Spectrum of a four-dimensional Yang-Mills theory

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For the propagator of Yang-Mills theory in the infrared limit, one generally exploits a current expansion [0, 39]. Instead to start from the action, we prefer the equations of motion [10]

\[ \partial^\mu \partial_\mu A^a_\nu - \left(1 - \frac{1}{\alpha} \right) \partial_\nu (\partial^\mu A^a_\mu) + g f^{abc} A^{b\mu}_\nu (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) + g^2 f^{abc} f^{def} A^{b\mu}_\nu A^{d\nu}_\rho A^{e^\rho}_\sigma = j^a_\nu. \] (1)

Then, given a functional form \( A^a_\nu = A^a_\nu [j] \) and doing a Taylor expansion around a given asymptotic solution \( A^a_\nu [0] \), one has

\[ A^a_\nu [j(x)] = A^a_\nu [0] + \int d^4x' \frac{\delta A^a_\nu}{\delta j^b_\mu (x')} j^b_\mu (x') + \int d^4x' d^4x'' \frac{\delta^2 A^a_\nu}{\delta j^b_\mu (x') \delta j^c_\kappa (x'')} j^b_\mu (x') j^c_\kappa (x'') + \ldots. \] (2)

Exact solutions can be obtained in this way [11] for the Landau gauge. The set of solutions we will start with are

\[ A^a_\nu [0] = \eta^a_\nu \chi (x) \] (3)

being \( \chi (x) \) a solution of the equation

\[ \partial^2 \chi + 3 N g^2 \chi^3 = 0 \] (4)

and this is given by

\[ \chi (x) = \mu (2 / N g^2)^{1/3} \text{sn} (p \cdot x + \theta, -1). \] (5)

being \( \mu \) an arbitrary integration constant having the dimensions of a mass and the “momenta” \( p \) satisfy the dispersion relation

\[ p^2 = \mu^2 \sqrt{N g^2 / 2}. \] (6)
We take these solutions as the ground state of the theory. Then, the propagator of the theory will be

$$G_{\mu\nu}(x, x') = \frac{\delta A_{\mu}^a(x)}{\delta J_{\mu}^a(x')} \bigg|_{j=0}. \quad (7)$$

Setting $j = 0$ one gets for the Green function of Yang-Mills theory

$$\partial^2 G^{\mu\nu}_{\nu\rho}(x, x') - \left( 1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu G^{\mu\nu}_{\nu\rho}(x, x') + g f^{abc} G^{\mu\nu}_{\nu\rho}(x, x') \left( \partial_\mu A_{\nu}^b - \partial_\nu A_{\mu}^b(x) \right) + g f^{abc} A_{\mu}^b \left( \partial_\mu G^{\mu\nu}_{\nu\rho}(x, x') - \partial_\nu G^{\mu\nu}_{\nu\rho}(x, x') \right) + g f^{abc} \partial_\mu \left( A_{\nu}^b G^{\mu\nu}_{\nu\rho}(x, x') \right) + g f^{abc} \partial_\nu \left( A_{\mu}^b G^{\mu\nu}_{\nu\rho}(x, x') \right) + g^2 f^{abc} f^{\alpha\beta\delta} G^{\mu\nu}_{\nu\rho}(x, x') A_{\alpha \mu} A_{\beta \nu} + g^2 f^{abc} f^{\alpha\beta\delta} G^{\mu\nu}_{\nu\rho}(x, x') A_{\alpha \mu} + g^2 f^{abc} f^{\alpha\beta\delta} G^{\mu\nu}_{\nu\rho}(x, x') \left( \delta_{\alpha\delta} \eta_{\mu\nu} \delta^4(x - x') \right) \quad (8)

or

$$\partial^2 G^{\mu\nu}_{\nu\rho}(x, x') - \left( 1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu G^{\mu\nu}_{\nu\rho}(x, x') + g f^{abc} \partial_\mu \left( \eta_{\mu}^b \chi(x) \right) \left( \delta_{\nu\rho} \eta_{\mu}^b \chi(x) \right) + g f^{abc} \partial_\nu \left( \eta_{\mu}^b \chi(x) \right) \left( \delta_{\nu\rho} \eta_{\mu}^b \chi(x) \right) + g f^{abc} \partial_\mu \left( \eta_{\mu}^b \chi(x) G^{\mu\nu}_{\nu\rho}(x, x') \right) + g f^{abc} \partial_\nu \left( \eta_{\mu}^b \chi(x) G^{\mu\nu}_{\nu\rho}(x, x') \right) + g^2 f^{abc} f^{\alpha\beta\delta} \partial_\mu \left( \eta_{\mu}^b \chi(x) G^{\mu\nu}_{\nu\rho}(x, x') \right) + g^2 f^{abc} f^{\alpha\beta\delta} \partial_\nu \left( \eta_{\mu}^b \chi(x) G^{\mu\nu}_{\nu\rho}(x, x') \right) + g^2 f^{abc} f^{\alpha\beta\delta} \eta_{\mu}^b \eta_{\delta}^a \delta^4(x - x'). \quad (9)$$

We fix the gauge to the Landau gauge ($\alpha = 1$) that also grants that we are using exact formulas rather than asymptotic ones. Then,

$$G_{\mu\nu}(x, x') = \delta_{\mu\nu} \left( g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) \Delta(x, x') \quad (10)$$

being $p_\mu$ the momentum vector. So, we have to solve the equation

$$\partial^2 \Delta(x, x') + 3 N g^2 \chi^2(x) \Delta(x, x') = \delta^4(x - x'). \quad (11)$$

This equation with that of the Green function for the scalar field obtained in [8]. Then, the propagator can be immediately written down as [11]. One has

$$G(p) = \sum_{n=0}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon} \quad (12)$$

with

$$B_n = (2n + 1)^2 \frac{\pi^3}{4K(1-1)} e^{-\left(n + \frac{1}{2}\right)\pi} \frac{3}{1 + e^{-2(n + 1)\pi}} \quad (13)$$

being $K(-1)$ the complete elliptic integral of the first kind and we get the “mass spectrum”

$$m_n = 2n + 1 \frac{\pi}{2K(-1)} (N g^2/2)^n. \quad (14)$$

This spectrum is kept in quantum field theory but we also obtain higher order corrections. So, our final result for the Green function is

$$G_{\mu\nu}(p) = \delta_{\mu\nu} \left( g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) G(p). \quad (15)$$

We see that the Yang-Mills theory shows up a mass gap. We will use the equation for the spectrum to fit with lattice data with a proper quantum correction.

The quantum theory can be studied using Dyson-Schwinger equations for 1- and 2-point functions. These were already discussed in our recent work [11]. One has

$$\partial^2 G^{\mu_1\nu_1}_{\nu_2\rho_2}(x) + g f^{\alpha\beta\gamma} \left( \partial_\mu G^{\alpha\beta}_{\mu_{\nu_1}}(0) + \partial_\mu G^{\beta\mu}_{\mu_{\nu_1}}(0) + \partial_\mu G^{\gamma\mu}_{\mu_{\nu_1}}(0) \right) - \partial_{\nu_1} G^{\alpha\beta}_{\mu_{\nu_2}}(0) - \partial_{\nu_2} G^{\beta\mu}_{\mu_{\nu_1}}(0) - \partial_{\nu_1} G^{\nu_2\rho_2}_{\mu_{\nu_1}}(0) = 0 \quad (16)$$
The Dyson-Schwinger equations for the two-point functions are
\[
\begin{align*}
\partial^2 G_{2\nu}(x - y) + g f^{abc} (\partial^\mu G_{2\mu}^{bc}(0, x - y)) \\
+ g f^{abc} (\partial^\mu G_{2\mu}^{bc}(0, x - y)) \\
- \partial_\nu G_{2\nu}(0, x - y) - \partial_\nu G_{2\nu}(0, x - y) G_{1\nu}(x) \\
- \partial_\nu G_{1\nu}(x) G_{2\nu}(x - y) \\
+ \partial^2 G_{2\nu}(x - y) + \partial^2 G_{2\nu}(x - y) G_{1\nu}(x) \\
+ \partial^2 G_{1\nu}(x) G_{2\nu}(x - y) \\
+ \gamma^2 f^{acd} G_{2\mu}(0, x - y) + G_{2\nu}(0, x - y) G_{1\nu}(x) \\
+ g f^{abc} (\partial^\mu G_{2\mu}^{bc}(0, x - y) + G_{2\nu}(0, x - y) G_{1\nu}(x)) \\
+ \partial^2 G_{2\nu}(x - y) + \partial^2 G_{2\nu}(x - y) G_{1\nu}(x) \\
+ \partial^2 G_{1\nu}(x) G_{2\nu}(x - y) \\
= g f^{abc} (\partial^\mu K_{2\mu}^{bc}(0, x - y) + \partial_\nu (P_{2\nu}(x) K_{2\nu}^{bc}(x - y))) \\
+ \partial_\nu (K_{2\nu}^{bc}(x - y) P_{2\nu}(x)) \\
+ \delta_{\mu\rho} G_{2\nu}(0, x - y) G_{1\nu}(x) \\
+ \partial^\rho G_{2\nu}(x - y) G_{1\nu}(x) \\
+ \partial^\rho (K_{2\nu}(x - y) P_{2\nu}(x)) \\
= g f^{abc} (\partial^\mu K_{2\mu}^{bc}(0, x - y) + \partial_\nu (P_{2\nu}(x) K_{2\nu}^{bc}(x - y))) \\
+ \partial_\nu (K_{2\nu}^{bc}(x - y) P_{2\nu}(x)) \\
+ \delta_{\mu\rho} G_{2\nu}(0, x - y) G_{1\nu}(x) \\
+ \partial^\rho G_{2\nu}(x - y) G_{1\nu}(x) \\
+ \partial^\rho (K_{2\nu}(x - y) P_{2\nu}(x)) \\
= g f^{abc} (\partial^\mu K_{2\mu}^{bc}(0, x - y) + \partial_\nu (P_{2\nu}(x) K_{2\nu}^{bc}(x - y))) \\
+ \partial_\nu (K_{2\nu}^{bc}(x - y) P_{2\nu}(x)) \\
+ \delta_{\mu\rho} G_{2\nu}(0, x - y) G_{1\nu}(x) \\
+ \partial^\rho G_{2\nu}(x - y) G_{1\nu}(x) \\
+ \partial^\rho (K_{2\nu}(x - y) P_{2\nu}(x)) \\
= 0.
\end{align*}
\]

The solutions to this set of equations can be obtained by choosing
\[
G_{2\nu}(x) = \eta_\nu^0 \phi(x)
\]
being \(\eta_\nu^0\) a set of constants and \(\phi(x)\) the solution of a differential equation we are going to determine. Besides, for the Fourier transform of the 2-point function is
\[
G_{2\nu}(p) = \delta_{\mu\nu} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Delta(p)
\]
with the equation for \(\Delta(x - y)\) given below. This set can be solved by taking for the ghost 2-point function
\[
P_{2\nu}(p) = \frac{\delta_{\mu\nu}}{p^2 + \mu^2}
\]
The ghost field decouples in this case and is free. Then, the 1-point function is obtained by the equation
\[
\eta_\nu^0 \partial^2 \phi(x) + g f^{abc} (\partial^\mu G_{2\mu}^{bc}(0, x - y)) \\
+ G_{2\mu}(0, x - y) \phi(x) + G_{2\mu}(0, x - y) \phi(x) \\
+ \eta_\mu \eta_\nu \phi^3(x) = 0.
\]
This becomes the equation for the scalar field, given SU(N) for the gauge group,
\[
\partial^2 \phi(x) + 2 N g^2 \delta \mu^2 \phi(x) + N g^2 \phi^3(x) = 0
\]
and we see that the mass correction \(\delta \mu^2\) is here too. This will provide an equation for the renormalization of mass. Solving eqs. (23), one has
\[
G_{1}(x) = \sqrt{\frac{2 \mu^2}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}}} \\
\times \left( p \cdot x + \chi, \frac{m^2 - \sqrt{m^4 + 2 N g^2 \mu^4}}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}} \right)
\]
being \(\mu\) and \(\chi\) two arbitrary integration constants and we have set \(m^2 = 2 N g^2 G_2(0), G_3(0, 0) = 0\) and taken the momenta \(p\) in such a way that
\[
p^2 = m^2 + \frac{N g^2 \mu^4}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}},
\]
From these results, we can obtain the correction to the mass spectrum by changing the modulus of the Jacobi elliptic functions and integrals going from \(k^2 = -1\) to \(k^2 = m^2 - \sqrt{m^2 + 2 N g^2 \mu^4}\) so that, given the dispersion relation in eq. (24), we will get
\[
m_{\nu}(m) = (2n + 1)^2 \pi \frac{N g^2 \mu^4}{2 K(k^2)} \left( m^2 + \frac{N g^2 \mu^4}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}} \right)
\]
and, as usual, \(\mu\) is an integration constant, having the dimensions of a mass, coming from the integration of the classical theory. This implies that the equation for \(\delta \mu^2\) will have the unknown on both sides. So, one can solve it iteratively by taking at the leading order \(\delta \mu^2 = 0\). One has from eq. (25)
\[
m^2 = 2 N g^2 \int \frac{d^4 p}{(2\pi)^4} \sum_{n=0}^{\infty} (2n + 1)^2 \frac{\pi^3}{4K(1)^{-1}} e^{-(n+\frac{1}{2})\pi} \times
\]
\[
\frac{1}{p^2 - m^2_0(0) + i0},
\]
(26)
This is the first iterate. In this way, we can evaluate a first correction to the mass spectrum. We just observe that this integral diverges. Indeed, this integral is very well-known in literature and can be evaluated by dimensional regularization. One will get for the finite part
\[
m^2 = N g^2 (\gamma - 1) \sum_{n=1}^{\infty} \frac{(2n+1)^2 \pi}{32 K^3(-1)} \frac{e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} m_n^2(0)
\]
being \(\gamma\) the Euler-Mascheroni constant that we will use to evaluate the full spectrum of the theory. It is interesting to note that this contribution is small and negative.

We can write it as
\[
m^2 = m^2 N g^2 \sigma
\]
where we have fixed the string tension to \(\sigma = \sqrt{N g^2/2 \mu^2}\) as usual. Then, \(m^2 = -0.03212775693 \ldots\) Considering this a small correction to the classical result, the spectrum \([27]\) can be expressed through the formula
\[
m_n \approx (2n + 1) \frac{\pi}{2 K(-1)} \times \left[ 1 + \left( \frac{1}{4} - \frac{1}{2} \frac{K(\sqrt{2}/2) - E(\sqrt{2}/2)}{K(\sqrt{2}/2)} \right) \frac{m^2 N g^2}{\beta} \right].
\]
Now, we can compare this spectrum with the result from lattice computations given in [1]. In order to do this, we can fix the value of \(N g^2\) with the value of \(\beta = 2N/g^2\) used in [1] to compute the spectrum. So, we rewrite
\[
m_n \approx (2n + 1) \frac{\pi}{2 K(-1)} \times \left[ 1 + \left( \frac{1}{4} - \frac{1}{2} \frac{K(\sqrt{2}/2) - E(\sqrt{2}/2)}{K(\sqrt{2}/2)} \right) \frac{m^2 N^2}{\beta} \right].
\]
This yields the comparison table [1] for the ground state 0++ of the theory as seen on lattice computations.

| \(N\) | Lattice | Theoretical | \(\beta\) | Error |
|-------|---------|-------------|--------|--------|
| 2     | 3.78    | 3.5509927197 | 2.4265 | 6%     |
| 3     | 3.55(7) | 3.556252384  | 6.0625 | 0.1%   |
| 4     | 3.56(6) | 3.556378900  | 11.085 | 0.1%   |
| 6     | 3.25(9) | 3.557102106  | 25.452 | 8.6%   |
| 8     | 3.56(12)| 3.558412808  | 45.70  | 0.2%   |

Table 1: Comparison for the ground state at varying \(N\). The lattice data are obtained from Ref. [1] for the continuum limit.

Also in this case the agreement is really stunning. We notice that, in our computations, the dependence on the degree of the group is weak but otherwise noticeable. This is due to the need to perform the comparison exactly for the same \(\beta\) as in [1] for consistency reasons.

We have shown how the spectrum of a Yang-Mills theory in four dimensions can be predicted with an exquisite precision granting strict agreement with lattice computations. This could pave the way for similar precise computations in QCD for the properties of hadrons and, more generally, to get predictions in the spectrum of this theory for exotic states.

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Table 2: Comparison for the 2++ state at varying \(N\). The lattice data are obtained from Ref. [1] for the continuum limit.