Eling-Oz Formula for Exotic Hairy Black Holes

Alexander Patrushev\textsuperscript{1}

\textsuperscript{1}Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada

E-mail: apatrush@uwo.ca

ABSTRACT: We checked Eling-Oz formula [1] for the bulk viscosity of the holographic fluid dual to the exotic black holes [2]. Initially, Eling and Oz argued that the formula is valid in the high temperature and adiabatic limits. In [3] the validity of the formula for $N = 2^*$ plasma and cascading gauge theory was pushed forward for arbitrary temperatures. We successfully verified the formula with the computations of the bulk viscosity [4, 5] for a wide range of the temperatures. Moreover, it correctly reproduces the critical behavior in the vicinity of the critical point, where the bulk-to-shear viscosity diverges.

KEYWORDS: Gauge-gravity correspondence, Strongly coupled holographic plasmas, Exotic black holes

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1 Introduction and Summary

A new formula for bulk-to-shear viscosity of strongly coupled gauge theory plasma was proposed by Eling and Oz (EO) \cite{1}. The wide class of the gauge theories are dual (in the context of AdS/CFT \cite{6}) to the following \((d+1)\) gravitational action

\[
S = \frac{1}{16\pi} \int \sqrt{-g} d^{d+1}x \left( R - \frac{1}{2} \sum_i (\partial \phi_i)^2 - V(\phi_i) \right) + S_{\text{gauge}}. \tag{1.1}
\]

They used the null focusing (Raychaudhuri) equation describing the evolution of the horizon entropy, which is equivalent to the viscous fluid entropy balance law. In the absence of chemical potentials for the conserved charges, the formula for the bulk viscosity of the plasma dual to (1.1) takes the following form

\[
\frac{\zeta}{\eta} = \sum_i c_s^4 T^2 \left( \frac{d\phi_i^H}{dT} \right)^2, \tag{1.2}
\]

where \(\phi_i^H\) are the scalar field values evaluated at the horizon of the black brain, \(T\) is the temperature of plasma dual to the black brane, \(c_s\) is the speed of sound waves in plasma. In the same paper the EO formula was verified for a large number of gauge theories dual to string theory at high temperature limit \cite{7–10} and some phenomenological models of gauge/gravity correspondence \cite{11, 12}.

The expression for the bulk-to-shear viscosity employs the values of scalar fields only at the horizon. It is an intriguing result as the bulk viscosity in general depends on the energy scale; the boundary data is essential to capture microscopic scales of the theory. That is in contrast with the universality of the shear viscosity calculations \cite{13–15}. In \cite{3} the validity of (1.2) was extended for cascading gauge plasma \cite{16, 17} and \(\mathcal{N} = 2^*\) gauge theory plasma \cite{18–20} for all the temperatures.

The correctness of the EO formula for the phenomenological models of gauge/gravity correspondence was also verified in \cite{21}. Particularly, bulk viscosity obtained from the Gubser, Pufu and Rocha (GPR) formula for the GPR model \cite{22} and the Improved Holographic QCD model \cite{23} coincides with the EO formula. The essential feature of the GPR formula (extracted from the holographic Kubo formula) is that it is suitable only for the models with one gravitational scalar field acting as the new radial coordinate. The exotic
black hole model is the example of the phenomenological model with several gravitational scalar fields.

We checked the validity of the Eling-Oz formula analytically for the exotic black holes in the high-temperature (conformal) limit. The formula is correct for the intermediate temperatures, the vicinity of the phase transition and for the temperatures up to $m_{T} < 2.75$, where $m$ is the mass associated with breaking of the conformal symmetry. $^{1}$ The correctness of the formula for exotic black holes extends the number of models for which the EO formula is valid for all energy scales. It would be interesting to explain this kind of universality of (1.2) (see also footnote [4] in the text). While preparing this manuscript, I learned that the authors of [1] proved such a universality of the transport coefficients of the holographic plasmas [24]. They showed that the transport coefficients depend on the boundary conditions, but they are independent of the RG running from UV to IR.

2 Bulk viscosity for the exotic hairy black holes

The exotic black hole model is defined by the following effective (3+1)-dimensional gravitational action $^{[2, 25]}$:

$$S_4 = \frac{1}{2\kappa^2} \int dx^4 \sqrt{-\gamma} \left[ R + 6 - \frac{1}{2} (\nabla \phi)^2 + \phi^2 - \frac{1}{2} (\nabla \chi)^2 - 2 \chi^2 - g \phi^2 \chi^2 \right],$$ (2.1)

where $g$ is a coupling constant. $^{2}$ Note that $\phi$ induces a relevant deformation of the dual CFT by an operator $O_r$ and $\chi$ is associated with an irrelevant operator $O_i$ in the dual gauge theory. The last term in (2.1) involves mixing of $O_i$ with $O_r$ under RG dynamics.

The central charge of the UV fixed point is defined as $^{[2]}$

$$c = \frac{192}{\kappa^2},$$ (2.2)

This central charge should be understood as a measure of the degrees of freedom in the CFT, which is defined thermodynamically or via two-point correlation functions $^{[26]}$. We demand the solution to be AdS$_4$ asymptotically with translationary invariant horizon. For this purpose only the normalizable mode of $O_i$ is nonzero near the boundary.

The background geometry is defined as

$$ds^2_4 = -c_1(r)^2 dt^2 + c_2(r)^2 \left[ dx^2_1 + dx^2_2 \right] + c_3(r)^2 dr^2, \quad \phi = \phi(r), \quad \chi = \chi(r),$$ (2.3)

where $r \to \infty$ corresponds to the AdS boundary. Then one can introduce a new radial coordinate $x$ as follows

$$1 - x \equiv \frac{c_1(r)}{c_2(r)},$$ (2.4)

so that $x \to 0$ corresponds to the AdS boundary, and $y \equiv 1 - x \to 0$ corresponds to a horizon asymptotic. Afterwards, we introduce $a(x)$ as

$$c_2(x) = \left( \frac{a(x)}{2x - x^2} \right)^{1/3},$$ (2.5)

$^{1}$Original calculations of the bulk viscosity were done in [4, 5].

$^{2}$In numerical analysis we set $g = -100$. 

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The equations of motion (EOMs) obtained from 2.1, with the background ansatz (2.3), define the following expansion of the model parameters

\[
a = \alpha \left( 1 - \frac{1}{40} p_1^2 x^{2/3} - \frac{1}{18} p_1 p_2 x + \mathcal{O}(x^{4/3}) \right),
\]

\[
\phi = p_1 x^{1/3} + p_2 x^{2/3} + \frac{3}{20} p_1^3 x + \mathcal{O}(x^{4/3}),
\]

\[
\chi = \chi_4 \left( x^{4/3} + \left( \frac{1}{7} - \frac{3}{70} \right) p_1^2 x^2 + \mathcal{O}(x^{7/3}) \right),
\]

(2.6)

near the boundary \( x \to 0_+ \), and

\[
a = \alpha \left( a_0^h + a_1^h y^2 + \mathcal{O}(y^4) \right), \quad \phi = \phi_0^h + \mathcal{O}(y^2), \quad \chi = \chi_0^h + \mathcal{O}(y^2),
\]

(2.7)

near the horizon. Up to the overall scaling factor \( \alpha \) the thermodynamics of the black branes can be uniquely specified with 3 UV coefficients \( \{ p_1, p_2, \chi_4 \} \) and 4 IR coefficients \( \{ a_0^h, a_1^h, \phi_0^h, \chi_0^h \} \).

We use the integral of motion \([25]\)

\[
(a_0^h)^2 \sqrt{(6a_0^h)(6 + (p_1^h)^2 - 2(\chi_0^h)^2 - g(p_0^h)^2(\chi_0^h)^2)} = 6,
\]

(2.8)

which arises after integration of the EOMs, to find the temperature \( T \) and the entropy density \( s \) of the black brane solution (2.3):

\[
T = \frac{3\alpha}{4\pi(a_0^h)^2},
\]

(2.9)

\[
\hat{s} \equiv \frac{384}{c} s = 4\pi \alpha^2 (a_0^h)^2,
\]

(2.10)

In the dual picture \( p_1 \) can be interpreted as the coupling of the operator \( \mathcal{O}_r \), \( p_2 \) as the expectation value of \( \mathcal{O}_r \) and \( \chi_4 \) as \( < \mathcal{O}_i > \) (see [27] for the argumentation). Without the loss of generality, we choose the model with \( \text{dim}[\mathcal{O}_r] = 2 \). Which means the combination \( p_1 \alpha \) should be fixed.

For a given set of \( \{ \alpha, p_1 \} \) there is a discrete set of the remaining parameters

\[
\{ a_0^h, \phi_0^h, \chi_0^h \}
\]

allowing us to characterize the thermodynamics of black branes suitable for Eling-Oz formula. One solution with \( \chi_0^h = 0 \) describes the black brane without the condensate of the \( \chi \) field. All the other solutions have \( \chi_0^h \neq 0 \) and describe the “exotic black branes” [2]. This model is interesting because the transition occurs at the high temperatures (rather than the low temperatures). That is the irrelevant operator \( \mathcal{O}_i \) obtains nonzero vacuum expectation value for \( T > T_c \), spontaneously breaking a discrete \( \mathbb{Z}_2 \) symmetry of the model. Also, it was shown in [5] that all the exotic black branes contain a tachyonic quasinormal mode. Thus, they are dynamically unstable but thermodynamically stable, thereby, violating the correlated stability conjecture [28–30].

For the exotic black hole model formula (1.2) takes the following simple form

\[
\zeta \bigg|_{EO} = c_s T^2 \left( \left( \frac{dp_0^h}{d\tau} \right)^2 + \left( \frac{dc_0^h}{d\tau} \right)^2 \right),
\]

(2.11)
where $c_s$ is a speed of sound defined by

$$c_s^2 = \frac{d(\ln T)}{d(\ln s)},$$

and $\tau$ is an inversed dimensionless temperature, e.g. $\tau = \frac{T_c}{T}$ or $\tau = \frac{\alpha_p}{T}$. Further, we will check the validity of the formula for the different temperature regimes.

2.1 Explicit analytical check of (1.2) in the conformal limit of a symmetric phase

In this section we briefly repeat the main results for the thermodynamics of the model in the high temperature limit as presented in [25]. These results can be readily applied to supply evidence of (2.11) in the conformal limit. We demonstrate this below. For the sake of simplicity, we consider the symmetric phase only ($\chi = 0$). If we introduce the small deformation parameter $\delta$ such that

$$\delta = \frac{m}{T} \ll 1,$$

where $m$ is the mass associated with the deformation of CFT. Then one can expand the scalar field $\phi$ in the EOMs to leading order in $\delta$ as

$$\phi(y) = \delta \tilde{\phi}(y).$$

Afterwards, we solve the EOMs demanding regularity of the scalar field at the horizon and the first law of thermodynamics to fix the integration constants.

The expansion of the scalar field $\phi$ near horizon (2.7) assumes that

$$p^h_0 = \delta.$$
Eventually, it leads to the following expression for the speed of sound in the conformal limit

$$c_s^2 = \frac{1}{2} - \sqrt{\frac{3}{5\pi}}\delta^2 + \mathcal{O}(\delta^4).$$  \hspace{1cm} (2.16)

The results for the bulk viscosity in the conformal limit were discussed in [4]. For a symmetric phase at the high temperatures we have

$$\left. \frac{\zeta}{\eta} \right|_{\text{ordered}} = \frac{2\pi}{\sqrt{3}} \left( \frac{1}{2} - c_s^2 \right) + \mathcal{O} \left( \left( \frac{1}{2} - c_s^2 \right)^2 \right). \hspace{1cm} (2.17)$$

As a result, the bulk viscosity should be proportional to the square of the deformation parameter, i.e.,

$$\frac{\zeta}{\eta} = \frac{1}{4}\delta^2. \hspace{1cm} (2.18)$$

If we substitute (2.15) and (2.16) into (2.11) we recover exactly the same relation ($\tau = \delta$ in this case). This justifies the validity of (2.11) in the conformal limit.

One can see that in the conformal limit the value of the scalar field at the horizon is proportional to $\frac{m}{\mathcal{M}}$. In Fig. 1 we use a linear approximation for the scalar field at the horizon to establish the connection between $\delta$ and $\frac{m}{\mathcal{M}}$. In addition, on the same figure we compare the bulk viscosity evaluated from the sound waves attenuation coefficient with analytical result (2.18) in the conformal limit.

### 2.2 Comparison of (1.2) away from criticality for symmetric and unstable phases

Now it is possible to check the formula for an intermediate temperature regime. The original calculations of the thermodynamics and the bulk viscosity were done in [2, 4]. We use a cubic spline approximation for the values of the scalar fields at the horizon to get their derivatives with respect to the inverted temperature. Then we compare the validity of the Eling-Oz formula with respect to two arguments $\delta$ and $\frac{m}{\mathcal{M}}$.

Fig. 2 illustrates the absolute value of relative error between the bulk viscosity obtained from quasinormal mode method and the Eling-Oz formula. The error is slightly big for large
temperatures due to small values of the scalar fields (which increases numerical errors). But we have the analytic results from the previous section for this region of temperatures. The agreement is also worse in the vicinity of the critical point (due to large values of $\frac{dh_0}{d\tau}$). In the next subsection we will improve the agreement in the critical regime.

2.3 Comparison of (1.2) at criticality

The critical behavior of the exotic black branes was discussed in [4]. The peculiar thing in the model is that the bulk-to-shear viscosity diverges in the broken phase at criticality. One can use more detailed data to check the formula close to criticality as it is done in the Fig. 2. Alternatively, we can proceed the semi-numerical analysis.

In [4] the authors constructed an exotic model of the second order transition in $d = 3$ at finite temperature and zero chemical potentials. The corresponding conformal field theory in $2 + 1$ dimensions is deformed by a relevant operator $\mathcal{O}_r$. The expectation value of $\mathcal{O}_r$ acts as the order parameter of the phase transition and it scales as $|t|^{1/2}$ in the vicinity of the critical point, where $t = T - T_c$. In Fig. 3 we plot $\ln \left( \frac{\zeta}{\eta} \right)$ versus $\ln \left( c_0^h \right)$ with a dashed green line which fits the data

$$y = -1.999(8)x - 9.427(8).$$

(2.19)

It is clear from the Fig. 3 that parameter $c_0$ has the same critical exponent as $\langle \mathcal{O}_r \rangle$

$$c_0 \propto |t|^{1/2}.$$  

(2.20)

For the speed of sound and bulk viscosity it was shown that

$$c_s \propto |t|^0, \quad \frac{\zeta}{\eta}_{\text{disordered}} \propto |t|^{-1}.  \hspace{1cm} (2.21)$$

Whereas, formula (1.2) suggests that at criticality

$$\left. \frac{\zeta}{\eta} \right|_{EO} = c_4 T_c^2 \left( \frac{c_0}{T - T_c} \right)^2 \propto |t|^{-1},$$

(2.22)

which confirms the correctness of the EO formula at criticality. Also, we expect that the formula will be valid for other symmetry-broken phases.
2.4 Validity of (1.2) for the low temperatures

Let us proceed to the check of the Eling-Oz formula in the case when the stable phase is driven into the low temperature regime. First note that data for the thermodynamic parameters is more discrete, while the number of points for viscosity is significantly reduced (100 points altogether). Furthermore, Fig. 4 presents a plot of the relative error with respect to $\delta$ and $\frac{T_c}{T}$. One can readily see that formula (2.11) is valid up to $\delta < 2.75$. Therefore, combining the results of [3], we expect that the formula is valid for the whole range of the temperatures.

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