RADIAL LENGTH, RADIAL JOHN DISKS AND $K$-QUASICONFORMAL HARMONIC MAPPINGS

SHAOLIN CHEN AND SAMINATHAN PONNUSAMY

ABSTRACT. In this article, we continue our investigations of the boundary behavior of harmonic mappings. We first discuss the classical problem on the growth of radial length and obtain a sharp growth theorem of the radial length of $K$-quasiconformal harmonic mappings. Then we present an alternate characterization of radial John disks. In addition, we investigate the linear measure distortion and the Lipschitz continuity on $K$-quasiconformal harmonic mappings of the unit disk onto a radial John disk. Finally, using Pommerenke interior domains, we characterize certain differential properties of $K$-quasiconformal harmonic mappings.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

This paper continues the study of previous work of the authors [6] and is mainly motivated by the articles of Beardon and Carne [3], Carroll and Twomey [4], Chuaqui et al. [10], Pommerenke [29], and the monograph of Pommerenke [30]. In order to state our first result concerning the growth of the radial length of $K$-quasiconformal harmonic mappings (see Theorem 1), we need to recall some basic definitions and some results which motivate the present work.

Let $f$ be a complex-valued and continuously differentiable function defined in the unit disk $D = \{ z : |z| < 1 \}$ and let $\ell_f(\theta, r)$ be the length of the $f$-image (with counting multiplicity) of the radial line segment $[0, z]$ from 0 to $z = re^{i\theta} \in D$, where $\theta \in [0, 2\pi]$ is fixed and $r \in [0, 1)$. Then (cf. [5])

$$
\ell_f(\theta, r) := \ell(f([0, z])) = \int_0^r |df(re^{i\theta})| = \int_0^r |f_z(re^{i\theta}) + e^{-2i\theta}f_x(re^{i\theta})| \, d\rho.
$$

In [21], Keogh showed that if $f$ is a bounded, analytic and univalent function in $D$, then, for each $\theta \in [0, 2\pi]$,

$$
(1.1) \quad \ell_f(\theta, r) = O \left( (\log(1/(1 - r)))^{1/2} \right) \text{ as } r \to 1^-.
$$

Throughout the discussion, we let

$$
(1.2) \quad \psi(r) = (\log(1/(1 - r)))^{1/2} \text{ for } 0 < r < 1.
$$

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This second author is on leave from IIT Madras.
Keogh also gave some examples to show that the exponent 1/2 in (1.1) cannot be decreased. Jenkins improved on these examples in [16], and Kennedy [20] presented further examples by showing that

$$\ell_f(\theta, r) = O(\mu(r)\psi(r)) \text{ as } r \to 1^-$$

is false in general for every positive function \( \mu \) in \([0, 1) \) satisfying \( \mu(r) \to 0 \) as \( r \to 1^- \).

In [4], Carroll and Twomey established certain refinements and extension of these results without the boundedness condition in the following form.

**Theorem A.** Suppose that \( f(z) = a_1z + a_2z^2 + \cdots \) is univalent in \( \mathbb{D} \). Then, for any fixed \( \theta \in [0, 2\pi] \), there is a constant \( C_1 > 0 \) such that

\[
(1.3) \quad \ell_f(\theta, r) \leq C_1 \max_{r \in [0, r]} |f(re^{i\theta})|\psi(r) \quad \text{for } r \in (0, 1).
\]

If, further, \( f(re^{i\theta}) = O(1) \) as \( r \to 1^- \), then (1.1) holds.

Later, Beardon and Carne [3] gave a relatively simple argument to Theorem A in hyperbolic geometry and provided with further examples. It is worth pointing out here two results which strengthened (1.3) and was inspired by the work of Sheil-Small [33] and Hall [14]. If \( f \in \mathcal{S} \) is starlike, i.e. \( f(\mathbb{D}) \) contains the line segment \([0, w] \) whenever it contains \( w \), then (see [19])

\[
\ell_f(\theta, r) \leq |f(re^{i\theta}|(1 + r) < 2|f(re^{i\theta}| \quad \text{for } r \in (0, 1)
\]

and the inequality of course is not sharp for all \( r \), but the bound 2 sharp as the Koebe function \( k(z) = z/(1 - z)^2 \) shows and is attained when \( r \) approaches 1 (see [14, 33]). Later in 1993, Balasubramanian et al. [1] showed that if \( f \in \mathcal{S} \) is convex, i.e. \( f(\mathbb{D}) \) is a convex domain, then

\[
\ell_f(\theta, r) \leq |f(re^{i\theta}| r^{-1} \arcsin r \quad \text{for } r \in (0, 1)
\]

and the inequality is sharp as the convex function \( f(z) = z/(1 - z) \) shows. Note that \( \varphi(r) = r^{-1} \arcsin r \) is increasing on \((0, 1)\) and \( \varphi(r) \leq \lim_{r \to 1^-} \varphi(r) = \pi/2 \) and thus, the conjecture of Hall [15] was settled (see also [2]).

The first aim of this paper is to extend Theorem A for the case of harmonic quasiconformal mappings (see Theorem 1 below). We need some preparation to state this result.

For a real \( 2 \times 2 \) matrix \( A \), we use the matrix norm \( \|A\| = \sup\{|Az| : |z| = 1\} \) and the matrix function \( l(A) = \inf\{|Az| : |z| = 1\} \). For \( z = x + iy \in \mathbb{C} \), the formal derivative of the complex-valued function \( f = u + iv \) is given by the Jacobian matrix

\[
D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},
\]

so that

\[
\|D_f\| = |f_x| + |f_x| \quad \text{and} \quad l(D_f) = ||f_x| - |f_x||,
\]

where \( f_x = (1/2)(f_x - if_y) \) and \( f_y = (1/2)(f_x + if_y) \). Let \( \Omega \) be a domain in \( \mathbb{C} \), with non-empty boundary. A sense-preserving homeomorphism \( f \) from a domain \( \Omega \)
onto $\Omega'$, contained in the Sobolev class $W_{loc}^{1,2}(\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega$,

$$\|D_f(z)\|^2 \leq K|\det D_f(z)|,$$

i.e., $\|D_f(z)\| \leq K\ell(D_f(z))$,

where $K \geq 1$ and $\det D_f$ is the determinant of $D_f$ (cf. [18, 22, 35, 36]).

Let $\mathcal{S}_H$ denote the family of sense-preserving planar harmonic univalent mappings $f = h + \overline{g}$ in $\mathbb{D}$, with the normalization $h(0) = g(0) = 0$ and $h'(0) = 1$. Recall that $f$ is sense-preserving if the Jacobian $J_f$ of $f$ given by

$$J_f := \det D_f = |f_z|^2 - |f_w|^2 = |h'|^2 - |g'|^2$$

is positive. Thus, $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$; or equivalently if $h' \neq 0$ in $\mathbb{D}$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in $\mathbb{D}$ (see [11, 12, 23]). The family $\mathcal{S}_H$ together with a few other geometric subclasses, originally investigated in detail by [11, 34], became instrumental in the study of univalent harmonic mappings (see [12, 31]) and has attracted the attention of many function theorists. If the co-analytic part $g$ is identically zero in the decomposition of $f = h + \overline{g}$, then the class $\mathcal{S}_H$ reduces to the classical family $\mathcal{S}$ of all normalized analytic univalent functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$. If $\mathcal{S}_H^0 = \{ f = h + \overline{g} \in \mathcal{S}_H : g'(0) = 0 \}$, then the family $\mathcal{S}_H^0$ is both normal and compact. See [11] and also [8, 6, 12, 31].

**Theorem 1.** For $K \geq 1$, let $f \in \mathcal{S}_H$ be a $K$-quasiconformal harmonic mapping. Then, for any fixed $\theta \in [0, 2\pi]$, there is a constant $C_2 > 0$ such that

$$\ell_f(\theta, r) \leq C_2 \max_{\rho \in [0, r]} |f(\rho e^{i\theta})|\psi(r) \text{ for } r \in (0.5, 1).$$

If, further, $f(\rho e^{i\theta}) = O(1)$ as $r \to 1^-$, then

$$\ell_f(\theta, r) = O(\psi(r)) \text{ as } r \to 1^-,$$

and the exponent $1/2$ in $\psi(r)$ defined by (1.2) cannot be replaced by a smaller number.

First we remark that if $K = 1$, then Theorem 1 coincides with Theorem A. Secondly, the proof of Theorem 1 is substantially harder than the proof of Theorem A. This is because Beardon and Carne’s argument of Theorem A in [3] is not applicable in the proof of Theorem 1.

We need further notation and terminology before stating our second result. Let $d_\Omega(z)$ be the Euclidean distance from $z$ to the boundary $\partial \Omega$ of $\Omega$. If $\Omega = \mathbb{D}$, then we set $d(z) := d_\mathbb{D}(z)$.

**Definition 1.** A bounded simply connected plane domain $G$ is called a $c$-John disk for $c \geq 1$ with John center $w_0 \in G$ if for each $w_1 \in G$ there is a rectifiable arc $\gamma$, called a John curve, in $G$ with end points $w_1$ and $w_0$ such that

$$\sigma_\ell(w) \leq cd_G(w)$$

for all $w$ on $\gamma$, where $\gamma[w_1, w]$ is the subarc of $\gamma$ between $w_1$ and $w$, and $\sigma_\ell(w)$ is the Euclidean length of $\gamma[w_1, w]$ (see [6, 13, 17, 28, 30]).
Remark 1. If \( f \) is a complex-valued and univalent mapping in \( \mathbb{D} \), \( G = f(\mathbb{D}) \) and, for \( z \in \mathbb{D} \), \( \gamma = f([0, z]) \) in Definition 1, then we call \( c \)-John disk a \textit{radial} \( c \)-John disk, where \( w_0 = f(0) \) and \( w = f(z) \). In particular, if \( f \) is a conformal mapping, then we call \( c \)-John disk a \textit{hyperbolic} \( c \)-John disk. It is well known that any point \( w_0 \in G \) can be chosen as a John center by modifying the constant \( c \) if necessary. When we do not wish to emphasize the role of \( c \), then we regard the \( c \)-John disk simply as a John disk in the natural way (cf. [6, 13, 17, 28]).

Unless otherwise stated, throughout the discussion we consider the following terminology. Denote by \( \mathcal{F}(K) \) if \( f \in \mathcal{F} \) and is a \( K \)-quasiconformal harmonic mapping in \( \mathbb{D} \), where \( K \geq 1 \). Also, we denote by \( \mathcal{F}(K, \Omega) \) if \( f \in \mathcal{F}(K) \) and \( f \) maps \( \mathbb{D} \) onto \( \Omega \). We prove several results mainly when \( \mathcal{F} \) equals one of \( S_H \), \( S_H^0 \), and \( S_{H_2} \), and \( \Omega \) equals either radial John disk or Pommerenke interior domain.

Further, for \( z \in \mathbb{D} \), we define

\[
B(z) := \{ \zeta : |z| \leq |\zeta| < 1, \ |\arg z - \arg \zeta| \leq \pi(1 - |z|) \}.
\]

In the following, we continue our previous work of [6], and give another characterization of the radial John disk.

**Theorem 2.** Let \( f \in S_H^0(K) \). Then the following are equivalent:

(i) \( \Omega = f(\mathbb{D}) \) is a radial John disk.

(ii) There is an \( x \in (0, 1) \) such that

\[
\sup_{|\zeta| = 1} \sup_{\rho \in (0, 1)} \frac{(1 - \rho^2)}{(1 - \tau^2)} \frac{\|Df(\rho \zeta)\|}{\|Df(r \zeta)\|} < 1 \quad \text{for } \rho = \frac{x + r}{1 + xr}.
\]

(iii) \( \sup_{z \in B(z), w \in B(z)} \frac{|f(z) - f(w)|}{(1 - |z|^2)} \|Df(z)\| < \infty \).

Next, we establish the linear measure distortion on \( K \)-quasiconformal harmonic mappings of \( \mathbb{D} \) onto a radial John disk.

**Theorem 3.** Let \( f = h + \overline{g} \in S_H^0(K, \Omega) \), where \( \Omega \) is a radial John disk. Then, for \( a_1, a_2 \in \mathbb{D} \) with \( B(a_1) \subset B(a_2) \), there is a positive constant \( C_3 \) such that

\[
\frac{\text{diam} f(B(a_1))}{\text{diam} f(B(a_2))} \leq C_3 \left( \frac{\ell(B(a_1) \cap \partial \mathbb{D})}{\ell(B(a_2) \cap \partial \mathbb{D})} \right)^{\alpha}.
\]

where \( \alpha = \sup_{f \in S_H} \frac{|K'(-1)|}{2} \) and \( B(z) \) is defined by (1.4).

We remark that \( 2 \leq \alpha = \sup_{f \in S_H} \frac{|K'(-1)|}{2} < \infty \), but the sharp value of \( \alpha \) is still unknown (see [6, 9, 12, 34]). We discuss the Lipschitz continuity on \( K \)-quasiconformal harmonic mappings of \( \mathbb{D} \) onto a radial John disk, which is as follows.

**Theorem 4.** Let \( f = h + \overline{g} \in S_H^0(K, \Omega) \), where \( \Omega \) is a radial John disk. Then, for \( z \in \mathbb{D} \) with \( |z| \geq \frac{1}{2} \) and \( \zeta_1, \zeta_2 \in B(z) \), there are constants \( \delta_1 \in (0, 1) \) and \( C_4 > 0 \) such that

\[
|f(\zeta_1) - f(\zeta_2)| \leq C_4 d_{\Omega}(f(z)) \left( \frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^{\delta_1}.
\]
Let \( f \in S_H(K, G) \), where \( G \) is domain. For \( 0 < r < 1 \), let \( \mathbb{D}_r = \{ z : |z| < r \} \) and \( \partial \mathbb{D}_r \) denote the boundary of \( \mathbb{D}_r \). Now, for \( w_1, w_2 \in f(\partial \mathbb{D}_r) \), let \( \gamma_r \) be the smaller subarc of \( f(\partial \mathbb{D}_r) \) between \( w_1 \) and \( w_2 \), and let

\[
d_{G_r}(w_1, w_2) = \inf_{\Gamma} \text{diam} \Gamma,
\]

where \( \Gamma \) runs through all arcs from \( w_1 \) to \( w_2 \) that lie in \( G_r = f(\mathbb{D}_r) \) except for their endpoints. If

\[
\sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_{G_r}(w_1, w_2)} \right\} < \infty,
\]

then we call \( G \) a Pommerenke interior domain (cf. \([6, 29]\)). In particular, if \( G \) is bounded, then we call \( G \) as a bounded Pommerenke interior domain.

Given a sense-preserving harmonic mapping \( f = h + \overline{g} \) in \( \mathbb{D} \), fix \( \zeta \in \mathbb{D} \) and perform a disk automorphism (also called Koebe transform \( F \) of \( f \)) to obtain

\[
F(z) = \frac{f \left( \frac{z + \zeta}{1 + \zeta \bar{z}} \right) - f(\zeta)}{h'(\zeta)(1 - |\zeta|^2)} =: H(z) + \overline{G(z)}.
\]

A calculation gives,

\[
\frac{H''(0)}{2} = \frac{1}{2} \left\{ (1 - |\zeta|^2) \frac{h''(\zeta)}{h'(\zeta)} - 2\zeta \right\}.
\]

Now, we consider the class \( S_{H_2} \) of all harmonic mappings \( f = h + \overline{g} \in S_H \) satisfying

\[
\sup_{z \in \mathbb{D}} \left| (1 - |z|^2) \frac{h''(z)}{h'(z)} - 2\zeta \right| < 4.
\]

This inequality obviously holds if \( h \in S \) and \( h \) is not the Koebe function \( z/(1 - e^{i\theta}z)^2 \), \( \theta \in \mathbb{R} \). Note that for the Koebe function the supremum turns out to be 4. Our next two results are extension of \([29, \text{Theorem 3}]\).

**Theorem 5.** Let \( f \in S_{H_2}(K, G) \), where \( G \) is a bounded Pommerenke interior domain. If there are positive constants \( \delta_2 \in (0, 1) \) and \( C_5 \) such that, for each \( \zeta \in \partial \mathbb{D} \) and for \( 0 \leq \rho_1 \leq \rho_2 < 1 \),

\[
\|D_f(\rho_2\zeta)\| \leq C_5 \left( \frac{1 - \rho_2}{1 - \rho_1} \right)^{\delta_2 - 1} \|D_f(\rho_1\zeta)\|,
\]

then

\[
\sup_{\zeta \in \mathbb{D}} \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{\|D_f(\zeta)\|}{\|D_f(\zeta')\|} \frac{1 - |\zeta|^2}{|\xi - \zeta|^2} |d\xi| < \infty.
\]

We remark that if \( K = 1 \), then Theorem 5 coincides with \([29, \text{Theorem 3}]\).

By using similar reasoning as in the proof of Theorem 5, one can easily get the following result which replaces the assumption \( f \in S_{H_2} \) by a more general condition \( f \in \tilde{S}_H \) and thus, we omit its proof.
Theorem 6. Let \( f \in \mathcal{S}_H(K, G) \), where \( G \) is a bounded Pommerenke interior domain. If there are constants \( C_6 > 0, C_7 > 0, \delta_3 > 0 \) and \( \delta_4 \in (0, 1) \) such that, for each \( \zeta \in \partial \mathbb{D} \) and for \( 0 \leq \rho_1 \leq \rho_2 < 1 \),

\[
C_6 \left( \frac{1 - \rho_1}{1 - \rho_2} \right)^{\delta_3 - 1} \| D_f(\rho_1 \zeta) \| \leq \| D_f(\rho_2 \zeta) \| \leq C_7 \left( \frac{1 - \rho_2}{1 - \rho_1} \right)^{\delta_4 - 1} \| D_f(\rho_1 \zeta) \|
\]

then

\[
\sup_{\zeta \in \partial \mathbb{D}} \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{\| D_f(\xi) \|}{\| D_f(\zeta) \|} \frac{1 - |\xi|^2}{|\xi - \zeta|^2} d\xi < \infty.
\]

Also, the following result easily follows from Theorem 5 and [6, Theorem 1].

Corollary 1. For \( K \geq 1 \), let \( f \in \mathcal{S}_{H_2} \cap \mathcal{S}_{H}^0 \) be a \( K \)-quasiconformal harmonic mapping from \( \mathbb{D} \) onto a bounded Pommerenke interior domain \( G \). If \( G \) is a radial John disk, then

\[
\sup_{\zeta \in \partial \mathbb{D}} \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{\| D_f(\xi) \|}{\| D_f(\zeta) \|} \frac{1 - |\xi|^2}{|\xi - \zeta|^2} d\xi < \infty.
\]

The proofs of Theorems 1-5 will be presented in Section 2.

2. The proofs of the main results

Let \( \lambda_\mathbb{D} \) stand for the hyperbolic distance (or Poincaré distance) on the unit disk \( \mathbb{D} \). We have

\[
\lambda_\mathbb{D}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2} = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|,
\]

where the infimum is taken over all smooth curves \( \gamma \) in \( \mathbb{D} \) connecting \( z_1 \in \mathbb{D} \) and \( z_2 \in \mathbb{D} \) (cf. [30]). In [34], Sheil-Small proved that if \( f = h + \overline{g} \in \mathcal{S}_H \), then

\[
(1 - |z|)^{\alpha - 1} \leq |h'(z)| \leq (1 + |z|)^{\alpha - 1}
\]

and

\[
\alpha := \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2} < \infty.
\]

Unless otherwise stated, the number \( \alpha \) will be used throughout the discussion and is indeed called the order of the linear invariant family \( \mathcal{S}_H \) (see [34]).

Lemma 1. Suppose that \( f \in \mathcal{S}_H(K) \). Then, for \( z_0, z_1 \in \mathbb{D} \),

\[
\frac{1}{\alpha(1 + K)} \left[ 1 - e^{-2\alpha\lambda_\mathbb{D}(z_1, z_2)} \right] \leq \frac{|f(z_1) - f(z_0)|}{(1 - |z_0|^2)|f_z(z_0)|} \leq \frac{K}{\alpha(1 + K)} \left[ e^{2\alpha\lambda_\mathbb{D}(z_1, z_2)} - 1 \right].
\]

In particular,

\[
\frac{1}{\alpha(1 + K)} \left[ 1 - e^{-2\alpha\lambda_\mathbb{D}(z, 0)} \right] \leq |f(z)| \leq \frac{K}{\alpha(1 + K)} \left[ e^{2\alpha\lambda_\mathbb{D}(z, 0)} - 1 \right], \quad z \in \mathbb{D}.
\]
Proof. By assumption \( f = h + \overline{g} \in S_H \) is a \( K \)-quasiconformal harmonic mapping, where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Thus, by (2.1), we have

\[
\|D_f(z)\| \leq \frac{2K}{K+1} |h'(z)| \leq \frac{2K}{K+1} (1 + |z|)^{\alpha-1}
\]

and thus, for \( z \in \mathbb{D} \), we obtain

\[
|f(z)| \leq \int_{[0,z]} \|D_f(\zeta)\| |d\zeta|
\]

(2.4)

\[
\leq \frac{2K}{K+1} \int_0^{|z|} \frac{(1+\rho)^{\alpha-1}}{(1-\rho)^{\alpha+1}} d\rho = \frac{K}{\alpha(K+1)} \left[ \left( \frac{1+|z|}{1-|z|} \right)^\alpha - 1 \right].
\]

On the other hand, let \( \Gamma \) be the preimage under \( f \) of the radial segment from 0 to \( f(z) \). Again, because

\[
l(D_f(z)) \geq \frac{2}{K+1} |h'(z)| \geq \frac{2}{K+1} (1 - |z|)^{\alpha-1},
\]

it follows that

(2.5)

\[
|f(z)| \geq \int_{\Gamma} l(D_f(\zeta)) |d\zeta| \geq \frac{1}{\alpha(K+1)} \left[ 1 - \left( \frac{1-|z|}{1+|z|} \right)^\alpha \right].
\]

Let \( z = \frac{z_1 - z_0}{1 - \overline{z}_0 z_1} \) so that \( z_1 = \frac{z + z_0}{1 + \overline{z}_0 z} \), where \( z_0, z_1 \in \mathbb{D} \). Then, by assumption,

\[
F(z) = \frac{f(z) - f(z_0)}{(1 - |z_0|^2)h'(z_0)} \in S_H
\]

and is a \( K \)-quasiconformal harmonic mapping, i.e. \( F \in S_H(K) \). Applying (2.4) and (2.5) to \( F \) gives us the desired result if we take into account of the fact that

\[
\left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right| = \frac{e^{2\lambda \beta(z_1,z_2)} - 1}{e^{2\lambda \beta(z_1,z_2)} + 1} = \tanh \lambda_D(z_1,z_2).
\]

The proof of the lemma is complete. \( \square \)

**Lemma 2.** Assume that \( f \in S_H(K) \). Then

\[
\|D_f(z)\| |z| \leq \frac{C_8 |f(z)|}{1 - |z|} \quad \text{for } z \in \mathbb{D},
\]

where

(2.6)

\[
C_8 = 2\alpha K \sup_{z \in \mathbb{D}} \left\{ \frac{|z|(1+|z|)^{\alpha-1}}{|(1+|z|)^\alpha - (1-|z|)^\alpha} \right\} \geq K.
\]

Proof. Suppose that \( f = h + \overline{g} \in S_H(K) \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Next, for fixed \( \zeta \in \mathbb{D} \), consider the Koebe transform \( F \) of \( f \) given by (1.6). By assumption, \( F \in S_H \) and is also a \( K \)-quasiconformal harmonic mapping. By letting \( z = -\zeta \) in (1.6) and applying (2.2) to \( F \), we obtain (since \( f(0) = 0 \))

\[
|F(-\zeta)| = \frac{|f(\zeta)|}{(1 - |\zeta|^2)|h'(\zeta)|} \geq \frac{1}{\alpha(1+K)} \frac{[(1+|\zeta|)^\alpha - (1-|\zeta|)^\alpha]}{(1+|\zeta|)^\alpha}
\]
which gives
\[ \frac{|h'(\zeta)|}{(1 + K)|f(\zeta)|} \leq \frac{(1 + |\zeta|)^{\alpha - 1}}{[(1 + |\zeta|)^\alpha - (1 - |\zeta|)^\alpha]} \cdot \frac{\alpha}{1 - |\zeta|}. \]
Since this follows for each \( \zeta \in \mathbb{D} \), by the first inequality in (2.3), we easily have
\[ \|D_f(z)\| \frac{|z|}{|f(z)|} \leq \frac{2K}{K + 1} \frac{|h'(z)||z|}{|f(z)|} \leq \frac{C_8}{1 - |z|}, \]
where \( C_8 \) is given by (2.6).

Lemma 3. Let \( f \in \mathcal{S}_H(K) \) and, for any fixed \( \theta \in [0, 2\pi] \), set
\[ m_f(r, \theta) = \max_{\rho \in [0, r]} |f(\rho e^{i\theta})|, \]
where \( r \in [0, 1] \). Then, for \( 0 < \rho_0 \leq r < 1 \) and \( 0 \leq \rho \leq r \), there is a constant \( C_9 > 0 \) which depends only on \( \rho_0 \) such that
\[ (2.7) \quad \frac{|f(\rho e^{i\theta})|}{\rho} \leq C_9 m_f(r, \theta), \]
where \( \rho_0 \) is a constant.

Proof. Without loss of generality, we assume that \( \theta = 0 \). Clearly, (2.2) yields that
\[ \lim_{\rho \to 0^+} \frac{|f(\rho)|}{\rho} \leq \frac{K}{\alpha(1 + K)} \lim_{\rho \to 0^+} \frac{e^{2\alpha \lambda_0(\rho, 0)} - 1}{\rho} = \frac{2K}{1 + K}, \]
which implies that \( f(\rho)/\rho \) is bounded in \([0, \rho_0]\), where \( \rho_0 \) is a constant such that \( 0 < \rho_0 \leq r < 1 \). Hence there is a constant \( C_{10} > 0 \) such that
\[ (2.8) \quad \frac{f(\rho)}{\rho} \leq C_{10} m_f(r, 0) \text{ for } \rho \in [0, \rho_0], \]
where \( r \in [\rho_0, 1] \). For \( r \in [0, 1] \), let
\[ T(r) = \frac{K}{\alpha(1 + K)} \left[ 1 - e^{-2\alpha \lambda_0(r, 0)} \right]. \]
Then \( T \) is increasing in \([0, 1]\), which, together with (2.2), yields that
\[ (2.9) \quad 0 < T(\rho_0) \leq T(\rho) \leq f(\rho) \leq \frac{f(\rho)}{\rho} \leq \frac{f(\rho)}{\rho_0} \leq \frac{1}{\rho_0} m_f(r, 0) \text{ for } \rho \in [\rho_0, r], \]
where \( r \in [\rho_0, 1] \). Therefore, (2.7) follows from (2.8) and (2.9).

Lemma 4. For \( r \in (0, 1) \), let \( \Omega_r \) be the Stolz-type domain consisting of the interior of the convex hull of the point \( r \) and the disk \( \mathbb{D}_{r/4} \). Then, for \( z = \rho e^{i\eta} \in \Omega_r \setminus \mathbb{D}_{r/4} \),
\[ |\eta| \leq \frac{4\pi}{r\sqrt{15}}(r - \rho) < \frac{4\pi}{r\sqrt{15}}(1 - \rho). \]
Proof. Assume without loss of generality that $\eta \geq 0$. Let $A$, $D$, and $E$ represent the points $r$, $r/4$, and $\rho_1 e^{i\eta}$ (see Figure 1), respectively. As $\angle OCA = \pi/2$, it is clear that

$$\sin \angle COA = \frac{\sqrt{15}}{4}, \quad \cos \angle COA = \frac{1}{4}, \quad \sin \angle COE = \frac{\sqrt{\rho_1^2 - r^2}}{\rho_1}, \quad \cos \angle COE = \frac{r}{4\rho_1}.$$  

Then, because $|\sin \eta| / \eta \geq 2/\pi$ for $|\eta| < \pi/2$, it follows that for $\eta \geq 0$

$$\frac{2\eta}{\pi} \leq \sin \eta = \sin (\angle COA - \angle COE)$$
$$= \sin \angle COA \cos \angle COE - \cos \angle COA \sin \angle COE$$
$$= \frac{\sqrt{15r^2} - \sqrt{16\rho_1^2 - r^2}}{16\rho_1}$$
$$= \frac{r^2 - \rho_1^2}{\rho_1 \left( \sqrt{15r} + \sqrt{16\rho_1^2 - r^2} \right)}$$

Note that $r^4 < \rho < \rho_1 < r$ and, because

$$\frac{r^2 - \rho_1^2}{\rho_1} < \frac{4}{r} \left( r^2 - \rho_1^2 \right) < 8(r - \rho_1) < 8(r - \varrho),$$

the last above relation clearly implies that

$$\frac{2\eta}{\pi} < \frac{8(r - \varrho)}{r\sqrt{15}}$$

which gives the desired conclusion. Observe that $\frac{8}{r\sqrt{15}}(r - \varrho)$ is less than $6/\sqrt{15}$ from which we also deduce that $|\eta| < 3\varpi/\sqrt{15}$. □
Proof of Theorem 1. Assume without loss of generality that θ = 0. For r ∈ (0, 1), we use Ω_r to denote the Stolz-type domain, where Ω_r is same as in Lemma 4. Let $z = re^{i\eta} \in \Omega_r \setminus \mathbb{D}_{r/4}$. Then, by Lemma 4, there is a constant $C_{11} > 0$ which depends only on r such that

$$|\eta| < C_{11}(1 - \rho).$$  \hspace{1cm} (2.10)

Suppose that $f = h + \overline{g} \in S_H(K)$. By calculations, we get

$$\log \frac{f(ze^{i\eta})}{\rho e^{i\eta}} - \log \frac{f(\rho)}{\rho} = i \int_0^\eta \left( \frac{pe^{it}h'(pe^{it}) - pe^{-it}g'(pe^{it})}{f(pe^{it})} - 1 \right) dt. \hspace{1cm} (2.11)$$

Taking real part of (2.11) on both sides, and then using (2.10), (2.11) and Lemma 2, we see that there is a constant $C_{12}$ such that

$$\log \frac{|f(ze^{i\eta})|}{\rho} - \log \frac{|f(\rho)|}{\rho} \leq \int_0^\eta \frac{\rho \|D_f(\rho e^{it})\|}{|f(\rho e^{it})|} dt \leq C_{12} \int_0^\eta \frac{dt}{1 - \rho} \leq C_{11}C_{12},$$

which gives that

$$|f(z)| = |f(ze^{i\eta})| \leq e^{C_{11}C_{12}} |f(\rho)|. \hspace{1cm} (2.12)$$

By (2.2), we see that (2.12) also holds for $z \in \mathbb{D}_{r/4}$. Then, by (2.12), there is constant $C_{13}$ such that $f(\Omega_r)$ is contained in $\mathbb{D}_{C_{13}m_f(r, 0)}$, which yields that

$$\int_{\Omega_r} J_f(\zeta) dA(\zeta) \leq C_{13}^2 m_f^2(r, 0), \hspace{1cm} (2.13)$$

where $\zeta = x + iy$, $dA = dxdy/\pi$ and $m_f(r, \theta)$ is defined as in Lemma 3.

By [24, Theorem 2], there is a constant $C_{14}$ such that

$$\int_0^1 (1 - \rho)|H'(\rho)|^2 d\rho \leq C_{14} \int_{\Omega_1} |H'(z)|^2 dA(z), \hspace{1cm} (2.14)$$

where $H(z)$ is analytic in $\mathbb{D}$. For $r \in (0, 1)$, let $H(z) = h(rz)$ for $z \in \mathbb{D}$. Then, by (2.14), we obtain

$$\int_0^r (r - \rho)|h'(\rho)|^2 d\rho \leq C_{15} \int_{\Omega_r} |h'(z)|^2 dA(z),$$

which implies that

$$\int_0^r (r - \rho)\|D_f(\rho)\|^2 d\rho \leq C_{15}K \int_{\Omega_r} J_f(z) dA(z),$$

$$\leq C_{13}^2C_{15}Km_f^2(r, 0) \hspace{1cm} (by \ (2.13)), \hspace{1cm} (2.15)$$

where $C_{15} > 0$ is a constant.

By Lemmas 2 and 3, for $r \in (1/2, 1)$ and $\rho \in [0, r]$, there is a constant $C_{16}$ such that

$$\int_0^r \|D_f(\rho)\|^2 d\rho \leq C_{16} \int_0^r \frac{m_f^2(r, 0)}{(1 - \rho)^2} d\rho = C_{16}m_f^2(r, 0) \frac{r}{1 - r}. \hspace{1cm} (2.16)$$
Writing $1 - \rho = (1 - r) + (r - \rho)$ and then, applying (2.15) and (2.16), it follows that

$$\int_0^r (1 - \rho) \| D_f(\rho) \|^2 \, d\rho \leq (C_{13}^2 C_{15} K + C_{16}) m_f^2(r, 0).$$

Therefore, by (2.17), we conclude that

$$\ell_f(0, r) \leq \int_0^r \| D_f(\rho) \| \, d\rho$$

$$\leq \left( \int_0^r (1 - \rho) \| D_f(\rho) \|^2 \, d\rho \right)^{1/2} \left( \int_0^r \frac{d\rho}{1 - \rho} \right)^{1/2}$$

$$\leq (C_{13}^2 C_{15} K + C_{16})^{1/2} m_f(r, 0) \left( \log \frac{1}{1 - r} \right)^{1/2}.$$

Now we prove the sharpness part. For any $\tau \in (0, 1/2)$, by [16, 21], there is a function $h_0 \in S$ such that,

$$\ell_{h_0}(0, r) > C_{17} \left( \log \frac{1}{1 - r} \right)^{\tau} \text{ as } r \to 1^-,$$

where $C_{17}$ is a positive constant. Finally, consider

$$f_0(z) = h_0(z) + \frac{K - 1}{K + 1} \overline{h_0(z)}, \quad z \in \mathbb{D},$$

and observe that $f_0 \in S_H$ and is a $K$-quasiconformal harmonic mapping. Consequently,

$$\ell_{f_0}(0, r) = \int_0^r \left| h'_0(\rho) + \frac{K - 1}{K + 1} \overline{h'_0(\rho)} \right| \, d\rho \geq 2 \frac{K - 1}{K + 1} \int_0^r |h'_0(\rho)| \, d\rho = \frac{2}{K + 1} \ell_{h_0}(0, r),$$

which, together with (2.18), implies that

$$\ell_{f_0}(0, r) > \frac{2C_{17}}{K + 1} \left( \log \frac{1}{1 - r} \right)^{\tau} \text{ as } r \to 1^-.$$

The proof of this theorem is complete. \qed

**Lemma 5.** Let $f \in S_H^0$. Then, for $\xi \in \partial \mathbb{D}$ and $0 \leq \rho \leq r < 1$,

$$\frac{(1 - \rho^2) \| D_f(\rho \xi) \|}{(1 - r^2) \| D_f(r \xi) \|} \leq e^{2\alpha \lambda_0(\rho, r)}.$$

**Proof.** Let $f = h + \overline{g} \in S_H^0$, where $h$ and $g$ are analytic in $\mathbb{D}$. For every $\mu \in \mathbb{D}$, consider the affine mapping

$$f_\mu = f + \mu \overline{g} = (h + \mu g) + \overline{(g + \mu h)}.$$ 

Clearly, $f_\mu \in S_H$. For a fixed $\zeta \in \mathbb{D}$, we consider the Koebe transform $F_\mu$ of $f_\mu$ as given by (1.6). Then we can write $F_\mu = H_\mu + \overline{G_\mu}$ which again belongs to $S_H$ and obviously,

$$\frac{H''_\mu(0)}{2} = A_2(\zeta) = \frac{1}{2} (1 - |\zeta|^2) \frac{h''(\zeta) + \mu g''(\zeta)}{h'(\zeta) + \mu g'(\zeta)} - \overline{\zeta}.$$
Since $|A_2| \leq \alpha$, we see that
\[
\left| \frac{\partial}{\partial \rho} \log \left( (1 - \rho^2) (h'(\rho \xi) + \mu g'(\rho \xi)) \right) \right| = \left| \frac{h''(\rho \xi) + \mu g''(\rho \xi)}{h'(\rho \xi) + \mu g'(\rho \xi)} - \frac{2\rho \tilde{q}}{1 - \rho^2} \right| \leq \frac{2\alpha}{1 - \rho^2},
\]
where $\xi \in \partial \mathbb{D}$. Integration leads to
\[
(1 - r^2)|h'(r \xi) + \mu g'(r \xi)| \geq \left( 1 - r \cdot \frac{1 + \rho}{1 + r \cdot 1 - \rho} \right)^\alpha,
\]
which gives
\[
(1 - \rho^2)|h'(\rho \xi) + \mu g'(\rho \xi)| \leq e^{2\alpha \lambda \Omega(r, \rho)} (1 - r^2)|h'(r \xi) + \mu g'(r \xi)|
\]
and the desired inequality (2.19) follows from this and the arbitrariness of $\mu$. \qed

We remark that Mateljević [26] (see also [27, 25]) proved the following lemma for $f \in S^0_H(K)$ instead of $f \in S_H(K)$. That is, the normalization condition on $f$, namely, $f_{\Omega}(0) = 0$, is not necessary.

**Lemma 6.** If $f \in S_H(K)$ and $\Omega = f(\mathbb{D})$, then
\[
d_H(f(z)) \geq \frac{\|D_f(z)\|(1 - |z|^2)}{16K} \quad \text{for } z \in \mathbb{D}.
\]

**Proof.** Let $f = h + \overline{g} \in S_H(K)$, where $h$ and $g$ are analytic in $\mathbb{D}$. Then the affine mapping $f_0$ defined by
\[
f_0(z) = \frac{f(z) - g'(0)f(z)}{1 - |g'(0)|^2}
\]
belongs to $S^0_H$. By [11, Theorem 4.4], we have
\[
\frac{|f(z)|}{1 - |g'(0)|^2} \geq |f_0(z)| = \frac{|f(z) - g'(0)f(z)|}{1 - |g'(0)|^2} \geq \frac{|z|}{4(1 + |z|)^2}, \quad z \in \mathbb{D}.
\]
Recall again, for any fixed $\zeta \in \mathbb{D}$, the Koebe transform $F$ of $f$ given by (1.6) belongs to $S_H$ and $F$ is again a $K$-quasiconformal harmonic mapping. As a result, (2.21) applied to $F$ shows that
\[
\left| f \left( \frac{z + \zeta}{1 + \zeta z} \right) - f(\zeta) \right| \geq (1 - |\zeta|^2)|h'(\zeta)|(1 - |F_{\Omega}(0)|) \frac{|z|}{4(1 + |z|)^2}
\]
\[
\geq (1 - |\zeta|^2)|h'(\zeta)| \left( \frac{2}{K + 1} \right) \frac{|z|}{4(1 + |z|)^2}
\]
\[
\geq (1 - |\zeta|^2)\|D_f(\zeta)\| \frac{|z|}{4(1 + |z|)^2},
\]
which implies that
\[
d_\Omega(f(\zeta)) = \liminf_{|z| \to 1^-} \frac{|f \left( \frac{z + \zeta}{1 + \zeta z} \right) - f(\zeta)|}{|z|} \geq \frac{\|D_f(\zeta)\|(1 - |z|^2)}{16K}.
\]
The proof of this Lemma is complete. □

**Lemma B.** ([6, Lemma 2]) Let \(a_1, a_2\) and \(a_3\) be positive constants and let \(0 < \|z_0\| = 1 - \delta_5\), where \(\delta_5 \in (0, 1)\). If \(f \in \mathcal{S}_H\), \(0 \leq 1 - a_2 \delta_5 \leq |z| \leq 1 - a_1 \delta_5\) and \(\arg z - \arg z_0 \leq a_3 \delta_5\), then

\[
\frac{1}{M(a_1, a_2, a_3)} \|Df(z_0)\| \leq \|Df(z)\| \leq M(a_1, a_2, a_3) \|Df(z_0)\|,
\]

where \(M(a_1, a_2, a_3) = 2e^{(1+\alpha)(a_3 + \frac{1}{2} \log \frac{a_2-a_1}{a_1})}\).

**Proof of Theorem 2.** Let \(f \in \mathcal{S}_H^0(K)\). First we show that (ii) \(\Rightarrow\) (i). We assume that

\[
(2.22) \quad \frac{(1 - \rho^2)}{(1 - r^2)} \|Df(\rho \zeta)\| \leq \|Df(r \zeta)\| \leq \beta < 1 \quad \text{for} \quad \rho = \frac{x + r}{1 + xr}, \quad |\zeta| = 1,
\]

uniformly on \(r\) and \(\zeta\). Define \(x_1 = x\) and \(x_k\) for \(k = 2, 3, \ldots\), by

\[
1 + x_k \frac{1}{1 - x_k} = \left(1 + x \frac{1}{1 - x}\right)^k, \quad \text{i.e.,} \quad x_{k+1} = \frac{x + x_k}{1 + x x_k}.
\]

Note that \(\rho > r\) and thus, \(x_{k+1} > x_k\). Consequently, by (2.22), we have

\[
(2.23) \quad \frac{(1 - x_{k+1}^2)}{(1 - x_k^2)} \|Df(x_{k+1})\| \leq \beta < 1.
\]

Let \(\delta_6 \in (0, 1)\) such that

\[
\beta < \left(\frac{1 - x}{1 + x}\right)^{\delta_6}.
\]

Then, for \(j < k\), by (2.23),

\[
\frac{(1 - x_k^2)}{(1 - x_j^2)} \|Df(x_k)\| \leq \frac{(1 - x_k^2)}{(1 - x_{k-1}^2)} \|Df(x_{k-1})\| \times \frac{(1 - x_{k-1}^2)}{(1 - x_{k-2}^2)} \|Df(x_{k-2})\| \times \cdots \times \frac{(1 - x_{j+1}^2)}{(1 - x_j^2)} \|Df(x_{j+1})\| \leq \beta^{k-j} < \left(\frac{1 - x_k}{1 + x_k}\right)^{\delta_6} \left(\frac{1 - x_j}{1 + x_j}\right)^{-\delta_6}
\]

(2.24)

By calculations, for \(k = \{1, 2, \ldots\}\),

\[
\lambda_{\mathcal{D}}(x_k, x_{k+1}) = \lambda_{\mathcal{D}}(0, x),
\]
which, together with (2.24) and Lemma 5, yields that there is a constant $C_{18} > 0$ such that

$$\frac{\|D_f(\rho \zeta)\|}{\|D_f(r \zeta)\|} \leq C_{18} \left(\frac{1 - \rho}{1 - r}\right)^{\delta_6 - 1}.$$  

(2.25)

Hence, by (2.25) and [6, Theorem 1], we conclude that $\Omega$ is a radial John disk.

(i) $\Rightarrow$ (ii). Suppose that $\Omega = f(\mathbb{D})$ is a radial John disk. Then, by [6, Theorem 1], there are constants $C_{19} > 0$ and $\delta_7 \in (0, 1)$ such that, for each $\zeta \in \partial \mathbb{D}$ and for $0 \leq \rho \leq r < 1$,

$$\frac{(1 - \rho^2)\|D_f(\rho \zeta)\|}{(1 - r^2)\|D_f(r \zeta)\|} \leq C_{19} \left(\frac{1 - \rho}{1 - r}\right)^{\delta_7} = C_{19} \left(\frac{1 - x}{1 + rx}\right)^{\delta_7} \leq C_{19}(1 - x)^{\delta_7}.$$  

It is not difficult to see that $C_{19}(1 - x)^{\delta_7} < 1$ by taking $x$ sufficiently close to 1.

Next we show that (i) $\Rightarrow$ (iii). For $z = re^{i\theta} \in \mathbb{D}$ and $w = r_1e^{i\theta_1} \in B(z)$, by [6, Theorem 1] and Lemma B, we see that there are positive constants $C_{20}, C_{21}$ and $\delta_8 \in (0, 1)$ such that

$$|f(z) - f(w)| \leq |f(re^{i\theta}) - f(re^{i\theta_1})| + |f(re^{i\theta_1}) - f(r_1e^{i\theta_1})| \leq r \int_{\gamma'} \|D_f(re^{it})\| dt + \int_{r_1}^r \|D_f(\rho e^{i\theta_1})\| d\rho \leq C_{20}r \int_{\gamma'} \|D_f(re^{i\theta})\| dt + C_{21} \int_{r_1}^r \|D_f(re^{i\theta})\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta_8 - 1} d\rho \leq C_{20}r \|D_f(re^{i\theta})\| \ell(\gamma') + \frac{C_{21}}{\delta_8} \|D_f(re^{i\theta})\|(1 - r) \leq \left(\frac{2\pi C_{20} + C_{21}}{\delta_8}\right) \|D_f(re^{i\theta})\|(1 - r),$$

which gives that

$$\sup_{z \in \mathbb{D}, \ w \in B(z)} \frac{|f(z) - f(w)|}{(1 - |z|^2)\|D_f(z)\|} < \infty,$$

where $\gamma'$ is the smaller subarc of $\partial \mathbb{D}_r$ between $re^{i\theta}$ and $re^{i\theta_1}$.

Finally, we prove (iii) $\Rightarrow$ (i). For $z \in \mathbb{D}$ and $w_1, w_2 \in B(z)$, there is a positive constant $C_{22}$ such that

$$|f(w_1) - f(w_2)| \leq |f(w_1) - f(z)| + |f(w_2) - f(z)| \leq C_{22}(1 - |z|^2)\|D_f(z)\| \leq 16KC_{22}d_\Omega(f(z))$$

(by Lemma 6),

which implies that

$$\text{diam} f(B(z)) \leq 16KC_{22}d_\Omega(f(z)).$$  

(2.26)

By (2.26) and [6, Theorem 2], we conclude that $\Omega$ is a radial John disk. The proof of this theorem is complete.
Proof of Theorem 3. Let \( f = h + \overline{g} \in S^0_d(K, \Omega) \), where \( \Omega \) is a radial John disk. Assume that \( a_1 = re^{i\theta} \) and \( r_1e^{i\theta_1}, r_2e^{i\theta_2} \in B(a_1) \) with \( r_1 \leq r_2 \), where \( r = |a_1| \). Since \( \Omega \) is a radial John disk \( \Omega \), by [6, Theorem 1], we see that there are constants \( C_{23} > 0 \) and \( \delta_9 \in (0, 1) \) such that for each \( \zeta \in \partial \mathbb{D} \) and for \( 0 \leq r \leq \rho < 1 \),

\[
(2.27) \quad \|D_f(\rho \zeta)\| \leq C_{23}\|D_f(r \zeta)\| \left( \frac{1-\rho}{1-r} \right)^{\delta_9-1}.
\]

Then, by (2.27) and Lemma B, there is a positive constant \( C_{24} \) such that

\[
|f(r_2e^{i\theta_2}) - f(r_1e^{i\theta_1})| \leq |f(r_2e^{i\theta_2}) - f(re^{i\theta_2})| + |f(r_1e^{i\theta_1}) - f(re^{i\theta_1})| + |f(re^{i\theta_2} - f(re^{i\theta_1})|
\]

\[
= \int_r^{r_2} \|D_f(\rho e^{i\theta_2})\|d\rho + \int_r^{r_1} \|D_f(\rho e^{i\theta_1})\|d\rho + J
\]

\[
\leq C_{23} \left[ \int_r^{r_2} \|D_f(re^{i\theta})\| \left( \frac{1-\rho}{1-r} \right)^{\delta_9-1} d\rho + \int_r^{r_1} \|D_f(re^{i\theta})\| \left( \frac{1-\rho}{1-r} \right)^{\delta_9-1} d\rho \right] + J
\]

\[
\leq \frac{2C_{23}}{\delta_9}\|D_f(re^{i\theta})\|(1-r) + J,
\]

where

\[
J = r \int_{\gamma_0} \|D_f(re^{it})\| dt \leq C_{24}r \int_{\gamma_0} \|D_f(re^{i\theta})\| dt
\]

\[
\leq C_{24} |\theta_2 - \theta_1|\|D_f(re^{i\theta})\|
\]

\[
\leq 2\pi C_{24}\|D_f(re^{i\theta})\|(1-r),
\]

where \( \gamma_0 \) is the smaller subarc of \( \partial \mathbb{D}_r \) between \( re^{i\theta_1} \) and \( re^{i\theta_2} \). Combining the last two inequalities shows that

\[
|f(r_2e^{i\theta_2}) - f(r_1e^{i\theta_1})| \leq \left( \frac{2C_{23}}{\delta_9} + 2\pi C_{24} \right)\|D_f(re^{i\theta})\|(1-r)
\]

Hence there is a constant \( C_{25} > 0 \) such that

\[
(2.28) \quad \text{diam} B(a_1) \leq C_{25}(1 - |a_1|)\|D_f(a_1)\|.
\]

Moreover, by Lemmas 6 and B, we see that there is a constant \( C_{26} > 0 \) such that

\[
\text{diam} f(B(a_2)) \geq d_{\Omega}(f(a_2))
\]

\[
\geq \frac{1}{16K}(1 - |a_2|^2)\|D_f(a_2)\|
\]

\[
\geq \frac{1}{16K}(1 - |a_2|)\|D_f(a_2)\|
\]

\[
\geq \frac{C_{26}}{16K}(1 - |a_2|)\|D_f(|a_2|e^{i\theta})\|.
\]

(2.29)
By (2.28), (2.29) and Lemma 5, we conclude that
\[
\frac{\text{diam} f(B(a_1))}{\text{diam} f(B(a_2))} \leq \frac{16KC_{25}}{C_{26}} \frac{(1 - |a_1|) \| D_f(a_1) \|}{(1 - |a_2|) \| D_f(a_2e^{\theta}) \|} \\
\leq \frac{2^{5+\alpha}KC_{25}}{C_{26}} \left( \frac{1 - |a_1|}{1 - |a_2|} \right)^{\alpha - 1} \\
= \frac{2^{5+\alpha}KC_{25}}{C_{26}} \left( \frac{1 - |a_1|}{1 - |a_2|} \right)^\alpha
\]
and the proof of the theorem is complete. □

**Lemma 7.** For \( K \geq 1 \), suppose that \( f \in S_H(K) \). Let \( a_1, a_2 \) and \( a_3 \) be positive constants and let \( 0 < |z_0| = 1 - \delta \), where \( \delta \in (0, 1) \). Suppose further that \( 0 \leq 1 - a_2\delta \leq |z| \leq 1 - a_1\delta \) and \( |\arg z - \arg z_0| \leq a_3\delta \). Then
\[
|f(z) - f(z_0)| \leq \frac{K}{\alpha(1 + K)} \left[ \left( \frac{M(a_1, a_2, a_3)}{2} \right)^{\frac{1}{\alpha}} - 1 \right] (1 - |z_0|^2)|f_z(z_0)|,
\]
where \( M(a_1, a_2, a_3) \) is defined in Lemma B.

**Proof.** Follows from [6, Lemma 2], but for the sake of completeness, we include certain details.

Let \( \angle AOB = 2a_3\delta \) and \( z_1, z_2, z_3 \) line in the line \( OB \) with \( |z_1| \leq |z_2| = |z_0| \leq |z_3| \) (see Figure 2). Clearly the distance from \( O \) to \( B \) is less than 1. Then the length of

\[
\text{Figure 2}
\]
the circular arc from \( z_0 \) to \( z_2 \) is less than \( a_3\delta \). As in [6, Lemma 2], it is easy to see that
\[
\lambda_{\mathbb{D}}(z_0, z_2) < a_3, \quad \frac{|z_3 - z_1|}{1 - z_1 \overline{z_3}} \leq \frac{a_2 - a_1}{a_2} \quad \text{and} \quad \lambda_{\mathbb{D}}(z_0, z) \leq a_3 + \frac{1}{2} \log \frac{2a_2 - a_1}{a_1}.
\]
The desired conclusion follows if we use Lemma 1. □

The following result is an easy consequence of Lemmas 6 and 7.

**Corollary 2.** Under the hypotheses of Lemma 7, we also have
\[
|f(z) - f(z_0)| \leq \frac{16K^2}{\alpha(1 + K)} \left[ \left( \frac{M(a_1, a_2, a_3)}{2} \right)^{\frac{1}{\alpha}} - 1 \right] d_{f(\mathbb{D})}(f(z_0)),
\]
where \( M(a_1, a_2, a_3) \) is defined in Lemma B.

**Proof of Theorem 4.** Let \( z = re^{i\theta} \), \( \sigma = |\zeta_1 - \zeta_2| \) and \( \zeta_j = r_je^{i\theta_j} \) \((j = 1, 2)\) with \( r_1 \leq r_2 \).

**Step 1.** If \( r \leq \rho = 1 - 2\sigma \leq r_1 \leq r_2 \), then

\[
|\zeta_1 - \zeta_2| = |r_1 e^{i\theta_1} - r_2 e^{i\theta_2}|
= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}
= \sqrt{(r_1 - r_2)^2 + 4r_1r_2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)}
\geq 2\sqrt{r_1r_2} \left| \sin \frac{\theta_1 - \theta_2}{2} \right|
\geq \frac{2\rho |\theta_1 - \theta_2|}{\pi},
\]

which, together with [6, Theorem 2], Lemmas 6 and 7, imply that there are positive constants \( C_{27}, C_{28}, C_{29}, C_{30} \), and \( \delta_{10} \in (0, 1) \) such that

\[
|f(\zeta_1) - f(\zeta_2)| \leq |f(\zeta_1) - f(\rho e^{i\theta_1})| + |f(\zeta_2) - f(\rho e^{i\theta_2})| + |f(\rho e^{i\theta_1}) - f(\rho e^{i\theta_2})|
\leq C_{27} \left[ (1 - \rho) \| D_f(\rho e^{i\theta_1}) \| + (1 - \rho) \| D_f(\rho e^{i\theta_2}) \| \right]
+ \rho \int_{\gamma_1} \| D_f(\rho e^{it}) \| dt \quad \text{(by Lemma 7)}
\leq C_{27} \left[ (1 - \rho) \| D_f(\rho e^{i\theta_1}) \| + (1 - \rho) \| D_f(\rho e^{i\theta_2}) \| \right]
+ \rho \int_{\gamma_1} \| D_f(z) \| \left( \frac{1 - \rho}{1 - |z|} \right)^{\delta_{10} - 1} dt \quad \text{(by [6, Theorem 2])}
\leq \| D_f(z) \| \left( \frac{1 - \rho}{1 - |z|} \right)^{\delta_{10} - 1} \left[ C_{28}(1 - \rho) + C_{29} \ell(\gamma_1) \right]
\leq \| D_f(z) \| \left( \frac{1 - \rho}{1 - |z|} \right)^{\delta_{10} - 1} \left[ C_{28}(1 - \rho) + \frac{C_{29} \pi \sigma}{2} \right]
\leq C_{30} d_f(z) \left( \frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^{\delta_{10}} \quad \text{(by Lemma 6)}
\]

where \( \gamma_1 \) is the smaller subarc of \( \partial \mathbb{D}_\rho \) between \( \rho e^{i\theta_1} \) and \( \rho e^{i\theta_2} \), and

\[
\ell(\gamma_1) = \rho |\theta_1 - \theta_2| \leq \frac{\pi \sigma}{2}.
\]

**Step 2.** If \( r_1 < \rho = 1 - 2\sigma \), then, by Lemma 7, there are positive constants \( C_{31} \) and \( C_{32} \) such that

\[
(2.30) \quad \| D_f(\zeta) \| \leq C_{31} \| D_f(\zeta_1) \| \leq C_{32} \| D_f(\rho e^{i\theta_1}) \|,
\]
where $|\zeta - \zeta_1| \leq \sigma$. We see that there are positive constants $C_{33}$ and $\delta_{11} \in (0,1)$ such that

$$|f(\zeta_1) - f(\zeta_2)| \leq \int_{[\zeta_1, \zeta_2]} \|D_f(\zeta)\| |d\zeta| \leq C_{32}\|D_f(\rho e^{i\theta_1})\| |\zeta_1 - \zeta_2| \quad \text{(by (2.30))}$$

$$\leq C_{33}\|D_f(z)\| |\zeta_1 - \zeta_2| \left(\frac{1 - \rho}{1 - r}\right)^{\delta_{11}-1} \quad \text{([6, Theorem 2])}$$

$$\leq 2^{3+\delta_{11}}KC_{33}d_G(f(z)) \left(\frac{|\zeta_1 - \zeta_2|}{1 - |z|}\right)^{\delta_{11}} \quad \text{(by Lemma 6).}$$

**Step 3.** If $1 - 2\sigma < r$, then, by [6, Theorem 2], we conclude that there are constants $C_{34} > 0$ and $\delta_{12} \in (0,1)$ such that

$$|f(\zeta_1) - f(\zeta_2)| \leq 2^{\delta_{12}}C_{34}d_G(f(z)) \left(\frac{|\zeta_1 - \zeta_2|}{1 - |z|}\right)^{\delta_{12}}.$$ 

The proof of this theorem is complete. \qed

The following result is an improvement of [6, Lemma 3].

**Lemma 8.** Let $f \in SH(K, \Omega)$, where $G = f(\mathbb{D})$ is a bounded domain. If there are constants $C_{35} > 0$ and $\delta_{13} \in (0,1)$ such that for each $\zeta \in \partial \mathbb{D}$ and for $0 \leq r \leq \rho < 1$,

$$(2.31) \quad \|D_f(\rho \zeta')\| \leq C_{35}\|D_f(\rho \zeta)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta_{13}-1},$$

then, for $a \in \mathbb{D}$, we have

$$\text{diam } f(I(a)) \leq 32KC_{36}d_G(a), \quad C_{36} = 2\pi e^{(1+\alpha)\pi} + \frac{2C_{35}e^{(1+\alpha)\pi} + C_{35}}{\delta_{13}},$$

where $I(a) = \{z \in \partial \mathbb{D} : |\arg z - \arg a| \leq \pi(1 - |a|)\}$.

**Proof.** For $a \in \mathbb{D}$, let $a = \rho \zeta$ with $\rho = |a|$. For $z \in I(a)$, by Lemma B, we have

$$|f(z \rho) - f(\rho \zeta)| \leq \int_{\gamma'} \rho\|D_f(\rho \zeta)\| |d\xi| \leq 2e^{(1+\alpha)\pi} \rho\|D_f(\rho \zeta)\| \ell(\gamma'),$$

where $\gamma'$ is the smaller subarc of $\partial \mathbb{D}$ between $\rho z$ and $\rho \zeta$, so that

$$\ell(\gamma') = \int_{\gamma'} |d\xi| = \rho|\arg(\rho \zeta) - \arg z| \leq \pi \rho(1 - \rho) \leq \pi(1 - \rho).$$

Therefore,

$$(2.32) \quad |f(z \rho) - f(\rho \zeta)| \leq 2\pi e^{(1+\alpha)\pi}(1 - \rho)\|D_f(\rho \zeta)\|.$$
Next, we have
\[
|f(z\rho) - f(z)| \leq \int_{\rho}^{1} \|D_f(tz)\| dt
\]
\[
\leq C_{35} \int_{\rho}^{1} \|D_f(\rho z)\| \left( \frac{1-t}{1-\rho} \right)^{\delta_{13}-1} dt \quad \text{(by (2.31))}
\]
\[
= \frac{C_{35}}{\delta_{13}} (1-\rho) \|D_f(\rho z)\|
\]
and, finally,
\[
|f(\zeta\rho) - f(\zeta)| \leq \int_{\rho}^{1} \|D_f(t\zeta)\| dt
\]
\[
\leq C_{35} \int_{\rho}^{1} \|D_f(\rho \zeta)\| \left( \frac{1-t}{1-\rho} \right)^{\delta_{13}-1} dt \quad \text{(by (2.31))}
\]
\[
= \frac{C_{35}}{\delta_{13}} (1-\rho) \|D_f(\rho \zeta)\|.
\]
Again, for \( z \in I(a) \), by (2.32), (2.33), (2.34) and the triangle inequality, we obtain
\[
|f(\zeta) - f(z)| \leq |f(\zeta - f(\rho z))| + |f(z) - f(\rho z)| + |f(\rho \zeta) - f(\zeta)|
\]
\[
\leq C_{36} (1-\rho) \|D_f(\rho \zeta)\|
\]
\[
\leq 16KC_{36}d_G(a) \quad \text{(by Lemma 6),}
\]
which in turn implies that \( \text{diam} f(I(a)) \leq 32KC_{36}d_G(a) \) and the proof of the lemma is complete.

For \( p \in (0, \infty] \), the \textit{generalized Hardy space} \( H^p_\bar{g}(\mathbb{D}) \) consists of all those functions \( f : \mathbb{D} \to \mathbb{C} \) such that \( f \) is measurable, \( M_p(r, f) \) exists for all \( r \in (0, 1) \) and \( \|f\|_p < \infty \), where
\[
\|f\|_p = \begin{cases} 
\sup_{0<r<1} M_p(r, f) & \text{if } p \in (0, \infty) \\
\sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty
\end{cases}
\]
and
\[
M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.
\]
We refer to [7] for more details on \( H^p_\bar{g}(\mathbb{D}) \).

\textbf{Proof of Theorem 5. Case 1.} Let \( f = h + \bar{g} \in S_{H^2}(K, G) \), where \( G \) is a bounded Pommerenke interior domain. Then, by definition, (1.7) holds and thus (see for example, [29, Proof of Theorem 3]), there are constants \( \rho_0 \in (0, 1) \) and \( \beta_1 > 0 \) such that, for \( \rho_0 \leq \rho < 1 \) and \( \theta \in [0, 2\pi] \),
\[
\text{Re} \left[ e^{i\theta} \frac{h''(\rho e^{i\theta})}{h'(\rho e^{i\theta})} \right] \geq \frac{1 - \beta_1}{1 - \rho}.
\]
For $\rho_0 \leq r \leq \rho < 1$, by integrating both sides of (2.35), we have

$$(1 - r)^{\beta_1 - 1} |h'(re^{i\theta})| \leq (1 - \rho)^{\beta_1 - 1} |h'(\rho e^{i\theta})|;$$

which, by (2.3), deduces that

$$(2.36) \quad (1 - r)^{\beta_1 - 1} \|D_f(re^{i\theta})\| \leq \frac{2K}{1 + K} (1 - \rho)^{\beta_1 - 1} \|D_f(\rho e^{i\theta})\|.$$

For $\rho \in (\rho_0, 1)$, we choose a positive integer $N$ and $r_0, \ldots, r_N$ with $r_N = \rho_0 < r_{N-1} < \cdots < r_1 < r_0 = \rho$ such that, for $n \in \{0, 1, \ldots, N - 1\}$,

$$2^n (1 - \rho) \leq 1 - r_n < 2^{n+1}(1 - \rho).$$

For $\theta \in [0, 2\pi)$, let

$$I(r_n e^{i\theta}) = \{ \zeta \in \partial \mathbb{D} : |\arg \zeta - \theta| \leq \pi(1 - r_n) \}.$$  

For $2 \leq n \leq N$ and $e^{it} \in I(r_n e^{i\theta}) \setminus I(r_{n-1} e^{i\theta})$, let $\varphi = t - \theta$. Then, for $2 \leq n \leq N$,

$$(2.37) \quad \pi(1 - r_{n-1}) \leq |\varphi| \leq \pi(1 - r_n).$$

By the assumption, we let

$$(2.38) \quad c_p = \sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_G(w_1, w_2)} \right\} < \infty,$$

where $\gamma_r$ is given by (1.5). Then, by (1.8), (2.38) and [6, Theorem 4], $\|D_f\| \in H^1_0(\mathbb{D})$. Hence, for $n \in \{0, 1, \ldots, N - 1\}$, by (1.8), (2.38), Lemma 8 and [6, Inequality (2.3)], there is a positive constant $C_{37}$ such that

$$(2.39) \quad \frac{1}{K} \int_{I(r_n e^{i\theta})} \|D_f(e^{it})\| dt \leq \int_{I(r_n e^{i\theta})} \ell(D_f(e^{it})) dt$$

$$\leq \int_{I(r_n e^{i\theta})} |df(e^{it})| = \ell(I(r_n e^{i\theta}))$$

$$\leq c_p \text{diam}(f(I(r_n e^{i\theta}))) \quad \text{(by (2.38))}$$

$$\leq C_{37} c_p d_G(r_n e^{i\theta}) \quad \text{(by (1.8) and Lemma 8)}$$

$$\leq \frac{2KC_{37}c_p}{1 + K}(1 - r_n) \|D_f(r_n e^{i\theta})\| \quad \text{(by [6, Inequality (2.3)])}.$$  

Let $I_n(\theta) = I(r_n e^{i\theta})$. Since $\partial \mathbb{D} = I_0(\theta) \cup (I_1(\theta) \setminus I_0(\theta)) \cap \cdots \cap (I_N(\theta) \setminus I_{N-1}(\theta))$, by (2.36) and (2.37), we see that

$$(2.40) \quad \Lambda_f = \int_0^{2\pi} \|D_f(e^{it})\| \frac{1 - \rho^2}{|e^{it} - \rho e^{i\theta}|^2} dt \leq J_0 + \sum_{n=1}^{N} J_n$$

where, by (2.39),

$$(2.41) \quad J_0 = \frac{2}{1 - \rho} \int_{I_0(\theta)} \|D_f(e^{it})\| dt \leq \frac{4K^2C_{37}c_p}{1 + K} \|D_f(\rho e^{i\theta})\|.$$
In this case, Theorem 5 follows from the last two inequalities.

On the other hand, for

Thus,

By (2.36), (2.40), (2.41) and the last inequality, we conclude that

and

By (2.36), (2.40), (2.41) and the last inequality, we conclude that

Thus,

Case 2. For \( \rho \in [0, \rho_0] \) and \( \theta \in [0, 2\pi] \), by [6, Theorem 4], we have

On the other hand, for \( \theta \in [0, 2\pi] \),

In this case, Theorem 5 follows from the last two inequalities.

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S. L. CHEN, COLLEGE OF MATHEMATICS AND STATISTICS, HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE’S REPUBLIC OF CHINA.
E-mail address: mathechen@126.com

S. PONNUSAMY, INDIAN STATISTICAL INSTITUTE (ISI), CHENNAI CENTRE, SETS (SOCIETY FOR ELECTRONIC TRANSACTIONS AND SECURITY), MGR KNOWLEDGE CITY, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.
E-mail address: samy@isichennai.res.in, samy@iitm.ac.in