A NOTE ON SCHWARZ-PICK LEMMA FOR BOUNDED COMPLEX-VALUED 
HARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{R}^n$

SHAOYU DAI AND YIFEI PAN

Abstract. In this paper we prove a Schwarz-Pick lemma for bounded complex-valued harmonic functions in the unit ball of $\mathbb{R}^n$.

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1. Introduction

This paper is a note about Chen’s paper [1]. Using the same method in [1], we obtain a result Theorem 1, which extends the Schwarz-Pick lemma [1] for planar harmonic mappings to bounded complex-valued harmonic functions in the unit ball of $\mathbb{R}^n$. In addition, motivated by [1] and this paper, we consider a Schwarz lemma for harmonic mappings between real unit balls in another paper. Now we introduce some denotation and the background.

Let $n$ be a positive integer greater than 1. $\mathbb{R}^n$ is the real space of dimension $n$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $|x| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball of $\mathbb{R}^n$. The unit sphere, the boundary of $\mathbb{B}^n$ is denoted by $S$; normalized surface-area measure on $S$ is denoted by $\sigma$ (so that $\sigma(S) = 1$). Let $S^+$ denote the northern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n > 0\}$ and let $S^-$ denote the southern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n < 0\}$. $N = (0, \cdots, 0, 1)$ denotes the north pole of $S$. $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ denotes the open ball centered at origin of radius $r$; its closure is the closed ball $\overline{B}_r$. A twice continuously differentiable, complex-valued function $F$ defined on $\mathbb{B}^n$ is harmonic on $\mathbb{B}^n$ if and only if $\Delta F = 0$, where $\Delta = D_2^2 + \cdots + D_n^2$ and $D_j^2$ denotes the second partial derivative with respect to the $j^{th}$ coordinate variable $x_j$. By $\Omega_n$, we denote the class of all complex-valued harmonic functions $F(x)$ on $\mathbb{B}^n$ with $|F(x)| < 1$ for $x \in \mathbb{B}^n$.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote the disk $\{z \in \mathbb{C} : |z| < r\}$ by $D_r$; its closure is the closed disk $\overline{D}_r$.

For a holomorphic function $f$ from $\mathbb{D}$ into $\mathbb{D}$, the classical Schwarz lemma says that if $f(0) = 0$, then

$$|f(z)| \leq |z|$$

holds for $z \in \mathbb{D}$. For $0 < r < 1$, (1.1) may be written in the following form:

$$f(D_r) \subset D_r.$$ 

So the classical Schwarz lemma can be regarded as considering the region of $f(D_r)$. If the condition $f(0) = 0$ is relaxed, then what the region of $f(\overline{D}_r)$ is. The answer can be found in the classical Schwarz-Pick lemma. By Schwarz-Pick lemma [2], it is known that

$$\frac{|f(z_1) - f(z_2)|}{|1 - f(z_2)f(z_1)|} \leq \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|}$$

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holds for \( z_1, z_2 \in \mathbb{D} \). Using the notations
\[
d_p(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|}
\]
for the pseudo-distance between \( z_1, z_2 \in \mathbb{D} \), we know that
\[
(1.4) \quad d_p(f(z_1), f(z_2)) \leq d_p(z_1, z_2)
\]
for \( z_1, z_2 \in \mathbb{D} \) by (1.3). Denote \( \Delta(z, r) = \{ \zeta \in \mathbb{D} : d_p(\zeta, z) \leq r, z \in \mathbb{D}, 0 < r < 1 \} \) for the closed pseudo-disk with center at \( z \) and pseudo-radius \( r \). Then (1.4) may be written in the following form:
\[
f(\Delta(z, r)) \subset \overline{\Delta}(f(z), r)
\]
for \( z \in \mathbb{D} \) and \( 0 < r < 1 \). Note that \( \Delta(0, r) = \overline{D}_r \). So for \( f \) without the assumption \( f(0) = 0 \), we know
\[
(1.5) \quad f(D_r) \subset \overline{\Delta}(f(0), r).
\]
When \( f(0) = 0 \), (1.5) becomes (1.2).

For a complex-valued harmonic function \( F \) on \( \mathbb{D} \) such that \( F(\mathbb{D}) \subset \mathbb{D} \) and \( F(0) = 0 \), it is known [3] that
\[
(1.6) \quad F(z) \leq \frac{4}{\pi} \arctan |z|
\]
holds for \( z \in \mathbb{D} \). For \( 0 < r < 1 \), (1.6) may be written in the following form:
\[
(1.7) \quad F(D_r) \subset D_{\frac{4}{\pi} \arctan r}.
\]
If the condition \( F(0) = 0 \) is relaxed, then what the region of \( F(D_r) \) is. Unfortunately, the composition \( f \circ F \) of a harmonic function \( F \) and a holomorphic function \( f \) do not need to be harmonic, so it is a serious problem to seek the estimate corresponding to (1.5) for a harmonic function \( F \) without the assumption \( F(0) = 0 \). Fortunately, Chen resolved this problem in [1]. In [1], for any \( 0 < r < 1 \) and \( 0 \leq \rho < 1 \), the author construct a closed domain \( E_{r,\rho} \), which contains \( \rho \) and is symmetric to the real axis, with the following properties: Let \( z \in \mathbb{D} \) and \( w = \rho e^{i\alpha} \) be given. For every complex-valued harmonic function \( F \) with \( F(\mathbb{D}) \subset \mathbb{D} \) and \( F(z) = w \), the author has \( F(\Delta(z, r)) \subset e^{i\alpha}E_{r,\rho} = \{ e^{i\alpha} \zeta : \zeta \in E_{r,\rho} \} \); conversely, for every \( w' \in e^{i\alpha}E_{r,\rho} \), there exists a complex-valued harmonic function \( F \) such that \( F(\mathbb{D}) \subset \mathbb{D} \), \( F(z) = w \) and \( F(z') = w' \) for some \( z' \in \partial \Delta(z, r) \). Obviously, by Chen’s result, we know that for a complex-valued harmonic function \( F \) on \( \mathbb{D} \) such that \( F(\mathbb{D}) \subset \mathbb{D} \) without the assumption \( F(0) = 0 \), if \( F(0) = \rho e^{i\alpha} \), then
\[
(1.8) \quad F(D_r) \subset e^{i\alpha}E_{r,\rho},
\]
which is sharp. (1.8) is the estimate for complex-valued harmonic functions corresponding to (1.5). Note that a complex-valued harmonic function \( F \) on \( \mathbb{D} \) such that \( F(\mathbb{D}) \subset \mathbb{D} \) can be seen as \( F \in \Omega_2 \). So it is natural to consider the same problem in \( \Omega_n \).

For \( F \in \Omega_n \), harmonic Schwarz lemma [4] says that if \( F(0) = 0 \), then
\[
(1.9) \quad |F(x)| \leq U(|x|N)
\]
holds for \( x \in \mathbb{B}^n \), where \( U \) is the Poisson integral of the function that equals 1 on \( S^+ \) and -1 on \( S^- \). For \( 0 < r < 1 \), (1.9) may be written in the following form:
\[
(1.10) \quad F(D_r) \subset D_{U(rN)}.
\]
If the condition \( F(0) = 0 \) is relaxed, then what the region of \( F(D_r) \) is. This problem will be solved in this paper.

In this paper, by the same method in [1], we obtain the following theorem about the region of \( F(D_r) \). The result is sharp. When \( n = 2 \), our result is coincident with (1.8). And when \( F(0) = 0 \), our result is coincident with (1.10). Note that in the following theorem, \( E_{r,\rho} \) is defined as (3.1).
Theorem 1. Let $0 \leq \rho < 1$, $\alpha \in \mathbb{R}$ and $0 < r < 1$ be given. Then for every harmonic function $F$ with $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = pe^{i\alpha}$, we have $F(\mathbb{B}_r) \subset e^{i\alpha}E_{r,0} = \{e^{i\alpha} \zeta : \zeta \in E_{r,0}\}$; conversely, for every $w' \in e^{i\alpha}E_{r,0}$, there exists a harmonic function $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = pe^{i\alpha}$ and $F(rN) = w'$.

The theorem above will be proved by three steps as follows.

Step 1: find the extremal line of $F(\mathbb{B}_r)$ in the normal direction of $e^{0i}$, which is related to the value of $F(0)$.

Step 2: find the extremal line of $F(\mathbb{B}_r)$ in the normal direction of a given direction. For a given direction of $e^{i\beta}$ with $-\pi \leq \beta \leq \pi$, construct a new harmonic function $F_{\beta} = e^{-i\beta}F$ through rotating $F(\mathbb{B}_r)$ by an anti-clockwise rotation of angle $\beta$. Using the result of Step 1, we will have the extremal line of $F_{\beta}(\mathbb{B}_r)$ in the normal direction of $e^{0i}$, which is denoted by $l_{\beta}$. Note that $F(\mathbb{B}_r)$ can be obtained from $F_{\beta}(\mathbb{B}_r)$ by a clockwise rotation of angle $\beta$. Then the extremal line of $F(\mathbb{B}_r)$ in the normal direction of $e^{i\beta}$, which is denoted by $l_{\beta}$, can be obtained from $l_{\beta}$ by a clockwise rotation of angle $\beta$.

Step 3: using the result of Step 2, we will obtain all the extremal lines of $F(\mathbb{B}_r)$ in every normal direction, with which we can wrap $F(\mathbb{B}_r)$ and obtain the region of $F(\mathbb{B}_r)$.

Step 1 will be solved in Section 2. Step 2 and Step 3 will be solved in Section 3.

2. Some lemmas

In this section, we will introduce some lemmas, which are important for the proof of Theorem 2. Lemma 1 will be used in Lemma 2. Lemma 2 will be used in Lemma 3. Lemma 3 and Lemma 4 will be used in Theorem 2.

Now we give Lemma 1 first. Lemma 1 constructs a bijection $(R, I)$ from $\mathbb{R} \times \mathbb{R}^+$ onto the upper half disk $\{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}, a^2 + b^2 < 1, b > 0\}$, which will be used to construct $u_{a,b,r}$ in Lemma 2 for the case that $b > 0$.

For $0 < r < 1$, $\mu > 0$ and real number $\lambda$, define

$$A_{r,\lambda,\mu}(\omega) = \frac{1}{\mu} \left( \frac{1}{|rN - \omega|^n} - \lambda \right), \quad \omega \in S,$$

and

$$R(r, \lambda, \mu) = \int_S A_{r,\lambda,\mu}(\omega) \frac{1}{\sqrt{1 + A^2_{r,\lambda,\mu}(\omega)}} d\sigma, \quad I(r, \lambda, \mu) = \int_S \frac{1}{\sqrt{1 + A^2_{r,\lambda,\mu}(\omega)}} d\sigma.$$

The idea of the conformation of $A_{r,\lambda,\mu}(\omega)$, $R(r, \lambda, \mu)$ and $I(r, \lambda, \mu)$ originates from the needs of (2.16) and (2.21).

Lemma 1. Let $0 < r < 1$ be fixed. Then, there exist a unique pair of real functions $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b) > 0$, defined on the upper half disk $\{(a, b) : a^2 + b^2 < 1, b > 0\}$ and analytic in the real sense, such that $R(r, \lambda(r, a, b), \mu(r, a, b)) = a$ and $I(r, \lambda(r, a, b), \mu(r, a, b)) = b$ for any point $(a, b)$ in the half disk.

Proof. A simple calculation gives

$$\frac{\partial R(r, \lambda, \mu)}{\partial \lambda} = -\frac{1}{\mu} \int_S \frac{1}{(1 + A^2_{r,\lambda,\mu}(\omega))^{3/2}} d\sigma,$$

$$\frac{\partial R(r, \lambda, \mu)}{\partial \mu} = -\frac{1}{\mu} \int_S \frac{A_{r,\lambda,\mu}(\omega)}{(1 + A^2_{r,\lambda,\mu}(\omega))^{3/2}} d\sigma,$$

$$\frac{\partial I(r, \lambda, \mu)}{\partial \lambda} = \frac{1}{\mu} \int_S \frac{A_{r,\lambda,\mu}(\omega)}{(1 + A^2_{r,\lambda,\mu}(\omega))^{3/2}} d\sigma.$$
\[
(2.6) \quad \frac{\partial I(r, \lambda, \mu)}{\partial \mu} = \frac{1}{\mu} \int_S \frac{A_{r,\lambda,\mu}^2(\omega)}{(1 + A_{r,\lambda,\mu}^2(\omega))^{3/2}} d\sigma.
\]

It is easy to see that

(i) by (2.3), \( \frac{\partial R(r, \lambda, \mu)}{\partial \lambda} < 0 \) for any \( \lambda \) and \( \mu > 0 \), and \( R(r, \lambda, \mu) \) is strictly decreasing as a function of \( \lambda \) for a fixed \( \mu \);

(ii) by (2.2), for a fixed \( \mu \), \( R(r, \lambda, \mu) \to -1 \) or \( 1 \) according to \( \lambda \to +\infty \) or \( \lambda \to -\infty \);

(iii) by (2.3) - (2.6) and the convexity of the square function,

\[
\frac{\partial R(r, \lambda, \mu)}{\partial \lambda} \frac{\partial I(r, \lambda, \mu)}{\partial \mu} - \frac{\partial R(r, \lambda, \mu)}{\partial \mu} \frac{\partial I(r, \lambda, \mu)}{\partial \lambda} < 0
\]

for any \( \lambda \) and \( \mu > 0 \);

(iii) by (2.2), \( 0 < I(r, \lambda, \mu) < 1 \) for any \( \lambda \) and \( \mu > 0 \).

By (i) and (ii), we know that for fixed \( \mu \), \( R(r, \lambda, \mu) \) is strictly decreasing from \( -\infty \) to \( +\infty \). Then for any \( -1 < a < 1 \) and fixed \( \mu \), there exists a unique real number \( \lambda(\mu, a) \) such that

\[
(2.7) \quad R(r, \lambda, \mu)\big|_{\lambda=a(\mu,a)} = a.
\]

Further, using the implicit function theorem, we have that the function \( \lambda = \lambda(\mu, a) \) defined on \( \{ (\mu, a) : \mu > 0, -1 < a < 1 \} \) is a continuous function and

\[
\frac{\partial \lambda(\mu, a)}{\partial \mu} = -\left( \frac{\partial I(r, \lambda, \mu)}{\partial \mu} - \frac{\partial R(r, \lambda, \mu)}{\partial \lambda} \right)_{\lambda=a(\mu,a)}.
\]

Next, we consider the function \( I(r, \lambda(\mu, a), \mu) \) for \( \mu > 0 \).

\[
\begin{align*}
\frac{\partial I(r, \lambda(\mu, a), \mu)}{\partial \mu} &= \left( \frac{\partial I(r, \lambda, \mu)}{\partial \lambda} \frac{\partial \lambda(\mu, a)}{\partial \mu} + \frac{\partial I(r, \lambda, \mu)}{\partial \mu} \right)_{\lambda=a(\mu,a)} \\
&= \left( \left( \frac{\partial R(r, \lambda, \mu)}{\partial \lambda} \frac{\partial I(r, \lambda, \mu)}{\partial \mu} - \frac{\partial R(r, \lambda, \mu)}{\partial \mu} \frac{\partial I(r, \lambda, \mu)}{\partial \lambda} \right) + \frac{\partial R(r, \lambda, \mu)}{\partial \lambda} \right)_{\lambda=a(\mu,a)}.
\end{align*}
\]

By (i) and (iii), we have \( \frac{\partial I(r, \lambda(\mu, a), \mu)}{\partial \mu} > 0 \), which shows that \( I(r, \lambda(\mu, a), \mu) \) is strictly increasing as a function of \( \mu \) on \( (0, +\infty) \) for a fixed \( a \). Note that (iii). Thus, for a fixed \( a \), \( I(r, \lambda(\mu, a), \mu) \) respectively has finite limit as \( \mu \to 0 \) and as \( \mu \to +\infty \).

For a fixed \( a \), we claim that \( I(r, \lambda(\mu, a), \mu) \to 0 \) as \( \mu \to 0 \), and \( I(r, \lambda(\mu, a), \mu) \to \sqrt{1-a^2} \) as \( \mu \to +\infty \).

As \( \mu \to 0 \), there exists a subsequence \( \mu_k \to 0 \) such that \( \lambda(\mu_k, a) \) has a finite limit \( t \) or tend to \( \infty \). We only need to prove that \( I(r, \lambda(\mu_k, a), \mu_k) \to 0 \) as \( k \to \infty \). Since \( I(r, \lambda(\mu_k, a), \mu_k) = \int_S \frac{1}{\sqrt{1+A_{r,\lambda(\mu_k, a),\mu_k}^2(\omega)}} d\sigma \), we only need to prove that \( |A_{r,\lambda(\mu_k, a),\mu_k}(\omega)| \to +\infty \) almost everywhere on \( S \). Note that

\[
|A_{r,\lambda(\mu_k, a),\mu_k}(\omega)| = \frac{1}{\mu_k} \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a)
\]

and

\[
\frac{1}{(1+r)^n} \leq \frac{1}{|rN - \omega|^n} \leq \frac{1}{(1-r)^n}.
\]

If \( \lambda(\mu_k, a) \to t \) as \( k \to \infty \), then \( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \) is bounded and \( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \neq 0 \) almost everywhere on \( S \). Thus \( |A_{r,\lambda(\mu_k, a),\mu_k}(\omega)| \to +\infty \) almost everywhere on \( S \). If \( \lambda(\mu_k, a) \to \infty \) as \( k \to \infty \), then it is obvious that \( |A_{r,\lambda(\mu_k, a),\mu_k}(\omega)| \to +\infty \) uniformly for \( \omega \in S \). The first claim is proved.
As $\mu \to +\infty$, $\frac{1}{\mu} |rN - \omega|^n \to 0$ uniformly for $\omega \in S$. If there exists a subsequence $\mu_k \to +\infty$ such that $\lambda(\mu_k, a)/\mu_k \to \infty$, then $|A_{r, \lambda(\mu_k, a), \mu_k}(\omega)| \to +\infty$ uniformly for $\omega \in S$, and $I(r, \lambda(\mu_k, a), \mu_k) \to 0$, a contradiction. This shows that for $\lambda(\mu, a)/\mu$ is bounded as $\mu \to +\infty$. Thus there exists a subsequence $\mu_k \to +\infty$ such that $-\lambda(\mu_k, a)/\mu_k$ tend to a finite limit $t$. That is

$$\lim_{k \to \infty} -\lambda(\mu_k, a)/\mu_k = t.$$  

we only need to prove that $I(r, \lambda(\mu_k, a), \mu_k) \to \sqrt{1 - a^2}$ as $k \to \infty$. Let $(A(\omega))_k = A_{r, \lambda(\mu_k, a), \mu_k}(\omega)$. By (2.1), (2.8) and $\mu_k \to +\infty$, we obtain

$$\lim_{k \to \infty} \frac{(A(\omega))_k}{\sqrt{1 + ((A(\omega))_k)^2}} = \lim_{k \to \infty} \frac{1}{\mu_k} \left( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \right) \sqrt{1 + \frac{1}{\mu_k} \left( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \right)^2}$$

$$= \lim_{k \to \infty} \frac{1}{\mu_k} \left( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \right) \sqrt{1 + \left( \frac{\lambda(\mu_k, a)}{\mu_k} \right)^2} = \frac{t}{\sqrt{1 + t^2}}$$

uniformly for $\omega \in S$, and

$$\lim_{k \to \infty} \frac{1}{\sqrt{1 + ((A(\omega))_k)^2}} = \lim_{k \to \infty} \frac{1}{\sqrt{1 + \frac{1}{\mu_k} \left( \frac{1}{|rN - \omega|^n} - \lambda(\mu_k, a) \right)^2}}$$

$$= \lim_{k \to \infty} \frac{1}{\sqrt{1 + \left( \frac{\lambda(\mu_k, a)}{\mu_k} \right)^2}} = \frac{1}{\sqrt{1 + t^2}}$$

uniformly for $\omega \in S$. By the Lebesgue’s dominated convergence theorem and (2.2), (2.9), (2.10) we have

$$\lim_{k \to \infty} R(r, \lambda(\mu_k, a), \mu_k) = \lim_{k \to \infty} \int_S \frac{(A(\omega))_k}{\sqrt{1 + ((A(\omega))_k)^2}} d\sigma$$

$$= \int_S \lim_{k \to \infty} \frac{(A(\omega))_k}{\sqrt{1 + ((A(\omega))_k)^2}} d\sigma$$

$$= \int_S \frac{t}{\sqrt{1 + t^2}} d\sigma$$

and

$$\lim_{k \to \infty} I(r, \lambda(\mu_k, a), \mu_k) = \lim_{k \to \infty} \int_S \frac{1}{\sqrt{1 + ((A(\omega))_k)^2}} d\sigma$$

$$= \int_S \lim_{k \to \infty} \frac{1}{\sqrt{1 + ((A(\omega))_k)^2}} d\sigma$$

$$= \int \frac{1}{\sqrt{1 + t^2}} d\sigma$$

Note that $R(r, \lambda(\mu_k, a), \mu_k) \equiv a$ by (2.7), and $\left( \frac{t}{\sqrt{1 + t^2}} \right)^2 + \left( \frac{1}{\sqrt{1 + t^2}} \right)^2 = 1$. Then by (2.11) we obtain that $\frac{t}{\sqrt{1 + t^2}} = a$ and $\frac{1}{\sqrt{1 + t^2}} = \sqrt{1 - a^2}$. Consequently by (2.12),

$$\lim_{k \to \infty} I(r, \lambda(\mu_k, a), \mu_k) = \sqrt{1 - a^2}.$$

The second claim is proved.
It is proved that $I(r, \lambda(\mu, a), \mu)$ is continuous and strictly increasing from 0 to $\sqrt{1-a^2}$ as $\mu$ increasing from 0 to $+\infty$. Thus, for any $0 < b < \sqrt{1-a^2}$ and $-1 < a < 1$, there exists a unique real number $\mu(a, b)$ such that

\begin{equation}
I(r, \lambda(\mu(a, b), a), \mu(a, b)) = b.
\end{equation}

Further, using the implicit function theorem, we have the function $\mu(a, b)$ defined on $\{(a, b) : a^2 + b^2 < 1, b > 0\}$ is a continuous function.

Denote $\lambda(\mu(a, b), a)$ by $\lambda(r, a, b)$. Denote $\mu(a, b)$ by $\mu(r, a, b)$. We have proved that there exist a unique pair of functions $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b)$ such that

\[
R(r, \lambda(r, a, b), \mu(r, a, b)) = a, \quad I(r, \lambda(r, a, b), \mu(r, a, b)) = b
\]
on the upper half disk. The real analyticity of $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b)$ is asserted by the implicit function theorem. The lemma is proved. \hfill \Box

Let $a$ and $b$ be two numbers such that $0 \leq b < 1$, $-1 < a < 1$ and $a^2 + b^2 < 1$. Let $U_{a, b}$ denote the class of real-valued functions $u \in L^\infty(S)$ satisfying the following conditions:

\begin{equation}
\|u\|_{\infty} \leq 1, \quad \int_S u(\omega) d\sigma = a, \quad \int_S \sqrt{1 - u^2(\omega)} d\sigma \geq b.
\end{equation}

Every function $u \in L^\infty(S)$ defines a harmonic function

\[
U(x) = \int_S \frac{1 - |x|^2}{|x - \omega|^n} u(\omega) d\sigma \quad \text{for} \quad x \in \mathbb{B}^n.
\]

Let $0 < r < 1$ and define a functional $L_r$ on $L^\infty(S)$ by

\begin{equation}
L_r(u) = U(rN) = \int_S \frac{1 - r^2}{|rN - \omega|^n} u(\omega) d\sigma.
\end{equation}

Obviously, $U_{a, b}$ is a closed set, and $L_r$ is a continuous functional on $U_{a, b}$. Then there exists an extremal function such that $L_r$ attains its maximum on $U_{a, b}$ at the extremal function. We will claim in the following lemma that the extremal function is unique. In the proof of the following lemma, we will construct a function $u_0$ first and then prove that $u_0$ is the unique extremal function, which will be denoted by $u_{a, b, r}$.

**Lemma 2.** For any $a$, $b$ and $r$ satisfying the above conditions, there exists a unique extremal function $u_{a, b, r} \in U_{a, b}$ such that $L_r$ attains its maximum on $U_{a, b}$ at $u_{a, b, r}$.

**Proof.** Let $a$, $b$ and $r$ be fixed. First assume that $b > 0$. From Lemma 1 we have $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b) > 0$ such that $R(r, \lambda, \mu) = a$ and $I(r, \lambda, \mu) = b$. For the need of (2.21), let

\begin{equation}
u_0(\omega) = \frac{A_{r, \lambda, \mu}(\omega)}{\sqrt{1 + A^2_{r, \lambda, \mu}(\omega)}},
\end{equation}

where $A_{r, \lambda, \mu}(\omega)$ is defined as (2.1). Then $||u_0||_{\infty} < 1$ and by (2.2), we know

\begin{equation}
\int_S u_0(\omega) d\sigma = R(r, \lambda, \mu) = a, \quad \int_S \sqrt{1 - u_0^2(\omega)} d\sigma = I(r, \lambda, \mu) = b.
\end{equation}

This means that $u_0 \in U_{a, b}$.

Let $u \in U_{a, b}$. By (2.14) and (2.17), we have

\begin{equation}\lambda \int_S (u_0(\omega) - u(\omega)) d\sigma = 0,
\end{equation}

\begin{equation}\mu \int_S (\sqrt{1 - u_0^2(\omega)} - \sqrt{1 - u^2(\omega)}) d\sigma \leq 0.
\end{equation}
By the Taylor formula of the function $\sqrt{1-x^2}$, we have
\[
\frac{\sqrt{1-u^2(\omega)} - \sqrt{1-u_0^2(\omega)}}{\sqrt{1-u_0^2(\omega)}} = \frac{u_0(\omega)(u_0(\omega) - u(\omega))}{\sqrt{1-u_0^2(\omega)}} - \frac{(u_0(\omega) - u(\omega))^2}{2(1-\xi^2)^{3/2}}
\]
(2.20)
where $\xi$ is a real number between $u_0(\omega)$ and $u(\omega)$. By (2.16) and (2.1), we have
\[
\frac{1}{|rN-\omega|^n} - \frac{\mu u_0(\omega)}{\sqrt{1-u_0^2(\omega)}} = 0.
\]
(2.21)
Then by (2.15) and (2.18) and (2.21), we obtain that
\[
\frac{L_r(u_0) - L_r(u)}{1-r^2} = \int_S \frac{u_0(\omega) - u(\omega)}{|rN-\omega|^n} d\sigma
\]
\[
\geq \int_S \frac{u_0(\omega) - u(\omega)}{|rN-\omega|^n} d\sigma - \lambda \int_S (u_0(\omega) - u(\omega)) d\sigma - \mu \int_S \left(\frac{u_0(\omega)(u_0(\omega) - u(\omega))}{\sqrt{1-u_0^2(\omega)}}\right) d\sigma
\]
\[
= \int_S \frac{u_0(\omega) - u(\omega)}{|rN-\omega|^n} d\sigma - \lambda \int_S (u_0(\omega) - u(\omega)) d\sigma - \mu \int_S \frac{u_0(\omega)(u_0(\omega) - u(\omega))}{\sqrt{1-u_0^2(\omega)}} d\sigma
\]
\[
+ \mu \int_S \left(\frac{u_0(\omega) - u(\omega))^2}{(1-\xi^2)^{3/2}}\right) d\sigma = 0.
\]
(2.22)
Thus $L_r(u_0) \geq L_r(u)$ with equality if and only if $\mu \int_S \frac{(u_0(\omega) - u(\omega))^2}{(1-\xi^2)^{3/2}} d\sigma = 0$. Therefore $L_r(u_0) \geq L_r(u)$ with equality if and only if $u(\omega) = u_0(\omega)$ almost everywhere. This shows that $u_0(\omega)$ is the unique extremal function, which will be denoted by $u_{a,b,r}(\omega)$.

Next we consider the case that $b = 0$. For a real number $d$, let
\[
S_d = \{ x \in S : |N-x| = d \},
\]
(2.23)
\[
S^+_d = \{ x \in S : |N-x| < d \},
\]
(2.24)
\[
S^-_d = \{ x \in S : |N-x| > d \}.
\]
For a fixed real number $a$ such that $-1 < a < 1$, there exists a unique real number $d_a$ such that $\sigma(S^+_d) = \frac{1+a}{2}$ and $\sigma(S^-_d) = \frac{1-a}{2}$. Let
\[
u_0(\omega) = \begin{cases} 
1, & \omega \in S^+_d; \\
0, & \omega \in S_d; \\
-1, & \omega \in S^-_d.
\end{cases}
\]
(2.25)
We want to prove that $u_0$ is just the unique extremal function, which will be denoted by $u_{a,0,r}(\omega)$.

It is obvious that $u_0 \in U_{a,0}$. Let $u \in U_{a,0}$. By (2.14) and (2.25), we have
\[
\int_S (u_0(\omega) - u(\omega)) d\sigma = 0,
\]
(2.26)
\[
u_0(\omega) - u(\omega) \geq 0 \quad \text{for} \quad \omega \in S^+_d,
\]
(2.27)
\[
u_0(\omega) - u(\omega) \leq 0 \quad \text{for} \quad \omega \in S^-_d,
\]
(2.28)
Let
\begin{equation}
J_a = |rN - x_0|, \quad \text{where } x_0 \in S_{da}.
\end{equation}

Note that
\begin{equation}
|rN - \omega| < J_a \quad \text{for } \omega \in S_{da}^+, \tag{2.30}
\end{equation}
\begin{equation}
|rN - \omega| > J_a \quad \text{for } \omega \in S_{da}^-.
\end{equation}

Then by (2.15) and (2.26)-(2.31), we obtain that
\begin{equation}
\frac{L_r(u_0) - L_r(u)}{1 - r^2} = \int_S \frac{u_0(\omega) - u(\omega)}{|rN - \omega|^n} d\sigma
= \int_S \left( \frac{1}{|rN - \omega|^n} - \frac{1}{J_a^n} \right) (u_0(\omega) - u(\omega)) d\sigma
= \int_{S_{da}^+} \left( \frac{1}{|rN - \omega|^n} - \frac{1}{J_a^n} \right) (u_0(\omega) - u(\omega)) d\sigma + \int_{S_{da}^-} \left( \frac{1}{|rN - \omega|^n} - \frac{1}{J_a^n} \right) (u_0(\omega) - u(\omega)) d\sigma
\geq 0.
\end{equation}

Thus \( L_r(u_0) \geq L_r(u) \) with equality if and only if \( u(\omega) = u_0(\omega) \) almost everywhere. The lemma is proved. \( \square \)

Let \( a \) and \( b \) be two real numbers with \( a^2 + b^2 < 1 \), and \( 0 < r < 1 \). If \( b \geq 0 \), \( u_{a,b,r} \) has been defined in Lemma 2. Now, define
\begin{equation}
v_{a,b,r}(\omega) = \sqrt{1 - u_{a,b,r}^2(\omega)} \quad \text{for } \omega \in S,
\end{equation}
and
\begin{equation}
U_{a,b,r}(x) = \int_S \frac{1 - |x|^2}{|x - \omega|^n} u_{a,b,r}(\omega) d\sigma,
\end{equation}
\begin{equation}
V_{a,b,r}(x) = \int_S \frac{1 - |x|^2}{|x - \omega|^n} v_{a,b,r}(\omega) d\sigma.
\end{equation}

For \( b < 0 \), let
\begin{equation}
U_{a,b,r}(x) = U_{a,-b,r}(x), \quad V_{a,b,r}(x) = -V_{a,-b,r}(x).
\end{equation}
Then for any \( a \in \mathbb{R} \), \( b \in \mathbb{R} \) and \( a^2 + b^2 < 1 \), let
\begin{equation}
F_{a,b,r}(x) = U_{a,b,r}(x) + iV_{a,b,r}(x) \quad \text{for } x \in \mathbb{B}^n.
\end{equation}
The harmonic function \( F_{a,b,r}(x) = U_{a,b,r}(x) + iV_{a,b,r}(x) \) satisfies \( F_{a,b,r}(0) = a + bi \) and \( F_{a,b,r}(\mathbb{B}^n) \subset \mathbb{D} \), since we will show that \( |U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 < 1 \). By the convexity of the square function,
\begin{equation}
|U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 \leq \int_S \frac{1 - |x|^2}{|x - \omega|^n} (u_{a,b,r}^2(\omega) + v_{a,b,r}^2(\omega)) d\sigma = 1
\end{equation}
with equality if and only if \( u_{a,b,r}(\omega) \) and \( v_{a,b,r}(\omega) \) are constants almost everywhere on \( S \). However \( u_{a,b,r}(\omega) \) is not possible a constant almost everywhere on \( S \). Thus \( |U_{a,b,r}(x)|^2 + |V_{a,b,r}(x)|^2 < 1 \).

The functions \( F_{a,b,r} \) are the extremal functions in the following lemma.

\textbf{Lemma 3.} \textit{Let } \( F(x) = U(x) + iV(x) \text{ be a harmonic function such that } F(\mathbb{B}^n) \subset \mathbb{D}, \ F(0) = a + bi. \quad \text{Then, for } 0 < r < 1 \text{ and } \omega \in S,}
\begin{equation}
U(r\omega) \leq U_{a,b,r}(rN)
\end{equation}
with equality at some point \( r\omega \) if and only if \( F(x) = F_{a,b,r}(xA) \), where \( A \) is an orthogonal matrix such that \( r\omega A = rN \), \( U_{a,b,r} \) is defined as (2.33) and (2.35), \( F_{a,b,r} \) is defined as (2.36). Further, \( U(x) < U_{a,b,r}(rN) \) for \( |x| < r \).

**Proof.** Step 1: First the case that \( r\omega = rN \) will be proved. Let \( 0 < \tilde{r} < 1 \) be fixed. Construct function

\[
G(x) = F(\tilde{r}x) \quad \text{for} \quad x \in \mathbb{B}^n.
\]

\( G(x) \) is harmonic on \( \mathbb{B}^n \) and \( G(0) = a + bi \). Let \( G(x) = u(x) + iv(x) \). Then

\[
\begin{align*}
(2.37) \quad &||u||_{\infty} \leq 1, \quad \int_S u(\omega)d\sigma = a, \quad \int_S \sqrt{1 - u^2(\omega)}d\sigma \geq \int_S |v(\omega)|d\sigma \geq \left| \int_S v(\omega)d\sigma \right| = |b|.
\end{align*}
\]

So by (2.14) we know that \( u \in U_{a,|b|} \) and by Lemma 2 we have \( u(rN) \leq U_{a,|b|,r}(rN) \) with equality if and only if \( u(\omega) = u_{a,|b|,r}(\omega) \) almost everywhere on \( S \). For \( u_{a,|b|,r}(\omega) \), by (2.17) and (2.23) we have

\[
\begin{align*}
(2.38) \quad &\int_S \sqrt{1 - u_{a,|b|,r}^2(\omega)}d\sigma = |b|.
\end{align*}
\]

If \( u(\omega) = u_{a,|b|,r}(\omega) \) almost everywhere on \( S \), then by (2.33) and (2.35), we have

\[
\begin{align*}
&u(x) = U_{a,|b|,r}(x) = U_{a,b,r}(x) \quad \text{for} \quad x \in \mathbb{B}^n;
\end{align*}
\]

and by (2.32), we have

\[
(2.39) \quad \sqrt{1 - u_{a,|b|,r}^2(\omega)} = \sqrt{1 - u^2(\omega)}.
\]

Note that by (2.37), (2.38) and (2.39) we have

\[
|b| = \int_S u_{a,|b|,r}(\omega)d\sigma \geq \int_S |v(\omega)|d\sigma \geq \left| \int_S v(\omega)d\sigma \right| = |b|.
\]

Then

\[
\begin{align*}
&v(\omega) = v_{a,|b|,r}(\omega) \quad \text{almost everywhere on} \quad S \quad \text{when} \quad b \geq 0,
\end{align*}
\]

\[
\begin{align*}
&v(\omega) = -v_{a,|b|,r}(\omega) \quad \text{almost everywhere on} \quad S \quad \text{when} \quad b < 0.
\end{align*}
\]

So

\[
v(x) = V_{a,b,r}(x) \quad \text{for} \quad x \in \mathbb{B}^n.
\]

For \( G(x) = u(x) + iv(x) \), it is proved that \( u(rN) \leq U_{a,b,r}(rN) \) with equality if and only if \( G(x) = F_{a,b,r}(x) \). Now let \( \tilde{r} \to 1 \). Note that

\[
\lim_{\tilde{r} \to 1} G(x) = \lim_{\tilde{r} \to 1} F(\tilde{r}x) = F(x), \quad \lim_{\tilde{r} \to 1} u(rN) = U(rN).
\]

Then by the result for \( G(x) \), we have \( U(rN) \leq U_{a,b,r}(rN) \) with equality if and only if \( F(x) = F_{a,b,r}(x) \).

Step 2: Now we prove the case that \( r\omega \neq rN \). Construct function

\[
\tilde{F}(x) = F(xA^{-1}) \quad \text{for} \quad x \in \mathbb{B}^n,
\]

where \( A \) is an orthogonal matrix such that \( r\omega A = rN \) and \( A^{-1} \) is the inverse matrix of \( A \). By [4], we know that \( \tilde{F}(x) \) is also a harmonic function. Let \( \tilde{F}(x) = \tilde{U}(x) + i\tilde{V}(x) \). Note that \( \tilde{F}(0) = a + bi \). Then by the result of step 1, we have \( \tilde{U}(rN) \leq U_{a,b,r}(rN) \) with equality if and only if \( \tilde{F}(x) = F_{a,b,r}(x) \). Note that \( \tilde{U}(rN) = U(rNA^{-1}) = U(r\omega) \) and \( \tilde{F}(x) = F(xA^{-1}) \). Thus \( U(r\omega) \leq U_{a,b,r}(rN) \) with equality if and only if \( F(xA^{-1}) = F_{a,b,r}(x) \). It is just that \( U(r\omega) \leq U_{a,b,r}(rN) \) with equality if and only if \( F(x) = F_{a,b,r}(xA) \).

Step 3: We will show that \( U(x) < U_{a,b,r}(rN) \) for \( |x| < r \). By the result of step 2 and the maximum principle, we have \( U(x) \leq U_{a,b,r}(rN) \) for \( |x| \leq r \). If the equality holds for some \( x_0 \) with \( |x_0| < r \), then \( U(x) \) must be equal to \( U_{a,b,r}(rN) \) identically for \( |x| \leq r \). Note that if \( U(rN) = U_{a,b,r}(rN) \), then by the result of step 1, we have \( U(x) = U_{a,b,r}(x) \). Thus \( U_{a,b,r}(x) \equiv U_{a,b,r}(rN) \) for \( |x| \leq r \). However, it is impossible since \( U_{a,b,r} \) is not a constant. The proof of the lemma is complete. \( \square \)
Lemma 4. For fixed $0 < r < 1$ and $x \in \mathbb{B}^n$, $F_{a,b,r}(x)$ is defined as (2.36). Then $F_{a,b,r}(x)$, as a function of variables $a$ and $b$, is analytic in the real sense on the open half disk $\{ (a,b) : b > 0, a^2 + b^2 < 1 \}$ and is continuous to the real diameter.

Proof. Let $0 < r < 1$ and $x \in \mathbb{B}^n$ be fixed. It is obvious that $F_{a,b,r}(x)$ is analytic in the real sense on the open half disk, since it is determined there by the functions $\lambda(r,a,b)$ and $\mu(r,a,b)$ formulated in Lemma 1, which are analytic in the real sense on the open half disk $\{ (a,b) : b > 0, a^2 + b^2 < 1 \}$.

We only need to prove that $F_{a,b,r}(x)$ is continuous at the points of the real diameter. Note that (2.36). Then we only need to prove that $U_{a,b,r}(x)$ and $V_{a,b,r}(x)$ are continuous at the points of the real diameter.

Let $-1 < a_0 < 0$ be given. We want to prove that $U_{a,b,r}(x)$ and $V_{a,b,r}(x)$ is continuous at $(a_0, 0)$.

It is just to prove that $U_{a,b,r}(x) \to U_{a_0,0,r}(x)$ and $V_{a,b,r}(x) \to V_{a_0,0,r}(x)$ as $(a,b) \to (a_0,0)$.

Step 1: For the case that $(a,b) \to (a_0,0)$ with $b \to 0$, by (2.33) and (2.34), we only need to prove $u_{a,0,r}(\omega) \to u_{a_0,0,r}(\omega)$ almost everywhere on $S$ as $(a,b) \to (a_0,0)$. Recall that

$$u_{a,0,r}(\omega) = \begin{cases} 1, & \omega \in S^+_d; \\ 0, & \omega \in S_d^-; \\ -1, & \omega \in S_d^-, \end{cases}$$

where $S^+_d$, $S_d$ and $S_d^-$ are defined as (2.22), (2.23) and (2.24). This shows that $u_{a,0,r}(\omega) \to u_{a_0,0,r}(\omega)$ almost everywhere on $S$ as $(a,b) \to (a_0,0)$.

Step 2: For the case that $(a,b) \to (a_0,0)$ with $b > 0$, by (2.33) and (2.34), we only need to prove $u_{a,b,r}(\omega) \to u_{a_0,0,r}(\omega)$ for any $\omega \in S$ as $(a,b) \to (a_0,0)$ with $b > 0$.

First we want to prove that $\mu(r,a_0,b) \to 0$ as $(a,b) \to (a_0,0)$ with $b > 0$, where $\mu(r,a,b)$ is defined as $\mu(a,b)$ in (2.13). Assume that $\mu(r,a_0,b) \to 0$ as $(a,b) \to (a_0,0)$ with $b > 0$. Then there exists a sequence $(ak,bk) \to (a_0,0)$ with $bk > 0$ such that $\mu_k = \mu(r,ak,bk)$ has a positive lower bound since $\mu(r,a,b) > 0$. Then by (2.13) and (2.2), we have

$$\int_S \left( 1 + \frac{1}{\mu_k^2} \left( \frac{1}{|r^N - \omega|^n} - \lambda_k \right)^2 \right)^{-1/2} d\sigma = I(r, \lambda_k, \mu_k) = bk \to 0,$$

where $\lambda_k = \lambda(r,ak,bk)$, $\lambda(r,a,b)$ is defined as $\lambda(\mu(a,b),a)$ in (2.13). Thus $\lambda_k \to \infty$. Assume that $\lambda_k \to +\infty$. Then by (2.16), (2.1) and (2.17), we obtain that

$$u_{ak,bk,r}(\omega) = \frac{1}{\mu_k} \left( \frac{1}{|r^N - \omega|^n} - \lambda_k \right) \left( 1 + \frac{1}{\mu_k^2} \left( \frac{1}{|r^N - \omega|^n} - \lambda_k \right)^2 \right)^{1/2} \to -1,$$

uniformly for $\omega \in S$, and $ak \to -1$, a contradiction.

Now we want to prove that

$$\lambda(r,a,b) \to \lambda_0 = \frac{1}{J_{a_0}^n},$$

as $(a,b) \to (a_0,0)$ with $b > 0$, where $J_{a_0}^n$ is defined as (2.29). In contrary, assume that $\lambda(r,a,b) \to \lambda_0$ as $(a,b) \to (a_0,0)$ with $b > 0$. Then there is a sequence $(ak,bk) \to (a_0,0)$ with $bk > 0$ such that $\lambda_k = \lambda(r,ak,bk) \to \lambda' \neq \lambda_0$. If $\lambda' = \infty$, then, as above, $|ak| \to 1$, a contradiction. In the case that $\lambda'$ is finite, by (2.16), (2.1) and (2.17) we have

$$u_{ak,bk,r}(\omega) = \frac{1}{\mu_k} \left( \frac{1}{|r^N - \omega|^n} - \lambda_k \right) \left( \mu_k^2 \left( \frac{1}{|r^N - \omega|^n} - \lambda_k \right)^2 \right)^{1/2} \to \text{sgn} \left\{ \frac{1}{|r^N - \omega|^n} - \lambda' \right\},$$

and

$$a_k = \int_S u_{ak,bk,r}(\omega) d\sigma \to \int_S \text{sgn} \left\{ \frac{1}{|r^N - \omega|^n} - \lambda' \right\} d\sigma.$$
\[ = \begin{cases} 
-1, & \lambda' \geq 1/(1-r)^n; \\
1, & \lambda' \leq 1/(1+r)^n; \\
da', & \lambda' = 1/J_{a'}^n, -1 < a' < 1, \ a' \neq a_0.
\end{cases} \]

This contradicts \( a_k \to a_0 \).

It is proved that \( \mu(r,a,b) \to 0 \) and \( \lambda(r,a,b) \to \lambda_0 \) as \((a,b) \to (a_0,0)\) with \( b > 0 \). Thus,
\[ u_{a,b,r}(\omega) \to \text{sgn} \left( \frac{1}{|rN-\omega|^n} - \lambda_0 \right) = u_{a_0,0,r}(\omega). \]

Step 3: For the case that \((a,b) \to (a_0,0)\) with \( b < 0 \), by the result of step 2, we know that 
\[ U_{a,b,r}(x) \to U_{a_0,0,r}(x) \text{ and } V_{a,b,r}(x) \to V_{a_0,0,r}(x) \text{ as } (a,b) \to (a_0,0) \text{ with } -b > 0. \]

Then we have \( U_{a,b,r}(x) \to U_{a_0,0,r}(x) \) and \( V_{a,b,r}(x) \to V_{a_0,0,r}(x) = 0 \) as \((a,b) \to (a_0,0)\) with \( b < 0 \).

It is proved that \( U_{a,b,r}(x) \) and \( V_{a,b,r}(x) \) is continuous at \((a_0,0)\). The lemma is proved. \( \square \)

3. Main results

For \(-\pi \leq \beta \leq \pi\) and real number \( \delta \), denote the straight line \( l(\beta, \delta) \) and closed half plane \( P(\beta, \delta) \) by
\[ l(\beta, \delta) = \{ w = u + iv : \text{Re}\{we^{-i\beta}\} = u \cos \beta + v \sin \beta = \delta \} \]
and
\[ P(\beta, \delta) = \{ w = u + iv : \text{Re}\{we^{-i\beta}\} = u \cos \beta + v \sin \beta \leq \delta \}. \]

**Theorem 2.** Let \( 0 < r < 1 \) and \( 0 \leq \rho < 1 \). Denote
\[ P_\beta = P(\beta, U_{\rho \cos \beta, -\rho \sin \beta, r}(rN)), \quad l_\beta = l(\beta, U_{\rho \cos \beta, -\rho \sin \beta, r}(rN)), \]
and define
\[ E_{\rho, \beta} = \bigcap_{-\pi \leq \beta \leq \pi} P_\beta, \]
\[ \Gamma_{\rho, \beta} = \{ w : w = f_{\rho, \beta}(\beta) = e^{i\beta} F_{\rho \cos \beta, -\rho \sin \beta, r}(rN), -\pi \leq \beta \leq \pi \}, \]
where \( U_{\rho \cos \beta, -\rho \sin \beta, r} \) is defined as \((2.33)\) and \((2.35)\), \( F_{\rho \cos \beta, -\rho \sin \beta, r} \) is defined as \((2.36)\). Then:

1. For any harmonic function \( F \) such that \( F(\mathbb{B}^n) \subset \mathbb{D} \) and \( F(0) = \rho \), we have \( F(\overline{\mathbb{B}}_r) \subset E_{\rho, \beta}; \)
2. \( E_{\rho, \beta} \) is a closed convex domain and symmetrical with respect to the real axis, and \( \rho \) is an interior point of \( E_{\rho, \beta}; \)
3. \( \Gamma_{\rho, \beta} \) is a convex Jordan closed curve and \( \partial E_{\rho, \beta} = \Gamma_{\rho, \beta}; \)
4. For any \( w' \in E_{\rho, \beta} \), there is a harmonic function \( F \) such that \( F(\mathbb{B}^n) \subset \mathbb{D}, F(0) = \rho \) and \( F(rN) = w' \).

**Proof.** (1) Denote
\[ P'_\beta = P(0, U_{\rho \cos \beta, -\rho \sin \beta, r}(rN)), \quad l'_\beta = l(0, U_{\rho \cos \beta, -\rho \sin \beta, r}(rN)). \]
\( P_\beta \) and \( l_\beta \) are obtained from \( P'_\beta \) and \( l'_\beta \) by an anti-clockwise rotation of angle \( \beta \).

Let \( F \) be a harmonic function such that \( F(\mathbb{B}^n) \subset \mathbb{D} \) and \( F(0) = \rho \). For \(-\pi \leq \beta \leq \pi\), let \( F_\beta = e^{-i\beta} F \). Then, \( F_\beta(\mathbb{B}^n) \subset \mathbb{D} \) and \( F_\beta(0) = \rho \cos \beta - i \sin \beta \). Using lemma 3 to the harmonic function \( F_\beta \), we have \( F_\beta(\overline{\mathbb{B}}_r) \subset P'_\beta \) and, consequently, \( F(\overline{\mathbb{B}}_r) \subset P_\beta \). This shows (1).

(2) It is obvious that \( E_{\rho, \beta} \) is a closed convex set and symmetrical with respect to the real axis.

We only need to prove that \( \rho \) is an interior point of \( E_{\rho, \beta}. \)

First we want to prove that \( f_{\rho, \beta}(\beta) \in \partial E_{\rho, \beta} \) for \(-\pi \leq \beta \leq \pi\). \( f_{\rho, \beta}(\beta) \in l_\beta \) since \( F_{\rho \cos \beta, -\rho \sin \beta, r}(rN) \in l'_\beta \). Let \( G(x) = e^{i\beta} F_{\rho \cos \beta, -\rho \sin \beta, r}(x) \). The harmonic function \( G \) satisfies the conditions \( G(\mathbb{B}^n) \subset \mathbb{D} \) and \( G(0) = \rho \). By (1), \( f_{\rho, \beta}(\beta) = G(rN) \in E_{\rho, \beta} \). Note that \( E_{\rho, \beta} \subset P_\beta, l_\beta = \partial P_\beta \) and \( f_{\rho, \beta}(\beta) \in l_\beta \) proved above. Then we have \( f_{\rho, \beta}(\beta) \in \partial E_{\rho, \beta}. \)
For $f_{r,\rho}(0), f_{r,\rho}(\pi), f_{r,\rho}(\pi/2)$ and $f_{r,\rho}(-\pi/2)$, by lemma 3, we have

$$f_{r,\rho}(0) = F_{r,0,r}(rN) = U_{r,0,r}(rN) > U_{r,0,0}(0) = \rho,$$

$$f_{r,\rho}(\pi) = -F_{-r,0,r}(rN) = -U_{-r,0,r}(rN) < -U_{-r,0,0}(0) = \rho,$$

$$\text{Im} f_{r,\rho}(\pi/2) = U_{-r,-r,r}(rN) = U_{0,0,r}(rN) > U_{0,0,0}(0) = 0,$$

$$\text{Im} f_{r,\rho}(-\pi/2) = -U_{-r,0,r}(rN) < -U_{0,0,r}(0) = 0.\tag{3.3}$$

Then $\rho$ is an interior point of $E_{r,\rho}$ since $E_{r,\rho}$ is a convex set.

(3) First we want to prove that $\Gamma_{r,\rho}$ is a Jordan closed curve. $\Gamma_{r,\rho}$ is close and continuous by Lemma 4. Assume that there exist $0 < \beta_1 < \beta_2 < \pi$ such that $\omega_0 = f_{r,\rho}(\beta_1) = f_{r,\rho}(\beta_2)$. Then $\beta_2 - \beta_1 < \pi$ and $\omega_0$ is the vertex of the angular domain $P_{\beta_1} \cap P_{\beta_2}$. Further, it is easy to see that $f_{r,\rho}(\beta) = \omega_0$ for $\beta_1 < \beta < \beta_2$, since $l_\beta \cap \partial E_{r,\rho} = \omega_0$ and $f_{r,\rho}(\beta) \in l_\beta \cap \partial E_{r,\rho}$. $f_{r,\rho}(\beta)$ is analytic on $(0, \pi)$ in the real sense by Lemma 4. Then we have that $f_{r,\rho}(\beta) = \omega_0$ for $0 < \beta < \pi$ and, by the continuity, $f_{r,\rho}(0) = f_{r,\rho}(\pi) = \omega_0$. A contraction, since $f_{r,\rho}(0) > f_{r,\rho}(\pi)$ by (3.2) and (3.3). This shows that $\Gamma_{r,\rho}^f = \{ w = f_{r,\rho}(\beta) : 0 \leq \beta \leq \pi \}$ is a Jordan curve. By the same reason, $\Gamma_{r,\rho}^- = \{ w = f_{r,\rho}(\beta) : -\pi \leq \beta \leq 0 \}$ is also a Jordan curve. Then $\Gamma_{r,\rho}$ is a Jordan closed curve.

For $-\pi \leq \beta \leq \pi$, it is proved in (2) that $f_{r,\rho}(\beta) \in \partial E_{r,\rho}$. Then $\Gamma_{r,\rho} \subset \partial E_{r,\rho}$. Note that $\partial E_{r,\rho}$ must be a convex Jordan closed domain. Thus $\partial E_{r,\rho} = \Gamma_{r,\rho}$.

(4) For $w' \in E_{r,\rho}$, draw a straight line $l$ passing through $w'$ and intersect $\partial E_{r,\rho}$ at $w_1$ and $w_2$. Let $w' = k_1 w_1 + k_2 w_2$ with $k_1, k_2 \geq 0$ and $k_1 + k_2 = 1$. There are two real numbers $\beta_1$ and $\beta_2$ such that $f_{r,\rho}(\beta_1) = w_1$ and $f_{r,\rho}(\beta_2) = w_2$. Then the harmonic function $F = k_1 e^{i\beta_1} F_{r} \cos\beta_1 - r \sin\beta_1, r + k_2 e^{i\beta_2} F_{r} \cos\beta_2 - r \sin\beta_2, r$ satisfies $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = \rho$ and $F(rN) = w'$. The theorem is proved. □

When $\rho = 0$, we have a corollary as follows, which is coincident with (1.10).

**Corollary 1.** Let $0 < r < 1$. For any harmonic mapping $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = 0$, we have

$$F(\overline{B}_r) \subset \mathcal{D}_{U(rN)},$$

where $U$ is the Poisson integral of the function that equals 1 on $S^+$ and -1 on $S^-$.

**Proof.** By Theorem 2, we only need to prove that $E_{r,0} = \mathcal{D}_{U(rN)}$. Further, by the definition of $E_{r,\rho}$ in Theorem 2, we only need to prove that $U_{0,0,r}(rN) = U(rN)$. Note that by (2.25),

$$u_{0,0,r}(\omega) = \begin{cases} 1, & \omega \in S^+; \\ 0, & \omega \in S; \\ -1, & \omega \in S^- . \end{cases}\tag{2.33}$$

Then by (2.33) we know that $U_{0,0,r}(rN) = U(rN)$. The corollary is proved. □

From Theorem 2, we obtain Theorem 1, which is the general version of the above Theorem 2.

**References**

[1] Chen H. H., The Schwarz-Pick lemma for planar harmonic mappings, SCIENCE CHINA Mathematics, 2011, 54(6):1101-1118.

[2] Ahlfors L. V., Conformal invariants: Topics in geometric function theory, New York: McGraw-Hill, 1973, 1-3.

[3] Heinz E., On one-to-one harmonic mappings, Pacific J Math, 1959, 9: 101-105.

[4] Axler S., Bourdon P., Wade R., Harmonic function theory, Second Edition, New York: Springer-Verlag, 2001.
A NOTE ON SCHWARZ-PICK LEMMA

Department of General Study Program, Jinling Institute of Technology, Nanjing 211169, China

E-mail address: dymdsey@163.com

Department of Mathematical Sciences, Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA

E-mail address: pan@ipfw.edu