Fate of the area-law after partial measurement in quantum field theories

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We calculate numerically the Rényi bipartite entanglement entropy of the ground state of Klein-Gordon field theory (coupled harmonic oscillators) after fixing the position (partial measurement) of some of the oscillators in \( d = 1, 2 \) and 3 dimensions. We show that after partial measurement as far as the two parts are connected the area-law is valid in generic dimensions except in \( d = 1 \) conformal field theories. When due to the measured region the two parts are disconnected, the entanglement entropy decreases exponentially with respect to the minimum distance between the regions. In the massless case the decay is power-law with a universal dimension-dependent exponent.

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I. INTRODUCTION

Entanglement entropy has been playing an important role both in high-energy and in condensed matter physics for many years. One of the interesting aspects of the entanglement entropy in quantum field theory and many body physics is the validity of the area-law first discussed in [1], for review see [2,3]. Based on the area-law the bipartite entanglement entropy of the the ground state of a system is given by the common area shared by the two subsystems. Most of the quantum field theories and many body systems follow the area-law [2], for more recent works see [4]. However, there are also many examples of quantum systems that do not follow the area-law, the most famous examples are critical systems in \( 1 + 1 \) dimension that can be described with conformal field theories (CFT) [5,6] and the free fermionic system in generic dimensions [7]. Although the violation of the area-law is more common in long-range interacting systems, see [8], there are many long-range systems which follow the area-law [9]. Although now there are many interesting proofs regarding the validity of the area-law in short-range gapped \( 1 + 1 \) dimensional systems, see [12–14], there are not many analytical results valid for generic systems in higher dimensions. Investigating the validity or the violation of the area-law in different quantum systems in different conditions is now a common task.

One natural question that arises in this context regards the extent that the area-law is valid after partial measurement in the system. In other words consider the ground state of a many body system and make a local projective measurement in a subsystem and then look to the bipartite entanglement entropy of the remaining part. In general an answer to such kind of question can be dependent on the basis that one chooses to perform the measurement and of course also to the outcome of the measurement. For example, by considering a simple interacting spin system one can immediately realize that there are infinite possibilities to perform the measurement and so naturally the general solution to such kind of problem looks difficult. However, recently we were able to show [10] that for quantum chains that can be described with CFT there are some “natural” basis [18–20], see also [21,22] that do not destroy CFT structure of the system and so one can calculate analytically the bipartite entanglement entropy. In [16] it was shown that the ground state after partial measurement still follows the logarithmic law but the formulas change slightly. In this work we want to pursue the same idea and investigate mostly systems that follow the area-law. Of course one should note that since after partial measurement the particles inside the measured domain are disentangled from the rest the post measurement wave function can be the wave function of a system which some parts are disconnected from the rest. This opens an interesting issue of not having common area between subsystems. Since it is not even possible to define the area in this kind of conditions one would naturally seek for a generalization of the area-law. Note that since the setup that we follow is a tripartite one it is reminiscent of the quantum entanglement negativity studied in the context of CFT in [17]. In this work we will investigate the post measurement bipartite entanglement entropy in the Klein-Gordon field theory numerically. In the next section we first describe the set up of the problem carefully and define our system of interest. Then in section III we will discuss post measurement bipartite entanglement entropy in \( 1 + 1 \) dimensional massless and massive Klein-Gordon field theory. In section IV we extend our results to \( 2 + 1 \) and \( 3 + 1 \) dimensions. In section V we will present some arguments regarding possible extensions of our results to more general settings. Finally in section VI we summarize our results.

II. SETUP AND DEFINITIONS

Consider a quantum system in generic dimension and divide the system to two subsystems \( D \) and \( \bar{D} \). The von Neumann entanglement entropy of \( D \) with respect to \( \bar{D} \) is defined by the following formula

\[
S = -\text{tr} \rho_D \ln \rho_D, \tag{1}
\]
where $\rho_D$ is the reduced density matrix of the subsystem $D$. A simple generalization of the von Neumann entropy which is also a measure of entanglement is called Rényi entropy and is defined as

$$S_\alpha = -\frac{1}{1-\alpha} \ln \text{tr} \rho_D^\alpha. \quad (2)$$

The limit $\alpha \to 1$ gives back the von Neumann entropy. What we are interested to calculate in this paper is the following: consider a generic quantum system in its ground state and make local projective measurements in a subsystem $A$ of this system. After measurement the system will collapse to a wave function which the subsystem $A$ is disentangled from its complement $\bar{A}$. However, the subsystem $\bar{A}$ has a wave function which is highly entangled. Now divide the subsystem $\bar{A}$ to two subsystems $B$ and $\bar{B}$. What we are interested is the entanglement entropy between $B$ and $\bar{B}$. This setup was first studied in [16] in the most simple case of $1+1$ dimensional CFT’s when $B$ and $\bar{B}$ are connected and the measurement is done in the conformal basis. In principle there are many different ways to arrange $A$, $B$ and $\bar{B}$ and the results can be very different. We list here those arrangements that we are interested in this paper:

1. Case I: $A$, $B$ and $\bar{B}$ are connected regions and in addition $B \cup \bar{B}$ is also connected.
2. Case II: $B$ and $\bar{B}$ are connected regions but $A$ is not a connected region.
3. Case III: $A$, $B$ and $\bar{B}$ are connected regions and in addition $B$ and $\bar{B}$ are disconnected.

Notice that just for the case I and case II ($d > 1 + 1$) one can define common area between regions $B$ and $\bar{B}$ and hope to recover the area-law. For the case II ($d = 1 + 1$) there is no concept of common area and so one naturally expects different behavior. In this work we will study the cases I, II and III in $1+1$-dimension. In higher dimensions we will just study simple examples of just the cases I and III.

**Notation:** In this work $s$ and $l$ represent the characteristic sizes of $A$ and $B$.

Although some of our conclusions can be extended to generic quantum field theories we will mainly focus on Klein-Gordon field theory defined by the following action:

$$\frac{1}{2} \int \{(\partial \phi(x))^2 + m^2 \phi^2(x)\} dx. \quad (3)$$

The ground state of the above Hamiltonian has the following form

$$\Psi_0 = \left(\frac{\det K^{1/2}}{\pi L}\right)^{\frac{1}{2}} e^{-\frac{1}{2} \langle \phi | K^{1/2} | \phi \rangle}. \quad (5)$$

The Hamiltonian of $L$-coupled harmonic oscillators, with coordinates $\phi_1, \ldots, \phi_L$ and conjugated momenta $\pi_1, \ldots, \pi_L$:

$$H = \frac{1}{2} \sum_{n=1}^L \pi_n^2 + \frac{1}{2} \sum_{n, n'=1}^L \phi_n K_{nn'} \phi_{n'}. \quad (4)$$
One can calculate the two point correlators $X_D = \text{tr} \left( \rho_D \phi_i \phi_j \right)$ and $P_D = \text{tr} \left( \rho_D \pi_i \pi_j \right)$ using the $K$ matrix, where $\rho_D$ is the reduced density matrix of the domain $D$. The squared root of this matrix, as well as its inverse, can be split up into coordinates of the subsystems $D$ and $\bar{D}$, i.e.,

$$K^{-1/2} = \begin{pmatrix} X_D & X_{\bar{D}} \\ X_D^T & X_{\bar{D}} \end{pmatrix}, \quad K^{1/2} = \begin{pmatrix} P_D & P_{\bar{D}} \\ P_D^T & P_{\bar{D}} \end{pmatrix}.$$ 

The spectra of the matrix $2C = \sqrt{X_D P_D}$, can be used to calculate the von Neumann and Rényi entanglement entropies,

$$S = \text{tr} \left[ \left( C + \frac{1}{2} \right) \ln\left( C + \frac{1}{2} \right) - \left( C - \frac{1}{2} \right) \ln\left( C - \frac{1}{2} \right) \right],$$

$$S_\alpha = \frac{1}{\alpha - 1} \text{tr} \left[ \left( C + \frac{1}{2} \right)^\alpha - \left( C - \frac{1}{2} \right)^\alpha \right].$$

Now if we do measurement on the position of all the oscillators $\{\phi_i\} \in A$ they will take some definite values and eventually will get decoupled from the rest of the oscillators. In other words the final state will be the same as $\bar{A}$ but instead of $K^{1/2}$ we need to consider $(K^{1/2})_\bar{A}$ which is a subblock of the matrix $K^{1/2}$ corresponding to the oscillators in the subsystem $\bar{A}$. This means that we now have a new Gaussian wave function and one can calculate its bipartite entanglement entropy with the formulas (6) and (7). There is a simple interpretation for the wave function after measurement. It is just the ground state of the Hamiltonian $\tilde{H}$ with $H = -\tilde{K} = (K^{1/2})_A^2$. It is not difficult to see that this Hamiltonian is highly non-local, for example, if we start with a discrete Laplacian in one dimension $K_{ij} = -\delta_{i,i-1} - \delta_{i,i+1} + 2\delta_{i,i}$ the $\tilde{K}$ will have the form shown in Figure 1. Note that the $\tilde{K}_{ii}$ is usually a number very close to $-2$ and $\tilde{K}_{i,i+1} = \tilde{K}_{i+1,i}$ is very close to 1 and the rest of the couplings are much smaller than these three couplings. As it is shown in figure 1 for site $i = 1$ the couplings decrease like a power-law but after reaching to the middle of the original system they start to increase. It seems that this behavior is generic for all the oscillators up to the middle of the system. The sites beyond the middle of the system just show reverse behavior. The behavior for massive case $K_{ij} = -\delta_{i,i-1} - \delta_{i,i+1} + (2 + m^2)\delta_{i,i}$ is a bit different. In this case $\tilde{K}_{i,j}$ decreases exponentially as it is shown in Figure 1. In this case the only significant elements of the matrix $\tilde{K}$ are $\tilde{K}_{i,i+1} = \tilde{K}_{i+1,i} = 1$ and $\tilde{K}_{ii} = -2 - m^2$ which means that $\tilde{K}$ is actually almost identical to the $K$ matrix of the local massive Laplacian. This result will have significant consequences in the next sections when we discuss post measurement entanglement entropy.

### III. POST MEASUREMENT ENTANGLEMENT ENTROPY IN 1+1 DIMENSION

In this section we would like to investigate bipartite entanglement entropy after partial measurement in 1+1 dimensional Klein-Gordon field theory for the arrangements depicted in the Figure 2. In the first part we study massive case and in the second part we will focus on the massless case.

#### A. 1+1 dimensional massive QFT

When the field theory is massive the bipartite entanglement entropy follows the area-law [3, 4]. There are also many concrete proofs regarding the validity of the area-law in the 1+1 dimensional gapped systems, see [12–15]. In massive local quantum field theories it is shown [6] that the entanglement entropy of the system is given
by

\[ S_\alpha = -\kappa \frac{c}{12}(1 + \frac{1}{\alpha}) \ln m + ..., \tag{8} \]

where \( c \) is the central charge of the systems which is equal to 1 for Klein-Gordon field theory, \( m \) is the mass and \( \kappa \) is the number of contact points between two subsystems. For example, for bipartite entanglement entropy we have \( \kappa = 2 \) and \( \kappa = 1 \) in periodic and open systems respectively. We calculated the same quantity after partial measurement, as we discussed before, for different cases. Interestingly we found that the equation (8) is valid also after we decouple the region \( A \) from the system. One just needs to consider \( \kappa \) as the number of contact points between \( B \) and \( \bar{B} \). We checked this for periodic boundary conditions for the cases that \( B \) and \( \bar{B} \) have one and two contact points. The results are shown in the Figure 3. This result looks consistent with this picture that the entanglement between two regions are related to the couplings just around the interfaces of the two regions. Since as we discussed in the section II some local measurements in other parts of the system disturb very little the couplings around the interfaces one might find it natural to have still the area-law after the partial measurement. Although post measurement area-law is probably a generic behavior of gapped systems the equation (8) is valid just in those cases that we make our measurements in the conformal basis. We notice here that since in our system the correlations \( \text{tr} (\rho_D \pi_2 \pi_1) \) and \( \text{tr} (\rho_D \phi_1 \phi_2) \) decay exponentially even after the measurement based on [14] one naturally expect the area-law. This might be a good starting point for analytical approach to the problem in the most general basis.

We also calculated the entanglement entropy in the cases II and III for the massive field theory and we found that the entanglement entropy decays exponentially in both cases, see Figure 4. In other words we have

\[ S_\alpha \approx e^{-\gamma(\alpha)ms}, \tag{9} \]

where \( \gamma(\alpha) \) is a discretization dependent number.

B. 1 + 1 dimensional CFT

It is now well-known that the entanglement entropy in 1 + 1 dimensional CFT’s does not follow the area-law [3, 4]. Bipartite entanglement entropy of an infinite system with the subsystem size \( l \) follows the following logarithmic formula

\[ S_\alpha = \frac{c}{6}(1 + \frac{1}{\alpha}) \ln \frac{l}{a} + ..., \tag{10} \]

where \( c \) is the central charge of the system, \( a \) is the lattice spacing and dots are subleading terms. For periodic system with finite size \( L \) one just needs to replace \( l \) with the chord length \( \frac{L}{2} \sin \frac{\pi}{L} \). If we now make measurement in some parts of the system based on the type of the measurement and the measurement region naturally the above formulas change. Here we always assume that we measure the position of the oscillators and so we have always conformally invariant setup. In a recent paper [10] we already discussed the case I in detail and showed that if the size of the region \( A \) is \( s \) and the size of the region \( B \) is equal to \( l \) then the Rényi entanglement entropy is given by

\[ S_\alpha = \frac{c}{12}(1 + \frac{1}{\alpha}) \ln \left( \frac{L \sin \frac{\pi}{L}(l + s) \sin \frac{\pi}{L}s}{a \sin \frac{\pi}{L}l} \right) + \gamma + ... \tag{11} \]

Here we do not discuss this case anymore. Now we study the case II when the size of \( B \) is much bigger than the size of \( A \) and \( B \). Following the numerical procedure of the last section one can easily calculate the entanglement entropy. We took the size of the whole system equal to \( L = 1000 \) and the size of \( B \) equal to \( l = 10 \). The results depicted in figure 5 show the following behavior for the entanglement entropy

\[ S_\alpha \sim \left( \frac{\alpha}{s} \right)^{\delta(\alpha)}, \tag{12} \]

where \( s \) is the distance between the two regions \( B \) and \( \bar{B} \). The exponent \( \delta(\alpha) \) is a universal number and depends on the dimensionality of the system and \( \alpha \). Numerical results shown in Figure 5 indicates that \( \delta(1) \) is around 2.0. For larger \( \alpha \)’s the exponent increases very slowly and eventually saturates after \( \alpha = 2 \) [31], see Figure 5. In conclusion our numerical results show that whenever the two regions \( B \) and \( \bar{B} \) are connected the entanglement entropy follows a complicated logarithmic behavior but for the disjoint regions it shows power-law behavior with respect to the distance between regions with an exponent which is \( \alpha \) dependent. In principle the above result should be calculable with CFT techniques because based on the strategy followed in [10].
FIG. 5: (Color online) Top: log-log plot of the Rényi entropy for the case II. From top to bottom the full lines correspond to equation (12) with δ(α) = 1.1, 1.6, 2.01, 2.3 and 2.3. Down: The exponent of the power-law prefactor in the equation (12), i.e. δ(α), with respect to α. The size of the total periodic system L = 1000 and the size of the region B is l = 10.

the above cases can be translated to the two point functions of twist operators on the annulus. Unfortunately there are not much results available regarding the two point functions of twist operators on the annulus.

IV. POST MEASUREMENT ENTANGLEMENT ENTROPY IN HIGHER DIMENSIONS

In this section we study the post measurement entanglement entropy in 2 + 1 and 3 + 1 dimensions. Since even for massless systems the area-law is valid in higher dimensions we will just consider the two massless cases: case I) the region A is a ball with radius s and B an annulus and spherical shell surrounding A in 2 + 1 and 3 + 1 dimensions respectively with thickness l. Case II) The same as case I we just exchange the domains A and B and the role of s and l. The two dimensional case is shown in Figure 6.

FIG. 6: (Color online) Different arrangements of measured region in 2 + 1 dimensional quantum system. Case I: D1 = A, D2 = B and D3 = B. Case II: D1 = B, D2 = A and D3 = B.

FIG. 7: (Color online) Post measurement area-law for the case I in 2 + 1 dimensions for α = 1 and 2. The total size L = 100 and the radius of the measurement disc is s = 10.

A. 2 + 1 dimensions

To calculate the entanglement entropy in two dimensions one can simply first go to the radial coordinates and then discretize the Hamiltonian [24]. The Hamiltonian after discretization has the following form:

$$H = \sum_n H^{(n)} = \sum_n \left( \frac{1}{2} \sum_{i=1}^{L} (n_i^{(n)})^2 + \frac{1}{2} \sum_{i,j=1}^{L} \phi_i K_{ij}^{(n)} \phi_j \right),$$

(13)
where
\[ K_{11}^{(n)} = \frac{3}{2} + n^2; \quad K_{ii}^{(n)} = 2 + \frac{n^2}{i^2}, \quad i > 1, \]
(14)
\[ K_{i,i+1}^{(n)} = K_{i+1,i}^{(n)} = -\frac{i + 1/2}{\sqrt{i(i + 1)}}, \]
(15)
Finally the entanglement entropy can be calculated by the following formula
\[
S_{\alpha} = S_{\alpha}^{(0)} + \sum_{n=1}^{\infty} 2S_{\alpha}^{(n)},
\]
(16)
\[
S_{\alpha}^{(n)} = \frac{1}{\alpha - 1} \text{tr} \left[ \ln \left( (C^{(n)} + \frac{1}{2})^{\alpha} - (C^{(n)} - \frac{1}{2})^{\alpha} \right) \right],
\]
(17)
where \(2C^{(n)} = \sqrt{X_D^{(n)} P_D^{(n)}}\). Although as we already discussed in 1 + 1 dimensional case there is a power-law decaying couplings \(K_{ij}\) in our system (see Figure A2 in the appendix) it is already well-known \([10, 11]\) that the power-law decaying couplings by themselves are not enough to change the area-law. Indeed our numerical results demonstrated in figure 7 show that the area-law is still valid.

When \(B\) and \(\bar{B}\) are disconnected the entanglement entropy decreases like a power-law. The same equation as \((12)\) is also valid in 2 + 1 dimensions with \(\delta(1)\) around 2.8.

The numerical result for \(\alpha = 1\) is shown in the figure 8. In the massive case the entanglement entropy decreases exponentially as we had in the one dimensional case.

\section*{B. 3 + 1 dimensions}

The calculation of entanglement entropy in 3 + 1 dimensions is very similar to 2 + 1 dimensions. One needs to first go to the radial coordinates and first write the Hamiltonian as
\[
\mathcal{H} = \sum_{lm} H^{(lm)};
\]
(18)
\[
H^{(lm)} = \frac{1}{2} \sum_{i=1}^{L} (\Pi^{(lm)}_i)^2 + \frac{1}{2} \sum_{i,j=1}^{L} \phi_i K^{(lm)}_{ij} \phi_j;
\]
(19)
where
\[
K^{(lm)}_{11} = \frac{9}{4} + l(l + 1),
\]
(20)
\[
K^{(lm)}_{ii} = 2 + \frac{1}{i^2} \left( \frac{1}{2} + l(l + 1) \right), \quad i > 1,
\]
(21)
\[
K^{(lm)}_{i,i+1} = K^{(n)}_{i+1,i} = -\frac{i + 1/2}{\sqrt{i(i + 1)}}.
\]
(22)
Finally the entanglement entropy can be calculated by the following formula
\[
S_{\alpha} = \sum_{l=0}^{\infty} (2l + 1) S_{\alpha}^{(l)},
\]
(23)
\[
S_{\alpha}^{(l)} = \frac{1}{\alpha - 1} \text{tr} \left[ \ln \left( (C^{(l)} + \frac{1}{2})^{\alpha} - (C^{(l)} - \frac{1}{2})^{\alpha} \right) \right].
\]
(24)

The couplings after measurement show the same behavior as Figure A2. Using the above equations one can also simply calculate the entanglement entropy for the case I when \(A\) is a sphere and \(B\) is a spherical shell surrounding \(A\). The numerical results demonstrated in Figure 9 show that the area-law is valid in this case. When we exchange the role of \(A\) and \(B\) as we did in the 2 + 1 dimensional case the entanglement between separated regions decays as the equation \((12)\) with \(\delta(1)\) around 3.9. The numerical result for \(\alpha = 1\) is shown in the figure 8.
FIG. 10: (Color online) Different arrangements of measurement region in 1 + 1 and 2 + 1 dimensional quantum system with disconnected $B$ and $\bar{B}$ regions.

V. DISCUSSION ON MORE GENERAL SETTINGS

In this section we summarize our findings and we will also give some plausible arguments regarding possible extension of the results.

In 1 + 1 dimensional CFT we showed that as far as $B$ and $\bar{B}$ are connected the logarithmic behavior is valid. When the regions are disconnected the entanglement entropy decays like a power-law, see equation (12). For the massive case when the regions $B$ and $\bar{B}$ are connected we showed that the area-law is intact and when they are separated bipartite entanglement entropy decreases exponentially with respect to the distance.

The natural question to ask is if the two regions are separated with two different measurement regions which one plays more important role? For example in Figure 10 we depicted a case which we have two measurement regions with lengths $s_{\text{min}}$ and $s_{\text{max}}$. Our numerical results demonstrated in Figure 11 show that the more important role is always played by the smaller regions.

In figure 11 we consider two measurement regions with sizes $s_1$ fixed and $s_2$ varies. The entanglement entropy first decreases rapidly with $s_2$ up to $s_2 = s_1$ but after that it is almost a constant. This just means as far as the minimum distance between the regions $B$ and $\bar{B}$ is constant changing the other parts of the system does not make significant difference in the amount of the entanglement entropy. Although numerically very difficult it seems like the same argument should be true also in 2 + 1 dimensions. For the configuration demonstrated in Figure 10 the entanglement entropy changes significantly when $s_{\text{min}}$ changes. In other words the leading term of the entanglement entropy changes as

$$S_\alpha \propto e^{-\gamma(\alpha)m s_{\text{min}}} + \ldots$$

(25)

FIG. 11: (Color online) Entanglement entropy for the massive 1 + 1 dimensional system with total length $L = 200$ and $m = 0.5$ for the arrangement depicted in Figure 10.

In 2 + 1 and 3 + 1 dimensions as far as $B$ and $\bar{B}$ are connected the area-law is still valid. Although we showed this just for the case of spherical shells we believe that this is a more general rule. Note that here the area-law means that we need to consider just the common area between $B$ and $\bar{B}$. This seems perfectly consistent with also this observation that only the minimum distances between $B$ and $\bar{B}$ gives the highest contribution to the entanglement entropy.

VI. CONCLUSIONS

In this paper we discussed the entanglement entropy of the ground state of the coupled harmonic oscillators after fixing the position of some of the oscillators in $d = 1, 2$ and 3 dimensions. We showed that the bipartite entanglement entropy of the unmeasured oscillators follows the area-law (except in 1 + 1 dimensional CFT) as far as the two parts are connected. For the massless systems; when after the measurement the two parts are disconnected; the bipartite entanglement entropy decreases like a power-law with respect to the distance between the two regions. The exponent of the power-law depends on the dimension and $\alpha$. In the massive case the decay is exponential with respect to the minimum distance between the two regions. Although our numerical calculations were fairly straightforward and included just coupled harmonic oscillators (discrete version of the Klein-Gordon field theory) we believe that some of our results can be generalized to other field theories [32]. One of the future directions could be analytical derivation of our results especially [12], using well-known techniques of CFT in two dimensions [23, 26] and also in higher dimensions [27, 28]. Holographic techniques can also shed light on the possible analytical calculation of post measurement bipartite entanglement entropy [30].

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Appendix A: Couplings $\tilde{K}_{ij}$ in different conditions

In this appendix we list few figures regarding how $\tilde{K}_{ij}$ changes with the distance in different circumstances. The first case is in $1+1$ dimensional massless system with measurement domain in the center. We classified this as case III, the $\tilde{K}_{ij}$ is shown in Figure 1A. Increase in the couplings strength by getting closer to the measurement domain is a general feature of all the systems. We put all the couplings with respect to the oscillators in the measured domain equal to zero. $\tilde{K}_{ij}$ for $j$’s that are in the disconnected domain (in this case the right hand side of the system) are smaller than the couplings with those that are in the connected part. In the right hand side the couplings decay like a power-law with a very large exponent. For $i = 45$ the exponent is almost 7.

The second case that we are interested here is $\tilde{K}_{ij}$ in $2+1$ dimensions when the measurement domain is a disc. In Figure A2 we depicted the couplings for different sites for $\tilde{K}_{ij}^{(0)}$. As we already observed in the $1+1$ dimensional case, for example, the couplings of the first site decay up to almost the middle of the system like a power-law with exponent around 2. Since the original system is not periodic here we don’t see increase in the couplings after the middle of the system. Interestingly the trend for all the sites are the same and independent of $n$. In other words for sites far from the measurement region the couplings get stronger when we approach to the measurement domain, see figure A2. Note that the nearest neighbor couplings are exceptions and they always take much bigger values (close to what we have for discrete Laplacian).

![Figure A1](image1.png)

**FIG. A1:** (Color online) $\tilde{K}_{ij}$ with respect to $j$ for case III in $1+1$ dimension. The size of the original system and the measured domain were $L = 100$ and $s = 10$ respectively.

![Figure A2](image2.png)

**FIG. A2:** (Color online) Top: $\tilde{K}_{ij}$ with respect to $j$ for $n = 0$ after performing measurements on 10 sites for radial discrete Laplacian in $2+1$ dimensions. Bottom: The same quantity in the log-log plot up to $j = 30$. The size of the original system is $L = 100$.

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