This homotopy equivalence is a continuous version of Goodwillie’s isomorphism
\( K(R, J) \sim \rightarrow CC(R, J)[1] \) valid for any \( \mathbb{Q} \)-algebra \( R \) and a nilpotent two-sided ideal \( J \), see [G] or [Lo] 11.3. The construction follows closely that of Goodwillie, with
a characteristic \( p \) simplification (the difference between \( CC(A) \) and \( CC(A, I) \),
being of torsion, disappears) and the Malcev theory input replaced by a version of
Lazard’s theory.

0.2. Here is a geometric application (see 2.4): Let \( E \) be a \( p \)-adic field, \( O_E \) its ring
of integers, \( X \) be a proper \( O_E \)-scheme with smooth generic fiber \( X_E \), \( Y \subset X \) be
any closed subscheme whose support equals the closed fiber. Set \( X_i := X \otimes \mathbb{Z}/p^i \).
Then (0.1.1) yields a canonical identification of \( \mathbb{Q}_p \)-vector spaces

\[
\mathbb{Q} \otimes \lim_{n \to \infty} K_{n-1}^B(X_i, Y) \sim \sim \mathbb{Q} \otimes R \lim_{n \to \infty} K_{n-1}^B(X_i, Y) \sim \sim \oplus_a H_{\text{dR}}^{2a-n}(X_E)/F^a.
\]

Here \( K^B \) is the non-connective \( K \)-theory (see [TT] §6) and \( F^\cdot \) is the Hodge filtration
on the de Rham cohomology.

The subjects left untouched: (i) comparison of the composition of (0.2.1) and
the boundary map \( K^B_n(Y) \to \lim_{n \to \infty} K_{n-1}^B(X_i, Y) \) with the crystalline characteristic
class map for \( Y \) (here \( Y \) can be arbitrary singular, see [Cr] for the crystalline story
in that setting), and (ii) comparison with the \( p \)-adic regulators theory (see [NN]).

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0.3. This note comes from an attempt to understand the work of Bloch, Esnault, and Kerz [BEK] where an identification similar to (0.2.1) was constructed under extra assumptions on $X$. The method of [BEK] is different (and it remains to be checked that the two isomorphisms coincide). Its input is McCarthy’s identification of $K$-theory of an arbitrary ring relative to a nilpotent ideal with the relative topological cyclic homology, see [DGM]. One has then to identify the topological cyclic homology with truncated de Rham cohomology. Unlike the classical cyclic homology case, this presents a problem, and the passage taken in [BEK] - with étale cohomology of $X_E$ as a way station - is not easy.

It would be nice to find conditions on $(R,J)$, where $p$ is nilpotent in $R$ and $J$ nilpotent, that would imply $K_i(R,J) = H_{i-1}CC(R,J)$ for $i < p$, cf. [BEK] 8.5.

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1. Spectral preliminaries

Given a sequence $\ldots \rightarrow A_1 \rightarrow A_0$ of abelian groups, then $(\varprojlim A_i) \otimes \mathbb{Q}$ is not determined if we know merely $A_i \otimes \mathbb{Q}$, but we are in good shape if $A_i$ are known up to subgroups of torsion whose exponent is bounded by a constant independent of $i$. We will see that some natural maps known to be isomorphisms in rational homotopy theory are, in fact, invertible up to universally bounded denominators. Thus they remain to be isomorphisms in the rational homotopy theory of pro-spaces.

1.1. Spectra up to isogeny. (a) Let $\mathcal{C}$ be an additive category. We say that a nonzero integer $n$ kills an object $X$ of $\mathcal{C}$ if $n\cdot 1_X = 0$; $X$ is a bounded torsion object if it is killed by some $n$ as above. A map $f : X \rightarrow Y$ in $\mathcal{C}$ is an isogeny if there is $g : Y \rightarrow X$ such that $fg = n\cdot 1_Y$ and $gf = n\cdot 1_X$ for some nonzero integer $n$.

We denote by $\mathcal{C} \otimes \mathbb{Q}$ the category equipped with functor $\mathcal{C} \rightarrow \mathcal{C} \otimes \mathbb{Q}$, $X \mapsto X \otimes \mathbb{Q}$, that is bijective on objects and yields identification $\text{Hom}(X \otimes \mathbb{Q}, Y \otimes \mathbb{Q}) = \text{Hom}(X, Y) \otimes \mathbb{Q}$. We call $X \otimes \mathbb{Q}$ the object up to isogeny that corresponds to $X$. Notice that $f$ is an isogeny if and only if $f \otimes \mathbb{Q}$ is invertible, and $\mathcal{C} \otimes \mathbb{Q}$ is the localization of $\mathcal{C}$ with respect to isogenies; one has $X \otimes \mathbb{Q} = 0$, i.e., $X$ is isogenous to 0, if and only if $X$ is a bounded torsion object.

If $\mathcal{I}$ is a category and $X, Y : \mathcal{I} \rightarrow \mathcal{C}$ are functors, then a morphism of functors $f : X \rightarrow Y$ is said to be an isogeny if there is a morphism $g : G \rightarrow F$ such that $fg = n\cdot 1_Y$ and $gf = n\cdot 1_X$ for some nonzero integer $n$. (If $\mathcal{I}$ is essentially small, this means that $f$ is an isogeny in the category $\mathcal{C}^\mathcal{I}$.)

If $\mathcal{C}$ is abelian then $\mathcal{C} \otimes \mathbb{Q}$ is abelian as well, and it coincides with the quotient $\mathcal{C}_\mathbb{Q}$ of $\mathcal{C}$ modulo the Serre subcategory of bounded torsion objects. If $\mathcal{C}$ carries a tensor structure then bounded torsion objects form an ideal, hence $\mathcal{C}_\mathbb{Q}$ is a tensor category and $\mathcal{C} \rightarrow \mathcal{C}_\mathbb{Q}$ is a tensor functor. In particular, the category $\text{Ab}$ of abelian groups yields the tensor abelian category $\text{Ab}_\mathbb{Q}$.

Remarks. The functor $\text{Ab}_\mathbb{Q} \rightarrow \text{Vect}_\mathbb{Q}$, $X \mapsto X \otimes \mathbb{Q}$, is not an equivalence of categories (for the category $\text{Vect}_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces is the quotient of $\text{Ab}$ modulo the Serre subcategory of groups all of whose elements are torsion). One has
This functor is an equivalence of tensor triangulated categories.

\[ \text{Proposition.} \]

Let \( \mathcal{S}p\) be the stable \( \infty\)-category of spectra; by abuse of notation, we denote by \( \mathcal{S}p\) its homotopy category as well. This is a t-category with heart \( \mathcal{A}b\): the subcategory \( \mathcal{S}p_{\geq 0}\) is formed by connective spectra, the t-structure homology functor is the (stable) homotopy groups functor \( X \mapsto \pi(X)\), the t-structure truncations \( X \mapsto \tau_{\leq n}X\) are Postnikov’s truncations. The t-structure is non-degenerate: if all \( \pi_n(X) = 0\) then \( X = 0\). \( \mathcal{S}p\) is a symmetric tensor \( \infty\)-category with respect to the smash-product \( \wedge\), see [Lu2] 6.3.2. It is right t-exact and induces the usual tensor product on the heart \( \mathcal{A}b\); the unit object is the sphere spectrum \( S\).

Let \( \mathcal{D}(\mathcal{A}b)\) be the stable \( \infty\)-category of chain complexes of abelian groups, see [Lu2] 1.3.5.3, 8.1.2.8, 8.1.2.9; we denote again by \( \mathcal{D}(\mathcal{A}b)\) its homotopy category which is the derived category of \( \mathcal{A}b\) with its standard t-structure. \( \mathcal{D}(\mathcal{A}b)\) is a symmetric tensor \( \infty\)-category with the usual tensor product \( \otimes\).

The Eilenberg-MacLane functor \( \text{EM}: \mathcal{D}(\mathcal{A}b) \rightarrow \mathcal{S}p\) that identifies complexes with abelian spectra is naturally an \( \infty\)-category functor. \( \text{EM}\) is t-exact and equals \( \text{Id}_{\mathcal{A}b}\) on the heart. It sends rings to rings and modules to modules, so the 0th Eilenberg-Maclane spectrum \( \mathbb{Z}_{\mathcal{S}p} := \text{EM}(\mathbb{Z})\) is a unital ring spectrum and \( \text{EM}\) lifts to a functor \( \mathcal{D}(\mathcal{A}b) \rightarrow \mathbb{Z}_{\mathcal{S}p}\)-mod (:= the \( \infty\)-category of \( \mathbb{Z}_{\mathcal{S}p}\)-modules in \( \mathcal{S}p\)). The latter functor is an equivalence of categories, see [Lu2] 8.1.2.13.

Let \( \mathcal{S}p^- := \bigcup_n \mathcal{S}p_{\geq 0}[-n]\) be the stable \( \infty\)-subcategory of eventually connective spectra; this is a tensor subcategory of \( \mathcal{S}p\). A map \( X \rightarrow Y\) in \( \mathcal{S}p^-\) is called quasi-isogeny if all maps \( \pi_n(X) \rightarrow \pi_n(Y)\) are isogenies, or, equivalently, all maps \( \tau_{\leq n}X \rightarrow \tau_{\leq n}Y\) are isogenies in the homotopy category of spectra. Thus \( X \in \mathcal{S}p^-\) is quasi-isogenous to 0 if each \( \pi_n(X)\) is a bounded torsion group, or, equivalently, every \( \tau_{\leq n}X\) is a bounded torsion spectrum; such \( X\) form a thick subcategory.

**Lemma.** This subcategory is a \( \wedge\)-ideal in \( \mathcal{S}p^-\).

**Proof.** Suppose \( X, Y \in \mathcal{S}p^-\) and \( Y\) quasi-isogenous to 0; let us check that \( X \wedge Y\) is quasi-isogenous to 0. We can assume, replacing \( X, Y\) by their shifts, that \( X\) and \( Y\) are connective. Then \( \tau_{\leq n}(X \wedge Y) = \tau_{\leq n}(X \wedge \tau_{\leq n}Y)\), and we are done. \( \square\)

Let \( \mathcal{S}p^-_{\mathbb{Q}}\) be the corresponding Verdier quotient category of \( \mathcal{S}p^-\); this is a symmetric tensor t-category. We call its objects spectra up to quasi-isogeny, for a spectrum \( X\) we denote by \( X_{\mathbb{Q}}\) the corresponding spectrum up to quasi-isogeny.

Consider the derived categories \( \mathcal{D}^- (\mathcal{A}b), \mathcal{D}^- (\mathcal{A}b_{\mathbb{Q}})\) of bounded above complexes of abelian groups. Then \( \mathcal{D}^- (\mathcal{A}b_{\mathbb{Q}})\) is the quotient of \( \mathcal{D}^- (\mathcal{A}b)\) modulo the thick subcategory of complexes with bounded torsion homology. \( \text{EM}\) sends \( \mathcal{D}^- (\mathcal{A}b)\) to \( \mathcal{S}p^-\); passing to the quotients, we get a t-exact functor

\[ \mathcal{D}^- (\mathcal{A}b_{\mathbb{Q}}) \rightarrow \mathcal{S}p^-_{\mathbb{Q}}. \]

**Proposition.** This functor is an equivalence of tensor triangulated categories.

**Proof.** Let \( \mathbb{Z}_{\mathcal{S}p^-}\)-mod\( ^{\sim} \) be the category of eventually connective \( \mathbb{Z}_{\mathcal{S}p^-}\)-modules and \( \mathbb{Z}_{\mathcal{S}p^-}\)-mod\( ^{\sim}_{\mathbb{Q}}\) be its quotient modulo objects which are quasi-isogenous to 0. The
forgetful functor \( Z_{S^p}-\mathrm{mod}^- \to S^p^- \) admits left adjoint \( X \mapsto Z_{S^p} \wedge X \) which is a tensor functor. The adjoint functors \( Z_{S^p}-\mathrm{mod}^- \cong S^p^- \) yield, by passing to the quotients, the adjoint functors \( Z_{S^p}-\mathrm{mod}^- \cong S_{pQ}^- \). By the above, it is enough to show that the latter functors are mutually inverse, i.e., that for \( X \in S^p^- \), \( Y \in Z_{S^p}-\mathrm{mod}^- \) the adjunction maps \( a_X : X \to Z_{S^p} \wedge X \) and \( a_Y : Z_{S^p} \wedge Y \to Y \) are quasi-isogenies.

One has \( a_X = a_S \wedge \text{id}_X \) where \( a_S : S \to Z_{S^p} \) is the unit map. Since \( \pi_0(a_S) = \text{id}_Z \), the groups \( \pi_i(\text{Cone}(a_S)) \) (the stable homotopy groups of spheres) are all finite, so \( \text{Cone}(a_S) \), hence \( \text{Cone}(a_X) = \text{Cone}(a_S) \wedge X \), is quasi-isogenous to 0, i.e., \( a_X \) is a quasi-isogeny. Now \( a_Y^\vee : Z_{S^p} \wedge Y \to Y \) is a quasi-isogeny since \( a_Y^\vee a_Y = \text{id}_Y \). □

**Remark.** Let \( c_n \) be an integer that kills \( \tau_{\leq n} \text{Cone}(a_S) \) (e.g. the product of exponents of the first \( n \) stable homotopy groups of spheres). Then for every \( X \) such that \( \tau_{\leq n} X = 0 \), \( c_n \) kills \( \tau_{\leq n} \text{Cone}(a_X) = \tau_{\leq n}((\tau_{\leq n} \text{Cone}(a_S)) \wedge X) \). Therefore \( \pi_n(a_X) : \pi_n X \to \pi_n(Z_{S^p} \wedge X) \) is an isogeny of \( \mathcal{A}b \)-valued functors on (every shift of) the category of connective spectra (see 1.1(a)).

(c) We will need an \( \mathcal{A}b_Q \)-refinement of (a corollary of) the Milnor-Moore theorem:

For a simplicial set (often referred to as topological space below) \( P \) we denote by \( C(P,Z) \) the chain complex of \( P \) and by \( C(P,Z) \) the reduced chain complex which is the kernel of the augmentation map \( C(P,Z) \to Z \), so for \( p_0 \in P \) one has \( C(P,Z) \to C(P,\{p_0\};Z) := \text{the relative chain complex} \). Suppose \( P \) is connected. We say that \( a \in H_n(C(P,Z)) \) is primitive if \( \Delta_n(a) \in H_n(C(P \times P,Z) = H_n(C(P,Z)^\otimes 2) \) equals \( 1 \otimes a + a \otimes 1 \), where 1 is the generator of \( H_0(C(P,Z) \) and \( \Delta : P \mapsto P \times P \) is the diagonal map. We denote by \( \text{Prim}_n C(P,Z) \) the subgroup of primitive elements.

**Exercises.** (i) Show that the projections \( P \times P' \to P, P' \) yield isomorphisms \( \text{Prim}_n C(P \times P',Z) \cong \text{Prim}_n C(P,Z) \oplus \text{Prim}_n C(P',Z) \).

(ii) If \( \pi_i(P) = 0 \) for \( i > 1 \) and \( \pi_1(P) \) is abelian then \( \text{Prim}_n C(P,Z) = 0 \) for \( n > 1 \).

Denote by \( \text{Top}^\circ \) the category of pointed topological spaces. We have the adjoint infinite suspension and infinite loop functors \( S^\infty : \text{Top}^\circ \cong S^p^- : \Omega^\infty \). For \( X \in S^p^- \) and \( P = (P,p_0) \in \text{Top}^\circ \) let \( b_X : S^\infty \Omega^\infty X \to X \) and \( c_P : P \to \Omega^\infty S^\infty P \) be the adjunction maps, so one has \( \Omega^\infty(b_X)c_{\Omega^\infty X} = \text{id}_{\Omega^\infty X} \) and \( b_{S^p \Omega^\infty}c_{S^p \Omega^\infty} \) \( = \text{id}_{S^p \Omega^\infty} \). Let \( h_P \) be the composition \( P \to \Omega^\infty S^\infty P \to \Omega^\infty(Z_{S^p} \wedge S^\infty P) = \Omega^\infty \text{EM}(C(P,\{p_0\};Z)) \), the first arrow is \( c_P \), the second arrow is \( \Omega^\infty(a_{S^p}) \) (see the proof of the proposition in 1.1(b)). Then \( \pi_n(h_P) : \pi_n(P,p_0) \to H_n(C(P,\{p_0\};Z) = H_n(C(P,Z), n \geq 1, is the classical Hurewicz map; its image lies in \( \text{Prim}_n C(P,Z) \).

**Theorem.** For every connected spectrum \( X \) the Hurewicz maps \( \pi_n(h_{\Omega^\infty X}) : \pi_n(X) = \pi_n(\Omega^\infty X,0) \to \text{Prim}_n C(\Omega^\infty X,Z) \) are isogenies. Moreover, their kernel and cokernel are killed by nonzero integers that depend only on \( n \) (and not on \( X \)), i.e., \( \pi_n(h_{\Omega^\infty}) \) are isogenies of \( \mathcal{A}b \)-valued functors on the category of connective spectra (see 1.1(a)).

**Proof.** (i) The functor \( X \mapsto \text{Prim}_n C(\Omega^\infty X,Z) \) is additive: To see this, we need to check that for any map of spectra \( f,g : X \to Y \) and \( \alpha \in \text{Prim}_n C(\Omega^\infty X,Z) \) one has \( (\Omega^\infty(f+g))_n(\alpha) = (\Omega^\infty f)_n(\alpha) + (\Omega^\infty g)_n(\alpha) \), which follows since \( \Omega^\infty(f+g) \)

\[1\text{I am grateful to Nick Rozenblyum for the help with the proof.} \]
equals the composition $\Omega^\infty X \to \Omega^\infty X \times \Omega^\infty X \to \Omega^\infty Y \times \Omega^\infty Y \to \Omega^\infty Y$, the first arrow is $\Delta$, the second one is $\Omega^\infty f \times \Omega^\infty g$, the third one is the sum operation $+$.\footnote{Use the fact that the restriction of $+$ to $\{0\} \times \Omega^\infty Y$ and $\Omega^\infty Y \times \{0\}$ is the identity map.}

(ii) Consider a commutative diagram of spectra

$$S^\infty \Omega^\infty X \to \mathbb{Z}_S \wedge S^\infty \Omega^\infty X$$

(1.1.2)

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$X \to \mathbb{Z}_S \wedge X$$

the horizontal arrows are $a_{S^\infty \Omega^\infty X}$ and $a_X$, the vertical arrows are $b_X$ and $\text{id}_{\mathbb{Z}_S} \wedge b_X$. It implies that $\Omega^\infty (\text{id}_{\mathbb{Z}_S} \wedge b_X)\Omega^\infty = \Omega^\infty (\text{id}_{\mathbb{Z}_S} \wedge b_X)\Omega^\infty (a_{S^\infty \Omega^\infty X})\Omega^\infty = \Omega^\infty (a_X)\Omega^\infty (b_X)\Omega^\infty = \Omega^\infty (a_X)$, hence $\pi_n (\text{id}_{\mathbb{Z}_S} \wedge b_X)\pi_n (\Omega^\infty) = \pi_n (a_X)$. Let

$$r_{nX} : \text{Prim} H_n C(\Omega^\infty X, \mathbb{Z}) \to \pi_n (\mathbb{Z}_S \wedge X)$$

be the restriction of $\pi_n (\text{id}_{\mathbb{Z}_S} \wedge b_X) : H_n C(\Omega^\infty X, \mathbb{Z}) \to \pi_n (\mathbb{Z}_S \wedge X)$ to primitive classes. Since $r_{nX} \pi_n (\Omega^\infty) = \pi_n (a_X)$ is an isogeny of $\mathbb{A}B$-valued functors on connected spectra (see Remark in 1.1(b)), we see that $\pi_n (\Omega^\infty)$ is an isogeny if and only if $r_n$ is, and to check this it is enough to show that $\ker r_{nX}$ is killed by a nonzero integer $c_n$ that does not depend on $X$.

(iii) We first prove that $r_n$ is an isogeny on the subcategory of suspension spectra: Let us show that $\ker r_n S^p X$ for connected $P = (P, p_0) \in \text{Top}^+$ is killed by $n!$.

As in [M], $\Omega^\infty S^\infty P$ identifies naturally with a free $E_\infty$-space $F$ generated by $(P, p_0)$. Let $\Gamma_i$ be our $E_\infty$-operad. Then $F$ is a union of closed subspaces $\{0\} = F_0 \subset F_1 \subset F_2 \subset \ldots$ where $(F_i, F_0) = (P, \{p_0\})$ and $F_i$ is the image of the operad action map $(\Gamma_i \times P^i)/\Sigma_i \to F$ where $\Sigma_i$ is the symmetric group. Notice that $(\Gamma_i \times P^i)/\Sigma_i$ is the homotopy quotient of $P^i$ modulo the action of $\Sigma_i$. The pointed space $F_i/F_{i-1}$ equals $(\Gamma_i \times (P, p_0)^{\wedge i})/\Sigma_i$, which is the homotopy quotient of $(P, p_0)^{\wedge i}$ modulo $\Sigma_i$, so $C(F_i, F_{i-1}; Z) = C(\Sigma_i, C(P, \{p_0\}; Z)^{\wedge i})$ where $C(\Sigma_i, \cdot)$ is the complex of group chains. Thus the compositions $C(\Sigma_i, C(P, Z)^{\wedge i}) \to C(\Sigma_i, C(P, Z)^{\wedge i}) \to C(F_i, \{0\}; Z)$ split the filtration $C(F_i, \{0\}; Z)$ on $C(F, \{0\}; Z)$ (see the beginning of 1.1(c) for the notation), so we have a canonical quasi-isomorphism

(1.1.4) \[ \oplus_{i \geq 1} C(\Sigma_i, C(P, Z)^{\wedge i}) \rightarrow C(F, Z) = C(F, \{0\}, Z). \]

The map $\text{id}_{\mathbb{Z}_S} \wedge b_{S^p X} : \mathbb{Z}_S \wedge S^\infty \Omega^\infty S^\infty P \to \mathbb{Z}_S \wedge S^\infty P$ of abelian spectra amounts to a morphism $\nu_P : C(F, \{0\}; Z) \to C(P, \{p_0\}; Z)$ of complexes that can be described as follows. Consider the chain complex $C(P, Z)$ as a simplicial abelian group freely generated by $P$, and $\bar{C}(P, Z)$ as the kernel of the map of simplicial groups $C(P, Z) \to C(\{\text{point}\}, Z)$; ditto for $F$, etc. Then $\nu_P$ is $\mathbb{Z}$-linear extension of the map of simplicial sets $F \to C(P, Z)$ that extends the standard embedding $P \hookrightarrow C(P, Z)$ and transforms the $\Gamma_-$-operations into addition. Therefore $\nu_P$ is the identity map on the first summand of (1.1.4) and it kills the rest of the sum. Thus it remains to check that $H_n (\oplus_{i \geq 2} C(\Sigma_i, C(P, Z)^{\wedge i})) \cap \text{Prim} H_n (F, Z)$ is killed by $n!$.

Set $\Phi^m := \oplus_{i \geq m} C(\Sigma_i, C(P, Z)^{\wedge i})$. We view $\Phi$ as a split filtration on $C(F, Z)$ via (1.1.4). Let us show that $m(H_n \Phi^m \cap \text{Prim} H_n (F, Z)) \subset H_n \Phi^{m+1}$ for $m \geq 2$. Since $H_n \Phi^{n+1} = 0$, this proves our assertion.
We equip \( C(F \times F, \mathbb{Z}) = C(F, \mathbb{Z}) \otimes C(F, \mathbb{Z}) \) with the tensor product of the filtrations \( \Phi \). Since \( \Delta_\ast : C(F, \mathbb{Z}) \to C(F \times F, \mathbb{Z}) \) is compatible with the operad product, \( \Delta_\ast \) is compatible with the filtrations and \( \text{gr}^n \Phi = C(S_m, C(P, \mathbb{Z})^\otimes m) \to \oplus_{a+b=m} C(S_a, C(P, \mathbb{Z})^\otimes a) \otimes C(S_b, C(P, \mathbb{Z})^\otimes b) \) has \((a,b)\)-component equal to the transfer map \( \rho_{a,b} : C(S_m, C(P, \mathbb{Z})^\otimes m) \to C(S_a \times S_b, C(P, \mathbb{Z})^\otimes m) \) for the usual embedding \( S_a \times S_b \to S_m \). Now if \( \alpha \in \text{Prim} H_n(F, \mathbb{Z}) \) lies in \( \Phi^m \), \( m \geq 2 \), then the image \( \bar{\alpha} \) of \( \alpha \) in \( \text{gr}^m \Phi H_n(F, \mathbb{Z}) \) satisfies \( \rho_{a,b}(\bar{\alpha}) = 0 \) if \( a, b \geq 1 \). In particular, \( \rho_{1,m-1}(\bar{\alpha}) = 0 \), and we are done since \( \ker \rho_{1,m-1} \) is killed by \( |\Sigma_m/\Sigma_{m-1}| = n \).

(iv) To finish the proof, let us show that \( \ker r_{n,X} \) for any connected spectrum \( X \) is killed by \( e_n = n! \ell_n^2 \) where \( e_n \) is as in Remark in 1.1(b).

To that end we will find a map of spectra \( s_{X_n} : \tau_{\leq n} X \to \tau_{\leq n} S^\infty \Omega^n X \) such that \( \pi_i(\tau_{\leq n}(b_X) s_{X_n}) = \ell_n \text{id}_{\pi_i(X)} \) for \( i \leq n \) where \( \ell_n = e_n^2 \). Assuming we have \( s_{X_n} \), let us finish the proof.

One has \( \tau_{\leq n}(b_X) s_{X_n} = \ell_n \text{id}_{\tau_{\leq n} X} + \epsilon \) where \( \pi_i(\epsilon) = 0 \). Since the canonical filtration on \( \tau_{\leq n} X \) has length \( n \), one has \( e^n = 0 \). So, by (i), \( \tau_{\leq n}(b_X) s_{X_n} \) acts on \( \text{Prim} H_n C(\Omega^n X, \mathbb{Z}) \) as the sum of the multiplication by \( \ell_n \) map and an operator \( \epsilon \) such that \( \pi_i(\epsilon) = 0 \). The kernel \( K \) of this action is killed by \( \ell_n^2 \) for \( \ell_n \text{id}_K = -\epsilon(K) \).

We are done since, by (iii) applied to \( P = \Omega^n X \), the map \( n! \tau_{\leq n}(b_X) s_{X_n} \) kills \( \ker r_{n,X} = \ker r_{n,\tau_{\leq n} X} \).²

1It remains to construct the promised \( s_{X_n} \). For any connected spectrum \( Y \) let \( \eta : \tau_{\leq n} (\mathbb{Z} \otimes \mathbb{A} Y) \to \tau_{\leq n} Y \) be a map of spectra such that \( \eta \tau_{\leq n}(aY) = e_n \text{id}_{\tau_{\leq n} Y} \).³ Let \( u_i, i \in [1, n] \), be the composition \( \pi_i(\mathbb{Z} \otimes X) \to \pi_i(X) \to \pi_i(\mathbb{Z} \otimes S^\infty \Omega^n X) \), the first arrow is \( \pi_i(\eta_X) \), the second one is \( \pi_i(\partial_{\leq n} X) = \pi_i(a_{\leq n} \Omega^n X) \pi_i(\epsilon_{\leq n} X) \). Let \( u : \tau_{\leq n} (\mathbb{Z} \otimes X) \to \tau_{\leq n} (\mathbb{Z} \otimes S^\infty \Omega^n X) \) be a map of spectra such that \( \pi_i(u) = u_i \).⁴ We define \( s_{X_n} \) as the composition of maps \( \tau_{\leq n} X \to \tau_{\leq n} (\mathbb{Z} \otimes X) \to \tau_{\leq n} (\mathbb{Z} \otimes S^\infty \Omega^n X) \) where the first arrow is \( \tau_{\leq n}(a_X) \) and the last arrow is \( q_{S^\infty \Omega^n X} \). Finally, for \( i \leq n \) one has \( \pi_i(\tau_{\leq n}(b_X) s_{X_n}) = \pi_i(b_X) \pi_i(q_{S^\infty \Omega^n X}) \pi_i(a_{\leq n} \Omega^n X) \pi_i(\epsilon_{\leq n} X) \pi_i(\epsilon_X) = \tau_n^2 \pi_i(b_X) \pi_i(\epsilon_X) = e_n^2 \pi_i(X) \), and we are done. □

Remarks. (i) I do not know if the theorem remains true for connected H-spaces that are not infinite loop spaces.

(ii) The map \( s_{X_n} \) from (iv) of the proof was constructed in an artificial manner; it need not commute with morphisms of \( X \). One can look for a natural \( s_{X_n} \) (may be, with a better constant \( \ell_n \)). A possible approach: The ideal answer would be to find a natural section \( s_X : X \to S^\infty \Omega^n X \) of \( b_X \), and take \( s_X := \tau_{\leq n}(s_X) \) with \( \ell_n = 1 \). Such an \( s_X \) does not exist, but we have a natural section \( \epsilon_X : \Omega^n X \to \Omega^n S^\infty \Omega^n X \) of \( \Omega^n(b_X) \) which is not an \( E_\infty \)-map for the standard \( E_\infty \)-structures on our spaces. Notice though that \( \Omega^n S^\infty \Omega^n X \) is naturally an \( E_\infty \)-ring space with the product operation coming from the usual \( E_\infty \)-structure on \( \Omega^n X \), and \( \epsilon_X \) is an \( E_\infty \)-map for the product \( E_\infty \)-structure. A natural candidate for \( s_X \) might be the logarithm of \( \epsilon_X \). It is ill defined due (at least) to non-integrality of the logarithm series, but one can look for \( n! \log \tau_{\leq n}(\epsilon_X) \) that would be an \( E_\infty \)-map for the standard \( E_\infty \)-structures, hence providing a natural \( s_{X_n} \) with \( \ell_n = n! \).

³Notice that \( \text{Prim} H_n C(\Omega^n X, \mathbb{Z}) \xrightarrow{n!} \text{Prim} H_n C(\Omega^n \tau_{\leq n} X, \mathbb{Z}) \).

⁴Such a \( \eta \) exists since \( e_n \) kills \( \text{cone}(\tau_{\leq n}(aY)) \).

⁵Our \( u \) exists since \( \tau_{\leq n}(\mathbb{Z} \otimes X) \) and \( \tau_{\leq n}(\mathbb{Z} \otimes S^\infty \Omega^n X) \), being abelian spectra, are homotopy equivalent to products of the Eilenberg-MacLane spectra corresponding to their homotopy groups.
1.2. The pro-setting. (a) The basic reference for pro-objects is [GV] §8.

As in loc. cit., one assigns to any category $\mathcal{C}$ its category of pro-objects $\text{pro-}\mathcal{C}$ together with a fully faithful functor $\iota : \mathcal{C} \hookrightarrow \text{pro-}\mathcal{C}$, $\mathcal{X} \mapsto \mathcal{X}$. The category $\text{pro-}\mathcal{C}$ is closed under codirected limits (i.e., for any $I$-diagram $Z_i$ in $\text{pro-}\mathcal{C}$, $I$ a directed category, $\lim_i Z_i$ exists), and $\iota$ is a universal functor from $\mathcal{C}$ to such a category (i.e., every functor $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is closed under codirected limits, extends in a unique way to a functor $\text{pro-}\mathcal{C} \to \mathcal{D}$ that commutes with codirected limits). For $\mathcal{C}$ small, the Yoneda embedding $\mathcal{C} \to \text{Funct}(\mathcal{C}, \text{Sets})^{\text{op}}$ extends naturally to a functor $\text{pro-}\mathcal{C} \to \text{Funct}(\mathcal{C}, \text{Sets})^{\text{op}}$; if $\mathcal{C}$ is closed under finite limits, it identifies $\text{pro-}\mathcal{C}$ with the subcategory of functors that commute with finite limits. Each pro-object can be realized as “$\lim^\mathcal{C} X_i := \lim_i X_i$ for some $I^\mathcal{C}$-diagram $X_i$ in $\mathcal{C}$, $I$ is directed, and $\text{Hom}(\text{“}$lim$^\mathcal{C} X_i, \text{“}$lim$^\mathcal{C} Y_j)$ = $\lim_j \lim_i \text{Hom}(X_i, Y_j)$.” If $\mathcal{C}$ is abelian, then $\text{pro-}\mathcal{C}$ is abelian and $\iota$ is exact. More generally, for any $I$ as above the functor $\mathcal{C}^I \to \text{pro-}\mathcal{C}$, $(X_i) \mapsto \text{“}$lim$^\mathcal{C} X_i$, is exact. E.g. we have abelian category $\text{pro-}\text{Ab}$ of pro-abelian groups. The tensor product “$\lim$”$X_i \otimes \text{“}$lim$^\mathcal{C} Y_j := \text{“}$lim$^\mathcal{C} X_i \otimes Y_j$ makes it a symmetric tensor category.

Remarks. (i) If $\mathcal{C}$ is closed under arbitrary limits or colimits, then so is $\text{pro-}\mathcal{C}$.

(ii) For $\mathcal{C}$ abelian, an object “$\lim$”$X_i$ vanishes if and only if for every $i \in I$ one of the maps $X_i \to X_i$ of the diagram equals to zero.

(b) An $\infty$-category version of the story of [GV] is explained in Chapter 5 of [Lu1], the $\infty$-category of pro-objects is treated in section 5.3.

Let $\text{pro-}\mathcal{S}p$ be the $\infty$-category of pro-spectra, i.e., pro-objects of the $\infty$-category $\mathcal{S}p$. This is a stable $\infty$-category by [Lu2] 1.1.3.6. By abuse of notation, we denote its homotopy category by $\text{pro-}\mathcal{S}p$ as well. It carries a t-structure with the category of connective objects $\text{pro-}\mathcal{S}p_{\geq 0}$ equal to $\text{pro-}(\mathcal{S}p_{\geq 0})$. The heart of the t-structure equals $\text{pro-}\text{Ab}$, the homotopy functor is $X = \text{“}$lim$^\mathcal{C} X_i \mapsto \pi_\ast X := \text{“}$lim$^\mathcal{C} \pi_\ast (X_i)$, and the truncations are $\tau_{\geq n}$, $\text{“}$lim$^\mathcal{C} X_i = \text{“}$lim$^\mathcal{C} \tau_{\geq n} X_i$, $\tau_{\leq n}$, “lim$^\mathcal{C} X_i = \text{“}$lim$^\mathcal{C} \tau_{\leq n} X_i$. Let $\text{pro-}\mathcal{S}p^{-} := \cup_n \text{pro-}\mathcal{S}p_{>-n}$ be the t-subcategory of eventually connective pro-spectra.

The $\infty$-category space of morphisms of pro-spectra $\text{Hom}(\text{“}$lim$^\mathcal{C} X_i, \text{“}$lim$^\mathcal{C} Y_j)$ is equal to $\text{holim}_n \text{hocolim}_i \text{Hom}(X_i, Y_j)$; passing to $\pi_0$ we get the set of morphisms in the homotopy category.

By [Lu2] 6.3.1.13, $\text{pro-}\mathcal{S}p$ is a symmetric tensor $\infty$-category with tensor product “$\lim$”$X_i \otimes \text{“}$lim$^\mathcal{C} Y_j = \text{“}$lim$^\mathcal{C} X_i \otimes Y_j$. The tensor product is right t-exact; it induces the usual tensor product on the heart $\text{pro-}\text{Ab}$.

The above discussion can be repeated with “spectrum” replaced by “chain complex of abelian groups”: we have stable $\infty$-category $\text{pro-}\mathcal{D}(\text{Ab})$ of pro-complexes, its subcategory $\text{pro-}\mathcal{D}(\text{Ab})^{-}$ of eventually connective pro-complexes, etc. For a pro-complex $Z$ we denote by $H Z$ its homology pro-abelian groups. We have the $t$-exact functor $\text{pro-EM}: \text{pro-}\mathcal{D}(\text{Ab}) \to \text{pro-}\mathcal{S}p$ which is identity on the hearts.

A map $X \to Y$ in $\text{pro-}\mathcal{S}p^{-}$ is called quasi-isomorphism, resp. quasi-isogeny, if the maps $\pi_n(X) \to \pi_n(Y)$ are isomorphisms, resp. isogenies, for all $n$. So a pro-spectrum $X$ is quasi-isomorphic, resp. quasi-isogenous, to $0$ if $\pi_n(X)$, resp. $\pi_n(X)_Q$, vanish for all $n$. Such $X$ form thick subcategories of $\text{pro-}\mathcal{S}p^{-}$ that are $\wedge$-ideals (repeat the proof of the lemma in 1.1(b)). We denote by $\text{pro-}\mathcal{S}p_{>0}^{-}$ and $\text{pro-}\mathcal{S}p_{<0}^{-}$ the corresponding Verdier quotients; these are symmetric tensor t-categories with hearts $\text{pro-}\text{Ab}$ and $\text{pro-}\text{Ab}_Q$, the localizations $\text{pro-}\mathcal{S}p^{-} \to \text{pro-}\mathcal{S}p_{>0}^{-} \to \text{pro-}\mathcal{S}p_{<0}^{-}, X \mapsto X_n \mapsto$
\[ X_{\mathbb{Q}}, \] are t-exact tensor functors. The discussion can be repeated for \( \text{pro-} D(Ab)^{-} \).

\textbf{Exercise. } For \( X = \text{"lim"} X_i \) and \( Y = \text{"lim"} Y_j \) in \( \text{pro-} \mathcal{S}p^{-} \) one has \( \text{Hom} (X_\tau, Y_\tau) = \text{holim}_n \text{Hom} (\tau_{\leq n} X, \tau_{\leq n} Y) \), \( \text{Hom} (X_\mathbb{Q}, Y_\mathbb{Q}) = \text{holim}_n \text{Hom} (\tau_{\leq n} X, \tau_{\leq n} Y) \otimes \mathbb{Q} \).

\textbf{Remarks.} (i) For any \( Y \in \mathcal{S}p^{-} \) the pro-spectrum \( X := \text{"lim"} \tau_{\geq n} Y \) is quasi-isomorphic to zero; if \( Y \) is not eventually coconnective, then \( X \neq 0 \).

(ii) Probably \( \text{pro-} \mathcal{S}p^{-}_n \) coincides with the eventually connective part of the homotopy category of pro-spectra as defined in [FI].

Passing to the quotients, \( \text{pro-EM} \) yields the functor

\[
\text{pro-} D(Ab)^{-}_Q \to \text{pro-} \mathcal{S}p^{-}_Q.
\]

\textbf{Proposition. } This functor is an equivalence of tensor triangulated categories.

\textbf{Proof.} Repeat the proof of the proposition in 1.1(b) using the remark after the proof to check that \( \text{Cone}(a_X) \) is quasi-isogenous to 0 for every \( X \in \text{pro-} \mathcal{S}p^{-} \). \( \square \)

(c) The theorem in 1.1(c) has an immediate pro-version as well. It will be used in \( \S 6 \) in the following manner:

For \( X = \text{"lim"} X_i \in \text{pro-} \mathcal{S}p^{-} \) one has a canonical map \( \nu_X : C(\Omega^\infty X, Z)_{\mathbb{Q}} \to X_{\mathbb{Q}} \) defined as the composition \( C(\Omega^\infty X, Z) = \mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X \to \mathbb{Z}_{\mathcal{S}p} \wedge X \leftarrow X \). Here \( C(\Omega^\infty X, Z) = \text{"lim"} C(\Omega^\infty X_i, Z) \), the first arrow is \( \text{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X, b_X = \text{"lim"} b_{X_i} \), the second one is quasi-isogenous \( a_X = \text{"lim"} a_{X_i} \), see (1.1.2).

Suppose \( X \) is connected, i.e., \( \pi_{\leq 0}(X) = 0 \). Let \( V \) be a pro-complex such that \( H_{\leq 0}(V) = 0 \) and \( \psi : V_Q \to C(\Omega^\infty X, Z)_Q \) be a morphism such that for every \( n \) the map \( H_n(\psi) : H_n V_Q \to H_n C(\Omega^\infty X, Z)_Q \) is injective with image \( \text{Prim} H_n C(\Omega^\infty X_i, Z)_Q := (\text{"lim"} \text{Prim} H_n C(\Omega^\infty X_i, Z))_Q \).

\textbf{Proposition. } The map \( \nu_X \psi : V_Q \to X_Q \) is a quasi-isogeny.

\textbf{Proof.} It is enough to check that \( a_X \nu_X \psi : V \to \mathbb{Z}_{\mathcal{S}p} \wedge X \) is a quasi-isogeny. One has \( H_n(a_X \nu_X \psi) = r_{nX} H_n(\psi), \) see (1.1.3). We are done since \( r_{nX} \) is an isogeny (see the proof of the theorem in 1.1(c)). \( \square \)

(d) One has a natural functor of stable \( \infty \)-categories \( \text{holim} : \text{pro-} \mathcal{S}p \to \mathcal{S}p, \) \( \text{"lim"} X_i \mapsto \text{holim} X_i \), which is left t-exact. Its restriction to the subcategory \( \text{pro}^{\text{ho}} \mathcal{S}p^{-} \) of those \( \text{"lim"} X_i \) that the set \( I \) of indices \( i \) is countable, has homological dimension 1,\footnote{I am grateful to Jacob Lurie for that example.} so we have triangulated functor \( \text{holim} : \text{pro}^{\text{ho}} \mathcal{S}p^{-} \to \mathcal{S}p^{-} \) and, passing to the quotient categories, holim: \( \text{pro}^{\text{ho}} \mathcal{S}p_Q \to \mathcal{S}p_Q \). A similar functor \( \text{holim} : \text{pro}^{\text{ho}} D(Ab)^{-} \to D(Ab) \) is the right derived functor of the projective limit functor \( \text{"lim"} C_i \mapsto \text{lim} C_i \), and the two \( \text{holim} \)'s are compatible with the (pro-) EM functors.

2. THE MAIN THEOREM AND A GEOMETRIC APPLICATION

2.1. For an associative unital \( V \)-algebra \( R, V \) is a commutative ring, we denote by \( C(R/V) \) the relative Hochschild complex (see [Lo] 1.1.3), so \( C(R/V)_n = R \otimes_{\otimes V}^{\otimes n+1} \) for \( n \geq 0 \) (here \( \otimes = \otimes_V \)), \( C(R/V)_n = 0 \) otherwise. If \( R \) is \( V \)-flat, then \( C(R/V) \)

\footnote{For a precise homological dimension assertion if \( I \) is arbitrary, see [Mi].}
equals $R \otimes_{R \otimes \epsilon R} R$. Let $CC(R/V)$ be the cyclic complex as in [Lo] 2.1.2: this is the total complex of the cyclic bicomplex $CC(R/R)$; here $CC(R/V)_{m,n} = R^\otimes_{m+1}$ for $m,n \geq 0$ and $C(R/V)_{m,n} = 0$ otherwise; the $n$th row complex is the chain complex for the $Z/n+1$-action $r_n \otimes \cdots r_n \mapsto (-1)^n r_n \otimes \cdots \otimes r_{n-1}$ on $R^\otimes_{m+1}$; the even column complexes equal $C(R/V)$ and the odd ones are acyclic. We denote by $\Phi_m$ the increasing filtration on $CC(R/V)$ that comes from the filtration $CC(R/V)_{\leq 2m}$. Thus $\Phi_{-1} = 0$ and the projection $\text{gr}_m^p CC(R) \to CC(R/V)_{2m} = C(R/V)[2m], m \geq 0$, is a quasi-isomorphism.

Apart from 2.3–2.4, we consider only Hochschild and cyclic complexes with $V = Z$ and denote them simply $C(R), CC(R)$.

2.2. The main theorem. Let $A, I$ be as in 0.1; as in loc. cit., $A_i := A/p^i$ and $I_i$ is the image of $I$ in $A_i$. Consider the relative $K$-theory pro-spectrum $K(A, I) := \text{“lim”} K(A, I_i)$ and the cyclic homology pro-complex $CC(A) := \text{“lim”} CC(A_i)$ (see 3.1 below for a reminder and the notation).

**Theorem.** Suppose $A$ has bounded $p$-torsion and $A_1$ has finite stable rank (see [Su]). Then there is a natural quasi-isogeny

\[(2.2.1) \quad K(A, I) \sim \rightarrow CC(A)[-1]Q.\]

The proof takes §§3–6. The outline of the construction, which is a continuous version of Goodwillie’s story [G] or [Lo] 11.3.2, is as follows:

- Consider the pro-complex $C(\mathfrak{g}_t(A)) := \text{“lim”} C(\mathfrak{g}_t(A_i))$ of Lie algebra chains; here $\mathfrak{g}_t$ is the Lie algebra of matrices. By a version of the Loday-Quillen-Tsygan theorem [Lo] 10.2.4, there is a canonical map of pro-complexes $C(\mathfrak{g}_t(A)) \to \text{Sym}(CC(A)[-1])$ which is a quasi-isogeny in degrees $\leq r$.

- Let $GL_r(A)^{(m)} := \text{Ker}(GL_r(A) \to GL_r(A_m))$ be the congruence subgroup of level $m \geq 2$, and $C(GL_r(A)^{(m)}) := \text{“lim”} C(GL_r(A)^{(m)}/GL_r(A)^{(m+i)}, Z)$ be the pro-complex of its group chains. A version of Lazard’s theory from Ch. V of [L] provides (we use the bounded $p$-torsion condition here) a canonical quasi-isogeny $C(GL_r(A)^{(m)}) Q \sim \rightarrow C(\mathfrak{g}_t(A)) Q$.

- We realize $K(A, I)$ using Volodin’s construction. By Suslin’s results [Su] (we use the finiteness of stable rank condition here) one can compute the relative $K_i$ pro-groups using $GL_r$’s with bounded $r$ depending on $i$. Viewed up to isogeny, they coincide with the primitive part of $H_i C(GL_r(A)^{(m)})$.

Combining the above quasi-isogenies and using 1.2(c), we get (2.2.1).

2.3. In the next subsection we explain a geometric application of the main theorem. We need three technical lemmas; the reader can skip them at the moment to return when necessary.

(a) Let $R$ be a ring and $I$ its two-sided ideal. Let $K^B_i$ be the non-connective relative $K$-theory spectrum from [TT] §6, so $K^B_i(R) = K_n(R)$ for $n \geq 0$ and there is a long exact sequence $\ldots \to K^B_i(R, I) \to K^B_n(R) \to K^B_n(R/I) \to K^B_{n-1}(R, I) \to \ldots$

**Lemma.** If $I$ is nilpotent then $K^B_i(R, I)$ is connected, i.e., $K^B_i(R, I) = 0$ for $n \leq 0$.

**Proof.** The assertion of the lemma amounts to surjectivity of the map $K_1(R) \to K_1(R/I)$ and bijectivity of the maps $K^B_i(R) \to K^B_i(R/I)$ for all $n \leq 0$.

---

8I.e., the $p$-torsion subgroup of $A$ is killed by some $p^n$. 

---
Every idempotent in Mat\(_n(R/I)\) lifts to an idempotent in Mat\(_n(R)\), and the map \(\text{Isom}(P, P') \to \text{Isom}(P/IPP', P'/IPP')\) is surjective for all projective \(R\)-modules \(P, P'\). Therefore \(K_1(R) \to K_1(R/I)\) and \(K_0(R) \to K_0(R/I)\).

Induction by \(-n\): Suppose \(K^B_n(R) \to K^B_n(R/I)\) for all \((R, I)\) with \(I\) nilpotent. Then \(K^B_n(R) \to K^B_n(R/I)\). Indeed, the map from the canonical exact sequence \(0 \to K^B_n(R) \to K^B_n(R[t]) \oplus K^B_n(R[t^{-1}]) \to K^B_n(R[t, t^{-1}]) \to K^B_{n-1}(R) \to 0\) of \([TT]\) 6.6 to the similar exact sequence for \(R/I\) is an isomorphism at each term but the last one by the induction assumption. So it is an isomorphism everywhere, q.e.d. □

(b) Let \(X^*\) be a formal scheme, so \(X^*\) is direct limit of a directed family of closed subschemes \(X_i\) where the ideals of the embeddings \(X_i \hookrightarrow X_j\) are nilpotent. Let \(Y\) be a closed subscheme of \(X^*\) such that \(X_i\) contain \(Y\) and the ideal of \(Y\) in \(X_i\) is nilpotent. One has the projective system of the non-connective relative \(K\)-theory spectra \(K^B(X_i, Y)\).

**Lemma.** If \(Y\) is quasi-compact and separated then the pro-spectrum “\(\text{ lim}^n\)” \(K^B(X_i, Y)\) is eventually connective.

**Proof.** Pick a finite open affine Zariski covering \(\{Y_\alpha\}_{\alpha \in A}\) of \(Y\); let \(\{X_{\alpha}\}\) be the corresponding covering of \(X_i\). Let \(S\) be the set of non-empty subsets \(S\) of \(A\) ordered by inclusion. We have an \(S\)-diagram \(S \to Y_S := \cap_{\alpha \in S} Y_\alpha\) of affine schemes and open embeddings, and similar \(S\)-diagrams \(S \to X_iS\). By \([TT]\) 8.4, one has \(K^B(X_i, Y) \to \text{holim}_S K^B(X_iS, Y_S)\). Thus \(K^B(X_i, Y)\) has a finite filtration with \(\text{gr}^n K^B(X_i, Y) = \Pi_{|S|=n} K^B(X_iS, Y_S)[1 - n]\). Since \(X_iS\) is affine and the ideal of \(Y_S\) in \(X_iS\) is nilpotent, the above lemma shows that the spectrum \(K^B(X_iS, Y_S)\) is connected. Therefore \(\pi_{-n} K^B(X_i, Y) = 0\) for \(n \geq |A| - 1\), and we are done. □

(c) Let \(k\) be a perfect field of characteristic \(p\), \(W = W(k)\) be the Witt vectors ring, and \(R\) be a unital associative flat \(W_\tau\)-algebra where \(W_\tau := W/p^\infty\).

**Lemma.** The evident map \(\tau : CC(R) \to CC(R/W_i)\) is a quasi-isomorphism.

**Proof.** The map \(\tau\) is compatible with the filtrations \(\Phi\) (see 2.1), so it is enough to check that \(\tau\) is a filtered quasi-isomorphism, i.e., that the map of Hochschild complexes \(C(R) \to C(R/W_i)\) is a quasi-isomorphism. Both \(C(R)\) and \(C(R/W_i)\) are \(\mathbb{Z}/p^i\)-flat (since \(R = W_{\tau}\)-flat), so it is enough to check that the map \(C(R) \oplus \mathbb{Z}/p \to C(R/W_i) \oplus \mathbb{Z}/p\) is a quasi-isomorphism. The latter is the map \(C(R_1) \to C(R_1/k)\), \(R_1 := R/pR\), so replacing \(R\) by \(R_1\), we can assume that \(i = 1\).

We check first that \(C(k) = C(k/F_p) \to k\) is a quasi-isomorphism. It is enough to consider the case when \(k\) is the perfectionization of a field \(k'\) finitely generated over \(F_p\). Then \(k'\) is a separable extension of a purely transcendental extension of \(F_p\), and \(H^iC(k'/F_p) \sim \Omega^i(k'/F_p)\) by [Lo] 3.4.4. Since Frobenius kills \(\Omega^i(k'/F_p)\), one has \(H_{>0}C(k) = 0\), q.e.d.

If \(R\) is an arbitrary \(k\)-algebra, then let \(P\) be an \(R \otimes_{F_p} R^\bullet\)-flat resolution of \(R\). The terms of \(P\) are \(k \otimes_{F_p} k\)-flat, hence \(P_k := P \otimes_{k \otimes_{F_p}} k\) is again a resolution of \(R\) (indeed, \(P_k \to R \otimes_{k \otimes_{F_p}} k = R \otimes_k (k \otimes_{F_p} k) \to R \otimes_k C(k)\)). Thus \(P_k\) is an \(R \otimes_{F_p} R^\bullet\)-flat resolution of \(R\), hence \(C(R) \to C(R/k, R) = P_k \otimes_{R \otimes_{F_p} R^\bullet} R \to C(R/k)\), and we are done. □

2.4. Let \(E\) be a \(p\)-adic field, \(O_E\) be its ring of integers, so \(O_E\) is a complete mixed characteristic dvr with perfect residue field \(k\) of characteristic \(p\). We denote by
the induced covering of $Y$ Here the target is seen via (2.4.1) as an object of 2.4.3)

Proof. up to quasi-isogeny.

Let $D_{id}^-(\text{Vect}_E)$ be the derived category of bounded above complexes of $E$-vector spaces with finite-dimensional homology. The functor $D_{id}^-(O_E\text{-mod}) \to D_{id}^-\text{(Vect}_E)$, $M \mapsto M \otimes \mathbb{Z}/p^i$, identifies the target with the Verdier quotient of $D_{id}^-D_{id}(O_E\text{-mod})$ modulo thick subcategory of complexes with torsion cohomology. The functor $\cdot \otimes \mathbb{Z}/p^i$ sends the latter subcategory to pro-complexes quasi-isogenous to zero, hence it yields a t-exact functor

\[(2.4.1) \quad D_{id}^-\text{(Vect}_E) \to \text{pro-}D_{id}(\mathbb{A}b)_{\mathbb{Q}}^-
\]

which is evidently faithful (fully faithful if $E = \mathbb{Q}_p$).

Let $X$ be a proper $O_E$-scheme with smooth generic fiber $X_\mathcal{E}$. Let $R\Gamma_{\text{dr}}(X_\mathcal{E}) := R\Gamma(X_\mathcal{E}, \Omega^i_{X/\mathcal{E}})$ be the de Rham complex of $X_\mathcal{E}$ and $F^a$ be its Hodge filtration. The complexes $R\Gamma_{\text{dr}}(X_\mathcal{E})/F^a$ are bounded and have finite-dimensional cohomology.

Let $Y \subset X$ be a closed subscheme whose support equals the closed fiber. Set $X_i := X \otimes \mathbb{Z}/p^i$, so $Y \subset X_i$ for large enough $i$. By 2.3(b) we have the pro-spectrum $K^B(X,Y) := \text{lim}^\leftarrow K^B(X_i,Y) \in \text{pro-Sp}^-\mathcal{E}$.

**Theorem.** There is a natural quasi-isogeny of pro-spectra

\[(2.4.2) \quad K^B(X,Y)_{\mathbb{Q}} \sim \oplus_a (R\Gamma_{\text{dr}}(X_\mathcal{E})/F^a)[2a - 1].
\]

Here the target is seen via (2.4.1) as an object of $\text{pro-}D_{id}(\mathbb{A}b)_{\mathbb{Q}}^-$, hence a pro-spectrum up to quasi-isogeny.

Applying holim from 1.2(d), we get identification (0.2.1).

**Proof.** (i) Pick a finite open affine covering $\{X_\alpha\}$ of $X$; let $\{Y_\alpha := Y \cap X_\alpha\}$ be the induced covering of $Y$. As in the proof of the lemma in 2.3(b), we write $X_S := \cap_{\alpha \in S} X_\alpha = \text{Spec} R_S$, $Y_S = \text{Spec} R_S/I_S$ for $S \in \mathcal{S}$, etc. By loc. cit. and the lemma in 2.3(a), $K^B(X,Y) := \text{lim}^\leftarrow \text{holim}_{S} K(X_{IS},Y_S) = \text{holim}_{S} K(X_S,Y_S)$. Passing to pro-spectra up to isogeny, we get $K^B(X,Y)_{\mathbb{Q}} \sim \oplus_a \text{holim}_{S} K(X_S,Y_S)_{\mathbb{Q}}^a$. Set $CC(X) := \text{holim}_{S} CC(X_{IS})$, $CC(X_S) := \text{lim}^\rightarrow CC(X_{IS})$, and $CC(X) := \text{lim}^\leftarrow CC(X_{iS}) = \text{holim}_{S} CC(X_S)$. The rings $A_S = \lim_{S} R_{IS}$ satisfy the conditions of the theorem in 2.2, so (2.2.1) applied to $A_S$ and ideals $I_{S}A_S$ provides then a canonical identification

\[(2.4.3) \quad K^B(X,Y)_{\mathbb{Q}} \sim CC(X)_{\mathbb{Q}}[1].
\]

(ii) The terms of (2.4.2) do not change if we replace $E$ by the field of fractions of $W = W(k)$, so we can assume that $O_E = W$.

Let $X_{\text{red}} \subset X$ be the reduced scheme. Since $X_\mathcal{E}$ is smooth and $X$ has finite type, $\mathcal{O}_{X_{\text{red}}}$ is the quotient of $\mathcal{O}_X$ modulo ideal killed by high enough power of $p$,
so the maps $CC(X_{iS}) \to CC(X_{\text{red}iS})$ are quasi-isogeneies. Therefore, by (2.4.3), $K^B(X,Y) \otimes \mathbb{Q} \sim K^B(X_{\text{red}},Y_{\text{red}})\otimes \mathbb{Q}$. Thus the terms of (2.4.2) do not change if we replace $X$ by $X_{\text{red}}$, so we can assume that $X$ is flat over $W$.

(iii) Consider $W$-complexes $CC(X_S/W) := CC(R_S/W)$; they are $W$-flat since $R_S$ is $W$-flat. Set $CC(X/W) := \text{holim}_S CC(X_S/W)$. By 2.3(c) one has $CC(X_{iS}) \sim CC(X_{iS}/W_i) = CC(X_S/W) \otimes \mathbb{Z}/p^i$. Therefore

\[(2.4.4) \quad CC(X) \sim \lim \text{“lim”} CC(X/W) \otimes \mathbb{Z}/p^i.\]

The complex $CC(X/W)$ lies in $D_{fg}(W-\text{mod})$, i.e., the homology of $CC(X/W)$ is finitely generated $W$-modules: Indeed, since filtration $\Phi$ on $CC$ (see 2.1) is finite on every homology group and $S$ is finite, it suffices to show that $\text{gr}_m^\Phi CC(X/W) := \text{holim}_S \text{gr}_m^\Phi CC(X_S/W) \in D_{fg}(W-\text{mod})$. Since $\text{gr}_m^\Phi CC(X_S/W) \sim C(X_S/W)[2m]$, it is enough to check that $\text{holim}_S H_n C(X_S/W) \in D_{fg}(W-\text{mod})$. Now $H_n C(X_S/W) = \Gamma(X,S_H)$, $S_H := H_n L\Delta^* \Delta_s O_X$ where $\Delta : X \to X \times \text{Spec} W X$ is the diagonal embedding. Since $S_H$ is a coherent $O_X$-module and $X/W$ is proper, the homology of $\text{holim}_S H_n C(X_S/W) = \Gamma(X,S_H)$ are finitely generated $W$-modules, q.e.d.

(iv) Combining (2.4.3) and (2.4.4) we get a canonical identification

\[(2.4.5) \quad K^B(X,Y) \otimes \mathbb{Q} \sim (\text{“lim”} CC(X/W) \otimes \mathbb{Z}/p^i)_\mathbb{Q}[1].\]

By (iii) and the definition of (2.4.1), to get (2.4.2) it remains to produce a natural quasi-isomorphism

\[(2.4.6) \quad CC(X,W) \otimes \mathbb{Q} \sim \oplus_n (R\Gamma_{dR}(X_E)/F^n)[2a - 2].\]

Now $CC(X/W) \otimes \mathbb{Q} = \text{holim}_S CC(X_{SE}/E)$, and (2.4.6) comes from a canonical quasi-isomorphism $CC(R/E) \sim \oplus_n (\Omega (R/E)/F^n)[2a - 2]$ from [Lo] 3.4.12\footnote{By loc.cit., this quasi-isomorphism is defined as the composition of quasi-isomorphisms $CC(R/E) \sim B(R/E) \sim \oplus_n (\Omega (R/E)/F^n)[2a - 2]$ from [Lo] 2.1.7, 2.1.8 and [Lo] 2.3.6, 2.3.7.} valid for any smooth commutative algebra $R$ over $E \supset \mathbb{Q}$, and applied to algebras $R = R_S \otimes \mathbb{Q}$, and we are done. □

3. The Loday-Quillen-Tsygan isomorphism

The Loday-Quillen-Tsygan theorem, see [Lo] 10.2.4, provides a canonical quasi-isomorphism $C(\mathfrak{gl}(R)) \sim \text{Sym}(CC(R)[1])$ for any $\mathbb{Q}$-algebra $R$. We adapt the argument of loc. cit. for our continuous setting.

3.1. For an associative unital ring $R$ let $C^\lambda(R)$ be the Connes complex (see [Lo] 2.1.4). This is the quotient complex of the Hochschild complex $C(R)$: namely, $C^\lambda(R)_n$ is the coinvariants of the $\mathbb{Z}/n + 1$-action on $R^{\otimes n+1}$ (see 2.1). There is an evident projection $\pi_R : CC(R) \to C^\lambda(R)$ that equals the projection $C(R) \to C^\lambda(R)$ on $CC(R)_0$, and kills the rest of the bicomplex $CC(R)$.

**Lemma.** The homology group $H_a \text{Ker}(\pi_R)$ is killed by $n!$.

**Proof.** Consider the increasing filtration by the bicomplex row number on $CC(R)$, and its subcomplex $\text{Ker}(\pi_R)$. Then $H_a \text{gr}_m \text{Ker}(\pi_R)$ equals $H_{n-m}(\mathbb{Z}/m + 1, R^{\otimes m+1})$ for $n > m > 0$ and is 0 otherwise. It is killed by $m + 1$, hence the assertion. □
3.2. Let $A$ be any $p$-adic unital ring, so $A = \varprojlim A_i$ where $A_i := A/p^i$. We have pro-complexes $C(A)^\gamma := \text{“lim”} C(A_i)$ and, similarly, $C^\lambda(A)^\gamma$, $CC(A)^\gamma$. Consider the projection $\pi_A^{-1} : CC(A)^\gamma \to C^\lambda(A)^\gamma$.

**Corollary.** $\pi_A^{-1}$ is a quasi-isomorphism.

**Proof.** Consider the pro-complex $\text{Ker}(\pi_A^{-1})$. The pro-group $H_\infty \text{Ker}(\pi_A^{-1})$ is equal to $\text{“lim”} H_n \text{Ker}(\pi_A)$, so it is killed by $n!$ according to the above lemma. We are done by the exact triangle $\text{Ker}(\pi_A^{-1}) \to CC(A)^\gamma \to C^\lambda(A)^\gamma$.

3.3. For a Lie algebra $\mathfrak{g}$ we denote by $C(\mathfrak{g})$ its Chevalley complex with trivial coefficients (see [Lo] 10.1.3), so $C(\mathfrak{g})_n = \Lambda^n \mathfrak{g}$ (the exterior power over $\mathbb{Z}$) and $H_n(\mathfrak{g}) := H_n C(\mathfrak{g})$ are the Lie algebra homology groups. Our $C(\mathfrak{g})$ is a cocommutative counital dg coalgebra, the coproduct map $\delta : C(\mathfrak{g}) \to C(\mathfrak{g} \otimes \mathfrak{g}) = C(\mathfrak{g}) \otimes C(\mathfrak{g})$ comes from the diagonal map $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$.

Consider the symmetric dg algebra $\text{Sym}(C^\lambda(R)[1])$. This is a commutative and cocommutative (unital and counital) Hopf dg algebra; the coproduct comes from the diagonal map $C^\lambda(R)[1] \to C^\lambda(R)[1] \times C^\lambda(R)[1]$.

We recall the key construction of Loday-Quillen and Tsygan, cf. [Lo] 10.2. Consider the matrix Lie algebra $\mathfrak{gl}_r(R)$, $r \geq 1$.

**Theorem-construction.** There is a canonical morphism of dg coalgebras

$$\kappa_R = \kappa_R^* : C(\mathfrak{gl}_r(R)) \to \text{Sym}(C^\lambda(R)[1]).$$

The maps $C(\mathfrak{gl}_r(R))_n \to (\text{Sym}(C^\lambda(R)[1]))_n$ are surjective for $n \leq r$.

**Proof.** We need a preliminary construction from [Lo] 9.2. Let $V$ be a free abelian group of rank $r$ and $I$ be a finite set of order $n$. The symmetric group $\Sigma_I$ acts on $V^\otimes I$ transposing the factors. Let $\theta_I^* : \mathbb{Z}[\Sigma_I] \to \text{End}(V)^{\otimes I}$ be the composition $\mathbb{Z}[\Sigma_I] \to \text{End}(V^\otimes I) \cong \text{End}(V)^{\otimes I}$ of the $\Sigma_I$-action map and the inverse to the tensor product map. The source and target of $\theta_I^*$ are naturally self-dual: the elements $\sigma$ of $\Sigma_I$ form an orthonormal base of the source, the duality for the target is $f, g \mapsto \text{tr}(fg)$. Thus we have the dual map $\theta_I^* : \Sigma_I \to \text{End}(V)^{\otimes I}$.

An explicit formula for $\theta_I^*$: Let $\{I_\alpha\}$ be the orbits of $\sigma$. So $I = \sqcup I_\alpha$ and $I_\alpha$ are naturally cyclically ordered: we write $I_\alpha = \{i^\alpha_1, \ldots, i^\alpha_{n_\alpha}\}$ where $\sigma(i^\alpha_j) = j_{i-1}$ for $1 < j \leq n_\alpha$. Then (see [Lo] 9.2.2)

$$\theta_I^*(\otimes f_i) = \Pi_\alpha \text{tr}(f_i^{\alpha_1} \ldots f_i^{\alpha_{n_\alpha}}).$$

The map $\theta_I^*$ is surjective if $r \geq n$: We need to find for every $\sigma \in \Sigma_I$ some $f_i^\alpha \in \text{End}(V)$, $i \in I$, such that $\theta_I^*(\otimes f_i^\alpha) = \sigma$. Pick a base $\{v_a\}$ of $V$ indexed by elements $\alpha$ of $\cup I$, and define $f_i^\alpha(v_a) = v_{\sigma(i)}$, $f_i^\alpha(v_{\alpha}) = 0$ for $i \neq \alpha$.

The map $\theta_R := \theta_I \otimes \text{id}(R[1])^{\otimes I} : \text{End}(V)^{\otimes I} \otimes (R[1])^{\otimes I} \to \mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I}$ commutes with the $\Sigma_I$-actions (where $\Sigma_I$ acts on $(R[1])^{\otimes I}$ transposing the factors). One has $\text{End}(V)^{\otimes R} = \text{End}(V)^{\otimes (R[1])^{\otimes I}} = (\text{End}(V) \otimes R)^{\otimes I}$ for $V = \mathbb{Z}$. So for $V = \mathbb{Z}$ we get a $\Sigma_I$-equivariant map

$$\theta_R^* : \mathfrak{gl}_r(R)[1]^{\otimes I} \to \mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I}.$$
Lemma. The coinvariants $\left( \mathbb{Z}[\Sigma_l] \otimes (R[1])^{\otimes I} \right)_{\Sigma_l}$ identify naturally with the degree $n$ component of $\text{Sym} (C^\lambda (R)[1])$.

Proof of Lemma. Since $\Sigma_l$ acts on $\mathbb{Z}[\Sigma_l]$ by conjugation, $(\mathbb{Z}[\Sigma_l] \otimes (R[1])^{\otimes I})_{\Sigma_l}$ equals the direct sum, labeled by conjugacy classes of $\sigma \in \Sigma_l$, of coinvariants $((R[1])^{\otimes I})_{S_\sigma}$; here $S_\sigma$ is the centralizer of $\sigma$ in $\Sigma_l$. For an orbit $I_\alpha$ of $\sigma$ let $\sigma_\alpha \in S_\sigma$ be equal to $\sigma$ on $I_\alpha$ and identity outside; set $n_\alpha := |I_\alpha|$. The subgroup $N_\sigma = \Pi_\alpha \mathbb{Z}/n_\alpha$ generated by $\sigma_\alpha$'s is normal in $S_\sigma$. The action of $S_\sigma$ on the set of $I_\alpha$'s yields an identification $S_\sigma/N_\sigma = \Pi_I \Sigma_I$; here $J_I$ is the set of orbits $I_\alpha$ with $n_\alpha = \ell$. One has $((R[1])^{\otimes I})_{N_\sigma} = \otimes_\alpha (C^{\lambda}_{\alpha - 1}(R)[n_\alpha])$, hence $((R[1])^{\otimes I})_{S_\sigma} = \otimes_\ell ((C^{\lambda}_{\ell - 1}(R)[\ell])^{\otimes J_I})_{\Sigma_{J_I}}$, q.e.d. \hfill $\square$

Since $((\mathfrak{gl}_r (R)[1])^{\otimes I})_{\Sigma_I} = \Lambda^n(\mathfrak{gl}_r (R))[n]$ and by the lemma, the $\Sigma_I$-coinvariants map $(\theta_{\Sigma_I})_{\Sigma_I}$ can be rewritten as $\kappa_{\ell r}^{\Sigma_I} : C(\mathfrak{gl}_r (R))_n \to (\text{Sym} (C^{\lambda}(R)[1]))_n$. One checks using (3.3.2) that $\kappa_{\ell r}^{\Sigma_I}$ commutes with the differential; it commutes with the coproducts by construction. The surjectivity assertion follows from surjectivity of $\theta_I$. We are done. \hfill $\square$

3.4. Consider the subcomplex $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$ of $C(\mathfrak{gl}_r (R))$.

Theorem. For every $r \geq 0$ there is a nonzero integer $c_r$ that kills the homology group $H_n \text{Ker}(\kappa_{\ell r}^{\Sigma_I})$ for every ring $R$ and $n \leq r$.

Proof. As in 3.3, set $V = Z^\ell$. Consider the action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}(V) = \mathfrak{gl}_r (\mathbb{Z})$ on $\text{End}(V)^{\otimes n}$. The map $\theta_n : \text{End}(V)^{\otimes n} \to \mathbb{Z}[\Sigma_n]$ from 3.3 commutes with the $\mathfrak{g}$-actions, so $\text{Ker}(\theta_n)$ is a free $\mathbb{Z}$-module with $\mathfrak{g}$-action. By the invariant theory [Lo] 9.2.8 and reductivity of $\mathfrak{g}_Q$, $\text{Ker}(\theta_n)_Q$ is a direct sum of finitely many nontrivial irreducible $\mathfrak{g}_Q$-modules. Let $\epsilon \in U(\mathfrak{g})$ be the Casimir element. Recall that $\epsilon$ lies in the center of the enveloping algebra $U(\mathfrak{g})$, it kills the trivial representation, i.e., $\epsilon \in U(\mathfrak{g})$ and its action on any nontrivial irreducible finite-dimensional representation of $\mathfrak{g}_Q$ is nontrivial. So we can find $h(t) \in t \mathbb{Q}[t]$ such that $h(\epsilon)$ acts as identity on $\text{Ker}(\theta_n)_Q$. Let $c$ be an integer such that $c \epsilon g(t) \in t \mathbb{Z}[t]$.

We check that $(n!c)^2$ kills $H_n \text{Ker}(\kappa_{\ell r}^{\Sigma_I})$; then $c_r := (c^r \Pi_1^{2 r n} r!)^2$ satisfies the condition of the theorem. Let $\pi : (\mathfrak{gl}_r (R)[1])^{\otimes n} \to (\mathfrak{gl}_r (R)[1]) \otimes n = C(\mathfrak{gl}_r (R))[n]$ be the projection and $s : C(\mathfrak{gl}_r (R))[n] \to (\mathfrak{gl}_r (R)[1])^{\otimes n}$ be the map such that $s \pi = \Sigma_{\sigma \in \Sigma_n} \sigma$. Then $\pi$ and $s$ commute with $\mathfrak{g}$-action, $\pi s$ is multiplication by $n!$. Since $\theta_n$ is surjective, one has $s(\text{Ker}(\kappa_{\ell r}^{\Sigma_I})) \subset \text{Ker}(\theta_n) = \text{Ker}(\theta_n)_Q \otimes (R[1])^{\otimes n}$. Thus $h(\epsilon)$'s equals $c$ on $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$; composing with $\pi$, we see that $n!h(\epsilon)$ equals $n!c$ on $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$. The adjoint action of $\mathfrak{gl}_r (R)$ on the Chevalley complex $C(\mathfrak{gl}_r (R))$ is homotopically trivial, so such is the action of the subalgebra $\mathfrak{g}$. So, since $h(\epsilon) \in U(\mathfrak{g})$, its action on $C(\mathfrak{gl}_r (R))$ is homotopic to zero; since $h(\epsilon)$ sends $C(\mathfrak{gl}_r (R))$ to $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$, the action of $h(\epsilon)$ on $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$ is homotopic to zero. The multiplication by $(n!c)^2$ on $\text{Ker}(\kappa_{\ell r}^{\Sigma_I})$ equals $(n!)^2 h(\epsilon)^2$, hence it kills the homology. \hfill $\square$

3.5. For a $p$-adic Lie algebra $\mathfrak{g} = \bigcup_{i \in \mathbb{Q}/p\mathbb{Q}} \mathfrak{g}/p^i \mathfrak{g}$ we have the Chevalley pro-complex $C(\mathfrak{g}') := \text{" limit"} C(\mathfrak{g}/p^i \mathfrak{g})$, etc. So for a $p$-adic $A$ as in 3.2 we have pro-complexes $C(\mathfrak{gl}_r (A))' := \text{" limit"} C(\mathfrak{gl}_r (A))$, $\text{Sym}(C^{\lambda}(A)[1]) := \text{" limit"} \text{Sym}(C^{\lambda}(A)[1])$, and a morphism $\kappa_{\ell r}^{A} = \text{" limit"} \kappa_{\ell r}^{A} : C(\mathfrak{gl}_r (A))' \to \text{Sym}(C^{\lambda}(A)[1])$.

Corollary. The map $\tau_{\leq r} \kappa_{\ell r}^{A} : \tau_{\leq r} C(\mathfrak{gl}_r (A))' \to \tau_{\leq r} \text{Sym}(C^{\lambda}(A)[1])$ is a quasi-isogeny. \hfill $\square$
Remark. Suppose $A$ has bounded $p$-torsion. Then the terms of $C(\mathfrak{gl}(A)^{\gamma})$ and $\text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1])$ have bounded $p$-torsion as well, so the evident maps $C(\mathfrak{gl}(A)^{\gamma}) \otimes^L \mathcal{L} C(\mathfrak{gl}(A)^{\gamma}) \rightarrow C(\mathfrak{gl}(A)^{\gamma}) \otimes C(\mathfrak{gl}(A)^{\gamma})$, $\text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1]) \otimes^L \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1]) \rightarrow \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1]) \otimes \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1])$ are isogenies. So the coproducts on $C(\mathfrak{gl}(A)^{\gamma})$ and $\text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1])$ yield maps $\Delta_C : C(\mathfrak{gl}(A)^{\gamma}) \rightarrow C(\mathfrak{gl}(A)^{\gamma}) \otimes C(\mathfrak{gl}(A)^{\gamma})$, $\Delta_S : \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1]) \rightarrow \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1]) \otimes \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1])$ in pro-$\mathcal{D}(\mathfrak{Ab})$, and one has $(\kappa^p_A)^{\otimes 2} \Delta_C = \Delta_S \kappa^p_A$.

3.6. We want to pass to the limit $r \rightarrow \infty$ in the above corollary to get rid of the truncation $\tau_{<r}$. Since the torsion exponent $c_r$ of the theorem in 3.4 depends on $r$, we cannot pass to the limit directly, and proceed as follows:

In the setting of 3.3 consider the standard embeddings $\mathfrak{gl}_1(R) \subset \mathfrak{gl}_2(R) \subset \ldots$; set $\mathfrak{gl}(R) := \cup \mathfrak{gl}_i(R)$. Maps $\kappa^p_{aR}$ are mutually compatible, so they yield a morphism $\kappa_R : C(\mathfrak{gl}(R)) \rightarrow \text{Sym}(\mathcal{L}^\lambda(R)^{[1]}[1])$. The complexes $C(\mathfrak{gl}_i(R))$ form an increasing filtration on $C(\mathfrak{gl}(R))$. Denote by $C(\mathfrak{gl}(R))_{(a)}$ the décalage of that filtration, so the $n$th component of $C(\mathfrak{gl}(R))_{(a)}$ equals $\{ c \in C(\mathfrak{gl}_{a+n}(R))_n : \partial(c) \in C(\mathfrak{gl}_{a+n-1}(R))_{n-1} \}$. Let $\kappa_{aR}$ be the restriction of $\kappa_R$ to $C(\mathfrak{gl}(R))_{(a)}$.

Corollary. For $a \geq 0$ the map $\kappa_{aR} : C(\mathfrak{gl}(R))_{(a)} \rightarrow \text{Sym}(\mathcal{L}^\lambda(R)^{[1]}[1])$ is surjective and the groups $H_n \text{Ker}(\kappa_{aR})$ are killed by nonzero integers that depend on $a$ and $n$ but not on $R$. \hfill $\Box$

Notice that $C(\mathfrak{gl}(R))_{(a)}$ are not subcoalgebras of $C(\mathfrak{gl}(R))$.

3.7. For a $p$-adic $A$ set $C(\mathfrak{gl}(A)^{\gamma})_{(a)} := \text{"lim"} C(\mathfrak{gl}(A))_{(a)}$. We have the maps

\begin{equation}
(3.7.1) \quad \kappa_{aA} := \text{"lim"} \kappa_{aA} : C(\mathfrak{gl}(A)^{\gamma})_{(a)} \rightarrow \text{Sym}(\mathcal{L}^\lambda(A)^{[1]}[1])
\end{equation}

that are compatible with the embeddings $C(\mathfrak{gl}(A)^{\gamma})_{(a)} \hookrightarrow C(\mathfrak{gl}(A)^{\gamma})_{(a+1)}$.

Corollary. Maps (3.7.1) are quasi-isogenies for $a \geq 0$. \hfill $\Box$

3.8. Consider the adjoint action of $GL_r(R)$ on $\mathfrak{gl}_r(R)$ and the corresponding $GL_r(R)$-action on $C(\mathfrak{gl}_r(R))$.

Corollary. For every $r \geq 0$ there is a nonzero integer $d_r$ such that for each ring $R$ and $n \leq r$ the $GL_r(R)$-action on the subgroup $d_r H_n C(\mathfrak{gl}_r(R))$ of $H_n C(\mathfrak{gl}_r(R))$ is trivial.

Proof. Consider the standard embedding $\mathfrak{gl}_r(R) \times \mathfrak{gl}_r(R) \hookrightarrow \mathfrak{gl}_{2r}(R)$ and the corresponding two embeddings $i_1, i_2 : \mathfrak{gl}_r(R) \hookrightarrow \mathfrak{gl}_{2r}(R)$. Consider the adjoint action of $GL_r(R)$ on $\mathfrak{gl}_{2r}(R)$ via the standard embedding $GL_r(R) \hookrightarrow GL_{2r}(R)$. The restriction of this action on the image of $i_1$ is the adjoint action on $\mathfrak{gl}_r(R)$, and on the image of $i_2$ it is trivial. The theorem in 3.4 implies that the kernel and cokernel of both maps $i_1, i_2 : H_n C(\mathfrak{gl}_r(R)) \rightarrow H_n C(\mathfrak{gl}_{2r}(R))$ are killed by some nonzero integer $b_r$, so we can take $b_r^2$ for $d_r$. \hfill $\Box$

4. The Lazard isomorphism

In chapter V of [L] (see [HKN] for a digest) Lazard identifies the continuous group and Lie algebra cohomology for saturated groups of finite rank. We adapt his argument to the setting of pro-complexes where the finite rank condition becomes irrelevant.
4.1. Let $G$ be a topological group such that open normal subgroups $\{G_α\}$ of $G$ form a base of the topology and the topology is complete, i.e., $G = \varprojlim G/α$. We denote by $C(G')$ the group chain pro-complex “lim” $C(G/α, \mathbb{Z})$. The diagonal embedding $G \rightarrow G \times G$ yields a cocommutative coproduct on $C(G')$.

The example we need: Let $A = \varprojlim A_i$, $A_i := A/p^i$, be a $p$-adic unital ring. Then $GL_r(A)$ carries a topology as above whose base are congruence subgroups $GL_r(A)^{(i)} := \text{Ker}(GL_r(A) \rightarrow GL_r(A_i))$. Thus for any open subgroup $G$ of $GL_r(A)$ we have the pro-complex $C(G^r)$. As in 3.5, we have the Chevalley pro-complex $C(gl_r(A)^r) := \text{lim}^r C(gl_r(A_i))$.

**Theorem.** Suppose $A$ has bounded $p$-torsion. Then for any open $G$ that lies in $GL_r(A)^{(1)}$ if $p$ is odd and in $GL_r(A)^{(2)}$ if $p = 2$ there is a natural quasi-isogeny

$$(4.1.1) \quad \zeta = \zeta_G : C(G^r)Q \rightarrow C(gl_r(A)^r)Q.$$

These $\zeta_G$ are compatible with the coproduct, embeddings of $G$, the adjoint action of $GL_r(A)$, the embeddings $GL_r(A) \rightarrow GL_{r+1}(A)$, and the morphisms of $A$.

The proof takes 4.2–4.10. We start with general remarks about the homology of topological groups of relevant kind (4.2–4.4), then we recast the theorem, following Drinfeld’s suggestion, as a general assertion about $p$-adic Lie algebras (4.5–4.6). The map $\zeta$ is constructed in 4.7, and we check that it is a quasi-isogeny in 4.8–4.10.

4.2. Let $K$ be a discrete group. Its finite filtration $K = K_n \supset \ldots \supset K_1 \supset K_0 = \{1\}$ by normal subgroups is called $p$-special if every successive quotient $K_i/K_{i-1}$ is an abelian group killed by $p$. We say that $K$ is $p$-special if it admits a $p$-special filtration.

**Lemma.** Suppose that $K$ admits an $n$-step $p$-special filtration. Let $F$ be an abelian group viewed as a $K$-module with trivial $K$-action. Then $H_a(K, F)$, $a > 0$, is killed by $f(a, n) = p^{(n-1)}$.

**Proof.** Since $C(K, F) = C(K, \mathbb{Z}) \otimes F$, the group $H_a(K, F)$ is isomorphic to a direct sum of $H_a(K, \mathbb{Z}) \otimes F$ and $\text{Tor}^F_a(H_a-1(K, \mathbb{Z}), F)$. So it is enough to consider the case $F = \mathbb{Z}$.

Let $K' := K_{n-1}$ be the next to the top term of the filtration, so the lemma for $K'$ is known by induction. Let us compute $H_a(K, \mathbb{Z})$ using Hochschild-Serre spectral sequence $E^2_{i,j} = H_i(K/K', H_j(K', \mathbb{Z}))$. Term $E^2_{i,j}$ is killed by $p^{(j-2)}$ if $i + j > 0$: indeed, for $j > 0$ this follows by the induction assumption, and $E^2_{i,0} = H_i(K/K', \mathbb{Z})$ is killed by $p$ for $i > 0$ since $K/K'$ is a direct sum of copies of $\mathbb{Z}/p$ (use Künneth formula and the standard computation of $H_*(\mathbb{Z}/p, \mathbb{Z})$). Thus $H_a(K, \mathbb{Z})$, $a > 0$, is killed by $p\sum_{0 \leq j \leq n}(i+j-2) = p^{(n-1)}$, q.e.d. □

**Remarks.** (i) If $K'$ is a normal subgroup of $K$, then $K$ is $p$-special if and only if both $K'$ and $K/K'$ are $p$-special.

(ii) The assertion of the lemma need not be true if $F$ is an arbitrary $K$-module, as follows from the next exercise:

**Exercise.** For any finite group $K$ and any $a > 0$ find a $K$-module $F$ such that $H_a(K, F) = \mathbb{Z}/|K|\mathbb{Z}$. 
4.3. We say that a topological group $G = \lim G/G_\alpha$ as in 4.1 is $p$-special if every $G/G_\alpha$ is a $p$-special discrete group.

**Example.** $GL_r(A)^{(1)}$, hence every its open subgroup, is $p$-special.

**Remark.** If $G'$ is a closed normal subgroup of $G$ then $G$ is $p$-special if and only if both $G'$ and $G/G'$ are $p$-special.

**Corollary.** Suppose $G$ is $p$-special. If the map $C(G'') \to C(G')$ is a quasi-isogeny for all $G'$ in some base of open normal subgroups of $G$, then the same is true for every open $G' \subset G$.

**Proof.** Pick a normal open $G'' \subset G'$ such that $C(G'') \to C(G')$ is a quasi-isogeny. Our $G'$ is an extension of $K := G'/G''$ by $G''$, and we compute $H.C(G'')$ using the Hochschild-Serre spectral sequence. It is enough to check that $H^i\alpha, i$ for all $i$.

**Corollary.** Suppose $G$ is $p$-special. If the map $C(G'') \to C(G')$ is a quasi-isogeny for all $G'$ in some base of open normal subgroups of $G$, then the same is true for every open $G' \subset G$.

**Proof.** Replacing $\mathbb{Z}/p^i$ by $\text{Cone}(p^i : \mathbb{Z} \to \mathbb{Z})$, we see that the homotopy fiber of the arrow is a pro-complex $\text{lim}_{\alpha} \tau_0 C(G/G_\alpha, \mathbb{Z}_i)$ where $\mathbb{Z}_i$ are copies of $\mathbb{Z}$ arranged into a projective system $\ldots \mathbb{Z} \to \mathbb{Z}$. We want to show that its homology $\text{lim}_{\alpha} H_0(G/G_\alpha, \mathbb{Z}_i)$ is killed by a power of $p$ if $a > 0$ and that the map $H_0C(G'') \to H_0(K, H_0C(G''))$ is an isogeny. We can replace $H_0C(G'')$ by $H_0C(G')$ since the map $H_0C(G'') \to H_0C(G')$ is an isogeny. Now the action of $K$ on $H_0(K, H_0C(G', \mathbb{Z})$ is trivial, and we are done by the lemma in 4.2.

4.4. It will be convenient to replace integral chains by $p$-adic ones: For $G$ as in 4.1 set $C(G)^{-} := \text{lim}_{\alpha} C(G/G_\alpha, \mathbb{Z}/p^i)$. There is an evident map $C(G)^{-} \to C(G)^{+}$.

**Corollary.** If $G$ is $p$-special then $\tau_0 C(G)^{-} \approx \tau_0 C(G)^{+}$.

4.5. Let us recast the theorem in 4.1 as a general result about $p$-adic Lie algebras:

Let $\mathfrak{g}$ be any $p$-adic Lie algebra, so $\mathfrak{g} = \lim \mathfrak{g}/p^i\mathfrak{g}$. Suppose $\mathfrak{g}$ has no $p$-torsion and its Lie bracket is divisible by $p$ if $p \neq 2$ and by 4 if $p = 2$. Then the Campbell-Hausdorff series converges and defines a topological group structure on $\mathfrak{g}$. Denote the corresponding group by $G_\mathfrak{g}$, and let

$$\log : G_\mathfrak{g} \to \mathfrak{g} : \exp$$

be the mutually inverse “identity” identifications. Consider the pro-complexes $C(G_\mathfrak{g}), C(\mathfrak{g}^{-})$ of group and Lie algebra chains (see 4.1, 3.5).

**Theorem.** There is a canonical quasi-isogeny

$$\zeta = \zeta_\mathfrak{g} : C(G_\mathfrak{g})_\mathbb{Q} \approx C(\mathfrak{g}^{-})_\mathbb{Q}$$

functorial with respect to morphisms of $\mathfrak{g}$’s.

**Remarks.** (i) The topological group $G_\mathfrak{g}$ is $p$-special. More precisely, set $\mathfrak{g}^{(i)} := p^{i-1}\mathfrak{g}$ where $m = 1, i \geq 1$ if $p$ is odd, and $m = 2, i \geq 2$ if $p = 2$ (the reason for that normalization will be clear later). Then $G^{(i)}_\mathfrak{g} := \exp(\mathfrak{g}^{(i)})$ are normal subgroups that form a basis of the topology on $G_\mathfrak{g}$, the successive quotients $G^{(i)}_\mathfrak{g}$ are abelian,
and one has a canonical identification of abelian groups \( \text{gr}^{(i)} \exp : \text{gr}^{(i)} g \rightarrow \text{gr}^{(i)} G_g \).

(ii) Since \( \exp(pa) = \exp(a)p \), the map \( g \mapsto g^p \) yields bijections \( G_g^{(p)} \rightarrow G_g^{(i+1)} \) and \( \mathbb{F}_p \)-vector space isomorphisms \( \text{gr}^{(i)} G_g \rightarrow \text{gr}^{(i+1)} G_g \).

(iii) Consider the \( p \)-adic Chevalley pro-complex \( C(g)^\ast := \text{"lim"}(C(g) \otimes \mathbb{Z}/p^i = \text{"lim"}(C(g/p^i g, \mathbb{Z}/p^i))_. \) One has \( \tau_{>0} C(g)^\ast = \tau_{>0} C(g)^{\ast} \). Since the chain complexes in (4.5.2) are direct sums of \( \mathbb{Z} \) and their \( \tau_{>0} \) truncations, the corollary in 4.4 implies that (4.5.2) amounts to a natural quasi-isogeny

\[
(4.5.3) \quad \zeta = \zeta_g : C(G_g)^\ast \rightarrow C(g)^\ast.
\]

**Question** (Drinfeld). Is there an assertion for \( p \)-torsion Lie algebras behind the theorem? More precisely, let \( g \) be a \( p \)-torsion Lie algebra, and suppose we have another Lie bracket on \( g \) such that the original bracket equals \( p \) times the new one if \( p \) is odd or 4 times the new one if \( p = 2 \). Then we have the group \( G_g \) as above, and there is a natural map \( C(G_g) \rightarrow C(g)^\ast \), where \( g^\ast \) is \( g \) with the new Lie bracket, defined as in 4.7 below. What can one say about this map?

### 4.6. Lemma

The theorem in 4.5 implies that in 4.1.

**Proof.**

(a) It is enough to consider the case when \( A \) has no torsion: Indeed, for an arbitrary \( A \) let \( A_{\text{tors}} \subset A \) be the ideal of \( p \)-torsion elements, \( \hat{A} := A/A_{\text{tors}} \), and let \( \hat{G} \) be the image of \( G \) in \( \text{GL}_r(\hat{A}) \). Then \( G \) is an extension of \( \hat{G} \) by \( K := G \cap (1 + \text{Mat}_r(A_{\text{tors}})) \), and \( K \) is discrete since \( A \) has bounded \( p \)-torsion. Since \( K \) is semi-special (see the remark in 4.3), the Hochschild-Serre spectral sequence and 4.2 show that the map \( C(G^\ast) \rightarrow C(\hat{G})^\ast \) is a quasi-isogeny. The map \( C(p^m g_r(\hat{A})^\ast) \rightarrow C(p^m g_r(\hat{A})^\ast) \) is a quasi-isogeny as well, and we are done.

(b) It is enough to construct quasi-isogeny \( \zeta_G \) of (4.1.1) when \( G \) is a congruence subgroup: Indeed, by the corollary in 4.3, we know then that embeddings of \( G \)'s quotient Lie algebras induce (4.4.1) as composition of the above quasi-isogenies, and we are done. The compatibility assertions in the theorem 4.1 follow since \( \zeta_G \) is functorial with respect to morphisms of \( g \)’s.

4.7. **Proof of the theorem in 4.5.** From now on \( g \) is a Lie algebra as in 4.5 and \( G = G_g \). Let us construct the map \( \zeta_g : G(G)^\ast \rightarrow C(g)^\ast \) of (4.5.3).

Let \( U(g)^\ast := \text{"lim"} U(g)_i \), where \( U(g)_i = U(g)/p^i U(g) = U_{\mathbb{Z}/p^i}(g/p^i g) \), be the \( p \)-adic completion of the enveloping algebra \( U(g) \). One has

\[
(4.7.1) \quad C(g)^\ast = \mathbb{Z}_p \hat{\otimes}_{U(g)} \mathbb{Z}_p := \text{"lim"}(\mathbb{Z}/p^i) \hat{\otimes}_{U(g)} (\mathbb{Z}/p^i).
\]
Set \( g' := p^{-m}g \subset g \otimes \mathbb{Q} \) where \( m = 1 \) if \( p \neq 2 \) and \( m = 2 \) if \( p = 2 \). The conditions on the Lie bracket on \( g' \) mean that it extends to the Lie bracket on \( g' \supset g \). We have same objects as above for \( g' \), and

\[
C(g') = Z_p \otimes_{U(g')} \mathbb{Z}_p := \text{"lim"} (\mathbb{Z}/p^i) \otimes_{U(g')} (\mathbb{Z}/p^i).
\]

Similarly, consider the Iwasawa algebra \( Z_p[[G]] := \text{lim}(\mathbb{Z}/p^i)[G/G^{(i)}] \) where \( G^{(i)} \) are as in Remark (i) in 4.5. Then

\[
C(G) = \mathbb{Z}_p \otimes_{Z_p[[G]]} \mathbb{Z}_p := \text{"lim"} (\mathbb{Z}/p^i) \otimes_{Z_p[[G]/p^{(i)}]} (\mathbb{Z}/p^i).
\]

One has natural maps of topological algebras

\[
Z_p[[G]] \xrightarrow{\eta} U(g')^- \xleftarrow{i} U(g)^-
\]

where \( i \) comes from the embedding \( g \hookrightarrow g' \) and \( \eta \) is defined as follows. The exponential series in \( U(g')^- \) converges on \( p^mU(g')^- \) and yields a map

\[
\exp_U : p^mU(g')^- \to (1 + p^mU(g')^-) \subset U(g')^-^\times
\]

that satisfies the Campbell-Hausdorff formula. We get a continuous group homomorphism (here \( \log(g) \in g \subset g' \) is as in (4.5.1))

\[
\eta : G \to U(g')^-^\times, \quad \xi(g) := \exp_U(\log(g)).
\]

We define \( \eta \) in (4.7.4) as the morphism of topological rings that extends (4.7.6). Notice that \( \eta = \lim \eta_i \) where \( \eta_i : (\mathbb{Z}/p^i)[G/G^{(i)}] \to U(g')^- = U\mathbb{Z}/p^i(g'/p^i g') \).

Applying (4.7.4) to the Tor pro-complexes of (4.7.1)–(4.7.3) we get the maps

\[
G(G)^- \xrightarrow{\xi} C(g')^- \xleftarrow{i} C(g)^-.
\]

The map \( i : C(g)^- \to C(g')^- \) is evidently a quasi-isogeny since it identifies \( C(g)^- \) with \( p^mC(g')^- \). We define \( \zeta_\alpha : G(G)^- \otimes \mathbb{Q} \to C(g)^- \otimes \mathbb{Q} \) of (4.5.3) as the composition \( i^{-1}\xi \) of (4.7.7).

It remains to prove that the Tor map \( \xi : G(G)^- \to C(g')^- \) for \( \eta \) is a quasi-isogeny.

4.8. Let \( I_p := \text{Ker}(\mathbb{Z}[G] \to \mathbb{Z}/p) \) be the \( p \)-augmentation ideal; set \( Z_p[[G]]^- := \text{lim} \mathbb{Z}[G]/I_p^i \). Remark (ii) in 4.5 implies that the \( I_p \)-adic topology is weaker than the Iwasawa algebra topology from 4.7,\(^{10}\) so we have a continuous map \( \alpha : Z_p[[G]] \to Z_p[[G]]^- \). \( \alpha(g) = g \). Since \( \eta(I_p) \subset pU(g')^- \), \( \eta \) is continuous for the \( I_p \)-adic topology, i.e., we have a continuous map \( \tilde{\eta} : Z_p[[G]]^- \to U(g')^- \) such that \( \eta = \tilde{\eta}\alpha \). Set

\[
C(G)^- = Z_p \otimes_{Z_p[[G]]^-} \mathbb{Z}_p := \text{"lim"} (\mathbb{Z}/p^i) \otimes_{Z_p[[G]/I_p]} (\mathbb{Z}/p^i),
\]

and let \( \tau, \tilde{\xi} \) be the Tor maps for \( \alpha \) and \( \tilde{\eta} \). Since \( \xi = \tilde{\xi}\tau \), the theorem in 4.5, hence that in 4.1, follows from the next result to be proven in 4.9–4.10 below:

\(^{10}\)By loc. cit., it is enough to find for any \( n > 0 \) some \( a = a(n) \) such that \( g^{p^n} \) equals 1 in \( \mathbb{Z}[G]/I_p^n \) for every \( g \in G \). If \( p^k > n \) then \( g^{p^k} \) equals 1 in \( \mathbb{Z}[G]/I_p^n + p\mathbb{Z}[G] \), so we can take \( a = b + n \).
Theorem. (i) The map $\xi : C(G^-) \to C(g')^-$ is a quasi-isogeny.
(ii) The map $\tau : C(G') \to C(G^-)$ is a quasi-isomorphism.

Remark. The map $\alpha : \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]]^-$ is not an isomorphism unless $g$ is a finitely generated $\mathbb{Z}_p$-module: Indeed, $\mathbb{Z}_p[[G]]$ has discrete quotient $\mathbb{F}_p[G/G(2)] = \mathbb{F}_p[g/pg]$ (we assume $p$ is odd), while the corresponding quotient of $\mathbb{Z}_p[[G]]^-$ is the completion of that group algebra with respect to powers of the augmentation ideal.

4.9. Proof of 4.8(i). We adapt Lazard's argument from chapter V of [L]:
Below we identify the graded ring $gr\mathbb{Z}_p$ for the $p$-adic filtration with $\mathbb{F}_p[t]$, deg $t = 1$, where $t$ is class of $p$ in $p\mathbb{Z}/p^2\mathbb{Z} = gr^1\mathbb{Z}_p$. For every flat $\mathbb{Z}_p$-algebra $B$ the associated graded ring $gr\mathbb{B}$ for the $p$-adic filtration on $B$ identifies in the evident manner with $B \otimes \mathbb{F}_p[t] = B_1[t]$ where $B_1 := B/pB$. For example, we have $gr U(g')^- = U_{\mathbb{F}_p}(h)(t) = U_{\mathbb{F}_p}[t](b(t))$ where $b := g'/pg'$. Recall that $\bar{g} = p^m g'$.

Let $R$ be the closure of the image of $\bar{\eta}$ (or $\eta$) in $U(g')^-$. Let $R^{[n]} := R \cap p^n U(g')^-$ be the ring filtration on $R$ induced by the $p$-adic filtration on $U(g')^-$ and $gr U(g')^-$ be the associated graded ring.

Lemma. (i) The map $\bar{\eta} : \mathbb{Z}_p[[G]]^-> R$ is a homeomorphism of topological rings.
(ii) $gr^{[1]} R$ is the $\mathbb{F}_p[t]$-subalgebra of $U_{\mathbb{F}_p}(h)(t)$ generated by $t^m h$ which is $U_{\mathbb{F}_p}[t](t^m h(t)) \subset U_{\mathbb{F}_p}[t](b(t))$.

Proof. Let $I := \text{Ker}(\mathbb{Z}[G] \to \mathbb{Z})$ be the augmentation ideal, and let $\mathbb{Z}[G]^{[n]}$ be the sum of the ideals $p^n I^n$ where $a, b \geq 0$ and $a + mb \geq n \geq 0$. This is a ring filtration on $\mathbb{Z}[G]$, and the topology it defines equals the $I_p$-adic topology. The ring $gr^{[1]} \mathbb{Z}[G]$ is a graded $\mathbb{F}_p[t]$-algebra generated by $G := I/(I^2 + pI) \subset gr^{[n]} \mathbb{Z}[G]$.

One has $\eta(\mathbb{Z}[G]^{[1]} \subset R^{[1]}$, so we have the map of graded rings $gr^{[1]} \eta : gr^{[1]} \mathbb{Z}[G] \to gr U(g')^-$. To prove the lemma it is enough to show that $gr^{[1]} \eta$ is injective.

Recall that $I/I^2 = G/[G, G]$, so $G = G/[G, G]$ that identifies canonically with $g/pg$ by Remarks (i), (ii) in 4.5. The map $gr^{[1]} \eta|_G$ is the embedding $g/pg = t^m \mathbb{Z} \hookrightarrow U_{\mathbb{F}_p}(h)(t)$. Therefore the $\mathbb{F}_p[t]$-submodule of $gr^{[1]} \mathbb{Z}[G]$ generated by $G$ equals $G[t] = \mathbb{F}_p[t] \otimes G$ and $gr^{[1]} \eta$ identifies it with $t^m b(t)$.

Notice that $[G, G]$ lies in $t^m G \subset gr^{[2n]} \mathbb{Z}[G]$. Thus $G[t]$ is a graded Lie $\mathbb{F}_p[t]$-subalgebra of $gr^{[1]} \mathbb{Z}[G]$, so we have a surjective map of graded associative algebras $\nu : U_{\mathbb{F}_p}[t](G[t]) \to gr^{[1]} \mathbb{Z}[G]$. The composition $(gr^{[1]} \eta|_G) \circ \nu : U_{\mathbb{F}_p}[t](G[t]) \to U_{\mathbb{F}_p}(b)(t)$ is the morphism of enveloping algebras that corresponds to the embedding of Lie algebras $G[t] \hookrightarrow t^m h(t) \hookrightarrow h(t)$. It is injective, hence $gr^{[1]} \eta$ is injective, q.e.d. □

Set $C(R^-) := \lim^\to \big((\mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n}(\mathbb{Z}/p^n)\big)$, so the embedding $R \hookrightarrow U(g')^-$ yields a map of pro-complexes $C(R^-) \to C(g')^-$ (see (4.7.2)). By (i) of the lemma, 4.8(i) amounts to the next assertion:

Proposition. The morphism $C(R^-) \to C(g')^-$ is a quasi-isogeny.

Proof. Let $K$ be the Chevalley chain complex of the Lie $\mathbb{F}_p[t]$-algebra $t^m h(t)$ with coefficients in the free module $U_{\mathbb{F}_p}[t](t^m h(t))$. This is a graded free $U_{\mathbb{F}_p}[t](t^m h(t))$-resolution of $\mathbb{F}_p[t]$ (with trivial Lie algebra action) with terms $K_n = t^m \Lambda^2 h(t) \otimes_{\mathbb{F}_p}$

\[11\] Indeed, one has $\mathbb{Z}[G]^{[mn]} \subset I^m \subset \mathbb{Z}[G]^n$. 


Thus $P_n$ is a complete filtered free $R$-module with generators in filtered degree $mn$.

Let $S$ be the induced complete free $R$-module with generators in filtered degree $mn$.

Using resolutions $P$ and $T$, one can realize the homotopy map of the proposition as composition $C(R) \xrightarrow{\sim} \text{"lim"}(\mathbb{Z}/p^i) \otimes_{R/R[i]} (P/P[i]) \xrightarrow{\sim} \text{"lim"}(\mathbb{Z}/p^i) \otimes_R (P/P[i]) = \text{"lim"}(\mathbb{Z}/p^i) \otimes_{U[g]} (S/S[i])$.

To finish the proof, we show that $\alpha$ is a quasi-isomorphism. It is enough to check that $\text{"lim"}_i \text{Tor}_a^{R/R[i]}(\mathbb{Z}/p^i, P_n/P_n[i]) = 0$ for every $n, j$ and any $a > 0$. This follows since the map $\text{Tor}_a^{R/R[0]}(\mathbb{Z}/p^i, P_n/P_n[i]) \rightarrow \text{Tor}_a^{R/R[i]}(\mathbb{Z}/p^i, P_n/P_n[i])$ factors through $\text{Tor}_a^{R/R[i]}(\mathbb{Z}/p^i, P_n/P_n[i])$ and $P_n/P_n[i]$ is a free $R/R[i]$-module. \hfill $\square$

4.10. Proof of 4.8(ii). (a) Set $C(G, \mathbb{Z}/p^i)^\sim := \text{"lim"}(\mathbb{Z}/p^i) \otimes_{U[G]} (\mathbb{Z}/p^i) = C(G, \mathbb{Z}/p^i)^{\sim} := \text{"lim"}(\mathbb{Z}/p^i) \otimes_{Z[G]} (\mathbb{Z}/p^i)$. Since $C(G)^{\sim}$ and $C(G)^{\sim}$ are homotopy limits of pro-complexes $C(G, \mathbb{Z}/p^i)^\sim$ and $C(G, \mathbb{Z}/p^i)^{\sim}$, see (4.7.3) and (4.8.1), assertion 4.8(ii) would follow if we show that the natural maps $\tau : C(G, \mathbb{Z}/p^i)^\sim \rightarrow C(G, \mathbb{Z}/p^i)^{\sim}$ are quasi-isomorphisms. By devissage, it is enough to check that $\tau : C(G, \mathbb{Z}/p^i) \rightarrow C(G, \mathbb{Z}/p^i)$ is a quasi-isomorphism.

(b) We compute $C(G, \mathbb{Z}/p^i)^{\sim}$ using resolution $P$ from the proof of the proposition in 4.9: As in loc. cit., we have quasi-isomorphisms $C(G, \mathbb{Z}/p^i)^{\sim} \xrightarrow{\sim} \text{"lim"}(\mathbb{Z}/p) \otimes_{U[G]} (P/P[i])$ where $\alpha$ is a quasi-isomorphism by the argument at the end of 4.9. By the construction, $\text{"lim"}(\mathbb{Z}/p) \otimes_R (P/P[i])$ equals $(\mathbb{Z}/p) \otimes_{U[p]}(t^m \mathbb{h}[t]/t^m+1 \mathbb{h}[t])$, which is the same as the Chevalley complex of the abelian Lie $\mathbb{F}_p$-algebra $t^m \mathbb{h}[t]/t^m+1 \mathbb{h}[t]$ with coefficients in $\mathbb{F}_p$. Thus, using the multiplication by $t^m$ isomorphism $\mathbb{h} \rightarrow t^m \mathbb{h}[t]/t^m+1 \mathbb{h}[t]$, we get identification $H_n C(G, \mathbb{Z}/p^{\sim}) = \Lambda_{\mathbb{F}_p}^n (\mathbb{h})$.

(c) Let us compute $C(G, \mathbb{Z}/p)^{\sim} = \text{"lim"}C(G/G^{(i)}, \mathbb{Z}/p)$. This is a cocommutative counital $\mathbb{F}_p$-coalgebra in the usual way.

Proposition. There is a canonical isomorphism of coalgebras

$$H C(G, \mathbb{Z}/p)^{\sim} \xrightarrow{\sim} \Lambda_{\mathbb{F}_p}^n (\mathbb{h}).$$

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12One constructs silly truncations of $P$ by induction using $\text{gr}^{[-1]} \text{Ker}(\partial) = \text{Ker}(\text{gr}^{[-1]} \partial)$ where $\partial$ is the differential of $P$. 

Proof. Set $G_{i,n} := G_i^{(n)}/G_{i+1}^{(n+1)}$. Below we write $H_\cdot(\zeta/p)$ for $H_\cdot(\zeta/p)$.  

(i) By Remark (ii) in 4.5 the projection $G_{i,n} \to G_{1,n} = \mathfrak{h}$, exp$(p^na) \mapsto a \bmod p^\ell$, yields an isomorphism $\varepsilon_{i,n} : H_1(G_{i,n}) \to H_1(\mathfrak{h}) = \mathfrak{h}$; here we view $\mathfrak{h}$ as mere abelian group.

(ii) The kernel of the projection $G_{i+1,n} \to G_{i,n}$ is $G_{1,n+i} = \mathfrak{h}$, and the adjoint action of $G_{i+1,n}$ on it is trivial. Thus the corresponding Hochschild-Serre spectral sequence converging to $H_\cdot(G_{i+1,n})$ has $E^2_{p,q} = H_p(G_{i,n}) \otimes H_q(G_{1,n+i})$. By (i) for $G_{i+1,n}$, the component $\nu_{i,n} : H_2(G_{i,n}) \to H_1(G_{1,n+i}) = \mathfrak{h}$ of the differential in $E^2$ is surjective.

(iii)\footnote{The reader may find it more convenient to follow the computation in (iii) and (iv) using the dual language of cohomology (the coalgebra structure turns then into the algebra one).} Case $p \neq 2$: Then $\Lambda_{\mathfrak{g}_p}(\mathfrak{h})$ is the free cocommutative counital graded $\mathbb{F}_p$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 1. Since $H_\cdot(G_{i,n})$ are cocommutative counital graded $\mathbb{F}_p$-coalgebras, we have compatible morphisms of graded coalgebras $\tilde{\varepsilon}_{i,n} : H_\cdot(G_{i,n}) \to \Lambda_{\mathfrak{g}_p}(\mathfrak{h})$ that equal $\varepsilon_{i,n}$ in degree 1. We define (4.10.1) as $\lim_{\nu_{i,n}} \tilde{\varepsilon}_{i,n}$. To see that it is an isomorphism, let us compute the homology of $G_{i,n}$.

The free cocommutative counital graded $\mathbb{F}_p$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 2 equals $\Gamma_{\mathfrak{g}_p}^2(\mathfrak{h})$.\footnote{Here $\Gamma_{\mathfrak{g}_p}(\mathfrak{h})$ is the divided powers Hopf $\mathbb{F}_p$-algebra generated by a copy of $\mathfrak{h}$.} Thus $\nu_{i,n}$ extends to a morphism of graded coalgebras $\tilde{\nu}_{i,n} : H_\cdot(G_{i,n}) \to \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h})$. Let us show that the morphism

$$
\tilde{\varepsilon}_{i,n} \otimes \tilde{\nu}_{i,n} : H_\cdot(G_{i,n}) \to \Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \otimes \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h})
$$

of graded coalgebras is an isomorphism.

For $i = 1$ one has $G_{1,n} = \mathfrak{h}$, and the assertion follows from the standard calculation of $H_\cdot(\mathbb{Z}/p)$ combined with the Künneth formula.\footnote{Use the fact that for every homomorphism $\epsilon_a : \mathbb{Z}/p \to G_{1,n}$, $\epsilon_a(x) = (1 + ap^\ell)^x$, $a \in \mathfrak{h}$, one has $\nu_{i,n}(\epsilon_a(\xi)) = 0$, where $\xi$ is the standard generator of $H_2(\mathbb{Z}/p)$ (which follows since the map $x \mapsto x^p$ on $G_2\mathbb{Z}$ yields an isomorphism $G_{1,n} \cong G_{1,n+i}$).}

We proceed by induction by $i$ using the spectral sequence from (ii) to pass from $i$ to $i+1$: By the induction assumption and since we know that (4.10.2) is an isomorphism for $G_{1,n+i}$, one has $E_{p,q}^2 = H_p(G_{i,n}) \otimes H_q(G_{1,n+i}) = (\Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \otimes \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h})) \otimes (\Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \otimes \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h}))$. By the construction of $\nu_{i,n}$, the differential in $E^2$ identifies the second copy of $\mathfrak{h}$ (which lies in $H_2(G_{1,n})$) with the third copy of $\mathfrak{h}$ (which is in $H_3(G_{1,n+i})$). Since the composition of $\mathfrak{h} \to H_2(G_{1,n+i}) \to H_2(G_{1,n+1})$ is injective (indeed, its composition with $\nu_{i+1,n}$ equals $\text{id}_\mathfrak{h}$), the map from the fourth copy of $\mathfrak{h}$ (which lies in $H_2(G_{1,n+i})$) to $E_0^\infty$ is injective. The compatibility of the differentials with the coproduct implies then that $E^3_{p,q} = E^2_{p,q} \otimes \Gamma_{\mathfrak{g}_p}^3(\mathfrak{h}) = E^\infty_{p,q}$, therefore $\tilde{\varepsilon}_{i+1,n} \otimes \tilde{\nu}_{i+1,n}$ is an isomorphism.

Now, since $\nu_{i,n}$ vanishes on the image of $H_2(G_{i+1,n}) \to H_2(G_{1,n})$, the transition map $H_\cdot(G_{i+1,n}) \to H_\cdot(G_{1,n})$ written in terms of (4.10.2) kills the copy of $\mathfrak{h}$ in degree 2. Thus, since the transition map is a map of coalgebras, it equals the composition $\Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \otimes \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h}) \to \Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \to \Lambda_{\mathfrak{g}_p}(\mathfrak{h}) \otimes \Gamma_{\mathfrak{g}_p}^2(\mathfrak{h})$. This implies that (4.10.1) is an isomorphism, q.e.d.

(iv) Case $p = 2$: The free cocommutative counital graded $\mathbb{F}_2$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 1 equals $\Gamma_{\mathfrak{g}_2}(\mathfrak{h})$. The morphism of graded coalgebras $\tilde{\varepsilon}_{1,m} : H_\cdot(G_{1,n}) \to \Gamma_{\mathfrak{g}_2}(\mathfrak{h})$ that extends $\varepsilon_{1,n}$ is an isomorphism (by the standard computation of $H_\cdot(\mathbb{Z}/2)$ and Künneth formula).
As in (iii), the free cocommutative counital graded $F_2$-coalgebra cogenerated by a copy of $h$ in degree 2 equals $\Gamma_{F_2}^2(h)$, and $\nu_{i,n}$ extends to a morphism of graded coalgebras $\tilde{\nu}_{i,n}: H_2(G_{i,n}) \to \Gamma_{F_2}^2(h)$. Since $\nu_{1,m}$ vanishes on the image of $H_2(G_{2,n}) \to H_2(G_{1,n})$ and $\Lambda_{F_2}(h) \subset \Gamma_{F_2}^2(h)$ is the largest graded subcoalgebra of $H_2(G_{1,n})$ on which $\nu_{1,n}$ vanishes, we see that $\tilde{\nu}_{i,n}$ takes values in $\Lambda_{F_2}(h)$ if $i \geq 2$, i.e., $\tilde{\nu}_{i,n}$ yields $\tilde{\nu}_{i,n}: H_2(G_{i,n}) \to \Lambda_{F_2}(h)$. We define (4.10.1) as $\lim_{i \to \infty} \tilde{\nu}_{i,n}$. 

Now for $i \geq 1$ we have the map $\tilde{\nu}_{i,n} \otimes \tilde{\nu}_{i,n}: H_2(G_{i,n}) \to \Lambda_{F_2}(h) \otimes \Gamma_{F_2}^2(h)$ of graded coalgebras. One checks that it is an isomorphism in the same way as we did in (iii), and this implies that (4.10.1) is an isomorphism, q.e.d. □

(d) To finish the proof of 4.8(ii), hence of the theorems in 4.5 and in 4.1, it remains to notice that $H.C(G,\mathbb{Z}/p)$ is a cocommutative coalgebra in the same way as $H.C(G,\mathbb{Z}/p)$ is, and our map $\tau: H.C(G,\mathbb{Z}/p) \to H.C(G,\mathbb{Z}/p)$ is a morphism of graded coalgebras. We have identified both coalgebras with $\Lambda_{\mathbb{Z}/p}(h)$ in (b) and (c). Since $\tau$ equals $id_h$ in degree 1, it equals $id_{\Lambda_{\mathbb{Z}/p}(h)}$ in all degrees, q.e.d. □

4.11. We are in the setting of 4.1. The theorem in 4.1 can be partially extended to some other subgroups of $GL_r(A)$ due to the next proposition:

Let $G' \subset GL_r(A)$ be any open p-special subgroup (see 4.3). E.g. the latter condition holds if the image of $G'$ in $GL_r(A_1)$ consists of upper triangular matrices with 1 on the diagonal. Let $G \subset G'$ be a small enough congruence subgroup.

**Proposition.** The map $H_nC(G') \to H_nC(G''')$ is an isogeny if $n \leq r$.

**Proof.** Consider the adjoint action of $GL_r(A)$ on $H_nC(G')$. Combining the theorem in 4.1 with the corollary in 3.8, we see that for some nonzero integer $c$, this action is trivial on $cH_nC(G')$. Now compute $H_nC(G''')$ using the Hochschild-Serre spectral sequence for $G \subset G'$, and use the lemma in 4.2. □

**Question.** Is it true that the adjoint action of $GL_r(A)$ on $C(\mathfrak{g}_r(A_1))$ is trivial? If yes, this can replace the reference to 3.8 in the above proof, and the assumption $n \leq r$ can be dropped.

5. **Relative K-groups and Suslin’s stabilization**

5.1. We identify the symmetric group $\Sigma_r$ with a subgroup of $GL_r$ in the usual way (the transpositions of standard base vectors). For every $\sigma \in \Sigma_r$ let $U^\sigma \subset GL_r$ be the $\sigma$-conjugate of the subgroup of unipotent upper triangular matrices. For a ring $R$ we have Volodin’s simplicial set $X_r(R) := \cup_{\sigma \in \Sigma_r} B_{U^\sigma}(R) \subset B_{GL_r}(R)$. It is clear that $X_r(R)$ is connected, $\pi_1(X_r(R)) = St_r(R)$. As usually, $GL(R)$ is the union of $GL_1(R) \subset GL_2(R) \subset \ldots$, $X(R) := \cup X_r(R)$, etc.

For a cell complex $P$ let $P \to \tau \leq P$ be the $n$th Postnikov truncation of $P$, so $\pi_i(P) \to \pi_i(\tau \leq n P)$ for $i \leq n$ and $\pi_{> n}(\tau \leq n P) = 0$. Below $+$ is Quillen’s’ construction. Let $t$ be the stable rank of $R$. The next result is due to Suslin [Su]:

**Theorem.** (i) The reduced homology $\tilde{H}_n(X_r(R),\mathbb{Z})$ vanish for $r \geq 2n + 1$. Therefore $X(R)$ is acyclic.

(ii) $X(R)$ identifies naturally with the homotopy fiber of $B_{GL_r}(R) \to B_{GL_r}(R)^+$. 

(iii) If $r \geq \max(2n+1, t+n)$, then $H_n(\tilde{H}_n(X_r(R),\mathbb{Z})) \to H_n(GL(R),\mathbb{Z})$ and $\tau_{\leq n} X_r(R)$ identifies naturally with the homotopy fiber of $B_{GL_r}(R) \to \tau_{\leq n} B_{GL_r}(R)^+$. 


Proof. (i) is [Su] 7.1. (ii) The composition $X(R) \to B_{GL(R)} \to B_{GL(R)}^+$ factors through $X(R)^+$ which is contractible by (i). We get the map from $X(R)$ to the homotopy fiber of $B_{GL(R)} \to B_{GL(R)}^+$. It is a homotopy equivalence by [Lo] 11.3.6. (iii) The composition of maps $\tau_{\leq n} X_r(R) \to B_{GL_r(R)} \to \tau_{\leq n} B_{GL_r(R)}^+$ factors through $\tau_{\leq n} X_r(R)^+$ which is contractible if $r \geq 2n + 1$ by (i), hence comes the map from $\tau_{\leq n} X_r(R)$ to the homotopy fiber of $B_{GL_r(R)} \to \tau_{\leq n} B_{GL_r(R)}^+$: it is a homotopy equivalence by [Su] 8.1. The rest is [Su] 8.2. □

5.2. If $J \subset R$ is a two-sided ideal, then the relative Volodin’s simplicial set $X_r(R, J)$ is defined as the preimage of $X_r(R/J)$ by the projection $B_{GL_r(R)} \to B_{GL_r(R/J)}$, i.e., $X_r(R, J) = \cup_{\sigma \in \Sigma_r} B_{U^\sigma(R, J)} \subset B_{GL_r(R)}$ where $U^\sigma(R, J)$ is the preimage of $U^\sigma(R/J)$ in $GL_r(R)$. So $X_r(R, J)$ is connected, $\pi_1(X_r(R, J)) = St_r(R/J) \times_{GL_r(R/J)} GL_r(R)$.

From now on we assume that $J$ is nilpotent. Then $K_0(R) \to K_0(R/J)$ and $GL_r(R) \to GL_r(R/J)$, hence $K_1(R) \to K_1(R/J)$. Thus the relative $K$-spectrum $K(R, J)$ is connected and $\Omega^\infty K(R, J)$ is the homotopy fiber of $B_{GL_r(R)}^+ \to B_{GL_r(R/J)}^+$. Let $r$ be the stable rank of $R$ or of $R/J$ (they coincide since $J$ is nilpotent).

Proposition. (i) $X(R, J)^+ \sim \Omega^\infty K(R, J)$, so $C(X(R, J), Z) \sim C(\Omega^\infty K(R, J), Z)$. (ii) $H_n C(X_r(R, J), Z) \sim H_n C(X_r(R, J), Z)$ for $r \geq \max(2n + 1, r + n)$.

Proof. (i) (see [Lo] 11.3.6) The projection $B_{GL(R)} \to B_{GL(R/J)}$ is Kan’s fibration. So, by 5.1(ii) applied to $R/J$, $X(R, J)$ is the homotopy fiber of $B_{GL(R)} \to B_{GL(R/J)}$. The map $B_{GL(R)} \to B_{GL(R/J)}^+$ over $B_{GL(R/J)}^+$ yields the map of the homotopy fibers $X(R, J) \to \Omega^\infty K(R, J)$. By 5.1(ii), the homotopy fiber of the latter map equals $X(R)$. Thus, by 5.1(i), $C(X(R, J), Z) \sim C(\Omega^\infty K(R, J), Z)$, hence the assertion. (ii) Repeat the argument of (i) with $B_{GL_r(?)}^+$ replaced by $\tau_{\leq n} B_{GL_r(?)^+}$ and $X(R, J)$ replaced by $\tau_{\leq n} X_r(R, J)$. □

Question. Can it be that (ii) actually holds for $r > n$? If yes, this would eliminate the assumption of finiteness of stable rank from the corollary in 5.4, hence from the theorem in 2.2.

5.3. Suppose $A$ and $I$ are as in 2.2 and $A$ has bounded $p$-torsion. Choose $m$ such that $p^m \in I$. Then $GL_r(A)^{(m)} := \text{Ker}(GL_r(A) \to GL_r(A_{\text{min}(t, m)})) \subset U^\sigma(A, I)$ for any $\sigma$, hence $B_{GL_r(A)^{(m)}} \subset X_r(A, I) \subset B_{GL_r(A)}$. Consider the maps of chain complexes $C(GL_r(A)^{(m)}, Z) \to C(X_r(A, I), Z)$. Applying “lim”, we get a map of pro-complexes

\begin{equation}
C(GL_r(A)^{(m)}) \to C(X_r(A, I)^{-}, Z).\end{equation}

Proposition. For $n \leq r$ the map $H_n C(GL_r(A)^{(m)}) \to H_n C(X_r(A, I)^{-}, Z)$ is an isogeny.

Proof. $X_r(A, I)$ is covered by simplicial subsets $B_{U^\sigma(A, I)}$. Let $B_{U(A, I)}$ be the intersection of several of these subsets. One has $GL_r(A)^{(m)} \subset U(A, I)$, so $G := GL_r(A)^{(m)} \subset G^r := \varprojlim U(A, I)$. The proposition follows since the maps $H_n C(G^r) \to H_n C(G^r)$ for $n \leq r$ are isogenies by 4.11. □

16If $R$ is a $Q$-algebra, the positive answer follows from Goodwillie’s theorem [G], [Lo] 11.3 combined with the Loday-Quillen stabilization [Lo] 10.3.2.
5.4. For a ring $R$ consider the standard embeddings $GL_1(R) \subset GL_2(R) \subset \ldots$. The complexes $C(GL_n(R), \mathbb{Z})$ form an increasing filtration on $C(GL(R), \mathbb{Z})$. As in 3.6, denote by $C(GL(R), \mathbb{Z})_{\langle a \rangle}$ the décalage of that filtration, so the $n$th component of $C(GL(R), \mathbb{Z})_{\langle a \rangle}$ equals $\{ c \in C(GL_{n+a}(R), \mathbb{Z}) : \partial(c) \in C(GL_{n+a+1}(R), \mathbb{Z}) \}_{n-1}$.

Replacing $GL_r$ by its congruence subgroup we get projective systems of complexes $C(GL(A)^{(m)}_{\langle a \rangle}, \mathbb{Z})_{\langle a \rangle}$. Set $C(GL(A)^{(m)}_{\langle a \rangle}) := \lim^* C(GL(A)^{(m)}_{\langle a \rangle}, \mathbb{Z})_{\langle a \rangle}$.

Maps (5.3.1) for different $r$'s are compatible, so, by 5.2(i), they produce maps

\[(5.4.1)\quad C(GL(A)^{(m)}_{\langle a \rangle}) \to C(\Omega^\infty K(A, I)^*, \mathbb{Z})\]

that are compatible with the embeddings $C(GL(A)^{(m)}_{\langle a \rangle}) \to C(GL(A)^{(m)}_{\langle a+1 \rangle})$ and $C(GL(A)^{(m+1)}_{\langle a \rangle}) \to C.GL(A)^{(m)}_{\langle a \rangle}$.

Notice that the theorem in 4.1 yields quasi-isomorphisms (see 3.7)

\[(5.4.2)\quad C(GL(A)^{(m)}_{\langle a \rangle}) \otimes \to C(gl(A)^{\wedge})_{\langle a \rangle}.\]

**Corollary.** Suppose the stable rank of $A_1$ is finite. Then maps (5.4.1) are quasi-isomorphies for $a \geq 0$.

**Proof.** The stable rank of $A_1$ equals that of $A/I$, so, by 5.3 and 5.2(ii), we know that for given $n$ the map $H_n C(GL_n(A)^{(m)}_{\langle a \rangle}) \to H_n C(\Omega^\infty K(A, I)^*, \mathbb{Z})$ is an isogeny if $r$ is large enough. Now (5.4.2) together with the corollary in 3.7 implies that $C(GL(A)^{(m)}_{\langle a \rangle}) \to C(GL(A)^{(m)}_{\langle a \rangle}) \to \ldots$ are all quasi-isomorphies, q.e.d. □

6. Coda

It remains to tie up the segments of the tail:

6.1. Let $A$ and $I$ be as in the theorem in 2.2. Denote by $\chi$ the composition of the chain of quasi-isogenies $Sym(CC(A)^{\wedge})[1]_Q \xrightarrow{3.2} Sym(CC^\wedge(A)^{\wedge})[1]_Q \xrightarrow{(3.5.1)} C(gl(A)^{\wedge})_{\langle a \rangle}Q$.

\[(5.4.2)\quad C(GL(A)^{(m)}_{\langle a \rangle})_Q \xrightarrow{(5.4.1)} C(\Omega^\infty K(A, I)^*, \mathbb{Z})_Q \to C(K(A, I)^*, Q)\]

where the first arrow is the evident embedding, the second arrow is $\chi$, and the third one is $\nu_{K(A, I)}$ - from 1.2(c).

**Proposition.** The composition $CC(A)^{\wedge}[1]_Q \to K(A, I)_Q$ is a quasi-isogeny.

The promised quasi-isogeny (2.2.1) is its inverse.

**Proof.** Let $\psi : CC(A)^{\wedge}[1]_Q \to C(\Omega^\infty K(A, I)^*, \mathbb{Z})_Q$ be the composition of the first two arrows in (6.1.1). According to the proposition in 1.2(c), it is enough to check that $H_n(\psi)$ identifies $H_n(CC(A)^{\wedge}[1]_Q)$ with $\text{Prim}H_n C(\Omega^\infty K(A, I)^*, \mathbb{Z})_Q$. Since $H_n(CC(A)^{\wedge}[1]_Q)$ equals $\text{Prim}H_n Sym(CC(A)^{\wedge})[1]_Q$, we need to show that $\chi$ identifies the primitive parts of the homology. The only problem is that the terms of the segment $C(gl(A)^{\wedge})_{\langle a \rangle}Q$ of the chain that defines $\chi$ are not coalgebras. To solve it, pick any $a \geq r \geq n$. By 3.5, 3.7 the embedding $C(gl_r(A)^{\wedge}) \to C(gl(A)^{\wedge})_{\langle a \rangle}$ yields isogenies between the homology in degrees $\leq n$, so $\tau_{\leq n} \chi$ can be computed replacing the above segment by $C(gl_r(A)^{\wedge})_{\langle a \rangle}Q \xrightarrow{(4.1.1)} C(GL_r(A)^{\wedge})_{\langle a \rangle}Q$. Now all terms of the chain are coalgebras, the maps are compatible with the coproducts (see Remark in 3.5), and we are done. □
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