A search for resonances in the $p\mu e$ system

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Abstract
The charge radius of the proton has recently been determined from $2s$–$2p$ spectroscopy of muonic hydrogen, giving a result which significantly deviates from earlier measurements. One hypothesis is that this discrepancy could arise because a metastable $p\mu e$ state is formed under the $p\mu(2s)$ threshold. We search for such a state by considering the three-body $p\mu e$ and $p\mu x$ systems, where $x$ is a particle with charge $–e$ and a mass which is varied between the electron and muon mass. We identify a new class of resonances, which extends the mass range allowing formation of resonances compared to an earlier work. We conclude that no $p\mu e$ resonances relevant for $2s$–$2p$ spectroscopy in muonic hydrogen can exist.

Keywords: proton radius, muonic hydrogen, three-body systems, resonances

1. Introduction
Over the past few years an experimental contradiction regarding the charge radius of the proton has arisen. Older measurements of this radius, which were based on hydrogen spectroscopy and electron–proton scattering data, were compiled into the CODATA 2006 value $r_p = 0.8768(69) \text{ fm}$ [4]. In contrast, a recent measurement using laser spectroscopy of the $2s$–$2p$ transition in muonic hydrogen, where the finite proton size yields an important correction, gave the result $r_p = 0.84184(67) \text{ fm}$ [1]—a difference of five standard deviations. Very recently this discrepancy was further reinforced by a new measurement carried out by the same group giving $r_p = 0.84037(39) \text{ fm}$ [2], now seven standard deviations from the CODATA 2006 value.

At present there is no explanation for this discrepancy. However, Jentschura [6] speculated that there may be a $p\mu e$ three-body resonance just under the $p\mu(2s)$ threshold. Muonic hydrogen is formed at high principal quantum numbers by stopping a low-energy muon beam in a hydrogen target. The muonic atoms cascade down to more tightly bound states, and a fraction of them will reach the metastable $2s$ state, which is the initial state of the spectroscopy. If a $p\mu e$ resonance exists, it may be formed by charge exchange collisions with the ordinary hydrogen present. Given a sufficiently long lifetime, this state could be the true lower state of the spectroscopic transition in the experiments. One would thus observe a transition frequency which is shifted by the binding energy of the electron. In order to be relevant for the proton radius problem this binding energy would have to be about $\geq 0.3 \text{ meV}$. The binding energy could also be larger than this value if the final $p\mu(2p_{3/2})$ is similarly shifted by the presence of the electron.

Furthermore, the lifetime of the initial metastable state has to be at least $\tau \sim 1 \mu s$, in order to survive the delay until the spectroscopic laser pulse [1]. This puts an upper limit $\Gamma = \hbar/\tau \lesssim 6.6 \times 10^{-10} \text{ eV}$ to the width of the metastable state. If also the final state of the transition is metastable, its lifetime is limited through the width of the observed line (which is consistent with the lifetime $8.5 \text{ ps}$ of $p\mu(2p)$ [1]).

The purpose of this paper is to search numerically for such a resonance. A similar investigation has previously been published by Karr and Hilico [3]. They studied how the binding energy of the $p\mu\mu$ resonance below the $p\mu(2s)$ state evolved as the mass of one of the muons was artificially reduced from its real value $m_\mu = 207m_e$ towards the value for the electron $m_e$. They concluded that there is no resonance with binding energy in the relevant regime, for masses lighter than about $48m_e$ (or $32m_e$ if the effect of vacuum polarization is not taken into account).

Our paper is complementary to the investigation by Karr and Hilico. In our work we use a method based on explicitly correlated Gaussian wave functions, as well as a Gaussian representation of the vacuum polarization potential. We search for $p\mu e$ resonances using mass scaling, starting both...
from the $p_{\mu\mu}$ system and from $pee$ (or H$^-$). In addition to the resonances calculated in [3] we also identify a new class of resonances. These resonances appear at the crossings of scattering thresholds belonging to bound states of different sub-systems.

2. Method

The three-body systems were calculated using the Coupled Rearrangement Channels method developed by Kamimura and co-workers [7, 8]. In this method the wave function is expanded using Jacobi coordinates ($\mathbf{r}_i$, $\mathbf{R}_i$), $i = 1, 2, 3$ in all three possible rearrangement channels (see figure 1). This gives a very versatile basis set, capable of adapting to states close to break-up thresholds of any pair of particles. We also note that by using Jacobi coordinates we automatically include all so-called mass-polarization terms which, in atomic calculations using coordinates centred on the nuclei, arise as corrections which are non-trivial to calculate.

Within this coordinate system we represent the wave function using a partial wave expansion of the angular variables and Gaussians in the radial variables. That is, for a state with total orbital angular momentum $J$, $M$ the wave function has the form

$$\Psi_{JM} = \sum_{a=1}^{3} \sum_{l_{a}=0}^{l_{a}^{\text{max}}} \sum_{\ell=0}^{\ell_{a}^{\text{max}}} \sum_{L_{a}=0}^{L_{a}^{\text{max}}} C_{a\ell L_{a}JM} \phi_{a\ell L_{a}JM},$$

where $a$ is the index of a particular rearrangement channel, $l_a$ and $L_a$ are the angular momenta along $\mathbf{r}_a$ and $\mathbf{R}_a$, respectively, and $i, I$ numbers the Gaussians along the two radial coordinates. The angular momenta $l_i$ and $L_i$ are chosen consistent with the total $J$ (i.e. $|l_i - L_i| \leq J \leq l_i + L_i$), which since in the calculations in this paper $J = 0$ implies $l_i = L_i$, up to some maximum values $l_{a}^{\text{max}}$ and $L_{a}^{\text{max}}$, which may be different for different rearrangement channels. Usually the maximum angular momentum used in our calculations was $l_i = L_i = 4$.

The total number of Gaussian trial functions for each rearrangement channel and angular momentum is given by $l_{a}^{\text{max}}$ and $L_{a}^{\text{max}}$. $N_{a\ell L_{a}JM}$ is a normalization constant ensuring that $\langle \phi_{a\ell L_{a}JM} | \phi_{a\ell L_{a}JM} \rangle = 1$. The widths of the Gaussians $r_{a\ell L_{a}JM}$ and $R_{a\ell L_{a}JM}$ are, for each channel and set of angular momenta, chosen as geometric progressions

$$r_{a\ell L_{a}JM} = r_{a\ell L_{a}JM}^{0} \left( \frac{r_{a\ell L_{a}JM}^{1}}{r_{a\ell L_{a}JM}^{0}} \right)^{l_{a}^{\text{max}}-1},$$

$$R_{a\ell L_{a}JM} = R_{a\ell L_{a}JM}^{0} \left( \frac{R_{a\ell L_{a}JM}^{1}}{R_{a\ell L_{a}JM}^{0}} \right)^{L_{a}^{\text{max}}-1},$$

where the smallest and largest values $\{r_{a\ell L_{a}JM}^{0}, r_{a\ell L_{a}JM}^{1}, R_{a\ell L_{a}JM}^{0}, R_{a\ell L_{a}JM}^{1}\}$ are set explicitly and used as non-linear variational parameters. In this way most Gaussians will span the short-to medium range, while a few more diffuse Gaussians capture the long range part of the wave function. Below, resonances with very small binding energies are calculated, and hence it was essential to set a large enough value for the outer radius $R_{a\ell L_{a}JM}$.

Resonances are calculated using the complex scaling method [9]. The complex dilation operator $U(\theta)$ acting on a function $f(\mathbf{r})$ is defined through

$$U(\theta)f(\mathbf{r}) = e^{3i\theta/2}f(e^{i\theta} \mathbf{r}),$$

where the exponential prefactor ensures that the complex scaled function satisfies the normalization condition $\int U(\theta)f^*(\mathbf{r})U(\theta)f(\mathbf{r})d^3r = 1$. The corresponding transformation of the Hamiltonian is, for the special case $V \propto 1/r$,

$$H(\theta) = U(\theta)HU^{-1}(\theta) = e^{-2i\theta T} + e^{-i\theta V}.$$  

Stationary eigenvalues of the complex scaled generalized eigenvalue problem

$$H(\theta)\psi(\theta) = E(\theta)\tilde{S}(\theta)\psi(\theta), \quad \tilde{S}(\theta) = \langle \psi(H(\theta)\tilde{\psi}) \rangle,$$

correspond to complex energies $E = E_{r} - i\Gamma/2$ of the system, where $E_{r}$ is the resonance energy and $\Gamma$ the width.

For atoms containing muons we must also include the effect of vacuum polarization. This is a very short-range correction to the pure Coulomb interaction due to the production of virtual electron–positron pairs by the electromagnetic field. For ordinary hydrogen this effect is very small, even compared to other QED corrections. However, due to the tighter binding of a muon to the proton, this effect is observable in muonic hydrogen. It leads to a splitting of the 2s and 2p states in muonic hydrogen of order 0.2 eV (see table 1). To first order, and assuming a point nucleus, the vacuum polarization can be calculated using the Uehling
potential \[ V_{vp}(r) = \frac{Z_1 Z_2 \alpha}{r} \int_0^1 e^{-2x^{2/3}}(2 + x) \left( \frac{1}{x^2} - 1 \right) dx, \tag{7} \]

where \( Z_1 \) and \( Z_2 \) are the charges of the two particles, \( \alpha \) the fine-structure constant and \( c \) the speed of light in vacuum. The potential \( V_{vp}(r) \) diverges in the limit \( r \to 0 \), which makes it difficult to treat numerically, and moreover, is difficult to use with complex scaling. We instead use the method of Walein and Kamimura \cite{11} and fit the potential to 20 Gaussians. Since the divergence is rather mild \( \propto \ln r \), such a fit can still give accurate results \cite{5}. The fitted potential is compared to the analytical form in figure 2. As shown in table 1 this does not introduce any severe approximations. With this form, all matrix elements can be calculated analytically, and complex scaling can be used.

3. Results

3.1. The pex system

The negative hydrogen ion has several resonances under the \( n = 2 \) threshold. In table 2 we compare our results to the best results found in literature. To be able to make a direct comparison to published results, these results were calculated using an infinite nuclear mass, meaning that the mass-polarization term was neglected. Our results were obtained using 5700 Gaussian basis functions with a maximum \( l = L = 4 \). We conclude that our calculation of resonances in \( ^{-}H \) is good to at least about \( 10^{-7} \) atomic units. An even better accuracy could probably be obtained by using symmetry adapted basis functions. However, since, we mostly consider systems without any particle exchange symmetry, we have chosen not to use symmetry adaptation in any of the calculations. In

| Table 1. Corrections due to vacuum polarization of the 2s and 2p states in muonic hydrogen, calculated in first-order perturbation theory. Energies are in meV. The experimental 2s–2p splitting is dominated by vacuum polarization, but also includes various other small terms, including terms dependent on the proton radius. |
| --- |
| \( E(2s) \) | \( E(2p) \) | \( E(2p) - E(2s) \) |
| Experimental | 206.295 |  |
| Uehling potential | -219.58 | -14.58 | 205.01 |
| Fitted Uehling potential | -219.57 | -14.57 | 205.00 |

table 3 present our results for the same states calculated using the physical proton mass.

Next, we follow the energy and width of the most tightly bound resonance \( 2s^2 \, ^{1}S_0 \) (now using the real proton mass) as the mass of one of the electrons is gradually increased. The results are summarized in table 4, and in figure 3. (The bound ground state of the system disappears for \( m_x > 1.2 m_e \).) Since we are interested in whether a resonance exists under the \( p\mu(2s) \) threshold, we look for resonances under the \( px(n = 2) \) threshold, where \( x \) is a negatively charged particle with mass \( m_x \). The energy of this threshold is in atomic units \( E_n^x = -\frac{1}{2n^2} \frac{m_x m_p}{m_x + m_p} \), where \( m_p \) is the proton mass. This energy is shown as a solid black line in figure 3.

We find that the resonance still exists at \( m_x = 3.8 m_e \), where the binding energy is 0.81 meV and the width 0.54 meV. Vacuum polarization has a negligible effect for particles with masses in this regime. As the mass is increased to \( m_x = 3 m_e \), we observe that a new resonance appears under the \( H(1s) \) threshold. (This state is a resonance since it lies above the \( px(1s) \) threshold.) As \( m_x \) grows further the binding energy and width of this resonance increases. Around \( m_x = 4m_e \), where the two thresholds cross, the resonance becomes associated to the \( px(n = 2) \) threshold instead. In the transition region between the two thresholds, where the binding energy relative the nearest threshold reaches its maximum, also the width of the resonance has a maximum. We were able to trace this resonance up to \( m_x = 6.7 m_e \), where the binding energy is 0.14 meV. Though the resonance may exist for slightly larger mass values its binding energy would be too small to be relevant for the proton radius problem. For masses \( m_x > 6 m_e \), the width was too small to be determined accurately.

3.2. \( p\mu\mu \)

Our second approach was to start instead from the \( p\mu\mu \) system, and progressively reduce the mass of one of the muons while following the binding energy of the resonance under the \( p\mu(n = 2) \) threshold. That is, we look for resonances in the \( p\mu\mu \) system, where \( m_e \leq m_x \leq m_{\mu} = 207.6 \, m_e \). This is the same approach as was followed by Karr and Hilico \cite{3}. For the \( p\mu \) ion we obtained a ground-state energy of \( -97.566 \, 9903 \, au \), which slightly improves to result \( -97.566 \, 9834 \, au \) obtained
Table 2. Ground state and resonances in H⁺, energies and widths in atomic units, compared with results from Bürgers and Lindroth [12], Frolov [13], Chen [14] and Bylicki and Nicolaides [15]. These calculations assume an infinite nuclear mass.

| State | Energy (au) | \( \Gamma / 2 \) (au) | Literature | Energy (au) | \( \Gamma / 2 \) (au) | Reference |
|-------|-------------|-----------------------|------------|-------------|-----------------------|-----------|
| \( 1s^2 \ 1S' \) | -0.527 751 016 21 | — | — | | | [13] |
| \( 2s^2 \ 1S' \) | -0.148 776 253 73 | 0.000 866 1793 | | -0.148 776 253 94 | 0.000 866 618 17 | [12] |
| \( 2s3s \ 3S' \) | -0.127 104 276 | 5.0 \times 10^{-7} | | -0.127 102 4 | 3.42 \times 10^{-7} | [14] |
| \( 2s3s \ 3S' \) | -0.126 020 061 | 0.000 045 25 | | -0.126 020 063 74 | 0.000 045 264 86 | [12] |
| \( 2s4s \ 4S' \) | -0.125 118 188 | 3 \times 10^{-6} | | -0.125 118 18 | 2 \times 10^{-8} | [14] |
| \( 2p^2 \ 1P' \) | -0.142 134 807 | 0.000 106 73 | | -0.142 134 84 | 0.000 106 9 | [15] |
| \( 2p^3 \ 1P' \) | -0.126 049 804 7 | 6.6 \times 10^{-7} | | -0.126 049 859 48 | 6.8419 \times 10^{-7} | [12] |
| \( 2p^3 \ 1P' \) | -0.125 410 127 | 0.000 002 30 | | -0.125 410 17 | 0.000 002 49 | [15] |

Table 3. Ground state and resonances in H⁺, energies and widths in atomic units, calculated using the physical proton mass. Our results are compared to the results in [12], including the mass corrections tabulated there. For the ground state there is a very accurate calculation by Frolov [13] giving -0.527 445 881 114 104. For the triplet states, we know of no other calculation.

| State | Energy (au) | \( \Gamma / 2 \) (au) | Literature | Energy (au) | \( \Gamma / 2 \) (au) | Reference |
|-------|-------------|-----------------------|------------|-------------|-----------------------|-----------|
| \( 1s^2 \ 1S' \) | -0.527 445 881 08 | — | — | | | |
| \( 2s^2 \ 1S' \) | -0.148 694 749 81 | 0.000 865 363 | | -0.148 694 749 68 | 0.000 865 377 32 | |
| \( 2s3s \ 3S' \) | -0.127 034 277 | 3.3 \times 10^{-7} | | — | — | |
| \( 2s3s \ 3S' \) | -0.125 951 442 | 0.000 045 20 | | -0.125 951 443 84 | 0.000 045 214 11 | |
| \( 2s4s \ 4S' \) | -0.125 050 021 | 4 \times 10^{-8} | | — | — | |
| \( 2p^2 \ 1P' \) | -0.142 058 510 | 0.000 105 91 | | — | — | |
| \( 2p^3 \ 1P' \) | -0.125 980 80 | 5.3 \times 10^{-7} | | -0.125 980 867 20 | 6.8632 \times 10^{-7} | |
| \( 2p^3 \ 1P' \) | -0.125 341 89 | 0.000 002 23 | | — | — | |

Table 4. Binding energies and widths of pex resonance \( 2s^2 \ 1S' \), under the \( px(n = 2) \) threshold as a function of the mass \( m_e \) of the particle \( x \). The binding energies of resonance 2 are given relative the nearest atomic threshold, i.e. relative the H \( (n = 1) \) threshold for \( m_e \leq 4m_e \) and relative the \( px(n = 2) \) threshold for \( m_e > 4m_e \). (The binding energy for \( m_e = 1 \) corresponds to the second line in table 3, after subtracting the threshold energy given by equation (8), and converting the energy unit to eV, i.e. \( E_B = [ -0.148 694 749 81 +m_p m_e/(m_p + m_e)]/8 ] \times 27.211 385 05 \) eV.) For \( m_e > 4m_e \), the widths of resonance 2 were too small to be determined accurately.

| \( m_e \) | \( E_B \) (eV) | \( \Gamma \) (eV) | \( E_B \) (eV) | \( \Gamma \) (eV) |
|-------|-----------------|-----------------|-----------------|-----------------|
| 1.0   | 0.646 62        | 0.047 11        | —               | —               |
| 2.0   | 0.063 38        | 0.025 70        | —               | —               |
| 3.0   | 0.004 86        | 0.003 90        | 0.011 58        | 0.016 61        |
| 3.6   | 0.001 24        | 0.009 29        | 0.460 07        | 0.044 88        |
| 3.8   | 0.000 81        | 0.000 54        | 0.746 85        | 0.043 88        |
| 4.0   | —               | —               | 1.096 03        | 0.039 90        |
| 5.0   | —               | —               | 0.220 60        | 0.012 14        |
| 6.0   | —               | —               | 0.014 46        | 0.000 23        |
| 6.5   | —               | —               | 0.001 11        | —               |
| 6.6   | —               | —               | 0.000 29        | —               |
| 6.7   | —               | —               | 0.000 14        | —               |

in [16]. For the first resonance in \( p\mu \), we obtained the binding energy 120.471 654 eV and width 7.146 072 583 eV, in good agreement with the values 120.471 6427 eV and 7.146 063 657 eV obtained in [3]. In table 5 we compare our results to [3]. We find that while rather different numerical approaches were used, the results agree very well both with and without including the correction due to vacuum polarization.

As was concluded in [3], this resonance disappears, if vacuum polarization is included, for \( m_e < 48m_e \). However, similarly to the pex system we find that another resonance appears, as shown in figure 4 and table 6. This time, the resonance appears under the \( px(n = 1) \) threshold at \( m_e \approx 65m_e \). As \( m_e \) is further reduced the binding energy and width of this resonance grows and reaches a maximum close to the crossing of the thresholds at \( m_e \approx 47.7m_e \). The resonance then follows the \( p\mu(n = 2) \) threshold, until it disappears at \( m_e \approx 30m_e \).

4. Discussion

The appearance of the additional resonances can be understood qualitatively. In the pex system the new resonance appears under the ground-state threshold of ordinary hydrogen. The long-range potential from the hydrogen atom is \(-\alpha/(2r^3)\), where \( \alpha = 4.5 \) is the polarizability of ground-state hydrogen. This is the potential experienced by the \( x \) particle.
bound close to threshold. This long-range potential does not change as the mass of the particle is scaled. As its mass is increased the zero-point energy of the -particle will decrease as \( m < m_x \), and it will therefore become more prone to binding in a weak potential. According to our calculations, binding is possible when \( m_x > 3m_e \). As \( m_x \) is increased further the binding energy relative to the H(1s) threshold continues to increase. In the region 3.0m_e < m_x < 3.8m_e resonances exist both under the H(n = 1) threshold and the p\( s(n = 1) \) threshold. While the two thresholds cross at \( m_x = 4m_e \) the resonances change in the \( p\mu \) system, where the particle has charge \( -e \) and mass according to the x-axis. The upper frame shows the energies on an absolute scale, along with the \( p\mu(n = 2) \) threshold (dashed line) and \( p\mu(n = 1) \) threshold (solid line) inlaid. (The plotted results do not include vacuum polarization, however, the difference is too small to be resolved by eye.)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{\( m_x/m_e \)} & \textbf{Present work} & \textbf{Karr and Hilico [3]}
\hline
\textbf{Without v.pol.} & \textbf{Including v.pol.} & \\
\textbf{Without v.pol.} & \textbf{Including v.pol.} & \\
\hline
80 & 1.6730 & 0.9078 & 1.6122 & 0.9075 & 1.6730 & 0.9077 & 1.6110 & 0.9071 \\
70 & 0.6236 & 0.4240 & 0.5676 & 0.4228 & 0.6235 & 0.4239 & 0.5671 & 0.4226 \\
60 & 0.1651 & 0.1335 & 0.1182 & 0.1291 & 0.1652 & 0.1335 & 0.1182 & 0.1291 \\
55 & 0.0703 & 0.0577 & 0.0320 & 0.0498 & 0.0702 & 0.0577 & 0.0322 & 0.0501 \\
50 & 0.0253 & 0.0169 & 0.0024 & 0.0093 & 0.0253 & 0.0171 & 0.0016 & 0.0081 \\
48 & 0.0168 & 0.0064 & \( < 10^{-4} \) & \( < 10^{-4} \) & 0.0168 & 0.0064 & \( 7(6) \times 10^{-5} \) & \( 7(6) \times 10^{-5} \) \\
\hline
\end{tabular}
\end{table}

\textbf{Table 5.} Binding energies and width of \( \mu \) resonances under the \( \mu\mu(2\Sigma) \) threshold as a function of the mass \( m_x \) of the particle \( x \). The binding energy at \( m_x = 48m_e \), including vacuum polarization, was too small to be reliably determined.
resonance energies cannot do so. Instead one of the resonances is pushed above the \( px(n = 2) \) threshold, while the other resonance takes its place and now follows this threshold instead of the \( H(n = 1) \) threshold.

Also in the \( p\mu x \) system we find resonances following two different thresholds. At large \( m_e \) (to the right in figure 4), the only resonance we find is the \( x \) particle loosely bound to the \( p\mu(n = 2) \) atom. The \( px(n = 1) \) system also gives rise to an attractive long-range potential, which, including the appropriate mass scaling, has the form \(-a/(2m_x^3r^4)\). At large \( m_x \) values this potential is to weak to bind the heavier muon. However, as \( m_x \) is reduced (moving left in figure 4), this potential becomes deeper, and thus it becomes easier to bind a muon. We find that this happens for \( m_x \lesssim 65m_e \).

### 5. Conclusions

We find that the three-body system consisting of a proton and a particle \( x \) with charge \(-e\) can bind an electron under its \( 2s \) threshold only if \( m_x < 6.7m_e \). Similarly, the \( p\mu(2s) \) atom can bind a third negatively charged particle only if its mass is greater than \( 30m_e \). This slightly extends the allowed mass-range compared to the results in [3] due to the appearance of an additional resonance. However, we conclude, in agreement with [3], that there can be no resonances under the \( 2s \) threshold in the \( px \) system.

Similar results can be found through a simple estimate. We note that if the splitting due to vacuum polarization and other QED effects is neglected, the muonic hydrogen atom has a non-vanishing linear Stark shift. The long-range potential experienced by an electron at distance \( r \) from the muonic hydrogen atom would therefore be \(-\mu L/r^2\), where \( \mu L \) is the electric dipole moment of the atom. The smallest electric dipole moment which can bind an electron has been determined to be 0.639\( a_0 \) [17]. The dipole moment of a hydrogenic atom can be estimated as \( \mu \propto n^2 a_0/m_x \), where \( \mu \) is the muon–proton reduced mass and \( n \) the principal quantum number. The largest reduced mass \( m_x \) of the atom which would bind an electron under the \( n = 2 \) threshold is then \( \mu \propto 6.3m_e \), in fair agreement with our numerical result 6.7\( m_e \). In a similar way we can estimate the smallest possible mass \( m_x \) of a particle which binds to \( p\mu(n = 2) \), giving \( m_x \approx 30m_e \), also in good agreement with our calculation above.

Finally, for the actual \( p\mu e \) system, we can estimate the excitation of the \( p\mu \) atom required to bind an electron, which gives \( n \gtrsim 11 \). Based on this estimate it is therefore not surprising that no \( p\mu e \) state under the \( n = 2 \) threshold could be found.

We note that various corrections, such as the strong magnetic field present in the trap, have not been included in this calculation. However, since both estimates and numerical results indicate that we are quite far from binding, we deem it unlikely that this perturbation will induce any new \( p\mu e \) resonances in the relevant energy region. Thus the solution for the proton radius puzzle must be sought from different sources.

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