It is a common practice to characterize anisotropic superconductors by a single anisotropy parameter defined as \( \gamma = \xi_a / \xi_c \equiv \lambda_c / \lambda_a \) (\( \xi \) is the coherence length, \( \lambda \) is the penetration depth, and \( a, c \) are principal directions of a uniaxial crystal of the interest here). The practice emerged after the anisotropic Ginzburg-Landau (GL) equations were derived for arbitrary gap and Fermi surface anisotropies by Gor'kov and Melik-Barkhudarov [1]. Formally, this came out because in the GL domain, the anisotropies of these quantities are not necessarily common with evaluation of Eq. (1). This implies that the analysis of the torque data might be misleading if Eq. (1) is employed formally the expression [2] with \( H_{c2}(\theta) = H_{c2,0}/\sin(\theta) \) having the anisotropy different from that of \( \lambda \), one may obtain qualitatively different angular behavior of \( \tau(\theta) \).

Expression [2] for the torque has been derived within the London approach by employing the cutoff at distances on the order of the coherence length \( \xi \) where this approach fails; that is how the upper critical field \( H_{c2} \sim \phi_0 / \xi^2 \) enters the London formula. The formula, however, has been confirmed experimentally with a good accuracy as far as the angular dependencies of quantities involved are concerned [10]. Uncertainties of the London approach are incorporated in the parameter \( \eta \sim 1 \); discussion of those can be found, e.g., in Ref. [1].

Below, the free energy of the mixed state and the torque are derived for the general case of \( \gamma_H \neq \gamma_\lambda \). The torque expression is shown to acquire new terms which describe a more complicated behavior as compared to that of Eq. (1). This implies that the analysis of the torque data might be misleading if Eq. (1) is employed to materials like MgB$_2$ [11, 12, 13].

Let us start with the London expression for the free energy valid for intermediate fields \( H_{c1} \ll H \ll H_{c2} \) along \( c \) tilted with respect to the \( c \) crystal axis over the angle \( \theta \) toward the crystal direction \( a \) [14]:

\[
F = \frac{B^2}{8\pi} + \frac{B^2m_{zz}}{8\pi\lambda^2m_a} \sum_G \frac{1}{m_{zz}G_x^2 + m_cG_y^2};
\]

here, \( m_{zz} = m_a \sin^2 \theta + m_c \cos^2 \theta, m_c/m_a = \gamma^2, m_a^2 = 1 \) for uniaxial crystals, and \( G \) form the reciprocal vortex lattice. The summation is extended over all nonzero \( G \).
As usual we evaluate the sum here by replacing it with an integral over the reciprocal plane: \( \sum G \to (\phi_0/4\pi^2B) \int dG_x dG_y \):

\[
\tilde{F} = F - \frac{B^2}{8\pi} = \frac{\phi_0 B m_{zz}^2}{32\pi^3 \lambda^2 m_a} \int \frac{dG_x dG_y}{m_{zz} G_x^2 + m_c G_y^2} = \frac{\phi_0 B \sqrt{m_{zz}}}{2\pi^2 \lambda^2} \int_0^{2\pi} d\varphi \int \frac{dq}{g},
\]

(4) where \( g_x = \sqrt{m_{zz}} G_x \), \( g_y = \sqrt{m_{c}} G_y \), and we use polar coordinates: \( g_x = g \sin \varphi \), \( g_y = g \cos \varphi \).

To determine the limits of integration over \( g \), one can recall that in fields \( B \ll H_{c2} \) the vortex lattice structure is fixed by the anisotropy parameter \( \gamma_\lambda \) [4]. In intermedium fields, for all possible equilibrium London structures, the distance in the reciprocal space \( g \) from the origin to the nearest neighbors is given by [5]

\[
g_0^2 = m_{zz} G_x^2 + m_c G_y^2 = \frac{8\pi^2 B}{\sqrt{3} \phi_0} \sqrt{m_c m_{zz}}.
\]

(5) Therefore, one can take \( g_0 \) (multiplied by a number of order unity) as the lower limit in the logarithmically divergent integral over \( g \).

The upper limit in this integral is affected by the form of the vortex core. To determine the core shape we note the microscopic evaluation of \( H_{c2} \) (see Ref. 4) shows that with a good accuracy the angular dependence of \( H_{c2} \) is given by the standard GL form at any \( T \):

\[
H_{c2} = \frac{\phi_0}{2\pi \xi \sqrt{\mu_a \sin \theta + \mu_c \cos \theta}} = \frac{\phi_0}{2\pi \xi \sqrt{\mu_a}}.
\]

(6) This can be written as

\[
H_{c2} = \frac{\phi_0}{2\pi \xi \sqrt{\mu_a}} \xi_x = \xi \sqrt{\mu_a \mu_z}, \quad \xi_y = \frac{\xi}{\sqrt{\mu_a}}.
\]

(7) where \( \xi_x, \xi_y \) are semiaxes of the elliptical core. We stress that the “masses” \( \mu_k \) are different from \( m_k \) which determine the anisotropy of \( \lambda \); in particular, \( \mu_a = \gamma_{H/2}^{-2/3}, \quad \mu_c = \gamma_{H}^{-4/3} \), whereas \( m_a = \gamma_{H}^{-2/3}, \quad m_c = \gamma_{H}^{4/3} \).

Equation (6) gives maximum values of \( G_x \) and \( G_y \):

\[
G_{x,m} = \frac{2\pi}{\xi \sqrt{\mu_a \mu_z}} \quad G_{y,m} = \frac{2\pi \mu_a}{\xi}.
\]

(8) Thus, the domain of integration in the \( G \) plane is bound by an ellipse

\[
\frac{G_x^2}{G_{x,m}^2} + \frac{G_y^2}{G_{y,m}^2} = 1.
\]

(9) In other words, at a given \( \varphi \), the upper limit in the integral over \( g \) is

\[
g_m(\varphi) = \frac{2\pi \sqrt{\mu_a \mu_c}}{\xi \sqrt{\beta^2 \cos^2 \varphi + \sin^2 \varphi}}, \quad \beta^2 = \frac{m_c \mu_z}{m_z \mu_c},
\]

(10) and we obtain:

\[
\tilde{F} = \frac{\phi_0 B \sqrt{m_{zz}}}{32\pi^3 \lambda^2} \int_0^{2\pi} d\varphi \ln \frac{g_m(\varphi)}{g_0}
\]

\[
= \frac{\phi_0 B \sqrt{m_{zz}}}{32\pi^2 \lambda^2} \left[ \ln \frac{3m_c \mu_a \phi_0}{2 B \sqrt{m_{zz}}} - \frac{1}{2\pi} \int_0^{2\pi} d\varphi \ln (\beta^2 \cos^2 \varphi + \sin^2 \varphi) \right].
\]

(11) Note that \( \beta = 1 \) for coexisting anisotropies of \( H_{c2} \) and \( \lambda \); the integral \( J(\beta) \) over \( \varphi \) then vanishes and we have the standard expression for the energy. To evaluate \( J(\beta) \), we observe that the integral for \( dJ/d\beta \) is a rational function of \( \cos^2 \varphi \) and therefore can be calculated by going to the complex plane with the help of residues: \( dJ/d\beta = 4\pi/(\beta + 1) \). Since \( J(1) = 0 \) we obtain:

\[
J = 4\pi \ln \frac{1 + \beta}{2}.
\]

(12) Thus, we have

\[
\tilde{F} = \frac{\phi_0 B \sqrt{m_{zz}}}{32\pi^3 \lambda^2} \ln \frac{2\sqrt{3} \mu_a \phi_0}{\xi^2 B \sqrt{m_{zz}} (1 + \beta)^2}.
\]

(13) To write explicitly the angular dependence of \( F \), it is convenient to use the angular functions

\[
\Theta_{\lambda,H}(\theta) = \varepsilon_{\lambda,H}(\theta)/\gamma_{\lambda,H}
\]

(14) where \( \varepsilon_{\lambda,H}(\theta) \) is defined in Eq. (3) with corresponding \( \gamma \)’s. In terms of these functions, \( \beta = \Theta_H/\Theta_\lambda \) and

\[
\tilde{F} = \frac{\phi_0 B \Theta_\lambda}{32\pi^3 \lambda^2_{ab}} \ln \frac{2\sqrt{3} \mu_a \phi_0 \Theta_\lambda}{\xi^2 B (\Theta_\lambda + \Theta_H)^2}.
\]

(15) The torque density follows:

\[
\tau = \frac{-\partial \tilde{F}}{\partial \theta} = -\frac{\phi_0 B}{32\pi^3 \lambda^2_{ab}} \left[ \Theta_\lambda' \ln \frac{2\sqrt{3} \mu_a \phi_0 \Theta_\lambda}{\xi^2 B (\Theta_\lambda + \Theta_H)^2} - 2\Theta_\lambda' \Theta_H/\Theta_\lambda + \Theta_H \right],
\]

(16) where \( e = 2.718... \). The torque is zero at \( \theta = 0, \pi/2 \) because

\[
\Theta' = -\frac{(\gamma^2 - 1) \sin 2\theta}{2 \gamma^2 \Theta}.
\]

(17) for both \( \Theta_\lambda \) and \( \Theta_H \). In the standard case of \( \gamma_H = \gamma_\lambda = \gamma \), Eq. (16) reduces to the result [10] if we set \( \eta = \pi \sqrt{3}/e \approx 2 \). We then can rewrite the torque in the form:

\[
\tau = \frac{\phi_0 B (\gamma_\lambda^2 - 1) \sin 2\theta}{64\pi^2 \lambda^2_{ab} \lambda^2_3 \Theta_\lambda} \left[ \ln \left( \frac{\eta H_{c2,c}}{B} \frac{4e^2 \Theta_\lambda}{(\Theta_\lambda + \Theta_H)^2} \right) - \frac{2\Theta_\lambda}{\Theta_\lambda + \Theta_H} \left( 1 + \frac{\Theta_H'}{\Theta_\lambda'} \right) \right].
\]

(18)
Since $\Theta_{\lambda, H}^f < 0$, the second contribution to the torque \((18)\) is negative whereas the first one is positive. The positive torque implies that the system energy decreases with increasing $\theta$, as in the case of $\gamma_H = \gamma_\lambda$ for which $\theta = \pi/2$ is the stable equilibrium.

The competing roles of these two contributions can be demonstrated by considering stability of equilibrium states at $\theta = 0$ and $\theta = \pi/2$. To do this one notes that $\Theta(0) = 1$, $\Theta(\pi/2) = 1/\gamma$, $\Theta^f(0) = \Theta^f(\pi/2) = 0$, and

$$\Theta^f(0) = -\frac{\gamma^2 - 1}{\gamma}, \quad \Theta^f(\pi/2) = \frac{\gamma^2 - 1}{\gamma}$$

(19)

(for both $\Theta_\lambda$ and $\Theta_H$). Then, one obtains:

$$F''\left(\frac{\pi}{2}\right) = \frac{\gamma^2 - 1}{\gamma_\lambda} \ln \frac{4\pi e \sqrt{3} H_{d_{2, 0}} \gamma_H \gamma_\lambda}{B (\gamma_H + \gamma_\lambda)^2} - 2 \frac{\gamma_H \gamma_\lambda - 1}{\gamma_\lambda},$$

$$F''(0) = -\frac{\gamma^2 - 1}{\gamma_\lambda} \ln \frac{\pi \sqrt{3} H_{d_{2, 0}}}{B} + \frac{\gamma_\lambda^2 - 1}{\gamma_H},$$

(20)

where the constant positive prefactor is omitted since we are interested only in the sign of $F''$. Clearly, $\theta = \pi/2$ corresponds to the stable equilibrium for $\gamma_H = \gamma_\lambda$. In the general case, however, there is no such a clear-cut result: for a fixed $\gamma_\lambda$ and large enough $\gamma_H$, $\theta = \pi/2$ may become unstable. E.g., for $\gamma_\lambda = 1$ and $\gamma_H > 1$, $F''(\pi/2) < 0$ whereas $F''(0) > 0$.

To illustrate how the angular dependence of the torque varies with anisotropies of $H_{d_{2}}$ and $\lambda$, we evaluate numerically the torque density \((18)\) for parameters in the range of those for MgB$_2$. Fig. 1 shows $\Theta(\gamma)$ for $\gamma_\lambda = 2.2$ and $\gamma_H = 3$, the values expected for temperatures somewhat below $T_c$. Qualitatively, the dependence is standard; the torque is positive in the whole angular domain, i.e., $\theta = \pi/2$ is the stable equilibrium.

With decreasing $T$, $\gamma_H$ of MgB$_2$ increases whereas $\gamma_\lambda$ decreases. In Fig. 2 the torque \((18)\) is plotted for $\gamma_\lambda = 2$, $\gamma_H = 5$ (the upper curve) and for $\gamma_\lambda = 1.7$, $\gamma_H = 5.3$ (the lower curve). These values correspond roughly to 0.7 and 0.6$T_c$ according to Ref. [2]. Clearly, $\theta = \pi/2$ as well as $\theta = 0$ are unstable; the stable equilibrium is shifted to $0 < \theta < \pi/2$. Interestingly enough, Angst et al. observe that a strong “peak-effect-like” irreversibility develops in the torque data at $\approx 77^{\circ}$ at $T = 15$K and $H = 7.5$T [2]. In layered materials this effect is commonly interpreted as manifestation of the “intrinsic pinning” in the small vicinity of the equilibrium orientation at $\theta = \pi/2$. From the point of view of this article, the peak should move to a position of the stable equilibrium, i.e., to lower angles. Moreover, the data show a negative torque above this angle ($77^{\circ} < \theta < 90^{\circ}$) in agreement with Fig. 2. Still, interpretation of these data within the London model should not be taken too seriously: the applied field $H = 7.5$T exceeds $H_{d_{2, 0}}(0)$ and in the $\theta$-domain where $H < H_{d_{2}}(\theta)$, the London model should not be trusted.

Finally, we plot in Fig. 3 the torque density for parameters which correspond to low temperatures, where $\gamma_\lambda \approx 1.1$ and $\gamma_H \approx 6$. The torque is negative for all angles implying the stable equilibrium at $\theta = 0$.

Physically, the large low temperature anisotropy of $H_{d_{2}}$ in MgB$_2$ is caused by the large superconducting gap on the nearly two-dimensional sheets of the Fermi surface of this material [7,8,9]. With increasing $T$, thermal mixing with the states at the three-dimensional part of the Fermi surface suppresses this anisotropy to about $\gamma_H(T_c) \approx 2.6$. The anisotropy of the London $\lambda$ (or of the superfluid density) at $T = 0$ of clean materials does not depend on the gap at all (“Galilean invariance of the superflow”) and therefore is determined by the whole Fermi surface, i.e., it is weak for MgB$_2$ (see discussions in Ref. [2,9]). The calculation [8] shows that $\gamma_\lambda(T = 0) \approx 1.1$. 

**FIG. 1:** The torque in units of $\phi_0 B/32 \pi^3 \lambda_0^2$ versus angle $0 < \theta < \pi/2$ for $\gamma_\lambda = 2.2$, $\gamma_H = 3$, and $4\pi^2 \eta H_{d_{2, 0}}/B = 100$.

**FIG. 2:** The same as in Fig. 1. The upper curve is calculated with Eq. (18) for $\gamma_\lambda = 2$ and $\gamma_H = 5$; at the lower curve $\gamma_\lambda = 1.7$ and $\gamma_H = 5.3$.

**FIG. 3:** The same as in Fig. 1 but $\gamma_\lambda = 1.1$ and $\gamma_H = 6$. 

and grows to $\approx 2.6$ as $T \to T_c$.

Thus, different gaps at different Fermi surface pieces (or generally, anisotropic gaps on anisotropic Fermi surface) may lead to profound macroscopic consequences such as those considered above. Certainly, the equilibrium magnetization and the flux lattice structure should be strongly affected by differences in anisotropies of the upper critical field and of the London penetration depth. Occurrence of a peak-effect near the field orientation other than $H \parallel ab$ is an example of peculiar dynamic phenomena which call for further study.

I thank P. Miranović and M. Angst for numerous discussions which led me to consider the problem of this paper. Ames Laboratory is operated for the U. S. Department of Energy by Iowa State University under Contract No. W-7405-Eng-82. This work is supported by the Office of Basic Energy Sciences.

[1] L.P. Gor’kov and T.K. Melik-Barkhudarov, Sov. Phys. JETP, 18, 1031 (1964).
[2] S.L. Bud’ko and P.C. Canfield, Phys. Rev. 65, 212501 (2002).
[3] M. Angst, R. Puzniak, A. Wisniewski, J. Jun, S.M. Kazakov, J. Karpinski, J. Roos, and H. Keller, Phys. Rev. Lett. 88, 167004 (2002).
[4] U. Welp, G. Karapetrov, W.K. Kwok, G.W. Crabtree, Ch. Marcenat, L. Paulius, T. Klein, J. Marcus, K.H.P. Kim, C.U. Jung, S.-S. Lee, B. Kang, and S.-I. Lee, cond-mat/0203337.
[5] M. Zehetmayer, M. Eisterer, H.W. Weber, J. Jun, S.M. Kazakov, J. Karpinski, and A. Wisniewski, cond-mat/0204199.
[6] V.G. Kogan, Phys. Rev. B 66, 020509 (2002).
[7] P. Miranović, K. Machida, and V.G. Kogan, cond-mat/0207146.
[8] V.G. Kogan, Phys. Rev. B, 38, 7049 (1988).
[9] D.E. Farrell, C.M. Williams, S.A. Wolf, N.P. Bansal, and V.G. Kogan, Phys. Rev. Lett. 61, 2805 (1988).
[10] D.E. Farrell, J.P. Rice, D.M. Ginzb erg, and J.Z. Liu, Phys. Rev. Lett. 64, 1573 (1990).
[11] V.G. Kogan, A. Gurevich, J.H. Cho, D.C. Johnston, Ming Xu, J.R. Thompson, and A. Martynovich, Phys. Rev. B, 54, 12386 (1996).
[12] K. Takahashi, T. Atsumi, N. Yamamoto, M. Xu, H. Kitazawa, and T. Ishida, Phys. Rev. B 66, 012501 (2002).
[13] G.K. Perkins, J. Moore, Y. Bugoslavsky, L.F. Cohen, J. Jun, S.M. Kazakov, J. Karpinski, and A.D. Caplin, cond-mat/0205030.
[14] L.J. Campbell, M.M. Doria, and V.G. Kogan, Phys. Rev. B 38, 2439 (1988).
[15] This result comes about because the vortex lattice hexagon of the isotropic case have the same energy for all possible orientations; the nearest neighbors are all situated at a circle of a certain radius. The anisotropy amounts to rescaling of $x$ and $y$ directions by different factors, i.e., transforms the the nearest neighbor circle to an ellipse. All possible structures so obtained have the same energy in the approximation used here [14].
[16] J. Karpinski, M. Angst, J. Jun, S.M. Kazakov, R. Puzniak, A. Wisniewski, J. Roos, H. Keller, A. Perrucchi, L. Degiorgi, M. Eskildsen, P. Border, L. Vinnikov, and A. Mironov, cond-mat/0207263.
[17] K.D. Belashchenko, M. van Schilfgaarde, and V.P. Antropov, Phys. Rev. B 64, 092503 (2001).
[18] A.Y. Liu, I.I. Mazin, and J. Kortus, Phys. Rev. Lett. 87, 087705 (2001).
[19] H.J. Choi, D. Roundy, H. Sun, M.L. Cohen, and S.G. Louie, cond-mat/0111183.