ON A CERTAIN APPROACH TO QUANTUM HOMOGENEOUS SPACES

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Abstract. We propose a definition of a quantum homogeneous space of a locally compact quantum group. We show that classically it reduces to the notion of homogeneous spaces, giving rise to an operator algebraic characterization of the transitive group actions. On the quantum level our definition goes beyond the quotient case providing a framework which, besides the Vaes' quotient of a locally compact quantum group by its closed quantum subgroup (our main motivation) is also compatible with, generically non-quotient, quantum homogeneous spaces of a compact quantum group studied by P. Podleś as well as the Rieffel deformation of \( G \)-homogeneous spaces. Finally, our definition rules out the paradoxical examples of the non-compact quantum homogeneous spaces of a compact quantum group.

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1. Introduction

Although the theory of locally compact quantum groups is already well established (see [14], [18]), the quantum counterpart of the transitive group actions is not yet known. One aspect of quantum homogeneous spaces which is not thoroughly understood is related with the Vaes' construction [26] of the quotient of a locally compact quantum group group by its closed quantum subgroup: in order to prove the existence of the quantum quotient, the regularity of a given quantum group is needed. The second difficulty is related with the non-quotient type of a generic quantum homogeneous space which is expected since the construction of Podleś spheres in [19].

In the case of compact quantum groups the situation considerably simplifies. It easy to realize that the quantum counterpart of a transitive group action is provided by the concept of ergodic coaction (see [3], [19]). Adopting this view point, the harmonic analysis on the homogeneous spaces may be extended to the quantum setting. A quantum counterpart of a given classical result may happen to be not totally straightforward which is exemplified by the following surprising fact: the multiplicity of an irreducible subrepresentation entering a quantum homogeneous space, though always finite, may exceed the classical dimension of this representation - for the explicit examples we refer to [2]. In order to formulate the quantum version of the discussed inequality the quantum...
dimension was introduced in \cite{3}. In spite of some subtleties, the theory of quantum homogeneous spaces of compact quantum groups is satisfactory.

As soon as we drop the compactness assumption we are immediately challenged with a problem of formulating an appropriate definition of a quantum homogeneous space. $C^*$-algebraic methods that prove their strength in the compact case seem to be not efficient enough to capture the homogeneity of a non-compact type classical space. To be more precise, suppose that $X$ is a homogeneous space of a locally compact but non-compact group $G$. The problem that one is confronted with is to express the transitivity of the action of $G$ on $X$ by some properties of the $G$-$C^*$-algebra $C_0(X)$. Even such a basic question has so far not been answered.

In this paper we propose a definition of a quantum homogeneous space (QHS) which goes beyond the compact and the quotient case. Our approach is motivated by the Vaes’ construction of a quantum quotient of a locally compact quantum group by its closed quantum subgroup and is based on the interplay between the $C^*$-algebraic and the von Neumann version of a QHS. Classically, our definition reduces to the notion of homogeneous spaces, providing an operator algebraic characterization of the transitive group actions which does not invoke the identification with a quotient space. Our definition is compatible with different classes of examples of quantum homogeneous spaces such as: quantum homogeneous spaces of the quotient type due to S. Vaes \cite{26}, quantum homogeneous spaces of a compact quantum group studied by P. Podleś \cite{20} and the Rieffel deformation of $G$-homogeneous spaces - see \cite{10}. On the other hand we are able to rule out the paradoxical examples of non-compact quantum homogeneous spaces of a compact quantum group provided by S.L. Woronowicz \cite{32}.

Although the above advantages, the proposed theory of QHS is not yet satisfactory. It is expected that in the non-regular quotient case there should exist examples which does not fit into it, not mentioning that dropping the regularity assumption one does not even know how to construct the quantum quotient. Some of the experts suggested to base the construction of the quotient-type QHS on the integration along the quantum subgroup (private communication by S.L. Woronowicz, R. Meyer). The mathematical tools for this construction already exist (see \cite{4, 23}) but the theory of QHS which uses it is not formulated.

Let us describe the contents of the paper. In Section \ref{sec:2} we recall the basic definitions and fix a notation concerning locally compact quantum groups. In particular we specify the category $C(G)$ of $G$-$C^*$-algebras. In Section \ref{sec:3} the definition of a quantum homogeneous space is formulated. As was already mentioned our approach is based on the interplay between the $C^*$- and the von Neumann algebraic version of a given QHS. We prove that they uniquely determine each other. In Section \ref{sec:4} we show that the $C^*$-algebraic version of a QHS is a $G$-simple object in $C(G)$. This implication is a quantum counterpart of the classically trivial fact that the homogeneity of a space is a stronger property than the density of each orbit. It may be surprising that in order to prove the quantum version of this implication one has to use the Tomita-Takesaki theory employed in the construction of the canonical implementation of a coaction whose existence was proved by S. Vaes \cite{27}. In Section \ref{sec:5} we prove that there is a 1-1 correspondence between the homogeneous $G$-spaces and quantum homogeneous $G$-spaces with the underlying von Neumann algebra being commutative. In Section \ref{sec:6} we show that our notion of a QHS when restricted to compact quantum groups boils down to the standard notion of a quantum homogeneous space introduced by P. Podleś \cite{20}. Section \ref{sec:7} contains the proof of the fact that the Rieffel deformation of a homogeneous space is a quantum homogeneous space. It is a generalization of the result of \cite{10} which deals with the case when the deformed QHS is of the quotient type.

Throughout the paper we freely use the $C^*$-algebraic concepts such as multipliers $M(A)$ of a given $C^*$-algebra $A$ and morphisms $\text{Mor}(A, B)$ from $A$ to a $C^*$-algebra $B$. Our main reference for the $C^*$-algebraic notions is \cite{25}. For the theory of von Neumann algebras we refer to one of the standard textbooks (e.g. \cite{25}). We often use the \textit{leg numbering notation}. Consider a tensor triple of Hilbert spaces $H_1 \otimes H_2 \otimes H_3$. For $X \in B(H_1 \otimes H_2)$ we define $X_{12} = X \otimes 1$. We analogously define $Y_{23}$ and $Z_{13}$. The leg numbering notation passes naturally to the context of elements of the triple tensor products of $C^*$- or von Neumann algebras.

I would like to express my gratitude to A. Skalski for reading a preliminary version of this paper and giving the hints that helped to improve it.
We say that $\alpha$ coaction of and let Definition 2.2. $\phi$ be a normal, semi-finite, faithful (n.s.f.) weight on a von Neumann algebra $M$. Then we shall denote:

$$M_\phi^+ = \{x \in M^+ | \phi(x) < \infty\} \text{ and } N_\phi = \{x \in M | \phi(x^*x) < \infty\}.$$

**Definition 2.1.** A pair $G = (M, \Delta)$ is called a locally compact quantum group when

- $M$ is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation: $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$;
- there exist n.s.f. weights $\phi$ and $\psi$ on $M$ such that
  - $\phi$ is left invariant: $\phi((\omega \otimes 1)\Delta(x)) = \phi(x)\omega(1)$ for all $x \in M_\phi^+$ and $\omega \in M_\psi^+$,
  - $\psi$ is right invariant: $\psi((1 \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$ for all $x \in M_\psi^+$ and $\omega \in M_\phi^+$.

In the first step of the development of the theory one moves on to the Hilbert space level. Let $(L^2(G), \pi, \Lambda_\phi)$ be the GNS-triple corresponding to the weight $\phi$, where $L^2(G)$ is the completion of $N_\phi$ in the norm $||x||_2^2 = \phi(x^*x)$. The von Neumann algebra $\hat{M}$ is the completion of $\{\phi \otimes \pi(W) | \phi \in N_\phi \}$, also referred as the left regular corepresentation, is defined by the formula:

$$W^*(\Lambda_\phi(a) \otimes \Lambda_\phi(b)) = (\Lambda_\phi \otimes \Lambda_\phi)(\Delta(b)(a \otimes 1)) \text{ for any } a, b \in N_\phi.$$

One proves that the von Neumann algebra $M$ may be recovered as the strong closure of the algebra $\{\phi \otimes \pi(W) | \omega \in B(L^2(G))\}$. In turn, the comultiplication $\Delta$ is implemented by $W$.

$$\Delta(x) = W^*(1 \otimes x)W \text{ for any } x \in \hat{M}.$$

The operator $W$ encodes the pair $(M, \Delta)$.

One of the trademarks of the LCQG theory is a well established duality - a broad generalization of the Pontryagin duality for locally compact abelian groups. Starting with $G$ one constructs the dual quantum group $\hat{G} = (\hat{M}, \hat{\Delta})$. The duality, on the level of the multiplicative units, is manifested as follows: the multiplicative unitary related with $\hat{G}$ is given by $\hat{W} = \Sigma W^*$ where $\Sigma : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$ is the flip operator. The von Neumann algebra $\hat{M}$ is the strong closure of the algebra $\{\phi \otimes \pi(W) | \omega \in B(L^2(G))\}$ and the comultiplication $\hat{\Delta}$ is implemented by $\hat{W}$.

**Definition 2.2.** Let $G = (M, \Delta)$ be a locally compact quantum group, $N$ a von Neumann algebra and let $\alpha : N \to M \otimes N$ be a normal, injective, unital $*$-homomorphism. We say that $\alpha$ is a left coaction of $G$ on $N$ when $(\iota \otimes \alpha) \circ \alpha = (\Delta \otimes \iota) \circ \alpha$. We define the algebra of coinvariants $N^\alpha$:

$$N^\alpha = \{x \in N | \alpha(x) = 1 \otimes x\}.$$

We say that $\alpha$ is ergodic when $N^\alpha = \mathbb{C}1$. 2. Preliminaries

In order to define quantum homogeneous spaces we shall first recall basic definitions and facts concerning locally compact quantum groups (LCQG). Our main reference for this subject is [14] and [15]. This section to some extents follows Section 2 of [26].
Let \( \alpha : N \to M \otimes N \) be a coaction of \( G \), \( \theta : N^+ \to [0, \infty] \) a n.s.f. weight and let \( K \) be the Hilbert space of the GNS-construction based on \( \theta \). The antilinear Tomita-Takesaki conjugation will be denoted by \( J_\theta : K \to K \). Combining Proposition 3.7, Proposition 3.12 and Theorem 4.4 of [27] one may prove the quantum counterpart of the Haagerup’s theorem on the existence of the canonical unitary implementation of an action of a locally compact group on a von Neumann algebra.

**Theorem 2.3.** Let us adopt the notation introduced above. There exists a unitary operator \( U \in M \otimes B(K) \) such that

\[
\begin{align*}
(\Delta \otimes \iota)U &= U_{23}U_{13}; \\
\alpha(x) &= U(1 \otimes x)U^*; \\
(J \otimes J_\theta)U(J \otimes J_\theta) &= U^*.
\end{align*}
\]

In order to describe the coactions of \( G \) on the \( C^* \)-algebras we shall introduce the category \( C(G) \) of \( G \)-\( C^* \)-algebras - a quantum counterpart of the category \( C(G) \) of \( G \)-\( C^* \)-algebras (see e.g. [8]). We adopt the following notation: the closed linear span of a subset \( \mathcal{W} \subset \mathcal{V} \) of a Banach space \( \mathcal{V} \) will be denoted by \( [\mathcal{W}] \). The definition of \( G \)-\( C^* \)-algebras may be traced back to [3] and [20].

**Definition 2.4.** Let \( G \) be a LCQG. A \( G \)-\( C^* \)-algebra is a pair \( \mathcal{D} = (D, \Delta_D) \) consisting of a \( C^* \)-algebra \( D \) and an injective coaction \( \Delta_D \in Mor(D, A \otimes D) \):

\[
\begin{align*}
(\iota \otimes \Delta_D) \circ \Delta_D &= (\Delta \otimes \iota) \circ \Delta_D,
\end{align*}
\]

which is continuous: \( [\Delta_D(D)(A \otimes 1)] = C_0(G) \otimes D \). The \( C^* \)-algebra \( D \) will also be denoted by \( C_0(G) \) and the coaction \( \Delta_D \) will be denoted by \( \Delta_G \).

In accordance with the above definition we shall use the notation \( C_0(G) \) and \( \Delta_G \). We shall also denote \( M \) by \( L^\infty(G) \). In order to specify the morphism in the category of \( G \)-\( C^* \)-algebra we adopt the following definition.

**Definition 2.5.** Let \( G \) be a LCQG and suppose that \( B \) and \( D \) are \( G \)-\( C^* \)-algebras. We say that a morphism \( \pi \in Mor(\mathcal{C}(B), \mathcal{C}(D)) \) is a \( G \)-morphism if \( \Delta_B \circ \pi = (\iota \otimes \pi) \circ \Delta_B \). The set of \( G \)-morphism from \( B \) to \( D \) will be denoted by \( Mor_G(B, D) \).

Let \( \mathcal{D} \) be a \( G \)-\( C^* \)-algebra. One may extend the classical crossed product construction to this case defining the reduced crossed product \( \hat{\mathcal{G}} \)-\( C^* \)-algebra \( \mathcal{G} \ltimes \mathcal{C}_0(\mathcal{D}) \), where \( \hat{\mathcal{G}} \)-\( C^* \)-algebra is a pair \( \hat{\mathcal{G}} \)-\( C^* \)-algebra (see e.g. [8]). For the details of this construction we refer to [26]. Let us only recall the definition of the \( C^* \)-algebra \( \mathcal{G} \ltimes \mathcal{C}_0(\mathcal{D}) \): \( \mathcal{G} \ltimes \mathcal{C}_0(\mathcal{D}) = [\Delta_D(\mathcal{C}_0(\mathcal{D}))(\mathcal{C}_0(\hat{\mathcal{G}}) \otimes 1)] \).

In the course of the paper we shall represent the von Neumann algebras on the \( C^* \)-Hilbert modules - the reference for the subject is the standard textbook [16]. Let us remind that for any \( C^* \)-Hilbert module \( \mathcal{E} \) we may define the \( C^* \)-algebra of adjointable operators:

\[
\mathcal{L}(\mathcal{E}) = \{ T : \mathcal{E} \to \mathcal{E} \mid \exists T^* : \mathcal{E} \to \mathcal{E} \text{ s.t. } (e_1 | T e_2) = (T^* e_1 | e_2) \text{ for any } e_1, e_2 \in \mathcal{E} \}.
\]

Let \( \mathcal{K}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E}) \) be the ideal of compact operators. We shall often identify \( \mathcal{L}(\mathcal{E}) \) with \( M(\mathcal{K}(\mathcal{E})) \). The following definition is due to S. Vaes - see Definition 3.1 [29].

**Definition 2.6.** Let \( N \) be a von Neumann algebra and \( \mathcal{E} \) a \( C^* \)-Hilbert module. A unital \( * \)-homomorphism \( \pi : N \to \mathcal{L}(\mathcal{E}) \) is said to be strict if it is \( * \)-strongly continuous on the unit ball of \( N \).

The strictness of a \( * \)-homomorphism \( \pi : N \to \mathcal{L}(\mathcal{E}) \) means that for any \( * \)-strongly convergent, uniformly bounded net \( x_i \in N \) and any \( v \in \mathcal{E} \), the nets \( \pi(x_i)v \) and \( \pi(x_i^*)v \) are norm convergent in \( \mathcal{E} \).

**Remark 2.7.** Let \( N \) be a von Neumann algebra and \( p \in N \) a central projection in \( N \). We may define a weakly closed 2-sided ideal generated by \( p \): \( I = pN \). This establishes a 1-1 correspondence between the central projections in \( N \) and its 2-sided ideals.
3. Quantum Homogeneous Spaces - Definition and the Uniqueness Results

Let $\mathbb{G}$ be a locally compact quantum group. Our definition of a quantum homogeneous space is motivated by the Vaes’ construction of the quotient of a locally compact quantum group by a closed quantum subgroup and is based on the interplay between the C*-algebraic and the von Neumann version of it. Let us first formulate the definition leaving the explanation of motivation for later.

**Definition 3.1.** Let $N$ be a von Neumann algebra and let $\Delta_N : N \to L^\infty(\mathbb{G}) \hat{\otimes} N$ be an ergodic coaction of a locally compact quantum group $\mathbb{G}$. We say that $(N, \Delta_N)$ is a quantum homogeneous space (QHS) if there exists a $\mathbb{G}$-$\ast$-algebra $\mathbb{D}$ such that:

- $C_0(\mathbb{D})$ is a strongly dense C*-subalgebra of $N$;
- $\Delta_\mathbb{D}$ is given by the restriction of $\Delta_N$ to $C_0(\mathbb{D})$;
- $\Delta_N(N) \subseteq M(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}))$ and the map $\Delta_N : N \to \mathcal{L}(L^2(\mathbb{G}) \otimes C_0(\mathbb{D}))$ is strict.

**Remark 3.2.** It turns out that the form of Definition 3.1 is not optimal. Indeed, one may replace the last condition by an apparently weaker condition:

- For any uniformly bounded net $x_i \in N$ converging *-strongly to $x \in N$ and for any $y \in K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D})$, the net $\Delta_N(x_i)y$ converges in the norm topology to $\Delta_N(x)y$.

In order to see that the above condition implies the third condition of Definition 3.1 let us fix $x \in N$ and a uniformly bounded net $d_i \in C_0(\mathbb{D})$ that *-converges to $x$. Then $\Delta_\mathbb{D}(d_i) \in M(C_0(\mathbb{G}) \otimes C_0(\mathbb{D})) \subseteq M(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}))$ and the above condition implies that $\Delta_N(d_i)$ converges to $\Delta_N(x)$ in the strict topology of $M(K(L^2(\mathbb{G})) \otimes D)$. In particular $\Delta_N(x) \in M(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}))$ and $\Delta_N$ gives rise to a strict map $\Delta_N : N \to \mathcal{L}(L^2(\mathbb{G}) \otimes C_0(\mathbb{D}))$.

**Remark 3.3.** Let $G_1$ be a closed quantum subgroup of $\mathbb{G}$ in the sense of Definition 2.5 of [20]. Using the results of Section 4 of [20] we may construct the von Neumann version of a quantum quotient space $Q \subset L^\infty(\mathbb{G})$. Definition 3.1 was motivated by Theorem 6.1 of [20] where it was shown that under the regularity assumption imposed on $\mathbb{G}$ there exists a unique C*-algebraic version $\mathbb{G}/G_0$ of the quantum quotient satisfying the conditions of Definition 3.1 (for the notion of regularity we refer to [1]). The uniqueness of $\mathbb{G}/G_0$ does not require the regularity of $\mathbb{G}$ which was already noted by S. Vaes in the course of his proof. This suggests that the uniqueness of $\mathbb{D}$ should hold in the context of Definition 3.1 which we shall prove in the next proposition. As we show in [12], in the case of the quantum quotient by a compact quantum subgroup also the existence of $\mathbb{G}/G_0$ does not require the regularity of $\mathbb{G}$. For the C*-algebraic account of this construction we refer to [24].

**Proposition 3.4.** Let $(N, \Delta_N)$ be a quantum homogeneous space in the sense of Definition 3.1. Then the $\mathbb{G}$-$\ast$-algebra $\mathbb{D}$ is uniquely determined by the conditions of Definition 3.1.

**Proof.** The below proof is a straightforward generalization of the Vaes’ uniqueness argument - see Theorem 6.1 of [20]. Suppose that $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Definition 3.1. Using the continuity of the coaction $\Delta_{\mathbb{D}_i}$ and the strictness of $\Delta_N$ we get

$C_0(\mathbb{D}_1) = [C_0(\mathbb{D}_1) C_0(\mathbb{D}_1)]$

$= [(\omega \otimes 1)(\Delta_{\mathbb{D}_1}(C_0(\mathbb{D}_1))(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}_1))) : \omega \in B(L^2(\mathbb{G})),]\$

$= [(\omega \otimes 1)(\Delta_N(N)(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}_1))) : \omega \in B(L^2(\mathbb{G})),]\$

$= [(\omega \otimes 1)(\Delta_N(C_0(\mathbb{D}_2))(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{D}_1))) : \omega \in B(L^2(\mathbb{G})),] = [C_0(\mathbb{D}_2) C_0(\mathbb{D}_1)].$

In the forth equality we used the strong density of $C_0(\mathbb{D}_2) \subseteq N$ and again the strictness of $\Delta_N$. By symmetry we have $C_0(\mathbb{D}_2) = [C_0(\mathbb{D}_1) C_0(\mathbb{D}_2)]$. Taking the adjoint we get $C_0(\mathbb{D}_1) = C_0(\mathbb{D}_2)$. \[\Box\]

Let $(N, \Delta_N)$ be a QHS. Our next aim is to prove that the C*-algebraic version $\mathbb{D}$ uniquely determines $(N, \Delta_N)$. This justifies the following notation: $N = L^\infty(\mathbb{D})$, $\Delta_N = \Delta_{L^\infty(\mathbb{D})}$.

**Proposition 3.5.** Let $\mathbb{G}$ be a locally compact quantum group and let $(N_1, \Delta_{N_1})$ and $(N_2, \Delta_{N_2})$ be quantum homogeneous $\mathbb{G}$-spaces. Suppose that there exists a $\mathbb{G}$-isomorphism $\pi \in \text{Mor}_\mathbb{G}(\mathbb{D}_1, \mathbb{D}_2)$. Then $\pi$ extends to a $\mathbb{G}$-isomorphism $\pi : N_1 \to N_2$, $\Delta_{N_1} \circ \pi = (\pi \otimes \pi) \circ \Delta_{N_2}$.
Proof. Using the fact that $\Delta_{N_1}(N_1) \subset M(K(L^2(G)) \otimes C_0(\mathbb{D}_1))$ we may define an injective map $(\iota \otimes \pi) \circ \Delta_{N_1} : N_1 \to M(K(L^2(G)) \otimes C_0(\mathbb{D}_2))$. The $G$-covariance of $\pi$ and the strictness of $\Delta_{N_1}$ and $\Delta_{N_2}$ implies that $(\iota \otimes \pi) \circ \Delta_{N_1}(N_1) = \Delta_{N_2}(N_2)$. Let $U_{N_2}$ be the canonical implementation of $\Delta_{N_2}$ (see Theorem 2.3). The extension $\pi : N_1 \to N_2$ of $\pi \in \text{Mor}_G(\mathbb{D}_1, \mathbb{D}_2)$ is given by the following explicit formula: $1 \otimes \pi(x) = U_{N_2}^* ((\iota \otimes \pi)\Delta_{N_1}(x)) U_{N_2}$ for any $x \in N_1$.

4. $G$-simplicity

Let $G$ be a locally compact group and let $X$ be a $G$-space. The action of $G$ on $X$ is said to be minimal if $Gx = X$ for any $x \in X$. In other words, the action of $G$ on $X$ is minimal if there does not exist a non-trivial $G$-invariant closed subset of $X$. This condition may be translated into the non-existence of a non-trivial $G$-invariant ideal in $C_0(X)$ which leads to the following notion of a $G$-simple $C^*$-algebra (see Definition 2.4 [10]).

Definition 4.1. Let $G$ be a locally compact quantum group and $\mathbb{D}$ a $G$-$C^*$-algebra. We say that $\mathbb{D}$ is $G$-simple if for any $G$-$C^*$-algebra $\mathbb{B}$ and any $G$-morphism $\pi \in \text{Mor}_G(\mathbb{D}, \mathbb{B})$ we have $\ker \pi = \{0\}$.

In the next theorem we shall provide the quantum counterpart of the classically trivial implication: transitivity $\Rightarrow$ minimality. It may be surprising that the proof relies on the Tomita-Takesaki theory employed in the construction of the canonical implementation of the coaction.

Theorem 4.2. Let $G$ be a locally compact quantum group and let $(N, \Delta_N)$ be a QHS in the sense of Definition 2.4. The $C^*$-algebraic version $\mathbb{D}$ of $(N, \Delta_N)$ is $G$-simple.

Proof. Let $\pi \in \text{Mor}_G(\mathbb{D}, \mathbb{B})$ and let $I_0$ denote the kernel of $\pi$. Let $I \subset N$ be the strong closure of $I_0$. The strictness of $\Delta_N : N \to \mathcal{L}(L^2(G)) \otimes C_0(\mathbb{D})$ implies that for any $x \in I$ and any $y \in \mathcal{K}(L^2(G)) \otimes C_0(\mathbb{D})$ we have $(\iota \otimes \pi)|\Delta(x)y| = 0$. Using the nuclearity of $\mathcal{K}(L^2(G))$ we see that $\Delta(x)y \in \mathcal{K}(L^2(G)) \otimes I_0$. Replacing $y$ by the elements $e_i$ of approximate unit for $\mathcal{K}(L^2(G)) \otimes I_0$ and taking the limit we see that

$$\Delta(x)y \in \mathcal{B}(L^2(G)) \otimes I.$$ (3)

Let $p \in N$ be the central projection in $N$ corresponding to $I$, $I = pN$ (see Remark 2.7). Note that (3) implies the following inequality

$$\Delta_N(p) \leq 1 \otimes p.$$ (4)

In order to prove the opposite inequality let us use the canonical implementation of $\Delta_N$. Adopting the notation of Theorem 2.3 we express (4) by

$$U(1 \otimes p)U^* \leq 1 \otimes p.$$ This easily implies that $(1 \otimes p) \leq U^*(1 \otimes p)U$. Conjugating both sides with $J \otimes J_0$ and using the centrality of $p$ we get $(1 \otimes p) \leq U(1 \otimes p)U^* = \Delta_N(p)$ which together with (4) gives rise to the equality

$$\Delta_N(p) = 1 \otimes p.$$ (5)

Using the ergodicity of $\Delta_N$ we get $p = 0$ or $p = 1$. In the case $p = 0$ we have $\ker \pi = \{0\}$ which proves the $G$-simplicity of $\mathbb{D}$.

The case $p = 1$ can be translated into the strong density of $I_0$ inside $N$. Repeating the argument from the first paragraph of the proof we may see that

$$[\Delta_{\mathbb{D}}(I_0)(\mathcal{K}(L^2(G)) \otimes C_0(\mathbb{D}))] \subset \mathcal{K}(L^2(G)) \otimes I_0.$$ (6)

Let us fix an element $d \in C_0(\mathbb{D})$ and $y \in \mathcal{K}(L^2(G)) \otimes C_0(\mathbb{D})$. There exists a uniformly bounded net $x_i \in I_0$ that *-strongly converges to $d$. Using the strictness condition for $\Delta_N$ and inclusion (6) we get the norm convergence

$$\Delta_{\mathbb{D}}(d)y = \lim_i \Delta_{\mathbb{D}}(x_i)y.$$ This together with (6) shows that

$$\mathcal{K}(L^2(G)) \otimes C_0(\mathbb{D}) = [\Delta_{\mathbb{D}}(C_0(\mathbb{D}))(\mathcal{K}(L^2(G)) \otimes C_0(\mathbb{D}))] \subset \mathcal{K}(L^2(G)) \otimes I_0$$

and immediately gives $C_0(\mathbb{D}) \subset I_0$. Since the opposite inclusion is clear we get $\ker \pi = C_0(\mathbb{D})$ which contradicts the non-degenerateness of $\pi$ and rules out the case $p = 1$. \qed
5. Classical case

Let $G$ be a locally compact group and $\mathcal{G}$ the corresponding locally compact quantum group: $M = L^\infty(G)$ and $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ is the standard comultiplication. Let $X$ be a homogeneous $G$-space. The action of $G$ on $X$ gives rise to the coaction of $\mathcal{G}$ on $L^\infty(X)$: $\Delta_N : L^\infty(X) \to L^\infty(X) \otimes L^\infty(X)$, $\Delta_N (f)(g, x) = f(gx)$ for any $g \in G$, $x \in X$ and $f \in L^\infty(X)$. It turns out that $(L^\infty(X), \Delta_N)$ is a QHS in the sense of Definition [3.1] with the $C^*$-algebraic version $X = \{(0, x), (x)\}$. The fact that $X$ satisfies all the conditions of Definition [3.1] may be concluded from Theorem 6.1 [20]. Our aim is to prove the opposite implication which together with the above provides a 1-1 correspondence between the quantum homogeneous $G$-spaces with the underlying commutative von Neumann algebra $N$ on one side and homogeneous $G$-spaces on the other. This correspondence is the operator algebraic characterization of a locally compact actions of a groups mentioned in the introduction.

Theorem 5.1. Let $G$ be a locally compact group and $\mathcal{G}$ the corresponding LCQG. Let $(N, \Delta_N)$ be a quantum homogeneous space in the sense of Definition [3.1] with $N$ being commutative. Then the spectrum $X = \text{Sp}(\mathcal{D})$ is a homogeneous $G$-space and $(N, \Delta_N) = (L^\infty(X), \Delta_X)$.

Proof. Let us fix a point $\tau \in X$ and let $ev_\tau : C_0(\mathcal{D}) \to \mathbb{C}$ be the corresponding character. Consider the morphism $\pi_\mathcal{D} \in \text{Mor}_G(\mathcal{D}, \mathcal{G})$ given by the formula $\pi_\mathcal{D}(d) = (\iota \otimes ev_\tau)\Delta(d)$ for any $d \in C_0(\mathcal{D})$. Using the $G$-simplicity of $\mathcal{D}$ (see Theorem 6.2) we see that $\pi_\mathcal{D}$ is injective.

In what follows we shall show that $\pi_\mathcal{D}$ may be extended to an injective normal $*$-homomorphism of $\pi_N : N \to L^\infty(\mathcal{G})$. The strictness of $\Delta_N$ gives $\Delta_N(N) \subset M(\mathcal{K}(L^2(\mathcal{G})) \otimes C_0(\mathcal{D}))$, which enables us to define $\pi_N(x) = (\iota \otimes ev_\tau)\Delta_N(x)$ for any $x \in N$. Let $I = \ker \pi_N$ and let $p \in N$ be the central projection generating $I: I = pN$. Using the equality $(\iota \otimes \pi_N) \circ \Delta_N = \Delta_G \circ \pi_N$ we may see that $\Delta_N(I) \subset L^\infty(\mathcal{G}) \otimes I$. In particular $\Delta_N(p) \leq I \otimes p$. Proceeding as in the proof of Theorem 4.2 we get the equality $\Delta_N(p) = 1 \otimes p$. The ergodicity of $\Delta_N$ implies that either $p = 0$ or $p = 1$. The case $p = 1$ is ruled out by the injectivity of $\pi_\mathcal{D}$ hence $p = 0$ and $\pi_N$ is injective. In what follows we shall identify $N$ with its image in $L^\infty(\mathcal{G})$ under $\pi_N$.

Let $\pi_N : G \to X$ be the continuous map induced by a $\mathcal{G}$-morphism $\pi_\mathcal{D}$, where the relation between $\pi_N$ and $\pi_\mathcal{D}$ is established by the Gelfand correspondence: $\pi_\mathcal{D}(f) = f \circ \pi_N$ for any $f \in C_0(X)$. Note the $G$-map property of $\pi_N$

$$\pi_N(gg') = g\pi_N(g')$$

for any $g, g' \in G$. The injectivity of $\pi_\mathcal{D}$ is reflected by the density of the image of $\pi_N$. Using we may see that in order to show the homogeneity of $X$ it is enough to prove the surjectivity of $\pi_N$.

Suppose that $Y = X \setminus \pi_N(G)$ is a non-empty set. Let $y \in Y$ and let $U \subset G$ be a compact neighborhood of $y \in G$. Let us consider a compact set $Uy \subset Y$:

$$Uy = \{x \in X \mid \exists g \in U : x = gy\}.$$ 

Let $O_i \subset X$ be the net of open neighborhoods of $Uy$ directed by inclusions. The net of the corresponding characteristic functions $\chi_{O_i} : X \to \{0, 1\}$ converges pointwisely to the characteristic function of $Uy$. Using the embedding $N \subset L^\infty(\mathcal{G})$ we may define the net $x_i = \chi_{O_i} \circ \pi_N \in N$. Note that $||x_i|| \leq 1$ and for any compactly supported function $f \in L^2(G)$ we have $\lim ||x_i f||_2 = 0$. The above inequality is clear and the convergence follows from the fact that for any pair of disjoint compact subset of $X$ we may find their neighborhoods that are disjoint. We conclude that $x_i$ converges strongly to zero and the third condition of Definition [3.1] implies that for any $k \in \mathcal{K}(L^2(\mathcal{G}))$ and any $d \in C_0(\mathcal{D})$ we have

$$\lim_i (\iota \otimes ev_y)(\Delta_N(x_i)(k \otimes d)) = 0$$

(note that in the above formula we used the fact $\Delta_N(x_i)(k \otimes d) \in M(\mathcal{K}(L^2(\mathcal{G})) \otimes C_0(\mathcal{D})))$. It is easy to see that

$$\langle \iota \otimes ev_y, (\Delta_N(x_i)(k \otimes d)) \rangle = x_i(y)kd(y)$$

where $x_i(y)$ is interpreted as the operator on $L^2(G)$ of multiplication by the function $g \mapsto x_i(gy)$. Let $k_U \in \mathcal{K}(L^2(\mathcal{G}))$ be the 1-dimensional projection onto the subspace of $L^2(G)$ spanned by the
characteristic function $\chi_\mathcal{U} \in L^2(G)$ of $\mathcal{U} \subset G$ and let us fix $d \in C_0(\mathbb{D})$ such that $d(y) = 1$. Using Eq. 4 together with the strong convergence of $x_i(\cdot, y)$ to the operator of the multiplication by $\chi_\mathcal{U}$ we may conclude the norm convergence: $\lim_i (\iota \otimes ev_y)((\Delta_N(x_i)(k_\mathcal{U} \otimes d)) = k_\mathcal{U}$. This contradicts 8 and shows that $\pi_X$ is surjective. \hfill $\Box$

6. QUANTUM HOMOGENEOUS SPACES OF COMPACT QUANTUM GROUPS

The aim of this section is to show that our definition of a quantum homogeneous space is compatible with the definition of quantum homogeneous space of a compact quantum group introduced by P. Podleś [20]. Let us first recall the notion of a compact quantum group (see [31]).

Definition 6.1. A compact quantum group $G = (A, \Delta)$ consists of a unital $C^*$-algebra $A$ together with a unital $*$-homomorphism $\Delta : A \to A \otimes A$ satisfying the coassociativity relation

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$$

and the cancellation properties

$$[\Delta(A)(A \otimes 1)] = A \otimes A = [\Delta(A)(1 \otimes A)].$$

If $G$ is a compact quantum group then there exists a unique Haar state $\phi \in A^*$ on it:

$$(\phi \otimes \iota)\Delta(a) = \phi(a)1 = (\iota \otimes \phi)\Delta(a).$$

In what follows we shall assume that $\phi$ is faithful.

Definition 6.2. Let $H$ be a Hilbert space. A unitary corepresentation $u \in G$ on $H$ is a unitary element of $M(C_0(\hat{G}) \otimes K(H))$ such that $(\Delta \otimes \iota)u = u_{13}u_{23}$. The dimension of the underlying Hilbert space $H$ is called the dimension of $u$ and denoted by $\dim u$.

The tensor product of the unitary corepresentation $u$ and $v$ is defined by $u \otimes v := u_{12}v_{13}$. As was proved by S.L. Woronowicz, a unitary corepresentation may be decomposed into a direct sum of the irreducible ones and the irreducible corepresentations are finite dimensional. In the course of this section we shall write $\hat{G}$ for the set of equivalence classes of irreducible corepresentations of $G$. For all $\alpha \in \hat{G}$ we choose unitary representatives $u^\alpha \in C_0(\hat{G}) \otimes B(\mathcal{H}^\alpha)$.

Definition 6.3. Let $G$ be a compact quantum group and $\mathbb{D}$ a unital $G$-$C^*$-algebra. We say that $\mathbb{D}$ is a quantum homogeneous space in the sense of Podleś (QHSP) if $\mathbb{D}$ is ergodic:

$$\{d \in C_0(\mathbb{D}) \mid \Delta_\mathbb{D}(d) = 1 \otimes d\} = C \cdot 1.$$ 

Notation 6.4. Let $x, y \in L^2(G)$. We define $\omega_{x,y} : L^2(G) \to L^2(G)$ by the formula $\omega_{x,y}(T) = (x | T y)$ for any $T \in B(L^2(G))$. For any $x, y \in C_0(G) \subset L^2(G)$ (note the identification of $C_0(G)$ with its image under the GNS-map $\Lambda_\phi$) and any $\alpha \in C_0(\hat{G})$ we have $\omega_{x,y}(a) = \phi(x^*ay)$. In this context we shall use the notation $\omega_{x,y}(\cdot) = \phi(x^* \cdot y)$.

Let $\mathbb{D}$ be a QHSP and let us fix an irreducible corepresentation $\alpha \in \hat{G}$. Let $W^\alpha_\mathbb{D} \subset C_0(\mathbb{D})$ be the spectral subspace corresponding to $\alpha$. Choosing a basis in $\mathcal{H}^\alpha$ we may identify $u^\alpha$ with the matrix of elements $u^\alpha ij \in C_0(G)$, $i, j = 1, \ldots, \dim \alpha$. Then $W^\alpha_\mathbb{D} = \operatorname{span}\{((\phi(u^\alpha ij^* \cdot \iota)\Delta_\mathbb{D}(d) | d \in C_0(\mathbb{D}), i, j = 1, \ldots, \dim \alpha\}$. For a canonical definition of $W^\alpha_\mathcal{U}$ we refer to Definition 2.3 of [2] (see also [3] and [20]). It turns out that the dim $W^\alpha_\mathbb{D} < \infty$. The precise estimation of dim $W^\alpha_\mathbb{D}$ in terms of the quantum dimension is the subject of Theorem 17 of [3] and Theorem 2.5 of [2]. We shall write $\mathcal{D} \subset C_0(\mathbb{D})$ for the subspace generated by the spectral subspaces:

$$\mathcal{D} = \oplus_{\alpha \in \hat{G}} W^\alpha_\mathbb{D}. \quad (10)$$

Treating $G$ as a $G$-$C^*$-algebra we may consider the spectral subspaces $W^\alpha_\mathcal{D}$. The following finite dimensionality result will be used in the main theorem of this section.

Lemma 6.5. Let $\alpha \in \hat{G}$ be an irreducible corepresentation of $G$, $y \in W^\alpha_\mathcal{D}$ and let $\mathbb{D}$ be a QHSP. Then $\{(\phi(y^* \cdot y) \otimes \iota)\Delta_\mathbb{D}(d) | d \in C_0(\mathbb{D})\}$ is a finite dimensional subspace of $C_0(\mathbb{D})$. 

bounded net \(x^*y \otimes \iota)\Delta_D(d) = (\phi((y\sigma_{-i}(y^*)^* \cdot 1) \otimes \iota)\Delta_D(d). \quad (11)

Let \(\alpha_1, \ldots, \alpha_n\) be the irreducible corespresentations entering the decomposition of \(\alpha \widehat{\otimes} \alpha^*\) onto the irreducible components. Using Equation (11) and the fact that \(\sigma_{-i}(y^*) \in W^\alpha_D\) we get \((\phi(y^* \cdot y \otimes \iota)\Delta_D(d) \in W^\alpha_D \oplus \ldots \oplus W^\alpha_D^*\). The finite dimensionality of \(W^\alpha_D\) implies that \(\dim \{(\phi(y^* \cdot y \otimes \iota)\Delta_D(d) \mid d \in C_0(D)\} < \infty \)

Let us move on to the von Neumann version of \(D\). For the details of the following standard construction we refer to \([3]\) or \([28]\). The ergodicity of \(\Delta_D\) implies that \((\phi \otimes \iota)\Delta_D(d) \in \mathbb{C} \cdot 1\). This enables us to define a state \(\rho : C_0(D) \rightarrow \mathbb{C} : (\phi \otimes \iota)\Delta_D(d) = \rho(d)1\). Using the injectivity of \(\Delta_D\) and the faithfulness of \(\phi\) we may see that \(\rho\) is faithful. In order to check the \(D\)-invariance of \(\rho\) we compute:

\[
(\iota \otimes \rho)\Delta_D(d) = (\phi \otimes \iota \otimes \iota)(\iota \otimes \Delta_D)\Delta_D(d) = (\phi \otimes \iota \otimes \iota)(\Delta_G \otimes \iota)\Delta_D(d) = (\phi \otimes \iota)\Delta_D(d) = \rho(d)1.
\]

Let \((L^2(D), \pi_\rho, \Lambda_\rho)\) be the GNS-construction corresponding to \(\rho\), i.e. \(\pi_\rho : C_0(D) \rightarrow B(L^2(D))\) is the GNS-representation and \(\Lambda_\rho : C_0(D) \rightarrow L^2(D)\) is the GNS-map. There exists a unitary operator \(U_\rho \in B(L^2(G \otimes L^2(D))\) such that \(U_\rho(\Lambda_\rho(a) \otimes \Lambda_\rho(d)) = (\Lambda_\rho(a) \otimes \Lambda_\rho(d))(\Delta_D(d)(a \otimes 1))\). In order to see that \(U_\rho\) is isometric one uses the \(D\)-invariance of \(\rho\). The fact that \(U_\rho\) is surjective follows from the continuity of \(\Delta_D : [\Delta_D(C_0(D)) \otimes C_0(G \otimes L^2(D))] = C_0(G) \otimes C_0(D)\). In what follows we shall treat \(C_0(G)\) and \(C_0(D)\) as concrete \(C^*\)-algebras acting on the Hilbert spaces \(L^2(G)\) and \(L^2(D)\) respectively. On the notational level it means that we shall skip \(\pi_\rho\) and \(\pi_\rho\) from all formula.

Let \(N \subseteq B(L^2(D))\) be the strong closure of \(C_0(D)\). Noting that \(\Delta_D(d) = U_\rho(1 \otimes d)U_\rho^*\) we may extend \(\Delta_D\) to \(\Delta_N : N \rightarrow L^\infty(G \otimes N) : \Delta_N(x) = U_\rho(1 \otimes x)U_\rho^*\) for any \(x \in N\). In what follows we shall refer to \((N, \Delta_N)\) as the von Neumann version of \(D\). Before we move on to the proof that \((N, \Delta_N)\) is a QHS in the sense of Definition \([3,1]\) let us extend Lemma \([6,6]\) to the pair \((N, \Delta_N)\).

**Lemma 6.6.** Let \((N, \Delta_N)\) be the von Neumann version of the quantum homogeneous space \(D\). Let \(\alpha \in \hat{G}\) be an irreducible representation and let \(y \in W_0^\alpha\). Then \(\{(\phi(y^* \cdot y \otimes \iota)\Delta_N(x) \mid x \in N\}\) is finite dimensional.

**Proof.** Let \(x \in N\) and let \(d_i \in C_0(D)\) be a net that strongly converges to \(x\). Using the notation of Lemma \([6,3]\) we may see that \(x_i \equiv (\phi(y^* \cdot y \otimes \iota)\Delta_N(d_i) \in W_0^\alpha \oplus \ldots \oplus W_0^\alpha\). The strong convergence of the net \(x_i\) and the fact that the subspace \(W_0^\alpha \oplus \ldots \oplus W_0^\alpha\) is closed in the strong topology (this follows from its finite dimensionality) imply that \(x \in W_0^\alpha \oplus \ldots \oplus W_0^\alpha\). In particular \(\dim \{(\phi(y^* \cdot y \otimes \iota)\Delta_N(x) \mid x \in N\} < \infty\). \(\square\)

**Theorem 6.7.** Let \(G\) be a compact quantum group with a faithful Haar state \(\phi\) and let \(D\) be a quantum homogeneous space of \(G\) in the sense of Podleś. Let \((N, \Delta_N)\) be the von Neumann version of \(D\) described above. Then \((N, \Delta_N)\) is a quantum homogeneous space in the sense of Definition \([3,1]\) and its \(C^*\)-algebraic version coincides with \(D\).

**Proof.** Let us first extend \(\rho\) to a n.s.f. state on \(N\). Using the extension of \(\phi\) to \(L^\infty(G)\) we define \(\rho(x)1 = (\phi \otimes \iota)\Delta_N(x)\). Note that \(\rho : N \rightarrow \mathbb{C}\) is \(\Delta_N\)-invariant. Suppose that \(\Delta_N(x) = 1 \otimes x\) for some \(x \in N\). The computation

\[
\rho(x)1 = (\phi \otimes \iota)\Delta_N(x) = (\phi \otimes \iota)(1 \otimes x) = x
\]

shows that \(\Delta_N\) is ergodic.

It is clear that \(C_0(D)\) is strongly dense in \(N\) and that the coaction \(\Delta_N\) when restricted to \(C_0(D)\) coincides with \(\Delta_D\). In order to prove the strictness of \(\Delta_N\) let us consider a uniformly bounded net \(x_i \in N\); \(||x_i|| \le C\) converging to 0 in the \(*\)-strong topology. Note that \(x_i^*x_i \in N\) \(*\)-strongly converges to 0. Indeed, \(||x_i^*x_i|| \le C||x_i|| \rightarrow 0\) for any \(v \in L^2(D)\). Let \(\alpha \in \hat{G}\) be an irreducible corepresentation and let us fix \(y \in W_0^\alpha\). Let \(P_y\) be the projection onto the 1-dimensional subspace spanned by \(y \in L^2(G)\). Using the uniform boundness of \(x_i\) and the unitality
of $C_0(\mathbb{D})$ one may see that in order to prove the strictness of $\Delta_N$ it is enough to show that $\Delta_N(x_i)(P_y \otimes 1)$ converges to zero in the norm sense. Indeed, using the fact that $x_i$ is uniformly bounded we see that the norm convergence of $\Delta_N(x_i)(P_y \otimes 1)$ implies the norm convergence of $\Delta_N(x_i)(k \otimes 1)$ for any $k \in K(L^2(G))$. This in turn implies that for any $b \in K(L^2(G)) \otimes C_0(\mathbb{D})$ the net $\Delta_N(x_i)b$ norm converges to 0. Finally, in order to see that for any $x \in N$ we have $\Delta_N(x) \in M(K(L^2(G)) \otimes C_0(\mathbb{D}))$ let us fix a uniformly bounded net $d_i \in C_0(\mathbb{D})$ that $*-strongly$ converges to $x$. The above argument shows that $\Delta_N(d_i)b$ norm converges to $\Delta_N(x)b$ which together with Remark 5.2 shows that $\Delta_N(x) \in M(K(L^2(G)) \otimes C_0(\mathbb{D}))$.

Let us move on to the proof of the fact that $\Delta_N(x_i)(P_y \otimes 1)$ norm converges to 0. We compute:

$$||\Delta_N(x_i)(P_y \otimes 1)||^2 = ||(P_y \otimes 1)\Delta_N(x_i^*x_i)(P_y \otimes 1)|| = ||(\phi(y^* \cdot y) \otimes i)\Delta_N(x_i^*x_i)||.$$  

The finite dimensionality of the subspace $\{(\phi(y^* \cdot y) \otimes i)\Delta_N(x_i^*x_i)| x \in N\}$ proved in Lemma 6.6 and the fact that $(\phi(y^* \cdot y) \otimes i)\Delta_N(x_i^*x_i)$ converges strongly to zero imply the norm convergence $\lim_i (\phi(y^* \cdot y) \otimes i)\Delta_N(x_i^*x_i) = 0$.  

In the next theorem we shall prove that an ergodic coaction of a compact quantum group on a von Neumann algebra is automatically a QHS in the sense of Definition 3.1.

**Theorem 6.8.** Let $G$ be a compact quantum group and let $\Delta_N : N \to L^\infty(G) \otimes N$ be an ergodic coaction of $G$ on a von Neumann algebra $N$. Then $(N, \Delta_N)$ is a QHS in the sense of Definition 3.1.

**Proof.** Let us only sketch the proof. Extending the definition of the spectral subspaces to the von Neumann setting we may define $D \subset N$ - the direct sum of the spectral subspaces of $\Delta_N$ (compare with (11)). It may be checked that the $C^*$-completion of $D$ gives rise to a $G$-$C^*$-algebra which we shall denote by $D$. Using the techniques of the proof of Theorem 6.7 one may see that $D$ satisfies all conditions of Definition 3.1.

**Remark 6.9.** The above theorem shows that $D$ is a unital $G$-$C^*$-algebra, where the unit is provided by the trivial representation entering $N$ under the form of $C_1 \subset N$. This observation rules out from our framework the paradoxical examples of non-compact quantum homogeneous spaces of a compact quantum group given in [22].

7. Rieffel deformation of homogeneous spaces

Rieffel deformation is a well established method of deforming $C^*$-algebras. In his original approach M. Rieffel starts from deformation data $(A, \rho, J)$ which consists of a $C^*$-algebra $A$, an action $\rho$ of $\mathbb{R}^n$ on $A$ and a skew symmetric matrix $J : \mathbb{R}^n \to \mathbb{R}^n$. Using these data M. Rieffel was able to deform the original product on the algebra of $\rho$-smooth elements: $A^\infty \subset A$. The deformed $C^*$-algebra $A'$ is defined as a $C^*$-algebraic completion of $A^\infty$ considered as an algebra with this deformed product. The Rieffel deformation which was originally developed to deform $C^*$-algebras (see [21]) may then be applied for the deformation of locally compact quantum groups (see [22]).

7.1. Rieffel deformation via crossed products. In our recent approach to the Rieffel deformation (see [11]) we formulate the deformation procedure in terms of the crossed product structure. The starting point is the choice of the deformation data $(A, \rho, \Psi)$ where $A$ is a $C^*$-algebra, $\rho$ is a continuous action of an abelian group $\Gamma$ on $A$ and $\Psi : \Gamma \times \Gamma \to T$ satisfying:

(i) $\Psi(e, \hat{\gamma}) = \Psi(\hat{\gamma}, e) = 1$ for all $\hat{\gamma} \in \hat{\Gamma}$;

(ii) $\Psi(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_1 + \hat{\gamma}_3)\Psi(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3)$ for all $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \in \hat{\Gamma}$.

For the theory of 2-cocycles we refer to [13].

The deformation procedure of $A$ consists of the following steps:

1. Let $B$ be the crossed product $C^*$-algebra $B = \Gamma \ltimes_\rho A$ and let $(B, \lambda, \tilde{\rho})$ be the $\Gamma$-product structure on this crossed product i.e. $\lambda : \Gamma \to M(B)$ is the representation of $\Gamma$ implementing the action $\rho$ and $\tilde{\rho}$ is the dual action on $B$. 


2. Let \( \lambda \in \text{Mor}(C^*(\Gamma), B) \) be the morphism corresponding to the representation \( \lambda \in \text{Rep}(\Gamma, B) \) and let \( \Psi_\gamma \in M(C^*(\Gamma)) \) be the family of functions given by \( \Psi_\gamma(\gamma') = \Psi(\gamma', \gamma) \). Applying \( \lambda \) to \( \Psi_\gamma \) (note the identification of \( C_0(\hat{\Gamma}) \) with \( C^*(\Gamma) \) via the Fourier transform), we get a \( \hat{\rho} \)-projective unitary 1-cocycle \( U_\gamma \in M(B) \):

\[
U_{\gamma_1 + \gamma_2} = \Psi(\hat{\gamma}_1, \hat{\gamma}_2)U_{\gamma_1}\hat{\rho}_{\gamma_1}(U_{\gamma_2}).
\]

We define the deformed dual action \( \hat{\rho}^\Psi : \hat{\Gamma} \to \text{Aut}(B) \) by the formula: \( \hat{\rho}_\gamma(b) = U_\gamma^*\hat{\rho}_\gamma(b)U_\gamma \), for any \( \hat{\gamma} \in \hat{\Gamma} \) and \( b \in B \).

3. The deformed \( C^* \)-algebra \( A^\Psi \) is defined as the Landstad algebra of the deformed \( \Gamma \)-product \((B, \lambda, \hat{\rho}^\Psi)\):

\[
A^\Psi = \left\{ b \in M(B) \middle| \begin{array}{l}
1. \hat{\rho}_\gamma(b) = b \\
2. \text{The map } \gamma \mapsto \lambda_a \lambda^*_b \in M(B) \text{ is norm-continuous} \\
3. \lambda(x)a, a\lambda(x) \in B \text{ for any } x \in C^*(\Gamma) \end{array} \right\}
\]

7.2. Rieffel deformation of \( G \)-\( C^* \)-algebras. Let \( G \) be a locally compact group, \( G \) the corresponding LCQG (see Section 3) and let \( \mathbb{D} \) be a \( G \)-\( C^* \)-algebra. As was shown in \([9]\), the Rieffel deformation can also be used for deforming \( \mathbb{D} \) where we simultaneously deform \( G \), \( C_0(\mathbb{D}) \) and the coaction \( \Delta_B \). In what follows we shall give a concise description of the deformation procedure.

The deformation of \( G \) described below is the dual version of the 2-cocycle twist of the comultiplication \( \Delta_G \) studied in \([6]\) and then developed in \([5]\) and \([7]\). The twisted coaction is given by the formula: \( \Delta_G^\Psi(x) = \Psi^*(\Delta_B(x))\Psi \) for any \( x \in C_0(\hat{G}) \). The \( C^* \)-algebra of the deformed LCQG \( \mathbb{C}^\Psi \) is constructed as follows. Let \( \Gamma \) be a closed abelian subgroup of \( G \), \( \Psi \) a continuous 2-cocycle on \( \hat{\Gamma} \) and let \( \rho : \Gamma^2 \to \text{Aut}(C_0(G)) \) be the continuous action given by the left and the right shifts along \( \Gamma \). Let us consider a 2-cocycle \( \Psi \otimes \hat{\Psi} \) on \( \Gamma^2 \) where \( \hat{\Psi} \) is defined by the formula:

\[
\hat{\Psi}(\hat{\gamma}_1, \hat{\gamma}_2) = \Psi(\hat{\gamma}_1, \hat{\gamma}_2).
\]

The deformation data \( (C_0(G), \rho, \Psi, \hat{\Psi}) \) gives rise to the \( C^* \)-algebra \( C_0(\mathbb{C}^\Psi) \), by means of the deformation procedure described in Section 7.2.

In order to construct the comultiplication \( \Delta_G^\Psi \) we employ the \( \rho \)-covariance properties of the the comultiplication \( \Delta_G \):

\[
(\rho_{\gamma, \gamma} \otimes \rho_{\gamma', e})(\Delta_G(f)) = \Delta_G(f),
\]

\[
\Delta_G(\rho_{\gamma, \gamma'})(f) = (\rho_{\gamma', e} \otimes \rho_{\gamma, e'})(\Delta_G(f)),
\]

for all \( f \in C_0(G), \gamma, \gamma' \in \Gamma \). Let \( (B, \lambda, \hat{\rho}) \) be the standard \( \Gamma^2 \)-product structure on \( B = \Gamma^2 \ltimes_{\rho} C_0(G) \). Using \([14]\) and the universal properties of the crossed product we may extend \( \Delta_G \) to a morphism \( \Delta_G^\rho : B \to M(B \otimes B) \). It is uniquely determined by the following two properties:

- the action of \( \Delta_G^\rho \) on \( C_0(\hat{G}) \subset M(B) \) coincides with \( \Delta_G \);
- the action of \( \Delta_G^\rho \) on \( \lambda_{\gamma, \gamma'} \in M(B) \) is given by \( \Delta_G^\rho(\lambda_{\gamma, \gamma'}) = \lambda_{\gamma, e} \otimes \lambda_{e, \gamma'} \).

Let us consider a unitary representation \( U : \Gamma^2 \to M(B \otimes B) \)

\[
U_{\gamma, \gamma'} = \lambda_{\gamma, \gamma} \otimes \lambda_{e, \gamma'}
\]

and a function \( \Psi^* \in M(C_0(\hat{G}) \otimes C_0(\hat{G})) \) given by \( \Psi^* = \Psi(\hat{\gamma}_1, \hat{\gamma}_1 + \hat{\gamma}_2) \). Extending \( U \) to the morphism \( \pi_U : C^*(\Gamma^2) \to M(B \otimes B) \) we may apply it to \( \Psi^* \) obtaining

\[
\Upsilon = \pi_U(\Psi^*).
\]

Let us define \( \Delta_G^\Psi : B \to M(B \otimes B) \)

\[
\Delta_G^\Psi(b) = \Upsilon \Delta_G^\rho(b) \Upsilon^*.
\]

Using \([13]\) and \([14]\) one may show that \( \Delta_G^\Psi \) when restricted to \( C_0(\mathbb{G}^\Psi) \subset M(B) \) defines the comultiplication: \( \Delta_G^\Psi : C_0(\mathbb{G}^\Psi) \to M(C_0(\mathbb{G}^\Psi) \otimes C_0(\mathbb{G}^\Psi)) \).

Suppose now that \( \mathbb{D} \) is a \( \mathbb{G} \)-\( C^* \)-algebra. Restricting the \( \mathbb{G} \)-coaction to the subgroup \( \Gamma \) we get an action \( \rho' : \Gamma \to C_0(\mathbb{D}) \). Performing the Rieffel deformation based on the deformation data
(C₀(𝒟), ρ′, Ψ) we get the C∗-algebra C₀(𝒟′ ′). In order to define Δ₂Ψ we first extend Δ₂ to a morphism Δ₂′ ∈ Mor(Γ ×⁻ C₀(D), B ⊗ Γ ×⁻ C₀(D′)). In the analogy with (10) we may define Δ₂Ψ ∈ Mor(Γ ⊗⁻ C₀(D), B ⊗ Γ ⊗⁻ C₀(D′)). Restricting Δ₂Ψ to C₀(D′ ′) ⊂ M(Γ ⊗⁻ C₀(D, B ⊗ Γ ⊗⁻ C₀(D′)) we get the coaction Δ₂Ψ ∈ Mor(C₀(D′ ′), C₀(G′) ⊗ C₀(D′ ′)). As was shown in [12] the above construction leads to a GΨ-C∗-algebra D′ ′. Actually, the reader may have noticed some differences between the above description of the duality relation between GΨ and D′ and the one given in [9]. They are due to the adopted duality relation between GΨ and D′ ′. As was described in the previous section we may define the dual action is given by:

\[ \hat{\rho} : \Gamma \rightarrow Aut(C_0(\hat{\Gamma})) \]

As was shown in [9] the above construction describes the GΨ-C∗-algebra XΨ. Let us move on to the construction of (L∞(XΨ), ΔL∞(XΨ)).

As may be expected by the reader, L∞(XΨ) is defined as a strong closure of C₀(XΨ). To be more precise we may take the strong closure inside the von Neumann crossed product Γ ×⁻ L∞(X). In order to keep the connection with the Rieffel deformation we need also the description of L∞(XΨ) in terms of the invariant element under the twisted dual action \( \hat{\rho}^Ψ : \Gamma \rightarrow Aut(G′) \) (see Section 7.3). For the later purposes we shall interpret the dual action \( \hat{x} : \Gamma \rightarrow Aut(G′) \) as a right coaction \( \hat{x} : \Gamma \rightarrow Aut(C_0(G′)) \). Under this interpretation the twisted dual action is given by:

\[ \hat{x}^Ψ(x) = \Psi^* \hat{x}'(x) \Psi. \quad (17) \]

The von Neumann algebra L∞(XΨ) introduced above coincides with the algebra of the \( \hat{x}^Ψ \)-coinvariants

\[ L∞(XΨ) = \{ x ∈ \Gamma \;| \; \hat{x}^Ψ(x) = x \otimes 1 \}. \]

Our next aim is to extend the C∗-algebraic coaction Δ₂Ψ to the von Neumann level and define ΔL∞(XΨ) : L∞(XΨ) → L∞(G′) ⊗ L∞(XΨ). In order to do it we make the following observations:

- The crossed product \( \Gamma \times⁻ L∞(X) \) may be faithfully represented on L²(G). Indeed, identifying X with a quotient space G/Gx where Gx is a stabilizing subgroup of a point x ∈ X and using the idea of Section 4 of [26] one may check that there exists an injective normal *-homomorphism that sends L∞(X) ⊂ \( \Gamma \times⁻ L∞(X) \) to L∞(G/Gx) ⊂ B(L²(G)) and \( \lambda_x \in \Gamma \times⁻ L∞(X) \) to the left shift \( L_{\lambda_x} \subset B(L²(G)) \).
- Let us note that the representation \( \Gamma \times⁻ (G \gamma_1, G \gamma_2) \rightarrow L_{\gamma_1} \otimes L_{\gamma_2} \subset B(L²(G) \otimes L²(G)) \) when extended to \( \Gamma \) provides an interpretation of the unitary \( \Upsilon \in M(C^∗(\Gamma²)) \) (see [14]) as an element of \( B(L²(G) \otimes L²(G)) \) in turn enables us to define the unitary

\[ U = \Upsilon W^* (JJ \otimes 1) V \in B(L²(G) \otimes L²(G)) \]

- for the definition of J and \( \hat{J} \) we refer to Section 2. It turns out that \( U \) implements \( \Delta_{\hat{x}^Ψ} : \Delta_{\hat{x}^Ψ}(x) = U(1 \otimes x)U^* \). In order to prove that it is enough to check the following two equalities:
  (i) \( U(1 \otimes L_{\gamma})U^* = L_{\gamma} \otimes 1 = \Delta_{\hat{x}^Ψ}(\lambda_x) \),
  (ii) \( U(1 \otimes f)U^* = \Upsilon \Delta_x(f)\Upsilon^* = \Delta_{\hat{x}^Ψ}(f) \) for any \( \gamma \in \Gamma \) and \( f \in C_0(X) \).

The sufficiency of that may be concluded from Section 6.2.

The above observations shows that we may define the von Neumann counterpart of \( \Delta_{\hat{x}^Ψ} \) by the formula \( \Delta_{L∞(XΨ)}(x) = U(1 \otimes x)U^* \) where \( x \in L∞(XΨ) \).

**Remark 7.1.** Using the embedding of \( \Gamma \times⁻ L∞(X) \) into \( B(L²(G)) \) described above one may express the \( \hat{x}^Ψ \)-invariance of \( x \in L∞(XΨ) \) by the equation

\[ W(x \otimes 1)W^* = \Psi(x \otimes 1)\Psi^*, \quad (18) \]

where \( W \) is the left regular corepresentation of \( G \). It follows from Eq. (17) and the fact that \( W \) implements the dual coaction \( \hat{x}'(x) = W(x \otimes 1)W^* \).
Let us now show that the coaction of $\Delta_{L^\infty(X^\Psi)}$ is ergodic. Suppose that $\Delta_{L^\infty(X^\Psi)} = 1 \otimes x$ for some $x \in L^\infty(X^\Psi)$. Note that $\Gamma$ is a closed subgroup of $G^\Psi$ in the sense of Definition 2.5 of [26]. In particular the $G^\Psi$-invariance of $x$ implies its $\Gamma$-invariance. In other words $x$ commutes with the image of the representation $\lambda: \Gamma \to \Gamma \ltimes L^\infty(K)$. This together with the $\rho^\Psi$-invariance of $x$ shows that $x \in L^\infty(X)$ and $\Delta_{L^\infty(X)}(x) = 1 \otimes x$. The ergodicity of $\Delta_{L^\infty(X)}$ implies that $x \in C1$ which proves the ergodicity of $\Delta_{L^\infty(X^\Psi)}$.

The case of the Rieffel deformation of the quotient space $G / G_0$ for which $\Gamma \subset G_0 \subset G$ was considered in [10]. It was proved there that under some modular assumption concerning the embedding $\Gamma \subset G$ and some assumptions on $\Psi$ we have $(G / G_0)^\Psi \cong G^\Psi / G_0^\Psi$. By the results of [26], the right hand side of this equality gives rise to a QHS. It turns out that the proof given in [10] may be simplified and adopted to the proof of the fact that $(L^\infty(X^\Psi), \Delta_{L^\infty(X^\Psi)})$ is a QHS where we do not assume that $\Gamma \subset G_0$, there are no modular assumption concerning the embedding $\Gamma \subset G$ and no assumption about a 2-cocycle $\Psi$ except its continuity. In this more general situation we expect to obtain generically a QHS of the non-quotient type. Let us first prove the following lemma.

Lemma 7.2. Let $X$ be a $G$-homogeneous space and $X^\Psi$ the corresponding $G$-$C^*$-algebra. Let $\Gamma \subset G$ be a closed abelian subgroup of $G$ and $\Psi$ a continuous 2-cocycle on the dual group $\hat{\Gamma}$. Let $X^\Psi$ be the Rieffel deformation of $X$. Then the corresponding crossed-product $C^*$-algebras are isomorphic: $G \ltimes C_0(X) \cong G^\Psi \ltimes C_0(X^\Psi)$.

Proof. Note that $C_0(\hat{G})$ and $C_0(X)$ may be embedded into $M(G \ltimes C_0(X))$ and they together generate it: $G \ltimes C_0(X) = [C_0(\hat{G}) C_0(X)]$. In order to embed $C_0(X)$ we use the coaction $\Delta_X$ - see [2]. The fact that $G \ltimes C_0(X)$ may also be embedded into $M(G \ltimes C_0(X))$, the equality $C_0(\hat{G}) = C_0(\hat{G}^\Psi)$ together with the isomorphism

$$\Gamma \ltimes C_0(X) \cong \Gamma \ltimes C_0(X^\Psi)$$

(proved in Proposition 3.2 of [11]) leads to

$$G \ltimes C_0(X) = [C_0(\hat{G}) C_0(X)] = [C_0(\hat{G}) C^*(\Gamma) C_0(X)] = [C_0(\hat{G}) (\Gamma \ltimes C_0(X))].$$

Note that in the second and the fourth equality we used the fact that $C_0(\hat{G}) = [C^*(\Gamma) C_0(\hat{G})]$ and in the fourth equality we used $C_0(\hat{G}^\Psi) = C_0(\hat{G})$. \hfill \Box

Remark 7.3. In this remark we give a concise description of the induction procedure of the regular representation $G_0 \ni g \mapsto L^G_0 < L(C_0(\hat{G}_0))$. Since our ultimate aim is to prove that $X^\Psi$ gives rise to a quantum homogeneous $G^\Psi$-space, we shall stick to the S. Vaes’ formulation and notation of the induction procedure for locally compact quantum groups given in [26]. We shall do so even in the case of the classical closed subgroup $G_0$ of a locally compact group $G$.

The induction procedure applied to the left regular representation $G_0 \ni g \mapsto L^G_0 < L(C_0(\hat{G}_0))$ gives rise to the induced $C_0(\hat{G}_0)$-module $\text{Ind}(C_0(\hat{G}_0))$ and the induced representation $\text{Ind}(L^G_0): G \to L(\text{Ind}(C_0(\hat{G}_0)))$. Let $\alpha: G \to \text{Aut}(L^\infty(X))$ be the action given by the left shifts: $\alpha_g(f)(x) = f(g^{-1}x)$. As is always the case for the induced representations, $\text{Ind}(C_0(\hat{G}_0))$ is equipped with a faithful strict $*$-homomorphism $\rho: L^\infty(X) \to L(\text{Ind}(C_0(\hat{G}_0)))$ which is covariant with respect to $\text{Ind}(L^G_0)$: $\rho(\alpha_g(f)) = \text{Ind}(L^G_0)(\rho(f)) \text{Ind}(L^G_0)(g)$.

In the course of the proof of Theorem 6.1 of [26] it was noted that we have the following identification

$$\mathcal{K}(\text{Ind}(C_0(\hat{G}_0))) \cong G \ltimes C_0(X).$$

In particular $\mathcal{K}(\text{Ind}(C_0(\hat{G}_0)))$ is equipped with the dual (right) coaction - the counterpart of the dual coaction on $G \ltimes C_0(X)$:

$$\gamma: \mathcal{K}(\text{Ind}(C_0(\hat{G}_0))) \to M(\mathcal{K}(\text{Ind}(C_0(\hat{G}_0)))) \otimes C_0(\hat{G})).$$

(21)
The data described above may be encoded in the bicovariant \(C_0(\hat{G}_0)\)-bicorrespondence \(F\) (see Section 3.1 of [20]). As a Hilbert \(C_0(\hat{G}_0)\)-module, \(F\) is the tensor product \(F = L^2(G) \otimes \text{Ind}(C_0(\hat{G}_0))\). The structure of the bicovariant \(C_0(\hat{G}_0)\)-bicorrespondence on \(F\) is given by:

- a strict \(*\)-homomorphism \(\pi_l : L^\infty(G) \rightarrow \mathcal{L}(F)\) which sends a generator \(L_g\) to \(\pi_l(L_g) = L_g \otimes \text{Ind}(\mathcal{L}(\hat{G}))\);
- a strict \(*\)-antihomomorphism \(\pi_r : L^\infty(G) \rightarrow \mathcal{L}(F)\) which sends a generator \(L_g\) to \(\pi_r(L_g) = R_g \otimes 1\);
- a strict \(*\)-homomorphism \(\pi : L^\infty(G) \rightarrow \mathcal{L}(F)\) given by \(\pi(f) = f \otimes 1\) for any \(f \in L^\infty(G)\).

The bicovariance relation linking \(\pi_l, \pi_r\) and \(\pi\) and the right regular corepresentation \(\hat{V} \in L^\infty(G) \otimes L^\infty(\hat{G})\) (see Eq. (1)) is described in Definition 3.5 of [20]. It may be shown that \(\pi_l\) together with \(\rho\) gives rise to a strict \(*\)-homomorphism \(\sigma : G \ltimes L^\infty(X) \rightarrow \mathcal{L}(F)\) such that \(\sigma(L_g) = \pi_l(L_g)\) and \(\sigma(f) = \rho(f)\) for any \(g \in G\) and \(f \in L^\infty(X)\). In what follows \(\sigma\) will be denoted by \(\pi_l\).

**Theorem 7.4.** Let \(X\) be a \(G\)-homogeneous space, \(\Gamma \subset G\) a closed abelian subgroup and \(\Psi\) a continuous 2-cocycle on the Pontryagin dual \(\hat{\Gamma}\). Let \(X^\Psi\) be the Rieffel deformation of \(X\) described above. Then the von Neumann version \((L^\infty(X^\Psi), \Delta_{X^\Psi})\) of \(X^\Psi\) is a QHS with the corresponding \(C^*\)-algebraic version coinciding with \(X^\Psi\).

**Proof.** In the course of the proof we shall use the notation introduced in Remark 7.2. Let us consider the \(C_0(\hat{G}_0)\)-module \(L^2(G) \otimes \text{Ind}(C_0(\hat{G}_0)) \otimes L^2(G)\). Note that under the identification \(\hat{X} = X\) we have

\[
\pi_l(x)_{12} = \sum_1 \gamma_{23}(x) \Sigma_{13}
\]

for any \(x \in M(\hat{G} \ltimes C_0(\hat{X}))\). On the other hand, using Eq. (18) we may see that

\[
\gamma(x) = \Psi(x \otimes 1)\Psi^*,
\]

for any \(x \in C_0(X^\Psi)\). This equation together with (22) shows that, for any \(x \in C_0(X^\Psi)\) we have \(\Psi\pi_l(x)\Phi^* = 1 \otimes x\) where \(\Phi \in M(C^*(\Gamma) \otimes C^*(\Gamma))\) is the unitary element given by \(\Phi(\gamma_1, \gamma_2) = \Psi(\gamma_1, \gamma_2)\). Using the strictness of \(\pi_l\) and Lemma 7.2 we conclude that the natural embedding of \(C_0(X^\Psi)\) into \(M(G^\Psi \ltimes C_0(X^\Psi))\) extends to the embedding \(\iota : L^\infty(X^\Psi) \rightarrow M(G^\Psi \ltimes C_0(X^\Psi))\).

Let us note that \(\iota\) inherits the following strictness property: for any uniformly bounded, \(*\)-strongly convergent net \(x_i \in L^\infty(X^\Psi)\) and any \(y \in G^\Psi \ltimes C_0(X^\Psi)\), the net \(\iota(x_i)\) is norm convergent. Composing \(\iota\) with the embedding

\[
G^\Psi \ltimes C_0(X^\Psi) = [\Delta_{X^\Psi}(C_0(X^\Psi))(C_0(X^\Psi) \otimes 1)] \hookrightarrow M(K(L^2(G)) \otimes C_0(X^\Psi))
\]

we conclude that the map \(\Delta_{\hat{X}^\Psi} : L^\infty(X^\Psi) \rightarrow L^\infty(G^\Psi) \otimes L^\infty(X^\Psi)\) gives rise to a strict \(*\)-homomorphism \(\Delta_{L^\infty(X^\Psi)} : L^\infty(X^\Psi) \rightarrow \mathcal{L}(L^2(G^\Psi) \otimes C_0(X^\Psi))\). This shows that third condition of Definition 3.1 is satisfied. The first and the second are trivially satisfied in our case. \(\square\)

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