BORCHERDS PRODUCTS ASSOCIATED WITH CERTAIN THOMPSON SERIES

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Abstract. We apply Zagier’s result for the traces of singular moduli to construct Borcherds products in higher level cases.

1. Introduction

Let $M_{1/2}^!$ be the additive group consisting of nearly holomorphic modular forms of weight $1/2$ for $\Gamma_0(4)$ whose Fourier coefficients are integers and satisfy the Kohnen’s “plus space” condition (i.e. $n$-th coefficients vanish unless $n \equiv 0$ or $1$ modulo $4$). We also let $\mathcal{B}$ be the multiplicative group consisting of meromorphic modular forms for some characters of $SL_2(\mathbb{Z})$ of integral weight with leading coefficient $1$ whose coefficients are integers and all of whose zeros and poles are either cusps or imaginary quadratic irrationals. Borcherds [3] gave an isomorphism between $M_{1/2}^!$ and $\mathcal{B}$ by means of infinite products which we call modular products or Borcherds products.

Let $d$ denote a positive integer congruent to $0$ or $3$ modulo $4$. We denote by $Q_d$ the set of positive definite binary quadratic forms $Q = [a, b, c] = aX^2 + bXY + cY^2$ ($a, b, c \in \mathbb{Z}$) of discriminant $-d$, with usual action of the modular group $\Gamma = PSL_2(\mathbb{Z})$. To each $Q \in Q_d$, we associate its unique root $\alpha_Q \in \mathbb{H}$ (=upper half plane). We define the Hurwitz-Kronecker class number $H(d)$ by $H(d) = \sum_{Q \in Q_d/\Gamma} \frac{1}{w_Q}$ where $w_Q = |\Gamma_Q|$. For instance, we have $H(3) = 1/3$, $H(4) = 1/2$, $H(7) = H(8) = H(11) = 1$, $H(12) = 4/3$, $H(15) = 2$, etc. For

Key words and phrases. modular product, generalized Hecke operator, Jacobi form, half integral form.

Mathematics Subject Classification : 11F03, 11F11, 11F22, 11F50
the modular invariant \( j(\tau) \), we define a function \( H_d(j(\tau)) \in \mathcal{B} \) by
\[
\prod_{Q \in \mathbb{Q}_d/\Gamma} \left( j(\tau) - j(\alpha_Q) \right)^{1/w_Q}.
\]
On the other hand, for each \( d \) there is a unique modular form \( f_{d,1} \in M_{1/2}^! \) having a Fourier development of the form
\[
f_{d,1} = q^{-d} + \sum_{D > 0} A(D, d) q^D, \quad q = e^{2\pi i \tau} (\tau \in \mathbb{H}).
\]
Then Borcherds’ theorem says that
\[
H_d(j(\tau)) = q^{-H_d} \prod_{u=1}^{\infty} (1 - q^u)^{A(u^2, d)}.
\]
Zagier [16] described the trace of a singular modulus of discriminant \(-d = \sum_{Q \in \mathbb{Q}_d/\Gamma} d/\Gamma (j(\alpha_Q) - 744)\) as the coefficient of \( q^d \) in a fixed modular form \(-g_{1,1}(\tau)\) of weight \( 3/2 \). By making use of this formula and considering Hecke operators in integral and half-integral weight Zagier reproved (\( \ast \)) (see [16] §6). Moreover he generalized the trace formula to the group \( \Gamma_0(N)^* \) (= the group generated by \( \Gamma_0(N) \) and all Atkin-Lehner involutions \( W_e \) for \( e || N \)) for \( 2 \leq N \leq 6 \) (see [16] §8).

In this article we find an analogue of (\( \ast \)) in higher level cases \( N = 2, 3, 5, 6 \) by applying Zagier’s Theorem 8 in [16]. Let \( M_{k-1/2}^+ (N)^! \) be the vector space consisting of nearly holomorphic modular forms of half-integral weight \( k - 1/2 \) on \( \Gamma_0(4N) \) whose \( n \)-th Fourier coefficient vanishes unless \(-1)^{k-1} n \) is a square modulo \( 4N \). There is a unique modular form \( f_{d,N} \in M_{1/2}^+ (N)^! \) having a Fourier expansion of the form
\[
f_{d,N} = q^{-d} + \sum_{D > 0} A(D, d) q^D.
\]
An explicit construction of \( f_{d,N} \) is given in the appendix and the uniqueness of \( f_{d,N} \) is shown in the end of §2. Let \( \mathbb{Q}_{d,N} \) be the set of forms \( Q = [a, b, c] \in \mathbb{Q}_d \) satisfying \( N | a \). Then \( \Gamma_0(N)^* \) naturally acts on \( \mathbb{Q}_{d,N} \) and the quotient \( \mathbb{Q}_{d,N}/\Gamma_0(N)^* \) has a bijection with \( \mathbb{Q}_d/\Gamma \) (see [16] §8). We can therefore define, for the Hauptmodul \( t(\tau) \) for \( \Gamma_0(N)^* \), a modular function \( H_d(t(\tau)) \) by
\[
\prod_{Q \in \mathbb{Q}_{d,N}/\Gamma_0(N)^*} (t(\tau) - t(\alpha_Q))^{1/w_Q}.
\]
In §3 we will prove the following theorem.
Theorem 1.1. Let $1 \leq N \leq 6$ other than 4 and $t$ be the Hauptmodul for $\Gamma_0(N)^*$. Let $-d$ be the discriminant corresponding to a Heegner point (i.e. the discriminant of $Q \in \mathbb{Q}_{d,N}$ with the condition that if $f^2$ divides $d$, then $(f, N) = 1$). Define $A^*(u^2, d) = 2^{s(u, N)} A(u^2, d)$ where $s(u, N)$ is the number of distinct prime factors dividing $(u, N)$. Then

$$
\mathcal{H}_d(t(\tau)) = q^{-H(d)} \prod_{u=1}^{\infty} (1 - q^u)^{A^*(u^2, d)}.
$$

We remark that this theorem is related to the problem of generalizing Borcherds’ theorem ([3] Theorem 14.1) to higher levels ([3] problem 10 in §17). In some sense Borcherds proved it himself in [4] Theorem 13.3. The vector valued modular forms he uses include the higher level case because a higher level form can be induced up to a vector valued form of level 1. An explicit infinite product is given in part 5 of Theorem 13.3 of [4]. But as he pointed out, it seems to take a bit of effort to unravel it to see what it says in the case of modular forms. Also Bruinier [5] proved that every automorphic forms with zeros on Heegner divisors can be written as modular products in the case that the lattice considered splits two hyperbolic planes over $\mathbb{Z}$.

Finally in §4 by using the idea given in [12] we derive a recursion formula which enables us to estimate all $A^*(u^2, d)$ for $u \geq 1$ from the Fourier coefficients of $\mathcal{H}_d(t(\tau))$.

2. Preliminaries

2.1. Generalized Hecke operator. Let $N$ be a positive integer and $e$ be any Hall divisor of $N$ (written $e\mid\mid N$), that is, a positive divisor of $N$ for which $(e, N/e) = 1$. We denote by $W_{e,N}$ a matrix \left(\begin{array}{cc} ae & b \\ cN & de \end{array}\right)$ with $\det W_{e,N} = e$ and $a, b, c, d \in \mathbb{Z}$, and call it an Atkin-Lehner involution. Let $S$ be a subset of Hall divisors of $N$ and let $N + S$ be the subgroup of $PSL_2(\mathbb{R})$ generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions $W_{e,N}$ for $e \in S$ (we may choose $S$ so that $1 \notin S$ and if $e_1, e_2 \in S$, then $e_1 e_2/(e_1, e_2)^2 \in S$ unless $e_1 = e_2$). We
assume that the genus of the group $N + S$ is zero. Then there exists a unique modular function $t$ with respect to $N + S$ satisfying

(i) $t$ is holomorphic on the complex upper half plane $\mathcal{H}$,

(ii) $t$ has the Fourier expansion at $\infty$ of the form

$$ t = q^{-1} + \sum_{k \geq 1} H_k q^k, \quad q = e^{2\pi i \tau} \ (\tau \in \mathcal{H}), $$

(iii) $t$ is holomorphic at all cusps which are not equivalent to $\infty$ under $N + S$. Such a function $t$ is called the Hauptmodul for $N + S$. By the result of Borcherds [2] $t$ becomes a monstrous function whose Fourier coefficients are related to representations of the monster group $\mathbb{M}$ except for the three cases (25−, 49 + 49 and 50 + 50). More precisely the $q$-series of $t$ coincides with a Thompson series $T_g(\tau) = \sum_{n \in \mathbb{Z}} \text{Tr}(g|V_n)q^n$ for some element $g$ of $M$. Here $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is the infinite dimensional graded representation of $\mathbb{M}$ constructed by Frenkel et al. [10, 11]. For a prime number $p$, let $t^{(p)}$ be the Hauptmodul for $N^{(p)} + S^{(p)}$ where $N^{(p)} = N/(p, N)$ and $S^{(p)}$ is the set of all $e$ in $S$ which divide $N^{(p)}$. Generally if $m = p_1 p_2 \cdots p_r$ is a product of primes $p_i$, then we define the $m$-th replicate $t^{(m)}$ of $t$ by

$$ t^{(m)} = (\cdots ((t^{(p_1)})^{(p_2)}) \cdots )^{(p_r)}. $$

For every positive integer $n$, let $t_n$ be a unique polynomial of $t$ satisfying $t_n \equiv q^{-n} \mod q\mathbb{C}[[q]]$. Define the $m$-th generalized Hecke operator $T(m)$ [1, 8, 9, 13] by

$$ t_n | T(m) = \sum_{\substack{a d = m \\ 0 \leq b < d}} t_n^{(a)} \left( \frac{a \tau + b}{d} \right). $$

The $m$-th replication formula [3, 13] says that $t_m = t | T(m)$. 
2.2. **Jacobi forms.** A *(holomorphic)* Jacobi form on $SL_2(\mathbb{Z})$ is defined to be a holomorphic function $\phi: \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the two transformation equations

$$
\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i N c z^2}{c\tau + d}} \phi(\tau, z) \quad ((a b c d) \in SL_2(\mathbb{Z}))
$$

$$
\phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i N(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z) \quad ((\lambda \mu) \in \mathbb{Z}^2)
$$

and having a Fourier expansion of the form

$$
\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}, 4Nn-r^2 \geq 0} c(n,r)q^n \zeta^r \quad (q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}). \quad (1)
$$

Here $k$ and $N$ are positive integers, called the *weight* and *index* of $\phi$, respectively. The coefficient $c(n,r)$ depends only on $4Nn-r^2$ and on $r (\text{mod } 2N)$ ([7] Theorem 2.2). In (1), if the condition $4Nn-r^2 \geq 0$ is deleted, we obtain a *nearly holomorphic Jacobi form*.

Let $J^!_{k,N}$ be the space of nearly holomorphic Jacobi forms of weight $k$ and index $N$. Let $J^!_{s,*}$ be the ring of all nearly holomorphic Jacobi forms and $J^!_{ev,*}$ its even weight subring. Then $J^!_{ev,*}$ is the free polynomial algebra over $M^!_4(\Gamma) = \mathbb{C}[E_4, E_6, \Delta^{-1}]/(E_4^3 - E_6^2 = 1728\Delta)$ on two generators $a = \tilde{\phi}_{-2,1}(\tau, z) \in J^!_{-2,1}$ and $b = \tilde{\phi}_{0,1}(\tau, z) \in J^!_{0,1}$ (for details, see [7] §9). Fix $k = 2$ and $1 \leq N \leq 6, \neq 4$. There are unique Jacobi forms $\phi_{D,N} \in J^!_{2,N}$ having Fourier coefficients $c(n,r) = B(D, 4Nn-r^2)$ which depend only on the discriminant $r^2 - 4Nn$ with $B(D, -D) = 1$ and $B(D, d) = 0$ if $d = 4Nn-r^2 < 0, \neq -D$. The uniqueness of $\phi_{D,N}$ is obvious since the difference of any two functions satisfying the definition of $\phi_{D,N}$ would be an element of $J^!_{2,N}$ (=the space of holomorphic Jacobi forms of weight 2 and index $N$), which is of dimension zero by [7], Theorem 9.1 (2). For the existence, we need an additional condition on Fourier coefficients that $B(D,0) = \begin{cases} -2, & \text{if } D \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$
The structure theorem then allows us to express \( \phi_{D,N} \) as a linear combination of \( a^i b^{N-i} (i = 0, \ldots, N) \) over \( M_1^*(\Gamma) \). Define

\[
g_{D,N} = q^{-D} + \sum_{d \geq 0} B(D, d) q^d.
\]

By the correspondence between Jacobi forms and half-integral forms ([7] Theorem 5.6), \( g_{D,N} \) lies in the space \( M_{3/2}^+(N) \) so that \( f_{d,N} g_{D,N} \) defines a modular form of weight 2 for \( \Gamma_0(4N) \). We write \( f_{d,N} g_{D,N} = \sum_{n \in \mathbb{Z}} c_n q^n \). The “plus” conditions imposed on \( f_{d,N} \) and \( g_{D,N} \) force \( (f_{d,N} g_{D,N})|_{U_{4N}} \) to be a modular form of weight 2 on \( SL_2(\mathbb{Z}) \). Here \( U_{4N} \) is the operator sending \( \sum_{n \in \mathbb{Z}} c_n q^n \) to \( \sum_{n \in \mathbb{Z}} c_{4Nn} q^n \). In fact, if we consider \( h = \sum_{i \in (\mathbb{Z}/4NZ)^\times} (f_{d,N} g_{D,N}) \left( \frac{\tau+i}{4N} \right) \), then \( h \) is invariant under the action of \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) and has the Fourier development of the form \( \varphi(4N) \sum_{n \in \mathbb{Z}} e^{2\pi in/N} c_{4n} q^n \) since \( c_n \) vanishes whenever \( n \equiv 2 \mod 4 \). \( \sum_{i=0}^{N-1} h(\tau+i) = N \varphi(4N)(f_{d,N} g_{D,N})|_{U_{4N}} \) is then invariant under the action of \( SL_2(\mathbb{Z}) \) with a pole only at \( \infty \). Thus \( (f_{d,N} g_{D,N})|_{U_{4N}} \) can be written as the derivative of some polynomial in \( j \). By comparing the constant terms we get \( A(D, d) = -B(D, d) \).

This also shows the uniqueness of \( f_{d,N} \).

Through the article we adopt the following notations:

- \( T(m) \): generalized Hecke operator
- \( T_m \): Hecke operator acting on Jacobi forms or half-integral forms ([7] §4 and §5)
- \( \phi_D = \phi_{D,N} \)
- \( g_D = g_{D,N} \)
- \( f_d = f_{d,N} \)
- \( \phi_D^{(p)} = \phi_{D,N(p)} \)
- \( g_D^{(p)} = g_{D,N(p)} \)
- \( B(d) = B(1, d) \)
For each positive integer $m$ and prime $p$, we define

$$J_m(d) = \sum_{Q \in \mathbb{Q}_{d,N}/\Gamma_0(N)^*} \frac{1}{w_Q} t_m(\alpha_Q)$$

and

$$J_m^{(p)}(d) = \sum_{Q \in \mathbb{Q}_{d,N(p)}/\Gamma_0(N(p))^*} \frac{1}{w_Q} t_m^{(p)}(\alpha_Q).$$

First we need two lemmas.

**Lemma 3.1.** Let $p$ be a prime dividing $N$. For $i \geq 0$ and $m$ coprime to $p$,

$$\phi_p^{(p)}|_{\mathcal{T}^i} = p \phi_{p^{2i+2}m^2} + \phi_{p^{2i}m^2}.$$  

Here $V_p$ is the Hecke operator on Jacobi forms defined by the formula (2) in [7].

**Proof.** According to [7] Theorem 4.1, the operator $V_p$ maps $J_{2,N/p}^i$ to $J_{2,N}^i$. From the formula (7) in [7], p.43, we find that

the coefficient of $q^n \zeta^r$ in $\phi_p^{(p)}|_{\mathcal{T}^i}|_{V_p} = \begin{cases} 
p, & \text{if } 4Nn - r^2 = -p^{2i+2}m^2 \\
1, & \text{if } 4Nn - r^2 = -p^{2i}m^2 \\
0, & \text{if } 4Nn - r^2 < 0, \neq -p^{2i+2}m^2, -p^{2i}m^2. \end{cases}$

From these observations and the uniqueness of $\phi_D$, the lemma immediately follows. \hfill $\Box$

**Lemma 3.2.** Let $l$ be a positive integer coprime to $N$ and $d = 4Nn - r^2$. Then

(i) $J_l(d) = -\text{coefficient of } q^n \zeta^r \text{ in } \phi_1|_{\mathcal{T}l}.$

(ii) $\phi_1|_{\mathcal{T}l} = \sum_{\nu \mid l} \nu \phi_{\nu^2}.$
Proof. (i) Let $p$ be a prime divisor of $l$. Then

$$J_p(d) = \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0(N)^*} \frac{1}{w_Q} t_p\left(\alpha_Q\right) = \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0(N)^*} \frac{1}{w_Q} t|_{T(p)}(\tau)|_{\tau=\alpha_Q}$$

$$= J_1(dp^2) + \left(-\frac{d}{p}\right) J_1(d) + pJ_1(d/p^2)$$

by a similar argument given in the proof of [10] Theorem 5-(ii)

$$= -[B(dp^2) + \left(-\frac{d}{p}\right) B(d) + pB(d/p^2)] \text{ by } [10] \text{ Theorem 8}$$

$$= -B_p(d).$$

Here $J_1(d/p^2)$ (resp. $B(d/p^2)$) is defined to be zero unless $d/p^2$ is an integer. And $B_p(d)$ denotes the coefficient of $q^d$ in $g_1|_{T_p}$, which is the same as the coefficient of $q^n\zeta^r$ in $\phi_1|_{T_p}$ ([7] Theorems 4.5 and 5.4). Now let $p^s||l$. Observe that $t|_{T(p^s)} = t_{p^s-1}|_{T(p)} - pt_{p^s-2}(\tau)$. Thus

$$J_{p^s}(d) = J_{p^s-1}(dp^2) + \left(-\frac{d}{p}\right) J_{p^s-1}(d) + pJ_{p^s-1}(d/p^2) - pJ_{p^s-2}(d)$$

$$= -[B_{p^s-1}(dp^2) + \left(-\frac{d}{p}\right) B_{p^s-1}(d) + pB_{p^s-1}(d/p^2)] + pB_{p^s-2}(d) \text{ by induction on } s$$

$$= -\text{coefficient of } q^n\zeta^r \text{ in } [(\phi_1|_{T_{p^s-1}})|_{T_p} - p\phi_1|_{T_{p^s-2}}]$$

$$= -\text{coefficient of } q^n\zeta^r \text{ in } \phi_1|_{T_{p^s}} \text{ by Corollary 1 in } [7] \text{ p.51.}$$

Now write $l = l'p^s$ with $(l', p) = 1$. Let $n(l)$ be the number of prime factors of $l$. We will use induction on $n(l)$. If $n(l) = 1$, it returns to the previous case. Now

$$t|_{T(l)} = t|_{T(l')}t|_{T(p^s)} = t_{l'}t|_{T(p^s)} = t_{l'p^s-1}|_{T(p)} - pt_{l'p^s-2} \text{ which yields that}$$

$$J_l(d) = J_{l'p^s-1}(dp^2) + \left(-\frac{d}{p}\right) J_{l'p^s-1}(d) + pJ_{l'p^s-1}(d/p^2) - pJ_{l'p^s-2}(d)$$

$$= -\text{coefficient of } q^n\zeta^r \text{ in } \phi_1|_{T_l} \text{ by induction on } s.$$
(ii) As before let \( p \) be a prime dividing \( l \) and \( p^s \divides l \). First we will show that \( \phi_1 | T_p^s = \sum_{i=0}^{s} p^i \phi_{p^2i} \). Let \( s = 1 \). Then the coefficient of \( q^d \) in \( g_1 | T_p \) is

\[
B(dp^2) + \left( -\frac{d}{p} \right) B(d) + pB(d/p^2) = \begin{cases} 
1, & \text{if } d = -1 \\
p, & \text{if } d = -p^2 \\
0, & \text{if } d < 0, \neq -1, -p^2. 
\end{cases}
\]

This implies \( g_1 | T_p = pg_{p^2} + g_1 \) and therefore \( \phi_1 | T_p = p\phi_{p^2} + \phi_1 \). Now let \( s \geq 2 \). Then

\[
\phi_1 | T_p^s = (\phi_1 | T_p^{s-1}) | T_p - p\phi_1 | T_p^{s-2} \\
= (\sum_{i=0}^{s-1} p^i \phi_{p^2i}) | T_p - p \sum_{i=0}^{s-2} p^i \phi_{p^2i} \text{ by induction on } s.
\]

For \( i > 0 \), the coefficient of \( q^d \) in \( g_{p^2i} | T_p \) is

\[
B(p^{2i}, dp^2) + \left( -\frac{d}{p} \right) B(p^{2i}, d) + pB(p^{2i}, d/p^2) = \begin{cases} 
1, & \text{if } d = -p^{2i-2} \\
p, & \text{if } d = -p^{2i+2} \\
0, & \text{if } d < 0, \neq -p^{2i-2}, -p^{2i+2}. 
\end{cases}
\]

This shows that \( \phi_{p^2i} | T_p = \begin{cases} 
\phi_{p^2i-2} + p\phi_{p^2i+2}, & \text{if } i > 0 \\
\phi_1 + p\phi_{p^2}, & \text{if } i = 0.
\end{cases} \)

Thus

\[
\phi_1 | T_p^s = (\sum_{i=0}^{s-1} p^i \phi_{p^2i}) | T_p - p \sum_{i=0}^{s-2} p^i \phi_{p^2i} = \sum_{i=1}^{s-1} p^i(\phi_{p^{2i-2}} + p\phi_{p^{2i+2}}) + \phi_1 + p\phi_{p^2} - p \sum_{i=0}^{s-2} p^i \phi_{p^2i} \\
= \sum_{i=0}^{s} p^i \phi_{p^2i}.
\]
As in the proof of (i), write $l = l'p^s$ with $(l', p) = 1$ and use induction on the number $n(l)$ of prime divisors of $l$. If $n(l) = 1$, the assertion is clear. If $n(l)$ is greater than 1, then
\[
\phi_1|_{T_l} = \phi_1|_{T_{l'}T_{p^s}} = \left(\sum_{\nu|l'} \nu \phi_{l',2}\right)|_{T_{p^s}} \text{ by induction on } n(l)
\]
\[
= \sum_{\nu|l} \nu \phi_{l',2} \text{ by induction on } s \text{ and applying the same argument as before.}
\]
\[
\square
\]

We claim that for $d = 4Nn - r^2$,
\[
J_m(d) = \text{coefficient of } q^n r^r \text{ in } \sum_{u|m} 2^{n(u,N)} u \phi_u^2. \tag{2}
\]

Let $p$ be a prime dividing $N$. By [13] Theorem 6.3 (2) (or [9] Proposition 2.6), the generalized Hecke operator $T(p)$ satisfies the following composition rule: for $k \geq 0$,
\[
T(p^k) \circ T(p) = T(p^{k+1}) + p I_p \circ T(p^{k-1})
\]
where $t_n|_{T_p} = t_n^{(p)}$ and $t_n$ is defined to be 0 if $n$ is not a rational integer. For $l$ coprime to $p$, we obtain
\[
t_{lp^{k+1}} = t_l|_{T(p^{k+1})} = (t_l|_{T(p^k)})|_{T(p)} - pt_l^{(p)}|_{T(p^{k-1})}
\]
\[
= t_{lp^k}|_{T(p)} - pt_{lp^{k-1}} = t_l^{(p)}(p\tau) + pt_{lp^k}|_{U_p} - pt_{lp^{k-1}}. \tag{3}
\]

Meanwhile [13] Theorem 3.1 Case I (or [9] Theorem 3.7 Case 1) provides the formula
\[
pt_{lp^k}|_{U_p} + t_{lp^k} = t_{lp^{k+1}}. \tag{4}
\]

Combining (3) with (4) we come up with $t_{lp^{k+1}}(\tau) = t_{lp^{k+1}}(p\tau) + t_{lp^k}(p\tau) - t_{lp^k}(\tau)$ and therefore
\[
\sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0(N)^*} t_{lp^{k+1}}(\alpha Q) = \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0(N)^*} \left(t_{lp^{k+1}}(p\tau) + t_{lp^k}(p\tau)|_{\tau = \alpha Q} - \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0(N)^*} t_{lp^k}(\alpha Q). \tag{5}
\]
The map which sends \([a, b, c] \in \mathbb{Q}_{d,N}\) to \([a/p, b, cp] \in \mathbb{Q}_{d,N/p}\) induces a bijection between \(\mathbb{Q}_{d,N}/\Gamma_0(N)^*\) and \(\mathbb{Q}_{d,N/p}/\Gamma_0(N/p)^*\). And the natural map from \(\mathbb{Q}_{d,N}/\Gamma_0(N)^*\) to \(\mathbb{Q}_{d,N/p}/\Gamma_0(N/p)^*\) also gives a bijection. Thus (3) is rewritten as

\[
\sum_{Q \in \mathbb{Q}_{d,N}/\Gamma_0(N)} t_{lp^{k+1}}(\alpha_Q) = 2 \sum_{Q \in \mathbb{Q}_{d,N}/\Gamma_0(N)} t_{lp^k}^{(p)}(\alpha_Q) - \sum_{Q \in \mathbb{Q}_{d,N}/\Gamma_0(N)} t_{lp^k}(\alpha_Q),
\]

which yields

\[
J_{lp^{k+1}}(d) = 2J_{lp^k}^{(p)}(d) - J_{lp^k}(d) \quad \text{for } k \geq 0. \tag{6}
\]

We divide \(N\) into two cases.

Case I. \(N = p = 2\) or \(3\) or \(5\)

In (2) we write \(m = lp^k\) with \((l, p) = 1\). We use induction on \(k\) to prove the claim. If \(k = 0\), the claim (2) follows from Lemma 3.2. Now assume the claim for \(k\).

\[
J_{lp^{k+1}}(d) = 2J_{lp^k}^{(p)}(d) - J_{lp^k}(d)
\]

by [16] Theorem 5-(ii) and induction hypothesis

\[
= -\text{coefficient of } q^n \zeta^r \text{ in } \left[2(\phi_1^{(p)})_{T_{lp^k}}|_{V_p} - \left(\sum_{i=1}^k \sum_{\nu|l} 2\nu p^i \phi_{\nu^2 p^2i} + \sum_{\nu|l} \nu \phi_{\nu^2}\right)\right]
\]

by [16] formula (19) and Theorem 5-(iii)

\[
= -\text{coefficient of } q^n \zeta^r \text{ in } \left[\sum_{i=0}^k \sum_{\nu|l} \nu p^i (p\phi_{\nu^2 p^2i + \phi_{\nu^2 p^2i}})\right] - \left(\sum_{i=1}^k \sum_{\nu|l} 2\nu p^i \phi_{\nu^2 p^2i} + \sum_{\nu|l} \nu \phi_{\nu^2}\right)\]
\]

by Lemma 3.1

\[
= -\text{coefficient of } q^n \zeta^r \text{ in } \left[\sum_{i=1}^{k+1} \sum_{\nu|l} 2\nu p^i \phi_{\nu^2 p^2i} + \sum_{\nu|l} \nu \phi_{\nu^2}\right] \quad \text{as desired.}
\]
Case II. $N = 6$

In (2) we write $m = l2^{k_1}3^{k_2}$ with $(l, 6) = 1$ and $k_1, k_2 \geq 0$. For simplicity, we put $\alpha(u) = 2^{s(u, 2)}u$ and $\beta(u) = 2^{s(u, 3)}u$. We will use induction on $k_1 + k_2$. If $k_1 + k_2 = 0$, the claim is immediate from Lemma 3.2. Now assume $k_1 + k_2 \geq 1$, say $k_2 \geq 1$.

$J_{l2^{k_1}3^{k_2}}(d) = 2J_{l2^{k_1}3^{k_2-1}}^{(3)}(d) - J_{l2^{k_1}3^{k_2-1}}^{(3)}(d)$

by (3)

$= -\text{coefficient of } q^n \zeta^r$ in $2\sum_{i=0}^{k_1} \sum_{j=0}^{k_2-1} \sum_{\nu | i} \alpha(\nu 2^i 3^j) \phi_{(\nu 2^i 3^j)^2}^{(3)} \nu_i - \sum_{i=0}^{k_1} \sum_{j=0}^{k_2-1} \sum_{\nu | i} \beta(\nu 2^i 3^j) \phi_{(\nu 2^i 3^j)^2}^{(3)}$

by the result in the case $N = 2$ and induction hypothesis

$= -\text{coefficient of } q^n \zeta^r$ in

$2\sum_{i=0}^{k_1} \sum_{j=0}^{k_2-1} \sum_{\nu | i} \alpha(\nu 2^i 3^j) (3 \phi_{(\nu 2^i 3^j)^2} + \phi_{(\nu 2^i 3^j)^2}) - \sum_{i=0}^{k_1} \sum_{j=0}^{k_2-1} \sum_{\nu | i} \beta(\nu 2^i 3^j) \phi_{(\nu 2^i 3^j)^2}$

by Lemma 3.1

$= -\text{coefficient of } q^n \zeta^r$ in

$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2-1} \sum_{\nu | i} 2\alpha(\nu 2^i 3^j) \cdot 3 \phi_{(\nu 2^i 3^j)^2}$

$+ \sum_{i=0}^{k_1} \sum_{j=1}^{k_2-1} \sum_{\nu | i} [2\alpha(\nu 2^i 3^{j-1}) \cdot 3 + 2\alpha(\nu 2^i 3^j) - \beta(\nu 2^i 3^j)] \phi_{(\nu 2^i 3^j)^2}$

$+ \sum_{i=0}^{k_1} \sum_{\nu | i} [2\alpha(\nu 2^i) - \beta(\nu 2^i)] \phi_{(\nu 2^i)^2}$

$= -\text{coefficient of } q^n \zeta^r$ in

$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \sum_{\nu | i} \beta(\nu 2^i 3^j) \phi_{(\nu 2^i 3^j)^2}$, as desired.

Let $z \in \mathfrak{F}$. Note that $\frac{1}{m} t_{m}(z)$ can be viewed as the coefficient of $q^m$-term in $- \log q - \log(t(\tau) - t(z))$ (see [14]). Thus $\log q^{-1} - \sum_{m > 0} \frac{1}{m} t_{m}(z) q^m = \log(t(\tau) - t(z))$. Taking exponential on both sides, we get

$q^{-1} \exp(- \sum_{m > 0} \frac{1}{m} t_{m}(z) q^m) = t(\tau) - t(z).$ (7)
Define $B^*(u^2, d) = 2s(u,N)B(u^2, d)$. By the claim (2), we obtain

$$J_m(d) = -\sum_{u|m} uB^*(u^2, d). \quad (8)$$

From (7) and (8) it follows that

$$H_d(t(\tau)) = q^{-H(d)} \exp(-\sum_{m=1}^\infty J_m(d)q^m/m) = q^{-H(d)} \exp(\sum_{m=1}^\infty \sum_{u|m} uB^*(u^2, d)q^m/m)$$

$$= q^{-H(d)} \exp(\sum_{m=1}^\infty \sum_{u=1}^\infty uB^*(u^2, d)q^{mu}/(mu))$$

$$= q^{-H(d)} \exp(\sum_{u=1}^\infty (-B^*(u^2, d))\sum_{m=1}^\infty -(q^u)^m/m)$$

$$= q^{-H(d)} \exp(\sum_{u=1}^\infty \log(1 - q^u)^{-B^*(u^2,d)}) = q^{-H(d)} \prod_{u=1}^\infty (1 - q^u)^{-B^*(u^2,d)}.$$ 

Now the fact $A(D, d) = -B(D, d)$ completes the proof of our theorem.

**Remark 3.3.** If $N = 4$, our proof does not apply since in this case the 2-plicate $t^{(2)}$ of $t$ is the Hauptmodul for $\Gamma_0(2)$ which is not $\Gamma_0(N)^*$-invariant for any $N$. In fact, we can numerically check that Theorem 1.1 fails when $N = 4$.

### 4. Some recursion formulas

Let $\delta$ be the denominator of $H(d)$. In the course of proving Theorem 1.1 we have seen that

$$H_d(t(\tau)) = q^{-H(d)} \prod_{m=1}^\infty \exp(-\sum_{u|m} uA^*(u^2, d)q^m/m).$$

Observe that $(q^H(d)H_d(t(\tau)))^\delta$ is of the form $1 + \sum_{m=1}^\infty c(m)q^m$ with $c(m) \in \mathbb{Z}$. Then

$$1 + \sum_{m=1}^\infty c(m)q^m = \prod_{m=1}^\infty \exp(-\sum_{u|m} \delta uA^*(u^2, d)q^m/m).$$
Put $V = \prod_{m=1}^{\infty} \exp(-\sum_{u|m\,\delta u A^*(u^2, d)q^m/m)$. The differential identity $(\log V)' = V'/V$ (here $'$ denotes $d\,d\eta/d\tau$) leads to

$$(-\sum_{m=1}^{\infty} \sum_{u|m\,\delta u A^*(u^2, d)q^m} \cdot (1 + \sum_{m=1}^{\infty} c(m)q^m) = \sum_{m=1}^{\infty} mc(m)q^m.$$  

Comparing the coefficients of $q^m$ on both sides we get

$$\sum_{u|m\,\delta u A^*(u^2, d)q^m} + \sum_{1\le k<m\,c(m-k)(\sum_{u|k}\delta u A^*(u^2, d)) = -mc(m).$$

Now we come up with the following recursion formula for $A^*(m^2, d)$: for $m \geq 1$,

$$A^*(m^2, d) = -\frac{1}{\delta}c(m) - \frac{1}{m}[\sum_{1\le u<m\,u A^*(u^2, d) + \sum_{1\le k<m\,c(m-k)(\sum_{u|k}u A^*(u^2, d))]. \quad (9)$$

Thus all $A^*(m^2, d)$ can be computed from the values of $c(m)$. Likewise all $c(m)$ can be estimated recursively from the values of $A^*(m^2, d)$.

**Example 4.1.** $N = 2, d = 4$ Theorem [1] yields the following product formula:

$$\left(t(\tau) - t\left(\frac{1 + \sqrt{-1}}{2}\right)\right)^{1/2} = q^{-1/2} \prod_{u=1}^{\infty}(1 - q^u)^{A^*(u^2, d)}. \quad (10)$$

Here the Hauptmodul $t$ for $\Gamma_0(2)^*$ can be described by means of Dedekind $\eta$-functions, i.e.,

$$t(\tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24} + 24 + \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24} = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + 10698752q^4 + 74428120q^5 + \cdots,$$

from which we obtain $t\left(\frac{1 + \sqrt{-1}}{2}\right) = -104$. The identity (10) is then rewritten as

$$1 + 104q + 4372q^2 + \cdots = \prod_{u=1}^{\infty}(1 - q^u)^{2A^*(u^2, d)} = \prod_{m=1}^{\infty} \exp(-\sum_{u|m\,2u A^*(u^2, d)q^m/m).$$
In (9) we take $\delta = 2$, $c(1) = 104$, $c(2) = 4372$, $c(3) = 96256$, etc. Then

\[
A^*(1, 4) = -\frac{1}{2} c(1) = -52,
\]

\[
A^*(4, 4) = -\frac{1}{2} c(2) - \frac{1}{2} [A^*(1, 4) + c(1) A^*(1, 4)] = 544,
\]

\[
A^*(9, 4) = -\frac{1}{2} c(3) - \frac{1}{3} [A^*(1, 4) + c(2) A^*(1, 4) + c(1) A^*(1, 4) + c(1) \cdot 2 \cdot A^*(4, 4)] = -8244,
\]

Appendix

Let $f_0 = \theta$. We found the initial $f_d$'s by expressing $[f_0, E_{12-2n}(4N\tau)]_n/\Delta(4N\tau)$ (if necessary, $[f_d, E_{12-2n}(4N\tau)]_n/\Delta(4N\tau)$) as linear combinations of them for $n = 1, 2, 3, 4$. Here $E_k$ is the normalized Eisenstein series of weight $k$, $\Delta$ is the modular discriminant and $[\cdot, \cdot]_n$ denotes the “Cohen bracket” ([3] §7 or [13] §1).

$N = 2$
\[
f_4 = q^{-4} - 52q + 272q^4 + 2600q^8 - 8244q^9 + 15300q^{12} + 71552q^{16} - 204800q^{17} + 282880q^{20} + \cdots,
\]

\[
f_7 = q^{-7} - 23q - 2048q^4 + 45056q^8 + 252q^9 - 516096q^{12} + 4145152q^{16} - 1771q^{17} - 26378240q^{20} + \cdots,
\]

$N = 3$
\[
f_3 = q^{-3} - 14q + 40q^4 - 78q^9 + 168q^{12} - 378q^{13} + 688q^{16} + \cdots,
\]

\[
f_8 = q^{-8} - 34q - 188q^4 + 2430q^9 + 8262q^{12} - 11968q^{13} - 34936q^{16} + \cdots,
\]

\[
f_{11} = q^{-11} + 22q - 552q^4 - 11178q^9 + 48600q^{12} + 76175q^{13} - 269744q^{16} + \cdots,
\]

$N = 5$
\[
f_4 = q^{-4} - 8q + q^4 + 10q^5 + 12q^9 - 62q^{16} + 65q^{20} + \cdots,
\]

\[
f_{11} = q^{-11} - 12q - 56q^4 - 45q^5 + 276q^9 + 672q^{16} + 2520q^{20} + \cdots,
\]

\[
f_{15} = q^{-15} - 38q + 112q^4 - 96q^5 - 988q^9 + 8512q^{16} + 11856q^{20} + \cdots,
\]

\[
f_{16} = q^{-16} - 6q - 132q^4 + 120q^5 - 1014q^9 + 3585q^{16} + 17030q^{20} + \cdots,
\]

\[
f_{19} = q^{-19} + 20q + 56q^4 - 210q^5 - 780q^9 - 23200q^{16} + 46760q^{20} + \cdots,
\]
\( N = 6 \)

\[
f_8 = q^{-8} - 10q - 12q^4 + 54q^9 + 54q^{12} - 88q^{16} + \cdots ,
\]

\[
f_{12} = q^{-12} - 28q + 26q^4 - 156q^9 + 168q^{12} + 728q^{16} + \cdots ,
\]

\[
f_{15} = q^{-15} - 10q - 64q^4 + 3q^9 - 320q^{12} + 1664q^{16} + \cdots ,
\]

\[
f_{20} = q^{-20} + 12q - 64q^4 - 756q^9 + 945q^{12} - 2912q^{16} + \cdots ,
\]

\[
f_{23} = q^{-23} - 13q + 64q^4 - 27q^9 - 1728q^{12} - 5760q^{16} + \cdots ,
\]

For the remaining \( f_d(\tau) \) we inductively obtain them by multiplying \( f_{d-4N}(\tau) \) by \( j(4N\tau) \) to get a “plus” form of weight 1/2 with leading coefficient \( q^{-d} \) and then subtracting a suitable linear combination of \( f_{d'}(\tau) \) with \( 0 \leq d' < d \).

**Acknowledgment**

I am grateful to Professor Don Zagier for introducing me to this subject. I would also like to take an opportunity to thank Professor Richard E. Borcherds, Professor Jan H. Bruinier and Professor Ja Kyung Koo for their kind and valuable comments.

**References**

[1] D. Alexander, C. Cummins, J. Mckay and C. Simons, *Completely replicable functions*. In: Groups, Combinatorics and Geometry, Cambridge Univ. Press, 87-95, 1992.

[2] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. math. **109** (1992), 405-444.

[3] R. Borcherds, *Automorphic forms on \( O_{s+2,2}(R) \) and infinite products*, Invent. Math. **120** (1995), 161-213.

[4] R. Borcherds, *Automorphic forms with singularities on Grassmanians*, Invent. Math. **132** (1998), 491-562.

[5] J. H. Bruinier, *Borcherds products on \( O(2, l) \) and Chern classes of Heegner Divisors*, Habilitationsschrift (Universität Heidelberg, May 2000) to appear in the Springer Lecture Notes of Mathematical series.
[6] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. 217 (1975), 271-285.

[7] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. 55, Bikhäuser-Verlag, Boston-Basel-Stuttgart, 1985.

[8] C. R. Ferenbaugh, *Lattices and generalized Hecke operators*. In Groups, Difference sets and Monster, (K. T. Arasu et al., eds) de Gruyter, 1995, pp. 363-368.

[9] C. R. Ferenbaugh, *Replication Formulae for n|h-Type Hauptmoduls*, J. Algebra 179 (1996), 808-837.

[10] I. B. Frenkel, J. Lepowsky, A. Meurman, *A natural representation of the Fischer-Griess monster with the modular function J as character*, Proc. Natl. Acad. Sci. USA 81 (1984), 3256-3260.

[11] I. B. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the monster*, Boston, MA Academic Press, 1988.

[12] S. J. Kang, C. H. Kim, J. K. Koo and Y. T. Oh, *Graded Lie superalgebras and super-replicable functions*, (preprint)

[13] M. Koike, *On replication formula and Hecke operators*, Nagoya University (preprint).

[14] S. P. Norton, *More on moonshine*. In: Computational Group Theory, 185-193, Academic Press, 1984.

[15] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. 104 (1994), 57-75.

[16] D. Zagier, *Traces of singular moduli*, Max-Planck-Institut für Mathematik, Preprint series 2000 (8).

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