1/$N_c$ expansion for the partition function in four fermion models*

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Abstract

We present a derivation of the bosonic contribution to the thermodynamical potential of four fermion models by means of a $1/N_c$-expansion of the functional integral defining the partition function. This expansion turns out to be particularly useful to correct the mean field approximation especially at low temperatures, where the relevant degrees of freedom are low-mass bosonic excitations (pseudogoldstones).

1 Introduction

Four fermion models, like the NJL [1] and the Gross-Neveu model [2], have been widely studied, especially in connection with chiral symmetry breaking and restoration at finite densities and temperatures. The thermodynamics of these models may offer useful hints in understanding the properties of thermal QCD, especially those related to the chiral symmetry. Such a knowledge is expected to be important to interpret heavy-ion collisions at relativistic energies and for the description of early phases of cosmology.

Usually such studies are carried out in the mean field approximation, for instance by using the Dolan-Jackiw formalism [3] of imaginary-time Green functions.

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A review of the applications of the NJL model to hadronic and nuclear physics can be found in the report by Vogl and Wise [4], to which we also refer for the vast literature on the subject. A calculation of the thermodynamical potential for the NJL model in the grand canonical ensemble, in the mean field approximation, can be found for instance in ref. [5].

The mean field approximation neglects fluctuations around it, and it excludes in such a way possible bosonic contributions to the thermodynamical potential. On the other hand such fluctuations, as they stand out from possible resonant or non resonant bosonic scattering states, can be expected to be of great importance in the physical problem. For instance the pions, which are pseudogoldstones related to spontaneous chiral symmetry breaking, are expected to dominate the low energy properties of the system, as they correspond to the lowest mass bosonic excitations. We recall that in non-relativistic statistical mechanics a resembling situation is the calculation of the second virial coefficient, which is related to time-delay in the collision, and, in particular, bound states (two molecules combining to form a new molecule) lead to a modification of the system pressure [6].

Unfortunately, even in a simple field theoretic model, such as NJL, exact analytical calculations appear unfeasible and one has to resort to approximate methods, such as large-N expansion. Such type of expansion has been used since a long time in statistical field theory. Well-known applications are to the $O(N)$ symmetric field theory [7] and to the Gross-Neveu model [8].

A calculation of fluctuations has been recently carried out [9] for the NJL model by evaluating the thermodynamical potential through the method of coupling constant integration, as developed in the book of Fetter and Walecka [10], by including terms to order $1/N_c$, where $N_c$ is the number of colors ($N_c = 3$ in QCD). The $1/N_c$ contributions are taken into account through diagrammatic method. The use of $1/N_c$ expansion goes back to Witten [11] and a review is given in the Erice lectures by Coleman [12].

The use of diagrammatic methods, besides its cumbersome features, has also a delicate aspect, as it may easily destroy general symmetries of the system by improper selection of the set of diagrams [13]. It is therefore strongly preferable to recur to procedures which automatically guarantee conservation of the system symmetries.

In this note we shall develop a method based on the integral functional formulation of field thermodynamics which leads to straightforward calculation of the $1/N_c$ expansion of the partition function. Within this method conservation of symmetries is consistently and naturally guaranteed.
2 $1/N_c$ expansion

Let us consider a model in $D = d + 1$ dimensions defined by the following four fermion lagrangian \[1, 2\]

$$
L = \bar{\psi} \left( i\hat{\partial} - M \right) \psi + \frac{g^2}{2} \left[ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \vec{\tau} \psi)^2 \right]
$$

where $M$ is the bare fermion mass, $\tau_i$ are the matrices of the fundamental representation of the flavor group $SU(N_f)$, and $\psi$ consists of $N_c$ separate $N_f$-dimensional representations of $SU(N_f)$.

The generating functional (evaluated at vanishing sources) is

$$
Z = \int D(\bar{\psi} \psi) e^{i \int d^D x L}
$$

By using the identities

$$
\int D(\sigma) e^{-\frac{i}{2} \int d^D x (\sigma - g\bar{\psi} \psi)^2} = \text{cost}
$$

$$
\int D(\vec{\pi}) e^{-\frac{i}{2} \int d^D x (\vec{\pi} - ig\bar{\psi} \gamma_5 \vec{\tau} \psi)^2} = \text{cost}'
$$

and by inserting them in eq. (2), we can write

$$
Z = \int D(\bar{\psi} \psi) D(\sigma) D(\vec{\pi}) e^{i \int d^D x \left[ \bar{\psi} \left( i\hat{\partial} - M + g\sigma + ig\gamma_5 \vec{\tau} \cdot \vec{\pi} \right) \psi - \frac{1}{2}(\sigma^2 + \vec{\pi}^2) \right]}
$$

By carrying out the integral over the fermion fields, we obtain an effective action in which only the bosonic degrees of freedom do appear

$$
Z = \int D(\sigma) D(\vec{\pi}) e^{i S_B} \equiv e^{i W}
$$

By redefining the $\sigma$ field with the following shift

$$
g\sigma \rightarrow g\sigma + M
$$

we can write

$$
S_B = \int d^D x \left[ -\frac{1}{2} (\sigma^2 + \vec{\pi}^2) (1 + \delta Z_D) - \frac{M\sigma}{g} \right] - i \log \det \left( i\hat{\partial} + g\sigma + ig\gamma_5 \vec{\tau} \cdot \vec{\pi} \right)
$$

where $\delta Z_D$ is an counterterm coming from the renormalization of the fermion loop [2]. In NJL $\delta Z_4 = 0$, since the theory is non-renormalizable. In this case the regularization of infinities is obtained by means of a momentum cut-off.
To simplify the notations we define

\[ N \equiv N_f N_c \quad ; \quad \lambda \equiv N g^2 \quad ; \quad \alpha \equiv \frac{M}{Ng^2} \] (8)

\[ \phi \equiv (g\sigma, g \vec{\pi}) \quad ; \quad a \equiv (1, i\gamma_5 \vec{\tau}) \] (9)

By using the formal identity \( \log \det = tr \log \), and carrying out explicitly the trace over colour indices, we can write

\[
S_B = \int d^Dx \left[ -\frac{N}{2\lambda} \phi^2 (1 + \delta Z_D) - N\alpha \phi^j \delta j_1 \right] - iNc \text{Tr} \log (i\hat{\partial} + \phi \cdot a)
\]

\[ \equiv \int d^Dx L_B(\phi) \] (10)

At this point, by following the method of ref.[14], we couple an external constant source \( J \) to the field \( \phi(x) \), and expand \( \phi(x) \) around its vacuum expectation value \( \bar{\phi} \)

\[ \phi(x) = \phi'(x) + \bar{\phi} \] (11)

\[ \bar{\phi}[J] = \langle 0^+ | \phi(x) | 0^- \rangle_J = \frac{\delta W}{\delta J} \] (12)

Since \( J \) is constant, from the Lorentz invariance of the vacuum, it follows that \( \bar{\phi} \) is constant too. By supposing \( \bar{\phi}[J] \) to be invertible, we can define the effective action \( \Gamma[\bar{\phi}] \) as the Legendre transform of \( W[J] \)

\[ \Gamma[\bar{\phi}] = W[J[\bar{\phi}]] - \int d^Dx J[\bar{\phi}] \bar{\phi} \equiv -\mathcal{V}(\bar{\phi}) \int d^Dx \] (13)

It can be easily shown that, at vanishing sources, \( \bar{\phi} \) must satisfy the stationary equation

\[ \frac{\delta \mathcal{V}(\phi)}{\delta \bar{\phi}} \bigg|_{\phi=\bar{\phi}} = 0 \] (14)

By using eq.(12), \( W[J] \) can be written as

\[ W[J] \equiv W_0[J] + W_1[J] \] (15)

where

\[ W_0[J] = \int d^Dx \left( L_B(\bar{\phi}) + J \bar{\phi} \right) \] (16)

and \( W_1[J] \) satisfies the integral equation [14]

\[ W_1[J] \equiv -i \log \int D(\phi') \exp \left[ i \int d^Dx \left( L_B^{(2)}(\phi', \bar{\phi}) - \phi' \frac{\delta W_1}{\delta \bar{\phi}} \right) \right] \] (17)

and \( L_B^{(2)} \) is obtained from the lagrangian \( L_B(\phi' + \bar{\phi}) \) after having subtracted constant and linear terms in \( \phi' \).
By using eq. (11), eq. (10) becomes
\[ S_B = \int d^Dx \left[ -\frac{N}{2\lambda} (\phi' + \bar{\varphi})^2 (1 + \delta Z_D) - N\alpha (\phi' + \bar{\varphi})^j \delta j_1 \right] \]
\[ -iN_c \text{Tr} \log (i\dot{\varphi} + \bar{\varphi} \cdot a + \phi' \cdot a) \] (18)

We now define
\[ (i\dot{\varphi} + \bar{\varphi} \cdot a)G(x_1 - x_2) = \delta^D(x_1 - x_2) \] (19)

The last term in eq. (18) becomes
\[ -iN_c \text{Tr} \log (i\dot{\varphi} + \bar{\varphi} \cdot a + \phi' \cdot a) = -iN_c \text{Tr} \log (i\dot{\varphi} + \bar{\varphi} \cdot a) - iN_c \text{Tr} \log (1 + G\phi \cdot a) \] (20)

Thus
\[ W_0[J] = \int d^Dx \left[ -\frac{N}{2\lambda} \bar{\varphi}^2 (1 + \delta Z_D) - N\alpha \bar{\varphi}^j \delta j_1 + J\bar{\varphi} \right] - iN_c \text{Tr} \log (i\dot{\varphi} + \bar{\varphi} \cdot a) \] (21)

and its final form, at \( J = 0 \), is the standard mean field fermionic term
\[ W_0 = \int d^Dx \left[ -\frac{N}{2\lambda} \bar{\varphi}^2 (1 + \delta Z_D) - N\alpha \bar{\varphi}^j \delta j_1 + iN D^2 \right] \int \frac{d^Dp}{(2\pi)^D} \log (p^2 + \bar{\varphi}^2) \] (22)

which is the only part which survives in the \( N_c \to +\infty \) limit, since \( W_1 \) is of order \( 1/N_c \) with respect to \( W_0 \).

To evaluate \( W_1 \), one has to expand \( S_B \) in eq. (18) in powers of \( \phi' \), starting from the quadratic terms. Now, by ordering the terms of the argument of the exponential in eq. (17) in powers of \( 1/N_c \) (instead of the \( \hbar \) formal series of ref. [14]), it turns out that the leading term is just the quadratic one (this can be verified by the rescaling \( \phi' \to \phi'/\sqrt{N_c} \)).

Thus, by keeping only the leading term of the 1/\( N_c \) expansion, \( W_1 \) is given by
\[ e^{iW_1} = \int \mathcal{D}(\phi') \exp \left\{ -\frac{iN}{2\lambda} \int d^Dx \phi'^2 (1 + \delta Z_D) \right. \]
\[ -iN_c \int d^Dx_1 d^Dx_2 \text{Tr} \left[ G(x_1 - x_2)\phi'(x_1) \cdot aG(x_2 - x_1)\phi'(x_2) \cdot a \right] \] (23)

By Fourier transforming and evaluating the traces in the previous equation we finally obtain
\[ W_1 = \sum_j \frac{i}{2} \int d^Dx \int \frac{d^Dp}{(2\pi)^D} \log \left[ \frac{iN}{2\lambda} D_j^{-1}(p) \right] \] (24)

where \( j \) labels each scalar/pseudoscalar field \( \phi'_j \) and \( D_j^{-1}(p) \) is given by
\[ D_j^{-1}(p) = i \left[ (1 + \delta Z_D) + i\Pi_j(p) \right] \]
\[ = i \left[ (1 + \delta Z_D) - iD\lambda \int \frac{d^Dq}{(2\pi)^D} \frac{q \cdot (q + p) \pm \bar{\varphi}^2}{(p + q)^2 - \bar{\varphi}^2(q^2 - \bar{\varphi}^2)} \right] \] (25)
where $\pm$ refer to the scalar and pseudoscalar self energy $\Pi_j(p)$ respectively.

Notice that, apart from the linear term in $\bar{\phi}_1$, both $W_0$ and $W_1$ depend on the chirally invariant combination $\bar{\phi}^2 = g^2(\bar{\sigma}^2 + \bar{\pi}^2)$, which has to be determined by the eq.(14). From this condition it turns out that the physical value is for zero pseudoscalar components, i.e.

$$\bar{\phi} = (g\bar{\sigma}, 0)$$  \hspace{1cm} (26)

3 \ Free energy density at order $1/N_c$

Taking into account the bosonic fluctuations is particularly interesting also as far as the thermodynamics is concerned. In fact, although depressed by a $1/N_c$ factor, pion excitations should dominate the low-temperature equilibrium properties of strong interacting matter (this is particularly expected in the chiral limit, although there are limitations in applying this method for zero current quark masses in low dimensions, due to well known infrared problems).

The finite temperature extension of the formalism is straightforward. In fact, for imaginary times, one has simply to replace

$$V_D \to -i\beta \int d^d x = -i\beta V_d \quad ; \quad \int \frac{d^D p}{(2\pi)^D} \to \frac{i}{\beta} \sum_{n=-\infty}^{n=+\infty} \int \frac{d^d p}{(2\pi)^d}$$  \hspace{1cm} (27)

and thus, at zero sources, the effective potential, defined by the eq.(13), is

$$V_\beta = -i\frac{W_\beta}{\beta V_d} = -\log \frac{Z_\beta}{\beta V_d}$$  \hspace{1cm} (28)

which is the free energy density $F$.

By means of the $1/N_c$ expansion discussed in the previous paragraph, $F$ can be separated in

$$F = F^F + \sum_j F_j^B$$  \hspace{1cm} (29)

with

$$F^F = \frac{N}{2\lambda} \bar{\phi}^2 (1 + \delta Z_D) + N\alpha \bar{\phi} - \frac{ND}{2\beta} \sum_n \int \frac{d^d p}{(2\pi)^d} \log \left( \bar{\phi}^2 + \bar{p}^2 + \omega_n^2 \right)$$

$$= V_0(\bar{\phi}) - \frac{ND}{2\beta} \int \frac{d^d p}{(2\pi)^d} \log \left( 1 + e^{-\beta \sqrt{\bar{p}^2 + \bar{\phi}^2}} \right)$$  \hspace{1cm} (30)

$$F_j^B = \frac{1}{2\beta} \sum_n \int \frac{d^d p}{(2\pi)^d} \log \left[ \frac{iN}{2\lambda} D_j^{-1}(p_n, \bar{p}) \right]$$  \hspace{1cm} (31)
where the inverse bosonic propagator can be calculated at finite temperature by means of standard methods \cite{15}

\begin{equation}
D^{-1}(p_0, \vec{p}) = D_0^{-1}(p_0, \vec{p}) - iD\lambda \int \frac{d^dq}{(2\pi)^d} \frac{n_F(E_q)}{E_q}
\end{equation}

\begin{equation}
+ \frac{D\lambda}{4} (p^2 - \epsilon^2_M) \int \frac{d^dq}{(2\pi)^d} \left\{ \frac{n_F(E_q)}{E_q} \left[ \frac{1}{(E_q - p_0)^2 - E^2_{q+p}} + \frac{1}{(E_q + p_0)^2 - E^2_{q+p}} \right] \right. \\
+ \frac{n_F(E_{q+p})}{E_{q+p}} \left[ \frac{1}{(E_{q+p} - p_0)^2 - E^2_q} + \frac{1}{(E_{q+p} + p_0)^2 - E^2_q} \right] \right\}
\end{equation}

where

\begin{equation}
p_0 = i\nu_n = 2n\pi i/\beta ; \quad n_F(E) = \frac{1}{e^{\beta E} + 1} ; \quad E_p = \sqrt{p^2 + \varphi^2}
\end{equation}

and $\epsilon^2_M = 4\varphi^2$ for the scalar and $\epsilon^2_M = 0$ for the pseudoscalars.

The term $F_F$ is the standard mean field term coming from the one-loop calculations and it is purely fermionic, whereas the second term, $F_B$, is the contribution of bosonic fluctuations to the free energy. Expression (31) and (30) put in eq.(29) give the equation of state when evaluated at the solution of the gap equation (14) at finite temperature, which is

\begin{equation}
\frac{\delta F(\varphi)}{\delta \varphi} \bigg|_{\varphi = \bar{\varphi}} = 0
\end{equation}

In particular, since the free energy $F(\varphi)$ is a convex function of $\varphi$, $\bar{\varphi}$ must be the absolute minimum of $F$ and this gives the evolution of the fermion condensate with temperature. The procedure adopted as well as the expressions obtained are general, although the bosonic part of the free energy may be difficult to evaluate in the form of eq. (31), because of the sum over discrete energies $p_0n$. It is more convenient for calculations to transform the sums into integrals by a two-steps standard procedure (see for instance \cite{15}). The first step is to use the residue theorem to trade the bosonic sums for an integral over a circuit around the imaginary axis, and the second step is rotating along a circuit around the real axis.

Namely one can use the relations $(n_B(z) = 1/[\exp(\beta z) - 1])$

\begin{equation}
\frac{1}{\beta} \sum_n f(i\nu_n) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \left[ \int_{-i\infty + \epsilon}^{+i\infty + \epsilon} dz f(z)n_B(z) + \int_{+i\infty - \epsilon}^{-i\infty - \epsilon} dz f(z)n_B(z) \right] \\
= \lim_{\epsilon \to 0} \frac{1}{\pi i} \int_0^{+\infty} d\omega n_B(\omega) [f(\omega + i\epsilon) - f(\omega - i\epsilon)] + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega f(i\omega)
\end{equation}

provided $f(z) \exp(-\beta|z|)$ vanishes sufficiently fast at infinity, $f(z)$ has no singularities on the complex plane apart from a cut along the real axis, and $f(z) = f(-z)$ (as it turns out to be the case for the inverse propagator $iD^{-1}(z, \vec{p})$).
Therefore, the bosonic term can be cast in the form

\[
F^B_j = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int \frac{d^d p}{(2\pi)^d} \int_0^{+\infty} d\omega \ n_B(\omega) \log \left[ \frac{i D^{-1}_j(\omega + i \epsilon, \vec{p})}{i D^{-1}_j(\omega - i \epsilon, \vec{p})} \right] 
\]

\[
+ \frac{1}{2\pi} \int \frac{d^d p}{(2\pi)^d} \int_0^{+\infty} d\omega \ \log \left[ i D^{-1}_j(i\omega) \right]
\]

(36)

apart from infinities independent on \( \beta \) and \( \bar{\varphi} \).

Furthermore, we notice that the function \( g(z) = iD^{-1}(z, \vec{p}) \) satisfies the Schwarz reflection principle, \( g^*(z) = g(z^*) \). Thus, by reintroducing the explicit dependence on \( \bar{\varphi} \) and \( \beta \), we can write

\[
F^B_j(\bar{\varphi}, \beta) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int \frac{d^d p}{(2\pi)^d} \int_0^{+\infty} d\omega \ n_B(\omega) \left[ \text{arg} \left[ i D^{-1}_j(\omega + i \epsilon, \vec{p}; \bar{\varphi}, \beta) \right] - \pi \right]
\]

\[
+ \frac{1}{2\pi} \int \frac{d^d p}{(2\pi)^d} \int_0^{+\infty} d\omega \ \log \left[ i D^{-1}_j(i\omega; \bar{\varphi}, \beta) \right]
\]

(37)

where \( \text{arg}[f] \) is the argument \( \theta \in [0, 2\pi) \) of the complex number \( f \equiv |f| \exp(i\theta) \).

We notice that (i) the second term of eq. (37) would be the zero temperature term for a free boson gas, whereas now it is temperature dependent since \( iD^{-1} \) is a function of \( \beta \); (ii) this term may need renormalization at \( T = 0 \), due to momentum divergencies (in \( d = 3 \) divergent integrals are understood to be regulated by a momentum cutoff). The first term, instead, is finite at any \( T \), and tends to zero for \( T \to 0 \).

4 Conclusions

We have presented a functional integral procedure to calculate the contribution of bosonic fluctuations to the partition function in theories with four-fermion interactions by means of a \( 1/N_c \) expansion. Such a procedure has the advantage, as compared to diagrammatic methods, to automatically guarantee conservation of the system symmetries, through the consistent use of the effective action formalism.

Our expression for the bosonic term in the free energy is given in eq.(37), as a sum of a finite term which vanishes at zero temperature and of a term which does not vanish in that limit, where it must be regularized.

Fluctuations around the mean field are of greatest physical importance in connection to chiral symmetry breaking, manifesting themselves through bosonic excitations, some of which, by virtue of the Goldstone theorem, correspond to the lowest bosonic mass excitations, and as such they dominate the thermodynamic behaviour at low temperatures where chiral symmetry is still far from being restored.
Note added: After completing this work we have seen a recent preprint by E.N.Nikolov et al. \[\text{[hep-ph/9602274]}\] where the functional formalism is applied to the NJL model at zero temperature.

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