Stability Analysis of the Observer Error of an In-Domain Actuated Vibrating String

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Abstract—In this letter, the behaviour of the observer error of an in-domain actuated vibrating string, where the observer system has been designed based on energy considerations exploiting a port-Hamiltonian system representation for infinite-dimensional systems, is analysed. Thus, the observer-error dynamics are reformulated as an abstract Cauchy problem, which enables to draw conclusions regarding the well-posedness of the observer-error system. Furthermore, we show that the observer error is asymptotically stable by applying LaSalle’s invariance principle.

Index Terms—Distributed parameter systems, stability of linear systems, observers for linear systems.

I. INTRODUCTION

A very popular research discipline in control theory is the extension of control methodologies originally developed for systems that are described by ordinary differential equations (ODEs) to systems governed by partial differential equations (PDEs); however, with regard to stability investigations this extension is accompanied by a significant rise of complexity, see, e.g., [1] for a comprehensive framework for the stability analysis of infinite-dimensional systems. Therefore, a lot of research effort has been invested in this topic, where for example the stability analysis of mechanical systems with certain boundary conditions has been addressed. For instance, in [2] the stability of an Euler-Bernoulli beam subjected to nonlinear damping and a nonlinear spring at the tip is analysed, whereas [3] is concerned with the stability behaviour of a gantry crane with heavy chain and payload. Furthermore, the proof of stability of a Lyapunov-based control law as well as a Lyapunov-based observer design for an in-domain actuated Euler-Bernoulli beam has been presented in [4].

A well-known methodology, that has also been extended to the infinite-dimensional scenario, is the combination of the port-Hamiltonian (pH) system representation with energy-based control. In this regard, in particular a pH-system representation based on an underlying jet-bundle structure, see, e.g., [5]–[7], as well as a formulation exploiting Stokes-Dirac structures, see, e.g., [8], [9], have turned out to be especially suitable. Both of these approaches, where the main difference is the choice of the variables, have their advantages, which are compared and discussed in [10] or [11] for instance. In fact, with respect to boundary-control systems, a lot of literature is available, see, e.g., [12], [13] and [14], [15], where boundary controllers based on the well-known energy-Casimir method are designed within the jet-bundle and the Stokes-Dirac approach, respectively. Moreover, recently the pH-system description has also been exploited with regard to the observer design, see, e.g., [16], where a pH-based observer design for boundary-control systems is presented within the Stokes-Dirac scenario. In light of the observer design, course stability investigations play an important role, since it must be ensured that the observer error tends to zero.

In [11], a control-design procedure based on the energy-Casimir method together with an observer design exploiting the pH-system representation has been presented within the jet-bundle framework as well as within the Stokes-Dirac scenario for infinite-dimensional systems with in-domain actuation. Furthermore, the design procedures have been demonstrated and compared by means of an in-domain actuated vibrating string, where it should be stressed that regarding the dissipation rate of the observer-error energy both approaches yield the same result; however, the investigation regarding the asymptotic stability of the observer error – which is of course essential – has only been sketched. Therefore, the aim of this letter is to carry out the stability investigation of the observer error of this system in detail. To this end, first of all, in Section II we summarise the observer design that exploits the pH-system representation based on a jet-bundle structure, while in Section III the observer design is explicitly demonstrated for an in-domain actuated vibrating string. Thus, the main contribution of this letter is to verify the asymptotic stability of the observer error, where it is necessary to i) investigate the well-posedness, see Section IV-A, and ii) apply LaSalle’s invariance principle for infinite-dimensional systems, see Section IV-B.

II. OBSERVER IDEA BASED ON A PH-FRAMEWORK

With respect to the observer design, see [11, Sec. V], we intend to exploit a pH-system description for
infinite-dimensional systems with 1-dimensional spatial domain, which is equipped with the spatial coordinate \( z \in [0, L] \). The system representation is based on an underlying jet-bundle structure and is particular suitable for systems that allow for a variational characterisation. Thus, first of all we introduce the bundle \( \pi : E \to B \), where the total manifold \( E \) is equipped with the coordinates \( (z, x^a) \), with \( x^a, \alpha = 1, \ldots, n \), denoting the dependent variables, while the base manifold \( B \) possesses the independent coordinate \( z \) solely. Next, we consider the vertical tangent bundle \( \nu_E : \nu(E) \to E \), equipped with \( (z, x^a, \dot{x}^a) \), which is a subbundle of the tangent bundle \( \tau_E : \mathcal{T}(E) \to E \), possessing the coordinates \( (z, x^a, \dot{z}, \dot{x}^a) \) together with the fibre bases \( \partial_z = \partial/\partial z \) and \( \partial_a = \partial/\partial x^a \). Thus, a vertical vector field \( \nu = E \to \nu(E) \), locally given as \( \nu = \nu^\alpha \partial_a \), with \( \nu^\alpha \in C^\infty(E) \), i.e., \( \nu^\alpha \) is a smooth function on \( E \), is defined as a section. A further important construction is the so-called co-tangent bundle \( \tau_E^* : \mathcal{T}^*(E) \to E \), possessing the coordinates \( (z, x^a, \dot{x}_a) \), together with the fibre bases \( dz \) and \( dx^a \), which allows to introduce a one-form \( w : E \to \mathcal{T}^*(E) \) as a section that can locally be given as \( w = \dot{x}_a dx^a + w_a dx^a \), with \( \dot{x}_a, w_a \in C^\infty(E) \). With respect to the pH-system representation, we are interested in densities \( \delta = H(dz) \), \( H \in C^\infty(J^1(E)) \), where these densities can be formed by sections of certain pullback bundles, whose use is omitted here for ease of presentation. That is, \( H \) is a smooth function on the first jet manifold \( J^1(E) \), which is equipped with \( (z, x^a, \dot{x}^a, \ddot{x}^a) \), where the 1st-order jet variable \( x^a_\alpha \) corresponds to the derivative of \( x^a \) with respect to \( z \). Moreover, the first prolongation of a vertical vector field reads as \( J^1(\nu) = \nu_\alpha \partial_a + d_\alpha \nu^\beta \partial_\beta \), with \( \nu_\beta = \partial/\partial x^\beta \), where we exploit the total derivative \( d_z = \partial_z + \dot{x}^a \partial_a + x^a \partial_\alpha \partial_z \), ... Next, together with appropriate boundary conditions, where it should be stressed that here we consider systems with trivial boundary conditions solely, we introduce the pH-system representation including inputs and outputs as
\[
\begin{align*}
\dot{x} &= (J - \mathcal{R})(\delta \delta \rightleftharpoons) + u \beta \mathcal{G}, \\
y &= G^* \beta \delta \delta \rightleftharpoons,
\end{align*}
\tag{1a}
\]
see, e.g., [5, 17, 18], where \( \delta \) denotes the so-called Hook operator allowing for the natural contraction between tensor fields. In (1a), the variational derivative \( \delta \delta \rightleftharpoons = \delta \delta = \delta \delta \), with \( \delta \delta \) denoting the interior (wedge) product, locally reads as \( \delta \delta \rightarrow \delta \delta \delta \rightarrow \delta \delta \delta \). Moreover, the linear operators \( J, \mathcal{R} : \mathcal{T}^*(E) \to \mathcal{T}^*(B) \to \mathcal{V}(E) \) describe the internal power flow and the dissipation effects of the system, respectively. The coefficients \( J^{\alpha \beta} \) of the interconnection tensor \( J \) meets \( J^{\alpha \beta} = -J^{\beta \alpha} \in C^\infty(J^2(E)) \), while we have \( \mathcal{R}^{\alpha \beta} = \mathcal{R}^{\beta \alpha} \in C^\infty(J^2(E)) \) and \( [J^{\alpha \beta}] \geq 0 \) for the coefficient matrix of the symmetric and positive semi-definite dissipation mapping \( R \). With respect to the dual input and output bundles \( \rho : \mathcal{U} \to J^2(E) \) and \( \varphi : \mathcal{Y} \to J^2(E) \), we have the input map and its adjoint output map \( \varphi : \mathcal{U} \to \mathcal{V}(E) \) and \( \mathcal{G}^* : \mathcal{T}^*(E) \to \mathcal{T}^*(B) \to \mathcal{Y} \), respectively, and thus, the relation \( (u | \mathcal{G}) \delta \delta \rightleftharpoons = u | (G^* \beta \delta \delta \rightleftharpoons) \) holds, see [5, Sec. 4]. To determine the formal change of the Hamiltonian functional \( \mathcal{H} = \int_0^L \mathcal{H} dz \) along solutions of (1a), we make use of the Lie-derivative \( L_{F(t)} \), where we set \( v = \dot{x} \) with (1a), see [12, Sec. IV-A], and thus, we obtain \( \mathcal{H} = \int_0^L \mathcal{H} dz \) where the last term denotes collocation restricted to the boundary and vanishes as we focus on systems with trivial boundary conditions here.

Next, the aim is to exploit the pH-formulation with respect to the observer design. That is, the copy of the plant (1a) is extended by an error-injection term, and thus, by means of the observer-energy density \( H \), the observer system reads as
\[
\begin{align*}
\dot{x} &= (J^{\alpha \beta} = -R^{\alpha \beta} \delta \delta \rightleftharpoons) + u \beta \mathcal{G} + \mathcal{K}^\alpha \beta \dot{u}^\alpha \rightleftharpoons, \\
\dot{y}_\xi &= G_\xi \delta \delta \rightleftharpoons H, 
\end{align*}
\tag{2a}
\tag{2b}
\]
with \( \alpha, \beta = 1, \ldots, n \) and \( \xi, \eta = 1, \ldots, m \), where we use Einstein’s convention on sums. In (2a), we have the additional input \( u^\alpha_\eta = \delta \delta \rightleftharpoons (\tilde{y}_\xi - \tilde{y}_\xi) \) - with the Kronecker-Delta symbol meeting \( \delta \delta \rightleftharpoons = 1 \) for \( \xi = \eta \) and \( \delta \delta \rightleftharpoons = 0 \) for \( \xi \neq \eta \), where \( \tilde{y}_\xi \) corresponds to the integrated output density of the plant according to \( \tilde{y}_\xi = \int_0^L \tilde{y}_\xi dz \), which is assumed to be available as measurement quantity, while \( \tilde{y}_\xi \) represents the copy of the plant-integrated output according to \( \tilde{y}_\xi = \int_0^L \tilde{y}_\xi dz \) with (2b). The aim is to design the observer gain \( \mathcal{K}^\alpha \beta \) such that the observer error \( \hat{x} = \hat{x} - \hat{x} \) tends to 0, where it is beneficial to rewrite the observer-error dynamics \( \dot{x} = \hat{x} - \hat{x} \) as pH-system according to
\[
\begin{align*}
\dot{x} &= (J^{\alpha \beta} = -R^{\alpha \beta} \delta \delta \rightleftharpoons) + \mathcal{K}^\alpha \beta \dot{u}^\alpha \rightleftharpoons, \\
\dot{y}_\xi &= -\mathcal{K}^\alpha \beta \delta \delta \rightleftharpoons H, 
\end{align*}
\tag{3a}
\tag{3b}
\]
with (3b) denoting the collocated output density. If we investigate the formal change of the error-Hamiltonian \( \mathcal{H} = \int_0^L \mathcal{H} dz \), which follows to \( \mathcal{H} = -\int_0^L \delta \delta \rightleftharpoons H \delta \delta \rightleftharpoons H dz \) - \( \int_0^L \delta \delta \rightleftharpoons H \delta \delta \rightleftharpoons H dz \), we find that by means of a proper choice for the components \( \mathcal{K}^\alpha \beta \) we are able to render \( \mathcal{H} \leq 0 \). Hence, the total energy of the observer error \( \mathcal{H} \) is an appropriate candidate for a Lyapunov functional and therefore serves as basis with respect to the stability analysis. Next, the observer-design procedure is demonstrated by an example.

### III. Observer Design for an In-Domain Actuated Vibrating String

In this chapter, we design an infinite-dimensional observer for an in-domain actuated vibrating string by exploiting energy considerations. The governing equation of motion, which can, e.g., be deduced by the calculus of variations, reads as
\[
\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial z^2} + f(z, t), \tag{4a}
\]
where \( w \) describes the vertical deflection of the string, \( \rho \) the mass density and \( T \) Young’s modulus. Regarding the boundary conditions, we have that the string is clamped at \( z = 0 \) and free at \( z = L \), i.e.,
\[
w(0, t) = 0, \quad T \frac{\partial w}{\partial z}(L, t) = 0. \tag{4b}
\]
In (4a), the distributed force \( f(z, t) = g(z) u(t) \) is generated by an actuator behaving like a piezoelectric patch, where the applied voltage \( u(t) \) serves as manipulated variable. The spatially dependent function \( g(z) = h(z - L_{p1}) - h(z - L_{p2}) \), where \( h(\cdot) \) denotes the Heaviside function, describes the placement of the actuator between \( z = L_{p1} \) and \( z = L_{p2} \). In fact, the
force-distribution on the domain $L_{p_1} \leq z \leq L_{p_2}$ is supposed to be constant and scaled by the system input $u(t)$.

First, the intention is to find a pH-system representation that can be exploited for the observer design. To this end, we introduce the underlying bundle structure based on $\pi : (z, w, p) \rightarrow (z)$ together with the generalised momenta $\hat{p} = \rho \hat{w}$, and thus, (4a) can be rewritten as $\hat{p} = Twz + g(z)u$. If we use the Hamiltonian density $H = \frac{1}{2}\rho \hat{p}^2 + \frac{1}{2}T(\hat{w})^2 \in C^\infty(N_l(E))$, we obtain the appropriate pH-system formulation

$$\begin{align*}
\dot{\hat{w}} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_\omega H + \begin{bmatrix} 0 \\ \frac{g(z)}{\rho} \end{bmatrix} u, \\
\dot{\hat{p}} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_\hat{p} H = g(z)\frac{p}{\rho}.
\end{align*}$$

(5a)

By means of the boundary conditions (4b), the formal change of the Hamiltonian functional $\mathcal{H}$ follows to $\dot{\mathcal{H}} = \int_0^L g(z)\frac{p}{\rho}dz$, i.e., we have a distributed port that can be used for control purposes. In fact, for the system under consideration, in [11] a dynamic controller based on the energy-Casimir method has been designed. However, with regard to this control methodology, it should be mentioned that it yields unsatisfactory results for uncertain initial conditions, see, e.g., [19], where this problem is briefly discussed for a boundary-control system. To overcome this obstacle, we design an infinite-dimensional observer in the following.

Concerning the observer design, it is assumed that the spatial integration of the distributed output density (5b) according to $\tilde{y} = \int_0^L g(z)\frac{p}{\rho}dz$, which can be interpreted as the current through the actuator and is the output of the plant, is available as measurement quantity. Thus, if we use $\mathcal{H} = \frac{1}{2}\rho \hat{p}^2 + \frac{1}{2}T(\hat{w})^2$ and the copy of the plant output $\tilde{y} = \int_0^L g(z)\frac{p}{\rho}dz$, we are able to introduce an observer for the in-domain actuated vibrating string in the form

$$\begin{align*}
\dot{\tilde{w}} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_\omega \tilde{H} + \begin{bmatrix} 0 \\ g(z) \end{bmatrix} u + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} (\tilde{y} - \check{y}),
\end{align*}$$

(5b)

where the governing equations are restricted to the boundary conditions $\tilde{w}(0) = 0$ and $T\tilde{w}_z(0) = 0$. Next, by means of the error coordinates $\tilde{w} = w - \tilde{w}$, $\tilde{p} = p - \check{p}$, the observer-error dynamics can be deduced to

$$\begin{align*}
\dot{\tilde{w}} &= \frac{1}{\rho} \tilde{p} - k_1 (\tilde{y} - \check{y}), \\
\dot{\tilde{p}} &= T\tilde{w}_z - k_2 (\tilde{y} - \check{y}),
\end{align*}$$

(6)

where the boundary conditions $\tilde{w}(0) = 0$ and $T\tilde{w}_z(0) = 0$ hold. With respect to the determination of $k_1$ and $k_2$, it is beneficial to reformulate (6) as the pH-system

$$\begin{align*}
\dot{\tilde{w}} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_\omega \tilde{H} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} (\tilde{y} - \check{y}), \\
\dot{\tilde{p}} &= -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \delta_\hat{p} \tilde{H} = k_1 T\tilde{w}_z - k_2 \frac{\tilde{p}}{\rho},
\end{align*}$$

(7a)

(7b)

where the energy density of the observer error reads as $\mathcal{H} = \frac{1}{2}\rho \tilde{p}^2 + \frac{1}{2}T(\tilde{w})^2$ and (7b) states the corresponding output density. If we investigate the formal change of the error-Hamiltonian functional $\mathcal{H}$, which can be deduced to $\dot{\mathcal{H}} = \int_0^L (T\tilde{w}_z) k_1 (\tilde{y} - \check{y}) - \frac{\tilde{p}}{\rho} k_2 (\tilde{y} - \check{y}) dz$, and take into account that $\frac{\tilde{y} - \check{y}}{\check{y}} = \int_0^L g(z)\frac{p}{\rho} dz$, we find that the choice $k_1 = 0$ and $k_2 = kg(z)$ with $k > 0$, yields

$$\dot{\mathcal{H}}(\tilde{w}, \tilde{p}) = -k(\tilde{y} - \check{y})^2 \leq 0.$$  (8)

However, the fact that $\mathcal{H} > 0$ and $\dot{\mathcal{H}} < 0$ hold is not sufficient for the convergence of the observer, and therefore, in the following, detailed stability investigations are carried out to verify that the observer error is asymptotically stable.

IV. OBSERVER CONVERGENCE

In this section, based on functional analysis the convergence of the observer error is proven in two steps. First, we address the well-posedness of the observer-error system making heavy use of the well-known Lumer-Phillips theorem, see, e.g., [20]. Afterwards, LaSalle’s invariance principle for infinite-dimensional systems is applied to show the asymptotic stability of the observer error, where beforehand it is necessary to verify the precompactness of the solution trajectories.

A. Well-Posedness of the Observer-Error System

Now, a careful investigation of the well-posedness of the observer-error system is carried out. To this end, we reformulate (6) for the particular case $k_1 = 0$ and $k_2 = \rho g(z)$ as an abstract Cauchy problem and show that the observer under consideration generates a $C_0$-semigroup of contractions.

First, we define the state vector $\chi = [\chi^1, \chi^2]^T = [\tilde{w}, \tilde{p}]^T$ together with the state space $X = H^1_0(0, L) \times L^2(0, L)$, where $H^1_0(0, L) = \{ \chi \in H^1(0, L) \mid \chi(0) = 0 \}$, with $H^1(0, L)$ denoting a Sobolev space of functions whose derivatives up to order 1 are square integrable, see [21] for a detailed introduction of Sobolev spaces. Thus, the state space $X$ is equipped with the standard norm

$$||\chi||_X^2 = (\tilde{w}, \tilde{w})_{L^2} + (\tilde{p}, \tilde{p})_{L^2}.$$  (9)

Next, to be able to rewrite the observer-error dynamics, where $k_1 = 0$ and $k_2 = \rho g(z)$ hold, as an abstract Cauchy problem of the form $\dot{\chi}(t) = A\chi(t)$ with $\chi(0) = \chi_0$, we introduce the linear operator $A : D(A) \subset X \rightarrow X$ according to

$$A : \begin{bmatrix} \tilde{w} \\ \tilde{p} \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{p} \\ T\tilde{w}_z - \rho g(z) \int_0^L g(z)\frac{p}{\rho}dz \end{bmatrix},$$

(10)

where the (dense) domain of $A$ is defined as

$$D(A) := \{ \chi \in X \mid \tilde{w} \in (H^2(0, L) \cap H^1_0(0, L)), \tilde{p} \in H^1_0(0, L), T\tilde{w}_z(0) = 0 \}.$$  (11)

Thus, the intention is to investigate the operator $A$ regarding some properties such that a variant of the well-known Lumer-Phillips theorem [20, Th. 1.2.4] can be applied. With respect to this forthcoming investigations, it is beneficial to introduce

$$||\chi||_X^2 = (\chi, \chi)_X = T(\tilde{w}, \tilde{w})_{L^2} + \frac{1}{\rho} (\tilde{p}, \tilde{p})_{L^2},$$

(12)

which is called energy norm due to the equivalence $\mathcal{H} = \frac{1}{2}||\chi||_X^2$. Since $\tilde{w}(0) = 0$ and further $\tilde{w}(z) = \int_0^z \tilde{w}_z(y)dy$ holds, we find constants $c_1, c_2$, which have to meet $0 < c_1 \leq \min\{\frac{T}{\rho}, \frac{1}{\rho} T\}$ and $c_2 \geq \max\{T, \frac{1}{\rho} T\} > 0$, such that $c_1 ||\chi||_X^2 \leq \frac{1}{2}||\chi||_X^2 \leq c_2 ||\chi||_X^2$ is fulfilled, and hence, the energy norm (12) is equivalent to the standard norm (9).
Theorem 1: The operator $A$ defined by (10) is the infinitesimal generator of a $C_0$-semigroup of contractions on $\mathcal{X}$.

Proof: To prove Theorem 1, we apply the well-known Lumer-Phillips theorem according to [20, Th. 1.2.4]. Therefore, it is necessary that $D(A)$ is dense in $\mathcal{X}$, the operator $A$ is dissipative and $0 \in \rho(A)$, the resolvent set of $A$. First, similar to the [3, proof of Lemma 2.2], where they exploit the dense inclusion $H^2(0, L) \subset H^1(0, L)$ and modify the boundary values of $\tilde{w}$ and its derivatives in a proper manner, it can be shown that the domain $D(A)$ given in (11) is dense in $\mathcal{X}$. Thus, according to [20, Definition 1.1.1], – since we have $\mathcal{H} = \frac{1}{2} \|\chi\|_{\mathcal{X}}^2$ – the relation (8) implies that $A$ is dissipative.

To prove that $0 \in \rho(A)$, we show that the inverse operator $A^{-1}$ exists and is bounded, i.e., for every $\tilde{\chi} = [f, h]^T \in \mathcal{X}$ and $\chi = [\tilde{w}, \tilde{p}]^T \in D(A)$, we can uniquely solve

$$A \begin{bmatrix} \tilde{w} \\ \tilde{p} \end{bmatrix} = \begin{bmatrix} T\tilde{w}_{zz} - kg(z) \int_0^L g(z) f(y_1) dy_1 \\ \int_0^L k g(y_2) \int_0^L g(y_1) f(y_1) dy_1 dy_2 \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix},$$

and prove that $A^{-1}$ maps bounded sets in $\mathcal{X}$ into bounded sets in $\mathcal{K} := (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L)$. The boundedness of $A^{-1}$ implies that $\lambda = 0$ cannot be an eigenvalue of $A$, and hence, it follows that $0 \in \rho(A)$. From the 1st line of (13) it follows that $\tilde{p} = \rho f \in H^1_0(0, L)$. Moreover, an integration of the 2nd line of (13) yields

$$\tilde{w}_{\tilde{z}}(z) = -\frac{1}{T} \left( \int_0^L h(y_2) dy_2 \right) + \int_0^L k g(y_2) \int_0^L g(y_1) f(y_1) dy_1 dy_2$$

as $\tilde{w}_{\tilde{z}}(L) = 0$ holds. If we further integrate (14), we obtain

$$\tilde{w}(z) = -\frac{1}{T} \left( \int_0^z \int_0^L h(y_2) dy_2 dy_3 \right) + \int_0^z \int_0^L k g(y_2) \int_0^L g(y_1) f(y_1) dy_1 dy_2 dy_3$$

as $\tilde{w}(0) = 0$, and thus, $\tilde{w}(z)$ is uniquely defined by $\tilde{\chi}$. Since we have shown that the inverse operator $A^{-1}$ exists, it remains to investigate the boundedness. To this end, it is verified that the norm of $\chi = A^{-1} \tilde{\chi}$ in $\mathcal{K}$ is bounded by $\|\tilde{\chi}\|_{\mathcal{X}}$. First, we state an inequality that is often used in the following; in fact, for a $-basically arbitrary – function $f$, by means of the Cauchy-Schwarz inequality we find the important relation

$$\left( \int_0^L f dz \right)^2 \leq C \int_0^L |f|^2 dz,$$

where it should be mentioned that here and in the following $C$ denotes positive, not necessarily equal constants. Next, we investigate the norm $\|\tilde{w}_{\tilde{z}}\|_{L^2}$. Therefore, we substitute (14) in $\|\tilde{w}_{\tilde{z}}\|_{L^2} = \left( \int_0^L \|\tilde{w}_{\tilde{z}}\|^2 dz \right)^{1/2}$ and apply the Triangle inequality, which yields

$$\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq \left( \int_0^L \frac{1}{T^2} \left( \int_0^L h(y_2) dy_2 \right)^2 dz \right)^{1/2} + \left( \int_0^L \frac{1}{T^2} \left( \int_0^L k g(y_2) \int_0^L g(y_1) f(y_1) dy_1 dy_2 dy_3 \right)^2 dz \right)^{1/2}.$$

Thus, by means of (16) and due to the fact that $\int_0^L h^2 dy_2 \leq \int_0^L h^2 dz = \|h\|_{L^2}$ holds, we obtain

$$\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq C(\|h\|_{L^2} \frac{1}{T} \int_0^L dz + \int_0^L \frac{1}{T^2} k^2 g^2(y_2) \int_0^L g(y_1) f(y_1) dy_1 dy_2 dy_3 dz)^{1/2}.$$

Next, we apply the Cauchy-Schwarz inequality to the second term of the right-hand side in (17), which enables us to find $\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq C(\|f\|_{H^1} + \|h\|_{L^2})$. Similarly, by means of the 2nd line of (13) we are able to deduce $\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq C(\|f\|_{H^1} + \|h\|_{L^2})$. Moreover, if we substitute (15) in $\|\tilde{w}_{\tilde{z}}\|_{L^2} = \left( \int_0^L \|\tilde{w}\|^2 dz \right)^{1/2}$, we find $\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq C(\|f\|_{H^1} + \|h\|_{L^2})$, and hence, we have $\|\tilde{w}_{\tilde{z}}\|_{L^2} \leq C(\|f\|_{H^1} + \|h\|_{L^2})$. Since from the first line in (13) we immediately get $\|\tilde{p}\|_{H^1} = \rho \|f\|_{H^1}$, we can state the important estimate $\|\tilde{w}_{\tilde{z}}\|_{L^2} + \|\tilde{p}\|_{H^1} \leq C(\|f\|_{H^1} + \|h\|_{L^2})$, which shows that $A^{-1}$ maps bounded sets in $\mathcal{X}$ into bounded sets in $\mathcal{K}$.

Since all requirements for the Lumer-Phillips theorem [20, Th. 1.2.4] are met, it follows that $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $\mathcal{X}$.

The findings of this subsection imply that the solution of the observer error remains bounded; however, of course it is necessary that it tends to 0 for $t \to \infty$.

B. Asymptotic Stability of the Observer-Error System

Now, the objective is to verify the asymptotic stability of the observer-error system in order to justify the application of the observer given in Section III.

Theorem 2: The $C_0$-semigroup with generator $A$ defined by (10) is asymptotically stable, i.e., $\lim_{t \to \infty} \|\chi(t)\|_{\mathcal{X}} = 0$.

Proof: To prove Theorem 2, we intend to apply LaSalle’s invariance principle, where the proof follows the intention of [22, Sec. 3]. However, the applicability of LaSalle’s invariance principle [1, Th. 3.64] requires the precompacness of the solution trajectories. Since in the previous section we have shown that $A^{-1}$ is bounded, by means of the Sobolev embedding theorem, it follows that $A^{-1}$ is compact (see [3, proof of Lemma 2.4] or [1, p. 211]), which further implies the precompacness of the trajectories, see [2, Rem. 4.2].

Thus, we investigate the set $\mathcal{S} = \{ \chi \in \mathcal{X}; \mathcal{H} = 0 \}$, where $\dot{\mathcal{H}}(\tilde{w}, \tilde{p}) = -k \int_0^L g(z) \frac{1}{\rho} \tilde{p} dz = 0$ implies $\int_0^L g(z) \frac{1}{\rho} \tilde{p} dz = 0$. In the set $\mathcal{S}$ we have

$$\tilde{w}_t = \frac{1}{\rho} \tilde{p}, \quad \tilde{p}_t = T \tilde{w}_{\tilde{z}},$$

$$\tilde{w}(0, t) = 0, \quad T \tilde{w}_{\tilde{z}}(L, t) = 0,$$

which is similar to the problem considered in [22, Sec. 3]; however, the restriction describing the set $\mathcal{S}$, which is constrained to the boundary there, is completely different. To show that the only possible solution in $\mathcal{S}$ is the trivial one, we investigate the general solution of (18). To this end, like in [22, Sec. 3], we first focus on determining the eigenvalues and eigenfunctions of (18), i.e., we consider

$$\tilde{A} = \begin{bmatrix} \phi \\ \kappa \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} \kappa \\ T \phi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \kappa \end{bmatrix},$$

where the linear operator $\tilde{A}$ can be introduced as $\tilde{A}[\begin{bmatrix} \tilde{w} \\ \tilde{p} \end{bmatrix}] = \begin{bmatrix} \frac{1}{\rho} \kappa \\ T \phi \end{bmatrix}$ by means of (18), $\lambda$ denotes
an eigenvalue of $\tilde{A}$ and $\phi$ and $\kappa$ the eigenfunctions for $\tilde{w}$ and $\tilde{p}$. From (19) we obtain $\kappa = \rho \beta \phi$ and furthermore

$$\phi_{zz} = \frac{\lambda^2}{\vartheta^2} \phi, \quad \phi(0) = 0, \quad \phi_z(L) = 0$$

where $\vartheta^2 = \frac{m}{\rho}$. To find the solution of (20), we have to investigate the three cases $\lambda^2 > 0$, $\lambda^2 = 0$ and $\lambda^2 < 0$ in the following. For $\lambda^2 > 0$ and $\lambda^2 > 0$, we have the ansatz $\phi(z) = Ae^{\vartheta z} + Be^{-\vartheta z}$ and $\phi(z) = Az + B$, respectively, where by means of the boundary conditions (20b), one can easily deduce that for both cases only the trivial solution $\phi(z) = 0$ exists. Thus, we focus on the case $\lambda^2 < 0$, and consequently, due to the fact that $\lambda$ has an imaginary character then, as ansatz for the eigenfunctions we have $\phi(z) = A \sin \left( \frac{\lambda}{\vartheta} z \right) + B \cos \left( \frac{\lambda}{\vartheta} z \right)$. To fulfil the boundary condition $\phi(0) = 0$, $B = 0$ must be valid, and hence, the ansatz simplifies to $\phi(z) = A \sin \left( \frac{\lambda}{\vartheta} z \right)$. Furthermore, by means of the boundary condition $\phi_z(L) = 0$, we find $\vartheta \phi(L) = \frac{\lambda}{\vartheta} A \cos \left( \frac{\lambda}{\vartheta} L \right) = 0$, which exhibits infinitely many non-trivial solutions for

$$|\lambda_k| = (k - \frac{1}{2}) \frac{\pi}{L}, \quad k = 1, 2, \ldots \quad (21)$$

with $\vartheta = \frac{m}{\rho}$. With regard to the investigation of the set $S$, the velocity of the vibrating string is of particular interest. Consequently, since we deduced the (imaginary) eigenvalues $\lambda_k = \pm \imath \omega_k \vartheta$ with $\omega_k = (k - \frac{1}{2}) \frac{\pi}{L}$, the ansatz for the general solution of the velocity can be given according to

$$\tilde{w}_i(z, t) = \sum_{k=1}^{\infty} \left( a_k \cos(\omega_k \vartheta t) + b_k \sin(\omega_k \vartheta t) \right) \varphi_k(z), \quad (22a)$$

where the coefficients $A_k$ are hidden in $a_k$ and $b_k$, and therefore, for the eigenfunctions we use $\varphi_k(z) = \sin(\omega_k \vartheta z)$ here and in the following. Hence, an integration of (22a) yields

$$\tilde{w}(z, t) = \sum_{k=1}^{\infty} \left( a_k \sin(\omega_k \vartheta t) - b_k \cos(\omega_k \vartheta t) \right) \frac{\varphi_k(z)}{\omega_k \vartheta}, \quad (22b)$$

By means of $\sin(x) = \frac{1}{2}(e^{ix} - e^{-ix})$ and $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$, after a straightforward computation we can beneficially rewrite (22) according to

$$\tilde{w}(z, t) = \sum_{k=1}^{\infty} c_k e^{i \omega_k \vartheta t} \left[ \frac{-i \vartheta}{\omega_k} \varphi_k \right] + \sum_{k=1}^{\infty} c_{-k} e^{-i \omega_k \vartheta t} \left[ \frac{i \vartheta}{\omega_k} \varphi_k \right],$$

where, the coefficients $c_k = \frac{1}{2} (a_k - ib_k)$ and $c_{-k} = \frac{1}{2} (a_k + ib_k)$ fulfill\footnote{For $a_k = \frac{1}{2} \int_0^{L_p} \tilde{w}_i(z) \varphi_k(z) dz$ and $b_k = \frac{1}{2} \int_0^{L_p} \tilde{w}_{i,0} \varphi_k(z) dz$, with $\tilde{w}_{i,0} = \partial_z \tilde{w}_i(z, 0)$ and $\tilde{w}_i(z) = \partial_z \tilde{w}(z, 0)$, by means of integration by parts it can be shown that they multiplicatively depend on $\frac{1}{2}$. Thus, in (23) the terms $\frac{1}{2}$ occur, often termed as the Basel problem, where it is known that it converges.} (see [22, eq. (3.19)])

$$\sum_{k=1}^{\infty} |c_{\pm k}|^2 = \sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right) < \infty, \quad (23)$$

which will play an important role later. Thus, we are able to write $\int_0^{L_p} g(z) \frac{1}{\rho} \tilde{p} \tilde{w} dz = \int_{L_p}^{L_p} \tilde{w}_i dz = 0$ as

$$\int_{L_p}^{L_p} \sum_{k=1}^{\infty} (c_k e^{i \omega_k \vartheta t} + c_{-k} e^{-i \omega_k \vartheta t}) \sin(\omega_k z) dz = 0. \quad (24)$$

Now, we show that $S$ admits the trivial solution only, i.e., $c_{\pm k} = 0 k \geq 1$ is valid. Otherwise, if there exists a $k_0$ with $|c_{k_0}| \neq 0$, due to (23) we can find a $K > k_0$ such that

$$\left| \int_{L_p}^{L_p} \sum_{k=K}^{\infty} c_{\pm k} \varphi_k dz \right| < \left| \frac{c_{k_0}}{4} \int_{L_p}^{L_p} \varphi_{k_0} dz \right| \quad (25)$$

holds, i.e., the sum of the coefficients from $K$ to $\infty$ multiplied with their corresponding eigenfunctions can be bounded by $c_{k_0}$ and $\varphi_{k_0}$. Here, it is assumed that $\int_{L_p}^{L_p} \sin(\omega_{k_0} z) dz \neq 0$, i.e., $\omega_{k_0} \neq \frac{2 \pi n}{L_p - L_p}$ with $j \in N_+$. However, if we consider the absolute value of the eigenvalues (21), we find that this is ensured for a proper choice of the length of the in-domain actuator according to $L_p - L_p \neq \frac{2 \pi n}{L_p}$. Thus, since $\omega_k \neq \omega_j \forall k \neq j$, for $t > 0$ we can reformulate (24) as

$$\int_{L_p}^{L_p} \varphi_{k_0} dz \quad \int_{L_p}^{L_p} \varphi_{k_0} dz \quad$$

Next, the idea is to integrate (26) with respect to $t$ and to investigate the absolute value. Thus, the right-hand side of

$$\left| c_{k_0} \int_{L_p}^{L_p} \varphi_{k_0} dz \right| \quad (27)$$

where we used (25) to obtain an estimation for the sums from $k = K + 1$ to $k = \infty$, is bounded for all $t \geq 0$. Since for an appropriate choice of the actuator-length it is ensured that the integral on the left-hand side cannot vanish, the only possibility that inequality (27) holds for $t \rightarrow \infty$ is that $c_{k_0} = 0$ is valid. Hence, it is shown that the only possible solution in $S$ is the trivial one, which finally proves the asymptotic stability of the observer error.

Thus, the asymptotic stability of the observer error justifies the application of the observer developed in [11]. In Fig. 1, the simulation result for the deflection error $\tilde{w}(z, t)$ is depicted. The plant and the observer have been implemented by means of the finite difference-coefficient method, where all parameters were set to 1. The actuator is placed between $L_p = 0.4$...
In this letter, the asymptotic stability of an observer error of an in-domain actuated vibrating string, where the observer has been developed in [11], was investigated. First, we showed that the linear operator, which describes the observer error as an abstract Cauchy problem, is the infinitesimal generator of a contraction semigroup. Second, by means of LaSalle’s invariance principle the asymptotic stability of the observer error was proven. In fact, by choosing the length of the actuator properly, it was shown that the only possible solution for $\dot{\mathcal{H}} = 0$ is the trivial one, which implies that the observer error tends to zero. Future-research tasks might deal with the stability analysis of the closed loop obtained by the controller design presented in [11], or even with the stability investigation of the combination of controller and observer. Moreover, note that in [9] and [23] a comprehensive framework for well-posedness and stability investigations for a broad class of boundary-control systems is available, which would be desirable for in-domain actuated systems as well. However, since the investigations heavily depend on the input function $g(z)$, such a generalisation is far from straightforward. Nevertheless, we aim to generalise the results and findings of this letter to spatially 1-dimensional systems with 1st-order Hamiltonian within future-research tasks.

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