Edge Burst in Non-Hermitian Quantum Walk

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We uncover a novel non-Hermitian phenomenon, dubbed edge burst, in non-Hermitian quantum dynamics. Specifically, in a typical class of non-Hermitian quantum walk, an exceptionally large portion of loss occurs at the system boundary. The physical origin of this edge burst is found to be an interplay between two unique non-Hermitian phenomena, non-Hermitian skin effect and imaginary gap closing. Furthermore, we establish a universal bulk-edge scaling relation underlying the edge burst.

Standard quantum mechanics postulates Hermiticity of Hamiltonian, yet non-Hermitian Hamiltonians are useful in many branches of physics. For example, open systems with gain and loss naturally exhibit non-Hermitian physics [1]. Recently, there have been growing interests in non-Hermitian topological physics. In particular, the bulk-boundary correspondence principle is drastically reshaped by the non-Hermitian skin effect (NHSE), namely the boundary localization of bulk-band eigenstates [2–10]. It indicates that the boundary plays an even more profound role in non-Hermitian systems compared to their Hermitian counterparts.

In this paper, we unveil a new boundary-induced dynamical phenomenon, dubbed “edge burst”, in a typical class of non-Hermitian systems. For concreteness, we consider quantum-mechanical time evolution of particles (called “quantum walkers”) in a lossy lattice. Intuitively, a walker starting from a certain site far from the boundary is expected to escape predominantly from nearby sites. However, an unexpected high peak in the loss probability is found at the edge, whose relative height grows with the distance from the initial site to the edge. Furthermore, we find that this edge burst exhibits a unique scaling behavior, originating from a universal bulk-edge scaling relation. This provides an underlying theory that not only tells the precise conditions for edge burst, but also have implications beyond.

We note that the appearance of an edge peak has been reported in a very recent work, though it was incorrectly attributed to topological edge states [11].

Edge burst. For concreteness, we consider a one-dimensional lossy lattice shown in Fig. 1(a). During the quantum walk, the walker can escape from B sites. The Schrödinger equation $i \frac{d}{dt} \psi(t) = H(t) \psi(t)$ reads

$$
\frac{d\psi_x}{dt} = i t_1 \psi_x + i \frac{t_2}{2} (\psi_{x-1} - \psi_{x+1}) + \frac{t_2}{2} (\psi_{x-1}^B + \psi_{x+1}^B),
$$

with loss rate $\gamma > 0$. The corresponding Bloch Hamiltonian is

$$
H(k) = (t_1 + t_2 \cos k) \sigma_x + (t_2 \sin k + i \frac{\gamma}{2}) \sigma_y - i \frac{\gamma}{2} I,
$$

where $\sigma_{x,y,z}$ are Pauli matrices and $I$ is the identity matrix. This model is similar to that of Ref. [12], except that it is purely lossy. It is equivalent to the non-Hermitian SSH model under a basis change[2], and therefore it also features the NHSE, which distinguishes it from earlier quantum-walk models without NHSE [13].

The wavefunction norm decreases as $\frac{d}{dt} \langle \psi(t) | \psi(t) \rangle = i \langle \psi(t) | (H^\dagger - H) | \psi(t) \rangle = -\sum_x 2\gamma |\psi_x^B(t)|^2$, and the probability that the walker escapes from location $x$ is

$$
P_x = 2\gamma \int_0^\infty dt |\psi_x^B(t)|^2.
$$

FIG. 1. (a) The model. Each unit cell, labeled by spatial coordinate $x$, contains two sites $A$ and $B$. (b) The spatially resolved loss probability $P_x$, shown for a walker initiated at $x_0 = 50$. The chain length $L = 60$, $t_1 = 0.4$. (c) The relative height $P_{\text{edge}}/P_{\text{min}}$ with varying $t_1$, for $x_0 = 50$ and 25. Here, $P_{\text{edge}} \equiv P_{x=1}$ and $P_{\text{min}} \equiv \min \{P_1, P_2, \ldots, P_6\}$. (d) Relative height with $x_0$ varying from 40 to 140, for $t_1 = 0.40$ (black triangle) and $t_1 = 0.63$ (green square) (marked in (c)).. $L = 150$. Throughout (b)(c)(d), $t_2 = 0.5, \gamma = 0.8$ are fixed.

Note that $\sum_x P_x = 1$ is satisfied under the initial-state normalization $\langle \psi(0) | \psi(0) \rangle = 1$. Let us consider a walker starting from $x = x_0$, with $\psi_x^A = \delta_{x,x_0}$ and $\psi_x^B = 0$. It
appears natural to expect that \( P_z \) would decay away from \( x_0 \), which is largely confirmed by numerical simulation using Runge-Kutta method [Fig. 1(b)]. We also notice that the \( P_z \) distribution is left-right asymmetric. The preference of walking left can be attributed to the NHSE, all eigenstates being localized at the left edge [2].

The most intriguing feature is the high peak at the left edge, namely the edge burst, which stands out from the almost invisible decaying tail [Fig. 1(b)]. It appears natural to attribute it to topological edge state, which turns out to be incorrect. To quantify the edge burst, we calculate the relative height, defined as \( P_{\text{edge}}/P_{\text{min}} \), where \( P_{\text{edge}} = P_1 \), and \( P_{\text{min}} \equiv \min\{P_1, P_2, \ldots, P_{x_0}\} \) is the minimum of \( P \) between the starting point and the edge. The existence and absence of edge burst manifests in \( P_{\text{edge}}/P_{\text{min}} \gg 1 \) and \( P_{\text{edge}}/P_{\text{min}} \sim 1 \), respectively. We see in Fig. 1(c) that the relative height increases with \( x_0 \) for \( t_1 \in (0, t_2) \) (approximately), and rapidly decreases to order of unity otherwise, with \( t_2 = 0.5 \). In Fig. 1(d), we plot the relative height for \( t_1 = 0.40 \) and 0.63, both within the topologically nontrivial regimes (i.e. there are topological edge modes [2]), yet the edge burst exists only for \( t_1 = 0.40 \). Thus, the edge burst does not stem from topological edge states. We also note that NHSE is present for all \( t_1 \neq 0 \), and therefore Fig. 1(c)(d) tells us that NHSE by itself does not guarantee edge burst.

To unveil the origin of edge burst, we plot both \( P_{\text{edge}} \) and bulk \( P_x \) in Fig. 2. Fig. 2 (a,b) indicate that \( P_{\text{edge}} \) follows a power law for \( |t_1| \leq 0.5 \),

\[
P_{\text{edge}} \sim |x_0|^{-\alpha_\epsilon},
\]

and an exponential law \( P_{\text{edge}} \sim (\lambda_\epsilon)^{x_0} \) for \( |t_1| > |t_2| \). Fig. 2 (c,d) indicate similar behaviors in the bulk,

\[
P_x \sim |x - x_0|^{-\alpha_b},
\]

for \( |t_1| \leq |t_2| \), and exponential law \( P_x \sim (\lambda_b)^{x_0-x} \) (\( \lambda_b < 1 \)) for \( |t_1| > |t_2| \). Note that Eq. (5) is valid only for \( x \) in the bulk, i.e. not too close to the edge; also note that \( \alpha_b \neq \alpha_\epsilon \). The algebraic (i.e. power-law) behavior of bulk \( P_x \) reflects the algebraic decay of wavefunction norm in the time domain, which originates from the Bloch energy spectrum touching the real axis, i.e. closing the imaginary gap [Fig. 2(e)]. In other words, algebraic decay corresponds to \( \max[\text{Im}(E(k))] = 0 \), with \( E \) denoting the eigen-spectrums of \( H \). It can be readily checked that the imaginary gap closes for \( |t_1| \leq |t_2| \) [14].

In the language of open quantum system, the algebraic behavior means that the dissipative gap (or Liouvillian gap) closes [15]. In fact, our non-Hermitian \( H \) in Eq. (1) can be reformulated in terms of the quantum master equation, \( \dot{\rho}/\Delta t = -i[H, \rho] + \sum_{ij} (L_{x_i} \rho L_{x_j}^\dagger - \frac{1}{2} L_{x_i}^\dagger L_{x_j} \rho) \), where \( H = \sum_{ij} c_i^\dagger h_{ij} c_j \), with \( h \) denoting the Hermitian part of \( H \) in Fig. 1(a), namely, \( h_{ij} = H_{ij}(\gamma = 0) \), and the dissipator \( L_x = \sqrt{2 \gamma} x \). Note that \( c_i \) can be either bosonic or fermionic, which does not affect the single-particle dynamics. The effective non-Hermitian Hamiltonian \( H_{\text{eff}} = H - \sum_{x_i} b_{x_i}^\dagger b_{x_i} = H - \gamma \sum_{x_i} c_{x_i}^\dagger c_{x_i} \) corresponds to closing the dissipative (imaginary) gap.

Given the imaginary gap closing, namely \( |t_1| \leq |t_2| \), we always see the edge burst except at \( t_1 = 0 \). The \( t_1 = 0 \) point is special in two aspects. First, NHSE is absent at this parameter value. Second, the periodic-boundary-condition (PBC) energy spectrum encloses zero area in complex plane [green triangle in Fig. 2(e)]. These two features are concurrent. In fact, a precise correspondence has been established between the existence (absence) of NHSE and the complex energy enclosing nonzero (zero) area [16, 17]. The zero and nonzero enclosed area is also known as having trivial and nontrivial point-gap topol-
ogy, respectively [18–20].

Summarizing the above numerical findings, we infer that the edge burst stems from the interplay between two prominent non-Hermitian phenomena, NHSE and imaginary gap closing. The latter is a non-Hermitian counterpart of being gapless in Hermitian systems. This imaginary gaplessness and NHSE jointly induce the edge burst.

**Bulk-edge scaling relation.**—The exponent \( \alpha_b \) in Eq. (4) and \( \alpha_e \) in Eq. (5) characterize the bulk and edge dynamics, respectively. One of our central results is the scaling relation

\[
\alpha_e = \alpha_b - 1
\]

(6)
in the presence of NHSE and imaginary gap closing. For our specific model, it holds true when \( |t_1| \leq |t_2| \) (such that imaginary gap closes) and \( t_1 \neq 0 \) (such that NHSE is present). At the NHSE-free point \( t_1 = 0 \), we have \( \alpha_0 - \alpha_e = 0 \) instead. Remarkably, although both \( \alpha_b \) and \( \alpha_e \) are model/parameter dependent, the relation Eq. (6) remains universal.

Before calculating \( \alpha_b, \alpha_e \) and proving Eq. (6), we observe that this equation implies edge burst. In fact, Eq. (5) implies that \( P_x \) takes the minimum near (but not too close to) the edge, and \( P_{\text{min}} \sim x_0^{-\alpha_b} \). Therefore, it follows from Eq. (6) that

\[
P_{\text{edge}}/P_{\text{min}} \sim x_0^{\alpha_b - \alpha_e} \sim x_0.
\]

(7)
Thus, as the starting point \( x_0 \) moves away from the edge, the relative height of edge peak increases. This is precisely the origin of edge burst illustrated in Fig. 1(b).

Now we calculate \( \alpha_b, \alpha_e \) and derive Eq. (6) using Green’s function, which has been a useful tool in non-Hermitian systems [21–26]. It is convenient to work in the frequency (energy) domain, defining

\[
|\psi(\omega)| = \int_{-\infty}^{+\infty} dt e^{i\omega t}|\psi(t)| = iG(\omega)|\psi(t = 0)|,
\]

(8)
where \( |\psi(t)| = \Theta(t)e^{-iHt}|\psi(t = 0)| \), \( \Theta(t) \) standing for the Heaviside step function, and the frequency-domain Green’s function \( G(\omega) \) reads

\[
G(\omega) = \frac{1}{\omega + i0^+ - H}.
\]

(9)
The infinitesimal imaginary part i0+ is included to ensure the convergence of integral at \( t = +\infty \). Now we can rewrite \( P_x \) of Eq. (3) as

\[
P_x = \frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d\omega \langle x, B|G(\omega)|x_0, A \rangle^2.
\]

(10)
where the initial wavefunction \( |\psi(t = 0)| = |x_0, A \rangle \) has been used. To calculate \( \alpha_b \), it is more convenient to consider an infinite chain. The relevant Green’s function reads

\[
\langle x, B|G(\omega)|x_0, A \rangle = \int_0^{2\pi} \frac{dk}{2\pi} e^{i(k-x_0)} \left( \frac{1}{\omega + i0^+ - H(k)} \right)_{BA} = \phi |_{|\beta| = \frac{1}{2}} \frac{d\beta}{2\pi i\beta} \delta_{x,x_0} \left( \frac{1}{\omega + i0^+ - H(\beta)} \right)_{BA}.
\]

(11)
where \( H(\beta) \) is the analytic continuation of \( H(k) \) in Eq. (2). \( H(\beta) \equiv H(k)_{|k| = \beta} \). For our specific model, \( H(\beta) = (t_1 + t_2 \frac{i0^+ + 1}{2})/\det[\omega + i0^+ - H(\beta)] \). This integration can be done by the residue theorem, and the asymptotic behavior at \( |x - x_0| \to \infty \) is determined by the roots of \( \det[\omega + i0^+ - H(\beta)] = 0 \) [22]. As a quadratic equation, it has two roots that we order as \( |\beta_L(\omega)| \geq |\beta_R(\omega)| \). Following Ref. [22], we have \( \langle x, B|G(\omega)|x_0, A \rangle \sim f_{L,R} \beta_{L,R}^{-x_0} \) for \( x < x_0 \), and \( \langle x, B|G(\omega)|x_0, A \rangle \sim f_{L,R} \beta_{L,R}^{-x_0} \) for \( x > x_0 \) \((|\beta_L(\omega)| \geq 1 \) is satisfied for real-valued \( \omega \), where \( f_{L,R} \) are \( x \)-independent and their precise values do not concern us [14]. Accordingly, \( P_x^\infty \), in which the superscript \( \infty \) stands for the infinite chain, is given by

\[
P_x^\infty = \frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d\omega |f_{L,R}(\omega)|^2 |\beta_{L,R}(\omega)|^{2|x-x_0|},
\]

(12)
where the subscript \( L \) and \( R \) corresponds to \( x < x_0 \) and \( x > x_0 \), respectively. For \( |x-x_0| \) large, the integral of Eq. (12) is dominated by \( |\beta| \) closest to 1. In fact, the existence (absence) of a real \( \omega \) satisfying \( |\beta(\omega)| = 1 \) determines the algebraic (exponential) behavior of \( P_x \). To satisfy \( |\beta(\omega)| = 1 \) for real-valued \( \omega \) is to close the imaginary gap of Bloch Hamiltonian, because the gap-closing point \( \omega_0 \) satisfies \( \det[\omega_0 - H(\beta)] = 0 \) with \( |\beta| = 1 \). These points are marked as \( A_1, A_2 \) and \( B \) in Fig. 2(e).

As we focus on \( x < x_0 \), the relevant root is \( \beta_L(\omega) \). Let us write \( \omega = \omega_0 + \delta \omega \), and then expand \( \beta_L(\omega) \), \( f_L(\omega) \) to the lowest order of \( \delta \omega \), so that \( |\beta_L(\omega)| \approx 1 + K \delta \omega \approx \exp(K \delta \omega) \), and \( |f(\omega)|^2 \approx \delta \omega^m \). Now \( P_x^\infty \sim \int d(\delta \omega) \delta \omega^m \exp(-2K \delta \omega^2|x-x_0|) \sim |x-x_0|^{-m+1}/n \), and therefore \( \alpha_b = (m+1)/n \). In contrast, when the imaginary gap opens, we have \( |\beta_L(\omega)| > 1 \) and exponential decay \( P_x^\infty \sim [\max(1/|\beta_L(\omega)|)]^{-2|x-x_0|} \). For our model, the imaginary gap closing regime is \( |t_1| \leq |t_2| \), in which the bulk \( P_x \) indeed exhibits algebraic behavior. Furthermore, taking \( t_2 > 0 \), we have \( (n,m) = (1/2,0), (2,2), \) and \( (4,4) \) for \( t_1 = 0, t_1 \in (0,t_2), t_1 = t_2, \) respectively, which leads to the bulk exponent [14]

\[
\alpha_b = \frac{m+1}{n} = 2, 3, 5, 7, 4,
\]

(13)
for these three cases.

Now let us consider a chain with open boundary condition (OBC) at \( x = 1 \) and \( L \) [Fig. 1(b)]. The NHSE of our model localizes all eigenstates exponentially to the edge. This effect can be precisely characterized by
the generalized Brillouin zone (GBZ), which is the trajectory of $\beta$ associated with OBC eigenstates [2, 27–31]. In our model, the GBZ is a circle with radius $|\beta| = \sqrt{|(t_1 - \gamma/2)/(t_1 + \gamma/2)|} < 1$ for $t_1 > 0$, indicating NHSE with skin modes localized at the left edge [2]. The NHSE induces leftward walking, and the walker becomes trapped at the left edge once it arrives there. We compare the $P_x$ of the (effectively) infinite chain and finite chain [Fig. 3(a)], which indicates that $P_x^\infty$ is almost the same as OBC $P_x$ for $x$ not too close to the edge. In view of the probability sum $\sum_x P_x = 1$ in both cases, we conclude that the missing part, namely the edge accumulation in the OBC case and $\sum_{x=-\infty}^0 P_x^\infty$ in the infinite-chain case, must be equal. This observation leads to the estimation

$$P_{\text{edge}} \sim \sum_{x=-\infty}^0 P_x^\infty \sim \int_{-\infty}^0 |x-x_0|^{-\alpha_b} dx$$

$$\sim \int_{x_0}^{\infty} x^{-\alpha_b} dx \sim (x_0)^{-\alpha_b+1}. \quad (14)$$

Therefore, we see that $\alpha_b$ in Eq. (4) equals $\alpha_b - 1$. As explained by Eq. (7), this “−1” in exponent means a dramatic enhancement of $P_{\text{edge}}$ compared to the decay tail of $P_x$, generating the edge burst. In contrast, when the imaginary gap is nonzero, we have $P_{\text{edge}} \sim \int_{-\infty}^{\infty} (\lambda_b)^{x_0-x} dx \sim \int_{x_0}^{\infty} (\lambda_b)^x dx \sim (\lambda_b)^{x_0}$, which is of the same order as the decay tail (taking $x = 0$ in $P_x \sim \lambda_b^{x_0-x}$), and therefore no edge burst exists.

Since $\alpha_b$ is a bulk-band quantity, Eq. (14) and Eq. (6) unambiguously tells that the edge burst is a bulk-band phenomenon independent of edge details. The bulk-band nature can also be seen in the long-time behavior of wavefunction. In fact, we can write $H = \sum_n E_n |n_R\rangle \langle n_L|$ in terms of the right and left eigenstates $|n_R\rangle$ and $|n_L\rangle$, then $\langle x, B|\psi(t)\rangle = \sum_n e^{-iE_n t} \langle x, B|n_R\rangle \langle n_L|x_0, A\rangle$. It follows that $\max|\text{Im}(E_n)|$ dominates the long-time behavior, and $|\langle x, B|\psi(t)\rangle| \sim e^{-\max|\text{Im}(E_n)|} t$ for $t \to \infty$. Under OBC, the bulk band consists of skin modes localized at the edge, and $E_n$ should be calculated from GBZ [2, 27]. According to Longhi [28], $\max|\text{Im}(E)|$ of bulk band occurs at a saddle point $\beta_s$ on GBZ, satisfying $(\partial E/\partial \beta)|_{\beta=\beta_s} = 0$. We numerically calculate the time dependence of edge-site wavefunction, which indeed follows an exponential law with exponent close to $-\max|\text{Im}(E)|$ [insets of Fig. 3(a)], confirming the bulk-band nature of edge burst. To further back up our results, we change the sign of $t_1$ so that the skin modes and edge burst are seen at the right edge; the results again support our picture [Fig. 3(b)].

**Conclusions.** We unveil a boundary-induced non-Hermitian dynamical phenomenon, dubbed the edge burst, which is an unexpected interplay between imaginary (dissipative) gap and NHSE. Its origin is identified as a universal bulk-edge scaling relation [Eq. (6)]. Our theory can be readily confirmed in various non-Hermitian platforms including, for example, the photon quantum walk in which NHSE has been realized and the dissipative gap can be conveniently tuned [8, 32].

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[1] Yuto Ashida, Zongping Gong, and Masahito Ueda, “Non-hermitian physics,” Advances in Physics 69, 249–435 (2020).
[2] Shunyu Yao and Zhong Wang, “Edge states and topological invariants of non-hermitian systems,” Phys. Rev. Lett. 121, 086803 (2018).
[3] Shunyu Yao, Fei Song, and Zhong Wang, “Non-hermitian chern bands,” Phys. Rev. Lett. 121, 136802 (2018).
[4] Flore K. Kunst, Elisabet Edvardsson, Jan Carl Budich, and Emil J. Bergholtz, “Biorthogonal bulk-boundary correspondence in non-hermitian systems,” Phys. Rev. Lett. 121, 026808 (2018).
[5] Ching Hua Lee and Ronny Thomale, “Anatomy of skin modes and topology in non-hermitian systems,” Phys. Rev. B 99, 201103 (2019).
[6] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, “Non-hermitian robust edge states in...
one dimension: Anomalous localization and eigenspace condensation at exceptional points,” Phys. Rev. B 97, 121401 (2018).

[7] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, “Generalized bulk–boundary correspondence in non-hermitian topological circuits,” Nature Physics 16, 747 (2020).

[8] Lei Xiao, Tianshu Deng, Kunkun Wang, Gaoyan Zhu, Zhong Wang, Wei Yi, and Peng Xue, “Non-Hermitian bulk-boundary correspondence in quantum dynamics,” Nature Physics 16, 761 (2020). [1907.12566 [cond-mat.mes-hall]].

[9] Ananya Ghatak, Martin Brandenbourger, Jasper van Wezel, and Corentin Coulais, “Observation of non-hermitian topology and its bulk–edge correspondence in an active mechanical metamaterial,” Proceedings of the National Academy of Sciences 117, 29561–29568 (2020).

[10] Emil J. Bergholtz, Jan Carl Budich, and Flore K. Kunst, “Exceptional topology of non-hermitian systems,” Rev. Mod. Phys. 93, 015005 (2021).

[11] Li Wang, Qing Liu, and Yunbo Zhang, “Quantum dynamics on a lossy non-hermitian lattice,” Chinese Physics B 30, 020506 (2021).

[12] Tony E. Lee, “Anomalous edge state in a non-hermitian lattice,” Phys. Rev. Lett. 116, 133903 (2016).

[13] M. S. Rudner and L. S. Levitov, “Topological transition in a non-hermitian quantum walk,” Phys. Rev. Lett. 102, 065703 (2009).

[14] See Supplemental Material.

[15] Zi Cai and Thomas Barthel, “Algebraic versus exponential decoherence in dissipative many-particle systems,” Phys. Rev. Lett. 111, 150403 (2013).

[16] Kai Zhang, Zhesen Yang, and Chen Fang, “Correspondence between winding numbers and skin modes in non-hermitian systems,” Phys. Rev. Lett. 125, 126402 (2020).

[17] Nobuyuki Okuma, Kohei Kawabata, Ken Shiozaki, and Masatoshi Sato, “Topological origin of non-hermitian skin effects,” Phys. Rev. Lett. 124, 086801 (2020).

[18] Kohei Kawabata, Ken Shiozaki, Masahito Ueda, and Masatoshi Sato, “Symmetry and topology in non-hermitian physics,” Phys. Rev. X 9, 041015 (2019).

[19] Zongping Gong, Yuto Ashida, Kohei Kawabata, Kazuki Takasan, Sho Hiigashikawa, and Masahito Ueda, “Topological phases of non-hermitian systems,” Phys. Rev. X 8, 031079 (2018).

[20] Huitao Shen, Bo Zhen, and Liang Fu, “Topological band theory for non-hermitian Hamiltonians,” Phys. Rev. Lett. 120, 146402 (2018).

[21] A. McDonald, T. Pereg-Barnea, and A. A. Clerk, “Phase-dependent chiral transport and effective non-hermitian dynamics in a bosonic kitaev-majorana chain,” Phys. Rev. X 8, 041031 (2018).

[22] Wen-Tan Xue, Ming-Rui Li, Yu-Min Hu, Fei Song, and Zhong Wang, “Simple formulas of directional amplification from non-bloch band theory,” Phys. Rev. B 103, L241408 (2021).

[23] Clara C Wanjura, Matteo Brunelli, and Andreas Nunnenkamp, “Topological framework for directional amplification in driven-dissipative cavity arrays,” Nature communications 11, 3149 (2020).

[24] Johan Carlström, “Correlations in non-hermitian systems and diagram techniques for the steady state,” Physical Review Research 2, 013078 (2020).

[25] Heinrich-Gregor Zirnstein, Gil Refael, and Bernd Rosenow, “Bulk-boundary correspondence for non-hermitian hamiltonians via green functions,” Phys. Rev. Lett. 126, 216407 (2021).

[26] Dan S. Borgnia, Alex Jura Kruchkov, and Robert-Jan Slager, “Non-hermitian boundary modes and topology,” Phys. Rev. Lett. 124, 056802 (2020).

[27] Kazuki Yokomizo and Shuichi Murakami, “Non-bloch band theory of non-hermitian systems,” Phys. Rev. Lett. 123, 066404 (2019).

[28] Stefano Longhi, “Probing non-hermitian skin effect and non-bloch phase transitions,” Phys. Rev. Research 1, 023013 (2019).

[29] Zhesen Yang, Kai Zhang, Chen Fang, and Jianguiping Hu, “Non-hermitian bulk-boundary correspondence and auxiliary generalized brillouin zone theory,” Phys. Rev. Lett. 125, 226402 (2020).

[30] Tian-Shu Deng and Wei Yi, “Non-bloch topological invariants in a non-hermitian domain wall system,” Phys. Rev. B 100, 035102 (2019).

[31] Kohei Kawabata, Nobuyuki Okuma, and Masatoshi Sato, “Non-bloch band theory of non-hermitian hamiltonians in the symplectic class,” Phys. Rev. B 101, 195147 (2020).

[32] Lei Xiao, Tianshu Deng, Kunkun Wang, Zhong Wang, Wei Yi, and Peng Xue, “Observation of non-bloch parity-time symmetry and exceptional points,” Phys. Rev. Lett. 126, 230402 (2021).
I. IMAGINARY GAP CLOSING CONDITIONS

We have stated in the main article that the imaginary gap closes for $|t_1| \leq |t_2|$. This can be checked by straightforward calculations. A less computational approach is as follows.

The model considered in our main article is reproduced as follows:

$$H(k) = (t_1 + t_2 \cos k)\sigma_x + (t_2 \sin k + i\frac{\gamma}{2})\sigma_z - i\frac{\gamma}{2}I.$$  \hfill (1)

For later use, we also consider a different model

$$H(k) = (t_1 + t_2 \cos k)\sigma_x + [t_3 \cos(k - \alpha) + i\frac{\gamma}{2}]\sigma_z - i\frac{\gamma}{2}I.$$  \hfill (2)

We can express the Hamiltonian in terms of their eigenvalues and bi-orthogonal eigenstates as

$$H(k) = \sum_n E_n(k)|u_{nR}(k)\rangle\langle u_{nL}(k)|.$$  \hfill (3)

Here, $\text{Im}E(k) \leq 0$ is satisfied because of the lossy nature of our models, and $\text{Im}E(k) = 0$ is the imaginary gap closing point. We start from the obvious expression

$$E_n(k) = \frac{\langle u_{nR}(k)|H(k)|u_{nR}(k)\rangle}{\langle u_{nR}(k)|u_{nR}(k)\rangle}.$$  \hfill (4)

Taking the imaginary part of this equation, we have

$$\text{Im}E(k) = \frac{\langle u_{nR}(k)|D(k)|u_{nR}(k)\rangle}{\langle u_{nR}(k)|u_{nR}(k)\rangle},$$  \hfill (5)

where the Hermitian matrix $D(k) = [H(k) - H^\dagger(k)]/2i$ is the anti-Hermitian part of $H$ with the “i” factor removed. It can be diagonalized as

$$D(k) = \sum_i \lambda_i(k)|d_i(k)\rangle\langle d_i(k)|,$$  \hfill (6)

with real-valued eigenvalues. Inserting this formula into the expression of $\text{Im}E(k)$, we see

$$\text{Im}E(k) = \frac{\sum_i \lambda_i(k)|\langle d_i(k)|u_{nR}(k)\rangle|^2}{\langle u_{nR}(k)|u_{nR}(k)\rangle}.$$  \hfill (7)

For the two models above, we observe that they both share a $k$-independent $D(k) = \frac{\gamma}{2}I$. Its eigenvalues and corresponding eigenstates are

$$\lambda_1(k) = 0, \quad |d_1(k)\rangle = (1, 0)^T;$$
$$\lambda_2(k) = -\gamma, \quad |d_2(k)\rangle = (0, 1)^T.$$  \hfill (8)

Eq. (7) tells us that $\text{Im}E(k) = 0$ requires $|\langle d_2(k)|u_{nR}(k)\rangle| = 0$, i.e. the right eigenstate at the imaginary gap closing point should be orthogonal to $|d_2(k)\rangle = (0, 1)^T$. Thus, $|u_{nR}(k)\rangle$ is parallel to $|d_1\rangle$ or, equivalently, $|d_1\rangle$ is a right eigenstate of $H(k)$ at the imaginary gap closing point. This is possible only when the coefficient of $\sigma_z$ in...
\[ H(k) \text{ vanishes. For our models, the } \sigma_x \text{ coefficient is } t_1 + t_2 \cos k, \text{ and therefore the imaginary gap closes at } k = k_0 \text{ determined by} \]
\[ \cos k_0 = -t_1/t_2. \]

Since \(|\cos(k)| \leq 1\) for real-valued \(k\), we obtain the imaginary gap closing condition \(|t_1| \leq |t_2|\). The energies of the imaginary gap closing points, where the \(E(k)\) curve touches the real axis, are given by

\[
\text{Model I : } \omega_0^+ = \pm \sqrt{t_2^2 - t_1^2}, \\
\text{Model II : } \omega_0^+ = \frac{t_3}{t_2} \left(-t_1 \cos \alpha \pm \sqrt{t_2^2 - t_1^2 \sin \alpha}\right). \tag{10}
\]

II. EXPLICIT CALCULATION OF THE BULK DECAY EXPONENT

In the main article, we have used the fact that the decay of bulk \(P_x\) follows an algebraic law \(P_x \sim |x - x_0|^{-\alpha_b}\) when the imaginary gap closes. Here, we provide an explicit calculation of the exponent \(\alpha_b\). To this end, we consider a long one-dimensional chain so that the boundary effect is negligible, and denote \(P_x\) by \(P_x^\infty\), meaning that the chain is effectively infinite. We start from the formula

\[
P_x^\infty = \frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d\omega |\langle x, B|G(\omega)|x_0, A \rangle|^2, \tag{11}
\]

with

\[
|\langle x, B|G(\omega)|x_0, A \rangle| = \int_0^{2\pi} \frac{dk}{2\pi} e^{ik(x-x_0)} \left( \frac{1}{\omega + i0^+ - H(k)} \right)_{BA} = \oint_{|\beta| = 1} \frac{d\beta}{2\pi i \beta} \beta^{x-x_0} \left( \frac{1}{\omega + i0^+ - H(\beta)} \right)_{BA}, \tag{12}
\]

where \(H(\beta) \equiv H(k)|_{e^{ik} \rightarrow \beta} [1, 2]\). For our specific model in the main article,

\[
\left( \frac{1}{\omega + i0^+ - H(\beta)} \right)_{BA} = \frac{t_1 + t_2 \frac{\beta^{x-x_0}}{2}}{\det[\omega + i0^+ - H(\beta)]}. \tag{13}
\]

The integral in Eq. (12) can be done by the residue theorem, and the asymptotic behavior at \(|x - x_0| \rightarrow \infty\) is determined by the roots of \(\det[\omega + i0^+ - H(\beta)] = 0 [2]\). As a quadratic equation, it has two roots \(\beta_L(\omega)\) and \(\beta_R(\omega)\), which satisfy \(|\beta_L(\omega)| \geq 1 \geq |\beta_R(\omega)|\) for real-valued \(\omega\). Following Ref. [2], we have \(|\langle x, B|G(\omega)|x_0, A \rangle| \sim f_{L/R}^{\beta^{x-x_0}}\) for \(x < x_0\), and \(|\langle x, B|G(\omega)|x_0, A \rangle| \sim f_{R}^{\beta^{x-x_0}}\) for \(x > x_0\), where the residue factors \(f_{L/R}\) are \(x\)-independent and their values are

\[
f_{L/R}(\omega) = \lim_{\beta \to \beta_{L/R}} (\beta - \beta_{L/R}) \frac{t_1 + t_2 \frac{\beta^{x-x_0}}{2}}{\beta \det[\omega + i0^+ - H(\beta)]}. \tag{14}
\]

Thus, Eq. (11) becomes

\[
P_x^\infty = \frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d\omega |f_{L/R}(\omega)|^2 |\beta_{L/R}(\omega)|^{2(x-x_0)}, \tag{15}
\]

where the subscript \(L\) and \(R\) corresponds to \(x < x_0\) and \(x > x_0\), respectively. As has been explained in the main article, the integral is dominated at large \(|x - x_0|\) by the neighborhood of imaginary gap closing point \(\omega_0\), where \(|\beta_{L/R}(\omega_0)| = 1\). To find the asymptotic behavior of the integral, we need the expansions of \(\beta_{L/R}(\omega)\) and \(f_{L/R}(\omega)\) near the imaginary gap closing point.

Hereafter, we shall focus on the \(x < x_0\) region and therefore only \(\beta_L(\omega)\) is relevant. Writing \(\omega = \omega_0 + \delta \omega\), and expanding \(f_L(\omega), \beta_L(\omega)\) to the lowest order of \(\delta \omega\), we have

\[
|f_L(\omega)|^2 \sim \delta \omega^n, \tag{16}
\]

and \(|\beta_L(\omega)| \approx 1 + K\delta \omega^n \approx \exp(K\delta \omega^n)\) or, equivalently

\[
\ln |\beta_L(\omega)| \approx K\delta \omega^n. \tag{17}
\]
It follows that

\[ P^\infty_x \sim \int d(\delta \omega) \delta \omega^m \exp(-2K \delta \omega^n |x - x_0|) \sim |x - x_0|^{-(m+1)/n}, \tag{18} \]

and therefore \( \alpha_b = (m+1)/n \). The rest part of this section is to calculate \( n \) and \( m \).

Let us calculate \( n \) first. Note that the \( i0^+ \) term is irrelevant in most cases and we shall discard it for the moment, so that the determinant equation \( \det [\omega + i0^+ - H(\beta)] = 0 \) becomes \( \det [\omega - H(\beta)] = 0 \). For model I, namely the Eq. (1)(2) of the main article, the equation \( \det [\omega - H(\beta)] = 0 \) can be explicitly written as

\[ t_2(t_1 + \frac{\gamma}{2}) \beta + t_2(t_1 - \frac{\gamma}{2}) \beta^{-1} + t_1^2 + t_2^2 - \omega^2 - i\gamma \omega = 0. \tag{19} \]

Its two roots are

\[ \beta_\pm(\omega) = \frac{-b(\omega) \pm \sqrt{b^2(\omega) - 4t_2^2(t_1^2 - \gamma^2/4)}}{2t_2(t_1 + \gamma/2)} \tag{20} \]

where \( b(\omega) = t_1^2 + t_2^2 - \omega^2 - i\gamma \omega \). At the imaginary gap closing point of the Bloch Hamiltonian, \( \omega_0^+ = \pm \sqrt{t_2^2 - t_1^2} \), there exists at least one root whose modulus is 1. To avoid excessive signs, we shall only focus on the positive frequency \( \omega_1^+ = \sqrt{t_2^2 - t_1^2} \) (We focus on \( t_2 \geq t_1 \geq 0 \) throughout this section). Under the ordering \( |\beta_L(\omega)| \geq 1 \geq |\beta_R(\omega)| \), the two roots read

\[ \beta_L(\omega_0^+) = \beta_+(\omega_0^+) = -t_1 \frac{t_1 - \gamma/2}{t_1 + \gamma/2}, \quad \beta_R(\omega_0^+) = \beta_-(\omega_0^+) = \frac{t_1 - \gamma/2}{t_1 + \gamma/2} \left( \frac{t_1 - \gamma/2}{t_1 + \gamma/2} \right). \tag{21} \]

We can rewrite it as

\[ \beta_L(\omega_0^+) = e^{i k_0}, \quad \beta_R(\omega_0^+) = \frac{t_1 - \gamma/2}{t_1 + \gamma/2} e^{-i k_0} \tag{22} \]

where \( k_0 = \arccos(-t_1/t_2) \). As the frequency slightly shifts away from the imaginary gap closing point \( \omega_0^+ \), namely \( \omega = \omega_0^+ + \delta \omega \), \( \beta_L(\omega_0^+ + \delta \omega) \) will also shift from \( e^{i k_0} \) to \( e^{i(k_0 + \delta k + i \delta k')} \), \( \delta k \) and \( \delta k' \) being functions of \( \delta \omega \). The leading order expansion of \( i(\delta k + i \delta k') = \ln[\beta_L(\omega)/\beta_R(\omega_0)] \) is

\[ i(\delta k + i \delta k') = \left( \frac{\partial \ln \beta_L}{\partial \omega} \right)_{\omega = \omega_0^+} \delta \omega + \left( \frac{\partial^2 \ln \beta_L}{\partial \omega^2} \right)_{\omega = \omega_0^+} \frac{\delta \omega^2}{2} + \cdots. \tag{23} \]

The real part of left hand side (LHS) is \( -\delta k' = \ln |\beta_L| \approx K \delta \omega^n \), and that of the right hand side (RHS) should tell us the value of \( n \) of Eq. (17). In fact, the lowest order nonzero real-valued coefficient of RHS should be identified as the \( K \delta \omega^n \) term. By explicit calculations, we obtain the derivatives

\[ \left( \frac{\partial \ln \beta_L}{\partial \omega} \right)_{\omega = \omega_0^+} = \frac{1}{\beta_L} \frac{\partial \beta_L}{\partial \omega} = \frac{i}{t_1}, \quad \left( \frac{\partial^2 \ln \beta_L}{\partial \omega^2} \right)_{\omega = \omega_0^+} = \frac{1}{\beta_L^2} \frac{\partial^2 \beta_L}{\partial \omega^2} - \frac{1}{\beta_L} \frac{\partial \beta_L}{\partial \omega} = -\frac{\gamma \sqrt{t_2^2 - t_1^2}}{t_1^3 (\omega_0^+ + i \gamma/2)} \right. \tag{24} \]

When \( t_1 = 0 \), the derivatives diverge and the expansion fails; let us focus on the \( t_1 \in (0, t_2] \) region for now and come back to the special case \( t_1 = 0 \) shortly. If \( t_1 \in (0, t_2) \), the leading order real-valued term at the RHS of Eq. (23) is the \( \delta \omega^2 \) term; therefore, we have \( -\delta k' = \ln |\beta_L| \propto \delta \omega^2 \), meaning that \( n = 2 \). On the other hand, if \( t_1 = t_2 \), the second-order term at RHS vanishes and higher order expansions is necessary. The lowest order contribution occurs at the fourth order of \( \delta \omega \):

\[ t_1 = t_2 : \left( \frac{\partial \ln \beta_L}{\partial \omega} \right)_{\omega = \omega_0^+} = \frac{i}{t_1}, \quad \left( \frac{\partial^2 \ln \beta_L}{\partial \omega^2} \right)_{\omega = \omega_0^+} = 0, \quad \left( \frac{\partial^3 \ln \beta_L}{\partial \omega^3} \right)_{\omega = \omega_0^+} = \frac{i}{t_1^3}, \quad \left( \frac{\partial^4 \ln \beta_L}{\partial \omega^4} \right)_{\omega = \omega_0^+} = -\frac{6}{t_1^4}, \tag{25} \]

and we have \( n = 4 \).

Now we turn to the calculation of the residue factor \( f_L(\omega) \), which reads

\[ f_L(\omega) = \lim_{\beta \to \beta_L} \frac{(\beta - \beta_L)}{\beta \det [\omega - H(\beta)]} \left( \frac{t_1 + t_2 \beta + \beta^{-1}}{2} \right) \tag{26} \]
The determinant in the denominator, \( \det[\omega - H(\beta)] \), can be expressed in terms of its two roots:

\[
\det[\omega - H(\beta)] = t_2(t_1 + \gamma/2)(\beta - \beta_L)(\beta - \beta_R)/\beta.
\]  

(27)

Thus, in terms of the roots, \( f_L(\omega) \) reads

\[
f_L(\omega) = \frac{t_1 + t_2 \beta_L + \beta_R^{-1}}{t_2(t_1 + \gamma/2)(\beta_L - \beta_R)}.
\]  

(28)

Making use of the results from the previous section, we have \( f_L(\omega) = 0 \) at the imaginary gap closing point, which follows from

\[
t_1 + t_2 \frac{\beta_L(\omega_0^+ + \delta\omega) + \beta_R^{-1}(\omega_0^-)}{2} = t_1 + t_2 \cos k_0 = 0.
\]  

(29)

Near the gap closing point, the leading order expansion of \( f_L(\omega) \) can be obtained by using the aforementioned expressions of \( \beta_L \) and \( \beta_R \). We have

\[
f_L(\omega_0^+ + \delta\omega) \approx -\frac{\sin k_0}{2\gamma \cos k_0 + 4it_1 \sin k_0} (\delta k + i\delta k')
\]  

(30)

when \( t_1 \neq t_2 \) and

\[
f_L(\omega_0^+ + \delta\omega) \approx -\frac{t_1}{4\gamma} (\delta k + i\delta k')^2
\]  

(31)

when \( t_1 = t_2 \) and \( k_0 = \pi \). It follows that \( |f_L(\omega_0^+ + \delta\omega)|^2 \propto |\delta k + i\delta k'|^2 \propto \delta\omega^2 \) when \( 0 < t_1 < t_2 \), and \( |f_L(\omega_0^+ + \delta\omega)|^2 \propto |\delta k + i\delta k'|^4 \propto \delta\omega^4 \) when \( t_1 = t_2 \). In other words, we have \( m = 2 \) for \( t_1 \in (0, t_2) \), and \( m = 4 \) for \( t_1 = t_2 \).

Now we come back to the special case \( t_1 = 0 \), for which the Taylor expansion fails due to divergent derivatives. Therefore, we follow a different approach. We insert the frequency \( \omega = \omega_0^+ + \delta\omega = t_2 + \delta\omega \) into the expression of \( \beta_\pm \) [Eq. (20)], which yields

\[
\beta_\pm(t_2 + \delta\omega) \approx i \pm 2 \sqrt{\frac{i(t_2 + i\gamma/2)\delta\omega}{\gamma t_2}},
\]  

(32)

and consequently,

\[
\ln |\beta_\pm(t_2 + \delta\omega)| \propto \sqrt{\delta\omega}.
\]  

(33)

This gives a fractional value \( n = 1/2 \) for \( t_1 = 0 \). Regarding \( f_L(\omega) \), we observe that \( \beta_+(\omega_0^+) = \beta_-(\omega_0^+) \) or, equivalently, \( \beta_L(\omega_0^+) = \beta_R(\omega_0^+) \) (recall that \( \beta_{L/R} \) are just \( \beta_\pm \) under the ordering \( |\beta_L| \geq |\beta_R| \)), at the imaginary gap closing point, so that Eq. (28) becomes ill-defined. To cope with this issue, we have to restore the infinitesimal \( i0^+ \) term in the determinant equation \( \det[\omega + i0^+ - H(\beta)] = 0 \), namely,

\[
f_L(\omega) = \lim_{\epsilon \to 0^+} \frac{\beta - \beta_L}{\beta - \beta_R} \frac{t_1 + t_2 \beta + \beta_R^{-1}}{2} \frac{\gamma}{\beta \det[\omega + i\epsilon - H(\beta)]}
\]  

\[
= \lim_{\epsilon \to 0^+} \beta_L(t_2 + i\epsilon) + \beta_R^{-1}(t_2 + i\epsilon) \frac{1}{\gamma}
\]  

(34)

where \( \beta_{L/R}(t_2 + i\epsilon) = i \left( 1 \pm 2\sqrt{t_2 + i\gamma/2}\epsilon/i\gamma \right) \). In the second line, we observe that the numerator \( \beta_L(t_2 + i\epsilon) + \beta_R^{-1}(t_2 + i\epsilon) \propto \sqrt{\epsilon} \) and the denominator \( \beta_L(t_2 + i\epsilon) - \beta_R(t_2 + i\epsilon) \propto \sqrt{\epsilon} \) in the same way, which causes their ratio

\[
\frac{\beta_L(t_2 + i\epsilon) + \beta_R^{-1}(t_2 + i\epsilon)}{\beta_L(t_2 + i\epsilon) - \beta_R(t_2 + i\epsilon)} \to 1 \quad \text{as} \quad \epsilon \to 0.
\]

The nonzero value of \( f_L \) at the imaginary gap closing point means that \( m = 0 \) for \( t_1 = 0 \).

The main results of this section can be summarized as

\[
t_1 = 0 : \quad \ln |\beta_L(\omega_0 + \delta\omega)| \propto \sqrt{\delta\omega}, \quad |f_L(\omega)|^2 \propto \delta\omega^0, \quad (n, m) = (\frac{1}{2}, 0);
\]

\[
t_1 \in (0, t_2) : \quad \ln |\beta_L(\omega_0 + \delta\omega)| \propto \delta\omega^2, \quad |f_L(\omega)|^2 \propto \delta\omega^2, \quad (n, m) = (2, 2);
\]

\[
t_1 = t_2 : \quad \ln |\beta_L(\omega_0 + \delta\omega)| \propto \delta\omega^4, \quad |f_L(\omega)|^2 \propto \delta\omega^4, \quad (n, m) = (4, 4).
\]

(35)
where $n, m$ stand for the exponents in the expansions $|\beta_L(\omega)| \approx 1 + K\delta \omega^n$, and $|f(\omega)|^2 \sim \delta \omega^m$. These analytic results are also confirmed numerically; see Fig. 1 of this Supplemental Material.

Inserting these expressions back into Eq. (15), we find that (for $x < x_0$) the large $|x - x_0|$ behavior is $P_x^\infty \sim \int d\delta \omega \delta \omega^m \exp(-2K\delta \omega^m(x - x_0)) \sim |x - x_0|^{-(m+1)/n}$ when $t_1 \leq t_2$. In other words, we have $P_x^\infty \sim |x - x_0|^{-\alpha_b}$ with exponent $\alpha_b = (m + 1)/n$ whose explicit value is

$$
t_1 = 0: \quad \alpha_b = 2; 
$$

$$
t_1 \in (0, t_2): \quad \alpha_b = \frac{3}{2}; 
$$

$$
t_1 = t_2: \quad \alpha_b = \frac{5}{4}.
$$

III. GREEN’S FUNCTION FORMULAS SATISFY THE SUM RULE

We denote the onsite loss probability by $\gamma_j$ for each site $j = (x, A/B)$; for our specific model, $\gamma_{x,A} = 0$ and $\gamma_{x,B} = \gamma > 0$. The wavefunction norm evolves as

$$
\frac{d}{dt} \langle \psi(t)|\psi(t)\rangle = -2 \sum_j \gamma_j \langle j|\psi(t)\rangle^2 < 0.
$$

Integration of both the left hand side (LHS) and right hand side (RHS) of this equation from $t = 0$ to $t = \infty$ leads to

$$
\langle \psi(0)|\psi(0)\rangle - \langle \psi(\infty)|\psi(\infty)\rangle = \sum_j P_j,
$$

where $P_j = 2\gamma_j \int_0^\infty dt \langle j|\psi(t)\rangle^2$ is the local loss. Under the standard normalization $\langle \psi(0)|\psi(0)\rangle = 1$ and $\langle \psi(\infty)|\psi(\infty)\rangle = 0$, Eq. (38) indeed gives $\sum_j P_j = 1$.

We would like to do a consistency check that the sum rule $\sum_j P_j = 1$ is satisfied in our Green’s function approach, in which

$$
P_j = \frac{\gamma_j}{\pi} \int_{-\infty}^{+\infty} d\omega \langle j|G(\omega)|\psi(0)\rangle^2 = \frac{\gamma_j}{\pi} \int_{-\infty}^{+\infty} d\omega (j) \frac{1}{\omega + i0^+ - H} |\psi(0)|^2.
$$
Now the sum is
\[ \sum_j P_j = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega \sum_j \gamma_j \langle \psi_0 | G^\dagger(\omega) | j \rangle \langle j | G(\omega) | \psi(0) \rangle \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \omega \langle \psi(0) | G^\dagger(\omega) (H^\dagger - H) G(\omega) | \psi(0) \rangle \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \omega \langle \psi(0) | G^\dagger(\omega) ((\omega - H) - (\omega - H^\dagger)) G(\omega) | \psi(0) \rangle \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \omega \langle \psi(0) | (G^\dagger(\omega) - G(\omega)) | \psi(0) \rangle \]
\[ = \sum_n \langle \psi(0) | nR \rangle \langle nR | \psi(0) \rangle \]
\[ = 1, \]
where we have used \( \sum_j \gamma_j | j \rangle \langle j | = (H^\dagger - H)/2i \), and the spectral representation \( G(\omega) = \sum_n | nR \rangle \langle nL | / (\omega + i0^+ - E_n) \) in terms of the eigenstates \( | nR \rangle, | nL \rangle \) and eigen-energies \( E_n \). Note that the integrand
\[ G^\dagger(\omega) - G(\omega) \rightarrow 1/\omega^2 \] (40)
as \( \omega \rightarrow \infty \), and therefore the \( \omega \) integration is well defined; taking contour in the upper or lower complex plane yields the same result.

The simple sum rule \( \sum_j P_j = 1 \) places a constraint that \( \alpha_b > 1 \) in the algebraic decay \( P_x \sim |x - x_0|^{-\alpha_b} \), otherwise the integral
\[ \int_{-\infty}^{0} |x - x_0|^{-\alpha_b} dx \]
(41)
is divergent and the sum rule cannot be satisfied in an infinite chain.

IV. Bipolar Non-Hermitian Skin Effect and Bipolar Edge Burst

Based on our present understanding of the edge burst, we may expect that the bipolar NHSE [3], namely, skin modes localized at both the left and right ends, might generate edge burst at both ends provided that the imaginary gap closes. We have confirmed this expectation using the model II, whose Hamiltonian is reproduced as:
\[ H(k) = (t_1 + t_2 \cos k) \sigma_x + [t_3 \cos(k - \alpha) + \frac{i\gamma}{2}] \sigma_z - \frac{i\gamma}{2} I, \]
(42)
and also illustrated in Fig. 2(a) of this Supplemental Material. As shown in Fig. 2(b), we have bipolar edge burst, namely, edge burst occurs at both ends of the chain.

Based on the bulk-edge scaling relation underlying the edge burst, we infer algebraic decay of \( P_x \) in the bulk towards both left and right ends. Notably, one of the imaginary gap closing point \( (A_1) \) is enclosed by the GBZ, and the other \( (A_2) \) is outside the GBZ [Fig. 2(d)]. This is in line with the GBZ-based formulas of Green’s function, in which leftward and rightward Green’s function is determined by roots outside and inside the GBZ [2].

V. Random Starting Point

In the main article, the walker starts from a fixed location \( x_0 \). Suppose that the initial location is randomly distributed in the chain of length \( L \), then we have
\[ P_x = \sum_s P(s \rightarrow x)p_s, \]
(43)
where \( p_s \) is the probability of starting from location \( s \), with the sum rule \( \sum_s p_s = 1 \) satisfied. In the main article, we have focused on the case \( p_s = \delta_{s, x_0} \). Another interesting choice of \( p_s \) is the uniform distribution \( p_s = 1/L \) that is location-independent. Such a distribution describes an incoherent input that has equal probability at everywhere
FIG. 2. (a) Pictorial illustration of the Hamiltonian with bipolar NHSE (Eq. 42). (b) $P_x$ profile of a finite chain with $L = 60$. $x_0 = 31$. (c) The PBC spectrum. (d) The GBZ. $t_1 = 0.8, t_2 = 2, t_3 = 2, \alpha = \pi/5; \gamma = 2$. The imaginary gap closing points $A_1$ and $A_2$ in (c) correspond to $\beta$ value $A_1$ and $A_2$ in (d), respectively. The point enclosed by the GBZ ($A_1$) corresponds to edge burst at the right end, while that outside the GBZ ($A_2$) corresponds to edge burst at the left end.

FIG. 3. The spatially resolved loss $P_x$ for a uniform distribution of starting point on an open-boundary chain. The model is the one from the main article (i.e. the model I of this Supplemental Material). The parameters are $t_1 = 0.4, t_2 = 0.5, \gamma = 0.8$. We take chain length $L = 100$, and consequently $p_s = 1/L = 1/100$. We find that the edge burst remains present for a generic $p_s$ distribution. Specifically, we have shown $P_x$ for the uniform $p_s$ distribution in Fig. 3 of this Supplemental Material, with a prominent edge burst. The relative height of the edge burst can be readily estimated. Because the typical distance between the starting point and the edge is of order $L$, the height of edge peak is roughly $P_{\text{edge}} \sim L^{-\alpha_b} \sim L^{-\alpha_b+1}$. On the other hand, the average height of $P_x$ in the bulk is $L^{-1}$. Therefore, the relative height of the edge peak is roughly $L^{-\alpha_b+1}/L^{-1} \sim L^{-\alpha_b+2}$. For example, with $\alpha_b = 3/2$, the relative height for $t_1 \in (0, t_2)$ is $\sqrt{L}$, which grows with $L$.

[1] Shunyu Yao and Zhong Wang, “Edge states and topological invariants of non-hermitian systems,” Phys. Rev. Lett. 121, 086803 (2018).
[2] Wen-Tan Xue, Ming-Rui Li, Yu-Min Hu, Fei Song, and Zhong Wang, “Simple formulas of directional amplification from non-bloch band theory,” Phys. Rev. B 103, L241408 (2021).
[3] Fei Song, Shunyu Yao, and Zhong Wang, “Non-hermitian topological invariants in real space,” Phys. Rev. Lett. 123, 246801 (2019).