Abstract: This note shows that the arithmetic function \( \psi(N) / N = \prod_{p \mid N} (1 + 1 / p) \), called the Dedekind psi function, achieves its extreme values on the subset of primorial integers \( N = 2^{v_2} \cdot 3^{v_3} \cdots p_k^{v_k} \), where \( p_i \) is the \( k \)th prime, and \( v_i \geq 1 \). In particular, the inequality \( \psi(N) / N > 6\pi^2 e^\gamma \log \log N \) holds for all large squarefree primorial integers \( N = 2 \cdot 3 \cdot 5 \cdots p_k \) unconditionally.

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1 Introduction
The psi function \( \psi(N) = N \prod_{p \mid N} (1 + 1 / p) \) and its normalized counterpart \( \psi(N) / N = \prod_{p \mid N} (1 + 1 / p) \) arise in various mathematic, and physic problems. Moreover, this function is entangled with other arithmetic functions. The values of the normalized psi function coincide with the squarefree kernel

\[
\sum_{d \mid N} \frac{\mu^2(d)}{d}
\]

of the sum of divisor function \( \sigma(N) = \sum_{d \mid N} d \). In particular, \( \psi(N) / N = \prod_{p \mid N} (1 + 1 / p) = \sigma(N) / N \) on the subset of square-free integers. This note proposes a new lower estimate of the Dedekind function.

Theorem 1. Let \( N \in \mathbb{N} \) be an integer, then \( \psi(N) / N > 6\pi^2 e^\gamma \log \log N \) holds unconditionally for all sufficiently large primorial integer \( N = 2 \cdot 3 \cdot 5 \cdots p_k \), where \( p_k \) is the \( k \)th prime.

An intuitive and clear-cut relationship between the Riemann hypothesis and the Dedekind psi function is established in Theorem 4 by means of the prime number theorem. Recall that the Riemann hypothesis claims that the nontrivial zeros of the zeta function \( \zeta(s) \) are located on the critical line \( \{ \Re(s) = 1/2 \} \), see [ES]. A survey of various strikingly different reformulations of the Riemann hypothesis appears in [AM]. And a derivation of the reformulation of the Riemann hypothesis in terms of the psi function was recently resolved in [SP].
This estimate is quite similar to the lower estimate of the totient function, namely,

\[ N_k / \varphi(N_k) > e^{\gamma} \log \log N_k \]  

(2)

for all large squarefree primorial integers \( N_k = 2 \cdot 3 \cdot 5 \cdots p_k \). This inequality, called the Nicolas inequality, is also a reformulation of the Riemann hypothesis, see [NS].

The next section discusses background information, and supplies the proof of Theorem 1 in Subsection 2.2. The derivation of this result unfolds from a result on the oscillations theorem of finite prime product.

2 Elementary Materials And The Proof
The basic concepts and results employed throughout this work are stated in this Section.

2.1 Sums and Products Over the Primes. The most basic finite sum over the prime numbers is the prime harmonic sum \( \sum_{p \leq x} p^{-1} \). The refined estimate of this finite sum, stated below, is a synthesis of various results due to various authors. The earliest version \( \sum_{p \leq x} p^{-1} = \log \log x + B_1 + O(1 / \log x) \) is due to Mertens.

**Theorem 2.** Let \( x \geq 2 \) be a sufficiently large number. Then

\[
\sum_{p \leq x} \frac{1}{p} = \begin{cases} 
\log \log x + B_1 + O(e^{-c(\log \log x)^{1/2}}), & \text{unconditionally,} \\
\log \log x + B_1 + O((\log x)^{1/2}), & \text{conditional on the Riemann hypothesis,} \\
\log \log x + B_1 + \Omega_{\delta}(\log x)^{-1/2} \log \log \log x / \log \log x), & \text{unconditional oscillations,}
\end{cases}
\]  

(3)

where \( B_1 = .2614972128 \ldots \).

Proof: Use the integral representation of the finite sum

\[
\sum_{p \leq x} \frac{1}{p} = \int_{c}^{x} \frac{d\pi(t)}{t},
\]  

(4)

where \( c > 1 \) is a small constant. Here, the prime counting function \( \pi(x) = \# \{ p \leq x : p \text{ is prime} \} \) has the form

\[
\pi(x) = \begin{cases} 
li(x) + O(x e^{-c(\log x)^{1/2}}), & \text{unconditionally,} \\
li(x) + O(x^{1/2} \log x), & \text{conditional on the Riemann hypothesis,} \\
li(x) + \Omega_{\delta}(x^{1/2} \log \log x / \log x), & \text{unconditional oscillations.}
\end{cases}
\]  

(5)

The unconditional part of the prime counting formula arises from the delaVallee Poussin form of the prime number theorem \( \pi(x) = li(x) + O(x e^{-c(\log x)^{1/2}}) \), see [MV, p. 179], the conditional part arises from the Riemann form of the prime number theorem \( \pi(x) = li(x) + O(x^{1/2} \log x) \), and the unconditional oscillations part arises from the Littlewood form of the prime number theorem \( \pi(x) = li(x) + \Omega_{\delta}(x^{1/2} \log \log x / \log x) \), consult [IV, p.
The quantity \( T \) from where the Euler constant is defined by \( e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!} \), and the appropriate prime counting measure \( d\pi(t) \), and simplify the integral.

The proof of the unconditional part of this result is widely available in the literature, see [HW], [MV], [TN], et cetera. As an application of the last result, there are the following interesting products:

The omega notation \( f(x) = g(x) + \Omega_x(h(x)) \) means that both \( f(x) > g(x) + c_0 h(x) \) and \( f(x) < g(x) - c_0 h(x) \) occur infinitely often as \( x \to \infty \), where \( c_0 > 0 \) is a constant, see [MV, p. 5], [WK].

**Theorem 3.** Let \( x \in \mathbb{R} \) be a large real number, then

\[
\prod_{p \leq x} \left(1 - \frac{1}{p} \right)^{-1} = \begin{cases} 
\exp \log x + O(e^{c (\log x)^{1/2}} \log x), & \text{unconditionally,} \\
\exp \log x + O(x^{-1/2} \log x), & \text{conditional on the Riemann hypothesis,} \\
\exp \log x + \Omega_x(x^{-1/2} \log \log \log x / \log x), & \text{unconditional oscillations,}
\end{cases}
\]

(6)

Proof: Consider the logarithm of the product

\[
\log \prod_{p \leq x} (1 - 1/p)^{-1} = \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{np^n} = \sum_{p \leq x} \frac{1}{p} + \gamma - B_1 + O(1/x),
\]

(7)

where the Euler constant is defined by \( \gamma = \lim_{x \to \infty} \sum_{n=1}^{x} (n^{-1} - \log n) = 0.577215665... \), and the Mertens constant is defined by \( B_1 = \gamma + \sum_{p \leq 2} (\log (1 - 1/p) + 1/p) = .2614972128... \), see [HW, p. 466]. The last equality in (7) stems from the power series expansion \( B_1 = \gamma - \sum_{p \leq 2} \sum_{n=2}^{\infty} (np^n)^{-1} \), which yields

\[
\sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{np^n} = \gamma - B_1 - \sum_{p > x} \sum_{n=2}^{\infty} \frac{1}{np^n} = \gamma - B_1 + O(1/x),
\]

(8)

The remaining steps follows from Theorem 2, and reversing the logarithm.

The third part above simplifies the proof given in [DP] of the following result:

The quantity

\[
x^{1/2} \left( \prod_{p \leq x} (1 - 1/p)^{-1} - \exp \log x \right)
\]

(9)

attains arbitrary large positive and negative values as \( x \to \infty \).
Theorem 4. Let \( x > x_0 \) be a real number, then
\[
\prod_{p \leq x} \left( 1 + \frac{1}{p} \right) = \begin{cases} 
6\pi^{-2} e^x \log x + O(e^{-c(\log x)^{1/2}} \log x), & \text{unconditionally,} \\
6\pi^{-2} e^x \log x + O(x^{-1/2} \log x), & \text{conditional on the Riemann hypothesis,} \\
6\pi^{-2} e^x \log x + \Omega_2(x^{-1/2} \log \log x / \log x), & \text{unconditional oscillations,}
\end{cases}
\]
(10)

Proof: For a large real number \( x \in \mathbb{R} \), rewrite the product as
\[
\prod_{p \leq x} \left( 1 + \frac{1}{p} \right) = \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \prod_{p \leq x} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1}.
\]
(11)

Replacing \( \prod_{p \leq x} (1 - 1/p^2) = \sum_{n \leq x} \mu(n)n^{-2} = 6\pi^{-2} \) in the first product on the right side, yields
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} = \frac{6}{\pi^2} - \sum_{n \leq x} \mu(n)n^{-2} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1}
\]
\[
= \frac{6}{\pi^2} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} + O\left( \frac{\log x}{x^c} \right),
\]
(12)

where \( z = O(x^c) \), \( c \geq 1 \) constant. Lastly, applying Theorem 3, to the last product above, yields the claim.

This result immediately gives an improved upper bound of the sum of divisors function; currently, the best upper bound of the sum of divisors function is
\[
\sigma(N) / N \leq e^\gamma \log N + O(1 / \log \log N)
\]
(13)
for any integer \( N \geq 1 \), see [RN].

Corollary 5. The sum of divisors function satisfies the followings inequalities unconditionally:
\[
\sum_{d \mid N} \frac{1}{d} = e^\gamma \log N + O(\log N^{-1/2} \log \log N),
\]

conditional on the RH,
\[
\sum_{d \mid N, \mu(d) \neq 0} \frac{1}{d} = 6\pi^{-1} e^\gamma \log N + O(e^{-c(\log N)^{1/2}}),
\]
unconditionally,
\[
6\pi^{-1} e^\gamma \log N + O((\log N)^{-1/2} \log \log N),
\]
conditional on the RH,
\[
6\pi^{-1} e^\gamma \log N + \Omega_2((\log N)^{-1/2} \log \log N / \log N),
\]
unconditional oscillations,
the sum of divisors function \( \sigma(N)/N \),

(14)
the squarefree kernel of \(\sigma(N)/N\),

\[
\sum_{d|N, \mu(d)=0} \frac{1}{d} = \begin{cases} 
(1-6\pi^{-1})e^{\gamma} \log \log N + O(e^{-\gamma \log \log N}^{1/2}), & \text{unconditionally,} \\
(1-6\pi^{-1}e^{\gamma}) \log \log N + O((\log N)^{-1/2} \log \log N), & \text{conditional on the RH,} \\
(1-6\pi^{-1}e^{\gamma}) \log \log N + \Omega_{\pm}((\log N)^{-1/2} \log \log \log N / \log \log N), & \text{unconditional oscillations,}
\end{cases}
\]

the nonsquarefree kernel of \(\sigma(N)/N\).

Proof: Utilize the decomposition of the sum of divisors function

\[
\sum_{d|N} \frac{1}{d} = \sum_{d|N, \mu(d)=0} \frac{1}{d} + \sum_{d|N, \mu(d)=0} \frac{1}{d}, \tag{15}
\]

and apply Theorem 4 to the identities

\[
\sum_{d|N, \mu(d)=0} \frac{1}{d} = \prod_{p \in \mathbb{P}} \left( 1 + \frac{1}{p} \right), \quad \text{and} \quad \sum_{d|N, \mu(d)=0} \frac{1}{d} = \left( \frac{\pi^2}{6} - 1 \right) \prod_{p \in \mathbb{P}} \left( 1 + \frac{1}{p} \right), \tag{16}
\]

where \(x = c_1 \log N\), and \(c_1 > 0\) is a constant.

The oscillating nature of the sum of divisor function is readily visible in the Ramanujan expansion

\[
\frac{\sigma(N)}{N} = \frac{\pi^2}{6} \left( 1 + \frac{\cos \pi N}{2^2} + \frac{2 \cos 2\pi N}{3^2} + \ldots \right). \tag{17}
\]

### 2.2 The Extreme Values of Dedekind Psi Function

The proof of Theorem 1 presented below relies on the oscillation theorem of the finite prime product; a completely elementary proof, not based on the oscillation theorem is also available, but it is longer.

From the elementary inequality

\[
\prod_{p|N} (1+1/p) = \prod_{p|N} \left( 1-1/p^2 \right) \prod_{p|N} (1-1/p)^{-1} > \frac{6}{\pi^2} \prod_{p|N} (1-1/p)^{-1}, \tag{18}
\]

and Theorems 3 and 4, it is plausible to expect that \(\prod_{p|N} (1+1/p) > 6e^{\gamma} \pi^{-2} \log \log N\) infinitely often as \(N \to \infty\).

**Theorem 1.** Let \(N \in \mathbb{N}\) be a primorial integer, then \(\psi(N)/N > 6\pi^{-2}e^{\gamma} \log \log N\) holds unconditionally for all sufficiently large \(N = 2 \cdot 3 \cdot 5 \cdots p_k\).

Proof: Theorem 4 implies that the product
A Dedekind Psi Function Inequality

\[
\prod_{p \in \mathbb{P}} (1 + 1/p) = \frac{6e^\gamma}{\pi^2} \log x + \Omega \left( \frac{\log \log x}{x^{1/2} \log \log x} \right). \tag{19}
\]

In particular, it follows that

\[
\prod_{p \in \mathbb{P}} (1 + 1/p) > \frac{6e^\gamma}{\pi^2} \log x + c_0 \frac{\log \log x}{x^{1/2} \log \log x} \tag{20}
\]

and

\[
\prod_{p \in \mathbb{P}} (1 + 1/p) < \frac{6e^\gamma}{\pi^2} \log x - c_0 \frac{\log \log x}{x^{1/2} \log \log x}
\]

occur infinitely often as \( x \to \infty \), where \( c_0, c_1, \) and \( c_2 > 0 \) are constants. It shows that \( \prod_{p \in \mathbb{P}} (1 + 1/p) \) oscillates infinitely often, symmetrically about the line \( 6e^\gamma \pi^2 \log x \) as \( x \to \infty \).

To rewrite the variable \( x \geq 1 \) in terms of the integer \( N \), recall that the Chebychev function \( \theta(x) = \sum_{p \leq x} \log p \), and

\[
\log N_k = \sum_{p \leq p_k} \log p = \theta(p_k), \quad \text{and} \quad \theta(p_k) = p_k + o(p_k) \leq c_1 \log N_k. \tag{21}
\]

So it readily follows that \( p_k \leq x = c_1 \log N_k \).

Moreover, since the maxima of the sum of divisor function \( \sigma(N) \geq \psi(N) \) occur at the colossally abundant integers \( N = 2^{v_1} \cdot 3^{v_2} \cdots p_k^{v_k} \), and \( v_1 \geq v_2 \geq \cdots \geq v_k \geq 1 \), see [AE], [BR], [LA], [RJ], it follows that the maxima of the Dedekind psi function \( \psi(N) \) occur at the squarefree primorial integers \( N_k = 2 \cdot 3 \cdot 5 \cdots p_k \). Therefore, expression (19) implies that

\[
\frac{\psi(N_k)}{N_k} = \prod_{p \in \mathbb{P}, \log N_k} (1 + 1/p) > \frac{6e^\gamma}{\pi^2} \log N_k + c_2 \frac{\log \log \log N_k}{(\log N_k)^{1/2} \log \log N_k} \tag{22}
\]

as the primorial integer \( N_k = 2 \cdot 3 \cdot 5 \cdots p_k \) tends to infinity. \( \blacksquare \)
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