Omni $n$-Lie algebras and linearization of higher analogues of Courant algebroids

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Abstract

In this paper, we introduce the notion of an omni $n$-Lie algebra and show that they are linearization of higher analogues of standard Courant algebroids. We also introduce the notion of a nonabelian omni $n$-Lie algebra and show that they are linearization of higher analogues of Courant algebroids associated to Nambu-Poisson manifolds.

1 Introduction

Courant algebroids were introduced in [20] (see also [22]), and have many applications. See [17] and references therein for more details. On $T^{n-1}M \triangleq TM \oplus \wedge^{n-1}T^*M$, define a symmetric nondegenerate $\wedge^{n-2}T^*M$-valued pairing $(\cdot, \cdot)_+: T^{n-1}M \times T^{n-1}M \rightarrow \wedge^{n-2}T^*M$ by

$$(X + \alpha, Y + \beta)_+ = i_X \beta + i_Y \alpha, \quad \forall X + \alpha, Y + \beta \in \mathfrak{X}(M) \oplus \Omega^{n-1}(M),$$

and define a bracket operation $[\cdot, \cdot]: \Gamma(T^{n-1}M) \times \Gamma(T^{n-1}M) \rightarrow \Gamma(T^{n-1}M)$ by

$$[X + \alpha, Y + \beta] = [X, Y] + L_X \beta - i_Y \alpha.$$

The quadruple $(TM \oplus \wedge^{n-1}T^*M, (\cdot, \cdot)_+, \{\cdot, \cdot\}, \text{pr}_{TM})$ is called the higher analogue of the standard Courant algebroid. In particular, if $n = 2$, we obtain the standard Courant algebroid. Recently, due to applications in multisymplectic geometry, Nambu-Poisson geometry, $L_\infty$-algebra theory and topological field theory, higher analogues of Courant algebroids are widely studied. See [2, 3, 10, 11, 12, 13, 27] for more details.

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The notion of an omni-Lie algebra was introduced by Weinstein in [26] to study the linearization of the standard Courant algebroid. Then it was studied from several aspects [13, 23, 25]. An omni-Lie algebra associated to a vector space $V$ is a triple $(\mathfrak{gl}(V) \oplus V, \{-,\}, \cdot)$, where $(\cdot,\cdot)_+$ is the $V$-valued pairing given by
\[
(A + u, B + v)_+ = Av + Bu, \quad \forall A + u, B + v \in \mathfrak{gl}(V) \oplus V,
\] and $(\cdot,\cdot)$ is the bilinear bracket operation given by
\[
\{A + u, B + v\} = [A, B] + Av.
\]

Even though $(\mathfrak{gl}(V) \oplus V, \{-,\})$ is not a Lie algebra, its Dirac structures characterize all Lie algebra structures on $V$. We can construct a Lie 2-algebra from an omni-Lie algebra. See [23] for more details.

In [19], the authors introduced the notion of a nonabelian omni-Lie algebra $(\mathfrak{g}(\mathfrak{g}) \oplus \mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_g)$ associated to a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_g)$, which originally comes from the study of homotopy Poisson manifolds [18]. In particular, they showed that it is the linearization of the Courant algebroid $T\mathfrak{g}^* \oplus T\pi^*_\mathfrak{g}^*$ associated to the linear Poisson manifold $(\mathfrak{g}^*, \pi_\mathfrak{g})$, where $\pi_\mathfrak{g}$ is the Lie-Poisson structure on $\mathfrak{g}^*$.

The purpose of this paper is to extend the above results to the $n$-ary case. First we introduce the notion of an omni-$n$-Lie algebra, which is a triple $(\mathfrak{gl}(V) \oplus \Lambda^{n-1}V, (\cdot,\cdot)_+, \{\cdot,\cdot\})$ including a bracket operation $(\cdot,\cdot)$ and a $(V \otimes \Lambda^{n-2}V)$-valued pairing $(\cdot,\cdot)_+$. Similar to the classical case, $(\mathfrak{gl}(V) \oplus \Lambda^{n-1}V, (\cdot,\cdot))$ is a Leibniz algebra. We show that a linear map $F : \Lambda^n V \rightarrow V$ defines an $n$-Lie algebra structure on $V$ only if the graph of $F^* : \Lambda^{n-1}V \rightarrow \mathfrak{gl}(V)$ is a sub-Leibniz algebra of $(\mathfrak{gl}(V) \oplus \Lambda^{n-1}V, (\cdot,\cdot))$. Note that this result is not totally parallel the classical case. Namely the condition that $F$ being skew-symmetric can not be simply described by being isotropic with respect to the $(V \otimes \Lambda^{n-2}V)$-valued pairing $(\cdot,\cdot)_+$. We further show that an omni-$n$-Lie algebra $(\mathfrak{gl}(V) \oplus \Lambda^{n-1}V, (\cdot,\cdot)_+, \{\cdot,\cdot\})$ can be viewed as the linearization of the higher analogue of the standard Courant algebroid $(TM \oplus \Lambda^{n-1}T^*M, (\cdot,\cdot)_+, \{\cdot,\cdot\}_M, \text{pr}_{TM})$ via letting $M = V^*$. Then we introduce the notion of a nonabelian omni-$n$-Lie algebra $(\mathfrak{gl}(\mathfrak{g}) \oplus \Lambda^{n-1}\mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_g)$ associated to an $n$-Lie algebra $\mathfrak{g}$ and study its algebraic properties. Finally, we give the notion of higher analogues of Courant algebroids associated to Nambu-Poisson manifolds and study their properties. Furthermore, we show that nonabelian omni-$n$-Lie algebras are linearization of higher analogues of Courant algebroids associated to Nambu-Poisson manifolds.

The paper is organized as follows. In Section 2, we recall $n$-Lie algebras and Nambu-Poisson manifolds. In Section 3, we introduce the notion of an omni $n$-Lie algebra associated to a vector space $V$ and characterize $n$-Lie algebra structures on $V$ via sub-Leibniz algebra structures of the omni $n$-Lie algebra. In Section 4, we show that an omni $n$-Lie algebra is the linearization of the higher analogue of the standard Courant algebroid. In Section 5, we introduce the notion of a nonabelian omni $n$-Lie algebra and study its algebraic properties. In Section 6, we introduce the notion of higher analogues of Courant algebroids associated to Nambu-Poisson manifolds and show that nonabelian omni $n$-Lie algebras are their linearization.

2 Preliminaries

In this section, we briefly recall the notions of $n$-Lie algebras and Nambu-Poisson manifolds. The notion of an $n$-Lie algebra, or a Filippov algebra, was introduced in [8] and have many applications in mathematical physics. See the review article [6] for more details. Nambu-Poisson structures were introduced in [24] by Takhtajan in order to find an axiomatic formalism for Nambu-mechanics which is a generalization of Hamiltonian mechanics.
Definition 2.1. An $n$-Lie algebra is a vector space $\mathfrak{g}$ together with an $n$-multilinear skew-symmetric bracket $[\cdot, \ldots, \cdot]_{\mathfrak{g}} : \wedge^n \mathfrak{g} \rightarrow \mathfrak{g}$ such that for all $u, v_i \in \mathfrak{g}$, the following Fundamental Identity is satisfied:

$$[u_1, u_2, \ldots, u_{n-1}, [v_1, v_2, \ldots, v_n]_{\mathfrak{g}}]_{\mathfrak{g}} = \sum_{i=1}^{n} [v_1, v_2, \ldots, [u_1, u_2, \ldots, u_{n-1}, v_i]_{\mathfrak{g}}, \ldots, v_n]_{\mathfrak{g}}. \quad (5)$$

For $u_1, u_2, \cdots, u_{n-1} \in \mathfrak{g}$, define $\text{ad} : \wedge^{n-1} \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad}_{u_1, u_2, \ldots, u_{n-1}} v = [u_1, u_2, \ldots, u_{n-1}, v]_{\mathfrak{g}}, \quad \forall \ v \in \mathfrak{g}.$$ 

Then Eq. (5) is equivalent to that $\text{ad}_{u_1, u_2, \ldots, u_{n-1}}$ is a derivation, i.e.

$$\text{ad}_{u}[v_1, v_2, \ldots, v_n]_{\mathfrak{g}} = \sum_{i=1}^{n} [v_1, v_2, \ldots, \text{ad}_{u}v_i, \ldots, v_n]_{\mathfrak{g}}, \quad \forall \ u = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \in \wedge^{n-1} \mathfrak{g}. \quad (6)$$

Elements in $\wedge^{n-1} \mathfrak{g}$ are called fundamental objects of the $n$-Lie algebra $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}})$. In the sequel, we will denote $\text{ad}_{u}v$ simply by $u \circ v$.

Define a bilinear operation on the set of fundamental objects $\circ : (\wedge^{n-1} \mathfrak{g}) \otimes (\wedge^{n-1} \mathfrak{g}) \rightarrow \wedge^{n-1} \mathfrak{g}$ by

$$u \circ v = \sum_{\pi \in \Sigma(n)} v_1 \wedge \cdots \wedge v_{i-1} \wedge u \circ v_i \wedge v_{i+1} \wedge \cdots \wedge v_{n-1}, \quad (7)$$

for all $u = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1}$ and $v = v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1}$. In [5], the authors proved that $(\wedge^{n-1} \mathfrak{g}, \circ)$ is a Leibniz algebra. See [21] for details about Leibniz algebras, which are also called Loday algebras. Moreover, the Fundamental Identity (5) is equivalent to

$$u \circ (v \circ w) - v \circ (u \circ w) = (u \circ v) \circ w, \quad \forall \ u, v, w \in \wedge^{n-1} \mathfrak{g}, w \in \mathfrak{g}. \quad (8)$$

Definition 2.2. [21] A Nambu-Poisson structure of order $n - 1$ on $M$ is an $n$-linear map $\{\cdot, \cdots, \cdot\} : C^\infty(M) \times \cdots \times C^\infty(M) \rightarrow C^\infty(M)$ satisfying the following properties:

1. skewsymmetry, i.e. for all $f_1, \cdots, f_n \in C^\infty(M)$ and $\sigma \in \text{Sym}(n)$,
   $$\{f_1, \cdots, f_n\} = (-1)^{|\sigma|} \{f_{\sigma(1)}, \cdots, f_{\sigma(n)}\};$$

2. the Leibniz rule, i.e. for all $g \in C^\infty(M)$, we have
   $$\{f_1 g, \cdots, f_n\} = f_1 \{g, \cdots, f_n\} + g \{f_1, \cdots, f_n\};$$

3. integrability condition, i.e. for all $f_1, \cdots, f_{n-1}, g_1, \cdots, g_n \in C^\infty(M),$
   $$\{f_1, \cdots, f_{n-1}, \{g_1, \cdots, g_n\}\} = \sum_{i=1}^{n} \{g_i, \cdots, \{f_1, \cdots, f_{n-1}, g_i\}, \cdots, g_n\}.$$ 

In particular, a Nambu-Poisson structure of order 1 is exactly a usual Poisson structure. Similar to the fact that a Poisson structure is determined by a bivector field, a Nambu-Poisson structure of order $n - 1$ is determined by an $n$-vector field $\pi \in \mathfrak{X}^n(M)$ such that

$$\{f_1, \cdots, f_n\} = \pi(df_1, \cdots, df_n).$$

3
An $n$-vector field $\pi \in \mathfrak{X}^n(M)$ is a Nambu-Poisson structure if and only if for all $f_1, \ldots, f_{n-1} \in C^\infty(M)$, we have
\[
L_{\pi} (df_1 \wedge \cdots \wedge df_{n-1}) \pi = 0,
\]
where $\pi^\sharp : \wedge^{n-1} T^* M \longrightarrow TM$ is defined by
\[
\langle \pi^\sharp (\xi_1 \wedge \cdots \wedge \xi_{n-1}), \xi_n \rangle = \pi (\xi_1 \wedge \cdots \wedge \xi_{n-1} \wedge \xi_n), \quad \forall \xi_1, \ldots, \xi_n \in \Omega^1(M).
\]
Let $\pi$ be a Nambu-Poisson structure on a manifold $M$. Then there is a natural operation $[,]_\pi$ on $\Omega^{n-1}(M)$ given by
\[
[\alpha, \beta]_\pi = L_{\pi^1(\alpha)} \beta - L_{\pi^1(\beta)} \alpha + di_{\pi^1(\alpha)} \beta, \quad \forall \alpha, \beta \in \Omega^{n-1}(M)
\]
such that $(\wedge^{n-1} T^* M, [,],_\pi, \pi^\sharp)$ is a Leibniz algebroid. See [2, 13] for more details.

3 Omni $n$-Lie algebras

Let $V$ be a finite dimensional vector space. For all $A \in \mathfrak{gl}(V)$, define $L_A : \otimes^{n-1} V \longrightarrow \otimes^{n-1} V$ by
\[
L_A (v_1 \otimes \cdots \otimes v_{n-1}) = \sum_{i=1}^{n-1} v_1 \otimes \cdots \otimes Av_i \otimes \cdots \otimes v_{n-1}.
\]

**Definition 3.1.** An omni $n$-Lie algebra associated to a vector space $V$ is a triple $(\mathfrak{gl}(V) \otimes \wedge^{n-1} V, (\cdot)_+, \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ is the bilinear bracket operation given by
\[
\{A + u, B + v\} = [A, B] + L_A v,
\]
and $(\cdot)_+$ is the $(V \otimes \wedge^{n-2} V)$-valued pairing given by
\[
(A + u, B + v)_+ = \sum_{i=1}^{n-1} (-1)^{i+1} (Av_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{n-1} + Bu_i \otimes u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_{n-1}),
\]
where $u = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1}$ and $v = v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1}$.

**Remark 3.2.** When $n = 2$, we recover Weinstein’s omni-Lie algebras [26].

**Proposition 3.3.** With the above notations, $(\mathfrak{gl}(V) \otimes \wedge^{n-1} V, \{\cdot,\cdot\})$ is a Leibniz algebra. Furthermore, the pairing $(\cdot)_+$ and the bracket $\{\cdot,\cdot\}$ are compatible in the sense that
\[
\langle \{e_1, e_2\}_+ + (e_2, \{e_1, e_3\}_+) = \rho_V (e_1)(e_2, e_3)_+,
\]
where $e_i \in \mathfrak{gl}(V) \otimes \wedge^{n-1} V$, $i = 1, 2, 3$, and $\rho_V : \mathfrak{gl}(V) \otimes \wedge^{n-1} V \longrightarrow \mathfrak{gl}(V \otimes \wedge^{n-2} V)$ is given by
\[
\rho_V (A + u)(w) = L_A w, \quad \forall w \in V \otimes \wedge^{n-2} V.
\]

**Proof.** Since $L_{[A, B]} = L_A L_B - L_B L_A$, we can deduce that $(\mathfrak{gl}(V) \otimes \wedge^{n-1} V, \{\cdot,\cdot\})$ is a Leibniz algebra directly.
For all $A + u, B + v, C + w \in \mathfrak{gl}(V) \oplus \wedge^{n-1}V,$ on one hand, we have
\[
(\{A + u, B + v\}, C + w) + (B + v, A + u, C + w)\]
\[
= (\{A, B\} + \mathcal{L}_A v, C + w) + (B + v, [A, C] + \mathcal{L}_A w) +
\]
\[
= \sum_{j=1}^{n-1} (-1)^{j+1}(ABw_j \wedge v_1 \wedge \cdots \wedge \hat{w}_{j} \wedge \cdots \wedge v_{n-1} + ACv_j \wedge v_1 \wedge \cdots \wedge \hat{v}_{j} \wedge \cdots \wedge v_{n-1})
\]
\[
+ \sum_{i \neq j} (-1)^{i+1}(Bw_i \wedge v_1 \wedge \cdots \wedge Aw_j \wedge \cdots \wedge v_{n-1}) + Cv_i \wedge v_1 \wedge \cdots \wedge Av_j \wedge \cdots \wedge v_{n-1}).
\]
On the other hand, we have
\[
\rho_V(A + u)(B + v, C + w) = \sum_{j=1}^{n-1} (-1)^{j+1} \rho_V(A + u)(Bw_j \wedge v_1 \wedge \cdots \wedge \hat{w}_{j} \wedge \cdots \wedge w_{n-1} + Cw_j \wedge v_1 \wedge \cdots \wedge \hat{v}_{j} \wedge \cdots \wedge v_{n-1})
\]
\[
= \sum_{j=1}^{n-1} (-1)^{j+1}(ABw_j \wedge v_1 \wedge \cdots \wedge \hat{w}_{j} \wedge \cdots \wedge w_{n-1} + ACv_j \wedge v_1 \wedge \cdots \wedge \hat{v}_{j} \wedge \cdots \wedge v_{n-1})
\]
\[
+ \sum_{i \neq j} (-1)^{i+1}(Bw_i \wedge v_1 \wedge \cdots \wedge Aw_j \wedge \cdots \wedge w_{n-1}) + Cv_i \wedge v_1 \wedge \cdots \wedge Av_j \wedge \cdots \wedge v_{n-1}),
\]
which implies that $[12]$ holds.

Let $F : \wedge^n V \to V$ be a linear map. Then $F$ induces a linear map $F^\sharp : \wedge^{n-1}V \to \mathfrak{gl}(V)$ by
\[
F^\sharp(u)(u) = F(u, u), \quad \forall u \in \wedge^{n-1}V, \quad u \in V.
\]
Denote by $G_F \subset \mathfrak{gl}(V) \oplus \wedge^{n-1}V$ the graph of $F^\sharp$.

**Theorem 3.4.** Let $F : \wedge^n V \to V$ be a linear map. Then $(V, F)$ is an $n$-Lie algebra if and only if $G_F$ is a Leibniz subalgebra of the Leibniz algebra $(\mathfrak{gl}(V) \oplus \wedge^{n-1}V, \{\cdot, \cdot\})$.

**Proof.** $G_F$ is a Leibniz subalgebra of the Leibniz algebra $(\mathfrak{gl}(V) \oplus \wedge^{n-1}V, \{\cdot, \cdot\})$ if and only if for all $u, v \in \wedge^{n-1}V$, $(F^\sharp(u) + u, F^\sharp(v) + v) \in G_F$, which is equivalent to
\[
F^\sharp(\mathcal{L}_{F^\sharp(u)}v) = [F^\sharp(u), F^\sharp(v)].
\]
Since $\mathcal{L}_{F^\sharp(u)}v = \sum_{i=1}^{n-1} v_i \wedge \cdots \wedge F(u, v_i) \wedge \cdots \wedge v_{n-1},$ thus the above equality can be written as
\[
F^\sharp(u \circ v) = [F^\sharp(u), F^\sharp(v)]
\]
which is equivalent to that $(V, F)$ is an $n$-Lie algebra.

4 Linearization of the higher analogue of the standard Courant algebroid

Let $V$ be an $m$-dimensional vector space and $V^*$ its dual space. We consider the direct sum bundle $T^{n-1}V^* = TV^* \oplus \wedge^{n-1}T^*V^*$. Denote the vector spaces of linear vector fields and constant $(n−1)$-forms
on \( V^* \) by \( \mathcal{X}_{\text{lin}}(V^*) \) and \( \Omega_{\text{con}}^{n-1}(V^*) \) respectively. It is obvious that \( \mathcal{X}_{\text{lin}}(V^*) \oplus \Omega_{\text{con}}^{n-1}(V^*) \cong \mathfrak{gl}(V) \oplus \wedge^{n-1} V \). To make this explicit, for any \( x \in V \), denote by \( l_x \) the corresponding linear function on \( V^* \). Let \( \{ x^i \} \) be a basis of the vector space \( V \). Then \( \{ l_x \} \) forms a coordinate chart for \( V^* \). So \( \{ \frac{\partial}{\partial x^i} \} \) constitutes a basis of vector fields on \( V^* \) and \( \{ dl_x \} \) constitutes a basis of 1-forms on \( V^* \). For \( A = (a_{ij}) \in \mathfrak{gl}(V) \), we get a linear vector field \( \hat{A} = \sum l_{A(x^i)} \frac{\partial}{\partial x^i} \) on \( V^* \). Also \( u = \sum i \leq m \xi_{i_1 \cdots i_{n-1}} x^{i_1} \wedge \cdots \wedge x^{i_{n-1}} \) defines a constant \( (n-1) \)-form \( \hat{u} = \sum i \leq m \xi_{i_1 \cdots i_{n-1}} dl_{x^{i_1}} \wedge \cdots \wedge dl_{x^{i_{n-1}}} \).

Define \( \Phi : \mathfrak{gl}(V) \oplus \wedge^{n-1} V \longrightarrow \mathcal{X}_{\text{lin}}(V^*) \oplus \Omega_{\text{con}}^{n-1}(V^*) \) by

\[
\Phi(A + u) = \hat{A} + \hat{u}.
\]

Obviously, \( \Phi \) is an isomorphism between vector spaces.

Any element \( v \otimes u \in V \otimes \wedge^{n-2} V \) will give rise to a linear \( (n-2) \)-form \( \bar{v} \otimes \bar{u} \) defined by

\[
\bar{v} \otimes \bar{u} = l_{u} \hat{u}.
\]

We give some useful formulas first.

**Lemma 4.1.** With the above notations, for all \( \hat{A} \in \mathfrak{gl}(V) \) and \( u \in \wedge^{n-1} V \), we have

\[
(A, \hat{u})_+ = \{A, u\}_+ ,
\]

(14)

\[
d(\hat{A}, \hat{u})_+ = \hat{L}_A u ,
\]

(15)

\[
L_{\hat{A}} \hat{u} = \hat{L}_A u ,
\]

(16)

\[
[\hat{A}, \hat{B}] = [A, B] .
\]

(17)

**Proof.** On one hand, for \( u = \sum_{i_1 < \cdots < i_{n-1} \leq m} \xi_{i_1 \cdots i_{n-1}} x^{i_1} \wedge \cdots \wedge x^{i_{n-1}} \in \wedge^{n-1} V \), we have

\[
(A, \hat{u})_+ = \sum \sum_{i_1 < \cdots < i_{n-1} \leq m} (-1)^{j+1} \xi_{i_1 \cdots i_{j} \cdots i_{n-1}} \hat{A} x^{i_1} \wedge \cdots \wedge \hat{A} x^{i_{j}} \wedge \cdots \wedge \hat{A} x^{i_{n-1}} .
\]

On the other hand, we have

\[
(A, u)_+ = \sum \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq m} (-1)^{j+1} \xi_{i_1 \cdots i_{j} \cdots i_{n-1}} \hat{A} x^{i_1} \wedge \cdots \wedge \hat{A} x^{i_{j}} \wedge \cdots \wedge \hat{A} x^{i_{n-1}} ,
\]

which implies that (14) holds.

By direct calculation, we have

\[
d(\hat{A}, \hat{u})_+ = \sum \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq m} (-1)^{j+1} \xi_{i_1 \cdots i_{j} \cdots i_{n-1}} \hat{A} x^{i_1} \wedge \cdots \wedge \hat{A} x^{i_{j}} \wedge \cdots \wedge \hat{A} x^{i_{n-1}} ,
\]

(16)
On the other hand, we have
\[ L_A u = A \left( \sum_{1 \leq i_1 < \cdots < i_m \leq n} \xi_{i_1} \cdots \xi_{i_m} x^{i_1} \wedge \cdots \wedge x^{i_m} \right) \]
\[ = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \xi_{i_1} \cdots \xi_{i_m} x^{i_1} \wedge \cdots \wedge Ax^{i_1} \wedge \cdots \wedge x^{i_m} \]
\[ = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \sum_{j=1}^{m} \xi_{i_1} \cdots \xi_{i_m} x^{i_1} \wedge \cdots \wedge x^k \wedge \cdots \wedge x^{i_m}. \]

Thus (15) follows immediately.

(16) follows from
\[ (\Phi(A + u), \Phi(B + v))_+ = (A + u, B + v)_+, \]
\[ [\Phi(A + u), \Phi(B + v)] = \Phi(A + u, B + v), \]
\[ L_{pr TV} \Phi(A + u)(w) = \rho_V(A + u)(w). \]

Proof. By (14), we have
\[ (\Phi(A + u), \Phi(B + v))_+ = (\hat{A}, \hat{B})_+ + (\hat{u}, \hat{B}) = (A, B)_+ + (u, B)_+ = (A + u, B + v)_+. \]

By (16) and (17), we have
\[
\begin{align*}
[\Phi(A + u), \Phi(B + v)] &= [\hat{A} + \hat{u}, \hat{B} + \hat{v}] = [\hat{A}, \hat{B}] + L_{\hat{A}} \hat{v} \\
&= [A, B] + L_A v = \Phi(A + u, B + v).
\end{align*}
\]

Finally, for all \( w = w_1 \otimes w_2 \wedge \cdots \wedge w_{n-1} \in V \otimes \wedge^{n-2} V \), we have
\[ L_{pr TV} \Phi(A + u)(w) = L_A (l_{w_1} dw_2 \wedge \cdots \wedge dw_{n-1}) \]
\[ = L_A (l_{w_1}) dw_2 \wedge \cdots \wedge dw_{n-1} + l_{w_1} \left( \sum_{i=2}^{n-1} dw_2 \wedge \cdots \wedge L_A dw_i \wedge \cdots \wedge dw_{n-1} \right) \]
\[ = l_{Aw_1} dw_2 \wedge \cdots \wedge dw_{n-1} + \sum_{i=2}^{n-1} l_{w_i} dw_2 \wedge \cdots \wedge d(Aw_i) \wedge \cdots \wedge dw_{n-1} \]
\[ = Aw_1 \otimes w_2 \wedge \cdots \wedge w_{n-1} + \sum_{i=2}^{n-1} w_1 \otimes w_2 \wedge \cdots \wedge Aw_i \wedge \cdots \wedge w_{n-1} \]
\[ = \rho_V(A + u)(w). \]

The proof is finished. \( \blacksquare \)
5 Nonabelian omni $n$-Lie algebras

Definition 5.1. A nonabelian omni $n$-Lie algebra associated to an $n$-Lie algebra $(\mathfrak{g}, \lbrack \cdot, \cdots, \cdot \rbrack_\mathfrak{g})$ is a triple $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \langle \cdot, \cdots, \cdot \rangle_\mathfrak{gl}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$, where $(\langle \cdot, \cdots, \cdot \rangle_\mathfrak{gl}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$ is the $(\mathfrak{gl} \otimes \wedge^{n-1} \mathfrak{g})$-valued pairing given by (21) and $\{\cdot, \cdots, \cdot \}_\mathfrak{g}$ is the bilinear bracket operation given by

\[
\{A + u, B + v\}_\mathfrak{g} = [A, B] + [A, ad_u] + [ad_u, B] - ad_{\mathcal{L}_A} v + \mathcal{L}_A v + u \circ v, \quad \forall A, B \in \mathfrak{gl}(\mathfrak{g}), u, v \in \wedge^{n-1} \mathfrak{g}. \tag{21}
\]

Theorem 5.2. Let $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \langle \cdot, \cdots, \cdot \rangle_\mathfrak{gl}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$ be a nonabelian omni $n$-Lie algebra. Then we have

(i) $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$ is a Leibniz algebra;

(ii) $\{A + u, A + u\}_\mathfrak{g} = - \text{ad}_{\mathcal{L}_A} u + \mathcal{L}_A u + u \circ u$;

(iii) the pairing $(\cdot, \cdots, \cdot)_+$ and the bracket $(\cdot, \cdots, \cdot)_\mathfrak{g}$ are compatible in the sense that

\[
\rho_\mathfrak{g}(e_1)(e_2, e_3)_+ = (\{e_1, e_2\}_\mathfrak{g} - ([A, ad_B] - \text{ad}_{\mathcal{L}_A} B), e_3)_+ + (e_2, [e_1, e_3]_\mathfrak{g} - ([A, \text{ad}_u] - \text{ad}_{\mathcal{L}_A} u))_+, \tag{22}
\]

where $e_1 = A + u, e_2 = B + v, e_3 = C + w \in \mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}$ and $\rho_\mathfrak{g} : \mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g} \otimes \wedge^{n-2} \mathfrak{g})$ is given by

\[
\rho_\mathfrak{g}(A + u)(w) = \mathcal{L}_A + \text{ad}_u w, \quad \forall w \in \mathfrak{g} \otimes \wedge^{n-2} \mathfrak{g}. \tag{23}
\]

Proof. (i) We can prove that $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$ is a Leibniz algebra directly by a complicated computation. In the sequel, we will show that $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \{\cdot, \cdots, \cdot \}_\mathfrak{g})$ is a trivial deformation of the omni $n$-Lie algebra $(\mathfrak{gl}(\mathfrak{g}) \otimes \wedge^{n-1} \mathfrak{g}, \{\cdot, \cdots, \cdot \})$. Thus, we omit details here.

(ii) It follows from (21) directly.

(iii) By straightforward computation, we have

\[
([A, B] + [ad_u, B] + \mathcal{L}_A v + u \circ v, C + w)_+ = \sum_{i=1}^{n-1} (-1)^{i+1} ([A, B] + [ad_u, B]) w_i \otimes w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_{n-1}
\]

\[
+ \sum_{i \neq j} (-1)^{i+1} C v_j \otimes w_1 \wedge \cdots \hat{v}_j \wedge \cdots \wedge (A v_i + u \circ v_i) \wedge \cdots \wedge v_{n-1}
\]

\[
+ \sum_{i=1}^{n-1} (-1)^{i+1} (C A v_i + C (u \circ v_i)) \otimes v_1 \wedge \cdots \hat{v}_i \wedge \cdots \wedge v_{n-1}.
\]

On the other hand, we have

\[
(B + v, [A, C] + [ad_u, C] + \mathcal{L}_A w + u \circ w)_+ = \sum_{i=1}^{n-1} (-1)^{i+1} ([A, C] + [ad_u, C]) v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{n-1}
\]

\[
+ \sum_{i \neq j} (-1)^{i+1} (B w_j \otimes w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge (A w_i + u \circ w_i) \wedge \cdots \wedge w_{n-1})
\]

\[
+ \sum_{i=1}^{n-1} (-1)^{i+1} (B A w_i + B (u \circ w_j)) \otimes w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_{n-1}.
\]
Thus we have

\[
([A, B] + [ad_u, B] + \mathcal{L}_A w + u \circ v, C + w) + (B + v, [A, C] + [ad_u, C] + \mathcal{L}_A w + u \circ w) + \\
= \sum_{i=1}^{n-1} (-1)^{i+1} (A \circ B + ad_u \circ B) w_i \otimes w_1 \wedge \cdots \wedge w_{i-1} \wedge w_{i+1} \wedge \cdots \wedge w_{n-1} \\
+ \sum_{i=1}^{n-1} (-1)^{i+1} (A \circ C + ad_u \circ C) v_i \otimes v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_{n-1} \\
+ \sum_{i \neq j} (-1)^{i+1} B v_j \otimes v_1 \wedge \cdots \wedge v_{i-1} \wedge (Av_i + u \circ v_i) \wedge \cdots \wedge v_{n-1} \\
+ \sum_{i \neq j} (-1)^{i+1} B w_j \otimes w_1 \wedge \cdots \wedge w_{i-1} \wedge (Aw_i + u \circ w_i) \wedge \cdots \wedge w_{n-1} \\
= \rho_g (A + u)(B + v, C + w) +.
\]

The proof is finished. ■

Obviously, for all \( A \in \text{Der}(g) \), we have \([A, ad_u] - ad_{\mathcal{L}_A u} = 0\). Thus, we have

**Corollary 5.3.** For all \( e_1, e_2, e_3 \in \text{Der}(g) \oplus \wedge^{n-1} g \), we have

\[
\rho_g (e_1)(e_2, e_3)_+ = ([e_1, e_2]_g, e_3)_+ + (e_2, [e_1, e_3]_g)_+.
\]

In the sequel we show that a nonabelian omni \( n \)-Lie algebra \((gl(g) \oplus \wedge^{n-1} g, \{\cdot, \cdot\}_g)\) can be viewed as a trivial deformation of the omni \( n \)-Lie algebra \((gl(g) \oplus \wedge^{n-1} g, \{\cdot, \cdot\})\). For details of deformations of Leibniz algebras, see [4][10].

Let \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L})\) be a Leibniz algebra. For an endomorphism \( N \) of \( \mathfrak{L} \), define

\[
[e_1, e_2]_N = [Ne_1, e_2]_\mathfrak{L} + [e_1, Ne_2]_\mathfrak{L} - N[e_1, e_2]_\mathfrak{L},
\]

and set

\[
TN(e_1, e_2) = [Ne_1, e_2]_\mathfrak{L} - N[e_1, e_2]_\mathfrak{L}.
\]

The endomorphism \( N \) is called a **Nijenhuis operator** if \( TN = 0 \).

A Nijenhuis operator gives a trivial deformation of the Leibniz algebra \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L})\).

**Proposition 5.4.** [4] Let \( N \) be a Nijenhuis operator on the Leibniz algebra \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L})\). Then we have

1. \((\mathfrak{L}, [\cdot, \cdot]_N)\) is a Leibniz algebra;
2. \( N \) is a morphism of Leibniz algebras from \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L})\) to \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L})\);
3. \((\mathfrak{L}, [\cdot, \cdot]_\mathfrak{L} + [\cdot, \cdot]_N)\) is a Leibniz algebra.

Let \((g, [\cdot, \cdots, \cdot]_g)\) be an \( n \)-Lie algebra. Then we can define a linear map \( N : gl(g) \oplus \wedge^{n-1} g \longrightarrow gl(g) \oplus \wedge^{n-1} g \) by

\[
N(A + u) = ad_u.
\]

**Lemma 5.5.** The linear map \( N \) given by \((24)\) is a Nijenhuis operator on the Leibniz algebra \((gl(g) \oplus \wedge^{n-1} g, \{\cdot, \cdot\})\), where the Leibniz bracket \( \{\cdot, \cdot\} \) is given by \((10)\).
Proof. First by definition, we have

\[
\{ A + u, B + v \}_N = \{ N(A + u), B + v \} + \{ A + u, N(B + v) \} - N\{ A + u, B + v \} \\
= \{ \text{ad}_u, B \} + \mathcal{L}_{\text{ad}_u} v + [A, \text{ad}_u] - \text{ad}_{\mathcal{L}_A v} \\
= \{ \text{ad}_u, B \} + u \circ v + [A, \text{ad}_u] - \text{ad}_{\mathcal{L}_A v}.
\]

Hence it is clear that

\[
N\{ A + u, B + v \}_N = \text{ad}_{u \circ v} = \{ \text{ad}_u, \text{ad}_v \} = \{ N(A + u), N(B + v) \},
\]

which says that \( N \) is a Nijenhuis operator.

It is straightforward to see that

\[
\{ A + u, B + v \}_N = \{ A + u, B + v \} + \{ A + u, B + v \}_N.
\]

Therefore, by Proposition 5.4 and Lemma 5.5 we have

**Theorem 5.6.** Let \( \mathfrak{g}, \{ [], \cdots, [] \}_g \) be an \( n \)-Lie algebra. Then the bracket \( \{ [], \cdot \}_g \) is a trivial deformation of the Leibniz bracket \( \{ [], \cdot \} \). In particular, \( (\mathfrak{gl}(\mathfrak{g}) \oplus \wedge^{n-1} \mathfrak{g}, \{ [], \cdot \}_g) \) is a Leibniz algebra.

**Remark 5.7.** If we view \( (\mathfrak{gl}(\mathfrak{g}), \{ [], \cdot \}) \) as a Leibniz algebra, then \( (\mathfrak{gl}(\mathfrak{g}), \{ [], \cdot \}) \) and \( (\wedge^{n-1} \mathfrak{g}, \circ) \) form a matched pair of Leibniz algebras and the Leibniz algebra \( (\mathfrak{gl}(\mathfrak{g}) \oplus \wedge^{n-1} \mathfrak{g}, \{ [], \cdot \}_g) \) is exactly their double. See [2] for more details about matched pairs of Leibniz algebras.

# 6 Linearization of higher analogues of Courant algebroids associated to Nambu-Poisson structures

Let \((M, \pi)\) be a Nambu-Poisson manifold. We introduce a bracket \(\left[ \cdot, \cdot \right]_\pi : \Gamma(T^{n-1}M) \times \Gamma(T^{n-1}M) \longrightarrow \Gamma(T^{n-1}M)\) by

\[
\left[ X + \alpha, Y + \beta \right]_\pi = [X, Y] + [X, \pi^t(\beta)] + [\pi^t(\alpha), Y] - \pi^t(L_X \beta) + \pi^t(i_Y d\alpha) + L_X \beta - i_Y d\alpha + [\alpha, \beta]_\pi,
\]

where \(X, Y \in \mathfrak{X}(M), \alpha, \beta \in \Omega^{n-1}(M)\) and \(\left[ \cdot, \cdot \right]_\pi\) is given by (25).

Let \(\rho_\pi : T^n M \longrightarrow TM\) be the bundle map defined by

\[
\rho_\pi(X + \alpha) = X + \pi^t(\alpha), \quad \forall X \in \mathfrak{X}(M), \alpha \in \Omega^{n-1}(M).
\]

We call the quadruple \((T^{n-1}M, \left[ \cdot, \cdot \right]_\pi, \rho_\pi)\) the higher analogue of the Courant algebroid associated to a Nambu-Poisson manifold and denote it by \(T^n M\). In the sequel, we will see that even though we call it the higher analogue of a Courant algebroid, some important properties for Courant algebroids do not hold anymore.

**Theorem 6.1.** Let \((T^{n-1}M, \left[ \cdot, \cdot \right]_\pi, \rho_\pi)\) be the higher analogue of the Courant algebroid associated to a Nambu-Poisson manifold. Then we have

(i) \((T^{n-1}M, \left[ \cdot, \cdot \right]_\pi, \rho_\pi)\) is a Leibniz algebroid.

(ii) The bracket \(\left[ \cdot, \cdot \right]_\pi\) is not skew-symmetric. Instead, we have

\[
\left[ X + \alpha, X + \alpha \right]_\pi = d(X, \alpha)_+ + d(\pi^t(\alpha), \alpha)_+ - \pi^t(d(X, \alpha)_+).
\]
The conclusion follows from this finishes the proof.

**Corollary 6.2.**
Proof. For all $c_1 = X + \alpha$, $c_2 = Y + \beta$, $c_3 = Z + \gamma \in \mathfrak{X}_H(M) \oplus \Omega^{n-1}_{\mathfrak{g}}(M)$, since $\alpha$ is closed, we have

$$i_{\pi^*(\gamma)}\gamma = i_{\pi^*(\gamma)}(\gamma) = 0.$$

For all $\xi, \eta \in \Omega^{n-1}(M)$, we have the following formula

$$\pi^*(L_{\pi^*(\xi)}\eta) = [\pi^*(\xi), \pi^*(\eta)] = (-1)^{n-1}(i_d\pi\pi^*(\xi)\eta).$$

Without loss of generality, let $X = \pi^*(df_1 \land \cdots \land df_{n-1})$, then we have

$$i_{[X,\pi^*(\beta)]\gamma} - i_{\pi^*(L_X\beta)\gamma} + i_{[X,\pi^*(\gamma)]\beta} - i_{\pi^*(L_X\gamma)\beta} = (-1)^n(i_{\pi^*(df_1 \land \cdots \land df_{n-1})}\pi^*(\beta)\gamma + (-1)^n(i_{\pi^*(df_1 \land \cdots \land df_{n-1})}\pi^*(\gamma)\beta = 0.$$

We finishes the proof. 

In the following, we show that the nonabelian omni $n$-Lie algebra is a linearization of the higher analogue of the Courant algebroid $(\mathcal{T}^{n-1}M, \{\cdot, \cdot\}, \{\cdot, \cdot, \cdot\}, \rho_n)$ associated to a Nambu-Poisson manifold $(M, \pi)$.

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \rho_3)$ be an $m$-dimensional $n$-Lie algebra such that it induces a linear Nambu-Poisson structure $\pi^*\mathfrak{g}$ on $\mathfrak{g}^*$. Then we obtain the higher analogue of the Courant algebroid $\mathcal{T}^{n-1}_\pi \mathfrak{g}^*$. Let $\{x^i\}$ be a basis of the vector space $\mathfrak{g}$. Using the same notations as in Section 4 we have

$$\pi^*\mathfrak{g} = \sum_{1 \leq i_1 < \cdots < i_n \leq m} l_{[x^{i_1}, \ldots, x^{i_n}]} \frac{\partial}{\partial x^{i_1}} \land \cdots \land \frac{\partial}{\partial x^{i_n}}.$$

Lemma 6.3. For all $A \in \mathfrak{g}(\mathfrak{g})$ and $u, v \in \land^{n-1} \mathfrak{g}$, we have

$$\pi^*\mathfrak{g}(\tilde{u}) = \overline{\text{ad}_u}, \quad (31)$$

$$\tilde{[u, v]}_{\pi^*\mathfrak{g}} = u \circ v, \quad (32)$$

$$\pi^*\mathfrak{g}(L_A \tilde{u}) = \overline{\text{ad}_{L_A u}}. \quad (33)$$

Proof. For $u = \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq m} \xi_{j_1, \ldots, j_{n-1}} x^{j_1} \land \cdots \land x^{j_{n-1}} \in \land^{n-1} \mathfrak{g}$ with the corresponding constant $(n-1)$-form $\tilde{u} = \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq m} \xi_{j_1, \ldots, j_{n-1}} \cdot \partial_{x^{j_1}} \land \cdots \land \partial_{x^{j_{n-1}}}$, we have

$$\pi^*\mathfrak{g}(\tilde{u}) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \sum_{k=1}^{n} (-1)^{n-k} \xi_{i_1, \ldots, i_k, \ldots, i_n} l_{[x^{i_1}, \ldots, x^{i_k}, \ldots, x^{i_n}]} \frac{\partial}{\partial x^{i_k}}$$

$$= \sum_{1 \leq i_1 < \cdots < i_n \leq m} \xi_{i_1, \ldots, i_n} l_{[x^{i_1}, \ldots, x^{i_n}]} \frac{\partial}{\partial x^{i_n}}$$

$$= \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq m} \sum_{l=1}^{m} \xi_{j_1, \ldots, j_{n-1}, l} l_{[x^{j_1}, \ldots, x^{j_{n-1}}, x^l]} \frac{\partial}{\partial x^{j_l}}$$

$$= \overline{\text{ad}_u}.$$

which implies that \( \mathfrak{g} \) holds.

\footnote{Not all $n$-Lie algebras can give linear Nambu-Poisson structures on dual spaces, see \cite{7} for details.}
Since $\hat{u}$ is a constant $(n - 1)$-form, by (16) and (31), we have
\[
[u, \tilde{v}]_{\pi_\theta} = L_{\pi_1(u)} \tilde{v} - L_{\pi_1(\tilde{u})} u + di_{\pi_1(u)} \tilde{u} \\
= L_{\pi_1(u)} \tilde{v} - i_{\pi_1(\tilde{u})} d\tilde{u} \\
= L_{\pi_1(u)} \tilde{v} - \mathcal{L}_{\mathcal{L}_u} \tilde{v} \\
= \mathcal{L}_{\mathcal{L}_u} v = u \circ v,
\]
which implies that (32) holds.

By (16) and (31), we have
\[
\pi^2(L_A \tilde{u}) = \pi^2(\mathcal{L}_A u) = \text{ad}_{\mathcal{L}_u}.
\]
This ends the proof. $\blacksquare$

**Theorem 6.4.** Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \{\cdot, \cdot\}_g)$ be an $m$-dimensional $n$-Lie algebra such that it induces a linear Nambu-Poisson structure $\pi_\theta$ on $\mathfrak{g}^*$. Then the nonabelian omni $n$-Lie algebra $(\mathfrak{g}(\mathfrak{g}) \oplus \wedge^{n-1} \mathfrak{g}, (\cdot, :)_g, \{\cdot, \cdot\}_g, \rho_{\pi_\theta})$ is induced from the higher analogue of the Courant algebroid $(T^{n-1} \mathfrak{g}^*, (\cdot, :)_g, \{\cdot, \cdot\}_g, \rho_{\pi_\theta})$ associated to the Nambu-Poisson manifold $(\mathfrak{g}^*, \pi_\theta)$ via restriction to $\mathfrak{X}_{\text{lin}}(\mathfrak{g}^*) \oplus \Omega_{\text{con}}^{n-1}(\mathfrak{g}^*)$. More precisely, we have
\[
(\Phi(A + u), \Phi(B + v))_+ = [A + u, B + v]^+, \\
\{\Phi(A + u), \Phi(B + v)\}_{\pi_\theta} = \Phi(A + u, B + v)_g, \\
L_{\rho_{\pi_\theta} \Phi(A + u)} \mathfrak{W} = \rho_{\mathfrak{g}}(A + u)(\mathfrak{W}).
\]

**Proof.** (34) has been proved in Theorem 1.2. By (15) - (17) and (31) - (33), we deduce that
\[
\left[\hat{A} + \hat{u}, \hat{B} + \hat{v}\right]_{\pi_\theta} = [\hat{A}, \hat{B}] + [\hat{A}, \pi^2(\hat{v})] + [\pi^2(\hat{u}), \hat{B}] - \pi^2(L_A \hat{v}) + \pi^2(i_B d\hat{u}) + L_A \hat{v} - i_B d\tilde{u} + [\hat{u}, \tilde{v}]_{\pi_\theta} \\
= [\hat{A}, \hat{B}] + [\hat{A}, \mathcal{L}_u] + [\mathcal{L}_u, \hat{B}] - \mathcal{L}_{\mathcal{L}_A} \mathcal{L}_v + \mathcal{L}_A \tilde{v} + u \circ \tilde{v} \\
= [\hat{A}, \hat{B}] + [\hat{A}, \mathcal{L}_u] + [\mathcal{L}_u, \hat{B}] - \mathcal{L}_{\mathcal{L}_A} \mathcal{L}_v + \mathcal{L}_A \tilde{v} + u \circ \tilde{v} \\
= \Phi(A + u, B + v)_{\mathfrak{g}},
\]
which implies that (35) holds. By (17), we have $L_{\hat{A} + i^2(\hat{u})} \mathfrak{W} = \rho_{\mathfrak{g}}(A)(\mathfrak{W})$. Thus
\[
L_{\hat{A} + i^2(\hat{u})} \mathfrak{W} = L_{\hat{A} + i^2(\hat{u})} \mathfrak{W} = L_{\hat{A} + i^2(\hat{u})} \mathfrak{W} + L_{\mathcal{L}_u} \mathfrak{W} \\
= \rho_{\mathfrak{g}}(A)(\mathfrak{W}) + \rho_{\mathfrak{g}}(\mathcal{L}_u)(\mathfrak{W}) = \rho_{\mathfrak{g}}(A + u)(\mathfrak{W}),
\]
which says that (36) holds. This ends the proof. $\blacksquare$

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