Quantum resource studied from the perspective of quantum state superposition

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Abstract
Quantum resources, such as discord and entanglement, are crucial in quantum information processing. In this paper, quantum resources are studied from the aspect of quantum state superposition. We define the local superposition (LS) as the superposition between basis of single part, and nonlocal superposition (NLS) as the superposition between product basis of multiple parts. For quantum resource with nonzero LS, quantum operation must be introduced to prepare it, and for quantum resource with nonzero NLS, nonlocal quantum operation must be introduced to prepare it. We prove that LS vanishes if and only if the state is classical and NLS vanishes if and only if the state is separable. From this superposition aspect, quantum resources are categorized as superpositions existing in different parts. These results are helpful to study quantum resources from a unified frame.

Keywords: quantum resource, quantum state superposition, local superposition, nonlocal superposition, multipartite system

1. Introduction
Quantum resources are crucial in quantum information processing [1–4]. Quantum entanglement, the first proposed quantum resource, has shown to be important in quantum computation [1–3], quantum teleportation [5, 6], superdense coding [7], quantum cryptography [8, 9], Bell test [10, 11], etc. However, states without quantum entanglement but with nonzero quantum discord [12] are also useful in some quantum information process, e.g., deterministic
quantum computation with one qubit [13, 14], remote quantum state preparation [15, 16], thus also being considered as quantum resource. These resources have been studied from several perspectives, such as Von Neumann entropy [12, 17, 18], distance [19–22], and so on [23–27]. Since all entangled states have nonzero discord, entangled states are considered as a subset of quantum discordant states [12], but whether entanglement and discord are different resources is still an open question.

In this paper, we study the quantum resources from the aspect of quantum state superposition. For a state in composite system, we define the local superposition (LS) as the superposition between basis of single part, and nonlocal superposition (NLS) as the superposition between product basis of multiple parts. In this state superposition framework, quantum resources can be categorized by the superposition between different kinds of basis including different parts. We show that a state is classical when and only when it has zero LS and a state is separable when and only when it has zero NLS. As for the relation of state superposition to the previously proposed quantum entanglement and quantum discord, quantum entanglement can be considered as the quantification of NLS but excluding the LS, while quantum discord includes the LS. However, whether quantum discord also includes NLS is not determined in this paper.

Moreover, the LS and NLS introduced hereinafter is generalized to multidimensional and multipartite systems. For multipartite states, we find that for pure states with Schmidt decomposition, NLS equals LS, while for pure states without Schmidt decomposition, NLS and LS might be different. This state superposition perspective might be useful in studying the multipartite entanglement. For multipartite systems, we show explicitly the different kinds of state superpositions existing in different single part of the system, some composite parts, or all parts of the system. These plural superpositions, thus plural resources, provide us with many ways to explore the quantum feature of quantum systems.

This paper is organized as follows. In part 2, we first introduce the LS in two state bipartite system and then generalize it to multidimensional bipartite system. In part 3, NLS for two partite system is introduced. In part 4, LS and NLS are defined and analyzed for multipartite system. In the 5 part, several discussions are given. In part 6, main conclusions are presented. In the appendix, some proofs of the statements in the paper are provided.

2. LS in bipartite system

When considering a quantum state with superposition in the \( \{ \ket{0}, \ket{1} \} \) basis, \( \ket{\psi} = \alpha \ket{0} + \beta \ket{1} \), the coherence of the state can be defined as

\[
C(\{ \ket{0}, \ket{1} \}) = 2 \left| \alpha \right| \left| \beta \right| = 2 \sqrt{P(\ket{0}) P(\ket{1})},
\]

where \( |\alpha|^2 = P(\ket{0}) = \langle \psi | \ket{0} \langle 0 | \psi \rangle \) , \( |\beta|^2 = P(\ket{1}) = \langle \psi | \ket{1} \langle 1 | \psi \rangle \) , and \( P(\ket{0}) \) and \( P(\ket{1}) \) are the probability for state \( \ket{\psi} \) in state \( \ket{0} \) and \( \ket{1} \) , respectively. Zero coherence indicates that there is no state superposition, and thus this state has no superposition between basis \( \{ \ket{0}, \ket{1} \} \). On the contrary, if the coherence gets the maximum value 1, there is perfect state superposition between basis \( \{ \ket{0}, \ket{1} \} \). There are other cases between these two extreme ones. Therefore, it is reasonable to define the amount of superposition of \( \ket{\psi} \) between basis \( \{ \ket{0}, \ket{1} \} \) as

\[
S(\{ \ket{\psi}, \{ \ket{0}, \ket{1} \} \} = 2 \sqrt{P(\ket{0}) P(\ket{1})}).
\]

Equation (1) indicates that when \( S(\{ \ket{\psi}, \{ \ket{0}, \ket{1} \} \} = 0 \), there must be state superposition between \( \ket{0} \) and \( \ket{1} \). Note that the value in equation (1) is dependent on the basis chosen, that is the amount of superposition defined above is basis specific.
Consider a system \( A \) in a composite system \( AB \), for example, a singlet state, \( |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \), no matter which basis for system \( A \) is used, it is hard to say that \( A \) is in a state superposition of that basis. Actually, in this case, it is impossible to assign a state description of system \( A \). However, for this state, when the basis \{0, 1\} is chosen, system \( A \) will behave exactly as if it is in state \( \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \), assuming that there is no operation on \( B \). (Note that, this assumption is true, since if there is some operation on \( B \), then this composite system should be described by a different state.) Thus, from this point, we generalize the state superposition in system \( A \) and define the amount of superposition of this state in system \( A \) in the same way as equation (1). Generally, for a two qubits pure state \( |\psi\rangle_{AB} \), we define the amount of superposition of \( A \) in \( \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \) basis (through this paper, when basis is mentioned, it refers to orthonormal basis) as

\[
S_A \left( |\psi\rangle_{AB}, \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \right) = 2 \left( P(|\varphi\rangle_A)P\left( |\varphi^{\perp}\rangle_A \right) \right)^{1/2},
\]

where \( P(|\varphi\rangle_A) = Tr_{\{ A \}}(\langle \varphi | \psi \rangle_{AB} \langle \psi | \varphi \rangle_A) \), which is the probability for system \( A \) in state \( |\varphi\rangle_A \). More generally, for a specific decomposition of mixed state \( \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i | \), we define the amount of superposition of \( A \) in \( \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \) basis as

\[
\sum_i p_i S_A \left( |\psi_i\rangle_{AB}, \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \right).
\]

The above superposition is introduced by the basis of part \( A \), and we define this as the LS with respect to part \( A \).

**Definition.** The measure of LS of \( \rho_{AB} \) for part \( A \) is defined as

\[
LS_A(\rho) = \min \sum_i p_i S_A \left( |\psi_i\rangle_{AB}, \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \right),
\]

where the minimum is taken over all decompositions \( \{ p_i, |\psi_i\rangle_{AB} \} \) and all basis \( \{ |\varphi\rangle_A, |\varphi^{\perp}\rangle_A \} \).

Note that the LS defined in equation (3) is basis independent. From the perspective of preparation, we have the choice to choose different basis. For \( LS_A = 0 \), there exists a certain decomposition and basis that no superposition presents, and thus part \( A \) can be prepared via classical operation (we use classical operation to mean the operation that no quantum superposition is introduced and use quantum operation to mean the other). For \( LS_A > 0 \), no matter which decomposition and basis are considered, there is superposition in part \( A \), and thus part \( A \) can only be prepared via quantum operation. For example: for state \( |\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)(|0\rangle_B + |1\rangle_B) \), the LS in the \{0\}_A, \{1\}_A \} basis is nonzero and quantum operation is needed for the preparation when this basis is chosen, while the LS between basis \( \left\{ \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A), \frac{1}{\sqrt{2}}(|0\rangle_A - |1\rangle_A) \right\} \) is zero and classical operation is enough for the preparation when this basis is chosen. Thus, from the aspect of preparation, equation (3) shows the minimum amount of superposition produced in part \( A \) when preparing this state.

**Theorem 1.** For pure state \( |\psi\rangle_{AB} \), \( LS_A = 0 \) iff the state is a product state.
Proof. For general pure state $|\psi\rangle_{AB} = a_{00} |00\rangle + a_{01} |01\rangle + a_{10} |10\rangle + a_{11} |11\rangle$, by Schmidt decomposition, it can be written as $|\psi\rangle_{AB} = \alpha |00\rangle + \beta |11\rangle$, where $\alpha$ and $\beta$ are the singular value of matrix $A$ (with $a_{ij}$ being its elements) and $\alpha, \beta \in [0, 1]$. By setting basis $|\varphi\rangle = \sin(\theta/2) |0\rangle + e^{i\phi}\cos(\theta/2) |1\rangle$, $|\varphi^+\rangle_A = -e^{-i\phi}\cos(\theta/2) |0\rangle + \sin(\theta/2) |1\rangle$ and considering equation (3), we can get $LS_A = 2 |\alpha| |\beta|$ (the detailed proof of this formula is given in appendix). If $LS_A = 0$ then $\alpha = 0$ or $\beta = 0$, in either case, $|\psi\rangle_{AB}$ is a product state. If $|\psi\rangle_{AB}$ is a product state, then $\alpha = 0$ or $\beta = 0$ thus $LS_A = 0$. Q.E.D.

Theorem 2. For state $\rho_{AB}$, the necessary and sufficient condition for $LS_A = 0$ is that $\rho_{AB}$ can be written in the following form

$$\rho_{AB} = p_1 |\varphi\rangle_A \langle \varphi| \otimes \rho^1_B + p_2 |\varphi^+\rangle_A \langle \varphi^+| \otimes \rho^2_B.$$  

(4)

Proof. For mixed state that can be written as the form $\rho_{AB} = p_1 |\varphi\rangle_A \langle \varphi| \otimes \rho^1_B + p_2 |\varphi^+\rangle_A \langle \varphi^+| \otimes \rho^2_B$, since basis $\{ |\varphi\rangle_A, |\varphi^+\rangle_A \}$ can be chosen, thus we have $LS_A = 0$.

For states with $LS_A = 0$, from the definition (see equation (3)), we know that there must exist a decomposition $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ and a basis $\{ |\varphi\rangle_A, |\varphi^+\rangle_A \}$ such that for each $|\psi_i\rangle_{AB}$, $S_A(|\psi_i\rangle_{AB} \{ |\varphi\rangle_A, |\varphi^+\rangle_A \}) = 0$. From equation (2), $|\psi_i\rangle_{AB}$ must be written as $|\psi_i\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, and $|\psi\rangle_A$ is either $|\varphi\rangle_A$ or $|\varphi^+\rangle_A$. Summing the corresponding terms with $|\varphi\rangle_A \langle \varphi|$ and $|\varphi^+\rangle_A \langle \varphi^+|$, respectively, we have $\rho_{AB} = p_1 |\varphi\rangle_A \langle \varphi| \otimes \rho^1_B + p_2 |\varphi^+\rangle_A \langle \varphi^+| \otimes \rho^2_B$.

Note that equation (4) is also the necessary and sufficient condition for the original definition of quantum discord (when measurement is done to part $A$) to be zero [12], and it is called a classical-quantum state [2].

For any state $\rho_{AB}$, the following inequality holds, $0 \leq LS_A(\rho_{AB}) \leq 1$. The proof is given in appendix.

In the same way, for part $B$,

$$LS_B(\rho_{AB}) = \min \sum_i p_i S_B\left( |\psi_i\rangle_{AB} \{ |\varphi\rangle_B, |\varphi^+\rangle_B \} \right).$$

where the minimum is taken over all decompositions $\{ p_i, |\psi_i\rangle_{AB} \}$ and all basis $\{ |\varphi\rangle_B, |\varphi^+\rangle_B \}$. The necessary and sufficient condition for $LS_B = 0$ is that $\rho_{AB}$ can be decomposed in the following form

$$\rho_{AB} = p_1 \rho^1_A \otimes |\varphi\rangle_B \langle \varphi| + p_2 \rho^2_A \otimes |\varphi^+\rangle_B \langle \varphi^+|,$$

which is called a quantum-classical state [2]. This is also the necessary and sufficient condition for the original definition of quantum discord (when measurement is done to part $B$) to be zero [12].
Note that LS\textsubscript{A} and LS\textsubscript{B} are neither symmetric. A symmetric one can be defined

\[
\text{LS} = \min_i \sum_p \left[ S_A(\ket{\psi_i}_AB, \ket{\varphi}_A, \ket{\varphi^\perp}_A) + S_B(\ket{\psi}_AB, \ket{\varphi}_B, \ket{\varphi^\perp}_B) \right],
\]

(5)

where the minimum is taken over all decompositions \(\{p_i, \ket{\psi_i}_AB\}\) and all basis \(\{\ket{\varphi}_A, \ket{\varphi^\perp}_A\}, \{\ket{\varphi}_B, \ket{\varphi^\perp}_B\}\).

**Theorem 3.** For any state \(\rho_{AB}\), \(\text{LS} = 0\) iff it can be written as

\[
\rho_{AB} = p_1 \ket{\varphi}_A \bra{\varphi} \otimes \ket{\varphi}_B \bra{\varphi} + p_2 \ket{\varphi^\perp}_A \bra{\varphi^\perp} \otimes \ket{\varphi^\perp}_B \bra{\varphi^\perp}.
\]

(6)

The proof is similar to theorem 2, and will be omitted. The state in the right side of equation (6) is a classical state [2]. This means that for LS = 0, the state can be prepared without quantum operation in either parts, while for LS \(\neq 0\), quantum operation must be introduced to prepare this state. It is obvious that a state is quantum discordant when and only when it has LS. Note that, generally, \(\text{LS} \geq (\text{LS}_A + \text{LS}_B)/2\) (the proof will be given in appendix). LS\textsubscript{A}, LS\textsubscript{B}, and LS range from 0 to 1.

This definition can be easily generalized to multidimensional case. Consider a two partite system with dimension \(d_A\) and \(d_B\) for each subsystem, for state \(\ket{\psi}_AB\), the amount of superposition between basis \(\ket{j}_A\) is defined as

\[
S_A(\ket{\psi}_AB, \ket{\varphi}_A) = 2 \left( \sum_{m < n} P(\ket{\varphi^m}_A) P(\ket{\varphi^n}_A) \right)^{1/2},
\]

(7)

where \(P(\ket{\varphi^m}_A) = \text{Tr}_B[\rho_{AB} \langle \varphi^m | \ket{\varphi^m}_A] \), \(m = 1, \ldots, d_A\). Hence the amount of LS in part \(A\) for \(\rho_{AB} = \sum_i p_i \ket{\psi_i}_AB \bra{\psi_i}\) is

\[
\text{LS}_A = \min_i \sum_p S_A(\ket{\psi_i}_AB, \ket{\varphi}_A).
\]

(8)

where the minimum is taken over all decompositions \(\{p_i, \ket{\psi_i}_AB\}\) and all basis \(\{\ket{\varphi}_A\}\). In the same way, LS\textsubscript{B} for part \(B\) and LS for both parts can be defined analogous to previous bipartite two-state definitions. Correspondingly, \(\text{LS} = 0\) holds iff \(\rho_{AB} = \sum_i p_i \ket{\varphi^i}_A \bra{\varphi^i} \otimes \ket{\varphi^i}_B \bra{\varphi^i}\), where \(\{\ket{\varphi^i}_A\}\) and \(\{\ket{\varphi^i}_B\}\) are orthonormal basis for system \(A\) and \(B\), respectively.

Example: For state in \(3 \times 3\) system, \(\ket{\psi}_AB = \sum_{i,j=1,2,3} a_{ij} \ket{ij}\), by using the parametrization of three-dimensional unitary matrix [28] thus running over all the basis, the LS in side \(A\) or \(B\) can be numerically obtained. We compare the numerical results with \(2(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^{1/2}\), where \(\lambda_1, \lambda_2, \lambda_3\) are the singular value (also called Schmidt coefficients) of matrix \(A\) (with \(a_{ij}\) being its elements). Specifically, for pure state

\[
\ket{\psi}_AB = \frac{\sqrt{2}}{3} \lambda \ket{00} + \frac{\sqrt{2}}{3} \lambda \ket{11} + \sqrt{1 - \lambda^2} \ket{22},
\]

where \(0 \leq \lambda \leq 1\), the value expressed by singular value is \(\sqrt{2/3} \lambda + 2(\sqrt{2} + 1)\lambda\sqrt{1 - \lambda^2}/\sqrt{3}\). As shown in figure 1 they are the same. For the fifth point in figure 1, \(\lambda = 0.2\), LS = 0.583 986 531 642 978, and results by Schmidt
coefficients equals 0.583 986 531 642 978. However, due to the complicate calculations, it is hard to strictly prove that $L = 2 \left( \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \right)^{1/2}$.

3. NLS in bipartite system

Consider, for pure state $|\psi\rangle_{AB} = a_{00} |00\rangle + a_{01} |01\rangle + a_{10} |10\rangle + a_{11} |11\rangle$, the state superposition in the product basis, i.e., $|\varphi^{00}\rangle_{AB} = |\varphi\rangle_A \otimes |\varphi\rangle_B$, $|\varphi^{10}\rangle_{AB} = |\varphi^+\rangle_A \otimes |\varphi\rangle_B$, $|\varphi^{01}\rangle_{AB} = |\varphi\rangle_A \otimes |\varphi^+\rangle_B$, $|\varphi^{11}\rangle_{AB} = |\varphi^+\rangle_A \otimes |\varphi^+\rangle_B$. Since all parts of the system are included in the product basis, we define the state superposition between the product basis as NLS. For state $|\psi\rangle_{AB}$, the amount of NLS in this set of basis is defined as

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{For pure state, $|\psi\rangle_{AB} = \lambda \sqrt{2}/\sqrt{3} |00\rangle + \lambda_1 \sqrt{3} |11\rangle + \sqrt{1 - \lambda^2} |22\rangle$, the LS (green star line) compared with $\sqrt{2}/3 \lambda^2 + 2(\sqrt{2} + 1) \lambda \sqrt{1 - \lambda^2}/\sqrt{3}$ (blue solid line).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{NLS (blue solid line) and Concurrence (asterisk marked) as the function of parameter $\alpha$.}
\end{figure}
\[
\text{NLS}\left( |\psi\rangle_{AB}, \left\{ |\varphi^{ij}\rangle_{AB} \right\} \right) = 2 \left( \sum_{2m+n<2k+l} P ( |\varphi^{mn}\rangle_{AB} ) P ( |\varphi^{kl}\rangle_{AB} ) \right)^{1/2},
\]

where \( P ( |\varphi^{mn}\rangle_{AB} ) = \langle \varphi^{mn} | \psi \rangle_{AB} \langle \psi | \varphi^{mn} \rangle_{AB} \), \( m = 0, 1 \), and the sum is taken over all \( m, n, k, l \) satisfying \( 2m+n < 2k+l \).

**Definition.** The amount of NLS of state \( |\psi\rangle_{AB} \) is defined as

\[
\text{NLS}( |\psi\rangle_{AB} ) = \text{min} \left\{ \text{NLS}( |\psi\rangle_{AB}, \left\{ |\varphi^{ij}\rangle_{AB} \right\} ) \right\},
\]

where the minimum is taken over all product basis \( \left\{ |\varphi^{ij}\rangle_{AB} \right\} \).

**Theorem 4.** For pure state, \( \text{NLS}( |\psi\rangle_{AB} ) = 0 \) iff \( |\psi\rangle_{AB} \) is a product state.

**Proof.** For product state \( |\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \), product basis \( \left\{ |\varphi^{ij}\rangle_{AB} \right\} \), in which \( |\varphi^{00}\rangle = |\varphi\rangle_A \otimes |\varphi\rangle_B \), can be chosen. Therefore, in this case, there is no state superposition between the chosen basis, and \( \text{NLS}( |\psi\rangle_{AB} ) = 0 \). If \( \text{NLS}( |\psi\rangle_{AB} ) = 0 \) (from equation (10) and equation (9)), there exists a product basis \( \left\{ |\varphi^{mn}\rangle_{AB} \right\} \) that only one \( P ( |\varphi^{mn}\rangle_{AB} ) \) is nonzero. Thus, we have \( |\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\varphi\rangle_B \), which is a product state. Q.E.D.

For \( \text{NLS}( |\psi\rangle_{AB} ) = 0 \), no matter which set of product basis is chosen, there is state superposition between the basis.

For any pure state \( |\psi\rangle_{AB} = a_{00} |00\rangle + a_{01} |01\rangle + a_{10} |10\rangle + a_{11} |11\rangle \), by Schmidt decomposition, it can be written as \( |\psi\rangle_{AB} = \alpha |00'\rangle + \beta |11'\rangle \), where \( \alpha \) and \( \beta \) are the singular value of matrix \( A \) (with \( a_{ij} \) being its elements) and \( \alpha, \beta \in [0, 1] \). The numeric result of NLS of this pure state is compared with concurrence [18], which in this case is \( 2\alpha\beta \). As shown in figure 2, NLS and concurrence are the same. Since for any pure state, \( \alpha \in [0, 1] \) and \( \beta = \sqrt{1 - \alpha^2} \), the results in figure 2 has compared all the pure states. For the fifth point in figure 2, \( \alpha = 4/19 \), NLS = 0.411 616 080 243 916, and Concurrence = 0.411 616 080 243 916. However, due to the complicate calculations, it is hard to strictly prove that \( \text{NLS}( |\psi\rangle_{AB} ) = 2\alpha\beta \).

The concurrence, when used in the definition of entanglement of formation, is only a mathematical expression. From this state superposition perspective, concurrence can also be directly considered as the amount of NLS existing in state \( |\psi\rangle_{AB} \).

**Definition.** The amount of NLS for mixed state \( \rho_{AB} \) is defined as, according to convex roof theory [29],

\[
\text{NLS}( \rho_{AB} ) = \text{min} \sum_i p_i \text{NLS}( |\psi_i\rangle_{AB} ),
\]

where the minimum is taken over all decompositions \( \left\{ p_i, |\psi_i\rangle_{AB} \right\} \).

Note that the minimum taken here is different from that in LS. In NLS, local quantum operation, which will not introduce any nonlocal superposition, should be allowed. By local unitary operation, any two different product basis can be changed to each other, e.g., for \( \left\{ |\varphi^{ij}\rangle_{AB} = |\varphi\rangle_A \otimes |\varphi\rangle_B \right\} \) and \( \left\{ |\varphi^{ij'}\rangle_{AB} = |\varphi\rangle_A \otimes |\varphi\rangle_B \right\} \), by local unitary operation in part \( A \), we can change \( \left\{ |\varphi^n\rangle_A \right\} \) to \( \left\{ |\varphi^{ij'}\rangle_A \right\} \), and the same way for part \( B \). Therefore, in NLS
(equation (11)) for each $|\psi_i\rangle_{AB}$, unlike the minimum in the definition of LS (equation (3)) where the basis is the same for every $|\psi_i\rangle_{AB}$, different product basis should be allowed.

From the perspective of preparation, as showed above, by local quantum operation, we have the choice to choose product basis (without introducing any NLS) for the preparation of each pure state ensemble $|\psi_i\rangle_{AB}$. Thus, equation (11) gives the minimum NLS produced in the preparation. When $\text{NLS}(\rho_{AB}) = 0$, by choosing certain decomposition and basis, local quantum operation is enough to prepare the state, while when $\text{NLS}(\rho_{AB}) \neq 0$, no matter which decomposition and product basis are chosen, nonlocal quantum operation must be introduced.

**Theorem 5.** For any state $\rho_{AB}$: $\text{NLS}(\rho_{AB}) = 0$ iff $\rho_{AB}$ is separable ($\rho_{AB}$ can be written as $\rho_{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B$).

**Proof.** If $\rho_{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B$, each $\rho_k^A$ and $\rho_k^B$ can be decomposed again into pure state ensemble, $\rho_k^A = \sum_l p_{kl}^A |\varphi_{kl}^A\rangle \langle \varphi_{kl}^A|$, $\rho_k^B = \sum_l p_{kl}^B |\varphi_{kl}^B\rangle \langle \varphi_{kl}^B|$, where $\{|\varphi_{kl}^A\rangle\}$ and $\{|\varphi_{kl}^B\rangle\}$ are orthogonal basis for each system. Thus, $\rho_{AB}$ can be decomposed as $\rho_{AB} = \sum_{kl} p_{kl}^A p_{kl}^B |\varphi_{kl}^A\rangle \langle \varphi_{kl}^A| \otimes |\varphi_{kl}^B\rangle \langle \varphi_{kl}^B|$. Under this decomposition, each pure state ensemble is a pure product state, which, according to theorem 4, has zero NLS. Thus, if $\rho_{AB}$ is separable, $\text{NLS}(\rho_{AB}) = 0$.

If $\text{NLS}(\rho_{AB}) = 0$, according to equation (11), there exists a certain decomposition $\{p_i, |\psi_i\rangle_{AB}\}$, that for each $|\psi_i\rangle_{AB}$, $\text{NLS}(|\psi_i\rangle_{AB}) = 0$. According to theorem 4, each $|\psi_i\rangle_{AB}$ is a product state, $|\psi_i\rangle_{AB} = |\tilde{\psi}_i^A\rangle \otimes |\tilde{\psi}_i^B\rangle$. Thus, $\rho_{AB}$ can be written as $\rho_{AB} = \sum_i p_i |\tilde{\psi}_i^A\rangle \langle \tilde{\psi}_i^A| \otimes |\tilde{\psi}_i^B\rangle \langle \tilde{\psi}_i^B|$, which is a separable state. Q.E.D.

Theorem 5 directly indicates that a state is entangled when and only when it has NLS. Thus, from this superposition perspective, entanglement can be considered as the NLS existing in the system. Besides, as we numerically showed above, for pure state, concurrence and NLS are the same. According to the way concurrence calculated for mixed states in [18], for any two qubits, NLS has the same value as concurrence.

This definition can be easily generalized to multidimensional states. Consider a bipartite system with dimension $d_A$ and $d_B$ for each subsystem, for state $|\psi\rangle_{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} a_{ij} |i\rangle_{A} |j\rangle_{B}$, the amount of NLS between product basis $|\varphi_{mn}\rangle_{AB} = |\varphi_{m}^A\rangle \otimes |\varphi_{n}^B\rangle$, where $m = 1, \ldots, d_A$, $n = 1, \ldots, d_B$ and $\{|\varphi_{m}^A\rangle\}, \{|\varphi_{n}^B\rangle\}$ are orthonormal basis of system $A$ and $B$, respectively, is defined as

$$\text{NLS}$$}}}

$$\text{NLS} \left( |\psi\rangle_{AB}; \left\{ |\varphi_i\rangle_{AB} \right\} \right) = 2 \left( \sum_{md_n + n < kd_l + l} P \left( |\varphi_{mn}\rangle_{AB} \right) \langle \varphi_{mn}\rangle_{AB} \right)^{1/2},$$

where $P \left( |\varphi_{mn}\rangle_{AB} \right) = AB \langle \varphi_{mn}| \langle \psi| \langle \varphi_{mn}\rangle_{AB}$ and the summation is taken over all $m, n, k, l$ satisfying $md_n + n < kd_l + l$. The amount of NLS of state $|\psi\rangle_{AB}$ is

$$\text{NLS} \left( |\psi\rangle_{AB} \right) = \min \text{NLS} \left( \left\{ |\varphi_i\rangle_{AB} \right\} \right),$$

where the minimum is taken over all product basis $\left\{ |\varphi_i\rangle_{AB} \right\}$. Using convex roof theory, NLS for mixed states can also be defined.
4. LS and NLS in multipartite system

The definition of LS and NLS can be generalized to multipartite case. Assuming an $n$-partite system with dimension $d_m$ for the $m$th subsystem, for pure state

$$|\psi_{i_m,...,i_n}\rangle = \sum_{i_m,...,i_n} a_{i_m,...,i_n} |i_1\rangle \otimes ... \otimes |i_n\rangle,$$

where $i_m = 1, ..., d_m$, the amount of LS for the $m$th part, when basis $\{|\varphi_{i_m}\rangle\}_m$ is chosen, reads

$$LS_m\left( |\psi_{i_1,..,i_n}\rangle, \{ |\varphi_{i_m}\rangle\}_m \right) = 2 \left( \sum_{k \leq l} P\left( |\varphi_{k_m}\rangle\right) P\left( |\varphi_{l_m}\rangle\right) \right)^{1/2},$$

where $P\left( |\varphi_{k_m}\rangle\right) = Tr_{i_{m+1},..,i_n} \{ \langle \varphi_{k_m}| \psi_{i_1,..,i_n} \rangle \langle \psi_{i_1,..,i_n}| \varphi_{k_m}\}$. For a mixed state $\rho_{i_1,..,i_n} = \sum_i |\psi_i\rangle_{i_1,..,i_n} \langle \psi_i|$, the amount of LS for the $m$th part is defined as

$$LS_m(\rho_{i_1,..,i_n}) = \min \sum_i p_i LS_m\left( |\psi_i\rangle_{i_1,..,i_n}, \{ |\varphi_{i_m}\rangle\}_m \right),$$

(13)

where the minimum is taken over all decompositions $\{p_i, |\psi_i\rangle_{i_1,..,i_n}\}$ and all basis $\{|\varphi_{i_m}\rangle\}_m$. The amount of LS including all parts is

$$LS = \min \sum_i p_i \frac{1}{n} \sum_m LS_m\left( |\psi_i\rangle_{i_1,..,i_n}, \{ |\varphi_{i_m}\rangle\}_m \right),$$

(14)

where the minimum is taken over all decompositions $\{p_i, |\psi_i\rangle_{i_1,..,i_n}\}$ and all basis $\{|\varphi_{i_m}\rangle\}_m$. Equation (14) indicates that LS = 0 iff $\rho_{i_1,..,i_n}$ can be written as

$$\rho_{i_1,..,i_n} = \sum_{l_1,...,l_n} p_{l_1,...,l_n} |l_1\rangle \langle l_1| \otimes ... \otimes |l_n\rangle \langle l_n|,$$

where $\{l_m\}$ is the orthonormal basis of part $m$.

For NLS, for pure state $|\psi_{i_1,..,i_n}\rangle$, when choosing product basis $|\varphi_{i_1,..,i_n}\rangle = \{|\varphi_1\rangle\} \otimes ... \otimes \{|\varphi_n\rangle\}$, where $\{|\varphi_m\rangle\}_m$ are the orthonormal basis for part $m$, the amount of NLS under this product basis is defined as

$$NLS\left( |\psi_{i_1,..,i_n}\rangle, \{ |\varphi_{i_m}\rangle\}_m \right) = 2 \left( \sum_{R \leq R'} P\left( |\varphi_{R_1,..,R_n}\rangle\right) P\left( |\varphi_{R'_1,..,R'_n}\rangle\right) \right)^{1/2},$$

where

$$R = \sum_{k} (i_k - 1) \times d_k + (i_{k-1} - 1) \times d_{k-1} + ... + d_1$$

and $P\left( |\varphi_{R_1,..,R_n}\rangle\right) = \langle \varphi_{R_1,..,R_n}| \psi_{i_1,..,i_n} \rangle \langle \psi_{i_1,..,i_n}| \varphi_{R_1,..,R_n}\}$. The amount of NLS for state $|\psi_{i_1,..,i_n}\rangle$ is

$$NLS\left( |\psi_{i_1,..,i_n}\rangle \right) = \min NLS\left( |\psi_{i_1,..,i_n}\rangle, \{ |\varphi_{i_m}\rangle\}_m \right),$$

(15)

where the minimum is taken over all product basis $\{|\varphi_{i_m}\rangle\}_m$.

For mixed state $\rho_{i_1,..,i_n}$, the amount of NLS is

$$NLS(\rho_{i_1,..,i_n}) = \min \sum_i p_i NLS\left( |\psi_i\rangle_{i_1,..,i_n}\right),$$

(16)

where the minimum is taken over all decompositions $\{p_i, |\psi_i\rangle_{i_1,..,i_n}\}$. Equation (16) indicates that NLS(\rho_{i_1,..,i_n}) = 0 iff $\rho_{i_1,..,i_n} = \sum_i p_i \rho'_i \otimes ... \otimes \rho'_n$, which is fully separable. The proof is similar to theorem 5.

LS and NLS can also be defined in partial separable ways. If the Hilbert space is divided as $\{i_1, ..., i_n\}$, where $I$ is independent subset of $I = \{1, ..., n\}$ and $|I| = k$, LS and NLS can be defined in the above ways by changing the system to this $k$-partite. NLS($\rho_{i_{k+1},..,i_n}$) = 0 if and only if the state is $k$-partite separable respect to the above partition.
As defined above in this section, the superpositions existing in multipartite system are plural. It can be in a single part, in some parts, or in all parts of the system. Table 1 shows all the superpositions in a three partite system. In the table, \( |A|B|C \) represents the partition of the system, e.g., \( |A|B|C \) represents taking part \( A \) and part \( B \) as a whole and dividing the system as part \( AB \) and part \( C \). For the examples, \( |\text{GHZ}\rangle = \frac{1}{\sqrt{2}} [ |0\rangle_A |0\rangle_B |0\rangle_C + |1\rangle_A |1\rangle_B |1\rangle_C ] \). \( |\text{W}\rangle = \frac{1}{\sqrt{3}} [ |1\rangle_A |0\rangle_B |0\rangle_C + \frac{1}{\sqrt{2}} |0\rangle_A |1\rangle_B |0\rangle_C + \frac{1}{\sqrt{3}} |0\rangle_A |0\rangle_B |1\rangle_C ] \). These can be easily generalized to \( n(n > 3) \) partite system.

Examples: For GHZ-like states \( |\psi\rangle = 2 |0\rangle_A |0\rangle_B |0\rangle_C + \sqrt{1 - \lambda^2} |1\rangle_A |1\rangle_B |1\rangle_C \). This state has Schmidt decomposition and is already written in that form. The LS and NLS for this state is shown in figure 3 compared with the expression calculated by Schmidt value \( 2\lambda \sqrt{1 - \lambda^2} \).

For \( \text{W}-\)like states \( |\psi\rangle = \lambda/2 |0\rangle_A |0\rangle_B |1\rangle_C + \sqrt{3} \lambda/2 |0\rangle_A |1\rangle_B |0\rangle_C + \sqrt{1 - \lambda^2} |1\rangle_A |0\rangle_B |0\rangle_C \). Unlike the two partite states and GHZ-like states, there is no Schmidt decomposition of this state. The LS\(_A\), LS\(_B\), LS\(_C\), LS, and NLS is shown in figure 4. It can be seen that in three partite case, when there is no Schmidt decomposition, the LS and NLS might be different.

### Table 1. Superpositions existing in three partite system.

| Superposition | Basis | \( |\text{GHZ}\rangle \) | \( |\text{W}\rangle \) |
|---------------|-------|-----------------|-----------------|
| NLS in \( A|B|C \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 1.155 | |
| NLS in \( A|B|C \) | \{ |\psi\rangle_A \otimes |\psi\rangle_C \} | 0.943 | |
| NLS in \( A|B|C \) | \{ |\psi\rangle_B \otimes |\psi\rangle_C \} | 0.943 | |
| NLS in \( A|B|C \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 0.943 | |
| LS in \( A \) | \{ |\psi\rangle_A \} | 1.043 | |
| LS in \( A \) | \{ |\psi\rangle_B \} | 1.043 | |
| LS in \( A \) | \{ |\psi\rangle_C \} | 1.043 | |
| LS in \( A \) | \{ |\psi\rangle_{AB} \} | 1.043 | |
| LS in \( A \) | \{ |\psi\rangle_{AC} \} | 1.043 | |
| LS in \( A \) | \{ |\psi\rangle_{BC} \} | 1.043 | |
| LS in \( B \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 1.155 | |
| LS in \( B \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 1.155 | |
| LS in \( C \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 1.155 | |
| LS in \( C \) | \{ |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \} | 1.155 | |
| LS in \( AB \) | \{ |\psi\rangle_{AB} \} | 1.043 | |
| LS in \( AC \) | \{ |\psi\rangle_{AC} \} | 1.043 | |
| LS in \( BC \) | \{ |\psi\rangle_{BC} \} | 1.043 | |

### 5. Discussion

The distinctive feature of quantum world different from classical world is state superposition. For composite system, the state superposition can exist between basis of a single part or product basis including two or more parts. These superpositions are the quantum resources used in quantum information processing. Quantum entanglement and quantum discord have both been considered as quantum resources. As we have showed in this paper, state with LS is equivalent to state nonclassical, and state with NLS is equivalent to state entangled. In this paper, the general definition of entangled state that state cannot be prepared by local operation and classical communication is used [17], and for discord, the original definition is used [12]. From this superposition perspective, the quantum entanglement resource can be considered as the NLS in system, while for quantum discord resource, it includes the superposition in single part. However, whether the several kinds of discord introduced before only includes the LS is...
still an open question. From this state superposition aspect, quantum resources are categorized by the superpositions in different parts, and quantum entanglement and quantum discord both capture some kind of specific superposition in the system. We think state superposition is a basic feature of quantum world, and the quantum entanglement and quantum discord are some aspect of this feature. Considering these quantum resources from this basic feature might be a promising way to better understand quantum correlations.

For pure states, we showed that when Schmidt decomposition exists, the amount of LS is the same as NLS. Thus, for these states, the quantification of entanglement can be reduced to the property of the reduced state of a single part, e.g. the entanglement of formation for two partite pure state is defined by the reduced density matrix of either part [17]. However, as we showed, for pure states without Schmidt decomposition, the amount of LS and NLS might be different. For these states, it is inconvenient to study entanglement by the reduced density matrix, and the state superposition view introduced in this paper might be useful.
It should be noted that, in equations (7) and (9), although one kind of measure of superposition is given in the expressions, the specific mathematical formula can be changed. For the mathematical form defined in this paper, we have not yet find the way to calculate the LS for mixed two partite states due to the difficulty of taking over all decompositions. Whether there is a way to calculate the LS with the mathematical form presented in this paper or there are some other reasonable mathematical definitions that can make all the local superposition and NLS easier to calculate is still an open question. Since we have not find the way to calculate the LS for mixed two partite states, elaborate comparison to quantum discord is not presented in this paper, and these might be done in the future work.

The applications of quantum resources in quantum information processes can also be seen from this state superposition aspect. As state entangled is equivalent to state with nonzero NLS, quantum process, for which quantum entanglement is necessary, such as entanglement swapping [30], can also be considered as having explored the NLS in the system. And the states with zero NLS but nonzero LS can also be useful in some quantum process. For example, the state \( \rho = \frac{1}{7} |0\rangle \langle 0| \otimes | - \rangle \langle - | + \frac{1}{3} | + \rangle \langle + | \otimes |1\rangle \langle 1| \), with no nonlocal state superposition but nonzero local state superposition (the proof that this state has nonzero LS is presented in appendix), can be used for remote state preparation, as shown in [16]. For multipartite case, the superpositions might be very rich, which might exist in any single part, any two partite, or any combination of them. These plural superpositions provide us tremendous ways to explore the quantum feature of quantum systems. Thus, from the perspective of state superposition, our results are useful for the consideration of resource for quantum process.

6. Conclusion

We have studied quantum resources from the perspective of quantum state superposition. We have given clear definition and quantification of LS and NLS. For states in composite system, the LS is defined as the superposition between basis of a single part and NLS as the superposition between product basis of all parts. From the quantum state superposition perspective, quantum resources are categorized by superpositions existing in different parts. From the aspect of preparation, when nonzero LS presented, quantum operation must be introduced to prepare this state, and when nonzero NLS presented, nonlocal quantum operation must be introduced. We showed that state with zero LS is equivalent to state classical and state with zero NLS is equivalent to state separable. From this aspect, the quantum entanglement resource can be considered as the NLS in system, while for quantum discord resource, we only know that it includes the superposition in single part, and whether it also includes some NLS is an open question. From this aspect, the difference between quantum entanglement and quantum discord appears clear.

Besides, LS and NLS are defined in multipartite case. For this case, the kinds of superpositions are plural. We show that for three partite pure state, when there is no Schmidt decomposition, the amount of LS and NLS might be different and this state superposition view might be useful in studying multipartite entanglement. All these results provide us a direction for the consideration of states proper for specific quantum information processes.
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Appendix

(1) Detailed proof for the equality $L_{S_A} = 2 |\alpha||\beta|$ presented in the proof of theorem 1.

First, write the state as $|\psi\rangle_{AB} = \alpha |00\rangle + \beta |11\rangle$. Since this is a pure state, there is only one kind of decomposition $|\psi\rangle_{AB} \langle \psi|$. Thus, to get the minimum in equation (3), we only need to take over all the basis. Consider the following form of basis:

$$|\varphi\rangle_A = \sin \frac{\theta}{2} |0\rangle + \cos \frac{\theta}{2} e^{i\phi} |1\rangle,$$

$$|\varphi^{-1}\rangle_A = -\cos \frac{\theta}{2} e^{-i\phi} |0\rangle + \sin \frac{\theta}{2} |1\rangle,$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. By taking all the values of $\theta$, $\phi$, all the basis are taken. According to the definition of $P(|\varphi\rangle_A)$ and $P(|\varphi^{-1}\rangle_A)$ in equation (2), we have

$$P(|\varphi\rangle_A) = |\alpha|^2 \sin^2 \frac{\theta}{2} + |\beta|^2 \cos^2 \frac{\theta}{2},$$

$$P(|\varphi^{-1}\rangle_A) = |\alpha|^2 \cos^2 \frac{\theta}{2} + |\beta|^2 \sin^2 \frac{\theta}{2}.$$ 

By multiplying them

$$P(|\varphi\rangle_A)P(|\varphi^{-1}\rangle_A) = (|\alpha|^4 + |\beta|^4) \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} + |\alpha|^2 |\beta|^2 \left( \sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2} \right).$$

Considering that $\cos^2 \frac{\theta}{2} = 1 - \sin^2 \frac{\theta}{2}$,

$$P(|\varphi\rangle_A)P(|\varphi^{-1}\rangle_A) = - \left( |\alpha|^2 - |\beta|^2 \right)^2 \left[ \left( \sin^2 \frac{\theta}{2} - \frac{1}{2} \right)^2 + \frac{1}{4} \right] + |\alpha|^2 |\beta|^2.$$ 

The above formula get the minimum when $\sin^2 \frac{\theta}{2}$ equals 1 or 0 (with basis $\{|0\rangle, |1\rangle\}$), and the minimum is $|\alpha|^2 |\beta|^2$. Thus, according to equation (3), we have $L_{S_A} = 2 |\alpha||\beta|$. Q.E.D.

(2) Proof for $0 \leq L_{S_A}(\rho_{AB}) \leq 1$.

For pure state, we have proved before that $L_{S_A}(|\psi\rangle_{AB}) = 2 |\alpha||\beta|$. Since $0 \leq 2 |\alpha||\beta| \leq 1$, thus, for pure state, $0 \leq L_{S_A}(|\psi\rangle_{AB}) \leq 1$.

For mixed state $\rho_{AB}$, we first prove that for $|\psi\rangle_{AB}$, the LS between any basis is less than or equal 1. For any basis $\{|\varphi\rangle_A, |\varphi^{-1}\rangle_A\}$, we have

$$2\sqrt{P(|\varphi\rangle_A)P(|\varphi^{-1}\rangle_A)} \leq \left( P(|\varphi\rangle_A) + P(|\varphi^{-1}\rangle_A) \right) = 1.$$ 

Thus, $S_A(|\psi\rangle_{AB}, \{|\varphi\rangle_A, |\varphi^{-1}\rangle_A\}) \leq 1$. For any decomposition $\{p_i, |\psi_i\rangle_{AB}\}$,

$$\sum_i p_i S_A(|\psi_i\rangle_{AB}, \{|\varphi\rangle_A, |\varphi^{-1}\rangle_A\}) \leq \sum_i p_i = 1.$$
Thus
\[ \min \sum_i p_i S_i \left( \left| \psi_i \right\rangle_{AB}, \{ |\varphi\rangle_A, |\varphi^{+}\rangle_A \} \right) \leq 1. \]
\[ \text{Q.E.D.} \]

(3) Proof for \( LS \geq (LS_A + LS_B)/2 \).

According to the definition of \( LS \) in equation (5),
\[ LS = \min \frac{1}{2} \sum_i p_i \left[ S_i \left( \left| \psi_i \right\rangle_{AB}, |\varphi\rangle_A, |\varphi^{+}\rangle_A \right) + S_i \left( \left| \psi_i \right\rangle_{AB}, |\varphi\rangle_B, |\varphi^{+}\rangle_B \right) \right], \]
where the minimum is taken over all decompositions \( \{ p_i, |\psi_i\rangle_{AB} \} \) and all basis \( \{ |\psi\rangle_A, |\psi^{+}\rangle_A \}, \{ |\varphi\rangle_B, |\varphi^{+}\rangle_B \} \). It is straightforward that
\[ LS \geq \frac{1}{2} \min \sum_i p_i S_i \left( \left| \psi_i \right\rangle_{AB}, |\varphi\rangle_A, |\varphi^{+}\rangle_A \right) + \frac{1}{2} \min \sum_i p'_i S_i \left( \left| \psi'_i \right\rangle_{AB}, |\varphi\rangle_B, |\varphi^{+}\rangle_B \right), \]
where the minimum in the first term is taken over all decompositions \( \{ p_i, |\psi_i\rangle_{AB} \} \) and all basis \( \{ |\psi\rangle_A, |\psi^{+}\rangle_A \} \), and the minimum in the second term is taken over all decompositions \( \{ p'_i, |\psi'_i\rangle_{AB} \} \) and all basis \( \{ |\varphi\rangle_B, |\varphi^{+}\rangle_B \} \). The last two terms are the definition for \( LS_A \) and \( LS_B \). Thus, \( LS \geq (LS_A + LS_B)/2 \). Q.E.D.

(4) Proof for nonzero \( LS \) in state
\[ \rho_{AB} = \frac{1}{2} |1\rangle_A \langle 1| \otimes |B\rangle \langle +| + \frac{1}{2} |1\rangle_A \langle +| \otimes |1\rangle_B \langle 1|, \]
where \( |B\rangle \langle +| = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) \).

Assume for this state, \( LS_A = 0 \). According to theorem 2, \( \rho_{AB} \) can be written as
\[ \rho_{AB} = p_1 |\varphi\rangle_A \langle \varphi| \otimes \rho_B^+ + p_2 |\varphi^{+}\rangle_A \langle \varphi^{+}| \otimes \rho_B^2. \]
By tracing part \( B \), that is \( \rho_A = \text{Tr}_B(\rho_{AB}) \), \( \rho_A \) reduces to
\[ \rho_A = p_1 |\varphi\rangle_A \langle \varphi| + p_2 |\varphi^{+}\rangle_A \langle \varphi^{+}|. \]
The above state is the diagonal form of \( \rho_A \). For the given state \( \rho_{AB} \) in equation (18), by tracing part \( B \)
\[ \rho_A = \frac{1}{2} |1\rangle_A \langle 1| + \frac{1}{2} |+\rangle_A \langle +|. \]
Diagonalizing it and comparing it to equation (19), we have
\[ p_1 = \frac{2 + \sqrt{2}}{4}, \quad p_2 = \frac{2 - \sqrt{2}}{4}, \]
\[ |\varphi\rangle_A = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \left[ |0\rangle_A + \left( 1 + \sqrt{2} \right) |1\rangle_A \right], \]
\[ |\varphi^{+}\rangle_A = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \left[ |0\rangle_A + \left( 1 - \sqrt{2} \right) |1\rangle_A \right]. \]
Thus, $|1\rangle_A$ and $| + \rangle_A$ can be written as

$$|1\rangle_A = \frac{\sqrt{2} + \sqrt{2}}{2} |\varphi\rangle_A - \frac{\sqrt{2} - \sqrt{2}}{2} |\varphi^\perp\rangle_A,$$

$$| + \rangle_A = \frac{\sqrt{2} + \sqrt{2}}{2} |\varphi\rangle_A + \frac{\sqrt{2} - \sqrt{2}}{2} |\varphi^\perp\rangle_A.$$

Taking them into equation (17),

$$\rho_{AB} = \frac{2 + \sqrt{2}}{4} |\varphi\rangle_A \langle \varphi| \otimes \frac{1}{2} (| + \rangle_B \langle + | + |1\rangle_B \langle 1|)$$

$$+ \frac{2 - \sqrt{2}}{4} |\varphi^\perp\rangle_A \langle \varphi^\perp| \otimes \frac{1}{2} (| + \rangle_B \langle + | + |1\rangle_B \langle 1|)$$

$$+ \frac{\sqrt{2}}{4} |\varphi\rangle_A \langle \varphi^\perp| \otimes \frac{1}{2} (|1\rangle_B \langle 1| - | + \rangle_B \langle + |)$$

$$+ \frac{\sqrt{2}}{4} |\varphi^\perp\rangle_A \langle \varphi^\perp| \otimes \frac{1}{2} (|1\rangle_B \langle 1| - | + \rangle_B \langle + |).$$

Comparing the above equation to the right side of equation (18), and noticing that $|\varphi\rangle_A \langle \varphi|, |\varphi^\perp\rangle_A \langle \varphi^\perp|, |\varphi\rangle_A \langle \varphi^\perp|, |\varphi^\perp\rangle_A \langle \varphi| \langle \varphi^\perp|$ are linearly independent, to make the right side of these two equations equal, the following equality must hold

$$\frac{1}{2} (|1\rangle_B \langle 1| - | + \rangle_B \langle + |) = 0.$$

However, the above equality obviously does not hold. Thus, the assumption is false. So, $L S_A > 0$. Since $LS \geq (LS_A + LS_B)/2$, thus, $LS > 0$ also holds. Q.E.D.

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