A CONSTRUCTION OF HORIKAWA SURFACE VIA Q-GORENSTEIN SMOOTHINGS

YONGNAM LEE AND JONGIL PARK

Abstract. In this article we prove that Fintushel-Stern’s construction of Horikawa surface, which is obtained from an elliptic surface via a rational blow-down surgery in smooth category, can be performed in complex category. The main technique involved is Q-Gorenstein smoothings.

1. Introduction

As an application of a rational blow-down surgery on 4-manifolds, R. Fintushel and R. Stern showed that Horikawa surface $H(n)$ can be obtained from an elliptic surface $E(n)$ via a rational blow-down surgery in smooth category [3]. Note that Horikawa surface $H(n)$ is defined as a double cover of a Hirzebruch surface $F_{n-3}$ branched over $|6C_0 + (4n - 8)f|$, where $C_0$ is a negative section and $f$ is a fiber of $F_{n-3}$.

In this article we show that a rational blow-down surgery to obtain Horikawa surface can be performed in fact in complex category. That is, we reinterpret algebraically Fintushel-Stern’s topological construction [3] of Horikawa surface $H(n)$ to give a complex structure on it. The main technique we use in this paper is Q-Gorenstein smoothings. Note that Q-Gorenstein smoothing theory developed in deformation theory in last thirty years is a very powerful tool to construct a non-singular surface of general type. The basic scheme is the following: Suppose that a projective surface contains several disjoint chains of curves representing the resolution graphs of special quotient singularities. Then, by contracting these chains of curves, we get a singular surface $X$ with special quotient singularities. And then we investigate the existence of a Q-Gorenstein smoothing of $X$. It is known that the cohomology $H^2(T_X^0)$ contains the obstruction space of a Q-Gorenstein smoothing of $X$. That is, if $H^2(T_X^0) = 0$, then there is a Q-Gorenstein smoothing of $X$. For example, we recently constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by proving the cohomology $H^2(T_X^0) = 0$ [3]. But, in general, the cohomology $H^2(T_X^0)$ is not zero and it is a very difficult problem to determine whether there exists a Q-Gorenstein smoothing of $X$. In this article we also give a family of examples which admit Q-Gorenstein smoothings even though the cohomology $H^2(T_X^0)$ does not vanish. Our main technique is a Q-Gorenstein smoothing theory with a cyclic group action. It is briefly reviewed and developed in Section 2.

The sketch of our construction whose details are given in Section 3 is as follows: We first construct a simply connected relatively minimal elliptic surface $E(n)$ ($n \geq 5$) with...
a special fiber, which contains two linear chains of configurations of $\mathbb{P}^1$'s

$$\frac{-n}{U_{n-3}} - \frac{-2}{U_{n-4}} - \frac{-2}{U_{n-5}} - \cdots - \frac{-2}{U_1}.$$

We construct this kind of an elliptic surface $E(n)$ explicitly by using a double cover of a blowing-up of Hirzebruch surface $\mathbb{F}_n$ branched over a special curve. The double cover of $\mathbb{F}_n$ has two rational double points $A_1$ and $A_{2n-9}$. Then its minimal resolution is an elliptic surface $E(n)$ which has an $I_{2n-6}$ as a special fiber. Now we contract these two linear chains of configurations of $\mathbb{P}^1$'s to produce a normal projective surface $X_n$ with two special quotient singularities, both singularities are of type $\frac{1}{(n-2)^2}(1,n-3)$. Finally, we apply $\mathbb{Q}$-Gorenstein smoothing theory with a cyclic group action developed in Section 2 for $X_n$ in order to get our main result which is following.

**Theorem 1.1.** The projective surface $X_n$ obtained by contracting two disjoint configurations $C_{n-2}$ from an elliptic surface $E(n)$ admits a $\mathbb{Q}$-Gorenstein smoothing of two quotient singularities all together, and a general fiber of the $\mathbb{Q}$-Gorenstein smoothing is Horikawa surface $H(n)$.

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2. $\mathbb{Q}$-Gorenstein smoothing

In this section we briefly review a theory of $\mathbb{Q}$-Gorenstein smoothing for projective surfaces with special quotient singularities, which is a key technical ingredient in our main construction.

**Definition.** Let $X$ be a normal projective surface with quotient singularities. Let $\mathcal{X} \to \Delta$ (or $\mathcal{X}/\Delta$) be a flat family of projective surfaces over a small disk $\Delta$. The one-parameter family of surfaces $\mathcal{X} \to \Delta$ is called a $\mathbb{Q}$-Gorenstein smoothing of $X$ if it satisfies the following three conditions;

(i) the general fiber $X_t$ is a smooth projective surface,
(ii) the central fiber $X_0$ is $X$,
(iii) the canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$-Cartier.

A $\mathbb{Q}$-Gorenstein smoothing for a germ of a quotient singularity $(X_0,0)$ is defined similarly. A quotient singularity which admits a $\mathbb{Q}$-Gorenstein smoothing is called a singularity of class $T$.

**Proposition 2.1** ([8] [12] [16]). Let $(X_0,0)$ be a germ of two dimensional quotient singularity. If $(X_0,0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then $(X_0,0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn^2}(1,dn-1)$ for some integers $a,n,d$ with $a$ and $n$ relatively prime.
Proposition 2.2 ([8] [12] [17]). (1) The singularities $-\mathfrak{b}_1 - \mathfrak{b}_2 - \cdots - \mathfrak{b}_r$ are of class $T$.

(2) If the singularity $-\mathfrak{b}_1 - \cdots - \mathfrak{b}_r$ is of class $T$, then so are

$$-\mathfrak{b}_1 - \cdots - \mathfrak{b}_r - \mathfrak{b}_i - \mathfrak{b}_j - \cdots - \mathfrak{b}_r$$

and

$$-\mathfrak{b}_1 - \mathfrak{b}_2 - \cdots - \mathfrak{b}_r - \mathfrak{b}_i - \mathfrak{b}_j - \cdots - \mathfrak{b}_r.$$

(3) Every singularity of class $T$ that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2).

Let $X$ be a normal projective surface with singularities of class $T$. Due to the result of Kollár and Shepherd-Barron [8], there is a $\mathbb{Q}$-Gorenstein smoothing locally for each singularity of class $T$ on $X$ (see Proposition 2.5). The natural question arises whether this local $\mathbb{Q}$-Gorenstein smoothing can be extended over the global surface $X$ or not. Roughly geometric interpretation is the following: Let $\bigcup \alpha V_\alpha$ be an open covering of $X$ such that each $V_\alpha$ has at most one singularity of class $T$. By the existence of a local $\mathbb{Q}$-Gorenstein smoothing, there is a $\mathbb{Q}$-Gorenstein smoothing $V_\alpha / \Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T^i_X = \text{Ext}^i_X(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if $X$ is a smooth surface, then $T^0_X$ is the usual holomorphic tangent sheaf $T^0_X$ and $T^1_X = T^2_X = 0$. By applying the standard result of deformations [10] [14] to a normal projective surface with quotient singularities, we get the following

Proposition 2.3 ([16], §4). Let $X$ be a normal projective surface with quotient singularities. Then

(1) The first order deformation space of $X$ is represented by the global $\text{Ext}$ 1-group $T^1_X = \text{Ext}^1_X(\Omega_X, \mathcal{O}_X)$.

(2) The obstruction lies in the global $\text{Ext}$ 2-group $T^2_X = \text{Ext}^2_X(\Omega_X, \mathcal{O}_X)$.

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves ([14], §3) to deformation theory of surfaces with quotient singularities so that $E_2^{p,q} = H^p(T^q_X) \Rightarrow T^{p+q}_X$, and by $H^j(T^0_X) = 0$ for $i, j \geq 1$, we also get

Proposition 2.4 ([12] [16]). Let $X$ be a normal projective surface with quotient singularities. Then

(1) We have the exact sequence

$$0 \to H^1(T^0_X) \to T^1_X \to \ker[H^0(T^0_X) \to H^2(T^0_X)] \to 0$$

where $H^1(T^0_X)$ represents the first order deformations of $X$ for which the singularities remain locally a product.

(2) If $H^2(T^0_X) = 0$, every local deformation of the singularities may be globalized.

The vanishing $H^2(T^0_X) = 0$ can be obtained via the vanishing of $H^2(T_V(- \log E))$, where $V$ is the minimal resolution of $X$ and $E$ is the reduced exceptional divisors. Note that every singularity of class $T$ has a local $\mathbb{Q}$-Gorenstein smoothing by Proposition 2.5 below.

Let $X$ be a normal projective surface with singularities of class $T$. Our concern is to understand $\mathbb{Q}$-Gorenstein smoothings in $T^1_X$, not the whole first order deformations. These special deformations can be constructed via local index one cover. Let $U \subset X$ be an analytic neighborhood with an index one cover $U'$. For the case of the field $\mathbb{C}$,
this index one cover is unique up to isomorphism. The first order deformations which associate $\mathbb{Q}$-Gorenstein smoothings can be realized as the invariant part of $T^1_U$. The sheaves $\tilde{T}_X^1$ are defined by the index one covering stack and by the étale sites $\tilde{\mathcal{X}}$. The first order deformation of a $\mathbb{Q}$-Gorenstein smoothing of singularities of class $T$ is expressed by the cohomology $H^0(\tilde{T}_X^1)$ [5, 7, 8]. By the help of the birational geometry in threefolds and their applications to deformations of surface singularities, the following proposition is obtained. Note that the cohomology $H^0(\tilde{T}_X^1)$ is given explicitly as follows.

**Proposition 2.5** ([8, 12]).

1. Let $a, d, n > 0$ be integers with $a, n$ relatively prime and consider a map $\pi : \mathcal{Y}/\mu_n \to \mathbb{C}^d$, where $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^d$ is the hypersurface of equation $uv - y^n = \sum_{k=0}^{d-1}t_k y^{kn}$; $t_0, \ldots, t_{d-1}$ are linear coordinates over $\mathbb{C}^d$, $\mu_n$ acts on $\mathcal{Y}$ by
   
   \[ \mu_n \ni \xi : (u, v, y, t_0, \ldots, t_{d-1}) \mapsto (\xi u, \xi^{-1} v, \xi^a y, t_0, \ldots, t_{d-1}) \]
   
   and $\pi$ is the factorization to the quotient of the projection $\mathcal{Y} \to \mathbb{C}^d$. Then $\pi$ is a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity of a germ $(X_0, 0)$ of type $\frac{1}{\text{deg}_{P, 1}}(1, dna - 1)$. Moreover every $\mathbb{Q}$-Gorenstein smoothing of $(X_0, 0)$ is isomorphic to the pull-back of $\pi$ for some germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$.

2. Let $X$ be a normal projective surface with singularities of class $T$. Then
   
   \[ H^0(\tilde{T}_X^1) = \sum_{p \in \text{singular points of } X} \mathbb{C}_{\sigma_p}^{d_p} \]
   
   where a singular point $p$ is of type $\frac{1}{\text{deg}_{P, p}}(1, d_p a - 1)$ with $(a, n) = 1$.

**Theorem 2.1** ([9]). Let $X$ be a normal projective surface with singularities of class $T$. Let $\pi : V \to X$ be the minimal resolution and let $E$ be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log E)) = 0$. Then $H^2(\tilde{T}_X^0) = 0$ and there is a $\mathbb{Q}$-Gorenstein smoothing of $X$.

As we see in Theorem 2.1 above, if $H^2(\tilde{T}_X^0) = 0$, then there is a $\mathbb{Q}$-Gorenstein smoothing of $X$. For example, we constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by proving the cohomology $H^2(\tilde{T}_X^0) = 0$ [9]. But, in general, the cohomology $H^2(\tilde{T}_X^0)$ is not zero and it is a very difficult problem to determine whether there exists a $\mathbb{Q}$-Gorenstein smoothing of $X$. Hence, in the case that $H^2(\tilde{T}_X^0) \neq 0$, we have to develop another technique in order to investigate the existence of $\mathbb{Q}$-Gorenstein smoothings. Even though we do not know whether such a technique exists in general, if $X$ is a normal projective surface with singularities of class $T$ which admits a cyclic group with some nice properties, then we are able to show that it admits a $\mathbb{Q}$-Gorenstein smoothing. Explicitly, we get the following theorem.

**Theorem 2.2.** Let $X$ be a normal projective surface with singularities of class $T$. Assume that a cyclic group $G$ acts on $X$ such that

1. $Y = X/G$ is a normal projective surface with singularities of $T$,
2. $p_g(Y) = q(Y) = 0$,
3. $Y$ has a $\mathbb{Q}$-Gorenstein smoothing,
4. the map $\sigma : X \to Y$ induced by a cyclic covering is flat, and the branch locus $D$ (resp. the ramification locus) of the map $\sigma : X \to Y$ is an irreducible nonsingular curve lying outside the singular locus of $Y$ (resp. of $X$), and
5. $H^1(Y, \mathcal{O}_Y(D)) = 0$. 


Then there exists a $\mathbb{Q}$-Gorenstein smoothing of $X$ that is compatible with a $\mathbb{Q}$-Gorenstein smoothing of $Y$. And the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing.

Proof. Let $\mathcal{Y} \to \Delta$ be a $\mathbb{Q}$-Gorenstein smoothing of $Y$, and let $Y_t$ be a general fiber of the $\mathbb{Q}$-Gorenstein smoothing. By the semi-continuity, we have $p_g(Y_t) = q(Y_t) = 0$. The base change theorem and Leray spectral sequence imply that $H^1(\mathcal{Y}, \mathcal{O}_Y) = H^2(\mathcal{Y}, \mathcal{O}_Y) = 0$. It gives an isomorphism $r_0: \text{Pic}(\mathcal{Y}) \to \text{Pic}(Y_t)$ and an injective map $r_1: \text{Pic}(\mathcal{Y}) \to \text{Pic}(Y_t)$ (Lemma 2 in [12]). The vanishing $H^1(Y, \mathcal{O}_Y(D)) = 0$ ensures that the deformation of $Y$ can be lifted to the deformation of the pair $(Y, D)$, i.e. the branch divisor $D$ is extended to $D_t$ in $Y_t$. Since the divisor $D$ is nonsingular, $D_t$ is also nonsingular. And the flatness of the map ensures that the divisor $L$ which is the data of the cyclic cover, i.e. $L^{\otimes |G|} \cong D$, is extended to $L_t$ with $L_t^{\otimes |G|} \cong D_t$. Hence, the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing of $Y$. \hfill $\Box$

3. A CONSTRUCTION OF HORIKAWA SURFACE

Let $E(n)$ be a simply connected relatively minimal elliptic surface with a section and with $c_2 = 12n$. Then there is only one up to diffeomorphism such an elliptic surface and the canonical class is given by $K_{E(n)} = (n-2)C$, where $C$ is a general fiber of an elliptic fibration. Hence each section is a nonsingular rational curve whose self-intersection number is $-n$. Assume that $n \geq 4$ and let $C_{n-2}$ denote a simply connected smooth 4-manifold obtained by plumbing the $(n-3)$ disk bundles over the 2-sphere according to the linear diagram

$$
\begin{array}{cccccc}
-2 & -2 & -2 & \cdots & -2 \\
\circ & \circ & \circ & \cdots & \circ \\
U_{n-3} & U_{n-4} & U_{n-5} & \cdots & U_1
\end{array}
$$

Assume that an elliptic surface $E(n)$ has two configurations $C_{n-2}$ such that all embedded 2-spheres $U_i$ are holomorphic curves (We show the existence of such an $E(n)$ later). Let $Y'_n$ be a normal projective surface obtained by contracting one configuration $C_{n-2}$ from $E(n)$. Then $Y'_n$ does not admit a $\mathbb{Q}$-Gorenstein smoothing because it violates Noether inequality (Corollary 7.5 in [3]). In fact, it does not satisfy the vanishing condition in the hypothesis of Theorem 2.1 that is, we have $H^2(E(n), T_{E(n)}) \neq 0$: Let $h: E(n) \to \mathbb{P}^1$ be an elliptic fibration. Assume that $C$ is a general fiber of the map $h$. We have an injective map $0 \to h^*\Omega_{\mathbb{P}^1} \to \Omega_{E(n)}$ and the map induces an injection $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(n-2)) \hookrightarrow H^0(E(n), \Omega_{E(n)}((n-2)C))$ by tensoring $(n-2)C$ on $0 \to h^*\Omega_{\mathbb{P}^1} \to \Omega_{E(n)}$. Since $K_{E(n)} = (n-2)C$, the cohomology $H^0(E(n), \Omega_{E(n)}(K_{E(n)}))$ is not zero. Hence the Serre duality implies that $H^2(E(n), T_{E(n)})$ is not zero.

Next, let $X_n$ be a normal projective surface obtained by contracting two disjoint configurations $C_{n-2}$ from $E(n)$, and we want to investigate the existence of a $\mathbb{Q}$-Gorenstein smoothing of $X_n$. As a warming-up, we first investigate $n = 4$ case.

Example. R. Gompf constructed a family of symplectic 4-manifolds by taking a fiber sum of other symplectic 4-manifolds [1]. To recall Gompf’s example briefly, we start with a simply connected relatively minimal elliptic surface $E(4)$ with a section and with $c_2 = 48$. It is known that $E(4)$ admits nine rational $(−4)$-curves as disjoint sections. Rationally blowing-down $n$ $(−4)$-curves of $E(4)$ is the same as the normal connected sum of $E(4)$ with $n$ copies of $\mathbb{P}^2$ by identifying a conic in each $\mathbb{P}^2$ with one $(−4)$-curve in $E(4)$. Let us denote this 4-manifold by $W_{4,n}$. Then the manifold $W_{4,1}$ does not admit
any complex structure because it violates the Noether inequality. But we will show that $W_{4,2}$ admits a complex structure using a $\mathbb{Q}$-Gorenstein smoothing theory. For this, let us first denote the singular projective surface obtained by contracting $n$ $(-4)$-sections from $E(4)$ by $W'_n$. And then we claim that $W'_{4,2}$ has a $\mathbb{Q}$-Gorenstein smoothing. The reason is following: Consider $E(4)$ as a double cover of Hirzebruch surface $\mathbb{F}_4$ branched over an irreducible nonsingular curve $D$ in the linear system $|4(C_0 + 4f)|$, where $C_0$ is a negative section and $f$ is a fiber of $\mathbb{F}_4$. Then $H^1(\mathbb{F}_4, \mathcal{O}_{\mathbb{F}_4}(D)) = 0$: Since $p_g(\mathbb{F}_4) = q(\mathbb{F}_4) = 0$, $H^1(\mathbb{F}_4, \mathcal{O}_{\mathbb{F}_4}(D)) \simeq H^1(D, \mathcal{O}_D(D)) \simeq H^0(D, \mathcal{O}_D(K_D - D))$. 

And $\deg K_D - D^2 = 4(C_0 + 4f)(2C_0 + 10f) - 16(C_0 + 4f)^2 = -24 < 0$ implies that $H^0(D, \mathcal{O}_D(K_D - D)) = 0$. Since $D$ does not intersect $C_0$, $W'_{4,2}$ is a double cover of a cone $\hat{\mathbb{F}}_4$ which is a contraction of $C_0$ from $\mathbb{F}_4$. This implies that the map $\sigma$ induced by a double cover is flat and $H^1(Y, \mathcal{O}_Y(D)) = 0$. Note that $\hat{\mathbb{F}}_4$ has a $\mathbb{Q}$-Gorenstein smoothing whose general fiber is $\mathbb{P}^2$. It is obtained by a pencil of hyperplane section of the cone of the Veronese surface imbedded in $\mathbb{P}^5$. Hence $W'_{4,2}$ has a $\mathbb{Q}$-Gorenstein smoothing by Theorem 2.2. Finally, since the rational blow-down manifold $W'_{4,2}$ is diffeomorphic to the general fiber of the $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,2}$, $W'_{4,2}$ admits a complex structure. Furthermore, using a triple cover of $\hat{\mathbb{F}}_4$ branched over $D$ in the linear system $|3(C_0 + 4f)|$, we can also prove that $W'_{4,3}$ has a $\mathbb{Q}$-Gorenstein smoothing by the similar proof as above. And, by extending Theorem 2.2 to a finite abelian group, it is possible to show that some other manifolds $W'_{4,n}$ has a $\mathbb{Q}$-Gorenstein smoothing, too. We leave it for a future research.

Now we investigate the general case. Assume that $n \geq 5$ and let $\mathbb{F}_n$ be a Hirzebruch surface. Let $C_0$ be a negative section with $C_0^2 = -n$ and $f$ be a fiber of $\mathbb{F}_n$. Consider the linear system $|4(C_0 + nf)|$. The surface $\mathbb{F}_n$ can be obtained from the cone over a rational normal curve of degree $n$ by blowing up the vertex. And a curve in the linear system $|4(C_0 + nf)|$ is the strict transform of the hyperplane section of the cone. By Bertini’s theorem, there is an irreducible nonsingular curve in the linear system $|4(C_0 + nf)|$. The double cover of $\mathbb{F}_n$ branched over an irreducible nonsingular member in $|4(C_0 + nf)|$ is an elliptic surface $E(n)$: Let $\sigma : \hat{X}_n \to \mathbb{F}_n$ be a double covering branched over an irreducible nonsingular member in the linear system $|4(C_0 + nf)|$. Then, by the invariants of a double covering ([2], Chapter V), we have $p_g(\hat{X}_n) = p_g(\mathbb{F}_n) + h^0(\mathbb{F}_n, K_{\mathbb{F}_n} + L) = h^0(\mathbb{F}_n, (n - 2)f) = n - 1$ and $\chi(\mathcal{O}_{\hat{X}_n}) = 2\chi(\mathcal{O}_{\mathbb{F}_n}) + \frac{1}{2}(L \cdot K_{\mathbb{F}_n}) + \frac{1}{2}(L \cdot L) = n$, where $L = 2(C_0 + nf)$. Therefore we have $q(\hat{X}_n) = 0$ and $K_{\hat{X}_n}^2 = 2(\sigma^*(K_{\mathbb{F}_n} + L))^2 = 2((n - 2)f)^2 = 0$.

In this article we want to choose a special irreducible (singular) curve $D$ in the linear system $|4(C_0 + nf)|$, which has a special intersection with one special fiber $f$: Note that $D \cdot f = 4$. We want $D$ to intersect with $f$ at two distinct points $p$ and $q$ that are not in $C_0$. Let $x = 0$ be the local equation of $f$ and $x, y$ be a coordinate at $p$ (resp. at $q$). We require that the local equation of $D$ at $p$ (resp. at $q$) is $(y - x)(y + x) = 0$ (resp. $(y - x^{n-4})(y + x^{n-4}) = 0$). These are $3(n - 4) + 3$-conditions: $1, x, x^2, \ldots, x^{2n-9}, y, yx, \ldots, yx^{n-5}$ terms should vanish to have the local analytic equation $(y - x^{n-4})(y + x^{n-4}) = 0$. By next lemmas and proposition, we have such a curve $D$ satisfying the conditions above.

**Lemma 3.1.** We have $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(D)) = 10n + 5$, where $D$ is a member in the linear system $|4(C_0 + nf)|$. 

by Bertini’s theorem, we conclude that $L_{\text{all}}$ double points by the local equations of $D$.

Note that $F$ is not necessarily irreducible and $F^2 = F_1^2 = K_{\mathbb{Z}_n} \cdot F = K_{\mathbb{Z}_n} \cdot F_i = -1$ for all $i = 1, \ldots, n - 4$. Let $f_0 = U_{n-4}$, which is a proper transform of the fiber, and let $L = \Delta - (\pi^*C_0 + f_0) - K_{\mathbb{Z}_n}$.

In Proposition 3.1 below, we prove that the linear system $|\Delta|$ is base point free. Then, by Bertini’s theorem, we conclude that $D$ is nonsingular except the two points $p$ and $q$.

**Lemma 3.2.** $L^2 \geq 5$ and $L$ is nef.
We note that $\Delta$ is the proper transform of
We conclude that the linear system
there is an effective divisor $H$.

Hence, using Lemma 3.3 above and the short exact sequence
By Lemma 3.2 above,
Proof.
The linear system
Lemma 3.3.
The linear system $|\Delta - (\pi^*C_0 + f_0)|$ on $Z_n$ is base point free.

Proof. By Lemma 3.2 above, $L$ is nef and $L^2 \geq 5$. Hence, applying to Reider’s theorem [15], if the adjoint linear series $|\Delta - (\pi^*C_0 + f_0)| = |K_{Z_n} + L|$ has a base point at $x$ then there is an effective divisor $G$ in $Z_n$ passing through $x$ such that either $G \cdot L = 0$ and $G^2 = -1$; or $G \cdot L = 1$ and $G^2 = 0$.

Assume that $G \cdot L = 0$ and $G^2 = -1$. Write $G = G_1 + \cdots + G_k$, where $G_k$ is an irreducible curve. Since $L$ is nef, $G \cdot L = 0$ implies that $G_i \cdot L = 0$ for all $i = 1, \ldots, k$. Then we get a contradiction by a similar argument to show that $-a + 5b \leq 0$ in the proof Lemma 3.2 above.

Assume that $G \cdot L = 1$ and $G^2 = 0$. By the same argument in the case $G \cdot L = 0$, $G = G_1$. Then we get a contradiction by a similar argument to show that $-a + 5b \leq 1$ unless $G = C_0$ or $f_0$. Furthermore, since $C_0^2 = -n$ and $f_0^2 = -2$, it also contradicts.

Proposition 3.1. The linear system $|\Delta|$ on $Z_n$ is base point free.

Proof. Note that $\Delta \cdot \pi^*C_0 = \Delta \cdot f_0 = 0$. Therefore we have $O_{\pi^*C_0 + f_0}(\Delta) = O_{\pi^*C_0 + f_0}$. Furthermore, by Lemma 3.2 above and by the vanishing theorem, we also have

$$H^1(Z_n, \Delta - (\pi^*C_0 + f_0)) = H^1(Z_n, K_{Z_n} + L) = 0.$$ 

Hence, using Lemma 3.3 above and the short exact sequence

$$0 \to O_{Z_n}(\Delta - (\pi^*C_0 + f_0)) \to O_{Z_n}(\Delta) \to O_{\pi^*C_0 + f_0} \to 0,$$

we conclude that the linear system $|\Delta|$ is base point free.

Next, by Artin’s criterion of contraction [11], we can contract a configuration $C_{n-2}$, which is a linear chain of $\mathbb{P}^1$'s

$$\frac{-n}{U_{n-3}} - \frac{-2}{U_{n-4}} - \frac{-2}{U_{n-5}} - \cdots - \frac{-2}{U_1},$$

so that it produces a singular normal projective surface. We denote this surface by $Y_n$. We note that $\Delta$ is the proper transform of $D$ in $Z_n$ and that $Y_n$ has a cyclic quotient singularity of type $\frac{1}{(n-2)t}(1, n-3)$, which is a singularity of class $T$.

Lemma 3.4. $H^1(Z_n, O_{Z_n}(\Delta)) = 0$. 
Proof. Since $p_g(Z_n) = q(Z_n) = 0$, we have
\[ H^1(Z_n, O_{Z_n}(\Delta)) \simeq H^1(\Delta, O_{\Delta}(\Delta)) \simeq H^0(\Delta, O_{\Delta}(K_{\Delta} - \Delta))^\vee. \]
We also have $\Delta^2 = D^2 - 4(n - 3) = 12n + 12$ and $\deg K_{\Delta} = \deg K_D - 2(n - 3) = 12n - 8 - 2(n - 3) = 10n - 2$. Therefore it satisfies
\[ \deg K_{\Delta} - \Delta^2 = 10n - 2 - 12n - 12 < 0, \]
and it implies that $H^0(\Delta, O_{\Delta}(K_{\Delta} - \Delta)) = 0$. \qed 

**Proposition 3.2.** The singular surface $Y_n$ admits a $\mathbb{Q}$-Gorenstein smoothing.

Proof. It is enough to show that $-K_{Y_n}$ is effective (Theorem 21 in [12]). Let $\pi : Z_n \to \mathbb{F}_n$ be a composition of blowing-ups, and $\psi : Z_n \to Y_n$ be a contraction. Then we have
\[ K_{Z_n} = \pi^*K_{\mathbb{F}_n} + E_1 + U_{n-5} + 2U_{n-6} + \cdots + (n-5)U_1 + (n-4)E_2. \]
Since $K_{\mathbb{F}_n} = -2C_0 - (n + 2)f$, $-K_{Z_n}$ is effective. Furthermore, since $h^0(-K_{Y_n}) = h^0(\psi_*(-K_{Z_n})) = h^0(-K_{Z_n})$ ($\S 3.9.2$ in [16]), $-K_{Y_n}$ is also effective. \qed 

Now we are in a position to prove our main theorem mentioned in the Introduction. First remind that Horikawa surface $H(n)$ is a double cover of $\mathbb{F}_{n-3}$ branched over a smooth curve $D_{n-2}$ in the linear system $|6C_0 + (4n - 8)f|$. R. Fintushel and R. Stern showed that Horikawa surface can be decomposed into
\[ H(n) = B_{n-2} \cup D_{n-2} \cup B_{n-2}, \]
where $B_{n-2}$ is the complement of a neighborhood of the pair of 2-spheres $(C_0 + (n-2)f)$ and $C_0$ in $\mathbb{F}_{n-3}$ (Lemma 2.1 in [3]), and they proved that an elliptic surface $E(n)$ is obtained from $H(n)$ by replacing two rational balls $B_{n-2}$ with two configurations $C_{n-2}$ (Lemma 7.3 in [3]). In other words, R. Fintushel and R. Stern proved that Horikawa surface $H(n)$ can be obtained from an elliptic surface $E(n)$ by rationally blowing-down two disjoint configurations $C_{n-2}$ lying in $E(n)$ in smooth category. The aim of this article is to prove that the rational blow-down surgery above can be performed in complex category, which is following.

**Proof of Theorem 3.1.** Note that $\tilde{X}_n$ is a double covering of $\mathbb{F}_n$ branched over $D$, and the minimal resolution of two rational double points of type $A_1$ and $A_{2n-9}$ in $\tilde{X}_n$ is $E(n)$, which is also a double cover of $Z_n$ branched over the proper transform of $D$. Since the proper transform of $D$ does not meet the contracted linear chain of $\mathbb{P}^1$'s, we have a double cover of $Y_n$ branched over the image of the proper transform of $D$ by the map $\psi$. We denote this surface by $X_n$. Then $X_n$ is the singular surface obtained by contracting two disjoint configurations $C_{n-2}$ from an elliptic surface $E(n)$ and it has two quotient singularities of class $T$, both are of type $\frac{1}{n-2}n(1, n-3)$. By the fact that the proper transform of $D$ is disjoint from the contracted liner chain of $\mathbb{P}^1$'s and Lemma 3.4 the map from $X_n$ to $Y_n$ is flat and $H^1(Y_n, O_{Y_n}(D_{Y_n})) = 0$, where $D_{Y_n}$ is the image of $\Delta$ in $Y_n$ under the contraction $C_{n-2}$ of the liner chain of $\mathbb{P}^1$'s. Therefore we have the following commutative diagram of maps
\[
\begin{array}{ccc}
\tilde{X}_n & \leftarrow & E(n) \rightarrow & X_n \\
\downarrow & & \downarrow & \\
\mathbb{F}_n & \leftarrow & Z_n & \rightarrow & Y_n
\end{array}
\]
where all vertical maps are double coverings. Then, by Theorem 2.2 and Proposition 3.2 above, the singular surface $X_n$ has a $\mathbb{Q}$-Gorenstein smoothing of two quotient singularities all together.

Finally, by applying the standard arguments about Milnor fibers (§5 in [11] or §1 in [13]), we know that a general fiber of a $\mathbb{Q}$-Gorenstein smoothing of $X_n$ is diffeomorphic to the 4-manifold obtained by rational blow-down of $E(n)$. And we also know that $H(n)$ has one deformation class ([2], Chapter VII). Therefore a general fiber of a $\mathbb{Q}$-Gorenstein smoothing of $X_n$ is a Horikawa surface $H(n)$ in complex category. □

References

[1] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, Amer. J. Math. 84 (1962), 485–496.
[2] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact complex surfaces, 2nd ed. Springer-Verlag, Berlin, 2004.
[3] R. Fintushel and R. Stern, Rational blowdowns of smooth 4-manifolds, Jour. Diff. Geom. 46 (1997), 181–235.
[4] R. Gompf, A new construction of symplectic manifolds, Ann. of Math. (2) 142 (1995), 527–595.
[5] P. Hacking, Compact moduli of plane curves, Duke Math. J. 124 (2004), 213–257.
[6] R. Hartshorne, Algebraic Geometry, Springer, 1977.
[7] B. Hassett, Stable log surfaces and limits of quartic plane curves, Manuscripta Math. 100 (1999), 469–497.
[8] J. Kollár and N.I. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. 91 (1988), 299–338.
[9] Y. Lee and J. Park, A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$, Invent. Math. 170 (2007), 483–505.
[10] S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, Trans. Amer. Math. Soc. 128 (1967), 41–70.
[11] E. Looijenga and J. Wahl, Quadratic functions and smoothing surface singularities, Topology 25 (1986), 261–291.
[12] M. Manetti, Normal degenerations of the complex projective plane, J. Reine Angew. Math. 419 (1991), 89–118.
[13] M. Manetti, On the moduli space of diffeomorphic algebraic surfaces, Invent. Math. 143 (2001), 29–76.
[14] V. P. Palamodov, Deformations of complex spaces, Russian Math. Surveys 31:3 (1976), 129–197.
[15] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. Math. 127 (1988), 309–316.
[16] J. Wahl, Smoothing of normal surface singularities, Topology 20 (1981), 219–246.
[17] J. Wahl, Elliptic deformations of minimally elliptic singularities, Math. Ann. 253 (1980), 241–262.