LIMITED REGULARITY OF SOLUTIONS TO FRACTIONAL HEAT AND SCHRÖDINGER EQUATIONS

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Abstract. When $P$ is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$, or a pseudodifferential generalization thereof, the Dirichlet problem for the associated heat equation over a smooth set $\Omega \subset \mathbb{R}^n$: $r^+ Pu(x,t) + \partial_t u(x,t) = f(x,t)$ on $\Omega \times [0,T]$, $u(x,t) = 0$ for $x \notin \Omega$, $u(x,0) = 0$, is known to be solvable in relatively low-order Sobolev or Hölder spaces. We now show that in contrast with differential operator cases, the regularity of $u$ in $x$ at $\partial \Omega$ when $f$ is very smooth cannot in general be improved beyond a certain estimate. An improvement requires the vanishing of a Neumann boundary value. — There is a similar result for the Schrödinger Dirichlet problem $r^+ P v(x) + V v(x) = g(x)$ on $\Omega$, supp $v \subset \overline{\Omega}$, with $V(x) \in C^\infty$. The proofs involve a precise description of the Dirichlet domains in terms of functions supported in $\Omega$ and functions pulled back from boundary values.

0. Introduction

The main purpose of the paper is to investigate limitations on the regularity of solutions to nonlocal parabolic Dirichlet problems for $x$ in a bounded smooth subset $\Omega$ of $\mathbb{R}^n$ and $t$ in an interval $I = ]0,T[$:

\begin{align}
 r^+ Pu(x,t) + \partial_t u(x,t) &= f(x,t) \quad \text{on } \Omega \times I, \\
 u(x,t) &= 0 \quad \text{for } x \notin \Omega, \\
 u(x,0) &= u_0(x),
\end{align}

(0.1)

where $P$ is the fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, or a pseudodifferential generalization (that can be $x$-dependent and non-symmetric); here $r^+$ denotes restriction to $\Omega$. For the stationary Dirichlet problem

\begin{equation}
 r^+ P v(x) = g(x) \quad \text{in } \Omega, \quad \text{supp } u \subset \overline{\Omega},
\end{equation}

(0.2)
it is known from works of Ros-Oton, Serra, Grubb [RS14,RS16,G15,G14], that the solution v bears a singularity at ∂Ω like d(x)^a, where d(x) = dist(x, ∂Ω), but that v/d^a is steadily more regular, the more regular g is. One has for example:

\[ g \in C^\sigma(\Omega) \implies v/d^a \in C^{\sigma+\sigma}(\Omega) \] for \( \sigma > 0 \) with \( \sigma, a + \sigma \notin \mathbb{N} \),

\[ g \in C^{\infty}(\Omega) \iff v/d^a \in C^{\infty}(\Omega). \]

(0.3) is shown in [RS14,RS16] for small \( \sigma \) (allowing low regularity of ∂Ω), and in [G14] for all \( \sigma \), and (0.4) is shown in [G15], drawing on early work of Hörmander.

To formulate results for the nonstationary problem (0.1), let us temporarily denote the domain spaces for the Dirichlet problem (0.2) with \( g \in \overline{H}_p^r(\Omega) \), resp. \( g \in \overline{C}^r(\Omega) \) by

\[ D_{r,H_p}(P) = \{ v \in \hat{H}_p^a(\Omega) \mid r^+Pv \in \overline{H}_p^r(\Omega) \} \] for \( r \geq 0 \);

\[ D_{r,C}(P) = \{ v \in \hat{H}_2^a(\Omega) \mid r^+Pv \in \overline{C}^r(\Omega) \}, \] for \( r \in \mathbb{R}_+ \setminus \mathbb{N} \).

These spaces, which identify with spaces \( H_p^{a(2a+r)}(\Omega) \) resp. \( C_s^{a(2a+r)}(\Omega) \) introduced in [G15,G14], will be described in detail in Section 2, explaining how the factor d^a enters.

Up to now, there have been shown some results for (0.1) in function spaces of relatively low order, such as for example:

\[ f \text{ is } C^\sigma \text{ in } x \text{ and } C_{2/\sigma}^\infty \text{ in } t \implies u/d^a \text{ is } C^{\sigma+\sigma} \text{ in } x \text{ and } C^{2+\sigma} \text{ in } t, \]

\[ f \in L_p(\Omega \times I) \iff u \in L_p(I; D_{0,H_p}(P)) \cap \overline{H}_p^1(I, L_p(\Omega)), \]

\[ f \in L_2(I; \overline{H}^r(\Omega)) \cap \overline{H}^k(\Omega; L_2(\Omega)) \text{ with } \partial^j t f(x, 0) = 0 \text{ for } j < k \]

\[ \implies u \in L_2(I; D_{r,H_2}(P)) \cap \overline{H}^{k+1}(I; L_2(\Omega)), \]

with \( \sigma \in ]0, a] \), \( a + \sigma \notin \mathbb{N} \), \( 1 < p < \infty \), \( k \in \mathbb{N} \), \( r \leq \min\{2a, a+\frac{1}{2} - \varepsilon\} \). Here (0.7) is shown by Ros-Oton and Vivas [RV18] building on Fernandez-Real and Ros-Oton [FR17], and (0.8), (0.9) are shown in [G18] resp. [G18b].

It is natural to ask whether the nonstationary results can be lifted to higher regularities in x like in the stationary case: Will solutions have a \( C^{\infty} \)-property if f is \( C^{\infty} \)? or e.g. a higher Hölder regularity, when f belongs to a higher Hölder space? Such rules holds for differential operator heat problems, and for interior regularity [G18], but, perhaps surprisingly, they do not hold up to the boundary in the present nonlocal cases.

A first counterexample to the \( C^{\infty} \)-lifting was given in [G18b], derived from a certain irregularity of the eigenfunctions of the Dirichlet realization of P. In the present study we show more precisely how the regularity of the solution is limited to \( u \in D_{a,C}(P) \) with respect to x (not \( D_{a+\delta,C}(P) \) with \( \delta > 0 \), unless the boundary value of u/d^a vanishes.

The heat equation result is based on an analysis of the solutions of the resolvent equation for \( \lambda \neq 0 \),

\[ (r^+P - \lambda)v = g \text{ in } \Omega, \text{ supp } v \subset \overline{\Omega}, \]

and a precise description of the spaces \( D_{r,C}(P) \).
We also study Schrödinger Dirichlet problems
\[(r^+ P + V)v = g \text{ in } \Omega, \quad \text{supp } v \subset \overline{\Omega},\]
with a $C^\infty$-potential $V$, and find a related limitation on the smoothness of solutions, when $V$ does not vanish on $\partial \Omega$.

About the operators $P$: The fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, can be described as a pseudodifferential operator ($\psi$-do) or as a singular integral operator:
\[
(-\Delta)^a u = \text{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a} \hat{u}(\xi))
\]
\[
= c_{n,a} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x + y)}{|y|^{n+2a}} \, dy.
\]
The operators we shall study are the following generalization of (0.12) to a large class of $\psi$-do’s: The classical strongly elliptic $\psi$-do’s $P = \text{Op}(p(x, \xi)$) of order $2a$, with symbol $p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ being even:
\[
p_j(x, -\xi) = (-1)^j p_j(x, \xi), \text{ all } j.
\]
(An example is $P = A(x, D)^a$, where $A(x, D)$ is a second-order strongly elliptic differential operator.)

The singular integral definition (0.13) can also be generalized, by replacement of the kernel function $|y|^{-n-2a}$ by other positive functions $K(y)$ homogeneous of degree $-n-2a$ and even, i.e. $K(-y) = K(y)$, and with possibly less smoothness, or nonhomogeneous but estimated above in terms of $|y|^{-n-2a}$ (see e.g. the survey [R16]); this gives translation-invariant symmetric operators. These are operators defining stable Lévy processes. The case where $K$ is homogeneous and $C^\infty$ for $y \neq 0$ is a special case of our $\psi$-do’s, with symbol $p(\xi) = \mathcal{F}K(y)$.

Let us mention some of the results. The general assumption is:

**Hypothesis 0.1.** For some $a > -1$, $P = \text{Op}(p(x, \xi))$ is a classical strongly elliptic $\psi$-do of order $2a$ on $\mathbb{R}^n$, with even symbol $p(x, \xi)$, cf. (0.14), and possibly with a smoothing term $\mathcal{R}$ added (continuous from $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$).

Firstly, we show in Section 2 an exact representation of the elements of the domain spaces $D_{r,H_p}(P)$ (0.5) and $D_{r,C}(P)$ (0.6) in terms of a component supported in $\overline{\Omega}$ with smoothness $2a + r$ and a component pulled back from the boundary $\partial \Omega$ by Poisson-like operators applied to weighted boundary values $\gamma_j^a v$,
\[
\gamma_j^a v = \Gamma(a + 1 + j) \gamma_j(v/d^a);
\]
this is a development of results from [G15,G14].

Based on this, we show in Sections 3 resp. 4:

**Theorem 0.2.** Let $0 < a < 1$, and let $V \in \overline{C}^\infty(\Omega)$. When $g \in \overline{C}^a(\Omega)$, the solutions of (0.11) are in $D_{a,C}(P)$.

Let $V \neq 0$ on an open subset $\Sigma$ of $\partial \Omega$, and let $v$ be a solution of (0.11). If there is a $\delta > 0$ such that $v \in D_{a+\delta,C}(P)$ with $g \in \overline{C}^{a+\delta}(\Omega)$, then $\gamma_0^a v$ vanishes on $\Sigma$.

Here $\gamma_0^a v$ can be regarded as the Neumann boundary value of $v$. 
Theorem 0.3. Let $0 < a < 1$. Let $u(x,t) \in \overline{W}^{1,1}(I; D_{a,C}(P))$, $f(x,t) \in L_1(I; C^a(\Omega))$, satisfying (0.1) with $u_0 = 0$.

If there is a $\delta > 0$ such that $u(x,t) \in \overline{W}^{1,1}(I; D_{a+\delta,C}(P))$ and $f(x,t) \in L_1(I; C^{a+\delta}(\Omega))$, then $\gamma_0^a u(x,t) = 0$.

Expressed in words, a necessary condition for lifting the regularity parameter $a$ to $a + \delta$ for $u$ and $f$ is that the Neumann boundary value vanishes.

There is a rich literature on the fractional Laplacian and its generalizations; let us mention some of the studies through the times: Blumenthal and Getoor [BG59], Landkof [L72], Hoh and Jacob [HJ96], Kulczycki [K97], Chen and Song [CS98], Jacob [J01], Jakubowski [J02], Bogdan, Burdzy and Chen [BBC03], Cont and Tankov [CT04], Silvestre [S07], Caffarelli and Silvestre [CS07], Musina and Nazarov [MN14], Frank and Geisinger [FG16], Ros-Oton and Serra [RS14,RS14a,RS16], Abatangelo [A15], Felsinger, Kassmann and Voigt [FKV15], Bonforte, Sire and Vazquez [BSV15], Servadei and Valdinoci [SV14], Ros-Oton [R16, R18]; there are many more papers referred to in these works, and numerous applications to nonlinear problems.

Further contributions to the regularity theory of fractional heat equations are given in Felsinger and Kassmann [FK13], Chang-Lara and Davila [CD14], Jin and Xiong [JX15], Ros-Oton and Serra [RS14, RS16], Abatangelo [A15], Felsinger, Kassmann and Voigt [FKV15], Bonforte, Sire and Vazquez [BSV15], Servadei and Valdinoci [SV14], Ros-Oton [R16, R18]; there are many more papers referred to in these works, and numerous applications to nonlinear problems.

1. Preliminaries

1.1 $L_p$-Sobolev spaces.

The domain spaces for homogeneous Dirichlet problems were described in many scales of spaces in [G15] and [G14]; we shall here just focus on two scales, namely the Bessel-potential scale $H^s_p$ ($1 < p < \infty$) (the main subject of [G15]), which serves to show general estimates in $L_p$ Sobolev spaces, and the Hölder-Zygmund scale $C^s$ (included in [G14] in a systematic way) leading to optimal Hölder estimates. We shall go through the various concepts somewhat rapidly, since they have already been explained in previous papers; the reader may consult e.g. [G15,G14] if more details are needed.

We recall that the standard Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$ and $s \geq 0$, have a different character according to whether $s$ is integer or not. Namely, for $s$ integer, they consist of $L_p$-functions with derivatives in $L_p$ up to order $s$, hence coincide with the Bessel-potential spaces $H^s_p(\mathbb{R}^n)$, defined for $s \in \mathbb{R}$ by

\begin{equation}
H^s_p(\mathbb{R}^n) = \{ u \in L^\infty(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_p(\mathbb{R}^n) \}.
\end{equation}

Here $\mathcal{F}$ is the Fourier transform $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx$, and the function $\langle \xi \rangle$ equals $(|\xi|^2 + 1)^{\frac{s}{2}}$. For noninteger $s$, the $W^{s,p}$-spaces coincide with the Besov spaces, defined e.g. as follows: For $0 < s < 2$,

\begin{equation}
f \in B^s_p(\mathbb{R}^n) \iff \| f \|_{L_p}^p + \int_{\mathbb{R}^{2n}} \frac{|f(x) + f(y) - 2f((x+y)/2)|^p}{|x+y|^{n+ps}} \, dx \, dy < \infty;
\end{equation}

and $B^{s+t}_p(\mathbb{R}^n) = (1 - \Delta)^{-t/2} B^s_p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$. The Bessel-potential spaces are important because they are most directly related to $L_p(\mathbb{R}^n)$; the Besov spaces have other
convenient properties, and are needed for boundary value problems in an $H^s_p$-context, because they are the correct range spaces for trace maps $\gamma_j u = (\partial^j_n u)|_{x_n = 0}$:

\begin{equation}
\gamma_j: \overline{H}^s_p(\mathbb{R}^n_+), \overline{B}^s_p(\mathbb{R}^n_+) \to B^{s-j-1/p}_p(\mathbb{R}^{n-1}), \text{ for } s - j - 1/p > 0,
\end{equation}

surjectively and with a continuous right inverse; see e.g. the overview in the introduction to [G90]. For $p = 2$, the two scales are identical, but for $p \neq 2$ they are related by strict inclusions: $H^s_p \subset B^s_p$ when $p > 2$, $H^s_p \supset B^s_p$ when $p < 2$.

The following subsets of $\mathbb{R}^n$ will be considered: $\mathbb{R}^n_\pm = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}$ (where $(x_1, \ldots, x_{n-1}) = x'$), and bounded $C^\infty$-subsets $\Omega$ with boundary $\partial \Omega$, and their complements. Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from $\mathbb{R}^n$ to $\Omega$ resp. $\Omega^c$) is denoted $r^\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from $\Omega$ resp. $\Omega^c$ to $\mathbb{R}^n$) is denoted $e^\pm$. Restriction from $\mathbb{R}^n_+$ or $\Omega^c$ to $\partial \mathbb{R}^n_+$ resp. $\partial \Omega$ is denoted $\gamma_0$.

We denote by $d(x)$ a function of the form $d(x) = \text{dist}(x, \partial \Omega)$ for $x \in \Omega$, $x$ near $\partial \Omega$, extended to a smooth positive function on $\Omega$; $d(x) = x_n$ in the case of $\mathbb{R}^n_+$.

Along with the spaces $H^s_p(\mathbb{R}^n)$ defined in (1.1), we have the two scales of spaces associated with $\Omega$ for $s \in \mathbb{R}$:

\begin{equation}
\overline{H}^s_p(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H^s_p(\mathbb{R}^n) \}, \text{ the restricted space,}
\end{equation}

\begin{equation}
\dot{H}^s_p(\Omega) = \{ u \in H^s_p(\mathbb{R}^n) \mid \text{supp } u \subset \Omega \}, \text{ the supported space;}
\end{equation}

here supp $u$ denotes the support of $u$. The definition is also used with $\Omega = \mathbb{R}^n_+$. $\overline{H}^s_p(\Omega)$ is in other texts often denoted $H^s_p(\Omega)$ or $H^s_p(\Omega^c)$, and $\dot{H}^s_p(\Omega)$ may be indicated with a ring, zero or twiddle; the current notation stems from Hörmander [H85], Appendix B2. There are similar spaces with $B^s_p$.

We recall that $\overline{H}^s_p(\Omega)$ and $\dot{H}^s_p(\Omega^c)$ are dual spaces with respect to a sesquilinear duality extending the $L_2(\Omega)$-scalar product; $\frac{1}{p} + \frac{1}{p'} = 1$.

### 1.2 Pseudodifferential operators.

A pseudodifferential operator (ψdo) $P$ on $\mathbb{R}^n$ is defined from a symbol $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

\begin{equation}
P u = \text{Op}(p(x, \xi)) u = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{u} d\xi = F_{\xi \to x}^{-1}(p(x, \xi) F u(x)),
\end{equation}

using the Fourier transform $F$. We refer to textbooks such as Hörmander [H85], Taylor [T81], Grubb [G09] for the rules of calculus (in particular the definition by oscillatory integrals in [H85]). The symbols $p$ of order $m \in \mathbb{R}$ we shall use are assumed to be classical: $p$ has an asymptotic expansion $p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ with $p_j$ homogeneous in $\xi$ of degree $m - j$ for $|\xi| \geq 1$, all $j$, such that

\begin{equation}
\partial^\alpha_x \partial^\beta_\xi (p(x, \xi) - \sum_{j < J} p_j(x, \xi)) = O(|\xi|^{m - \alpha - J}) \text{ for all } \alpha, \beta \in \mathbb{N}^n_0, J \in \mathbb{N}_0.
\end{equation}

$P$ (and $p$) is said to be strongly elliptic when Re $p_0(x, \xi) \geq c|\xi|^m$ for $|\xi| \geq 1$, with $c > 0$. These classical ψdo’s of order $m$ are continuous from $H^s_p(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

For a complete theory one adds to these operators the smoothing operators (mapping any $H^s_p(\mathbb{R}^n)$ into $\bigcap_1 H^j_p(\mathbb{R}^n)$), regarded as operators of order $-\infty$. (For example, $(-\Delta)^a$ fits into the calculus when it is written as $\text{Op}(1 - \zeta(\xi))|\xi|^{2a} + \text{Op}(\zeta(\xi)|\xi|^{2a})$, where $\zeta(\xi)$ is a $C^\infty$-function that equals 1 for $|\xi| \leq \frac{1}{2}$ and 0 for $|\xi| \geq 1$; the second term is smoothing.)
Remark 1.1. The operators we consider in this paper are moreover assumed to be even, cf. (0.14), for simplicity. In comparison, \( P \) of order \( 2a \) satisfies the \( a \)-transmission condition introduced by Hörmander [H66,H85,G15] relative to a given smooth set \( \Omega \subset \mathbb{R}^n \) when

\[
\partial_x^\beta \partial_\xi^\alpha p(x, -\nu(x)) = (-1)^{|\alpha|+j} \partial_x^\beta \partial_\xi^\alpha p(x, \nu(x)), \quad \text{all } \alpha, \beta, j,
\]
at all points \( x \in \partial \Omega \), with interior normal denoted \( \nu(x) \). The evenness means that this is satisfied for any choice of \( \Omega \). The results in the following hold also when one only assumes that the \( a \)-transmission condition is satisfied relative to the particular domain \( \Omega \) considered.

For our description of the solution spaces for (0.2) we must introduce order-reducing operators. There is a simple definition of operators \( \Xi_{\pm}^t \) on \( \mathbb{R}^n \), \( t \in \mathbb{R} \),

\[
\Xi_{\pm}^t = \text{OP}(\chi_{\pm}^t), \quad \chi_{\pm}^t(\xi) = (\langle \xi \rangle \pm i \xi_n)^t;
\]
they preserve support in \( \mathbb{R}^n_{\pm} \), respectively, because the symbols extend as holomorphic functions of \( \xi \) into \( \mathbb{C}_\pm \), respectively; \( \mathbb{C}_{\pm} = \{ z \in \mathbb{C} \mid \text{Im } z \geq 0 \} \). (The functions \( (\langle \xi \rangle \pm i \xi_n)^t \) satisfy only part of the estimates (1.6) with \( m = t \), but the \( \psi \)-do definition can be applied anyway.) There is a more refined choice \( \Lambda_{\pm}^t \) [G90, G15], with symbols \( \lambda_{\pm}^t(\xi) \) that do satisfy all the required estimates, and where \( \bar{\lambda}_{\pm}^t = \lambda_{\pm}^t \). These symbols likewise have holomorphic extensions in \( \xi \), to the complex halfspaces \( \mathbb{C}_\pm \), so that the operators preserve support in \( \mathbb{R}^n_{\pm} \), respectively. Operators with that property are called ”plus” resp. ”minus” operators. There is also a pseudodifferential definition \( \Lambda_{\pm}^{(t)} \) adapted to the situation of a smooth domain \( \Omega \), cf. [G15].

It is elementary to see by the definition of the spaces \( H^s_p(\mathbb{R}^n) \) in terms of Fourier transformation, that the operators define homeomorphisms for all \( s : \Xi_{\pm}^t : H^s_p(\mathbb{R}^n) \to H^{s-t}_p(\mathbb{R}^n) \). The special interest is that the ”plus”/”minus” operators also define homeomorphisms related to \( \mathbb{R}^n_{\pm} \) and \( \overline{\Omega} \), for all \( s \in \mathbb{R} \):

\[
\Xi_{\pm}^t : \hat{H}^s_p(\mathbb{R}^n_{\pm}) \to \hat{H}^{s-t}_p(\mathbb{R}^n_{\pm}), \quad r^+ \Xi_{\pm}^t e^+ : \overline{H}^s_p(\mathbb{R}^n_{\pm}) \to \overline{H}^{s-t}_p(\mathbb{R}^n_{\pm}),
\]

\[
\Lambda_{\pm}^{(t)} : \hat{H}^s(\overline{\Omega}) \to \hat{H}^{s-t}(\overline{\Omega}), \quad r^+ \Lambda_{\pm}^{(t)} e^+ : \overline{H}^s_p(\overline{\Omega}) \to \overline{H}^{s-t}_p(\overline{\Omega}),
\]

with similar rules for \( \Lambda_{\pm}^t \). Moreover, the operators \( \Xi_{\pm}^t \) and \( r^+ \Xi_{\pm}^t e^+ \) identify with each other’s adjoints over \( \mathbb{R}^n_{\pm} \), because of the support preserving properties. There is a similar statement for \( \Lambda_{\pm}^t \) and \( r^+ \Lambda_{\pm}^t e^+ \), and for \( \Lambda_{\pm}^{(t)} \) and \( r^+ \Lambda_{\pm}^{(t)} e^+ \) relative to the set \( \Omega \).

1.3 The \( a \)-transmission spaces.

The special \( a \)-transmission spaces were introduced by Hörmander [H66] for \( p = 2 \), cf. the account in [G15] with the definition for general \( p \):

\[
H^{a(s)}_p(\mathbb{R}^n_{\pm}) = \Xi_{\pm}^a e^+ \overline{H}^{s-a}_p(\mathbb{R}^n_{\pm}) = \Lambda_{\pm}^a e^+ \overline{H}^{s-a}_p(\mathbb{R}^n_{\pm}), \quad \text{for } s > a - 1/p',
\]

\[
H^{a(s)}_p(\overline{\Omega}) = \Lambda_{\pm}^{(-a)} e^+ \overline{H}^{s-a}_p(\overline{\Omega}), \quad \text{for } s > a - 1/p'.
\]

Observe in particular that

\[
H^{a(s)}_p(\overline{\Omega}) = \hat{H}^s_p(\overline{\Omega}) \text{ for } s - a \in ] - 1/p', 1/p[,}
\]
and that for \( s \geq a \),
\[
H_p^{a(s)}(\Omega) \subset H_p^{a(a)}(\Omega) = \dot{H}_p^a(\Omega).
\]

Recall also from [G15] Sect. 5 that there is a hierarchy: 
\[
H_p^{a(s)}(\Omega) \supset H_p^{(a+1)(s)}(\Omega) \supset \cdots \supset H_p^{(a+j)(s)}(\Omega) \quad \text{for } s > a + j - 1/p',
\]
where
\[
(1.13) \quad u \in H_p^{(a+j)(s)}(\Omega) \iff u \in H_p^{a(s)}(\Omega) \text{ with } \gamma_0^a u = \cdots = \gamma_{j-1}^a u = 0.
\]

Moreover,
\[
(1.14) \quad \bigcap_s H_p^{a(s)}(\Omega) = \mathcal{E}_a(\Omega) = e^+d^aC^\infty(\Omega).
\]

The great interest of these spaces is that they allow an exact description of the solution spaces for the Dirichlet problem (0.2), and are independent of \( P \). The following result comes from [G15]:

**Theorem 1.2.** Let \( P \) satisfy Hypothesis 0.1. Consider \( v \in \dot{H}_p^{a-1/p'+\varepsilon}(\Omega) \) (any \( \varepsilon > 0 \)). For \( r > -a - 1/p' \), the solutions of problem (0.2) with \( g \in \mathcal{P}_p(\Omega) \) satisfy \( v \in H_p^{a(r+2a)}(\Omega) \); in fact
\[
(1.15) \quad g \in \mathcal{P}_p(\Omega) \iff v \in H_p^{a(r+2a)}(\Omega),
\]
and the mapping \( r^+P: v \mapsto g \) is Fredholm between these spaces.

It follows that the Dirichlet domain \( D_{r,H_p}(P) \) defined in (0.5) for \( r \geq 0 \) satisfies
\[
(1.16) \quad D_{r,H_p}(P) = H_p^{a(r+2a)}(\Omega).
\]

**Proof.** The first statement is taken directly from Th. 4.4 of [G15], with \( \mu_0 = a \), \( m = 2a \); the factorization index is \( a \) because of the strong ellipticity, as shown in detail e.g. in [G16a].

The second statement specializes this to \( r \geq 0 \) (which is all we need in the present paper) where the prerequisite on \( v \) can be simplified to \( v \in \dot{H}_p^a(\Omega) \), since \( H_p^{a(r+2a)}(\Omega) \subset \dot{H}_p^a(\Omega) \) for small \( \varepsilon \). \( \Box \)

### 1.4 Results in Hölder-Zygmund spaces.

In [G14] (that was written after [G15]), the results were extended to many other scales of spaces, such as Besov spaces \( B^s_{p,q} \) for \( 1 \leq p,q \leq \infty \), and Triebel-Lizorkin spaces \( F^s_{p,q} \) for \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \). Of particular interest is the scale \( B^{s,\infty}_\infty \), also denoted \( C^s_\infty \), the Hölder-Zygmund scale. Here \( C^s_\infty \) identifies with the Hölder space \( C^s \) when \( s \in \mathbb{R}_+ \setminus \mathbb{N} \), and for positive integer \( k \) satisfies \( C^{k-\varepsilon} \supset C^k \supset C^{k-1,1} \supset C^k \) for small \( \varepsilon > 0 \); moreover, \( C^0_\infty \supset L_\infty \).

Here we define
\[
(1.17) \quad C^a_\infty(\Omega) = \Lambda_+^{(-a)} e^+C^{a-a}_\infty(\Omega), \text{ for } s > a - 1,
\]
(where the *'s can be omitted if \( s, s - a \notin \mathbb{N}_0 \)), and there are inclusions as described for \( H^{a(s)}(\Omega) \)-spaces in (1.13)–(1.14):
\[
(1.18) \quad u \in C^a(s+j)(\Omega) \iff u \in C^a_\infty(\Omega) \text{ with } \gamma_0^a u = \cdots = \gamma_{j-1}^a u = 0,
\]
\[
\bigcap_s C^a_\infty(\Omega) = \mathcal{E}_a(\Omega).
\]

There is a result similar to that of Theorem 1.2 with \( H^s_\infty \)-spaces replaced by \( C^s_\infty \)-spaces:
**Theorem 1.3.** Let \( P \) satisfy Hypothesis 1.1. Consider \( v \in \dot{H}^{a-1/p' + \varepsilon}(\Omega) \) (any \( \varepsilon > 0 \)). For \( r \geq 0 \), the solutions of problem (0.2) with \( g \in C^r_*(\Omega) \) satisfy \( v \in C^r_*(r+2a)(\Omega) \); in fact

\[
g \in C^r_*(\Omega) \iff v \in C^r_*(r+2a)(\Omega),
\]

and the mapping \( r^+ P; v \mapsto g \) is Fredholm between these spaces.

It follows that the Dirichlet domain \( D_{r,C}(P) \) defined as \( D_{r,C}(P) = \{ v \in \dot{H}^a_2(\Omega) \mid r^+ P v \in C^r_*(\Omega) \} \) for \( r \geq 0 \), satisfies

\[
D_{r,C}(P) = C^a_*(r+2a)(\Omega).
\]

For \( r \in \mathbb{R}_+ \setminus \mathbb{N} \), this is the domain defined in (0.6); and when \( a + r \) and \( 2a + r \) are noninteger, \( C^a_*(r+2a)(\Omega) \) identifies with \( C^a_*(r+2a)(\Omega) = \Lambda^{(a)}_+ e^{r+C^a}(\Omega) \) defined in terms of ordinary Hölder spaces. It is sometimes an advantage to keep the \( C_* \)-notation, since one does not have to make exceptions for integer indices all the time.

In applications of the above results it is important to get a better understanding of what the \( a \)-transmission spaces consist of. Such an analysis was carried out for the scale \( H^a_p(r+2a) \) in [G15] and for \( C^a_*(r+2a) \) in [G14], and in the next section we take it up again with more explicit results relative to the set \( \Omega \).

2. **Analysis of \( a \)-transmission spaces**

2.1 **Decompositions in terms of the first trace.**

In the following, we shall give a characterization of the \( a \)-transmission spaces showing an exact decomposition of the elements in the case of a general domain \( \Omega \); it is just a more detailed development of the decomposition principle described in Th. 5.4 of [G15]. For clarity, we begin with a decomposition with just one trace involved.

In the proofs we shall use a localization with particularly convenient coordinates, described in detail in [G16] Rem. 4.3 and Lem. 4.4 and recalled in [G18a] Rem. 4.3, which we also recall here:

**Remark 2.1.** \( \overline{\Omega} \) has a finite cover by bounded open sets \( U_0, \ldots, U_I \) with \( C^\infty \)-diffeomorphisms \( \kappa_i; U_i \to V_i \), \( V_i \) bounded open in \( \mathbb{R}^n \), such that \( U_i^+ = U_i \cap \Omega \) is mapped to \( V_i^+ = V_i \cap \mathbb{R}^n_+ \) and \( U_i' = U_i \cap \partial \Omega \) is mapped to \( V_i' = V_i \cap \partial \mathbb{R}^n_+ \); as usual we write \( \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \). For any such cover there exists an associated partition of unity, namely a family of functions \( \vartheta_i \in C^\infty_0(U_i) \) taking values in \([0,1]\) such that \( \sum_{i=0,\ldots,I} \vartheta_i = 1 \) on a neighborhood of \( \overline{\Omega} \).

When \( P \) is a \( \psi \)-do on \( \mathbb{R}^n \), its application to functions supported in \( U_i \) carries over to functions on \( V_i \) as a \( \psi \)-do \( \mathcal{P}^{(i)} \) defined by

\[
\mathcal{P}^{(i)} v = P(v \circ \kappa_i) \circ \kappa_i^{-1}, \quad v \in C^\infty_0(V_i).
\]

When \( u \) is a function on \( U_i \), we usually denote the resulting function \( u \circ \kappa_i^{-1} \) on \( V \) by \( \underline{u} \).

We shall use a convenient system of coordinate charts as described in [G16], Remark 4.3: Here \( \partial \Omega \) is covered with coordinate charts \( \kappa'_i; U'_i \to V'_i \subset \mathbb{R}^{n-1}, i = 1, \ldots, I \), and the \( \kappa_i \) will be defined on certain subsets of a tubular neighborhood \( \Sigma_r = \{ x' + t v(x') \mid x' \in \mathbb{R}^{n-1} \} \).
\( \partial \Omega, |t| < r \), where \( \nu(x') = (\nu_1(x'), \ldots, \nu_n(x')) \) is the interior normal to \( \partial \Omega \) at \( x' \in \partial \Omega \), and \( r \) is taken so small that the mapping \( x'+tv(x') \mapsto (x', t) \) is a diffeomorphism from \( \Sigma_r \) to \( \partial \Omega \times ]-r, r[ \). For each \( i \), \( \kappa_i \) is defined as the mapping \( \kappa_i: x'+tv(x') \mapsto (\kappa_i(x'), t) \) \((x' \in U'_i) \). \( \kappa_i \) goes from \( U_i \) to \( V_i \), where

\[
U_i = \{ x'+tv(x') \mid x' \in U'_i, |t| < r \}, \quad V_i = V'_i \times ]-r, r[ .
\]

These charts are supplied with a chart consisting of the identity mapping on an open set \( U_0 \) containing \( \Omega \setminus \sum r, + \), with \( \bigcup_0 \subset \Omega \), to get a full cover of \( \Omega \).

Note that the normal \( \nu(x') \) at \( x' \in \partial \Omega \) is carried over to the normal \((0, 1) \) at \((\kappa_i(x'), 0) \) when \( x' \in U'_i \). The halfline \( L_{x'} = \{ x'+tv(x') \mid t \geq 0 \} \) is the geodesic into \( \Omega \) orthogonal to \( \partial \Omega \) at \( x' \) (with respect to the Euclidean metric on \( \mathbb{R}^n \)), and there is a positive \( r' \leq r \) such that for \( 0 < t < r' \), the distance \( d(x) \) between \( x = x'+tv(x') \) and \( \partial \Omega \) equals \( t \). Then \( t \) plays the role of \( d \) in the definition of expansions and boundary values of \( u \in E_\alpha(\overline{\Omega}) \) in [G15] (5.3)ff.

\[
u(x') = \frac{1}{(a+1)} t^a u_0 + \frac{1}{(a+2)} t^{a+1} u_1 + \frac{1}{(a+3)} t^{a+2} u_2 + \ldots \quad \text{for } t > 0, \quad u = 0 \text{ for } t < 0,
\]

where the \( u_j \) are constant in \( t \) for \( t < r' \); this serves to define the boundary values

\[
\gamma_j^a u = \gamma_0 u_j (= u_j|_{t=0}), \quad j = 0, 1, 2, \ldots
\]

(denoted \( \gamma_{a,j} u \) in [G15]).

By comparison of (2.3) with \( t^a \) times the Taylor expansion of \( u/t^a \) in \( t \), we also have:

\[
\gamma_0^a u = \Gamma(a+1) \gamma_0 (u/t^a), \quad \gamma_1^a u = \Gamma(a+2) \gamma_1 (u/t^a) = \Gamma(a+2) \gamma_0 (\partial_t (u/t^a)), \quad \text{etc.}
\]

We first recall (and reprove) a result from [G15] for the case where the domain is \( \mathbb{R}^n_+ \):

**Lemma 2.2.** Let \( a > -1 \).

Let \( K_0 \) denote the Poisson operator from \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^n_+ \) with symbol \( (\langle \xi' \rangle + i \xi_n )^{-1} \).

When \( s > a+1/p \), the elements of \( H^a_p(\mathbb{R}^n_+) \) have a decomposition

\[
u(x) = v + w, \quad \text{where } w \in H^{a+1}(s)(\mathbb{R}^n_+), \quad \text{and}
\]

\[
u(x) = e^{t^a} \frac{1}{\Gamma(a+1)} x^a K_0 \gamma_0 u \in e^{t^a} x^a \mathcal{H}^{-a}_p(\mathbb{R}^n_+) \cap H^a_p(\mathbb{R}^n_+).
\]

In fact, the elements of \( H^a_p(\mathbb{R}^n_+) \) are parametrized as

\[
u(x) = e^{t^a} \frac{1}{\Gamma(a+1)} x^a K_0 \varphi + w,
\]

where \( \varphi \) runs through \( B^{s-a-1/p}_p(\mathbb{R}^{n-1}) \) and \( w \) runs through \( H^{a+1}(s)(\mathbb{R}^n_+) \); here \( \varphi \) equals \( \gamma_0^a u \).

**Proof.** In detail, \( K_0 \) is the elementary Poisson operator of order 0 in the Boutet de Monvel calculus (cf. e.g. [B71,G96,G09]):

\[
u(x) \mapsto \mathcal{F}_{\xi' \rightarrow x'}^{-1} (\hat{\varphi}(\xi')(\langle \xi' \rangle + i \xi_n)^{-1} = \mathcal{F}_{\xi' \rightarrow x'}^{-1} (\hat{\varphi}(\xi') r^+ e^{-x_n(\xi')}),
\]
which solves the Dirichlet problem \((1 - \Delta)u = 0\) on \(\mathbb{R}_+^n\), \(\gamma_0 u = \varphi\) on \(\mathbb{R}^{n-1}\). (An extension by 0 for \(x_n < 0\) is sometimes tacitly understood.) Define \(K_0^a\) by

\[
K_0^a \varphi = \Xi^a_+ e^+ K_0 \varphi = F_{\xi \to x}^{-1} (\hat{\varphi}(\xi')((\xi') + i \xi_n)^{-1-a}) = \frac{1}{i(a+1)} x_n^a e^+ K_0 \varphi;
\]

by the last expression, it is a right inverse of \(\gamma_0^a: u \mapsto \Gamma(a+1)\gamma_0(u/x_n^a)\). (These calculations played an important role in [G15], cf. (2.5) and the proofs of Cor. 5.3 and Th. 5.4 there; the constant called \(c_\mu\) in (5.16) is written explicitly here, equal to \(1/\Gamma(\mu + 1)\).

When \(u \in H_p^{a(s)}(\mathbb{R}_+^n)\), then \(\gamma_0^a u \in B_p^{s-a-1/p}(\mathbb{R}^{n-1})\) (cf. [G15] Th. 5.1), and \(w = u - K_0^a \gamma_0^a u\) has \(\gamma_0^a w = 0\). By the mapping properties of the Poisson operator \(K_0\) known from [G90], \(e^+ K_0 \gamma_0^a u\) lies in \(e^+ \overline{H}_p^{s-a}(\mathbb{R}_+^n)\). Then the last expression for \(K_0^a\) in (2.9) shows that \(K_0^a \gamma_0^a u\) lies in \(x_n^a e^+ \overline{H}_p^{s-a}(\mathbb{R}_+^n)\). Moreover, \(K_0^a \gamma_0^a u\) lies in \(H_p^{a(s)}(\mathbb{R}_+^n)\), since it is \(\Xi^a_+\) of a function in \(e^+ \overline{H}_p^{s-a}(\mathbb{R}_+^n)\). Then also \(w\) lies in \(H_p^{a(s)}(\mathbb{R}_+^n)\), so since \(\gamma_0^a w = 0\), it follows from (1.13) that \(w\) is in the subspace \(H_p^{(a+1)(s)}(\mathbb{R}_+^n)\).

All functions \(\varphi \in B_p^{s-a-1/p}(\mathbb{R}^{n-1})\) give rise to functions in \(H_p^{a(s)}(\mathbb{R}_+^n)\) by the mapping \(\Xi^a_+ e^+ K_0\), and \(H_p^{(a+1)(s)}(\mathbb{R}_+^n) \subset H_p^{a(s)}(\mathbb{R}_+^n)\), so all functions \(\varphi \in B_p^{s-a-1/p}(\mathbb{R}^{n-1})\) and \(w \in H_p^{(a+1)(s)}(\mathbb{R}_+^n)\) are reached in the decomposition. \(\Box\)

We shall now show a similar result for arbitrary smooth bounded sets \(\Omega\).

**Theorem 2.3.** Let \(\Omega \subset \mathbb{R}^n\), bounded, open with \(C^\infty\)-boundary, and let \(a > -1\).

There is a Poisson operator \(K_{\Omega(0)}\) of order 0 from \(\partial \Omega\) to \(\overline{\Omega}\) (in the Boutet de Monvel calculus) with principal symbol \(((\xi')^{a} + i \xi_n)^{-1}\) in local coordinates at the boundary, such that \(K_{\Omega(0)}\) is a right inverse of \(\gamma_0\), and the following holds:

When \(s > a + 1/p\), the elements of \(H_p^{a(s)}(\overline{\Omega})\) have a decomposition

\[
(10) \quad u = v + w, \text{ with } w \in H_p^{(a+1)(s)}(\overline{\Omega}) \text{ and } v = e^+ \frac{1}{i(a+1)} d^a K_{\Omega(0)} \gamma_0^a u \in e^+ d^a \overline{H}_p^{s-a}(\Omega) \cap H_p^{a(s)}(\overline{\Omega}).
\]

In fact, the elements of \(H_p^{a(s)}(\overline{\Omega})\) are parametrized as

\[
(11) \quad u = K_{\Omega(0)}^a \varphi + w, \quad K_{\Omega(0)}^a = \frac{1}{i(a+1)} d^a K_{\Omega(0)},
\]

where \(\varphi\) runs through \(B_p^{s-a-1/p}(\partial \Omega)\) and \(w\) runs through \(H_p^{(a+1)(s)}(\overline{\Omega})\). The latter space is equal to \(H_p^{a(s)}(\overline{\Omega})\) when \(s - a \in \mathbb{Z}\). When \(u\) is written in this way, \(\gamma_0^a u\) equals \(\varphi\).

Here \(K_{\Omega(0)}^a\) maps \(B_p^{s-a-1/p}(\partial \Omega)\) continuously into \(H_p^{a(s)}(\overline{\Omega})\) and satisfies

\[
(12) \quad \gamma_0^a K_{\Omega(0)}^a \varphi = \varphi, \text{ all } \varphi \in B_p^{s-a-1/p}(\partial \Omega).
\]

**Proof.** We use the local coordinates \(\kappa_i: U_i \to V_i, i = 0, 1, \ldots, I\), described in Remark 2.1, with an associated partition of unity \(\{\varrho_i\}_{i \leq I}\). We can moreover choose nonnegative
functions $\psi^k_i \in C^\infty(U_i)$, $k = 1, 2, 3$, such that $\psi^1_i \varrho_i = \varrho_i$, i.e., $\psi^1_i$ is 1 on supp $\varrho_i$, and similarly $\psi^2_i \varrho^1_i = \varrho^1_i$, $\psi^3_i \varrho^2_i = \varrho^2_i$.

Let $u \in H^{\alpha(s)}_p(\Omega)$, i.e., $u = \Lambda_+^{(-a)}$ for some $z \in e^+ H^{\alpha} \Omega$. Write

$$u = \Lambda_+^{(-a)} \sum_{i=0}^I \varrho_i z = \sum_{i} \psi^1_i \Lambda_+^{(-a)} \varrho_i z + \sum_{i} (1 - \psi^1_i) \Lambda_+^{(-a)} \varrho_i z. \tag{2.13}$$

Since $(1 - \psi^1_i) \varrho_i = 0$, $(1 - \psi^1_i) \Lambda_+^{(-a)} \varrho_i$ is a $\psi$do of order $-\infty$, so it maps $z$ into $C^\infty(\mathbb{R}^n)$; moreover, its symbol in local coordinates is holomorphic for $\operatorname{Im} \xi_n < 0$, so it preserves support in $\Omega$. Henceforth we can focus on the first sum

$$\sum_{i} u_i, \quad u_i = \psi^1_i \Lambda_+^{(-a)} \varrho_i z;$$

where $u_i$ is compactly supported in the set $U_i$ and belongs to $H^{\alpha(s)}_p(\Omega)$.

Consider one $u_i$. Here $u_i = u_i \circ \kappa_i^{-1}$ is in $H^{\alpha(s)}_p(\mathbb{R}^n_+)$ with support in $\operatorname{supp} \psi^1_i$.

By Lemma 2.2,

$$u_i = e^+ \frac{1}{1(a+1)} x_0^a K_0 \gamma^a_0 u_i + w_i, \quad w_i \in H^{(a+1)(s)}_p(\Omega),$$

where $\gamma^a_0 \frac{1}{1(a+1)} x_0^a K_0 = I$. Multiplication by $\psi^2_i$ or $\psi^3_i$ does not alter $w_i$; this gives the representation (where we denote $\gamma_0^a \psi^k_i = \psi^k_i$)

$$u_i = e^+ \frac{1}{1(a+1)} x_0^a K_0 \gamma^a_0 (\psi^2_i w_i) + \psi^3_i w_i = e^+ \frac{1}{1(a+1)} x_0^a \psi^3_i K_0 \gamma^a_0 u_i + \psi^3_i w_i.$$ 

Here $e^+ x_0^a K_0 \gamma^a_0 (\psi^2_i w_i) \in H^{\alpha(s)}_p(\mathbb{R}^n_+)$, since it is compactly supported and is the sum of $e^+ x_0^a K_0 \gamma^a_0 (\psi^2_i w_i) \in H^{\alpha(s)}_p(\mathbb{R}^n_+)$ and $(1 - \psi^3_i) e^+ x_0^a K_0 \gamma^a_0 (\psi^2_i w_i) \in E_a(\mathbb{R}^n_+)$, using that $(1 - \psi^3_i) K_0 \psi^2_i \gamma^a_0$ is a Poisson operator of order $-\infty$. Then also $\psi^3_i w_i$ is in $H^{\alpha(s)}_p(\mathbb{R}^n_+)$, and since its first boundary value $\gamma^a_0 (\psi^3_i w_i) = \psi^3_i \gamma^a_0 w_i$ vanishes, it is in fact in $H^{(a+1)(s)}_p(\mathbb{R}^n_+)$. Denote $(\psi^3_i w_i) \circ \kappa_i = \tilde{w}_i$, then we get the formula in the original coordinates:

$$u_i = e^+ \frac{1}{1(a+1)} d^a \tilde{K}_0 \gamma^a_0 u_i + \tilde{w}_i, \quad \text{where} \quad \tilde{K}_0 = (\psi^3_i K_0 \gamma^a_0) \circ \kappa_i,$$

the operator induced by $\psi^3_i K_0 \gamma^a_0$ in the original coordinates. Again $\gamma^a_0 \frac{1}{1(a+1)} d^a \tilde{K}_0 \gamma^a_0 u_i = \gamma^a_0 u_i$. Finally we find by summation over $i$ the formula

$$u = e^+ \frac{1}{1(a+1)} d^a K_0 \gamma^a_0 u + w = K_0^a \gamma^a_0 u + w,$$

with $K_0 = \sum_{i=0}^I \tilde{K}_0$, $K_0^a = e^+ \frac{1}{1(a+1)} d^a K_0$ and $w = \sum_{i} \tilde{w}_i$;

here

$$\gamma^a_0 K_0 = I \quad \text{and} \quad \gamma^a_0 K_0^a = I.$$ 

This ends the proof. □
2.2 Systems of traces.

For higher $s$, we also have representations in terms of consecutive sets of traces and Poisson operators.

Th. 5.4 of [G15] contains a formula (5.14) including higher-order traces of $u$. This formula lacks a sum of terms of the form $S_{jk} x^u_{\mu+k} e^+ K_0 (\gamma_j^\mu u)$ with $0 \leq k < j$, the $S_{jk}$ being constant-coefficient $\psi$do’s on $\mathbb{R}^{n-1}$ of order $j - k$. In fact, we had overlooked that $\gamma_j K_k = \delta_{jk}$ only holds for $j \leq k$ (cf. [G15] (1.7)), whereas it produces a nontrivial $\psi$do on $\mathbb{R}^{n-1}$ when $j > k$. This is a minor correction that does not change the outcome (5.15) of the theorem. The treatment of higher-order traces in the following theorems gives a more correct formula.

Theorem 2.4. Let $a > -1$ and let $M$ be a positive integer.

1° With $K_0$ defined in Theorem 2.3, denote

$$K(j) = \frac{1}{j!} d^j K(0),$$

then

$$\gamma_j K(k) = \delta_{jk} I \text{ for } j \leq k, \quad \gamma_j K(k) = \Psi_{jk} \text{ for } j > k,$$

where the $\Psi_{jk}$ are $\psi$do’s on $\partial \Omega$ of order $j - k$. With

$$\Theta_M = \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{M-1} \end{pmatrix}, \quad \mathcal{K}_M = (K(0) \ldots K_{(M-1)}),$$

the composition of $\Theta_M$ and $\mathcal{K}_M$ is a triangular invertible $M \times M$-matrix:

$$\Theta_M \mathcal{K}_M = \Psi_{(M)} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \Psi_{10} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{M-1,0} & \Psi_{M-1,1} & \ldots & 1 \end{pmatrix},$$

where $\Psi_{jk} = \delta_{jk} I$ for $j \leq k$.

Thus $\mathcal{K}_M \Psi_{(M)}^{-1}$ is a right inverse of $\Theta_M$. It maps $\prod_{0 \leq j < M} \mathcal{B}_p^{s-j-1/p} (\partial \Omega)$ into $\mathcal{H}_p^s(\overline{\Omega})$

for all $s \in \mathbb{R}$.

2° Define

$$K^a_{(j)} = \frac{1}{\Gamma(a+1+j)} d^a K(j), \quad j \in \mathbb{N}_0,$$

and, with $\gamma^a_j u = \Gamma(a + 1 + j) \gamma_j (u/d^a)$ (cf. (0.15)),

$$\Theta^a_M = \begin{pmatrix} \gamma^a_0 \\ \vdots \\ \gamma^a_{M-1} \end{pmatrix}, \quad \mathcal{K}^a_M = (K^a(0) \ldots K^a_{(M-1)}),$$
then

\begin{equation}
\phi_M^a \mathcal{K}_M^a = \Psi(M).
\end{equation}

Thus \( \mathcal{K}_M^a \Psi^{-1}(M) \) is a right inverse of \( \phi_M^a \).

Here \( K_{(j)}^a \) maps \( B_p^{s-a-j-1/p}(\partial \Omega) \) into \( H_p^{(a+j)(s)}(\Omega) \cap e^+ d^{a+j} H_p^{s-a-j}(\Omega) \) when \( s > a + j + 1/p \). Moreover, \( \mathcal{K}_M^a \Psi^{-1}(M) \) maps \( \prod_{0 \leq j < M} B_p^{s-a-j-1/p}(\partial \Omega) \) into \( H_p^{a(s)}(\Omega) \) when \( s > a + M - 1/p' \).

Proof. 1°. Since \( d \) identifies with \( t \) as in Remark 2.1 near \( \partial \Omega \), it is verified immediately that \( \gamma_j K(k) = \delta_{j,k} \) when \( j \leq k \). For \( j > k \) it is an elementary fact in the Boutet de Monvel calculus that the composition \( \gamma_j K(k) \) results in a \( \psi \)do on \( \partial \Omega \) of order \( j - k \). These facts lead to (2.17), where the triangular matrix is clearly invertible, continuous from \( \prod_{0 \leq j < M} B_p^{s-j-1/p}(\partial \Omega) \) to itself for all \( s \). The stated mapping property follows, since \( d^j K(0) \) is a Poisson operator of order \( -j \) in the Boutet de Monvel calculus.

For 2°, we note that by definition,

\[ \gamma_j^a K_{(k)}^a = \gamma_j K(k) \]

for all \( j, k \), so the formula (2.20) follows immediately from (2.17).

The mapping property of \( K_{(j)}^a \) follows from Theorem 2.3, since \( K_{(j)}^a \) is a constant times \( K_{(0)}^{a+j} \). In the collected statement on \( \mathcal{K}_M^a \Psi^{-1}(M) \), the case \( j = 0 \) gives the weakest range space. \( \square \)

Similar calculations hold with \( \Omega \) replaced by \( \mathbb{R}_+^n \), \( K_{(0)} \) replaced by \( K_0 \).

We can then generalize Theorem 2.3 to sets of traces as follows:

**Theorem 2.5.** With \( a > -1 \) and \( M \in \mathbb{N} \), let \( s > a + M - 1/p' \). Every \( u \in H_p^{a(s)}(\Omega) \) has a decomposition

\begin{equation}
\begin{aligned}
&u = v + w, \quad \text{with } w \in H_p^{(a+M)(s)}(\Omega) \quad \text{and} \\
v = \mathcal{K}_M^a \Psi^{-1}(M) \phi_M^a u \in e^+ d^a H_p^{s-a}(\Omega) \cap H_p^{a(s)}(\Omega).
\end{aligned}
\end{equation}

In fact, the elements of \( H_p^{a(s)}(\Omega) \) are parametrized as

\begin{equation}
\begin{aligned}
u = \mathcal{K}_M^a \Psi^{-1}(M) \varphi + w,
\end{aligned}
\end{equation}

where \( \varphi \) runs through \( \prod_{0 \leq j < M} B_p^{s-a-j-1/p}(\partial \Omega) \) and \( w \) runs through \( H_p^{(a+M)(s)}(\Omega) \). The latter space equals \( \tilde{H}_p^s(\Omega) \) if \( s - a \in ]M - 1/p', M + 1/p[ \). When \( u \) is written in this way, \( \phi_M^a u \) equals \( \varphi \).

Proof. For \( u \in H_p^{a(s)}(\Omega) \), set \( \varphi = \phi_M^a u \) and \( v = \mathcal{K}_M^a \Psi^{-1}(M) \varphi \). Then \( w = u - v \) belongs to \( H_p^{a(s)}(\Omega) \) and satisfies

\[ \phi_M^a (u - v) = \varphi - \varphi = 0; \]
hence lies in $H_p^{(a+M)(s)}(\Omega)$, in view of (1.13). This gives a representation of the elements of $H_p^{(s)}(\Omega)$ as desired. An application of $\varphi_M^a$ to (2.22) shows that $\varphi$ must necessarily equal $\varphi_M^a u$, since $K_{M,M}^a \Psi^{-1}(M)$ is a right inverse of $\varphi_M^a$. □

The progress in this theorem in comparison with Th. 5.4 of [G15] is that it gives a precise parametrization of the elements of the transmission space $H_p^{a(s)}(\Omega)$ in the case of arbitrary domains $\Omega$. In particular, it shows explicitly how the structure of the operators $K_{M,j}^a$ assures that they map into $H_p^{(a+j)(s)}(\Omega)$, not just into the general spaces $e^{+}d^{a+j}C_p^{a-j}(\Omega)$.

Systems of traces have recently been considered by Abatangelo, Jarohs and Saldana [AJS18] in the case of the fractional Laplacian on the ball, with explicit calculations.

Remark 2.6. When $P$ satisfies Hypothesis 0.1 with $a = b > 0$, then the space $H^{(b-1)(s)}(\Omega)$ is well-defined for $s > b-1/p'$. Here $H^{b(s)}(\Omega)$ is for $s > b-1/p'$ the subspace of elements $u \in H^{b-1}(\Omega)$ with $\gamma_0^{b-1} u = 0$, cf. (1.13). In fact, $\gamma_0^{b-1} u$ plays the role of the Dirichlet boundary value, and inhomogeneous Dirichlet problems can be considered on $H^{b-1}(\Omega)$ with good solvability properties, cf. [G15]ff. In this context, the next trace $\gamma_0^{b-1} u$ can be regarded as a Neumann boundary value; it also enters in nonhomogeneous boundary value problems ([G14,G16a,G18a]). It is easy to check that when $u \in H^{b(s)}(\Omega)$, then

$$\gamma_0^{b-1} u = \gamma_0^b u;$$

in other words, when $u$ satisfies the homogeneous Dirichlet condition, the lowest nontrivial boundary value $\gamma_0(u/d^b)$ is the Neumann value.

2.3 The result in H"older-Zygmund spaces.

As shown in [G14], the results carry over to many other interesting scales of function spaces. We shall here in particular consider the H"older-Zygmund scale, for which we get the following version of Theorem 2.5:

**Theorem 2.7.** Let $\Omega$ be a smooth, open bounded subset of $\mathbb{R}^n$ and let $a > 0$.

The operators $K_{M,j}^a$ defined in Theorem 2.3 map $C^{d-a-j}(\partial N) \to e^{+}d^{a+j}C^{d-a-j}(\Omega) \cap C^{(a+j)(t)}(\Omega)$, when $t > a + j$.

Let $M$ be a positive integer, and let $s > a + M - 1$. The elements of $C_p^{a(s)}(\Omega)$ have decompositions

$$u = K_{M,M}^a \Psi^{-1}(M) \varphi + w, \text{ with } \varphi \in \prod_{j < M} C^{d-a-j}(\partial N), \text{ w } \in C^{a+M}(\Omega),$$

described by operators as in Theorem 2.3; $\varphi = \varphi_M^a u$. Here, when $u$ runs through $C^{a(s)}(\Omega)$, $\varphi$ runs through $\prod_{j < M} C^{d-a-j-j}(\partial N)$, and $w$ runs through $C^{a+M}(\Omega)$. The latter space equals $C_n^*(\Omega)$ if $s-a \le M-1, M$.

Here $K_{M,M}^a \Psi^{-1}(M)$ maps $\prod_{0 \le j < M} C^{s-a-j}(\partial N)$ into $C^{a(s)}(\Omega)$ when $s > a + M - 1$.

**Proof.** We use that $C^* = B_{\infty, \infty}$ in the Besov scales $B_{p,q}^*$, where the $\psi$do’s and the boundary operators from the Boutet de Monvel calculus act similarly as in $H_s^*$, as shown in Johnsen [J96], the consequences for our calculations in the Besov scales being recalled in [G14].
Here \( p = \infty \), so \( p' = 1 \). The inequalities \( s > a + 1/p \) and \( s > a + M - 1/p' \) are here replaced by \( s > a \) and \( s > a + M - 1 \). □

We are of course primarily interested in the results for noninteger positive values of the exponents, where the spaces are ordinary Hölder spaces, \( C^s_\ast = C^s \) for \( s \in \mathbb{R}_+ \setminus \mathbb{N} \), but the \( C^s_\ast \) spaces are useful e.g. by having good interpolation properties — and of course by allowing statements without exceptional parameters. Recall moreover from [J96] and [G14] that the identification of spaces \( \dot{C}^s_\ast(\Omega) \) and \( e^+C^s_\ast(\Omega) \) takes place for \(-1 < s < 0\), another useful point.

3. The regularity of solutions of fractional Schrödinger Dirichlet problems

In preparation for the study of heat equation regularity, we shall consider a related problem for the Schrödinger equation, which is of interest in itself. Consider the Dirichlet problem for the fractional Schrödinger equation:

\[
(3.1) \quad r^+ Pu + Vu = f \text{ in } \Omega, \quad \text{supp} u \subset \overline{\Omega},
\]

where \( V \) is a \( C^\infty \)-function, and \( P \) satisfies Hypothesis 0.1 with \( 0 < a < 1 \).

Recall that for the usual Laplacian \( \Delta \), it makes no difference in the regularity of Dirichlet solutions whether \( a \in \mathbb{N} \), and operators \( P \) satisfying Hypothesis 0.1 with \( a / \in \mathbb{N} \), the regularity may be considerably restricted in comparison with the case \( V = 0 \). This is linked to the fact that the multiplication by a nonzero function \( V \) does not fit into the symbol sequence \( p \sim \sum_{j \in \mathbb{N}_0} p_j \), \( p_j(x, -t\xi) = t^{2a-j}(-1)^j p_j(x, \xi) \).

We first improve the regularity as far as we can by using the known regularity results for the Dirichlet problem

\[
(3.2) \quad r^+ Pu = g \text{ in } \Omega, \quad \text{supp} u \subset \overline{\Omega}.
\]

**Theorem 3.1.** Let \( P \) satisfy Hypothesis 0.1 for some \( a \in \mathbb{R}_+ \setminus \mathbb{N} \), and let \( V \in \overline{C}^\infty(\Omega) \). For a given \( f \in \overline{C}^\infty(\Omega) \), let \( u \in \mathcal{H}^a(\overline{\Omega}) \) satisfy (3.1). Then \( u \in C^a_\ast(\overline{\Omega}) \).

The conclusion also holds if merely \( f \in \overline{C}^a(\Omega) \). In fact, the solutions with \( f \in \overline{C}^a(\Omega) \) run through \( C^a_\ast(\overline{\Omega}) \), with \( \gamma_0 u \) running through \( C^{2a}_\ast(\partial \Omega) \).

**Proof.** Let \( V \in \overline{C}^\infty(\Omega) \) be given, and let \( f \in \overline{C}^\infty(\Omega) \). By variational theory, the operator \( r^+ P + V \) is Fredholm from \( \mathcal{H}^a(\overline{\Omega}) \) to \( \overline{H}^{-a}(\Omega) \), and from \( \{ u \in \mathcal{H}^a(\overline{\Omega}) \mid r^+ Pu \in L_2(\Omega) \} \) to \( L_2(\Omega) \).
Let \( u \in \dot{H}^a(\Omega) \) be a solution of (3.1) with \( f \in \mathcal{C}^\infty(\Omega) \). Using the regularity theory for (3.2), we shall improve the knowledge of the regularity of \( u \) in a finite number of iterative steps, as in a related situation in [G15a], pf. of Th. 2.3:

Recall the well-known general embedding properties for \( p, p_1 \in ]1, \infty[ \):

\[
\dot{H}^a_p(\Omega) \subset \epsilon^+ L_p^1(\Omega), \quad \text{when } \frac{n}{p_1} \geq \frac{n}{p} - a, \quad \dot{H}^a_p(\Omega) \subset \dot{C}^0(\Omega) \text{ when } a > \frac{n}{p}.
\]

We make a finite number of iterative steps to reach the information \( u \in C^0(\Omega) \), as follows: Define \( p_0, p_1, \ldots, \) with \( p_0 = 2 \) and \( q_j = \frac{n}{p_j} \) for all the relevant \( j \), such that

\[
q_j = q_{j-1} - a \quad \text{for } j \geq 1.
\]

This means that \( q_j = q_0 - ja \); we stop the sequence at \( j_0 \) the first time we reach a \( q_{j_0} \leq 0 \).

As a first step, we note that \( u \in \dot{H}^a(\Omega) \subset \epsilon^+ L_p^1(\Omega) \) implies that \( f - Vu \in L_p^1(\Omega) \), whence \( u \in H^{2a}(\Omega) \) by [G15] Th. 4.4 applied to \( r^+ Pu \in L_p^1(\Omega) \). Then \( u \in \dot{H}^a_p(\Omega) \) in view of (1.12). In the next step we use the embedding \( \dot{H}^a_p(\Omega) \subset \epsilon^+ L_p^2(\Omega) \) to conclude in a similar way that \( u \in \dot{H}^{2a}(\Omega) \), and so on. If \( j_0 < 0 \), we have that \( \frac{n}{p_{j_0}} < a \), so \( u \in \dot{H}^a_{j_0}(\Omega) \subset \dot{C}^0(\Omega) \). If \( j_0 = 0 \), the corresponding \( p_{j_0} \) would be \( +\infty \), and we see at least that \( u \in \epsilon^+ L_p^0(\Omega) \) for any large \( p \); then one step more gives that \( u \in \dot{C}^0(\Omega) \).

The rest of the argumentation relies on Hölder estimates, as [G14], Section 3. By the regularity results there,

\[
f - Vu \in \mathcal{C}^0(\Omega) \implies u \in C^{(2a)}_s(\Omega) \subset \epsilon^+ \dot{a}^C(\Omega) + \dot{C}^{2a-\epsilon}(\Omega) \subset \dot{C}^a(\Omega).
\]

Next, \( f - Vu \in \mathcal{C}^a(\Omega) \) implies \( u \in C^{(3a)}_s(\Omega) \).

Clearly, only the smoothness \( f \in \mathcal{C}^a(\Omega) \) is needed for the whole argumentation.

Conversely, if \( u \in C^{(3a)}_s(\Omega) \), then \( r^+ Pu \in \mathcal{C}^{2a}(\Omega) \subset \mathcal{C}^a(\Omega) \), and since \( u \in \dot{C}^{3a-\epsilon}(\Omega) + \epsilon^+ \dot{a}^C(\Omega) \subset \epsilon^+ \dot{C}^a(\Omega) \), also \( Vu \in \dot{C}^a(\Omega) \); here \( \gamma^a_0 u \) runs through \( C^{2a}_s(\partial \Omega) \) (cf. Theorem 2.7).

Since \( u \in \dot{C}^\infty(\Omega) \) implies \( r^+ Pu \in \mathcal{C}^\infty(\Omega) \), very high regularity of solutions to (3.1) is not completely excluded. But we shall now show that when \( V \) is nonvanishing on parts of the boundary, a higher regularity can only hold if \( \gamma^a_0 u \) vanishes there.

We show this for \( 0 < a < 1 \), leaving cases of higher \( a \) to the reader.

**Theorem 3.2.** Let \( P \) be as in Theorem 3.1 with \( 0 < a < 1 \), and let \( V \in \mathcal{C}^\infty(\Omega) \). For a given \( f \in \mathcal{C}^a(\Omega) \), let \( u \in C^{(3a)}_s(\Omega) \) be a solution of (3.1).

1° Assume that \( 1/V \in \mathcal{C}^\infty(\Omega) \). If \( u \in C^{(3a+\delta)}_s(\Omega) \) and \( f \in \mathcal{C}^{a+\delta}(\Omega) \) for some \( \delta > 0 \), then \( \gamma^a_0 u = 0 \).

2° Let \( V \neq 0 \) on an open subset \( \Sigma \) of the boundary \( \partial \Omega \). If \( u \in C^{(3a+\delta)}_s(\Omega) \) and \( f \in \mathcal{C}^{a+\delta}(\Omega) \) for some \( \delta > 0 \), then \( \gamma^a_0 u = 0 \) on \( \Sigma \).

**Proof.** 1°. First consider the case where \( 0 < a < \frac{1}{2} \). Then \( 2a < 1 \), and we can take \( \delta \in ]0, 1 - 2a[ \), so that also \( 2a + \delta < 1 \). Assume that \( u \in C^{(3a+\delta)}_s(\Omega) \) and \( f \in \mathcal{C}^{a+\delta}(\Omega) \).

By (2.24) with \( M = 1, s = 3a + \delta \),

\[
u = K^a(\delta)\gamma^a_0 u + w_0, \quad w_0 \in C^{(a+1)(3a+\delta)}_s(\Omega) = \dot{C}^{3a+\delta}(\Omega),
\]
where we used that $3a + \delta - a - 1 < 0$; here $K^a_{(0)}\gamma_0^a u = d^a z$ with $z = \frac{1}{1(a+1)}e^+K^a_{(0)}\gamma_0^a u \in e^+C^{2a+\delta}(\Omega)$. Since $\dot{C}^{3a+\delta}_{(0)}(\Omega) \subset d^a\dot{C}^{2a+\delta}_{(0)}(\Omega)$, we have that

$$u/d^a = z + w_0', \quad w_0 = w_0/d^a \in \dot{C}^{2a+\delta}_{(0)}(\Omega),$$

and hence, in local coordinates at the boundary, where $d$ is replaced by $x_n$,

$$u(x', x_n) = x_n^a(z(x', 0) + O(x_n^{2a+\delta})) = x_n^a(z(x', 0) + O(x_n^{3a+\delta})) \text{ for small } x_n > 0. \tag{3.4}$$

On the other hand, $u = V^{-1}(f - r^+Pu) \in C^{\alpha+\delta}(\Omega)$ and therefore has an expansion in local coordinates

$$u(x', x_n) = u(x', 0) + O(x_n^{\alpha+\delta}) \text{ for small } x_n > 0. \tag{3.5}$$

Comparing the two expansions, we first conclude that $u(x', 0) = 0$, and next, that $z(x', 0) = 0$. This shows that $\gamma_0^a u$ must be 0.

Now consider the case $a \in [\frac{3}{2}, 1]$. Here $2a \in [1, 2]$, and we consider $\delta > 0$ satisfying $\delta < 1 - a$; then also $\delta < 2 - 2a$, so that $2a + \delta \in ]1, 2[$ and $a + \delta < 1$. Assume that $u \in C^{(a+2)(3a+\delta)}(\Omega)$ and $f \in C^{\alpha+\delta}(\Omega)$. By (2.28) with $M = 2$, $s = 3a + \delta$,

$$u = K^a_{(0)}\gamma_0^a u + K^a_{(1)}(\gamma_1^a u - \Psi_{10}\gamma_0^a u) + w_1, \text{ with } w_1 \in C^{(a+2)(3a+\delta)}(\Omega) = \dot{C}^{3a+\delta}_{(0)}(\Omega),$$

where we used that $3a + \delta - a - 2 < 0$. Here

$$K^a_{(0)}\gamma_0^a u + K^a_{(1)}(\gamma_1^a u - \Psi_{10}\gamma_0^a u) = d^a z + d^{a+1}z_1,$$

with $z = \frac{1}{1(a+1)}e^+K^a_{(0)}\gamma_0^a u \in e^+C^{2a+\delta}(\Omega)$ and $z_1 = \frac{1}{1(a+1)}e^+K^a_{(1)}(\gamma_1^a u - \Psi_{10}\gamma_0^a u) \in e^+d\dot{C}^{2a+\delta-1}(\Omega)$. Then

$$u/d^a = z + dz_1 + w_1', \quad w_1' = w_1/d^a \in \dot{C}^{2a+\delta}_{(0)}(\Omega),$$

and hence, in local coordinates where $d$ is replaced by $x_n$,

$$u(x', x_n) = x_n^a(z(x', 0) + O(x_n^{a+1}) + O(x_n^{3a+\delta}) \text{ for small } x_n > 0. \tag{3.6}$$

By comparison with the expansion (3.5) we can again first conclude that $u(x', 0) = 0$, and next that $z(x', 0) = 0$, so we find again that $\gamma_0^a u$ must be 0. This shows 1°.

For $2^\circ$, we just carry the above argumentation through in coordinate patches intersecting the boundary in open subsets $\Sigma'$ of $\Sigma$ with $\overline{\Sigma'} \subset \Sigma$. \hfill $\square$

As a corollary we find for the resolvent equation:

**Corollary 3.3.** Let $P$ satisfy Hypothesis 0.1 with $0 < a < 1$, and let $\lambda \neq 0$. For a given $f \in C^{\alpha}(\Omega)$, let $u \in C^{a(3a)}(\Omega)$ be a solution of

$$r^+Pu - \lambda u = f \text{ in } \Omega, \quad \text{supp } u \subset \overline{\Omega}. \tag{3.7}$$

If $u \in C^{a(3a+\delta)}(\Omega)$ and $f \in C^{\alpha+\delta}(\Omega)$ for some $\delta > 0$, then $\gamma_0^a u = 0$.

**Proof.** This is the special case of Theorem 3.2 1° where $V = -\lambda$. \hfill $\square$

Recall from Remark 2.6 that for $u \in C^{a(s)}(\Omega)$, $\gamma_0^a u$ can be regarded as the Neumann boundary value.
Remark 3.4. What is it that happens when \( s \) in the parameter \( 2a+s \) passes from \( a \) to \( a+\delta \), \( \delta > 0 \)? Recall that we are dealing with the operator \( r^+ P \) going from \( E_1^s = C_*(2a+s)(\Omega) \) to \( E_0^s = C_*(\Omega) \), for \( s \geq 0 \). Here we always have that (with \( \varepsilon \) active if \( 2a+s \in \mathbb{N} \))

\[
E_1^s \subset C_*(2a+s(-\varepsilon)(\Omega) + e^+ \partial^a C_*(\Omega) \subset d^a[\dot{C}^a(\Omega) + e^+ C^a(\Omega)] \subset \dot{C}^a(\Omega).
\]

But note also that (cf. (1.18))

\[
E_1^s = C_*^{(2a+s)}(\Omega) \supset E_0(\Omega) = e^+ d^a C^\infty(\Omega),
\]

and for \( \delta > 0 \) there are elements of \( e^+ d^a C^\infty(\Omega) \) not lying in \( e^+ C^{a+\delta}(\Omega) \). So for \( s = a + \delta \), \( E_1^s \) contains nontrivial elements of \( \dot{C}^a(\Omega) \) \( \setminus e^+ C^{a+\delta}(\Omega) \). Briefly expressed,

\[
\begin{align*}
& (3.8) \quad E_1^s \subset E_0^s \text{ when } s \leq a, \\
& (3.9) \quad E_1^s \not\subset E_0^s \text{ when } s > a.
\end{align*}
\]

The inclusion \( E_1^s \subset E_0^s \) is necessary in order to define a resolvent acting in \( E_0^s \); this is not possible for \( s > a \).

Remark 3.5. When the Neumann value vanishes, we can get meaningful statements on smaller spaces. E.g., for \( \delta = 1 \), \( u \in C_*(3a+1)(\Omega) \) with \( \gamma_0^a u = 0 \) means that \( u \in C_*(a+1)(3a+1)(\Omega) \), in view of (1.18). This space is contained in \( e^+ C^{a+1}(\Omega) \). When \( u \in C_*(a+1)(3a+1)(\Omega) \), then \( r^+ Pu \in C^{a+1}(\Omega) \), and the equation (3.7) can be fulfilled with \( f \in C^{a+1}(\Omega) \), without contradiction.

4. The regularity of solutions of fractional heat Dirichlet problems

Corollary 3.3 can now be applied in a discussion of the regularity of solutions of the fractional heat equation.

Theorem 4.1. Let \( P \) satisfy Hypothesis 0.1 with \( 0 < a < 1 \). When \( u \in \overline{W^{1,1}}(\mathbb{R}; C_*(3a)(\Omega)) \), it satisfies

\[
(4.1) \quad r^+ Pu(x,t) + \partial_t u(x,t) = f(x,t) \text{ on } \Omega \times \mathbb{R},
\]

\[
 u(x,t) = 0 \text{ for } x \notin \Omega.
\]

with \( f(x,t) \in L^1(\mathbb{R}; C^a(\Omega)) \); here \( \gamma_0^a u \in \overline{W^{1,1}}(\mathbb{R}; C^2(\partial \Omega)) \) can take any value.

However, if for some \( \delta > 0 \), \( u(x,t) \in \overline{W^{1,1}}(\mathbb{R}; C_*(3a+\delta)(\Omega)) \) and \( f(x,t) \in L^1(\mathbb{R}; C^{a+\delta}(\Omega)) \), then \( \gamma_0^a u = 0 \).

Proof. For the first statement, we note that \( \partial_t u \in L^1(\mathbb{R}; C_*(3a)(\Omega)) \subset L^1(\mathbb{R}; \dot{C}^a(\Omega)) \), so \( r^+ Pu + \partial_t u \in L^1(\mathbb{R}; \dot{C}^a(\Omega)) \) as asserted.

For the second statement, we need only consider a small \( \delta > 0 \) with \( a + \delta < 1 \). The functions are sufficiently regular to allow Fourier transformation with respect to \( t \), leading to the equation (where we denote \( \mathcal{F}_{t \rightarrow \tau} g(x,t) = \hat{g}(x,\tau) \))

\[
(4.2) \quad r^+ P\hat{u}(x,\tau) + i\tau \hat{u}(x,\tau) = \hat{f}(x,\tau) \text{ on } \Omega \times \mathbb{R}.
\]
By assumption,

\[(4.3) \quad u \in W^{1,1}(\mathbb{R}; C^{a(3a+\delta)}_*(\Omega)), f \in L_1(\mathbb{R}; C^{a+\delta}_*(\Omega)), \gamma_0^a u \in W^{1,1}(\mathbb{R}; C^{2a+\delta}_*(\partial\Omega)), \]

so since the Fourier transform maps $L_1(\mathbb{R}; X)$ into $C^0(\mathbb{R}; X)$, the ingredients in (4.2) are in spaces:

\[(4.4) \quad r^+ Pu \in C^1(\mathbb{R}; C^{a+\delta}_* (\Omega)), \tau \hat{u}(x, \tau) \in C^0(\mathbb{R}; C^{a(3a+\delta)}_*(\Omega)), \hat{f} \in C^0(\mathbb{R}; C^{a+\delta}_*(\Omega)), \]

and the equation holds pointwise in $\tau$ (this use of Fourier transformation of functions valued in Banach spaces is justified by the analysis in Amann [A97]).

At each $\tau \neq 0$ we can use Corollary 3.3 to see that if $\delta > 0$, then $\hat{u}(x, \tau)$ cannot be in $C^{a(3a+\delta)}_*$ in $x$ unless $\gamma_0^a \hat{u} = 0$ for that value of $\tau$.

Observing this at all $\tau \neq 0$, we see in view of the continuity in $\tau$ that if $\delta > 0$, $\gamma_0^a \hat{u}(x, \tau)$ vanishes. By the injectiveness of the Fourier transform, also $\gamma_0^a u(x, t)$ vanishes. \(\Box\)

Note in particular that if $f$ is $C^\infty$ in all variables, $u$ cannot be better than $C^{a(3a)}_*$ in the $x$-variable without the vanishing of the Neumann value $\gamma_0^a u$.

As a corollary, we have a similar result for $t$ in a finite interval:

**Corollary 4.2.** With $I = ]0, T[$, let $u \in W^{1,1}(\mathbb{R}; C^{a(3a)}_*(\Omega))$ with $u(x, 0) = 0$; then it satisfies

\[(4.5) \quad r^+ Pu(x, t) + \partial_t u(x, t) = f(x, t) \text{ on } \Omega \times I, \]

\[\begin{align*}
u(x, t) &= 0 \text{ for } x \notin \Omega, \\
u(x, 0) &= 0, \end{align*}\]

with $f(x, t) \in L_1(I; C^0_*(\Omega))$ (here $\gamma_0^a u \in W^{1,1}(I; C^{2a}_*(\partial\Omega))$ can be freely prescribed for positive $t$).

However, if for some $\delta > 0$, $u(x, t) \in W^{1,1}(I; C^{a(3a+\delta)}_*(\Omega))$ and $f(x, t) \in L_1(I; C^{a+\delta}_*(\Omega))$, then $\gamma_0^a u = 0$.

**Proof.** First extend $u$ and $f$ by $0$ for $t < 0$, and next extend the resulting functions across $t = T$ by reflection in $t$. This results in functions $\tilde{u}$ resp. $\tilde{f}$ that satisfy the hypotheses of Theorem 4.1, so the conclusions from that theorem carry over. \(\Box\)

It should be noted that the regularity theorem of Ros-Oton and Vivas [RV18] shows that for solutions of (4.5),

\[(4.6) \quad f \text{ is } C^a \text{ in } x \text{ and } C^{\frac{1}{2}} \text{ in } t \implies u/d^a \text{ is } C^{2a} \text{ in } x \text{ and } C^1 \text{ in } t,\]

if $a \neq \frac{1}{2}$ (with a slightly weaker statement for $a = \frac{1}{2}$, cf. (0.7)); this is consistent with the first assertion in the corollary. But an extension of the upper indices from $a$ to $a + \delta$, resp. $2a$ to $2a + \delta$, may possibly need restrictive hypotheses as in the second assertion.

**Remark 4.3.** Under the hypothesis $\gamma_0^a u = 0$, the heat problem can be studied in smaller spaces for more regular $f$ without contradiction, cf. Remark 3.5. Then it is no longer a Dirichlet problem, but a problem with more boundary conditions.
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