Modeling the impulsive noise component and its effect on the operation of a simple coherent network algorithm for detecting unmodeled gravitational wave bursts

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Abstract
An analytic model à la Middleton of the impulsive noise component in the data of interferometric gravitational wave detectors is proposed, based on an atomic representation of glitches. A fully analytic characterization of the coherent network data analysis algorithm proposed by Rakhmanov and Klimenko is obtained, for the simplest relevant case of triggered detection of unmodeled gravitational wave bursts, using the above noise model. The detector’s performance is evaluated under a suitable central-limit hypothesis, and the effects of both the noisiness of the pseudo-templates and the presence of the impulsive noise component are highlighted.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Gravitational wave (henceforth GW) astronomy is expected to open an essentially new observational window on the physical universe. Several classes of GWs of cosmic origin are currently being sought for, including continuous, transient and stochastic ones. An essential distinction among these different signals concerns our ability in modeling the expected waveforms. GW bursts (henceforth GWB) are a paradigm of transient signals for which only a few physically-based models exist [1–3].

GW detectors (with specific reference to the present-day large baseline optical interferometers) are invariably affected by transient disturbances of various origins [4]. Using auxiliary channels to monitor the status of the instrument and its environment may help in identifying and vetoing these disturbances. Experimental evidence suggests that a residual...
impulsive component will nonetheless be present in the data. Distinguishing these spurious noise glitches from true GWB of cosmic origin will be almost impossible when only data from a single detector are available. It becomes feasible, in principle, if the outputs of several detectors are suitably combined. Using data from several detectors it is further possible to reconstruct the GW signal waveform, which encodes the relevant source physics, so as to capitalize on and refine astrophysical models.

As an historical heritage of acoustic GW detectors [5], various coincidence algorithms, based on consistency tests among candidate-events gathered by different detectors, have been studied and tested [6, 7]. These algorithms, while conceptually simple and computationally inexpensive, turn out to be less efficient, in general, compared to coherent techniques, where the output data from several sensors are combined to form a suitable detection statistic to be used in classical hypotheses tests [8]. Several coherent techniques have been hitherto proposed [9–16], but only a few (e.g., WAVEBURST [11], X-PIPELINE [15] and RIDGE [16]) have been fully implemented to date in the data analysis pipelines of running experiments. Considerable work is still needed for completely characterizing alternative coherent algorithms in terms of performance and computational cost.

This paper (the first in a suite, where we propose to investigate problems of increasing complexity) attempts to provide a quantitative answer to the rather fundamental question of how well a network of several GW detectors may discriminate true GWBs from local disturbances (glitches) using coherent detection statistics. Among the essential benefits provided by the coherent network operation, we mention (i) the ability of detecting unmodeled signals, (ii) the capability of rejecting local disturbances, the possibility of (iii) retrieving the source position on the celestial sphere and (iv) reconstructing the gravitational waveforms. Here we focus on the first two properties, assuming for simplicity that the GW direction of arrival (henceforth DOA) and the time of occurrence (henceforth TOO) of the event are known (triggered search) from observations of different nature (e.g., electromagnetic, neutrino, etc).

To this end, modeling the impulsive noise component (the glitches) is a key, and yet open, issue. In this paper we adopt, for the first time to the best of our knowledge, a general representation of impulsive noise proposed by Middleton in a series of seminal papers [17]. Glitches are accordingly modeled as time–frequency atoms, i.e., transients whose energy content is almost confined to a compact region in the time–frequency plane, and characterized in terms of a few (random) parameters; we adopt the possibly simplest (though observation-driven) model for such atoms: real-valued sine-Gaussian (SG) functions. The impulsive noise component in each detector is modeled as a random train of these atoms, each occurring independently on the other ones.

In order to keep the formal complexity to a minimum, while still capturing the key aspects of the problem, we make a number of simplifying assumptions summarized below.

We refer to the coherent cross-correlation method introduced by Rakhmanov and Klimenko (henceforth RK) in [18] and deduce its statistical properties in analytic form. The RK algorithm may be recognized as a natural extension of the matched-correlator concept to the detection of unmodeled GWBs with a redundant network of detectors.

We limit to the simplest (though realistic and up-to-date) case of a network composed of three interferometers of comparable sensitivity, with specific reference to the LIGO–Hanford (henceforth LH), LIGO–Livingston (henceforth LL) and Virgo (henceforth V) detectors, and restrict to the case where the incident wave is linearly polarized.

This paper is accordingly laid out as follows. In section 2 we recall the RK formalism, and deduce the distribution of its detection statistics under the $H_1$ hypothesis (GWB in the data). In section 3 the adopted atomic representation of glitches is briefly introduced, and the first two moments of the RK detection statistics under the $H_0$ hypothesis (only noise in the
data) are derived following a simple heuristic reasoning, making the simplifying assumption that in each detector no more than a single glitch may occur in the analysis window. This assumption is relaxed in section 4, where we propose the anticipated fairly general and rigorous approach à la Middleton to model the impulsive component of the interferometer noise; the results obtained in sections 3 and 4 are shown to coincide, under the appropriate simplifying assumptions. In section 5, based on the above results, we evaluate the RK correlator-based detector’s performance by numerical experiments. Conclusions and hints for future work follow under section 6.

2. The RK coherent analysis algorithm

Whenever the sought waveform is known a priori, optimal detection in additive stationary (band-limited) white Gaussian noise is achieved by matched-filtering the data with a template of the sought waveform. The output of the filter (also known as matched correlator) has to be compared to a properly chosen threshold in order to decide about the presence or absence of the signal in the data. When the signal shape is known, except for a finite number of parameters, a set of correlators corresponding to a suitably dense covering of the parameter space can be computed, and the largest one exceeding the threshold is selected, yielding an estimate for the signal parameters.

For unmodeled GWBs the matched-filtering technique cannot be adopted. It is thus basically impossible to distinguish a GWB from a spurious glitch surviving the auxiliary-channel-based vetos, in the data of a single interferometer.

One possible way to circumvent this difficulty using data from a (redundant) network of detectors has been proposed in [18], and will be shortly recalled hereinafter.

Let a (plane) gravitational wave with linearly polarized (TT gauge) components \( h_+ (t) \) and \( h_\times (t) \) impinge on a set of interferometric detectors located at \( \vec{r} = \vec{r}_i, i = 1, 2, 3 \), from a direction /Omega_1s. In the absence of noise, the detector outputs can be written as

\[
S_i(t) = F_+^i (\Omega_i) h_+ [t - \tau_i (\Omega_i)] + F_\times^i (\Omega_i) h_\times [t - \tau_i (\Omega_i)], \quad i = 1, 2, 3, \tag{1}
\]

where \( \tau_i (\Omega_i) = c^{-1} \hat{n} \cdot \vec{r}_i \) is the propagation delay of the (plane) wavefront referred to its arrival at the spatial origin (usually taken coincident with the Earth center), \( \hat{n} \) is the unit wave-vector and \( F_+^i (\Omega_i), F_\times^i (\Omega_i) \) are the detector’s pattern functions, describing their directional responses.

Equations (1) can be rewritten in matrix form as follows:

\[
\begin{pmatrix}
S_1(t) \\
S_2(t) \\
S_3(t)
\end{pmatrix}
= 
\begin{pmatrix}
F_+^1 (\Omega_i) & F_\times^1 (\Omega_i) \\
F_+^2 (\Omega_i) & F_\times^2 (\Omega_i) \\
F_+^3 (\Omega_i) & F_\times^3 (\Omega_i)
\end{pmatrix}
\begin{pmatrix}
h_+ (t) \\
h_\times (t)
\end{pmatrix}, \tag{2}
\]

where

\[
S_i (t) = S_i^0 [t + \tau_i (\Omega_i)], \quad i = 1, 2, 3 \tag{3}
\]

1 A network of detectors is redundant if, in the absence of noise, the output of each detector can be expressed in terms of the outputs of the others. Since a gravitational signal has only two independent polarization components, any network of three or more (differently located and oriented) detectors is redundant in the common observational band.

2 In writing equation (1), we make the usual (competing) assumptions that the gravitational wave signal is (i) short enough to make the reciprocal motion between source and detector negligible and (ii) spectrally narrow enough to make the response of the antenna practically instantaneous.
are the properly time-shifted noise-free detector outputs, and the matrix on the rhs is the network response matrix. The rank of this latter cannot exceed 2, hence,

$$\det \begin{pmatrix} S_1 F_1^* & F_1^* \\ S_2 F_2^* & F_2^* \\ S_3 F_3^* & F_3^* \end{pmatrix} = 0.$$  

(4)

Expanding the determinant in the elements of the first column, we obtain the null condition

$$A_1(\Omega_1s)S_1(t) + A_2(\Omega_1s)S_2(t) + A_3(\Omega_1s)S_3(t) = 0,$$

(5)

where

$$\begin{align*}
A_1 &= F_1^*F_3^* - F_3^*F_2^* \\
A_2 &= F_1^*F_1^* - F_2^*F_3^* \\
A_3 &= F_1^*F_2^* - F_2^*F_1^*.
\end{align*}$$

(6)

In (6) (and whenever possible, hereafter), the dependence on $\Omega_1s$ of $A_i$ and $F_i^*$, $i = 1, 2, 3$ is omitted for notational ease.

From equation (5) one may infer that, in the absence of noise, the output of each and any detector in the network is proportional to a linear combination of the outputs of the remaining two, namely,

$$\Sigma_i(t) = -A_j S_j(t) - A_\ell S_\ell(t) = A_i S_i(t), \quad i, j, \ell = 1, 2, 3, \quad j \neq \ell \neq i$$

(7)

and is thus a template for $S_i$. The actual interferometer outputs $V_i(t)$, however, differ from the $S_i(t)$ due to the presence of noise, namely,

$$V_i(t) = S_i(t) + n_i(t), \quad i = 1, 2, 3.$$  

(8)

Accordingly, by using the $V_i$ in place of $S_i$ in (7) we obtain a noisy template for $S_i$,

$$W_i(t) = -A_j V_j(t) - A_\ell V_\ell(t) = A_i [S_i(t) + v_i(t)],$$

(9)

where

$$v_i(t) = -\frac{1}{A_i} [A_j n_j(t) + A_\ell n_\ell(t)], \quad i, j, \ell = 1, 2, 3, \quad j \neq \ell \neq i.$$  

(10)

The $W_i(t)$ can thus be used, following RK [18], to compute the pseudo-matched-correlators

$$C_i = \langle V_i, W_i \rangle = \int_{\Theta_i} V_i(t) W_i(t) \, dt \approx f_s^{-1} \sum_{m=1}^{N_s} V_{im} W_{im} \equiv f_s^{-1} V_i \cdot W_i^T, \quad i = 1, 2, 3.$$  

(11)

In (11), $\Theta_i$ is the ($T$ seconds wide) analysis window, $f_s$ is the sampling frequency, $N_s = \lfloor f_s T \rfloor$ is the number of samples in $\Theta_i$, $V_{im}$ and $W_{im}$ are the time samples of $V(t)$ and $W(t)$, respectively, $V_i = \{V_{i1}, V_{i2}, \ldots, V_{iN_s}\}$ and $W_i = \{W_{i1}, W_{i2}, \ldots, W_{iN_s}\}$.

In view of the relatively large value of $N_s$ in (11), typically $\gtrsim 10^2$, in the following we shall make the working assumption that a suitable form of the (generalized) central-limit theorem may be invoked [20] to argue that the $C_i$ are normal distributed under both hypotheses $H_1$ (signal present) and $H_0$ (no signal), despite the presence of glitches, which makes the interferometer noises depart from Gaussianity. Only the first two moments will be accordingly needed to characterize them.

3 For all practical purposes, one may use as a template for $S_i$ any quantity differing from $S_i$ by an arbitrary multiplicative factor [19].

4
2.1. RK correlators distributions under $H_1$

In this section, we assume that the occurrence of a GWB and an instrumental glitch in the same analysis window can be neglected, being extremely unlikely\(^4\). Denoting the moments for the $H_1$ case with the subscript 1, one readily obtains ($i = 1, 2, 3$)

$$
\mu_1^{(i)} = E[C_i|H_1] = A_i(\Omega_2) \int_{\theta_i} S_i^2(t) \, dt \approx f_s^{-1} A_i(\Omega_1 s) S_i \cdot S_i^T,
$$

(12)

where $S_i = \{S_i(t_1), S_i(t_2), \ldots, S_i(t_N)\}$, and

$$
(\sigma_1^{(i)})^2 = \text{Var}[C_i|H_1] = A_i^2(\Omega_2) \left[ \frac{N_i + \tilde{N}_i}{2} \int_{\theta_i} S_i^2(t) \, dt + \frac{N_i \tilde{N}_i}{4} N_i \right]
$$

$$
\approx f_s^{-2} A_i^2(\Omega_1 s) \left[ \sigma_1^2 + \sigma_i^2 \right] S_i \cdot S_i^T + \sigma_i^2 \tilde{\sigma}_i^2 N_i \right],
$$

(13)

where $N_i$ and $\tilde{N}_i$ denote the one-sided power spectral densities of $n_i(t)$ and $\nu_i(t)$, respectively\(^5\).

In deriving (12) and (13), we capitalize on the statistical independence between $n_i$ and $\nu_i$, the obvious identities

$$
E[n_i] = E[\nu_i] = E[n_i \nu_i] = 0,
$$

(14)

and the relationship $N_i = 2\sigma_i^2 / f_s$, valid for the band-limited Gaussian white noise with standard deviation $\sigma_i$.

The performance of the RK correlator is described in terms of its deflection \(^{[19]}\), aka signal-to-noise ratio (SNR) defined as

$$
d(i) = \frac{\mu^{(i)}(S \cdot S^T)}{\sigma^{(i)}} \approx \frac{S_i \cdot S_i^T}{[\sigma_i^2 + \tilde{\sigma}_i^2]^{1/2}}.
$$

(15)

In the following, we shall assume for simplicity that all detectors in the network have comparable noise PSDs, thus letting $N_i = N$ and $\sigma_i = \sigma, \forall i$. Accordingly,

$$
\tilde{\sigma}_i^2 = \frac{A_i^2 + A_j^2 + A_\ell^2}{A_i^2} \sigma^2,
$$

(16)

so that equation (15) becomes

$$
d(i) = \frac{A_i (S_i \cdot S_i^T)^{1/2}}{\sigma \left[ A_i^2 + A_j^2 + A_\ell^2 + N_i (A_j^2 + A_\ell^2) \sigma^2 \right]^{1/2}}.
$$

(17)

The deflection (17) can be written in a more transparent form by introducing the quantities

$$
\delta_{S}^{(i)} = \left( \frac{f_{\theta_i} S_i^2(t) \, dt}{N/2} \right)^{1/2} \approx \frac{(S_i \cdot S_i^T)^{1/2}}{\sigma}
$$

(18)

\(^4\) This (reasonable) assumption may be relaxed using the moments of the total noise derived in section 4, in lieu of those of the Gaussian component alone. Results pertaining to this more general case will be presented elsewhere.

\(^5\) The last term in (13) is obtained from

$$
E \left[ \int_{[T]} \int_{[T]} dx n_i(t)n_j(s)\nu_j(t)\nu_j(s) \right]
$$

using the band-limited white-noise formula

$$
E[n(t)n(s)] = \frac{N}{2} \int_{-B}^{B} \exp[2\pi f (t-s)] df.
$$
representing the signal-to-noise ratio of a perfect matched filter applied to the actual data at the output of detector-i, and

$$\delta_h = \left( \frac{h_{\text{rss}}^2}{N/2} \right)^{1/2},$$

where

$$h_{\text{rss}}^2 = \int_0^{\Omega_1} [h_x(t)^2 + h_\times(t)^2] \text{d}t$$

is a frequently used measure of the GWB strength. For the simplest case of linearly polarized GWBs,

$$\delta_S^{(i)} = |F_i| \delta_h,$$

where $F_i = F_i^{\text{+,-}}$, depending on the wave polarization and $\delta_h$ represents the intrinsic signal-to-noise ratio of a perfect matched filter applied to the bare gravitational waveform embedded in the detector noise; such a deflection would be attained if the antenna were isotropic ($F_i^{\text{+,-}} = 1$) and the template noise-free.

The deflection $d^{(i)}$ in (15) can thus be conveniently written as

$$d^{(i)} = \delta_S^{0/1} \Xi_i(\Omega_S, N_s, F_i, \delta_h),$$

where

$$\Xi_i(\Omega_S, N_s, F_i, \delta_h) = \frac{A_i}{\left[ A_i^2 + A_\text{+}^2 + A_\text{\times}^2 + N_s(A_i^2 + A_\text{+}^2)(|F_i|\delta_h)^{-2} \right]^{1/2}}$$

measures the SNR degradation of the RK correlator $C_i$ w.r.t. the perfect matched filter acting on the output data of detector-i, due to the noisiness of the template. It is seen from (23) that $\Xi_i(\cdot)$ depends on the DOA, the number of samples $N_s$ in the analysis window, the polarization-dependent pattern function $F_i$ and the intrinsic deflection $\delta_h$.

Figures 1(a) and (b) display the sky maps of the function $\Xi_i(\cdot)$ in (23) for the L H, L L and V detectors for the two linear polarizations, for $N_s = 100$ and two extremal values of $\delta_h$, namely $\delta_h = 10$ and $\delta_h = 100$, respectively. The source position in figures 1(a) and (b) is parametrized in terms of the polar and azimuthal angles $\vartheta_s, \varphi_s$ in an Earth-centered coordinate system whose polar axis points to the North Pole and $\varphi_s = 0$ identifies the Prime Meridian.

3. Glitches

Available experimental evidence [21, 22] suggests that instrumental noise glitches can be efficiently modeled as atoms [23] in the time–frequency plane [24]. Atoms are waveforms with almost-compact time–frequency support. They can be characterized in terms of their energy content, and their first- and second-order moments, i.e., occurrence time $t_0$, center frequency $f_0$, effective duration $\sigma_t$ and bandwidth $\sigma_f$.

The choice of an atom family (technically called a dictionary) appropriate to modeling glitches in GW interferometers must be compliant to and derived from experimental evidence. In this connection, the work in [21, 22] aimed at classifying glitches and identifying glitch families (clusters in parameter space) is particularly relevant. It should be noted that atoms in general form overcomplete systems, and this fact must be taken properly into account in deducing the distributions of the atom parameters from observed glitch populations [25].

6 GWBs can also be modeled as atoms. For the detection technique adopted here, however, only the GWB energy is relevant, not its shape.
We adopt here the possibly simplest atom, the real-valued sine-Gaussian (SG) functions defined by
\[ \psi(t - t_0; g_0, f_0, \sigma_t) = g_0 \sin(2\pi f_0(t - t_0)) e^{-(t-t_0)^2/\sigma_t^2}. \] (24)
whose waveform and time–frequency representation are shown in figure 2. The SG atom is entirely characterized by its shape parameters \( g_0, f_0 \) and \( \sigma_t \) and occurrence (firing) time \( t_0 \).
Figure 2. SG atom with $g_0 = 1$, $t_0 = 0.5$ s, $f_0 = 100$ Hz, $\sigma_t = 0.02$ s. Top: time-domain waveform; bottom: time–frequency (Wigner–Ville) representation.

The choice of the SG dictionary is suggested by the fact that a wide variety of observed glitches in the data channel are well modeled as SG atoms [22], and is further motivated by its structural simplicity, minimum time–frequency spread, $(\sigma_t \sigma_f = (4\pi)^{-1})$ and positive-definiteness of its Wigner–Ville transform. These properties should likely permit us to represent the instrumental transients in a close-to-optimal (i.e., minimally redundant) way (see, e.g., [26, 27]).

3.1. RK correlators distributions under $H_0$—heuristic approach

In this section, we obtain a heuristic characterization of the RK correlator distribution under $H_0$ by considering the glitches as (spurious) signals with random parameters. Specifically, we derive the 'average' among the marginal distributions of the RK correlator corresponding to all possible glitch realizations in the network. To keep the analysis as simple as possible, we assume that no more than a single glitch may occur in the analysis window in each interferometer. This restriction will be removed in the following section, where a fairly general and rigorous model for the impulsive noise component will be proposed.

It is expedient to write the moments under $H_0$ (denoted with a subscript 0) as follows:

$$
\mu^{(i)}_0 = (1 - \Pi)^3 \mu^{(i)}_0 + \Pi(1 - \Pi)^2 \mu^{(i)}_{0,1} + \Pi^2(1 - \Pi)\mu^{(i)}_{0,2} + \Pi^3 \mu^{(i)}_{0,3},
$$

$$
\sigma^{(i)}_0 = (1 - \Pi)^3 (\sigma^{(i)}_{0,0})^2 + \Pi(1 - \Pi)^2 (\sigma^{(i)}_{0,1})^2 + \Pi^2(1 - \Pi)(\sigma^{(i)}_{0,2})^2 + \Pi^3 (\sigma^{(i)}_{0,3})^2,
$$

where $\mu^{(i)}_{0,k}$, $\sigma^{(i)}_{0,k}$ refer to the cases where $k$ detectors ($k = 0, 1, 2, 3$) in the network exhibit a glitch within the analysis window, and the corresponding factors in front of them are the related occurrence probabilities, $\Pi$ being the (known) probability of observing a single glitch in the analysis window.

The quantities $\mu^{(i)}_{0,k}$ and $\sigma^{(i)}_{0,k}$ in equations (25) and (26) can be computed for any glitch instance in the network, i.e. for any allowed set of (possibly null) SG atoms in the interferometers’ outputs. For each instance, these quantities identify the corresponding
conditional moments of the detection statistics $C_i$ under $H_0$. We are obviously interested in computing the same moments averaged over all possible glitch realizations in the network detectors, using the known prior distributions for the glitch parameters. After some tedious algebra, we accordingly get (under the usual discrete-time representation):

\begin{align}
\mu_{0,0}^{(i)} &= 0, \\
(\sigma_{0,0}^{(i)})^2 &= f_s^{-2}(A_j^2 + A_l^2)\sigma_s^4 N_s, \\
\mu_{0,1}^{(i)} &= 0, \\
(\sigma_{0,1}^{(i)})^2 &= 3(\sigma_{0,0}^{(i)})^2 + 2 f_s^{-2}\sigma_s^2(A_j^2 + A_l^2)E(\psi \cdot \psi^T), \\
(\sigma_{0,2}^{(i)})^2 &= 3(\sigma_{0,0}^{(i)})^2 + f_s^{-2}\sigma_s^2[4(A_j^2 + A_l^2)E(\psi \cdot \psi^T) + 2A_jA_lE(\psi \cdot \psi^{T,T})]. \\
(\sigma_{0,3}^{(i)})^2 &= (\sigma_{0,0}^{(i)})^2 + f_s^{-2}\sigma_s^2[2(A_j^2 + A_l^2)E(\psi \cdot \psi^T) + 2A_jA_lE(\psi \cdot \psi^{T,T})],
\end{align}

where $\psi = \{\psi(t_1), \psi(t_2), \ldots, \psi(t_N)\}$, and $E(\psi \cdot \psi^T)$ and $E(\psi \cdot \psi^{T,T})$ (multiplied by $f_s^{-1}$) are the expected glitch energy and the expected correlation between glitches occurring in different detectors, respectively, both expectations being taken over all possible glitch instances.

Note that equations (27) are nothing but the moments of $C_i$ under $H_0$ in the absence of glitches, i.e., due to the Gaussian noise floor only. It is therefore apparent that glitches have a twofold effect, making the expected value of $C_i$ nonzero, and increasing its variance.

Substituting equations (27)–(30) into equations (25) and (26), the (marginalized) first two moments of the detection statistic under $H_0$ can be written ($i = 1, 2, 3$) as

\begin{equation}
\mu_0^{(i)} = -\Pi^2 f_s^{-1}(A_j + A_l)E(\psi \cdot \psi^T)
\end{equation}

and

\begin{equation}
(\sigma_0^{(i)})^2 = (\sigma_{0,0}^{(i)})^2 \left\{1 + \frac{2}{N_s} \left[\Pi \frac{E(\psi \cdot \psi^T)}{\sigma_s^2} + \Pi^2\mathcal{H}(\Omega_s) \frac{E(\psi \cdot \psi^{T,T})}{\sigma^2}\right]\right\},
\end{equation}

where

\begin{equation}
\mathcal{H}(\Omega_s) = \frac{2A_jA_l}{(A_j^2 + A_l^2)}.
\end{equation}

4. The impulsive component—toward a rigorous approach

In this section, we relax the assumption made in the previous section that no more than a single glitch may occur in the analysis window in each interferometer, and adopt a general, fully rigorous approach to model the glitchy component, along the lines laid out by Middleton in a series of seminal papers [17].

The impulsive noise component in each interferometer is accordingly modeled as a random process consisting of a linear superposition of atoms, namely\(^7\),

\begin{equation}
g_i(t) = \sum_{k=1}^{K(\Gamma)} \psi(t - t_{0,i}^{(k)}, \tilde{\alpha}_i^{(k)}), \quad t \in \Theta_i, \quad i = 1, 2, 3.
\end{equation}

Here, $\psi(\cdot)$ is the chosen representation atom, $t_{0,i}^{(k)}$ are a set of random glitch firing times, $\tilde{\alpha}_i^{(k)}$ is a set of random (independent) shape parameters (e.g., amplitude, center frequency, duration, duration, duration).

\(^7\) A straightforward generalization is obtained by adding several terms like (34) using different atom families. The characteristic function of the resulting process will be the product of the characteristic functions of its terms.
bandwidth) and $K_i[T]$ is also a random variable, denoting the number of glitches occurring in the analysis window $\Theta_i$, whose time-width is denoted as $T$. The firing times and shape parameters are determined independently at each glitch occurrence (i.e., for each $k$).

The key modeling assumption is that the glitching component may be taken as stationary (homogeneous) on time scales sufficiently long compared to the analysis window. On such time scales typical glitches will show up with constant probabilities, and occur at a constant rate, which can be estimated from actual data.

The number of events $K_i[T]$ will be accordingly ruled by a Poisson distribution, i.e.

$$\text{prob}\{K_i[T] = K_i\} = \frac{\bar{N}_i^K}{K_i!} e^{-\bar{N}_i},$$

(35)

where $\bar{N}_i$ (aka, $\lambda_i T$, $\lambda_i$ being the glitch firing rate) is the average number of glitches occurring in interferometer-$i$ in the ($T$-seconds wide) analysis window. For simplicity we shall assume the above probability, as well as the distributions of the firing times and shape parameters in (34), to be the same for all instruments, and henceforth drop the index $i$.

The characteristic functions of process (34) can be computed exactly up to any order [17]. The first-order one can be written as

$$F_g(\xi, t) = \sum_{K=0}^{\infty} \text{prob}\{K[T] = K\} F_g(\xi, t|K),$$

(36)

where $F_g(\xi, t|K)$ is the conditional characteristic function, given $K$ glitches in the analysis window $\Theta$, namely,

$$F_g(\xi, t|K) = E\left\{\exp\left[\sum_{m=1}^{K} \psi(t - t_0^{(m)}; \vec{a}^{(m)})\right]\right\}.$$

(37)

The expectation in (37) is taken with respect to both the firing times, $t_0^{(m)}$, and the shape parameters, $\vec{a}^{(m)}$. The pertinent distributions being assumed as time-invariant in $\Theta$, and independent for each glitch occurrence, equations (37) and (36) become, respectively,

$$F_g(\xi, t|K) = E[\exp\{\sum_{m=1}^{K} \psi(t - t_0^{(m)}; \vec{a}^{(m)})\}]^K$$

(38)

and

$$F_g(\xi, t) = \exp[\bar{N}(E[\exp\{\psi(t - t_0; \vec{a})\}] - 1)].$$

(39)

From the characteristic function $F_g(\xi, t)$ it is straightforward to compute the moments of the process $g(t)$, representing the impulsive (glitch) noise component in each interferometer:

$$\mu_g^{(Q)} = (-t)^Q \frac{\partial^Q}{\partial \xi^Q} F_g(\xi, t) \bigg|_{\xi = 0},$$

(40)

yielding

$$\mu_g^{(1)} = E[g(t)] = \bar{N} E[\psi(t - t_0; \vec{a})]$$

(41)

and

$$\mu_g^{(2)} = E[g(t)^2] = \bar{N}^2 E[\psi(t - t_0; \vec{a})^2] + \bar{N} E[\psi^2(t - t_0; \vec{a})].$$

(42)

where the expectations are taken with respect to both $t_0$ and $\vec{a}$. The related distributions being assumed as time-invariant in $\Theta$, the moments (41) and (42) are also time-independent.
4.1. RK correlators distributions under $H_0$—rigorous approach

Using the model exploited in section 4 for the impulsive component of the instrument noise, it is possible to compute the first two moments of the distribution of $C_i$ under $H_0$ in a rigorous way. Formally, these are obtained by making the substitution

$$n(t) \rightarrow n(t) + g(t),$$

(43)

for the noise in each detector in computing $E[C_i|H_0]$ and $\text{Var}[C_i|H_0]$, thus obtaining (to second order in the noise moments of $g$)

$$\mu^{(i)}_0 = E[C_i|H_0] = -f_s^{-1} N_i(A_j + A_\ell) \left( \mu^{(1)}_g \right)^2$$

(44)

and

$$\left( \sigma^{(i)}_0 \right)^2 = \text{Var}[C_i|H_0] = \left( A^2_j + A^2_\ell \right) \sigma^4 f_s^{-2} N_i + \sigma^2 f_s^{-2} N_i \left[ 2(A^2_j + A^2_\ell) \mu^{(2)}_g + 2A_j A_\ell \left( \mu^{(1)}_g \right)^2 \right]$$

(45)

and then using equations (41) and (42) for the first two moments of the impulsive components $g_i(t)$ in (44) and (45) to get

$$\mu^{(i)}_0 = E[C_i|H_0] = -f_s^{-1} N_i(A_j + A_\ell) N^2 E^2[\psi(t - t_0; \bar{a})]$$

(46)

and

$$\left( \sigma^{(i)}_0 \right)^2 = \text{Var}[C_i|H_0]$$

(47)

It is now interesting to compare equations (46) and (47) to equations (31) and (32), obtained from the heuristic reasoning in the previous section. In order to do so, the sum in (36) should include only the $K = 0, 1$ terms, to match the assumption made there that no more than a single glitch may occur in the analysis window. This gives the following approximate expressions for the first two moments of the impulsive noise component,

$$E[g(t)|K = 0, 1] = \bar{N} e^{-\bar{N} E[\psi(t - t_0; \bar{a})]}.$$  

(48)

$$E[g^2(t)|K = 0, 1] = \bar{N} e^{-\bar{N} E[\psi^2(t - t_0; \bar{a})]}.$$  

(49)

yielding, upon substitution in (44) and (45),

$$E[C_i|H_0, K = 0, 1] = -f_s^{-1} N_i(A_j + A_\ell) \bar{N}^2 e^{-2\bar{N} E^2[\psi(t - t_0; \bar{a})]}$$

(50)

and

$$\text{Var}[C_i|H_0, K = 0, 1] = \left( \sigma^{(i)}_0 \right)^2 \left[ 1 + 2 \left( \frac{N e^{-N E[\psi^2(t - t_0; \bar{a})]} \bar{N} e^{-\bar{N} E[\psi^2(t - t_0; \bar{a})]} + \bar{N}^2 e^{-2\bar{N} E^2[\psi(t - t_0; \bar{a})]} }{\sigma^2} \right) \right].$$

(51)

Equations (50) and (51) reproduce equations (31) and (32) if

$$\Pi N_s^{-1} E(\psi \cdot \psi^T) = \bar{N} e^{-\bar{N} E[\psi^2(t - t_0; \bar{a})]}.$$  

(52)
Both equalities are trivially proven, noting that for Poissonian distributions $\Pi = \bar{N} e^{-\bar{N}}$.

In conclusion, the rigorous approach sketched above agrees, in the appropriate limit, with the result obtained in the previous section from a simple heuristic argument. The rigorous approach, on the other hand, allows any number of glitches in the analysis window in each interferometer, in a natural way.

For the special case of SG atoms, assuming a uniform distribution of the glitch firing time over the analysis window, we have

$$\mu^{(1)} = 0, \quad \mu^{(2)} = \frac{\bar{N} \sqrt{\pi}}{2 \sqrt{2T}} E \left[ g_0^2 \sigma \left( 1 - e^{-2\pi^2 \bar{N} \sigma^2} \right) \right].$$

(53)

The analytic results for the moments in equations (44), (45) and (53) were checked successfully against Monte Carlo simulations.

5. RK correlator-based detector performance

Under the made assumption of Gaussianity of the distributions of the $C_i$ under both $H_1$ and $H_0$, it is straightforward to obtain the receiver operating characteristics (ROCs) [19], which completely characterize the RK-correlator-based detector.

In the appropriate surveillance context, the detection thresholds are determined, according to the Neyman–Pearson criterion, from the prescribed false alarm probability $\alpha$ as

$$\gamma^{(i)} = \sigma_0^{(i)} \text{erfc}^{-1}(\alpha) + \mu_0^{(i)}, \quad i = 1, 2, 3.$$

(54)

Note that, in view of equations (46) and (47), the thresholds depend on the variance of the Gaussian noise floor, the DOA, the number of samples in the analysis window, and, in view of equations (41) and (42), the average glitch energy and firing rate.

The corresponding false dismissal probabilities are

$$\beta^{(i)} = 1 - P_D^{(i)} = 1 - \text{erfc} \left( \frac{\gamma^{(i)} - \mu_1^{(i)}}{\sigma_1^{(i)}} \right), \quad i = 1, 2, 3.$$

(55)

Combining equations (54) and (55), we get the explicit expression of the ROCs

$$P_D^{(i)} = \text{erfc} \left[ \frac{\sigma_0^{(i)}}{\sigma_1^{(i)}} \text{erfc}^{-1}(\alpha) + \frac{\mu_1^{(i)}}{\sigma_0^{(i)}} \frac{\mu_1^{(i)}}{\sigma_1^{(i)}} \right], \quad i = 1, 2, 3,$$

(56)

where $P_D^{(i)}$ is the detection probability.

In the numerical experiments illustrated below, we consider the three-detectors network including LH, LL and V, using these subscripts accordingly.

Equations (54) and (55) have been used to obtain the ROCs ($\beta$ versus $\alpha$ curves) shown in figures 3–8. All figures refer to the $+$-polarized case. Similar results are obtained for the $\times$-polarized case, and are not reported for brevity.

It is interesting first to illustrate how the mere noisiness of the pseudo-templates (9) spoils the performance of the RK detector compared to the perfect matched filter. To do so, we shall momentarily ignore the impulsive noise component, by letting $\bar{N} = 0$.

Figure 3 shows the ROCs of the RK correlators (best DOA assumed) for different values of the intrinsic SNR $\delta_h$ of the GWB. The ROC for the perfect matched filter corresponding to SNR = 7, which is conventionally considered as the lowest operational value for this latter, is also shown. The value $\delta_h = 7$ corresponds, in the pertinent best DOAs, to $\delta_S = 6.22$ (LH), 6.48 (LL) and 6.68 (V).

8 Strictly speaking, these are not ROCs, according to the usual definition, although ROCs can be trivially derived from them.
where we collected the values of the quantities $\Delta h \approx 50 \text{ m}, \delta h = 10 \text{ m}, \delta h = 50 \text{ m}$, several $\delta h$ values. Left: $C_{\text{LH}}$; mid: $C_{\text{LL}}$; right: $C_V$.

Figure 3. Performance in terms of ROCs of the RK pseudo-correlator. Optimal DOAs. $T = 100 \text{ ms}, f_i = 4096 \text{ Hz}$, several $\delta h$ values. Left: $C_{\text{LH}}$; mid: $C_{\text{LL}}$; right: $C_V$.

Figure 4. Performance in terms of ROCs of the RK pseudo-correlator. Optimal DOAs. $f_i = 4096 \text{ Hz}, \delta h = 10$; several $T$. $C_{\text{LH}}$ only.

Figure 5. All-sky averaged performance in terms of ROC curves of the RK pseudo-correlator. $T = 100 \text{ ms}, f_i = 4096 \text{ Hz}$. Left: $C_{\text{LH}}$; mid: $C_{\text{LL}}$; right: $C_V$.

It is seen that the RK detector’s performance is better when using $C_{\text{LH}}$ or $C_{\text{LL}}$ as a detection statistic. In this case, at a given false-alarm rate, one needs roughly to double the SNR to obtain the same false-dismissal level as the perfect matched filter. Using $C_V$, instead, the SNR must be larger by a factor $\sim 4$. This is due to the different directional response of Virgo compared to the two LIGOs, due to its different orientation. This is illustrated in table 1, where we collected the values of the quantities

$$
\hat{\rho}^{(i)} = \frac{\delta^{(i)}}{\delta h} = \left| F_{i}^{(i)} \right|, \quad \hat{\rho}^{(j)} = \frac{\delta^{(j)}}{\delta h} = \frac{|F_i A_i|}{(A_i^2 + A_j^2)^{1/2}},
$$

(57)
representing the signal-to-noise ratios, normalized to the intrinsic SNR of the incoming GWB, of the data ($\bar{\rho}$) and the noisy template ($\tilde{\rho}_T$), respectively. Table 1 shows that in the DOA ranges where Virgo exhibits the largest response (largest normalized SNR $\bar{\rho}$), the two LIGOs respond poorly, and the pseudo-template obtained from them has a low normalized SNR $\bar{\rho}_T$. Conversely, in the DOA ranges where either of the LIGOs has the largest response (largest $\tilde{\rho}$), the pseudo-template constructed from the other LIGO and Virgo has still a decent normalized SNR $\tilde{\rho}_T$. 

Figure 6. Histograms of SG-atom parameter distributions from one week of S5 data (29). Left: center frequency ($f_0$); right: effective duration ($\sigma_t$).

Figure 7. Performance in terms of ROCs of the RK pseudo-correlator for different glitch rates. Optimal DOAs. $T = 100$ ms; $f_s = 4096$ Hz; $\delta_h = 15$; SNR$_{\text{veto}} = 100$. Left: $C_{\text{LH}}$; mid: $C_{\text{LL}}$; right: $C_{\text{V}}$.

Figure 8. Performance in terms of ROCs of the RK pseudo-correlator for different SNR$_{\text{veto}}$ levels. Optimally oriented source. $T = 100$ ms; $f_s = 4096$ Hz; $\delta_h = 15$; $\bar{N} = 0.1$. $C_{\text{LH}}$ only.
Table 1. The quantities in (57) evaluated at optimal DOAs.

| Quantity | $\rho_{(H)}$ | $\rho_{(L)}$ | $\rho_{(V)}$ | $\rho_{(H)}$ | $\rho_{(L)}$ | $\rho_{(V)}$ |
|----------|--------------|--------------|--------------|--------------|--------------|--------------|
| $\Omega_{\text{best}}^{(H)}$ | 0.95 | 0.85 | 0.14 | 0.86 | 0.86 | 0.03 |
| $\Omega_{\text{best}}^{(L)}$ | 0.79 | 0.95 | 0.20 | 0.84 | 0.82 | 0.04 |
| $\Omega_{\text{best}}^{(V)}$ | 0.10 | 0.22 | 0.95 | 0.08 | 0.24 | 0.23 |

Figure 4 shows the ROCs for $C_{LH}$ for $\delta_h = 10$ and three typical durations of the analysis window (20, 40 and 100 ms). The ROCs for $C_{LL}$ and $C_{V}$ are similar, and are not shown for brevity.

Figure 5 is same as figure 3, except that here the performances are averaged over the whole celestial sphere. In this case, the RK correlator-based detector performs worse than the perfect matched filter roughly by a factor of 3 in terms of SNR.

The effect of instrumental glitches is illustrated in figures 7 and 8. In order to draw these figures, we estimated the parameter distributions of the SG-atoms to be used in (34) from (unclustered) triggers collected in one week of S5 data, kindly provided by S Chatterji [29]. The distributions obtained for $f_0$ and $\sigma_t$ are sketched in figure 6. The SG-atom amplitude distribution was assumed as uniform in an interval set by the maximum SNR in each detector, beyond which the data are vetoed-out, denoted as $\text{SNR}_{\text{veto}}$.

In figure 7, the ROCs for $C_{LH}$, $C_{LL}$ and $C_{V}$ are shown for different values of the glitch firing rate $\lambda$, and compared to the no-glitch case. Obviously, as $\lambda$ increases, the best achievable false-dismissal versus false-alarm probability trade-off deteriorates. Here the glitch amplitude is assumed as being uniformly distributed up to a level corresponding to $\text{SNR}_{\text{veto}} = 100$.

Finally, in figure 8 the way the chosen $\text{SNR}_{\text{veto}}$ value affects the performance is illustrated.

6. Conclusions and directions for future work

We modeled the impulsive noise component following Middleton, using an atomic representation for the glitch population. The proposed model allows us to describe analytically the detector’s performance in the presence of glitches.

Based on the above, we also presented a simple, fully analytic characterization of the RK coherent network data analysis algorithm for detecting unmodeled GWBs with known DOA and TOO. Under a reasonable central-limit hypothesis for the RK detection statistics distributions, we derived and discussed the detector’s performance, in terms of its operating characteristics.

Our main results can be summarized as follows. The presence of noise in the pseudo-templates spoils the deflection, compared to a perfect matched filter. The related degradation factor depends on the direction of arrival, the energy of the signal and the length of the analysis window. The detection threshold, on the other hand, depends on the variance of the Gaussian noise floor, the DOA, the number of samples in the analysis window, the average glitch firing rate and the maximum allowed (veto-dependent) glitch energy. Constant false alarm rate (CFAR) operation is possible, and the RK detector turns out to be reasonably robust against instrumental/environmental glitches.

More or less straightforward developments of this work include (i) using a better detection statistic, e.g., a linear combination of the RK correlators with (DOA-dependent) coefficients...
chosen so as to maximize the deflection and (ii) allowing for a time-varying glitch firing rate (Cox processes [30]).

As possible directions for future work, we mention (i) identifying a better atom dictionary, and characterizing more accurately the prior distributions of the relevant parameters using a systematic matching-pursuit-based analysis [25] of the available glitch databases and (ii) exploiting in full Middleton’s model to derive more efficient implementations of the detector. In this connection, we note that the straightforward extension of the matched correlator to unmodeled waveforms provided by the RK algorithm is likely to be not optimal in view of the non-Gaussian nature of the instruments noise, whereby some suitable pre-conditioning of the data will be most likely required [31].

Finally, we mention the possibility of integrating Middleton’s model in a full-fledged interferometer noise simulator including glitches. Work along these directions is in progress.

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