ON CLASSICAL SOLUTIONS OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH VACUUM

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ABSTRACT. In this paper, we consider the 3-D compressible isentropic MHD equations with infinity electric conductivity. The existence of unique local classical solutions is established when the initial data is arbitrarily large, contains vacuum and satisfies some initial layer compatibility condition. The initial mass density needs not be bounded away from zero and may vanish in some open set or decay at infinity. Moreover, we prove that the \( L^\infty \) norm of the deformation tensor of velocity gradients controls the possible blow-up (see [16, 22]) for classical (or strong) solutions, which means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of the deformation tensor as the critical time approaches. Our criterion (see (1.12)) is the same as Ponce’s criterion for 3-D incompressible Euler equations [15] and Huang-Li-Xin’s criterion for the 3-D compressible Navier-stokes equations [9].

1. Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. In 3-D space, the compressible isentropic magnetohydrodynamic equations in a domain \( \Omega \) of \( \mathbb{R}^3 \) can be written as

\[
\begin{align*}
H_t - \text{rot}(u \times H) &= -\text{rot}\left(\frac{1}{\sigma}\text{rot}H\right), \\
\text{div}H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}T + \mu_0\text{rot}H \times H.
\end{align*}
\]

In this system, \( x \in \Omega \) is the spatial coordinate; \( t \geq 0 \) is the time; \( H = (H^{(1)}, H^{(2)}, H^{(3)}) \) is the magnetic field; \( 0 < \sigma \leq \infty \) is the electric conductivity coefficient; \( \rho \) is the mass...
density; \( u = (u^{(1)}, u^{(2)}, u^{(3)}) \in \Omega \) is the velocity of fluids; \( P \) is the pressure law satisfying
\[
P = A \rho^\gamma, \quad \gamma > 1,
\] (1.2)
where \( A \) is a positive constant and \( \gamma \) is the adiabatic index; \( T \) is the stress tensor given by
\[
T = 2\mu D(u) + \lambda \text{div}\!u \mathbb{I}_3, \quad D(u) = \frac{\nabla u + (\nabla u)^\top}{2},
\] (1.3)
where \( D(u) \) is the deformation tensor, \( \mathbb{I}_3 \) is the \( 3 \times 3 \) unit matrix, \( \mu \) is the shear viscosity coefficient, \( \lambda \) is the bulk viscosity coefficient, \( \mu \) and \( \lambda \) are both real constants,
\[
\mu > 0, \quad \lambda + \frac{2}{3} \mu \geq 0,
\] (1.4)
which ensures the ellipticity of the Lamé operator. Although the electric field \( E \) doesn’t appear in system (1.1), it is indeed induced according to a relation \( E = -\mu_0 u \times H \) by moving the conductive flow in the magnetic field.

However, in this paper, when \( \sigma = +\infty \), system (1.1) can be written into
\[
\begin{align*}
H_t - \text{rot}(u \times H) &= 0, \\
\text{div}H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}T + \mu_0 \text{rot}H \times H
\end{align*}
\] (1.5)
with initial-boundary conditions
\[
(H, \rho, u)|_{t=0} = (H_0(x), \rho_0(x), u_0(x)), \quad x \in \Omega, \quad u|_{\partial \Omega} = 0,
\] (1.6)
\[
(H(t, x), \rho(t, x), u(t, x), P(t, x)) \to (0, \overline{\rho}, 0, \overline{P}) \quad \text{as} \quad |x| \to \infty, \quad t > 0,
\] (1.7)
where \( \overline{\rho} \geq 0 \) and \( \overline{P} = A \rho^\gamma \) are both constants, and \( \Omega \) can be a bounded domain in \( \mathbb{R}^3 \) with smooth boundary or the whole space \( \mathbb{R}^3 \). We have to point out that, if \( \Omega \) is a bounded domain (or \( \mathbb{R}^3 \)), then the condition (1.7) at infinity (or the boundary condition in (1.6) respectively) should be neglected.

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:
\[
D^{k,r} = \{ f \in L^1_{\text{loc}}(\Omega) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty \}, \quad D^k = D^{k,2},
\]
\[
D_1^0 = \{ f \in L^6(\Omega) : |f|_{D^1} = |\nabla f|_{L^2} < \infty \text{ and } f|_{\partial \Omega} = 0 \}, \quad \|(f, g)|_X = \|f\|_X + \|g\|_X,
\]
\[
\|f\|_{s} = \|f\|_{H^s(\Omega)}, \quad |f|_p = \|f\|_{L^p(\Omega)}, \quad |f|_{D^k} = \|f\|_{D^k(\Omega)}; \quad A : \mathbb{B} = (a_{ij}b_{ij})_{3 \times 3},
\]
\[
f \cdot \nabla g = \sum_{i=1}^{3} f_i \partial_i g, \quad f \cdot (\nabla g) = (\sum_{i=1}^{3} f_i \partial_i g_1, \sum_{i=1}^{3} f_i \partial_i g_2, \sum_{i=1}^{3} f_i \partial_i g_3)^\top,
\]
where \( f = (f_1, f_2, f_3)^\top \in \mathbb{R}^3 \) or \( f \in \mathbb{R} \), \( g = (g_1, g_2, g_3)^\top \in \mathbb{R}^3 \) or \( g \in \mathbb{R} \), \( X \) is some Sobolev space, \( A = (a_{ij})_{3 \times 3} \) and \( \mathbb{B} = (b_{ij})_{3 \times 3} \) are both \( 3 \times 3 \) matrixes. A detailed study of homogeneous Sobolev space may be found in [6].

As has been observed in [5], which proved the existence of unique local strong solution with initial vacuum, in order to make sure that the Cauchy problem or IBVP (1.5)-(1.7)
with vacuum is well-posed, the lack of a positive lower bound of the initial mass density \( \rho_0 \) should be compensated with some initial layer compatibility condition on the initial data \((H_0, \rho_0, u_0, P_0)\). For classical solution, it can be shown as

**Theorem 1.1.** Let constant \( q \in (3, 6] \). If the initial data \( (H_0, \rho_0, u_0, P_0) \) satisfies

\[
(H_0, \rho_0 - \mathbf{p}, P_0 - \mathbf{p}) \in H^2 \cap W^{2,q}, \rho_0 \geq 0, u_0 \in D_0^1 \cap D^2,
\]

and the compatibility condition

\[
Lu_0 + \nabla P_0 - \text{rot} H_0 \times H_0 = \sqrt{\rho_0} g_1
\]

for some \( g_1 \in L^2 \), where \( L \) is the Lamé operator defined via

\[
Lu = -\mu \Delta u - (\lambda + \mu) \nabla \text{div} u.
\]

Then there exists a small time \( T_0 \) and a unique solution \( (H, \rho, u, P) \) to IBVP \( (1.5)-(1.7) \) satisfying

\[
(H, \rho - \mathbf{p}, P - \mathbf{p}) \in C([0, T_0]; H^2 \cap W^{2,q}),
\]

\[
u \in C([0, T_0]; D_0^1 \cap D^2) \cap L^2([0, T_0]; D^3) \cap L^{p_0}([0, T_0]; D^{3,q}), u_t \in L^2([0, T_0]; D_0^1),
\]

\[
\sqrt{\rho u_t} \in L^\infty([0, T_0]; L^2), \ t^\frac{1}{5} u \in L^\infty([0, T_0]; D^3), \ t^\frac{1}{7} \sqrt{\rho u_{tt}} \in L^2([0, T_0]; L^2),
\]

\[
t^\frac{4}{3} u_t \in L^\infty([0, T_0]; D_0^1) \cap L^2([0, T_0]; D^3), \ tu \in L^\infty([0, T_0]; D^{3,q}),
\]

\[
tu_t \in L^\infty([0, T_0]; D^2), \ tu_{tt} \in L^2([0, T_0]; D_0^1), \ t^\sqrt{\rho u_{tt}} \in L^\infty([0, T_0]; L^2),
\]

where \( p_0 \) is a constant satisfying \( 1 \leq p_0 \leq \frac{4q}{3q-6} \in (1, 2) \).

**Remark 1.1.** The solution we obtained in Theorem 1.1 becomes a classical one for positive time. Some similar results have been obtained in \([5, 12]\), which give the local existence of strong solutions. So, the main purpose of this theorem is to give a better regularity for the solutions obtained in \([5, 12]\) when the initial mass density is nonnegative.

Though the smooth global solution near the constant state in one-dimensional case has been studied in \([10]\), however, in 3-D space, the non-global existence has been proved for the classical solution to isentropic magnetohydrodynamic equations in \([16]\) as follows:

**Theorem 1.2.** \([16]\) Assume that \( \gamma \geq \frac{6}{5} \), if the momentum \( \int_\Omega \rho ud\mathbf{x} \neq 0 \), then there exists no global classical solution to \( (1.5)-(1.7) \) with conserved mass and total energy.

So, naturally, we want to understand the mechanism of blow-up and the structure of possible singularities: what kinds of singularities will form in finite time and what is the main mechanism of possible breakdown of smooth solutions for the 3-D compressible MHD equations? Therefore, it is an interesting question to ask whether the same blow-up criterion in terms of \( D(u) \) in \([9, 15]\) still holds for the compressible MHD equations or not. However, the similar result has been obtained in Xu-Zhang \([24]\) for strong solutions obtained in \([5]\), which is in terms of \( \nabla u \):

\[
\lim \sup_{T \to T_1} \|\nabla u\|_{L^1([0, T]; L^\infty(\Omega))} = \infty.
\]

Based on a subtle estimate for the magnetic field, our main result in this paper answered this question for classical (or strong) solutions positively, which can be shown as
Theorem 1.3 (Blow-up criterion for the IBVP (1.5)–(1.7)).

Assume that $\Omega$ is a bounded domain and the initial data $(H_0, \rho_0, u_0, P_0)$ satisfies (1.8)-(1.9). Let $(H, \rho, u, P)$ is a classical solution to IBVP for (1.5)–(1.7). If $0 < T < \infty$ is the maximal time of existence, then

$$\limsup_{T \to T_0} |D(u)|_{L^1([0,T];L^\infty(\Omega))} = \infty.$$  

(1.12)

Moreover, our blow-up criterion also holds for the strong solutions obtained in [5].

Remark 1.2. When $H \equiv 0$ in 3-D space, the existence of unique local strong solution with vacuum has been solved by many papers, and we refer the readers to [2][3][4]. Huang-Li-Xin obtained the well-posedness of global classical solutions with small energy but possibly large oscillations and vacuum for Cauchy problem [7] or IBVP [8].

However, for compressible non-isentropic Navier-Stokes equations, the finite time blow-up has been proved in Olga [17] for classical solutions $(\rho, u, S)$ ($S$ is the entropy) with highly decreasing at infinity for the compressible non-isentropic Navier-stokes equations, but the local existence for the corresponding smooth solution is still open.

Recently, Xin-Yan [23] showed that if the initial vacuum only appears in some local domain, the smooth solution $(\rho, \theta, u)$ to the Cauchy problem (1.5)–(1.7) will blow-up in finite time regardless of the size of initial data, which has removed the key assumption that the vacuum must appear in the far field in [22].

Sun-Wang-Zhang [20][21] established a Beal-Kato-Majda blow-up criterion in terms of the upper bound of density for the strong solution with vacuum in 3-D or 2-D space, which is weaker than the blow-up criterions obtained in [9][15]. Then our result can not replace $\int_0^T |D(u)|_{\infty} dt$ by $|\rho|_{\infty}$ because of the coupling of $u$ and $H$ in magnetic equation and the lack of smooth mechanism of $H$.

Moreover, results presented above are essentially dependent of the strong ellipticity of Lamé operator. Compared with Euler equations [14], the velocity $u$ of fluids satisfies $Lu_0 = 0$ in the vacuum domain naturally due to the constant viscosity coefficients which makes sure that $u$ is well defined in the vacuum points without other assumptions as [14].

Recently, Li-Pan-Zhu [11] proved the local existence of regular solutions for the 2-D Shallow water equations with $\mathbb{T} = \rho \nabla u$ when initial mass density decays to zero, and the corresponding Beal-Kato-Majda blow-up criterion is also obtained.

The rest of this paper is organized as follows. In Section 2, we give some important lemmas which will be used frequently in our proof. In Section 3, via establishing a priori estimate (for the approximation solutions) which is independent of the lower bound of the initial mass density $\rho_0$, we can obtain the existence of unique local classical solution by the approximation process from non-vacuum to vacuum. In Section 4, we give the proof for the blow-up criterion (1.12) for the classical solutions obtained in Section 3. Firstly in Section 4.1, via assuming that the opposite of (1.12) holds, we show that the solution in $[0, \mathbb{T}]$ has the regularity that the strong solution has to satisfy obtained in [5]. Then secondly in Section 4.2, based on the estimates shown in Section 4.1, we improve the regularity of $(H, \rho, u, P)$ to make sure that it is also a classical one in $[0, \mathbb{T}]$, which contradicts our assumption.
2. Preliminary

Now we give some important Lemmas which will be used frequently in our proof.

**Lemma 2.1.** [13] Let constants $l$, $a$ and $b$ satisfy the relation $\frac{1}{l} = \frac{1}{a} + \frac{1}{b}$ and $1 \leq a$, $b$, $l \leq \infty$. \( \forall s \geq 1 \), if $f, g \in W^{s,a} \cap W^{s,b}(\Omega)$, then we have

$$
|D^l(fg) - fD^l g|_1 \leq C_s(|\nabla f|_a |D^{s-1} g|_b + |D^l f|_b |g|_a),
$$

(2.1)

$$
|D^l(fg) - fD^l g|_1 \leq C_s(|\nabla f|_a |D^{s-1} g|_b + |D^l f|_a |g|_b),
$$

(2.2)

where $C_s > 0$ is a constant only depending on $s$.

The proof can be seen in Majda [13], here we omit it. The following one is some Sobolev inequalities obtained from the well-known Gagliardo-Nirenberg inequality:

**Lemma 2.2.** For $n \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on $n$ such that for $f \in D^1_0(\Omega)$, $g \in D^1_0 \cap D^2(\Omega)$ and $h \in W^{1,n}(\Omega)$, we have

$$
|f|_6 \leq C|f|_{D^1_0}, \quad |g|_\infty \leq C|g|_{D^1_0 \cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{1,n}}.
$$

(2.3)

The next lemma is important in the derivation of our local a priori estimate for the higher order term of $u$, which can be seen in the Remark 1 of [1].

**Lemma 2.3.** If $h(t, x) \in L^2(0, T; L^2)$, then there exists a sequence $s_k$ such that $s_k \rightarrow 0$, and $s_k |h(s_k, x)|^2_2 \rightarrow 0$, as $k \rightarrow \infty$.

Based on Harmonic analysis, we introduce a regularity estimate result for Lamé operator

$$
-\mu \Delta u - (\mu + \lambda)\nabla \text{div} u = Lu = F \quad \text{in} \quad \Omega, \quad u \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.
$$

(2.4)

We define $u \in D^{1,q}_0(\Omega)$ means that $u \in D^{1,q}(\Omega)$ with $u|_{\partial \Omega} = 0$.

**Lemma 2.4.** [19] Let $u \in D^{1,l}_0$ with $1 < l < \infty$ be a weak solution to system (2.4), if $\Omega = \mathbb{R}^3$, we have

$$
|u|_{D^{k+2,l}(\mathbb{R}^3)} \leq C|F|_{D^{k+2,l}(\mathbb{R}^3)};
$$

if $\Omega$ is a bounded domain with smooth boundary, we have

$$
|u|_{D^{k+2,l}(\Omega)} \leq C\left(|F|_{D^{k+2,l}(\Omega)} + |u|_{D^{1,l}_0(\Omega)}\right),
$$

where the constant $C$ depending only on $\mu$, $\lambda$ and $l$.

**Proof.** The proof can be obtained via the classical estimates from Harmonic analysis, which can be seen in [2] [19] or [20].

We also show some results obtained via the Aubin-Lions Lemma.

**Lemma 2.5.** [18] Let $X_0$, $X$ and $X_1$ be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that $X_0$ is compactly embedded in $X$ and that $X$ is continuously embedded in $X_1$.

1) Let $G$ be bounded in $L^p(0, T; X_0)$ where $1 \leq p < \infty$, and $\frac{\partial G}{\partial t}$ be bounded in $L^1(0, T; X_1)$. Then $G$ is relatively compact in $L^p(0, T; X)$.

2) Let $F$ be bounded in $L^\infty(0, T; X_0)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^l(0, T; X_1)$ with $l > 1$. Then $F$ is relatively compact in $C(0, T; X)$. 

Finally, for \((H, u) \in C^2(\Omega)\), there are some formulas based on \(\text{div} H = 0\):
\[
\begin{align*}
\text{rot}(u \times H) &= (H \cdot \nabla)u - (u \cdot \nabla)H - H \text{div} u, \\
\text{rot}H \times H &= \text{div}(H \otimes H - \frac{1}{2}|H|^2 I_3) = -\frac{1}{2} \nabla |H|^2 + H \cdot \nabla H.
\end{align*}
\tag{2.5}
\]

3. Well-posedness of classical solutions

In order to prove the local existence of classical solutions to the original nonlinear problem, we need to consider the following linearized problem:
\[
\begin{align*}
H_t + v \cdot \nabla H + (\text{div} I_3 - \nabla v) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\text{div} H &= 0 \quad \text{in } (0, T) \times \Omega, \\
\rho_t + \text{div}(\rho v) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\rho u_t + \rho v \cdot \nabla v + \nabla P + Lu &= \mu_0 \text{rot} H \times H \quad \text{in } (0, T) \times \Omega, \\
(H, \rho, u)\big|_{t=0} &= (H_0(x), \rho_0(x), u_0(x)) \quad \text{in } \Omega,
\end{align*}
\]
\[
(H, \rho, u, P) \to (0, \overline{\rho}, 0, \overline{P}) \quad \text{as } |x| \to \infty, \quad t > 0,
\]
where \((H_0(x), \rho_0(x), u_0(x))\) satisfies (1.8)-(1.9) and \(v(t, x) \in \mathbb{R}^3\) is a known vector \(v \in C([0, T]; D_0^1 \cap D^2) \cap L^2([0, T]; D^3) \cap L^{p_0}([0, T]; D^{3,q}),\) \(v_t \in L^2([0, T]; D_0^1),\) \(t^\frac{1}{2}v \in L^\infty([0, T]; D^3),\) \(t^\frac{1}{2}v_t \in L^\infty([0, T]; D_0^1) \cap L^2([0, T]; D^2),\) \(tv \in L^\infty([0, T]; D^{3,q}),\) \(tv_t \in L^\infty([0, T]; D^2),\) \(tv_{tt} \in L^2([0, T]; D_0^1),\) \(v(0, x) = u_0.\)

3.1. Unique solvability of (3.1) away from vacuum

First we give the following existence of classical solution \((H, \rho, u)\) to (3.1) by the standard methods at least for the case that the initial mass density is away from vacuum.

**Lemma 3.1.** Assume in addition to (1.8)-(1.9) that \(\rho_0 \geq \delta\) for some constant \(\delta > 0.\) Then there exists a unique classical solution \((H, \rho, u)\) to (3.1) such that
\[
(H, \rho - \overline{\rho}, P - \overline{P}) \in C([0, T]; H^2 \cap W^{2,q}), \quad (H_t, \rho_t, P_t) \in C([0, T]; H^1),
\]
\[
t^\frac{1}{2}(H_t, \rho_t, P_t) \in L^\infty([0, T]; D^{1,q}), \quad u \in C([0, T]; H^2) \cap L^2([0, T]; D^3) \cap L^{p_0}([0, T]; D^{3,q}),
\]
\[
u_t \in L^2([0, T]; D_0^1) \cap L^\infty([0, T]; L^2), \quad t^\frac{1}{2}u_t \in L^\infty([0, T]; D^3),
\]
\[
t^\frac{1}{2}u_t \in L^\infty([0, T]; D_0^1), \quad t^\frac{1}{2}u_{tt} \in L^2([0, T]; L^2), \quad tu \in L^\infty([0, T]; D^{3,q}),
\]
\[
tu_t \in L^\infty([0, T]; L^2), \quad tu_{tt} \in L^2([0, T]; D_0^1) \cap L^\infty([0, T]; L^2), \quad tu_{ttt} \in L^\infty([0, T]; H^{-1}),
\]
and \(\rho \geq \overline{\delta}\) on \([0, T] \times \mathbb{R}^3\) for some positive constant \(\overline{\delta}\).

**Proof.** Firstly, we observe the magnetic equations (3.1), it has the form
\[
H_t + \sum_{j=1}^3 A_j \partial_j H + BH = 0,
\tag{3.3}
\]
where $A_j = v_j I_3$ ($j = 1, 2, 3$) are symmetric and $B = \text{div}v I_3 - \nabla v$. According to the regularity of $v$ and the standard theory for positive and symmetric hyperbolic system, we easily have the desired conclusions.

Secondly, the existence and regularity of a unique solution $\rho$ to \((3.1)_{3}\) can be obtained essentially according to Lemma 1 in \[4\]. Due to pressure $P$ satisfies the following problem

$$P_t + v \cdot \nabla P + \gamma P \text{div}v = 0, \quad P_0 - \overline{P} \in H^2 \cap W^{2,q},$$

so we easily have the same conclusions for $P$ via the similar argument as $\rho$.

Finally, the momentum equations \((3.1)_{4}\) can be written into

$$\rho u_t + Lu = -\nabla P - \rho v \cdot \nabla v + \mu_0 \text{rot}H \times H,$$

then from Lemma 3 in \[4\], the desired conclusions is easily obtained. \hfill \Box

### 3.2. A priori estimate to the linearized problem away from vacuum.

Now we want to get some a priori estimate for the classical solution $(H, \rho, u)$ to \((3.1)_{3}\) obtained in Lemma \[3.1\] which is independent of the lower bound of the initial mass density $\rho_0$. For simplicity, we first fix a positive constant $c_0$ sufficiently large that

$$2 + \overline{r} + \|(\rho_0 - \overline{r}, P_0 - \overline{P}, H_0)\|_{H^2 \cap W^{2,q}} + |u_0|_{D_0^1 \cap D^2} + |g_1|_2 \leq c_0,$$

and

$$\sup_{0 \leq t \leq T^*} \frac{|v(t)|_{D_0^1 \cap D^2}^2}{t} + \int_0^{T^*} \left(|\gamma|_{D^3}^2 + |v|_{D^3, q}^2 + |v_t|_{D_0^1}^2\right) dt \leq c_1,$$

$$\text{ess sup}_{0 \leq t \leq T^*} \left(t|v_t(t)|_{D_0^1}^2 + t|v(t)|_{D^3}^2 + \int_0^{T^*} t|v_t|_{D^2}^2 dt \right) \leq c_2,$$

$$\text{ess sup}_{0 \leq t \leq T^*} \left(t^2|v(t)|_{D^3, q}^2 + t^2|v_t(t)|_{D_0^1}^2 + \int_0^{T^*} t^2|v_t|_{D_0^1}^2 dt \right) \leq c_3$$

for some time $T^* \in (0, T)$ and constants $c_j$'s with $1 < c_0 \leq c_1 \leq c_2 \leq c_3$. Throughout this and next two sections, we denote by $C$ a generic positive constant depending only on fixed constants $\mu$, $\mu_0$, $T$ and $\lambda$.

Now we give some estimates for the magnetic field $H$.

**Lemma 3.2 (Estimates for magnetic field $H$).**

$$\|H(t)\|_{H^2 \cap W^{2,q}}^2 + \|H_t(t)\|_{D^1}^2 \leq Cc_1^4, \quad \int_0^t \|H_{tt}\|^2_{D^1} ds \leq Cc_1^3, \quad t|H_t(t)|_{D^1, q} \leq Cc_2^3 (3.8)$$

for $0 \leq t \leq T_1 = \min(T^*, (1 + c_1)^{-1})$.

**Proof.** Firstly, let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ($|\alpha| \leq 2$) and $\alpha_i = 0, 1, 2$, differentiating \((3.1)_{1}\) $\alpha$ times with respect to $x$, we have

$$D^\alpha H_t + \sum_{j=1}^3 A_j \partial_j D^\alpha H + BD^\alpha H$$

$$= (D^\alpha (BH) - BD^\alpha H) + \sum_{j=1}^3 (D^\alpha A_j \partial_j H - A_j \partial_j D^\alpha H) = \Theta_1 + \Theta_2.$$
Then multiplying (3.9) by \( rD^a H |D^a H|^{r-2} \) \((r \in [2, q])\) and integrating over \( \Omega \), we have
\[
\frac{d}{dt}|D^a H|_r \leq \left( \sum_{j=1}^{3} |\partial_{x_j} A_j|_\infty + |B|_\infty \right) |D^a H|_r + |\Theta_1|_r |D^a H|_r^{-1} + |\Theta_2|_r |D^a H|_r^{-1}. \tag{3.10}
\]

Secondly, let \( l = r = a, b = \infty \) and \( s = |\alpha| = 1 \) in (2.2) of Lemma 2.1 we easily have
\[
|\Theta_1|_r = |D^a (BH) - BD^a H|_r \leq C|\nabla^2 v|_r |H|_\infty \leq C|\nabla^2 v|_r |H|_2; \tag{3.11}
\]
let \( l = r = a, b = \infty \) and \( s = |\alpha| = 2 \) in (2.2) of Lemma 2.1 we have
\[
|\Theta_1|_r = |D^a (BH) - BD^a H|_r \leq C(\nabla^2 v|_r \nabla H|_\infty + |\nabla^3 v|_r |H|_\infty) \\
\leq C||\nabla^2 v||H^1_{\cap W^{1,q}} ||H||H^2_{\cap W^{2,q}}. \tag{3.12}
\]

And similarly, let \( b = \infty, l = r = a \) and \( s = |\alpha| = 1 \) in (2.2) of Lemma 2.1 we have
\[
|D^a (A_j \partial_j H) - A_j \partial_j D^a|_r \leq C(|\nabla v|_\infty |\nabla^2 H|_r + |\nabla^2 v|_r |\nabla H|_\infty) \\
\leq C|\nabla v|_2 ||H||H^2_{\cap W^{2,q}}. \tag{3.13}
\]

Then combining (3.10) to (3.14), according to Gronwall’s inequality, we have
\[
||H||H^2_{\cap W^{2,q}} \leq C||H_0||H^2_{\cap W^{2,q}} \exp \left( \int_0^t ||\nabla v(s)||^2_{H^2_{\cap W^{2,q}}} ds \right) \leq C_0. \tag{3.15}
\]
for \( 0 \leq t \leq T_1 \), where we have used the fact
\[
\int_0^t ||v(s)||_{D^a H} ds \leq t^{\frac{1}{p_0}} \left( \int_0^t ||v(s)||^2_{D^a H} ds \right)^{\frac{1}{p_0}} \leq C_1, \quad \text{and}
\]
\[
\int_0^t ||\nabla v(s)||_{2} ds \leq t^{\frac{1}{q}} \left( \int_0^t ||\nabla v(s)||^2_{2} ds \right)^{\frac{1}{q}} \leq C(c_1 t + (c_1 t)^{\frac{1}{2}}) \leq C_1, \tag{3.16}
\]
and \( \frac{1}{p_0} + \frac{1}{q_0} = 1 \). Finally, from the magnetic field equations (3.1): \( H_t = -v \cdot \nabla H - (\text{div} I_3 - \nabla v) H \)
we quickly get the desired estimates for \( H_t \) and \( H_{tt} \). \( \square \)

Next we give the estimates for the mass density \( \rho \) and pressure \( P \).

**Lemma 3.3 (Estimates for the mass density \( \rho \) and pressure \( P \)).**

\[
\|(\rho - \overline{\rho}, P - \overline{P})(t)\|_{H^2_{\cap W^{2,q}}} + \|(\rho_t, P_t)(t)\|_{H^1_{\cap L^q}} \leq C_1 \tag{3.17},
\]
\[
\int_0^t ||(\rho_t, P_t)||_{H^2_{\cap W^{2,q}}}^2 ds \leq C_1, \quad t ||(\rho_t, P_t)(t)||_{H^2_{\cap W^{2,q}}}^2 \leq C_2 \tag{3.18}
\]
for \( 0 \leq t \leq T_1 = \min(T, C(1 + c_1)^{-1}) \).
Proof. From (3.1)3 and the standard energy estimate shown in [3], for 2 \leq r \leq q, we have
\[
\|\rho(t) - \overline{\rho}\|_{W^{2,r}} \leq \left( \|\rho_0 - \overline{\rho}\|_{W^{2,r}} + \overline{\rho} \int_0^t \|\nabla v(s)\|_{W^{2,r}} \, ds \right) \exp \left( C \int_0^t \|\nabla v(s)\|_{H^{2,0}} \, ds \right).
\] (3.17)
Then from (3.16), the desired estimate for \(\|\rho(t)\|_{H^{2,0}}\) can be easily obtained via (3.17):
\[
\|\rho(t) - \overline{\rho}\|_{H^{2,0}} \leq C_{\rho_0}, \quad \text{for } 0 \leq t \leq T_1 = \min(T^*, (1 + c_1)^{-1}).
\] (3.18)
Secondly, the estimates for \((\rho_t, \rho u_t)\) follows immediately from the continuity equation
\[
\rho_t = -\rho \div v - v \cdot \nabla \rho.
\] (3.19)
Finally, due to pressure \(P\) satisfies (3.4), then the corresponding estimates for \(P\) can be obtained via the same method as \(\rho\).

Now we give the estimates for the lower order terms of the velocity \(u\).

Lemma 3.4 (Lower order estimate of the velocity \(u\)).
\[
|u(t)|_{D^{1} \cap D^2}^2 + |\sqrt{\rho} u_t(t)|_{D^3}^2 + \int_0^t (|u|_{D^3}^2 + |u_t|_{D^3}^2) \, ds \leq Cc_1^2
\] for \(0 \leq t \leq T_2 = \min(T^*, C(1 + c_1)^{-8})\).

Proof. Step 1: Multiplying (3.14) by \(u_t\) and integrating over \(\Omega\), we have
\[
\int_\Omega \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \left( \mu |\nabla u|^2 + (\lambda + \mu) (\div u)^2 \right) \, dx
= \int_\Omega \left( -\nabla P - \rho v \cdot \nabla v + (\rot H \times H) \right) \cdot u_t \, dx = \frac{d}{dt} \Lambda_1(t) - \Lambda_2(t),
\] (3.20)
where
\[
\Lambda_1(t) = \int_\Omega \left( (P - \overline{\rho}) \div u + (\rot H \times H) \cdot u \right) \, dx,
\]
\[
\Lambda_2(t) = \int_\Omega \left( P \div u + \rho (v \cdot \nabla v) \cdot u_t + (\rot H \times H_t) \cdot u \right) \, dx.
\]
According to Lemmas 3.2-3.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we easily deduce that
\[
\Lambda_1(t) \leq C(\|\nabla u\|_{L^2} |P - \overline{\rho}|_{L^2} + |\nabla H|_{L^2} |H|_{L^3} |\nabla u|_{L^2}) \leq \frac{\mu}{10} |\nabla u|_{L^2}^2 + Cc_1^8,
\]
\[
\Lambda_2(t) \leq C(\|\nabla u\|_{L^2} |P_t|_{L^2} + |\rho|^\frac{1}{50} |\sqrt{\rho} u_t|_{L^2} \|v\|_{L^\infty} |\nabla v|_{L^2} + \|H_t\|_{L^2} \|H\|_{L^1} |\nabla u|_{L^2})
\]
\[
\leq C |\nabla u|_{L^2}^2 + \frac{1}{10} |\sqrt{\rho} u_t|_{L^2}^2 + Cc_1^8
\]
for \(0 < t \leq T_1\). Then integrating (3.20) over \((0, t)\) with respect to \(t\), we have
\[
\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 \, ds + |\nabla u(t)|_{L^2}^2 \leq C \int_0^t |\nabla u(s)|_{L^2}^2 \, ds + Cc_1^8
\]
for $0 \leq t \leq T_1$, via Gronwall’s inequality, we have

$$\int_0^t |\sqrt{\rho u}(s)|^2 \text{d}s + \|\nabla u(t)\|^2 \leq Cc_1^8 \exp( Ct ) \leq Cc_1^8, \quad 0 \leq t \leq T_1. \tag{3.21}$$

Combining Lemmas 3.2-3.3 and Lemma 2.4 we easily have

$$\int_0^t |u|^2_{D_0} \text{d}s \leq C \int_0^t \left( |\rho u_t + \rho v \cdot \nabla v|^2 + |\nabla P|^2_2 + \|\text{rot} H \times H\|^2_2 + |u|^2_{D_0} \right) \text{d}s \leq Cc_1^{10}. \tag{3.22}$$

Step 2: Differentiating (3.23) with respect to $t$, we have

$$\rho u_{tt} + Lu_t = -\nabla P_t - \rho u_t - (\rho v \cdot \nabla v)_t + (\text{rot} H \times H)_t. \tag{3.23}$$

Multiplying (3.23) by $u_t$ and integrating (3.23) over $\Omega$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \text{d}x + \int_{\mathbb{R}^3} (\mu |\nabla u_t|^2 + (\lambda + \mu) (\text{div} u_t)^2) \text{d}x$$

$$= \int_{\Omega} \left( -\nabla P_t - (\rho v \cdot \nabla v)_t - \frac{1}{2} \rho u_t + (\text{rot} H \times H)_t \right) \cdot u_t \text{d}x \equiv: \sum_{i=1}^4 I_i. \tag{3.24}$$

According to Lemmas 3.2-3.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

$$I_1 = \int_{\Omega} \rho |u_t|^2 \text{d}x \leq C |P_t|_2 |\nabla u_t|_2 \leq \frac{\mu}{10} |\nabla u_t|^2 + Cc_1^4,$$

$$I_2 \leq C |\rho|^{\frac{3}{2}} |\nabla v|_2 |\sqrt{\rho} u_t|_2 + |\rho|^{\frac{1}{2}} |v|_\infty |\nabla v_t|_2 \leq C |\rho|^{\frac{1}{2}} |v|_D |\sqrt{\rho} u_t|_2 + C |\rho|_3 |v|_\infty |\nabla v_t|_2 \leq C |\sqrt{\rho} u_t|^2 + \frac{\mu}{10} |\nabla u_t|^2 + Cc_1^4 (1 + |\nabla v_t|^2_2),$$

$$I_3 = -\frac{1}{2} \int_{\Omega} \rho |u_t|^2 \text{d}x \leq \int_{\Omega} \rho u_t \cdot \nabla u_t \text{d}x \leq C |\rho|^{\frac{3}{2}} |v|_D |\sqrt{\rho} u_t|_3 |\nabla u_t|_2$$

$$\leq Cc_1^8 |\sqrt{\rho} u_t|^2 + \frac{\mu}{10} |\nabla u_t|^2,$$

$$I_4 = \int_{\Omega} \text{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) \cdot u_t \text{d}x = -\int_{\Omega} (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla u_t \text{d}x \leq C |\nabla u_t|_2 |H_t|_2 |H|_\infty \leq Cc_1^8 + \frac{\mu}{10} |\nabla u_t|^2.$$

Then combining the above estimate (3.25) and (3.21), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \text{d}x + \int_{\Omega} |\nabla u_t|^2 \text{d}x \leq Cc_1^8 |\sqrt{\rho} u_t|^2 + Cc_1^4 |\nabla v|^2_2 + Cc_1^8. \tag{3.26}$$

Integrating (3.26) over $(\tau, t) \ (\tau \in (0,t))$, for $\tau \leq t \leq T_1$, we have

$$|\sqrt{\rho} u_t(t)|_2^2 + \int_{\tau}^t |\nabla u_t|^2 \text{d}s \leq |\sqrt{\rho} u_t(\tau)|_2^2 + Cc_1^8 \int_{\tau}^t |\sqrt{\rho} u_t|^2 \text{d}s + Cc_1^8. \tag{3.27}$$

From the momentum equations (3.1), we easily have

$$|\sqrt{\rho} u_t(\tau)|_2^2 \leq C \int_{\Omega} \rho |v|^2 |\nabla v|^2 \text{d}x + C \int_{\Omega} \frac{|\nabla P + Lu - \text{rot} H \times H|^2}{\rho} \text{d}x, \tag{3.28}$$
due to the initial layer compatibility condition (1.9), letting $\tau \to 0$ in (3.27), we have

$$\limsup_{\tau \to 0} |\sqrt{\rho_0}u(t)\|_2^2 \leq C \int_0^\tau \rho_0 |u_0|^2 |\nabla u_0|^2 dx + C \int_0^\tau |g_1|^2 dx \leq C_0^4.$$  (3.29)

Then, letting $\tau \to 0$ in (3.27), we have

$$|\sqrt{\rho_0}u(t)\|_2^2 + \int_0^t |\nabla u_t|^2 ds \leq C C_1^8 + C C_2^8 \int_0^t |\sqrt{\rho_0}u_t\|_2^2 ds.$$  (3.30)

From Gronwall’s inequality, we deduce that

$$|\sqrt{\rho_0}u(t)\|_2^2 + \int_0^t |\nabla u_t|^2 ds \leq C C_1^8 \exp(C C_1^8 t) \leq C C_1^8, \quad 0 \leq t \leq T_2.$$  (3.31)

Finally, due to Lemmas 3.2, 3.3 and Lemma 2.4 for $0 \leq t \leq T_2$, we easily have

$$|u(t)|_{D^2} \leq (|\rho u(t) + \rho v \cdot \nabla v(t)|_2 + |\nabla P(t)|_2 + |\text{rot} H \times H(t)|_2 + |u(t)|_{D^1}) \leq C C_1^5,$$

$$\int_0^t |u_t|^2 ds \leq C \int_0^t \left( |\rho u_t + \rho v \cdot \nabla v|_{D^1}^2 + |\nabla P|_{D^1}^2 + |\text{rot} H \times H|_{D^1}^2 + |u_t|^2 \right) ds \leq C C_1^{12}.$$  

Now we will give some estimates for the higher order terms of the velocity $u$ in the following three Lemmas.

**Lemma 3.5 (Higher order estimate of the velocity $u$).**

$$t|u_t(t)|_{D^1}^2 + t|u(t)|_{D^3}^2 + \int_0^t s|u_t|^2 + |\sqrt{\rho_0}u_t|^2 |ds \leq C C_2^{24}, \quad 0 \leq t \leq T_2.$$  

**Proof.** Multiplying (3.23) by $u_{tt}$ and integrating over $\Omega$, we have

$$\int_\Omega \rho |u_{tt}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \left( \mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2 \right) dx$$

$$= \int_\Omega \left( - \nabla P_t - (\rho v \cdot \nabla v)_{tt} - \rho u_t + (\text{rot} H \times H)_t \right) \cdot u_{tt} dx = \frac{d}{dt} \Lambda_3(t) + \Lambda_4(t),$$

where

$$\Lambda_3(t) = \int_\Omega \left( P_t \text{div} u_t - \rho (v \cdot \nabla v) \cdot u_t - \frac{1}{2} \rho |u_t|^2 + (\text{rot} H \times H)_t \cdot u_t \right) dx,$$

$$\Lambda_4(t) = \int_\Omega \left( - P_t \text{div} u_t - \rho (v \cdot \nabla v)_t \cdot u_t + \rho u_t (v \cdot \nabla v) \cdot u_t + \rho (v \cdot \nabla v) \cdot u_t \right) dx$$

$$+ \int_\Omega \left( \frac{1}{2} \rho u_t |u_t|^2 - (\text{rot} H \times H)_t \cdot u_t \right) dx \equiv \sum_{i=5}^{10} I_i.$$  

Then almost same to (3.25), we also have

$$\Lambda_3(t) \leq \frac{\mu}{10} |\nabla u_t|^2 + C C_1^8 \sqrt{\rho_0}u_t|^2 + C C_1^8 \leq \frac{\mu}{10} |\nabla u|^2 + C C_1^{20}, \quad 0 \leq t \leq T_2.$$  (3.33)

Let we denote

$$\Lambda^*(t) = \frac{1}{2} \int_\Omega \mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2 dx - \Lambda_3(t),$$
then from (3.33), for $0 \leq t \leq T_2$, we quickly have

$$C|\nabla u_t|^2_2 - Cc_1^{20} \leq \Lambda^* (t) \leq C|\nabla u_t|^2_2 + Cc_1^{20}. \quad (3.34)$$

Similarly, from Holder’s inequality and Gagliardo-Nirenberg inequality, for $0 < t \leq T_2$, we deduce that

$$I_5 \leq C|P_t|_2 |\nabla u_t|_2, \quad I_6 \leq |\rho|^\frac{3}{2} |\sqrt{\rho u_t}|_2 (|v|_\infty |\nabla v_t|_2 + |\nabla v|_3 |\nabla v_t|_2),$$

$$I_7 \leq C|\rho u|_2 |\nabla u_t|_2 |\nabla v|_3 |v|_\infty, \quad I_8 \leq C|\rho|_2 |v_t|_6 |\nabla v_t|_2 + C|v|_\infty |v_t|_6 |\nabla u_t|_2 |\rho_t|_3,$$  

$$I_9 \leq C|\rho|_2 |\nabla u_t|_2 |v|_\infty |u_t|_6 + C|\rho|_\infty |\sqrt{\rho u_t}|_3 |v_t|_6 |\nabla u_t|_2,$$

where we have used the facts $\rho = -\text{div}(\rho v)$, and

$$I_{10} = - \int_\Omega (\text{rot} H \times H)_{tt} \cdot u_t \, dx = \int_\Omega (H \otimes H - \frac{1}{2} |H|^2 I_3)_{tt} : \nabla u_t \, dx \leq C|\nabla u_t|_2 |H|^4_{\ell_4^2} + C|\nabla u_t|_2 |H_{tt}|_2 |H|_\infty. \quad (3.36)$$

Combining (3.35)-(3.36) and Lemmas 3.2-3.4 from Young’s inequality, we have

$$\Lambda^*_4 (t) \leq \frac{1}{2} |\sqrt{\rho u_t} (t)|^2_2 + Cc_1^{8} (1 + |v_t|^2_{D_0^1}) |\nabla u_t|^2_2 + Cc_1^{4} (1 + |P_t|^2_2 + |\rho u|^2_2 + |H_{tt}|^2_2) + Cc_1^{8} |v|^2_2 |D_0^1. \quad (3.37)$$

Then multiplying (3.32) with $t$ and integrating over $(\tau, t)$ $(\tau \in (0, t))$, from (3.34) and (3.37), we have

$$\int_{\tau}^{t} s |\sqrt{\rho u_t} (s)|^2_2 ds + t|\nabla u_t (t)|^2_2 \leq \tau |u_t (\tau)|^2_{D_0^1} + Cc_1^{8} \int_{\tau}^{t} s (1 + |\nabla v_t|^2_2) |\nabla u_t|^2_2 ds + Cc_2^{20} \quad (3.38)$$

for $\tau \leq t \leq T_2$. From Lemma 3.4 we have $\nabla u_t \in L^2 ([0, T_2]; L^2)$, then according to Lemma 2.3, there exists a sequence $s_k$ such that

$$s_k \to 0, \quad \text{and} \quad s_k |\nabla u_t (s_k)|^2_2 \to 0, \quad \text{as} \quad k \to \infty.$$

Therefore, letting $\tau = s_k \to 0$ in (3.38), we conclude that

$$\int_0^t s |\sqrt{\rho u_t} (s)|^2_2 ds + t|\nabla u_t (t)|^2_2 \leq Cc_1^8 \int_0^t s (1 + |\nabla v_t|^2_2) |\nabla u_t|^2_2 ds + Cc_2^{20}. \quad (3.39)$$

Then from Gronwall’s inequality, we have

$$\int_0^t s |\sqrt{\rho u_t} (s)|^2_2 ds + t|u_t|^2_{D_0^1} \leq Cc_2^{20} \exp \left( Cc_1^8 \int_0^t s (1 + |\nabla v_t|^2_2) ds \right) \leq Cc_2^{20}.$$

Finally, from Lemma 2.4, for $0 \leq t \leq T_2$, we immediately have

$$t|u(t)|^2_{D_3^1} \leq t (|\rho u_t + \rho v \cdot \nabla v|^2_{D_1^1} + |\nabla P|^2_{D_1^1} + |\text{rot} H \times H|^2_{D_1^1} + |u|^2_{D_0^1}) \leq Cc_2^{24},$$

and similarly,

$$\int_0^t s |u_t|^2_{D_2} ds \leq C \int_0^t s (|\rho u_t + \rho v \cdot \nabla v|^2_2 + |\nabla P|^2_2 + |(\text{rot} H \times H)|^2_2 + |u|^2_{D_0^1}) ds \leq Cc_2^{22}.$$
Lemma 3.6 (Higher order estimate of the velocity $u$).

$$\int_0^t |u(s)|_{D^3,q}^{p_0} \, ds \leq Cc_2^{54} \quad \text{for} \quad 0 \leq t \leq T_2.$$

Proof. From (2.1), via Lemma 2.1 Holder’s inequality and Gagliardo-Nirenberg inequality, we easily deduce that

$$|u|_{D^3,q} \leq (|pu_t + pv \cdot \nabla v|_{D^2,q} + |\nabla P|_{D^1,q} + |\text{rot} H \times H|_{D^1,q} + |u|_{D^1,q}) \leq C(c_1^6 + c_1^2 |u|_\infty + c_1^5 |\nabla u_t|_q + c_1^3 |v|_{D^2,q}).$$ (3.40)

Due to the Sobolev inequality and Young’s inequality, we have

$$|u_t|_\infty \leq C|u_t|_{L^{\frac{3}{2} - \frac{3}{q}}(\Omega)} 2^{\frac{4}{3}} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q,$$ when $\Omega$ is bounded,

$$|u_t|_\infty \leq C|u_t|_6^{\frac{6(q-3)}{3q} + \frac{3q}{6(q-3)}} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q,$$ when $\Omega = \mathbb{R}^3$.

Then we quickly obtain

$$|u(t)|_{D^3,q} \leq Cc_2^2(|\nabla u_t|_2 + |\nabla u_t|_q) + Cc_1^3|v|_{D^2,q} + Cc_1^6.$$

According to Lemmas 3.2-3.5 we have

$$\int_0^t |u|_{D^3,q}^{p_0} \, ds \leq Cc_1^{12} + Cc_1^{6}\int_0^t \left(|v|_{D^2,q}^{p_0} + |\nabla u_t|_{D^0,q}^{p_0} + |\nabla u_t|_{D^0,q}^{p_0}ight) \, ds$$

$$\leq Cc_1^{12} + Cc_1^{6}\int_0^t |\nabla u_t|_2^{(6-g-\frac{3}{4})} \left|\nabla u_t\right|_6^{g_0(\frac{3q-6}{4q})} \, ds$$

$$\leq Cc_1^{12} + Cc_1^{6}\int_0^t s^{-\frac{p_0}{2}} \left(s|\nabla u_t|_2^{p_0(\frac{6-g-\frac{3}{4})}{4q}} \left(s|u_t|_{D^2}^{2 \frac{p_0(3q-6)}{4q}} \right) \, ds$$

$$\leq Cc_1^{12} + Cc_1^{6}\left(\sup_{[0,T_2]} s|\nabla u_t|_2 \right) \frac{\frac{p_0(3q-6)}{4q}}{4q-p_0(3q-6)} \, ds$$

$$\leq Cc_1^{12} + Cc_1^{6}\left(\int_0^t s^{-\frac{2p_0}{4q-p_0(3q-6)}} \, ds \right) \frac{\frac{p_0(3q-6)}{4q}}{4q-p_0(3q-6)} \, ds$$

$$\leq Cc_2^{54}$$

due to $0 < \frac{2p_0}{4q-p_0(3q-6)} < 1$ and $0 < \frac{p_0(3q-6)}{4q} < 1$. \qed

Lemma 3.7 (Higher order estimate of the velocity $u$).

$$t^2|u(t)|_{D^3,q} + t^2|u_t(t)|_{D^2}^2 + t^2|\text{rot} u(t)|_{D^0}^2 + \int_0^t s^2|u_t(s)|_{D^0}^2 \, ds \leq Cc_3^{34}$$

for $0 \leq t \leq T_3 = \min(T^*, (1 + c_3)^{-8}).$

Proof. Differentiating the equations (3.33) with respect to $t$, we have

$$\rho u_{tt} + Lu_{tt} = -\nabla P_{tt} - \rho(v \cdot \nabla v)_{tt} - 2\rho_t(v \cdot \nabla v + u_t)_t$$

$$- \rho_t(v \cdot \nabla v + u_t) + (\text{rot} H \times H)_{tt}.$$ (3.42)
Multiplying (3.42) by $u_{tt}$ and integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_{tt}|^2 \, dx + \int_{\Omega} (\mu |\nabla u_{tt}|^2 + (\lambda + \mu) (\text{div} u_{tt})^2) \, dx \\
= \int_{\Omega} \left( P_{tt} \text{div} u_{tt} - \rho (v \cdot \nabla v)_{tt} \cdot u_{tt} - 2 \rho_t (v \cdot \nabla v)_t \cdot u_{tt} - \rho_{tt} (v \cdot \nabla v) \cdot u_{tt} \right) \, dx \\
+ \int_{\Omega} \left( - \frac{3}{2} \rho_t |u_{tt}|^2 - \rho_t u_{tt} \cdot u_{tt} + (\text{rot} (\nabla \times H)_{tt} \cdot u_{tt}) \right) \, dx = \Delta_5(t) \equiv \sum_{i=1}^{17} I_i.
\] (3.43)

From Lemmas 3.2, 3.3, Holder’s inequality and Gagliardo-Nirenberg inequality, we obtain
\[
I_{11} \leq C |P_{tt}|_2 |u_{tt}|_2, \quad I_{12} \leq C |\rho|_{x\infty}^\frac{5}{2} \sqrt{|u_{tt}|_2} \left( |v_{tt}|_{D_0^3} |\nabla v|_1 + |v_t|_{D_0^3} |\nabla v_t|_1 \right), \\
I_{13} \leq C |\rho|_{3} |v_{tt}|_1 |v_t|_{D_0^3} |\nabla u_{tt}|_2, \quad I_{14} \leq C |\rho|_{tt}^2 |\nabla v|_2^2 |\nabla u_{tt}|_2, \\
I_{15} \leq C |\rho|_{x\infty}^\frac{3}{2} |v_{tt}|_1 |\sqrt{\rho} u_{tt}|_2 |\nabla u_{tt}|_2 |v_t|_{D_0^3} |\nabla u_{tt}|_2 + C |\rho|_{3} |v_{tt}|_1 |u_t|_{D_0^3} |\sqrt{\rho} u_{tt}|_2 |\nabla u_{tt}|_2^\frac{1}{2},
\] (3.44)

where we have used the fact that $\rho_t = \text{div}(\rho v)$, and
\[
I_{17} = - \int_{\Omega} (\text{rot} (\nabla \times H)_{tt} \cdot u_{tt} \, dx = \int_{\Omega} (H \otimes H - \frac{1}{2} |H|^2 I_3)_{tt} \cdot \nabla u_{tt} \, dx \leq C |\nabla u_{tt}|_2 |H|_{L^4}^2 + C |\nabla u_{tt}|_2 |H_{tt}|_2 |H|_{x\infty}.
\] (3.45)

Then from Young’s inequality, the above estimates (3.44) - (3.45) imply that
\[
t^2 \Delta_5(t) \leq \frac{1}{2} t^2 |u_{tt}|_{D_0^3}^2 + C (c_5^6 + c_5^3 |v_t|_{D_0^3}^2) t^2 |\sqrt{\rho} u_{tt}|_2^2 + C c_3 t^2 (|u_t|_{D_0^3}^2 + |P_{tt}|_2^2) \\
+ C c_4 t^2 |H_{tt}|_2^2 + C t^2 (|v_{tt}|_{D_0^3}^2 + |v_t|_{D_0^3}^2) + C c_3^2 |u_t|_{D_0^3}^2 + C c_3^{30}.
\] (3.46)

Then multiplying (3.43) by $t^2$ and integrating over $(\tau, t)$ ($\tau \in (0, t)$), we obtain
\[
t^2 |\sqrt{\rho} u_{tt}(t)|_2^2 + \int_{\tau}^{t} s^2 |\nabla u_{tt}|_2^2 \, ds \leq \tau^2 |\sqrt{\rho} u_{tt}(\tau)|_2^2 + C \int_{\tau}^{t} (c_3^6 + c_3^3 |v_t|_{D_0^3}^2) s^2 |\sqrt{\rho} u_{tt}|_2^2 \, ds + C c_3^{30}
\] (3.47)

for $\tau \leq t \leq T_2$. Due to Lemma 3.5, we have $t^2 |\sqrt{\rho} u_{tt} \in L^2([0, T_2]; L^2)$, then from Lemma 2.3 there exists a sequence $s_k$ such that
\[s_k \to 0, \quad \text{and} \quad s_k^\frac{1}{2} |\sqrt{\rho} u_{tt}(s_k)|_2^2 \to 0, \quad \text{as} \quad k \to \infty.
\]

Therefore, letting $\tau = s_k \to 0$ in (3.47), we conclude that
\[
t^2 |\sqrt{\rho} u_{tt}(t)|_2^2 + \int_{0}^{t} s^2 |\nabla u_{tt}|_2^2 \, ds \leq C \int_{\tau}^{t} (c_3^6 + c_3^3 |v_t|_{D_0^3}^2) s^2 |\sqrt{\rho} u_{tt}|_2^2 \, ds + C c_3^{30}.
\] (3.48)

Via the Gronwall’s inequality, for $0 \leq t \leq T_3$, we have
\[
t^2 |\sqrt{\rho} u_{tt}(t)|_2^2 + \int_{0}^{t} s^2 |\nabla u_{tt}|_2^2 \, ds \leq C c_3^{30} \exp \left( \int_{\tau}^{t} (c_3^6 + c_3^3 |v_t|_{D_0^3}^2) \, ds \right) \leq C c_3^{30}.
\]
Moreover, from Lemma 2.4 and (3.40), we quickly have
\[ t^2 |u_t|^2_{D_2} \leq C t^2 \left( (|\rho u_t + P \cdot \nabla v|)^2 + |\nabla P_1|^2 + |(\text{rot} H \times H) u_t|^2 + |u_t|_{D_0}^2 \right) \leq C c_3^2, \]
\[ t^2 |u|_{D^3,2} \leq C t^2 (c_1^2 + c_1^2 |u_t|_\infty + c_1^2 |\nabla u_t|_{D_2} + C c_3^3 |u|_{D^2,2}) \leq C c_3^4. \]
\[ \Box \]

Then combining the above lemmas, for \( 0 < t \leq T_* = \min(T^*, (1 + c_3)^{-8}) \), we have the following a priori estimate:
\[ \| (H, \rho - \overline{\rho}, P - \overline{P})(t) \|_{H^2 \cap W^{2,2}} + \| (H_t, \rho_t, P_t)(t) \|_{L^2 \cap L^q} \leq C c_1^2, \]
\[ \int_0^t |(H_{tt}, \rho_{tt}, P_{tt})|^2_{D_2} ds + t |(H_t, \rho_t, P_t)(t) |_{D_2}^2 \leq C c_1^2, \]
\[ |u(t)|_{D_1 \cap D^2}^2 + |\sqrt{\rho} u_t(t)|_{D_3}^2 + \int_0^t \left( |u_t|_{D_3}^2 + |u_{tt}|_{D_3}^2 \right) ds \leq C c_1^2, \]
\[ t |u(t)|_{D_0}^2 + t^2 |u(t)|_{D_2}^2 + \int_0^t \left( |u|_{D_{3,2}}^2 + s |u_t|_{D_2}^2 + |\sqrt{\rho} u_{tt}|_{D_3}^2 \right) ds \leq C c_3^2, \]
\[ t^2 |u(t)|_{D^3,2}^2 + t^2 |u(t)|_{D^2,2}^2 + \int_0^t s^2 |u_{tt}|_{D_3}^2 ds \leq C c_3^4. \]

3.3. Unique solvability of the IBVP (3.1) and (1.8)-(1.9) with vacuum.

In this section, we will construct a sequence of approximation solutions to the linearized problem (3.1) with vacuum.

**Lemma 3.8.** Let (3.2) and (3.6)-(3.7) hold. Assume \((H_0, \rho_0, u_0)\) satisfies (1.8)-(1.9). Then there exists a unique classical solution \((H, \rho, u)\) to (3.1) satisfying
\[ (H, \rho - \overline{\rho}, P - \overline{P}) \in C([0, T_*]; H^2 \cap W^{2,2}), \]
\[ u \in C([0, T_*]; D_0^{1/2} \cap D^2) \cap L^2([0, T_*]; D^2) \cap L^p([0, T_*]; D^3, q), \quad u_t \in L^2([0, T_*]; D_0), \]
\[ \sqrt{\rho} u_t \in L^\infty([0, T_*]; L^2), \quad t^{1/2} u \in L^\infty([0, T_*]; D^3), \quad t^{1/2} \sqrt{\rho} u_{tt} \in L^2([0, T_*]; L^2), \]
\[ t^{1/2} u_t \in L^\infty([0, T_*]; D_0^{1/2} \cap L^2([0, T_*]; D^2), \quad tu \in L^\infty([0, T_*]; D^3, q), \]
\[ tu_t \in L^2([0, T_*]; D_0), \quad tu_t \in L^\infty([0, T_*]; D^2), \quad t^2 \sqrt{\rho} u_{tt} \in L^\infty([0, T_*]; L^2). \]
Moreover, the solution \((H, \rho, u)\) also satisfies the estimate (3.50).

**Proof.** Step 1: Existence. We define \( \rho_\delta = \rho_0 + \delta \) for each \( \delta \in (0, 1) \). Then from the compatibility condition (1.9), we have
\[ Lu_0 + \nabla P(\rho_\delta) - \mu_0 \text{rot} H_0 \times H_0 = (\rho_\delta)^{1/2} u_0, \]
where
\[ g_\delta = \left( \frac{\rho_0}{\rho_\delta} \right)^{1/2} g_1 + \frac{\nabla (P(\rho_\delta) - P(\rho_0))}{(\rho_\delta)^{1/2}}. \]
Then according to assumption (3.6), for sufficiently small \( \delta > 0 \), we have
\[ 1 + \overline{\rho} + \delta + \| (\rho_\delta - (\overline{\rho} + \delta), P(\rho_\delta) - P(\overline{\rho} + \delta), H_0) \|_{H^2 \cap W^{2,2}} + |u_0|_{D_0 \cap D^2} + |g_\delta|_2 \leq c_0. \]
Therefore, corresponding to \((H_0, \rho_0^\delta, P(\rho_0^\delta), u_0^\delta)\), there exists a unique classical solution \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) satisfying (3.50). Then there exists a subsequence of solutions \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) converges to a limit \((H, \rho, P, u)\) in weak or weak* sense. And for any \(R > 0\), due to Lemma 2.5 there exists a subsequence of solutions \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) satisfying

\[(H^\delta, \rho^\delta, P^\delta, u^\delta) \to (H, \rho, P, u) \text{ in } C([0,T_*];H^1(\Omega_R)),\]  

(3.52)

where \(\Omega_R = \Omega \cap B_R\). Combining the lower semi-continuity of norms and (3.52), we know that \((H, \rho, P, u)\) also satisfies the local estimates (3.50). So it is easy to show that \((H, \rho, P, u)\) is a solution in distribution sense and satisfies the regularity

\[(H, \rho - \overline{\rho}, P - \overline{P}) \in L^\infty([0,T_*];H^2 \cap W^{2,q}),\]
\[u \in L^\infty([0,T_*];D_0^1 \cap D^2) \cap L^2([0,T_*];D^3) \cap L^p((0,T_*];D^3),\]
\[u_t \in L^2([0,T_*];D_0^0) \cap \sqrt{\rho}u_t \in L^\infty([0,T_*];L^2),\]
\[t^{\frac{1}{2}}u \in L^\infty([0,T_*];D^3), \quad t^{\frac{1}{2}}\sqrt{\rho}u_t \in L^2([0,T_*];L^2),\]
\[t^{\frac{1}{2}}u_t \in L^\infty([0,T_*];D_0^0) \cap L^2([0,T_*];D^2), \quad tu \in L^\infty([0,T_*];D^3),\]
\[tu_{tt} \in L^2([0,T_*];D_0^0) \cap L^2([0,T_*];D^2), \quad t^{2}\sqrt{\rho}u_{tt} \in L^\infty([0,T_*];L^2).\]

Step 2: Uniqueness. Let \((H_1, \rho_1, u_1)\) and \((H_2, \rho_2, u_2)\) be two solutions. Due to Lemma 3.1 in Section 3.1, we know \(\rho_1 = \rho_2\) and \(H_1 = H_2\). For the momentum equations (3.14), let \(\overline{u} = u_1 - u_2\), we have

\[\rho_t \overline{u} - \mu \Delta \overline{u} - (\lambda + \mu) \nabla \text{div} \overline{u} = 0,\]  

(3.54)

because we do not know whether \(\sqrt{\rho} \overline{u} \in L^\infty([0,T_*];L^2(\Omega))\) or not, so we consider this equation in bounded domain \(\Omega_R\). We define \(\varphi^R(x) = \varphi(x/R)\), where \(\varphi \in C_c^\infty(B_1)\) is a smooth cut-off function such that \(\varphi = 1\) in \(B_{1/2}\). Let \(\overline{u}^R = \varphi^R(t,x)u(t,x)\), we have

\[\rho_t \overline{u}_t^R - \mu \varphi^R \Delta \overline{u} - \varphi^R(\lambda + \mu) \nabla \text{div} \overline{u} = 0.\]  

(3.55)

Therefore, multiplying (3.55) by \(\overline{u}_t^R\) and integrating over \([0,t] \times \Omega_R\) \((t \in (0,T_*))\), we have

\[
\frac{1}{2} \int_{\Omega_R} \rho_t |\overline{u}^R|^2(t)dx + \int_0^t \int_{\Omega_R} \left( \mu(\varphi^R)^2 |\nabla \overline{u}|^2 + (\lambda + \mu)(\varphi^R)^2 | \nabla \text{div} \overline{u} |^2 \right)dxds,
\]
\[
= \int_0^t \int_{\Omega_R} \rho v \cdot \nabla \overline{u}_t^R \cdot \overline{u}^R dxds - 2\mu \int_0^t \int_{\Omega_R} \varphi^R(\nabla \overline{u} \cdot \nabla \varphi^R)dxds
\]
\[
- 2\int_0^t \int_{\Omega_R} (\lambda + \mu)\varphi^R \text{div} \overline{u} \nabla \varphi^R \cdot \overline{u} dxds = A_1 + A_2 + A_3.
\]
From Holder’s inequality and Sobolev’s imbedding theorem, we have
\begin{align*}
|A_1| & \leq \int_0^t \int_{\Omega_R} |\varphi R \rho v \cdot \nabla \overline{u}| |\nabla \overline{u}|_{2,2} dx ds + \int_0^t \int_{\Omega_R} |\overline{\rho u_R}| |\nabla \varphi R| v |\overline{u}| dx ds \\
& \leq C \int_0^t \int_{\Omega_R} |\varphi R |^2 |\nabla \overline{u}|_{2,2}^2 dx ds + \int_0^t \frac{\mu}{2} (\varphi R)^2 |\nabla \overline{u}|_{2,2}^2 dx ds + \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\overline{u}|^2 dx ds,
\end{align*}
\begin{align*}
|A_2| & \leq \frac{C}{R} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\overline{u}| |\nabla \overline{u}| dx ds \\
& \leq \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\overline{u}|^2 dx ds + C \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\nabla \overline{u}|^2 dx ds \\
& \leq \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} \frac{1}{2} \int_0^s \int_{(\Omega_R \setminus B_{R/2})} |\overline{u}|^2 dx ds + C \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\nabla \overline{u}|^2 dx ds \\
& \leq C \int_0^T \int_{(\Omega_R \setminus B_{R/2})} |\nabla \overline{u}(s)|_{L^2(\Omega_R \setminus B_{R/2})}^2 ds \to 0 \quad \text{as} \quad R \to \infty.
\end{align*}
Similarly, we can also obtain that
\begin{align*}
|A_3| \leq C \int_0^T \int_{(\Omega_R \setminus B_{R/2})} |\nabla \overline{u}(s)|_{L^2(\Omega_R \setminus B_{R/2})}^2 ds \to 0 \quad \text{as} \quad R \to \infty.
\end{align*}
Then from the above estimates, we deduce that
\begin{align}
\frac{1}{2} \int_{\Omega_R} \rho |\overline{u}|^2(t) dx + \int_0^t \int_{\Omega_R} \mu (\varphi R)^2 |\nabla \overline{u}|^2 dx ds & \leq C \int_0^t |\sqrt{\rho u_R}|^2 dx ds + Q_R, \quad (3.57)
\end{align}
where \(Q_R \to 0\) as \(R \to \infty\). Then letting \(R \to \infty\) in \(3.57\), via Gronwall’s inequality, we derive that \(\overline{u} \equiv 0\), which means that \(u_1 = u_2\).

**Step 3:** Time-continuity of the solution \((H, \rho, u, P)\). Firstly, the time-continuity of \(\rho, P\) and \(H\) can be obtained by Lemma 3.1. Secondly, from a classical embedding result (see \[6\]), we have \(u \in C([0, T_\ast]; D^1_0) \cap C([0, T_\ast]; D^2 - \text{weak})\). From the momentum equations \(3.4\), we know that \((\rho u_t) \in L^2([0, T_\ast]; H^{-1})\). Due to \(\rho u_t \in L^2([0, T_\ast]; D^1_0)\), we have immediately that \(\rho u_t \in C([0, T_\ast]; D^1_0)\). Similarly, from the following equations,
\[Lu = -\rho u_t - \rho (v \cdot \nabla) v - \nabla P + \text{rot} H \times H \equiv F,\]
where \(F \in C([0, T_\ast]; L^2)\), we can obtain \(u \in C([0, T_\ast]; D^2)\). \(\square\)

### 3.4. Proof of Theorem 1.1

Based on Lemma 3.8, now we give the proof of Theorem 1.1. We first fix a positive constant \(c_0\) sufficiently large such that
\begin{align}
2 + \overline{\rho} + \|(\rho_0 - \overline{\rho}, P_0 - \overline{P}, H_0)\|_{H^2 \cap W^{2,q}} + |u_0|_{D^1_0 \cap D^2} + |g_1|_2 & \leq c_0. \quad (3.58)
\end{align}
Then let \(u^0 \in C([0, +\infty); D^1_0 \cap D^2) \cap L^p([0, +\infty); D^{3,q})\) be the unique solution to the following linear parabolic problem
\[h_t - \Delta h = 0 \quad (0, +\infty) \times \Omega \quad \text{and} \quad h(0) = u_0 \quad \text{in} \quad \Omega.\]
We assume that the opposite holds, i.e., similar existence result can be obtained via the similar argument used in this Section.

Remark 3.1. For the case and time-continuity is also similar to those in [3][5] and so omitted. We can easily show that \((H, \rho, P, u)\) converges to a limit \((\bar{H}, \bar{\rho}, \bar{P}, \bar{u})\) in a strong sense. But this can be done by a slight modification of the arguments in [5]. We omit its details. Then adapting the proof of Lemma 3.8, we know that there exists a unique classical solution \((H, \rho, P, u)\) to the linearized problem (3.1) with \(v\) replaced by \(u^0\), which satisfies the estimate (3.50).

Similarly, we construct approximate solutions \((H^{k+1}, \rho^{k+1}, P^{k+1}, u^{k+1})\) inductively, as follows: assuming that \(u^k\) was defined for \(k \geq 1\), let \((H^{k+1}, \rho^{k+1}, P^{k+1}, u^{k+1})\) be the unique classical solutions to the problem (3.1) with \(v\) replaced by \(u^k\) as following

\[
\begin{align*}
H_t^{k+1} + u^k \cdot \nabla H^{k+1} + (\text{div} u^k I_3 - \nabla u^k) H^{k+1} &= 0, \\
\text{div} H^{k+1} &= 0, \\
\rho_t^{k+1} + \text{div}(\rho^{k+1} u^k) &= 0, \\
\rho^{k+1} u_t^{k+1} + \rho^{k+1} u^k \cdot \nabla u^k + \nabla P^{k+1} + L^{k+1} u &= \mu_0 \text{rot} H^{k+1} \times H^{k+1}, \\
(H^{k+1}, \rho^{k+1}, u^{k+1})|_{t=0} &= (H_0(x), \rho_0(x), u_0(x)) \quad x \in \Omega, \\
(H^{k+1}, \rho^{k+1}, u^{k+1}, P^{k+1}) &\rightarrow (0, \bar{\rho}, 0, \bar{P}) \quad \text{as} \quad |x| \rightarrow \infty, \quad t > 0.
\end{align*}
\]

(3.59)

Then from Lemma 3.8 that \((H^k, \rho^k, P^k, u^k)\) satisfies (3.50). Next, we show that \((H^k, \rho^k, P^k, u^k)\) converges to a limit \((H, \rho, P, u)\) in a strong sense. But this can be done by a slight modification of the arguments in [5]. We omit its details. Then adapting the proof of Lemma 3.8, we can easily show that \((H, \rho, P, u)\) is a solution to (1.5)-(1.7). The proof for uniqueness and time-continuity is also similar to those in [3][5] and so omitted.

\[\square\]

Remak 3.1. For the case \(0 < \sigma < +\infty\), if we add \(H|_{\partial \Omega} = 0\) to (1.3)-(1.7), then the similar existence result can be obtained via the similar argument used in this Section.

4. BLOW-UP CRITERION FOR CLASSICAL SOLUTIONS

Now we prove (1.12). Let \((H, \rho, u)\) be the unique classical solution to IBVP (1.5)-(1.7). We assume that the opposite holds, i.e.,

\[
\limsup_{T \to T^*} |D(u)|_{L^1([0,T];L^\infty(\Omega))} = C_0 < \infty.
\]

(4.1)

Due to \(P = A\rho^\gamma\), we quickly know that \(P\) satisfies

\[
P_t + u \nabla P + \gamma P \text{div} u = 0, \quad P_0 \in H^2 \cap W^{2,q}.
\]

(4.2)

We first give the standard energy estimate that
Lemma 4.1. 

\[(\sqrt{\rho}u(t)|^2 + |H|^2 + |P|^1) + \int_0^T |\nabla u(t)|_2^2 \, dt \leq C, \quad 0 \leq t < T,
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, \overline{T}]\)).

**Proof.** We first show that

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |u|^2 + \frac{P}{\gamma - 1} + \frac{1}{2} H^2 \right) \, dx + \int_{\Omega} \left( (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) \right) \, dx = 0. \tag{4.3}
\]

Actually, (4.3) is classical, which can be shown by multiplying (1.5) by \(u\), (1.5) by \(|u|^2\) and (1.5) by \(H\), then summing them together and integrating the result equation over \(\Omega\) by parts, where we have used the fact

\[
\int_{\Omega} \text{rot} H \times H \cdot u \, dx = -\int_{\Omega} \text{rot}(u \times H) \cdot H \, dx. \tag{4.4}
\]

Let \(f = (f^1, f^2, f^3)^\top \in \mathbb{R}^3\) and \(g = (g^1, g^2, g^3)^\top \in \mathbb{R}^3\), we denote \((f \otimes g)_{ij} = (f_i g_j)\). Next we need to show some lower order estimate for our classical solution \((H, \rho, u)\), which is the same as the regularity that the strong solution obtained in [5] has to satisfy.

### 4.1. Lower order estimate.

By assumption (4.1), we first show that both \(H\) and \(\rho\) are both uniform bounded.

**Lemma 4.2.**

\[(|\rho(t)|_\infty + |H(t)|_\infty) \leq C, \quad 0 \leq t < T,
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, \overline{T}]\)).

**Proof.** Multiplying (1.5) by \(q |H|^{q-2} H\) and integrating over \(\Omega\) by parts, then we have

\[
\frac{d}{dt} |H|^q = q \int_{\Omega} (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \cdot H |H|^{q-2} \, dx
\]

\[
= q \int_{\Omega} (H \cdot D(u) - u \cdot \nabla H - H \text{div} u) \cdot H |H|^{q-2} \, dx. \tag{4.5}
\]

By integrating by parts, the second term on the right-hand side can be written as

\[
-q \int_{\Omega} (u \cdot \nabla H) \cdot H |H|^{q-2} \, dx = \int_{\Omega} \text{div} u |H|^q \, dx,
\]

which, together with (4.5), immediately yields

\[
\frac{d}{dt} |H|^q \leq (2q + 1) \int_{\Omega} |D(u)||H|^q \, dx \leq (2q + 1)|D(u)|_\infty |H|^q, \tag{4.7}
\]

which means that

\[
\frac{d}{dt} |H|^q \leq \frac{(2q + 1)}{q} |D(u)|_\infty |H|^q, \tag{4.8}
\]

hence, it follows from (4.1) and (4.8) that

\[
\sup_{0 \leq t \leq \overline{T}} |H|^q \leq C, \quad 0 \leq T < \overline{T}.
\]

where $C > 0$ is independent of $q$. Therefore, letting $q \to \infty$ in the above inequality leads to the desired estimate of $|H|_{\infty}$. In the same way, we also obtains the bound of $|\rho|_{\infty}$ which indeed depends only on $\|\text{div} u\|_{L^1([0,T];L^\infty(\Omega))}$.

The next lemma will give a key estimate on $\nabla H$, $\nabla \rho$ and $\nabla u$.

**Lemma 4.3.**

$$\sup_{0 \leq t \leq T} (|\nabla u|^2 + |\nabla \rho|^2 + |\nabla H|^2) + \int_0^T |\nabla^2 u|^2 dt \leq C, \quad 0 \leq T < T,$$

where $C$ only depends on $C_0$ and $T$.

**Proof.** Firstly, multiplying (1.5) by $\rho^{-1}(-Lu - \nabla P - \nabla |H|^2 + H \cdot \nabla H)$ and integrating the result equation over $\Omega$, then we have

$$\frac{1}{2} \frac{d}{dt} (\frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda}{2} |\text{div} u|^2) + \int_{\Omega} \rho^{-1}(-Lu - \nabla P - \nabla |H|^2 + H \cdot \nabla H)^2 dx$$

$$= -\mu \int_{\Omega} (u \cdot \nabla u) \cdot \nabla \times (\text{rot} u) dx + (2\mu + \lambda) \int_{\Omega} (u \cdot \nabla u) \cdot \nabla \text{div} u dx$$

$$- \int_{\Omega} (u \cdot \nabla u) \cdot \nabla P(\rho) dx - \int_{\Omega} (u \cdot \nabla u) \left( \frac{1}{2} |\nabla |H|^2 - H \cdot \nabla H \right) dx$$

$$- \int_{\Omega} u \cdot \nabla P(\rho) dx - \int_{\Omega} u \cdot \left( \frac{1}{2} |\nabla |H|^2 - H \cdot \nabla H \right) dx \equiv \sum_{i=1}^6 L_i,$$

where we have used the fact that $\Delta u = \nabla \text{div} u - \nabla \times \text{rot} u$.

We now estimate each term in (4.9). Due to the fact that $\rho^{-1} \geq C^{-1} > 0$, we find the second term on the left hand side of (4.9) admits

$$\int_{\Omega} \rho^{-1}|-Lu + \nabla P + \nabla |H|^2 - H \cdot \nabla H|^2 dx$$

$$\geq C^{-1}|Lu|^2 - C(|\nabla P|^2 + |\nabla u|^2 + |H|_{\infty}^2 |\nabla H|^2)$$

$$\geq C^{-1}|u|^2_{H^2} - C(|\nabla \rho|^2 + |\nabla u|^2 + |\nabla H|^2),$$

where we have used the standard $L^2$- theory of elliptic system and Lemma 4.2. Note that $L$ is a strong elliptic operator. Next according to

$$\left\{ \begin{array}{l}
\nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + (\text{div} b)a - (\text{div} a)b, \\
\nabla \times (a \times b) = \frac{1}{2} \nabla (|u|^2) - u \cdot \nabla u,
\end{array} \right.$$

and Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce

$$|L_1| = \mu \left| \int_{\Omega} (u \cdot \nabla u) \cdot \nabla \times (\text{rot} u) dx \right| = \mu \left| \int_{\Omega} \nabla \times (u \cdot \nabla u) \cdot \text{rot} u dx \right|$$

$$= \mu \left| \int_{\Omega} \nabla \times (u \times \text{rot} u) \cdot \text{rot} u dx \right|$$

$$= \mu \left| \frac{1}{2} \int_{\Omega} (\text{rot} u)^2 dv u \cdot \text{rot} u dx \right| - \int_{\Omega} \text{rot} u \cdot D(u) \cdot \text{rot} u dx \leq C \|D(u)\|_{\infty} \|\nabla u\|_2^2,$$
Hence, similar to the proof of the above estimates for $L_2$ with the last three terms on the right-hand side of $H$

$|L_2| = (2\mu + \lambda) \left| \int_\Omega (u \cdot \nabla u) \cdot \nabla \text{div} u \right|$

$= (2\mu + \lambda) \left| - \int_\Omega \nabla u : (\nabla u)^\top \text{div} u + \frac{1}{2} \int_\Omega (\text{div} u)^3 \right|$

$\leq C|D(u)|_{\infty} |\nabla u|^2_2,$

$L_3 = - \int_\Omega (u \cdot \nabla u) \cdot \nabla P \, dx \leq C|\nabla u|_2 |\nabla u|_3 |\nabla P|_2$

$\leq C(\varepsilon)(|\nabla \rho|_2^2 + 1) |\nabla u|^2_2 + \varepsilon |u|^2_{D^2},$

$L_4 = - \int_\Omega (u \cdot \nabla u) \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \, dx \leq C|\nabla H|_2 |H|_\infty |\nabla u|_3 |u|_6$

$\leq C(\varepsilon)|H|_\infty^2 |\nabla H|_2^2 |\nabla u|_2^2 + \varepsilon |\nabla u|_1^2 \leq C(\varepsilon)(|\nabla H|_2^2 + 1) |\nabla u|^2_2 + \varepsilon |u|^2_{D^2},$

$L_5 = - \int_\Omega u_t \cdot \nabla P \, dx = \frac{d}{dt} \int_\Omega P \text{div} u - \int_\Omega P_t \text{div} u \tag{4.12}$

$= \frac{d}{dt} \int_\Omega P \text{div} u + \int_\Omega (u \cdot \nabla P \text{div} u + \gamma P(\text{div} u)^2) \, dx$

$\leq \frac{d}{dt} \int_\Omega P \text{div} u + C|\nabla P|_2 |u|_6 |\nabla u|_3 + C|P|_\infty |\nabla u|^2_2$

$= \frac{d}{dt} \int_\Omega P \text{div} u + C(\varepsilon)|\nabla u|^2_2 (1 + |\nabla \rho|^2_2) + \varepsilon |u|^2_{D^2},$

$L_6 = - \int_\Omega u_t \cdot \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \, dx$

$= \frac{1}{2} \frac{d}{dt} \int_\Omega |H|^2 \text{div} u - \frac{d}{dt} \int_\Omega H \cdot \nabla u \cdot H \, dx$

$- \int_\Omega \text{div} u H \cdot H_t \, dx + \int_\Omega H_t \cdot \nabla u \cdot H \, dx + \int_\Omega H \cdot \nabla u \cdot H_t \, dx.$

where we have used the fact $\text{div} H = 0$ and $\epsilon > 0$ is a sufficiently small constant. To deal with the last three terms on the right-hand side of $L_6$, we need to use

$H_t = H \cdot \nabla u - u \cdot \nabla H - H \text{div} u.$

Hence, similar to the proof of the above estimates for $L_i$, we also have

$\int_\Omega H_t \cdot \nabla u \cdot H \, dx = \int_\Omega -\text{div} H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \, dx$

$\leq C|H|_\infty^2 |\nabla u|^2_2 + |D(u)|_\infty |\nabla H|_2 |u|_6 |H|_3$

$\leq C(|D(u)|_\infty + 1)(|\nabla u|^2_2 + |\nabla H|^2_2),$

$\int_\Omega H_t \cdot \nabla u \cdot H \, dx + \int_\Omega H \cdot \nabla u \cdot H_t \, dx \tag{4.13}$

$\leq \int_\Omega |(H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \cdot \nabla u \cdot H| \, dx$

$\leq C|H|_\infty^2 |\nabla u|^2_2 + |u|_\infty |\nabla u|_2 |\nabla H|_2 |H|_\infty$

$\leq C(\varepsilon)(|\nabla H|_2 + 1)|\nabla u|^2_2 + \epsilon |u|^2_{D^2}.$
Then combining (4.9)-(4.13), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 - (P + \frac{1}{2} |H|^2) \text{div} u \cdot H ) \, dx + C |\nabla^2 u|^2 \leq C(|\nabla u|^2 + |\nabla H|^2 + 1)(|\nabla u|^2 + |D(u)|_\infty + 1).
\] (4.14)

Secondly, applying \( \nabla \) to (1.5), and multiplying the result equation by 2\( \nabla \rho \), we have

\[
\begin{align*}
( |\nabla \rho|^2 )_t + \text{div}(|\nabla \rho|^2 u) + |\nabla \rho|^2 \text{div} u &= -2(\nabla \rho)^T \nabla \nabla \rho - 2\rho \nabla \rho \cdot \nabla \text{div} u \\
&= -2(\nabla \rho)^T D(u) \nabla \rho - 2\rho \nabla \rho \cdot \nabla \text{div} u.
\end{align*}
\] (4.15)

Then integrating (4.15) over \( \Omega \), we have

\[
\frac{d}{dt} |\nabla \rho|^2 \leq C(|D(u)|_\infty + 1)|\nabla \rho|^2 + \epsilon |\nabla^2 u|^2.
\] (4.16)

Thirdly, applying \( \nabla \) to (1.5), due to

\[
\begin{align*}
A &= \nabla(H \cdot \nabla u) = (\partial_j H \cdot \nabla u^i)_i + (H \cdot \nabla \partial_j u^i)_i, \\
B &= \nabla(u \cdot \nabla H) = (\partial_j u \cdot \nabla H^i)_i + (u \cdot \nabla \partial_j H^i)_i, \\
C &= \nabla(H \text{div} u) = \nabla H \text{div} u + H \otimes \nabla \text{div} u,
\end{align*}
\] (4.17)

then multiplying the result equation \( \nabla (1.5)_1 \) by 2\( \nabla H \), we have

\[
(|\nabla H|^2)_t - 2A : \nabla H + 2B \nabla H - 2C : \nabla H = 0.
\] (4.18)

Then integrating (4.18) over \( \Omega \), due to

\[
\begin{align*}
\int_{\Omega} A : \nabla H \, dx &= \int_{\Omega} \sum_{j=1}^3 \left( \sum_{i=1}^3 \sum_{k=1}^3 \partial_j H^k \partial_k u^i \partial_j H^i \right) \, dx + \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H^k \partial_{ij} u^i \partial_j H^i \, dx \\
&= \int_{\Omega} \sum_{j=1}^3 \left( \sum_{i,k} \partial_j H^k (\partial_k u^i + \partial_k u^j) / 2 \partial_j H^i \right) \, dx + \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H^k \partial_{ij} u^i \partial_j H^i \, dx \\
&\leq C|D(u)|_\infty |\nabla H|^2 + C|H|_\infty |\nabla H|_2 |u|_{D^2},
\end{align*}
\] (4.19)

\[
\begin{align*}
\int_{\Omega} B : \nabla H \, dx &= \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 \partial_j u^k \partial_k H^i \partial_j H^i \, dx + \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 u^k \partial_{ij} H^i \partial_j H^i \, dx \\
&= \int_{\Omega} \sum_{i,j,k} \left( \sum_{j,k} \partial_k H^i (\partial_j u^k + \partial_j u^j) / 2 \partial_j H^i \right) \, dx + \frac{1}{2} \int_{\Omega} \sum_{j,k} \left( \sum_{i,j,k} u^k \partial_{ij} (\partial_j H^i)^2 \right) \, dx \\
&\leq C|D(u)|_\infty |\nabla H|^2,
\end{align*}
\]
\[
\int_{\Omega} C : \nabla H \, dx = \int_{\Omega} (\text{div} u |\nabla H|^2 + (H \otimes \nabla \text{div} u) : \nabla H) \, dx \\
\leq C|D(u)|_\infty |\nabla H|_2^2 + C|H|_\infty |\nabla H|_2 |u|_{D^2},
\] 

we quickly have the following estimate from (4.18)–(4.20):

\[
\frac{d}{dt} |\nabla H|_2^2 \leq C(|D(u)|_\infty + 1) |\nabla H|_2^2 + \epsilon |\nabla u|^2.
\] 

(4.21)

Adding (4.16) and (4.21) to (4.14), from Gronwall’s inequality we immediately obtain

\[
|\nabla u(t)|_2^2 + |\nabla \rho(t)|_2^2 + |\nabla H(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 \, ds \leq C, \quad 0 \leq t < T.
\]

Next, we proceed to improve the regularity of \(\rho, H\) and \(u\). To this end, we first drive some bounds on derivatives of \(u\) based on estimates above. Now we give the estimates for the lower order terms of the velocity \(u\).

**Lemma 4.4 (Lower order estimate of the velocity \(u\)).**

\[
|u(t)|_{D^2}^2 + |\sqrt{\rho} u_t(t)|_2^2 + \int_0^T |u_t|_{D^2}^2 \, dt \leq C, \quad 0 \leq t \leq T,
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, \overline{T})\)).

**Proof.** Via (1.3) and Lemmas 2.4, 4.1–4.3, we show that

\[
|u|_{D^2} \leq C(|\sqrt{\rho} u_t|_2 + 1).
\] 

(4.22)

Differentiating (1.5) with respect to \(t\), we have

\[
\rho u_{tt} + Lu_t = -\rho_t u_t - \rho u \cdot \nabla u_t - \rho_t u \cdot \nabla u - \rho u_t \cdot \nabla u - \nabla P_t + (\text{rot} H \times H)_t.
\] 

(4.23)

Multiplying (4.23) by \(u_t\) and integrating over \(\Omega\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2) \, dx \\
= \int_{\Omega} \rho u \cdot \nabla |u_t|^2 \, dx - \int_{\Omega} \rho u \nabla u \cdot \nabla u_t \, dx - \int_{\Omega} \rho u_t \cdot \nabla u \cdot u_t \, dx + \int_{\Omega} P_t \text{div} u_t \, dx \\
+ \int_{\Omega} H \cdot H_t \text{div} u_t \, dx - \int_{\Omega} (H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H) \, dx \equiv \sum_{i=7}^{12} L_i,
\]

(4.24)

where we have used the fact \(\text{div} H = 0\).
According to Lemmas 4.1-4.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

\[ L_7 = - \int_\Omega \rho u \cdot \nabla |u_t|^2 dx \leq C|\rho|^{\frac{1}{\infty}}|u|_\infty|\nabla \rho u_t|_2 |\nabla u_t|_2 \leq C\|\nabla u\|_2^2 |\nabla \rho u_t|_2 + \epsilon |\nabla u_t|^2, \]

\[ L_8 = - \int_\Omega \rho u \nabla (u \cdot \nabla u \cdot u_t) dx \leq C \int_\Omega (|u| |\nabla u|^2 |u_t| + |u|^2 |\nabla^2 u| |u_t| + |u|^2 |\nabla u| |\nabla u_t|) dx \]

\[ \leq C|u|_6 |\nabla u|^2 |u_t|_4 + C\|u\|_2^2 |\nabla^2 u| |u_t|_6 + C\|u\|_3^2 |\nabla u| |u_t|_2 \]

\[ \leq C\|\nabla u\|_1 |\nabla u_t|_2 \leq \epsilon |\nabla u_t|^2 + C(\epsilon)\|\nabla u\|^2, \quad (4.25) \]

where we have used the fact that

\[ |u|^2_3 \leq C|u|_6^2 \leq C|\nabla u|_3^2, \quad |\nabla u|^3 \leq C|\nabla u|_2 |\nabla u|_6 \leq C|\nabla u|_2 |\nabla u|_1. \quad (4.26) \]

And similarly, we also have

\[ L_9 = - \int_\Omega \rho u_t \cdot \nabla u \cdot u_t dx \leq C|\rho|^{\frac{1}{\infty}}|u_t|_6|\nabla u|_3 \]

\[ \leq \epsilon |\nabla u_t|^2_2 + C(\epsilon)|\nabla \rho u_t|_2 |\nabla u|^2_1, \]

\[ L_{10} = \int_\Omega P_t \div u_t dx \leq \int_\Omega |u \cdot \nabla P + \gamma P \div u | |\nabla u_t| dx \]

\[ \leq C|u|_\infty |\nabla P|_2 |\nabla u_t|_2 + C|P|_\infty |\div u|_2 |\nabla u_t|_2 \]

\[ \leq \epsilon |\nabla u_t|^2_2 + C(\epsilon)\|\nabla u\|^2_1, \quad (4.27) \]

\[ L_{11} + L_{12} = \int_\Omega H \cdot H_t \div u_t dx - \int_\Omega (H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H) dx \]

\[ \leq C|\nabla u_t|_2 |H_t|_2 |H|_\infty \leq C(|H|_\infty |\nabla u|_2 + |u|_\infty |\nabla H|_2) |\nabla u_t|_2 \]

\[ \leq \epsilon |\nabla u_t|^2_2 + C(\epsilon)\|\nabla u\|^2_1. \]

Then combining the above estimate (4.25)-(4.27), from (4.24), we have

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 dx + \int_\Omega |\nabla u_t|^2 dx \leq C(\|\sqrt{\rho} u_t\|_2^2 + 1)(\|\nabla u\|^2_1 + 1). \quad (4.28) \]

Then integrating (4.28) over \((\tau, t) \ (\tau \in (0, t))\), for \(\tau \leq t \leq T\), we have

\[ |\sqrt{\rho} u_t(\tau)|_2^2 + \int_\tau^t |\nabla u_t|_D^2 ds \leq |\sqrt{\rho} u_t(t)|_2^2 + C \int_\tau^t (\|\nabla u\|^2_1 + 1) |\sqrt{\rho} u_t|_2^2 ds + C. \quad (4.29) \]

From the momentum equations (1.5.4), we easily have

\[ |\sqrt{\rho} u_t(\tau)|_2^2 \leq C \int_\Omega \rho |u_t|^2 |\nabla u|^2 dx + C \int_\Omega \frac{|\nabla P + Lu - \rot H \times H|^2}{\rho} dx, \quad (4.30) \]

due to the initial layer compatibility condition (1.9), letting \(\tau \to 0\) in (4.30), we have

\[ \limsup_{\tau \to 0} |\sqrt{\rho} u_t(\tau)|_2^2 \leq C \int_\Omega \rho_0 |u_0|^2 |\nabla u_0|^2 dx + C \int_\Omega |g_1|^2 dx \leq C. \quad (4.31) \]
Then, letting \( \tau \to 0 \) in (4.29), from Gronwall’s inequality and (4.22), we deduce that
\[
|\sqrt{\rho} u_t(t)|^2 + |u(t)|^2_{D^2} + \int_0^t |\nabla u_t|^2_{D^1} ds \leq C, \quad 0 \leq t \leq T.
\] (4.32)

Finally, the following lemma gives bounds of \( \nabla \rho, \nabla H \) and \( \nabla^2 u \).

**Lemma 4.5.**
\[
\left( \| (\rho, H, P)(t) \|_{W^{1,q}} + |(\rho_t, H_t, P_t)(t)|_q \right) + \int_0^T |u(t)|^2_{D^2,q} dt \leq C, \quad 0 \leq t < T,
\] (4.33)
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)), and \( q \in (3, 6] \).

**Proof.** Via (1.5) and Lemmas 2.4, 4.1-4.4, we show that
\[
|\nabla^2 u|^q \leq C (|\rho u_t|_q + |\rho u \cdot \nabla u|_q + |\nabla P|_q + |\text{rot}H \times H|_q + |u|_{D^1,q}) 
\leq C (1 + |\nabla u_t|_2 + |\nabla P|_q + |\nabla H|_q). \] (4.34)

Firstly, applying \( \nabla \) to (1.5)\( _3 \), multiplying the result equations by \( q|\nabla \rho|^q - 2 \nabla \rho \), we have
\[
(|\nabla \rho|^q)_t + \text{div}(|\nabla \rho|^q u) + (q - 1)|\nabla \rho|^q \text{div} u = -q|\nabla \rho|^{q-2}(\nabla \rho)^\top D(u)(\nabla \rho) - q|\nabla \rho|^{q-2} \nabla \rho \cdot \nabla \text{div} u. \] (4.35)

Then integrating (4.35) over \( \Omega \), we immediately obtain
\[
\frac{d}{dt} |\nabla \rho|_q \leq C |D(u)|_\infty |\nabla \rho|_q + C |\nabla^2 u|_q. \] (4.36)

Secondly, applying \( \nabla \) to (1.5)\( _1 \), multiplying the result equations by \( q|\nabla H|^q - 2 \), we have
\[
(|\nabla H|^2)_t - qA : \nabla H |\nabla H|^{q-2} + qB \nabla H |\nabla H|^{q-2} + qC : \nabla H |\nabla H|^{q-2} = 0. \] (4.37)

Then integrating (4.37) over \( \Omega \), due to
\[
\int_{\Omega} A : \nabla H |\nabla H|^{q-2} dx
= \int_{\Omega} \sum_{j=1}^3 \left( \sum_{i,k} \partial_j H^k \partial_k u^i \partial_j H^i \right) |\nabla H|^{q-2} dx + \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H^k \partial_k u^i \partial_j H^i |\nabla H|^{q-2} dx
\leq C |D(u)|_\infty |\nabla H|_q^q + C |H|_\infty |\nabla H|_q^{q-1} |u|_{D^2,q},
\] (4.38)
\[ \int_\Omega B : \nabla H \nabla H|^{q-2} \mathrm{d}x \]

\[ = \int_\Omega \sum_{i=1}^3 \sum_{j,k=1}^3 \partial_j u^k \partial_k \partial_i u \nabla H|^{q-2} \mathrm{d}x + \int_\Omega \sum_{i=1}^3 \sum_{k=1}^3 u^k \partial_{kj} \partial_i u \nabla H|^{q-2} \mathrm{d}x \]

\[ = \int_\Omega \sum_{i=1}^3 \left( \sum_{j,k=1}^3 \partial_j u^k \partial_k \partial_i u \right) \nabla H|^{q-2} \mathrm{d}x + \frac{1}{2} \int_\Omega \sum_{k=1}^3 u^k \left( \sum_{j,i} \partial_k |\partial_j u| |\nabla H|^{q-2} \right) \mathrm{d}x \]

\[ = \int_\Omega \sum_{i=1}^3 \left( \sum_{j,k=1}^3 \partial_k \partial_j u^k \partial_i u \right) \nabla H|^{q-2} \mathrm{d}x + \frac{1}{2} \int_\Omega \sum_{k=1}^3 u^k \left( \sum_{j,i} \partial_k |\nabla H|^2 |\nabla H|^{q-2} \right) \mathrm{d}x \]

\[ = \int_\Omega \sum_{i=1}^3 \left( \sum_{j,k=1}^3 \partial_k \partial_j u^k \partial_i u \right) \nabla H|^{q-2} \mathrm{d}x + \frac{1}{q} \int_\Omega \sum_{k=1}^3 u^k \partial_k |\nabla H|^q \mathrm{d}x \]

\[ \leq C |\partial u|_\infty |\nabla H|_q \]

\[ \int_\Omega C : \nabla H \nabla H|^{q-2} \mathrm{d}x = \int_\Omega (\text{div} u |\nabla H|^q + (H \otimes \nabla \text{div} u) : \nabla H |\nabla H|^{q-2}) \mathrm{d}x \]

\[ \leq C |\partial u|_\infty |\nabla H|_q + C |H|_\infty |\nabla H|_q^{q-1} |u|_{D^2,q}. \quad (4.39) \]

we quickly obtain the following estimate:

\[ \frac{d}{dt} |\nabla H|_q \leq C (|\partial u|_\infty + 1) |\nabla H|_q + C |\partial u|_{D^2,q}. \quad (4.40) \]

Then from (4.34), (4.38), (4.40) and Gronwall’s inequality, we immediately have

\[ (|\nabla \rho(t)|_q + |\nabla H(t)|_q) \leq C \exp \left( \int_0^t (1 + |\partial u|_\infty) \mathrm{d}s \right) \leq C, \quad 0 \leq t \leq T. \]

Finally, via (4.34) and Lemma 4.1, we easily have

\[ \int_0^t |u(s)|_{D^2,q}^2 \mathrm{d}s \leq C \int_0^t (1 + |\nabla u(t)|_q^2) \mathrm{d}s \leq C, \quad 0 \leq t \leq T. \quad (4.41) \]

4.2. Improved regularity.

In this section, we will get some higher order regularity of \( H, \rho \) and \( u \) to make sure that this solution is a classical one in \([0, T]\). Based on the estimates obtained in the above section, in truth, we have already proved that \( \int_0^t |\nabla u|_\infty^2 \mathrm{d}s \leq C \).

Lemma 4.6 (Higher order estimate).

\[ (\langle \rho, P, H \rangle(t)|_{D^2}^2 + \langle (\rho_t, P_t, H_t)(t) \rangle|_{D^2}^2) + \int_0^T \left( |u|_{D^3}^2 + |(\rho u_t, P_{tt}, H_{tt})|_2^2 \right) \mathrm{d}t \leq C, \quad 0 \leq t < T, \]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).

Proof. Via (1.3) and Lemmas 2.4, 4.1, 4.3 we show that

\[ |u|_{D^2} \leq C (|\rho u_t|_{D^1} + |\rho u \cdot \nabla u|_{D^1} + |\nabla P|_{D^1} + |\text{rot} H \times H|_{D^1}) \]

\[ \leq C (1 + |u_t|_{D^1} + |P|_{D^2} + |H|_{D^2}). \quad (4.42) \]
Firstly, applying $\nabla^2$ to (1.5), and multiplying the result equation by $2\nabla^2\rho$, integrating over $\Omega$ we easily deduce that
\[
\frac{d}{dt}|\rho|_{D^2}^2 \leq C|\nabla u|_{\infty}^2 |\rho|_{D^2} + C|\rho|_{D^3} |\rho|_{D^2} + |\nabla \rho|_3 |\nabla^2 \rho|_2 \|\nabla^2 u\|_1, \tag{4.43}
\]
which, together with (4.42),
\[
\frac{d}{dt}|\rho|_{D^2} \leq C(|\nabla u|_{\infty} + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|_{\frac{3}{2}}^2. \tag{4.44}
\]
And similarly, we have
\[
\begin{align*}
\frac{d}{dt}|H|_{D^2} & \leq C(|\nabla u|_{\infty} + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|_{\frac{3}{2}}^2, \\
\frac{d}{dt}|P|_{D^2} & \leq C(|\nabla u|_{\infty}^2 + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|_{\frac{3}{2}}^2.
\end{align*} \tag{4.45}
\]
So combining (4.44)-(4.45), we quickly have
\[
\frac{d}{dt}(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) \leq C(1 + |\nabla u|_{\infty})(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) + C(1 + |\nabla u_t|_{\frac{3}{2}}^2). \tag{4.46}
\]
Then via Gronwall’s inequality and (4.46), we obtain
\[
|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2} + \int_0^t |u(s)|_{D^3}^2 dt \leq C, \quad 0 \leq t \leq T.
\]
Finally, due to the following relation
\[
\begin{align*}
H_t &= H \cdot \nabla u - u \cdot \nabla H - H \text{div} u, \\
p_t &= -u \cdot \nabla \rho - \rho \text{div} u, \quad \rho_t = -u \cdot \nabla P - \gamma P \text{div} u,
\end{align*} \tag{4.47}
\]
we immediately get the desired conclusions. \hfill \Box

Now we will give some estimates for the higher order terms of the velocity $u$ in the following three Lemmas.

**Lemma 4.7 (Higher order estimate of the velocity $u$).**

\[
t |u_t(t)|_{D^1}^2 + t |u(t)|_{D^3}^2 + \int_0^T t(|u|_{D^2}^2 + |\sqrt{\rho}u_t|^2) \, ds \leq C, \quad 0 \leq t \leq T,
\]
where $C$ only depends on $C_0$ and $T$ (any $T \in (0, \bar{T})$).

**Proof.** Firstly, multiplying (4.23) by $u_{tt}$ and integrating over $\Omega$, we have
\[
\int_\Omega \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \left( \mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 \right) \, dx
= \int_\Omega \left( - \nabla P_t - (\rho u \cdot \nabla) u_t - \rho_t u_t + (\text{rot} \times H) u_t \cdot u_t \right) \, dx = \frac{d}{dt} \Phi_1(t) + \Phi_2(t), \tag{4.48}
\]
where $\Phi_1(t)$ and $\Phi_2(t)$ are given in (4.4) and (4.7) respectively. The proof is completed. \hfill \Box
Then multiplying (4.48) by \( t \), there exists a sequence for \( \tau \) from (4.50) and (4.53), we have

\[
\Phi(t) = \int \rho(u \cdot \nabla u) \cdot u - \frac{1}{2} \rho_{|u|^2} + (\text{rot} H \times H) \cdot u_i \, dx,
\]

\[
\Phi_2(t) = \int ( - P_t \text{div} u_t - \rho(u \cdot \nabla u) \cdot u_t + \rho_{tt}(u \cdot \nabla u) \cdot u_t + \rho_t(u \cdot \nabla u) \cdot u_t) \, dx
\]

\[
+ \int \rho_{tt} |u_t|^2 - (\text{rot} H \times H) \cdot u_t \, dx \equiv \sum_{i=13}^{18} L_i.
\]

Then almost same to (4.25), we also have

\[
\Phi_1(t) \leq \frac{\mu}{10} |\nabla u_t|^2 + C.
\] (4.49)

Let we denote

\[
\Phi^*(t) = \frac{1}{2} \int \mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u)^2 \, dx - \Lambda_3(t),
\]

then from (4.49), for \( 0 \leq t \leq T \), we quickly have

\[
C |\nabla u_t|^2 - C \leq \Phi^*(t) \leq C |\nabla u_t|^2 + C.
\] (4.50)

Similarly, according to Lemmas (4.2) and Gagliardo-Nirenberg inequality, for \( 0 < t \leq T \), we deduce that

\[
L_{13} \leq C |P_{tt}| |\nabla u_t|, \quad L_{14} \leq \left| \frac{1}{\lambda} \right| |\sqrt{\rho} u_t| \, \left( |u|_\infty |\nabla u_t| + |\nabla u|_{3/2} |\nabla u_t|_2 \right),
\]

\[
L_{15} \leq C |\rho u| \, |\nabla u_t| |\nabla u|_{3/2} |\nabla u_t|_\infty,
\]

\[
L_{16} \leq C |\rho|_2 |u|_6 |\nabla u|_6 |\nabla u_t|_2 + C |u|_\infty |u|_6 |\nabla u_t|_2 |\rho|_3,
\]

\[
L_{17} \leq C |\rho|_3 |\nabla u_t|_2 |u|_\infty |u|_6 + C |\rho|_\infty |\sqrt{\rho} u_t|_3 |u|_6 |\nabla u_t|_2,
\] (4.51)

where we have used the facts \( \rho_t = -\text{div}(\rho u) \), and

\[
L_{18} = - \int (\text{rot} H \times H) : u_t \, dx = \int \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) : u_t \, dx
\]

\[
\leq C |\nabla u_t|_2 |H|_4^2 + C |\nabla u_t|_2 |H|_2 |H|_\infty.
\] (4.52)

Combining (4.51)–(4.52), from Young’s inequality, we have

\[
\Phi_2(t) \leq \frac{1}{2} |\sqrt{\rho} u_t(t)|^2 + C (1 + |\nabla u_t|^2_2 |\nabla u_t|^2_2 |\nabla u|^2_\infty + C (|P_{tt}|_2 + |\rho|_2 |H|_2^2). \quad (4.53)
\]

Then multiplying (4.48) by \( t \) and integrating the result inequality over \( (\tau, t) \) \( \tau \in (0, t) \), from (4.50) and (4.53), we have

\[
\int t s |\sqrt{\rho} u_t(s)|^2 ds + t |\nabla u_t(t)|_2^2 \leq t |u_t(\tau)|_2^2 + \int_0^t s (1 + |\nabla u_t|^2_2 |\nabla u_t|^2_2 ds + C
\]

\[
\text{for } \tau \leq t \leq T. \text{ From Lemma 4.4 we have } \nabla u_t \in L^2([0, T]; L^2), \text{ then according to Lemma 2.3, there exists a sequence } s_k \text{ such that}
\]

\[
s_k \to 0, \quad \text{and } s_k |\nabla u_t(s_k)|^2_2 \to 0, \quad \text{as } k \to \infty.
\]
Therefore, letting $\tau = s_k \to 0$ in (4.54), from Gronwall's inequality, we have
\[
\int_0^t \rho_u \frac{|\nabla u|^2}{2} ds + t|u|_{D^1}^2 \leq C \exp \left( \int_0^t (1 + |\nabla u|^2) ds \right) \leq C.
\]
From (4.42) (4.54), Lemmas 2.4 and 4.1-4.6 we immediately have
\[
t|u|_{D^2}^2 + \int_0^t s|u|_{D^2}^2 ds \leq C(t|u|_{D^0} + 1) + C \int_0^t (1 + \sqrt{\rho u})^2 ds \leq C.
\]

**Lemma 4.8 (Higher order estimate of the velocity $u$).**

\[
\left\| (\rho, P, H)(t) \right\|_{D^2} + t\left\| (\rho_t, P_t, H_t)(t) \right\|_{D^1} \leq \int_0^T |u|_{D^3}^p dt \leq C,
\]
where $C$ only depends on $\Omega$ and $T$ (any $T \in (0, T]$).

**Proof.** From Lemmas 2.4 and 4.1-4.7 we easily obtain
\[
|u|_{D^3} \leq C(|\rho u| + |\nabla u|_{D^1} + |\text{rot} H|_{D^1} + |P|_{D^2}) \leq C(|\rho u|_{\infty} + |\nabla u|_{q} + |u|_{D^2} + |H|_{D^2} + |P|_{D^2}).
\]
Due to the Sobolev inequality, Poincare inequality and Young's inequality, we have
\[
|u|_{\infty} \leq C|u|_{q} \leq C|\nabla u|_{2} + C|\nabla u|_{q},
\]
then we have
\[
|u(t)|_{D^3} \leq C(|\nabla u|_{2} + |\nabla u|_{q} + |u|_{D^2} + |H|_{D^2} + |P|_{D^2}).
\]
According to Lemmas 3.4-4.7 via the completely same argument in (3.31), we have
\[
\int_0^t C(|\nabla u|_{2} + |\nabla u|_{q} + |u|_{D^2})^p dt \leq C.
\]
Then, applying $\nabla^2$ to (1.5), and multiplying the result equation by $q \nabla^2 \rho \nabla^2 \rho^2$, integrating over $\Omega$ we easily deduce that
\[
\frac{d}{dt}|\rho|^q \leq C(|\nabla u|_{\infty} + 1 + F)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2} + C)\rho^2, \tag{4.57}
\]
which, together with (4.42),
\[
\frac{d}{dt}|\rho|_{D^2} \leq C(|\nabla u|_{\infty} + F + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + CF, \tag{4.58}
\]
where $F = |\nabla u|_{2} + |\nabla u|_{q} + |u|_{D^2}$. And similarly, we have
\[
\begin{cases}
\frac{d}{dt}|H|_{D^2} \leq C(|\nabla u|_{\infty} + F + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + F, \\
\frac{d}{dt}|P|_{D^2} \leq C(|\nabla u|_{\infty} + F + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + F.
\end{cases} \tag{4.59}
\]
So we combining (4.58) - (4.59), we quickly have
\[
\frac{d}{dt}(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) \leq C(1 + |\nabla u|_{\infty} + F)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + C(1 + F).
\]

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Then via Gronwall’s inequality, (4.56) and (4.60), we obtain
\[ |\rho|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}} + \int_0^t |u(s)|_{D^{3,q}}^2 \, ds \leq C, \quad 0 \leq t \leq T. \]

Finally, due to relation (4.47), we immediately get the desired conclusions. \hfill \Box

Finally, we have

**Lemma 4.9 (Higher order estimate of the velocity \(u\)).**

\[ t^2 |u(t)|_{D^{3,q}} + t^2 |u_t(t)|_{D^{2}} + t^2 \sqrt{\rho u_t(t)} \| + \int_0^T s^2 |u_{tt}(s)|_{D_0^1}^2 \, ds \leq C \]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0,T]\)).

This lemma can be easily proved via the method used in Lemma 4.7 here we omit it. And this will be enough to extend the regular solutions of \((H, \rho, u, P)\) beyond \(t \geq T\).

In truth, in view of the estimates obtained in Lemmas 4.1-4.8 we quickly know that the functions \((H, \rho, u, P)|_{t=T} = \lim_{t \to T} (H, \rho, u, P)\) satisfies the conditions imposed on the initial data (4.13) – (4.19). Therefore, we can take \((H, \rho, u, P)|_{t=T}\) as the initial data and apply the local existence Theorem 1.1 to extend our local classical solution beyond \(t \geq T\). This contradicts the assumption on \(T\).

**References**

[1] J. L. Boldrini, M. A. Rojas-Medar, E. Fernández-Cara, Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, *J.Math.Pures.Appl.* 82 (2003) 1499-1525.

[2] Y. Cho, H. J. Choe, and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J.Math.Anal.Appl.* 83 (2004) 243-275.

[3] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, *J.Differ.Equations* 228 (2006) 377-411.

[4] Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *manu.math* 120 (2006) 91-129.

[5] J. Fan, W. Yu, Strong solutions to the magnetohydrodynamic equations with vacuum, *Nonl. Anal.* 10 (2009) 392-409.

[6] G. P. Galdi, An introduction to the Mathematical Theory of the Navier-Stokes equations, Springer, New York, 1994.

[7] X.D. Huang, J. Li and Z.P. Xin, Global Well-posedness of classical solutions with large oscillations and vacuum, *Comm. Pure. Appl. Math* 65 (2012) 0549-0585.

[8] X.D. Huang, J. Li and Z.P. Xin, Global Well-posedness of classical solutions with vacuum on bounded domains, (2012) Preprint.
[9] X.D. Huang, J. Li and Z.P. Xin, Blow-up criterion for the compressible flows with vacuum states, *Comm. Math. Phys* **301** (2010) 23-35.

[10] S. Kawashima, M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Acad. Ser. A Math. Sci* **58** (1982) 384-387.

[11] Y. Li, R. Pan and S. Zhu, On regular solutions of the 2-D compressible flow with degenerate viscosities and vacuum, (2013) Preprint.

[12] X. Li, N Su and D. Wang, Local strong solution to the compressible magnetohydrodynamics flow with large data, *J. Hyperbolic Differential Equations* **3** (2011) 415-436.

[13] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Science **53**, Springer-Verlag: New York, Berlin Heidelberg, 1986.

[14] T. Makino, S. Ukai, S. Kawashima, Sur la solution à support compact de equations d’Euler compressible, *Japan J Appl Math* **33** (1986) 249-257.

[15] Ponce, G. Remarks on a paper: Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys* **98** (1985) 349-353.

[16] O. Rozanova, Blow-up of smooth solutions to the barotropic compressible magnetohydrodynamic equations with finite mass and energy, Preprint.

[17] O. Rozanova, Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes Equations, *J. Differ. Equations* **51** (1998) 0229-0240.

[18] J. Simon, Compact sets in $L^P(0,T;B)$, *Ann. Mat. Pur. Appl* **146** (1987) 65-96.

[19] E. M. Stein, *Singular integrals and Differentiability properties of Functions*, Princeton Univ. Press, Princeton NJ, 1970.

[20] Y. Sun, C. Wang and F. Zhang, A Beale-Kato-Majda blow-up criterion to the compressible Navier-Stokes Equation, *J. Math. Pure. Appl* **95** (2011) 36-47.

[21] Y. Sun, C. Wang and F. Zhang, A blow-up criterion of strong solutions to the 2-D compressible Navier-Stokes Equation, *Sci. Chi. Math* **54** (2011) 105-116.

[22] Z. P. Xin, Blow-up of smooth solutions to the compressible Navier-Stokes Equation with Compact Density, *Commun. Pure. App. Math* **51** (1998) 0229-0240.

[23] Z. P. Xin and W. Yan, On blow-up of classical solutions to the compressible Navier-Stokes Equations, *Commun. Math. Phys* **321** (2013) 529-541.

[24] X. Xu and J. Zhang, A blow-up criterion for the 3-D non-resistive compressible magnetohydrodynamic Equations with initial vacuum, *Nonl. Anal* **12** (2011) 3442-3451.

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