EXPONENTIAL STABILITY FOR THE LOCALLY DAMPED
DEFOCUSING SCHRÖDINGER EQUATION ON COMPACT
MANIFOLD

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Abstract. In this paper we study the asymptotic dynamics for semilinear
defocusing Schrödinger equation subject to a damping locally distributed on
a n-dimensional compact Riemannian manifold $M^n$ without boundary. The
proofs are based on a result of unique continuation property, in the construction
of a function $f$ whose Hessian is positive definite and $\Delta f = C_0$ in some region
contained in $M$ and about the smoothing effect due to Aloui adapted to the
present context.

1. Introduction. In general, if one consider the Schrödinger equation posed on
a compact Riemannian manifold without boundary we get a reasonable number of
contributors regarding well-posedness results. In fact, Burq, Gérard and Tzvetkov in
[8] (see also [9, 10]) established Strichartz estimates and consequently the solvability
for a general class of nonlineairies related to the following Schrödinger equation,
\[
\begin{cases}
    iu_t + \Delta u = P(|u|^2)u & \text{in } M \times \mathbb{R}, \\
    u(x, 0) = u_0(x), & \text{in } M,
\end{cases}
\]
where $P : \mathbb{R}_+ \to \mathbb{R}$ is a polynomial. Those estimates allowed them to estab-
lish global existence results for regular as well as mild solutions of the nonlinear
Schrödinger equations on surfaces in the case of defocusing polynomial nonlinear-
ities and, moreover, on three-manifolds in case of defocusing nonlinearities. Similar
considerations can be found also in Dehman, Gérard and Lebeau [17] (see Proposition 7). It is worth quoting other papers in connection with compact manifolds: [3, 9, 18].

Thomann in [32], has considered supercritical nonlinear Schrödinger equations in an analytic Riemannian manifold \((M, g)\) of dimension \(d \geq 3\), given by,

\[
\begin{cases}
  iu_t + \Delta u = \omega |u|^{p-1}u & \text{in } M \times \mathbb{R}, \\
  u(x, 0) = u_0(x), & \text{in } M,
\end{cases}
\]

and established results of ill-posedness in energy norms. Here, the metric \(g\) is analytic and with either \(\omega = -1\) (defocusing equation) or \(\omega = 1\) (focusing equation).

Using an analytic WKB method, it was possible to construct an Ansatz for the semiclassical equation for times independent of the small parameter. In Euclidean context, we have a large number of results which treat about ill-posedness phenomena for equation above, specially when the critical power is considered, see for instance, [27, 26].

Regarding results of stabilization and exact controllability for the Schrödinger equation posed on a compact Riemannian manifold without boundary we do not have a large numbers of contributors in the current literature. However, we can mention a nice work due to Dehman, Gérard and Lebeau in [17]. These authors established the stability and exact control in \(H^1\) norm for the following equation,

\[
\begin{cases}
  iu_t + \Delta u - a(x)(1 - \Delta_M)^{-1}a(x)u_t = P'(|u|^2)u & \text{in } M \times (0, +\infty), \\
  u(x, 0) = u^0(x), & \text{in } M,
\end{cases}
\]

where \(M\) is 2-dimensional and \(P\) is a polynomial function with real coefficients, satisfying \(P(0) = 0\) and the following defocusing assumption,

\[
P'(r) \to +\infty, \quad \text{as } r \to +\infty.
\]

The approach is based on a result of propagation of singularities and recent dispersion estimates (Strichartz type inequalities) due to Burq, Gérard and Tzvetkov in [8].

Next, when the Euclidean setting is considered, we have a recent work due to Cavalcanti, Domingos Cavalcanti, Fukuoka and Natali in [13] which establishes the exponential stability associated to equation

\[
iu_t + \Delta u - |u|^2u + ia(x)u = 0, \quad \text{in } \mathbb{R}^2 \times (0, +\infty).
\]

By using classical results of well-posedness on \(H^3\)-norm in bounded domains, the authors proved a result of unique continuation for the well-known Schrödinger equation with defocusing cubic nonlinearity and consequently the exponential stability in \(L^2\)-norm. In addition, Bortot and Corrêa [6] also establish the exponential decay in level \(L^2\)-norm to the similar problem in a Euclidean domain.

This paper is concerned the study of the unique continuation property associated with the defocusing semilinear Schrödinger equation

\[
\begin{cases}
  i\partial_t y + \Delta y - h(|y|^2) y = 0 & \text{in } M \times (0, \infty) \\
  y(0) = y_0 & \text{in } M.
\end{cases}
\]  

where \((M, g)\) is a compact, connected, orientable Riemannian manifold without boundary. The \(\Delta\) denotes the Laplace-Beltrami operator associated with Riemannian metric \(g\). Even more, we get exponential stability of the energy for the strong
locally distributed dissipation problem

\[
\begin{aligned}
& i \partial_t y + \Delta y - h(|y|^2)y + i a(x)(1 - \Delta)^{1/2} a(x) y = 0 \quad \text{in } M \times (0, \infty) \\
& y(0) = y_0 \quad \text{in } M,
\end{aligned}
\] (2)

The damping term \( i a(x)(1 - \Delta)^{1/2} a(x) y \) is specially fitted to the use the smoothing effect given in Aloui [1, 2].

We consider the standard assumptions:

(H1) Assumptions about the function \( a : M \to \mathbb{R} \):

(i) \( a \) is nonnegative function and \( a \in C^\infty(M) \);

(ii) \( a(x) \geq a_0 \geq 0 \) on \( M_\ast \), where \( M_\ast \) contains properly the complementary part \( \mathbb{V} \) strategically chosen in \( M \) (see comments before figure 1).

(H2) Assumptions about the function \( h : \mathbb{R}_+ \to \mathbb{R} \):

(i) \( h \in C^1(\mathbb{R}_+) \);

(ii) \( |h(s)| \leq M_1; \forall s \in \mathbb{R}_+ \);

(iii) \( 0 \leq h'(s) s \leq M_2; \forall s \in \mathbb{R}_+ \);

(iv) \( h(s) \geq 0; \forall s \in \mathbb{R}_+ \).

Remark 1. The assumption \((H2)\) are inspired by Brézis (see [7], Theorem 1, page 31). Some examples of functions that satisfies \((H2)\) can be found in [6].

The main goal of the present paper is a unique continuation property to mild solutions of the problem (1) in the class \( y \in L^\infty(0, T; L^2(M)) \).

In addition, we obtain exponential decay rates of the energy in \( L^2 \)-norm, that is, there exist positive constants \( C, \gamma \) such that

\[
|||y(t)|||_{L^2(M)}^2 \leq Ce^{-\gamma t}|||y_0|||_{L^2(M)}^2,
\]

for all regular solutions to problem (2) provided that the initial data \( y_0 \) is taken in bounded sets of \( L^2(\Omega) \).

In order to obtain the exponential stabilization, we need strongly a function \( f \), a subset \( \mathbb{V} \) of \( M \) and a constant \( C > 0 \) such that

\[
C \int_0^T \int_{\mathbb{V}} |\nabla y|^2 dM dt \leq 2 \text{Re} \int_0^T \int_{\mathbb{V}} \text{Hess } f(\nabla y, \nabla \bar{y}) dM dt.
\] (3)

We need \( \Delta f \equiv C_0 > 0 \) in \( \mathbb{V} \) and \( \text{meas}(\mathbb{V}) > \text{meas}(M) - \epsilon \). The dissipative effect is considered effective in the complementary of \( \mathbb{V} \) (see Figure 1).

Recently Cavalcanti et al. [16] obtain relevant results about this our problem on surfaces. In what follows, we would like the relevance of this paper compared to Cavalcanti. Indeed, in [16], the authors obtains the exponential employing assumptions and results patterned from Dehman, Gérard and Lebeau [17]. On the other hand, this paper presents the proof of its own unique continuation result. Although this proof comes from known multiplier techniques, to this end, we need of the tools from Riemannian geometry to construct the function \( f \) given in (3). The main difference between [16] and the present article is the unique continuation property, here has proved the property based on construction of function \( f \) satisfying the property (3), while which [16] assume unique continuation property (due Dehman, Gérard and Lebeau [17]).

There is some relevant works in connection with the subject of the present paper, see [5, 15, 21, 22, 23, 25, 30, 33].
In this paper, we will focus on the energy decay rates of the problem (2) in light of [6]. The main features of our work are summarized as follows.

1. We prove the existence of global unique solution for the semilinear defocusing Schrödinger equation on compact manifold. In [6] was investigated on a bounded domain.
2. For the problem (1) we can get unique continuation result in two steps regular and mild solutions. From Riemaniann geometry arguments and important results we construct the function $f$ whose Hessian is positive definite to conclude the unique continuation.
3. The main result, is the proof of uniform decay rates using energy methods.

Our paper is organized as follows. Section 2 is notations and some preliminaries. In Section 3 we present the corresponding results for the well-posedness of Problem (1) and (2). In Section 4 we establish a unique continuation property in connection with problem (1). In Section 5 we present the construction of $f$ satisfying (3), and finally, in Section 6 the proofs of the exponential stability.

2. Notation and organization. Here we will show some results and statements for differential calculus of tensor fields and Sobolev spaces on Riemannian manifolds, for more detail see [24, 29] and [31]. First, we consider the space $L^2(M)$ of complex valued function on $M$, with the following real inner product and norm

$$L^2(M) = \{ y : M \to \mathbb{C}; \int_M |y|^2 \, dM < \infty \}, \quad (y, z)_{L^2(M)} = \text{Re} \int_M y(x) \overline{z(x)} \, dM,$$

$$\|y\|^2_{L^2(M)} = (y, y)_{L^2(M)}.$$

In addition, we consider

$$H^1(M) = \{ y : M \to \mathbb{C}; y \in L^2(M); \nabla y \in L^2(M) \},$$

$$V := \{ y \in H^1(M); \int_M y(x) \, dM = 0 \},$$

$$H := \{ y \in L^2(M); \int_M y(x) \, dM = 0 \}. $$
equipped with the norms, respectively
\[ \|y\|_{H^1(M)}^2 = \|y\|_{L^2(M)}^2 + \|\nabla y\|_{L^2(M)}^2, \quad \|y\|_{H^2(M)}^2 = \|\nabla y\|_{L^2(M)}^2, \]
and \( H \) with the standard \( L^2 \)-norm.

Note that, from Poincaré's inequality,
\[ \|y\|_{L^2(M)}^2 \leq \lambda_1^{-1} \|\nabla y\|_{L^2(M)}^2; \quad \forall \ y \in V, \tag{4} \]
where \( \lambda_1 \) is the first eigenvalue of the Laplace-Beltrami operator. Thus it is possible to show that the norms in \( H^1(M) \) and \( V \) are equivalent.

Consider the following Hilbert space
\[ H^{2m}(M) = \{ y \in L^2(M); \Delta^m y \in L^2(M) \}, \]
equipped with norm
\[ \|y\|_{H^{2m}(M)}^2 = \|y\|_{L^2(M)}^2 + \|\Delta^m y\|_{L^2(M)}^2. \]

Let \( y : M \to \mathbb{C} \) a regular function and \( X \) a complex vector field on \( M \), namely, \( X = Y + iZ \), where \( Y \) and \( Z \) are real vector field. Denote by
\[
\langle \nabla y, X \rangle = \langle \nabla(\text{Re} \ y), Y \rangle_g - \langle \nabla(\text{Im} \ y), Z \rangle_g \\
+ i\langle \nabla(\text{Re} \ y), Z \rangle_g + i\langle \nabla(\text{Im} \ y), Y \rangle_g.
\]

Then,
\[ \langle \nabla y, \nabla \bar{y} \rangle = \langle \nabla(\text{Re} \ y), \nabla(\text{Re} \ y) \rangle_g + \langle \nabla(\text{Im} \ y), \nabla(\text{Im} \ y) \rangle_g = \| \nabla y \|^2. \]

In addition if \( f : M \to \mathbb{R} \) is a sufficiently regular real function and \( X_1 = Y_1 + iZ_1 \) is a complex vector field on \( M \) we have
\[
\text{Hess} \ f(X, X_1) = \text{Hess} \ f(Y, Y_1) - \text{Hess} \ f(Z, Z_1) \\
+ i \text{Hess} \ f(Y, Z_1) + \text{Hess} \ f(Z, Y_1).
\]
So
\[ \text{Hess} \ f(X, X) = \text{Hess} \ f(Y, Y) + \text{Hess} \ f(Z, Z), \]
and therefore using the following identity (see [14], p.22)
\[ \langle \nabla f, \nabla (H(f)) \rangle = \nabla H(\nabla f, \nabla f) + \frac{1}{2} \langle \nabla(|\nabla f|^2), H \rangle. \tag{5} \]

When \( X = \nabla y \) and \( y : M \to \mathbb{C} \), we can write
\[
\text{Hess} \ f(\nabla y, \nabla \bar{y}) = \langle \nabla(\text{Re} \ y), \nabla f, \nabla(\text{Re} \ y) \rangle_g \\
+ \langle \nabla(\text{Im} \ y), \nabla \nabla f, \nabla(\text{Im} \ y) \rangle_g + \frac{1}{2} (|\nabla y|^2 \Delta f) - \frac{1}{2} \text{div}(|\nabla y|^2 \nabla f),
\]
where \( |\nabla y|^2 = |\nabla(\text{Re} \ y)|^2 + |\nabla(\text{Im} \ y)|^2 \). Observe that
\[
\text{Re} \langle \nabla y, \nabla \nabla f, \nabla \bar{y} \rangle = \langle \nabla(\text{Re} \ y), \nabla \nabla f, \nabla(\text{Re} \ y) \rangle_g \\
+ \langle \nabla(\text{Im} \ y), \nabla \nabla f, \nabla(\text{Im} \ y) \rangle_g.
\]
Therefore we conclude that
\[ \text{Re} \text{Hess} \ f(\nabla y, \nabla \bar{y}) = \text{Re} \langle \nabla y, \nabla \nabla f, \nabla \bar{y} \rangle \\
+ \frac{1}{2} (|\nabla y|^2 \Delta f) - \frac{1}{2} \text{div}(|\nabla y|^2 \nabla f). \tag{6} \]
Replacing $\nabla f$ in the identity (6) by an arbitrary real vector field $H$ sufficiently regular, have the translation of the identity (5) for the complex case, namely:

$$\text{Re} \nabla H(\nabla y, \nabla y) = \text{Re} \langle \nabla y, \nabla \langle H, \nabla y \rangle \rangle - \frac{1}{2} \langle \nabla (|\nabla y|^2), H \rangle.$$  (7)

The following result is also true (see [11], Section 4.1): Let $M$ be a compact Riemannian manifold without boundary $y : M \to \mathbb{C} \in H^1(M)$ and $X$ a vector field of class $C^1$ on $M$. Then

$$\int_M \langle X, \nabla y \rangle \, dM = - \int_M (\text{div} X) \, y \, dM.$$  (8)

Consequently, if $y \in H^1(M)$ such that $\Delta y \in L^2(M)$ and $w \in H^1(M)$ then the following identity is valid:

$$\int_M \langle \nabla y, \nabla w \rangle \, dM = - \int_M \Delta y \, w \, dM.$$

3. Well-posedness. The well-posedness of the problems (1) and (2) was studied in [6], through semigroup theory, in the case of locally distributed damping on a bounded domain case. The similar result with additional the smoothing effect $ia(x)(1 - \Delta)^{1/2} a(x) y$ given by Aloui will follow with the same arguments. For the sake of completeness, we show the main steps of the proofs.

First, we write problem (2) as an equivalent Cauchy problem

$$\frac{dy}{dt} = Ay + F(y), \quad y(0) = y_0,$$

where

$$A : D(A) = H^2(M) \cap V \to L^2(M)$$

$$y \quad \mapsto \quad Ay = i \Delta y - a(x) (1 - \Delta)^{1/2} a(x) y$$

with

$$F(y) = -ih(|y|^2)y.$$

Note that the operator $A$ is maximal dissipative in $D(A)$ and $F$ is globally Lipschitz in $L^2(M)$. We can see the similar result in [6]. So using standard arguments from Pazy [28] follows next Theorem.

**Theorem 3.1.** Assume the hypotheses (H1) and (H2). Then, given $y_0 \in D(A)$, the problem (2) has a unique regular solution $y$ for the problem satisfying

$$y \in C([0, +\infty); H^2(M) \cap V) \cap C^1([0, +\infty); L^2(M)).$$

**Remark 2.** The operator $1 - \Delta$ is defined by $\{V, H, a(y, z)\}$, where

$$a(y, z) = \text{Re} \int_M y z + \langle \nabla y, \nabla z \rangle \, dM.$$

We know that $D(1 - \Delta) = H^2(M) \cap V$ and such operator can be extended to $V$. According to Lion-Magenes [24] we have $D((1 - \Delta)^{3/4}) = V$ and $D((1 - \Delta)^{1/2}) = V \to D((1 - \Delta)^{1/4}) \to L^2(M)$. Furthermore

$$\left( (1 - \Delta)^{1/2} y, (1 - \Delta)^{1/2} z \right)_{L^2(M)} = (y, z)_{H^1(M)}, \forall y, z \in V.$$
Remark 3. Writing \( E_0(t) := \|y(t)\|^2_{L^2(M)} \) we get an important energy estimate

\[
E_0(t_2) - E_0(t_1) = -2 \int_{t_1}^{t_2} \|(1 - \Delta)^{1/4} a(\cdot) y(t)\|^2_{L^2(M)} dt, \quad \forall \, t_2 \geq t_1 \geq 0. \tag{9}
\]

In fact, observe that the operator \((1 - \Delta)^{1/2}\) is positive and multiplying (2) by \(y\), we infer that

\[
\frac{1}{2} \frac{d}{dt} \int_M |y|^2 dM = -\text{Re} \left((1 - \Delta)^{1/2} [a(\cdot) y], a(\cdot) y \right)_{L^2(M)} \leq 0.
\]

Thus

\[
\|y(t)\|^2_{L^2(M)} \leq \|y_0\|^2_{L^2(M)}.
\]

Moreover, the operator \((1 - \Delta)^{1/4}\) is self-adjoint, this implies

\[
(1 - \Delta)^{1/4} a(\cdot) y, a(\cdot) y \right)_{L^2(M)} = \left((1 - \Delta)^{1/4} a(\cdot) y, (1 - \Delta)^{1/4} a(\cdot) y \right)_{L^2(M)}
\]

\[
= \|(1 - \Delta)^{1/4} a(\cdot) y(t)\|^2_{L^2(M)}.
\]

Consequently we obtain (9).

So, studying the problem (1), note that the energy is given by

\[
E_1(t) = \frac{1}{2} \left[ \int_M |\nabla y|^2 dM + \int_M G(|y|^2) dM \right]
\]

where

\[
G(s) = \int_0^s h(\sigma) d\sigma. \tag{10}
\]

Thanks to \((H2)(iv)\), we get

\[
G(s) \geq 0, \quad \forall \, s \in \mathbb{R}_+.
\]

Thus, multiplying (1) by \(\partial_t y\) and considering the real part, we have

\[
E_1(t) = E_1(0). \tag{11}
\]

Consequently \(\|y\|_V\) remains bounded for any \(t > 0\). After these observations the next theorem guarantees the existence, uniqueness of the problem (1).

**Theorem 3.2.** Assume that \(y_0 \in H^2(M) \cap V\). Then, problem (1) possesses a unique solution which belongs to

\[y \in C([0, +\infty); H^2(M) \cap V) \cap C^1([0, +\infty); L^2(M)).\]

The proof of Theorem 3.2 is similar to Theorem 3.1. Just note that operator

\[
B : D(B) = H^2(M) \cap V \to L^2(M)
\]

\[y \mapsto By = i \Delta y\]

is maximal dissipative in \(D(B)\) and the problem (1) is equivalent to an Cauchy problem, given by

\[
\frac{dy}{dt} = By + F(y), \quad y(0) = y_0,
\]

Consequently we have Theorem 3.2.
Remark 4. By Lummer-Phillip’s Theorem the operator $B$ is the infinitesimal generator of $C_0$ semigroup contractions $S(t)$ in $H$. So, given $y_0 \in H$ the problem (1) have a unique mild solution $y$ in class $L^\infty(0,T;L^2(M))$ which satisfies the follow integral equation

$$ y(t) = S(t)y_0 + \int_0^t S(t-s)F(y(s))ds. $$

4. Unique continuation. In this section we establish the unique continuation property of the problem (1). Before we need proof a technical lemma on the regular solutions.

Lemma 4.1. Let $M^n$ be a compact Riemannian manifold, without boundary, orientable and connected. Let $q \in \left[C^2(M)\right]^n$ be a real vector field. Then, for all regular solution of problem (1) the following identity holds

\[
\text{Re} \left( 2 \int_0^T \int_M \nabla q (\nabla y, \nabla \bar{y}) \, dM \, dt \right) = \text{Re} \left[ i \int_M \langle q, \nabla \bar{y} \rangle \, dM \right]_0^T \\
- \text{Re} \left( \int_0^T \int_M (\nabla \bar{y}, \nabla (\text{div } q)) y \, dM \, dt \right) + \int_0^T \int_M (\text{div } q)G(|y|^2) \, dM \, dt \\
- \int_0^T \int_M (\text{div } q)h(|y|^2) |y|^2 \, dM \, dt
\]

where $G$ is given in (10).

Proof. Indeed, multiplying equation (1) by $\langle q, \nabla \bar{y} \rangle$ and integrating over $M \times (0,T)$, we obtain

\[
0 = \int_0^T \int_M [i y_t + \Delta y - h(|y|^2) y] \langle q, \nabla \bar{y} \rangle \, dM \, dt. \\
0 = \int_0^T \int_M i y_t \langle q, \nabla \bar{y} \rangle \, dM \, dt + \int_0^T \int_M \Delta y \langle q, \nabla \bar{y} \rangle \, dM \, dt \\
- \int_0^T \int_M \text{div}(yq) y_t \, dM \, dt
\]

Integrating $I_1$ by parts,

\[
I_1 = \left[ i \int_M y(q, \nabla \bar{y}) \, dM \right]_0^T - i \int_0^T \int_M y \langle q, \nabla \bar{y}_t \rangle \, dM \, dt.
\]

On the other hand, from (8) we obtain an identity for the second term of (14),

\[
i \int_0^T \int_M y(q, \nabla \bar{y}_t) \, dM \, dt = i \int_0^T \int_M \langle yq, \nabla \bar{y}_t \rangle \, dM \, dt \\
= -i \int_0^T \int_M \text{div}(yq) \bar{y}_t \, dM \, dt \\
= -i \int_0^T \int_M (\text{div } q) y \bar{y}_t \, dM \, dt - i \int_0^T \int_M \langle q, \nabla y \rangle \, dM \, dt.
\]
Then
\[ I_1 = \left[ i \int_M y \langle q, \nabla \bar{y} \rangle \, dM \right]_0^T + i \int_0^T \int_M \langle q, \nabla y \rangle \bar{y} \, dM \, dt \]
\[ + i \int_0^T \int_M (\text{div } q) y \bar{y} \, dM \, dt, \]  

(15)

However, knowing that
\[ iy_t = -\Delta y + h(\|y\|^2) y \iff \bar{y}_t = -i \Delta \bar{y} + i h(\|y\|^2). \]

Now, using (15), we get
\[ I_1 = \left[ i \int_M y \langle q, \nabla \bar{y} \rangle \, dM \right]_0^T + \int_0^T \int_M \langle q, \nabla y \rangle \Delta \bar{y} \, dM \, dt \]
\[ - \int_0^T \int_M \langle q, \nabla y \rangle h(\|y\|^2) \bar{y} \, dM \, dt + \int_0^T \int_M (\text{div } q) \Delta \bar{y} \, dM \, dt \]
\[ - \int_0^T \int_M (\text{div } q) h(\|y\|^2) \|y\|^2 \, dM \, dt. \]  

(16)

Taking the real part of (13), having in mind the real part of (16), and observing that \( \text{Re}(z) = \text{Re}(\bar{z}) \), for all \( z \in \mathbb{C} \), we deduce that
\[ 0 = \text{Re} \left[ i \int_M y \langle q, \nabla \bar{y} \rangle \, dM \right]_0^T + 2 \text{Re} \int_0^T \int_M \Delta y \langle q, \nabla \bar{y} \rangle \, dM \, dt \]
\[ - 2 \text{Re} \int_0^T \int_M h(\|y\|^2) y \langle q, \nabla \bar{y} \rangle \, dM \, dt + \text{Re} \int_0^T \int_M (\text{div } q) \Delta \bar{y} \, dM \, dt \]
\[ - \int_0^T \int_M (\text{div } q) h(\|y\|^2) \|y\|^2 \, dM \, dt. \]  

(17)

In what follows let us analyze some terms on right-hand side of (17). According to (7) and (8) we get
\[ I_2 = -2 \text{Re} \int_0^T \int_M \langle \nabla y, \nabla (\langle q, \nabla \bar{y} \rangle) \rangle \, dM \, dt \]
\[ = -2 \text{Re} \int_0^T \int_M \nabla q (\nabla y, \nabla \bar{y}) \, dM \, dt - \int_0^T \int_M \langle \nabla (\|y\|^2), q \rangle \, dM \, dt \]
\[ = -2 \text{Re} \int_0^T \int_M \nabla q (\nabla y, \nabla \bar{y}) \, dM \, dt + \int_0^T \int_M \text{div } q \|y\|^2 \, dM \, dt \]  

(18)

Observe that
\[ 2 \text{Re} h(\|y\|^2)y \langle q, \nabla \bar{y} \rangle = h(\|y\|^2)\langle q, \nabla (\|y\|^2) \rangle = \langle q, h(\|y\|^2) \nabla (\|y\|^2) \rangle = \langle q, \nabla G(\|y\|^2) \rangle. \]

Then, employing (8), we deduce that
\[ I_3 = -2 \text{Re} \int_0^T \int_M h(\|y\|^2) y \langle q, \nabla \bar{y} \rangle \, dM \, dt \]
\[ = \int_0^T \int_M (\text{div } q) G(\|y\|^2) \, dM \, dt \]  

(19)
Applying (8), it results that
\[
\int_0^T \int_M (\text{div } q) \Delta y \, dM \, dt = - \int_0^T \int_M \langle \nabla y, \nabla ((\text{div } q) y) \rangle \, dM \, dt
\]
\[
= - \int_0^T \int_M \langle \nabla y, \nabla (\text{div } q) \rangle y \, dM \, dt - \int_0^T \int_M (\text{div } q) |\nabla y|^2 \, dM \, dt.
\]
Combining (17), (18), (19) and (20) we conclude that
\[
0 = \text{Re} \left[ i \int_0^T \int_M y \langle q, \nabla y \rangle \, dM \right]_0^T + \int_0^T \int_M (\text{div } q) G(|y|^2) \, dM \, dt
\]
\[
- \int_0^T \int_M (\text{div } q) h(|y|^2)|y|^2 \, dM \, dt - 2 \text{Re} \int_0^T \int_M \nabla q(\nabla y, \nabla \overline{y}) \, dM \, dt
\]
\[
- \text{Re} \int_0^T \int_M \langle \nabla y, \nabla (\text{div } q) \rangle y \, dM \, dt.
\]
Which finishes the proof. \(\square\)

The next theorem is a crucial result to obtain the exponential decay.

**Theorem 4.2 (Unique continuation).** Let \( y \) be a mild solution of problem (1) in the class

\[ y \in L^\infty(0, T; L^2(M)). \]

If \( y \equiv 0 \) on \( M_\ast \times (0, T) \) where \( M_\ast \) is a open subset of \( M \) satisfying (H1)(ii), then \( y \equiv 0 \) on \( M \times (0, T) \), for \( T \) large enough.

Now, we will show the proof sketch in two steps.

**Step 1:** Regular Solutions.

Let \( y \) be the regular solution of the problem (1). Substituting the vector field \( q \) by \( \nabla f \) in Lemma 4.1, where the function \( f : M \rightarrow \mathbb{R} \) to be fixed in next Section, we have

\[
\text{Re} \left( 2 \int_0^T \int_M \text{Hess } f(\nabla y, \nabla \overline{y}) \, dM \, dt \right) = \text{Re} \left[ i \int_0^T \int_M \langle y, \nabla f, \nabla \overline{y} \rangle \, dM \right]_0^T
\]
\[
- \text{Re} \left( \int_0^T \int_M \langle \nabla y, \nabla (\Delta f) \rangle y \, dM \, dt \right) + \int_0^T \int_M (\Delta f) G(|y|^2) \, dM \, dt
\]
\[
- \int_0^T \int_M (\Delta f) h(|y|^2)|y|^2 \, dM \, dt.
\]

**Remark 5.** In the next Section we find a subset \( \mathbb{V} \) of \( M \) and a constant \( C > 0 \) such that

\[
C \int_0^T \int_\mathbb{V} |\nabla y|^2 \, dM \, dt \leq 2 \text{Re} \int_0^T \int_\mathbb{V} \text{Hess } f(\nabla y, \nabla \overline{y}) \, dM \, dt.
\]

In addition, \( \Delta f \equiv C_0 > 0 \) in \( \mathbb{V} \) and \( \text{meas}(\mathbb{V}) > \text{meas}(M) - \epsilon. \)

If Remark 5 is valid, we obtain
\[
\int_0^T \int_\mathbb{V} (\Delta f)[G(|y|^2) - h(|y|^2)|y|^2] \, dM \, dt \leq 0,
\]

since \( H(s) := G(s) - h(s)s \) is non increasing by (H2)(ii).
Furthermore,
\[
\int_0^T \int_M |\nabla y|^2 dM dt = C \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + C \int_0^T \int_V |\nabla y|^2 dM dt
\]
\[
\leq \hat{C} \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + \left| \text{Re} \left[ i \int_M y(\nabla f, \nabla \overline{y}) dM \right]_0^T \right| + C_1 \int_0^T \int_{M\setminus V} |y|^2 dM dt
\]
\[
+ C_2 \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + C_3 \int_0^T \int_M |\nabla y|^2 dM dt.
\] (25)

From (21), (22), (23) and (24), we get
\[
C \int_0^T \int_M |\nabla y|^2 dM dt = C \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + C \int_0^T \int_V |\nabla y|^2 dM dt
\]
\[
\leq \hat{C} \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + \left| \text{Re} \left[ i \int_M y(\nabla f, \nabla \overline{y}) dM \right]_0^T \right| + C_1 \int_0^T \int_{M\setminus V} |y|^2 dM dt
\]
\[
+ C_2 \int_0^T \int_{M\setminus V} |\nabla y|^2 dM dt + C_3 \int_0^T \int_M |\nabla y|^2 dM dt.
\] (26)

Using the initial assumptions,
\[
|G(|y|^2)| = \left| \int_0^{|y|^2} h(s) ds \right| \leq M_1 |y|^2.
\]

Consequently,
\[
C \int_0^T \int_M |\nabla y|^2 dM dt \leq C_4 \left| \text{Re} \left[ i \int_M y(\nabla f, \nabla \overline{y}) dM \right]_0^T \right| + C_4 \int_0^T \int_{M\setminus V} |y|^2 dM dt.
\] (27)
Remember that 
\[ E_1(t) = \frac{1}{2} \left[ \int_M |\nabla y|^2 dM + \int_M G(|y|^2) dM \right]. \]

So, according to (11) and (27) we get the following estimate

\[ \left| \text{Re} \left[ i \int_M y(\nabla f, \nabla y) dM \right] \right| \leq C_7 E_1(t), \]  

(28)

where \( C_7 = C_7(\lambda_1, f, h) \). From (26) and (28), we have

\[ C \int_0^T \int_M |\nabla y|^2 dM dt \leq C_7 E_1(t) + C_4 \int_0^T \int_{M \setminus \mathcal{V}} |y|^2 + |\nabla y|^2 dM dt. \]  

(29)

From (H2)(ii) and (4),

\[ \int_0^T \int_M G(|y|^2) dM dt \leq M_1 \int_0^T \int_M |y|^2 dM dt \]

\[ \leq M_1 \lambda_1^{-1} \int_0^T \int_M |\nabla y|^2 dM dt. \]  

(30)

Combining (29) and (30) we conclude

\[ C \int_0^T E_1(t) dt \leq \frac{C}{2} \int_0^T |\nabla y|^2 dM dt + C \int_0^T \int_M G(|y|^2) dM dt \]

\[ \leq \frac{1}{2} \int_0^T \int_M |\nabla y|^2 dM dt + \frac{CM_1}{\lambda_1} \int_0^T \int_M |\nabla y|^2 dM dt \]

\[ = \left( \frac{1}{2} + \frac{M_1}{\lambda_1} \right) C \int_0^T \int_M |\nabla y|^2 dM dt \]

\[ \leq \left( \frac{1}{2} + \frac{M_1}{\lambda_1} \right) C_7 E_1(t) + C_4 \int_0^T \int_{M \setminus \mathcal{V}} |y|^2 + |\nabla y|^2 dM dt, \]

where \( C = C(h, f) \). Thus

\[ \left( TC - \frac{1}{2} - \frac{M_1}{\lambda_1} \right) E_1(t) \leq C_4 \int_0^T \int_{M \setminus \mathcal{V}} |y|^2 + |\nabla y|^2 dM dt. \]

Then, for all \( T > \frac{1}{2C} + \frac{M_1}{C\lambda_1} \), we have the following inverse inequality

\[ E_1(t) \leq \tilde{C} \int_0^T \int_{M \setminus \mathcal{V}} |y|^2 + |\nabla y|^2 dM dt. \]

Let \( M_{\epsilon} \subset M \) an open subset of \( M \), such that, \( M_{\epsilon} \supset M \setminus \mathcal{V} \). Therefore if \( y \equiv 0 \) in \( M_{\epsilon} \times (0, T) \), it result that

\[ 0 = E_1(t) \geq \|\nabla y(t)\|_2^2 \quad \text{a.e. in} \quad (0, T). \]

However,

\[ \|y(t)\|_2^2 \leq \lambda_1^{-1} \|\nabla y(t)\|_2^2 = 0, \]

So \( y \equiv 0 \) a.e. in \( M \times (0, T) \) proving the Unique continuation for regular solutions.

**Step 2:** Mild Solutions.

We can extend the Theorem 4.2 for mild solution \( y \in L^\infty(0, T; L^2(M)) \) of problem (1). That is, given a mild solution \( y \) of problem (1), such that

\[ y = 0 \quad \text{a.e. in} \quad M_{\epsilon} \times (0, T) \]
Then \( y = 0 \) a.e. in \( M \times (0, T) \). Observe that in [6] appears a similar result on bounded domain. In the case of the Riemannian manifold we proceed in a similar way, we must be careful with regions where there are damping effects.

Therefore, we conclude Theorem 4.2.

5. Construction of \( f \). Fix \( \epsilon > 0 \). The next sections are devoted to the construction of a smooth function \( f : M \rightarrow \mathbb{R} \) as well as an open subset \( V \subset M \) with smooth boundary such that \( \text{meas}(V) > \text{meas}(M) - \epsilon \) and Remark 5 is valid.

5.1. Construction of a function satisfying the Remark 5 locally. The general idea of the construction of a function \( f \) satisfying the Remark 5 locally is similar to [11].

Lemma 5.1. Let \( M \) be a Riemannian manifold and consider \( p \in M \). There exist a neighborhood \( V_p \) of \( p \) and smooth function \( f : V_p \rightarrow \mathbb{R} \) such that \( |\nabla f| \geq C_1 > 0 \), \( \text{Hess} f \) is positive definite and \( \Delta f \) is a positive constant.

The proof will be done in several steps. The objects on \( M \) will be denoted by usual symbols. The objects on the tangent space will be denoted by caligraphic symbols.

5.2. General idea of the proof. Let \( B(p, r) \subset M \) be a geodesic ball inside the injectivity radius of \( p \). The exponential map \( \exp : B(0, r) \rightarrow B(p, r) \) induces normal coordinates in \( B(p, r) \). Consider the homothety \( \hat{h}_r : B(0, 1) \rightarrow B(0, r) \) and \( h_r : B_r(p, 1) \rightarrow B(p, r) \), where \( B_r(p, 1) \) is the \( 1/r \) Riemannian dilation of \( B(p, r) \). Denote by \( \exp_r : B(0, 1) \rightarrow B_r(p, 1) \) the exponential map such that \( \exp \circ \hat{h}_r = h_r \circ \exp_r \). In this section we always use these coordinate systems on \( B(p, r) \) and \( B_r(p, 1) \). The symbols \( \nabla_r, \Delta_r \) and \( \text{Hess}_r \) denote respectively the gradient, Laplacian and the Hessian of \( B_r(p, 1) \). We identify \( f : B(p, r) \rightarrow \mathbb{R} \) with \( f \circ \exp \) when there is no possibility of misunderstandings.

For each \( r \), we consider the problem

\[
\begin{align*}
\Delta_r f_r(x) &= 2n & \text{in } B_r(p, 1) \\
f_r(x) &= 3x_1 & \text{on } \partial B_r(p, 1). 
\end{align*}
\]

We prove that as \( r \rightarrow 0 \), the solution \( f_r \) of this problem converges \( C^2 \) to the solution of the Euclidean problem, which is

\[
f_0(x) = \sum_{i=1}^{n} x_i^2 + 3x_1 - 1.
\]

Notice that \( f_0 \) satisfies all the required conditions stated in Lemma 5.1 for the Euclidean setting. Therefore there exist a positive number \( r > 0 \) such that (31) and the other conditions stated in Lemma 5.1 are satisfied. Finally, we conclude that the function \( f : B(p, r) \rightarrow \mathbb{R} \) defined as \( f(x) = f_r(x/r) \) satisfies the conditions stated in Lemma 5.1.

5.3. Some formulas in a coordinate system. Let \( M \) be a Riemannian manifold and let \( U \subset M \) be an open set. Let \( (x_1, \ldots, x_n) \) be a coordinate system on \( U \). Denote the components of the Riemannian metric with respect to this coordinate
system by $g_{ij}$. We denote the components of the inverse matrix of $(g_{ij})$ by $g^{ij}$. Given $f \in C^1(U)$, the gradient of $f$ is given by

$$(\nabla f)_i = \sum_{j=1}^{n} g^{ij} \frac{\partial f}{\partial x_j}.$$ 

Hess $f$ is given by

$$(\text{Hess } f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \Gamma^{k}_{ij}$$

where

$$\Gamma^{ij}_{k} = \frac{1}{2} \sum_{l} \left( \frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij} \right) g^{lk}$$

are the Christoffel symbols of $(U,(x_1,\ldots,x_n),g)$. Finally, the Laplacian of $f$ is the trace of the Hessian with respect to the metric $g$ and it is given by

$$\Delta f = \sum_{i=1}^{n} \left( \frac{\partial^2 f}{\partial x_i \partial x_i} + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \Gamma^{k}_{ii} \right) g^{ij}.$$  \hfill (32)

5.4. **Equation (31) in a coordinate system.** Consider $B_r(p,1)$ and $B(p,r)$ with their coordinate systems $\exp_p$ and $\exp$ respectively. If we denote their metric matrix by $(g_r)_{ij}$ and $g_{ij}$ respectively, then $(g_r)_{ij}(x) = g_{ij}(rx)$. It also follows that $(\Gamma_{r})_{ij}^{k}(x) = r^{k} (\Gamma_{ij}^{k})$. Observe that in normal coordinates we have that $\lim_{x \to 0} \Gamma_{ij}^{k}(x) = 0$ for every $i,j$ and $k$. Let $\kappa$ be a positive number strictly smaller than the injectivity radius of $M$ at $p$. For every $r \in (0,\kappa]$ the problems (31) with respect to the parametrization $\exp_r$ are given by

$$\begin{cases}
L_r f_r(x) = 2n & \text{in } B(0,1) \\
f_r(x) = 3x_1 & \text{on } \partial B(0,1).
\end{cases} \hfill (33)$$

where $L_r$ are given by (32). Therefore $\lim_{r \to 0} L_r f = L_0 f$ for every $f \in C^2(\overline{B}(0,1))$, where $L_0$ is the Laplacian in the Euclidean space.

5.5. **Solutions and a priori estimates.** Here we remember some classical a priori estimates of solutions of elliptic differential equations $L \phi = \varphi$ defined on a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. The elliptic operator is given by

$$L \phi = \sum_{i,j}^{n} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial \phi}{\partial x_i} + c(x) \phi, \quad a_{ij} = a_{ji},$$

where $a_{ij}, b_i, c, \varphi \in C^0(\overline{\Omega})$ and there exist positive real numbers $\lambda$ and $\Lambda$ such that the coefficients satisfy $0 < \lambda |\xi|^2 \leq a_{ij}(\xi)\xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$. The theorems stated below are valid under much more general conditions, but we present them with these conditions for the sake of simplicity. For more general cases, see [20].

**Lemma 5.2.** Let $L \phi = \varphi$ in a bounded domain $\Omega$, where $c \leq 0$ and $\phi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Then

$$\|\phi\|_{C^0} \leq \sup_{\partial \Omega} |\phi| + C \frac{\|\varphi\|_{C^0}}{\lambda},$$

where $C$ is a constant only depending on $\text{diam}(\Omega)$ and $\sup |b|/\lambda$.  

Lemma 5.3. Let \( \phi \in C^2(\bar{\Omega}) \) be a solution of \( L\phi = \varphi \) in \( \Omega \) and let \( \Theta \) be a constant such that
\[
|a_{ij}|c^0, |b_{i}|c^0, |c|c^0 \leq \Theta.
\]
Let \( \zeta \in C^2(\bar{\Omega}) \) and suppose \( \phi = \zeta \) on \( \partial\Omega \). Then
\[
\|\phi\|_{C^2} \leq C(\|\phi\|_{C^0} + \|\zeta\|_{C^2} + \|\varphi\|_{C^0})
\]
where \( C = C(n, \lambda, \Theta, \Omega) \).

Lemma 5.4. The Dirichlet problem
\[
\begin{cases}
L\phi = \varphi & \text{in } \mathcal{B}(0,1) \\
\phi = \zeta & \text{on } \partial\mathcal{B}(0,1)
\end{cases}
\]
with \( c \leq 0 \) has a unique solution \( \phi \in C^2(\mathcal{B}(0,1)) \).

5.6. Convergence of the one parameter family of solutions. Consider the family of Dirichlet problems (33). We prove in this subsection that \( \lim_{r \to 0} \|f_r - f_0\|_{C^2} = 0 \). First of all, we consider an equivalent problem making the transformation \( \hat{f}_r = f_r - 3x_1 \). The resulting family of equations is given by
\[
\begin{cases}
L_r\hat{f}_r = \hat{\varphi}_r & \text{in } \mathcal{B}(0,1) \\
\hat{f}_r = 0 & \text{on } \partial\mathcal{B}(0,1)
\end{cases}
\]
(34)
where \( \hat{\varphi}_r = 2n - L_r(3x_1) \). For each \( r \), (34) has a unique solution due to Lemma 5.4. Moreover due to Lemma 5.2 and 5.3, we have that
\[
\|f\|_{C^2} \leq C\|L_rf\|_{C^0}
\]
for every \( f \in C^2(\mathcal{B}(0,1)) \) that vanishes on the boundary. Here \( C = C(n, \sup_{r} \lambda, \sup_{r} \Theta, \sup_{r} |b|c^0) \) (notice that \( C \) does depend on \( r \)). We claim that for every \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that \( \|\hat{f}_{r_1} - \hat{f}_{r_2}\|_{C^2} < \varepsilon \) whenever \( r_1, r_2 \in [0, \delta] \). In particular we have that \( \lim_{r \to 0} \|\hat{f}_r - f_0\|_{C^2} = 0 \). Consider \( r_1, r_2 \geq 0 \). Then
\[
\|\hat{f}_{r_1} - \hat{f}_{r_2}\|_{C^2} \leq C\|L_{r_1}(\hat{f}_{r_1} - \hat{f}_{r_2})\|_{C^0} = C\|L_{r_1}(f_{r_1} - f_{r_2})\|_{C^0} = C\|L_{r_1} - L_{r_2}\|_{C^0} \|f_{r_2}\|_{C^0} = C\|(L_{r_2} - L_{r_1})f_{r_2}\|_{C^0}.
\]

Due to Lemma 5.2 there exist a constant \( C_1 \) such that \( \|f_r\|_{C^0} \leq C_1 \) for every \( r \in [0, \kappa] \) (Use \( L = L_r \) for each \( r \)). Using Lemma 5.3 we have that \( \|f_r\|_{C^2} \leq C_2 \) for every \( r \). Moreover notice that the indices of \( L_r \) varies continuously with respect to \( r \). Then we can find a \( \delta > 0 \) such that if \( r < \delta \), then \( \|\hat{f}_r - f_0\|_{C^2} < \varepsilon \) and the claim is proved. Now it is immediate that \( \lim_{r \to 0} \|\hat{f}_r - f_0\|_{C^2} = 0 \).

The \( C^2 \) convergence of \( f_r \) implies that there exist a \( \varepsilon > 0 \) such that
1. \( (\Delta_r)f_r = 2n \) for every \( r \in [0, \kappa] \);
2. \( \text{Hess}_r f_r \) is positive definite for every \( r \in [0, \varepsilon] \);
3. \( \|\nabla f_r\| \geq C > 0 \) for every \( r \in [0, \varepsilon] \).
5.7. **Proof of Lemma 5.1.** Consider the geodesic balls \( B(p, r) \) and its \( 1/r \) Riemannian dilation \( B_{r}(p, 1) \) as in Subsection 5.2. We use the coordinate systems induced by \( \exp \) and \( \exp_{r} \) on \( B(p, r) \) and \( B_{r}(p, 1) \) respectively. If \( f_{r}: B_{r}(p, 1) \to \mathbb{R} \) is a smooth function, then the function \( f: B(p, r) \to \mathbb{R} \) defined as \( f(x) = f_{r}(x/r) \) satisfies the following equalities

\[
(\nabla f(x))_{i} = \left( \frac{1}{r} (\nabla_{r} f_{r}(x/r)) \right)_{i}
\]

\[
\Delta f(x) = \frac{1}{r^{2}} (\Delta_{r} f_{r}(x/r))
\]

\[
(Hess f(x))_{ij} = \left( \frac{1}{r^{2}} Hess_{r} f_{r}(x/r) \right)_{ij}
\]

where \((\nabla f(x))_{i}\) stands for the \( i \)-th coordinate of \( \nabla f \) in terms of the parametrization \( \exp \) and so on. Therefore if choose a \( f_{r} \) satisfying the conditions stated at the end of the last subsection, \( f: B(p, r) \to \mathbb{R} \) defined as \( f(x) := f_{r}(x/r) \) satisfies the conditions of Lemma 5.1.

5.8. **A function that satisfies the Remark 5 in a wide domain.** The main aim of this section is to prove the following theorem:

**Theorem 5.5.** Let \((M^{n}, g)\) be a compact \( n \)-dimensional Riemannian manifold without boundary. Fix \( \epsilon > 0 \). Then there exist an open subset \( \mathbb{V} \subset M \) with smooth boundary, a smooth function \( f: M \to \mathbb{R} \) and \( C, C_{1} > 0 \) such that:

1. \( \text{meas}(\mathbb{V}) \geq \text{meas}(M) - \epsilon \);
2. \( |\nabla f| \geq C_{1} > 0 \) on \( \mathbb{V} \);
3. \( \text{Hess } f(v, v) \geq C|v|^2 \) for every vector \( v \) on a tangent space of \( \mathbb{V} \);
4. \( \Delta f \) is a positive constant on \( \mathbb{V} \);
5. \( |\nabla f| \) is bounded on \( M \).

We begin proving some preliminary results. The following lemma is classical and can be found in [34] (See the proof of Lemma 1.9).

**Lemma 5.6.** Let \( M \) be a topological space which is locally compact, Hausdorff and has countable basis. Then there exist a increasing sequence of open sets \( (V_{i})_{i \in \mathbb{N}} \) such that:

1. \( M = \bigcup_{i=1}^{\infty} V_{i} \).
2. \( \bar{V}_{i} \subset V_{i+1} \).
3. \( \bar{V}_{i} \) is compact.

Given a compact Riemannian manifold \( M \), the injectivity radius of \( M \) is given by \( \min_{x \in M} \text{inj}(x) \), where \( \text{inj}(x) \) is the injectivity radius of \( x \) in \( M \).

We want to define a mollifier smoothing \( f_{\epsilon}: M \to \mathbb{R} \) of a locally summable function \( f: M \to \mathbb{R} \). The bump function \( \hat{\eta}: M \to \mathbb{R} \) is defined similarly as in Euclidean case:

\[
\hat{\eta}(x, y, \epsilon) = \begin{cases} 
\exp \left( \frac{1}{(\text{dist}(x,y))^2 - 1} \right) & \text{if } \text{dist}(x,y) < \epsilon < \text{inj}(M) \\
0 & \text{if } \text{dist}(x,y) \geq \epsilon.
\end{cases}
\]

The function \( \hat{\eta} \) is clearly \( C^{\infty} \). We normalize \( \hat{\eta} \) and get

\[
\eta(x, y, \epsilon) = \frac{\hat{\eta}(x, y, \epsilon)}{\int_{M} \hat{\eta}(x, y, \epsilon) dM(y)}.
\]
Notice that $\eta$ is also smooth. We define the mollifier smoothing $f_{\varepsilon} : M \to \mathbb{R}$ by

$$f_{\varepsilon}(x) = \int_M \eta(x, y, \varepsilon) f(y) dM.$$  \hfill (36)

**Lemma 5.7.** Let $M$ be a compact Riemannian manifold, $f : M \to \mathbb{R}$ be a locally summable function and $\varepsilon \in (0, \text{inj}(M))$ be a strictly positive number. Then the mollifier smoothing $f_{\varepsilon} : M \to \mathbb{R}$ defined by (36) is a smooth function.

**Proof.** The theorem holds because a Riemannian manifold behaves like Euclidean domains inside the injectivity radius. For the complete proof, see [19]. \hfill \Box

Lemmas 5.8 and 5.9 are proved in [12].

**Lemma 5.8.** Let $M$ be a Riemannian manifold and consider two subsets $A$ and $B$ such that $\text{dist}(A, B) > 0$. Suppose that $A$ and $B$ are compact. Then there exist open subsets $O_A \supset \supset A$ and $O_B \supset \supset B$ with smooth boundaries such that $\text{dist}(O_A, O_B) > 0$. Moreover there exist a smooth (cut-off) function $\rho : M \to \mathbb{R}$ such that $\rho|_{O_A} = 1$, $\rho|_{O_B} = 0$ and $\rho(M) \subset [0, 1]$.

**Lemma 5.9.** The set $O_A$ constructed in Lemma 5.8 has a finite number of components and the closure of each component is a Riemannian manifold with smooth boundary.

Now we prove the main theorem of this section:

**Proof of Theorem 5.5.** According to Lemma 5.1, for every $p \in M$, there exist a neighborhood $V_p$ of $p$ and a smooth function $f_p : V_p \to \mathbb{R}$ such that $\text{Hess} f_p$ is positive definite, $|\nabla f_p| \geq C_1 > 0$ and $\Delta f_p$ is a positive constant. If we restrict $f_p$ to an open subset $\hat{W}_p \subset V_p$ with smooth boundary, then there exist $C_p > 0$ such that $\text{Hess} f_p(v, v) \geq C_p |v|^2$ for every vector $v$ on a tangent space of $\hat{W}_p$.

Using the compactness of $M$, we can choose a finite covering $\{W_i\}_{i=1}^k$ of $M$. For $i = 1, \ldots, k$, denote the respective functions by $f_i : \hat{W}_i \to \mathbb{R}$ and set $C = \min \{C_1, \ldots, C_k\}$. Denote $B = \bigcup_{i=1}^k \partial \hat{W}_i$. Notice that $M - B$ is an open subset of $M$. Denote the points of $M - B$ which are in $\hat{W}_i$ by $W_i$. For $i = 2, \ldots, k$, denote the points of $M - B$ which are in $\hat{W}_i - \bigcup_{j=1}^{i-1} \hat{W}_j$ by $W_i$. Observe that we have the disjoint union $M - B = \bigcup_{i=1}^k W_i$. Moreover, without loss of generality, we can suppose that $W_i \neq \emptyset$, $i = 1, \ldots, k$. We claim that $W_i$, $i = 1, \ldots, k$, are open subsets of $M$: In fact, $M - B$ is an open subset and it can be written as a countable union of connected components. Each connected component is either completely contained in $\hat{W}_i$ or it does not intersect $\hat{W}_i$. Therefore each $W_i$ is a union of connected components of $M - B$. For the sake of simplicity, we keep writing $f_i : W_i \to \mathbb{R}$ instead of $f_i|_{W_i}$.

Fix $i \in \{1, \ldots, k\}$. Using Lemma 5.6, we can find an open set $\hat{V}_i \subset W_i$ such that $\text{meas}(W_i \setminus \hat{V}_i) < \varepsilon/k$. Notice that $\text{dist} (\hat{V}_i, B) = d_i > 0$ due to the compactness of $B$ and $\hat{V}_i$. Using Lemma 5.8 there exist open subsets $V_i \supset \supset \hat{V}_i$ and $O_i \supset \supset M - W_i$ with smooth boundaries and a smooth (cut-off) function $\rho_i : M \to \mathbb{R}$ such that $\rho_i|_{V_i} \equiv 1$, $\rho_i|_{O_i} \equiv 0$ and $\rho_i(M) \subset [0, 1]$. Now set $\rho = \sum_{i=1}^k \rho_i$ and $V = \bigcup_{i=1}^k V_i$. We can see that

1. $\text{meas}(M) - \text{meas}(V) = \sum_{i=1}^k \text{meas}(W_i - V_i) \leq \sum_{i=1}^k \text{meas}(W_i - \hat{V}_i) < \varepsilon$ what implies that $\text{meas}(V) > \text{meas}(M) - \varepsilon.$
continuous and for every $i$

Let $\rho |_V \equiv 1$;

$\partial V$ is smooth.

Now we are in position to construct $f$. Define

$$f(x) = \begin{cases} f_i(x)\rho(x) & \text{if } x \in W_i \\ 0 & \text{if } x \in B. \end{cases}$$

Notice that $f$ is smooth because $f|_{W_i} = f_i\rho_i$ is smooth and $f|_{W_i} = 0$ near $B$ for every $i = 1, \ldots, k$. Moreover $f|_V$ satisfies Items 2, 3 and 4 because $\rho|_V \equiv 1$ and $f|_{V_i} = f_i$ for every $i = 1, \ldots, k$. Finally $|\nabla f|$ is bounded on $M$ because it is continuous and $M$ is compact. \hfill \Box

6. Stabilization. Let’s start the main section of this work with a technical Lemma whose the similar proof can be found in [6] and [4]. In fact we only join our ideas.

**Lemma 6.1.** Let $y \in C([0, T]; D(A)) \cap C^1([0, T]; L^2(M))$ be a regular solution associated to equation (2), obtained from Theorem 3.2 and such that $\|y_0\|_{L^2(M)} < L$, with $L > 0$. Then, there is a positive constant $C_0 = C_0(\|y_0\|_{L^2(M)})$ such that,

$$\int_0^T \int_{M \setminus M_*} |y|^2 dM dt \leq C_0 \int_0^T \|(1 - \Delta)^{1/4} a(\cdot) y(t)\|^2_{L^2(\Omega)} dt.$$

The proof follows by an argument of contradiction, constructing a sequence of solutions $\{y_k\}_k$ of (2) such that $\{y_k(0)\}_k$ is uniform bounded and

$$\lim_{k \to +\infty} \int_0^T \int_{M_*} |y_k(x, t)|^2 dM dt = 0.$$

In this step we need to use the smoothing effect due Aloui [1]. From the assumption (H2)(ii) we get $F(y_k) = ih(|y_k|^2)y_k \in L^2(0, T; L^2(M))$, noting that $E_0^k(t) := \|y_k(t)\|_{L^2(M)}^2$ is a non-increasing function and the sequence is uniform bounded. So smoothing effect mentioned above give us the following estimate:

$$\|(\theta y_k)(t)\|_{L^2(0, T; H^\frac{1}{2}(M))} \leq C \left( \|y_k^0\|^2_{L^2(M)} + \|F(y_k(t))\|_{L^2(0, T; L^2(M))} \right)
$$

$$\leq C \left( \|y_k^0\|^2_{L^2(M)} + M_1 \|y_k(t)\|_{L^2(0, T; L^2(M))} \right),$$

(37)

$$\leq C L (1 + M_1 T), \forall \theta \in C_0^\infty(0, T).$$

Thus, $y_k \longrightarrow \tilde{y}$ strongly in $L^2(0, T; L^2(M))$, where

$$\tilde{y} = \begin{cases} y, & \text{a.e. in } \Omega \setminus M_* \\ 0, & \text{a.e. in } M_* \end{cases}.$$

Therefore the lemma follows in two cases, when $y = 0$ and when $y \neq 0$, using Theorem 4.2 (Unique Continuation Property). Below is the main result and a sketch proof.

**Theorem 6.2 (Exponential decay).** Let $y$ be a regular solution to problem (2) according to Theorem 3.1. Then, for all $L > 0$, there exist $C, \gamma$, positive constants, such that the following exponential decay holds

$$E_0(t) \leq Ce^{-\gamma t}E_0(0)$$

such that $E_0(t) := \frac{1}{2} \|y(t)\|^2_{L^2(M)}$, provided that $\|y_0\|_{L^2(M)} \leq L$. 

Proof. Denote by $\beta$ the positive constant of embedding $D((1 - \Delta)^{1/4}) \hookrightarrow L^2(M)$. Then from (9) and ($H1$)(ii) we have

$$\int_0^T E_0(t) dt \leq -\frac{\beta^2}{a_0} \left[ \int_M |y|^2 dM \right]^T_0 + \int_0^T \int_{M \setminus M^*} |y|^2 dM dt.$$

From Lemma 6.1 we deduce that

$$\int_0^T E_0(t) dt \leq -\frac{\beta^2}{a_0} E_0(0) + C_0 \int_0^T \left( (1 - \Delta)^{1/4} a(\cdot) y(t) \right)^2_{L^2(M)} dt \leq \frac{\beta^2}{a_0} E_0(0) - \frac{\beta^2}{a_0} E_0(0) + \frac{C_0}{2} E_0(0) - \frac{C_0}{2} E_0(T),$$

Finally,

$$\int_0^T E_0(t) dt \leq C E_0(0), \quad \text{for all } T > T_0,$$

where $C$ is a positive constant. Therefore the main result follows from semigroup property (see [4]).

REFERENCES

[1] L. Aloui, Smoothing effect for regularized Schrödinger equation on compact manifolds, Collect. Math., 59 (2008), 53–62.
[2] L. Aloui, Smoothing effect for regularized Schrödinger equation on bounded domains, Asymptot. Anal., 59 (2008), 179–193.
[3] R. Anton, Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains, Bull. Soc. Math. France, 136 (2008), 27–65.
[4] C. A. Bortot and M. M. Cavalcanti, Asymptotic stability for the damped Schrödinger equation on noncompact Riemannian manifolds and exterior domains, Communications in Partial Differential Equations, 39 (2014), 1791–1820.
[5] C. A. Bortot, M. M. Cavalcanti, W. J. Corrêa and V. N. Domingos Cavalcanti, Uniform decay rate estimates for Schrödinger and plate equations with nonlinear locally distributed damping, Journal of Differential Equations, 254 (2013), 3729–3764.
[6] C. A. Bortot and W. J. Corrêa, Exponential stability for the defocusing Schrödinger equation subject to strong damping locally distributed, Differential and Integral Equations, 31 (2018), 273–300.
[7] H. Brézis, Nonlinear Evolution Equations. Autumn Course on Semigroups, Theory and Applications, International Centre for Theoretical Physics. Trieste, 1984.
[8] N. Burq, P. Gérard and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Am. J. Math., 126 (2004), 569–605.
[9] N. Burq, P. Gérard and N. Tzvetkov, The Schrödinger equation on a compact manifold: Strichartz estimates and applications, Journées Équations aux Dérivées Partielles, (2001), 1–18.
[10] N. Burq, P. Gérard and N. Tzvetkov, The Cauchy Problem for the Nonlinear Schrödinger Equation on a Compact Manifold, J. Nonlinear Math. Phys., 10 (2003), 12–27.
[11] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka and J. A. Soriano, Uniform stabilization of the wave equation on compact surfaces and locally distributed damping, Transactions of AMS, 361 (2009), 4561–4580.
[12] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka and J. A. Soriano, Uniform stabilization of the wave equation on compact manifolds and locally distributed damping - a sharp result, J. Math. Anal. Appl., 351 (2009), 661–674.
[13] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka and F. Natali, Exponential stability for the 2-D defocusing Schrödinger equation with locally distributed damping, Differential Integral Equations, 22 (2009), 617–636.
[14] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka and J. A. Soriano, Asymptotic stability of the wave equation on compact manifolds and locally distributed damping: A sharp result, Arch. Rational Mech. Anal., 197 (2010), 925–964.
[15] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano and F. Natali, Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: exponential and polynomial stabilization, J. Differential Equations, 248 (2010), 2955–2971.

[16] M. M. Cavalcanti, W. J. Corrêa, V. N. Domingos Cavalcanti and M. R. Astudillo et al. Z. Angew. Math. Phys., (2018) 69: 100. https://doi.org/10.1007/s00033-018-0985-y

[17] B. Dehman, P. Gérard and G. Lebeau, Stabilization and control for the nonlinear Schrödinger equation on a compact surface, Math. Z., 254 (2006), 729–749.

[18] S. Demoulini, Global existence for a nonlinear Schrödinger-Chern-Simons system on a surface, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), 207–225.

[19] R. Fukuoka, Mollifier smoothing of tensor fields on differentiable manifolds and applications to Riemannian Geometry, preprint, arXiv:math.DG/0608230.

[20] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag Berlin Heidelberg, 2001.

[21] I. Lasiecka and R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control, Differential Integral Equations, 5 (1992), 521–535.

[22] I. Lasiecka and R. Triggiani, Well-posedness and sharp uniform decay rates at the $L^2(\Omega)$ - level of the Schrödinger equation with nonlinear boundary dissipation, J. Evol. Equ., 6 (2006), 485–537.

[23] C. Laurent, Global controlabilty and stabilzation for the nonlinear Schrödinger equation on some compact manifolds of dimension 3, SIAM J. Math. Anal., 42 (2010), 785–832.

[24] J. L. Lions and E. Magenes, Problèmes aux Limites non Homogènes, Aplications, Dunod, Paris, 1968.

[25] E. Machtyngier and E. Zuazua, Stabilization of the Schrödinger equation, Portugaliae Mathematica, 51 (1994), 243–256.

[26] F. Merle and P. Raphael, On universality of blow-up profile for $L^2$ critical nonlinear Schrödinger equation, Invent. Math., 156 (2004), 565–672.

[27] C.E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math., 166 (2006), 645–675.

[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[29] M. Spivak, A Comprehensive Introduction to Differential Geometry, Publish or Perish, INC., Houston, 1999.

[30] W. Strauss and C. Bu, An inhomogeneous boundary value problem for nonlinear Schrödinger equations, Journal of Differential Equations, 173 (2001), 79–91.

[31] M. Taylor, Partial Differential Equations, Springer, Berlin, 1991.

[32] L. Thomann, Instabilities for supercritical Schrödinger equations in analytic manifolds, Journal of Differential Equations, 245 (2008), 249–280

[33] M. Tsutsumi, On global solutions to the initial boundary value problem for the damped nonlinear Schrödinger equations, J. Math. Anal. Appl., 145 (1990), 328–341.

[34] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Foresman and Company, Scott, 1971.

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