Memory-free dynamics for the TAP equations of Ising models with arbitrary rotation invariant ensembles of random coupling matrices

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We propose an iterative algorithm for solving the Thouless-Anderson-Palmer (TAP) equations of Ising models with arbitrary rotation invariant (random) coupling matrices. In the thermodynamic limit, we prove by means of the dynamical functional method that the proposed algorithm converges when the so-called de Almeida Thouless (AT) criterion is fulfilled. Moreover, we give exact analytical expressions for the rate of the convergence.

I. INTRODUCTION

TAP equations are sets of generalized mean field equations for computing approximate marginal moments in large probabilistic models with a dense connectivity of couplings with random strengths. Originally developed to compute the magnetisations of the Sherrington Kirkpatrick (SK) model of a spin-glass [1], there are now various generalizations especially to models in the area of statistical data science [2–5].

Hence, it is important to design and study the performance of iterative algorithms which can generate solutions to TAP equations. Several results for the dynamical properties of such algorithms have been derived for models with specific probability distributions of the couplings [6–8]. More recent research has attempted to extend such results to more general (and possibly more realistic) classes of probability distributions [9–13].

In this paper we will address the dynamics of algorithms for the original case of Ising spin models. The model in its simplest formulation (with constant external fields) discussed in this paper is less interesting for data science. But its generalizations have been applied e.g. to modeling the dependencies of spikes recorded from ensembles of neurons [14] [15], to protein structure determination [16] and gene expression analysis [17]. For a review, see [18]. In any case, we expect that a proper understanding of algorithms for the simplest setting will also be highly useful for treating more complex models.

We will consider TAP equations for the large class of rotation invariant random matrices [19], which allow for nontrivial dependencies between matrix elements. We will show in Section II A that simpler cases such as the independent Gaussian couplings of the SK model or those of the Hopfield model and its generalizations [20] are special cases of the rotation invariant ensembles studied in this work.

The analysis of algorithms for TAP equations can be described in terms of the nonlinear dynamics of a system of nodes coupled by a dense random matrix. In the thermodynamic limit of large systems, it is possible to derive an exact decoupling of the degrees of freedom for the statistical properties of the dynamics using the so-called dynamical functional (DF) method [21]. Unfortunately, the effective stochastic process of single variables contains couplings between the same variables at different times. This usually precludes closed form analytical results for the dynamical properties. In a previous paper, motivated by existing algorithms for the case of specific random matrices, we have shown that such temporal self couplings can be canceled for the entire class of rotation invariant random matrices, if one includes specific memory terms in the algorithm. In that case, the stochastic fields which drive the effective dynamics reduce to a Gaussian process.

Our previous construction of the so-called single-step memory (SSM) algorithm for solving TAP equations presented in [9] comes with two major drawbacks. First, the memory terms in the algorithm are based on the coefficients of the power series expansion of a specific function derived from the random matrix ensemble. Of course, these can be obtained easily, when an analytical expression of the function is known. But for any given large matrix which is assumed to be drawn at random from a rotation invariant (but otherwise unknown) ensemble it is not clear how to obtain such coefficients numerically in a simple way. This problem will make the algorithm not a good candidate for practical applications. Second, from a theoretical perspective our analytical analysis is still incomplete. We were able to show by a linear stability analysis that the well-known de Almeida Thouless (AT) [22] criterion which separates an ergodic region of the spin-model from a more complex phase, is necessary for convergence of the algorithm. Unfortunately, we were not able to get general analytical results for the asymptotic convergence rates.

In this paper we present a new construction for an algorithm for the Ising system which was motivated by similar approaches in the field of compressed sensing [10, 11]. Remarkably, the updates of the algorithm avoid the use of explicit memory terms and can be easily applied to an Ising model with a given matrix of couplings. The analysis by the DF method again shows a vanishing of the self-couplings. This leaves us with an effective dynamics that is driven by a Gaussian process but with a much simpler covariance structure compared to our pre-

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vorous method. We are able to prove convergence from a 
random initialization of the algorithm in the thermo-
dynamic limit provided that the AT criterion is fulfilled. 
Moreover, we also obtain explicit expressions for the rate 
of convergence. To our knowledge this is the first time 
that analytical results for the convergence of such algo-
rithms have been obtained.

The paper is organized as follows: In Section 2 we in-
troduce the model and present motivating examples for 
studying rotation invariant ensembles of coupling ma-
trices. Section 3 provides a brief presentation on the 
TAP equations. In Section 4 we present our new algo-
rithm for solving the TAP equations and in Section 5 we 
study its properties in the thermodynamic limit. Sec-
tion 6 provides convergence properties of the algorithm. 
In section 7 we show how to compute parameters which 
are needed by the algorithm before the iteration starts. 
For comparison, in Section 8 we give analytical expres-
sions for the convergence rates of the SSM algorithm of 
for the special cases of the SK and Hopfield models. 
Comparisons of the theory with simulations are given in 
Section 9. Section 10 presents a summary and outlook. 
Major derivations of our results are located at the Ap-
pendix.

II. ISING MODELS WITH RANDOM COUPLINGS

We consider Ising models with pairwise interactions of 
the spins \( s = (s_1, \ldots, s_N)^\top \in \{-1, 1\}^N \) described by the 
(conditional) Gibbs distribution

\[
p(s|J, h) = \frac{1}{Z} \exp \left( \frac{1}{2} s^\top J s + s^\top h \right)
\]

where \( Z \) stands for the normalization constant. Our con-
cern is to compute the vector of magnetizations

\[
m = \mathbb{E}[s]
\]

where the expectation is taken over the Gibbs distribu-
tion. For the sake of simplicity of the analysis we limit 
our attention to the case where all external fields are equal

\[
h_i = h \neq 0, \ \forall i.
\]

We assume that the coupling matrix \( J = J^\top \) is drawn 
from a rotation invariant matrix ensemble, i.e. \( J \) and 
\( VJV^\top \) have the same probability distributions for any 
orthogonal matrix \( V \) independent of \( J \). Equivalently, \( J \) 
has the spectral decomposition \( J = O^\top DO \)

\[
J = O^\top DO\quad (4)
\]

where \( O \) is a Haar (orthogonal) matrix that is indepen-
dent of a real-diagonal matrix \( D \). Though our analysis 
does not make reference to any specific coupling matrix 
model, it is interesting to note, that certain models de-
derived from products of simpler matrices which were origi-
nally introduced in the context of communication theory \([24]\) 
and recently reappeared in the statistical physics context \([20]\), belong to the rotation invariant family.

A. Motivation: Models with a product of random matrices

Consider the product of \( M \) matrices

\[
X = X_M X_{M-1} \cdots X_1
\]

where \( X_m \in \mathbb{R}^{N_m \times N_m} \) and \( N_0 \equiv N \). We will consider 
coupling matrices of the form

\[
J = \beta X^\top X - \frac{\beta}{N} \text{tr}(X^\top X)I
\]

where \( \beta \) stands for the inverse temperature parameter. 
With this modeling of the coupling matrix the Ising model 
becomes equivalent to an \( M \)-layer probabilistic feed-forward neural network with linear transfer 
functions. Specifically, for a set of hidden node vectors \( u \) 
we define an \( M \)-layer linear network by the distribution

\[
p(s, u|\lambda, h) = \frac{1}{Z} e^{-\frac{1}{2} s^\top \tilde{\beta} (u_0 + h) \times
\times \left( \prod_{m=1}^M \delta(u_{m-1} - X_m^\top u_m) \right) N(u_M; 0, 1/\beta)\}
\]

where \( X \equiv \{X_1, \ldots, X_M\} \), \( \tilde{\beta} = \beta \text{tr}(X^\top X) \) and 
\( N(\cdot; \mu, \Sigma) \) denotes the Gaussian density function with 
mean \( \mu \) and covariance \( \Sigma \). It is easy to see that by in-
tegrating out the Gaussian variables \( U \), we arrive at the 
Gibbs distribution

\[
p(s|J, h) = \int d\lambda p(s, \lambda|\lambda, h).
\]

We will next discuss some interesting cases for which 
this model leads to rotation invariant ensembles.

1. Multi-layer Hopfield model

Let \( J \) be as in \([6]\) and let all \( N_m \times N_m \) matrices \( X_m \) 
in \([5]\) be independent where the entries of \( X_m \) are indepen-
dent Gaussian random variables with zero mean and 
variance \( 1/N_m \). The case \( M = 1 \) is essentially the 
well-known Hopfield model \([22]\). Here, we ignore the fact 
that in the present case, the stored “patterns” are Gaussi-
\n
an not Ising variables. Since we are not interested in 
memory retrieval properties of the model, the difference 
is not relevant.

The case \( M = 2 \) coincides with Mezard’s combinatorial 
disorder Hopfield model \([20]\). In general, random 
matrixs \( J \) constructed in such a way belong to rotation 
Invariant ensembles and we call the model multi-layer 
Hopfield model.
2. Multi-layer random orthogonal model

Let $J$ be as in (6) and the $N_m \times N_m - 1$ matrices $X_m$ in (5) are defined as

$$X_m = \frac{1}{\sqrt{p_m}} P_{\rho_m}^L O_m (P_{\rho_m - 1}^L)^\top$$ (9)

where all $O_m$ are independent $L \times L$ Haar matrices, and $P_{\rho}^L$ is a $\rho L \times L$ matrix with $\rho \leq 1$ and

$$(P_{\rho})_{ij} = \delta_{ij}, \quad \forall i, j.$$ (10)

For $M = 1$ and $\rho_0 = 1$ we arrive at the random orthogonal model of Parisi and Potters [19]. In general, matrices $J$ are also rotation invariant and we call the model multi-layer random orthogonal model.

3. SK model as the universal small $\alpha$-limit

We next consider the scaling properties of the coupling matrix model (6) when the aspect ratio $\alpha = N_0/N_\rho$ of the $N_M \times N_\rho$ matrix $X$ in (5) becomes small. For the 1-layer Hopfield model it is well-known that by rescaling $J \to \sqrt{\alpha} J$ of the coupling matrix model the model becomes equivalent to the SK model [20] in the limit $\alpha \to 0$. To be precise, the limiting spectral distribution of $\sqrt{\alpha} J$ becomes a Wigner semicircle law in the limit $\alpha \to 0$. We next show that this small-$\alpha$ limit holds in a universal sense for both the multi-layer Hopfield and multi-layer random orthogonal models. Specifically, the coupling matrices for $\alpha \leq 1$ are in general of the form

$$J = P_\alpha^{N_M} Y (P_\alpha^{N_M})^\top.$$ (11)

Let $f_\alpha$ denote the density of the limiting spectral distribution of $\sqrt{\alpha} J$ and let $\sigma_Y^2 = \lim_{N \to \infty} \frac{1}{N} \text{tr}(Y^2)$. Then, we show in Appendix D that

$$\lim_{\alpha \to 0} f_\alpha(x) = \frac{\sqrt{4\alpha \sigma_Y^2 - x^2}}{2\pi \sigma_Y^2}, \quad |x| \leq 2\sigma_Y.$$ (12)

Actually, this result holds for any coupling matrix of the form (11) with rotation invariant $Y$ which has a compactly supported limiting spectral distribution with $\lim_{N \to \infty} \frac{1}{N} \text{tr}(Y) = 0$. We note that $\sigma_Y^2$ is a model-dependent parameter, e.g. for the multi-layer Hopfield model we have

$$\sigma_Y^2 = \beta^2 (1 + \sum_{m=1}^M \alpha_m)$$ (13)

with $\alpha_m = N_0/N_m$. For a convenient computation of the model parameter $\sigma_Y^2$ in general we refer the interested reader to Appendix E.

III. TAP EQUATIONS

The TAP equations are a set of nonlinear equations for approximating the magnetizations $m$ of the Ising model with random couplings. They are assumed (under certain conditions) to give the exact results in the thermodynamic limit [20] and were first proposed for the SK model in [22]. A generalization to rotation invariant matrices were originally derived in [19] using a free energy approach. An alternative derivation was given in [28] using the cavity method. We also refer to [29] for a rigorous approach on the self averaging assumptions made for cavity field variances in this approach. The TAP equations are given by

$$m = \text{Th}(\gamma)$$ (14a)

$$\gamma = Jm - R(\chi)m$$ (14b)

$$\chi = E_u[\text{Th}'(\sqrt{(1-\chi)R'(\chi)u})].$$ (14c)

Here, for convenience we define the function

$$\text{Th}(x) = \tanh(h + x).$$

The random variable $u$ is a standard (zero mean, unit variance) normal Gaussian.

The only dependency on the random matrix ensemble of $J$ is through the $R$-transform which is defined as [30]

$$R(\omega) = G^{-1}(\omega) - \frac{1}{\omega},$$ (15)

where $G^{-1}$ is the functional inverse of the Green-function

$$G(z) = \lim_{N \to \infty} \frac{1}{N} \text{tr}((z I - J)^{-1}).$$ (16)

$G(z)$ is bijective for real $z$ which are not in the support of the limiting spectral distribution of $J$. Hence, $R(\omega)$ is well-defined for $G_{\text{min}} < \omega < G_{\text{max}}$ where we have defined $G_{\text{max}} \equiv \lim_{z \to \lambda_{\text{max}}} G(z)$ (and similarly for $G_{\text{min}}$) with $\lambda_{\text{max},\text{min}}$ denoting the supremum and infimum of the support, respectively. One can show that $R(\omega)$ is strictly increasing [31] (unless $J = c I$ for some $c \in \mathbb{R}$), i.e. $R'(\omega) > 0$ where $R'$ stands for the derivative of $R$.

As an example, for the (1-layer) Hopfield model with $\alpha_1 \equiv N_0/N_1$ we have [3]

$$R(\omega) = \frac{\alpha_1^2 \beta^2 \omega}{1 - \alpha_1 \beta \omega}.$$ (17)

The corresponding equations [13] agree with the well-known TAP equations of the Hopfield model previously obtained e.g. in [26]. Moreover, for the 2-layer Hopfield model with $\alpha_2 \equiv N_0/N_2$ we have (see Appendix E)

$$R(\omega) = \frac{\frac{1}{\alpha_2} - \alpha_1 - \alpha_2 - \sqrt{\left(\frac{1}{\alpha_2} - \alpha_1 - \alpha_2\right)^2 - 4\alpha_1 \alpha_2}}{2\alpha_1 \alpha_2 \omega}.$$ (18)
and the resulting TAP equations \[14\] coincide with those derived in \[20\] by means of a mean field message passing method.

The TAP equations \[14\] are well-defined when

\[
\eta R'(\chi) < 1 \quad (\text{AT}) \quad \chi < G_{\text{max}}
\]

where we have set

\[
\eta \doteq E_u[(\text{Th}'(\sqrt{(1-\chi)R'(\chi)u}))^2].
\]

The condition (AT) is sufficient for \( \chi \) to have a unique solution in \[14\]. This result can be easily derived by following the arguments of the derivation of \[8\, \text{Lemma 2.2}\]. There, the result refers to the SK model, i.e. \( R(\omega) = \beta^2 \omega \).

Equality in \[19\], i.e. \( \eta R'(\chi) = 1 \), agrees with the well-known AT line of stability \[22\] for the Replica-symmetric ansatz for spin models with rotation invariant coupling matrices, see \[32\, \text{Eq. (46)}\]. In the region of stability, we can also expect that the solution of the TAP equations will give us the asymptotically correct magnetisations in the thermodynamic limit. The second condition \[20\] is needed to ensure that the Green-function has a unique inverse such that the R-transform can be defined properly. For the SK-model, e.g. it restricts the valid inverse temperature regions to \( \beta < \frac{1}{\sqrt{N}} \). Similarly, for the (1-layer) Hopfield model, one needs \( \beta < \frac{1}{\chi} \).

IV. ITERATIVE SOLUTION OF THE TAP EQUATIONS

In this section we define a new iterative algorithm for solving the TAP equations \[14\]. First, we recall that the only dependency on the random matrix ensemble in the TAP equations \[14\] is via the R-transform \( R \equiv R(\chi) \) (for short) and its derivative \( R' \equiv R'(\chi) \). Hence, if the limiting spectral distribution of the random matrix ensemble is known, these quantities can be computed exactly from the Green-function \[16\] via \[15\]. We will show in Section \[VII\] a practical computation of these quantities for concrete realizations of matrices \( J \).

We are looking for a solution to the TAP equations in terms of an iteration of a vector of auxiliary variables \( \gamma(t) \), where \( t \) denotes the discrete time index of the iteration. The initialization of this vector is given by \( \gamma(0) = \sqrt{(1-\chi)R'u} \) where \( u \) is a vector of independent normal Gaussian random variables. We then proceed by iterating

\[
\gamma(t) = A \tilde{\gamma}(t)
\]

for \( t = 1, 2, 3, \ldots, T \) with the \textit{time-independent} random matrix

\[
A \doteq \frac{1}{\chi}(\lambda I - J)^{-1} - I.
\]

The variable \( \lambda \) is defined as the (unique) solution of the scalar equation

\[
G(\lambda) = \chi
\]

or equivalent (see \[15\])

\[
R(\chi) = \lambda - \frac{1}{\chi}.
\]

It is easy to see that the fixed points of \( \gamma(t) \) coincide with the solution of the TAP equations for \( \gamma \). \[14\], if we identify the corresponding magnetisations by

\[
m = \chi(\tilde{\gamma} + \gamma).
\]

V. DYNAMICS IN THE THERMODYNAMIC LIMIT

Our goal is to analyze the convergence properties of the sequence \( \gamma(t) \) in the thermodynamic limit by studying the deviation between the dynamical variables at different times

\[
\Delta(t, s) \doteq \lim_{N \to \infty} \frac{1}{N} \mathbb{E}[(\gamma(t) - \gamma(s))^2]
\]

where the expectation is taken over the random matrix ensemble of \( A \) and the random initialization \( \gamma(0) \). In the thermodynamic limit, the normalized squared distance will become self averaging and represents the typical squared distance corresponding to a single realization of the dynamics \[22\]. We will show later, that as \( t, s \to \infty \), the deviation \( \Delta(t, s) \) will converge to 0 showing that the sequence of iterates \( \gamma(t) \) will have a limit which thus solves the TAP equations.

Dynamical properties of fully connected disordered systems can be computed by a discrete time version of the method of DF developed by Martin, Siggia and Rose \[21\] and also used extensively for studying spin-glass dynamics, see e.g. \[23, 34\]. This method provides us with exact results for the thermodynamic properties of the marginal distribution of a trajectory of arbitrary length \( T \), for \textit{single variables} (in our case \( \gamma_i(t) \) and \( \tilde{\gamma}_i(t) \)) as long as the large-system limit \( N \to \infty \) is taken before the long-time limit \( T \to \infty \). The method is based on averaging the moment \textit{generating functional} for the trajectory of \( \gamma_i(t) \) and \( \tilde{\gamma}_i(t) \)

\[
Z\{l_i(t)\} \doteq \int \prod_{t=1}^{T} d\gamma(t) d\tilde{\gamma}(t) \delta(\tilde{\gamma}(t) - f(\gamma(t-1))) \times \delta(\gamma(t) - A \tilde{\gamma}(t)) e^{\gamma(t)l_i(t)}
\]

over the random matrix \( A \) and the random initialization \( \gamma(0) \). Here, for short we introduce the function

\[
f(x) \doteq \frac{1}{\chi} \text{Th}(x) - x.
\]
By causality $\mathcal{G}$ is an upper triangular matrix (i.e. its $(t,s)$ indexed entries are zero for $s \geq t$). The R-transform of a matrix can be defined from the power series expansion of the R-transform \[ R_A(\omega) = \sum_{n=1}^{\infty} c_{A,n} \omega^{n-1} \]
in terms of the so-called free cumulants $c_{A,n}$ by inserting the powers of the matrix $\mathcal{G}^{-1}$ for $\omega^{-1}$. It follows that $\mathcal{G}$ is also upper triangular. From the definition of the matrix $A$, \[ (\mathcal{G}^{-1} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \]
and the $c_{A,n}$ are defined by the R-transform \[ c_{A,1} = \lim_{N \to \infty} \frac{1}{N} \text{tr}(A) = 0 \]
\[ c_{A,2} = \lim_{N \to \infty} \frac{1}{N} \text{tr}(A^2) = c_{A,1}^2 \]
\[ = \lim_{N \to \infty} \left( \frac{1}{N} \text{tr}((\lambda I - J)^{-2}) - 1 \right) \]
\[ = \frac{\chi^2 R'}{1 - \chi^2 R'} \]
where in \[ (37) \] we use $G'(\lambda) = -\lim_{N \to \infty} \frac{1}{N} \text{tr}((\lambda I - J)^{-2})$ and re-express the result in terms of the R-transform of the limiting spectral distribution of $J$.

Finally, the zero-mean Gaussian process $\{\phi(t)\}$ has a covariance matrix given by

\[ C_\phi = \sum_{n=1}^{\infty} c_{A,n} \sum_{k=0}^{n-2} \mathcal{G}^k \bar{\mathcal{G}} (\mathcal{G}^\top)^{n-2-k} \] (39)

where

\[ \bar{\mathcal{G}}(t,s) = \mathbb{E}[\gamma(t)\gamma(s)]. \] (40)

and the $c_{A,n}$ are defined by the R-transform \[ \mathcal{G} = 0. \]

It may seem that we have not gained much from this approach, because interactions between different dynamical variables are now replaced by couplings of single variables over time. In principle, all matrix elements of $\mathcal{G}$, $\mathcal{G}$ and $\mathcal{C}$ could be (approximately) computed sequentially in time, e.g. by lengthy Monte Carlo simulations of the process $\phi(t)$ \[ \mathcal{C} \] on the other hand, the memory terms preclude an analytical treatment in general. Surprisingly, as we shall show in Appendix \[ A \] the memory terms vanish, i.e. we get

\[ \mathcal{G} = 0. \]

The memory-free property \[ (11) \] implies that

\[ \gamma(t) = \phi(t) \]

for $t = 1, 2, \ldots, T$ are Gaussian random variables with a much simplified covariance structure that is given by

\[ \mathcal{C} = c_{A,2} \mathcal{G} \]

\[ = \frac{\chi^2 R'}{1 - \chi^2 R'} \bar{\mathcal{G}}. \] (44)
Explicit results can be computed by the recursion (see Appendix A41)
\[ C(t, s) = \frac{g(C(t - 1, s - 1))}{1/R' - \chi^2} \] (45a)
\[ C(t, t) = (1 - \chi)R' \] (45b)
where we have defined
\[ g(x) \equiv E[\Theta(\gamma_1)\Theta(\gamma_2)] - \chi^2 x \] (46)
such that the expectation is taken over the random variables \( \gamma_1 \) and \( \gamma_2 \) which are jointly Gaussian with zero mean and equal variance \((1 - \chi)R'\) and covariance
\[ x = E[\gamma_1 \gamma_2]. \] (47)
To initialize the recursion (45a), we note that by (42) we have \( C(t, s) = E[\gamma(t)\gamma(s)] \) and we can include the initialization \( \gamma(0) \) in the definition. It is easy to see that the random variable \( \gamma(0) \) is independent from \( \gamma(t) \) for \( t > 0 \). Hence, we get
\[ C(t, 0) = 0, \quad t > 0. \] (48)

VI. CONVERGENCE OF THE SINGLE-VARIABLE DYNAMICS

The convergence properties of the dynamics can now be obtained from the covariance using the equality
\[ \Delta(t, s) = C(t, t) + C(s, s) - 2C(t, s) \] (49)
\[ = 2(1 - \chi)R' - 2C(t, s). \] (50)
From these results, we are able to show that
\[ \Delta(t + 1, s + 1) < \Delta(t, s), \quad \forall t, s. \] (51)
This means that with increasing time, the distance between pairs of iterates separated by a fixed amount of time strictly decreases. We thus get
\[ \lim_{t, s \to \infty} \Delta(t, s) = 0 \] (52)
and the dynamics is convergent. Let us denote the convergence rate of \( \Delta(t, s) \) by
\[ \mu_{mf} \equiv \lim_{t \to \infty} \frac{\Delta(t + 1, s + 1)}{\Delta(t, s)} \] (53)
where for convenience we introduce the abbreviation “mf” in the notation to emphasize the thermodynamic memory-free property (41). We obtain
\[ \mu_{mf} = 1 - \frac{1 - \eta R'}{1 - \chi^2 R'}. \] (54)
Moreover, it turns out that \( \ln \mu_{mf} \) gives the exponential decay rate:
\[ \lim_{t \to \infty} \frac{1}{t} \ln \Delta(t, \infty) = \ln \mu_{mf} \] (55)
where \( \Delta(t, \infty) \equiv \lim_{s \to \infty} \Delta(t, s) \). Notice that when the model parameters approach the AT-line, i.e. \( \eta R' = 1 \), the convergence becomes arbitrarily slow. The derivations of (51), (54) and (55) are located at Appendix B5.

VII. ALGORITHMIC CONSIDERATIONS

So far we have assumed that the limiting spectral distribution of \( J \) is analytically known beforehand for computing the quantities \( R \) and \( R' \). In the sequel, we present a practical approach that bypasses this need.

Specifically, we first compute the spectral decomposition
\[ J = O^\top D O. \] (56)
The Green function and its derivative are then replaced by their finite-size approximations as
\[ G(z) = \frac{1}{N} \text{tr}((zI - D)^{-1}) \] (57)
\[ G'(z) = -\frac{1}{N} \text{tr}((zI - D)^{-2}). \] (58)
The necessary R-transform and its derivative are then obtained by solving the simple fixed-point equations
\[ \lambda = R + \frac{1}{\chi} \] (59a)
\[ R = \lambda - \frac{1}{G(\lambda)} \] (59b)
\[ R' = \frac{1}{G(\lambda)^2} + \frac{1}{G'(\lambda)} \] (59c)
\[ \chi = E_u[\Theta'(\sqrt{(1 - G(\lambda))R' u})]. \] (59d)
The matrix \( A \) is then computed as
\[ A = \frac{1}{\chi} O^\top (\lambda I - D)^{-1} O - I. \] (60)
These steps have to be performed before the main iterations and have \( O(N^3) \), i.e. cubic computational complexities. On the other hand, the iterations are just based on matrix vector multiplications and evaluations of scalar nonlinear functions. These have only quadratic computational complexity per iteration.

VIII. COMPARISON WITH A PREVIOUS ALGORITHM IN THE CASES OF SK AND HOPFIELD MODELS

In a previous paper [1], we have introduced the SSM iterative algorithm for solving TAP equations for models with general rotation invariant coupling matrices. This algorithm and its corresponding DF analysis is based on the coefficients of power series expansions related to R-transforms of the random matrix ensemble. This practically limits the applicability of the method to cases where such coefficients are known analytically. Nevertheless, both the algorithm and its analysis become simple for the SK and Hopfield models. Hence, it is interesting to derive the corresponding convergence rates of SSM algorithms for these models which is what we show in the sequel.
As for the SK model we first note that the necessary R-transform quantities are given by

\[ R = \beta^2 \chi \]  
\[ R' = \beta^2 \]  

where \( \chi \) is the solution of (14c). The SSM dynamics for the SK model is nothing but Bolthausen’s celebrated convergent dynamics [4]:

\[ m(t) = T_h(\rho(t - 1)) \]  
\[ \rho(t) = J m(t) - R m(t - 1). \]  

Here, \( m(0) = 0 \) and \( \rho(0) = \sqrt{(1 - \chi)} R^T u \) where \( u \) is a vector of independent normal Gaussian random variables.

Second, for the Hopfield model, the R-transform and its derivative are (see (14))

\[ R = \frac{\beta^2 \alpha_1 \chi}{1 - \beta \alpha_1 \chi} \]  
\[ R' = \frac{\beta^2 \alpha_1}{(1 - \beta \alpha_1 \chi)^2} \]

where \( \chi \) is the solution of (14c). In the spirit of Bolthausen’s dynamics [62], we introduce a SSM construction for the Hopfield model as

\[ m(t) = T_h(\rho(t - 1)) \]  
\[ \phi(t) = J m(t) - R m(t - 1) \]  
\[ \rho(t) = (1 + \frac{R}{\beta}) \phi(t) - \frac{R}{\beta} \rho(t - 1) \]

where \( m(0) = 0 \) and \( \rho(0) = \sqrt{(1 - \chi)} R^T u \).

Note that the variable \( \gamma(t) \) in the dynamics [22] has the same fixed points as the variables \( \rho(t) \) in the dynamics (62) and (64) for the SK and Hopfield models, respectively. In order to analyze the convergence properties of the SSM algorithms (62) and (64) in the thermodynamic limit we define

\[ \Delta_{ssm} = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\|\rho(t) - \rho(s)\|^2]. \]

For both the SK and Hopfield models, we have

\[ \Delta_{ssm}(t + 1, s + 1) < \Delta_{ssm}(t, s), \quad \forall t, s. \]

Thus, \( \Delta_{ssm}(t, s) \) converges to 0 as \( t, s \to \infty \). In particular, the rate of convergence is obtained as

\[ \mu_{ssm} \approx \lim_{t, s \to \infty} \frac{\Delta_{ssm}(t + 1, s + 1)}{\Delta_{ssm}(t, s)} = \eta R'. \]

The derivations of (60) and (67) are given in Appendix C.

From a computational point of view, for the cases of the SK model and the Hopfield model, the SSM algorithms (62) and (64) are more convenient than our new algorithm (22), because the latter requires the computation of a matrix inverse [23] (before the iteration starts). Nevertheless, it is interesting to note from (64) that

\[ \mu_{mf} - \mu_{ssm} = \chi^2 R'(\eta R' - 1) < 0. \]

Thus, in the thermodynamic limit the SSM algorithms for SK and Hopfield models have both a slower convergence compare to the algorithm (22).

FIG. 1. Discrepancy between theory and simulations for the covariance in case of a 2-layer Hopfield model with \( N/N_1 = 1 \) and \( N/N_2 = 2, N = 10^4 \) and \( \beta = 0.25 \).

IX. SIMULATION RESULTS

In this section, we compare our analytical results with simulations of the algorithm for large random matrices. We note from (49) that the dynamics (22) in the thermodynamic limit is described in terms of the limiting covariance expression

\[ C(t, s) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\gamma(t)^T \gamma(s)]. \]

whose explicit analytical expression is given in (10).

In Figure 1 we illustrate how well \( C(t, s) \) predicts \( \frac{1}{N} \mathbb{E}[\gamma(t)^T \gamma(s)] \) for a 2-layer Hopfield model with a single realization of the dynamics. Note that we assume the large-system limit \( N \to \infty \) is taken before the long-time limit \( t \to \infty \). Thus, we get excellent agreement between theoretical predictions and simulations on single instances for finite-time properties of large systems. The discrepancy between theory and simulations for large times increases as the model parameters approach the AT line. This can be seen in Figure 2 close to the AT line (i.e. \( \beta = 2.29 \)) where (51) fails to predict the exponential decay for a single-realization of dynamics. In fact the dynamics has started to lose its self-averaging properties in this regime, i.e. the dynamics for different realisations of the random matrix and random initial conditions yield remarkably different asymptotic convergence behavior, see also in Figure 3. In Figure 3 and Figure 4, we compute the necessary quantities \( R \) and \( R' \) by the algorithmic approach in Section VII.

Figure 3 refers to the SK and (1-layer) Hopfield models where we compare the convergence rates of the dynamics (22) with the SSM dynamics for a single-realization. The simulation results confirm the thermodynamic bound (65). However, as we illustrate in Figure 4 (for the Hopfield model) that when the model parameters approach
FIG. 2. Asymptotics of the algorithm for 2-layer Hopfield model with \( N/N_1 = 1 \) and \( N/N_2 = 2 \), \( N = 10^4 \) and \( h = 1 \). The inverse temperature \( \beta = 0.35 \) gives the AT line. The flat lines around \( 10^{-30} \) are the consequence of the machine precision of the computer which was used.

FIG. 3. Comparison of the convergence rates with SSM algorithms for the SK and Hopfield models. \( N = 10^4 \).

the AT line, after few iteration steps the dynamics (22) shows strong fluctuations with respect to different realizations of while the SSM dynamics shows negligible fluctuations.

So far we have shown examples of the performance of the algorithm on matrices drawn at random from spherical invariant distributions. On the other hand, all necessary parameters (R-transform and its derivative) of the algorithm can be computed from a given matrix without explicit knowledge of the distribution. The same parameters appear in the analytical results of the theory and depend only on the (limiting) spectrum of the matrix. What happens if we run the algorithm on a matrix which is not a typical realization from a spherical invariant random matrix ensemble? To get a first impression on the robustness of our results we compare the theoretical results (obtained from the spectrum) with simulations based on a model with randomly-signed Hadamard matrices. To understand this model we first consider the (spherical symmetric) random orthogonal model discussed by Parisi and Potters [19] (i.e. the 1-layer random-orthogonal model with \( \rho_0 = 1 \)). The coupling matrix can be written in the form

\[
J = \beta \tilde{O}^\top D_\rho \tilde{O}
\]

(70)

where \( \tilde{O} \) is a Haar matrix and \( D_\rho = \text{diag}(d_1, \cdots, d_N) \) has random binary elements \( d_i = \pm 1 \) with \( |\{d_i = 1\}| = \rho N \). Motivated by a recent study [35] in random matrix theory we now introduce a non-rotation invariant coupling matrix ensemble — called randomly-signed Hadamard by replacing the random Haar matrix by a different random orthogonal matrix \( \tilde{O} \):

\[
J = \beta \hat{O}^\top D_\rho \hat{O} \quad \text{with} \quad \hat{O} = \frac{1}{\sqrt{N}} H_N \tilde{Z}.
\]

(71)

Here, \( \tilde{Z} \) is an \( N \times N \) diagonal matrix whose diagonal entries are independent and composed of binary \( \mp 1 \) random variables with equal probabilities and \( H_N \) is the \( N \times N \) Hadamard matrix which is deterministically constructed [36] from the recursion

\[
H_{2k} = \begin{bmatrix} H_{2k-1} & H_{2k-1} \\ H_{2k-1} & -H_{2k-1} \end{bmatrix}, \quad k \geq 1
\]

(72)

with \( H_1 = 1 \). Note that \( \hat{O} \) is an orthogonal matrix with the binary entries \( \hat{O}_{ij} = \mp \frac{1}{\sqrt{N}} \). Both coupling matrices have the same spectra and thus the same R-transform which can be easily obtained as

\[
R(\omega) = \frac{(\beta \omega - \rho) + \sqrt{(\beta \omega - \rho)^2 + 4 \rho^2 \beta \omega}}{2 \rho \omega} - \beta.
\]

(73)
FIG. 5. Discrepancy between theory and simulations for the covariance in case of randomly-signed Hadamard model with $N = 2^{13}$, $h = 2$, $\alpha = 1/2$ and $\beta = 4$.

In Figure 5 and Figure 6 we illustrate how well our theoretical results describe the dynamics when the coupling matrix $J$ is modeled by a randomly-signed Hadamard ensemble. The simulation results suggest that our theoretical results can be valid for larger classes of coupling matrices than the rotation invariant matrix ensembles.

In this paper we have introduced an iterative algorithm for solving the TAP equations of Ising models with arbitrary rotation invariant coupling matrices. We have shown that the algorithm is convergent in the thermodynamic limit when the AT-line criteria is fulfilled. We have also obtained a compact analytical expression for the rate of convergence. The analytical results are supported by numerical simulations of large systems on single instances of random matrices. Preliminary simulations for random matrices which are not generated from rotation invariant ensembles but show similar weak dependencies between matrix elements indicate that our theory may extend to larger classes of matrices. Preliminary analytical calculations suggest that some of our results on convergence can be derived under the weaker property of \emph{asymptotic freeness} [30] (which is fulfilled by rotation invariant matrix ensembles). Such extensions of our approach would also be important for applications to TAP-style mean field equations for statistical inference in probabilistic models. A first step in this direction would be to consider Ising models with non-constant or random external fields. Such problem setting plays a role when the probability distribution of the spins is conditioned on observed spin variables. This is a relevant problem in machine learning e.g. in the training of restricted Boltzmann machines [5]. More complex extensions would be to the important case of models with continuous random variables. Of course, the latter case would require new and nontrivial generalizations of our basic algorithm. These generalizations might be motivated e.g. by the so-called \emph{expectation propagation algorithms} [37, 38] which are frequently used in machine learning. A different direction of research is in the optimization of algorithms. It will be interesting to investigate which properties makes our new method converge faster than the previously defined SSM algorithms. On the other hand it will also be important to get a better understanding of finite size fluctuations, because these will influence the robustness of convergence, when the dynamics is close to the AT instability.

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Appendix A: Vanishing of memory terms

Using the definition (33) and differentiating (31b) with respect to $\phi(\tau)$ for $\tau < t$, we obtain

$$\mathcal{G}(t, \tau) = \mathbb{E} \left[ f'(\gamma(t - 1)) \frac{\partial \gamma(t - 1)}{\partial \phi(\tau)} \right]$$

$$= \sum_{\tau < s < t} \mathcal{G}(t - 1, s) \mathbb{E} \left[ f'(\gamma(t - 1)) \frac{\partial \gamma(t - 1)}{\partial \phi(\tau)} \right]$$

$$+ \mathbb{E} [f'(\gamma(t - 1))] \delta_{t-1, \tau} \quad (A1)$$

for $t = 1, 2, \ldots, T$. From equations (A1) and the definitions of $\mathcal{G}$ and $C$, one can show that these quantities...
are uniquely defined recursively in time as expectations over the stochastic process $\phi$ and the random initial conditions. Hence, if we can show that the trivial solution $G = \tilde{G} = 0$ fulfills these equations, we will be done. Obviously, this solution fulfills the relation between $G$ and $\tilde{G}$ eq. (52). On the other hand, if $\tilde{G} = 0$, we obtain $G = 0$ from (A1), provided we can show that

$$E\left[f'(\gamma(t-1))\right] = 0 \quad (A2)$$

is consistent with these assumptions. The proof of this fact and the resulting form of the covariance matrix (A5) is deferred to Appendix A1.

1. The covariance matrix (A5)

By using the definition of $\tilde{C}$ in (40) and using (31a), we obtain the recursion

$$\tilde{C}(t, s) = \frac{1}{\chi^2} E[\text{Th}(\gamma(t-1))\text{Th}(\gamma(s-1))] +$$

$$\left(1 - \frac{1}{\chi} E[\text{Th}'(\gamma(t-1))] - \frac{1}{\chi} E[\text{Th}'(\gamma(s-1))]\right) \times$$

$$C(t-1, s-1). \quad (A3)$$

Note, that this result is also true for $s = t = 1$, if we set

$$C(0, 0) = E[\gamma^2(0)] = (1 - \chi)R'. \quad (A4)$$

Recall that by the AT line condition (19), the equation (13c) has a unique solution for $\chi$. Then, for the variance terms $C(t, t)$, we can show by induction that

$$C(s, s) = (1 - \chi)R' \quad s > 0 \quad (A5)$$

is independent of time. Specifically, assuming that (A5) is true for time $s = t - 1$. Hence, we have

$$\frac{1}{\chi} E[\text{Th}'(\gamma(t-1))] = 1 \quad (A6)$$

which in turn is equivalent to (A2). We then get at the next time step

$$C(t, t) = \frac{\chi^2R'}{1 - \chi^2R'} \left[1 - \frac{\chi}{\chi^2} - C(t-1, t-1)\right] \quad (A7)$$

$$= (1 - \chi)R'. \quad (A8)$$

Moreover, plugging (A7) into (A3) we have

$$\tilde{C}(t, s) = \frac{1}{\chi^2} E[\text{Th}(\gamma(t-1))\text{Th}(\gamma(s-1))] - C(t-1, s-1) \quad (A9)$$

which completes the derivation of (A5).

Appendix B: Derivations of (51), (53) and (55)

Remark 1 (A useful remark) Let $\gamma_1$ and $\gamma_2$ be Gaussian random variables and be identically distributed. Furthermore, let $C_{12}$ stand for the covariance between $\gamma_1$ and $\gamma_2$ such that $C_{12} > 0$. Moreover, let the function $f$ have derivatives of all orders in $R$. Then, the covariance between the random variables $f(\gamma_1)$ and $f(\gamma_2)$ is positive, too.

The derivation of Remark 1 is located in Appendix B3.

1. Derivation of (51)

Using the representation of the Gaussian density in terms of the characteristic function (see the argument of (B27)) one can show that

$$g'(x) = E[\text{Th}'(\gamma_1)\text{Th}'(\gamma_2)] - \chi^2 \quad (B1)$$

where $g'$ denotes the derivative of the function $g$ in (40). Note that $\chi = E[\text{Th}'(\gamma_1)]$. Thus, $g'(x)$ equals the covariance between the random variables $\text{Th}'(\gamma_1)$ and $\text{Th}'(\gamma_2)$. Thereby, from the Remark 1 it turns out that

$$g'(x) > 0, \quad x > 0. \quad (B2)$$

To prove (51) it is sufficient to show

$$C(t + 1, s + 1) > C(t, s), \quad \forall t, s. \quad (B3)$$

Since $g(x)$ is strictly increasing on the positive axis, assuming (B3) holds and $C(t, s) > 0$ we have

$$C(t + 2, s + 2) = \frac{R'}{1 - \chi^2R'} g(C(t + 1, s + 1)) \quad (B4)$$

$$> \frac{R'}{1 - \chi^2R'} g(C(t, s)) \quad (B5)$$

$$= C(t + 1, s + 1). \quad (B6)$$

Thus, we complete the derivation by showing the inequality $C(t + 1, s + 1) > C(t, 0), \forall t$. This follows from the fact that $g(0) > 0$ and (18), i.e. $C(t, 0) = 0$.

2. Derivations of (51)

From (15) and (50) we can write $\Delta(t + 1, s + 1)$ in the form

$$\Delta(t + 1, s + 1) = d(\Delta(t, s)) \quad (B7)$$

for an appropriately defined function $d$. In particular, we note from (50) that

$$\frac{\partial \Delta(t + 1, s + 1)}{\partial \Delta(t, s)} = \frac{\partial C(t + 1, s + 1)}{\partial C(t, s)} \quad (B8)$$

$$= \frac{R'}{1 - \chi^2R'} g'(C(t, s)) \quad (B9)$$

where the function $g'$ is as defined in (B1). Since $g'$ has derivative in all order (see Remark 2), for sufficiently large $t$ and $s$ we can expand $\Delta(t + 1, s + 1)$ around $0$ as

$$\Delta(t + 1, s + 1) = d'(0)\Delta(t, s) + O(\Delta(t, s)^2) \quad (B10)$$
with noting that \( d(0) = 0 \). Hence, we get
\[
\lim_{t,s \to \infty} \frac{\Delta(t+1,s+1)}{\Delta(t,s)} = \lim_{t,s \to \infty} \left( d'(0) + O(\Delta(t,s)) \right)
\]
\[
d'(0) = \frac{R'}{1 - \chi^2 R'} g'((1 - \chi) R')
\]
\[
= \frac{R'}{1 - \chi^2 R'} (E[\Delta(\gamma_1)])^2 - \chi^2
\]
\[
= 1 - \frac{1 - \eta R'}{1 - \chi^2 R'}
\]
This completes the derivation.

3. Derivations of \( \eta(x) \)

For short let
\[
\Delta(t) \doteq \lim_{s \to \infty} \Delta(t,s) = 2(1 - \chi) R' - 2C(t).
\]
where we have defined
\[
C(t) \doteq \lim_{s \to \infty} C(t,s).
\]
We are interested in the rate of the asymptotic decay
\[
\Delta(t) \simeq e^{\kappa t} \quad t \to \infty.
\]
The rate \( \kappa \) can be computed as
\[
\kappa = \lim_{t \to \infty} \ln \frac{\partial \Delta(t+1)}{\partial \Delta(t)}.
\]
The derivative is obtained from the recursion of \( C(t) \)
\[
C(t+1) = \frac{g(C(t))}{1/R' - \chi^2}
\]
where the function \( g \) is defined in \( \ref{gdef} \).
\[
\eta(x) = g'(x) + \chi^2.
\]
Following the steps of \( \ref{B1} \) we get \( C(t+1) > C(t), \forall t \). Moreover, by using Remark \( \ref{B1} \) we have that \( \eta(x) \) is a strictly increasing function on the positive axis. Thus, we have
\[
\eta(C(t)) < \eta(C(t+1)) < \eta, \quad \forall t.
\]
Hence, we have
\[
\lim_{t \to \infty} \eta(C(t)) = \eta
\]
which completes the derivation.

4. Derivation of Remark \ref{B1}

Remark \ref{B1} evidently follows from the following general result.

**Remark 2** Let \( \gamma_1 \) and \( \gamma_2 \) be Gaussian random variables with means \( \mu_1 \) and \( \mu_2 \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively. Moreover, let \( C_{12} \) stand for the covariance between \( \gamma_1 \) and \( \gamma_2 \). Also, let the function \( f \) have derivatives of all orders in \( \mathbb{R} \). Then, the covariance between the random variables \( f(\gamma_1) \) and \( f(\gamma_2) \) is given by
\[
\sum_{n=1}^{\infty} C_{12}^n \left[ \frac{d^n f(\gamma_1)}{d\gamma_1^n} \right] E \left[ \frac{d^n f(\gamma_2)}{d\gamma_2^n} \right].
\]
The derivation of Remark \ref{B2} is based on the representation of the Gaussian density in terms of the characteristic function:
\[
E[f(\gamma_1)f(\gamma_2)] = \frac{1}{(2\pi)^2} \int d\gamma_1 d\gamma_2 dk_1 dk_2 f(\gamma_1)f(\gamma_2) \times e^{ik_1(\mu_1 - \gamma_1) + ik_2(\mu_2 - \gamma_2) - \frac{1}{2}(\sigma_1^2 k_1^2 + \sigma_2^2 k_2^2)} e^{-C_{12} k_1 k_2}
\]
\[
= E[f(\gamma_1)]E[f(\gamma_2)] + \sum_{n=1}^{\infty} C_{12}^n \left[ \frac{1}{(2\pi)^2} \int d\gamma_1 d\gamma_2 dk_1 dk_2 \right] \times (-k_1 k_2)^n f(\gamma_1)f(\gamma_2) e^{ik_1(\mu_1 - \gamma_1) + ik_2(\mu_2 - \gamma_2) - \frac{1}{2}(\sigma_1^2 k_1^2 + \sigma_2^2 k_2^2)}
\]
\[
= E[f(\gamma_1)]E[f(\gamma_2)] + \sum_{n=1}^{\infty} C_{12}^n \left[ E\left[ \frac{d^n f(\gamma_1)}{d\gamma_1^n} \right] \right] E\left[ \frac{d^n f(\gamma_2)}{d\gamma_2^n} \right]
\]
where in \( \ref{B24} \) we use the Taylor expansion representation of \( e^{-C_{12} k_1 k_2} \) and in \( \ref{B29} \) we represent \( (-k_1 k_2)^n \) by means of the derivatives \( \frac{d^n f(\gamma_1)}{d\gamma_1^n} \).

Appendix C: Derivations of \( \ref{B24} \) and \( \ref{B29} \)

Let us introduce
\[
C_{\rho}(t,s) \doteq \lim_{N \to \infty} \frac{1}{N} E[\rho(t)^T \rho(s)]
\]
where the expectation is taken over the random matrix ensemble of $J$ and the random initialization $\rho(0)$. We will next show from the results of \cite{3} that for both the SK and Hopfield models we have

$$
C_\rho(t, s) = R'E[Th(\rho(t - 1))Th(\rho(s - 1))] \tag{C2}
$$

where the expectation is taken over the random variables $\rho(t)$ and $\rho(s)$ which are jointly zero-mean Gaussian with equal variances $(1 - \chi)R'$ and covariance $C_\rho(t, s)$. Also we note from (C2) that

$$
\frac{\partial C_\rho(t + 1, s + 1)}{\partial C_\rho(t, s)} = R'E\left[Th'(\rho(t))Th'(\rho(s))\right]. \tag{C3}
$$

Then, by following the steps of Appendix D\textsuperscript{1} and D\textsuperscript{2} one can obtain \cite{66} and \cite{67}, respectively.

To read off the result (C2) from \cite{3} for the SK model, we first write the SSM updates \cite{3} Eq. (35)–(38) for the SK model as

$$
m(t) = Th(\rho(t - 1)) \tag{C4a}
$$

where $m(0) = 0$ and we have defined

$$
\chi(t) = 1 - \lim_{N \to \infty} \frac{1}{N} E[m(t)'m(t)]. \tag{C5}
$$

In this case, the two-time covariance $C_\rho(t, s)$ is read off from \cite{3} Eq. (60) as

$$
C_\rho(t, s) = \beta^2E[Th(\rho(t - 1))Th(\rho(s - 1))] \tag{C6}
$$

where the expectation is taken over the random variables $\rho(t)$ and $\rho(s)$ which are jointly zero-mean Gaussian with covariance $C_\rho(t, s)$. Moreover, we have

$$
\chi(t) = E\left[Th'(\sqrt{C_\rho(t - 1, t - 1)}u)^2\right]. \tag{C7}
$$

From the initialization $\rho(0) = \sqrt{(1 - \chi)\beta^2}u$ we have $C_\rho(0, 0) = (1 - \chi)\beta^2$. Then, it follows inductively from (C6) and (C7) that $C_\rho(t, t) = (1 - \chi)\beta^2$ so that we have

$$
\chi(t) = \chi, \forall t. \tag{C8}
$$

Using this result in (C4) and (C6) leads to (D2) and (C2), respectively.

To read off the result (C2) from \cite{3} for the Hopfield model, we follow the definition of the SSM construction \cite{3} Section 4 and define the following dynamics

$$
m(t) = Th(\rho(t - 1)) \tag{C9a}
$$

$$
\phi(t) = Jm(t) - \frac{\chi(t)}{\chi(t - 1)} R(\chi(t - 1))m(t - 1) \tag{C9b}
$$

$$
\rho(t) = [1 + \frac{1}{\beta}R(\chi(t))]\phi(t) - \frac{1}{\beta}R(\chi(t))\rho(t - 1) \tag{C9c}
$$

where $m(0) = 0$, $\chi(t)$ is as in (C5). The R-transform for the Hopfield model is given by $R(\omega) = \frac{\alpha_1\beta^2\omega}{1 - \alpha_1\beta^2\omega}$. Basically, $\phi(t)$ and $\rho(t)$ play the roles of $\phi(t)$ and $\sum_{\tau=0}^{t} A(t, \tau)\phi(\tau)$ in (3), respectively. Thus, we have from \cite{3} Eq. (60)]

$$
C_\rho(t, s) = \frac{R(\chi(t))R(\chi(s))}{\beta^2\alpha_1\chi(t)\chi(s)} E[Th(\rho(t - 1))Th(\rho(s - 1))] \tag{C10}
$$

where $\chi(t)$ reads as in (C7). Again, from the initialization $\rho(0) = \sqrt{(1 - \chi)R'\omega}$ we have $C_\rho(0, 0) = (1 - \chi)R'$. It then follows inductively that $C_\rho(t, t) = (1 - \chi)R'$ so that we have

$$
\chi(t) = \chi, \forall t. \tag{C11}
$$

Using this result in (C4) and (C6) lead to (D4) and (C2), respectively.

**Appendix D: Derivation of (12)**

We first note the scaling property of the R-transform \cite{39}

$$
R_{cJ}(\omega) = cR_{J}(\omega) \tag{D1}
$$

where $R_{cJ}$ is the R-transform of the limiting spectral distribution of the matrix given in subscript. Second, by an appropriate reformulation of the result \cite{40} Corollary 1.14 — known as free compression of random matrices — we write

$$
R_{P_m^NY(P_m^N)'}(\omega) = R_Y(\alpha\omega). \tag{D2}
$$

Then, by using (D1) and (D2) we get

$$
R_{\frac{1}{\sqrt{\alpha}}J}(\omega) = \frac{1}{\sqrt{\alpha}}R_Y\left(\sqrt{\alpha}\omega\right) \tag{D3}
$$

$$
= \frac{1}{\sqrt{\alpha}} \sum_{n=1}^\infty c_n \alpha^{\frac{n-1}{2}} \omega^{n-1}. \tag{D4}
$$

Here, $c_n$ denotes the $n$th free cumulant of the limiting spectral distribution of $Y$ such that $c_1$ and $c_2$ are the mean and variance of the distribution, respectively, i.e. $c_1 = 0$ and $c_2 = \sigma_Y^2$. Thus, we get

$$
\lim_{\alpha \to 0} R_{\frac{1}{\sqrt{\alpha}}J}(\omega) = \sigma_Y^2 \omega \tag{D5}
$$

which is nothing but the R-transform of a Wigner semicircle distribution \cite{41}. This completes the derivation.

**Appendix E: Additivity of $\lim_{N \to \infty} \frac{1}{N} \text{tr}(J^2)$**

Let $J$ be modeled as in (6) and let all $N_m \times N_{m-1}$ matrices $X_m$ in (5) be asymptotically free of each others. Moreover, for convenience we assume the normalization

$$
\lim_{N \to \infty} \frac{1}{N} \text{tr}(X_m'X_m) = 1, \quad \forall m. \tag{E1}
$$
Then, we have

$$\sigma_2^2 = \beta^2 \sum_{m \leq M} \alpha_m \sigma_\omega^2 X_m^\top X_m$$ \hspace{1cm} (E2)

where $\alpha_m \equiv \frac{N_0}{N_m}$ and $\sigma_\omega^2$ denotes the variance of the limiting spectral distribution of the matrix given in subscript. Here, for the definition of asymptotic freeness of random matrices we refer the reader to [31, 32]. Nevertheless, we note that for the multi-layer Hopfield and multi-layer random orthogonal models the aforementioned asymptotic freeness condition holds (almost surely) [41].

The result (E2) follows from the properties of the (Voiculescu) S-transform [42]. Specifically, let $F$ be a probability distribution on real line with a non-zero mean and $R$ be its R-transform. The S-transform of $F$ is defined by the functional inverse of $\tilde{R}(\omega) \equiv \omega R(\omega)$ as [43]

$$S(s) \doteq \frac{1}{s} \tilde{R}^{-1}(s).$$ \hspace{1cm} (E3)

In particular, for $c_1$ and $c_2$ denoting the mean and variance of $F$, respectively, we have [43, Page 207]

$$S(0) = \frac{1}{c_1} \quad \text{and} \quad S'(0) = \frac{c_2}{c_1^2}.$$ \hspace{1cm} (E4)

Moreover, for asymptotically free matrices, say $A = A^\top$ and $B = B^\top$, we have $S_{AB}(s) = S_A(s)S_B(s)$ [42] where $S_{\cdot\cdot}$ denotes the S-transform of the limiting spectral distribution of the matrix given in subscript. Combining this property of the S-transform with [43, Lemma 4.3] one can show that

$$S_{X^\top X}(s) = \prod_{m \leq M} S_{X_m^\top X_m}(\alpha_m s).$$ \hspace{1cm} (E5)

Note that $\sigma_2^2 = \beta^2 \sigma_\omega^2 X^\top X$. Then, (E2) follows easily by invoking the properties of the S-transform [42] for (E5).

Appendix F: Derivation of (18)

From [24, Theorem 2] we write the S-transform (see the definition in [53]) of the limiting spectral distribution of $X^\top X$ as

$$S_{X^\top X}(s) = \frac{1}{\alpha_1 s + 1} \frac{1}{\alpha_2 s + 1}$$ \hspace{1cm} (F1)

Then, from (E3) we get the R-transform of the limiting spectral distribution of $X^\top X$

$$R_{X^\top X}(\omega) = \frac{1}{\omega - \alpha_1 - \alpha_2 + \sqrt{(\frac{1}{\omega} - \alpha_1 - \alpha_2)^2 - 4 \alpha_1 \alpha_2}}$$ \hspace{1cm} (F2)

Moreover, with the scaling property of the R-transform (see (D1)) we have

$$R_{\beta X^\top X}(\omega) = \beta R_{X^\top X}(\beta \omega) - \beta R_{X^\top X}(0)$$ \hspace{1cm} (F3)

$$= \beta R_{X^\top X}(\beta \omega) - \beta.$$ \hspace{1cm} (F4)

From (E2) we have two solutions of $R_{\beta X^\top X}(\omega)$. Only one fulfills the property $R_{\beta X^\top X}(\omega) > 0$. This completes the derivation.

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