Universal Inference with Composite Likelihoods

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Abstract

Wasserman et al. (2020, PNAS, vol. 117, pp. 16880–16890) constructed estimator agnostic and finite-sample valid confidence sets and hypothesis tests, using split-data likelihood ratio-based statistics. We demonstrate that the same approach extends to the use of split-data composite likelihood ratios as well, and thus establish universal methods for conducting multivariate inference when the data generating process is only known up to marginal and conditional relationships between the coordinates. Always-valid sequential inference is also considered.

1 Introduction

Let $X \in \mathbb{X} \subseteq \mathbb{R}^d$ ($d \in \mathbb{N}$) be a random variable arising from a parametric family of distributions $\mathcal{P}_\theta$ with probability density/mass functions (we shall use PDFs/PMFs) of form $p(x; \theta)$, for $\theta \in \Theta \subseteq \mathbb{R}^q$ ($q \in \mathbb{N}$). Let $X_{2n} = (X_1, \ldots, X_{2n})$ be a sample of $2n$ ($n \in \mathbb{N}$) independently and identically distributed replicates of $X$ and split the data into two subsamples $X_n^0 = (X_1, \ldots, X_n)$ and $X_n^1 = (X_{n+1}, \ldots, X_{2n}) = (X_n^0, \ldots, X_n^0)$ and $X_n^1 = (X_{n+1}, \ldots, X_{2n}) = (X_n^1, \ldots, X_n^1)$. Without causing confusion, we shall use PDF to mean PDF or PMF, throughout the text.

Suppose that the data generating process (DGP) of $X$ has distribution $\mathcal{P}_{\theta^*}$ for some $\theta^* \in \Theta$ and that $\hat{\theta}_n^k$ is some generic estimator of $\theta^*$, using data $X_n^k$ ($k \in \{0, 1\}$). Consider the split likelihood ratio statistics (LRSs)

$$U_n^k(\theta) = \frac{L(\hat{\theta}_n^{1-k}, X_n^k)}{L(\theta, X_n^k)},$$

for each $k$, and the swapped LRS

$$\bar{U}_n(\theta) = \frac{U_n^0(\theta) + U_n^1(\theta)}{2},$$

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and

\[ L(\theta; X_n^k) = \prod_{i=1}^{n} p(X_i^k; \theta) \]

is the likelihood of subsample \( X_n^k \), evaluated at parameter value \( \theta \).

Let \( E_{\theta^*} \) and \( \text{Pr}_{\theta^*} \) denote the expectation and probability operators with respect to the distribution \( P_{\theta^*} \), respectively. In Wasserman et al. (2020), the remarkable result that

\[ E_{\theta^*}[U_n^k(\theta^*)] \leq 1 \]  

is established and used to derive finite-sample validity of a number of simple universal confidence set estimators and hypothesis tests, using (1) and (2) (and variants), that are agnostic to the choice of parameter estimators \( \hat{\theta}_n^k \) and DGPs \( P_{\theta} \). The results are then extended from likelihood-based inference to misspecified likelihood, power likelihood, and smoothed likelihood-based inference, as per the works of White (1982), Royall & Tso (2003), and Seo & Lindsay (2013), respectively. Furthermore, Wasserman et al. (2020) prove results regarding always-valid tests, \( p \)-values and confidence sets, in the style of Johari et al. (2017).

In this note, we derive extensions to the results of Wasserman et al. (2020) for the context of composite likelihood-based (or equivalently, pseudo-likelihood-based) inference, as considered in Lindsay (1988), Arnold & Strauss (1991), Molenberghs & Verbeke (2005), Varin et al. (2011), Yi (2014), and Nguyen (2018), among numerous other texts. This includes results for batch inference as well as sequential inference.

We proceed as follows. In Section 2, we present the main results that extend upon the theorems of Wasserman et al. (2020). Proofs are then provided in Section 3. Technical requirements to prove our results are provided in the Appendix.

### 2 Main results

Let \( 2^d \) be the power set of \( [d] = \{1, \ldots, d\} \), and let \( S_d = 2^d \setminus \{\emptyset\} \). For each \( S \in S_d \), let \( S = \{s_1, \ldots, s_{|S|}\} \subseteq [d] \), where \( |S| \) is the size of \( S \). Further, let \( T_d \) be the set of all divisions of \( [d] \) into two non-empty subsets. For elements \( T \in T_d \), we write \( \vec{T} = \{\vec{t}_1, \ldots, \vec{t}_{|T|}\} \subset [d] \) and \( \vec{T} = \{\vec{t}_1, \ldots, \vec{t}_{|T|}\} \subset [d] \setminus \vec{T} \) to be the “left-hand” and “right-hand” subsets of the division \( T \), respectively. We note that \( |S_d| = 2^d - 1 \) and \( |T_d| = 3^d - 2^{d+1} + 1 \).

For each \( S \), let \( \alpha_S \geq 0 \) and for each \( T \), let \( \beta_T \geq 0 \). We shall call these coefficients weights. Put the weights \( \alpha_S \) and \( \beta_T \) in the vectors \( \alpha = (\alpha_S)_{S \in S_d} \) and \( \beta = (\beta_T)_{T \in T_d} \), respectively, and assume that

\[ \gamma = \sum_{S \in S_d} \alpha_S + \sum_{T \in T_d} \beta_T > 0. \]  

(4)
Given the set of weights $\alpha$ and $\beta$, we define the individual composite likelihood (CL) for $X$ as

$$p_{\alpha, \beta}(X; \theta) = \prod_{S \in S_d} [p(X_S; \theta)]^{\alpha_S / \gamma} \prod_{T \in T_d} [p(X_T^\gamma | X_T^\gamma; \theta)]^{\beta_T / \gamma},$$

where $X_S = (X_{s_1}, \ldots, X_{s_{|S|}})$, $X_T^\gamma = (X_{\ell_1}, \ldots, X_{\ell_{|T|}})$, and $X_T^\gamma = (X_{\ell_1}, \ldots, X_{\ell_{|T|}})$. That is, $p(x_S; \theta)$ is the marginal PDF with respect to the coordinates of $X$ corresponding to the subset $S$, and $p(x_T^\gamma | x_T^\gamma; \theta)$ is the conditional PDF of the coordinates corresponding to $T$, conditioned on the coordinates corresponding to $T^\gamma$.

Assume, as in the introduction, that the elements of $X_{2n}$ are sampled IID from a DGP with distribution $P_{\theta^*}$ and PDF $p(x; \theta^*)$, for some $\theta^* \in \Theta$. Further, $\tilde{\theta}_n^k$ are still generic estimators of $\theta^*$, for each $k \in \{0, 1\}$.

Let

$$L_{\alpha, \beta}(\theta; X_n^k) = \prod_{i=1}^n p_{\alpha, \beta}(X_i^k; \theta)$$

denote the composite likelihood of the subsample $X_n^k$, evaluated at $\theta \in \Theta$. We shall write the split composite likelihood ratio statistics (CLRSs) and the swapped CLRS as

$$U_{\alpha, \beta, n}^k(\theta) = \frac{L_{\alpha, \beta}(\tilde{\theta}_n^{1-k}; X_n^k)}{L_{\alpha, \beta}(\theta; X_n^k)},$$

for each $k \in \{0, 1\}$, and

$$\bar{U}_{\alpha, \beta, n}(\theta) = \frac{U_{\alpha, \beta, n}^0(\theta) + U_{\alpha, \beta, n}^1(\theta)}{2},$$

respectively.

Let

$$C_{n}^\alpha = \{ \theta \in \Theta : U_{\alpha, \beta, n}^0(\theta) \leq 1/\alpha \}$$

and

$$\bar{C}_{n}^\alpha = \{ \theta \in \Theta : \bar{U}_{\alpha, \beta, n}(\theta) \leq 1/\alpha \}$$

be universal confidence set estimators. We are now ready to establish our first result regarding finite-sample validity of $C_{n}^\alpha$ and $\bar{C}_{n}^\alpha$.

**Proposition 1.** The confidence set estimators $C_{n}^\alpha$ and $\bar{C}_{n}^\alpha$ are finite sample valid $100(1 - \alpha)$% confidence sets for $\theta^*$. That is,

$$\Pr_{\theta^*}(\theta^* \in C_{n}^\alpha) \geq 1 - \alpha,$$

and

$$\Pr_{\theta^*}(\theta^* \in \bar{C}_{n}^\alpha) \geq 1 - \alpha,$$
Consider the null and alternative hypotheses

\[ H_0 : \theta \in \Theta_0, \quad \text{and} \quad H_1 : \theta \in \Theta \setminus \Theta_0. \tag{5} \]

Due to the duality between confidence sets and hypothesis tests (cf. Thm. 2.3 of Hochberg & Tamhane, 1987, Appendix 1), Proposition 1 can be used to construct simple hypothesis tests using the rejection rules: reject \( H_0 \) if \( C_\alpha^n \cap \Theta_0 = \emptyset \) or if \( \bar{C}_\alpha^n \cap \Theta_0 = \emptyset \). Both of these tests control the Type I error at the correct level of significance \( \alpha \). However, these tests may be difficult to use when the shapes of \( \Theta_0 \), \( C_\alpha^n \), and \( \bar{C}_\alpha^n \) are complex and difficult to compute.

Let \( \hat{\theta}_n^k = \arg \max_{\theta \in \Theta_0} L_{\alpha, \beta}(\theta; X_n^k) \) denote the maximum CL estimator (MCLE) computed using the subset \( X_n^k \), for each \( k \in \{0, 1\} \). Using the MCLEs, we can construct tests that are more akin to the traditional likelihood ratio test or the pseudo-likelihood ratio test of Molenberghs & Verbeke (2005). To construct our tests, we require the split test statistics

\[ V_{\alpha, \beta, n}^k = \frac{L_{\alpha, \beta}(\hat{\theta}_n^{1-k}; X_n^k)}{L_{\alpha, \beta}(\hat{\theta}_n^k; X_n^k)}, \]

for each \( k \), and the swapped test statistic

\[ V_{\alpha, \beta, n} = \frac{V_{\alpha, \beta, n}^0 + V_{\alpha, \beta, n}^1}{2}. \]

We define the split composite likelihood ratio test (CLRT) and the swapped CLRT via the rules: reject \( H_0 \) if \( V_{\alpha, \beta, n}^0 > 1/\alpha \) or if \( V_{\alpha, \beta, n} > 1/\alpha \), respectively. The following result establishes the correctness of the split and swapped CLRTs.

**Proposition 2.** The split and the swapped CLRTs control the Type I error at the level \( \alpha \), for all \( n \in \mathbb{N} \). That is,

\[ \sup_{\theta^* \in \Theta_0} \Pr_{\theta^*}(V_{\alpha, \beta, n}^0 > 1/\alpha) \leq \alpha, \]

and

\[ \sup_{\theta^* \in \Theta_0} \Pr_{\theta^*}(V_{\alpha, \beta, n} > 1/\alpha) \leq \alpha. \]

**2.1 Always-valid inference**

Instead of observing \( X_n = X_n^0 \) in a single batch, we now consider that the IID elements of \( X_n \) (i.e., \( X_1, X_2, \ldots \)) arrive sequentially, from distribution \( \mathcal{P}_{\theta^*} \). For each \( n \in \mathbb{N} \), we wish to conduct a test
of the hypotheses \[^{[5]}\].

Let \(\hat{\theta}_{n-1}^1\) be a generic non-anticipating estimator of \(\theta^*\) (i.e., \(\hat{\theta}_{n-1}^1\) is only dependent on the data in \(X_{n-1}\)), and let \(\hat{\theta}_n^0\) be the same as it was defined in \(^{[6]}\). Further, define the running CLRT test statistic
\[
M_{\alpha,\beta,n} = \frac{\prod_{i=1}^{n} p_{\alpha,\beta}(X_i; \hat{\theta}_{i-1}^1)}{\prod_{i=1}^{n} p_{\alpha,\beta}(X_i; \hat{\theta}_n^0)}
\]
and at any time \(n\), reject \(H_0\) and stop the sequence of tests if \(M_{\alpha,\beta,n} > 1/\alpha\). If \(\nu_{\theta}^*\) denotes the time at which the test stops, under the rejection rule, given that the data arrises IID from \(P_{\theta}\), then we establish the fact that \(\nu_{\theta}^*\) is finite with probability at most \(\alpha\).

**Proposition 3.** The running CLRT has Type I error at most \(\alpha\). That is
\[
\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (\nu_{\theta^*} < \infty) \leq \alpha.
\]

Let \(P_n = 1/M_{\alpha,\beta,n}\) and \(\tilde{P}_n = \min_{s \leq n} (1/M_{\alpha,\beta,n})\) be \(p\)-values for the test of \(^{[5]}\) and let \(N \in \mathbb{N}\) be a random variable. The following result establishes that both \(P_N\) and \(\tilde{P}_N\) are valid.

**Proposition 4.** For any random \(N\), not necessarily a stopping time, \(P_N\) and \(\tilde{P}_N\) are valid \(p\)-values. That is
\[
\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (P_N \leq \alpha) \leq \alpha,
\]
and
\[
\sup_{\theta^* \in \Theta_0} \Pr_{\theta^*} (\tilde{P}_N \leq \alpha) \leq \alpha,
\]
for all \(\alpha \in [0, 1]\).

We define a confidence sequence for \(\theta^*\) as an infinite sequence of confidence sets that are all simultaneously valid. In the current context, such confidence sequence are \((\tilde{D}_n^\alpha)_{n \in \mathbb{N}}\) and \((\check{D}_n^\alpha)_{n \in \mathbb{N}}\), where
\[
\tilde{D}_n^\alpha = \{ \theta \in \Theta : R_{\alpha,\beta,n} (\theta) \leq 1/\alpha \},
\]
\[
\check{D}_n^\alpha = \bigcap_{m \leq n} D_m^\alpha,
\]
and
\[
R_{\alpha,\beta,n} (\theta) = \frac{\prod_{i=1}^{n} p_{\alpha,\beta}(X_i; \hat{\theta}_{i-1}^1)}{\prod_{i=1}^{n} p_{\alpha,\beta}(X_i; \hat{\theta}_n^0)}.
\]

The following result establishes the validity of \((\tilde{D}_n^\alpha)_{n \in \mathbb{N}}\) and \((\check{D}_n^\alpha)_{n \in \mathbb{N}}\).

**Proposition 5.** The confidence sequences \((\tilde{D}_n^\alpha)_{n \in \mathbb{N}}\) and \((\check{D}_n^\alpha)_{n \in \mathbb{N}}\) are valid. That is
\[
\Pr_{\theta^*} (\forall n \in \mathbb{N} : \theta^* \in \tilde{D}_n^\alpha) \geq 1 - \alpha.
\]
and

\[ \Pr_{\theta^*} \left( \forall n \in \mathbb{N} : \theta^* \in \tilde{D}_n^\alpha \right) \geq 1 - \alpha. \]

3 Proofs

The following result provides the primary mechanism under which Propositions 1 and 2 can be established, and is a direct analog to (3) for CLs.

**Lemma 1.** If \( X_{2n} \) is an IID sample from a DGP with distribution \( P_{\theta^*} \) and PDF \( f(x; \theta^*) \), then \( U_{\alpha, \beta, n}^k (\theta^*) \) has bounded expectation \( E_{\theta^*} \left[ U_{\alpha, \beta, n}^k (\theta^*) \right] \leq 1 \), for each \( k \in \{0, 1\} \) and for all \( n \in \mathbb{N} \).

**Proof.** We shall prove the \( k = 0 \) case. Let \( x_n = (x_1, \ldots, x_n) \) and write

\[
E_{\theta^*} \left[ U_{\alpha, \beta, n}^0 (\theta^*) | X_n^1 \right]
= \int_{X^n} \frac{L_{\alpha, \beta} \left( \tilde{\theta}_n^1, x_n \right)}{L_{\alpha, \beta} (\theta^*; x_n)} L (\theta^*; x_n) \, dx_n
= \int_{X^n} \frac{\prod_{i=1}^n p_{\alpha, \beta} (x_i; \tilde{\theta}_n^1)}{\prod_{i=1}^n p_{\alpha, \beta} (x_i; \theta^*)} \prod_{i=1}^n p (x_i; \theta^*) \, dx_n
= \int_{X^n} \prod_{i=1}^n \frac{p_{\alpha, \beta} (x_i; \tilde{\theta}_n^1)}{p_{\alpha, \beta} (x_i; \theta^*)} p (x_i; \theta^*) \, dx_n.
\]
Then, simplify the integrand by making the factorization

\[
\begin{align*}
\frac{p_{\alpha, \beta}(\mathbf{x}_i; \tilde{\theta}^1_n)}{p_{\alpha, \beta}(\mathbf{x}_i; \theta^*)} & \left\langle \begin{array}{l}
(i) \\
(ii) \\
(iii)
\end{array} \right. \\
\prod_{S \in \mathcal{S}_d} & \left[ p \left( \mathbf{x}_i; \tilde{\theta}^1_n \right) \right]^{\alpha_S/\gamma} \prod_{T \in \mathcal{T}_d} \left[ p \left( \mathbf{x}_{i, \overline{T}}; \tilde{\theta}^1_n \right) \right]^{\beta_T/\gamma} \\
& \times \prod_{S \in \mathcal{S}_d} \left[ p \left( \mathbf{x}_i; \theta^* \right) \right]^{\alpha_S/\gamma} \prod_{T \in \mathcal{T}_d} \left[ p \left( \mathbf{x}_{i, \overline{T}}; \theta^* \right) \right]^{\beta_T/\gamma} \\
& \prod_{T \in \mathcal{T}_d} \left[ p \left( \mathbf{x}_{i, \overline{T}}; \theta^* \right) \right]^{\beta_T/\gamma} \\
& \prod_{S \in \mathcal{S}_d} \left[ \tilde{p} \left( \mathbf{x}_i; \tilde{\theta}^1_n, \theta^* \right) \right]^{\alpha_S/\gamma} \prod_{T \in \mathcal{T}_d} \left[ \tilde{p} \left( \mathbf{x}_i; \tilde{\theta}^1_n, \theta^* \right) \right]^{\beta_T/\gamma}
\end{align*}
\]

where (i) is due to (8) and (ii) is due to the PDF decompositions

\[
p \left( \mathbf{x}_i; \theta^* \right) = p \left( \mathbf{x}_{i, [d] \setminus S} \mid \mathbf{x}_{i, S}; \theta^* \right) p \left( \mathbf{x}_{i, S}; \theta^* \right)
\]

and

\[
p \left( \mathbf{x}_i; \theta^* \right) = p \left( \mathbf{x}_{i, [d] \setminus (\overline{T} \cup \overline{T})} \mid \mathbf{x}_{i, \overline{T}}, \mathbf{x}_{i, \overline{T}}; \theta^* \right) p \left( \mathbf{x}_{i, \overline{T}} \mid \mathbf{x}_{i, \overline{T}}; \theta^* \right) p \left( \mathbf{x}_{i, \overline{T}}; \theta^* \right).
\]

The PDFs on line (iii) are then constructed as

\[
\tilde{p} \left( \mathbf{x}_i; \tilde{\theta}^1_n, \theta^* \right) = p \left( \mathbf{x}_{i, [d] \setminus S} \mid \mathbf{x}_{i, S}; \theta^* \right) p \left( \mathbf{x}_{i, S}; \tilde{\theta}^1_n \right)
\]

and

\[
\tilde{p} \left( \mathbf{x}_i; \tilde{\theta}^1_n, \theta^* \right) = p \left( \mathbf{x}_{i, [d] \setminus (\overline{T} \cup \overline{T})} \mid \mathbf{x}_{i, \overline{T}}, \mathbf{x}_{i, \overline{T}}; \theta^* \right) p \left( \mathbf{x}_{i, \overline{T}} \mid \mathbf{x}_{i, \overline{T}}; \tilde{\theta}^1_n \right) p \left( \mathbf{x}_{i, \overline{T}}; \theta^* \right).
\]
We then have

\[
E_{\theta^*} \left[ U_{\alpha,\beta,n}^0 (\theta^*) | X_n^1 \right]
= \int \prod_{i=1}^n \prod_{S \in \mathcal{S}_d} \prod_{T \in \mathcal{T}_d} \left[ \tilde{p} \left( x_i; \tilde{\theta}_n^1, \theta^* \right) \right]^{\alpha_S/\gamma} \prod_{T \in \mathcal{T}_d} \left[ \bar{p} \left( x_i; \bar{\theta}_n^1, \theta^* \right) \right]^{\beta_T/\gamma} dx_i
\]

\[
\leq \prod_{i=1}^n \prod_{S \in \mathcal{S}_d} \prod_{T \in \mathcal{T}_d} \left[ \int \tilde{p} \left( x_i; \tilde{\theta}_n^1, \theta^* \right) dx_i \right]^{\alpha_S/\gamma} \prod_{T \in \mathcal{T}_d} \left[ \int \bar{p} \left( x_i; \bar{\theta}_n^1, \theta^* \right) dx_i \right]^{\beta_T/\gamma}
\]

\[
= \prod_{i=1}^n \prod_{S \in \mathcal{S}_d} \prod_{T \in \mathcal{T}_d} 1 = 1,
\]

where (i) is due to separability, (ii) is due to the generalized Hölder’s inequality, and (iii) is due to the fact that (8) and (9) are PDFs. Finally, via the law of iterated expectations, we have

\[
E_{\theta^*} \left[ U_{\alpha,\beta,n}^0 (\theta^*) \right] = E_{\theta^*} E_{\theta^*} \left[ U_{\alpha,\beta,n}^0 (\theta^*) | X_n^1 \right] \leq 1.
\]

\[\blacksquare\]

### 3.1 Proof of Proposition 1

We shall prove the fact that \( \Pr_{\theta^*} (\theta^* \in \bar{C}_n^\alpha) \geq 1 - \alpha \) and not that the case for \( C_n^\alpha \) can be proved in an identical manner.

For any \( \theta^* \in \Theta \) and \( n \), we have

\[
\Pr_{\theta^*} (\theta^* \notin \bar{C}_n^\alpha) = \Pr_{\theta^*} \left( \frac{U_{\alpha,\beta,n}^0 (\theta^*) + U_{\alpha,\beta,n}^1 (\theta^*)}{2} > 1/\alpha \right)
\]

\[
\leq \alpha E_{\theta^*} \left[ \frac{U_{\alpha,\beta,n}^0 (\theta^*) + U_{\alpha,\beta,n}^1 (\theta^*)}{2} \right]
\]

\[
= \frac{\alpha}{2} E_{\theta^*} \left[ U_{\alpha,\beta,n}^0 (\theta^*) \right] + \alpha E_{\theta^*} \left[ U_{\alpha,\beta,n}^1 (\theta^*) \right]
\]

\[
\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,
\]

where (i) is due to Markov’s inequality and (ii) is due to Lemma 1. We obtain the desired result by computing the complement

\[
\Pr_{\theta^*} (\theta^* \in \bar{C}_n^\alpha) = 1 - \Pr_{\theta^*} (\theta^* \notin \bar{C}_n^\alpha) \geq 1 - \alpha.
\]
3.2 Proof of Proposition 2

We shall prove the result for the swapped CRLT and note that the split CLRT result can be proved in an identical manner.

For any $\theta^* \in \Theta_0$ and $n$, we have

$$
\Pr_{\theta^*} \left( \bar{V}_{\alpha,\beta,n} > \frac{1}{\alpha} \right) = \Pr_{\theta^*} \left( \frac{V_{\alpha,\beta,n}^0 + V_{\alpha,\beta,n}^1}{2} > \frac{1}{\alpha} \right)
$$

(i)

$$
\leq \alpha E_{\theta^*} \left[ \frac{V_{\alpha,\beta,n}^0 + V_{\alpha,\beta,n}^1}{2} \right]
$$

(ii)

$$
= \frac{\alpha}{2} E_{\theta^*} \left[ U_{\alpha,\beta,n}^0 (\theta^*) + \frac{\alpha}{2} E_{\theta^*} \left[ U_{\alpha,\beta,n}^1 (\theta^*) \right] \right]
$$

(iii)

$$
\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,
$$

where (i) is due to Markov’s inequality, (ii) is due to Lemma 1 (i.e., $L_{\alpha,\beta} (\hat{\theta}_n; X_n^k) \geq L_{\alpha,\beta} (\theta^*; X_n^k)$, for all $\theta^* \in \Theta_0$), and (iii) is due to Lemma 1. The desired result is thus obtained.

3.3 Proof of Proposition 3

Under $H_0$, observe that $M_{\alpha,\beta,n} \leq M_n^*$, where $M_n^* = R_{\alpha,\beta,n} (\theta^*)$ is as defined in (7), since $L_{\alpha,\beta} (\hat{\theta}_n; X_n) \geq L_{\alpha,\beta} (\theta^*; X_n)$, for $\theta^* \in \Theta_0$. Let $(F_n)_{n \in \mathbb{N} \cup \{0\}}$ be the natural filtration, where $F_n = \sigma (X_n)$. Upon defining $M_0^* = 1$, notice that

$$
E_{\theta^*} \left[ M_n^* | F_{n-1} \right] = E_{\theta^*} \left[ \frac{\prod_{i=1}^{n-1} p_{\alpha,\beta} (X_i; \hat{\theta}_{i-1}^{\beta}) p_{\alpha,\beta} (X_n; \hat{\theta}_{n-1}^{\beta})}{\prod_{i=1}^{n-1} p_{\alpha,\beta} (X_i; \theta^*) p_{\alpha,\beta} (X_n; \theta^*)} | F_{n-1} \right]
$$

$$
= E_{\theta^*} \left[ M_n^* \frac{p_{\alpha,\beta} (X_n; \hat{\theta}_{n-1}^{\beta})}{p_{\alpha,\beta} (X_n; \theta^*)} | F_{n-1} \right]
$$

$$
= M_{n-1}^* E_{\theta^*} \left[ \frac{p_{\alpha,\beta} (X_n; \hat{\theta}_{n-1}^{\beta})}{p_{\alpha,\beta} (X_n; \theta^*)} | F_{n-1} \right]
$$

(i)

$$
\leq M_{n-1}^*.
$$

where (i) is established using the same argument as used in Lemma 1. Thus, we have established that $(M_n^* \in \mathbb{N} \cup \{0\})$ is a supermartingale, adapted to $(F_n)_{n \in \mathbb{N} \cup \{0\}}$. Upon application of Lemma 2 we
have
\[ \Pr_{\theta^*} \left( \exists n \in \mathbb{N} : M_n^* \geq 1/\alpha \right) \leq \alpha M_0^* = \alpha. \] (10)

Note that
\[ \{ \nu_{\theta^*} = \infty \} = \{ \forall n \in \mathbb{N} : M_{\alpha,\beta,n} < 1/\alpha \} \]
and hence
\[ \{ \nu_{\theta^*} < \infty \} = \{ \exists n \in \mathbb{N} : M_{\alpha,\beta,n} \geq 1/\alpha \}. \]

We obtain the desired result, since
\[ \Pr_{\theta^*} (\nu_{\theta^*} < \infty) = \Pr_{\theta^*} (\exists n \in \mathbb{N} : M_{\alpha,\beta,n} \geq 1/\alpha) \]
\[ \leq \Pr_{\theta^*} (\exists n \in \mathbb{N} : M_n^* \geq 1/\alpha) \]
\[ \leq \alpha. \]

3.4 Proof of Proposition 4

Firstly note that
\[ \{ \exists n \in \mathbb{N} : M_{\alpha,\beta,n} \geq 1/\alpha \} = \{ \exists n \in \mathbb{N} : P_n \leq \alpha \} \]
\[ = \bigcup_{n \in \mathbb{N}} \{ P_n \leq \alpha \}, \]
and apply Lemma 3 to establish the validity of $P_N$.

In order to establish the validity of $\tilde{P}_N$, we note that
\[ \left\{ \tilde{P}_n \leq \alpha \right\} = \bigcup_{m \leq n} \{ P_m \leq \alpha \} \]
and hence
\[ \bigcup_{n \in \mathbb{N}} \left\{ \tilde{P}_n \leq \alpha \right\} = \bigcup_{n \in \mathbb{N}} \bigcup_{m \leq n} \{ P_m \leq \alpha \} \]
\[ = \bigcup_{n \in \mathbb{N}} \{ P_n \leq \alpha \}. \]
3.5 Proof of Proposition 5

Notice that \( R_{\alpha,\beta,n} (\theta^*) = M^*_n \), for each \( n \in \mathbb{N} \), where \( M^*_n \) is as defined in the proof of Proposition 3. Then

\[
\Pr_{\theta^*} (\exists n \in \mathbb{N} : \theta^* \notin D^\alpha_n) = \Pr_{\theta^*} (\exists n \in \mathbb{N} : R_{\alpha,\beta,n} (\theta^*) > 1/\alpha) \\
\leq \Pr_{\theta^*} (\exists n \in \mathbb{N} : M^*_n \geq 1/\alpha) \\
\leq \alpha,
\]
due to (10). Thus, we have demonstrated the validity of \((D^\alpha_n)_{n \in \mathbb{N}}\).

To prove the validity of \((\tilde{D}^\alpha_n)_{n \in \mathbb{N}}\), write

\[
\{ \theta^* \notin \tilde{D}^\alpha_n \} = \left\{ \theta^* \notin \bigcap_{m \leq n} D^\alpha_m \right\} \\
= \bigcup_{m \leq n} \{ \theta^* \notin D^\alpha_m \}.
\]

Thus

\[
\{ \exists n \in \mathbb{N} : \theta^* \notin \tilde{D}^\alpha_n \} = \bigcup_{n \in \mathbb{N}} \left\{ \theta^* \notin \tilde{D}^\alpha_n \right\} \\
= \bigcup_{n \in \mathbb{N}} \bigcup_{m \leq n} \{ \theta^* \notin D^\alpha_m \} \\
= \bigcup_{n \in \mathbb{N}} \{ \theta^* \notin D^\alpha_n \} \\
= \{ \exists n \in \mathbb{N} : \theta^* \notin D^\alpha_n \},
\]
as required.

Appendix

Technical requirements

We state some technical results that are required throughout the text. References for unproved results are provided at the end of the section.

Lemma 2 (Ville’s Inequality). If \((Y_n)_{n \in \mathbb{N} \cup \{0\}}\) is a non-negative supermartingale, adapted to the filtration \((\mathcal{F}_n)_{n \in \mathbb{N} \cup \{0\}}\). Then, for any \(\alpha > 0\), we have

\[
\Pr (\exists n \in \mathbb{N} : Y_n \geq 1/\alpha) \leq \alpha Y_0.
\]
Lemma 3. Let \((A_n)_{n\in\mathbb{N}}\) be a sequence of events in some filtered probability space, and let \(A_\infty = \limsup_{n\to\infty} A_n\). If \(\alpha \in [0,1]\), then the following statements are equivalent: (a) \(\Pr(\bigcup_{n=1}^\infty A_n) \leq \alpha\), (b) \(\Pr(A_N) \leq \alpha\) for all random (potentially not stopping times) \(N\), (c) \(\Pr(A_\nu) \leq \alpha\) for all stopping times \(\nu\) (possibly infinite).

Lemma 2 appears as Lemma 1 in Howard et al. (2020a) (see also Stout, 1973, Lem. 1.1). Lemma 3 appears as Lemma 3 in Howard et al. (2020b).

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