CHOQUARD EQUATIONS UNDER CONFINING EXTERNAL POTENTIALS

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ABSTRACT. We consider the nonlinear Choquard equation
$$\begin{align*}
-\Delta u + Vu &= (I_\alpha * |u|^p)|u|^{p-2}u \\
&\text{in } \mathbb{R}^N
\end{align*}$$

where $N \geq 1$, $I_\alpha$ is the Riesz potential integral operator of order $\alpha \in (0, N)$ and $p > 1$. If the potential $V \in C(\mathbb{R}^N; [0, +\infty))$ satisfies the confining condition
$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p-N}}} = +\infty,$$
and $\frac{1}{p} > \frac{N-2}{N+\alpha}$, we show the existence of a groundstate, of an infinite sequence of solutions of unbounded energy and, when $p \geq 2$ the existence of least energy nodal solution. The constructions are based on suitable weighted compact embedding theorems. The growth assumption is sharp in view of a Pohožaev identity that we establish.

1. INTRODUCTION AND MAIN RESULTS

We are interested in the following class of Choquard equations
$$\begin{align*}
-\Delta u + Vu &= (I_\alpha * |u|^p)|u|^{p-2}u \\
&\text{in } \mathbb{R}^N
\end{align*}$$

in the Euclidean space $\mathbb{R}^N$ of dimension $N \geq 1$, where $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$, which is defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by
$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N+\alpha}}, \text{ with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^\frac{N}{2}}$$
where $\Gamma$ denotes the classical Gamma function, and $p > 1$ is a given exponent.

When $N = 3$, $\alpha = 2$ and $p = 2$, the equation \((C)\) appears in several physical contexts, such as standing waves for the Hartree equation, the description by Pekar of the quantum physics of a polaron at rest [19], the description by Choquard of an electron trapped in its own hole [11] or the coupling of the Schrödinger equation under a classical Newtonian gravitational potential [6, 10, 15, 20].

When the potential $V$ is a positive constant function, groundstate solutions are known to exist [11, 13, 17] under the assumption that the exponent $p$ satisfies
$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}. \quad (1.1)$$

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Moreover, infinitely many geometrically distinct solutions can be constructed [13]. We refer the reader to the survey [18] for further discussion and references on the Choquard equation.

The goal of the present work is to examine how the presence of a confining potential $V$ changes and possibly improves the situation. Our first result is that groundstates can exist in a wider range of nonlinearities when the external potential $V$ is coercive enough.

**Theorem 1.1.** Let $N \geq 1$, $\alpha \in (0, N)$, $p \in (1, +\infty)$ and $V \in C(\mathbb{R}^N; [0, +\infty))$. If

$$\frac{1}{p} > \frac{N - 2}{N + \alpha}$$

and if

$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^\frac{N+\alpha}{p} - N} = +\infty,$$

then the Choquard equation (C) has a groundstate solution.

The solutions are groundstates in the sense that they minimize among nontrivial solutions the functional

$$J_p(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_* |u|^p)|u|^p;$$

solutions of the Choquard equation (C) are formally critical points of the functional $J_p$.

A striking feature of Theorem 1.1 is that the condition $p > \frac{N+\alpha}{N}$ in (1.1) can be loosened when $V$ grows fast enough at infinity. In particular, if $V(x) = |x|^\beta$ with $\beta > 0$, one can take $p > \max\{\frac{N+\alpha}{N+\beta}, 1\}$. The growth assumption is sharp. Indeed, if $V(x) = |x|^\beta$ and $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N)$ solves the Choquard equation (C) and then we have the Pohožaev identity (Theorem 5.1)

$$\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N + \beta}{2} \int_{\mathbb{R}^N} V|u|^2 = \frac{N + \alpha}{2p} \int_{\mathbb{R}^N} (I_* |u|^p)|u|^p,$$

provided that the integral on the right-hand side is finite. This condition cannot be satisfied when $p$ does not satisfy the assumption of Theorem 1.1.

Theorem 1.1 can be thought as counterpart for the nonlocal Choquard equation of results for the nonlinear Schrödinger equation with a coercive potential [21]. Radial positive solutions for the Choquard equation (C) had already been obtained in the quadratic case $p = 2$ when the potential $V$ is radial and radially increasing [3].

The core of the proof of Theorem 1.1 is to obtain the well-definiteness, the continuity and the compactness properties of the Riesz potential energy term in the definition of the functional $J_p$. This is done by combining a suitable Sobolev-type compact weighted embedding theorem together with the weighted estimates for fractional integrals of Stein and Weiss [22], which are a weighted counterpart of the more classical Hardy–Littlewood–Sobolev inequality.

We now turn on to the question whether the Choquard equation has, under the conditions of Theorem 1.1 more solutions. This is indeed the case and there are infinitely many solutions.
Theorem 1.2. Let \( N \geq 1 \), \( \alpha \in (0, N) \), \( p \in (1, +\infty) \) and \( V \in C(\mathbb{R}^N; [0, +\infty)) \). If
\[
\frac{1}{p} > \frac{N - 2}{N + \alpha}
\]
and if
\[
\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^N} = +\infty,
\]
then the Choquard equation \((C)\) has an infinite sequence of solutions whose energies do not remain bounded.

The solutions are constructed with the fountain theorem [2] (see also [25, Theorem 3.6]); thanks to the same weighted embedding and fractional integral estimates as in the proof of Theorem 1.1, the Palais–Smale condition for the functional \( J_p \) can be established by classical arguments.

Finally we investigate the question whether the Choquard equation \((C)\) has a least energy sign-changing solution, that is, a solution that changes sign and which minimizes the functional \( J_p \) among such solutions.

A natural way to construct such solutions is to minimize, as for the local semilinear elliptic problems [4, 5, 14, 23], the functional on the Nehari nodal set:
\[
\{ u \in H^1(\mathbb{R}^N) \mid u^+ \neq 0, u^- \neq 0, \langle J'_p(u), u^+ \rangle = 0 \text{ and } \langle J'_p(u), u^- \rangle = 0 \},
\]
where \( u^+ = \max(u, 0) \) and \( u^- = \min(u, 0) \). Such solutions of \((C)\) have been constructed when \( V = 1 \)
\[
\frac{N - 2}{N + \alpha} < \frac{1}{p} \leq \frac{1}{2},
\]
they were obtained by a new minimax principle and concentration-compactness method, and the minimization problem on the Nehari nodal set was observed to be degenerate when \( p < 2 \) [7, 8].

Sign-changing solutions have been constructed for the Schrödinger-Poisson system in \( \mathbb{R}^3 \) in which a nonlocal nonlinearity appears with opposite sign [1, 24].

Now, we are in a position to state our main results on the nodal solutions of equation \((C)\):

Theorem 1.3. Let \( N \geq 1 \), \( \alpha \in ((N - 4)_+, N) \), \( p \in [2, +\infty) \) and \( V \in C(\mathbb{R}^N; [0, +\infty)) \). If
\[
\frac{1}{p} > \frac{N - 2}{N + \alpha}
\]
and if
\[
\liminf_{|x| \to +\infty} V(x) = +\infty,
\]
then the Choquard equation \((C)\) has at least one least-energy sign-changing solution.

As before, the assumptions provide us with a functional with nice compactness properties. The situation is still more challenging than for a local semilinear elliptic equation on a bounded domain because some of the usual properties of the local nonlinear Schrödinger functional on negative and positive parts fail: in general \( J_p(u) \neq J_p(u^+) + J_p(u^-) \), and \( \langle J'_p(u), u^+ \rangle \neq \langle J'_p(u^+), u^+ \rangle \).
Theorem 1.3 was stated by Ye [27, Theorem 1.3]; it seems that he unfortunately overlooks the crucial question whether the proposed solution \( u \) changes sign, which is quite delicate when \( p = 2 \) (see the proof of Theorem 1.3 and [7]). We propose here a proof relying on tools similar to those for Theorems 1.1 and 1.2.

When \( p < 2 \), we prove that the energy functional does not achieve its minimum on the Nehari nodal set (see Proposition 4.4).

The remainder of this paper is organized as follows. In section 2, we first prove a weighted embedding theorem, then show that the function \( J_p \) is of \( C^1 \) on the natural Sobolev space \( H^1_{V}(\mathbb{R}^N) \) and satisfies the Palais–Smale condition. The proof of our main results will be postponed to the next two sections 3 and 4. In the last section 5, we will establish Pohožaev identity responding to equation (C), with which we can deduce some nonexistence results.

2. Function spaces and weighted embedding theorems

The linear part of the Choquard equation (C) naturally induces the Euclidean norm

\[
\| u \|_V := \left( \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \right)^{\frac{1}{2}}.
\]

We define \( H^1_{V}(\mathbb{R}^N) \) as the Hilbert space obtained by completion of the set of smooth test functions \( C^\infty_c(\mathbb{R}^N) \) with respect to the norm \( \| \cdot \|_V \). We first establish some embedding theorem from \( H^1_{V}(\mathbb{R}^N) \) into the weighted spaces \( L^2(|x|^{-\gamma} \, dx; \mathbb{R}^N) \) which is defined for \( \gamma \geq 0 \) by

\[
L^2(|x|^{-\gamma} \, dx; \mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-\gamma} |u(x)|^2 \, dx < +\infty \right\}.
\]

We begin by establishing the following embedding theorem.

**Proposition 2.1.** Let \( N \geq 1 \) and \( \gamma \in [0, +\infty) \). If \( V \in C(\mathbb{R}^N; [0, +\infty)) \) satisfies

\[
\liminf_{|x| \to +\infty} \frac{V(x)}{|x|^{\gamma}} > 0,
\]

then there exists a constant \( C > 0 \) such that for every \( u \in H^1_{V}(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} |x|^{-\gamma}|u(x)|^2 \, dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2.
\]

If moreover,

\[
\lim_{|x| \to +\infty} \frac{V(x)}{|x|^{\gamma}} = +\infty,
\]

then the corresponding embedding is compact. In particular, the embedding \( H^1_{V}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \) is compact for any \( q \) with \( \frac{1}{q} \in \left( \frac{1}{2} - \frac{\gamma}{N}, \frac{1}{2} \right) \) if \( \gamma = 0 \).

**Proof.** Given \( \lambda \in (0, +\infty) \) such that

\[
\lambda < \liminf_{|x| \to +\infty} \frac{V(x)}{|x|^{\gamma}},
\]

then...
there exists $\kappa > 0$ sufficiently large so that if $x \in \mathbb{R}^N \setminus B(0, \frac{\kappa}{2})$, we have $V(x) \geq \lambda |x|^\gamma$.

(Here and in the sequel, we use the notation $B(a, r)$ for the ball centered at $a$ of radius $r$ and in $\mathbb{R}^N$.) By integration, we have in particular,

$$\lambda \int_{\mathbb{R}^N \setminus B(0, \frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \, dx \leq \int_{\mathbb{R}^N \setminus B(0, \frac{\kappa}{2})} V|u|^2.$$

(2.1)

We take a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi(x) = 1$ for every $x \in B(0, \frac{\kappa}{2})$ and $\varphi(x) = 0$ for every $x \in \mathbb{R}^N \setminus B(0, \kappa)$. Then, it follows that

$$\int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2 \, dx \leq C_1 \kappa^\gamma \int_{B(0, \kappa)} |\nabla (\varphi u)|^2 + \int_{\mathbb{R}^N \setminus B(0, \frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \, dx$$

$$\leq 2C_1 \kappa^\gamma \int_{B(0, \kappa)} |\nabla u|^2 + 2C_1 \kappa^\gamma \int_{B(0, \kappa) \setminus B(0, \frac{\kappa}{2})} |\nabla \varphi|^2 |u|^2 + \int_{\mathbb{R}^N \setminus B(0, \frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \, dx$$

$$\leq 2C_1 \kappa^\gamma \int_{\mathbb{R}^N} |\nabla u|^2 + \left( 2C_1 \kappa^\gamma \frac{\|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^2}{(\kappa/2)^\gamma} + 1 \right) \int_{\mathbb{R}^N \setminus B(0, \frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \, dx,$$

where the constant $C_1$ comes from the Poincaré inequality with Dirichlet boundary conditions on the ball $B(0, \kappa)$, which is independent of the function $u$. We now apply the estimate (2.1) to the second term to obtain

$$\int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2 \, dx \leq 2C_1 \kappa^\gamma \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2^{\gamma+1}C_1\|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^2 + 1}{\lambda} \int_{\mathbb{R}^N} V|u|^2$$

$$\leq \max \left\{ 2C_1 \kappa^\gamma, \frac{2^{\gamma+1}C_1\|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^2 + 1}{\lambda} \right\} \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2,$$

(2.2)

and the first part of the conclusion follows.

For the compactness, without loss of generality, let $(v_n)_{n \in \mathbb{N}}$ be a sequence such that $v_n \to 0$ weakly as $n \to \infty$ in $H_1^1(\mathbb{R}^N)$. In particular, the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $H_1^1(\mathbb{R}^N)$. We are going to prove that $v_n \to 0$ strongly as $n \to \infty$ in $L^2(|x|^\gamma \, dx; \mathbb{R}^N)$. By assumption, for every $\varepsilon > 0$, there exists $R_1 > 0$, such that

$$\left( \sup_{|x| \geq R_1} \frac{|x|^\gamma}{V(x)} \right) \|v_n\|_V^2 \leq \varepsilon.$$

Since $\gamma \geq 0$, for any fixed $R > 0$, the weighted space $L^2(|x|^\gamma \, dx; B(0, R))$ is embedded into the classical Lebesgue space $L^2(B(0, R))$ defined on bounded domain $B(0, R)$. By the classical Sobolev embedding theorem, $v_n \to 0$ strongly in $L^2(B(0, R))$ as $n \to \infty$. Therefore, for fixed $R \geq R_1$, there exists $N_1 > 0$ such that

$$\int_{B(0, R)} |x|^\gamma |v_n(x)|^2 \, dx \leq \varepsilon \quad \text{for each } n \geq N_1.$$
Then for \( n \geq N_1 \), we have

\[
\int_{\mathbb{R}^N} |x|^{\gamma} |v_n(x)|^2 \, dx = \int_{B(0,R)} |x|^{\gamma} |v_n(x)|^2 \, dx + \int_{\mathbb{R}^N \setminus B(0,R)} |x|^{\gamma} |v_n(x)|^2 \, dx
\]

\[
\leq \int_{B(0,R)} |x|^{\gamma} |v_n(x)|^2 \, dx + \left( \sup_{|x| \geq R} \frac{|x|^{\gamma}}{V(x)} \right) \int_{\mathbb{R}^N \setminus B(0,R)} V(x) |v_n(x)|^2 \, dx
\]

\[
\leq \varepsilon + \left( \sup_{|x| \geq R} \frac{|x|^{\gamma}}{V(x)} \right) \|v_n\|_V^2 \leq 2\varepsilon.
\]

Finally, we interpolate to conclude our proof of the compact embedding \( H^1_V \subset L^q(\mathbb{R}^N) \) with \( \frac{1}{2} - \frac{1}{N} < \frac{1}{q} < \frac{1}{2} \) for the case \( \gamma = 0 \). Take \( \tilde{q} = \frac{2N}{N-2} \) if \( N \geq 3 \), or any \( \tilde{q} \in (q, +\infty) \) if \( N = 1, 2 \), there exists \( b \in (0, 1) \) such that

\[
\frac{1}{\tilde{q}} = b + \frac{1 - b}{q},
\]

it follows that as \( n \to \infty \),

\[
\|v_n\|_{L^\tilde{q}} \leq \|v_n\|_{L^2}^{b} \|v_n\|_{L^q}^{1-b} \leq C(1-b)\|v_n\|_{L^2}^{b} \|v_n\|_{L^q}^{1-b} \to 0.
\]

With the aid of the Stein–Weiss inequality [22], we show that the nonlocal Riesz potential energy term \( G_p \) of the functional \( J_p \) is well-defined and prove that the functional \( J_p \) is of class \( C^1 \) on the weighted Sobolev space \( H^1_V(\mathbb{R}^N) \). Finally, thanks to the compact embedding result, we close this section by verifying that the functional \( J_p \) satisfies the Palais–Smale condition.

**Proposition 2.2.** Let \( N \geq 1 \) and \( \alpha \in (0, N) \). If \( V \in C(\mathbb{R}^N; [0, +\infty)) \) satisfies

\[
\liminf_{|x| \to \infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p}}} > 0,
\]

then the mappings \( u \in H^1_V(\mathbb{R}^N) \to I_{\alpha/2} * |u|^p \in L^2(\mathbb{R}^N) \) and

\[
u \in H^1_V(\mathbb{R}^N) \to (I_{\alpha} * |u|^p)|u|^{p-2} u \in (H^1_V(\mathbb{R}^N))'
\]

are continuous for \( p > 1 \) and \( \frac{1}{p} > \frac{N-2}{N+\alpha} \).

If moreover

\[
\liminf_{|x| \to \infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p}}} = +\infty,
\]

the above mappings are weak to strong type, that is, they map weakly converging sequence to strongly converging sequence.

Here and in the sequel, \( X' \) denotes the topological dual space of the normed space \( X \).

**Proof of Proposition 2.2.** In the case \( p > \frac{N+\alpha}{N} \), the well-definiteness and the continuity follow from the continuous embedding \( H^1_V(\mathbb{R}^N) \subset H^1(\mathbb{R}^N) \), the classical Sobolev embedding and the Hardy–Littlewood–Sobolev inequality as in the case where \( V \) is constant [16]. If moreover \( \liminf_{|x| \to \infty} V(x) = +\infty \), the embedding \( H^1_V(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \) is compact for every \( q \in [2, +\infty) \) with \( \frac{1}{q} > \frac{1}{2} - \frac{1}{N} \) by Proposition 2.1 and then the weak to strong continuity property follows.
We assume now that $p \leq \frac{N+\alpha}{N-2} < 2$. We first show that the nonlocal term $G_p$ of the functional $J_p$ is well defined on the space $H^1_V(\mathbb{R}^N)$. By the Stein–Weiss inequality [22], together with the semi-group identity for the Riesz potential $I_\alpha = I_{\alpha/2} * I_{\alpha/2}$ [12], we have, since $\frac{2}{p} > 1$,

$$G_p(u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = \int_{\mathbb{R}^N} |I_{\frac{\alpha}{2}} * |u|^p|^2 \leq C \left( \int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p} - N} |u(x)|^2 \, dx \right)^p.$$ 

In view of the above continuous embedding Proposition 2.1, the functional $G_p$ is well defined on $H^1_V(\mathbb{R}^N)$. By Proposition 2.1 again, the superposition operator

$$u \in L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N) \mapsto |u|^p \in L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N)$$

(2.3)

is continuous. Taking into account the Stein–Weiss inequality [22] again, the Riesz potential integral operator

$$f \in L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N) \mapsto I_{\alpha/2} * f \in L^2(\mathbb{R}^N)$$

(2.4)

is a continuous linear operator. Thus the conclusion follows and the stronger conclusion follows directly from the compact embedding Proposition 2.1. In fact, suppose that $u_n \rightharpoonup u$ weakly in $H^1_V(\mathbb{R}^N)$, by compactness, we know that, as $n \to \infty$

$$u_n \to u \text{ strongly in } L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N),$$

thus, up to a subsequence, $u_n \to u$ almost everywhere in $\mathbb{R}^N$. From the continuity of the map defined by (2.3), we have that, as $n \to \infty$

$$|u_n|^p \to |u|^p \text{ strongly in } L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N),$$

and $|u_n|^{p-2} u_n \to |u|^{p-2} u$ strongly in $L^\frac{2}{p-1}(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N)$. By the Stein–Weiss inequality [22], we deduce that, as $n \to \infty$

$$I_\alpha * |u_n|^p \to I_\alpha * |u|^p \text{ strongly in } L^\frac{2}{p-1}(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N),$$

thus

$$(I_\alpha * |u_n|^p)|u_n|^{p-2} u_n \to (I_\alpha * |u|^p)|u|^{p-2} u$$

strongly in $L^2(|x|^{N-\frac{N+\alpha}{p}} \, dx; \mathbb{R}^N) = \left( L^2(|x|^{\frac{N+\alpha}{p} - N} \, dx; \mathbb{R}^N) \right)'$.

By the continuous embedding results again, we have

$$(I_\alpha * |u_n|^p)|u_n|^{p-2} u_n \to (I_\alpha * |u|^p)|u|^{p-2} u \text{ strongly in } (H^1_V(\mathbb{R}^N))'.$$

The compact embedding theorems imply straightforwardly that the functional $J_p$ is well-defined and satisfies the Palais–Smale condition.

**Lemma 2.3.** Let $N \geq 1$, $\alpha \in (0, N)$ and $p > 1$. If $\frac{1}{p} > \frac{N-2}{N+\alpha}$ and if

$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p} - N}} = +\infty,$$

then the functional $J_p$ is of class $C^1$ on $H^1_V(\mathbb{R}^N)$ and satisfies the Palais–Smale condition, that is, any sequence $(u_n)_{n \in \mathbb{N}}$ in $H^1_V(\mathbb{R}^N)$ such that $(J_p(u_n))_{n \in \mathbb{N}}$ is bounded, and
$J'_p(u_n) \to 0$ strongly in $(H^1_V(\mathbb{R}^N))'$ as $n \to \infty$ has a subsequence that converges strongly in $H^1_V(\mathbb{R}^N)$.

**Proof.** To prove that the functional $J_p$ is of continuously differentiable, we only need to consider the nonlocal term $G_p$ of $J_p$, that is,

$$G_p(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = \int_{\mathbb{R}^N} |I_{\alpha/2} * |u|^p|^2.$$

By Proposition 2.2 the functional $G_p$ is continuous on $H^1_V(\mathbb{R}^N)$ and then the functional $J_p$ is also continuous. For the continuous differentiability, we observe that by Proposition 2.2 again the map $G_p$ is Gâteaux-differentiable on that space [23, Proposition 1.3] and hence it is continuously Fréchet differentiable on that space [23, Proposition 1.3] and the first part of the conclusion follows.

Suppose now that $(u_n)_{n \in \mathbb{N}}$ is a Palais–Smale sequence for the functional $J_p$, that is, as $n \to \infty$

$$(J_p(u_n))_{n \in \mathbb{N}} \text{ is bounded and } J'_p(u_n) \to 0 \text{ strongly in } (H^1_V(\mathbb{R}^N))'.$$

First, we observe that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in the space $H^1_V(\mathbb{R}^N)$, because

$$\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_{V}^2 = J_p(u_n) - \frac{1}{2p} \langle J'_p(u_n), u_n \rangle = J_p(u_n) + o(\|u_n\|_V).$$

Up to a subsequence, we can assume that the sequence $(u_n)_{n \in \mathbb{N}}$ converges weakly to some function $u \in H^1_V(\mathbb{R}^N)$. By Proposition 2.2 we have $G'_p(u_n) \to G'_p(u)$ as $n \to \infty$ strongly in $(H^1_V(\mathbb{R}^N))'$ — that is, the map $G'_p$ is weak to strong type. It follows then that, as $n \to \infty$,

$$\|u_n - u\|_{V}^2 = \langle J'_p(u_n) - J'_p(u), u_n - u \rangle + \frac{1}{2p} \langle G'_p(u_n) - G'_p(u), (u_n - u) \rangle \to 0,$$

which concludes the proof. 

\[\square\]

3. Ground states and multiplicity solutions

We first give a proof of Theorem 1.1 by minimization of the Sobolev quotient and then prove the multiplicity result Theorem 1.2 by the fountain theorem at the end of this section.

**Proof of Theorem 1.1.** We are going to find a minimizer $u \in H^1_V(\mathbb{R}^N)$ for the infimum $\theta_p$, defined by

$$\theta_p := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \mid u \in H^1 V(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = 1 \right\},$$

once this will be done a nontrivial solution $v$ of equation (3) will be obtained after a rescaling, more precisely, by taking $v = \theta_p^{1/(2p-2)} u$.

Let $(u_n)_{n \in \mathbb{N}}$ in $H^1_V(\mathbb{R}^N)$ be a minimizing sequence for $\theta_p$, then

$$\|u_n\|_V^2 \to \theta_p \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \, dx = 1.$$
Since the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \(H^1_V(\mathbb{R}^N)\), we can assume without loss of generality that \(u_n \rightharpoonup u \) weakly in \(H^1_V(\mathbb{R}^N)\) as \(n \to \infty\). By the weakly lower semi-continuity of the norm, we know that
\[
\|u\|_V \leq \liminf_{n \to \infty} \|u_n\|_V. \tag{3.1}
\]
On the other hand, by Proposition 2.2, we deduce that
\[
I_{\alpha/2} * |u_n|^p \to I_{\alpha/2} * |u|^p \text{ strongly in } L^2(\mathbb{R}^N),
\]
and thus, as \(n \to \infty\),
\[
\mathcal{G}_p(u_n) = \int_{\mathbb{R}^N} |I_{\alpha/2} * |u_n|^p|^2 \to \int_{\mathbb{R}^N} |I_{\alpha/2} * |u|^p|^2 = \mathcal{G}_p(u).
\]
Therefore,
\[
\int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p \, dx = 1,
\]
which leads to \(u \neq 0\) and \(\|u\|_V^2 \geq \theta_p\) by the definition of \(\theta_p\). This, together with the inequality (3.1), implies that \(\|u\|_V^2 = \theta_p\). Therefore, \(u\) is a minimizer for \(\theta_p\). \(\square\)

**Remark 3.1.** In fact, the nontrivial solution obtained above is a positive solution with least energy, that is, a groundstate, see [25].

In the remainder of this section, we prove Theorem 1.2 on the multiplicity results by the fountain theorem.

**Proof of Theorem 1.2.** We are going to apply Bartsch’s fountain theorem [2] as it is presented in [25, Theorem 3.6]. We consider the action of the group \(\mathbb{Z}/2\mathbb{Z} = \{-1,1\}\) on the space \(H^1_V(\mathbb{R}^N)\) defined for \(g \in \{-1,1\}\) and \(u \in H^1_V(\mathbb{R}^N)\) by multiplication. This action is continuous and isometric: for every \(g \in \{-1,1\}\) and \(u \in H^1_V(\mathbb{R}^N)\), \(\|gu\|_V = \|u\|_V\). The functional \(J_p\) is invariant under this action of the group \(\{-1,1\}\) since it is an even functional. Moreover, by the Borsuk–Ulam theorem [25, Theorem D.17], the action of \(\{-1,1\}\) on \(\mathbb{R}\) is admissible for the fountain theorem, that is, every continuous odd map \(f : \partial U \to \mathbb{R}^{k-1}\) has a zero, where \(k \geq 2\) and \(U\) is an open bounded symmetric neighborhood of 0 in \(\mathbb{R}^k\).

We choose an orthonormal basis \((e_j)_{j \geq 0}\) of \(H^1_V(\mathbb{R}^N)\) and define \(X_j := \mathbb{R}e_j\). Then the spaces \(X_j\) are invariant, that is for every \(g \in \{-1,1\}\), \(gX_j = X_j\), and \(H^1_V(\mathbb{R}^N) = Y_k \oplus Z_k\), where \(Y_k := \bigoplus_{j=0}^k X_j\) and \(Z_k := \bigoplus_{j=k}^{\infty} X_j\). For every \(k \geq 1\), we set
\[
B_k := \{u \in Y_k \mid \|u\|_V \leq \rho_k\}, \quad N_k := \{u \in Z_k \mid \|u\|_V = r_k\}
\]
with \(\rho_k > r_k > 0\) and we define a sequence of minimax level,
\[
c_k := \inf_{\varphi \in \Gamma_k} \max_{u \in B_k} J_p(\varphi(u)),
\]
where
\[
\Gamma_k := \{\varphi \in C(B_k, H^1_V(\mathbb{R}^N)) \mid \varphi \text{ is even and } \varphi = \text{id on } \partial B_k\}.
\]
To apply the fountain theorem, we still need to verify the following conditions \((A_1)\) and \((A_2)\):
\[
(a_k := \max_{u \in \partial B_k} J_p(u) \leq 0 \quad (A_1)
\]
We show that \( \beta A \) the condition \((A_1)\) holds, it thus follows from the fountain theorem that there exists \( u_k \in Z_k \) such that
\[
\|u_k\|_V = 1 \text{ and } \|I_{\alpha/2} * |u_k|^p\|_{L^2} > \frac{\beta_k}{2}.
\]
By definition of \( Z_k \), we have \( u_k \rightharpoonup 0 \) weakly in \( H^1(\mathbb{R}^N) \). Thus by the weak to strong convergence property of Proposition 2.2, we deduce that \( I_{\alpha/2} * |u_k|^p \to I_{\alpha/2} * |u|^p \) as \( k \to \infty \) strongly in \( L^2(\mathbb{R}^N) \). Therefore \( \beta = 0 \).

For every \( u \in Z_k \),
\[
J_p(u) \geq \frac{1}{2} \|u\|_V^2 - \frac{\beta_k^2}{2p} \|u\|_V^{2p}.
\]
Set \( r_k := 1/(\beta_k)^{1/(p-1)} \), then we have
\[
J_p(u) \geq \left( \frac{1}{2} - \frac{1}{2p} \right) \frac{1}{r_k^p} \to +\infty \text{ as } k \to \infty.
\]
The condition \((A_2)\) holds, it thus follows from the fountain theorem that \( (c_k)_{k \in \mathbb{N}} \) is a sequence of critical values of \( J_p \) tending to \(+\infty\). This concludes the proof. \( \square \)

4. Existence of nodal solution with least energy

In this section, we shall prove the existence of nodal solutions by minimization method on the Nehari nodal set defined by
\[
\mathcal{M}_p = \{ u \in H^1(\mathbb{R}^N) \mid u^+ \neq 0 \neq u^- \text{ and } \langle \nabla J_p(u), u^+ \rangle = \langle J_p'(u), u^- \rangle = 0 \}.
\]
It is obvious that all the sign-changing solutions are contained in \( \mathcal{M}_p \). We are going to study whether it is possible to obtain a least energy nodal solution by finding a minimizer for
\[
c_p := \inf_{u \in \mathcal{M}_p} J_p(u).
\]
The following lemma plays an essential role in showing the existence of the minimizer for \( c_p \). The proof follows the strategy of \[8\] Proof of proposition 3.2; \[27\] Lemma 3.2.

Lemma 4.1. Let \( p > 2 \). For any \( u \in H^1(\mathbb{R}^N) \) with \( u^\pm \neq 0 \), there exists a unique pair \((t, s) \in (0, +\infty)^2\) such that \( tu^+ + su^- \in \mathcal{M}_p \) and if \( u \in \mathcal{M}_p \), then \( J_p(u) \geq J_p(tu^+ + su^-) \) for any \( t \geq 0, s \geq 0 \).
Therefore, $\Phi_w$ Then, we prove the minimizer $u$ is a maximum point and there is at most one maximum point.

Proof of Theorem 1.3 when $p > 2$. For the case of $p > 2$, our proof, in fact, relies on the compact embedding: $H^1_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ with $\frac{1}{q} \in \left(\frac{1}{2} - \frac{1}{N}, \frac{1}{2}\right]$ and can be carried out into two steps. First, we show that $c_p > 0$ is attained by some minimizer $w \in \mathcal{M}_p$. Then, we prove the minimizer $w$ for $c_p$ is indeed a critical point of $J_p$, thus being a nodal solution of (\text{Q}).

Step 1 The energy level $c_p > 0$ is achieved by some minimizer $w \in \mathcal{M}_p$.

Let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence for $c_p$ in $\mathcal{M}_p$, namely, $\lim_{n \to \infty} J_p(u_n) = c_p$. We first observe that

$$
\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 = J_p(u_n) - \frac{1}{2p} J'_p(u_n), u_n) = J_p(u_n) \to c_p, \quad (4.2)
$$

\[\Phi_p(t, s) := J_p(t^{\frac{2}{p}} u^+ + s^{\frac{2}{p}} u^-) = \frac{t^2}{2} \|u^+\|_V^2 + \frac{s^2}{2} \|u^-\|_V^2 - \frac{t^2}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * (t|u^+|^p + s|u^-|^p)|^2,
\]

where $u = u^+ + u^-$ with $u^\pm \neq 0$. The condition $t^{\frac{2}{p}} u^+ + s^{\frac{2}{p}} u^- \in \mathcal{M}_p$ is equivalent to $\nabla \Phi_p(t, s) = 0$ with $t > 0, s > 0$. It is sufficient to prove that there exists a unique critical point for the function $\Phi_p$ on the domain $(0, +\infty)^2$.

By the definition of $\Phi_p$,

$$
\Phi_p(t, s) \leq \frac{t^2}{2} \|u^+\|_V^2 + \frac{s^2}{2} \|u^-\|_V^2 - \frac{t^2}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * |u^+|^p|^2 - \frac{s^2}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * |u^-|^p|^2;
$$

from which we can get that

$$
\lim_{t^2 + s^2 \to +\infty} \Phi_p(t, s) \leq \lim_{t^2 + s^2 \to +\infty} \left( \frac{t^2}{2} \|u^+\|_V^2 - \frac{t^2}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * |u^+|^p|^2 + \frac{s^2}{2} \|u^-\|_V^2 - \frac{s^2}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * |u^-|^p|^2 \right) = -\infty.
$$

Therefore, $\Phi_p$ must have at least one global maximum point on $[0, \infty) \times [0, \infty)$.

Since the quadratic form

$$(t, s) \mapsto \int_{\mathbb{R}^N} |I_{\alpha/2} * (t|u^+|^p + s|u^-|^p)|^2$$

is positive definite, the function $\Phi_p$ is strictly concave. In particular, any critical point is a maximum point and there is at most one maximum point.

The conclusion follows provided that we can rule out that this maximum point is on the boundary of $[0, +\infty)^2$. Suppose that $(t_0, 0)$ with $t_0 \geq 0$ is the global maximum point of $\Phi_p$, then $\frac{\partial \Phi_p(t_0, 0)}{\partial t} \leq 0$. However, a direct computation shows that

$$
\frac{\partial \Phi_p(t_0, s)}{\partial s}\bigg|_{s=0} = +\infty,
$$

Similarly, $\Phi_p$ cannot achieve its global maximum on $(0, s)$ for any $s \geq 0$.

Proof of Theorem 1.3 when $p > 2$. For the case of $p > 2$, our proof, in fact, relies on the compact embedding: $H^1_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ with $\frac{1}{q} \in \left(\frac{1}{2} - \frac{1}{N}, \frac{1}{2}\right]$ and can be carried out into two steps. First, we show that $c_p > 0$ is attained by some minimizer $w \in \mathcal{M}_p$. Then, we prove the minimizer $w$ for $c_p$ is indeed a critical point of $J_p$, thus being a nodal solution of (\text{Q}).
We choose a continuous function \( \eta \). This argument seems simpler than previous deformation arguments [1, 24].

We deduce that \( u \) is defined by

\[
\langle J, u \rangle = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p \, dx
\]

which yields, since by our constraint again \( u_n^{\pm} \neq 0 \), that

\[
\lim_{n \to \infty} \|u_n^{\pm}\|_{L^{2N/p}} > 0.
\]

Since the embedding \( H^1_0(\mathbb{R}^N) \hookrightarrow L^{2N/p}(\mathbb{R}^N) \) is compact, we have

\[
u_n^{\pm} \to u^{\pm} \text{ strongly in } L^{2N/p}(\mathbb{R}^N),
\]

and then we deduce from (4.4) that \( u^{\pm} \neq 0 \). Next, by the Hardy–Littlewood–Sobolev inequality, we see that

\[
\langle J, w \rangle = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|w_n^{\pm}|^p \, dx
\]

and

\[
\langle J, w \rangle = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|w_n^{\pm}|^p \, dx
\]

Hence, by Lemma [4.1] there exists a unique pair \((t_0, s_0)\) with \( t_0, s_0 > 0 \) such that \( t_0u^+ + s_0u^- \in \mathcal{M}_p \). Moreover, we have

\[
c_p \leq J_p(t_0u^+ + s_0u^-) \leq \liminf_{n \to \infty} J_p(t_0u_n^+ + s_0u_n^-) \leq \limsup_{n \to \infty} J_p(t_0u_n^+ + s_0u_n^-) \leq \lim_{n \to \infty} J_p(u_n) = c_p.
\]

The second inequality above follows from the weakly lower semi-continuity of the norm and from (4.5) and (4.6). We conclude by setting \( w = t_0u^+ + s_0u^- \).

**Step 2** \( J'_p(w) = 0 \).

To complete this, we follow the idea of perturbing the functional in one direction [14]. This argument seems simpler than previous deformation arguments [11,24].

Suppose that \( w \) is not a critical point, then there exists a function \( v \in C_0^\infty(\mathbb{R}^N) \) such that \( \langle J'_p(w), v \rangle = -2 \). Since \( J_p \) is continuously differentiable, there exists \( \delta > 0 \) small enough such that

\[
\langle J'_p(tu^+ + su^- + \epsilon v), w \rangle \leq -1, \quad \text{if } |t - t_0| + |s - s_0| \leq \delta \text{ and } 0 \leq \epsilon \leq \delta.
\]

We choose a continuous function \( \eta : D \to [0, 1] \), where \( D \) being a bounded domain and is defined by

\[
D := \{(t, s) \in \mathbb{R}^2 : |t - t_0| \leq \delta, |s - s_0| \leq \delta\},
\]
such that
\[ \eta(t, s) = \begin{cases} 
1 & \text{if } |t - t_0| \leq \frac{\delta}{4} \text{ and } |s - s_0| \leq \frac{\delta}{2}, \\
0 & \text{if } |t - t_0| \geq \frac{\delta}{2} \text{ or } |s - s_0| \geq \frac{\delta}{2}.
\end{cases} \]

We define \( Q \in C(D, H^1_V(\mathbb{R}^N)) \) for \((t, s) \in D\) by
\[ Q(t, s) = tu^+ + su^- + \delta \eta(t, s)v. \]
and \( h : D \to \mathbb{R}^2 \) for \((t, s) \in D\) as
\[ h(t, s) := ((J''_p(Q(t, s)), Q(t, s)^+), (J''_p(Q(t, s)), Q(t, s)^-)). \]
The map \( h \) is continuous because the map \( u \mapsto u^+ \) is continuous in \( H^1_V(\mathbb{R}^N) \). If \( |t - t_0| = \delta \), or \( |s - s_0| = \delta \), then \( \eta = 0 \) by its definition, therefore \( Q(t, s) = tu^+ + su^- \), which implies that \( h(t, s) \neq (0, 0) \) by Lemma 4.1. As a consequence, the Brouwer topological degree \( \deg(h, \text{int}(D), 0) \) is well defined and \( \deg(h, \text{int}(D), 0) = 1 \), thus there exists a pair \((t_1, s_1) \in \text{int}(D)\) such that \( h(t_1, s_1) = (0, 0) \). Thus \( Q(t_1, s_1) \in M_p \), and then, it follows from the definition of \( c_p \) that
\[ J_p(Q(t_1, s_1)) \geq c_p. \]  
On the other hand, from equation (4.7) we arrive at
\[ J_p(Q(t_1, s_1)) = J_p(t_1 u^+ + s_1 u^-) + \int_0^1 \langle J'_p(t_1 u^+ + s_1 u^- + \rho \delta \eta(t_1, s_1)v), \delta \eta(t_1, s_1)v \rangle \, d\rho 
\leq J_p(t_1 u^+ + s_1 u^-) - \delta \eta(t_1, s_1). \]  
If \((t_1, s_1) \neq (t_0, s_0)\), we know from Lemma 4.1 that \( J_p(t_1 u^+ + s_1 u^-) < J_p(t_0 u^+ + s_0 u^-) = c_p \), thus from inequality (4.9)
\[ J_p(Q(t_1, s_1)) \leq J_p(t_1 u^+ + s_1 u^-) < c_p. \]  
If \((t_1, s_1) = (t_0, s_0)\), then \( \eta(t_1, s_1) = 1 \), follows from (4.9) we also have
\[ J_p(Q(t_1, s_1)) \leq c_p - \delta < c_p, \]  
which contradicts inequality (4.8) in any case. \( \square \)

We bring to the attention of the reader that the assumptions on the potential \( V \) are only used to ensure the compactness of the embedding \( H^1_V(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{2p_0}_{2p_0}(\mathbb{R}^N). \)

The case \( p = 2 \) is more complicated since we have neither a property similar to Lemma 4.1 nor an estimate like (4.3) to guarantee \( u^\pm \neq 0 \), where \( u = u^+ + u^- \) is the weak limit of a minimizing sequence. To find a nodal solution with least energy for the quadratic case, we follow the idea of [7] of employing equation (4) with \( p > 2 \) as a regularisation for the quadratic equation (2) and then pass to the limit as \( p \searrow 2 \). We start our proof by showing that Nehari nodal set \( \mathcal{M}_2 \) is not empty.

**Lemma 4.2.** One has \( \mathcal{M}_2 \neq \emptyset \). In particular, \( c_2 < +\infty \).

**Proof.** We are going to construct a function \( w \in H^1_V(\mathbb{R}^N) \) with \( w^\pm \neq 0 \) such that the following linear system admits a solution \((t, s)\) with \( t, s > 0 \),
\[ \begin{aligned}
& \left( \int_{\mathbb{R}^N} (I_\alpha * |w^+|^2) |w^+|^2 \right) t^2 \left( \int_{\mathbb{R}^N} (I_\alpha * |w^-|^2) |w^-|^2 \right) s^2 \\
& \left( \int_{\mathbb{R}^N} (I_\alpha * |w^+|^2) |w^-|^2 \right) t^2 \left( \int_{\mathbb{R}^N} (I_\alpha * |w^-|^2) |w^-|^2 \right) s^2 = \left( \|w^+\|^4 \right) t^4 \left( \|w^-\|^4 \right) .
\end{aligned} \]  
(4.10)
The conclusion will then follow since \( tw^+ + sw^- \in \mathcal{M}_p \). By Cramer’s Rule, it is sufficient to find a function \( w \in H^1_0(\mathbb{R}^N) \) with \( w^\pm \neq 0 \) such that

\[
\frac{\int_{\mathbb{R}^N} (I_\alpha \ast |w^+|^2)w^+ \, \text{d}x}{\int_{\mathbb{R}^N} (I_\alpha \ast |w^-|^2)w^- \, \text{d}x} < \frac{\|w^+\|_V^2}{\|w^-\|_V^2} < \frac{\int_{\mathbb{R}^N} (I_\alpha \ast |w^+|^2)w^+ \, \text{d}x}{\int_{\mathbb{R}^N} (I_\alpha \ast |w^-|^2)w^- \, \text{d}x}.
\] (4.11)

Let \( U \in C^1(\mathbb{R}^N) \setminus \{0\} \) such that \( U \geq 0 \) and \( \text{supp} \, U \subset B(0,1) \). We choose \( a_+, a_- \notin \text{supp} \, U \) and define

\[
w_\sigma(x) := U(\frac{x-a_+}{\sigma}) - U(\frac{x-a_-}{\sigma}).
\]

Since the function \( U \) has compact support, we know that \( w_\sigma^+(x) = U(\frac{x-a_+}{\sigma}) \) and \( w_\sigma^-(x) = -U(\frac{x-a_-}{\sigma}) \) for sufficiently small \( \sigma \). To end the proof, we show that the estimate (4.11) holds as \( \sigma \) becomes small enough. In fact,

\[
\|w_\sigma^\pm\|_V^2 = \int_{\mathbb{R}^N} \sigma^{N-2} |\nabla U(x)|^2 + \sigma^N V(a_\pm + \sigma x) U^2(x) \, \text{d}x
\]

\[= \sigma^{N-2} \left( \int_{\mathbb{R}^N} |\nabla U|^2 + O(\sigma^2) \right),\]

\[
\int_{\mathbb{R}^N} (I_\alpha \ast |w_\sigma^\pm|^2)|w_\sigma^\pm|^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A_\alpha |U(\frac{y-a_\pm}{\sigma})|^2 |U(\frac{x-a_\pm}{\sigma})|^2 \frac{\|x-y\|^{N-\alpha}}{|x-y|^{N-\alpha}} \, \text{d}y \, \text{d}x
\]

\[= \sigma^{N+\alpha} \int_{\mathbb{R}^N} (I_\alpha \ast |U|^2)|U|^2,\]

and when \( \sigma \leq |a_+ - a_-|/4 \),

\[
\int_{\mathbb{R}^N} (I_\alpha \ast |w_\sigma^\pm|^2)|w_\sigma^\pm|^2 \leq \sigma^{2N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A_\alpha |U(y)|^2 |U(x)|^2 \frac{\|(a_\pm + \sigma x) - (a_\pm + \sigma y)\|^{N-\alpha}}{|a_\pm - a_\pm|^{N-\alpha}} \, \text{d}y \, \text{d}x
\]

\[\leq \sigma^{2N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{N-\alpha} A_\alpha |U(y)|^2 |U(x)|^2}{|a_\pm - a_\pm|^{N-\alpha}} \, \text{d}y \, \text{d}x.
\]

We observe that in (4.11), since \( \alpha < N \), the left-hand side goes to 0 as \( \sigma \to 0 \), the middle term converges to a positive constant and the right-hand side diverges to +\( \infty \). The inequality (4.11) holds for sufficiently small \( \sigma \) and thus the system (4.10) has a solution \( (t, s) \) such that \( t, s > 0 \), that is, \( M_2 \neq \emptyset \).

**Proof of Theorem 1.3** when \( p = 2 \). Let \( (u_{pn})_{n \in \mathbb{N}} \subset H^1_0(\mathbb{R}^N) \) be a sequence of least energy nodal solution for the equation \( \mathcal{C} \) with \( \frac{1}{p_n} > \frac{N-2}{N+\alpha} \) and \( p_n \to 2 \) as \( n \to \infty \). In particular, we have \( J_{pn}(u_{pn}) = c_{pn} \), and the function \( u_{pn} \) satisfies the equation

\[-\Delta u_{pn} + V u_{pn} = (I_\alpha \ast |u_{pn}|^{p_n})|u_{pn}|^{p_n-2}u_{pn}.\]

We first show that \( \|u_{pn}\|_V \) is bounded both from below and above. In fact, by a direct computation, we see that for every \( n \in \mathbb{N} \), by the Hardy–Littlewood–Sobolev and by the Sobolev inequality,

\[
\|u_{pn}\|_V^2 = \int_{\mathbb{R}^N} (I_\alpha \ast |u_{pn}|^{p_n})|u_{pn}|^{p_n} \leq C \left( \int_{\mathbb{R}^N} |u_{pn}|^{\frac{2N}{N+\alpha}} \right)^{N+\alpha} \leq C_1 \|u_{pn}\|_V^{2p_n},
\]

where the constant \( C_1 \) can be taken independently of \( p_n \) since \( (p_n)_{n \in \mathbb{N}} \) remains bounded. It follows that

\[
\liminf_{n \to \infty} \|u_{pn}\|_V > 0.
\] (4.12)
On the other hand, thanks to Lemma 4.2 above, we can take $w \in M_2$, and define $w_n = t^{1/p_n}u^+ + s^{1/p_n}u^-$, where $(t^{1/p_n}, s^{1/p_n})$ is given by Lemma 4.1. Then, $w_n \in M_{p_n}$, and $J(w_n) \geq c_{p_n}$. Since $\Phi_p(t, s) \to -\infty$ as $(t, s) \to +\infty$ uniformly in $p$ in bounded sets and $\Phi_p \to \Phi_2$ as $p \to 2$ uniformly over compact subsets of $[0, +\infty)^2$, we have $t^{1/p_n}, s^{1/p_n} \to 1$, and therefore $J_{p_n}(w_n) \to J_2(w)$. Since $w$ is an arbitrary function in $M_2$, we deduce that
\[
\lim_{n \to +\infty} c_{p_n} \leq c_2 < +\infty,
\] (4.13)
and thus
\[
\|u_n\|_V^2 = \frac{1}{2} - \frac{1}{2p_n}(J_{p_n}(u_n) - \frac{1}{2p_n}(J'_{p_n}(u_n), u_n)) = \frac{2p_n c_{p_n}}{p_n - 1} \leq 4c_2 + o(1).
\]
In particular, $(\|u_n\|_V)_{n \in \mathbb{N}}$ is bounded from above. It follows that there exists some function $u \in H^1_V(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^1_V(\mathbb{R}^N)$ as $n \to \infty$. By the compactness of the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^{4N/3}(\mathbb{R}^N)$ (Proposition 2.1), we have
\[
u_n \to u \text{ strongly in } L^{4N/3}(\mathbb{R}^N)
\]
and $u_n \to u$ almost everywhere in $\mathbb{R}^N$, so that $(\nu_n)_{n \in \mathbb{N}}$ is bounded in $L^{4N/3}(\mathbb{R}^N)$. Moreover, by interpolation through Hölder’s inequality, we have
\[
\|u_n\|_{L^{4N/(3\alpha)}(\mathbb{R}^N)} \leq \|u_n\|_{L^{2N/(\alpha+2)}(\mathbb{R}^N)}^{1/\lambda_n} \leq \|u_n\|_{L^{2N/(\alpha+2)}(\mathbb{R}^N)}^{\lambda_n} \leq \|u_n\|_{L^{2N/(\alpha+2)}(\mathbb{R}^N)}^{1-\lambda_n} (C \|u_n\|_V)^{\lambda_n},
\] (4.14)
where $\lambda_n \in (0, 1)$ satisfies
\[
1 = \frac{1}{4N/(\alpha+2)} = \frac{\lambda_n}{2N/(\alpha+2)} + \frac{1 - \lambda_n}{2},
\]
that is, $\lambda_n = \frac{N - \alpha}{N - \alpha} \to 1$ as $n \to \infty$ and the constant $C$ can be chosen independently of $u_n$. Similarly, taking $q = \frac{2N}{N-2}$ for $N \geq 3$, and $\frac{4N}{N+\alpha} < q < +\infty$ for $N = 1, 2$, we have
\[
\|u_n\|_{L^{2N/(\alpha+2)}(\mathbb{R}^N)} \leq \|u_n\|_{L^{2N/(\alpha+2)}(\mathbb{R}^N)}^{1/\mu_n} \|u_n\|_{L^{4N/(\alpha+2)}(\mathbb{R}^N)}^{\mu_n} \leq (C \|u_n\|_V)^{1-\mu_n} \|u_n\|_{L^{4N/(\alpha+2)}(\mathbb{R}^N)}^{\mu_n}.
\] (4.15)
with
\[
\mu_n = \frac{N + \alpha}{4N/(\alpha+2)} - \frac{2}{N - \alpha} \to 1, \quad \text{as } n \to \infty.
\]
Taking limit on the both sides of (4.14) and (4.15) and combining the boundedness of $\|u_n\|_V$ from below and above, we obtain that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^{2N/(\alpha+2)} = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^{4N/(\alpha+2)} = \int_{\mathbb{R}^N} |u|^{4N/(\alpha+2)}.
\]
Thus, we get that (see for example [25, Proposition 4.2.6])
\[
|u_n|^{p_n} \to |u|^2 \text{ strongly in } L^{2N/(\alpha+2)}(\mathbb{R}^N),
\] (4.16)
which, together with the Stein–Weiss inequality [22], yields that
\[
I_\alpha * |u_n|^{p_n} \to I_\alpha * |u|^2 \text{ strongly in } L^{2N/(\alpha+2)}(\mathbb{R}^N).
\] (4.17)
Similarly to (1.3), we have
\[ C_1 \|u_{p_n}\|_{L^{4N/(N-α)}(\mathbb{R}^N)}^2 \leq \|u_{p_n}\|_V^2 = \int_{\mathbb{R}^N} (I_α * |u_{p_n}|^{p_n}) |u_{p_n}|^{p_n} \leq C \|u_{p_n}\|_{L^{4N/(N-α)}(\mathbb{R}^N)}^2, \]
which implies, by taking limit on both sides, that \( \|u\|_{L^{4N/(N-α)}(\mathbb{R}^N)} \geq C > 0 \), that is \( u \neq 0 \).

For large \( n \), we choose \( q \) as in (4.15), we employ the interpolation inequalities again, and we get
\[ \|u_{p_n}\|_{L^{(p_n-1)4N/(N-α)}(\mathbb{R}^N)} \leq \|u_{p_n}\|_{L^p(\mathbb{R}^N)}^{1-\lambda_n} \|u_{p_n}\|_{L^{4N/(N-α)}(\mathbb{R}^N)}^{\lambda_n} \leq (C \|u\|_V)^{1-\lambda_n} \|u_{p_n}\|_{L^{4N/(N-α)}(\mathbb{R}^N)}^{\lambda_n}, \]
where
\[ \lambda_n = \frac{4N/(p_n-1)\alpha}{4N\alpha - \frac{1}{q}} \to 1, \]
that is \( |u_{p_n}|^{p_n-2}u_{p_n} \) is bounded in \( L^{4N/(N-α)}(\mathbb{R}^N) \), it converges to \( u \) weakly in the space \( L^{4N/(N-α)}(\mathbb{R}^N) \) [26] Proposition 5.4.7]. Therefore,
\[ \langle J'(u), \psi \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \psi + Vu\psi - (I_α * |u|^2)u\psi \]
\[ = \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla u_{p_n} \cdot \nabla \psi + Vu_{p_n}\psi - (I_α * |u_{p_n}|^{p_n})|u_{p_n}|^{p_n-2}u_{p_n}\psi \]
\[ = \langle J'_n(u_{p_n}), \psi \rangle = 0. \]
which means that \( u \) is a weak solution of the quadratic Choquard equation (2) since the function \( \psi \in H^1_V(\mathbb{R}^N) \) is arbitrary. Moreover, we deduce from the convergences (4.16) and (4.17) that
\[ \|u_{p_n}\|^2_V = \int_{\mathbb{R}^N} (I_α * |u_{p_n}|^{p_n}) |u_{p_n}|^{p_n} \to \int_{\mathbb{R}^N} (I_α * |u|^2)|u|^2 = \|u\|^2_V, \quad (4.18) \]
and thus \( u_{p_n} \to u \) strongly in \( H^1_V(\mathbb{R}^N) \) as \( n \to \infty \).

We are now in a position to finish our proof by showing that \( u^+ \neq 0 \) and \( J_2(u) \) is the least among all the nodal solutions of the quadratic Choquard equation. By (1.12) and (4.18), we have \( u \neq 0 \). Without loss of generality, let us assume by contradiction that \( u^+ \neq 0 \) and that \( u^- = 0 \). Set for each \( n \in \mathbb{N} \)
\[ v_{p_n} := \frac{u_{p_n}^-}{\|u_{p_n}^-\|_{V}^{2/p_n}}, \]
then from the equality \( \langle J'_n(u_{p_n}), v_{p_n}^- \rangle = 0 \), we get that for each \( n \in \mathbb{N} \)
\[ \int_{\mathbb{R}^N} (I_α * |u_{p_n}|^{p_n}) |v_{p_n}^+|^{p_n} = 1. \]
Since we have assumed that \( u^- = 0 \), we have \( u_{p_n}^- \to 0 \) strongly in \( H^1_V(\mathbb{R}^N) \). By Young’s inequality, we know that
\[ \|u_{p_n}\|_V = \|u_{p_n}^-\|_V^{1-\frac{2}{p_n}} \leq \left(1 - \frac{2}{p_n}\right)\|u_{p_n}^-\|_V + \frac{2}{p_n}, \]
which yields that the sequence \((u_{p_n})_{n \in \mathbb{N}}\) is bounded in \( H^1_V(\mathbb{R}^N) \). The compactness of the embedding \( H^1_V(\mathbb{R}^N) \hookrightarrow L^{4N/(N-α)}(\mathbb{R}^N) \) (Proposition 2.1) implies in turn that \((v_{p_n})_{n \in \mathbb{N}}\)
converges to some \( v \) strongly in \( L^{\frac{4N}{N+4}}(\mathbb{R}^N) \), which, together with (4.15) by replacing \( u_{pn} \) with \( v_{pn} \) and the boundness of \( \|v_{pn}\|_1 \), implies that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |v_{pn}|^{2^*} < +\infty,
\]
thus, it follows from [26, Proposition 5.4.7] again that
\[
|v_{pn}|^{2^*} \to |v|^2 \text{ in } L^{\frac{2N}{N+4}}(\mathbb{R}^N).
\]
Combining the strong convergence of (4.17), we deduce that
\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^2)|v|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_{pn}|^2)|v_{pn}|^2 = 1. \tag{4.19}
\]

On the other hand, by the definition of \( v_{pn} \) and by the strong convergence of \((u_{pn}^+)_n \in \mathbb{N}\) to \( u \) in \( H^1(\mathbb{R}^N) \), we have \( uv = 0 \) almost everywhere on \( \mathbb{R}^N \). Since \( u \) is a nontrivial negative weak solution to the Choquard equation, it is a classical solution (following [19, Theorem 3]) and thus, by the classical strong maximum principle for second order elliptic operators, \( u > 0 \) everywhere on \( \mathbb{R}^N \) and thus \( v = 0 \), which is a contradiction with (4.19).

In particular, we have \( u \in \mathcal{M}_2 \), and thus \( J_2(u) \geq c_2 \). On the other hand, by (4.13) and the strong convergence of \((u_{pn})_n \in \mathbb{N}\) we have
\[
J_2(u) = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2
= \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2p_n} \right) \int_{\mathbb{R}^N} |\nabla u_{pn}|^2 + V|u_{pn}|^2 = \limsup_{n \to \infty} J_{pn}(u_{pn}) = \limsup_{n \to \infty} c_{pn} \leq c_2;
\]
this concludes the proof. \( \square \)

**Remark 4.3.** In fact, in the case of \( p \geq 2 \), we have \( c_p > c_{0,p} \) where \( c_{0,p} \) is the energy level of the groundstates, since any groundstate solution should have constant sign. However, the question of whether or not or when the estimate \( c_p > 2c_{0,p} \) holds is open; in the case of constant potential this estimate was crucial for the compactness.

Finally, we prove that the energy level \( c_p \) is degenerate when \( p < 2 \).

**Proposition 4.4.** For \( p < 2 \), we have \( c_p = c_{0,p} \). Then the energy functional \( J_p \) does not achieve its minimum on the Nehari nodal set.

**Proof.** We observe that if \( u \in \mathcal{N}_p \), then \( |u| \in \mathcal{N}_p \), where \( \mathcal{N}_p \) denote the Nehari manifold, that is,
\[
\mathcal{N}_p := \{ u \in H^1(\mathbb{R}^N) : u \neq 0 \text{ and } \langle J_p(u), u \rangle = 0 \}.
\]
With this notation, we know that \( c_{0,p} = \inf_{u \in \mathcal{N}_p} J_p(u) \) (see Remark 4.3 and [25]). Since \( \mathcal{M}_p \subset \mathcal{N}_p \), thus we get that \( c_p \geq c_{0,p} \). In fact, we shall show the reverse inequality holds. By a density argument, it follows that
\[
c_{0,p} = \inf \{ J_p(u) : u \in \mathcal{N}_p \cap C^1_0(\mathbb{R}^N) \text{ and } u \geq 0 \text{ on } \mathbb{R}^N \}.
\]
Let \( u \in \mathcal{N}_p \cap C^1_0(\mathbb{R}^N) \) and \( u \geq 0 \) on \( \mathbb{R}^N \). We choose a point \( a \notin \text{supp } u \) and a function \( \psi \in C^1(\mathbb{R}^N) \backslash \{0\} \) such that \( \psi \geq 0 \) and we define as in \( \mathbb{R} \) for each \( \sigma > 0 \) the function \( u_{\sigma} : \mathbb{R}^N \to \mathbb{R} \) by
\[
u_{\sigma}(x) = u(x) - \sigma^{\frac{2}{p-1}} \psi\left(\frac{x-a}{\sigma}\right).
\]
Then, \( u^+_\sigma = u \) for sufficiently small \( \sigma \). By a direct computation, \( tu^+_\sigma + su^-_\sigma \in \mathcal{M}_p \) if and only if

\[
\begin{aligned}
& (t^2-p-t^p) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = s^p \sigma^{N+2p-2} K_\sigma, \\
& s^{2-p} \int_{\mathbb{R}^N} |\nabla \psi(y)|^2 + \sigma^2 V(a+\sigma y)\psi^2(y) \, dy = t^p K_\sigma + s^p \sigma^\alpha \int_{\mathbb{R}^N} (I_\alpha * |\psi|^p)|\psi|^p
\end{aligned}
\]

where \( K_\sigma = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(a+\sigma y)|\psi(y)|^p \, dy \).

Observe that the system (4.20) has a unique solution when \( \sigma = 0 \). By the implicit function theorem, for \( \sigma > 0 \) small enough there exists a pair \( (t_\sigma, s_\sigma) \in (0, +\infty)^2 \) that solves the system (4.20) and

\[
\lim_{\sigma \to 0} t_\sigma = 1, \quad \lim_{\sigma \to 0} s_\sigma = \left( \frac{(I_\alpha * |u|^p)(a) \int_{\mathbb{R}^N} |\psi|^p}{\int_{\mathbb{R}^N} |\nabla \psi|^2} \right)^{\frac{1}{2-p}}.
\]

Since \( \frac{4}{2-p} + N - 2 > 0 \), we have \( u^-_\sigma \to 0 \) in \( H^1_V(\mathbb{R}^N) \) as \( \sigma \to 0 \), and thus \( t_\sigma u^+_\sigma + s_\sigma u^-_\sigma \to u \) in \( H^1_V(\mathbb{R}^N) \). Hence, we have that

\[
c_p \leq J_p(t_\sigma u^+_\sigma + s_\sigma u^-_\sigma) \to J_p(u), \quad \text{as } \sigma \to 0,
\]

which implies that \( c_p \leq c_{0,p} \) since \( u \in \mathcal{N}_p \) is arbitrary.

We assume that \( u \in \mathcal{M}_p \) minimizes the functional \( J_p \) on the Nehari nodal set \( \mathcal{M}_p \).

Since \( c_{0,p} = c_p \), thus \( u \) also minimizes \( J_p \) on the Nehari manifold \( \mathcal{N}_p \).

By regularity theory for the Choquard equation and by the strong maximum principle, either \( u > 0 \) or \( u < 0 \), which contradicts with \( u \in \mathcal{M}_p \).

\[ \square \]

5. Pohožaev identity

This section is devoted to the proof of a Pohožaev identity for the Choquard equation \( \Box \).

**Theorem 5.1** (Pohožaev identity). Let \( N \geq 3 \), \( V \in C^1(\mathbb{R}^N, [0, \infty)) \). If the function \( u \in W^{2,2}_\text{loc}(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N) \) is a solution to the Choquard equation \( \Box \) such that

\[
\int_{\mathbb{R}^N} \|x \cdot \nabla V(x)\| |u(x)|^2 \, dx + \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p < +\infty,
\]

then

\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (NV(x)+x \cdot \nabla V(x))|u(x)|^2 \, dx = \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p. \tag{5.1}
\]

Here, \( \eta \cdot \zeta \) denotes the canonical scalar product of vectors \( \eta, \zeta \in \mathbb{R}^N \).

**Proof of Theorem 5.1** We take \( \varphi \in C^1_c(\mathbb{R}^N) \) such that \( \varphi = 1 \) on \( B(0,1) \). Since the function \( \varphi \) has compact support, we can define a function \( v_\lambda \in H^1_V(\mathbb{R}^N) \) for \( \lambda \in (0, +\infty) \) by

\[
v_\lambda(x) := \varphi(\lambda x) x \cdot \nabla u(x).
\]

By testing the Choquard equation \( \Box \) against the function \( v_\lambda \), we have

\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda + Vu v_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uv_\lambda.
\]
We compute the square term for $\lambda > 0$. By the definition of $v_\lambda$, the chain rule and by the Gauss integral formula, we get that

$$\int_{\mathbb{R}^N} V u v_\lambda = \int_{\mathbb{R}^N} V(x) u(x) \varphi(\lambda x) x \cdot \nabla u(x) \, dx$$

$$= \int_{\mathbb{R}^N} V(x) \varphi(\lambda x) x \cdot \nabla \left( \frac{1}{2} |u|^2 \right)(x) \, dx$$

$$= - \int_{\mathbb{R}^N} (NV(x) \varphi(\lambda x) + V(x) (\lambda x \cdot \nabla \varphi(\lambda x) + x \cdot \nabla V(x) \varphi(\lambda x)) \frac{|u(x)|^2}{2} \, dx.$$

(5.2)

In view of the various boundedness assumptions, Lebesgue’s dominated convergence theorem applies and gives us

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} V(x) u v_\lambda = - \frac{1}{2} \int_{\mathbb{R}^N} (NV(x) + x \cdot \nabla V(x)) |u(x)|^2 \, dx$$

In view of the assumption $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)$, we can perform an integration by parts

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda = \int_{\mathbb{R}^N} \varphi(\lambda x) \left( |\nabla u(x)|^2 + x \cdot \nabla \left( \frac{|u|^2}{2} \right)(x) \right) \, dx$$

$$+ \int_{\mathbb{R}^N} (\lambda \nabla u(x) \cdot \nabla \varphi(\lambda x))(x \cdot \nabla u(x)) \, dx$$

$$= - \int_{\mathbb{R}^N} \left( (N - 2) \varphi(\lambda x) + \lambda x \cdot \nabla \varphi(\lambda x) \right) \frac{|\nabla u(x)|^2}{2} \, dx$$

$$+ \int_{\mathbb{R}^N} (\nabla u(x) \cdot \nabla \varphi(\lambda x))(\lambda x \cdot \nabla u(x)) \, dx$$

Since $|(\eta \cdot \zeta)(\eta \cdot \xi)| \leq |\eta|^2 |\zeta||\xi|$ for any $\zeta, \xi, \eta \in \mathbb{R}^N$, we have for each $x \in \mathbb{R}^N$

$$|(|\nabla u(x) \cdot \nabla \varphi(\lambda x)) (\lambda x \cdot \nabla u(x))| \leq |\nabla u(x)|^2 |\lambda x||\nabla \varphi(\lambda x)| \leq |\nabla u(x)|^2 \sup_{z \in \mathbb{R}^N} |z||\nabla \varphi(z)|.$$

By Lebesgue’s dominated convergence theorem again, we have, since $u \in H^1_0(\mathbb{R}^N)$,

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda = - \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2.$$

Finally, by symmetry and integration by parts

$$\int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^{p-2} u v_\lambda$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y) |u(y)|^p \varphi(\lambda x) x \cdot \nabla \left( \frac{|u|^p}{p} \right)(x) \, dx \, dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y) \left( |u(y)|^p \varphi(\lambda x) x \cdot \nabla \left( \frac{|u|^p}{p} \right)(x) \right.$$

$$+ |u(x)|^p \varphi(\lambda y) y \cdot \nabla \left( \frac{|u|^p}{p} \right)(y) \big) \, dx \, dy$$

$$= - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y) |u(y)|^p (NV(\lambda x) + \lambda x \cdot \nabla \varphi(\lambda x)) \frac{|u(x)|^p}{p} \, dx \, dy$$

$$+ \frac{N - \alpha}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y) |u(y)|^p \frac{(x - y) \cdot (x \varphi(\lambda x) - y \varphi(\lambda y))}{|x - y|^2} |u(x)|^p \, dx \, dy.$$
For any $\lambda > 0$ and $x, y \in \mathbb{R}^N$,

$$
\left| \frac{(x - y) \cdot (x\varphi(\lambda x) - y\varphi(\lambda y))}{|x - y|^2} \right| = \left| \frac{(\lambda x - \lambda y) \cdot (\lambda x\varphi(\lambda x) - \lambda y\varphi(\lambda y))}{|\lambda x - \lambda y|^2} \right| 
\leq \sup_{z, w \in \mathbb{R}^N} \left| \frac{(w - z) \cdot (w\varphi(w) - z\varphi(z))}{|w - z|^2} \right| < +\infty.
$$

(5.3)

We can thus apply Lebesgue’s dominated convergence theorem to conclude that

$$
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^{p-2} u v_\lambda = -\frac{N + \alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha |u|^p) |u|^p.
$$

(5.3)

Hence, the identity (5.1) holds. \qed

Remark 5.2. The Pohožaev identity implies some nonexistence results for the Choquard equation (C). In general, if

$$
(2V(x) + x \cdot \nabla V(x))(N - 2 - \frac{N + \alpha}{p}) \geq 0,
$$

(5.4)

or

$$
((N - \frac{N + \alpha}{p})V(x) + x \cdot \nabla V(x))(N - 2 - \frac{N + \alpha}{p}) \geq 0,
$$

(5.5)

then the Choquard equation (C) has no nontrivial solutions satisfying the regularity and boundedness assumptions of Theorem 5.1. In particular, if $V(x) = |x|^\beta$ is homogeneous, then the Choquard equation (C) has no such solution if $p \in (1, \max\{1, \frac{N + \alpha}{N + \beta}\}] \cup [\frac{N + \alpha}{N + \beta}, +\infty)$.

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