Resonant cavity photon creation via the dynamical Casimir effect

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Motivated by a recent proposal for an experimental verification of the dynamical Casimir effect, the macroscopic electromagnetic field within a perfect cavity containing a thin slab with a time-dependent dielectric permittivity is quantized in terms of the dual potentials. For the resonance case, the number of photons created out of the vacuum due to the dynamical Casimir effect is calculated for both polarizations (TE and TM).

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One of the most impressive manifestations of the non-trivial structure of the vacuum is the static Casimir effect, i.e., the attraction of two perfectly conducting plates, for example, generated by the corresponding distortion of the electromagnetic vacuum state [1]. The non-inertial motion of a mirror can even create particles (i.e., photons) out of the vacuum [2] due to the time-dependent disturbance – which is called (in analogy) dynamical Casimir effect (see, e.g., [3] for review). Unfortunately, in contrast to the former (static) effect, the latter (dynamical) prediction has not been experimentally verified yet. To this end, it is probably advantageous to exploit the drastic enhancement of the number of created photons within a cavity occurring if the frequency of the wall vibration is in resonance with one of the (discrete) cavity modes. The difficulty of accomplishing mechanical vibrations of the wall with high frequencies (and appropriate amplitudes) experimentally has lead to the idea of simulating the wall motion by manipulating the dielectric permittivity (or magnetic permeability) of some medium filling the whole cavity (which can be done much faster). E.g., filling the whole cavity with a homogeneous medium described by a time-dependent permittivity \( \varepsilon(t) \) is analogous to introducing an effective length of the cavity via \( L_{\text{eff}}(t) = \sqrt{\varepsilon(t)} L \). However, since it is rather difficult to influence a medium filling the complete cavity, a new proposal [4] (see also [5]) for an experimental verification of the dynamical Casimir effect envisions a small slab with a fixed thickness \( a \) and a time-dependent permittivity \( \varepsilon(t) \) located at one of the walls of the cavity, cf. Fig. 1.

The question of whether and how the motion of the cavity wall can be simulated by such a small dielectric slab – especially in view of the number of created photons – will be the subject of the subsequent considerations (see also [6] for a 1+1 dimensional scalar field model).

In this Letter, we present an \textit{ab initio} derivation of the dynamical Casimir effect based on the quantization of the full macroscopic electromagnetic field within a (perfect) cavity with space-time dependent dielectric properties – superseding previous effectively 1+1 dimensional calculations (scalar field model, see, e.g., [6,7]) and approaches based on special factorization assumptions (see, e.g., [8]). For 3+1 dimensional cavities with moving walls, there exist various calculations for scalar fields, but very few taking into account the full electromagnetic field. E.g., in [9], the electromagnetic field is effectively split into two independent scalar fields obeying different boundary conditions via introducing different potentials for the TE and TM modes – which leads to a decoupling of the polarizations (TE and TM) per construction. However, in the most general situation, TE and TM modes can mix – hence their coupling should not be excluded \textit{a priori} but investigated for each special case.

Since we are considering low-frequency (e.g., microwave) photons only, we start from the macroscopic source-free Maxwell equations (\( \varepsilon_0 = \mu_0 = \hbar = 1 \))

\[
\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{D} = 0, \quad \dot{\mathbf{D}} = \nabla \times \mathbf{H}, \quad \dot{\mathbf{B}} = -\nabla \times \mathbf{E}, \quad (1)
\]

with \( \mathbf{H}(t, \mathbf{r}) = \mathbf{B}(t, \mathbf{r}) \) and \( \mathbf{D}(t, \mathbf{r}) = \varepsilon(t, \mathbf{r}) \mathbf{E}(t, \mathbf{r}) \). If we were to use the usual vector potential \( \mathbf{A} \) in temporal gauge (\( \Phi = 0 \))

\[
\mathbf{E} = \dot{\mathbf{A}}, \quad \mathbf{B} = -\nabla \times \mathbf{A}, \quad (2)
\]

the constraint \( \nabla \cdot [\varepsilon(t, \mathbf{r}) \dot{\mathbf{A}}(t, \mathbf{r})] = 0 \) would render the usual canonical quantization

\[
\left[ \dot{A}_i(t, \mathbf{r}), \dot{D}_j(t, \mathbf{r}') \right] = C_{ij}(\mathbf{r}, \mathbf{r}'), \quad (3)
\]

in connection with eliminating the longitudinal degree of freedom rather tedious (cf. also [10]) because, in this case, \( \nabla \cdot \mathbf{D} = 0 \) implies \( \partial_t C_{ij} = 0 \) but \( \partial_t C_{ij} \neq 0 \) in general.

Therefore, we avoid these difficulty with the well-known trick of introducing the dual vector potential (see, e.g., [9]).
\[ H = \hat{\Lambda} \cdot D = \nabla \times \Lambda, \]  

which applies in this form to the source-free Maxwell equations (1) only. In terms of the dual vector potential, the constraint simply reads \( \nabla \cdot \Lambda = 0 \). After the duality transformation [11], the Lagrangian is still the usual Larmor invariant – but with the opposite sign

\[ \mathcal{L} = \frac{1}{2} \int d^3 r \left[ B \cdot H - E \cdot D \right], \quad (5) \]

and the Hamiltonian is again the total energy

\[ \mathcal{H} = \frac{1}{2} \int d^3 r \left[ \Lambda^2 + \frac{1}{\varepsilon} (\nabla \times \Lambda)^2 \right]. \quad (6) \]

The continuity conditions \( (\Delta \Lambda = \Lambda^I - \Lambda^II) \) for the dual vector potential at the interface between the regions \( I \) and \( II \) of the cavity can be derived from the Maxwell equations (1)

\[ \Delta \Lambda = \Delta (\nabla \times \Lambda)_{\perp} = \Delta \left( \frac{1}{\varepsilon} \nabla \times \Lambda \right) || = 0. \quad (7) \]

Assuming that the walls of the cavity are perfectly conducting, for example, the boundary conditions read

\[ E_{\parallel} = 0 \leadsto (\nabla \times \Lambda)_{\parallel} = 0 \leadsto (\Lambda_{\parallel} \times [\nabla \times \Lambda])_{\perp} = 0. \quad (8) \]

Consequently, the boundary term arising from the integration by parts (as in the Poynting theorem) of the term \( (\nabla \times \Lambda)^2 \) in Eq. (6) vanishes. Hence we can introduce a non-negative and self-adjoint operator \( K \) via

\[ K f_\alpha = \nabla \times \left( \frac{1}{\varepsilon} \nabla \times f_\alpha \right) = \Omega^2_\alpha f_\alpha, \]

with eigenfunctions \( f_\alpha \) and eigenvalues \( \Omega^2_\alpha \). Note that we consider a lossless (ideal) dielectric medium resulting in a real permittivity \( \varepsilon \in \mathbb{R} \) (and hence a self-adjoint operator \( K \)). Owing to the time-dependence of the dielectric permittivity \( \varepsilon(t, r) \), the operator \( K(t) \) and consequently its eigenfunctions \( f_\alpha(t, r) \) as well as eigenvalues \( \Omega^2_\alpha(t) \) are also explicitly time-dependent in general.

The longitudinal modes \( f_{\alpha}^\parallel \) form the (orthogonal) eigenspace with zero eigenvalue \( \nabla \times f_{\alpha}^\parallel = 0 \) and hence we can restrict the operator \( K \) to the constraint sub-space \( \nabla \cdot f_\alpha = 0 \). Since \( K \) is a real operator, we can choose its eigenfunctions to be real as well \( f_\alpha = f_\alpha^* \); and because \( K \) is self-adjoint, its eigenfunctions are orthonormal (for equal times)

\[ \int d^3 r \, f_\alpha(t) \cdot f_\beta(t) = \delta_{\alpha \beta}, \quad (10) \]

and complete

\[ \sum_\alpha f_\alpha^I (t, r) f_\alpha^J (t, r') = \delta^J_I (r - r'), \quad (11) \]

with \( \delta^J_I (r - r') \) denoting the transversal Dirac \( \delta \)-distribution \( \partial_t \delta^J_I (r - r') = 0 \). Hence a corresponding normal mode expansion of the Lagrangian and the Hamiltonian in terms of the dual potentials into the instantaneous basis

\[ \Lambda(t, r) = \sum_\alpha Q_\alpha(t) f_\alpha(t, r), \quad (12) \]

leads to (see also [12])

\[ \mathcal{H}(t) = \frac{1}{2} \sum_\alpha \left( P^2_\alpha + \Omega^2_\alpha \right) + \sum_{\alpha \beta} P_{\alpha} Q_{\beta} M_{\alpha \beta}(t). \quad (13) \]

From now on, we shall drop the summation signs for convenience by declaring a corresponding (Einstein-like) sum convention. The canonical conjugated momenta are given by \( P_\alpha = Q_\alpha + M_{\alpha \beta}(t) Q_\beta \) and the anti-symmetric inter-mode coupling matrix reads

\[ M_{\alpha \beta}(t) = \int d^3 r \, f_\alpha(t) \cdot \dot{f}_\beta(t). \quad (14) \]

The usual equal-time canonical commutation relations, e.g., \([Q_\alpha, P_\beta] = i \delta_{\alpha \beta} \) are equivalent to the commutators for the fields, such as \([\hat{\Lambda}^I(t, r), \hat{\Pi}^J(t, r')] = i \delta^J_I (r - r') \).

It will be convenient to classify the eigenmodes with respect to their polarization at the interface into TE (transversal electric) and TM (transversal magnetic) modes

\[ E^TE_{\perp} = 0, \quad B^TM_{\perp} = 0. \quad (15) \]

In terms of the dual potentials, these conditions read \( (\nabla \times \Lambda)^E_{\perp} = 0 \) and \( (\nabla \times \Lambda)^M_{\perp} = 0 \). Assuming the absence of any static fields and fields outside the cavity (the walls are supposed to be perfectly reflecting), the boundary condition \( B_{\parallel} = 0 \) implies \( \Lambda_{\parallel} = 0 \) and hence Eq. (8) imposes the condition \( \nabla \cdot \Lambda_{\perp} = 0 \). As a result, we can make the following separation \textit{ansatz} in the homogeneous region \( I \)

\[ f^I_{k\sigma} = \begin{pmatrix} \sin(k_x x) \cos(k_y y) \cos(k_z z) \epsilon_{k\sigma}^I x \\ \cos(k_x x) \sin(k_y y) \cos(k_z z) \epsilon_{k\sigma}^I y \\ - \cos(k_x x) \cos(k_y y) \sin(k_z z) \epsilon_{k\sigma}^I z \end{pmatrix}, \quad (16) \]

and analogously for region \( I \) with \( x \) being replaced by \( x - L \).

The wave-numbers \( k_y \) and \( k_z \) are simply determined by the perpendicular cavity dimensions \( L_y, L_z \) via \( k_y = n_y \pi / L_y \) and \( k_z = n_z \pi / L_z \), respectively, with integers \( n_y \) and \( n_z \). The remaining polarization factors \( \epsilon_{k\sigma}^{I/III} \) as well as the \( k_{x/III}^{I/III} \) -values have to be determined according to the continuity conditions in Eq. (7), the polarization
condition (TE or TM) in Eq. (15), the transversality condition \( \nabla \cdot \mathbf{A} = 0 \), the overall normalization in Eq. (10), and, finally, the eigenvalue equation (for a fixed time)

\[
\Omega^2 = \frac{(k_x^I)^2 + k_y^I + k_z^I}{\varepsilon^I} = \frac{(k_x^{II})^2 + k_y^I + k_z^I}{\varepsilon^{II}},
\]

which provides a relation between \( k_x^I \) and \( k_x^{II} \). Using the conditions mentioned above, we arrive at the transcendental equations

\[
\text{TE : } \tan(ak_x^I) = \frac{\tan(k_x^{II}[a - L])}{k_x^{II}},
\]

\[
\text{TM : } \frac{k_x^I \tan(ak_x^I)}{\varepsilon^I} = \frac{k_x^I \tan(k_x^{II}[a - L])}{\varepsilon^{II}},
\]

which have to be satisfied simultaneously to the eigenvalue equation (17).

Assuming the slab to be sufficiently small \( a \ll L \), we can find approximate solutions for the TE modes

\[
k_x^{II} = \frac{n_x \pi}{L} + O\left(\frac{a^3}{L^3}\right),
\]

and for the TM modes (for \( n_x > 0 \))

\[
k_x^{II} = \frac{n_x \pi}{L} \left(1 + \frac{a}{L} \left[\frac{\varepsilon^{II}}{\varepsilon^I} - 1\right] k_x^I k_x^{II} \right) + O\left(\frac{a^2}{L^2}\right),
\]

with \( k_x^I = k_x^I + k_z^I \) and \( n_x = \frac{n_x \pi}{L} / \varepsilon^I \).

We observe that the first-order (in \( a/L \ll 1 \)) contributions to the eigenvalues \( \Omega^2 \) of the TE modes are independent of \( \varepsilon^{II}/\varepsilon^I \) (t). Only for the TM modes, a variation of the permittivities \( \varepsilon^{II}/\varepsilon^I \) induces a change of the eigenvalues (with the label \( \alpha = \{n, \text{TM}\} \))

\[
\Delta \Omega^2_{\alpha, \text{TM}}(t) = \frac{2}{\varepsilon^{II}} \frac{a}{L} \left[\frac{\varepsilon^{II}}{\varepsilon^I}(t) - 1\right] + O\left(\frac{a^2}{L^2}\right). \tag{21}
\]

The first-order term of the coupling matrix can be derived in complete analogy: \( \mathcal{M}_{\alpha, \beta}(t) \propto (a/L) \partial_t (\varepsilon^{II}/\varepsilon^I) \).

In order to simulate an oscillation of the wall, we assume a harmonic time-dependence of the ratio

\[
\frac{\varepsilon^{II}}{\varepsilon^I}(t) = \xi + \chi \sin(\omega t), \tag{22}
\]

with the amplitude \( \chi \) and an irrelevant additive constant \( \xi \) (which just induces a constant shift of the eigenfrequencies). A small harmonic perturbation over a relatively long time duration (i.e., many oscillations) enables us to employ the rotating wave approximation, which neglects all non-resonant terms. According to Eq. (13) with \( \Omega^2_{\alpha}(t) = (\Omega^0_{\alpha})^2 + \Delta \Omega^2_{\alpha}(t) \), the perturbation Hamiltonian can be split up into two parts, the diagonal (so-called squeezing) term \( \Delta \Omega^2_{\alpha}(t) \hat{Q}_\alpha^2 / 2 \) and the off-diagonal (so-called velocity) contribution \( \hat{P}_\alpha \hat{Q}_\beta \mathcal{M}_{\alpha, \beta}(t) \), cf. [12]. The resonance condition for the former (squeezing) term reads \( \omega = 2\Omega^0_{\alpha} \) and for the latter inter-mode coupling (velocity) contribution \( \omega = |\Omega^0_{\alpha} \pm \Omega^0_{\beta}| \). In the following, we shall assume a cavity with well-separated eigenfrequencies where the external oscillation frequency \( \omega \) matches the diagonal resonance condition \( \omega = 2\Omega^0_{\alpha} \) for a certain TM mode only (no resonant inter-mode coupling). In this case, the effective Hamiltonian reads

\[
\hat{\mathcal{S}}_{\text{eff}}(t) = \frac{i}{2} \frac{k_x^I}{\omega} \frac{\chi}{\varepsilon^{II}} \left[|\hat{a}_\alpha|^2 - |\hat{a}_\alpha|^2\right] \frac{a}{L} + O\left(\frac{a^2}{L^2}\right), \tag{23}
\]

for the resonant TM mode \( \alpha \). Accordingly, the time-dependence of the dielectric permittivity of the thin slab induces the creation of an exponentially increasing number of particles (photons) out of the vacuum (dynamical Casimir effect)

\[
\langle \hat{N} \rangle(t) = \langle 0 | \exp\{i\hat{\mathcal{S}}_{\text{eff}}(t)\hat{a}^\dagger \hat{a} \exp\{-i\hat{\mathcal{S}}_{\text{eff}}(t)\} | 0 \rangle = \sinh^2 \left(\frac{k_x^I}{\omega} \frac{\chi}{\varepsilon^{II}} \frac{a}{L} t \right). \tag{24}
\]

Let us state the physical assumptions entering the above derivation. Firstly, by starting from the source-free macroscopic Maxwell equations with perfectly conducting boundary conditions we assumed an ideal cavity and omitted losses and decoherence etc. Of course, the applicability of this assumption has to be checked (e.g., whether the Q-factor of the cavity is large enough) before conducting a corresponding experiment. Secondly, the external oscillation was assumed to be harmonic with the frequency matching the resonance condition exactly. Other periodic time-dependences could lead to the contribution of higher harmonics (2\( \omega \), etc.) and one has to make sure that the possibly resulting inter-mode coupling does not spoil the main contribution in Eq. (24). A deviation (detuning) from the exact resonance \( \omega = 2\Omega^0_{\alpha}(1 + \delta) \) is also not critical as long as the relative detuning \( \delta \) is smaller than the relative perturbation amplitude, cf. [13]. Thirdly, the neglect of the higher-order terms in the Taylor expansion of the transcendental matching equations (18) leading to Eqs. (19) and (20) assuming a small slab \( a \ll L \) is only justified if all other involved quantities are not too large. If the ratio \( \varepsilon^{II}/\varepsilon^I(t) \) changes drastically, this approximation breaks down as soon as the smallness of the expansion parameter \( a/L \) is compensated by a huge variation in \( \varepsilon^{II}/\varepsilon^I(t) \).

Let us study the two limiting cases: For \( \varepsilon^I \gg \varepsilon^{II} \), the wave-numbers behave as \( k_x^I \gg k_x^{II} \) according to Eq. (17). Hence the poles of the functions \( \tan(ak_x^I) \) in Eq. (18) induce drastic changes of \( k_x^{II} \) and thus \( \Omega \) - for both, TE and TM modes. In this case, the eigenmodes are 'pulled' into the small slab (which is not desirable since the time-varying material properties will entail the danger of dissipation and decoherence). In the opposite case \( \varepsilon^I \ll \varepsilon^{II} \), the wave-number in the small slab becomes imaginary, cf. Eq. (17). In some sense, the modes are...
changing the dielectric properties, e.g., laser illumination depth [4]. Note that $a/L = 1/100$ is still by far larger than the relative amplitudes that can be achieved by mechanical vibrations (at these frequencies). Provided that the assumption of a perfect cavity (e.g., Q-factor) is appropriate during such time-scales, one would create a significant amount of photons after a few microseconds.

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