The Recoils of the Accelerated Detector and the Decoherence of its Fluxes

R. Parentani

Laboratoire de Physique Théorique de l’Ecole Normale Supérieure
24 rue Lhomond 75.231 Paris CEDEX 05, France.

Abstract  The recoils of an accelerated system caused by the transitions characterizing thermal equilibrium are taken into account through the quantum character of the position of the system. The specific model is that of a two level ion accelerated by a constant electric field. The introduction of the wave function of the center of mass position clarifies the role of the classical trajectory in the thermalization process. Then the effects of the recoils on the properties of the emitted fluxes are analyzed. The decoherence of successive emissions induced by the recoils is manifest and simplifies the properties of the fluxes. The fundamental reason for which one cannot neglect the recoils upon considering accelerated systems is stressed and put in parallel with the emergence of ”transplanckian” frequencies in Hawking radiation.

1e-mail: parenta@physique.ens.fr
2Unité propre de recherche du C.N.R.S. associée à l’ENS et à l’Université de Paris Sud.
1 Introduction

It is now well known that an accelerated system perceives Minkowski vacuum as a thermal bath\cite{1}. This can be understood from the fact that the inner degrees of freedom of the system interact with the local quanta of the radiation field. Indeed the accelerated system interacts with the Rindler quanta and these quanta are related to the usual Minkowski ones by a non trivial Bogoliubov transformation\cite{2}.

It also known that this Bogoliubov transformation is very similar in character to the one that describes the scattering of modes in the time dependent geometry of a collapsing star and leads to Hawking radiation\cite{3}: a steady thermal flux.

In both cases, the kinematical agent of the thermalization is treated classically. Indeed in the accelerated situation, the uniformly accelerated trajectory $z^2 - t^2 = 1/a^2$ is a priori given and unaffected by the recoils of the system when it interchanges quanta during the transitions characterizing thermal equilibrium. Similarly, in the black hole situation, the background geometry is treated classically and unaffected by the individual emissions of Hawking quanta. Up to now the backreaction of Hawking radiation onto the geometry has been achieved\cite{4}\cite{5}\cite{6} only in the mean i.e. in the semi classical approximation wherein only the mean value of the energy momentum tensor enters in the Einstein equations treated classically. This averaging procedure over individual emission acts appears to be doubtful for the following reason\cite{7}\cite{8}\cite{9}. The frequencies involved in the conversion from vacuum fluctuations to Hawking quanta rapidly exceed the Planck mass. The reason for this is precisely what leads to the thermalization i.e. an exponentially varying Doppler factor relating incoming scattered modes to out-modes specified in the infinity future. To understand the consequences of those transplanckian frequencies for both the Hawking flux at latter times and the notion of a classical geometry is the major problem of black hole physics.

In the present article we shall investigate the much more modest problem of the effects of recoils of the accelerated system when one treats its trajectory as a dynamical variable. That is, one introduces an external field which causes the acceleration and one quantizes the center of mass motion of the accelerated system. The specific example treated here is that of a two level ion in a constant electric field\cite{10}. The consequences of the recoils are the following.

The recoils do not modify the thermalization of the inner degrees of freedom of the ion. The formulation in which the center of mass is quantized permits even to extend this thermalization to more ”quantum” situations wherein the accelerated system is described by a completely delocalized wave function\cite{11}.

Secondly the recoils do modify the properties of the flux emitted by the accelerated system. We first recall its properties in the absence of recoil\cite{12}. Once thermal equilibrium is achieved, no mean flux is emitted\cite{13}\cite{14}\cite{15} nor absorbed by the accelerated system, in perfect analogy with the usual inertial thermal equilibrium. Nevertheless upon considering the total energy emitted one finds a diverging energy\cite{16}\cite{17} whose origin is that each internal transition is accompanied by the emission of a Minkowski quantum. The reconciliation between these seemingly contradictory results lies in an analysis of the transients\cite{18} which shows how the interferences among all the Minkowski photons
destructively hide their presence in the intermediate equilibrium situation. These interferences are automatically washed out upon considering global quantities such as the total energy emitted.

When the recoils of the accelerated system are taken into account, one finds a great simplification since the Minkowski photons decohere after a logarithmically short proper time, concomitant with the exponential Doppler shift mentioned above. Then the thermalized accelerated system emits a steady positive flux directly associated with the produced Minkowski quanta. This steady emission of a positive flux comes from the steady conversion of potential energy in the electric field into radiation.

The paper is organized as follow. In section 1, we briefly recall the properties of the transition amplitudes and the fluxes in the no-recoil situation. In section 2 we present the model of the two level ion. In section 3 we analyze the modifications of the transition amplitudes introduced by the strict momentum conservation which accounts for the recoil. Finally in section 4 we show how the finiteness of the rest mass of the accelerated system leads, after a logarithmically short time, to the decoherence of the emitted fluxes. We conclude the paper by some remarks mentioning other situations wherein recoil effects play or might play an important role.

2 The Fluxes in the Absence of Recoil

In this section, we present the relevant properties of the transition amplitudes which lead to the thermalization of the accelerated system\cite{1}. We then analyze the properties of the flux emitted by the accelerated atom\cite{12,13,14,15}. Since this material is now well understood and already published\cite{16,17,18}, we shall restrict ourselves to the presentation of the various properties which will be compared to the ones obtained in the situation wherein the recoil of the atom is taken into account.

We consider a two level atom maintained, for all times, exactly on the uniformly accelerated trajectory

$$t_a(\tau) = a^{-1}\sinh a\tau, \quad z_a(\tau) = a^{-1}\cosh a\tau \quad (1)$$

where $a$ is the acceleration and $\tau$ is the proper time of the accelerated atom. We work for simplicity in Minkowski space time in $1+1$ dimensions. The two levels of the atom are designated by $|->$ and $|+>$ for the ground state and the excited state respectively. The transitions from one state to another are induced by the operators $A, A^\dagger$

$$A|-> = 0, \quad A|+> = |->$$
$$A^\dagger|-> = |+>, \quad A^\dagger|+> = 0 \quad (2)$$

The atom is coupled to a massless scalar field $\phi(t, z)$. The Klein-Gordon equation is

$$(\partial_t^2 - \partial_z^2)\phi = 0$$

and the general solution is thus

$$\phi(U,V) = \phi(U) + \phi(V) \quad (3)$$
where \(U, V\) are the light like coordinates given by \(U = t - z, V = t + z\). The general solution of the right moving part may be decomposed into the orthonormal plane waves:

\[
\varphi_\omega(U) = \frac{e^{-iωU}}{\sqrt{4πω}}
\]

where \(ω\) is the Minkowski energy. The Hamiltonian of the right moving modes is given by

\[
H_M = \int_{-∞}^{+∞} dUT_{UU} = \int_0^{∞} dωω(a_ω^†a_ω)
\]

where \(T_{UU} = (\partial_t \phi)^2\) is the normal ordered (with respect to the Minkowski operators) flux and where the Heisemberg field operator \(\phi\) has been decomposed into the Minkowski operators of destruction and creation \(a_ω, a_ω^†\) as

\[
\phi(U) = \int_0^{∞} dω \left( a_ω\varphi_ω(U) + a_ω^†\varphi_ω^*(U) \right)
\]

The operators \(a_ω\) annihilates Minkowski vacuum \(|Mink>\), the ground state of the Hamiltonian \(H_M\). In this article we shall restrict ourselves to the \(U\)-part of the \(φ\) field only. (Note however that the coupling with the atom introduces some mixing among the \(U\) and the \(V\)-parts; but this mixing does not modify the properties of the flux nor the conclusions of this article).

The coupling between the atom and the field is taken to be

\[
\int dtdx \ H_{\text{int}}(\mathbb{I}, \frac{\partial}{\partial \mathbb{I}}) = \int \left[ \tau H_{\text{int}}(\tau) = \int \left[ \tau \left[ \left( A^{-}\right)^{\hat{A}} + A^+^{\hat{A}} \right) \phi(U_a(\tau)) \right] \right] \]

where \(U_a(\tau) = t_a(\tau) - z_a(\tau) = -e^{-aτ}/a\) and where \(H_{\text{int}}(\tau)\) is

\[
H_{\text{int}}(\tau) = ga \int_0^{∞} dω \sqrt{4πω} \left[ \left( Ae^{-iΔmτ} + A^+ e^{iΔmτ} \right) \left( a_ω e^{-aτ}/a + a_ω^† e^{-ωe^{-aτ}/a} \right) \right]
\]

By the locality of the coupling, \(\phi(U_a(\tau))\) evaluated only along the classical trajectory enters into the interaction.

Then the transition amplitude (spontaneous excitation) from \(|- >|Mink>\) to the excited state \(|+ >|1_ω >\) (where \(|1_ω > = a_ω^†|Mink>\) is the one particle state of energy-momentum \(ω\)) is

\[
B(ω, Δm) = < 1_ω | < +| e^{-i \int dτ H_{\text{int}}} |- >|Mink>
\]

To first order in \(g\) it is given by

\[
B(ω, Δm) = -iga \int_{-∞}^{+∞} dτ e^{iΔmτ} e^{-\frac{ωe^{-aτ}/a}{\sqrt{4πω}}} = ig\Gamma(-iΔm/a) \frac{(ω/a)^{iΔm/a}}{\sqrt{4πω}} e^{-πΔm^2/2a}
\]

where \(Γ(x)\) is the Euler function. This transition amplitude is closely related to the β coefficient of the Bogoliubov transformation[3] which relates the Minkowski operators
to the Rindler operators associated with the eigenmodes of $-i\partial_\tau$ (hence given by $\varphi_{\text{Rindler}}(\tau) = e^{-i\lambda\tau}$). This relation indicates that the accelerated atom absorbs (or emits) Rindler quanta.

The transition amplitude (disintegration) from $|+\rangle|Mink\rangle$ to the state $|-\rangle|1\omega\rangle$ is given by $A(\omega, \Delta m) = B(\omega, -\Delta m)$. To first order in $g$ one finds

$$A(\omega, \Delta m) = -B^*(\omega, \Delta m)e^{\pi\Delta m/a}$$

Thus for all $\omega$ one has

$$\left|\frac{B(\omega, \Delta m)}{A(\omega, \Delta m)}\right|^2 = e^{-2\pi\Delta m/a}$$

Since this ratio is independent of $\omega$, the ratio of the probabilities of transitions (excitation and disintegration) is also given by eq. (12). Hence at equilibrium, the ratio of the probabilities $P_-, P_+$ to find the atom in the ground or excited state satisfy

$$\frac{P_+}{P_-} = \left|\frac{B(\omega, \Delta m)}{A(\omega, \Delta m)}\right|^2 = e^{-2\pi\Delta m/a}$$

This is the Unruh effect \[1\]: at equilibrium, the probabilities of occupation are thermally distributed with temperature $a/2\pi$.

In preparation for the "recoil case" and following ref. \[19\], we evaluate the amplitude $B(\omega, \Delta m)$ at the saddle point approximation in order to localized around which value of $\tau$ does the amplitude acquire its value. The stationary phase of the integrand of eq. (10) is at

$$\Delta m = -\omega e^{-a\tau^*(\omega)}$$

hence at

$$\tau^*(\omega) = \frac{1}{a} \ln \frac{\omega}{\Delta m} + i\pi/a$$

where the imaginary part is fixed by an analysis in the complex $\tau$ plane. Similarly the stationary phase for the disintegration amplitude $A(\omega, \Delta m)$ gives $\Delta m = \omega e^{-a\tilde{\tau}(\omega)}$. Thus the saddle $\tilde{\tau}$ is given by the real part of eq. (15) (i.e. the saddle $\tilde{\tau}$ is the time at which the exponentially varying Doppler factor $e^{-a\tau}$ brings $\omega$ in resonance with $\Delta m$). In both cases, the width around the saddle is independent of $\omega$ and equal to $(\Delta ma)^{-1/2}$. This allows for the establishment of a rate through successive resonances with different $\omega$ \[13\] \[20\]. Very important also is the fact that the difference ($= i\pi/a$) between the saddles times for $A$ and $B$ is imaginary and independent of $\omega$. Indeed, it leads to the fact that $B(\omega, \Delta m)$ and $A(\omega, \Delta m)$, evaluated at the saddle point (s.p.) approximation, satisfy automatically

$$B_{\text{s.p.}}(\omega, \Delta m) = -ig(\omega/\Delta m)^{-i\Delta m/a} e^{-\pi\Delta m/a} \sqrt{\frac{2\pi a}{i\Delta m}} e^{i\Delta m/a}$$

Thus eq. (12) is satisfied and therefore, at equilibrium, eq. (13) as well. One should nevertheless add that the validity of the saddle point approximation requires $\Delta m \gg a$. 

\[1\]
Thus, strictly speaking, this method is valid only in the Boltzman regime where the density of particles is low. We recall that the norm of $B(\omega, \Delta m)$ given in eq. (14) is proportional to the Bose Einstein distribution: $(e^{2\pi \Delta m/a} - 1)^{-1}$.

We now analyze the properties of the mean flux emitted by the atom. When the initial state is $| - > | Mink >$, the mean flux emitted at $V = \infty$ is given by

$$\langle T_{UU} \rangle = < Mink | < - | e^{i \int d\tau H_{int} (\partial U \phi)^2} e^{-i \int d\tau H_{int}} | - > | Mink > \quad (17)$$

To order $g^2$, the state $e^{-i \int d\tau H_{int}} | - > | Mink >$ is

$$e^{-i \int d\tau H_{int}} | - > | Mink > = | - > | Mink > + \int_0^\infty d\omega B(\omega, \Delta m)|+ > | 1_\omega > + \int_0^\infty d\omega \int_0^\infty d\omega' C(\omega, \omega', \Delta m)| - > | 1_\omega 1_{\omega'} > \quad (18)$$

where

$$C(\omega, \omega', \Delta m) = < 1_\omega 1_{\omega'} | - | e^{-i \int d\tau H_{int}} | - > | Mink > \quad (19)$$

is the amplitude to emit two Minkowski photons. Thus to order $g^2$, the mean flux is

$$\langle T_{UU} \rangle = \langle T_{UU} \rangle_1 + \langle T_{UU} \rangle_2 \quad (20)$$

where

$$\langle T_{UU} \rangle_1 = 2 \int_0^\infty d\omega \int_0^\infty d\omega' B(\omega, \Delta m) B^*(\omega', \Delta m) \frac{\sqrt{\omega' \omega}}{4\pi} e^{i(\omega' - \omega)U} \quad (21)$$

comes from the square of the linear term in $g$ of eq. (18), and where

$$\langle T_{UU} \rangle_2 = -2 \text{Re} \left[ \int_0^\infty d\omega \int_0^\infty d\omega' C(\omega, \omega', \Delta m) \frac{\sqrt{\omega' \omega}}{4\pi} e^{-i(\omega' + \omega)U} \right] \quad (22)$$

comes from the interference between the (unperturbed) first term of eq. (18) and the third term containing two particles.

Two important properties should be noted. The first one is that $\langle T_{UU} \rangle_2$ does not contribute to the total energy emitted since $\int_{-\infty}^{+\infty} dU e^{-i(\omega' + \omega)U} = \delta(\omega' + \omega)$ and $\omega, \omega' > 0$. The total energy emitted is thus given by $\langle T_{UU} \rangle_1$ only

$$\langle H_M \rangle = \int_{-\infty}^{+\infty} dU \langle T_{UU} \rangle_1 = \int_0^\infty d\omega \omega |B(\omega, \Delta m)|^2 \quad (23)$$

As remarked in [10] [21] [13], this energy diverges due to the U.V. character of the amplitudes $B(\omega)$ (We shall not be bothered by the I.R. behavior since the contribution for the total energy is always finite). Thus a cut off is needed in order to obtain a finite energy. The simplest cut off consist on multiplying $B(\omega)$ by the regulator $e^{-\varepsilon \omega}$. By virtue of the locality of the resonance condition eq. (13), the introduction of the regulator which damps the $\omega > 1/\varepsilon$ mimics a switch off function around

$$\tau_\varepsilon = \frac{1}{a \ln(\frac{1}{\varepsilon \Delta m})} \quad (24)$$
where $\varepsilon$ is such that $\tau_\varepsilon >> 1/a$ (indeed the interaction should last many $1/a$ times in order for the atom to properly thermalize\[18\]).

The second important property concerns the relative importance of $\langle T_{UU} \rangle_1$ and $\langle T_{UU} \rangle_2$ in the stationary regime (when the Golden Rule is applicable i.e. when the transition probability grows linearly with $\tau$). This intermediate behavior is manifest, for $|U| > \varepsilon$, when one compute explicitly $\langle T_{UU} \rangle_1$ and $\langle T_{UU} \rangle_2$ using the regulator $\varepsilon$. Indeed one finds

$$\langle T_{UU} \rangle_1 = +2 \left( \frac{g}{4\pi} \right)^2 |\Gamma(i\Delta m/a)|^4 \left[ \frac{1}{U^2 + \varepsilon^2} \right] \left[ \theta(U) + e^{-2\pi\Delta m/a}\theta(-U) \right]$$

$$\langle T_{UU} \rangle_2 = -2 \left( \frac{g}{4\pi} \right)^2 |\Gamma(i\Delta m/a)|^4 \Re \left[ \frac{1}{(U+i\varepsilon)^2} \right]$$

(The essential simplifying step in the computation of $\langle T_{UU} \rangle_2$ is the replacement of $C(\omega,\omega',\Delta m)$ by $A(\omega',\Delta m)B(\omega,\Delta m)$. This is legitimate when the Golden Rule conditions are satisfied. See ref. \[18\] for the details.)

For positive $U$ (in the left quadrant where the atom isn’t), $\langle T_{UU} \rangle_1 + \langle T_{UU} \rangle_2 = 0$ as causality requires\[12\]. For negative $U$, one finds (for $-U > \varepsilon$) a steady absorption of Rindler energy\[13\] associated with the steady increase of the probability to find the atom in the excited state

$$\langle T_{UU} \rangle_2 = -2 \left( \frac{g}{4\pi} \right)^2 |\Gamma(i\Delta m/a)|^4 \frac{1}{(U+i\varepsilon)^2}$$

Thus one has two very different regimes. In the intermediate period $\langle T_{UU} \rangle_2$ dominates the flux in accordance with the Golden Rule description in the accelerated frame in which one sees the absorption of a Rindler quantum. But when integrated, to find the total Minkowski emitted, see eq. \(23\), $\langle T_{UU} \rangle_2$ gives no contribution at all. The reconciliation of the two pictures arises from a detailed analysis of the transients\[18\]. For instance one sees from eq. \(26\) that $\langle T_{UU} \rangle_2$ is positive for $|U| < \varepsilon$ and one verifies that its integral over $U$ vanishes.

In a similar fashion one may study the equilibrium regime. One includes the flux associated with the transition from $|-\rangle$ to $|+\rangle$ and weights the two contributions with the thermal occupation probabilities given in eq. \(13\). One finds that in thermal equilibrium, there is no mean flux as first pointed out by Grove\[13\]. Nevertheless, in a global description, when one compute the total energy radiated or the total number of quanta radiated during the interacting period, one finds again a contribution which seems to occur at a constant rate\[18\]. Again, the reconciliation between the two descriptions necessitates the analysis of the transients when one switches off the interaction (i.e. for $|U| \simeq \varepsilon$).

This concludes the analysis when the recoil of the atom is not taken into account. We emphasize that all the emitted Minkowski quanta interfere constructively in the $\langle T_{UU} \rangle_2$ term so as to maintain a negative mean flux (eq. \(27\)) in the Golden Rule regime. The extremely well tuned phases which lead to this non vanishing character of $\langle T_{UU} \rangle_2$ will be washed out after a finite proper time when recoils will be taken into account through momentum conservation.
3 The Two Level Ion in a Constant Electric Field

In this section, we present the model introduced in ref. [10] (which is similar to the one used by Bell and Leinaas[22]) which will allow us to take automatically into account the recoils of the ion caused by the transitions characterizing the accelerated thermal situation. This model will also allow us to prove that the thermalization of the inner degrees of freedom of an accelerated system does not require a well defined classical trajectory. Indeed, even when one deals with delocalized waves for the position of the ion, the ratio eq. (12) is obtained[11] and therefore the thermal equilibrium ratio eq. (13) is obtained as well.

The model consist on two scalar charged fields ($\psi_M$ and $\psi_m$) of slightly different masses ($M$ and $m$) which will play the role of the former states of the atom: $|+>$ and $|->$. The quanta of these fields are accelerated by an external classical constant electric field $E$. One has

$$\frac{E}{M} = a \simeq \frac{E}{m}$$

because one imposes

$$\Delta m = M - m << M$$

(29)

to have the mass gap well separated from the rest mass of the ion.

We work in the homogeneous gauge ($A_t = 0, A_z = -Et$). In that gauge, the momentum $k$ is a conserved quantity and the energy $p$ of a relativistic particle of mass $M$ is given by the mass shell constraint $(p_\mu - A_\mu)^2 = M^2$ i.e.

$$p^2(M, k, t) = M^2 + (k + Et)^2$$

(30)

The classical equations of motion are easily obtained from this equation and are given in terms of the proper time $\tau$ by

$$p(M, k, t) = M \cosh a \tau$$

$$t + k/E = (1/a) \sinh a \tau$$

$$z - z_0 = (1/a) \cosh a \tau$$

(31)

Thus at fixed $k$, the time of the turning point (i.e. $dt/d\tau = 1$) is fixed whereas its position is arbitrary.

From eq. (30), the Klein Gordon equation for a mode $\psi_{k,M}(t, z) = e^{ikz} \chi_{k,M}(t)$ is

$$\left[\partial_t^2 + M^2 + (k + Et)^2\right] \chi_{k,M}(t) = 0$$

(32)

When $\Delta m \simeq a$ and when eq. (29) is satisfied, one has $M^2/E >> 1$. Then the Schwinger pair production amplitude[23] may be completely ignored (since the mean density of produced pairs scales like $e^{-\pi M^2/E}$). Furthermore, in this case, the W.K.B. approximation for the modes $\chi_{k,M}(t)$ is valid for all $t$. Indeed, the corrections to this approximation are smaller than $(M^2/E)^{-1}$. 

8
The modes $\psi_{k,M}(t,z)$ can be thus correctly approximated by

$$
\psi_{k,M}(t,z) = \frac{e^{ikz}}{\sqrt{2\pi}} e^{-i \int p(M,k,t') dt'} \frac{\sqrt{\pi} e^{-\int p(M,k,t)}}{\sqrt{p(M,k,t)}}
$$

(33)

where $p(M,k,t')$ is the classical energy at fixed $k$ given in eq. (30).

As emphasized in refs. [10] [24], the wave packets of the form

$$
\Psi_{k,M}(t,z) = \int dk' \frac{e^{-(k' - k)^2/2\sigma}}{(\pi \sigma)^{1/4}} \psi_{k',M}(t,z)
$$

(34)

do not spread if $\sigma \simeq E$. In that case, the spread (in $z$ at fixed $t$) is of the order of $E^{-1/2} = (Ma)^{-1/2}$ for all times and thus much smaller than the acceleration length $1/a$ characterizing the classical trajectory eq. (2). (This is readily seen by evaluating the time dependence of phase of $\Psi_{k,M}(t,z)$ at large $t$ when the W.K.B. approximation becomes exact.) Since the stationary phase condition of the $\Psi_{k,M}(t,z)$ modes gives back the accelerated trajectory and since the wave packets do not spread, one has thus a center of mass position quantized version of the accelerated system, which furthermore tends uniformly to the classical limit when $M \to \infty, E \to \infty$ with $E/M = a$ fixed.

The interacting Hamiltonian which induces transitions between the quanta of mass $M$ and $m$ by the emission or absorption of a massless neutral quantum of the $\phi$ field is simply

$$
H_{\psi \phi} = \tilde{g} M^2 \int dz \left[ \psi_M^\dagger(t,z) \psi_m(t,z) + \psi_M(t,z) \psi_m^\dagger(t,z) \right] \phi(t,z)
$$

(35)

where $\tilde{g}$ is dimensionless. In momentum representation, by limiting ourselves to the right moving modes of the $\phi$ field, one obtains

$$
H_{\psi \phi} = \tilde{g} M^2 \int_{-\infty}^{+\infty} dk \int_0^{\infty} \frac{d\omega}{\sqrt{4\pi \omega}} \left[ b_{M,k-\omega} \chi_{M,k-\omega}(t) b_{m,k}^\dagger \chi_{m,k}^*(t) + \text{h.c.} \right] a_{\omega} e^{-i\omega t} + \left[ b_{M,k+\omega} \chi_{M,k+\omega}(t) b_{m,k}^\dagger \chi_{m,k}^*(t) + \text{h.c.} \right] a_{\omega}^\dagger e^{+i\omega t}
$$

(36)

where the operator $b_{M,k-\omega}$ detroys a quantum of mass $M$ and momentum $k - \omega$ and the operator $b_{m,k}^\dagger$ creates a quantum of mass $m$ and momentum $k$. Therefore the product $b_{M,k-\omega} b_{m,k}^\dagger$ plays the role of the operator $A$ (see eq. (2)) with, in addition, a strict conservation of momentum. (We have not introduced anti-ion creation operators in the Hamiltonian $H_{\psi \phi}$. This is a legitimate truncation when $M^2/E >> 1$.)

Contrary to what happens in the original Unruh model, the interaction between the radiation field $\phi$ and the two levels of the accelerated system is no longer restricted a priori to a classical trajectory (see eqs. (7) and (8)). It is now in the behavior of the wave functions $\chi_{M,k}(t)$ that the accelerated properties (and the thermalization after effects) are encoded.
4 The Transition Amplitudes for the Two Level Ion

As in the no-recoil model, we shall compute the transition amplitudes as well as the properties of the fluxes caused by these transitions. The analysis of the fluxes is presented in the next section.

We first prove that the behavior of the $\chi_M$ modes is sufficient to obtain the Unruh effect and this without having to localize the ion [11] i.e. without having to deal with well localized wave packets. Thus we compute the amplitude to jump from the state $|1_k >_m |0 >_M |Mink >$ to the state $|0 >_m |1_k' >_M |1 >_M$. (Where $|0 >_M$ designates the vacuum state for the $\psi_M$ field, and where $|1_k >_m = b^\dagger_{m,k} |0 >_m$ is the one particle state of the $\psi_m$ field of momentum $k$). This amplitude is given by

$$B(||, ||', \omega) = \langle 0 | e^{-i\int dtH_{\psi\phi}} |1_k >_m |0 >_M |Mink > = 2\pi \delta(k - k' - \omega) \tilde{B}(\Delta m, k, \omega)$$

where momentum conservation occurs owing to the homogeneous character of the electric field. To first order in $\tilde{g}$, $\tilde{B}(\Delta m, k, \omega)$ is

$$\tilde{B}(\Delta m, k, \omega) = -i\tilde{g}M^2 \int_{-\infty}^{+\infty} dt \chi^*_{M,k-\omega}(t) \chi_{m,k}(t) e^{i\omega t} \sqrt{4\pi \omega}$$

where we have used the W.K.B approximation eq. (33) for the $\chi$ modes.

To grasp the content of this amplitude, it is useful to first develop the integrand in powers of $\omega$ and $\Delta m$. To first order in $\Delta m$ and $\omega$, the phase $\varphi(\tau)$ of the integrand is

$$\varphi(\tau) = \int^\tau [p(M, k - \omega, t') - p(m, k, t')] + \omega t$$

$$= \Delta m \int^\tau dt' \partial_M p(M, k, t') - \omega \int^\tau dt' \partial_k p(M, k, t') + \omega t$$

$$= \Delta m \Delta \tau(t) - \omega(\Delta z_a(t) - t) + C$$

$$= \Delta m \Delta \tau(t) - \frac{\omega}{a} e^{-a\Delta \tau(t)} - \frac{\omega k}{E} + C$$

(39)

where $C$ is a constant and where $\Delta \tau(t)$ and $\Delta z_a(t)$ are the classical relations, eqs. (31), between the lapses of proper time and of $z$ and $t + k/E$ evaluated along any uniformly accelerated trajectory. Eq. (39) may be checked explicitly by computing the integrals using eq. (33). It is perhaps more instructive to realize that those relations are nothing but the Hamilton-Jacobi relations between the classical action $S_{cl}$ and proper time or momentum: $\partial_M S_{cl} = -\Delta \tau$ and $\partial_k S_{cl} = -\Delta z$. Hence, whatever is the nature of the external field which brings the system into constant acceleration, the first two terms of eq. (39) will always be found.

In addition, in that approximation, one can neglect the dependence in $\omega$ and $\Delta m$ in the denominator of eq. (38). Hence the measure is $dt/p(M, k, t) = d\tau/M$. Thus to first
order in $\omega$ and $\Delta m$, one has

$$\tilde{B}(\Delta m, k, \omega) = -ig M \int_{-\infty}^{+\infty} d\tau e^{i\Delta m \tau} e^{-i\omega \epsilon^{\text{a}}/\alpha} e^{-i(\omega k/E - C)}$$

$$= \left[ \frac{\tilde{g} M}{ga} \right] B(\Delta m, \omega) e^{-i(\omega k/E - C)}$$

(40)

where $B(\Delta m, \omega)$ is the amplitude in the no-recoil original situation given in eq. (10).

Very important is the fact that the initial momentum $k$ introduces only a phase in the amplitude $\tilde{B}(\Delta m, k, \omega)$. Thus any superposition of modes $\psi_{m,k}$ will give rise to the same probability to emit of photon of energy $\omega$. And the norm of the ratio of $\tilde{B}(\Delta m, k, \omega)$ over $\tilde{A}(\Delta m, k, \omega)$ satisfies eq. (11). Therefore, in that approximation, the two level ion thermalizes exactly as in the no-recoil case.

Having understood the role of the various factors of the amplitude $\tilde{B}(\Delta m, k, \omega)$, one may now refine the derivation and compute $\tilde{B}(\Delta m, k, \omega)$ at the saddle point approximation without developing a priori the integrand in powers of $\omega$ and $\Delta m$. The upshot of this calculation is that eq. (40) is correct up to an additional phase factor and correction terms of the form $(\Delta m/M)^n O(1)$. In order for this to be true, it is necessary to verify that terms in $(\Delta m/M)^n$ are not multiplied by factors scaling like $e^{a\tau}$, which one might have feared owing to the scaling given in eq. (14) of the resonant energy $\omega$ with $\tau$. This is an important condition because the proper time necessary to get a rate formula, which gives rise to the thermal distribution, requires $\tau >> 1/\alpha$. A scaling factor $e^{a\tau}$ would then ruin the condition that relies on $\Delta m/M << 1$.

The stationary point $t^*$ of the integrand of eq. (38) is at

$$p(M, k - \omega, t^*) - p(m, k, t^*) + \omega = 0$$

(41)

i.e. conservation of the Minkowski energy (contrary to the condition eq. (14) which was the resonance condition in the accelerated frame, i.e. conservation of Rindler energy). Taking the square of eq. (11) and using eq. (30) one gets

$$\frac{M^2 - m^2}{2} = \omega [(k - \omega + Et^*) - p(M, k - \omega, t^*)]$$

(42)

Introducing once more the proper time eq. (34) one finds

$$\Delta m(1 - \Delta m/2M) = -\omega e^{-a\tau}$$

(43)

Thus, in our situation with $\Delta m/M << 1$, the stationarity condition gives back the resonance time eq. (15) (as well as its imaginary part) of the no-recoil model. A similar analysis of the inverse transition $\tilde{A}(\Delta m, k, \omega)$ (i.e. where the initial state $M$ has momentum $k$) leads back to the saddle point $\tilde{\tau}$ given by eq. (13) with a $+$ sign on the r.h.s.

When evaluated at the saddle time $t^*$, the total phase $\tilde{\varphi}(t^*)$ of $\tilde{B}$, eq. (38), is

$$\tilde{\varphi}(t^*) = \omega t^* + \int_{0}^{t^*+(k-\omega)/E} dt' p(M, 0, t') - \int_{0}^{t^*+(k-\omega)/E} dt' p(m, 0, t')$$

$$= \omega t^* + \int_{0}^{t^*+(k-\omega)/E} dt' [p(M, 0, t') - p(m, 0, t')] - \int_{t^*+(k-\omega)/E}^{t^*+(k-\omega)/E} dt' p(M, 0, t')$$

$$= \varphi_1(t^*) + \varphi_2(t^*) + \varphi_3(t^*)$$

(44)
We have fixed the constant $C$ of eq. (39) by this choice of the lower bounds for the two integrals. The same choice of the phase at different $k$ is available when one construct a wave packet as in eq. (34). One develops the second term in powers of $\Delta m/M$ and gets

$$\varphi_2(t^*) = \frac{M^2}{E} \left[ \frac{\Delta m}{M} (1 - \frac{\Delta m}{2M}) (a\tau^*) + (\frac{\Delta m}{M})^2 O(1) \right]$$

where all the higher powers of $\Delta m/M$ are multiplied by $O(1)$. Thus $\varphi_2(t^*) = \Delta m \tau^*$. The third term is developed in powers of $\omega$ and reads

$$\varphi_3(t^*) = -\frac{\omega}{E} [p(M,k,t^*) - \frac{\omega}{2} \frac{k + Et^*}{p(M,k,t^*)} + \frac{\omega^2}{p(M,k,t^*)} O(1)]$$

$$= -\omega \Delta z(t^*) + \frac{\omega^2}{2E} \tanh(a\tau^*) \left( 1 + \frac{\Delta m}{M} O(1) \right)$$

where we have used eq. (31) to replace $p/E$ by $\Delta z$. We have also used eq. (43) to replace $\omega/p(M,k,t^*)$ by $(\Delta m/M) O(1)$. In addition, one verifies that all the higher powers of $\omega$ appear in the form $(\omega/p)^n$ only, hence, by the same replacement, are expressible as $(\Delta m/M)^n O(1)$ and are therefore negligible. The fact that all powers of $\omega$ appear in the ration $\omega/p$ is not an accident. It means that in the boosted frame at rest at $\tau = \tau^*$, the recoil effects are controlled by $\Delta m/M$. We believe that this property will be found whatever is the accelerating external field.

Only the second term cannot be neglected since the resonant $\omega (= \Delta m e^{a\tau^*})$ is bigger than $E^{1/2}$ after a proper time $\tau_{recoil}$ equal to

$$\tau_{recoil} = \frac{1}{2a} \ln \frac{M}{\Delta m} + \frac{1}{2a} \ln \frac{a}{\Delta m}$$

This is related to the fact that it is kinematically inherent for uniformly accelerated systems that the energy $\omega$ exchanged during transitions exceeds the rest mass of the system in a logarithmicly short proper time given by $a\tau = \ln M/\Delta m$. In the collapsing black hole situation, for exactly the same kinematical reasons (i.e. the formation of an horizon giving rise to an exponentially varying Doppler factor), one finds that the resonant frequencies $\omega$ which give rise to the Hawking quanta exceed the Planck mass\cite{7}\cite{8}\cite{9}\cite{18} after a logarithmicly short time (measured at spatial infinity).

The quadratic spread $\langle \Delta t \rangle^2$ around the saddle $t^*$, which is given by the inverse of the second derivative of the phase, is (for both $\tilde{B}$ and $\tilde{A}$)

$$\langle \Delta t \rangle^{-2} = E \left[ \frac{k - \omega + Et^*}{p(M,k - \omega,t^*)} - \frac{k + Et^*}{p(m,k,t^*)} \right] \left[ \frac{(M^2 - m^2)/2}{p(M,k - \omega,t^*)p(m,k,t^*)} \right]$$

$$= \langle \Delta \tau \rangle^{-2} (\frac{dt}{d\tau})^{-2} \left[ 1 + \frac{\Delta m}{M} O(1) \right]$$

(48)
where $\langle \Delta \tau \rangle$ is the spread of proper time in the no-recoil situation. Similarly the denominator of eq. (38) is

$$\frac{dt}{\sqrt{p(M, k - \omega, t^*)p(m, k, t^*)}} = d\tau \left[ 1 + \frac{\Delta m}{M} O(1) \right]$$  (49)

where powers of $\omega/p(M, k, t^*)$ are powers of $\Delta m/M$ hence negligible.

By collecting the various terms one obtains that $\tilde{B}(\Delta m, k, \omega)$ is given, at the saddle point approximation by

$$\tilde{B}_{s.p.}(\Delta m, k, \omega) = \left[ \frac{\tilde{g} M}{ga} \right] B_{s.p.}(\Delta m, \omega) e^{-i\left[2\omega k - \omega^2 \tanh a \tau^*/2\right]/2E}$$  (50)

where $B_{s.p.}(\Delta m, \omega)$ is given in eq. (16). By a similar analysis, one easily shows that

$$\tilde{A}_{s.p.}(\Delta m, k, \omega) = \left[ \frac{\tilde{g} M}{ga} \right] A_{s.p.}(\Delta m, \omega) e^{-i\left[2\omega k - \omega^2 \tanh a \tilde{\tau}^*/2\right]/2E}$$  (51)

where $\tilde{\tau} = \text{Re}[\tau^*]$. Since the modification for both amplitudes is a pure phase, eq. (11) is still satisfied. Hence at equilibrium one will find the thermal population eq. (13) as well. Strictly speaking we have shown the thermal equilibrium only when $\Delta m >> a$ which is a necessary condition to have a good saddle point approximation. But the fact that the differences of the integrands of the amplitudes, between the present case and the no-recoil case, scale like $\Delta m/M$ and not like $\Delta m/a$ indicates that when $\Delta m \simeq a$ the square of the amplitude $\tilde{B}$ will be proportional to the full Planck distribution as well\(^3\) (see discussion after eq. (16)).

## 5 The Flux Emitted by the Two Level Ion

We now compare the flux emitted by the two level ion, with the former no-recoil flux given in eqs. (17→22). To this end we have to use well localized wave packets. To be the closest to the no-recoil case, we use the ”minimal” wave packet\(^{[10],[24]}\) given in eq. (34) with $\sigma = E$ to describe the initial state of the ion of mass $m$. For simplicity we shall center its mean momentum at $k = 0$ and works with the phase specified in eq. (44). Then the position of the turning point of the mean trajectory encoded in the initial wave function is at $t = 0, z_0 = 0$, see eq. (31).

To order $\tilde{g}^2$, the flux is, in total analogy with eqs. (20), (21) and (22), given by two terms

$$\langle \tilde{T}_{UU} \rangle = \langle \tilde{T}_{UU} \rangle_1 + \langle \tilde{T}_{UU} \rangle_2$$  (52)

\(^3\)This has been verified explicitly by S. Massar who used integral representations (see ref. [24]) of the exact solutions of eq. (32). He found that the norm of $\tilde{B}$ is, as in the no-recoil case, proportional to the Planck distribution. Hence, the saddle point restriction $\Delta m >> a$ can be waived. Furthermore, since he worked with the exact solutions of eq. (32) rather than W.K.B. approximations, it proves that, at least for the ion in a constant electric field, the thermalization of the inner degrees of freedom does not even require the semi-classical limit: $M^2 >> E$, see ref. [10].
With the initial state of the ion \( m \) given as specified just above, \( \langle \tilde{T}_{UU} \rangle_1 \) is

\[
\langle \tilde{T}_{UU} \rangle_1 = \int dk_1 \int dk_2 \frac{e^{-(k_1)^2/2E} e^{-(k_2)^2/2E}}{(\pi E)^{1/4}} \int_0^\infty d\omega \int_0^\infty d\omega' \delta(k_1 - \omega - (k_2 - \omega'))
\]

\[
2 \tilde{B}(\Delta m, k_1, \omega) \tilde{B}^*(\Delta m, k_2, \omega') \frac{\sqrt{\omega \omega'}}{4\pi} e^{i(\omega' - \omega)U}
\]

(53)

where the \( \delta \) of Dirac comes from momentum conservation of the exchanged ion of mass \( M \). Similarly \( \langle \tilde{T}_{UU} \rangle_2 \) is

\[
\langle \tilde{T}_{UU} \rangle_2 = \int dk_1 \int dk_2 \frac{e^{-(k_1)^2/2E} e^{-(k_2)^2/2E}}{(\pi E)^{1/4}} \int_0^\infty d\omega \int_0^\infty d\omega' \delta(k_1 - \omega - \omega' - (k_2))
\]

\[
(-2) \text{Re} \tilde{C}(k_1, \omega, \omega', \Delta m) \frac{\sqrt{\omega \omega'}}{4\pi} e^{-i(\omega' + \omega)U}
\]

(54)

where \( \tilde{C}(k_1, \omega, \omega', \Delta m) \) is the amplitude to emit two photons of frequencies \( \omega \) and \( \omega' \) starting from the ion in the state \( |k_1 > m \). Notice that the argument of the \( \delta \) of Dirac is not the same as in eq. (53). This is due to the fact that it comes now from the overlap between the twice scattered ion \( m \) of momentum \( k_1 - \omega - \omega' \) with the unperturbed one of momentum \( k_2 \). This difference of arguments will have a determinative importance in the sequel.

Performing the \( k_2 \) integration one obtains

\[
\langle \tilde{T}_{UU} \rangle_1 = 2 \int_0^\infty d\omega \int_0^\infty d\omega' \int dk_1 \frac{e^{-(k_1)^2/2E} e^{-(k_1-\omega+\omega')^2/2E}}{(\pi E)^{1/4}}
\]

\[
\tilde{B}(\Delta m, k_1, \omega) \tilde{B}^*(\Delta m, k_1 - \omega + \omega', \omega') \frac{\sqrt{\omega \omega'}}{4\pi} e^{i(\omega' - \omega)U}
\]

(55)

and

\[
\langle \tilde{T}_{UU} \rangle_2 = -2 \int_0^\infty d\omega \int_0^\infty d\omega' \int dk_1 \frac{e^{-(k_1)^2/2E} e^{-(k_1-\omega-\omega')^2/2E}}{(\pi E)^{1/4}}
\]

\[
\text{Re} \tilde{B}(\Delta m, k_1, \omega) \tilde{A}(\Delta m, k_1 - \omega + \omega', \omega') \frac{\sqrt{\omega \omega'}}{4\pi} e^{-i(\omega' + \omega)U}
\]

(56)

(where we have replaced \( \tilde{C}(k_1, \omega, \omega', \Delta m) \) by \( \tilde{B}(\Delta m, k_1, \omega) \tilde{A}(\Delta m, k_1 - \omega, \omega') \) c.f. the remark after eq. (26)). Again one sees the difference of the recoil effects in the different arguments of the second gaussian factor in eqs. (55) and (56).

Analyzing the content of these fluxes, we note, first of all, that, as in the no recoil case, \( \langle \tilde{T}_{UU} \rangle_2 \) does not contribute to the total energy emitted (see eq. (26)). Furthermore since the total energy is a function of the norm of \( \tilde{B}(\Delta m, k_1, \omega) \) only (this is because only \( \omega = \omega' \) contributes in eq. (23)) and since the amplitudes \( \tilde{B}(\Delta m, k_1, \omega) \) differ from \( \tilde{B}(\Delta m, \omega) \) by a phase only, see eq. (60), the energy emitted is identical to the one in the no-recoil case. The effect of the recoil is therefore to modify the repartition of the energy density at most.
To verify that the recoil does modify the repartition of the energy density, we shall use the saddle point approximation expressions, eqs. (50, 51), for $\tilde{B}(\Delta m, k_1, \omega)$ and for $\tilde{A}(\Delta m, k_1 - \omega, \omega')$. This leads to a tedious but straightforward computation.

We first simplify the $k$ dependence of the additional phase factor $i(\omega^2/2E)\tanh a\tau^*$ of eqs. (50, 51) where $\tau^*$ is a function $k$ through eq. (31). We drop this $k$ dependence and replace this phase by $i\omega^2/2E$. This is legitimate since for large $\tau$ the $\tanh a\tau^*$ is exponentially close to 1, and for small $\tau^*$, the resonant frequency $\omega$ is of the order of $\Delta m$, hence this phase is negligible anyway.

With this simplification, we can perform the $k_1$ integration since it is now gaussian, and we obtain

$$\langle \tilde{T}_{UU} \rangle_1 = 2 \int_0^\infty d\omega \int_0^\infty d\omega' e^{-(\omega-\omega')^2/2E} \left[ B_{s.p}(\Delta m, \omega)B_{s.p}^*(\Delta m, \omega') \frac{\sqrt{\omega\omega'}}{4\pi} e^{i(\omega'-\omega)U} e^{-(\omega'+\omega)\varepsilon} \right]$$

where the quantities which appear in the brackets are computed in the no-recoil case (see eqs. (21, 22)). The regulator $\varepsilon$ is given in eq. (24) and we have set $[\tilde{g}M/ga] = 1$ for simplicity. One sees that the effect of the combined gaussian weights of the wave packet describing the initial state of the ion and the $k$ dependence of the phase factors of the amplitudes $\tilde{B}$ and $\tilde{A}$ is to erase the $\omega^2$ phase factors of eqs. (50, 51) and to introduce a gaussian factor into the integrand of the no-recoil case. (One trivially verifies that in the limit $M \to \infty$, $E \to \infty$ with $E/M = a$ fixed, eqs. (57, 58) give back identically eqs. (25, 26) with $B$ and $A$ evaluated at the saddle point approximation.) These gaussian factors are the sole effect of the recoil on the energy repartition. They have the following consequences.

To understand the $U$ dependence of $\langle \tilde{T}_{UU} \rangle_1$ and $\langle \tilde{T}_{UU} \rangle_2$, let us first make clear the hierarchy of characteristic proper times and, by virtue of the resonance condition eq. (43), the hierarchy of the energies $\omega$ exchanged. One has

$$\tau_0 < \tau_{recoil} + \tau_0 < \tau_\varepsilon + \tau_0$$

$\tau_0$ is the time at which one prepares the initial wave packet, it marks therefore the beginning of the interaction and effectively defines the "laboratory Minkowski frame" in which the frequencies $\omega$ have an absolute meaning. This is possible because the construction of the wave packets breaks the Rindler invariance present in the no-recoil case. Indeed one can explicitly check that the wave packets given in eq. (34) are not invariant under boost. Using the wave packet given in eq. (34) with $k = 0$ and with the phase given in eq. (44) means that the construction is made at (around) $t = 0$, hence in this case $\tau_0 = 0$. 

15
\( \tau_{\text{recoil}} \) designates the lapse of proper time which starts at \( \tau_0 \) and indicates the moment at which the resonant frequency \( \omega \) (measured in the Minkowski frame where \( \tau_0 = 0 \)) is equal to \( (Ma)^{1/2} \) (see eq. (54)), i.e. when the energy exchanged equals the spread in \( k \) of the wave packets of eq. (54) with \( \sigma = E \). \( \tau_{\text{recoil}} \) characterizes, as we shall see, the time from which recoil effects dominate the physics. Furthermore, when \( \Delta m \approx a \), it is half the time at which \( \omega \) equals the rest mass of the ion. Had we used therefore another external field to put the system into acceleration, we would have found a similar proper time associated with the width of the wave packet.

\( \tau_{s} \) designates the end of the interaction. We recall that it is necessary to limit the frequency \( \omega \) in the U.V.

One has two different regimes. For \( -U = e^{-\sigma}a > > E^{-1/2} \) (i.e. \( \tau \) smaller than \( \tau_{\text{recoil}} \)) the resonant energy \( \omega \) is much smaller than \( (aM)^{1/2} \), hence the gaussian weights in eqs. (54, 55) play no role and the energy repartition \( \langle \tilde{T}_{UU} \rangle_1 + \langle \tilde{T}_{UU} \rangle_2 \) is still the same as in the no-recoil case hence given by eq. (27). This can be immediately verified by computing the \( \omega, \omega' \) integrals at the saddle point approximation. One finds indeed that the saddle frequency \( \omega^* \) is at \( \omega^* = -\Delta m/aU \).

Instead, for \( -U < E^{-1/2} \) (i.e. \( \tau > \tau_{\text{recoil}} \)), the effects of the gaussian factors on \( \langle \tilde{T}_{UU} \rangle_1 \) and \( \langle \tilde{T}_{UU} \rangle_2 \) are totally differently. \( \langle \tilde{T}_{UU} \rangle_2 \) does no longer scale like \( 1/U^2 \) (which expresses a constant rate in Rindler time). Indeed, using eq. (10), eq. (58) reads

\[
\langle \tilde{T}_{UU} \rangle_2 = -2 \int_0^\infty d\omega \int_0^\infty d\omega' e^{-\omega^2/2E} e^{-i\Delta m/a} \left( g \left( \frac{a}{\Delta m} \right) e^{-\pi \Delta m/a} \right) \text{Re} \left[ \left( \frac{\omega}{\omega'} \right)^{i\Delta m/a} \right] e^{-i(\omega' + \omega)U/V - (\omega' + \omega)\epsilon} \]

\[
= +2g^2 \left( \frac{\Delta m}{a} \right) e^{-2\pi \Delta m/a} \int_0^\infty d\omega_1 \omega_1 \cos(\omega_1 U) e^{-\omega_1^2/2E} e^{-\omega_1 \epsilon} \]  

(60)

where we have defined \( \omega_1 = \omega + \omega' \) and integrated over the angle \( \arctan(\omega/\omega') \). The behavior of \( \langle \tilde{T}_{UU} \rangle_2 \) at small \( U \) is now governed by \( E \) and no longer by \( \epsilon \). Indeed the development of the integral at small \( U \) gives \( E(1-EU^2 + O(EU^2)^2) \) instead of \(-\text{Re}[1/(U+i\epsilon)^2]\) obtained in the no recoil case in eq. (26).

On the contrary, the behavior of \( \langle \tilde{T}_{UU} \rangle_1 \) has to be still like \( 1/U^2 \) since the total energy emitted is unaffected by the recoil and scales like \( 1/\epsilon \). This is confirmed by the saddle point approximation of the \( \omega, \omega' \) integrations.

Thus for \( -U < E^{-1/2} \), the negative contribution of \( \langle \tilde{T}_{UU} \rangle_2 \) becomes completely negligible as compared to the unaffected \( \langle \tilde{T}_{UU} \rangle_1 \) contribution. Therefore one finds that the flux emitted by the accelerated system is now positive and given by the contribution of the first term in the Born series (linear in \( g \), i.e. the \( B \) term in eq. (15)).

Similarly, in the equilibrium situation, for \( \tau > \tau_{\text{recoil}} \), the flux is given by the sum of the positive contributions of the excitation and disexcitation processes. One has then that all transitions characterizing thermal equilibrium lead to a positive flux in situ contrariwise of what happened in the no recoil case owing to the interferences among all emitted quanta. Therefore, the decoherence greatly simplifies the resulting flux which is now much closer to what one might have naively expected (i.e. one should no longer be bothered by the formerly interfering \( \langle T_{UU} \rangle_2 \) terms).
Furthermore, there is a strict relation between this decoherence and the conservation of momentum, eq. (37), and energy, eq. (41). Indeed, the two level ion constantly loses energy and momentum in accordance with these conservation laws. To understand how momentum conservation modifies the mean trajectory of the ion, compare the orbit where there is no recoil \( g \to 0 \) with the orbit of the emitting two level ion. There is, in the second case, a succession of hyperbolae such that their turning points (see eqs. (31)) drift in \( t \) and \( z \) corresponding to later times and greater \( z \). The total change in the time of the turning point is \( E \Delta t_{t.p.} = \sum_i \omega_i \) (i.e. the total momentum lost is the sum of the successive losses due to each transition). One also verifies that the total change in position is \( \sum_i \omega_i/E \). These successive changes of hyperbolae lead to the decoherence of the emissions causing these changes. This can be seen directly from the argument of the \( \delta \) function of eq. (54) and the spread in \( \omega \) \( (= E^{1/2} \) when \( \omega + \omega' \gg E^{1/2} \) the overlap of the two wave packets vanishes. This is not the case for \( \langle T_{UU} \rangle_1 \) since the argument of the \( \delta \) in eq. (53) contains \( \omega - \omega' \).

We conclude this paper by three remarks.
1. We mention the work\[25\] of Chung and Verlinde who attempt to take into account the recoils of a non inertial mirror which follows, in the mean, the trajectory \( aV = -e^{-aU} \), (compare with eq. (15)), which leads to a thermal flux as in the case of a collapsing black hole\[3\]. The difficulty of their approach is compounded by the fact that the quantum source of the recoils is not the individual quantum emission acts (as in this paper, see eq. (37)) but rather the quantized version of the mean energy momentum (i.e. the mean flux expressed in terms of the quantized trajectory).

2. We point out that, in the semiclassical treatment\[4\][5] the black hole loses mass through the absorption of a negative mean flux which crosses its future horizon. Furthermore, this mean negative flux can be viewed as arising from the interferences of states with different local particle number\[26\]. We now recall that in the accelerating situation, upon considering the quantum recoils of the system, the decoherence of successive emissions washes out the interferences leading to a negative mean flux after a logarithmicly short time. Since the "transplanckian" frequencies\[7\][8] encountered in black hole evaporation have their origin in the exponentially varying Doppler shift, in a manner similar\[19\] to that of the accelerated detector, one may call into question the validity of the mean negative flux crossing the future horizon when quantum gravitational recoils will be taken into account.

4 P. Grove and D. Raine emphasize that one should consider a "Rindler rigid" accelerated external field (i.e. an external classical field such that the resulting W.K.B. trajectories will be at constant Rindler position and thus invariant under boosts) instead of the homogeneous field which gives all trajectories with the same acceleration. In such a case, they claim that the recoils of the accelerated system would not destroy the coherence of the emissions since one would still have a Rindler Killing vector guaranteeing Rindler energy conservation. My objections to such a construction are the following:
1. The total Minkowski energy emitted is independent of the recoil.
2. If one includes the accelerating external field into the dynamics its own recoil would not be negligible after a proper time given by eq. (47) with \( M \) replaced by the mass of the total system. For these reasons I do not consider that the "Rindler rigid" construction conceptually differs from the one presented here.
3. We remark that the recoil of the accelerated system which follows from strict momentum conservation bears some similarities with the recoil of the gravitational part of the universe induced by some matter change and taken into account through the quantum character of the wave-function of gravity + matter (solution of the Wheeler de Witt equation in a mini superspace reduction). In particular, the emergence of the ”Banks” time [27] [28] [10] [11] occurs for exactly the same reasons that have lead to the proper time parametrization of the $\varphi_2$ term in eq. (45), i.e. $\varphi_2(t^*) = \Delta m \tau^*$.

Acknowledgments I thank S. Massar for numerous clarifying discussions and for the precious remark that the norm of the amplitude can be exactly computed by making no appeal to the W.K.B nor to the saddle point approximation. I also wish to thank D. Sciama for convincing encouragements to complete this work and P. Grove and D. Raine for many useful discussions during two weeks spent at SISSA in September 1994. I also thank O. Fonarev for help to resolve some algebraic difficulties. I'm grateful to SISSA for their hospitality during which I enjoyed discussions with D. Sciama, P. Grove and D. Raine.

References

[1] W. G. Unruh, Phys. Rev. D 14 (1976) 870
[2] S. A. Fulling, Phys. Rev. D 7 (1973) 2850
[3] S. W. Hawking, Nature 248 (1974) 30
   Commun. Math. Phys. 43 (1975) 199
[4] J. M. Bardeen, Phys. Rev. Lett. 46 (1981) 382
[5] S. Massar, *The semi classical back reaction to black hole evaporation*
   preprint ULB-TH 94/19, gr-qc/9411039
[6] R. Parentani and T. Piran, Phys. Rev. Lett. 73 (1994) 2805
[7] G. ’t Hooft, Nucl. Phys. B 256 (1985) 727
[8] T. Jacobson, Phys. Rev. D 44 (1991) 1731, Phys. Rev. D 48 (1993) 728
[9] F. Englert, S. Massar and R. Parentani, Class. Quantum Grav. 11 (1994) 2919
[10] R. Brout, R. Parentani and Ph. Spindel, Nucl. Phys. B353 (1991) 209
[11] R. Parentani, PhD. Thesis (unpublished) (1992)

[12] W. G. Unruh and R. M. Wald, Phys. Rev. D 29 (1984) 1047

[13] P. Grove, Class. Quant. Grav. 3 (1986) 801

[14] D. Raine, D. Sciama and P. Grove, Proc. R. Soc. A 435 (1991) 205

[15] S. Massar, R. Parentani and R. Brout, Class. Quant. Grav. 10 (1993) 385

[16] W. G. Unruh, Phys. Rev. D 46 (1992) 3271

[17] J. Audretsch and R. Müller, Phys. Rev. D 49 (1994) 4056; Phys. Rev. D 49 (1994) 6566; Phys. Rev A 50 (1994) 1755

[18] S. Massar and R. Parentani From Vacuum Fluctuations to Radiation: Accelerated Detectors and Black Holes. (2), preprint ULB-TH 94/02, LPTENS 95/07 (1995) gr-qc/9502024

[19] R. Parentani and R. Brout, Int. Journ. of Mod. Phys. D 1 (1992) 169

[20] R. Brout, S. Massar, R. Parentani and Ph. Spindel, A Primer for Black Hole Quantum Physics ULB-TH 95/02, ULM-MG 95/01, LPTENS 95/03 (1995) Phys. Rep. in the press.

[21] R. Parentani, Class. Quantum Grav. 10 (1993) 1409

[22] J. S. Bell and J. M. Leinaas, Nucl. Phys. B 212 (1983) 131

[23] J. Schwinger, Phys. Rev. 82 (1951) 664

[24] R. Brout, S. Massar, R. Parentani, S. Popescu and Ph. Spindel, Quantum Source of the Back Reaction on a Classical Field, preprint ULB-TH 93/16 UMH-MG 93/03 (1993) hep-th/9311019

[25] T. D. Chung and H. Verlinde, Nucl. Phys. B 418 (1994) 305

[26] S. Massar, R. Parentani and R. Brout, Class. Quant. Grav. 10 (1993) 2431

[27] T. Banks, Nucl. Phys. B 249 (1985) 332

[28] R. Brout and G. Venturi, Phys. Rev. D 39 (1989) 2436