COORDINATE-IN Variant INCREMENTAL LYAPUNOV FUNCTIONS

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Abstract. In this note, we propose coordinate-invariant notions of incremental Lyapunov function and provide characterizations of incremental stability in terms of existence of the proposed Lyapunov functions.

1. Control Systems and Stability Notions

1.1. Notation. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, real, positive, and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by $x_i$ the $i$-th element of $x$, and by $\|x\|$ the Euclidean norm of $x$; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of $f$ is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (\text{ess}\sup\{\|f(t)\| : t \geq 0\})$. Function $f$ is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define $f_\tau$ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f_\tau(t) = 0$ elsewhere; $f$ is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, $f_\tau$ is essentially bounded. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $K$ if it is strictly increasing and $\gamma(0) = 0$; $\gamma$ is said to belong to class $K_\infty$ if $\gamma \in K$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $K\mathcal{L}$ if, for each fixed $s$, the map $\beta(r,s)$ belongs to class $K_\infty$ with respect to $r$ and, for each fixed nonzero $r$, the map $\beta(r,s)$ is decreasing with respect to $s$ and $\beta(r,s) \to 0$ as $s \to \infty$. A function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $d(x,y) = 0$ if and only if $x = y$; ii) $d(x,y) = d(y,x)$; and iii) $d(x,z) \leq d(x,y) + d(y,z)$. For a set $\mathcal{A} \subseteq \mathbb{R}^n$, and any $x \in \mathbb{R}^n$, $d(x,\mathcal{A})$ denotes the point-to-set distance, defined by $d(x,\mathcal{A}) = \inf_{y \in \mathcal{A}} d(x,y)$.

1.2. Control Systems. The class of control systems with which we deal in this note is formalized in the following definition.

Definition 1.1. A control system is a quadruple:

$\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f),$

where:

- $\mathbb{R}^n$ is the state space;
- $U \subseteq \mathbb{R}^n$ is the input set;
- $\mathcal{U}$ is the set of all measurable functions of time from intervals of the form $]a,b[ \subseteq \mathbb{R}$ to $U$ with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x,u) - f(y,u)\| \leq Z \|x - y\|$ for all $x, y \in Q$ and all $u \in U$.

A curve $\xi : ]a,b[ \to \mathbb{R}^n$ is said to be a trajectory of $\Sigma$ if there exists $\nu \in \mathcal{U}$ satisfying:

(1.1) $\xi(t) = f(\xi(t), \nu(t)),$

for almost all $t \in ]a,b[$. We also write $\xi_{\nu}(t)$ to denote the point reached at time $t$ under the input $\nu$ from initial condition $x = \xi_{\nu}(0)$; this point is uniquely determined, since the assumptions on $f$ ensure existence and uniqueness of trajectories [Son98]. A control system $\Sigma$ is said to be forward complete if every trajectory is defined on an interval of the form $]a,\infty[$. We refer the interested readers to [AS99] for sufficient and necessary
conditions for a system to be forward complete. A control system \( \Sigma \) is said to be smooth if \( f \) is an infinitely differentiable function of its arguments.

1.3. Stability notions. Here, we recall the notions of incremental global asymptotic stability (\( \delta_3 \)-GAS) and incremental input-to-state stability (\( \delta_3 \)-ISS), presented in [ZTT1].

**Definition 1.2** ([ZTT1]). A control system \( \Sigma \) is incrementally globally asymptotically stable (\( \delta_3 \)-GAS) if it is forward complete and there exist a metric \( d \) and a KL function \( \beta \) such that for any \( t \in \mathbb{R}_0^+ \), any \( x, x' \in \mathbb{R}^n \) and any \( v \in U \) the following condition is satisfied:

\[
(1.2) \quad d(\xi_{xv}(t), \xi_{x'v}(t)) \leq \beta(d(x, x'), t).
\]

As defined in [Ang02], \( \delta \)-GAS requires the metric \( d \) to be the Euclidean metric. However, Definition 1.2 only requires the existence of a metric. We note that while \( \delta \)-GAS is not generally invariant under changes of coordinates, \( \delta_3 \)-GAS is.

**Definition 1.3** ([ZTT1]). A control system \( \Sigma \) is incrementally input-to-state stable (\( \delta_3 \)-ISS) if it is forward complete and there exist a metric \( d \), a KL function \( \beta \), and a \( K_\infty \) function \( \gamma \) such that for any \( t \in \mathbb{R}_0^+ \), any \( x, x' \in \mathbb{R}^n \), and any \( v, v' \in U \) the following condition is satisfied:

\[
(1.3) \quad d(\xi_{xv}(t), \xi_{x'v'}(t)) \leq \beta(d(x, x'), t) + \gamma(\|v - v'\|_\infty).
\]

By observing (1.2) and (1.3), it is readily seen that \( \delta_3 \)-ISS implies \( \delta_3 \)-GAS while the converse is not true in general. Moreover, whenever the metric \( d \) is the Euclidean metric, \( \delta_3 \)-ISS becomes \( \delta \)-ISS as defined in [Ang02]. We note that while \( \delta \)-ISS is not generally invariant under changes of coordinates, \( \delta_3 \)-ISS is.

Here, we introduce the following definition which was inspired by the notion of uniform global asymptotic stability with respect to sets in [LSW96].

**Definition 1.4.** A control system \( \Sigma \) is uniformly globally asymptotically stable (U\( \_\_ \)GAS) with respect to a set \( \mathcal{A} \) if it is forward complete and there exist a metric \( d \), and a KL function \( \beta \) such that for any \( t \in \mathbb{R}_0^+ \), any \( x \in \mathbb{R}^n \) and any \( v \in U \) the following condition is satisfied:

\[
(1.4) \quad d(\xi_{xv}(t), \mathcal{A}) \leq \beta(d(x, \mathcal{A}), t).
\]

We discuss in the next section characterizations of \( \delta_3 \)-GAS and \( \delta_3 \)-ISS in terms of existence of incremental Lyapunov functions.

1.4. Characterizations of incremental stability. This section contains characterizations of \( \delta_3 \)-GAS and \( \delta_3 \)-ISS in terms of existence of incremental Lyapunov functions. We start by defining the new notions of \( \delta_3 \)-GAS and \( \delta_3 \)-ISS Lyapunov functions.

**Definition 1.5.** Consider a control system \( \Sigma = (\mathbb{R}^n, U, \mathcal{U}, f) \) and a smooth function \( V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+ \). Function \( V \) is called a \( \delta_3 \)-GAS Lyapunov function for \( \Sigma \), if there exist a metric \( d \), \( K_\infty \) functions \( \underline{\alpha}, \overline{\alpha}, \underline{\gamma}, \overline{\gamma} \), and \( \kappa \in \mathbb{R}_+^\times \) such that:

1. for any \( x, x' \in \mathbb{R}^n \)
   \[
   \underline{\alpha}(d(x, x')) \leq V(x, x') \leq \overline{\alpha}(d(x, x'));
   \]
2. for any \( x, x' \in \mathbb{R}^n \) and any \( u \in U \)
   \[
   \frac{\partial}{\partial x} f(x, u) + \frac{\partial}{\partial x} f(x', u) \leq -\kappa V(x, x').
   \]

Function \( V \) is called a \( \delta_3 \)-ISS Lyapunov function for \( \Sigma \), if there exist a metric \( d \), \( K_\infty \) functions \( \underline{\alpha}, \overline{\alpha}, \underline{\gamma}, \overline{\gamma}, \sigma \), and \( \kappa \in \mathbb{R}_+^\times \) satisfying conditions (i) and:

1. for any \( x, x' \in \mathbb{R}^n \) and for any \( u, u' \in U \)
   \[
   \frac{\partial}{\partial x} f(x, u) + \frac{\partial}{\partial x} f(x', u') \leq -\kappa V(x, x') + \sigma(\|u - u'\|).
   \]
Remark 1.6. Condition (iii) of Definition 1.5 can be replaced by:
\[ \frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u') \leq -\rho(d(x,x')) + \sigma(||u-u'||), \]
where \( \rho \) is a \( K_\infty \) function. It is known that there is no loss of generality in considering \( \rho(d(x,x')) = \kappa V(x,y) \), by appropriately modifying the \( \delta_3\)-ISS Lyapunov function \( V \) (see Lemma 11 in [PW96]).

While \( \delta\)-GAS and \( \delta\)-ISS Lyapunov functions, as defined in [Ang02], require the metric \( d \) in condition (i) in Definition 1.5 to be the Euclidean metric, Definition 1.5 only requires the existence of a metric. We note that while \( \delta\)-GAS and \( \delta\)-ISS Lyapunov functions are not invariant under changes of coordinates in general, \( \delta_3\)-GAS and \( \delta_3\)-ISS Lyapunov functions are.

We now introduce the following definition which was inspired by the notion of uniform global asymptotic stability (UGAS) Lyapunov function in [LSW96].

Definition 1.7. Consider a control system \( \Sigma \), a set \( A \), and a smooth function \( V : \mathbb{R}^n \to \mathbb{R}_+^+ \). Function \( V \) is called a \( U_3 \) GAS Lyapunov function, with respect to \( A \), for \( \Sigma \), if there exist a metric \( d \), \( K_\infty \) functions \( \underline{\sigma}, \overline{\sigma} \), and \( \kappa \in \mathbb{R}_+^+ \) such that:

(i) for any \( x \in \mathbb{R}^n \)
\[ \underline{d}(d(x,A)) \leq V(x) \leq \overline{d}(d(x,A)); \]
(ii) for any \( x \in \mathbb{R}^n \) and any \( u \in U \)
\[ \frac{\partial V}{\partial x} f(x,u) \leq -\kappa V(x). \]

The following theorem characterizes \( U_3 \) GAS in terms of existence of a \( U_3 \) GAS Lyapunov function.

Theorem 1.8. Consider a control system \( \Sigma \) and a set \( A \). If \( U \) is compact and \( d \) is a metric such that the function \( \psi(x) = d(x,y) \) is continuous\(^{1}\) for any \( y \in \mathbb{R}^n \) then the following statements are equivalent:

1. \( \Sigma \) is forward complete and there exists a \( U_3 \) GAS Lyapunov function with respect to \( A \), equipped with the metric \( d \).
2. \( \Sigma \) is \( U_3 \) GAS with respect to \( A \), equipped with the metric \( d \).

Proof. First we show that the function \( \phi(x) = d(x,A) \) is a continuous function with respect to the Euclidean metric. Assume \( \{x_n\}_{n=1}^\infty \) is a converging sequence in \( \mathbb{R}^n \) with respect to the Euclidean metric, implying: \( x_n \to x^* \) as \( n \to \infty \) for some \( x^* \in \mathbb{R}^n \). By triangle inequality, we have:
\[ d(x^*,y) \leq d(x^*,x_n) + d(y,x_n), \]
for any \( n \in \mathbb{N} \) and \( y \in A \). Using inequality (1.5), we obtain:
\[ \phi(x^*) = \inf_{y \in A} d(x^*,y) = \inf_{y \in A} \{ d(x^*,x_n) + d(y,x_n) \} \]
\[ \leq \inf_{y \in A} d(y,x_n) + \inf_{y \in A} d(x^*,x_n) = \phi(x_n) + d(x^*,x_n). \]

Using inequality (1.6) and the continuity assumption on \( d \), we obtain:
\[ \phi(x^*) \leq \lim_{n \to \infty} \inf \phi(x_n), \]
for any \( n \in \mathbb{N} \), where limit inferior exists because of greatest lower bound property of real numbers [RRA09].

By doing the same analysis, we have:
\[ \phi(x^*) \geq \lim_{n \to \infty} \sup \phi(x_n), \]

1Here, continuity is understood with respect to the Euclidean metric.
for any \( n \in \mathbb{N} \). Using inequalities (1.7) and (1.8), we obtain:

\[
\phi(x^*) = \lim_{n \to \infty} \phi(x_n),
\]

implying that \( \phi \) is a continuous function. Since \( \phi(x) = d(x, A) \) is a continuous function, by choosing \( \omega_1(x) = \omega_2(x) = d(x, A) \) and using Theorem 1 in \cite{TP00}, the proof completes.

Before showing the main results, we need the following technical lemma, inspired by Lemma 2.3 in \cite{Ang02}.

**Lemma 1.9.** Consider a control system \( \Sigma = (\mathbb{R}^n, U, \mathcal{U}, f) \). If \( \Sigma \) is \( \delta_3 \)-GAS, then the control system \( \hat{\Sigma} = (\mathbb{R}^{2n}, U, \mathcal{U}, \hat{f}) \), where \( \hat{f}(\zeta, \upsilon) = [f(\xi_1, \upsilon)^T, f(\xi_2, \upsilon)^T]^T \), and \( \zeta = [\xi_1^T, \xi_2^T]^T \), is \( U_3 \)-GAS with respect to the diagonal set \( \Delta \), defined by:

\[
\Delta = \left\{ z \in \mathbb{R}^{2n} \mid \exists x \in \mathbb{R}^n : z = [x^T, x'^T]^T \right\}.
\]

**Proof.** Since \( \Sigma \) is \( \delta_3 \)-GAS, there exists a metric \( d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+ \) such that property (1.2) is satisfied. Now we define a new metric \( \hat{d} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}_0^+ \) by:

\[
\hat{d}(z, z') = d(x_1, x_1') + d(x_2, x_2'),
\]

for any \( z = [x_1^T, x_2^T]^T \in \mathbb{R}^{2n} \) and \( z' = [x_1'^T, x_2'^T]^T \in \mathbb{R}^{2n} \). It can be readily checked that \( \hat{d} \) satisfies all three conditions of a metric. Now we need to show that \( \hat{d}(z, \Delta) \), for any \( z = [x_1^T, x_2^T]^T \in \mathbb{R}^{2n} \), is proportional to \( d(x_1, x_2) \). We have:

\[
\hat{d}(z, \Delta) = \inf_{z' \in \Delta} \hat{d}(z, z') = \inf_{x' \in \mathbb{R}^n} \hat{d}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x' \\ x' \end{bmatrix}\right)
\]

\[
= \inf_{x' \in \mathbb{R}^n} (d(x_1, x') + d(x_2, x')) \leq d(x_1, x_2).
\]

Since \( d \) is a metric, by using the triangle inequality, we have: \( d(x_1, x_2) \leq d(x_1, x') + d(x_2, x') \) for any \( x' \in \mathbb{R}^n \), implying that \( d(x_1, x_2) \leq \hat{d}(z, \Delta) \). Hence, using (1.12), one obtains:

\[
d(x_1, x_2) \leq \hat{d}(z, \Delta) \Rightarrow d(x_1, x_2) = \hat{d}(z, \Delta).
\]

Using equality (1.13) and property (1.2), we have:

\[
\hat{d}(\zeta_{z\upsilon}(t), \Delta) = d(\xi_{x_1\upsilon}(t), \xi_{x_2\upsilon}(t)) \leq \beta(d(x_1, x_2), t) = \beta\left(\hat{d}(z, \Delta), t\right),
\]

for any \( t \in \mathbb{R}_0^+ \), and \( \upsilon \in \mathcal{U} \), where \( \zeta_{z\upsilon} = [\xi_{x_1\upsilon}^T, \xi_{x_2\upsilon}^T]^T \), and \( z = [x_1^T, x_2^T]^T \). Hence, \( \hat{\Sigma} \) is \( U_3 \)-GAS with respect to \( \Delta \). \( \square \)

We can now state one of the main results, providing characterization of \( \delta_3 \)-GAS in terms of existence of a \( \delta_3 \)-GAS Lyapunov function.

**Theorem 1.10.** Consider a control system \( \Sigma \). If \( U \) is compact and \( d \) is a metric such that the function \( \psi(x) = d(x, y) \) is continuous\footnote{Here, continuity is understood with respect to the Euclidean metric.} for any \( y \in \mathbb{R}^n \) then the following statements are equivalent:

1. \( \Sigma \) is forward complete and there exists a \( \delta_3 \)-GAS Lyapunov function, equipped with the metric \( d \).
2. \( \Sigma \) is \( \delta_3 \)-GAS, equipped with the metric \( d \).
Without loss of generality we can assume \( \alpha \). Using Lemma 1.9 and the proposed metric \( \hat{d} \) function \( \hat{d}(z, z') \) is also continuous for any \( z' \in \mathbb{R}^{2n} \), where the metric \( \hat{d} \) was defined in Lemma 1.9. Using Theorem 1.8, we conclude that there exists a \( \mathcal{U}_3 \)GAS Lyapunov function \( V : \mathbb{R}^{2n} \to \mathbb{R}_0^+ \), with respect to \( \Delta \), for \( \hat{d} \). Thanks to the special form of the function \( \hat{d} \) and using the equality (1.13), the function \( V \) satisfies:

1. \( \alpha(d(x, x')) \leq V(x, x') \leq \alpha(d(x, x')) \);
2. \( \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq -\kappa V(x, x') \),

for some \( K_{\infty} \) functions \( \alpha, \beta \) and some \( \kappa \in \mathbb{R}_0^+ \). Hence, \( V \) is a \( \mathcal{U}_3 \)-GAS Lyapunov function for \( \Sigma \). \( \square \)

Before providing characterization of \( \mathcal{U}_3 \)-ISS in terms of existence of a \( \mathcal{U}_3 \)-ISS Lyapunov function, we need the following technical lemma, inspired by Proposition 5.3 in [Ang02]. To state the following results, we need to define the function:

\[
\text{sat}_U(u) = \begin{cases} u & \text{if } u \in U, \\ \arg\min_{u' \in U} \|u' - u\| & \text{if } u \notin U. \end{cases}
\]

As explained in [Ang02], by assuming \( U \) is closed and convex and since \( ||\cdot|| : \mathbb{R}^m \to \mathbb{R}_0^+ \) is a proper, convex function, the definition (1.15) is well-defined and the minimizer of \( ||u' - u|| \) with \( u' \in U \) is unique. Moreover, by convexity of \( U \) we have:

\[
||\text{sat}_U(u_1) - \text{sat}_U(u_2)|| \leq ||u_1 - u_2||, \forall u_1, u_2 \in \mathbb{R}^m.
\]

**Lemma 1.11.** Consider a control system \( \Sigma = (\mathbb{R}^n, U, U, f) \), where \( U \) is closed and convex. If \( \Sigma \) is \( \mathcal{U}_3 \)-ISS, equipped with a metric \( d \) such that \( \psi(x) = d(x, y) \) is continuous for any \( y \in \mathbb{R}^n \), then there exists a \( K_{\infty} \) function \( \rho \) such that the control system \( \hat{\Sigma} = (\mathbb{R}^{2n}, D, D, \hat{f}) \) is \( \mathcal{U}_3 \)GAS with respect to the diagonal set \( \Delta \), where:

\[
\hat{f}(\xi, \omega) = \begin{bmatrix} f(\xi_1, \text{sat}_U(\omega_1 + \rho(d(\xi_1, \xi_2)\omega_2))) \\ f(\xi_2, \text{sat}_U(\omega_1 - \rho(d(\xi_1, \xi_2)\omega_2))) \end{bmatrix},
\]

\( \xi = [\xi_1^T, \xi_2^T]^T \), \( D = U \times \mathcal{B}(0) \), and \( \omega = [\omega_1^T, \omega_2^T]^T \).

**Proof.** The proof was inspired by the proof of Proposition 5.3 in [Ang02]. Since \( \Sigma \) is \( \mathcal{U}_3 \)-ISS, equipped with the metric \( d \), there exists some \( KL \) function \( \beta \) and \( K_{\infty} \) function \( \gamma \) such that:

\[
d(\xi_{x,t}(\cdot), \xi_{x',t}(\cdot)) \leq \max\{\beta(d(x, x'), t), \gamma(||v - v'||\infty)\}
\]

Note that inequality (1.18) is a straightforward consequence of inequality (1.13) (see Remark 2.5 in [SW95]). Using Lemma 1.9 and the proposed metric \( \hat{d} \) in (1.11), we have: \( d(x, x') = \hat{d}(z, \Delta) \), where \( z = [x^T, x'^T]^T \). Without loss of generality we can assume \( \alpha(r) = \beta(r, 0) > r \) for any \( r \in \mathbb{R}_0^+ \). Let \( \rho \) be a \( K_{\infty} \) function satisfying \( \rho(r) \leq \frac{1}{2}r^{-1} \circ (\alpha^{-1}(r)/4) \).

Now we show that

\[
\gamma\left(\left\|2\omega_2(t)\rho(\hat{d}(\xi_{x\omega}(t), \Delta))\right\|\right) \leq \hat{d}(z, \Delta)/2,
\]

for any \( t \in \mathbb{R}_0^+ \), any \( z \in \mathbb{R}^{2n} \), and any \( \omega \in D \). Since \( \gamma \) is a \( K_{\infty} \) function and \( \omega_2(t) \in \mathcal{B}(0) \), it is enough to show

\[
\gamma\left(2\rho(\hat{d}(\xi_{x\omega}(t), \Delta))\right) \leq \hat{d}(z, \Delta)/2.
\]

Since

\[
\gamma\left(2\rho(\hat{d}(\xi_{x\omega}(0), \Delta))\right) = \gamma\left(2\rho(\hat{d}(\xi_{x}(0), \Delta))\right) \leq \alpha^{-1}\left(\hat{d}(z, \Delta)/4\right) \leq \hat{d}(z, \Delta)/4,
\]

\( \mathcal{D} \) is the set of all measurable, locally essentially bounded functions of time from intervals of the form \( [a, b] \subseteq \mathbb{R} \) to \( D \) with \( a < 0 \) and \( b > 0 \).
and \( \varphi(z) = \hat{d}(z, \Delta) \) is a continuous function (see proof of Theorem 1.8), then for all \( t \in \mathbb{R}_0^+ \) small enough, we have \( \gamma \left( 2\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \right) \leq \hat{d}(z, \Delta)/4 \). Now, let

\[
(1.22) \quad t_1 = \inf \left\{ t > 0 \mid \gamma \left( 2\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \right) > \hat{d}(z, \Delta)/2 \right\}.
\]

Clearly \( t_1 > 0 \). We will show that \( t_1 = \infty \). Now, assume by contradiction that \( t_1 < \infty \). Therefore, the inequality (1.20) holds for all \( t \in [0, t_1) \). Hence, for almost all \( t \in [0, t_1) \), one obtains:

\[
(1.23) \quad \gamma \left( \left\| 2\omega_2(t)\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \right\| \right) \leq \gamma \left( 2\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \right) \leq \hat{d}(z, \Delta)/2 < \alpha \left( \hat{d}(z, \Delta) \right)/2.
\]

Let \( v \) and \( v' \) be defined as:

\[
v(t) = \text{sat}_0 \left( \omega_1(t) + \rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \omega_2(t) \right),
\]

\[
v'(t) = \text{sat}_0 \left( \omega_1(t) - \rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \omega_2(t) \right).
\]

By using (1.10), we obtain: \( \| v(t) - v'(t) \| \leq \| 2\omega_2(t)\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \| \). Using (1.18) and (1.23), we have:

\[
(1.24) \quad \hat{d}(\zeta_{\omega}(t), \Delta) = d \left( \xi_{xv}(t), \xi_{x'v}(t) \right) \leq \beta \left( d(x, x'), 0 \right) = \beta \left( \hat{d}(z, \Delta), 0 \right) = \alpha \left( \hat{d}(z, \Delta) \right),
\]

for any \( t \in [0, t_1] \) and any \( z = [x^T, x'^T]^T \in \mathbb{R}_0^{2n} \) which implies that \( \gamma \left( 2\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \right) \leq \hat{d}(z, \Delta)/4 \), contradicting the definition of \( t_1 \). Therefore, \( t_1 = \infty \) and inequality (1.19) is proved for all \( t \in \mathbb{R}_0^+ \). Therefore, using (1.18) and (1.19), we obtain:

\[
(1.25) \quad \hat{d}(\zeta_{\omega}(t), \Delta) = d \left( \xi_{xv}(t), \xi_{x'v}(t) \right) \leq \max \left\{ \beta \left( d(x, x'), t \right), \gamma \left( \| v - v' \|_\infty \right) \right\}
\]

\[
\leq \max \left\{ \beta \left( d(x, x'), t \right), \gamma \left( \| 2\omega_2(t)\rho \left( \hat{d}(\zeta_{\omega}(t), \Delta) \right) \|_\infty \right) \right\}
\]

\[
\leq \max \left\{ \beta \left( \hat{d}(z, \Delta), t \right), \hat{d}(z, \Delta)/2 \right\},
\]

for any \( z = [x^T, x'^T]^T \in \mathbb{R}_0^{2n} \), any \( \omega \in \mathcal{D} \), and any \( t \in \mathbb{R}_0^+ \). Since \( \beta \) is a \( K \mathcal{L} \) function, it can be readily seen that for each \( r > 0 \) if \( \hat{d}(z, \Delta) \leq r \), then there exists some \( T_r \geq 0 \) such that for any \( t \geq T_r \), \( \beta \left( \hat{d}(z, \Delta), t \right) \leq r/2 \) and, hence, \( \hat{d}(\zeta_{\omega}(t), \Delta) \leq r/2 \). For any \( \varepsilon \in \mathbb{R}^+ \), let \( k \) be a positive integer such that \( 2^{-k}r < \varepsilon \). Let \( r_1 = r \) and \( r_i = r_{i-1}/2 \) for \( i \geq 2 \), and let \( \tau = T_{r_1} + T_{r_2} + \cdots + T_{r_k} \). Then, for \( t \geq \tau \), we have \( \hat{d}(\zeta_{\omega}(t), \Delta) \leq 2^{-k}r < \varepsilon \) for all \( \hat{d}(z, \Delta) \leq r \), all \( \omega \in \mathcal{D} \), and all \( t \geq \tau \). Therefore, it can be concluded that the set \( \Delta \) is a uniform global attractor for the control system \( \hat{\Sigma} \). Furthermore, since \( \hat{d}(\zeta_{\omega}(t), \Delta) \leq \beta \left( \hat{d}(z, \Delta), 0 \right) \) for all \( t \in \mathbb{R}_0^+ \), all \( z \in \mathbb{R}_0^{2n} \), and all \( \omega \in \mathcal{D} \), the control system \( \hat{\Sigma} \) is uniformly globally stable and as showed in [TP00], it is \( U_{3}\text{GAS} \).

The next theorem provide characterization of \( \delta_3\text{-ISS} \) in terms of existence of a \( \delta_3\text{-ISS} \) Lyapunov function.

**Theorem 1.12.** Consider a control system \( \Sigma \). If \( U \) is compact and convex and \( d \) is a metric such that the function \( \psi(x) = d(x, y) \) is continuous\footnote{Here, continuity is understood with respect to the Euclidean metric.} for any \( y \in \mathbb{R}^n \) then the following statements are equivalent:

1. \( \Sigma \) is forward complete and there exists a \( \delta_3\text{-ISS} \) Lyapunov function, equipped with metric \( d \).
2. \( \Sigma \) is \( \delta_3\text{-ISS} \), equipped with metric \( d \).

**Proof.** The proof from (1) to (2) has been showed in Theorem 2.6 in [ZM11], even in the absence of the compactness and convexity assumptions on \( U \) and the continuity assumption on \( d \). We now prove that (2) implies (1). As we proved in Lemma 1.11 since \( \Sigma \) is \( \delta_3\text{-ISS} \), it implies that the control system \( \hat{\Sigma} \), defined in Lemma 1.11 is \( U_{3}\text{GAS} \). Since \( \psi(x) = d(x, y) \) is continuous for any \( y \in \mathbb{R}^n \), it can be easily verified that
\( \hat{\psi}(z) = \hat{d}(z, z') \) is continuous for any \( z' \in \mathbb{R}^n \), where the metric \( \hat{d} \) was defined in Lemma 1.9. Using Theorem 1.8, we conclude that there exists a \( U \_3 \) GAS Lyapunov function \( V \), with respect to \( \Delta \), for \( \hat{\Sigma} \). Thanks to the special form of \( \hat{\Sigma} \) and using the equality (1.13), the function \( V \) satisfies:

(1.26) \[ \alpha(d(x, x')) \leq V(x, x') \leq \beta(d(x, x')) \]

for some \( K_\infty \) functions \( \alpha, \beta \), any \( x, x' \in \mathbb{R}^n \), and

(1.27) \[ \frac{\partial V}{\partial x} f(x, \text{sat}_0(d_1 + \rho(d(x, x')))d_2) + \frac{\partial V}{\partial x'} f(x', \text{sat}_0(d_1 - \rho(d(x, x')))d_2) \leq -\kappa V(x, x') \]

for some \( \kappa \in \mathbb{R}^+ \) and any \( [d_1', d_2']^T \in D \). By choosing \( d_1 = (u + u')/2 \) and \( d_2 = (u - u')/(2\rho(d(x, x'))) \) for any \( u, u' \in U \), it can be readily checked that \( [d_1', d_2']^T \in U \times B_1(0) \), whenever \( 2\rho(d(x, x')) \geq \|u - u'\| \). Hence, using (1.27), we have:

(1.28) \[ \varphi(d(x, x')) \geq \|u - u'\| \Rightarrow \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\kappa V(x, x') \]

where \( \varphi(r) = 2\rho(r) \). As showed in Remark 2.4 in [SW95], there is no loss of generality in modifying inequality (1.28) to

(1.29) \[ \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\hat{\kappa} V(x, x') + \gamma(\|u - u'\|) \]

for some \( K_\infty \) function \( \gamma \) and some \( \hat{\kappa} \in \mathbb{R}^+ \), which completes the proof. \( \square \)

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