Entanglement and objectivity in pure dephasing models

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(Dated: March 10, 2022)

We study the relation between the emergence of objectivity and qubit-environment entanglement generation. We find that although entanglement with the unobserved environments is irrelevant (since sufficiently strong decoherence can occur regardless), entanglement with the observed environments is crucial. In fact, the appearance of an objective qubit-observed-environment state is strictly impossible if their joint evolution does not lead to entanglement. Furthermore, if a single observer has access to a single environment (no macrofractions) then the required orthogonality of the observed environmental states comes only as a consequence of the system-environment state becoming strongly entangled (maximally entangled for the given initial occupation of the qubit if the environmental state is initially pure).

\section{I. INTRODUCTION}

The idea of objectivity is related to the notion that classicality should emerge naturally out of the quantum description to allow unrelated observations of a system by different parties, which would neither destroy the state of the observed system nor leave ambiguity in the information obtained about its state. Such observations are to be performed not by direct measurements of the system of interest (which would obviously not satisfy the requirements of objectivity in quantum mechanics), but rather by measurements of different environments which can gain information about the system state during their joint evolution with said system.

Recently, the notion of such a system-environment state has been has been specified and its mathematical structure proposed. The so called spectrum broadcast structure (SBS) states are zero-discord states from the point of view of the system of interest and from each environment to be observed, which guarantees that repeated, independent measurements on individual environments do not disturb the state of the system nor of other environments:

\begin{equation}
\hat{\rho}_{SBS} = \sum_{i} p_i |i\rangle \langle i| \otimes \hat{\rho}_1^i \cdots \otimes \hat{\rho}_N^i ,
\end{equation}

\begin{equation}
\hat{\rho}_k^i \perp \hat{\rho}_k^{i'} \text{ for every } i' \neq i \text{ and } k = 1, \ldots, N.
\end{equation}

Here \{ |i\rangle \} is the so-called pointer basis of the central system to which it decoheres, \( p_i \) are initial pointer probabilities, \( k \) enumerates the environments, and \( \hat{\rho}_k^i \) are some states of the observed parts of the environment of which we require only that they have mutually orthogonal supports for different pointer index \( i \) or in other words are perfectly distinguishable in one shot.

To study situations when the emergence of objectivity is possible in the above sense of SBS creation, typically system-environment interactions are taken into account which lead to pure dephasing of the qubit after the environmental degrees of freedom are traced out. These types of interactions describe situations when the \( T_2^* \) times are much shorter than the \( T_1 \) times (the decay of phase coherence is much faster than relaxation) are fairly common in solid state scenarios. It is true that for the two-level systems earliest studied as qubits in context of quantum information processing, such as two-level atoms manipulated coherently with a narrow linewidth laser field, energy exchange with an environment (spontaneous emission) was often the most relevant decoherence mechanism. However, this was due to the fact that the only environment of such atoms was electromagnetic vacuum. In order to advance the development of quantum circuits to the multi-qubit stage, it turned out to be necessary to move to architectures in which the relevant environment was much more structured. All the solid-state based qubits (both spin based and charge-based in semiconductors, flux and many charge-based qubits in superconductors) are exposed to structured environments lattice vibrations, charge noise from various sources, nuclear spin fluctuations etc. which typically have slow dynamics (correlation times possibly longer than typical qubit decoherence time), and as such are more efficient at dephasing of the qubit state that is in a superposition of pointer states, than at changing the populations of these pointer states. It is important to note that the same applies to ion trap qubits. All these systems the dephasing of the qubit occurs on timescales orders of magnitude shorter than the timescale of energy exchange with the environment, and decoherence channels such as amplitude damping are irrelevant when the initial state of the qubit is a superposition of pointer states.

Such interactions do not disturb the occupations of the system described in the basis of so called pointer states, but are detrimental to qubit coherence. When a system-environment state is initially in a pure (product) state, the decoherence corresponds directly to the buildup of system-environment entanglement, and is in fact accompanied by the transfer of information about the system state into the environment. This information transfer is the basis for the possible formation of SBS states, since for information about the system state to
be read out of an environment, it must first be transferred there by some physical process. The formation of SBS states further requires a source of decoherence (and environment or environments which are not observed) as to lose the coherence and correlations between the system and the observed environments, which is necessary to get the zero-discord form.

In realistic situations, pure initial states of any environments are rare. Hence, the study of mixed initial environmental states is necessary to assess if SBS states are likely to occur. For mixed states the correlation between system-environment entanglement buildup and system decoherence is less direct; although such entanglement is always accompanied by decoherence, decoherence is also possible without said entanglement [27, 31]. In the following we use the methods devised in Refs. [30, 31] to study the interrelations between system-environment entanglement generation and the possibility of the emergence of SBS states. We show that entanglement is a necessary, but not sufficient condition for objectivity. In fact, if no entanglement between the observed environments and the system is generated, there is no possibility to distinguish between the system pointer states by performing any measurements on the environments. Hence, entanglement is directly responsible for the transfer of information about the system state into an environment and its lack is equivalent to the complete absence of such transfer, so no methods which can be used to enhance distinguishability (such as assigning more environments to a single observer [6–13]) will work.

We further study the conditions which need to be fulfilled by the interaction with the observed environments for environmental states corresponding to qubit pointer states to become fully distinguishable (this part is limited to the system consisting of a single qubit). We find that the amount of entanglement necessary is surprisingly small and that, consequently, there are constraints on the initial purity of the environments for such full distinguishability to be able to manifest itself.

The article is organized as follows. In Sec. II we introduce the studied model in detail, including the Hamiltonian of the system, observed environments, and unobserved environments, the resulting evolution operator and the structure of the density matrix of the whole system at time t. We then show the necessary constraints on the evolution of the observed and unobserved environments conditional on the states of the system for the emergence of objectivity. In Sec. III we first show that separability of the qubit and its observed environments excludes the emergence of SBS states, and later generalize the results to a system of any size. Sec. IV is devoted to the study of when observations of an environment can be used to completely distinguish between qubit pointer states. We find that this leads to a strict condition for qubit-environment entanglement, but also excludes small values of initial purity of the environment. Sec. V concludes the article.

II. PURE DEPHASING EVOLUTIONS AND SBS STATES

The measurement Hamiltonian which is used to study the emergence of spectrum broadcast structure states is, in fact a special case of the general Hamiltonian which leads to pure dephasing evolutions [6–13] in a system-environment scenario [30, 31]. The Hamiltonian is given by

\[ \hat{H} = \sum_i \varepsilon_i |i\rangle\langle i| + |i\rangle\langle i| \otimes \hat{V}_{\text{not}}^i + \sum_k |i\rangle\langle i| \otimes \hat{V}_k^i, \]

where the first term on the right describes the free Hamiltonian of the system in its eigenbasis \(|i\rangle\}, while the other two terms describe the interaction with the environment. The environmental coupling is divided into two parts: The second term on the right is responsible for the interaction with the part of the environment which is not observed and its free evolution, \(\hat{V}_{\text{not}}^i = \hat{H}_{\text{not}}^i + \hat{V}_{\text{not}}^i\). This Hamiltonian may be further resolved into parts describing different unobserved environments. We do not do it explicitly here, as it has no bearing on the described results. The third term is responsible for the interaction of the system with the observed part of the environment and its free evolution, \(\hat{V}_k^i = \hat{H}_k^i + \hat{V}_k^i\). This part of the Hamiltonian is resolved with respect to the individual observers, which are labeled by the index k. Since all terms in the Hamiltonian [6] commute, the evolution operator for the whole system can be written as

\[ \hat{U}(t) = \sum_i e^{-i\varepsilon_i t} |i\rangle\langle i| \otimes e^{-i\hat{V}_{\text{not}}^i t} \otimes \bigotimes_k e^{-i\hat{V}_k^i t}. \]

Let us stress here that the free evolution of the environments is present in the environmental coupling operators \(\hat{V}_{\text{not}}^i\) and \(\hat{V}_k^i\) and no assumptions are made on the commutation relations between different parts of the Hamiltonian pertaining to the same environment (neither the free Hamiltonian must commute with the interaction, nor the different elements of the interaction with themselves). The difference between Hamiltonian [6] and the general pure-dephasing Hamiltonian amounts to the lack of interaction between the different environments.

We assume (following the papers on objectivity [6–13]) that the initial state of the system and environments is a product state with respect to the system, the unobserved environment, and each observed environment. This translates into the physical situation, when the state of the system can be prepared without disturbing any of the environments, and the observers each are truly separated. Furthermore we assume (as to be able to use the results on system-environment entanglement for pure dephasing evolutions [30, 31]) that the initial state of the qubit is pure. This means that the initial state of the system and its environments can be written as

\[ \hat{\sigma}(0) = |\psi_0\rangle\langle \psi_0| \otimes \hat{R}_{\text{not}}(0) \otimes \bigotimes_k \hat{\rho}_k^i(0), \]
where $|\psi_0\rangle = \sum_i a_i |i\rangle$ is a superposition of pointer states of the system, $\hat R_{\text{nat}}(0)$ is the initial state of the unobserved environment, and $\hat \rho_{ii}(0)$ are the individual initial states of the observed environments.

\[ \hat \rho_{ii}(0) = \begin{pmatrix} \rho_{ii}(0) & 0 \\ 0 & 0 \end{pmatrix} \]

The density matrix (9) is given by the off-diagonal terms in eq. (9). Under such an assumption, non-entangling evolutions [27–31], and we will implicitly consider this condition, since its fulfillment is not related to the qubit being in one of its pointer states are the same. This in turn is equivalent to the qubit initially in a pure state, as in the initial state or the states of the environments themselves [4, 5].

**III. ENTANGLEMENT AND THE EMERGENCE OF OBJECTIVITY**

Let us begin with the study of the situation when the system of interest is only a qubit (so $i = 0, 1$). In Ref. (31) it is shown that for an evolution governed by a Hamiltonian which leads to pure dephasing of the qubit after the environmental degrees of freedom are traced out, of which the Hamiltonian given in eq. (4) is a special case, and a product initial qubit–environment state, with the qubit initially in a pure state, as in the initial state of eq. (4), the qubit–environment state is separable if and only if the evolutions of the environment conditional on the qubit being in one of its pointer states are the same. In the studied scenario, this condition translates into

\[ \hat R_{ii}(t) \otimes \hat R_{jj}(t) = \hat R_{jj}(t) \otimes \hat R_{ii}(t) \]

for every pair of system pointer states, $i \neq j$, and for all observed environments $k$. Only then can the system states be uniquely determined by measurements on any of the environments, without damaging either the system state or the states of the environments themselves [4, 5].

Taking the initial state (5) and using the evolution operator given by eq. (4) yields the density matrix of the system and its environments at any time $t$. If the system is only a qubit, $i = 0, 1$, this can be explicitly written as

\begin{align*}
\hat \sigma(t) = & \left( \begin{array}{cc}
|a_0|^2 \hat R_{00}(t) \otimes \hat \rho_{00}(t) & a_0 a_1^* (t) \hat \rho_{01}(t) \\
\hat a_1(t) a_0^* \hat R_{10}(t) \otimes \hat \rho_{10}(t) & |a_1|^2 \hat R_{11}(t) \otimes \hat \rho_{11}(t)
\end{array} \right),
\end{align*}

where $a_i(t) = e^{-i \Delta \varepsilon t}$ and $\Delta \varepsilon = \varepsilon_1 - \varepsilon_0$,

\begin{align*}
\hat R_{ii}(t) & = \hat \omega_{ii}^\text{not}(t) \hat \rho_{\text{not}}(0) \hat \omega_{ii}^\text{not}_\dagger, \\
\hat \rho_{ii}(t) & = \hat \omega_i^k(t) \hat \rho_{ii}(0) \hat \omega_i^k(t),
\end{align*}

with the environmental evolution operators conditional on the state of the qubit given by

\begin{align*}
\hat \omega_{ii}^\text{not}(t) & = e^{-i \hat \Gamma_{ii}(t)} \\
\hat \omega_i^k(t) & = e^{-i \hat \Gamma_i^k(t)}.
\end{align*}

This structure is preserved, and it can be easily generalized to a system of any size [31].

The next step in trying to obtain the SBS state is tracing out over the unobserved environment, which for a qubit yields

\[ \hat \sigma(t) = \left( \begin{array}{cc}
|a_0|^2 \hat \rho_{00}(t) & a_0 a_1^* (t) \hat \rho_{01}(t) \\
\hat a_1(t) a_0^* \hat \rho_{10}(t) & |a_1|^2 \hat \rho_{11}(t)
\end{array} \right),
\]

since $\text{Tr} \hat R_{ii}(t) = 1$ (the matrices $\hat R_{ii}(t)$ are density matrices, as they are obtained via a unitary operation from the initial density matrix of the unobserved environment) and where

\[ \hat \Gamma_{01}(t) = \text{Tr} \hat \rho_{01}(t) = \text{Tr} \left[ \hat \omega_{i1}^\text{not}(t) \hat \omega_{0i}^\text{not}(t) \hat \rho_{\text{not}}(0) \right]
\]

is the decoherence factor.

For the state (4) to become an SBS state in the \{|0\rangle, |1\rangle\} pointer basis of the qubit, two conditions need to be met. Firstly, the decoherence function stemming from the unobserved part of the environment has to decay to zero, $\hat \Gamma_{01}(t) = 0$. In the following we will not be studying this condition, since its fulfillment is not related to the generation of qubit–environment entanglement, but only to the strength of the interaction between the qubit and the unobserved part of the environment. In fact, the condition can be met both in case of entangling and non-entangling evolutions [27–31], and we will implicitly assume that the interaction is strong enough to cancel the off-diagonal terms in eq. (4). Under such an assumption, the density matrix (9) is given by

\[ \hat \sigma(t) = \sum_i |a_i|^2 |i\rangle \langle i| \otimes \hat \rho_{ii}(t). \]
Hence, if there is no entanglement generated between the qubit and the environment, then
\[ \forall k \rho_{ii}^k(t) = \rho_{jj}^k(t), \]  
(14)

Obviously this implies that the state (11) cannot be an SBS state, as states \( \rho_{ij}^k(t) \) are then identical for different central qubit states \( i \).

This result can be easily generalized to a system of any size with the help of Ref. (31), where it is shown that the necessary condition for separability (but not sufficient) is exactly of the form as eq. (14), but it must be fulfilled for all pairs of system pointer states, \( i \neq j \). This translates into a family of conditions for different environments \( k \), as in eq. (13). Hence, for systems larger than a qubit, objectivity cannot emerge if the the system-environment evolution for state \( k \) is non-entangling, but also for entangling evolutions that satisfy the necessary separability condition (14).

This leads us to our first result:

**Proposition 1** If the evolution governed by a pure dephasing Hamiltonian (3) with an initial state of the form (4) does not generate a qubit-environment entanglement, then SBS states will not be formed.

Thus we obtain an interesting result that qubit-environment entanglement (QEE) is a necessary condition for the emergence of objectivity. This is somewhat intriguing since the purely quantum property of entanglement appears to be necessary for the emergence of the classical property of objectivity (for a similar conclusion but obtained in a very different context see Ref. (32)). But QEE alone is not sufficient for the emergence of objectivity even for qubit systems, since the condition for QEE generation, \( \rho_{ii}^k(t) \neq \rho_{jj}^k(t) \), is much weaker than the condition (12). We study the latter in more detail in the next Section.

### IV. STRICT DISTINGUISHABILITY

Let us now study the situation when the orthogonality condition (12) is strictly fulfilled. In realistic situations one can hardly expect such strict fulfillment (see e.g. [6–13]) and some measure of distinguishability [33, 34], like the state fidelity, must be used. Nevertheless it is interesting to study the ideal situation and use it to infer about the behavior of systems which do not show strict orthogonality, but do exhibit the generation of system-environment entanglement. The following will be restricted to the system of interest composed of a single qubit.

The fulfillment of condition (12) means that the environmental density matrices conditional on different qubit states \(|0\rangle\) and \(|1\rangle\) must be defined on separate subspaces of the Hilbert space of the corresponding environment. Hence, it imposes strong limitations on the evolution of the qubit-environment state. Since the condition (12) must be fulfilled separately for any environment \( k \), the study of the condition for a single environment leads to results which must be fulfilled by all observed environments separately. Hence, we will study one environment labeled by the index \( k \) without any limitations on its dimension \( d_k \).

It is now most convenient to express all of the \( \rho_{ij}^k(t) \) matrices, eq. (7), with the help of the \( \hat{\rho}_{00}^k(t) \) conditional environmental density matrix,
\[ \hat{\rho}_{11}^k(t) = \hat{w}^k(t)\hat{\rho}_{00}^k(t)\hat{w}^k(t), \]  
(15a)
\[ \hat{\rho}_{01}^k(t) = \hat{\rho}_{00}^k(t)\hat{w}^k(t), \]  
(15b)
\[ \hat{\rho}_{10}^k(t) = \hat{w}^k(t)\hat{\rho}_{00}^k(t), \]  
(15c)

where \( \hat{w}^k(t) = \hat{w}^k(t)\hat{w}^k(t) \).

We can write the density matrix of the \( k \)-th environment conditional on the qubit state being zero at time \( t \) in its eigenbasis
\[ \hat{\rho}_{00}^k(t) = \sum_{n_k} c_{n_k}|n_k\rangle\langle n_k(t)| \]  
(17)

(\( c_{n_k} \) do not depend on \( t \) because \( \hat{\rho}_{00}^k(t) \) is obtained from \( \hat{\rho}^k(0) \) by a unitary rotation). The basis of the density matrix \( \hat{\rho}_{00}^k(t) \) is time-dependent, but in what follows we will not write the time-dependences explicitly, both for clarity and to save space. Equally well, one can perform the following analysis in the eigenbasis of \( \hat{\rho}^k(0) \) or \( \hat{\rho}_{11}^k(t) \) and we pick \( \hat{\rho}_{00}^k(t) \) for definiteness and convenience.

The orthogonality condition (12) can be written in this notation
\[ \hat{\rho}_{00}^k(t)\hat{\rho}_{11}^k(t) = \hat{\rho}_{11}^k(t)\hat{w}^k(t)\hat{\rho}_{00}^k(t)\hat{w}^k(t) \]  
(18)
\[ = \sum_{n_k, m_k} c_{n_k}c_{n_k}|n_k\rangle\langle n_k|\hat{w}^k(t)|m_k\rangle\langle m_k|\hat{w}^k(t) = 0. \]  
(19)

For (18) to be fulfilled, all of the elements of the matrix \( \hat{\rho}_{00}^k(t)\hat{\rho}_{11}^k(t) \) must be equal to zero, so for all \( p_k \) and \( q_k \),
\[ \langle p_k|\hat{\rho}_{00}^k\hat{\rho}_{11}^k|q_k\rangle = \sum_{n_k, m_k} c_{n_k}c_{m_k}\langle p_k|n_k\rangle\langle n_k|\hat{w}^k|m_k\rangle\langle m_k|\hat{w}^k|q_k\rangle = c_{p_k}\sum_{m_k} c_{m_k}\langle p_k|\hat{w}^k|m_k\rangle\langle m_k|\hat{w}^k|q_k\rangle = 0. \]  
(19)

Here the second line is obtained assuming that the states \(|p_k\rangle\) and \(|q_k\rangle\) are eigenstates of \( \hat{\rho}_{00}^k(t) \). In this case, let us look at the diagonal elements, \( p_k = q_k \), for which we must have
\[ \langle p_k|\hat{\rho}_{00}^k|\hat{\rho}_{11}^k(t)|p_k\rangle = \sum_{m_k} c_{m_k}\langle p_k|\hat{w}^k|\hat{w}^k|\hat{w}^k|m_k\rangle\langle m_k|\hat{w}^k|q_k\rangle = 0, \]  
(20)

for all \( p_k \). Since \( c_{p_k} \geq 0 \), \( c_{m_k} \geq 0 \), and \( |\langle p_k|\hat{w}^k|\hat{w}^k|m_k\rangle|^2 \geq 0 \), the only situation when (20) is fulfilled, is either when \( c_{p_k} = 0 \) or when \( c_{m_k} = 0 \) or \( \langle p_k|\hat{w}^k|m_k\rangle = 0 \) for all \( m_k \).
Note that this automatically implies that the condition (19) for off-diagonal elements is also met. The last relevant, if somewhat trivial, observation here is that for \( m_k = p_k \) (the sum spans over all \( \hat{\rho}_{00}(t) \) eigenstates) we get that either \( c_{p_k} = 0 \) or \( (p_k | \hat{w}^k | p_k) \neq 0 \), which must hold for any \( p_k \).

If the orthogonality condition is to be met for environment \( k \), there must exist separate subspaces in the Hilbert space of the environment for \( \hat{\rho}'_{00}(t) \) and \( \hat{\rho}'_{11}(t) \), so there must exist eigenvalues of the conditional matrix \( \hat{\rho}'_{00}(t) \) which are equal to zero, \( c_{p_k} = 0 \), since \( \hat{\rho}'_{11}(t) \) is a density matrix and cannot have all diagonal elements equal to zero. In other words \( \hat{\rho}'_{00}(t) \) must have a non-trivial kernel. We denote the states corresponding to this kernel as \( |q_k \rangle \).

Since \( \hat{w}^k(t) \) is unitary, we can always write

\[
\hat{w}^k(t) | n_k \rangle = \sum_{m_k} b_{m_k} | m_k \rangle = b_{n_k} | n_k \rangle + \sum_{m_k \neq n_k} b_{m_k} | m_k \rangle,
\]

with \( \sum_{m_k} | b_{m_k} |^2 = 1 \). As shown previously, orthogonality implies that either \( b_{n_k} = \langle n_k | \hat{w}^k(t) | n_k \rangle = 0 \), or \( c_{n_k} = 0 \), so for all \( n_k \neq q_k \) \( (c_{n_k} \neq 0) \), we have \( b_{n_k} = 0 \) and

\[
\hat{w}^k(t) | n_k \rangle = \sum_{m_k \neq n_k} b_{m_k} | m_k \rangle = | n_{k,\perp} \rangle,
\]

(22)

where \( | n_{k,\perp} \rangle \) is some state orthogonal to \( | n_k \rangle \).

From (25a) we have

\[
\hat{\rho}'_{11}(t) = \sum_{n_k} c_{n_k} | n_{k,\perp} \rangle \langle n_{k,\perp} |,
\]

(23)

and the orthonormality condition yields

\[
\hat{\rho}'_{00}(t) \hat{\rho}'_{11}(t) = \sum_{n_k} c_{n_k} | n_k \rangle \langle n_k | \sum_{m_k} b_{m_k} | m_k \rangle \langle m_k | = \sum_{n_k \neq m_k} c_{n_k} b_{m_k} | n_k \rangle \langle m_k | = 0.
\]

(24)

Since the condition (24) means that all of the elements of the matrix must be equal to zero, it is equivalent to statement that for all \( n, m \neq q_k \),

\[
\langle n_k | \hat{w}^k(t) | m_k \rangle = \langle n_k | m_{k,\perp} \rangle = 0,
\]

(25)

which means that the states \( | n_{k,\perp} \rangle \) must be orthogonal not only to \( | n_k \rangle \), but also to all other states in the support of \( \hat{\rho}'_{00}(t) \) (eigenstates with non-zero occupations). This is not in the least surprising, because it simply means that the operator \( \hat{w}^k(t) \) takes the eigenstates of \( \hat{\rho}'_{00}(t) \) into a different subspace, as it should. Moreover, the states \( | n_{k,\perp} \rangle \) are orthogonal with respect to each other, since

\[
\langle n_{k,\perp} | m_{k,\perp} \rangle = \langle n_{k,\perp} | \hat{w}^k(t) \hat{w}^k(t) | m_{k,\perp} \rangle = \langle n_k | m_k \rangle = \delta_{n_k m_k},
\]

(26)

Hence, they constitute a basis in the \( \hat{\rho}'_{11}(t) \) subspace and belong to the kernel of \( \hat{\rho}'_{00}(t) \) (our \( | q_k \rangle \) states, for which \( c_{q_k} = 0 \)). There can of course exist other states in \( \text{Ker} \hat{\rho}'_{00}(t) \) but they play no role in our analysis. Furthermore, the number of \( | n_{k,\perp} \rangle \) states has to be the same as the number of \( | n_k \rangle \) states, so the conditional density matrices of the environment \( \hat{\rho}'^k(t) \) are symmetric with respect to each other, in the sense, that they have the same occupations \( c_{n_k} \), but for a different set of orthogonal eigenstates in different subspaces, which immediately follows from (17a). This further implies that for an environment of dimension \( d_k \), the dimension of the support of \( \hat{\rho}'_i(t) \) cannot exceed \( d_k/2 \) for even \( d_k \) and \( (d_k - 1)/2 \) for odd \( d_k \).

The above analysis can be compactly summarized by writing matrix elements of \( \hat{w}^k(t) \) in the basis \( \{ | r_k \rangle \} = \{ | n_k \rangle \} \cup \{ | w^k(t) | m_k \rangle \} \cup \{ | t_k \rangle \} \), where the last vectors correspond to the part of the kernel of \( \hat{\rho}'_{00}(t) \) which is not of the form \( \hat{w}^k(t) | m_k \rangle \). Strict orthogonality at time \( t \) implies that in such a chosen and ordered basis, we have the following matrix structure,

\[
\langle r_k | \hat{w}^k(t) | r'_k \rangle = \begin{pmatrix}
0 & * & 0 \\
1 & 0 & 0 \\
0 & * & *
\end{pmatrix}
\]

(27)

In reality, strict orthogonality at time \( t \) is a strong constraint on the free parameters of the model: The conditional evolutions \( \hat{V}_t \), the initial environment state \( \hat{\rho}'^k(0) \), and the time moment \( t \). Indeed, coming back the definitions (36b,fc) we have that \( | n_k(t) \rangle = w^k_0(t) | n_k(0) \rangle \), where \( | n_k(0) \rangle \) is the eigenbasis of \( \hat{\rho}^k(0) \). Defining a new basis in the environment Hilbert space by \( | r_k(t) \rangle = w_0(t)^t | r_k(0) \rangle \), we have that

\[
\langle r_k(t) | \hat{w}^k(t) | r'_k(t) \rangle = \langle r_k(0) | e^{i \hat{V}_t} e^{-i \hat{V}_t} | r'_k(0) \rangle
\]

(28)

and this matrix must have form (27) at the time \( t \).

\[\text{A. Purity}\]

Quite surprisingly, the emergence of objectivity puts some constraint on the initial purity of the environment. The constraint comes from the orthogonality for the conditional evolution of each environmental state \( \hat{\rho}'_{01}(t) \). It is straightforward to show that the initial purity of the \( k \)-th environment

\[
P^k(0) = \text{Tr} \hat{\rho}^k(0)^2 = \text{Tr} \hat{\rho}'_{00}(t)^2 = \text{Tr} \hat{\rho}'_{11}(t)^2,
\]

(29)

using the definitions (7b), the fact that the operators \( \hat{w}^k(t) \) are unitary, and the properties of the trace.

In general the purity of a normalized state is bounded from below by \( 1/d_k \) (where \( d_k \) is the dimension), but we have previously shown that the dimensionality of the support of the states \( \hat{\rho}'_{01}(t) \) cannot exceed \( d_k/2 \) for even
$d_k$ and $(d_k - 1)/2$ for odd $d_k$. Hence, the minimal purity of $\hat{\rho}_t^{(j)}(t)$ is $2/d_k$ or $2/(d_k - 1)$, respectively. This means, as we see from eq. (24), that orthogonality is possible only when

$$P^k(0) \geq \frac{2}{d_k} \text{ or } \frac{2}{d_k - 1}. \quad (30)$$

This doubly exceeds the minimum value for a mixed state purity and has non-negligible consequences, especially for small environments. In the extreme case, when each environment consists only of a qubit, it means that the emergence of objectivity is only possible if the whole environment is initially pure. The same conclusion is drawn for qudit environments.

### B. Consequences for qubit-environment entanglement

Let us examine the consequences of strict orthogonality for QEE. To this end, we will study the full system density matrix (6) without tracing out the degrees of freedom of the unobserved environment. We do so since the resulting lack of coherence would conceal the level of qubit-environment entanglement necessary for the emergence of strict orthogonality without any advantage to our understanding of the studied processes. This matrix can be written at time $t$ using the eigenstates of the unobserved environmental matrix $\hat{R}_0(t)$ and each unobserved environmental matrix $\hat{\rho}_0(t)$ at time $t$. $\{|r\rangle\}$ and $\{|n_k\rangle\}$, respectively. To do this we introduce the joint matrices describing the evolution of the observed environments $\hat{\rho}_{ij}(t) = \bigotimes_k \hat{\rho}_{ij}^k(t)$ and the joint conditional evolution operator of the observed environment $\hat{\tilde{w}}(t) = \bigotimes_k \hat{\tilde{w}}^k(t)$. Using equations analogous to eqs (15) we obtain

$$\hat{\hat{\tilde{w}}}(t) = \bigotimes_k \hat{\tilde{w}}^k(t). \quad (31)$$

Inserting eqs (33) and (35) into eq. (31) yields

$$\hat{\hat{\tilde{w}}}(t) = \bigotimes_k \hat{\tilde{w}}^k(t). \quad (36)$$

where $\hat{\tilde{w}}^n(t) = \hat{\tilde{w}}_1^n(t) \hat{\tilde{w}}_0^n(t)$. We now introduce eigenstates of the conditional density matrix of all observed environments

$$|n\rangle = \bigotimes_k |n_k\rangle = |n_1 \ldots n_N\rangle, \quad (32)$$

where $N$ is the number of such environments, so the their joint conditional density matrix must be of the form

$$\hat{\rho}_{0}(t) = \sum_n C_n |n\rangle \langle n|, \quad (33)$$

where the eigenvalues corresponding to each eigenstate $|n\rangle$ are (cf. (17))

$$C_n = \prod_{k=1}^N c_{n_k}, \quad (34)$$

The summation in eq. (33) is over all possible values of $n$, so it is over all possible combinations of the eigenstates $|n_k\rangle$ for different observed environments. We can similarly decompose the conditional density matrix of the unobserved environment,

$$\hat{R}_0(t) = \sum_r c_r |r\rangle \langle r|, \quad (35)$$

where $c_r$ and $|r\rangle$ are its eigenvalues and eigenvectors at time $t$, respectively.

$$\hat{\hat{\tilde{w}}}(t) = \sum_{r,n} c_r C_n |\psi_{r,n}\rangle \langle \psi_{r,n}|, \quad (36)$$

where

$$\hat{\hat{\tilde{w}}}(t) = a_0|0\rangle \langle 0| + a_1|1\rangle \langle 1|, \quad (38)$$

and the rest of the system in a state $|\psi_{r,n}\rangle$, since the states $|r\rangle \otimes |n\rangle$ and $\hat{\tilde{w}}^n(t)|r\rangle \otimes \hat{\tilde{w}}(t)|n\rangle$ are orthogonal. In fact, the state has maximum possible entanglement between any subspace containing the qubit and any subspace containing the environment that fulfills the orthogonality condition.

Another consequence of the orthogonality condition being fulfilled for even one environment is the guaranteed full dephasing of the qubit due to the qubit-environment interaction. Tracing out over the environmental degrees of freedom yields

$$\text{Tr}_E \hat{\hat{\tilde{w}}}(t) = |a_0|^2|0\rangle \langle 0| + |a_1|^2|1\rangle \langle 1|,$$
because the orthogonality condition \(25\) for environment \(k\) gives \(\text{Tr}[\hat{\psi}_k^\dagger(t)\hat{\rho}_0(t)] = 0\) which kills the off-diagonal elements in \(31\) regardless of the other environments. This is a manifestation of a fact that if considered in the same subspace, orthogonality is a stronger condition than decoherence. Indeed one can easily show that the modulus of the decoherence factor for the \(k\)-th environment is not greater than the generalized overlap between the states \(\hat{\rho}_0(t)\) and \(\hat{\psi}_k^\dagger(t)\):

\[
\left|\text{Tr}\left[\hat{\psi}_k^\dagger(t)\hat{\rho}_0(t)\right]\right| \leq \sqrt{\text{Tr}\left[\hat{\psi}_k^\dagger(t)\hat{\rho}_0(t)\hat{\psi}_k^\dagger(t)\right]}^{1/2}.
\]

(39)

For an SBS state to emerge, strict orthogonality must be fulfilled for all observed environments (for all \(k\)). As is straightforward to see, all of the states \(32\) which enter the decomposition \(33\) must be defined on different subspaces of the full Hilbert space, so it is straightforward to find entanglement (as measured by negativity \(N\)) between the qubit and all of the environment. This is because the qubit-environment density matrix \(34\) is in this case block-diagonal in subspaces of \(\{|0\rangle\otimes|n\rangle, |1\rangle\otimes|n\rangle\}\) for a given \(|n\rangle\) and \(\{|1\rangle\otimes|n\rangle\}\) from the support of \(\rho_0(t)\) and an arbitrary \(|r\rangle\). This means that the block diagonal form persists after partial transposition. Since Negativity \(37\) is the absolute value of the sum of all negative eigenvalues of a density matrix after partial transposition with respect to one of the (potentially) entangled subsystems, for such block diagonal matrices, negativity will be the sum of the Negativities of \(|\psi_{r,n}\rangle\langle\psi_{r,n}|\) weighted by the coefficients \(c_rC_n\), as \(|\psi_{r,n}\rangle\) are all orthogonal to each other.

The Negativity of each matrix \(|\psi_{r,n}\rangle\langle\psi_{r,n}|\) is the same and is equal to \(|a_0a_1|\), cf. \(37\), so the Negativity of the density matrix \(35\) is also equal to \(|a_0a_1|\)

\[
N(\hat{\sigma}(t)) = |a_0a_1|,
\]

(40)
since \(\sum c_r\sum C_n = 1\). This is fairly surprising, since the value of Negativity which allows for the formation of SBS states is set only by the occupation of qubit pointer states, and does not depend on the dimension of the observed environments. It also does not depend on the purity of the state, as long as the purity is high enough that it allows for the formation of states which fulfill the orthonormality criterion (see previous subsection). Note, that this is the same value of Negativity that one would have, if the environment would be a qubit. In this case the qubit-environment state has to be pure (again see previous subsection) and the qubit-environment state in question has the largest possible amount of entanglement for a given initial qubit superposition state.

An important observation here is that if the value of Negativity, \(|a_0a_1|\), between the qubit and its environments is reached, this does not automatically mean that all of the observed environments fulfill the strict orthogonality criterion. It only means that at least one of the environments does. Hence, to probe orthogonality via the created entanglement in a system with many observed environments, it would be necessary to find the Negativity between each observed environment and the rest of the system separately. The fulfillment of the strict objectivity criterion would be accompanied by the maximization of such Negativity, when it would reach the value of \(|a_0a_1|\).

V. CONCLUSIONS

We have studied the situation when a system interacts with multiple observed and unobserved environments to determine the interdependencies between system-environment entanglement generation and the possibility of the emergence of objectivity via SBS states \(4\). Such entanglement does not change the properties of decoherence \(27–31\) and hence is irrelevant for the decoherence function, so the effect of the unobserved environments is independent of qubit-environment entanglement formation. On the other hand, the entanglement turns out to be crucial when it comes to the ability of the observed environments to store information about the state of the decohering qubit.

Firstly, we have shown that the lack of system-environment entanglement generation is synonymous with the impossibility of an SBS state formation with respect to the pointer basis of the qubit. This is true regardless of the number of environments in each observers’ macrofraction, since separability translates into the environmental states conditioned on the qubit state being completely indistinguishable.

For a qubit system, we have further studied case of ideal distinguishability, when the supports of the environmental states conditioned on the qubit state are strictly orthogonal. It turns out that such orthogonality requires the qubit-observed-environment state to take an extreme entangled form, which is characterized by the same amount of Negativity as would be found in the pure state \(a_0|0\rangle + a_1|1\rangle\) \((N = |a_0a_1|)\) for an environment consisting of a single qubit. Additionally, we have found that such states cannot appear if the initial qubit-observed-environment state is of too low a purity. In fact, the limitations on the initial purity of the environment are quite strong (the purity must be twice as big as the minimum possible purity of a system of a given size), and are especially important for small environments. This limitation in the extreme case of qubit or qutrit environments leads to the requirement of a pure initial state of all environments. Otherwise the emergence of objectivity (SBS) is impossible.

It is known that in many situations \(6–13\), a single environment being observed by a single observer is insufficient for the emergence of objectivity, but the addition of more environments to the observers disposal (forming a so called macrofraction) can lead to SBS states in the limit of a large number of environments. Since we have here qualified the correspondence between
qubit-environment entanglement in the extreme cases and shown that separability excludes objectivity, while the natural emergence of SBS states for an observer limited to a single environment requires the qubit and the observed environment to entangle strongly, we conjecture that the amount of the generated QEE manifests itself in the way that the SBS state is approached with the growing number of environments in a single macrofraction. Since weakly entangled systems feature weakly distinguishable conditional environmental states, it follows that the more qubit-environment entanglement present in the system, the faster a near-SBS state should be approached with growing macrofractions. Consequently, not only is entanglement a necessary condition for objectivity, but objectivity is to be expected more readily in systems which strongly entangle during their joint evolutions with their environments.

ACKNOWLEDGEMENTS

The authors would like to thank Łukasz Cywiński for helpful discussions about qubit decoherence. This work is supported by funds from Polish National Science Center (NCN) Grant No. DEC-2015/19/B/ST3/03152.

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