DANTE: Deep AlterNations for Training nEural networks

Sneha Kudugunta, Vaibhav B Sinha, Adepu Ravi Sankar, Surya Teja Chavali, Purushottam Kar, Vineeth N Balasubramanian

Abstract—We present DANTE, a novel method for training neural networks using the alternating minimization principle. DANTE provides an alternate perspective to traditional gradient-based backpropagation techniques commonly used to train deep networks. It utilizes an adaptation of quasi-convexity to cast training a neural network as a bi-quasi-convex optimization problem. We show that for neural network configurations with both differentiable (e.g. sigmoid) and non-differentiable (e.g. ReLU) activation functions, we can perform the alternations very effectively. DANTE can also be extended to networks with multiple hidden layers. In experiments on standard datasets, neural networks trained using the proposed method were found to be very promising and competitive to traditional backpropagation techniques, both in terms of quality of the solution, as well as training speed.

Index Terms—Neural nets, Deep Learning, Backpropagation, Machine Learning.

1 INTRODUCTION

For much of the recent march of deep learning, gradient-based backpropagation methods, e.g. Stochastic Gradient Descent (SGD) and its variants, have been the mainstay of practitioners. The use of these methods, especially on vast amounts of data, has led to unprecedented progress in several areas of artificial intelligence. The intense focus on these techniques has led to an intimate understanding of hardware requirements and code optimizations needed to execute these routines on large datasets in a scalable manner. Today, myriad off-the-shelf and highly optimized packages exist that can churn reasonably large datasets on GPU architectures with relatively mild human involvement and little bootstrap effort.

However, this surge of success of backpropagation-based methods in recent years has somewhat overshadowed the need to continue to look for options beyond backpropagation to train deep networks. Despite several advancements in deep learning with respect to novel architectures such as encoder-decoder networks and generative adversarial models, the reliance on backpropagation methods remains. While reinforcement learning methods are becoming increasingly popular, their scope is limited to a particular family of settings such as agent-based systems or reward-based learning. Several works have studied the limitations of SGD-based backpropagation, whether it be vanishing gradients, especially for certain activation functions [1]; the tendency of SGD to face difficulties with saddle points [2] - even for simple architectures [3] or even more subtle issues as seen in [4].

From another perspective, there has been marked progress in recent years in the broader area of non-convex optimization. Several alternate algorithms with provable guarantees, such as iterative hard thresholding [5], alternating minimization [6], [7] and [8]. Some recent methods such as [9], [10] and [11] use tensor methods to prove bounds for neural networks under specific settings. In this work, we investigate a non-backpropagation strategy to train neural networks, leveraging recent advances in quasi-convex optimization. Our method is called DANTE (Deep AlterNations for Training nEural networks), and it offers an alternating minimization-based technique for training neural networks.

DANTE is based on the simple but useful observation that the problem of training a single hidden-layer neural network can be cast as a bi-quasiconvex optimization problem (described in Section 3.1). This observation allows us to use an alternating optimization strategy to train the neural network, where each step involves solving relatively simpler quasi-convex problems. DANTE then uses efficient solvers for quasi-convex problems such as stochastic normalized gradient descent [12] to train the neural network using alternating minimization. The key contributions of this work are summarized below:

- We show that the error in each layer of an neural network can, in fact, be viewed as a quasi-convex function, thus allowing us to treat a single hidden-layer neural network as a bi-quasi-convex optimization problem. This allows us to propose an alternating minimization strategy, DANTE, where each quasi-convex optimization problem can be solved effectively (using Stochastic Normalized Gradient Descent (SNGD) [12]).

- S. Kudugunta was with the Department of Computer Science and Engineering, Indian Institute of Technology Hyderabad, India when she worked on this project.

- V. Sinha, A. Sankar and V. Balasubramanian are with the Department of Computer Science and Engineering, Indian Institute of Technology Hyderabad, India.

- S. Chavali is with Department of Computer Sciences, University of Wisconsin-Madison.

- P. Kar is with the Department of Computer Science and Engineering, Indian Institute of Technology Kanpur, India.
• While earlier results on the effectiveness of SNGD for solving a quasi-convex problem was restricted to a simple sigmoid Generalized Linear Model (GLM), we show that SNGD can converge in high probability to an ε-suboptimal solution even in case of layers of a neural network. We also expand the scope to include Rectified Linear Units (ReLU) activation functions and its variants by introducing a Generalized ReLU activation function.

• We show DANTE can be extended to train deep neural networks with multiple hidden layers.

• We empirically validate DANTE with both the generalized ReLU and sigmoid activations and establish that DANTE provides competitive or better performance on several standard datasets, when compared to standard mini-batch SGD-based backpropagation.

We now review earlier related efforts, before presenting details of the proposed methodology.

2 Related Work

Backpropagation-based techniques date back to the early days of neural network research [13], but remain to this day, the most commonly used methods for training a variety of neural networks including multi-layer perceptrons, convolutional neural networks, autoencoders, recurrent networks and the like.

In recent years, Taylor et al. proposed a method to train neural networks using the Alternating Direction Method of Multipliers (ADMM) and Bregman iterations [15]. The focus of this method, however, was on scaling the training of neural networks to a distributed setting on multiple cores across a computing cluster. Jaderberg also proposed the idea of ‘synthetic gradients’ in [16]. While this approach is interesting, this work is more focused towards a more efficient way to carry out gradient-based parameter updates in a neural network. More recently, Jagatap and Hegde [17] proposed a method to train single hidden layer ReLU networks using an alternating minimization technique. Unlike our method, this method alternates between updating weights, and state variables which indicate which ReLU activations are on, and so is very specific to ReLU activations.

In our work, we focus on an entirely new approach to training neural networks using alternating optimization and quasi-convexity (different from the abovementioned methods), and show that this approach shows promising results on a range of datasets. Although alternating minimization has found much appeal in areas such as matrix factorization [6], to the best of our knowledge, this is the one of the early efforts in using alternating principles to train feedforward neural networks effectively.

Some other efforts have also explored target propagation based methods, such as in [18], Difference Target Propagation [19] and target propagation in a Bayesian setting [20]. There are also efforts that use random feedback weights such as feedback-alignment [21] and direct/indirect feedback-alignment [22] where the weights used for propagation need not be symmetric with the weights used for forward propagation. We however do not focus on credit assignment in this work. However, one could view the proposed method as carrying out ‘implicit’ credit assignment using partial derivatives, but there is no defined model for credit assignment in our work.

3 Deep Alternations for Training Neural Networks (DANTE)

3.1 Problem Formulation

Consider a neural network with \(L\) layers. Each layer \(l \in \{1, 2, \ldots, L\}\) has \(n_l\) nodes and is characterized by a linear operator \(W_l \in \mathbb{R}^{n_{l-1} \times n_l}\) and a non-linear activation function defined as \(\phi_l : \mathbb{R}^{n_l} \rightarrow \mathbb{R}^{n_l}\). The activations generated by layer \(l\) are denoted by \(a_l \in \mathbb{R}^{n_l}\). We denote by \(a_0\), the input activations and \(n_0\) to be the number of input activations i.e. \(a_0 \in \mathbb{R}^{n_0}\). Each layer uses activations being fed into it to compute its own activations as \(a_l = \phi_l(W_l,a_{l-1}) \in \mathbb{R}^{n_l}\), where \(\phi(\cdot,\cdot)\) denotes \(\phi((\cdot,\cdot))\) for simplicity of notation. A multi-layer neural network is formed by nesting such layers to form a composite function \(f\) given as follows:

\[
f(W;x) = \phi_L(W_L, \phi_{L-1}(W_{L-1}, \ldots, \phi_1(W_1, x)))
\]

where \(W = \{W_l\}\) is the collection of all the weights through the network, and \(x = a_0\) contains the input activations for each training sample.

Given \(m\) data samples \(\{(x_i,y_i)\}_{i=1}^m\) from a distribution \(D\), the network is trained by tuning the weights \(W\) to minimize a given loss function, \(J\):

\[
\min_W \mathbb{E}_{(x,y) \sim D}[J(f(W;x),y)]
\]

For purpose of simplicity and convenience, we first consider the case of a single hidden layer neural network, represented as \(f(W;x) = \phi_2(W_2, \phi_1(W_1,x))\) to describe our methodology. We later describe how this can be extended to multi-layer neural networks. A common loss function used to train neural networks is the squared loss function which yields the following objective:

\[
\min_W \mathbb{E}_{(x,y) \sim D}\|f(W;x) - y\|^2_2
\]

where:

\[
\|f(W;x) - y\|^2_2 = \|\phi_2(W_2, \phi_1(W_1,x)) - y\|^2_2
\]

An important observation here is that if we fix \(W_1\), then Eqn (4) turns into a set of Generalized Linear Model (GLM) problems with \(\phi_2\) as the activation function, i.e.

\[
\min_{W_2} \mathbb{E}_{(x,y) \sim D}\|\phi_2(W_2,z) - y\|^2_2
\]

where \(z = \phi_1(W_1,x)\). In particular, a GLM with differentiable activation functions such as sigmoid satisfy a property called Strict Locally Quasi-Convexity (SLQC), which allows techniques such as SNGD to solve the GLM problem effectively [12]. We exploit this observation, and generalize this result in a few ways: (i) we firstly show that a GLM with non-differentiable activation functions such as ReLUs (and its variants) also satisfy the SLQC property; (ii) we show that a set of GLMs, such as in a layer of a neural network, also satisfy the SLQC property; and (iii) we leverage these generalizations to develop an alternating minimization methodology to train neural networks. Section 3.2 describes these generalizations further.

Optimizing for \(W_1\) in Equation 4 cannot be viewed as a set of SLQC GLMs. To this end, we provide a generalization of local quasi-convexity in Section 3.2 and show that fixing \(W_2\) does indeed turn the problem below into yet another
SLQC problem, this time with $W_1$ as the parameter (note that $\phi_{W_2}(\cdot) = \phi_2(W_2, \phi_1(\cdot))$):

$$\min_{W_1} \mathbb{E}_{(x, y) \sim D} \|\phi_{W_2}(W_1, x) - y\|_2^2$$  \hspace{1cm} (6)

Putting Equations 5 and 6 together now gives us a single-layer neural network setup, where each layer is individually SLQC, and can be efficiently solved using SNGD. This allows us to propose our alternating minimization strategy to train a neural network in an effective manner.

### 3.2 Background and Preliminaries

Let $\| \cdot \|$ denote the $L_2$ (Euclidean) norm for vectors, and $\| \cdot \|_F$ denote the Frobenius norm of a matrix. We sometimes drop the subscript $F$ from $\| \cdot \|_F$ for brevity and clarity of understanding (the appropriate norm can be identified from the context). $B(x, r)$ denotes a Euclidean ball of radius $r$ with $x$ as centre and $B$ denotes $B(0, 1)$. We begin with the formal definitions of Local Quasi-Convexity and Generalized Linear Model (GLM).

**Definition 1 (Local-Quasi-Convexity [12]).** Let $x, z \in \mathbb{R}^d, \kappa, \epsilon > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Then $f$ is said to be $(\epsilon, \kappa, z)$-Strictly-Locally-Quasi-Convex (SLQC) in $x$, if at least one of the following applies:

1. $f(x) - f(z) \leq \epsilon$
2. $\langle \nabla f(x), y \rangle > 0$, and $\forall y \in B(z, \epsilon/\kappa), \langle \nabla f(x), y - x \rangle \leq 0$

where $B(z, \epsilon/\kappa)$ is a ball centered at $z$ with radius $\epsilon/\kappa$.

**Definition 2 (Idealized and Noisy Generalized Linear Model [12]).** In the idealized GLM setting, we are given $m$ samples $\{(x_i, y_i)\}_{i=1}^m \in \mathbb{B} \times [0, 1]$ and an activation function $\phi : \mathbb{R} \to \mathbb{R}$. Moreover, there exists $w^* \in \mathbb{R}^d$ such that $y_i = \phi(w^*, x_i) \forall i \in \{1, \cdots, m\}$ where $w^*$ is the global minimizer of the empirical error function:

$$err(w) = \frac{1}{m} \sum_{i=1}^m (y_i - \phi(w, x_i))^2$$

In the noisy GLM setting, we are given $m$ samples $\{(x_i, y_i)\}_{i=1}^m \in \mathbb{B}_d \times [0, 1]$ drawn i.i.d. from an unknown distribution $\mathcal{D}$. There exists a $w^* \in \mathbb{R}^d$ such that $E_{(x,y) \sim \mathcal{D}}[y|x] = \phi(w^*, x)$, and $w^*$ is the global minimizer of:

$$err(w) = E_{(x,y) \sim \mathcal{D}}[(y - \phi(w, x))^2]$$

Hazan et al. showed in [12] that the idealized GLM problem with the sigmoid activation function is $(\epsilon, \epsilon \|w^*\|, w^*)$-SLQC in $w$, $\forall w \in B(0, \|w^*\|)$ and $\forall \epsilon > 0$; and that if we draw $m \geq \Omega\left(\exp(2\|w^*\|^2) \log \frac{1}{\delta}\right)$ i.i.d. samples from $\mathcal{D}$, the empirical error function $err$ with sigmoid activation is $(\epsilon, \epsilon \|w^*\|, w^*)$-SLQC in $w$ for any $w \in B(0, \|w^*\|)$ with probability at least $1 - \delta$. However, these results are fundamentally restrictive, since they are firstly for a single GLM. Besides, their proofs rely on properties of the sigmoid function, which restricts us from using these (and any following) results to contemporary neural networks which use other activation functions such as the ReLU. Hence, we begin our description of the proposed methodology by showing that a GLM with a ReLU activation function is also SLQC. To this end, we introduce a new generalized ReLU activation function, defined as follows.

**Definition 3. (Generalized ReLU)** The generalized ReLU function $f : \mathbb{R} \to \mathbb{R}$, $0 < a \leq b$, $a, b \in \mathbb{R}$ is defined as:

$$f(x) = \left\{ \begin{array}{ll} ax & \text{if } x \leq 0 \\ bx & \text{if } a > 0 \end{array} \right.$$

Note that this definition subsumes variants of ReLU such as the Leaky ReLU [23] or PReLU [24]. This function is differentiable at every point except 0. We define the function $g$ that provides a valid subgradient for the generalized ReLU at all $x$ to be:

$$g(x) = \left\{ \begin{array}{ll} a & \text{if } x < 0 \\ b & \text{if } x \geq 0 \end{array} \right.$$
Theorem 3. Let an idealized single-layer multi-output neural network be characterized by a linear operator \( \mathbf{W} \in \mathbb{R}^{d \times d} = \{ \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_d \} \) and a generalized ReLU activation function applied element-wise \( \phi : \mathbb{R}^d \rightarrow \mathbb{R}^d \). Let the output of the layer be \( \phi(\mathbf{W}, \mathbf{x}) \in \mathbb{R}^d \) where \( \mathbf{x} \in \mathbb{R}^d \) is the input. Assuming \( \| \mathbf{W}^* \|_F \leq W, e\|W\|_F \) is \( \left( \epsilon, \frac{2\phi/\alpha}{W^*} \right) - \text{SLQC} \) in \( \mathbf{W} \) for all \( \mathbf{W} \in \mathbb{B}_d(0, W) \) and \( \epsilon > 0 \).

Proof Sketch. To show this result we use definition 4. Let \( \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_d] \) be a point \( \epsilon/\kappa \)-close to minima \( \mathbf{W}^* \) with \( \kappa = \frac{1}{2d} \). Let \( G(\mathbf{W}) \) be the subgradient of \( e\|W\|_F \). Then we show that \( \langle G(\mathbf{W}), \mathbf{W} - \mathbf{V} \rangle \geq 0 \), thus proving the result. See section 5.4 for a complete proof.

Taking this further, we next consider a single-hidden-layer neural network. While the outer layer (layer 2) of a single-hidden-layer neural network can be viewed as a set of GLMs (see section 5.3), the inner layer cannot be viewed the same way. We hence show that given a fixed \( \mathbf{w}_2 \), the error \( e\|W\|_F(\mathbf{W}_1, \mathbf{w}_2) \) is also SLQC in \( \mathbf{W}_1 \). In this case the empirical error function is:

\[
e\|W\|_F(\mathbf{W}_1, \mathbf{w}_2) = \frac{1}{m} \sum_{i=1}^{m} \|y_i - \phi_2(\mathbf{w}_2, \phi(\mathbf{W}_1, x_i))\|^2
\]

when \( x_i \in \mathbb{R}^d \) is the input, \( y_i \in \mathbb{R} \) is the corresponding correct output, \( \mathbf{W}_1 \in \mathbb{R}^{d \times d} \), \( \mathbf{w}_2 \in \mathbb{R}^d \) are the weights of the inner and outer layers respectively. Let the global minimizer of \( e\|W\|_F \) be \( (\mathbf{W}_1^*, \mathbf{w}_2^*) \). This setting corresponds to the inner layer of single-hidden-layer single-layer neural network.

Theorem 4. Let an idealized two-layer neural network be characterized by linear operators \( \mathbf{W}_1 \in \mathbb{R}^{d \times d}, \mathbf{w}_2 \in \mathbb{R}^d \) and generalized ReLU activation functions \( \phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d, \phi_2 : \mathbb{R}^d \rightarrow \mathbb{R}^d \). Assuming \( \| W_1^* \|_F \leq W_1, \| W_2^* \|_F \leq W_2, e\|W\|_F \) is \( \left( \epsilon, \left( \frac{\alpha}{\sqrt{d}W_2 W_1} - \frac{1}{\epsilon} \right) \right) - \text{SLQC} \) in \( \mathbf{W}_1, \forall \mathbf{W}_1 \in \mathbb{B}(0, W_1) \) and \( \epsilon > 0 \).

Proof Sketch. We again use definition 4 to prove the result. Let \( \mathbf{V} \) be a point \( \epsilon/\kappa \) close to minima. We show that \( \langle G(\mathbf{W}), \mathbf{W} - \mathbf{V} \rangle \geq 0 \). As a consequence of definition 4, this proves the result. See section 5.5 for a complete proof.

The above results together postulate that a single-hidden-layer neural network is layer-wise SLQC. We use the above result to show that error \( e\|W\|_F(\mathbf{W}_1, \mathbf{W}_2) \) is SLQC in \( \mathbf{W}_1 \) for single-hidden-layer neural network, even with multiple outputs. The empirical error function in this setting is:

\[
e\|W\|_F(\mathbf{W}_1, \mathbf{W}_2) = \frac{1}{m} \sum_{i=1}^{m} \|y_i - \phi_2(\mathbf{W}_2, \phi(\mathbf{W}_1, x_i))\|^2
\]

where \( x_i \in \mathbb{R}^d \) is the input, \( y_i \in \mathbb{R}^d \) is the corresponding correct output, \( \mathbf{W}_1 \in \mathbb{R}^{d \times d}, \mathbf{W}_2 \in \mathbb{R}^{d \times d} \) are the weights of the inner and outer layers respectively. Let the global minimizer of \( e\|W\|_F \) be \( (\mathbf{W}_1^*, \mathbf{W}_2^*) \). This setting corresponds to the inner layer of a multi-output single-hidden-layer neural network.

Theorem 5. Let an idealized two-layer neural network be characterized by linear operators \( \mathbf{W}_1 \in \mathbb{R}^{d \times d}, \mathbf{W}_2 \in \mathbb{R}^{d \times d} \) and generalized ReLU activation functions \( \phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d, \phi_2 : \mathbb{R}^d \rightarrow \mathbb{R}^d \). Assuming \( \| W_1^* \|_F \leq W_1, \| W_2^* \|_F \leq W_2, e\|W\|_F \) is \( \left( \epsilon, \left( \frac{\alpha}{\sqrt{d}W_2 W_1} - \frac{1}{\epsilon} \right) \right) - \text{SLQC} \) in \( \mathbf{W}_1, \forall \mathbf{W}_1 \in \mathbb{B}(0, W_1) \) and \( \epsilon > 0 \).

Proof Sketch. We use Theorem 4 to prove this result. The error \( e\|W\|_F(\mathbf{W}_1, \mathbf{W}_2) \) of a multi-output single-hidden layer network can be seen as the sum of errors of \( d \) single-output single-hidden layer networks. This observation combined with Theorem 4 is used to prove the result. See section 5.6 for a complete proof.

While the above results have been shown with the generalized ReLU, each of these results also holds for sigmoid activation functions. Moreover, most other widely used error functions such as cross-entropy loss are convex (and thus SLQC) as well as Lipschitz. We believe that our results can be extended to all such error functions, and this will form an important direction of our future work.

3.3 Methodology

We begin our methodology with connecting the above understanding (Section 3.2) with an earlier result which shows that Stochastic Normalized Gradient Descent (SNGD) converges with high probability to the optimum for SLQC functions [12]. To this end, we begin by briefly reviewing the Stochastic Normalized Gradient Descent (SNGD) method, and then state the relevant result.

3.3.1 Stochastic Normalized Gradient Descent (SNGD)

Normalized Gradient Descent (NGD) is an adaptation of traditional Gradient Descent, where the updates in each iteration are based only on the direction of the gradients. This is achieved by normalizing the gradients. SNGD is the stochastic version of NGD, where weight updates are performed using individual (randomly chosen) training samples, instead of the complete set of samples. Mini-batch SNGD generalizes this by applying updates to the parameters at the end of every mini-batch of samples, as does mini-batch Stochastic Gradient Descent (SGD). In the remainder of this paper, we refer to mini-batch SNGD as SNGD itself, as is common for SGD. Algorithm 1 describes the SNGD methodology for a generic problem.

Algorithm 1 Stochastic Normalized Gradient Descent (SNGD)

Input: Number of iterations \( T \), training data \( S = \{(x_i, y_i)\}_{i=1}^{m} \in \mathbb{R}^d \times \mathbb{R} \), learning rate \( \eta \), minibatch size \( b \), Initialization parameters \( w_0 \)

For \( t = 1 \) to \( T \) do

Select a random mini-batch of training points by sampling \( \{(x_i, y_i)\}_{i=1}^{b} \sim \text{Uniform}(S) \)

Let \( f_i(\mathbf{w}) = \frac{1}{b} \sum_{i=1}^{b} (y_i - \phi(\mathbf{w}, x_i))^2 \)

Let \( g_i = \nabla f_i(\mathbf{w}) \), and \( \mathbf{g}(t) = \frac{\mathbf{g}_i}{\|\mathbf{g}_i\|} \)

\[ w_{t+1} = w_t - \eta \cdot \mathbf{g}_t \]

End for

Output: Model given by \( \mathbf{w}_T \)
Theorem 6 ([12]). Let $\epsilon, \delta, G, M, \kappa > 0$, let $f : \mathbb{R}^d \to \mathbb{R}$ and $w^* = \arg\min_w f(w)$. Assume that for $b \geq b_0(\epsilon, \delta, T)$, with probability $1 - \delta$, $f_t$ defined in Algorithm [2] is $(\epsilon, \kappa, (w^*))$-SLQC $\forall w$, and $|f_i| \leq M^i t^i \in \{1, \cdots, T\}$. If we run SNGD with $T \geq \frac{\kappa^2 W^2}{2\epsilon^2} \log \frac{4KL}{\delta}$ and $\eta = \frac{\kappa^2}{4}$, and $b \geq \max \left\{ \frac{M^i W^2}{2\epsilon^2}, b_0(\epsilon, \delta, T) \right\}$, with probability $1 - 2\delta$, $f(w) - f(w^*) \leq 3\epsilon$. 

Importantly, note that the convergence rate of SNGD depends on the $\kappa$ parameter. While the GLM error function with sigmoid activation has $\kappa = e^W$ (stated earlier in the section), we obtain $\kappa = \frac{2^k W^2}{\alpha}$ (i.e. linear in $W$) for the generalized ReLU setting for both GLMs and layers, which is an exponential improvement. This is significant as the number of iterations $T$ in Theorem 6 depends on $\kappa^2$. In other words, SNGD offers accelerated convergence with the proposed generalized ReLU layers as compared to sigmoid GLMs.

3.3.2 DANTE

We have thus far shown that each layer of the considered one-hidden-layer neural network comprises of a set of SLQC problems, each independent in its parameters. Also, SNGD provides an effective method for each such SLQC problem to converge to its respective $\epsilon$-suboptimal solution with high probability, as shown in Theorem 6. This allows us to propose an alternating strategy, DANTE, where each individual SLQC problem is effectively solved (or each individual layer is effectively trained) using SNGD. Now present our proposed method, DANTE. We note that although DANTE uses stochastic gradient-style methods internally (such as SNGD), the overall strategy adopted by DANTE is not necessarily a descent-based strategy, but an alternating-minimization strategy.

Consider the optimization problem below for a single hidden layer network:

$$
\min_w f(W_1, W_2) = E_{x \sim D} \|\phi_2(W_2, \phi_1(W_1, x)) - y\|^2_2 
$$

As seen in Section 3.2 on fixing each of $W_1$ and $W_2$, we have an SLQC problem. On fixing $W_1$, we have the SLQC problem:

$$
\min_w E_{x \sim D} \|\phi_2(W_2, z) - y\|^2_2, 
$$

where $z = \phi_1(W_1, x)$. On fixing $W_2$, we have the following SLQC problem:

$$
\min_w E_{x \sim D} \|\phi(W_1, x) - y\|^2_2, 
$$

DANTE solves each of these intermediate problems using SNGD steps by sampling several mini-batches of data points and performing updates as in Algorithm 1. Algorithm 2 provides the complete algorithm for the proposed method.

3.4 Extending to a Multi-Layer Neural Network

In the previous sections, we illustrated that a single hidden-layer neural network can be cast as a set of SLQC problems and proposed an alternating minimization method, 1. Replacing inner product with Frobenius inner product in the proof for this result in [12] allows us to extend this result to our definition of SLQC for matrices.
Fig. 1: An illustration of the proposed multi-layer DANTE (best viewed in color). In each training phase, the outer pairs of weights (shaded in gold) are treated as a single-hidden-layer neural network to be trained using single-layer DANTE, followed by the inner single-hidden-layer neural network (shaded in black).

after the other in the sequence in which the layers appear in the network. Our experiments however found the stacked approach discussed in Figure 1 to work better in practice.

We note that the convergence of the SLQC function is provided by Theorem 6. It is however non-trivial to provide overall convergence guarantees for the alternating minimization strategy, just as it is challenging to prove convergence for any other training methodology for deep neural networks. We do not show overall convergence in this work, similar to [15] [16], and this will form an important pursuit in our future work.

4 EXPERIMENTS AND RESULTS

We validated DANTE by training feedforward neural networks on standard datasets from the UCI machine learning repository. The standard benchmarking setup of training and test data, as in the repository, was used on each of these datasets. In particular, we studied the training as well as test errors on these experiments, and then computed the classification accuracy using features extracted from a trained model on test data of the datasets considered. We used vanilla SGD-based backpropagation (henceforth, called SGD in the experiments) as the baseline method. In addition to running multiple trials of SGD and DANTE with different learning rates to ensure fair comparison, we used advanced variants of SGD (e.g., Adagrad, RMSProp or Adam) in these experiments, and similar variants of DANTE in the SNGD steps.

4.1 Experiments on MNIST

In this experiment, we studied the training and test error of a single hidden-layer autoencoder on the 32 × 32 MNIST dataset [25] with both sigmoid and a generalized ReLU activation, trained using SGD and DANTE.

**Sigmoid Activation:** A single-hidden-layer autoencoder with a sigmoid activation was trained using DANTE as well as standard backprop-SGD using the standard Mean-Squared Error loss function. The experiments considered 400 hidden units, a learning rate of 0.001 (chosen after many empirical studies, described later), and a minibatch size of 20 (for SGD and the SNGD used inside DANTE). The results are shown in Figure 2a. DANTE obtains both lower training and test error consistently better than SGD, corroborating the goodness of the solutions obtained in each iteration of DANTE using SNGD. (The time taken for the iterations were comparable across both DANTE and backprop-SGD, viz. ≈ 10^3 secs on a standard PC without a GPU for training).

**ReLU Activation:** Similar to the above experiment, a single-hidden-layer autoencoder with a leaky ReLU activation was trained using DANTE and backprop-SGD using the Mean-Squared Error loss function. The experiments considered 400 hidden units, a leakiness parameter of 0.01 for the leaky ReLU, a learning rate of 0.001, and a minibatch size of 400. The results are shown in Figure 2b. Barring the initial phase, DANTE once again performed better than SGD in this setting.

Similar to autoencoders, we used DANTE to train feedforward neural networks on the MNIST dataset for multi layer architectures. These results are shown in Figure 3. Similar to the results on autoencoders, DANTE provides competitive or better performance than SGD.

4.2 Experiments on Other Datasets

We studied the performance of DANTE on other standard datasets, viz. Ionosphere (34 dimensions, 351 datapoints), SVMGuide4 (10 dimensions, 300 datapoints), Vehicle (18 dimensions, 846 datapoints), and USPS (256 dimensions, 7291 datapoints). Figure 4 shows the test error plots of DANTE vs SGD on the abovementioned datasets. Barring the USPS dataset where SGD does marginally better, DANTE demonstrates better performance on all the other datasets. We
also performed experiments on other datasets from the UCI repository and the final test error values at convergence are noted in Table 1, further supporting the promise of the proposed method.

4.3 Studying Classification Error

| Dataset    | DANTE (%) | SGD (%) |
|------------|-----------|---------|
| MNIST      | 93.6%     | 92.44%  |
| Ionosphere | 88.45%    | 86.15%  |
| SVMGuide4  | 87.65%    | 70.37%  |
| USPS       | 90.43%    | 81.49%  |
| Vehicle    | 77.02%    | 74.80%  |

Table 2: Classification accuracies using ReLU autoencoder features on different datasets.

Going further, we conducted experiments to study the effectiveness of the feature representations learned using the autoencoder models trained using DANTE and SGD. After training, we passed the dataset through the autoencoder, extracted the hidden layer representations, and then trained a linear SVM. The classification accuracy results using the hidden representations are given in Table 2. The table clearly highlights the improved performance of DANTE on this task. In case of the SVMGuide4 dataset, DANTE showed a significant improvement of over 17% on the classification accuracy.

4.4 Other Empirical Studies

4.4.1 Comparison with SGD and Alternating Minimization

A natural question one could ask is the relevance of SNGD to train each layer of the proposed methodology. To study this empirically, we compared our algorithm to an analogous algorithm that uses SGD for the inner loop. Table 3 presents these results, allowing different learning rates for the SGD variant, but DANTE provides a better performance than any of these variants.

| Algorithm  | Learning Rate | Loss     |
|------------|---------------|----------|
| DANTE      | 0.001         | 0.02075  |
| SGD        | 0.001         | 0.02216  |
| SGD+AltMin | 0.001         | 0.02351  |
| SGD+AltMin | 0.0005        | 0.02491  |
| SGD+AltMin | 0.0005        | 0.03567  |

Table 3: Loss on using SGD+AltMin to learn the MNIST dataset.

4.4.2 Impact of Learning Rate and Adaptive Learning Rate Methods

In order to see the impact of changing the learning rate in the proposed methodology, we varied the learning rates for SGD in the setup in Figure 2(b) for a more thorough comparison between DANTE and SGD. Table 4 shows these results; DANTE continues to performs better than SGD, even with different learning rate values.

We also studied several adaptive learning schemes with both SGD and DANTE. The results of these studies are presented in Table 5. DANTE with some momentum is able to outperform SGD with all the popular adaptive learning rate schemes.
TABLE 4: Loss on varying the learning rate for both SGD and DANTE.

| Algorithm      | Learning Rate | Loss       |
|----------------|---------------|------------|
| DANTE          | 0.001         | 0.030775   |
| SGD            | 0.001         | 0.031126   |
| SGD            | 0.0001        | 0.031510   |
| SGD            | 0.0001        | 0.031346   |
| SGD            | 0.0005        | 0.031263   |
| SGD            | 0.1           | 0.032676   |

TABLE 5: Loss on using various adaptive learning schemes with DANTE and SGD. (LR = Learning Rate; MP = Momentum Parameter)

| Algorithm                  | Parameter | Loss       |
|----------------------------|-----------|------------|
| DANTE                      | LR = 0.001| 0.030775   |
| SGD                        | LR = 0.001| 0.031126   |
| SGD+Adam                   | 0.0001    | 0.021704   |
| SGD+Adagrad                | 0.0001    | 0.021896   |
| SGD+RMSProp                | 0.0001    | 0.021953   |
| SGD+Momentum               | MP=0.9    | 0.021497   |
| DANTE +Momentum            | MP=0.0005 | 0.020816   |

4.4.3 Varying Number of Hidden Neurons

Given the decomposable nature of the proposed solution to learning neural networks, we also finally studied the effect of varying hyperparameters across the layers, in particular, the number of hidden neurons in a single-layer autoencoder. The results of these experiments on MNIST are shown in Figure 5. The plots show that when the number of hidden neurons is low, DANTE reaches its minumum value sooner (considering this is a subgradient method, one can always choose the best iterate over training) than SGD, although SGD finds a better solution over the iterations. However, when the number of hidden neurons increases, DANTE starts getting consistently better. This can be attributed to the fact that the subproblem is relatively more challenging for an alternating optimization setting when the number of hidden neurons is lesser. While not conclusive, this could also convey that DANTE performs well when the complexity of the problem increases (in terms of the number of parameters in the network). We plan to study this observation in more detail in our future work.

5 Proofs

5.1 Proof of Theorem 1

Theorem. In the idealized GLM with generalized ReLU activation, assuming \( \|w^*\| \leq W \), \( e\mathcal{R}_m(w) \) is \( (\epsilon, \frac{2kW}{\alpha}, w^*) - SLQC in w, \forall w \in \mathbb{B}(0,W) \) and \( \forall \epsilon > 0 \).

Proof: Consider \( w \in \mathbb{B}(0,W) \), \( \|w\| \leq W \) such that \( e\mathcal{R}_m(w) = \frac{1}{m} \sum_{i=1}^{m} (y_i - \phi(w, x_i))^2 \geq \epsilon \), where \( m \) is the total number of samples. Also let \( v \) be a point \( \epsilon/\kappa \)-close to minima \( w^* \) with \( \kappa = \frac{2kW}{\alpha} \). Let \( g \) be the subgradient of the generalized ReLU activation and \( G \) be the subgradient of \( e\mathcal{R}_m(w) \). (Note that as before, \( g(\cdot, \cdot) \) denotes \( g(\cdot, \cdot) \)). Then:

\[
    \langle G(w), w - v \rangle
    = \frac{2}{m} \sum_{i=1}^{m} g(w, x_i) (\phi(w, x_i) - y_i) (x_i, (w - v))
    \]

\[
    = \frac{2}{m} \sum_{i=1}^{m} g(w, x_i) (\phi(w, x_i) - \phi(w^*, x_i))
    \]

\[
    \geq \frac{2}{m} \sum_{i=1}^{m} g(w, x_i) \left[ b^{-1} (\phi(w, x_i) - \phi(w^*, x_i))^2 + (\phi(w, x_i) - \phi(w^*, x_i))(x_i, (w^* - v)) \right]
    \]
Consider $||w|| \leq W$ such that $\epsilon \hat{r}_m(w) - \epsilon \hat{r}_m(w^*) \geq \epsilon$.
Also, let $v$ be a point $\epsilon/k$-close to minima $w^*$ with $k = \frac{2bW}{m}$.
Let $g$ be the subgradient of the generalized ReLU activation and $G$ be the subgradient of $\epsilon \hat{r}_m(w)$, as before. Then:

$$\langle G(w), w - v \rangle$$

$$= \frac{2m}{m} \sum_{i=1}^m g(w, x_i) \left( \phi(w, x_i) - y_i \right) (x_i, (w - v))$$

$$= \frac{2m}{m} \sum_{i=1}^m g(w, x_i) \phi(w, x_i) - \phi(w^*, x_i) - \xi_i$$

$$\left[ (x_i, w - w^*) + (x_i, w^* - v) \right]$$

$$\geq \frac{2b^{-1}}{m} \sum_{i=1}^m g(w, x_i) \phi(w^*, x_i) - \phi(w, x_i)^2$$

$$- \frac{2m}{m} \sum_{i=1}^m g(w, x_i) \xi_i (x_i, (w, x_i) - (w^*, x_i))$$

$$+ \frac{2m}{m} \sum_{i=1}^m g(w, x_i) \cdot (\phi(w, x_i) - \phi(w^*, x_i) - \xi_i)(w^* - v, x_i)$$

$$\geq \frac{2b^{-1}}{m} \sum_{i=1}^m g(w, x_i) \phi(w^*, x_i) - \phi(w, x_i)^2$$

$$- \frac{2m}{m} \sum_{i=1}^m g(w, x_i) \xi_i (x_i, (w, x_i) - (w^*, x_i))$$

$$- 2\epsilon \frac{\epsilon^2}{\kappa} (||w - w^*|| + \frac{m}{m} \sum_{i=1}^m |\xi_i|)$$

$$\geq 2ab^{-1}\epsilon - 2\epsilon\frac{\epsilon^2}{\kappa} (||w - w^*|| + \frac{m}{m} \sum_{i=1}^m |\xi_i|)$$

$$\geq 2ab^{-1}\epsilon - 2ab^{-1}\epsilon (||w - w^*|| + \frac{m}{m} \sum_{i=1}^m |\xi_i|)$$

$$\geq -ab^{-1}\epsilon W^{-1} + \frac{m}{m} \sum_{i=1}^m |\xi_i|$$

Here, $\lambda_i(w) = 2g(w, x_i)((w, x_i) - (w^*, x_i)) - 4ab^{-1}(\phi(w, x_i) - \phi(w^*, x_i))$, and
network be characterized by a linear operator

Let an idealized single-layer multi-output neural

Here we first restate the theorem 3 and then prove the result.

5.4 Proof of Theorem 3

We restate theorem 4, and then prove the result.

\begin{equation}
\frac{1}{m} \sum_{i=1}^{m} \xi_i \lambda_i(w) \geq ab^{-1}W^{-1}\epsilon
\end{equation}

our model is SLQC. By simply using the Höftling’s bound, we get that the theorem statement holds for $m \geq \frac{288b^4W^4}{a^2}\log(1/\delta)/\epsilon^2$.

\section{5.3 Viewing the Outer Layer of a Neural Network as a Set of GLMs}

Given an (unknown) distribution $D$, let the layer be characterized by a linear operator $W \in \mathbb{R}^{d \times d'}$ and a non-linear activation function defined by $\phi : \mathbb{R} \rightarrow \mathbb{R}$. Let the layer output be defined by $\phi(Wx)$, where $x \in \mathbb{R}^d$ is the input, and $\phi$ is used element-wise in this function.

Consider the mean squared error loss, commonly used in neural networks, given by:

\begin{equation}
\min_W \text{err}(W) = \min_W \mathbb{E}_{x \sim D}\|\phi(Wx) - y\|_2^2
\end{equation}

Each of these sub-problems above is a GLM, which can be solved effectively using SNGD as seen in Theorem 5 which we leverage in this work.

\section{5.4 Proof of Theorem3}

Here we first state the theorem 3 and then prove the result.

\textbf{Theorem.} Let an idealized single-layer multi-output neural network be characterized by a linear operator $W \in \mathbb{R}^{d \times d'} = [w_1 w_2 \cdots w_{d'}]$ and a generalized ReLU activation function applied element-wise $\phi : \mathbb{R}^{d'} \rightarrow \mathbb{R}$. Let the output of the layer be $\phi(Wx)$, where $x \in \mathbb{R}^d$ is the input. Assuming $\|W^*\|_F \leq W$, $\text{err}(W) = \left(\epsilon, \frac{2bW}{a}, W^*\right) - \text{SLQC in W for all W} \in \mathbb{R}(0, W)$ and $\epsilon > 0$.

\textbf{Proof.} Consider $W \in \mathbb{B}(0, W)$, $\|W\| \leq W$ such that $\text{err}_m(W) = \frac{1}{m} \sum_{i=1}^{m} \|y_i - \phi(Wx_i)\|_2^2 \geq \epsilon$, where $m$ is the total number of samples. Also let $V = \{v_1, v_2, \cdots, v_{d'}\}$ be a point $\epsilon/\kappa$-close to minima $W^*$ with $\kappa = \frac{2bW}{a}$. Let $g$ be the subgradient of the generalized ReLU activation, $G(W)$ be the subgradient of $\text{err}_m(W)$ and $G(w_j)$ be the subgradient of $\text{err}_m(w_j)$. (Note that as before, $g(.,.)$ denotes $g((.,.))$.

\begin{equation}
\langle G(W), W - V \rangle_F
\end{equation}

By defn of Frobenius inner product

\begin{equation}
\frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{d'} (\phi(w_j, x_i) - y_{ij}) \left(\frac{\partial(\phi(w_j, x_i))}{\partial w_j}, (w_j - v_j)\right)_F
\end{equation}

\begin{equation}
\geq \frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{d'} (b^{-1} \phi(w_j, x_i) - \phi(w_j^*, x_i))^2 - b\|\phi(w_j, x_i) - \phi(w_j^*, x_i)\|_2 \|w_j^* - v_j\|
\end{equation}

\begin{equation}
\geq \frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{d'} (b^{-1} \phi(w_j, x_i) - \phi(w_j^*, x_i))^2 - b^2\|\phi(w_j, x_i) - \phi(w_j^*, x_i)\|_2^2
\end{equation}

\begin{equation}
\geq \frac{2}{m} \sum_{i=1}^{m} \sum_{j=1}^{d'} (\phi(w_j, x_i) - \phi(w_j^*, x_i))^2 - \frac{a}{W}\|\phi(w_j, x_i) - \phi(w_j^*, x_i)\|_2^2\epsilon\|x_i\|
\end{equation}

\begin{equation}
\geq 0
\end{equation}

In step 6, $\|\langle w, x_i \rangle - \langle w^*, x_i \rangle \| = \max \|\langle w_j, x_i \rangle - \langle w_j^*, x_i \rangle \|

To simplify from Step 7 we use the fact that $\|W^*\| \leq W \implies \|W^*\| \leq W$. The remainder of the proof proceeds precisely as in Theorem 4.

\section{5.5 Proof of Theorem4}

We restate theorem 4 and then prove the result.

\textbf{Theorem.} Let an idealized two-layer neural network be characterized by linear operators $W_1 \in \mathbb{R}^{d \times d'}$, $w_2 \in \mathbb{R}^{d'}$ and generalized ReLU activation functions $\phi_1 : \mathbb{R}^{d'} \rightarrow \mathbb{R}^d$, $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$. Assuming $\|W_1\|_F \leq W_1$, $\|w_2\|_F \leq W_2$, $\text{err}(W_1, w_2)$ is $\left(\epsilon, \frac{a}{2bW_2W_1}, W_1^*\right) - \text{SLQC in W_1, \forall W_1 \in \mathbb{B}(0, W_1) and \forall \epsilon > 0}$.

\textbf{Proof.} In this case the prediction of the network on $x$ is $f(W_1; w_2, x)$. Consider $W_1 \in \mathbb{B}(0, W_1)$, $\|W_1\| \leq W_1$, $\|w_2\| \leq W_2$ such
that \( e r (W_1, W_2) \geq \epsilon \). Let \( V_1 \) be a point \( \frac{x}{\kappa} \) close to minima \( W_1 \), where \( \kappa = \left( \frac{4a^p W_2^2}{e^2} \right)^{-1} \).

Let \( \|f(W_1; w_2; x) - y\|_2^2 = \|f_2(w_2, \phi_1(W_1, x)) - y\|_2^2 \) and \( \langle \cdot, \cdot \rangle_F \) be the Frobenius inner product.

\[
\langle \nabla_{W_1} e r (W_1, W_2), W_1 - V_1 \rangle_F = \frac{2}{m} \sum_{i=1}^{m} \langle \phi_2(w_2, \phi_1(W_1, x_i)) - y_i \rangle \cdot \frac{\partial(\phi_2(w_2, \phi_1(W_1, x_i)))}{\partial W_1} \cdot (W_1 - V_1) \rangle_F \tag{Step 1}
\]

Using chain rule, we can simplify \( \frac{\partial(\phi_2(w_2, \phi_1(W_1, x)))}{\partial w_1} \) as

\[
\left[ \frac{\partial(\phi_2(w_2, \phi_1(W_1, x)))}{\partial W_1} \right]^T = \frac{\partial(\phi_2(w_2, \phi_1(W_1, x)))}{\partial \phi_1(W_1, x)} \cdot \frac{\partial(\phi_1(W_1, x))}{\partial W_1} \cdot \frac{\partial(W_1, x)}{\partial W_1}^T \cdot \frac{\partial(W_1, x)}{\partial W_1}^T\right] \cdot \left[ \frac{\partial(w_2, \phi_1(W_1, x))}{\partial W_1} \right] \cdot \left[ \frac{\partial(\phi_2(w_2, \phi_1(W_1, x)))}{\partial W_1} \right]^T
\]

Continuing from Step 1:

\[
= \frac{2}{m} \sum_{i=1}^{m} g_2(W_1, w_2, x_i) \cdot (\phi_2(w_2, \phi_1(W_1, x_i)) - y_i) \cdot g_1(W_1, x) \cdot w_2 \cdot x^T
\]

Therefore, on setting \( W = W_1 \) and using the fact that the generalized ReLU is \( b \)-Lipschitz and monotonically increasing, we have:

\[
\langle \phi_2(w_2, \phi_1(W_1, x)) - \phi_2(w_2, \phi_1(W_1, x)) \rangle^2 \\
\leq b \langle \phi_2(w_2, \phi_1(W_1, x)) - \phi_2(w_2, \phi_1(W_1, x)) \rangle \\
\cdot \langle \phi_2(w_2, \phi_1(W_1, x)) - \phi_2(w_2, \phi_1(W_1, x)) \rangle \\
= b \langle \phi_2(w_2, \phi_1(W_1, x)) - \phi_2(w_2, \phi_1(W_1, x)) \rangle \\
\cdot (Tr(g_1(W_1, x)w_2x^T W_1) - Tr(g_1(W_1, x)w_2x^T W_1))
\]

Plugging this result into Step 2:

\[
\geq 2 \frac{b}{\epsilon} \sum_{i=1}^{m} g_2(W_1, w_2, x_i) \cdot (\langle \phi_2(w_2, \phi_1(W_1, x_i)) - |W_1 - V_1|_F + |W_1 - V_1|_F)
\]

From triangle inequality, \( |W_1 - V_1|_F \leq \frac{\epsilon}{\kappa} \).

\[
|Tr(g_1(W_1, x_i)w_2x^T W_1) - Tr(g_1(W_1, x_i)w_2x^T V_1)| \\
\leq |Tr(g_1(W_1, x_i)w_2x^T W_1)| + |Tr(g_1(W_1, x_i)w_2x^T V_1)| \\
\leq b \cdot |w_2|_F ||W_1|| + b \cdot |w_2|_F ||V_1|| \\
\leq 1 \cdot b \cdot |W_1|_F ||W_1|| + ||V_1|| \\
\leq b \cdot W_2 \cdot ||W_1||_F + ||W_1||
\]

Now consider the term

\[
|\phi_2(w_2, \phi_1(W_1, x_i)) - \phi_2(w_2, \phi_1(W_1, x_i))| \tag{Step 2}
\]

\[
\langle w_2, \phi_1(W, x) \rangle_F = Tr(g_1(W, x)w_2x^T W) \\
= Tr(g_1(W, x)w_2x^T W) - Tr(g_1(W, x)w_2x^T W)
\]

Using these and the fact that \( g_2(W_1, w_2, x) \leq b \) in Step 3.
Observe here that $\|\phi W_{minima} \|$ is characterized by linear operators $W$. Let an idealized two-layer neural network be characterized by $W$. Theorem. We restate theorem 5, and then prove the result.

The proof uses the fact that the minimum value of the $a^2 b^2 W_{minima}$ is $a^2 b^2 W_{minima}$. Hence, we get that, $a^2 b^2 W_{minima} \geq 0$.

The idea of the proof is similar to that of previous theorems. The proof uses the fact that the minimum value of the quasigradient of $g$ is $a$.

5.6 Proof of Theorem [5]

We restate theorem [5] and then prove the result.

**Theorem.** Let an idealized two-layer neural network be characterized by linear operators $W_1 \in \mathbb{R}^{d \times d'}$, $W_2 \in \mathbb{R}^{d' \times d''}$ and generalized ReLU activation functions $\phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\phi_2 : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$. Assuming $\|W_1\|_F \leq W_1$, $\|W_2\|_F \leq W_2$, $err(W_1, W_2) \geq \epsilon$. Let $W_1$ be a point $\frac{\epsilon}{\epsilon}$ close to minima $W$, where $\epsilon = \left(\frac{a}{25 b^2 W_{\minima} - \frac{W_1}{\epsilon}}\right)$. Let $W_2 \in \mathbb{R}^{d' \times d''} = [w_1^2, w_2^2, \ldots w_d^2]$. Note that,

$$
\nabla W_1 err(W_1, W_2) = \nabla W_1 \frac{1}{m} \sum_{i=1}^{m} \|y_i - \phi_2(W_2, \phi_1(W_1, x_i))\|^2
$$

$$
= \nabla W_1 \frac{1}{m} \sum_{j=1}^{d'} \sum_{i=1}^{m} \|y_j - \phi_2(w_j^2, \phi_1(W_1, x_i))\|^2
$$

$$
= \sum_{j=1}^{d'} \nabla W_1 \frac{1}{m} \sum_{i=1}^{m} \|y_j - \phi_2(w_j^2, \phi_1(W_1, x_i))\|^2
$$

$$
= \sum_{j=1}^{d'} \nabla W_1 \nabla W_1 err(W_1, w_j^2)
$$

Now,

$$
\langle \nabla W_1 err(W_1, W_2), W_1 - V_1 \rangle_F = \sum_{j=1}^{d'} \langle \nabla W_1 err(W_1, w_j^2), W_1 - V_1 \rangle_F
$$

Observe here that $\|W_2\|_F \leq W_2 \Rightarrow \|w_j^2\|_F \leq W_2 y_j$. Using this and the result from theorem [6] we get that each term $\langle \nabla W_1 err(W_1, w_j^2), W_1 - V_1 \rangle_F \geq \frac{a}{b} \epsilon$. Hence, we get that,

$$
\langle \nabla W_1 err(W_1, W_2), W_1 - V_1 \rangle_F \geq \frac{a}{b} \epsilon d' \geq 0
$$

6 Conclusion and Future Work

In this work, we presented a novel methodology, Deep Alternatives for Training Neural networks (DANTE), to effectively train neural networks using alternating minimization, thus providing a competitive alternative to standard backpropagation. We formulated the task of training each layer of a neural network (in particular, an autoencoder without loss of generality) as a Strictly Locally Quasi-Convex (SLQC) problem, and leveraged recent results to use Stochastic Normalized Gradient Descent (SNGD) as an effective method to train each layer of the network. While recent work was restricted to sigmoidal GLMs, we introduced a new generalized ReLU activation, and showed that a multi-output layer satisfies this SLQC property, thus allowing us to expand the applicability of the proposed method to networks with both sigmoid and ReLU family of activation functions. In particular, we extended the definitions of local quasi-convexity in order to prove that a one hidden-layer neural network with generalized ReLU activation is $(\epsilon, (2a W_1 - W_2) - \epsilon)$ in $W_1$, which improves the convergence bound for SLQC in the GLM with the generalized ReLU (as compared to a GLM with sigmoid). We also showed how DANTE can be extended to train multi-layer neural networks. We empirically validated DANTE with both sigmoidal and ReLU activations on standard datasets as well as in a multi-layer setting, and observed that it provides a competitive alternative to standard backprop-SGD, as evidenced in the experimental results.

Future Work and Extensions

DANTE can not only be used to train multi-layer neural networks from scratch, but can also be combined with backprop-SGD, which can be used to finetune the network end-to-end periodically. Our future work will involve a more careful study of the proposed method for deeper neural networks, as well as in studying performance bounds for the proposed alternating minimization strategy.

References

[1] S. Hochreiter and J. Schmidhuber, “Long Short-term Memory,” Neural Computation, vol. 8, no. 9, pp. 1735–1780, 1997.

[2] Y. N. Dauphin, R. Pascanu, C. Gulcehre, K. Cho, S. Ganguli, and Y. Bengio, “Identifying and attacking the saddle point problem in high-dimensional non-convex optimization,” in Advances in neural information processing systems, 2014, pp. 2933–2941.

[3] Y. Tian, “Symmetry-breaking convergence analysis of certain two-layered neural networks with relu nonlinearity,” 2016.

[4] S. Shalev-Shwartz, O. Shamir, and S. Shammah, “Failures of gradient-based deep learning,” in Proceedings of the 34th International Conference on Machine Learning, 2017, pp. 3067–3075.

[5] T. Blumschat and M. E. Davies, “Iterative Hard Thresholding for Compressed Sensing,” Applied and Computational Harmonic Analysis, vol. 27, no. 3, pp. 265–274, 2009.

[6] P. Jain, P. Netrapalli, and S. Sanghavi, “Low-rank Matrix Completion using Alternating Minimization,” in 45th Annual ACM Symposium on Theory of Computing (STOC), 2013.

[7] A. Anandkumar and R. Ge, “Efficient Approaches for Escaping Higher Order Saddle Points in Non-Convex Optimization,” in 29th Conference on Learning Theory (COLT), 2016.
Adepu Ravi Sankar is a PhD candidate at Indian Institute of Technology, Hyderabad, India. He also holds a Masters degree in Computer Science and Engineering from IIT-H in 2014. His research areas include Deep learning, Optimization, Machine Learning. He is the recipient of prestigious Intel India PhD fellowship 2015 for the proposal "How quickly can we make Deep Architectures learn?". He has also received the Best Paper Award at the IEEE HiPC Student Research Symposium 2014, as well as the Microsoft Research travel grant to attend ACM 2015. He has also interned with the Amazon Machine Learning lab at Bangalore during summer 2016. He has served as the local organizing committee for the Indo-UK Conformal Predictions workshop in Dec 2015. He is a member of ACM. He is currently working on understanding the error surfaces of deep neural networks.

Surya Teja Chavalli is a second-year graduate student at the University of Wisconsin-Madison. His current interests lie at the intersection of machine learning and formal methods: specifically, using formal Methods to prove and certify improved robustness and correctness of machine learning models in safety-critical applications. He also works on systems problems endemic to machine learning Applications.

Purushottam Kar is an Assistant Professor in the Department of Computer Science and Engineering at Indian Institute of Technology Kanpur (IIT Kanpur). He is also the Dr. Deep Singh and Daljeet Kaur Faculty Fellow at the department. He holds a bachelor’s and doctoral degrees in computer science (2008, 2013) from IIT Kanpur. He was a Post-doctoral Researcher at Microsoft Research India Pvt. Ltd between Jul 2013 - August 2015. His research interests include online and stochastic optimization techniques, and non convex optimization, statistical learning theory, high dimensional statistics, as well as applications to machine learning problems such as kernel methods and extreme multi-label learning. His research work is published in top tier conferences including ICML, NIPS, JMLR. He extended his service as a reviewer to many journals viz TNNLS, TPAMI, JMLR, TSMC, TOCS. For more information, please see [https://www.cse.iitk.ac.in/users/purushot/index.php](https://www.cse.iitk.ac.in/users/purushot/index.php).

Vineeth N Balasubramanian is an Associate Professor in the Department of Computer Science and Engineering at the Indian Institute of Technology, Hyderabad, India. Until July 2013, he was an Assistant Research Professor at the Center for Cognitive Ubiquitous Computing (CUBiC) at Arizona State University (ASU), USA, where he also served as an Associate Director between 2012-13. He holds dual Masters degrees in Mathematics (2001) and Computer Science (2003) from Sri Sathyai Sai Institute of Higher Learning, India, and worked at Oracle Corporation for two years until 2005. His PhD dissertation (completed in 2010) on the Conformal Prediction framework was nominated for the Outstanding PhD Dissertation at the Department of Computer Science at ASU. He was also awarded the Gold Medals for Academic Excellence in the Bachelors program in Math in 1999, and for his Masters program in Computer Science in 2003. His research interests include pattern recognition, machine learning, computer vision and multimedia computing, and he is passionate about pursuing research that brings together theory and application in these areas. He has over several published articles in premier peer-reviewed venues (such as IEEE TPAMI, IEEE TNNLS, ACM KDD, IEEE CVPR, IEEE ICDM), and an edited book on a recent development in machine learning called Conformal Prediction. He is an active reviewer of journals including IEEE TPAMI, TNNLS, and JMLR. He is a member of the IEEE, ACM and currently serves as the Secretary of the AAAI India chapter. For more information, please see [http://www.iith.ac.in/~vineethnb/index.html](http://www.iith.ac.in/~vineethnb/index.html).

Sneha Kudugunta received her Bachelors in Computer Science and Engineering from Indian Institute of Technology, Hyderabad. While there, she worked on viewing neural networks in terms of non-convex optimization problems. She has also spent time at the Institute for Pure and Applied Mathematics, UCLA modeling cryptographic side-channel attacks, and at the Information Sciences Institute, USC using machine learning to detect social bots. Her interests include understanding language - especially in less structured contexts - and using mathematics to understand machine learning. She is currently a part of the Google AI Residency, where she is working on understanding various aspects of multi-task learning in the context of natural language understanding.

Vaibhav B Sinha Vaibhav Sinha is a fourth-year undergraduate student in the Department of Computer Science and Engineering at Indian Institute of Technology, Hyderabad. He is primarily interested in optimization methods for deep learning. He enjoys working on theoretical aspects of machine learning in general.