The Complexity of the Evolution of Graph Labelings

Geir Agnarsson
Department of Mathematics
George Mason University, MS 3F2
4400 University Drive
Fairfax, Virginia 22030
geir@math.gmu.edu

Raymond Greenlaw∗
Department of Computer Science
Armstrong Atlantic State University
11935 Abercorn Street
Savannah, Georgia 31419-1997
raymond.greenlaw@gmail.com

Sanpawat Kantabutra
The Theory of Computation Group
Computer Science Department
Chiang Mai University
Chiang Mai, 50200, Thailand
sanpawat@alumni.tufts.edu

Abstract

We study the Graph Relabeling Problem—given an undirected, connected, simple graph \( G = (V, E) \), two labelings \( L \) and \( L' \) of \( G \), and label flip or mutation functions determine the complexity of transforming or evolving the labeling \( L \) into \( L' \). The transformation of \( L \) into \( L' \) can be viewed as an evolutionary process governed by the types of flips or mutations allowed. The number of applications of the function is the duration of the evolutionary period. The labels may reside on the vertices or the edges. We prove that vertex and edge relabelings have closely related computational complexities. Upper and lower bounds on the number of mutations required to evolve one labeling into another in a general graph are given. Exact bounds for the number of mutations required to evolve paths and stars are given. This corresponds to computing the exact distance between two vertices in the corresponding Cayley graph. We finally explore both vertex and edge relabeling with privileged labels, and resolve some open problems by providing precise characterizations of when these problems are solvable. Many of our results include algorithms for solving the problems, and in all cases the algorithms are polynomial-time. The problems studied have applications in areas such as bioinformatics, networks, and VLSI.

1 Introduction

Graph labeling is a well-studied subject in computer science and mathematics, and a problem that has widespread applications, including in many other disciplines. Here we explore a variant of graph labeling called the Graph Relabeling Problem that was first explored by Kantabutra [17] and later by the authors of this paper in [2]. A shorter preliminary version of this paper appeared in [3]. Here we present some new results and extend the results given in [17, 2, 3]. In particular, we \( NC^1 \) reduce the Vertex Relabeling Problem to the Edge Relabeling Problem and vice versa, and provide upper and lower bounds on the complexity of the Vertex and Edge Relabeling

∗Ray gratefully acknowledges Chiang Mai University for supporting this research.
Problems, give tight bounds on relabeling a path, and provide precise characterizations of when instances of relabeling with privileged labels are solvable. The paper also includes a number of related open problems.

The problem of graph labeling has a rich and long history, and we recommend Gallian’s extensive survey for an introduction to this topic and for a cataloging of the many different variants of labeling that have been studied [9]. Puzzles have always intrigued computer scientists and mathematicians alike, and a number of puzzles can be viewed as relabeled graphs (for example, see [34]). One of the most famous of these puzzles is the so-called 15-Puzzle [30]. The 15-Puzzle consists of 15 tiles numbered from 1 to 15 that are placed on a 4 \times 4 board leaving one position empty. The goal is to reposition the tiles of an arbitrary arrangement into increasing order from left-to-right and from top-to-bottom by shifting tiles around while making use of the open hole. In [17], a generalized version of this puzzle called the \((n \times n)\)-Puzzle was used to show a variant of the \textbf{Vertex Relabeling Problem with Privileged Labels} is NP-complete.

Graph labeling has been studied in the context of cartography [16, 21]. And, of course, there are many special types of labelings which are of great interest—codings [24], colorings [5], and rankings [19] to name but three. In these cases we are typically interested in placing labels on the vertices or edges of a graph in some constrained manner so that certain properties are met. Such problems are usually not stated in terms of the evolutionary process that our labeling problems fall under. In August of 2008 Google searches of graph coloring, graph labeling, graph coding and graph ranking turned up 339,000 hits, 10,700,000 hits, 4,060,000 hits and 5,640,000 hits respectively. All of these fields have ongoing research.

The \textbf{Graph Relabeling Problem} is not only interesting in its own right but also has applications in several areas such as bioinformatics, networks, and VLSI. New applications for such work are constantly emerging, and sometimes in unexpected contexts. For instance, the \textbf{Graph Relabeling Problem} can be used to model a \textit{wormhole routing} in processor networks in which one-byte messages called \textit{flits} [33] are sent among processors. In this example each processor has a limited buffer, one byte, and the only way to send a message is by exchanging it with another processor. Other well-known problems, for example, the \textbf{Pancake Flipping Problem}, can be modeled as a special case of our problem [11].

This paper is organized as follows: \S 2 contains preliminaries and definitions; \S 3 shows the Ver-
TEX Relabeling Problem and the Edge Relabeling Problem are $NC^1$ reducible to each other; \cite{4} proves upper and lower bounds for general graphs for both the Vertex Relabeling Problem and the Edge Relabeling Problem; \cite{5} contains exact bounds for relabeling a path and a star; \cite{6} resolves several open problems and includes results about the Vertex Relabeling Problem with Privileged Labels and the Edge Relabeling Problem with Privileged Labels; \cite{7} presents concluding remarks and open problems. For background material on algorithms we refer the reader to \cite{6}, for graph theory to \cite{1}, and for basic notations of complexity theory including reducibility to $NC$.

2 Preliminaries and Problem Definitions

Let $\mathbb{N} = \{1, 2, \ldots \}$ denote the set of the natural numbers. Throughout the paper let $G = (V, E)$ be a simple, undirected, and connected graph. Let $n = |V|$ and $m = |E|$. Let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The line graph of $G = (V, E)$ is the graph $L(G) = (E, E')$, where $E' = \{(e_1, e_2) \mid e_1, e_2 \in E$ and $e_1 \cap e_2 \neq \emptyset\}$ that is, in the line graph edges from the original graph become vertices and two of these new vertices are connected if they share an endpoint in the original graph. Sometimes we use $V(G)$ and $E(G)$ to denote the vertices and edges of the graph $G$ respectively.

Let $S_V, S_E \subseteq \mathbb{N}$. A labeling $L_V$ of $V$ is a mapping $L_V : V \mapsto S_V$. A labeling $L_E$ of $E$ is a mapping $L_E : E \mapsto S_E$. In this paper we are usually interested in $S_V = \{1, 2, \ldots, n\}$ and $S_E = \{1, 2, \ldots, m\}$. We associate a graph $G$ with labelings using angle bracket notation, for example, $(G, L_V, L_E)$ denotes the graph $G$ with vertex labeling $L_V$ and edge labeling $L_E$. A flip or mutation function $f$ maps triples $(G, L_V, L_E)$ to triples $(G, L_V', L_E')$, or ordered pairs if we are only interested in one labeling.

We study both vertex and edge mutation functions. In general, the mutation function $f$ will be defined based on various properties of $G$. Here we study just restricted classes of mutation functions. First, we define the consecutive vertex mutation function, where $f$ maps a pair $(G, L_V)$ to a pair $(G, L_V')$, so $(f \circ L_V)(v_i) = f(L_V(v_i)) = L_V'(v_i)$ for each $i$, with the following conditions:

1. $L_V = L_V'$, except on two vertices $u$ and $w$
2. $\{u, w\} \in E$
3. $L_V(u) = L_V'(w)$ and $L_V(w) = L_V'(u)$
4. $S_V = \{1, 2, \ldots, n\}$
5. $f$ is a bijection

That is, the labels on the adjacent nodes $u$ and $w$ are swapped, while all other labels remain the same. In addition, the set of labels are chosen from $\{1, 2, \ldots, n\}$, and since the definition requires $f$ to be a bijection, labels are used exactly once. It would be interesting to study other types of mutation functions where, for example, labels along an entire path are mutated, or where labels can be reused. One application of the function $f$ is called a flip or mutation. We next define a
decision problem that captures the complexity of the evolution of one labeling into another via the consecutive vertex mutation function.

**Definition 2.1 (Vertex Relabeling Problem)**

**Instance:** A graph $G$, labelings $L_V$ and $L'_V$, and $t \in \mathbb{N}$.

**Question:** Can labeling $L_V$ evolve into $L'_V$ in $t$ or fewer vertex mutations?

We can similarly define the consecutive edge mutation function, where $L_E = L'_E$ except for two edges whose labels have been swapped, and $S_E = \{1, 2, \ldots, m\}$. Note, when employing the consecutive edge mutation function, the edges whose labels are swapped must share an endpoint.

We have the following analogous decision problem for edge relabelings.

**Definition 2.2 (Edge Relabeling Problem)**

**Instance:** A graph $G$, labelings $L_E$ and $L'_E$, and $t \in \mathbb{N}$.

**Question:** Can labeling $L_E$ evolve into $L'_E$ in $t$ or fewer edge mutations?

In the remainder of the paper we focus on the consecutive versions of the mutation functions.

The word ‘consecutive’ refers to the fact that only neighbors can be mutated, that is, labels to be swapped appear consecutively in the graph.

### 3 Relating Vertex and Edge Relabeling

The following theorem shows that the computational complexities of the Vertex Relabeling Problem and the Edge Relabeling Problem are closely related. In the theorem we use $NC^1$ many-one reducibility—a weak form of reducibility; and therefore, one that shows a very close relationship between problems—to relate the Vertex and Edge Relabeling Problems.

**Theorem 3.1 (Vertex/Edge Relabeling Related)**

The Vertex Relabeling Problem is $NC^1$ many-one reducible to the Edge Relabeling Problem, and the Edge Relabeling Problem is $NC^1$ many-one reducible to the Vertex Relabeling Problem.

**Proof.** We first show that the Vertex Relabeling Problem is $NC^1$ many-one reducible to the Edge Relabeling Problem. Consider an instance $G = (V, E), L_V$ and $L'_V$, and $t \in \mathbb{N}$ of the Vertex Relabeling Problem. Let $v'_i, 1 \leq i \leq n$, be $n$ new vertices. We construct a corresponding instance of the Edge Relabeling Problem $G' = (V \cup \{v'_1, \ldots, v'_n\}, E \cup \{\{v_i, v'_i\} | 1 \leq i \leq n\})$, $L_E$ is such that $\{v_i, v'_i\}$ gets $L_V(v_i)$ for $1 \leq i \leq n$ and $L_E(e_i) = i + m$ for $e_i \in E$, $L'_E$ is such that $\{v_i, v'_i\}$ gets $L'_V(v_i)$ for $1 \leq i \leq n$ and $L'_E(e_i) = i + m$ for $e_i \in E$, and the mutation bound is $3t$.

We now argue the correctness of this reduction. If we have a yes instance of the Vertex Relabeling Problem, then it is clear that the answer to the resulting instance of the Edge Relabeling Problem is also yes since each mutation in $G$ can be mimicked by three mutations in $G'$. That is, suppose labels $L(v_k)$ and $L(v_l)$ are mutated. The following three mutations mimic this swap:
1. $L_E(\{v_k, v'_k\})$ with $L_E(\{v_k, v_l\})$

2. new label of $\{v_k, v_l\}$, which is $L_E(\{v_k, v'_k\})$, with $L_E(\{v_l, v'_l\})$

3. new label of $\{v_k, v_l\}$, which is $L_E(\{v_k, v'_k\})$, with new label of $\{v_k, v'_k\}$, which is $L_E(\{v_k, v_l\})$

In the other direction, suppose we have a yes instance of the Edge Relabeling Problem. By construction of $L'_E$, the labels on the original edges of $G$ remain the same in $L_E$ and $L'_E$. Thus, any movement of labels in $G'$ between the edges $\{v_k, v'_k\}$ and $\{v_l, v'_l\}$, where the edge $\{v_k, v_l\} \in E'$, requires a minimum of three mutations to swap the labels on these two edges and to restore the label on $\{v_k, v_l\}$. Thus, the corresponding instance of the Vertex Relabeling Problem also has a yes answer.

It is not hard to see that if each edge knows its number as part of the input, then the reduction can be accomplished in $NC^1$ because addition is in the class $AC^0$ which is contained in $NC^1$.

Now we show that the Edge Relabeling Problem is $NC^1$ many-one reducible to the Vertex Relabeling Problem. Consider an instance $I_E$ of the Edge Relabeling Problem, where $G = (V, E)$, $L_E$ and $L'_E$ are labelings, and $t \in \mathbb{N}$. We construct an instance $I_V$ of the Vertex Relabeling Problem using the line graph $L(G) = (E, E')$, $L_{V(L(G))}(e_i) = L_E(e_i)$ for $1 \leq i \leq m$, $L'_{V(L(G))}(e_i) = L'_E(e_i)$ for $1 \leq i \leq m$, and the bound $t$.

We argue the correctness of the reduction. Since for each mutation in the instance $I_E$ of the edges, there is a corresponding mutation of the vertices in the instance $I_V$, we see that $I_E$ is a yes instance of the Edge Relabeling Problem if and only if $I_V$ is a yes instance of the Vertex Relabeling Problem.

The reduction can be accomplished in $NC^1$. This completes the proof of the theorem.

Notice in the first reduction, we relied on the input being coded so that each edge “knows” its own number. Without having the input encoded in some suitable fashion that provides this information, it is not clear that the reduction is in $NC^1$, however, the reduction could still be performed in $NC^2$. Theorem 3.1 demonstrates a close relationship between the Vertex Relabeling Problem and the Edge Relabeling Problem, when the mutation functions are the consecutive versions. The theorem comes in handy when proving results about the Edge Relabeling Problem based on facts about the Vertex Relabeling Problem.

### 4 Tight Bounds for the Relabeling Problem

This section contains several theorems about the time complexity of the Vertex/Edge Relabeling Problems. Theorem 4.1 shows that for an arbitrary graph and two arbitrary labelings at most $n(n-1)/2$ mutations are required to evolve one vertex labeling into another. Corollary 4.2 shows that a similar statement can be made about the Edge Relabeling Problem. Observation 4.3 yields a lower bound on the number of mutations required in evolving either vertex-labeled graphs or edge-labeled graphs.
We begin with the upper bound on the number of flips required to evolve any given vertex labeling into any other labeling.

**Theorem 4.1 (Vertex Relabeling Upper Bound)**

Let $G = (V, E)$ be a graph, $L_V$ and $L'_V$ vertex labelings, and $t = n(n - 1)/2$, then the answer to the Vertex Relabeling Problem is yes. That is, any labeled graph can evolve into any other labeled graph in at most $n(n - 1)/2$ mutations.

**Proof.** Let $G = (V, E)$ be any graph. We need to consider the number of mutations required to change an arbitrary labeling $L_V$ into an arbitrary labeling $L'_V$.

We first construct a spanning tree $T$ of $G$. Let $v_1, v_2, \ldots, v_n$ be the fixed numbering of the vertices (not labels) that denotes the Prüfer code order when the leaves of $T$ are deleted during the process of constructing a Prüfer code; note, $v_j \in \{v_i \mid 1 \leq i \leq n\}$ for $1 \leq j \leq n$. The Prüfer code iteratively requires the lowest numbered vertex of degree one to be removed. Here we are not interested in the actual Prüfer code itself but rather just the leaf elimination order given by the Prüfer code (see [12] for more on the background and complexity of computing Prüfer codes).

The idea is to transform labels from $L_V$ into their positions in $L'_V$ in the order specified by the $v_i$’s and along the path in the spanning tree from their starting position in $L_V$ to their final position in $L'_V$.

Let $\pi$ be a permutation of $\{1, \ldots, n\}$ (presented as $\pi_1, \ldots, \pi_n$) such that $L_V(v_{\pi_i}) = L'_V(v_{\pi_i})$ for each $i \in \{1, \ldots, n\}$.

To move $L_V(v_{\pi_1}) = L'_V(v_{\pi_1})$ from the initial labeling to its final position can take at most $n - 1$ mutations. Note, $v_1$ is an initial leaf in $T$, and $T$ contains exactly $n - 1$ edges.

To move $L_V(v_{\pi_2}) = L'_V(v_{\pi_2})$ from the initial labeling to its final position, we need at most $n - 2$ mutations, since $L'_V(v_{\pi_1})$ is already in its rightful place.

In general, after $i$ iterations, where all of the labels $L'_V(v_{\pi_1})$ through and including $L'_V(v_{\pi_i})$ are in their correct places, then, to move $L_V(v_{\pi_{i+1}}) = L'_V(v_{\pi_i+1})$ to its correct place, we need at most $n - i - 1$ flips, since the remaining spanning tree induced by the vertex set, $V(T) - \{v_{\ell} \mid 1 \leq \ell \leq i\}$, has exactly $n - i - 1$ edges. Note, we do not perform any flips in locations of the tree that have already been completed.

All in all, we use at most $(n - 1) + (n - 2) + \cdots + 1 = n(n - 1)/2$ flips to obtain $L'_V$ from $L_V$. 

Note that the proof of Theorem 4.1 is constructive and provides the sequence of flips to evolve one labeling into another. We chose to use the well-known Prüfer code ordering to place the labels into leaves first, but any other such leaf ordering would work as well. The complexity of the algorithm in Theorem 4.1 is the complexity of computing a spanning tree, $\theta(n + m)$, plus the complexity of computing the Prüfer code elimination order, $\theta(n)$, plus the complexity of the flips, $\theta(n(n - 1)/2)$, which overall is therefore $\theta(n^2)$. It is interesting to consider that in the parallel setting we might be able to compute the sequence of flips required for the evolution much more quickly than we could actually execute them sequentially. We leave this as an open problem.
Corollary 4.2 contains the analogous result to Theorem 4.1 but for the Edge Relabeling Problem.

**Corollary 4.2 (Edge Relabeling Upper Bound)**

Let $G = (V, E)$ be a graph, $L_E$ and $L'_E$ edge labelings, and $t = m(m-1)/2$, then the answer to the Edge Relabeling Problem is yes. That is, any labeled graph can evolve into any other labeled graph in at most $m(m-1)/2$ flips.

**Proof.** The result follows directly from Theorems 4.1 and 3.1. $\Box$

We now discuss the matching lower bounds for the bounds of $t$ given in Theorem 4.1 and Corollary 4.2, together with some well-known folklorish but relevant results.

Consider the path $P_n$ on $n$ vertices. For convenience we represent a vertex labeling of $P_n$ by a permutation $\pi$ of $\{1, 2, \ldots, n\}$ which we can view as a string $s = \pi_1 \pi_2 \ldots \pi_n$. For each such string $s$ let $p(s)$ be the number of inversions (also known as inversion pairs) of $s$, that is, $p(s) = |\{(i, j) : 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|$. Note that each mutation reduces or increases the value of $p(\cdot)$ by exactly one. In other words, if $s'$ is the string obtained from $s$ by some mutation, then $|p(s') - p(s)| = 1$. This well-known observation is stated as a lemma in the original treatise [25, p. 27] on determinants. From this we see that $p(s)$ is the number of flips or mutations necessary to obtain $\pi_1 \pi_2 \ldots \pi_n$ from $1 2 \ldots n$. This shows that the bound of Theorem 4.1 is tight.

**Observation 4.3 (Lower Bounds for Relabeling Graphs)**

There is a graph $G = (V, E)$, labelings $L_V$ and $L'_V$, and $t = (n(n-1)/2) - 1$ such that the Vertex Relabeling Problem has an answer of no. That is, there exist two labelings that require $n(n-1)/2$ mutations to evolve one into the other. There is a graph $H = (V', E')$, labelings $L_{E'}$ and $L'_{E'}$, and $t = (m(m-1)/2) - 1$ such that the Edge Relabeling Problem has an answer of no.

**Proof.** For the permutations $1 2 \ldots n$ and $n (n - 1) \ldots 1$ (viewed as strings), we clearly have $p(1 2 \ldots n) = 0$ and $p(n (n - 1) \ldots 1) = \left(\begin{array}{c} n \\ 2 \end{array}\right) = n(n - 1)/2$. Hence, at least $n(n - 1)/2$ consecutive flips are needed to obtain $n (n - 1) \ldots 1$ from $1 2 \ldots n$. The case for edges is similar. $\Box$

**Remark:** When we view a labeling of the path $P_n$ on $n$ vertices as a string $s = \pi_1 \pi_2 \ldots \pi_n$, we note that the transformation of $s$ to $1 2 \ldots n$ strongly resembles standard bubble sort—the simplest of the sorting algorithms on $n$ elements (see [18, p. 108] for discussion and analysis). In the case when evolving the string $n (n - 1) \ldots 1$ to $1 2 \ldots n$, the sequence of flips or mutations is precisely the procedure of bubble sort, except for the very last iteration.

## 5 Exact Computations for the Star

In this section we determine exactly how many flips are needed to transform one vertex labeling $L_V$ of $G = (V, E)$ to another vertex labeling $L'_V$ when $G = K_{1,n-1}$ is the star on $n$ vertices. Considering
the cases where $G$ is firstly a path and secondly a star seems like a good starting point since these constitute the simplest trees: the path having the largest diameter (of $n-1$, and smallest maximum degree of two) and the star having the smallest diameter (of two, and the largest maximum degree of $n-1$).

The case when $G = P_n$, the simple path on $n$ vertices, is a well-known classic result. Although the statements of these well-known results for the path are contained in the original work by Thomas Muir \cite{25}, and the expanded and edited version \cite{26}, the proofs are folklorish or scattered throughout the literature at best. Hence, in what follows we provide self-contained proofs of them in our notation. Later on these methods for the path will also be referred to in the case when $G$ is the $n$-star. The case for the star has also been investigated before in this context, in particular, in \cite{4} and from an algorithmic point of view in \cite{22} and \cite{23}, all nice and interesting papers on how this applies to connectivity in computer networks. In this section we will generalize these results and show how some of their results follow from ours as special cases.

Consider the transformation of one labeling of the path $P_n$ into another. It is clear that the minimum number of mutations needed to evolve $s = \pi_1 \pi_2 \ldots \pi_n$ into $s' = 1 2 \ldots n$ is the same as the minimum number of evolving $s'$ into $s$. Hence, for the sake of simplicity, we will assume that we are to evolve $s$ into $s'$. A flip or mutation sequence $(s_i)_{i=0}^m$ is a sequence of strings with $s_0 = s$, $s_m = s'$, and where $s_{i+1}$ is obtained from $s_i$ by a single mutation, $0 \leq i \leq m-1$. In this case we see that for an arbitrary labeling $s = \pi_1 \pi_2 \ldots \pi_n$, we have

$$p(s) = |p(s_0) - p(s_m)| = \sum_{i=0}^{m-1} (p(s_i) - p(s_{i+1})) \leq \sum_{i=0}^{m-1} |p(s_i) - p(s_{i+1})| = m, \quad (1)$$

reestablishing what we know that at least $p(s)$ mutations are needed to evolve $s$ into $s'$.

By induction on $n$, it is easy to see that $p(s)$ mutations suffice to evolve $s$ to $s'$: this claim is clearly true for $n = 2$.

Assume that this assertion is true for length $(n-1)$-strings, and let $s = \pi_1 \pi_2 \ldots \pi_n$ be such that $n = \pi_i$, for a fixed $i$, $1 \leq i \leq n$. In this case we have $p(s) = n - i + p(\hat{s})$, where $\hat{s} = \pi_1 \ldots \pi_{i-1} \pi_{i+1} \ldots \pi_n$. Clearly, in $s$ we can move $n = \pi_i$ to the rightmost position by precisely $n - i$ mutations. By induction, we can obtain $1 2 \ldots (n-1)$ from $\hat{s}$ by $p(\hat{s})$ mutations. Hence, we are able to evolve $s$ into $s'$ using $p(s)$ mutations.

Finally, we note that if we have two vertex labelings $L_V$ and $L'_V$ of the vertices of the path $P_n$, we can define the corresponding relative parameter $p(L_V, L'_V)$ as $p(s)$, where $s$ is the unique permutation obtained from $L_V$ by renaming the labels in $L'_V$ from left-to-right as $1, 2, \ldots, n$ and reflecting these new names in $L_V$. By our previous comment, we have the symmetry $p(L_V, L'_V) = p(L'_V, L_V)$. This well-known result can now be stated in our notation as follows.

**Observation 5.1 (Tight Bound on Path Relabeling Complexity)**

Let $P_n$ be the path on $n$ vertices, $L_V$ and $L'_V$ vertex labelings, and $t \in \mathbb{N}$. Then the answer to the Vertex Relabeling Problem for $P_n$ is yes if and only if $t \geq p(L_V, L'_V)$. 

Finally, note that by Observation [5.1] we can always evolve $L_V$ into $L'_V$ using the minimum of $p(L_V, L'_V)$ mutations, and repeating the last mutation (or any fixed mutation for that matter!) $2k$ times is not going to alter $L'_V$, since repeating a fixed mutation an even number of times corresponds to the identity (or neutral) relabeling. Hence, for any nonnegative integer $k$ one can always evolve $L_V$ into $L'_V$ using $t = p(L_V, L'_V) + 2k$ mutations.

We will now verify that if $L_V$ can evolve into $L'_V$ in $t$ mutations, then $t - p(L_V, L'_V)$ must be even. By renaming the labels, we may assume $L_V$ is given by the string $s = \pi_1 \pi_2 \ldots \pi_n$ and $L'_V$ by the string $s' = 1 \, 2 \, \ldots \, n$. Now let $(s_i)_{i=0}^m$ and $(s'_i)_{i=0}^{m'}$ be two mutation sequences with $s_0 = s'_0 = s$ and $s_m = s'_{m'} = s'$. Since $p(s_0) = p(s'_0) = p(s)$ and $p(s_m) = p(s'_m) = 0$, we have

$$p(s) = p(s_0) - p(s_m) = \sum_{i=0}^{m-1} (p(s_i) - p(s_{i+1})) = P_+ - P_-,$$

and

$$p(s) = p(s'_0) - p(s'_{m'}) = \sum_{i=0}^{m'-1} (p(s'_i) - p(s'_{i+1})) = P'_+ - P'_-,$$

where

$$P_+ = |\{i \in \{0, \ldots, m-1\} : p(s_i) - p(s_{i+1}) = 1\}|,$$

$$P_- = |\{i \in \{0, \ldots, m-1\} : p(s_i) - p(s_{i+1}) = -1\}|,$$

$$P'_+ = |\{i \in \{0, \ldots, m'-1\} : p(s'_i) - p(s'_{i+1}) = 1\}|,$$

and

$$P'_- = |\{i \in \{0, \ldots, m'-1\} : p(s'_i) - p(s'_{i+1}) = -1\}|.$$

In particular, we have $P'_+ - P'_- = P_+ - P_-$. Since $m = P_+ + P_-$ and $m' = P'_+ + P'_-$, we obtain

$$m' - m = (P'_+ + P'_-) - (P_+ + P_-) = (P'_+ - P_+) + (P'_- - P_-) = 2(P'_+ - P'_-),$$

and thus $m$ and $m'$ must have the same parity. This result shows that if $L_V$ is evolved into $L'_V$ in exactly $t$ mutations, then $t - p(s)$ must be even. This proves the following well-known fact about permutations, which in our setting reads as follows.

**Theorem 5.2 (Muir)** Let $P_n$ be the path on $n$ vertices, $L_V$ and $L'_V$ vertex labelings, and $t \in \mathbb{N}$. Then we can evolve the labeling $L_V$ into $L'_V$ using $t$ mutations if and only if $t = p(L_V, L'_V) + 2k$ for some nonnegative integer $k$.

**Remark:** In many places in the literature (especially in books on abstract algebra), a permutation of $\{1, 2, \ldots, n\}$ that swaps two elements $i \leftrightarrow j$ is called a transposition or a 2-cycle and is denoted by $(i, j)$. If $i < j$, then a flip or mutation in our context is a transposition where $j = i + 1$. In general, by first moving $j$ to the place of $i$ and then moving $i$ up to the place of $j$, we see that $(i, j)$ can be obtained by exactly $2(j - i) - 1$ mutations. Since every permutation $\pi$ of $\{1, 2, \ldots, n\}$ is a composition of transpositions, say $t$ of them, then $\pi$ can be obtained from $1 \ldots n$ by $N$ mutations, where $N$ is a sum of $t$ odd numbers. By Theorem 5.2 we therefore have that $p(\pi) \equiv N \equiv t$. 

(mod 2). This result gives an alternative and more quantitative proof of the classic group-theoretic fact that the parity of the number of transpositions in a composition that yields a given permutation is unique and only depends on the permutation itself (see [15, p. 48] for the classic proof).

We now discuss the case $G = K_{1,n-1}$, the star on $n$ vertices. For our general setup, let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G) = \{\{v_0, v_i\} : i = 1, 2, \ldots, n-1\}$, so we assume that $v_0$ is the center vertex of our star $G$. If $L_V$ and $L'_V$ are two vertex labelings of $G$, we may (by renaming the vertices) assume $L'_V(v_i) = i$ for each $i \in \{0, 1, \ldots, n-1\}$. In this case the initial labeling is given by $L_V(v_i) = \pi(i)$, where $\pi$ is a permutation of $\{0, 1, \ldots, n-1\}$, and so $\pi \in S_n$, the symmetric group on $n$ symbols $\{0, 1, \ldots, n-1\}$ in our case here. Call the set of the elements moved by $\pi$ the support of $\pi$, denote this set by $\text{Sp}(\pi)$, and let $|\text{Sp}(\pi)| = |\pi|$ be its cardinality. If $\pi$ has the set $S$ as it support, then we say that $\pi$ is a permutation on $S$ (as supposed to a permutation of $S$).

Recall that a cycle $\sigma \in S_n$ is a permutation such that $\sigma(i_{\ell}) = i_{\ell+1}$ for all $\ell = 1, \ldots, c-1$, and $\sigma(i_c) = i_1$, where $\text{Sp}(\sigma) = \{i_1, \ldots, i_c\} \subseteq \{0, 1, \ldots, n-1\}$ is the support of the cycle, so $|\sigma| = c$ here. Such a cycle $\sigma$ is denoted by $(i_1, \ldots, i_c)$. Each permutation $\pi \in S_n$ is a product of disjoint cycles $\pi = \sigma_1 \sigma_2 \cdots \sigma_k$ (see [15, p. 47]), and this product/composition is unique. (Note that every two disjoint cycles commute as compositions of maps $\{0, 1, \ldots, n-1\} \rightarrow \{0, 1, \ldots, n-1\}$). For each permutation $\pi$, denote its number of disjoint cycles by $\varsigma(\pi)$. Note that for the star $G$ every mutation or flip has the form $f_i$, where $f_i$ swaps the labels on $v_0$ and $v_i$ for $i \in \{1, 2, \ldots, n-1\}$. Hence, we have $f_i = (0, i)$, the 2-cycle transposing 0 and $i$.

**Lemma 5.3** Let $G = K_{1,n-1}$ be the star on $n$ vertices. Let $L_V$ and $L'_V$ be vertex labelings such that $L_V(v_i) = \sigma(i)$ and $L'_V(v_i) = i$, where $\sigma$ is a cycle with $\text{Sp}(\sigma) \subseteq \{1, 2, \ldots, n-1\}$. In this case the labeling $L_V$ can be transformed into $L'_V$ in $|\sigma| + 1$ or fewer flips.

**Proof.** If $\sigma = (i_1, \ldots, i_c)$, where $\{i_1, \ldots, i_c\} \subseteq \{1, 2, \ldots, n-1\}$, then apply the composition $f_{\sigma} := f_{i_1} f_{i_2} \cdots f_{i_c} f_{i_1}$ to the labeling $L_V$ and obtain $L'_V$ since

$$f_{i_1} f_{i_2} \cdots f_{i_c} f_{i_1} \sigma = (0, i_1)(0, i_2) \cdots (0, i_c)(0, i_1)(i_1, \ldots, i_c)$$

is the identity permutation. Since $f_{\sigma}$ consists of $c + 1$ flips altogether, we have the lemma. \qed

For a cycle $\sigma$ with $\text{Sp}(\sigma) \subseteq \{1, 2, \ldots, n-1\}$, let $f_{\sigma}$ denote the composition of the $|\sigma| + 1$ label flip functions as in the previous proof. By Lemma 5.3 we have the following corollary.

**Corollary 5.4** Let $G = K_{1,n-1}$ be the star on $n$ vertices. Let $L_V$ and $L'_V$ be vertex labelings such that $L_V(v_i) = \pi(i)$ and $L'_V(v_i) = i$ for $i \in \{0, 1, \ldots, n-1\}$ where $\pi(0) = 0$. In this case the labeling $L_V$ can be transformed into $L'_V$ in $|\pi| + \varsigma(\pi)$ or fewer flips.

**Proof.** If $\pi = 1 \cdots 1 \cdots k$, a product of $k$ disjoint cycles each having its support in $\{1, 2, \ldots, n-1\}$, then apply the composition $f_{\sigma_k} f_{\sigma_{k-1}} \cdots f_{\sigma_1}$ to the labeling $L_V$ and obtain $L'_V$. This composition consists of $\sum_{i=1}^k (|\sigma_i| + 1) = |\pi| + k = |\pi| + \varsigma(\pi)$ flips altogether. \qed
Corollary 5.4 establishes an upper bound on how many flips are needed to transform one labeling into another. This upper bound is the easier part and coincides with [4] Lemma 1, p. 561.

We now consider the harder case. In order to obtain the tight lower bound, we will define a parameter \( q(\cdot) \), a function from the set of all possible labelings of \( G \) into the set of nonnegative integers, such that each flip either reduces or increases the parameter by exactly one, just like the number \( p(\cdot) \) of inversions of a permutation on the path. Before we present the formal definition of the parameter \( q \), we need some notation. For each permutation \( \pi \) on \( \{0, 1, \ldots, n-1\} \), we define a corresponding permutation \( \pi^0 \) on the same set in the following way:

1. If \( \pi(0) = 0 \), then \( \pi^0 := \pi \).
2. If \( \pi(0) = i \neq 0 \), then let \( j \in \{1, 2, \ldots, n-1\} \) be the unique element with \( \pi(j) = 0 \). In this case we let \( \pi^0 := \pi(0, j) \).

Note that for any permutation \( \pi \) on \( \{0, 1, \ldots, n-1\} \) we always have \( \pi^0(0) = 0 \). If \( L_V \) is a vertex labeling of the star \( G \) such that \( L_V(v_i) = \pi(i) \) for each \( i \in \{0, 1, \ldots, n-1\} \), then let \( L'_V \) be the vertex labeling corresponding to the permutation \( \pi^0 \), so \( L'_V(v_i) = \pi^0(i) \) for each \( i \in \{0, 1, \ldots, n-1\} \).

With this preliminary notation we can now define our parameter.

**Definition 5.5** Let \( L_V : V(G) \rightarrow \{0, 1, \ldots, n-1\} \) be a vertex labeling of the star \( G = K_{1,n-1} \) given by \( L_V(v_i) = \pi(i) \), where \( \pi \) is some permutation of \( \{0, 1, \ldots, n-1\} \).

1. If \( \pi(0) = 0 \), then let \( q(L_V) = |\pi| + \varsigma(\pi) \).
2. Otherwise, if \( \pi(0) = i \neq 0 \) and hence \( \pi(j) = 0 \) for some \( j \), then let

\[
q(L_V) = \begin{cases} 
q(L'_V) + 1 & \text{if } i = j, \\
q(L'_V) - 1 & \text{if } i \neq j.
\end{cases}
\]

Note that \( L_V(v_i) = i \) for each \( i \in \{0, 1, \ldots, n-1\} \) if and only if \( q(L_V) = 0 \).

We now want to show that if \( L_V \) is a vertex labeling of the star \( G \), and \( L'_V \) is obtained from \( L_V \) by a single flip, then \( |q(L_V) - q(L'_V)| = 1 \). First we note that if one of the labels swapped by the single flip is zero, then we either have \( L'_V = L^0_V \) or vice versa \( L_V = L^0_V \). Hence, in this case we have directly by Definition 5.5 that \( |q(L_V) - q(L'_V)| = 1 \).

Assume now that neither labels \( i \) nor \( j \) swapped by the flip is zero. In this case we have \( L_V(v_0) = i \) and \( L'_V(v_0) = j \), and hence \( L_V(v_\ell) = L'_V(v_\ell) = 0 \) for some \( \ell \in \{1, 2, \ldots, n-1\} \). Let the labelings \( L_V \) and \( L'_V \) on \( \{0, 1, \ldots, n-1\} \) be given by the permutations \( \pi \) and \( \pi' \), respectively. Since \( \pi(k) = j \) and \( \pi'(k) = i \) for some \( k \neq \ell \) and \( \pi'(\ell) = \pi(\ell) = 0 \), we have \( \pi' = \pi(0, k) \). Using the notation introduced earlier, we have \( \pi^0 = \pi(0, \ell) \) and \( \pi'^0 = \pi'(0, \ell) \). Since \( \pi = \pi(0, \ell)(0, \ell) = \pi^0(0, \ell) \) and \( (0, \ell)(0, k)(0, \ell) = (k, \ell) \), we have

\[
\pi'^0 = \pi'(0, \ell) = \pi(0, k)(0, \ell) = \pi^0(0, \ell)(0, k)(0, \ell) = \pi^0(k, \ell).
\] (3)

Note that (3) also implies that \( \pi'^0(k, \ell) = \pi^0 \), and so this observation yields a symmetry \( \pi^0 \leftrightarrow \pi'^0 \) that we will use later. Also, since \( \pi^0(k) = \pi(k) = j \), \( \pi^0(\ell) = \pi(0) = i \), \( \pi'^0(k) = \pi'(k) = i \), and
We now consider the following cases for Proposition 5.8 part 2.

Let \( \sigma \) be a cycle and \( i, j \) be distinct. Then one of the following holds for \( \sigma(i_1, i_2) \):

1. \( \text{Sp}(\sigma(i_1, i_2)) = \text{Sp}(\sigma) \) and \( \sigma(i_1, i_2) = \sigma_1\sigma_2 - \text{a product of disjoint cycles with} \text{Sp}(\sigma_1) \cup \text{Sp}(\sigma_2) = \text{Sp}(\sigma) \).
2. \( \text{Sp}(\sigma(i_1, i_2)) = \text{Sp}(\sigma) \setminus \{i^*\} \), where \( i^* \in \{i_1, i_2\} \) and \( \sigma(i_1, i_2) \) is a cycle on \( \text{Sp}(\sigma) \setminus \{i^*\} \).
3. \( \text{Sp}(\sigma(i_1, i_2)) = \emptyset \) and \( \sigma = (i_1, i_2) \).

Proof. Let \( \sigma = (a_1, \ldots, a_h) \), where \( h \geq 2 \). We may assume \( (i_1, i_2) = (a_i, a_i) \) for some \( i \in \{2, \ldots, h\} \).

We now consider the following cases for \( h \) and \( i \):

If \( h = 2 \), then \( i = 2 \) and \( \sigma = (a_1, a_2) = (i_1, i_2) \), and we have part 3.

If \( h \geq 3 \) and \( i = 2 \), then \( \sigma(i_1, i_2) = (a_1, \ldots, a_h)(a_1, a_2) = (a_1, a_3, \ldots, a_h) \), and we have part 2.

If \( h \geq 3 \) and \( i = h \), then \( \sigma(i_1, i_2) = (a_1, \ldots, a_h)(a_1, a_h) = (a_2, a_3, \ldots, a_h) \), and again we have part 2.

Finally, if \( h \geq 3 \) and \( i \not\in \{2, h\} \), then \( i \in \{3, \ldots, h - 1\} \) (and hence \( h \geq 4 \)), and \( \sigma(i_1, i_2) = (a_1, \ldots, a_h)(a_1, a_i) = (a_1, a_{i+1}, \ldots, a_h)(a_2, \ldots, a_i) \), and we have part 1. 

We are now ready to consider the cases of whether \( \ell \in \{i, j\} \) or not.

First case: \( \ell \not\in \{i, j\} \). Directly by definition we have here that \( q(L_V) = q(L'_V) - 1 \) and \( q(L'_V) = q(L'_V) - 1 \), and hence \( q(L_V) - q(L'_V) = q(L'_V) - q(L'_V) \).

Proposition 5.8 If \( \ell \not\in \{i, j\} \), then \( q(L_V) - q(L'_V) = q(L'_V) - q(L'_V) = \pm 1 \).
Proof. Assuming \( \ell \not\in \{i,j\} \), we have \( \pi^0(\ell) = i \) and \( \pi^0(\ell) = j \), and hence \( \ell \) is in the support of both \( \pi^0 \) and \( \pi^0 \).

If \( k \not\in \{i,j\} \), then \( \{k,\ell\} \) is contained in both \( \text{Sp}(\pi^0) \) and \( \text{Sp}(\pi^0) \), and hence by definition we have that \( |\pi^0(k,\ell)| = |\pi^0| \). Since \( \pi^0 \) is a product of disjoint cycles, then, either (i) there are two

cycles \( \sigma_1 \) and \( \sigma_2 \) of \( \pi^0 \) such that \( k \in \text{Sp}(\sigma_1) \) and \( \ell \in \text{Sp}(\sigma_2) \), or (ii) there is one cycle \( \sigma \) of \( \pi^0 \) such that \( \{k,\ell\} \subseteq \text{Sp}(\sigma) \).

Since the cycles of \( \pi \) commute, we have by Claim 5.6 in case (i) that

\[ \varsigma(\pi^0(k,\ell)) = \varsigma(\pi^0) - 1, \]

and by Claim 5.7 in case (ii) part 1 that \( \varsigma(\pi^0(k,\ell)) = \varsigma(\pi^0) + 1 \). By (4) this completes the argument when \( k \not\in \{i,j\} \).

If \( k \in \{i,j\} \), we may by symmetry \( \pi^0 \leftrightarrow \pi^0 \) assume that \( k = i \neq j \). In this case we have \( \pi^0(k) = j \) so \( k \in \text{Sp}(\pi^0) \), and \( \pi^0(k) = i \) so \( k \not\in \text{Sp}(\pi^0) \). Hence, we have \( |\pi^0(k,\ell)| = |\pi^0| - 1 \). Since \( \pi^0(\ell) = i = k \), we see that both \( k \) and \( \ell \) are contained in the same cycle \( \sigma \) of \( \pi^0 \) in its disjoint cycle decomposition, and they are consecutive. Moreover, since \( \pi^0(k) = j \neq i \), we see that \( |\sigma| \geq 3 \).

Again, since disjoint cycles commute, we have by Claim 5.7 part 2 that \( \varsigma(\pi^0(k,\ell)) = \varsigma(\pi^0) \). By (4) this fact completes the argument when \( k = i \), and hence the proof of the proposition.

**SECOND CASE:** \( \ell \in \{i,j\} \). By symmetry we may assume \( \ell = i \). In this case we have directly by definition that \( q(L_V) = q(L^0_V) + 1 \) and \( q(L_V') = q(L^0_V) - 1 \). Before continuing we need one more basic observation about permutations.

**Claim 5.9** Let \( \sigma \) be a cycle. If \( i_1 \in \text{Sp}(\pi) \) and \( i_2 \not\in \text{Sp}(\pi) \), then \( \sigma(i_1,i_2) \) is a cycle on \( \text{Sp}(\pi) \cup \{i_2\} \).

**Proof.** Let \( \sigma = (a_1,\ldots,a_h) \). We may assume \( (i_1,i_2) = (a_1,b) \), where \( b \not\in \{a_1,\ldots,a_h\} \), and so we get \( \sigma(i_1,i_2) = (a_1,\ldots,a_h)(a_1,b) = (a_1,b,a_2,\ldots,a_h) \). \(\square\)

**Proposition 5.10** If \( \ell = i \), then \( q(L_V) - q(L_V') = q(L^0_V) - q(L^0_V) + 2 = \pm 1 \).

**Proof.** Assuming \( \ell = i \), we have \( \pi^0(\ell) = i \) and \( \pi^0(\ell) = j \), and hence \( \ell \in \text{Sp}(\pi^0) \setminus \text{Sp}(\pi^0) \).

If \( k \in \{i,j\} \), then since \( k \neq \ell \), we have \( k = j \). Also, since \( \pi^0(k) = j \) and \( \pi^0(k) = i \), we have \( k \in \text{Sp}(\pi^0) \setminus \text{Sp}(\pi^0) \). Since \( \pi^0 \) and \( \pi^0 \) only differ on \( k \) and \( \ell \), we have \( \text{Sp}(\pi^0) = \text{Sp}(\pi^0) \cup \{k,\ell\} \), this union being disjoint. From this fact it is immediate that \( |\pi^0| = |\pi^0(k,\ell)| = |\pi^0| + 2 \) and \( \varsigma(\pi^0(k,\ell)) = \varsigma(\pi^0) + 1 \), and hence by (4), we have the following:

\[ q(L^0_V) = |\pi^0(k,\ell)| + \varsigma(\pi^0(k,\ell)) = |\pi^0| + \varsigma(\pi^0) + 3 = q(L^0_V) + 3, \]

and hence \( q(L^0_V) - q(L^0_V) + 2 = q(L^0_V) - (q(L^0_V) + 3) + 2 = -1 \), which completes the argument when \( k \in \{i,j\} \).

If \( k \not\in \{i,j\} \), then since \( \pi^0(k) = j \) and \( \pi^0(k) = i \), we have that \( k \) is contained in both \( \text{Sp}(\pi^0) \) and \( \text{Sp}(\pi^0) \), and therefore \( |\pi^0(k,\ell)| = |\pi^0| + 1 \). Since \( \pi^0 \) is a product of disjoint cycles, there is a unique cycle \( \sigma \) of \( \pi^0 \) whose support contains \( k \). By Claim 5.9 \( \sigma(k,\ell) \) is a cycle on \( \text{Sp}(\pi) \cup \{\ell\} \), and hence \( \varsigma(\pi^0(k,\ell)) = \varsigma(\pi^0) \). By (4) we therefore have

\[ q(L^0_V) = |\pi^0(k,\ell)| + \varsigma(\pi^0(k,\ell)) = |\pi^0| + \varsigma(\pi^0) + 1 = q(L^0_V) + 1, \]

and hence \( q(L^0_V) - q(L^0_V) + 2 = q(L^0_V) - (q(L^0_V) + 1) + 2 = 1 \), which completes the argument when \( k \not\in \{i,j\} \). This result completes the proof. \( \square \)
Corollary 5.11 Let \( G = K_{1,n-1} \) be the star on \( n \) vertices. If \( L_V \) is a vertex labeling of \( G \) and \( L'_V \) is a vertex labeling obtained from \( L_V \) by a single flip, then \( |q(L'_V) - q(L_V)| = 1 \).

Corollary 5.11 shows that the upper bound given in Corollary 5.4 is also a lower bound. We summarize these results in the following.

Proposition 5.12 Let \( G = K_{1,n-1} \) be the star on \( n \) vertices. Let \( L_V \) and \( L'_V \) be vertex labelings such that \( L'_V(v_1) = i \) and \( L_V(v_1) = \pi(i) \) for \( i \in \{0,1,\ldots,n-1\} \), where \( \pi(0) = 0 \). In this case the labeling \( L_V \) can be transformed into \( L'_V \) in \( t \) flips if and only if \( t \geq |\pi| + \varsigma(\pi) \).

In Proposition 5.12 we restricted to labelings \( L_V \) and \( L'_V \) with \( L_V(v_0) = L'_V(v_0) = 0 \), which by Definition 5.5 is the fundamental case for defining the parameter \( q(L_V) \). Just as we summarized for the case of the path \( G = P_n \) in the beginning of this section, we can likewise define the relative star parameter \( q(L_V,L'_V) \) for any two vertex labelings \( L_V \) and \( L'_V \) of the \( n \)-star \( G = K_{1,n-1} \) to be \( q(L'_V,L'_V) \), where \( L'_V \) is the unique vertex labeling obtained from \( L_V \) by renaming the labels of \( L'_V \) so that \( L'_V(v_i) = i \) for all \( i \). (Strictly speaking, if \( L_V \) and \( L'_V \) are given by permutations \( \pi \) and \( \pi' \) of \( \{0,1,\ldots,n-1\} \), then \( L'_V \) is given by the permutation \( \pi'' = \pi(\pi'^{-1}) \).) Clearly, this relative parameter \( q \) is symmetric, \( q(L_V,L'_V) = q(L'_V,L_V) \), as was the case for the path.

As with the path \( P_n \), where the parameter \( \rho(\cdot) \) increased or decreased by exactly one with each mutation or flip, by Corollary 5.11 so does \( q(\cdot) \) for the star \( G = K_{1,n-1} \). Hence, exactly the same arguments used for (1) and (2) can be used to obtain the following theorem, our main result of this section.

Theorem 5.13 Let \( G = K_{1,n-1} \) be the star on \( n \) vertices, \( L_V \) and \( L'_V \) vertex labelings, and \( t \in \mathbb{N} \). Then we can transform the labeling \( L_V \) into \( L'_V \) using \( t \) flips if and only if \( t = q(L_V,L'_V) + 2k \) for some nonnegative integer \( k \), where \( q \) is the relative parameter corresponding to the one in Definition 5.5.

Theorem 5.13 generalizes the results both from [4] and [23].

Consider the graph \( C \) where its vertex set \( V(C) \) consists of all the \( n! \) vertex labelings of the star \( G = K_{1,n-1} \), so each vertex \( v_\pi \) of \( C \) corresponds to a permutation \( \pi \in S_n \), and where two vertices \( v_\pi \) and \( v_{\pi'} \) are connected in \( C \) if and only if \( \pi' = \pi(i) \) for some \( i \in \{1,2,\ldots,n-1\} \). Here \( C = (V(C),E(C)) \) is an example of a Cayley graph, and this particular one is sometimes ambiguously also referred to as the star graph in the literature [4, p. 561], [22], and [23, p. 374]. In terms of Cayley graphs, we can interpret Theorem 5.13 as follows:

Corollary 5.14 Let \( C \) be the Cayley graph of the \( n \)-star \( G = K_{1,n-1} \). For any \( \pi, \pi' \in S_n \), let \( v_\pi, v_{\pi'} \in V(C) \) be the corresponding vertices of \( C \), and \( L_V \) and \( L'_V \) the corresponding vertex labelings of \( G \). Then the following holds:

1. The distance between \( v_\pi \) and \( v_{\pi'} \) in \( C \) is precisely \( q(L_V,L'_V) \).

2. There is a walk between \( v_\pi \) and \( v_{\pi'} \) in \( C \) of length \( d \) if and only if \( d = q(L_V,L'_V) + 2k \) for some nonnegative integer \( k \).
Other related results regarding the Cayley graph of the star can be found in [32] where the distance distribution among the vertices of the star graph is computed, and in [27] where the cycle structure of the Cayley graph of the star is investigated.

Let \( n \in \mathbb{N} \) be given. Among all permutations \( \pi \) on \( \{0, 1, \ldots, n-1\} \) with \( \pi(0) = 0 \), clearly a maximum value of \( |\pi| \) is \( n-1 \), obtained when \( \text{Sp}(\pi) = \{1, 2, \ldots, n-1\} \). Also, the maximum value of \( \varsigma(\pi) \) is \( \lfloor (n-1)/2 \rfloor \), obtained when every cycle of \( \pi \) has support of two when \( n-1 \) is even, or when every cycle except one (with support of three) has support of two when \( n-1 \) is odd. Hence, among all permutations \( \pi \) on \( \{0, 1, \ldots, n-1\} \), the maximum value of \( |\pi^0| + \varsigma(\pi^0) \) is always \( n-1 + \lfloor (n-1)/2 \rfloor = \lfloor 3(n-1)/2 \rfloor \).

Consider the star \( G = K_{1,n-1} \) and a vertex labeling \( L_V \) of \( G \) with \( q(L_V) \) at maximum. Let \( \pi \) be the permutation on \( \{0, 1, \ldots, n-1\} \) corresponding to \( L_V \), so \( L_V(v_i) = \pi(i) \). If \( \pi(0) = 0 \), then \( \pi = \pi^0 \) and by Definition [5,5] the value \( q(L_V) \) is at most \( \lfloor 3(n-1)/2 \rfloor \). Assume now that \( \pi(0) = i \neq 0 \), and hence \( \pi(j) = 0 \) for some \( j \). If \( i \neq j \), then by Definition [5,5] and previous remarks \( q(L_V) = q(L_V^0) - 1 \leq \lfloor 3(n-1)/2 \rfloor - 1 \). Finally if \( i = j \), then \( q(L_V) = q(L_V^0) + 1 \). Since \( \pi^0 = \pi(0,j) \), we obtain in this case that

\[
\pi^0(i) = [\pi(0,j)](i) = [\pi(0,i)](i) = i,
\]

and hence \( i \notin \text{Sp}(\pi^0) \). Therefore, \( |\pi^0| \leq n-2 \) and \( \varsigma(\pi^0) \leq \lfloor 3/2(n-2) \rfloor \), and so \( q(L_V) = q(L_V^0) + 1 = |\pi^0| + \varsigma(\pi^0) + 1 \leq n-2 + \lfloor (n-2)/2 \rfloor + 1 = \lfloor (3n-4)/2 \rfloor \leq \lfloor 3(n-1)/2 \rfloor \).

From this inequality we see, in particular, that for a given \( n \in \mathbb{N} \), the maximum value of \( q(L_V) \) among all vertex labelings of the star on \( n \) vertices is \( \lfloor 3(n-1)/2 \rfloor \). Hence, we obtain the next observation as a special case. This special case was also observed both in [4] p. 561 and in [23] p. 378. In our setting we can state the following.

**Observation 5.15** Let \( G = K_{1,n-1} \) be the star on \( n \) vertices, \( L_V \) and \( L_V' \) vertex labelings, and \( t = \lfloor 3(n-1)/2 \rfloor \), then the answer to the Vertex Relabeling Problem is yes. That is, any labeled star on \( n \) vertices can evolve into any other labeled star in \( t = \lfloor 3(n-1)/2 \rfloor \) mutations. Moreover, this value of \( t \) is the smallest possible with this property.

A group-theoretical interpretation of this result is as follows.

**Corollary 5.16** Let \( T \subseteq S_n \) be a set of \( n-1 \) transpositions, all of which move a given element. Then every permutation \( \pi \in S_n \) is a composition of at most \( t = \lfloor 3(n-1)/2 \rfloor \) transpositions from \( T \), and this value of \( t \) is the least \( t \) with this property.

We conclude this section by some observations that generalize even further what we have done for the path and the star, but first we need some additional notation and basic results.

For \( n \in \mathbb{N} \) let \( K_n \) be the complete graph on \( n \) vertices, and let \( V(K_n) = \{v_1, \ldots, v_n\} \) be a fixed numbering of the vertices. Clearly, for each edge \( e = \{v_i, v_j\} \) of \( K_n \) there is a corresponding transposition \( \tau_e = (i,j) \) in the symmetric group \( S_n \) on \( \{1,2,\ldots,n\} \), and vice versa, for each
transposition \( \tau = (i, j) \in S_n \) yields an edge \( e_\tau = \{v_i, v_j\} \) of \( K_n \). This correspondence is 1-1 in the sense that \( e_\tau = e \) and \( \tau_e = \tau \) for every \( e \) and every \( \tau \). For edges \( e_1, \ldots, e_m \) of \( K_n \) let \( G[e_1, \ldots, e_m] \) be the simple graph induced (or formed) by these edges. In light of Theorem 4.1 the following observation is clear.

**Observation 5.17** The transpositions \( \tau_1, \ldots, \tau_m \in S_n \) generate the symmetric group \( S_n \) if and only if the graph \( G[e_\tau_1, \ldots, e_\tau_m] \) contains a spanning tree of \( K_n \). In particular, \( m \geq n - 1 \) must hold.

Consider now a connected simple graph \( G = (V(G), E(G)) \) on \( n \) vertices, where \( V(G) = \{v_1, \ldots, v_n\} \) is a fixed numbering. As before, a vertex labeling \( L_V : V(G) \to \{1, 2, \ldots, n\} \) corresponds to a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) in \( S_n \). Since \( G \) is connected, it contains a spanning tree; and hence, each vertex labeling \( L_V \) of \( G \) can be transformed to any other labeling \( L'_V \) of \( G \) by a sequence of edge flips or mutations. As for the path and star, we have in general the following.

**Theorem 5.18** Let \( G \) be a connected simple graph with \( V(G) = \{v_1, \ldots, v_n\} \). For vertex labelings \( L_V, L'_V : V(G) \to \{1, 2, \ldots, n\} \) there exists a symmetric nonnegative parameter \( p_G(L_V, L'_V) \) and a function \( p_G(n) \) such that we have the following:

1. The labeling \( L_V \) can be transformed into \( L'_V \) in exactly \( t \) edge flips if and only if \( t = p_G(L_V, L'_V) + 2k \) for some nonnegative integer \( k \).

2. Every labeling \( L_V \) can be transformed into another labeling \( L'_V \) in at most \( t \) edge flips if and only if \( t \geq p_G(n) \).

**Proof.** For a given graph \( G \) and given vertex labelings \( L_V \) and \( L'_V \) of \( G \), we define the parameter \( p_G(L_V, L'_V) \) as the minimum number of edge flips needed to transform \( L_V \) into \( L'_V \). This existence is guaranteed since every nonempty subset of \( N \cup \{0\} \) contains a least element. If \( L_V \) can be transformed into \( L'_V \) in \( t \) edge flips, then by reversing the process \( L'_V \) can be transformed into \( L_V \) in \( t \) edge flips as well, so \( p_G(L_V, L'_V) \) is clearly symmetric. By repeating the last edge flip an even number of times, it is clear that \( L_V \) can be transformed into \( L'_V \) in \( t + 2k \) edge flips. Assume that \( L_V \) can be transformed into \( L'_V \) in \( t' \) edge flips. By viewing the edge flips of \( t \) and \( t' \) as permutations of \( S_n \), they must have the same parity, so \( t - t' \) must be even. This completes the proof of the first part.

By Theorem 5.14 we have that \( p_G(L_V, L'_V) \leq n(n - 1)/2 \) for all vertex labelings \( L_V \) and \( L'_V \) of \( G \). Hence, the maximum of \( p_G(L_V, L'_V) \) among all pairs of vertex labelings \( L_V \) and \( L'_V \) is also at most \( n(n - 1)/2 \). Letting \( p_G(n) \) be this very maximum, the second part clearly follows. \( \square \)

**Remark:** Using the notation of Theorem 5.18 what we have in particular is (i) \( p_G(n) \leq n(n - 1)/2 \) for every connected graph \( G \) on \( n \) vertices, (ii) \( p_K_n(n) = n(n - 1)/2 \), the classical result on the number of inversions by Muir [26], and (iii) \( p_{K_{1, n - 1}}(n) = \lfloor 3(n - 1)/2 \rfloor \) for the star.
6 Relabeling with Privileged Labels

In this section we describe the last variants of the relabeling problem that we consider in this paper. We impose an additional restriction on the flip or mutate operation. Some labels are designated as privileged. Our restricted mutations can only take place if at least one label of the pair to be mutated is a privileged label. The problem can be defined for vertices and for edges as follows.

**Definition 6.1 (Vertex Relabeling with Privileged Labels Problem)**
Instance: A graph $G$, labelings $L_V$ and $L'_V$, a nonempty set $S \subseteq \{1, 2, \ldots, n\}$ of privileged labels, and $t \in \mathbb{N}$.
Question: Can labeling $L_V$ evolve into $L'_V$ in $t$ or fewer restricted vertex mutations?

**Definition 6.2 (Edge Relabeling with Privileged Labels Problem)**
Instance: A graph $G$, labelings $L_E$ and $L'_E$, a nonempty set $S \subseteq \{1, 2, \ldots, m\}$ of privileged labels, and $t \in \mathbb{N}$.
Question: Can labeling $L_E$ evolve into $L'_E$ in $t$ or fewer restricted edge mutations?

The problems in Definitions 6.1 and 6.2 are increasingly restricted as the number of privileged labels decreases. Of course, one question is whether the problems are solvable at all. If $|S| = 1$, the Vertex Relabeling with Privileged Labels Problem can be reduced to the $(n \times n)$-Puzzle Problem, in which half of the starting configurations are not solvable [31]. This result proved in [17] shows that the Vertex Relabeling with Privileged Labels Problem is NP-complete.

**Definition 6.3 ($(n \times n)$-Puzzle Problem)**
Instance: Two $n \times n$ board configurations $B_1$ and $B_2$, and $k \in \mathbb{N}$.
Question: Is there a sequence of at most $k$ moves that transforms $B_1$ into $B_2$?

By reducing the $(n \times n)$-Puzzle Problem to the Vertex Graph Relabeling with Privileged Labels Problem, by taking $G$ as an $n \times n$ mesh, $L_V$ corresponding to $B_1$, $L'_V$ corresponding to $B_2$, $T = \{n^2\}$ corresponding to the blank space, and $t = k$, it is not hard to see that the instance of the $(n \times n)$-Puzzle Problem is “yes” if and only if the answer to the constructed instance of the Vertex Graph Relabeling with Privileged Labels Problem is also “yes”. We summarize in the following.

**Observation 6.4 (Intractability, Privileged Labels)** The Vertex Graph Relabeling with Privileged Labels Problem is NP-complete.

In Theorem [31] we proved that the Vertex Relabeling Problem is NC$^1$ many-one reducible to the Edge Relabeling Problem, however, that reduction does not suffice when talking about the versions of the problems involving privileged labels. We do not yet know if the Edge Relabeling with Privileged Labels Problem with $|S| = 1$ is NP-complete. It is interesting to note that many other similar games and puzzles such as the Generalized Hex Problem [7],...
(n × n)-Checkers Problem [8], (n × n)-Go Problem [20], and the Generalized Geography Problem [29] are also NP-complete.

Prior to Observation 6.4 it was still open whether some other unsolvable instances of the Vertex/Edge Relabeling with Privileged Labels Problems existed. However, we provide some simple examples of unsolvable instances in this section and provide some interesting characterizations of both solvable and unsolvable instances of these problems. We begin with an example.

Example A: Let \(n \geq 2\) and consider two vertex labelings \(L_V\) and \(L'_V\) of the path \(P_n\), where we have precisely \(k\) privileged labels \(p_1, \ldots, p_k\), where \(k \in \{0, 1, \ldots, n-2\}\). For a fixed horizontal embedding of \(P_n\) in the plane, assume the labelings are given in the following left-to-right order:

\[
L_V : (p_1, \ldots, p_k, 1, 2, 3, \ldots, n-k), \text{ and} \\
L'_V : (p_1, \ldots, p_k, 2, 1, 3, \ldots, n-k).
\]

Note that by any restricted mutation, where one of the labels are among \(\{p_1, \ldots, p_k\}\), the relative left/right order of the non-privileged labels will remain unchanged. Since the order of the two non-privileged labels 1 and 2 in \(L'_V\) is different from the one of \(L_V\), we see that it is impossible to evolve \(L_V\) to \(L'_V\) by restricted mutations only. Note that we can push these labels onto the edges by adding one more edge to the path. This example yields the following theorem.

**Theorem 6.5 (General Insolubility, Privileged Labels)**

Among all connected vertex labeled graphs on \(n\) vertices with \(k\) privileged labels where \(k \in \{0, 1, \ldots, n-2\}\), the Vertex Relabeling with Privileged Labels Problem is, in general, unsolvable. Among all connected edge labeled graphs on \(m\) edges with \(k\) privileged labels where \(k \in \{0, 1, \ldots, m-2\}\), the Edge Relabeling with Privileged Labels Problem is, in general, unsolvable.

Note that it is clear that for any connected graph \(G\) with all labels but one being privileged, any mutation is a legitimate transformation, since for any edge \(e = \{u, v\}\) either the label on \(u\) or \(v\) is privileged. Hence, among all connected graphs on \(n\) vertices with \(n-1\) privileged labels, the Vertex Relabeling with Privileged Labels Problem is solvable and in \(P\). A similar observation holds for the Edge Relabeling with Privileged Labels Problem.

Restricting now to the class of 2-connected simple graphs, we consider a slight variation of Example A.

Example B: Let \(n \geq 3\) and consider two vertex labelings \(L_V\) and \(L'_V\) of the cycle \(C_n\), where we have precisely \(k\) privileged labels \(p_1, \ldots, p_k\), where \(k \in \{0, 1, \ldots, n-3\}\). For a fixed planar embedding of \(C_n\), assume the labelings are given cyclically in clockwise order as follows:

\[
L_V : (p_1, \ldots, p_k, 1, 2, 3, \ldots, n-k), \text{ and} \\
L'_V : (p_1, \ldots, p_k, 2, 1, 3, \ldots, n-k).
\]

Note that by any restricted mutation, where one of the labels are among \(\{p_1, \ldots, p_k\}\), the relative orientation (clockwise or anti-clockwise) of the non-privileged labels 1, 2, and 3 will remain unchanged. Since the orientation of 1, 2, and 3 in \(L'_V\) is anti-clockwise, and the opposite of the
clockwise order of 1, 2, and 3 in \( L_V \), we see again that it is impossible to evolve \( L_V \) to \( L'_V \) by restricted mutations. Notice that we can push the labels onto the edges.

We summarize the implication of Example B in the following theorem.

**Theorem 6.6 (2-Connected Insolubility, Privileged Labels)**

Among all 2-connected vertex labeled graphs on \( n \) vertices with \( k \) privileged labels where \( k \in \{0, 1, \ldots, n-3\} \), the Vertex Relabeling with Privileged Labels Problem is, in general, unsolvable. Among all 2-connected edge labeled graphs on \( m \) edges with \( k \) privileged labels where \( k \in \{0, 1, \ldots, m-3\} \), the Edge Relabeling with Privileged Labels Problem is, in general, unsolvable.

We will now fully analyze the case where \( G \) is connected and all but two of the labels are privileged.

**Claim 6.7** If a simple graph is neither a path nor a cycle, then it has a spanning tree that is not a path (and hence contains a vertex of degree at least three).

**Proof.** Let \( G \) be a graph that is neither a path nor a cycle. Then \( G \) contains a vertex \( u \) of degree greater than or equal to three. Assigning the weight of one to each edge, we start by choosing three edges with \( u \) as an end-vertex and complete the construction of our spanning tree using Kruskal’s algorithm.

**Claim 6.8** Among vertex labeled trees, which are not paths, with exactly two non-privileged labels, any two labels can be swapped using restricted mutations.

**Proof.** Let \( G = (V, E) \) be a tree that is not a path, and \( L_V \) a labeling of the vertices. For any two distinct vertices \( x \) and \( y \) denote the unique path between them by \( P(x, y) \).

Assume that we want to swap the labels \( L_V(u) \) and \( L_V(v) \) on vertices \( u \) and \( v \). We first consider the case where all labels, except possibly one, on \( P(u, v) \), are privileged. Restricting to \( P(u, v) \), there are \( 2\partial(u, v) - 1 \) legitimate mutations that swap the labels on \( u \) and \( v \). (Here \( \partial(u, v) \) denotes the distance between \( u \) and \( v \) in the tree, or the length of \( P(u, v) \). This fact was noted in the remark right after the proof of Theorem 5.2.) Let us denote such a privileged swap by \( SW(u, v) \).

Consider next the case where the labels of \( u \) and \( v \) are both non-privileged. Let \( u' \) and \( v' \) be vertices such that the \((u', v')\)-path \( P^* \) is of maximum length in the tree and such that it contains \( P(u, v) \) as a sub-path. Hence, the three paths \( P(u', u) \), \( P(u, v) \), and \( P(v, v') \) make up this maximum length path \( P^* \). By the maximality of \( P^* \) and our assumption on the tree, there is an internal vertex \( w \) on \( P^* \) (note \( w \notin \{u', v'\} \)) of degree three or more, and hence that has a neighbor \( w' \) not on \( P^* \). We now perform the following procedure of legitimate swaps:

1. \( SW(u, u') \) and \( SW(v, v') \),
2. \( SW(u', w') \),
3. \( SW(u', v') \),
4. \( SW(v', w') \), and

5. \( SW(u, u') \) and \( SW(v, v') \).

This procedure has legitimately swapped the labels on \( u \) and \( v \).

If at least one of the labels of \( u \) and \( v \) is privileged, but both of the non-privileged labels do lie on \( P(u, v) \), say \( x \) and \( y \), then we can perform at least one of the swaps \( SW(u, x) \) or \( SW(y, v) \), say \( SW(u, x) \), after which we perform the swaps \( SW(x, v) \) and \( SW(u, x) \) to complete the legitimate swap. The case where \( SW(y, v) \) was performed first is handled similarly. This completes the proof. \( \square \)

We can now state the following lemma.

**Lemma 6.9** Among vertex labeled trees, which are not paths, with exactly two non-privileged labels, the Vertex Relabeling with Privileged Labels Problem is solvable and in \( P \).

**Proof.** Since any transformation from one labeling \( L_V \) to another \( L'_V \) is a composition of transpositions, this follows from Claim \( 6.8 \). \( \square \)

We now have the following summarizing theorem.

**Theorem 6.10 (Vertex Solubility, Two Privileged Labels)**

Among all connected vertex labeled graphs \( G \) on \( n \geq 4 \) vertices with all but two vertex labels privileged, the Vertex Relabeling with Privileged Labels Problem is solvable if and only if \( G \) is not a path.

**Proof.** We see from Example A that for \( n \geq 2 \) there are labelings of the vertices of the path \( P_n \) that cannot evolve into one another using restricted mutations.

If \( G \) is a cycle on \( n \geq 4 \) vertices, we can first move the labels of the non-privileged labels to their desired places by using appropriate clockwise and/or anti-clockwise sequences of mutations, and then move all the privileged labels to their places using mutations as on a path.

If \( G \) is neither a path nor a cycle, then by Claim \( 6.7 \) \( G \) has a spanning tree \( T \) that is not a path. Restricting to \( T \) we can by Lemma \( 6.9 \) move all the labels to their desired places within \( T \) and hence within \( G \). This completes our proof. \( \square \)

We obtain the following corollary as a consequence of Theorems \( 6.10 \) and \( 8.1 \).

**Theorem 6.11 (Edge Solubility, Two Privileged Labels)**

Among all connected edge labeled graphs \( G \) on \( n \geq 4 \) edges with all but two edge labels privileged, the Edge Relabeling with Privileged Labels Problem is solvable if and only if \( G \) is not a path.
7 Conclusions and Open Problems

We have defined several versions of a graph relabeling problem, including variants involving vertices, edges, and privileged labels, and proved numerous results about the complexity of these problems, answering several open problems along the way. A number of interesting open problems remain as follows:

▷ Study other types of mutation functions where, for example, labels along an entire path are mutated, or where labels can be reused.

▷ In the parallel setting, compute the sequence of mutations required for the transformation of one labeling into another. The parallel time for computing the sequence could be much smaller than the sequential time to execute the mutation sequence.

One result of interest in this direction is the problem of given a labeled graph, a prescribed flipping sequence, and two designated labels $l_1$ and $l_2$ are $l_1$ and $l_2$ flipped? A prescribed flipping sequence is an ordering of edges in which each succeeding edge’s labels may be flipped if and only if neither of its labels has already been flipped. This problem is NC-equivalent to the Lexicographically First Maximal Matching Problem, and so $CC$-complete; see [14] for a list of $CC$-complete problems.

▷ For various classes of graphs determine the probability of one labelings evolving naturally into another. Such an evolution of a labeling could be used to model mutation periods.

▷ Study the properties of the graphs of all labelings. In this graph all labelings of a given graph are vertices and two vertices are connected if they are one mutation apart. Other conditions for edge placement may also be worthwhile to examine.

▷ Determine if there is a version of the Edge Relabeling with Privileged Labels Problem that is $NP$-complete.

▷ Define the cost of a mutation sequence to be the sum of the weights on all edges that are mutated. Determine mutation sequences that minimize the cost of evolving one labeling into another. Explore other cost functions.

References

[1] Agnarsson, G. and Greenlaw, R. Graph Theory: Modeling, Applications, and Algorithms. Prentice Hall, 2007.

[2] Agnarsson, G., Greenlaw, R., and Kantabutra, S. The Graph Relabeling Problem and Its Variants. Proceedings of the Fifth International Conference in Electrical Engineering/Electronics, Computer, Telecommunications, and Information Technology, Krabi, Thailand, pages 49–52, May 2008.
[3] Agnarsson, G., Greenlaw, R., and Kantabutra, S. The Complexity of the Evolution of Graph Labelings. Proceedings of the Ninth ACIS International Conference on Software Engineering, Artificial Intelligence, Networking and Parallel/Distributed Computing (SNPD 2008), Phuket, Thailand, pages 79–84, August 2008.

[4] Akers, S. B. and Krishnamurthy, B. A Group-Theoretic Model for Symmetric Interconnection Networks. IEEE Transaction on Computers, 38:(4)555–565, 1989.

[5] Appel, K. and Haken, W. Every Map is Four Colorable. Providence, RI: American Mathematical Society, first edition, 1989.

[6] Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. Introduction to Algorithms, second edition, MIT Press, 2001.

[7] Even, S. and Tarjan, R. E. A Combinatorial Problem Which Is Complete in Polynomial Space. Journal of the Association for Computing Machinery, 23:710–719, 1976.

[8] Fraenkel, A. S., Garey, M. R., Johnson, D. S., Schaefer, T., and Yesha, Y. The Complexity of Checkers on an $N \times N$ Board—Preliminary Report, Proceedings of the 19th Annual Symposium on the Foundations of Computer Science, IEEE Computer Society, Long Beach, CA, pages 55–64, 1978.

[9] Gallian, J. A Dynamic Survey of Graph Labeling (tenth edition). Electronic Journal of Combinatorics, 14:1–180, 2007.

[10] Gardner, M. Mathematical Puzzles of Sam Loyd, Dover Publications, 1959.

[11] Gates, W. H. and Papadimitriou, C. H. Bounds for Sorting by Reversal. Discrete Mathematics, 27:47–57, 1979.

[12] Greenlaw, R., Halldórsson, M., and Petreschi, R. On Computing Prüfer Codes and Their Corresponding Trees Optimally in Parallel (Extended Abstract). Proceedings of Journées de l’Informatique Messine (JIM 2000), Université de Metz, France, Laboratoire d’Informatique Théorique et Appliquée, editor D. Kratsch, pages 125–130, 2000.

[13] Greenlaw, R., Hoover, H. J., and Ruzzo, W. L. Limits to Parallel Computation: P-Completeness Theory, Oxford University Press, 1995.

[14] Greenlaw, R. and Kantabutra, S. On the Parallel Complexity of Hierarchical Clustering and CC-Complete Problems. Complexity, Wiley, DOI 10.1002/cplx.20238, 2008.

[15] Hungerford, T. W. Algebra, Graduate Texts in Mathematics GTM—73, Springer Verlag, 1987.

[16] Kakoulis, K. G. and Tollis, I. G. On the Complexity of the Edge Label Placement Problem. Computational Geometry, 18(1):1–17, 2001.
[17] Kantabutra, S. The complexity of label relocation problems on graphs. Proceedings of the 8th Asian Symposium on Computer Mathematics, National University of Singapore, Singapore, December 2007.

[18] Knuth, D. E. The Art of Computer Programming, Volume 3, second edition, Addison Wesley, 1998.

[19] Lam, T. W. and Yue, F. L. Optimal Edge Ranking of Trees in Linear Time. Algorithmica, 30(1): 12–33, 2001.

[20] Lichtenstein, D. and Sipser, M. GO is Pspace hard. Proceedings of the 19th Annual Symposium on the Foundations of Computer Science, IEEE Computer Society, Long Beach, CA, pages 48–54, 1978.

[21] Marks, J. and Shieber, S. The Computational Complexity of Cartographic Label Placement. Technical Reports TR-05-91, Advanced Research in Computing Technology, Harvard University, 1991.

[22] Misic, J. Multicomputer Interconnection Network Based on the Star Graph. PhD thesis, University of Belgrade, October 1993.

[23] Misic, J. Multicomputer Interconnection Network Based on a Star Graph. Proceedings of the Twenty-Fourth Annual Hawaii International Conference on System Sciences. IEEE Computer Society, 2:373–381, 1991.

[24] Moon, J. W. Counting Labelled Trees. Canadian Mathematical Monographs, William Clowes and Sons, Limited, 1970.

[25] Muir, T. A Treatise on the Theory of Determinants. McMillan and Co., London, 1882.

[26] Muir, T. A Treatise on the Theory of Determinants. Revised and enlarged by W. H. Metzler. Dover Publications Inc., New York, 1960.

[27] Qiu, K., Meijer, H., and Akl, S. G. On the Cycle Structure of Star Graphs. Congressum Numerantium, 96:123–141, 1993.

[28] Ratner, D. and Warmuth, M. The \((n^2 - 1)\)-Puzzle and Related Relocation Problems. Journal of Symbolic Computation, 10:111–137, 1990.

[29] Schaefer, T. J. Complexity of Some Two-Person Perfect-Information Games. Journal of Computer and System Science, 16:185–225, 1978.

[30] Slocum, J. and Sonneveld, D. The 15 Puzzle: How It Drove the World Crazy. The puzzle that started the craze of 1880. How America’s greatest puzzle designer, Sam Loyd, fooled everyone for 115 years. Beverly Hills, CA, Slocum Puzzle Foundation, 2006.
[31] Storey, W. E. Notes on the 15 Puzzle 2. American Journal of Mathematics, 2(4):399–404, 1879.

[32] Wang, L., Subramanian, S., Latifi, S., and Srimani, P. K. Distance Distribution of Nodes in Star Graphs. Applied Mathematics Letters, 19(8):780–784, 2006.

[33] Wilkinson, B. and Allen, M. Parallel Programming: Techniques and Applications Using Networked Workstations and Parallel Computers, Prentice Hall, 1999.

[34] Wilson, R. M. Graph Puzzles, Homotopy, and the Alternating Group. Journal of Combinatorial Theory B, 16:86–96, 1974.