start with the even terms. Substituting the expression for \( c_{r,s}^{(2m)} \) into (49) and introducing a new summation index \( 2k = s - r - 1 \) we obtain
\[ \tilde{\chi}^{(2m)}(x, y) = \]
\[ = \frac{1}{g_{BHL}^{2m-1}} \sum_{r=2}^{\infty} \sum_{k=0}^{\infty} \frac{-2\zeta(2m)}{2(2\pi)^{2m-1} \Gamma(2m-1)} \frac{\Gamma(k + r + m - 1)}{\Gamma(k + r - m + 1)} \frac{\Gamma(k + m)}{\Gamma(k + m + 2)} \frac{1}{(xy)^r y^{2k+1}} \]
\[ = \frac{-1}{g_{BHL}^{2m-1}} \frac{\zeta(2m)}{(2\pi)^{2m-1}} \frac{1}{\Gamma(2m-1)} \frac{1}{y^2} W\left( \frac{1}{xy}; \frac{1}{y^2} \right) \]
where
\[ W(z, w) = \sum_{k=0}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k - m + 2)} w^k \sum_{r=2}^{\infty} \frac{\Gamma(k + r + m - 1)}{\Gamma(k + r - m + 1)} z^{r-1} \]  \tag{116}

The second sum can be expressed as a hypergeometric function
\[ \sum_{r=2}^{\infty} \frac{\Gamma(k + r + m - 1)}{\Gamma(k + r - m + 1)} z^{r-1} = \frac{\Gamma(k + m + 1)}{\Gamma(k - m + 3)} \,_2F_1(1, k + m + 1, k - m + 3, z) \]

At strong coupling, \( z = \frac{1}{x g} \) is close to 1. Indeed we have
\[ z = 1 - \frac{a + b}{g_{BHL} \sin \frac{p}{2}} \]  \tag{117}

Using relations for hypergeometric functions we can change argument \( z \to 1 - z \) and expand the result around \( 1 - z = 0 \). Leaving only the leading term we get
\[ _2F_1(1, k + m + 1, k - m + 3, z) \sim \frac{\Gamma(k - m + 3)\Gamma(2m - 1)}{\Gamma(k + m + 1)} (1 - z)^{1-2m} \]

Thus the sum over \( r \) is independent of \( k \) at strong coupling and equals
\[ \sum_{r=2}^{\infty} \frac{\Gamma(k + r + m - 1)}{\Gamma(k + r - m + 1)} z^{r-1} \sim \frac{\Gamma(2m - 1)}{(1 - z)^{2m-1}} \]  \tag{118}

At this stage we have
\[ W(z, w) = \frac{\Gamma(2m - 1)}{(1 - z)^{2m-1}} \sum_{k=0}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k - m + 2)} w^k \]  \tag{119}

Let us now focus on the remaining sum over \( k \)
\[ \sum_{k=0}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k - m + 2)} w^k = \sum_{k=0}^{\infty} (k + m - 1) \cdots (k - m + 2) w^{k-m+1} w^{m-1} \]
\[ = w^{m-1} \sum_{k=0}^{\infty} \frac{d^{2m-2}}{dw^{2m-2}} w^{k+1} = w^{m-1} \frac{d^{2m-2}}{dw^{2m-2}} \frac{w^{m-1}}{1 - w} \]
\[ = w^{m-1} \frac{\Gamma(2m - 1)}{(1 - w)^{2m-1}} \]
Putting these results together and plugging in the relation (117), we have

$$\tilde{\chi}^{(2m)}(x, y) = -\Gamma(2m - 1) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{y - \frac{1}{y}} 2^{m-1}$$ \quad (120)$$

The leading term of the \( y \) expansion at strong coupling is equal to \( e^{-ip} \) then

$$y - \frac{1}{y} = e^{-i\frac{p}{2}} - e^{i\frac{p}{2}} = -2i \sin(\frac{p}{2})$$ \quad (121)$$

Summing up

$$\tilde{\chi}^{(2m)}(x, y) = -\Gamma(2m - 1) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{2^{m-1}}$$ \quad (122)$$

Analogously we can derive

$$\tilde{\chi}^{(2m)}(y, x) = -\Gamma(2m - 1) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{2^{m-1}}$$ \quad (123)$$

where \( x = e^{ip} \) in the leading approximation. Then

$$\tilde{\chi}^{(2m)}(y, x) = -i \Gamma(2m - 1) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{2^{m-1}}$$ \quad (124)$$

and we can see that functions \( \tilde{\chi}^{(2m)} \) are antisymmetric. At the end we get the relation

$$\chi^{(2m)}(y, x) = i \Gamma(2m - 1) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{2^{m-1}}$$ \quad (125)$$

which is in perfect agreement with (85).

For the odd term analogical derivation can be done and the result is

$$\tilde{\chi}^{(2m+1)}(x, y) = -\Gamma(2m) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{y - \frac{1}{y}} 2^{m}$$

$$= -\Gamma(2m) \frac{\zeta(2m)}{2 \pi^{2m}} \frac{1}{(a + b)2m-1} \frac{1}{x - \frac{1}{x}} 2^{m} = \tilde{\chi}^{(2m+1)}(y, x)$$

thus the functions \( \tilde{\chi}^{(2m+1)} \) are symmetric and the contribution to the dressing factor from the odd terms vanishes

$$\chi^{(2m+1)}(x, y) = 0$$ \quad (126)$$
D Borel resummation of $\sigma^2_{n \geq 2}$

In this appendix we will perform a Borel resummation of the asymptotic series

$$\sigma^2_{n \geq 2} = \exp \left( 2i \sum_{n=1}^{\infty} (\chi^{(2n)}(\frac{1}{2}) - \chi^{(2n)}(1)) \right)$$  \hspace{1cm} (127)

where

$$\chi^{(2n)}(a + b) = i \Gamma(2n - 1) \frac{\zeta(2n)}{(i \pi)^{2n}} \frac{1}{(a + b)^{2n-1}} \frac{1}{2^{4n-2}}$$  \hspace{1cm} (128)

Firstly let us assume that $a + b = \frac{1}{2}$. Then

$$\sum_{n=1}^{\infty} \frac{(2n - 2)! \zeta(2n)}{2^{2n-1} (i \pi)^{2n}} = i \Re \left( \sum_{k=2}^{\infty} \frac{(k - 2)! \zeta(k)}{2k-1} \frac{1}{(i \pi)^k} \right) = \text{Borel}$$

$$= i \Re \left( \sum_{k=2}^{\infty} \int_{0}^{\infty} e^{-t t^{k-2}} \frac{\zeta(k)}{2k-1} \frac{1}{(i \pi)^k} dt \right)$$

Using the definition of the $\zeta$ function, $\zeta(k) = \sum_{m=1}^{\infty} \frac{1}{m^k}$ we can rewrite the last expression as

$$i \Re \left( \frac{-1}{2 \pi^2} \int_{0}^{\infty} e^{-t \left( \sum_{m=1}^{\infty} \frac{1}{m(m \frac{2 \pi t}{\pi})} \right)} dt \right) = i \Re \left( \frac{i}{\pi} \sum_{m=1}^{\infty} \frac{\Gamma(0, -2im \pi)}{k} \right).$$  \hspace{1cm} (129)

where $\Gamma(x, y)$ is the incomplete $\Gamma$ function.

Analogously for $a + b = 1$ we get

$$\sum_{n=1}^{\infty} \frac{(2n - 2)! \zeta(2n)}{2^{4n-2} (i \pi)^{2n}} = i \Re \left( \frac{i}{\pi} \sum_{m=1}^{\infty} \frac{\Gamma(0, -4im \pi)}{k} \right).$$  \hspace{1cm} (130)

Then

$$\sigma^2_{n \geq 2} = \exp \left( \frac{2}{\pi} \Re \left( -i \sum_{m=1}^{\infty} \frac{\Gamma(0, -2im \pi) - \Gamma(0, -4im \pi)}{m} \right) \right)$$

$$= \exp \left( \frac{2}{\pi} \Re \left( i \int_{1}^{2} \frac{\log(1 - e^{2\pi t})}{t} dt \right) \right) = \exp \left( \frac{2}{\pi} \Re \left( i \int_{1}^{2} \left( \frac{1}{t} \log(-16 \sin(\pi t)) dt - \pi + 3 \frac{\pi}{2} \log(2) \right) \right) \right)$$

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where \( \int_1^2 (4 \ln(-16 \sin(\pi t)))dt \) is real-valued and well-defined integral and thus does not contribute. In this way we obtain the final result

\[
\sigma_{n \geq 2}^2 = \exp(-2 + 3 \log(2)) = \frac{8}{e^2} \tag{131}
\]

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