REPRESENTATIONS OF $\omega$-LIE ALGEBRAS AND TAILED DERIVATIONS

RUNXUAN ZHANG

Abstract. We study the representation theory of finite-dimensional $\omega$-Lie algebras over the complex field. We derive an $\omega$-Lie version of the classical Lie’s theorem, i.e., any finite-dimensional irreducible module of a soluble $\omega$-Lie algebra is one-dimensional. We also prove that indecomposable modules of some three-dimensional $\omega$-Lie algebras could be parametrized by the complex field and nilpotent matrices. We introduce the notion of a tailed derivation of a nonassociative algebra $g$ and prove that if $g$ is a Lie algebra, then there exists a one-to-one correspondence between tailed derivations of $g$ and one-dimensional $\omega$-extensions of $g$.

1. Introduction

In 2007, Nurowski introduced the notion of $\omega$-Lie algebras for which the original motivation stems from some geometry considerations, see [8], [2] and [9]. More specifically, a vector space $L$ over a field $F$ equipped with a skew-symmetric bracket $[−, −]: L \times L \rightarrow L$ and a bilinear form $\omega: L \times L \rightarrow F$ is called an $\omega$-Lie algebra provided that

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \omega(x, y)z + \omega(y, z)x + \omega(z, x)y$$

($\omega$-Jacobi identity)

for all $x, y, z \in L$. Clearly, $\omega$-Lie algebras with $\omega = 0$ are nothing but ordinary Lie algebras, which means that the notion of $\omega$-Lie algebras extends that of Lie algebras.

The present article is devoted to a study of the representation theory of finite-dimensional $\omega$-algebras over the complex field. Let’s recall some development on this subject. In 2010, Zusmanovich in [10]. Section 9, Theorem 1] proved an important result on the structure of $\omega$-Lie algebras, which says that all finite-dimensional non-Lie $\omega$-Lie algebras are either low-dimensional or have a quite degenerate structure. By the $\omega$-Jacobi identity one sees that there are no non-Lie $\omega$-Lie algebras of dimensions one and two. In our previous works [4] and [5], we derived a rough classification of three- and four-dimensional complex $\omega$-Lie algebras. With the classification, we recently calculated the automorphism groups and the derivation algebras of low-dimensional $\omega$-Lie algebras over the complex field, reformulated elementary facts about the representation theory of $\omega$-Lie algebras, and we also proved that all finite-dimensional irreducible representations of the family $C_{\alpha}$ of $\omega$-Lie algebras are one-dimensional; see [6]. Sections 6 and 7].

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The first purpose of this article is to generalize the classical Lie’s theorem of complex soluble Lie algebras to the case of $\omega$-Lie algebras. We introduce the following notion of degree of $\omega$-Lie algebras.

**Definition 1.1.** Suppose that $L$ is a finite-dimensional $\omega$-Lie algebra. The positive integer
\[ \text{deg}(L) := \min \{ \dim(L) - \dim(I) \mid I \subset L \text{ is a proper ideal} \} \]
is called the **degree** of $L$.

We will show that soluble $\omega$-Lie algebras are of degree 1; see Proposition 2.2 below. Our first main result can be formulated as follows.

**Theorem 1.2.** Let $L$ be a non-simple complex $\omega$-Lie algebra of degree 1 with a soluble ideal $\mathfrak{g}$ of maximal dimension $\dim(L) - 1$ and $V$ be a finite-dimensional irreducible $L$-module. Then $\dim(V) = 1$.

Proposition 2.2 and Theorem 1.2 combine to a direct consequence which could be regarded as an $\omega$-Lie version of the classical Lie’s theorem.

**Corollary 1.3** (The $\omega$-Lie version of Lie’s theorem). Let $L$ be a finite-dimensional soluble $\omega$-Lie algebra over the complex field and $V$ be a finite-dimensional irreducible $L$-module. Then $\dim(V) = 1$.

We also give some applications of Theorem 1.2 and fundamental properties of $\omega$-Lie algebra modules in Section 2.

The second goal of this paper is to study indecomposable representations of some three-dimensional non-Lie $\omega$-Lie algebras. Note that we have already classified these $\omega$-Lie algebras in [5, Theorem 2] into $\mathcal{L} := \{ L_1, L_2, A_\alpha, B, C_\alpha \}$, see Section 3 for details. Let $L \in \{ L_1, A_\alpha \}$ and $\mathcal{R}_n(\mathbb{C})$ be the set of all indecomposable $L$-modules on $\mathbb{C}^n$. Section 3 is devoted to a proof of the following second main result.

**Theorem 1.4.** The equivalence classes in $\mathcal{R}_n(\mathbb{C})$ could be parametrized by the complex field $\mathbb{C}$, the conjugacy classes of $n \times n$ nilpotent matrices and an affine variety.

Our third purpose is to study one-dimensional $\omega$-extensions of Lie algebras. Note that one-dimensional extensions of a Lie algebra $\mathfrak{g}$ can be parameterized by the set of all twisted derivations of $\mathfrak{g}$; see [1, Proposition 5.4]. Let $\mathfrak{g}$ be a Lie algebra and $L = \mathfrak{g} \oplus \mathbb{C}x$ be the vector space of dimension $\dim(\mathfrak{g}) + 1$. Then $L$ is called a one-dimensional $\omega$-extension of $\mathfrak{g}$ through $\mathbb{C}x$ if there exists an $\omega$-Lie algebra structure on $L$ containing $\mathfrak{g}$ as an ideal and $\omega(\mathfrak{g}, \mathfrak{g}) = 0$. To describe the set $\text{Ext}^1_\omega(\mathfrak{g})$ of all one-dimensional $\omega$-extensions of $\mathfrak{g}$, we introduce the notion of tailed derivations of nonassociative algebras.
Definition 1.5. Let $A$ be a nonassociative algebra. A linear map $D : A \to A$ is called a tailed derivation of $A$ if there exists a linear form $d : A \to \mathbb{F}$ ($y \mapsto d_y$) such that

$$D([y, z]) = [D(y), z] + [y, D(z)] + d_z y - d_y z$$

for all $y, z \in A$.

We observe that for a tailed derivation $D$, such linear form $d$ is unique; and moreover, in [10], Section 6, Definition], tailed derivations of an anti-commutative algebra have appeared as a special kind of $(\alpha, \lambda)$-derivations with $\lambda = 0$. Clearly, all derivations of $A$ are tailed derivations with trivial tails, i.e., $d_y = d_z = 0$ for all $y, z \in A$. We denote by $\text{TDer}(A)$ the set of all tailed derivations of $A$. We will show that $\text{TDer}(A)$ is a Lie subalgebra of the general linear Lie algebra $\mathfrak{gl}(A)$; see Proposition 4.1. Thus $\text{Der}(A) \subseteq \text{TDer}(A) \subseteq \mathfrak{gl}(A)$ as Lie subalgebras, with the containment might be strict; see Example 4.2. Now the third main result can be stated as follows.

**Theorem 1.6.** Let $\mathfrak{g}$ be a Lie algebra and $\omega$ be a skew-symmetric bilinear form on $\mathfrak{g} \oplus \mathbb{C}x$. Then there exists a one-to-one correspondence between $\text{Ext}^1_\omega(\mathfrak{g})$ and $\text{TDer}(\mathfrak{g})$.

We also provide an example that demonstrate that $\omega$-Lie algebras could be constructed by Lie algebras and their tailed derivations; see Example 4.6.

**Conventions.** The Lie algebra notions that do not involve the form $\omega$ in their definitions are extended verbatim to $\omega$-Lie algebras: for example, subalgebras, ideals, simple, soluble and abelian algebras.

Throughout this article we assume that the ground field is the complex field $\mathbb{C}$. All representations (modules), vector spaces and algebras are finite-dimensional over $\mathbb{C}$. We use $z_V$ to denote the linear transformation of an abstract element $z$ acting on a vector space $V$. We use $\mathbb{Z}^+$ and $\mathbb{Z}_{\geq 0}$ to denote the sets of positive and non-negative integers, respectively.

2. The $\omega$-Lie version of Lie’s theorem

In this section, we show Theorem 1.2 and provide some applications. To begin with, we present two examples of non-simple $\omega$-Lie algebras.

**Example 2.1.** The following three-dimensional $\omega$-Lie algebras are of degree 1:

1. $L_1 : [x, z] = 0, [y, z] = z, [x, y] = y$ and $\omega(y, z) = \omega(x, z) = 0, \omega(x, y) = 1$;
2. $L_2 : [x, y] = 0, [x, z] = y, [y, z] = z$ and $\omega(x, y) = 0, \omega(x, z) = 1, \omega(y, z) = 0$.

Here $\{x, y, z\}$ denotes a basis of the underlying vector space. We observe that the subspace spanned by $y$ and $z$ is a proper ideal, so $L_1$ and $L_2$ are non-simple and of degree 1.
Note that $L_1$ and $L_2$ in Example 2.1 are both soluble $\omega$-Lie algebras. Moreover, we have the following more general result.

**Proposition 2.2.** Soluble $\omega$-Lie algebras are of degree 1.

**Proof.** Let $L$ be an $n$-dimensional soluble $\omega$-Lie algebra. Then $[L, L] \neq L$ and so it is not simple. To show that $L$ has degree 1, we may find an $(n-1)$-dimensional subspace $I$ of $L$ such that $[L, L] \subseteq I \subseteq L$. As $[I, L] \subseteq [L, L] \subseteq I$, we see that $I$ is an ideal of $L$. Clearly, $I$ is a proper ideal with the maximal dimension $n-1$. Hence, $L$ has degree 1. □

We also present some examples of three-dimensional simple $\omega$-Lie algebras.

**Example 2.3.** Let $\{x, y, z\}$ be a basis of $\mathbb{C}^3$. The following $\omega$-Lie algebras are simple:

1. $A_\alpha : [y, z] = z, [x, z] = y - z, [x, y] = x + \alpha z, \omega(y, z) = \omega(x, z) = 0, \omega(x, y) = -1$;
2. $B : [y, z] = z, [x, y] = z - x, [x, z] = y, \omega(y, z) = \omega(x, y) = 0, \omega(x, z) = 2$;
3. $C_\alpha : [y, z] = z, [y, x] = \alpha x, [z, x] = y, \omega(y, z) = \omega(x, y) = 0, \omega(z, x) = 1 + \alpha$,

where $\alpha \in \mathbb{C}$. See [3, Proposition 7.1] for the details. Comparing with [5, Theorem 2] or [6, Theorem 1.4], we see that in this example the generating relations actually have been reformulated by choosing a suitable basis.

**Remark 2.4.** In fact, [5, Theorem 2] indicates that every three-dimensional non-Lie $\omega$-Lie algebra over $\mathbb{C}$ must be isomorphic to one of $L = \{L_1, L_2, A_\alpha, B, C_\alpha\}$.

Here we provide an example of a four-dimensional non-simple $\omega$-Lie algebra of degree $> 1$.

**Example 2.5.** Let $\{x, y, z, e\}$ be a basis of $\mathbb{C}^4$. In the following $\omega$-Lie algebra

$$\tilde{B} : \quad [x, y] = y, [x, z] = y + z, [y, z] = x, [e, x] = -2e, [e, y] = 0, [e, z] = 0,$$

and $\omega(x, y) = \omega(x, z) = 0, \omega(y, z) = 2, \omega(e, x) = \omega(e, y) = \omega(e, z) = 0$,

the subspace spanned by $\{e\}$ is a proper ideal of $\tilde{B}$ with the maximal dimension 1, i.e., there are no proper ideals in $\tilde{B}$ with dimension $> 1$. Hence $\tilde{B}$ is a non-simple $\omega$-Lie algebra of degree 3.

Let $L$ be an $\omega$-Lie algebra and $V$ be a finite-dimensional vector space. Recall that $V$ is called an $L$-module if there exists a bilinear map $L \times V \rightarrow V, (x, v) \mapsto x \cdot v$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) + \omega(x, y)v$$

for all $x, y \in L$ and $v \in V$.

To derive an $\omega$-Lie version of the classical Lie’s theorem, we concentrate on the class of non-simple $\omega$-Lie algebras of degree 1, and we give a proof of Theorem 1.2.
Proof of Theorem \[\text{[3]}\]. If \(\dim(L) \leq 2\), then \(L\) is a soluble Lie algebra. It follows from the classical Lie’s theorem that \(\dim(V) = 1\). Thus we may suppose \(\dim(L) \geq 3\) and regard \(V\) as a \(g\)-module. By \([11]\), Corollary 3.2, we see that a proper soluble ideal \(g\) of \(L\) is a soluble Lie algebra. If \(V\) is an irreducible \(g\)-module, then classical Lie’s theorem implies \(\dim(V) = 1\), and we are done.

Now we assume that \(V\) is a reducible \(g\)-module and there exists an irreducible \(g\)-submodule \(W \subset V\). Applying the classical Lie’s theorem again we see that \(\dim(W) = 1\). Fix a nonzero vector \(w_0 \in W\), there exists a one-dimensional representation \(\lambda\) of \(g\) given by \(W\) such that \(g \cdot w_0 = \lambda(g)w_0\) for all \(g \in g\). Define

\[
U := \{v \in V \mid g \cdot v = \lambda(g)v \text{ for all } g \in g\}.
\]

Then \(W \subseteq U \subseteq V\). We claim that \(U\) is also an \(L\)-module. If this claim holds, the irreducibility of \(V\) as an \(L\)-module, implies that \(V = U\); thus \(g \cdot v = \lambda(g)v\) for all \(g \in g\) and \(v \in V\). Moreover, for any vector \(\ell \in L\) but not in \(g\), let \(J\) denote the one-dimensional subspace spanned by \(\ell\). Then \(L\) can be decomposed into the direct sum \(g \oplus J\) as vector spaces. Let \(v_0\) be an eigenvector of \(\ell_V\) and let \(V_0\) denote the one-dimensional subspace spanned by \(v_0\). Then \(\ell \cdot v_0 \in V_0 \subseteq V\), which together with the fact that \(g \cdot v = \lambda(g)v\) for all \(g \in g\) and \(v \in V\), implies that \(V_0\) is an \(L\)-submodule of \(V\). As \(V\) is irreducible, we have \(V = V_0\). Hence, \(\dim(V) = \dim(V_0) = 1\).

Therefore, to accomplish the proof, it is sufficient to prove the claim that \(U\) is an \(L\)-module. For all \(g, g' \in g\) and \(v \in U\), we see that \(g \cdot (g' \cdot v) = g \cdot (\lambda(g')v) = (g') \cdot v = \lambda(g')\lambda(g) v = \lambda(g)\lambda(g')v = \lambda(g)(g' \cdot v)\), i.e., \(g' \cdot U \subseteq U\) for all \(g' \in g\). Thus it suffices to show that \(\ell \cdot U \subseteq U\); in other words, we have to prove that \(g \cdot (\ell \cdot v) = \lambda(g)(\ell \cdot v)\) for all \(g \in g\) and \(v \in U\). As \([g, J] \subseteq g\), we see that

\[
\lambda([g, \ell])v = [g, \ell] \cdot v = g \cdot (\ell \cdot v) - \ell \cdot (g \cdot v) + \omega(g, \ell)v = g \cdot (\ell \cdot v) - \lambda(g)(\ell \cdot v) + \omega(g, \ell)v.
\]

Thus it suffices to show that

\[
\lambda([g, \ell]) = \omega(g, \ell).
\]

To do this, we let \(0 \neq u \in U\) and define \(u_i := \ell \cdot u_{i-1}\) for \(i \in \mathbb{Z}^+\), starting with \(u_0 := u\) and \(u_1 := \ell \cdot u\). Let \(V'\) be the subspace spanned by \(\{u_i \mid i \in \mathbb{Z}_{\geq 0}\}\). Since \(V' \subseteq V\) and \(\dim(V')\) is finite, there exists some \(k \in \mathbb{Z}_{\geq 0}\) such that \(\dim(V') = 1\), which together with the fact that \(\dim(V) = 1\) implies \(\dim(V'') = 1\). Clearly, \(\ell \cdot V' \subseteq V'\). Let \(V_j'\) denote the subspace spanned by \(u_0, u_1, \ldots, u_j\) for \(j = 0, 1, \ldots, k\). Induction on \(j\) shows that \(g \cdot u_j - \lambda(g)u_j \in V_{j-1}'\) for all \(g \in g\). This means that \(V'\) is an \(L\)-submodule of \(V\). As \(V\) is irreducible, we have \(V = V'\), and the resulting matrix \(g_V\) can be written as an upper triangular matrix with the diagonals \(\lambda(g)\). Thus \(\text{Tr}(g_V) = (k + 1)\lambda(g)\) for all \(g \in g\); in particular, \(\text{Tr}([g, \ell]V) = (k + 1)\lambda([g, \ell])\). Since \([g, \ell]V = g_V \circ \ell_V - \ell_V \circ g_V + \omega(g, \ell)\), it
follows that \( \text{Tr}([g, \ell]_V) = \text{Tr}(\omega(g, \ell)1) = (k + 1)\omega(g, \ell) \). This implies that \( \lambda([g, \ell]) = \omega(g, \ell) \) and the proof is completed. \( \square \)

We provide two applications of Theorem 1.2.

**Proof of Corollary 1.3.** As any ideal of a soluble \( \omega \)-Lie algebra is soluble, this corollary could be obtained directly from Theorem 1.2 and Proposition 2.2. \( \square \)

Recall that an \( \omega \)-Lie algebra \( L \) is said to be **multiplicative** if there exists a linear form \( \lambda : L \to \mathbb{C} \) such that \( \omega(x, y) = \lambda([x, y]) \) for all \( x, y \in L \); see [9, Section 2] and [7, Section 6] for more results on multiplicative \( \omega \)-Lie algebras.

**Lemma 2.6.** Let \( L \) be an \( \omega \)-Lie algebra. Then \( \ker(\omega) = \{ x \in L \mid \omega(x, y) = 0 \text{ for all } y \in L \} \) is an \( L \)-module via the adjoint action.

**Proof.** Indeed, for all \( x \in \ker(\omega) \) and \( y, z \in L \), the \( \omega \)-Jacobi identity gives 

\[
[[y, z], x] + [[z, x], y] + [[x, y], z] = \omega(y, z)x + \omega(z, x)y + \omega(x, y)z = \omega(y, z)x.
\]

Then \( [[y, z], x] = [y, [z, x]] - [z, [y, x]] + \omega(y, z)x \) and hence \( \ker(\omega) \) is an \( L \)-module. \( \square \)

**Proposition 2.7.** Let \( L \) be a non-simple \( \omega \)-Lie algebra of degree 1 with a soluble ideal \( g \) of maximal dimension \( \dim(L) - 1 \). If \( \dim(L) > 2 \), then \( L \) is multiplicative.

**Proof.** As \( \dim(L) > 2 \), it follows from [10, Lemma 8.1] that \( \omega \) is degenerate. Then \( \ker(\omega) \) is a nonzero \( L \)-module. Let \( W \) be an irreducible \( L \)-submodule of \( \ker(\omega) \). By Theorem 1.2 we see that \( \dim(W) = 1 \). It follows from [10, Lemma 2.1] that \( L \) is multiplicative. \( \square \)

We give some remarks on modules and cohomology of \( \omega \)-Lie algebras. We refer to [11, Section 6] for some fundamental properties of modules for \( \omega \)-Lie algebras. The following example shows that the cohomology groups \( \mathfrak{H}^n(L, V) \) of an \( \omega \)-Lie algebra \( L \) with coefficients in an \( L \)-module \( V \) cannot be defined by the same formula for the differential as for ordinary Lie algebras via the way of Chevalley–Eilenberg complex; compared with [4].

**Example 2.8.** Suppose \( L \) is an \( \omega \)-Lie algebra and \( V \) is an \( L \)-module. As in the Chevalley–Eilenberg complex, we define the \( \mathbb{C} \)-vector space of \( k \)-cochains of \( L \) with coefficients in \( V \) to be \( C^0(L, V) := V \) and \( C^k(L, V) := \text{Hom}_\mathbb{C}(\Lambda^k L, V) \) for \( k \geq 1 \). The differential \( d_k : C^k(L, V) \to C^{k+1}(L, V) \) is defined as

\[
d_k(f)(x_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} x_i \cdot f(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1}) + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}).
\]
In particular, if \( v \in C^0(L, V) = V \), then \( d_0(v) : L \rightarrow V \) is given by \( d_0(v)(x) = x \cdot v \) for all \( x \in L \). For \( f \in C^1(L, V) \), \( d_1(f) \in C^2(L, V) \) is given by

\[
d_1(f)(x, y) = x \cdot f(y) - y \cdot f(x) - f([x, y])
\]

for all \( x, y \in L \). We observe that the map \( d_1 \circ d_0 \) is not zero, unless \( L \) is a Lie algebra. In fact, for \( v \in V \) and \( x, y \in L \),

\[
(d_1 \circ d_0)(v)(x, y) = d_1(d_0(v))(x, y) \\
= x \cdot d_0(v)(y) - y \cdot d_0(v)(x) - d_0(v)([x, y]) \\
= x \cdot (y \cdot v) - y \cdot (x \cdot v) - [x, y] \cdot v \\
= -\omega(x, y)v.
\]

The last equality follows from Eq. (3).

Moreover, let \( L \) be an \( \omega \)-Lie algebra and \( V, W \) be two \( L \)-modules. We also note that unlike the situation of ordinary Lie algebras, the map defined by

\[
(x, v \otimes w) \mapsto x \cdot v \otimes w + v \otimes x \cdot w
\]

would not give an \( L \)-module structure on the tensor product \( V \otimes W \), where \( x \in L, v \in V \) and \( w \in W \). However, for multiplicative \( \omega \)-Lie algebras we have the following proposition.

**Proposition 2.9.** Let \( L \) be a multiplicative \( \omega \)-Lie algebra with the linear form \( \lambda \) and \( V, W \) be \( L \)-modules. Then \( V \otimes W \) is an \( L \)-module defined by

\[
x \cdot (v \otimes w) := x \cdot v \otimes w + v \otimes x \cdot w - \lambda(x)v \otimes w,
\]

where \( x \in L, v \in V, \) and \( w \in W \).

**Proof.** For an arbitrary element \( y \in L \), we have \( [y, x] \cdot (v \otimes w) = [y, x] \cdot v \otimes w + v \otimes [y, x] \cdot w - \lambda([y, x])v \otimes w \).

\[
y \cdot (x \cdot (v \otimes w)) = y \cdot (x \cdot v \otimes w + v \otimes x \cdot w - \lambda(x)v \otimes w)
\]

\[
= y \cdot x \cdot v \otimes w + x \cdot v \otimes y \cdot w - \lambda(y)x \cdot v \otimes w \\
+ y \cdot v \otimes x \cdot w + v \otimes y \cdot (x \cdot w) - \lambda(y)v \otimes x \cdot w \\
- \lambda(x)y \cdot v \otimes w - \lambda(x)v \otimes y \cdot w + \lambda(x)\lambda(y)v \otimes w,
\]

\[
x \cdot (y \cdot (v \otimes w)) = x \cdot (y \cdot v \otimes w + v \otimes (y \cdot w) - \lambda(y)v \otimes w)
\]

\[
= x \cdot (y \cdot v) \otimes w + y \cdot v \otimes x \cdot w - \lambda(x)y \cdot v \otimes w \\
+ x \cdot v \otimes y \cdot w + v \otimes x \cdot (y \cdot w) - \lambda(x)v \otimes y \cdot w.
\]
\[-\lambda(y)x \cdot v \otimes w - \lambda(y)v \otimes x \cdot w + \lambda(y)\lambda(x)v \otimes w.\]

Then \(y \cdot (x \cdot (v \otimes w)) - x \cdot (y \cdot (v \otimes w)) + \omega(y, x)v \otimes w = [y, x] \cdot (v \otimes w),\) which implies that \(V \otimes W\) is an \(L\)-module. \[
\]

Note that the adjoint map does not give an \(L\)-module structure on \(L,\) unless \(L\) is a Lie algebra. The following example demonstrates that for \(k \in \mathbb{Z}^+,\) the space of \(k\)-cochains \(C^k(L, V)\) might not be an \(L\)-module via the formula

\[
(x \cdot f)(z_1, \cdots, z_k) := x \cdot (f(z_1, \cdots, z_k)) - \sum_{i=1}^{k} f(z_1, \cdots, z_{i-1}, [x, z_i], \cdots, z_k), \tag{6}
\]

where \(x, z_1, \ldots, z_k \in L\) and \(f \in C^k(L, V).\)

**Example 2.10.** Consider \(k = 1\) and \(C^1(L, V) = \text{Hom}_C(L, V).\) For \(x, y, z \in L,\) the formula (6) reads to \((x \cdot f)(z) = x \cdot f(z) - f([x, z]).\) Thus \(([x, y] \cdot f)(z) = [x, y] \cdot f(z) - f([x, y], z]) = x \cdot (y \cdot f(z)) - y \cdot (x \cdot f(z)) + \omega(x, y)f(z) - f([x, y], z)).\) On the other hand, we note that

\[
\begin{align*}
x \cdot (y \cdot f)(z) &= x \cdot ((y \cdot f)(z)) - (y \cdot f)([x, z]) \\
&= x \cdot (y \cdot f(z)) - x \cdot f([y, z]) - y \cdot f([x, z]) + f([y, [x, z]])
\end{align*}
\]

and

\[
y \cdot (x \cdot f)(z) = y \cdot (x \cdot f(z)) - y \cdot f([x, z]) - x \cdot f([y, z]) + f([x, [y, z]]).
\]

Thus

\[
([x, y] \cdot f)(z) - x \cdot (y \cdot f)(z) + y \cdot (x \cdot f)(z) - \omega(x, y)f(z)
= f([x, [y, z]]) - f([x, y], z]) - f([y, [x, z]])
= -f(\omega(y, z)x + \omega(x, y)z + \omega(z, x)y),
\]

which does not vanish in general, unless \(L\) is a Lie algebra.

3. Indecomposable modules

In this section we study indecomposable modules of some three-dimensional \(\omega\)-Lie algebras and give a proof of Theorem 2.4.

Let \(L \in \mathcal{L}\) be a three-dimensional non-Lie \(\omega\)-Lie algebra over \(\mathbb{C}\) with a basis \(\{x, y, z\}.\)

It follows from [4, Theorem 2] that there always exists a two-dimensional Lie subalgebra \(g \subset L,\) spanned by \(y\) and \(z\) such that \([y, z] = z.\) Define \(\mathfrak{h}\) to be the subspace spanned by \(z.\) Clearly, \(g\) is isomorphic to the unique two-dimensional nonabelian Lie algebra over \(\mathbb{C}\) and \(\mathfrak{h}\) can be viewed as an abelian Lie algebra. Throughout this section we assume that the element \(z\) belongs to \(\ker(\omega);\) namely, \(L \in \{L_1, A_3\}.\)
Suppose $V$ is a finite-dimensional indecomposable $L$-module. Since $V$ is also an $\mathfrak{h}$-module, there exists a finite set $\{\lambda_1, \ldots, \lambda_k\}$ of weights of $\mathfrak{h}$ such that

$$V = \bigoplus_{i=1}^{k} V_{\lambda_i},$$

where $V_{\lambda_i} := \{v \in V \mid \text{for each } h \in \mathfrak{h}, \text{ there exists } n_h \text{ such that } (h_V - \lambda_i(h)1)^{n_h}(v) = 0\} \neq \{0\}$. Further, these $V_{\lambda_i}$ are $\mathfrak{h}$-modules; see [3, Theorem 2.9].

Note that $\mathfrak{h} \subset \mathfrak{g} \subset L$ and $V$ is also a $\mathfrak{g}$-module. With above notations and conventions, we obtain several helpful lemmas.

**Lemma 3.1.** For $1 \leq i \leq k$, $V_{\lambda_i}$ is a $\mathfrak{g}$-module.

**Proof.** It suffices to show that $y \cdot v \in V_{\lambda_i}$ for all $y \in \mathfrak{g}$ and $v \in V_{\lambda_i}$. Consider the Lie algebra $\mathfrak{g}$ and the $\mathfrak{g}$-module $V$. Since $\omega(y, z) = 0$ in $L$, an analogous argument with [3, Proposition 2.7] implies that for $h \in \mathfrak{h}$, $\lambda_i(h) \in \mathbb{C}$ and $v \in V_{\lambda_i}$, we have

$$\begin{align*}
(h_V - \lambda_i(h)1)^{n}(y \cdot v) = \sum_{j=0}^{n} \binom{n}{j} ((\text{ad}_{h})^j(y))(h_V - \lambda_i(h)1)^{n-j}(v)
\end{align*}$$

for $n \in \mathbb{Z}^+$. Note that $h = az$ for some $a \in \mathbb{C}$ and $[y, z] = z$. Setting $n = n_h + 1$ in Eq. (3), we see that $(h_V - \lambda_i(h)1)^{n_h+1}(y \cdot v) = 0$. This means $y \cdot v \in V_{\lambda_i}$ and thus $V_{\lambda_i}$ is a $\mathfrak{g}$-module. □

Let $\mathcal{D} := (L \oplus V, \Omega)$ be the semi-direct product of an $\omega$-Lie algebra $(L, \omega)$ and an $L$-module $V$, where $\Omega$ extends $\omega$ trivially; see [4, Proposition 6.3] for the definition of the semi-direct product of an $\omega$-Lie algebra and its module.

**Lemma 3.2.** There is an abelian Lie subalgebra $H$ of $\mathcal{D}$ such that $H \subseteq \ker(\Omega)$ and $\dim(H) > 1$.

**Proof.** If $V$ is a trivial $\mathfrak{h}$-module, i.e., $z \cdot v = 0$ for all $v \in V$, then $H = \mathfrak{h} \oplus V$ is what we want. Now assume that $V$ is a nontrivial $\mathfrak{h}$-module and consider the Lie subalgebra $\mathfrak{g} \oplus V$ of $\mathcal{D}$. We observe that $\mathfrak{g} \oplus V$ is a soluble Lie algebra, thus $[\mathfrak{g} \oplus V, \mathfrak{g} \oplus V]$ is nilpotent. Since $[y, z] = z$, we have $\mathfrak{h} \oplus \{0\} \subseteq [\mathfrak{g} \oplus V, \mathfrak{g} \oplus V] \subseteq \mathfrak{h} \oplus V$. As $V$ is not a trivial $\mathfrak{h}$-module, we can find a vector $v_0 \in V$ such that $z \cdot v_0 \neq 0$. Thus $(0, z \cdot v_0) = [(z, 0), (z, v_0)] \in [\mathfrak{g} \oplus V, \mathfrak{g} \oplus V]$ but not in $\mathfrak{h} \oplus \{0\}$. This implies that $\dim([\mathfrak{g} \oplus V, \mathfrak{g} \oplus V]) > \dim(\mathfrak{h}) = 1$. Let $V' \subseteq V$ be the subspace such that $[\mathfrak{g} \oplus V, \mathfrak{g} \oplus V] = \mathfrak{h} \oplus V'$. Then $\dim(V') \geq 1$. By Engel’s theorem, $\text{ad}_{(z, 0)} : \mathfrak{h} \oplus V' \to \mathfrak{h} \oplus V'$ is nilpotent, and it also restricts to a nilpotent linear map on $V'$. We use $V_1$ to denote the kernel of $\text{ad}_{(z, 0)}$ in $V'$. Then $V_1 \neq \{0\}$ and so $\dim(V_1) \geq 1$. Note that for any $v \in V_1$, the fact that $0 = \text{ad}_{(z, 0)}(0, v) = [(z, 0), (0, v)] = (0, z \cdot v)$ implies $z \cdot v = 0$, thus the action of $\mathfrak{h}$ on $V_1$ is trivial. Let $H = \mathfrak{h} \oplus V_1$. Observe that $H$ is an abelian Lie subalgebra of $\mathcal{D}$ such that $H \subseteq \ker(\Omega)$ and $\dim(H) > 1$. The proof is completed. □
**Lemma 3.3.** Let \( \text{ad} : \mathcal{D} \rightarrow \mathcal{D} \) be the adjoint map. Then

\[
\sum_{j=0}^{n} \binom{n}{j} \left[ (\text{ad}_h + \alpha 1)^{n-j}(u), (\text{ad}_h + \beta 1)^j(v) \right] = (\text{ad}_h + (\alpha + \beta)1)^n([u, v]) - n(\alpha + \beta)^{n-1}\Omega(u, v)h
\]

for all \( n \in \mathbb{Z}^+, u, v \in \mathcal{D}, h \in H \) and \( \alpha, \beta \in \mathbb{C} \).

**Proof.** We apply [10, Lemma 4.4] for \( \mathcal{D} = (L \oplus V, \Omega) \) with \( H \) defined in Lemma 3.2. \( \square \)

We identify \( L \) with \( L \oplus \{0\} \) and identify \( V \) with \( \{0\} \oplus V \) in \( \mathcal{D} \). With this two identifications, we are working on \( \mathcal{D} \). In Eq. (3), setting \( \alpha = 0, h \in \mathfrak{h} = \mathfrak{h} \oplus \{0\}, u = x \in L \) and \( v \in V \), we obtain the following lemma.

**Lemma 3.4.** For any \( n \in \mathbb{Z}^+ \) and \( \beta \in \mathbb{C} \),

\[
(\text{ad}_h + \beta 1)^n([x, v]) = \sum_{j=0}^{n} \binom{n}{j} \left[ (\text{ad}_h)^{n-j}(x), (\text{ad}_h + \beta 1)^j(v) \right]. \tag{10}
\]

Finally, we prove the following key lemma.

**Lemma 3.5.** \( V_{\lambda_i} \) is an \( L \)-module for \( 1 \leq i \leq k \).

**Proof.** By Lemma 3.3 it suffices to show that \( x \cdot v \in V_{\lambda_i} \) for all \( v \in V_{\lambda_i} \). We observe that \([x, v] = [(x, 0), (0, v)] = (0, x \cdot v) = x \cdot v \) and for \( w \in V \), \( (\text{ad}_h + \beta 1)(w) = \text{ad}_h(w) + \beta 1(w) = [h, w] + \beta 1(w) = [(h, 0), (0, w)] + \beta 1(0, w) = (0, h \cdot w) + (0, \beta 1(w)) = (h_V + \beta 1)_w \). Thus \( (\text{ad}_h + \beta 1)^n(w) = (h_V + \beta 1)^n(w) \) for all \( w \in V \) and \( n \in \mathbb{Z}^+ \). These observations, together with setting \( \beta = -\lambda_i(h) \) in Eq. (10), imply that

\[
(h_V - \lambda_i(h)1)^n(x \cdot v) = \sum_{j=0}^{n} \binom{n}{j} \left[ (\text{ad}_h)^{n-j}(x), (h_V - \lambda_i(h)1)^j(v) \right]. \tag{11}
\]

Recall that \( h = az \) for some \( a \in \mathbb{C} \) and \([z, [z, [z, x]]] = 0 \) in \( L \). Thus \( \text{ad}_h^j(x) = 0 \) for \( j \geq 3 \). Taking \( n = n_h + 2 \) in Eq. (11), we obtain \((h_V - \lambda_i(h)1)^{n_h}(x \cdot v) = 0 \). Hence, \( V_{\lambda_i} \) is an \( L \)-module. \( \square \)

An important consequence has been derived.

**Corollary 3.6.** \( k = 1 \) in Eq. (4).

**Proof.** As \( V \) is indecomposable, Lemma 3.3 implies \( k = 1 \). \( \square \)

Suppose \( n \in \mathbb{Z}^+ \) and \( M_n(\mathbb{C}) \) denotes the \( n^2 \)-dimensional vector space of all \( n \times n \) matrices over \( \mathbb{C} \). Let \( N_n(\mathbb{C}) \) be the set of all nilpotent matrices in \( M_n(\mathbb{C}) \) and \( D_n(\mathbb{C}) = \{ \lambda I_n \mid \lambda \in \mathbb{C} \} \).
be the subspace spanned by the identity matrix $I_n$ in $M_n(\mathbb{C})$. Clearly, $N_n(\mathbb{C}) \cap D_n(\mathbb{C}) = \{0\}$.

Define

$$P_n(\mathbb{C}) := D_n(\mathbb{C}) \times N_n(\mathbb{C}).$$

There exists a natural conjugacy action of the general linear group $GL(n, \mathbb{C})$ on $P_n(\mathbb{C})$ given by

$$\sigma(\lambda I_n, A) := (\sigma(\lambda I_n)\sigma^{-1}, \sigma A\sigma^{-1}) = (\lambda I_n, \sigma A\sigma^{-1}),$$

where $\sigma \in GL(n, \mathbb{C})$, $\lambda \in \mathbb{C}$ and $A \in N_n(\mathbb{C})$.

We use $\mathcal{R}_n^0(\mathbb{C})$ to denote the set of all indecomposable $L$-modules on $\mathbb{C}^n$ such that the actions of $x$ and $y$ on $\mathbb{C}^n$ are determined by the action of $z$. Let $\mathcal{B}_n(\mathbb{C})$ be the set of all $\mathfrak{h}$-modules on $\mathbb{C}^n$ and $\mathcal{A}_n(\mathbb{C})$ be the subset of $\mathcal{B}_n(\mathbb{C})$ consisting of all $\mathfrak{h}$-modules for which the resulting matrix of $z$ on $\mathbb{C}^n$ can be written as the sum of two matrices from the components of $P_n(\mathbb{C})$.

**Proposition 3.7.** There exists an injective map $\phi$ from $\mathcal{R}_n^0(\mathbb{C})$ to $\mathcal{A}_n(\mathbb{C})$.

**Proof.** For each $V \in \mathcal{R}_n^0(\mathbb{C})$, it is also a $\mathfrak{h}$-module. Corollary 3.6 shows that $V$ is isomorphic to some $V_\lambda$ for $\lambda \in \text{Hom}(\mathfrak{h}, \mathbb{C})$. Note that $\dim(\mathfrak{h}) = 1$ and $\mathfrak{h}$ is spanned by $z$, so $\lambda$ is determined by the complex number $\lambda(z)$. Since $z_{V_\lambda} - \lambda(z)I_n \in N_n(\mathbb{C})$, we have $z_{V_\lambda} = \lambda(z)I_n + (z_{V_\lambda} - \lambda(z)I_n)$, where $(\lambda(z)I_n, z_{V_\lambda} - \lambda(z)I_n) \in P_n(\mathbb{C})$. Now we define

$$\phi : \mathcal{R}_n^0(\mathbb{C}) \rightarrow \mathcal{A}_n(\mathbb{C})$$

by $V \mapsto \phi(V)$, where $\phi(V)$ is determined uniquely by $z_{\phi(V)} = \lambda(z)I_n + (z_{V_\lambda} - \lambda(z)I_n)$. For any $V_1, V_2 \in \mathcal{R}_n^0(\mathbb{C})$, there exist $\lambda_1, \lambda_2 \in \text{Hom}(\mathfrak{h}, \mathbb{C})$ such that $V_i = V_{\lambda_i}$ for $i = 1, 2$. As $\lambda(z)I_n$ and $z_{V_\lambda} - \lambda(z)I_n$ are the semisimple and nilpotent parts respectively in the Jordan-Chevalley decomposition in $z_{\phi(V)}$, the uniqueness of the decomposition implies that if $z_{\phi(V_1)} = z_{\phi(V_2)}$, then $\lambda_1 = \lambda_2$. Thus $V_1 = V_{\lambda_1} = V_{\lambda_2} = V_2$. This means that $\phi$ is injective. \hfill \Box

**Proposition 3.8.** There exists a bijection between $\mathcal{A}_n(\mathbb{C})$ and $P_n(\mathbb{C})$. Moreover, the equivalence classes in $\mathcal{A}_n(\mathbb{C})$ are in one-to-one correspondence with the conjugacy classes in $P_n(\mathbb{C})$.

**Proof.** Since $\mathfrak{h}$ is one-dimensional and spanned by $z$, any $\mathfrak{h}$-module $V$ in $\mathcal{A}_n(\mathbb{C})$ is determined by the matrix $z_V = \lambda(z)I_n + (z_{V_\lambda} - \lambda(z)I_n)$, where $(\lambda(z)I_n, z_{V_\lambda} - \lambda(z)I_n) \in P_n(\mathbb{C})$. If $V \in \mathcal{A}_n(\mathbb{C})$, then $\varphi(V) := (\lambda(z)I_n, z_{V_\lambda} - \lambda(z)I_n)$ gives rise to a map from $\mathcal{A}_n(\mathbb{C})$ to $P_n(\mathbb{C})$. Conversely, as $\mathfrak{h}$ is one-dimensional, any matrix $B \in M_n(\mathbb{C})$ could define an $\mathfrak{h}$-module $V_B$ by $z_{V_B} = B$. If $(\lambda(z)I_n, B - \lambda(z)I_n) \in P_n(\mathbb{C})$, then $V_B \in \mathcal{A}_n(\mathbb{C})$. Let $\varphi' : P_n(\mathbb{C}) \rightarrow \mathcal{A}_n(\mathbb{C})$ be the map given by $\varphi'(B) = V_B$. Clearly, $\varphi \circ \varphi' = 1_{P_n(\mathbb{C})}$ and $\varphi' \circ \varphi = 1_{\mathcal{A}_n(\mathbb{C})}$. Hence, $\varphi$ is a bijection between $\mathcal{A}_n(\mathbb{C})$ and $P_n(\mathbb{C})$. Note that $V_1$ is equivalent to $V_2$ in $\mathcal{A}_n(\mathbb{C})$ if and only
if \(z_{V_1}\) and \(z_{V_2}\) are similar, if and only if \(z_{V_1}\) is conjugate with \(z_{V_2}\) in \(P_n(\mathbb{C})\). This proves the second statement. \(\square\)

\textbf{Proof of Theorem 1.4.} Combining Propositions 3.7 and 3.8 together with the fact that if \(V_1\) is equivalent to \(V_2\) in \(R_n(\mathbb{C})\) then \(\phi(V_1)\) and \(\phi(V_2)\) are also equivalent in \(A_n(\mathbb{C})\), we see that the actions of \(z \in L\) in two equivalent representations can be parameterized by the complex field and conjugacy classes of nilpotent \(n \times n\)-matrices. By the nonzero generating relations in \(L\), we see that the actions of \(x\) and \(y\) on \(\mathbb{C}^n\) can be determined by finitely many polynomial equations. Thus an arbitrary action of \(L\) on \(\mathbb{C}^n\) can be determined by a complex number, a nilpotent matrix and two elements of an affine variety. This completes the proof. \(\square\)

As an application, we conclude with the following example.

\textbf{Example 3.9.} We can completely determine all 2-dimensional indecomposable \(L_1\)-modules. Suppose \(V\) is such a module. Recall that any \(2 \times 2\) nilpotent matrix is similar to \((0\ 0)\) or \((0\ 1)\).

For the first case, we may assume

\[
z_V = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},
y_V = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}
\text{ and } x_V = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}
\]

with respect to a basis \(\{e_1, e_2\}\) of \(V\), where \(a, b, c, \in \mathbb{C}, 1 \leq i \leq 4\). By Eq. (3), we obtain two subcases:

1. \(z_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
y_V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) and \(x_V = \begin{pmatrix} c + 1 & b \\ 0 & c \end{pmatrix}\);
2. \(z_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
y_V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(x_V = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}\),

where \(b, c \in \mathbb{C}\).

For the second case, we assume

\[
z_V = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},
y_V = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}\) and \(x_V = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}\)

with respect to a basis \(\{e_1, e_2\}\) of \(V\), where \(a, b, c, \in \mathbb{C}, 1 \leq i \leq 4\). A direct calculation leads to \(3/2 = b_1 = 1\), which is a contradiction. It also shows that the map \(\phi\) in Proposition 3.7 is not surjective.

4. Tailed derivations of Lie algebras

The last section is mainly to study relations between one-dimensional \(\omega\)-extensions of a Lie algebra \(g\) and tailed derivations of \(g\), focusing on fundamental properties and examples on tailed derivations of Lie algebras and giving a proof of Theorem 1.6.
Proposition 4.1. Let \( A \) be a nonassociative algebra. Then \( \text{TDer}(A) \) is a Lie subalgebra of \( \mathfrak{gl}(A) \).

Proof. Suppose \( D, T \in \text{TDer}(A) \) are arbitrary tailed derivations. For \( y, z \in A \), we have

\[
(D + T)([y, z]) = D([y, z]) + T([y, z]) = [D(y), z] + [y, D(z)] + d_y y - d_y z + [T(y), z] + [y, T(z)] + t_y y - t_y z
\]

where \( d_y, t_y, d_z, t_z \in \mathbb{C} \). For \( a \in \mathbb{C} \), we see that \( (aD)([y, z]) = a(D[y, z]) = a([D(y), z] + [y, D(z)] + d_y y - d_y z) \). This means that \( \text{TDer}(A) \) is a subspace of \( \mathfrak{gl}(A) \). To show \( \text{TDer}(A) \) is a Lie subalgebra of \( \mathfrak{gl}(A) \), it suffices to show that \( [D, T] = DT - TD \) is also a tailed derivation. Indeed, since

\[
DT([y, z]) = D([T(y), z] + [y, T(z)] + t_y y - t_y z) = [DT(y), z] + [T(y), D(z)] + d_z T(y) - d_T(y) z + [D(y), T(z)] + [y, DT(z)] + d_T(z) y - d_y T(z) + t_z D(y) - t_y D(z),
\]

\[
TD([y, z]) = T([D(y), z] + [y, D(z)] + d_y y - d_y z) = [TD(y), z] + [D(y), T(z)] + t_z D(y) - t_D(y) z + [T(y), D(z)] + [y, TD(z)] + d_T(z) y - t_y D(z) + d_y T(y) - d_y T(z),
\]

we have

\[
[D, T]([y, z]) = [[D, T][y, z] + [y, [D, T](z)] + (d_T(z) - t_D(z)) y - (d_T(y) - t_D(y)) z. \quad (13)
\]

Note that \( d_{T(-)} - t_{D(-)} = (d \circ T - t \circ D)(-) \) is a linear form of \( A \). Thus \( [D, T] \) is a tailed derivation of \( A \). This shows that \( \text{TDer}(A) \) is a Lie algebra. \( \square \)

Example 4.2. Let \( \mathfrak{g} \) be the two-dimensional nonabelian Lie algebra defined by \( [y, z] = z \) and \( D = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \) be a linear map on \( \mathfrak{g} \) with respect to the basis \( \{y, z\} \), where \( a, b, c, e \in \mathbb{C} \). A direct calculation shows that if \( D \in \text{Der}(\mathfrak{g}) \), then \( a = c = 0 \). Thus \( \dim \text{Der}(\mathfrak{g}) = 2 \).

Moreover, consider the linear form \( d \) which sends \( y \) to \( a \) and \( z \) to \( c \). Then together with the linear form \( d \), every \( D = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \) is a tailed derivation of \( \mathfrak{g} \). This means that \( \text{TDer}(\mathfrak{g}) = \mathfrak{gl}_2(\mathbb{C}) \), strictly containing \( \text{Der}(\mathfrak{g}) \).

Proposition 4.3. Let \( L \) be an \( \omega \)-Lie algebra with a nonzero proper ideal \( \mathfrak{g} \). Suppose \( L = \mathfrak{g} \oplus \mathfrak{h} \) denotes a decomposition of vector spaces. Then \( \text{ad}_x \) restricted to \( \mathfrak{g} \) is a tailed derivation of \( \mathfrak{g} \) for all \( x \in \mathfrak{h} \).
indicates that the problem of finding all one-dimensional simple Lie algebra could be transformed to calculate tailed derivations of \( g \).

**Lemma 4.4.** Let \( g \) be a nonzero Lie subalgebra of an \( \omega \)-Lie algebra \( L \) of dimension \( \dim(L) - 1 \). Let \( x \in L \setminus g \) be an arbitrary nonzero vector. Then \( \text{ad}_x \) restricted to \( g \) is a tailed derivation of \( g \).

**Proof.** Let \( y, z \in g \). As \( \omega(y, z) = 0 \), it follows from the \( \omega \)-Jacobi identity that \( \text{ad}_x([y, z]) = [\text{ad}_x(y), z] + [y, \text{ad}_x(z)] + \omega(x, z)y - \omega(x, y)z \). Thus \( \text{ad}_x \) restricted to \( g \) is a tailed derivation of \( g \).

Now we are ready to prove Theorem 4.4.

**Proof of Theorem 4.4.** Let \( L_x \in \text{Ext}_1^{\omega}(g) \) be a one-dimensional \( \omega \)-extension of \( g \) through \( \mathbb{C}x \). Lemma 4.4 shows that the adjoint map \( \text{ad}_x : L_x \to L_x \) restricted to \( g \) is an element of \( \text{TDer}(g) \). We can define a map \( \varphi : \text{Ext}_1^{\omega}(g) \to \text{TDer}(g) \) by carrying \( L_x \) to \( \text{ad}_x|_g \). Conversely, if \( D \) is a tailed derivation of \( g \), then there exists a linear form \( d \) of \( g \) such that \( D([y, z]) = [D(y), z] + [y, D(z)] + dz - dy \) for all \( y, z \in g \). We define an \( \omega \)-Lie algebra \( L_x = g \oplus \mathbb{C}x \) by

\[
L_x : [x, y] = D(y), [x, x] = 0 \text{ and } \omega(x, y) = dy, \omega(x, x) = 0
\]

for all \( y \in g \); the remaining bracket product \([y, z]\) in \( L_x \) matches with that in \( g \) and \( \omega(y, z) = 0 \) for all \( y, z \in g \). Note that \( dy \) only depends upon \( y \) so \( \omega(x, y) = dy \) does make sense. Thus \( L_x \) is a well-defined \( \omega \)-Lie algebra. We also define a map \( \phi : \text{TDer}(g) \to \text{Ext}_1^{\omega}(g) \) by \( \phi(D) = L_x \). Furthermore, note that \( g \) is an ideal of \( L_x \) and by the previous construction we see that \( \phi \circ \varphi = 1_{\text{Ext}_1^{\omega}(g)} \) and \( \varphi \circ \phi = 1_{\text{TDer}(g)} \). This completes the proof.

Theorem 4.4 indicates that the problem of finding all one-dimensional \( \omega \)-extensions of a Lie algebra \( g \) could be transformed to calculate tailed derivations of \( g \). As a direct application, the following example illustrates how to determine all one-dimensional \( \omega \)-extensions of three-dimensional simple Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \).

**Example 4.5.** Suppose that \( \mathfrak{sl}_2(\mathbb{C}) \) has a basis \( \{e_1, e_2, e_3\} \) with \([e_1, e_2] = -e_1, [e_1, e_3] = 2e_2 \) and \([e_2, e_3] = -e_3 \). A tedious but direct calculation shows that \( \text{Der}(\mathfrak{sl}_2(\mathbb{C})) = \text{TDer}(\mathfrak{sl}_2(\mathbb{C})) \) has dimension 3 and the element \( D \in \text{Der}(\mathfrak{sl}_2(\mathbb{C})) \) is of the form:

\[
D = \begin{pmatrix} a & b & 0 \\ -2c & 0 & -2b \\ 0 & c & -a \end{pmatrix},
\]
where \(a, b, c \in \mathbb{C}\). Hence, any one-dimensional \(\omega\)-extension of \(\mathfrak{sl}_2(\mathbb{C})\) can be determined by at most three parameters.

We present an example of a non-Lie \(\omega\)-Lie algebra that can be obtained by a Lie algebra \(\mathfrak{g}\) and a tailed derivation \(D\) of \(\mathfrak{g}\).

**Example 4.6.** Let \(\mathfrak{g}\) be the two-dimensional nonabelian Lie algebra defined by \([y, z] = z\) and \(D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) be a linear map on \(\mathfrak{g}\) with respect to the basis \(\{y, z\}\). Let \(\{y^*, z^*\}\) be the dual basis. Then \(y^*: \mathfrak{g} \rightarrow \mathbb{C}\) is a linear form such that \(D\) becomes a tailed derivation of \(\mathfrak{g}\). By the construction in the proof of Theorem 1.6 we eventually derive a three-dimensional non-Lie \(\omega\)-Lie algebra which is actually the \(\omega\)-Lie algebra \(L_1\) in Example 2.1. The \(\omega\)-Lie algebra \(L_2\) in Example 2.1 can also be obtained in a similar way.

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School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R. China

Email address: zhangrx728@nenu.edu.cn