CAR FLOWS ON TYPE III FACTORS AND ITS EXTENDABILITY

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Abstract. In this paper using one of the necessary conditions obtained for extendability in [BISSar], we prove that the CAR flows ([Amo01]) on type III factors arising from most quasi-free states are not extendable. As a consequence we find the super product system of CAR flows. We know from [Arv03] that CCR flows and CAR flows on type I factors with the same Arveson index are cocycle conjugate. But our result together with [BISSar] will show that CCR flows and CAR flows on type III factors are not cocycle conjugate.

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1. Introduction

A weak-* continuous semigroup of of unital $*$-endomorphisms on a von Neumann algebra is called an $E_0$-semigroup. $E_0$-semigroups on type I factors have received much attention (see the monograph [Arv03] for an extensive reference). The study of $E_0$-semigroups on type II$_1$ factors was initiated by Powers in 1998 (see [Pow88]). There was little progress on $E_0$-semigroups on type II$_1$ factors until the results independently obtained recently in [Ale04] and [MS12]).

On the other hand, $E_0$-semigroups on type III factors had not received too much attention.

In 2001, G.G. Amosov, A.V Bulinski and Shirkov initiated a study of $E_0$-semigroups on arbitrary factors (see [ABS01]). They were interested in a special kind of $E_0$-semigroups which they called “regular $E_0$-semigroups”, which are those that can be extended to type I factors in a canonical way. Then in 2013, in [BISSar], we studied a certain class of endomorphisms and $E_0$-semigroups on arbitrary factors which we call extendable. Unfortunately [ABS01] has some errors: for example they claim in section 5 of their paper, that CAR flows arising from quasi-free state of $\frac{1}{2}$ are regular semigroups. We proved in [BISSar] that these CAR flows are not regular semigroups (although we refer to them as “extendable” $E_0$-semigroups).

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Section 2 is the preliminary section. Where we will briefly discuss the meaning of extendability of an \( E_0 \)-semigroup. We also mention the definition of supper product system and then we write some results regarding the supper product system.

In section 3 we study CAR flows on type \( III \) factors and prove that they are not extendable. Also we point out an error in [ABS01] and we find out the supper product system of CAR flows.

In section 4 we study the relations between CCR and CAR flows on type \( III \) factors and then we distinguish them.

2. Preliminaries

2.1. Extendable \( E_0 \)-semigroup. We begin by recalling some facts from [BISSar] that will be used often in the sequel. Assume that \( \phi \) is a faithful normal state on a factor \( M \). Let \( \lambda_M \) be the left regular representation of \( M \) onto \( B(L^2(M, \phi)) \). Identify \( x \in M \) with \( \lambda_M(x) \). For the modular conjugation operator, we simply write \( \mathcal{J} \) for \( (\mathcal{J}_\phi) \). Thanks to the Tomita-Takesaki theorem, we know that

\[
\mathcal{J} = \mathcal{J}^J, \quad \mathcal{J} \text{ is a } *\text{-preserving conjugate-linear isomorphism of } B(L^2(M, \phi)) \text{ onto itself, which maps } M \text{ and } M' \text{ onto one another, and that}
\]

- \( \tilde{\Gamma}_M \) is a cyclic and separating vector for \( M' \).

We assume that \( \theta \) is a normal unital \(*\)-endomorphism, which preserves \( \phi \). The invariance assumption \( \phi \circ \theta = \phi \) implies that there exists a unique isometry \( u_\theta \) on \( L^2(M, \phi) \) such that \( u_\theta x \tilde{\Gamma}_M = \theta(x) \tilde{\Gamma}_M \), which in turn implies that \( u_\theta x = \theta(x)u_\theta \forall x \in M \). Recall the following definition from [BISSar].

**Definition 2.1.** If \( M, \phi, \theta \) are as above, and if the associated isometry \( u_\theta \) of \( L^2(M, \phi) \) commutes with the modular conjugation operator \( \mathcal{J} = \mathcal{J}_\phi \), then \( \theta \) is called an equi-modular endomorphism of the factorial non-commutative probability space \((M, \phi)\).

Suppose \( \theta \) is an equi-modular endomorphism of a factorial non-commutative probability space \((M, \phi)\). Then the equation \( \theta'(x') = \mathcal{J} \theta(\mathcal{J} x' \mathcal{J}) \mathcal{J} \) defines a unital normal \(*\)-endomorphism of \( M' \), which preserves the state given by \( \phi'(x') = \phi(\mathcal{J} x' \mathcal{J}) \); and we have the identifications \( L^2(M', \theta') = L^2(M, \phi) \), \( \tilde{\Gamma}_M = \tilde{\Gamma}_{M'} \), and \( u_{\theta'} = u_\theta \) (for details see [BISSar]). For convenience of reference, we include this definition from [BISSar].

**Definition 2.2.** Let \( \theta \) be an equi-modular endomorphism of a factorial non-commutative probability space \((M, \phi)\) in standard form (i.e., viewed as embedded in \( B(L^2(M, \phi)) \) as above). Then \( \theta \) is called extendable if there exists a unital normal \(*\)-endomorphism \( \theta^{(2)}(x) = \theta(x) \) and \( \theta^{(2)}(j(x)) = j(\theta(x)) \) for all \( x \in M \).

**Definition 2.3.** \( \{\alpha_t : t \geq 0\} \) is said to be an \( E_0 \)-semigroup on a von Neumann probability space \((M, \phi)\), if

1. \( \alpha_t \) is a \( \phi \)-preserving normal unital \(*\)-homomorphism of \( M \) for each \( t \geq 0 \);
2. \( \alpha_t \circ \alpha_s = \alpha_{t+s} \forall s, t' \geq 0 \);
3. \( \alpha_0 = id_M \); and
4. \( \rho(\alpha_t(x)) \) is continuous for each \( x \in M, \rho \in M_* \).

It is called extendable \( E_0 \)-semigroup if for all \( t \geq 0 \), \( \alpha_t \) is extendable.

2.2. Super product system. The notion of super product system is already introduced in [MS12]. It is a generalization of the product systems introduced by Arveson, and may help to analyse \( E_0 \)-semigroups on non-type \( I \) factors ([MS12]).

**Definition 2.4.** A super product system of Hilbert spaces is a one parameter family of separable Hilbert spaces \( \{H_t : t \geq 0\} \), together with isometries \( U_{s, t} : H_s \otimes H_t \mapsto H_{s+t}, \)
for $s, t \in (0, \infty)$, which satisfy the following requirements of associativity and measurability:

(i) (Associativity) For any $s_1, s_2, s_3 \in (0, \infty)$

$$U_{s_1, s_2 + s_3}(1_{H_{s_1}} \otimes U_{s_2, s_3}) = U_{s_1, s_2, s_3}(U_{s_1, s_2} \otimes 1_{H_{s_3}}).$$

(ii) (Measurability) The space $H = \{(t, \xi) : t(0, \infty), \xi \in H_t\}$ is equipped with a structure of standard Borel space that is compatible with the projection $p : H \mapsto (0, \infty)$, given by $p((t, \xi)) = t$.

Given an equi-modular $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on a factorial non-commutative probability space $(M, \phi)$, we can always associate a super product system corresponding to the $E_0$-semigroup $\alpha$. Assume $M$ is acting standardly on $H = L^2(M, \phi)$. We consider, for every $t > 0$, the interwiner space

$$E^{\alpha_t} = \{T \in B(L^2(M, \phi)) : \alpha_t(x)T = Tx, \forall x \in M\}.$$ 

$\alpha' = \{\alpha_t' = \gamma \circ \alpha_t \circ \gamma : t \geq 0\}$ defines an $E_0$-semigroup on the commutant $M'$; and similarly we have $E^{\alpha'_t}$.

We first focus on the ‘fundamental unit’ $\{u_t : t \geq 0\}$ - which will establish the fact that $E^{\alpha_t} \cap E^{\alpha'_t} \neq \emptyset \quad \forall t \geq 0$. For $t \geq 0$, the fact that ‘$\phi$’ is preserved by $\alpha_t$ implies the existence of a unique family (necessarily a one-parameter semigroup) $\{u_t : t \geq 0\}$ of isometries on $L^2(M)$ such that $u_t x 1 = \alpha_t(x) 1$ $\forall x \in M$, and consequently $u_t \in E^{\alpha_t}$. As $\alpha_t$ is a equi-modular $^*$-endomorphism of $M$, it follows - see [BISSar] - that $u_t$ also ‘implements’ $\alpha'_t$, i.e., also $u_t x 1 = \alpha'_t(x') 1$ $\forall x' \in M'$, and consequently that $u_t \in E^{\alpha'_t}$. Thus,

$$u_t \in E^{\alpha_t} \cap E^{\alpha'_t} \quad \forall t \geq 0.$$ 

Now for every $t > 0$, let us write $H(t) = E^{\alpha_t} \cap E^{\alpha'_t}$. In fact, $H(t)$ is actually a Hilbert space; if $S, T \in H(t)$, then

$$T^* S \in (E^{\alpha_t})^* E^{\alpha_t} \cap (E^{\alpha'_t})^* E^{\alpha'_t} \subset M' \cap M = \mathbb{C}$$

and we find that $T^* S$ is a scalar multiple of the identity and the value of that scalar defines an inner product by way of

$$T^* S = \langle S, T \rangle I.$$ 

Now $H = \{(t, \xi) : \xi \in H_t\}$ is a super product system with the family of isometries

$$U_{s, t} : H_s \otimes H_t \mapsto H_{s+t},$$

uniquely determined by $U_{s, t}(S \otimes T) = ST$, for $S \in H_s, T \in H_t$.

We collect the following explicit description of these intertwiner spaces which will be useful in the sequel.

**Theorem 2.5.** $E^{\alpha_t} = [M' u_t] = \alpha_t(M)' u_t$.

**Proof.** We know that $E^{\alpha_t}$-see [Ale04]- is a Hilbert von Neumann $M' - M'$-bimodule. In particular $E^{\alpha_t}$ is Hilbert von Neumann $M'$ module. Now we shall verify that $[M' u_t]$ is Hilbert von Neumann submodule of $E^{\alpha_t}$. For that we need to check that $[M' u_t]$ is Hilbert von Neumann $M'$ module and $[M' u_t] \subset E^{\alpha_t}$. For the first assertion notice that

$$[(m_1' u_t)^* m_2' u_t : m_1', m_2' \in M'] = [u_t^* m_1' m_2' u_t] = [u_t^* M' u_t],$$
so it suffices to check that $u_i^*M'u_t \subset M'$, i.e., that $u_i^*m'u_tx = xu_i^*m'u_t \forall m' \in M', x \in M$; but

$$u_i^*m'u_tx = u_i^*m'\alpha_t(x)u_t \ (s \ ince \ u_t \in E_t^\alpha)$$

$$= u_i^*\alpha_t(x)m'u_t$$

$$= (\alpha_t(x^*)u_t)^*m'u_t$$

$$= (u_tx^*)^*m'u_t$$

$$= xu_i^*m'u_t.$$ 

Conversely, $m' = u_i^*u_t m' = u_i^*\alpha_t'(m')u_t$ so $M' \subset u_i^*\alpha_t'(M')u_t \subset u_i^*M'u_t$, and hence we do have $M' = u_i^*M'u_t$.

For the second assertion observe that

$$\alpha_t(m)m'u_t = m'\alpha_t(m)u_t$$

$$= m'u_t m,$$

for all $m \in M$ and $m' \in M'$, thus showing that $M'u_t \subset E^\alpha_t$, and hence also that $[M'u_t] \subset E^\alpha_t$.

Now suppose that there exist $T \in E^\alpha_t$ such that $T \in [M'u_t]^\perp$, i.e., $T^*m'u_t = 0$ for all $m' \in M'$. Now notice that $T^*m'u_t 1_M = T^*m'1_M = T^*m'$, and hence conclude that $T = 0$. Deduce then from the Riesz lemma that $E^\alpha_t = [M'u_t]$.

Observe next that for $m \in M$ and $x \in \alpha_t(M)'$, we have

$$\alpha_t(m)xu_t = x\alpha_t(m)u_t$$

$$= xu_t m,$$

and deduce that $\alpha_t(M)'u_t \subset E^{\alpha_t}$. On the other if $T \in E^{\alpha_t}$ observe that

$$T = Tu_t^*u_t$$

$$= yu_t$$

where $y = Tu_t^* \subset [E^{\alpha_t}, E^{\alpha_t^*}] = \alpha_t(M)'$. That is $T \in \alpha_t(M)'u_t$. So we have $E^{\alpha_t} \subset \alpha_t(M)'u_t$, yielding $E^{\alpha_t} = \alpha_t(M)'u_t$, as desired. \qed

**Remark 2.6.**

(1) We have already seen that $E^{\alpha_t}$ is Hilbert von Neumann $M' - M'$-bimodule, so $[M'u_t]$ and $\alpha_t(M)'u_t$ are also a Hilbert von Neumann $M' - M'$-bimodule.

(2) Replacing $\alpha'$ by $\alpha$ and $M'$ by $M$ in Proposition 2.5, we get $E_t^{\alpha'} = [Mu_t] = \alpha_t'(M)'u_t$.

Let $P(t) = \alpha_t(M)$ and $P_1(t)$ is the Jones basic construction. Then we have $P_1(t) = JP(t)J$ (see [BISSar]). But we know that $\alpha_t'(M)' = J\alpha_t(M)'J = P_1(t)$, we may summarize thus:

(3) $E^{\alpha_t} = [M'u_t] = P(t)'u_t$.

and

(4) $E^{\alpha_t'} = [Mu_t] = P_1(t)u_t$.

3. **CAR Flow**

Let $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$, where $\mathcal{K}$ is any Hilbert space. Let $\mathcal{F}_-(\mathcal{H})$ denote the anti-symmetric Fock space. For given $f \in \mathcal{H}$, let $a(f)$ be the creation operator in $B(\mathcal{F}_-(\mathcal{H}))$;

(1) $\mathcal{H} \ni f \mapsto a(f)$ is $\mathbb{C}$-linear,

(2) **(CAR)**

$$a(f)a(g) + a(g)a(f) = 0 \text{ and } a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle 1,$$
where $f, g \in \mathcal{H}$. Let $\mathcal{A}$ be the unital $C^*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$ in $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$. We note that $|a(f)| = |f|$ for $f \in \mathcal{H}$. Now suppose $R \in \mathcal{B}(\mathcal{H})$ satisfies $0 \leq R \leq 1$, where of course 1 is the identity operator $i\mathcal{H}$. The operator $R$ determines the so-called quasi-free state $\omega_R$ on $\mathcal{A}$ which satisfies the condition:

$$\omega_R(a^*(f_m) \cdots a^*(f_1)a(g_1) \cdots a(g_n)) = \delta_{nn} \det((Rg_i, f_j)).$$

It is known - see [BR81], [Amo01] - that there exists a representation $\pi_R$ of the $C^*$-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}_R = \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H})$ defined by the formulae

$$\pi_R(a(f)) = a((1 - R)^{1/2} f) \otimes \Gamma + 1 \otimes a^*(qR^{1/2} f),$$

$$\pi_R(a^*(f)) = a^*((1 - R)^{1/2} f) \otimes \Gamma + 1 \otimes a(qR^{1/2} f),$$

$$\pi_R(1) = 1,$$

where $f \in \mathcal{H}$. Here $\Omega$ is the ‘vacuum vector’ for the antisymmetric Fock space $\mathcal{F}_-(\mathcal{H})$, $q$ is an anti-unitary operator on $\mathcal{H}$ with $q^2 = 1$, and $\Gamma$ is the unique unitary operator on $\mathcal{F}_-(\mathcal{H})$ satisfying the conditions $\Gamma a(f) = -a(f)\Gamma$, $f \in \mathcal{H}$, and $\Gamma \Omega = \Omega$. In this representation, the state $\omega_R$ becomes the vector state

$$\omega_R(x) = \langle \Omega \otimes \Omega, \pi_R(x)\Omega \otimes \Omega \rangle,$$

for $x \in \mathcal{A}$, and $\mathcal{H}_R = \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H}) = \pi_R(\mathcal{A})\Omega \otimes \Omega$ becomes the GNS Hilbert space, under the assumption that both $R$ and $1 - R$ are injective (and hence also have dense range). So $(\pi_R, \mathcal{H}_R, \Omega \otimes \Omega)$ is the GNS triple for the $C^*$-algebra $\mathcal{A}$ with respect to the state $\omega_R$. We write $M_R = \{\pi_R(\mathcal{A})\}''$, which is always a factor, most often of type III (see [PS70] Theorem 5.1 and Lemma 5.3).

Let $\{s_t\}_{t \geq 0}$ be the shift semigroup on $\mathcal{H}$. Assume $s_t^* R s_t = R$ for all $t \geq 0$. Then, by [Arv03] Proposition 13.2.3 and [PS70] Lemma 5.3, there exists an $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on $M_R$, where $\alpha_t$ is uniquely determined by the following condition:

$$\alpha_t(\pi_R(a(f))) = \pi_R(a(s_t f)),$$

for all $f \in \mathcal{H}, t \geq 0$. This $E_0$-semigroup is called the CAR flow of rank dim $\mathcal{K}$ (on $M_R$).

### 3.1. Extendability of CAR Flows

For the remainder of this paper, we shall assume the following:

1. $qs_t = s_t q$ for all $t \geq 0$. (Such a $q$ always exists.)
2. We write $a_R(f)$ for $\pi_R(a(f))$ whenever $f \in \mathcal{H}$, and write $\mathcal{F}$ for the modular conjugation operator of $M_R$.
3. Both $R$ and $1 - R$ are invertible; i.e., $\exists \epsilon > 0$ such that $\epsilon \leq R \leq 1 - \epsilon$.
4. $R$ is diagonalisable; in fact, there exists an orthonormal basis $\{f_i\}$ for $\mathcal{K}$ with $Rf_i = \lambda_i f_i$ for some $\lambda_i \in [\epsilon, 1 - \epsilon] \setminus \left\{\frac{1}{2}\right\}$.
5. $R s_t = s_t R$ $\forall t \geq 0$. (Clearly then, also the Toeplitz condition $s_t^* R s_t = R$ is met.)

As we are unaware of whether, and if so where, these details may be found in the literature, we shall explicitly determine the modular operators in this case, and eventually ascertain (in Remark 3.5) the equi-modularity of the CAR flow.

For any (usually orthonormal) set $\{w_i\}_{i \in \mathbb{N}}$ in $\mathcal{H}$, we shall use the following notation for the rest of the paper: if $I = (i_1, i_2, \cdots, i_n)$ and $J = (j_1, j_2, \cdots, j_m)$ are ordered subsets of $\mathbb{N}$, then

1. $w_I = w_{i_1} \wedge \cdots \wedge w_{i_n}$,
2. $w_{IJ} = w_{i_1} \wedge \cdots \wedge w_{i_n} \wedge w_{j_1} \wedge \cdots \wedge w_{j_m}$,
3. $T w_I = T w_{i_1} \wedge \cdots \wedge T w_{i_n}$ for any operator $T \in \mathcal{B}(\mathcal{H})$;
4. $\hat{I} = \{i_n, \cdots, i_1\}$ so $w_{\hat{I}} = w_{i_n} \wedge \cdots \wedge w_{i_1}$;
(5) $a_R (w_I) = a_R (w_{i_1}) \cdots a_R (w_{i_n})$,

(6) $a_R^* (f) = (a_R (f))^*$, so $a_R^* (w_I) = a_R^* (w_{i_1}) \cdots a_R^* (w_{i_n}) = (a_R (w_I))^*$.

For a while, to simplify the notations, we write $A = (1 - R)^{1/2}$, $B = qR^{1/2}$ and notice that

$\langle Bh_i, Bh_j \rangle = \langle qR^{1/2} h_i, qR^{1/2} h_j \rangle$

$= \langle R^{1/2} h_j, R^{1/2} h_i \rangle$ since $q$ is anti-unitary

$= \langle Rh_j, h_i \rangle$

$= \delta_{i,j} \lambda_i$.

Now we write the following Lemmas without proof. Most of the proof follows from the use induction of cardinality of $L$ and our strong Toeplitz assumption (that $s_t$ commutes with $R$ and hence also with $A$ and $B$)

**Lemma 3.1.** Let $L = \{l_1 < \cdots < l_p\}$ be an ordered subset\(^1\) of $\mathbb{N}$. Then we have

$(5)$

$$a_R (h_L) a_R^* (h_L) \Omega \otimes \Omega = \sum c (L_i) Ah_{L_i} \otimes Bh_{L_i},$$

where the summation is taken over all ordered (possibly empty) subsets $L_i$ of $L$ and the $c (L_i)$ are all non-zero real numbers - with $Ah_0$ and $Bh_0$ being interpreted as $\Omega$.

**Lemma 3.2.** Let $L = \{l_1 < \cdots < l_p\}$ so that, by Lemma 3.1, equation 5 is satisfied. Then we have

(i) $a_R (s_I h_L) a_R^* (s_I h_L) \Omega \otimes \Omega = \sum c (L_i) As_I h_{L_i} \otimes Bs_I h_{L_i}, \forall t \geq 0$;

(ii) $a_R (h_I) a_R (h_L) a_R^* (h_J) a_R^* (h_J) \Omega \otimes \Omega$

$= \sum (-1)^{||J||+||L_i||||I||+||J||} c (L_i) Ah_I \wedge Ah_{L_i} \otimes Bh_{L_i} \wedge Bh_J$

(iii) $a_R (s_I h_I) a_R (s_I h_L) a_R^* (s_I h_J) a_R^* (s_I h_J) \Omega \otimes \Omega$

$= \sum (-1)^{||J||+||L_i||||I||+||J||} c (L_i) As_I h_I \wedge As_I h_{L_i} \otimes Bs_I h_{L_i} \wedge Bh_J, \forall t \geq 0$

where $I$ and $J$ are finite ordered subsets of $\mathbb{N}$ with $I \cap J = I \cap L = L \cap J = \phi$, and the summation is taken over all ordered subsets $L_i$ of $L$.

The fact that $s_t$ commutes with $R$ is seen to imply that the state $\omega_R$ is preserved by the CAR flow $\{\omega_t : t \geq 0\}$ and hence there exists a canonical semi-group $\{S_t : t \geq 0\}$ of isometries on $\mathcal{H}$ such that

$$S_t (x \Omega \otimes \Omega) = \omega_t (x) (\Omega \otimes \Omega) \forall x \in M_R.$$

The next lemma relates this semigroup $\{S_t : t \geq 0\}$ of isometries on $\mathcal{H}$ and the shift semigroup $\{s_t : t \geq 0\}$ of isometries on $\mathcal{H}$.

**Lemma 3.3.** Let $\{h_i\}_{i \in \mathbb{N}}$ be the orthonormal basis of $\mathcal{H}$ as above. Then for every $t \geq 0$, we have,

$$S_t (h_L \wedge h_I \otimes qh_L \wedge qh_J) = s_I h_L \wedge s_I h_I \otimes qs_I h_L \wedge qs_I h_J,$$

where $I$, $J$, and $L$ are ordered subsets of $\mathbb{N}$ with $I \cap J = I \cap L = L \cap J = \phi$.

Now the following lemma describes the action of the modular conjugation $J$ and the commutant of $M_R$.

\(^1\)For us, an ordered subset of $\mathbb{N}$ will always mean a finite subset of $\mathbb{N}$ with elements ordered in increasing order.
Lemma 3.4. With the above notation,

\begin{align}
(i) & \quad \mathcal{J}(h_I \land h_L \otimes qh_L \land qh_J) = h_J \land h_L \otimes qh_L \land qh_I \\
(ii) & \quad \mathcal{M}_R \mathcal{J} = M'_R = \{ \Gamma \otimes \Gamma b_R(h_i), b_R'(h_j)\Gamma \otimes \Gamma : i, j \in \mathbb{N} \}'' \\
(iii) & \quad \mathcal{J} a_R(h_i) \mathcal{J} = \Gamma \otimes \Gamma b_R'(h_i)
\end{align}

where \( b_R(h) = a(R^{1/2}h) \otimes \Gamma - 1 \otimes a^*(q(1 - R)^{1/2}h) \).

Proof. Recall the definition of the anti-linear (Tomita) operator \( S \), given by \( Sx\Omega \otimes \Omega = x^*\Omega \otimes \Omega \), \( x \in M_R \). We want to show the following expression for \( S \):

\[
S(Ah_I \land Ah_L \otimes Bh_L \land Bh_J)
\]

\[
= Ah_J \land Ah_L \otimes Bh_L \land Bh_I
\]

(6)

The proof is again by induction on the cardinality of \( L \). For \(|L| = 0\), the above assertion follows from

\[
S((1 - R)^{1/2}h_I \otimes qR^{1/2}h_J)
\]

\[
= Sa_R(h_I)a_R^*(h_J)(\Omega \otimes \Omega)
\]

\[
= a_R(h_J)a_R^*(h_I)(\Omega \otimes \Omega)
\]

\[
= (1 - R)^{1/2}h_J \otimes qR^{1/2}h_I
\]

Assume now that \(|L| = n\) and that we know the validity of equation (6) whenever \(|L| < 1\).

The point to be noticed is that Corollary 3.2(ii) may be re-written - in view of (i) each \( c(L_1) \) (and \( c(L) \) in particular) being non-zero- as:

\[
Ah_I \land Ah_L \otimes Bh_L \land Bh_J
\]

\[
= da_R(h_I)a_R(h_L)a_R^*(h_J)(\Omega \otimes \Omega)
\]

\[
+ \sum_{L_1 \subseteq L} d(L_1)Ah_I \land Ah_{L_1} \otimes Bh_{L_1} \land Bh_J,
\]

where the constants \( d, d(L_1) \) are all real and remain unchanged under changing \((I, J)\) to \((\hat{J}, \hat{I})\).

Now apply \( S \) to both sides of the above equation. Then the two terms on the right side get replaced by the terms obtained by replacing \((I, J)\) by \((\hat{J}, \hat{I})\) (8 by definition of \( S \) and 9 by the induction hypothesis regarding 6), thereby completing the proof of equation 6.

Equation (6) clearly implies that

\[
S(h_I \otimes qh_J) = ((1 - R)R^{-1})^{1/2}h_J \otimes q((R(1 - R)^{-1})^{1/2}h_I
\]

(even if \( I \cap J \neq \emptyset \); consideration of their intersections was needed essentially in order to establish Lemma 3.1 and thereby deduce the foregoing conclusions.)

Let \( D \) be the linear subspace spanned by \( \{ h_I \otimes qh_J : |I|, |J| \geq 0 \} \). Thus \( D \) is an obviously dense subspace of \( H_R \) which is contained in the domain of the Tomita conjugation operator \( S \), where its action is given by equation 10. We now wish to show that \( D \) is also contained in \( \text{dom}(S^*) \) and that \( S^*|_D \) is the operator \( F \) defined by the equation

\[
F(h_I \otimes qh_J) = ((R(1 - R)^{-1})^{1/2}h_J \otimes q((1 - R)R^{-1})^{1/2}h_I
\]
Indeed, notice that
\[ \langle S(h_I \otimes q h_J, h_I' \otimes q h_J') \rangle \]
\[ = \langle ((1 - R)R^{-1})^{\frac{1}{2}} h_j \otimes q (R(1 - R)^{-1})^{\frac{1}{2}} h_I, h_I' \otimes q h_J' \rangle \]
\[ = \langle ((1 - R)R^{-1})^{\frac{1}{2}} h_I, h_J' \rangle \delta(h_I, (R(1 - R)^{-1})^{\frac{1}{2}} h_I) \]
\[ = \langle (R(1 - R)^{-1})^{\frac{1}{2}} h_J, h_I \rangle \delta(h_I, (R(1 - R)^{-1})^{\frac{1}{2}} h_I) \]
\[ = \langle (R(1 - R)^{-1})^{\frac{1}{2}} h_J, \otimes q ((1 - R)R^{-1})^{\frac{1}{2}} h_I, h_I \otimes q h_J \rangle \]
\[ = \langle F(h_I' \otimes q h_J), h_I \otimes q h_J \rangle \]

Then, as \( S \) and \( F \) leave \( \mathcal{D} \) invariant, we see that
\[ FS(h_I \otimes q h_J) \]
\[ = F((1 - R)R^{-1})^{\frac{1}{2}} h_I \otimes q (R(1 - R)^{-1})^{\frac{1}{2}} h_J \]
\[ = R(1 - R)^{-1} h_I \otimes q(1 - R)R^{-1} h_J \]

If \( S = \mathcal{J} \Delta \mathcal{J}^{1/2} \) is its polar decomposition, with \( \mathcal{J} \) the modular conjugation and \( \Delta \) the modular operator for \( M_R \), the action of \( \mathcal{J} \) and \( \Delta \) on \( \mathcal{D} \) are thus seen to be given by the following rules respectively:
\[ \mathcal{J}(h_I \wedge h_L \otimes q h_L \wedge q h_J) = h_J \wedge h_L \otimes q h_L \wedge q h_J \]
\[ \mathcal{J}^2(A \otimes B) = BA = \Delta(A \otimes B) \]

and
\[ \Delta(h_I \wedge h_L \otimes q h_L \wedge q h_J) \]
\[ = R(1 - R)^{-1} h_I \wedge R(1 - R)^{-1} h_L \otimes q(1 - R)R^{-1} h_L \wedge q(1 - R)R^{-1} h_J \]

This proves part (i) of the Lemma, while the proof of parts (ii) and (iii) only involve of the following facts:

1. Lemma 3.1 and Corollary 3.2 imply that \( M_R(\Omega \otimes \Omega) \) is dense in \( \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H}) \);
2. Lemma 3.4 (i) implies that \( \mathcal{J}(\mathcal{D}) = \mathcal{D} \)
3. A painful but not difficult case-by-case computation reveals that \( \mathcal{J} \alpha_R(f) \mathcal{J} = (\Gamma \otimes \Gamma) b_R^*(f) \in M_R' \forall f \)

**Remark 3.5.** Using the definition of \( S_t \) and \( \mathcal{J} \), it easily follows that \( S_t \mathcal{J} = \mathcal{J} S_t \) for all \( t > 0 \), which implies that \( \alpha_t \) is equi-modular endomorphism for every \( t > 0 \). So now we are in the perfect situation to talk about the extendability of the CAR flow and under the above assumptions on \( R \), we prove that CAR flows are not extendable.

Now our aim is to explicitly determine \( (\alpha_t(M_R)' \cap M_R)(\Omega \otimes \Omega) \) for the CAR flow \( \alpha = \{\alpha_t : t \geq 0\} \).

Let \( \mathcal{P} \) and \( \mathcal{F} \) denote copies of \( \mathbb{N} \) - where we wish to think of \( \mathcal{F} \) and \( \mathcal{P} \) as signifying the future and past respectively. Let us write \( f_t = s_t h_j \), so \( \{f_j\}_{j \in \mathcal{F}} \) is an orthonormal basis for \( L^2(t, \infty) \otimes \mathcal{K} \). Also consider an orthonormal basis \( \{e_i\}_{i \in \mathcal{P}} \) of \( L^2(0, t) \otimes \mathcal{K} \). Then clearly \( \{e_i\}_{i \in \mathcal{P}} \cup \{f_j\}_{j \in \mathcal{F}} \) is an orthonormal basis for \( L^2(0, \infty) \otimes \mathcal{K} \).
Let $F(\mathcal{F})$ and $F(\mathcal{P})$ denote the collections of all finite ordered subsets of $\mathcal{F}$ and $\mathcal{P}$ respectively. Then $\mathcal{J}\mathcal{L} = \{ v_{i_1,j_1} \otimes q v_{i_2,j_2} : I_1, I_2 \in F(\mathcal{P}), J_1, J_2 \in F(\mathcal{F}) \}$ is an orthonormal basis for $F_-(\mathcal{H}) \otimes F_-(\mathcal{H})$, where $v_{i_1,j_1} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \wedge f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_m}$, with $I = \{ i_1 < i_2 \cdots < i_n \} \subset \mathcal{P}$ and $J = \{ j_1 < j_2 \cdots j_m \subset \mathcal{F} \}$.

Now if $T \in \mathcal{B}(\mathcal{H}_R)$, we will be working with the expansion of $T(\Omega \otimes \Omega)$ with respect to above orthonormal basis. Let us fix an $l \in \mathcal{F}$. We shall write $T(\Omega \otimes \Omega)$ in the following fashion, paying special attention to the occurrence or not of $l$ in the first and/or second tensor factor:

$$T(\Omega \otimes \Omega) = \sum (p_{00}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q v_{i_2,j_2}$$

$$+ p_{11}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2})$$

$$+ \sum u_{00}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q v_{i_2,j_2}$$

$$+ \sum u_{10}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q v_{i_2,j_2}$$

$$+ \sum u_{01}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2}$$

$$+ \sum u_{11}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2}.$$ (12)

Here and in the sequel, it will be tacitly assumed that the sums range over $((I_1 J_1), (I_2 J_2)) \in (F(\mathcal{P}) \times F(\mathcal{F}))^2$ - where we write $F_l(\mathcal{F}) = F(\mathcal{F} \setminus \{ l \}$ - and $p_{mn}, u_{mn} : \{(I_1 J_1, I_2 J_2) : I_1 \in F(\mathcal{P}), J_1 \in F(\mathcal{F}) \} \rightarrow \mathbb{C}, m, n \in \{0, 1\}$ where it is demanded that $spt(p_{00}) = spt(p_{11})$ and that $spt(p_{11})$, $spt(u_{00}), spt(u_{10}), spt(u_{01})$ and $spt(u_{11})$ are all disjoint sets - where we write $spt(f)$ for the subset of its domain where the function $f$ is non-zero. When necessary to show their dependence on the index $l$, we shall anoint these functions with an appropriate superscript, as in: $p_{11}^l(I J, I' J')$.

The letters $p$ and $u$ are meant to signify ‘paired’ and ‘unpaired’. Thus, suppose $l \in \mathcal{F}$, $I, L \in F(\mathcal{P})$, and $J, K \in F(I(\mathcal{F})$. If both $v_{i_1,j_1} \otimes q v_{i_1,k_1}$ and $f_1 \wedge v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_1,k_1}$ appear in the representation of $T(\Omega \otimes \Omega)$ with non-zero coefficients, then we shall think of $(I J, K L)$ as being an $l$-paired ordered pair. Thus $spt(p_{00}) = spt(p_{11})$ is the collection of $l$-paired ordered pairs, while $\cup_{m,n=0}^{spt(u_{mn})}$ is the collection of $l$-unpaired ordered pairs.

Note that in such an expression of $T(\Omega \otimes \Omega)$ with respect to different $l$, the type of a summand may change but the coefficients remain the same up to sign, since two vectors anti-commute under wedge product. We also note that $T(\Omega \otimes \Omega)$ has been written with respect to the basis $\mathcal{J}\mathcal{L} = \{ v_{i_1,j_1} \otimes q v_{i_2,j_2} : I_1, I_2 \in F(\mathcal{P}), J_1, J_2 \in F(\mathcal{F}) \}$. There are five types of sums in the representation of $T(\Omega \otimes \Omega)$. For simplicity of notation, let us write:

\[
(i) \quad \xi_T(p) = \sum (p_{00}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q v_{i_2,j_2} \\
+ p_{11}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2})
\]

\[
(ii) \quad \xi_T(u_{00}) = \sum u_{00}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q v_{i_2,j_2}
\]

\[
(iii) \quad \xi_T(u_{10}) = \sum u_{10}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q v_{i_2,j_2}
\]

\[
(iv) \quad \xi_T(u_{01}) = \sum u_{01}(I_1 J_1, I_2 J_2) v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2}
\]

\[
(v) \quad \xi_T(u_{11}) = \sum u_{11}(I_1 J_1, I_2 J_2) f_1 \wedge v_{i_1,j_1} \otimes q f_1 \wedge q v_{i_2,j_2},
\]

and $S = \{ p, u_{00}, u_{10}, u_{01}, u_{11} \}$. So we have:

$$T(\Omega \otimes \Omega) = \sum_{x \in S} \xi_T(x).$$
We also write:

(i) \( A_1 = \frac{1}{(1 - \lambda_l)^{1/2}} a_R(f_l) \),

(ii) \( A_2 = -\frac{1}{\lambda_l^{1/2}} \Gamma \otimes \Gamma b_R(f_l) \),

(iii) \( B_1 = \frac{1}{\lambda_l^{1/2}} a_R^*(f_l) \),

(iv) \( B_2 = -\frac{1}{(1 - \lambda_l)^{1/2}} b_R^*(f_l)(\Gamma \otimes \Gamma) \).

There is an implicit dependence in the definition of the \( A_i \)'s and \( B_i \)'s of the preceding equations on the arbitrarily chosen \( l \in \mathcal{F} \). When we wish to make this dependence explicit (as in Theorem 3.6 below), we shall adopt the following notational device: \( A_i = \{A_1, A_2\} \) and \( B_i = \{B_1, B_2\} \). We shall frequently use the following facts in the sequel:

1. \( R^{1/2} f_l = R^{1/2} s_h \ell_l = \lambda_l^{1/2} f_l \);
2. \( (1 - R)^{1/2} f_l = (1 - R)^{1/2} s_h \ell_l = (1 - \lambda_l)^{1/2} f_l \); and
3. \( f_l \otimes \Omega, \Omega \otimes f_l \in \text{ran}(S_l) \forall l \in \mathcal{F} \).

**Theorem 3.6.** If \( T \in \mathcal{B}(\mathcal{H}_R) \) satisfies \( A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega) \) and \( B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega) \) for \( A_1, A_2 \in \mathcal{A}_1, B_1, B_2 \in \mathcal{B}_1 \) and for all \( l \in \mathcal{F} \), then

\[
\Omega \otimes \Omega \subset \{v_{I_1} \otimes q v_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}\},
\]

where \([\ ]\) denotes span closure.

We start with a \( T \in \mathcal{B}(\mathcal{H}_R) \), which satisfies the hypothesis of the above Theorem 3.6 and write \( T(\Omega \otimes \Omega) \) as in 12 , for an arbitrary choice of index \( l \). Then we go through the following Lemmas and prove that the coefficient functions \( p_{00}, p_{11}, u_{10}, u_{01}, u_{11} \) are identically zero, while the support of \( u_{00} \) is contained in the set \( \{(I_1 J_1, I_2 J_2) : J_1 \cup J_2 = \emptyset, (-1)^{|I_1|} = (-1)^{|I_2|}\} \). The truth of this assertion for all choices of \( l \) will prove our Theorem 3.6. We go through the following Lemmas regarding the representation of \( T(\Omega \otimes \Omega) \) whose proofs elementary and simple. We may sometime omit the details.

**Lemma 3.7.** Let \( \eta(x) \) (resp., \( \eta(y) \)) be a summand of the sum \( \xi_T(x) \) (resp., \( \eta(y) \)), where \( x, y \in \mathcal{S} \). Then \( \langle \eta(x), \eta(y) \rangle = 0 \) implies that \( \langle X \eta(x), Y \eta(y) \rangle = 0 \), for all \( x, y \in \mathcal{S} \) and \( X, Y \in \mathcal{A} \) or \( X, Y \in \mathcal{B} \).

**Proof.** This follows from (i) the assumptions that \( \text{spt}(p_{00}) = \text{spt}(p_{11}), \) (ii) \( \text{spt}(p_{11}), \text{spt}(u_{01}), \text{spt}(u_{11}) \) are all disjoint sets and (iii) the definition of the action of \( X, Y \) on \( \eta(x) \).

**Lemma 3.8.** If \( A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega) \), then \( A_1 \xi_T(x) = A_2 \xi_T(x) \) for all \( x \in \mathcal{S} \). Similarly if \( B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega) \), then \( B_1 \xi_T(x) = B_2 \xi_T(x) \) for all \( x \in \mathcal{S} \).

**Proof.** This follows from

\[
\|(A_1 - A_2)T(\Omega \otimes \Omega)\|^2 = \sum_{x \in \mathcal{S}} \|(A_1 - A_2)\xi_T(x)\|^2,
\]

which is a consequence of Lemma 3.7.

Now onwards we assume that \( T \) satisfies the hypothesis of the Theorem 3.6 and with the foregoing notations we have the following Lemma regarding the coefficients of the representation of \( T(\Omega \otimes \Omega) \).

---

2By a summand of \( \xi_T(x) \) we shall mean a ‘paired term’ of the form \( (p_{00}(I_1 J_1, I_2 J_2) \ v_{I_1 J_1} \otimes q v_{I_2 J_2} + p_{11}(I_1 J_1, I_2 J_2) \ f_i \otimes q f_i \otimes q v_{I_2 J_2}) \) rather than an individual term of such a pair.
Lemma 3.9. \( p_{00}, \ p_{11}, \ u_{00}, \ u_{10}, \ u_{01}, \ u_{11} \) satisfy the following equations:

\[
\begin{align*}
(i) \quad & \sigma(I_2, J_2) p_{00}(I_1 J_1, I_2 J_2) + p_{11}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} \\
& = \sigma(I_1, J_1) p_{00}(I_1 J_1, I_2 J_2) + \rho(I_1 J_1, I_2 J_2) p_{11}(I_1 J_1, I_2 J_2) \frac{(1 - \lambda_i)^{1/2}}{\lambda_i^{1/2}}, \\
(ii) \quad & \sigma(I_2, J_2) u_{00}(I_1 J_1, I_2 J_2) = \sigma(I_1, J_1) u_{00}(I_1 J_1, I_2 J_2), \\
(iii) \quad & u_{01}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} = * u_{01}(I_1 J_1, I_2 J_2) \frac{(1 - \lambda_i)^{1/2}}{\lambda_i^{1/2}}, \\
(iv) \quad & * u_{11}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} = * u_{11}(I_1 J_1, I_2 J_2) \frac{(1 - \lambda_i)^{1/2}}{\lambda_i^{1/2}}, \\
(v) \quad & * \frac{(1 - \lambda_i)^{1/2}}{\lambda_i^{1/2}} u_{10}(I_1 J_1, I_2 J_2) = \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} u_{10}(I_1 J_1, I_2 J_2)
\end{align*}
\]

where \( \sigma : F(P) \times F(F) \to \{1, -1\} \) is defined by \( \sigma(I, J) = (-1)^{|I|+|J|} \), \( \rho : \{(I_1 J_1, I_2 J_2) : I_k \in F(P), J_l \in F(F)\} \to \{1, -1\} \), defined by \( \rho(I_1 J_1, I_2 J_2) = (-1)^{|I_1|+|I_2|+|J_1|+|J_2|} \), and \(* = \pm 1\).

Proof. \( T \) satisfies \( A_1(T \Omega \otimes \Omega) = A_2(T \Omega \otimes \Omega) \). So from the Lemma 3.8, we have \( A_1 \xi_T(x) = A_2 \xi_T(x) \) for every \( x \in S \). Now for every \( x \in S \), we separately compute \( A_1 \xi_T(x) \) and \( A_2 \xi_T(x) \) and compare their coefficients.

(i) If \( A_1 \xi_T(p) = A_2 \xi_T(p) \), observe that

\[
A_1 \xi_T(p) = \frac{1}{(1 - \lambda_i)^{1/2}} a_R(f_i) \xi_T(p)
\]

\[
= \sum \left( p_{00}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} a_R(f_i) v_{I_1, J_1} \otimes q v_{I_2, J_2} \\
+ p_{11}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} a_R(f_i) f_i \wedge v_{I_1, J_1} \otimes q f_i \wedge q v_{I_2, J_2} \right)
\]

while

\[
A_2 \xi_T(p) = \left( -\frac{1}{\lambda_i^{1/2}} \Gamma \otimes \Gamma b(f_i) \xi_T(p) \right)
\]

\[
= \sum \left( p_{00}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} \Gamma \otimes \Gamma b(f_i) v_{I_1, J_1} \otimes q v_{I_2, J_2} \\
+ p_{11}(I_1 J_1, I_2 J_2) \frac{\lambda_i^{1/2}}{(1 - \lambda_i)^{1/2}} \Gamma \otimes \Gamma b(f_i) f_i \wedge v_{I_1, J_1} \otimes q f_i \wedge q v_{I_2, J_2} \right)
\]

\[
= \sum (\sigma(I_1, J_1) p_{00}(I_1 J_1, I_2 J_2) \\
+ \rho(I_1 J_1, I_2 J_2) p_{11}(I_1 J_1, I_2 J_2) \frac{(1 - \lambda_i)^{1/2}}{\lambda_i^{1/2}} f_i \wedge v_{I_1 J_1} \otimes q v_{I_2, J_2})
\]
and (i) follows upon comparing coefficients in the two equations above.

Equations (ii), (iii) and (iv) are proved by arguing exactly as for (ii) above.

As for (v), we also have $B_1T(\Omega \otimes \Omega) = B_2T(\Omega \otimes \Omega)$. So from Lemma 3.8, we have $B_1\xi_T(x) = B_2\xi_T(x)$ for all $x \in \mathcal{S}$. In particular we have $B_1\xi_T(u_{10}) = B_2\xi_T(u_{10})$. Then $v$ follows from the comparing coefficients of $B_1\xi_T(u_{10})$ and $B_2\xi_T(u_{10})$.

\[ \square \]

With foregoing notation and the assumptions on $T$, we have the following Corollary.

**Corollary 3.10.** If we represent $T(\Omega \otimes \Omega)$ las in eqn. 12, then

\[ u_{01} = u_{11} = u_{10} = 0. \]

That is the functions $u_{01}, u_{11}$ and $u_{10}$ are identically zero.

**Proof.** This is an immediate consequence of Lemma 3.9(iii), 3.9(iv) and 3.9(v) and the assumption that $\lambda_l \neq 1/2 \forall l$.

We continue to assume that an operator $T \in \mathcal{B}(\mathcal{H}_R)$ satisfies the hypothesis of the Theorem 3.6 and proceed to analyse the representation of $T(\Omega \otimes \Omega)$ as in eqn. (12).

**Remark 3.11.**

1. Lemma 3.9(i) implies that if $(IJ, KL)$ are l-paired, then $\sigma(I, J) \neq \sigma(K, L)$. (Reason: Otherwise, since $\rho(IJ, KL) = \pm 1$, and $|p_{11}(IJ, KL)| \neq 0$, we must have $\lambda_l = \frac{1}{2}$.)

2. Lemma 3.9(ii) implies that if $u_{00}(I_1J_1, I_2J_2) \neq 0$, then $\sigma(I_2, J_2) = \sigma(I_1, J_1)$, i.e.

\[ (-1)^{|I_1|+|J_1|} = (-1)^{|I_2|+|J_2|}. \]

Now we wish to compare the representations of $T(\Omega \otimes \Omega)$ for different $l$'s.

**Lemma 3.12.** Let $I, K \in F(\mathcal{P})$ and $J, L \in F(\mathcal{F})$. If a term of the form $v_{IJ} \otimes qv_{KL}$ appears in $T(\Omega \otimes \Omega)$ with non-zero coefficient, then $(IJ, KL)$ can be w-paired for at most finitely many $w \in \mathcal{F}$ with $w \notin J \cup L$.

**Proof.** Suppose, if possible, that \{ $l_n : n \in \mathcal{F}$ \} is an infinite sequence of distinct indices such that $(IJ, KL)$ is $l_n$-paired for each $n \in \mathbb{N}$. Then we may, by Remark 3.11(1), conclude that \{ $\sigma(I, J), \sigma(K, L)$ \} = \{ $1, -1$ \}.

Deduce now from Lemma 3.9(i) that

\[ \sigma(K, L)p_{00}(IJ, KL) + *p_{11}^{l_n}(IJ, KL) \frac{\lambda_n^{1/2}}{(1 - \lambda_n)^{1/2}} = \sigma(I, J)p_{00}(IJ, KL) + *p_{11}^{l_n}(IJ, KL) \frac{1 - \lambda_n}{\lambda_n^{1/2}} \]

where $* \in \{ +, - \}$. Since $\lambda_n \in (\epsilon, 1 - \epsilon) \setminus \{ 1/2 \}$ for all $n$, we see that \{ $\lambda_n^{1/2} : n \in \mathbb{N}$ \} and \{ $(1 - \lambda_n)^{1/2} : n \in \mathbb{N}$ \} are bounded sequences. As $p_{11}^{l_n}(IJ, KL)$ are Fourier coefficients, the sequence \{ $p_{11}^{l_n}(IJ, KL)$ \} converges to 0, as $n \to \infty$. Clearly then \{ $\frac{\lambda_n^{1/2}}{(1 - \lambda_n)^{1/2}} p_{11}^{l_n}(IJ, KL) : n \in \mathbb{N}$ \} and \{ $(1 - \lambda_n)^{1/2} p_{11}^{l_n}(IJ, KL) : n \in \mathbb{N}$ \} are sequences converges to 0, as $n \to \infty$. So from the above equation we get $p_{00}(IJ, KL) = 0$. But we had assumed that $p_{00}(IJ, KL)$ is non-zero. Hence $v_{IJ} \otimes qv_{KL}$ can not be l-paired for infinitely many $l \in \mathcal{F}$ with $l \notin J \cup L$. \[ \square \]
Lemma 3.13. Let $I, K ∈ F(Π)$ and $J, L ∈ F(ℱ)$ with $l不属于J∪L$. Suppose an element of the form $v_{I,J} ⊗ qv_{KL}$, appearing in $TΩ ⊗ Ω$ with a non-zero coefficient. Then we have $(-1)^{|I|+|J|} = (-1)^{|K|+|L|}$.

Proof. From Lemma 3.12, we can find a $l₀ ∈ ℱ$ such that $l₀不属于J∪L$ and $v_{I,J} ⊗ qv_{KL}$ is not $l₀$-paired. If we write $TΩ ⊗ Ω$ with respect to $l₀$, we see that $v_{I,J} ⊗ qv_{KL}$ appears with the same coefficient as in the third type of sum. So by observing the Remark 3.11 with respect to $l₀$, we see that $(-1)^{|I|+|J|} = (-1)^{|K|+|L|}$.

Again with the foregoing notations, we have the following Lemma about the coefficients of the representation of $T(Ω ⊗ Ω)$.

Lemma 3.14. $p_{00} = p_{11} = 0$.

Proof. Recall the equation 3.9(i) from Lemma 3.9:

$$\sigma(I_2, J_2)p_{00}(I_1J_1, I_2J_2) + p_{11}(I_1J_1, I_2J_2) \frac{λ^{1/2}}{(1 - λ)} = \sigma(I_1, J_1)p_{00}(I_1J_1, I_2J_2) + \rho(I_1J_1, I_2J_2)p_{11}(I_1J_1, I_2J_2) \frac{λ^{1/2}}{(1 - λ)}$$

where $\sigma(I_2, J_2) = (-1)^{|I_2|+|J_2|} and \sigma(I_1, J_1) = (-1)^{|I_1|+|J_1|}$. But from 3.13 we have $(-1)^{|I_2|+|J_2|} = (-1)^{|I_1|+|J_1|}$. Since $λ_l ≠ 1/2$, from the above equation we get $p_{11}(I_1J_1, I_2J_2) = 0$, which implies that $p_{00}(I_1J_1, I_2J_2) = 0$, since $spt(P_{00}) = spt(p_{11})$, i.e. they have the same support.

Lemma 3.15. $TΩ ⊗ Ω = \sum x(I_1J_1) v_{I_1} ⊗ qv_{I_2},$ where the summation is taken over $I_1, I_2 ∈ F(Π)$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$.

Proof. So we started with a representation of $TΩ ⊗ Ω$ like 12 and by using the Corollary 3.10 and Lemma 3.14 ended up with the following conclusions;

$$p_{00} = p_{11} = u_{10} = u_{01} = u_{11} = 0.$$

Thus finally the representation of $TΩ ⊗ Ω$ will be of the form

$$T(Ω ⊗ Ω) = \sum u_{00}(I_1J_1, I_2J_2) v_{I_1J_1} ⊗ qv_{I_2J_2},$$

where the summation is taken over $I_1J_1, I_2J_2 ∈ F(Π), J_1, J_2 ∈ F(ℱ)$ with $(-1)^{|I_2|+|J_2|} = (-1)^{|I_1|+|J_1|}$ and $l不属于J_1∪J_2$, for $l ∈ ℱ$. Since this is true for all $l ∈ ℱ$, $J_1, J_2$ are empty sets, i.e.

$$T(Ω ⊗ Ω) = \sum x(I_1J_2) v_{I_1} ⊗ qv_{I_2},$$

where the summation is taken over $I_1J_2 ∈ F(Π)$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$ and $x(I_1J_2) = u_{00}(I_1∅, I_2∅)$ are complex numbers.

So finally the above Lemma 3.15 proves our theorem 3.6.

Theorem 3.16. Let $T ∈ α(M_R)' ∩ M_R$, then

$$T(Ω ⊗ Ω) ⊂ \{v_{I_1} ⊗ qv_{I_2} : I_1, I_2 ∈ F(Π), (-1)^{|I_1|} = (-1)^{|I_2|}\},$$

Proof. It is enough to prove that $A_1T(Ω ⊗ Ω) = A_2T(Ω ⊗ Ω)$ and $B_1T(Ω ⊗ Ω) = B_2T(Ω ⊗ Ω)$, then it follows from the Theorem 3.19.
Observe that,
\begin{align*}
A_1 T(\Omega \otimes \Omega) &= \frac{1}{(1 - \lambda_1)^{1/2}} a_R(f_i) T \Omega \otimes \Omega \\
&= \frac{1}{(1 - \lambda_1)^{1/2}} T a_R(f_i) \Omega \otimes \Omega \quad \text{since } T \in \alpha_t(M_R)'
\end{align*}

\begin{align*}
&= Tf_i \otimes \Omega \\
&= -\frac{1}{\lambda_1^{1/2}} TB \otimes \Gamma b_R(f_i) \Omega \otimes \Omega \\
&= -\frac{1}{\lambda_1^{1/2}} \Gamma \otimes \Gamma b_R(f_i) T \Omega \otimes \Omega \quad \text{since } T \in M_R \text{ and } \Gamma \otimes \Gamma b_R(f_i) \in M'_R \\
&= A_2 T(\Omega \otimes \Omega).
\end{align*}

So we have \( A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega) \). Again observe that,
\begin{align*}
B_1 T(\Omega \otimes \Omega) &= \frac{1}{\lambda_1^{1/2}} a_R^*(f_i) T \Omega \otimes \Omega \\
&= \frac{1}{\lambda_1^{1/2}} T a_R^*(f_i) \Omega \otimes \Omega \quad \text{since } T \in \alpha_t(M_R)'
\end{align*}

\begin{align*}
&= T \Omega \otimes f_i \\
&= -\frac{1}{(1 - \lambda_1)^{1/2}} T b_R^*(f_i) \Gamma \otimes \Gamma \Omega \otimes \Omega \\
&= -\frac{1}{(1 - \lambda_1)^{1/2}} b_R^*(f_i) \Gamma \otimes \Gamma T \Omega \otimes \Omega, \quad \text{since } T \in M_R \text{ and } b_R^*(f_i) \Gamma \otimes \Gamma \in M'_R \\
&= B_2 T(\Omega \otimes \Omega).
\end{align*}

That is \( B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega) \).

\[\square\]

**Theorem 3.17.** CAR flow \( \alpha = \{\alpha_t : t \geq 0\} \) is not not extendable.

**Proof.** It is enough to show that for some \( t > 0 \), \( \alpha_t : M_R \to M_R \) is not extendable. To prove \( \alpha_t \) is not extendable, we use the Theorem 3.7 of [BISSar]. We observe that
\begin{align*}
\{(y\alpha_t(x) \Omega \otimes \Omega : x \in M_R, y \in \alpha_t(M_R)' \cap M_R)\} &= \{(\alpha_t(x)y \Omega \otimes \Omega : x \in M_R, y \in \alpha_t(M_R)' \cap M_R)\} \\
&= \{((a_R(f_j)a_R^*(f_L))^* T \Omega \otimes \Omega : J, L \in F(\mathcal{F}), T \in \alpha_t(M_R)' \cap M_R)\}.
\end{align*}

Now if \( T \in \alpha_t(M_R)' \cap M_R \), then from the Theorem 3.19 we have, \( T(\Omega \otimes \Omega) \in \{v_1 \otimes qv_2 : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}\} \). If \( g \in \mathcal{P} \), we notice that
\begin{align*}
\langle (a_R(f_j)a_R^*(f_L))^* T \Omega \otimes \Omega, e_g \otimes \Omega \rangle &= \langle T \Omega \otimes \Omega, a_R(f_j)a_R^*(f_L)e_g \otimes \Omega \rangle \\
&= 0.
\end{align*}

So from the above, we conclude that \( \{y\alpha_t(x) \Omega \otimes \Omega : x \in M_R, y \in \alpha_t(M_R)' \cap M_R\} \) is orthogonal to the vector \( e_g \otimes \Omega \), i.e. \( \{y\alpha_t(x) : x \in M_R, y \in \alpha_t(M_R)' \cap M_R\} \) cannot be weakly total in \( M_R \), so by the Theorem 3.7 of [BISSar] \( \alpha_t \) cannot be extendable. \(\square\)
Remark 3.18. It has been proved in section 5[ABS01] that CAR flows, arising from quasi-free state for scalar, on type III factors are extendable. But we have proved that CAR arising from quasi-free state for diagonalisable positive contractions (in particular scalars) are not extendable. So our result shows that there is some error in section 5 [ABS01] regarding the conclusion of extendability of CAR flows. In fact, we think that there is a mistake in the theorem 4 of section 5 [ABS01] and for that their conclusion regarding the extendability of CAR flows went wrong.

3.2. Super Product System for CAR flows. Now we recall the definition of fundamental unit. Since for every $t \geq 0$, quasi-free state is invariant under $\alpha_t$, we get the fundamental unit. Let $\{S_t\}_{t \geq 0}$ be the fundamental unit for the CAR flow. Then recall from the Theorem 2.5 that the common intertwiner space for the CAR flow is $H_t = \alpha_t(M_R)''S_t \cap J \alpha_t(M_R)''J S_t$. Now our aim is to find explicitly $H_t(\Omega \otimes \Omega)$.

Theorem 3.19. Let $T \in H_t$,

$$T(\Omega \otimes \Omega) \subset \{[v_{I_1} \otimes v_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}]\},$$

Proof. It is enough to prove that $A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega)$ and $B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega)$, then it follows from the Theorem 3.6.

For $T \in H_t$, we may - by Theorem 3.6, as the $u_t$ there is our $S_t$, when $M = M_R$ and $\alpha$ is the CAR flow - write $T = T_1 S_t = T_2 S_t$, where $T_1 \in \alpha_t(M_R)'$ and $T_2 \in J \alpha_t(M_R)'J$. As $T_1$ and $T_2$ agree on the range of $S_t$, observe that,

$$T(\Omega \otimes \Omega) = T_1 (\Omega \otimes \Omega) = T_2 (\Omega \otimes \Omega),$$

since $S_t(\Omega \otimes \Omega) = (\Omega \otimes \Omega)$. We note that,

$$A_1 T(\Omega \otimes \Omega) = \frac{1}{(1 - \lambda_t)^{1/2}} a_R(f_1) T \Omega \otimes \Omega$$

$$= \frac{1}{(1 - \lambda_t)^{1/2}} a_R(f_1) T_1 \Omega \otimes \Omega$$

$$= \frac{1}{(1 - \lambda_t)^{1/2}} T_1 a_R(f_1) \Omega \otimes \Omega \text{ since } T_1 \in \alpha_t(M_R)'$$

$$= T_1 f_1 \otimes \Omega \text{ by (3) above, since } T_1 S_t = T_2 S_t$$

$$= -\frac{1}{\lambda_t^{1/2}} T_2 \Gamma \otimes \Gamma b_R(f_1) \Omega \otimes \Omega$$

$$= -\frac{1}{\lambda_t^{1/2}} \Gamma \otimes \Gamma b_R(f_1) T_2 \Omega \otimes \Omega \text{ since } T_2 \in J \alpha_t(M_R)'J$$

$$= -\frac{1}{\lambda_t^{1/2}} \Gamma \otimes \Gamma b_R(f_1) T \Omega \otimes \Omega$$

$$= A_2 T(\Omega \otimes \Omega).$$

So we have $A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega)$. Again by similar kind of computation as above, we observe that $B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega)$.

Let us recall that $\{h_i\}_{i \in \mathbb{N}}$ and $\{e_i\}_{i \in \mathcal{P}}$ are orthonormal bases of $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$ and $L^2(0, t) \otimes \mathcal{K}$ respectively. Now with respect to the fix orthonormal basis $\{e_i\}_{i \in \mathcal{P}}$ of $L^2(0, t) \otimes \mathcal{K}$, we define the following operator on $\mathcal{H}_R$. If $I_1, I_2 \in F(\mathcal{P})$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$, then there exists an operator $T_{I_1 I_2} : \mathcal{H}_R \to \mathcal{H}_R$ which is defined by the following rule,

$$T_{I_1 I_2}(h_{I_1} \otimes qh_{I_2}) = (-1)^{|I_1|} |I_1| s_I h_{I_1} \wedge e_{I_1} \otimes q s_I h_{I_2} \wedge q e_{I_2}.$$
where $J_1, J_2$ are finite ordered subset of $\mathbb{N}$. With the above notation we have the following theorem.

**Theorem 3.20.** For $t > 0$, we have

$$H_t = \{ [T_{I_1, I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \}.$$ 

**Proof.** If $I_1, I_2 \in F(\mathcal{P})$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$, we check that $T_{I_1, I_2} \in H_t$. We want to prove $\alpha_t(a_R(h_I))T_{I_1, I_2} = T_{I_1, I_2}a_R(h_I)$ for all $l \in \mathbb{N}$. We observe that

$$\alpha_t(a_R(h_I))T_{I_1, I_2}(h_{I_1} \otimes qh_{I_2}) = a_R(s_t h_I)T_{I_1, I_2}(h_{I_1} \otimes qh_{I_2})$$

$$= (-1)^{|I_1||I_2|}a_R(s_t h_I)(s_t h_{I_1} \wedge e_{I_1} \otimes qs_t h_{I_2} \wedge qe_{I_2})$$

$$= \frac{(-1)^{|I_1||I_2|+|I_2|}}{(1 - \lambda_I)^{1/2}}(s_t h_{I_1} \wedge e_{I_1} \otimes qs_t h_{I_2} \wedge qe_{I_2})$$

$$+ \frac{(-1)^{|I_1||I_2|}}{\lambda_I^{1/2}}s_t h_{I_1} \wedge e_{I_1} \otimes a^*(s_t q h_I)qs_t h_{I_2} \wedge qe_{I_2})$$

$$= \frac{(-1)^{|I_1|(|I_1|+1)+|I_2|}}{(1 - \lambda_I)^{1/2}}(s_t h_{I_1} \wedge e_{I_1} \otimes qs_t h_{I_2} \wedge qe_{I_2})$$

$$+ \frac{(-1)^{|I_1||I_2|+1}}{\lambda_I^{1/2}}s_t h_{I_1} \wedge e_{I_1} \otimes a^*(s_t q h_I)qs_t h_{I_2} \wedge qe_{I_2}).$$

In the above to write second last to last equation, we have used $(-1)^{|I_1|} = (-1)^{|I_2|}$. On the other hand observe that

$$T_{I_1, I_2}a_R(h_I)(h_{I_1} \otimes qh_{I_2})$$

$$= T_{I_1, I_2}((-1)^{|I_2|}(1 - \lambda_I)^{1/2}(h_I \wedge h_{I_1} \otimes qh_{I_2})$$

$$+ \lambda_I^{1/2}h_{I_1} \otimes a^*(q h_I)qh_{I_2})$$

$$= \frac{(-1)^{|I_1||I_1|+|I_2|}}{(1 - \lambda_I)^{1/2}}(s_t h_{I_1} \wedge s_t h_{I_2} \wedge e_{I_1} \otimes qs_t h_{I_2} \wedge qe_{I_2})$$

$$+ \frac{(-1)^{|I_1||I_2|+1}}{\lambda_I^{1/2}}s_t h_{I_1} \wedge e_{I_1} \otimes a^*(s_t q h_I)qs_t h_{I_2} \wedge qe_{I_2})$$.

Note that in the above equation we have used $s_t q = qs_t$. Finally above computations show that $\alpha_t(a_R(h_I))T_{I_1, I_2} = T_{I_1, I_2}a_R(h_I)$. Again similar computation will show that $\alpha_t(a_R^*(h_I))T_{I_1, I_2} = T_{I_1, I_2}a_R^*(h_I)$, for all $l \in \mathbb{N}$. So we conclude that $T_{I_1, I_2} \in E^{\alpha_I}$. From theorem 2.5 we have $E^{\alpha_I} = \alpha_t(M)_t$. So we have $T_{I_1, I_2} \in \alpha_t(M)_t$. Now recall the definition of $\mathcal{F}$ from lemma 3.4 and observe that

$$\mathcal{J}T_{I_1, I_2}(h_{I_1} \otimes qh_{I_2}) = \mathcal{J}T_{I_1, I_2}(q^2 h_{I_2} \otimes qh_{I_1})$$

$$= \mathcal{J}T_{I_1, I_2}(h_{I_2} \otimes qh_{I_1})$$

$$= \mathcal{J}((-1)^{|I_1|} |I_2|s_t h_{I_2} \wedge e_{I_1} \otimes s_t q h_{I_1} \wedge qe_{I_2})$$

$$= (-1)^{|I_1|} |I_2|e_{I_2} \wedge s_t h_{I_1} \wedge qe_{I_2}$$

$$+ (-1)^{|I_2|} |I_1|s_t h_{I_1} \wedge e_{I_2} \otimes s_t q h_{I_1} \wedge qe_{I_1}$$

$$= T_{I_2, I_1}(h_{I_1} \otimes qh_{I_2}).$$

In the above equation we have used the property that two vector anti commute under wedge product. The above equation says that $\mathcal{J}T_{I_1, I_2} = T_{I_2, I_1} \in \alpha_t(M)_t$. That is
$T_{I_1I_2} \in \mathcal{J} \alpha_t(M)'S_t = \mathcal{J} \alpha_t(M)' \mathcal{J} S_t$, since $\mathcal{J}$ commute with $S_t$. So from the theorem 2.5 it is clear that $T_{I_1I_2} \in H_t$, i.e.

$$\{[T_{I_1I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \} \subset H_t.$$  

To prove the equality we show that if $T \in H_t$ and $T \in \{[T_{I_1I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \}^\perp$, then $T = 0$. $T \in H_t$ implies that $\alpha_t(x)T = Tx$ for all $x \in M_R$, i.e. $\alpha_t(x)T\Omega \otimes \Omega = T\Omega \otimes \Omega$. As $\Omega \otimes \Omega$ is cyclic for $M_R$, conclude that $T$ is determined by its action on $\Omega \otimes \Omega$. So to prove $T = 0$, it is enough to prove that $T(\Omega \otimes \Omega) = 0$. If $I_1, I_2 \in F(\mathcal{F})$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$, then we notice that

$$\langle T(\Omega \otimes \Omega), e_{I_1} \otimes qe_{I_2} \rangle = \langle T(\Omega \otimes \Omega), T_{I_1I_2}(\Omega \otimes \Omega) \rangle = \langle T_{I_1I_2}, T((\Omega \otimes \Omega), T_{I_1I_2}(\Omega \otimes \Omega)) \rangle = 0 \text{ since } \langle T_{I_1I_2}, T \rangle = 0.$$  

So we have $T(\Omega \otimes \Omega) \in \{[v_{I_1} \otimes qv_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \}^\perp$, but Theorem 3.6 says that $T(\Omega \otimes \Omega) \in \{[v_{I_1} \otimes qv_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \}^\perp$. So $T\Omega \otimes \Omega$ has to be zero. So we get

$$H_t = \{[T_{I_1I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}] \}.$$  

\[\square\]

Now write $H = \{(t, x_t) : t > 0, x_t \in H_t\}$. Obviously $H$ is the super product system for CAR flow.

4. CCR and CAR flow

We have already described CAR flow on type III factors. Let us describe CCR flow as follows.

Let $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$, where $\mathcal{K}$ is a Hilbert space. For $n = 0, 1, 2, \cdots$ we will write $\mathcal{H}^n$ for the symmetric tensor product of $n$ copies of $\mathcal{H}$ for $n \geq 1$ with $\mathcal{H}^0 = \mathbb{C}$. The symmetric Fock space over $H$ is defined as the direct sum of Hilbert spaces

$$\mathcal{F}_+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$$

The exponential map $\exp : \mathcal{H} \to \mathcal{F}_+(\mathcal{H})$ is defined by

$$\exp(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}$$

The symmetric Fock space $\mathcal{F}_+(\mathcal{H})$ is span closure of the vectors of the form $\exp(f)$, $f \in \mathcal{H}$. Now for every vector $f \in \mathcal{H}$ there is a unique unitary operator $W(f)$ on $\mathcal{F}_+(\mathcal{H})$ satisfies

$$W(\xi) \exp(\eta) = e^{-1/2||f||}(g.f) \exp(g + f)$$

Let $CCR(\mathcal{H})$ be the unital $C^*$-algebra generated by $\{W(f) : f \in \mathcal{H}\}$ in $\mathcal{B}(\mathcal{F}_+(\mathcal{H}))$. Let $T$ be a positive operator on $\mathcal{B}(\mathcal{H})$. Then the operator $T$ determines the a state on $CCR(\mathcal{H})$ which satisfies the conditions;

$$\varphi_T(W(f)) = e^{-1/2||(1+2T)^{1/2}f||}.$$  

This is called the quasi-free state with symbol $T$.

Consider the Hilbert space $\mathcal{H}_T = \mathcal{F}_+(\mathcal{H}) \otimes \mathcal{F}_+(\mathcal{H})$. There exists a representation $\pi_T$ of the $C^*$-algebra $CCR(\mathcal{H})$ on the Hilbert space $\mathcal{H}_T = \mathcal{F}_+(\mathcal{H}) \otimes \mathcal{F}_+(\mathcal{H})$, defined by the formula

$$\pi_T(W(f)) = W((1 + T)^{1/2}f) \otimes W(qT^{1/2}f),$$
where \( f \in \mathcal{H} \) and \( q \) is an anti-unitary operator on \( \mathcal{H} \) with \( q^2 = 1 \) (see [BR81], or [AW69]). In this representation, the state \( \varphi_T \) becomes the vector state

\[
\varphi_T(x) = \langle \Omega \otimes \Omega, \pi_T(x) \Omega \otimes \Omega \rangle,
\]

for \( x \in CCR(\mathcal{H}) \), and \( \mathcal{H}_T = F_+(\mathcal{H}) \otimes F_+(\mathcal{H}) = \pi_T(\mathcal{CCR}(\mathcal{H})) \Omega \otimes \Omega \) is the GNS Hilbert space, under the assumption that \( T \) is injective (and hence also has dense range). So \((\pi_T, \mathcal{H}_T, \Omega \otimes \Omega)\) is the GNS triple for the \( C^* \)-algebra \( CCR(\mathcal{H}) \) with respect to the state \( \varphi_T \). We write \( M_T = \{ \pi_T(\mathcal{CCR}(\mathcal{H})) \}' \).

Let \( \{ s_t \}_{t \geq 0} \) be the shift semigroup on \( \mathcal{H} \) and suppose that \( T \) commutes with \( s_t \) for all \( t \geq 0 \). Then \( M_T \) is a type III factor (see [Hol71]) and the CCR flow [Arv03] restricts to an \( E_0 \)-semigroup on \( M_T \), \( \alpha = \{ \alpha_t : t \geq 0 \} \) uniquely determined by the following condition:

\[
\alpha_t(\pi_T(W(f))) = \pi_T(W(s_t f)),
\]

for all \( f \in \mathcal{H}, t \geq 0 \). This \( E_0 \)-semigroup is called \textbf{CCR flow of rank dim} \( K \).

Note that if \( T = \lambda \mathbb{1} - \lambda \mathbb{1} \) with \( \lambda \in (0, 1) \), then it is well-known that \( M_\lambda = M_T \) is a type \( III_\lambda \) factor. Further, it has been mentioned in the section of examples of [BISSar] together with [MS12] that \( \{ \alpha_t : t \geq 0 \} \) is equi-modular and all these \( E_0 \)-semigroups on type \( III_\lambda \) factors are extendable.

**Remark 4.1.** Type \( III \) factors arising from quasi-free representation of \( CCR \) and \( CAR \) algebras with respect to the quasi-free states will always be hyperfinite factors (see [AW69]). In particular in both the cases we find hyperfinite \( III_\lambda \) factors for \( \lambda \in (0, 1) \) \( \setminus \{ \frac{1}{2} \} \). Since \( III_\lambda \) factors are unique for every \( \lambda \in (0, 1) \) \( \setminus \{ \frac{1}{2} \} \), so we have two families of \( E_0 \)-semigroups namely \( CAR \) flows and \( CCR \) flows on the same factor.

Now we have the following Corollary to the Theorem 3.17 regarding the cocycle conjugacy of \( CAR \) flows and \( CCR \) flows.

**Corollary 4.2.** The \( CAR \) and \( CCR \) flows arising from quasi-free states are not cocycle conjugate.

**Proof.** It has been proved in [BISSar] that \( CCR \) flows arising from these quasi-free states are extendable. By Theorem 3.17, \( CAR \) flows arising from quasi-free states are not extendable. But extendability of \( E_0 \)-semigroup is a cocycle conjugacy invariant, so the result follows. \( \square \)

**Remark 4.3.** This result is surprising, since on the type I factor \( CCR \) and \( CAR \) flows of same Arveson index are cocycle conjugate ([Arv03]).

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