GENERALIZED DERIVATIONS ON CERTAIN BANACH ALGEBRAS

ALI EBRAHIMZADEH ESFAHANI AND MEHDI NEMATI

Abstract. Let $\mathcal{A}$ be a Banach algebra with the properties that $\text{rad}(\mathcal{A}) = \text{rann}(\mathcal{A})$ and the algebra $\mathcal{A}/\text{rad}(\mathcal{A})$ is commutative. We show that a derivation of $\mathcal{A}$ maps $\mathcal{A}$ into $\text{rad}(\mathcal{A})$. Using this, we determine among other things when a generalized derivation of $\mathcal{A}$ maps $\mathcal{A}$ into $\text{rad}(\mathcal{A})$. We also study $k$-centralizing generalized derivations of $\mathcal{A}$. Then, for a generalized derivation $(\delta, d)$ of $\mathcal{A}$ we obtain a necessary and sufficient condition for $(\delta^2, d^2)$ to be still a generalized derivation of $\mathcal{A}$. The main applications are concerned with the algebras over locally compact groups. In particular, we deduce these results for bidual of Fourier algebras of discrete amenable groups as an application of our approach.

1. Introduction and preliminaries

For a Banach algebra $\mathcal{A}$, we denote by $\text{rad}(\mathcal{A})$ the (Jacobson) radical of $\mathcal{A}$ and by $\text{rann}(\mathcal{A})$ the right annihilator of $\mathcal{A}$; i.e. the set of all $c \in \mathcal{A}$ such that $ac = 0$ for all $a \in \mathcal{A}$. Note that $\text{rann}(\mathcal{A})$ is nilpotant and so $\text{rann}(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$; see for example [4, Proposition 1.5.6]. A linear map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation of $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. For each $a \in \mathcal{A}$, the map $d_a : \mathcal{A} \to \mathcal{A}$ defined by $d_a(b) = [b, a] = ba - ab$ for all $b \in \mathcal{A}$, is a derivation which is called the inner derivation induced by $a$. A linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a generalized derivation of $\mathcal{A}$ if there exists a derivation $d$ of $\mathcal{A}$ such that $\delta(ab) = a\delta(b) + d(a)b$ for all $a, b \in \mathcal{A}$. This coupling is denoted by $(\delta, d)$.

A well-known theorem of Singer and Wermer states that every bounded derivation on a commutative Banach algebra has its image in the radical [18]. About 30 years later, Thomas extended the Singer-Wermer theorem to arbitrary, not necessarily bounded, derivations [19]. A number of authors have generalized this theorem in several ways; see [2, 13, 14, 15]. For example, a well-known result in [13] states that if $d$ is a bounded derivation of a Banach algebra $\mathcal{A}$ such that $[a, d(a)] \in \mathbb{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$. Later in [14] the authors prove the following: If $d$ is a derivation of a Banach algebra $\mathcal{A}$ such that $[a, d(a)] \in \mathbb{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$.

Along these lines of research, Breasar and Mathieu [2, Theorem 2.8] obtained a necessary and sufficient condition for a derivation to be spectrally bounded on a unital Banach algebra.

2010 Mathematics Subject Classification. Primary: 47B47, 47B48 Secondary: 46H05, 43A30.

Key words and phrases. Generalized derivation, Radical, Singer-Wermer conjecture, Orthogonal generalized derivation.
Throughout the paper \( \mathcal{A} \) denotes a Banach algebra with the properties that \( \text{rad}(\mathcal{A}) = \text{rann}(\mathcal{A}) \) and the algebra \( \mathcal{A}/\text{rad}(\mathcal{A}) \) is commutative. In harmonic analysis, some examples of (classes of) Banach algebras have been discovered for which \( \text{rad}(\mathcal{A}) = \text{rann}(\mathcal{A}) \) and the algebra \( \mathcal{A}/\text{rad}(\mathcal{A}) \) is commutative. The most notable cases are \( L_0^\infty(G)^* \) when \( G \) is an abelian locally compact group \([12]\) and \( VN(G)^* \) when \( G \) is a discrete group \([11]\).

As we shall see, the situation where the Banach algebra \( \mathcal{A} \) satisfying these conditions provides a framework which allows us to apply the well-known results concerning derivations and generalized derivations to \( \mathcal{A} \) and investigate the truth of all previous results for \( \mathcal{A} \).

The organization of this paper is as follows. In Section 2, we consider an introverted subspace \( X \) of \( VN(G) \), the von Neumann algebra generated by the left regular representation of \( G \), with the following properties

\[
C^*_\lambda(G) \subseteq X, \quad A(G) \cdot X \subseteq C^*_\lambda(G). \quad (1.1)
\]

For the Banach algebra \( X^* \), equipped with the Arens product, we show that \( \text{rad}(X^*) = \text{rann}(X^*) \) and the algebra \( X^*/\text{rad}(X^*) \) is commutative. We also prove that \( X^* \) is neither commutative nor semiprime when \( X \neq C^*_\lambda(G) \).

In Section 3, we first show that every derivation of the Banach algebra \( \mathcal{A} \) satisfying the needed conditions, has its image in the radical. Using this, we prove that when \( \mathcal{A} \) has a right identity every generalized derivation of \( \mathcal{A} \) is spectrally bounded. We also show that a generalized derivation \((\delta, d)\) of \( \mathcal{A} \) has its image in the radical if and only if \( \delta = d \) and this holds if and only if \( \delta \) is spectrally infinitesimal.

In Section 4, we investigate Posner’s second theorem \([16]\) and show that the zero map is the only centralizing derivation of \( \mathcal{A} \). We characterize the space of all inner derivations of \( \mathcal{A} \) and prove that \( \mathcal{A} \) is commutative if and only if any derivation of \( \mathcal{A} \) is zero. We also investigate our results for \( k \)-centralizing generalized derivations and prove similar results. For a generalized derivation \((\delta, d)\) of \( \mathcal{A} \) we obtain a necessary and sufficient condition for \((\delta^2, d^2)\) to be still a generalized derivation of \( \mathcal{A} \).

In Section 5, the problem of \( k \)-skew centralizing generalized derivations of \( \mathcal{A} \) are discussed.

We shall now fix some notation. Throughout this paper, \( G \) denotes a locally compact group. Let \( B(G) \) denote the Fourier-Stieltjes algebra of \( G \) consisting of all coefficient functions arising from all the weakly continuous unitary representations of \( G \). Then \( B(G) \) can be identified with the dual of \( C^*(G) \), the group \( C^* \)-algebra of \( G \). Also, \( B(G) \) with pointwise multiplication and the dual norm is a commutative semisimple Banach algebra.

Let \( P(G) \) be the set of all continuous positive definite functions on \( G \). Let \( P_\lambda(G) \) denote the closure of \( P(G) \cap C_c(G) \) in the compact open topology, where \( C_c(G) \) is the set of all continuous functions on \( G \) with compact support, and let \( B_\lambda(G) \) denote the linear span of \( P_\lambda(G) \). Then \( B_\lambda(G) \) is a closed ideal in \( B(G) \) and \( B_\lambda(G) \) is precisely the dual of \( C^*_\lambda(G) \), where \( C^*_\lambda(G) \) is the norm closure of \( \{\lambda(f) : f \in L^1(G)\} \) in the operator algebra \( B(L^2(G)) \), and \( \lambda(f)(g) = f \ast g \) for all \( g \in L^2(G) \).
The Fourier algebra $A(G)$ is the norm-closed linear span of $P(G) \cap C_c(G)$ in $B(G)$. Then $A(G)$ is a closed ideal in $B(G)$ and $A(G) \subseteq B_\lambda(G)$; see [6, Page 208]. Furthermore, the dual of $A(G)$ is isometrically isomorphic to $VN(G)$, the group von Neumann algebra which is generated by $\lambda$ in $B(L^2(G))$. It is known from [17, Theorem 1.4.1] that $B_\lambda(G) = B(G)$ if and only if $G$ is amenable, or equivalently $A(G)$ has a bounded approximate identity. See [6] and [9] for more information on $B(G)$, $A(G)$ and $VN(G)$.

Let $\mathcal{A}$ be Banach algebra. Then $\mathcal{A}^*$ is a Banach $\mathcal{A}$-bimodule in a natural way. Let $X$ be a norm closed $\mathcal{A}$-subbimodule of $\mathcal{A}^*$. For $m \in X^*$ and $x \in X$, define $m \cdot x \in \mathcal{A}^*$ by

$$\langle m \cdot x, a \rangle = \langle m, x \cdot a \rangle \quad (a \in \mathcal{A}).$$

If $m \cdot x \in X$ for all $m \in X^*$ and $x \in X$, then $X$ is called a (left) introverted subspace of $\mathcal{A}^*$. The dual of an introverted subspace can be turned into a Banach algebra if for all $m, n \in X^*$ we define $m \cdot n \in X^*$ by $\langle m \cdot n, x \rangle = \langle m, n \cdot x \rangle$. In particular, we have the first Arens product on $\mathcal{A}^{**}$ by taking $X = \mathcal{A}^*$; see [5] for more details. If $X$ is faithful (that is, $a = 0$ whenever $\langle x, a \rangle = 0$ for all $x \in X$), then the natural map of $\mathcal{A}$ into $X^*$ is injective. The space $X^*$ can be equipped with the weak$^*$-topology $\sigma(X^*, X)$. It is well known that for each $m \in X^*$, the map $m \mapsto m \cdot n, X^* \rightarrow X^*$, is weak$^*$ continuous. In addition to $\mathcal{A}^*$, another example of introverted subspace of $\mathcal{A}^*$ is the closed linear span of $\mathcal{A}^* \cdot \mathcal{A}$ in $\mathcal{A}^*$ which is denoted by $LUC(\mathcal{A})$. If $G$ is a locally compact group, then $LUC(\mathcal{A}(G))$ coincides with $UCB(\hat{G})$, the space of uniformly continuous functionals on $A(G)$; see [7].

Note that even if $\mathcal{A}$ is commutative, $X^*$ may not be commutative in general. For example, $L^\infty(G)^*$ for $G$ abelian and infinite [3] and $VN(G)^*$ for $G$ amenable and infinite [11] are not commutative.

A linear map $T : \mathcal{A} \rightarrow \mathcal{A}$ is called a right (resp. left) multiplier of $\mathcal{A}$ if it satisfies

$$T(ab) = aT(b) \quad (\text{resp. } T(ab) = T(a)b) \quad (a, b \in \mathcal{A}).$$

For each $a \in \mathcal{A}$, let $R_a : \mathcal{A} \rightarrow \mathcal{A}$ (resp. $L_a : \mathcal{A} \rightarrow \mathcal{A}$) be the multiplication map defined by

$$R_a(b) = ba \quad (\text{resp. } L_a(b) = ab) \quad (b \in \mathcal{A}).$$

Then it is easy to see that $R_a$ (resp. $L_a$) is a right (resp. left) multiplier of $\mathcal{A}$.

2. Remark and Examples

Throughout this section, $X$ will denote an introverted subspace of $VN(G)$ satisfying (1.1). In the sequel we give some introverted subspaces of $VN(G)$ satisfying these conditions.

**Example 2.1.** Let $G$ be a locally compact group.

(i) It is known that $X = C_\lambda^*(G)$ is an introverted subspace of $VN(G)$ satisfying (1.1); see [10, Proposition 5.2].

(ii) If $G$ is discrete, then by [10, Proposition 4.5], we have $UCB(\hat{G}) = C_\lambda^*(G)$ and so $X = VN(G)$ satisfying (1.1).
Let \( \iota : C^*_\lambda(G) \to X \) be the natural embedding. Then it is easy to see that \( \pi := \iota^* : X^* \to C^*_\lambda(G)^* \) is the natural restriction map. A simple computation shows that \( \pi \) is the identity on \( A(G) \) and it is a weak*-weak* continuous algebra homomorphism when \( C^*_\lambda(G)^* \) is equipped with the induced Arens product. Moreover, we recall that the Arens product on \( C^*_\lambda(G)^* = B_\lambda(G) \) is precisely the pointwise multiplication; see [10, Proposition 5.3].

Before giving the following results, note that \( X \cdot A(G) \subseteq C^*_\lambda(G) \). Hence, for every \( n \in X^* \) and \( x \in X \), we may define the functional \( \pi(n) \cdot x \in VN(G) \) as follows

\[
\langle \pi(n) \cdot x, \varphi \rangle = \langle \pi(n), x \cdot \varphi \rangle \quad (\varphi \in A(G)).
\]

It follows that \( \pi(n) \cdot x = n \cdot x \in X \). Thus, for every \( m, n \in X^* \) we can define the functional \( m \cdot \pi(n) \in X^* \) as follows

\[
\langle m \cdot \pi(n), x \rangle = \langle m, \pi(n) \cdot x \rangle \quad (x \in X).
\]

**Lemma 2.2.** Let the map \( \pi : X^* \to B_\lambda(G) \) be the natural restriction map. Then the following statements hold.

(i) \( m \cdot n = m \cdot \pi(n) \) for all \( m, n \in X^* \).

(ii) \( \ker \pi = \text{rann}(X^*) \).

(iii) \( \text{rad}(X^*) = \text{rann}(X^*) \).

**Proof.** (i). Let \( m, n \in X^* \). Then there is a net \((\varphi_\alpha)\) in \( A(G) \) such that \( \varphi_\alpha \to m \) in the weak*-topology of \( X^* \). For each \( x \in X \), since \( x \cdot \varphi_\alpha \in C^*_\lambda(G) \) we have

\[
\langle m \cdot n, x \rangle = \lim_\alpha \langle \varphi_\alpha \cdot n, x \rangle = \lim_\alpha \langle n, x \cdot \varphi_\alpha \rangle = \lim_\alpha \langle \pi(n), x \cdot \varphi_\alpha \rangle = \lim_\alpha \langle \pi(n) \cdot x, \varphi_\alpha \rangle = \langle m, \pi(n) \cdot x \rangle = \langle m \cdot \pi(n), x \rangle.
\]

This shows that \( m \cdot n = m \cdot \pi(n) \).

(ii). Let \( n \in \ker \pi \) and \( m \in X^* \). Then there is a bounded net \((\varphi_\alpha)\) in \( A(G) \) such that \( \varphi_\alpha \to m \) in the weak*-topology of \( X^* \). Thus, for each \( x \in X \) we have

\[
\langle m \cdot n, x \rangle = \lim_\alpha \langle \pi(n), x \cdot \varphi_\alpha \rangle = 0,
\]

since \( x \cdot \varphi_\alpha \in X \cdot A(G) \subseteq C^*_\lambda(G) \) for all \( \alpha \). Therefore, \( \ker \pi \subseteq \text{rann}(X^*) \). To prove the reverse inclusion, let \( r \in \text{rann}(X^*) \). Then \( m \cdot r = 0 \) for all \( m \in X^* \). It follows that

\[
\langle m, r \cdot x \rangle = \langle m \cdot r, x \rangle = 0 \quad (m \in X^*, x \in X).
\]

Therefore, \( r \cdot x = 0 \) for all \( x \in X \). Thus, for each \( \varphi \in A(G) \) and \( x \in C^*_\lambda(G) \) we have

\[
\langle \pi(r), x \cdot \varphi \rangle = \langle r \cdot x, \varphi \rangle = 0.
\]

Since \( C^*_\lambda(G) \cdot A(G) \) is dense in \( C^*_\lambda(G) \); see [10, Proposition 4.4], we conclude that \( \pi(r) = 0 \).

(iii). Since \( \pi \) is an epimorphism from \( X^* \) onto semisimple Banach algebra \( B_\lambda(G) \), it follows from [1, Chapter 25, Proposition 10] that

\[
\text{rad}(X^*) \subseteq \ker \pi = \text{rann}(X^*).
\]
Moreover, \( \text{rann}(X^*) \) is nil and must be included in \( \text{rad}(X^*) \); see [4, Proposition 1.5.6]. Hence, \( \text{rad}(X^*) = \text{rann}(X^*) \).

As an immediate consequence of previous lemma we have the following result.

**Corollary 2.3.** The quotient Banach algebra \( X^*/\text{rad}(X^*) \) is isometrically isomorphic to \( B_\lambda(G) \).

We recall that, since \( C^*_\lambda(G) \subseteq X \), then \( X \) is faithful and the natural map of \( A(G) \) into \( X^* \) is an embedding, and we will regard \( A(G) \) as a subalgebra of \( X^* \). In fact, for each \( \varphi \in A(G) \), we have

\[
\|\varphi\|_{A(G)} \geq \|\varphi\|_{X^*} = \sup \{ |\langle \varphi, x \rangle| : x \in X, \|x\| \leq 1 \} \\
\geq \sup \{ |\langle \varphi, x \rangle| : x \in C^*_\lambda(G), \|x\| \leq 1 \} \\
= \|\varphi\|_{B_\lambda(G)} = \|\varphi\|_{A(G)};
\]

that is \( \|\varphi\|_{A(G)} = \|\varphi\|_{X^*} \).

**Corollary 2.4.** \( A(G) \) is an ideal in \( X^* \).

*Proof.* Suppose that \( \varphi \in A(G) \) and \( m \in X^* \). Then \( m \cdot \varphi = \varphi \cdot m = \varphi \cdot \pi(m) = \varphi \pi(m) \in B_\lambda(G) \), since \( A(G) \) is an ideal in \( B_\lambda(G) \) and \( \pi(m) \in B_\lambda(G) \). It follows that \( A(G) \) is an ideal in \( X^* \). □

If \( G \) is an amenable locally compact group, then \( A(G) \) has a bounded approximate identity \( (\varphi_\alpha) \) such that \( \|\varphi_\alpha\| = 1 \). It is easy to see that any weak*-cluster point \( E \) of \( (\varphi_\alpha) \) is a right identity for \( X^* \) with \( \|E\| = 1 \).

**Corollary 2.5.** The following statements hold.

(i) \( X^* \) has a right identity if and only if \( G \) is amenable.

(ii) \( X^* \) has a left identity if and only if \( X^* = B(G) \).

*Proof.* (i). Suppose that \( G \) is amenable. Then \( A(G) \) has a bounded approximate identity, say \( (\varphi_\alpha) \). It is easy to see that any weak*-cluster point \( E \) of \( (\varphi_\alpha) \) is a right identity of the Banach algebra \( X^* \). For the converse, suppose that \( E \) is a right identity of \( X^* \). Then it is easy to see that \( \pi(E) \) is the identity element of \( B_\lambda(G) \). Therefore, \( G \) is amenable.

(ii). Suppose that \( E \) is a left identity of \( X^* \). Then Lemma 2.2(ii) implies that \( \ker \pi = \{0\} \). Thus, \( \pi \) is an algebra isomorphism, that is, \( X^* = B_\lambda(G) \). Now, let \( \mu \in B_\lambda(G) \) and \( m \in X^* \) be any Hahn-Banach extension of \( \mu \). Then it follows that

\[
\mu \pi(E) = \pi(E) \mu = \pi(E) \cdot \mu = \pi(E) \cdot \pi(m) = \pi(E \cdot m) = \pi(m) = \mu.
\]

Hence, \( B_\lambda(G) \) is unital and so \( X^* = B_\lambda(G) = B(G) \). The converse, is trivial. □

An algebra \( A \) is called semiprime if for any \( a \in A \), \( aAa \) implies \( a = 0 \). It is known that every semisimple algebra is semiprime.

**Lemma 2.6.** Let \( G \) be amenable. Then the following statements are equivalent.

(i) \( X^* \) is semisimple.

(ii) \( X^* \) is semiprime.

(iii) \( X = C^*_\lambda(G) \)
(iv) $X^*$ is commutative.
(v) $X^*$ is unital.

Proof. The implications (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(v) are trivial. The implication (v)$\Rightarrow$(i) follows from Corollary 2.5(ii). It suffice to prove (ii)$\Rightarrow$(iii). Suppose that $X^*$ is semiprime. Then it is easy to see that $\ker \pi = \rann(X^*) = \{0\}$. Now, by Hahn-Banach theorem we conclude that $X = C^\lambda_\chi(G)$. \hfill $\Box$

3. SINGER-WERMER CONJECTURE FOR GENERALIZED DERIVATIONS

Throughout the rest, $\mathcal{A}$ denotes a Banach algebra with the properties that $\rad(\mathcal{A}) = \rann(\mathcal{A})$ and the algebra $\mathcal{A}/\rad(\mathcal{A})$ is commutative.

Remark 3.1. (i) Let $G$ be an infinite discrete amenable group. Then it follows from [8, Theorem 3] that $VN(G) \neq C^\lambda_\chi(G) = UCB(G)$. Since $G$ is discrete, by Example 2.1(ii), $VN(G)$ satisfies the conditions of (1.1) and so $\rad(VN(G)^*) = \rann(VN(G)^*)$ and the algebra $VN(G)^*/\rad(VN(G)^*)$ is commutative. Furthermore, the algebra $VN(G)^*$ has a right identity and it is never commutative by Lemma 2.6.

(ii) Let $G$ be an abelian non-discrete locally compact group. Then $\rad(L^\infty_0(G)^*) = \rann(L^\infty_0(G)^*)$ and the algebra $L^\infty_0(G)^*/\rad(L^\infty_0(G)^*)$ is commutative; see [12]. Furthermore, the algebra $L^\infty_0(G)^*$ has a right identity and it is never commutative.

Theorem 3.2. Let $d$ be a derivation of $\mathcal{A}$. Then $d(\mathcal{A}) \subseteq \rad(\mathcal{A})$.

Proof. Let $c \in \rad(\mathcal{A}) = \rann(\mathcal{A})$ and $a \in \mathcal{A}$. Then

$$0 = d(ac) = ad(c) + d(a)c = ad(c).$$

This shows that $d(\rad(\mathcal{A})) \subseteq \rad(\mathcal{A})$. It is easy to see that the map $\tilde{d} : \mathcal{A}/\rad(\mathcal{A}) \rightarrow \mathcal{A}/\rad(\mathcal{A})$ defined by $\tilde{d}(a) = d(a) + \rad(\mathcal{A})$ is a derivation. Moreover, by assumption $\mathcal{A}/\rad(\mathcal{A})$ is a commutative semisimple Banach algebra. Therefore, by [19] we obtain $\tilde{d} = 0$. This implies that $d(\mathcal{A}) \subseteq \rad(\mathcal{A})$. \hfill $\Box$

Before giving the following consequence of Theorem 3.2, let us recall that a linear mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ is called spectrally bounded if there exists $M \geq 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in \mathcal{A}$, where $r(a)$ denotes the spectral radius of $a \in \mathcal{A}$. In addition, if $M = 0$, $T$ is called spectrally infinitesimal.

Corollary 3.3. The following statements hold.

(i) The composition of two derivations of $\mathcal{A}$ is always a derivation of $\mathcal{A}$.
(ii) Every derivation of $\mathcal{A}$ is spectrally infinitesimal.

Proof. (i). This is an immediate consequence of Theorem 3.2.

(ii). Let $d$ be a derivation on $\mathcal{A}$. Since $d(\mathcal{A}) \subseteq \rad(\mathcal{A})$ and $\rad(\mathcal{A}) = \rann(\mathcal{A})$, it follows that $(d(a))^i = 0$ for all $i \geq 2$ and $a \in \mathcal{A}$. Therefore, by spectral radius theorem, $r(d(a)) = 0$ and hence $d$ is spectrally infinitesimal. \hfill $\Box$

Theorem 3.4. If $\mathcal{A}$ has a right identity, then every generalized derivation of $\mathcal{A}$ is spectrally bounded.
Proof. Let \((\delta, d)\) be a generalized derivation of \(A\). Since the Banach algebra \(A/\text{rad}(A)\) is commutative, it follows from [20] that the spectral radius is subadditive and submultiplicative on \(A\). Suppose that \(e \in A\) is a right identity for \(A\). Then there are positive numbers \(\alpha\) and \(\beta\) such that

\[
\begin{align*}
\gamma(\delta(a)) &= \gamma(\delta(ae)) \\
&= \gamma(a\delta(e) + d(a)) \\
&\leq \alpha(\gamma(a\delta(e)) + \gamma(d(a))) \\
&\leq \alpha(\beta\gamma(\delta(e))\gamma(a) + \gamma(d(a))),
\end{align*}
\]

for all \(a \in A\). By Corollary 3.3 we have \(\gamma(d(a)) = 0\) for all \(a \in A\). Therefore, we obtain

\[
\gamma(\delta(a)) \leq \alpha \beta \gamma(\delta(e))\gamma(a),
\]

for all \(a \in A\). That is, \(\delta\) is spectrally bounded. \(\Box\)

**Theorem 3.5.** Let \((\delta, d)\) be a generalized derivation of \(A\). If \(A\) has a right identity, then the following statements are equivalent.

(i) \(\delta(A) \subseteq \text{rad}(A)\)

(ii) \(\delta\) is spectrally infinitesimal.

(iii) \(\delta = d\).

Proof. (i)\(\Rightarrow\)(ii). Suppose that \(\delta(A) \subseteq \text{rad}(A)\). Since \(\text{rad}(A) = \text{rann}(A)\), we conclude that \(\delta\) is spectrally infinitesimal.

(ii)\(\Rightarrow\)(iii). First we note that the Banach algebra \(A/\text{rad}(A)\) is commutative. Thus, it follows from [20] that the spectral radius is submultiplicative on \(A\). Therefore, there exists \(\alpha > 0\) such that for each \(a \in A\) we have

\[
\gamma(a\delta(e)) \leq \alpha \gamma(a)\gamma(\delta(e)) = 0,
\]

where \(e \in A\) is a right identity for \(A\). Hence, by [1, Proposition 1(ii)], we have \(\delta(e) \in \text{rad}(A) = \text{rann}(A)\). Thus

\[
\delta(a) = \delta(ae) = a\delta(e) + d(a)e = d(a),
\]

for all \(a \in A\). This shows that \(\delta = d\). The implication (iii)\(\Rightarrow\)(i) follows from Theorem 3.2. \(\Box\)

Let \(T\) be a right multiplier of \(A\) and let 0 denotes the zero derivation of \(A\). Then it is easy to see that \((T, 0)\) is a generalized derivation of \(A\). Thus, we have the following result as an immediate consequence of Theorem 3.5.

**Corollary 3.6.** Let \(T\) be a right multiplier of \(A\). If \(A\) has a right identity, then the following statements are equivalent.

(i) \(T(A) \subseteq \text{rad}(A)\).

(ii) \(T\) is spectrally infinitesimal.

(iii) \(T\) is a derivation on \(A\).

(iv) \(T = 0\).
4. $k$-Centralizing Generalized Derivations

In this section, we investigate Posner's second theorem and show that the zero map is the only centralizing derivation of $\mathcal{A}$. We also investigate our results for centralizing generalized derivations and prove similar results.

We recall that the algebraic center of $\mathcal{A}$ is denoted by $Z(\mathcal{A})$. Let $a, b \in \mathcal{A}$ and $k$ be a fixed positive integer. Set
\[
[a, b]^k := [[a, b]_{k-1}, b],
\]
where
\[
[a, b]^0 := a, \quad [a, b] := ab - ba.
\]
A mapping $T : \mathcal{A} \to \mathcal{A}$ is called $k$-centralizing (resp. $k$-commuting) if $[T(a), a^k] \in Z(\mathcal{A})$ (resp. $[T(a), a^k] = 0$) for all $a \in \mathcal{A}$. In the case $k = 1$, $T$ is said to be centralizing (resp. commuting).

**Theorem 4.1.** Let $d$ be a derivation of $\mathcal{A}$ and let $k$ be a positive integer. If $\mathcal{A}$ has a right identity, then the following statements are equivalent.

(i) $d = 0$.

(ii) $d$ is $k$-centralizing.

(iii) $d$ is $k$-commuting.

**Proof.** The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. By assumption and Theorem 3.2, we obtain
\[
[d(a), a^k] = d(a)a^k = d(a^{k+1}) \in \text{rann}(\mathcal{A}) \cap Z(\mathcal{A}).
\]
Thus, $[d(a), a^k] = 0$.

(iii) $\Rightarrow$ (i). Let $d$ be $k$-commuting and let $e$ be a right identity of $\mathcal{A}$. Then
\[
d(e) = [d(e), e] = [d(e), e^k] = 0. \quad (4.1)
\]
Given $c \in \text{rann}(\mathcal{A})$, we obtain $(c + e) = (c + e)^k$. Thus,
\[
d(c) = [d(c + e), (c + e)] = [d(c + e), (c + e)^k] = 0. \quad (4.2)
\]
From (4.1) and (4.2), we get
\[
d(a) = d(ea) + d(a - ea) = d(e)a + d(a - ea) = 0,
\]
for all $a \in \mathcal{A}$. \hfill \Box

**Corollary 4.2.** Let $d$ be a derivation of $\mathcal{A}$. If $\mathcal{A}$ has a right identity, then the following statements are equivalent.

(i) $d = 0$.

(ii) $d$ is centralizing.

(iii) For every $k \in \mathbb{N}$, $d$ is $k$-centralizing.

(iv) There exists $k \in \mathbb{N}$ such that $d$ is $k$-centralizing.

(v) For every $k \in \mathbb{N}$, $d$ is $k$-commuting.

(vi) There exists $k \in \mathbb{N}$ such that $d$ is $k$-commuting.

**Theorem 4.3.** If $\mathcal{A}$ has a right identity, then the following statements are equivalents.

(i) $\mathcal{A}$ has an identity.
(ii) Any derivation of \( \mathcal{A} \) is zero.
(iii) Any inner derivation of \( \mathcal{A} \) is zero.
(iv) \( \mathcal{A} \) is commutative.

**Proof.** The implications (iii) \( \Rightarrow \) (iv) and (v) \( \Rightarrow \) (i) are obvious.

(i) \( \Rightarrow \) (ii). Let \( e \) be an identity for \( \mathcal{A} \) and let \( d \) be a derivation of \( \mathcal{A} \). Then by Theorem 3.2, we have \( d(a) = ed(a) = 0 \) for all \( a \in \mathcal{A} \).

(iv) \( \Rightarrow \) (v). Suppose that any inner derivation of \( \mathcal{A} \) is zero. Then \( d_a(b) = ba - ab = 0 \) for all \( a, b \in \mathcal{A} \). This implies that \( \mathcal{A} \) is commutative. \( \square \)

**Lemma 4.4.** Let \( a, b, c \in \mathcal{A} \). Then \( [a, b] \in \text{rann}(\mathcal{A}) \) and consequently \( cab = cba \).

**Proof.** Consider the inner derivation \( d_b \) of \( \mathcal{A} \). Then by Theorem 3.2, \( [a, b] = d_b(a) \in \text{rann}(\mathcal{A}) \). Thus, \( cab - cba = cd_b(a) = 0 \). It follows that \( cab = cba \). \( \square \)

**Theorem 4.5.** Let \( d \) be a derivation of \( \mathcal{A} \). If \( \mathcal{A} \) has a right identity, then the following statements are equivalent.

(i) \( d \) is an inner derivation.
(ii) There exists \( b_0 \in \mathcal{A} \) such that for each \( k \in \mathbb{N} \), the mapping \( a \mapsto d(a) + b_0a \) is \( k \)-commuting.
(iii) There exist \( b_0 \in \mathcal{A} \) and \( k \in \mathbb{N} \) such that the mapping \( a \mapsto d(a) + b_0a \) is \( k \)-commuting.
(iv) There exist \( b_0 \in \mathcal{A} \) and \( k \in \mathbb{N} \) such that the mapping \( a \mapsto d(b) + b_0a \) is \( k \)-centralizing.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( d \) is inner. Then there exists \( b_0 \in \mathcal{A} \) such that \( d(a) = ab_0 - b_0a \), for all \( a \in \mathcal{A} \). By Lemma 4.4, for each \( k \in \mathbb{N} \), we have

\[
[d(a) + b_0a, a^k] = [ab_0, a^k] = ab_0a^k - a^kab_0 = aa^kb_0 - a^kab_0 = a^{k+1}b_0 - a^{k+1}b_0 = 0,
\]

which implies (ii).

The implications (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are trivial.

Suppose now that (iv) holds. Consider the map \( D : \mathcal{A} \to \mathcal{A} \) defined by \( D(a) := d(a) - [a, b_0] \). It is easy to see that \( D \) is a derivation of \( \mathcal{A} \). Thus

\[
[D(a), a^k] = D(a)a^k = [d(a) + b_0a, a^k] \in Z(\mathcal{A}).
\]

We now invoke Theorem 4.1 to conclude that \( D = 0 \), which implies that \( d \) is inner. \( \square \)

We denote by InnD(\( \mathcal{A} \)) the space of all inner derivations of \( \mathcal{A} \).

**Theorem 4.6.** Suppose that \( \mathcal{A} \) has a right identity. Then InnD(\( \mathcal{A} \)) is continuously linearly isomorphic to \( \mathcal{A}/Z(\mathcal{A}) \).

**Proof.** Consider the mapping \( \Gamma \) from \( \mathcal{A}/Z(\mathcal{A}) \) into InnD(\( \mathcal{A} \)) defined by \( \Gamma(a + Z(\mathcal{A})) = d_a \). It is clear that \( \Gamma \) is onto. To check the injectivity, suppose that \( \Gamma(a + Z(\mathcal{A})) = 0 \) for some \( a \in \mathcal{A} \). Then \( d_a(b) = ba - ab = 0 \), for all \( b \in \mathcal{A} \), which
implies that \( a \in Z(\mathcal{A}) \). Thus, \( \Gamma \) is injective. Now, let \( a, b \in \mathcal{A} \) and \( z \in Z(\mathcal{A}) \). Then
\[
\|d_a(b)\| = \|ba - ab\|
\leq \|ba - zb\| + \|zb - ab\|
= \|ba - bz\| + \|zb - ab\|
\leq \|b\|\|a - z\| + \|z - a\|\|b\|
= 2\|b\|\|a - z\|.
\]
This implies that
\[
\|\Gamma(a + Z(\mathcal{A}))\| = \|d_a\| \leq 2\|a - z\| \quad (a \in \mathcal{A}, z \in Z(\mathcal{A})).
\]
Therefore,
\[
\|\Gamma(a + Z(\mathcal{A}))\| \leq 2\|a + Z(\mathcal{A})\|.
\]
This shows that the map is continuous.

**Theorem 4.7.** Let \((\delta, d)\) be a generalized derivation of \(\mathcal{A}\) and \(k \in \mathbb{N}\). If \(\mathcal{A}\) has a right identity, then the following statements are equivalent.

(i) \(\delta\) is \(k\)-commuting.

(ii) \(\delta\) is \(k\)-centralizing.

(iii) \(\delta\) is a right multiplier.

(iv) There exists \(b \in \mathcal{A}\) such that \(\delta = R_b\).

**Proof.** (i)\(\Rightarrow\)(ii). This is trivial.

(ii) \(\Rightarrow\) (i). By induction and paying attention to the fact that \([a, b] \in \text{rann}(\mathcal{A}) \) for all \(a, b \in \mathcal{A}\), we have that
\[
[[\delta(a), a]_{i-1}, a] = [\delta(a), a^i] \quad (a \in \mathcal{A}, i \in \mathbb{N}). \tag{4.3}
\]
Hence,
\[
[[\delta(a), a]_{k-1}, a] = [\delta(a), a^k] \in Z(\mathcal{A}) \quad (a \in \mathcal{A}).
\]
This implies that the mapping \(a \mapsto [\delta(a), a]_{k-1}\) is centralizing. Using (4.3), we obtain
\[
[\delta(a), a^k] = [[\delta(a), a]_{k-1}, a] \in Z(\mathcal{A}) \cap \text{rann}(\mathcal{A}) \quad (a \in \mathcal{A}).
\]
This implies that \([\delta(a), a^k] = 0\) for all \(a \in \mathcal{A}\). Therefore, \(\delta\) is \(k\)-commuting.

(i)\(\Rightarrow\)(iii). Let \(e\) be a right identity of \(\mathcal{A}\) and \(c \in \text{rann}(\mathcal{A})\). Since \((e + c)^i = e + c\) for all \(i \in \mathbb{N}\), it follows that
\[
\delta(e + c) = \delta(e + c)(e + c)^k = (e + c)^k\delta(e + c)
= \delta((e + c)^{k+1}) - d((e + c)^k)(e + c)
= \delta(e + c) - d(e + c).
\]
This shows that \(d(c) = -d(e)\). Furthermore,
\[
\delta(e) = e\delta(e) + d(e) = \delta(e)e + d(e) = \delta(e) + d(e).
\]
Thus \(d(e) = 0\). Hence \(d(c) = 0\) for all \(a \in \mathcal{A}\). Thus
\[
d(a) = d(ea) = d(e)a + ed(a) = 0 \quad (a \in \mathcal{A}).
\]
This implies that
\[ \delta(ab) = a\delta(b) + d(a)b = a\delta(b) \quad (a, b \in A); \]
that is \( \delta \) is a right multiplier.

(iii)\(\Rightarrow\)(iv). Let \( \delta \) be a right multiplier and let \( e \) be a right identity of \( A \). Then it is straightforward to show that \( \delta = R_{\delta(e)} \).

(iv)\(\Rightarrow\)(i). Using Lemma 4.4, we can see that
\[
[\delta(a), a^k] = \delta(a)a^k - a^k\delta(a) \\
= aba^k - a^kab \\
= a^{k+1}b - a^{k+1}b = 0 \quad (a \in A).
\]

Thus, \( \delta \) is \( k \)-commuting.

As an immediate consequence of Theorem 4.7, we have the following result.

**Corollary 4.8.** Let \( (\delta, d) \) be a generalized derivation of \( A \). If \( A \) has a right identity, then the following statements are equivalent.

1. \( \delta \) is centralizing.
2. For every \( k \in \mathbb{N} \), \( \delta \) is \( k \)-centralizing.
3. There exists \( k \in \mathbb{N} \) such that \( \delta \) is \( k \)-centralizing.
4. For every \( k \in \mathbb{N} \), \( \delta \) is \( k \)-commuting.
5. There exists \( k \in \mathbb{N} \) such that \( \delta \) is \( k \)-commuting.

By \( \mathcal{C}(A) \) we denote the set of all centralizing generalized derivations of \( A \). By Theorem 4.7, we conclude that the product of two elements of \( \mathcal{C}(A) \) is always a centralizing generalized derivation of \( A \). Hence the space \( \mathcal{C}(A) \) is a Banach algebra as a subalgebra of \( B(A) \), the space of all bounded linear maps on \( A \).

**Theorem 4.9.** Suppose that \( A \) has a right identity. Then the Banach algebra \( \mathcal{C}(A) \) is isomorphic to \( A/\text{rad}(A) \).

**Proof.** Let \( e \) be right identity of \( A \). One can introduce a bounded linear map \( \Gamma : \mathcal{C}(A) \rightarrow A/\text{rad}(A) \) by \( \Gamma(\delta) := \pi(\delta(e)) \), where \( \pi : A \rightarrow A/\text{rad}(A) \) is the natural homomorphism. Using Theorem 4.7(iv) and Lemma 4.4, we conclude that \( \delta_1 \circ \delta_2 = \delta_2 \circ \delta_1 \) for all \( \delta_1, \delta_2 \in \mathcal{C}(A) \). Hence, \( \mathcal{C}(A) \) is a commutative Banach algebra. Therefore, for every \( \delta_1, \delta_2 \in \mathcal{C}(A) \), we have
\[
\Gamma(\delta_1 \circ \delta_2) = \pi(\delta_2(\delta_1(e)e)) = \pi(\delta_1(e)\delta_2(e)) \\
= \pi(\delta_1(e))\pi(\delta_2(e)) = \Gamma(\delta_1)\Gamma(\delta_2).
\]

Hence, \( \Gamma \) is a homomorphism. Now we show that \( \Gamma \) is onto. Given \( \pi(b) = b + \text{rad}(A) \in A/\text{rad}(A) \), define \( \delta \in \mathcal{C}(A) \) by \( \delta(a) = ab \). Then
\[
\Gamma(\delta) = \pi(\delta(e)) = \pi(eb) \\
= \pi(e)\pi(b) = \pi(b)\pi(e) \\
= \pi(be) = \pi(b).
\]

This proves theorem. \( \square \)

The maps \( T \) and \( S \) from \( A \) into \( A \) are called orthogonal, denoted by \( T \perp S \), if \( T(a)cS(b) = S(b)cT(a) = 0 \) for all \( a, b, c \in A \).
Theorem 4.10. Let \((\delta, d)\) be a generalized derivation of \(\mathcal{A}\). If \(\mathcal{A}\) has a right identity, then the following statements are equivalent.

(i) \([\delta(a), a] = 0\) for all \(a \in \mathcal{A}\).

(ii) \(d \perp \delta\).

(iii) \((\delta^2, d^2)\) is a generalized derivation of \(\mathcal{A}\).

Proof. (i) \(\Rightarrow\) (ii). Given \(a \in \mathcal{A}\), by assumption and Lemma 4.4, we have

\[ [\delta(a), a] = 0. \]

Moreover, it is easy to see that \(\delta(a) = a\delta(e) + d(a)\), where \(e\) is a right identity for \(\mathcal{A}\). Now, Theorem 3.2 and Lemma 4.4 imply that

\[ [\delta(a), a] = d(a)a^2\delta(e). \]  

(4.4)

Thus,

\[ d(a)a^2\delta(e) = 0. \]  

(4.5)

Replacing \(a\) by \(e\) in (4.5), we get

\[ d(e)\delta(e) = 0. \]  

(4.6)

Replacing \(a\) by \(a + e\) in (4.5) and using (4.6), we obtain

\[ d(a)\delta(e) + 2d(a)a\delta(e) = 0. \]  

(4.7)

Putting \(-a\) for \(a\) in the relation above we have

\[ -d(a)\delta(e) + 2d(a)a\delta(e) = 0. \]  

(4.8)

It follows from (4.7) and (4.8) that \(d(a)\delta(e) = 0\). Therefore,

\[ d(a)\delta(b) = d(a)b\delta(e) = d(a)\delta(e)b = 0, \]

for all \(a, b \in \mathcal{A}\). This implies that

\[ d(a)c\delta(b) = d(a)\delta(b)c = 0, \]  

(4.9)

for all \(a, b, c \in \mathcal{A}\). Since \(d(\mathcal{A}) \subseteq \text{rann}(\mathcal{A})\), it follows that

\[ \delta(b)cd(a) = 0, \]  

(4.10)

for all \(a, b, c \in \mathcal{A}\). Thus, by (4.9) and (4.10) the statement (ii) holds.

(ii) \(\Rightarrow\) (iii). Suppose that \(d\) and \(\delta\) are orthogonal. Then for every \(a, b \in \mathcal{A}\), we have

\[
\delta^2(ab) = \delta(\delta(ab)) = \delta(a\delta(b) + d(a)b) \\
= a\delta^2(b) + 2d(a)\delta(b) + d^2(a)b \\
= a\delta^2(b) + d^2(a)b.
\]

This shows that \((\delta^2, d^2)\) is a generalized derivation.

(iii) \(\Rightarrow\) (i). Assume that \((\delta^2, d^2)\) is a generalized derivation of \(\mathcal{A}\). Then

\[ \delta^2(ab) = a\delta^2(b) + d^2(a)b, \]  

(4.11)
for all \(a, b \in \mathcal{A}\). Moreover, 
\[
\delta^2(ab) = \delta(\delta(ab)) = \delta(a\delta(b) + d(a)b) = a\delta^2(b) + d(a)\delta(b) + d(a)\delta(b) + d^2(a)b = a\delta^2(b) + 2d(a)\delta(b) + d^2(a)b,
\]
for all \(a, b \in \mathcal{A}\). Comparing (4.11) and (4.12), we get that
\[
d(a)\delta(b) = 0,
\]
for all \(a, b \in \mathcal{A}\). Using (4.4) and (4.13), we obtain 
\[
[[\delta(a), a], \delta(a)] = d(a)a^2\delta(e) = d(a)\delta(e)a^2 = 0,
\]
for all \(a \in \mathcal{A}\) and therefore (i) holds. 

Obviously the product of two right multipliers is a right multiplier, and the product of two derivations of \(\mathcal{A}\) is a derivation. The next example shows that this fact does not hold for all generalized derivations of \(\mathcal{A}\).

**Example 4.11.** Let \(G = \mathbb{Z}\). Then \(G\) is an infinite discrete amenable group. Thus, \(VN(G)^\ast\) is a non-commutative Banach algebra with a right identity, say \(e\). It is easy to see that \((\delta, d)\) defined as \(d = d_e\) and \(\delta = R_e + d_e\) is a generalized derivation of \(VN(G)^\ast\). Now, by Lemma 2.6, there exists \(c \in \text{rad}(VN(G)^\ast) = \text{rann}(VN(G)^\ast)\) such that \(c \neq 0\). It follows that 
\[
\delta^2(c) = 4c, \quad c\delta^2(e) + d^2(c) = 2c.
\]
But \((\delta^2, d^2)\) fails to be a generalized derivation. Indeed, if \((\delta^2, d^2)\) is a generalized derivation, then by (4.14) we deduce that \(c = 0\) which is a contradiction.

5. **k-Skew Centralizing Generalized Derivations**

Let \(a, b \in \mathcal{A}\) and \(k\) be a fixed positive integer. Set 
\[
\langle a, b \rangle_k = \langle\langle a, b \rangle_{k-1}, b \rangle,
\]
where 
\[
\langle a, b \rangle_0 = a, \quad \langle a, b \rangle = ab + ba.
\]
A mapping \(T : \mathcal{A} \rightarrow \mathcal{A}\) is called \(k\)-skew centralizing (resp. \(k\)-skew commuting) if it satisfies 
\[
\langle T(a), a^k \rangle \in Z(\mathcal{A}) \quad (\text{resp.} \langle T(a), a^k \rangle = 0) \quad (a \in \mathcal{A}).
\]
In the case \(k = 1\), \(T\) is called skew centralizing (resp. skew commuting).

**Theorem 5.1.** Let \((\delta, d)\) be a generalized derivation of \(\mathcal{A}\) and \(k \in \mathbb{N}\). If \(\mathcal{A}\) has a right identity, then the following statements hold.

(i) If \(\delta\) is \(k\)-skew centralizing, then there exists \(z \in Z(\mathcal{A})\) such that \(\delta = R_z\).
(ii) If \(\delta\) is \(k\)-skew commuting, then \(\delta = 0\).
Proof. (i). By induction, we obtain
\[
[\langle \delta(a), a^i \rangle, a] = [\delta(a), a^{i+1}],
\]  
for all \( a \in \mathcal{A} \) and \( i \in \mathbb{N} \). Now suppose that \( \delta \) is \( k \)-skew centralizing. Then \( \langle \delta(a), a^k \rangle \in Z(\mathcal{A}) \) for all \( a \in \mathcal{A} \). Using (5.1), we have
\[
0 = \langle \langle \delta(a), a^k \rangle, a \rangle = \langle \delta(a), a^{k+1} \rangle,
\]  
for all \( a \in \mathcal{A} \). This shows that \( \delta \) is \((k + 1)\)-commuting. It follows from Theorem 4.7, that there exists \( b \in \mathcal{A} \) such that \( \delta = R_b \). Let \( e \) be a right identity of \( \mathcal{A} \). Then
\[
2eb = \langle \delta(e), e^k \rangle \in Z(\mathcal{A}).
\]  
This implies that \( \delta(a) = az = 0 \) for all \( a \in \mathcal{A} \). □

Lemma 5.2. : \( Z(\mathcal{A}) \) is a closed ideal of \( \mathcal{A} \).

Proof. It is easy to check that \( Z(\mathcal{A}) \) is a closed subspace of \( \mathcal{A} \). To complete the proof, we show that \( Z(\mathcal{A}) \) is an ideal in \( \mathcal{A} \). By lemma 4.4, we obtain
\[
(zb)a = zab = a(zb),
\]  
for all \( z \in Z(\mathcal{A}) \) and \( a, b \in \mathcal{A} \). Similarly,
\[
(bz)a = zba = zab = azb = a(bz),
\]  
for all \( z \in Z(\mathcal{A}) \) and \( a, b \in \mathcal{A} \), which implies that \( Z(\mathcal{A}) \) is an ideal in \( \mathcal{A} \). □

Corollary 5.3. Let \( \langle \delta, d \rangle \) be a generalized derivation of \( \mathcal{A} \). If \( \mathcal{A} \) has a right identity, then the following statements hold.

(i) \( \delta \) is skew centralizing.

(ii) For every \( k \in \mathbb{N} \), \( \delta \) is \( k \)-skew centralizing.

(iii) There exists \( k \in \mathbb{N} \) such that \( \delta \) is \( k \)-skew centralizing.

Proof. (i) ⇒ (ii) By Theorem 5.1, there exists \( z \in Z(\mathcal{A}) \) such that \( \delta(a) = az \) for all \( a \in \mathcal{A} \). Hence by Lemma 5.2, we have
\[
\langle \delta(a), a^k \rangle = \langle az, a^k \rangle = aza^k + a^ka = az \in Z(\mathcal{A}),
\]  
for all \( a \in \mathcal{A}, k \in \mathbb{N} \).

(ii) ⇒ (iii). This is obvious.

(iii) ⇒ (i). By Theorem 5.1, there exists \( z \in Z(\mathcal{A}) \) such that \( \delta(a) = az \) for all \( a \in \mathcal{A} \). Therefore by Lemma 5.2, we have
\[
\langle \delta(a), a \rangle = \langle az, a \rangle = aza + aza \in Z(\mathcal{A}),
\]  
for all \( a \in \mathcal{A} \). □
By $SC(A)$ we denote the set of all skew centralizing generalized derivations on $A$. By Theorem 5.1 and Lemma 5.2 we conclude that the space $SC(A)$ is a closed ideal of $C(A)$, the space of all centralizing generalized derivations of $A$.

**Corollary 5.4.** Suppose that $A$ has a right identity. Then the Banach algebra $SC(A)$ is isomorphic to $Z(A)$.

**Proof.** We show that the mapping $\Gamma : SC(A) \rightarrow Z(A)$ defined by $\Gamma(\delta) := z$, where $\delta = R_z$ for some $z \in Z(A)$, is an isomorphism. To prove this, let $\delta_1, \delta_2 \in SC(A)$ such that $\delta_1 = \delta_2$. Then by Theorem 5.1 there exist $z_1, z_2 \in Z(A)$ such that $\delta_1 = R_{z_1}$ and $\delta_2 = R_{z_2}$. Now, suppose that $e$ is a right identity of $A$. Then $z_1 = \delta_1(e) = \delta_2(e) = z_2$, whence $\Gamma$ is well-defined. One can easily check that $\Gamma$ is an isomorphism. $\square$

Before giving the following result, recall that $(L_b, d_{-b})$ is a generalized derivation of $A$ for all $b \in A$.

**Theorem 5.5.** Let $z \in A$ and $A$ has a right identity. Then the following statements are equivalent.

(i) $L_z$ is centralizing.
(ii) $L_z$ is skew centralizing.
(iii) $z \in Z(A)$.

**Proof.** (i) $\Rightarrow$ (iii). By assumption and Theorem 4.7 we obtain that $L_z$ is a right multiplier. Thus $za = L_z(a) = L_z(ae) = aL_z(e) = a(ze) = az,$ for all $a \in A$. This implies $z \in Z(A)$.

(ii) $\Rightarrow$ (iii). Suppose that $L_z$ is skew centralizing. Then by Theorem 5.1, there exists $z' \in Z(A)$ such that $L_z(a) = az'$ for all $a \in A$. Thus $z = ze = L_z(e) = ez' = z'e = z' \in Z(A)$.

(iii) $\Rightarrow$ (i). Let $z \in Z(A)$. Then $[L_z(a), a] = [za, a] = za - az = 0 \in Z(A)$, for all $a \in A$.

(iii) $\Rightarrow$ (ii). Let $z \in Z(A)$. By Lemma 5.2, we have $za^2 \in Z(A)$ for all $a \in A$. Thus $\langle L_z(a), a \rangle = 2za^2 \in Z(A)$, for all $a \in A$. $\square$

**Theorem 5.6.** Let $T$ be a left multiplier of $A$. Then $T(A) \subseteq \text{rad}(A)$ if and only if $T$ is spectrally infinitesimal.

**Proof.** Suppose that $T(A) \subseteq \text{rad}(A)$. By [1, Proposition 1(i)], we have $r(T(a)) = 0$ for all $a \in A$. This shows that $T$ is spectrally infinitesimal. Conversely, assume that $T$ is spectrally infinitesimal. Then $r(T(ab)) = 0$. 


for all $a, b \in \mathcal{A}$. Hence, [1, Proposition 1(ii)] implies that $T(a) \in \text{rad}({\mathcal{A}})$ for all $a \in \mathcal{A}$. □

By Theorem 5.5 we deduce that, the left multiplier $L_z$ is both centralizing and skew centralizing on $\mathcal{A}$ for all $z \in \mathcal{Z}(\mathcal{A})$. But this assertion is not true for all $z \in \mathcal{A}$ as shown in the following example.

**Example 5.7.** Let $\mathcal{A} = VN(\mathbb{Z})^*$. Then there exists an element $z \in \mathcal{A}$ such that $L_z$ is neither centralizing nor skew centralizing. In fact, by Lemma 2.6 there exists $z \in \text{ran}(\mathcal{A})$ such that $z \neq 0$. If $L_z$ is either centralizing or skew centralizing, then $z \in \mathcal{Z}(\mathcal{A})$, by Theorem 5.5. Now let $e$ be a right identity of $\mathcal{A}$. Then $z = ze = ez = 0$. This contradiction will show that $L_z$ is neither centralizing nor skew centralizing.

**References**

1. F. F. Bonsall and J. Duncan, Complete Normed Algebras, Springer, New York, 1973.
2. M. Bresar and M. Mathieu, Derivations mapping into the radical III, *J. Funct. Anal.* 133 (1995), 21-29.
3. P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pac. J. Math.* 11 (1961), 847-870.
4. H. G. Dales, Banach Algebras and Automatic Continuity, Oxford University Press, New York (2000).
5. H. G. Dales and A. T.-M. Lau, The second duals of Beurling algebras, Memoirs American Math. Soc. 177, 2005, no. 836, vi+191, DOI 10.1090/memo/0836.
6. P. Eymard, L’algèbre de Fourier d’un groupe localement compact, *Bull. Soc. Math. France* 92 (1964), 181-236.
7. E. E. Granirer, Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group, *Trans. Amer. Math. Soc.* 189 (1974), 371-382.
8. E. E. Granirer, On group representations whose C*-algebra is an ideal in its von Neumann algebra, *Ann. Inst. Fourier (Grenoble)*, 29 (1979), 37-52.
9. K. Kaniuth, and A. T.-M. Lau, Fourier and Fourier-Stieltjes Algebras on locally compact Groups, Math Surveys and Monographs, vol 231 American Math. Society, 2018.
10. A. T.-M. Lau, Uniformly continuous functionals on the Fourier algebra of any locally compact group, *Trans. Amer. Math. Soc.* 251 (1979), 39-59.
11. A. T.-M. Lau, The second conjugate algebra of the Fourier algebra of a locally compact group, *Trans. Amer. Math. Soc.* 267 (1981), 53-63.
12. S. Maghsoudi and R. Nasr-Isfahani, On the maximal and minimal left ideals of certain Banach algebras on locally compact groups, *Results Math.* 62 (2012), 157-165.
13. M. Mathieu, and G. J. Murphy, Derivations mapping into the radical, *Arch. Math.* 57 (1991), 469-474.
14. M. Mathieu, and V. Runde, Derivations mapping into the radical II, *Bull. Lond. Math. Soc.* 24 (1992), 485-487.
15. M. J. Mehdipour, and Z. Saeedi, Derivations on group algebras of a locally compact abelian group, *Monatsh. Math.* 180 (2016), 595-605.
16. E. C. Posner, Derivations in prime rings, *Proc. Am. Math. Soc.* 8 (1957), 1093-1100.
17. V. Runde, *Amenable Banach Algebras*, Springer Monographs in Mathematics, 2010.
18. I. M. Singer, and J. Wermer, Derivations on commutative normed algebras, *Math. Ann.* 129 (1955), 260-264.
19. M. Thomas, The image of a derivation is contained in the radical, *Ann. Math.* 128 (1988), 435-460.
20. J. Zemanek, Spectral radius characterizations of commutativity in Banach algebras, *Stud. Math.* 61 (1977), 257–268.
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran;
Email address: ali.ebrahimzadeh@math.iut.ac.ir

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran;

and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box: 19395–5746, Tehran, Iran.
Email address: m.nemati@iut.ac.ir