The cosmic no-hair theorem and the nonlinear stability of homogeneous Newtonian cosmological models

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Abstract
The validity of the cosmic no-hair theorem is investigated in the context of Newtonian cosmology with a perfect fluid matter model and a positive cosmological constant. It is shown that if the initial data for an expanding cosmological model of this type is subjected to a small perturbation then the corresponding solution exists globally in the future and the perturbation decays in a way which can be described precisely. It is emphasized that no linearization of the equations or special symmetry assumptions are needed. The result can also be interpreted as a proof of the nonlinear stability of the homogeneous models. In order to prove the theorem we write the general solution as the sum of a homogeneous background and a perturbation. As a by-product of the analysis it is found that there is an invariant sense in which an inhomogeneous model can be regarded as a perturbation of a unique homogeneous model. A method is given for associating uniquely to each Newtonian cosmological model with compact spatial sections a spatially homogeneous model which incorporates its large-scale dynamics. This procedure appears very natural in the Newton-Cartan theory which we take as the starting point for Newtonian cosmology.

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1. Introduction

Explicit solutions of the Einstein equations can only be found under restrictive assumptions such as high symmetry. The physical significance of these solutions depends on the existence of a class of solutions which is general enough to have a direct physical application and whose qualitative behaviour resembles that of the exact solutions. These remarks apply in particular to the homogeneous and isotropic models used in cosmology. Thus it is of interest to know which properties of these models are stable under small perturbations. There is a large literature concerned with linearized perturbations of such models but this can at best give a rough indication of the behaviour of small but finite perturbations. (For a general discussion of the relations between different concepts of stability see [9].) The basic question studied in this paper is the following. Given a homogeneous cosmological model, assumed for simplicity to have compact spatial sections, what is the behaviour of solutions which develop from initial data which are, in an appropriate sense, close to being homogeneous? Do they remain almost homogeneous? Do they perhaps in some sense become more homogeneous in the course of time?

At present no rigorous results exist on the question just raised concerning non-vacuum solutions of the Einstein equations with almost homogeneous initial data. It appears that the only results on the nonlinear stability of any solution of the Einstein equations are those of Friedrich [7] for de Sitter space and Christodoulou and Klainerman [4] for Minkowski space. In this paper we study the analogous question for Newtonian cosmological models. There are at least two motivations for doing this. Firstly, Newtonian cosmological models are of interest for applications since they are used to model structure formation in the early universe [15]. Secondly, they may provide useful insights for the relativistic case.

As matter model we take either an isentropic perfect fluid with polytropic equation of state or dust. The models considered expand for ever and have positive cosmological constant $\Lambda$. (Some special solutions with $\Lambda = 0$ are also considered for comparison.) The main results concern those solutions with isotropic expansion. (The meaning of this terminology will be explained below.) In the case of a positive cosmological constant it is shown that the solutions corresponding to nearly homogeneous initial data exist for all positive times and that the difference between the inhomogeneous and homogeneous solutions tends to zero in a strong sense as $t \to \infty$. On the other hand it is shown by an example using dust that the quantity $(\rho - \bar{\rho})/\bar{\rho}$ often does not tend to zero. Here $\rho$ denotes the density and $\bar{\rho}$ the mean density. Another example using dust shows that when the cosmological constant is zero there are solutions which develop singularities in finite time and which evolve from data which are arbitrarily close to homogeneity.

To put these results in context it is useful to compare with the case of a non-relativistic fluid without gravitation. In that case the spatially homogeneous solutions are time independent. It has been shown by Sideris [17] that if the fluid is polytropic rather general small perturbations of homogeneous initial data lead to singularities in finite time. Intuitively, one of the mechanisms leading to singularities in a fluid with non-vanishing pressure is the formation of shock waves. In the case of dust shell-crossing singularities will often occur. The local effects of gravitation cannot be expected to improve this situation and may even aggravate it. There is, however, a global effect which can be helpful. This is
that the background solution may be expanding and that this expansion may ‘pull apart’ potential shock waves and shell-crossing singularities, thus preventing them from forming. Our results show that when $\Lambda = 0$ the expansion is not powerful enough to achieve this but when $\Lambda > 0$ it is, at least for small initial perturbations. Another comparison which is of some interest is that with Newtonian cosmological models where the matter is collisionless and described by the Vlasov equation. This kind of matter does not have the same kind of tendency to form singularities as a fluid and it is possible to prove global existence without any restriction on the size of the perturbations\cite{16}. On the other hand less is known about the asymptotic properties of the solutions in that case and it is possible that the existing estimates for the asymptotic behaviour are not optimal.

The method of proof of the existence theorem is inspired by the following observation, which is of more general interest. Consider the symmetric hyperbolic system

$$A^0(u) \partial_t u + A^i(u) \partial_i u = 0$$  \hspace{1cm} (1.1)

for a vector-valued function $u$ in $n$ space dimensions. One solution of this equation is given by $u = 0$. It is a well known fact (see e.g. \cite{10, 13}) that even arbitrarily small initial data for (1.1) often leads to a solution which develops a singularity in finite time. In this sense the solution $u = 0$ is unstable. If, however, this equation is replaced by

$$A^0(u) \partial_t u + A^i(u) \partial_i u + ku = 0,$$  \hspace{1cm} (1.2)

where $k$ is a positive constant then the situation changes dramatically. (This is shown explicitly for the simple scalar equation $u_t + uu_x + u = 0$ in \cite{10}.) The solutions corresponding to any initial data for (1.2) which are small enough exist for all positive time and, in fact, converge to zero exponentially as $t \to \infty$. This can be proved using the fact that energy estimates for (1.2) show that as long as the solution is small its norm in the Sobolev space $H^s, s \geq 4,$ is decreasing and a bootstrap argument. Below a proof of this kind in a situation which is slightly more complicated (due to the presence of the Newtonian potential term) will be presented in detail. In the cosmological problem the role of the constant $k$ is played by $\dot{a}/a$ where $a$ is the scale factor of a homogeneous model. For the argument to work this quantity must be bounded away from zero (or at least tend to zero sufficiently slowly as $t \to \infty$) and it is the positive cosmological constant which ensures this. There is also a term representing the perturbed gravitational field which must be controlled.

The starting point for the study of Newtonian cosmology in this paper is the Newton-Cartan theory, which is a geometrical version and at the same time a slight generalization of ordinary Newtonian gravitation theory. It is shown that we can split off from the full equations a system of ordinary differential equations which describe the evolution of the mean density, a flat metric and a connection on a torus. These are identical to the equations which describe the evolution of spatially homogeneous Newtonian cosmological models. This means that it is possible to associate uniquely to each solution of the Newton-Cartan theory with compact spatial sections a ‘background solution’ which is spatially homogeneous. It is then useful for the analysis of the dynamics to regard the given solution as a perturbation of the corresponding background. In the literature on Newtonian cosmology it is usual to split general solutions into the sum of a background and a
perturbation. The analysis in this paper clarifies the significance of a splitting of this kind and shows that it can be invariantly defined. This contrasts with the situation in general relativity where the problem of associating a homogeneous cosmological model with an inhomogeneous one so as to give a mathematical formulation of the idea that the universe is on average homogeneous is a difficult one[6]. The ‘solutions with isotropic expansion’ referred to above are those where the evolution of the flat metric on the torus is just a rescaling by a spatially constant conformal factor. If the expansion is not isotropic then certain parts of the geometry can be given arbitrarily and are not fixed by the field equations. Thus, as we will see shortly, the field equations of the Newton-Cartan theory, unlike the Einstein equations, are not strong enough to determine a solution uniquely in terms of initial data.

The results of this paper on the global dynamics of Newtonian cosmological models are linked to two questions of interest in cosmology. The first is the cosmic no-hair conjecture. This says that in a cosmological model with positive cosmological constant perturbations should be strongly damped. This conjecture refers originally to an inflationary period in general relativity. It is interesting to consider whether the analogous statement is true in Newtonian theory. In particular this may give insight into the mathematical mechanism responsible for the damping. Because of the quantities which can be given freely in the case where the expansion is not isotropic, it is clear that not all perturbations can be damped in the Newtonian case. This fact has been remarked upon by Barrow and Götz [1]. (These authors did not use the Newton-Cartan theory.) Our results show that in the isotropic case small perturbations are strongly damped and at the same time set limits on how strong the damping can be. The second question is that of structure formation in cosmology. This has been studied in the context of Newtonian cosmology with a positive cosmological constant and dust as a matter model by Bildhauer, Buchert and Kasai[2]. They use the formation of singularities in a dust solution as a signal for structure formation in a more realistic model. Their results are consistent with those of the present paper and indicate that singularities are likely to occur in finite time for large initial perturbations.

The paper is organized as follows. Section 2 describes the Newton-Cartan theory and sets up the basic equations needed in the sequel. In section 3 the results on global existence and asymptotic behaviour are proved. The examples which show the sharpness of these results are presented in section 4.

2. Newtonian cosmology

In this section we present some general information about spatially compact cosmological models in the Newtonian theory of gravity. More precisely, since Newtonian gravitation as it is normally presented is only applicable to isolated systems, we use a slight generalization of the standard Newtonian formalism, the Newton-Cartan theory. The original treatment of Newtonian cosmology, due to Heckmann and Schücking [8], is more elementary but does not bring out the underlying geometrical structures. The definition which will be used here is that which has been discussed by Ehlers[5] in the context of the Newtonian limit of general relativity. The basic objects of the theory are a covariant symmetric tensor $g_{\alpha\beta}$ (the
time metric), a contravariant tensor \( h^{\alpha\beta} \) (the space metric), a torsion-free connection \( \Gamma^\alpha_{\beta\gamma} \) and a symmetric tensor \( T^{\alpha\beta} \) (the matter tensor) defined on a four-dimensional manifold (spacetime). They are supposed to satisfy the following conditions:

1) At each point of spacetime there exists at least one vector \( V^\alpha \) with \( g_{\alpha\beta}V^\alpha V^\beta > 0 \). Vectors with this property are called timelike.

2) If \( V^\alpha \) is timelike then the restriction of \( h^{\alpha\beta} \) to the space of covectors \( \omega_\alpha \) with \( \omega_\alpha V^\alpha = 0 \) is positive definite.

3) \( g_{\alpha\beta}h^{\beta\gamma} = 0 \).

4) \( g_{\alpha\beta} \) and \( h^{\alpha\beta} \) are covariantly constant with respect to \( \Gamma^\alpha_{\beta\gamma} \).

5) The curvature tensor \( R^\alpha_{\beta\gamma\delta} \) has the symmetry property \( h^{\sigma\gamma}R^\alpha_{\beta\sigma\delta} = h^{\sigma\alpha}R^\gamma_{\beta\sigma\delta} \).

6) \( R_{\alpha\beta} = 8\pi(g_{\alpha\sigma}g_{\beta\tau}T^{\sigma\tau} - \frac{1}{2}g_{\sigma\tau}T^{\tau\gamma}g_{\alpha\beta}) - \Lambda g_{\alpha\beta} \) where \( R_{\alpha\beta} \) is the Ricci tensor of \( \Gamma^\alpha_{\beta\gamma} \) and \( \Lambda \) is the cosmological constant.

7) \( \nabla_\alpha T^{\alpha\beta} = 0 \).

It is a known consequence of these conditions that the 3-dimensional distribution defined as the kernel of \( g_{\alpha\beta} \) is integrable. The integral manifolds are the surfaces of constant absolute time \( t \). A spatially compact cosmological model in this theory can be defined as one where the hypersurfaces of constant time are compact. The tensor \( h^{\alpha\beta} \) defines a Riemannian metric on each time slice and the field equation (condition 6) above) implies that this metric is flat. This means in particular that if a time slice is compact it must be isometric to a quotient of a flat torus by a finite group of isometries[12].

By analogy with general relativity, a Newtonian spacetime \( M \) will be called time-orientable if there exists a global smooth timelike vector field \( V^\alpha \). It can be assumed without loss of generality that \( g_{\alpha\beta}V^\alpha V^\beta = 1 \). Let \( S \) be one of the time slices and define a smooth mapping \( f \) from an open subset \( U \) of \( S \times \mathbb{R} \) to \( M \) by the condition that \( f(x,s) \) is the point of \( M \) obtained by starting at \( x \) and following an integral curve of \( V^\alpha \) for a parameter distance \( s \). The set \( U \) is supposed to be the maximal one for which such a mapping is defined i.e. each integral curve is followed to the end. The normalization condition ensures that the hypersurfaces \( t=\text{const.} \) coincide with the images of the hypersurfaces \( s=\text{const.} \) under \( f \). This in turn implies that \( U \) is of the form \( S \times I \) for some open interval \( I \), since otherwise the time slices would fail to be compact. The mapping \( f \) is a local diffeomorphism. In particular \( f(U) \) is open. This construction can be started from any slice and so if \( f \) were not onto it would be possible to write \( M \) as a disjoint union of non-empty open sets. If spacetime is assumed to be connected this is impossible and so \( f \) is surjective. To get the injectivity of \( f \) it seems to be necessary to make another causality assumption. Call a subset of a Newtonian spacetime achronal if there exists no timelike curve starting and ending there. The following lemma can now be stated.

**Lemma 2.1**

*Let \( M \) be a Newtonian spacetime with the following properties:*

(i) it is time orientable
(ii) each time slice is compact and achronal

Then $M$ is diffeomorphic to $S \times I$, where $S$ is a compact manifold admitting a flat metric and $I$ is an open interval.

As already remarked, such a manifold $S$ is covered by a torus. Passing to this covering does not change the dynamics and so when investigating dynamical questions it may be assumed without loss of generality that $S$ is a torus. This will be done in the following.

Represent $S$ in the form $\mathbb{R}^3/\sim$, where the projection from $\mathbb{R}^3$ to $S$ is smooth. The tensor $h_{ab}$ on $M$ defines a tensor on $\mathbb{R}^3 \times I$ which will be denoted in the same way. Let $h_{ab}(t)$ be its restriction to $\mathbb{R}^3 \times \{t\}$. This is a complete flat metric on $\mathbb{R}^3$ and so must be isometric to the standard metric on $\mathbb{R}^3$[12]. This implies that given any orthonormal frame at the origin there exist coordinates on $\mathbb{R}^3 \times \{t\}$ such that the metric takes the form $\delta_{ab}$ and the given frame takes the form $\partial/\partial x^a$. Introduce coordinates of this type on each slice. Since these can be constructed by following geodesics the $x^a$ are smooth functions on $\mathbb{R}^3 \times I$. In these coordinates the components $h_{ab} = \delta_{ab}$. The identification which leads from $\mathbb{R}^3$ to $S$ is of the form that $x^a$ is identified with $x^a + n^i e^a_i(t)$ for each triple of integers $n^i$. What is not a priori clear is that $e^a_i(t)$ is smooth (or even continuous). A covering transformation of $\mathbb{R}^3 \times I$ is smooth and must be of the form $x^a \mapsto x^a + n^i e^a_i(t)$ for each fixed $t$. This is only possible if $e^a(t)$ is smooth. Doing a time dependent linear transformation of the coordinates leads to a coordinate system where the identifications are time independent and the components $h_{ab}$ only depend on $t$.

The picture is now that the spacetime is the product of a torus with an interval and there are periodic coordinates such that the time and space metrics take the form

$$
\begin{align*}
g_{00} &= 1, \ g_{0a} = 0, \ g_{ab} = 0, \\
 h^{0a} &= 0, \ h^{ab} = h^{ab}(t). 
\end{align*}
$$

Certain restrictions on $\Gamma^\alpha_{\beta\gamma}$ follow from (2.1) and assumption 4) above. They are

$$
\begin{align*}
\Gamma^0_{\alpha\beta} &= 0, \ \Gamma^a_{bc} = 0, \\
 \partial_t h^{ab} + \Gamma^a_{s0} h^{sb} + \Gamma^b_{s0} h^{sa} &= 0 
\end{align*}
$$

Names will now be assigned to the parts of the connection which do not vanish as a consequence of (2.2).

$$
\begin{align*}
 E^a &= \Gamma^a_{00}, \\
 \Theta_{ab} &= \frac{1}{2}(h_{ac} \Gamma^c_{b0} + h_{bc} \Gamma^c_{a0}), \\
 \Omega_{ab} &= \frac{1}{2}(h_{ac} \Gamma^c_{b0} - h_{bc} \Gamma^c_{a0}). 
\end{align*}
$$

Combining (2.2) and (2.4) gives

$$
\partial_t h_{ab} = 2\Theta_{ab}.
$$
This means in particular that $\Theta_{ab}$ only depends on $t$ and not on the spatial coordinates. Let $\rho = T^{00}$. Condition 6) gives the equations

$$\nabla_a E^a = h^{ab} \partial_t \Theta_{ab} - \Theta^{ab} \Theta_{ab} - \Omega^{ab} \Omega_{ab} + 4\pi \rho - \Lambda,$$

$$h^{bc} \nabla_c \Omega_{ab} = 0,$$  \hspace{1cm} (2.7, 2.8)

where indices have been raised with $h^{ab}$ while condition 5) implies the equations

$$\nabla_c [\Omega_{ab}] = 0,$$

$$\partial_t \Omega_{ab} = \nabla_{[b} E_{a]}.$$  \hspace{1cm} (2.9, 2.10)

Combining (2.8) and (2.9) shows that $\Omega_{ab}$ is also independent of the spatial coordinates. Integrating (2.10) over the torus and applying Stokes’ theorem gives

$$\partial_t \Omega_{ab} = 0.$$  \hspace{1cm} (2.11)

Substituting this back into (2.10) shows that the one-form $E_a$ is closed. Thus

$$E_a = \eta_a - \nabla_a U,$$  \hspace{1cm} (2.12)

where $\eta_a$ depends only on time and the function $U$, which is determined only up to addition of a function of $t$ by (2.12), is supposed to satisfy $\int U = 0$. Consider now the effect of doing a coordinate transformation of the form $x'^a = x^a + y^a$, $t' = t$ where the $y^a$ depend only on $t$. This preserves the conditions (2.1) and changes $E_a$ by $h_{ab} \ddot{x}^b$. Thus it can be used to eliminate $\eta^a$ without disturbing the assumptions made up to now. It follows that it can be assumed without loss of generality that $\eta^a = 0$. Before (2.7) is analysed $\Theta_{ab}$ will be split into trace and tracefree parts:

$$\Theta_{ab} = \Sigma_{ab} + \frac{1}{3} \theta h_{ab},$$  \hspace{1cm} (2.13)

where $h^{ab} \Sigma_{ab} = 0$ and $\theta = h^{ab} \Theta_{ab}$. Then (2.7) becomes

$$\Delta_h U = -[\dot{\theta} + \frac{1}{3} \theta^2 + \Sigma_{ab} \Sigma^{ab} - \Omega_{ab} \Omega^{ab} + 4\pi \rho - \Lambda]$$  \hspace{1cm} (2.14)

where $\Delta_h U = h^{ab} \nabla_a \nabla_b U$. Let $\bar{\rho}(t)$ denote the average density at time $t$, i.e. $\bar{\rho}(t) = \int \rho(t,x) dx / \int dx$. Then averaging (2.14) over the torus and using the fact that $\theta, \Sigma_{ab}$ and $\Omega_{ab}$ are spatially constant gives

$$\dot{\theta} + \frac{1}{3} \theta^2 + \Sigma_{ab} \Sigma^{ab} - \Omega_{ab} \Omega^{ab} + 4\pi \bar{\rho} - \Lambda = 0.$$  \hspace{1cm} (2.15)

Putting this back in (2.14) shows that

$$\Delta_h U = -4\pi (\rho - \bar{\rho}).$$  \hspace{1cm} (2.16)

Thus the Poisson equation has been recovered in a familiar form.
It will now be discussed how the solutions of the equations can usefully be parametrized. Taking the zeroth component of the equation $\nabla_\alpha T^{\alpha\beta} = 0$ of condition 7) and averaging over the torus gives

$$\partial_t \bar{\rho} + \theta \bar{\rho} = 0.$$ \hfill (2.17)

To specify a solution we need to give the following objects: the matter variables (which in particular allow $\rho$ to be calculated), $\theta$ and $\Omega_{ab}$ on the initial hypersurface $t = 0$ and $\Sigma_{ab}$ everywhere. Then, using (2.6), (2.11), (2.15) and (2.17) the quantities $h_{ab}$, $\bar{\rho}$, $\theta$ and $\Omega_{ab}$ are determined everywhere. All Christoffel symbols can be calculated from $h_{ab}$, $\theta$, $\Sigma_{ab}$, $\Omega_{ab}$ and $\nabla_a U$. The only one of these which is not already known is $\nabla_a U$. For reasonable matter fields the system of matter fields coupled to (2.16) should have a well posed initial value problem. In the case of a perfect fluid the equations look as follows:

$$\partial_t \rho + \nabla_a (\rho u^a) + \theta \rho = 0$$ \hfill (2.18)

$$\partial_t u^a + u^b \nabla_b u^a + \rho^{-1} h^{ab} \nabla_b p + \frac{2}{3} \theta u^a - \nabla^a U + 2(\Sigma^a_b + \Omega^a_b) u^b = 0$$ \hfill (2.19)

assuming that the density $\rho$ does not vanish anywhere. These equations are obtained by choosing the matter tensor

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + ph^{\alpha\beta}$$ \hfill (2.20)

with $u^0 = 1$ and computing the condition 7) explicitly. That this procedure does in fact work for a fluid will be shown in the next section (Theorem 3.1).

The simplest solutions of these equations are the Friedmann-like solutions which are obtained by taking $\rho = \bar{\rho}$, $u^a = 0$, $E_a = 0$, $\Sigma_{ab} = 0$ and $\Omega_{ab} = 0$. In this case (2.6) reduces to $\partial_t h_{ab} = \frac{2}{3} \theta h_{ab}$, which implies that $h_{ab} = a^2(t) h_{ab}(0)$ for some function $a(t)$. Suppose without loss of generality that $a(0) = 1$. This function satisfies the equation $\dot{a} = \frac{1}{3} \theta a$. Substituting this into (2.17) gives

$$\rho a^3 = \rho(0).$$ \hfill (2.21)

Combining (2.15), (2.17) and (2.21) results in the equation

$$\ddot{a} = -\frac{4\pi}{3} \rho(0) a^{-2} + \frac{1}{3} \Lambda a.$$ \hfill (2.22)

For solutions of (2.22) the quantity $E = \frac{1}{2} \dot{a}^2 - \frac{4\pi}{3} \rho(0) a^{-1} - \frac{1}{6} \Lambda a^2$ is constant and so the qualitative behaviour of its solution can easily be analysed using an effective potential. (The equation (2.22) also describes homogeneous and isotropic dust solutions in general relativity and in that case $E = -k/2$ where $k$ is the curvature of the space sections.) This analysis shows that in the case $\Lambda > 0$, which is the one of greatest interest in the following, $a$ tends to infinity as $t \to \infty$ precisely in the cases:

(i) $E > -\frac{1}{2} (4\pi)^{2/3} \Lambda^{1/3} \rho(0)^{2/3}$

(ii) $E \leq -\frac{1}{2} (4\pi)^{2/3} \Lambda^{1/3} \rho(0)^{2/3}$ and $a(0) > (4\pi \rho(0)/\Lambda)^{1/3}$

The constancy of $E$ implies that for these solutions $\dot{a}/a \to \sqrt{\Lambda/3}$ as $t \to \infty$ so that asymptotically the expansion becomes exponential.
3. The main theorems

Now general solutions of the equations of Newtonian cosmology of the previous section with a perfect fluid as a matter model will be considered. The first question to be asked is whether the system of equations consisting of (2.16), (2.18) and (2.19) has a well posed initial value problem. There are various ways in which (2.18) and (2.19) can be written as a symmetric hyperbolic system for fixed $U$ if an equation of state $p = f(\rho)$ with $f' > 0$ is given. In the following only polytropic equations of state will be considered i.e. $p = K \rho^{\frac{n+1}{n+2}}$, where $K$ and $n$ are positive real numbers. It will be convenient to use the variable $w = \sqrt{2n(2n+2)K \rho^{1/2n}}$ introduced by Makino [14]. Equations (2.18) and (2.19) can then be written as follows.

$$
\frac{\partial}{\partial t} w + u^a \partial_a w + \frac{1}{2n} w \partial_a u^a + \frac{1}{2n} \partial w = 0 \tag{3.1}
$$

$$
\frac{\partial}{\partial t} u^a + u^b \partial_b u^a + \frac{1}{2n} w h^{ab} \partial_b w - h^{ab} \nabla_b U + \frac{2}{3} \theta u^a + 2(\Sigma^a_b + \Omega^a_b) u^b = 0 \tag{3.2}
$$

The term in this system involving $U$ is $-\nabla^a U$ and this can be rewritten formally as $4\pi \nabla^a \Delta_{h}^{-1} (\rho - \bar{\rho})$. This should be thought of as a lower order term, since it is formally of order -1. Accepting this for the moment, we see that contracting (3.2) with $h_{ac}$ and taking the result together with (3.1) gives a system which is exactly like a symmetric hyperbolic system except that it contains one lower order term which is non-local. It is then natural to prove local in time existence for (2.16), (2.18) and (2.19) by following the steps in the standard local existence proof for symmetric hyperbolic systems as given in [13] for instance and checking that the non-local term does not cause any problems. This has been done in [3] for the case $\Sigma_{ab} = 0$, $\Omega_{ab} = 0$. The general case is not significantly different. There results the following theorem:

**Theorem 3.1**

Let $\Sigma_{ab}$ be a given $C^\infty$ function of t. Let $\Omega^0_{ab}$ be an antisymmetric matrix, $h^0_{ab}$ a symmetric matrix, $\theta_0$ a real number and $\rho_0$, $u^0_a$ functions in $H^s(T^3)$ for some $s \geq 3$. Suppose that $\rho$ is positive. Then there exists a $T > 0$ and a unique solution of equations (2.6) (2.11), (2.15), (2.16), (2.18) and (2.19) on $[0, T)$ such that

(i) $\Omega_{ab}$, $h_{ab}$ and $\theta$ are $C^\infty$ functions of t which take on the values $\Omega^0_{ab}$, $h^0_{ab}$ and $\theta_0$ respectively for $t=0$

(ii) $\rho$ and $u^a$ are $C^{0}$ with values in $H^s(T^3)$ and $C^{1}$ with values in $H^{s-1}(T^3)$ and take on the values $\rho_0$ and $u^0_a$ respectively for $t=0$

Moreover for this solution:

(iii) $U$ is $C^{0}$ with values in $H^{s+1}(T^3)$ and $C^{1}$ with values in $H^s(T^3)$

(iv) if $\rho$ is uniformly bounded away from zero and the $C^{1}$ norms of $\rho$ and $u^a$ are bounded on the interval $[0, T]$ then the solution can be extended to an interval $[0, T_1)$ with $T_1 > T$

Next the global behaviour of solutions with almost homogeneous initial data will be discussed. At this stage it will be assumed that $h_{ab}(0) = \delta_{ab}$, $\Sigma_{ab} = 0$ and $\Omega_{ab} = 0$. Some remarks will be made later on what happens when these quantities are allowed to be non-zero. Under these restrictions the spatial metric can be written in the form $h_{ab}(t) = a^2(t)\delta_{ab}$. Then $\theta = 3\dot{a}/a$. The equations (3.1) and (3.2) simplify and it is possible
to obtain a symmetric hyperbolic system from them in a slightly different way from what was done previously. If \( v^a = au^a \) is taken as a new variable then the system is symmetric hyperbolic without contracting (3.2) with the metric. This has the advantage that the coefficient matrix of the time derivatives in the resulting system is the identity and this simplifies the derivation of energy estimates. Writing \( V \) for the pair \((w - \bar{w}, v^a)\), the system is now of the form

\[
\partial_t V + A^a \partial_a V + BV + PV = 0.
\]  

(3.3)

Here \( B = \text{diag}\{3\dot{a}/2na, \dot{a}/a, \dot{a}/a, \dot{a}/a\} \) and \( PV = (0, aP_i V) \), where \( P_i V \) is the gradient of the potential \( U \) of the density corresponding to \( w \). All the matrices \( A^i \) have a similar form. For example

\[
A^1 = \begin{pmatrix}
v^1/a & w/2na & 0 & 0 \\
w/2na & v^1/a & 0 & 0 \\
0 & 0 & v^1/a & 0 \\
0 & 0 & 0 & v^1/a
\end{pmatrix}
\]

In deriving estimates for these equations we will repeatedly use the following Moser inequalities (see [11] for a derivation).

\[
\|D^\alpha (fg)\|_2 \leq C(\|f\|_\infty \|D^s g\|_2 + \|D^s f\|_2 \|g\|_\infty)
\]

\[
\|D^\alpha (fg) - fD^\alpha g\|_2 \leq C(\|Df\|_\infty \|D^{s-1} g\|_2 + \|D^s f\|_2 \|g\|_\infty)
\]

Here the subscripts 2 and \( \infty \) refer to the \( L^2 \) and \( L^\infty \) norms respectively, \( \alpha \) is a multi-index and \( s = |\alpha| \).

The basic energy estimate for (3.3) goes as follows:

\[
\frac{d}{dt} \|V\|_2^2 = 2 \int \langle V, \partial_t V \rangle dx
\]

\[
= -2 \int \langle V, A^i \partial_i V + BV + PV \rangle dx
\]

\[
\leq (\|\partial_i A^i\|_\infty - 2 \min(\frac{3}{2n}, 1) \frac{4}{s}) \|V\|_2^2 + 2 \|V\|_2 \|PV\|_2
\]

(3.4)

Now apply a spatial derivative \( D^\alpha \) (\( \alpha \) a multi-index) to (3.3) to get

\[
\partial_t (D^\alpha V) + A^i \partial_i (D^\alpha V) + BD^\alpha V + D^\alpha PV = -[D^\alpha (A^i \partial_i V) - A^i D^\alpha (\partial_i V)]
\]

(3.5)

Applying the Moser estimates to the term on the right hand side gives

\[
\|D^\alpha (A^i (V) \partial_i V) - A^i (V) D^\alpha (\partial_i V)\|_2 \leq C \|\partial A/\partial V\|_{C^{s'}} (1 + \|V\|^{s-1}_\infty) \|DV\|_\infty \|D^s V\|_2
\]

(3.6)

where \( s = |\alpha| \) and \( s' = \max(1, s - 1) \). Thus

\[
\frac{d}{dt} \|D^s V\|_2^2 \leq C \|\partial A/\partial V\|_{C^{s'}} (1 + \|V\|^{s-1}_\infty) \|DV\|_\infty - 2 \min(\frac{3}{2n}, 1) \frac{4}{s} \|D^s V\|_2^2
\]

\[
+ 2 \|D^s V\|_2 \|D^s (PV)\|_2
\]

(3.7)
It remains to estimate $PV$. This will be done not in terms of $w$ but in terms of $\rho - \bar{\rho}$. Standard elliptic theory gives the estimate

$$\|D^s(PV)\|_2 \leq Ca\|\rho - \bar{\rho}\|_{H^{s-1}}$$

(3.8)

This inequality is only helpful if we have an estimate for $\rho - \bar{\rho}$ independent of (3.7). To obtain this, write the continuity equation in terms of $a^k(\rho - \bar{\rho})$, where $k$ is some number $\leq 3$ to be fixed later. The result is

$$\partial_t (a^k(\rho - \bar{\rho})) + a^k u_i \nabla_i (\rho - \bar{\rho}) - \frac{(k-3)}{a}a^k(\rho - \bar{\rho}) = a^k \rho \nabla_i u^i$$

(3.9)

Multiplying this by $a^k(\rho - \bar{\rho})$ gives an energy type estimate for $\rho - \bar{\rho}$. The equation (3.9) can be differentiated in space to get higher energy estimates. The final result is

$$\frac{d}{dt}\|a^k D^{s-1}(\rho - \bar{\rho})\|_2^2 \leq (C\|Du\|_\infty + \frac{(k-3)}{a})\|a^k D^{s-1}(\rho - \bar{\rho})\|_2^2$$

$$+ C[\|D(a^k(\rho - \bar{\rho}))\|_\infty + \|a^k(\rho - \bar{\rho})\|_\infty + \|a^k \bar{\rho}\|_\infty]\|a^k D^{s-1}(\rho - \bar{\rho})\|_2 \|D^s u\|_2.$$  

(3.10)

The inequalities (3.7) and (3.10) are almost what is needed for the global existence theorem but still need to be altered slightly. First note that the expression $C(\partial A/\partial V|_{C^r})(1 + \|V\|_\infty^{s-1})$ can be thought of as a constant depending continuously on $\|V\|_\infty$. Next, the dynamics of the scale factor $a$ implies that there exists a constant $H_0 > 0$, only depending on the initial data, such that $\dot{a}/a \geq H_0$ as long as a solution exists. It is also obviously possible to replace $L^2$ norms of derivatives of order $s$ of $V$ and $PV$ by $H^s$ norms of these functions themselves. The $H^s$ norm of $PV$ can be eliminated in favour of $\rho - \bar{\rho}$ using (3.8). Finally the whole inequality can be divided by $\|V\|_{H^s}$. Hence

$$\frac{d}{dt}\|V\|_{H^s} \leq [C(\|V\|_\infty)\|DV\|_\infty - \min(\frac{3}{2n}, 1)H_0]\|V\|_{H^s} + ca^{1-k}\|a^k(\rho - \bar{\rho})\|_{H^{s-1}}$$

(3.11)

In (3.10) the quantity $H_0$ can be introduced in the same way. The inequality can be divided by $\|a^k(\rho - \bar{\rho})\|_{H^{s-1}}$, giving

$$\frac{d}{dt}\|a^k(\rho - \bar{\rho})\|_{H^{s-1}} \leq [C\|DV\|_\infty + (k-3)H_0]\|a^k(\rho - \bar{\rho})\|_{H^{s-1}}$$

$$+ \|V\|_{H^s}[\|D(a^k(\rho - \bar{\rho}))\|_\infty + \|a^k(\rho - \bar{\rho})\|_\infty + \|a^k \bar{\rho}\|_\infty].$$

(3.12)

The inequalities (3.11) and (3.12) are the main tools needed to prove the main theorem.

**Theorem 3.2**

*Let initial data be given for (3.3) as in Theorem 3.1 with $s \geq 4$. If $a_0$, $\theta_0$, $\bar{\rho}_0$ and $\Lambda > 0$ are fixed then there exists a positive number $\epsilon$ such that if*

$$\|\rho_0 - \bar{\rho}_0\|_{H^s} + \|v_0\|_{H^s} < \epsilon$$

(3.13)
the corresponding solution (whose local existence is guaranteed by Theorem 3.1) exists globally in time. Moreover the quantity \( \|v\|_{H^s} \) converges exponentially to zero and \((\rho - \bar{\rho})/\bar{\rho} \) tends to a limit in \( H^{s-2} \) as \( t \to \infty \).

**Proof**

Choose some \( k \) lying strictly between 1 and 3 and define \( X = \|a^k(\rho - \bar{\rho})\|_{H^{s-1}} + \|V\|_{H^s} \). Let \( h = \min((3-k)H_0, 3/2n, 1) \). Then it follows from (3.11) and (3.12) that

\[
\frac{dX}{dt} \leq [C(X)X - h]X + Ca^lX. \tag{3.14}
\]

Here \( C(X) \) can be chosen to depend continuously on \( X \) (and hence to be bounded on bounded subsets) and \( l = \max(1-k, k-4) \). The assumption \( s \geq 4 \) is necessary so that the expression \( \|D(a^k(\rho - \bar{\rho}))\|_\infty \) occurring in (3.12) can be estimated by \( \|a^k(\rho - \bar{\rho})\|_{H^{s-1}} \). The essential idea of the proof is now as follows. If \( X \) is sufficiently small then the first term on the right hand side of (3.14) will be negative and so will tend to cause \( X \) to become smaller. However more information is needed in order to conclude that \( X \) actually gets smaller since this term must be negative enough to outweigh the effect of the second term. This will be the case if \( t \) is large enough (and hence \( a^l \) is small enough). Thus it is appropriate to do a bootstrap argument on a sufficiently long time interval.

Let \( \delta_1 \) be a positive number which is small enough so that \( X < \delta_1 \) implies that \( C(X)X < h/3 \). If for a given solution of (3.3) on an interval \([0, T]\) the quantity \( X \) is bounded then the solution can be extended to a larger time interval. Thus the inequality (3.14) shows that given any \( T_1 > 0 \) there is some \( \delta_2 > 0 \) such that if \( X \) is less than \( \delta_2 \) for \( t = 0 \) the solution exists up to time \( T_1 \) and \( X < \delta_1 \) on the whole interval \([0, T_1]\). Now choose \( T_1 \) so that \( Ca^l(t) < h/3 \) for all \( t \geq T_1 \). (This is possible since \( a \to \infty \) as \( t \to \infty \).) Choose \( \epsilon > 0 \) so that the inequality (3.13) implies that the initial value of \( X \) is smaller than \( \delta_2 \). Now consider an initial datum for which (3.13) is satisfied. Let \( T^* \) be the largest time (possibly infinite) for which the corresponding solution exists on \([0, T^*]\) and \( X \leq \delta_1 \) for \( t < T^* \). Clearly \( T^* > T_1 \). On the interval \([T_1, T^*]\) (3.14) implies that \( dX/dt \leq -\frac{1}{3}hX \). It follows in particular that \( X \) is strictly decreasing there. Hence \( X(t) < X(T_1) \) on \([T_1, T^*]\). If \( T^* \) were finite we could conclude firstly that the solution extends to a longer time interval (since \( X \) is bounded) and secondly that the inequality \( X(t) \leq \delta_1 \) holds on a longer time interval. The second point follows from the fact that \( X(t) < X(T_1) < \delta_1 \) on \([T_1, T^*]\) and continuity. This contradicts the definition of \( T^* \). The conclusion is that \( T^* = \infty \). So the solution exists globally and \( dX/dt \leq -\frac{1}{3}hX \) for ever. A simple comparison argument then shows that \( X(t) \leq X(T_1) \exp(-\frac{1}{3}ht) \) for all \( t \geq T_1 \). Going back to the definitions shows that \( \|v\|_{H^s} = a\|u\|_{H^s} = O(\exp(-\frac{1}{3}ht)) \) as \( t \to \infty \). Since \( a \) itself tends to infinity it follows that \( \|v\|_{H^s} \) and \( \|u\|_{H^s} \) converge to zero exponentially as \( t \to \infty \). To get the remaining statement of the theorem, consider the inequality (3.12) in the case \( k = 3 \). Writing \( Y = \|(\rho - \bar{\rho})/\bar{\rho}\|_{H^{s-1}} \) this leads to an estimate of the form

\[
\frac{dY}{dt} \leq CX(Y + 1) \tag{3.15}
\]
Applying Gronwall’s inequality and using the fact that \( \int_0^\infty X(t)dt < \infty \) shows that \( Y \) is bounded. Using equation (3.9) in the case \( k = 3 \) gives the inequality

\[
\left\| \frac{(\rho - \bar{\rho})}{\bar{\rho}}(t_2) - \frac{(\rho - \bar{\rho})}{\bar{\rho}}(t_1) \right\|_{H^{s-2}} \leq C \int_{t_1}^{t_2} \| V \|_{H^s} dt \\
\leq Ce^{-\frac{1}{3}h(t_2-t_1)}
\]

for any two times \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \). This implies that there exists a function \( f \) such that \( (\rho - \bar{\rho})/\bar{\rho} \to f \) in \( H^{s-2} \) as \( t \to \infty \).

Theorems 3.1 and 3.2 concern fluids with polytropic equation of state. Very similar results can be proved for dust using the same methods. The differences will now be indicated briefly. Firstly, no Makino variable is necessary in the dust case; the density can be used directly. Secondly, the equations (2.18) and (2.19) are only coupled via the potential \( U \) when the pressure vanishes. Hence instead of (2.18) and (2.19) together forming a system which can be written in symmetric hyperbolic form, the equation (2.19) is symmetric hyperbolic by itself. Thus \( V \) can be taken to be just \( v^a \) in this case. Apart from this the argument works just as for a polytropic fluid.

If \( \Sigma_{ab} \) is arbitrary then the analogue of the proof of Theorem 3.2 breaks down in general. However it is possible to generalize somewhat. Firstly it may be observed that \( \Omega_{ab} \) makes no contribution in the energy estimates. Hence the analogue of Theorem 3.2 remains true if an arbitrary \( \Omega_{ab} \) is allowed. In the case of non-zero \( \Sigma_{ab} \) it turns out that the condition needed to make the argument go through is that \( \Theta_{ab} \) should be positive definite. To ensure this it suffices to ensure a sufficiently small uniform bound on \( \Sigma^a_b \). In general relativity it turns out that the shear, which is analogous to \( \Sigma_{ab} \), decays exponentially in homogeneous cosmological models with positive cosmological constant[18]. Thus the results found here can be regarded as indirect positive evidence for the validity of the cosmic no-hair conjecture in general relativity.

4. Sharpness of the results

In this section some examples will be presented which demonstrate the sharpness of the results obtained in the previous section. They all concern dust solutions and are proved by studying the paths of dust particles i.e. the integral curves of \( u^a \). The solutions studied are such that \( \rho \) and \( u^a \) only depend on \( t \) and one spatial coordinate, which will be denoted by \( x \). Moreover it is assumed that the velocity is of the form \( u \partial/\partial x \) for some function \( u \). Under these assumptions the equations reduce to

\[
\begin{align*}
\rho_t + (\rho u)_x + \theta \rho &= 0 \\
u_t + uu_x + \frac{2}{3} \theta u - a^{-2}U_x &= 0 \\
U_{xx} &= -4\pi a^2 (\rho - \bar{\rho})
\end{align*}
\]

(4.1)

The equation of motion of a dust particle is \( dx/dt = u(x(t), t) \). Differentiating this with respect to \( t \) and using (4.1) gives:

\[
d^2x/dt^2 = \left( -\frac{2}{3} \theta u + a^{-2}U_x \right)(x).
\]

(4.2)
Now consider two integral curves $x_1(t)$ and $x_2(t)$.

\[
\frac{d^2}{dt^2}(x_2 - x_1) = -\frac{2}{3} \theta \frac{d}{dt} (x_2 - x_1) + \int_{x_1(t)}^{x_2(t)} a^{-2} U_{xx} dx = -\frac{2}{3} \theta \frac{d}{dt} (x_2 - x_1) - 4\pi \int_{x_1(t)}^{x_2(t)} (\rho - \bar{\rho}) dx.
\]

(4.3)

Since $\bar{\rho} a^3$ is independent of time it follows that

\[
\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \bar{\rho} a^3 dx = (\bar{\rho} a^3) \bigg|_{t=0} \left( \frac{dx_2}{dt} - \frac{dx_1}{dt} \right).
\]

(4.4)

On the other hand (4.1) and the relation $\theta = 3\dot{a}/a$ give

\[
\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \rho a^3 dx = 0,
\]

(4.5)

which just expresses the conservation of mass. Thus

\[
\frac{d^2}{dt^2}(x_2 - x_1) = (Aa^{-3} - 2\dot{a}/a)\frac{d}{dt}(x_2 - x_1) - Ba^{-3},
\]

(4.6)

for some positive constants $A$ and $B$. If $d/dt(x_2 - x_1)$ is initially non-positive then it will remain so. To see this note that if it is originally strictly negative it will remain so at least for a short time by continuity. If it is originally zero then (4.6) shows that it will immediately become negative and the same conclusion holds. Now suppose that $d/dt(x_2 - x_1)$ becomes zero at some later time $t_1$. Then the second derivative of $x_2 - x_1$ at $t_1$ is non-negative and this contradicts (4.6). Thus $d/dt(x_2 - x_1)$ remains negative as long as the solution exists.

It will now be shown that for certain solutions of (4.1) the quantity $(\rho - \bar{\rho})/\bar{\rho}$ cannot converge uniformly to zero as $t \to \infty$. This shows that the fall-off property of the density perturbation proved in the last section is optimal. If the initial datum for $\rho$ is not constant it is possible to choose starting values for $x_1$ and $x_2$ so that

\[
\int_{x_1(0)}^{x_2(0)} \rho - \bar{\rho} dx > 0
\]

(4.7)

Now choose the initial velocity to be such that $u(x_2) - u(x_1) \leq 0$. Then $u(x_2(t)) - u(x_1(t))$ remains non-positive and so $x_2(t) - x_1(t) \leq x_2(0) - x_1(0)$. Let

\[
M(t) = (a(t))^3 \int_{x_1(0)}^{x_2(0)} \bar{\rho}(t) dx = 2\pi (a(t))^2 \bar{\rho}(t)
\]

(4.8)
Then $M(t)$ is constant. Now

$$\int_{x_1(t)}^{x_2(t)} (\rho - \bar{\rho}) dx/\bar{\rho} > C(\int_{x_1(t)}^{x_2(t)} (\rho - \bar{\rho}) dx) a^3$$

$$= C[(\int_{x_1(t)}^{x_2(t)} \rho dx) a^3 - \frac{1}{2\pi} (x_2(t) - x_1(t)) M(t)]$$

$$> C[(\int_{x_1(t)}^{x_2(t)} \rho dx) a^3 - \frac{1}{2\pi} (x_2(0) - x_1(0)) M(0)]$$

Hence $\int_{x_1(t)}^{x_2(t)} (\rho - \bar{\rho}) dx/\bar{\rho}$ is bounded below by a positive constant and this is enough to show that $(\rho - \bar{\rho})/\bar{\rho}$ does not converge uniformly to zero.

The argument just given applies for any value of the cosmological constant. It will now be shown that the analogues of the results of section 3 fail when the cosmological constant is zero. First the solution of (2.22) for $\Lambda = 0$ and $E = 0$ will be computed. In that case $\dot{a} = \sqrt{8\pi \rho(0)/3a^{-1/2}}$ and hence, after a translation in time, the solution is of the form $a = Kt^{2/3}$. In particular the solution expands for ever. Now choose data once again for the fluid and for two dust particles so that $dx_2/dt - dx_1/dt \leq 0$ as long as the solution exists. Then (4.6) implies that

$$d^2/dt^2(x_2 - x_1) \leq -\frac{4}{3} t^{-1} d/dt(x_2 - x_1) - BK^{-3}t^{-2}$$

(4.10)

Consider now the corresponding differential equation

$$d^2 y/dt^2 + \frac{4}{3} t^{-1} dy/dt = -BK^{-3}t^{-2}$$

(4.11)

Its general solution is

$$y = -3BK^{-3}\log t - 3C_1 t^{-1/3} + C_2$$

(4.12)

The qualitative behaviour of $(x_1 - x_2)(t)$ can be studied by comparing it with the solution $y(t)$ of (4.11) with the same initial data. Equation (4.11) is a first order ODE for $dy/dt$ and so standard comparison arguments for solutions of first order ODE’s give $\frac{d}{dt}(x_2 - x_1) \leq \frac{dy}{dt}$. This inequality can then be integrated again to show that $x_2 - x_1 \leq y$. Now (4.12) implies that $y \to 0$ in finite time if the solution of (4.1) exists that long. The trajectories of two dust particles in a regular solution of (4.1) cannot cross and so we have found data for (4.1) with the property that the corresponding solutions break down in finite time. Moreover these data can be arbitrarily close to homogeneity.

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