General formulation of cosmological perturbations in scalar-tensor dark energy coupled to dark matter

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Received June 1, 2020
Accepted September 29, 2020
Published November 16, 2020

\textbf{Abstract.} For a scalar field $\phi$ coupled to cold dark matter (CDM), we provide a general framework for studying the background and perturbation dynamics on the isotropic cosmological background. The dark energy sector is described by a Horndeski Lagrangian with the speed of gravitational waves equivalent to that of light, whereas CDM is dealt as a perfect fluid characterized by the number density $n_c$ and four-velocity $u^\mu_c$. For a very general interacting Lagrangian $f(n_c, \phi, X, Z)$, where $f$ depends on $n_c$, $\phi$, $X = -\partial^\mu \phi \partial_\mu \phi / 2$, and $Z = u^\mu_c \partial_\mu \phi$, we derive the full linear perturbation equations of motion without fixing any gauge conditions. To realize a vanishing CDM sound speed for the successful structure formation, the interacting function needs to be of the form $f = -f_1(\phi, X, Z)n_c + f_2(\phi, X, Z)$. Employing a quasi-static approximation for the modes deep inside the sound horizon, we obtain analytic formulas for the effective gravitational couplings of CDM and baryon density perturbations as well as gravitational and weak lensing potentials. We apply our general formulas to several interacting theories and show that, in many cases, the CDM gravitational coupling around the quasi de-Sitter background can be smaller than the Newton constant $G$ due to a momentum transfer induced by the $Z$-dependence in $f_2$.

\textbf{Keywords:} dark energy theory, modified gravity

\textbf{ArXiv ePrint:} 2005.13809v1
Today’s universe is dominated by two unknown components — dark energy (DE) and dark matter (DM). The late-time cosmic acceleration is driven by a negative pressure of DE, whereas DM is the main source for gravitational clusterings. The standard cosmological paradigm is known as the Λ-cold-dark-matter (ΛCDM) model [1, 2], in which the origin of DE is the cosmological constant with nonrelativistic DM fluids. The cosmological constant has no dynamical propagating degrees of freedom, so it does not couple to CDM directly. On the theoretical side, it is generally difficult to reconcile the observed energy scale of Λ with the vacuum energy arising from particle physics [3]. In recent observations, the ΛCDM model is plagued by tensions of today’s expansion rate $H_0$ as well as the amplitude of matter density contrast $\sigma_8$ between the Cosmic-Microwave-Background (CMB) and low-redshift measurements [4–9].

The cosmological constant is not the only possibility for the origin of DE, but there are other dynamical models of DE proposed in the literature (see refs. [10–13] for reviews). The simple example of dynamical DE models is a scalar field $\phi$ with associated potential and kinetic energies [14–23]. Theories with a single propagating scalar degree of freedom include Horndeski gravity [24–27] and its extension to theories containing derivatives higher than second order — such as GLPV [28] and DHOST theories [29–31]. In Horndeski theories the field equations of motion are strictly of second order on any curved background. After the
gravitational-wave event GW170817 [32] together with the electromagnetic counterpart [33], the speed of tensor perturbations $c_t$ is constrained to be very close to that of light $c$ for the redshift $z < 0.009$. If we strictly demand that $c_t = c$, the allowed Horndeski Lagrangian is of the form $\mathcal{L}_H = G_1(\phi)R + G_2(\phi, X) + G_3(\phi, X)\Box\phi$ [34–41], where $G_1$ depends only on $\phi$, and $G_{2,3}$ are functions of $\phi$ and $X = -\partial^\mu\phi\partial_\mu\phi/2$.

In this sub-class of Horndeski theories where the field $\phi$ is uncoupled to CDM, the gravitational coupling of CDM and baryon perturbations relevant to the scale of galaxy clusterings is usually larger than the Newton constant $G$ [13, 39, 42]. This property is attributed to the fact that the scalar-matter interaction is attractive under the absence of ghost and Laplacian instabilities for perturbations deep inside the sound horizon. Then the growth of CDM and baryon density perturbations is enhanced with respect to the ΛCDM model, so the discrepancy of $\sigma_8$ between CMB and low-redshift measurements tends to be even worse. This means that, even with the cosmological background reducing the tension of $H_0$ in comparison to the ΛCDM, the problem of $\sigma_8$ discrepancy still persists at the level of perturbations [43].

If the scalar field is coupled to CDM, there are intriguing possibilities that the gravitational couplings for CDM are smaller than $G$ for linear perturbations associated with the growth of large-scale structures. In particular, the derivative interaction between the CDM four-velocity $u^\mu_c$ and the field derivative $\partial_\mu\phi$, which is weighed by the scalar product $Z = u^\mu_c\partial_\mu\phi$, allows the possibility of weak cosmic growth through a momentum transfer [44–56]. Besides this momentum transfer, the energy exchange between CDM and the scalar field can be also accommodated by implementing the CDM number density $n_c$ coupled to $\phi$ and $X$ [44, 45, 52, 55]. This dependence of $n_c$ in the interacting Lagrangian can be interpreted as the dependence of CDM density $\rho_c = m_c n_c$, where $m_c$ is the mass of CDM particles. These coupled DE and DM theories in the Lagrangian formulation have a theoretical advantage over the phenomenological approaches taken in refs. [57–75], in that the background and perturbation equations of motion in former theories unambiguously follow from the interacting Lagrangian [76].

The interacting Lagrangian containing the effect of both energy and momentum transfers is generally expressed in the form $f(n_c, \phi, X, Z)$, where $f$ is a function of $n_c$, $\phi$, $X$, and $Z$. So far, the background and perturbation dynamics of such interacting theories have been studied for several sub-classes of the coupling $f(n_c, \phi, X, Z)$, with the DE sector restricting to quintessence [44–51]. In ref. [52], the present authors considered the Lagrangian of Horndeski theories for the DE scalar field, but the interacting function is restricted to be of the form $f = -f_1(\phi, X)n_c + f_2(n_c, \phi, X)Z$. In ref. [55], the dynamics of perturbations for an interacting Lagrangian $f = -f_1(\phi, X, Z)n_c + f_2(\phi, X, Z)$ was studied by choosing specific functional forms of $f_1$ and $f_2$.

In this paper, we study the cosmology of coupled DE and DM theories characterized by the interacting function $f(n_c, \phi, X, Z)$, with the aforementioned Horndeski Lagrangian $\mathcal{L}_H$ in the DE sector. The CDM is dealt as a perfect fluid, which is described by a Schutz-Sorkin action [77–79]. We obtain the field equations of motion in a covariant form and apply them to the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background. We then expand the action up to second-order in scalar perturbations without fixing gauge conditions and derive the full linear perturbation equations of motion in a gauge-ready form. We also identify conditions for the absence of ghost and Laplacian instabilities and show that the condition $\partial^2 f/\partial n_c^2 = 0$ must be satisfied to realize the vanishing effective CDM sound speed. In this class of theories, we obtain the effective gravitational couplings of CDM and baryons by
employing the quasi-static approximation for perturbations deep inside the sound horizon. Finally, we apply our general formulas to several concrete theories of coupled DE and DM and show that the weak cosmic growth around the quasi de Sitter background is possible in many cases by the momentum exchange.

Throughout the paper, we use the natural units for which the speed of light \( c \), the reduced Planck constant \( \hbar \), and the Boltzmann constant \( k_B \) are set to unity. We also adopt the metric signature \((- , +, +, +)\), with the Greek and Latin indices representing components in four-dimensional space-time and in three-dimensional space, respectively.

2 Horndeski scalar coupled to DM

We study a general Lagrangian formulation of coupled DE in which a scalar field \( \phi \) interacts with CDM through both energy and momentum exchanges. We assume that the scalar field couples to neither baryons nor radiation. For the DE sector, we consider a sub-class of Horndeski theories in which the speed of gravitational waves is identical to that of light [34–41].

We deal with CDM, baryons, and radiation as perfect fluids.

The total action is given by

\[
S = \int d^4x \sqrt{-g} \mathcal{L}_H - \sum_{I=c,b,r} \int d^4x \left[ \sqrt{-g} \rho_I(n_I) + J^\mu_I \partial_\mu \ell_I \right] + \int d^4x \sqrt{-g} f(n_c, \phi, X, Z),
\]

(2.1)

where

\[
\mathcal{L}_H = G_4(\phi) R + G_2(\phi, X) + G_3(\phi, X) \Box \phi.
\]

(2.2)

Here, \( g \) is the determinant of metric tensor \( g_{\mu\nu} \), \( R \) is the Ricci scalar, \( G_4 \) is a function of \( \phi \), and \( G_{2,3} \) depend on both \( \phi \) and its kinetic term

\[
X = -\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi,
\]

(2.3)

with the covariant derivative operator \( \nabla_\mu \) and the d’Alembertian \( \Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \).

The second integral on the right-hand-side of eq. (2.1), which corresponds to the Schutz-Sorkin action [77–79], describes the perfect fluids of CDM, baryons, and radiation (labeled by \( c, b, r \), respectively). The energy density \( \rho_I \) is a function of each fluid number density \( n_I \). The vector field \( J^\mu_I \) in the Schutz-Sorkin action is related to the fluid four-velocity \( u^\mu_I \), as

\[
u^\mu_I = \frac{J^\mu_I}{n_I \sqrt{-g}}.
\]

(2.4)

Since the four-velocity obeys \( u^\mu_I u_{I\mu} = -1 \), the explicit relation between \( n_I \) and \( J^\mu_I \) is given by

\[
n_I = \sqrt{\frac{J^\mu_I J_I^\mu}{g}}.
\]

(2.5)

The scalar quantity \( \ell_I \) in the Schutz-Sorkin action is a Lagrange multiplier, with the notation \( \partial_\mu \ell_I = \partial \ell_I / \partial x^\mu \). Since the product \( J^\mu_I \partial_\mu \ell_I \) is not multiplied by the volume factor \( \sqrt{-g} \) in the action, we do not deal with \( J^\mu_I \) as a covariant four vector. Alternatively one can introduce the four vector \( \bar{J}^\mu_I = J^\mu_I / \sqrt{-g} \) and consider the covariant action \( -\int d^4x \sqrt{-g} \bar{J}^\mu_I \nabla_\mu \ell_I \) [80], but we do not take this latter approach in this paper. There are also additional contributions associated with the dynamical vector degrees of freedom to the above Schutz-Sorkin
Since vector perturbations are non-dynamical in scalar-tensor theories, we do not take them into account.

The third integral on the right-hand-side of eq. (2.1) represents the interaction between the scalar field and CDM [44]. The function \( f \) depends on \( n_c, \phi, X, \) and

\[
Z = u_c^\mu \nabla_\mu \phi = \frac{J_\mu}{n_c \sqrt{-g}} \nabla_\mu \phi .
\]  

(2.6)

If we consider CDM with mass \( m_c \), the corresponding density is given by \( \rho_c = m_c n_c \) and hence \( n_c \) is directly related to its density \( \rho_c \). Then the \( n_c \) dependence in \( f \), along with the coupling with \( \phi \) and \( X \), accommodates the energy transfer between CDM and DE. The \( Z \) dependence characterizes a derivative interaction between the CDM four-velocity and the scalar derivative \( \nabla_\mu \phi \), which mediates the momentum exchange. The scalar products \( X \) and \( Z \) are constructed by the first derivative of \( \phi \). One can also consider more general scalar products containing the derivatives higher than the first order, but we will not do so in this paper to keep the equations of motion up to second order.

The interacting Lagrangian \( f(n_c, \phi, X, Z) \) may be related to CDM conformally and disformally coupled to the metric \( \bar{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\nabla_\mu \phi \nabla_\nu \phi \) [81–83]. Under the disformal transformation, Horndeski theories are mapped to GLPV or DHOST theories (see, e.g., ref. [84]). It may be possible to seek for the frame in which CDM is minimally coupled, i.e., the Jordan frame for CDM. In doing so, we need to caution that the product \( J_\mu \partial_\mu \ell_I \) in the action (2.1), which is associated with the conservation of total particle number for each fluid on the cosmological background, is subject to modifications under the disformal transformation. It is of interest to study how the theories described by eq. (2.1) can be expressed in such a transformed frame, but this is beyond the scope of our paper. We focus on the frame where the theories are given by the action (2.1), i.e., the frame in which the total particle number of each fluid is conserved.

### 2.1 Covariant equations of motion

We derive the covariant equations of motion for the theories given by eq. (2.1). First, we vary the action (2.1) with respect to the Lagrangian multiplier \( \ell_I \) and obtain the following constraint,

\[
\partial_\mu J_\mu^I = 0 \quad \text{(for } I = c, b, r) .
\]  

(2.7)

In terms of the four velocity \( u_\mu^I \), this equation translates to \( \partial_\mu (\sqrt{-\bar{g}} n_I u_\mu^I) = 0 \). On using the relation \( \partial_\mu (\sqrt{-\bar{g}} u_\mu^I) = \sqrt{-g} \nabla_\mu u_\mu^I \), it follows that

\[
u_\mu^I \partial_\mu n_I + n_I \nabla_\mu u_\mu^I = 0 .
\]  

(2.8)

Since the density \( \rho_I \) depends on \( n_I \) alone, it is straightforward to relate \( \partial_\mu n_I \) with the partial derivative \( \partial_\mu \rho_I \), such that

\[
\rho_{I,n_I} u_\mu^I \partial_\mu n_I = u_\mu^I \partial_\mu \rho_I ,
\]  

(2.9)

where \( \rho_{I,n_I} = \partial \rho_I / \partial n_I \). Then, we can express eq. (2.8) in the form,

\[
u_\mu^I \partial_\mu \rho_I + (\rho_I + P_I) \nabla_\mu u_\mu^I = 0 ,
\]  

(2.10)

where \( P_I \) is the fluid pressure defined by

\[
P_I = n_I \rho_{I,n_I} - \rho_I .
\]  

(2.11)
As we will see below, eq. (2.10) corresponds to the conservation (continuity) equation for the fluid energy-momentum tensor.

Second, we vary the action (2.1) with respect to the vector fields \( J_{I}^{\mu} \). Taking note that the \( J_{I}^{\mu} \) dependence in the action (2.1) appears through \( \rho_{c}(n_{c}) \) as well as the \( n_{c} \) and \( Z \) dependence in \( f \), the variation with respect to the CDM vector field \( J_{c}^{\mu} \) leads to

\[
- \sqrt{-g} \rho_{c,n_{c}} \frac{\partial n_{c}}{\partial J_{c}^{\mu}} - \partial_{\mu} \ell_{c} + \sqrt{-g} \left( f_{,n_{c}} \frac{\partial n_{c}}{\partial J_{c}^{\mu}} + f_{,Z} \frac{\partial Z}{\partial J_{c}^{\mu}} \right) = 0 ,
\]

where

\[
\frac{\partial n_{c}}{\partial J_{c}^{\mu}} = \frac{J_{c\mu}}{n_{c} g} = - \frac{u_{c\mu}}{\sqrt{-g}} ,
\]

\[
\frac{\partial Z}{\partial J_{c}^{\mu}} = \frac{1}{n_{c} \sqrt{-g}} (\nabla_{\mu} \phi + Z u_{c\mu}) .
\]

Then, we obtain

\[
\partial_{\mu} \ell_{c} = (\rho_{c,n_{c}} - f_{,n_{c}}) u_{c\mu} + \frac{f_{,Z}}{n_{c}} (\nabla_{\mu} \phi + Z u_{c\mu}) .
\]

For baryons and radiation, the relations analogous to eq. (2.15) are

\[
\partial_{\mu} \ell_{I} = \rho_{I,n_{I}} u_{I\mu} \quad \text{(for} I = b, r) .
\]

We will exploit eqs. (2.15) and (2.16) to eliminate the Lagrange multipliers \( \ell_{I} \) from the covariant equations of motion.

Third, we vary the action (2.1) with respect to \( g_{\mu\nu} \) to derive the gravitational equations of motion. The variation of the Horndeski Lagrangian \( L_{H} = \sqrt{-g} L_{H} \) is given by \cite{26}

\[
\frac{2}{\sqrt{-g}} \frac{\delta L_{H}}{\delta g_{\mu\nu}} = M_{pl}^{2} G_{\mu\nu} - T_{\mu\nu}^{(H)} ,
\]

where \( G_{\mu\nu} \) is the Einstein tensor, \( M_{pl} \) is the reduced Planck mass, and

\[
T_{\mu\nu}^{(H)} = G_{2g_{\mu\nu}} + G_{2,X} \nabla_{\mu} \phi \nabla_{\nu} \phi + G_{3,X} \nabla_{\mu} \phi \nabla_{\nu} \phi \nabla_{\nu} \phi - g_{\mu\nu} \nabla_{\lambda} G_{3} \nabla^{\lambda} \phi + \nabla_{\mu} G_{3} \nabla_{\nu} \phi + \nabla_{\nu} G_{3} \nabla_{\mu} \phi + (M_{pl}^{2} - 2G_{4}) G_{\mu\nu} + 2G_{4,\phi} \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \nabla_{\phi} \nabla_{\phi} - 2X g_{\mu\nu} .
\]

In eq. (2.17), we have separated the term \( M_{pl}^{2} G_{\mu\nu} \) from the other contributions.

For the second and third integrals in the action (2.1), we express them in the form

\[
S_{F} = \int d^{4}x L_{F} ,
\]

where

\[
L_{F} = - \sum_{I = c,b,r} \left[ \sqrt{-g} \rho_{I}(n_{I}) + J_{I}^{\mu} \partial_{\mu} \ell_{I} \right] + \sqrt{-g} f(n_{c}, \phi, X, Z) .
\]

Variation of this Lagrangian with respect to \( g_{\mu\nu} \) leads to

\[
\delta L_{F} = - \sum_{I = c,b,r} \left[ \delta \sqrt{-g} \rho_{I} + \sqrt{-g} \rho_{I,n_{I}} \delta n_{I} + J_{I}^{\mu} \partial_{\mu} \delta \ell_{I} \right] + \delta \sqrt{-g} f + \sqrt{-g} (f_{,n_{c}} \delta n_{c} + f_{,X} \delta X + f_{,Z} \delta Z) ,
\]
where

\[
\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu},
\]

\[
\delta n_I = \frac{\eta I}{2} (g_{\mu\nu} - u_{I\mu} u_{I\nu}) \delta g^{\mu\nu},
\]

\[
\delta X = -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi \delta g^{\mu\nu},
\]

\[
\delta Z = \left( \frac{1}{2} Z u_{cm} u_{cv} + u_{cm} \nabla_\nu \phi \right) \delta g^{\mu\nu}.
\]

On using the properties (2.15) and (2.16), it follows that

\[
-2 \sqrt{-g} \delta L_F \delta g^{\mu\nu} = \sum_{I=c,b,r} T^{(I)}_{\mu\nu} + T^{(\text{int})}_{\mu\nu},
\]

where

\[
T^{(I)}_{\mu\nu} = (\rho_I + P_I) u_{I\mu} u_{I\nu} + P_I g_{\mu\nu},
\]

\[
T^{(\text{int})}_{\mu\nu} = f g_{\mu\nu} - n_c f n_c (g_{\mu\nu} + u_{cm} u_{cv}) + f, X \nabla_\mu \phi \nabla_\nu \phi + Z f, Z u_{cm} u_{cv}.
\]

From eqs. (2.17) and (2.25), the gravitational equations of motion are given by

\[
M^2_{\text{pl}} G_{\mu\nu} = T^{(\text{H})}_{\mu\nu} + \sum_{I=c,b,r} T^{(I)}_{\mu\nu} + T^{(\text{int})}_{\mu\nu}.
\]

The energy-momentum tensors $T^{(\text{H})}_{\mu\nu}$, $T^{(I)}_{\mu\nu}$, and $T^{(\text{int})}_{\mu\nu}$ correspond to those arising from the Horndeski sector, perfect fluids, and the coupling $f$, respectively. Taking the covariant derivative of eq. (2.28), we obtain

\[
\nabla^\mu T^{(\text{H})}_{\mu\nu} + \sum_{I=c,b,r} \nabla^\mu T^{(I)}_{\mu\nu} + \nabla^\mu T^{(\text{int})}_{\mu\nu} = 0.
\]

On using eq. (2.10), the perfect-fluid energy-momentum tensor $T^{(I)}_{\mu\nu}$ obeys

\[
\nabla^\mu T^{(I)}_{\mu\nu} = - \left[ u^\mu \partial_\mu \rho_I + (\rho_I + P_I) \nabla_\mu u^\mu \right] = 0,
\]

which is equivalent to the continuity equation (2.7). If the four-velocities of CDM, baryons, and radiation are identical to each other (which is the case for the FLRW background) or there is only one fluid component characterized by the four-velocity $u^\mu$, then the continuity equation $u^\mu \nabla_\mu T^{(I)}_{\mu\nu} = 0$ holds for each fluid or a single fluid. In this case, eq. (2.29) gives

\[
\nabla^\nu \left( T^{(\text{H})}_{\mu\nu} + T^{(\text{int})}_{\mu\nu} \right) = 0,
\]

which corresponds to the continuity equation for the scalar field.

We note that the function $f$ in $T^{(\text{int})}_{\mu\nu}$ contains the dependence of CDM density $\rho_c$ through the number density $n_c$. It is then possible to absorb such $\rho_c$-dependent terms into the standard CDM energy-momentum tensor $T^{(c)}_{\mu\nu}$. By defining the modified CDM energy-momentum tensor $\hat{T}^{(c)}_{\mu\nu}$ in this way, the continuity equations of CDM and scalar field possess explicit interacting terms associated with the energy transfer [52]. In section 2.2, we will explicitly see this for the coupling $f$ separable into $n_c$ and other variables.
2.2 Background equations of motion

We derive the background equations of motion on the flat FLRW background described by the line element
\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j , \] (2.32)
where \( a(t) \) is the time-dependent scale factor. On this background, the scalar field \( \phi \) depends only on \( t \). Each perfect fluid in the rest frame has the four-velocity \( u^\mu_I = (1, 0, 0, 0) \), with \( I = c, b, r \). From eq. (2.4), the temporal component of \( J^\mu_I \) is equivalent to \( J^0_I \equiv N_I = n_I a^3 \). Due to the constraint (2.7), we have
\[ N_I = \text{constant} , \] (2.33)
which corresponds to the conservation of total particle number of each fluid. This relation is equivalent to the continuity eq. (2.10). On the background (2.32), eq. (2.10) reduces to
\[ \dot{\rho}_I + 3H (\rho_I + P_I) = 0 , \quad \text{for} \quad I = c, b, r , \] (2.34)
where a dot represents the derivative with respect to \( t \), and \( H = \dot{a}/a \) is the Hubble-Lemaître expansion rate.

The (00) and \((ii)\) components of the gravitational eq. (2.28) are given, respectively, by
\[ 3M^2_{\text{pl}} H^2 = \rho_{\text{DE}} + \sum_{I=c,b,r} \rho_I , \] (2.35)
\[ M^2_{\text{pl}} \left( 2\dot{H} + 3H^2 \right) = -P_{\text{DE}} - \sum_{I=c,b,r} P_I , \] (2.36)
where
\[
\rho_{\text{DE}} = -G_2 + \phi^2 G_{2,X} + \phi^3 \left( G_{3,\phi} - 3H \dot{\phi} G_{3,X} \right) \\
+ 3H^2 \left( M^2_{\text{pl}} - 2G_4 \right) - 6H \dot{\phi} G_{4,\phi} - f + \phi^2 f_{X} + \dot{\phi} f_{Z} ,
\] (2.37)
\[
P_{\text{DE}} = G_2 + \phi^2 \left( G_{3,\phi} + \ddot{\phi} G_{3,X} \right) + \left( 2\dot{H} + 3H^2 \right) \left( 2G_4 - M^2_{\text{pl}} \right) \\
+ 2 \left( \ddot{\phi} + 2H \dot{\phi} \right) G_{4,\phi} + 2\phi^2 G_{4,\phi\phi} + f - n_c f_{n_c} .
\] (2.38)

Defining the density parameters,
\[ \Omega_{\text{DE}} = \frac{\rho_{\text{DE}}}{3M^2_{\text{pl}} H^2} , \quad \Omega_I = \frac{\rho_I}{3M^2_{\text{pl}} H^2} , \] (2.39)
the Hamiltonian constraint (2.35) is expressed in the form,
\[ \Omega_{\text{DE}} + \sum_{I=c,b,r} \Omega_I = 1 . \] (2.40)

Taking the time derivative of eq. (2.35) and using eqs. (2.34) and (2.36), we obtain
\[ \dot{\rho}_{\text{DE}} + 3H (\rho_{\text{DE}} + P_{\text{DE}}) = 0 , \] (2.41)
which corresponds to eq. (2.31). We define the equations of state of dark energy and perfect fluids, as
\[ w_{\text{DE}} = \frac{P_{\text{DE}}}{\rho_{\text{DE}}} , \quad w_I = \frac{P_I}{\rho_I} . \] (2.42)
For given \( w_I \) (\( I = c, b, r \)) and initial conditions, the background dynamics is determined by integrating eqs. (2.34), (2.36) and (2.41) together with the constraint eq. (2.40). The evolution of \( \Omega_{\text{DE}} \) and \( w_{\text{DE}} \) is known accordingly.
Let us consider the interacting theories given by the function
\[ f = -f_1(\phi, X, Z)\rho_c(n_c) + f_2(\phi, X, Z), \tag{2.43} \]
where \(f_1\) and \(f_2\) depend on \(\phi, X, Z\). In this case, the coupling \(f_1\) gives rise to the terms \(f_1\rho_c\) and \(f_1P_c\) in \(\rho_{DE}\) and \(P_{DE}\), respectively. If these terms are absorbed into \(\rho_c\) and \(P_c\) appearing in eqs. (2.35) and (2.36), respectively, then we can define the effective CDM density and pressure, as
\[ \hat{\rho}_c = (1 + f_1)\rho_c, \quad \hat{P}_c = (1 + f_1)P_c, \tag{2.44} \]

Then, from the continuity eqs. (2.34) and (2.41), it follows that
\[ \dot{\hat{\rho}}_c + 3H(\hat{\rho}_c + \hat{P}_c) = \frac{\dot{f}_1}{1 + f_1}\rho_c, \tag{2.46} \]
\[ \dot{\hat{\rho}}_{DE} + 3H(\hat{\rho}_{DE} + \hat{P}_{DE}) = -\frac{\dot{f}_1}{1 + f_1}\hat{\rho}_c, \tag{2.47} \]
whose right-hand-sides are opposite to each other. Hence the energy exchange between CDM and DE is explicit with the definitions (2.44) and (2.45). The CDM density \(\rho_c\) and pressure \(P_c\) are those associated with the conservation of CDM particle number \(J_0^c = N_c\), so the standard continuity equation (2.34) holds for them. The CDM acquires a field-dependent effective mass through the energy exchange with the scalar field. This results in the modified continuity eq. (2.46).

3 Second-order action and perturbation equations

In this section, we first derive the second-order action of tensor perturbations on the flat FLRW background for the coupled DE and DM theories given by the action (2.1). Then, we proceed to the derivation of the second-order action of scalar perturbations. There are no dynamical vector degrees of freedom for the theories under consideration, so we do not take vector perturbations into account.

The matter sector is dealt as perfect fluids described by the Schutz-Sorkin action, so there are no anisotropic shear and viscosity. The matter components interact with each other only through gravity except for the interaction between the scalar field and CDM. This perfect-fluid treatment for CDM and baryons is sufficient to study the late-time dynamics of perturbations associated with the cosmic growth rate.

3.1 Tensor perturbations

The perturbed line element containing the tensor perturbation \(h_{ij}\) on the flat FLRW background is given by
\[ ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij}) \, dx^i dx^j, \tag{3.1} \]
where \(h_{ij}\) obeys the transverse and traceless conditions \(\partial_i h_{ij} = 0\) and \(h^i_i = 0\). We consider the gravitational waves propagating along the \(z\) direction, in which case the nonvanishing components of \(h_{ij}\) can be chosen as \(h_{11} = h_1(t,z)\), \(h_{22} = -h_1(t,z)\), and \(h_{12} = h_{21} = h_2(t,z)\). Expanding the action (2.1) up to second order in \(h_1\) and \(h_2\), integrating by parts, and using the background equations of motion, the second-order action of tensor perturbations reduces to
\[ S_t^{(2)} = \int dt \, q^3 \sum_{i=1}^2 \frac{a^3}{4} q_t \left[ h^2_i - \frac{c_s^2}{a^2} (\partial h_i)^2 \right], \tag{3.2} \]
where
\[ q_t = 2G_4, \quad \text{and} \quad c_t^2 = 1. \] (3.3)
The tensor ghost is absent under the condition \( G_4 > 0 \). Since the speed of gravitational waves is equivalent to that of light, the theories given by the action (2.1) are consistent with the observational bound of \( c_t^2 \) derived from the GW170817 event [32].

### 3.2 Scalar perturbations

The line element containing four scalar perturbations \( \alpha, \chi, \zeta \) and \( E \) is given by [85]
\[ ds^2 = -(1 + 2\alpha)dt^2 + 2\partial_i\chi dt dx^i + a^2(t) [(1 + 2\zeta)\delta_{ij} + 2\partial_i\partial_j E] dx^i dx^j, \] \[ \text{(3.4)} \]
where the perturbed quantities depend on both cosmic time \( t \) and spatial coordinates \( x^i \).

The scalar field is decomposed into the time-dependent background part \( \bar{\phi}(t) \) and the perturbed part \( \delta \phi \), as
\[ \phi = \bar{\phi}(t) + \delta \phi, \] \[ \text{(3.5)} \]
where we will omit the bar in the following discussion. We also decompose the temporal and spatial components of \( J_{\mu}^i \) in the forms,
\[ J_{\mu}^0 = N_I + \delta J_I, \quad J_{\mu}^i = \frac{1}{a^2(t)} \delta ^{ik} \partial_k \delta j_I, \] \[ \text{(3.6)} \]
where \( N_I \) is the background conserved particle number, while \( \delta J_I \) and \( \delta j_I \) are the scalar perturbations.

Substituting eq. (3.6) into the definition of number density (2.5) and expanding \( n_I \) up to second order in scalar perturbations, it follows that
\[ n_I = \frac{N_I}{a^3} \left[ 1 + \frac{\delta \rho_I}{\rho_I + \rho_t} (1 - 3\zeta - \partial^2 E) - \frac{(\partial \delta j_I + N_I \partial \chi)^2}{2N_I^2 a^2} - \frac{1}{2} \left( \zeta + \partial^2 E \right) (3\zeta - \partial^2 E) \right] + \mathcal{O}(\varepsilon^3), \] \[ \text{(3.7)} \]
where \( \varepsilon \) represents the order of perturbations, and \( \delta \rho_I \) is the density perturbation defined by
\[ \delta \rho_I = \frac{\rho_I + \rho_t}{N_I} \left[ \delta J_I - N_I \left( 3\zeta + \partial^2 E \right) \right]. \] \[ \text{(3.8)} \]
At linear order, the perturbation \( \delta n_I \) of number density is related to \( \delta \rho_I \) according to \( \delta n_I = n_I \delta \rho_I \). By using eqs. (3.6) and (3.7), the four velocity \( u_{\mu} = J_{\mu}/(n_I \sqrt{-g}) \), which is expanded up to linear order, is given by
\[ u_{\mu} = -1 - \alpha, \quad u_i = -\partial_iv_I, \] \[ \text{(3.9)} \]
where
\[ \partial_i v_I = -\partial_i \left( \chi + \frac{\delta j_I}{N_I} \right). \] \[ \text{(3.10)} \]
Note that \( v_I \) corresponds to the velocity potential of each fluid. In the following, we express \( \delta j_I \) and \( \delta J_I \) in terms of \( \delta \rho_I, v_I, \) and metric perturbations. On using eqs. (3.5) and (3.9), the spatial component of eq. (2.15) expanded up to linear order in perturbations is given by
\[ \partial_i \ell_c = - \left( \rho_{c,n_c} - f_{,n_c} \right) \partial_i v_c + \frac{a^3 f_Z}{N_c} \left( \partial_i \delta \phi - \dot{\phi} \partial_i v_c \right), \] \[ \text{(3.11)} \]
where the coefficients \( \rho_{c,n_c}, f_{,n_c}, \) and \( f_{,Z} \) should be evaluated on the background. Integrating eq. (3.11) with respect to \( x^i \) and using the property \( \ell_c = - \left( \rho_{c,n_c} - f_{,n_c} \right) \) on the background, we obtain
\[ \ell_c = - \int^t \left[ \rho_{c,n_c}(\tilde{t}) - f_{,n_c}(\tilde{t}) \right] \dot{\tilde{t}} - \left( \rho_{c,n_c} - f_{,n_c} \right)v_c + \frac{a^3 f_Z}{N_c} \left( \delta \phi - \dot{\phi} \right) v_c. \] \[ \text{(3.12)} \]
This relation will be used to eliminate the Lagrange multiplier \( \ell_c \) from the action (2.1).
3.2.1 Second-order action

Since the energy density $\rho_I$ depends on $n_I$, it can be expanded in the form,

$$\rho_I(n_I) = \rho_I + (\rho_I + P_I) \frac{\delta n_I}{n_I} + \frac{1}{2} (\rho_I + P_I) c_I^2 \left( \frac{\delta n_I}{n_I} \right)^2 + O(\varepsilon^3),$$  \hspace{1cm} (3.13)

where $c_I^2$ is the fluid sound speed squared defined by

$$c_I^2 = \frac{n_I \rho_I n_{nI}}{\rho_I n_I}.$$  \hspace{1cm} (14)

We also express the interacting Lagrangian $f(n_c, \phi, X, Z)$ as

$$f(n_c, \phi, X, Z) = f + f_{,n_c} \delta n_c + f_{,\phi} \delta \phi + f_{,X} \delta X + f_{,Z} \delta Z + \frac{1}{2} f_{,n_c n_c} \delta n_c^2 + \frac{1}{2} f_{,\phi \phi} \delta \phi^2 + \frac{1}{2} f_{,XX} \delta X^2 + \frac{1}{2} f_{,ZZ} \delta Z^2$$

$$+ f_{,n_c \phi} \delta n_c \delta \phi + f_{,n_c X} \delta n_c \delta X + f_{,n_c Z} \delta n_c \delta Z + f_{,X \phi} \delta \phi \delta X + f_{,Z \phi} \delta \phi \delta Z + f_{,XZ} \delta X \delta Z + O(\varepsilon^3),$$  \hspace{1cm} (3.15)

where $\delta n_c$ is the perturbed part of eq. (3.7) with $I = c$, and

$$\delta X = \hat{\phi}(\delta \phi - \hat{\phi} \alpha) + \frac{1}{2a^2} \left[ (\delta \phi - 2 \hat{\phi} \alpha)^2 - \frac{1}{a^2} (\partial \delta \phi + \hat{\phi} \partial \chi)^2 \right] + O(\varepsilon^3),$$  \hspace{1cm} (3.16)

$$\delta Z = \hat{\phi}(\delta \phi - \hat{\phi} \alpha) + \frac{1}{2a^2} \left[ \hat{\phi} \left\{ 3a^2 \alpha^2 - (\partial_i \chi)^2 + (\partial_i v_c)^2 \right\} - 2a^2 \hat{\phi} \alpha \right] + O(\varepsilon^3).$$  \hspace{1cm} (3.17)

We expand the action (2.1) up to quadratic order in scalar perturbations, integrate it by parts, and use the background equations of motion. Then, the resulting second-order action is expressed in the form,

$$S_s^{(2)} = \int dt \, d^3x \left( L_0 + L_f \right),$$  \hspace{1cm} (3.18)

where

$$L_0 = a^3 \left( D_1 \delta \phi^2 + D_2 \frac{(\delta \phi)^2}{a^2} + D_3 \delta \phi^2 + (D_4 \delta \phi + D_5 \delta \phi + D_6 \frac{\partial^2 \delta \phi}{a^2}) \alpha - \left( \frac{\dot{D}_0 \delta \phi - \dot{D}_7 \delta \phi}{a^2} \right) \partial^2 \chi \right.$$  \hspace{1cm}

$$+ \left( \dot{\phi} D_6 - 2H q_I \right) \alpha \frac{\partial^2 \chi}{a^2} + \left( \dot{\phi}^2 D_1 + 3H \dot{\phi} D_6 - 3H^2 q_I \right) \alpha^2$$  \hspace{1cm}

$$+ \sum_{I=c,b,r} \left\{ \left( \rho_I + P_I \right) v_I \frac{\partial^2 X}{a^2} - v_I \delta \rho_I - 3H(1+c_I^2) v_I \delta \rho_I - \frac{\rho_I + P_I}{2a^2} (\partial v_I)^2 - \frac{c_I^2}{2(\rho_I + P_I)} \delta \rho_I^2 - \alpha \delta \rho_I \right\}$$  \hspace{1cm}

$$+ \left\{ 3D_0 \delta \phi^2 - 3D_7 \delta \phi^2 - 3 \left( \dot{\phi} D_6 - 2H q_I \right) \alpha - \sum_{I=c,b,r} 3(\rho_I + P_I) v_I + 2q_I \frac{\partial^2 \chi}{a^2} \right\} \dot{\chi} - 3q_I \dot{\xi}^2 + q_I \frac{\partial \xi}{a^2}$$  \hspace{1cm}

$$- 2 \left( \frac{q_I}{\dot{\phi}} \delta \phi + q_I \alpha \right) \frac{\partial^2 \xi}{a^2} + \left[ D_6 \delta \phi - 2 q_I \dot{\xi} - D_7 \delta \phi - (\dot{\phi} D_6 - 2H q_I) \alpha - \sum_{I=c,b,r} (\rho_I + P_I) v_I \right] \partial^2 \dot{E},$$  \hspace{1cm} (3.19)
and

\[
L_f = a^3 \left\{ \frac{1}{2} (f_{,X} + \dot{\phi} f_{,XX} + 2 \dot{\phi} f_{,XZ} + f_{,ZZ}) (\dot{\phi} - \phi \dot{\phi})^2 - \frac{1}{2} \left[ (f_{,X} + \dot{\phi} f_{,X} + 2 \dot{\phi} f_{,XX} + f_{,ZZ}) \dot{\phi} + \phi f_{,X} \varphi + 3H \dot{\phi} (f_{,X} - n_c f_{,n_c X}) + \dot{\phi} f_{,Z} \varphi + 3H (f_{,Z} - n_c f_{,n_c Z}) - f_{,\phi} \right] \delta \phi^2 \\
- \frac{f_{,X}}{2a^2} \left[ (\delta \delta \phi)^2 - 2 \delta \phi \dot{\phi} \delta^2 \chi \right] + \left( f_{,\phi} - \dot{\phi} f_{,X} - \dot{\phi} f_{,Z} \right) \alpha \delta \phi \\
- \frac{n_c f_{,n_c X} - \dot{\phi} f_{,Z}}{\rho_c + P_c} \left[ \frac{\rho_c}{\alpha} \dot{\rho}_c - v_c \delta \rho_c - 3H (1 + \alpha^2) v_c \delta \rho_c - \frac{\rho_c + P_c}{2a^2} (\partial v_c)^2 \right] \\
+ \frac{n_c^2 f_{,n_c X} n_c - \dot{\phi} f_{,n_c X}}{\rho_c + P_c} \left[ \frac{\rho_c}{\alpha} \dot{\rho}_c - v_c \delta \rho_c - 3H (1 + \alpha^2) v_c \delta \rho_c - \frac{\rho_c + P_c}{2a^2} (\partial v_c)^2 \right] \\
- \frac{1}{\rho_c + P_c} \left[ (\dot{\phi} f_{,X} + f_{,ZZ}) \dot{\phi} + \dot{\phi} f_{,Z} + 3H (f_{,Z} - n_c f_{,n_c Z}) - n_c f_{,n_c X} \right] \delta \rho_c \delta \phi \\
+ \frac{n_c f_{,n_c X} - \dot{\phi} f_{,Z}}{\rho_c + P_c} \delta \rho_c \delta \phi \left( 3 \dot{\chi} + \partial^2 \dot{E} \right) \right\}, \tag{3.20}
\]

where \( n_c \) is evaluated on the background, i.e., \( n_c = \mathcal{N}_c / a^3 \). The Lagrangian \( L_f \) arises from the coupling \( f \). The quantity \( q_f \) is given in eq. (3.3). As we derived in section 3.1, we require the condition \( q_f > 0 \) to avoid the tensor ghost. The other coefficients \( D_i \) (\( i = 1, \ldots, 7 \)) are given by

\[
D_1 = \frac{1}{2} G_{2,X} + G_{3,\phi} + \frac{1}{2} \delta^2 \left( G_{2,X} + G_{3,XX} \right) - \frac{3}{2} H \dot{\phi} \left( 2G_{3,X} + \dot{\phi}^2 G_{3,XX} \right), \tag{3.21}
\]
\[
D_2 = \frac{1}{2} G_{2,X} - G_{3,\phi} + 2H \phi G_{3,X} + \frac{1}{2} \delta^2 G_{3,XX} + \frac{1}{2} \left( 2G_{3,X} + G_{3,XX} \right) \delta \phi, \tag{3.22}
\]
\[
D_3 = \frac{1}{2} G_{2,\phi} - \frac{1}{2} \left( G_{2,X,\phi} + G_{3,\phi,\phi} \right) \dot{\phi}^2 + \frac{3}{2} \left( G_{3,XX} \dot{\phi}^2 - 3G_{3,X} \dot{\phi} \right) H \dot{\phi} \\
+ \frac{3}{2} \left( \dot{\phi}^2 G_{3,X} + 4G_{4,\phi} \right) H \dot{\phi} \\
- \left[ \frac{1}{2} G_{2,X} + G_{3,\phi} - \frac{3}{2} \left( G_{3,XX} \dot{\phi}^2 + 2G_{3,X} \right) H \dot{\phi} + \frac{1}{2} \left( G_{2,X,\phi} + G_{3,XX} \right) \dot{\phi}^2 \right] \delta \phi, \tag{3.23}
\]
\[
D_4 = - \left( G_{2,X} + 2G_{3,\phi} \right) \phi - \left( G_{2,XX} + G_{3,XX} \right) \delta \phi^3 + \frac{3}{2} \left( 3G_{3,XX} \dot{\phi}^2 + 2G_{3,X} \phi \right) H \dot{\phi} + \frac{1}{2} \left( G_{2,X,\phi} + G_{3,XX} \right) \dot{\phi}^2 \right] \delta \phi, \tag{3.24}
\]
\[
D_5 = G_{2,\phi} - \dot{\phi}^2 \left( G_{2,X} + G_{3,\phi} \right) + 3H \dot{\phi} \left( \dot{\phi}^2 G_{3,X} + 2G_{4,\phi} \right) + 6H^2 G_{4,\phi}, \tag{3.25}
\]
\[
D_6 = - \dot{\phi}^2 G_{3,X} - 2G_{4,\phi}, \tag{3.26}
\]
\[
D_7 = \dot{\phi} \left( G_{2,X} + 2G_{3,\phi} + 2G_{4,\phi} \right) - H \left( 3\dot{\phi}^2 G_{3,X} + 2G_{4,\phi} \right). \tag{3.27}
\]

Among these coefficients, there are following four conditions:

\[
2\dot{\phi}^2 D_2 = -2H \dot{q}_t - \dot{\phi} \left( \dot{D}_6 + HD_6 + D_7 \right), \tag{3.28}
\]
\[
D_4 = -2\dot{\phi} D_1 - 3HD_6, \tag{3.29}
\]
\[
2\dot{q}_t H - D_6 \dot{\phi} + f_{,X} \dot{\phi}^2 + (D_7 + f_{,Z}) \dot{\phi} + \sum_{I = e, b, r} (\rho_I + P_I) - n_c f_{,n_c} = 0, \tag{3.30}
\]
\[
\left( 2D_1 + f_{,X} + \dot{\phi}^2 f_{,XX} + 2\dot{\phi} f_{,XZ} + f_{,ZZ} \right) \phi + 3D_6 \dot{H} - D_5 + 3HD_7 - f_{,\phi} + \phi \left[ 3 \dot{H} (f_{,X} - n_c f_{,n_c X}) + f_{,Z} \dot{\phi} + \dot{\phi} f_{,X} \phi \right] + 3H \left( f_{,Z} - n_c f_{,n_c Z} \right) = 0, \tag{3.31}
\]
where eq. (3.30) is equivalent to the subtraction of eq. (2.36) from eq. (2.35), and eq. (3.31) corresponds to the scalar-field eq. (2.41). In section 3.2.2, we will use these relations for simplifying the perturbation equations of motion.

Among scalar perturbations in the second-order action (3.18), the variables $\alpha, \chi, v_c, v_b, v_r$ and $E$ are non-dynamical. We introduce the comoving wavenumber $k$ and derive the perturbation equations for these non-dynamical variables in Fourier space. Variations of the action (3.18) with respect to $\alpha, \chi, v_c, v_b, v_r, E$ lead to

$$
\begin{align*}
&\left[D_4 - \dot{\phi}(f_{,X} + \dot{\phi}^2 f_{,XX} + 2\dot{f}_{,XZ} + \dot{f}_{,ZZ}) \right] (\delta \phi - \dot{\phi} \alpha) - 3 \left( \dot{\phi} D_6 - 2H q_t \right) (\zeta - H\alpha) \\
&+ \left(D_5 + f_{,\phi} - \dot{\phi}^2 f_{,\phi X} - \dot{\phi} f_{,\phi Z} \right) \delta \phi + \frac{k^2}{a^2} \left[ 2q\zeta - \left( \dot{\phi} D_6 - 2H q_t \right) (\chi - a^2 \dot{E}) - D_6 \delta \phi \right] \\
&- \sum_{I=c,b,r} \delta \rho_I + \frac{n_c(f_{,n_c} - \dot{\phi}^2 f_{,n_c X} - \dot{\phi} f_{,n_c Z})}{\rho_c + P_c} \delta \rho_c = 0, \\
&\frac{\partial}{\partial \chi} \left[ X^2 \right] = 0
\end{align*}
$$

(3.32)

$$
D_6 \delta \phi - 2q\dot{\zeta} - \left( D_7 + \dot{\phi} f_{,X} \right) \delta \phi - \left( \dot{\phi} D_6 - 2H q_t \right) \alpha - \sum_{I=c,b,r} (\rho_I + P_I) v_I + \left(n_c f_{,n_c} - \dot{\phi} f_{,Z} \right) v_c = 0,
$$

(3.33)

$$
\frac{\partial}{\partial \chi} \left( 3H \dot{Z} \right) + 3 \left( D_7 + \dot{\phi} f_{,X} \right) \dot{\zeta} + \frac{k^2}{a^2} (\rho_I + P_I) \left( v_I + \chi - a^2 \dot{E} \right) = 0, \quad \text{for} \quad I = c, b, r,
$$

(3.34)

$$
\ddot{\mathcal{W}} + 3H \mathcal{W} = 0,
$$

(3.35)

respectively, where

$$
\mathcal{W} = 2q\dot{\zeta} - D_6 \delta \phi + \left( D_7 + \dot{\phi} f_{,X} \right) \delta \phi + \left( \dot{\phi} D_6 - 2H q_t \right) \alpha + \sum_{I=c,b,r} (\rho_I + P_I) v_I - \left(n_c f_{,n_c} - \dot{\phi} f_{,Z} \right) v_c.
$$

(3.36)

Varying the action (3.18) with respect to the dynamical perturbations $\delta \phi, \delta \rho_b, \delta \rho_c, \delta \rho_r, \zeta$, we obtain

$$
\begin{align*}
\hat{\dot{\zeta}} + 3H \dot{Z} + 3 \left( D_7 + \dot{\phi} f_{,X} \right) \dot{\zeta} &+ M_5^2 \delta \phi - \left( D_5 + f_{,\phi} - \dot{\phi}^2 f_{,\phi X} - \dot{\phi} f_{,\phi Z} \right) \alpha \\
&+ \frac{1}{\rho_c + P_c} \left[ \left( \dot{\phi} f_{,XZ} + \dot{\phi} f_{,ZZ} \right) \dot{\phi} + \dot{\phi} f_{,\phi Z} + 3H (f_{,Z} - n_c f_{,n_c Z}) - n_c f_{,n_c \phi} \right] \delta \rho_c \\
&- \frac{k^2}{a^2} \left[ 2D_2 \delta \phi - D_6 \alpha - D_7 \chi + \frac{2q\dot{\zeta}}{\phi} - a^2 \left(D_7 + 3H D_7\right) E - f_{,X} \left\{ \delta \phi + \dot{\phi} \left( \chi - a^2 \dot{E} \right) \right\} \right] = 0,
\end{align*}
$$

(3.37)

$$
\begin{align*}
&\left(1 + \frac{\dot{\phi} f_{,n_c} - n_c f_{,n_c}}{\rho_c + P_c} \right) \dot{v}_c - \left( c^2 - \frac{n_c^2}{\rho_c + P_c} \right) \left( 3H v_c + \frac{\delta \rho_c}{\rho_c + P_c} \right) \\
&- \frac{1}{\rho_c + P_c} \left[ \left( \dot{\phi} f_{,XZ} + \dot{\phi} f_{,ZZ} \right) \dot{\phi} + \dot{\phi} f_{,\phi Z} + 3H (f_{,Z,n_c f_{,n_c Z}) - n_c f_{,n_c \phi} \right) \left( \delta \phi - \dot{\phi} v_c \right) \\
&- \frac{f_{,Z,n_c} \delta \phi - n_c f_{,n_c X} - f_{,n_c Z}}{\rho_c + P_c} \delta \phi - \dot{\phi} v_c \right) \right] - \left[ 1 - n_c f_{,n_c X} - f_{,n_c Z} \right] \alpha = 0, \\
&\dot{v}_I - 3H c^2 v_I - \frac{c^2}{\rho_I + P_I} \delta \rho_I - \alpha = 0, \quad \text{for} \quad I = b, r,
\end{align*}
$$

(3.38)

$$
\begin{align*}
\dot{\mathcal{W}} + 3H \mathcal{W} + \frac{2k^2}{3a^2} \left\{ q \left[ \alpha + \dot{\chi} + \zeta + H \chi - a^2 \left( \dot{E} + 3H \dot{E} \right) \right] + \dot{q} \left( \chi - a^2 \dot{E} + \frac{\delta \phi}{\phi} \right) \right\} = 0,
\end{align*}
$$

(3.39)
where
\[
 M_\phi^2 = -2D_3 - f_{,\phi\phi} + \left( f_{,XX} + 2\dot{\phi}f_{,XX\phi} + 2\dot{\phi}f_{,XZ\phi} + f_{,ZZ\phi} \right) \dot{\phi} + \dot{\phi}^2 f_{,X\phi\phi} + \dot{\phi} f_{,Z\phi\phi} \\
 + 3H\dot{\phi} \left( f_{,XX} - n_c f_{,n_c X\phi} \right) + 3H \left( f_{,Z\phi} - n_c f_{,n_c Z\phi} \right)
\]
(3.41)
\[
 Z = 2D_1 \delta\phi + \left( f_{,X} + \dot{\phi}^2 f_{,XX} + 2\dot{\phi}f_{,XZ} + f_{,ZZ} \right) \left( \delta\phi - \dot{\phi}\alpha \right) + 3D_6 \zeta + D_4 \alpha
\]
- \frac{f_{,Z} - n_c (f_{,n_c X} + f_{,n_c Z})}{\rho_c + P_c} \delta\rho_c + \frac{k^2}{a^2} \left[ D_6 \chi - a^2 \left( D_6 \dot{E} + D_7 \dot{E} \right) \right].
\]
(3.42)
The quantity \( M_\phi^2 \) corresponds to the effective mass squared of scalar field perturbation. On using eqs. (3.35) and (3.40), it follows that
\[
 q_t \left[ \alpha + \dot{\chi} + \zeta + H\chi - a^2 \left( \dot{E} + 3H\dot{E} \right) \right] + \dot{\zeta} \left( \chi - a^2 \dot{E} + \frac{\delta\phi}{\dot{\phi}} \right) = 0.
\]
(3.43)
Since we have not yet fixed the gauge degrees of freedom, the perturbation equations (3.32)–(3.40) can be applied to any choices of gauges. Namely, they are written in a gauge-ready form [86, 87].

### 3.2.2 Perturbation equations with gauge-invariant variables

In this subsection, we rewrite the perturbation equations of motion in terms of gauge-invariant variables. Let us consider the infinitesimal transformation given by
\[
 \tilde{t} = t + \xi^0 \quad \text{and} \quad \tilde{x}^i = x^i + \delta^i_j \partial_j \xi
\]
(3.44)
where \( \xi^0 \) and \( \xi \) are scalar variables. Then, the metric perturbations in eq. (3.4) transform as
\[
 \tilde{a} = a - \xi^0, \quad \tilde{\chi} = \chi + \xi^0 - a^2 \xi, \quad \tilde{\zeta} = \zeta - H\xi^0, \quad \tilde{E} = E - \xi,
\]
(3.45)
while the perturbations associated with the scalar field and fluids transform as
\[
 \tilde{\delta}\phi = \delta\phi - \dot{\phi}\xi^0, \quad \tilde{\delta}\rho_I = \delta\rho_I - \dot{\rho}_I \xi^0, \quad \tilde{\dot{v}}_I = \dot{v}_I - \xi^0.
\]
(3.46)
We introduce the following variables invariant under the transformation (3.44),
\[
 \Psi = \alpha + \frac{d}{dt} \left( \chi - a^2 \dot{E} \right), \quad \Phi = \zeta + H \left( \chi - a^2 \dot{E} \right),
\]
\[
 \delta\phi_N = \delta\phi + \dot{\phi} \left( \chi - a^2 \dot{E} \right), \quad \delta\rho_{IN} = \delta\rho_I + \dot{\rho}_I \left( \chi - a^2 \dot{E} \right), \quad v_{IN} = v_I + \chi - a^2 \dot{E}
\]
(3.47)
where \( \Psi \) and \( \Phi \) are Bardeen gravitational potentials [85]. To simplify the perturbation equations of motion, we also define the dimensionless variables,
\[
 \alpha_K = \frac{2\dot{\phi}^2 D_1}{H^2 q_t}, \quad \alpha_B = -\frac{\dot{\phi} D_6}{2H q_t}, \quad \alpha_M = \frac{q_t}{H q_t},
\]
\[
 \beta_K = \frac{\dot{\phi}^2 (f_{,XX} + 2\dot{\phi} f_{,XX\phi} + 2\dot{\phi} f_{,XZ\phi} + f_{,ZZ\phi})}{H^2 q_t}, \quad \beta_{nc} = \frac{n_c (f_{,n_c X} - \dot{\phi} f_{,n_c X\phi} - \dot{\phi} f_{,n_c Z\phi})}{\rho_c + P_c}
\]
(3.48)
and
\[
 \epsilon_{\alpha_K} = \frac{\dot{\alpha}_K}{H \alpha_K}, \quad \epsilon_{\alpha_B} = \frac{\dot{\alpha}_B}{H \alpha_B}, \quad \epsilon_{\beta_K} = \frac{\dot{\beta}_K}{H \beta_K}, \quad \epsilon_{\beta_{nc}} = \frac{\dot{\beta}_{nc}}{H \beta_{nc}}, \quad \epsilon_H = \frac{\dot{H}}{H^2}, \quad \epsilon_\phi = \frac{\dot{\phi}}{H \dot{\phi}}.
\]
(3.49)
The quantity \( \alpha_B \) is related to \( \alpha_B^{(BS)} \) introduced by Bellini and Sawicki [88], as \( \alpha_B = -\alpha_B^{(BS)}/2 \), while \( \alpha_K \) and \( \alpha_M \) are the same as those given in ref. [88]. The quantities \( \beta_K \) and \( \beta_{nc} \) are new dimensionless variables arising from the coupling \( f \).
In the following, we eliminate $D_2, D_4, D_5, D_7$ by using the relations (3.28)–(3.31), and replace $D_1, D_3, D_6$ with $\alpha_K, M_2^2, \alpha_B$, respectively. On using the gauge-invariant variables given in eq. (3.47), the equations of motion for the non-dynamical perturbations $\alpha, \chi, v_c, v_s, v_t, E$, i.e., eqs. (3.32)–(3.35), and for the dynamical perturbations $\delta \phi, \delta \phi_c, \delta \phi_b, \delta \rho_r$, i.e., eqs. (3.37)–(3.39), are expressed as

$$
6(1+\alpha_B)\frac{\Phi}{H} + (6\alpha_B - \alpha_K - \beta_K) \frac{\delta \phi_N}{\phi} + 2 \left( \frac{k}{aH} \right)^2 \Phi - (6 + 12\alpha_B - \alpha_K - \beta_K) \Psi - (1 - \beta_n) \frac{\delta \rho_N}{H^2 q_t} - \sum_{l=b,r} \frac{\delta \rho_N}{H^2 q_t} \\
+ \left[ 2 \left( \frac{k}{aH} \right)^2 \alpha_B - 6(1+\alpha_B)\epsilon_h - (6\alpha_B - \alpha_K - \beta_K)\epsilon_\phi - \frac{3(\rho_c + P_c)}{H^2 q_t} (1 - \beta_n) - \sum_{l=b,r} \frac{3(\rho_l + P_l)}{H^2 q_t} \right] \frac{H}{H^2} \delta \phi_N = 0,
$$

(3.50)

$$
\frac{\Phi}{H} + \alpha_B \frac{\delta \phi_N}{\phi} - (1 + \alpha_B) \Psi + \frac{q_c(\rho_c + P_c)}{2H q_t} \left( v_c - \frac{\delta \phi_N}{\phi} \right) + \sum_{l=b,r} \frac{\rho_l + P_l}{2H q_t} \left( v_{lN} - \frac{\delta \phi_N}{\phi} \right) - (\epsilon_h + \epsilon_\phi \alpha_B) \frac{H}{H^2} \delta \phi_N = 0,
$$

(3.51)

$$
\delta \rho_N + 3H(1 + c_2^2)\delta \rho_N + (\rho_l + P_l) \left( 3 \Phi + \frac{k^2}{a^2} v_{IN} \right) = 0, \quad \text{for } I = c, b, r,
$$

(3.52)

$$
\dot{W} + 3H \dot{W} = 0,
$$

(3.53)

and

$$
(\alpha_K + \beta_K) \frac{\delta \phi_N}{H \phi} - 6\alpha_B \frac{\Phi}{H^2} + [\epsilon_{c\phi} \alpha_K + \epsilon_{b\phi} \beta_K + (3 + \alpha_M + 2\epsilon_h - 2\epsilon_\phi)(\alpha_K + \beta_K)] \frac{\dot{\phi}_{IN}}{\phi} + (6\alpha_B - \alpha_K - \beta_K) \frac{\Psi}{H} \\
- 3 \left[ 2H + 2(3 + \epsilon_h + 3\alpha_M + \epsilon_{c\phi})\alpha_B + \frac{q_c(\rho_c + P_c)}{H^2 q_t} + \sum_{l=b,r} \frac{\rho_l + P_l}{2H q_t} \right] \frac{\Phi}{H} - 2\alpha_M \left( \frac{k}{aH} \right)^2 \Phi \\
- \left[ 2\alpha_B \left( \frac{k}{aH} \right)^2 + \epsilon_{c\phi} \alpha_K + \epsilon_{b\phi} \beta_K - 6\epsilon_{c\phi} \alpha_B + (3 + 2\epsilon_h + 3\alpha_M)(\alpha_K + \beta_K) - 6\epsilon_h (1 + \alpha_B) \right] \\
- 3(1 - \beta_n) \frac{\rho_c + P_c}{H^2 q_t} - 3 \sum_{l=b,r} \frac{\rho_l + P_l}{2H q_t} \right] \Psi + \left[ \frac{k}{aH} \right]^2 \left( \frac{2\alpha_B (\alpha_B - 2\alpha_M) + \dot{\phi}_N^2 q_c^2}{2H q_t^2} + \dot{\phi}_N^2 M_B^2 a^2 H^2 q_t \right] \frac{H \delta \phi_N}{H^2 q_t} \\
+ (1 - \beta_n) q_c \frac{\delta \rho_N}{H^2 q_t} + \left[ 3 - 3q_c (1 - c_2^2 + c_3^2) - (3 + \epsilon_{c\phi})\beta_n \right] \frac{\delta \rho_N}{H^2 q_t} = 0,
$$

(3.54)

$$
\dot{v}_{cN} - H (3c_2^2 - \epsilon_{c\phi}) v_{cN} = \frac{c_2^2 \dot{\phi}_{cN}}{c_2^2 - \frac{\rho_c + P_c}{\rho_c + P_c}} \left( 1 - \beta_n \right) \frac{\Psi}{q_c} + \frac{1 - \beta_n - q_c}{q_c} \left( \delta \phi_N - H \epsilon_{c\phi} \delta \phi_N \right) - (3c_2^2 - 3c_3^2) \frac{H \delta \phi_N}{H^2 q_t} = 0,
$$

(3.55)

$$
\dot{v}_{IN} - 3H c_3^2 v_{IN} = \frac{c_3^2}{\rho_l + P_l} \delta \rho_N - \Psi = 0, \quad \text{for } I = b, r,
$$

(3.56)

where

$$
q_c = 1 + \dot{\phi}_f Z - n_c \dot{f}_{c,se},
$$

(3.57)

$$
q_s = 2H^2 q_2^2 (\alpha_K + \beta_K + 6\alpha_B^2),
$$

(3.58)

$$
\epsilon_c^2 = \frac{1}{q_c} \left( \frac{c_2^2}{c_2^2 - \frac{n_c^2 f_{c,se}}{\rho_c + P_c}} \right),
$$

(3.59)

$$
\epsilon_s^2 = -\frac{4H^2 q_2^2}{\dot{\phi}_s^2 q_s} \left[ \epsilon_h - \alpha_M + \alpha_B (1 + \epsilon_h - 3\alpha_M + \epsilon_{c\phi}) + \frac{q_c(\rho_c + P_c)}{2H^2 q_t} + \sum_{l=b,r} \frac{\rho_l + P_l}{2H^2 q_t} \right],
$$

(3.60)

$$
\epsilon_{q_c} = \frac{\dot{q}_c}{H q_c},
$$

(3.61)
and
\[
\frac{\mathcal{W}}{2Hq_t} = \alpha_B \frac{\dot{\delta \phi_N}}{\phi} + \frac{\dot{\Phi}}{H} - (\epsilon_H + \epsilon_B \alpha_B) \frac{H}{\phi} \delta \phi_N - (1 + \alpha_B) \Psi
\]
\[
+ \sum_{l=b,r} \frac{P_l + P_r}{2Hq_t} \left( v_{lN} - \frac{\delta \phi_N}{\phi} \right) + \frac{q_c (\rho_c + P_c)}{2Hq_t} \left( \frac{v_{cN} - \delta \phi_N}{\phi} \right).
\]
(3.62)

As we will see later in section 4, the quantities \(q_c, q_s, c_c^2, \) and \(c_s^2\) are related to the stability conditions of CDM and scalar-field perturbations. We note that eq. (3.53) is written in terms of the gauge-invariant variable \(\mathcal{W}\) given by eq. (3.62). Finally, eq. (3.43) is expressed as
\[
\Psi + \Phi + \alpha_M \frac{H}{\phi} \delta \phi_N = 0.
\]
(3.63)

The equation of motion for \(\zeta\), which is given by eq. (3.40), is the combination of eqs. (3.53) and (3.63). From eq. (3.63), it follows that there is an anisotropic stress (\(\Psi \neq -\Phi\)) for the theories with \(\alpha_M \neq 0\).

### 4 Stability conditions

In this section, we derive the stability conditions for scalar perturbations deep inside the sound horizon. Since these conditions are independent of the choice of gauges [13], the residual gauge degrees of freedom can be fixed by choosing a particular gauge. Let us choose the unitary gauge characterized by
\[
\delta \phi = 0, \quad E = 0,
\]
(4.1)
which is realized by setting \(\xi = E\) and \(\xi^0 = \delta \phi / \dot{\phi}\) in eqs. (3.45) and (3.46), respectively. We also introduce the following gauge-invariant variables,
\[
\mathcal{R} = \zeta - \frac{H}{\phi} \delta \phi, \quad \delta \rho_{Iu} = \delta \rho_I - \frac{\dot{\rho}_I}{\phi} \delta \phi,
\]
(4.2)
which reduce to \(\mathcal{R} = \zeta\) and \(\delta \rho_{Iu} = \delta \rho_I\) in the unitary gauge.

We solve eqs. (3.32)–(3.34) for \(\alpha, \chi, v_c, v_b, \) and \(v_r\) to eliminate the non-dynamical variables from the second-order action (3.18). After the integration by parts, the resulting second-order action for dynamical perturbations \(\mathcal{R}, \delta \rho_{cu}, \delta \rho_{ba}, \delta \rho_{ru}\) is expressed in the form,
\[
S^{(2)} = \int dt d^3x a^3 \left( \mathcal{X}^2 K \mathcal{X} - \frac{k^2}{a^2} \mathcal{X}^2 G \mathcal{X} - \mathcal{X}^2 M \mathcal{X} - \frac{k}{a} \mathcal{X}^2 B \mathcal{X} \right),
\]
(4.3)
where \(K, G, M, B\) are 4 × 4 matrices, and
\[
\mathcal{X}^2 = (\mathcal{R}, \delta \rho_{cu}/k, \delta \rho_{ba}/k, \delta \rho_{ru}/k).
\]
(4.4)

Taking the small-scale limit, the leading-order matrix components for \(K, G, B\) are given, respectively, by
\[
K_{11} = \frac{q_c a^2}{4H^2 q_t (1 + \alpha_B)^2}, \quad K_{22} = \frac{q_c a^2}{2(\rho_c + P_c)}, \quad K_{33} = \frac{a^2}{2(\rho_b + P_b)}, \quad K_{44} = \frac{a^2}{2(\rho_r + P_r)},
\]
(4.5)
\[
G_{11} = \frac{q_c c^2 a^2}{4H^2 q_t (1 + \alpha_B)^2}, \quad G_{22} = \frac{c^2 q_c a^2}{2(\rho_c + P_c)}, \quad G_{33} = \frac{c^2 a^2}{2(\rho_b + P_b)}, \quad G_{44} = \frac{c^2 a^2}{2(\rho_r + P_r)},
\]
(4.6)
\[
B_{12} = -B_{21} = -\frac{a (1 - \beta_n - q_c)}{2H (1 + \alpha_B)},
\]
(4.7)
where \(q_c, q_s, c_c^2, \) and \(c_s^2\) are defined in eqs. (3.57)–(3.60).
To avoid the scalar ghosts, the components of $K$ in eq. (4.5) must be positive. As long as the ghost is absent in the tensor sector ($q_t > 0$) and the weak energy conditions $\rho_I + P_I > 0$ hold for $I = c, b, r$, the no-ghost conditions are given by

$$q_s > 0 \quad \text{and} \quad q_c > 0.$$ (4.8)

The dispersion relations for baryons and radiation are not affected by the off-diagonal components of matrix $B$, so their propagation speed squares are given, respectively, by

$$c_{2b}^2 = G_{33}/K_{33}$$ and $$c_{2r}^2 = G_{44}/K_{44}.$$ The off-diagonal components (4.7) can modify the propagation of perturbations $X_1 \equiv \mathcal{R}$ and $X_2 \equiv \delta \rho_k/k$. We vary the second-order action (4.3) with respect to the variables $X_j$ (where $j = 1, 2$) and then substitute the solutions of the form $X_j = \tilde{X}_j e^{i(\omega t - kx)}$ into their equations of motion. In the small-scale limit, the dominant contributions to the dispersion relation are those containing $\omega^2$, $\omega k$, and $k^2$. Then, it follows that

$$\omega^2 \tilde{X}_1 - c_s^2 k^2 \tilde{X}_1 - i \omega k \frac{B_{12}}{K_{11}} \tilde{X}_2 \simeq 0,$$ (4.9)

$$\omega^2 \tilde{X}_2 - c_c^2 k^2 \tilde{X}_2 - i \omega k \frac{B_{21}}{K_{22}} \tilde{X}_1 \simeq 0.$$ (4.10)

We will focus on the case in which the bare CDM sound speed squared vanishes, i.e.,

$$c_{2c}^2 = \frac{\rho_{c,n} c_{2c}}{\rho_{c,n}} \to 0.$$ (4.11)

For the coupling $f$ obeying the condition,

$$f_{n,nc} = 0,$$ (4.12)

we have $c_{2c}^2 = 0$ from eq. (3.59). In this case, we obtain the two separable solutions to eq. (4.10), as

$$\omega = 0,$$ (4.13)

$$\omega \tilde{X}_2 - i \frac{k}{a} \frac{B_{21}}{K_{22}} \tilde{X}_1 = 0.$$ (4.14)

The dispersion relation (4.13) is that of CDM, so the resulting CDM effective sound speed squared is given by

$$c_{2\text{CDM}}^2 = \frac{\omega^2 a^2}{k^2} = 0.$$ (4.15)

Substituting the other solution (4.14) to eq. (4.9), the dispersion relation for the perturbation $\mathcal{R}$ is expressed in the form $\omega^2 = c_s^2 k^2 / a^2$, where

$$c_s^2 = c_s^2 + \Delta c_s^2,$$ (4.16)

with

$$\Delta c_s^2 = \frac{B_{12}^2}{K_{11} K_{22}} = \frac{2q_t (\rho_{c} + P_{c})(1 - \beta_{nc} - q_c)}{q_s q_c c_s^2}.$$ (4.17)

We recall that $c_s^2$ is given by eq. (3.60). The off-diagonal components of $B$ give rise to the modification $\Delta c_s^2$ to $c_s^2$. The Laplacian instability of the perturbation $\mathcal{R}$ is absent for

$$c_{2s}^2 \geq 0.$$ (4.18)
Under the absence of scalar and tensor ghosts, it follows that $\Delta c_s^2 \geq 0$. This means that even the negative value of $\hat{c}_s^2$ in the range $\hat{c}_s^2 \geq -\Delta c_s^2$ can satisfy the condition (4.18). However, in the regime where the scalar field dominates over CDM, the term $\Delta c_s^2$ can be negligible relative to $\hat{c}_s^2$, so it is necessary to satisfy the condition $\hat{c}_s^2 \geq 0$ to ensure the stability during the whole cosmic expansion history.

The theories obeying the condition (4.12) corresponds to the coupling $f$ containing the linear dependence of $n_c$, i.e.,

$$f = -\tilde{f}_1(\phi, X, Z)n_c + f_2(\phi, X, Z),$$  \hspace{1cm} (4.19)

where $\tilde{f}_1(\phi, X, Z)$ and $f_2(\phi, X, Z)$ are arbitrary functions of $\phi$, $X$, and $Z$. The CDM with mass $m_c$ has the density $\rho_c = m_c n_c$, so the coupling (4.19) is equivalent to

$$f = -f_1(\phi, X, Z)\rho_c + f_2(\phi, X, Z),$$  \hspace{1cm} (4.20)

where $f_1 = \tilde{f}_1/m_c$. Since $c_{CDM}^2 = 0$ in this case, there is no additional pressure which prevents or enhances the gravitational clustering of CDM density perturbations. In refs. [44, 46, 48, 49, 51–55], the authors studied the cosmology for several sub-classes of couplings which belong to the general form (4.20).

5 Effective gravitational couplings

We derive the effective gravitational couplings of CDM and baryons for the interacting theories satisfying

$$f_{n_c n_c} = 0,$$  \hspace{1cm} (5.1)

and the conditions,

$$P_c = 0, \quad c_c^2 = 0, \quad P_b = 0, \quad c_b^2 = 0.$$  \hspace{1cm} (5.2)

In this case, the CDM effective sound speed squared $c_{CDM}^2$ vanishes. We also neglect the contribution of radiation to the dynamics of both background and perturbations.

To study the evolution of CDM and baryon density perturbations, we introduce the gauge-invariant matter density contrast,

$$\delta_{IN} = \frac{\delta \rho_{IN}}{\rho_I},$$  \hspace{1cm} (5.3)

where $I = c, b$. From eqs. (3.52) and (3.55), the CDM density contrast $\delta_c$ and velocity potential $v_c$ obey

$$\dot{\delta}_c + 3\dot{\Phi} + \frac{k^2}{a^2} v_c = 0,$$  \hspace{1cm} (5.4)

$$\dot{v}_c + H \epsilon_q v_c = \frac{1 - \beta_{n_c}}{q_c} \Psi + \frac{1 - \beta_{n_c} - q_c}{q_c} \delta \phi_N - \frac{H}{q_c} \left[ (1 - \beta_{n_c} - q_c) \epsilon_\phi + q_c \epsilon_q \right] \delta \phi_N = 0.$$  \hspace{1cm} (5.5)

Differentiating eq. (5.4) with respect to $t$ and using eq. (5.5), it follows that

$$\ddot{\delta}_c + (2 + \epsilon_q) H \dot{\delta}_c + \frac{k^2}{a^2} v_c - \frac{k^2}{a^2} \frac{1 - \beta_{n_c}}{q_c} \Psi - \frac{k^2}{a^2} \frac{1 - \beta_{n_c} - q_c}{q_c} \delta \phi_N = \frac{k^2}{a^2} \frac{(1 - \beta_{n_c} - q_c) \epsilon_\phi + q_c \epsilon_q}{q_c} H \delta \phi_N = -3\ddot{\Phi} - 3(2 + \epsilon_q) H \ddot{\Phi}.$$  \hspace{1cm} (5.6)
Let us employ the quasi-static approximation for the perturbations deep inside the sound horizon, under which the dominant contributions to the perturbation equations are those containing \(k^2\), \(\delta c_N\), \(\delta s_N\), and \(\delta s_{NN}\) \cite{90–91}. We ignore the mass squared \(M_\phi^2\) of the scalar degree of freedom arising from a scalar potential \(V(\phi)\). This is a good approximation to study the evolution of perturbations in the late Universe, apart from dark energy models in which \(M_\phi\) is much larger than \(H\) until recently. For later convenience, we introduce the following combinations,

\[
\Delta_1 = \alpha_B - \alpha_M, \quad \Delta_2 = \frac{\delta^2 q_c c_s^2}{4H^2 q_l^2}, \quad \Delta_3 = (1 - \beta_{nc})\Delta_1 - 3\beta_{nc}\beta_{nc}.
\]  

(5.7)

Applying the quasi-static approximation to eqs. (3.50) and (3.54), it follows that

\[
2q_\ell k^2 \left( \frac{\Phi + H\alpha_B}{\phi} \delta \phi_N \right) - (1 - \beta_{nc})\rho c \delta_{cN} - \rho_b \delta_{bN} = 0,
\]

(5.8)

\[
2H q_\ell k^2 \left[ (1 - \alpha_B)\Phi - \alpha_B \Psi + (2\alpha_B H_1 - \alpha_B^2 + \Delta_2) \frac{H \delta \phi_N}{\phi} \right]
\]

\[
\quad + (1 - \beta_{nc} - q_c)\rho c \delta_{cN} - H \rho \beta_{nc} \beta_{nc} \delta_{cN} = 0.
\]

(5.9)

Solving these equations and eq. (3.63) for \(\Psi\), \(\Phi\), and \(\delta \phi_N\), we obtain

\[
\Psi = \frac{a^2}{2q_\ell \Delta_2 k^2} \left[ (\Delta_1 \Delta_3 + (1 - \beta_{nc})\Delta_2) \rho c \delta_{cN} + (\Delta_2^2 + \Delta_2) \rho_b \delta_{bN} + (1 - \beta_{nc} - q_c)\Delta_1 \rho c \frac{\delta_{cN}}{H} \right],
\]

(5.10)

\[
\Phi = \frac{a^2}{2q_\ell \Delta_2 k^2} \left[ \alpha_B \Delta_3 + (1 - \beta_{nc})\Delta_2 \rho c \delta_{cN} + (\alpha_B \Delta_1 + \Delta_2) \rho_b \delta_{bN} + (1 - \beta_{nc} - q_c)\alpha_B \rho c \frac{\delta_{cN}}{H} \right],
\]

(5.11)

\[
\delta \phi_N = \frac{a^2 \delta \phi}{2H q_\ell \Delta_2 k^2} \left[ \Delta_3 \rho c \delta_{cN} + \Delta_1 \rho_b \delta_{bN} + (1 - \beta_{nc} - q_c)\rho c \frac{\delta_{cN}}{H} \right].
\]

(5.12)

Under the quasi-static approximation, the terms on the right-hand-side of eq. (5.6) can be neglected relative to those on the left-hand-side. Substituting eq. (5.12) and its time derivative as well as eq. (5.10) into the right-hand-side of eq. (5.6), we obtain

\[
\ddot{\delta}_{cN} + c_1 H \dot{\delta}_{cN} + c_2 H \dot{\delta}_{bN} - \frac{3H^2}{2G} (G_{cc} \Omega c \delta_{cN} + G_{cb} \Omega b \delta_{bN}) = 0,
\]

(5.13)

where \(G = 1/(8\pi M_\phi^2)\) is the Newton gravitational constant, and

\[
c_1 = (2 + \epsilon_{qc}) \frac{c_s^2}{c_s} + \left[ \frac{2\Delta_3 - 2q_c(\Delta_1 + \epsilon_{qc})}{1 - \beta_{nc} - q_c} - 1 - \Delta_1 - \alpha_B - \epsilon_{\Delta_2} - 2\epsilon_H \right] \left( 1 - \frac{c_s^2}{c_s^2} \right),
\]

(5.14)

\[
c_2 = \frac{3(1 - \beta_{nc} - q_c)\Omega_b \Delta_1 c_s^2}{2Q_c q_\ell \Delta_2 c_s^2},
\]

(5.15)

\[
G_{cc} = \frac{\Delta_1 \Delta_3 q_c + \Delta_2 (1 - \beta_{nc})^2 + \Delta_3 [q_c \epsilon_{qc} + (1 - \beta_{nc} - q_c)(1 + \alpha_B + \epsilon_H + \epsilon_{\Delta_2} - \epsilon_{\Delta_2})]}{q_c Q_\ell \Delta_2} c_s^2 G,
\]

(5.16)

\[
G_{cb} = \frac{\Delta_2^2 q_c + \Delta_2 (1 - \beta_{nc}) + \Delta_1 [q_c \epsilon_{qc} + (1 - \beta_{nc} - q_c)(1 + \alpha_B + \epsilon_H - \epsilon_{\Delta_2} + \epsilon_{\Delta_2})]}{q_c Q_\ell \Delta_2} c_s^2 G,
\]

(5.17)
with

\[ Q_t = \frac{q_t}{M_{\text{pl}}}, \quad \epsilon_{\Delta_i} = \frac{\Delta_i}{H\Delta_i}, \quad \text{for} \quad i = 1, 2, 3. \] (5.18)

The CDM and baryon density parameters are defined in eq. (2.39), i.e., \( \Omega_c = \rho_c / (3M_{\text{pl}}^2H^2) \) and \( \Omega_b = \rho_b / (3M_{\text{pl}}^2H^2) \). The relation between \( c_s^2 \) and \( \hat{c}_s^2 \) is given by

\[ c_s^2 = \hat{c}_s^2 \left[ 1 + \frac{3\Omega_c(1 - \beta_{nc} - q_c)^2}{2q_cQ_t\Delta_2} \right]. \] (5.19)

In eq. (5.13), there is no effective pressure of the form \( c_{\text{CDM}}^2 (k^2/a^2) \delta c_n \) as expected. This property is attributed to the assumption (5.1) of the coupling \( f \) as well as the vanishing value of \( c_s^2 \). The clustering of CDM density perturbations occurs by the gravitational couplings \( G_{cc} \) and \( G_{cb} \), both of which generally differ from \( G \).

The baryon density contrast \( \delta_{bN} \) and velocity potential \( v_{bN} \) obey

\[ \dot{\delta}_{bN} + 3\dot{\Phi} + \frac{k^2}{a^2} v_{bN} = 0, \] (5.20)

\[ \dot{\delta}_{bN} - \Psi = 0, \] (5.21)

so that the second-order equation for \( \delta_{bN} \) is

\[ \ddot{\delta}_{bN} + 2\dot{H} \delta_{bN} + \frac{k^2}{a^2} \Psi = -3\ddot{\Phi} - 6H \dot{\Phi}. \] (5.22)

Neglecting the right-hand-side of eq. (5.22) and substituting eq. (5.10) into eq. (5.22), it follows that

\[ \ddot{\delta}_{bN} + 2\dot{H} \delta_{bN} - \frac{3\Delta_1(1 - \beta_{nc} - q_c)\Omega_c}{2Q_t\Delta_2} \dot{H} \delta_{cN} - \frac{3H^2}{2G} (G_{bc} \Omega_c \delta_{cN} + G_{bb} \Omega_b \delta_{bN}) = 0, \] (5.23)

where

\[ G_{bc} = \frac{\Delta_1 \Delta_3 + (1 - \beta_{nc}) \Delta_2}{Q_t \Delta_2} G, \] (5.24)

\[ G_{bb} = \frac{\Delta_2^2 + \Delta_2}{Q_t \Delta_2} G. \] (5.25)

The baryon density perturbation is directly affected by the evolution of gravitational potential \( \Psi \). The difference from uncoupled Horndeski theories is that the time derivative \( \dot{\delta}_{bN} \) appears in the expression of \( \Psi \) given by eq. (5.10). By defining the dimensionless quantities,

\[ f_c = \frac{\delta_{cN}}{H \delta_{cN}}, \quad \mu_{bc} = \frac{G_{bc}}{G}, \quad \mu_{bb} = \frac{G_{bb}}{G}, \] (5.26)

one can express eq. (5.10) in the form of Poisson equation,

\[ \frac{k^2}{a^2} \Psi = -4\pi G \left\{ \mu_{bc} + \frac{f_c(1 - \beta_{nc} - q_c) \Delta_1}{Q_t \Delta_2} \right\} \rho_c \delta_{cN} + \mu_{bb} \rho_b \delta_{bN} \right\}. \] (5.27)

The gravitational potential associated with the observations of weak lensing is defined by [92]

\[ \psi_{WL} = \frac{1}{2} (\Psi - \Phi). \] (5.28)
On using eqs. (5.10) and (5.11) together with the relation $\alpha_B = \Delta_1 + \alpha_M$, it follows that

$$
\frac{k^2}{a^2} \psi_{WL} = -4\pi G \left\{ \mu_{bc} + \frac{\alpha_M \Delta_3 + f_c(1 - \beta_{ac} - q_c)(2\Delta_1 + \alpha_M)}{2Q_t \Delta_2} \right\} \rho_b \delta_{cN} + \left( \mu_{bb} + \frac{\alpha_M \Delta_1}{2Q_t \Delta_2} \right) \rho_b \delta_{bN}.
$$

(5.29)

If $\alpha_M = 0$, then the right-hand-sides of eqs. (5.27) and (5.29) coincide with each other, so that $\psi_{WL} = \Psi = -\Phi$. This is the consequence of the absence of anisotropic stress in eq. (3.63). If the anisotropic stress is present, there are contributions to $\psi_{WL}$ arising from a nonvanishing value of $\alpha_M$. The CDM growth rate $f_c$ also appears on the right-hand-side of eq. (5.29). The dynamics of $\delta_{cN}$ and $\delta_{bN}$ for perturbations deep inside the sound horizon is known by solving eqs. (5.13) and (5.23) with the gravitational couplings (5.16)–(5.17) and (5.24)–(5.25). The modified evolution of $\delta_{cN}$ and $\delta_{bN}$ in comparison to the theories with $f = 0$ affects the dynamics of gravitational potentials $\Psi$ and $\psi_{WL}$ through eqs. (5.27) and (5.29).

The effect of DE and DM interactions on $G_{cc}$, $G_{cb}$, and $G_{bc}$ appears through the two quantities $q_c$ and $\beta_{ac}$. The $Z$ and $n_c$ dependence in $q_c$ leads to the deviation of $q_c$ from 1. If there is no $n_c$ dependence in $f$, the quantity $\beta_{ac}$ vanishes. This means that the deviation of $\beta_{ac}$ from 0 occurs through the energy transfer associated with the change of $n_c$. Now, we are considering the interacting theories satisfying the condition (5.1), under which the coupling $f$ is constrained to be of the form (4.20) with the linear dependence $\rho_c \propto n_c$. In this case, we have

$$
q_c = 1 + f_1 - \dot{\phi}_f f_{1,Z} + \frac{\dot{\phi}_f f_{2,Z}}{\rho_c},
$$

(5.30)

$$
\beta_{ac} = -f_1 + \dot{\phi}_f f_{1,X} + \dot{\phi}_f f_{1,Z}.
$$

(5.31)

This means that $\beta_{ac}$ depends on $f_1$ alone, while $q_c$ contains the dependence of both $f_1$ and $f_2$. For the theories with $f_1 = 0$, we have $q_c = 1 + \dot{\phi}_f f_{2,Z}/\rho_c$ and $\beta_{ac} = 0$. In this case, the interaction between CDM and $\phi$ occurs through the momentum transfer characterized by the $Z$ dependence in $f_2$. In the limit that $q_c \to 1$, $\epsilon_{q_c} \to 0$, $\beta_{ac} \to 0$, and $\epsilon_{\beta_{ac}} \to 0$, one can conform that $G_{cc}$, $G_{cb}$, and $G_{bc}$ reduce to $G_{bb}$ given by eq. (5.25). This value of $G_{bb}$ is identical to the gravitational coupling of baryons and CDM derived for uncoupled Horndeski theories [13, 91], which is larger than $G/Q_t$, the absence of ghosts and Laplacian instabilities. The existence of coupling $f$ generally leads to the values of $G_{cc}$, $G_{cb}$, and $G_{bc}$ different from $G_{bb}$.

### 6 Gravitational couplings in concrete theories

In concrete interacting theories of DE and DM, we compute the gravitational couplings $G_{cc}$, $G_{cb}$, $G_{bc}$, and $G_{bb}$ derived in section 5. For this purpose, we will focus on the coupling function of the form,

$$
f = -f_1(\phi, X, Z) \rho_c + f_2(\phi, X, Z),
$$

(6.1)

which satisfies the condition $f_{\Delta_{n_c}} = 0$. We classify the theories into two classes: (i) $f_1 = 0$, and (ii) $f_1 \neq 0$. In each class, we estimate the values of $G_{cc}$, $G_{cb}$, $G_{bc}$, and $G_{bb}$ for theories which belong to the coupling (6.1).

#### 6.1 $f_1 = 0$ and $f_2 \neq 0$

We begin with interacting theories in which the coupling $f_1$ is absent, i.e.,

$$
f = f_2(\phi, X, Z).
$$

(6.2)
In this case, the quantities (5.30) and (5.31) reduce, respectively, to

\[ q_c = 1 + \frac{\dot{\phi} f_2 Z}{\rho_c}, \quad \beta_{nc} = 0. \] (6.3)

With this latter relation, we have

\[ \Delta_3 = \Delta_1, \quad \epsilon \Delta_3 = \epsilon \Delta_1. \] (6.4)

Then, the gravitational couplings (5.16)–(5.17) and (5.24)–(5.25) yield

\[ G_{cc} = G_{cb} = \frac{\Delta_1^2 q_c + \Delta_1 q_c \epsilon q_c - \Delta_1 (q_c - 1)(1 + \alpha_B + \epsilon H - \epsilon \Delta_1 + \epsilon \Delta_2)}{q_c Q_1 \Delta_2} \frac{c_s^2}{c_s^2} G, \] (6.5)

\[ G_{bb} = G_{bc} = \frac{\Delta_1^2 + \Delta_2}{Q_1 \Delta_2} G, \] (6.6)

where

\[ \frac{c_s^2}{c_s^2} = 1 + \frac{3 \Omega_c (1 - q_c)^2}{2 q_c Q_1 \Delta_2}. \] (6.7)

Hence \( G_{cc} \) and \( G_{bb} \) are equivalent to \( G_{cb} \) and \( G_{bc} \), respectively. In the limit that \( q_c \to 1 \) and \( \epsilon q_c \to 0 \), \( G_{cc} \) reduces to \( G_{bb} \).

The quantity \( \Delta_1 = \alpha_B - \alpha_M \) is different depending on the choice of Horndeski Lagrangian (2.2). In the following, we will consider three different cases: (a) k-essence, (b) extended Galileons, and (c) nonminimal couplings.

### 6.1.1 k-essence

Let us first consider minimally coupled k-essence theories [21–23] given by the Lagrangian

\[ \mathcal{L}_H = \frac{M^2_{pl}}{2} R + G_2(\phi, X). \] (6.8)

Since \( \alpha_B = 0 \) and \( \alpha_M = 0 \) in this case, it follows that

\[ \Delta_1 = 0, \] (6.9)

with \( Q_1 = 1 \). Then, eqs. (6.5) and (6.6) reduce, respectively, to

\[ G_{cc} = G_{cb} = \frac{c_s^2}{c_s^2} G, \] (6.10)

\[ G_{bb} = G_{bc} = G. \] (6.11)

The baryon gravitational couplings \( G_{bb} \) and \( G_{bc} \) are equivalent to the Newton constant \( G \), but \( G_{cc} \) and \( G_{cb} \) are different from \( G \). Since

\[ \Delta_2 = \frac{\dot{\phi}^2 (G_{2,X} + f_{2,X})}{2 M^2_{pl} H^2}, \] (6.12)

the ratio (6.7) is expressed as

\[ \frac{c_s^2}{c_s^2} = 1 + \frac{f_{2,Z}}{(G_{2,X} + f_{2,X})(\dot{\phi} f_{2,Z} + \rho_c)}, \] (6.13)
where we used $\rho_c = 3M_{\text{pl}}^2H^2 \Omega_c$ instead of $\Omega_c$. Then, from eq. (6.10), we obtain

$$G_{cc} = G_{\phi} = \frac{G}{1 + r_{f_2}}, \tag{6.14}$$

where

$$r_{f_2} = \frac{(G_{2,X} + f_{2,X})\dot{\phi}f_{2,Z} + f_{2,Z}^2}{(G_{2,X} + f_{2,X})\rho_c}. \tag{6.15}$$

In quintessence given by the Lagrangian $G_2(\phi, X) = X - V(\phi)$, we have $G_{2,X} = 1$ in eq. (6.15), which coincides with the result derived in ref. [53]. Now, we showed that the generalized expression (6.14) with (6.15) holds for k-essence.

Provided that $\rho_c$ dominates over the density associated with the coupling $f_2$ in the early matter era, we have $r_{f_2} \ll 1$ and hence $G_{cc} \approx G$. The deviation of $G_{cc}$ from $G$ starts to occur after the dominance of DE, around which the term $(G_{2,X} + f_{2,X})\dot{\phi}f_{2,Z} + f_{2,Z}^2$ becomes the same order as $(G_{2,X} + f_{2,X})\rho_c$. In this epoch, the CDM gravitational interaction weaker than the Newton constant $G$ can be realized for $r_{f_2} > 0$. In the DE sector, the ghost and Laplacian instabilities are absent for

$$q_s = 2M_{\text{pl}}^2 \left( G_{2,X} + \phi^2 G_{2,XX} + f_{2,X} + \phi^2 f_{2,XX} + f_{2,ZZ} + 2\dot{\phi}f_{2,Z} \right) > 0, \tag{6.16}$$

$$q_s c_s^2 = 2M_{\text{pl}}^2 (G_{2,X} + f_{2,X}) > 0. \tag{6.17}$$

Under the requirement (6.17), the condition $G_{cc} < G$ translates to

$$(G_{2,X} + f_{2,X})\dot{\phi}f_{2,Z} + f_{2,Z}^2 > 0, \tag{6.18}$$

which is satisfied for $\dot{\phi}f_{2,Z} > 0$. If we consider the interaction $f_2 = \beta Z^2$ [44, 49, 53], for example, the positive coupling constant $\beta$ always leads to the weak CDM gravitational interaction.

The evolution of $G_{cc}$ in the asymptotic future depends on the scalar time-derivative $\dot{\phi}$. For quintessence with an exponential potential, i.e., $G_2 = X - V_0 e^{-\lambda \phi/M_{\text{pl}}}$ with $\lambda^2 < 2$, there exists a future accelerating fixed point along which $\dot{\phi}$ is proportional to $H$, with the DE equation of state $w_{\text{DE}} = -1 + \lambda^2/3$ and density parameter $\Omega_{\text{DE}} = 1$ [10, 93]. In this case, the $\dot{\phi}$-dependent terms in $r_{f_2}$ slowly decrease in comparison to $\rho_c (\propto a^{-3})$ and hence $r_{f_2}$ grows continuously. Then, $G_{cc}$ and $G_{cb}$ approach 0 toward the future accelerating fixed point [53]. This is also the case for k-essence allowing for the existence of a future de Sitter solution characterized by $\dot{\phi} = $ constant and $H = $ constant. We also note that the quantity $q_c$ asymptotically behaves as $q_c \approx \dot{\phi}f_{2,Z}/\rho_c$, so the absence of ghosts in the CDM sector requires that $\dot{\phi}f_{2,Z} > 0$. In this case the condition (6.18) is satisfied, so the weak gravitational interaction for CDM is naturally realized after the dominance of DE.

In summary, k-essence with the $Z$-dependent contributions to $f_2$ leads to $G_{cc}$ smaller than $G$ at low redshifts, with $G_{cc}$ approaching 0 in the future.

### 6.1.2 Extended Galileons

The minimally coupled extended Galileon is given by the Lagrangian

$$\mathcal{L}_H = \frac{M_{\text{pl}}^2}{2} R + G_2(X) + G_3(X) \Box \phi, \tag{6.19}$$
where $G_2$ and $G_3$ depend on $X$ alone. The cubic Galileon \cite{94,95} is characterized by the functions $G_2(X) = c_2 X$ and $G_3(X) = c_3 X$, where $c_2$ and $c_3$ are constants. The extended Galileon \cite{96,97} corresponds to the theories with arbitrary functions of $G_2(X)$ and $G_3(X)$.

For the Lagrangian (6.19), we have

$$\alpha_M = 0, \quad \Delta_1 = \alpha_B = \frac{\dot{\phi}^2 G_{3,X}}{2 H M_{pl}^2} , \quad (6.20)$$

and $Q_t = 1$. Then, the gravitational couplings (6.5) and (6.6) reduce to

$$G_{cc} = G_{cb} = \alpha_B^2 + \Delta_2 + \frac{\alpha_B q_c \epsilon_{qc} - \alpha_B (q_c - 1) (1 + \epsilon_H - \epsilon_{\Delta_1} + \epsilon_{\Delta_2}) c_s^2}{q_c \Delta_2} G, \quad (6.21)$$

$$G_{bb} = G_{bc} = \alpha_B^2 + \Delta_2 G , \quad (6.22)$$

where

$$\Delta_2 = \frac{\dot{\phi}^2 [2 M_{pl}^2 (f_{2,X} + G_{2,X} - 2 \phi G_{3,X} - 4 H \phi G_{3,X} - \phi \phi^2 G_{3,X} X) - \phi^4 G_{3,X}^2]}{4 H^2 M_{pl}^4} , \quad (6.23)$$

$$\frac{c_s^2}{c_s^2} = 1 + \frac{\phi^2 f_{2,Z}^2}{2 H^2 M_{pl}^2 (\phi f_{2,Z} + \rho_c) \Delta_2} . \quad (6.24)$$

Since $\Delta_2 > 0$ for the absence of ghosts and Laplacian instabilities, $G_{bb}$ and $G_{bc}$ are larger than $G$. The braiding term $\alpha_B^2$, which arises from the cubic coupling $G_3(X)$, leads to the enhancement of baryon gravitational couplings. In contrast, the CDM perturbation is affected not only by the term $\alpha_B^2$ but also by the deviation of $q_c$ from 1 (which is induced by the $Z$-dependence in $f_2$). The latter term allows a possibility for realizing $G_{cc}$ smaller than $G$.

Subtracting eq. (2.35) from eq. (2.36), we obtain

$$-2 M_{pl}^2 \ddot{H} = \dot{\phi} \left( \phi f_{2,X} + f_{2,Z} + \phi G_{2,X} - 3 H \phi^2 G_{3,X} + \phi \phi G_{3,X} \right) + \sum_{l=c,b} (\rho_l + P_l) . \quad (6.25)$$

Provided that $f_2$ does not contain the $\phi$ dependence, i.e.,

$$f = f_2(X, Z) , \quad (6.26)$$

there exists a de Sitter solution characterized by

$$H = \text{constant}, \quad \dot{\phi} = \text{constant}, \quad \rho_l = 0 = P_l . \quad (6.27)$$

Along this solution, the nonvanishing time derivative $\dot{\phi}$ obeys

$$f_{2,Z} + \dot{\phi} \left( f_{2,X} + G_{2,X} - 3 H \phi G_{3,X} \right) = 0 . \quad (6.28)$$

Then, the quantities (6.23) and (6.24) reduce, respectively, to

$$\Delta_2 = -\alpha_B (\alpha_B + 1) - g_2 , \quad (6.29)$$

$$\frac{c_s^2}{c_s^2} = \frac{\alpha_B (\alpha_B + 1)}{\alpha_B (\alpha_B + 1) + g_2} , \quad (6.30)$$

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where
\[ g_2 = \frac{\dot{\phi} f_{2, Z}}{2H^2 M_{\text{pl}}^2}. \] (6.31)

Requiring that the quantity \( q_c \) around the de Sitter solution (\( q_c \simeq \dot{\phi} f_{2, \rho_c} \)) is positive, it follows that \( g_2 > 0 \). Since \( \Delta_2 > 0 \) to avoid the ghost and Laplacian instabilities, it is at least necessary to satisfy the inequality \(-\alpha_B (\alpha_B + 1) > 0\), i.e.,
\[ -1 < \alpha_B < 0. \] (6.32)

On the de Sitter fixed point characterized by eq. (6.27), the quantities appearing in eq. (6.5) satisfy
\[ \epsilon_{q_c} = 3, \quad \epsilon_H = 0, \quad \epsilon_{\Delta_1} = 0, \quad \epsilon_{\Delta_2} = 0. \] (6.33)

Taking the limit \( q_c \to \infty \) in eq. (6.5), the CDM gravitational coupling on the de Sitter solution yields
\[ (G_{cc})_{\text{dS}} = \frac{2\alpha_B \epsilon_{q_c}^2 G}{\Delta_2 \epsilon_{q_c}} = -\frac{2}{\alpha_B + 1} G. \] (6.34)

Since \((G_{cc})_{\text{dS}}\) is negative under the condition (6.32), the CDM gravitational interaction is repulsive. This peculiar behavior results from the momentum exchange between CDM and the self-accelerating scalar field. As we observe in eq. (6.21), the terms \( \alpha_B^2 + \Delta_2 \), which also appear in the numerator of eq. (6.22), are completely dominated by the \( q_c \)-dependent contributions to \( G_{cc} \) on the de Sitter solution. This means that the \( Z \)-dependence in \( f_2 \) gives the value of \((G_{cc})_{\text{dS}}\) very different from \((G_{bb})_{\text{dS}}\). We also note that the result (6.34) agrees with the weak vector-field coupling limit of the CDM gravitational coupling derived for generalized Proca theories [98] (with the change of notation \( \alpha_B \to -\alpha_B \)).

Provided that \( q_c \simeq 1 \) in the early matter era, \( G_{cc} \) is close to the value \( G_{bb} (> G) \). The evolution of \( G_{cc} \) just after the dominance of DE depends on the forms of \( f_2 \) and \( G_3 \). Since \((G_{cc})_{\text{dS}} < 0\), the CDM perturbation should eventually cross the point \((G_{cc})_{\text{dS}} = 0\) on the way of approaching the future de Sitter fixed point. The moment at which this transition occurs depends on the chosen model parameters.

### 6.1.3 Nonminimal couplings

We proceed to nonminimally coupled k-essence theories characterized by the Lagrangian
\[ \mathcal{L}_H = G_4(\phi) R + G_2(\phi, X). \] (6.35)

In this case, we have
\[ \alpha_M = 2\alpha_B = \frac{\dot{\phi} G_{4, \phi}}{HG_4}, \quad \Delta_1 = -\alpha_B = -\frac{\dot{\phi} G_{4, \phi}}{2HG_4}, \quad Q_t = \frac{2G_4}{M_{\text{pl}}^2}. \] (6.36)

Then, the gravitational couplings are given by
\[ G_{cc} = G_{cb} = \frac{(2q_c - 1)\alpha_B^2 + \Delta_2 - \alpha_B q_c \epsilon_{q_c} + \alpha_B (q_c - 1)(1 + \epsilon_H - \epsilon_{\Delta_1} + \epsilon_{\Delta_2}) \epsilon_{q_c}^2 c_s^2 G}{q_cQ_t \Delta_2}, \] (6.37)
\[ G_{bb} = G_{bc} = \frac{\alpha_B^2 + \Delta_2}{Q_t \Delta_2} G, \] (6.38)
where
\begin{equation}
\Delta_2 = \frac{\phi^2 |G_4(G_{2,X} + f_2 X) + 3G^2_{4,\phi}|}{4H^2G^2_4}, \tag{6.39}
\end{equation}
\begin{equation}
\frac{c_s^2}{c_s^2} = 1 + \frac{G_4f_{2,Z}^2}{|G_4(G_{2,X} + f_2 X) + 3G^2_{4,\phi}|(\phi f_{2,Z} + \rho_c)}.
\tag{6.40}
\end{equation}

Provided that \(q_c\) and \(c_s^2/c_s^2\) are close to 1 during the early matter era due to the smallness of \(f_{2,Z}\), \(G_{cc}\) reduces to the value \(G_{bb}\) in eq. (6.38). As \(q_c\) and \(c_s^2/c_s^2\) start to deviate from 1 at low redshifts, \(G_{cc}\) exhibits the different evolution from \(G_{bb}\). If the term \(\dot{\phi}f_{2,Z}\) decreases slowly in comparison to \(\rho_c\) in the late Universe, then \(q_c\) continuously grows toward infinity.

Taking the limit \(q_c \to \infty\) in eq. (6.37), it follows that
\begin{equation}
(G_{cc})_{\text{late}} = \frac{\alpha_B(1 - \epsilon_{q_c} + \epsilon_H + 2\alpha_B - \epsilon_{\Delta_1} + \epsilon_{\Delta_2})c_s^2}{Q_l\Delta_2}G.
\tag{6.41}
\end{equation}

If the scalar field \(\phi\) evolves slowly on a quasi de-Sitter background, the terms \(\epsilon_H\), \(\alpha_B\), \(\epsilon_{\Delta_1}\), and \(\epsilon_{\Delta_2}\) should be much smaller than 1, with \(\epsilon_{q_c} \approx 3\). In this case, eq. (6.41) approximately reduces to
\begin{equation}
(G_{cc})_{\text{late}} \approx -\frac{2\alpha_B\Delta_s}{Q_l\Delta_2}G.
\tag{6.42}
\end{equation}

The dominant contributions to \((G_{cc})_{\text{late}}\) arise from the terms \(-\alpha_Bq_c\epsilon_{q_c}\) and \(\alpha_B(q_c - 1)\) in the numerator of eq. (6.37). This means that the momentum transfer between CDM and the scalar field completely dominates over the terms associated with nonminimal couplings on the quasi de-Sitter background.

### 6.2 \(f_1 \neq 0\) and \(f_2 \neq 0\)

Finally, we study the interacting theories in which the coupling \(f_1\) is present besides \(f_2\). We focus on the simple case in which \(f_1\) depends on \(\phi\) alone, i.e.,
\begin{equation}
f = -f_1(\phi)\rho_c + f_2(\phi, X, Z).
\tag{6.43}
\end{equation}

For the DE sector, we consider the minimally coupled k-essence given by the Lagrangian,
\begin{equation}
\mathcal{L}_H = \frac{M^2_{pl}}{2}R + G_2(\phi, X),
\tag{6.44}
\end{equation}
under which \(\alpha_B = 0\), \(\alpha_M = 0\), and \(Q_l = 1\). In such theories, we have
\begin{equation}
\Delta_1 = 0, \quad \Delta_2 = \frac{\dot{\phi}^2(G_{2,X} + f_2 X)}{2H^2M^2_{pl}}, \quad \Delta_3 = \frac{\dot{\phi}f_{1,\phi}}{H},
\tag{6.45}
\end{equation}
and
\begin{equation}
q_c = 1 + \frac{\phi f_{2,Z}}{f_1\rho_c}, \quad \frac{c_s^2}{c_s^2} = 1 + \frac{f_{2,Z}^2}{(G_{2,X} + f_2 X)(1 + f_1)\rho_c + \phi f_{2,Z}}.
\tag{6.46}
\end{equation}

Then, the gravitational couplings (5.16)–(5.17) and (5.24)–(5.25) are expressed as
\begin{align*}
G_{cc} &= \frac{1 + f_1 + r_1}{1 + r_2} G, \tag{6.47} \\
G_{cb} &= \frac{1}{1 + r_2} G, \tag{6.48} \\
G_{bc} &= (1 + f_1) G, \tag{6.49} \\
G_{bb} &= G, \tag{6.50}
\end{align*}
where

\[
\begin{align*}
 r_1 &= -\frac{2HM_{\text{pl}}^2f_{1,\phi}}{(G_{2,X} + f_{2,X})(1 + f_1)\rho_c} \left[ f_{2,Z} \left( 1 - \epsilon_q + \epsilon_H + \epsilon_{A2} - \epsilon_{A1} \right) - (1 + f_1) \frac{\rho_c \epsilon_q}{\dot{\phi}} \right], \\
 r_2 &= \frac{(G_{2,X} + f_{2,X})\dot{\phi}f_{2,Z} + f_{2,Z}^2}{(G_{2,X} + f_{2,X})(1 + f_1)\rho_c}.
\end{align*}
\]

Taking the limit \( f_1 \to 0 \), these gravitational couplings coincide with those derived in section 6.1.1. The nonvanishing function \( f_1 \) leads to the difference between \( G_{cc} \) and \( G_{cb} \), and also between \( G_{bc} \) and \( G_{bb} \). The \( \phi \)-dependence in \( f_1 \) gives rise to a new contribution \( r_1 \) to the numerator of \( G_{cc} \) in eq. (6.47). This contribution arises through the energy exchange between the scalar field and CDM. The momentum transfer between \( \phi \) and CDM, which appears as the \( Z \) dependence in \( f_2 \), occurs through the term \( r_2 \) in the denominators of \( G_{cc} \) and \( G_{cb} \). From eq. (6.48), we find that \( G_{cb} \) is affected only by the momentum exchange. The coupling \( f_1 \) modifies the amplitude of \( G_{bc} \), but \( G_{bb} \) is equivalent to \( G \).

As long as the conditions \(|r_1| \ll 1\) and \(|r_2| \ll 1\) are satisfied in the early matter era, \( G_{cc} \) and \( G_{cb} \) are close to \( G_{bc} = (1 + f_1)G \) and \( G_{bb} = G \), respectively. After the DE density dominates over the CDM density, \( G_{cc} \) and \( G_{cb} \) start to deviate from their initial values. Let us consider the case in which the scalar field evolves slowly after the dominance of DE. At a sufficiently late epoch in which \( \rho_c \) becomes negligibly small relative to the DE density, one can take the limit \( \rho_c \to 0 \) in eqs. (6.47) and (6.48), with eqs. (6.51) and (6.52). In this regime, the CDM gravitational couplings reduce to

\[
\begin{align*}
(G_{cc})_{\text{late}} &\simeq -\frac{2f_{1,\phi}HM_{\text{pl}}^2(1 - \epsilon_q + \epsilon_H + \epsilon_{A2} - \epsilon_{A1})}{(G_{2,X} + f_{2,X})\dot{\phi} + f_{2,Z}}G, \\
(G_{cb})_{\text{late}} &\simeq 0.
\end{align*}
\]

The \( \phi \)-dependence in \( f_1 \) renders \( (G_{cc})_{\text{late}} \) different from 0, while \( G_{cb} \) asymptotically approaches 0.

For concreteness, let us consider the coupling function,

\[
f = -\left( e^{Q\phi/M_{\text{pl}}} - 1 \right) \rho_c + \beta (2X)^{1-m/2} Z^m,
\]

and quintessence with an exponential potential,

\[
G_2 = X - V_0 e^{-\lambda\phi/M_{\text{pl}}},
\]

where \( Q, \beta, m, V_0, \lambda \) are constants. In this model, there exists the scalar-field dominated fixed point satisfying \[55\]

\[
\frac{\dot{\phi}}{HM_{\text{pl}}} = \frac{\lambda}{1 + 2\beta}; \quad \epsilon_H = -\frac{\lambda^2}{2(1 + 2\beta)}; \quad \Omega_c = 0,
\]

at which we have

\[
\epsilon_q = 3 - \frac{\lambda^2}{1 + 2\beta}, \quad \epsilon_{A2} = 0; \quad \epsilon_{A3} = \frac{Q\lambda}{1 + 2\beta};
\]

Then, eq. (6.53) reduces to

\[
(G_{cc})_{\text{late}} = \frac{4(1 + 2\beta) + \lambda(2Q - \lambda) Q}{1 + 2\beta} \frac{e^{Q\phi/M_{\text{pl}}}G}{\lambda}.
\]
which coincides with that derived in ref. [55]. ¹ For $Q$ close to 0, $(G_{cc})_{\text{late}}$ can be much smaller than $G$. In ref. [55], it was shown that $G_{cc}$ can enter the region $G_{cc} < G$ by today and it finally approaches the asymptotic value (6.59). Thus, even in the presence of the energy transfer arising from the coupling $f_1(\phi)\rho_c$, there are models in which the realization of weak cosmic growth at low redshifts is possible.

The CDM gravitational couplings (6.47)–(6.48) and their asymptotic values (6.53)–(6.54) can be applicable to arbitrary functions $f_1(\phi)$, $f_2(\phi, X, Z)$ and the k-essence Lagrangian $G_2(\phi, X)$. Further extensions to theories with the coupling $f_1(\phi, X, Z)\rho_c$ and the Horndeski Lagrangian (2.2) are straightforward by using our most general formulas (5.16)–(5.17) and (5.24)–(5.25) of CDM and baryon gravitational couplings.

Before closing this section, we explicitly show how the weak gravitational CDM coupling can be realized in concrete interacting theories. In figure 1, we plot the evolution of $G_{cc}$ versus the redshift $z$ for two concrete models 1 and 2 presented in refs. [53] and [55], respectively. The model 1 corresponds to the coupling $f_1 = 0$ and $f_2 = \beta Z^2$, so it belongs to the class (A). The model 2, which belongs to the class (B), is characterized by the coupling (6.55) with $m = 3$. In both models the Horndeski functions are given by $G_2 = X - V_0 e^{-\lambda \phi/M_{pl}}$, $G_3 = 0$, and $G_4 = M_{pl}^2/2$, with the model parameters $\lambda = 1$, $\beta = 1/4$ (model 1) and $\lambda = 1$, $\beta = 0.5$, and $Q = 0.02$ (model 2). In figure 1, we observe that $G_{cc}$ is smaller than $G$ at low redshifts, so the weak CDM gravitational interaction is indeed realized in these models. In models 1 and 2, the asymptotic values of $G_{cc}$ in future are 0 and (6.59), respectively, which are confirmed numerically.

¹In ref. [55], the factor $e^{Q \phi/M_{pl}}$ is absorbed into the definition of $\Omega_c$ in eq. (5.13), such that $\Omega_c \to (1 + f_1)\Omega_c$. 

Figure 1. Evolution of $G_{cc}$ versus the redshift $z$ in three different models. The red line corresponds to the model 1 with $\lambda = 1$ and $\beta = 1/4$, while the blue line to the model 2 with $\lambda = 1$, $\beta = 0.5$, and $Q = 0.02$. We also plot the case without the coupling $f$, i.e., $G_{cc} = G$, as a black line.
7 Conclusions

We studied very general interacting theories of DE and DM given by the action (2.1), by paying particular attention to the gravitational couplings of CDM and baryon density perturbations. The DE sector is described by a scalar field \( \phi \) with the Horndeski Lagrangian (2.2), whereas the CDM and baryons are dealt as perfect fluids characterized by the second integral in eq. (2.1). The Lagrangian \( f(n_c, \phi, X, Z) \) accommodates the interaction between DE and CDM. In particular, the \( Z \) dependence in \( f \) mediates the momentum transfer besides the energy exchange associated with the CDM number density \( n_c \) coupled to the scalar field.

In section 2, we derived the gravitational and CDM equations of motion in the covariant forms (2.28) and (2.30). On the flat FLRW background, they reduce to eqs. (2.35) and (2.36), with the continuity eqs. (2.34) and (2.41). If we consider the interacting Lagrangian (2.43) and define the effective CDM density and pressure as eq. (2.44), the energy exchange between CDM and DE induced by the coupling \( f_1 \rho_c \) can be explicitly seen in eqs. (2.46) and (2.47). The momentum transfer associated with the coupling \( f_2 \) does not appear on the right-hand-sides of CDM and DE continuity equations at the background level.

In section 3, we obtained the second-order action of scalar perturbations for the perturbed line element (3.4) without fixing any particular gauge conditions. The resulting full linear perturbation equations of motion are given by eqs. (3.32)–(3.43), which are written in the gauge-ready form. By introducing several gauge-invariant perturbations in eq. (3.47) and dimensionless variables in eq. (3.48), we showed that all the perturbation equations are expressed in terms of gauge-invariant combinations without residual gauge degrees of freedom, see eqs. (3.50)–(3.63).

In section 4, we identified stability conditions under which neither ghost nor Laplacian instabilities are present for scalar perturbations deep inside the sound horizon. The tensor perturbation does not have a ghost for \( q_t = 2G_4 > 0 \), with the propagation speed \( c_t \) equivalent to that of light. As long as \( q_c \) and \( q_s \) given by eqs. (3.57) and (3.58) are positive, the ghosts are absent in the CDM and DE sectors. If the coupling \( f \) satisfies the condition \( f_{n_c n_c} = 0 \) with \( c^2_c = 0 \), we showed that the effective CDM sound speed squared \( c^2_{\text{CDM}} \) vanishes. This includes the interacting theories given by the coupling (4.20), for which there are no additional pressures preventing or enhancing the gravitational instability of CDM perturbations. The mixing between DE and CDM adds a contribution \( \Delta c^2_s \) to the scalar propagation speed squared \( \hat{c}^2_s \) given by eq. (3.60), so the Laplacian instability in the DE sector is absent for \( \hat{c}^2_s = c^2_s + \Delta c^2_s \geq 0 \).

In section 5, we employed the quasi-static approximation for perturbations deep inside the sound horizon to derive the effective gravitational couplings of CDM and baryons for the interacting theories satisfying \( f_{n_c n_c} = 0 \). The CDM density contrast \( \delta_{cN} \) obeys the second-order differential eq. (5.13), with \( G_{cc} \) and \( G_{cb} \) given by eqs. (5.16) and (5.17) respectively. On the other hand, the effective gravitational couplings \( G_{bc} \) and \( G_{bb} \) for baryons are of the forms (5.24) and (5.25). Unlike the standard uncoupled Horndeski theories, the growth rate \( f_c = \dot{\delta}_{cN}/(H\delta_{cN}) \) of CDM perturbations appears in the Poisson eq. (5.27) and eq. (5.29) of the weak lensing potential \( \psi_{\text{WL}} \).

In section 6, we applied our general formulas of \( G_{cc} \), \( G_{cb} \), \( G_{bc} \), and \( G_{bb} \) for concrete interacting theories which belong to the coupling (6.1). For the theories with \( f_1 = 0 \) and \( f_2 \neq 0 \), the momentum exchange between CDM and DE generally gives the values of \( G_{cc} \) and \( G_{cb} \), very different from \( G_{cb} \) and \( G_{bc} \), at late cosmological epochs. This property is attributed to the fact that the \( Z \) dependence in \( f_2 \) leads to the increase of the quantity \( q_c = 1 + \phi f_{2, Z}/\rho_c \).
If the DE sector is described by k-essence or extended Galileons, we showed that $G_{cc}$ smaller than $G$ can be naturally realized after the dominance of DE. The presence of nonvanishing coupling $-f_1(\phi)\rho_c$ besides $f_2$ gives rise to additional contributions to $G_{cc}$, but the momentum transfer arising from the $Z$ dependence in $f_2$ plays an important role to suppress the CDM gravitational couplings at late times. We showed that our general formulas of gravitational couplings reproduce the results for specific interacting theories known in the literature.

It will be of interest to apply our general Lagrangian formulation of coupled DE and DM to place observational constraints on concrete models. In particular, the implementation of the perturbation equations of motion into the (hi-)CLASS [99, 100] or EFTCAMB code [101, 102] is the first step for confronting interacting models with numerous observational data. The perturbation equations derived in this paper are suitable for this purpose, as they are written in a gauge-ready/gauge-invariant form with the EFT-like parameters $\beta_K, \beta_{an}$ besides $\alpha_K, \alpha_B, \alpha_M$. We hope that the interacting DE and DM models allow the possibility for alleviating the observational tensions of $H_0$ and $\sigma_8$.

Acknowledgments

RK is supported by the Grant-in-Aid for Young Scientists B of the JSPS No. 17K14297. ST is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 19K03854 and MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas “Cosmic Acceleration” (No. 15H05890).

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