Linear sets in the projective line over the endomorphism ring of a finite field

Hans Havlicek     Corrado Zanella

March 1, 2022

Abstract

Let PG(1, E) be the projective line over the endomorphism ring \( E = \text{End}_q(\mathbb{F}_{q^t}) \) of the \( \mathbb{F}_{q^t} \)-vector space \( \mathbb{F}_{q^t} \). As is well known there is a bijection \( \Psi : \text{PG}(1, E) \rightarrow G_{2t, t, q} \) with the Grassmannian of the \((t - 1)\)-subspaces in PG(2t - 1, q). In this paper along with any \( \mathbb{F}_q \)-linear set \( L \) of rank \( t \) in PG(1, qt), determined by a \((t - 1)\)-dimensional subspace \( T^{\Psi} \) of PG(2t - 1, q), a subset \( L_T \) of PG(1, E) is investigated. Some properties of linear sets are expressed in terms of the projective line over the ring \( E \). In particular the attention is focused on the relationship between \( L_T \) and the set \( L'_T \), corresponding via \( \Psi \) to a collection of pairwise skew \((t - 1)\)-dimensional subspaces, with \( T \in L'_T \), each of which determine \( L \). This leads among other things to a characterization of the linear sets of pseudoregulus type. It is proved that a scattered linear set \( L \) related to \( T \in \text{PG}(1, E) \) is of pseudoregulus type if and only if there exists a projectivity \( \varphi \) of PG(1, E) such that \( L_T^\varphi = L'_T \).

Mathematics subject classification (2010): 51E20, 51C05, 51A45, 51B05.

Keywords: scattered linear set; linear set of pseudoregulus type; projective line over a finite field; projective line over a ring.

1 Introduction

1.1 Motivation

In this paper linear sets of rank \( t \) in the projective line PG(1, qt) are investigated, where \( q \) is a power of a prime \( p \). Such linear sets can be

\*This work was supported by GNSAGA of Istituto Nazionale di Alta Matematica “F. Severi” (Rome) and partly done while the first author was Visiting Professor at the University of Padua, Vicenza, Italy.
described by means of the field reduction map \( F = F_{2,t,q} \) [14] mapping any point \( ((a, b))_{q^t} \in PG_q(F_{d^t}) \cong PG(1, q^t) \) to the \((t - 1)\)-subspace\(^{\dagger}\) of \( PG_q(F_{d^t}) \cong PG(2t - 1, q) \) associated with \( ((a, b))_{q^t} \) (considered here as a \( t \)-dimensional \( F_q \)-vector subspace). A point set \( L \subseteq PG(1, q^t) \) is said to be \( F_q \)-linear (or just linear) of rank \( n \) if \( L = B(T') \), where \( T' \) is an \((n - 1)\)-subspace of \( PG_q(F_{d^t}) \), and

\[
B(T') = \{ \langle(u, v)\rangle_{q^t} \mid \langle(u, v)\rangle_q \in T' \} = \{ P \in PG(1, q^t) \mid P^F \cap T' \neq \emptyset \}. \tag{1}
\]

Additionally, each such \( T' \) gives rise to the set \( U(T') = B(T')^F = L^F \), which is a collection of \((t - 1)\)-subspaces belonging to the standard Desarguesian spread \( D = PG(1, q^t)^F \) of \( PG_q(F_{d^t}) \).

If a linear set \( L \) of rank \( n \) in \( PG(1, q^t) \) has size \( \theta_{n-1} = (q^n - 1)/(q - 1) \) (which is the maximum size for a linear set of rank \( n \)), then \( L \) is a scattered linear set. For generalities on the linear sets the reader is referred to [13, 14, 15, 16, and 19].

As it has been pointed out in [12] Prop. 2, if \( L = B(T') \) is a scattered linear set of rank \( t \) in \( PG(1, q^t) \), then the union of all subspaces in \( U(T') = L^F \) is a hypersurface \( Q \) of degree \( t \) in \( PG(2t - 1, q) \), and an embedded product space isomorphic to \( PG(t - 1, q) \times PG(t - 1, q) \). So, \( Q \) has two partitions in \((t - 1)\)-subspaces. The first one is \( U(T') \), the second one is \( U'(T') = \{ T'h \mid h \in F_q^t \} \), where \( T'h = \{ \langle(hu, hv)\rangle_q \mid \langle(u, v)\rangle_q \in T' \} \). By Prop. 3.2, the family \( U'(T') \) can be recovered uniquely from \( U(T') \) and \( T' \) (disregarding that \( F_{d^t}^t \) is the underlying vector space of our \( PG(2t - 1, q) \)).

For \( t = n \) there is alternative approach to \( B(T') \) and \( U(T') \) irrespective of whether \( T' \) is scattered or not. It is based on the \( F_q \)-endomorphism ring \( E \) of \( F_q \) and the projective line \( PG(1, E) \) over this ring. On the one hand, there is a bijection \( \Psi \) between the projective line \( PG(1, E) \) and the Grassmannian \( \mathcal{G}_{2t,t,q} \) of \((t - 1)\)-subspaces of \( PG_q(F_{d^t}) \). So, instead of \( T' \) we may consider its image under \( \Psi^{-1} \), which is a point \( T \) of \( PG(1, E) \). On the other hand, we have a natural embedding \( \iota : PG(1, q^t) \to PG(1, E) \). It maps the linear set \( B(T') \) to a subset \( B(T')^\Psi =: L_T \) of \( PG(1, E) \), which in turn is the preimage under \( \Psi \) of \( U(T') \). In Section 2 we take up these ideas, but we start with an equivalent definition, which is in terms of \( PG(1, E) \) only, of the set \( L_T \). There we also define a second set \( L_T' \subseteq PG(1, E) \) in such a way that \((L_T')^\Psi \) equals the set \( U'(T')^\Psi \) from above in the scattered case. Furthermore, since \( T \) will play a predominant role, \( B(T') = B(T'^\Psi) \) will frequently also be denoted by \( B(T) \); \textit{mutatis mutandis} this applies also to \( U(T') \) and \( U'(T') \).

\(^{\dagger}\)Abbreviation for \((t - 1)\)-dimensional subspace.
A special example of a scattered linear set \( L = B(T) \) in \( \text{PG}(1, q^t) \) is a linear set of pseudoregulus type, defined in [6], [17], and further investigated in [5]. In our setting it is obtained by taking \( T = E(1, \tau) \), where \( \tau \) is a generator of the Galois group \( \text{Gal}(F_{q^t}/F_q) \). The related hypersurface \( Q \) in \( \text{PG}(2t - 1, q) \) has been studied in [12], revealing a high degree of symmetry. As a matter of fact there are \( t \) families \( S_0, S_1, \ldots, S_{t-1} \) of \((t - 1)\)-subspaces partitioning \( Q \) [12, Thm. 6] where \( S_0 = \mathcal{U}(T) = L_T^\Psi \) and \( S_1 = \mathcal{U}'(T) = (L_T')^\Psi \) are defined above. Furthermore, in [12, Cor. 18] it is proved that the stabilizer of \( Q \) inside \( \text{PGL}_2(\mathbb{F}_{q^t}) \) contains a dihedral subgroup of order \( 2t \) acting on such \( t \) families of \((t - 1)\)-subspaces. A consequence thereof is the following result, for which we give a short direct proof in Prop. 3.1:

There is a projectivity of \( \text{PG}(1, E) \) mapping \( L_T \) onto \( L_T' \). For \( t \geq 3 \) this turns out to be a characteristic property of the linear sets of pseudoregulus type (Thm. 3.5). As the projectivities of \( \text{PG}(1, E) \) and the projectivities of \( \text{PG}_{q^2}(\mathbb{F}_{q^2}) \) are in one-to-one correspondence, this leads to the following:

**Theorem.** Let \( L = B(T') \) be a scattered linear set of rank \( t \) in \( \text{PG}(1, q^t) \), with \( T' \) a \((t - 1)\)-dimensional subspace of \( \text{PG}(2t - 1, q) \), and \( t \geq 3 \). Then \( L \) is a linear set of pseudoregulus type if, and only if, a projectivity of \( \text{PG}(2t - 1, q) \) exists mapping the first family \( \mathcal{U}(T') \) of subspaces of the related embedded product space to the second one \( \mathcal{U}'(T') \).

### 1.2 Notation

Let \( E = \text{End}_q(\mathbb{F}_{q^t}) \) with \( t \geq 2 \) be the ring of \( \mathbb{F}_q \)-linear endomorphisms of \( \mathbb{F}_{q^t} \). The ring \( E \) has the identity \( 1 \in E \) as its unit element. The multiplicative group comprising all invertible elements of \( E \) will be denoted as \( E^* \).

Let us briefly recall the definition of the projective line over the ring \( E \), which will be denoted by \( \text{PG}(1, E) \), and several basic notions; see [3, 1.3], [8, 3.2], and [9, 1.3]. We start with \( E^2 \), which is regarded as a left module over \( E \) in the usual way. Elements of \( E^2 \) are written as rows. This module has the standard basis \(((1, 0), (0, 1))\), and so it is a free module of rank 2. All invertible \( 2 \times 2 \) matrices with entries in \( E \) constitute the general linear group \( \text{GL}_2(E) \), which acts in a natural way on the elements of \( E^2 \) from the right hand side. Now \( \text{PG}(1, E) \), whose elements will be called points, is defined as the orbit of the cyclic submodule \( E(1, 0) \) (the “starter point”) under the action of the group \( \text{GL}_2(E) \) on \( E^2 \). Therefore, any point of \( \text{PG}(1, E) \) can be written in the form \( E(\alpha, \beta) \), where the pair \((\alpha, \beta) \in E^2\) is admissible, i.e., it is the first row of a matrix from \( \text{GL}_2(E) \). Furthermore, if \((\alpha', \beta')\)
is any element of $E^2$ then $E(\alpha', \beta') = E(\alpha, \beta)$ holds precisely when there is an element $\gamma \in E^*$ such that $(\alpha', \beta') = (\gamma \alpha, \gamma \beta)$. In this case $(\alpha', \beta')$ is admissible too.

The projective line $\text{PG}(1, E)$ is endowed with a binary distant relation $\triangle$ as follows: The relation $\triangle$ is the orbit of the pair $((1, 0), (0, 1))$ under the (componentwise) action of $\text{GL}_2(E)$. Thus $E(\alpha, \beta) \triangle E(\gamma, \delta)$ holds if, and only if, $(\alpha \gamma, \beta \delta) \in \text{GL}_2(E)$.

The map
\[
\Psi : \text{PG}(1, E) \rightarrow \mathcal{G}_{2t,t,q} : E(\alpha, \beta) \mapsto \left\{ (\frac{u^\alpha, u^\beta}{q} : u \in F_q^* \right\}
\]
(2)
is a bijection of $\text{PG}(1, E)$ onto the Grassmannian $\mathcal{G}_{2t,t,q}$ of $(t-1)$-subspaces of $\text{PG}_q(F_q^2) \cong \text{PG}(2t-1, q)$. Any two points of $\text{PG}(1, E)$ are distant if, and only if, their images under $\Psi$ are disjoint (or, said differently, complementary) [1, Thm. 2.4]. For versions of the previous results in terms of matrix rings we refer to [3, 10.2], [8, 5.2.3], and [9, 4.5]. See also [20, 123ff.], even though the terminology used there is quite different from ours.

Let $\phi$ denote a projectivity of $\text{PG}(1, E)$, i.e., $\phi$ is given by a matrix
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in \text{GL}_2(E)
\]
(3)
acting on $E^2$. Then the mapping
\[
\hat{\phi} : \text{PG}_q(F_q^2) \rightarrow \text{PG}_q(F_q^2) : ((u, v)) \mapsto ((u^\alpha + v^\gamma, u^\beta + v^\delta))
\]
(4)
is a projective collineation. The action of $\hat{\phi}$ on the Grassmannian $\mathcal{G}_{2t,t,q}$ is given by $\Psi^{-1} \phi \Psi$. By [11, 642–643], every projective collineation of $\text{PG}_q(F_q^2)$ can be written as in (4) for some matrix from $\text{GL}_2(E)$.

Under any projectivity of $\text{PG}(1, E)$ the distant relation $\triangle$ is preserved. The obvious counterpart of this observation is the fact that under any projective collineation of $\text{PG}_q(F_q^2)$ the complementarity of subspaces from $\mathcal{G}_{2t,t,q}$ is preserved.

If $a \in F_q^t$ then $\rho_a \in E$ is defined by $x^{\rho_a} = ax$ for all $x \in F_q^t$. The mapping
\[
F_q^t \rightarrow E : a \mapsto \rho_a
\]
is a monomorphism of rings taking $1 \in F_q^t$ to the identity $1 \in E$. The image of this monomorphism will be denoted by $F$. We now consider $\text{PG}(1, F)$ as a subset of $\text{PG}(1, E)$ by identifying $F(\rho_a, \rho_b)$ with $E(\rho_a, \rho_b)$ for all $(a, b) \in F_q^t \times F_q^t$. 

4
\( F_2^q \setminus \{(0,0)\} \). This allows us to embed the projective line \( \text{PG}(1, q^t) \) in the projective line \( \text{PG}(1, E) \) as follows:

\[
\iota : \text{PG}(1, q^t) \to \text{PG}(1, E) : \langle (a, b) \rangle_{q^t} \mapsto E(\rho_a, \rho_b). \tag{5}\]

Following \cite{2}, the image of \( \text{PG}(1, F) \) under any projectivity of \( \text{PG}(1, E) \) is called an \( F\text{-chain} \) of \( \text{PG}(1, E) \). In particular, \( \text{PG}(1, q^t) \) is an \( F\text{-chain} \) of \( \text{PG}(1, E) \).

Any two distinct points of \( \text{PG}(1, E) \) are distant precisely when they belong to a common \( F\text{-chain} \) \cite[Lemma 2.1]{2}. From this we obtain the following result \cite[Thm. 3.4.7]{8}, which is a slightly modified version of \cite[Thm. 2.3]{2}:

**Proposition 1.1.** Given three distinct points \( P_1, Q_1, R_1 \) on an \( F\text{-chain} \) \( C_1 \) and three distinct points \( P_2, Q_2, R_2 \) on an \( F\text{-chain} \) \( C_2 \) there is at least one projectivity \( \pi \) of \( \text{PG}(1, E) \) with \( \pi P_1 = P_2, \pi Q_1 = Q_2, \pi R_1 = R_2 \) and \( \pi C_1 = C_2 \).

2 Scattered points

**Definition 2.1.** For any point \( T = E(\alpha, \beta) \in \text{PG}(1, E) \) define:

\[
L_T = \left\{ E(\rho_a, \rho_b) \mid (a, b) \in (F^2_q)^* \text{ s.t. } E(\rho_a, \rho_b) \not\Delta T \right\};
\]

\[
L'_T = \left\{ T \cdot \text{diag}(\rho_h, \rho_h) \mid h \in F^*_{q^t} \right\}.
\]

Also, we introduce the shorthand \( Th := T \cdot \text{diag}(\rho_h, \rho_h) \), where \( h \) is as above. By the proof of Prop. 2.11 below, the point set \( L'_T \) is the orbit of \( T \) under the group of all projectivities of \( \text{PG}(1, E) \) that fix \( \text{PG}(1, F) \) pointwise.

The following diagram describes the relationships involving some objects defined so far. (Note that the right hand side of \( 1 \) gives \( B(T)^F = L'_T \).)

\[
\begin{array}{ccc}
B(T) \subset \text{PG}(1, q^t) & \xrightarrow{\iota} & L_T \subset \text{PG}(1, E) \\
\downarrow F & & \downarrow \Psi \\
B(T)^F \subset D & \leftarrow & L'_T \subset G_{2t,t,q}
\end{array}
\]

**Definition 2.2.** A scattered point of \( \text{PG}(1, E) \) is a point \( T \) such that \( \#L_T = \theta_{t-1} \).

\(^2\)Our \( F\text{-chains} \) are different from the chains in \cite{3} and \cite{9}, since \( F \) is not contained in the centre of \( E \).
A point $T \in \PG(1, E)$ is scattered if, and only if, $B(T)$ is a scattered linear set. For a point $X = E(\rho_a, \rho_b)$ is distant from $T$ if, and only if, the $(t-1)$-subspace $X^\Psi$ defined by the vector subspace $\langle (a, b) \rangle_{q'}$ is disjoint from $T^\Psi$.

**Example 2.3.** If $\tau$ is a generator of $\text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and $T_0 = E(1, \tau)$, then $B(T_0)$ is a scattered linear set of pseudoregulus type $[17]$. Hence $T_0$ is a scattered point of $\PG(1, E)$.

**Definition 2.4.** For $T \in \PG(1, E)$, the set $L_T$ will be said of pseudoregulus type when $L_{\Psi^{-1}} T$ is a linear set of pseudoregulus type.

Any two linear sets of pseudoregulus type are projectively equivalent $[6], [17]$. So $L_T$ is of pseudoregulus type if and only if $L_T = L_{\pi} E(1, \tau)$ where $\tau$ is a generator of $\text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and $\pi$ is a projectivity of $\PG(1, F)$.

**Example 2.5.** The point $T_1 = E(1, \sigma \rho^g + \sigma t^{-1})$ with $\sigma : u \mapsto u^g, g \in \mathbb{F}_{q^t}^*$, and $g^{t-1} \neq 1$ is scattered $[18, \text{Thm. 2}]$.

**Proposition 2.6.** Let $T \in \PG(1, E) \setminus \PG(1, F)$. Then $Th = Tk$ for any $h, k \in \mathbb{F}_{q^t}^*$ such that $h^{-1} k \in \mathbb{F}_q$. Furthermore, if $T$ is scattered, then $Th \triangle Tk$ for any $h, k \in \mathbb{F}_{q^t}^*$ such that $h^{-1} k \notin \mathbb{F}_q$.

**Proof.** If $h^{-1} k \in \mathbb{F}_q$ and $T = E(\alpha, \beta)$, then

$$Th = E(\alpha \rho_h, \beta \rho_h) = E(\rho_{h^{-1}} \alpha \rho_h, \rho_{h^{-1}} \beta \rho_h) = E(\alpha \rho_k, \beta \rho_k) = Tk.$$

Let $\mathcal{P}'$ be the set of all points of $\PG_q(\mathbb{F}_{q^t}^2) \cong \PG(2t-1, q)$ belonging to the $(t-1)$-subspaces of $L_T^\Psi$. By the previous paragraph, it follows $\# \mathcal{P}' \leq \theta_{t-1}^2$, and the equality holds if, and only if, for any $h, k$ the relation $h^{-1} k \in \mathbb{F}_{q^t} \setminus \mathbb{F}_q$ implies $Th \triangle Tk$.

Let $\mathcal{P}$ be the set of all points of $\PG_q(\mathbb{F}_{q^t}^2)$ belonging to the $(t-1)$-subspaces in $L_T^\Psi$. Then $\mathcal{P} \subseteq \mathcal{P}'$.

If $T$ is scattered, then $\# \mathcal{P} = \theta_{t-1}^2$. \hfill \Box

**Proposition 2.7.** Let $T = E(1, \beta)$ be a scattered point of $\PG(1, E)$. Then the following assertions hold:

(i) A point $P \in \PG(1, E)$ belongs to $L_T$ if, and only if, a $u \in \mathbb{F}_{q^t}^*$ exists such that $P = E(1, \rho_{u^g} u)$;

(ii) the size of the set $I = \{ u^\beta/u \mid u \in \mathbb{F}_{q^t}^* \}$ is $\theta_{t-1}$. 

6
(iii) For any \( u, v \in F_q^* \), \( u \) and \( v \) are \( F_q \)-linearly dependent if, and only if, \( u^\beta / u = v^\beta / v \);

(iv) The dimension of \( \ker \beta \) is at most one;

(v) \( \beta \) is a singular endomorphism if, and only if, \( E(1, 0) \in L_T \).

Proof. Let \( P = E(\rho_a, \rho_b) \) be a point. Then \( P \in L_T \) holds precisely when the \((t-1)\)-subspaces \( P^\Psi \) and \( T^\Psi \) are not disjoint; that is, there are two nonzero elements of \( F_q^* \), say \( u \) and \( v \), such that \( u = va \) and \( u^\beta = vb \). This is equivalent to \( a \neq 0 \) and \( u^\beta = a^{-1} bu \). This implies (i), and consequently (v).

The size of \( I \) equals the size of \( L_T \), and this implies (ii).

If \( r \in F_q^* \) and \( u \in F_q^* \), then \( (ru)^\beta / (ru) = u^\beta / u \). This implies that the size of the image of the map \( u \in F_q^* \mapsto u^\beta / u \in F_q^* \) is at most \( \theta_t - 1 \), and the equality holds only if condition (iii) is satisfied. The last condition implies (iv).

Take notice that (i) and (v) hold irrespective of whether the point \( T \in PG(1, E) \) is scattered or not.

The following result is merely a reformulation of [12, Prop. 2], with \( \beta \in E \) playing the role of the matrix \( A \) from there.

Proposition 2.8. Let \( T = E(1, \beta) \) be a scattered point. The map

\[
\epsilon : \left( \langle h \rangle_q, \langle (u, u^\beta) \rangle_q \right) \mapsto \langle (hu, hu^\beta) \rangle_q
\]

(6)

is a projective embedding of the product space \( PG_q(F_q^*) \times T^\Psi \) into \( PG_q(F_q^2) \), that is, is an injective mapping such that the image of any line of the product space is a line of \( PG_q(F_q^2) \).

Remark 2.9. In the case of non-scattered linear sets, the map \( \epsilon \) is not an embedding, but the image of \( \epsilon \) is still a non-injective projection of a Segre variety [16].

In [7, Thm. 1] a result similar to the following one is proved in terms of the matrix group \( \text{GL}_2(q^t) \).

Proposition 2.10. Let \( \kappa \in \text{PGL}_2(q^t) \) be a collineation of \( PG(1, q^t) \) whose accompanying automorphism \( \eta \) is in \( \text{Gal}(F_{q^t}/F_q) \). After embedding \( PG(1, q^t) \) in \( PG(1, E) \) according to (5), the collineation \( \iota^{-1} \kappa \iota \) of \( PG(1, F) \) can be extended to at least one projectivity of \( PG(1, E) \). Conversely, the restriction to \( PG(1, F) \) of any projectivity of \( PG(1, E) \) that fixes \( PG(1, F) \) as a set is a collineation with accompanying automorphism in \( \text{Gal}(F_{q^t}/F_q) \).
Proof. There is a matrix \((m_{ij}) \in \text{GL}_2(q^t)\) such that
\[
\langle (a, b) \rangle_{q^t} \mapsto \langle (a^n, b^n) \rangle_{q^t} \cdot (m_{ij}) \quad \text{for all } \langle (a, b) \rangle_{q^t} \in \text{PG}(1, q^t).
\]
For all \(x, a \in \mathbb{F}_{q^t}\) we have \(xa^n = (x^{n^{-1}} a)^n\) and so \(\rho_{an} = \eta^{-1} \rho_{a} \eta\). The permutation of \(\text{PG}(1, E)\) given by
\[
E(\alpha, \beta) \mapsto E(\eta^{-1} \alpha \eta, \eta^{-1} \beta \eta) \cdot (\rho_{m_{ij}}) \quad \text{for all } E(\alpha, \beta) \in \text{PG}(1, E)
\]
is a projectivity, since the automorphism of \(E\) acting on \(\alpha\) and \(\beta\) is inner. By construction, this projectivity extends the collineation \(\iota^{-1} \kappa \nu\) of \(\text{PG}(1, F)\).

Conversely, let \(\pi\) be a projectivity of \(\text{PG}(1, E)\) that fixes \(\text{PG}(1, F)\) as a set. Since \(\text{PGL}(2, q^t)\) acts (sharply) 3-transitively on \(\text{PG}(1, q^t)\) there is a (unique) projectivity \(\lambda\) of \(\text{PG}(1, q^t)\) such that the images of \(E(1, 0)\), \(E(0, 1)\), \(E(1, 1)\) under \(\iota^{-1} \kappa \lambda \nu\) and \(\pi\) are the same. We choose matrices \((\pi_{ij}) \in \text{GL}_2(E)\) and \((\nu_{ij}) \in \text{GL}_2(q^t)\) that describe \(\pi\) and \(\lambda\), respectively. Then \((\pi_{ij}) \cdot (\nu_{ij})^{-1}\) induces a projectivity of \(\text{PG}(1, E)\) that fixes each of the points \(E(1, 0)\), \(E(0, 1)\), \(E(1, 1)\) and also the \(F\)-chain \(\text{PG}(1, F)\). So there is a \(\delta \in E^*\) with
\[
(\pi_{ij}) = \text{diag}(\delta, \delta) \cdot (\rho_{\nu_{ij}})
\]
and for each \(b \in \mathbb{F}_{q^t}\) there is a unique \(b' \in \mathbb{F}_{q^t}\) such that
\[
E(1, \rho_b) \cdot \text{diag}(\delta, \delta) = E(\delta, \rho_{b'} \delta) = E(1, \rho_{b'}) = E(\delta, \delta \rho_{b'}).
\]
This leads us to \(\delta^{-1} \rho_b \delta = \rho_{b'}\) for all \(b \in \mathbb{F}_{q^t}\). Thus the inner automorphism of \(E\) given by \(\delta\) restricts to an automorphism of the field \(F\). Going back to \(\mathbb{F}_{q^t}\) shows that \(\eta : \mathbb{F}_{q^t} \mapsto \mathbb{F}_{q^t} : b \mapsto b'\) is an automorphism of \(\mathbb{F}_{q^t}\). Furthermore, we read off \(\eta \in \text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)\) from \(\rho_b\) being in the centre of \(E\) for all \(b \in \mathbb{F}_{q^t}\).

Let \(d := 1^\delta \in \mathbb{F}_{q^t}^*\) and choose any \(b \in \mathbb{F}_{q^t}\). Calculating the image of \(d\) under \(\delta^{-1} \rho_b \delta = \rho_{b'}\) in two ways gives \(b'^t = db^n\), whence \(\delta = \eta \rho_d\). This leads us finally to
\[
(\pi_{ij}) = \text{diag}(\eta, \eta) \cdot (\rho_{d \nu_{ij}}) = \text{diag}(\eta, \eta) \cdot (\rho_{d \nu_{ij}}).
\]
We now repeat the first part of the proof with \((dc_{ij})\) instead of \((m_{ij})\). This gives the projectivity
\[
E(\alpha, \beta) \mapsto E(\eta^{-1} \alpha \eta, \eta^{-1} \beta \eta) \cdot (\rho_{dc_{ij}}) \quad \text{for all } E(\alpha, \beta) \in \text{PG}(1, E),
\]
which obviously coincides with \(\pi\). \qed

8
The collineation \( \kappa \) from the previous proposition can be extended in precisely \( \theta_{t-1} \) different ways to a projectivity of \( \text{PG}(1, E) \). Even though this could be derived easily from well known results about spreads [10, Sect. 1], we give a short direct proof.

**Proposition 2.11.** There are precisely \( \theta_{t-1} \) projectivities of \( \text{PG}(1, E) \) that fix \( \text{PG}(1, F) \) pointwise.

**Proof.** Let \( \pi \) be a projectivity of \( \text{PG}(1, E) \) that fixes \( \text{PG}(1, F) \) pointwise. We repeat the second part of the proof of Prop. 2.10 under this stronger assumption, while maintaining all notations from there. However, in our current setting we may choose \((c_{ij}) = \text{diag}(1, 1) \in \text{GL}_2(q^t)\). We are thus led to \((\pi_{ij}) = \text{diag}(\delta, \delta)\) for some \( \delta \in E^* \). This \( \delta \) has to satisfy now \( \delta^{-1} \rho_b \delta = \rho_b \) for all \( b \in F_{q^t} \), which in turn gives that \( \eta \) is the trivial automorphism of \( F_{q^t} \). Letting \( d := 1^\delta \), as we did before, gives therefore that \( \pi \) is given by the matrix
\[
(\pi_{ij}) = \text{diag}(\rho_d, \rho_d) \quad \text{with} \quad d \in F_{q^t}^*.
\]

Conversely, for all \( d \in F_{q^t}^* \) the matrix \( \text{diag}(\rho_d, \rho_d) \) determines a projectivity of \( \text{PG}(1, E) \) that fixes \( \text{PG}(1, F) \) pointwise. Two matrices of this kind give rise to the same projectivity if, and only if, they differ by a factor \( \text{diag}(\gamma, \gamma) \), where \( \gamma \) is an invertible element from the centre of \( E \). This condition for \( \gamma \) is equivalent to \( \gamma = \rho_f \) with \( f \in F_{q^t}^* \), whence the assertion follows from \( \theta_{t-1} = (q^t-1)/(q-1) = (#F_{q^t}^*)/(#F_{q^t}) \).

**Proposition 2.12.** If \( \pi \in \text{PGL}_2(E) \) stabilizes \( \text{PG}(1, F) \), then \( L_T^\pi = (L_T)^\pi \) and \( L_T'^\pi = (L_T')^\pi \) for each \( T \in \text{PG}(1, E) \).

**Proof.** The first equation can be derived as follows:
\[
L_T^\pi = \{ P \in \text{PG}(1, F) \text{ s.t. } P \not\Delta T^\pi \} = \{ P^\pi \text{ s.t. } P \in \text{PG}(1, F), P^\pi \not\Delta T^\pi \}
= \{ P^\pi \text{ s.t. } P \in \text{PG}(1, F), P \not\Delta T \} = L_T^\pi.
\]

Let \( h \in F_{q^t}^* \). Taking into account the structure of the projectivity \( \pi \), described in [10], \( T^\pi h = (Th^{n^{-1}})^\pi \), whence \( L_T'^\pi = (L_T')^\pi \).

**Remark 2.13.** The investigation of scattered points can be restricted taking into account that if \( T \) is a scattered point of \( \text{PG}(1, E) \), then there exist a projectivity \( \pi \) of \( \text{PG}(1, E) \) and an element \( \beta \in E^* \), such that \( \text{PG}(1, F)^\pi = \text{PG}(1, F), 1 \in \text{Spec}(\beta) \), and \( T^\pi = E(1, \beta) \) (cf. [17, Rem. 4.2]).

**Lemma 2.14.** Let \( s \) be the greatest element of \( \{1, 2, \ldots, t-1\} \) that divides \( t \). Then any two distinct \( F \)-chains of \( \text{PG}(1, E) \) have at most \( q^s+1 \) common points.
Proof. Due to $3 = 2^1 + 1 \leq q^s + 1$ it is enough to consider two distinct $F$-chains $C_1$ and $C_2$ with at least three distinct common points. By Prop. 1.1 we may assume these points to be $E(1,0), E(0,1), E(1,1)$ and $C_1 = \text{PG}(1,F)$. Applying Prop. 1.1 once more shows that there is a projectivity $\pi$ of $\text{PG}(1,E)$ that fixes each of the points $E(1,0), E(0,1), E(1,1)$ and takes $C_1$ to $C_2$. Consequently, there is a $\delta \in E^*$ such that

$$E(\alpha, \beta)^\pi = E(\alpha, \beta) \text{diag}(\delta, \delta) = E(\delta^{-1}\alpha\delta, \delta^{-1}\beta\delta)$$

for all $E(\alpha, \beta) \in \text{PG}(1,E)$.

This gives

$$C_1 \cap C_1^\pi = \{E(1, \rho) \mid \rho \in F \cap (\delta^{-1}F\delta)\} \cup \{E(0,1)\}.$$ 

The intersection $F \cap \delta^{-1}F\delta$ is a proper subfield of $F$, since $\delta^{-1}F\delta$ is an isomorphic copy of $F$ in $E$ and $C_1 \neq C_2$ implies $F \neq \delta^{-1}F\delta$. This gives

$$\#(C_1 \cap C_2) = \#(F \cap \delta^{-1}F\delta) + 1 \leq q^s + 1. \quad \Box$$

**Proposition 2.15.** Let $T$ be a scattered point of $\text{PG}(1,E)$ and assume that $t \geq 3$. Then $L_T$ is contained in no $F$-chain other than $\text{PG}(1,F)$.

**Proof.** From $t \geq 3$ follows $\#L_T = \theta_{t-1} > q^{t-1} + 1$. The assertion is now immediate from Lemma 2.14. \quad \Box

**Remark 2.16.** For $t = 2$ the set $L_T^F$ is a regulus in $\text{PG}(3,q)$ and the $F$-chains are precisely the regular spreads in $\text{PG}(3,q)$. Choose a point that is off the hyperbolic quadric $\mathcal{H}$ that carries $L_T^F$. Then there are as many regular spreads through $L_T^F$ as there are external lines to $\mathcal{H}$ through the chosen point. A straightforward counting shows that the number of these lines is $\frac{1}{2}(q^2 - q)$. Thus, unless $q = 2$, there is more than one $F$-chain through $L_T$.

**Theorem 2.17.** Let $T$ and $U$ be points of $\text{PG}(1,E)$, with $T$ a scattered point. Then the following assertions are equivalent:

(i) A collineation $\kappa \in \text{PGL}_2(q^t)$ with companion automorphism $\eta \in \text{Gal}(F_{q^t}/F_q)$ exists, such that $B(T)^\kappa = B(U)$;

(ii) $L_T$ and $L_U$ are projectively equivalent in $\text{PG}(1,E)$.

**Proof.** If $\kappa$ is given as in (i) then, by the first part of Prop. 2.10 there is a projectivity of $\text{PG}(1,E)$ that takes $B(T)^\kappa = L_T$ to $B(U)^\kappa = L_U$.

Conversely, let $\pi$ be a projectivity of $\text{PG}(1,E)$ that takes $L_T$ to $L_U$. There are two cases:
Case 1: \( t \geq 3 \). From \( \#L_T = \#L_U \) the point \( U \) is scattered. By Prop. 2.15 each of the sets \( L_T \) and \( L_U \) is contained in no \( F \)-chain other than \( \text{PG}(1,F) \). Hence \( \text{PG}(1,F) \) is invariant under \( \pi \) so that the second part of Prop. 2.10 shows the existence of a collineation of \( \text{PG}(1,q^t) \) with the required properties.

Case 2: \( t = 2 \). The sets \( B(T) \) and \( B(U) \) are two linear sets of rank 2 and cardinality \( q + 1 \), i.e. two Baer sublines of \( \text{PG}(1,q^2) \). These are well known to be projectively equivalent.

Proposition 2.18. If \( g \neq 0 \) and \( t, q > 3 \), the sets \( L_T^0 \) and \( L_T^1 \), where \( T^0 \) and \( T^1 \) are defined in Examples 2.3 and 2.5, are not projectively equivalent in \( \text{PG}(1,E) \).

Proof. The sets \( B(T^0) \) and \( B(T^1) \) are not projectively equivalent [17, Example 4.6]. Since any linear set in the \( \text{PGL}_2(q^t) \)-orbit of a linear set of pseudoregulus type is again of pseudoregulus type, \( B(T^0) \) and \( B(T^1) \) also are not equivalent up to collineations. Then the assertion follows from Thm. 2.17.

3 Characterization of the linear sets of pseudoregulus type

Proposition 3.1. Let \( T \in \text{PG}(1,E) \) be such that \( L_T \) is a set of pseudoregulus type. Then there is a \( \varphi \in \text{PGL}_2(E) \) such that \( L_T^\varphi = L_T' \).

Proof. Since up to projectivities in \( \text{PG}(1,q^t) \) there is a unique linear set of pseudoregulus type \([6,17]\), it may be assumed that \( T = E(1,\tau) \) with \( \tau \) a generator of \( \text{Gal}(F_{q^t}/F_q) \). A projectivity \( \varphi \) satisfying the thesis is given by the matrix \( \text{diag}(1,\tau) \in \text{GL}_2(E) \). As a matter of fact, for any \( u \in F_{q^t}^* \),

\[
E(\rho_1/u^\tau, \tau \rho_1/u^\tau) = E(1,\rho_1/u) = E(1,\tau \rho_1/u) = E(1,\varphi),
\]

and by Prop. 2.7 this implies that \( \varphi \) maps \( L_T \) onto \( L_T' \).

The goal of this section is to prove the converse of Prop. 3.1

Proposition 3.2. Let \( \varepsilon : \Pi_1 \times \Pi_2 \to \Pi_3 \) be a projective embedding, where \( \Pi_j \) is a projective space of finite dimension \( d_j \geq 1 \) for \( j = 1, 2, 3 \). Let \( \mathcal{U}_1 \) be the set of all \( d_1 \)-subspaces of type \( \Pi_1 \times Q \) for \( Q \) a point in \( \Pi_2 \), and \( \mathcal{U}_2 \) the set of all \( d_2 \)-subspaces of type \( P \times \Pi_2 \) for \( P \) a point in \( \Pi_1 \). Fix any point \( S \in \Pi_2 \) and consider the subspace \( (\Pi_1 \times S) \) in \( \mathcal{U}_1 \). Under these assumptions the following assertions hold:

\[\]
1. A line $y$ of $\Pi_3$ is contained in some subspace belonging to $U_1$ if, and only if, there is a $d_2$-regulus $R_y$ in $\Pi_3$ subject to the following three conditions:

(a) $R_y$ has a transversal line in $(\Pi_1 \times S)^\varepsilon$.
(b) $R_y$ is contained in $U_2$.
(c) $y$ is a transversal line of $R_y$.

2. A subspace $V$ of $\Pi_3$ belongs to $U_1$ if, and only if, $V$ has dimension $d_1$ and for any line $y$ of $V$ there is a $d_2$-regulus $R_y$ subject to the conditions (a), (b), and (c) from above.

Proof. Our reasoning will be based on the following three facts: (i) due to the injectivity of $\varepsilon$, each point in the image of $\varepsilon$ is incident with a unique subspace from $U_1$ and a unique subspace from $U_2$; (ii) for any line $\ell \subset \Pi_1$ the set

$$\{(P \times \Pi_2)^\varepsilon \mid P \in \ell\} \subset U_2$$

(10)

is a $d_2$-regulus; (iii) the transversal lines of this regulus are precisely the lines of the form $(\ell \times Q)^\varepsilon$ with $Q$ varying in $\Pi_2$.

Ad 1. Let $y$ be a line such that $y \subset (\Pi_1 \times R)^\varepsilon \in U_1$, where $R \in \Pi_2$. Since the restriction of $\varepsilon$ to $\Pi_1 \times R$ is a collineation onto the subspace $(\Pi_1 \times R)^\varepsilon$, there is a unique line $\ell_y \subset \Pi_1$ such that $y = (\ell_y \times R)^\varepsilon$. By (10), the $d_2$-regulus $R_y := \{(P \times \Pi_2)^\varepsilon \mid P \in \ell_y\}$ satisfies (b). Furthermore, both $y$ and $(\ell_y \times S)^\varepsilon \subset (\Pi_1 \times S)^\varepsilon$ are transversal lines of $R_y$, whence conditions (a) and (c) are satisfied too.

Conversely, assume that for a line $y \subset \Pi_3$ there is a regulus $R_y$ satisfying the three conditions from above. By (a), there is a line $\ell_y \subset \Pi_1$ for which $(\ell_y \times S)^\varepsilon$ is a transversal line of $R_y$. Now (b) implies that $R_y$ can be written as in (10) with $\ell$ to be replaced with $\ell_y$. Thus (c) shows that there is a point $R \in \Pi_2$ such that $y = (\ell_y \times R)^\varepsilon$. This in turn gives $y \subset \Pi_1 \times R \in U_1$.

Ad 2. Let $V \in U_1$. The dimension of $V$ obviously is $d_1$. Given any line $y \subset V$ there is a regulus $R_y$ with the required properties by the first part of the proposition.

For a proof of the converse, we fix a point $Y \in V$ and consider an arbitrary line $y \subset V$ through $Y$. By the first part of the proposition, there is at least one point $R \in \Pi_2$ such that $y \subset (\Pi_1 \times R)^\varepsilon \in U_1$. This implies $Y = (X, R)^\varepsilon$ for some point $X \in \Pi_1$. The point $R$ does not depend on the choice of the line $y$ on $Y$. Consequently, $V \subset (\Pi_1 \times R)^\varepsilon$ and, due to $d_1$ being the dimension of $V$, these two subspaces are identical.\[\square\]
Clearly, analogous statements hold, where the roles of $U_1$ and $U_2$ are interchanged. We will refer to $U_1$ and $U_2$ in Prop. 3.2 as to the maximal subspaces of the embedded product space $(\Pi_1 \times \Pi_2)^\varepsilon$. Notice that such subspaces are defined with respect to the embedding, since $(\Pi_1 \times \Pi_2)^\varepsilon$ may contain further maximal subspaces. Prop. 3.2 implies:

**Proposition 3.3.** Let $U_1$ and $V$ be a collection of $d_1$-subspaces and a $d_2$-subspace in a projective space $\Pi$, respectively, where $d_1$ and $d_2$ are positive integers. There exists at most one collection $U_2$ of $d_2$-subspaces of $\Pi$ with $V \in U_2$, and such that $U_1$ and $U_2$ are the collections of maximal subspaces of an embedded product space.

**Remark 3.4.** By Prop. 3.2, a point $P \in \text{PG}(1,E)$ is in $L_T$ if, and only if, no line of $P^\Psi$ is irregular as defined in [13] with respect to the scattered subspace $T^\Psi$.

**Theorem 3.5.** Let $T = E(1, \beta)$ be a scattered point, where $\beta \in E^*$, let $t \geq 3$ and suppose that there exists a projectivity $\varphi \in \text{PGL}_2(E)$ such that $L^\varphi_T = L'_T$. Then $L_T$ is of pseudoregulus type.

**Proof.** We split the proof into four steps.

**Step 1.** First we fix some matrix $(\varphi_{ij}) \in \text{GL}_2(E)$ that describes $\varphi$. Then we choose any point $U \in \text{PG}(1,E)$ such that $U^\varphi \in L_T$. The point $U^\varphi$ is non-distant to all points of $L'_T$, whence $U$ is non-distant to all (namely $\theta_{t-1}$) points of $L_T$. So $U$ is scattered and $L_T = L_U$. Also, there is a $\gamma \in E^*$ satisfying

$$U = E(1, \gamma).$$

According to [14], the matrix $(\varphi_{ij})$ describes also that projective collineation $\hat{\varphi}$ of $\text{PG}(\mathbb{F}_2^t)$ whose action on the Grassmannian $\text{PG}(1,E)^\Psi$ coincides with $\Psi^{-1}\varphi\Psi$. It can be deduced from Prop. 2.8 that for any scattered point $X \in \text{PG}(1,E)$, $L_X^\Psi$ and $(L'_X)^\Psi$ are the two collections of maximal subspaces of an embedded product space. So, a repeated application of Prop. 3.3 implies:

(a) $L^\varphi_U$ is the unique collection of maximal subspaces of an embedded product space, containing $U^\varphi\Psi$, the other collection being $(L'_T)^\Psi$.

(b) $(L'_U)^\Psi$ is the unique collection of maximal subspaces of an embedded product space, containing $U^\Psi$, the other one being $L^\Psi_U = L^\Psi_T$.

By applying $\hat{\varphi}$ to (b), one obtains:
(c) \((L_U')^\varphi\Psi\) is the unique collection of maximal subspaces of an embedded product space, containing \(U^\varphi\Psi\), the other one being \(L_T^\varphi\Psi = (L_T')^\Psi\).

By (a) and (c)
\[
(L_U')^\varphi = L_T = L_U. \tag{11}
\]

**Step 2.** Let \(H\) be the subgroup \(\text{GL}_2(E)\) formed by all matrices \(\text{diag}(\rho_h, \rho_h)\) with \(h \in \mathbb{F}_q^*\). Also, let \(\Lambda\) be the associated group of projectivities of \(\text{PG}(1, E)\). Both \(H \cong \mathbb{F}_q^*\) and \(\Lambda \cong \mathbb{F}_q^*/\mathbb{F}_q^*\) are cyclic. Now, for all \(h, k \in \mathbb{F}_q^*\), the equality \(E(\rho_h, \gamma \rho_h) = E(\rho_k, \gamma \rho_k)\) holds precisely when \(h\) and \(k\) are \(\mathbb{F}_q\)-linearly dependent (cf. Prop. 2.6). Consequently we have shown: the cyclic group \(\Lambda\) acts regularly on \(L_U'\) and fixes \(L_U = L_T\) pointwise.

Consider now the subgroup \(H' := (\varphi_{ij})^{-1} \cdot H \cdot (\varphi_{ij})\) of \(\text{GL}_2(E)\) and the corresponding group \(\Lambda' := \varphi^{-1}\Lambda\varphi\) of projectivities. By the above, the group \(\Lambda'\) is cyclic, acts regularly on \(L_T\), and fixes \(L_T'\) pointwise. From \(t \geq 3\) and Prop. 2.15 the \(F\)-chain \(\text{PG}(1, F)\) is invariant under the action of \(\Lambda'\). Since \(\Lambda'\) has order \(\theta_{t-1}\), which is the size of the orbit \(L_T\), the group \(\Lambda'\) acts faithfully on \(\text{PG}(1, F)\).

**Step 3.** Let us choose any element \(h \in \mathbb{F}_q^*\). The projectivity \(\pi\) given by the matrix
\[
(\pi_{ij}) := (\varphi_{ij})^{-1} \cdot \text{diag}(\rho_h, \rho_h) \cdot (\varphi_{ij}) \in \text{GL}_2(E)
\]
fixes \(\text{PG}(1, F)\), as a set. We therefore can repeat the second part of the proof of Prop. 2.10 up to (8). By this formula\(^3\), there exists an automorphism \(\eta \in \text{Gal}(\mathbb{F}_q^*/\mathbb{F}_q)\) and an invertible matrix \((c_{ij}) \in \text{GL}_2(q^t)\) such that
\[
(\pi_{ij}) = \text{diag}(\eta, \eta) \cdot (\rho_{c_{ij}}) \in H'. \tag{12}
\]

We claim that \(\eta = 1\). In order to verify this assertion, we fix any element \(u \in \mathbb{F}_q^*\). The point \(E(\rho_u, \rho_{u^3})\) is in \(L_T\), which is invariant under \(\pi \in \Lambda'\) by Step 2. This implies that there is a \(v \in \mathbb{F}_q^*\) such that \(E(\rho_u, \rho_{u^3})^\pi = E(\rho_v, \rho_{u^3})\). Furthermore, \(L_T'\) is fixed pointwise under \(\pi\). So, the associated projective collineation \(\hat{\pi}\) of \(\text{PG}_q(\mathbb{F}_q^2)\) sends for all \(k \in \mathbb{F}_q^*\) the point
\[
\langle (uk, u^3 k) \rangle_q = E(\rho_u, \rho_{u^3})^\Psi \cap E(\rho_k, \beta \rho_k)^\Psi
\]
\(^3\)Take notice that the elements \(c_{ij}\) that are used now play the role of the elements \(d c_{ij}\) that appear in [8].
to the point
\[
E(\rho_v, \rho_v) \cap E(\rho_k, \rho_k) = \langle (vk, v^3k) \rangle_q.
\]
Therefore, for each \( k \in \mathbb{F}_q^* \) there is an element \( x_k \in \mathbb{F}_q^* \) such that
\[
((uk)^9, (u^3k)^9) \cdot (c_{ij}) = x_k(vk, v^3k).
\]
As \( \mathbb{F}_q^2 \) is also a vector space over \( \mathbb{F}_q \), this can be rewritten in the form
\[
k^9(u^9, u^9) \cdot (c_{ij}) = x_kk(v,v^3).
\]
Letting \( k := 1 \) gives \((u^9, u^9) \cdot (c_{ij}) = x_1(v,v^3)\) so that \( k^9x_1(v,v^3) = x_kk(v,v^3) \). We thus have arrived at
\[
k^9 = (x_k/x_1)k \text{ for all } k \in \mathbb{F}_q^*.
\]
This means that each \( k \in \mathbb{F}_q^* \) is an eigenvector of the \( \mathbb{F}_q \)-linear mapping \( \eta \in E \). Thus \( \eta \) has a unique eigenvalue, i.e., \( x_k/x_1 \) is a constant that does not depend on \( k \). Finally, \( 1^9 = 1 \) shows that this constant is equal to 1.

Step 4. We maintain all notions from the previous step (with \( \eta = 1 \)), under the additional assumption that \( h \) is a generator of the multiplicative group \( \mathbb{F}_q^* \). There are three possibilities for the matrix \((c_{ij}) \in \text{GL}_2(q^2)\):

First, assume that \((c_{ij})\) has a single eigenvalue in \( \mathbb{F}_q^2 \). Then, since \((c_{ij})\) cannot be a scalar multiple of the identity matrix, it is similar to a matrix of the form
\[
a \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\]
with \( a, b \in \mathbb{F}_q^* \).

Due to \( \text{Char } \mathbb{F}_q^4 = p \), the \( p \)-th power of this matrix is \( \text{diag}(a^p, a^p) \), which gives that \( \Lambda' \) acts on PG(1, \( F \)) as a cyclic permutation group of order \( p \). Since \( L_T \) is an orbit under this action, we obtain the contradiction \( \#L_T = \theta_{t-1} \leq p \).

Next, assume that \((c_{ij})\) has no eigenvalue in \( \mathbb{F}_q^4 \), whence it has two distinct eigenvalues \( z \neq \overline{z} \) in \( \mathbb{F}_{q^2} \supset \mathbb{F}_q^4 \). Here \( \overline{z} \) denotes the unique non-trivial automorphism in \( \text{Gal} (\mathbb{F}_{q^2}/\mathbb{F}_q^4) \). So, over \( \mathbb{F}_{q^2} \), the matrix \((c_{ij})\) is similar to \( \text{diag}(z, \overline{z}) \). Let \( w \) be a generator of the multiplicative group \( \mathbb{F}_{q^2}^* \).

We consider the matrix group \( \{ \text{diag}(w^s, \overline{w}^s) \mid s = 1, \ldots, q^2t - 1 \} \) of order \( q^2t - 1 \). It acts as a group \( \Omega \) of projectivities on PG(1, \( q^2t \)). The kernel of this action comprises all matrices \( \text{diag}(w^s, \overline{w}^s) \) with \( w^s = \overline{w}^s = \overline{w}^s \) or, equivalently, with \( w^s \in \mathbb{F}_q^* \). Hence we have
\[
\#\Omega = \frac{q^{2t} - 1}{q^2 - 1} = q^t + 1.
\]
The powers of \( \text{diag}(z, z) \) constitute a matrix group and give rise to a subgroup of \( \Omega \). This subgroup is isomorphic to \( \Lambda' \), and therefore its order is \( \theta_{t-1} \). This gives that \( \theta_{t-1} \) divides \( q^t + 1 \), which is impossible due to
\[
(q - 1) \theta_{t-1} = q^t - 1 < q^t + 1 < q^t + q^{t-1} + \cdots + q = q \theta_{t-1}.
\]

By the above, \( (c_{ij}) \) has two distinct eigenvalues \( a, b \) in \( \mathbb{F}_q^* \) and so there is a matrix \( (m_{ij}) \in \text{GL}_2(q^t) \) such that
\[
\text{diag}(a, b) = (m_{ij})^{-1} \cdot (c_{ij}) \cdot (m_{ij}).
\]

Let \( (\mu_{ij}) = (\rho_{m_{ij}}) \in \text{GL}_2(E) \) and let \( \mu \in \text{PGL}_2(E) \) be the projectivity given by \( (\mu_{ij}) \). It fixes the \( F \)-chain \( \text{PG}(1, F) \) as a set. We therefore obtain that \( T^\mu \) is a scattered point and the equality \( L^\mu_{T} = L_{T^\mu} \) (cf. Prop. 2.12). The group
\[
H'' := (\mu_{ij})^{-1} \cdot H' \cdot (\mu_{ij}) = \{ \text{diag}(\rho_a, \rho_b)^s \mid s = 1, 2, \ldots, q^t - 1 \}
\]
induces the cyclic group \( \Lambda'' := \mu^{-1} \Lambda' \mu \) of projectivities, which acts regularly on \( L^\mu_{T} \) and fixes the points \( E(1, 0) \) and \( E(0, 1) \). So \( L^\mu_{T} \) contains none of these points, whence we have
\[
T^\mu = E(1, \delta) \text{ with } \delta \in E^*.
\]

The matrix group generated by \( \text{diag}(1, \rho_{b/a}) \) also induces the group \( \Lambda'' \). So the order of \( b/a \) in the multiplicative group \( \mathbb{F}_q^* \) is \( \theta_{t-1} \). There is an element \( e \in \mathbb{F}_q^* \) such that
\[
b/a = e^{q^t-1}.
\]

This allows us to give another group of matrices that induces \( \Lambda'' \), namely the group generated by
\[
\rho_e \text{ diag}(1, \rho_{b/a}) = \text{diag}(\rho_e, \rho_{e^s}), \quad (13)
\]
where \( \sigma \in \text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q) \) is given by \( x \mapsto x^q \).

Finally, let \( d := 1^\delta \). Then \( E(1, \rho_d) \in L^\mu_T \) and, by writing the action of \( \Lambda'' \) in terms of the powers of the matrix from (13), we obtain
\[
L^\mu_{T} = \{ E(1, \rho_u) \text{ diag}(\rho_u, \rho_{u^s}) = E(\rho_u, \rho_{u^s d}) \mid u = e^s, \ s = 1, 2, \ldots, \theta_{t-1} \}.
\]

Thus \( L^\mu_{T} \) is contained in \( L_W \), where
\[
W := E(1, \sigma d).
\]

This implies, due to \( \#L^\mu_{T} = \#L_W \), that \( T_T \) is projectively equivalent to \( L_W \), which is of pseudoregulus type. \[\square\]
It should be noted that in Step 4 in the proof above the choice of a generator \( h \) of the multiplicative group \( \mathbb{F}_{q^t}^* \) serves the sole purpose that the projectivity related to the matrix \( (\rho_{c_{ij}}) \) is a generator of the group \( \Lambda' \) acting regularly on \( L_T \). Bearing this in mind, the following proposition can be extrapolated:

**Proposition 3.6.** Let \( L \) be a scattered linear set of rank \( t \) in \( \text{PG}(1,q^t) \). If there exists a cyclic subgroup of \( \text{PGL}_2(q^t) \) acting regularly on \( L \), then \( L \) is of pseudoregulus type.

On the other hand, any linear set of pseudoregulus type is projectively equivalent to \( \{(1,u^{q-1})_{q^t} \mid u \in \mathbb{F}_{q^t}^* \} \) \cite{6}, \cite{17}. Hence the converse of Prop. 3.6 can be proved directly.

**References**

1. A. Blunck: Regular spreads and chain geometries. Bull. Belg. Math. Soc. 6 (1999), 589–603.
2. A. Blunck - H. Havlicek: Extending the concept of chain geometry, Geom. Dedicata, 83 (2000), 119–130.
3. A. Blunck - A. Herzer: Kettengeometrien – Eine Einführung, Shaker Verlag, Aachen, 2005.
4. B. Csajbók - C. Zanella: On the equivalence of linear sets. Des. Codes Cryptogr., DOI:10.1007/s10623-015-0141-z.
5. B. Csajbók - C. Zanella: On scattered linear sets of pseudoregulus type in PG(1,q^t). arXiv:1506.08875v1
6. G. Donati - N. Durante: Scattered linear sets generated by collineations between pencils of lines, J. Algebraic Combin. 40 (2014), 1121–1134.
7. R.H. Dye: Spreads and classes of maximal subgroups of GL_n(q), SL_n(q), PGL_n(q) and PSL_n(q). Ann. Mat. Pura Appl. (4) 158 (1991), 33–50.
8. H. Havlicek: Divisible designs, Laguerre geometry, and beyond, J. Math. Sci. (N.Y.) 186 (2012), 882–926.
9. A. Herzer: Chain geometries, In: F. Buekenhout (ed.), Handbook of Incidence Geometry, Elsevier, Amsterdam, 1995, 781–842.
10. N. Knarr: Translation Planes, Lecture Notes in Mathematics, 1611, Springer, Berlin, 1995.
[11] S. Lang: Algebra, 3rd ed., Addison-Wesley, Reading, MA, 1993.
[12] M. Lavrauw - J. Sheekey - C. Zanella: On embeddings of minimum dimension of PG(n,q) × PG(n,q), Des. Codes Cryptogr. 74 (2015), 427–440.
[13] M. Lavrauw - G. Van de Voorde: On linear sets on a projective line. Des. Codes Cryptogr. 56 (2010), 89–104.
[14] M. Lavrauw - G. Van de Voorde: Field reduction and linear sets in finite geometry, Topics in finite fields, 271–293, Contemp. Math., 632, Amer. Math. Soc., Providence, RI, 2015.
[15] M. Lavrauw - C. Zanella: Subgeometries and linear sets on a projective line. Finite Fields Appl. 34 (2015), 95–106.
[16] M. Lavrauw - C. Zanella: Subspaces intersecting each element of a regulus in one point, André-Bruck-Bose representation and clubs. Electron. J. Combin. 23 (2016), Paper 1.37, 11 pp.
[17] G. Lunardon - G. Marino - O. Polverino - R. Trombetti: Maximum scattered linear sets of pseudoregulus type and the Segre variety S_{n,n}. J. Algebraic Combin. 39 (2014), 807–831.
[18] G. Lunardon - O. Polverino: Blocking sets and derivable partial spreads. J. Algebraic Combin. 14 (2001), 49–56.
[19] O. Polverino: Linear sets in finite projective spaces, Discrete Math. 310, 3096–3107 (2010).
[20] Z.-X. Wan: Geometry of Matrices, World Scientific, Singapore, 1996.

Authors’ addresses:

Hans Havlicek
Institut für Diskrete Mathematik und Geometrie,
Technische Universität Wien,
Wiedner Hauptstraße 8–10,
A-1040 Wien,
Austria

Corrado Zanella
Dipartimento di Tecnica e Gestione dei Sistemi Industriali,
Università di Padova,
Stradella S. Nicola, 3,
I-36100 Vicenza,
Italy