Introduction. — The specific heat (SH) jump $\Delta C$ is the most well known thermodynamic signature of the second order phase transition and hence contains the generic information of the transition as well as the material specific information. For example, the BCS theory of superconductivity predicts the universal ratio $\Delta C/C_{el} \sim T_c^4$, hence $\Delta C/T_c = 1.43\gamma$ is a temperature independent constant and tells us the material specific quantity $\gamma$, the Sommerfeld coefficient of the normal state $\gamma = C_{el,n}/T$. In view of this BCS prediction, $\Delta C/T_c = \text{const.}$, the experimental observation by Bud’ko, Ni, and Canfield (BNC)\cite{ref1}, $\Delta C/T_c \approx T_c^2$ for a family of doped Ba(Fe$_{1-x}$TM$_x$)$_2$As$_2$ compounds with TM=Co, Ni is a very intriguing behavior and stimulated active investigations both experimentally and theoretically. After the work of Ref.\cite{ref1}, this so-called BNC scaling relation was expanded with an increasing list of the iron pnictide and iron chalcogenide (FePn/Ch) superconducting (SC) compounds\cite{ref2-9}, hence strengthens the speculation that some generic mechanism must exist behind this unusual scaling behavior. However, the recent observation of a strong deviation from the BNC scaling in a series of K-doped Ba$_{1-x}$K$_x$Fe$_2$As$_2$ for $0.7 < x \leq 1$\cite{ref10} is confusingly contrasted to the Na-doped Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ ($0.1 \leq x \leq 0.7$)\cite{ref8} which displays an excellent BNC scaling.

For the theoretical investigations, there are three attempted explanations. Kogan\cite{ref11} argued that strong pair-breaking can cause $\Delta C/T_c \propto T_c^2$. The essence of this theory is a dimensional counting. The free energy difference near $T_c$, $\Delta F = F_S - F_N$, can be expanded in powers of $\Delta^2$ ($\Delta$: the SC order parameter (OP)). In the BCS theory, $\Delta F \propto -N(0)\Delta^2$. Using the BCS result of $\Delta^2(T) \sim T_c^2(1 - \frac{T}{T_c})$, we get $\Delta C/T_c \propto \frac{\partial^2 \Delta F}{\partial T^2} \propto N(0)$, the well known BCS prediction. In the case of the strong pair-breaking limit, $\Gamma_e \gg T_c$ ($\Gamma_e$ = pair-breaking rate), considered by Kogan, $\Delta F \propto -N(0)\frac{T^2}{T_c}$ by a dimensional counting. Substituting the same BCS behavior of $\Delta^2(T) \sim T_c^2(1 - \frac{T}{T_c})$, we recover the Kogan’s result $\Delta C/T_c \propto N(0)\frac{T^2}{T_c^2}$. However, we believe that this result is the consequence of an inconsistent approximation\cite{ref13}. The theory of Vavilov et al.\cite{ref14} mainly studied the coexistence region with magnetic order $M$ and SC order $\Delta$. It is a plausible theory that the coexisting magnetic order over the SC order can substantially reduce $\Delta C$, hence develops a steep variation of $\Delta C$ vs. $T_c$. However this theory didn’t reveal a specific reason as to why $\Delta C/T_c$ follows the BNC scaling $\sim T_c^\alpha$ with $\alpha \approx 2$. Finally, Zannen\cite{ref15} attributed the behavior $\Delta C \propto T_c^3$ to the normal state electronic SH with the scaling form $C_{el}^0 \propto \text{const.}$ due to the critical fluctuations near the quantum critical point (QCP). A problem of this theory is that there is no evidence of $C_{el}^0 \propto \text{const.}$ for a wide doping range of the FePn/Ch superconductors. All three theories mentioned above are single band theories and do not particularly utilize the unique properties of the FePn/Ch superconductors. In this paper, we propose a theory in which the multi-band nature of the FePn/Ch superconductors is the root cause for producing the BNC scaling behavior.

Two Band model for the SH jump $\Delta C$. — For a multi-band superconductor, the SH jump formula is generalized as

$$\Delta C = \sum_{i=h,e} N_i(0) \left(-\frac{d\Delta^2}{dT}\right)|_{T_c}$$

(1)

where the band index "i" counts the different bands and we specify it as the hole and electron band typical in the Fe-based superconductors. $N_{h,e}$ are the DOSs, and $\Delta_{h,e}$ are the SC OPs of each band. In the one band BCS superconductor, using $\Delta^2(T) \sim T_c^2(1 - \frac{T}{T_c})$, the above equation gives $\Delta C/T_c \propto N(0) = \text{const}$. However, in the case of a multi-band superconductor, Eq.(1) can reveal more information for the pairing mechanism as well as the pairing state. At present the most widely accepted pairing state in the Fe-based superconductors is the sign-changing S-wave state ($\pm$S-wave) mediated by a dominant interband repulsive interaction ($V_{inter} > V_{intra}$)\cite{ref16}. The essential physics of this $\pm$S-wave state can be studied with the two coupled gap equations\cite{ref18}

$$\Delta_h = -[V_{hh}N_h\chi_h]\Delta_h - [V_{he}N_e\chi_e]\Delta_e,$$

(2)
\[ \Delta_c = -[V_{ee} N_c \chi_e] \Delta_c - [V_{eh} N_h \chi_h] \Delta_h, \]

where the pair susceptibility at \( T_c \) is defined as

\[ \chi_{h,e}(T_c) = T_c \sum_n \int_{-\Lambda_{hi}}^{\Lambda_{hi}} \frac{1}{\omega^2_n + \xi^2} \approx \ln \left[ \frac{1.14 \Lambda_{hi}}{T_c} \right], \quad (3) \]

where \( \omega_n = \pi T_c (2n + 1) \) and \( \Lambda_{hi} \) is a pairing energy cut-off. The pairing potentials \( V_{ab} (a, b = h, e) \) are all positive and further simplified in this paper as \( V_{he} = V_{eh} = V_{\text{inter}} \) and \( V_{hh} = V_{ee} = V_{\text{intra}} \) without loss of generality.

In the limit \( \frac{V_{\text{intra}}}{V_{\text{inter}}} \to 0 \), Eq.(1) can be analytically solved and provides the interesting kinematic constraint relation\([17]\]

\[ \frac{\Delta_h}{\Delta_c} \approx \sqrt{\frac{N_c}{N_h}} \quad \text{as} \quad T \to T_c, \quad (4) \]

and the critical temperature is given by

\[ T_c \approx 1.14 \Lambda_{hi} \exp \left[ -1/(V_{\text{inter}} \sqrt{N_c N_h}) \right]. \quad (5) \]

For further modeling the calculation of the experimental data of \( \Delta C \) vs. \( T_c \) for a Fe-122 compound with a series of doping, we first notice that the undoped parent compound such as BaFe\(_2\)As\(_2\) is a compensated metal, hence has the same number of electrons and holes, i.e. \( n_h = n_e \). Therefore it is a reasonable approximation for our model to take \( \tilde{N}_h = N_c \) at no doping and then the doping of holes (K, Na, etc.) or electrons (Co, Ni, etc.) is simulated by varying \( N_h \) or \( N_e \) while keeping \( N_c + N_h = N_{\text{tot}} = \text{const.} \). For the rest of this paper, it is convenient to use the normalized DOSs as \( \tilde{N}_{h,e} = N_{h,e}/N_{\text{tot}} \) and \( N_{\text{tot}} \) is combined to define the dimensionless coupling constants as \( \tilde{V}_{\text{intra/inter}} = N_{\text{tot}} \cdot V_{\text{intra/inter}} \).

Expanding the gap equations Eq.(2) near \( T_c \) and using Eq.(4), we obtain \( \tilde{\Delta}_{h,e}(T) \) near \( T_c \) as

\[ \Delta_h^2(T) \approx \frac{2}{1 + \tilde{N}_h/N_c} \Delta_{BCS}^2(T), \quad (6) \]

\[ \Delta_e^2(T) \approx \frac{2}{1 + N_e/N_h} \Delta_{BCS}^2(T) \]

with \( \Delta_{BCS}^2(T) = \pi^2 \frac{8}{\chi(\alpha)} T_c^2 (1 - T/T_c) \). Combining the results of Eq.(4) and (6), Eq.(1) provides

\[ \frac{\Delta C}{T_c} \approx 4 \times (3.06)^2 N_{\text{tot}} \cdot (\tilde{N}_h \tilde{N}_e). \quad (7) \]

This is our key result. In contrast to the one band BCS superconductor, Eq.(7) clearly shows that \( \Delta C/T_c \) can have a strong \( T_c \) dependence through \( \tilde{N}_h \tilde{N}_e \) even with a constant \( N_{\text{tot}} \) (see Eq.(5)). With doping in a given FePn/Ch compound, \( \tilde{N}_h \) and \( \tilde{N}_e (= 1 - \tilde{N}_h) \) varies over the range of \([0, 1]\) of \([19]\). As such if \( (\tilde{N}_h \tilde{N}_e) \sim T_c^2 \) for some region of \( \tilde{N}_{h,e} \), we would obtain the BNC scaling.

Having analyzed the ideal case (\( V_{\text{intra}} = 0 \)), we numerically study the more realistic cases, including the impurity scattering effect. We solve the coupled gap equations Eq.(2) for \( \Delta_{h,e}(T) \) near \( T_c \) and directly calculate \( \Delta C \) using Eq.(1). We find that the kinematic constraint of the two band pairing model discovered above is robust. However in order to explain the ideal BNC scaling \( \Delta C/T_c \propto T_c^2 \) in Ba(Fe\(_{1-x}\)TM\(_x\))\(_2\)As\(_2\) (TM=Co,Ni) as well as its strong deviation in Ba\(_{1-x}\)K\(_x\)Fe\(_2\)As\(_2\)\([10]\), we find that the non-pair-breaking impurity scattering plays a crucial role.

**Numerical results.**—In Fig.1(A), we calculated \( T_c \) vs. \( \tilde{N}_h \) of the two band model Eq.(2) for \( V_{\text{intra}} = 1.0 \) and 2.0, respectively, with \( V_{\text{intra}} = 0.0 \) for both cases. (B) Plots of \( N_h \tilde{N}_e \) vs. \( T_c \) with the same data of (A). Solid lines are of \( \sim T_c \) and \( \sim T_c^2 \) respectively. (C) Numerical calculations of \( \Delta C/T_c \) vs. \( T_c \) for \( V_{\text{inter}} = 1.0, 1.5, 2.0 \), and 3.0, respectively, with \( V_{\text{intra}} = 0.0 \) for all cases. Horizontal dashed line is the BCS limit of 9.36\( N_{\text{tot}} \) and the dotted lines of \( \sim T_c \) and \( \sim T_c^2 \) (BNC scaling) are guides for the eyes.

![Fig. 1: Color online](image-url) (A) Numerical calculations of \( T_c \) vs. \( \tilde{N}_h \) of the two band model for \( V_{\text{inter}} = 1.0 \), and 2.0, respectively, with \( V_{\text{intra}} = 0.0 \) for both cases. (B) Plots of \( N_h \tilde{N}_e \) vs. \( T_c \) with the same data of (A). Solid lines are of \( \sim T_c \) and \( \sim T_c^2 \); respectively. (C) Numerical calculations of \( \Delta C/T_c \) vs. \( T_c \) for \( V_{\text{inter}} = 1.0, 1.5, 2.0 \), and 3.0, respectively, with \( V_{\text{intra}} = 0.0 \) for all cases. Horizontal dashed line is the BCS limit of 9.36\( N_{\text{tot}} \) and the dotted lines of \( \sim T_c \) and \( \sim T_c^2 \) (BNC scaling) are guides for the eyes.
this strength of pairing potential is unrealistically large. (2) Including \( V_{\text{intra}} \neq 0.0 \) does not change the general behavior shown in Fig.1(C) as long as \( V_{\text{intra}} < V_{\text{inter}}/2 \).

While we have found that the BNC scaling can be realized in a region near the maximum \( T_c \) with the generic two band model, we still need an extra mechanism to enhance the BNC scaling for the wider region of \( T_c \). As shown in Fig.1(A) and Fig.1(B), \( T_c \) is maximum when \( \tilde{N}_e = \tilde{N}_h = 0.5 \) and it quickly decreases with doping as \( \tilde{N}_h \tilde{N}_e \ll 0.25 \) and accordingly one of the OPs, either \( \Delta \lambda \) or \( \Delta_c \), becomes tiny. Hence, the effect of impurity scattering on the tiny gap becomes increasingly stronger for the lower \( T_c \) region where the ratio \( \tilde{N}_e/\tilde{N}_h \) is far from 1. We found that this doping-dependent, therefore \( T_c \)-dependent, impurity effect changes the generic \( \Delta C/T_c \) vs. \( T_c \) relation to a steeper relation at the lower \( T_c \) region, hence enhances the region of the BNC scaling even with a moderate strength of \( V_{\text{inter}} \).

Phenomenologically we introduce two parameters of the impurity scattering in the two band model: \( \Gamma_0 \) (intra-band scattering) and \( \Gamma_\pi \) (inter-band scattering). As we assumed the \( \pm S \)-wave state, \( \Gamma_\pi \) causes strong pair-breaking effect (e.g. suppression of \( T_c \) and reduction of \( \Delta_{\text{h,c}} \)), but \( \Gamma_0 \) doesn’t affect the superconductivity itself[20]. However, the quasiparticle broadening is governed by the sum \( \Gamma_{\text{tot}} = \Gamma_0 + \Gamma_\pi \) and the calculations of \( \Delta C \) from Eq.(1) should be generalized with this broadening of the quasiparticle spectra as follows[21],

\[
\Delta C = \sum_{i=h,c} N_i \left( -\frac{d\Delta_i^2}{dT} \right) T_c \int_0^\infty dx \frac{x^2}{2 \cosh^2(\frac{x}{2})} \left[ 1 + \left( \frac{\omega_n}{\Delta^0} \right) \right] \left[ 1 + \left( \frac{\omega_n}{\Delta} \right) \right]^2
\]

(8)

where \( x = \omega/T_c \). The standard pair-breaking effect of \( \Gamma_\pi \) still enters the pair-susceptibility \( \chi_{h,c}(T_c) = T_c \sum_n \frac{\Delta_{h,c}^0 \xi_n}{\omega_n + \Gamma_\pi} \) instead of Eq.(3) with \( \omega_n = \omega_n + \Gamma_\pi \), therefore \( \Gamma_\pi \) directly affects \( T_c \) and \( \left( \frac{\omega_n}{\Delta^0} \right) \) in Eq.(8). However, increasing \( \Gamma_\pi \) only[11] doesn’t help for producing the BNC scaling as discussed in the Introduction[13]. On the other hand, Eq.(8) above shows that the total quasiparticle damping rate \( \Gamma_{\text{tot}} \) entering the thermodynamic average part in Eq.(8) is more important to determine \( \Delta C/T_c \) than \( T_c \).

In Fig.2, we show the numerical results of \( \Delta C/T_c \) vs. \( T_c \) in log-log scale with a choice of a moderate strength of the pairing potentials, \( V_{\text{inter}} = 2.0 \) and \( V_{\text{intra}} = 0.5 \), and varied the impurity scattering rates \( \Gamma_0 \) and \( \Gamma_\pi \). Without impurity scattering (red ” × ” symbols, \( \Gamma_0 = \Gamma_\pi = 0.0 \)), \( \Delta C/T_c \) shows the \( T_c^3 \) scaling only for the limited region near the maximum \( T_c \) and it quickly becomes flatter and slower than \( T_c \). Interestingly, this behavior looks very similar to the experimental data of Ba1-xKxFexAs2[10]. Therefore, we speculate that the K-doping in Ba1-xKxFexAs2 doesn’t introduce many impurity scatterers. Next, a small increase of impurities (green ” + ” symbols, \( \Gamma_0 = \Gamma_\pi = 0.02 \) in unit of \( \Delta_{\text{h,c}} \)) immediately changes \( \Delta C/T_c \) closer to \( T_c^2 \) over the whole \( T_c \) range, and the case with \( \Gamma_0 = 0.1 \) and \( \Gamma_\pi = 0.05 \) (pink ” ◦ ” symbols) displays an ideal BNC scaling \( \Delta C/T_c \sim T_c^2 \) for the entire range of \( T_c \). Finally, for demonstration purposes, we also show the case with unrealistically large impurity scattering rates, \( \Gamma_0 = 0.5 \) and \( \Gamma_\pi = 0.05 \) (dark green ” o ” symbols), which displays \( \Delta C/T_c \sim T_c^3 \), a super-strong scaling.

**Coexistence region with magnetic and SC orders.** — Experiments showed that the BNC scaling continues to be valid even when the spin density wave (SDW) order coexists with the SC order in the underdoped regime. Now we would like to extend our model including the magnetic order in the underdoped regime. We took a simple phenomenological approach ignoring the self-consistence between two OPs. We consider only the hole doped region \( \tilde{N}_h \in [0.5,1] \), because our model is symmetric with the hole and electron doping. We arbitrarily chose the coexistence region for \( 0.5 < \tilde{N}_h < 0.7 \), just for the sake of demonstration, and then we introduced the magnetic order \( M(N_h) \) for this region. The magnetic OP \( M(N_h) \) linearly grows from zero at \( \tilde{N}_h = 0.7 \) to a
maximum value $M_{\text{max}}$ at $N_h = 0.5$ as shown in Fig.3(A). When a finite $M$ exists, it affects the superconductivity in two important ways: (1) it weakens the SC pair susceptibility and we take the simplest approximation as $\chi_{h,e}(T_c) = T_c \sum \int_0^{\Lambda_{h,e}} \frac{d\mathbf{k}}{2\pi^2} \sqrt{\sum_{i,j} \Delta^2}$ [14]. (2) The presence of SDW order $M$ also removes a part of the FSs. Phenomenologically, we mimic this effect by linearly reducing the total DOS $N_{\text{tot}}$ starting from $N_h = 0.7$ to a maximum reduction at $N_h = 0.5$ as $N_{\text{tot}}(N_h) = N_{\text{tot}}^0(1 - a \frac{M(N_h)}{\Lambda_{h,e}})$ ($a = 0.5$ was chosen for calculations in Fig.3). With these phenomenological Ansätze, we solved the $T_c$-equations from Eq.(2) with fixed pairing interactions and damping ($\bar{V}_{\text{inter}} = 2.0, \bar{V}_{\text{intra}} = 0.5$; and $\Gamma_0 = 0.10, \Gamma_\pi = 0.05$) for three different strengths of $M_{\text{max}}$ in Fig.3(A). The results qualitatively simulate the experimental phase diagram: $T_c$ starts decreasing when $M(N_h)$ starts developing from $N_h = 0.7$ and the reduction of $T_c$ is faster with larger magnetic order.

In Fig.3(B), $\Delta C/T_c$ vs. $T_c$ is calculated for the corresponding three cases of Fig.3(A). The case of $M = 0$ (dark yellow "X" symbols), displaying the $T^2$ BNC scaling, is the same calculation as in Fig.2 with $\Gamma_0 = 0.10$ and $\Gamma_\pi = 0.05$ but only over $N_h \in [0.7, 1]$. Then the three other solid symbols are the calculation results for the region of $N_h \in [0.5, 0.7]$ with three different strengths of magnetic order $M(N_h)$ of Fig.3(A). The results of Fig.3(B) reveal an interesting behavior; namely, although it is more natural to expect that $\Delta C/T_c$ vs. $T_c$ with a coexisting magnetic order should behave differently from the one without a magnetic order [14], the calculations of Fig.3(B) with a crude phenomenological treatment of the coexisting magnetic and SC orders show that it is quite robust to follow the BNC scaling even with widely different strengths of $M$. We trace the origin of this surprising result to the fact that the underdoped region (i.e. where $N_h$ is near 0.5 and $T_c$ is maximum), when the magnetic order is absent, is the region where the BNC scaling is best obeyed due to the kinematic constraint of the multiband superconductor (see Fig.1(c) and Fig.2). Therefore, even if the magnetic order modifies the pair susceptibility $\chi_{h,e}(M)$ and cuts out a part of DOS from $N_{\text{tot}}^0$, the generic kinematic constraint of the multiband superconductor dominated by $\bar{V}_{\text{inter}}$ is still operative.

Summary and Conclusions — We showed that the puzzling BNC scaling relation $\Delta C/T_c \sim T_c^2$ [11] observed in a wide range of the FePn/Ch SC compounds [10,12] is a manifestation of the generic property of the multiband superconductor paired by a dominant inter-band pairing potential $\bar{V}_{\text{inter}} > \bar{V}_{\text{intra}}$. The underlying mechanism is the kinematic constraint $\frac{\Delta c}{\Delta c} \sim \sqrt{\frac{N_h}{N_e}}$ near $T_c$, and the subsequent relations of $\Delta C \sim N_h N_c$ and $T_c(\sqrt{N_h N_c})$. A consideration of the non-pair-breaking impurity effect which broadens the quasiparticle spectra near $T_c$ also explains the evolution from the ideal BNC scaling to its strong deviation as found in Ba$_{1-x}$K$_x$Fe$_2$As$_2$ [10].

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[1] S. L. Budko, N. Ni, and P. C. Canfield, Phys. Rev. B 79, 220516 (2009).
[2] J. S. Kim, G. R. Stewart, S. Kasahara, T. Shibauchi, T. Terashima, and Y. Matsuda, J. Phys.: Condens. Matter 23, 222201 (2011).
[3] F. Hardy et al., Phys. Rev. B 81, 060501(R) (2010); F. Hardy et al., Europhys. Lett. 91, 47008 (2010).
[4] K. Gofryk et al., Phys. Rev. B 81, 184518 (2010); K. Gofryk, A. B. Vorontsov, I. Vekhter, A. S. Sefat, T. Imai, E. D. Bauer, J. D. Thompson, and F. Ronning, ibid. 83, 064513 (2011).
[5] J. S. Kim, B. D. Faeth, and G. R. Stewart, Phys. Rev. B 86, 054509 (2012).
[6] C. Chaparro et al., Phys. Rev. B 85, 184525 (2012).
[7] P. Walmsley et al., Phys. Rev. Lett. 110, 257002 (2013).
[8] S. L. Bud’ko et al., Phys. Rev. B 89, 014510 (2014).
[9] J. Xing et al., Phys. Rev. B 89, 140503 (2014).
[10] S. L. Bud’ko, M. Sturza, D. Y. Chung, M. G. Kanatzidis, and P. C. Canfield, Phys. Rev. B 87, 100509 (2013).
[11] V. G. Kogan, Phys. Rev. B 80, 214532 (2009); V. G. Kogan, ibid. 81, 184528 (2010).
[12] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics, (Dover, New York, 1963), Eq.(36.9) in p.306.
[13] In the strong pair-breaking limit ($\Gamma_\pi > T_c$) where $\Delta F \sim -N(0) \frac{\Delta c}{\Delta c}$ is valid, the behavior of $\Delta F(T)$ should also change to $\Delta^2(T) \propto \Gamma_\pi^2(1 - \frac{\pi}{2})$ instead of the BCS behavior $\Delta^2(T) \propto T_c^2(1 - \frac{T}{T_c})$. Then the SH jump in this limit becomes $\Delta C/T_c \sim N(0) \frac{\Delta c}{\Delta c}$, which is the same as the BNC scaling.
[14] M. G. Vavilov, A. V. Chubukov, and A. B. Vorontsov, Phys. Rev. B 84, 140502(R) (2011); M. G. Vavilov and A. V. Chubukov, ibid., 84, 214521 (2011).
[15] J. Zaanen, Phys. Rev. B 80, 212502(R) (2009).
[16] I.I. Mazin, D.J. Singh, M.D. Johannes, M.H. Du, Phys. Rev. Lett. 101, 057003 (2008); K. Kuroki, S. Onari, R. Arita, H. Usui, Y. Tanaka, H. Kontani, and H. Aoki , ibid., 101, 087004 (2008).
[17] Y. Bang and H.-Y. Choi, Phys. Rev. B, 78, 134523 (2008).
[18] Y. Bang, H.-Y. Choi, and H. Won, Phys. Rev. B, 84, 054529 (2009).
[19] For the undoped parent compound BaFe$_2$As$_2$, our model assumes $N_h = N_e = 0.5$. For electron doping in Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$, hole pocket(s) disappears ($N_h = 0$) around $x \approx 0.15$. For the hole doping cases, electron pockets(s) disappears ($N_e = 0$) in (Ba$_{1-x}$K)$_x$Fe$_2$As$_2$ when $x > 0.7$ but in (Ba$_{1-x}$Na)$_x$Fe$_2$As$_2$ the electron pocket still remains when $x \approx 1$. Therefore, the actual doping $x^{\text{eq}}$ and our model doping parameter $N_h,e$ should be properly rescaled according to the $T_c$ variation vs. doping.
[20] A. A. Abrikosov and L. P. Gorkov, Sov. Phys. JETP 12, 1243 (1961).
[21] S. Skalski, O. Betbeder-Matibet, and P.R. Weiss, Phys. Rev. 136, A1500 (1957), see Eq.(5.19).