\[ \pi^+ - \pi^0 \text{ mass difference} \]
from the Bethe-Salpeter equation

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Abstract

In the framework of the Schwinger-Dyson equation and the Bethe-Salpeter equation in the improved ladder approximation, we calculate the \( \pi^+ - \pi^0 \) mass difference on the same footing as the pion decay constant and the QCD \( S \) parameter (or \( L_{10} \)) through the difference between the vector current correlator \( \Pi_{VV} \) and the axial-vector current correlator \( \Pi_{AA} \). We find that all the results can be fit to the experimental values for rather large \( \Lambda_{QCD} \sim 700 \text{MeV} \) which reflects the “scale ambiguity”. By fitting to the calculated data using the pole saturated form of \( \Pi_{VV} - \Pi_{AA} \), we also derive masses and decay constants of \( \rho \) meson and \( a_1 \) meson, which we found are consistent with the experiments rather insensitively to the “scale ambiguity”.

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1 Introduction

The $\pi^+ - \pi^0$ mass difference $\Delta m^2_\pi \equiv m_{\pi^+}^2 - m_{\pi^0}^2$ is an interesting quantity to measure an explicit breaking of the chiral symmetry by the gauge coupling ($U(1)_{em}$) in the spontaneously broken phase of the chiral symmetry. It in fact has been a prototype of the mass calculation of pseudo Nambu-Goldstone (NG) bosons in strong coupling gauge theories such as those in the technicolor theories [1] and more recently in the little Higgs models [2]. The sign as well as the absolute value of $\Delta m^2_\pi$ is an important issue (“vacuum alignment problem”), since the negative sign would imply that the vacuum would align so as to break spontaneously the gauge symmetry ($U(1)_{em}$ in the case of pion) which explicitly breaks the spontaneously broken symmetry. Such a situation never happens in the real-life QCD but may do in other theories. Hence this type of calculation plays a central role of the model buildings.

The first successful calculation of $\Delta m^2_\pi$ [3] was done by the current algebra in conjunction with the Weinberg spectral function sum rules [4] saturated by the $\pi$, $\rho$ and $a_1$ meson poles. Somewhat more elaborate calculation on this line was done combined with the QCD information [5]. Recently, an effective field theory calculation without $a_1$ meson has been successfully done [6], based on the Hidden Local Symmetry (HLS) model [7, 8, 9] at loop level [10, 11] and the Wilsonian matching [12] of the HLS model with the QCD (for reviews of the HLS approach, see [13, 14]). Ref. [6] demonstrated that the HLS model is a little Higgs model with two sites and two links (open moose) whose locality of theory space forbids quadratic divergence in $\Delta m^2_\pi$ even without $a_1$ meson.

On the other hand, direct QCD calculation of the $\Delta m^2_\pi$ has never been done except for the lattice simulation [15]. Although the lattice QCD is a powerful method, it is still important to establish alternative QCD-based methods which are more intuitive and less dependent on the power of computers. Such methods will be crucial to the subjects which are difficult to be studied in the framework of the lattice simulation. Most relevant subjects are the dynamical origin of the electroweak symmetry breaking, QCD at finite density, etc.

In this paper, we calculate in the chiral symmetric limit the $\Delta m^2_\pi$ simultaneously with the the pion decay constant $f_\pi$ and the QCD $S$ parameter [16] (or the Gasser-Leutwyler parameter $L_{10}$ [17]) within the framework of the the Schwinger-Dyson (SD) equation and the Bethe-Salpeter (BS) equation in the the improved ladder approximation. We note that all these quantities are given by “generalized” Weinberg spectral function sum rules for the difference between the vector current correlator $\Pi_{VV}$ and the axial-vector current correlator $\Pi_{AA}$: $f^2_\pi$ is given by the Weinberg first sum rule, the $S$ parameter by the Das-Mathur-Okubo (DMO) sum rule [18] or the “zeroth Weinberg sum rule”, and $\Delta m^2_\pi$ by the Das-Guralnik-Mathur-Low-Young (DGMLY) sum rule [3] or the “third Weinberg sum rule”. We then rewrite them in terms of the current correlators instead of the spectral functions and directly calculate $\Delta m^2_\pi$ on the same footing as $f_\pi$ and the QCD $S$ parameter through $\Pi_{VV} - \Pi_{AA}$ in the spacelike momentum region. Such correlators are written in terms of the BS amplitudes which are given by solving the inhomogeneous BS (IBS) equation and the SD equation.
with the improved ladder approximation in the Landau gauge both in the space-like region. In contrast to our case, solving the bound state spectra by the homogenous BS equation in the time-like region is usually difficult, since it would need analytic continuation of the running coupling which is not an analytic function.  

This kind of method has been extensively used to investigate masses and decay constants of low lying mesons \cite{20,21,22} in good agreement with the experiments. On the other hand, the same method failed to reproduce the QCD $S$ parameter consistently with the experiment \cite{23}.

However, we find that the QCD $S$ parameter is rather sensitive to the infrared (IR) cutoff parameter of the QCD running coupling. We actually reproduce the QCD $S$ parameter consistently with the experiment by setting the IR cutoff parameter in the region never investigated in the previous works. Our parameter choice corresponds to taking the QCD scale parameter as $\Lambda_{\text{QCD}} \approx 724$ MeV, while it was chosen as $\Lambda_{\text{QCD}} \approx 500$ MeV in the previous results of the decay constants and the masses \cite{20,21} and of the QCD $S$ parameter \cite{23}, both values being considerably higher than the conventional $\Lambda_{\text{QCD}}^{(3)} = 300$-$450$ MeV obtained in the $\overline{\text{MS}}$ scheme \cite{23}. This actually corresponds to exploiting the “scale ambiguity” \cite{25} of $\Lambda_{\text{QCD}}$ in the SD and BS equations recently emphasized by Hashimoto and Tanabashi \cite{26}.

We then show that $\pi^+ - \pi^0$ mass difference $\Delta m_\pi^2$ is calculated in rough agreement with the experimental value for the same parameter region as that reproducing the QCD $S$ parameter and $f_\pi$. Actually, although infrared dynamics is quite important for determination of $f_\pi$ and $S$, it is not so important for determination of $\Delta m_\pi^2$. In other words, $f_\pi$ and $S$ are sensitive to the infrared structure of the running coupling, while $\Delta m_\pi^2$ is not.

We also derive the masses and decay constants of $\rho$ meson and $a_1$ meson by fitting them to the calculated $\Pi_{VV} - \Pi_{AA}$ within our method using the pole saturated form of $\Pi_{VV} - \Pi_{AA}$. These quantities are also compared with experiments, which is another check of the validity of the calculation in the present analysis.

The calculation presented in this paper turns out to be the first example which derives $\Delta m_\pi^2$ directly from QCD without depending on the lattice simulation. Once we have checked the reliability of such a method in the ordinary QCD (with three light flavors), we may apply it to the mass of pseudo NG bosons and vacuum alignment problem in other strong coupling gauge theories such as the large $N_f$ QCD, walking technicolor, little Higgs, hot/dense QCD, etc..

This paper is organized as follows. In section 2 we introduce spectral function sum rules: The DMO sum rule or the zeroth Weinberg sum rule for $S$ parameter, the first Weinberg sum rule for $f_\pi^2$, the second Weinberg sum rule, and the DGMLY sum rule or the third Weinberg sum rule for $\Delta m_\pi^2$. We then rewrite them in terms of the current correlators $\Pi_{VV} - \Pi_{AA}$. In section 3 we show how the current correlators are obtained from the BS amplitude which is calculated from the IBS equation in the space-like region. In section 4 we introduce the IBS equation and show how to solve it numerically. Section 5 is the main part of this paper: We first show the result of the

\footnote{See an exception in Ref. 19.}
IBS equation for QCD, and discuss the dependence of $f_\pi$, $S$ and $\Delta m_\pi^2$ on the infrared structure of the running coupling. Then we compare them with the experimental values. We also show results for masses and decay constants of $\rho$ and $a_1$ mesons. In section 6 we give summary and discussions on applications to the large $N_f$ QCD and the electroweak symmetry breaking.

2 Spectral function sum rules

In this section we introduce the spectral function sum rules, which express the pion decay constant $f_\pi$, the QCD $S$ parameter and $\Delta m_\pi^2$ in terms of $\Pi_{VV} - \Pi_{AA}$ in the space-like momentum region. Here we consider only the chiral symmetric limit of massless three flavors ($N_f = 3$), the corrections of the finite quark mass being expected to be small for the quantities we consider.

Let us begin with introducing the vector and axial-vector currents as

\begin{equation}
V_\mu^a(x) = \bar{\psi}(x)T^a\gamma_\mu\psi(x),
\end{equation}

\begin{equation}
A_\mu^a(x) = \bar{\psi}(x)T^a\gamma_\mu\gamma_5\psi(x),
\end{equation}

where $T^a$ is the generator of $SU(N_f)$ normalized as $\text{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ and we consider $a, b = 1, 2, 3$. In the chiral symmetric limit the current correlator $\Pi_{JJ}$ is given as

\begin{equation}
\delta^{ab}\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)\Pi_{JJ}(q^2) = i\int d^4x \ e^{iqx} \langle 0|TJ_\mu^a(x)J_\nu^b(0)|0\rangle,
\end{equation}

\begin{equation}
(J_\mu^a(x) = V_\mu^a(x), A_\mu^a(x)).
\end{equation}

The Umezawa-Kamefuchi-Källen-Lehmann spectral representation for the current correlators are expressed as

\begin{align}
&i\int d^4x \ e^{iqx} \langle 0|TJ_\mu^a(x)J_\nu^b(0)|0\rangle \\
&= -\delta^{ab}\int_0^\infty ds \frac{1}{s - q^2 - i\epsilon} \left\{ \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{s} \right) \rho_j^{(1)}(s) - q_\mu q_\nu \rho_j^{(0)}(s) \right\}. \tag{2.4}
\end{align}

Here, we decomposed the spectral function into the spin-one part $\rho_j^{(1)}(s)$ and spin-zero part $\rho_j^{(0)}(s)$. Since the axial-vector current $A_\mu^a$ couples to the massless NG boson, the pion, while no massless particles couple to the vector current $V_\mu^a$, we have

\begin{align}
\rho_{V}^{(0)}(s) &= 0, \\
\rho_{A}^{(0)}(s) &= f_\pi^2\delta(s),
\end{align}

where $f_\pi$ is the decay constant of the NG boson $\pi$ defined by

\begin{equation}
\langle 0|A_\mu^a(0)|\pi^b(q)\rangle = iq_\mu f_\pi\delta^{ab}.
\end{equation}
Using the spectral functions, we can write down the following “generalized” Weinberg sum rules:

\[\int_0^\infty \frac{ds}{s^2} \left[ \rho_V^{(1)}(s) - \rho_A^{(1)}(s) \right] = \frac{S}{4\pi}, \quad (2.8)\]

\[\int_0^\infty \frac{ds}{s} \left[ \rho_V^{(1)}(s) - \rho_A^{(1)}(s) \right] = f_\pi^2, \quad (2.9)\]

\[\int_0^\infty ds \left[ \rho_V^{(1)}(s) - \rho_A^{(1)}(s) \right] = 0, \quad (2.10)\]

\[-\frac{3\alpha_{em}}{4\pi f_\pi^2} \int_0^\infty ds \log s \left[ \rho_V^{(1)}(s) - \rho_A^{(1)}(s) \right] = \Delta m_\pi^2, \quad (2.11)\]

where, \(S\) is the QCD \(S\) parameter [16], which in the case of \(N_f = 3\) is related to the Gasser-Leutwyler parameter \(L_{10}\) [17] as \(S = -16\pi L_{10}\). \(\alpha_{em}(=\frac{e^2}{4\pi} = \frac{1}{137})\) is the coupling constant of the electromagnetic interaction, and \(\Delta m_\pi^2\) is the \(\pi^+ - \pi^0\) mass difference defined by \(\Delta m_\pi^2 \equiv m_\pi^2 - m_{\pi^0}^2\). Equations (2.9) and (2.10) are the first and second Weinberg sum rules [4], respectively, and Eqs. (2.8) is the DMO sum rule [18] or often called the “zeroth Weinberg sum rule”. Equation (2.11) was derived by Das et al. [3] and may be called the DGMLY sum rule or the “third Weinberg sum rule”.

Of course these relations in Eqs. (2.8), (2.9), (2.10), and (2.11) make sense only if the integrals converge. For example, convergence of the second sum rule (2.10) requires that the difference between \(\Pi_{VV}\) and \(\Pi_{AA}\) must satisfy

\[Q^2 \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right] \xrightarrow{(Q^2 \to \infty)} 0, \quad (2.12)\]

where \(Q^2\) is related to the space-like momentum as \(Q^2 = -q^2(>0)\).

The asymptotic behavior of the current correlators can be calculated by the operator product expansion (OPE) technique. In the chiral limit, the form of \(\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)\) in the ultraviolet region is estimated as follows [27, 28]:

\[\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \xrightarrow{(Q^2 \to \infty)} \frac{4\pi(N_c^2 - 1)}{N_c^2} \frac{\alpha_s<\bar{q}q>^2}{Q^4}, \quad (2.13)\]

up to logarithm. This means that \(\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)\) indeed satisfies the condition (2.12) and the relation in Eq. (2.10) makes sense. When the integral in the second sum rule converges, those in the first and zeroth sum rules are also convergent.

Now we rewrite the zeroth and the first sum rules in terms of the current correlators as

\[S = -4\pi \frac{d}{dQ^2} \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right] \bigg|_{Q^2=0}, \quad (2.14)\]

\[f_\pi^2 = \Pi_{VV}(0) - \Pi_{AA}(0), \quad (2.15)\]

and also the third sum rule: [29]

\[\Delta m_\pi^2 = \frac{3\alpha_{em}}{4\pi f_\pi^2} \int_0^\infty dQ^2 \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right]. \quad (2.16)\]
From Eq. (2.16), we can see that the asymptotic behavior (2.13) also guarantees convergence of the third sum rule.

In Ref. [23] the pion decay constant $f_\pi$ and the QCD $S$ parameter were calculated by the use of the above formulas. However, $\Delta m_\pi^2$ was not calculated so far. So the calculation presented in this paper turns out to be the first example which derives $\Delta m_\pi^2$ directly from QCD without depending on the lattice simulation. Moreover, there is another interest regarding $\Delta m_\pi^2$ related to the structure of the QCD vacuum, as we stressed in the Introduction. When we switch off the electromagnetic interaction and set masses of quarks to be zero, pions are identified with the exact NG bosons associated with the spontaneous breaking of $SU(2)_L \otimes SU(2)_R$ chiral symmetry down to $SU(2)_V$ symmetry. The existence of the electromagnetic interaction explicitly breaks the $SU(2)_L \otimes SU(2)_R$ chiral symmetry, which makes $\pi^+$ and $\pi^-$ be the pseudo NG bosons. On the other hand, $\pi^0$ remains massless since the photon does not interact with $\pi^0$. The interesting point here is whether $\Delta m_\pi^2 \equiv m_{\pi^+}^2 - m_{\pi^0}^2$ (in the chiral limit) becomes positive or negative, which is called “vacuum alignment problem”. Negative $\Delta m_\pi^2$ means that fluctuation of $\pi^+$ field around $\langle \pi^+ \rangle = 0$ is unstable and the vacuum with $\langle \pi^+ \rangle = 0$ is not a true vacuum. If this is the case, $\pi^+$ has non-zero vacuum expectation value and $U(1)_{em}$ symmetry is broken. We know that, in the real world, $\Delta m_\pi^2$ is positive and the vacuum with $\langle \pi^+ \rangle = 0$ is the true vacuum. $U(1)_{em}$ symmetry is not broken in the real world. This is quite nontrivial fact resulting from the nonperturbative dynamics of the strong interaction. So it is interesting to investigate whether we can reproduce positive $\Delta m_\pi^2$ which is realized in the real-life QCD.

3 Current correlators from BS amplitudes

In the previous section we have written down the QCD $S$ parameter, $f_\pi$, and $\Delta m_\pi^2$ in terms of the current correlators. Then, in this section, we show how the current correlators are obtained from the BS amplitude which will be calculated from the IBS equation.

To derive properties of hadrons as boundstates, it is straightforward to perform calculations directly in the time-like momentum region. However, it is difficult to solve the BS equation and the SD equation in the time-like region since we have to carry out the analytic continuation of the running coupling from the space-like region to the time-like region. In the case of QCD, the one-loop running coupling is not an analytic function, so that we have to approximate it by some analytic function in order to perform the calculations in the time-like region. However, here we need the BS amplitude only for the space-like region in order to calculate the current correlators for $\Delta m_\pi^2$, $f_\pi$ and the QCD $S$ parameter.

The BS amplitude $\chi^{(J)} (J = V, A)$ is defined in terms of the three-point vertex
function as follows:

\[ \delta^i_j \left( \frac{\chi^a_i}{2} \right)^f_g \int \frac{d^4p}{(2\pi)^4} e^{-ipr} \chi^{(J)}_{\alpha\beta}(p;q,\epsilon) = \epsilon^\mu \int d^4xe^{iqx}\langle 0 | T \psi_{\alpha f}(r/2) \tilde{\psi}^{f'}_{\beta'}(-r/2) J^a_\mu(x) | 0 \rangle, \]  

(3.1)

where \( q^\mu \) is the total momentum of the fermion and anti-fermion and \( p^\mu \) is the relative one. \( \epsilon^\mu \) is the polarization vector defined by \( \epsilon \cdot q = 0, \epsilon \cdot \epsilon = -1 \), and \( f, f', (i, j), (\alpha, \beta) \) are flavor, color and spinor indices, respectively. Closing the fermion legs of the above three-point vertex function and taking the limit \( r \to 0 \), we can express the current correlator in terms of the BS amplitude as follows:

\[ \Pi_{JJ}(q^2) = \frac{1}{3} \sum_{\epsilon} \int \frac{d^4p}{i(2\pi)^4} N_c \left[ (\epsilon \cdot G^{(J)}) \chi^{(J)}(p;q,\epsilon) \right] \]  

(3.2)

where

\[ G^{(V)}_\mu = \gamma_\mu, \quad G^{(A)}_\mu = \gamma_\mu \gamma_5, \]  

(3.3)

and \( N_c = 3 \) is the number of colors. In the above expression we averaged over the polarizations so that \( \Pi_{JJ}(q^2) \) does not depend on the polarization.

We expand the BS amplitude \( \chi^{(J)}_{\alpha\beta}(p;q,\epsilon) \) in terms of the bispinor bases \( \Gamma_i^{(J)} \) and the invariant amplitudes \( \chi_i^{(J)} \) as

\[ \left[ \chi^{(J)}(p;q,\epsilon) \right]_{\alpha\beta} = \sum_{i=1}^8 \left[ \Gamma_i^{(J)}(p;\hat{q},\epsilon) \right]_{\alpha\beta} \chi_i^{(J)}(p;q), \]  

(3.4)

where \( \hat{q}_\mu = q_\mu/\sqrt{Q^2} \). The bispinor bases can be chosen in a way that they have the same properties of spin, parity and charge conjugation as the corresponding current \( J^a_\mu(x) \) has. We adopt the following bispinor bases for the vector vertex:

\[ \Gamma_1^{(V)} = \epsilon, \quad \Gamma_2^{(V)} = \frac{1}{2}[\epsilon, \hat{p}](p \cdot \hat{q}), \quad \Gamma_3^{(V)} = \frac{1}{2}[\epsilon, \hat{q}], \quad \Gamma_4^{(V)} = \frac{1}{3!}[\epsilon, \hat{p}, \hat{q}] \]  

\[ \Gamma_5^{(V)} = (\epsilon \cdot p), \quad \Gamma_6^{(V)} = \hat{p}(\epsilon \cdot p), \quad \Gamma_7^{(V)} = \hat{q}(p \cdot \hat{q})(\epsilon \cdot p), \quad \Gamma_8^{(V)} = \frac{1}{2}[\hat{p}, \hat{q}](\epsilon \cdot p), \]  

(3.5)

where \([a, b, c] \equiv a[b, c] + b[c, a] + c[a, b]\). For the axial-vector vertex we use

\[ \Gamma_1^{(A)} = \epsilon \gamma_5, \quad \Gamma_2^{(A)} = \frac{1}{2}[\epsilon, \hat{p}]\gamma_5, \quad \Gamma_3^{(A)} = \frac{1}{2}[\epsilon, \hat{q}](p \cdot \hat{q}) \gamma_5, \]  

\[ \Gamma_4^{(A)} = \frac{1}{3!}[\epsilon, \hat{p}, \hat{q}] \gamma_5, \quad \Gamma_5^{(A)} = (\epsilon \cdot p)(p \cdot \hat{q}) \gamma_5, \quad \Gamma_6^{(A)} = \hat{p}(\epsilon \cdot p) \gamma_5, \]  

\[ \Gamma_7^{(A)} = \hat{q}(\epsilon \cdot p)(p \cdot \hat{q}) \gamma_5, \quad \Gamma_8^{(A)} = \frac{1}{2}[\hat{p}, \hat{q}](\epsilon \cdot p)(p \cdot \hat{q}) \gamma_5. \]  

(3.6)

From the above choice of the bases, we can easily show that all the invariant amplitudes \( \chi_i^{(J)} \) are the even functions of \((p \cdot \hat{q})\) using the charge conjugation property of the current.

In the present analysis we fix the frame of reference in such a way that only the zero component of the total momentum \( q^\mu \) becomes non-zero. Furthermore, we study
the case where \( q^\mu \) is in the space-like region. Then, it is convenient to parameterize the total momentum \( q^\mu \) as
\[
q^\mu = (iQ, 0, 0, 0). \tag{3.7}
\]
For the relative momentum \( p^\mu \), we perform the Wick rotation, and parameterize it by the real variables \( u \) and \( x \) as
\[
p \cdot q = -Q \ u, \quad p^2 = -u^2 - x^2. \tag{3.8}
\]
Consequently, the invariant amplitudes \( \chi_i^{(J)} \) become functions in \( u \) and \( x \):
\[
\chi_i^{(J)}(u, x; Q). \tag{3.9}
\]
From the charge conjugation properties for the BS amplitude \( \chi^{(J)} \) and the bispinor bases defined above, the invariant amplitudes \( \chi_i^{(J)}(u, x) \) are shown to satisfy
\[
\chi_i^{(J)}(u, x; Q) = \chi_i^{(J)}(-u, x; Q). \tag{3.10}
\]
Using this property of the invariant amplitudes, we rewrite Eq. (3.2) as
\[
\Pi_{VV}(Q^2) = \frac{N_c}{\pi^3} \int_0^\infty du \int_0^\infty dx \int_0^\infty x^2 \left[ \chi_1^{(V)}(u, x; Q) + \frac{x^2}{3} \chi_6^{(V)}(u, x; Q) \right], \tag{3.11}
\]
\[
\Pi_{AA}(Q^2) = \frac{N_c}{\pi^3} \int_0^\infty du \int_0^\infty dx \int_0^\infty x^2 \left[ \chi_1^{(A)}(u, x; Q) - \frac{x^2}{3} \chi_6^{(A)}(u, x; Q) \right]. \tag{3.12}
\]
Here, we used the expanded form of the BS amplitude shown in Eq. (3.4) and carried out the three dimensional angle integration.

From Eqs. (3.11) and (3.12), the quantity \( \Pi_{VV} - \Pi_{AA} \) is expressed as
\[
\Pi_{VV} - \Pi_{AA} = \frac{1}{3} \sum \epsilon \int \frac{d^3p}{i(2\pi)^4} \frac{N_c}{2} \text{tr} \left[ \gamma_5 \chi^{(A)}(p; q, \epsilon) - \gamma_5 \chi^{(V)}(p; q, \epsilon) \right],
\]
\[
= \frac{N_c}{\pi^3} \int_0^\infty du \int_0^\infty dx x^2 \left[ \left( \chi_1^{(V)}(u, x; Q) + \chi_1^{(A)}(u, x; Q) \right) + \frac{x^2}{3} \left( \chi_6^{(V)}(u, x; Q) + \chi_6^{(A)}(u, x; Q) \right) \right]. \tag{3.13}
\]
We note that, although either \( \Pi_{VV} \) or \( \Pi_{AA} \) is logarithmically divergent quantity, the difference \( \Pi_{VV} - \Pi_{AA} \) becomes finite due to the cancellation of the divergence ensured by the chiral symmetry.

### 4 Inhomogeneous Bethe-Salpeter equation

In this section we introduce the inhomogeneous Bethe-Salpeter (IBS) equation from which we calculate the BS amplitude defined in the previous section. We also show the numerical method for solving the IBS equation.
4.1 IBS equation

The IBS equation is the self-consistent equation for the BS amplitude $\chi^{(J)}$, and it is expressed as (see Fig. 1 for graphical expression)

$$T(p; q) \chi^{(J)}(p; q, \epsilon) = \epsilon \cdot G^{(J)} + K(p; k) \ast \chi^{(J)}(k; q, \epsilon). \quad (4.1)$$

The kinetic part $T$ is given by

$$T(p; q) = S^{-1}_F(p + q/2) \otimes S^{-1}_F(p - q/2), \quad (4.2)$$

where $S_F$ is the full fermion propagator $iS^{-1}_F(p) = \not{p} - \Sigma(p)$. (Note that wave function renormalization factor $A(p)$ becomes unity when we use the Landau gauge.) The BS kernel $K$ in the improved ladder approximation is expressed as

$$K(p; k) = \frac{N_c^2 - 1}{2N_c} \frac{\bar{g}^2(p, k)}{(p - k)^2} \left( g_{\mu\nu} - \frac{(p - k)_{\mu}(p - k)_{\nu}}{(p - k)^2} \right) \gamma^\mu \otimes \gamma^\nu, \quad (4.3)$$

where $\bar{g}(p, k)$ is the running coupling of QCD whose explicit form will be shown later. In the above expressions we used the tensor product notation

$$(A \otimes B) \chi = A \chi B, \quad (4.4)$$

and the inner product notation

$$K(p; k) \ast \chi^{(J)}(k; q, \epsilon) = \int d^4k \frac{K(p, k) \chi(k; q)}{(2\pi)^4} \quad (4.5)$$

The mass function of the quark propagator is obtained from the SD equation:

$$\Sigma(p) = K(p, k) \ast iS_F(p). \quad (4.6)$$

It should be stressed that we must use the same kernel $K(p, k)$ as that used in the IBS equation for consistency with the chiral symmetry \cite{30, 31, 32, 33}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{A graphical expression of the IBS equation in the (improved) ladder approximation.}
\end{figure}
4.2 Numerical method for solving the IBS equation

In this subsection we transform the IBS equation in Eq. (4.1) into the form by which we can solve it numerically.

First, we introduce the conjugate bispinor bases defined by

$$\bar{\Gamma}_i^{(J)}(p; q, \epsilon) \equiv \gamma_0 \Gamma_i^{(J)}(p^*; q, \epsilon)^\dagger \gamma_0 . \tag{4.7}$$

Multiplying these conjugate bispinor bases from left, taking the trace of spinor indices and summing over the polarizations, we rewrite Eq. (4.1) into the following form:

$$T_{ij}^{(J)}(u, x) \chi_j^{(J)}(u, x) - \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dv \int_{0}^{\infty} dyy^2 K_{ij}^{(J)}(u, x; v, y) \chi_j^{(J)}(v, y) = I_i^{(J)}(u, x), \tag{4.8}$$

where the summation over the index $j$ is understood, and

$$I_i^{(J)} = \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_i^{(J)}(p; q, \epsilon) \left( \epsilon \cdot G^{(J)} \right) \right], \tag{4.9}$$

$$T_{ij}^{(J)}(u, x) = \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_i^{(J)}(p; q, \epsilon) T(p; q) \Gamma_j^{(J)}(p; q, \epsilon) \right], \tag{4.10}$$

$$K_{ij}^{(J)}(u; v, y) = \int_{-1}^{1} d\cos \theta \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_i^{(J)}(p; q, \epsilon) K(p, k) \Gamma_j^{(J)}(k; q, \epsilon) \right], \tag{4.11}$$

with the real variables $v$ and $y$ introduced as

$$k \cdot q = -v Q, \quad k \cdot p = -uv - xy \cos \theta. \tag{4.12}$$

Here $\theta$ is the angle between the spatial components of $p_\mu$ and $k_\mu$.

Using the property of $\chi_i^{(J)}$ in Eq. (3.10), we restrict the integration range as $v > 0$:

$$\int dv K_{ij}^{(J)}(u, x; v, y) \chi_j^{(J)}(v, y) = \int_{v>0} dv [K_{ij}^{(J)}(u, x; v, y) + K_{ij}^{(J)}(u, x; -v, y)] \chi_j^{(J)}(v, y). \tag{4.13}$$

Then, in the following, we treat all the variables $u$, $x$, $v$ and $y$ as positive values.

To discretize the variables $u$, $x$, $v$ and $y$ we introduce new variables $U$, $X$, $V$ and $Y$ as

$$u = e^U, \quad x = e^X,$$

$$v = e^V, \quad y = e^Y, \tag{4.14}$$

and set ultraviolet (UV) and infrared (IR) cutoffs as

$$U, V \in [\lambda_U, \Lambda_U], \quad X, Y \in [\lambda_X, \Lambda_X]. \tag{4.15}$$

We discretize the variables $U$ and $V$ into $N_{BS,U}$ points evenly, and $X$ and $Y$ into $N_{BS,X}$ points. Then, the original variables are labeled as

$$u_{[u]} = \exp [\lambda_U + D_U I_U], \quad x_{[x]} = \exp [\lambda_X + D_X I_X],$$

$$v_{[v]} = \exp [\lambda_U + D_U I_V], \quad y_{[y]} = \exp [\lambda_X + D_X I_Y],$$

$$I_{[u]} = \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_{i_{[u]}}^{(J)}(p_{[u]}; q, \epsilon) \left( \epsilon \cdot G^{(J)} \right) \right],$$

$$T_{ij}^{(J)}(u, x) = \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_{i_{[u]}}^{(J)}(p_{[u]}; q, \epsilon) T(p_{[u]}; q) \Gamma_j^{(J)}(p_{[u]}; q, \epsilon) \right],$$

$$K_{ij}^{(J)}(u; v, y) = \int_{-1}^{1} d\cos \theta \sum_\epsilon \text{tr} \left[ \bar{\Gamma}_{i_{[u]}}^{(J)}(p_{[u]}; q, \epsilon) K(p_{[u]}, k) \Gamma_j^{(J)}(k_{[u]}; q, \epsilon) \right],$$

$$K_{ij}^{(J)}(u; v, y) = \int_{v>0} dv [K_{ij}^{(J)}(u, x; v, y) + K_{ij}^{(J)}(u, x; -v, y)] \chi_j^{(J)}(v, y).$$
where \( I_U, I_V = 0, 1, 2, \cdots (N_{BS,U} - 1) \) and \( I_X, I_Y = 0, 1, 2, \cdots (N_{BS,X} - 1) \). The measures \( D_U \) and \( D_X \) are defined as

\[
D_U = \frac{\Lambda_U - \lambda_U}{N_{BS,U} - 1}, \quad D_X = \frac{\Lambda_X - \lambda_X}{N_{BS,X} - 1}.
\]

As a result, the integration is converted into the summation:

\[
\int_{v>0} y^2 \, dv \cdot \cdots \implies D_U D_V \sum_{I_V, I_Y} v y^3 \cdot \cdots.
\]

In order to avoid integrable singularities in the kernel \( K(u, x; v, y) \) at \((u, x) = (v, y)\), we adopt the following four-splitting prescription [21]:

\[
K_{ij}(u, x; v, y) \implies \frac{1}{4} \left[ K_{ij}(u, x, v_+, y_+) + K_{ij}(u, x, v_+, y_-) + K_{ij}(u, x, v_-, y_+) + K_{ij}(u, x, v_-, y_-) \right],
\]

where

\[
v_\pm = \exp \left[ V \pm \frac{D_U}{4} \right], \quad y_\pm = \exp \left[ Y \pm \frac{D_X}{4} \right].
\]

Now that all the variables have become discrete and the original integral equation (4.1) has turned into a linear algebraic one, we are able to deal it numerically.

5 Results and discussion

In this section, we show the results of calculations and give some discussions. First, we discuss the dependence of the results on the infrared structure of the running coupling. Second, we compare the results with experimental values.

For solving the IBS and SD equations, we have to fix the form of the running coupling \( \bar{g}^2(p, k) \) which appears in the IBS and the SD equations. We use the solution of the renormalization group equation for the QCD running coupling with one-loop approximation. We regularize the infrared divergence of the one-loop running coupling by introducing the IR cutoff parameter \( t_F > 0 \) as follows:

\[
\alpha(p, k) \equiv \frac{\bar{g}^2(p, k)}{4\pi} = \alpha_0 \frac{1}{\max(t_F, t)}, \quad t = \ln \left[ (p_E^2 + k_E^2)/\Lambda^2_{QCD} \right]
\]

where \( \alpha_0 = 12\pi/(11N_c - 2N_f) \) with \( N_f \) being the number of flavors. As for the argument of the running coupling \( \bar{g}^2(p, k) \), we used the angle averaged form \( p_E^2 + k_E^2 \) so that we can analytically carry out the angle integration in the SD and the IBS equations.

In Fig. 2, we plot the solutions of the SD equation for several values of \( t_F \). For smaller value of \( t_F \), the value of mass functions \( \Sigma(x) \) in the infrared energy region becomes larger. This is natural because smaller value of \( t_F \) means larger running coupling in the infrared region.
Figure 2: Mass functions obtained from the SD equation for several values of $t_F$.

When we solve the IBS equation, we use the following parameters:

$$[\lambda_U, \Lambda_U] = [-5.5, 2.5], \quad (5.2)$$

$$[\lambda_X, \Lambda_X] = [-2.5, 2.5], \quad (5.3)$$

$$N_{BS,U} = N_{BS,X} = 30. \quad (5.4)$$

These parameters are chosen so that the dominant supports always lie within the energy region between UV and IR cutoffs for all values of $t_F$ which we use in the present analysis ($t_F = 0.04-0.5$).

### 5.1 $t_F$ dependence of $\Pi_{VV} - \Pi_{AA}$

In Fig. 3 we plot the resultant values of $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ for several values of $t_F$. Horizontal axis shows the square of the Euclidean momentum $Q$. In this figure, $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ and $Q^2$ are normalized by $\Lambda^2_{QCD}$, and plotted as the dimensionless quantities. As we mentioned before, either $\Pi_{VV}(Q^2)$ or $\Pi_{AA}(Q^2)$ is logarithmically divergent. These divergences are expected to be canceled in $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ ensured by the chiral symmetry. Figure 3 shows this cancellation actually occurs in the present calculation. ² From the results shown in Fig. 3 we can see that $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ is dependent on the value of $t_F$. In other words, $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ is sensitive to the infrared structure of the QCD running coupling.

Once we obtain $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$, we can calculate the QCD $S$ parameter, $f_\pi$ and $\Delta m_\pi^2$ by using Eqs. (2.14), (2.15) and (2.16). In table II we list the resultant

²If the quadratic divergences were not canceled, $\Pi_{VV} - \Pi_{AA}$ would become $\frac{\Lambda_{QCD}^2}{8\pi^2} \log \Lambda^2 \sim 0.1\Lambda_{QCD}^2$. Figure 3 shows that $\Pi_{VV} - \Pi_{AA}$ is about 0.03$\Lambda_{QCD}^2$ at biggest, so that the divergences are actually canceled with each other.
$$\Pi_{VV}(q_E^2) - \Pi_{AA}(q_E^2) / \Lambda^2_{QCD}$$

Figure 3: Resultant values of $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ for several values of $t_F$. Horizontal axis shows the square of the Euclidean momentum normalized by the scale $\Lambda^2_{QCD}$. $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ in the vertical axis is also normalized by $\Lambda^2_{QCD}$. 

$t_f = 0.5$ ◇
$t_f = 0.3$ +
$t_f = 0.1$ □
$t_f = 0.05$ ×
$t_f = 0.04$ △

$q_E^2 / \Lambda^2_{QCD}$
| $t_F$  | 0.04 | 0.05 | 0.1 | 0.2 | 0.3 | 0.5 |
|-------|------|------|-----|-----|-----|-----|
| $S$   | 0.33 | 0.33 | 0.35 | 0.39 | 0.43 | 0.47 |
| $f_\pi / \Lambda_{QCD}$ | 0.128 | 0.132 | 0.145 | 0.162 | 0.171 | 0.171 |
| $\Delta m_{\pi}^2 / \Lambda_{QCD}^2$ | 0.00201 | 0.00203 | 0.00212 | 0.00215 | 0.00212 | 0.00201 |

Table 1: Resultant values of the QCD $S$ parameter, $f_\pi$ and $\Delta m_{\pi}^2$ for several values of $t_F$. Dimensionfull quantities are normalized by $\Lambda_{QCD}$ and expressed as dimensionless quantities.

values of these quantities for several values of $t_F$. Here, we normalized dimensionfull quantities by $\Lambda_{QCD}$, and expressed them as dimensionless quantities. These results show that the QCD $S$ parameter and $f_\pi / \Lambda_{QCD}$ are sensitive to the value of IR cutoff parameter $t_F$ while $\Delta m_{\pi}^2 / \Lambda_{QCD}^2$ is less sensitive to it. This is natural because the QCD $S$ parameter and $f_\pi$ are directly related to the infrared quantities as can be easily seen from Eqs. (2.14) and (2.15), while the infrared dependence of the integration of $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ in Eq. (2.16) is compensated by $f_\pi^2 (= \Pi_{VV}(0) - \Pi_{AA}(0))$ in the denominator.

Table 1 shows that, when we take the value of $t_F$ smaller, the value of $f_\pi / \Lambda_{QCD}$ becomes smaller. Here, smaller value of $t_F$ means larger value of infrared running coupling, or, in other words, strong infrared dynamics. At first sight, this behavior seems strange because $f_\pi$ is the order parameter of the chiral symmetry breaking, and it is expected to reflect the magnitude of the chiral symmetry breaking. One might think that $f_\pi / \Lambda_{QCD}$ becomes larger for smaller value of $t_F$. However, it is not the case. In the present analysis, $f_\pi$ is not necessarily proportional to the magnitude of the chiral symmetry breaking because QCD with $N_f = 3$ in the vacuum is very far from the chiral phase transition point. To understand this behavior, let us see the Pagels-Stokar (PS) formula [34]:

$$f_\pi^2 = \frac{N_c}{4\pi^2} \int_0^\infty dx \ x \ \frac{1}{[x + \Sigma^2(x)]^2} \left[ \Sigma^2(x) - \frac{4}{x} \frac{d}{dx} \left[ \Sigma^2(x) \right] \right].$$

(5.5)

As shown in Fig. 2, the mass functions $\Sigma(x)$ are almost constant for $0 < x = Q^2 < \Lambda_{QCD}^2$ and suddenly drops at $x = \Lambda_{QCD}^2$ for all values of $t_F$. So we can cut the integration in Eq. (5.5) at $x = \Lambda_{QCD}^2$ and drop the derivative of mass function. We also drop $x$ in the denominator since $x$ satisfies $x < \Sigma(x)$ for the relevant integral region ($0 < x < \Lambda_{QCD}^2$). Then we can approximate Eq. (5.5) as

$$f_\pi^2 \simeq \frac{N_c}{4\pi^2} \int_0^{\Lambda_{QCD}^2} dx \ x \ \frac{1}{\Sigma^2(0)} \propto \frac{\Lambda_{QCD}^4}{\Sigma^2(0)}.$$  

(5.6)

From this, we can easily understand that, when infrared dynamics becomes strong, i.e., $\Sigma(0)$ becomes large, $f_\pi$ becomes small.

---

3 We checked that the difference between $f_\pi$ calculated from the IBS equation and the PS formula is less than 10% for all values of $t_F$. We also checked that qualitative feature of $t_F$ dependence of $f_\pi$ calculated from the PS formula is not different from $f_\pi$ obtained from the IBS equation.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$ t_F $ & 0.04 & 0.05 & 0.1 & 0.2 & 0.3 & 0.5 & Exp. \\
\hline
$ \Delta m_{\pi}^2 $ (MeV$^2$) & 1050 & 1003 & 855 & 698 & 620 & 585 & 1261.1 \\
\hline
$ S $ & 0.33 & 0.33 & 0.35 & 0.39 & 0.43 & 0.47 & 0.32 $\pm$ 0.04 \\
\hline
$ f_\pi $ (MeV) (input) & 92.4 & 92.4 & 92.4 & 92.4 & 92.4 & 92.4 & 92.4 \\
\hline
\end{tabular}
\caption{Resultant values of $ \Delta m_{\pi}^2 $, $ S $ and $ f_\pi $ for several values of $ t_F $. Here, we used $ f_\pi = 92.4 $ MeV as an input to introduce the physical energy scale. Experimental values of $ \Delta m_{\pi}^2 $ and $ f_\pi $ are given in Ref. [35], and $ S $ is obtained from the value of the $ L_{10} $ given in Ref. [14] through $ S = -16\pi \left[ L_{10}(\mu) + \frac{1}{192\pi^2} \left( \ln \frac{m_{\pi}^2}{\mu^2} + 1 \right) \right] $.}
\end{table}

For comparison, we consider the decay constant near the chiral phase transition point. In this case, $ \Sigma(x) $ becomes very small and Eq. (5.5) is well approximated by

$$ f_\pi^2 = \frac{N_c}{4\pi^2} \int_0^\infty \frac{dx}{x} \Sigma^2(x). \quad (5.7) $$

From this, we can see that if the running coupling, and then the mass function becomes smaller, $ f_\pi $ also becomes smaller. Thus near the critical point, order parameter $ f_\pi $ is actually proportional to the magnitude of the chiral symmetry breaking. We can actually see this behavior in the large $ N_f $ QCD near the critical point (see Figs. 4, 5 in Ref. [19]).

### 5.2 Comparison with experiments

In this subsection, we compare our results with experiments. So far in this paper, all the dimensionfull quantities were normalized by $ \Lambda_{QCD} $ and dealt as dimensionless quantities. Here, we introduce the “physical energy scale” by setting $ \Lambda_{QCD} $ in a way that it reproduces the experimental value of $ f_\pi $. Since $ f_\pi $ is sensitive to the infrared parameter, fixing $ f_\pi $ implies that $ \Lambda_{QCD} $ in turn appears sensitive to the infrared parameter and so does the $ \Delta m_{\pi}^2 $ which was shown insensitive to the infrared parameter as far as $ \Lambda_{QCD} $ is fixed. We use $ f_\pi = 92.4 $ MeV as an input to fix $ \Lambda_{QCD} $ in the present analysis.\(^4\)

Table 2 shows comparison of the resultant values of $ \Delta m_{\pi}^2 $ and $ S $ for several values of $ t_F $ with experimental values. $ \Delta m_{\pi}^2 $, $ S $ and $ f_\pi $ are directly calculated from $ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) $ by using Eqs. (2.16), (2.14) and (2.15). From table 2 we can see that if we change the value of $ t_F $, values of $ \Delta m_{\pi}^2 $ and $ S $ move to opposite directions: When we decrease the value of $ t_F $, $ \Delta m_{\pi}^2 $ becomes large, while $ S $ becomes small. For the choice $ t_F = 0.5 $, which was adopted in Ref. [36] [20] [21] as a good regularization in the calculations of the masses and decay constants of $ \rho $ and $ a_1 $ mesons, we obtained $ \Delta m_{\pi}^2 = 585 \text{ MeV}^2 $ and $ S = 0.47 $. Both of these values are far from their experimental values $ \Delta m_{\pi}^2 = 1261.1 \text{ MeV}^2 $ and $ S = 0.29 - 0.36 $.

\(^4\)In the chiral symmetry limit of three flavors $ m_u = m_d = m_s = 0 $, the pion decay constant is estimated as $ f_\pi = 86.4 \pm 0.26 \text{ MeV} \quad [14] $. In this paper we use instead the physical value only for fixing $ \Lambda_{QCD} $ as a reference scale, which is up to the “scale ambiguity” in the SD and BS approach.
However, as we decrease the value of $t_F$, both $\Delta m_\pi^2$ and $S$ move toward their experimental values. At $t_F = 0.04$, they become

\[
\begin{align*}
\Delta m_\pi^2 &= 1050 \text{MeV}^2, \\
S &= 0.33,
\end{align*}
\]

in good agreement with the experimental value of $S$ and in rough agreement with that of $\Delta m_\pi^2$. In the case of $t_F = 0.04$ we take

\[\Lambda_{QCD} = 724 \text{MeV}\] (5.10)

in order to set $f_\pi = 92.4 \text{MeV}$. This value is much larger than the conventional value of $\Lambda_{QCD}$ in the MS scheme [24]:

\[\Lambda_{QCD}^{(3)} = 300 - 450 \text{MeV}.\] (5.11)

Such a problem already occurred in the previous works [36, 20, 21],

\[\Lambda_{QCD} \simeq 500 \text{MeV},\] (5.12)

which roughly corresponds to $t_F = 0.5$ in the present analysis. This reflects the "scale ambiguity" [25], as was recently pointed out in Ref. [26] where an interesting method was proposed to solve it by using the effective coupling.

From these results, we conclude that, when we calculate $f_\pi$, $\Delta m_\pi^2$ and $S$ by using the BS equation with the improved ladder approximation, the infrared cutoff parameter $t_F$ should be taken smaller than values used so far in the previous works [36, 20, 21]. This means that the running coupling in the infrared energy region should be taken larger in order to reproduce experimental values of $f_\pi$, $\Delta m_\pi^2$ and $S$ at the same time in the calculations based on the BS equation with the improved ladder approximation. (When we take $t_F = 0.04$, the value of the running coupling in the infrared energy region becomes about $\alpha \sim 35$.)

Now let us look at what other quantities can be obtained from $\Pi_{VV} - \Pi_{AA}$. As for $\rho$ meson and $a_1$ meson, we derive their masses and decay constants by fitting them to the calculated $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ using the pole saturated form of $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$. Here, we use the following simplest version of the pole saturated form as a fitting function:

\[
\left[\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)\right]_{\text{fit}} = -\frac{Q^2 f_\rho^2}{M_\rho^2 + Q^2} + f_\pi^2 + \frac{Q^2 f_{a_1}^2}{M_{a_1}^2 + Q^2}.
\] (5.13)

Resultant best fitted values are listed in Table 3 together with the experimental values. From the results shown in this table, we can see that the masses and the decay constants of $\rho$ and $a_1$ mesons are not so much dependent on the infrared cutoff parameter $t_F$ compared with $t_F$ dependence of $\Delta m_\pi^2$ and $S$. When we choose $t_F = 0.04$, for which the values of $\Delta m_\pi^2$ and $S$ are predicted in good agreement with experiments,
Table 3: Resultant best fitted values of masses and decay constants of $\rho$ and $a_1$ mesons for several values of $t_F$. Experimental values for each quantities are also listed. Experimental values of $M_\rho$, $f_\rho$ and $M_{a_1}$ are given in Ref. [35], and $f_{a_1}$ is given in Ref. [37].

| $t_F$ | 0.04 | 0.05 | 0.1 | 0.2 | 0.3 | 0.5 | Exp. |
|-------|------|------|-----|-----|-----|-----|-----|
| $M_\rho$ (MeV) | 730  | 710  | 716 | 645 | 613 | 612 | 770 - 772 |
| $f_\rho$ (MeV)  | 150  | 146  | 151 | 150 | 150 | 151 | 154 |
| $M_{a_1}$ (MeV) | 1186 | 1146 | 1012 | 908 | 861 | 859 | 1190 - 1270 |
| $f_{a_1}$ (MeV) | 115  | 109  | 119 | 118 | 119 | 123 | 144 |

we obtained

$$M_\rho = 730 \text{ MeV}, \quad (5.14)$$
$$f_\rho = 150 \text{ MeV}, \quad (5.15)$$
$$M_{a_1} = 1186 \text{ MeV}, \quad (5.16)$$
$$f_{a_1} = 115 \text{ MeV}. \quad (5.17)$$

These are also in good agreement with experiments. This confirms the reliability of calculations in the present analysis based on the improved ladder BS equation with small infrared cutoff parameter ($t_F \sim 0.04$).

## 6 Summary and Discussions

We have calculated the $\pi^+ - \pi^0$ mass difference $\Delta m_\pi^2$ on the same footing as the pion decay constant $f_\pi$ and the QCD $S$ parameter through the vector and the axial-vector current correlators. For the calculations we solved inhomogeneous BS equations for the vector and axial-vector vertex functions together with SD equation for the quark mass function within the improved ladder approximation in the Landau gauge. We also obtained the masses and the decay constants of $\rho$ and $a_1$ mesons by fitting them to $\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)$ using the pole saturated form.

We showed that all of these quantities are simultaneously fit in agreement with the experiments when we take the IR cutoff parameter $t_F = 0.04$, which corresponds to taking

$$\Lambda_{\text{QCD}} \approx 724 \text{ MeV}. \quad (6.1)$$

This parameter choice of $t_F$ is fairly smaller than the ones investigated in the previous works [20, 21, 22, 23], which corresponds to

$$\Lambda_{\text{QCD}} \approx 500 \text{ MeV}. \quad (6.2)$$

Either value of $\Lambda_{\text{QCD}}$ above is substantially larger than that in the $\overline{\text{MS}}$ scheme [24]:

$$\Lambda_{\text{QCD}}^{(3)} = 300 - 450 \text{ MeV}. \quad (6.3)$$

Our results imply that the running coupling of QCD in the IR region should be taken larger than that considered so far, when we calculate not only $f_\pi$ but also the QCD $S$...
parameter and $\Delta m_\pi^2$ in the framework of the BS and the SD equations in the improved ladder approximation. Apparently, choosing the IR cutoff parameter corresponds to exploiting the “scale ambiguity” [25] of $\Lambda_{\text{QCD}}$ in the SD and BS equations as was recently emphasized in Ref. [26] which proposed an interesting method to resolve the scale ambiguity in the calculation of $f_\pi$.

In order to establish the approach of the SD and the BS equations in the improved ladder approximation, it is certainly desirable to apply the method of Ref. [26] to our case, namely the calculations of $\Delta m_\pi^2$ and QCD $S$ parameter as well as $f_\pi$. This will be done in future work.

The success of the analysis based on the BS and the SD equations in the real-life QCD in this paper well motivate us to apply this method for studies of other strong coupling gauge theories. One of the most interesting examples of such is the large $N_f$ QCD (see, e.g., Refs. [38, 39, 40, 41, 42, 43, 19]), which is a QCD with large number of massless flavors (but not too large number as to destroy the asymptotic freedom). It is expected that the chiral symmetry gets restored for a certain large number of massless flavors, which was in fact confirmed by the lattice simulations [11].

It was argued [44, 14] based on the effective field theory of HLS [7, 8, 9] that this chiral restoration of the large $N_f$ QCD is accompanied by the massless vector meson degenerate with the NG boson (pseudoscalar meson) as the chiral partners ("Vector Manifestation" of chiral symmetry). It was further argued [45] that the HLS model even in the real-life QCD behaves as a little Higgs model [2] with two sites and two links, whose locality of theory space forbids the quadratic divergence in $\Delta m_\pi^2$ which is nothing but the (mass)$^2$ of the Higgs in the framework of the Little Higgs models. In the large $N_f$ QCD $\Delta m_\pi^2$ becomes small compared with $f_\pi^2$ near the chiral restoration point, which may suggest that the large $N_f$ QCD may be considered as a UV completion of the Little Higgs model.

On the other hand, we found in the previous paper [19], by the explicit calculation based on the homogeneous BS and the SD equations in the improved ladder approximation, that masses of the scalar, vector, and axial-vector mesons in the large $N_f$ QCD are proportional to $f_\pi$ and vanish at the chiral restoration point, which implies somewhat different manifestation than the Vector Manifestation in the HLS model. (This behavior was also conjectured in Ref. [45].)

Thus it would be very useful to clarify the situation to calculate the $S$ parameter and $\Delta m_\pi^2$ in large $N_f$ QCD which have not yet been calculated directly from QCD. In the forthcoming paper we shall calculate $\Delta m_\pi^2$ and $S$ parameter in large $N_f$ QCD by the same method presented in this paper.

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References

[1] See for reviews, e.g., E. Farhi and L. Susskind, Phys. Rept. 74, 277 (1981); C. T. Hill and E. H. Simmons, Phys. Rept. 381, 235 (2003) [Erratum-ibid. 390, 553 (2004)] and references therein.

[2] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Lett. B 513, 232 (2001) N. Arkani-Hamed, A. G. Cohen, E. Katz, A. E. Nelson, T. Gregoire and J. G. Wacker, JHEP 0208, 021 (2002) N. Arkani-Hamed, A. G. Cohen, E. Katz and A. E. Nelson, JHEP 0207, 034 (2002)

[3] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low and J. E. Young, Phys. Rev. Lett. 18, 759 (1967)

[4] S. Weinberg, Phys. Rev. Lett. 18, 507 (1967).

[5] W. A. Bardeen, J. Bijnens and J. M. Gerard, Phys. Rev. Lett. 62, 1343 (1989).

[6] M. Harada, M. Tanabashi and K. Yamawaki, Phys. Lett. B 568, 103 (2003)

[7] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. 54, 1215 (1985);

[8] M. Bando, T. Kugo and K. Yamawaki, Nucl. Phys. B 259, 493 (1985).

[9] M. Bando, T. Kugo and K. Yamawaki, Prog. Theor. Phys. 73, 1541 (1985).

[10] M. Harada and K. Yamawaki, Phys. Lett. B 297, 151 (1992)

[11] M. Tanabashi, Phys. Lett. B 316, 534 (1993)

[12] M. Harada and K. Yamawaki, Phys. Rev. D 64, 014023 (2001)

[13] M. Bando, T. Kugo and K. Yamawaki, Phys. Rept. 164, 217 (1988).

[14] M. Harada and K. Yamawaki, Phys. Rept. 381, 1 (2003)

[15] A. Duncan, E. Eichten and H. Thacker, Phys. Rev. Lett. 76, 3894 (1996) A. Duncan, E. Eichten and H. Thacker, Nucl. Phys. Proc. Suppl. 53, 295 (1997)

[16] M. E. Peskin and T. Takeuchi, Phys. Rev. Lett. 65, 964 (1990); Phys. Rev. D 46, 381 (1992).

[17] J. Gasser and H. Leutwyler, Annals Phys. 158, 142 (1984). Nucl. Phys. B 250, 465 (1985).
[18] T. Das, V. S. Mathur and S. Okubo, Phys. Rev. Lett. 19, 859 (1967).

[19] M. Harada, M. Kurachi and K. Yamawaki, Phys. Rev. D 68, 076001 (2003)

[20] K. I. Aoki, M. Bando, T. Kugo, M. G. Mitchard and H. Nakatani, Prog. Theor. Phys. 84, 683 (1990).

[21] K. I. Aoki, T. Kugo and M. G. Mitchard, Phys. Lett. B 266, 467 (1991).

[22] C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. 45, S1 (2000)

[23] M. Harada and Y. Yoshida, Phys. Rev. D 50, 6902 (1994)

[24] A. J. Buras, arXiv:hep-ph/9806471.

[25] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D 28, 228 (1983).

[26] M. Hashimoto and M. Tanabashi, arXiv:hep-ph/0210115. M. Hashimoto and M. Tanabashi, in Proceedings of the 2002 International Workshop on Strong Coupling Gauge Theories and Effective Field Theories (SCGT’02), Nagoya, Japan, Dec 10-13, 2002, ed. M. Harada, Y. Kikukawa, and K. Yamawaki (World Scientific Pub. Co., Singapore 2003), p.96-102.

[27] C. W. Bernard, A. Duncan, J. LoSecco and S. Weinberg, Phys. Rev. D 12, 792 (1975).

[28] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979); M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 448 (1979).

[29] K. Yamawaki, Phys. Lett. B 118, 145 (1982).

[30] T. Maskawa and H. Nakajima, Prog. Theor. Phys. 52, 1326 (1974); T. Maskawa and H. Nakajima, Prog. Theor. Phys. 54, 860 (1975).

[31] T. Kugo and M. G. Mitchard, Phys. Lett. B 282, 162 (1992).

[32] T. Kugo and M. G. Mitchard, Phys. Lett. B 286, 355 (1992).

[33] M. Bando, M. Harada and T. Kugo, Prog. Theor. Phys. 91, 927 (1994)

[34] H. Pagels and S. Stokar, Phys. Rev. D 20, 2947 (1979).

[35] K. Hagiwara et al. [Particle Data Group Collaboration], Phys. Rev. D 66, 010001 (2002).

[36] T. Kugo, in Proc. of 1991 Nagoya Spring School on Dynamical Symmetry Breaking, Nakatsuigawa, Japan, 1991, ed. K. Yamawaki (World Scientific, Singapore, 1992).

[37] N. Isgur, C. Morningstar and C. Reader, Phys. Rev. D 39, 1357 (1989).
[38] T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).

[39] T. Appelquist, J. Terning and L. C. Wijewardhana, Phys. Rev. Lett. 77, 1214 (1996) T. Appelquist, A. Ratnaweera, J. Terning and L. C. Wijewardhana, Phys. Rev. D 58, 105017 (1998)

[40] V. A. Miransky and K. Yamawaki, Phys. Rev. D 55, 5051 (1997) [Erratum-ibid. D 56, 3768 (1997)]

[41] J. B. Kogut and D. K. Sinclair, Nucl. Phys. B 295, 465 (1988); F. R. Brown, H. Chen, N. H. Christ, Z. Dong, R. D. Mawhinney, W. Schaffer and A. Vaccarino, Phys. Rev. D 46, 5655 (1992); Y. Iwasaki, K. Kanaya, S. Sakai and T. Yoshie, Phys. Rev. Lett. 69, 21 (1992); Y. Iwasaki, K. Kanaya, S. Kaya, S. Sakai and T. Yoshie, Nucl. Phys. Proc. Suppl. 53, 449 (1997); Prog. Theor. Phys. Suppl. 131, 415 (1998).

[42] R. Oehme and W. Zimmermann, Phys. Rev. D 21, 471 (1980).

[43] M. Velkovsky and E. Shuryak, Phys. Lett. B 437, 398 (1998).

[44] M. Harada and K. Yamawaki, Phys. Rev. Lett. 86, 757 (2001)

[45] R. S. Chivukula, Phys. Rev. D 55, 5238 (1997)