Separable Kernel of Nucleon-Nucleon Interaction in the Bethe-Salpeter Approach for $J = 0, 1$ \(^*\)

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The solution for the nucleon-nucleon ($NN$) $T$ matrix in the framework of the covariant Bethe-Salpeter approach for a two spin-one-half particle system with a separable kernel of interaction is analyzed. The explicit analytical connection between parameters of the separable kernel and low energy scattering parameters, deuteron binding energy and phase shifts is established. Covariant separable kernels for positive-energy partial channels with total angular momentum $J = 0$ ($^1S_0^+, ~^3P_0^+$) and $J = 1$ ($^3S_1^+, ~^3D_1^+, ~^1P_1^+, ~^3P_1^+$) are constructed by using obtained relations.

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1. Formalism

We start with the partial-wave decomposed Bethe-Salpeter equation for the nucleon-nucleon $T$ matrix (in the rest frame of two-nucleon system):

\[
t_{L' L}(p_0', p_0, p_0, p; s) = v_{L' L}(p_0', p_0, p_0, p; s) + \frac{i}{4\pi^3} \sum_{L''} \int dk_0 \int k^2 dk \frac{v_{L'' L''}(p_0', k_0, k; s) t_{L'' L}(k_0, k, p_0, p; s)}{(\sqrt{s}/2 - e_k + i0)^2 - k_0^2}.
\]  

Here $t$ is the partial-wave decomposed $T$ matrix and $v$ is the kernel of the $NN$ interaction, $e_k = \sqrt{k^2 + m^2}$. There is only one term in the sum for the singlet (uncoupled triplet) case ($L = J$) and there are two terms for the coupled triplet case ($L = J \pm 1$). We introduced square of the total momentum $s = P^2 = (p_1 + p_2)^2$ and the relative momentum $p = (p_1 - p_2)/2 \ [p' = (p_1' - p_2')/2]$ (for details, see reference [1]).

Assuming the separable form (rank I) for the partial-wave decomposed kernels of $NN$ interactions

\[
v_{L' L}(p_0', p_0, p_0, p; s) = \lambda g^{[L']}(p_0', p') g^{[L]}(p_0, p),
\]  

we can solve eq. (1) and write for the $T$ matrix:

\[
t_{L' L}(p_0', p_0, p_0, p; s) = \tau(s) g^{[L']}(p_0', p') g^{[L]}(p_0, p),
\]  

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with the function \( \tau(s) \) being:

\[
\tau(s) = 1/(\lambda^{-1} + h(s)).
\]

Function \( h(s) \) has the following form:

\[
h(s) = \sum_{L} h_{L}(s) = -\frac{i}{4\pi^{3}} \int dk_{0} \int k^{2} dk \sum_{L} \frac{[g^{[L]}(k_{0}, k)]^{2}}{(\sqrt{s}/2 - e_{k} + i0)^{2} - k_{0}^{2}}.
\]

We use following normalization condition for the on-mass-shell \( T \) matrix for the singlet (uncoupled triplet) case:

\[
t(s) \equiv t(0, \bar{p}, 0, \bar{p}, s) = -\frac{16\pi}{\sqrt{s}\sqrt{s - 4m^{2}}} e^{i\delta} \sin \delta,
\]

and for the coupled triplet case:

\[
t(s) = \frac{8\pi i}{\sqrt{s}\sqrt{s - 4m^{2}}} \begin{pmatrix} \cos 2\epsilon e^{2i\delta_{<}} - 1 & i \sin 2\epsilon e^{i(\delta_{<} + \delta_{>})} \\ i \sin 2\epsilon e^{i(\delta_{>})} & \cos 2\epsilon e^{2i\delta_{>} - 1} \end{pmatrix},
\]

with \( \bar{p} = \sqrt{s/4 - m^{2}} = \sqrt{2mT_{ab}} \). We introduced phase shifts \( \delta \equiv \delta_{L=J}, \delta_{\leq} \equiv \delta_{L=J+1} \) and mixing parameter \( \epsilon \).

Bound state, if exist, is described by simple pole in the \( T \) matrix. Using eq. (4) we can write (\( M_{b} = 2m - E_{b} \), \( E_{b} \) is the energy of bound state):

\[
\lambda^{-1} = -h(s = M_{b}^{2}).
\]

We introduce also the low-energy parameters – scattering length \( a_{L} \) and effective range \( r_{L} \) – by the following equation:

\[
\bar{p}^{2L+1} \cot \delta_{L}(s) = -1/a_{L} + \frac{r_{L}}{2} \bar{p}^{2} + \mathcal{O}(\bar{p}^{3})
\]

At this point by using eq. (3) and calculating \( T \) matrix on the mass-shell (\( p_{0} = p'_{0} = 0, p = p' = \bar{p} \)) we can connect internal parameters of the \( NN \) kernel and observables – phase shifts, bound state energy and low-energy parameters.

We use covariant generalization of the Yamaguchi \cite{2} functions for \( g^{[L]}(k_{0}, k) \):

\[
g^{[S]}(k_{0}, k) = \frac{1}{k_{0}^{2} - k^{2} - \beta_{0}^{2} + i0}, \quad (10)
\]

\[
g^{[P]}(k_{0}, k) = \frac{\sqrt{\left| -k_{0}^{2} + k^{2} \right|}}{(k_{0}^{2} - k^{2} - \beta_{1}^{2} + i0)^{2}}, \quad (11)
\]

\[
g^{[D]}(k_{0}, k) = \frac{C(k_{0}^{2} - k^{2})}{(k_{0}^{2} - k^{2} - \beta_{2}^{2} + i0)^{2}}, \quad (12)
\]

Let us consider function \( h_{0}(s, \beta) \) defined in the following way (we introduce explicit dependence on \( \beta \)):

\[
h_{0}(s, \beta) = -\frac{i}{4\pi^{3}} \partial_{\beta^{2}} \int dk_{0} \int k^{2} dk \frac{1}{(\sqrt{s}/2 - e_{k} + i0)^{2} - k_{0}^{2}} \frac{1}{k_{0}^{2} - e_{\beta}^{2} + i0}, \quad (13)
\]
here $e_\beta = \sqrt{k^2 + \beta^2}$ and $\partial_\beta^2 = \partial/\partial \beta^2$.

Analyzing the analytic structure and properties of the function $h_0(s, \beta)$ on $s$ we can rewrite it in the dispersion form:

$$h_0(s, \beta) = \int_{4m^2}^{+\infty} \frac{\rho(s', \beta)}{s' - s - i\epsilon} ds',$$

(14)

$$\rho(s', \beta) = \theta(s' - 4m^2)\rho_{\text{el}}(s', \beta) + \theta(s' - 4(m + \beta)^2)\rho_{\text{in}}(s', \beta),$$

with two spectral functions $\rho_{\text{el,in}}$ ($\text{el}$ stands for elastic and $\text{in}$ for inelastic) which are connected with the imaginary parts as

$$\rho(s', \beta) = \frac{1}{\pi} \text{Im} h_0(s', \beta) = \frac{1}{2\pi i}(h_0 - h_0^*).$$

(15)

Calculating integral (14) we can obtain analytic expressions for function $h_0(s, \beta)$ and taking into account definitions (10-12) can connect all other functions $h^{[L]}$ with $h_0$ as:

$$h^{[S]}(s, \beta_0) = h_0(s, \beta_0),$$

(16)

$$h^{[P]}(s, \beta_1) = -\frac{1}{2} \left[ \partial_{\beta_1^2} + \frac{1}{3} \beta_1^2 \partial_{\beta_1^2} \right] h_0(s, \beta_1),$$

(17)

$$h^{[D]}(s, \beta_2) = C^2 \left[ 1 + \beta_2^2 \partial_{\beta_2^2} + \frac{1}{6} \beta_2^4 \partial_{\beta_2^4} \right] h_0(s, \beta_2).$$

(18)

2. Calculations and results

We can now calculate internal parameters of the $NN$ kernel by using obtained above equations to reproduce experimental values for the phase shifts (data are taken using SAID program [http://gwdac.phys.gwu.edu/]), deuteron energy and quadrupole moment, low-energy parameters (data are from ref. [3]).

1. To find parameters $\lambda$ and $\beta$ in $^1S_0^+$ channel we solve system of the nonlinear equations (exp stands for experimental, $s$ - for singlet):

$$a_s^{\exp} = a_s(\lambda, \beta), \quad r_s^{\exp} = r_s(\lambda, \beta).$$

(19)

2. To find parameters $\lambda, \beta_0, \beta_2$ and $C$ in $^3S_1^- - 3^1D_1^+$ coupled channel we solve system of the nonlinear equations ($t$ stands for triplet):

$$a_t^{\exp} = a_t(\lambda, \beta_0, \beta_2, C), \quad E_d^{\exp} = E_0(\lambda, \beta_0, \beta_2, C),$$

(20)

$$p_d = p_d(\lambda, \beta_0, \beta_2, C), \quad q_d^{\exp} = q_d(\lambda, \beta_0, \beta_2, C).$$

Here we introduced $D$-wave pseudoprobability $p_d$.

3. To find parameters $\lambda$ and $\beta$ in uncoupled $^3P_0^+$, $^1P_1^+$ and $^3P_1^+$ channels we use procedure to minimize function:

$$\chi^2 = \sum_{i=1}^n \frac{(\delta^{\exp}(s_i) - \delta(s_i))^2}{\Delta\delta^{\exp}(s_i)^2},$$

(21)

where $n$ is the number of the experimental points taking into account.

The results of calculations are given in tables 1 and 2 and figs. 1-4.
Table 1. Parameters for $^1S_0^+$ and $^3S_1^+ - ^3D_1^+$ channels.

| Channel       | $^1S_0^+$ | $^3S_1^+ - ^3D_1^+$ (pd = 4%) | $^3S_1^+ - ^3D_1^+$ (pd = 5%) | $^3S_1^+ - ^3D_1^+$ (pd = 6%) |
|---------------|-----------|-------------------------------|-------------------------------|-------------------------------|
| $\lambda$ (GeV$^2$) | -0.294254 | -0.499045                     | -0.425235                     | -0.359856                     |
| $\beta_0$ (GeV)  | 0.224129  | 0.251248                      | 0.246713                      | 0.242291                      |
| $\beta_2$ (GeV)  | 0.294096  | 0.324494                      | 0.350217                      |                                |
| $C$            | 1.6489    | 2.4109                        | 3.2801                        |                                |

Table 2. Parameters for $^3P_0^+$, $^1P_1^+$ and $^3P_1^+$ channels.

| Channel       | $^3P_0^+$ | $^1P_1^+$ | $^3P_1^+$ | $^3P_1^+$ |
|---------------|-----------|-----------|-----------|-----------|
| $\lambda$ (GeV$^2$) | -0.0295720 | 0.0915850 | 0.195296  | 0.312975  |
| $\beta_1$ (GeV)  | 0.238515  | 0.276724  | 0.308907  | 0.338898  |

Figure 1. $^1S_0^+$ and $^3S_1^+$ channels phase shifts.

Figure 2. $^3P_0^+$ channel phase shifts.

Figure 3. $^1P_1^+$ channel phase shifts.

Figure 4. $^3P_1^+$ channel phase shifts.

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