A SDFEM for Singularly Perturbed System of Convection-Diffusion Delay Differential Equations

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Abstract. In this article, we consider a system of convection-diffusion equations with delay terms. When a parameter multiplying the second order derivatives in the equations is small, boundary layers as well as interior layers appear in their solutions. A numerical method based on finite element and Shishkin mesh is presented. We derive an error estimate of order \(O(N^{-1} \log^2 N)\) in the energy norm with respect to the parameter using the streamline-diffusion finite element method. Numerical experiments are also presented to support our theoretical results.

1. Introduction

Singularly Perturbed Delay Differential Equations (SPDDEs) appear in several branches of applied mathematics. Many researchers\([1, 2, 3, 4]\) are attracted by the analytical and numerical treatment of these equations. A good number of articles have been developed in the past three decades on nonclassical methods which cover mostly second order singular perturbation problems. A few number of authors\([5, 6, 7, 8, 9]\) developed numerical methods for singularly perturbed system of ordinary differential equations. A numerical method for a system of singularly perturbed reaction diffusion equations with discontinuous source term is discussed by\([10]\). In\([11]\) authors claimed that the hp version of finite element method on an approximately designed mesh yields robust exponential convergence rate. Several finite element methods are proposed by\([12]\) to solve singularly perturbed system of reaction diffusion problems having two overlapping subdomains. In this paper we focus on the following Dirichlet’s boundary value problem (BVP) for singularly perturbed system of convection-diffusion delay equations:

\[
P_{\varepsilon}\overline{y}: \begin{cases} 
-\varepsilon y''_1(x) + a_{11}y'_1 + \sum_{k=1}^{2} [b_{1k}(x)y_k(x) + c_{1k}y_k(x-1)] = f_1(x), \forall x \in \Omega, \\
-\varepsilon y''_2(x) + a_{22}y'_2 + \sum_{k=1}^{2} [b_{2k}(x)y_k(x) + c_{2k}y_k(x-1)] = f_2(x),
\end{cases}
\]  

(1.1)

where \(P_{\varepsilon} = (P_{1,\varepsilon}, P_{2,\varepsilon})\) and \(f_1, f_2\) are sufficiently differentiable functions on \(\overline{\Omega}\), \(\Omega = [0, 2]\), \(\Omega^- = (0, 1), \Omega^+ = (1, 2)\) and \(y_i(x) = 0, x \in [-1, 0]\), for \(i = 1, 2\), \(y_i(0) = 0, y_i(2) = 0\), \(0 \leq \rho_1 \leq (b_{11} - a_{11}^2/2), 0 \leq \rho_2 \leq (b_{22} - a_{22}^2/2)\). Throughout this paper, \(C, C_1\) denote the generic constants that are independent of the parameter \(\varepsilon\) and \(N\). We also assume \(\varepsilon \leq CN^{-1}\). For our later analysis, it is useful to have a decomposition of \(\overline{y}\), \(\overline{y} = \overline{v} + \overline{w}\), where \(\overline{v} = (v_1, v_2)\) and \(\overline{w} = (w_1, w_2)\).
2. Analytical Results

2.1. Equivalent Problem

Consider the following BVP: Find \( \pi = (u_1, u_2)^T \) such that

\[
P_{i,\varepsilon} \overline{y} := \begin{cases}
-\varepsilon y''_1(x) + a_{11} y'_1(x) + \sum_{k=1}^{2} b_{1k}(x) y_k(x) = f_1(x), x \in \Omega^-, \\
-\varepsilon y'_1(x) + a_{11} y_1 + \sum_{k=1}^{2} [b_{1k}(x) y_k(x) + c_{1k}(x) y_k(x-1)] = f_1(x), x \in \Omega^+,
\end{cases}
\]

(2.1)

\[
P_{2,\varepsilon} \overline{y} := \begin{cases}
-\varepsilon y''_2(x) + a_{22} y'_2(x) + \sum_{k=1}^{2} b_{2k}(x) y_k(x) = f_2(x), x \in \Omega^-, \\
-\varepsilon y'_2(x) + a_{22} y_2 + \sum_{k=1}^{2} [b_{2k}(x) y_k(x) + c_{2k}(x) y_k(x-1)] = f_2(x), x \in \Omega^+,
\end{cases}
\]

(2.2)

Then the reduced problem is as follows:

\[
P_{i,0} \overline{y}_0 := \begin{cases}
a_{11} y'_1,0(x) + \sum_{k=1}^{2} b_{1k}(x) y_k,0(x) = f_1, x \in \Omega^- \\
a_{11} y_1,0 + \sum_{k=1}^{2} b_{1k}(x) y_k,0(x) + c_{1k}(x) y_k,0(x-1) = f_1, x \in \Omega^+,
\end{cases}
\]

(2.3)

\[
P_{2,0} \overline{y}_0 := \begin{cases}
a_{22} y'_2,0(x) + \sum_{k=1}^{2} b_{2k}(x) y_k,0(x) = f_2, x \in \Omega^- \\
a_{22} y_2,0 + \sum_{k=1}^{2} b_{2k}(x) y_k,0(x) + c_{2k}(x) y_k,0(x-1) = f_2, x \in \Omega^+,
\end{cases}
\]

(2.4)

Theorem 2.1. (Maximum Principle)[13] If \( \overline{y} = (y_1, y_2) \in C^0(\Omega) \cap C^2(\Omega^*), i = 1, 2 \) is any function satisfying conditions \( y_i(0) \geq 0, y_i(2) \geq 0, P_{i,\varepsilon} \overline{y}(x) \geq 0 \ \forall x \in \Omega^* \), then \( y_i(1+) - y_i(1-) \geq 0 \), \( y_i(x) \geq 0, \forall x \in \Omega, i = 1, 2 \).

Note: Since the operator \( P_{\varepsilon} \) satisfies the above maximum principle, the solution of the BVP(2.1)-(2.1) is unique if it exists. Further if \( \overline{y} \) is the solution of the problem (1.1), then it is also the solution of (2.1)-(2.1)

Corollary 2.2. [13] If the function \( y_i \in Y \), then \( |y_i| \leq C \sup \{ |y_i(0)|, |y_i(2)|, \sup_{\Omega^*} |P_{i,\varepsilon} \overline{y}(x)| \} \), for \( i = 1, 2 \).

Theorem 2.3 (Derivative Estimates). [16] The solution \( \overline{y} \) of the BVP (1.1) can be decomposed as \( \overline{y} = \overline{v} + \overline{w} \), where \( \overline{v} \) is the regular component and \( \overline{w} \) is the singular component. Further the singular component can be decomposed as \( \overline{w} = \overline{w}_1 + \overline{w}_2 \) and the regular component satisfies

\[ |v_i^{(k)}(x)| \leq C \varepsilon^{2-k}, k = 0, 1, 2, 3, i = 1, 2. \]

where the singular components \( w_1 \) and \( w_2 \) satisfy

\[ |w_1^{(k)}(x)| \leq C \varepsilon^{k} \exp\left(\frac{-\beta(2 - x)}{\varepsilon}\right), \ x \in \Omega^*, \ k = 0, 1, 2, 3, i = 1, 2 \]

\[ |w_2^{(k)}(x)| \leq \begin{cases} C \varepsilon^{-k+1} \exp\left(\frac{-\beta(1-x)}{\varepsilon}\right), & x \in \Omega^- \\ C \varepsilon^{-k+1}, & x \in \Omega^+, \end{cases} \ k = 0, 1, 2, 3, i = 1, 2 \]

2.2. Auxiliary Problem

An auxiliary problem of (1.1) is as follows: find \( \overline{\pi} = (\overline{y}_1, \overline{y}_2) \) such that

\[
\begin{cases}
-\varepsilon \overline{y}_1''(x) + a_{11} \overline{y}_1'(x) + b_{11}(x) \overline{y}_1(x) + b_{12} \overline{y}_2(x) = g_1^i(x), \ \forall x \in \Omega \\
-\varepsilon \overline{y}_2''(x) + a_{22} \overline{y}_2'(x) + b_{21}(x) \overline{y}_1(x) + b_{22} \overline{y}_2(x) = g_2^i(x), \ \forall x \in \Omega
\end{cases}
\]

(2.5)

where

\[
g_1^i(x) = \begin{cases} f_1(x), & x \in \Omega^- \\
f_1(x) - c_{11}(x)y_{1,0}(x-1) - c_{12}y_{2,0}(x-1), & x \in \Omega^+
\end{cases}
\]

\[
g_2^i(x) = \begin{cases} f_2(x), & x \in \Omega^- \\
f_1(x) - c_{21}(x)y_{1,0}(x-1) - c_{22}y_{2,0}(x-1), & x \in \Omega^+
\end{cases}
\]

Theorem 2.4. [14, Theorem 4.4] If \( \overline{y} \) and \( \overline{\pi} \) are solutions of (1.1) and (2.5), then \( |y_i(x) - \overline{y}_i(x)| \leq C \varepsilon \), for \( i = 1, 2 \).
3. Finite Element Formulation

The weak formulation of the above problem (2.5) is as follows: Find $\tilde{y} \in V^2 = (H_0^1(\Omega))^2$ such that

$$ A_1(\tilde{y}, v) = f_1^*(v), \ \forall v \in (H_0^1(\Omega))^2 $$

$$ A_2(\tilde{y}, v) = f_2^*(v) $$

where

$$ A_1(\tilde{y}, v) = \langle \varepsilon \tilde{y}_1, v'_1 \rangle + \langle a_{11}\tilde{y}_1, v_1 \rangle + \langle b_{11}\tilde{y}_1(x), v_1 \rangle + \langle b_{12}(x)\tilde{y}_2, v_2 \rangle, $$

$$ A_2(\tilde{y}, v) = \langle \varepsilon \tilde{y}_2, v'_2 \rangle + \langle a_{22}\tilde{y}_2, v_2 \rangle + \langle b_{21}\tilde{y}_1(x), v_1 \rangle + \langle b_{22}\tilde{y}_2, v_2 \rangle, $$

$$ f_1^*(v) = \langle g_1, v_1 \rangle, \quad f_2^*(v) = \langle g_2, v_2 \rangle. $$

Here $H_0^1(\Omega)$ denotes the usual Sobolev space and $\langle *, * \rangle$ is the inner product on $L^2(\Omega)$. Now we combine (3.1) and (3.2) and get a single bilinear form as follows:

$$ A(\tilde{y}, v) = f^*(v) $$

with $A(\tilde{y}, v) = A_1(\tilde{y}, v) + A_2(\tilde{y}, v), f^*(v) = f_1^*(v) + f_2^*(v)$. Now we define a norm on $(H_0^1(\Omega))^2$ associated with the bilinear form $A(\tilde{y}, v)$ called energy norm as

$$ |||\tilde{y}||| = \varepsilon(|||\tilde{y}_1\tilde{y}_1||| + ||\tilde{y}_2\tilde{y}_2||| + ||\varepsilon\tilde{y}_1\tilde{y}_1||| + ||\varepsilon\tilde{y}_2\tilde{y}_2|||)^{1/2} $$

where $||y||_0 = ||y||^{1/2}$ is the standard norm on $L^2(\Omega), ||y|| = ||y||_0$ is usual semi norm on $(H_0^1(\Omega))^2$. We also use the notation $||\tilde{y}||_0 = (||\tilde{y}_1||_0^2 + ||\tilde{y}_2||_0^2)^{1/2}$. A is a bilinear functional defined on $(H_0^1(\Omega))^2$. Further we have to prove that it is coercive with respect to $|||*|||.$

**Lemma 3.1.** A bilinear functional $A$ satisfies the coercive property with respect to $|||*|||.$

**Proof.**

$$ A(\tilde{y}, v) = \varepsilon\tilde{y}_1\tilde{y}_1 + \tilde{y}_2\tilde{y}_2 + b_{11}\tilde{y}_1, v_1 + b_{12}(x)\tilde{y}_2, v_2 $$

$$ + \varepsilon\tilde{y}_2\tilde{y}_2 + b_{21}\tilde{y}_1(x), v_1 + b_{22}\tilde{y}_2, v_2 $$

$$ = \varepsilon(|||\tilde{y}_1\tilde{y}_1||| + ||\tilde{y}_2\tilde{y}_2||| + \int_0^2 b_{11}\frac{\tilde{y}_1^2}{2}dx + \int_0^2 b_{12}\tilde{y}_1\tilde{y}_2dx $$

$$ + \int_0^2 b_{22}\frac{\tilde{y}_2^2}{2}dx + \int_0^2 b_{22}\tilde{y}_1\tilde{y}_2dx $$

$$ \geq \varepsilon(|||\tilde{y}_1\tilde{y}_1||| + ||\tilde{y}_2\tilde{y}_2||| + \rho_1\int_0^2 \tilde{y}_1^2dx + \rho_2\int_0^2 \tilde{y}_2^2dx $$

$$ \geq \varepsilon(|||\tilde{y}_1\tilde{y}_1||| + ||\tilde{y}_2\tilde{y}_2||| + \rho(|||\tilde{y}_1\tilde{y}_1||| + ||\tilde{y}_2\tilde{y}_2|||)) \geq |||\tilde{y}\tilde{y}|||^{1/2}. $$

where $\rho = \min\{\rho_1, \rho_2\}.$

A is coercive. Moreover it is continuous in the energy norm and $f^*$ is a bounded linear functional on $(H_0^1(\Omega))^2$. By Lax-milgram theorem, we conclude that the problem (3.3) has a unique solution.
4. Discretization

4.1. Mesh points and properties

Let $N$ be a positive integer and $\lambda = \min\{ \frac{1}{2}, \frac{\epsilon_{\text{th}} \log N}{\beta} \}$, $\tau_0 \geq 1$. Further, $\Omega_{\epsilon} = \{ x_i \}_{i=0}^N$. The mesh will be equidistant on $\Omega_s$ where $\Omega_s = (0, 1 - \lambda) \cup (1, 2 - \lambda)$ and graded on $\Omega_0$ where $\Omega_0 = (1 - \lambda, 1) \cup (2 - \lambda, 2)$. We choose the transition points to be $x_{N/4} = 1 - \lambda$, $x_{3N/4} = 2 - \lambda$. The mesh points are

$$x_i = \begin{cases} \frac{4i(1-\lambda)}{N}, & i = 0, \ldots, N/4 \\ 1 - \frac{7\epsilon}{N} \phi(t_i), & i = N/4 + 1, \ldots, N/2 \\ 1 + \frac{4(1-\lambda)(i-N/2)}{N}, & i = N/2 + 1, \ldots, 3N/4 \\ 2 - \frac{7\epsilon}{N} \phi(t_i), & i = 3N/4 + 1, \ldots, N \end{cases}$$

where $t_i = iN^{-1}$, $\phi = -\log(\psi)$ and $\psi = e^{-2(1-2\lambda) \log N}$. On coarse part $\Omega_s$ we have $h_i \leq CN^{-1}$ where $h_i = x_i - x_{i-1}$. It is obvious that on the layer part $\Omega_0$ of the Shishkin mesh, $h_i \leq C\epsilon N^{-1} \log N$. Let $\Omega_{\epsilon} = \{ x_0, x_1, \ldots, x_N \}$ be a set of mesh points for some positive integer $N$. For $i \in \{ 1, 2, \ldots, N \}$, we set $h_i = x_i - x_{i-1}$ to be the local mesh step size. Let $h_i = (h_i + h_{i+1})/2$. Let $v_h \subset H_0^1(\Omega)$ be the space of piecewise linear functions of $V_h$ given by

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & x \in (x_{i-1}, x_i), \\ \frac{x_{i+1}-x}{h_{i+1}}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}). \end{cases}$$

4.2. Discretization of weak problem

Then our discretization of (3.3) in this section is: find $\tilde{y}_h \in V_h^0$ such that

$$A_h(\tilde{y}_h, v) = f^*(v), \forall v \in V_h, \hspace{1cm} (4.1)$$

$$A_h(\tilde{y}_h, v) = <\varepsilon \tilde{y}_h', v'> + \int_0^2 a_{11}(x)\tilde{y}_h v_1 dx + \int_0^2 b_{11} \tilde{y}_h v_1 dx + \int_0^2 b_{12} \tilde{y}_h v_2 dx$$

$$+ \sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_{1k}(a_{11}\tilde{y}_h' + b_{11} \tilde{y}_h + b_{12} \tilde{y}_h)v_1' dx$$

$$+ <\varepsilon \tilde{y}_h', v'> + \int_0^2 a_{22}(x)\tilde{y}_h v_2 dx + \int_0^2 b_{21} \tilde{y}_h v_2 dx + \int_0^2 b_{22} \tilde{y}_h v_2 dx$$

$$+ \sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_{2k}(a_{22}\tilde{y}_h' + b_{21} \tilde{y}_h + b_{22} \tilde{y}_h)v_2' dx$$

$$f^*(v) = f_1^*(v) + f_2^*(v),$$

$$f_1^*(v) = \int_0^2 g_{11}(x)v_1 dx + \sum_{k=1}^N \int_{x_{k-1}}^{x_k} g_{11}(x)\delta_{1k}a_{11}v_1' dx,$$

$$f_2^*(v) = \int_0^2 g_{22}(x)v_2 dx + \sum_{k=1}^N \int_{x_{k-1}}^{x_k} g_{22}(x)\delta_{2k}a_{22}v_2' dx.$$
where \( \|y_h\|_{0,\Omega}^2 = \sum_{k=1}^{N-1} h_k y_h^2 \) and \( \|\nabla y_h\|_{0,\Omega}^2 = (\|y_{1h}\|_{0,\Omega}^2 + \|y_{2h}\|_{0,\Omega}^2)^{1/2} \) is a discrete norm on \((\mathcal{V}_h)^2\). We can easily prove that \( A_h \) is coercive with respect to \( \|\cdot\|_{\mathcal{V}_h} \). Moreover it is continuous and \( f_h \) is bounded linear functional on \((\mathcal{V}_h)^2\). By Lax-Milgram theorem, we conclude that the discrete problem (5.2) possesses a unique solution. Then the corresponding difference scheme is

\[
P_1^N \tilde{y}_h := -\varepsilon (D^+ \tilde{y}_{1h} - D^- \tilde{y}_{1h}) + D^+ \tilde{y}_{1h} + D^+ \tilde{y}_{2h} + \tilde{y}_{1h} \gamma_{1h} \\
+y_{2h-1} \left( \frac{h_k}{2} b_{12,k-1/2} + \frac{k}{2} b_{12,k-1} a_{11,k-1/2} \right) + y_{2h} \left( \frac{h_k}{2} b_{12,k-1/2} \right)
+ \frac{h_k+1}{2} b_{12,k+1/2} + \frac{k}{2} b_{12,k+1} a_{11,k+1/2} - \frac{k}{2} b_{12,k+1/2} a_{11,k+1/2}
+ \frac{h_k+1}{2} b_{12,k+1/2} - \frac{k}{2} b_{12,k+1/2} a_{11,k+1/2} = f_1^*(\phi_h)
\]

\[
F_{1h}^*(\phi_h) = \int_{x_{k-1}}^{x_{k+1}} g_1^*(x) \phi_h dx + \int_{x_{k-1}}^{x_{k+1}} g_1^*(x) \delta_{11} a_{11} \phi_h dx
\]

\[
P_2^N \tilde{y}_h := -\varepsilon (D^+ \tilde{y}_{2h} - D^- \tilde{y}_{2h}) + D^+ \tilde{y}_{2h} + D^+ \tilde{y}_{2h} + \tilde{y}_{2h} \gamma_{2h,k} \\
+y_{1h-1} \left( \frac{h_k}{2} b_{21,k-1/2} + \frac{k}{2} b_{21,k-1} a_{22,k-1/2} \right) + y_{1h} \left( \frac{h_k}{2} b_{21,k-1/2} \right)
+ \frac{h_k+1}{2} b_{21,k+1/2} + \frac{k}{2} b_{21,k+1} a_{22,k+1/2} - \frac{k}{2} b_{21,k+1/2} a_{22,k+1/2}
+ \frac{h_k+1}{2} b_{21,k+1/2} - \frac{k}{2} b_{21,k+1/2} a_{22,k+1/2} = f_2^*(\phi_h)
\]

\[
F_{2h}^*(\phi_h) = \int_{x_{k-1}}^{x_{k+1}} g_2^*(x) \phi_h dx + \int_{x_{k-1}}^{x_{k+1}} g_2^*(x) \delta_{11} a_{22} \phi_h dx
\]

where \( \alpha_{1,k} = \frac{h_k+1}{2} a_{11,k+1/2} + \frac{h_k^2}{4} b_{11,k+1/2} \)

\[\alpha_{2,k} = \frac{h_k+1}{2} a_{22,k+1/2} + \frac{h_k^2}{4} b_{22,k+1/2} \]

\[\beta_{1,k} = \frac{h_k a_{11,k+1/2}}{2} - \frac{h_k^2}{4} b_{11,k+1/2} + \frac{h_k a_{11,k+1/2}}{2} - \frac{h_k b_{11,k+1/2} a_{11,k+1/2}}{2} \]

\[\beta_{2,k} = \frac{h_k a_{22,k+1/2}}{2} - \frac{h_k^2}{4} b_{22,k+1/2} + \frac{h_k a_{22,k+1/2}}{2} - \frac{h_k b_{22,k+1/2} a_{22,k+1/2}}{2} \]

\[\gamma_{1,k} = \frac{h_k^2 a_{11,k+1/2}}{2} + \frac{h_k^2 b_{11,k+1/2}}{2} + \frac{h_k^2 b_{11,k+1/2}}{2} - \frac{h_k b_{11,k+1/2} a_{11,k+1/2}}{2} \]

\[\gamma_{2,k} = \frac{h_k^2 a_{22,k+1/2}}{2} + \frac{h_k^2 b_{22,k+1/2}}{2} + \frac{h_k^2 b_{22,k+1/2}}{2} - \frac{h_k b_{22,k+1/2} a_{22,k+1/2}}{2} \]

\( \delta_{1,i} \) is chosen as zero, when the local mesh step size is small enough, otherwise \( \delta_{1,i} \) is derived from the condition that \( \alpha_{1,i-1} = 0 \). In a similar way \( \delta_{2,i} \) is found from \( \alpha_{2,i-1} = 0 \). Thus we get

\[
\delta_{1,i} = \begin{cases} 
0 & \text{if } h_k \leq \frac{2}{|\alpha_{1,i}(x)|_{\infty}}, \\
\left[ \frac{h_k a_{11,k+1/2} + h_k^2 b_{11,k+1/2}}{2} \right] & \text{otherwise,}
\end{cases}
\]
\[
\delta_{2,i} = \begin{cases} 
0 & \text{if } h_k \leq \frac{2\varepsilon}{|||\partial_{22}(\bar{u})|||_\infty}, \\
\frac{h_k a_{22,k-1/2} \lambda^2_{1} + h_k a_{22,k-1/2} \lambda^2_{2}}{a^2_{22,k-1/2} + h_k a_{22,k-1/2} a_{11,k-1/2}} & \text{otherwise.}
\end{cases}
\]

5. Error Analysis

A general problem of the form

\[ A(\bar{y}, \bar{v}) = f^*(\bar{v}), \forall \bar{v} \in V^2, \]

is discretized by: find \( \bar{y}_h \in V^2 \cap (H^1_0(\Omega))^2 \) such that

\[ A_h(\bar{y}_h, \bar{v}_h) = f^*_h(\bar{v}_h), \forall \bar{v}_h \in V^2_h. \]

We define a biorthogonal basis of \( V^2_h \) with respect to \( A_h \) to be the set of functions \( \{\bar{v}^j\}^N_{j=1} \)

\[
A_h(\bar{y}_h, \bar{v}^j) = (\delta_{ij}, \delta_{ij}), \text{ for } i,j=1\ldots 2(n-1),
\]

\[ A_{1h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^j, \lambda_2^j) = \delta_{ij}, \] \( A_{2h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^j, \lambda_2^j) = \delta_{ij}. \)

Then, the components \( \bar{y}_{1h} \) and \( \bar{y}_{2h} \) can be uniquely represented as

\[
\bar{y}_{1h} = \sum_{i=1}^{N-1} A_{1h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^j, \lambda_2^j) \phi_i
\]

\[
\bar{y}_{2h} = \sum_{i=1}^{N-1} A_{2h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}) \phi_i
\]

\[
P_1, P_2 : (H^1_0(\Omega))^2 \to V_h \text{ such that }
\]

\[
P_1 \bar{y} = \sum_{i=1}^{N-1} A_{1h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^j, \lambda_2^j) \phi_i
\]

\[
P_2 \bar{y} = \sum_{i=1}^{N-1} A_{2h}(\bar{y}_{1h}, \bar{y}_{2h}), (\lambda_1^j, \lambda_2^j) \phi_i
\]

The error \( \bar{y} - \bar{y}_h \) can be represented as

\[
\bar{y} - \bar{y}_h = \bar{y} - \tilde{\bar{y}} + P(\tilde{\bar{y}}' - \bar{y}) + P \tilde{\bar{y}} - \bar{y}_h, \text{ where } P = (P_1, P_2)
\]

(5.1)

Now, \( K = \bar{P}(\bar{y}' - \bar{y}) \) is called the consistency error. The convergence analysis of the numerical scheme begins at the triangular inequality.

\[
|||\bar{y} - \bar{y}_h||| \leq |||\bar{y} - \tilde{\bar{y}}||| + |||\tilde{\bar{y}} - \bar{y}||| + |||\bar{y}' - \bar{y}_h|||, \]

(5.2)

where \( \bar{y}' \) denotes the piecewise linear interpolant to \( \bar{y} \) on \( \Omega \).
Let $f \in C^2(x_{k-1}, x_k)$ be an arbitrary function and $f^I$, a piecewise linear interpolant to $f$ on $\Omega$. Then from classical theory, we have

$$|(f^I - f)(x)| \leq 2 \int_{x_{i-1}}^{x_i} |f''(t)|(t - x_{i-1})dt$$

**Lemma 6.1.** [18, Theorem 3.1] For the Sishkin mesh, we have

$$|\tilde{y}_i(x) - \tilde{y}_i^I(x)| \leq \begin{cases} CN^{-2}(\ln N)^2, & x \in \Omega_0 \\ CN^{-2}, & x \in \Omega_a, i = 1, 2. \end{cases}$$

### 6.1 Projection error

Let $x_i \in \mathbb{N}^n_x$ be a mesh point. Then from the error representation, the projection error at the mesh points is

$$\mathcal{P}(\tilde{y}^I - \tilde{y})(x_i) = (P_1(\tilde{y}^I - \tilde{y}), P_2(\tilde{y}^I - \tilde{y}))$$

Now, each of the component will be estimated separately.

$$P_1(\tilde{y}^I - \tilde{y}) = A_{1k}((\tilde{y}^I_1 - \tilde{y}_1, \tilde{y}^I_2 - \tilde{y}_2), (\lambda_1^I, \lambda_2^I))$$

$$= \varepsilon < (\tilde{y}^I_1 - \tilde{y}_1), \lambda_1^I > + a_{11}(\tilde{y}^I_1 - \tilde{y}_1),\lambda_1^I > + \sum_{j=1}^{N-1} h_j b_{11}(x_j((\tilde{y}^I_{1,j} - \tilde{y}_{1,j})))\lambda_{1,j}^I$$

$$+ \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} \delta_{1,j}(\varepsilon\tilde{y}^I_1 + a_{11}(\tilde{y}^I_1 - \tilde{y}_1),\tilde{y}^I_1 + a_{11}(\tilde{y}^I_1 - \tilde{y}_1) + b_{11}(\tilde{y}^I_2 - \tilde{y}_2))a_{11}(\lambda_1^I)dx$$

$$\varepsilon < (\tilde{y}^I_1 - \tilde{y}_1), \lambda_1^I > = 0$$

$$\left| \int a_{11}(\tilde{y}^I_1 - \tilde{y}_1)\lambda_1^I dx \right| \leq \left| \int (\tilde{y}^I_1 - \tilde{y}_1)(a_{11}\lambda_1^I) dx \right| \leq CN^{-1} ||(\tilde{y}^I_1 - \tilde{y}_1)||_{L^\infty(\Omega)}.$$
Now, for estimating \( \sum \int \delta_{1,j} \varepsilon \hat{y}_{11}(\lambda_1^j) \, dx \), \( \hat{y}_1 \) is decomposed, \( \hat{y}_1 = v_1 + w_1 \). Further, \( w_1 = w_{1,1} + w_{1,2} \). Then for the regular part of the solution, we have, \( \left| \sum \int \delta_{1,j} \varepsilon \hat{y}_{11}(\lambda_1^j) \, dx \right| \leq C \varepsilon N^{-1} \).

For the singular part of the solution, we have
\[
\left| \sum \int \delta_{1,j} \varepsilon \hat{w}_{11}(\lambda_1^j) \, dx \right| \leq C \varepsilon \left[ \sum_{j=N/4+1}^{N/2} \int_{x_{j-1}}^{x_j} |w_{1,1}''| \, dx + \sum_{j=3N/4+1}^{N} \int_{x_{j-1}}^{x_j} |w_{1,2}''| \, dx \right] \leq C N^{-2},
\]
\[
\left| P_1(\tilde{y} - \tilde{y_1})(x) \right| \leq C N^{-2} + C N^{-1}(||\tilde{y}_1 - \tilde{y}_1|| + ||\tilde{y}_2 - \tilde{y}_2||)
\]
\[
\leq C N^{-2} + C N^{-2}(\log N)^2.
\]

6.2. Consistency error
Let \( K = (K_1, K_2) = ((P_1 \tilde{y} - \tilde{y}_{1h}),(P_2 \tilde{y} - \tilde{y}_{2h})) \). Let \( f_{1h,k} = \tilde{h}_k f_k \), then , \( K_1 = \sum (A_{1h} - A_1)(\tilde{y}_1, \lambda^i) \phi_i + \sum (f_1 - f_{1h})(\lambda^i \phi_i) \), for a some fixed point \( x_i \in \Omega^N \). Then , \( K_1(x_i) = (A_{1h} - A_1)(\tilde{y}_1, \lambda^i) + (f_1 - f_{1h})(\lambda^i) \)

\[
K_1(x_i) = \sum \tilde{h}_k b_{11,k} \tilde{y}_1, \lambda^i_k - \int_0^2 b_{11,\tilde{y}_1} \lambda^i \, dx - \int_0^2 (b_{11,\tilde{y}_1})^I \lambda^i \, dx
\]
\[
+ \sum \tilde{h}_k h_{12,k} \tilde{y}_2, \lambda^i_k - \int_0^2 b_{12,\tilde{y}_2} \lambda^i \, dx - \int_0^2 (b_{12,\tilde{y}_2})^I \lambda^i \, dx
\]
\[
+ \int_0^2 f_1 \lambda^i \, dx - \sum f_{1h,k} \lambda^i_k + \int_0^2 f_1^I \lambda^i \, dx - \int_0^2 f_1^I \lambda^i \, dx
\]
\[
+ \sum \tilde{h}_k c_{11,k} y_{01}(x - 1) - \int_1^2 c_{11} y_{01}(x - 1) \lambda^i
\]
\[
+ \int_0^2 (c_{11} y_{01}(x - 1))^I \lambda^i \, dx - \int_0^2 (c_{11} y_{01}(x - 1))^I \lambda^i \, dx
\]
\[
+ \sum \tilde{h}_k c_{12,k} y_{02}(x - 1) - \int_1^2 c_{12} y_{02}(x - 1) \lambda^i
\]
\[
+ \int_0^2 (c_{12} y_{02}(x - 1))^I \lambda^i \, dx - \int_0^2 (c_{12} y_{02}(x - 1))^I \lambda^i \, dx
\]

Let \( K_1^*(x_i) = \sum \tilde{h}_k b_{11,k} \tilde{y}_1, \lambda^i_k - \int_0^2 (b_{11,\tilde{y}_1})^I \lambda^i \, dx + \sum \tilde{h}_k b_{12,k} \tilde{y}_2, \lambda^i_k - \int_0^2 (b_{12,\tilde{y}_2})^I \lambda^i \, dx
\]
\[
+ \int_0^2 f_1 \lambda^i \, dx - \sum f_{1h,k} \lambda^i_k + \sum \tilde{h}_k c_{11,k} y_{01}(x - 1) - \int_1^2 (c_{11} y_{01}(x - 1))^I \lambda^i \, dx
\]
\[
+ \sum \tilde{h}_k c_{12,k} y_{02}(x - 1) - \int_1^2 c_{12} y_{02}(x - 1) \lambda^i
\]

Then, \( K_1(x_i) = K_1^*(x_i) + \int_0^2 ((b_{11,\tilde{y}_1})^I - b_{11,\tilde{y}_1}) \lambda^i + \int_0^2 ((b_{12,\tilde{y}_2})^I - b_{12,\tilde{y}_2}) \lambda^i + \int_0^2 (f_1 - f_1^I) \lambda^i
\]
\[
+ \int_0^2 ((c_{11} y_{01}(x - 1))^I - c_{11} y_{01}(x - 1)) \lambda^i
\]
As $\int g^T \lambda^i - \sum \overline{K}_k g_k^T \lambda^i_k \leq CN^{-2} \|g^T\|_{L_\infty(\Omega)} \|\lambda^i\| \leq CN^{-2}$, we have $|K^*_1(x_i)| \leq CN^{-2}.$

As $\int (f - f^T) \lambda^i dx \leq C(\|f - f^T\|_{L_\infty(\Omega)} + N^{-2})\|f\|_{L_1(\Omega)} \leq CN^{-2}$.

Therefore, we have $|K_1(x_i)| \leq CN^{-2}$. In a similar manner one can prove that $|K_2(x_i)| \leq CN^{-2}$, $|\overline{K}(x_i)| \leq \max\{K_1(x_i), |K_2(x_i)|\} \leq |K_1| + |K_2| \leq CN^{-2}$.

Therefore, $|\overline{g}(x_i) - \overline{g}_h(x_i)| \leq |\overline{g}(x_i) - \overline{g}^T(x_i)| + |P(\overline{f} - \overline{f})(x_i)| + |(P\overline{g} - \overline{g}_h)| \leq CN^{-2}(\log N)^2 + CN^{-2} + CN^{-2} \leq CN^{-2}(\log N)^2$.

**Lemma 6.2.** [10, Lemma 4.3] Let $\overline{g}$ be the solution of (2.5) and $\overline{g}_h$ be the interpolant, respectively. Then, $|||\overline{g} - \overline{g}_l||| \leq CN^{-1} \log N$.

**Lemma 6.3.** The following estimate holds true for Shishkin mesh $|||\overline{g}^T - \overline{g}_h||| \leq CN^{-1} \log N$.

**Proof.**

\[
|||\overline{g}^T - \overline{g}_h|||^2 \leq C(|||\overline{g}^T - \overline{g}_h|||^2) + C|A(\overline{g}^T - \overline{g}_h, \overline{g}^T - \overline{g}_h)|
\]

\[
\leq C(|A(\overline{g}^T - \overline{g}, \overline{g}^T - \overline{g}_h)| + |A(\overline{g}^T, \overline{g}^T - \overline{g}_h)| + A_h(\overline{g}^T, \overline{g}^T - \overline{g}_h)) + |f^*(\overline{g}^T - \overline{g}_h) - f_h^*(\overline{g}^T - \overline{g}_h))|
\]

Now we estimate the first term of the above expression. $\varepsilon((\overline{g}^T_1 - \overline{g}_1), (\overline{g}^T_2 - \overline{g}_2)) = 0 = \varepsilon((\overline{g}^T_3 - \overline{g}_3), (\overline{g}^T_2 - \overline{g}_2))$, $<a_{11}(\overline{g}^T_1 - \overline{g}_1), (\overline{g}^T_2 - \overline{g}_2)> = <a_{22}(\overline{g}^T_3 - \overline{g}_3), (\overline{g}^T_2 - \overline{g}_2)>

|A(\overline{g}^T - \overline{g}, \overline{g}^T - \overline{g}_h)| = \left| <b_{11}(\overline{g}^T_1 - \overline{g}_1), (\overline{g}^T_2 - \overline{g}_2)> + <b_{12}(\overline{g}^T_2 - \overline{g}_2), (\overline{g}^T_1 - \overline{g}_2)>ight|

\leq C|||\overline{g}^T - \overline{g}_h|||_0|||\overline{g}^T - \overline{g}_h|||_0.

\[
|||\overline{g}^T - \overline{g}_h|||^2 \leq C(|||\overline{g}^T - \overline{g}_h|||_0|||\overline{g}^T - \overline{g}_h|||_0 + |A(\overline{g}^T, \overline{g}^T - \overline{g}_h))| + |f^*(\overline{g}^T - \overline{g}_h) - f_h^*(\overline{g}^T - \overline{g}_h))|
\]

\[
|A_h(\overline{g}^T, \overline{g}^T - \overline{g}_h)) - A(\overline{g}^T, \overline{g}^T - \overline{g}_h))| = \left| <b_{11}(\overline{g}^T_1, (\overline{g}^T_2 - \overline{g}_2) >> + <b_{12}(\overline{g}^T_2, (\overline{g}^T_1 - \overline{g}_2)>
\]

\[
+ \sum_{j=1}^{x_{k+1}} \delta_{1,j}(a_{11}\overline{y}^*_1 + b_{11}\overline{y}_1 + b_{12}\overline{y}_2)(\overline{g}^T_1 - \overline{g}_1) a_{11} dx + <b_{21}\overline{y}^*_1, \overline{g}^T_2 - \overline{g}_2)>
\]

\[
+ <b_{22}\overline{g}^T_2, \overline{g}^T_2 - \overline{g}_2) > + \sum_{j=1}^{x_{k+1}} \delta_{2,j}(a_{22}\overline{y}^*_2 + b_{21}\overline{y}_1 + b_{22}\overline{y}_2)(\overline{g}^T_1 - \overline{g}_1) a_{22} dx
\]

\[
- \int_{0}^{2} b_{11}\overline{y}^*_1(\overline{g}^T_1 - \overline{g}_1) dx - \int_{0}^{2} b_{12}\overline{g}^T_2(\overline{g}^T_1 - \overline{g}_1) dx - \int_{0}^{2} b_{21}\overline{y}^*_2(\overline{g}^T_2 - \overline{g}_2) dx
\]

\[
- \int_{0}^{2} b_{22}\overline{g}^T_2(\overline{g}^T_2 - \overline{g}_2) dx \leq \left| \sum_{k=0}^{N-1} \overline{h}_k b_{11,k}\overline{y}^*_1, \overline{g}_1 - \overline{g}_1, k) + \sum_{k=0}^{N-1} \overline{h}_k b_{12,k}\overline{g}^T_2, \overline{g}_2, k) - \overline{g}_1, k) NEXT \right|
\]
\[ + \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f (\tilde{y}_2^f - \tilde{y}_{2h}) + \sum_{k=1}^{N-1} \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f (\tilde{y}_1^f - \tilde{y}_{1h}) \right) dx \]

\[ - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f (\tilde{y}_1^f - \tilde{y}_{1h}) dx - \int_{x_{k-1}}^{x_k} b_{21}(x) \tilde{y}_1^f (\tilde{y}_2^f - \tilde{y}_{2h}) dx - \int_{x_{k-1}}^{x_k} b_{22}(x) \tilde{y}_2^f (\tilde{y}_2^f - \tilde{y}_{2h}) dx \]

\[ + \sum_{k=1}^{N-1} \int_{x_{k-1}}^{x_k} \delta_1(k) (a_{11} \tilde{y}_1^f + b_{11} \tilde{y}_{1h} + b_{12} \tilde{y}_{2h}) (\tilde{y}_1^f - \tilde{y}_{1h}) a_{11} dx \]

\[ + \sum_{k=1}^{N-1} \int_{x_{k-1}}^{x_k} \delta_2(k) (a_{22} \tilde{y}_2^f + b_{21} \tilde{y}_{1h} + b_{22} \tilde{y}_{2h}) (\tilde{y}_1^f - \tilde{y}_{1h}) a_{22} dx \]

\[ \leq \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| + \sum_{k=1}^{N-1} \left| \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right| (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{11,k} \tilde{y}_1^f - \int_{x_{k-1}}^{x_k} b_{11}(x) \tilde{y}_1^f \phi_k dx \right) (\tilde{y}_1^f - \tilde{y}_{1h}) (x_k) \right| \]

\[ + \left| \sum_{k=1}^{N-1} \left( \frac{N-1}{2} b_{12,k} \tilde{y}_2^f - \int_{x_{k-1}}^{x_k} b_{12}(x) \tilde{y}_2^f \phi_k dx \right) (\tilde{y}_2^f - \tilde{y}_{2h}) (x_k) \right| \]

\[ + C N^{-1} \vert (\tilde{y}_1^f - \tilde{y}_{1h}) \vert + C N^{-1} \vert (\tilde{y}_2^f - \tilde{y}_{2h}) \vert \]
Let $p_{i,x} = \frac{h}{2}b_{i,k}\tilde{y}_i^j - \int_{x_{i-1}}^{x_i} b_{i,k}\tilde{y}_i^j(k)dx = \int_{x_{i-1}}^{x_i} (b_{i,k}\tilde{y}_i^j - b_{i,k}\tilde{y}_i^j(k))\phi_k dx$, $\overline{p}_{12,k} = \frac{h}{2}b_{12,k}\tilde{y}_i^j - \int_{x_{i-1}}^{x_i} b_{12,k}\tilde{y}_i^j(k)dx = \int_{x_{i-1}}^{x_i} (b_{12,k}\tilde{y}_i^j - b_{12,k}\tilde{y}_i^j(k))\phi_k dx$.

Now we use the decomposition of the solution $\tilde{y}_1 = \tilde{v}_1 + \tilde{w}_1$.

$$|b_{i,k}\tilde{y}_i^j - b_{i,k}\tilde{y}_i^j(k)| \leq \left| \int_{x_{i-1}}^{x_i} (b_{i}(s)\tilde{v}_i^j(s))' ds \right|$$

$$\leq \left| \int_{x_{i-1}}^{x_i} (b_{i}(s)\tilde{v}_i^j(s))' ds \right| + \left| \int_{x_{i-1}}^{x_i} (b_{i}(s)\tilde{w}_i^j(s))' ds \right|.$$

On Shishkin mesh, when $x \in (x_{i-1}, x_i) \subset \Omega^+ \cap \Omega_0$, $\sum_{k=1}^{N} k \| \tilde{y}_i^j - \tilde{y}_i^j(k) \|_0^{N} \leq C \frac{1}{N^{1/2}} \log N$.

Similarly, if $x \in (x_{i-1}, x_i) \subset \Omega^- \cap \Omega_0$, we can show that $|b_{i,k}\tilde{y}_i^j - b_{i,k}\tilde{y}_i^j(k)| \leq C \frac{1}{N^{1/2}} \log N$.

Then, $|A_h(\tilde{y}_i^j, \tilde{y}_i^j - \tilde{y}_i^j) - A_h(\tilde{y}_i^j, \tilde{y}_i^j - \tilde{y}_i^j) \tilde{y}_i^j - \tilde{y}_i^j)| \leq C \frac{1}{N^{1/2}} \log N$. Now, the third term will be considered.
Therefore, 

\[ \sum_{N=1}^{N/2-1} \int_{x_{k-1}}^{x_k} \delta_{1,k}(f_1 a_{11})'(\tilde{y}_1 - \tilde{y}_{1h})dx \]

\[ \sum_{N=1}^{N/2-1} \int_{x_{k-1}}^{x_k} \delta_{1,k}(f_1 a_{11})(\tilde{y}_1 - \tilde{y}_{1h})dx \]

\[ \leq CN^{-1} \log N \left( \|f_1'\|_{L_\infty} \sum_k \tilde{h}_k \|\tilde{y}_1 - \tilde{y}_{1h}\|(x_k) \right) \]

\[ + CN^{-1} \log N \left( \|f_1 - c_{11}y_1 - c_{12}y_0\|_{L_\infty} \sum_k \tilde{h}_k \|\tilde{y}_1 - \tilde{y}_{1h}\|(x_k) \right) \]

\[ + CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| + CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| \]

\[ \leq CN^{-1} \log N \left( \|f_1'\|_{L_\infty} + \|(f_1 - c_{11}y_1 - c_{12}y_0)'\|_{L_\infty} \right) \sum_k \tilde{h}_k \|\tilde{y}_1 - \tilde{y}_{1h}\|(x_k) \]

\[ + CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| + CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| \leq CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\|. \]

\[ |f_2(\tilde{y}_2 - \tilde{y}_{2h}) - f_2(\tilde{y}_2 - \tilde{y}_{2h})| \leq |f_2(\tilde{y}_2 - \tilde{y}_{2h}) - f_2(\tilde{y}_2 - \tilde{y}_{2h})| \]

\[ \leq CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| + CN^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\| \]

\[ \leq CN^{-1} \log N \left( \|\tilde{y}_1 - \tilde{y}_{1h}\| + \|\tilde{y}_2 - \tilde{y}_{2h}\| \right) \leq CN^{-1} \log N \left( \|\tilde{y}_1 - \tilde{y}_{1h}\| \right) \]

Therefore,

\[ \|\tilde{y} - \tilde{y}_h\|^2 \leq C \left( \|\tilde{y} - \tilde{y}_h\| \|\tilde{y} - \tilde{y}_h\|_0 + N^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\|_0 + N^{-1} \log N \|\tilde{y}_1 - \tilde{y}_{1h}\|_0 \right) \]

\[ \leq C \|\tilde{y} - \tilde{y}_h\| \|\tilde{y} - \tilde{y}_h\|_0 + N^{-1} \log N \]

\[ \leq C \|\tilde{y} - \tilde{y}_h\| \|\tilde{y} - \tilde{y}_h\|_0 + N^{-1} \log N \]

\[ \|\tilde{y} - \tilde{y}_h\| \leq CN^{-1} \log N. \]

**Theorem 6.4.** Let \( \tilde{y} \) be the solution of the original problem and \( \tilde{y}_h \) be the solution of the equivalent problem. Then \( \|\tilde{y} - \tilde{y}_h\| \leq \|\tilde{y} - \tilde{y}_h\| + \|\tilde{y} - \tilde{y}_h\| + \|\tilde{y} - \tilde{y}_h\| \).

**Proof.**

\[ -\varepsilon(y_1 - \tilde{y}_1)' + a_{11}(x)(y_1 - \tilde{y}_1)' + b_{11}(x)(y_1 - \tilde{y}_1) + b_1(y_2 - \tilde{y}_2) = \]

\[ \begin{cases} 0, & \text{in } \Omega^- \\ c_{11}(y_1 - y_0)(x - 1) + c_{12}(y_2 - y_0)(x - 1), & \text{in } \Omega^+ \end{cases} \]
\[-\varepsilon(y_2 - \tilde{y}_2)'' + a_{11}(x)(y_2 - \tilde{y}_2)' + b_{11}(x)(y_1 - \tilde{y}_1) + b_{12}(y_2 - \tilde{y}_2) =
\begin{cases}
0, & \text{in } \Omega^- \\
c_{11}(y_1 - y_{01})(x - 1) + c_{12}(y_2 - y_{02})(x - 1), & \text{in } \Omega^+
\end{cases}\]

Integrating over \(\Omega\), we get
\[-\varepsilon(y_1 - \tilde{y}_1)' + \int_{\Omega} a_{11}(x)(y_1 - \tilde{y}_1)' + \int_{\Omega} b_{11}(x)(y_1 - \tilde{y}_1) + \int_{\Omega} b_{12}(y_2 - \tilde{y}_2) = \begin{cases} 0 \\
\int_{\Omega^+} c_{11}(y_1 - y_{01})(x - 1) + c_{12}(y_2 - y_{02})(x - 1) \end{cases}\]

Let \(x \in I_i = [x_{i-1}, x_i] \subset \Omega^-\)

Taking modulus on both sides, we get
\[
\varepsilon|(y_1 - \tilde{y}_1)'| = \left| \int_{\Omega} a_{11}(x)(y_1 - \tilde{y}_1)' + \int_{\Omega} b_{11}(x)(y_1 - \tilde{y}_1) + \int_{\Omega} b_{12}(y_2 - \tilde{y}_2) \right|
\]
Integrating by parts
\[
\leq \left| \int_{\Omega} a_{11}'(y_1 - \tilde{y}_1) + \int_{\Omega} b_{11}(y_1 - \tilde{y}_1) + \int_{\Omega} b_{12}(y_2 - \tilde{y}_2) \right|
\leq \max\{a_{11}'\}|y_1 - \tilde{y}_1|h_i + \max\{b_{11}\}|y_1 - \tilde{y}_1|h_i + \max\{b_{12}\}|y_1 - \tilde{y}_1|h_i
\]

\[
\varepsilon|(y_1 - \tilde{y}_1)'| \leq C\varepsilon N^{-1}
\]
\[
||y_1 - \tilde{y}_1|| \leq CN^{-1}.
\]

\[
||y_1 - \tilde{y}_1||^2 = \int_{I_i} \|(y_1 - \tilde{y}_1)'\|^2 dx \leq C N^{-5}(\log N)^2
\]

Similarly, one can prove that \(||y_2 - \tilde{y}_2||^2 \leq C N^{-3}, ||y_1 - \tilde{y}_1||^2 \leq C N^{-5}(\log N)^2\)

\[
||\tilde{y} - \tilde{\tilde{y}}||^2 = \left[ \varepsilon \|y_1 - \tilde{y}_1\|^2 + \|y_2 - \tilde{y}_2\|^2 + \rho(\|y_1 - \tilde{y}_1\|^2_{L^2(I_i)} + ||y_2 - \tilde{y}_2||^2_{L^2(I_i)}) \right]
\]

\[
||\tilde{y} - \tilde{\tilde{y}}||^2 \leq C N^{-3} + C N^{-5}(\log N)^4 \leq C N^{-3}
\]

\[
||\tilde{y} - \tilde{\tilde{y}}|| \leq C N^{-3/2}(\log N)^2.
\]

Similar result can be proved for \(\Omega^+\), \(||\tilde{y} - \tilde{\tilde{y}}_h|| \leq ||\tilde{y} - \tilde{\tilde{y}}|| + ||\tilde{y} - \tilde{\tilde{y}}' || + ||\tilde{\tilde{y}}' - \tilde{\tilde{y}}_h||\)

\[
||\tilde{y} - \tilde{\tilde{y}}|| \leq C N^{-3/2}(\log N)^2 + C N^{-1} \log N + C N^{-1} \log N
\]

\[
||\tilde{y} - \tilde{\tilde{y}}|| \leq C N^{-1} \log N.
\]

7. Numerical experiment
In this section, we experimentally verify our theoretical results proved in the previous section by considering the following BVP. We apply two mesh principle given in [13] to estimate maximum error and computational convergence order.

We use \(D^M_{\varepsilon} = \max_{0 \leq i \leq M} | Y_{2i}^M - Y_i^M |\), where \(Y_i^M\) and \(Y_{2i}^M\) are numerical solutions at same node \(x_i\) with number of mesh points \(M\) and \(2M\) respectively. Further, \(D^M = \max_{\varepsilon} D^M_{\varepsilon}\) and \(p^M = \log_2(\frac{D^M_{\varepsilon}}{D^{2M}_{\varepsilon}})\). Here \(D^M\) is the uniform error and \(p^M\) is computational convergence order. The perturbation parameter \(\varepsilon\) ranges from \(2^{-6}\) to \(2^{-10}\) for the problem considered.
Example 7.1. In this example, we considered 
\( a_{11} = 10 + x^2, \ a_{22} = 10 + e^{-x}, \ b_{11} = b_{22} = 2, \ c_{11} = -x, \ c_{12} = -x/2, \ c_{21} = -x, \ c_{22} = -x, \ f_1 = 0, \ f_2 = 1. \) The \( L^2 \) and \( H_1 \) errors are given in Tables 1, 2, 3. The numerical solution is plotted in Figure 1.

![Graph of two components \( y_{1h}, \ y_{2h} \) of the numerical solution of the BVP for \( \varepsilon = 2^{-5} \) and \( N = 2^6 \)](image)

\[ \text{Table 1.} \ L^2 \text{ Error for } y_1 \text{-component} \]

| \( \varepsilon \)       | 2\(^6\) | 2\(^7\) | 2\(^8\) | 2\(^9\) | 2\(^{10}\) |
|------------------------|---------|---------|---------|---------|---------|
| \( \varepsilon = 2^{-6} \) | 8.1000e-5 | 3.9433e-5 | 2.1984e-5 | 1.4327e-5 | 3.6633e-3 |
| \( \varepsilon = 2^{-7} \) | 7.3855e-5 | 3.4753e-5 | 1.8106e-5 | 1.0537e-5 | 7.0136e-6 |
| \( \varepsilon = 2^{-8} \) | 6.8952e-5 | 3.2497e-5 | 1.6299e-5 | 8.7691e-6 | 5.1794e-6 |
| \( \varepsilon = 2^{-9} \) | 6.6831e-5 | 3.1378e-5 | 1.5432e-5 | 7.9414e-6 | 4.3298e-6 |
| \( \varepsilon = 2^{-10} \) | 6.5755e-5 | 3.0821e-5 | 1.5009e-5 | 7.5441e-6 | 3.9277e-6 |
| Max error \( (D^{max}) \) | 8.1000e-5 | 3.4753e-5 | 1.6299e-5 | 7.9414e-6 | 4.3298e-6 |
| Rate of convergence \( (p^\text{ex}) \) | 1.7613 | 1.5739 | 1.4965 | 1.4654 | - |

\[ \text{Table 2.} \ L^2 \text{ Error for } y_2 \text{-component} \]

| \( \varepsilon \)       | 2\(^6\) | 2\(^7\) | 2\(^8\) | 2\(^9\) | 2\(^{10}\) |
|------------------------|---------|---------|---------|---------|---------|
| \( \varepsilon = 2^{-6} \) | 1.0488e-3 | 4.7819e-4 | 2.2153e-4 | 1.0098e-4 | 3.1890e-2 |
| \( \varepsilon = 2^{-7} \) | 1.0640e-3 | 4.9111e-4 | 2.3278e-4 | 1.1033e-4 | 5.1001e-5 |
| \( \varepsilon = 2^{-8} \) | 1.0715e-3 | 4.9706e-4 | 2.3781e-4 | 1.1491e-4 | 5.5087e-5 |
| \( \varepsilon = 2^{-9} \) | 1.0752e-3 | 4.9988e-4 | 2.4011e-4 | 1.1700e-4 | 5.7080e-5 |
| \( \varepsilon = 2^{-10} \) | 1.0770e-3 | 5.0124e-4 | 2.4119e-4 | 1.1796e-4 | 5.8011e-5 |
| Max error \( (D^{max}) \) | 1.0770e-3 | 5.0124e-4 | 2.4119e-4 | 1.1796e-4 | 5.8011e-5 |
| Rate of convergence \( (p^\text{ex}) \) | 1.592 | 1.5225 | 1.4886 | 1.4773 | - |
Table 3. Error in $H_1$ norm for $Y = (y_1, y_2)$

| $N$ (Number of mesh points) | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ |
|-----------------------------|-------|-------|-------|-------|-------|
| $\varepsilon = 2^{-10}$    | 5.6754e-3 | 3.6126e-3 | 2.4953e-3 | 1.8028e-3 | 7.3841e-3 |
| $\varepsilon = 2^{-9}$     | 4.5108e-3 | 2.3859e-3 | 1.6415e-3 | 1.3189e-3 | 9.2098e-4 |
| $\varepsilon = 2^{-8}$     | 3.1025e-3 | 2.2829e-3 | 1.8002e-3 | 1.0002e-3 | 6.9245e-4 |
| $\varepsilon = 2^{-7}$     | 7.0255e-3 | 2.3174e-3 | 1.2239e-3 | 7.9882e-4 | 5.2190e-4 |
| Max error($D_N$)            | 1.1943e-2 | 2.6417e-3 | 1.3628e-3 | 7.9882e-4 | 4.1397e-4 |
| Rate of convergence($p_N$)  | 3.1402   | 1.3777   | 1.1117   | 1.3727   | -       |

8. Discussion
A SDFEM method on shishkin mesh is presented for convection-diffusion singularly perturbed delay differential equations in this article. It is established that the order of convergence is almost one with respect to the perturbation parameter.

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