Some Problems in Logic:

APPLICATIONS OF KRIPKE’S NOTION OF FULFILMENT

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Edited April 2019: The original thesis was type-written with hand-drawn symbols. This is a transcription of into LaTeX. No changes have been made, except for the correction of some typos, and the replacing of the Peano-Russell dot notation in formulas by parentheses.
This work is a study of S. Kripke’s notion of fulfilment. Motivated by the result of [5], Kripke was looking for a proof of Gödel’s Incompleteness Theorem which was model-theoretic, natural (that is, without self-reference), and easy. The resulting notion of fulfilment is very simple and it provides a versatile tool for deriving a large number of results, both new and old (indeed, the oldest), and in both Proof Theory and Model Theory.

In Chapter I, we give short and elegant proofs to a number of classical results; most of these are due to Kripke. There are two new results. One, due to Kripke, is that there exists an easily definable subring $R$ of the ring of primitive recursive functions such that for any non-principal ultrafilter $D$ on $\omega$, $R/D$ is a recursively saturated model of Peano arithmetic. The other is that for any r.e. theory $T$ extending $\text{PRA}$ and for any given r.e. set, we can feasibly find a $\Sigma^0_1$ formula which semi-represents in $T$ the given set; and if $T$ is not $\Sigma^0_1$-sound, we can choose the formula to be $\Delta^0_1(T)$.

Chapter II contains two distinct results. One answers a problem of [3] by showing that 

$$\{ \forall \phi \in \Pi^0_k : \phi \text{ is } \Sigma^0_k \text{-conservative over } \text{PA} \}$$

is a complete $\Pi^0_2$ set. The second, when combined with the results of III, gives a version of Herbrand’s Theorem and the relationship between the notions of proof and fulfilment.

III gives an exposition and extension of the Hilbert-Ackermann method of proving the consistency of $\text{PA}$; our account is largely based on that of [6].

IV is an exposition and extension of the method in [1] for obtaining conservation results of the form: $\Sigma^1_2\text{-AC}$ is $\Pi^1_3$-conservative over $(\Pi^1_1\text{-CA})_{<\varepsilon_0}$. We give a general version of this and from it derive a number of results for arithmetic, analysis, and set theory. Using III, we may also obtain uniform versions: e.g.

$$\forall \alpha < \varepsilon_0 \ (\Pi^1_1\text{-CA})_{\alpha} \vdash \text{RFN}_{\Pi^1_3}(\Sigma^1_2\text{-AC})$$

In V and VI we give some model-theoretic applications of fulfilment. V deals with non-$\omega$-models and is based on [4]. We also prove an extension of the theorem of D. Scott and [2] involving Weak König’s Lemma, and we describe the order types of some sets of elementary initial segments of recursively saturated models of certain theories. VI is an extension of [2]’s theorem on minimal models of analysis. We develop the notion of indicator for countable fragments of $L_{\infty\omega}$, and obtain a close parallel between this and the first-order case. The chapter concludes with some representability results in $\omega$-logic and the analogue of ($\ast$).

VII gives an exposition of the Paris-Harrington statement, deriving the sharp negative results established by others, and we also give a strengthened version of a key combinatorial lemma of [5].

[1] H. Friedman, Iterated inductive definitions and $\Sigma^1_2\text{-AC}. \text{Buffalo Conf. 1970}$
[2] ————, Countable models of set theories, Cambridge Summer School 1973
[3] D. Guaspari, Partially conservative extensions of arithmetic, TAMS 1979
[4] L. Kirby & J. Paris, Initial segments of models of PA, LNM 619 1977
[5] J. Paris & L. Harrington, A mathematical incompleteness in PA, Handbook 1977
[6] T. M. Scanlon, The consistency of number theory via Herbrand’s Thm, JSL 1973
Table of Contents

Introduction ................................................................. 1
I  The Main Definition .................................................. 7
II  Herbrand’s Theorem and Reflection Principles ............... 19
III The Substitution Method ........................................... 29
IV  Some Conservation Results ....................................... 37
V  Some Model-theoretic Applications: Non-\(\omega\)-models ...... 49
VI  More Model-theoretic Applications: \(\omega\)-models .......... 59
VII The Paris-Harrington Statement ................................. 71
Bibliography ................................................................. 79
Introduction

The notion of fulfilment\(^1\) is due to S. Kripke\(^2\). Motivated by the Paris-Harrington result, he was looking for a proof of Gödel’s Incompleteness Theorem which was model-theoretic, natural (that is, without self-reference) and easy. The notion of fulfilment is very simple, and it is implicit in much of mathematical logic. (Indeed, it is more or less stated in some of the early works of Skolem, Herbrand and Gödel.) In making it explicit, we obtain a unifying notion with applications in both Proof Theory and Model Theory.

Let us first consider model-theoretic proofs of proof-theoretic results. Fulfilment enables us to give enlightening proofs of some classical results—proofs which do not use the fixed-point theorem. Included here are:

1.4 Peano’s arithmetic, PA, is not finitely axiomatizable;
1.5 PA is not complete; and 1.10 the theory of \(\mathbb{N}\) is not arithmetical. To dispense with the fixed-point theorem, however, we must occasionally pay the price by considering only theories which include enough Induction. This is well illustrated by 2.7 where we give two proofs of the Essential Unboundedness Theorem of G. Kreisel and A. Levy \cite{68}. Let us consider our proof of the Incompleteness Theorem. When we apply it to theories not in the language of arithmetic, say in the language of set theory or analysis, the independent sentences we first obtain are quite complex as measured in terms of the number and types of quantifiers, e.g. \(\Pi_2\). But by weakening the notion of fulfilment, we may obtain independent \(\Pi_0^{\omega_1}\) sentences. By weakening the notion still further and using the fixed-point theorem, we may obtain independent \(\Pi_0^1\) sentences.

Next let us consider some results of a more recursion-theoretic flavour. Given any sufficiently strong r.e. theory \(T\) and any \(\Sigma^0_1\) formula \(\theta(x)\),\(^3\) we show in 1.12 how to feasibly (without the fixed-point theorem) semi-represent the r.e. set \(\{n \in \omega : \theta n\}\) in \(T\) by a \(\Sigma^0_1\) formula \(\psi(x)\); indeed, if \(T\) is not \(\Sigma^0_1\)-sound, we may choose \(\psi\) to be \(\Delta^0_1(T)\). In 2.8 we solve an open problem of D. Guaspari \cite{79} and R. Solovay by showing

\[
\{ \theta \in \Pi^0_k : \theta \text{ is } \Sigma^0_k \text{-conservative over } PA \}
\]

to be a complete \(\Pi^0_k\) set for each \(k \geq 2\); we also give a complete (in terms of quantifier complexity) classification of the analogous sets for theories other than PA.

Thirdly, consider some proof-theoretic applications of fulfilment. In Chapter III we give an exposition and slight generalization of the Hilbert-Ackermann method of proving the consistency of theories such as PA; our account is largely based on that of T. Scanlon \cite{73}. From our result we may easily derive Herbrand’s Theorem 2.3, 2.5 and the Reflexiveness Theorem 2.4:

\[
S + \text{Induction } \vdash \forall n \left( \Pr_{pc}(\Gamma \varphi \Pi) \supset \varphi n \right),
\]

where \(S\) is any weak theory which is sufficient to perform the required coding, \(\varphi n\) is any formula in the language of \(S\) with a single free variable of integer type, and the schema of Induction ranges over all formulae in the language of \(S\). Also from our proof we may obtain the sharp

---

\(^1\) The term “fulfillability” for “the notion of fulfilment” has the advantage of being one word as opposed to four. But I will not use it, and instead shall usually abbreviate “the notion of fulfilment” by “fulfilment”.

\(^2\) I am greatly indebted to Professor Kripke for informing me of some of his results. In Chapter I, results 1.3, 1.4, 1.5, 1.7, 1.8, and 1.13 are due to him, as is the equivalence \(\text{RFN}(T) \equiv \text{FUL}(T)\) given in Chapter II. Professor Kripke also informed me of the possibility of the exposition of the Herbrand-Ackermann-Scanlon method given in Chapter III.

\(^3\) Footnote added 2019: The notation here means that \(\theta\) is a formula in the language of \(T\) which can be interpreted as being \(\Sigma^0_1\) in the language of arithmetic. Also recall semi-represents here means that for all \(n \in \omega, \mathbb{N} \models \theta n \text{ iff } T \vdash \varphi \Pi\).
bounds of G. Minc [71] concerning subsystems of arithmetic, and the “No-Counter-Example Interpretation” of G. Kriesel. Our result also gives, for example, the following. In $\text{KP} + \text{Infinity}$ we may define by a $\Delta_0$ predicate an ordering which is intuitively a well-ordering of order-type $\varepsilon_{0n+1}$, the least $\varepsilon$-number greater than the class of ordinals. Then $\text{KP}$ plus $V = L$ plus the schema of Foundation on this ordering implies the schema of Uniform Reflection for $\text{KP}$. We conclude Chapter III with some conservation results for the schema of Induction over various theories of analysis and set theory, and for the language of arithmetic augmented with an extra constant $c$, conservation results for the schema of Induction up to $c$ over various theories of arithmetic.

In Chapter IV we give an exposition and extension of H. Friedman’s [70] method of obtaining conservation results of the form, e.g. $\Sigma_2^1 - \text{AC}$ is $\Pi_3^1$-conservative over $\Pi^1_1 - \text{CA}$, and $\Sigma_2^1 - \text{AC}$ is $\Pi_3^1$-conservative over $\langle \Pi^1_1 - \text{CA} \rangle_{< \varepsilon_0}$, that is, axioms asserting that the relativized hyperjump may be iterated $\alpha$ times for all $\alpha < \varepsilon_0$ but without the schema of Induction. The new results here are uniform versions of the above: for example

$$\forall n (\langle \Pi^1_1 - \text{CA} \rangle_n \models \text{RFN}_{\Pi_3^1}(\Sigma_2^1-\text{AC}))$$

$$\forall \alpha < \varepsilon_0 (\langle \Pi^1_1 - \text{CA} \rangle_\alpha \models \text{RFN}_{\Pi_3^1}(\Sigma_2^1-\text{AC}))$$

where e.g. $\forall n (\langle \Pi^1_1 - \text{CA} \rangle_n)$ is an axiom asserting that the relativized hyperjump may be iterated $n$ times for each integer $n$. We give a large number of applications of our general theorems to set theory, analysis and arithmetic; these results are for the most part known.

Next let us consider some model-theoretic applications. Theorem 1.13, due to S. Kripke, is interesting and very simple: there exists an easily definable subring $F$ of the ring of rudimentary (or primitive recursive, or recursive, etc.) functions such that for any non-principle ultrafilter $D$ on $\omega$, $F/D$ is a recursively saturated model of $\text{PA}$. Chapter V is concerned with non-$\omega$-models of first-order theories. (This is a rapidly expanding field, especially with regard to countable models of arithmetic, and undoubtedly many of the results contained herein are known to other workers in the field. I have tried to give complete references, and, for my own results, acknowledge any work independent of my own. But let me say that most of the results of V were inspired by, or are due to, J. Paris and L. Kirby.) We first note that many results concerning nonstandard models of arithmetic have nothing whatsoever to do with arithmetic, so we for the most part give our results in more generality; that is, they apply to any model with sufficient coding abilities and satisfying the appropriate Collection axioms. Theorem 5.3 gives necessary and sufficient conditions for the existence of certain initial segments which are models of coded theories, and indicators for the same. We use this to give necessary and sufficient conditions for the existence of initial segments which are models of a given complete theory. In 5.11 we give a description of the order-types of certain sets of elementary initial segments of a recursively saturated model of $\text{PA}$. In 5.9 we give an extension of a result of L. Kirby, K. McAloon and R. Murawski concerning indicators in models of arithmetic for models of analysis; the lemma 5.9 required is a common generalization of a theorem of D. Scott and H. Friedman and the generalization of the MacDowell-Specker Theorem (for countable models) by R.G. Phillips and H. Gaifman. It concerns finding extensions of a countable model $M$ of arithmetic which code a given countable class $\mathcal{X}$ of subsets of $M$; the main requirement is that $\langle M, \mathcal{X} \rangle$ satisfy $\text{WKL}$: every infinite binary tree has an infinite branch.

In Chapter VI we consider analogues for the results of V for $\omega$-models, or more generally, models of theories contained in countable fragments of the infinitary language $\mathcal{L}_{\omega_1\omega}$. We start by considering an extension of a result of H. Friedman [73]: there are no minimal models of analysis, where analysis is assumed to contain the full schema of $\text{AC}$. We present a proof of this, one which is essentially that of Friedman [73], but simpler and more general. The main idea is
that in a non-\(\beta\)-model we can construct trees which are well-founded inside the model but not in the real world; the desired substructure is then obtained from any infinite branch of such a tree. Then we shall modify this construction using the notion of fulfilment to obtain the following improvement: there are no minimal models of \(\Sigma^1_1\text{-BI}\). Indeed, in 6.8 we show that a sufficient condition for a model \(A\) of a given \(\Pi^1_1\) theory \(T\) not to be a minimal model of \(T\) is that the notion of a well-founded linear order is not \(\Sigma^1_1/A\), and this is also a necessary condition for \(A\) to be non-minimal in a certain strong sense.

We develop the ideas contained in the above proof into a theory of indicators in, for example, non-\(\beta\)-models of \(\Sigma^1_1\text{-AC}\) and non-well-founded models of KP for theories contained in countable fragments of \(L_{\omega_1^\omega}\). We obtain a striking parallel between the \(\omega\)- and non-\(\omega\)-model cases. Typical instances of our main theorem 6.6 for, say, the language of set theory are as follows. If \(T\) is an r.e. theory extending \(ZF^-\), then any non-\(\beta\)-model of \(T\) has, for each \(k \in \omega\), a \(k\)-elementary transitive substructure which is a model of \(T\). If \(T\) is an r.e. theory extending \(KP_1^\omega + \Pi^1_1\text{-Foundation}\), then any countable (or, more generally, locally countable) nonstandard model of \(T\) has a transitive substructure which is a model of \(T\), and moreover, we can choose the substructure to have certain saturation properties. Theorem 6.6 also provides model-theoretic proofs of results in Feferman [68].

Chapter VI is concluded with an analogue of 2.8. We show, for example, that if \(T\) is an r.e. theory in the language of analysis extending \(\Sigma^1_1\text{-AC}\) with \(k \geq 1\), then

\[
\{ \theta \in \Sigma^1_{k+1} : \theta \text{ is } \Pi^1_{k+1}\text{-conservative over } T \text{ with the } \omega\text{-rule} \}
\]

and, if \(k \geq 2\),

\[
\{ \theta \in \Pi^1_k : \theta \text{ is } \Sigma^1_k\text{-conservative over } T \text{ with the } \omega\text{-rule} \}
\]

are both \(\Pi^1_1\) in Kleene’s \(O\), and are complete for this class of sets. We also prove a lemma concerning the semi-representation of \(\Pi^1_1\) sets in r.e. theories \(T\) with the \(\omega\)-rule: for example, if \(T\) is not \(\Pi^1_1\)-sound, then any \(\Pi^1_1\) set may be semi-represented in \(T\) with the \(\omega\)-rule by a formula in \(\Delta^1_1(T)\).

The final chapter is an exposition of the Paris-Harrington result. This chapter is presented as an introduction for the general reader, and we give proofs of the truth and independence of the Paris-Harrington statement, and we also show, by fairly simple model-theoretic means, the sharp negative results established by the work of J. Ketonen, L. Kirby, G. Mills, J. Paris, and R. Solovay. We also prove an optimal result, namely that for \(c = 2\),

\[\text{PA} \nvdash \forall e \exists n ((e, n) \rightarrow (e + 1)^c)\]

(this has also been proved independently by others) and we give the equivalence of the Paris-Harrington statement with the \(\Sigma_1\text{-Uniform Reflection Principle of PA}\).

\ *

We shall use the following notation and abuses of notation: areas of logic where our abuses would lead to trouble are, while important, not of interest to us here. We shall often not distinguish between formulae, etc., and their codes, and it will often be implicitly assumed that some object \(x\) codes a finite sequence \(\langle (x)_0, (x)_1, \ldots, (x)_{|x|} \rangle\).\(^4\) \(\bar{x} \in y\) means that \((x)_i \in y\) for all \(i \leq |x|\); \(\bar{x} < y, \bar{x} \subseteq y\) are defined similarly.

\(^4\) Footnote added 2019: See page 7 for the notation \(|x|^+\). In the 1980 thesis, this was denoted \(\lambda x\).
We shall often let the same metavariable represent both formal variables and parameter variables; for example, we might say "if \( \exists x \theta x \) holds, choose a witness \( x \)". When we say, for example, that Collection is the schema

\[
\forall x \in a \exists y \theta \supseteq \exists b \forall x \in a \exists y \in b \theta,
\]

we mean that Collection is the set of universal closure of sentences of this form, where \( \theta \) ranges over all formulae of the language under consideration, and where suitable precautions are taken to avoid collision of variables. A theory is a set of sentences.

When there is no risk of confusion, the same symbol "\( \epsilon \)" will have three different uses: as a symbol of our formal language, as a binary relation of some model under consideration, and with its usual informal meaning. Likewise, we shall usually not need to distinguish between constant or function symbols and their interpretation in some model. When we write \( f \overline{x} \) we implicitly assume that the length of \( \overline{x} \) is equal to the arity of \( f \). If \( f \) is of arity \( n \), let \( f^{\prime\prime}A = f[A^n] \).

We shall be interested in models \( A = \langle A, e, \ldots \rangle \) of a weak set theory. \( \operatorname{wf}(A) \) is the well-founded part of \( A \); we shall often implicitly suppose that \( \operatorname{wf}(A) \) is equal to its transitive collapse. \( A \) is nonstandard if it contains linear orderings which are internally well-founded but are not so in the real world, and \( A \) is an \( \omega \)-model if it does not contain non-standard integers. We shall usually try to distinguish between "contains" and "includes". A subset \( B \) of \( \operatorname{wf}(A) \) is coded in \( A \) if there exists \( b \in A \) such that \( B = \{ a \in \operatorname{wf}(A) : A \models a \in b \} \). If there is no risk of confusion, the structure in which sentences, especially quantifier-free ones, are to be interpreted will only be implicitly mentioned; for example, we should usually write the previous equation as \( B = \{ a \in \operatorname{wf}(A) : a \in b \} \).

\( A \) is a model of overspill (\( \Sigma_k \)-overspill) if for each (\( \Sigma_k \)) formula \( \theta x \), possibly with parameters from \( A \), if \( \theta a \) holds in \( A \) for each standard ordinal \( a \) of \( A \), then \( \theta a \) holds of some nonstandard ordinal of \( A \), where here ordinal refers to any element of any internally well-founded linear ordering. A similar notion which is useful when dealing with weaker theories is this: \( A \) is recursively saturated (\( \Sigma(k) \)-recursively saturated) if for any recursive set of (\( \Sigma(k) \)) formulae, with only a finite number of free variables and perhaps a finite number of parameters from \( A \), if each finite subset is realized in \( A \), then the whole set is realized in \( A \). (\( \Sigma(k) \) is the closure of \( \Pi_{k-1} \) under conjunction, disjunction, and existential and bounded universal quantification; in the presence of \( \Sigma_k \)-Collection and the pairing axiom, each \( \Sigma(k) \) formula is equivalent to a \( \Sigma_1 \) formula.) A substructure \( B \) of \( A \) is an initial segment (and \( A \) is an end-extension of \( B \)) if \( x \in A \) and \( y \in B \) with \( x \in y \) implies \( x \in B \); and a sub-structure \( B \) of \( A \) is cofinal in \( A \) if for all \( x \in A \) there exist \( n \) and \( x_1, \ldots, x_n \in A \), \( y \in B \) such that \( x \in x_1 \in \ldots \in x_n \in y \). \( A \) is locally countable if \( \{ y \in A : y \in x \} \) is countable for all \( x \in A \).

* * *

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\(^5\) Robin Gandy, 1919-1995.
\(^6\) Alan Adamson, 1949-2012.
ready to lend an ear and offer suggestions; and I should like also to thank my fellow students at the Mathematical Institute for patiently putting up with my often foolish questions and comments. David Guaspari and Lawrence Kirby, among others, generously sent me preprints of their work. I should also like to thank all those who have kindly set aside an hour or so to talk to me. I have left my greatest debt till last: because of the constant, cheerful and uncomplaining support of my dearest wife Katherine, I have dedicated this thesis to her.

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1 The Main Definition

In this first chapter we define the notion of fulfilment, along with several of its variants. After a lemma giving some simple properties of this notion, we discuss its relationship to some of the early work in Mathematical Logic by Skolem, Herbrand, and Gödel. Then, using fulfilment, we give results on the completeness and finite axiomatizability of theories such as Peano’s arithmetic—results whose proofs require only very simple model-theoretic arguments. Our independent statements, however, suffer from the defect of being at best \( \Pi^0_2 \) whereas Gödel’s independent sentences are \( \Pi^0_1 \). However, by making use of the fixed-point theorem, we shall generate interesting independent \( \Pi^0_1 \) (but self-referential) statements of a model-theoretic nature.

We shall close the chapter with three miscellaneous results. The first is a model-theoretic proof (without self-reference) of Tarski’s Theorem: the theory of \( \mathbb{N} \) is not arithmetical. The second shows how we may give a feasible (without the use of the fixed-point theorem or the use of such notions as pairs of recursively inseparable sets) semi-representation of any r.e. set in an r.e. theory which is not necessarily \( \Sigma^0_1 \)-sound. Finally, we give a method of constructing models of various theories in the language of arithmetic as quotients of certain “constructively given” function rings.

Let \( \mathcal{L} \) be any finite first-order language. Let \((I, <)\) be any linearly ordered set with a minimal element, which will be denoted by 0, and is such that each element \( i \) of \( I \) (except the maximal element, if one exists) has an immediate successor, which will be denoted by \( i + 1 \).\(^7\) Let \( \sigma = (A_i)_{i \in I} \) be a family of new unary predicate symbols, let \(|\sigma|^\prec\) denote \( I \) less (if it exists) the maximal element of \( I \), and let \(|\sigma| = I \). For each formula \( \varphi \) of \( \mathcal{L} \) and for each \( i \) in \(|\sigma|\) define the (not necessarily first-order) formula \( \varphi_i^\sigma \) inductively as follows. (Here \( i \) and \( j \) are understood to range over \(|\sigma|\). When the context is clear, the superscript \( \sigma \) will be omitted.)

i. If \( \theta \) is atomic, let \( \theta_i = \theta \) and \((\neg \theta)_i = \neg \theta \).

ii. \((\exists x \theta)_i = \exists x \in A_{i+1} \theta_i \).

iii. \((\forall x \theta)_i = \forall j \geq i \forall x \in A_j \theta_j \), where \( j \) is a new variable.

iv. \((\neg \forall x \theta)_i = (\exists x \neg \theta)_i \); \((\neg \exists x \theta)_i = (\forall x \neg \theta)_i \).

v. Otherwise, \( \theta_i \) is obtained from \( \theta \) by replacing each positive instance of a Boolean atom \( \psi \) of \( \theta \) by \( \psi_i \) and each negative instance by \( \neg(\neg \psi)_i \).

For example, if \( \theta \) is quantifier-free, then \((\forall x \exists y \forall u \exists v \theta)^\sigma \) is just

\[
\forall i \geq 0 \forall x \in A_i \exists y \in A_{i+1} \forall j \geq i \forall u \in A_j \exists v \in A_{j+1} \theta_i ;
\]

and \((\varphi \supset \psi)_i^\sigma = \neg(\neg \varphi)_i^\sigma \supset \psi_i^\sigma \). \( \sigma \) is increasing if for all \( i \) in \(|\sigma|\), \( A_i \subseteq A_{i+1} \), and \( \sigma \) is closed if for each \( i \) in \(|\sigma|\) and for each function \( f \) of \( \mathcal{L} \), \( f''A_i \subseteq A_{i+1} \), and if each constant of \( \mathcal{L} \) is contained in \( A_0 \). Let \( A = \langle A, \ldots \rangle \) be a structure for \( \mathcal{L} \) and for each \( i \) in \( I \), let \( A_i \) be interpreted as a non-empty subset of \( A \). Let \( \varphi \) be any formula of \( \mathcal{L} \) and let \( \bar{a} \) be a valuation in \( A \) of the free variables of \( \varphi \). Say \( \sigma \) fulfils \( \varphi \) \( \bar{a} \), and write \( A \models (\varphi \bar{a})^\sigma \), if:

\[
A \models (\varphi \bar{a})^\sigma \text{ and } \sigma \text{ is increasing and closed.}
\]

---

\(^7\) Footnote added 2019: Here 0 and + are overloaded notations, and we should perhaps write \( 0_T \) and \( +_T \); for example, if \( I \) is the set of positive odd integers, then \( 0_T \) and \( +_T \) are interpreted as stated, and the first element of \( \sigma \) would be \( \sigma_0_{\sigma_T} \). We will also abuse the von Neumann ordinal notation: when we write \(|\sigma|^\prec = 3\), we mean \(|\sigma|^\prec = \{0, 1, 2\} \) (as an ordered set), and so \(|\sigma|\) must be \( \{0, 1, 2, 2+1\} \).
(The last two requirements ensure that \( \sigma \) fulfils \( \forall x \exists y (x = y) \) and \( \forall x \exists y (f x = y) \) for every function symbol \( f \).) We shall occasionally use the phrase \( |\sigma|^-\)-fulfil when we wish to indicate \( |\sigma|^--\). It will be convenient to allow a closed sequence of length one \( \langle A_0 \rangle \) to vacuously fulfil any statement.

The motivation behind this definition should be clear. For example, suppose \( \varphi \) is a sentence which is true in a structure \( A \). Then there exist many sequences of subsets of \( A \) which fulfil \( \varphi \); for example, the trivial sequence \( \langle A, A, A, \ldots \rangle \). A more useful construction—one which is due to Skolem—is as follows. Choose (using the axiom of choice) a set \( F \) of satisfaction functions for \( \varphi \): for example, if \( \varphi = \forall x \exists y \forall u \exists v \theta(x,y,u,v) \) where \( \theta \) is quantifier-free, choose \( f \) and \( g \) such that

\[
A \models \forall x \forall u \theta(x,f(x),u,g(x))
\]

and set \( F = \{ f, g \} \). Let \( A_0 \neq \emptyset \) contain the (interpretations of the) constants of \( \mathcal{L} \) and let

\[
A_{i+1} = \bigcup_{f \in \mathcal{L} \cup F} f''A_i
\]

for each \( i \) in \( \omega \). Then \( \langle A_i \rangle_{i<\omega} \) fulfils \( \varphi \), as does every finite subsequence.

In Chapter III it will be convenient to use a slight variant of the above definition: let \( i \) and \( j \) range over \( |\sigma|^-- \) rather than \( |\sigma|^- \), and change clause (ii) above to read

\[
(\exists x \theta)_i = \begin{cases} 
\exists x \in A_{i+1} \theta_{i+1}, & \text{if } i \in |\sigma|^- \\
\text{true,} & \text{otherwise.}
\end{cases}
\]

The motivation behind this definition is also simple. In the above example, choose \( F \) to be a set of so-called Skolem functions for the existential quantifiers of \( \varphi \); that is, if \( \varphi \) is as before, choose \( f \) and \( g \) such that

\[
A \models \forall x \forall u \exists v \theta(x,f(x),u,v)
\]

and

\[
A \models \forall x \forall y \forall u (\exists v \theta(x,y,u,v) \supset \theta(x,y,u,g(x,y,u))).
\]

Now if we define \( A_0, A_1, A_2, \ldots \), as before, but using these Skolem functions rather than the satisfaction functions, we obtain a sequence which fulfils \( \varphi \) according to this second notion of fulfilment.

It is clear that if \( \sigma \) fulfils a sentence \( \varphi \) according to the first definition, then it also fulfils \( \varphi \) according to the second. Furthermore, a moment’s thought shows that if \( \sigma = (\sigma_i)_{i \leq |\sigma|^-} \) fulfils \( \varphi \) according to the second definition, where \( |\sigma|^-- \leq \omega \), and if \( \varphi \) has \( k \) existential quantifiers, then the sequence

\[
(\sigma ik)_{i \leq |\sigma|^-}
\]

fulfils \( \varphi \) according to the first. Thus for our purposes the two notions are equivalent, and we shall always use the first except in Chapters III and VII.

There are three more variants of the notion of fulfilment which naturally occur. The definitions are listed here below for ease of reference, even though it would perhaps be less daunting to the reader, and clearer, if they were deferred until they were actually needed.

For the first variant, we do not start with a given fixed structure, but rather we consider pairs \( (\sigma,A) \) where \( (\sigma_i)_{i \leq |\sigma|} \) is an increasing sequence of sets of new constant symbols, and \( A \)
is a structure for $\mathcal{L}$ with domain $\bigcup_{i \in |\sigma|} \sigma_i$, and where $\sigma$ is closed under the functions of $\mathcal{L}$ as interpreted in $\mathcal{A}$. (We do not need the function symbols of $\mathcal{L}$ to be interpreted as total functions; it suffices that they be defined wherever needed.) In our notation we shall not mention the structure $\mathcal{A}$ explicitly. Say $\sigma$ *fulfils* a sentence of $\mathcal{L}$, and write $\varphi^{\sigma}$, if

$$\mathcal{A} \models \varphi^{\sigma}.$$ 

The *raison d’être* of the notion of *fulfilment* is this. If $\varphi$ is a sentence of any finite language $\mathcal{L}$, then the sentence

$$\text{for all } n \in \omega, \text{there exists } \sigma \text{ with } |\sigma| = n \text{ which *fulfils* } \varphi$$

may, by the usual coding techniques, be expressed by a $\Pi^0_1$ sentence of arithmetic (for we can easily estimate an upper bound on the cardinality of $\sigma_n$, that is, of the structure $\mathcal{A}$; the bound is in fact an $\mathcal{E}_3$ function of $n$ and the number of quantifiers in $\varphi$); whereas the sentence

$$\text{for all } n \in \omega, \text{there exists } \sigma \text{ with } |\sigma| = n \text{ which fulfils } \varphi$$

may not be expressed in arithmetic at all except when $\varphi$ itself is in the language of arithmetic, and even then it must in general be expressed by a $\Pi^0_2$ sentence.

For our next definition we consider a language $\mathcal{L}$ which contains (or, at least, some definitional extension of it contains) a type (or a unary predicate symbol) $\overline{0}$, constant symbols $0$ and $\overline{1}$, and function symbols $\overline{+}$ and $\overline{\times}$ whose domain is $\overline{\omega}$. A sequence $\sigma$ (or more precisely, an ordered pair $(\sigma, \mathcal{A})$) *half-fulfils* a sentence $\varphi$, and we write $\varphi^{\frac{1}{2}+\sigma}$, if: $\sigma = \langle \sigma_i \rangle_{i \in |\sigma|}$ is an increasing sequence of $\mathcal{L}$ sets which contain both constant symbols and integers; $\mathcal{A}$ is a structure with domain $\bigcup_{i \in |\sigma|} \sigma_i$ in which the interpretation of $\overline{\omega}$ consists of integers, and in which the interpretation of the symbols $\overline{0}$, $\overline{1}$, $\overline{+}$, and $\overline{\times}$ is the standard one; and

$$\mathcal{A} \models \varphi^{\sigma}.$$ 

Thus if $\varphi$ is a sentence in the language of, say, set theory or analysis, we may express

$$\text{for all } n \in \omega, \text{there exists } \sigma \text{ with } |\sigma| = n \text{ which } \frac{1}{2}+\text{fulfils } \varphi$$

by a $\Pi^0_2$ sentence of arithmetic. Note that we may also allow $\varphi$ to have free variables of integer type.

In our final variation of the notion of fulfilment, we consider structures $\mathcal{A} = \langle A, \ldots \rangle$ for the language of either set theory or arithmetic. The notion of *i-fulfilment* (for initial) is like that of fulfilment, except we require of our sequences $\sigma = \langle \sigma_i \rangle_{i \in |\sigma|}$, in the case of arithmetic, that each $\sigma_i$ be an initial segment of $A$ (i.e. $x < y \in \sigma_i$ implies $x \in \sigma_i$), and in the case of set theory, that for all $i$ in $|\sigma|$, $x \in y \in \sigma_i$ implies $x \in \sigma_{i+1}$. In the case of arithmetic, (the internalized version of) i-fulfilment gives a particularly elegant formulation of fulfilment, for rather than considering sequences of *sets* of integers we can just consider sequences of *integers*, and say that $\sigma = \langle \sigma_i \rangle_{i \leq n}$ i-fulfils e.g. $\forall x \exists y \theta$ iff

$$\forall i < n \forall x < \sigma_i \exists y < \sigma_{i+1} \theta.$$ 

We shall use this observation to simplify the statements or proofs of two or three results. One could combine the notions of i-fulfilment and $\frac{1}{2}+\text{fulfilment}$, but we shall have little need for this.

Next we shall consider a few simple properties of the notion of fulfilment. The first two parts of the following lemma are due to Kripke.

**1.1 Lemma** Let $\mathcal{A} = \langle A, \ldots \rangle$ be any structure for $\mathcal{L}$, let $\varphi$ be any formula of $\mathcal{L}$ (which may contain parameters from $A$) and let $\sigma = \langle A_i \rangle_{i \in |\sigma|}$ be a sequence of subsets of $A$. 
i. If $|\sigma|$ is unbounded and if $\sigma$ fulfils $\varphi$, then $\varphi$ is true in
\[ B = A \upharpoonright \bigcup_{i \in |\sigma|} A_i \]

ii. Let $J \subseteq |\sigma|$ have a minimal element and be such that each element of $J$ (except the maximal element, if one exists) has an immediate successor in $J$, and let $\sigma \upharpoonright J = \langle A_i \rangle_{i \in J}$. If $\sigma$ fulfills $\varphi$, then $\sigma \upharpoonright J$ fulfills $\varphi$.

iii. If $|\sigma| = \omega$, if $\langle A_i \rangle_{i \leq n}$ fulfills $\varphi$ for each $n \in \omega$, and if each $A_i$ is finite, then $\langle A_i \rangle_{i < \omega}$ fulfills $\varphi$.

Furthermore, i, ii, and iii also hold for $\ast, \frac{1}{2} \ast, \text{ and } i\text{-fulfilment.}$

Proof: (i) Let $\psi$ be obtained from $\varphi$ by first eliminating all occurrences of "⊃" by use of "¬" and "∨", and then "pushing" all negations inside. From the definition of fulfilment, $\psi^\sigma$ iff $\varphi^\sigma$. Suppose $\varphi^\sigma$ and so $\sigma^\sigma$.

Then $\chi$ holds in $B$, where $\chi$ is obtained from $\psi^\sigma$ by removing the bounds on the existential quantifiers. Working from the innermost quantifiers of $\chi$ outwards, replace each quantifier pair of the form "∀ $j \geq i \forall x \in A_j$" by "∀ $x$": each successive alteration maintains the truth in $B$. Hence $\psi, \varphi$, hold in $B$.

(ii) Again we may suppose that $\varphi$ is in negation normal form. The result is now obvious.

(iii) The proof is by induction on the length of $\varphi$, which we can assume is in negation normal form. There is nothing to prove if $\varphi$ is quantifier-free, and the conjunction and disjunction steps are trivial. Let $\sigma \upharpoonright (n + 1) = \langle A_i \rangle_{i \leq n}$. For the induction to work, we need to show a bit more: for all $k \geq 0$ and all $\theta$ with any evaluation of its free variables, $\theta^\sigma_k$ iff $\forall n > k \theta^\sigma_k[n+1]$.

For the universal quantifier we have
\[
\forall n > k \ (\forall x \varphi)^\sigma_k[n+1] \\
\text{iff} \quad \forall n > k \ \forall i \geq k \ \forall x \in A_i \ \varphi^\sigma_i[n+1], \text{ by definition, with } i \text{ new, } i < n \\
\text{iff} \quad \forall i \geq k \ \forall x \in A_i \ \forall n > i \ \varphi^\sigma_i[n+1] \\
\text{iff} \quad \forall i \geq k \ \forall x \in A_i \ \varphi^\sigma_i, \text{ by our induction hypothesis} \\
\text{iff} \quad (\forall x \varphi)^\sigma_k, \text{ by definition.}
\]

For the existential quantifier,
\[
\forall n > k \ (\exists x \varphi)^\sigma_k[n+1] \\
\text{iff} \quad \forall n > k \ \exists x \in A_{k+1} \ \varphi^\sigma_k[n+1], \text{ by definition}
\]
and because $A_{k+1}$ is finite,\footnote{Footnote added 2019: The finiteness is generally needed. For example, consider the sentence "the odd numbers are bounded" in the language $L = \{\text{even}, <\}$ given by $\exists x \forall y \{x < y \supset \text{even}(y)\}$. Let $A_0$ be the even natural numbers. Add the first odd number to $A_1$, the second to $A_2$, etc. Each finite subsequence fulfills the sentence, but the whole does not.} by the pigeon-hole principle this holds
\[
\text{iff} \quad \exists x \in A_{k+1} \text{ such that for infinitely many } n > k, \varphi^\sigma_k[n+1] \\
\text{iff} \quad \exists x \in A_{k+1} \forall n > k \varphi^\sigma_k[n+1], \text{ by (ii)} \\
\text{iff} \quad \exists x \in A_{k+1} \varphi^\sigma_k, \text{ by the induction hypothesis} \\
\text{iff} \quad (\exists x \varphi)^\sigma_k, \text{ by definition.} \]

Now that the above machinery has been set up, we can give a succinct account of some of the earliest work in Mathematical Logic. As already mentioned, Skolem in [20], in his proof of
Löwenheim’s Theorem, observed that if a sentence $\theta$ is true in a structure $\mathcal{A}$, then by the axiom of choice we can find a sequence $\langle \sigma_i \rangle_{i \in \omega}$ of finite subsets of $\mathcal{A}$ which fulfills $\theta$; $\mathcal{A} \models \bigcup_i \sigma_i$ will then be a countable substructure of $\mathcal{A}$ which is a model of $\theta$.

In [22], [28], and [29], Skolem saw that this shows

$$\text{if } \theta \text{ is satisfiable, then } \forall n \exists \sigma (|\sigma| = n \land \theta^{\sigma^*}),$$

(1)

and, moreover, the converse is also true, by a simple argument using König’s Infinity Lemma. Skolem pointed out (in [28]) that this gives a proof procedure which, as we see, is cut-free and has the subformula property. (Aside: Kripke has pointed out an elegant, nonstandard argument for the converse of (1). Suppose $\forall n \exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$. Then in any proper elementary extension of $\mathbb{N}$ we can find (the code of) a sequence $\sigma$ of nonstandard length which $*$fulfils $\theta$. Then by 1.1.i and ii, $\theta$ is true in the structure determined by $\sigma$ with domain $\bigcup_n \text{standard } \sigma_n$.)

In [30], Herbrand gave an (incomplete) proof of

$$\vdash \neg \theta \implies \neg \forall n \exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$$

(2)

and its converse; moreover, his proof is effective in that we may obtain primitive recursively a witness $n$ from any proof of $\neg \theta$, and conversely, a (bound on the) proof from any witness $n$.

From the converses of (1) and (2) we may immediately obtain, as many people have remarked, the Completeness Theorem of Gödel [30]. Gödel’s proof was, very roughly, as follows. He proved the converse of (1) as Skolem did, so let us consider the converse of (2). First note that for each formula $\theta$ of $\mathcal{L}$ and each integer $n$, we may express the notion

$$\exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$$

(3)

in the language of $\mathcal{L}$ by writing out the elements of the terms in explicitly. For example, if $\theta = \forall x \exists y \psi xy$, with $\psi$ quantifier-free, and if the language $\mathcal{L}$ has no function symbols or constants, we may write (3) as

$$\exists x_0, x_1, \ldots, x_n \bigwedge_{i < n} \psi x_i x_{i+1},$$

With this understanding, we can show that, in any axiomatization of the predicate calculus which we may wish to consider, for each $n \in \omega$,

$$\vdash \theta \supset \exists \sigma (|\sigma| = \bar{n} \land \theta^{\sigma^*}).$$

(4)

So now to show the converse of (2), suppose for some $n$, $\neg \exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$. Then $\vdash \neg \exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$. But this is essentially a propositional formula, and as the propositional calculus is complete, we have $\vdash \neg \exists \sigma (|\sigma| = n \land \theta^{\sigma^*})$. Hence by (4), $\vdash \neg \theta$. Note that this proof is also effective.

This completes our discussion of early logic. We shall next consider various incompleteness results.

To obtain these results, we shall need to be able to formalize the notion of fulfilment. We wish our results to be applicable to many different areas—arithmetic, analysis, and set theory—and so to state them in the necessary generality, we shall usually consider some fixed but arbitrary r.e. theory $S$ of $\mathcal{L}$ in which a notion of membership $\epsilon$ may be defined so that $S$ proves

Pair: $\forall x, y \exists z (x \in z \land y \in z \land \forall u \in z (u = x \lor u = y))$, and

Union: $\forall x \exists y (\forall z \in x \forall u \in z \in y \land \forall u \in y \exists z \in x \in z)$. 

(For example, in arithmetic, say \( x \in y \) if
\[
\exists s, t \left( t < 2^x \land y = s2^{x+1} + 2^x + t \right),
\]
and in analysis, extend the usual notion of membership \( \in \subseteq \omega \times \mathcal{P}_\omega \) by the class
\[
\{ (X, Y) : \exists n \forall m (m \in X \equiv (n, m) \in Y) \}
\]
where \( \lambda \) (natural number) that some notion of membership may be defined in \( S \), and that \( S \) proves \( \Delta_1(S) \)-Induction, where as usual we have
\[
\Delta_0 = \Pi_0 = \Sigma_0 = \text{closure of atomic formulae under } \land, \lor, -, \forall x \in y, \exists x \in y
\]
\[
\Sigma_{k+1} = \text{closure of } \Pi_k \text{ under } \exists x \text{ and equivalences in the PC}
\]
\[
\Pi_{k+1} = \text{dual of } \Sigma_{k+1}
\]
\[
\Delta_{k+1}(T) = \text{formulae provably equivalent in } T \text{ to a } \Sigma_{k+1} \text{ and a } \Pi_{k+1} \text{ formula .}
\]
Let \( \bar{0} \) denote the first natural number as represented in \( S \), \( \bar{T} \) the second, etc. Finally, we shall suppose that \( S \) proves that some \( \Sigma_1 \) formula is a satisfaction predicate for \( \Delta_0 \) formulae. This latter assumption may often be omitted, but it makes our exposition somewhat easier. In particular, it enables us to treat various restricted schemata as single sentences, which saves us the trouble of picking the appropriate instances. Also, for certain proof-theoretic results we shall need that \( S \) proves the fourth Grzegorczyk function \( E_4 \) is total.

Among the schemata which we will find useful to consider are:

Induction: \( \theta \bar{0} \land \forall n (\theta n \supset \theta (n + 1)) \supset \forall n \theta n \),

Foundation: \( \exists x \theta x \supset \exists x (\theta x \land \forall y \in x - \theta y) \),

Separation: \( \exists b \forall x \in a (x \in b \equiv \theta) \),

Collection: \( \forall \bar{x} \in a \exists \bar{y} \theta \supset \exists b \forall \bar{x} \in a \exists \bar{y} \in b \theta \), and

Bounding: \( \exists b \forall x \in a (\exists y \theta \supset \exists y \in b \theta) \)

where \( b \) does not occur free in \( \theta \). \( \Lambda \)-Induction, etc., is the schema of Induction with \( \theta \) restricted to \( \Lambda \). Infinity is the axiom asserting that the integers form a set. The variables \( n \) and \( m \) in will always be restricted to the integers.

One base theory we shall occasionally consider is \( PA_{\bar{\omega}} \), that is, arithmetic with axioms for addition, multiplication, exponentiation and membership, and with induction limited to \( \Delta_0 \) formulae.

The relation
\[
(\varphi \bar{a})^\gamma
\]
may be expressed as a \( \Delta_1(S) \) predicate of \( \sigma, i \), (the code of the sequence) \( \bar{a} \), and (the Gödel number of) \( \varphi \), and henceforth this notation will implicitly refer to some such suitable representation. If \( \sigma \) satisfies the extra requirement for i-fulfilment, we write \( i(\sigma) \). For a formula \( T(x) \) (which, as the notation suggests, is to be considered as representing a class of sentences) let \( (T)^{\sigma} \) denote
\[
\forall [\theta \bar{x}] \in |\sigma|^{-1} \left( T([\vartheta \bar{x}]) \supset \theta^{\sigma} \right).
\]
For a formula \( \theta \bar{x} \), let \( (\text{Tr} \theta \bar{x})^{\sigma} \) denote
\[
\forall i \in |\sigma|^{-1} \forall \bar{x} \in \sigma_i \left( \theta \bar{x} \supset (\theta \bar{x})^{i \gamma} \right),
\]
Footnote added 2019: To clarify, for a formula $T(x)$ in the language of $S$, we denote $T = \{x : T(x)\}$, and let
\[
T^\sigma := \forall \theta \in T \cap |\sigma|^{-} \theta^\sigma
\]
\[
\text{Tr} T^\sigma := \forall i \in |\sigma|^{-} \forall \theta \in T \cap \sigma_i (\theta \in T_i)^\sigma.
\]
The notation “Tr” in the second phrase is perhaps unfortunate: it has little to do with truth, and merely ensures that the notion of being an axiom of $T$ persists in the submodels we create.
This contradicts the theorem. \(\square\)

**1.5 Corollary (Kripke)** Let \(T\) be any consistent theory extending \(S + \Delta_0\text{-Foundation}\) which is semi-representable in \(T\). Then

\[
\forall n \exists \sigma (|\sigma|^\ominus = n \wedge n \in \sigma_0 \wedge \text{Tr} \Delta_0^{\sigma} \wedge \text{Tr} T^{\sigma} \wedge T^{\sigma})
\]

is not provable in \(T\), where “\(T\)” in this sentence is understood to be any formula which semi-represents \(T\) in \(T\). In particular, there is a true \(\Pi_1^0\) sentence not provable in \(PA\), and a true \(\Sigma_1\) sentence not provable in \(ZFC\) (assuming \(ZFC\) has a standard model). \(\square\)

This corollary may either be proved by the same method or, if “\(T\)” is \(\Pi_1\), be derived from 1.3; we omit the proof. (See, however, 1.10 below.)

Corollary 1.5 is of course weaker the Gödel’s First Incompleteness Theorem \([31]\) which gives (true) \(\Pi_1^0\) sentences which are not provable in \(PA\) and \(ZF\); furthermore, 1.5 does not apply to theories without the schema of \(\Delta_0\text{-Foundation}\). But we can improve our results as follows. By use of the notion of \(\downarrow\) fulfilment, the proofs of 1.3 and 1.5 yield \(\Pi_1^0\) sentences which independent of, say \(A_2\) and \(ZF\), respectively, as stated in 1.6 below, and we also have that \(A_2\) is not finitely axiomatizable. By making use of the fixed-point theorem, however, we may obtain independent \(\Pi_1^0\) sentences as in 1.8 below.

**1.6 Theorem** If \(\varphi\) is a sentence consistent with \(S\) then

\[
\varphi \supset \forall n \exists \sigma (|\sigma|^\ominus = n \wedge n \in \sigma_0 \wedge \text{Tr} \Delta_0^{\sigma} \wedge \varphi^{\downarrow^0\sigma})
\]

is not a theorem of the predicate calculus, and so no consistent extension of \(S + \text{Induction}\) is finitely axiomatizable. (Here it is immaterial whether we write \(\text{Tr} \Delta_0^{\sigma}\) or \(\text{Tr} \Delta_0^{\downarrow^0\sigma}\).) And if \(T\) is a consistent theory extending \(S\) which is semi-representable in (some consistent extension of) \(T\), then

\[
\forall n \exists \sigma (|\sigma|^\ominus = n \wedge n \in \sigma_0 \wedge \text{Tr} \Delta_0^{\sigma} \wedge \text{Tr} T^{\sigma} \wedge T^{\sigma})
\]

is not provable in \(T\), where “\(T\)” in this sentence is understood to be the representation of \(T\). \(\square\)

The proof of 1.6 is almost word-for-word the same as those of 1.3 and 1.4 and is left to the reader.

We have exhibited sentences which are not provable in various theories \(T\), but without some extra assumptions on \(T\) their negations may very well be provable. Let us consider, for example, an r.e. theory \(T\) in the language of arithmetic. If \(T\) is \(\Sigma_1\) sound, the sentence of 1.5 is a true \(\Pi_1^0\) not provable in \(T\), and so if \(T\) is also \(\Sigma_2^0\) sound, its negation is not provable. By use of the fixed-point theorem, however, we may obtain independent sentences without this soundness hypothesis.

**1.7 Theorem (Rosser; this proof, Kripke)** Suppose \(T\) extends \(S\) and that \(T\) is binumerated in (some consistent extension of) \(T\). Then \(T\) is not complete.

**Proof:** Choose \(\theta\) so that \(T\) proves:

\[
\theta \equiv \forall n \left( \exists \sigma (|\sigma|^\ominus = n \wedge T^{\sigma} \wedge \theta^{\sigma}) \supset \exists \sigma (|\sigma|^\ominus = n \wedge T^{\sigma} \wedge \neg \theta^{\sigma}) \right),
\]

where “\(T\)” in this sentence is to be understood as above. We claim that \(\theta\) is neither provable nor refutable in \(T\). Let \(A\) be any model of \(T\) (plus the theory \(\{ T(\theta^\theta) : \theta \in T \} \)).

Suppose \(\theta\) holds in \(A\). We can assume that overspill holds in \(A\), for otherwise we just consider some recursively saturated elementary extension of \(A\). For each \(n\) in \(\omega\), \(\exists \sigma (|\sigma|^\ominus = n \wedge T^{\sigma} \wedge \theta^{\sigma})\) holds in \(A\) and so it must hold for some nonstandard \(n\) in \(A\). Then \(\exists \sigma (|\sigma|^\ominus = n \wedge T^{\sigma} \wedge \neg \theta^{\sigma})\)
also holds for this \( n \). Let \( \sigma \) be any witness to this and let \( \mathcal{B} = \mathcal{A} \upharpoonright \bigcup_{i \in \omega} \sigma_i \). Then \( \mathcal{B} \) is a model of \( \mathcal{T} \) and \( \neg \theta \).

Suppose \( \neg \theta \) holds in \( \mathcal{A} \). Let \( n \) be any witness to \( \neg \theta \); \( n \) is necessarily nonstandard. Then \( \exists \sigma (|\sigma| = n \land \mathcal{T}^{\sigma} \land \theta^{\sigma}) \). Let \( \sigma \) be any witness to this and let \( \mathcal{B} = \mathcal{A} \upharpoonright \bigcup_{i \in \omega} \sigma_i \). Then \( \mathcal{B} \) is a model of \( \mathcal{T} \) and \( \theta \).

If \( \mathcal{T} \) is r.e., the above independent sentence is (equivalent to one which is) \( \Pi_2 \). By using \( \frac{1}{2} \) fulfilment instead of fulfilment, we may obtain an independent \( \Pi_2^0 \) sentence. However, with \( \frac{1}{2} \) fulfilment we can obtain an independent \( \Pi_1^0 \) sentence, namely:

**1.8 Corollary** (Kripke)\(^{11}\) If \( \mathcal{T} \) is a consistent r.e. theory extending \( \mathcal{S} \) and if

\[
\forall n \exists \sigma \left( |\sigma| = n \land \mathcal{T}^{\sigma} \land \theta^{\sigma} \right)
\]

then \( \theta \) is (equivalent in \( \mathcal{T} \) to) a true \( \Pi_1^0 \) sentence not provable in \( \mathcal{T} \). \( \Box \)

The proof is identical to that of 1.7, and is omitted. We remark that these \( \theta \)'s are not, however, very novel independent sentences: by corollary 2.5 below, they are provably equivalent in \( \mathcal{S} \) to a Rosser sentence and to \( \text{Con}(\mathcal{T}) \), respectively.

**1.9 Corollary** If \( \mathcal{T} \) is, say, an r.e. extension of \( \mathcal{S} \) and if \( k \geq 1 \), there is a \( \Pi_{k+1} \) sentence \( \psi \) which is \( \Sigma_{k+1} \)-conservative over \( \mathcal{T} \), and whose negation is \( \Pi_{k+1} \)-conservative over \( \mathcal{T} \); namely, choose any \( \theta \) for which \( \mathcal{T} \) proves

\[
\exists \sigma \left( |\sigma| = n \land \mathcal{T}^{\sigma} \land \theta^{\sigma} \right) \supset \forall x \exists \sigma \left( |\sigma| = n \land x \in \sigma_0 \land \mathcal{T}^{\sigma} \land \mathcal{Tr} \Sigma_{k-1}^{\sigma} \land \mathcal{Tr} \Delta_{0}^{\sigma} \right),
\]

and let \( \psi \) be the right-hand-side of this equivalence. \( \Box \)

For the definitions of the concepts involved here, and for an indication of the proof, see Chapter II, pages 25ff.

For the remainder of the chapter we shall eschew the use of the fixed-point theorem. The next theorem may also be obtained as a corollary of 1.5.

**1.10 Theorem** (Tarski)\(^{12}\) The theory of \( \mathbb{N} \) is not arithmetical.

**Proof:** Suppose an arithmetical formula \( \mathcal{T}(x) \) represents the theory of \( \mathbb{N} \) in \( \mathbb{N} \), and let \( \theta(n, \sigma) \) be

\[
|\sigma| = n \land n \in \sigma_0 \land \mathcal{T}^{\sigma} \land (\mathcal{Tr} \mathcal{T})^{\sigma} \land \mathcal{Tr} \Delta_0^{\sigma}.
\]

Then

\[
\mathbb{N} \models \forall n \exists \sigma \theta(n, \sigma).
\]

Let \( \mathcal{A} \) be a proper elementary extension of \( \mathbb{N} \). Choose a nonstandard \( n \) in \( \mathcal{A} \) and let \( \sigma' \) in \( \mathcal{A} \) be the least witness to \( \exists \sigma \theta(n, \sigma) \). Then \( \mathcal{B} = \mathcal{A} \upharpoonright \bigcup_{i \in \omega} \sigma'_i \) is also a model of the theory of \( \mathbb{N} \) which contains \( n \). The formula \( \mathcal{T} \) is absolute between \( \mathcal{A} \) and \( \mathcal{B} \), and hence so is \( \theta \). But by our choice of \( \sigma' \), there is no witness in \( \mathcal{B} \) for \( \exists \sigma \theta(n, \sigma) \), which contradicts (6).

---

\(^{11}\) This corollary, along with the notion of \( \frac{1}{2} \) fulfilment, was also discovered independently by the author.

\(^{12}\) Kripke has also obtained a similar proof.
By using the notion of fulfilment, the above proof immediately yields:

1.11 Corollary If $A$ is any $\omega$-model of $S$, the theory of $A$ is not definable over $A$ by any parameter-free formula of $\mathcal{L}$. \qed

D. Scott [62] was probably the first to use nonstandard models to prove Tarski's Theorem. His proof, which is related to arguments of Feferman and Tennenbaum, is as follows. By Henkin's proof of the Completeness Theorem, if $\text{Th}(\mathbb{N})$ is arithmetical there exists a non-standard arithmetical model $A = \langle \omega, 0, 1, \oplus, \otimes \rangle$ of $\text{Th}(\mathbb{N})$. Let

$$\delta(n) = 1 \oplus 1 \oplus \cdots \oplus 1 \quad (n \text{ times});$$

this is primitive recursive in the function $\oplus$. Choose a formula $\psi$ such that $\psi(n)$ holds in $\mathbb{N}$ iff $A \models \delta(n) \not\in n$. Because $A$ and $\mathbb{N}$ are elementarily equivalent, $\mathbb{N} \models \psi(n)$ iff $A \models \psi(\overline{n})$. Let $m$ be a (nonstandard) element of $A$ coding the set $\{ n : A \models \psi(n) \}$. Then

$$A \models \overline{n} \in m \; \text{iff} \; A \models \psi(\overline{n}) \; \text{iff} \; \mathbb{N} \models \psi(n) \; \text{iff} \; A \models \overline{n} \not\in n.$$ 

The substitution of $m$ for $n$ gives a contradiction. This proof may also be generalized to give 1.11 above.

1.12 Theorem Let $T \supseteq S$ be any consistent r.e. theory and let $\{ n : \mathbb{N} \models \exists m \chi mn \}$ be any r.e. set, where $\chi$ is $\Delta^0_0$. Then we can feasibly semi-represent this set in $T$ by the $\Sigma^0_1$ formula

$$\psi = \exists m \exists \sigma \left( |\sigma| = m \land T^{\downarrow_{\sigma^0}} \land \text{Tr} \Delta^0_0 \sigma^0 \land \chi mn \right),$$

where “$T$” here is understood to be $\Delta^0_0$ formula which gives an axiomatization of $T$. \qed

The proof contains no new ideas, and is left to the reader. I am not sure of the correct notion of feasible; we have that

the length of $\psi = \text{the length of } \chi + \text{ a constant independent of } \chi$;

whereas a result of Parikh [71] seems to suggest that any semi-representation obtained in the usual manner via the fixed-point theorem would not have this property. An extension of 1.12, giving a $\psi$ in $\Delta^0_0(T)$ for non-$\Sigma^0_1$-sound $T$ is contained in the proof of 2.8 in the next chapter.

Our next result complements the discussion in Scott [61].

1.13 Theorem (Kripke, Kochen, Friedman) Let $G$ be any collection of functions from $\omega$ to $\omega$ containing an unbounded non-decreasing function and which is closed under composition and bounded recursion. Let $T$ be any $\Sigma^0_1$-sound set of sentences extending $\text{PA}$ whose characteristic function is in $G$. There exists a function $g(n, x)$ in $G$ such that if $F$ is any collection of functions including $G$ and which is closed under composition and bounded recursion and if $D$ is any non-principal ultrafilter on $\omega$, then

$$B = \{ f \in F : f \text{ unary and } \exists n \forall x (f x < g(n x)) \} / D$$

is a (recursively saturated) model of $T$. 
Sketch of proof: Let $h(x)$ be an unbounded, non-decreasing function in $G$. Define:

$$m(x) = \begin{cases} \max \{ m < h x : \exists \sigma < h x (|\sigma| = m \wedge i \sigma \wedge T^\sigma) \}, & \text{if this exists,} \\ 0, & \text{otherwise;} \end{cases}$$

$$\sigma(x) = \begin{cases} \min \{ \sigma < h x : |\sigma| = m x \wedge i \sigma \wedge T^\sigma \}, & \text{if } m x \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$g(n, x) = \begin{cases} (\sigma x)_n, & \text{if } n \leq m x, \\ 0, & \text{otherwise,} \end{cases}$$

where for this definition we are using the notion of i-fulfilment as described on page 9. For any non-principal ultrafilter $D$, consider the structure $A = \{ f \in F : f \text{ unary} \}/D$. Then $\sigma/D$ is (the code of) a sequence in $A$ of nonstandard length $m/D$ which i-fulfils each standard sentence of $T$. Hence $\bigcup_{n \in \omega} \{ x \in A : x < (\sigma/D)_n \}$ is a (recursively saturated) model of $T$. But this is exactly the structure $B$ in question.

The above result is essentially due to Kripke. From the embedding technique of Friedman [73], one may obtain:

**Theorem** If $A$ is a model of $\Delta^0_0$-overspill and $T$ is a $\Sigma^0_1$-sound set of sentences extending PA coded in $A$, then there exists a recursively saturated initial segment $B$ of $A$ which is a model of $T$.

However, in an unpublished typescript (of which Kripke was not aware), Friedman [7x] shows a result which to my mind is much more interesting: one can canonically define the initial segment. This, together with the observation that we may take $A$ to be $F/D$, will yield Theorem 1.13.
II Herbrand’s Theorem and Reflection Principles

In this second chapter we begin a more detailed study of the notion of fulfilment. Using a combinatorial result from the next chapter, we use fulfilment to derive a version of Herbrand’s Theorem. As a corollary, we can show that the notion of fulfilment is fairly stable: for example, if

\[ \vdash \theta \equiv \varphi \]

then in any structure \( A \), for all \( n \) there exists a sequence of subsets of \( A \) which \( n \)-fulfils \( \theta \) if and only if for all \( n \) there exists such a sequence which \( n \)-fulfils \( \varphi \); and moreover, there exists a primitive recursive function \( g(n) \) such that there is a constructive method of obtaining a sequence \( n \)-fulfilling \( \theta \) from one \( g(n) \)-fulfilling \( \varphi \). As another corollary, we obtain the Reflexiveness Theorem for theories such as \( \text{PA} \). We go on to compare a number of minor variants of schemata involving fulfilment with various proof-theoretic reflection principles, and then we give two proofs of the main theorem of Kreisel and Levy [68]. We conclude the chapter by solving a problem of Guaspari and Solovay concerning the complexity of

\[ \{ \theta \in \Lambda_1 : \theta \text{ is } \Lambda_2 \text{-conservative over } T \} \]

In this chapter \( \Gamma \) will always denote a finite set of formulae and we shall say that a sequence \( \sigma \) fulfils \( \Gamma \lor \neg \Gamma \), and write \( (\Gamma \lor \neg \Gamma)\sigma \), if \( \sigma \) fulfils \( \forall \overline{x_\theta} (\theta \lor \neg \theta) \) for each \( \theta \) in \( \Gamma \), where \( \overline{x_\theta} \) is a list of the free variables of \( \theta \). Let \( S \) be as on page 11; the language of \( T \) here can be unrelated to that of \( S \).

The following is the main lemma of this chapter.

2.1 Lemma If a sentence \( \varphi \) is provable in a universal theory \( T \), then there exists \( n \) and a finite set of formulae \( \Gamma \), depending on the proof, such that in any model of \( T \) there exists no sequence which \( n \)-fulfils \( \neg \varphi \) and \( \Gamma \lor \neg \Gamma \).

If we ignore the reference to \( \Gamma \) (as we shall be able to do using 3.6 below) this is just a version of Herbrand’s Theorem.

First we need a simple lemma. Define the rank of a formula as follows. \( \theta \) is of rank 0 if it is quantifier-free, \( \theta \) is of rank \( k \) if it is a Boolean combination of formulae of rank at most \( k \), and \( Qx\theta \) is of rank \( k + 1 \) if \( \theta \) is of rank \( k \).

2.2 Lemma If \( \theta \) is of rank \( k \), then \( \theta \land \neg \theta \) is not \( k + 1 \)-fulfillable in any structure and for any valuation of its free variables.

The proof of 2.2 is an easy induction on the complexity of \( \theta \), but because such a proof obscures the reason why 2.2 is true, we shall consider instead a simple example and leave the complete proof to the reader. Let \( \theta = \forall x \exists y \forall u \exists v \psi \), where \( \psi \) is quantifier-free, and let \( \sigma = \langle A_0, A_1, \ldots, A_5 \rangle \).

Then \( \theta^\sigma \) implies

\[ \forall x \in A_1 \exists y \in A_2 \forall u \in A_3 \exists v \in A_4 \psi \]

while \( (\neg \theta)^\sigma \) implies the negation. Hence not \( (\theta \land \neg \theta)^\sigma \).

Proof of 2.1: We use induction on the length of proof, and we shall specify what \( n \) and \( \Gamma \) should be as we carry out our induction. First we need to select some formal first-order system: for example, consider the following Hilbert-style system.
Axiom schemata:

1. All tautologies.
2. All equality axioms.
3. All formulae of either the forms

\[(\forall x \theta(x)) \supset \theta t, \quad \theta t \supset (\exists x \theta(x))\]

where \(t\) is any term free for \(x\) in \(\theta\).

Rules of Inference:

1. Modus Ponens: 
   \[
   \frac{\theta \quad \theta \supset \psi}{\psi}
   \]

2. Generalization: 
   \[
   \frac{\theta \supset \psi x}{\exists x \psi x \supset \theta} \quad \text{where } x \text{ is not free in } \theta.
   \]

This system has the advantage that the only difficult step (i.e., the only step involving \(\Gamma\)) in our induction corresponds to Modus Ponens. (This is not to say that using, for example, a cut-free sequent calculus would be any easier; in that case it would be the \(\exists\)-introduction rule which would be difficult.)

Let \(T\) be any set of universal sentences. By induction on the length of proof, we shall show that for any formula \(\theta\) there exists \(n\) and \(\Gamma\) such that \(\Gamma \lor \neg \Gamma\) plus the negation of the universal closure of \(\theta\) is not \(n\)-fulfiable in any structure for \(T\).

If \(\theta\) is an axiom of \(T\), then \(\neg(\neg \theta)^\sigma\) for any \(\sigma\) because \(\theta\) is universal; likewise for the equality axioms.

Let \(\theta\) be a tautology. As fulfilment is preserved under truth preserving Boolean transformations, we may consider the normal form of \(\neg \theta\) consisting of a disjunction of conjuncts. Then each of these conjuncts must contain both \(\psi\) and \(\neg \psi\) for some formula \(\psi\). Let \(n\) be greater than the ranks of all such \(\psi\). By 2.2, the existential closure of \(\theta\) is not \(n\)-fulfiable.

Next consider the axiom \(\theta t \supset \exists x \theta\). Let the height of a constant or variable be 0, and for each function symbol \(f\), let the height of \(f\bar{u}\) be 1 plus the maximum of the heights of its arguments. Let \(\bar{v}\) be a list of the free variables of \(\theta t \supset \exists x \theta\) and let \(\sigma = \langle A_0, \ldots, A_n \rangle\). Unravelling the definition, \(\neg(\neg \forall \bar{v} (\theta t \supset \exists x \theta))^{\sigma}\) if \(\forall \bar{v} \in A_1 ((\theta t)_0 \supset \exists i < n \exists x \in A_i (\neg(\neg \theta(x)))\). Fix \(\bar{v} \in A_1\), and suppose \((\theta t)_0\). By Lemma 1.1.ii, \((\theta t)_k\) for all \(k < n\). In particular, \((\theta t)_h\) for \(h\) the height of \(t\). If \(n\) is greater than \(h\) plus the rank of \(\theta\), then \(\neg(\neg \theta t)_h\) by Lemma 2.2. Thus \(\exists i < n \exists x \in A_i (\neg(\neg \theta(x)))\), and we are done. Since fulfilment is preserved by “moving negations in or out,” the dual form follows.

For Generalization, let \(\sigma = \langle A_0, \ldots, A_n \rangle\) and note that

\[-(\neg \forall \bar{v}, x (\theta \supset \psi x))^{\sigma}\quad \text{iff} \quad \forall \bar{v}, x \in A_1 ((\theta_0 \supset \neg(\neg \psi))^{\sigma})\]

\[-(\neg \forall \bar{v}, x (\theta \supset \psi x))^{\sigma}\quad \text{iff} \quad \forall \bar{v} \in A_1 ((\theta_0 \supset \neg(\neg \forall \bar{v}, x \psi x))^{\sigma})\]

\[-(\neg \forall \bar{v}, x (\theta \supset \psi x))^{\sigma}\quad \text{iff} \quad \neg(\neg \forall \bar{v}, x (\theta \supset \psi x))^{\sigma}\]

For Modus Ponens, let \(\bar{u}, \bar{v}, \) and \(\bar{w}\) be listings of the free variables of \(\psi, \theta,\) and \(\theta \supset \psi\), respectively. Suppose that \(\theta\) is in \(\Gamma\) and that we have a sequence \(\tau\) which \(n + 1\)-fulfils \(\neg \forall \bar{u} \psi \bar{u}\) and \(\Gamma \lor \neg \Gamma\). If \(n\) is large enough then by 2.2, \(\tau\) fulfils \(\forall \bar{v} \theta\). If we let \(\sigma\) be \(\tau\) less its first term,
say $\sigma = \langle A_0, \ldots, A_n \rangle$, then $\exists \bar{w} \in A_0 (\neg \psi)_0^\sigma$ and $\forall \bar{v} \in A_0 \theta^\sigma_0$. Hence $\exists \bar{w} \in A_1 (\theta^\sigma_0 \land (\neg \psi)_0^\sigma)$, that is, $\sigma$-$n$-fulfills $\forall \bar{w} \, (\theta \supset \psi)$ and $\Gamma \land \neg \Gamma$, contradicting our induction hypothesis.

This concludes the proof of 2.1. \hfill \Box

Next we shall quote a result, paraphrased, from the next chapter page 33, which allows us to eliminate the mention of $\Gamma$ in Lemma 2.1.

3.6 Theorem For any structure $A$ for $\mathcal{L}$ and any sentence $\theta$ of $\mathcal{L}$, for all $n$ there exists a sequence of subsets of $A$ which $n$-fulfills $\theta$ if and only if for all $n$ and for all finite sets $\Gamma$ of formulae of $\mathcal{L}$ there exists such a sequence which $n$-fulfills $\theta$ and $\Gamma \land \neg \Gamma$. Moreover, the proof of this is effective in the following sense. Given $n$, $\Gamma$, and $\theta$, we may obtain an $m$ primitive recursively (in fact, by a function in $\mathcal{E}_4$) from $n$ and the number of quantifiers in $\theta$ and $\Gamma$ such that if in some structure we have a sequence $\sigma$ which $m$-fulfills $\theta$, we may constructively obtain a sequence $\sigma'$ from $\sigma$ which $n$-fulfills $\theta$ and $\Gamma \land \neg \Gamma$. Thus we also have the formal versions:

$$\text{PRA} \vdash \forall \theta \, (\forall n \, \exists \sigma \, (|\sigma|^\theta = n \land \theta^\sigma) \equiv \forall n, \Gamma \exists \sigma \, (|\sigma|^\theta = n \land \theta^\sigma \land (\Gamma \land \neg \Gamma)^\sigma))$$

where $\theta \, (\Gamma)$ ranges over (the codes of) sentences (finite sets of formulae, respectively) of $\mathcal{L}$, and assuming that $S$ proves the functions in $\mathcal{E}_4$ are total,

$$S \vdash \forall \theta \, (\forall n \, \exists \sigma \, (|\sigma|^\theta = n \land \theta^\sigma) \equiv \forall n, \Gamma \exists \sigma \, (|\sigma|^\theta = n \land \theta^\sigma \land (\Gamma \land \neg \Gamma)^\sigma))$$

where $\theta$ and $\Gamma$ are as before but with $\mathcal{L}$ the language of $S$. \hfill \Box

From 2.1 and 3.6 we may immediately obtain Herbrand’s Theorem. A formalized version of this is:

2.3 Corollary (Herbrand)

$$S \vdash \forall \theta \, (\text{Pr}_{\text{PC}}(\theta) \supset \neg \forall n \, \exists \sigma \, (|\sigma|^\theta = n \land (\neg \theta)^\sigma))$$

where $\text{Pr}_{\text{PC}}$ is any natural proof predicate for the Predicate Calculus and $\theta$ ranges over (the codes of) sentences in the language of $S$. \hfill \Box

Results 1.2 and 2.3 immediately give:

2.4 The Reflexiveness Theorem For each formula $\varphi \, x$ in the language of $S$

$$S + \text{Induction} \vdash \forall n \, (\text{Pr}_{\text{PC}}(\varphi \hat{n}) \supset \varphi \hat{n})$$

where $\varphi \hat{n}$ is (the code of) the sentence obtained by substituting the term $\bar{p}$ for $x$ in $\varphi \, x$. \hfill \Box

For any formula $T(x)$ let $\text{Pr}_t(\theta) = \exists$ finite $X \subseteq T \text{Pr}_{\text{PC}}(\land X \supset \theta)$. An alternative formalization of Herbrand’s Theorem is:

2.5 Corollary If $T$ is a formula in the language of $S$,

$$S \vdash \forall \theta \, (\text{Pr}_t(\theta) \supset \neg \forall n \, \exists \sigma \, (|\sigma|^\theta = n \land T^\sigma \land (\neg \theta)^\sigma)),$$

and if $T$ is a formula in the language of arithmetic,

$$\text{PRA} \vdash \forall \theta \, (\text{Pr}_t(\theta) \equiv \neg \forall n \, \exists \sigma \, (|\sigma|^\theta = n \land T^\sigma \land (\neg \theta)^\sigma)),$$
where \( \theta \) has the appropriate ranges.

Proof: That the converse direction also holds for \( \ast \)-fulfilment is given by Gödel's proof of the Completeness Theorem as discussed on page 11. \( \square \)

We may restate 2.5 perhaps more elegantly by using the fact that \( S \) proves:

\[
\forall \theta \left( \neg \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land (\neg \theta)^\sigma \right) \equiv \forall \varphi \left( \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land \varphi^\sigma \land (\neg \theta)^\sigma \right) \right) \right)
\]

and PRA proves the \( \ast \)-fulfilment version. The left to right direction is immediate from 3.6 by taking \( \Gamma = \{ \neg \theta \} \), and the converse is obtained from (the formalized version of) 2.2 by taking \( \varphi = \neg \theta \).

If we wished to avoid the use of 3.6 we could, for example, restate 2.5 as

\[
S \vdash \forall \theta \left( \Pr_r(\theta) \equiv \neg \forall n, \Gamma \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land (\neg \theta)^\sigma \land (\Gamma \lor \neg \Gamma)^\sigma \right) \right)
\]

and similarly for \( \ast \)-fulfilment. Likewise, in the next theorem we could avoid the use of 3.6 by making analogous changes to all the following schemata which involve fulfilment. Of course, if \( S \) proves that the theory \( T \) contains all sentences of the form \( \forall \vec{x} \ (\theta \lor \neg \theta) \), then this alteration has no effect, and it also has no effect for (a natural axiomatization of) a theory such as PA, where the induction schema contains sentences which are essentially just \( \forall \vec{x} \ (\theta \lor \theta) \).

Next we shall use 2.5 to explore minor variants of different sentences and schemata and to compare these with various reflection principles. To concentrate on the case which is of most interest to us, let us suppose that all formulae below are in the language of \( S \), that \( S \) has a \( \Delta^0_k \) axiomatization, and that \( S \) proves that \( T \) is an extension of \( S \). First consider the schemata:

\[
\begin{align*}
\text{Ful}(T): & \quad \varphi \supset \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land \varphi^\sigma \right) \\
\text{FUL}_1(T): & \quad \forall m \left( \varphi m \supset \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land (\varphi m)^\sigma \right) \right) \\
\text{FUL}_2(T): & \quad \forall x \left( \varphi x \supset \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land (\varphi x)^\sigma \right) \right)
\end{align*}
\]

and the sentences

\[
\begin{align*}
\text{FUL}^3_k(T): & \quad \forall n \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land \text{Tr} \Sigma_k^\sigma \right) \\
\text{FUL}^4_k(T): & \quad \forall n, x \exists \sigma \left( \left| \sigma \right| = n \land T^\sigma \land \text{Tr} \Sigma_k^\sigma \land x \in \sigma_0 \right)
\end{align*}
\]

and similar sentences for \( \Pi_k \). Let \( i\text{-Ful} \), \( i\text{-FUL}_k^1 \), \ldots, \( i\text{-FUL}_k^4 \) be the analogous notions for \( i \)-fulfilment, and we can also consider the schemata \( \ast \text{Ful} \) and \( \ast \text{FUL}_1 \) for \( \ast \)-fulfilment. Let \( \text{ful} \), \( \text{ful}_1 \), \ldots, \( \text{ful}_k^4 \) be the schemata

\[
\begin{align*}
\text{ful}(T): & \quad \varphi \supset \exists \sigma \left( \left| \sigma \right| = \pi \land T^\sigma \land \varphi^\sigma \right), \quad n \in \omega, \\
\text{ful}_1(T): & \quad \forall m \left( \varphi m \supset \exists \sigma \left( \left| \sigma \right| = \pi \land T^\sigma \land (\varphi m)^\sigma \right) \right), \quad n \in \omega, \text{ etc.}
\end{align*}
\]

Finally consider

\[
\begin{align*}
\text{RFN}(T): & \quad \forall m \left( \Pr_r(\varphi m) \supset \varphi m \right) \\
\text{Rfn}(T): & \quad \Pr_r(\varphi) \supset \varphi, \text{ and} \\
\text{rfn}(T): & \quad \text{Proof}_r(\overline{\varphi}, \varphi) \supset \varphi, \ p \in \omega.
\end{align*}
\]

The main feature of the following result, that \( \text{FUL}(T) \) is equivalent to \( \text{RFN}(T) \), is due to Kripke. Suppose, for simplicity, that \( T \) is \( \Delta^0_0 \).

**2.6 Corollary** (a) In \( S \), for all \( k \) in \( \omega \), with “(T)” omitted for brevity:
i. $\text{FUL}^3_{\Sigma_k} \equiv \text{FUL}^3_{\Pi_{k+1}} \equiv \text{FUL}^1_{\Pi_{k+1}} \equiv \text{RFN}_{\Sigma_{k+1}}$

ii. $\text{FUL}^4_{\Sigma_k} \equiv \text{FUL}^4_{\Pi_{k+1}} \equiv \text{FUL}^2_{\Pi_{k+1}} \equiv \text{FUL}^2_{\Sigma_{k+2}} \equiv \text{FUL}^1_{\Sigma_{k+2}} \equiv \text{RFN}_{\Pi_{k+2}}$

iii. $\text{FUL}^2_{\Delta_0} \equiv \text{FUL}^2_{\Sigma_1} \equiv \text{FUL}^1_{\Sigma_1} \equiv \text{RFN}_{\Pi_1}$

iv. $\forall n \exists \sigma ((|\sigma| = n \land \mathcal{T}^\sigma) \lor \forall n \exists \sigma ((|\sigma| = n \land \mathcal{T}^\sigma^\sigma) \equiv \text{Con}(\mathcal{T})$

v. $\text{Ful}_{\Pi_k} \supset \text{Ful}_{\Pi_k} \equiv \text{RFN}_{\Sigma_k}$

vi. $\text{Ful}_{\Sigma_k} \supset \text{Ful}_{\Sigma_k} \equiv \text{RFN}_{\Pi_k}$

vii. $\text{ful}^3_{\Sigma_k} \equiv \text{ful}^3_{\Pi_{k+1}} \equiv \text{ful}^1_{\Pi_{k+1}} \equiv \text{RFN}_{\Sigma_{k+1}} \equiv \text{ful}^1_{\Pi_{k+1}}$

viii. $\text{ful}^4_{\Sigma_k} \equiv \text{ful}^4_{\Pi_{k+1}} \equiv \text{ful}^2_{\Pi_{k+1}} \equiv \text{ful}^2_{\Sigma_{k+2}} \equiv \text{ful}^1_{\Sigma_{k+2}} \equiv \text{ful}^1_{\Sigma_{k+2}}$

ix. $\text{ful}^3_{\Delta_0} \equiv \text{ful}^3_{\Sigma_1} \equiv \text{ful}^1_{\Sigma_1} \equiv \text{RFN}_{\Pi_1} \equiv \text{ful}_{\Sigma_1} \equiv \text{ful}^1_{\Sigma_1}$

x. $\{ \exists \sigma (|\sigma| = n \land \mathcal{T}^\sigma) : n \in \omega \}$

Thus in the presence of $\Sigma + \text{"Collection } \subseteq \mathcal{T}\text{"}”, \text{fulfilment} and \text{i-fulfilment} are essentially the same notions, except in the $\Sigma_1$ schemata.

(c) The interest in (vii), (viii) and (ix) lies in the fact that $\text{RFN}_{\Pi_{k+1}}(\mathcal{T})$ and $\text{RFN}_{\Sigma_{k+1}}(\mathcal{T})$ axiomatize the $\Pi_{k+1}$ and $\Sigma_{k+1}$ consequences of $\mathcal{T}$, respectively. The following are, in the presence of $\Sigma + \text{"Induction } \subseteq \mathcal{T}\text{"}$, further examples of such axiomatizations. First, for the $\Pi_{k+1}$ consequences, we have:

xi. $\{ \text{RFN}_{\Pi_{k+1}}(\mathcal{T} \cap \overline{m}) : m \in \omega \}$

For the $\Sigma_{k+1}$ consequences, we have:

xii. $\{ \text{RFN}_{\Sigma_{k+1}}(\mathcal{T} \cap \overline{m}) : m \in \omega \}$

and in the presence of “Infinity $\in \mathcal{T}$” and “$\Sigma_{k+1}$-Collection $\subseteq \mathcal{T}$”, we also have the $\Sigma_{k+1}$ form of

\[ \{ \text{FUL}^3_{\Sigma_{k+1}}(\mathcal{T} \cap \overline{m}) : m \in \omega \} \]

(d) There are similar results for $\frac{1}{2}$*fulfilment. In particular,

$\frac{1}{2} \text{ful}^3_{\Sigma_k}(\mathcal{T}): \exists \sigma (|\sigma| = n \land \mathcal{T}^\sigma \land (\text{Tr}_{k}^{\sigma^\sigma}) \land \mathcal{T}^\sigma), \ n \in \omega$,

axiomatizes the $\Sigma^0_{k+1}$ consequences of $\mathcal{T}$, and

$\frac{1}{2} \text{ful}^4_{\Sigma_k}(\mathcal{T}): \forall m \exists \sigma (|\sigma| = n \land \mathcal{T}^\sigma \land (\text{Tr}_{k}^{\sigma^\sigma}) \land n \in \sigma_0), \ n \in \omega$,
axiomatizes the $\Pi^0_{k+2}$ consequences of $T$. Also, if Induction is included in $T$, then $\{ \forall n \exists \sigma (|\sigma|=n \land T \land \exists \sigma') : m \in \omega \}$ axiomatizes the $\Pi^0_k$ consequences of $T$.

Proof: We shall only consider typical examples of nontrivial cases. First let us consider $\text{FUL}_{\Sigma_k}^4(T) \supset \text{FUL}_{\Sigma_k}^4(T)$). Suppose $\exists z \varphi z$ is true in some structure, with $\varphi \Pi_1$. Choose any witness $a$ and let $b = \{a\}$. Then any sequence which fulfils the true $\Pi_1$ formula

$$\exists x \in b \, (x = a \land \varphi a)$$

will fulfil $\exists z \varphi z$.

That $\ast \text{FUL}(T)$ implies $\text{RFN}(T)$ (and conversely) is an immediate consequence of 2.5.

Finally, suppose $\text{RFN}_{\Pi_{k+1}}(T)$ and that we wish to show $\text{FUL}_{\Sigma_{k-1}}^4(T)$, say. For each $n$ in $\omega$ we have

$$T \vdash \forall x \exists \sigma (|\sigma|=n \land x = \sigma_0 \land T^\sigma \land \text{Tr} \Sigma_{k-1} \sigma).$$

Moreover, this fact can be formalized, and so we have

$$S \vdash \forall n \text{Pr}_T \left( \forall x \exists \sigma (|\sigma|=n \land x = \sigma_0 \land T^\sigma \land \text{Tr} \Sigma_{k-1} \sigma) \right).$$

Hence by $\text{RFN}_{\Pi_{k+1}}(T)$ we obtain $\text{FUL}_{\Sigma_{k-1}}^4(T)$.

We cannot in general improve (iv), (v) or (vi). A result of Feferman [60] is that

$$S + \text{RFN}_{\Sigma_{k+1}}(T) \vdash \text{RFN}_{\Sigma_{k+1}}(T + \text{Rfn}(T)).$$

(Aside: this can be easily generalized from $\Sigma_k^0$ to arbitrary $\Lambda \supseteq \Sigma_k^0$. A simple proof is given in Smoryński [77]; or by an equally simple proof, using 2.5 we can show

$$S + \text{FUL}_{\Lambda_k}^4(T) \vdash \text{FUL}_{\Lambda_k}^4(T + \text{Rfn}(T)).$$

Now by Matijasevic’s Theorem and 2.5 we have that for any arithmetical theory $T$ in the language of arithmetic,

$$S + \text{“}\Sigma_k\text{-Induction} \subseteq T\text{”} \vdash \left( \forall n \exists \sigma (|\sigma|=n \land T^\sigma) \right) \equiv \text{FUL}_{\Pi_k}^4(T)$$

and so if $T$ is $\Delta_0$, then by either Gödel’s Second Incompleteness Theorem or 1.5,

$$S + \text{Rfn}(T) \not\vdash \forall n \exists \sigma (|\sigma|=n \land T^\sigma).$$

Our next result is a version of the main lemma of Kreisel and Levy [68].

**2.7 Corollary** Let $U$ extend $S$ and let $T$ be a subset of $U$ which is representable in (some consistent extension of) $U$. Suppose that $U$ implies $\text{RFN}(T)$, where “$T$” in this schema is understood to be some representation of $T$. Then no consistent extension of $U$ (plus $\{ T(\theta^\sigma) : \theta \in T \}$) may be obtained by adding a set of $\Sigma_k$ sentences to $T$ for any $k$.

Proof: We shall sketch two proofs, the second uses the fixed-point theorem while the first does not, but the first proof requires the further assumption that $U$ extends Foundation. Fix $k$ in $\omega$.

By 2.6 we have that $U$ implies

$$\forall n \exists \sigma (|\sigma|=n \land n \in \sigma_0 \land T^\sigma \land \text{Tr} \Sigma_k \sigma).$$

(1)
We can suppose that $\mathcal{T}(x)$ is $\Pi_k$. Then by fixing a nonstandard $n$ and choosing a minimal witness $\sigma$ as in 1.3, we see by our usual method that any nonstandard model of $U + \{ \mathcal{T}(\theta^T) : \theta \in \mathcal{T} \}$ has a $k$-elementary substructure which is a model of $\mathcal{T}$ and in which (1) is false.

For our second proof, let $U_0$ be a finite subset of $U$ which suffices to prove that some formula $\text{Tr} \Sigma_k(x)$ is a $\Sigma_k$ truth predicate. By 2.6, for all sentences $\chi$

$$U \vdash \chi \supset \forall \varphi \left( \text{Tr} \Sigma_k(\varphi) \supset \forall \sigma \exists \tau \left( (|\sigma|=n \land (\text{Tr} \Sigma_k(\varphi))^\sigma \land U_0^\sigma \land \chi^\tau) \right) \right).$$

Choose $\chi$ such that

$$U \vdash \chi \equiv \forall \varphi \left( \text{Tr} \Sigma_k(\varphi) \supset \forall \sigma \exists \tau \left( (|\sigma|=n \land (\text{Tr} \Sigma_k(\varphi))^\sigma \land U_0^\sigma \land \neg \chi^\tau) \right) \right).$$

Then $U \vdash \chi$, for by (2), $U \vdash \neg \chi \supset \chi$. But for any $\Sigma_k$ sentence $\varphi$ consistent with $U + \{ \mathcal{T}(\theta^T) : \theta \in \mathcal{T} \}$, we may find a model of $\text{Tr} \Sigma_k(\varphi)$, $\mathcal{T}$, $U_0$, and $\neg \chi$. Hence $\mathcal{T} + \varphi \not\vdash \varphi$. □

We conclude the chapter by considering a problem posed by D. Guaspari [79]. Given a class of sentences $\Lambda$ and a set of sentences $\mathcal{T}$, say that a sentence $\varphi$ is $\Lambda$-conservative over $\mathcal{T}$ if for all $\theta$ in $\Lambda$, if $\mathcal{T} + \varphi \vdash \theta$ then $\mathcal{T} \vdash \theta$ already. For a theory $\mathcal{T}$ let

$$(\Lambda_1, \Lambda_2) = \{ \langle \varphi \rangle \in \Lambda_1 : \varphi \text{ is } \Lambda_2\text{-conservative over } \mathcal{T} \}.$$ 

R. Solovay and P. Hájek have shown that if $\mathcal{T}$ is a consistent r.e. reflexive theory then $(\Sigma_k, \Pi_k)$ is a complete $\Pi^0_2$ set for all $k \geq 1$. We shall prove this without the hypothesis of reflexivity and we shall also consider the dual case. (In retrospect, I see that simple modifications of Hájek’s [78] proof will give most of the results in (b) below: just use the equivalence of the notions of reflection and fulfilment as given in 2.6.) Part (a) below is due to Guaspari [79] who uses a construction due to Scott [62] and Friedman [73]. He requires, however, that $\mathcal{T}$ contain the power-set axiom.

**2.8 Corollary** (a) Let $\mathcal{T}$ be a consistent extension of $\mathcal{S}$. A sentence $\varphi$ is $\Sigma_{k+1}$-conservative ($\Pi_{k+2}$-conservative) over $\mathcal{T}$ iff all models of $\mathcal{T} + \Sigma_{k+1}$-overspill which code $\mathcal{T}$ have (for any element $x$) a $k$-elementary substructure (containing $x$) which is a model of $\mathcal{T} + \varphi$.

(b) Let $\mathcal{T}$ be an r.e. extension of $\mathcal{S}$. Then:

i. For all $k \geq 1$, $(\Sigma_k, \Pi_k)$ is a complete $\Pi^0_2$ set.

ii. For all $k \geq 2$, $(\Pi_k, \Sigma_k)$ is a complete $\Pi^0_2$ set, as are $(\Pi_2, \Sigma_1)$ and $(\Sigma_2, \Sigma_1)$, and if $\mathcal{T}$ proves infinity so is $(\Pi_1, \Sigma_1)$.

iii. $(\Delta^0_2(\mathcal{T}), \Sigma^0_2)$ is a complete $\Pi^0_2$ set if $\mathcal{T}$ is not $\Sigma^0_2$-sound, and is $\Delta^0_2$ otherwise.

iv. $(\Pi^0_1, \Sigma^0_1)$ and $(\Pi^0_1, \Delta^0_1(\mathcal{T}))$ are complete $\Pi^0_2$ sets if $\mathcal{T}$ is not $\Sigma^0_1$-sound, and are complete $\Pi^0_1$ sets otherwise.

Proof: Part (a) is immediate from 2.6, so let us consider (b). It is clear by their definition that these sets are $\Pi^0_2$. It is also easy to check that if $\mathcal{T}$ is $\Sigma^0_1$-sound, a sentence is $\Delta^0_1(\mathcal{T})$-conservative over $\mathcal{T}$ iff it is consistent with $\mathcal{T}$, a $\Pi^0_1$ sentence is $\Sigma^0_1$-conservative iff it is true, and if $\mathcal{T}$ is also $\Sigma^0_2$-sound, a $\Pi^0_2$ sentence is $\Sigma^0_2$-conservative over $\mathcal{T}$ iff it is true. This establishes the negative parts of (iii) and (iv).

We shall next show that $(\Pi_2, \Sigma_1)$ and $(\Pi_{k+1}, \Sigma_{k+1})$ are complete $\Pi^0_2$ sets for $k \geq 1$. We shall in fact give two proofs: the first avoids the use of the fixed-point theorem but requires the further assumption that $\mathcal{T}$ extends Foundation. Let $k \geq 1$, let $\psi_n$ be $\Sigma_1$, and let $\Phi$ be the $\Pi_{k+1}$ sentence

$$\Phi = \forall n \left( \exists \sigma \left( (|\sigma|=n \land \mathcal{T}^\sigma \land \text{Tr} \Sigma_k^\sigma) \supset \psi_n \right) \right).$$

Consider the following statements:
(a) for all \( n \in \omega \), \( T \vdash \psi^\perp \);
(b) \( \Phi \) is \( \Sigma_{k+1} \)-conservative over \( T \);
(c) \( \Phi \) is \( \Pi_k \)-conservative over \( T \);
(d) \( \Phi \) is \( \Sigma^0_1 \)-conservative over \( T \);
(e) \( \Phi \) is \( \Delta^0_1(T) \)-conservative over \( T \).

We claim that (a), (b) and (c) are equivalent. The result follows from this claim, for it is easy to show by the reader’s favourite method (e.g. 1.12) that

\[
\{ \forall x \in \Sigma_1^0 : \forall n \in \omega \cap \psi^\perp \}
\]

is a complete \( \Pi^0_2 \) set.

To prove the claim, note that (b) \( \supset \) (c) is trivial, and that (c) \( \supset \) (a) is easy. So consider (a) \( \supset \) (b).

Suppose (a). Let \( A \) be any non-\( \omega \)-model of \( T \) (+Foundation). It suffices to show there exists a \( k \)-elementary substructure \( B \) of \( A \) which is a model of \( T \) and \( \Phi \). If \( \Phi \) holds in \( A \), we are done. If not, choose the least witness \( m \) of \( \neg \Phi \); \( m \) is necessarily nonstandard by (a). Now choose a minimal witness \( \sigma \) as in 1.3, and let \( B = \bigcup_{i \in \omega} \sigma^i \). Then \( B \prec_k A \), and \( B \models T \). It remains to show that \( \Phi \) holds in \( B \). Consider \( n \in B \). If \( n < m \), then \( \psi n \) holds in \( A \) by the choice of \( m \), and so \( \psi n \) holds in \( B \). If \( n \geq m \), then by the choice of \( \sigma \) the antecedent in \( \Phi \) must be false at \( n \) in \( B \). Hence \( \Phi \) holds in \( B \).

To prove that \( (\Pi_{\max(k+1,2)}, \Sigma_{k+1}) \) is a complete \( \Pi^0_2 \) set if \( T \) does not necessarily extend Foundation, choose \( \Phi \) such that \( T \) proves:

\[
\Phi \equiv \forall n \left( \exists \sigma \left( |\sigma| = n \land T^{\sigma} \land \text{Tr} \Sigma^0_k \land \Phi^{\sigma} \right) \supset \psi n \right).
\]

(3)

Then, as before, (a), (b) and (c) are equivalent. Since the choice of \( \Phi \) can be made recursive in \( \psi^\perp \), we are done. Likewise, if \( T \) proves Infinity, \( (\Pi_1, \Sigma_1) \) is a complete \( \Pi^0_2 \) set. (Note that the analogous fixed-point and argument using the notion of *fulfilment does not give as sharp a result: we obtain that \( (\forall n \exists m < n \Pi_{k+1}, \Sigma_{k+1}) \) is a complete \( \Pi^0_2 \) set, where this first class is \( \Pi_{k+1} \) prefixed by a universal and a bounded existential numerical quantifier.)

To show that \( (\Sigma_2, \Sigma_1) \) is a complete \( \Pi^0_2 \) set, let \( \psi m = \exists l \psi_0 ml \) be \( \Sigma^0_1 \) and consider the fixed point in \( T \):

\[
\Phi \equiv \neg \forall n \left( \exists l \forall m < n \psi_0 ml \supset \exists \sigma \left( |\sigma| = n \land T^{\sigma} \land \text{Tr} \Delta^0_0 \land \Phi^{\sigma} \right) \right).
\]

\( \Phi \) is (equivalent to) a \( \Sigma_2 \) sentence, and as above, (a), (b) (with \( k = 0 \)) and (c) are equivalent.

Next suppose \( T \) is not \( \Sigma^0_2 \)-sound, and choose some true \( \Pi^0_2 \) sentence \( \forall m \exists l \theta ml \) which \( T \) refutes. Let \( \psi m = \exists l \psi_0 ml \) be \( \Sigma^0_1 \), and let \( \chi n = \exists l \forall m < n \psi_0 ml \land \exists l \forall m \leq n \theta ml m \), and suppose \( T \) proves:

\[
\Phi \equiv \forall n \left( \exists \sigma \left( |\sigma| = n \land T^{\sigma} \land \text{Tr} \Delta^0_0 \land \Phi^{\sigma} \right) \supset \chi n \right),
\]

Then (a) and (d) are equivalent. Since \( T \) proves \( \neg \forall n \chi n \), \( T \) proves:

\[
\Phi \equiv \exists n \left( \neg \exists \sigma \left( |\sigma| = n \land T^{\sigma} \land \text{Tr} \Delta^0_0 \land \Phi^{\sigma} \right) \land \chi(n - 1) \right).
\]

Thus \( \Phi \) is \( \Delta^0_2(T) \), and so \( (\Delta^0_2(T), \Sigma^0_1) \) is a complete \( \Pi^0_2 \) set.
Suppose now that $T$ is not $\Sigma^0_1$-sound. D. Jensen and A. Ehrenfeucht [76] and D. Guaspari [79] have shown by the fixed-point theorem that an r.e. theory $T$ is $\Sigma^0_1$-sound iff it decides every $\Delta^0_1(T)$ sentence. Using a similar calculation, we can show that if an r.e. theory $T$ is not $\Sigma^0_1$-sound, then any r.e. set may be semi-represented in $T$ by a formula which is $\Delta^0_1$ in $T$. We can, however, do better by giving a feasible representation as in 1.12. Let $\exists k \chi k$ be a false $\Sigma^0_1$ sentence provable in $T$, with $\chi \Delta^0_1$, let $\theta mn$ be any $\Delta^0_0$ formula, and let

$$
\psi n = \exists m, \sigma \left( |\sigma| = m \land T^{\frac{1}{2}+\sigma} \land \text{Tr} \Delta^0_0 \frac{1}{2}+\sigma \land \theta mn \land \forall k < |\sigma| \neg \chi k \right).
$$

Then, since $S$ is assumed to include $\Delta^0_1(S)$-Induction, it is easy to show that for all $n \in \omega$, $T \vdash \psi n$ iff $\exists m \theta mn$ is true. Moreover, because $T$ proves $\exists k \chi k$, $T$ also proves

$$
\forall n \left( \neg \psi n \equiv \exists k \left( \chi k \land \neg \exists m, \sigma \leq k \left( |\sigma| = m \land T^{\frac{1}{2}+\sigma} \land \text{Tr} \Delta^0_0 \frac{1}{2}+\sigma \land \theta mn \right) \right) \right).
$$

and so $\psi$ is $\Delta^0_1(T)$. Now if $T$ contains $\Delta^0_1(T)$-Induction, consider

$$
\Phi = \forall n \left( \exists \sigma \left( |\sigma| = n \land T^{\frac{1}{2}+\sigma} \land \text{Tr} \Delta^0_0 \frac{1}{2}+\sigma \right) \supset \psi n \right);
$$

and if not, consider a fixed point similar to (3). Then we have that (a), (d) and (e) are equivalent, and so $(\Pi^0_1, \Sigma^0_1)$ and $(\Pi^0_1, \Delta^0_1(T))$ are complete $\Pi^0_2$ sets.

Finally, for the sets $(\Sigma^0_{k+1}, \Pi^0_{k+1})$ consider either the fixed points:

$$
\Phi \equiv \neg \forall n \left( \forall m < n \psi m \supset \forall x \exists \sigma \left( |\sigma| = n \land x \in \sigma_0 \land T^{\sigma} \land \text{Tr} \Sigma_{k-1}^{\sigma} \land \Phi^{\sigma} \right) \right);
$$

(which work for all $k \geq 1$); or the fixed points:

$$
\Phi \equiv \neg \forall n \left( \forall m < n \psi m \supset \forall \theta \left( \text{Tr} \Sigma_{k+1}^{\theta} \supset \exists \sigma \left( |\sigma| = n \land T^{\sigma} \land \theta^{\sigma} \land \Phi^{\sigma} \right) \right) \right),
$$

(which work for all $k$). The details are left to the reader. □

Theorem 6.12 on page 69 gives an analogue of this last result for the $\omega$-rule.
II Herbrand’s Theorem and Reflection Principles
III The Substitution Method

Hilbert posed the following problem: can one prove the consistency of Peano arithmetic in a constructive manner? He was interested in this question because a positive answer implies that the (nonconstructive) PA is a conservative extension of the constructive techniques used with respect to real (that is, $\Pi^0_1$) sentences. (See the introduction of Smoryński [77] for a further discussion.) As is well-known, solutions to this problem were given by Gentzen, in [36] and [38], who used as his main tool (in the second paper at least) cut-elimination, and by Ackermann in [40], who used the Hilbert substitution method. As Dreben and Denton [70] point out, the Gentzen approach, with its purely syntactical transformations of formal proofs, has little use for the interpretations of formulae; the Ackermann approach, on the other hand, “finitistically” exploits the oldest and most naive idea in proof theory: a set of axioms is consistent if it has a model. An elegant extension of Ackermann’s proof was given in Tait [65a], [65b] using functionals: Tait’s proof applies to systems of arithmetic with the schema of foundation on some arbitrary primitive recursive linear ordering. Scanlon [73] building on the work of Dreben and Denton [70] obtains the same results for linear orderings without the use of functionals; unfortunately, Scanlon’s paper is very long and intricate, and contains, as Kripke has pointed out, some unnecessary detours. This chapter is essentially an exposition of the above-mentioned work of Ackermann, Dreben, Denton and Scanlon.

The notion of fulfilment is of course not necessary in our treatment (indeed one could regard it as superfluous since all the work is still done with finite versions of Skolem functions) but I believe that it is very helpful conceptually—especially as it provides a clear separation between the logical and combinatorial parts of our argument.

We shall assume that $S$ on page 11 is finite. Let be $\preceq$ any formula of $L$, the language of $S$, with two free variables; this should be thought of as representing a pre-well-foundeded relation, i.e., the non-linear analogue of a pre-well-ordering. Let $\preceq^{\text{strict}}$ denote $x \preceq y \land y \not\preceq x$; this will also be denoted by $\prec$. Let $\text{Foundation}(\preceq)$ be the schema

$$\text{Foundation}(\preceq): \quad \exists x \theta x \supset \exists x \left( \theta x \land \forall y (\theta y \supset y \not\preceq x) \right).$$

Consider the consistency of the theory $S + \text{Foundation}(\preceq)$. By 2.6, it suffices to show that each finite subset of this theory plus any true sentence is $n$-fulfillable for all $n$; and we would like to do this in as constructive a manner as possible. Of course, by the Second Incompleteness Theorem or by 1.5, we must assume a quite strong notion of constructiveness; our proof may be formalized in an extension $\text{Foundation}(\preceq_{\Sigma})$ defined below. If the axioms of $S$ are $\Sigma_3$ and the relation $\preceq$ is $\Sigma_2$, we need only the schema $\text{Foundation}(\preceq_{\Sigma})$ restricted to $\Sigma_1$ formulae, and if the axioms of $S$ have the form

$$\exists x \forall y \exists! z \theta$$

with $\theta \Delta_0$, and if $\preceq$ has the form $\forall y \exists! z \theta$, with $\theta \Delta_0$, then we need only the schema $\text{Foundation}(\preceq_{\Sigma})$ restricted to $\Delta_1(S)$ predicates.

We need some general definitions. Given a binary relation $\preceq$, let $\leq_{\preceq^{\text{strict}}}$ be as above. Define a binary relation $\preceq^{\text{lex}}$ on nonempty finite sequences by $\langle a_0, \ldots, a_m \rangle \preceq^{\text{lex}} \langle b_0, \ldots, b_n \rangle$ iff either there exists $k \leq \min(m,n)$ such that $a_l = b_l$ for all $l < k$ and $a_k < b_k$, or $m < n$ and $a_l = b_l$ for all $l \leq m$. Let $\preceq^\omega$ be the disjoint union over $n$ of $\preceq^{\text{lex}}$ restricted to sequences of length $n$; that is, $x \preceq^\omega y$ iff $|x| = |y|$ and $x \preceq^{\text{lex}} y$. A finite sequence $x$ is $\preceq$-descending if $x_{i+1} < x_i$ for all $i < |x|$; let $\Sigma^\preceq$ denote $\preceq^{\text{lex}}$ restricted to $\preceq$-descending sequences. If $\preceq$ is well-founded and
nontrivial, $\preceq_{\text{lex}}$ is not well-founded, but the usual proofs (see, for example Feferman [77]) show that Foundation($\preceq$) implies Foundation($\preceq_{\text{lex}}$) and Foundation($2^\preceq$).

Let us go back to our given $\preceq$. Fix any element and denote it by $\infty$. Define the relations:

$$
x \preceq_+ y \text{ if } y = \infty \text{ or both } x \neq \infty \text{ and } x \preceq y,
$$

$$
x \preceq_0 y \text{ if } x (\preceq_+)^\omicron y,
$$

$$
x \preceq_{n+1} y \text{ if } x (2^\preceq)^n y,
$$

and let $\preceq_E$ be a disjoint union over $n$ of $\preceq_n$: e.g.

$$(n,x) \preceq_E (m,y) \text{ if } n = m \text{ and } x \preceq_n y.$$  

We shall suppose that this relation may be represented in the language of $S$ in some natural way. We note that if $\preceq$ is $\Delta_k(S)$ then $\preceq_E$ is $\Delta_k(S + \Sigma_{k+1-}\text{Induction})$.

Recall one of the twenty-odd schemata given on page 22, there labelled FUL.$^2$:

FUL(T): \hspace{1cm} \forall x \left( \varphi x \supset \forall n \exists \sigma \left( |\sigma| = n \land T^\sigma \land (\varphi x)^\sigma \right) \right).

3.1 Theorem $S + \text{Foundation}(\preceq) \vdash \text{FUL}(S + \text{Foundation}(\preceq))$.

We shall give an informal proof, which will be divided into four lemmas.

With each formula $\theta$ with $n$ free variables associate an $n$-ary function symbol $f_\theta$. Define the Skolemization $S(\theta)$ of $\theta$ by induction on length:

- i. if $\theta$ is an atom, let $S(\theta) = \theta$;
- ii. let $S(\theta \supset \psi) = S(\theta) \supset S(\psi)$, and similarly for the other Boolean connectives; and
- iii. let $S(Qx\theta) = S(\theta)(x/f_{Qx\theta} \vec{v})$ where $\vec{v}$ is a listing of the free variables of Qx$\theta$.

For the remainder of the proof fix an integer $n$ and a finite set $H$ of formulae of the form $\exists x\theta$.

We shall first consider the problem of finding a sequence which $n$-fulfils the universal closure of $\text{Foundation}(\preceq)$

$$
\exists x\theta x \supset \exists x \left( \theta x \land \forall y (\theta y \supset y \neq x) \right)
$$

(1)

for each $\exists x\theta$ in $H$.

Let sub($H$) be the collection of subformulae of formulae of $H$. If Qx$\theta$ in sub($H$) \ H is of rank $r$ (as defined in Chapter II page 19), let the rank of $f_{Qx\theta}$ be $r$. Let $q - 1$ be the maximum of the ranks of formulae in sub($H$) \ H, and for all $\exists x\theta$ in $H$ let the rank of $f_{\exists x\theta}$ be $q$. Let the functions of $\mathcal{L}$ have rank 1. Let $\mathcal{F} = \{ f_{Qx\theta} : Qx\theta \in \text{sub}(H) \}$.

3.2 Lemma Suppose we are given an interpretation of the functions of $\mathcal{F}$ and a set $A_0$ which contains (the interpretations of) the constants of $\mathcal{L}$. Define by $\sigma = \langle A_i \rangle_{i \leq n}$ by

$$
A_{i+1} = A_i \cup \bigcup_{f \in \mathcal{L} \cup \mathcal{F} f''} A_i.
$$

Suppose:

- i. for each $\exists \theta$ in $H$ with free variables $\vec{u}$,

$$
\forall \vec{u}, x (S(\theta) \supset x \neq f_{\exists x\theta} \vec{u})
$$

where here, as in (ii) and (iii) below, the quantifiers $\forall \vec{u}$, $\exists x$, and $\forall x$ are restricted to $A_{n-1}$;
II. The Substitution Method

3.3 Lemma There exists a constructive interpretation of $\mathcal{F}$ which satisfies the premises of Lemma 3.2.

Proof: We shall first consider only the premises (i), (ii) and (iii). A simple modification of our algorithm will then take care of (iv); this is discussed following Lemma 3.5.
We shall define a sequence $\langle F^i \rangle_{i \in \Omega}$ (where $\Omega \leq \omega$) of interpretations of $F$, where $F^i = \{ f^i : f \in F \}$. Each $f^i$ will have constant value $\infty$ except at a finite number of arguments. For each $F^i$, define $\sigma^i$ and $\sigma^i$ as above, with $A_0$ any finite set containing the constants and $\infty$ and $A_0$ a listing of $A_0$. Let $i, j$ range over $\Omega$.

For each $f$ in $F$, let $f^0$ be the constant function of appropriate arity with value $\infty$. Suppose we have defined $F^i$. If $F^i$ satisfies the premises (i) to (iii), stop. Otherwise, choose some $Q \times \theta$ in sub$(H)$ whose corresponding condition is false, and choose $u, x$ in $A^i_{n-1}$ as witnesses. Define $F^{i+1}$ by:

i. if $f \in F$, $f \neq Q \times \theta$ and rank$(f) \leq r$, let $f^{i+1} = f^i$;
ii. if $f \in F$ and rank$(f) > r$, let $f^{i+1} = f^0$; and
iii. let $f^{i+1}$ equal $f^i$ everywhere except at $u$, where we let $f^{i+1}(u) = x$.

We shall call $(f_Q \times \theta, u)$ the $(i + 1)^{st}$-critical pair. We shall suppose that the choice of the $(i + 1)^{st}$-critical pair depends only on $\sigma^i$. The rank of $\sigma^{i+1}$ will be $r$; let $\sigma^0$ have rank $0$.

We must show that our algorithm halts.

First note that if $(f, u)$ is the $(i + 1)^{st}$-critical pair, then $f^{i+1} < f^i u$. If $f^i u = \infty$ this is trivial, so suppose $f^i u \neq \infty$. Then there exists a greatest $j \leq i$ such that $(f, u)$ is the $j^{th}$ critical pair. Let $f = f_Q \times \theta$, and let $\pm \theta$ be $\theta$ or $-\theta$ according to whether $Q \in \exists$ or $\forall$. Then $S^i(\pm \theta)(f^j u, u)$ is true, and for all $k$ with $j \leq k \leq i$, $S^k(\pm \theta)(f^k u, u)$ remains true since nothing has altered. Thus, for $(f, u)$ to be the $(i + 1)^{st}$ critical pair, the rank of $f$ must be $q$ and $f^{i+1} u < f^i u$. Finally, $f^{i+1} u \neq \infty$, for otherwise there would exist a loop.

Next note that if $i < j$, if the rank of $\sigma^j$ is not less than that of $\sigma^i$, and if the rank of $\sigma^k$ is greater than that of $\sigma^j$ for all $k$ between $i$ and $j$, then for all $f$ in $F$, and all $u$, $f^j u = f^i u$ except when $(f, u)$ is the $j^{th}$ critical pair, in which case $f^j u < f^i u$.

Finally note that if for all $f$ in $F$ and all $u$, $f^j u < f^i u$ or $f^j u = f^i u$, then $\sigma^j = \sigma^i$ or $\sigma^j_n \leq \sigma^i_n$, where $\leq$ is the ordering $(<^*)^\varnothing$.

Let $(i, \nu)$ denote $\langle \sigma^j \rangle_{i \leq j < \nu}$, where $\nu \leq \Omega$. $(i, \nu)$ is an $r$-subroutine if the rank of $\sigma^i$ is $\leq r$, the rank of $\sigma^j$ is greater than $r$ for all $j$ between $i$ and $\nu$, and if $\nu < \Omega$, the rank of $\sigma^\nu$ is $\leq r$.\[^{13}\]

**3.4 Lemma** Let $(i, i + s)$, $(j, j + s)$ be two consecutive (i.e. $i + s = j$) $r$-subroutines and suppose that the rank of $\sigma^j$ is not less than that of $\sigma^i$. Then there exists $p < \min(s, t)$ such that for all $i < p$, $\sigma^i p \leq \sigma^i$ and $\sigma^i p = \sigma^i$. Proof: By the above remarks, $\sigma^i n \leq \sigma^i$ or $\sigma^j = \sigma^i$. In the first case, let $p = 0$ and we are done. If $\sigma^j = \sigma^i$, then $\sigma^j p = \sigma^i$. In the first case we let $p = 1$, and in the second we repeat our argument again and again. We do in fact find a $p < \min(s, t)$, for suppose otherwise. If $s < t$ (or $s < t$), then $\sigma^{i+s-1} = \sigma^{j+s-1}$. This implies that the rank of $\sigma^j (\sigma^{i+s})$ is greater than $r$. And if $s = t$, then $\sigma^{i+s-1} = \sigma^{j+s-1}$ implies the existence of a common $j^{th}$ and $(j + t)^{th}$ critical pair, contradicting the remarks above.

For each $r \leq q$ define an ordering $\leq_r$ on the finite $r$-subroutines inductively as follows. For each $r \leq q$ define an ordering $\leq_r$ on the finite $r$-subroutines inductively as follows.

\[^{13}\] I have reversed Scannlon’s terminology here because my labelling seems to be more natural if one hopes to extend this method to stronger systems.
each \( r \)-subroutine is the concatenation of a unique sequence of \( r + 1 \)-subroutines: let \( \ll_r \) be the lexicographic ordering based on \( \ll_{r+1} \). The next lemma is purely combinatorial.

**3.5 Lemma** Let \( \langle i, i + s \rangle, \langle j, j + t \rangle \) be two \( r \)-subroutines such that for some \( p < \min(s, t) \), \( \bar{\sigma}_{i+p} \ll \bar{\sigma}_{n+p} \) and \( \bar{\sigma}_{j+l} = \bar{\sigma}_{j+l+1} \) for all \( l < p \). Then \( \langle j, j + t \rangle \ll_r \langle i, i + s \rangle \).

Proof: By induction on \( r \). If \( r = q \) there is nothing to prove. Suppose the lemma holds for \( r + 1 \), and let \( \langle i, i + s \rangle, \langle j, j + t \rangle \) satisfy the hypotheses. Choose the greatest \( k \leq p \) such that \( \sigma_{i+k} \) is of rank \( \leq r + 1 \). (Such a \( k \) exists because \( \sigma \) of rank \( r \) has rank \( r \).) Then \( \sigma_{i+k} \) has the same rank. By the inductive hypothesis, the \( r + 1 \)-subroutine beginning with \( \sigma_{i+k} \) is less than (in the order \( \ll_{r+1} \)) the one beginning with \( \sigma_{i+k} \). Since for all \( l < k, \bar{\sigma}_{i+l+1} \) and \( \bar{\sigma}_{j+l+1} \) are equal and have equal ranks, we may conclude that \( \langle j, j + t \rangle \ll_r \langle i, i + s \rangle \). \( \square \)

By induction on \( r < q \) we see that each \( r \)-subroutine is a concatenation of a \( \ll_{r+1} \)-descending sequence of finite \( r + 1 \)-subroutines, and so each \( r \)-subroutine is finite. In particular, the (unique) 0-subroutine is finite, i.e. our algorithm terminates.

For any formula \( \varphi \) (perhaps with parameters) the above algorithm is easily modified to produce a \( \sigma \) which also fulfils \( \varphi \) if \( \varphi \) is true. If \( \varphi \) has definable satisfaction functions, we merely add these to our language. If \( \varphi \) does not have (globally) definable satisfaction functions, we may still define some finite approximations as we go along. More precisely, to each partial computation, \( \langle \bar{\sigma}^0, \bar{\sigma}^1, \ldots, \bar{\sigma}^k \rangle \) we assign an index as follows. For each \( r \leq q \) let \( \infty \) be a new constant symbol and extend the ordering \( \ll \) to the ordering \( \ll^+_r \) which has \( \infty \) as its maximal element. Let the index of \( \langle \bar{\sigma}^0, \ldots, \bar{\sigma}^k \rangle \) be \( \langle a_0, a_1, \ldots, a_r \rangle \), where \( r \) is the rank of \( \sigma^k \) for all \( s \leq r \).

\[
\alpha_s = \begin{cases} 
\langle j, l \rangle, & \text{where } l \leq k \text{ and } j \text{ is the greatest } j < k \text{ for which this is an } s\text{-subroutine, if this exists,} \\
\infty, & \text{otherwise.}
\end{cases}
\]

Order these indices lexicographically, that is, \( \langle a_0, \ldots, a_s \rangle \ll \langle b_0, \ldots, b_t \rangle \) iff either there exists \( r < \min(s, t) \) such that \( a_p = b_p \) for all \( p < r \) and \( a_r \ll_r b_r \) or \( s < t \) and \( a_r = b_r \) for all \( r \leq s \).

Now consider the set

\[ \{ \alpha : \alpha \text{ is the index of some partial computation } \langle \sigma_0, \ldots, \sigma^k \rangle \text{ which is s.t. each } \sigma^k \text{ fulfils } \Tr \Sigma_l \} . \]

By Foundation(\( \preceq_{\varphi} \)) choose a minimal \( \alpha \) in this set and some partial computation \( \langle \sigma_0, \ldots, \sigma^k \rangle \) which is a witness for this \( \alpha \). Then \( \sigma^k \) is in the halting state, for suppose not. As Foundation(\( \preceq_{\varphi} \)) implies Induction, we may choose a further finite approximation to the satisfaction functions for \( \Tr \Sigma_l \) and then using these perform the next step of our algorithm. This produces a \( \sigma_{k+1} \) which also fulfils \( \Tr \Sigma_l \). But \( \langle \sigma_0, \ldots, \sigma^{k+1} \rangle \) has index less than \( \alpha \), a contradiction.

In particular, we can ensure that \( \sigma \) fulfils the axioms of \( S \), and similarly we can have \( (\Tr x \neq y)^\sigma \). This completes the proof of 3.1. \( \square \)

Next we fulfil a promise made in Chapter II on page 21.

**3.6 Corollary** \( S \vdash \forall \varphi \left( \forall m \exists \tau (|\tau|^m = m \land \varphi^\tau) \supset \forall n, \Gamma \exists \sigma \left( |\sigma|^n = n \land \varphi^{\sigma} \land (\Gamma \lor \neg \Gamma)^\sigma \right) \right) \), and PRA proves the *fulfilment version.*

Proof: We shall sketch an informal proof. Let \( \varphi, \Gamma, \) and \( n \) be given, and suppose that Skolem function symbols for the existential quantifiers of \( \varphi \) are added to the language. Consider the proof of 3.1 when \( \preceq \) is the empty relation and \( \Gamma = H \). Then it is easy to calculate an upper bound on the length of any \( \ll_{\preceq} \)-descending sequence. Hence we can calculate an upper bound \( b \) on the number of iterations of our algorithm requires to find a sequence \( n \)-fulfilling \( \Gamma \lor \neg \Gamma \) and which is
closed under the functions of the language—and the bound is independent of how these functions are interpreted. Now choose \( m = (b + 1)(n + 1) \), choose a \( \tau \) which \( m \)-fulfills \( \varphi \), and interpret the Skolem functions for the existential quantifiers of \( \varphi \) in accordance with the winning strategy for the game associated with \( \varphi^\tau \). We have chosen \( m \) large enough so that we may carry out our computation without encountering any undefined values of these Skolem functions.

If \( S + \text{Foundation}(\prec) = \text{PA} \), our proof gives the sharp bounds of Minc \([71]\).

### 3.7 Corollary
For any natural ordering \( \prec \) of order type \( 2\omega^2 \), the consistency of \( S + \Sigma_{k+1}^- \text{-Induction} \) is provable in \( \text{Foundation}(2\omega^2) \) with \( \text{Foundation}(2\omega^2) \) restricted to primitive recursive predicates, where \( 2\alpha^0 = \alpha, 2\alpha^1 = 2\alpha^2 \).

Sketch of Proof: Note that \( \text{PA}^- \) implies

\[
\Sigma_k^- \text{-Induction} \equiv \Pi_k^- \text{-Induction} \equiv \Sigma_k^- \text{-Foundation}(\omega).
\]

Also note that that \( \omega^2 \) is essentially just the lexicographic ordering of pairs of integers and so for any “natural” ordering of order type \( \omega^2 \) we have

\[
\Pi_k^- \text{-Foundation}(\omega^2) \supset \Sigma_{k+1}^- \text{-Foundation}(\omega).
\]

Inspection of the ordinals in the proof of 3.1 yields that

\[
\text{PR-Foundation}(2\omega^2) \vdash \forall n \exists \sigma \left( \left| \sigma \right| = n \land (\Pi_k^- \text{-Foundation}(\omega^2))^\sigma \land (\text{Tr} \Delta_0)^\sigma \right),
\]

where we obtain “\( \text{Tr} \Delta_0^\sigma \)” by considering \( \Delta_0 \) matrices rather than quantifier-free ones, and adding the appropriate Skolem functions to the language. This in turn implies, by 2.6 and the formalized versions of (3) and (4),

\[
\text{PR-Foundation}(2\omega^2) \vdash \text{RFN}_{\Sigma_1}(\text{PA}^- + \Sigma_{k+1}^- \text{-Induction}).
\]

It remains to note:

\[
(\omega^2 + 1)^\omega \leq (\omega^2 + 1)^\omega \leq \omega^\omega = 2\omega^2.
\]

Our next corollary is the “no-counter-example” interpretation due to Kreisel. Consider for example the sentence \( \varphi = \exists x \forall y \exists u \forall v \psi xyuv \), where \( \psi \) is \( \Delta_0 \). Third-order logic easily yields:

\[
\varphi \equiv \neg \forall x \exists y \forall u \exists u \neg \psi
\]

\[
\equiv \neg \exists f, g \forall x, u \neg \psi(x, f, x, u, g, xu)
\]

\[
\equiv \forall f, g \exists x, u \psi(x, f, x, u, g, xu)
\]

\[
\equiv \exists H, K \forall f, g \psi(Hfg, f(Hfg), Kfg, g(Hfg)(Kfg)).
\]

If \( \varphi \) is true in \( \mathbb{N} \), we may choose \( \langle H, G \rangle \) to be recursive: let \( \langle Hfg, Kfg \rangle \) be the least pair \( \langle x, u \rangle \) such that \( \psi(x, f, x, u, g, xu) \).

### 3.8 Corollary
If \( \text{PA} \vdash \varphi \), we may choose \( \langle H, K \rangle \) above to be \( <_{\varepsilon_0} \)-recursive.

Sketch of proof: We shall not define the notion of \( <_{\varepsilon_0} \)-recursiveness, but merely indicate how to obtain \( H \) and \( K \). Suppose \( \text{PA}^- + \Sigma_k^- \text{-Induction} \vdash \varphi \). By the proofs of 3.1 and 2.5, by \( 2\omega^2 \)-recursion we may find, uniformly in \( f \) and \( g \), a sequence \( \sigma \) which \( 5 \)-fulfills \( \varphi \) and which is closed under \( f \) and \( g \). Let \( \langle Hfg, Kfg \rangle \) be the least pair \( \langle x, u \rangle \) from \( \sigma_1 \times \sigma_2 \) such that \( \varphi(x, f, x, u, g, xu) \).
We shall end this chapter with an application which will be useful in the next. First we give a general corollary. Let \( S \) consist of a finite number of \( \Sigma_3 \) sentences.

**3.9 Corollary**  (i) Let \( k \geq 1 \) and suppose for some \( \Sigma_{k+2} \) formula \( \chi(x) \)

\[
S \vdash \exists! x \chi(x) \wedge \text{“} x \text{ is a binary relation”}
\]

Denote this relation by \( < \). Then the schema of \( \text{Foundation}(<) \) is a \( \Pi_{k+2} \)-conservative extension of \( \nabla_k \)-\( \text{Foundation}(<) \) over \( S \), where

\[
\nabla_k = \text{closure of } \Sigma_k \text{ under Boolean operations and bounded quantification.}
\]

(ii) More generally we have the following. Consider the schema of so-called Bar-Induction:

\[
\Gamma \text{-BI}_k : \quad \WF(x) \supset \psi x \\
\Gamma \text{-BI} : \quad \forall x (\WF(x) \supset \psi x)
\]

where \( \psi x \) ranges over \( \Gamma \)-\( \text{Foundation}(x) \). Let \( \theta x m \) be \( \Pi_{k+2} \) with \( k \geq 1 \). Then for any instance \( \varphi x \) of \( \text{BI}_k \), there is an instance \( \varphi_0 x \) of \( \nabla_k \text{-BI}_x \) such that

\[
S \vdash \forall m \left( \Pr_{\varphi_0 x} (\forall x (\varphi x \supset \theta x m)) \supset \forall x (\varphi_0 x \supset \theta x m) \right).
\]

Sketch of Proof: Note that (ii) implies (i), so let us consider (ii). Choose an instance \( \varphi_0 x \) of \( \nabla_k \text{-BI}_x \) so that

\[
S \vdash \forall x \left( \varphi_0 x \supset \varphi_1 (2^{(x+1)^x}) \right)
\]

where \( k \in \omega \) and \( \varphi_1 y \in \Pi_k \text{-BI}_y \) are such that

\[
S \vdash \forall x \left( \varphi_1 (2^{(x+1)^x}) \supset \forall y (\neg \theta y \supset \forall n \exists \sigma (|\sigma|^\omega = n \wedge x \in \sigma_0 \wedge (\neg \theta y)^\sigma \wedge (\varphi x)^\sigma)) \right).
\]

In Chapter VI page 68 we shall, however, prove a result (essentially due to Friedman [76]) which in most applications is stronger, as follows. Let \( \text{countable} \) bar-induction be the schema:

\[
\text{cBI} : \quad \forall x (\text{“} x \text{ countable”} \wedge \WF(x) \supset \psi x), \text{ where } \psi x \text{ ranges over } \text{Foundation}(x).
\]

**Fact**: Let \( k > 1 \) and let \( \theta x m \) be \( \Pi_{k+2} \). Then for any instance \( \varphi x \) of \( \text{cBI}_k \), there is an instance \( \varphi_0 x \) of \( \Pi_k \text{-cBI} \) such that

\[
S + \Delta_0 \text{-cBI} \vdash \forall m \left( \Pr_{\varphi_0 x} (\forall x (\varphi x \supset \theta x m)) \supset \forall x (\varphi_0 x \supset \theta x m) \right).
\]

We shall also mention, for the purposes of comparison, the following well-known result.

**Fact**: If \( k > 1 \) and \( \theta m \) is \( \Pi_{k+2} \) then

\[
S + \Sigma_k \text{-DC} \vdash \forall m \left( \Pr_{\varphi_0 x} (\theta m) \supset \theta m \right).
\]
for each \( \Pi_1^1 \) formula \( \theta XY \) there exists a \( \Pi_1^1 \) formula \( \psi XY \) such that
\[
\Pi_1^1 \text{-CA} \vdash \forall X \left( \exists Y \theta XY \supset \exists!Y (\theta XY \land \psi XY) \right).
\]

We note that the right-hand-side is \( \Pi_1^4 \) and that the schema of \( \Pi_1^1 \text{-CA} \) is (equivalent in \( \Pi_1^1 \text{-CA} \) to a schema which is) \( \Pi_1^3 \). Since \( \Pi_1^2 \text{-CA} \) implies \( \forall_2 \text{-Induction} \), we have that
\[
\Pi_1^2 \text{-CA} \vdash \text{"}\Pi_1^1 \text{-Uniformization"}
\]
and so by the usual arguments,
\[
\Pi_1^2 \text{-CA} \vdash \Sigma_1^1 \text{-AC}.
\]

A similar application is as follows. If \( k \geq 3 \), then for any \( \Sigma_k \) formula \( \theta XY \),
\[
\Pi_{k-1}^1 \text{-CA} \vdash \forall X, W \left( V = L(W) \supset \left( \exists Y \theta XY \supset \exists Y (\theta XY \land \forall Z (\theta XZ \supset Y \prec Z)) \right) \right).
\]

where \( \prec \) is a \( \Pi_1^1 \) formula with parameter \( W \) representing Addison’s well-ordering of the reals constructible from \( W \). (The \( \Sigma_1^1 \)-BI needed for this proof is derivable from \( \Pi_1^1 \text{-CA} \).) Now the right-hand-side is \( \Pi_k^1 \) and \( \Pi_k^1 \text{-CA} \) is (essentially) \( \Pi_k^1 \), and so by 3.9 we have that this is provable in \( \Pi_k^1 \text{-CA} \) and so:
\[
\Pi_k^1 \text{-CA} + \exists W. V = L(W) \vdash \Sigma_1^1 \text{-AC}.
\]

Our final application is a conservation result for arithmetic; let \( S = \text{PA}_{\omega} \), as on page 12. Add a new constant \( c \) to the language of arithmetic, and consider the schema \( \text{Induction}(c) \) of induction up to \( c \). From the proof of 3.1 we see that for each \( k, n \in \omega \) and each finite subset \( H \) of \( S + \text{Induction}(c) \) there exists \( l \in \omega \) such that
\[
S + \text{Induction}(2^c_l) \vdash \forall m \exists \sigma (|\sigma|^= \Pi \land m \in \sigma_0 \land H^\sigma \land \text{Tr} \Sigma_k^\sigma).
\]

Hence we may obtain results of the following form, where \emph{conservative extension} is as defined on page 37.

3.10 Corollary With respect to the language of arithmetic plus \( c \), \( \text{Induction}(c) \) is:

i. \( \Pi_2 \)-conservative over \( S \)

ii. \( \Pi_3 \)-conservative over \( S + \Delta_1 \text{-Induction}(c) \)

iii. \( \Pi_{k+2} \)-conservative over \( S + \Sigma_{k-1} \text{-Collection} + \Sigma_k \text{-Induction}(2^c_l) \) \( l \in \omega \), and is

iv. \( \Pi_{k+2} \)-conservative over \( S + \Sigma_{k+2} \text{-Induction}(2^c_l) \) \( l \in \omega \).

\( \square \)
IV Some Conservation Results

This chapter consists, in part, of an exposition of Friedman’s [70] method of obtaining conservation results of the form e.g. \( \Sigma^1_{k+1} \text{-AC} \) is \( \Pi^1_k \)-conservative over \( \Pi^1_k \text{-AC} \), for various \( k \) and \( l \), and \( \Sigma^1_k \text{-AC} \) is \( \Pi^1_1 \)-conservative over \( \{ (\Pi^1 \text{-CA})_\alpha : \alpha < \varepsilon_0 \} \) (i.e., axioms asserting that the \( \Pi^1_1 \)-jump may be iterated \( \alpha \) times for all \( \alpha < \varepsilon_0 \)). The former type of result—the so-called restricted case—is not, however, explicitly stated in Friedman [70]. This is probably because at that time nobody was interested in this case, for it seems to me improbable that one could discover the unrestricted versions without being aware of the restricted. Let us briefly, then, indicate the work done by others.

Barwise and Schlippers [75] gave a model theoretic proof for the \( \Sigma^1_1 \text{-AC} \) case, which was considerably simplified by Feferman [76] and, independently, Stavi. Feferman [76] also considered \( \Sigma^1_2 \text{-AC} \). These latter proofs are, in essence, the same as those given below. Proof-theoretic proofs have been given by Tait ([68] & [70]) for \( \Sigma^1_1 \text{-AC} \) and \( \Sigma^1_2 \text{-AC} \), and by Feferman [77] and Feferman and Sieg [80] for both \( \Sigma^1_k \text{-AC} \) and \( \Sigma^1_1 \text{-AC} \) for all \( k \).

The other part of this chapter deals with uniform versions of the above: for example

\[
\forall n (\Pi^1_1 \text{-AC})_n \vdash \text{RFN}_{\Pi^1_1}(\Sigma^1_2 \text{-AC})
\]

and

\[
\forall \alpha < \varepsilon_0 (\Pi^1_1 \text{-AC})_\alpha \vdash \text{RFN}_{\Pi^1_1}(\Sigma^1_2 \text{-AC}),
\]

where e.g. \( \forall \alpha < \varepsilon_0 (\Pi^1_1 \text{-AC})_\alpha \vdash \text{RFN}_{\Pi^1_1}(\Sigma^1_2 \text{-AC}) \) consists of the axiom asserting that for all \( \alpha < \varepsilon_0 \) and for all \( X \), there exists a hyperjump hierarchy of length \( \alpha \) relativized to \( X \) (plus other simple axioms).

Let me briefly discuss the papers of Feferman [77] and Feferman and Sieg [80]. The methods in these papers—namely, the normalization of infinite terms followed by a Gödel-style functional interpretation, and cut-elimination arguments, respectively—are stronger than those used here in that they give conservation results for certain extensions of the theory of types, \( \mathbb{Z}^\omega + \text{QF-AC} \), whereas mine do not. But if one is only interested in, say, second-order conservation results, then these proof-theoretic arguments are, to my mind, much more complicated than the corresponding model-theoretic arguments. It seems likely, however, that the uniform versions may also be obtained from Feferman and Sieg [80], and in this case the two methods would be of about the same complexity.

The plan of this chapter is as follows. After some preliminary definitions, we shall consider the restricted case, then give some applications, followed by the unrestricted case. It is hoped that the presentation here of Friedman’s construction in a more general setting (thus making it necessary to state explicitly the principles used) helps rather than hinders the reader.

Given a class of sentences \( \Lambda \) and sets of sentences \( A \), \( B \), and \( C \), say that \( A \) is a \( \Lambda \)-conservative extension of \( B \) over \( C \) if \( A + C \) extends \( B + C \) but for all \( \theta \) in \( \Lambda \), if \( A + C \vdash \theta \) then \( B + C \vdash \theta \).

Let \( S \) be, as usual, a theory suitable for elementary set theory as on page 11. For convenience, we shall assume that the axioms of \( S \) are \( \Sigma_3 \). In Chapter I we assumed, also mostly for convenience, that \( S \) proves the existence of a \( \Sigma_1 \) satisfaction predicate for \( \Delta_0 \) formulae. There is, however, an interesting application of Theorem 4.2 (Application (vii) below) in which this is not the case, and so we note that this assumption is not necessary for this theorem. Consider the schema Bounding
given on page 12. It is easy to show:

4.1 Fact In the presence of $S + \Delta_0$-Separation, the following are equivalent:

$$\Pi_k$$-Bounding

$$\Sigma_{k+1}$$-Bounding

$$\Sigma_{k+1}$$-Collection + $$\Sigma_{k+1}$$-Separation.

Let $S_1$ be the set of formulae of the form

$$\exists \bar{x} \forall a \exists b \forall \bar{y} \in a \exists \bar{z} \in b \theta,$$

where $\exists \bar{x} \forall \bar{y} \exists \bar{z} \theta$ is an axiom of $S$. Obviously,

$$S + \Sigma_1$$-Collection $\vdash S_1;$$

indeed, in many natural examples, $S \vdash S_1$.

4.2 Theorem $\Sigma_1$-Collection is $\Pi_2$-conservative over $S_1$, and for all $k \geq 1$, $\Sigma_{k+1}$-Collection is a $\Pi_{k+2}$-conservative extension of $\Sigma_k$-Bounding over $S$.

We shall give two proofs which are essentially the same but the second is more readily formalized. After a corollary of this second proof, we shall give a long list of applications.

Proof: Fix $k \geq 0$. Let $A = \langle A, \ldots \rangle$ be a model of $S_1$ and, if $k \geq 1$, $\Sigma_k$-Bounding, and suppose that $S$ is coded in $A$ and that $A$ is recursively saturated. It suffices to show that for any $c \in A$ there exists a $k$-elementary substructure $B = \langle B, \ldots \rangle$ of $A$ containing $c$ which is a model of $S + \Sigma_{k+1}$-Collection.

Fix $c \in A$. For each $n \in \omega$ there exists in $A$ a sequence $\sigma = \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle$ such that:

i. $|\sigma|^{\bar{c}} \geq n$;

ii. $c \in \sigma_0$, and the (interpretations of) the constants of the language are contained in $\sigma_0$;

iii. $\sigma_i \subseteq \sigma_{i+1}$, and $x \in y \in \sigma_i \supseteq x \in \sigma_{i+1}$, for all $i < |\sigma|^{\bar{c}}$;

iv. if $\exists \bar{x} \forall \bar{y} \exists \bar{z} \theta$ is among the first $n$ axioms of $S$, then there exists $\bar{x} \in \sigma_1$ such that for all $i < |\sigma|^{\bar{c}}$,

$$\forall \bar{y} \in \sigma_i \exists \bar{z} \in \sigma_{i+1} \theta;$$

and

v. if $k \geq 1$ and if $\exists \bar{y} \theta(\bar{x}, \bar{y})$ is among the first $n$ $\Sigma_k$ formulae, then for all $i < |\sigma|^{\bar{c}}$,

$$\forall \bar{x} \in \sigma_i (\exists \bar{y} \theta \supset \exists \bar{y} \in \sigma_{i+1} \theta).$$

By recursive saturation, there exists $\sigma$ satisfying (i) to (v) for all $n \in \omega$. Let $B = \bigcup_{i \in \omega} \sigma_i$. It is clear that $B \models S$ and that $B \prec_k A$. (If $k = 0$, then (iii) ensures that $B$ is a transitive substructure, and so $\Delta_0$ formulae are absolute.) A simple application of underspill will show that $\Sigma_{k+1}$-Collection holds in $B$, as follows.

Suppose $B \models \forall \bar{x} \in a \exists \bar{y} \theta$ where $\theta$ is $\Pi_k$. Since $B$ is a transitive substructure, the set $a$ has no more elements in $A$ than in $B$ and so for any nonstandard $i < |\sigma|^{\bar{c}}$,

$$A \models \forall \bar{x} \in a \exists \bar{y} \in \sigma_i \theta.$$
IV Some Conservation Results

By underspill, this must hold for some standard \( i \). Since \( \sigma_i \in B \) for all standard \( i \), we may conclude
\[
\mathcal{B} \models \exists b \forall \bar{x} \in a \exists \bar{y} \in b \theta ,
\]
and this completes the first proof of theorem 4.2.

Our second proof is, for each fixed value of the parameter \( n \) purely internal. Fix \( n \in \omega \), and suppose \( \sigma \) satisfies (i) to (v). We claim that \( \sigma \) fulfils the first \( n \) axioms of \( S \), that \( \sigma \) satisfies
\[
\forall i < |\sigma^-| \forall \bar{x} \in \sigma_i (\theta \bar{x} \supset (\theta \bar{x})^\sigma)
\]
for \( \theta \) among the first few \( \Sigma_k \) formulae (where here \( f \)ew depends on \( n \), on the Gödel numbering, etc.), and that \( \sigma \) fulfils the first few sentences of the form
\[
\forall a, \bar{z} \exists \bar{x} \in a \forall \bar{y} \exists b (\theta \bar{x} \bar{y} \bar{z} \supset \forall \bar{x}' \in a \exists \bar{y}' \in b \theta \bar{x}' \bar{y}' \bar{z})
\]
(1)
where \( \theta \) is \( \Pi_k \).

We need only check the latter part of this claim. First observe that for any \( \Pi_k \) formula \( \theta \bar{x} \bar{y} \bar{z} \) whose Gödel number is sufficiently small with regard to \( n \), we have for all \( i, j, l \), with \( i \leq j, l < |\sigma^-| \), and for all \( \bar{x}, \bar{y}, \bar{z} \in \sigma_i \),
\[
\theta^\sigma_j \equiv \theta^\sigma_i \equiv \neg (\neg \theta)^\sigma
\]
(2)
because each of these holds if and only if \( \theta \) does. To show that \( \sigma \) fulfils (1), we use the terminology of game theory. Suppose Player I has chosen \( i < |\sigma^-| \) and \( a, \bar{z} \in \sigma_i \). Let
\[
j_o = \begin{cases} 
\min j < |\sigma^-| \text{ s.t. } \forall \bar{x} \in a \exists \bar{y} \in \sigma_j \theta \bar{x} \bar{y} \bar{z}, & \text{if this exists,} \\
|\sigma^-|, & \text{otherwise.} 
\end{cases}
\]
Let Player II choose \( \bar{x} \in a \) such that if \( j_o \neq 0 \), then
\[
\neg \exists \bar{y} \in \sigma_{j_0-1} \theta \bar{x} \bar{y} \bar{z}.
\]
Now Player II has essentially won, for if Player I chooses \( j < j_o \) and \( \bar{y} \in \sigma_i \), then
\[
\neg (\neg \theta \bar{x} \bar{y} \bar{z})^\sigma_j
\]
is \textit{false}, and if Player I chooses \( j \geq j_o \), then II picks \( b \) to be \( \sigma_{j_0} \). This proves the claim. The theorem now follows from 2.6.

It is clear that for \textit{finite} \( S \) the second proof may be formalized to obtain by 2.6:

4.3 Corollary \( S_1 + \Sigma_1\)-Induction \( \vdash \text{RFN}_{\Pi_2}(S + \Sigma_1\text{-Collection}) \), and for \( k \geq 1 \), \( S + \Sigma_{k+1}\)-Induction + \( \Sigma_k\)-Bounding \( \vdash \text{RFN}_{\Pi_{k+2}}(S + \Sigma_{k+1}\text{-Collection}) \).

Further observing that we do not need to work with models of \( \Sigma_k\)-Bounding but only with arbitrarily large “finite approximations” of such models, we have, for example:
\[
PRA \vdash \forall k \forall \theta \in \Sigma_{k+2} \left( \forall n \exists \sigma (|\sigma^-| = n \land (S + \Sigma_k\text{-Bounding} + \theta)^\sigma) \supset \forall n \exists \sigma (|\sigma^-| = n \land (S + \Sigma_{k+1}\text{-Collection} + \theta)^\sigma) \right),
\]
which immediately translates into:
\[
PRA \vdash \forall k \forall \theta \in \Pi_{k+2} \left( P_{\text{PRA}}(\Sigma_{k+1}\text{-Collection}(\theta) \supset P_{\text{PRA}}(\Sigma_k\text{-Bounding}(\theta)) \right).
\]
Similarly, we may obtain e.g. for $k \geq 1$, 
\[ S \vdash \forall l \geq k \left( \text{RFN}_{\Pi_{k+2}}(S + \Sigma_l\text{-Bounding}) \supset \text{RFN}_{\Pi_{k+2}}(S + \Sigma_{l+1}\text{-Collection}) \right). \]

Next we shall consider some applications of 4.2, beginning first with set theory. The following are immediate from our results.

**Application (i)** Let \( \text{KP} \) be \( \text{KP} \) with Infinity but with the Foundation schema replaced by the Foundation axiom. Then \( \text{KP} \) is a \( \Pi_2 \)-conservative extension of \( \text{KP} \setminus \Delta_0\text{-Collection} \). Hence \( \text{KP} \) is a very weak set theory indeed: \( L_{\omega + \omega} \) is a model of its \( \Pi_2 \) consequences.

**Application (ii)** For \( k \geq 1 \), \( \Sigma_{k+1}\text{-Replacement} \) is a \( \Pi_{k+2} \)-conservative extension of \( \Sigma_k\text{-Separation} \) over \( \text{KP} \). For any non-projectable \( \gamma > \omega \), \( L_\gamma \) is a model of the \( \Pi_3 \) consequences of \( \Sigma_2\text{-Collection} + \text{KP} \) (because \( \gamma \) is the limit of \( \gamma \)-stable ordinals).

**Application (iii)** \( \Sigma_1\text{-Induction} + \text{KP} \setminus \Delta_0\text{-Collection} \vdash \text{RFN}_{\Pi_1}(\text{KP}) \), and for \( k \geq 1 \),

\[ \text{KP} \vdash \Sigma_{k+1}\text{-Induction} + \Sigma_k\text{-Collection} + \Sigma_k\text{-Separation} \]

\[ \vdash \text{RFN}_{\Pi_{k+2}}(\text{KP} \vdash \Delta_{k+1}\text{-Induction} + \Sigma_{k+1}\text{-Collection} + \Delta_{k+1}\text{-Separation}). \]

Next let us briefly consider two-sorted theories with \( x, y, \ldots \) intended to range over sets and \( X, Y, \ldots \) over classes. We obtain, for example, that:

**Application (iv)** GB \( \setminus \{ \text{Power, Choice} \} \) plus the schema \( \Sigma_1\text{-wAC} \)

weak-\( \Sigma_1\text{-AC} \): 
\[ \forall x \exists Y \theta(x, Y) \supset \exists Z \forall x \exists y \theta(x, (Z)_y), \theta \text{ in } \Sigma_1, \]
is conservative over ZF\(^-\): for \( \Sigma_1\text{-wAC} \) is \( \Pi_1\)-conservative over GB \( \setminus \{ \text{Power, Choice} \} \), and any model of ZF\(^-\) can obviously be extended to a model of GB \( \setminus \{ \text{Power, Choice} \} \). (A weaker but similar result was obtained by Moschovakis, c.1971.)

Solovay has shown that any countable model \( A \) of ZFC may be extended to a model \( (A, \leq) \) of ZFC in the language with an extra binary predicate which also satisfies
\[ \forall x (\leq \cap x^2 \text{ is a well-ordering}). \]

Feferman [76] observed that this yields:

**Application (v)** (Schlipf [78]) GB plus the schema
\( \Sigma_1\text{-AC} \): 
\[ \forall x \exists Y \theta(x, Y) \supset \exists Z \forall x \theta(x, (Z)_x), \theta \text{ in } \Sigma_1, \]
is conservative over ZFC.

Now let us consider analysis; our language is that for \( \langle \omega, \mathcal{P} \omega; +, \times, 0, 1, \epsilon \rangle \). Here \( S \) consists of the axiom of Induction, \( \Delta^0_0\text{-CA}_\Lambda \), (where \( \Delta^0_0 \) is the closure of the atomic formulae under \( \land, \lor, \neg, \exists n < m, \forall n < m \)), and simple axioms for addition and multiplication. As usual, \( S \) plus a schema such as
\( \Gamma\text{-AC} \):
\[ \forall n \exists Y \theta(n, Y) \supset \exists Y \forall n \theta(n, (Y)_n), \theta \in \Gamma, \]
will be denoted \( \Gamma\text{-AC} \), while \( \Gamma\text{-AC} \) will denote \( \Gamma\text{-AC} \ plus the full schema of Induction. Then 4.2 gives:

**Application (vi)** (Feferman and Sieg [80]) \( \Sigma_1\text{-wAC} \) is \( \Pi^1_2\)-conservative over \( \Delta^0_0\text{-CA} \), over PR-AC\(^+\), and also over \( \Pi^1_0\text{-AC} \). Thus:

**Application (vii)** (ibid.) \( \Sigma_1\text{-wAC} \) is conservative over PRA, and

**Application (viii)** (Friedman [76], Barwise and Schlipf [75]) \( \Sigma_1\text{-AC} \) is conservative over PA.
Natural instances of Bounding are just various basis theorems. For example, the Kleene and the Kondo-Addison Basis Theorems immediately give (upon checking that the proofs go through in the appropriate theory or by using 4.1 and the discussion at the end of the last chapter) that:

**Application (ix)** (essentially Friedman [70], as are (x)-(xii) below) $\Sigma_2^1$-$AC\upharpoonright$ is a $\Pi_3^1$-conservative extension of $\Pi_3^1$-$CA\upharpoonright$, and

**Application (x)** (ibid.) $\Sigma_3^1$-$AC\upharpoonright$ is a $\Pi_4^1$-conservative extension of $\Pi_4^1$-$CA\upharpoonright$.

For further basis theorems it seems that further assumptions are needed. For example, using the relativized version of Addison’s well-ordering of the constructible reals, and that his proof may be formalized in the appropriate theory, we have:

**Application (xi)** (ibid.) For $k \geq 3$, $\Sigma_{k+1}^1$-$AC\upharpoonright$ is a $\Pi_{k+2}^1$-conservative extension of $\Pi_{k+2}^1$-$CA\upharpoonright$.

Since $\exists X.V = L(X)$ is $\Pi_4^1$-conservative over $\Pi_4^1$-$CA\upharpoonright$ for $k \geq 3$ (and this is easy to check), (xi) gives:

**Application (xii)** (ibid.) For $k \geq 3$, $\Sigma_{k+1}^1$-$AC\upharpoonright$ is a $\Pi_4^1$-conservative extension of $\Pi_4^1$-$CA\upharpoonright$.

Uniform versions of (viii), (ix) and (x) are:

**Application (xiii)** For $k = 0, 1,$ or $2$

$$\Pi_k^1$-$CA\upharpoonright$+$\Sigma_{k+1}^1$-Induction $\vdash$ RFN$_{\Pi_{k+1}^1}(\Sigma_{k+1}^1$-$AC\upharpoonright)$, (where here $\Pi_0^1 = \Pi_1^0$)

and uniform versions may also be concocted for the general case.

Next let us consider arithmetic. Here $S$ consists of $PA^-_{\infty}$, as given on page 12; all theories below are assumed to contain $S$. As is well-known, many schemata which are very different in set theory become equivalent in arithmetic. In particular,

**Fact** In $PA^-_{\infty}$ for $k \geq 1$,

$$\Sigma_k$-Induction $\equiv$ $\Pi_k$-Induction $\equiv$ $\Sigma_k$-Separation $\equiv$ $\Sigma_k$-Bounding

$$\Sigma_k$-Induction $\supset$ $\Sigma_k$-Collection \hspace{1cm} (3)$$

These results, due to Ch. Parsons ([70] and [72]) (and independently rediscovered by many others), are all straightforward to prove, except for

$$\Sigma_k$-Induction $\supset$ $\Sigma_k$-Separation

for which we shall now give a proof using an elegant trick of H. Friedman’s [79]. Suppose we have already shown (3). It is easy to check that $\Sigma_k$-Collection implies that any formula of the form $\forall x < y \theta$, with $\theta$ in $\Sigma_k$ is equivalent to a $\Sigma_k$ formula. Using this, it is easy to see that $\Sigma_k$-Induction implies any class of the form

$$\{ x < b : \varphi x \},$$

with $\varphi$ in $\Sigma_k$, has a maximum element. So given any element $a$ and any $\Sigma_k$ formula $\theta$, the class $\{ x < a : \theta x \}$ is realized by the maximum element of

$$\{ x < 2^a : \forall y \in x \theta y \}, \hspace{1cm} Q.E.F.$$
Now these equivalences immediately give for all $k \geq 0$,

**Application (xiv)** (Paris & Kirby, Friedman) $\Sigma_{k+1}$-Collection is a $\Pi_{k+2}$-conservative extension of $\Sigma_k$-Induction, and

**Application (xv)** (cf. Parsons [71], [72]) $\Sigma_{k+1}$-Induction $\vdash$ RFN$_{\Pi_{k+2}}$($\Sigma_{k+1}$-Collection).

Finally let us note, as Friedman [70] remarks at the end of his paper, that the first proof of 4.2 may be used to yield interesting results concerning certain infinitary theories. We replace the notion of recursive saturation by $\mathbb{A}$-saturation for the appropriate admissible set $\mathbb{A}$, and obtain, for example:

**Application (xvi)** In $\omega$-logic, $\Sigma_{k+1}^1$-AC is a $\Pi_{k+1}^1$-conservative extension of $(\Pi_{k+1}^1$-CA)$_{<\omega_1^C}$, where $l = 2, 3, 4$ according to whether $k = 0, 1, 2, 3$. (This notion of iterated $\Pi_{k+1}^1$-CA is described below; we mention this result here only to compare it with Application (xxi).)

Theorem 4.2 may be generalized to the so-called unrestricted case as follows. We shall assume that $S$ contains $\Delta_0$-Separation. Suppose that for some sufficiently absolute formula $\psi(x, y)$, $S$ proves $\exists! \Omega < (\psi(\Omega, <))$ and that $< \in$ is a strict linear well-ordering of its domain $\Omega$. We shall usually refer to $(\Omega, <)$ by $\Omega$, leaving the ordering implicit. As in Chapter III, we may define in $S$ and in a natural fashion from $\Omega$, the linear orderings $(\Omega)^\omega$, $2^{(\Omega+1)^\omega}$, and $\varepsilon(\Omega)$. In dealing with these orderings, it will be convenient to use the same terminology and notations which one uses for von Neumann ordinals with $\alpha, \beta, \ldots$ ranging over $\varepsilon(\Omega)$. For $\alpha = \langle \sigma_{\beta} \rangle_{\beta < \alpha}$, let $\sigma_{< \beta} = \bigcup_{\gamma < \beta} \sigma_{\gamma}$. Say $\text{good}(\sigma)$ if

$$\forall \beta \in |\sigma| \left( \bigcup \sigma_{< \beta} \cup \sigma_{< \beta} \cup \{\sigma_{< \beta}\} \cup \{\sigma \upharpoonright \beta\} \right) \subseteq \sigma_{\beta};$$

this is the analogue of clause (iii) in the proof of 4.2. For any definable $\alpha \subseteq \varepsilon(\Omega)$, consider the schemata:

- **Bounding$_\alpha$:**
  $$\forall b \exists \sigma \left( |\sigma| = \alpha \land b \in \sigma_0 \land \text{good}(\sigma) \land \forall \beta < |\sigma| \forall \beta \in \sigma_{\beta} (\exists \overline{x} \theta \supset \exists \overline{y} \in \sigma_{\beta+1} \theta) \right)$$

- **DC$_\alpha$:**
  $$\forall \overline{x} \exists \overline{y} \theta \supset \forall b \exists \sigma \left( |\sigma| = \alpha \land b \in \sigma_0 \land \text{good}(\sigma) \land \forall \beta < |\sigma| \forall \beta \in \sigma_{\beta} \exists \overline{y} \in \sigma_{\beta+1} \theta \right)$$

- **S$_\alpha$:**
  $$\forall b \exists \sigma \left( |\sigma| = \alpha \land b \in \sigma_0 \land \text{good}(\sigma) \land \exists \overline{x} \exists \overline{y} \theta \in \mathcal{S}_{\alpha} \exists \overline{x} \in \sigma_1 \forall \beta \leq |\sigma| \forall \overline{y} \in \sigma_{\beta} \exists \overline{z} \in \sigma_{\beta+1} \theta \right)$$

Let $\text{Bounding}_{< \varepsilon(\Omega)} = \bigcup_{\alpha < \varepsilon(\Omega)} \text{Bounding}(2^{(\omega+1)^\omega})$, and similarly for the other schemata.

Note that by taking the $\theta \overline{x} \overline{y}$ in DC$_\alpha$ be

$$\exists \overline{z} \theta \overline{x} \overline{z} \supset \theta \overline{x} \overline{y},$$

we have immediately:

$$\Sigma_1\text{-DC}_\alpha \vdash \text{S}_\alpha,$$

and

$$\Sigma_{k+1}\text{-DC}_\alpha \vdash \Sigma_k\text{-Bounding}_\alpha,$$

for $k \geq 1$.

Before stating our next theorem, we shall indicate, as briefly as possible, some derivations of $\Sigma_{k+1}\text{-DC}_\alpha$ from more familiar principles. For example, if WO is the axiom asserting that every set can be well-ordered, then

$$S + \text{WO} + (\text{card}(\alpha) = \beta) + \Sigma_{k+1}\text{-DC}_\beta + \Sigma_{k+2}\text{-Foundation}(\alpha) \vdash \Sigma_{k+1}\text{-DC}_\alpha.$$
Alternatively, define a \( \Sigma_{k+1} \)-Uniform Collection schema to be one which has the form:

\[
\forall a, z \left( \forall x \in a \exists y \theta x y z \supset \exists ! b (\psi_\theta abx z \land \forall x \in a \exists y \in b \theta x y z) \right)
\]

where \( \theta \mapsto \psi_\theta \) is a map from \( \Pi_k \) formulæ to \( \Sigma_{k+1} \) formulæ. (For example, in set theory we may choose \( b \) to be the least \( V_\alpha \) (or perhaps the least \( L_\alpha \)) such that \( \forall x \in a \exists y \in b \theta x y z \), and in analysis there are the Uniformization results.) Then we have:

\[
S + \Sigma_{k+1} \text{-Uniform Collection} + \Sigma_{k+1} \text{-Foundation}(\alpha) \vdash \Sigma_{k+1} \text{-DC}_\alpha.
\]

Finally, note that in analysis the standard notion of \( \Sigma^1_k \text{-DC} \) is equivalent to our notion \( \Sigma^1_k \text{-DC}_\omega \), which somewhat justifies our notation. For suppose \( \Sigma^1_k \text{-DC} \). This implies the schema:

\[
\Sigma^1_k \text{-GDC}:
\forall n, X \exists y \theta nXY \supset \forall U (\exists X \theta (U \land \forall n \theta n(V_n(V_n(U_{n+1})))).
\]

Suppose \( \forall X \exists y \theta \), where \( \theta \) is \( \Pi^1_k \)-Consistent, and we wish to show

\[
\forall A \exists \sigma (|\sigma| = \omega \land A = \sigma_0 \land \forall n \forall X \exists n \exists Y \in \sigma_n \exists \sigma_{n+1} \theta).
\]

Let \( \varphi nXY \) be:

- if \( n = 0 \), then \( Y = X \);
- if \( n = 3(k+1) \) for some \( k \in \omega \), then \( Y = (X)_k \);
- if \( n = 3(k, l) + 1 \) for some \( k, l \in \omega \), then \( Y = (X)_l \);
- if \( n = 3(k+1) + 2 \) for some \( k \in \omega \), then \( \theta(X)_kY \),

and let \( \psi UV \) be \((U = V_0) \land \forall n \varphi n V_n V_{n+1}\). Then \( \forall n \forall X \exists Y \psi n X Y \), and so by \( \Sigma^1_k \text{-GDC}, \forall U \exists Y \psi \).

Now by \( \Sigma^1_k \text{-DC}, \forall A \exists \sigma (\sigma_0 = A \land \forall n \psi \sigma_n \sigma_{n+1}) \). This implies (4).

These observations will be used when we give applications of:

**4.4 Theorem** Suppose \( S \) includes \( \Delta_0 \)-Separation. Then \( \Sigma_1 \text{-DC}_{\omega_1} \) + Foundation(\( \Omega \)) is a \( \Pi_2 \)-Conservative extension of \( S_{\omega_1} \), and for \( k \geq 1 \), \( \Sigma_{k+1} \text{-DC}_{\omega_1} \) + Foundation(\( \Omega \)) is a \( \Pi_{k+2} \)-Conservative extension of \( \Sigma_k \text{-Bounding}_{\omega_1} \) over \( S \).

Proof: As before, we shall give two proofs. The first is mostly model theoretic, but at one point it uses 3.1, namely that

\[
S + \text{Foundation}(\omega_1) \vdash \text{RFN(Foundation}(\Omega));
\]

whereas the second proof is purely internal, but it involves details of the Hilbert-Ackermann-Scanlon method for proving 3.1. The first proof is essentially that of Friedman [70], and it seems to require the further assumption that \( S \) proves \( \Omega \) is infinite. The proof is in four stages.

For the first, note that Foundation(\( \Omega \)) is \( \Pi_{k+2} \)-Conservative over \( S + \Sigma_k \text{-Bounding}_{\omega_1} \) (or, if \( k = 0 \), \( S_{\omega_1} \)). Indeed, because \( S \) proves \( \Delta_0 \)-Foundation, it is clear that any \( \Sigma_{k+2} \) sentence consistent with \( S + \Sigma_k \text{-Bounding}_{\omega} \) is consistent with \( S + \Sigma_k \text{-Bounding}_{\omega} + \text{Foundation}(\Omega) \) (and similarly for \( S_{\omega_1} \)), and that \( 2^{(\omega_1)^\omega} < 2^{(\omega_1)^\omega} \) for large enough \( k \).

For the second stage, add new constant symbols \( \alpha, \lambda \) to the language, and consider the theory:

\[
T_\alpha = \lambda \in \omega + (l > \omega)\in \omega + \alpha = 2^{(\omega_1)^\omega} + \Sigma_k \text{-Bounding}_{\alpha} \text{ (or, if } k = 0, S_\alpha \).
\]

It is clear that \( T_\alpha \) is conservative over \( S = \Sigma_k \text{-Bounding}_{\omega_1} + \text{Foundation}(\Omega) \).
For the third stage, by 3.1 we have
\[ T_\alpha + \text{Foundation}(\alpha) \vdash \text{RFN}("T_\alpha + \Sigma_\omega\text{-Foundation}(\Omega)") \]
and so our proof of the Essential Unboundedness Theorem 2.7 gives that there must be some instance \( \varphi \) of \( \text{Foundation}(\alpha) \) such that \( \neg \varphi \) is \( \Pi_{k+2} \)-conservative over the extensional theory represented by "\( T_\alpha + \Sigma_\omega\text{-Foundation}(\Omega) \)"; i.e., \( T_\alpha + \text{Foundation}(\Omega) \). (Indeed, a more careful inspection yields that we may take \( \varphi \) to be an instance of \( \Sigma_{k+1}\text{-Foundation}(\alpha) \), but we shall not need this fact.)

For the last stage, the theorem now follows immediately from:

**4.5 Lemma** For any instance \( \varphi \) of \( \text{Foundation}(\alpha) \), any \( \Sigma_{k+2} \) sentence of \( \mathcal{L} \) consistent with \( T_\alpha + \text{Foundation}(\Omega) + \neg \varphi \) is consistent with \( S + \Sigma_{k+1}\text{-DC}_{\leq \varepsilon}(\Omega) + \text{Foundation}(\Omega) \).

Proof: Let \( \mathcal{A} \) be any model of \( T_\alpha + \text{Foundation}(\Omega) + \neg \varphi \). Fix \( m \in \omega \) sufficiently large so that \( \Sigma_{m+1}\text{-Foundation}(\alpha) \) is false, and choose \( 2^\gamma \leq \alpha \) so that \( \Sigma_m\text{-Foundation}(2^\gamma) \) holds but not \( \Sigma_{m+1}\text{-Foundation}(2^\gamma) \). This we can do by choosing the least (nonstandard) \( n \) for which
\[ \Sigma_{m+1}\text{-Foundation}(2^\gamma) \]
is false, and setting \( \gamma = 2^{(\Omega+1)^\omega} \); then \( \Sigma_m\text{-Foundation}(2^\gamma) \) is implied by \( \Sigma_{m+1}\text{-Foundation}(\gamma) \).

Now we may choose a \( \Pi_{m+1} \) formula \( P \in \mathcal{L} \) (possibly with parameters) such that the class
\[ I = \{ \delta < \alpha : P\delta \} \]
is a proper initial segment of \( \alpha \) for which the following overspill property holds:
\[ \text{if } \theta x \text{ is } \Sigma_m \text{, possibly with parameters, and if } \theta \delta \text{ holds for all } \delta \in I \text{ then } \theta \delta \text{ holds for some } \delta \in \alpha \setminus I. \tag{5} \]

Now let \( a \) be an arbitrary element of \( A \) and let \( \sigma \) witness \( k\text{-Bounding}_a \) (or, if \( k = 0 \), \( S_a \)) with \( a \in \sigma_0 \). Let \( B = \bigcup_{\varepsilon < \omega} \sigma_\varepsilon \) and let \( B = \mathcal{A} \upharpoonright B \). It is clear that \( B \) is a \( k \)-elementary transitive substructure of \( \mathcal{A} \) and, as it is definable in \( \mathcal{A} \), \( \text{Foundation}(\Omega) \) must hold in it. If \( m \geq k + 2 \), it is easy to see that \( \Sigma_k\text{-Collection} \) also holds in \( B \) using the overspill property (5) just as in the proof of 4.2. However, we must show that \( \Sigma_{k+1}\text{-DC}_{\leq \varepsilon}(\Omega) \) holds in \( B \).

Suppose \( B \models \forall x \exists y \theta \), where \( \theta \) is \( \Pi_k \), and let \( b \in B \). It suffices to show that
\[ \mathcal{A} \models \exists! f \in \sigma_\delta \mathcal{H}(f, \beta, \delta) \tag{6} \]
for suitable \( \beta \), where
\[ \mathcal{H}(f, \beta, \delta) = "f : \beta \mapsto \delta \text{ is increasing}" \land b \in \sigma_f \land \forall \gamma < \beta \left( \forall x \in \sigma_f \exists y (x \in \sigma_f(y) \land \neg \exists \xi (\gamma < \xi < f(\gamma + 1) \land \forall \tilde{x} \in \sigma_f \exists \tilde{y} (\tilde{y} \in \sigma_\xi)) \right). \]

This implies \( B \models \Sigma_{k+1}\text{-DC}_\beta \), as follows. Choose \( \delta \in I \) and \( f \in \sigma_\delta \) to witness (6); then as \( S \) includes \( \Delta_0\text{-Separation} \) and as \( \langle \sigma_\xi \rangle_{\xi < \delta} \in \sigma_\delta \), we have that \( \langle \sigma_\xi \rangle_{\xi < \beta} \in \sigma_{\delta+1} \subseteq B \). Now as \( \mathcal{H} \) is absolute between \( \mathcal{A} \) and \( B \), this is our required witness to \( \Sigma_{k+1}\text{-DC}_\beta \).

To prove (6) we use induction on \( \beta \). We may suppose that \( \beta \) is a limit ordinal, for the successor case is the same as for \( \Sigma_{k+1}\text{-Collection} \). If (6) is false, then by the induction hypothesis,
\[ \alpha \setminus I = \{ \delta : \mathcal{A} \models \forall \gamma < \beta \exists f \in \sigma_\gamma \mathcal{H}(f, \beta, \delta) \}. \]
This gives us a $\Delta_0$ definition of $I$, for if $\lambda$ is a limit ordinal, for all $\delta < \lambda$ and $\bar{x}, \bar{y} \in \sigma_\delta$, 
$$\theta \bar{x} \bar{y} \equiv (\theta \bar{x} \bar{y})^{(\sigma_{\delta+1},\ldots,\sigma_{\delta+k})}. \text{ But this contradicts (5).}$$

This concludes our first proof of 4.4. \hfill $\square$

Our second proof of 4.4 is purely internal, that is, it involves no external model-theoretic considerations. We shall suppose that $k \geq 1$; the $k=0$ case is similar and easier. Consider the following:

4.6 Lemma Suppose $S$ contains the axiom of Infinity. Let $l \in \omega$ with $l \geq 1$, and let $\beta_0 = \omega^{2^{(l+1)^\omega}+1}$. Then

$$S+\Sigma_k\text{-Bounding}_{\beta_0+l} \vdash \forall n, x \exists \sigma (|\sigma|^2 = n \land x \in \sigma_0 \land \Sigma_l\text{-Foundation}(\Omega) \land \Sigma_{k+1}\text{-Collection} \land \Sigma_k \sigma).$$

Comment before proof: If $S$ includes the negation of Infinity, then the above ordinal is too large for $\Sigma_k\text{-Bounding}_{\beta_0+l}$ to make sense. We have, however the following extension of 3.10. Add a new constant $c$ to the language of arithmetic.

4.7 Lemma In arithmetic, for each $n, l \in \omega$ there exists $m \in \omega$, such that:

$$S+\Sigma_k\text{-Bounding}_{\omega^m} \vdash \forall x \exists \sigma (|\sigma|^2 = n \land x \in \sigma_0 \land \Sigma_l\text{-Induction}(c) \land \Sigma_{k+1}\text{-Collection} \land \Sigma_k \sigma). \quad \square$$

The proof of 4.7 is similar to that of 4.6, and is omitted. After sketching an informal proof of 4.6, we shall briefly describe how one could modify the proof to show the existence of a sequence which fulfils $\Sigma_{k+1}\text{-DC}_\beta$ for suitable $\beta$ rather than $\Sigma_{k+1}\text{-Collection}$. From this amended version, theorem 4.4 follows immediately. For the applications which we have in mind, however, 4.6 suffices.

Proof of 4.6: Recall from Chapter III that for any finite subset $\Gamma$ of $\text{Foundation}(\Omega)$, any $n \in \omega$, and any finite sets $C, G$ of constant and function symbols, there is an algorithm $A_{\Gamma,n}(C, G)$ which, upon inputting any (interpretation of $C$ as) elements $C$ and any (interpretation of the function symbols $G$ as) functions $G$ (of the appropriate arity), produces a sequence $\sigma$ which $n$-fulfils $\Gamma$, has $C \subseteq \sigma_0$, and is closed under $G$. Further recall that each state of $A$ (i.e. each iterate of the single loop of which $A$ consists) depends upon only finite bits of $G$. More specifically, the $i$th (initial) state depends only upon the values of terms built up from $G$ and $C$ (and the constants and functions of the language) whose height is less than $n$, and in general, to calculate the $i$th state requires, at most, knowledge of the values of the terms whose height is less than $n(i+1)$.

Also recall that with each state of $A$ is associated an ordinal such that

$$\alpha_0 > \alpha_1 > \alpha_2 > \ldots$$

where $\alpha_i$ is the ordinal of the $i$th state. These $\alpha_i$ in general depend upon $G$, but we may obtain an upper bound for $\alpha_0$ depending only on $\Gamma$ and $n$: $\alpha_0 < 2^{(\Omega+1)^m}$, for some $m$ if $\Gamma$ is included in $\Sigma_l\text{-Foundation}(\Omega)$.

Switching topics slightly, let $\psi$ be any sentence and suppose some sequence $\tau = \langle \tau_0, \ldots, \tau_{|\tau|} \rangle$ fulfils $\psi$. Let $G$ be a set of satisfaction function symbols for $\psi$. Then, as in 3.6, for any finite $C$ there exists a finite interpretation of $G$ (also denoted by $G$) by partial functions such that:

i. all terms of depth $\leq|\tau|$ built up from $C$ and $G$ have a value;
ii. if \( g \in \mathcal{G}, i < |\tau|^{-}, \) and \( \bar{x} \in \tau_i \cap \text{domain}(g) \), then \( g\bar{x} \in \tau_{i+1} \); and

iii. \( \mathcal{G} \) is, in so far as it is defined, a set of satisfaction functions for \( \psi \).

We just choose the interpretation for \( \mathcal{G} \) via a winning strategy for the game associated with \( \psi^\sigma \). Also, if \( \tau \) satisfies

\[
\forall i < |\tau|^{-} \forall \bar{x} \in \tau_i (\varphi \bar{x} \supset (\varphi \bar{x})_i^{\tau}), \quad \text{(i.e. \( (\text{Tr} \varphi \bar{x})^\tau \))},
\]

we could make a similar remark for the satisfaction functions for \( \varphi \).

Now after the above preliminaries, let us proceed with the sketch of the proof of the lemma. Fix any universal instance

\[
\forall a, \bar{x} (\forall x \in a \exists y \theta \supset \exists b \forall x \in a \exists y \in b \theta)
\]

of \( \Sigma_{k+1} \)-Collection with \( \theta \) in \( \Pi_k \), and let

\[
\psi = \forall a, \bar{x} \exists x \in a \forall y \exists b (\theta \supset \forall x' \in a \exists y' \in b \theta x' y').
\]

Fix a finite subset \( \Gamma \) of \( \Pi_i \)-Foundation(\( \Omega \)) and let \( \varphi x \) be any complete \( \Pi_k \) formula. Fix \( n \) and \( x \).

We shall show

\[
\exists \sigma (|\sigma| = n \land x \in \sigma_0 \land \psi^\sigma \land \Gamma^\sigma \land \text{Tr} \varphi^\sigma).
\]

Let \( \tau \) be a witness to \( \Sigma_i \)-Bounding\( \beta_0 \), with \( x \in \tau_0 \). Let \( C = \{x\} \) and let \( \mathcal{G} \) consist of satisfaction function symbols for \( \psi \) and \( \varphi \). Observe that for each finite subset \( M \) of \( \beta_0 \), the sequence \( (\tau_y)_{y \in M} \) fulfils \( \psi \). (And if the reader first proves this as an exercise, the remainder of the proof should be clear.) Furthermore, the remark (2) on page 39 also holds, and so there exists some interpretation of \( \mathcal{G} \) so that (i), (ii) and (iii) above hold for \( (\tau_y)_{y \in M} \) and \( \psi \) and \( \varphi x \). The problem is to choose \( M \) and \( \mathcal{G} \) so that, with this choice, the computation \( \mathcal{A} = \mathcal{A}_{\Gamma, n}(C, \mathcal{G}) \) may be carried out, thus producing the required sequence. But we cannot make an \( a \text{ priori} \) choice of \( M \) and \( \mathcal{G} \) as we did in 3.6; instead we shall choose them according to the algorithm \( C \) incorporating \( \mathcal{A} \) and another subroutine \( \mathcal{B} \), which we shall now describe.

Let TBC (to be considered) be a subclass of the ordinals less than \( \beta_0 \), and suppose initially that all are TBC. Also initially set \( M = \{0\} \) and let all the functions in \( \mathcal{G} \) have empty domains. Set \( \alpha = 2^{(\Omega+1)^m} \); \( \beta \) will always be the ordinal of the order type of TBC.

Our algorithm \( C \) is this: repeat subroutine \( \mathcal{B} \) \( n \) times, and then perform one step of \( \mathcal{A} \). Set \( \alpha \) equal to the ordinal associated with the present state of \( \mathcal{A} \). If \( \mathcal{A} \) is not in its terminal state, repeat \( \mathcal{B} \) \( n \) times again, and so on.

The subroutine \( \mathcal{B} \) is as follows. Suppose we are given \( M = \{\gamma_0 < \cdots < \gamma_j\}, \mathcal{G}, \text{TBC} (= \text{TBC}_{old}) \), and \( \alpha \). Now there exists a (unique) sequence

\[
\mu_0 \subseteq \cdots \subseteq \mu_j
\]

of finite sets such that for all \( i < j \), \( \mu_i \subseteq \tau_{\gamma_i} \), and \( \mu_j \) is included in the domains of all the functions \( \mathcal{G} \), and \( \mu_{i+1} = \bigcup_{g \in \mathcal{G} \in C} g'' \mu_i \). We extend the domains of the functions of \( \mathcal{G} \) to include \( \mu_j \) and we reduce TBC_{old} to TBC_{new}, as follows. If \( g \in \mathcal{G} \) is some satisfaction function for \( \theta \) or \( \varphi \), and \( g \bar{y} \in \mu_j \), it is clear that we can choose the appropriate \( g \bar{y} \in \tau_{\delta_j+1} \), and TBC remains unaffected by this. The interesting cases are the satisfaction functions for the quantifiers “\( \exists x \in a \)” and “\( \exists b \)” in \( \psi \). By the exercise just mentioned on page 46, it should be clear that we can define \( g_{\exists x \in a} \) and TBC_{new} so that:
i. \( \beta_{\text{new}} \), the order type of TBC_{new}, is sufficiently large (we shall explicitly state this requirement below), and if
\[
\gamma^+ = \text{least ordinal in } \text{TBC}_{\text{new}} \cup \{\beta_0\} \text{ greater than } \gamma,
\]
we have:
ii. \( \forall a, \bar{z} \in \mu_j \forall \gamma \left( \gamma_j \leq \gamma < \beta_0 \supset \forall y \in \tau_\gamma \left( \theta(g_{\exists \in a}(a, \bar{z}), y, \bar{z}) \supset \forall x' \in a \exists y' \in \tau_{\gamma^+} \theta(x'y'\bar{z}) \right) \right) \).

Now we can choose \( g_{\exists \in a} \) so that for all \( a, \bar{z} \) in \( \mu_j \), if
\[
y = \text{least } \gamma \text{ s.t. } \gamma_j \leq \gamma < \beta_0 \land \exists y \in \tau_\gamma \theta(g_{\exists \in a}(a, \bar{z}), y, \bar{z})
\]
then \( g_{\exists \in a}(a, \bar{z}, y) = \tau_{\gamma^+} \) for all \( y \in \tau_{\beta_0} \). Add \( \gamma^+_j \) to \( M \).

In requirement (i), we wish to have \( \beta \) sufficiently large so that immediately after each \( n \)-fold iteration of \( B \), we have
\[
\beta \geq \omega^\alpha.
\]
That it is possible to satisfy this requirement follows from the easy combinatorial Lemma 4.8 given below. In satisfying it, we ensure that we will always have enough TBC ordinals to continue extending the domains of the partial functions interpreting \( \Box \) until such time as the algorithm A halts.

This completes the description of the subroutine \( B \).

4.8 Lemma Let \( \delta \geq \omega^{\alpha + 1} \) and let \( N \in \omega \). Consider the following game. Set \( \delta_0 = \delta \). On the \( (i + 1) \)-th move, Player I chooses a partition
\[
\delta_i = \eta + \nu
\]
and Player II chooses \( \mu < \eta \). Set \( \delta_{i+1} = \mu + \nu \). Player II wins if \( \delta_N \geq \omega^\alpha \). We claim that Player II has a winning strategy.

Proof: Let Player II play as follows: if \( \omega^\alpha j < \eta < \omega^\alpha(j + 1) \) for \( j \in \omega \), let \( \mu = \omega^\alpha j \), and if \( \eta \geq \omega^{\alpha + 1} \), set \( \mu = \omega^\alpha N \). It is easy to check that this works. \( \square \)

This concludes our sketch of the proof of Lemma 4.6. We hope our description has been clear enough for the interested reader to fill in the missing details.

Theorem 4.4 for \( k \geq 1 \) follows from:

4.9 Lemma For \( l \geq 1 \) and for suitable \( \beta \), we have
\[
S + \Sigma_k\text{-Bounding}(\beta^{\omega(\alpha + 1)/\omega}) \vdash \forall n, x \exists \sigma \left( |\sigma| = n \land x \in \sigma \land \Sigma_l\text{-Foundation}(\Omega) \sigma \land \Sigma_{k+1}\text{-DC} \beta^\sigma \land \text{Tr } \Sigma_k \sigma \right). \square
\]

The proof of 4.9 is similar to that of 4.6, but as we shall not make use of this result, we shall omit all of it except for the statement of the analogous (equally easy) combinatorial lemma.

4.10 Lemma Let \( \delta \geq \beta^{(\alpha + 1)\omega} \) and let \( N \in \omega \). Let \( \delta_0 = \delta \) and consider the game where on the \( (i + 1) \)-th move Player I presents a partition
\[
\delta_i = \sum_{\gamma < \beta} \eta_\gamma + \nu
\]
and Player II chooses \( \delta_{i+1} = \eta_\gamma + \nu \) for some \( \gamma < \beta \). Player II wins if \( \delta_N \geq \beta^{\alpha \omega} \). We claim that II has a winning strategy. \( \square \)
As before, we may use the effective proofs to obtain uniform versions:

4.11 Corollary If $S$ includes $\Delta_0$-Separation,

$$\forall \alpha < \varepsilon(\Omega) S_\alpha \vdash RFN_{\Pi_2}(S + \Sigma_1-DC_{<\varepsilon(\Omega)} + \text{Foundation}(\Omega)),$$

and for $k \geq 1$,

$$S + \forall \alpha < \varepsilon(\Omega) \Sigma_k\text{-Bounding}_\alpha \vdash RFN_{\Pi_{k+2}}(S + \Sigma_{k+1}-DC_{<\varepsilon(\Omega)} + \text{Foundation}(\Omega)).$$

Note that in this terminology, we can restate the content of Corollary 4.3 as:

$$\forall n S_n \vdash RFN_{\Pi_2}(S + \Sigma_1\text{-Collection}),$$

and for $k \geq 1$,

$$S + \forall n \Sigma_k\text{-Bounding}_n \vdash RFN_{\Pi_{k+2}}(S + \Sigma_{k+1}\text{-Collection}).$$

Let us now briefly consider some applications of 4.4. Our notation is as before.

Application (xvii) (Jäger [78]) In set theory we have, for example, that $KP \vdash \text{Induction}$ is a $\Pi_2$-conservative extension of $KP \setminus \Delta_0\text{-Collection} + \{ \forall x \exists y y = L_\alpha(x) : \alpha < \varepsilon_0 \}$; hence $L_{\varepsilon_0}$ is a model of the $\Pi_2$ consequences of $KP \vdash \text{Induction}$. □

Application (xviii) Also if $\alpha$ is the $\alpha^{th}$ nonprojectible, and if $\beta < \alpha$ is $\Sigma_1$-definable, then $L_\alpha$ is a model of the $\Pi_3$ consequences of $KP \vdash \text{Foundation}(\beta) + \Sigma_2\text{-Collection}$. □

In analysis we have, for example:

Application (xix) (Friedman [70]) For $k = 0, 1$, or $2$, $\Sigma_{k+1}\text{-AC}$ is a $\Pi^1_{k+2}$-conservative extension of $(\Pi^1_{k+2}\text{-AC})_{<\varepsilon_0} \vdash$; □

Application (xx) For $k = 0, 1$, or $2$, $\forall \alpha < \varepsilon_0 (\Pi^1_k\text{-CA})_\alpha \vdash RFN_{\Pi_{k+2}}(\Sigma_{k+1}\text{-AC})$; and

Application (xxi) For $k = 0, 1$, or $2$, $\Sigma_{k+1}\text{-AC}$ plus the schema of

$$BL_\prec$$

where $\prec$ ranges over primitive recursive orderings or well-orderings is $\Pi_{k+2}$-conservative over the schema

$$\text{wf}(\prec) \supset (\Pi^1_k\text{-CA})_{\prec} \vdash,$$

where $\prec$ has the same range. □

Application (xxii) In arithmetic, extend the usual language $\mathcal{L}$ to $\mathcal{L}(c)$ by adding a new constant $c$. Then from 4.7 we have that for all $k \geq 0$, $\Sigma_k\text{-Collection} + \text{Induction}(c)$ is a $\Pi_{k+2}$-conservative (w.r.t. the language $\mathcal{L}(c)$) extension of $\Sigma_k\text{-Induction}$. □

The proofs of 4.4 also give the following result (which, however, does not follow from our present statement of 4.4):

Application (xxiii) $\Sigma^1_1\text{-wAC}$ is $\Pi^1_2$-conservative over $\text{PR-CA} \vdash$ + the induction schema restricted to first order formulae with second order parameters + the schema

$$\forall X \exists H \text{ "} H \text{ is the Wainer hierarchy relativized to } X \text{ up to } \alpha^{\circ}, \alpha < \varepsilon_0 \text{.}$$
V Some Model-theoretic Applications: Non-$\omega$-models

This and the following chapter contain some model-theoretic applications of the notion of fulfillment. In this chapter we shall consider only non-$\omega$-models. Let us say at the outset that much of this chapter was inspired by the work of J. Paris and L. Kirby.

Our first theorem gives necessary and sufficient conditions for the existence of certain initial segments which model a given coded theory, and indicators for the same. By an iteration of this, we are able to give necessary and sufficient conditions for the existence of an initial segment which models a given complete theory in Corollary 5.5. We next consider a result of Kirby, McAlloon, and Murawski [79]: they noticed that for any countable model $M$ of $\text{PRA}$ and for any theory $T$ in the language of analysis extending $\Pi^0_1$-$\text{CA}$ which is coded in $M$ there exists an indicator for the initial segments $I$ of $M$ which are such that $\langle I, R_I(M) \rangle$ is a model of $T$. (These terms are defined below.) We shall extend this result by weakening the condition on $T$: namely, in 5.8 we only require that $T$ extend $\Delta^0_1$-$\text{CA}$, $\text{WKL}$ (weak König’s lemma) plus (something weaker than) $\Sigma^0_1$-Induction. Our last result 5.11 gives a description of the order type of the set of elementary initial segments of a recursively saturated (r.s.) model of $\text{PA}$, and it extends results which were known to hold for countable r.s. models (or, more generally, resplendent models).

By use of the arithmetized Completeness Theorem of Hilbert and Bernays, we can obtain the following well-known result.

5.1 Result Let $M$ be a nonstandard model of $\text{PA}^-$ $+ \Delta_2$-Induction, and let $T$ be any theory coded in $M$ extending $\text{PA}$. There exists a r.s. end-extension of $M$ which is a model of $T$ iff the $\Sigma_1$ theory of $M$ is consistent with $T$. □

By the use of fulfilment we may easily obtain the dual of 5.1, that is, with “end-extension” replaced by “initial segment” and “$\Sigma_1$” by “$\Pi_1$”. Our first theorem is an elaboration upon this idea.

But first we must consider the question: what is an initial segment? For models of arithmetic the answer is clear, and for set theory we have the notion of transitive substructure. For analysis, however, there is no obvious choice. If we use the notion determined by the membership relation (as defined on page 11) and have $\bar{\omega}$ as a constant of our language, then the notion of an initial segment which is a model of $\text{QF-AC}$ is the same as that of an $\omega$-absolute substructure which is a model of $\text{QF-AC}$, which is indeed a very natural one. Or we could base the notion of initial segment on some quite different ordering, such as $x$ is constructible before $y$, or $x$ has hyper-(or Turing or Wadge)-degree less than that of $y$. We shall, however, always assume that the notion of initial segment is the one based on the membership relation, that is, $B$ is an initial segment of a structure $\mathcal{A}$, denoted by $B \subseteq^{\text{end}} \mathcal{A}$, if for all $x, y \in A$, $x \in y \in B$ implies $x \in B$ (where here, of course, the relation $x \in y$ is to be interpreted in $\mathcal{A}$). While this assumption is partially for simplicity, it is also because of doubts about the model-theoretic importance of these other notions. At the same time, we would like to emphasize that our constructions are for the most part perfectly general, and work for any appropriate interpretation of “bounded quantifier”.

Let $\mathcal{A}$ be a model of $\mathcal{S}$ as on page 11, let $B \subseteq A$ and let $b \in A$. Say $B \subseteq b$ if for all $a \in B$, $A \models a \in b$. A formula $\theta(xyz)$, possibly with parameters from $\mathcal{A}$, is an indicator for a collection $\mathcal{S}$ of initial segments of $\mathcal{A}$ if for all $a, b \in A$, there exists $I \in \mathcal{S}$ such that $a \in I \subseteq b$ iff for all $n \in \omega$, $A \models \theta n a b$. Two sets, $\mathcal{S}$ and $\mathcal{S}'$, are symbiotic if for all $a, b \in A$, there exists $I \in \mathcal{S}$ with $a \in I \subseteq b$ iff there exists $I' \in \mathcal{S}'$ with $a \in I' \subseteq b$. Say $\omega$ codes an initial segment $I$ if there exists a sequence $\sigma$ of nonstandard length such that $I = \bigcup_{i \in \omega} \sigma_i$. $\mathcal{S}$ is cofinal in $\mathcal{A}$ iff for all $a \in A$, there exists...
$I \in S$ with $a \in I$.

The next lemma is well-known (perhaps due to Gaifman or Puritz), and we omit the proof. Given structures $A, C$, with $A \subseteq C$, there exists a unique set $B$ that $A \cocod B \subseteq^\text{end} C$. If $A \models \Sigma_1$-Collection and $A \not\leq_0 C$ we have that $B = C \restriction B$ is a structure. Moreover, if $A \not\models \Delta_0$-Separation:

5.2 Lemma i. If $A \not\leq_k C$ and $\Sigma_{k+1}$-Collection holds in $A$, then $B \not\leq_k C$.

ii. If $A \not\leq_0 B$ and $\Sigma_{k+1}$-Collection holds in $A$, then $A \not\leq_{k+1} B$. \hfill $\square$

5.3 Theorem Let $k \in \omega$, let $A$ be a non-$\omega$-model of $S$, let $T$ be any theory coded in $A$, and let $S = \{ I \not\leq^\text{end}_k A : I \models T \}$.

i. Suppose $T$ extends $S$ and Collection and that $\Sigma_{k+1}$-overspill holds in $A$. Then

- $S$ has an indicator, and
- $S$ is symbiotic with $\{ I \in S : I \text{ r.s. and } \omega \text{ codes } I \}$.
- $S \setminus \{ A \}$ is nonempty iff the $\Pi_{k+1}$ theory of $A$ is consistent with $T$, and
- $S$ is cofinal in $A$ iff the $\Sigma_{k+2}$ theory of $A$ is consistent with $T$.

ii. Suppose $A$ is $\Sigma_{(k+1)}$-recursively saturated and is locally countable. Then

- $S$ is symbiotic with $\{ I \in S : I \text{ r.s. } \}$.
- $S$ is nonempty iff the $\Pi_{(k+1)}$ theory of $A$ is consistent with $T$, and
- $S$ equals $\{ A \}$ or is cofinal in $A$ iff the $\Sigma_{(k+2)}$ theory of $A$ is consistent with $T$.
- If $\Sigma_{k+1}$-Collection either holds in $A$ or is included in $T$, $S$ has an indicator, and if we also have either QF-Foundation or QF-Separation in $A$ or $T$, then $S \neq \{ A \}$.

Before the proof, let us first make several remarks.

The notion of indicator in arithmetic is due to J. Paris and L. Kirby; see Kirby [77]. The existence of the indicators for arithmetic indicated above is also due to them: they use a game-theoretic argument.

It will be clear from our proof that if the power-set axiom holds in $A$, we only need $\Pi_k$-overspill and $\Pi_{(k)}$-recursive saturation in order to show the existence of the indicators and the symbiosis.

For information concerning the order types of these sets of initial segments, see theorem 5.11 below.

Part (i) (without the condition of $\omega$ coding $I$ and with $T$ also assumed to contain $\Delta_0$-Separation) actually follows from (ii) by taking countable elementary submodels of $A$ and using 5.2 and a result of Smoryński and Stavi [79], namely that cofinal elementary extensions of r.s. models of $S + \text{Separation}$ are recursively saturated. However, as there are short elegant proofs using i-fulfilment, we shall prove (i) directly.

It is easy to check that if $\sigma \in A$, if $i \sigma$ holds and if $\sigma$ fulfils the universal closure of $\theta \lor \neg \theta$, then it also fulfils the universal closure of

$$\exists x \in z \theta \supset \exists x \in z \left( \theta \land \forall y \in x \neg \theta(y/x) \right) .$$

This places a restriction (at least in the arithmetical case) on the ease with which we may construct r.s. initial segments in which Collection does not hold—for example, our method requires the countability condition.
Finally, let us remark that in a pre-print (but not in the published) version of Paris and Kirby [78], a result is (essentially) proved which is almost the dual of ours; let $\mathcal{M}$ be a countable model of $\text{PA}^*_{\omega}$ as given on page 12, let $T$ be any r.e. theory extending $\text{PA}$; then there exists a $k$-elementary end-extension of $\mathcal{M}$ which is a model of $T$ iff $\mathcal{M}$ is a model of $\Sigma_{k+1}$-Collection, the $\Pi_{k+1}$ theory of $T$, and $T$-provable-$\Pi_{k+1}$-overspill, that is, if $\varphi x$ is $\Pi_{k+1}$ and for all $n \in \omega$, $T$ proves $\varphi \bar{n}$, then $\varphi \bar{m}$ holds in $\mathcal{M}$ for some nonstandard $m$.

Proof of 5.3: Let us first quickly check the assertions of (i) in the order given.

Let $a, b \in A$ and suppose $a \in I \subseteq b$ for some $I \in S$. Then for each $n \in \omega$,

$$\exists \sigma \left( |\sigma| = n \land a \in \sigma_0 \land \overline{\sigma} \subseteq b \land i \sigma \land T^{\sigma} \land Tr \Sigma_k^\sigma \right)$$

(1)

is true in $A$, where “$\exists$” here is represented by any code for $T$. Conversely, if (1) holds for each $n \in \omega$ then, using overspill, there is an $I$ in $S$ with $a \in I \subseteq b$: just let $I = \bigcup_{i \in \omega} \sigma_i$ where $\sigma$ witnesses (1) for some nonstandard $n$. As $\sigma$ also fulfills the universal closure of $\theta \forall \sigma$ for all formulae $\theta$, it provides a global satisfaction predicate for $I$ in $A$, and so $I$ is recursively saturated.

If $S$ is nonempty (respectively, cofinal), then it is clear that $A$ is a model of the $\Sigma_{k+1}$ ($\Pi_{k+2}$) theory of $T$. Conversely, if $A$ is a model of the $\Sigma_{k+1}$ ($\Pi_{k+2}$) theory of $T$, then (1), with references to $a$ and $b$ deleted, holds for each $n \in \omega$. So, using overspill, $S$ is not empty. If $A$ is a model of the $\Pi_{k+1}$ theory of $T$, then (1), prefixed by a universal quantification of $a$ and with $b$ deleted, holds for each $n \in \omega$. Hence $S$ is cofinal in $A$. This establishes (i).

Now consider 5.3.ii. Given $n \in \omega$ and two closed, increasing sequences $\sigma, \tau$, say $\sigma \leq_n \tau$ if $|\sigma| = |\tau|$, for all $i < |\sigma|$, $\sigma_i \subseteq \tau$, and for all $i < |\sigma|$, all $\varphi \overline{\nu} \nu < n$, and all $\bar{\tau} \epsilon \sigma_t$, $(\varphi \overline{\nu})^{\sigma_t} \supset (\varphi \overline{\nu})^{\tau_i}$. Fix $a, b \in A$. Choose some code $t$ for $T$: we shall suppose that $T$ contains the universal closure of $\theta \forall \sigma$ for all formulae $\theta$. For each $n \in \omega$, define a $\Sigma_{k+1}$ predicate $\text{extendible}_n(\sigma)$, with parameters $a, b$, and $t$, as follows. Let $\text{extendible}_0(\sigma)$ be $0 = 0$, and let $\text{extendible}_n(\sigma)$ hold iff

$$\langle |\sigma| > \bar{n} \land a \in \sigma_0 \land \overline{\sigma} \subseteq b \land i \sigma \land Tr \Sigma_k^\sigma \land \forall i < |\sigma| \forall x \in \sigma_t \forall y \in x \exists \tau \left( \gamma \epsilon \tau_0 \land \sigma \leq_n \tau \land \text{extendible}_n(\tau) \right).$$

Let $n$-extendible be the corresponding informal property: with this notion we are trying to capture some of the properties of a sequence defined (externally) via satisfaction functions.

If there exists $I$ in $S$ such that $a \in I \subseteq b$, then it is clear that for each $n \in \omega$ there exists an $n$-extendible sequence in $A$. By $\Sigma_{k+1}$-recursive saturation, we may choose a sequence $\sigma$ in $A$ which is $n$-extendible for all $n \in \omega$. Let $\sigma^0 = \sigma$. Let $y \in A$ be such that there exist $i \in \omega$ and $x \in \sigma_t$ with $y \in x$. Then for each $n \in \omega$ there exists $\tau$ such that

$$y \in \tau_0 \land \sigma \leq_n \tau \land \text{extendible}_n(\tau).$$

By $\Sigma_{k+1}$-recursive saturation, choose $\tau$ such that this holds for all $n \in \omega$; let $\sigma^1 = \tau$. Continuing in some such manner, define a sequence $\langle \sigma^i \rangle_{i \in \omega}$. Let $B_i = \bigcup_{j \in \omega} \langle \sigma^j \rangle_i$. The $B_i$'s form an ascending elementary chain of $k$-elementary substructures of $A$ which are recursively saturated models of $T$, with $B_i \subseteq b$ for all $i \in \omega$. Let $I = \bigcup_{i \in \omega} B_i$. Then $I$ is also a $k$-elementary substructure of $A$ which is a r.s. model of $T$, with $I \subseteq b$. If we make the successive choices of $x$ and $y$ in a sufficiently orderly manner and use the local countability of $A$, we may ensure that $I$ is also an initial segment. Thus $S$ is symbiotic with its r.s. members.

Suppose that the $\Sigma_{k+1}$ theory of $T$ holds in $A$. Redefine $\text{extendible}_n$ by deleting the references to $a$ and $b$. In any model of $T$, for each $n \in \omega$, there exists a sequence which is $n$-extendible. Just as in our proof of the Completeness Theorem on page 11, this underlined statement may be
expressed as a $\Sigma_{k+1}$ sentence without the coding of either sequences or formulae and without the use of the $\Sigma_k$ satisfaction predicate, and, so expressed, it is provable in $T$. Hence it must hold in $A$, and so for each $n \in \omega$, $\exists \sigma$ extendible in $\omega$. By $\Sigma_{k+1}$-recursive saturation and the above construction, we may conclude that $S$ is nonempty. The cofinal case is argued similarly.

If $\Sigma_{k+1}$-Collection holds in either $A$ or $T$, we may express the notion of $n$-extendible uniformly in $n$ to obtain an indicator. Also in this case, we can find $b \in A$ such that $B \subseteq b$ for the $B$ constructed above. If we have QF-Foundation, then $b \not\in b$, and if we have QF-Separation, consider $c = \{ x \in b : x \not\in x \}$. Obviously $c \not\in b$. The next result shows that, in general, $\Sigma_{k+1}$-Collection is both a necessary and sufficient condition to ensure $S \neq \{ A \}$. □

5.4 Lemma Let $k \in \omega$ and let $A$ be a countable model of $S + \Sigma_{k+1}$-overspill. The following are equivalent.

i. For all formulae $\varphi \bar{a}$ with parameters $\bar{a} \in A$, there exists $b \in A$ and $B \subseteq b$ such that $\bar{a} \in B$, $B \prec_k A$, and $\varphi \bar{a}$ is absolute between $A$ and $B$.

ii. $\Sigma_{k+1}$-Collection holds in $A$.

Proof: Suppose (ii). Replace “$T$” with “$\varphi \bar{a}$” in the above definition of $n$-extendible. By $\Sigma_{k+1}$-Collection we may bound the quantifiers “$\exists \tau$”, and then (i) follows by the above construction.

Suppose (i), and suppose $\forall \bar{x} \in a \exists \bar{y} \theta$ holds in $A$, where $\theta \in \Sigma_{k+1}$. Choose $b \in A, B \subseteq b$ such that $B \prec_k A$ and $B \models \forall \bar{x} \in a \exists \bar{y} \theta$. Then $A \models \forall \bar{x} \in a \exists \bar{y} \in b \theta$. Thus (ii). □

5.5 Corollary Let $A$ be a model of $S$ and $\Sigma_{k+1}$-overspill, and let $T$ be a complete theory extending $S$ plus Collection plus Separation. The following are equivalent.

i. There exists a $k$-elementary initial segment of $A$ which is a nonstandard model of $T$.

ii. $A$ is a model of $T \cap \Sigma_{k+1}$, and for all $n \in \omega$, $T \cap \Sigma_n$ is coded in $A$.

Before the proof, let us make a number of remarks.

This result is (in the arithmetical case) the dual of a result of Wilkie [77]. (The proof below is modelled on Lessan’s [78] proof of this dual.) Together these results answer a question in the introduction of Friedman [73]: are the standard systems of the nonstandard models of $PA$ which contain all arithmetic sets the same as those of models of the theory of $\mathbb{N}$?

Let us consider more closely the arithmetical case when $T$ extends $PA$: we shall quickly sketch two alternative proofs of 5.5. First, by 5.3.i, we can construct a chain of nonstandard initial segments

$$A \succ_k B_0 \succ_{k+1} B_1 \succ_{k+2} B_2 \succ_{k+3} \cdots$$

(2)

where $B_k$ is a model of $T \cap \Sigma_{n+k+2} \cup PA$. Let $I = \bigcap_{i \in \omega} B_i$. As $T$ has definable satisfaction functions, it is easy to check that $I \prec_k A$ and that $I$ is a model of $T$. It is also easy to ensure that $I$ is nonstandard.

For the second proof, let $B$ be the minimal model for $T$, or, if $T = Th(\mathbb{N})$, let $B$ be a conservative extension of $\mathbb{N}$ (that is, its standard system consists of exactly the arithmetical sets; see Phillips [74] or 5.9 below). A subset of $\omega$ is coded in $B$ iff it is recursive in $T \cap \Sigma_n$ for some $n \in \omega$. Thus the set of reals of $B$ is included in that of $A$, and so we may, by Friedman [73], construct a $k$-elementary embedding of $B$ into $A$. Then the initial segment of $A$ determined by the image of $B$ is by 5.2 a $k$-elementary substructure of $A$ and is a model of $T$. 

Third proof of 5.5: The left to right implication is easy, for $S + \text{Collection} \vdash \text{Induction}$, which provides the overspill required to code the true $\Sigma_n$ sentences. For the converse, by 5.2 we may suppose $A$ is countable. Let $(c_i)_{i \in \omega}$ be an infinite sequence of new constants. Let $(\varphi_i)_{i \in \omega}$ be a listing of all sentences of $L \cup (c_i)_{i \in \omega}$ of the form $\exists x \theta$, and suppose is $(c_i)$ is in $\Sigma_i + k$ and only contains the constants $(c_i)_{i \leq i}$. We shall define a descending chain of initial segments in (2) above, and an interpretation of $(c_i)_{i \in \omega}$ as follows. (We shall denote the interpretation of $c_i$ also by $c_i$.) Choose $B_0 \preceq A$ to be any r.s. initial segment which is a model of $T \cap \Pi_{k+3}$, and let $c_0$ be any nonstandard integer of $B_0$. Suppose we have defined $B_i$, a r.s. model of $T \cap \Pi_{i+k+3}$, and have interpreted $(c_j)_{j \leq i}$ in $B_i$. If $\varphi_i = \exists x \theta$ holds in $B_i$, let $c_{i+1}$ be interpreted by a witness for $\varphi_i$; otherwise, let $c_{i+1}$ be the first element of some fixed listing of $A$ which is a member of $B_i \setminus \{c_0, \ldots, c_i\}$. Now choose $B_{i+1}$ to be any $(k + i)$-elementary r.s. initial segment of $B_i$ which includes $\{c_0, \ldots, c_{i+1}\}$ and is a model of $T \cap \Pi_{i+k+4}$.

Let $I = \cap_{i \in \omega} B_i = \{ c_i : i \in \omega \}$. By construction, $I$ is a nonstandard model of $T$. □

5.5.ii Corollary Let $A$ be a countable model of $S$ plus $\Sigma_{(k+1)}$-recursive saturation, and let $T$ be a complete theory extending $S$. Then the following are equivalent.

i. There exists a $k$-elementary initial segment $B$ of $A$ which is a model of $T$ and which is $\Sigma_{(n)}$-recursively saturated for all $n \in \omega$.

ii. $A$ is a model of $T \cap \Sigma_{(k+1)}$, and for all $n \in \omega$, $T \cap \Sigma_{(n)}$ is coded in $A$.

If $A$ is also a model of $\Sigma_{k+1}$-Collection, we may ensure that $B \neq A$.

Proof: The proof is as above but, as the intersection of recursively saturated initial segments may not be recursively saturated, we must in our construction also add witnesses to ensure $\Sigma_{(n)}$-recursive saturation. □

We can extend 5.3 as follows. Let $T$ be a theory in any language in which the language of arithmetic may be interpreted. Let $M$ be a nonstandard model of $\text{PRA}$ and suppose $T$ is coded in $M$. Since (an axiomatization of) the arithmetical consequences of $T$, $\text{arith}(T)$, will also be coded in $M$, we may use 5.3 to consider, say,

$$S = \{ I =^k_A : I \models \text{arith}(T) \}.$$

There is however another, perhaps more elegant approach using $\frac{1}{2}$-fulfilment, and this also allows us to consider

$$\{ I \in S : I \text{ is expandable to a model of } T \}.$$

In the countable case and for many theories $T$ (e.g. any theory extending $\Delta^1_1$-$\text{CA}$), this coincides with $\{ I \in S : I \text{ r.s.} \}$, so we do not have any advantage over 5.3.ii. But we may improve 4.3.i to the following, also obtained independently by Kirby, McAloon, and Murawski [79] for theories extending $\Pi^0_1$-$\text{CA}$.

5.6 Theorem Let $k \in \omega$ and let $M$ be a model of $\text{PRA}$ plus $\Sigma^0_{k+1}$-overspill. Let $T$ be any theory coded in $M$ extending $S$ and also extending the schema:

$$\text{Arith-Collection:} \quad \forall m < n \exists p \theta \to \exists q \forall m < n \exists p < q \theta.$$

Let $S = \{ I =^k_A : I \models \text{arith}(T) \}$. Then $S$ has an indicator, and is symbiotic with

$$\{ I \in S : I \text{ r.s., } \omega \text{ codes } I, \text{ and } I \text{ is expandable to a model of } T \}.$$
$S \setminus \{M\}$ is nonempty iff the $\Pi^0_{k+1}$ theory of $M$ is consistent with $T$, and $S$ is cofinal in $M$ iff the $\Sigma^0_{k+2}$ theory of $M$ is consistent with $T$. \hfill \Box$

The proof of 5.6 is clear using the notion of $\frac{1}{2}$-fulfilment, and we omit it, only pausing to remark that the schema of Induction implies that of Arith-Collection.

We shall consider two more model-theoretic “tricks”. The first is closely related to $\frac{1}{2}$-fulfilment and 5.6. Let $\mathbb{PA}_e$ be the version of Peano arithmetic formulated in the language with a single binary relation $\in$. Let $T$ be any consistent, recursive theory in the language $\{\in\}$ which proves that there exists no $\in$-loops, For example, $T$ might be ZFC or some theory of analysis over $HF$. A well-known folklore result, easily proved, using the compactness theorem, is that any model of $T$ may be embedded in a model of $\mathbb{PA}_e$. Using fulfilment, we easily obtain the dual result:

5.7 Theorem Any nonstandard model $A$ of $\mathbb{PA}_e$ has a substructure $B$ which is a recursively saturated model of $T$. Moreover, if we assume that $T$ is strong enough to perform some coding, that $T$ includes Arith-Collection, and that $A$ is a model of the theory of $T$, then we may choose $B$ so that $HF^B$ (i.e. $\{ x \in B : x \in^B HF^B \}$) is an initial segment of $A$. And the usual hierarchical and cofinal variants also hold.

Proof: We simply note that any finite poset with no loops is isomorphic to a subset of $\langle HF, \in \rangle$. \hfill \Box

For our our second “trick”, let $T$ be a theory in any finite language in which we may interpret arithmetic, and let be a nonstandard model of $\mathbb{PRA}$ which codes $T$. In $T$, add a constant $\bar{P\omega}$ for the definable type consisting of subsets of $\omega$. We have been considering models $A$ of $T$ “coded” in $M$ in which $\overline{\omega}^A$ (i.e. $\{ x \in A : x \in^A \overline{\omega}^A \}$) is an initial segment of $M$. In such a case we have

$$\{ \{ x \in \overline{\omega}^A : x \in^A X \} : X \in^A \bar{P\omega}^A \} \subseteq \mathcal{R}(\overline{\omega}^A, M),$$

where this latter set is the collection of those subsets of $\overline{\omega}^A$ which are coded in $M$. Kirby, McAlloon, and Murawski \cite{79} noticed if $T$ is a second-order theory extending $\Pi^0_1$-CA$|$ and if $M$ is countable, then the collection of initial segments $I$ of $M$ which are expandable to a model $A$ of $T$ is symbiotic with its subset consisting of those $I = \overline{\omega}^A$ for which the relation (3) is in fact equality. We may extend this result as follows.\footnote{Footnote added 2019: This next result and its two lemmas have nothing to do with fulfillability.}

5.8 Theorem Let $k \geq 1$, and suppose $A$ is a model of

- $\Delta^0_0$-CA$|$,$\Delta^0_0$-Arith-Collection (where these two schemata are allowed to have second-order parameters),
- an axiom asserting that the $k$th Turing jump of the empty set exists, and
- WKL (an axiom asserting that every infinite binary tree has an infinite branch).
- Further suppose that $\Pi^0_1$-overspill holds in $A$, and that
- $\bar{P\omega}^A$ is countable.

Then for each $c \in \overline{\omega}^A$ there exists a structure $B$

- isomorphic to $A$ such that $\overline{\omega}^B$ is a $(k - 1)$-elementary initial segment of $\overline{\omega}^A$,
- $\bar{P\omega}^B = \mathcal{R}(\overline{\omega}^A, \overline{\omega}^B)$, and
- the isomorphism fixes $c$. 


If in addition we have that $\Sigma^1_1$-overspill holds in $\mathcal{A}$ and that $\exists X. X = \emptyset^{\mathcal{A}}$ holds in $\mathcal{A}$ for all $k \in \omega$, then we may require that $\overline{\omega^B} < \overline{\omega^A}$.

Before the proof, we need a definition and two lemmas. First note that without loss of generality we can assume that $\mathcal{A}$ is a structure for the language of analysis, say $\mathcal{A} = \langle M, X \rangle$. A pseudo-standard formula of $\mathcal{M}$ is one of the form $\varphi(a, b, \ldots, u, v, \ldots)$ where $\varphi(x, y, \ldots, u, v, \ldots)$ is a formula and $a, b, \ldots$ are elements of $\mathcal{M}$. If we add new constants $\{ d_a : a \in M \}$ to our metalanguage, with each pseudo-standard formula $\varphi(a, b, \ldots)$ we may associate a formula $\varphi(d_a, d_y, \ldots)$. For any $T \subseteq M$, let p.s.(T) be the collection of those $\varphi(d_a, d_y, \ldots)$ for which $\varphi(a, b, \ldots)$ is a pseudo-standard formula whose code is in $T$.

The first of the two lemmas, which does not require that $\mathcal{M}$ be nonstandard, is a common generalization of Scott [62] and Friedman [73] and the generalizations of the MacDowell-Specker Theorem [61] by Phillips [74] and Gaifman [76]. A slightly less general version was first observed by Kirby, McAloon, and Murawski [79]; their proof is an ultrapower construction which requires that $\mathcal{A}$ be a model of $\Pi^1_1$-$\text{CA}$.  

5.9 Lemma Let $\langle M, X \rangle$ be a countable model of $\text{PA}_{\omega}^\omega$, $\Delta^0_0$-Induction, $\Delta^0_0$-Arith-Collection, and WKL.

i. Let $T \in X$ be such that in $\langle M, X \rangle$ $T$ is a consistent set of sentences of arithmetic. Then there exists an end-extension $\mathcal{N'}$ of $\mathcal{M}$ which is a model of p.s.$(T)$ in the natural sense with $d_a$ being interpreted by $a$ for all $a \in M$, and is such that $R(\mathcal{N'}, M) = X$. If $M = \mathbb{N}$ or if $\langle M, X \rangle$ is a model of $\Pi^1_1$-overspill, we may choose $\mathcal{N'}$ to be recursively saturated.

ii. Let $T \subseteq M$ be such that p.s.$(T)$ is a complete theory in the language of arithmetic and for all $k \in \omega$, $(T \cap \Sigma^0_k) \in X$ and $M \models \text{Con}(T \cap \Sigma^0_k)$. Then there exists an end-extension $\mathcal{N'}$ which is a model of p.s.$(T)$ and is such that $R(\mathcal{M}, \mathcal{N'}) = X$.

Proof: For (i) we simply have to check that the usual Henkin construction works for $M \neq \mathbb{N}$. Since $T$ proves each true quantifier-free pseudo-standard sentence of $M$, we can suppose that these are included in $T$. Let $\langle c_i \rangle_{i \in \omega}$ be a list of new constants, and let $L(c_0, \ldots, c_i)$ be the language of arithmetic augmented with the constants $c_0, \ldots, c_i$. We shall define an increasing sequence $\langle T_n \rangle_{n \in \omega}$, where for each $n \in \omega$, $T_n \in X$ is a theory of $L(c_0, \ldots, c_i)$ for some $i \in \omega$ which is consistent in $M$. Let $T_0 = T \cup \{ \langle c > \overline{a} : a \in M \}$. This is consistent, because given any proof of a contradiction from $T_0$ we can replace $c_0$ by $\overline{a}$ for some large $a \in M$, obtaining a contradiction from $T$. Define $T_n$ inductively as follows.

Suppose $n = 4m + 1$ and suppose our language has only $\exists, \forall$ and $\neg$ as logical symbols. Let $\varphi$ be the $m$th member of some enumeration of the formulae of $\bigcup_{i \in \omega} L(c_0, \ldots, c_i)$. Let $T_{n+1}$ be $T_n + \neg \varphi$, if this is consistent; otherwise, if $\varphi = \exists x \theta x$ let $T_{n+1}$ be $T_n + \varphi + \theta c$ for some $c$ not yet considered, and for $\varphi$ in other forms simply let $T_{n+1} = T_n + \varphi$.

Suppose $n = 4m + 2$. Let $X$ be the first element of some enumeration of $X$ not yet considered, and let $c$ be some constant not yet considered. Let $T_{n+1} = T_n \cup \{ \overline{a} \in c : a \in X \} \cup \{ \overline{c} : a \in X \}$.

Suppose $n = 4m + 3$. Since we are already using “$\in$” to denote membership between $M$ and $X$, we shall let $\in$ denote the membership relation between integers, as defined on page 11. Choose $X \in X$ so that

$$T_{n+1} = T_n \cup \{ \overline{a} \in c : a \in X \} \cup \{ \overline{c} : a \in X \}$$

is consistent. We can do so, for consider the tree of binary sequences $\{ b \in M : \text{there is no proof of } 0=1 \text{ with code } < |b|^\} \text{ from } T_n + \{ \overline{a} \in c : b_a = 1 \} + \{ \overline{c} : b_a = 0 \}$.
This is in \( \mathcal{X} \), and it is infinite in \( \langle M, \mathcal{X} \rangle \), because otherwise by \( \Delta_0^0 \)-Arith-Collection we could piece together proofs to obtain a contradiction from \( T_n \). By WKL, we can choose an infinite branch in \( \mathcal{X} \) to obtain \( X \).

Suppose \( n = 4m + 4 \). It is at this stage that we ensure that the end-extension is r.s. We can suppose \( M \neq \mathbb{N} \), for otherwise we simply use the standard argument. Let \( \theta x \) be the \( m \)th \( \Delta_0^0 \) formula, and suppose that for some \( i \in \omega \) the set \( \{ x \in \omega : \theta x \} \) consists of the codes of formulae of the form \( \varphi c_0 \ldots c_i \). Suppose for each \( p \in \omega \), \( \exists x \bigwedge \{ \varphi x : \theta \exists \varphi^3 \land \exists \varphi^3 < p \} \) is consistent in \( M \) with \( T_n \). Then by \( \Pi_1^0 \)-overspill, this is consistent for some nonstandard \( p \in M \). Let \( c \) be some constant not yet considered, and let \( T_{n+1} = T_n \cup \{ \varphi c : \theta \exists \varphi^3 \land \exists \varphi^3 < p \} \).

This completes our construction. The remainder of the proof is standard and is left to the reader.

The proof of (ii) is obtained by a simple modification of the above, essentially due to D. Jensen and A. Ehrenfeucht [76] and, independently, Guaspari [79]. But since we shall not require (ii), the proof is omitted.

The next lemma is an immediate corollary of the embedding techniques of Friedman [73], Wilkie [7x], and Wilmers [77].

5.10 Lemma Let \( M \) and \( N \) be countable.

i. Suppose \( M \) and \( N \) are models of \( \text{PA}_{\omega} + \Sigma_{k+1} \)-overspill with the same standard systems and with \( \text{Th}_{\Sigma_{k+1}} (N) \subseteq \text{Th}_{\Sigma_{k+1}} (M) \). Also suppose \( N \models \Sigma_{k+1} \)-Collection. Then \( N \) is isomorphic to a \( k \)-elementary initial segment of \( M \).

ii. If \( N \) and \( M \) are elementarily equivalent, recursively saturated models of \( \text{PA} \) with the same standard system, then \( N \) is isomorphic to a proper elementary initial segment of \( M \).

Proof of 5.8: By \( \exists X. X = \emptyset^{(k)} \), we have that the set of true \( \Pi_k^0 \) sentences of \( M \) is an element of \( \mathcal{X} \). By Application (xv) on page 42 of Chapter IV, \( \Sigma_{k}^0 \)-Induction implies the consistency of these true \( \Pi_k^0 \) sentences plus \( \Sigma_k \)-Collection, and so by 5.9 we may choose a recursively saturated \( k \)-elementary end-extension \( N \) of \( M \) which is a model of \( \Sigma_k \)-Collection and is such that \( \mathcal{R}(M, N) = \mathcal{X} \). Now for any \( c \in M \), \( \text{Th}_{\Sigma_k} (N, c) = \text{Th}_{\Sigma_k} (M, c) \) and so by 5.10 there is an embedding of \( N \) into a proper initial segment of \( M \) fixing \( c \).

For 5.8.ii, we note that there is a recursive predicate \( P(x, k, X) \) such that \( \{ x : P(x, k, \emptyset^{(k)}) \} \) is the set of true \( \Pi_k^0 \) sentences. So if \( M \) is a model of \( \text{PA} \) and if \( \Sigma_k^1 \)-overspill and \( \exists X. X = \emptyset^{(k)} \) for all \( k \in \omega \) hold in \( \langle M, \mathcal{X} \rangle \), then there is some nonstandard \( k \in M \) for which

\[
\exists X \left( X = \emptyset^{(k)} \land \text{Con}(\{ x : P(x, k, X) \}) \right)
\]

holds. Now by 5.9, we may choose a r.s. elementary end-extension \( N \) of \( M \) such that \( \mathcal{R}(M, N) = \mathcal{X} \), and then by 5.10, embed \( N \) in \( M \).

This completes the proof of 5.8.

Our final theorem considers the order types of various classes of elementary initial segments of r.s. models of \( \text{PA} \).

5.11 Theorem Let \( M \) be a recursively saturated model of \( \text{PA} \), and let

\[
A = \{ I \leq^\text{end} M : I \text{ not r.s.} \}
\]
\[
B = \{ I \leq^\text{end} M : I \text{ r.s. and } \omega \text{ codes } I \}
\]
\[
C = \{ I \leq^\text{end} M : I \text{ r.s.} \}
\]
Then:

i. For all $I \leq \mathfrak{M}$, $I \in A$ iff there exists $a \in I$ such that the elements of $M$ definable from $a$ are cofinal in $I$.

ii. There exists an order-preserving map from $M$ into $A$ (ordered by inclusion), and so by (i), the cardinality of $A$ is equal to that of $M$.

iii. For all $I \in A$, $I = \bigcap \{ J \in B : I \subseteq J \}$, and so $A$ is densely ordered.

iv. $C = \{ \bigcup X : \emptyset \neq X \subseteq A, X$ has no greatest element $\}$.

v. There exists an order-preserving map from $M$ into $B$, and so the cardinality of $B$ equals that of $M$.

vi. For all $I \in B$, $I = \bigcap \{ J \in B : I \subseteq J \} = \bigcup \{ J \in B : J \subseteq I \}$, and so $B$ is densely ordered.

vii. $C = \{ \bigcup X : \emptyset \neq X \subseteq B \}$, and so by (vi), if $M$ is countable then $C \setminus \{ M \}$ has the order-type of the real numbers.

Let us first make some remarks. Many of these results are well-known for countable $M$: see, for example, Barwise [75] or Schlif [78]. Part (i) is essentially due to W. Marek and H. Kotlarski; see Kotlarski [78]. For the case when $M$ is countable, much of (ii) to (vii) has also been independently obtained by Kotlarski [78] and Murawski [78].

We remark that Paris and Kirby [78] show that if $M \leq \mathfrak{N}$ with $M \models S$, then Collection holds in $M$ and $\mathfrak{N}$, and so it is necessary for us to consider models of $\mathfrak{PA}$ in this result. It will be obvious, however, that an analogue of 5.11 could be given for rank extensions of r.s. models of ZF or for L-extensions of r.s. models of $\text{ZF}^+ + V = L$. The set-theoretic case has in addition another natural class of initial segments symbiotic with those above: those internal sets $B \in A$ for which $B < A$.

Proof of 5.11: We shall only consider i-fulfilment in this proof, as given on page 9. Let $\text{True}(\sigma)$ be the schema

$$\forall i < |\sigma| \exists x \langle \varphi x \supset (\varphi x)^i \rangle,$$

where $\varphi$ has the free variables indicated.

We shall first prove (vii). Suppose $I \in C$. For each $a \in I$ choose a $\sigma \in I$ to realize the recursive type

$$\{ a < \sigma_0 \} \cup \{ |\sigma| > \overrightarrow{n} : n \in \omega \} \cup \text{True}(\sigma),$$

and let $J_a = \bigcup_{i \in \omega} \sigma_i$. Then $J_a \in B$ and $I = \bigcup_{a \in I} J_a$. Conversely, it is clear that recursive saturation is preserved under unions of elementary chains.

This also gives the second part of (vi). For the first, let $I \in B$ and let $s$ be a sequence coded in $M$ such that $I = \bigcup_{i \in \omega} s_i$. Let $a \in M \setminus I$; by recursive saturation choose $t \in M$ to be an increasing sequence such that $|t|$ is nonstandard, $t|_t \leq |s|$, $s|_{|t|} < a$, and so that $\text{True}((s_i)_{i \leq |t|})$ holds; and, let $J_a = \bigcup_{i \in \omega} s_i$. Then $I = \bigcap_{a \in M \setminus I} J_a$, and (vi) is proved.

For each $a \in M$, let $I_a$ be the initial segment determined by those elements definable from $a$. Then $I_a \leq \mathfrak{M}$ by Lemma 5.2, and $I_a$ is not recursively saturated, for the type $\{ \exists y \theta ay \supset \exists y < x \theta ay : \theta y \text{ any formula} \}$ is not realized in $I_a$. $A$ and $B$ are easily seen to be symbiotic, because for any $X \subseteq A$ with no greatest element there exists $Y \subseteq B$ such that $\bigcup X = \bigcup Y$, and so $\bigcup X$ is r.s. Thus each member of $A$ has the form $I_a$, and so (i) holds.
Let $I = I_a$, let $b \in M \setminus I$, and let $c$ satisfy

\[
\exists y \theta xy \supset \exists y < b \theta xy \quad \text{and} \quad \exists y \theta ay \supset \exists y < x \theta ay
\]

for all formulae $\theta xy$. Then $I \subsetneq I_c < b$, and as $b$ was arbitrary, this proves (iii). (iv) is easy: if $J \in C$, $J = \bigcup_{a \in J} I_a$ where $I_a$ is as above.

Only (ii) and (v) remain to be proved. Imagine, for a moment, that there exists an $M$-infinite sequence $\sigma = \langle \sigma_i \rangle_{i \in M}$ satisfying $\text{True}(\sigma)$. (If $M$ is resplendent then such a sequence exists: we can let $\sigma$ be a cofinal set of indiscernibles for which $\langle M, \sigma \rangle$ is a model of full induction; see Schlipf [78].) Then we could define our required maps easily. Define $M \to A$ by letting $i \mapsto I_{\sigma_i}$. This is monotone.

Define the map $M \to B$ by $i \mapsto \bigcup_{j \in \omega} \sigma_{k+i+j}$, where $k$ is some fixed nonstandard element of $M$. This is clearly monotone.

I do not know whether such a sequence exists in general, but we shall construct a suitable alternative. Let $\Gamma(\sigma, x) = \{|\sigma| = x\} \cup \{x < \sigma_0\} \cup \text{True}(\sigma)$. Let $\alpha, \beta, \gamma$ range over the (real) ordinals, and let $\alpha$ be the cofinality of $M$, that is, $\alpha$ is the least cardinal such that there exists a cofinal subset of $M$ of cardinality $\alpha$. Choose a sequence of internal sequences $\langle \sigma_\beta \rangle_{\beta < \alpha}$ as follows. Let $\sigma_0$ satisfy $\Gamma(\sigma_0, 1)$, let $\sigma_{\beta+1}$ satisfy $\Gamma(\sigma_{\beta+1}, \sigma_\beta)$, and if $\gamma < \alpha$ is a limit ordinal, choose $x \in M$ greater than $\sigma_\beta$ for all $\beta < \gamma$, and let $\sigma_{\gamma}$ satisfy $\Gamma(\sigma_{\gamma}, x)$. Define an increasing sequence of functions $\langle f_\beta \rangle_{\beta < \alpha}$, each mapping an initial segment of $M$ into $M$, as follows. Let $f_0$ be empty; let

\[
f_{\beta+1}(x) = \begin{cases} f_\beta(x), & \text{if } x \in \text{domain}(f_\beta), \\ (\sigma_{\beta+1})_x, & \text{if } x \leq |\sigma_{\beta+1}|^- \text{ and } x \notin \text{domain}(f_\beta), \\ \text{undefined,} & \text{otherwise;}
\end{cases}
\]

and if $\gamma$ is a limit ordinal, let $f_\gamma = \bigcup_{\beta < \gamma} f_\beta$. Now define maps $M \to A$ and $M \to B$ as before, but replacing by $\sigma$ by $f_\alpha$.

This completes the proof of 5.11. \qed
VI More Model-theoretic Applications: $\omega$-models

We shall study in this chapter $\omega$-models of various theories, (including theories in the infinitary language $L_{\omega_{\omega}}$) which extend our base theory $S$ given on page 11. To give an illustration of our techniques, to compare them with other known techniques, and to help the reader understand the more general results given later, we shall first consider the example of $\omega$-models of the language of second-order arithmetic, that is, analysis.

Let $\text{BI}$, Bar-Induction, be the schema:

$$\text{BI}: \forall X \left( \text{wf}(\prec_X) \supset \left( \forall n \left( (\forall m \prec_X n \theta m) \supset \theta n \right) \supset \forall n \theta n \right) \right)$$

where $\prec_X = \{ (m, n) : \langle m, n \rangle \in X \}$. Let us consider a theory $T$ in the language of analysis extending $S + \Sigma^1_1$-$\text{BI}$ which has an $\omega$-model. To keep things simple, suppose $T$ is r.e., although for the following results it suffices to let $T$ be $\Pi^1_1$.

Gandy, Kreisel, and Tait [60] showed:

6.1 Theorem $T$ does not have a minimum model. $\square$

Here we are considering $\omega$-models ordered by inclusion. Briefly, they showed that the intersection, of all models of $T$ consists of the hyper-arithmetic sets, $\text{HYP}$, and as $\Sigma^1_1$-$\text{BI}$ is false in $\text{HYP}$, $T$ does not have a minimum model.

Friedman [73] proves that:

6.2 Theorem If $T$ extends the full schema of $\text{AC}$, $T$ does not have a minimal model.

Simpson [73] asks whether weaker conditions on $T$ suffice for this result. Indeed, we shall show:

6.3 Theorem $T$ does not have a minimal model.

We shall first give a very quick sketch of Friedman’s result 6.2 (since the existing versions in print are unnecessarily complicated) and then briefly explain the modification which allows us to remove $\text{AC}$.

First note that as $\exists X \left( \{(X)_n : n \in \omega \} \models T \right)$ is true in any $\beta$-model, we can restrict our attention to non-$\beta$-models. Fix some non-$\beta$-model $A$ of $T$ (i.e. $A \subseteq P\omega$ and $\langle \omega, A, \ldots \rangle \models T$) and choose some non-well-founded linear ordering $\prec \in A$ for which $A \models \text{wf}(\prec)$. Let $\alpha$ be the ordinal of the well-founded part of $\prec$.

First, to give Friedman’s proof, suppose $A$ is a model of $\text{AC}$. We consider $\tau \in A$ such that $\tau$ is (the code of) a tree of finite sequences $\sigma = \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle$ of sets for which

i. $\overline{\sigma_i} \models (T \cap i)$, for all $i \leq |\sigma|$, and

ii. $\overline{\sigma_i} \preceq_i \overline{\sigma_{i+1}}$, for all $i < |\sigma|$,

where in general in this chapter $X = \{ (X)_n : n \in \omega \}$, and $\preceq_i$ means $\Sigma^1_i$-elementary substructure.

Claim 1 For all $\beta < \alpha$ there exists such a $\tau \in A$ of rank $\beta$.

Theorem 6.2 follows from this claim. For as $T$ includes $\Sigma^1_1$-$\text{BI}$, we may apply overspill to obtain a tree $\tau$ of nonstandard rank. $\tau$ has (in the real world) an infinite branch $\langle \sigma_0, \sigma_1, \sigma_2, \ldots \rangle$. Let

Footnote added 2019: That is, for $\omega$-models of analysis, the same notation is used for both the model and its $P\omega$ component.
Let (i′) given a good sequence \( \tau \) for it is sufficient (and, by the definition (i′)) that each sequence in \( \tau \) is straightforward.

Then Claim 2 also holds for this new version of goodness, and the proof is as before. But now, given a good sequence \( \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle \), we do not need any strong axioms to extend it to another good sequence
\[
\langle \sigma_0, \ldots, \sigma_{|\sigma|}, A \rangle,
\]
for it is sufficient (and, by the definition (i′), necessary) to take \( A \) to be a finite collection of sets.

Finally, we note the well-known result that \( \Sigma^1_1 \)-Bl implies \( \Sigma^1_1 \)-AC. Thus we have strong enough axioms to carry out the inductive step at limit ordinals.

\[
B = \bigcup_{i \in \omega} \overline{\sigma_i}. \text{ Then it is clear that } B \subseteq A \text{ and } B \vdash T. \text{ That } A \neq B \text{ follows by Cantor's diagonal argument.}
\]

The proof of the claim is by induction on \( \beta \), and to be able to carry out the inductive step, we need to prove a bit more. Say a finite sequence \( \sigma \) is good if (i) and (ii) hold, and

iii. \( \overline{\sigma_i} \preceq_i A \), for all \( i \leq |\sigma| \).

**Claim 2** For each \( \beta < \alpha \) and each good sequence \( \sigma \in A \) there exists such a tree \( \tau \in A \) of rank \( \beta + |\sigma| \) which is such that each sequence in \( \tau \) extends \( \sigma \).

Proof of Claim 2: By induction on \( \beta \). Suppose the claim holds for \( \beta \) and that we wish to show that it holds for \( \beta + 1 \). This is easy but requires the essential use of AC. Suppose \( \sigma \in A \) is good, say \( \sigma = \langle \sigma_0, \ldots, \sigma_n \rangle \). By AC, we can find a set \( A \in A \) such that \( \overline{A} \preceq_n A \) and \( \overline{\sigma_n} \subseteq \overline{A} \). The sequence
\[
\langle \sigma_0, \ldots, \sigma_n, A \rangle
\]
is also good, and so we may apply the inductive hypothesis to it to obtain the appropriate \( \tau \) for \( \sigma \).

At limit \( \beta \), we in addition need to piece together the \( \tau \) of appropriate ranks using \( \Sigma^1_1 \)-AC. This is straightforward.

This concludes our sketch of the proof of the claim and of 6.2. \( \square \)

Turning next to the proof of 6.3, let \( A \) be a non-\( \beta \)-model of \( T \), and let \( < \) be as before. We now consider trees \( \tau \in A \) of finite sequences \( \sigma = \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle \) such that:

i′. \( \overline{\sigma_i} \) is finite for all \( i \leq |\sigma| \), and

ii′. \( \langle \overline{\sigma_i}, \overline{\sigma_{i+1}}, \ldots, \overline{\sigma_{|\sigma|}} \rangle \) fulfils the \( i \)th axiom of \( T \) for all \( i < |\sigma| \).

(More precisely, condition (i′) of course means this. \( \sigma \) is a subset of \( \omega \); \( \sigma_i \) is the set \( \sigma_i = \{ m : \langle m, i \rangle \in \sigma \} \); and \( \sigma_i \) codes the collection \( \{ X_j : j \in \omega \} \) where \( X_j = \{ x : \langle x, j \rangle \in \sigma_i \} \). We require in (i′) that \( X_j = \emptyset \) for all but finitely many \( j \).)

As before, it suffices to show that **Claim 1** holds. For if it does, then by overspill there exists such a \( \tau \in A \) of nonstandard rank. Choose (externally) an infinite branch \( \langle \sigma_0, \sigma_1, \ldots \rangle \) of \( \tau \), and let \( B = \bigcup_{i \in \omega} \overline{\sigma_i} \). Then \( B \subseteq A \); \( B \) is a model of \( T \) by Lemma 1.1; and \( A \neq B \) as before.

Say that the new version of **Claim 1** holds, we need, once again, to prove a bit more. Say a sequence \( \sigma = \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle \) is good if (i′) and (ii′) hold, and

iii′. for all \( i < |\sigma| \), there exist satisfaction functions \( f, g, \ldots, h \) for the \( i \)th axiom of \( T \) such that for all \( j \) with \( i < j < |\sigma| \),
\[
\overline{\sigma_{j+1}} \supseteq f''\overline{\sigma_j} \cup g''\overline{\sigma_j} \cup \cdots \cup h''\overline{\sigma_j}.
\]

Then **Claim 2** also holds for this new version of goodness, and the proof is as before. But now, given a good sequence \( \langle \sigma_0, \ldots, \sigma_{|\sigma|} \rangle \), we do not need any strong axioms to extend it to another good sequence
\[
\langle \sigma_0, \ldots, \sigma_{|\sigma|}, A \rangle,
\]
for it is sufficient (and, by the definition (i′), necessary) to take \( A \) to be a finite collection of sets.
This concludes the sketch of the proof of Claim 2, and of 6.3.

Next we shall give a more general development of the ideas contained in the above proof, starting with an extension of the definition of fulfilment to the language $L_{\omega \omega}$. It will be convenient to suppose that $L_{\omega \omega}$ is defined so that all formulae have only finitely many free variables. We shall consider sequences $\sigma = \langle (A_i, B_i) \rangle_{i \in I}$ of ordered pairs, where each $A_i$ is, as before, a subset of the domain of individuals, and each $B_i$ is a subset of $L_{\omega \omega}$. Often we shall write $\sigma = \langle (\sigma^0_i, \sigma^1_i) \rangle_{i \in I}$.

For reasons which will become clear, we will henceforth suppose that $\forall$ under the functions of $L$ just the premise of the lemma. Consider the

\begin{align*}
\langle (\forall \Phi) \rangle^\sigma_i &= \forall j \geq i \setminus \{ \Phi^\sigma_j : \Phi \in \Phi \cap B_j \}, \quad j \text{ a new variable,} \\
\langle (\exists \Phi) \rangle^\sigma_i &= \exists \{ \Phi^\sigma_j : \Phi \in \Phi \cap B_{i+1} \}.
\end{align*}

This choice of definition is rather arbitrary, and other choices are possible. This is because each conjunction or disjunction can be regarded as having a domain unique to itself, whereas all the quantifiers $\exists$ and $\forall$ have a common domain. (In this chapter, the words “conjunction” and “disjunction” refer to both the finitary and infinitary kinds.)

The analogue of Lemma 1.1 from page 9 holds; we shall combine parts (i) and (ii) of 1.1, as this is all we shall require. Recall that $\sigma^i(n+1) = \langle \sigma^i \rangle_{i \leq n}$.

**6.4 Lemma** Let $\sigma = \langle (A_i, B_i) \rangle_{i \in \omega}$ and let $\varphi$ be a sentence of $L_{\omega \omega}$. Suppose that $\cup B_i$ includes all subformulae of $\varphi$, that $\mathcal{A} = \cup_i A_i$, and that $\sigma^i(n+1)$ fulfils $\varphi$ for all $n \in \omega$. Then $\varphi$ is true in $\mathcal{A}$.

Proof: We shall sketch a game-theoretic argument. We can suppose that $\varphi$ is in negation-normal-form. The set of subformulae of $\varphi$ may be considered as a tree using the subformula relation. Two players, $\exists$ and $\forall$, play the following game (called the $\varphi$-game), in the process choosing a branch $\langle \varphi = \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_\Omega \rangle$ of this tree and an increasing sequence $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_\Omega$, with each $s_j$ a valuation in $\mathcal{A}$ of the free variables of $\varphi_j$. $\exists$ wins iff $\varphi_\Omega(s_\Omega)$ is true in $\mathcal{A}$. Suppose $\varphi_j$ and $s_j$ have been chosen, and consider the $(j+1)^{st}$ stage:

(a) If $\varphi_j$ is atomic or the negation of an atomic formula, let $\Omega = j$, and the game halts.

(b) If $\varphi_j = Qx \theta$, let $\varphi_{j+1} = \theta$ and let $s_{j+1} = s_j \cup \{(x, a)\}$, where $a \in \mathcal{A}$ is chosen by $Q$.

(c) If $\varphi_j = \theta \lor \psi$ (or $\theta \land \psi$), let $s_{j+1} = s_j$ and let $\varphi_{j+1}$ be $\theta$ or $\psi$, as chosen by $\exists$ (or $\forall$, respectively).

(d) If $\varphi_j = \bigvee \Phi$ (or $\bigwedge \Phi$), let $s_{j+1} = s_j$ and let $\varphi_{j+1} \in \Phi$ be chosen by $\exists$ (or $\forall$, respectively).

Then $\varphi$ is true in $\mathcal{A}$ iff $\exists$ has a winning strategy for the $\varphi$-game. We shall show that $\exists$ in fact has a winning strategy such that for any play according to this strategy,

\begin{equation}
\text{if for any } j \leq \Omega, \text{ if } \forall \text{ has made all of their choices up to the } (j+1)^{st} \text{ stage from } A_i \cup B_i, \text{ then } (\varphi_j s_j)^\sigma_i \cap (n+1) \text{ holds of all } n > i. \tag{1}
\end{equation}

We shall simultaneously define this strategy and verify (1), using induction on $j$. If $j = 0$, (1) is just the premise of the lemma. Consider the $(j+1)^{st}$ stage. If it is $\forall$’s move, then any choice by
them will preserve (1). Suppose it is \(\exists\)’s move, that \(\varphi_j = \exists x \varphi_{j+1} x\), say, and that \(\forall\) has made all of their choices from \(A_i \cup B_i\). Then by the inductive hypothesis \((\varphi_i)_j^{\alpha}[n]^{(n+1)}\) holds for all \(n > i\). As \(A_{i+1}\) is finite, there exists an \(x \in A_{i+1}\) such that \((\varphi_{i+1} j(x), s_{j})^{\alpha}[n]^{(n+1)}\) holds for infinitely many (and hence all) \(n > i\); let \(\exists\) choose such an \(x\). Cases (c) and (d) are similar. \(\square\)

In this chapter, the base theory \(S\) will be assumed to include \(\Sigma_1\)-Collection and \(\Delta_0\)-Separation. The reader is reminded of the simple notions and conventions regarding non-standard models given on page 3. We shall be considering non-\(\beta\)-models \(\mathcal{A} = (A, \ldots)\) of \(S\). In each such structure we shall (usually implicitly) fix some linear ordering \(< \in A\) which is well-founded internally but not externally, and let \(\alpha, \beta, \ldots\) range over its domain. \(\alpha\) is a standard ordinal of \(\mathcal{A}\) if it is in the well-founded part of this linear ordering. \(\)The\) standard ordinal of \(\mathcal{A}\) is the order-type of its standard ordinals; because \(\mathcal{A} \models \Sigma_1\)-Replacement, this is independent of the choice of \(<\). As usual, it will be convenient to use the notations and conventions associated with von Neumann ordinals.

We shall consider trees \(\tau\) of finite sequences \(\sigma\). Say a tree \(\tau\) fulfills a sentence \(\varphi\) of \(\mathcal{L}_{\omega \omega}\) if \(\sigma\) fulfills \(\varphi\) for each \(\sigma \in \tau\). Given \(\alpha\) in the domain of \(<\), say \(\tau\) is of rank \(\alpha\) if there is a map \(\text{rk}\) in \(A\) from \(\tau\) to \(\alpha + 1 = \{ \beta : \beta < \alpha \}\) satisfying

\[
\text{rk}(\sigma) = \sup\{ \text{rk}(\sigma') + 1 : \sigma' \in \tau \text{ and } \sigma' \text{ extends } \sigma \}.
\]

with \(\text{rk}(\text{root of } \tau) = \alpha\); we shall write \(\text{rk}(\tau) = \alpha\).

The next result is the analogue of Lemma 1.2 on page 13.

6.5 Lemma Let \(\varphi\) be a sentence of \(\mathcal{L}_{\omega \omega}\) and let \(\mathcal{A}\) be a model of \(S\) with \(\varphi \in A\) and \(\varphi\) true in \(\mathcal{A}\). Then for each standard ordinal \(\alpha\) of \(\mathcal{A}\) there exists a tree in \(\mathcal{A}\) of rank \(\alpha\) which fulfills \(\varphi\), and if we have sufficient Bar-Induction, the construction can be formalized.

Proof: In Lemma 1.2, we considered finite pieces of satisfaction functions for \(\varphi\). The infinitary analogue of satisfaction functions is a winning strategy for the \(\varphi\)-game, and we shall consider finite pieces of the same.

A \((\text{winning})\) strategy for \(\exists\) is a partial function \(\mathcal{S}\) from \((A \cup \text{sub}(\varphi))^<\omega\) into \(A \cup \text{sub}(\varphi)\) such that \(\exists\) plays (and wins, respectively) the \(\varphi\)-game by choosing at each turn \(\mathcal{S}\), where \(\mathcal{S}\) is the listing of \(\forall\)'s choices up to that point. If \(\mathcal{S}\) is a strategy for \(\exists\), let \(\mathcal{S}_0\) be \(\mathcal{S}\) with its domain restricted to arguments \(\mathcal{S}\) for which \(\exists\) actually needs the value \(\mathcal{S}\) in the course of some play; \(\mathcal{S}_0\) is a \(\text{minimal}\) strategy. Say \(\sigma = \langle (A_i, B_i) \rangle_{i \leq |\sigma|} <\omega\) is closed under a strategy \(\mathcal{S}\) for \(\exists\) if

\[
\mathcal{S}''(A_i \cup B_i)^{<\omega} \subseteq A_{i+1} \cup B_{i+1}
\]

for all \(i < |\sigma|\).

Claim 1 Given finite sets \(A_0, B_0\), a function \(F : \omega \rightarrow \mathcal{L}_{\omega \omega}\), and a strategy \(\mathcal{S}\) for \(\exists\), there exists \(\sigma = \langle (A_i, B_i) \rangle_{i <\omega}\) which is closed under \(\mathcal{S}\) and the functions of \(\mathcal{L}\), and satisfies \(Fi \in B_{i+1}\) for all \(i < \omega\).

This claim is not quite obvious, because \(\mathcal{S}_0\) may contain arbitrarily large \(n\)-tuples in its domain. Suppose \(A_i\) and \(B_i\) are finite. Let \(C\) be the smallest set of formulae including \(B_i\) and containing \(\psi\) and \(\psi'\) if it contains \(\exists x \psi, \forall x \psi, \psi \vee \psi', \text{ or } \psi \wedge \psi'\). By König’s lemma, \(C\) is finite, and the lengths of the sequences \(\mathcal{S}\) from \(A_i \cup B_i\) which must considered may be bounded by the cardinality of \(\mathcal{C}\). This establishes the claim.

Let \(\sigma = \langle (A_i, B_i) \rangle_{i \leq n}\). The \(\sigma\)-game (for \(\varphi\)) is like the \(\varphi\)-game, except that \(\forall\)'s choices are restricted to \(A_{n-1} \cup B_{n-1}\), and at each turn of \(\exists\), if \(\forall\) has made all of their previous choices from
Let \( A_i \cup B_i \), then \( \exists \) must choose from \( A_{i+1} \cup B_{i+1} \); and if either \( \forall \) or \( \exists \) cannot make a choice, the game is drawn. A strategy for \( \exists \) for the \( \sigma \)-game is good if for each play according to this strategy, \( \varphi_i ; s_i \) is true for all \( i \in \Omega \), where \( \varphi_i \), \( s_i \), and \( \Omega \) are as in 6.4 on page 61. The following is an easy consequence of these definitions.

**Claim 2** If \( \mathcal{S} \) is a winning strategy for \( \exists \) for the \( \varphi \)-game and if \( \mathcal{S} \) is closed under \( \mathcal{S} \), then \( \mathcal{S} \) is a good strategy for the \( \sigma \)-game. Conversely, any good, minimal strategy for a \( \sigma \)-game may be extended to a winning strategy \( \mathcal{S} \) for \( \exists \) for the \( \varphi \)-game which is such that \( \mathcal{S} \) is closed under \( \mathcal{S} \).

(We may now see the reasons for the above definitions; the notion of a winning strategy for \( \exists \) for the \( \varphi \)-game cannot be expressed directly in our language, while the notion of a good strategy for \( \exists \) for the \( \sigma \)-game can be so expressed by the use of partial truth definitions because such strategies are essentially finite; this will be required when we give an internal version of 6.5.)

Given a sequence \( \sigma \) and a tree \( \tau \), let \( \sigma^- \tau \) be the tree \( \{ \sigma^- \sigma' : \sigma' \in \tau \} \), where the latter \( \_^- \) is sequence concatenation.

**Claim 3** For all \( \sigma \) for which there exists a good strategy for \( \exists \) for the \( \sigma \)-game of \( \varphi \), and for all standard ordinals \( \alpha \) of \( A \), there exists a tree \( \tau \) of rank \( \alpha \) such that \( \sigma^- \tau \) fulfils \( \varphi \).

The lemma follows immediately from this claim. We use induction on \( \alpha \). If \( \alpha = 0 \), take \( \tau \) to be empty. Suppose the claim holds for all \( \alpha < \beta \). Let \( \sigma \) satisfy the premise. By **Claim 2**, there exists \( \sigma^- = \sigma^- (\langle A', B' \rangle) \) for which there exists a good strategy for the \( \sigma' \)-game. By the inductive hypothesis, for all \( \alpha < \beta \) there exists a \( \tau \) of rank \( \alpha \) such that \( \sigma^- \tau \) fulfils \( \varphi \). By \( \Sigma_1 \)-Strong Replacement, there exists a set \( K \) such that for all \( \tau' \in K \), \( \sigma^- \tau' \) fulfils \( \varphi \), and for all \( \alpha < \beta \) there exists \( \tau' \in K \) of rank \( \geq \alpha \). (If \( \beta \) is a successor, \( K \) may be taken to be a singleton.) Then \( \tau = \langle \langle A', B' \rangle \rangle^- \cup K \) is the required tree.

We shall want to consider an extension of the formula hierarchy into the transfinite. The details here do not matter; all we require of, say, \( \Sigma_\alpha \), is that it be closed under finitary conjunction, finitary disjunction, and existential quantification (and, if \( \alpha = 0 \), infinitary conjunctions and disjunctions), and that there exists a suitable universal formula, e.g., a formula \( \varphi_\alpha(e, x) \in L_{\alpha^+} \cap \Sigma_\alpha \) (where \( \alpha^+ \) = the next admissible) such that in any model \( A \) of \( \mathcal{S} \), for all \( \Sigma_\alpha \) formulae \( \psi x \in A \), \( A \vDash \forall x \langle \varphi x \equiv \varphi_\alpha(\psi x) \rangle \). For any subclass \( \Gamma \) of \( L_{\omega \omega} \), let \( \Gamma^\Sigma = \Gamma \cap \Sigma \). Let \( \Sigma^\alpha(\Sigma) \) be the closure of \( \Sigma^\alpha \) under conjunction, disjunction, existential and bounded universal quantification. We note that in the presence of the schema of

\( \Sigma_\alpha \)-**(Strong) Replacement**: \( \forall x \in a \exists y \theta \supset \exists f \forall x \in a (fx \neq \emptyset \land \forall y \in fx \theta) \), \( \theta \in \Sigma_\alpha \),

every formula of the form \( \forall x < y \theta \) with \( \theta \in \Sigma_\alpha \) is equivalent to a \( \Sigma_\alpha \) formula.

As a corollary of the above proof, we have the following extension.

**6.5(ii) Corollary** Let \( A \) be a model of \( \mathcal{S} \), and let \( T \) and \( B \) be coded in \( A \), with \( T \) a set of sentences of \( L_{\omega \omega} \) and \( B \) a transitive set satisfying some weak closure condition (e.g., \( L_B \) is a fragment of \( L_{\omega \omega} \)). Let \( F \in A \) be, in \( A \), a map from \( \omega \) into \( L_{\omega \omega} \), and suppose its range includes \( \Sigma^\beta \) and all subformulae of \( T \). For an ordinal \( \beta \) of \( A \), say

\[ \tau \beta \text{-fulfils } T + \text{Tr } \Sigma^\beta \] (with respect to \( F \))

if \( \tau \in A \) is a tree of rank \( \geq \beta \), and for all \( \sigma \in \tau \), all \( i < |\sigma| \),

i. \( F_{i} \in \sigma^1_i \)
ii. for all \( \varphi \in \sigma^1_i \cap T \), \( \varphi'^\sigma \),
iii. for all \( \varphi x \in \sigma^1_i \cap \Sigma_\alpha \cap B \), all \( x \in \sigma^0_i \), \( \varphi x \supset (\varphi x)^{i'} \).

VI More Model-theoretic Applications: \( \omega \)-models
Say $\tau \beta$-fulfils . . . if it does so w.r.t. some such $F$.

Then if $\mathcal{A}$ is a model of $T + \Sigma^B_{\alpha+1}$-Collection, then for each standard ordinal $\beta$ of $\mathcal{A}$, there exists a tree $\tau$ in $\mathcal{A}$ which $\beta$-fulfils $T + \text{Tr} \Sigma^B_\alpha$. \qed

Our next theorem is the generalization of 6.3 and the analogue of 5.3 from page 50.

6.6 Theorem Let $\mathcal{A}$ be a model of $\mathcal{S}$; let $\mathcal{L}_B$ be a fragment of $\mathcal{L}_{\omega\omega}$ coded in $\mathcal{A}$ by a code which is countable in $\mathcal{A}$, and let $T$ be a theory of $\mathcal{L}_B$ coded in $\mathcal{A}$. Let $\alpha$ be a standard ordinal of $\mathcal{A}$ and suppose that $\Sigma^B_{\alpha+1}$-overspill holds in $\mathcal{A}$. Consider the set

$$S = \{ B \leq^\text{end}_{\Sigma^B_\alpha} \mathcal{A} : B \models T \}$$

(a) Suppose that $\alpha$ is finite, that $T$ is included in the closure of the set of finitary formula under conjunction, disjunction, and existential quantification, and that $T$ extends $\mathcal{S}$ plus the first-order schema of Collection. Then

- $S$ has an indicator.
- $S \setminus \{\mathcal{A}\}$ is non-empty iff $\mathcal{A}$ is a model of the $\Sigma^{(HF)}_{\alpha+1}$ theory of $T$, and
- $S$ is cofinal in $\mathcal{A}$ iff $\mathcal{A}$ is a model of the $\Pi^{(HF)}_{\alpha+2}$ theory of $T$.

(b) Suppose $\mathcal{A}$ is locally countable and either that $\mathcal{A}$ is a model of $\Sigma^B_{\alpha+1}$-Strong-Replacement or that $T$ extends $\mathcal{S} + \Sigma^B_{\alpha+1}$-Strong-Replacement. Then

- $S$ has an indicator.
- $S \setminus \{\mathcal{A}\}$ is non-empty iff $\mathcal{A}$ is a model of the $\Sigma^{(B)}_{\alpha+1}$ theory of $T$, and
- $S$ is cofinal in $\mathcal{A}$ iff $\mathcal{A}$ is a model of the $\Pi^{(B)}_{\alpha+2}$ theory of $T$.
- If $B$ is a resolvable admissible set with height equal to the ordinal of $\mathcal{A}$, then $S$ is symbiotic with

$$\{ B \in S : B \text{ } B\text{-saturated } \},$$

that is, those $B$ which realize a $B$-r.e. type (which may have parameters from $B$) if every $B$-finite subset of the type is realized.

Proof: We shall only consider the condition for $S \setminus \{\mathcal{A}\}$ being nonempty, and the symbiosis. The remainder of the theorem will then be clear from our constructions.

The necessity of the condition is clear, so let us first consider its sufficiency in (a) when $0 \neq \alpha$. We need one preliminary definition: for each (real) set $a$, define the $\Delta_0$ formula $\theta_a(x)$ by

$$\theta_a(x) = \forall y \in x \bigvee_{b \in a} \theta_b(y) \land \bigwedge_{b \in a} \exists y \in x \theta_b(y).$$

The formal sentence $\exists x \theta_a(x)$ will often be paraphrased by $a \text{ exists}$.

Let $T$ be as given in (a). Then by 6.5.ii for each standard ordinal $\beta$ of $\mathcal{A}$, $T$ proves

1. if $T$ exists and is hereditarily countable and if $\beta$ exists,
2. then there exists $\tau$ which $\beta$-fulfils $T + \text{Tr} \Sigma_\alpha$.

(If $\alpha = 0$, this cannot be expressed as a $\Sigma_1$ sentence, and so we shall postpone this case until later.) If $\mathcal{A}$ is a model of the $\Sigma^{(HF)}_{\alpha+1}$ consequences of $T$, we have that (2) holds in $\mathcal{A}$ for all standard $\beta$. By overspill we may find a witness $\tau$ of nonstandard rank. Choose any infinite branch $\sigma = \langle \rho_i \rangle_{i \in \omega}$ of $\tau$, and let $C = \mathcal{A} \upharpoonright \cup_{i \in \omega} \sigma_i^0$. $C$ is a $\Sigma^{HF}_\alpha$-elementary substructure of $\mathcal{A}$ which is a model of $T$. Let $B$ be the initial segment of $\mathcal{A}$ determined by $C$. Then $B$ is as required, for Lemma 6.4 also shows that each axiom of $T$ is absolute between $\mathcal{B}$ and $C$. Finally $\mathcal{A} \neq \mathcal{B}$ as follows. We have
\[ A \models \forall n \exists! \langle b_0, \ldots, b_n \rangle \left( b_0 = \bigcup_{\sigma \in_T} \sigma_0^0 \land \forall i < n (b_{i+1} = \bigcup b_i) \right). \]

The relation \( x = \bigcup y \) is \( \Delta_0 \) and so by \( \Sigma_1 \)-Collection we have that \( b = \bigcup_{n \in \omega} b_n \) is an element of \( A \). Let \( c = \{ x \in b : x \notin x \} \). Then \( c \notin b \), and as \( B \subseteq b \), we have \( c \notin B \). This proves (a) for \( \alpha \neq 0 \).

Before we proceed with (b), let us consider the easier problem of finding a \( \Sigma^B_\alpha \)-elementary substructure of \( A \) which is a model of \( T \) when \( \Sigma^B_{\alpha+1} \)-Replacement holds in \( A \) and \( T \) is arbitrary. (We are not assuming \( T \) has any coding ability.) We shall show that this exists if (and only if) \( A \) is a model of those sentences contained in the closure of \( \Sigma^B_{\alpha+1} \), under conjunction, disjunction and existential quantification which are implied by \( T \). Let \( \beta \) be a standard ordinal of \( A \) and define the following sets in \( A \):

\[ B_\beta = \{ x \in B : \text{rk}(x) < \beta \}, \quad T_\beta = T \cap B_\beta. \]

Let \( G \in A \) be a surjection from \( \omega \) onto \( L_{B_\beta} \). For each \( n \in \omega \) and each standard ordinal \( \gamma \) of \( A \), define the formula (where the middle terms refer to i,ii in the definition of \( \beta \)-fulfil on page 63)

\[ F_{\gamma,n}(\sigma_0, \ldots, \sigma_{n-1}) = \left( \bigwedge_{i \in n} G i \in \gamma_i \right) \land \gamma_{\beta}^{(\sigma_0, \ldots, \sigma_{n-1})} \land \left( \text{Tr } \Sigma_{\alpha} \right)^{\left(\sigma_0, \ldots, \sigma_{n-1}\right)} \land \exists \sigma_n \bigwedge_{\delta < \gamma} F_{\delta,n+1}(\sigma_0, \ldots, \sigma_n) \]

where this is assumed to be defined without any coding as in our proof of the Completeness Theorem on page 11 and of 5.3.ii on page 51. Then for all \( \gamma, n, T \) proves

\[ \exists \sigma_0, \ldots, \sigma_{n-1} F_{\gamma,n}(\sigma_0, \ldots, \sigma_{n-1}) \]

since we can see (nonconstructively) that this is true in any model of \( T \). Now suppose \( F_{\gamma,n}(\sigma_0, \ldots, \sigma_{n-1}) \) holds in \( A \) for some \( \sigma = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in A \). We claim that there exists in \( A \) a tree \( \tau \) of rank \( \gamma \) such that \( \sigma \prec \tau \) fulfils \( T_\beta \) and \( \text{Tr } \Sigma_{\alpha} \) (with respect to \( G \)). We use induction on \( \gamma \). By the inductive hypothesis, there exists \( \sigma_n \in A \) so that for all \( \delta < \gamma \) there exists \( \tau_\delta \) of rank \( \geq \delta \) so that \( \sigma \prec (\sigma_n) \prec \tau_\delta \) fulfils \( T_\beta \) and \( \text{Tr } \Sigma_{\alpha} \) (w.r.t. \( G \)). By \( \Sigma^B_{\alpha+1} \)-Replacement, we may form a collection \( K \) of such \( \tau_\delta \)’s. Then \( \tau = \langle \sigma_n \rangle \cup K \) is as required. Hence if \( A \) is a model of the part of the theory of \( T \) specified above, by setting \( \gamma = \beta \) and applying overspill and 6.4, we may obtain a proper substructure of \( A \) which is a model of \( T \).

We may use this construction for (a) for the case \( \alpha = 0 \), by taking the appropriate initial segment.

Next consider the case where \( T \) extends \( S + \Sigma^B_{\alpha+1} \)-Replacement where \( A \) is a locally countable model of the \( \Sigma^B_{\alpha+1} \) theory of \( T \), and we wish to find a \( \Sigma^B_{\alpha+1} \)-elementary initial segment of \( A \) which is a model of \( T \). We need a definition analogous to that given in the non-\( \omega \)-model case, one which captures some of the properties of trees defined non-constructively via satisfaction functions. These further properties must be local in that they are preserved under the union of trees. Consider the following. Let \( \beta \) be a standard ordinal of \( A \). Fix some surjection \( G : \omega \to L_{B_\beta} \). For two sequences, \( \sigma, \sigma' \), with \( |\sigma| = |\sigma'| = n \), say \( \sigma \preceq \sigma' \) if for all \( i < n, \sigma_i \subseteq \sigma'_i \) and if \( Gi \) has free variables \( \bar{x} \), then for all \( \bar{x} \in \sigma_i, (Gi)(\bar{x})' \equiv (Gi)(\bar{x})' \). For any ordinal \( \gamma \) say extendible \( \mu, \gamma (\tau) \) (w.r.t. \( G \)) if \( \tau \) fulfils \( T_\beta + \text{Tr } \Sigma_{\alpha}^B \), and for all \( \delta < \gamma, \sigma \in \tau, x \in \sigma^0_{|\sigma|}, y \in x \),

- there exists a tree \( \tau' \) of sequences each extending, or a subsequence of, \( \sigma \), and
- there exists a tree isomorphism \( f \) from \( \tau \models \sigma = \{ s \in \tau : s \supseteq \sigma \lor s \subseteq \sigma \} \) to \( \tau' \) such that for all \( s \in \tau, s \preceq fs \), and

\[ \text{It is sufficient to suppose } f \sigma = \sigma \text{ rather than } s \preceq fs \text{ for all } s \in \tau; \text{ this latter requirement is needed, however, to construct a } B \text{-saturated initial segment.} \]
VI More Model-theoretic Applications: $\omega$-models

- for all $s \in \tau'$, if $|s| > |\sigma|$ then $y \in s_{|\sigma|}'$, and finally
- extendible$_{\beta,\delta}(\tau')$ holds.

Suppose $T$ includes $\Sigma^{B}_{\alpha+1}$-Replacement. Then by the proof of 6.5, it is easy to see by induction on $\gamma$ that in any model of $T$ containing $\beta, \gamma, T_\beta$, and $B_\beta$, for any “good” sequence $\sigma$ there is a tree $\tau$ of rank $\gamma$ such that $\sigma_\tau$ is a $(\beta, \gamma)$-extendible tree, and so in particular,

$$\text{if } \beta, T_\beta, \text{ and } B_\beta \text{ exist, there is a } (\beta, \beta)\text{-extendible tree of rank } \beta. \tag{3}$$

If $\alpha \neq 0$, (3) may be expressed in $T$ by a $\Sigma^{(B)}_{\alpha+1}$ sentence. So if $A$ is a model of the $\Sigma^{(B)}_{\alpha+1}$ consequences of $T$, then by overspill there exists a tree $\tau$ of nonstandard rank. We now construct our initial segment. Let $\tau = \tau^0$, and choose $\sigma^0 \in \tau^0$ of nonstandard rank. Pick $x \in (\sigma^0)^0_{|\sigma^0|}$- and $y \in x$. Then there is a tree $\tau^1$ satisfying the four points in the definition of extendible which is $(\gamma, \gamma)$-extendible of some nonstandard $\gamma$. Choose $\sigma^1 \in \tau^1$ of nonstandard rank and $x \in (\sigma^1)^0_{|\sigma^1|}$- and $y \in x$. Continuing in this way we have an increasing sequence of finite sequences

$$\sigma^0 \subseteq \sigma^1 \subseteq \sigma^2 \subseteq \ldots.$$

Let $\sigma = \langle \sigma_i \rangle$ be the limit. Then $\sigma$ fulfills $T + Tr^{B}_{\sigma}$, and so $B = A \upharpoonright \bigcup_{i \in \omega} \sigma^0_i$ is a $\Sigma^B_{\alpha+1}$-elementary substructure of $A$ which is a model of $T$. By judicious choices of the $x$’s and $y$’s, and using the local countability of $A$, we may ensure that $B$ is an initial segment. Finally, because $T$ includes $\Sigma^{B}_{\alpha+1}$-Collection, we can specify that all the $\tau$’s are contained in some $b \in A$, and in this way ensure that $B \neq A$.

If $\alpha = 0$, or if $T$ does not necessarily include $\Sigma^{B}_{\alpha+1}$-Replacement but $A$ is a model of $\Sigma^{B}_{\alpha+1}$- Replacement, then we can combine the above two constructions. That is, we can express (3) directly without coding, and so expressed, it is provable in $T$. So if $A$ is a model of the $\Sigma^{(B)}_{\alpha+1}$ consequences of $T$, then (3) holds in $A$, and moreover we may express (3) uniformly in $A$ using $\Sigma^{B}_{\alpha+1}$-Replacement in $A$ and codes of $T$ and $B$. Now use overspill and construct the initial segment as before.

Next let us consider the following problem. Suppose $A$ is a model of $T + \Sigma^{B}_{\alpha+1}$-Replacement + $\Sigma^{B}_{\alpha+1}$-overspill which codes $T$ and a resolvable countable admissible set $B$, and that we wish to find a proper $\Sigma^B_{\alpha}$-elementary substructure of $A$ which is a $B$-saturated model of $T$. We may suppose that $T$ contains $\forall \bar{x} (\theta \lor \neg \theta)$ for all $\theta \bar{x} \in L_B$. Let $R : ord(B) \to B$ be a resolution of $B$. Let $\tau \in A$ be any tree which fulfills $T + Tr^{\sigma}$ of nonstandard rank and let $C = \bigcup_{\sigma \in \tau} \sigma^0_{|\sigma|}$. Fix some (external) listing of all pairs $\langle \Phi, \bar{a} \rangle$, where $\Phi = \Phi(x, \bar{a})$ is a $B$-recursive subset of $L_B$ with only the free variables listed and $\bar{a} \in C$. Choose an infinite branch of $\tau$ in the following manner. Suppose we have chosen $\sigma \in \tau$ nonstandard rank. Let $\langle \Phi, \bar{a} \rangle$ be the first pair in our listing which we have not yet considered such that $\bar{a} \in \sigma^0_{|\sigma|}$. Suppose

$$\text{for all standard } \beta, \text{ there exists } i \in \omega \text{ and an extension } \sigma' \in \tau \text{ of } \sigma \text{ of rank } \beta \text{ such that for all extensions } \sigma'' \in \tau \text{ of } \sigma', \bigl( \exists x \bigl( (\Phi \land R\beta) \bigr) \bigr)^{\sigma''}_{i} \tag{4}.$$  

Then by overspill, we may choose $i \in \omega$ and a $\sigma'$ of nonstandard rank, and let this $\sigma'$ be in the branch. If not (4), then choose $\sigma' \in \tau$ to be any extension of $\sigma$ of nonstandard rank.

Let $\langle \sigma_i \rangle_{i \in \omega}$ be any infinite branch obtained in this way, and let $B = \bigcup_{i \in \omega} \sigma^0_i$. Then $B$ is $B$-saturated, for let $\Phi(x, \bar{a})$ be any $B$-recursive type. If (4) held when we came to consider the pair $\langle \Phi, \bar{x} \rangle$, then $\Phi$ is clearly realizable in $B$. Suppose (4) did not hold, and at that point we had already chosen $\sigma = \langle \sigma_i \rangle_{i \leq n}$. Then there exists a standard $\beta$ such that for all $i \in \omega$, all extensions
VI  More Model-theoretic Applications: \( \omega \)-models 67

\[ \sigma' \in \tau \text{ of } \sigma \text{ of rank } > \beta, \text{there is an extension } \sigma'' \text{ of } \sigma' \text{ that } -\theta^{\sigma''}_i \text{ holds, where } \theta = \exists x \bigwedge (\Phi \cap R\beta). \]

Choose \( i \) so that the sentence \( \forall \alpha (\theta \lor -\theta) \) is the \( i \)-th element of our implicit listing of \( T \). Then \( -\theta_i^{\sigma''} \) and so \( -\theta_i^{\sigma''} \). Thus for all extensions \( \sigma' \in \tau \) of nonstandard rank, \( -\theta_i^{\sigma''} \), and so \( \theta \) is false in \( B \), that is, \( \Phi \) is not \( \mathcal{B} \)-finitely realizable.

To obtain a \( \mathcal{B} \)-saturated initial segment, we in addition ensure that the tree \( \tau \) be \((\beta, \beta)\)-extendible for some nonstandard \( \beta \), and then in choosing an infinite branch, we alternately the two constructions given above.

If \( T \) does not necessarily extend \( S + \Sigma^B_{\alpha+1} \)-Replacement but \( A \) is a model of \( \Sigma^B_{\alpha+1} \)-Replacement, we combine all three of the above techniques to obtain the required result.

This completes our proof of 6.6. \qed

As a corollary we have:

**6.7 Corollary** Let \( T \) be any r.e. extension of \( \text{ZF}^- \). For any \( k \in \omega \), each nonstandard model of \( T \) has a proper \( k \)-elementary initial segment which is a model of \( T \). \qed

A weaker version of 6.7 (requiring the power-set axiom) appeared in the unpublished Friedman [7y] (but 6.7 was obtained independently).

Another corollary concerns \( \omega \)-models of analysis.

**6.8 Corollary** Let \( A \) be an \( \omega \)-model of \( \Sigma^1_1 \)-AC. Then the following are equivalent. (Also, see addendum on page 78.)

i. Let \( T = \{ \varphi x : \varphi \in T \} \) be a \( \Pi^1_1 \) collection of formulae of analysis, where the \( \Pi^1_1 \) definition may have parameters from \( A \). Let \( Z \in A \) and suppose \( A \models \varphi Z \) for all \( \varphi \in T \). Then there exists \( B \in A \) and \( B \subseteq \{ (B)n : n \in \omega \} \) such that \( Z \in B \) and \( B \models \varphi Z \) for all \( \varphi \in T \).

ii. The set \( W = \{ X \in A : <_x \text{ well-founded } \} \) is not \( \Sigma^1_1 / A \).

Proof: Suppose (i). If \( A \) is a \( \beta \)-model, then (i) holds, so suppose \( A \) is not a \( \beta \)-model. Then there is some pseudo-well-ordering \( < \in A \) and, moreover, we have \( < \)-overspill holding on this ordering, for otherwise we would have that \( W \) is \( \Sigma^1_1 / A \).

Let \( T = \{ \varphi : \forall y \exists n \theta(y, f(n \varphi^{-})) \} \) be any \( \Pi^1_1 \) set of sentences true in \( A \) (where we are suppressing the parameter \( Z \) and the parameters in \( \theta \)). Let \( <_\varphi \) be the Kleene-Brouwer ordering on the non-past-secured sequences of of \( \{ s : \theta(s, f \varphi^{-}) \} \). Now by 6.5, for each ‘ordinal’ \( \alpha \) in the standard part of \( <_\varphi \), we have

\[
\exists \tau \left( \text{rk}(\tau) \geq \alpha \land \forall \varphi ( \text{rk}(<_\varphi) < \alpha \implies \tau \text{ fulfils } \varphi ) \right).
\]

By overspill, this holds for some non-standard \( \alpha \), and we may obtain the required \( B \) from any infinite branch of any witness \( \tau \).

Now suppose (i) but not (ii). Then \( A \) cannot be a \( \beta \)-model. Fix some pseudo-well-ordering \( < \in A \) and suppose

the well-founded part of \( < = \{ n : A \models \varphi nZ \} \)

for some \( \Sigma^1_1 \) formula \( \varphi \) with parameter \( Z \in A \). Now

\[
T = \{ \varphi n x : n \in \text{well-founded part of } < \}
\]

is a \( \Pi^1_1 \) set with parameter \( < \in A \), and \( T(Z) \) is true in \( A \). So by (i) there exists \( X \in A \) such that for all \( \varphi n x \in T \), \( (\varphi nZ)^X \). But then the well-founded part of \( < \) is \( \Delta^1_1 \) in \( A \), and as \( \Delta^1_1 \)-CA holds in \( A \), this contradicts that \( < \) is a well-ordering in \( A \). \qed
We were careful enough in our proof of 6.5 so that it may be readily formalized to yield:

**6.9 Corollary** For all $k > 1$, and for each $\varphi \in \Sigma_{k+2}$,
\[
S + \Pi_k \text{-BI} \vdash \forall \varphi \in (\text{wf}(\varphi) \supset \forall x (\varphi x \supset \exists \tau (\text{rk}(\tau) = \text{rk}(\varphi) \land \tau \text{ fulfils } \varphi x))).
\]

For $k = 1$, we require, say, $\Pi_2 \text{-BI}$ or $\Sigma_1 \text{-DC}$. (In analysis we note that $\Pi_1^1 \text{-BI} \equiv \Sigma_1^1 \text{-DC}$ by S.D. Friedman [79].)

Consider the schema
\[
\omega \text{-RFN: } \forall X \subseteq \omega (\varphi X \supset \exists \omega \text{-model of } \varphi X \text{ containing } X).
\]

Our next result is due to H. Friedman [75]. His argument, using the completeness of the cut-free sequent calculus for $\omega$-logic, is probably simpler than ours. We note the next result immediately gives the Fact on page 35.

**6.10 Corollary** $S \vdash \Sigma_{k+2} \text{-RFN} \equiv \Pi_k \text{-cBI}$, for all $k \geq 1$.

Proof: We shall first show the right to left implication. Consider the version of 6.9 for $\frac{1}{2}$-$\text{fulfilment}$, namely for $\varphi \in \Sigma_{k+2}$
\[
S + \Pi_k \text{-cBI} \vdash \forall \varphi \in (\forall n \in \omega^2 (\varphi n \supset \exists \tau (\text{rk}(\tau) = \text{rk}(\varphi n) \land \forall \sigma \in \tau \varphi^{\frac{1}{2}} \sigma))).
\]

By $\Pi_1^0 \text{-CA}$ we can consider the set:
\[
Y = \{ \sigma \in \omega : \sigma \text{ codes a finite sequence which } \frac{1}{2} \text{-fulfils } \varphi \}.
\]

$Y$ is not well-founded, for otherwise by (5) we may find a tree $\tau \subseteq Y$ of rank greater than that of $Y$. Since $Y \subseteq \omega$, we may choose an infinite branch internally; the union of this branch is the required $\omega$-model. This argument relativizes to an arbitrary $X \subseteq \omega$ by adding a constant $\bar{X}$ and axioms $\{ \bar{n} \in X : n \in X \}$ and so we have $\Sigma_{k+2} \text{-RFN}$.

For the converse, note that for $\Pi_k \text{-cBI}$ it suffices to consider orderings on $\omega$. Let $\prec = \{ (m, n) : (m, n) \in \prec \}$ be such an ordering, and suppose
\[
\exists \text{ parameters } (\forall n ((\forall m < n \varphi m) \supset \varphi n) \land \forall n \varphi n)
\]

where $\varphi$ is in $\Pi_k$. Then by $\Sigma_{k+2} \text{-RFN}$ there is an $\omega$-model of (6), and so $\text{wf}(\prec)$ is false.

With regard to 6.6, in certain circumstances we need only look at first-order sentences. Call an admissible ordinal $\alpha$ self-definable if for no $\beta < \alpha$, $L_\beta \prec L_\alpha$.

**6.11 Corollary** Let $S$ be a model of $S + \Sigma_1$-overspill of standard ordinal $\alpha$, and suppose $\alpha$ is self-definable. Let $T \in L_\alpha$ be any theory of $L_\alpha$, extending $S$. Then there exists a proper 0-elementary substructure of $A$ which includes $\alpha$ and is a model of $T$ iff $A$ is a model of those first-order $\Sigma_1$ sentences $\theta$ for which $T \models \theta$, i.e., those $\theta$ which are true in all models of $T$ with standard ordinal $\geq \alpha$.

This follows easily from the following well-known result of Kripke and Platek:
\[
\alpha \text{ is self-definable iff for each } a \in L_\alpha, \text{ there exists a } \Sigma_1 \text{ formula } \theta x \text{ such that for any structure } B \text{ end-extendin} \qquad\text{such that for any structure } \qquad\text{such that for any structure } \qquad\text{such that for any structure } \qquad\text{such that for any structure } B \text{ end-extending } \alpha, B \models \exists x \theta x \land \theta a.
\]

For a proof, see V.7.8 of Barwise [75]. For readers who, like myself, know very little generalized recursion theory, I shall give a brief indication of the extent of the self-definable ordinals. The
least admissible ordinal which is not self-definable is greater than the first recursively inaccessible, the first recursively Mahlo, the first recursively hyper-Mahlo, etc.; see Cenzer [74]. The largest self-definable ordinal is the least stable ordinal, that is, the least \( \alpha \) such that \( L_\alpha <_1 L \).

As our final result, we shall consider the analogue of 2.8 on page 25. Let \( T \) be a consistent first-order r.e. theory extending \( \mathbb{S} + \text{Infinity} \), and let

\[(\Gamma_1, \Gamma_2)_\omega = \{ \varphi \in \Gamma_1 : \varphi \text{ is } \Gamma_2 \text{-conservative over } T \text{ with the } \omega \text{-rule} \}.
\]

Say \( T \) is \( \Pi^1_1 \)-sound if for all \( \Pi^1_1 \) sentences \( \varphi \), if \( T \vdash \varphi \), then \( \varphi \) is true.

The following result is not, as 2.8 was, the best possible, and further work remains to be done. (But see page 78.)

6.12 Theorem Let \( k \in \omega \) and let \( T \) be a consistent r.e. extension of \( S \).

i. If \( T \) extends \( \Sigma^k_{k+1} \)-Collection, a sentence \( \varphi \) is \( \Sigma^k_{k+1} \)-conservative (\( \Pi^k_{k+2} \)-conservative, respectively) over \( T \) with the \( \omega \)-rule if and only if all \( L_{\omega^k} \)-\( \omega \)-saturated models of \( T \) have (for any element \( x \)) a \( k \)-elementary substructure (containing \( x \)) which is a model of \( T + \neg \varphi \).

ii. If \( k \geq 0 \) and \( T \) extends \( \Sigma^k_{k+1} \)-Collection, then

\[(\Pi^k_{k+1}, \Sigma^k_{k+1})_\omega \text{ (if } k \geq 1 \text{), and } (\Sigma^k_{k+2}, \Pi^k_{k+2})_\omega
\]

are \( \Pi^1_0 \) in Kleene’s \( \mathcal{O} \), and are complete for this class of sets.

iii. If either \( T \) is not \( \Pi^1_1 \)-sound and includes \( \Sigma^1_1 \)-Bl or if \( T \) is strong enough to prove that every \( \Pi^1_1 \) formula is equivalent to a \( \Sigma^1_1 \) formula, then

\[(\Pi^1_1, \Sigma^1_1)_\omega \text{ and } (\Pi^1_1, \Delta^1_1(T))_\omega
\]

are also complete for this class.

Proof: The proof of (i) is clear if we can show that for any sentence \( \theta \) consistent with \( T \) in \( \omega \)-logic there exists a \( L_{\omega^k} \)-\( \omega \)-saturated model of \( T + \theta \). But this is a well-known result, closely related to the Gandy Basis Theorem, and is due to Ressayre [77] (and perhaps others).

For (ii), we first need a result concerning semi-representability.

6.13 Lemma Fix some recursive set \( W \), and suppose that for all \( e \in \omega \),

\[<_e = \{ \langle x, y \rangle : \langle x, y, e \rangle \in W \}\]

is a linear ordering and that \( \{ <_e : e \in \omega \} \) contains all primitive recursive well-orderings.

i. There is a \( \Sigma^1_1 \) formula \( \psi(x) \) such that

\[<_e \text{ is well-founded} \iff T \vdash e \varphi \underline{\psi}\]

ii. If \( T \) is \( \Pi^1_1 \)-sound, we may choose \( \psi \) above to be \( \Pi^1_1 \).

iii. If \( T \) is not \( \Pi^1_1 \)-sound and if \( T \) extends \( \Sigma^1_1 \)-Bl, we may choose \( \psi \in \Delta^1_1(T) \).

iv. The set

\[\{ e : \forall n \left( \text{wf}(<_n) \supset \text{wf}(<_e, n) \right) \}\]

is \( \Pi^0_1 \) in \( \mathcal{O} \), and is complete for this class of sets, and hence so is

\[\{ \psi : \forall n \left( \text{wf}(<_n) \supset (T \vdash e \varphi \underline{\psi}) \right) \}\]

where \( \psi \) ranges over \( \Sigma^1_1 \) formulae, or over \( \Delta^1_1(T) \) formulae if \( T \) is not \( \Pi^1_1 \)-sound and extends \( \Sigma^1_1 \)-Bl.
VI More Model-theoretic Applications: $\omega$-models

Proof of 6.13: (i) Choose $\psi$ so that $T$ proves for all $n \in \omega$,

$$\psi \equiv \exists T \left( \text{rk}(T) = \prec_n \land \forall \sigma \in T \left( T^\prec \land (\neg \psi \sigma) \right) \right).$$

Then $\prec_n$ well-founded implies $T \vdash T^\prec \psi \sigma \supset \psi \sigma$, which implies $T \vdash T^\prec \psi \sigma$. And $T \vdash T^\prec \psi \sigma$ implies $\prec_n$ well-founded.

(ii) If $T$ is $\Pi^1_1$-sound, we merely consider the $\Pi^1_1$ predicate

$$\exists T \left( \text{rk}(T) = \prec_n \land \forall \sigma \in T \left( T^\prec \land (\neg \psi \sigma) \right) \right)$$

where $\psi$ is as above. Alternatively, consider the predicate $\exists T \left( \text{rk}(T) = \prec_n \land \forall \sigma \in T \left( T^\prec \land (\neg \psi \sigma) \right) \right)$.

(iii) Let $\prec$ be some recursive non-well-ordering which $T$ proves is a well-ordering. Now $\Sigma^1_1$-BI implies that there is a hyperarithmetic hierarchy $H$ along $\prec$. Let $\psi$ be such that $T$ proves for all $n \in \omega$,

$$\psi \equiv \exists H \left( \text{rk}(H) = \prec_n \land \exists \omega \exists H \left( \text{rk}(H) = \prec_n \land \forall \sigma \in T \left( T^\prec \land (\neg \psi \sigma) \right) \right) \right).$$

We claim that $T \vdash T^\prec \psi \sigma$ if $\prec_n$ is a well-ordering. The only novel point here is to see that if $T + \psi \sigma$ is consistent in $\omega$-logic, then for any ordinal $\alpha < \omega^ck$, there exists a tree $\tau \in \text{HYP}$ of rank $\alpha$ which $\frac{1}{2}$-fulfils $T + \psi \sigma$; this is clear from the proof of 6.5.

(iv) Finally, (iv) is obtained immediately from the above by looking at the definition of the dual class, the collection of sets r.e. in $O$.

This concludes the proof of Lemma 6.13.

Continuation of proof 6.12: Fix $k \geq 0$, Let $\psi, \sigma$ be $\Sigma^1_1$, and consider the fixed point

$$T \vdash \left( \forall n \left( \exists T \left( \text{rk}(T) = \prec_n \land \forall \sigma \in T \left( T^\prec \land \text{Tr}(\Sigma^\sigma_k \land (\neg \psi \sigma)) \right) \right) \right) \right) \ (7)$$

or, alternatively, the fixed point

$$T \vdash \left( \forall n \left( \exists \left( \text{rk}(T) = \prec_n \land \forall \sigma \in T \left( T^\prec \land \text{Tr}(\Sigma^\sigma_k \land (\neg \psi \sigma)) \right) \right) \right) \right).$$

Consider:

(a) $\forall n \in \omega \left( \text{wf}(\prec_n) \supset (T \vdash \psi \sigma) \right)$
(b) $\Phi$ is $\Sigma^1_1$-conservative over $T + \omega$-rule
(c) $\Phi$ is $\Sigma^1_1$-conservative over $T + \omega$-rule
(d) $\Phi$ is $\Delta^0_1(T)$-conservative over $T + \omega$-rule

We claim that (a), (b) and (c) are equivalent.

First suppose (a), and let $A$ be any model of $T$. We shall show that there is a $k$-elementary substructure of $A$ which is a model of $T + \Phi$. If $A \models \Phi$, we are done. If not, choose some witnesses $n, \tau \in A$ for $\neg \Phi$. Now $\prec_n$ is necessarily non-well-founded, and so $\tau$ has (in the real world) an infinite branch, from which we may obtain the required structure. Hence (b).

The implication (b) $\supset$ (c) is trivial, and (c) $\supset$ (a) follows from 6.5.ii. By the lemma, this gives the first part of (ii). For the second, we merely alter (7) to ensure that there exists $\tau$ containing any arbitrary element.

For (iii), we simply choose $\psi$ to be $\Delta^1_1(T)$ in (7). Then we have that (a), (b), and (d) are equivalent.

This completes the proof of 6.12.
VII The Paris-Harrington Statement

Since the original definition of fulfilment was motivated by the Paris-Harrington statement, PHS, we thought that it would be appropriate to conclude this thesis with an exposition of this. For newcomers, we mention that PHS (defined below) is a natural combinatorial sentence which is true but not provable in PA: in fact, the instance

$$\forall e \exists n \rightarrow (e + 1)^c_3$$

is not provable in PA, even with the set of true $\Pi_1$ sentences as additional axioms. Furthermore, if we let

$$\sigma(e, c) = \mu n : n \rightarrow (e + 1)^c_3$$

then for each function $f$ provably recursive in PA there exists an $e$ such that

$$f(x) < \sigma(e, x)$$

for all $x$ (although for each $e$ the function $\lambda x. \sigma(e, x)$ is provably recursive), and

$$f(x) < \sigma(e, 3)$$

for all large $x$.

Let $k, e, c$ be (non-negative) integers and let $X$ be a (finite) set of integers. Say the partition relation

$$X \rightarrow (k)^e_c$$

holds if $|X| \geq e$ and for each function $f : [X]^e \rightarrow c$, where $[X]^e$ is the set of increasing $e$-tuples from $X$ and where $c = \{0, 1, \ldots, c - 1\}$, there exists a large subset $Y$ of $X$ of cardinality at least $k$ which is homogeneous for $f$, where $Y$ is large if $|Y| \geq \min Y$ and where $Y$ is homogeneous for $f$ if $f$ is constant on $[Y]^e$. This is a primitive recursive relation (in fact elementary), and so it is expressible by a formula in the language of arithmetic which is $\Delta_1$ in PRA. The Paris-Harrington statement is

PHS:

$$\forall k, e, c \exists n \rightarrow (k)^e_c.$$ 

The plan of this chapter is as follows. First the PHS is shown to be true, and then various independence results are established, with the PHS being shown to be equivalent to the $\Sigma_1$ Reflection Principle for PA. (Only this latter fact makes any use of the notion of fulfilment.) Then we consider the rate of growth of the function $\sigma$.

The exposition given below is an amalgam from many sources, but all the main ideas are due to J. Paris and L. Harrington. The work of L. Kirby, G. Mills and J. Paris [79] and of J. Ketonen and R. Solovay [79] has provided sharp results, namely that

$$\text{PA}^-_{ex} + \Sigma_{e+1} \text{-Collection} \vdash \forall k \exists n [k, n] \rightarrow (e + 2)^{c^e+1}_c \quad (1)$$

for each integer $c$, and that

$$\text{true } \Pi_1 \text{ sentences} + \Sigma_{e+1} \text{-Collection} \not\vdash \forall c \exists n n \rightarrow (e + 2)^{c^e+1}_c. \quad (2)$$

The proofs below give (2) but not (1). On the other hand, they are much simpler than those of the above-mentioned works, replacing complex model-theoretic and combinatorial arguments by naïve ones.

7.1 Theorem PHS is true.
Proof: Suppose not, and let \( k, e, c \) be chosen to violate the theorem. Call \( f \) \textit{good} if \( f : [n]^e \to c \) for some \( n \) and \( f \) has no large homogeneous subset of cardinality \( \geq k \). Let

\[
T = \{ g : g : [n]^e \to c \text{ for some } n \text{ and } g \subseteq f \text{ for some good } f \}.
\]

Then \( T \) ordered by inclusion is a finitely branching tree, and by our initial assumption, it is infinite. By König’s Lemma, it has an infinite branch \( B \). Let \( F = \bigcup B \). Then \( F : [\omega]^e \to c \), and by the infinite version of Ramsey’s Theorem, there exists an infinite \( Y \subseteq \omega \) which is homogeneous for \( F \). Let \( X \subseteq Y \) be a finite large set of cardinality \( \geq k \). Then \( F \upharpoonright X \subseteq f \) for some good function \( f \). But then \( X \) is homogeneous for \( f \), which contradicts the goodness of \( f \). \( \square \)

In fact, one can show that for each \( e \), where \( \text{PA}^-_{\text{ex}} \) is as given on 12,

\[
\text{PA}^-_{\text{ex}} + \Sigma_{2e+4}\text{-Collection} \vdash \forall k, c \exists n [k, n] \to (e+2)^c_{e+1}.
\] (3)

For in the above proof, we can choose the branch \( B \) so that the graph of \( F \) is definable: \( \Delta_3 \) in fact. By a direct analysis of a suitable proof of the infinite Ramsey’s Theorem (see Jockusch [72], page 275), it can be seen in \( \text{PA} \) that \( F \) has an infinite definable (it can be shown to be \( \Delta_{2e+2} \)) homogeneous subset. The contradiction follows as above. Thus for each \( e \) the function \( \lambda x . \sigma(e, x) \) is provably recursive in \( \text{PA} \).

The above argument will not work if \( e \) is a free variable, for Jockusch [72] proves, for instance, that for each \( e \geq 2 \), there exists a \textit{recursive} partition of \( [\omega]^e \) into two classes with no infinite \( \Delta_e \) homogeneous set.

7.2 Lemma For all \( e \geq 1 \)

\[
\text{PA}^-_{\text{ex}} + \Sigma_{e}\text{-Collection} \not\vdash \forall c \exists n n \to (e+2)^c_{e+1}.
\]

Proof: (We shall in fact prove more.) Let \( M \) be a nonstandard model of \( \text{PA}^-_{\text{ex}} \). Suppose that for some nonstandard \( c \in M \),

\[
\exists n n \to (e+2)^c_{e+1}.
\]

Fix \( n \) to be the least witness. Let \( 2^{b+2} \) be any nonstandard power of 2 less than \( c \), and let \( \langle \varphi_i \rangle \) be any natural listing of all \( \Delta_0 \) formulae whose free variables are among \( v_0, v_1, \ldots, v_e \). Define \( F : [n]^{e+1} \to c \) by

\[
F(\overline{v}) = \begin{cases} 2v_0 + 1, & \text{if } v_0 < 2^b, \\ \sum_{i<b} 2^{i+1} F_i(\overline{v}), & \text{otherwise}, \end{cases}
\]

where \( \overline{v} \) is (the code of) an \( e+1 \)-tuple from \( n \), and

\[
F_i(\overline{v}) = \begin{cases} 1, & \text{if } \text{Sat}_{\Delta_0}(\langle \varphi_i \rangle, \overline{v}), \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \text{Sat}_{\Delta_0} \) is the usual satisfaction predicate for \( \Delta_0 \) formulae.

Since \( F \) can be defined internally, there exists in \( M \) a large subset \( \{c_0 < c_1 < \ldots \} \) of \( n \) of cardinality greater than \( e+1 \) which is homogeneous for \( F \). Since \( F(c_0, \ldots, c_e) = F(c_1, \ldots, c_{e+1}) \), it must be the case that \( c_0 \geq 2^b \). Thus the set is of nonstandard cardinality, and so we may consider the initial segment \( I = \bigcup_{i \in \omega} c_i \). If we can show that \( I \) is a model of \( \text{PA}^-_{\text{ex}} + \Sigma_c\text{-Collection} \), we are done, for the partition relation is then absolute between \( I \) and \( M \), and obviously \( n \in I \). We proceed as follows. For a \( t \)-tuple \( \overline{t} = (i_0, \ldots, i_{t-1}) \), let \( c_{\overline{t}} = (c_{i_0}, \ldots, c_{i_{t-1}}) \).
(i) First note that for any $\Delta_0$ formula $\varphi\vec{v}$ with $t \leq e + 1$ free variables and for any two increasing $\tau$-tuples $0 \leq \vec{t} < \vec{t}$, $\varphi_{\vec{t}} \equiv \varphi_{\vec{t}}$ holds in $\mathcal{M}$. If $t \neq 0$, this is because $F(c_{\vec{t} - \vec{t}}) = F(c_{\vec{t} - \vec{t}})$ for $\vec{z} = (c_{t+1-t-1}, c_{t+1-t-2}, \ldots, c_{t})$ (where $\vec{z} = \langle \rangle$ when $t = e + 1$).

(ii) $I$ is closed under addition and multiplication, as follows. $c_0 + i \leq c_i$ for all $i \leq c_0 - e$, and so $2c_0 - e \leq c_{e+1}$. As $c_0 > 2e$, we have $\frac{c_0}{2} - \frac{e}{2} \leq c_{e+1}$, and hence by (i), $\frac{c_0}{2} - \frac{e}{2} < c_f$ for all $i \leq c_0 - e$ and so $\frac{c_0}{2} - \frac{e}{2} < c_{e+1} - \frac{e}{2}$. Thus $(\frac{c_0}{2})^2 < c_{e+1}$, and by (i) again, $(c_f)^2 < c_{e+1}$.

Define the class $V_k$ of formula inductively as follows. Let $V_0 = \Delta_0$, and let $V_{k+1}$ be the closure of $V_k \cup \{ \exists \vec{x} \theta : \theta \in V_k \} \cup \{ \forall \vec{x} \theta : \theta \in V_k \}$ under conjunction and disjunction. (NB: we do not close under bounded quantification.) For each $V_k$ formula $\varphi(\vec{x})$, define a $\Delta_0$ formula $\varphi^V(\vec{x}, y_0, \ldots, y_{k-1})$ inductively as follows. If $\varphi$ is $\Delta_0$, let $\varphi^V = \varphi$; let $(\varphi \land \theta)^V = \varphi^V \land \theta^V$, and similarly for conjunction; and let $(\neg \varphi)^V = \neg \varphi$ for $\varphi \in V_k$ and for $Q \exists \forall$.

Let $d_k$ be the $k$-tuple $(c_{e+1-k}, \ldots, c_{e+1})$.

(iii) We claim that if $\varphi\vec{x} \in \mathcal{V}_k$ with $k < e$, then for each $l$, each increasing $l$-tuple $\vec{t}$ with $l < \vec{t} < c_0 - e$,

$$\exists \vec{x} < c_l (\varphi^V(\vec{x}, y_0, \ldots, y_{k-1}) \land \exists \vec{y} \psi^V(\vec{x}, \vec{y})),$$

that is, $\exists \vec{x} < c_l (\varphi^V(\vec{x}, y_0, \ldots, y_{k-1}) \land \exists \vec{y} \psi^V(\vec{x}, \vec{y}))$,

where $\vec{t} = \langle t_0 \rangle \vec{t}$ and $d' = c_0 - e - k$. The inductive hypothesis gives that for all $\vec{x}, \vec{y} < c_{l_0}$,

$$\forall \vec{x} < c_{l_0} (\psi^V(\vec{x}, \vec{y})) \land \exists \vec{y} < c_{d'} \psi^V(\vec{x}, \vec{y})$$

so if $Q = \exists$, $\exists \vec{x} < c_{l_0} (\exists \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y}))$,

or if $Q = \forall$, $\exists \vec{x} < c_{l_0} (\forall \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y}))$.

Say $Q = \exists$. Since the number of $c$'s is $3 + (k - 1) \leq e + 1$, (i) gives

$$\exists \vec{x} < c_0 (\exists \vec{y} < c_2 \psi^V(\vec{x}, \vec{y})) \land \exists \vec{y} < c_3 \psi^V(\vec{x}, \vec{y})$$

Let $f(z) = \lceil \sqrt{z} - e - k - 2 \rceil$, where $s$ is the length of $\vec{x}$. By (ii), $c_0 < f(c_1)$, and so we may change the latest bound on $\vec{x}$ from $c_0$ to $f(c_1)$. So by (i), for all $i$ with $0 < i < d'$,

$$\exists \vec{x} < f(c_0) (\exists \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y})) \land \exists \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y})$$

Consider in $\mathcal{M}$ a map from $\{ i : 0 < i < d' \}$ to $\{ \vec{x} : \vec{x} < c(f_0) \}$, where for each $i$ we choose a witness $\vec{x}$ to the body of $(*)$. The size of the codomain is $f(c_0)^s \leq c_0 - e - k - 2 = d' - 2$, less than that of the domain. So by the pigeon-hole principle in $\mathcal{M}$, there exist $i, j$ with $0 < i < j < d'$ with the same witness $\vec{x}$. That is,

$$\exists \vec{x} < f(c_0) (\exists \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y})) \land \exists \vec{y} < c_{l_0} \psi^V(\vec{x}, \vec{y})$$

But the middle two conjuncts are contradictory. The $Q = \exists$ case is identical, except that the positions of the negations are changed. This proves the claim.
Using (iii), by an easy induction on complexity, we see that for \( k < e \), for any \( \nabla_k \) formula \( \varphi \bar{x} \) and any \( \bar{x} \in I \),

\[
I \models \varphi \bar{x} \quad \text{iff} \quad \varphi^A(\bar{x}, \bar{d}_k).
\]

(4)

Also for any \( \nabla_e \) formula \( \varphi \bar{x} \) and any \( \bar{x} \in I \),

\[
I \models \varphi \bar{x} \quad \text{iff} \quad \forall \text{ large } j \in \omega \ \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}).
\]

(4')

That \( \Sigma_e \text{-Collection} \) holds in \( I \) now follows by a simple application of underspill: suppose

\[
I \models \forall \bar{x} < a \exists \bar{y} \theta \bar{x} \bar{y}
\]

where \( a \in I \) and where \( \theta \) is a \( \Pi_{e-1} \) formula which may have parameters from \( I \). Then for any nonstandard \( i < c_0 + 1 - 2e \),

\[
\forall \bar{x} < a \exists \bar{y} < c_i \theta^A(\bar{x}, \bar{y}, \bar{d}_{e-1}).
\]

By underspill, there exists a standard \( i \) for which this holds, and by (4) we are done.

We have completed the proof of the lemma, but we shall also show that parameter-free \( \nabla_e \text{-Foundation} \) holds in \( I \), as follows. Let \( \forall x \) be \( \nabla_e \), with no other parameters, and suppose that

\[
I \models \exists x \varphi x, \text{ and that we wish to find a least witness. Then}
\]

\[
I \models \exists x \varphi x \quad \text{iff} \quad \exists i \exists x < c_i \ (I \models \varphi x), \text{ i standard}
\]

\[
\text{iff} \quad \exists i \exists x < c_i \forall j \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}), \text{ i, j standard, by (4')}
\]

\[
\text{iff} \quad \exists x < c_i \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}), \text{ for a standard i and non-standard j (or for } \Rightarrow, \text{ take } j = i + 1)
\]

\[
\text{iff} \quad \exists x < c_i \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}) \text{ for all } i, j \in M, \text{ with } 0 < i < j < d' = c_0 - 2e + 1, \text{ by (i)}
\]

\[
\text{iff} \quad \exists x < c_0 \varphi^A(\bar{x}, c_2, \bar{d}_{e-1}), \text{ by (i)}
\]

which implies \( \exists x < c_1/4 \varphi^A(\bar{x}, c_2, \bar{d}_{e-1}) \), by (ii)

\[
\text{iff} \quad \exists x < c_0/4 \varphi^A(\bar{x}, c_2, \bar{d}_{e-1}), \text{ by (i) again, for all i with } 0 < i < d'.
\]

Consider in \( M \) the map \( i \mapsto \mu x < c_0/4 \cdot \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}) \), \( 0 < i < d' \). Since \( d' - 1 > c_0/4 \), by the pigeon-hole principle there exist \( x_0 < c_0/4 \) and \( i, j \) with \( 0 < i < j < d' \) such that

\[
\varphi^A(\bar{x}_0, c_j, \bar{d}_{e-1}) \land \forall y < x_0 \neg \varphi^A(\bar{y}, c_i, \bar{d}_{e-1}) \land \varphi^A(\bar{x}_0, c_j, \bar{d}_{e-1}) \land \forall y < x_0 \neg \varphi^A(\bar{y}, c_j, \bar{d}_{e-1}).
\]

And so

\[
\exists x < c_1/4 \left( \varphi^A(\bar{x}, c_i, \bar{d}_{e-1}) \land \forall y < x \neg \varphi^A(\bar{y}, c_i, \bar{d}_{e-1}) \land \varphi^A(\bar{x}, c_j, \bar{d}_{e-1}) \land \forall y < x \neg \varphi^A(\bar{y}, c_j, \bar{d}_{e-1}) \right).
\]

By (i), this must hold for all \( i < j < d' \). Setting \( i = 0 \) shows the given map is constant. Thus \( \varphi^A(\bar{x}_0, c_i, \bar{d}_{e-1}) \) holds for all \( i < \omega \), and so \( I \models \varphi(x_0) \). If \( I \models \varphi(y) \) for some \( y < x \), then for some \( i < c_0 \) (in fact, for all sufficiently large \( i < \omega \), \( \varphi^A(\bar{y}, c_i, \bar{d}_{e-1}) \) holds, a contradiction.

\[\square\]

**7.3 Lemma** For all \( e \geq 0 \),

i. \( \Sigma_{e+1} \text{-Collection} \) is a \( \Pi_{e+2} \) conservative extension of \( \Sigma_e \text{-Induction} \), and

ii. \( \Sigma_e \text{-Induction} \) is a \( \Sigma_{e+2} \) conservative extension of \( \Sigma_e \text{-Collection plus parameter-free } \nabla_e \text{-Foundation} \) (with \( \nabla_e \) defined on page 73), all over \( \text{PA}^- \).
Proof: The first part of the lemma is just Application (xiv) on page 42, so let us consider the second. Let $\mathcal{M}$ be any model of $\text{PA}^- + \Sigma_e$-Collection and parameter-free $\nabla_e$-Foundation, and let $N \subseteq \mathcal{M}$ consist of those elements $a$ of $\mathcal{M}$ for which there is a $\Sigma_{e+1}$ formula $\psi x$ such that 

$$\mathcal{M} \vdash \exists! x \psi x \land \psi a.$$ 

Then $N \preceq_{e+1} \mathcal{M}$, as follows. $N$ is clearly closed under addition and multiplication. Suppose $\mathcal{M} \models \exists x \psi$, where $\psi$ is $\Pi_e$. Then by parameter-free $\nabla_e$-Foundation,

$$\exists! x \in \mathcal{M} \left( \mathcal{M} \models \psi x \land \forall y < x \neg \psi y \right).$$

By $\Sigma_e$-Collection, this latter formula is equivalent to a $\Sigma_{e+1}$ formula, and so $\psi$ has a witness in $N$. Thus $N \preceq_{e+1} \mathcal{M}$.

To complete the lemma, it suffices to show that $N$ is a model of $\Sigma_e$-Foundation. Suppose 

$$N \models \exists x \varphi x a,$$

where $\varphi$ is $\Sigma_e$ and for notational simplicity we only allow a single parameter $a \in N$. Then 

$$\mathcal{M} \models \exists x, u, z \left( \varphi xu \land \psi_a zu \right),$$

where $\psi_a$ is a $\Pi_e$ formula such that 

$$\mathcal{M} \models \exists! u \exists z \psi_a zu \land \exists z \psi_a za.$$ 

Now by parameter-free $\nabla_e$-Foundation, consider the least triple $(x_0, u_0, z_0)$ which is a witness to (5). Because $u_0$ is unique, $x_0$ and $z_0$ are independent of each other, and so $x_0$ is the unique element satisfying 

$$\mathcal{M} \models \exists u, z \left( \varphi xu \land \psi_a zu \land \forall y < x \neg \varphi yu \right).$$

Now by $\Sigma_e$-Collection, this is equivalent to a $\Sigma_{e+1}$ formula, and so $x_0 \in N$. Since $N \preceq_{e+1} \mathcal{M}$, $x_0$ is the least witness to $\varphi$ in $N$ also.\[\square\]

From 7.2 with parameter-free $\nabla_e$-Foundation and from 7.3 as the PHS is $\Pi_2$, we immediately have:

**7.4.i Theorem** True $\Pi_1$ sentences + $\Sigma_{e+1}$-Collection $\vdash \forall c \exists n n \rightarrow (e + 2)^c)^{e+1}.$  

From the proof of 7.2 we see that in $\text{PA}^\neg_c$ the sentence $\forall c \exists n n \rightarrow (e + 2)^c)^{e+1}$ implies that for each $m$ there exists a sequence of length $m$ which fulfils the first $m$ axioms of $\Sigma_{e-1}$-Induction, where the most convenient notion of fulfilment to use here is the combination of the second notion motivated by Skolem functions as given on page 8 with the version of i-fulfilment as given on page 9. The proofs are readily formalized to yield:

$$\text{PA}^\neg_c + \forall c \exists n n \rightarrow (e + 2)^c)^{e+1} \vdash \text{RFN}_{\Sigma_1}(\Sigma_{e-1}\text{-Induction}),$$

and so by the remarks following 4.3, we also have 

$$\text{PA}^\neg_c + \forall c \exists n n \rightarrow (e + 2)^c)^{e+1} \vdash \text{RFN}_{\Sigma_1}(\Sigma_e\text{-Collection}).$$

Moreover, the proofs are uniform in $e$, and so we may obtain 

$$\text{PA}^\neg_c + \forall e, c \exists n n \rightarrow (e + 1)^c) \vdash \text{RFN}_{\Sigma_1}(\text{PA}).$$
Furthermore, by formalizing the proof of (3) we have (where the dot notation is defined on page 21)

\[ \text{PA}^- \vdash \forall e \Pr_{pa}(\forall c \exists n n \rightarrow (e + 1)\langle c \rangle) \]

and so we have

**7.4.ii Corollary** \( \text{PA}^- \vdash \forall e, c \exists n n \rightarrow (e + 1)\langle c \rangle \equiv \text{RFN}_\Sigma_1(\text{PA}) \).

We may take a weaker instance of the left-hand-side of the above equivalence as follows. First we need an lemma of Paris and Harrington [77].

**7.5 Lemma** A set \( Y \subseteq X \) is homogeneous for \( F : [X]^e \rightarrow c \) iff every subset of \( Y \) of cardinality \( e + 1 \) is homogeneous for \( F \).

**Proof:** Let \( \vec{x} = \langle x_0, \ldots, x_{e-1} \rangle \) be the first \( e \) elements of \( Y \). Pick \( \vec{y} = \langle y_0, \ldots, y_{e-1} \rangle \) from \( Y \) so that \( F(\vec{x}) \neq F(\vec{y}) \) and \( y_0 + y_1 + \cdots + y_{e-1} \) is minimal. If \( i \) is the least index such that \( x_i \neq y_i \), then \( \{x_0, \ldots, x_i, y_i, \ldots, y_{e-1} \} \) is of cardinality \( e + 1 \) but not homogeneous.

Recall the finite version of Ramsey’s Theorem, which is provable in PRA:

\[ \forall k, e, c \exists n n \rightarrow (k)\langle c \rangle. \]

**7.6 Lemma** If \( m \geq 3 \), if \( m + e + 7 \rightarrow (e + 1)\langle c \rangle \), and if one of the following holds:

i. \( \text{N} \rightarrow (m + 1)_3 \)

ii. \( \text{N} \rightarrow (m + 2)_2 \)

iii. \( (m, N) \rightarrow (m + 1)_2 \), where \( (m, N) = \{ x : m < x < N \} \);

then \( (m, N) \rightarrow (m + e + 7)\langle c \rangle \).

**Proof:** We can suppose that \( e, c \geq 2 \). Let \( F : [N]^2 \rightarrow c \) be given. Let \( s = m - e - 1 \) and \( t = 2m + 6 \). Since \( t \rightarrow (m + 1)_2 \) (see comment at end of chapter), we can choose \( g : [t]^m \rightarrow 2 \) with no homogeneous set of cardinality \( m + 1 \). Define \( f : [N]^{e+1} \rightarrow 2 \) by

\[ f(v_0, \ldots, v_e) = \begin{cases} 
1 & \text{if } \{v_0, \ldots, v_e\} \text{ is homogeneous for } F \\
0 & \text{otherwise.} 
\end{cases} \]

Define \( G : [N]^m \rightarrow 3 \) as follows. Let \( \vec{v} = \langle v_0, \ldots, v_{m-1} \rangle < N \) be given and let \( i \) be the least \( i \) such that \( v_i \geq t \), if this exists, and \( m \) otherwise. If (i) holds let

\[ G(\vec{v}) = \begin{cases} 
g(\vec{v}) & \text{if } i = m \\
2 & \text{if } i = 1 \text{ or } m - 1 \\
\text{parity}(i) & \text{if } 1 < i < m - 1 \\
f(v_0 - s, \ldots, v_e - s) & \text{if } i = 0. 
\end{cases} \]

If (ii) holds, let

\[ G(\vec{v}) = \begin{cases} 
g(\vec{v}) & \text{if } i = m \\
\text{parity}(i) & \text{if } m - e \leq i < m \\
\text{parity}(i + f(v_i - s, \ldots, v_{i+e} - s)) & \text{if } i + e < m. 
\end{cases} \]

And if (iii) holds, let

\[ G(\vec{v}) = \begin{cases} 
\text{parity}(i + f(v_i - s, \ldots, v_{i+e} - s)) & \text{if } i + e < m. \\
0 \text{ or } 1 & \text{otherwise.} 
\end{cases} \]
Let $X \subseteq N$ be homogeneous for $G$ and such that $X$ satisfies the appropriate conditions of (i), (ii) or (iii). It is straightforward to check that in each case, $|X| \geq \min X \geq t$, and so we can suppose $X = \{c_0 < c_1 < \ldots < c_{e_0-1}\}$. Let $Y$ be the first $m + e + 7$ elements of $X$, and define $H : [Y]^e \rightarrow c$ by

$$H(v_0, \ldots, v_{e-1}) = F(v_0 - s, \ldots, v_{e-1} - s).$$

By our hypothesis, we can choose $Z \subseteq Y$ homogeneous of cardinality $e + 1$. Now the cardinality of $Z \cup \{c_{m+e+7}, \ldots, c_{e-1}\}$ is $(e + 1) + (t - (m + 7 - e)) = m$, and the value of $G$ on this set is 0, and so $G[X]^m = \{0\}$. Let $X' = \{c_0 - s, \ldots, c_{e_0-s-1} - s\}$. Obviously $m + e + 7 \leq \min X' \leq |X'|$. We claim that $X'$ is homogeneous for $F$. For if $\{x_0 - s, \ldots, x_e - s\}$ is any subset of $X'$ of cardinality $e + 1$, then the cardinality of $\{x_0, \ldots, x_e\} \cup \{c_{e_0-s}, \ldots, c_{e_0-1}\}$ is $e + 1 + s = m$, and $G(x_0, \ldots, x_e, c_{e_0-s}, \ldots, c_{e_0-1}) = 0$. By Lemma 7.5, we are done. \hfill \Box

7.7 Theorem (i) If $f : \omega \rightarrow \omega$ is provably recursive in $\text{PA}^- + \Sigma_{e+1}^e$-Collection, then

$$f(x) < \sigma(e + 1, x)$$

for all large $x$, for $e \geq 1$.

(ii) If $f : \omega \rightarrow \omega$ is provably recursive in PA, then

$$f(x) < \sigma(x, 3)$$

for all large $x$.

Proof: (i) Suppose $f$ is provably recursive in $\text{PA}^- + \Sigma_{e+1}^e$-Collection. Let $M$ be any proper elementary extension of $\mathbb{N}$, and fix $c$ nonstandard. Let $I$ be as in Lemma 7.2, but with a constant for $c$ added to the language. Now by the proof of Lemma 7.3, relativized to $c$, we have that $f(c) \in I$ whereas $\sigma(e, c) \notin I$. That is,

$$M \models \text{for all large } x, f(x) < \sigma(e + 1, x).$$

Now the same must be true in $\mathbb{N}$.

(ii) Let $M$ be any proper elementary extension of $\mathbb{N}$, and fix $e$ nonstandard. By 7.6 there is a primitive recursive function $g$ such that

$$\sigma(g(e, c), 3) > \sigma(e, c).$$

If $c$ is nonstandard, we may construct an initial segment which is a model of PA containing $c$ and $e$, and so $g(e, c)$ and $f(g(e, c))$, but not $\sigma(e, c)$, and so not $\sigma(g(e, c), 3)$. The theorem follows as before. \hfill \Box

By Lemma 7.6, we see that PA does not prove $\forall e \exists n \rightarrow (e + 1)^{\xi}, \forall e \exists n \rightarrow (e + 2)^{\xi}$, and $\forall e \exists n (e, n) \rightarrow (e + 1)^{\xi}$. But we have left unanswered the question of whether or not PA proves

$$\forall e \exists n \rightarrow (e + 1)^{\xi}.$$ 

This question does not appear to be answerable using our simple-minded techniques. There is an analogous (apparently very difficult) problem concerning the ordinary Ramsey partition relation: reasonable bounds are not known for the function

$$\lambda k : \mu n : n \rightarrow (k + 1)^{\xi}.$$
One can easily show that
\[(2k + 1) \not\rightarrow (k + 1)^2_2 \quad (k \geq 2)\].
Isbell [69] has proved that \(12 \not\rightarrow (4)^3_2\), and using this we may obtain\(^{17}\)
\[(2k + 6) \not\rightarrow (k + 1)^2_2 \quad (k \geq 3)\].

These lower bounds appear to me to be very low, but I cannot improve on them. Known upper bounds are also very poor.

Addendum

Corollary 6.8 on page 67 may be strengthened by requiring that \(\mathcal{A}\) be only an \(\omega\)-model of \(\Delta^1_1\)-CA.

Using the same technique, we may also greatly improve part (ii) of Theorem 6.12: e.g. for any \(\omega\)-consistent r.e. theory \(T\) extending \(\Delta^1_1\)-CA and for any \(k \geq 3\), the sets

\[(\exists m \Pi_k, \Sigma_k)_\omega \text{ and } (\Sigma_k, \Pi_k)_\omega\]

are \(\Pi^0_1\) in Kleene’s \(\mathcal{O}\), and are complete for this class of sets. (Here a sentence is in “\(\exists m \Pi_k\)” if it is a \(\Pi_k\) formula prefaced by existential numerical quantifiers.)

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\(^{17}\) Proof of \((2k + 6) \not\rightarrow (k + 1)^2_2\) for \(k \geq 3\). We can suppose \(k > 3\). Let:

- green = \(\{ n : n < 12 \}\), the first twelve numbers
- red = \(\{ n : 12 \leq n < k - 3 + 12 \}\), the next \(k - 3\) numbers
- blue = \(\{ n : k - 3 + 12 \leq n < 2(k - 3) + 12 \}\), and the next \(k - 3\) numbers

Choose \(f: [12]^3 \rightarrow 2\) to witness \(12 \not\rightarrow (4)^3_2\). Define \(F: [2k + 6]^k \rightarrow 2\) by

\[F(X) = \text{parity}(D + E)\]

where

\[D(X) = \begin{cases} \text{index of first blue number in natural order, if it exists} \\ k, \text{ otherwise} \end{cases}\]

and

\[E(X) = \begin{cases} f(Y), \text{ if } |X \cap 12| \geq 3, \text{ where } Y \text{ consists of the three least elements of } X \\ 0, \text{ otherwise} \end{cases}\]

Suppose \(F\) has a homogeneous set \(Z \subseteq 2k + 6\) of cardinality \(k + 1\). Then \(|Z \cap 12| \leq 3\). Hence \(Z\) must contain both red and blue numbers. But \(F(Z \setminus \{\text{any red number}\})\) cannot equal \(F(Z \setminus \{\text{any blue number}\})\). \(\square\)
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