Optimization methods in the design of heat-saving functional devices

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Abstract. Inverse problems associated with designing cylindrical heat manipulation devices are studied. Using the optimization method these inverse problems are reduced to corresponding control problems. Thermal conductivities of device materials play the role of passive controls. Numerical algorithm based on the particle swarm optimization is proposed. Results of numerical experiments for shielding and heat concentration problems are presented and discussed.

1. Introduction

In the polar latitudes the heat-saving and heat-shielding problems for industrial and residential objects play an important role [1, 2]. An innovative solution to these problems leads to the problems of heat flux manipulation. One of the ways to solve these problems is based on the extensive use of special materials, so-called thermal metamaterials. The need to use metamaterials for physical fields manipulation became clear after the publication of pioneer works [3, 4, 5] devoted to design of invisibility devices. The idea of cloaking objects by placing them inside a three-dimensional cloaking shell filled with an inhomogeneous anisotropic medium with specially selected parameters was proposed in these works.

Based on this idea, a method to design invisibility devices was developed first for electromagnetic fields, and then for acoustic, magnetic, electrical, thermal and other static fields [6, 7, 8]. Further, this method was also used to design other functional devices such as concentrators, rotators, inverters, etc. (see [9]). It should be noted that technical implementation of these devices is associated with great difficulties due to the lack of materials that possess the above properties. Therefore another design strategy began develop recently. It obtained the name of inverse design because it is related with solving inverse problems (see [10, 11]). The optimization method forms the core of the inverse design methodology. A growing number of papers is devoted to applying the inverse design for physical fields manipulation. Among them we mention [12, 13] where numerical optimization algorithms are applied for finding the unknown material parameters of cloak and papers [14, 15, 16] devoted to theoretical analysis of cloaking problems using the optimization approach.

In recent years numerical design of heat manipulation devices is associated with the use of topology optimization and discrete material optimization. In [17] topology optimization approach was applied to design anisotropic thermal-composite materials for heat flux shielding, focusing, and reversal. In [18] topology optimization was used to design thermal cloaking shell composed of iron and aluminium. A series of papers is devoted to the application of discrete
material optimization approach for numerical designing heat flux manipulation devices. Among them we mention works where a laminate of predefined materials with controlled orientation was used to facilitate the fabrication of designed thermal cloaks and concentrators. In [19] the material at a given point of the heat flux manipulation device is chosen between two prescribed materials with highly different conductivity.

In this paper optimization method is applied to solve inverse problems for the 2D stationary model of heat transfer. These problems arise in designing heat flux manipulation devices that have the form of the cylindrical shell. It is assumed that the desired shell consists of finite number of subdomains filled with homogeneous isotropic medium. Thermal conductivities of the medium filling subdomains play the role of controls. As a result the heat flux manipulation problems are reduced to solving the corresponding finite-dimensional extremum problems. We propose a numerical algorithm based on the particle swarm optimization (see [20]) according to the scheme of [21, 22] and discuss some results of numerical experiments.

2. Statement of direct problem

We begin with statement of the direct heat conductivity problem, considered in a rectangle $D = \{ x \equiv (x, y) : |x| < x_0, |y| < y_0 \}$ with specified numbers $x_0 > 0$ and $y_0 > 0$ (see Figure 1a). We will assume that an external temperature field $T_e$ is created by two vertical plates $x = \pm x_0$ which are kept at different values $T_1$ and $T_2$, while the upper and lower plates $y = \pm y_0$ are thermally insulated. We assume that there is a material shell $(\Omega, \kappa)$ inside $D$. Here, $\Omega$ is a circular layer $a < |x| < b$ where $b < \min \{x_0, y_0\}$ and $\kappa$ is the thermal conductivity of the inhomogeneous medium filling the domain $\Omega$. We assume also that the interior $\Omega_i : |x| < a$ and exterior $\Omega_e : |x| > b$ of $\Omega$ are filled with the same homogeneous medium having constant thermal conductivity $k_0 > 0$ (see Figure 1b).

In this case, the direct conduction problem consists of determination of a triplet of functions, namely $T_i$ in $\Omega_i$, $T$ in $\Omega$, and $T_e$ in $\Omega_e$, which satisfy the equations

\begin{align*}
  k_0 \Delta T_i & = 0 \quad \text{in } \Omega_i, \quad (1) \\
  \text{div} \ (\kappa \ \text{grad} \ T) & = 0 \quad \text{in } \Omega, \quad (2) \\
  k_0 \Delta T_e & = 0 \quad \text{in } \Omega_e, \quad (3)
\end{align*}

and obey the following boundary conditions on the boundary $\partial D$ of $D$

\begin{equation}
  T_e |_{x=-x_0} = T_1, \quad T_e |_{x=x_0} = T_2, \quad \frac{\partial T_e}{\partial y} |_{y=\pm y_0} = 0 \quad (4)
\end{equation}

and the matching conditions on internal $\Gamma_i$ and external $\Gamma_e$ components of the boundary $\Gamma$ of the shell $\Omega$ having the form

\begin{equation}
  T_i = T, \quad k_0 \frac{\partial T_i}{\partial n} = (\kappa \nabla T) \cdot n \quad \text{on } \Gamma_i, \quad T_e = T, \quad k_0 \frac{\partial T_e}{\partial n} = (\kappa \nabla T) \cdot n \quad \text{on } \Gamma_e. \quad (5)
\end{equation}

In the particular case when $\kappa = k_0$ (so that the homogeneous and isotropic medium filling the domain $D$ as shown in Figure 1a) the solution of the problem (1)–(5) is described by the temperature field $T^e$ and the constant heat flux $q^e$ by the formulas

\begin{equation}
  T^e (x) = \frac{T_2 - T_1}{2} \frac{x}{x_0} + \frac{T_1 + T_2}{2}, \quad q^e \equiv -k_0 \nabla T^e = \left( \frac{T_1 - T_2}{2} \frac{k_0}{x_0}, 0 \right). \quad (6)
\end{equation}
3. Control problems for complex shell

Now we assume that \((\Omega, \kappa)\) is a shell consisting of \(M\) subdomains \(\Omega_j, j = 1, 2, \ldots, M\). Each of these subdomains is filled with a homogeneous and isotropic medium described by constant thermal conductivity \(k_j > 0, j = 1, 2, \ldots, M\). The parameter \(\kappa\) of this shell is given by

\[
\kappa(x) = \sum_{j=1}^{M} k_j \chi_j(x). \tag{7}
\]

Here \(\chi_j(x)\) is a characteristic function of subdomain \(\Omega_j\) which is equal to 1 inside of \(\Omega_j\) and to 0 outside of \(\Omega_j\) while \(k_j\) are some constant coefficients.

We remind that our purpose is the numerical analysis of inverse problems for model (1)–(5) arising in designing heat flux manipulation devices. In the case of a composed shell these problems consist of finding unknown coefficients \(k_j, j = 1, 2, \ldots, M\) in (7) forming a \(M\)-dimensional vector \(k = (k_1, k_2, \ldots, k_M)\). We will refer to \(k\) as the conductivity vector.

By optimization method our inverse problems are reduced to extremum problems of minimization of certain cost functionals which adequately correspond to the inverse problems under consideration. In order to formulate these control problems we denote by \(T[k]\) \(\equiv T[k_1, k_2, \ldots, k_M]\) the solution to the direct problem (1)–(5) corresponding to parameters \(k_j\) in \(\Omega_j, j = 1, 2, \ldots, M\), and to conductivity \(k_0\) in \(\Omega_i\) and \(\Omega_e\). By \(q[k]\) we define corresponding heat flux \(q[k] = -k_0 \nabla T[k]\) in \(\Omega_i\). We will assume below that the vector \(k\) belongs to the bounded control set

\[
S = \{k \equiv (k_1, k_2, \ldots, k_M) : k_{\text{min}} \leq k_j \leq k_{\text{max}}, j = 1, 2, \ldots, M\}. \tag{8}
\]

Here given positive constants \(k_{\text{min}}\) and \(k_{\text{max}}\) are lower and upper bounds of the control set \(S\). Let us define the cost functional

\[
J(k) = \frac{||q[k] - q^d||_{L^2(\Omega_i)}}{||q^d||_{L^2(\Omega_i)}} \tag{9}
\]

where

\[
||q[k] - q^d||_{L^2(\Omega_i)}^2 = \int_{\Omega_i} |q[k] - q^d|^2 dx, \quad ||q^d||_{L^2(\Omega_i)}^2 = \int_{\Omega_i} |q^d|^2 dx.
\]
and formulate following control problem:

\[ J(k) \rightarrow \text{inf}, \ k \in S. \]  

(10)

Here \( q^d \) is the desired heat flux in domain \( \Omega_i \). In particular, we assume that \( q^d = \alpha q^e \) where heat flux \( q^e \) is defined in (6) while the constant \( \alpha = 0 \) in shielding problem, \( \alpha = -1 \) in the heat flux inversion problem and \( \alpha > 1 \) in heat concentration problem.

To solve problem (10) we use an algorithm based on the particle swarm optimization [20]. Within this method, the design parameters are presented as coordinates of the radius-vector \( k \equiv (k_1, k_2, \ldots, k_M) \) of some abstract particle. A particle swarm is considered to be any finite set of particles \( k_1, \ldots, k_N \). Within the particle swarm optimization, one sets the initial swarm position \( k^0_j, j = 1, 2, \ldots, N \) and the iterative displacement procedure \( k^{i+1}_j = k^i_j + v^i_j \) for all particles \( k^i_j \), which is described by the formulas

\[ v^{i+1}_j = w v^i_j + c_1 d_1 (p^i_j - k^i_j) + c_2 d_2 (p^g - k^i_j). \]  

(11)

After each displacement we calculate the value \( J(k^{i+1}_j) \) of the functional \( J \) for the new position \( k^{i+1}_j \), compare it to the current minimum value and, if necessary, update the personal and global best positions \( p^j \) and \( p^g \). In the end of this iteration process all particles have to come at the global minimum point.

Here \( v^i_j \) is the displacement vector; \( w, c_1 \) and \( c_2 \) are constant parameters of algorithm; \( d_1 \) and \( d_2 \) are random variables which are uniformly distributed over the interval \((0, 1)\). The subscript \( j \in \{1, 2, \ldots, N\} \) in (11) denotes the particle number, and the superscript \( i \in \{0, 1, \ldots, L\} \) indicates the iteration number. The choice of main parameters was considered in more detail in [20, 21, 22].

4. Numerical results

In this section we discuss numerical results obtained by the particle swarm optimization algorithm for some model heat flux manipulation problems. The most time-consuming part of the above-described algorithm is the calculation of the values \( J(k^i_j) \) of the functional \( J \) for the particle position \( k^i_j \) at different \( i \) and \( j \) values. This procedure includes two stages. At the first stage, the solution \( T[k^i_j] \) to direct problem (1)−(5) is calculated using the FreeFEM++ software package (www.freefem.org). After determining \( T[k^i_j] \), at the second stage we calculate the mean squared integral norms entering the definition of the functional \( J \) in (9) using the formulas of numerical integration.

Numerical experiments were performed for values \( N = 25, L = 50, w = 0.4, c_1 = 1, c_2 = 1.5 \) of the particle swarm optimization parameters and at the following given data:

\[ x_0 = y_0 = 4.5 \, \text{cm}, \ a = 1 \, \text{cm}, \ b = 3.25 \, \text{cm}, \ T_1 = 321.85 \, \text{K}, \ T_2 = 283.15 \, \text{K}, \ k_0 = 1 \, \text{W}/(\text{m} \cdot \text{K}). \]

The role of the external field was played by the field \( T^e \) determined in (6).

Below we assume that the background material in \( \Omega_e \) and \( \Omega_s \) is glass \((k_0 = 1 \, \text{W}/(\text{m} \cdot \text{K}))\) while lower and upper bounds in (8) are determined by \( k_{\text{min}} = 3 \times 10^{-2} k_0 \) (polystyrene) and \( k_{\text{max}} = 403 k_0 \) (copper).

Different types of inverse problems require different partitioning of the shell into subdomains \( \Omega_j \). In our first group of tests we solve the shielding problem for multilayer shell with isotropic layers. Denote by \( M \) the even number of layers \( \Omega_j \) and by \( k_j \) the thermal conductivity of the \( j \)-th layer, \( j = 1, \ldots, M \). Optimal values \( k^1_{\text{opt}}, k^2_{\text{opt}} \) and corresponding value \( J_{\text{opt}} \) of the cost functional \( J \) obtained by solving problem (10) with \( q^d = 0 \) using particle swarm optimization are presented in Table 1 for five different scenarios corresponding to \( M = 2, 4, 6, 8 \) and 10.
Table 1. Numerical results for shielding problem.

| $M$ | $k_{1}^{opt}/k_0$ | $k_{2}^{opt}/k_0$ | $J^{opt}$       |
|-----|-------------------|-------------------|----------------|
| 2   | $3 \times 10^{-2}$| 403               | $2.44 \times 10^{-3}$ |
| 4   | $3 \times 10^{-2}$| 403               | $8.55 \times 10^{-6}$ |
| 6   | $3 \times 10^{-2}$| 403               | $9.81 \times 10^{-8}$ |
| 8   | $3 \times 10^{-2}$| 403               | $2.31 \times 10^{-9}$ |
| 10  | $3 \times 10^{-2}$| 403               | $8.23 \times 10^{-11}$ |

Isolines of corresponding field $T^{opt}$ for a four-layer shell ($M = 4$) are plotted in Figure 2a. It is very interesting that all found parameters $k_{j}^{opt}$ with odd indices $3, 5, \ldots, M - 1$ coincide with $k_{1}^{opt}$ which is equal to $k_{min}$ while optimal parameters $k_{1}^{opt}, \ldots, k_{M}^{opt}$ with even indices coincide with $k_{2}^{opt}$ which is equal to $k_{max}$. Thus we have

$$k_{1}^{opt} = k_{3}^{opt} = \ldots = k_{M-1}^{opt} = 3 \times 10^{-2} k_0 \quad \text{and} \quad k_{2}^{opt} = k_{4}^{opt} = \ldots = k_{M}^{opt} = 403 k_0.$$

Analysis of Table 1 shows that the optimal values $J^{opt}$ of the cost functional $J$ decrease with increasing the number of layers $M$. In particular, $J^{opt} = 8.23 \times 10^{-11}$ at $M = 10$ that clearly corresponds to very high shielding efficiency. In this case the heat flux inside the domain $\Omega_i$ is close to zero. Such a shell allows us to protect this area from very high or very low temperatures using only two materials.

Another important problem is related to the heat concentration. First, we try to solve the problem (10) with $q^d = \alpha q^c$, $\alpha = b/a = 3.25$ using a four-layer shell ($M = 4$). In this case we obtain following optimal values of controls $k_j$ and the functional $J$:

$$k_{1}^{opt} = 0.775, \quad k_{2}^{opt} = 1.108, \quad k_{3}^{opt} = 1.098, \quad k_{4}^{opt} = 1.588, \quad J^{opt} = 2.208.$$

![Figure 2. Isolines of temperature field for a four-layer shell with isotropic layers: a) shielding problem; b) concentration problem.](image-url)
Figure 3. Isolines of temperature field for the heat concentration problem: a) two materials; b) four materials.

Isolines of corresponding temperature field $T^{opt}$ are presented in Figure 2b. It is clearly seen that the layered shell has a low shielding efficiency due to improper partition on subdomains. Therefore in our second group of tests we will use sectors as subdomains $\Omega_j, j = 1, \ldots, M$. We assume that the distribution of materials in the shell is symmetric.

First we will try to design a concentration shell using only two materials. For this purpose we split the shell into four sectors (left, right, upper and lower) using lines with angles $\varphi_1$ and $\varphi_2$ so that $\tan \varphi_1 = 0.75$ and $\tan \varphi_2 = -0.75$. Taking into account the symmetry, we assume that the thermal conductivity in the left and right sectors is equal to $k_1$ while the thermal conductivity in the upper and lower sectors is equal to $k_2$. Numerical solution of concentration problem by particle swarm optimization give us the following optimal values of controls $k_j$ and the functional $J$:

$$k_1^{opt} = 41.66, \quad k_2^{opt} = 0.03, \quad J^{opt} = 0.99.$$  

Isolines of corresponding temperature field $T^{opt}$ are presented in Figure 3a. Let us note that this version of the shell has a higher concentration efficiency compared to the layered shell. We can see high thermal conductivity in the left and right sectors and low value of the thermal conductivity in the upper and lower sectors. This distribution of materials in the shell allow us to concentrate the heat in domain $\Omega_i$.

Now we split the shell into twelve sectors using lines with angles $\varphi_1, \ldots, \varphi_6$ so that

$$\tan \varphi_1 = 0.5, \quad \tan \varphi_2 = 0.75, \quad \tan \varphi_3 = 1, \quad \tan \varphi_4 = -0.5, \quad \tan \varphi_5 = -0.75, \quad \tan \varphi_6 = -1.$$  

In this case we have four controls $k_1, \ldots, k_4$. Solving problem (10) with $q^d = 3.25 q^e$ we obtain following optimal values:

$$k_1^{opt} = 225, \quad k_2^{opt} = 0.03, \quad k_3^{opt} = 188, \quad k_4^{opt} = 0.03, \quad J^{opt} = 0.81.$$  

Isolines of corresponding temperature field $T^{opt}$ are plotted in Figure 3b. As in the previous case, left and right sectors have high thermal conductivity to ensure the heat concentration in domain $\Omega_i$. 

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Using a larger number of controls has increased the efficiency of the concentration shell. Unfortunately, it is not always possible to find materials corresponding to the optimal values of thermal conductivity. Therefore, a more promising approach is related to the situation when we control the geometric characteristics of the shell for a given set of available materials. This new approach will be considered in the future work of the authors.

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