Cuspidal modules for solenoidal Lie algebras over rational quantum tori

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Abstract: In this paper we classify all irreducible cuspidal modules over a solenoidal Lie algebra over a rational quantum torus, generalizing the results in [BF2, Su] and [Xu2].

Keywords: solenoidal Lie algebra; cuspidal module; quantum torus; module of tensor fields.

1 Introduction

The classification of irreducible Harish-Chandra modules for infinite dimensional Lie algebras is a classical problem in the representation theory, and was solved for the Virasoro algebra $Vir_{[M]}$ and its various generalizations, such as the twisted Heisenberg-Virasoro algebra $[LZ2]$, the gap-$p$ Virasoro algebras $Vir_{p}[Xu2]$, the Witt algebras $W_d[BF1]$, the higher rank Virasoro algebras(or solenoidal Lie algebras) $W_\mu[LZ1]$, and so on.

In order to achieve the classification result for $W_d$ in $[BF1]$, the authors applied a new technique, called cover method. Using this cover method, they also classified all irreducible cuspidal modules for $W_\mu$ in $[BF2]$, which was originally done in $[Su]$, but in a much more computational way. Let $A = \mathbb{C}[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$ be the Laurent polynomial ring with $d$ variables, where $d$ is a positive integer, and let $\mathcal{G}$ stand for $W_d$ or $W_\mu$. An $A\mathcal{G}$-module is defined to be a module over the Lie algebra $\mathcal{G} \ltimes A$ with an associative $A$-action. Then the $A$-cover $\hat{M}$ of a $\mathcal{G}$-module $M$ is a particular $A\mathcal{G}$-quotient of the module $\mathcal{G} \otimes M$. Here

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A plays the role of a coordinate algebra. It turns out that $\hat{M}$ is cuspidal if $M$ is, and $M$ becomes a $\mathcal{G}$-quotient of $\hat{M}$ provided that $M$ is irreducible. This reduces the classification of irreducible cuspidal $\mathcal{G}$-modules to the classification of irreducible cuspidal $A\mathcal{G}$-modules. It is well known that irreducible cuspidal $AW_d$-modules are the so-called modules of tensor fields, originally constructed by Shen [S], Larsson [Lar], and further studied in [R].

While for $W_\mu$, it was proved in [BF2] that cuspidal $AW_\mu$-modules with the support being one coset of $\mathbb{Z}^d$ in $\mathbb{C}^d$ are in a one-to-one correspondence to finite dimensional modules over an subalgebra of $\text{Der}\mathbb{C}[x_1, \ldots, x_d]$. These finite dimensional modules must be one dimensional if the corresponding $AW_\mu$-module is irreducible, which makes the irreducible cuspidal $AW_\mu$-module has weight multiplicities no more than one.

In the present paper we use a modified cover method to classify all irreducible cuspidal modules for a solenoidal Lie algebra $\mathfrak{g}$ over a rational quantum torus $\mathbb{C}_Q = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, which is a subalgebra of the algebra $\text{Der}(\mathbb{C}_Q)$ of derivations over $\mathbb{C}_Q$. The universal central extension of $\mathfrak{g}$ was computed in [Xu1]. When $d = 1$, the algebra $\mathfrak{g}$ is exactly the gap-$p$ Virasoro algebra studied in [Xu2]. When constructing the cover for a cuspidal module $M$ for $\mathfrak{g}$, we choose the centre $Z$ of $\mathbb{C}_Q$, instead of the whole $\mathbb{C}_Q$, as the coordinate algebra and define the $Z\mathfrak{g}$-cover of $M$ to be some $Z\mathfrak{g}$-quotient of the module $\mathfrak{g}'_R \otimes M$, where $\mathfrak{g}'_R$ is an ideal of $\mathfrak{g}$. This construction is a bit different with the $W_\mu$ or the $W_d$ case.

We establish a one-to-one correspondence between cuspidal modules over the solenoidal Lie algebra $\mathfrak{g}$ with support lying in one coset of $\mathbb{Z}^d$ in $\mathbb{C}^d$ and finite dimensional $\Gamma$-graded modules over an subquotient algebra $\mathcal{L}$ of the Lie algebra $\text{Der}\mathbb{C}[x_1, \ldots, x_d, t_1, \ldots, t_d]$, where $\Gamma$ is a finite group closely related to $\mathbb{C}_Q$. Through this correspondence we show that any irreducible cuspidal $Z\mathfrak{g}$-module is isomorphic to the module of tensor fields $\mathcal{V}(\alpha, \beta, W)$, for some $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$ and finite dimensional $\Gamma$-graded irreducible $\mathfrak{gl}_N$-module $W$, where $N$ is the order of $\Gamma$. These modules $\mathcal{V}(\alpha, \beta, W)$ may have weight multiplicities larger than one. The Lie algebra $\mathcal{L}$ has a quotient isomorphic to the direct sum of $\mathfrak{gl}_N$ and a solvable subalgebra of $\mathfrak{gl}_N$. This is how the seemingly peculiar algebra $\mathfrak{gl}_N$ comes into the picture. Furthermore, using the modified cover method we prove that any irreducible cuspidal $\mathfrak{g}$-module is isomorphic to the unique irreducible subquotient of some $\mathcal{V}(\alpha, \beta, W)$.
Since the relation between the algebra \( g \) and \( \text{Der}(\mathbb{C}Q) \) is similar to that between the solenoidal Lie algebra \( W_\mu \) and the Witt algebra \( W_d \), we believe our result may play a role in classification of irreducible Harish-Chandra \( \text{Der}(\mathbb{C}Q) \)-modules. We mention that in [LiuZ] irreducible cuspidal \( \text{Der}(\mathbb{C}Q) \)-modules were classified, also using a cover method.

The paper is arranged as follows. In Section 2 we recall some results about the gap-\( p \) Virasoro algebra \( V\text{ir}_p \), the solenoidal Lie algebra \( W_\mu, \mathbb{C}Q \) and \( \mathfrak{gl}_N \). In section 3 we give the construction of modules of tensor fields \( \mathcal{V}(\alpha, \beta, W) \) over \( g \), and their irreducibility criterion. Section 4 is devoted to finite dimensional \( \Gamma \)-graded \( \mathcal{L} \)-modules. In Section 5 we classify all irreducible cuspidal \( \mathbb{Z}g \)-modules, and in the last section all irreducible cuspidal \( g \)-modules are classified using the cover method.

Throughout this paper, \( \mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}_+ \) refer to the set of complex numbers, integers, nonnegative integers and positive integers respectively. For a Lie algebra \( \mathcal{G} \), we denote by \( \mathcal{U}(\mathcal{G}) \) the universal enveloping algebra of \( \mathcal{G} \). Let \( d \in \mathbb{Z}_+ \) and we fix a standard basis \( \epsilon_1, \ldots, \epsilon_d \) for the space \( \mathbb{C}^d \). Denote by \( (\cdot | \cdot) \) the inner product on \( \mathbb{C}^d \).

## 2 Notations and Preliminaries

For a Lie algebra, a weight module is called a Harish-Chandra module if all weight spaces are finite dimensional, and called a cuspidal module if all weight spaces are uniformly bounded. The support of a weight module is defined to be the set of all weights.

### 2.1 The gap-\( p \) Virasoro algebra \( V\text{ir}_p \)

Let \( p \) be a positive integer and \( \mathbb{C}[x^{\pm 1}] \) denote the Laurent polynomial ring in one variable \( x \). The gap-\( p \) Virasoro algebra \( V\text{ir}_p \) is a Lie algebra with a basis

\[
\{ x^{m+1} \frac{\partial}{\partial x}, x^s, C_i \mid m \in p\mathbb{Z}, s \notin p\mathbb{Z}, 0 \leq i \leq p-1 \},
\]

and Lie brackets

\[
[x^{m+1} \frac{\partial}{\partial x}, x^{n+1} \frac{\partial}{\partial x}] = (n - m)x^{m+n+1} \frac{\partial}{\partial x} + \delta_{m+n,0} \frac{1}{12} \left( \frac{m}{p} \right)^3 - \left( \frac{m}{p} \right) C_0;
\]

\[
[x^{m+1} \frac{\partial}{\partial x}, x^r] = r x^{m+r}; \quad [x^r, x^s] = \delta_{r+s,0} r C_r,
\]
where \( m, n \in p\mathbb{Z}, r, s \notin p\mathbb{Z} \) and we use \( \overline{r} \) to represent the residue of \( r \) by \( p \).

We recall from [Xu2] the module of intermediate series over \( \text{Vir}_p \). Let \( F = (F_{i,j}) \) be a \((p - 1) \times p\) complex matrix, with index \( 1 \leq i \leq p - 1, \ 0 \leq j \leq p - 1 \), satisfying the following three conditions

(I) \( 0 \in o(F) = \{ j \mid F_{i,j} \neq 0 \text{ for some } i \} \);

(II) if \( F_{i,j} \neq 0 \) then \( F_{s,j+i} \neq 0 \) for some \( 1 \leq s \leq p - 1 \);

(III) \( F_{r,i+s}F_{s,i} = F_{s,i}F_{r,i} \) for any \( 0 \leq i \leq p - 1 \) and \( 1 \leq r, s \leq p - 1 \);

For any \( j \in o(F) \) denote the space \( V(j) = \text{span}_\mathbb{C} \{ v_{j+k} \mid k \in \mathbb{Z} \} \). Let \( a, b \in \mathbb{C} \) and define the \( \text{Vir}_p \)-module structure on \( \bigoplus_{j \in o(F)} V(j) \) by

\[
\begin{align*}
x^{m+1}\frac{\partial}{\partial x} v_{j+n} & = (a + j + n + mb)v_{j+n+m}; \\
x^s v_{j+n} & = F_{s,j}v_{j+n+s}; \quad C_i v_{j+n} = 0,
\end{align*}
\]

where \( m, n \in p\mathbb{Z}, s \notin p\mathbb{Z}, j \in o(F) \) and \( i = 0, 1, \ldots, p - 1 \). We denote this module by \( V(a, b, F) \), and call it a module of intermediate series over \( \text{Vir}_p \). Roughly speaking, the module \( V(a, b, F) \) is a sum of several modules of intermediate series over the subalgebra \( \text{span}_\mathbb{C} \{ x^{m+1}\frac{\partial}{\partial x}, C_0 \mid m \in p\mathbb{Z} \} \), which is isomorphic to the normal Virasoro algebra. Let \( P = \text{span}_\mathbb{C} \{ x^m \mid m \in p\mathbb{Z} \} \). In [Xu2], it was proved that any irreducible cuspidal modules over \( \text{Vir}_p \ltimes P \) with an associative \( P \)-action must be of the form \( V(a, b, F) \).

2.2 The solenoidal Lie algebra \( W_\mu \)

Let \( d \in \mathbb{Z}_+ \) and \( \mu = (\mu_1, \ldots, \mu_d)^T \in \mathbb{C}^d \) is called generic if \( \mu_1, \ldots, \mu_d \) are linearly independent over the field of rational numbers. Let \( A = \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \) be the Laurent polynomial ring with \( d \) variables. For any \( \mathbf{m} = (m_1, \ldots, m_d)^T \in \mathbb{Z}^d \) denote \( x^\mathbf{m} = x_1^{m_1} \cdots x_d^{m_d} \).

Set \( \partial_x = \sum_{i=1}^d \mu_i x_i \frac{\partial}{\partial x_i} \). Then the solenoidal Lie algebra \( W_\mu \) over \( A \) has a basis

\[
\{ x^\mathbf{m} \partial_x \mid \mathbf{m} \in \mathbb{Z}^d \}
\]

and Lie bracket

\[
[x^\mathbf{m} \partial_x, x^n \partial_x] = (\mu \mid \mathbf{n} - \mathbf{m}) x^{m+n} \partial_x.
\]
Let $\alpha \in \mathbb{C}^d$, $\beta \in \mathbb{C}$. The modules $T(\alpha, \beta)$ of tensor fields over $W_\mu$ have bases $\{v_s \mid s \in \mathbb{Z}^d\}$ and the $W_\mu$-action

$$(x^m \partial_x) \cdot v_s = (\mu \mid \alpha + s + \beta m)v_{m+s}.$$ 

It is well known that the module $T(\alpha, \beta)$ is reducible if and only if $\alpha \in \mathbb{Z}^d$ and $\beta \in \{0, 1\}$. For $\alpha \in \mathbb{Z}^d$, $T(\alpha, 0)$ has a unique irreducible quotient $T(\alpha, 0)/Cv_\alpha$, and $T(\alpha, 1)$ has a unique submodule span$_C \{v_s \mid s \neq -\alpha\}$ of codimension one. Moreover, the module $T(\alpha, \beta)$ may be equipped with an additional associative $A$-action $x^m v_s = v_{m+s}$.

**Theorem 2.1** (Su, BF2). Any irreducible cuspidal module over $W_\mu \ltimes A$ with an associative $A$-action must be of the form $T(\alpha, \beta)$ for some $\alpha \in \mathbb{C}^d$ and $\beta \in \mathbb{C}$.

### 2.3 The quantum torus $\mathbb{C}_Q$ and the Lie algebra $\mathfrak{gl}_N$

Let $Q = (q_{ij})$ be a $d \times d$ complex matrix with all $q_{ij}$ being roots of unity and satisfying

$$q_{ii} = 1, \quad q_{ij}q_{ji} = 1 \text{ for all } 1 \leq i, j \leq d.$$ 

The rational quantum torus relative to $Q$ is the unital associative algebra $\mathbb{C}_Q = \mathbb{C}[t_1^{\frac{1}{k_i}}, \ldots, t_d^{\frac{1}{k_i}}]$ with multiplication

$$t_i t_j = q_{ij} t_j t_i \text{ for all } 1 \leq i, j \leq d.$$ 

For an element $m = (m_1, \ldots, m_d)^T \in \mathbb{Z}^d$ we denote $t^m = t_1^{m_1} \cdots t_d^{m_d}$.

For $m, n \in \mathbb{Z}^d$, denote

$$\sigma(m, n) = \prod_{1 \leq i < j \leq d} q_{ji}^{n_i m_j} \text{ and } R = \{m \mathbb{Z}^d \mid \sigma(m, n) = \sigma(n, m) \text{ for any } n \in \mathbb{Z}^d\}.$$ 

Clearly, the center $Z$ of $\mathbb{C}_Q$ is spanned by $\{t^m \mid m \in R\}$. From Theorem 4.5 in [N], up to an isomorphism of $\mathbb{C}_Q$, we may always assume that $q_{2i,2i-1} = q_i, q_{2i-2i} = q_i^{-1}$ for $1 \leq i \leq z$, and other entries of $Q$ are all 1, where $z \in \mathbb{Z}_+$ with $2z \leq d$ and the orders $k_i$ of $q_i$ as roots of unity satisfy $k_{i+1} \mid k_i$ for $1 \leq i \leq z$. Then the subgroup $R$ of $\mathbb{Z}^d$ has a simple form

$$R = \bigoplus_{i=1}^z (Zk_i \epsilon_{2i-1} \oplus Zk_i \epsilon_{2i}) \oplus \bigoplus_{l>2z} Z\epsilon_l.$$ 

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Moreover, we have

\[ \sigma(m, r) = \sigma(r, m), \quad t^m t^r = t^{m+r} \text{ for all } m \in R, r \in \mathbb{Z}^d. \]

Let \( I = \text{span}_c \{ t^{m+r} - t^r \mid n \in R, r \in \mathbb{Z}^d \} \), which is an ideal of the associative algebra \( \mathbb{C}_Q \). By \([N]\) and \([Z]\), we have \( \mathbb{C}_Q/I \cong \bigotimes_{i=1}^z M_{k_i}(\mathbb{C}) \cong M_N(\mathbb{C}) \), where \( N = \prod_{i=1}^z k_i \) and \( M_n(\mathbb{C}) \) is the associative algebra of all \( n \times n \) complex matrices. It is well known that \( M_{k_i}(\mathbb{C}) \) can be generated by

\[
X_{2i-1} = E_{1,1} + q_i E_{2,2} + \cdots + q_i^{k_i-1} E_{k_i,k_i}, \\
X_{2i} = E_{1,2} + E_{2,3} + \cdots + E_{k_i-1,k_i} + E_{k_i,1},
\]

where \( E_{kl} \) represents the \( k_i \times k_i \) matrix with 1 in the \((k,l)\)-entry and 0 elsewhere. Let \( E \) denote the identity matrix of suitable order. It is easy to see that \( X_{2i}^{k_i} = X_{2i-1}^{k_i} = E \) and \( X_{2i} X_{2i-1} = q_i X_{2i-1} X_{2i} \). Denote \( X^n = \bigotimes_{i=1}^z X_{2i-1}^{n_{2i-1}} X_{2i}^{n_{2i}} \) for \( n \in \mathbb{Z}^d \). Notice that for any \( n \in R, X^n \) is the identity matrix in \( M_N(\mathbb{C}) \), and \( X^r X^s = \sigma(r, s) X^{r+s} \).

Define the Lie bracket \([a, b] = ab - ba\) on \( M_N(\mathbb{C}) \), and we write \( \mathfrak{gl}_N \) for \( M_N(\mathbb{C}) \) as a Lie algebra. We have the Lie bracket for \( \mathfrak{gl}_N \)

\[
[X^m, X^n] = (\sigma(m, n) - \sigma(n, m)) X^{m+n}.
\]

Set \( \Gamma = \mathbb{Z}^d/R \) and let \( \overline{n} \) denote the image of \( n \). Clearly, \( \Gamma \) has order \( N \), and \( \mathfrak{gl}_N \) is a \( \Gamma \)-graded Lie algebra with homogeneous spaces \( (\mathfrak{gl}_N)\overline{n} = \text{span}_c \{ X^n \} \). In this paper by a \( \Gamma \)-gradation on \( \mathfrak{gl}_N \) we always mean this gradation. Set

\[
\Gamma_0 = \{ n \in \mathbb{Z}^d \mid 0 \leq n_{2i-1} < k_i, 0 \leq n_{2i} < k_i, 1 \leq i \leq z, \text{ and } n_l = 0, 2z < l \leq d \},
\]

which is a complete set of representatives for \( \Gamma \). When a representative of \( \overline{n} \) is needed we always choose the one \( n \in \Gamma_0 \).

3 Modules of tensor fields for \( g \)

In this section we construct modules of tensor fields \( \mathcal{V}(\alpha, \beta, W) \) for the solenoidal Lie algebra \( g \) over the quantum torus \( \mathbb{C}_Q \), and give an irreducibility criterion for \( \mathcal{V}(\alpha, \beta, W) \).
Let $\gamma = (\gamma_1, \cdots, \gamma_d)^T \in \mathbb{C}^d$ be generic and set

$$L_m = \begin{cases} t^m \sum_{i=1}^d \gamma_i \frac{\partial}{\partial t^i} & \text{if } m \in R; \\ t^m & \text{if } m \notin R, \end{cases}$$

where $t^m$ is the inner derivation on $\mathbb{C}_Q$ defined by $t^m(t^r) = \sigma(m, r)t^{m+r}$. The Lie algebra $\mathfrak{g}$ we consider in this paper has a basis $\{L_m \mid m \in \mathbb{Z}^d\}$, subject to the Lie brackets

$$[L_m, L_n] = (\gamma \mid n - m)L_{m+n};$$
$$[L_m, L_s] = (\gamma \mid s)L_{m+s}; \quad [L_r, L_s] = (\sigma(r, s) - \sigma(s, r))L_{r+s},$$

for $m, n \in R, r, s \notin R$. The subalgebra $\mathfrak{g}_R$ of $\mathfrak{g}$ spanned by $\{L_m \mid m \in R\}$ is isomorphic to the solenoidal Lie algebra $W_\mu$, where $\mu = B\gamma$, $B = \text{diag}\{k_1, k_1, k_2, k_2, \cdots, k_z, k_z, 1, \cdots, 1\}$. This isomorphism is given by $L_m \mapsto x^n \sum_{i=1}^d \mu_i x_i \frac{\partial}{\partial x_i}$, where $m = Bn$. The Lie algebra $\mathfrak{g}$ may be considered as a quantum version of $W_\mu$.

Now we give the construction of the module of tensor fields for $\mathfrak{g}$. Let $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$ and $W = \bigoplus_{s \in \Gamma} W_s$ be a $\Gamma$-graded $\mathfrak{gl}_N$-module. Recall the center $Z$ of $\mathbb{C}_Q$. Define a $\mathfrak{g}$-module structure on

$$\mathcal{V}(\alpha, \beta, W) = \bigoplus_{\mathfrak{g} \in \Gamma} W_\mathfrak{g} \otimes t^s Z$$

by (here $s$ is the representative of $\mathfrak{g}$ in $\Gamma_0$)

$$L_m(w_\mathfrak{g} \otimes t^{m+s}) = (\gamma \mid \alpha + n + s + \beta m)w_\mathfrak{g} \otimes t^{m+n+s};$$
$$L_r(w_\mathfrak{g} \otimes t^{n+s}) = (X^r w_\mathfrak{g}) \otimes t^{r+n+s},$$

where $m, n \in R, r \notin R, s \in \mathbb{Z}^d$ and $w_\mathfrak{g} \in W_\mathfrak{g}$. We call $\mathcal{V}(\alpha, \beta, W)$ a module of tensor fields over $\mathfrak{g}$.

**Lemma 3.1.** If the $\Gamma$-graded $\mathfrak{gl}_N$-module is irreducible and $\dim W > 1$, then the $\mathfrak{g}$-module $\mathcal{V}(\alpha, \beta, W)$ is irreducible for any $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$.

**Proof.** Let $U$ be a nonzero $\mathfrak{g}$-submodule of $\mathcal{V}(\alpha, \beta, W)$ and $0 \neq w_\mathfrak{g} \otimes t^s \in U$ for some $w_\mathfrak{g} \in W_\mathfrak{g}, s \in \mathbb{Z}^d$. We claim that

$$w_\mathfrak{k} \otimes t^{k+m} \in U \text{ for any } s \neq k \in \mathbb{Z}^d \text{ and } m \in R.$$
Since $W$ is a $\Gamma$-graded irreducible $\mathfrak{gl}_N$-module, we see that $w_\mathfrak{g} = aw_\mathfrak{g}$ for some homogeneous element $a$ in $\mathcal{U}(\mathfrak{gl}_N)$ of the form

$$a = \sum c_{s_1 \cdots s_p} X^{s_1} \cdots X^{s_p},$$

where $c_{s_1 \cdots s_p} \in \mathbb{C}$ and the sum takes over finitely many $(s_1, \cdots, s_p)$ such that $s_1 + \cdots + s_p = k - s$. Denote $b = \sum c_{s_1 \cdots s_p} L_{s_1} \cdots L_{s_1} \in \mathcal{U}(\mathfrak{g})$. Then we have

$$w_\mathfrak{g} \otimes t^k = aw_\mathfrak{g} \otimes t^k = b(w_\mathfrak{g} \otimes t^s).$$

Since $L_m(w_\mathfrak{g} \otimes t^{m+k}) = (\gamma | \alpha + k + \beta m)(w_\mathfrak{g} \otimes t^{m+k}) \in U$, we get that

$$w_\mathfrak{g} \otimes t^{m+k} \in U \text{ if } \alpha + k + \beta m \neq 0.$$

Assume $\alpha + k + \beta m = 0$. Since

$$L_m(w_\mathfrak{g} \otimes t^s) = (\gamma | \alpha + s + \beta m)(w_\mathfrak{g} \otimes t^{m+s}) = (\gamma | s - k)(w_\mathfrak{g} \otimes t^{m+s}) \in U$$

we obtain $w_\mathfrak{g} \otimes t^{m+s} \in U$. Then we still have $w_\mathfrak{g} \otimes t^{m+k} = b(w_\mathfrak{g} \otimes t^{m+s}) \in U$.

We still need to show $w_\mathfrak{g} \otimes t^{m+s} \in U$ for any $m \in \mathbb{R}$ and $w_\mathfrak{g} \in W_{\mathfrak{g}}$. But this is just a replicate of the proof of the above claim starting with a vector $w_\mathfrak{g} \otimes t^k \in U$ with $k \neq s$. So $U = \mathcal{V}(\alpha, \beta, W)$ and the lemma stands. $\square$

Furthermore, we have the following

**Theorem 3.2.** Let $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$ and $W$ be a $\Gamma$-graded irreducible $\mathfrak{gl}_N$-module. The $\mathfrak{g}$-module $\mathcal{V}(\alpha, \beta, W)$ is reducible if and only if $\dim W = 1, \alpha \in \mathbb{Z}^d$ and $\beta \in \{0, 1\}$.

**Proof.** By Lemma 3.1 we only need consider when $\dim W = 1$. In this case $L_s \mathcal{V}(\alpha, \beta, W) = 0$ for any $s \notin R$, and hence $\mathcal{V}(\alpha, \beta, W)$ reduces to a module over the subalgebra $\mathfrak{g}_R$ of $\mathfrak{g}$, which is isomorphic to the solenoidal Lie algebra $W_{B,\gamma}$. Then the theorem follows from the irreducibility criterion of the $W_{B,\gamma}$-module $T(\alpha, \beta)$. $\square$

## 4 The algebra $\mathcal{L}$ and finite dimensional modules

In this section we introduce the related algebra $\mathcal{L}$ and study finite dimensional modules over $\mathcal{L}$. 
Consider the associative algebra \( AC = \mathbb{C}[x_1, \cdots, x_d] \oplus (\mathbb{C}_Q / \mathbb{Z}) \) with \( x_it_j = t_jx_i \) for all \( 1 \leq i, j \leq d \). We simply write \( t^\Gamma \) for the image of \( t^s \) in \( \mathbb{C}_Q / \mathbb{Z} \). Denote by \( \mathcal{L} \) the subalgebra of \( \text{Der}(AC) \) spanned by
\[
\{ x^m d_\gamma, x^n t^\Gamma \mid m \in \mathbb{N}^d \setminus \{0\}, n \in \mathbb{N}^d, \Gamma \in \Gamma \},
\]
where \( d_\gamma = \sum_{i=1}^{d} \gamma_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{d} \gamma_i t_i \frac{\partial}{\partial t_i} \). The Lie bracket of \( \mathcal{L} \) is
\[
[x^m d_\gamma, x^n d_{\delta}] = \sum_{i=1}^{d} \gamma_i (n_i - m_i) x^{m+n-\epsilon_i} d_\gamma;
\]
\[
[x^m d_\gamma, x^n t^\Gamma] = \sum_{i=1}^{d} \gamma_i d_i x^{m+\epsilon_i} t^\Gamma + (\gamma | s) x^{m+1} t^\Gamma;
\]
\[
[x^m t^\Gamma, x^n t^\Gamma] = (\sigma(s, r) - \sigma(s, r)) x^{m+1} t^{\Gamma + \Gamma'},
\]
where \( m, n \in \mathbb{N}^d \setminus \{0\}, p, l \in \mathbb{N}^d \) and \( \Gamma, \tilde{\Gamma} \in \Gamma \). Here we have chosen the representative for \( \tilde{\Gamma} \in \Gamma \) in \( \Gamma_0 \) as usual. The algebra \( \mathcal{L} \) has a \( \Gamma \)-gradation \( \mathcal{L} = \bigoplus_{\Gamma \in \Gamma} \mathcal{L}(\tilde{\Gamma}) \) where
\[
\mathcal{L}(\tilde{\Gamma}) = \text{span}_\mathbb{C} \left\{ x^n t^\tilde{\Gamma} \mid n \in \mathbb{N}^d \right\} ;
\]
\[
\mathcal{L}(\Gamma) = \text{span}_\mathbb{C} \left\{ x^m d_\gamma, x^n t^\Gamma \mid m \in \mathbb{N}^d \setminus \{0\}, n \in \mathbb{N}^d \right\} .
\]
Denote two subalgebras of \( \mathcal{L} \)
\[
\mathcal{L}^{(x)} = \text{span}_\mathbb{C} \left\{ x^m d_\gamma \mid m \in \mathbb{N}^d \setminus \{0\} \right\} , \quad \mathcal{L}^{(t)} = \text{span}_\mathbb{C} \left\{ x^n t^\Gamma \mid n \in \mathbb{N}^d, \Gamma \in \Gamma \right\} .
\]
Set \( \text{deg}(x_i) = 1, \text{deg}(d_\gamma) = -1 \) and \( |m| = \sum_{j=1}^{d} m_j \) for all \( 1 \leq i \leq d \) and \( m \in \mathbb{N}^d \). We get \( \mathbb{Z} \)-gradations for the algebras \( \mathcal{L}^{(x)} = \bigoplus_{i \geq 0} \mathcal{L}^{(x)}_i \) and \( \mathcal{L}^{(t)} = \bigoplus_{i \geq 0} \mathcal{L}^{(t)}_i \) such that
\[
\mathcal{L}^{(x)}_i = \text{span}_\mathbb{C} \left\{ x^m d_\gamma \mid |m| = i + 1 \right\} , \quad \mathcal{L}^{(t)}_i = \text{span}_\mathbb{C} \left\{ x^n t^\Gamma \mid n = i, \Gamma \in \Gamma \right\} .
\]
Set \( \mathcal{L}_i = \mathcal{L}^{(x)}_i \oplus \mathcal{L}^{(t)}_i \). Notice that the derivation \( d_\gamma \) is not homogeneous, hence the algebra \( \mathcal{L} \) is not \( \mathbb{Z} \)-graded. But the subspace \( \mathcal{L}_+ = \bigoplus_{i \geq 1} \mathcal{L}_i \) still makes an ideal of \( \mathcal{L} \). The main result in this section is the following

**Theorem 4.1.** (1) The commutator \( [\mathcal{L}^{(x)}, \mathcal{L}^{(x)}] = [\mathcal{L}^{(x)}_0, \mathcal{L}^{(x)}_0] \oplus \left( \bigoplus_{j \geq 1} \mathcal{L}^{(x)}_j \right) \); (2) Every finite dimensional representation \((U, \rho)\) for \( \mathcal{L}^{(x)} \) satisfies \( \rho(\mathcal{L}^{(x)}_p) = 0 \) for \( p \gg 0 \); (3) Every finite dimensional irreducible \( \mathcal{L}^{(x)} \)-module is one dimensional, and parametrized
by some $\beta \in \mathbb{C}$ such that $x_i \frac{\partial}{\partial x_i} \mapsto \beta \gamma_i$, $1 \leq i \leq d$ and $L_j^{(x)} \mapsto 0$ for $j \geq 1$;

(4) Every finite dimensional representation $(U, \rho)$ for $L$ satisfies $\rho(L_p) = 0$ for $p \gg 0$;

(5) The ideal $L_+$ annihilates every finite dimensional irreducible $L$-module.

**Proof.** The first three statements are exactly the Theorem 3.1 from [BF2]. For (4), consider $U$ as a $L^{(x)}$-module. By (3) we have $\rho(L_p^{(x)}) = 0$ for $p \gg 0$. Then it follows from \([x^m \gamma, t^s] = (\gamma | s)x^m t^s\) that $\rho(\gamma x^m t^s) = 0$ if $|m| > p$. This proves (4).

Let $(V, \rho)$ be a finite dimensional irreducible representation of $L$. By (4) and equation (4.2) we see that $\rho(L^+) = 0$ for all $a \in L^+$. This means $V' \neq 0$. So $V' = V$ by the irreducibility of $V$. Hence $L_+ V = 0$. \[\square\]

For later use we mention that there is an isomorphism from the quotient algebra $L/L_+$ to $\mathfrak{gl}_d(\gamma) \oplus \mathfrak{gl}_N$, where $\mathfrak{gl}_d(\gamma) = \text{span}_\mathbb{C} \{ \epsilon_i \gamma^T \in \mathfrak{gl}_d \mid 1 \leq i \leq d \}$, defined by

\[x_i d_\gamma + L_+ \mapsto \epsilon_i \gamma^T, \quad t^s + L_+ \mapsto X^s. \tag{4.2}\]

Notice that $\mathfrak{gl}_d(\gamma)$ is solvable since $[\mathfrak{gl}_d(\gamma), \mathfrak{gl}_d(\gamma)]$ is abelian.

## 5 Cuspidal $Z\mathfrak{g}$-modules

In this section we study a specific class of cuspidal modules over $\mathfrak{g}$, and prove that the irreducible ones are exactly those modules of tensor fields.

Recall the centre $Z$ of $\mathbb{C}_Q$. We may form an extended Lie algebra $\mathfrak{g} \ltimes Z$. We call a module $V$ for $\mathfrak{g} \ltimes Z$ a $Z\mathfrak{g}$-module provided that the $Z$-action on $V$ is associative.

The following are two examples of $Z\mathfrak{g}$-module. Set $\mathfrak{g}_R' = \text{span}_\mathbb{C} \{ L_s \mid s \notin R \}$, which is an ideal of $\mathfrak{g}$. Then $\mathfrak{g}_R'$ makes a $Z\mathfrak{g}$-module if we define

\[L_m \cdot L_s = [L_m, L_s], \quad t^n \cdot L_s = L_{n+s} \quad \text{for} \quad m \in \mathbb{Z}^d, n \in R \quad \text{and} \quad s \notin R.\]

The second example is the module $V(\alpha, \beta, W) = \bigoplus_{p \in \mathbb{F}} W_p \otimes t^p Z$ of tensor fields for $\mathfrak{g}$ with a $Z$-action given by

\[t^m (w_p \otimes t^{n+s}) = w_p \otimes t^{m+n+s},\]
for $m,n \in R, s \not\in R$ and $w_{\mathfrak{s}} \in W_{\mathfrak{s}}$. Clearly, if $W$ is a $\Gamma$-graded irreducible $\mathfrak{gl}_\mathbb{R}$-module, then $\mathcal{V}(\alpha, \beta, W)$ is irreducible as a $\mathbb{Z}\mathfrak{g}$-module for any $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$.

Now let $M$ be a cuspidal $\mathbb{Z}\mathfrak{g}$-module with support lying in $\alpha + \mathbb{Z}^d$. Since the $\mathbb{Z}$-action is associative, $M$ can be represented as

$$M \cong \bigoplus_{\mathfrak{s} \in \Gamma} U_{\mathfrak{s}} \otimes \mathbb{Z},$$

where $U_{\mathfrak{s}} = M_{\alpha+s}$ for $s \in \Gamma_0$. Set $U = \bigoplus_{\mathfrak{s} \in \Gamma} U_{\mathfrak{s}}$.

For later use we should need some operators on $M$. For $m \in R, r \not\in R$, consider

$$D(m) = t^{-m}L_m, \quad D(m,r) = t^{-m}L_{m+r}.$$  

The operator $D(m)$ may be restricted to each $U_{\mathfrak{s}}$, and $D(m,r)$ may be restricted to an operator $U_{\mathfrak{s}} \longrightarrow U_{\mathfrak{s}+s}$. The operators $D(m), D(m,r)$ completely determine the $\mathfrak{g}$-action on $M$, since  

$$L_m(t^n v_{\mathfrak{s}}) = (\gamma | n) t^{m+n} v_{\mathfrak{s}} + t^{m+n} D(m) v_{\mathfrak{s}}$$

$$L_{m+r}(t^n v_{\mathfrak{s}}) = t^n (L_{m+r} v_{\mathfrak{s}}) = t^{m+n} (D(m,r) v_{\mathfrak{s}}).$$

The following lemma is easy to check.

**Lemma 5.1.** For $m,n \in R$ and $r,s \not\in R$, we have

$$[D(m), D(n)] = (\gamma | m)(D(m+n) - D(m)) - (\gamma | n)(D(m+n) - D(n));$$

$$[D(m), D(n,s)] = (\gamma | n+s)D(m+n,s) - (\gamma | n)D(n,s);$$

$$[D(m,r), D(n,s)] = (\sigma(r,s) - \sigma(s,r))D(m + n, r + s).$$

**Proposition 5.2.** Let $M$ be a cuspidal $\mathbb{Z}\mathfrak{g}$-module with support lying in $\alpha + \mathbb{Z}^d$. We may write $M \cong \bigoplus_{\mathfrak{s} \in \Gamma} U_{\mathfrak{s}} \otimes \mathbb{Z}$, where $U_{\mathfrak{s}} = M_{\alpha+s}$ for $s \in \Gamma_0$. Then the action of $\mathfrak{g}$ on $M$ is given by

$$L_m(v_{\mathfrak{s}} \otimes t^n) = ((\gamma | n) + D(m)) v_{\mathfrak{s}} \otimes t^{m+n};$$

$$L_{m+r}(v_{\mathfrak{s}} \otimes t^n) = D(m,r) \cdot v_{\mathfrak{s}} \otimes t^{m+n},$$

for $m,n \in R, r \in \Gamma_0 \setminus \{0\}$ and $v_{\mathfrak{s}} \in U_{\mathfrak{s}}$. Here the operators $D(m) : U_{\mathfrak{s}} \longrightarrow U_{\mathfrak{s}}$ for each $\mathfrak{s} \in \Gamma$, can be expressed as End($U_{\mathfrak{s}}$)-valued polynomial in $m$ with constant term $D(0) = (\gamma | \alpha + s) Id$, and the operators $D(m,r) : U_{\mathfrak{s}} \longrightarrow U_{\mathfrak{s}+r}$ for each $\mathfrak{s} \in \Gamma$, can be expressed as Hom($U_{\mathfrak{s}}, U_{\mathfrak{s}+r}$)-valued polynomial in $m$ whose constant term $D(0,r)$ has the form of an upper triangular matrix relative to suitable bases of $U_{\mathfrak{s}}$ and $U_{\mathfrak{s}+r}$. 

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Proof. Notice that $g_R$ is isomorphic to the solenoidal Lie algebra $W_{B\gamma}$. The part concerning the action of $g_R$ and $D(m)$ is just what was stated in Theorem 4.5 in [BF1]. Next we prove the $g_R$-action by induction on $d$. If $d = 1$, then the algebra $g$ shrinks either to a gap-$p$ Virasoro algebra (for suitable $p$), in which case the result follows from [Xu2], or to a normal Virasoro algebra, in which case $g'_R = 0$ and there is nothing to prove.

Now let us establish the induction step as follows. By induction assumption the operators $D(m - m_i e_i, r - r_i e_i)$ have polynomial dependence on $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_d$, and $D(m_i e_i, r_i e_i)$ is a polynomial on $m_i$, for all $i = 1, 2, \cdots, d$ and $r \notin R, r_i e_i \notin R$ and $r - r_i e_i \notin R$. Note that $m, m_i e_i \in R$. From

$$[D(0, r_i e_i), D(m - m_i e_i, r - r_i e_i)] = (\sigma(r_i e_i, r - r_i e_i) - \sigma(r - r_i e_i, r_i e_i))D(m - m_i e_i, r)$$

we see that $D(m - m_i e_i, r)$ is a polynomial on $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_d$. Then consider

$$[D(m_i e_i, D(m - m_i e_i, r)] = (\gamma | m - m_i e_i + r)D(m, r) - (\gamma | m - m_i e_i)D(m - m_i e_i, r)$$

we see that $D(m, r)$ may be expressed as a rational function $\frac{P_1(m)}{P_2(m)}$, where

$$F_i(m) = (\gamma | m - m_i e_i + r) = \gamma_i r_i + \sum_{j \neq i} \gamma_j (m_j + r_j) \neq 0$$

since $\gamma$ is generic and $m_j + r_j \neq 0$ for all $j \neq i$. Especially we have $\frac{P_1(m)}{P_2(m)} = \frac{P_1(m)}{P_2(m)}$ in $\text{Hom}(U_g, \overline{U_{r + \alpha}}) \otimes \mathbb{C}(m_1, \cdots, m_d)$. Hence

$$P_1(m)F_2(m) = P_2(m)F_1(m).$$

Notice that $F_1(m), F_2(m)$ are coprime to each other and the ring $\mathbb{C}[m_1, \cdots, m_d]$ is a unique factorization domain. It follows that $F_1(m)$ divides $P_1(m)$. So $D(m, r)$ is a polynomial.

It remains to show that the value of the polynomial $D(m, r)$ at $m = 0$ coincides with $D(0, r)$ on $U_r$. Let $k$ be the order of the image $\overline{F} \in \Gamma$ of $r$, and we have $k r \in R$. Denote by $g(r)$ the Lie subalgebra of $g$ generated by $\{L_{lr} | l \in \mathbb{Z}\}$, which is a gap-$k$ Virasoro algebra. Consider $M$ as a $g(r)$-module with composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M.$$

Each quotient $M_i/M_{i-1}$ is an irreducible cuspidal $g(r)$-module, and by [Xu2] has weight multiplicities no more than one. It is clear that $l$ is the maximal dimension of the weight
spaces of $M$. Then, according to this composition series, one can choose a basis of $M$, with respect to which the operator $D(0, r) = L_r$ has the form of upper triangular matrices of order $l$ on all weight spaces of $M$, hence on $U_\pi$. Moreover, from the construction of module of intermediate series from [Xu2], we see that $D(kr, r)$ has the same action as $D(0, r)$ on $U_\pi$. Notice that the constant term $D(0)$ of $D(m - kr)$ acts as $(\gamma | \alpha + s)\text{Id}$ on $U_\pi$, and as $(\gamma | \alpha + r + s)\text{Id}$ on $U_{\pi+s}$. By considering the constant term in both sides of

$$[D(m - kr), D(kr, r)] = (k + 1)(\gamma | r)D(m, r) - k(\gamma | r)D(kr, r),$$

we see that the value of $D(m, r)$ at $m = 0$ equals to $D(0, r)$ on $U_\pi$. □

Define operators $\partial^p_0 \in \text{End}(U_\pi)$ and $\partial^p_r \in \text{Hom}(U_\pi, U_{\pi+s})$ for $r \in \Gamma_0 \setminus \{0\}$, by expansions of the polynomials $D(m)$ and $D(m, r)$ in $m$:

$$D(m) = \sum_{p \in \mathbb{N}^d} \frac{mp}{p!} \partial^p_0; \quad D(m, r) = \sum_{p \in \mathbb{N}^d} \frac{mp}{p!} \partial^p_r,$$

with only finite number of the operators $\partial^p_0, \partial^p_r$ being nonzero. Here $p! = p_1! \cdots p_d!$ and $m^p = m_1^{p_1} \cdots m_d^{p_d}$.

Now we expand the commutator $[D(m), D(n)]$ in $m, n$,

$$\sum_{p, l \in \mathbb{N}^d} \frac{mp}{p!} \frac{nl_1}{l_1!} [\partial^p_0, \partial^l_0] = (\gamma | m) \sum_{q \in \mathbb{N}^d} \frac{(m + n)^q - m^q}{q!} \partial^q_0 - (\gamma | n) \sum_{q \in \mathbb{N}^d} \frac{(m + n)^q - n^q}{q!} \partial^q_0.$$

Comparing the coefficients at $\frac{mp}{p!} \frac{nl_1}{l_1!}$ in both sides, we get the Lie bracket of $\partial^p_0$

$$[\partial^p_0, \partial^l_0] = \begin{cases} \sum_{i=1}^d \gamma_i (l_i - p_i) \partial^{p+1}_{0-\epsilon_i} & \text{if } p, l \neq 0; \\ 0 & \text{if } p = 0 \text{ or } l = 0. \end{cases} \tag{5.1}$$

Similarly by expanding the commutators $[D(m), D(n, s)]$ and $[D(m, r), D(n, s)]$, then comparing coefficients at $\frac{mp}{p!} \frac{nl_1}{l_1!}$ in both sides, we get the Lie bracket of $[\partial^p_0, \partial^l_s]$

$$[\partial^p_0, \partial^l_s] = \begin{cases} (\gamma | s) \partial^{p}_{s} + \sum_{i=1}^d \gamma_i l_i \partial^{p+1}_{s-\epsilon_i} & \text{if } p \neq 0; \\ (\gamma | s) \partial^{l}_{s} & \text{if } p = 0, \end{cases} \tag{5.2}$$

and the Lie bracket of $[\partial^p_r, \partial^l_s]$

$$[\partial^p_r, \partial^l_s] = (\sigma(r, s) - \sigma(s, r)) \partial^{p+1}_{r+s}. \tag{5.3}$$
Recall the Lie bracket of the algebra \( \mathcal{L} \). The equations (5.1), (5.2) and (5.3) imply that the operators \( \{ \partial^p_0, \partial^s_l \mid p \in \mathbb{N}^d \setminus \{0\}, s \in \Gamma_0 \setminus \{0\}, l \in \mathbb{N}^d \} \) yield a finite dimensional \( \Gamma \)-graded representation of the algebra \( \mathcal{L} \) on the space \( U \). Denote by \( \rho \) this representation map. Combining with Proposition 5.2 we get the following

Theorem 5.3. There exists an equivalence between the category of finite dimensional \( \Gamma \)-graded \( \mathcal{L} \)-modules and the category of cuspidal \( \mathcal{Z}_\mathfrak{g} \)-modules with support lying in some coset \( \alpha + \mathbb{Z}^d \). The equivalence functor associates to a finite dimensional \( \Gamma \)-graded \( \mathcal{L} \)-module \( U = \bigoplus_{\pi \in \Gamma} U_\pi \) an \( \mathcal{Z}_\mathfrak{g} \)-module

\[
M = \bigoplus_{\pi \in \Gamma} U_\pi \otimes t^s \mathcal{Z}
\]

with the \( \mathfrak{g} \)-action

\[
L_m(v_\pi \otimes t^{s+n}) = \left( (\gamma | \alpha + n + s) \text{Id} + \sum_{p \in \mathbb{N}^d \setminus \{0\}} \frac{m^p}{p!} \rho(x^p d_\gamma) \right) v_\pi \otimes t^{s+n+m} ;
\]

\[
L_{m+r}(v_\pi \otimes t^{s+n}) = \sum_{p \in \mathbb{N}^d} \frac{m^p}{p!} \rho(x^p t^r) \cdot v_\pi \otimes t^{s+n+m+r},
\]

where \( m, n \in \mathbb{R}, r \notin \mathbb{R}, s \in \Gamma_0 \) and \( v_\pi \in U_\pi \).

Clearly, the subset of \( s \in \Gamma_0 \) with \( U_\pi \neq 0 \) equals to the subset of \( s \in \Gamma_0 \) with \( M_{\alpha+s} \neq 0 \). Furthermore, we have

Theorem 5.4. Every irreducible cuspidal \( \mathcal{Z}_\mathfrak{g} \)-module is isomorphic to some \( \mathcal{V}(\alpha, \beta, W) \), where \( \alpha \in \mathbb{C}^d, \beta \in \mathbb{C} \) and \( W \) is a finite dimensional \( \Gamma \)-graded irreducible \( \mathfrak{gl}_N \)-module.

Proof. By Theorem 5.3 \( U = \bigoplus_{\pi \in \Gamma} U_\pi \) is a finite dimensional \( \Gamma \)-graded irreducible \( \mathcal{L} \)-module. Hence \( \mathcal{L} U = 0 \) by Theorem 4.1(5). Notice that \( \mathcal{L}_+ \) is also \( \Gamma \)-graded. It reduces \( U \) to a \( \Gamma \)-graded irreducible module over \( \mathcal{L}/\mathcal{L}_+ \), which is isomorphic to \( \mathfrak{gl}_d(\gamma) \oplus \mathfrak{gl}_N \).

On the other hand, since \( \mathcal{L}^{(t)} \) is an ideal of \( \mathcal{L} \), we obtain \( \mathcal{L}^{(t)} U = 0 \) or \( U \) by the irreducibility of \( U \). If \( \mathcal{L}^{(t)} U = 0 \), then \( L_s M = 0 \) for all \( s \notin R \). Hence \( M \) is a \( \mathfrak{g}_R \)-module, and the result follows from [BF2].

Suppose \( \mathcal{L}^{(t)} U = U \). Notice that the ideal \( \mathcal{L}_+ \) is \( \Gamma \)-graded and annihilates \( U \). Therefore, \( U \) becomes a \( \Gamma \)-graded module over \( \mathcal{L}^{(t)}/\mathcal{L}_+ \), which is isomorphic to \( \mathfrak{gl}_N \). Then the equation (5.4) implies

\[
L_{m+r}(v_\pi \otimes t^{s+n}) = X^r v_\pi \otimes t^{s+n+m+r}.
\]
We identify elements of \( \mathfrak{gl}_d(\gamma) \) and that of \( L^{(x)}/L_+ \) through the isomorphism given in \([4,2]\). Notice that \( \mathfrak{gl}_d(\gamma) \) is solvable. By Lie’s Theorem, there exists a common eigenvector \( v \in U \) for \( \mathfrak{gl}_d(\gamma) \) such that
\[
(\epsilon_i \gamma^T)v = (x_id_\gamma)v = \beta_i v, \quad \beta_i \in \mathbb{C}, \quad \text{for all } 1 \leq i \leq d.
\]
Set \( \beta = \frac{\beta_i}{\gamma_1} \). Since \( 0 = [x_i d_\gamma, x_1 d_\gamma]v = (\beta_i \gamma_1 - \beta_1 \gamma_i)v \), we obtain \( \beta_i = \gamma_i \beta \) for all \( 1 \leq i \leq d \).

Since \( U \) is irreducible as a \( L/L_+ \)-module, \( v \) can generate any vector in \( U \) only by applying \( t^\mathfrak{g} \) and \( \epsilon_i \gamma^T \). For any \( s \in \Gamma \), we have
\[
(\epsilon_i \gamma^T)(t^\mathfrak{g}v) = (x_i d_\gamma)(t^\mathfrak{g}v) = [x_i d_\gamma, t^\mathfrak{g}](x_i d_\gamma)v = (\gamma | s)(x_i t^\mathfrak{g})v + \beta_i t^\mathfrak{g}v = \beta_i \gamma t^\mathfrak{g}v.
\]
This implies that all vectors in \( U \) are common eigenvectors for \( \mathfrak{gl}_d(\gamma) \), and that
\[
(m \gamma^T)w = (\sum_{i=1}^d m_i x_i d_\gamma)w = \beta(\gamma | m)w, \quad \text{for any } w \in U, m \in \mathbb{Z}^d.
\]
It remains only to show that \( U \) is irreducible as a \( \Gamma \)-graded \( \mathfrak{gl}_N \)-module. Let \( V \) be a nonzero \( \Gamma \)-graded \( \mathfrak{gl}_N \)-submodule. Since all vectors in \( V \) are common eigenvectors for \( \mathfrak{gl}_d(\gamma) \), \( V \) becomes a \( \mathfrak{gl}_d(\gamma) \oplus \mathfrak{gl}_N \)-module, hence equals to \( U \) by the irreducibility of \( U \). This completes the proof. \( \square \)

6 Classification of irreducible cuspidal \( \mathfrak{g} \)-modules

In this last section we use the modified cover method to prove the following

**Theorem 6.1.** Let \( M \) be an irreducible cuspidal \( \mathfrak{g} \)-module. Then \( M \) is isomorphic to an irreducible subquotient of some \( \mathcal{V}(\alpha, \beta, W) \), where \( \alpha \in \mathbb{C}^d, \beta \in \mathbb{C} \) and \( W \) is a finite dimensional \( \Gamma \)-graded irreducible \( \mathfrak{gl}_N \)-module.

From now on we fix an cuspidal \( \mathfrak{g} \)-module \( M \)(not necessarily irreducible). Recall the notion of \( Z\mathfrak{g} \)-module. The proof of the following Lemma is just a elementary check, and we omit it.
Lemma 6.2. (1) The tensor product $g'_R \otimes M$ of $g$-modules admits a $Zg$-module structure if we define the $Z$-action by

\[ t^n(L_s \otimes v) = L_{n+s} \otimes v \quad \text{for} \quad n \in R, s \notin R, v \in M. \]

(2) The map $\pi : g'_R \otimes M \rightarrow M$ defined by $y \otimes v \mapsto y \cdot v$ is a $g$-module homomorphism, and it is surjective if $g'_R M = M$.

(3) The subspace $J$ of $g'_R \otimes M$ spanned by vectors with the form of a finite sum $\sum L_s \otimes v_s$, where $s \notin R, v_s \in M$, and $\sum L_{n+s}v_s = 0$ for all $n \in R$, is a $Zg$-submodule of $g'_R \otimes M$, and lies in $\ker \pi$.

Denote $\hat{M} = (g'_R \otimes M)/J$. We call this quotient $Zg$-module the $Z$-cover of $M$.

Proposition 6.3. The $Zg$-module $\hat{M}$ is cuspidal.

Proof. Notice that $\Gamma$ is a finite group. Consider $M$ as a cuspidal $g_R$-module. Then $\hat{M}$ is a cuspidal $Zg_R$-module, and this proposition follows from [BF2]. \[ \square \]

Proof of Theorem 6.1. Now assume further that $M$ is irreducible. If $g'_R M = 0$, then $M$ is irreducible as a $g_R$-module. The result follows from [BF2] or [Su].

Assume $g'_R M \neq 0$. Then $g'_R M = M$ by the irreducibility of $M$. Since $J \subseteq \ker \pi$, we get a surjective $g$-module homomorphism $\hat{\pi} : \hat{M} \rightarrow M$. Consider the composition series for the $Zg$-module $\hat{M}$,

\[ 0 = \hat{M}_0 \subset \hat{M}_1 \subset \cdots \subset \hat{M}_t = \hat{M}, \]

where all $\hat{M}_i/\hat{M}_{i-1}$ are irreducible $Zg$-modules. Let $k$ be the smallest integer such that $\hat{\pi}(\hat{M}_k) \neq 0$. Then we have $\hat{\pi}(\hat{M}_k) = M$ and $\hat{\pi}(\hat{M}_{k-1}) = 0$, which reduces $\hat{\pi}$ to a $g$-module epimorphism from $\hat{M}_k/\hat{M}_{k-1}$ to $M$. Then Theorem 6.1 follows from Theorem 5.4.

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