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Attraction of like-charged walls with counterions only: Exact results for the 2D cylinder geometry

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Abstract We study a 2D system of identical mobile particles on the surface of a cylinder of finite length $d$ and circumference $W$, immersed in a medium of dielectric constant $\varepsilon$. The two end-circles of the cylinder are like-charged with the fixed uniform charge densities, the particles of opposite charge $-e$ ($e$ being the elementary charge) are coined as “counterions”; the system as a whole is electroneutral. Such a geometry is well defined also for finite numbers of counterions $N$. Our task is to derive an effective interaction between the end-circles mediated by the counterions in thermal equilibrium at the inverse temperature $\beta$. The exact solution of the system at the free-fermion coupling $\Gamma \equiv \beta e^2/\varepsilon = 2$ is used to test the convergence of the pressure as the (even) number of particles increases from $N = 2$ to $\infty$. The pressure as a function of distance $d$ is always positive (effective repulsion between the like-charged circles), decaying monotonously; the numerical results for $N = 8$ counterions are very close to those in the thermodynamic limit $N \to \infty$. For the couplings $\Gamma = 2\gamma$ with $\gamma = 1, 2, \ldots$, there exists a mapping of the continuous two-dimensional (2D) Coulomb system with $N$ particles onto the one-dimensional (1D) lattice model of $N$ sites with interacting sets of anticommuting variables. This allows one to treat exactly the density profile, two-body density and the pressure for the couplings $\Gamma = 4$ and 6, up to $N = 8$ particles. Our main finding is that the pressure becomes negative at large enough distances $d$ if and only if both like-charged walls carry a nonzero charge density. This indicates a like-attraction in the thermodynamic limit $N \to \infty$ as well, starting from a relatively weak coupling constant $\Gamma$ in between 2 and 4. As a by-product of the formalism, we derive specific sum rules which have direct impact on characteristics of the long-range decay of 2D two-body densities along the two walls.

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1 Introduction

The study of equilibrium statistical mechanics of classical (i.e. nonquantum) systems of particles interacting pairwisely by Coulomb potential is of particular importance in condensed matter and soft matter physics. In the real 3-dimensional (3D) space of practical interest, the Coulomb potential in vacuum of dielectric constant \( \varepsilon = 1 \) has in Gauss units the standard form \( \phi(r) = 1/r \) with \( r \) being the modulus of \( \mathbf{r} \). The definition of the Coulomb potential can be extended to any dimension \( \nu = 1, 2, \ldots \) as the solution of the Poisson equation

\[
\Delta \phi(r) = -s_\nu \delta(r),
\]

where \( s_\nu = 2\pi^{\nu/2}/\Gamma(\nu/2) \) (\( \Gamma \) being the Gamma function) is the surface area of the \( \nu \)-dimensional unit sphere. In an infinite space, the solution of (1.1), subject to the boundary condition \( \nabla \phi(r) \to 0 \) as \( r \to \infty \), reads as \( r \) in 1D, \(-\ln(r/L)\) (\( L \) is a free length scale) in 2D and \( r^{2-\nu}/(\nu-2) \) in spatial dimensions \( \nu \geq 3 \). The Fourier component of the Coulomb potential exhibits the singular \( 1/k^2 \) behavior in any dimension; this maintains many generic properties of 3D Coulomb systems like screening [56].

In one-component Coulomb models, the system of mobile (pointlike) particles of the same (say elementary) charge \( -e \) is neutralized by a fixed “background” charge density. The most known system of this kind is the jellium model of real materials in which the homogeneous background charge density of heavy nucleus ions is spread over the whole space of the domain mobile electrons are confined to [7]. Since due to the electroneutrality requirement the particle number is proportional to the domain’s volume, the system is “dense” and therefore exhibits good screening properties in thermal equilibrium, i.e., the particle correlation functions exhibit a short-range, usually exponential, decay at asymptotically large distances. The 1D jellium model, treated by using a transfer matrix method [51] and a generating function method [22], is exactly solvable for any temperature and particle density. The system exhibits the translational symmetry breaking of the particle number density which oscillates periodically in the thermodynamic limit [8,50]. Boundary effects are important in 1D [19]. In 2D characterized by the logarithmic Coulomb potential, the relevant coupling constant is \( \Gamma \equiv \beta e^2/\varepsilon \) where \( \beta \) is the inverse temperature and \( \varepsilon \) is the dielectric constant of the medium the particles are immersed in. These systems are especially important because they are exactly solvable, besides the mean-field \( \Gamma \to 0 \) limit, also at a special finite temperature. The exact solution of the 2D jellium model at the “free-fermion” coupling \( \Gamma = 2 \) involves the bulk case [13,33] as well as semi-infinite and fully finite geometries, see reviews [23,37].

A series of works was devoted to the study of thermal equilibrium of 2D one-component Coulomb systems for a series of couplings \( \Gamma = 2\gamma \) where \( \gamma = 1, 2, 3, \ldots \) is a (positive) integer. There are two basic approaches how to express integer powers of the Vandermonde determinants. The method using
a mapping of the 2D Coulomb system onto a 1D lattice anticommuting-field theory was initiated in \[72\] and subsequently used in a series of works \[73,74,75,81,83,84\] dealing with sum rules, finite-size corrections, asymptotic decay of two-body correlations along domain’s boundaries, etc. Another method using Jack polynomials was developed in \[92,93\]. The relation between the two methods was established in \[26\].

In this paper, another version of the one-component Coulomb systems is studied, with the background charge density spread over the boundary of the constraining domain. Due to the electroneutrality, the number of mobile charges is proportional to the domain’s boundary and the screening properties of the “sparse” system are not good. This kind of models describes biological experiments with macromolecules (colloids, polyelectrolytes) which are performed in polar solvents like water. Through the dissociation of functional surface groups, the surface of macromolecule releases micro-ions into the polar solvent, acquiring in this way a fixed surface charge density \[2,53\]. Since the charge of micro-ions is opposite to that of the surface charge density, they are coined as “counterions”. In thermal equilibrium, the charged surface of the macromolecule and the surrounding counterions form a neutral entity known as the electric double layer \[3,4,29,58\]. The effective interaction between two like-charged walls, mediated by counterions, is of primary experimental and theoretical interest \[30\]. At small enough temperatures, a counter-intuitive attraction of like-charged macromolecules was observed experimentally \[10,21,46,49,70\] as well as by computer simulations \[13,27,29,48\]; for more recent numerical and analytical advances in this field, see reviews \[12,53,64\].

For large macromolecules with the surface charge of order of thousands elementary charges, the curved surface can be replaced by an infinite rectilinear wall. Thermal equilibrium of charged surfaces with counterions only is usually considered in the canonical ensemble at the inverse temperature \[\beta = 1/(k_B T)\]. Two basic geometries are studied. In the case of one wall with counterions constrained to the semi-infinite (half) space, the particle density profile is of interest. The particle density at the wall is related to the wall’s surface charge density via the contact-value theorem \[11,16,17,31,32\]. To obtain the effective interaction of two parallel walls at distance \[d\], one calculates the pressure, either from the derivative of the free energy with respect to \[d\] or from contact densities. Since the background charge is confined to the surfaces of the walls, it stays inside the system when changing infinitesimally \[d\] and so, in contrast to one-component jellium systems \[17\], there is no ambiguity in the definition of the pressure.

From a theoretical point of view, models of charged wall surfaces with counterions only are probably the simplest ones to study the equilibrium properties of Coulomb fluids. The weak-coupling (high-temperature, WC) limit is described by the mean-field Poisson-Boltzmann (PB) approach \[2\] and by its systematic improvement within the field-theoretical formulation via the loop expansion \[3,65,69\]. In a single pure solvent, two symmetrically charged walls always repel one another in the WC limit; this is no longer true for a mixture of polar solvents when the medium becomes inhomogeneous due to solvation-related forces \[9,85\].
The strong-coupling (low-temperature, SC) limit of the fluid regime is more controversial. Within the virial SC theory put forward by Moreira and Netz [59,60,61,66], the leading SC term of the counterion density corresponds to a single (noninteracting) particle theory in the electric potential of charged wall(s) which has been confirmed by Monte Carlo (MC) simulations [20,40,41,59,60,61,62]. Next correction orders in inverse powers of the coupling constant, obtained within a virial fugacity expansion, require a renormalization of infrared divergencies via the electroneutrality condition. Comparison with MC simulations shows that the first correction term has the correct functional form in space, but the wrong prefactor. A dressed-ion version of the virial SC theory was applied to realistic Coulomb fluids in the presence of salt [42,43,44]; such an approach has been tested against simulations therein and against experiments in [45].

Another type of SC approaches is based on the formation of the classical Wigner crystal of counterions on the wall surfaces at zero temperature [28,52,88]. Based on a harmonic expansion of the interaction energy in particle deviations from their ground-state Wigner positions [76,77], the leading single-particle picture of the virial SC approach was reproduced. The first correction term to the counterion density is in excellent agreement with MC data for strong and intermediate Coulombic couplings. Although the first correction term is small relative to the leading one for small distances between the parallel walls, its precise form is important when calculating the pressure between the charged walls via the contact-value theorem at larger distances and specifying regions of the couplings and of the walls distances where the pressure is attractive.

The crucial problem with the Wigner SC approach is that Wigner crystals become unstable at extremely large values of the coupling constant; the melting of the single-layer and double-layer Wigner structures to their fluid phases is described in references [91] and [25,87], respectively. In spite of this taking the Wigner lattice as a reference provides an adequate description of the fluid phase up to intermediate couplings. The strong Coulomb repulsion of identical charges causes that their pair correlation function almost vanishes at small distances. The idea of a correlation hole was applied successfully in various ways to go beyond the PB theory [5,6,24,67,71]. To adapt the quantitatively correct Wigner SC approach to the fluid phase, the Wigner structure was substituted by a correlation hole in [82]. In the case of one wall with counterions only, another correlation-hole theory of the self-consistent nature [68] leads to a modified PB integral equation which implies the exact density profiles in both WC and SC limits. In contrast to similar attempts to establish a universal theory working well for any coupling [14,80], the density profile satisfies the contact-value theorem and provides a crossover from a short-distance exponential to a large-distance algebraic PB decay from the charged wall via a large density plateau.

The WC and SC analyses were done explicitly on the exactly solvable 1D gas of counterions [20]. As concerns the 2D problem of one line-charged wall with counterions only, the density profile at \( \Gamma = 2 \) was derived by Jancovici [36]. The pressure for two parallel walls at distance \( d \) was obtained in the symmetric and nonsymmetric cases in references [78] and [80], respectively.
In the case of like-charged walls, the pressure is always positive and decays monotonously from infinity at $d \to 0$ to 0 as $d \to \infty$. Another type of exact results concerns the Manning condensation of counterions at the charged surface of the 3D cylinder \cite{15,63}.

In equilibrium statistical mechanics of fluid systems it is generally believed that, except for phase transitions, a few particles are able to reproduce adequately statistical quantities of large systems \cite{55}. The primary motivation for the present work is the absence of exact results for 2D one-component models with the coupling constants $\Gamma > 2$ where one expects the counterintuitive phenomenon of the attraction between like-charged walls. We consider the cylinder of circumference $W$ and finite length $d$ with the charged circle ends, the counterions are allowed to move freely on the cylinder surface; such a model is well defined also for finite numbers of particles $N$. As is shown in this paper for the exactly solvable free-fermion coupling $\Gamma = 2$, the results for the pressure as the function of $d$ for $N = 8$ particles turn out to be very close to those for $N \to \infty$ particles. This fact justifies the exact treatment of the couplings $\Gamma = 4$ and 6 up to $N = 8$ particles by using the anticommuting-field formalism \cite{72} which can be done with modest computational efforts. It turns out that the attraction phenomenon of like-charged walls is observed for these relatively small couplings. As a by-product of the anticommuting-field formalism, we derive within the cylinder geometry the exact constraints (sum rules) for the particle one-body and two-body densities which have direct impact on characteristics of the long-range decay of two-body densities along the two walls in the pure 2D limit $W \to \infty$.

The paper is organized as follows. In Sect. 2 we review the general formalism for Coulomb systems confined to the surface of a cylinder and their mapping onto the 1D lattice model of interacting anticommuting fields for the coupling constant $\Gamma = 2\gamma$ with $\gamma$ a positive integer. The exact cylinder sum rules for the particle one-body and two-body densities are derived in Sect. 3. The impact of these sum rules on the long-range decay of 2D two-body densities along the two walls is explained in Sect. 4. Sect. 5 deals with the exactly solvable $\Gamma = 2$ case. The coupling constants $\Gamma = 4$ and 6 are treated for a finite number of particles in Sect. 6. The concluding Sect. 7 is a short recapitulation.

2 General formalism for cylinder geometry

2.1 Cylinder geometry

We consider the system of $N$ mobile pointlike particles with the elementary charge $-e$, confined to the surface of a cylinder of circumference $W$ and length $d$. The surface of the cylinder can be represented equivalently as a 2D semiperiodic rectangle domain $\Lambda$ of points $r = (x,y)$ with coordinates $x \in [0,d]$ (no restricting conditions at the end-points $x = 0, d$) and $y \in [0,W]$ (periodic boundary conditions at $y = 0, W$), see Fig. 1. It is useful to introduce the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. There are the fixed uniform line charge densities $\sigma e$ and $\sigma' e$ $(\sigma, \sigma'$ having dimension [length]$^{-1}$) along the $y$-axis at the end-points $x = 0$ and $x = d$, respectively.
We restrict ourselves to the like-charged line segments (circles), i.e., without any loss of generality, \( 0 \leq \sigma' \leq \sigma \). Introducing the asymmetry parameter

\[
\eta = \frac{\sigma'}{\sigma}, \quad \eta \in [0, 1],
\]

(2.1)

the symmetric case \( \sigma = \sigma' \) corresponds to \( \eta = 1 \). The overall electroneutrality condition is expressed as

\[
N = (\sigma + \sigma')W.
\]

(2.2)

The thermodynamic limit corresponds to the limits \( N, W \to \infty \), keeping the ratio \( N/W = \sigma + \sigma' \) fixed. The system possesses the obvious exchange symmetry \( \sigma \leftrightarrow \sigma' \) under the coordinate transformation \( x \rightarrow d - x \). The dielectric constants of the walls \( \varepsilon_W \) and of the medium the particles are immersed in \( \varepsilon \) are considered to be the same, \( \varepsilon_W = \varepsilon \), i.e., there are no image charges.

![Fig. 1](image)

**Fig. 1** The cylinder geometry with the periodic boundary conditions (period \( W \)) along the \( y \)-axis. Two parallel lines (circles) with the fixed charge densities \( \sigma e \) and \( \sigma' e \) are localized at the end points \( x = 0 \) and \( x = d \), respectively. Pointlike counterions of charge \( -e \) are allowed to move freely between the two charged lines.

The Coulomb potential \( \phi \) at a spatial position \( r \in \Lambda \), induced by a unit charge at the origin \( \mathbf{0} \), is defined as the solution of the 2D Poisson equation

\[
\Delta \phi(r) = -\frac{2\pi \delta(r)}{\varepsilon},
\]

under the periodicity requirement along the \( y \)-axis with period \( W \). Considering the potential as a Fourier series in \( y \), it is obtained in the form [18]

\[
\phi(r) = \frac{1}{\varepsilon W} \sum_{k_y} \int_{-\infty}^{\infty} dk_x \frac{1}{k_x^2 + k_y^2} e^{i(k_x x + k_y y)}, \quad k_y \in \frac{2\pi n}{W}
\]

(2.3)

with \( n = 0, \pm 1, \ldots \) being any integer. It is seen that also in the mixed discrete-continuous Fourier representation of the Coulomb potential has the characteristic \( 1/k^2 \) form. After integration over \( k_x \) and summation over \( k_y \), the
The periodic Coulomb potential \(\phi(r)\) takes the form
\[
\phi(r) = -\frac{1}{\varepsilon} \ln \left| 2 \sinh \left( \frac{\pi z}{W} \right) \right| = -\frac{1}{2\varepsilon} \ln \left( 2 \cosh \left( \frac{2\pi x}{W} \right) - 2 \cos \left( \frac{2\pi y}{W} \right) \right).
\]
(2.4)

For small distances \(r \ll W\), this potential reduces to the 2D Coulomb one \(-\frac{1}{\varepsilon} \ln \frac{2\pi r}{W}\). At large distances along the cylinder \(x \gg W\), this potential behaves like the 1D Coulomb one \(-\pi |x|/\varepsilon W\). For the calculation of the Coulomb interaction between charge line densities and particle \(s\), the following formula is important:
\[
\int_0^W dy \phi(r) = -\frac{\pi}{\varepsilon} x.
\]
(2.5)

According to the analysis made in \([80]\), the Coulomb energy of \(N\) particles at spatial positions \(\{r_1, \cdots, r_N\}\) plus the fixed line charge densities \(\sigma e\) and \(\sigma' e\) consists of the self and mutual interactions of the line charge densities \(E_{ll} = -\pi \sigma \sigma' W e^2 / \varepsilon\), of the interaction of particles with line charge densities \(E_{pl} = \sum_{j=1}^{N} \pi (\sigma - \sigma') x_j e^2 / \varepsilon + N \pi \sigma' e^2 / \varepsilon\) and the pair interactions of the particles \(E_{pp} = \sum_{j<k=1}^{N} e^2 \phi(|r_j - r_k|)\). At inverse temperature \(\beta = 1/(k_BT)\), the Boltzmann factor of the total energy \(E_N = E_{ll} + E_{pl} + E_{pp}\) reads as
\[
e^{-\beta E_N(\{r\})} = e^{-\pi \Gamma (\sigma - \sigma') x} \prod_{j=1}^{N} e^{-\beta v(x_j)} \prod_{(j<k)=1}^{N} \left| 2 \sinh \frac{\pi (z_j - z_k)}{W} \right| \Gamma,
\]
(2.6)

where \(v(x)\) is the one-body potential energy given by
\[
\beta v(x) = \pi \Gamma (\sigma - \sigma') x
\]
(2.7)

and \(\Gamma = \beta e^2 / \varepsilon\) is the coupling constant. Within the canonical ensemble, the partition function is defined as
\[
Z_N(\gamma) = \frac{1}{N!} \int_A (\frac{dr_1}{\lambda^2}) \cdots \int_A (\frac{dr_N}{\lambda^2}) e^{-\beta E_N(\{r\})},
\]
(2.8)

where \(\lambda\) is the thermal de Broglie wavelength.

There exist two possible representations of the partition function.

Firstly, applying the formula
\[
\left| 2 \sinh \frac{\pi (z - z')}{W} \right| = e^{\frac{\pi}{2} \left( x + x' \right)} \left| e^{-\frac{\pi}{2} x} - e^{-\frac{\pi}{2} x'} \right|
\]
(2.9)
to each two-particle interaction Boltzmann factor the partition function can be reexpressed as
\[
Z_N(\gamma) = \left( \frac{W^2}{4\pi \lambda^2} \right)^N \exp \left[ -\pi \Gamma (\sigma')^2 W d \right] Q_N(\gamma),
\]
(2.10)
where

\[ Q_N(\gamma) = \frac{1}{N!} \int \prod_{j=1}^{N} \left[ d^2 z_j \, w_{\text{ren}}(r_j) \right] \prod_{j<k} |e^{2\pi z_j} - e^{2\pi z_k}|^\Gamma \]  

(2.11)

with the renormalized one-body Boltzmann factor \( w_{\text{ren}}(r) \equiv w_{\text{ren}}(x) \) given by

\[ w_{\text{ren}}(x) = \frac{4\pi}{W^2} \exp \left[ -\beta v(x) + \frac{\pi\Gamma}{W} (N-1)x \right]. \]  

(2.12)

The second representation of the partition function follows from another version of the formula (2.9):

\[ 2 \sinh \frac{\pi(z - z')}{W} = e^{-\pi \sigma(x + x')} \left| e^{2\pi z} - e^{2\pi z'} \right| \]  

(2.13)

Then the partition function is still given by (2.10) where

\[ Q_N(\gamma) = \frac{1}{N!} \int \prod_{j=1}^{N} \left[ d^2 z_j \, w_{\text{ren}}(r_j) \right] \prod_{j<k} |e^{2\pi z_j} - e^{2\pi z_k}|^\Gamma \]  

(2.14)

with the renormalized one-body Boltzmann factor

\[ w_{\text{ren}}(x) = \frac{4\pi}{W^2} \exp \left[ -\beta v(x) + \frac{\pi\Gamma}{W} (N-1)x \right]. \]  

(2.15)

In what follows, we shall use mainly the first representation (2.9)-(2.12). The free energy \( F_N \), defined by

\[ -\beta F_N = \ln Z_N, \]

is expressible in both cases as

\[ -\beta F_N(\gamma) = N \ln \left( \frac{W^2}{4\pi\lambda^2} \right) - \pi\Gamma(\sigma')^2 Wd + \ln Q_N(\gamma). \]  

(2.16)

The particle density at point \( r \in \Lambda \) is given by

\[ n(r) = \langle \hat{n}(r) \rangle, \quad \hat{n}(r) = \sum_{j=1}^{N} \delta(r - r_j), \]  

(2.17)

where \( \langle \cdot \cdot \cdot \rangle \) denotes the statistical average over the canonical ensemble and \( \hat{n}(r) \) is the microscopic particle number density. The particle density can be obtained in the standard way as the functional derivative

\[ n(r) = w_{\text{ren}}(r) \frac{1}{Q_N} \frac{\delta Q_N}{\delta w_{\text{ren}}(r)}. \]  

(2.18)

Since the one-body potential (2.7) depends on the \( x \)-coordinate only and due to the cylinder geometry, it holds that \( n(r) \equiv n(x) \).

The two-body density

\[ n^{(2)}(r, r') = \left\langle \sum_{(j \neq k)=1}^{N} \delta(r - r_j)\delta(r' - r_k) \right\rangle \]  

(2.19)
can be calculated as
\[ n^{(2)}(\mathbf{r}, \mathbf{r}') = w_{\text{ren}}(\mathbf{r})w_{\text{ren}}(\mathbf{r}') \frac{1}{Q_N} \frac{\delta^2 Q_N}{\delta w_{\text{ren}}(\mathbf{r}) \delta w_{\text{ren}}(\mathbf{r}')} \]  \tag{2.20}

The corresponding (truncated) Ursell function \( U \) and the density structure function \( S \) are defined by
\[ U(\mathbf{r}, \mathbf{r}') = n^{(2)}(\mathbf{r}, \mathbf{r}') - n(\mathbf{r})n(\mathbf{r}'), \]  \tag{2.21}
\[ S(\mathbf{r}, \mathbf{r}') = \langle \hat{n}(\mathbf{r})\hat{n}(\mathbf{r}') \rangle - n(\mathbf{r})n(\mathbf{r}') = U(\mathbf{r}, \mathbf{r}') + n(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'), \]  \tag{2.22}
respectively. Due to the cylinder geometry, the two-point functions \( n^{(2)} \), \( U \) and \( S \) depend on \( x, x' \) and \(|y - y'|\).

2.2 Mapping onto the 1D lattice anticommuting-field theory

For \( \Gamma = 2\gamma \) (\( \gamma = 1, 2, 3, \ldots \) a positive integer), the technique of anticommuting variables [72,74] allows us to express \( Q_N \) \tag{2.11} as an integral over Grassman variables; for the cylinder geometry the mapping is established in [75,80]. Let us consider a discrete chain of \( N \) sites \( j = 0, 1, \ldots, N - 1 \). At each site \( j \), there is \( \gamma \) variables of type \( \{\xi_j^{(\alpha)}\} \) and \( \gamma \) variables of type \( \{\psi_j^{(\alpha)}\} \) (\( \alpha = 1, \ldots, \gamma \)), all variables anticommute with each other. The multi-dimensional integral of the form \tag{2.11} can be expressed as the integral over anticommuting variables:
\[ Q_N(\gamma) = \int \mathcal{D}\psi \mathcal{D}\xi e^{S(\Xi, \Psi)}, \quad S(\Xi, \Psi) = \sum_{j=0}^{\gamma(N-1)} \Xi_j w_j \Psi_j. \]  \tag{2.23}

Here, \( \mathcal{D}\psi \mathcal{D}\xi \equiv \prod_{j=0}^{N-1} d\psi_j^{(\gamma)} \cdots d\psi_j^{(1)} d\xi_j^{(\gamma)} \cdots d\xi_j^{(1)} \) and the action \( S(\Xi, \Psi) \) involves pair interactions of composite operators
\[ \Xi_j = \sum_{j_1 + \cdots + j_\gamma = j, j_1, \ldots, j_\gamma \geq 0} \xi_j^{(1)} \cdots \xi_j^{(\gamma)}, \quad \Psi_j = \sum_{j_1 + \cdots + j_\gamma = j, j_1, \ldots, j_\gamma \geq 0} \psi_j^{(1)} \cdots \psi_j^{(\gamma)}, \]  \tag{2.24}
i.e. the products of \( \gamma \) anticommuting variables of one type with the prescribed sum of site indices. The interaction strengths \( w_j \) \( [j = 0, 1, \ldots, \gamma(N-1)] \) are given by
\[ w_j = W \int_0^d dx w_{\text{ren}}(x) \exp \left( -\frac{4\pi}{W} jx \right) = 1 - \exp \left[ -\frac{4\pi d}{j - \gamma W \sigma' + \frac{d}{2}} \right]. \]  \tag{2.25}
The main advantage of the present formalism is that the one-body and two-body particle densities are expressible explicitly in terms of averages over the anticommuting variables

$$\langle \cdots \rangle \equiv \frac{1}{Q_N(\gamma)} \int D\psi D\xi e^{S(\Xi,\Psi)} \cdots$$  \hspace{1cm} (2.26)

describing certain products of composite operators. Namely, the particle density at \( x \) is given by

$$n(x) = w_{\text{ren}}(x) \sum_{j=0}^{\gamma(N-1)} \langle \Xi_j \Psi_j \rangle \exp \left( -\frac{4\pi}{W} jx \right),$$  \hspace{1cm} (2.27)

the two-body density between points \( r_1 = (z_1, \bar{z}_1) \) and \( r_2 = (z_2, \bar{z}_2) \) is expressible as

$$n^{(2)}(z_1, \bar{z}_1; z_2, \bar{z}_2) = w_{\text{ren}}(x_1)w_{\text{ren}}(x_2) \sum_{j_1 k_1, j_2 k_2 = 0}^{\gamma(N-1)} \langle \Xi_{j_1} \Psi_{k_1} \Xi_{j_2} \Psi_{k_2} \rangle$$

$$\times \exp \left[ -\frac{2\pi}{W} (j_1 z_1 + k_1 \bar{z}_1 + j_2 z_2 + k_2 \bar{z}_2) \right].$$  \hspace{1cm} (2.28)

As a trivial application of the formalism, we derive the basic formula of the contact-value theorem. The pressure \( P_N \) is the force between the charged circles, calculated per unit length of one of the circles:

$$\beta P_N = \frac{\partial}{\partial d} \left( -\frac{\beta F_N}{W} \right).$$  \hspace{1cm} (2.29)

The two circles repel (attract) one another if the pressure is positive (negative). Using the formula (2.16) for \( F_N \), the anticommuting representation (2.23) of \( Q_N \) and the relation

$$\frac{\partial w_j}{\partial d} = W w_{\text{ren}}(d) \exp \left( -\frac{4\pi}{W} jd \right),$$  \hspace{1cm} (2.30)

we arrive at the relationships given by the contact-value theorem

$$\beta P_N = n(d) - 2\pi \gamma (\sigma')^2 = n(0) - 2\pi \gamma \sigma^2,$$  \hspace{1cm} (2.31)

where the second equality follows directly from the invariance of the pressure with respect to the exchange symmetry \( \sigma \leftrightarrow \sigma' \) under the coordinate transformation \( x \rightarrow d - x \). Note that if \( \sigma' = 0 \) it holds that \( \beta P_N = n(d) \) and since the particle density is a positive quantity the pressure cannot be negative. We shall often use the notation

$$\tilde{P} \equiv \frac{\beta P}{2\pi \gamma \sigma^2} = \frac{n(0)}{2\pi \gamma \sigma^2} - 1$$  \hspace{1cm} (2.32)

for the dimensionless pressure. Since the particle density \( n(0) \) is positive, it holds that \( \tilde{P} \geq -1 \); the value \( \tilde{P} = -1 \) corresponds to the strongest possible attraction between the two charged walls.
3 Sum rules for the cylinder geometry

As was shown in [73], there exist specific linear transformations of anticommuting variables which keep the composite form of the composite operators \( (2.24) \). Most of transformations consist in a simple rescaling of one or all anticommuting components of a given \( \xi \) or \( \psi \) type, however, there is one nontrivial transformation which mixes all anticommuting-field components of a given type. Each transformation leads to the exact constraints (sum rules) for the correlation functions of the composite operators whose forms depend on the particular form of the interaction strengths \( \{ w_j \}_{j=0}^{(N-1)} \). These sum rules can be used to derive integral/differential equations for the one- and two-body densities whose forms depend on the particular geometry of the Coulomb problem. The one-body Boltzmann factor \( w_{\text{ren}}(x) \) will be considered in the general form \( (2.12) \) with \( \beta v(x) \) given by \( (2.7) \) for the present case of counterions in the potential of charged lines with density \( \sigma e \) at \( x = 0 \) and \( \sigma' e \) at \( x = d \).

3.1 Scaling transformations

• Rescaling by a constant \( \mu \) one of the anticommuting field components, say

\[
\xi_j^{(1)} \rightarrow \mu \xi_j^{(1)}, \quad j = 0, 1, \ldots, N - 1, \quad (3.1)
\]

the composite operators \( \Xi_j \) get the same factor \( \mu \) and the action in \( (2.23) \) transforms itself as \( S(\Xi, \Psi) \rightarrow \mu S(\Xi, \Psi) \). The Jacobian of the transformation \( (3.1) \) equals to \( \mu^N \).

Under the transformation \( (3.1) \), the quantity \( Q_N \) \( (2.23) \) takes the form

\[
Q_N = \mu^{-N} \int \mathcal{D}\psi \mathcal{D}\xi \exp \left( \mu \sum_{j=0}^{\gamma(N-1)} \Xi_j w_j \Psi_j \right). \quad (3.2)
\]

Since \( Q_N \) does not depend on \( \mu \), it holds that \( \partial \ln Q_N / \partial \mu |_{\mu=1} = 0 \) or, equivalently,

\[
\sum_{j=0}^{\gamma(N-1)} w_j \langle \Xi_j \Psi_j \rangle = N. \quad (3.3)
\]

Consequently,

\[
W \int_0^d dx \, n(x) = N, \quad (3.4)
\]

where we have substituted \( n(x) \) from \( (2.27) \) and used the definition of the interaction strength \( w_j \), \( (2.25) \). This equation provides the trivial information that there are \( N \) particles inside the cylinder domain \( \Lambda \).

Applying the transformation \( (3.1) \) to the quantity \( Q_N \langle \Xi_j \Psi_j \rangle \), one gets

\[
Q_N \langle \Xi_j \Psi_j \rangle = \mu^{-N+1} \int \mathcal{D}\psi \mathcal{D}\xi \Xi_j \Psi_j \exp \left( \mu \sum_{k=0}^{\gamma(N-1)} \Xi_k w_k \Psi_k \right). \quad (3.5)
\]
The equality $\partial (Q_N \langle \Xi_j \Psi_j \rangle) / \partial \mu |_{\mu=1} = 0$ implies that

$$\sum_{k=0}^{\gamma(N-1)} w_k \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle = (N-1) \langle \Xi_j \Psi_j \rangle. \quad (3.6)$$

This sum rule can be transformed into an integral equation by considering

$$\int_0^d dx' \int_0^W dy \, n^{(2)}(x, x'; y) = w_{\text{ren}}(x) \sum_{j,k=0}^{\gamma(N-1)} w_k \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle \times \exp \left( -\frac{4\pi}{W} j x \right), \quad (3.7)$$

where we have inserted the representation (2.28) of the two-body density and applied the orthogonality relation

$$\int_0^W dy \, \exp \left[ -\frac{2\pi}{W} (j - k) y \right] = W \delta_{j,k}. \quad (3.8)$$

Considering the sum rule (3.6) for the sum over $k$ on the rhs of (3.7), the rhs becomes equal to $(N-1)n(x)$ and

$$\int_0^d dx' \int_0^W dy \, U(x, x'; y) = (N-1)n(x) - Nn(x) = -n(x). \quad (3.9)$$

Consequently,

$$\int_0^d dx' \int_0^W dy \, S(x, x'; y) = 0. \quad (3.10)$$

This relation represents a generalization of the zeroth-moment Stillinger-Lovett condition \cite{89,90} to the cylinder geometry.

- Let us rescale all anticommuting field $\xi$-components as follows

$$\xi^\alpha_j \rightarrow \lambda_j \xi^\alpha_j, \quad j = 0, 1, \ldots, N-1, \quad \alpha = 1, \ldots, \gamma. \quad (3.11)$$

The composite operators $\Xi_j$ acquire the factor $\lambda_j$ and the action in (2.23) transforms itself as $S(\Xi, \Psi) \rightarrow \sum_{j=0}^{\gamma(N-1)} \lambda_j \Xi_j \Psi_j$. The Jacobian of the transformation (3.11) equals to $\lambda^{\gamma N(N-1)/2}$.

Under the transformation (3.11), the quantity $Q_N$ (2.23) is rewritten as

$$Q_N = \lambda^{-\gamma N(N-1)/2} \int \mathcal{D}\psi \mathcal{D}\xi \exp \left( \sum_{j=0}^{\gamma(N-1)} \lambda_j^2 \Xi_j \Psi_j \right). \quad (3.12)$$

The requirement $\partial \ln Q_N / \partial \lambda |_{\lambda=1} = 0$ is equivalent to the sum rule

$$\sum_{j=0}^{\gamma(N-1)} j w_j \langle \Xi_j \Psi_j \rangle = \frac{1}{2} \gamma N(N-1). \quad (3.13)$$
To make use of this relation, we consider the integral
\[ W \int_0^d dx \, w_{\text{ren}}(x) \frac{\partial}{\partial x} \left[ \frac{n(x)}{w_{\text{ren}}(x)} \right] = \sum_{j=0}^{\gamma(N-1)} \langle \Xi_j \Psi_j \rangle W \int_0^d dx \, w_{\text{ren}}(x) \]
\[ \times \left( -\frac{4\pi j}{W} \right) \exp \left( -\frac{4\pi j x}{W} \right). \]  \hspace{1cm} (3.14)

With regard to the definition of the interaction strengths \( \{w_j\} \) (2.25), the rhs of this equation equals to \(-4\pi/W\) times the lhs of Eq. (3.13), so that
\[ \int_0^d dx \frac{\partial}{\partial x} n(x) - \int_0^d dx n(x) \frac{\partial}{\partial x} \ln w_{\text{ren}}(x) = -\frac{2\pi}{W^2} \gamma N(N-1). \]  \hspace{1cm} (3.15)

Since
\[ \frac{\partial}{\partial x} \ln w_{\text{ren}}(x) = -\frac{\partial}{\partial x} [\beta v(x)] + \frac{2\pi\gamma}{W}, \]  \hspace{1cm} (3.16)
we finally end up with the relation
\[ n(d) - n(0) = \int_0^d dx n(x) \frac{\partial}{\partial x} \left[ \beta v(x) \right]. \]  \hspace{1cm} (3.17)

For our system of counterions with \( \beta v(x) \) given by (2.7), one obtains the equality
\[ n(d) - 2\pi\gamma \left( \frac{\sigma'}{2} \right)^2 = n(0) - 2\pi\gamma \sigma^2. \]  \hspace{1cm} (3.18)

The application of the transformation (3.11) to \( Q_N \langle \Xi_j \Psi_j \rangle \) results in
\[ Q_N \langle \Xi_j \Psi_j \rangle = \lambda^{-\gamma N(N-1)/2} \int \mathcal{D}\psi \mathcal{D}\xi \, \Xi_j \Psi_j \exp \left( \sum_{k=0}^{\gamma(N-1)} \lambda^k \Xi_k w_k \Psi_k \right). \]  \hspace{1cm} (3.19)

The equality \( \partial(Q_N \langle \Xi_j \Psi_j \rangle)/\partial\lambda|_{\lambda=1} = 0 \) leads to
\[ \sum_{k=0}^{\gamma(N-1)} k w_k \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle = \left[ \frac{1}{2} \gamma N(N-1) - j \right] \langle \Xi_j \Psi_j \rangle. \]  \hspace{1cm} (3.20)

This equation can be rewritten with the aid of the sum rule (3.13) as follows
\[ \sum_{k=0}^{\gamma(N-1)} k w_k \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle^T = -j \langle \Xi_j \Psi_j \rangle, \]  \hspace{1cm} (3.21)
where the truncated correlators \( \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle^T \equiv \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle - \langle \Xi_j \Psi_j \rangle \langle \Xi_k \Psi_k \rangle \).

Let us consider the integral
\[ \int_0^d dx' \int_0^W dy \, w_{\text{ren}}(x') \frac{\partial}{\partial x'} \left[ \frac{U(x,x';y)}{w_{\text{ren}}(x')} \right] \]
\[ = w_{\text{ren}}(x) \sum_{j,k=0}^{\gamma(N-1)} w_k \langle \Xi_j \Psi_j \Xi_k \Psi_k \rangle^T \left( -\frac{4\pi k}{W} \right) \exp \left( -\frac{4\pi j x}{W} \right). \]  \hspace{1cm} (3.22)
With regard to the sum rule (3.21), the rhs of this equation is written as
\[-w_{\text{ren}}(x) \frac{\partial}{\partial x} \left[ \frac{n(x)}{w_{\text{ren}}(x)} \right] = - \frac{\partial n(x)}{\partial x} + n(x) \frac{\partial}{\partial x} [\ln w_{\text{ren}}(x)]. \tag{3.23}\]

The lhs of Eq. (3.22) can be expressed as
\[
\int_0^d dx' \int_0^W dy \left\{ \frac{\partial}{\partial x'} U(x, x'; y) - U(x, x'; y) \frac{\partial}{\partial x'} [\ln w_{\text{ren}}(x')] \right\}. \tag{3.24}\]

Using the relations (3.16) and (3.9), we end up with
\[
\int_0^W dy \left[ U(x, 0; y) - U(x, d; y) \right] \tag{3.25}\]
which is a generalization of the 2D Wertheim-Lovett-Mou-Buff (WLMB) equation \cite{54,97} to the surface of cylinder constrained by two charged lines.

### 3.2 Transformation mixing all anticommuting components

It was shown in \cite{73} that there exists a nontrivial transformation of anticommuting variables, say $\xi$’s,
\[
\xi_j^{(\alpha)}(t) = \sum_{k=j}^{N-1} \binom{k}{j} t^{k-j} \xi_k^{(\alpha)}, \quad j = 0, 1, \ldots, N - 1, \quad \alpha = 1, \ldots, \gamma, \tag{3.27}\]
which keeps the composite form of the transformed composite operators:
\[
\Xi_j(t) = \sum_{k=j}^{\gamma(N-1)} \binom{k}{j} t^{k-j} \Xi_k, \quad j = 0, 1, \ldots, \gamma(N - 1). \tag{3.28}\]

Here, $t$ is a free parameter; the case $t = 0$ corresponds to the identity mapping. The Jacobian of the transformation equals to 1.

Under the transformation (3.27), $Q_N$ (2.23) takes the form
\[
Q_N = \int D\psi D\xi \exp \left[ \sum_{j=0}^{\gamma(N-1)} \Xi_j(t) w_j \Psi_j \right] = \int D\psi D\xi \exp \left\{ \sum_{j=0}^{\gamma(N-1)} \left[ \Xi_j + t(j + 1)\Xi_{j+1} + O(t^2) \right] w_j \Psi_j \right\}. \tag{3.29}\]
The condition $\frac{\partial \ln Q_N}{\partial t}|_{t=0} = 0$ implies the sum rule

$$\sum_{j=0}^{\gamma(N-1)-1} (j+1) w_j \langle \Xi_{j+1} \psi_j \rangle = 0. \quad (3.30)$$

This sum rule is trivial because the diagonalized action (2.23) implies that every correlator $\langle \Xi_{j+1} \psi_j \rangle = 0$.

Applying the transformation (3.27) to $Q_N \langle \Xi_{j-1} \psi_j \rangle$ implies

$$Q_N \langle \Xi_{j-1} \psi_j \rangle = \int \mathcal{D} \psi \mathcal{D} \xi \langle \Xi_{j-1}(t) \psi_j \rangle \exp \left[ \sum_{k=0}^{\gamma(N-1)} \xi_k(t) w_k \psi_k \right]$$

$$= \int \mathcal{D} \psi \mathcal{D} \xi \left[ \xi_{j-1} + tj \xi_j + O(t^2) \right] \psi_j$$

$$\times \exp \left\{ \sum_{k=0}^{\gamma(N-1)} \left[ \xi_k + t(k+1) \xi_{k+1} + O(t^2) \right] w_k \psi_k \right\}. \quad (3.31)$$

The requirement $\frac{\partial (Q_N \langle \Xi \psi \rangle)}{\partial t}|_{t=0} = 0$ leads to

$$\sum_{k=0}^{\gamma(N-1)-1} (k+1) w_k \langle \Xi_{j-1} \psi_j \Xi_{k+1} \psi_k \rangle = -j \langle \Xi \psi \rangle. \quad (3.32)$$

To make use of this sum rule, let us consider the integral

$$\int_0^W dy e^{-\frac{2\pi}{W} \langle \Xi \psi \rangle} n^{(2)}(x, x'; y) = w_{\text{ren}}(x) w_{\text{ren}}(x') W \sum_{j,k} \langle \Xi_{j-1} \psi_j \Xi_{k+1} \psi_k \rangle$$

$$\times e^{-\frac{2\pi}{W} (2j-1)x} e^{-\frac{2\pi}{W} (2k+1)x'}. \quad (3.33)$$

Note that $n^{(2)}(x, x'; y)$ can be substituted by $U(x, x'; y)$ in this relation and since $U(x, x'; y) = U(x', x; -y)$ only the real part of $e^{-\frac{2\pi}{W} \langle \Xi \psi \rangle} = \cos \left( \frac{2\pi}{W} y \right) - i \sin \left( \frac{2\pi}{W} y \right)$ survives. Consequently,

$$\int_0^d dx' \int_0^W dy \cos \left( \frac{2\pi}{W} y \right) w_{\text{ren}}(x') e^{\frac{2\pi}{W} x'} \frac{\partial}{\partial x'} \left[ U(x, x'; y) e^{-\frac{2\pi}{W} x'} \right]$$

$$= w_{\text{ren}}(x) \left( -\frac{4\pi}{W} \right) \sum_{j,k} (k+1) w_k \langle \Xi_{j-1} \psi_j \Xi_{k+1} \psi_k \rangle e^{-\frac{2\pi}{W} (2j-1)x}. \quad (3.34)$$

Applying the sum rule (3.32), the rhs of this equation can be expressed as

$$-w_{\text{ren}}(x) e^{\frac{2\pi}{W} x} \frac{\partial}{\partial x} \left[ \frac{n(x)}{w_{\text{ren}}(x)} \right] = e^{\frac{2\pi}{W} x} \left\{ -\frac{\partial n(x)}{\partial x} + n(x) \frac{\partial}{\partial x} \ln w_{\text{ren}}(x) \right\}. \quad (3.35)$$
After simple algebra we finally arrive at

\[
\int_0^W dy \cos \left( \frac{2\pi}{W} y \right) \left[ e^{i2\pi y U(x, d; y)} - U(x, 0; y) \right] + \int_0^d dx' \int_0^W dy \\
\times \cos \left( \frac{2\pi}{W} y \right) e^{i2\pi x'} S(x, x'; y) \left\{ \frac{\partial}{\partial x'} [\beta v(x')] - \frac{\pi}{W} [\Gamma(N-1) + 4] \right\} \\
= -e^{i2\pi x} \left[ \frac{\partial n(x)}{\partial x} + \frac{4\pi}{W} n(x) \right]. 
\] (3.36)

For the one-body potential (2.7) it holds that

\[
\frac{\partial}{\partial x'} [\beta v(x')] = -\frac{\pi}{W} [\Gamma(N-1) + 4] = - \left[ 2\pi \Gamma'(x') + \frac{4\pi}{W} (4 - \Gamma) \right]. 
\] (3.37)

Another version of the above sum rule can be derived by using the alternative representation of the Coulomb system on the cylinder surface given by Eqs. (2.13)-(2.15), with the renormalized one-body Boltzmann factor \( w_{\text{ren}}(x) \) defined by (2.15). Within this representation, the interaction strengths \( w_j \) \( j = 0, 1, \ldots, \gamma(N-1) \) are given by

\[
w_j = W \int_0^d dx w_{\text{ren}}(x) \exp \left( \frac{4\pi}{W} j x \right), 
\] (3.38)

the particle density by

\[
n(x) = w_{\text{ren}}(x) \sum_{j=0}^{\gamma(N-1)} \langle \Xi_j \Psi_j \rangle \exp \left( \frac{4\pi}{W} j x \right) 
\] (3.39)

and the two-body density by

\[
n^{(2)}(z_1, \bar{z}_1; z_2, \bar{z}_2) = w_{\text{ren}}(x_1) w_{\text{ren}}(x_2) \sum_{j_1, k_1, j_2, k_2 = 0}^{\gamma(N-1)} \langle \Xi_{j_1} \Psi_{k_1} \Xi_{j_2} \Psi_{k_2} \rangle \\
\times \exp \left[ \frac{2\pi}{W} \left( j_1 z_1 + k_1 \bar{z}_1 + j_2 z_2 + k_2 \bar{z}_2 \right) \right]. 
\] (3.40)

The counterpart of the relation (3.34) reads as

\[
\int_0^d dx' \int_0^W dy \cos \left( \frac{2\pi}{W} y \right) w_{\text{ren}}(x') e^{-i2\pi x' \frac{\partial}{\partial x'} \left[ U(x, x'; y) e^{i2\pi x'} \right]} \frac{U(x, x'; y)}{w_{\text{ren}}(x')} \\
= w_{\text{ren}}(x) \sum_{j,k} (k+1) w_j \langle \Xi_{j-1} \Psi_{j+k+1} \rangle \exp \left( \frac{4\pi}{W} (2j-1) x \right). 
\] (3.41)
Considering the sum rule (3.32) and following the preceding algebra leads to

\[
\int_0^W dy \cos \left( \frac{2\pi}{W} y \right) \left[ e^{-\frac{2\pi}{W} y} U(x, d; y) - U(x, 0; y) \right] + \int_0^d dx' \int_0^W dy \\
\times \cos \left( \frac{2\pi}{W} y \right) e^{-\frac{2\pi}{W} x'} S(x, x'; y) \left\{ \frac{\partial}{\partial x'} [\beta v(x')] + \frac{\pi}{W} [\Gamma(N-1) + 4] \right\} \\
= -e^{-\frac{2\pi}{W} x} \left[ \frac{\partial n(x)}{\partial x} - \frac{4\pi}{W} n(x) \right]. \quad (3.42)
\]

For the one-body potential (2.7) it holds that

\[
\frac{\partial}{\partial x'} [\beta v(x')] + \frac{\pi}{W} [\Gamma(N-1) + 4] = 2\pi \Gamma \sigma + \frac{\pi}{W} (4 - \Gamma). \quad (3.43)
\]

The physical content of the exact sum rules (3.36), (3.37) or (3.42), (3.43) is not obvious for a finite value of \( W \) due to the presence of the slowly changing factor \( \cos(2\pi y/W) \) along the integration path over \( y \in [0, W] \). On the other hand, these sum rules will be very useful in the limit \( W \to \infty \) to derive certain exact relations among relevant statistical quantities, see the next section.

4 Asymptotic decay of pair correlations along the walls

4.1 One-wall geometry

Let us first consider the 2D geometry of one wall (infinite line) localized at \( x = 0 \) and charged by the fixed charge density \( \sigma e \). The mobile counterions of charge \( -e \) are constrained to the half-space \( x > 0 \). Their number density \( n(x) \) fulfills the electroneutrality condition

\[
\int_0^d dx n(x) = \sigma. \quad (4.1)
\]

Near a hard wall, the screening cloud around a test charge is asymmetric and therefore the Ursell function exhibits a long-range (inverse-power law) decay at asymptotically large distances along the wall [34,35,95]. In 2D, the Ursell function between the points \( (x, y) \) and \( (x', y') \) behaves as

\[
U(x, x'; |y - y'|) \sim f^{(1)}(x, x') \frac{1}{(y - y')^2}, \quad (4.2)
\]

where the superscript 1 in \( f^{(1)} \) means that there is just one charged wall at \( x = 0 \). The function \( f^{(1)}(x, x') = f^{(1)}(x', x) \) obeys the sum rule [35,38,39]

\[
\int_0^\infty dx \int_0^\infty dx' f^{(1)}(x, x') = -\frac{1}{\pi^2 \Gamma}. \quad (4.3)
\]

Note that this sum rule does not depend on \( \sigma \).

Applying the Möbius conformal transformation to particle coordinates in a disc geometry and going from the disc to an infinite line [81,83], it was
found for the present model with counterions only that the function $f^{(1)}$ satisfies the following equations:

$$f^{(1)}(x,0) = -\frac{1}{\pi} \left[ x \frac{\partial}{\partial x} n(x) + 2n(x) \right], \quad (4.4)$$

$$f^{(1)}(x,0) = 2\pi \Gamma \sigma \int_{0}^{\infty} dx' f^{(1)}(x,x'). \quad (4.5)$$

Applying $\int_{0}^{\infty} dx$ to both sides of (4.5) and taking into account the sum rule (4.3), one finds that

$$\int_{0}^{\infty} dx' f^{(1)}(0,x') + \frac{\sigma}{\pi} = 0. \quad (4.6)$$

Finally, setting $x = 0$ in (4.5) the $f^{(1)}$-function with both points at the boundary is given by

$$f^{(1)}(0,0) = -2\Gamma \sigma^2. \quad (4.7)$$

For the single line charge density $\sigma e$, with counterions only, the particle density profile and the asymptotic function $f^{(1)}(x,x')$ were obtained in the PB limit $\Gamma \to 0 \quad [79]$, with $b = 1/(\Gamma \pi \sigma)$, and at the free-fermion coupling $\Gamma = 2 \quad [36,79]$, 

$$n(x) = -\frac{\sigma b}{(x+b)^2}, \quad f^{(1)}(x,x') = -\frac{2}{\pi^2 \Gamma} \frac{b^4}{(x+b)^3(x'+b)^3} \quad (4.8)$$

with $b = 1/(\Gamma \pi \sigma)$, and at the free-fermion coupling $\Gamma = 2 \quad [36,79]$. 

$$n(x) = \frac{1}{4\pi x^2} \left[ 1 - (1 + 4\pi \sigma x)e^{-4\pi \sigma x} \right], \quad f^{(1)}(x,x') = -4\sigma^2 e^{-4\pi \sigma x} e^{-4\pi \sigma x'}. \quad (4.9)$$

Note that while in the PB limit both $n(x)$ and $f^{(1)}(x,x')$ are long-ranged, $n(x)$ is long-ranged but $f^{(1)}(x,x')$ is short-ranged at $\Gamma = 2$. It is simple to check that the sum rules (4.4)-(4.7) are fulfilled at both exactly solvable $\Gamma$’s.

4.2 Two-walls geometry

In the presence of two walls, the Ursell functions are supposed to exhibit the same asymptotic behavior as in the one-wall case (4.2), i.e.,

$$U(x,x';|y-y'|) \sim \frac{f^{(2)}(x,x')}{(y-y')^2}, \quad (4.10)$$

where the superscript 2 in $f^{(2)}$ means that there are two parallel charged walls, the one with the charge density $\sigma e$ at $x = 0$ and the other with the charge density $\sigma' e$ at $x = d$. The aim of this part is to investigate the thermodynamic $W \to \infty$ limit of the sum rules (3.36), (3.37) and (3.42), (3.43).
Let us start with the analysis of the first sum rule (3.36), (3.37) in the limit \( W \to \infty \). Using the zeroth-moment condition (3.10) and the WLMB equation (3.26), the sum rule can be rewritten as

\[
\int_0^W dy \cos \left( \frac{2\pi}{W} y \right) \left( e^{\frac{2\pi}{W} d} - 1 \right) U(x, d; y) + \int_0^W dy \left[ \cos \left( \frac{2\pi}{W} y \right) - 1 \right] \left( U(x, d; y) - U(x, 0; y) \right) \\
- \left[ 2\pi \Gamma + \frac{\pi}{W} (4 - \Gamma) \right] \int_0^d dx' \int_0^W dy \left[ \cos \left( \frac{2\pi}{W} y \right) e^{\frac{2\pi}{W} x'} - 1 \right] S(x, x'; y) \\
= - \left( e^{\frac{2\pi}{W} x} - 1 \right) \frac{\partial n(x)}{\partial x} - 4\pi W n(x) e^{\frac{2\pi}{W} x}.
\]

(4.11)

In the limit \( W \to \infty \), one expands

\[
e^{\frac{2\pi}{W} x} \sim 1 + \frac{2\pi}{W} x + O \left( \frac{1}{W^2} \right), \quad \cos \left( \frac{2\pi}{W} y \right) \sim 1 - \frac{1}{2!} \left( \frac{2\pi}{W} \right)^2 y^2 + O \left( \frac{1}{W^4} \right).
\]

(4.12)

The integrals of the Ursell functions \( U \) (or the structure function \( S \)) over \( y \) can be done in the following way

\[
\int_0^W dy \left[ \cos \left( \frac{2\pi}{W} y \right) - 1 \right] U(x, x'; y) \sim W \to \infty \quad - \frac{1}{2!} \left( \frac{2\pi}{W} \right)^2 \int_0^W dy' y^2 \frac{f^{(2)}(x, x')}{y^2} \\
= - \frac{2\pi^2}{W} f^{(2)}(x, x').
\]

(4.13)

Comparing in (4.11) the terms proportional to \( 1/W \) implies the equality among the 2D statistical quantities:

\[
d \int_{-\infty}^\infty dy U(x, d; y) - 2\pi \Gamma \sigma' \int_0^d dx' \int_{-\infty}^\infty dy S(x, x'; y) \\
+ \pi \left[ f^{(2)}(x, 0) - f^{(2)}(x, d) \right] + 2\pi^2 \Gamma \sigma' \int_0^d dx' f^{(2)}(x, x') \\
= - \left[ x \frac{\partial n(x)}{\partial x} + 2n(x) \right].
\]

(4.14)

We proceed analogously with the second sum rule (3.42), (3.43). With the aid of the zeroth-moment condition (3.10) and the WLMB equation (3.26), one gets

\[
\int_0^W dy \cos \left( \frac{2\pi}{W} y \right) \left( e^{-\frac{2\pi}{W} d} - 1 \right) U(x, d; y) + \int_0^W dy \left[ \cos \left( \frac{2\pi}{W} y \right) - 1 \right] \left( U(x, d; y) - U(x, 0; y) \right) \\
- \left[ 2\pi \Gamma \sigma' + \frac{\pi}{W} (4 - \Gamma) \right] \int_0^d dx' \int_0^W dy \left[ \cos \left( \frac{2\pi}{W} y \right) e^{\frac{2\pi}{W} x'} - 1 \right] S(x, x'; y) \\
= - \left( e^{-\frac{2\pi}{W} x} - 1 \right) \frac{\partial n(x)}{\partial x} - 4\pi W n(x) e^{-\frac{2\pi}{W} x}.
\]

(4.11)
Using (4.12) and (4.13) and comparing in (4.15) the terms proportional to \(1/W\) leads to the equality
\[
-\pi \Gamma \sigma \int_0^d dx' \int_\infty^{-\infty} dy \left[ \frac{x \partial n(x)}{\partial x} + \frac{W}{\pi} n(x) e^{-\frac{\pi}{W} x} \right] + \frac{\pi}{W} \int_0^d dx' \int_\infty^{-\infty} dy S(x, x'; y) = -\left( e^{-\frac{\pi}{W} x} - 1 \right) \frac{\partial n(x)}{\partial x} + \frac{4\pi}{W} n(x) e^{-\frac{\pi}{W} x}.
\]

The crucial 2D Eqs. (4.14) and (4.16) are valid for any coupling \(\Gamma = 2\gamma\) with \(\gamma\) a positive integer. It is natural to extend their validity to all real \(\Gamma\) in the fluid region. We can obtain a couple of simpler relations by considering specific combinations of the two equations. The summation of Eqs. (4.14) and (4.16) results in
\[
-\Gamma (\sigma + \sigma') \int_0^d dx' \int_\infty^{-\infty} dy S(x, x'; y) + f^{(2)}(x, 0) - f^{(2)}(x, d)
\]
\[
+ \pi \Gamma (\sigma' - \sigma) \int_0^d dx' f^{(2)}(x, x') = 0.
\]

The subtraction of Eqs. (4.14) and (4.16) implies that
\[
\frac{d}{\pi} \int_\infty^{-\infty} dy U(x, d; y) + \Gamma (\sigma - \sigma') \int_0^d dx' \int_\infty^{-\infty} dy S(x, x'; y)
\]
\[
+ \pi \Gamma (\sigma + \sigma') \int_0^d dx' f^{(2)}(x, x') = -\frac{1}{\pi} \left[ x \frac{\partial n(x)}{\partial x} + 2n(x) \right].
\]

Integrating both sides of Eq. (4.15) over \(x \in [0, d]\), the integration of \(S(x, x'; y)\) over \(x'\) can be interchanged with the integration over \(x\) for a finite value of \(d\) and the corresponding term vanishes due to the counterpart of the zeroth-moment condition (3.10).

We emphasize that the interchange of the integrations cannot be performed in the one-wall limit \(d \to \infty\) because, as is known, the integral over \(x'\) is not absolutely convergent. The integral over \(x\) of the rhs of Eq. (4.15) can be simplified by applying the integration by parts:
\[
-\frac{1}{\pi} \int_0^d dx \left[ x \frac{\partial n(x)}{\partial x} + 2n(x) \right] = -\frac{d}{\pi} n(d) - \frac{1}{\pi} \int_0^d dx n(x).
\]
The term \((d/\pi)n(d)\) can be paired with the one \((d/\pi)\int_0^d dx f(x; d)\) to get 0 due to (4.19). Expressing the integral \(\int_0^d dx n(x)\) by using (4.1), we end up with the sum rule
\[
\int_0^d dx \int_0^d dx' f^{(2)}(x, x') = -\frac{1}{\pi^2 \Gamma}.
\] (4.21)

Note that this sum rule does not depend neither on the surface charge densities \(\sigma e\) and \(\sigma' e\), nor on the distance between the walls \(d\). In comparison with the analogous formula for the one-wall geometry (4.3), the factor \(1/2\) is missing on the rhs of (4.21). To explain this fact, let us consider the special limit \(d \to \infty\) of two independent walls when, for finite values of \(x\) and \(x'\),
\[
f^{(2)}(x, x') \sim f^{(1)}(x, x'; \sigma) + f^{(1)}(d - x, d - x'; \sigma').
\] (4.22)

Integrating over coordinates \(x\) and \(x'\) and changing the integration variables to \((d - x, d - x')\) when integrating the second term, one finds that the double integral of \(f^{(2)}\) must be twice the double integral of \(f^{(1)}\); note that the argument works because the sum rule (4.21) does not depend on \(d\).

Another sum rule can be obtained by integrating both sides of Eq. (4.17) over \(x \in [0, d]\). The integral of \(S(x, x'; y)\) vanishes once more and using (4.21) one gets
\[
\int_0^d dx' f^{(2)}(0, x') + \frac{\sigma}{\pi} = \int_0^d dx' f^{(2)}(d, x') + \frac{\sigma'}{\pi}.
\] (4.23)

In other words, for each of the walls the combination (4.6), which is equal to zero for one-wall geometry, acquires the same value in the two-walls geometry.

4.3 Small-distance behavior

In the limit \(d \to 0\), the particle density, the pressure and the asymptotic function \(f^{(2)}(x, x')\) exhibit singularities. As is evident from the electroneutrality condition (4.11), the particle density behaves as
\[
n(x) \sim \frac{\sigma + \sigma'}{d}.
\] (4.24)

Since the pressure is determined by the contact particle density, we have likewise
\[
\beta P \sim \frac{\sigma + \sigma'}{d}.
\] (4.25)

The asymptotic function \(f^{(2)}(x, x')\) is searched in the ansatz form
\[
f^{(2)}(x, x') \sim \frac{1}{d^2} [a + b(x + x') + \cdots].
\] (4.26)

Inserting this expansion into the sum rules (4.21) and (4.23), the expansion coefficients are found to be
\[
a = -\frac{1}{\pi^2 \Gamma}, \quad b = \frac{1}{\pi}(\sigma - \sigma').
\] (4.27)
5 The free-fermion coupling

At the free-fermion coupling $\Gamma = 2\ (\gamma = 1)$, the composite operators $\Xi_j$ and $\Psi_j$ become the ordinary anticommuting variables $\xi_j$ and $\psi_j$, respectively. Having the diagonalized action $S = \sum_{j=0}^{N-1} \xi_j \psi_j$, the integral over anticommuting variables (2.23) reads as

$$Q_N(1) = \prod_{j=0}^{N-1} w_j,$$

where the interaction strengths (2.25) take for $\gamma = 1$ the form

$$w_j = \frac{1 - \exp \left[ -\frac{4\pi d}{W} (j - \sigma'W + \frac{1}{2}) \right]}{j - \sigma'W + \frac{1}{2}}.$$  

The simplest correlators of anticommuting variables are given by

$$\langle \xi_j \psi_j \rangle = \frac{1}{w_j}, \quad j = 0, 1, \ldots, N - 1.$$

More complicated correlators can be obtained by using the Wick theorem. Like for instance,

$$\langle \xi_j \psi_k \xi_j' \psi_k' \rangle = \frac{1}{w_j w_j'} (\delta_{jk} \delta_{j'k'} - \delta_{jk'} \delta_{j'k}).$$

5.1 Particle density and pressure

Inserting (5.3) into the formula (2.27) for the particle density, one gets

$$n(x) = 4\pi \frac{N}{W^2} \sum_{j=0}^{N-1} \frac{j - \sigma'W + \frac{1}{2}}{1 - \exp \left[ -\frac{4\pi d}{W} (j - \sigma'W + \frac{1}{2}) \right]} \times \exp \left[ -\frac{4\pi x}{W} \left( j - \sigma'W + \frac{1}{2} \right) \right].$$

To obtain the explicit results for the 2D geometry of two parallel lines charged by the line charge densities $\sigma e$ and $\sigma' e$, at distance $d$ with counters only in between, we consider the thermodynamic limit $N, W \to \infty$, keeping the ratio $N/W = \sigma + \sigma'$ fixed. Choosing $t = (j - W\sigma' + \frac{1}{2})/N$ as the continuous variable, the particle density (5.5) can be expressed as

$$n(x) = 4\pi \left( \frac{N}{W} \right)^2 \int_{-\pi\sigma}^{\pi\sigma} \frac{ds}{2\pi \sigma'} \int dt \frac{e^{-4\pi(\sigma+\sigma')tx}}{1 - e^{-4\pi(\sigma+\sigma')td}}$$

$$= \frac{1}{4\pi} \int_{-4\pi\sigma'}^{4\pi\sigma} \frac{ds}{e^{-s} - e^{-sd}}$$

$$= n_0(x; \sigma) + n_0(d - x; \sigma'),$$
where
\[ n_0(x; \sigma) = \frac{1}{4\pi} \int_{4\pi}^{4\pi\sigma} ds s \frac{e^{-s x}}{1 - e^{-s d}} \] (5.7)
is the density of counterions between two parallel lines, the one at \( x = 0 \) charged with the line charge density \( \sigma e \) and the neutral one at \( x = d \). It is trivial to verify that \( \int_0^d dx n_0(x; \sigma) = \sigma \) as it should be. The separation form of the density profile (5.6) as the sum of two terms, the one depending only on \( \sigma \) and the other depending only on \( \sigma' \), is the special feature of the free-fermion point.

The pressure is given by the contact relations (2.31). Choosing the one
\[ P_N = n(0) - 2\pi\sigma^2, \] from (5.5) one gets
\[ \beta P_N(d) = 4\pi W^2 N - \sum_{j=0}^{N-1} \left( j - \sigma W + \frac{1}{2} \right) - 2\pi\sigma^2. \] (5.8)
The pressure is expected to vanish in the limit of an infinite distance between the lines, but this is not the case for a finite odd value of \( N \). Let us document this fact on the pair of symmetrically charged lines \( \sigma' = \sigma \), i.e. \( N = 2\sigma W \). In the limit \( d \to \infty \), only terms with \( j - \sigma W + \frac{1}{2} > 0 \) contribute to the pressure:
\[ \lim_{d \to \infty} \beta P_N(d) = 4\pi W^2 N - \sum_{j=0}^{N-1} \left( j - \sigma W + \frac{1}{2} \right) - 2\pi\sigma^2. \] (5.9)
If \( N \) is an even integer, \( \sigma W \) is an integer and one has
\[ \lim_{d \to \infty} \beta P_N(d) = 4\pi W^2 \left( 2\sigma W - 1 \right) - 2\pi\sigma^2 = 0. \] (5.10)
If \( N \) is an odd integer, \( \sigma W \) is a half-integer and one has
\[ \lim_{d \to \infty} \beta P_N(d) = 4\pi W^2 \left( 2\sigma W - 1 \right) - 2\pi\sigma^2 = -\frac{2\pi\sigma^2}{N^2}. \] (5.11)
A nonzero asymptotic force for an odd number of counterions between two charges occurs also in 1D [94,96].

Let us express the pressure by using a symmetric combination of the contact relations (2.31),
\[ P(d) = \frac{1}{2} [n(0) + n(d)] - \pi (\sigma^2 + \sigma'^2). \] (5.12)

In the thermodynamic limit (pure 2D geometry), using the explicit results (5.6) and (5.7) one obtains the separable solution
\[ P(d) = P(d; \sigma) + P(d; \sigma'), \] (5.13)
where
\[ \beta P(d; \sigma) = \frac{1}{2\pi d^2} \int_0^{2\pi \sigma d} dt \frac{t}{\sinh t} e^{-t} \]  
(5.14)
is the pressure between two parallel lines, the one at \( x = 0 \) charged with the line charge density \( \sigma e \) and another neutral one at \( x = d \).

For the studied case of like-charged lines \( 0 < \sigma' < \sigma \), \( \beta P \) is always positive, i.e. the two lines repel each other for an arbitrary distance \( d \). Using the substitution \( t = ds \) in (5.14) it can be shown that
\[ \frac{\partial}{\partial d} \beta P = -\frac{1}{2\pi} \left( \int_0^{2\pi \sigma} + \int_0^{2\pi \sigma'} \right) ds \left[ \frac{s}{\sinh(ds)} \right]^2 < 0. \]  
(5.15)
The pressure (5.14) diverges at small distances in agreement with the general formula (4.25) and decays monotonously to 0 at \( d \to \infty \). If \( 0 < \sigma' \leq \sigma \), the asymptotic decay
\[ \beta P(d) \sim \frac{1}{\pi d^2} \int_0^\infty dt \frac{t}{\sinh t} e^{-t} = \frac{\pi}{12} \frac{1}{d^2} \]  
(5.16)
is universal in the sense that the prefactor to \( 1/d^2 \) is independent of the (positive) line charge densities.

Fig. 2  The dimensionless pressure \( \tilde{P} \) versus the dimensionless distance \( d \) (measured in units of \( \sigma^{-1} \)) between the symmetrically charged walls with the asymmetry parameter \( \eta = 1 \) for the free-fermion coupling \( \Gamma = 2 \) (\( \gamma = 1 \)). The dotted (blue) curve corresponds to \( N = 2 \) particles, the dashed (orange) curve to \( N = 8 \) and the solid (black) curve to the thermodynamic limit \( N \to \infty \).

The data for the dimensionless pressure \( \tilde{P} \) (2.32) versus the dimensionless distance \( d \) (measured in units of \( \sigma^{-1} \)) between the symmetrically charged...
walls \((\eta = 1)\), calculated by using Eqs. (5.13) and (5.14), are pictured in Fig. 2. The dotted curve corresponds to \(N = 2\) particles, the dashed curve to \(N = 8\) particles and the solid curve to \(N \to \infty\) particles. The \(N = 8\) and \(N \to \infty\) curves are almost indistinguishable, so that the equation of state for \(N = 8\) particles is practically identical to the one in the thermodynamic limit. The same behavior is observed for all values of the asymmetry parameter \(\eta \in [0, 1]\). We anticipate that the quick convergence of data with the particle number \(N\), observed for \(\Gamma = 2\), is maintained also for the higher couplings \(\Gamma\).

### 5.2 Two-body density

Inserting the correlators (5.4) into the formula for the two-body density (2.28), the Ursell function (2.21) is expressible as

\[
U(z_1, \bar{z}_1; z_2, \bar{z}_2) = -w_{\text{ren}}(x_1)w_{\text{ren}}(x_2) \sum_{j,j'}^{N-1} \frac{1}{w_j w_{j'}} \times \exp \left[ -\frac{2\pi}{W} j (z_1 + \bar{z}_2) \right] \exp \left[ -\frac{2\pi}{W} j'(\bar{z}_1 + z_2) \right]. \tag{5.17}
\]

Let us denote \(y \equiv y_1 - y_2\) and note that our \(w_{\text{ren}}(x)\) (2.12) with \(\beta v(x)\) given by (2.7) satisfies the relation

\[
w_{\text{ren}}(x_1)w_{\text{ren}}(x_2) = w_{\text{ren}} \left( \frac{x_1 + x_2}{2} + i\frac{y}{2} \right) w_{\text{ren}} \left( \frac{x_1 + x_2}{2} - i\frac{y}{2} \right). \tag{5.18}
\]

The Ursell function (5.17) is thus expressible in terms of the particle density as

\[
U(x_1, x_2; y) = -n \left( \frac{x_1 + x_2}{2} + i\frac{y}{2} \right) n \left( \frac{x_1 + x_2}{2} - i\frac{y}{2} \right). \tag{5.19}
\]

The availability of the explicit formula for the Ursell function (5.19) enables us to investigate the effect of the two-wall geometry on the prefactor function \(f^{(2)}(x_1, x_2)\) defined by Eq. (4.10). The particle density (5.6) consists of two similar terms. The first term (5.7) can be expanded as follows

\[
n_0(x; \sigma) = -\frac{1}{4\pi} \frac{\partial}{\partial x} \int_0^{4\pi} ds e^{-sx} \sum_{j=0}^{\infty} e^{-jd} = -\frac{1}{4\pi} \frac{\partial}{\partial x} \sum_{j=0}^{\infty} \frac{1 - e^{-4\pi(x+jd)}}{x+jd} = \sum_{j=0}^{\infty} \left[ \frac{1}{4\pi} \frac{1 - e^{-4\pi(x+jd)}}{(x+jd)^2} - \sigma \frac{e^{-4\pi(x+jd)}}{x+jd} \right]. \tag{5.20}
\]

Substituting \(x\) by \((x_1 + x_2 \pm iy)/2\) in the last line of this expression and considering the limit \(y \to \infty\), one finds that

\[
n_0 \left( \frac{x_1 + x_2}{2} \pm i\frac{y}{2}; \sigma \right) \sim \pm i \frac{2\sigma}{y} \frac{e^{-2\pi \sigma (|x_1 + x_2| \pm iy)}}{1 - e^{-4\pi \sigma d}}. \tag{5.21}
\]
Analogously,
\[ n_0 \left( d - \frac{1}{2} \frac{y}{2} \pm i \frac{y}{2} \sigma' \right) \sim \frac{2 \sigma' e^{-2 \pi \sigma' \left(2d-(x_1+x_2)\mp iy\right)}}{y - e^{-4 \pi \sigma d}}. \] (5.22)

We conclude that
\[ n \left( \frac{x_1 + x_2}{2} \pm i \frac{y}{2} \right) \sim \pm \frac{i y}{2} \left\{ \frac{e^{-2 \pi \sigma \left[2d-(x_1+x_2)\pm iy\right]}}{1 - e^{-4 \pi \sigma d}} \right\}. \] (5.23)

The Ursell function then exhibits the asymptotic behavior
\[ U(x_1, x_2) \sim \frac{4}{y^2} \left( \frac{2 \sigma e^{-4 \pi \sigma (x_1+x_2)}}{(1 - e^{-4 \pi \sigma d})^2} + \frac{\sigma'^2 e^{-4 \pi \sigma' [2d-(x_1+x_2)]}}{(1 - e^{-4 \pi \sigma' d})^2} \right) \cos \left[ \frac{2\pi \sigma \sigma'}{2(1 - e^{-4 \pi \sigma d})(1 - e^{-4 \pi \sigma' d})} \right]. \] (5.24)

This asymptotic result is of type with
\[ f^{(2)}(x_1, x_2) = -4 \left( \frac{\sigma^2 e^{-4 \pi \sigma (x_1+x_2)}}{(1 - e^{-4 \pi \sigma d})^2} + \frac{\sigma'^2 e^{-4 \pi \sigma' [2d-(x_1+x_2)]}}{(1 - e^{-4 \pi \sigma' d})^2} \right); \] (5.25)

note that the oscillating term in (5.24) does not contribute to this function. It is simple to check that \( f^{(2)}(x, x') \) satisfies both sum rules (4.21) and (4.23).

In the limit \( d \to \infty \), keeping the coordinates \( x \) and \( x' \) finite, the two-wall formula reduces itself to the semi-infinite one-wall result as it should be. For a finite \( d \), \( f^{(2)}(x, x') \) cannot be written in the factorized form \(-g(x_1)g(x_2)\) as in the one-wall case.

Using the formulas
\[ \int_{-\infty}^\infty dy U(x, x'; y) = \frac{1}{4 \pi} \int_{-4 \pi \sigma}^{4 \pi \sigma} ds s^2 e^{-s(x+x')} \] (5.26)
\[ \int_0^d dx' \int_{-\infty}^\infty dy S(x, x'; y) = \frac{\sigma e^{-4 \pi \sigma x}}{1 - e^{-4 \pi \sigma d}} + \frac{\sigma' e^{-4 \pi \sigma' (d-x)}}{1 - e^{-4 \pi \sigma' d}} \] (5.27)

with the explicit forms of the particle density \( n(x) \) and the function \( f^{(2)}(x, x') \), it can be straightforwardly shown that Eqs. (4.17) and (4.18) hold.
6 Couplings $\Gamma = 2\gamma$ ($\gamma = 2, 3, \ldots$)

The two-wall problem can be solved also for higher couplings $\Gamma = 2\gamma$ ($\gamma = 2, 3, \ldots$) by expressing $Q_N(\gamma)$, the integral over anticommuting variables \cite{22,23}, as a function of the interaction strengths $w_j$ [$j = 0, 1, \ldots, \gamma(N - 1)$]. This can be done for lower values of $N$ \cite{74}. For $\gamma = 2$, one has

\[
\begin{align*}
Q_2(2) &= w_0 w_2 + 2 w_1^2, \\
Q_3(2) &= w_0 w_2 w_4 + 2 w_0 w_3^2 + 2 w_1^2 w_4 + 4 w_1 w_2 w_3 + 6 w_2^3, \\
Q_4(2) &= w_0 w_2 w_4 w_6 + 2 w_0 w_2 w_5^2 + 2 w_0 w_3 w_6 + 2 w_1^2 w_3 w_6 + 4 w_0 w_3 w_4 w_5 + 6 w_1 w_2 w_4 w_5 + 4 w_1^2 w_3^2 + 6 w_2^2 w_4 w_5 + 4 w_1 w_2 w_3 w_5 + 4 w_2 w_3^2 + 6 w_3^3 + 6 w_1 w_2 w_3 w_4 + 8 w_1 w_3 w_4^2 + 8 w_2 w_3 w_4 + 8 w_4^3 + 24 w_3^4, \\
\end{align*}
\]

e etc. For $\gamma = 3$, one has

\[
\begin{align*}
Q_2(3) &= w_0 w_3 + 3 w_1 w_2, \\
Q_3(3) &= w_0 w_3 w_6 + 3 w_0 w_4 w_5 + 3 w_1 w_2 w_6 \\
&\quad + 6 w_2 w_3 w_5 + 15 w_2 w_3 w_4, \\
Q_4(3) &= w_0 w_3 w_6 w_9 + 3 w_0 w_4 w_5 w_9 + 3 w_1 w_2 w_3 w_9 \\
&\quad + 6 w_2 w_3 w_4 w_5 + 9 w_2 w_3 w_4 w_7 + 9 w_1 w_2 w_3 w_7 + 9 w_0 w_4 w_5 w_7 + 15 w_0 w_4 w_5 w_8 + 15 w_2 w_3 w_4 w_9 \\
&\quad + 27 w_1 w_3 w_4 w_6 + 27 w_1 w_3 w_4 w_6 + 45 w_2 w_3 w_4 w_7 + 105 w_2 w_3 w_4 w_8 + 27 w_2 w_3 w_4 w_9 + 105 w_3 w_4 w_5 w_9, \\
\end{align*}
\]

e etc.\footnote{1} For the specific case of the one-component system constrained to a unit circle with all $w_j = 1$ it was proved that \cite{57}

\[
Q_N(\gamma) = \frac{(\gamma N)!}{(\gamma!)^N N!} \quad \text{for all $w_j = 1$.}
\]

The expressions \cite{6.1} and \cite{6.2} pass this test of validity.

Another possibility is to express explicitly for $N = 2$ particles $Q_2(\gamma)$ with an arbitrary integer value of $\gamma$:

\[
Q_2(\gamma) = \frac{1}{2} \sum_{j=0}^{\gamma} \binom{\gamma}{j}^2 w_j w_{\gamma-j}.
\]

The free energy $F_N(\gamma)$ is expressed in terms of $Q_N(\gamma)$ in Eq. \cite{2.10}, the pressure is calculated by using Eq. \cite{2.29} and the interaction strengths are given by \cite{2.26}. Even for a relatively large number of particles $N = 8$, the calculation of the pressure from the exact formulas by using \textit{Mathematica} takes a few seconds of CPU time on the standard PC.

\footnote{1 The explicit formulas for $Q_N(2)$ and $Q_N(3)$ up to $N = 10$ will be sent upon request by the author.}
Fig. 3 The pressure $\tilde{P}$ versus the distance $d$ for the coupling $\Gamma = 4$ ($\gamma = 2$). The dotted (blue) curve corresponds to the asymmetry parameter $\eta = 0$ and $N = 8$ particles, the dashed (orange) curve to $\eta = 1$ and $N = 2$, the solid (green) curve to $\eta = 1$ and $N = 8$.

For the coupling $\Gamma = 4$, the exact data for the (dimensionless) pressure $\tilde{P}$ as the function of the (dimensionless) distance between the walls $d$ are presented in Fig. 3. If the wall at $x = d$ does not carry any charge, i.e. $\sigma' = 0$ or $\eta = 0$, $\tilde{P}$ is always positive for finite $d$, in agreement with the remark after Eq. (2.31), and its decay to zero at asymptotically large $d$ is monotonous for any number of particles $N$; this fact is documented for $N = 8$ in Fig. 3 by the dotted curve. On the other hand, in the symmetric case $\eta = 1$, for any $N$ there is a point at which $\tilde{P}$ intersects the $d$-axis and the pressure becomes negative, reaches a global minimum and stays to be negative up to $d \to \infty$. For the particle numbers $N = 2$ and $N = 8$ this fact is documented in Fig. 3 by the dashed and solid curves, respectively; note that the two curves are very close to one another which confirms the expected quick convergence of data with increasing $N$. We conclude that the attraction phenomenon arises in 2D starting from a relatively small coupling constant $\Gamma$, somewhere between 2 and 4.

The exact data for the pressure $\tilde{P}$ versus the distance $d$ for the coupling $\Gamma = 6$ are presented in Fig. 4. As before, the dotted curve corresponds to $\eta = 0$ and $N = 8$, the dashed curve to $\eta = 1$ and $N = 2$ and the solid curve to $\eta = 1$ and $N = 8$. The results for $\Gamma = 4$ and $\Gamma = 6$ are similar qualitatively, the global minima for $\eta = 1$ are quantitatively more profound at $\Gamma = 6$.

A natural question is at which value of the asymmetry parameter $\eta$ the monotonous decay of the positive $\tilde{P}$, observed at $\eta = 0$, changes to a non-monotonous plot (with one negative global minimum), observed at $\eta = 1$. The answer to this question is presented for the coupling $\Gamma = 6$ and $N = 6$ particles in Fig. 5. It turns out that as soon as $\eta$ is nonzero $\tilde{P}$ exhibits a non-
Fig. 4 The pressure $\tilde{P}$ versus the distance between the walls $d$ for the coupling $\Gamma = 6$ ($\gamma = 3$). The dotted (blue) curve corresponds to the asymmetry parameter $\eta = 0$ and $N = 8$ particles, the dashed (orange) curve to $\eta = 1$ and $N = 2$, the solid (green) curve to $\eta = 1$ and $N = 8$.

monotonous behavior with one negative (global) minimum. Consequently, the necessary and sufficient condition for the attraction phenomenon is the presence of a nonzero charge density on both walls. We suggest that the same condition applies to the analogous 3D models with counterions only.

The formula for $N = 2$ particle system (6.4), valid for any integer $\gamma$, is used in Fig. 6 to visualize the effect of increasing the coupling on the dependence of the pressure $\tilde{P}$ on the distance $d$. The chosen values of $\gamma$ are 4 (dotted curve), 8 (dashed curve) and 20 (solid curve). For the asymmetry parameter $\eta = 0$, all plots decay monotonously to 0 at $d \to \infty$, as is expected from the previous analysis. For $\eta = 1$, all plots exhibit a global (negative) minimum and goes to 0 at $d \to \infty$ from below. It is seen that by increasing $\gamma$ the global minimum of $\tilde{P}$ goes down; for $\gamma = 20$ it approaches the lower bound $-1$.

7 Conclusion

The neutral system of identical pointlike charges moving on the surface of a cylinder of circumference $W$ and length $d$, with the like-charged (symmetrically or asymmetrically) end-circles, was of interest in this paper. Like any 2D one-component model, it admits a 1D anticommuting-field representation which permits one to express the one-body, two-body, etc., densities of particles in terms of the anticommuting-field correlators. Specific transformations of the anticommuting variables, which preserve the composite form of the operators (2.24), imply specific sum rules for the statistical quantities which are basically of two types. The sum rules (3.11) and (3.26) are the
obvious finite-$W$ generalizations of the 2D zeroth-moment Stillinger-Lovett and WLMB conditions, respectively. Another sum rules, the one given by Eqs. (3.36), (3.37) and the other given by Eqs. (3.36), (3.37), provide in the limit $W \to \infty$ the new exact constraints (4.21) and (4.23) for the prefactor function $f^{(2)}(x,x')$ of the asymptotic behavior of the Ursell function along the two walls (4.10).

The possibility of an effective attraction between like-charged walls was another important subject investigated in this paper. The exactly solvable case of the free-fermion coupling $\Gamma = 2$ was studied in Sect. 5. For the symmetrically charged walls ($\eta = 1$), the monotonous dependence of the (dimensionless) pressure $\tilde{P}$ on the (dimensionless) distance between the walls $d$ in Fig. 2 shows that the results for $N = 8$ and $N \to \infty$ particles are practically indistinguishable. We expect that also for higher values of $\Gamma$ the results for $N = 8$ particles describe adequately those in the thermodynamic limit. For $\Gamma = 4$ and 6 we were able to derive the exact expressions for the dependence $\tilde{P}(d)$ up to $N = 8$ particles, see Figs. 3 and 4. If there is no charge on one of the walls ($\eta = 0$), the pressure $\tilde{P}$ is always positive and decreases monotonously with the distance $d$. In the case of the symmetrically charged walls ($\eta = 1$), the plot of $\tilde{P}$ versus $d$ has one global minimum and goes to 0 at $d \to \infty$ from below. In other words, the repulsion between the walls in the region of small $d$ changes at a specific distance to the attraction which lasts up to $d \to \infty$. Note a small difference between data for $N = 2$ and $N = 8$. As is shown in Fig. 5 for $\Gamma = 6$ and $N = 6$, the change from the monotonous to nonmonotonous behavior of $\tilde{P}$ occurs as soon as $\eta > 0$, i.e., when the two walls are like-charged by a nonzero line charge density the attraction takes.

Fig. 5 The pressure $\tilde{P}$ versus the distance between the walls $d$ for the coupling $\Gamma = 6$ and $N = 6$ particles. The asymmetry parameter $\eta$ takes successively the values 0, 0.1, 0.2, 0.4, 0.6, 0.8, 1, the corresponding curves go at $d = 0.3$ from up to down.
The pressure $\tilde{P}$ versus the distance between the walls $d$ for $N = 2$ particles. The dotted (blue) curves correspond to $\gamma = 4$, the dashed (orange) curves to $\gamma = 8$ and the solid (black) curves to $\gamma = 20$. The curves located only above the $d$-axis are evaluated with the asymmetry parameter $\eta = 0$ and those going also below the $d$-axis with $\eta = 1$.

The attraction phenomena between like-charged lines occurs starting from a relatively small coupling, somewhere between $\Gamma = 2$ and $\Gamma = 4$, is surprising.

As concerns our future plans, it might be interesting to extend the present analysis to other one-component Coulomb systems like the jellium model.

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