Critical Topology for Optimization on the Symplectic Group

Rebing Wu, Raj Chakrabarti, and Herschel Rabitz

Abstract

Optimization problems over compact Lie groups have been extensively studied due to their broad applications in linear programming and optimal control. This paper analyzes least square problems over a noncompact Lie group, the symplectic group $\text{Sp}(2N, \mathbb{R})$, which can be used to assess the optimality of control over dynamical transformations in classical mechanics and quantum optics. The critical topology for minimizing the Frobenius distance from a target symplectic transformation is solved. It is shown that the critical points include a unique local minimum and a number of saddle points. The topology is more complicated than those of previously studied problems on compact Lie groups such as the orthogonal and unitary groups because the incompatibility of the Frobenius norm with the pseudo-Riemannian structure on the symplectic group brings significant nonlinearity to the problem. Nonetheless, the lack of traps guarantees the global convergence of local optimization algorithms.

*Department of Chemistry, Princeton University, Princeton, New Jersey 08544, USA
I. INTRODUCTION

The topology of solution sets to problems in the calculus of variations is the subject of considerable interest in mathematical physics and optimization theory \[1\]. It is of particular importance in theory of optimal control, where this topology can affect the efficiency of the search for effective control Hamiltonians \[2\]. Whereas in general it is very difficult to characterize these features for arbitrary functionals, when the objective or Lagrangian functional is defined on a Lie group, it is often possible to apply techniques from the theory of Lie groups and differential geometry to simplify the extraction of critical topology.

The topology of the critical submanifolds of classical Lie groups was originally studied by Frankel \[3\], who characterized the number of critical points and associated Morse indices of the trace function on compact classical Lie groups $U(N)$, $O(N)$, and $Sp(N)$. Dynnikov and Vesselo\[4\] subsequently identified these functions as perfect Morse-Bott functions and showed that they afford a cell decomposition of the associated groups. Recently, the equivalence of the trace function to that of a least-square matrix function for the distance between a real and target transformation led to the application of these results to optimization and control theory. Brockett \[5\] showed that a wide range of combinatorial optimization problems arising in linear programming can be framed as matrix least squares optimizations on compact Lie groups. In \[6\], the critical topology of the trace function on $U(N)$ was analyzed in light of its connection to the optimal control problem of implementing a quantum logic gate over discrete variables with maximal fidelity.

A unifying feature of these problems is the fact that the domain of the objective functional, being a compact Lie group, can always be endowed with the structure of a differential manifold with a bi-invariant Riemannian metric. In this paper, we attempt to extend such studies to the investigation of critical submanifolds of least squares
objective functions on noncompact Lie groups \cite{7}, in particular the symplectic group $\text{Sp}(2N, \mathbb{R})$. Although the geometry and topology of symplectic manifolds, and functions defined on those manifolds, have been the subject of extensive investigations in mathematical physics, functions defined on the symplectic group itself have received far less attention.

Specifically, we are concerned with the least-square distance function on the space of symplectic matrices,

$$ J(S) = \| S - W \|^2 = \text{Tr}(S - W)^T(S - W), \quad S \in \text{Sp}(2N, \mathbb{R}). $$

This cost function has recently been shown to have fundamental applications in the assessment of the fidelity of dynamical gates in quantum analog computation when implemented through optimal control theory \cite{8}, where $W$ represents the target quantum gate to be realized. Another potential important motivation for studying this problem comes from the control of beam systems in particle accelerators \cite{9}. As shown before, the cost function (1) on compact Lie groups (e.g., $\mathcal{U}(N)$, $\mathcal{O}(N)$) is equivalent to a linear trace function. However, this no longer holds on the symplectic group, because the corresponding Riemannian metric is not bi-invariant under symplectic transformations. This feature is caused by incompatibility of the Frobenius norm defined in (1) with the geometric structure of the symplectic group, and greatly complexifies the critical topology, as we will show below. On the other hand, this group can be treated as a pseudo-Riemannian manifold with a bi-invariant pseudo-Riemannian metric. Although it is possible to introduce an objective function that is compatible with this pseudo-Riemannian metric \cite{10}, such function is not positive definite and cannot be interpreted as a distance function. However, the corresponding critical topology is equivalent to that of a linear trace function on $\text{Sp}(2N, \mathbb{R})$, as well as those on $\mathcal{U}(N)$ and $\mathcal{O}(N)$. In contrast to the objective functionals (1), such compatibility leads to a simple critical topology as the effects of the pseudo-Riemannian geometry of noncompact Lie groups.

Existing works on control of classical mechanical systems generally do not require
direct control of the system propagators except some special cases (e.g., robotic motion planning on the Euclidean group $SE(3)$ \cite{11}). It is usually sufficient to attain the control over state vector with fewer degrees of freedom. However, since any given state vector in phase space is associated with an infinite number of symplectic matrices that propagate the initial state of the system to the desired final state, it is generally impossible to predict which of these symplectic matrices will be reached by the time-dependent control obtained through the optimization procedure. As such, the efficiency of control optimization will be highly system-dependent, with optimization algorithms traversing longer trajectories in the symplectic group for certain classes of Hamiltonians. In contrast, if the control problem is cast in terms of symplectic propagator optimization, it is possible to choose the shortest path in the symplectic group from the initial condition to the target \cite{21}. As such, gradient control algorithms based on propagator optimization may outperform those based on state vector optimization. Optimization algorithms of this type are currently the subject of intense study in the context of quantum control \cite{12, 13, 14, 15}.

In a study of optimization algorithms on noncompact Lie groups, Mahony indicated that (local) quadratic convergence can still be achieved by using the Newton method adapted for the curved manifold under local coordinates of the first kind \cite{7}, which results in no essential differences compared to algorithms on compact Lie groups. However, the global topology of the optima and suboptima may play a fundamental role in the overall efficiency of the optimizations, and will be the major concern of our studies here. This paper is organized as follows. Section II summarizes the definition and properties of symplectic groups. Section III derives the canonical form of landscape critical points. Section IV analyzes the Hessian quadratic form for each of these critical points. Section V studies the constrained landscape over the compact subgroup. Section VI provides an illustrative example. Finally, Section VII draws the conclusion.
II. PRELIMINARIES ON THE SYMPLECTIC GROUP

In classical mechanics, a transformation for a system described by \(N\) pairs of coordinate and momentum variables \(z_{\alpha,\beta} = (x_{\alpha,\beta}^1, \cdots, x_{\alpha,\beta}^N; p_{\alpha,\beta}^1, \cdots, p_{\alpha,\beta}^N)\) is called symplectic if it preserves the (skew-symmetric) symplectic form

\[
\omega(z_\alpha, z_\beta) = \sum_{i=1}^{N} (x_{\alpha}^i p_{\beta}^i - x_{\beta}^i p_{\alpha}^i).
\]

With this coordinate system, a symplectic transformation can be represented by a \(2N \times 2N\) dimensional real matrix that satisfies \(S^T J S = J\), where \(S^T\) is the transpose of \(S\) and

\[
J = \begin{pmatrix}
I_N \\
-I_N
\end{pmatrix}.
\]

The set of symplectic matrices forms a noncompact Lie group \(\text{Sp}(2N, \mathbb{R})\), and its Lie algebra \(\text{sp}(2N, \mathbb{R}) = \{JB| B^T = B, \ B \in \mathbb{R}^{2N \times 2N}\}\), from which it is easy to see that the dimension of \(\text{Sp}(2N, \mathbb{R})\) is \(N(2N + 1)\). As a linear vector space, \(\text{sp}(2N, \mathbb{R})\) can be decomposed into two mutually orthogonal subspaces \(\text{sp}(2N, \mathbb{R}) = \mathcal{L}_1 \oplus \mathcal{L}_2\), where

\[
\mathcal{L}_1 = \{JB \in \text{sp}(2N, \mathbb{R})| JB = -BJ\}.
\]

\[
\mathcal{L}_2 = \{JB \in \text{sp}(2N, \mathbb{R})| JB = BJ\}.
\]

The subspace \(\mathcal{L}_2\) is a Lie subalgebra of \(\text{sp}(2N, \mathbb{R})\). It generates the orthogonal symplectic group \(\text{OSp}(2N, \mathbb{R})\) as the maximal compact Lie subgroup of \(\text{Sp}(2N, \mathbb{R})\), as it is the intersection of the symplectic group \(\text{Sp}(2N, \mathbb{R})\) with the orthogonal group \(O(2N)\). There is an interesting isomorphism between \(\text{OSp}(2N, \mathbb{R})\) and the unitary group \(U(N)\) via the following mapping:

\[
X - iY \rightarrow \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}, \quad X - iY \in U(N).
\]

Here we briefly summarize some properties of symplectic matrices and symplectic groups that will be used in the following analysis. Readers of interests are referred to [9] for more details.
**Property 1** As the analog of the property $O^{-1} = O^T$ for any orthogonal matrix $O$, $S^{-1} = J^T S^T J$ for any symplectic matrix $S$.

**Property 2** The eigenvalues of a symplectic matrix always appear in reciprocal pairs, i.e., if $\omega$ is an eigenvalue of $S$, then so is $\omega^{-1}$, and they have identical degeneracy degrees.

**Property 3** There always exists a symplectic singular value decomposition (SVD) $S = UDV$, where $U$ and $V$ are orthogonal symplectic matrices. $D$ is a diagonal symplectic matrix whose diagonal elements are the singular values of $S$.

**Property 4** Denote by $\text{Stab}(D) = \{ R \in \text{OSp}(2N, \mathbb{R}) : R^T DR = D \}$ the stabilizer of the diagonal symplectic matrix $D$ in the group $\text{OSp}(2N, \mathbb{R})$. The stabilizer of a diagonal symplectic matrix

$$D = \text{diag}\{a_0 I_{n_0}, a_1 I_{n_1}, \cdots, a_r I_{n_r} ; a_0 I_{n_0}, a_1^{-1} I_{n_1}, \cdots, a_r^{-1} I_{n_r}\},$$

where $1 = a_0 < a_1 < a_2 \cdots < a_r$, is the direct product of subgroups $\text{Stab}(D) = \text{OSp}(2n_0, \mathbb{R}) \times O(n_1) \times \cdots \times O(n_r)$.

The above features can be demonstrated by the example of the 2-qunit SUM gate in continuous quantum computation [8, 16, 17], which acts on the quadratic vector $(q_1, q_2; p_1, p_2)$ as follows

$$\text{SUM} : \quad q_1 \rightarrow q_1, \quad q_2 \rightarrow q_1 + q_2, \quad p_1 \rightarrow p_1 - p_2, \quad p_2 \rightarrow p_2.$$

The matrix form of the SUM gate is

$$\text{SUM} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (3)$$
whose singular value decomposition $SUM = UEV$ can be found to be

$$U = \begin{pmatrix} -\xi & -\eta & 0 & 0 \\ -\eta & \xi & 0 & 0 \\ 0 & 0 & -\xi & -\eta \\ 0 & 0 & -\eta & \xi \end{pmatrix}, \quad E = \begin{pmatrix} \omega & \omega^{-1} \\ \omega^{-1} & \omega \end{pmatrix}, \quad V = \begin{pmatrix} -\eta & -\xi & 0 & 0 \\ \xi & \eta & 0 & 0 \\ 0 & 0 & -\eta & -\xi \\ 0 & 0 & \xi & \eta \end{pmatrix},$$

where

$$\xi = \sqrt{\frac{5 - \sqrt{5}}{10}}, \quad \eta = \sqrt{\frac{5 + \sqrt{5}}{10}}, \quad \omega = \frac{\sqrt{5} + 1}{2}.$$}

Since there is a two-fold degeneracy of the singular value $\omega$, the stabilizer of $E$ is isomorphic to the $O(2)$ group.

**III. CANONICAL FORM OF THE CRITICAL SUBMANIFOLDS**

Any candidate solution to the optimization problem must be one of its critical points, defined as a $S^* \in \text{Sp}(2N, \mathbb{R})$ such that the gradient $\nabla J(S^*)$ of the cost function vanishes. Since there generally exist multiple-solutions for the critical points, a complete understanding of the critical topology is of essential importance to assess the complexity of searching for the global optimal solution.

The basic idea to determine the set of critical solutions is, at an arbitrary fixed point $S \in \text{Sp}(2N, \mathbb{R})$, to perturb the cost function along an arbitrary direction in the tangent space (isomorphic to $\text{sp}(2N, \mathbb{R})$) and find those points where the directional derivation vanishes along all directions. For example, taking the parametrization $Se^{tJY}$ with $Y^T = Y$ (here $JY$ represents the local Cartesian coordinates in the tangent space at $S$), the critical condition can be obtained by forcing the derivative along $JY$ to be zero at $t = 0$ for arbitrary $Y$, i.e.,

$$\frac{d}{dt} J(Se^{tJY}) \bigg|_{t=0} = \text{Tr}[JY(S^TS - W^TS)] = 0, \quad \forall \ Y^T = Y,$$
which implies that the matrix \((S^T S - W^T S)J\) has to be skew-symmetric, i.e.,
\[
(S^T S - W^T S)J = J(S^T S - S^T W).\tag{4}
\]
Left multiplying a constant matrix \(J^T\) and applying the property \(J^T S^T J = S^{-1}\), we get a simpler form:
\[
S^T S - (S^T S)^{-1} = S^T W - (S^T W)^{-1}.	ag{5}
\]
Although (5) is a nonlinear (fourth-order in \(S\)) equation, its highly symmetric form makes it still solvable. Let \(S = U_1 D V_1\) be a symplectic SVD of \(S\), and \(W = U_0 E_d V_0\) be that of \(W\). Substituting these SVDs into the equation (5), we can simplify the condition as
\[
D^2 - D^{-2} = DE - (DE)^{-1},
\]
where \(E = U E_d V\) with \(U = U_1^T U_0\) and \(V = V_0 V_1^T\). Followed by a commutation with \(DE\) on both sides, this equation is further transformed as \([DE, D^2 - D^{-2}] = 0\), or equivalently, \([E, D^2 - D^{-2}] = 0\). We will show that this relation implies \([E, D] = 0\).

Let \(d_r > \cdots > d_1 \geq 1 \geq d_1^{-1} > \cdots > d_r^{-1}\) be the distinct eigenvalues of \(D\), where the degeneracy of \(d_i\) (or \(d_i^{-1}\)) is \(M_i\), \(i = 1, \cdots, r\); then \(D\) can be decomposed into diagonal blocks \(D_i = d_i I_{M_i}\), \(i = 1, \cdots, r\), and their inverses. The commutativity of \(E\) and \(D^2 - D^{-2}\) implies that \(E\) is simultaneously block-diagonal with \(D^2 - D^{-2}\), corresponding to sub-blocks \((d_i^2 - d_i^{-2}) I_{M_i}\). Since \(D\) is positive definite, two distinct eigenvalues of \(D\) must correspond to two distinct eigenvalues of \(D^2 - D^{-2}\), and vice versa. So \(D\) shares the same eigenspace decomposition with \(D^2 - D^{-2}\), as well as that of \(E\). Hence \(E\) commutes with \(D\).

Therefore, for each \(D_i\) in \(D\), there corresponds a diagonal block \(E_i\) of \(E\). Let \(E_i = U_i E_{0i} V_i\) be a SVD of \(E_i\), then the following block-diagonal symplectic orthogonal matrices
\[
U_d = \text{diag}\{U_1, \cdots, U_r; U_1^T, \cdots, U_r^T\},
\]
\[
V_d = \text{diag}\{V_1, \cdots, V_r; V_1^T, \cdots, V_r^T\},
\]
\[
E_d = \text{diag}\{E_{01}, \cdots, E_{0r}; E_{01}^{-1}, \cdots, E_{0r}^{-1}\}\]
define a SVD $E = U_d E_d V_d$ of $E$. Hence, for any general SVD $E = U E_d V$, the following relationship

$$(U_d^T U) E_d (V V_d^T) = E_d,$$

implies the existence of a symplectic orthogonal matrix $R \in \text{Stab}(E_d)$ such that $U = U_d R$ and $V = R^T V_d$.

Going back to the original symplectic matrix $S^*$, we then have a uniform expression for the critical solutions:

$$S^* = U_1 D V_1 = U_0 U^T D V^T V_0 = U_0 R^T (U_d^T D V_d^T) R V_0.$$ 

Notice that $U_d^T$ commutes with $D$, by which we can denote $L = U_d^T V_d^T$, and then simplify the representation of the critical points as follows:

$$S^* = U_0 R^T P V_0, \quad P = DL, \quad R \in \text{Stab}(E_d). \quad (6)$$

This canonical form shows that the critical manifold consists of orbits of admissible matrices $P$ under the action of $\text{Stab}(E_d)$, which can be represented by the quotient set $\mathcal{M} = \text{Stab}(E_d)/\text{Stab}(P)$. It is then sufficient to characterize the set of critical points by specify all possible values of the characteristic matrix $P$ that involves the singular values $d_i$ and their corresponding orthogonal matrix blocks $L_i = U_i V_i^T$.

For simplicity, owing to the reciprocal properties of singular values of symplectic matrices, we only need to analyze the singular values that are no less than 1. Restricting the matrix equation (5) on the eigenspace of each $d_i \geq 1$, and substituting the canonical form into (5), we have

$$(d_i^2 - d_i^{-2}) I_{M_i} = d_i E_{0i} L_i - d_i^{-1} L_i^T E_{0i}^{-1}.$$

Let $e_1, \cdots, e_{M_i}$ be the singular values of $E_{0i}$. This equation can be decomposed as

$$d_i e_\alpha L_{i,\alpha \beta} - d_i^{-1} e^\beta_{\beta} L_{i,\beta \alpha} = (d_i^2 - d_i^{-2}) \delta_{\alpha \beta}, \quad \alpha, \beta = 1, \cdots, M_i,$$ 

$$ (7)$$

$$ (8)$$
where $L_{i,\alpha\beta}$ is the $\alpha\beta$-th matrix element of $L_i$. Using equation (8), we classify the sub-blocks in $P$ into the following different types.

Firstly, suppose that the orthogonal matrix $L_i$ is diagonal, then each element $L_{i,\alpha\alpha}$ has to be unimodular. When $L_{i,\alpha\alpha} = 1$, we get

$$(d_i - e_\gamma)(d_i^5 + e_{-1}) = 0,$$

whose only admissible positive root is $d_i = e_\gamma$, where $e_\gamma \geq 1$. The case $L_{i,\alpha\alpha} = -1$ corresponds to

$$(d_i + e_\gamma)(d_i^3 - e_{-1}) = 0,$$

of which the only admissible positive root is $d_i = e_{-1/3}$ where $e_\gamma < 1$.

In such cases, each block $E_i$ allows for only one singular value so that all eigenvalues of $D_i$ are identical. Corresponding the case $L_i = I_{M_i}$, the block $D_i$ is called type I; for the case $L_i = -I_{M_i}$, the block $D_i$ is called type II.

For the more general case that $L_i$ is not diagonal, any pair of nonzero off-diagonal matrix elements must satisfy

$$
\begin{pmatrix}
  d_i e_\alpha & -d_i^{-1} e_{-\beta} \\
  -d_i^{-1} e_{-\alpha} & d_i e_\beta
\end{pmatrix}
\begin{pmatrix}
  L_{i,\alpha\beta} \\
  L_{i,\beta\alpha}
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}, \quad \alpha \neq \beta,
$$

(9)

in which the determinant of the coefficient matrix has to vanish, and this solves the eigenvalue $d_i$, i.e.,

$$d_i^2 e_\alpha e_\beta - (d_i^2 e_\alpha e_\beta)^{-1} = 0 \iff d_i = (e_\alpha e_\beta)^{-1/2}.
$$

(10)

Consequently, substituting (10) back into the equation (9), we find that the resulting nonzero off-diagonal matrix element $L_{i,\alpha\beta} = L_{i,\beta\alpha}$, i.e., the matrix $L_i$ must be symmetric. Obviously, in such cases each minimal block $D_i$ allows for exactly two distinct singular values of $E_i$ (otherwise $D_i$ will have non-unique singular values), and their repeating number are both $k_i = M_i/2$. 

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Without loss of generality, we assume that \( e_\alpha \geq e_\beta \) and \( d_i = (e_\alpha e_\beta)^{-1/2} \geq 1 \). Then the use of (10) solves the corresponding diagonal elements from (8) as follows

\[
L_{i,\alpha\alpha} = -L_{i,\beta\beta} = \frac{d_i^2 - d_i^{-2}}{d_i e_\alpha - d_i^{-1} e_\alpha^{-1}} = \frac{(e_\alpha e_\beta)^{-1} - e_\alpha e_\beta}{(e_\alpha/e_\beta)^{1/2} - (e_\beta/e_\alpha)^{-1/2}} \geq 0.
\]

As \( L_i \) is orthogonal, each of its matrix elements must satisfy \( 0 < L_{i,\alpha\beta} \leq 1 \), which set additional constraints on the admissible pairs \( e_\alpha \) and \( e_\beta \) that generate \( d_i \):

\[
e_\beta \leq 1 \leq e_\alpha, \quad e_\beta^{-1/3} \leq e_\alpha \leq e_\beta^{-1}.
\]

(11)

Under such conditions, the corresponding block \( D_i \) is called a type III block. Suppose that \( E_i = diag\{e_\alpha I_{k_i}, e_\beta I_{k_i}\} \), the corresponding matrix \( L_i \) must be in the following form:

\[
L_i = \begin{pmatrix}
\cos x_i & I_{k_i} \\
\sin x_i & O_{k_i}^T
\end{pmatrix},
\]

where \( O \) is some orthogonal matrix and the angle \( x_i = \arccos \frac{(e_\alpha e_\beta)^{-1} - e_\alpha e_\beta}{(e_\alpha/e_\beta)^{1/2} - (e_\beta/e_\alpha)^{-1/2}} \in [0, \frac{\pi}{2}] \).

Let \( T = diag\{O, O\} \), then \( P^T L_i P \) is in the following canonical form

\[
L_i = \begin{pmatrix}
\cos x_i I_{k_i} & \sin x_i I_{k_i} \\
\sin x_i I_{k_i} & -\cos x_i I_{k_i}
\end{pmatrix}.
\]

Since the transformation matrix \( T \) is in the stabilizer \( Stab(E_d) \), so \( L_i \) can be always represented by the above standard form.

In conclusion, suppose that \( W \) has \( s \) greater-than-1 singular values \( 1 < \omega_1 < \omega_2 < \cdots < \omega_s \) with degeneracy degrees \( n_1, \cdots, n_s \). From the above analyses, each given singular value \( \omega_\alpha > 1 \) can be used to produce a singular value \( d_i \) in the canonical form \( P \) through the following three ways:

I. \( d_i = \omega_\alpha \) with multiplicity \( m'_\alpha \) and a corresponding matrix block \( P'_\alpha = \omega_\alpha I_{m'_\alpha} \) in \( P \);

II. \( d_i = \omega_\alpha^{-\frac{1}{3}} < 1 \) with multiplicity \( m''_\alpha \) and a corresponding matrix block \( P''_\alpha = -\omega_\alpha^{-\frac{1}{3}} I_{m''_\alpha} \) in \( P \);

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III. \( d_i = (\omega_\alpha / \omega_\beta)^{-1/2} \) with multiplicity \( 2m''_{\alpha \beta} \), where \( \omega_\alpha \leq \omega_\beta \) (defined as that \( \omega_\beta^{1/3} \leq \omega_\alpha \leq \omega_\beta \)), which are from \( m''_{\alpha \beta} \) eigenvalues \( \omega_\alpha \) and \( m''_{\alpha \beta} \) eigenvalues \( \omega_\beta \).

The corresponding block is

\[
P''_{\alpha \beta} = - \left( \frac{\omega_\alpha}{\omega_\beta} \right)^{-\frac{1}{2}} \begin{pmatrix}
\cos x_{\alpha \beta} I_{m''_{\alpha \beta}} & \sin x_{\alpha \beta} I_{m''_{\alpha \beta}} \\
\sin x_{\alpha \beta} I_{m''_{\alpha \beta}} & -\cos x_{\alpha \beta} I_{m''_{\alpha \beta}}
\end{pmatrix},
\]

where \( x_{\alpha \beta} = \arccos \left( \frac{(\omega_\alpha/\omega_\beta)^{1-\omega_\alpha/\omega_\beta}}{(\omega_\alpha/\omega_\beta)^{1/2}-(\omega_\alpha/\omega_\beta)^{-1/2}} \right) \).

The singular value \( \omega_0 = 1 \) can be treated separately. Since the condition (11) can never be satisfied with \( \omega_0 = 1 \) and any other singular values of \( E \), \( \omega_0 = 1 \) is always incapable of generating a type III singular value \( d_i \neq 1 \) via (10). Thus, they only contributes to type I or II singular values \( d_i = 1 \) of \( D \). Let the degeneracy number of \( \omega_0 = 1 \), which must be even, be \( 2n_0 \), and the number of type I singular values is \( m_0 \), the possible characteristic matrix block \( P_0 \) is

\[
P_0 = \begin{pmatrix}
I_{m_0} & \\
-I_{n_0-m_0}
\end{pmatrix}, \quad m_0 = 0, \ldots, n_0.
\]

Any group of admissible indices \( \{m_0; m'_\alpha, m''_\alpha, m''_{\alpha \beta}; \alpha, \beta = 1, \ldots, s\} \) labels an orbit of \( \text{Stab}(E_d) \), which equivalently labels a unique critical submanifold of the set of critical points. The value of the cost function at these critical submanifolds are:

\[
J(S^*) = 8(n_0 - m_0)^2 + \sum_{\mu} m''_\mu (\omega_\mu^2 + \omega_\mu^{-2} + 3\omega_\mu^{2/3} + 3\omega_\mu^{-2/3})
\]

\[
+ \sum_{\alpha \leq \beta} m''_{\alpha \beta} [(\omega_\alpha + \omega_\beta^{-1})^2 + (\omega_\alpha^{-1} + \omega_\beta)^2].
\]

IV. TOPOLOGY ANALYSIS OF CRITICAL SUBMANIFOLDS

This section will delve into more intrinsic topological details of the critical manifolds, including (1) their connectedness, determined by counting the number of separate submanifolds, (2) their dimensions and local optimality status (i.e, local maximum, minimum
or saddle point), determined via Hessian analysis. Such information provides a global picture of the distribution of possible solutions and their influences on the actual search for optimal solutions to the optimization problem.

The number of critical submanifolds can be enumerated by counting all admissible combinations of indices \( \{m_0; m'_\alpha, m''_\alpha, m'''_{\alpha\beta}; \alpha, \beta = 1, \cdots, s\} \), each of which corresponds to a unique characteristic matrix \( P \), and hence labels a critical submanifold as the orbit of \( P \) under the action of \( \text{Stab}(E_d) \) [22]. This number is dependent with the degenerate structure of the singular values of the target transformation \( W \). For example, the simplest case is that \( W \) is an orthogonal symplectic matrix, where \( E_d = I_{2N} \) and \( P \) has only \( \pm 1 \) singular values. The total number of critical submanifolds is \( N + 1 \) corresponding to \( m = 0, 1, \cdots, N \), the repeating number of the \( -1 \) eigenvalues in \( P_0 \). When \( W \) has a fully degenerate singular value \( \omega > 1 \), the admissible characteristic matrices \( P \) may contain either of the I-III types of sub-blocks, and hence there are more critical submanifolds. Let \( m''' \) be the number of pairs that generate \( 2m''' \) type III singular values \( d = 1 \) in \( D \), \( m' \) for \( d = \omega \) and \( m'' \) for \( d = \omega^{1/3} \). Since \( m' + m'' + 2m''' = N \), counting such admissible combinations gives the number of critical submanifolds as a quadratic function of \( N \):

\[
N = \begin{cases} 
\frac{(N + 2)^2}{2}, & N \text{ even;} \\
\frac{(N + 1)(N + 3)}{2}, & N \text{ odd;} 
\end{cases}
\]  

(12)

The number of critical submanifolds shoots up when the degeneracy in \( W \) is broken up. The extremal case is that \( E_d \) is fully non-degenerate and the singular values of \( E_d \) are not far apart from each other such that \( \omega_\alpha \preceq \omega_\beta \) for any \( \alpha < \beta \), i.e., any two distinct singular values of \( W \) are allowed to produce a pair of type III singular values of \( S^* \). Let \( m \) be the number of pairs of singular values of \( W \) that generate type III singular values of \( S^* \); then there are \( N!/(N - 2m)!m!2^m \) different choices. Moreover, for each fixed \( m \), the possibilities of using the remaining singular values to generate I or II type singular
values of $S^*$ is $2^{N-2m}$. These set an upper bound

$$\mathcal{N} = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2^{N-3m}N!}{m!(N-2m)!},$$

which is super-exponential in $N$, on the maximal number of critical submanifolds in all cases.

The dimensions of the critical submanifolds are generally difficult to calculate. However, are simple for those that contains only type I and II blocks (i.e., $m''_{\alpha\beta} = 0$, for all $\alpha, \beta$) via their geometrical expression $\mathcal{M} = \text{Stab}(E_d)/\text{Stab}(P)$, i.e.,

$$\dim \mathcal{M} = \dim \text{Stab}(E_d) - \dim \text{Stab}(P).$$

As stated in Section II, $\text{Stab}(E_d)$ is the product of orthogonal subgroups $O(n_\mu)$ and a symplectic orthogonal group $\text{OSp}(2n_\mu, \mathbb{R})$ (for $\omega_0 = 1$). The stabilizer of $P$ is a Lie subgroup of $\text{Stab}(E_d)$, which is the product of $O(m'_\mu)$ and $O(m''_\mu)$ for type I and II singular values. Therefore, such critical submanifolds can be represented as

$$\mathcal{M} = \frac{\text{OSp}(2n_0, \mathbb{R})}{\text{OSp}(2m_0, \mathbb{R}) \times \text{OSp}(2n_0-2m_0, \mathbb{R})} \times \prod_{\mu=1}^{s} \frac{O(n_\mu)}{O(m'_\mu) \times O(m''_\mu)},$$

and their dimensions can be easily evaluated as

$$\mathcal{D} = 2m_0(n_0 - m_0) + \sum_{\mu=1}^{s} m'_\mu m''_\mu. \quad (13)$$

The optimality status of these critical submanifolds can be acquired from analysis of the local geometric structure for each of the critical submanifolds via their Hessian quadratic form (HQF). The numbers of positive, negative and zero Hessian eigenvalues determine the optimality status, i.e., a critical point is a local minimum (maximum) if all the eigenvalues are positive (negative), otherwise it is a saddle point. The HQF is defined as the second-order term of $Y$ in the Taylor expansion of the parametrization $Se^{tJY}$, which is dominant in the neighborhood of $S^*$ while the first-order term vanishes.
\[ H(Y) = \text{Tr} [JY(S^T S - W^T S)JY + JYS^T S(JY)^T] \\
= \text{Tr} [JYV_0^T R^T (P^2 - E_d P)RV_0 JY + JYV_0^T R^T P^2 RV_0 (JY)^T] \]

Notice that (1) \( P^2 = D^2 \) because \( L \) commutes with \( D \) and \( L \) is symmetric orthogonal; (2) \( JRV_0 = RV_0 J \), we may transform \( Y \) into \( X = (RV_0)Y(RV_0)^T \) and rewrite the HQF as

\[ H(X) = \text{Tr} [JX(D^2 - DE_d L_d) JX + JXD^2(JX)^T]. \tag{14} \]

Let \( x \) be the vector of independent variables in \( X \), then \( H(X) \) can be written as a quadratic form \( x^T Q x \), where \( Q \) is a symmetric \( N(2N + 1) \times N(2N + 1) \) matrix. The Hessian eigenvalues are defined as the eigenvalues of the matrix \( Q \).

Let \( D = \text{diag}\{\Theta, \Theta^{-1}\}, \ DE_d = \text{diag}\{\Omega, \Omega^{-1}\} \) and \( L = \text{diag}\{\Phi, \Phi\} \). Dividing the symmetric matrix \( X \) as

\[ X = \begin{pmatrix} A & C^T \\ C & B \end{pmatrix}, \]

where \( A \) and \( B \) are symmetric, we may rewrite the HQF as the function of \( A, B \) and \( C \), i.e.,

\[ H(A, B, C) = \text{Tr}(A\Theta^2 A - 2A\Sigma B + B\Theta^{-2} B) + \text{Tr}(C\Theta^2 C^T + 2C\Sigma C + C^T \Theta^{-2} C), \]

where \( \Sigma = (\Theta^2 + \Theta^{-2} - \Omega \Phi - \Phi \Omega^{-1})/2 = \Theta^2 - \Omega \Phi \) (the proof of the second \( "=" \) is nontrivial but will be omitted here).

For illustration, we carry out the Hessian analysis for critical submanifolds that contain only type I and II singular values, where the corresponding \( \Theta \) and \( \Sigma \) are diagonal. Moreover, we assume that \( \omega_i > 1 \) for all \( i = 1, \cdots, N \), and the spectrum of \( E_d \) is so widely spaced that \( \omega_i \neq \omega_j \) for any \( \omega_i < \omega_j \) (the other cases not involving type III singular values can be dealt with as well but are relatively cumbersome). Now suppose that the diagonal elements in \( \Theta \) are ordered as

\[ \text{diag}\{\omega_1 I_{m_1'}, \omega_1^{1/3} I_{m_1''}, \cdots; \omega_s I_{m_s'}, \omega_s^{1/3} I_{m_s''}\}. \]
Then the Hessian form can be decomposed into $\mathcal{H}(A, B, C) = \mathcal{H}_1(A, B, C) + \mathcal{H}_2(A, B, C)$ with

$$
\mathcal{H}_1(A, B, C) = \sum_{j=1}^{N} \left[ (d_j^2 + 2\sigma_j + d_j^2)c_{jj} + (d_ja_{jj} - \sigma_jd_j^{-1}b_{jj})^2 \right]
+ \sum_{1 \leq i < j \leq N} \left\{ \left[ (d_i^{-2} + d_j^{-2})\frac{\sigma_i + \sigma_j}{d_i^{-2} + d_j^{-2}} \right] c_{ij}^2 + \left[ (d_i^{-2} + d_j^{-2})\frac{\sigma_i + \sigma_j}{d_i^{-2} + d_j^{-2}} \right] a_{ij}^2 + \left[ (d_i^{-2} + d_j^{-2})\frac{\sigma_i + \sigma_j}{d_i^{-2} + d_j^{-2}} \right] b_{ij}^2 \right\}.
$$

$$
\mathcal{H}_2(A, B, C) = \sum_{j=1}^{N} (1 - \sigma_j^2)d_j^{-2}b_{jj}
+ \sum_{1 \leq i < j \leq N} \left\{ \left[ (d_i^{-2} + d_j^{-2}) - \frac{(\sigma_i + \sigma_j)^2}{d_i^{-2} + d_j^{-2}} \right] c_{ij}^2 + \left[ (d_i^{-2} + d_j^{-2}) - \frac{(\sigma_i + \sigma_j)^2}{d_i^{-2} + d_j^{-2}} \right] a_{ij}^2 + \left[ (d_i^{-2} + d_j^{-2}) - \frac{(\sigma_i + \sigma_j)^2}{d_i^{-2} + d_j^{-2}} \right] b_{ij}^2 \right\},
$$

where $a_{ij}$, $b_{ij}$, and $c_{ij}$ are matrix elements of $A$, $B$, and $C$; $d_j$ and $\sigma_j$ are diagonal matrix elements of $\Theta$ and $\Sigma$. The expressions consisting of square terms of independent variables actually represent the local coordinate system in which the HQF is diagonalized. This can be used to count the number of positive (negative or zero) Hessian eigenvalues by examining the signs of these square terms.

The first part $\mathcal{H}_1(X)$ contains $N^2 + N$ positive definite terms with respect to any choice of $X$, and hence it provides $N^2 + N$ positive Hessian eigenvalues. The $N^2$ terms in the (positive indefinite) second part $\mathcal{H}_2(X)$ needs to be further analyzed. It is easy to see that the coefficients of the first $N$ terms $(1 - \sigma_j^2)d_j^{-2}b_{jj}$, $j = 1, \cdots, N$, are positive for type I singular values $d_j$ where $\sigma_j = 0$, and negative for type II singular values $d_j$ where $\sigma_j = d_j^2 + d_j^{-2} \geq 2$. They provide $N' = \sum_{i=1}^{s} m'_i$ positive and $N'' = \sum_{i=1}^{s} m''_i$ Hessian eigenvalues.

The signs for the remaining terms are determined by the discriminants $\Delta'_{ij} = (d_i^2 + d_j^{-2})(d_i^{-2} + d_j^2) - (\sigma_i + \sigma_j)^2$ and $\Delta''_{ij} = (d_i^2 + d_j^2)(d_i^{-2} + d_j^{-2}) - (\sigma_i + \sigma_j)^2$, where $i < j$, whose signs correspond to that of Hessian eigenvalues in the coordinates of $c_{ij}$ and $b_{ij}$, respectively.

(1) When both $d_i$ and $d_j$ are of type I, $\sigma_i = \sigma_j = 0$ and hence both $\Delta'_{ij}$ and $\Delta''_{ij}$ are positive. This produces $N'(N' - 1)$ positive Hessian eigenvalues.
(2) When both \(d_i\) and \(d_j\) are of type II, the value of \(\Delta'_{ij}\) is:

\[
\Delta'_{ij} = (d_i^2 + d_j^{-2})(d_i^{-2} + d_j^2) - (d_i^2 + d_i^{-2} + d_j^2 + d_j^{-2})^2 \\
= -[(d_i^{-2} + d_j^2)^2 + (d_i^2 + d_j^{-2})(d_i^{-2} + d_j^2) + (d_i^2 + d_j^{-2})^2] < 0,
\]

from which we can see that the corresponding Hessian eigenvalues are all negative. The same holds for \(\Delta''_{ij}\). In total, this produces \(N''(N'' - 1)\) negative Hessian eigenvalues.

(3) When \(d_i\) is of type I and \(d_j\) is of type II, the discriminants become:

\[
\Delta'_{ij} = (d_i^2 + d_j^{-2})(d_i^{-2} + d_j^2) - (d_i^2 + d_i^{-2}) = d_j^{-4}(d_i^2 - d_j^2)(d_i^6 - d_j^{-2}), \\
\Delta''_{ij} = (d_i^2 + d_j^2)(d_i^{-2} + d_j^{-2}) - (d_i^2 + d_j^{-2})^2 = d_i^4(d_i^2 - d_j^{-2})(d_i^{-6} - d_j^{-2}).
\]

Because the \(d_i\) and \(d_j\) are always chosen to be greater than 1, it is easy to see that \(d_j^6 - d_i^{-2} > 0\) and \(d_i^2 - d_j^{-2} > 0\) except when \(d_i = d_j = 1\). Hence the signs of the discriminants are determined by

\[
\Delta'_{ij} \sim d_i^2 - d_j^2 = \omega_i^2 - \omega_j^{2/3}, \quad \Delta''_{ij} \sim d_j^{-6} - d_i^{-2} = \omega_j^{-2} - \omega_i^{-2}.
\]

\(\Delta'_{ij}\) is positive only when \(\omega_i < \omega_j\), and otherwise negative. However, by assumption this holds only when \(\omega_i = \omega_j\), which brings \(\sum_{\alpha} m'_\alpha m''_\alpha\) positive Hessian eigenvalues and \(\sum_{\alpha<\beta} m'_\alpha m''_\beta\) negative Hessian eigenvalues. The discriminant \(\Delta'_i\leq 0\) for \(\omega_i \leq \omega_j\), which brings \(\sum_{\alpha<\beta} m'_\alpha m''_\beta\) negative, and \(\sum_{\alpha} m'_\alpha m''_\alpha\) zero Hessian eigenvalues.

(4) When \(d_i\) is of type II and \(d_j\) is of type I, we may derive:

\[
\Delta'_{ij} = (d_i^2 + d_j^{-2})(d_i^{-2} + d_j^2) - (d_i^2 + d_j^{-2}) = d_i^{-4}(d_j^2 - d_i^2)(d_i^6 - d_j^{-2}), \\
\Delta''_{ij} = (d_i^2 + d_j^2)(d_i^{-2} + d_j^{-2}) - (d_i^2 + d_j^{-2})^2 = d_i^4(d_j^2 - d_i^{-2})(d_j^{-6} - d_i^{-2}),
\]

which lead to the similar criteria

\[
\Delta'_{ij} \sim d_j^2 - d_i^2 = \omega_j^2 - \omega_i^{2/3} > 0, \quad \Delta''_{ij} \sim d_j^{-6} - d_i^{-2} = \omega_j^{-2} - \omega_i^{-2} > 0.
\]

The Hessian eigenvalues in this case are all positive and its number is \(2 \sum_{\alpha<\beta} m''_\alpha m'_\beta\).
In conclusion, the total number of positive, negative and null Hessian eigenvalues can be summated as follows:

\[ D_0 = \sum_{\alpha=1}^{r} m'_\alpha m''_\alpha, \]  
\[ D_+ = N^2 + N + N'^2 + \sum_{\alpha=1}^{r} m'_\alpha m''_\alpha + 2 \sum_{\alpha<\beta} m'_\alpha m''_\beta, \]  
\[ D_- = N'^2 + 2 \sum_{\alpha<\beta} m''_\alpha m'_\beta. \]  

From these formulas, it is easy to see that, among these critical submanifolds, there is only one local minimum \( S^* = W \) in the landscape whose singular values are all of type I. The rest of them are all saddle submanifolds because both \( D_+ \) and \( D_- \) are nonzero. The same conclusion can be drawn for other critical submanifolds that have no type III blocks.

The Hessian analysis for critical submanifolds involving type III singular values is more complicated and any analytic formula is not available so far. However, it is not difficult to prove that they are all saddle submanifolds. So we may conclude the main theorem in this paper:

**Theorem 1** The optimization problem (1) has a unique minimum \( S^* = W \) and the rest of the critical submanifolds are all saddles.

**Proof:** It is sufficient to prove that critical submanifolds involving type III singular values have saddle structures, i.e., the corresponding Hessian form is neither positive or negative definite, or equivalently, there exist some \( X' \neq 0 \) and \( X'' \neq 0 \) such that \( \mathcal{H}(X') > 0 \) and \( \mathcal{H}(X'') < 0 \). Let \( \mathcal{M} \) be such a critical submanifold, and \( \Sigma = diag\{\Sigma_0, \Sigma_1\} \), where \( \Sigma_0 \) is a \( 2k_i \times 2k_i \) type III block and \( \Sigma_2 \) contains the rest blocks. Choose a particular \( A = diag\{A_0; 0\} \) (similarly for \( B \) and \( C \)), where \( A_0 \) corresponds to \( \Sigma_0 \) and 0 to \( \Sigma_1 \), such that the resulting Hessian quadratic form are irrelevant to \( \Sigma_1 \), i.e.,

\[ \mathcal{H}(A_0, B_0, C_0) = Tr(A_0\Theta_0^2 A_0 - 2A_0\Sigma_0 B_0 + B_0\Theta_0^{-2} B_0) + Tr(C_0\Theta_0^2 C_0^T + 2C_0\Sigma_0 C_0 + C_0^T \Theta_0^{-2} C_0). \]
Here the sub-block
\[
\Sigma_0 = \begin{pmatrix}
(d_i^2 - \gamma_i \cos x_i)I_{k_i} & -\gamma_i \sin x_i I_{k_i} \\
-\gamma_i^{-1} \sin x_i I_{k_i} & (d_i^2 + \gamma_i^{-1} \cos x_i)I_{k_i}
\end{pmatrix}
\]
where \(\gamma_i = (\omega_\alpha \omega_\beta)^{1/2}\) with \(\omega_\alpha\) and \(\omega_\beta\) being the pair of singular values of \(W\) that generates the singular value \(d_i = (\omega_\beta / \omega_\alpha)^{1/2}\). The matrix \(\Theta_0 = d_i I_{2k_i}\).

Now choose \(A_0 = I_2\), \(B_0 = \lambda I_2\) and \(C_0 = 0\), where \(\lambda\) is to be determined. Then
\[
\mathcal{H}(A_0, B_0, C_0) = \text{Tr}(\Theta_0^2 + 2\lambda \Sigma_0 + \lambda^2 \Theta_0^{-2}) = 2k_i[(d_i^2 + \lambda d_i^{-2}) - \lambda(2d_i^2 - (\gamma_i - \gamma_i^{-1}) \cos x_i)].
\]
According to the definition \(\cos x_i = (d_i^2 - d_i^{-2})/(\gamma_i - \gamma_i^{-1})\), the Hessian can be simplified as \(\mathcal{H}(A'_0, B'_0, C'_0) = 2k_i(1 - \lambda)d_i^2\). So the corresponding \(\mathcal{H}(X_0)\) is positive (resp., negative) when \(\lambda < 1\) (resp., \(\lambda > 1\)), which implies that the Hessian is neither positive nor negative definite. End of proof.

V. CRITICAL LANDSCAPE TOPOLOGY CONSTRAINED ON THE COMPACT SYMPLECTIC GROUP

Carrying out optimal control field searches over only the compact subgroup OSp\((2N, \mathbb{R})\) is also important in many circumstances, e.g., using only linear quantum optics to search for a symplectic quantum gate [8]. The derivation of the topology is similar to that for the landscapes on \(U(N)\) described above, since OSp\((2N, \mathbb{R})\) is isomorphic to \(U(N)\) [18]. The Lie algebra of OSp\((2N, \mathbb{R})\) consists of matrices of the form
\[
\text{osp}(2N, \mathbb{R}) = \{A = JY \mid Y^T = Y, \; JY = YJ\}.
\]
The condition for \(S\) to be a critical point in the constrained landscape is
\[
\text{Tr}(W^T S JY) = 0, \quad \forall JY \in \text{osp}(2N, \mathbb{R}),
\]
which can only hold if the matrix \(W^T S J\) is an element of the space complementary to that of \(B\), which is equivalent to requiring that \(W^T S\) is an element of the Jordan algebra.
of $\text{OSp}(2N, \mathbb{R})$. $W^T SJ$ must then simultaneously satisfy the two conditions

$$W^T SJ = -(W^T SJ)^T = JS^TW, \quad J(W^T SJ) = (W^T SJ)J,$$

which implies that $W^T S = S^TW$. This equation can be rearranged to give $S = W\sqrt{I_{2N}}$, where $\sqrt{I_{2N}}$ must lie within the group $\text{OSp}(2N, \mathbb{R})$. Because the cost functional is invariant with respect to the conjugation action of $\text{OSp}(2N, \mathbb{R})$, the solutions correspond to a set of $\text{OSp}(2N, \mathbb{R})$ orbits, i.e., $S^* = WR^TD_mR$, where $R \in \text{OSp}(2N, \mathbb{R})$ and

$$D_m = \begin{pmatrix} -I_m & 0 \\ 0 & I_{N-m} \\ -I_m & 0 \\ 0 & I_{N-m} \end{pmatrix}, \quad 0 \leq m \leq N.$$

There are then $N + 1$ solutions, with values of the cost functional $J = 0, 8, 16, ..., 8N$. The minimum and maximum values of $J$ correspond to $S = W$ and $S = -W$, respectively. The critical manifolds can be expressed as Grassmannian cosets $\mathcal{M} = \text{OSp}(2N, \mathbb{R})/\text{Stab}(D_m)$, where $\text{Stab}(D_m) = \text{OSp}(2m, \mathbb{R}) \times \text{OSp}(2(N-m), \mathbb{R})$, so that

$$G(m, N) = \frac{\text{OSp}(2N, \mathbb{R})}{\text{OSp}(2m, \mathbb{R}) \times \text{OSp}(2(N-m), \mathbb{R})}.$$

The HQF can be calculated by parameterizing the argument of $J(S) = \text{Tr}(W^TS)$ via $Se^{JY}$ as in the above. Taylor expanding the landscape function and keeping only the second-order term, we get the HQF,

$$\mathcal{H}(Y) = \text{Tr}[W^TS^*(JY)^2] = \text{Tr}[R^TD_mR(JY)^2] = \text{Tr}[(JRYR^T)^TD_m(JRYR^T)].$$

Let $X = RYR^T$, which still satisfies the conditions $X^T = X$ and $XJ = JX$. $X$ can be expressed in the form

$$X = \begin{pmatrix} A & C \\ -C & A \end{pmatrix}$$
where \( A^T = A \) and \( C^T = -C \) are \( N \)-dimensional matrices. Let \( a_{ij} = a_{ji} \) and \( c_{ij} = -c_{ji} \) are the matrix elements of \( A \) and \( C \). Since any \( S^* \) is represented by a corresponding matrix \( D \), we obtain the following polynomial expression for the HQF:

\[
H(X) = -2 \sum_{j=1}^{N} a_{jj}^2 \delta_j - 2 \sum_{1 \leq i < j \leq N} (a_{kl}^2 + c_{kl}^2) (\delta_k + \delta_l),
\]

where \( \delta_j \) is the \( j \)-th diagonal element of \( D_m \). It can then be verified that the landscape on the homogenous compact symplectic group has identical critical topology to the (unitary) transformation landscape on \( U(N) \) \[19\], with the following breakdown of Hessian eigenvalues for \( m = 0, \cdots, N \):

\[
\mathcal{D}_+ = (N - m)^2, \quad \mathcal{D}_- = m^2, \quad \mathcal{D}_0 = 2m(N - m).
\]

As in the case of the optimization over the full symplectic group, the critical topology for compact target symplectic gates also consists of orbits of orthogonal symplectic groups, whose numbers of Hessian eigenvalues are

\[
\mathcal{D}_+ = N^2 + N + (N - m)^2, \quad \mathcal{D}_- = m^2, \quad \mathcal{D}_0 = 2m(N - m),
\]

where \( m \) is defined in the standardized block \( P_0 \) in Section III. This shows that the critical topologies are very close between the full symplectic group and its compact subgroup, except for the \( N^2 + N \) difference in the number of positive Hessian eigenvalues. By this difference, the non-optimal critical points for \( \text{Sp}(2N, \mathbb{R}) \) are all saddle points, while one of \( \text{OSp}(2N, \mathbb{R}) \) is a minimal point.

**VI. EXAMPLES**

Consider the SUM gate that has one two-fold degenerate singular value \( \omega = (\sqrt{5} + 1)/2 \). The analysis predicts that there are 4 critical submanifolds for this gate. The first one is the global minimum point \( S_1^* = \text{SUM} \), whose characteristic matrix \( P_1 = \text{diag}\{\omega, \omega^{-1}, \omega^{-1}, \omega\} \) contains one type I block \((m' = 2, m'' = 0)\).
The second critical submanifold is an isolated saddle point, whose characteristic matrix $P_2 = \text{diag}\{-\omega^{-1/3}, -\omega^{1/3}, -\omega^{1/3}, -\omega^{-1/3}\}$ contains one type II block ($m' = 0$, $m'' = 2$), and the corresponding critical point is

$$S^*_2 = U P_2 V = \begin{pmatrix} -0.906 & 0.614 & 0 & 0 \\ -0.292 & -0.906 & 0 & 0 \\ 0 & 0 & -0.906 & 0.292 \\ 0 & 0 & -0.614 & -0.906 \end{pmatrix}.$$ 

As to the third one, where $P_3 = \text{diag}\{\omega, -\omega^{1/3}, -\omega^{-1}, -\omega^{-1/3}\}$ contains one type I and one type II blocks ($m' = 1$, $m'' = 1$), the corresponding critical submanifold is one-dimensional as the orbit of the $O(2)$ symmetry group of $E$. Parameterize $\text{Stab}(E_d)$ as:

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ \mp \sin \theta & \pm \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \mp \sin \theta \\ 0 & 0 & \sin \theta & \pm \cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi),$$

where the signs $\pm$ correspond to the two disjoint parts of $O(2)$. The critical submanifold can be expressed as an orbit of $O(2)$ group:

$$S^*_3 = \begin{pmatrix} 0.152 \cos \theta + 0.047 & 0.990 \cos \theta + 0.307 & 0.307 \sin \theta & 0.953 \sin \theta \\ 1.141 \cos \theta + 0.354 & 0.152 \cos \theta + 0.047 & -0.953 \sin \theta & 0.646 \sin \theta \\ -0.646 \sin \theta & -0.953 \sin \theta & 0.047 - 0.152 \cos \theta & -0.354 + 1.141 \cos \theta \\ 0.953 \sin \theta & -0.307 \sin \theta & -0.307 + 0.990 \cos \theta & 0.047 - 0.152 \cos \theta \end{pmatrix},$$

where the orbits of the two disjoint parts coincide with each other.

The last critical submanifold contains a type III block ($m' = 0$, $m'' = 0$ and $m''' = 1$). This block and its corresponding critical matrix are given by:
\[ P_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_4^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

The Hessian analysis can be done for the first three critical points (submanifolds) with formula given in Section III. Here we exemplify the Hessian analysis with the critical point \( S_4^* \), for which we don’t have an explicit counting formula yet. Using expression (14) and the decomposition of \( X \) into \( (A, B, C) \), we have

\[
\mathcal{H}(X) = \mathcal{H}_1(A, B) + \mathcal{H}_2(C),
\]

where

\[
\mathcal{H}_1(A, B) = a_{11}^2 + 2a_{12}^2 + a_{22}^2 + b_{11}^2 + 2b_{12}^2 + b_{22}^2 - 2a_{11}b_{11} - 4a_{12}b_{12} + 2\omega(a_{11}b_{12} + a_{12}b_{22}) + 2\omega^{-1}(a_{12}b_{11} + a_{22}b_{12}) - 2a_{22}b_{22}
\]

\[
\mathcal{H}_2(C) = 4c_{11}^2 + 2c_{12}^2 + 2c_{21}^2 + 4c_{22}^2 - 2\omega(c_{11}c_{21} + c_{22}c_{21}) - 2\omega^{-1}(c_{12}c_{22} + c_{11}c_{12}) + 4c_{21}c_{12}
\]

Denoting by \( x = (a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}, c_{11}, c_{12}, c_{21}, c_{22}) \) the vector of independent variables, the Hessian form can be expressed as a quadratic polynomial \( x^T Q x \), where

\[ Q = \begin{pmatrix}
1 & 0 & 0 & -1 & \omega & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \omega^{-1} & -2 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \omega^{-1} & -1 & 0 & 0 & 0 & 0 \\
-1 & \omega^{-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega & -2 & \omega^{-1} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -\omega^{-1} & -\omega & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 2 & 2 & -\omega^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega & 2 & 2 & -\omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & -\omega & 4 & 0
\end{pmatrix}
\]

is a block-diagonal 10-dimensional symmetric matrix. Numerical calculation shows that upper block corresponding to \( \mathcal{H}_1(A, B) \) offers 4 positive and 2 negative Hessian eigen-
values; the lower block corresponding to $\mathcal{H}_2(C)$ offers 2 positive and 1 negative Hessian eigenvalues. Hence the HQF has 7 positive and 3 negative eigenvalues.

In summary, there are a total of six critical submanifolds including 3 isolated points and two one-dimensional manifolds. The Hessian analyses are summarized in Table I.

| No. | Critical value | $D_0$ | $D_+$ | $D_-$ | type       |
|-----|----------------|-------|-------|-------|------------|
| 1   | 0              | 0     | 10    | 0     | minimum    |
| 2   | 18.623         | 0     | 6     | 4     | saddle     |
| 3   | 9.311          | 1     | 8     | 1     | saddle     |
| 4   | 10             | 0     | 7     | 3     | saddle     |

TABLE I: Landscape characteristics for the SUM gate.

VII. CONCLUSION

We have resolved the critical solutions for least square problems on the symplectic group. The critical topology of this nonlinear optimization problem over a noncompact Lie group was shown to be of high complexity compared to that of analogous problems on compact Lie groups. However, the topology is still devoid of multiple local extrema, and the critical solutions consist of a finite number of critical submanifolds which are within a bounded region. These results have important applications to the study of control landscapes [20] for classical mechanical systems or continuous variable quantum computation systems, implying that the search of optimal controls would encounter no essential obstructions.

Due to the noncompactness of the symplectic group, the optimal implementation of symplectic transformations (or symplectic gates in continuous variable quantum computation) might be more inefficient than that of unitary transformations (e.g., those applied
in discrete variable quantum computation). Nonetheless, recent OCT simulations using this objective function [8] verify the prediction that local gradient-based algorithms will converge due to the lack of local traps in the landscape.

APPENDIX A: STABILIZERS OF SYMPLECTIC MATRICES

Here we give the structures of stabilizers of several kinds of symplectic matrices encountered in this paper. The blocks $D_a = \text{diag}\{aI_n; a^{-1}I_n\}$ will be frequently encountered corresponding to a reciprocal pair of singular values $a$ and $a^{-1}$. We can substitute the standard form (2) of $R$ into the definition, which gives

$$\begin{pmatrix} aX & aY \\ -a^{-1}Y & a^{-1}X \end{pmatrix} = \begin{pmatrix} aX & a^{-1}Y \\ -aY & a^{-1}X \end{pmatrix}.$$  

It is easy to see that, when $a = 1$, any matrix $R \in \text{OSp}(2n, \mathbb{R})$ is in the stabilizer. For $a \neq 1$, $Y$ has to be zero, and hence leaves $R = \text{diag}\{X; X\}$ where $X \in O(n)$. So, the stabilizer for such $D$ is isomorphic to $O(n)$.

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[21] In this approach, distance is measured in terms of the length of the geodesic joining these two matrices in the group.

[22] Note that Stab($E_d$) can be a disconnected manifold because its subgroups $O(n_i)$ are not connected. However, the orbit of $P$ under the actions of different branches of Stab($E_d$) coincide with each other. So the orbit as a critical submanifold is still connected.