Starlikeness of a product of starlike functions with non-vanishing polynomials

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Abstract
For a function \( f \) starlike of order \( \alpha \), \( 0 \leq \alpha < 1 \), a non-constant polynomial \( Q \) of degree \( n \) which is non-vanishing in the unit disc \( D \) and \( \beta > 0 \), we consider the function \( F : D \to \mathbb{C} \) defined by
\[
F(z) = f(z)(Q(z))^{\beta/n}
\]
and find the largest value of \( r \in (0, 1] \) such that \( r^{-1}F(rz) \) lies in various known subclasses of starlike functions such as the class of starlike functions of order \( \lambda \), the classes of starlike functions associated with the exponential function, cardioid, a rational function, nephroid domain and modified sigmoid function. Our radii results are sharp. We also discuss the correlation with known radii results as special cases.

Keywords Univalence · Starlikeness · Convexity · Polynomials · Subordination · Radius problem

1 Introduction

We consider the class \( \mathcal{A} \) of all analytic functions defined on the unit disc \( D := \{ z : z < 1 \} \) and normalised by the condition \( f(0) = f'(0) - 1 = 0 \) as well as its subclass \( \mathcal{S} \) consisting of all univalent functions. For any two subclasses \( \mathcal{F} \) and \( \mathcal{G} \) of \( \mathcal{A} \), the \( \mathcal{G} \)-radius of the class \( \mathcal{F} \), denoted as \( R_{\mathcal{G}}(\mathcal{F}) \) is the largest number \( R_{\mathcal{G}} \in (0, 1) \) such that \( r^{-1}f(rz) \in \mathcal{G} \) for all \( f \in \mathcal{F} \) and \( 0 < r < R_{\mathcal{G}} \). In 1969, Başgöze [2] studied the radii of starlikeness and convexity for polynomial functions which are non-vanishing in the unit disc \( D \). This study was motivated by the work of Alexander [1] who showed that

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the radius of starlikeness and hence the radius of univalence for the function $f$ defined by $f(z) = zP(z)$, where $P$ is a polynomial of degree $n > 0$ with zeros outside the unit disc, is $(n + 1)^{-1}$. Further, Başgöze [3] also studied the radius results related to $\alpha$-spirallike and $\alpha$-convex spirallike functions of order $\lambda$ for various kinds of functions obtained from polynomials, such as $zP(z), P(z)^{\beta/n}, z(P(z))^{\beta/n}, zM(z)/N(z)$ where $P, M,$ and $N$ are polynomials with zeros outside the unit disc. In 2000, Gangadharan et al. [7] (see also Kwon and Owa [12]) determined the radius of $p$-valently strongly starlikeness of order $\gamma$ for the function $F : \mathbb{D} \to \mathbb{C}$ defined by $F(z) := f(z)(Q(z))^{\beta/n}$, where $f$ is a $p$-valent analytic function, $Q$ has properties similar to that of the polynomials considered in the paper by Başgöze, and $\beta$ is a positive real number. The present study continues this investigation to several recently studied subclasses of starlike functions defined by subordination.

For two analytic functions $f$ and $g$, $f$ is said to be subordinate to $g$, denoted by $f \prec g$, if $f = g \circ w$ for some analytic function $w : \mathbb{D} \to \mathbb{D}$, with $w(0) = 0$. When the function $g$ is univalent, the subordination $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Several subclasses of the class $A$ are defined using the concept of subordination. For an analytic function $\varphi : \mathbb{D} \to \mathbb{C}$, the class $S^*(\varphi)$ consists of all functions $f \in A$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

Shanmugam [18] studied convolution theorems for more general classes but with stronger assumption of convexity of $\varphi$. Ma and Minda [14] later gave a unified treatment of growth, distortion, rotation and coefficient inequalities for the class $S^*(\varphi)$ when the superordinate function $\varphi$ is a function with positive real part, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$. The class of starlike functions of order $\alpha, ST(\alpha)$ is a special case when $\varphi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$; for $\alpha = 0$, the usual class of starlike functions is obtained. Other subclasses of the class of starlike functions can also be derived for different choices of $\varphi$. For $\varphi_{\text{par}}(z) = 1 + (2/(\pi^2))(\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2$, the class $S_p = S^*(\varphi_{\text{par}})$ is the class of starlike functions associated with a parabola, the image of the unit disc under the function $\varphi_{\text{par}}$. The set $\varphi_{\text{par}}(\mathbb{D}) = \{w = u + iv : v^2 < 2u - 1\} = \{w : w - 1 < \text{Re } w\}$. This class was introduced by Rønning [17]. Mendiratta et al. [16] investigated the class of starlike functions associated with the exponential function $S^*_e = S^*(e^z)$. Similarly, various properties of the class of starlike functions associated with a cardioid, $S^*_c = S^*(\varphi_c(z))$ for $\varphi_c(z) = 1 + (4/3)z + (2/3)z^2$ are studied by Sharma et al. [19]. A function $f \in S^*_c$ if $zf'(z)/f(z)$ lies in the region bounded by the cardioid $\{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\}$. Kumar and Ravichandran [11] discussed the class $S^*_R = S^*(\psi(z))$ of starlike functions associated with a rational function for $\psi(z) = 1 + ((z^2 + kz)/(k^2 - kz)), k = \sqrt{2} + 1$. In 2020, Wani and Swaminathan [21] studied the class of starlike functions associated with a nephroid domain, $S^*_N = S^*(\varphi_N e^z)$ with $\varphi_N e(z) = 1 + z - z^3/3$. Goel and Kumar [8] explored various properties of the class $S^*_SG = S^*(2/(1 + e^{-z}))$ known as the class of starlike functions associated with modified sigmoid function. Radius problems relating to the ratio of analytic functions are recently considered in [13, 15].
In 2020, Wani and Swaminathan [20] discussed the radius problems for the functions associated with the nephroid domain. Cho and others have also investigated some interesting radius problems, see [4, 5].

For a given function $f$ starlike of order $\alpha$, a polynomial of degree $n$ whose zeros are outside the unit disk $\mathbb{D}$ (in other words, $Q$ is non-vanishing in $\mathbb{D}$) and $\beta$ a positive real number, we determine the $M$-radius of the function $F : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$F(z) := f(z)(Q(z))^{\beta/n}$$

for various choices of the class $M$. In Sect. 2, we determine the radius of starlikeness of order $\lambda$ for the function $F$, and obtain, as a special case, the radius of starlikeness for $F$. This is done by studying a mapping property of the function $w : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$w(z) := \frac{zF'(z)}{F(z)}.$$ 

Indeed, we find the smallest disc containing $w(D_r)$ where $D_r := \{z \in \mathbb{C} : z < r\}$. This disc is used in the investigation of all the radius problems. In Sects. 3 and 4, we respectively compute the values of the radius of starlikeness associated with the exponential function and the radius of starlikeness associated with a cardioid for the function $F$. In Sects. 5–7, we determine the radius of starlikeness associated with a particular rational function, the radius of starlikeness associated with nephroid domain and the radius of starlikeness associated with modified sigmoid function for the function $F$. All the radii obtained are shown to be sharp. Several known radii results are obtained as special cases for specific values of $\alpha$ and $\beta$.

## 2 Starlike functions of order $\lambda$

For $0 \leq \lambda < 1$, the class $ST(\lambda)$ of starlike functions of order $\lambda$ contains all functions $f \in A$, satisfying $\text{Re}(zf'(z)/f(z)) > \lambda$. Let $f \in ST(\alpha)$ and $Q$ be a polynomial of degree $n > 0$ with all of its zeros in the region $\mathbb{C} \setminus \mathbb{D}$. Since $Q$ is non-vanishing in $\mathbb{D}$, the function $F : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$F(z) := f(z)(Q(z))^{\beta/n}, \quad \beta > 0$$ 

(2.1)

is analytic in $\mathbb{D}$. We assume $Q(0) = 1$ throughout this paper so that $F \in A$. The following theorem gives the $ST(\lambda)$-radius for the function $F$ and it is independent of the degree of the polynomial $Q$.

**Theorem 1** If the function $f \in ST(\alpha)$, then the radius of starlikeness of order $\lambda$ for the function $F$ defined in (2.1) is given by

$$R_{ST(\lambda)}(F) = \frac{2(1 - \lambda)}{2 - 2\alpha + \beta + \sqrt{(2 - 2\alpha + \beta)^2 + 4(1 - \lambda)(2\alpha + \beta - 1 - \lambda)}}. \quad (2.2)$$

**Proof** We start with finding the disc in which $zF'(z)/F(z)$ lies for $z \in \overline{D}_r$, then using this disc, we determine the radius of starlikeness of order $\lambda$ for $F$.

For the function $F$ given by (2.1), a calculation shows that

$$\frac{zF'(z)}{F(z)} = \frac{zf'(z)}{f(z)} + \frac{\beta}{n} \frac{zQ'(z)}{Q(z)}. \quad (2.2)$$
Since \( f \in ST(\alpha) \), it is well-known that \( \frac{zf'(z)}{f(z)} \) has positive real part and so
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2}, \quad z \leq r. \tag{2.3}
\]
or equivalently \( \frac{zf'(z)}{f(z)} \) lies in the disc \( \mathbb{D}(a_f(r); c_f(r)) \) for \( z \leq r \) where
\[
a_f(r) = \frac{1 + (1 - 2\alpha)r^2}{1 - r^2}, \quad c_f(r) = \frac{2(1 - \alpha)r}{1 - r^2}. \tag{2.4}
\]
Let \( z_k, \ k = 1, 2, \ldots, n \) denote the zeros of the polynomial \( Q \), then the polynomial \( Q \) is a constant multiple of \( \prod_{k=1}^n (z - z_k) \) and so
\[
\frac{zQ'(z)}{Q(z)} = \sum_{k=1}^n \frac{z}{z - z_k}. \tag{2.5}
\]
Since \( z_k \in \mathbb{C}\setminus\mathbb{D} \) for every \( k \), the bilinear transformation \( z/(z - z_k) \) maps \( \overline{\mathbb{D}}_r \) to a disc. Indeed, [6, Lemma 3.2] shows that
\[
\left| \frac{z}{z - z_k} + \frac{r^2}{1 - r^2} \right| \leq \frac{r}{1 - r^2}, \quad z \leq r
\]
for every \( k \) and hence, using (2.5), we have
\[
\left| \frac{zQ'(z)}{Q(z)} + \frac{nr^2}{1 - r^2} \right| \leq \frac{nr}{1 - r^2}, \quad z \leq r. \tag{2.6}
\]
Using (2.2), we get
\[
\left| \frac{zF'(z)}{F(z)} - \frac{1 - (2\alpha - 1 + \beta)r^2}{1 - r^2} \right|
= \left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} + \frac{\beta zQ'(z)}{nQ(z)} + \frac{\beta r^2}{1 - r^2} \right|
\leq \left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| + \left| \frac{\beta zQ'(z)}{nQ(z)} + \frac{\beta r^2}{1 - r^2} \right|, \tag{2.7}
\]
for \( z \leq r \). Since \( \beta \in \mathbb{R} \) is positive, using the Eqs. (2.3) and (2.6), the inequality (2.7) gives
\[
\left| \frac{zF'(z)}{F(z)} - \frac{1 - (2\alpha - 1 + \beta)r^2}{1 - r^2} \right| \leq \frac{(2 - 2\alpha + \beta)r}{1 - r^2}, \quad z \leq r. \tag{2.8}
\]
Define the functions \( a_F \) and \( c_F \) by
\[
a_F(r) := \frac{1 - (2\alpha - 1 + \beta)r^2}{1 - r^2} \quad \text{and} \quad c_F(r) := \frac{(2 - 2\alpha + \beta)r}{1 - r^2},
\]
so that \( zF'(z)/F(z) \in \mathbb{D}(a_F(r); c_F(r)) \). It is observed that the center \( a_F \) is an increasing function of \( r \) for \( 2\alpha + \beta - 2 < 0 \), and is a decreasing function of \( r \) for \( 2\alpha + \beta - 2 \geq 0 \).

From (2.8), it follows that
\[
\operatorname{Re} \frac{zF'(z)}{F(z)} \geq a_F(r) - c_F(r)
\]
\[
= \frac{1 - (2\alpha - 1 + \beta)r^2}{1 - r^2} - \frac{(2 - 2\alpha + \beta)r}{1 - r^2}
\]
\[
= \frac{1 - (1 - 2\alpha)r}{1 + r} - \frac{\beta r}{1 - r} =: \psi(r).
\]

The equation \( \psi(\alpha, \beta, r) = \lambda \) is simplifies to
\[
(1 - 2\alpha - \beta + \lambda)r^2 - (2 - 2\alpha + \beta)r + 1 - \lambda = 0 \quad (2.10)
\]
and so the smallest positive root of the equation \( \psi(\alpha, \beta, r) = \lambda \) in the interval (0, 1) is given by
\[
\sigma_0 := \frac{2(1 - \lambda)}{2 - 2\alpha + \beta + \sqrt{(2 - 2\alpha + \beta)^2 + 4(1 - \lambda)(2\alpha + \beta - 1 - \lambda)}}.
\]

It can be seen that
\[
\psi'(r) = -\frac{2(1 - \alpha)(1 - r)^2 + \beta(1 + r)^2}{(1 - r^2)^2} < 0,
\]
which shows that \( \psi \) is a decreasing function of \( r \in (0, 1) \). Therefore, for \( r < \sigma_0 \), we have \( \psi(\alpha, \beta, r) > \psi(\alpha, \beta, \sigma_0) = \lambda \) and so (2.9) implies that \( \operatorname{Re}(zF'(z)/F(z)) \geq \psi(\alpha, \beta, r) > \lambda \) for all \( r < \sigma_0 \), or in other words, the radius of starlikeness of order \( \lambda \) of the function \( F \) is at least \( \sigma_0 \).

To show that the radius obtained is the best possible, take \( f(z) = z(1 - z)^{2\alpha - 2} \in ST(\alpha) \) and the polynomial \( Q(z) = (1 + z)^n \). For these choices of the functions \( f \) and \( Q \), we have
\[
F(z) = z(1 - z)^{2\alpha - 2}(1 + z)^{\beta},
\]
which implies
\begin{align}
  \frac{zF'(z)}{F(z)} &= 1 + \frac{(2 - 2\alpha)z}{1 - z} + \frac{\beta z}{1 + z} \\
  &= \frac{(1 - 2\alpha - \beta)z^2 + (2 - 2\alpha + \beta)z + 1}{1 - z^2} \\
  &= \lambda + \frac{(1 - 2\alpha - \beta + \lambda)z^2 + (2 - 2\alpha + \beta)z + 1 - \lambda}{1 - z^2}.
\end{align}
(2.11)

Using the fact that \( \sigma_0 \) is the positive root of the polynomial in (2.10), the equation (2.11) shows that \( \Re(zF'(z)/F(z)) = \lambda \) for \( z = -\sigma_0 \) proving sharpness of \( \sigma_0 \).

For \( \lambda = 0 \), the radius of starlikeness for the function \( F \) given by (2.1) is
\[ R_{ST}(F) = \frac{2}{2 - 2\alpha + \beta + \sqrt{(2 - 2\alpha + \beta)^2 + 4(2\alpha + \beta - 1)}}. \]

When \( \alpha \) goes to 1 in Theorem 1 we get that the radius of starlikeness of order \( \lambda \) for the function \( F(z) = z(Q(z))^{\beta/n} \), where \( Q \) is a non-constant polynomial of degree \( n \) non-vanishing on the unit disc and \( \beta > 0 \), comes out to be \( (1 - \lambda)/(\beta + 1 - \lambda) \). This result for \( \beta = n \) coincides with the one obtained by Başgöze in [2, Theorem 3]. Moreover, by letting \( \beta \to 0 \) in Theorem 1, we obtain the radius of starlikeness of order \( \lambda \) for the class of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) (see [9, p. 88]).

### 3 Starlike functions associated with the exponential function

The class \( S^*_e \) of starlike functions associated with the exponential functions consists of all the functions \( f \in A \) which satisfy \( zf'(z)/f(z) < e^z \). This class was introduced and studied by Mendiratta et al. [16] in 2015. The subordination definition is equivalent to the inequality \( |\log(zf'(z)/f(z))| < 1 \).

The main result of the section provides the \( S^*_e \) radius for the function \( F \) given by (2.1).

**Lemma 2** [16, Lemma 2.2] For \( 1/e < a < e \), let \( r_a \) be given by
\[ r_a = \begin{cases} 
  a - \frac{1}{e} & \text{if } \frac{1}{e} < a \leq \frac{e + e^{-1}}{2} \\
  e - a & \text{if } \frac{e + e^{-1}}{2} \leq a < e.
\end{cases} \]

Then, \( \{ w : w - a < r_a \} \subset \{ w : |\log w| < 1 \} = \Omega_e \), where \( \Omega_e \) is the image of the unit disc \( \mathbb{D} \) under the exponential function.
Theorem 3 \textit{If the function }$f \in ST(\alpha)$, \textit{then the radius of starlikeness associated with the exponential function for the function }$F$\textit{ defined in (2.1) is given by}

$$R_{S_{e}^{+}}(F) = \begin{cases} \sigma_{0} & \text{if } 2\alpha + \beta - 2 \geq 0 \\ \sigma_{0} & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) \leq 0 \\ \tilde{\sigma}_{0} & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) > 0, \end{cases}$$

where

$$\sigma_{0} = \frac{2(e - 1)}{e(2 - 2\alpha + \beta) + \sqrt{(e(2 - 2\alpha + \beta))^{2} - 4(e - 1)(1 - e(2\alpha - 1 + \beta))}},$$

$$\tilde{\sigma}_{0} = \frac{2(e - 1)}{(2 - 2\alpha + \beta) + \sqrt{(2 - 2\alpha + \beta)^{2} - 4(e - 1)(2\alpha - 1 + \beta - e)}}.$$ 

and

$$X(\alpha, \beta) = 2(\beta - 2\alpha)(1 + e^{2} - 2e(2\alpha + \beta - 1))((2\alpha + 2 \beta)$$

$$- \sqrt{(2\alpha + 2 \beta)^{2} - 4(e - 1)(2\alpha - 1 + \beta - e)})$$

$$+ 4(2\alpha + \beta - 1 - e)(e^{2} - 1)(2\alpha + \beta - 2). \quad (3.1)$$

\textbf{Proof} \textit{Our aim is to show that the }$\mathbb{D}(a_{F}(r); c_{F}(r)) \subset \Omega_{e}$\textit{ for all }$0 < r \leq R_{S_{e}^{+}}(F)$. \textit{Let}

$$\sigma_{0} := \frac{2(e - 1)}{e(2 - 2\alpha + \beta) + \sqrt{(e(2 - 2\alpha + \beta))^{2} - 4(e - 1)(1 - e(2\alpha - 1 + \beta))}},$$

and

$$\tilde{\sigma}_{0} := \frac{2(e - 1)}{(2 - 2\alpha + \beta) + \sqrt{(2 - 2\alpha + \beta)^{2} - 4(e - 1)(2\alpha - 1 + \beta - e)}}.$$ 

It is clear that $\sigma_{0}$ and $\tilde{\sigma}_{0}$ are both positive as both $2 - 2\alpha + \beta$ and $e - 1$ are positive. \textit{For the polynomial}

$$\phi(r) := (1 - e(-1 + 2\alpha + \beta))r^{2} - e(2 - 2\alpha + \beta)r + (e - 1) \quad (3.2)$$

obtained from the equivalent form $\phi(r) = 0$ of the equation $c_{F}(r) = a_{F}(r) - 1/e$, \textit{it is observed that }$\phi(0) = e - 1 > 0$, $\phi(1) = -2e\beta < 0$, \textit{showing that there is a zero for }$\phi$\textit{ in the interval }$(0, 1)$, \textit{namely }$\sigma_{0}$. \textit{Also, the positive root of the equation }$c_{F}(r) = e - a_{F}(r)$\textit{ or }$\psi(r) = 0$\textit{ with}

$$\psi(r) := (-1 + 2\alpha + \beta - e)r^{2} - (2 - 2\alpha + \beta)r + (e - 1) \quad (3.3)$$

is $\tilde{\sigma}_{0}$. \textit{To verify that }$\psi$\textit{ has a zero in the interval }$(0, 1)$, \textit{it is seen that }$\psi(0) = e - 1 > 0$ \textit{and }$\psi(1) = 4(\alpha - 1) < 0$. 
Case (i): $2\alpha + \beta - 2 \geq 0$. Since the center $a_F$ in (2.8) is a decreasing function of $r$, $a_F(r) > a_F(\sigma_0)$, for $r \in (0, \sigma_0)$. By definition, $\sigma_0$ is the solution of the equation $c_r(\sigma) = a_F(\sigma) - 1/e$, also, the radius in (2.8) satisfies $c_r(\sigma) > 0$, $r \in (0, 1)$, together imply that $a_F(r) > a_F(\sigma_0) > 1/e$, for $r \in (0, \sigma_0)$. Further, $a_F(r) < a_F(0) = 1 < (e + e^{-1})/2 \approx 1.54308$, for $r \in (0, \sigma_0)$. Thus, it is established that

$$1/e < a_F(r) \leq \frac{e + e^{-1}}{2e}, \text{ for } r \in (0, \sigma_0).$$

Applying Lemma 2 we get $\mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subset \Omega_e$ that is, the radius of starlikeness associated with the exponential function for the function $F$ is atleast $\sigma_0$.

Case (ii): $2\alpha + \beta - 2 < 0$ and $X(\alpha, \beta) \leq 0$. The number

$$\tilde{\sigma}_1 = \sqrt{\frac{1 - 2e + e^2}{1 + 2e + e^2 - 4e\alpha - 2e\beta}} < 1$$

is the positive root of the equation $a_F(r) = (e + e^{-1})/2$, or equivalently, $\zeta(r) = 0$ with $\zeta(r) := (1 + e^2 - 2e(2\alpha + \beta - 1))r^2 + 2e - e^2 - 1$. Here, $\zeta(0) = 2e - e^2 - 1 \approx -2.9525 < 0$ and $\zeta(1) = -2e(2\alpha + \beta - 2) > 0$ justifies the existence of a zero for the polynomial $\zeta$ in the interval $(0, 1)$. Further,

$$\zeta(\tilde{\sigma}_0) = \frac{1}{4(-1 - e + 2\alpha + \beta)^2} \left(2(2 + \beta - 2\alpha)(1 + e^2 - 2e(2\alpha + \beta - 1)) \right.$$

$$\left.\times ((-2\alpha + 2 + \beta) - \sqrt{(-2\alpha + 2 + \beta)^2 - 4(1 - e^2)(2\alpha - 1 + \beta - e)} + 4(2\alpha + \beta - 1 - e)(e^2 - 1)(2\alpha + \beta - 2)) \right),$$

(3.4)

and from this with (3.1) we infer that $X(\alpha, \beta) \leq 0$ implies $\zeta(\tilde{\sigma}_0) \leq 0$. Also, $\zeta(0) < 0$, $\zeta(\tilde{\sigma}_0) \leq 0$ along with the fact $\zeta(\tilde{\sigma}_1) = 0$ gives $\tilde{\sigma}_0 \leq \tilde{\sigma}_1$, which implies that $a_F(\tilde{\sigma}_0) \leq a_F(\tilde{\sigma}_1) = (e + e^{-1})/2$. The application of Lemma 2 gives $\mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subset \Omega_e$ or in other words, the radius of starlikeness associated with the exponential function for $F$ is atleast $\sigma_0$.

Case (iii): $2\alpha + \beta - 2 < 0$ and $X(\alpha, \beta) > 0$. Here following the same line of thought as in Case (ii), $\zeta(\tilde{\sigma}_0) > 0$ implies $a_F(\tilde{\sigma}_0) > (e + e^{-1})/2$, and thus Lemma 3.1 gives the required radius to be at least $\tilde{\sigma}_0$.

To show that the obtained radius values are the best possible, take $f(z) = z/(1 - z)^{2\alpha + 2} \in ST(\alpha)$ and the polynomial $Q(z) = (1 + z)^n$, these choices give the expression for $zF'(z)/F(z)$, as already shown in the proof of Theorem 1, to be

$$\frac{zF'(z)}{F(z)} = \frac{(1 - 2\alpha - \beta)z^2 + (1 + (1 - 2\alpha + \beta)z + 1}{1 - z^2} = \frac{e - ((1 + 2\alpha + \beta - e)z^2 - (2 - 2\alpha + \beta)z + e - 1}{1 - z^2}. \quad (3.5)$$
It is seen that (3.5) can also be written as
\[
\frac{zF'(z)}{F(z)} = \frac{1}{e} + \frac{(1 - e(-1 + 2\alpha + \beta))z^2 + e(2 - 2\alpha + \beta)z + e - 1}{e(1 - z^2)}. \tag{3.6}
\]
The definition of the polynomial \( \varphi \) in (3.2) for \( r = \sigma_0 \) together with (3.6) gives
\[
\left| \log \frac{-\sigma_0 F'(-\sigma_0)}{F(-\sigma_0)} \right| = \left| \log \frac{1}{e} \right| = 1,
\]
proving sharpness for \( \sigma_0 \). Further, the polynomial \( \psi \) in (3.3) for \( r = \tilde{\sigma}_0 \) and (3.5) provide
\[
\left| \log \frac{\tilde{\sigma}_0 F' (\tilde{\sigma}_0)}{F (\tilde{\sigma}_0)} \right| = |\log e| = 1.
\]
This proves sharpness for \( \tilde{\sigma}_0 \). \( \Box \)

If we let \( \alpha \) goes to 1 in Theorem 3, then the radius of starlikeness associated with the exponential function for the function \( F(z) = z(Q(z))^{\beta/n} \) where \( Q \) is a non-constant polynomial of degree \( n \) non-vanishing on the unit disc and \( \beta > 0 \), comes out to be \((e - 1)/(e\beta + e - 1)\). Moreover, when \( \beta \to 0 \) in Theorem 3, we obtain the radius of starlikeness associated with the exponential function for the class of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) obtained by Mendiratta et al. in [16, Theorem 3.4] and also by Khatter et al. in [10, Theorem 2.17 (2)] for \( S_{0,e}^* \) with \( A = 1 - 2\alpha \) and \( B = -1 \).

### 4 Starlike functions associated with a cardioid

Sharma et al. [19] studied the class \( S^*_{c} = S^*(\varphi_c), \varphi_c(z) = (1 + (4/3)z + (2/3)z^2) \) of starlike functions associated with a cardioid and proved the following lemma.

**Lemma 4** [19, lemma 2.5] For \( 1/3 < a < 3 \),
\[
r_a = \begin{cases} 
\frac{3a-1}{3} & \text{if } \frac{1}{3} < a \leq \frac{5}{3} \\
3 - a & \text{if } \frac{5}{3} \leq a < 3.
\end{cases}
\]

Then \{ \( w : w-a < r_a \} \subset \Omega_c \). Here \( \Omega_c \) is the region bounded by the cardioid \( \{ x + iy : (9x^2 + 9y^2 - 16x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0 \} \).

The following theorem gives the radius of starlikeness associated with a cardioid for the function \( F \) given in (2.1).

**Theorem 5** Let the function \( f \in ST(\alpha) \), the \( S^*_c \) radius for the function \( F \) defined in (2.1) is
\[
R_{S^*_c}(F) = \begin{cases} 
\sigma_0 & \text{if } 2\alpha + \beta - 2 \geq 0 \\
\sigma_0 & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) \leq 0 \\
\tilde{\sigma}_0 & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) > 0,
\end{cases}
\]
where
\[
\sigma_0 = \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{(6 - 6\alpha + 3\beta)^2 + 8(6\alpha + 3\beta - 4)}},
\]
\[
\tilde{\sigma}_0 = \frac{4}{(2 - 2\alpha + \beta) + \sqrt{(2 - 2\alpha + \beta)^2 - 8(2\alpha + \beta - 4)}}
\]
and
\[
X(\alpha, \beta) = 2(8 - 6\alpha - 3\beta)(6\alpha - 6 - 3\beta)
\]
\[
+ \sqrt{(6 - 6\alpha + 3\beta)^2 + 8(6\alpha + 3\beta - 4))}
\]
\[
+ 48(6\alpha + 3\beta - 4)(2 - 2\alpha - \beta).
\] (4.1)

**Proof** Consider the disc mentioned in (2.8), it needs to be shown that this disc lies in the region \(\Omega_1\) for all \(0 < r \leq R_{S^*}(F)\). Take the equations
\[
c_F(r) = a_F(r) - 1/3
\]
and
\[
a_F(\sigma_0) = 3 - a_F(r);
\]
which are equivalent to \(\phi(r) = 0\) and \(\psi(r) = 0\) respectively, with the corresponding polynomials \(\phi\) and \(\psi\) given by
\[
\phi(r) := (3(2\alpha - 1 + \beta) - 1)r^2 + 3(2 - 2\alpha + \beta)r - 2,
\] (4.2)
and
\[
\psi(r) := (2\alpha - 1 + \beta - 3)r^2 - (2 - 2\alpha + \beta)r + 2.
\] (4.3)

For the polynomial \(\phi\), \(\phi(0) = -2 < 0, \ \phi(1) = 6\beta > 0;\) thus \(\phi\) has a zero in the interval \((0, 1)\), let it be denoted by \(\sigma_0\). Also, considering the polynomial \(\psi\), it is seen that \(\psi(0) = 2 > 0\) and \(\psi(1) = 4(\alpha - 1) < 0\), let the zero of \(\psi\) in the interval \((0, 1)\) be denoted by \(\tilde{\sigma}_0\). Then it is obtained that
\[
\sigma_0 := \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{(6 - 6\alpha + 3\beta)^2 + 8(6\alpha + 3\beta - 4)}},
\]
and
\[
\tilde{\sigma}_0 := \frac{4}{(2 - 2\alpha + \beta) + \sqrt{(2 - 2\alpha + \beta)^2 - 8(2\alpha + \beta - 4)}}
\]

Here, it is evident from their values that both \(\sigma_0\) and \(\tilde{\sigma}_0\) are indeed positive.

Case (i): \(2\alpha + \beta - 2 \geq 0\). First it is shown that the center \(a_F\) in (2.8) satisfies
\[
\frac{1}{3} < a_F(r) < \frac{5}{3}, \ r \in (0, \sigma_0).
\] (4.4)

The fact that the center is a decreasing function of \(r\), implies \(a_F(r) > a_F(\sigma_0)\) for \(r \in (0, \sigma_0)\). Also, \(\sigma_0\) is the root of the equation \(a_F(r) - 1/3 = c_F(r)\), along with the fact that
the radius satisfies $c_F(r) > 0$, $r \in (0, 1)$ gives $a_F(r) > a_F(\sigma_0) > 1/3$ for $r \in (0, \sigma_0)$. Further, $a_F(r) < a_F(0) = 1 < 5/3$, $r \in (0, \sigma_0)$. This proves (4.4) and applying Lemma 4, infers that $\mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subset \Omega_c$, that is, the radius of starlikeness associated with a cardioid for the function $F$ is at least $\sigma_0$.

Case (ii): $2\alpha + \beta - 2 < 0$ and $X(\alpha, \beta) \leq 0$. The equation $a_F(r) = 5/3$ from Lemma 4, takes the form $\zeta(r) = 0$, with $\zeta(r) := (8 - 6\alpha - 3\beta)r^2 - 2$. Then, it is seen that $\zeta(0) = -2 < 0$ and $\zeta(1) = -3(2\alpha + \beta - 2) > 0$. This shows that $\zeta$ has a zero in the interval $(0, 1)$, let this be denoted by $\tilde{\sigma}_1$, then

$$\tilde{\sigma}_1 = \sqrt{\frac{2}{8 - 6\alpha - 3\beta}}.$$

Also since

$$\zeta(\sigma_0) = \frac{1}{4(-4 + 6\alpha + 3\beta)^2}(2(8 - 6\alpha - 3\beta)(6\alpha - 6 - 3\beta)(6\alpha - 6 - 3\beta)$$

$$+ \sqrt{(6 - 6\alpha + 3\beta)^2 + 8(6\alpha + 3\beta - 4))} + 48(6\alpha + 3\beta - 4)(2 - 2\alpha - \beta), \quad (4.5)$$

from (4.1) and (4.5) it is evident that $X(\alpha, \beta) \leq 0$ is equivalent to saying $\zeta(\sigma_0) \leq 0$. The facts that $\zeta(0) < 0$, $\zeta(\sigma_0) \leq 0$ and $\zeta(\tilde{\sigma}_1) = 0$ together imply that $\sigma_0 \leq \tilde{\sigma}_1$. Further, $\sigma_0 \leq \tilde{\sigma}_1$ will imply that $a_F(\sigma_0) \leq a_F(\tilde{\sigma}_1) = 5/3$, and thus using Lemma 4, $\mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subset \Omega_c$. Thus, proving that the required radius value is at least $\sigma_0$.

Case (iii): $2\alpha + \beta - 2 < 0$ and $X(\alpha, \beta) > 0$. On the similar lines as in Case (ii), $X(\alpha, \beta) > 0$ implies $\zeta(\sigma_0) > 0$, which gives that $a_F(\sigma_0) > 5/3$, this in turn, after another application of Lemma 4 concludes that the radius of starlikeness associated with a cardioid for the function $F$ is at least $\tilde{\sigma}_0$.

To verify the sharpness of the obtained radius values, take $f(z) = z/(1-z)^{-2\alpha+2} \in \mathcal{S}(\alpha)$ and the polynomial $Q$ as $Q(z) = (1 + z)^n$. Thus, the expression for $zF'(z)/F(z)$ as seen in the proof for Theorem 1, becomes

$$\frac{zF'(z)}{F(z)} = (1 - 2\alpha - \beta)z^2 + (2 - 2\alpha + \beta)z + 1. \quad (4.6)$$

The polynomial $\phi$ in (4.2) gives that for $r = \sigma_0$,

$$3((1 - 2\alpha - \beta)r^2 - (2 - 2\alpha + \beta)r + 1) = 1 - r^2. \quad (4.7)$$

Thus, the sharpness for $\sigma_0$ is proved by using (4.6) and (4.7) which gives that $zF'(z)/F(z) = 1/3 = \varphi_c(-1)$ for $z = -\sigma_0$. Also, the polynomial $\psi$ in (4.3) for $r = \tilde{\sigma}_0$ gives

$$(1 - 2\alpha - \beta)r^2 + (2 - 2\alpha + \beta)r + 1 = 3(1 - r^2). \quad (4.8)$$
Thus, using (4.8) in (4.6) it is seen that $\tilde{\sigma}_0 F'(\tilde{\sigma}_0) / F(\tilde{\sigma}_0) = 3 = \varphi_\varepsilon(1)$. This proves the sharpness of $\tilde{\sigma}_0$. \hfill \Box

If we let $\alpha$ goes to 1 in Theorem 5, then we obtain that the radius of starlikeness associated with a cardioid for the function $F(z) = z(Q(z))^{\beta/n}$, where $Q$ is a non-constant polynomial of degree $n$ non-vanishing on the unit disc and $\beta > 0$, comes out to be $2/(2 + 3\beta)$. Moreover, by letting $\beta \to 0$ in Theorem 5, we get the radius of starlikeness associated with a cardioid ([19, Theorem 4.7] with $A = 1 - 2\alpha$ and $B = -1$) for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$.

5 Starlike functions associated with a rational function

Kumar and Ravichandran [11] introduced the class of starlike functions associated with the rational function $\varphi_R(z) = 1 + ((z^2 + kz)/(k^2 - kz))$, with $k = \sqrt{2} + 1$. This class is represented by $S^*_R = S^*_R(\varphi_R(z))$. They also showed the following result, which is used in finding the $S^*_R$ radius for the function $F$ defined in (2.1).

Lemma 6 [11, lemma 2.2] For $2(\sqrt{2} - 1) < a < 2$,

$$r_a = \begin{cases} a - 2(\sqrt{2} - 1) & \text{if } 2(\sqrt{2} - 1) < a \leq \sqrt{2} \\ 2 - a & \text{if } \sqrt{2} \leq a < 2. \end{cases}$$

Then $\{ w : w - a < r_a \} \subset \varphi_R(\mathbb{D})$.

Theorem 7 If the function $f \in ST(\alpha)$, then the $S^*_R$ radius for the function $F$ defined in (2.1) is given by

$$R_{S^*_R}(F) = \begin{cases} \sigma_0 & \text{if } 2\alpha + \beta - 2 \geq 0 \\ \sigma_0 & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) \leq 0 \\ \tilde{\sigma}_0 & \text{if } 2\alpha + \beta - 2 < 0 \text{ and } X(\alpha, \beta) > 0, \end{cases}$$

where

$$\sigma_0 = \frac{2(3 - 2\sqrt{2})}{(2 - 2\alpha + \beta) + \sqrt{(-2 + 2\alpha - \beta)^2 - 4(3 - 2\sqrt{2})(2\sqrt{2} - 1 - 2\alpha - \beta)}},$$

$$\tilde{\sigma}_0 = \frac{2}{(2 - 2\alpha + \beta) + \sqrt{(-2 + 2\alpha - \beta)^2 - 4(2\alpha - 3 + \beta)}},$$

and

$$X(\alpha, \beta) = 2(2\alpha - 2 - \beta)(1 + \sqrt{2} - 2\alpha - \beta)((2\alpha - 2 - \beta)\sqrt{2} - 1 - 2\alpha - \beta) + \sqrt{(2\alpha - 2 - \beta)^2 - 4(3 - 2\sqrt{2})(2\sqrt{2} - 1 - 2\alpha - \beta)} + 4(1 - 2\sqrt{2} + 2\alpha + \beta)((3 - 2\sqrt{2})(1 + \sqrt{2} - 2\alpha - \beta) + (1 - \sqrt{2})(1 - 2\sqrt{2} + 2\alpha + \beta)).$$

(5.1)
Proof We show that the disc mentioned in (2.8) satisfies \( \mathbb{D}(a_F(r); c_F(r)) \subset \varphi_R(\mathbb{D}) \) for all \( 0 < r \leq R_{S_F^r}(F) \). Lemma 6 gives that the two possible values of the radius are the smallest positive roots of the equations \( c_F(r) = a_F(r) - 2(\sqrt{2} - 1) \) and \( c_F(r) = 2 - a_F(r) \); which are equivalent to \( \phi(r) = 0 \) and \( \psi(r) = 0 \) respectively, where the polynomials in \( r \) are of the form

\[
\phi(r) : = (2(\sqrt{2} - 1) - (2\alpha - 1 + \beta))r^2 - (2 - 2\alpha + \beta)r + 3 - 2\sqrt{2}, \tag{5.2}
\]

and

\[
\psi(r) : = (2\alpha - 3 + \beta)r^2 - (2 - 2\alpha + \beta)r + 1 \tag{5.3}
\]

respectively. The fact that both the polynomials \( \phi \) and \( \psi \) possess zeros in the interval \((0, 1)\) can be easily verified as \( \phi(0) = 3 - 2\sqrt{2} > 0 \) and \( \phi(1) = -2\beta < 0 \), also, \( \psi(0) = 1 > 0, \ \psi(1) = 4(\alpha - 1) < 0 \). The respective positive zeros of \( \phi \) and \( \psi \), denoted by \( \sigma_0 \) and \( \tilde{\sigma}_0 \), are given by

\[
\sigma_0 := \frac{2(3 - 2\sqrt{2})}{(2 - 2\alpha + \beta) + \sqrt{(-2 + 2\alpha - \beta)^2 - 4(3 - 2\sqrt{2})(2\sqrt{2} - 1 - 2\alpha - \beta)}}.
\]

and

\[
\tilde{\sigma}_0 := \frac{2}{(2 - 2\alpha + \beta) + \sqrt{(-2 + 2\alpha - \beta)^2 - 4(2\alpha - 3 + \beta)}}.
\]

Case (i): \( 2\alpha + \beta - 2 \geq 0 \). By using the fact that the center \( a_F(r) \) is decreasing function of \( r \), it will be proved that

\[
2(\sqrt{2} - 1) < a_F(r) \leq \sqrt{2}, \ r \in (0, \sigma_0). \tag{5.4}
\]

This after the application of Lemma 5.1 will directly imply \( \mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subset \varphi_R(\mathbb{D}) \), that is, the required radius is atleast \( \sigma_0 \). So, to prove (5.4), observe that \( a_F(r) > a_F(\sigma_0) \) for \( r \in (0, \sigma_0) \), and \( \sigma_0 \) is the positive root of the equation \( c_F(r) = a_F(r) - 2(\sqrt{2} - 1) \), also since the radius \( c_F(r) \) in (2.8) is positive for all \( r \in (0, 1) \), \( a_F(\sigma_0) - 2(\sqrt{2} - 1) > 0 \). Lastly, \( a_F(r) < a_F(0) = 1 < \sqrt{2}, \ r \in (0, \sigma_0) \), thus proving (5.4), and also the required result for this case.

Case (ii): \( 2\alpha + \beta - 2 < 0 \) and \( X(\alpha, \beta) \leq 0 \). Here, again from the Lemma 6, consider the equation \( a_F(r) = \sqrt{2} \), which gets simplified to \( \zeta(r) = 0 \), with \( \zeta(r) = (\sqrt{2} + 1 - 2\alpha - \beta)r^2 + 1 - \sqrt{2} \). Then, for the polynomial \( \zeta \), the positive root is denoted by \( \tilde{\sigma}_1 \), where

\[
\tilde{\sigma}_1 = \sqrt[2]{\frac{\sqrt{2} - 1}{\sqrt{2} + 1 - 2\alpha - \beta}}.
\]
It is observed that $\zeta(1) = -(2\alpha + \beta - 2) > 0$ and $\zeta(0) = 1 - \sqrt{2} < 0$, justifying the fact that $\tilde{\sigma}_1 \in (0, 1)$. This also gives that $\sigma_0 \leq \tilde{\sigma}_1$ if and only if $\zeta(\sigma_0) \leq 0$. It is seen that

$$
\zeta(\sigma_0) = \frac{1}{4(2\sqrt{2} - 1 - 2\alpha - \beta)^2} \left(2(2\alpha - 2 - \beta)(1 + \sqrt{2} - 2\alpha - \beta)((2\alpha - 2 - \beta)
+ \sqrt{(2\alpha - 2 - \beta)^2 - 4(3 - 2\sqrt{2})(2\sqrt{2} - 1 - 2\alpha - \beta))
+ 4(1 - 2\sqrt{2} + 2\alpha + \beta)((3 - 2\sqrt{2})(1 + \sqrt{2} - 2\alpha - \beta)
+ (1 - \sqrt{2})(1 - 2\sqrt{2} + 2\alpha + \beta)) \right),
$$

so, combining (5.1) and (5.5), $X(\alpha, \beta) \leq 0$ is same as saying $\zeta(\sigma_0) \leq 0$. Thus, in this case, $\sigma_0 \leq \tilde{\sigma}_1$ which implies $a_F(\sigma_0) \leq a_F(\tilde{\sigma}_1) = \sqrt{2}$, and now, Lemma 6 gives that the radius of starlikeness associated with the rational function for the function $F$ is at least $\sigma_0$.

Case (iii): $2\alpha + \beta - 2 < 0$ and $X(\alpha, \beta) > 0$. Following similar arguments as in Case (ii), $\zeta(\sigma_0) > 0$ gives $a_F(\sigma_0) > \sqrt{2}$, and then Lemma 6 implies that the required radius is at least $\tilde{\sigma}_0$.

Take $f(z) = z/(1 - z)^{-2\alpha + 2} \in ST(\alpha)$ and the polynomial $Q$ as $Q(z) = (1 + z)^n$. Using these choices for the function $f$ and the polynomial $Q$, the expression for $zF'(z)/F(z)$ as in the proof for Theorem 1, becomes,

$$
zF'(z)/F(z) = \frac{(1 - 2\alpha - \beta)z^2 + (2 - 2\alpha + \beta)z + 1}{1 - z^2}.
$$

The polynomials $\phi$ and $\psi$ in (5.2), (5.3) for $r = \sigma_0$ and $r = \tilde{\sigma}_0$ respectively imply that

$$
(1 - 2\alpha - \beta)r^2 - (2 - 2\alpha + \beta)r + 1 = 2(\sqrt{2} - 1)(1 - r^2),
$$

and

$$
(1 - 2\alpha - \beta)r^2 + (2 - 2\alpha + \beta)r + 1 = 2(1 - r^2).
$$

Now, observe that using (5.7) and putting $z = -\sigma_0$ in (5.6), it is obtained that

$$
\frac{(-\sigma_0)F'(-\sigma_0)}{F(-\sigma_0)} = 2(\sqrt{2} - 1) = \varphi_R(-1)
$$

this proves the sharpness for the radius $\sigma_0$. Also, considering (5.8), and replacing $z = \tilde{\sigma}_0$ in (5.6), it is seen that $zF'(z)/F(z)$ assumes the value $2 = \varphi_R(1)$ thus proving the sharpness for $\tilde{\sigma}_0$.

When $\alpha$ goes to 1 in Theorem 7 we get that the radius of starlikeness associated with a rational function for the function $F(z) = z(Q(z))^{\beta/n}$ where $Q$ is a non-constant polynomial of degree $n$ non-vanishing on the unit disc and $\beta > 0$, comes out to be $(3 - 2\sqrt{2})/(3 - 2\sqrt{2} + \beta)$. Further, when $\beta \to 0$, we obtain the radius of
starlikeness associated with a rational function for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$ obtained by Kumar and Ravichandran in [11, Theorem 3.2] (when $A = 1 - 2\alpha$ and $B = -1$).

6 Starlike functions associated with a nephroid domain

Wani and Swaminathan [21] in 2020 studied the class of starlike functions associated with a nephroid domain $S^*_{Ne} = S^*(\varphi_{Ne})$ with $\varphi_{Ne}(z) = 1 + z - z^3/3$. The function $\varphi_{Ne}$ maps the unit disc onto the interior of the nephroid, a 2-cusped curve,

$$\left((u - 1)^2 + v^2 - \frac{4}{9}\right)^{\frac{3}{2}} - \frac{4v^2}{3} = 0.$$ 

In this section, the radius of starlikeness associated with the nephroid is discussed for the function $F$ defined in (2.1), using the following lemma due to Wani and Swaminathan.

Lemma 8 [20, lemma 2.2] For $1/3 < a < 5/3$,

$$r_a = \begin{cases} 
a - \frac{1}{3} & \text{if } \frac{1}{3} < a \leq 1 \\
\frac{5}{3} - a & \text{if } 1 \leq a < \frac{5}{3}. \end{cases}$$

Then $\{w : w - a < r_a\} \subset \Omega_{Ne}$, where $\Omega_{Ne}$ is the region bounded by the nephroid, that is

$$\Omega_{Ne} := \left\{ \left((u - 1)^2 + v^2 - \frac{4}{9}\right)^{\frac{3}{2}} - \frac{4v^2}{3} < 0 \right\}.$$ 

Theorem 9 If the function $f \in ST(\alpha)$, then the radius of starlikeness associated with a nephroid domain for the function $F$ defined in (2.1) is given by

$$R_{S^*_{Ne}}(F) = \begin{cases} 
\sigma_0 & \text{if } 2\alpha + \beta - 2 \geq 0 \\
\tilde{\sigma}_0 & \text{if } 2\alpha + \beta - 2 < 0, \end{cases}$$

where

$$\sigma_0 = \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{9(-2 + 2\alpha - \beta)^2 - 8(4 - 6\alpha - 3\beta)}}.$$ 

and

$$\tilde{\sigma}_0 = \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{9(-2 + 2\alpha - \beta)^2 - 8(6\alpha - 8 + 3\beta)}}.$$
**Proof** The proof aims to show that the disc in (2.8) satisfies the following condition:

\[ \mathbb{D}(a_F(r); c_F(r)) \subseteq \Omega_{N_e}, \quad 0 < r \leq R_{S^*_e(F)}. \]

Let

\[ \sigma_0 := \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{9(-2 + 2\alpha - \beta)^2 - 8(4 - 6\alpha - 3\beta)}}. \]

The number \( \sigma_0 \) is the positive solution to the equation \( c_F(r) = a_F(r) - 1/3 \), which transforms into \( \phi(r) = 0 \) with the polynomial \( \phi \) in \( r \) given by

\[ \phi(r) := (4 - 6\alpha - 3\beta)r^2 - 3(2 - 2\alpha + \beta)r + 2. \] (6.1)

For the polynomial \( \phi, \phi(0) = 2 > 0 \) and \( \phi(1) = -6\beta < 0 \) justifying the existence of a zero for \( \phi \) in the interval \((0, 1)\). The number

\[ \tilde{\sigma}_0 := \frac{4}{3(2 - 2\alpha + \beta) + \sqrt{9(-2 + 2\alpha - \beta)^2 - 8(6\alpha - 8 + 3\beta)}}. \]

is the positive zero of the polynomial \( \psi \) in \( r \) is given by

\[ \psi(r) := (6\alpha - 8 + 3\beta)r^2 - 3(2 - 2\alpha + \beta)r + 2. \] (6.2)

Here, the equation \( \psi(r) = 0 \) is the the simplified form of the equation \( c_F(r) = 5/3 - a_F(r) \). The polynomial \( \psi \) indeed has a zero in the interval \((0, 1)\) as \( \psi(0) = 2 > 0 \) and \( \psi(1) = 12(\alpha - 1) < 0 \). It is known that the center \( a_F \) in (2.8) has the property that \( a_F(r) \leq 1 \) for \( 2\alpha + \beta - 2 \geq 0 \); and \( a_F(r) > 1 \) for \( 2\alpha + \beta - 2 < 0 \).

Case (i): \( 2\alpha + \beta - 2 \geq 0 \). In this case, the center \( a_F(r) \leq 1 \), thus Lemma 8 implies that \( \mathbb{D}(a_F(\sigma_0); c_F(\sigma_0)) \subseteq \Omega_{N_e} \). This proves that the \( S^*_e \) radius for the function \( F \) is at least \( \sigma_0 \).

To verify the sharpness of \( \sigma_0 \), take \( f(z) = z/(1 - z)^{2\alpha+2} \in ST(\alpha) \) and the polynomial \( Q \) as \( Q(z) = (1 + z)^n \). Using these choices for the function \( f \) and the polynomial \( Q \), the expression for \( zF'(z)/F(z) \) as in the proof for Theorem 1 is,

\[ \frac{zF'(z)}{F(z)} = \frac{(1 - 2\alpha - \beta)z^2 + (2 - 2\alpha + \beta)z + 1}{1 - z^2}. \] (6.3)

The polynomial \( \phi \) in (6.1) provides that for \( r = \sigma_0 \),

\[ 3((1 - 2\alpha - \beta)r^2 - (2 - 2\alpha + \beta)r + 1) = (1 - r^2). \] (6.4)

Thus using (6.4), and replacing \( z = -\sigma_0 \), (6.3) gives \( ((-\sigma_0)F'(-\sigma_0))/F(-\sigma_0) = 1/3 = \psi_{N_e}(-1) \). This proves the sharpness for \( \sigma_0 \).

Case (ii): \( 2\alpha + \beta - 2 < 0 \). Here, it is known that \( a_F(r) > 1 \), so, Lemma 8 directly gives that the required radius is at least the positive solution of the equation \( c_F(r) = 5/3 - a_F(r) \) that is \( \tilde{\sigma}_0 \).
To verify sharpness in this case, take \( f(z) = z/(1 - z)^{-2\alpha + 2} \in ST(\alpha) \) and the polynomial \( Q(z) = (1 + z)^n \). These expressions transform \( zF'(z)/F(z) \) into the form given in (6.3). The polynomial \( \psi \) given in (6.2) implies that for \( r = \tilde{\sigma}_0 \),

\[
3((1 - 2\alpha - \beta)r^2 + (2 - 2\alpha + \beta)r + 1) = 5(1 - r^2).
\]

Thus, the radius \( \tilde{\sigma}_0 \) is the best possible since an application of (6.5) in (6.3) gives that the expression for \( zF'(z)/F(z) \) takes the value \( 5/3 = \varphi_{Ne}(1) \) for \( z = \tilde{\sigma}_0 \). Thus, proving the sharpness for the radius \( \tilde{\sigma}_0 \). \( \square \)

When \( \alpha \) goes to 1 in Theorem 9 we get: The radius of starlikeness associated with a nephroid domain for the function \( F(z) = z(Q(z))^{\beta/n} \) where \( Q \) is a non-constant polynomial of degree \( n \) non-vanishing on the unit disc and \( \beta > 0 \), comes out to be \( 2/(2 + 3\beta) \). When \( \beta \to 0 \) in Theorem 9, we obtain the radius of starlikeness associated with a nephroid domain for the class of starlike functions of order \( \alpha, 0 \leq \alpha < 1 \) obtained by Wani and Swaminathan [20, Theorem 3.1(ii)] when \( A = 1 - 2\alpha \) and \( B = -1 \).

7 Starlike functions associated with modified sigmoid function

In 2020, Goel and Kumar [8] introduced the class \( S^*_{SG} \) of functions mapping \( \mathbb{D} \) onto the domain \( \Delta_{SG} = \{ w : |\log w/(2 - w)| < 1 \} \), with \( S^*_{SG} = S^*(2/(1 + e^{-z})) \). They also proved the following lemma.

**Lemma 10** [8, lemma 2.2] Let \( 2/(1 + e) < a < 2e/(1 + e) \). If

\[
r_a = \frac{e - 1}{e + 1} - a - 1,
\]

then \( \{ w : w < r_a \} \subset \Delta_{SG} \).

In the next result, we find the radius of starlikeness associated with modified sigmoid function for the function \( F \) defined in (2.1).

**Theorem 11** If the function \( f \in ST(\alpha) \), then the \( S^*_{SG} \) radius for the function \( F \) given in (2.1) is given by

\[
R_{S^*_{SG}}(F) = \begin{cases} 
\sigma_0 & \text{if } 2\alpha + \beta - 2 \geq 0 \\
\tilde{\sigma}_0 & \text{if } 2\alpha + \beta - 2 < 0,
\end{cases}
\]

where

\[
\sigma_0 = 2(e - 1)((2 - 2\alpha + \beta)(e + 1) + \sqrt{(e + 1)(2 - 2\alpha + \beta)^2 - 4(e - 1)(3 + e - 2\alpha - 2\alpha e - \beta - \beta e)^{-1}})
\]
\[ \sigma_0 = 2(e - 1)((2 - 2\alpha + \beta)(e + 1) + \sqrt{(e + 1)(2 - 2\alpha + \beta)^2 + 4(e - 1)(1 + 3e - 2\alpha - 2ae - \beta - \beta e)}^{-1}. \]

**Proof** We will show that \( \mathbb{D}(a_F(r); c_F(r)) \subset \Delta_{S}\alpha \) for \( 0 < r \leq R_{S}\alpha \).

Case (i): \( 2\alpha + \beta - 2 \geq 0 \). It is known that in this case, the center \( a_F \) satisfies the inequality \( a_F(r) \leq 1 \). Consequently, the equation \( c_F(r) = ((e - 1)/(e + 1)) - a_F - F''(r) - 1 \) becomes \( c_F(r) = ((e - 1)/(e + 1)) - 1 + a_F(r) \). This equation is simplified into the form \( \phi(r) = 0 \), with

\[ \phi(r) := ((2 - 2\alpha - \beta)(e + 1) - (e - 1))r^2 - (2 - 2\alpha + \beta)(e + 1)r + (e - 1). \]

(7.1)

The polynomial \( \phi \) has a zero in the interval \((0, 1)\) since, \( \phi(0) = (e - 1) > 0 \), and \( \phi(1) = -2\beta(e + 1) < 0 \), and this positive number is

\[ \sigma_0 = 2(e - 1)((2 - 2\alpha + \beta)(e + 1) + \sqrt{(e + 1)(2 - 2\alpha + \beta)^2 - 4(e - 1)(3 + e - 2\alpha - 2ae - \beta - \beta e)}^{-1}. \]

Now, to verify the sharpness of the radius \( \sigma_0 \), we take \( f(z) = z/(1-z)^{-2\alpha+2} \in ST(\alpha) \) and polynomial \( Q \) as \( Q(z) = (1 + z)^n \). Using these choices for the function \( f \) and the polynomial \( Q \), the expression for \( zF'(z)/F(z) \) as seen in the proof for Theorem 1 is obtained to be

\[ \frac{zF'(z)}{F(z)} = \frac{1 - 2\alpha - \beta}{{z}^2 + (2 - 2\alpha + \beta)z + 1}{1 - {z}^2}, \]

(7.2)

which implies that

\[ 2 - \frac{zF'(z)}{F(z)} = \frac{(-3 + 2\alpha + \beta)z^2 - (2 - 2\alpha + \beta)z + 1}{1 - z^2}. \]

(7.3)

Thus, (7.2) and (7.3) give that

\[ \left( \frac{zF'(z)}{F(z)} \right) \left( 2 - \frac{zF'(z)}{F(z)} \right)^{-1} = \frac{(1 - 2\alpha - \beta)z^2 + (2 - 2\alpha + \beta)z + 1}{(-3 + 2\alpha + \beta)z^2 - (2 - 2\alpha + \beta)z + 1}. \]

(7.4)

Further, the polynomial \( \phi \) in (7.1) gets reduced to

\[ (((1 - 2\alpha - \beta)r^2 - (2 - 2\alpha + \beta)r + 1)e = (-3 + 2\alpha + \beta)r^2 + (2 - 2\alpha + \beta)r + 1. \]

(7.5)
for $r = \sigma_0$. Thus using (7.5) in (7.4) for $z = -\sigma_0$, it is seen that

$$\left| \log \left( \frac{zF'(z)}{F(z)} \right) \left( 2 - \frac{zF'(z)}{F(z)} \right)^{-1} \right| = \left| \log \frac{1}{e} \right| = 1,$$

thus proving the sharpness for $\sigma_0$.

Case (ii): $2 \alpha + \beta - 2 < 0$. Here, the center $a_F(r) > 1$, which implies that the equation $c_F(r) = ((e - 1)/(e + 1)) - a_0 - F''(r) - 1$ converts to $c_F(r) = ((e - 1)/(e + 1)) + 1 - a_F(r)$. This is equivalent to $\psi(r) = 0$ for

$$\psi(r) := ((2 - 2\alpha - \beta)(e + 1) + (e - 1)r^2 + (2 - 2\alpha + \beta)(e + 1)r - (e - 1).$$

(7.6)

The number

$$\hat{\sigma}_0 = 2(e - 1)(2 - 2\alpha + \beta)(e + 1)
+ \sqrt{((e + 1)(2 - 2\alpha + \beta))^2 + 4(e - 1)(1 + 3e - 2\alpha - 2\alpha e - \beta - \beta e))^{-1}$$

is the smallest positive zero of the polynomial $\psi$, and the observations $\psi(0) = -(e - 1) < 0$, and $\psi(1) = 4(1 - \alpha)(e + 1) > 0$ justify the existence of a zero in $(0, 1)$.

To verify that the radius $\hat{\sigma}_0$ is the best possible, take the values of the function $f$ and the polynomial $Q$ same as in Case (i), thus the expression for $(zF'(z)/F(z))/(2 - (zf'(z)/F(z)))$ is same as given in (7.4). The polynomial $\psi$ in (7.6) gives

$$1 - 2\alpha - \beta)r^2 + (2 - 2\alpha + \beta)r + 1
= ((-3 + 2\alpha + \beta)r^2 + (2 - 2\alpha + \beta)r + 1)e.$$  

(7.7)

for $r = \hat{\sigma}_0$. Thus, putting $z = \hat{\sigma}_0$ in (7.4) and then using (7.7), it is obtained that

$$\left| \log \left( \frac{\hat{\sigma}_0 F'(\hat{\sigma}_0)}{F(\hat{\sigma}_0)} \right) \left( 2 - \frac{\hat{\sigma}_0 F'(\hat{\sigma}_0)}{F(\hat{\sigma}_0)} \right)^{-1} \right| = \left| \log e \right| = 1.$$

This proves sharpness for $\hat{\sigma}_0$. □

When $\alpha$ goes to 1 in Theorem 11 we get that the radius of starlikeness associated with modified sigmoid function for the function $F(z) = z(Q(z))^{\beta/n}$ where $Q$ is a non-constant polynomial of degree $n$ non-vanishing on the unit disc and $\beta > 0$, is $(e - 1)/(e - 1 + \beta(e + 1))$. When $\beta \to 0$ in Theorem 11, we obtain the radius of starlikeness associated with the modified sigmoid function for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$. The radius of parabolic starlikeness and the radii of other related starlikeness including the one related to the lemniscate of Bernoulli can be investigated.

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