SURFACES IN $S^3$ AND $H^3$ VIA SPINORS

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Abstract. We generalize the spinorial characterization of isometric immersions of surfaces in $\mathbb{R}^3$ given in [4] by T. Friedrich to surfaces in $S^3$ and $H^3$. The main argument is the interpretation of the energy-momentum tensor associated with a special spinor field as a second fundamental form. It turns out that such a characterization of isometric immersions in terms of a special section of the spinor bundle also holds in the case of hypersurfaces in the Euclidean 4-space.

1. Introduction

It is well known that a description of a conformal immersion of an arbitrary surface $M^2 \hookrightarrow \mathbb{R}^3$ by a spinor field $\varphi$ on $M^2$ satisfying the inhomogenous Dirac equation

$$D \varphi = H \varphi,$$

(1)

(where $D$ stands for the Dirac operator and $H$ for the mean curvature of the surface), is possible. Recently, many authors investigated such a description (see for example [7], [12]).

In fact, it is clear that any oriented immersed surface $M^2 \hookrightarrow \mathbb{R}^3$ inherits from $\mathbb{R}^3$ a solution of Equation (1), the surface $M$ being endowed with the induced metric and the induced spin structure. Moreover, the solution has constant length. This solution is obtained by the restriction to the surface of a parallel spinor field on $\mathbb{R}^3$. In [4], T. Friedrich clarifies the above-mentioned representation of surfaces in $\mathbb{R}^3$ in a geometrically invariant way by proving the following:

Theorem 1.1 (Friedrich [4]). Let $(M^2, g)$ be an oriented, 2-dimensional manifold and $H : M \to \mathbb{R}$ a smooth function. Then the following data are equivalent:

1. An isometric immersion $(\widetilde{M}^2, g) \to \mathbb{R}^3$ of the universal covering $\widetilde{M}^2$ into the Euclidean space $\mathbb{R}^3$ with mean curvature $H$.
2. A solution $\varphi$ of the Dirac equation

$$D \varphi = H \varphi,$$

with constant length $|\varphi| \equiv 1$.
3. A pair $(\varphi, T)$ consisting of a symmetric endomorphism $T$ of the tangent bundle $TM$ such that $\text{tr}(T) = H$ and a spinor field $\varphi$ satisfying, for any $X \in \Gamma(TM)$, the equation

$$\nabla_X \varphi + T(X) \cdot \varphi = 0.$$
In this paper, we prove the analogous characterizations for surfaces in $S^3$ and $H^3$ (Theorems 4.1 and 4.2). They are obtained by studying the equation of restrictions to a surface of real and imaginary Killing spinor fields (compare with [4]).

We note that the involved symmetric endomorphism $T$ is the energy-momentum tensor associated with the restricted Killing spinor which describes the immersion.

Finally, the case of the hypersurfaces of $R^4$ is treated (Theorem 5.3).

2. Restricting Killing spinor fields to a surface

Let $N^3$ be a 3-dimensional oriented Riemannian manifold, with a fixed spin structure. Denote by $\Sigma N$ the spinor bundle associated with this spin structure. If $M^2$ is an oriented surface isometrically immersed into $N^3$, denote by $\nu$ its unit normal vector field. Then $M^2$ is endowed with a spin structure, canonically induced by that of $N^3$. Denote by $\Sigma M$ the corresponding spinor bundle. The following proposition is essential for what follows (see for example [1],[3],[9],[13]):

**Proposition 2.1.** There exists an identification of $\Sigma N|_M$ with $\Sigma M$, which after restriction to $M$, sends every spinor field $\psi \in \Gamma(\Sigma N)$ to the spinor field denoted by $\psi^* \in \Gamma(\Sigma M)$. Moreover, if $\cdot_{\Sigma N}$ (resp. $\cdot_{\Sigma M}$) stands for Clifford multiplication on $\Sigma N$ (resp. $\Sigma M$), then one has

$$ (X \cdot_{\Sigma N} \nu \cdot_{\Sigma N} \psi)^* = X \cdot \psi^* , $$

(2)

for any vector field $X$ tangent to $M$.

Another important formula is the well-known spinorial Gauss formula: if $\nabla^N$ and $\nabla$ stand for the covariant derivatives on $\Gamma(\Sigma N)$ and $\Gamma(\Sigma M)$ respectively, then, for all $X \in TM$ and $\psi \in \Gamma(\Sigma N)$

$$ (\nabla^N_X \psi)^* = \nabla_X \psi^* + \frac{1}{2} h(X) \cdot \psi^* , $$

(3)

where $h$ is the second fundamental form of the immersion $M \hookrightarrow N$ viewed as a symmetric endomorphism of the tangent bundle of $M$.

Assume now that $N^3$ admits a non-trivial Killing spinor field of Killing constant $\eta \in \mathbb{C}$, i.e., a spinor field $\Phi \in \Gamma(\Sigma N)$ satisfying

$$ \nabla^N_Y \Phi = \eta Y \cdot_{\Sigma N} \Phi $$

(4)

for all vector field $Y$ on $N$. Recall that $\eta$ has to be real or pure imaginary and that $\Phi$ never vanishes on $N$, as a non-trivial parallel section for a modified connection (see [2],[3]). In what follows, we will consider the model spaces, with their standard metrics, $\mathbb{R}^3$ with $\eta = 0$, $S^3$ with $\eta = 1/2$, and $H^3$ with $\eta = i/2$ which are characterized by the fact that they admit a maximal number of linearly independant Killing spinor fields with constant $\eta$. 
Let \((e_1, e_2)\) be a positively oriented local orthonormal basis of \(\Gamma(TM)\) such that \((e_1, e_2, \nu)\) is a positively oriented local orthonormal basis of \(\Gamma(TN)\). Denote by
\[
\omega_3 = -e_1 \cdot e_2 \cdot \nu
\]
the complex volume form on the complex Clifford bundle \(\mathbb{C}lN\) and \(\omega = e_1 \cdot e_2\) the real volume form on \(\mathbb{C}lM\). Recall that \(\omega_3\) acts by Clifford multiplication as the identity on \(\Sigma^N\). Therefore, denoting \(\varphi := \Phi^*\), formula (2) yields
\[
(e_1 \cdot \Phi)^* = (-e_1 \cdot e_1 \cdot e_2 \cdot \nu \cdot \Phi)^* = e_2 \cdot \varphi = -e_1 \cdot \omega \cdot \varphi
\]
and
\[
(e_2 \cdot \Phi)^* = (-e_2 \cdot e_1 \cdot e_2 \cdot \nu \cdot \Phi)^* = -e_1 \cdot \varphi = -e_2 \cdot \omega \cdot \varphi
\]
and
\[
(\nu \cdot \Phi)^* = (-\nu \cdot e_1 \cdot e_2 \cdot \nu \cdot \Phi)^* = \omega \cdot \varphi
\]
Then, these last relations with Equations (3) and (4) show that
\[
\forall X \in TM, \quad \nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0 \tag{5}
\]
Recall that the spinor bundle \(\Sigma M\) splits into
\[
\Sigma M = \Sigma^+ M \oplus \Sigma^- M
\]
where \(\Sigma^\pm M\) is the \(\pm 1\)-eigenspace for the action of the complex volume form \(\omega_2 = i \omega\). Under this decomposition, we will denote \(\varphi = \varphi^+ + \varphi^\mp\), and define \(\overline{\varphi} := \varphi^+ - \varphi^-\). Therefore Equation (5) is equivalent to
\[
\nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi - i\eta X \cdot \overline{\varphi} = 0 .
\]
The ambient spinor bundle \(\Sigma N\) can be endowed with a Hermitian inner product \((\ , \ )_N\) for which Clifford multiplication by any vector tangent to \(N\) is skew-symmetric. This product induces another Hermitian inner product on \(\Sigma M\), denoted by \((\ , \ )\) making the identification of Proposition 2.1 an isometry. Now, relation (2) shows that Clifford multiplication by any vector tangent to \(M\) is skew-symmetric with respect to \((\ , \ )\).

**Proposition 2.2.** If \(\eta \in \mathbb{R}\), then \(\varphi\) has constant length. If \(\eta \in i\mathbb{R}^*\), then for all vector \(X\) tangent to \(M\),
\[
|X|\varphi|^2 = 2\Re(i\eta X \cdot \overline{\varphi}, \varphi) .
\]

**Proof.** Since Clifford multiplication by any vector tangent to \(M\) is skew-symmetric with respect to \((\ , \ )\), we have \(\Re(Y \cdot \varphi, \varphi) = 0\) for all \(Y \in TM\). Taking this fact into account and computing
\[
|X|\varphi|^2 = 2\Re(\nabla_X \varphi, \varphi)
\]
with the help of formula (5), completes the proof. \(\square\)

Recalling that the Dirac operator \(D\) is defined on \(\Gamma(\Sigma M)\) by
\[
D = e_1 \cdot \nabla_{e_1} + e_2 \cdot \nabla_{e_2} ,
\]
we compute directly that
\[ D\varphi = H\varphi + 2\eta \omega \cdot \varphi = H\varphi - 2i\eta \varphi \]
where \( H \) is the mean curvature of the immersion \( M \hookrightarrow N \). It is well known that the action of the Dirac operator satisfies \((D\varphi)^\pm = D\varphi^\mp\) (see [3],[8]). Therefore, we note that
\[ D(\varphi^\pm) = (H \pm 2i\eta)\varphi^\mp. \quad (6) \]

We have as in [4] the following

Proposition 2.3. Let \( M^2 \) be a minimal surface in \( N^3 \). Then the restriction of any Killing spinor \( \Phi \) with constant \( \eta \) on \( N^3 \) restricts to an eigenspinor \( \varphi^* \) on the surface \( M \):
\[ D\varphi^* = 2\eta\varphi^* \]
Moreover, if \( \eta \) is real, then \( \varphi^* \) has constant length.

Proof. Since \( H = 0 \), we have
\[ D(\varphi^\pm) = \pm 2i\eta\varphi^\mp. \]
Therefore, it suffices to define \( \varphi^* = \varphi^+ + i\varphi^- \). \( \square \)

3. Solutions of the restricted Killing spinor equation

Let \( (M^2, g) \) be an oriented, 2-dimensional Riemannian manifold with a spin structure. We endow the spinor bundle \( \Sigma M \) with a Hermitian inner product \((\ ,\ )\) for which Clifford multiplication by any vector tangent to \( M \) is skew-symmetric.

We study now some properties of a given solution \( \varphi \in \Gamma(\Sigma M) \) of the following equation
\[ \nabla_X \varphi + T(X) \cdot \varphi - i\eta X \cdot \varphi = 0, \quad (7) \]
or equivalently
\[ \nabla_X \varphi + T(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0, \quad (8) \]
where \( T \) stand for a symmetric endomorphism of the tangent bundle of \( M \), and \( \eta \in \mathbb{R} \cup i\mathbb{R} \).

In view of the preceding section and for reasons which will become clearer later, we will call this equation the restricted Killing spinor equation. The following proposition shows the role of solutions of the restricted Killing spinor equation in the theory of surfaces in \( \mathbb{R}^3, \mathbb{S}^3 \) and \( \mathbb{H}^3 \). In fact, we see that the integrability conditions for such sections of the spinor bundle are exactly the Gauß and Codazzi-Mainardi equations.

In the following, \((e_1, e_2)\) denotes a positively oriented local orthonormal basis of \( \Gamma(TM) \).

Proposition 3.1. Assume that \( (M^2, g) \) admits a non trivial solution of Equation \( (7) \) and let \( S = 2T \), then
\[ (\nabla_X S)(Y) = (\nabla_Y S)(X) \quad \text{(Codazzi-Mainardi Equation)}, \]
and
\[ R_{1212} - \det(S) = 4\eta^2 \quad \text{(Gauß Equation)}, \]
where \( R_{1212} = g(R(e_1, e_2)e_2, e_1) \), and \( R \) is the Riemann tensor of \( M \).
**Proof.** Let \( \varphi \) a non-trivial solution of (7). We compute the action of the spinorial curvature tensor \( \mathcal{R} \) on \( \varphi \) defined for all \( X, Y \in TM \) by

\[
\mathcal{R}(X, Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X, Y]} \varphi.
\]

Since it is skew-symmetric and \( \dim M = 2 \), we only compute

\[
\nabla_{e_1} \nabla_{e_2} \varphi = \nabla_{e_1} (-T(e_2) \cdot \varphi - \eta e_2 \cdot \omega \cdot \varphi)
= \nabla_{e_1} (-T(e_2) \cdot \varphi - \eta e_1 \cdot \varphi)
= -\nabla_{e_1} T(e_2) \cdot \varphi - T(e_2) \cdot \nabla_{e_1} \varphi - \eta \nabla_{e_1} e_1 \cdot \varphi - \eta e_1 \cdot \nabla_{e_1} \varphi
= -\nabla_{e_1} T(e_2) \cdot \varphi + T(e_2) \cdot T(e_1) \cdot \varphi - \eta T(e_2) \cdot e_2 \cdot \varphi
- \eta \nabla_{e_1} e_1 \cdot \varphi + \eta e_1 \cdot T(e_1) \cdot \varphi - \eta^2 e_1 \cdot e_2 \cdot \varphi
\]

as well as

\[
\nabla_{e_2} \nabla_{e_1} \varphi = -\nabla_{e_2} T(e_1) \cdot \varphi + T(e_1) \cdot T(e_2) \cdot \varphi + \eta T(e_1) \cdot e_1 \cdot \varphi
+ \eta \nabla_{e_2} e_2 \cdot \varphi - \eta e_2 \cdot T(e_2) \cdot \varphi + \eta^2 e_1 \cdot e_2 \cdot \varphi.
\]

So, taking into account that \([e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1\), a straightforward computation gives

\[
\mathcal{R}(e_1, e_2)\varphi = \left( (\nabla_{e_2} T)(e_1) - (\nabla_{e_1} T)(e_2) \right) \cdot \varphi
- \left( T(e_1) \cdot T(e_2) - T(e_2) \cdot T(e_1) \right) \cdot \varphi
- 2\eta^2 e_1 \cdot e_2 \cdot \varphi.
\]

Equations (9) and (10) yield

\[
C \cdot \varphi = Ge_1 \cdot e_2 \cdot \varphi.
\]

Note that \( e_1 \cdot e_2 \cdot \varphi = -i \varphi \), hence

\[
C \cdot \varphi^\pm = \pm i G \varphi^\mp.
\]

Applying two times this relation, it suffices to note that

\[
||C||^2 \varphi^\pm = -G^2 \varphi^\pm,
\]

and so \( C = 0 \) and \( G = 0 \). \( \square \)
Note that up to rescaling, we can take \( \eta = 0, 1/2, \) or \( i/2. \) The case \( \eta = 0 \) is treated in [4] and is the starting point of the proof of Theorem 1.1. We will discuss the cases \( \eta = 1/2 \) and \( \eta = i/2 \) separately. We begin by

**Lemma 3.2.** Let \( \varphi \) be a non-trivial solution of the restricted Killing spinor equation \([7]\). Then

- if \( \eta = 1/2, \ \varphi \) has constant norm and the symmetric endomorphism \( T, \) viewed as a covariant symmetric 2-tensor, is given by
  \[
  T(X,Y) = \frac{1}{2} \Re(X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi/|\varphi|^2)
  \]
- if \( \eta = i/2, \ \varphi \) satisfies \( X|\varphi|^2 = -\Re(X \cdot \overline{\varphi}, \varphi) \) and one has
  \[
  T(X,Y)|\varphi|^2 = \frac{1}{2} \Re(X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi) + \frac{1}{2} \left(|\varphi^-|^2 - |\varphi^+|^2\right)g(X,Y)
  \]

**Proof.** The first claim of each case is proved in Proposition 2.2. Let \( T_{jk} = g(T(e_j), e_k), \) then, for \( j = 1, 2, \)

\[
\nabla_{e_j} \varphi = -\sum_{k=1}^2 T_{jk} e_k \cdot \varphi + i \eta e_j \cdot \overline{\varphi}.
\]

Taking Clifford multiplication by \( e_1 \) and the scalar product with \( \varphi, \) we get

\[
\Re(e_1 \cdot \nabla_{e_j} \varphi, \varphi) = -\sum_{k=1}^2 T_{jk} \Re(e_1 \cdot e_k \cdot \varphi, \varphi) + \Re(i \eta e_1 \cdot e_j \cdot \overline{\varphi}, \varphi).
\]

Since \( \Re(e_1 \cdot e_k \cdot \varphi, \varphi) = -\delta_{jk}|\varphi|^2, \) it follows, by symmetry of \( T\)

\[
\Re(e_1 \cdot \nabla_{e_j} \varphi + e_j \cdot \nabla_{e_1} \varphi, \varphi) = 2T_{ij}|\varphi|^2 - 2\Re(i \eta \overline{\varphi}, \varphi)\delta_{ij}.
\]

This completes the proof by taking \( \eta = 1/2 \) or \( \eta = i/2. \) \( \square \)

Now, we prove that the necessary conditions on a spinor field \( \psi \in \Gamma(\Sigma M) \) obtained in the previous section (i.e. Proposition 2.2 and Equation (6)) are enough to prove that \( \psi \) is a solution of the restricted Killing spinor equation.

**The case \( \eta = 1/2: \)** Consider a non-trivial spinor field \( \psi \) of constant length, which satisfies \( D\psi^\pm = (H \pm i)\psi^\mp. \) Define the following 2-tensors on \((M^2, g)\)

\[
T^\pm(X,Y) = \Re(\nabla_X \psi^\pm, Y \cdot \psi^\mp).
\]

First note that

\[
\text{tr} T^\pm = -\Re(D\psi^\pm, \psi^\mp) = -\Re((H \pm i)\psi^\mp, \psi^\mp) = -H|\psi^\mp|^2. \tag{11}
\]

We have the following relations

\[
T^\pm(e_1, e_2) = \Re(\nabla_{e_1} \psi^\pm, e_2 \cdot \psi^\mp) = \Re(e_1 \cdot \nabla_{e_1} \psi^\pm, e_1 \cdot e_2 \cdot \psi^\mp) = \Re(e_2 \cdot \nabla_{e_2} \psi^\pm, e_1 \cdot e_2 \cdot \psi^\mp) = \Re((H \pm i)\psi^\mp, e_1 \cdot e_2 \cdot \psi^\mp) + \Re(\nabla_{e_2} \psi^\pm, e_1 \cdot \psi^\mp) = |\psi^\mp|^2 + T^\pm(e_2, e_1). \tag{12}
\]
Lemma 3.3. The 2-tensors $T^\pm$ are related by the equation

$$|\psi^+|^2 T^+ = |\psi^-|^2 T^-$$

Proof. This relation is trivial at any point $p \in M$ where $|\psi^+|^2$ or $|\psi^-|^2$ vanishes. Therefore we can assume in the following that both spinors $\psi^+$ and $\psi^-$ are not zero in the neighbourhood of a point in $M$.

With respect to the scalar product $\Re(\cdot, \cdot)$, the spinors $e_1 \cdot \frac{\psi^-}{|\psi^-|}$ and $e_2 \cdot \frac{\psi^-}{|\psi^-|}$ form a local orthonormal basis of $\Gamma(\Sigma^+M)$. Hence, in this basis, we can write

$$\nabla_X \psi^+ = \Re(\nabla_X \psi^+, e_1 \cdot \frac{\psi^-}{|\psi^-|}) e_1 \cdot \frac{\psi^-}{|\psi^-|} + \Re(\nabla_X \psi^+, e_2 \cdot \frac{\psi^-}{|\psi^-|}) e_2 \cdot \frac{\psi^-}{|\psi^-|}$$

where the vector field $T^+(X)$ is defined by

$$g(T^+(X), Y) = T^+(X, Y), \quad \forall Y \in TM.$$ 

In the same manner, we can show that

$$\nabla_X \psi^- = \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+.$$

Since $\psi$ has constant length, for all vector $X$ tangent to $M$, we have

$$0 = X|\psi|^2 = X(|\psi^+|^2 + |\psi^-|^2) = 2\Re(\nabla_X \psi^+, \psi^+) + 2\Re(\nabla_X \psi^-, \psi^-) = 2\Re(W(X) \cdot \psi^-, \psi^+)$$

with

$$W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2}.$$ 

To conclude, it suffices to note that Equations (11) and (12) imply $W$ is traceless and symmetric, and that Equation (13) implies that $W$ has rank less or equal to 1. This obviously implies $W = 0$. \qed

Proposition 3.4. Assume that there exists on $(M^2, g)$ a non-trivial solution $\psi$ of the equation $D\psi = H\psi - i\psi$ with constant length. Then such a solution satisfies the restricted Killing spinor equation with $\eta = 1/2$.

Proof. Let $F := T^+ + T^-$. Lemma 3.3 and the beginning of its proof imply

$$\frac{F}{|\psi|^2} = \frac{T^+}{|\psi^-|^2} = \frac{T^-}{|\psi^+|^2}.$$
Hence $F/|\psi|^2$ is well defined on the whole surface $M$, and
\[ \nabla_X\psi = \nabla_X\psi^+ + \nabla_X\psi^- = \frac{F(X)}{|\psi|^2} \cdot \psi \] (14)
where the vector field $F(X)$ is defined by $g(F(X), Y) = F(X, Y)$, $\forall Y \in TM$. Note that by Equation (12), the 2-tensor $F$ is not symmetric. Define now the symmetric 2-tensor
\[ T(X, Y) = -\frac{1}{2|\psi|^2} (F(X, Y) + F(Y, X)) . \]
Observe that $T$ is defined as in Lemma 3.2. It is straightforward to show that
\[ T(e_1, e_1) = -F(e_1, e_1)/|\psi|^2 , \quad T(e_2, e_2) = -F(e_2, e_2)/|\psi|^2 , \]
\[ T(e_1, e_2) = -F(e_1, e_2)/|\psi|^2 + \frac{1}{2} \quad \text{and} \quad T(e_2, e_1) = -F(e_2, e_1)/|\psi|^2 - \frac{1}{2} \]
once more by Equation (12). Taking into account these last relations in Equation (14), we conclude
\[ \nabla_X\psi = -T(X) \cdot \psi - \frac{1}{2}X \cdot \omega \cdot \psi . \]
\[ \square \]

The case $\eta = i/2$:

**Proposition 3.5.** Assume that there exists on $(M^2, g)$ a nowhere vanishing solution $\psi$ of the equation $D\psi = H\psi + \bar{\psi}$. Then, if this solution satisfies
\[ X|\psi|^2 = -\Re(X \cdot \bar{\psi}, \psi) , \quad \forall X \in \Gamma(TM), \]
then it is solution of the restricted Killing spinor equation with $\eta = i/2$.

**Proof.** Defining the 2-tensors $T^\pm$ as in the previous case, we get
\[ \text{tr}T^\pm = -(H \mp 1)|\psi^\mp|^2 , \quad (15) \]
and
\[ T^\pm(e_1, e_1) = T^\pm(e_2, e_2) . \quad (16) \]
First note that
\[ -\Re(X \cdot \bar{\psi}, \psi) = -\Re(X \cdot \psi^+, \psi^-) + \Re(X \cdot \psi^-, \psi^+) = 2\Re(X \cdot \psi^-, \psi^+) . \]
Therefore, following the proof of Lemma 3.3, we get
\[ \Re(X \cdot \psi^- \cdot \psi^+) = \Re(W(X) \cdot \psi^- \cdot \psi^+) \quad (17) \]
with
\[ W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2} . \]
As in the previous case, Equations (15), (16) and (17) imply that $W - \text{Id}_{TM}$ is a symmetric, traceless endomorphism of rank not greater than 1, hence $W = \text{Id}_{TM}$ and we have the relation
\[ |\psi^+|^2T^+ - |\psi^-|^2T^- = |\psi^+|^2|\psi^-|^2g . \]
Therefore, if we define the symmetric 2-tensor $F = T^+ + T^- + \frac{1}{2}(|\psi^+|^2 - |\psi^-|^2)g$, we have on the whole surface $M$

$$F = \frac{T^+ + T^- + (|\psi^+|^2 - |\psi^-|^2)g}{|\psi^+|^2 + |\psi^-|^2} = \frac{T^- - 1}{2}g = \frac{T^+}{|\psi^-|^2} - \frac{1}{2}g.$$  

On the other hand, we get

$$\nabla_X\psi = \nabla_X\psi^+ + \nabla_X\psi^- = \frac{T^+(X)}{|\psi^-|^2} \cdot \psi^- + \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+.$$  

These two last equations imply

$$\nabla_X\psi = \frac{F(X)}{|\psi|^2} \cdot (\psi^+ + \psi^-) + \frac{1}{2}X \cdot \psi^- - \frac{1}{2}X \cdot \psi^+,$$

which is equivalent to

$$\nabla_X\psi = -T(X) \cdot \psi - \frac{1}{2}X \cdot \overline{\psi}.$$  

Naturally, we put $T = -\frac{F}{|\psi|^2}$ and note that $T$ is defined as in Lemma 3.2. □

4. Surfaces in $S^3$ or $H^3$

We are now able to generalize Theorem 1.1 to surfaces in $S^3$ or $H^3$. In section 2, we saw that an oriented, immersed surface $M^2 \hookrightarrow S^3$ (resp. $H^3$) inherits an induced metric $g$, a spin structure, and a solution $\varphi$ of

$$D\varphi = H\varphi - i\overline{\varphi} \quad \text{ (resp. } D\varphi = H\varphi + \overline{\varphi}) \quad \text{ (18)}$$

with constant length (resp. with $X|\varphi|^2 = -\Re(X \cdot \overline{\varphi}, \varphi)$ for all vector $X$ tangent to $M$). This spinor field $\varphi$ on $M^2$ is the restriction of a real (resp. imaginary) Killing spinor field in $S^3$ (resp. $H^3$). Section 3 shows that at least locally the converse is true. Assume that there exists a solution of Equation (18) on an oriented, 2-dimensional Riemannian manifold $(M^2, g)$ endowed with a spin structure, for a given function $H : M \to \mathbb{R}$. Then this solution satisfies the restricted Killing spinor equation with a well defined endomorphism $T : TM \to TM$ with $\text{tr}T = H$. Moreover, there exists an isometric immersion $(M^2, g) \hookrightarrow S^3$ (resp. $H^3$) with second fundamental form $S = 2T$.

**Theorem 4.1.** Let $(M^2, g)$ be an oriented, 2-dimensional manifold and $H : M \to \mathbb{R}$ a smooth function. Then the following data are equivalent:

1. An isometric immersion $(\tilde{M}^2, g) \to S^3$ of the universal covering $\tilde{M}^2$ into the 3-dimensional round sphere $S^3$ with mean curvature $H$.
2. A solution $\varphi$ of the Dirac equation

$$D\varphi = H\varphi - i\overline{\varphi}$$

with constant length.
3. A pair $(\varphi, T)$ consisting of a symmetric endomorphism $T$ such that $\text{tr}(T) = H$ and a spinor field $\varphi$ satisfying the equation

$$\nabla_X\varphi + T(X) \cdot \varphi - \frac{i}{2}X \cdot \overline{\varphi} = 0.$$
**Theorem 4.2.** Let \((M^2, g)\) be an oriented, 2-dimensional manifold and \(H : M \to \mathbb{R}\) a smooth function. Then the following data are equivalent:

1. An isometric immersion \((\tilde{M}^2, g) \to \mathbb{H}^3\) of the universal covering \(\tilde{M}^2\) into the 3-dimensional hyperbolic space \(\mathbb{H}^3\) with mean curvature \(H\).
2. A nowhere vanishing solution \(\varphi\) of the Dirac equation
   \[
   D\varphi = H\varphi + \overline{\varphi}
   \]
satisfying
   \[
   X|\varphi|^2 = -\Re(X \cdot \overline{\varphi}, \varphi) \quad \forall X \in \Gamma(TM).
   \]
3. A pair \((\varphi, T)\) consisting of a symmetric endomorphism \(T\) such that \(\text{tr}(T) = H\) and a spinor field \(\varphi\) satisfying the equation
   \[
   \nabla X\varphi + T(X) \cdot \varphi + \frac{1}{2} X \cdot \overline{\varphi} = 0 \quad \forall X \in \Gamma(TM).
   \]

**Remark 4.3.** It has been pointed out to us that the case of surfaces in \(\mathbb{S}^3\) has already been treated by Leonard Voss (Diplomarbeit, Humboldt-Universität zu Berlin, unpublished).

5. **Hypersurfaces in \(\mathbb{R}^4\)**

We conclude by giving a characterization of hypersurfaces in the Euclidean 4-space in terms of a special section of the intrinsic spinor bundle of the hypersurface, in a very similar way to that of Theorem 1.1.

Let \(M^3\) be an oriented hypersurface isometrically immersed into \(\mathbb{R}^4\), denote by \(\nu\) its unit normal vector field. Then \(M^3\) is endowed with a spin structure, canonically induced by that of \(\mathbb{R}^4\). Denote by \(\Sigma M\) the corresponding spinor bundle and \(\Sigma^+ \mathbb{R}^4\) the bundle of positive spinors in \(\mathbb{R}^4\). We then have the analogous result of Proposition 2.1:

**Proposition 5.1.** There exists an identification of \(\Sigma^+ \mathbb{R}^4\) with \(\Sigma M\), which after restriction to \(M\), sends every spinor field \(\psi \in \Gamma(\Sigma^+ \mathbb{R}^4)\) to the spinor field denoted by \(\psi^* \in \Gamma(\Sigma M)\). Moreover, if \(\cdot\) (resp. \(\cdot\)) stands for Clifford multiplication on \(\Sigma^+ \mathbb{R}^4\) (resp. \(\Sigma M\)), then one has

\[
(X \cdot_{\mathbb{R}^4} \nu \cdot_{\mathbb{R}^4} \psi)^* = X \cdot \psi^*,
\]
for any vector field \(X\) tangent to \(M\).

Recall the following definition

**Definition 5.2.** A symmetric 2-tensor \(T \in S^2(M)\) is called a Codazzi tensor if it satisfies the Codazzi-Mainardi equation, i.e.

\[
(\nabla_X T)(Y) = (\nabla_Y T)(X) \quad \forall X, Y \in \Gamma(TM),
\]

\((T\) being viewed in this formula via the metric \(g\) as a symmetric endomorphism of the tangent bundle).

We now prove the following
Theorem 5.3. Let \((M^3, g)\) be an oriented, 3-dimensional Riemannian manifold. Then the following data are equivalent:

1. An isometric immersion \((\tilde{M}^3, g) \to \mathbb{R}^4\) of the universal covering \(\tilde{M}^3\) into the Euclidean space \(\mathbb{R}^4\) with second fundamental form \(h\).
2. A pair \((\varphi, T)\) consisting of a Codazzi tensor \(T\) such that \(2T = h\) and a non-trivial spinor field \(\varphi\) satisfying, for any \(X \in \Gamma(TM)\), the equation

\[
\nabla_X \varphi + T(X) \cdot \varphi = 0.
\]

Proof. Let \((M^3, g)\) be an oriented hypersurface isometrically immersed into \(\mathbb{R}^4\) with second fundamental form \(h\). Let \(\psi\) be any parallel positive spinor field on \(\mathbb{R}^4\). Denote by \(\varphi := \psi^* \in \Gamma(\Sigma M)\) the restriction of \(\psi\) given by Proposition 5.1. Then Gauß formula (3) yields

\[
\nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi = 0.
\]

Since \(h\) is a second fundamental form, it is clear that \(T = \frac{1}{2} h\) is a Codazzi tensor and that \((\varphi, T)\) give the desired pair.

Conversely, if \((M^3, g)\) is an oriented, 3-dimensional Riemannian manifold admitting such a pair \((\varphi, T)\), then obviously Codazzi-Mainardi equation holds for \(h = 2T\).

Therefore, the action of the spinorial curvature tensor on the spinor \(\varphi\) is given by

\[
\mathcal{R}(X, Y) \varphi = \left( T(Y) \cdot T(X) - T(X) \cdot T(Y) \right) \cdot \varphi
\]

(20)

Let \((e_1, e_2, e_3)\) be a positively oriented local orthonormal basis of \(\Gamma(TM)\). Then Equation (20) yields

\[
\sum_{k \neq l} \left( \mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl} \right) e_k \cdot e_l \cdot \varphi = 0
\]

which imply in dimension 3 that each component

\[
\mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl}
\]

is zero, since for \(1 \leq k < l \leq 3\) and \(1 \leq k' < l' \leq 3\),

\[
\mathcal{R}(e_k \cdot e_l \cdot \varphi, e_{k'} \cdot e_{l'} \cdot \varphi) = \pm \delta_{kk'} \delta_{ll'} |\varphi|^2.
\]

Therefore \(h = 2T\) satisfies the Gauß equation. \(\square\)

Remark 5.4. Let \((\varphi, T)\) be a pair as in Theorem 5.3 (2). Then necessarily the Codazzi tensor \(T\) has to be defined as the energy-momentum tensor associated with the spinor field \(\varphi\) (see for example [6], [3] or [11]). Such a special spinor field is then called a Codazzi Energy-Momentum spinor, and generalizes the notion of Killing spinors (see [10] for a study of these particular spinor fields).
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