Levinson theorem for Dirac particles in two dimensions

Qiong-gui Lin

China Center of Advanced Science and Technology (World Laboratory),
P.O.Box 8730, Beijing 100080, People’s Republic of China

and

Department of Physics, Zhongshan University, Guangzhou 510275,
People’s Republic of China

Abstract

The Levinson theorem for nonrelativistic quantum mechanics in two spatial dimensions is generalized to Dirac particles moving in a central field. The theorem relates the total number of bound states with angular momentum $j$ ($j = \pm 1/2, \pm 3/2, \ldots$), $n_j$, to the phase shifts $\eta_j(\pm E_k)$ of scattering states at zero momentum as follows: $\eta_j(\mu) + \eta_j(-\mu) = n_j \pi$.

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1Mailing address
I. Introduction

In 1949, Levinson established a theorem in nonrelativistic quantum mechanics[1]. The theorem gives a relation between bound states and scattering states in a given angular momentum channel $l$, i.e., the total number of bound states $n_l$ is related to the phase shift $\delta_l(k)$ at threshold ($k = 0$):

$$\delta_l(0) = n_l\pi, \quad l = 0, 1, 2, \ldots$$

The case $l = 0$ should be modified as

$$\delta_0(0) = (n_0 + 1/2)\pi$$

when there exists a zero-energy resonance (a half bound state)[2]. This is one of the most interesting and beautiful results in nonrelativistic quantum theory. The subject has been studied by many authors (some are listed in the Refs.[2-8]) and generalized to relativistic quantum mechanics[6,9-14]. However, most of these authors deal with the problem in ordinary three-dimensional space. A two-dimensional version of Levinson’s theorem does not appear to have been discussed by previous authors. In view of the wide interest in lower-dimensional field theories in recent years, e.g., Chern-Simons theory in 2+1 dimensions, and the previous applications of Levinson’s theorem to field theories[15], it seems of interest to study the theorem in two-dimensional space. On the other hand, as the problem exhibits some new features in two spatial dimensions, it may also be of interest in its own right. We are thus led to consider the problem.

In a recent work[16] we have established the Levinson theorem in two spatial dimensions, which takes the following form:

$$\eta_m(0) = n_m\pi, \quad m = 0, 1, 2, \ldots$$
where \( \eta_m(0) \) is the phase shift of the \( m \)th partial wave at threshold, and \( n_m \) is the total number of bound states with angular momentum \( m \) (it also equals the total number of bound states with angular momentum \( -m \) when \( m \neq 0 \)). As in three dimensions, the modulo-\( \pi \) ambiguity in the definition of \( \eta_m(k) \) has been resolved by setting \( \eta_m(\infty) = 0 \) (rather than a multiple of \( \pi \)) which can be freely done in nonrelativistic theory. The theorem is similar to the three-dimensional one but simpler in that the existence of half bound states (possible for \( m = 0, 1 \)) does not alter the form of the theorem.

In this paper we extend the previous work to the relativistic case and establish the Levinson theorem for Dirac particles moving in an external central field in two spatial dimensions. In a given angular momentum channel \( j \) (\( j = \pm 1/2, \pm 3/2, \ldots \)), the theorem relates the total number of bound states \( n_j \) to the phase shifts \( \eta_j(\pm E_k) \) at zero momentum:

\[
\eta_j(\mu) + \eta_j(-\mu) = n_j \pi. \tag{1}
\]

In the relativistic theory, one is not allowed to set \( \eta_j(\pm \infty) = 0 \). But the modulo-\( \pi \) ambiguity in the definition of \( \eta_j(\pm E_k) \) may be appropriately resolved (see Sec. V). As in the nonrelativistic case, the theorem is similar to the three-dimensional one[10] but somewhat simpler. In three dimensions, the theorem should be modified when there exists a half bound state, but here we have no such trouble.

Throughout this paper natural units where \( \hbar = c = 1 \) are employed. In the next section we first discuss various aspects of the solutions of the Dirac equation in an external central field in two spatial dimensions. Then we give a brief formulation of the partial-wave method for potential scattering of Dirac particles. In Sec. III the behavior of the phase shifts near \( k = 0 \) is analyzed. In Sec. IV we establish the Levinson theorem using the Green function method[3,4,6,10]. Sec. V is devoted to some discussions relevant to the theorem.
II. Dirac particles in an external central field in two dimensions

We work in (2+1)-dimensional space-time. The Dirac equation in an external vector field \( A_\nu(t, \mathbf{r}) \) reads

\[
(i\gamma^\nu D_\nu - \mu)\Psi = 0,
\]

where \( \mu \) is the mass of the particle, \( D_\nu = \partial_\nu + ieA_\nu \), \( e \) is the coupling constant, and summation over the repeated Greek index \( \nu (\nu = 0, 1, 2) \) is implied. The \( \gamma^\mu \) are Dirac matrices satisfying the Clifford algebra:

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},
\]

where \( g^{\mu\nu} = \text{diag}(1, -1, -1) \) is the Minkowskian metric. In this paper we only consider the zero component of \( A_\nu \), which is cylindrically symmetric, namely, we consider the special case where

\[
A = 0, \quad eA_0 = V(r),
\]

where \( r \) is one of the polar coordinates \((r, \theta)\) in two-dimensional space. In this case we may set

\[
\Psi(t, \mathbf{r}) = e^{-iEt}\psi(r),
\]

and get a stationary equation for \( \psi(r) \):

\[
H\psi = E\psi,
\]

where the Hamiltonian

\[
H = \alpha \cdot \mathbf{p} + \gamma^0\mu + V(r),
\]

where \( \mathbf{p} = -i\nabla \), \( \alpha = \gamma^0\gamma \) or \( \alpha^i = \gamma^0\gamma^i \) \((i = 1, 2)\).

In two dimensional space, the orbital angular momentum has only one component:

\[
L = e^{ij}x^i p^j,
\]
where \( \epsilon^{ij} \) is antisymmetric in \( i, j \) and \( \epsilon^{12} = 1 \), and summation over the repeated Latin indices \( i, j \) is implied. It is easy to show that \([L, H] = i\epsilon^{ij}\alpha^i p^j \neq 0\), thus \( L \) is not a constant of motion even when \( V = 0 \). Let

\[
S = \frac{i}{4} \epsilon^{ij} \gamma^i \gamma^j. \tag{9}
\]

It is easy to show that \([S, H] = -i\epsilon^{ij}\alpha^i p^j\), then \([L + S, H] = 0\), and the quantity

\[
J = L + S \tag{10}
\]

is a constant of motion. It is natural to regard \( J \) as the total angular momentum and \( S \) as the spin angular momentum in two dimensions.

II.A. Solutions in the external central field

To solve Eq.(6) a representation of the Dirac matrices is necessary. This can be realized by the Pauli matrices:

\[
\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2. \tag{11}
\]

In this representation \( S = \sigma^3/2 \). Let

\[
\psi_j(r, \theta) = \begin{pmatrix}
F(r)e^{i(j-1/2)\theta}/\sqrt{2\pi} \\
G(r)e^{i(j+1/2)\theta}/\sqrt{2\pi}
\end{pmatrix} = \begin{pmatrix}
v_1(r)e^{i(j-1/2)\theta}/\sqrt{2\pi r} \\
v_2(r)e^{i(j+1/2)\theta}/\sqrt{2\pi r}
\end{pmatrix}, \quad j = \pm 1/2, \pm 3/2, \ldots. \tag{12}
\]

It is easy to show that \( \psi_j \) is an eigenfunction of \( J \):

\[
J\psi_j = j\psi_j. \tag{13}
\]

So that \( \psi_j \) has angular momentum \( j \). (There is no danger of confusing the angular momentum with the Latin index \( j \) used above, as the latter will not appear henceforth.) The radial wave functions \( F \) and \( G \) satisfy the system of equations

\[
F' - \frac{j}{r}F + (E + \mu - V)G = 0, \tag{14a}
\]
\[ G' + \frac{j_+}{r}G - (E - \mu - V)F = 0, \]  
(14b)

where \( j_\pm = j \pm 1/2 \), and primes denote differentiation with respect to argument. We will have occasions to use the system of equations for \( v_1 \) and \( v_2 \), so we also write down it here:

\[ v_1' - \frac{j}{r}v_1 + (E + \mu - V)v_2 = 0, \]  
(15a)

\[ v_2' + \frac{j}{r}v_2 - (E - \mu - V)v_1 = 0. \]  
(15b)

From Eq.(14a), we have

\[ G = -\frac{F' - (j_-/r)F}{E + \mu - V}. \]  
(16)

Substituting this into Eq.(14b) we get an equation for \( F \) alone:

\[ F'' + \left( \frac{1}{r} + \frac{V''}{E + \mu - V} \right) F' + \left[ E^2 - \mu^2 - \frac{j^2}{r^2} - 2EV + V\frac{j-V'}{r(E + \mu - V)} \right] F = 0. \]  
(17)

With a given potential \( V(r) \) that is regular everywhere except possibly at \( r = 0 \) and appropriate boundary conditions one can in principle solve Eq.(17) for \( F \), and get \( G \) from Eq.(16). Equivalent to Eqs.(16) and (17), we may have

\[ F = \frac{G' + (j_+/r)G}{E - \mu - V}, \]  
(18)

\[ G'' + \left( \frac{1}{r} + \frac{V''}{E - \mu - V} \right) G' + \left[ E^2 - \mu^2 - \frac{j_+^2}{r^2} - 2EV + V^2 + \frac{j_+V'}{r(E - \mu - V)} \right] G = 0. \]  
(19)

If in some region or at some point \( E + \mu - V = 0 \), we can not use Eqs.(16) and (17). But then \( E - \mu - V \neq 0 \) in that region or in the neighbourhood of that point, and we can use Eqs.(18) and (19). Indeed, the fundamental equation is Eq. (14) or Eq. (15). They are regular everywhere except at \( r = 0 \). In principle one can solve them by direct integration without the help of Eqs. (16-19). Eqs. (16-17) or Eqs. (18-19) are to be employed when convenient. The possible singularities in these equations due to the vanishing denominator \( E + \mu - V \) or \( E - \mu - V \) should not cause any trouble in principle. Nevertheless, attention
should be paid to these possible singularities when we have to use Eq. (17) or Eq. (19) in the whole range of \( r \).

For free particles, \( V = 0 \), then Eq.(17) becomes
\[
F'' + \frac{1}{r} F' + \left( \frac{E^2 - \mu^2 - \frac{j^2}{r^2}}{r^2} \right) F = 0. \tag{20}
\]
In order to get well behaved solutions one should have \( E^2 - \mu^2 \geq 0 \). Thus we have positive-energy solutions with \( E \geq \mu \) and negative-energy solutions with \( E \leq -\mu \). Let us define
\[
k = \sqrt{E^2 - \mu^2} \geq 0,
\]
and denote positive-(negative-)energy solutions by the subscript \( k\) \((-k)\), thus we have, say,
\[
E_{\pm k} = \pm E_k = \pm \sqrt{k^2 + \mu^2}. \tag{21}
\]
It is not difficult to find the following solutions for free particles:
\[
F_{\pm kj}^{(0)}(r) = \sqrt{\frac{(E_k \mp \mu)k}{2E_k}} J_m(kr), \tag{22a}
\]
\[
G_{\pm kj}^{(0)}(r) = \pm \epsilon(j) \sqrt{\frac{(E_k \mp \mu)k}{2E_k}} J_{m+\epsilon(j)}(kr), \tag{22b}
\]
where \( m = |j_{-}| \), \( J_m(kr) \) is the Bessel function, \( \epsilon(j) = 1(-1) \) when \( j > 0(<0) \), and the normalization factors are chosen such that the orthonormal relation takes the form
\[
\int dr \psi_{\pm kj}^{(0)\dagger}(r) \psi_{\pm kj}^{(0)}(r) = \delta(k - k') \delta_{j,j'}, \tag{23}
\]
\[
\int dr \psi_{\pm kj}^{(0)\dagger}(r) \psi_{\pm kj}^{(0)}(r) = 0. \tag{23'}
\]
The completeness of these solutions is ensured by the following relation which can be verified straightforwardly:
\[
\sum_j \int_0^\infty dk [\psi_{kj}^{(0)}(r) \psi_{kj}^{(0)\dagger}(r') + \psi_{-kj}^{(0)}(r) \psi_{-kj}^{(0)\dagger}(r')] = \delta(r - r'). \tag{24}
\]
When \( r \to \infty \), the radial wave functions have the asymptotic form:
\[
v_{\pm kj}^{(0)}(r) \to \sqrt{\frac{E_k \pm \mu}{\pi E_k}} \cos \left( k r - \frac{m\pi}{2} - \frac{\pi}{4} \right), \tag{25a}
\]
$$v_{\pm kj}^{(0)}(r) \rightarrow \pm \sqrt{\frac{E_k \mp \mu}{\pi E_k}} \sin \left( kr - \frac{m\pi}{2} - \frac{\pi}{4} \right).$$  \tag{25b}

This can be obtained by using Eq.(22) and the asymptotic formula for the Bessel function:

$$J_n(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right).$$  \tag{26}

Now we consider particles moving in the external central potential $V(r)$. We assume that $V(r) \to 0$ more rapidly than $r^{-2}$ when $r \to \infty$, and is less singular than $r^{-1}$ when $r \to 0$. Then, for very large $r$, Eq.(17) takes the same form as Eq.(20). It is easy to see that $E^2 \geq \mu^2$ gives scattering solutions while $E^2 < \mu^2$ gives bound state solutions (bound states with $E = \pm \mu$ are also possible, see Sec.V). Scattering states will be denoted as above, while bound states will be denoted by a subscript $\kappa$ which takes discrete values.

The orthonormal relations are given by

$$\int dr \psi_{\pm k'j'}^\dagger(r) \psi_{\pm kj}(r) = \delta(k - k')\delta_{jj'},$$  \tag{27}

$$\int dr \psi_{\kappa'j'}^\dagger(r) \psi_{\kappa j}(r) = \delta_{\kappa\kappa'}\delta_{jj'},$$  \tag{27'}

and vanishing ones similar to Eq.(23'). The completeness relation is similar to Eq.(24) but has an additional term on the left-hand side (lhs):

$$\sum_j \int_0^\infty dk \left[ \psi_{kj}(\mathbf{r}) \psi_{k'j'}^\dagger(\mathbf{r}') + \psi_{-kj}(\mathbf{r}) \psi_{-k'j'}^\dagger(\mathbf{r}') \right] + \sum_{\kappa j} \psi_{\kappa j}(\mathbf{r}) \psi_{\kappa j}^\dagger(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$  \tag{28}

As pointed out above, Eq.(17) takes the form of Eq.(20) at large $r$, so the solution $F_{kj}(r)$ is given by a linear combination of $J_m(kr)$ and $N_m(kr)$, the Neumann function, at large $r$. Using Eq.(26) and

$$N_n(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right),$$  \tag{29}

and Eq.(16), the asymptotic forms for the radial wave functions can be shown to be

$$v_{\pm kj1}(r) \to \sqrt{\frac{E_k \mp \mu}{\pi E_k}} \cos \left[ kr - \frac{m\pi}{2} - \frac{\pi}{4} + \eta_j(\pm E_k) \right],$$  \tag{30a}
\[ v_{\pm kj_2}(r) \to \pm \sqrt{\frac{E_k + \mu}{\pi E_k}} \sin \left[ kr - \frac{m\pi}{2} - \frac{\pi}{4} + \eta_j(\pm E_k) \right], \]  

(30b)

when \( r \to \infty \), where \( m = |j| \) as before, \( \eta_j(\pm E_k) \) are the phase shifts. They depend on \( j \) rather than \( |j| \), and also depend on the sign (not only the magnitude) of the energy, as Eq.(17) does. Compared with Eq.(25), the asymptotic forms in the external central field are distorted by the phase shifts. But it should be remarked that the normalization factors in Eq.(30) are the same as in Eq.(25).

II.B. Partial-wave analysis of scattering by the central field

It is well known that positive-(negative-)energy solutions correspond to particles (antiparticles) after second quantization. In this subsection we discuss the scattering of positive-energy solutions by the central field \( V(r) \) described above. The scattering of negative-energy solutions can be formally discussed in a similar way.

The probability current density associated with Eq.(2) is given by

\[ j = \psi^\dagger \alpha \psi. \]  

(31)

The incident wave may be chosen as

\[ \psi_{in} = \begin{pmatrix} i \sqrt{\frac{E_k + \mu}{2E_k}} e^{ikx}, & k > 0, \\ i \sqrt{\frac{E_k - \mu}{2E_k}} e^{ikx}, & k < 0 \end{pmatrix}, \]  

(32)

which is a solution of Eq.(6) with positive energy \( E_k \) when \( x \to -\infty \). The incident probability current density is

\[ j_{in} = e_x k/E_k = e_x v, \]  

(33)

where \( e_x \) is the unit vector in the \( x \) direction, the incident direction in this case. The
scattered wave should have the asymptotic form when \( r \to \infty \):

\[
\psi_{sc} \to \sqrt{\frac{i}{r}} \begin{pmatrix} f_1(\theta) \\ f_2(\theta) \end{pmatrix} e^{ikr},
\]

(34)

where the factor \( \sqrt{i} = e^{i\pi/4} \) is introduced for latter convenience. Then the \( r \) and \( \theta \) components of \( j_{sc} \) at large \( r \) can be shown to be

\[
j_{scr} = \frac{2}{r} \text{Im}(e^{i\theta} f_1^{*} f_2), \quad j_{sc\theta} = \frac{2}{r} \text{Re}(e^{i\theta} f_1^{*} f_2^{*}).
\]

(35)

However, the expression (34) does not solve the Dirac equation at large \( r \) if \( f_1(\theta) \) and \( f_2(\theta) \) are independent of each other. In order to satisfy Eq.(6) at large \( r \), one should have

\[
f_2(\theta) = -\frac{ik}{E_k + \mu} e^{i\theta} f_1(\theta).
\]

(36)

Setting

\[
f_1(\theta) = i \sqrt{\frac{E_k + \mu}{2E_k}} f(\theta),
\]

(37)

it is easy to show that

\[
j_{sc} = e_r v |f(\theta)|^2 / r
\]

(38)

when \( r \to \infty \), where \( e_r \) is the unit vector in the radial direction. Thus the differential cross section (in two dimensions the cross section may be more appropriately called cross width) is given by

\[
\sigma(\theta) = |f(\theta)|^2.
\]

(39)

The outgoing wave comprises Eqs.(32) and (34). With the relation (36) and the definition (37), it takes the following asymptotic form when \( r \to \infty \):

\[
\psi \to \begin{pmatrix} i \sqrt{\frac{E_k + \mu}{2E_k}} \left[ e^{ikx} + \frac{1}{r} e^{ikr} f(\theta) \right] \\ \sqrt{\frac{E_k - \mu}{2E_k}} \left[ e^{ikx} + \frac{i}{r} e^{ikr} e^{i\theta} f(\theta) \right] \end{pmatrix}.
\]

(40)
On the other hand, the solution of Eq.(6) with definite energy $E_k > 0$ has the form

$$\psi(r, \theta) = \sum_j a_j \psi_{kj}(r, \theta), \quad (41)$$

where the summation is made over all $j = \pm 1/2, \pm 3/2, \ldots$. The asymptotic form of Eq.(41) can be obtained from Eq.(30). By using the formula

$$e^{ikx} = \sum_{n=-\infty}^{+\infty} i^n |J_n(kr)| e^{in\theta} \quad (42)$$

and Eq.(26), one can compare Eq.(40) with the asymptotic form of Eq.(41). They must coincide with each other for appropriately chosen $a_j$’s. In this way one finds all $a_j$ in terms of $\eta_j(E_k)$ and

$$f(\theta) = \sum_j \sqrt{\frac{2}{\pi k}} e^{i\eta_j(E_k)} \sin \eta_j(E_k) e^{ij-\theta}. \quad (43)$$

The total cross section $\sigma_t$ turns out to be

$$\sigma_t = \int_0^{2\pi} d\theta \sigma(\theta) = \frac{4}{k} \sum_j \sin^2 \eta_j(E_k). \quad (44)$$

One easily realizes that all information of the scattering process is contained in the phase shifts. The latter are determined by solving the system of equations (15) with the boundary conditions (30). The purpose of the Levinson theorem is to establish a relation between scattering states and bound states, specifically, to establish a relation between the phase shifts and the total number of bound states in a given angular momentum channel $j$.

III. Phase shifts near threshold

In this section we discuss the behavior of the phase shifts $\eta_j(\pm E_k)$ near $k = 0$. This will be employed in the next section. For exact analysis let us cut off the potential. That is, we consider potentials that satisfy $V(r) = 0$ when $r > a > 0$. Such potentials will be denoted by $V_a(r)$ in the following. In the region $r > a$, then, Eq.(17) reduces to the form
of Eq.(20), and the solution may take the form

\[
F_{\pm kj}(r) = \sqrt{\frac{(E_k \pm \mu)k}{2E_k}} [\cos \eta_j(\pm E_k)J_m(kr) - \sin \eta_j(\pm E_k)N_m(kr)],
\]  

(45)

where the superscript “>” indicates \( r > a \). Using Eqs.(16), (26), and (29), one can obtain the expected asymptotic forms (30). In the region \( r < a \), Eq.(17) cannot be simplified.

Let us consider the behavior of the solution near \( r = 0 \). We have assumed that \( V(r) \) is less singular than \( r^{-1} \) when \( r \to 0 \). Thus, when \( r \to 0 \), \( V(r) \) is regular or behaves like \( U_0/r^\delta \) where \( U_0 \) is a constant and \( 0 < \delta < 1 \). Accordingly, Eq.(17) becomes to leading terms

\[
F'' + \frac{1}{r} F' - j_2 F = 0
\]  

(46a)

in the first case or

\[
F'' + \frac{1 + \delta}{r} F' - j_2(j + \delta) F = 0
\]  

(46b)

in the second case. Therefore, the regular solution of Eq.(17) may have the following power dependence on \( r \) when \( r \to 0 \):

\[
f_j^\pm(r, k) \to r^m
\]  

(47a)

in the first case or

\[
f_j^\pm(r, k) \to \begin{cases} r^m, & j > 0 \\ r^{m-\delta}, & j < 0 \end{cases}
\]  

(47b)

in the second case, where we have denoted \( F_{\pm kj}(r) \) with these boundary conditions by \( f_j^\pm(r, k) \), and \( m = |j_-| \) as before. The solution of Eq.(17) in the region \( r < a \) is

\[
F_{\pm kj}^\leq(r) = A_j^\pm(k) f_j^\pm(r, k),
\]  

(48)

where the superscript “\( \leq \)” indicates \( r < a \). In general the coefficient \( A_j^\pm \) depends on \( k \) such that the two parts of \( F_{\pm kj}(r) \) can be appropriately connected at \( r = a \). Obviously, \( F_{\pm kj}(r) \) and \( G_{\pm kj}(r) \) should be continuous at \( r = a \), so that the probability density and
the probability current density are continuous at \( r = a \). For simplicity, we assume that \( V_a(r) \) is continuous at \( r = a \), which means \( V_a(r) \to 0 \) when \( r \to a^- \) and \( V_a(a) = 0 \). Then from Eq.(14) we see that \( F'_{\pm k_j}(r) \) is also continuous at \( r = a \). Therefore, \( F'_{\pm k_j}(r)/F_{\pm k_j}(r) \) is continuous at \( r = a \). This leads to

\[
\tan \eta_j(\pm E_k) = \frac{\xi J_m^\prime(\xi) - \beta_j^{\pm}(\xi) J_m(\xi)}{\xi N_m^\prime(\xi) - \beta_j^{\pm}(\xi) N_m(\xi)},
\]

(49)

where \( \xi = ka \) and

\[
\beta_j^{\pm}(\xi) = \frac{af_j^{\pm}(a, k)}{f_j^{\pm}(a, k)},
\]

(50)

where the prime indicates differentiation with respect to \( r \). The above result shows that the behavior of \( \eta_j(\pm E_k) \) is determined by that of \( \beta_j^{\pm}(\xi) \) and ultimately by that of \( f_j^{\pm}(a, k) \).

The general dependence of \( f_j^{\pm}(r, k) \) on \( k \) may be very complicated since Eq.(17) depends on \( k \) in a rather complicated way. This is quite different from the nonrelativistic case where the function \( f_m(r, k) \), the counterpart of \( f_j^{\pm}(r, k) \), is an integral function of \( k \)[16]. Fortunately, only the property of \( f_j^{\pm}(r, k) \) near \( k = 0 \) is necessary for our purpose.

We consider the limit \( E \to \mu \) of Eq.(17). In this limit it takes the following form to the first order in \( k^2 \):

\[
F'' + P(r, k) F' + Q(r, k) F = 0,
\]

(51)

where

\[
P(r, k) = \frac{1}{r} + \frac{V'}{2\mu - V} - \frac{V'}{2\mu(2\mu - V)^2} k^2,
\]

(51')

\[
Q(r, k) = -\frac{j^2}{r^2} - 2\mu V + V^2 - \frac{j V'}{r(2\mu - V)} + \left[ 1 - \frac{V}{\mu} + \frac{j V'}{2\mu r(2\mu - V)} \right] k^2.
\]

(51'b)

We denote the solution of this equation that satisfies the boundary condition (47) by \( \tilde{f}_j^{\pm}(r, k) \). Note that both \( P(r, k) \) and \( Q(r, k) \) are integral functions of \( k \), and the boundary condition (47) is independent of \( k \). Then a theorem of Poincaré tells us that \( \tilde{f}_j^{\pm}(r, k) \) is also an integral function of \( k \). On the other hand, Eq.(17) coincides with Eq.(51) in the
limit $E \to \mu$. Therefore, $f_j^+(r,k)$ must coincide with $\tilde{f}_j^+(r,k)$ in the limit $k \to 0$ since they satisfy the same boundary condition. Hence we conclude that $f_j^+(r,k)$ is an analytic function of $k$ in the neighbourhood of $k = 0$. Moreover, $f_j^+(r,k)$ is an even function of $k$, since Eq.(17) is invariant under the change $k \to -k$, and the boundary condition is independent of $k$. The above conclusion holds regardless of whether $V(r) = V_a(r)$ or not.

We can now proceed as in the nonrelativistic case[16] and arrive at the result

$$
\tan \eta_j(E_k) \to b_j^+ \xi^{2p_j^+} \quad \text{or} \quad \frac{\pi}{2 \ln \xi} \quad (k \to 0),
$$

(52a)

where $b_j^+ \neq 0$ is a constant and $p_j^+$ is a natural number. For a strong repulsive potential Eq. (51) may become singular at the points where $V = 2\mu$. Then the above analysis is not reasonable. In this case, however, we may consider $G_{k_j}(r)$ instead of $F_{k_j}(r)$ and study the limit $E \to \mu$ of Eq. (19). The same result (52a) can be attained in a similar way. A similar analysis leads to

$$
\tan \eta_j(-E_k) \to b_j^- \xi^{2p_j^-} \quad \text{or} \quad \frac{\pi}{2 \ln \xi} \quad (k \to 0),
$$

(52b)

where $b_j^- \neq 0$ is a constant and $p_j^-$ is a natural number.

In the above analysis we have assumed that $V_a(r)$ is continuous at $r = a$. This is, however, not mandatory. Using the condition that both $F_{\pm kj}(r)$ and $G_{\pm kj}(r)$ are continuous at $r = a$, one can arrive at the same result (52) even when $V(r)$ is not continuous at $r = a$.

IV. The Levinson theorem

With the above preparations, we now proceed to establish the Levinson theorem by the Green function method. We define the retarded Green function $G(r,r',E)$ of the Dirac
equation in an external potential $V(r)$ by

$$G(r, r', E) = \sum_\tau \frac{\psi_\tau(r)\psi_\tau^\dagger(r')}{E - E_\tau + i\epsilon},$$

(53)

where $\{\psi_\tau(r)\}$ is a complete set of orthonormal solutions to Eq.(6) where $V$ may be a noncentral potential, $E_\tau$ is the energy eigenvalue associated with the solution $\psi_\tau(r)$, and $\epsilon = 0^+$. $G(r, r', E)$ satisfies the equation

$$(E - H + i\epsilon)G(r, r', E) = \delta(r - r').$$

(54)

For a free particle, $V = 0$. We denote the Hamiltonian by $H_0$ and the solutions to the free Dirac equation by $\psi^{(0)}_\tau(r)$. The retarded Green function in this case is defined as

$$G^{(0)}(r, r', E) = \sum_\tau \frac{\psi^{(0)}_\tau(r)\psi^{(0)}_\tau^\dagger(r')}{E - E^{(0)}_\tau + i\epsilon},$$

(55)

where $E^{(0)}_\tau$ is the energy corresponding to the solution $\psi^{(0)}_\tau(r)$. $G^{(0)}(r, r', E)$ satisfies

$$(E - H_0 + i\epsilon)G^{(0)}(r, r', E) = \delta(r - r').$$

(56)

We have the integral equation for $G(r, r', E)$:

$$G(r, r', E) - G^{(0)}(r, r', E) = \int dr'' G^{(0)}(r, r'', E)V(r'')G(r'', r', E).$$

(57)

In a central field $V(r) = V(r)$ [not necessarily $V_a(r)$], we have discussed the solutions $\psi_\tau(r)$ of the Dirac equation (6) in Sec.II. They contain two different classes: $\psi_{\pm kj}(r, \theta)$ with continuous energy spectrum $E_{\pm k}$ and $\psi_{k\eta}(r, \theta)$ with discrete energy spectrum $E_{\kappa\eta}$. However, we may discretize the continuous part of the spectrum by requiring the radial wave function $F_{\pm kj}(r)$ or $v_{\pm kj1}(r)$ to vanish at a sufficiently large radius $R \ [R \gg a$ for $V_a(r)$]. In this case we will denote all the solutions by $\psi_{k\eta}(r, \theta)$ and the corresponding energies by $E_{\kappa\eta}$. We also apply the same prescription to the free Dirac particle to turn
the continuous energies $E^{(0)}_{\pm k}$ into discrete ones $E^{(0)}_{\kappa j}$. The waves functions will be denoted by $\psi^{(0)}_{\kappa j}(r, \theta)$ in the free case. We have then

$$G(r, r', E) = \sum_j \frac{e^{ij-(\theta-\theta')}}{2\pi} \begin{pmatrix} G_{j11}(r, r', E) & G_{j12}(r, r', E) e^{-i\theta'} \\ G_{j21}(r, r', E) e^{i\theta} & G_{j22}(r, r', E) e^{i(\theta-\theta')} \end{pmatrix},$$  \hfill (58)

where

$$G_{j\sigma}(r, r', E) = \sum_\kappa \frac{v_{\kappa jp}(r) v_{\kappa jq}(r')}{\sqrt{rr'}(E - E_{\kappa j} + i\epsilon)},$$  \hfill (58')

where $p, q = 1, 2$ are used to denote spinor indices. For a free particle we have similar relations. The integral equation for $G_{j\sigma}(r, r', E)$ may be derived from Eq.(57). The result turns out to be

$$G_{j\sigma}(r, r', E) - G^{(0)}_{j\sigma}(r, r', E) = \int dr'' r'' \sum_s G^{(0)}_{j\sigma s}(r, r'', E) V(r') G_{jsq}(r'', r', E).$$  \hfill (59)

Since the wave function should be finite at $r = 0$, we have the boundary conditions

$$v_{\kappa jp}(0) = 0, \quad v^{(0)}_{\kappa jp}(0) = 0. \hfill (60)$$

For bound states with $|E_{\kappa j}| < \mu$, we have $v_{\kappa jp}(\infty) = 0$. For scattering states with $|E_{\kappa j}| > \mu$ we may set $v_{\kappa jp}(r) = 0$ when $r > R$. However, it should be remarked that in general $v_{\kappa j2}(R) \neq 0$ when we require $v_{\kappa j1}(R) = 0$. In any case, we have

$$v_{\kappa jp}(\infty) = 0, \quad v^{(0)}_{\kappa jp}(\infty) = 0. \hfill (60')$$

Now the orthonormal relation takes the form (27') for both bound and scattering states, from which we have

$$(v_{\kappa' j}, v_{\kappa j}) = \delta_{\kappa\kappa'}, \quad \forall j,$$  \hfill (61)

where

$$(v_{\kappa' j}, v_{\kappa j}) \equiv \int_0^\infty dr \sum_p v^*_{\kappa'jp}(r)v_{\kappa jp}(r).$$  \hfill (62)
As we have set \( v_{\kappa j p}(r) = 0 \) when \( r > R \) for scattering states, the upper bound of integration in Eq.(62) need not be replaced by \( R \) for these states. For the free particle we have

\[
(v_{\kappa' j}^{(0)}, v_{\kappa j}^{(0)}) = \delta_{\kappa \kappa'}, \quad \forall j.
\]

Using Eqs.(58') and (61), it is easy to show that

\[
\int dr \sum_p G_{jpp}(r, r, E) = \sum_{\kappa} \frac{1}{E - E_{\kappa j} + i\epsilon}.
\] (64)

Employing the mathematical formula

\[
\frac{1}{x + i\epsilon} = P \frac{1}{x} - i\pi \delta(x),
\] (65)

and taking the imaginary part of the above equation, we have

\[
\text{Im} \int dr \sum_p G_{jpp}(r, r, E) = -\pi \sum_{\kappa} \delta(E - E_{\kappa j}).
\] (66)

Integrating this equation over \( E \) from \(-\mu^-\) to \( \mu^-\) yields

\[
\text{Im} \int_{-\mu^-}^{\mu^-} dE \int dr \sum_p G_{jpp}(r, r, E) = -n^-_j \pi,
\] (67)

where \( n^-_j \) is the number of bound states in the angular momentum channel \( j \) with \( |E_{\kappa j}| < \mu, \mu^- = \mu - 0, \) and \(-\mu^- = -\mu + 0\). The existence of bound states with critical energies \( E_{\kappa j} = \pm \mu \) does not alter this result. Similarly, we can show that

\[
\text{Im} \int_{-\mu^-}^{\mu^-} dE \int dr \sum_p G_{jpp}^{(0)}(r, r, E) = 0.
\] (68)

Combining Eqs.(67) and (68) we obtain

\[
\text{Im} \int_{-\mu^-}^{\mu^-} dE \int dr \sum_p [G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E)] = -n^-_j \pi.
\] (69)

On the other hand, substituting Eq.(58') and a similar one for the free particle into the right-hand side (rhs) of Eq.(59) we get

\[
\int dr \sum_p [G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E)] = \sum_{\kappa \kappa'} (v_{\kappa' j}^{(0)}, V v_{\kappa j}^{(0)}) (v_{\kappa' j}^{(0)}, v_{\kappa j}^{(0)}) \frac{(E - E_{\kappa j}^{(0)} + i\epsilon)(E - E_{\kappa' j}^{(0)} + i\epsilon)}{E - E_{\kappa j}^{(0)} + i\epsilon}.
\] (70)
where the definition of \( (v_{\kappa'j}, v_{\kappa j}^{(0)}) \) is similar to Eq.(62) and

\[
(v_{\kappa j}^{(0)}, V_{v_{\kappa'j}}) \equiv \int_0^\infty dr \sum_p v_{v_{\kappa'j}p}^0(r) V(r) v_{\kappa'j}p(r).
\]

(71)

Using the system of equations (15) and a similar one for the free case, and employing the boundary conditions (60) and (60'), one can show that

\[
(v_{\kappa j}^{(0)}, V_{v_{\kappa'j}}) = (E_{\kappa'j} - E_{\kappa j}^{(0)})(v_{\kappa j}^{(0)}, v_{\kappa'j}^j).
\]

(72)

Substituting this result into Eq.(70) and taking the imaginary part, we get

\[
\text{Im} \int dr \sum_p [G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E)] = \pi \sum_{\kappa\kappa'} \delta(E - E_{\kappa'j}^{(0)}) - \delta(E - E_{\kappa'j}) |(v_{\kappa'j}, v_{\kappa'j})|^2.
\]

(73)

Integrating this equation over \( E \) from \(-\infty\) to \(+\infty\) it turns out that

\[
\text{Im} \int_{-\infty}^{+\infty} dE \int dr \sum_p [G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E)] = 0.
\]

(74)

As in the nonrelativistic case, this means that the total number of states in a specific angular momentum channel is not altered by an external field, except that some scattering states are “pulled down” into the bound-state region. Here the external field may be either attractive or repulsive. This is different from the nonrelativistic case, where bound states exist only in attractive fields. Another new character of Dirac particles is that when the potential, say, an attractive potential, becomes strong enough, some bound states may even be pulled down into the region of negative-energy scattering states. This new character, however, does not alter the above conclusion that the total number of states in a specific angular momentum channel remains unchanged, since the conclusion is a consequence of the Dirac equation (15) and the completeness of the whole set of states.

If the integration over \( E \) in Eq.(74) is performed from \(-\mu^-\) to \(\mu^-\), we have

\[
\text{Im} \int_{-\mu^-}^{\mu^-} dE \int dr \sum_p [G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E)] = -\pi \sum_{\kappa'} \sum_{\kappa} |(v_{\kappa'j}, v_{\kappa j}^{(0)})|^2.
\]

(75)
where we have taken into account the fact that $|E_{\kappa j}^{(0)}| > \mu$, and the prime in $\sum'_{\kappa'}$ indicates that the summation over $\kappa'$ is performed only for bound states with $|E_{\kappa'j}| < \mu$. Using the completeness relation (28), which now takes the form

$$\sum_{\kappa j} \psi_{\kappa j}(r)\psi_{\kappa j}^*(r') = \delta(r - r'), \quad (28')$$

where the summation over $\kappa$ is performed for all states, we can show that

$$\sum_{\kappa} v_{\kappa j p}(r)v_{\kappa j q}^*(r') = \delta_{pq}\delta(r - r'), \quad \forall j. \quad (76)$$

This in turn leads to

$$\sum_{\kappa}|(v_{\kappa' j}, v_{\kappa j}^{(0)\dagger})|^2 = 1, \quad \forall \kappa', j. \quad (77)$$

Substituting Eq.(77) into Eq.(75) we recover Eq.(69). Combining Eqs.(69) and (74) we obtain

$$\text{Im} \left[ \int_{-\infty}^{-\mu^-} + \int_{\mu^-}^{+\infty} \right] dE \int dr r \sum_p \left[ G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E) \right] = n_{j^-} \pi. \quad (78)$$

We have thereupon finished the first step in our establishment of the Levinson theorem.

The next step is to calculate the lhs of Eq.(78) in another way. In the above treatment we have discretized the continuous spectrum of $E_{\kappa j}^{(0)}$ and the continuous part of $E_{\kappa j}$. In the following we will directly deal with these continuous spectra, and use the notations of Sec.II. Then the retarded Green function $G(r, r', E)$ is given by Eq.(58) but where $G_{jpq}(r, r', E)$ is given by

$$G_{jpq}(r, r', E) = \int_0^\infty dk \frac{v_{kjp}(r)v_{kj q}^*(r')}{\sqrt{rr'}(E - E_{kj} + i\epsilon)} + \frac{v_{-kjp}(r)v_{-kj q}^*(r')}{\sqrt{rr'}(E - E_{-kj} + i\epsilon)} + \sum_{\kappa} \frac{v_{\kappa j p}(r)v_{\kappa j q}^*(r')}{\sqrt{rr'}(E - E_{\kappa j} + i\epsilon)}, \quad (79)$$

where the integration is performed over scattering states while the summation over bound states, and $E_{\pm kj} = E_{\pm k}$ [cf. Eq.(21)] is independent of $j$. For $G_{jpq}^{(0)}(r, r', E)$ we have a
similar expression but without the last term. Using the formula (65) it is easy to show
that
\[
\Im \int dr \ r \ \sum \ G_{jpp}(r, r, E) \\
= -\pi \int_0^\infty dk \ [\delta(E - E_{kj})(v_{kj}, v_{kj}) + \delta(E - E_{-kj})(v_{-kj}, v_{-kj})] - \pi \sum \delta(E - E_{kj}). \quad (80)
\]

Here the inner products \((v_{kj}, v_{kj})\) etc. are defined in the same way as Eq.(62). Integrating
this equation over \(E\) as follows we have
\[
\Im \left[ \int_{-\infty}^{-\mu} + \int_{\mu}^{+\infty} \right] dE \ \int dr \ r \ \sum \ G_{jpp}(r, r, E) \\
= -\pi \int_0^\infty dk \ [(v_{kj}, v_{kj}) + (v_{-kj}, v_{-kj})] - \pi (\delta_{\mu j} + \delta_{-\mu j}), \quad (81)
\]
where \(\delta_{\mu j} = 1 (\delta_{-\mu j} = 1)\) if there exists a bound state in the angular momentum channel
\(j\) with critical energy \(E_{kj} = \mu (E_{kj} = -\mu)\), otherwise \(\delta_{\mu j} = 0 (\delta_{-\mu j} = 0)\). (See Sec.V for
further discussions.) Similar to Eq.(81) we have for the free case
\[
\Im \left[ \int_{-\infty}^{-\mu} + \int_{\mu}^{+\infty} \right] dE \ \int dr \ r \ \sum \ G_{jpp}(0)(r, r, E) = -\pi \int_0^\infty dk \ [(v_{kj}^{(0)}, v_{kj}^{(0)}) + (v_{-kj}^{(0)}, v_{-kj}^{(0)})]. \quad (82)
\]
Combining Eqs.(81) and (82) we obtain
\[
\Im \left[ \int_{-\infty}^{-\mu} + \int_{\mu}^{+\infty} \right] dE \ \int dr \ r \ \sum \ [G_{jpp}(r, r, E) - G_{jpp}(0)(r, r, E)] \\
= -\pi \int_0^\infty dk \ [(v_{kj}, v_{kj}) - (v_{kj}^{(0)}, v_{kj}^{(0)})] - \pi \int_0^\infty dk \ [(v_{-kj}, v_{-kj}) - (v_{-kj}^{(0)}, v_{-kj}^{(0)})] \\
- \pi (\delta_{\mu j} + \delta_{-\mu j}). \quad (83)
\]
From the orthonormal relations (23) and (27) one can show that
\[
(v_{\pm k'j}, v_{\pm kj}) = \delta(k - k'), \quad (v_{\pm k'j}^{(0)}, v_{\pm kj}^{(0)}) = \delta(k - k'), \quad \forall j. \quad (84)
\]
This implies that both integrands in the rhs of Eq.(83) are \(\delta(0) - \delta(0)\). There is, however,
a subtle difference between these two \(\delta(0)\)'s, and it is this difference that leads to the
Levinson theorem. To get rid of the difficulty of infiniteness, we define

\[(v_{lj}, v_{kj})_r \equiv \int_0^r \, dr \sum_p v_{ljp}^*(r)v_{kjp}(r), \quad (85)\]

and obtain \((v_{kj}, v_{kj})\) in the limit \(l \to k\) and \(r_0 \to \infty\). Using two systems of equations satisfied by \(v_{kjp}\) and \(v_{ljp}\) [cf. Eq.(15)], and the boundary conditions [cf. Eq.(60)]

\[v_{kjp}(0) = 0, \quad v_{kjp}^{(0)}(0) = 0, \quad (86)\]

it can be shown that

\[(E_{lj} - E_{kj})(v_{lj}, v_{kj})_r = v_{lj2}^*(r_0)v_{kj1}(r_0) - v_{lj1}^*(r_0)v_{kj2}(r_0). \quad (87)\]

Since \(r_0\) is large, we can use Eq.(30) to evaluate the rhs and in the limit \(l \to k\) we get

\[(v_{kj}, v_{kj})_r = \frac{r_0}{\pi} + \frac{1}{\pi} \frac{d\eta_j(E_k)}{dk} - (-)^m \frac{\mu}{2\pi k E_k} \cos(2kr_0 + 2\eta_j(E_k)). \quad (88)\]

Similarly, we have

\[(v_{kj}^{(0)}, v_{kj}^{(0)})_r = \frac{r_0}{\pi} - (-)^m \frac{\mu}{2\pi k E_k} \cos 2kr_0. \quad (89)\]

Obviously, the infiniteness lies in the first term \(r_0/\pi\) when we take the limit \(r_0 \to \infty\). This disappears when we subtract Eq.(89) from Eq.(88). Using the well-known formulas

\[\lim_{r_0 \to \infty} \frac{\sin 2kr_0}{\pi k} = \delta(k), \quad (90)\]

and \(g(k)\delta(k) = g(0)\delta(k)\) for any continuous function \(g(k)\), we obtain

\[\begin{align*}
(v_{kj}, v_{kj})_r - (v_{kj}^{(0)}, v_{kj}^{(0)})_r & = \frac{1}{\pi} \frac{d\eta_j(E_k)}{dk} + (-)^m \frac{\mu}{2\pi k E_k} \delta(k) \sin 2\eta_j(\mu) + (-)^m \frac{\mu}{\pi k E_k} \cos 2kr_0 \sin^2 \eta_j(E_k). \quad (91)
\end{align*}\]

In a similar way, we can show that

\[\begin{align*}
(v_{kj}, v_{kj})_r - (v_{kj}^{(0)}, v_{kj}^{(0)})_r & = \frac{1}{\pi} \frac{d\eta_j(-E_k)}{dk} - (-)^m \frac{\mu}{2\pi k E_k} \delta(k) \sin 2\eta_j(-\mu) - (-)^m \frac{\mu}{\pi k E_k} \cos 2kr_0 \sin^2 \eta_j(-E_k). \quad (92)
\end{align*}\]
So far in this section $V(r)$ need not be $V_a(r)$. In the following we set $V(r) = V_a(r)$. Then Eq.(52) holds. Therefore $\sin 2\eta_j(\mu) = \sin 2\eta_j(-\mu) = 0$. Integrating Eq.(91) over $k$ (from 0 to $+\infty$) and taking the limit $r_0 \to \infty$ we have

$$\int_0^\infty dk \left[ (v_{kj}, v_{k}) - (v_{kj}^{(0)}, v_{k}^{(0)}) \right] = \frac{1}{\pi} [\eta_j(+\infty) - \eta_j(\mu)] + (-)^m \frac{\mu}{\pi} \lim_{r_0 \to \infty} \int_0^\infty dk \frac{\sin^2 \eta_j(E_k)}{k E_k} \cos 2k r_0. \quad (93)$$

The last term in this equation can be decomposed into two integrals, the first from 0 to $\varepsilon = 0^+$, while the second from $\varepsilon$ to $+\infty$. The second integral vanishes in the limit $r_0 \to \infty$ since the factor $\cos 2k r_0$ oscillates very rapidly. For the first integral, we have

$$\int_0^\varepsilon dk \frac{\sin^2 \eta_j(E_k)}{k E_k} \cos 2k r_0 = \frac{1}{\mu} \int_0^\varepsilon dk \frac{\sin^2 \eta_j(E_k)}{k} = \frac{1}{\mu} \int_0^\varepsilon d\xi \frac{\sin^2 \eta_j(E_k)}{\xi}$$

as $k$ is very small. For the same reason we can use Eq. (52a). In the first case, $\tan \eta_j(E_k) = b_j^+ \xi^{2p_j^+}$, we have $\sin^2 \eta_j(E_k) = b_j^{+2} \xi^{4p_j^+}$, so that

$$\frac{1}{\mu} \int_0^\varepsilon d\xi \frac{\sin^2 \eta_j(E_k)}{\xi} = \frac{b_j^{+2}}{\mu} \int_0^\varepsilon d\xi \xi^{4p_j^+ - 1} = \frac{b_j^{+2}}{4\mu p_j^+} (\varepsilon a)^{4p_j^+} \to 0, \quad (\varepsilon \to 0^+).$$

In the second case, $\tan \eta_j(E_k) = \pi/2 \ln \xi$, we have $\sin^2 \eta_j(E_k) = \pi^2/4 \ln^2 \xi$, so that

$$\frac{1}{\mu} \int_0^\varepsilon d\xi \frac{\sin^2 \eta_j(E_k)}{\xi} = \frac{\pi^2}{4\mu} \int_0^\varepsilon d\xi \frac{1}{\ln^2 \xi} = -\frac{\pi^2}{4\mu} \frac{1}{\ln \varepsilon a} \to 0, \quad (\varepsilon \to 0^+),$$

where the final integral is calculated by setting $\ln \xi = w$. Thus the first integral vanishes as well. Therefore, we have

$$\int_0^\infty dk \left[ (v_{kj}, v_{k}) - (v_{kj}^{(0)}, v_{k}^{(0)}) \right] = \frac{1}{\pi} [\eta_j(+\infty) - \eta_j(\mu)]. \quad (94)$$

On the basis of Eqs.(92) and (52b) we can show that

$$\int_0^\infty dk \left[ (v_{-kj}, v_{-k}) - (v_{-kj}^{(0)}, v_{-k}^{(0)}) \right] = \frac{1}{\pi} [\eta_j(-\infty) - \eta_j(-\mu)]. \quad (95)$$
Substituting Eqs.(94) and (95) into Eq.(83) we obtain

$$\text{Im} \left[ \int_{-\infty}^{-\mu} dE \int dr \sum_p \left[ G_{jpp}(r, r, E) - G_{jpp}^{(0)}(r, r, E) \right] \right]$$

$$= [\eta_j(\mu) - \eta_j(+\infty)] + [\eta_j(-\mu) - \eta_j(-\infty)] - \pi(\delta_{\mu j} + \delta_{-\mu j}).$$

(96)

Combining this result with Eq.(78) we arrive at

$$[\eta_j(\mu) - \eta_j(+\infty)] + [\eta_j(-\mu) - \eta_j(-\infty)] = n_j \pi,$$

(97)

where

$$n_j = n_j^- + \delta_{\mu j} + \delta_{-\mu j}$$

(98)

is the total number of bound states, including the possible ones with critical energies \(\pm \mu\), in the angular momentum channel \(j\). Eq.(97) is the Levinson theorem for Dirac particles in an external central field in two dimensions. In the next section we will discuss some relevant points of the theorem.

**V. Discussions**

1. **On bound states with critical energies.** We consider the cut-off potential \(V_a(r)\). In the region \(r > a\), \(V_a(r) = 0\). Equations (16) and (17) are applicable for \(E = \mu\), while for \(E = -\mu\) Eqs.(18) and (19) should be employed.

In the region \(r > a\) and for \(E = \mu\), Eq.(17) becomes

$$F'' + \frac{1}{r} F' - \frac{m^2}{r^2} F = 0,$$

(99)

where \(m = |j_-|\) as before. The well behaved solution is

$$F^>_{\mu j}(r) = r^{-m}.$$  

(100)

The corresponding solution \(G^>_{\mu j}(r)\) obtained from Eq.(16) decreases more rapidly than \(F^>_{\mu j}(r)\) when \(r \to \infty\). Thus the normalizability of the solution is determined by the
behavior of $F_{\mu j}^>(r)$ at large $r$. Obviously, the solution can be normalized only when $m > 1$. It is well behaved when $m = 0, 1$, but cannot be normalized. In other words, the solution with energy $E = \mu$ is a bound state only when $j > 3/2$ or $j < -1/2$. The cases $j = \pm 1/2, 3/2$ do not correspond to bound states. Of course, a bound state with $E = \mu$ actually exists only when the potential $V_a(r)$ has a specific form such that the solution is continuous at $r = a$. For a square-well potential with depth $V_0$, this leads to

$$k_0 a J_{m+1}(k_0 a) = \left(2m + \frac{mV_0}{\mu}\right) J_m(k_0 a) \quad (101a)$$

for $j > 3/2$ or

$$J_{m-1}(k_0 a) = 0 \quad (101b)$$

for $j < -1/2$, where $k_0 = \sqrt{V_0^2 + 2\mu V_0}$. For a given $j$, this is satisfied only for some specified depth $V_0$ or radius $a$.

For $E = -\mu$, we consider Eq.(19). In the region $r > a$ it reduces to

$$G'' + \frac{1}{r}G' - \frac{m'^2}{r^2}G = 0, \quad (102)$$

where $m' = |j_+|$. The well behaved solution is

$$G_{-\mu j}^>(r) = r^{-m'} \quad (103)$$

One can get $F_{-\mu j}^>(r)$ from Eq.(18). It decreases more rapidly than $G_{-\mu j}^>(r)$ when $r \to \infty$. Therefore the solution is normalizable when $m' > 1$. In other words, the solution is a bound state when $j > 1/2$ or $j < -3/2$. The cases $j = \pm 1/2, -3/2$ do not correspond to bound states. For the square-well potential, a bound state with $E = -\mu$ really exists only when $V_0 > 2\mu$ and satisfies

$$\tilde{k}_0 a J_{m'+1}(\tilde{k}_0 a) = \left(2m' - \frac{m'V_0}{\mu}\right) J_{m'}(\tilde{k}_0 a) \quad (104a)$$
for $j < -3/2$ or

$$J_{n'-1}(\tilde{k}_0a) = 0 \quad (104b)$$

for $j > 1/2$, where $\tilde{k}_0 = \sqrt{V_0^2 - 2\mu V_0}$. Given $j$, say $j = -5/2$, there exist infinitely many solutions of $V_0$ to Eq. (101b). They are functions of $a$. In general most of them cannot satisfy Eq. (104a) at the same time as the two equations are independent, thus the two critical energy bound states do not appear simultaneously. However, the solutions of Eq. (104a) are also functions of $a$. By varying the parameter $a$, it may be possible to match some specific solution of Eq. (101b) with some specific one of Eq. (104a). When this really happens, the two critical energy bound states can appear simultaneously. Anyway, it cannot be asserted that the two critical energy bound states never appear simultaneously for any potential.

The form of the Levinson theorem (97) is not modified by the existence of the critical energy states, regardless of whether they are bound states or not. The existence of critical energy bound states just changes $n_j$ from $n_j^-$ to $n_j^- + 1$ (when there is one with $E = \mu$ or $E = -\mu$) or $n_j^- + 2$ (when the two appear simultaneously), and does not alter the form of Eq. (97). In three dimensions, the theorem involves additional terms that vanish except when there exist half bound states[10]. In two dimensions there is no such term, which is clear from the result (97). Thus the theorem in two dimensions is not affected by the existence of half bound states. The reason for the difference between two- and three-dimensional cases is similar to that in the nonrelativistic theory. This has been discussed in detail in Ref. [16].

2. About $\eta_j(\pm\infty)$. We write down two systems of equations for the radial wave functions $v_{kjp}^U(r)$ and $\tilde{v}_{kjp}^U(r)$ in two external fields $U(r)$ and $\tilde{U}(r)$, respectively. Using the
boundary condition (86) and the asymptotic form (30), it is not difficult to show that

\[ \sin[\tilde{\eta}_j^U(E_k) - \eta_j^U(E_k)] = -\frac{\pi E_k}{k}(\tilde{v}_{kj}^U, (\tilde{U} - U)v_{kj}). \]  

(105)

Here the superscript \( U \) is used to distinguish the wave functions and phase shifts from those in the external field \( V(r) \). Now we set \( U(r) = \lambda V(r) \) [\( V(r) \) not necessarily be \( V_a(r) \)], \( \tilde{U}(r) = (\lambda + \Delta \lambda)V(r) \), and denote \( v_{kjp}^U(r) = v_{kjp}(r, \lambda), \eta_j^U(E_k) = \eta_j(E_k, \lambda) \). When \( \Delta \lambda \to 0 \), \( \tilde{v}_{kjp}^U(r) = v_{kjp}(r, \lambda + \Delta \lambda) \) can be replaced by \( v_{kjp}(r, \lambda) \), and Eq.(105) becomes

\[ \sin[\Delta \eta_j(E_k, \lambda)] = -\frac{\pi E_k}{k}\Delta \lambda(v_{kj}(r, \lambda), V(r)v_{kj}(r, \lambda)), \]  

(106)

where \( \Delta \eta_j(E_k, \lambda) = \eta_j(E_k, \lambda + \Delta \lambda) - \eta_j(E_k, \lambda) \). Obviously, \( v_{kjp}(r, 0) = v_{kjp}^{(0)}(r), v_{kjp}(r, 1) = v_{kjp}(r), \eta_j(E_k, 1) = \eta_j(E_k) \). It is natural to define \( \eta_j(E_k, 0) = 0 \) in the absence of an external field. It is also natural to require that for any finite \( k, \eta_j(E_k, \lambda) \) be continuous functions of \( \lambda \) when \( \lambda \) varies continuously from 0 to 1, as all quantities in the equation and boundary conditions are continuous in \( \lambda \). With this requirement and the above definition \( \eta_j(E_k, 0) = 0 \) one can determine \( \eta_j(E_k) \). It should be remarked, however, that \( \eta_j(\mu, \lambda) \equiv \lim_{k \to 0} \eta_j(E_k, \lambda) \) is not continuous in \( \lambda \). Otherwise the Levinson theorem would be impossible. To see this, let us have a look at the original Levinson theorem in an attractive potential \( \lambda V(r) \) in three dimensions:

\[ \delta_l(0, \lambda) - \delta_l(\infty, \lambda) = n_l(\lambda)\pi, \quad l = 0, 1, 2, \ldots, \]  

(107)

where we have not included the modified \( l = 0 \) case. If \( \delta_l(0, \lambda) \) is a continuous function of \( \lambda \), then the lhs of Eq.(107) is continuous in \( \lambda \). On the other hand, \( n_l(\lambda) \) is a nonnegative integer. When \( \lambda \) varies continuously from 0 to 1, \( n_l(\lambda) \) varies from 0 to \( n_l \) by discontinuous jumps. The rhs of Eq.(107) is obviously not a continuous function of \( \lambda \). This is a contradiction. So that \( \delta_l(0, \lambda) \) cannot be continuous in \( \lambda \). The case for \( \eta_j(\mu, \lambda) \) is similar.
Since \( \eta_j(E_k, \lambda) \) is a continuous function of \( \lambda \), the \( \sin[\Delta \eta_j(E_k, \lambda)] \) on the l.h.s of Eq.(106) can be safely replaced by \( \Delta \eta_j(E_k, \lambda) \), and we have

\[
\frac{d\eta_j(E_k, \lambda)}{d\lambda} = -\frac{\pi E_k}{k} (v_{kj}(r, \lambda), V(r)v_{kj}(r, \lambda)).
\] (108)

When \( E_k \) is infinitely large, since \( V(r) \) is not very singular at \( r = 0 \) and is regular elsewhere, we may ignore \( \lambda V(r) \) in the system of radial equations and approximately replace \( v_{kj}(r, \lambda) \) by \( v_{kj}^{(0)}(r) \) in the above equation. Then Eq.(108) can be easily integrated over \( \lambda \) from 0 to 1 and results in

\[
\eta_j(E_k) = -\frac{\pi E_k}{k} (v_{kj}^{(0)}, Vv_{kj}^{(0)}).
\] (109)

Substituting the exact solution (22) into Eq.(109) and then replacing the Bessel functions by their asymptotic forms as \( k \) is large, we arrive at

\[
\eta_j(+\infty) = -\int_0^\infty dr V(r).
\] (110a)

In a similar way we can show that

\[
\eta_j(-\infty) = \int_0^\infty dr V(r).
\] (110b)

As we have assumed that \( V(r) \) is less singular than \( r^{-1} \) when \( r \to 0 \) and decreases more rapidly than \( r^{-2} \) when \( r \to \infty \), the above integrals converge. These results have the same form as those in three dimensions[9,17,18]. Of course, Eq.(110) holds in the special case \( V(r) = V_a(r) \). As a consequence of the above results, we have

\[
\eta_j(+\infty) + \eta_j(-\infty) = 0.
\] (111)

This reduces the Levinson theorem (97) to the form of Eq.(1). It means that the sum of the phase shifts at the two thresholds serves as a counter for the bound states in a
specific angular momentum channel. This is similar to the case in three dimensions, but
is somewhat simpler.

The Levinson theorem for Dirac particles could not be separated into two parts, each
of which likes that for Schrödinger particles [6,9,10,19]. Essentially this is because that
positive-energy solutions or negative-energy solutions alone do not form a complete set.
An evidence can be seen as follows. In an attractive potential, say, Eq. (108) shows that
\( \eta_j(E_k, \lambda) \) is positive and increases with \( \lambda \). Similarly, it can be shown that \( \eta_j(-E_k, \lambda) \)
is negative and decreases when \( \lambda \) increases. Thus \( \eta_j(-\mu) \) may be negative when the
attractive potential becomes strong enough. In three dimensions this has been verified
by numerical calculations[10,18]. As \( \eta_j(-\mu) \) may be negative, it cannot always equal a
nonnegative integer (in unit of \( \pi \)) and thus cannot serve as a counter.

3. Extension to more general potentials. Throughout this paper we have assumed that
\( V(r) \) is less singular than \( r^{-1} \) when \( r \to 0 \) and decreases more rapidly than \( r^{-2} \) when
\( r \to \infty \). In the development of the Levinson theorem, we further cut off \( V(r) \) to the
special case \( V_a(r) \) for the sake of exact analysis. However, the radius \( a \) beyond which
\( V_a(r) \) vanishes is not specified in our discussion. Though both sides of Eq.(1) depend
on the particular form of \( V_a(r) \) and thus depend on \( a \), the equality between them does
not. Thus we expect that Eq.(1) remains valid when \( V_a(r) \) is varied continuously to the
limit \( a \to \infty \) provided that the asymptotic form (30) holds and \( n_j \) remains finite in the
process. This requires that \( V(r) \) decreases rapidly enough when \( r \to \infty \). It seems that
the Levinson theorem holds at least for short-range potentials that decrease more rapidly
than \( r^{-2} \) when \( r \to \infty \).
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