Abstract.

The effective potential approach for composite operators has been generalized to non-zero temperatures in order to derive the analytical equation of state for pure SU(3) Yang-Mills fields from first principles. In the absence of external sources this is nothing but the vacuum energy density. The key element of this derivation is the introduction of a temperature dependence into the expression for the bag constant evaluated in the previous publications. The non-perturbative part of the analytical equation of state does not depend on the coupling constant, but instead introduces a dependence on the mass gap. This is responsible for the large-scale structure of the QCD ground state. Its perturbative part does analytically depend on the fine-structure constant of strong interactions as well. As it follows from our equation of state, the two massive gluonic excitations with the effective masses $m'_{\text{eff}} = 1.17$ GeV and $\bar{m}_{\text{eff}} = 0.585$ GeV, as well as the different types of massless gluonic excitations, are present in the SU(3) gluon plasma. Important thermodynamic quantities such as the pressure, energy and entropy densities, etc., have been calculated. We show explicitly that the pressure may continuously change around $T_c = 266.5$ MeV in order to achieve its Stefan-Boltzmann limit at high temperatures. All other thermodynamic quantities change drastically at this point. The entropy and energy densities have jump discontinuities at $T_c$. This is a firm evidence of the first-order phase transition in SU(3) pure gluon plasma. Our value for the latent heat is $\epsilon_{L,H} = 1.54$ (in dimensionless units). The heat capacity has a $\delta$-type singularity (an essential discontinuity) at $T_c$, so that the speed of light squared becomes zero at this point. The proposed NP analytical approach makes it possible to control for the first time the thermodynamics of the gluon plasma at low temperatures, below $T_c$. We have also calculated the gluon condensate, and hence the trace anomaly relation, as a function of temperature. Properly scaled they decrease not as $1/T^4$ but as $1/T^2$ at high temperatures due to the explicit presence of the mass gap in the equation of state. All our numerical results are in very good agreement with corresponding lattice data at $T \geq 2T_c$. 

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1. Introduction

Up to now, lattice QCD remained the only practical method to investigate QCD at finite temperature and baryon density [1, 2, 3, 4]. Recently it underwent a rapid progress ([1, 5, 6, 7, 8, 9] and references therein). However, lattice QCD, being a very specific regularization scheme, is primary aimed at obtaining well-defined corresponding expressions in order to get realistic numbers for physical quantities. One may therefore get numbers and curves without understanding what the physics is behind them. Such an understanding can only come from the dynamical theory, which is continuous QCD. For example, any description of the quark-gluon plasma (QGP) has to be formulated within the framework of a dynamical theory. The need for an analytical equation of state (EoS) remains, but, of course it should be essentially non-perturbative (NP), approaching the so-called Stefan-Boltzmann (SB) limit at very high temperatures. Thus the approaches of analytic NP QCD and lattice QCD to finite-temperature QCD do not exclude each other; on the contrary, they should be complementary. This is especially true at low temperatures where the thermal QCD lattice calculations suffer from big uncertainties [3, 4, 5, 6, 7, 8, 9]. On the other hand, any analytic NP approach has to correctly reproduce thermal QCD lattice results at high temperatures (see papers cited above). There already exist interesting phenomenological models based on the quasi-particle picture [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] (and references therein) to analyze results of $SU(3)$ lattice QCD calculations for the QGP EoS.

The main purpose of this paper is to complete the derivation of the NP analytical EoS for the gluon/glue plasma (GP), i.e., a system consisting purely of $SU(3)$ Yang-Mills (YM) fields without quark degrees of freedom. We have started it in our previous publication [20] (part I). This gluon medium was called as gluon matter (GM) in part I. However, here we prefer to call it GP, since we are going to include the free gluon contribution as well as the SB term in the present investigation, where the further analytical and numerical evaluation of the perturbative (PT) part of the gluon pressure will be also performed. The general formalism we use to generalize it to non-zero temperatures is the effective potential approach for composite operators [21]. In the absence of external sources it is nothing but the vacuum energy density (VED). This approach is NP from the very beginning, since it deals with the expansion of the corresponding skeleton vacuum loop diagrams, and thus allows one to calculate the VED from first principles [20, 21, 22]. The key element in this programme was the extension of our paper [22] to non-zero temperatures. This makes it possible to introduce the temperature-dependent bag constant (pressure) as a function of the mass gap. It is this which is responsible for the large-scale structure of the QCD ground state. The confining dynamics in the GP will therefore be nontrivially taken into account directly through the mass gap and via the temperature-dependent bag constant itself, but other NP effects will also be present.

The paper is organized as follows. In sections 2 and 3 we summarize our results obtained in [20] for the gluon pressure at zero (i.e., the above-mentioned VED) and
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non-zero temperatures, respectively. In section 3 the PT part of the gluon pressure is analytically derived as well. In section 4 we formulate a procedure how to include the free gluon contribution in the self-consistent way into the full GP pressure. A method of the simulating functions is proposed and analytical formulae for numerical simulations are introduced. This makes it possible to perform analytical and numerical simulations in order to determine the full GP pressure. In section 5 we display our numerical results for all the thermodynamic quantities, such as the entropy and energy densities, the heat capacity, etc., calculated with the help of the obtained GP pressure. We present also a brief discussion. In section 6 a description of the dynamical structure of the GP is given. In section 7 we summarize our conclusions. In appendix A the general expressions are given for the main thermodynamic quantities as functions of the pressure. In appendix B some analytical formulae for the GP thermodynamic quantities are derived, while in appendix C the latent heat is analytically and numerically evaluated. In appendix D the corresponding $\beta$-function for the confining effective charge is explicitly derived and briefly discussed.

2. The gluon pressure at zero temperature

In order to derive the gluon pressure at zero temperature $P_g$ in the first part of our investigation, we have used the effective potential approach for composite operators [21] to leading order, the so-called log-loop level. Analytically the gluon pressure looks like [20]

$$P_g = P_{NP} + P_{PT} = B_{YM} + P_{YM} + P_{PT},$$

so that $P_{NP} = B_{YM} + P_{YM}$, while

$$B_{YM} = 16 \int_{q_{eff}^2}^{q^2} \frac{d^4 q}{(2\pi)^4} \left[ \ln[1 + 3\alpha_{INP}(q^2)] - \frac{3}{4} \alpha_{INP}(q^2) \right],$$

$$P_{YM} = -16 \int \frac{d^4 q}{(2\pi)^4} \left[ \ln[1 + 3\alpha_{INP}(q^2)] - \frac{3}{4} \alpha_{INP}(q^2) \right],$$

and

$$P_{PT} = -16 \int_{\Lambda_{YM}^2}^{q_{eff}^2} \frac{d^4 q}{(2\pi)^4} \left[ \ln \left( 1 + \frac{3\alpha_{AF}(q^2)}{4 + 3\alpha_{INP}(q^2)} \right) - \frac{3}{4} \alpha_{AF}(q^2) \right].$$

In the expression for the bag constant (2) symbolically shown $q_{eff}^2$ is the effective scale squared, separating the soft momenta from the hard ones in the integration over $q^2$, that’s $0 \leq q^2 \leq q_{eff}^2$.

The intrinsically NP (INP) effective charge is

$$\alpha_{INP}(q^2) = \frac{\Delta^2}{q^2},$$

where $\Delta^2 \equiv \Delta_{JW}^2$ is the Jaffe-Witten (JW) mass gap [23], mentioned above, which is responsible for the large-scale structure of the QCD vacuum, and thus for its INP
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dynamics. Let us note that how the regularized mass gap appears in QCD is shown in [24]. It is worth emphasizing that in our recent works [25, 26] it has been proven that the INP effective charge (5) exactly reproduces the non-linear iteration solution for the full gluon propagator after the renormalization of the regularized mass gap is performed, so it is not an ansatz. The expression for the bag constant given in Eq. (2) is free of all types of the PT contributions ("contaminations"). Details of its derivation can be found in [22]. In the YM part of the NP gluon pressure $P_{YM}$, given in Eq. (3), the integration over variable $q^2$ goes from zero to infinity.

The PT part $P_{PT}$ of the gluon pressure shown in Eq. (4) contains the contribution which is mainly determined by the asymptotic freedom (AF) part of the PT effective charge, though the dependence on the INP effective charge is also present. The AF running effective charge $\alpha_{AF}(q^2)$ is given by the renormalization group equation solution, the so-called sum of the main PT logarithms [26, 27, 28], namely

$$\alpha_{AF}(q^2) = \frac{\alpha_s}{1 + \alpha_s b \ln(q^2/\Lambda_{YM}^2)},$$

(6)

and thus like the confining effective charge (5) this is not an ansatz either. Here $\Lambda_{YM}^2 = 0.09$ GeV$^2$ [29] is the asymptotic scale parameter for SU(3) YM fields, and $b = (11/4\pi)$ for these fields, while the strong fine structure constant is $\alpha_s = 0.1187$ [30]. Let us also note that in Eq. (6) $q^2$ cannot go below $\Lambda_{YM}^2$, which has been already symbolically shown in Eq. (4). In the $q^2 \to \infty$ limit from Eq. (6) one recovers the well known AF expression $\alpha_s^{AF}(q^2) = 1/b \ln(q^2/\Lambda_{YM}^2)$ [27, 28]. The gluon pressure $P_g$ in Eq. (1) is normalized to the free PT vacuum to be zero, i.e., when the interaction is switched formally off by putting $\alpha_{AF}(q^2) = \alpha_{INP}^{AF}(q^2) = 0$, then $P_{NP} = P_{PT} = 0$, so that $P_g = 0$ as well. It comes from the initial normalization condition of the free PT vacuum to be zero in the effective potential approach up to leading order [20, 21, 22].

3. The gluon pressure at non-zero temperatures

In the imaginary-time formalism [31, 32], all the four-dimensional integrals can be easily generalized to non-zero temperatures $T$ according to the prescription (let us remind that in part I [20] and in the present investigation the signature is Euclidean from the very beginning)

$$\int \frac{dq_0}{(2\pi)} \to T \sum_{n=-\infty}^{+\infty}, \quad q^2 = q_0^2 + \omega_n^2 = q_0^2 + \omega_n^2 = \omega^2 + \omega_n^2, \quad \omega_n = 2n\pi T. \quad (7)$$

In other words, each integral over $q_0$ of a loop momentum is to be replaced by the sum over the Matsubara frequencies labeled by $n$, which obviously assumes the replacement $q_0 \to \omega_n = 2n\pi T$ for bosons (gluons). In frequency-momentum space the effective charges (5) and (6) become

$$\alpha_{s}^{INP}(q^2) = \alpha_{s}^{INP}(q_0^2, \omega_n^2) = \alpha_{s}^{INP}(\omega^2, \omega_n^2) = \frac{\Lambda^2}{\omega^2 + \omega_n^2},$$

(8)
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and

\[ \alpha^{AF}(q^2) = \alpha^{AF}(\omega, \omega_n^2) = \alpha^{AF}(\omega, \omega_n^2) = \frac{\alpha_s}{1 + \alpha_s b \ln(\omega^2 + \omega_n^2/\Lambda_{YM}^2)}, \]

respectively. It is also convenient to introduce the following notations:

\[ T^{-1} = \beta, \quad \omega = \sqrt{q^2}, \]

where, evidently, in all the expressions here and below \( q^2 \) is the square of the three-dimensional loop momentum, in complete agreement with the relations (7).

Introducing the temperature dependence into the both sides of the relation (1), we finally obtain

\[ P_g(T) = P_{NP}(T) + P_{PT}(T) = B_{YM}(T) + P_{YM}(T) + P_{PT}(T), \]

where \( P_{NP}(T) \) and \( P_{PT}(T) \) are the NP and PT parts of the gluon pressure \( P_g(T) \), respectively. In what follows we will call them either the NP and PT gluon pressures or, for simplicity, the NP and PT pressures.

3.1. NP contribution

One of the attractive features of the confining effective charge (8) is that it allows an exact summation over the Matsubara frequencies in the NP pressure \( P_{NP}(T) \). Collecting all our analytical results obtained in our previous work [20], we can write

\[ P_{NP}(T) = B_{YM}(T) + P_{YM}(T) \]

\[ = \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} T[P_2(T) + P_3(T) - P_4(T)], \]

and

\[ P_1(T) = \int_{\omega_{eff}}^{\infty} d\omega \frac{\omega}{e^{\beta \omega} - 1} = \frac{\pi^2}{6} T^2 - \int_{0}^{\omega_{eff}} d\omega \frac{\omega}{e^{\beta \omega} - 1}, \]

while

\[ P_2(T) = \int_{\omega_{eff}}^{\infty} d\omega \omega^2 \ln \left( 1 - e^{-\beta \omega} \right), \]

\[ P_3(T) = \int_{0}^{\omega_{eff}} d\omega \omega^2 \ln \left( 1 - e^{-\beta \omega'} \right), \]

\[ P_4(T) = \int_{0}^{\infty} d\omega \omega^2 \ln \left( 1 - e^{-\beta \omega} \right), \]

where \( \omega_{eff} = 1 \) GeV and the mass gap \( \Delta^2 = 0.4564 \) GeV\(^2\) are fixed [20] [22]. Then \( \omega' \) and \( \bar{\omega} \) are given by the relations

\[ \omega' = \sqrt{q^2 + 3\Delta^2} = \sqrt{\omega^2 + m_{eff}'^2}, \quad m_{eff}' = \sqrt{3\Delta} = 1.17 \text{ GeV}, \]

and

\[ \bar{\omega} = \sqrt{q^2 + \frac{3}{4}\Delta^2} = \sqrt{\omega^2 + \bar{m}_{eff}^2}, \quad \bar{m}_{eff} = \frac{\sqrt{3}}{2} \Delta = 0.585 \text{ GeV}, \]

respectively. The NP part of the gluon pressure \( P_{NP}(T) \) is shown in Fig. 1.
3.2. PT contribution

In frequency-momentum space the PT part (4) of the gluon pressure (11) is

\[
P_{PT}(T) = -\frac{8}{\pi^2} \int_{\Lambda_{YM}}^{\infty} \frac{d\omega \omega^2 T}{\Lambda_{YM}^4} \sum_{n=\infty}^{\infty} \ln \left(1 + \frac{3\alpha^AF_1(w^2, \omega^2_n)}{4 + 3\alpha^{1NP}(w^2, \omega^2_n)} \right) - \frac{3}{4} \alpha^AF(w^2, \omega^2_n),
\]

However, it is instructive to begin the evaluation of the PT contribution to the gluon pressure (11) not from this expression, but from the very beginning, i.e., from Eq. (4). There is an interesting observation concerning the PT part (4). Let us consider the argument of its logarithm \(\ln[1 + x(q^2)]\), where

\[
x(q^2) = \frac{3\alpha^AF_1(q^2)}{4 + 3\alpha^{1NP}(q^2)} = \frac{3\alpha_s q^2}{(q^2 + 3\Delta^2)[1 + \alpha_s b \ln(q^2/\Lambda_{YM}^2)]},
\]

with the help of the expressions (5) and (6). In the integral (4) \(q^2 \geq \Lambda_{YM}^2\), thus at its lower limit \(q^2 = \Lambda_{YM}^2\) the argument (18) numerically becomes

\[
x(\Lambda_{YM}^2) = \frac{3\alpha_s \Lambda_{YM}^2}{(4\Lambda_{YM}^2 + 3\Delta^2)} = 0.0185,
\]

and the numerical values of \(\Delta^2\), \(\alpha_s\) and \(\Lambda_{YM}^2\) are as given above. The argument of the logarithm is really small and it will become smaller and smaller with \(q^2\) going to infinity. This means that the logarithm \(\ln[1 + x(q^2)]\) in the integral (4) is legitimated to expand in powers of \(\alpha^AF(q^2)\) or, equivalently, \(\alpha_s\). To the leading order one finally obtains

\[
P_{PT} = \alpha_s \Delta^2 9 \int_{\Lambda_{YM}^2}^{\infty} \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + \Delta^2)[1 + \alpha_s b \ln(q^2/\Lambda_{YM}^2)]} + O(\alpha_s^2),
\]

To say that this PT term is ”contaminated” by the NP contributions or to say that this NP term is ”contaminated” by the PT contributions will be equally correct. However, in any case it is suppressed by the order of \(\alpha_s\) in comparison with the NP part of the gluon pressure. In frequency-momentum space this term looks like

\[
P_{PT}(T) = \alpha_s \Delta^2 9 \int_{\Lambda_{YM}^2}^{\infty} \frac{d\omega \omega^2 T}{\Lambda_{YM}^4} \sum_{n=\infty}^{\infty} \ln \left[\frac{\bar{\omega}^2 + \omega^2_n}{1 + \alpha_s b \ln(\bar{\omega}^2 + \omega^2_n/\Lambda_{YM}^2)}\right],
\]

where \(\bar{\omega}\) is given in the relation (16). For convenience, we omit the term \(O(\alpha_s^2)\). Equivalently this integral can be presented as a sum of the two integrals, namely

\[
P_{PT}(T) = P^{(1)}_{PT}(T) + P^{(2)}_{PT}(T),
\]

where

\[
P^{(1)}_{PT}(T) = \alpha_s \Delta^2 9 \int_{\Lambda_{YM}^2}^{\infty} \frac{d\omega \omega^2 T}{\Lambda_{YM}^4} \sum_{n=\infty}^{\infty} \ln \left[\frac{\bar{\omega}^2 + \omega^2_n}{1 + \alpha_s b \ln(\bar{\omega}^2 + \omega^2_n/\Lambda_{YM}^2)}\right],
\]

and

\[
P^{(2)}_{PT}(T) = -\alpha_s^2 \Delta^2 9 \int_{\Lambda_{YM}^2}^{\infty} \frac{d\omega \omega^2 T}{\Lambda_{YM}^4} \sum_{n=\infty}^{\infty} \ln \left[\frac{\bar{\omega}^2 + \omega^2_n}{1 + \alpha_s b \ln(\bar{\omega}^2 + \omega^2_n/\Lambda_{YM}^2)}\right].
\]
In the integral (23) the summation over the Matsubara frequencies can be performed analytically (i.e., exactly) with the help of formula from part I of our investigation [20]. Omitting all the derivation and dropping the $\beta$-independent terms [31], one obtains

$$P_{PT}^{(1)}(T) = \alpha_s \Delta^2 \frac{9}{2\pi^2} \int_{\Lambda_{YM}}^\infty d\omega \omega^2 \frac{1}{\bar{\omega} e^{\beta \bar{\omega}} - 1},$$

(25)

where $\bar{\omega}$ is given in the relation (17).

Thus the so-called PT contribution (22) to the gluon pressure is

$$P_{PT}(T) = \alpha_s \Delta^2 \frac{9}{2\pi^2} \int_{\Lambda_{YM}}^\infty d\omega \omega^2 \frac{1}{\bar{\omega} e^{\beta \bar{\omega}} - 1} + O(\alpha_s^2),$$

(26)

and it is numerically small in the whole temperature range, indeed (see Fig. 1).

3.3. The gluon pressure $P_g(T)$

Summing up the expressions (12) and (26) up to leading order, for the gluon pressure

$$P_g(T) = P_{NP}(T) + P_{PT}(T),$$

(27)

defined in Eq. (11), one finally obtains

$$P_g(T) = \frac{6}{\pi^2} \Delta^2 \left[ P_1(T) + \frac{3}{4} \alpha_s P_1'(T) \right] + \frac{16}{\pi^2} T [P_2(T) + P_3(T) - P_4(T)],$$

(28)

where the terms of the $\alpha_s^2$ order are not shown, the integral $P_1'(T)$ is

$$P_1'(T) = \int_{\Lambda_{YM}}^\infty d\omega \omega^2 \frac{1}{\bar{\omega} e^{\beta \bar{\omega}} - 1},$$

(29)

and all other integrals are given in Eqs. (13) and (14).

Let us recall that the gluon pressure (28) is normalized to be zero when the interaction is switched formally off, i.e., putting $\alpha_s = \Delta^2 = 0$. Hence each of its terms in Eq. (27) becomes also zero in this case. This is due to the initial normalization of the free PT vacuum to zero. The confining dynamics in Eq. (28) is implemented via the bag pressure $B_{YM}(T)$ and the mass gap itself. However, other NP effects are also present via $P_{YM}(T)$ and $P_{PT}(T)$, though in the PT part they are suppressed in the powers of $\alpha_s$ in Eq. (26), and, consequently, in Eq. (28). The gluon pressure (28) is shown in Fig. 1.

4. The full GP EoS

From our numerical results it follows that the properly scaled (i.e., divided by $T^4/3$) gluon pressure (28) as a function of $T$ has a maximum at $T = 266.7$ MeV, while the PT pressure (26) has a maximum at $T = 269.4$ MeV. The NP pressure (12) has a maximum at some finite temperature, $T = T_c = 266.5$ MeV. In what follows we will call this value "characteristic", since the NP pressure has been calculated exactly, whereas the PT pressure, and hence the gluon pressure, has been calculated up to the
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Figure 1. The gluon pressure (28), the NP pressure (12) and the PT pressure (26) all properly scaled as functions of $T/T_c$. Effectively, all curves have a maximum at $T_c = 266.5$ MeV (vertical solid line). As functions of $T$ they have slightly different maxima (see text below).

leading order in powers of $\alpha_s$. Moreover, all the pressures properly scaled as functions of $T/T_c$ effectively have maxima just at $T_c$, see Fig. 1. From this figure it clearly follows that the gluon pressure (28) will never achieve the thermodynamic SB limit (A.7) at high temperatures. This is not surprising, since it has been normalized to the free PT pressure to be zero from the very beginning [20, 21, 22]. First of all, this means that the SB term is non-explicitly present in the gluon pressure $P_g(T)$ (27) through the NP pressure $P_{NP}(T)$ (12) in the whole temperature range. From the integral $P_2(T)$ in Eq. (14) it can be explicitly extracted as follows: $P_2(T) = \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 \ln \left(1 - e^{-\beta\omega}\right) = \int_{0}^{\infty} d\omega \, \omega^2 \ln \left(1 - e^{-\beta\omega}\right) - \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 \ln \left(1 - e^{-\beta\omega}\right) = -\left(\pi^2/8T\right)P_{SB}(T) - P_2'(T)$, where the integral $P_2'(T) = \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 \ln \left(1 - e^{-\beta\omega}\right)$. Multiplied further by $(16/\pi^2)T$, one finally obtains the contribution $-2P_{SB}(T)$ (along with terms describing another types of free gluons) to the NP pressure and hence to the gluon pressure. From the integrals $P_3(T)$ and $P_4(T)$ in Eq. (14) it cannot be extracted explicitly in a such simple form. However, in the $T \to \infty$ limit we can neglect the difference between massless $\omega$ and massive $\omega'$, $\bar{\omega}$ gluon excitations, and all the composition $[P_2(T) + P_3(T) - P_4(T)]$ in Eq. (12) goes to zero in this limit. This means that the SB pressure (A.7) has been already subtracted from it, but in a very specific way. It is not surprising that the NP effects are important below $T_c$. From Fig. 1, however, it follows that they are still important above $T_c$, at least, up to $3.75T_c = 1$ GeV = $\omega_{eff}$. The two other remarkable features of the gluon pressure (28) are to be also underlined. Firstly, below $T_c$ it goes exponentially down and up with $T$ going down and up, respectively. Secondly, above $T_c$ with increasing $T$ it goes down as $1/T^2$ and not as $1/T^4$ due to the explicit presence of the mass gap term $\Delta^2 T^2$ (see Eq. (28) and the integral (13)), while the composition $[P_2(T) + P_3(T) - P_4(T)]$ goes to zero at high temperatures as pointed out above. In
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general, this means that the gluon pressure (28), and hence the NP pressure (12), is at high temperatures a polynomial in integer powers of $T$ up to $T^2$. All these features have been discussed in more detail in [20]. The term $\sim T^2$ in the phenomenological EoS, which leads to the above-mentioned fall off at high temperatures, has been first discussed in [33] (see also [34, 35, 36]).

The gluon pressure (28) may change continuously its regime only in the near neighborhood of $T_c$ in order for its full counterpart to reach the corresponding SB limit at high temperatures. However, the SB term cannot be simply added to Eq. (28) above $T_c$, multiplied by the corresponding $\Theta$-function. The problem is that the different types of free gluons described by the NP pressure (12) are present in the GP in the whole temperature range, as it was discussed above. Thus in the right-hand-side of Eq. (28) the free gluon contribution has to appear in a more sophisticated way. The most general form how this can be done is to add to Eq. (28) the term $\Theta((T/T_c) - 1)Y(T) + \Theta((T_c/T - 1))X(T)$, valid in the whole temperature range, and the unknown yet functions $Y(T)$ and $X(T)$ are to be expressed in terms of $P_{SB}(T)$ and $P_{NP}(T)$ (see subsection below).

The previous Eq. (28) then becomes

$$P_{GP}(T) = P_g(T) + \Theta\left(\frac{T_c}{T} - 1\right)X(T) + \Theta\left(\frac{T}{T_c} - 1\right)Y(T),$$

(30)

and its left-hand-side here and below is denoted as $P_{GP}(T)$ (the above-mentioned full counterpart). The GP pressure (30) is continuous at $T_c$ if and only if

$$X(T_c) = Y(T_c),$$

(31)

which can be easily checked. Due to the condition (31), the dependence on the corresponding $\Theta$-functions disappears at $T_c$, and the GP pressure (30) remains continuous at any point of its domain. The role of the function $X(T)$ is to change the behavior of $P_{GP}(T)$ from $P_g(T)$ below $T_c$ especially in its near neighborhood. The function $Y(T)$ is aimed to change the behavior of $P_{GP}(T)$ from $P_g(T)$, as well as to introduce the SB term itself and its modification due to AF above and near $T_c$, respectively. These changes are inevitable indeed, since $P_{GP}(T)$ is different from $P_g(T)$ in the whole temperature range. The reason is that in the gluon pressure $P_g(T)$ the SB term is missing. However, as we saw in the GP pressure it cannot be restored in a trivial way. That is why the appearance of the corresponding $\Theta$-functions in the GP pressure (30) is also inevitable, playing, nevertheless, only an auxiliary role.

4.1. Analytical simulations

Actual analytical and numerical simulations are the main subject of this paper in order to derive the NP analytical EoS for the GP valid in the whole temperature range. Completing this program we will be able to compare our results and achieve the agreement with those of the thermal QCD lattice calculations above $T_c$ [36, 37].
The space of test functions, in terms of which the functions $X(T)$ and $Y(T)$ should be found, has already been established. On a general ground therefore we can put

$$
X(T) = f_1(T)P_{SB}(T) + \phi_1(T)P_{NP}(T),
Y(T) = f_2(T)P_{SB}(T) + \phi_2(T)P_{NP}(T),
$$

(32)

where all the dimensionless functions $f_n(T)$, $\phi_n(T)$, $n = 1, 2$ will be called the simulating functions. The test functions $P_{NP}(T)$ and $P_{SB}(T)$ are independent from each other, and they are exactly known. This also makes it possible to use in what follows the exact relations, derived in our previous work [20], namely

$$
\left[ P_{SB}(T) - 2P_{NP}(T) \right]_{T=T_c} = 0, \quad \left\{ \frac{\partial}{\partial T} \left[ P_{SB}(T) - 2P_{NP}(T) \right] \right\}_{T=T_c} = 0. \quad (33)
$$

In order to suppress the deep penetration of the SB term $P_{SB}(T)$ and the additional NP pressure $P_{NP}(T)$ below $T_c$, the simulating functions $f_1(T)$ and $\phi_1(T)$ are then convenient to choose as follows:

$$
f_1(T) = Ae^{-\alpha(T_c/T)}, \quad \alpha > 0, \quad \phi_1(T) = A_1e^{-\alpha_1(T_c/T)}, \quad \alpha_1 \geq 0,
$$

(34)

and $A$, $A_1$ are the arbitrary constants at this stage. Evidently, these simulating functions mimic the asymptotic of the so-called gluon mean number $N_g = (e^{\beta\omega} - 1)^{-1}$ [31] in the $\beta \to \infty$ ($T \to 0$) limit. Replacing $\omega$ by $\alpha T_c$ and $\alpha_1 T_c$, the different numerical factors $A$, $A_1$ should appear in the corresponding simulating functions shown in the relation (34). From the GP pressure (30) below $T_c$ then it follows

$$
P_{GP}(T) = P_g(T) + f_1(T)P_{SB}(T) + \phi_1(T)P_{NP}(T)
= P_{PT}(T) + \left[ 1 + A_1e^{-\alpha_1(T_c/T)} \right]P_{NP}(T) + Ae^{-\alpha(T_c/T)}P_{SB}(T),
$$

(35)

where the relation (27) has already been used. In the $T \to 0$ limit the additional contributions are exponentially suppressed, and the asymptotic of the GP pressure $P_{GP}(T)$ is determined by the gluon pressure $P_g(T)$, as it should be. At the same time, these terms allow to change the value of the GP pressure from the gluon pressure near $T_c$, as it is expected.

For the simulating function $f_2(T)$ our choice is the standard, empirical one [10, 38], recalculated only at $T_c = 266.5$ MeV, so it is

$$
f_2(T) = 1 - \alpha_s(T) = 1 - \frac{0.36}{1 + 0.55 \ln(T/T_c)}, \quad T \geq T_c.
$$

(36)

Evidently, the second term in this expression mimics the AF formula for the effective charge (6) as a function of temperature, while the first one leads to the correct SB limit for the GP pressure (30). The simulating function $\phi_2(T)$ has to be a regular function of $T$ as it goes to infinity; otherwise it remains arbitrary at this stage. The asymptotic of the GP pressure (30) at high temperatures thus becomes
\[ P_{GP}(T) = P_g(T) + f_2(T)P_{SB}(T) + \phi_2(T)P_{NP}(T) \]
\[ \sim \left(1 - \frac{0.36}{1 + 0.55 \ln(T/T_c)}\right)P_{SB}(T) \]
\[ + [1 + \phi_2(T)]P_{NP}(T) + P_{PT}(T) \rightarrow P_{SB}(T), \quad T \rightarrow \infty, \] (37)
and we again use here the relation (27). Since the \( P_{PT}(T) \) is very small, \([1 + \phi_2(T)]\) has to be negative above \( T_c \), so the GP pressure approaches the SB limit from below, as it should be. The growth of the NP pressure \( P_{NP}(T) \) at high temperatures is suppressed in comparison with the growth of the SB term \( P_{SB}(T) \) itself. The first one increases as \( \Delta^2 T^2 \) because of the explicit presence of the mass gap in the EoS (see Eq. (12) and integral (13)). The SB pressure increases as \( T^4 \) in this limit. This means that the sum \([1 + \phi_2(T)]P_{NP}(T)\) describes the leading and next-to-leading order term as \( T \rightarrow \infty \).

The explicit expressions for the \( X(T) \) and \( Y(T) \) functions (32), via the chosen simulating functions (34) and (36), are
\[ X(T) = Ae^{-\alpha(T/T_c)}P_{SB}(T) + A_1e^{-\alpha_1(T/T_c)}P_{NP}(T), \] (38)
and
\[ Y(T) = \left(1 - \frac{0.36}{1 + 0.55 \ln(T/T_c)}\right)P_{SB}(T) + \phi_2(T)P_{NP}(T). \] (39)

From the relations (31) and (38)-(39), and using the relations (33), it follows that
\[ 2Ae^{-\alpha} + A_1e^{-\alpha_1} = 2(1 - 0.36) + \phi_2(T_c) = 1.28 + \phi_2(T_c). \] (40)

The GP pressure (30), on account of the relations (38) and (39), then looks like
\[ P_{GP}(T) = P_g(T) + \Theta \left(\frac{T_c}{T} - 1\right) \left[Ae^{-\alpha(T/T_c)}P_{SB}(T) + A_1e^{-\alpha_1(T/T_c)}P_{NP}(T)\right] \]
\[ + \Theta \left(\frac{T}{T_c} - 1\right) \left[\left(1 - \frac{0.36}{1 + 0.55 \ln(T/T_c)}\right)P_{SB}(T) + \phi_2(T)P_{NP}(T)\right]. \] (41)

Due the relation (40) from the previous expression at \( T = T_c \), one obtains
\[ P_{GP}(T_c) = P_g(T_c) + [1.28 + \phi_2(T_c)]P_{NP}(T_c) \]
\[ = P_g(T_c) + [2Ae^{-\alpha} + A_1e^{-\alpha_1}]P_{NP}(T_c), \] (42)
so that it depends on the number \( \phi_2(T_c) \) only, since the values of \( P_{NP}(T_c) \) and \( P_g(T_c) \) are known from our calculations (see, for example Fig. 1).
4.2. Analytical and numerical simulation of $\phi_2(T)$

Our aim here is to find the simulating function $\phi_2(T)$ by fitting lattice data at high temperatures above $T_c$ \[36\]. For this, let us derive from the GP EoS (41) its values at $a = T/T_c$, $a > 1$ as follows:

$$P_{GP}(a) = B_a P_{SB}(a) + P_{PT}(a) + [1 + \phi_2(a)] P_{NP}(a),$$

(43)

where the relation (27) has been already used, and

$$B_a = 1 - \alpha_s(a) = \left(1 - \frac{0.36}{1 + 0.55 \ln a}\right).$$

(44)

Adjusting our parametrization of the GP pressure (43) to that used in recent lattice simulations for the YM $SU(3)$ case at $T = aT_c$ in \[36\], one obtains

$$\frac{3P_{GP}(T)}{T^4} = \frac{P_p(T)}{T^4} \times 5.26,$$

(45)

where 5.26 is the general SB number shown in Eq. (A.7) up to two digits only after the point. Let us note that, for simplicity, we will keep this for all our numbers throughout this paper. So that if the calculated number, for example is 1.177..., we will approximate it by 1.18, if it is 1.171..., we will approximate it by 1.17, and finally if it is 1.175..., we will keep this value. At the same time, let us stress that we have carried out all our calculations keeping six digits after the point (thus we can calculate all numbers to any requested accuracy). $P_p(T)/T^4$ describes lattice data taken from the above-mentioned Panero’s paper \[36\]. They are normalized to the SB limit for the pressure (see Eq. (A.7)), for example $3P_{GP}(2T_c)/(2T_c)^4 = (P_p(2T_c)/(2T_c)^4) \times 5.26 = 0.61 \times 5.26$ and so on. Then from the relation (45) and Eq. (43), one gets an equation for determining the numbers $\phi_2(a)$, namely

$$\{p\}_a = 5.26B_a + [pt]_a + (np)_a[1 + \phi_2(a)],$$

(46)

where the numbers

$$\{p\}_a = \frac{P_p(aT_c)}{(aT_c)^4} \times 5.26, \quad (np)_a = \frac{3P_{NP}(aT_c)}{(aT_c)^4}, \quad [pt]_a = \frac{3P_{PT}(aT_c)}{(aT_c)^4}$$

(47)

are exactly known from our calculations (see, for example Fig. 1 for the approximate values of the numbers $(np)_a$ and $[pt]_a$ and $3P_{SB}(T)/T^4 = 5.26$ at any high temperature (see Eqs. (A.7)). The numbers $\{p\}_a$ are to be taken as described above. From Eq. (46) then it follows that

$$\phi_2(a) = -1 + \frac{\{p\}_a - 5.26B_a - [pt]_a}{(np)_a}.$$

(48)

As we already know, the best way to choose the appropriate expression for the simulating function $\phi_2(a)$ is to mimic again the asymptotic of the gluon mean number
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\[ N_g = (e^{\beta\omega} - 1)^{-1} \] but in the \( \beta \to 0 \) \( (T \to \infty) \) limit, which is equivalent to the \( a \to \infty \) limit. Indeed, replacing \( \omega \) by \( \mu T \), one can write

\[ \phi_2(a) = \frac{B_2a^{-1}}{e^{(\mu/a) - 1}} = \frac{B_2a^{-1}}{\sum_{m=1}^{\infty} (1/n!) (\mu/a)^n} = \sum_{n=0}^{\infty} c_n a^{-n}, \] (49)

so that this simulating function above \( T_c \) becomes the series in inverse powers of \( a = T/T_c \).

First of all, we are interested in

\[ \phi_2(1) = \sum_{n=0}^{\infty} c_n, \] (50)

as it follows from the previous Eq. (49), since \( \phi_2(1) = \phi_2(T_c) \), which appears in Eq. (42). At values \( a \approx 1 \), we can put

\[ \phi_2(1) \approx -1 + \{q\} - 5.26B_a - [pt] (np)_a \] (51)
in Eq. (48), and the values for \( a \) in this equation are to be chosen from the low temperatures region, \( a = 1.010530, ..., 1.244705 \). The mean value of the numbers which appear in the right-hand-side of Eq. (51) is \(-2.12\), and thus

\[ \phi_2(1) = \sum_{n=0}^{\infty} c_n = -2.12. \] (52)

On the other hand, we can write

\[ \phi_2(a) = \sum_{n=0}^{\infty} c_n a^{-n} = \sum_{n=0}^{m} c_n a^{-n} + \sum_{n=m+1}^{\infty} c_n a^{-n} = \sum_{n=0}^{m} c_n a^{-n} + O(a^{-m-1}), \] (53)

so that when \( m \) and \( a \) are big enough, we can approximate Eq. (53) up to leading order

\[ \phi_2(a) = \sum_{n=0}^{m} c_n a^{-n}, \] (54)

and the values for \( a \) in this equation are to be chosen from the high temperature region, \( a = 3.006657, ..., 3.436657 \). The mean value of the numbers which appear in the right-hand-side of Eq. (48) is \(-1.53\), and thus

\[ -1.53 = \sum_{n=0}^{m} c_n a^{-n}. \] (55)

Then Eq. (53) can be re-written as follows:

\[ \phi_2(a) = \sum_{n=0}^{\infty} c_n a^{-n} = \sum_{n=0}^{m} c_n a^{-n} + \sum_{n=m+1}^{\infty} c_n a^{-n} = -1.53 + c_{m+1} a^{-m-1} + \sum_{n=m+2}^{\infty} c_n a^{-n}, \] (56)
which makes it possible to estimate the contribution of the remaining terms at $a \approx 1$. Indeed, in this case Eq. (56), due to the value (52), yields

$$c_{m+1} + \sum_{n=m+2}^{\infty} c_n = -0.59. \quad (57)$$

Substituting it back to the previous equation, one obtains

$$\phi_2(a) = -1.53 - 0.59a^{-m-1} - a^{-m-1} \sum_{n=m+2}^{\infty} c_n + \sum_{n=m+2}^{\infty} c_n a^{-n}. \quad (58)$$

This means that the whole expansion (49) or, equivalently, (58) can be effectively replaced by the polynomial as follows:

$$\phi_2(a) = -1.53 - 0.59a^{-m-1}, \quad (59)$$

which obviously has correct values at $a = 1$ and when $a$ goes to infinity ($T \to \infty$).

The best fit to lattice data [36] above $T_c$ is achieved at $m = 3$ in the relation (59). When $m \leq 2$ the GM pressure curve lies slightly below the lattice curve in the high temperature limit. At $m \geq 4$ the GM pressure curve lies substantially above the lattice curve in the moderate temperatures region $T_c \leq T \leq 2T_c$, see Fig. 2.

The final expression for the simulating function $\phi_2(T)$ is the two-term polynomial

$$\phi_2(T) = -1.53 - 0.59 \left(\frac{T_c}{T}\right)^4, \quad \phi_2(T_c) = -2.12, \quad (60)$$

determining this function up to leading and next-to-leading orders in the $T \to \infty$ limit.

Due to the value (60), from Eq. (42) one obtains

$$P_{GP}(T_c) = P_g(T_c) - 0.84P_{NP}(T_c) = 0.656 \quad (61)$$

in dimensionless units (45) and (47). At the same time, the relation (40) becomes

$$A = -[0.42 + 0.5A_1 e^{-\alpha_1}] e^\alpha. \quad (62)$$

In the connection with the choice of the simulating functions a few general things should be made perfectly clear. In principle, all the simulating functions can be chosen in many different ways. However, the request that the pressure is a growing continuous (class $C^0$) function of temperature in the whole range is a rather strong restriction (this means that it is differentiable at any point of its domain, while at $T_c$ derivative itself may not be continuous, i.e., it may have a discontinuity at this point). The other strong condition which should be imposed on the simulating functions is that they should not undermine analytical and asymptotical properties of the corresponding test functions. The simulating functions may produce only small numerical corrections to the values determined by the corresponding test functions. The justified choice of the simulating functions can only simplify the actual numerical calculations. i.e., they play only an auxiliary role. Let us remind that our choice of the simulating functions $f_1(T)$ and $\phi_1(T)$ has been made in accordance with the asymptotic of the gluon mean number $N_g = (e^{\beta_0} - 1)^{-1}$ in the $\beta \to \infty$ ($T \to 0$) limit. The simulating function $\phi_2(T)$
4.3. Numerical simulation of the GP pressure

After the substitution of the relation (62), the GP pressure (41) becomes

\[
P_{GP}(T) = P_g(T) - \Theta \left( \frac{T_c}{T} - 1 \right) \left[ 0.42 + 0.5A_1e^{-\alpha_1}e^{-\alpha(T_c/T)-1}P_{SB}(T) - A_1e^{-\alpha_1(T_c/T)}P_{NP}(T) \right] \\
+ \Theta \left( \frac{T}{T_c} - 1 \right) \left[ \left( 1 - \frac{0.36}{1 + 0.55\ln(T/T_c)} \right) P_{SB}(T) + \phi_2(T)P_{NP}(T) \right],
\]

(63)

where \(\phi_2(T)\) is completely determined and given in Eq. (60). This expression is convenient to simulate in order to continuously (and as much smoothly as possible) adjust both terms at \(T_c\), which are associated with the corresponding \(\Theta\)-functions.
It depends on the three independent (free) parameters, $A_1$, $\alpha_1$ and $\alpha$. The fourth parameter $A$ is to be calculated through the relation (62), and thus it plays no independent role. The simulation procedure consists of the numerical variation of these parameters in order to achieve the adjustment in the requested way. In principle, there are no other constraints on these constants apart from that all the derivatives of the GP pressure (63) should not gain negative values at low temperatures below $T_c$, i.e., they should exponentially approach zero from above.

Our best values for them are:

$$A_1 = -1.1, \quad \alpha_1 = 0.006, \quad \alpha = 4.$$ (64)

The value $\alpha = 4$ means that the deep penetration of the SB term below $T_c$ is indeed suppressed, as expected. The value $\alpha_1 = 0.006$ means that the NP pressure is not suppressed, but only the slope of its curve is decreasing below $T_c$, which is also expected. The number $A_1$ is, in principle, a dimensionless combination of all our dimensional parameters: $\Delta^2$, $\omega_{eff}$, $T_c$, $\Lambda^2_{YM}$. Let us remind that only $\Delta^2$ or, equivalently, $\omega_{eff}$ and $\Lambda^2_{YM}$ are independent, while the latter one is suppressed by $\alpha_s$, see Eq. (26). However, how to assign to it a physical meaning (if any) is not clear; anyway this is not important.

Thus the GP pressure (63) finally becomes

$$P_{GP}(T) = P_g(T) + \Theta\left(\frac{T}{T_c} - 1\right) \left[0.146e^{-4(\frac{T_c}{T}) - 1})P_{SB}(T) - 1.1e^{-0.006(\frac{T_c}{T})}P_{NP}(T)\right]$$

$$+ \Theta\left(\frac{T}{T_c} - 1\right) \left[0.36 \left(1 + 0.55 \ln\left(\frac{T}{T_c}\right)\right) P_{SB}(T) + \phi_2(T)P_{NP}(T)\right],$$ (65)

and it is completely determined now, since $\phi_2(T)$ is also determined and given by the relation (60). All other thermodynamic quantities will be calculated just via this expression.

5. Numerical results and discussion

In Fig. 3 we present the GP pressure (65) as well as the lattice pressure [36]. The main difference between them lies in the region of low temperatures below $T_c$. Within the NP analytic approach the confining effective charge (8) is mainly responsible for the structure of the GP in this region. It gives rise to the gluonic excitations of the dynamical origin which cannot be taken into account by the lattice simulations. The corresponding $\beta$-function (see appendix D) is always negative, as it is required by confinement [26, 27]. At the same time, the lattice $\beta$-function is always positive and almost constant [36, 37]. The discrepancy between NP analytic GP and lattice pressures seen in Fig. 3 just above $T_c$ up to $2T_c$ is also due to other NP effects which are still important in this region as pointed out above (see discussion in connection with Fig. 1).
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The GP pressure (65) as a function of $T/T_c$ (upper thick line). The lattice data [36] for SU(3) GP pressure is also shown (lower thin line). Both pressures are scaled in the same way in accordance with Eq. (45).

Figure 3. The GP pressure (65) as a function of $T/T_c$ (upper thick line). The lattice data [36] for SU(3) GP pressure is also shown (lower thin line). Both pressures are scaled in the same way in accordance with Eq. (45).

The GP entropy and energy densities are shown in Fig. 4. The size of the discontinuity in the energy density, the so-called latent heat (LH) is

$$\epsilon_{LH} = 1.54$$

in dimensionless units (see appendices B and C for its definition and analytical/numerical derivation, respectively). This means that the first-order phase transition in the GP is analytically confirmed for the first time, in complete agreement with thermal SU(3) YM lattice simulations. The physical reason of such sharp changes at $T_c$ in the derivatives of the GP pressure is that its exponential rise below $T_c$ is changing to the polynomial rise above $T_c$ in order to reach finally the thermodynamic SB limit. Our value (66) for the LH is consistent with various thermal SU(3) YM lattice calculations: in [37] $\epsilon_{LH} \approx 2$, in [36] $\epsilon_{LH} = 1.4$, in [39] $\epsilon_{LH} = 1.413$ and in [40] $\epsilon_{LH} = 1.54$. Note, that our value is slightly bigger than the recent lattice values in [36, 39], and coincides with the value in [40] (for simplicity, we do not specify the lattice parameters such as $N_t$, $N_s$, etc.). This agreement is rather remarkable, taking into account the completely different natures of the NP analytical and lattice approaches to thermal QCD.

The trace anomaly relation, which is especially sensitive to the NP effects including the conformal symmetry breaking, is shown in Fig. 5. Its two most interesting features are: the rapid rise of the peak due to the LH in the energy density (see Fig. 4) is exactly placed at $T_c$, while in lattice calculations it peaks at about 1.17$T_c$ [35, 36, 37, 40]. We interpret this lattice artefact as a clear evidence that by lattice simulations one cannot get correct numbers for all the thermodynamic quantities near $T_c$ and at $T_c$ itself. As it follows from Fig. 3, the lattice pressure below $T_c$ is almost zero and at $T_c$ it is about 0.1, while our pressure is 0.656 (in dimensionless units). That was the reason why we simulate the function $\phi_2(T)$ in subsection (4.2) not using the lattice value at $T_c$. It is worth underlining once more that lattice simulations cannot take into account the NP
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Figure 4. The GP pressure, the entropy and energy densities all properly scaled are shown as functions of $T/T_c$. The finite jumps in densities are clearly seen, and the LH is $\epsilon_{LH} = 1.54$. Their common SB limit (A.7) is approaching rather slowly.

Figure 5. The trace anomaly properly scaled is shown as a function of $T/T_c$ (solid line). The gluon condensate at zero temperature scaled in the same way is also shown (dot-dashed line).

dynamical effects, especially confinement, neither at zero nor at non-zero temperatures. They dominate the structure of the QCD vacuum at zero temperature and will remain important at non-zero temperatures as well (up to $3.75T_c$ within our approach [20]). In other words, they cannot be so negligible small below $T_c$ as it is shown by all lattice results for all the thermodynamic quantities. However, above $T_c$ at sufficiently high temperatures $T \geq 2T_c$ our results coincide because of the adjusted procedure necessary performed in subsections 4.2 and 4.3. The second interesting feature is that like in all lattice simulations (see for example [35] [36] [37] [40]) the trace anomaly falls off as $1/T^2$ and not as $1/T^4$ at high temperatures. As emphasized above in section 4 of the present investigation and in [20], this is due to the explicit presence of the mass gap term $\Delta^2T^2$.
in the EoS (65). The terms proportional to $T^4$ in the trace anomaly cancel each other at high temperatures due to Eq. (A.9), and hence the mass gap term comes out into play in this limit. It is the mass gap which violates the conformal symmetry from the very beginning within our approach.

In close connection with the trace anomaly is the gluon condensate defined by Eq. (A.6) and shown in Fig. 6. It approaches zero from below, so it gains negative values at high temperatures (fixed also by lattice simulations in [37]) due to the behavior of the trace anomaly in this region. In this connection, a few remarks are in order. The gluon condensate at zero temperature has been defined and calculated by the subtraction of

\[ \langle G^2 \rangle / T^4 \]

Figure 6. The properly scaled gluon condensate at non-zero temperatures is shown as a function of $T/T_c$. It approaches zero from below at high temperatures.

Figure 7. The NP gluon condensate scaled by $T^4$ as a function of $T/T_c$. The zero temperature gluon condensate scaled by $T^4$ is also shown. At high temperatures the NP gluon condensate scales as $1/T^2$. 
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Figure 8. $P(\epsilon)$ EoS (solid line) and approach to conformality $= (1/3)$ (thin line).

Figure 9. Conformity $C(T) = P(T)/\epsilon(T)$ as a function of $T/T_c$. It is zero at $T = 0$, and has a jump at $T_c$ due to a jump in the energy density at this point. It shows the non-trivial dependence on $T$ below $T_c$, and approaches the SB limit $1/3$ at high temperatures rather quickly.

all types of the PT contributions ("contaminations") in [22]. Here the temperature-dependent gluon condensate has been calculated with the help of the GP pressure (65), which contains explicitly the PT SB terms in different combinations. If the temperature-dependent gluon condensate is to be defined in the same way as its zero-temperature counterpart, then only the NP pressure (12) should be used for its calculation. This has been already done in [20] and shown in Fig. 7. It is always positive and at high temperatures it scales as $1/T^2$.

The GP pressure versus the GP energy density, the so-called EoS $P(\epsilon)$, is present in Fig. 8. The size of the LH and the rather quick approach to conformality are clearly seen. Let us note in advance that we distinguish between the conformality here and conformity defined in Eq. (A.4), though numerically in the limit of high temperatures
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Figure 10. The heat capacity as a function of $T/T_c$. It has the $\delta$-type singularity (an essential discontinuity) at $T_c$. It approaches its SB limit (A.7) (horizontal dashed line) at high temperatures very slowly.

Figure 11. The velocity of sound squared as a function of $T/T_c$. It shows the non-trivial dependence on $T$ below $T_c$, while at $T = 0$ and at $T = T_c$ it is zero. It approaches the SB limit $1/3$ at high temperatures rather quickly.

they are the same. The conformity itself is shown in Fig. 9. It has a finite jump at $T_c$ because of a jump in the energy density at this point, and approaches its SB limit (A.8) at high temperatures rather quickly. However, its most interesting feature is its non-trivial dependence on $T$ below $T_c$, which has been fixed explicitly in $SU(3)$ GP for the first time. It is due to the fact that the conformity is the ratio of the independent thermodynamic quantities, and therefore is not obliged to be always a growing function of $T$ below $T_c$, though above $T_c$ it is always growing.

The last independent thermodynamic quantity, the heat capacity defined in Eq. (A.2), is shown in Fig. 10. It is always a growing function of $T$ below and above $T_c$, while at $T_c$ it has a $\delta$-type singularity (an essential discontinuity) due to the expression...
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It approaches its SB limit (A.7) at high temperatures very slowly. The velocity of sound squared is shown in Fig. 11. Like conformity it is the ratio of the independent thermodynamic variables, see Eq. (A.3). It behaves very similar to conformity (Fig. 9), since the latter one mimics its properties. The principal difference from conformity is that at $T_c$ it is zero because of the heat capacity having the above-mentioned $\delta$-type singularity at this point. It approaches its SB limit (A.7) rather quickly.

6. The dynamical structure of SU(3) GP. A brief description

A rich dynamical structure of the GP emerges in the framework of our approach. The NP part (12) of the GP pressure (65) tells us that in the whole temperature range we have two different massive gluonic excitations (or, equivalently, effective gluonic degrees of freedom $[10, 11]$) $\omega'$ and $\bar{\omega}$ with the effective masses $m'_{\text{eff}} = 1.17$ GeV and $\bar{m}_{\text{eff}} = 0.585$ GeV, respectively. Both effective masses are due to the mass gap $\Delta^2$, which is responsible for the large-scale structure of the QCD ground state. It is dynamically generated by the nonlinear interaction of massless gluon modes $[24, 25, 26]$. The first massive excitations can be interpreted as glueballs, since $m'_{\text{eff}}$ is compared to the masses of scalar glueballs $[42]$. The second one $\bar{m}_{\text{eff}}$, might be identified with an effective gluon mass of about $(500-800)$ MeV, which arises in different quasi-particle models (see again $[42]$ and references therein).

We also have two different massless gluonic excitations $\omega$: the NP massless gluons, propagating in accordance with the integral $P_1(T)$ in Eq. (13), and almost SB massless gluons, propagating in accordance with the integral $P_2(T)$ in Eqs. (14). Let us remind that the integral (13) should be multiplied by $(6/\pi^2)\Delta^2$, and all other integrals (14) are to be multiplied by $(16/\pi^2)T$, when one speaks about different NP contributions to the pressure (12). However, this is not the whole story yet. The NP pressure (12) enters the GP pressure (65) in three different places, one through the gluon pressure (27). This means that all the above-mentioned massive and massless gluonic excitations propagate in the GP in two different ways: below and above $T_c$ they propagate in accordance with the terms in Eqs. (35) and (37) with parameters and the function $\phi_2(T)$ fixed by the relations (64) and (60), respectively.

The PT part (26) of the gluon pressure (27) assumes the presence of the NP massive gluonic excitations $\bar{\omega}$ in the GP in the whole temperature range. Having the same effective masses, nevertheless, they propagate in a completely different way, which is determined by the mass gap $\Delta^2$ and $\Lambda_{YM}$ (compare the integrals (26) and (14)). However, their propagation is suppressed by the order of $\alpha_s$. The terms of the order of $\alpha_s^2$ and higher have been left undetermined. In any case, numerically they are very small. Let us emphasize once more that all these NP massive and massless gluonic excitations have not been introduce by hand; contrary, they are of the dynamical origin, and only they (i.e., of the dynamical origin) can be described and accounted for within our approach to thermal QCD.

In the GP in the whole temperature range the free gluons context within our
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approach is to be described as follows: we definitely have the SB free massless gluons far away from $T_c$, while near $T_c$ their propagation is corrected by AF. Below $T_c$ the deep penetration of the SB term itself is suppressed. So we have three different types of the PT free gluons, apart from those discussed above and at the beginning of section 4.

The exponential rise of all the independent thermodynamic quantities in the transition region $(0.6 - 1)T_c$ clearly seen in Figs. 4 and 10 indicates that near $T_c$ a dramatic increase in the number of effective gluonic degrees of freedom will appear (for example, the glueballs will begin to dissolve rapidly). This will lead to drastic changes in the structure of the GP. A change in this number is enough to generate pressure gradients, but not enough to affect the pressure itself. It varies slowly and therefore remains continuous in this region. At the same time, the pressure gradients such as the energy and entropy densities, etc., undergo sharp changes in their behavior, having different types of discontinuities at $T_c$. Thus SU(3) GP has a first-order phase transition within our approach, in complete agreement with the thermal QCD lattice calculations (see [36, 37] and references therein). Of course, not all the glueballs will be dissolved in the transition region. Some of them will remain above $T_c$, along with other massive and massless gluonic excitations, forming thus a possible mixed and extended mixed phases around $T_c$ from its both sides [20, 43]. One can conclude that the NP physics of the mixed phases (the temperature interval approximately $(0.6 - 3.75)T_c$ in $SU(3)$ GP is well understood within our approach. At very high temperatures starting with $3.75T_c$ the NP effects become very small, and the structure of the GP will be mainly determined by the SB relations (A.7)-(A.9) between all the thermodynamic quantities.

7. Conclusions

The effective potential approach for composite operators [21] has been generalized to non-zero temperatures in order to derive the NP analytical EoS for pure $SU(3)$ YM fields from first principles. In its NP part (12) there is no dependence on the coupling constant, only a dependence on the mass gap, which is responsible for the large-scale structure of the QCD ground state. A key element of this part is the generalization of the expression for the bag constant at zero temperature [20, 22] to non-zero temperatures. Its PT part does analytically depend on the fine-structure constant of strong interactions as well. We also propose and formulate a method how to include the free gluon contribution into the GP pressure (65) in a self-consistent way.

Our main quantitative and qualitative results in this investigation are:

(i). In the GP EoS (65) the confining dynamics at non-zero temperatures (8) is taken into account through the $T$-dependent bag constant and the mass gap $\Delta^2$ itself. The GP pressure (65) compared to lattice pressure [36] is shown in Fig. 3.

(ii). Other NP effects are also taken into account via the YM and PT parts of the GP pressure, though the latter one is suppressed by the order of $\alpha_s$.

(iii). The mass gap $\Delta^2$ or, equivalently, $\omega_{eff}$ and the asymptotic scale parameter for YM fields $\Lambda^2_{YM}$ are only two independent input scale parameters in our approach.
(iv). The characteristic temperature $T_c = 266.5$ MeV is a temperature at which the maximum of the NP part of the GP pressure is achieved (Fig. 1).

(v). The presence of the mass gap $\Delta^2 T^2$ term in the GP pressure (65).

(vi). Due to this the trace anomaly and the NP gluon condensate will go down as $1/T^2$ at high temperatures. The trace anomaly has a peak just at $T_c$ (Figs. 5, 6 and 7).

(vii). The presence of the two different types of massive gluonic excitations of the NP origin with an effective masses 0.585 GeV and 1.17 GeV. They propagate in different ways below and above $T_c$. There is no place for some other types of massive gluonic excitations in our picture.

(viii). The presence of the different types of massless gluonic excitations including the free PT or, equivalently, SB gluons far away from $T_c$.

(ix). Below $T_c$ all the independent thermodynamic quantities exponentially go down and up with $T$ going down and up, respectively (Figs. 4 and 10). Above $T_c$ this regime is changed to a polynomial growing one.

(x). The thermodynamic quantities being the ratios of the corresponding independent ones such as conformity and the velocity of sound squared demonstrate the non-trivial similar dependence on $T$ below $T_c$ (Figs. 9 and 11). At high temperatures their approach to the SB limit (A.8) is accelerated by a factor of 3.

(xi). At $T_c$ the GP pressure is continuous, while the energy and entropy densities have finite jumps. The size of the discontinuity in the energy density is: $\epsilon_{LH} = 1.54$ (in dimensionless units), see Fig. 4. The heat capacity has an essential discontinuity of a $\delta$-type singularity at $T_c$, see Fig. 10.

(xii). The first-order phase transition at $T_c$ in SU(3) GP is thus analytically proven.

(xiii). Since the NP effects are still important up to $3.75T_c$, all the independent thermodynamic quantities approach their perspective SB limits rather slowly.

(xiv). Because of this the behavior of SU(3) GP in this region is substantially different from the behavior of a gas of free massless gluons.

The NP analytical approach to thermal QCD provides a detailed numerical and dynamical description of SU(3) GP in the whole temperature range, and especially in the region of low temperatures below $T_c$. Something that’s not accessible for and missing by lattice approach to thermal QCD. Our next and final step is to include the quark degrees of freedom into the formalism of the effective potential approach for composite operators at non-zero temperatures. This will make it possible to derive the NP analytical EoS for the QGP.

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Appendix A. Main thermodynamic quantities

Together with the pressure $P(T)$, the main thermodynamic quantities are the entropy density $s(T)$ and the energy density $ε(T)$. The general formulae which connect them are [31]

\[
s(T) = \frac{∂P(T)}{∂T},
\]

\[
ε(T) = T \left( \frac{∂P(T)}{∂T} \right) - P(T) = Ts(T) - P(T) \tag{A.1}
\]

for pure YM fields, i.e., when the chemical potential is equal to zero. Let us note that in quantum statistics the pressure $P(T)$ is nothing but the thermodynamic potential $Ω(T)$ apart from the sign, i.e., $P(T) = -Ω(T) > 0$.

Other thermodynamic quantities of interest are the heat capacity $c_V(T)$ and the velocity of sound $c_s^2(T)$, which are defined as follows:

\[
c_V(T) = \frac{∂ε(T)}{∂T} = T \left( \frac{∂s(T)}{∂T} \right), \tag{A.2}
\]

and

\[
c_s^2(T) = \frac{∂P(T)}{∂ε(T)} = \frac{s(T)}{c_V(T)}, \tag{A.3}
\]

i.e., they are defined through the second derivative of the pressure. The so-called conformity

\[
C(T) = \frac{P(T)}{ε(T)} \tag{A.4}
\]

mimics the behavior of the velocity of sound squared (A.3) but without involving such a differentiation.

A thermodynamic quantity of special interest is the thermal expectation value of the trace of the energy momentum tensor. This trace anomaly relation measures the deviation of the difference

\[
ε(T) - 3P(T) \tag{A.5}
\]

from zero at finite temperatures, in the high temperature limit it must vanish. As a consequence it is very sensitive to the NP contributions to the EoS. It also assists in the temperature dependence of the gluon condensate [44] (see [37] as well), namely

\[
< G^2 >_T = < G^2 >_0 - [ε(T) - 3P(T)], \tag{A.6}
\]

where $< G^2 >_0 = < G^2 >_{T=0}$ denotes the gluon condensate at zero temperature, whose numerical value is discussed in appendix B of [20].
Appendix A.1. The SB limit

The high-temperature behavior of all the thermodynamic quantities is governed by the SB ideal gas limit, when the matter can be described in terms of non-interacting massless particles (gluons). In this limit these quantities satisfy special relations

\[
\frac{3P_{SB}(T)}{T^4} = \frac{\epsilon_{SB}(T)}{T^4} = \frac{3s_{SB}(T)}{4T^3} = \frac{c_v(SB)(T)}{4T^3} = \frac{24}{45}\pi^2 \approx 5.26, \quad T \to \infty, \quad (A.7)
\]

and

\[
C_{SB}(T) = c^2_{s(SB)}(T) = \frac{1}{3}, \quad T \to \infty, \quad (A.8)
\]
on account of the previous relations and their definitions in Eqs. (A.3) and (A.4). The trace anomaly relation (A.5) also satisfies the SB limit, namely

\[
\epsilon_{SB}(T) - 3P_{SB}(T) = 0, \quad T \to \infty, \quad (A.9)
\]
as it comes out from the relations (A.7). In what follows its right-hand-side will be called the general SB number.

Appendix B. Analytical formulae for the GP thermodynamic quantities

It is instructive to derive analytically all the necessary formulae for the thermodynamic quantities using the GP pressure (30). Differentiating it in accordance with the definition (A.1), on account of the relation (31), one obtains

\[
s(T) = \frac{\partial P_g(T)}{\partial T} + \Theta \left( \frac{T_c}{T} - 1 \right) \frac{\partial X(T)}{\partial T} + \Theta \left( \frac{T}{T_c} - 1 \right) \frac{\partial Y(T)}{\partial T}, \quad (B.1)
\]

for convenience, the subscript "GP" in all the thermodynamic quantities will be omitted here and below. It is easy to see that the entropy density has a jump at \( T_c \),

\[
\Delta s(T_c) = [s(T > T_c) - s(T < T_c)]_{T \to T_c} = \left[ \frac{\partial Y(T)}{\partial T} - \frac{\partial X(T)}{\partial T} \right]_{T=T_c}, \quad (B.2)
\]

where the difference in the right-hand-side of this equation has to be positive.

In the same way for the energy density, one obtains

\[
\epsilon(T) = T\frac{\partial P_g(T)}{\partial T} - P_g(T) + \Theta \left( \frac{T_c}{T} - 1 \right) \left[ T\frac{\partial X(T)}{\partial T} - X(T) \right] + \Theta \left( \frac{T}{T_c} - 1 \right) \left[ T\frac{\partial Y(T)}{\partial T} - Y(T) \right]. \quad (B.3)
\]
The size of the discontinuity in the energy density (latent heat (LH)) is

\[
\epsilon_{LH}(T_c) = \Delta \epsilon(T_c) = [\epsilon(T > T_c) - \epsilon(T < T_c)]_{T \to T_c} = T_c \left[ \frac{\partial Y(T)}{\partial T} - \frac{\partial X(T)}{\partial T} \right]_{T=T_c}, \quad (B.4)
\]
and thus it is in agreement with the discontinuity in the entropy density, since from Eqs. (B.3) and (B.5) it follows that

$$\epsilon_{LH}(T_c) = \Delta \epsilon(T_c) = T_c \Delta s(T_c).$$

The next independent thermodynamic quantity is the heat capacity defined in Eq. (A.2). Differentiating the entropy density (B.1), one finally obtains

$$c_V(T) = T \frac{\partial^2 P_s(T)}{\partial T^2} + \Theta \left( \frac{T}{T_c} - 1 \right) T \frac{\partial^2 X(T)}{\partial T^2} + \Theta \left( \frac{T}{T_c} - 1 \right) T \frac{\partial^2 Y(T)}{\partial T^2} - \frac{T_c}{T} \delta \left( \frac{T}{T_c} - 1 \right) \frac{\partial X(T)}{\partial T} + \frac{T}{T_c} \delta \left( \frac{T}{T_c} - 1 \right) \frac{\partial Y(T)}{\partial T}.$$  \hspace{1cm} (B.6)

The important observation is that the heat capacity has a $\delta$-type singularity (an essential discontinuity) at $T = T_c$, so that the speed of light squared (A.3) at this point is zero

$$c_s^2(T_c) = \frac{s(T_c)}{c_V(T_c)} = 0,$$

indeed. All other thermodynamic quantities can be analytically derived by using these formulae and the corresponding definitions given in the previous appendix A.

**Appendix C. Analytical and numerical evaluation of the latent heat**

From the expression (38) it follows that

$$\frac{\partial X(T)}{\partial T} = \alpha A \frac{T_c}{T^2} e^{-\alpha(T_c/T)} P_{SB}(T) + A e^{-\alpha(T_c/T)} \frac{\partial P_{SB}(T)}{\partial T} + \alpha_1 A_1 \frac{T_c}{T} e^{-\alpha_1(T_c/T)} P_{NP}(T) + A_1 e^{-\alpha_1(T_c/T)} \frac{\partial P_{NP}(T)}{\partial T},$$

and at $T = T_c$ it is

$$\left( \frac{\partial X(T)}{\partial T} \right)_{T_c} = \alpha A \frac{T_c}{T_c^2} e^{-\alpha} P_{SB}(T_c) + A e^{-\alpha} \left( \frac{\partial P_{SB}(T)}{\partial T} \right)_{T_c} + \alpha_1 A_1 \frac{T_c}{T_c} e^{-\alpha_1} P_{NP}(T_c) + A_1 e^{-\alpha_1} \left( \frac{\partial P_{NP}(T)}{\partial T} \right)_{T_c}.$$  \hspace{1cm} (C.1)

In the same way from the expressions (39) and (60) it follows that

$$\frac{\partial Y(T)}{\partial T} = \frac{0.36}{(1 + 0.55 \ln(T/T_c))^2} \frac{0.55}{T} P_{SB}(T) + \left( 1 - \frac{0.36}{1 + 0.55 \ln(T/T_c)} \right) \frac{\partial P_{SB}(T)}{\partial T} + 2.36 \frac{T_c}{T} \frac{4}{P_{NP}(T)} + \phi_2(T) \frac{\partial P_{NP}(T)}{\partial T},$$  \hspace{1cm} (C.2)

and at $T = T_c$ it is
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\[
\left(\frac{\partial Y(T)}{\partial T}\right)_{T_c} = \frac{0.198}{T_c} P_{SB}(T_c) + 0.64 \left(\frac{\partial P_{SB}(T)}{\partial T}\right)_{T_c} + \frac{2.36}{T_c} P_{NP}(T_c) + \phi_2(T_c) \left(\frac{\partial P_{NP}(T)}{\partial T}\right)_{T_c}. \tag{C.4}
\]

Taking into account these relations and the relations (33) and (40), the latent heat (B.4) thus becomes

\[
\epsilon_{LH}(T_c) = \left[2.756 - 2\alpha A e^{-\alpha} - \alpha_1 A e^{-\alpha_1}\right] P_{NP}(T_c), \tag{C.5}
\]

and in dimensionless units \(\epsilon_{LH} \equiv (\epsilon_{LH}(T_c)/T_c^4)\) it is

\[
\epsilon_{LH} = \left[2.756 - 2\alpha A e^{-\alpha} - \alpha_1 A e^{-\alpha_1}\right] \times 0.875, \tag{C.6}
\]

where the number \(P_{NP}(T_c)/T_c^4\) has been already substituted. Using further the numbers (64), one finally obtains

\[
\epsilon_{LH} = 1.54. \tag{C.7}
\]

Appendix D. The \(\beta\)-function for the confining effective charge at non-zero temperatures

It is instructive to show explicitly the corresponding \(\beta\)-function for the INP effective charge (5). From the renormalization group equation,

\[
q^2 \frac{d\alpha_s^{INP}(q^2; \Delta^2)}{dq^2} = \beta(\alpha_s^{INP}(q^2; \Delta^2)), \tag{D.1}
\]

it simply follows that

\[
\beta(\alpha_s^{INP}(q^2; \Delta^2)) = -\alpha_s^{INP}(q^2; \Delta^2) = -\frac{\Delta^2}{q^2}. \tag{D.2}
\]

Thus, the corresponding \(\beta\)-function as a function of its argument is always in the domain of attraction (i.e., negative). So it has no IR stable fixed point indeed as it is required for the confining theory [27]. In frequency-momentum space from Eqs. (8) and (D.2) one gets

\[
\beta_s^{INP}(\omega^2, \omega_n^2) = -\alpha_s^{INP}(\omega^2, \omega_n^2) = -\frac{\Delta^2}{\omega^2 + \omega_n^2}. \tag{D.3}
\]

The confining effective charge with the corresponding \(\beta\)-function (D.3) determines the structure of \(SU(3)\) GP at low temperatures as well as in the extended mixed phase up to \(3.75 T_c\) [20, 43] within our approach.
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