Finite sample breakdown point of Tukey’s halfspace median

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Tukey’s halfspace median ($HM$), servicing as the multivariate counterpart of the univariate median, has been introduced and extensively studied in the literature. It is supposed and expected to preserve robustness property (the most outstanding property) of the univariate median. One of prevalent quantitative assessments of robustness is finite sample breakdown point (FSBP). Indeed, the FSBP of many multivariate medians have been identified, except for the most prevailing one—the Tukey’s halfspace median. This paper presents a precise result on FSBP for Tukey’s halfspace median. The result here depicts the complete prospect of the global robustness of $HM$ in the finite sample practical scenario, revealing the dimension effect on the breakdown point robustness and complimenting the existing asymptotic breakdown point result.

**Key words:** Tukey depth; Tukey median; Breakdown point; In general position

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1 Introduction

Robustness (as an insurance) is one of the most desirable properties for any statistical procedures. The most outstanding feature of univariate median is its robustness. Indeed, among all translation equivariant location estimators, it has the best possible breakdown point (Donoho, 1982) (and minimum maximum bias if underlying distribution has unimodal symmetric density (Huber, 1964)).

It is very much desirable to extend the univariate median to multidimensional settings and meanwhile inherit/preserve its outstanding robustness for multidimensional data. In fact, the earliest attempt of this type of extension was made at least one century ago (Weber, 1909). Oja’s median (Oja, 1983) is another promising extension.

On the other hand, defining the multi-dimensional median as the deepest point of the underlying multidimensional data is an obvious natural approach. Serving this purpose, general notions of data depth have been proposed and studied (Zuo and Serfling, 2000). The main goal of data depth is to provide a center-outward ordering of multidimensional observations. Multivariate medians as the deepest point (the generalization of the univariate median) therefore have been naturally introduced and examined. Among the depth induced multidimensional medians, Tukey’s halfspace median (Tukey, 1975) is the most prominent and prevailing. Robustness is of course the main targeted property to be shared by all depth induced medians.

There are many ways to measure the robustness of a statistical procedure (especially location estimators). Among others, maximum bias, influence function and finite sample breakdown point (FSBP) are the most standard gauges. FSBP by far is the most prevailing quantitative assess-
ment of robustness due to its plain definition (without involvement of probability/randomness concept).

The concept of breakdown point was introduced by Hodges (1967) and Hampel (Ph. D. dissertation (1968), Univ. California, Berkeley) and extended by Hampel (1971) and developed further by, among others, Huber (1981). It has proved most successful in the context of location, scale and regression problems. Finite sample version of breakdown point has been proposed, promoted and popularized by Donoho (1982) and Donoho and Huber (1983) (DH83).

The seminar paper of Donoho and Gasko (1992) (hereafter DG92) was devoted to extensively study the FSBP of multivariate location estimators including Tukey’s halfspace depth induced location estimators, especially the halfspace median (HM). Specifically, DG92 established FSBP for many location estimators, including the lower bound of FSBP for halfspace median. However, lower bound contains much scarce information about FSBP of HM. What is the exact FSBP of HM is still an open question.

Adrover and Yohai (2002) and Chen and Tyler (2002) have pioneered in studying the maximum bias of HM. It is found that the asymptotic breakdown point of HM is 1/3, as also given in DG92 (see also Chen (1995) and Mizera (2002)). The latter result however is obtained restricted to a sub-class of distributions (the absolutely continuous centrosymemetric ones) in the maximum bias definition, and when sample size n approaches to the infinity. Ironically, DG92 only provided the asymptotic breakdown point for HM and uncharacteristically left its FSBP open. Furthermore, the former does not provide any clue of the dimensional effect on the breakdown robustness and its behavior in finite sample practical scenario. To address this issue and provide a definite answer is the main objective of this manuscript.

Let’s end this section with some definitions. A location statistical functional T in \( \mathcal{R}^d \) \( (d \geq 1) \) is said to be affine equivariant if

\[
T(\Sigma \mathcal{X}^n + b) = \Sigma T(\mathcal{X}^n) + b,
\]

for any \( d \times d \) nonsingular matrix \( \Sigma \) and \( b \in \mathcal{R}^d \), where \( \Sigma \mathcal{X}^n + b = \{ \Sigma X_1 + b, \cdots, \Sigma X_n + b \} \) and \( X_1, \cdots, X_n \) is a given random sample in \( \mathcal{R}^d \) (denote \( \mathcal{X}^n = \{ X_1, \cdots, X_n \} \) hereafter). When \( d = 1 \) and \( \Sigma = 1 \), we call T is translation equivariant.

Define the depth of a point \( x \) with respect to \( \mathcal{X}^n \subset \mathcal{R}^d \) as

\[
D(x, \mathcal{X}^n) = \inf_{u \in S^{d-1}} P_n(u^\top X \leq u^\top x),
\]
where $S^{d-1} = \{ z \in \mathcal{R}^d : \|z\| = 1 \}$, and $P_n$ denotes the empirical probability measure and $\| \cdot \|$ stands for Euclidean norm. Denote

$$
\mathcal{M}(\mathcal{X}^n) = \{ x : D(x, \mathcal{X}^n) = \lambda^*(\mathcal{X}^n) \},
$$

where $\lambda^*(\mathcal{X}^n) = \sup_x D(x, \mathcal{X}^n)$. Tukey’s halfspace median is defined as,

$$
T^*(\mathcal{X}^n) = \text{Ave} \{ x \in \mathcal{M}(\mathcal{X}^n) \},
$$

i.e., the average of all points that maximize $D(x, \mathcal{X}^n)$. Clearly, when $d = 1$, $T^*(\mathcal{X}^n)$ reduces to the univariate sample median.

The finite sample additional breakdown point (ABP) of a location estimator $T$ at the given sample $\mathcal{X}^n$ is defined as

$$
\text{ABP}(T, \mathcal{X}^n) = \min_{1 \leq m \leq n} \left\{ \frac{m}{n+m} : \sup_{\mathcal{Y}^m} \| T(\mathcal{X}^n \cup \mathcal{Y}^m) - T(\mathcal{X}^n) \| = \infty \right\},
$$

where $\mathcal{Y}^m$ denotes an arbitrary contaminating sample of size $m$, adjoining to $\mathcal{X}^n \subset \mathcal{R}^d$. Namely, the ABP of an estimator is the minimum additional fraction which could drive the estimator beyond any bound. It is readily seen that the ABP of the sample mean and the univariate median are $1/(n+1)$ and $1/2$, respectively. The latter is the best that one can expect for any translation equivariant location estimator (Donoho, 1982).

Additional breakdown point is one of the forms of the finite sample breakdown point notion, replacement breakdown point (RBP) is the other one (DH83), where instead of adding contaminating points to $\mathcal{X}^n$, replacing $m$ points of $\mathcal{X}^n$ by $m$ arbitrary points. Some prefer RBP since it is arguably more close to the contamination in reality. The two are actually equivalent in the sense of Zuo (2001). Further discussions on FSBP concept could be found in DHS3 and Lopuhaä and Rousseeuw (1991). By FSBP we mean ABP in the sequel.

We anticipate that the approach and results here may be extended for the investigating the FSBP of estimators that are related to Tukey’s halfspace depth function such as multiple-output quantile regression estimators (Hallin et al., 2010), the maximum regression depth estimator (Rousseeuw and Hubert, 1999), and probably the functional halfspace depth estimators (López-Pintado and Romo, 2009).

The remainder of this article is organized as follows. Section 2 presents three preliminary lemmas for the main results, which will be proved in Section 3. The article ends in Section 4 with some concluding remarks.
2 Three preliminary lemmas

Since the proof of the main result is rather complicated and long, we divide it into several parts and present some of them as lemmas. In this section, three preliminary lemmas are established. They play important roles in the proof of the main results.

Without loss of generality, we assume that $X^n$ is in general position (IGP hereafter) throughout this paper. That is, no more than $d$ sample points lie in a $(d-1)$-dimensional hyperplane. This assumption is common in the literature involving statistical depth functions and breakdown point robustness (Donoho and Gasko, 1992; Mosler et al., 2009). Since $HM$ reduces to the sample median and its FSBP is known as $1/2$ for $d = 1$, we only focus on $d \geq 2$ in the sequel.

For $X^n$, under the IGP assumption, there must exist $N_d^n = \binom{n}{d}$ unit vectors, say $\mu_j \in S^{d-1}$, $j = 1, 2, \cdots, N_d^n$, such that they are respectively normal to $N_d^n$ hyperplanes with each of which passing through $d$ observations. Since $N_d^n$ is finite when $n$ and $d$ are fixed, we can find a unit vector, say $u$, satisfying

$$u^\top \mu_j \neq 0, \quad \text{for } \forall j \in \{1, 2, \cdots, N_d^n\}. \quad (1)$$

For simplicity, for any given $u \in S^{d-1}$, in the sequel we denote

$$A_u = (u_1, u_2, \cdots, u_{d-1}), \quad (2)$$

where $u_1, u_2, \cdots, u_{d-1}$ are orthogonal to $u$, and together with $u$, they form a set of standard basis vectors of $R^d$. Remarkably, although the choice of $u_1, u_2, \cdots, u_{d-1}$ is not unique when $d \geq 2$, this fact does not effect the proofs presented in the rest of this paper due to the affine equivariance of $HM$ and its related Tukey depth, nevertheless. Hence, we pretend that $A_u$ is unique in the sequel.

Write $x_i = A_u^\top x_i$, $i = 1, 2, \cdots, n$, and $X_u^n = \{x_1, x_2, \cdots, x_n\}$, call it $A_u$-projections of $X^n$ hereafter. It will greatly facilitate our discussion, if $X_u^n$ is still in general position. Fortunately, Lemma 1 provides a positive answer.

**Lemma 1.** Suppose $X^n$ is IGP, and $u \in S^{d-1}$ satisfies display (1). Then $X_u^n = \{x_1, x_2, \cdots, x_n\} \subset R^{d-1}$ is in general position too when $d \geq 2$.

**Proof.** If $X_u^n$ is not IGP, then there must exist a $(d-2)$-dimensional hyperplane $P_1$ containing at least $d$ $A_u$-projections, say $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ ($k \geq d$). Let $v \in S^{d-2}$ be the normal vector of $P_1$. Then, we have

$$v^\top x_{i_1} = v^\top x_{i_2} = \cdots = v^\top x_{i_k}.$$
Recalling the definition of $x_i$, we further obtain

$$(A_\mathbf{u} \mathbf{v})^\top X_{i_1} = (A_\mathbf{u} \mathbf{v})^\top X_{i_2} = \cdots = (A_\mathbf{u} \mathbf{v})^\top X_{i_k}.$$  

(3)

Write $\tilde{\mathbf{v}} = A_\mathbf{u} \mathbf{v}$. Clearly, $\tilde{\mathbf{v}} \in \mathcal{R}^d$ and $(\tilde{\mathbf{v}})^\top \tilde{\mathbf{v}} = 1$, namely, $\tilde{\mathbf{v}} \in S^{d-1}$. Hence, (3) implies that one can find a $(d-1)$-dimensional hyperplane, with $\tilde{\mathbf{v}}$ being its normal vector, that passes through $k$ observations. This contradicts with the IGP assumption of $\mathcal{X}^n$ if $k > d$. When $k = d$, (3) implies $\tilde{\mathbf{v}} \in \{\mu_j\}_{j=1}^{N_d}$. This obviously contradicts with the fact that $\mathbf{u}$ satisfies (1) due to $\tilde{\mathbf{v}}^\top \mathbf{u} = \mathbf{v}^\top A^\top \mathbf{u} \mathbf{u} = 0$. This completes the proof of this lemma. □

Remark 2.1 In fact, $\mathcal{X}^n$ is IGP if and only if $X^n_\mathbf{u}$ is IGP for $\mathbf{u}$ satisfying (1).

To derive the FSBP of HM, we need to investigate the maximum Tukey depth with respect to the $A_\mathbf{u}$-projections of $\mathcal{X}^n$. The following Lemma 2 will play an important role during this process.

We formally introduce some additional necessary notations as follows. For $\forall \mathbf{x}, \mathbf{y} \in \mathcal{R}^d$, let

\[ U_{\mathbf{x}} = \{ \mathbf{u} \in S^{d-1} : P_n(\mathbf{u}^\top \mathbf{X} \leq \mathbf{u}^\top \mathbf{x}) = D(\mathbf{x}, \mathcal{X}^n) \} \]

be the set of all optimal vectors of $\mathbf{x}$ which realize the depth at $\mathbf{x}$ with respect to $\mathcal{X}^n$, and

\[ \mathcal{H}_{\mathbf{x}, \mathbf{y}} = \{ \mathbf{u} \in S^{d-1} : \mathbf{u}^\top \mathbf{x} < \mathbf{u}^\top \mathbf{y} \} \]

the hemisphere determined by $\{\mathbf{x}, \mathbf{y}\}$. Furthermore, for $\forall \mathbf{z} \in \mathcal{M}(\mathcal{X}^n)$, let

\[ A_{\mathbf{z}} = \{ \mathbf{x} \in \mathcal{M}(\mathcal{X}^n) : U_{\mathbf{x}} \cap \mathcal{H}_{\mathbf{x}, \mathbf{z}} \neq \emptyset \}, \]

\[ B_{\mathbf{z}} = \{ \mathbf{x} \in \mathcal{M}(\mathcal{X}^n) : U_{\mathbf{x}} \cap \mathcal{H}_{\mathbf{x}, \mathbf{z}} = \emptyset \}, \]

\[ \tilde{B}_{\mathbf{z}} = B_{\mathbf{z}} \setminus \{ \mathbf{z} \}. \]

Obviously, (i) $\mathcal{M}(\mathcal{X}^n) = A_{\mathbf{z}} \cup B_{\mathbf{z}}$, (ii) $A_{\mathbf{z}} \cap B_{\mathbf{z}} = \emptyset$, (iii) $\mathbf{z} \in B_{\mathbf{z}}$ because $\mathcal{H}_{\mathbf{z}, \mathbf{z}} = \emptyset$.

For $U_{\mathbf{x}}, B_{\mathbf{x}}$ and $\tilde{B}_{\mathbf{z}}$, Lemma 2 below depicts several important properties of them.

Lemma 2. Suppose $\mathcal{X}^n$ is IGP and $\mathcal{M}(\mathcal{X}^n)$ is of affine dimension $d$. For $\forall \mathbf{z}_1$ lying in the interior of $\mathcal{M}(\mathcal{X}^n)$, if $\tilde{B}_{\mathbf{z}_1} \neq \emptyset$, then:

\[(o1)\] for $\forall \mathbf{u} \in U_{\mathbf{z}_1}$ and $\forall \mathbf{z} \in \tilde{B}_{\mathbf{z}_1}$, we have $\mathbf{u}^\top \mathbf{z} \geq \mathbf{u}^\top \mathbf{z}_1$.

\[(o2)\] for $\forall \mathbf{z} \in B_{\mathbf{z}_1}$, we have $U_{\mathbf{z}} \subset U_{\mathbf{z}_1}$.

\[(o3)\] for $\forall \mathbf{z} \in \tilde{B}_{\mathbf{z}_1}$, we have $B_{\mathbf{z}} \subset B_{\mathbf{z}_1}$, but $\mathbf{z}_1 \notin B_{\mathbf{z}}$. That is, $B_{\mathbf{z}} \subset \tilde{B}_{\mathbf{z}_1}$.  

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Proof. (o1). If not, $u^\top z < u^\top z_1$. Then $\lambda^*(\mathcal{X}^n) \leq P_n(u^\top X \leq u^\top z) \leq P_n(u^\top X \leq u^\top z_1) = \lambda^*(\mathcal{X}^n)$, resulting in $u \in U_z \cap H_{x,z_1}$, and hence contradicting with $z \in B_{z_1}$.

(o2). By definition, $z \in B_{z_1}$ implies $U_z \cap H_{x,z_1} = \emptyset$. Hence, for $\forall u \in U_z$, we have $u^\top z \geq u^\top z_1$. Then $\lambda^*(\mathcal{X}^n) \leq P_n(u^\top X \leq u^\top z) \leq P_n(u^\top X \leq u^\top z_1) = \lambda^*(\mathcal{X}^n)$. That is, $u \in U_z$, and hence $U_z \subset U_{z_1}$.

(o3). For $\forall x \in B_z$, $U_z \cap H_{x,z} = \emptyset$. Hence, $u^\top (x - z) \geq 0$ for $\forall u_x \in U_x$. Next, by (o2), we have $U_x \subset U_z \subset U_{z_1}$, which implies $u^\top x \geq u^\top z \geq u^\top z_1$ by (o1). That is, $U_z \cap H_{x,z_1} = \emptyset$, which implies $x \in B_{z_1}$, and hence $B_z \subset B_{z_1}$.

Next, since $z_1$ lies in the interior of $\mathcal{M}(\mathcal{X}^n)$, we can find a small enough $\varepsilon > 0$ such that $\{x : \|x - z_1\| < \varepsilon\} \subset \mathcal{M}(\mathcal{X}^n)$. For $\forall u_{z_1} \in U_{z_1}$, if the hyperplane $\{x : u^\top_{z_1}(x - z_1) = 0\}$ contains a sample point, say $X_1$, then let $\tilde{z}_1 = -\frac{1}{\varepsilon}u_{z_1} + z_1$. Clearly, $\tilde{z}_1 \in \mathcal{M}(\mathcal{X}^n)$, and $u^\top_{z_1} \tilde{z}_1 < u^\top_{z_1} z_1 = u^\top_{z_1} X_1$. As a result, $D(\tilde{z}_1, \mathcal{X}^n) \leq P_n(u^\top_{z_1} X \leq u^\top_{z_1} \tilde{z}_1) = P_n(u^\top_{z_1} X \leq u^\top_{z_1} z_1) - 1/n = \lambda^*(\mathcal{X}^n) - 1/n$, contradicting with $\tilde{z}_1 \in \mathcal{M}(\mathcal{X}^n)$.

Hence, $\forall u_{z_1} \in U_{z_1}$, $\{x : u^\top_{z_1}(x - z_1) = 0\}$ contains no sample point. This fact implies that there exists a permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ satisfying

$$
\begin{align*}
    u^\top_{z_1} X_{i_1} & \leq u^\top_{z_1} X_{i_2} \leq \cdots \leq u^\top_{z_1} X_{i_k} < u^\top_{z_1} z_1 < \\
    u^\top_{z_1} X_{i_k+1} & \leq \cdots \leq u^\top_{z_1} X_{i_n},
\end{align*}
$$

where $k^* = n\lambda^*(\mathcal{X}^n)$. Then similar to Liu et al. (2013), we have that a direction vector $u$ should belong to $U_{z_1}$ if it satisfies

$$
\begin{align*}
    u^\top (X_{i_1} - z_1) & < 0 \\
    u^\top (X_{i_2} - z_1) & < 0 \\
    & \vdots \\
    u^\top (X_{i_k} - z_1) & < 0 \\
    u^\top (z_1 - X_{i_k+1}) & < 0 \\
    & \vdots \\
    u^\top (z_1 - X_{i_n}) & < 0.
\end{align*}
$$

Using this, it is trivial to check that $U_{z_1}$ is non-coplanar when $z_1$ lies in the interior of $\mathcal{M}(\mathcal{X}^n)$. Hence, there must exist $v_{11}, v_{12}, \cdots, v_{1d} \in U_{z_1}$, which are of affine dimension $d$.

Observe that, $\forall z \in \overline{B}_{z_1}$, there $\exists \tilde{v} \in \{v_{11}, v_{12}, \cdots, v_{1d}\}$ satisfying

$$
\tilde{v}^\top (z - z_1) > 0.
$$

(If not, $v_{1l}^\top (z - z_1) = 0$ for $l = 1, 2, \cdots, d$, which lead to $z = z_1$. This is impossible due to $z \in \overline{B}_{z_1}$.) Hence, $U_{z_1} \cap H_{z_1,z} \neq \emptyset$, and then $z_1 \notin B_z$. This completes the proof of (o3). \hfill \Box
Relying on Lemma 2, we are able to find a point \( x_0 \) in the interior of \( \mathcal{M}(\mathcal{X}^n) \), which lies outside of at least one optimal halfspace of any \( x \neq x_0 \). Here by optimal halfspace of \( x \) we mean the halfspace realizing the depth at \( x \). That is, we have the following lemma.

**Lemma 3.** When \( \mathcal{X}^n \) is IGP, there must exist an \( x_0 \in \mathcal{M}(\mathcal{X}^n) \) such that \( U_x \cap H_{x,x_0} \neq \emptyset \) for \( \forall x \in \mathbb{R}^d \setminus \{x_0\} \), i.e., we can find a \( u \in U_x \) satisfying \( u^\top x < u^\top x_0 \).

In the sequel the major task is to prove: \( U_x \cap H_{x,x_0} \neq \emptyset \) for \( \forall x \in \mathcal{M}(\mathcal{X}^n) \setminus \{x_0\} \) when \( d \geq 2 \). It consists of three parts, i.e., (A), (B) and (C), related respectively to three scenarios of the affine dimension \( \text{dim}(\mathcal{M}) \) of \( \mathcal{M}(\mathcal{X}^n) \). Both (A) and (B) indicates that taking \( x_0 = T^*(\mathcal{X}^n) \) is valid, while (C) is technically much more difficult and the resulted \( x_0 \) may \( \neq T^*(\mathcal{X}^n) \). In (C), we first obtain a candidate point, say \( \bar{z}_0 \), through an iterative procedure consisting of three steps, i.e., (a), (b) and (c), and then show that \( \bar{z}_0 \) can serve as \( x_0 \). For convenience, we use the same notations (e.g., \( \mathcal{X}^n \)) as before in Lemma 2 though. Its result can be applied to any other IGP data set, nevertheless.

**Proof.** When \( d = 1 \), by letting \( x_0 \) be the sample median, the proof is trivial. When \( d \geq 2 \), for \( \forall x \notin \mathcal{M}(\mathcal{X}^n) \) and \( \forall z \in \mathcal{M}(\mathcal{X}^n) \), we claim \( U_x \cap H_{x,z} \neq \emptyset \). If not, for \( \forall u \in U_x \), we have \( u^\top x \geq u^\top z \), which leads to \( \lambda^*(\mathcal{X}^n) \leq P_n(u^\top X \leq u^\top z) \leq P_n(u^\top X \leq u^\top x) = D(x, \mathcal{X}^n) \), contradicting the definition of \( \lambda^* \) and \( \mathcal{M}(\mathcal{X}^n) \). In the sequel we show that there \( \exists x_0 \in \mathcal{M}(\mathcal{X}^n) \) satisfying \( U_x \cap H_{x,x_0} \neq \emptyset \) for \( \forall x \in \mathcal{M}(\mathcal{X}^n) \setminus \{x_0\} \).

(A) **Scenario \( \text{dim}(\mathcal{M}) = 0 \).** Since \( \mathcal{M}(\mathcal{X}^n) = \{T^*(\mathcal{X}^n)\} \), Lemma 3 already holds by letting \( x_0 = T^*(\mathcal{X}^n) \).

(B) **Scenario \( 0 < \text{dim}(\mathcal{M}) < d \).** We now show that taking \( x_0 = T^*(\mathcal{X}^n) \) is valid.

Relying on Theorem 4.2 of Paindaveine and Šiman (2011), it is easy to check that there \( \exists \mu_s \in \{\mu_j\}_{j=1}^{N^d} \) normal to the hyperplane \( \Pi_0 = \{z \in \mathbb{R}^d : \mu_s^\top z = q_s\} \) such that: (i) \( \Pi_0 \supset \mathcal{M}(\mathcal{X}^n) \), (ii) \( \Pi_0 \) contains \( d \) observations, say \( Z_d := \{X_{k_1}, X_{k_2}, \ldots, X_{k_d}\} \). Here \( q_s = \inf\{t \in \mathbb{R}^1 : P_n(\mu_s^\top X \leq t) \geq \lambda^*(\mathcal{X}^n)\} \).

Obviously, \( \mathcal{M}(\mathcal{X}^n) \subset \text{cov}(Z_d) \), i.e., the convex hull of \( Z_d \). If not, one may deviate \( \Pi_0 \) around a point \( x \in \mathcal{M}(\mathcal{X}^n) \setminus \text{cov}(Z_d) \), similar to Theorem 1 of Liu et al. (2015), to get rid of \( Z_d \) to obtain a contradiction.

For \( i = 1, 2, \ldots, d \), let \( W_i \) be the \((d-2)\)-dimensional hyperplane passing through \( Z_d \setminus \{X_{k_i}\} \) (\( W_i \) is a singleton when \( d = 2 \)), and let \( \nu_i \in S^{d-1} \) be the vector orthogonal to both \( \mu_s \) and \( W_i \),
and satisfying \( \nu_i^\top (X_{k_i} - X_{k_i}) > 0 \) for all \( l \in \{1, 2, \cdots, d\} \setminus \{i\} \). In the following, we show that

\[
\mathcal{M}(\mathcal{X}^n) \setminus \{x_0\} = \bigcup_{i=1}^{d} D_i,
\]

(6)

where \( D_i = \{x \in \mathcal{M}(\mathcal{X}^n) : \nu_i^\top x < \nu_i^\top x_0\} \).

The \( \supset \) part is trivial. We only show the \( \subset \) part. In fact, if \( \exists z \in \mathcal{M}(\mathcal{X}^n) \setminus \{x_0\} \) but \( z \notin \bigcup_{i=1}^{d} D_i \), then we have \( \nu_i^\top (x_0 - z) \geq 0 \) for \( i = 1, 2, \cdots, d \). Using this and the fact \( \mathcal{M}(\mathcal{X}^n) \subset \text{cov}(\mathcal{Z}_d) \), we obtain, for all \( \delta > 0 \),

\[
\delta(x_0 - z) + z \in \bigcap_{i=1}^{d} \left\{ x \in \Pi_0 : \nu_i^\top x \geq \nu_i^\top z \right\}
\]

\[
\subset \bigcap_{i=1}^{d} \left\{ x \in \Pi_0 : \nu_i^\top x \geq \nu_i^\top X_{k_i}, i \in \{1, 2, \cdots, d\} \setminus \{i\} \right\} = \text{cov}(\mathcal{Z}_d),
\]

contradicting with the boundedness of \( \text{cov}(\mathcal{Z}_d) \). Hence, (6) is true.

Relying on (6), it is easy to find a \( \nu \in \{\nu_j\}_{j=1}^{d} \) and \( \epsilon > 0 \) such that \( \bar{\mu}_* = \mu_* - \epsilon \nu \) with \( \bar{\mu}_* \) satisfying \( P_n(\bar{\mu}_*^\top X \leq \bar{\mu}_*^\top x) = \lambda^* (\mathcal{X}^n) \) and \( \bar{\mu}_*^\top x < \bar{\mu}_*^\top x_0 \) for all \( x \in \mathcal{M}(\mathcal{X}^n) \setminus \{x_0\} \).

(C) Scenario \( \text{dim} (\mathcal{M}) = d \). Let \( z_1 = T^* (\mathcal{X}^n) \). Clearly, \( z_1 \) lies in the interior of \( \mathcal{M}(\mathcal{X}^n) \). In the sequel we first obtain a candidate point, say \( \bar{z}_0 \), through an iterative procedure, and then prove that it can serve as \( x_0 \).

**Step (a)** If \( \mathcal{B}_{z_1} = \{z_1\} \), let \( x_0 = z_1 \). This lemma already holds. Otherwise, \( \bar{B}_{z_1} \neq \emptyset \), and let

\[
h(z_1) = \sup_{v \in \mathcal{U}_{z_1}, \ z \in \bar{B}_{z_1}} v^\top (z - z_1).
\]

By the property of the supremum, for \( \epsilon_2 = 1/2 \), there must exist \( \bar{z}_2 \in \bar{B}_{z_1} \) and \( \bar{v}_1 \in \mathcal{U}_{z_1} \) satisfying

\[
h(z_1) - \epsilon_2 < \bar{v}_1^\top (z_2 - z_1) \leq h(z_1).
\]

**Step (b)** Similar to (a), if \( \mathcal{B}_{z_2} = \{z_2\} \), let \( x_0 = z_2 \) and end the proof of this lemma. Otherwise, for \( \epsilon_3 = 1/3 \), we similarly have a \( z_3 \in \bar{B}_{z_2} \subset \mathcal{B}_{z_1} \) and a \( \bar{v}_2 \in \mathcal{U}_{z_2} \subset \mathcal{U}_{z_1} \), by Lemma 2, satisfying

\[
h(z_2) - \epsilon_3 < \bar{v}_2^\top (z_3 - z_2) \leq h(z_2).
\]

**Step (c)** If there is a finite \( m \ (m \geq 3) \) such that \( \mathcal{B}_{z_m} = \{z_m\} \), then let \( x_0 = z_m \) and end the proof of this lemma. Otherwise, by repeating (a) and (b), we can obtain a series of different points \( \{z_i\}_{i=1}^{\infty} \subset \mathcal{M}(\mathcal{X}^n) \) satisfying: for all \( m > 1 \), \( \mathbf{p1} \) \( z_m \in \bar{B}_{z_{m-1}} \), \( \mathbf{p2} \) \( \mathcal{U}_{z_m} \subset \mathcal{U}_{z_{m-1}} \), \( \mathbf{p3} \)
\[ \mathbf{u}^{\top} z_m \geq \cdots \geq \mathbf{u}^{\top} z_2 \geq \mathbf{u}^{\top} z_1 \quad \text{for all } \mathbf{u} \in \mathcal{U}_m, \] (p4) there exists a \( \bar{v}_{m-1} \in \mathcal{U}_{z_{m-1}} \) such that \( h(z_{m-1}) - \varepsilon_m < \bar{v}_{m-1}^{\top} (z_m - z_{m-1}) \leq h(z_{m-1}) \), where \( \varepsilon_m = 1/m \).

Since \( \mathcal{M}(X^n) \) is bounded, \( \{z_i\}_{i=1}^{\infty} \) must contain a convergent subsequence, say \( \{z_{k_m}\}_{m=1}^{\infty} \). Without confusion, suppose \( \lim_{m \to \infty} z_{k_m} = \bar{z}_0 \). Obviously, \( \bar{z}_0 \) should lie in the interior of \( \mathcal{M}(X^n) \), because for any point \( x \) on the boundary of \( \mathcal{M}(X^n) \), it is easy to find a \( \mathbf{u}_x \in \mathcal{U}_x \cap \mathcal{H}_{x,z} \) for some inner points \( z \) of \( \mathcal{M}(X^n) \).

**Now we show that \( \bar{z}_0 \) can serve as \( x_0 \).** By (p3), the fact \( k_m - 1 \geq k_{m-1} \) implies \( \mathbf{u}^{\top} z_{k_m-1} \geq \mathbf{u}^{\top} z_{k_{m-1}} \) for all \( \mathbf{u} \in \mathcal{U}_{z_{k_{m-1}}} \). Hence, for \( \bar{v}_{m-1} \in \mathcal{U}_{z_{k_{m-1}}} \) given in (p4), we have

\[
 h(z_{k_m-1}) - \varepsilon_{k_m} < \bar{v}_{k_{m-1}}^{\top} (z_{k_m} - z_{k_{m-1}}) \leq \bar{v}_{k_{m-1}}^{\top} (z_{k_m} - z_{k_{m-1}}) \leq \|z_{k_m} - z_{k_{m-1}}\|.
\]

This, together with the convergence of \( \{z_{k_m}\}_{m=1}^{\infty} \), leads to

\[
 h(z_{k_m-1}) \to 0, \quad \text{as } k_m \to +\infty. \tag{7}
\]

**Based on this, we can show that \( \mathcal{B}_{\bar{z}_0} = \{\bar{z}_0\} \) through two steps as follows.**

**Firstly, for all \( k \in \{k_m\}_{m=1}^{\infty} \), we have \( \bar{z}_0 \in \mathcal{B}_{z_{k-1}} \).** If not, there must exist a \( \tilde{\mathbf{u}} \in \mathcal{U}_{\bar{z}_0} \) satisfying \( \tilde{\mathbf{u}}^{\top} \bar{z}_0 < \tilde{\mathbf{u}}^{\top} z_{k-1} \). For \( \tilde{\mathbf{u}} \), similar to (4), there exists a permutation \( (i'_1, i'_2, \ldots, i'_n) \) of \((1, 2, \ldots, n)\) such that

\[
 \tilde{\mathbf{u}}^{\top} X_{i'_1} \leq \tilde{\mathbf{u}}^{\top} X_{i'_2} \leq \cdots \leq \tilde{\mathbf{u}}^{\top} X_{i'_{k'_m}} < \tilde{\mathbf{u}}^{\top} \bar{z}_0 < \tilde{\mathbf{u}}^{\top} X_{i'_{k'_m+1}} \leq \cdots \leq \tilde{\mathbf{u}}^{\top} X_{i'_n}.
\]

Let \( \delta_0 = \frac{1}{2} \min \left\{ \mathbf{u}^{\top} (z_0 - X_{i'_{k'_m}}), \mathbf{u}^{\top} (X_{i'_{k'_m+1}} - z_0), \mathbf{u}^{\top} (z_{k-1} - z_0) \right\} \). By the convergence of \( \{z_{k_m}\}_{m=1}^{\infty} \), we can find a \( k'_m > k \) among \( \{k_m\}_{m=1}^{\infty} \) such that \( \|z_{k'_m} - z_0\| < \delta_0 \). This, combined with \( \|\tilde{\mathbf{u}}^{\top} (z_{k'_m} - z_0)\| \leq \|z_{k'_m} - z_0\| \), leads to

\[
 \tilde{\mathbf{u}}^{\top} X_{i'_1} \leq \tilde{\mathbf{u}}^{\top} X_{i'_2} \leq \cdots \leq \tilde{\mathbf{u}}^{\top} X_{i'_{k'_m}} < \tilde{\mathbf{u}}^{\top} z_{k'_m} < \tilde{\mathbf{u}}^{\top} X_{i'_{k'_m+1}} \leq \cdots \leq \tilde{\mathbf{u}}^{\top} X_{i'_n}. \tag{8}
\]

That is, \( \tilde{\mathbf{u}} \in \mathcal{U}_{z_{k'_m}} \) (\( \subset \mathcal{U}_{z_{k-1}} \)). On the other hand, following a similar fashion to (8), we have

\[
 \tilde{\mathbf{u}}^{\top} z_{k'_m} < \tilde{\mathbf{u}}^{\top} z_{k-1}.
\]

This nevertheless contradicts with (o1) and (o2) of Lemma 2 due to \( z_{k'_m} \in \mathcal{B}_{z_{k-1}} \) when \( k'_m > k \). Hence, \( \bar{z}_0 \in \mathcal{B}_{z_{k-1}} \).

**Secondly, we show \( \mathcal{B}_{\bar{z}_0} \setminus \{\bar{z}_0\} = \emptyset \) based on the fact \( \bar{z}_0 \in \mathcal{B}_{z_{k-1}} \) for all \( k \in \{k_m\}_{m=1}^{\infty} \).** If not, suppose \( x \in \mathcal{B}_{\bar{z}_0} \setminus \{\bar{z}_0\} \) without loss of generality. Then, similar to (5), we can find a \( \mathbf{v}_0 \in \mathcal{U}_x \) such that \( \mathbf{v}_0^{\top} x - \mathbf{v}_0^{\top} \bar{z}_0 > 0 \). On the other hand, by (o2)-(o3) of Lemma 2, the facts
\( x \in B_{z_0} \) and \( \bar{z}_0 \in B_{z_{k-1}} \) together imply \( x \in B_{z_{k-1}} \setminus \{z_{k-1}\} \) and \( v_0 \in U_{z_{k-1}} \). These, combined with (o1) of Lemma 2 and the property of the supremum, lead to

\[
h(z_{k-1}) \geq v_0^\top x - v_0^\top z_{k-1} \geq v_0^\top x - v_0^\top \bar{z}_0 > 0.
\]

Nevertheless, \( v_0^\top x - v_0^\top \bar{z}_0 \) does not depend on \( \tilde{k} \), contradicting with (7).

This completes the proof of this lemma. \( \square \)

3 FSBP of Tukey’s halfspace median (Main results)

Note that for \( u \in S^{d-1} \), its \( A_u \)-projections \( X_n^u \) is not IGP if (1) is violated, while in the proof of our main theorem, we have to handle such situations that \( X_n^u \) is not IGP. Hence, in addition to three preliminary lemmas above, we need three more lemmas as follows.

**Lemma 4.** There exists a \( u_0 \in S^{d-1} \) such that: (s1) \( u_0 \) satisfies (1), and (s2)

\[
\lambda^*(X_{u_0}^n) = \inf_{u \in S^{d-1}} \lambda^*(X_u^n).
\]

**Proof.** Note that \( \lambda^*(X_{u}^n) \in \{0, 1/n, \cdots, 1\} \) for \( \forall u \in S^{d-1} \). Hence, there must exist a \( u_0 \in S^{d-1} \) satisfying (9). This completes the proof of (s2).

Now we show (s1). For simplicity, let \( N_0 = \{1, 2, 3, \cdots\} \) and denote \( \inf_{k \in I} t_k \) as the infimum of the set \( \{t : t = t_k, k \in I\} \) related to \( \{t_k\}_{k=1}^\infty \subset \mathbb{R}^1 \), where \( I \) denotes some subscript sets that \( I \subset N_0 \).

By noting that \( S^{d-1} \) is of affine dimension \( d \), it is easy to check that \( N_d \) hyperplanes \( \Pi_j = \{x \in \mathbb{R}^d : \mu_j^\top x = 0\}, j = 1, 2, \cdots, N_d \), together divide \( S^{d-1} \) into only a finite number of non-coplanar fragments. Hence, for any \( \vartheta_0 \in S^{d-1} \) such that \( \mu_j^\top \vartheta_0 = 0 \) for some \( j \in \{1, 2, \cdots, N_d\} \), we can find a sequence \( \{\vartheta_k\}_{k=1}^\infty \subset S^{d-1} \) satisfying: (i) each \( \vartheta_k \) lies in the interior of a non-coplanar fragment of \( S^{d-1} \) and satisfies display (1), and (ii) \( \lim_{k \to \infty} \vartheta_k = \vartheta_0 \). Now we show that

\[
\lambda^*(X_{\vartheta_0}^n) \geq \inf_{k \in N_0} \lambda^*(X_{\vartheta_k}^n).
\]

Since \( \vartheta_k \) can be obtained through rotating \( \vartheta_0 \), there must exist a unique orthogonal matrix \( Q_k \) such that \( \vartheta_k = Q_k \vartheta_0 \). Obviously, \( \lim_{k \to \infty} \vartheta_k = \vartheta_0 \) implies \( \lim_{k \to \infty} Q_k = I_d \), which is the \( d \times d \) identical matrix. Denote \( A_0 := A_{\vartheta_0} \). For \( \vartheta_k \), since \( Q_k A_0 \) satisfies display (2), it can serve as \( A_{\vartheta_k} \). For simplicity, hereafter denote \( A_k := A_{\vartheta_k} = Q_k A_0 \), and \( \theta_k = T^*(X_{\vartheta_k}^n) \) for \( k \in N_0 \). Note that, for any \( u \in S^{d-1} \), \( |u^\top X_i| \leq \max_{1 \leq j \leq n} \|X_j\| \). Hence, \( \{\theta_k\}_{k=1}^\infty \) is bounded, and it should contain a convergent subsequence. Without confusion, suppose \( \{\theta_k\}_{k=1}^\infty \) is convergent with \( \lim_{k \to \infty} \theta_k = \theta_0 \). (If not, use the convergent subsequence as \( \{\theta_{k}'\}_{k=1}^\infty \) instead).
Suppose \( v_0 \in S^{d-2} \) satisfies that \( p_n(v_0^T X_0 \leq v_0^T \theta_0) = D(\theta_0, X_{0n}^0) \), where \( X_0 = A_0^T X \), and hereafter \( p_n \) denotes the empirical probability measure in the \((d - 1)\)-dimensional space. For convenience, let

\[
\mathcal{J}^0 = \{ j : v_0^T (A_0^T X_j) = v_0^T \theta_0, \; j = 1, 2, \ldots, n \},
\]

\[
\mathcal{J}^- = \{ j : v_0^T (A_0^T X_j) < v_0^T \theta_0, \; j = 1, 2, \ldots, n \},
\]

\[
\mathcal{J}^+ = \{ j : v_0^T (A_0^T X_j) > v_0^T \theta_0, \; j = 1, 2, \ldots, n \}.
\]

Obviously, (i) \( n p_n(v_0^T X_0 \leq v_0^T \theta_0) = \#(\mathcal{J}^0 \cup \mathcal{J}^-) \), where \( \#(A) \) denotes the cardinal number of a set \( A \), and (ii) \( \#(\mathcal{J}^0) \leq d \) when \( X_n \) is IGP. Without loss of generality, write \( \mathcal{J}^0 = \{ j_1, j_2, \ldots, j_q \} \), where \( 0 \leq q \leq d \).

(i) When \( q = 0 \), i.e., \( \mathcal{J}^0 = \emptyset \), the facts \( \lim_{k \to \infty} Q_k = I_p \) and \( \lim_{k \to \infty} \theta_k = \theta_0 \) together lead to

\[
\lim_{k \to \infty} I \left( v_0^T (A_k^T X_i) \leq v_0^T \theta_k \right) = \lim_{k \to \infty} I \left( v_0^T (A_0^T Q_k X_i) \leq v_0^T \theta_k \right) = I \left( v_0^T (A_0^T X_i) \leq v_0^T \theta_0 \right)
\]

for each \( i = 1, 2, \ldots, n \), where \( I(\cdot) \) denotes the indicative function. Using this, we obtain

\[
\lim_{k \to \infty} p_n(v_0^T X_k \leq v_0^T \theta_k) = p_n(v_0^T X_0 \leq v_0^T \theta_0),
\]

where \( X_k = A_k^T X \).

(ii) When \( q > 0 \), i.e., \( \mathcal{J}^0 \neq \emptyset \), we have the following results.

For \( l = 1, 2, \ldots, q \), denote

\[
\mathcal{N}_{l-1}^- = \left\{ k \in \mathcal{N}_{l-1} : v_0^T (A_k^T X_{j_l}) \leq v_0^T \theta_k \right\},
\]

\[
\mathcal{N}_{l+1}^- = \left\{ k \in \mathcal{N}_{l-1} : v_0^T (A_k^T X_{j_l}) > v_0^T \theta_k \right\}.
\]

Check whether or not \( \#(\mathcal{N}_{l-1}^-) < \infty \). If not, set \( \mathcal{N}_l = \mathcal{N}_{l-1}^- \); Otherwise, \( \mathcal{N}_l = \mathcal{N}_{l+1}^- \).

Then \( \max\{\#(\mathcal{N}_{l-1}^-), \#(\mathcal{N}_{l-1}^-)\} = \infty \) due to \( \#(\mathcal{N}_{l-1}) = \infty \) for \( l = 1, 2, \ldots, q \). Hence, \( \#(\mathcal{N}_q) = \infty \) because \( q \leq d \).

Using this, we claim that for each \( l = 1, 2, \ldots, q \), either

\[
I \left( v_0^T (A_k^T X_{j_l}) \leq v_0^T \theta_k \right) = 0, \quad \text{for all } k \in \mathcal{N}_q,
\]

or

\[
I \left( v_0^T (A_k^T X_{j_l}) \leq v_0^T \theta_k \right) = 1, \quad \text{for all } k \in \mathcal{N}_q
\]
is true by the construction of $\mathcal{N}_q$. Hence,

$$\lim_{k \in \mathcal{N}_q, \ k \to \infty} I \left( v_0^T (A_k^T X_i) \leq v_0^T \theta_k \right) = \begin{cases} 0, & \text{if (12) is true} \\ 1, & \text{if (13) is true}. \end{cases}$$

This, together with $v_0^T (A_0^T X_i) = v_0^T \theta_0$ ($i \in J^0$), leads to

$$\lim_{k \in \mathcal{N}_q, \ k \to \infty} I \left( v_0^T (A_k^T X_i) \leq v_0^T \theta_k \right) \leq I \left( v_0^T (A_0^T X_i) \leq v_0^T \theta_0 \right) = 1,$$  \quad (14)

for $\forall i \in J^0$.

On the other hand, similar to (10), we have

$$\lim_{k \in \mathcal{N}_q, \ k \to \infty} I \left( v_0^T (A_k^T X_i) \leq v_0^T \theta_k \right) = I \left( v_0^T (A_0^T X_i) \leq v_0^T \theta_0 \right)$$

for $\forall i \notin J^0$. This, together with (14), shows

$$\lim_{k \in \mathcal{N}_q, \ k \to \infty} p_n(v_0^T X_k \leq v_0^T \theta_k) \leq p_n(v_0^T X_0 \leq v_0^T \theta_0).$$  \quad (15)

Next, by observing

$$p_n(v_0^T X_k \leq v_0^T \theta_k) \geq \inf_{v \in S^{d-2}} p_n(v_0^T X_k \leq v^T \theta_k) = D(\theta_k, X_{\theta_k}^n) = \lambda^*(X_{\theta_k}^n), \ k = 1, 2, \ldots,$$

and the fact that $p_n(\cdot) \in \{0, 1/n, 2/n, \ldots, 1\}$, we have that

$$\lim_{k \in \mathcal{I}, \ k \to \infty} p_n(v_0^T X_k \leq v_0^T \theta_k) \geq \inf_{l \in \mathcal{I}} \lambda^*(X_{\theta_l}^n) \geq \inf_{j \in \mathcal{N}_0} \lambda^*(X_{\theta_j}^n),$$

where $\mathcal{I} = \mathcal{N}_0$ if $J^0 = \emptyset$, otherwise $\mathcal{I} = \mathcal{N}_q$. This, combined with (11) and (15), implies

$$\lambda^*(X_{\theta_0}^n) \geq D(\theta_0, X_{\theta_0}^n) = p_n(v_0^T X_0 \leq v_0^T \theta_0) \geq \inf_{k \in \mathcal{N}_0} \lambda^*(X_{\theta_k}^n).$$

Finally, by noting that the image of $\lambda^*(X_{\theta_k}^n)$ takes only a finite set of values, we claim that there must exist a $k_0 > 0$ such that $\lambda^*(X_{\theta_{k_0}}^n) = \inf_{k \in \mathcal{N}_0} \lambda^*(X_{\theta_k}^n)$. This lemma then follows immediately. \hfill \Box

The aforementioned four lemmas are important in proving the upper bound parts of the main theorem, while the following two lemmas play a key role in obtaining the lower bound of the FSBP of HM.

**Lemma 5.** For any given $y \in \mathcal{R}^d \setminus \text{cov}(X^n)$ ($d \geq 2$), there exists a $u_0 \in S^{d-1}$ such that

$$D(A_{u_0}^T y, X_{u_0}^n) = \lambda^*(X_{u_0}^n).$$
Proof. For \( \forall y \in \mathbb{R}^d \setminus \text{cov}(\mathbb{X}^n) \), \( \|X_i - y\| \neq 0 \) for \( i = 1, 2, \cdots, n \). Hence, we may let \( \mathcal{W}^n = \{W_1, W_2, \cdots, W_n\} \), where \( W_i = (X_i - y)/\|X_i - y\| \).

Next, by observing the facts that (i) \( I(u^\top X_i \leq u^\top y) = I(u^\top W_i \leq 0) \), and (ii) \( v^\top A_u^\top u = 0 \) and \( \|A_u v\| = 1 \) hold true for \( \forall u \in \mathcal{S}^{d-1} \) and \( \forall v \in \mathcal{S}^{d-2} \), we obtain

\[
D(A_u^\top y, X_u^n) = \inf_{v \in \mathcal{S}^{d-2}} P_n \left( v^\top X \leq v^\top (A_u y) \right)
= \inf_{u \in \mathcal{S}^{d-1}, u \perp u} P_n (u^\top X \leq u^\top y)
= \inf_{u \in \mathcal{S}^{d-1}, u \perp u} \frac{1}{n} \sum_{i=1}^{n} I(u^\top W_i \leq 0), \tag{16}
\]

where by \( \alpha \perp \beta \) we mean that \( \alpha \) is normal to \( \beta \) hereafter.

Note that \( \mathcal{W}^n \in \mathcal{S}^{d-1} \) and \( u \) belongs to the closed hemisphere \( \{v \in \mathcal{S}^{d-1} : \bar{u}^\top v \leq 0\} \).

According to Liu and Singh (1992), (16) is in fact the angular Tukey’s depth of \( u \) with respect to \( \mathcal{W}^n \) on the sphere \( \mathcal{S}^{d-1} \). Let \( u_0 \) be the corresponding angular Tukey’s median of \( \mathcal{W}^n \). Then this lemma follows immediately. \( \square \)

**Lemma 6.** Let \( \mathcal{B}(\mathbb{X}^n) \) be the boundary of the convex hull \( \text{cov}(\mathbb{X}^n) \) of \( \mathbb{X}^n \). Then for any given \( u_\ell \in \mathcal{S}^{d-1} \) \( (d \geq 2) \), we have that

\[
D(z, \mathbb{X}^n \cup \mathcal{Y}^m) \geq \frac{\min\{n\lambda^*(X_u^n), m + 1\}}{n + m},
\]

where \( \mathcal{Y}^m \) denotes the data set containing exactly \( m \) repetitions of \( y \) with \( y \in \ell \setminus \text{cov}(\mathbb{X}^n) \), and \( z \) the closer to \( y \) intersection of \( \ell \) and \( \mathcal{B}(\mathbb{X}^n) \), where \( \ell = \{x : x = \Lambda_x u + \delta u_\ell, \delta \in \mathbb{R}^1\} \) with \( x \in \mathcal{M}(X_u^n) \).

Proof. For any \( u \in \mathcal{S}^{d-1} \), we have the following results.

(i) If \( u^\top z \geq u^\top y \), by observing that \( z \in \text{cov}(\mathbb{X}^n) \) implies \( nP_n(u^\top X \leq u^\top z) \geq 1 \), we have \( (n + m)P_{n+m}(u^\top X \leq u^\top z) \geq m + 1 \), where \( P_{n+m} \) denotes the empirical probability measure related to the data set \( \mathbb{X}^n \cup \mathcal{Y}^m \).

(ii) If \( u^\top z < u^\top y \), it is trivial that \( (n + m)P_{n+m}(u^\top X \leq u^\top z) = nP_n(u^\top X \leq u^\top z) \).

Now we prove that if there exist a \( v_0 \in \mathcal{S}^{d-1} \) such that \( P_n(v_0^\top X \leq v_0^\top z) < \lambda^*(X_u^n) \), we can obtain a contradiction.

Without confusion, let \( n_\ell \in \mathcal{S}^{d-1} \) be a normal vector of \( \ell \). Denote \( Q_{n_\ell}^1 = \{x : n_\ell^\top (x - z) > 0, v_0^\top (x - z) > 0\} \), \( Q_{n_\ell}^2 = \{x : n_\ell^\top (x - z) < 0, v_0^\top (x - z) > 0\} \), \( Q_{n_\ell}^3 = \{x : n_\ell^\top (x - z) < 0, v_0^\top (x - z) < 0\} \) and \( Q_{n_\ell}^4 = \{x : n_\ell^\top (x - z) > 0, v_0^\top (x - z) < 0\} \).
Clearly, $\mathbf{n}_\ell \neq \pm \mathbf{v}_0$ when $P_n(\mathbf{v}_0^T X \leq \mathbf{v}_0^T z) < \lambda^*(\mathbf{X}_{u_\ell})$ because of

$$
\lambda^*(\mathbf{X}_{u_\ell}) = \inf_{\mathbf{v} \in \mathbb{S}^{d-1}, \mathbf{v} \perp \ell} P_n(\mathbf{v}^T X \leq \mathbf{v}^T z).
$$

Among all normal vectors of $\ell$, there must exist at least one $\mathbf{n}_\ell$ satisfying (c1): \[\min\{\mathbb{N}(\mathbb{Q}^1_{n_\ell}), \mathbb{N}(\mathbb{Q}^2_{n_\ell})\} = 0\] with $\mathbb{N}(\mathcal{A}) = \sum_{i=1}^n I(X_i \in \mathcal{A})$ for a set $\mathcal{A}$. If not, there will exist a contradiction with the facts that $\mathbf{y} \notin \text{cov}(\mathcal{X}^n)$ and $\mathbf{z}$ is the closer intersection of $\ell$ and $\text{B}(\mathcal{X}^n)$ to $\mathbf{y}$.

Without loss of generality, suppose $\mathbb{N}(\mathbb{Q}^2_{n_\ell}) = 0$. Then, (c1), together with the fact $\mathbb{N}(\ell \cap \{\mathbf{x} : \mathbf{v}_0^T (\mathbf{x} - \mathbf{z}) > 0\}) = 0$, easily leads to $\mathbb{N}(\mathbb{Q}_{n_\ell}^2 \cup \mathbb{Q}_{n_\ell}^3) = \mathbb{N}(\tilde{A}_{n_\ell})$, where $\tilde{A}$ denotes the closure of $\mathcal{A}$. Note that: $\mathbb{N}(\mathbb{Q}_{n_\ell}^2 \cup \mathbb{Q}_{n_\ell}^3) = n P_n(\mathbf{n}_\ell^T X \leq \mathbf{n}_\ell^T \mathbf{z}) \geq n \lambda^*(\mathbf{X}_{u_\ell})$.

Hence, $\mathbb{N}(\mathbb{Q}_{n_\ell}^3) \geq n \lambda^*(\mathbf{X}_{u_\ell})$. Obviously, this contradicts with the assumption such that $n \lambda^*(\mathbf{X}_{u_\ell}) > n P_n(\mathbf{v}_0^T X \leq \mathbf{v}_0^T \mathbf{z}) = \mathbb{N}(\mathbb{Q}_{n_\ell}^2 \cup \mathbb{Q}_{n_\ell}^3) \geq \mathbb{N}(\mathbb{Q}_{n_\ell}^3)$.

Combined with (i) and (ii), we obtain this lemma immediately. \[\square\]

With Lemmas 1-6 at hand, we now are able to prove our main theorem as follows, in which we obtain a precise result on the FSBP for $HM$.

**Theorem 1.** Suppose $\mathcal{X}^n$ are in general position. When $d \geq 2$, the FSBP of Tukey’s halfspace median $T^*$ is

$$
\varepsilon(T^*, \mathcal{X}^n) = \frac{\inf_{\mathbf{u} \in \mathbb{S}^{d-1}} \lambda^*(\mathbf{X}_{u_0})}{1 + \inf_{\mathbf{u} \in \mathbb{S}^{d-1}} \lambda^*(\mathbf{X}_{u_0})}.
$$

**Proof.** Let $\mathbf{y}$ be an arbitrary datum, and assume that $\mathcal{Y}^m$ contains exactly $m$ repetitions of $\mathbf{y}$. Clearly, for $\mathcal{X}^n$ and $\mathcal{X}^n \cup \mathcal{Y}^m$, $\text{cov}(\mathcal{X}^n) \subset \text{cov}(\mathcal{X}^n \cup \mathcal{Y}^m)$.

By Lemma 4, there is a $\mathbf{u}_0 \in \mathbb{S}^{d-1}$ satisfying display (1), and simultaneously

$$
\lambda^*(\mathbf{X}_{u_0}) = \inf_{\mathbf{u} \in \mathbb{S}^{d-1}} \lambda^*(\mathbf{X}_{u}).
$$

By Lemma 1, $\mathbf{X}_{u_0}$ is still in general position under the current assumptions. This, combined with Lemma 2, indicates that there $\exists \mathbf{x}_0 \in \mathcal{M}(\mathbf{X}_{u_0})$ such that: there $\exists \mathbf{u} \in \mathcal{U}_X$ satisfying $\mathbf{u}^T \mathbf{x} < \mathbf{u}^T \mathbf{x}_0$ for $\forall \mathbf{x} \neq \mathbf{x}_0$.

Let $\mathbf{x}_0 = A_{u_0} \mathbf{x}_0$, and $\ell_0 = \{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \delta \mathbf{u}_0, \delta \in \mathbb{R}^1\}$. Obviously, for any $\mathbf{x} \in \ell_0$ and $\delta \in \mathbb{R}^1$, we have $A_{u_0}^T \mathbf{x} = A_{u_0}^T A_{u_0} \mathbf{x}_0 + \delta A_{u_0}^T \mathbf{u}_0 = \mathbf{x}_0$. That is, the $A_{u_0}$-projection of any $\mathbf{x} \in \ell_0$ is $\mathbf{x}_0$.

As $\mathbf{y}$ is arbitrary, we suppose $\mathbf{y} \in \ell_0 \setminus \text{cov}(\mathcal{X}^n)$. Now we show that $n \lambda^*(\mathbf{X}_{u_0})$ such $\mathbf{y}$ suffice for breaking down $T^*$.

Decompose $\text{cov}(\mathcal{X}^n) = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1 = \ell_0 \cap \text{cov}(\mathcal{X}^n)$ and $\mathcal{D}_2 = \text{cov}(\mathcal{X}^n) \setminus \ell_0$. \[14\]
(i) For $\forall x \in D_1$, its $A_{u_0}$-projection is $x_0 \in \mathcal{R}^{d-1}$. Hence, there $\exists v \in U_{x_0}$ satisfying $P_n(v^T x \leq v^T x_0) = \lambda^*(X_{u_0}^n)$, where $X = (A_{u_0}^T X)$.

Next, for $v$, similar to Dyckerhoff and Mozharovskyi (2016), by making $\varepsilon > 0$ small enough, we have that $\bar{v} = v - \varepsilon A_{u_0}^T (y - x)$ still satisfies $\lambda^*(X_{u_0}^n) = P_n(v^T x \leq \bar{v}^T x_0)$.

Using this and the fact that $\bar{u}^T (y - x) = v^T (A_{u_0}^T (y - x)) - \varepsilon||A_{u_0}^T (y - x)|| = -\varepsilon||A_{u_0}^T (y - x)|| < 0$, where $\bar{u} = A_{u_0} \bar{v}$, we obtain

$$
D(x, X^m \cup Y^m) \leq \frac{n}{n + m} P_n \left( u^T X \leq u^T x \right) = \frac{n}{n + m} P_n \left( v^T (A_{u_0} X) \leq v^T (A_{u_0} x) \right) = \frac{n}{n + m} P_n \left( v^T X \leq v^T x \right) = \frac{n}{n + m} \lambda^*(X_{u_0}^n).
$$

(ii) For $\forall x \in D_2$, denote $x = A_{u_0}^T x$. Since $x \neq x_0$, by Lemma 2, we can find a $v \in U_X$ such that $v^T x < v^T x_0$. By noting $A_{u_0} y = x_0$, we have $v^T x < v^T (A_{u_0}^T y)$. Using this, a similar derivation to (i) leads to

$$
D(x, X^m \cup Y^m) \leq \frac{n}{n + m} P_n \left( u^T X \leq u^T x \right) \leq \frac{n}{n + m} \lambda^*(X_{u_0}^n),
$$

where $u = A_{u_0} v$.

(i) and (ii) lead to

$$
\sup_{x \in \text{cov}(X^n)} D(x, X^m \cup Y^m) \leq \frac{n}{n + m} \lambda^*(X_{u_0}^n). \quad (17)
$$

Next, for any $x \not\in \text{cov}(X^n)$, there must exist a $u_x$ such that $P_n(u_x^T X \leq u_x^T x) = 0$ by the convexity of $\text{cov}(X^n)$. Using this, we claim that

$$
u_x^T y \leq u_x^T x < u_x^T x_1, u_x^T x_2, \ldots, u_x^T x_n
$$

hold true for any $x \in \text{cov}(X^n \cup Y^m) \setminus \text{cov}(X^n)$ but $x \neq y$. (The fact that $u_x^T x < u_x^T y, u_x^T x_1, u_x^T x_2, \ldots, u_x^T x_n$ contradicts with the convexity of $\text{cov}(X^n \cup Y^m)$.) Hence, $D(x, X^m \cup Y^m) \leq P_{n+m} (u_x^T X \leq u_x^T) = m/(n + m)$. While for $y$, $P_{n+m} (u^T X \leq u^T y) \geq m/(n + m)$ for any $u \in S^{d-1}$. Finally, we obtain

$$
D(y, X^m \cup Y^m) = \sup_{z \in \text{cov}(X^n \cup Y^m) \setminus \text{cov}(X^n)} D(z, X^m \cup Y^m).
$$

This, together with (17), implies that $y \in \mathcal{M}(X^n \cup Y^m)$ when $m = n \lambda^*(X_{u_0}^n)$. Note that: (a) $T^*(X^n \cup Y^m)$ is by definition the average of all points contained in $\mathcal{M}(X^n \cup Y^m)$, (b) $y$ is
arbitrary, it may belong to any bounded region. Hence, \( n\lambda^*(X^n_{u_0}) \) such \( y \) can make \( T^*(\mathcal{X}^n \cup \mathcal{Y}^m) \) outside the convex hull of \( \mathcal{X}^m \), and in turn break down \( T^* \).

This completes the first part of this theorem. Now we proceed to the second part. By Lemma 5, for \( \forall y \in \mathcal{R}^d \setminus \text{cov}(\mathcal{X}^n) \), there must exist a \( u_y \in S^{d-1} \) such that \( A_{u_y} \mathcal{X}^n \subseteq \mathcal{M}(X^n_{u_y}) \). Using this and Lemma 6, there \( \exists \) \( z \) on the boundary of \( \text{cov}(\mathcal{X}^n) \), and hence \( z \in \text{cov}(\mathcal{X}^n) \), such that

\[
D(z, \mathcal{X}^n \cup \mathcal{Y}^m) \geq \frac{\min\{n\lambda^*(X^n_{u_y}), m+1\}}{n+m} \geq \frac{\min\{n\lambda^*(X^n_{u_y}), m+1\}}{n+m} > \frac{m}{n+m} = D(y, \mathcal{X}^n \cup \mathcal{Y}^m) = \sup_{z \in \text{cov}(\mathcal{X}^n \cup \mathcal{Y}^m) \setminus \text{cov}(\mathcal{X}^n)} D(z, \mathcal{X}^n \cup \mathcal{Y}^m)
\]

when \( m \leq n\lambda^*(X^n_{u_0}) - 1 \) for \( u_0 \) given in the earlier paragraph of the proof of this theorem. Hence, \( T^*(\mathcal{X}^n \cup \mathcal{Y}^m) \in \text{cov}(\mathcal{X}^n) \). That is, less than \( n\lambda^*(X^n_{u_0}) \) repetitions of an arbitrary \( y \) could not break down \( T^* \), no matter where \( y \) locates at.

This completes the whole proof of this theorem. \( \square \)

**Remark 3.1.** When \( d = 2 \), \( \lambda^*(X^n_{u_0}) = \lceil n/2 \rceil \) for \( u_0 \in S^1 \) given in this theorem. Hence, Theorem 1 reduces to the following special case:

\[
\varepsilon(T^*, \mathcal{X}^n) = \frac{\lceil \frac{n}{2} \rceil}{n + \lceil \frac{n}{2} \rceil}.
\]

The key step of Theorem 1 is to locate the new maximizers of Tukey’s halfspace depth function after adding \( \mathcal{Y}^m \) to \( \mathcal{X}^n \). Considering the \( A_{u_y} \)-projections of the original observations is a helpful way to achieve this goal of identifying the maximizer. It turns out that the point \( z \) on the boundary of \( \text{cov}(\mathcal{X}^n) \), that determines a unit vector \( u_y = (y - z) / \|y - z\| \) such that the \( A_{u_y} \)-projections of \( y \) lies in the interior of \( \mathcal{M}(X^n_{u_y}) \), plays a key role in the whole proof of Theorem 1.

To gain an intuitive understanding of this, we provide a 2-dimensional illustration in Figure 1, where \( X_1, X_2, X_3 \) denote the data points, and \( x_1, x_2, x_3 \) the corresponding \( A_{u_y} \)-projections. When \( m = 1 \), the Tukey depth of the point \( z \) with respect to \( \mathcal{X}^n \cup \mathcal{Y}^m = \{X_1, X_2, X_3, y\} \) is clearly \( 1/2 \), greater than that of any point outside the convex hull of \( \{X_1, X_2, X_3\} \). On the other hand, the depth of any \( x \in \text{cov}(\mathcal{X}^n) \setminus \{z\} \) is smaller than \( 1/2 \) (see \( z_1, z_2 \) for example).
Figure 1: Shown is a 2-dimensional illustration for Theorem 1.

Note that when \( u_0 \in S^{d-1} \) satisfies (1), \( X_{n u_0} \) is in general position from Lemma 1. Hence, relying on Proposition 2.3 in DG92, Theorem 1 in Liu et al. (2015) and Theorem 1 above, we can easily obtain the following proposition. Since the proof is trivial, we omit it here.

**Proposition 1.** Suppose \( X^n \) is in general position. When \( d \geq 2 \), the FSBP of Tukey’s halfspace median \( T^* \) satisfies that

\[
\left\lceil \frac{n}{d} \right\rceil n + \left\lceil \frac{n}{d} \right\rceil \leq \varepsilon(T^*, \mathcal{X}^n) \leq \begin{cases} 
\left\lceil \frac{n-d+3}{2} \right\rceil, & \text{if } \exists u_0 \in S^{d-1} \text{ satisfying (1),} \\
\left\lceil \frac{n-d+2}{2} \right\rceil, & \text{and } \mathcal{M}(X_{u_0}^n) \text{ is singleton,} \\
\left\lceil \frac{n-d+1}{2} \right\rceil, & \text{otherwise.}
\end{cases}
\]

**Remark 3.2.** For \( d = 2 \), (a) when \( n \) is even, \( \mathcal{M}(X_{u_0}^n) \) is of affine dimension 1, we have \( \left\lceil \frac{n-d+2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil \); (b) when \( n \) is odd, \( \mathcal{M}(X_{u_0}^n) \) is singleton, we have \( \left\lceil \frac{n-d+3}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil \). Both scenarios indicate that \( \varepsilon(T^*, \mathcal{X}^n) \) attains the upper bound \( \left\lceil \frac{n}{2} \right\rceil/(n + \left\lceil \frac{n}{2} \right\rceil) \).
(a) The scatter plot of the data set.

(b) The first scenario of the $A_u$-projections.

(c) The second scenario of the $A_u$-projections.

(d) The third scenario of the $A_u$-projections.

(e) The fourth scenario of the $A_u$-projections.

Figure 2: Shown is an example for the upper bound for Proposition 1.

Both the upper and low bound given in Proposition 1 is \textit{attained} if the data set is strategically choosed. Let’s first see an illustration for the upper bound. The data points are plotted in 2(a). The scatter plot of the $A_u$-projections of this data set has four scenarios, though, as shown in Figures 2(b)-2(e). The maximum Tukey depth $\lambda^*(X^n_u)$ is equal to $1/2$ for any $u \in S^{d-1}$, nevertheless. Clearly, $1/2 = \left\lfloor \frac{\frac{1}{2} + \frac{3}{2}}{4} \right\rfloor$, and hence $\varepsilon(T^*, X^n) = 1/3$ attains the upper bound given Proposition 1 for this data set.
As to the low bound, we have an example shown in Figure 3(a). Since we can find a $\mathbf{u}$ such that the $\mathcal{A}_u$-projections of the original data set is a data set of points at the vertices of a collection of nested simplices; See Figure 3(b). The maximum Tukey depth with respect to these projections is only $2/6 = 1/d$ when $d = 3$. Hence, similar to DG92, the low bound of Proposition 1 is also attained, with $\varepsilon(T^*, X^n) = 1/4$ for this example.

Compared to the asymptotic result $1/3$, Proposition 1 indicates that the dimension $d$ indeed affects the finite sample breakdown point robustness of Tukey’s halfspace median. In detail, when $d$ increases, $\varepsilon(T^*, X^n)$ tends to decrease for fixed $n$. In fact, the true FSBP of Tukey’s halfspace median may be less than $1/3$ under the IGP assumption, and this gap may be very great in practice when $d$ is large relative to $n$.

4 Concluding remarks

In the literature, it has long been a open question as to the exact finite sample breakdown point of Tukey’s halfspace median. In this paper, we resolved this question through taking account of the $\mathcal{A}_u$-projections of the original observations when they are in general position. A precise result was provided for fixed sample size $n$. The current results revealed that, complimenting the asymptotic result $(1/3)$ obtained by DG92, the finite sample breakdown point robustness of $\text{HM}$ may be affected greatly by the dimension $d$, especially when $d$ is large relative to $n$. Since many offsprings, such as regression depth and multiple output regression, originated directly from Tukey’s halfspace depth function with the finite sample breakdown point of their median-like estimators unsolved, we wish that the developed results have the potential to facilitate the investigation of their finite sample breakdown point robustness.
Observe that $\inf_{u \in S^{d-1}} \lambda^*(X^n_u)$ involves an infinite number of maximum Tukey depths $\lambda^*(X^n_u)$. It computation is not trivial, and would be very time-consuming. Quite fortunately, there has been much progress in the computation of Tukey's halfspace median and its related depth; See, for example, Rousseeuw and Ruts (1998), Struyf and Rousseeuw (2000) and Liu et al. (2015) and reference therein.

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