The ‘Multifractal Model’ of Turbulence and A Priori Estimates in Large-Eddy Simulation, I. Subgrid Flux and Locality of Energy Transfer

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Abstract

We establish and discuss a priori estimates on subgrid stress and subgrid flux for filtering schemes used in the turbulence modelling method of Large-Eddy Simulation (LES). Our estimates are derived as rigorous consequences of the exact subgrid stress formulae from Navier-Stokes equations under realistic conditions for inertial-range velocity fields, those conjectured in the Parisi-Frisch “multifractal model.” The estimates are shown to be an expression of “local energy cascade,” i.e. the dominance of local wavevector triads in the energy transfer. We prove that for nearly any reasonable filter function the LES method defines an energy flux in which local triads dominate in individual realizations, due to cancellation of distant triadic contributions by detailed conservation. A somewhat similar observation of Leslie and Quarini on graded filters in the EDQNM closure is shown to be unrelated to the cancellation we establish in Navier-Stokes solutions. The sharp Fourier cutoff filter is one example which does not satisfy the modest conditions of our proof and, in fact, we show that with that filter the energy transfer in individual realizations at arbitrarily high Reynolds number will be dominated by nonlocal, convective sweeping.

Key words: Navier-Stokes, turbulence, large-eddy simulation, multifractal model

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1 Introduction

A fundamental hypothesis in the Kolmogorov phenomenology of developed turbulence is the “local energy cascade” [1, 2, 3]. In the first of his 1941 papers, Kolmogorov expressed this in spatial terms as a process of energy transport to small scales by successive transfer from “pulsations” of one size to daughter “pulsations” of size about half as large. The parallel works of Heisenberg [4], von Weizsäcker [5] and Onsager [6, 7] stated this in spectral terms as a process in which energy is transported through Fourier space along chains of wavevector triads in which typically two wavevectors of the same size transfer energy to a third wavevector about twice as big. Recently, the dominance of these local wavevector triads to turbulent energy transfer has been the subject of much debate. The papers [8]-[20] in chronological order are a representative sample containing many further references. The cited studies are based mostly upon direct numerical simulation of the Navier-Stokes equations or upon the solution of analytical closures, such as the EDQNM transfer equation. The first method is severely constrained by the low Reynolds numbers attainable with current computer resources. The numerical solution of analytical closures, on the other hand, is feasible at high Reynolds number—at least in the case of homogeneous and isotropic turbulence—but is subject to doubt concerning the uncontrolled closure hypotheses invoked.

However, we have in fact demonstrated in our past work [21] locality of instantaneous energy transfer for individual turbulent solutions of the Navier-Stokes equations based upon a direct, rigorous analysis of the inertial terms and thus valid for Reynolds numbers arbitrarily large without any assumption of statistical homogeneity or isotropy. Our original study involved mathematical considerations on so-called “weak solutions” which might make it inaccessible to fluid-dynamicists without a good grounding in modern analysis. Therefore, we shall present here the basic arguments of our earlier work with an emphasis on their physical interpretation in order to place them in the context of the existing turbulence literature. In addition, we shall discuss stronger results on simultaneous space-scale locality of transfer which we subsequently
obtained \cite{22} and new material exposing the various relations between both of our two works and the previous literature.

We have found that the filtering technique commonly used in the Large-Eddy Simulation or LES modelling method (see \cite{23,24}) provides the most theoretically well-founded definition of the intuitive idea of “scales of motion” in turbulent flow. In this method a filtering function $G_\ell(\mathbf{r}) = \ell^{-d}G_\ell(\mathbf{r}/\ell)$ is convoluted with the velocity field, as

$$\mathbf{\nabla}_\ell(\mathbf{r}) \equiv \int d^d \mathbf{r}' G_\ell(\mathbf{r} - \mathbf{r}') \mathbf{v}(\mathbf{r}')$$

(1)

to define the “large-scale velocity” or “resolved field” $\mathbf{v}_\ell$. The “small-scale velocity” or “subgrid field” is then simply defined as the complementary component:

$$\mathbf{v}_\ell'(\mathbf{r}) \equiv \mathbf{v}(\mathbf{r}) - \mathbf{v}_\ell(\mathbf{r}).$$

(2)

When the filtering operation is applied to the Navier-Stokes system an equation is obtained for $\mathbf{v}_\ell$:

$$\partial_t \mathbf{v}_\ell + \nabla \cdot (\mathbf{v}_\ell \mathbf{v}_\ell + \tau_\ell) = -\nabla \mathbf{p}_\ell + \nu_0 \Delta \mathbf{v}_\ell,$$

(3)

in which $\mathbf{p}_\ell$ is the filtered pressure field (required to maintain $\nabla \cdot \mathbf{v}_\ell = 0$) and

$$\tau_\ell = (\mathbf{v} \mathbf{v})_\ell - \mathbf{\nabla}_\ell \mathbf{\nabla}_\ell,$$

(4)

is a tensor representing the “subscale stress” of the eliminated turbulent eddies.

We shall use the same filtering technique as in conventional LES but in order to derive rigorous mathematical estimates for the exact subgrid stress $\tau_\ell$ which follows from Navier-Stokes. We show under very mild assumptions on the filter function $G$ that the energy flux which appears in the LES method is dominated by local wavevector triads, in agreement with the conventional wisdom. However, a few commonly used filters—including the sharp cutoff filter in Fourier space—will be shown to violate these assumptions and to lead to formulae for subscale stress which are not appropriate to model as an effect of the small-scales alone. The LES technique with a “good” filter will in fact be shown to simply and naturally resolve
certain subtleties of the locality issue, discussed in [8]-[20], which arise purely as an artefact of the conventional use of Fourier series.

Our results have also some further implications. First, the derived estimates place an a priori constraint on subgrid models and filters and may be used to evaluate the various choices. This is the subject of a companion paper [25], hereafter referred to as II. Second, the estimates imply restrictions on the space-regularity of turbulent velocity fields in the inertial range of scales in order to be consistent with the basic phenomenology of constant mean energy flux. Roughly speaking, the velocity must appear on those scales as a continuous, nowhere-differentiable field of the type represented by the famous Weierstrass function or a typical path of Brownian motion. This was noted already by Onsager in 1949 [7].

In fact, all of the estimates we derive are based upon the regularity of inertial-range velocity fields postulated in the Parisi-Frisch “multifractal model” [26], who generalized Onsager’s observation. See [27], [28] for recent accounts. Let us just recall here that those authors proposed that in the limit of zero viscosity the velocity field in turbulence should be Hölder continuous at each point, that is, for every point \( \mathbf{r} \) in the flow domain an inequality

\[
|\mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r})| \leq C|\mathbf{l}|^h
\]

should be satisfied for some real \( h \) and all \( |\mathbf{l}| \leq 1 \). (This definition must be modified for negative \( h \).) If so, then to each point may be assigned the largest value of \( h \) for which Eq. (5) holds and it was furthermore postulated that the points where the value \( < h \) occurs should be a fractal set \( S(h) \) with dimension \( D(h) \). This is the origin of the word “multifractal.” Notice that the original 1941 Kolmogorov theory corresponds in this language to \( D(1/3) = 3 \) and \( D(h) = -\infty \) otherwise, that is, \( h = 1/3 \) everywhere. \(^1\) Although the model must be considered to be conjectural, we

\[
S_p(\ell) \equiv \langle [\mathbf{I} \cdot (\mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r}))]^p \rangle \\
\sim v_0^p \left( \frac{\ell}{\gamma L} \right)^{\zeta_p},
\]

\(^1\)The original motivation for this “multifractal hypothesis" was the experimental observation [29] that longitudinal velocity-difference moments (structure functions) scale as powers of the separation length \( \ell = |\mathbf{l}| \).
shall see that it has some mathematical support from the Navier-Stokes equations. In fact, the “multifractal” velocities are the most regular velocity-fields possible which are still compatible with the basic physics of constant mean energy flux through inertial-range scales. In particular, it is not possible for the velocity to be everywhere space-differentiable on inertial-range length-scales (which would imply that Eq. (5) hold at all points \( r \) with \( h \geq 1 \).) Just to be clear, we emphasize that the conclusion is that Eq. (5) cannot hold for Navier-Stokes solutions with \( h > 1/3 \) at all \( r \) and inertial-range \( \ell \). The argument for this is simple, but somewhat off the main line of the paper, so that we include it as an Appendix.

The contents of our paper are as follows: In the next Section II we shall derive \textit{a priori} estimates on subgrid stress and flux as defined in the LES method, which are \textit{exact} consequences of Navier-Stokes assuming the “multifractal” regularity of velocities. In Section III we explain how the derived estimates express the “locality” of the instantaneous energy-transfer. It is shown by explicit analysis that for a “good” choice of filter the LES flux gets a negligible contribution from distant wavenumber triads and we thereby resolve some of the controversies regarding “locality” of transfer. We contrast our result with an earlier observation of Leslie and Quarini \[30\]. We also show that there are “bad” choices of filter for which the flux is dominated by distant triadic interactions of convective type. Finally, in Section IV we make some closing remarks and conclusions on the “multifractal model,” the locality of transfer, and the filtering approach.

over inertial-range scales \( \eta \ll \ell \ll L \) and that \( \zeta_p \neq p/3 \), the prediction of K41 theory. Parisi and Frisch explained this latter fact heuristically by noting that, with their hypotheses,

\[
S_p(\ell) \sim v_0^p \int d\rho(h) \left( \frac{\ell}{L} \right)^{ph} \left( \frac{\ell}{L} \right)^{3-D(h)},
\]

where the second factor measures the fraction of boxes of size \( \ell \) in the domain of size \( L \) in which the exponent \( h \) occurs. Applying steepest descent to evaluate this integral for \( \ell \ll L \) they obtained Eq. (6) with

\[
\zeta_p = \inf_h [ph + (3 - D(h))].
\]
2  *A Priori* Estimates and the “Multifractal Model”

We have already introduced the concept of “subscale stress” \( \tau_\ell \) in the LES filtering technique. It is also natural to write down a *local energy balance* for the resolved kinetic energy

\[
\tau_\ell(r, t) \equiv \frac{1}{2} \nabla_\ell^2(r, t),
\]

as

\[
\partial_t \tau_\ell(r, t) + \nabla \cdot j_\ell(r, t) = -\Pi_\ell(r, t) - \varepsilon_\ell(r, t).
\]

For example, see Section 6.1 of [24]. Each of the terms in the microscopic balance relation Eq.(10) has a precise physical interpretation. The term \( \varepsilon_\ell(r, t) \equiv \nu_0 (\nabla_\ell)^2 \) represents the local dissipation of energy in the length-scales \( > \ell \) through the action of molecular viscosity. The term \( j_\ell(r, t) \equiv (p_\ell + \tau_\ell)\nabla_\ell + \tau_\ell \cdot \nabla_\ell - \nu_0 \nabla \tau_\ell \) is a spatial current of energy in the scales \( > \ell \).

Finally, the term of main interest for us

\[
\Pi_\ell(r) \equiv - (\nabla_\ell) : \tau_\ell
\]

represents a *local energy flux* from the large-scales to the small-scales. It gives the “effective dissipation” of energy in the large-scales \( > \ell \) due to the action of the eddy-stress of the small-scales \( < \ell \) on the gradients of the large-scale motion.

In this section we shall derive some simple *a priori* estimates for the quantities \( \tau_\ell(r) \) and \( \Pi_\ell(r) \) assuming the H"{o}lder regularity Eq.(5) at point \( r \), a condition usually denoted \( \mathbf{v} \in C^h(r) \).

Only afterward will we consider the consequences of the estimates and the justification for the assumed regularity. A key identity which was noted, in a different language, in a paper of Constantin et al. [31], is

\[
\tau_\ell(r) = \langle \Delta_\ell \mathbf{v}(r) \Delta_\ell \mathbf{v}(r) \rangle_\ell - \mathbf{v}'_\ell(r) \mathbf{v}'_\ell(r).
\]

where

\[
\Delta_s \mathbf{v}(r) \equiv \mathbf{v}(r) - \mathbf{v}(r - s)
\]
is the “backward difference” and
\[
\langle f \rangle_\ell \equiv \int ds \, G_\ell(s)f(s),
\]
denotes an “average” over the separation distance with respect to the filter function. The proof is elementary. Writing the righthand side of Eq. (12) as
\[
\int ds \, G_\ell(s)[v(r) - v(r - s)][v(r) - v(r - s)] - [v(r) - v_\ell(r)][v(r) - v_\ell(r)],
\]
using the normalization condition
\[
\int d^d r \, G_\ell(r) = 1
\]
and some shifting of integration variables and cancelling terms gives \( \langle \Delta v \rangle_\ell \), which is the definition of the subscale stress.

A more symmetric and suggestive form of the same identity is
\[
\tau_\ell(r) = \langle \Delta v(r) \Delta v(r) \rangle_\ell - \langle \Delta v(r) \rangle_\ell \langle \Delta v(r) \rangle_\ell.
\]
The proof is instructive. By definition
\[
\nabla_\ell(r) = \int d^d r' \, G_\ell(r - r')v(r').
\]
Therefore, using \( v_\ell' = v - \nabla_\ell \) and the normalization condition Eq. (16) for \( G_\ell \),
\[
v_\ell'(r) = \int d^d r' \, G_\ell(r - r')[v(r) - v(r')]
= \int d^d s \, G_\ell(s)\Delta s v(r) \equiv \langle \Delta v(r) \rangle_\ell.
\]
Therefore, the subgrid velocity \( v_\ell' \) is represented by an “average” over velocity-differences with separation < \( \ell \). Substituting into the previous formula Eq. (12) for \( \tau_\ell \) gives Eq. (17). In this expression for the subscale stress the velocity again appears only through its difference \( \Delta v \) over length-scales < \( \ell \). If the filter function \( G \) is positive as well as normalized, then Eq. (14) defines a true average and, from Eq. (17), \( \tau_\ell \) is a positive, symmetric tensor. An equivalent identity was discovered and used to establish positivity of the stress in a recent work of Vreman et al. [32], which appeared after the first submission of this paper.
We can now straightforwardly derive our \textit{a priori} estimates using the condition $v \in C^h(r)$, i.e. $|\Delta_s v(r)| \leq (\text{const.})|s|^h$. For example, it follows directly from Eq.(19) that

\begin{equation}
|v'_\ell(r)| \leq (\text{const.}) \int d^d s \, G_\ell(s)|s|^h = O\left(\ell^h\right).
\end{equation}

if the filter function satisfies the modest condition $\int d\mathbf{x} \, |G(\mathbf{x})| \cdot |\mathbf{x}|^h < \infty$. Likewise,

\begin{equation}
\tau_\ell(r) = O\left(\ell^{2h}\right)
\end{equation}

by using Eq.(17) if $\int d\mathbf{x} \, |G(\mathbf{x})| \cdot |\mathbf{x}|^{2h} < \infty$. This is our main estimate on the subscale stress.

To obtain an estimate on the subgrid flux $\Pi_\ell$ we also require a formula for $\nabla v_\ell$ expressing it in terms of velocity-differences over length-scales $< \ell$. Using $\int \nabla G = 0$ we find

\begin{equation}
\nabla v_\ell(r) = \int d^d r' \, (\nabla G_\ell)(r-r') [v(r') - v(r)] \\
= - \int d^d s \, (\nabla G_\ell)(s) \Delta_s v(r).
\end{equation}

Making the same estimations as before, we then find that

\begin{equation}
\nabla v_\ell(r) = O\left(\ell^{h-1}\right),
\end{equation}

if $\int d\mathbf{x} \, |\nabla G(\mathbf{x})| \cdot |\mathbf{x}|^h < \infty$. From this result and the formula $\Pi_\ell = -\nabla v_\ell : \tau_\ell$ it follows that

\begin{equation}
\Pi_\ell(r) = O\left(\ell^{3h-1}\right).
\end{equation}

This is our main estimate for the flux.

If we gather together all of the conditions on the filter function $G$, we see that it is sufficient for the previous estimates to apply that it obey the moment condition

\begin{equation}
\int d\mathbf{x} \, [|G(\mathbf{x})| + |\nabla G(\mathbf{x})|] \cdot |\mathbf{x}|^2 < \infty,
\end{equation}

when the Hölder indices lie in the range $0 < h < 1$. The condition is quite modest. Of filters commonly used in practical LES, the \textit{Gaussian filter} is a “good” example which easily satisfies these constraints. On the other hand, another common filter, the \textit{sharp cutoff filter}, violates
condition Eq. (22), because of slow spatial decay in space $\sim |x|^{-d}$ for large $x$. In fact, we will see by an example from [21] that the previous bounds for $\tau_\ell$ and $\Pi_\ell$ may actually be violated for this filter. Smoothly cutting off the tails of the filter in physical space would cure this problem in principle, but there are definite signs of the bad behavior in practical use of the sharp-cutoff filter in current LES simulations. This is discussed at length in II.

So far we have employed the hypothesis of local regularity made in the Parisi-Frisch “multifractal model,” without considering its validity. Since the assumption of local Hölder regularity cannot presently be derived \textit{a priori} as a theorem for solutions of the 3D Euler equations, it does require some justification. Mathematical results—presented in [33]—in conjunction with basic facts of turbulence phenomenology strongly suggest the “multifractal model” as the correct candidate to describe the space variation of inertial-range velocity fields. We must emphasize, however, that the key elements of the “multifractal model” used in our previous analysis do not depend upon the questionable experimental results on anomalous scaling [29]. On the contrary, much weaker conditions guarantee the validity of the local Hölder regularity we employed. A particular condition which suffices is that the inertial-range energy spectrum be power-law $E(k) \sim k^{-n}$ with any spectral exponent whatsoever. In fact, we have proved elsewhere [33] that \textit{any homogeneous random field with such a power-law spectrum has realizations with local Hölder regularity with probability one}. We just emphasize here that “intermittency” is irrelevant for this issue and that the spectral exponent could be the Kolmogorov one, $n = 5/3$, or anything else. Furthermore, local Hölder regularity is supported by the estimate Eq. (24), which implies $\Pi_\ell \to 0$ as $\ell \to 0$ if $h > \frac{1}{3}$. It is impossible to account for mean energy flux through the inertial range if that condition holds (e.g. if the velocities are spatially smooth) at all points of the domain on inertial-range scales. Therefore, the “multifractal model” hypothesis of local Hölder regularity is vital to account for turbulent energy balance in the inertial-range.

\footnote{If the energy spectral exponent is exactly $n = \frac{5}{3}$, then “negative Hölder singularities” may occur but can occupy a space set of Hausdorff dimension equal to at most $2\frac{1}{3}$ [33]. If $n > \frac{5}{3}$, then the Hausdorff dimension of the “negative exponent” set is even smaller.}
3 Locality of Energy Transfer

3.1 The Estimates and Local Transfer in Wavenumber

The \textit{a priori} estimates on subscale stress and energy flux in LES were derived in the previous section just as mathematical upper bounds. However, they have a very intuitive physical interpretation in terms of \textit{locality of energy transfer}, which we discuss in this section. An essential point in the LES method is that it defines, via Eq. (11), an energy flux which is proportional only to the \textit{gradient} of the large-scale velocity field. The physics of this expression seems transparent: it describes the energy transfer due to the stretching action of the large-scale strain \( \mathbf{\sigma}_\ell \equiv \frac{1}{2} \left( \nabla \mathbf{v}_\ell + \nabla \mathbf{v}_\ell^T \right) \) upon the stress field \( \boldsymbol{\tau}_\ell(\mathbf{r}) \) due to the random distribution of vorticity elements in the small-scales \( < \ell \). In fact, it is often convenient to regard the fine-grained turbulence as a “tangle” of vortex filaments with a viscoelastic response to imposed large-scale strain \[34\]. All of the \textit{explicit} contributions of convective processes to the rate of change of the resolved energy \( e(\mathbf{r}, t) \) at the point \( \mathbf{r} \) in the LES method are incorporated into the term \( (\tau_\ell + p_\ell) \nabla_\ell \) of the current \( j_\ell \) and can be accounted as contributions to transport of energy in space rather than “downward” to smaller scales. Since it is the large-scale convective processes which can produce apparent violations of locality in energy transfer in the wavenumber description, it is rather natural to find that \( \Pi_\ell(\mathbf{r}) \) obeys a “local estimate” \( \sim \ell^{3h-1} \) which arises from \( \nabla \nabla_\ell \sim \ell^{h-1} \) and \( \boldsymbol{\tau}_\ell \sim \mathbf{v}_\ell' \mathbf{v}_\ell' \sim \ell^{2h} \).

However, we shall show that this simple picture actually depends upon the technical-looking condition Eq. (25) we imposed upon the filter function \( G \) in the previous section. Contrary to naive expectations, the LES flux can retain \textit{implicit} contributions from purely convective processes if that (mild) condition is violated. We shall show this by a concrete example. In such cases it is inappropriate to estimate \( \tau_\ell \sim \mathbf{v}_\ell' \mathbf{v}_\ell' \) because the “subscale stress” is actually still dependent upon the large-scale motion! In the course of our analysis we will make contact with the traditional discussion of the locality issue in Fourier space and resolve the subtle points which arise in that description.
In the Fourier representation, the conventional definition of “(subscale) energy flux” $\Pi(k, t)$
goes back at least to the work of Heisenberg [4] and von Weizsäcker [5], who defined it as the
total output power from the Fourier modes in the sphere of radius $k$ in wavenumber space:

$$
\Pi(k, t) = - \left. \frac{dE^{<k}(t)}{dt} \right|_{\text{Euler}}.
$$

(26)

Here $E^{<k}(t)$ is the instantaneous energy in modes of wavenumber $< k$. If the sharp cutoff filter
is used in LES in a rotationally symmetric form, as

$$
\hat{G}(k) = \begin{cases} 
1 & \text{if } |k| < 2\pi \\
0 & \text{otherwise.}
\end{cases}
$$

(27)

then the global flux thereby defined

$$
\Pi_\ell = \int_\Omega dr \, \Pi_\ell(r)
$$

(28)

coincides with the traditional flux, through the relation $\Pi_\ell = \Pi(2\pi/\ell)$.

For a “good” filter obeying the condition Eq.(25) it follows from the (stronger) local estimate
of the previous section that $\Pi_\ell = O\left(\ell^{3h-1}\right)$ if the minimum Hölder exponent of the velocity is $h$. However, it was found in [21] that the global flux for the sharp Fourier filter may be much
larger, and, in general, only the estimate

$$
\Pi(k) = O\left( k^{1-2h} \right),
$$

(29)

may be true. In Example 1 of [21] we constructed in 3D a velocity field $\mathbf{v} \in C^h(\Omega)$, $0 < h < 1$,
for which $\Pi(k) = (\text{const.})k^{1-2h}$ at a set of k’s of arbitrarily large magnitude (namely, $k = 2^\Lambda k_0$
with integer $\Lambda > 0$.) It is important here just to recall the salient features of that example.

We have already seen that the estimate $O(2^{\Lambda(1-3h)})$ is true if the local interactions dominate
in the energy flux. Therefore, the example constructed was of the opposite sort, for which
highly nonlocal triads are dominant. The idea of the construction was very simple: we used a
Fourier series in which every octave band of wavevectors contained one conjugate pair of modes
at the very bottom and one pair just below the top, in just such a way that the top pair of
each band and the bottom pair of the next highest band differ by a fixed pair of wavevectors in the lowest band. This pair of low-wavenumber modes induce an instantaneous transfer of energy between the high-wavenumber bands from the pairs at the top of one band into those at the bottom of the next higher band, without losing any energy themselves. This type of “catalytic” effect of the low-wavenumber modes for transfer between high-wavenumbers was, to our knowledge, first discussed in detail by Brasseur and Corssin in [8]. The same effect was observed in the numerical study of Domaradzki and Rogallo [10], who described it as “local transfer by nonlocal triads” (although their simulation was at a rather low Reynolds number and did not make clear whether such interactions might persist as an inertial-range effect). The triads involved correspond to an “R-interaction” in the classification of Waleffe [14].

A remarkable feature of the example is that the flux is proportional to the amplitude $U$ of the low-wavenumber modes rather than to their gradient. It therefore corresponds to an interaction of strictly convective type. This interpretation of the “R-interactions” was also argued by Waleffe in [14], whereas Domaradzki [17] has put forward a very different picture “...where the role of the large scales is to produce intermittent regions of relatively strong, internal shears characterized by smaller length scales, which serve as regions of efficient small-scale transfer.” The explicit verification of the convective nature of these processes renders the latter explanation untenable for the inertial-range at high Reynolds number. In fact, the basic physics of these processes was already discussed by Kraichnan in 1966 in the context of the energy transfer equation of ALHDIA [35]. He noted that: “Convection of high-wavenumber structures by strongly excited low-wavenumber velocity components implies a rapid change of phase of the high-wavenumber Fourier amplitudes; that is to say, a rapid exchange of energy between sine and cosine components of the high wavenumbers. This exchange is represented in (2.6) [his transfer equation] by the large, cancelling input and output contributions. The net contribution $T_{<q}(k, t)$ represents the effect of straining alone.” In other words, the convection of structures at a high wavenumber $\sim k$ by the largest, most energetic eddies will result in a rapid transfer of energy by a small distance in wavenumber back and forth across the sharp spectral
boundary at $k$. In a time-average of the flux $\langle \Pi(k) \rangle$ such effects may be expected to cancel, not only within the DIA closure, but in actual fact. However, for an individual ensemble realization at a given instant in time such effects will be present in $\Pi(k)$, and, indeed, will entirely dominate in $\Pi(k)$ at large $k$, swampng the comparatively small contribution of local triads.

Although the nonlocal, convective effects may be expected to cancel in a time- or ensemble-average of $\Pi(k)$, this would hardly suffice to justify Kolmogorov’s strong statement of statistical independence of the small-scale modes. In fact, the previous observations have raised some doubts, voiced in [10, 11, 15], whether the small scales will really achieve such independence, e.g. of the external forcing or of anisotropy of the large scales. However, it seems clear from our discussion above that the nonlocal contributions to $\Pi(k)$ are an artefact of the sharp spectral boundary, and do not represent an actual physical mechanism for transfer of excitation from large scales to small ones. We have already seen that the LES flux, with a “good” filter choice, gets no net contribution from such terms even at a single instant in a fixed flow realization (when the Hölder regularity is satisfied). Instead, this flux properly represents energy transfer as due to the large-scale strain. It is worthwhile to study in more detail the wavevector contributions to the LES flux in order to understand the origin of cancellations in the nonlocal triads.

The relation of the LES flux to the conventional spectral flux $\Pi(k)$ is obtained very easily when the filter function $G$ is spherically symmetric, as we hereafter assume. If the total resolved kinetic energy is represented as

$$
\overline{E}_\ell(t) \equiv \int \mathrm{d}r \frac{1}{2} \mathbf{v}_\ell^2(r, t) = \int \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{2} \hat{v}(k, t)|^2|\hat{G}(k\ell)|^2,
$$

then using $\Pi_\ell = -\left. \frac{d\overline{E}_\ell}{dt} \right|_{\text{Euler}}$, it follows that

$$
\Pi_\ell = \int_0^\infty \mathrm{d}k \frac{\partial \Pi(k)}{\partial k} |\hat{G}(k\ell)|^2
= \int_0^\infty \mathrm{d}k \Pi(k) D_\ell(k),
$$

where

$$
D_\ell(k) \equiv -\frac{\partial}{\partial k} |\hat{G}(k\ell)|^2.
$$
Observe that
\[
\int_0^\infty dk \, D_\ell(k) = |\hat{G}(0)|^2 - |\hat{G}(\infty)|^2 \\
= 1 - 0 = 1,
\]
and from its definition Eq.(32) that
\[
D_\ell(k) \geq 0,
\]
if $|\hat{G}(k)|^2$ decays monotonically in the spectral radius $k$. Under these modest assumptions, therefore, $D_\ell(k)$ is a normalized density in $k$ and the LES flux $\Pi_\ell$ is a “scale-average” of the spectral flux $\Pi(k)$. The nature of the averaging can be better seen by considering a typical example $|\hat{G}(k)|^2$:

A nice example, as pictured, will fall smoothly and monotonically from its value 1 near $k = 0$ to its value 0 at $k = +\infty$ in an interval of width $\approx 1/\ell$ centered at $2\pi/\ell$. From this it follows that $D_\ell$ will appear as:
Therefore, for a reasonable choice of filter, the function $D_\ell$ will give an average over the interval of width $\approx 1/\ell$ centered at $2\pi/\ell$.

To study the nature of the cancellations involved with such an average, it is convenient to adopt a simple model filter $G$ representative of this class. We take

$$|\hat{G}(k)|^2 = \begin{cases} 
1 & 0 \leq k \leq 2\pi \\
2 - \frac{k}{2\pi} & 2\pi \leq k \leq 4\pi \\
0 & 4\pi \leq k < \infty
\end{cases} \quad (35)$$

which is pictured as:

In this instance the averaging function $D_\ell$ is just the uniform distribution over the interval $[2\pi/\ell, 4\pi/\ell]$, so that

$$\Pi_\ell = \frac{\ell}{2\pi} \cdot \int_{2\pi/\ell}^{4\pi/\ell} dk \, \Pi(k). \quad (36)$$

It was this particular example which we studied in detail in [21] and it is essential to the arguments of this paper that the reader be familiar with that work.

We showed in [21] that the conventional measure of spectral energy flux, $\Pi(k)$, is inadequate to detect the distance energy moves in wavenumber space past the cutoff wavenumber $k$. By wavenumber conservation of the triadic interactions, energy may move to at most a wavenumber of magnitude $2k$. Efficient energy transfer of this type is achieved by the local triadic interactions, in which two wavenumbers of magnitude near $k$ transfer energy to a mode.
with wavenumber near $2k$. By contrast, the Example 1 of [21] achieved a transfer to wavenumbers $> k$ by simply moving the energy a fixed, small distance $k_0$ outside the sphere of radius $k$, for arbitrarily large $k$. The defect of the traditional flux $\Pi(k)$ is that it does not distinguish between these two transfers, whereas a proper measure of flux should take into account the relative distance the energy is moved outside the sphere. Independent of any connection with LES, it was these ideas that led us in [21] to introduce the “band-averaged flux”

$$\Pi_\Lambda = \frac{1}{2^\Lambda} \int_{2^\Lambda}^{2^{\Lambda+1}} dk \, \Pi(k),$$  

(37)

as a proper measure of energy transport. This is equivalent to the flux in Eq.(36) if the filter length is chosen there as $\ell = 2^{-\Lambda}(2\pi)$, corresponding to a wavenumber $2^\Lambda$ at the bottom of the $\Lambda$th octave band.

By a direct argument, it was shown for this quantity in [21] that the “local estimate” holds

$$\Pi_\Lambda = O \left( 2^{\Lambda(1-3h)} \right),$$

corresponding to transfer by local triads. The main property of the Navier-Stokes dynamics which was used is so-called detailed energy conservation. “Partial fluxes” may be defined, measuring transfer from the $N$th band to the $L$th, induced by the $M$th, as

$$\Pi_{N \rightarrow L}^{(M)} = \int_\Omega dr \, \mathbf{v}_N(r) \cdot [\mathbf{v}_M(r) \cdot \nabla] \mathbf{v}_L(r),$$  

(38)

in which $\mathbf{v}_N(r)$ is the component of the full velocity field obtained by summing only over wavenumber modes in the $N$th octave band. Simple integration by parts shows that the partial fluxes satisfy

$$\Pi_{N \rightarrow L}^{(M)} + \Pi_{L \rightarrow N}^{(M)} = 0,$$  

(39)

which is the “detailed conservation” law. Because of this property, one may anticipate that many cancellations will occur in the expression Eq.(37) between the small transfers backward and forward between neighboring small subbands of the $\Lambda$th octave band. The heuristic picture may be explained by the following figure showing the detailed transfer of energy between subbands of size $2^M$ for the contribution $\Pi^{(M-1)}(k)$ to the flux induced by the $M$th octave.
This picture suggests that the “local transfers by nonlocal triads” noted by Domaradzki and Rogallo in [10] will cancel \textit{instantaneously} in the quantity $\Pi_\Lambda^{(M-1)}$. The analytical verification of these cancellations was given in detail in Section 3 of [21]. The idea that such contributions should cancel with an appropriate combination of triadic contributions was also pointed out independently by Waleffe [14] and Zhou [19]. What is important here is that the LES method leads \textit{automatically} to an appropriate weighted average of the flux $\Pi(k)$ for which such cancellations occur.

Our previous analysis for the model filter Eq.(35) actually gives somewhat more than just the “local estimate” for $\Pi_\ell = \Pi_\Lambda$. In fact, our estimates allow us to put precise bounds on the fraction of the flux due to distant triads. The crucial equations are Eqs.(34) & (44) of [21] which represent the two contributions to the flux as sums over the “deviation parameters” $\Delta, \Delta', \Delta''$ which measure the distance in shellnumber of the contributing shells $N, M, L$ from $\Lambda$. In fact, it is easy to see from those equations that the fractional contribution to $\Pi_\Lambda$ from triads with smallest wavenumber in a shell lower than the $(\Lambda - S)$th is

$$\frac{\Pi_\Lambda(S, <)}{\Pi_\Lambda} = O \left(2^{-(1-h)S}\right), \quad (40)$$

while the contribution from triads with largest wavenumber in higher than the $(\Lambda + S)$th shell
\[ \frac{\Pi_A(S, >)}{\Pi_A} = O\left(2^{-2hS}\right). \] (41)

For the latter estimate, note that the contributions arise from terms with both \( L > \Lambda \) and \( M > \Lambda \), for which \( \Delta' \) and \( \Delta'' \) can differ at most by one because of the wavenumber selection rule. This gives the factor of 2 in the exponent since \( \Delta' \approx \Delta'' \approx S \). The parameter \( S \) measures the degree of nonlocalness of the triad (in IR or UV direction), i.e. the minimum number of “cascade steps” \( S \) between the smallest and largest wavenumbers.

Hence, we see that distant triads in fact make an exponentially small contribution in shell-number to instantaneous flux when the velocity field has Hölder type regularity with \( 0 < h < 1 \). This is a fact well-known for mean-flux as calculated within various analytical closures when the energy spectral exponent \( n \) lies in the range \( 1 < n < 3 \): see [36] and Appendix 1 [37]. (Note the Hölder and spectral exponents are related heuristically by \( n = 1 + 2h \).) However, our results here were derived for instantaneous flux in an individual flow realization without any closure approximation or statistical assumption. The exponent \( (1 - h) \) gives the “infrared locality” whereas the exponent \( 2h \) gives the “ultraviolet locality.” The bound \( h < 1 \) seems rather secure but \( h > 0 \) might be violated due to the appearance of “singular structures.” It should be appreciated, however, that while the contribution of distant triads is exponentially small in shellnumber this corresponds only to a slow, algebraic decay in wavenumber.

Our accounting of the fractional contribution of distant triads is very close to that made by Zhou in [12, 19]. Our parameter \( S \) is essentially the same as his “scale disparity” \( s \) (more properly, it corresponds to \( \log_2 s \)) and \( \Pi_A(S, <), \Pi_A(S, >) \) are the same as his \( \Pi^e(k, s), \Pi^e(k, s) \), respectively. In his work, however, he studied mean flux using direct numerical simulation at modest Reynolds number or LES to generate the raw statistics. If we take in our bounds Eqs. (40), (41) above \( h = \frac{1}{3} \) (the K41 mean value), then we obtain in both cases

\[ \frac{\Pi_A(S)}{\Pi_A} = O\left(2^{-2S/3}\right), \] (42)

which Zhou referred to as “Obukhov scaling” in the disparity. On the other hand, he referred
to an estimate $O \left( 2^{-4S/3} \right)$ as “Heisenberg scaling.” In his study, both scalings were observed in certain cases, although, in our opinion, the inertial-ranges were too short to convincingly demonstrate either exponent. In any case, it might not be appropriate to directly compare our estimates with his since he dealt with an average transfer, while our flux estimates are for instantaneous transfer in an individual realization. Additional cancellations of distant triads could occur in time- or ensemble-averaging, so that better bounds on the nonlocal contributions could exist. What is important in our result is that, given the required Hölder regularity, the “locality” of the energy transfer holds at each instant for a fixed turbulent flow field.

3.2 Comparison With the EDQNM Study of Leslie-Quarini

Superficially similar distinctions as ours above between transfer for sharp Fourier vs. graded filters were observed already for mean statistics within analytic closure by Leslie and Quarini [30]. In fact, it was observed in their work that without a low-wavenumber cutoff divergences appear in the sharp Fourier case for both forward transfer —“drain”—and backward transfer—“backscatter”—as the grid-wavenumber $K_1$ is approached, verifying an earlier observation of Kraichnan based upon the TFM closure [38]. These divergences are due to convection by vanishingly low wavenumbers, and indicate a breakdown of locality for each of the two separate terms. However, the divergent parts cancel identically in the net transfer, leaving just a finite “cusp.” It is this cancellation of the divergences due to distant triads which justifies our remark above that transfer across $K_1$ in the closures is dominated in magnitude by the local triads, and not by the distant triads which dominate in individual realizations for the sharp Fourier filter. Furthermore, for the Gaussian filter Leslie and Quarini found that neither forward nor backward scatter has separately the divergence (and even the finite “cusp” essentially disappears.) This last result is apparently in agreement with our conclusions, but a closer examination shows that there is no real correspondence.

3 In fact, they are the same divergences as those observed in Kraichnan’s earlier study of energy transfer in the ALHDIA closure cited before [33].
closure models. We shall follow the notations of Leslie and Quarini [30]. Within the analytical closures, the mean energy transfer into wavenumber $k$ due to interaction of wavenumbers $p, r$ is expressed as

$$S(k|p, r) = S^{(r)}_{p\to k} + S^{(p)}_{r\to k}$$

with

$$S^{(r)}_{p\to k} = 16\pi^2 \cdot \theta_{kp} \cdot k^2 p^2 r (x y + z^3)q(r)[q(p) - q(k)].$$

Here, $q(k) = E(k)/4\pi k^2$ is the energy spectrum per Fourier mode, $x, y, z$ are geometric factors associated to each triad (direction cosines) of order one, and $\theta_{kp} = [\eta(k) + \eta(p) + \eta(r)]^{-1}$ is an interaction time-scale associated to the triad. Different closure models are distinguished by their choice of $\eta(k)$, but a typical one is the EDQNM choice of $\eta(k) = \nu k^2 + \lambda (k^3 E(k))^{1/2}$.

The average flux is expressed in the closures by

$$\Pi(k) = \int_{k}^{\infty} dk' \int_{\Delta_{k'}} dp dr S(k'|p, r).$$

For the locality analysis, the dangerous contribution to consider is

$$\Pi^{(r)}(k) = \int_{k}^{\infty} dk' \int_{\Delta_{k'}} dp S^{(r)}_{p\to k'}$$

for $r \ll k$, which arises from convection by the low wavenumber $r$. It is very easy to make a simple estimate of this term as

$$\Pi^{(r)}(k) \sim \int_{k}^{\infty} dk' \int_{\Delta_{k'}} dp \theta_{kp} \cdot k^2 p^2 r q(r)[q(p) - q(k')],$$

$$\sim r^2 \cdot k^{4} \cdot r \theta_{rkk} q(r) \cdot \left(\frac{r}{k}\right) q(k).$$

The factor of $r^2$ arose from the fact that $k', p$ can only take values in a range $r$ around $k$. The crucial factor is the $(r/k)$ which arises from expanding $q(k \pm r)$ with a power-law spectrum and exploiting the cancellation between input and output terms in Eq.(44).

The energy scaling corresponding to our Hölder exponent $h$ is $E(k) \sim k^{-(1+2h)}$. This leads also to $\theta_{rkk} \sim r^{h-1}$ since the time scale for $h < 1$ is clearly dominated by the lowest wavenumber. Hence,

$$\Pi^{(r)}(k) \sim k^{-2h} \cdot r^{-h}.$$
Since the integral \( \int_{k_0}^{k} dr \, r^{-h} \) is dominated by the UV region when \( h < 1 \) and is then \( \sim k^{1-h} \), it follows finally that
\[
\Pi^{(\langle)}(k) \equiv \int_{k_0}^{k} dr \, \Pi^{(r)}(k) \sim k^{1-3h},
\]
(49)

exactly as expected. However, if the cancelling in-out contributions were not observed and the factor of \( r/k \) missed, then the scaling in Eq.(48) would be replaced by \( \Pi^{(r)}(k) \sim k^{1-2h}.r^{-(1+h)} \).

Since the integral of \( r^{-(1+h)} \) is dominated by the IR region for \( h > 0 \), the false result \( \Pi^{(\langle)} \sim k^{1-2h} \) would then be obtained. Notice here that, as is well-known, the proper locality estimates are obtained \textit{without any such scale-averaging as we performed previously}. It would be possible to make such an additional scale average in the closure context, and the same cancellations will occur since the only requirement for our argument was detailed conservation, which is satisfied by the “partial transfers”:
\[
S_{p \to k}^{(r)} + S_{k \to p}^{(r)} = 0.
\]
(50)

However, it is clear that the origin of these cancellations will be exactly the same as the “in-out cancellations” without averaging, since in the closures the detailed conservation condition Eq.(50) is just another consequence of the in-out cancelling terms. The scale-average is superfluous for the closure flux, which is already local. Intuitively, this is clear, because the closure flux is supposed to correspond to an ensemble-averaged value, which will show appropriate cancellations between individual realizations with positive and negative values of flux.

We can now comment on the Leslie-Quarini results on graded filters [30]. In their case, generalizing upon Kraichnan in [38], they defined a wavenumber-dependent eddy viscosity \( \nu_n(k) = \nu_n(k|K_1) \) (corresponding to Kraichnan’s \( \nu(k|k_m) \)) as
\[
\nu_n(k) = -\overline{S}(k)/2k^2\overline{E}(k),
\]
(51)
where they took \( \overline{E}(k) = G^2(k)E(k) \) to be the energy spectrum of the filtered velocity and
\[
\overline{S}(k) = \frac{1}{2} \int \int_{\Delta k} dpdr \, G^2(k) \{1 - G(p)G(r)\} S(k|p,r)
\]
(52)
was what they defined as the “true subgrid transfer.” Again following Kraichnan, they decomposed the transfer into two terms the “drain” part,

\[ 2k^2 \nu_d(k)E(k) = \sum_{2kp} \theta_{kpr} b_{kp} E(r) E(k) \cdot G^2(k) [1 - G(p)G(r)] \]  

and the “backscatter” part

\[ U(k) = \sum \frac{k^3}{pr} \theta_{kpr} b_{kp} E(p) E(r) \cdot G^2(k) [1 - G(p)G(r)]. \]  

We emphasize that these two terms arise exactly from the output and input terms of \( S(k|p, r) \), respectively. In particular, they do not satisfy separately a detailed conservation property such as Eq. (50), which arises precisely from a cancellation between those terms. Leslie and Quarini then made a corresponding division of the eddy viscosity \( \nu_n(k) \) into \( \nu_d(k) \), the “drain” part, and \( \nu_b(k) \), the “backscatter” part. They numerically reproduced the result of Kraichnan that these two separate terms for the Fourier filter each diverges as \( k \) approaches the grid wavenumber \( K_1 \), and that the divergence cancels in the net viscosity \( \nu_n(k) \). However, they furthermore showed from their numerical work that with the Gaussian filter even the separate terms, \( \nu_d(k) \) and \( \nu_b(k) \) are finite for \( k \) approaching \( K_1 \).

It is clear that this is what one would expect from our analysis. In fact, what we have established is much stronger than this, because (assuming Hölder regularity) we showed that the local estimate holds for every individual realization with a “good” filter, whether instantaneous transfer is forward or backward. Since the divergences observed by Kraichnan in mean transfer for the sharp Fourier case are a consequence of locality breakdown, it ought to be anticipated that they will not occur for a “good” choice of filter.

However, the kind of cancellations which we established for \( \Pi_\ell \) in the solutions of Navier-Stokes cannot occur separately in the quantities \( \nu_d \) and \( \nu_b \) defined by Leslie-Quarini. In fact, there is no possibility to exploit detailed conservation of energy, as in our microscopic argument, because in the closure that depends precisely upon the cancellation between those terms! Therefore, there is no equivalence of our results and Leslie-Quarini’s. Formally, the disappearance of
divergences in $\nu_d, \nu_b$ observed in [30] appear to arise in a quite different way, due to their being simply “smoothed out” by the averaging functions $G$ in Eq.(52). Note that Leslie-Quarini did not give a theoretical explanation of their effect—as we have for the corresponding phenomenon in Navier-Stokes—but only observed it numerically. It seems to be, in fact, a different effect. The fact that our argument based upon detailed conservation does not work in the closures just illustrates the point that the closures do not necessarily reflect reality. In fact, it is well known that energy conservation does not hold realization by realization in the Langevin models of the closures, so that the cancellations we established due to detailed conservation in Navier-Stokes solutions will not occur there.

In our opinion part of the difficulty of making a comparison of our result and that of Leslie-Quarini lies in the different definitions of “backscatter” in the two works. In ours here, we define a “backscatter” event— as in current LES practice—as a spacetime point $(r,t)$ where $\Pi_t(r,t) < 0$. However, the Leslie-Quarini definition of “backscatter,” while employing the filter function $G$, is still essentially Fourier-based. In other words, they define a “backscatter event” as one where, roughly speaking, the mean Fourier flux $\Pi(k) < 0$ and then take a suitable weighting with the filter function. (Even this involves an assumption that the “input” term to transfer arises from averaging conditioned on $\Pi(k) < 0$ while the “output” term arises from averaging conditioned on $\Pi(k) > 0$: this is not obviously true.) One may expect that these two definitions are globally equivalent, for space-averages, but even this does not appear easy to demonstrate.

4 Conclusions

4.1 On the “Multifractal Model”

We believe that the present work has demonstrated that the Parisi-Frisch “multifractal model” is not just an esoteric theoretical speculation relevant only to exotic “intermittency” phenomena in turbulence. In fact, it gets considerable support from the fluid equations them-
selves when coupled with some basic facts of turbulence phenomenology. Furthermore, the “multifractal model” provides a coherent framework which rationalizes and explains many observations of experiment and simulation. Local Hölder continuity is the maximal regularity consistent with the phenomenon of constant mean energy flux through the inertial interval. It is also a sufficient condition for local energy transfer, necessary for the validity of the Kolmogorov universality hypothesis. While it must be regarded as tentative and so far unproved, we believe the “multifractal model” is valuable at least as a working hypothesis. Further investigation, especially mathematical and theoretical, should clarify its status.

4.2 On Locality of Transfer

The condition that the Hölder exponent lie in the range $0 < h < 1$ is sufficient to guarantee locality of the instantaneous energy transfer. Nonlocal triads with wavevectors in distant bands from each other contribute an exponentially small fraction of the total flux. However, this implies only a slow, algebraic decay in Fourier space, so that very high Reynolds numbers might be necessary for a locally-dominated inertial-range to occur (see also 5.4 below). Furthermore, while local transfer is certainly necessary for statistical independence of the small scales, it is not sufficient. A good counterexample is Burgers equation, for which all of our analysis applies and in which instantaneous energy transfer is likewise local. Nevertheless, the small scales of Burgers turbulence do not have statistics independent of the large scales but, rather, largely determined by them. The key difference between Burgers and Navier-Stokes equation is that the former lacks the pressure force of the latter which tends to stochastically accelerate Lagrangian fluid particles. On the contrary, for Burgers equation the velocity of a fluid particle is unchanged until it is captured by a shock. The difference may be characterized as a lack of dynamic locality for Burgers equation, since Lagrangian time correlations will not be intrinsic and locally-determined in scale as they are for Navier-Stokes, but will rather be associated to the overall time of evolution $[39]$. These considerations show that some chaotic features of the Navier-Stokes dynamics—still insufficiently well-understood—will be necessary to produce independence of successive “cascade steps” and universality of the small scales.
4.3 On the Filtering Approach

The filtering scheme used in the LES modelling technique turns out to be a uniquely satisfactory method of scale decomposition for turbulent flows, at least in the case where a “good” filter is employed. (For an overview of the “filtering approach” to turbulence, see the interesting paper of Germano [40].) The requirements for a “good” filter are concrete and objective, as well as very easily satisfied. With such a filtering technique the basic physics of turbulent energy transport is made transparent. In contrast, the method of Fourier series—despite its traditional status in the subject—introduces many subtle and opaque effects which are just artefacts of using a basis of plane-waves ill-suited to describe the physical processes involved. A set of interactions like vortex-stretching which are naturally understood in physical space become difficult to comprehend in a wavenumber representation and are masked by other effects.

These same criticisms apply to some apparently more sophisticated representations, such as wavelet bases (e.g. see [41].) Since such bases were designed to provide a simultaneous space-scale resolution, they might appear ideal for turbulence applications. However, we have found that the wavelet methods that have so far been proposed have the same problems as the Fourier basis and in an even more severe form! We hope to discuss the wavelet method in detail in a future work [42].

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Appendix: Multifractal Model for Navier-Stokes

In Section 2 we reached the conclusion that Hölder exponents $h \leq 1/3$ are required to allow for constant mean energy flux in the inertial interval. However, it may seem to some that we are engaged in a mathematical “hair-splitting.” After all—it might be argued—in any real flow there is a finite viscosity, no matter how small, and the actual velocity field, the solution of viscous Navier-Stokes equations, will (presumably) be smooth. However, we would like to emphasize that our considerations apply even for finite $\nu_0 > 0$ over the long inertial-range of length-scales, when $\nu_0$ is small. It should be appreciated that the estimates we have made refer only to the range of scales $\sim \ell$ and they are unchanged if the velocity-field is smooth over much smaller length-scales. To see this clearly, consider the difference over length-scales $\sim \ell$ of a velocity-field with small-scale components $< a$ removed by filtering, when $a \ll \ell$. If $v \in C^\eta$, $\eta > 0$, then the filtered field $v_a$ is $C^\infty$, or even analytic, for a nice choice of the filter function, and $\Delta_l v_a$ will decay rapidly in $\ell$ for $\ell \ll a$. However,

$$\Delta_l v_a(r) \approx \Delta_l v(r),$$

for $\ell \gg a$. This may be demonstrated analytically using

$$\Delta_l v_a(r) = \int ds \, G_a(s) \left[ \Delta_{l+s} v(r) - \Delta_s v(r) \right].$$

Since only $s \leq a$ contribute to the integral in a good approximation (exactly so, if $G_a$ has compact support in the ball $B_a(r)$ of radius $a$ centered at $r$) and $a \ll \ell$, $\Delta_{l+s} v(r) \approx \Delta_l v(r)$ and comes out of the integral, whereas the second term is small. A quantitative estimate shows that

$$\Delta_l v_a(r) = \Delta_l v(r) + O(a^\eta),$$

where $\eta$ may be taken to be the minimum Hölder exponent of $v$ over the ball $B_a(r)$. Therefore, the remainder is negligible if $\Delta_l v(r) \sim \ell^h$, $a \ll \ell$, and $h$ and $\eta$ are comparable. These arguments just support the intuitive idea that $\Delta_l v$ will not be much changed if the velocity field is smoothed over much smaller scales than $\ell$, whether by a mathematical operation of filtering or by physical
regularization due to viscosity. In a similar situation, physical Brownian paths are smooth, because the size of atoms $a$ is finite, but over length-scales $\ell \gg a$ they “look” like nowhere-differentiable curves of Hölder index $\frac{1}{2}$, which only becomes exactly true in the idealized limit $a \to 0$.

Our previous arguments on energy flux may be given in a form which applies to the real situation where Reynolds number is finite, but large. Consider a case where turbulence is produced in a Navier-Stokes fluid with viscosity $\nu_0$ in a box of size $L$ by random forcing at the length-scales of the box-size. Suppose, in fact, that in every flow realization of the turbulent ensemble

$$|\Delta_1 \mathbf{v}(\mathbf{r})| \leq v_0 \left( \frac{\ell}{L} \right)^h,$$

with some $h > \frac{1}{3} + \delta$ at each point $\mathbf{r}$. The constant $v_0$ has an interpretation as the typical variation of the velocity over the largest eddies of size $\sim L$. Our previous estimates then imply that

$$|\Pi_\ell(\mathbf{r})| \leq \frac{v_0^3}{L} \left( \frac{\ell}{L} \right)^{3h-1},$$

at each point $\mathbf{r}$ with probability one. However, in that case,

$$|\langle \Pi_\ell \rangle| \leq \frac{\langle v_0^3 \rangle}{L} \left( \frac{\ell}{L} \right)^{3\delta}.$$  

(60)

Let us take $\ell_\nu$ to be a scale intermediate between $L$ and $\lambda$, e.g. $\ell_\nu \equiv \sqrt{\lambda L}$. Then $\ell_\nu/L \sim (\text{Re})^{-1/4}$ in the limit $\nu_0 \to 0$. However, according to our previous discussion, $\langle \Pi_{\ell_\nu} \rangle$ tends to a constant $\tau$ in the limit $\nu_0 \to 0$. This is not consistent with the previous estimate unless $\langle v_0^3 \rangle \sim (\text{Re})^{3\delta/4}$. We emphasize that nothing like this is observed. (At least not in three dimensions; in 2D simulations the level of energy fluctuations is indeed observed to rise as $\nu_0 \to 0$.) Therefore, the initial assumption Eq.(58) cannot hold with $\delta > 0$ for the range of scales $\lambda \ll \ell \ll L$.

The “multifractal model” accounts for the constant mean flux over this range of scales, with $\langle v_0^3 \rangle$ remaining finite, by postulating a suitable distribution of Hölder exponents $h$ in an interval $[h_{\text{min}}, h_{\text{max}}]$, so that

$$\inf_{h \in [h_{\text{min}}, h_{\text{max}}]} \left[ (3h - 1) + (3 - D(h)) \right] = 0.$$  

(61)
This is consistent with the requirement from Kolmogorov’s “\(\frac{4}{5}\)-law” [3], which implies \(\zeta_3 = 1\). The exponent \(h^*_r\) for which the infimum is achieved is believed to be \(\approx \frac{1}{3}\). As stated before, the velocity field could possibly be less regular than this, but not more so.

These arguments imply that actual singularities will occur with some finite probability in the turbulent ensemble in the limit \(r_0 \to 0\), if the currently observed finiteness of velocity fluctuations persists asymptotically to infinite Reynolds number. However, the present arguments do not require that singularities should appear in finite time for Euler dynamics starting from smooth initial data. In fact, we consider a stationary turbulent ensemble in which the singularities may have been developed possibly over an infinite period of time. Even the consideration of decaying turbulence in 3D does not require true singularities in finite time. Suppose in that case a quasi-equilibrium state is achieved with a long inertial range of constant mean flux over the range of length-scales \(L(t) \gg \ell \gg a(t)\), where \(L(t)\) is the (time-dependent) integral scale and \(a(t)\) is a time-dependent lower cutoff to this range. If \(a(t)\) were to decrease at a faster rate than the rate of decay of energy in the large length-scales \(\sim L(t)\), say, \(a(t) \sim e^{-\gamma t}\), then there would be little observable difference from the scenario where \(a(t) \to 0\) at a finite time \(t = T^*_r\). Our previous considerations would still imply for this case that “quasi-singularities” must occur in the velocity field over the range of lengths \(L(t) \gg \ell \gg a(t)\), to be consistent with the constant mean flux in that interval.

A good example of this is the enstrophy cascade range for Navier-Stokes turbulence in 2D. Considerations of the same type as those above show that the enstrophy flux \(Z_\ell(\mathbf{r})\) in 2D obeys an estimate

\[
Z_\ell(\mathbf{r}) = O \left( \ell^{2h} \right)
\]

in terms of the local Hölder index of the vorticity field \(\omega = \nabla \times \mathbf{v}\) at point \(\mathbf{r}\). In the 2D case the natural “multifractal” picture is in fact in terms of the Hölder spectrum of the vorticity field. Already in 1967 Kraichnan had proposed that the ultraviolet range of 2D turbulence will be an
“enstrophy cascade” with a constant mean flux,

$$\langle Z_\ell \rangle = \mathcal{F},$$  \hspace{1cm} (63)

for $\eta_{Kr} \ll \ell \ll L$ (here $L$ = the length-scale at which enstrophy is injected, $\eta_{Kr}$ = the Kraichnan dissipation length) [43]. From the previous estimate we see that $h \leq 0$ will therefore be required. This argument is carried out in a mathematically rigorous way in a separate paper [44], especially the last two parts of Section 4. The corresponding argument for the velocity regularity $h \leq 1/3$ of Navier-Stokes solutions in the 3D energy inertial-range is entirely the same, so that we may refer the reader to [44] for details.

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