A RELlich type theorem for discrete Schrödinger operators

HIROSHI ISOZAKI and HISASHI MORIOKA

Abstract. An analogue of Rellich’s theorem is proved for discrete Laplacian on square lattice, and applied to show unique continuation property on certain domains as well as non-existence of embedded eigenvalues for discrete Schrödinger operators.

1. Introduction

The Rellich type theorem for the Helmholtz equation is the following assertion (20): Suppose \( u \in H^2_{\text{loc}}(\mathbb{R}^d) \) satisfies
\[
(-\Delta - \lambda) u = 0, \quad |x| > R_0,
\]
for some constants \( \lambda, R_0 > 0 \), and
\[
u(x) = o(|x|^{-(d-1)/2}), \quad |x| \to \infty.
\]
Then \( u(x) = 0 \) on \( \{|x| > R_0\} \).

This theorem has been extended to a broad class of Schrödinger operators, since it implies the non-existence of eigenvalues embedded in the continuous spectrum (see e.g. 14, 22, 1), and also plays an important role in the proof of limiting absorption principle which yields the absolute continuity of the continuous spectrum (see e.g. 5, 11). The Rellich type theorem states a local property at infinity of solutions. Namely, it proves \( u(x) = 0 \) on \( \{|x| > R_1\} \) for some \( R_1 > R_0 \). By the unique continuation property, it then follows that \( u(x) = 0 \) for \( |x| > R_0 \). In the theory of linear partial differential equations (PDE), the Rellich type theorem can be regarded as the problem of division in the momentum space. In fact, given a linear PDE with constant coefficients \( P(D)u = f \), \( f \) being compactly supported, Fourier transformation leads to the algebraic equation \( P(\xi)\tilde{u}(\xi) = \tilde{f}(\xi) \), where \( \tilde{u}(\xi) \) denotes the Fourier transform of \( u(x) \). If \( P(\xi) \) divides \( \tilde{f}(\xi) \), \( u \) is compactly supported due to the Paley-Wiener theorem. This approach was pursued by Treves 25, and then developed by Littman 17, 18, Hörmander 9 and Murata 19. One should note that Besov spaces appear naturally through these works. In this paper, we shall consider its extension to the discrete case.

2000 Mathematics Subject Classification. Primary 81U40, Secondary 47A40.

Key words and phrases. Schrödinger operator, square lattice, Rellich’s theorem.
Throughout the paper, we shall assume that $d \geq 2$. Let $\mathbb{Z}^d = \{n = (n_1, \cdots, n_d) \mid n_i \in \mathbb{Z}\}$ be the square lattice, and $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ the standard bases of $\mathbb{Z}^d$. The discrete Laplacian $\Delta_{\text{disc}}$ is defined by

$$-(\Delta_{\text{disc}} \hat{u})(n) = -\frac{1}{4} \sum_{j=1}^{d} \{\hat{u}(n + e_j) + \hat{u}(n - e_j)\} + \frac{d}{2} \hat{u}(n)$$

for a sequence $\{\hat{u}(n)\}_{n \in \mathbb{Z}^d}$. Our main theorem is the following.

**Theorem 1.1.** Let $\lambda \in (0, d) \setminus \mathbb{Z}$ and $R_0 > 0$. Suppose that a sequence $\{\hat{u}(n)\}$, defined for $|n| \geq R_0$, satisfies

$$(-\Delta_{\text{disc}} - \lambda) \hat{u} = 0, \quad |n| > R_0,$$

$$\lim_{R \to \infty} \frac{1}{R} \sum_{R_0 < |n| < R} |\hat{u}(n)|^2 = 0.$$  

Then there exists $R_1 > R_0$ such that $\hat{u}(n) = 0$ for $|n| > R_1$.

A precursor of this theorem is given in the proof of Theorem 9 of Shaban-Vainberg [23]. Their purpose is to compute the asymptotic expansion of the resolvent $\hat{R}_0(\lambda \pm i0) = (-\Delta_{\text{disc}} - \lambda \mp i0)^{-1}$ on $\mathbb{Z}^d$ and to find the associated radiation condition. Let

$$T^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d = [-\pi, \pi]^d$$

be the standard $d$-dimensional torus and put

$$h(x) = \frac{1}{2} \left(d - \sum_{j=1}^{d} \cos x_j\right), \quad M_\lambda = \{x \in T^d \mid h(x) = \lambda\}.$$

Passing to the Fourier series, $-\Delta_{\text{disc}}$ is transformed to the operator of multiplication by $h(x)$ on $T^d$. Therefore, the computation of the behavior of $\hat{R}_0(\lambda \pm i0)$ boils down to that for an integral on $M_\lambda$. For a compactly supported function $\hat{f} \in \ell^2(\mathbb{Z}^d)$ and $\lambda \in (0, d) \setminus \mathcal{E}$, $\mathcal{E}$ being a set of some exceptional points, the stationary phase method gives the following asymptotic expansion as $|k| \to \infty$:

$$\hat{R}_0(\lambda \pm i0) \hat{f}(k) = \sum_j |k|^{-(d-1)/2} e^\pm ik \cdot x(\lambda, \omega_k; j) a_\pm(\lambda, \omega_k; j) + O(|k|^{-(d+1)/2}),$$

where $\omega_k = k/|k|$, and the summation ranges over all stationary phase points $x(\lambda, \omega_k; j) \in M_\lambda$ at which the normal of $M_\lambda$ is parallel to $\omega_k$ and the Gaussian curvature does not vanish. It is then natural to define the radiation condition by using the first term of the above asymptotic expansion (1.5). To show the uniqueness of solutions to the discrete Schrödinger equation satisfying the radiation condition, they proved the assertion (which is buried in the proof actually):

(SV) The solution of $(-\Delta_{\text{disc}} - \lambda) \hat{u} = 0$ on $|n| > R_0$, satisfying $\hat{u}(n) = O(|n|^{-(d+1)/2})$ as $|n| \to \infty$, vanishes on $\{|n| > R_1\}$ for large $R_1$. 

The new ingredient in the present paper is the following fact to be proved in §4: Consider the equation
\[(1.6) \quad (h(x) - \lambda)u(x) = f(x), \quad \text{on } \mathbb{T}^d.\]
If the Fourier coefficients $\hat{u}(n)$ of the distribution $u$ satisfy (1.2) and $\hat{f}(n)$ is compactly supported, then $u \in C^\infty(\mathbb{T}^d)$, hence $f(x) = 0$ on $M_\lambda$.

Once we establish this fact, we can follow the arguments for proving the assertion (SV) without any change to show that $\hat{u}(n)$ is compactly supported. For the sake of completeness, in §4, we will also reproduce the proof of this part, which makes use of basic facts in theories of functions of several complex variables and algebraic geometry.

As applications of Theorem 1.1, we show in §2 non-existence of eigenvalues embedded in the continuous spectrum (except for threshold energies $0, 1, \cdots, d$) for $-\Delta_{\text{disc}} + \hat{V}$ in the whole space as well as in exterior domains.

The result of the present paper is used as a key step in [13] on the inverse scattering from the scattering matrix of a fixed energy for discrete Schrödinger operators with compactly supported potentials. In [12], the inverse scattering from all energies was studied by using complex Born approximation (see also [4]).

Function theory of several complex variables and algebraic geometry have already been utilized as powerful tools not only in linear PDE but also in the study of spectral properties for discrete Schrödinger operators or periodic problems. See e.g. Eskina [4], Kuchment-Vainberg [16], Gérard-Nier [7].

We give some remarks about notation in this paper. For $x, y \in \mathbb{R}^d$, $x \cdot y = x_1y_1 + \cdots + x_dy_d$ denotes the ordinary scalar product, and $|x| = (x \cdot x)^{1/2}$ is the Euclidean norm. Note that even for $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, we use $|n| = (\sum_{i=1}^d |n_i|^2)^{1/2}$. For two Banach spaces $X$ and $Y$, $\mathcal{B}(X,Y)$ denotes the totality of bounded operators from $X$ to $Y$. For a self-adjoint operator $A$ on a Hilbert space, $\sigma(A)$, $\sigma_{\text{ess}}(A)$, $\sigma_{\text{ac}}(A)$ and $\sigma_{\text{p}}(A)$ denote the spectrum, the essential spectrum, the absolutely continuous spectrum and the point spectrum of $A$, respectively. For a set $S$, $\#S$ denotes the number of elements in $S$. We use the notation\[ \langle t \rangle = (1 + t^2)^{1/2}, \quad t \in \mathbb{R}.\]

1.1. Acknowledgement. The authors are indebted to Evgeny Korotyaev for useful discussions and encouragements. The second author is supported by the Japan Society for the Promotion of Science under the Grant-in-Aid for Research Fellow (DC2) No. 23110.

2. Some Applications of Theorem 1.1

2.1. Absence of embedded eigenvalues in the whole space. Granting Theorem 1.1, we state its applications in this section. The Schrödinger operator $\hat{H}$ on $\mathbb{Z}^d$ is defined by\[ \hat{H} = -\Delta_{\text{disc}} + \hat{V}, \]
where $\hat{V}$ is the multiplication operator:
$$(\hat{V} \, \hat{u})(n) = \hat{V}(n) \, \hat{u}(n).$$

**Theorem 2.1.** If $\hat{V}(n) \in \mathbb{R}$ for all $n$, and there exists $R_0 > 0$ such that $\hat{V}(n) = 0$ for $|n| > R_0$, we have $\sigma_p(\hat{H}) \cap ((0, d) \setminus Z) = \emptyset$.

The assumption of the theorem yields $\sigma_{ess}(\hat{H}) = [0, d]$ (see [12]). Therefore, Theorem 2.1 asserts the non-existence of eigenvalues embedded in the continuous spectrum except for the set of thresholds $Z \cap [0, d]$.

**Proof of Theorem 2.1.** Assume $\lambda \in ((0, d) \setminus Z) \cap \sigma_p(\hat{H})$ and $\hat{u} \in \ell^2(Z^d)$ is the associated eigenfunction i.e. $(-\Delta_{disc} + \hat{V} - \lambda)\hat{u} = 0$. Putting $\hat{f} = -\hat{V}\hat{u}$, which is compactly supported by the assumpton of $\hat{V}$, we have the equation
$$(-\Delta_{disc} - \lambda)\hat{u} = \hat{f} \quad \text{on} \quad Z^d.$$Since $\hat{u} \in \ell^2(Z^d)$, the condition [12] is satisfied. Theorem 1.1 then implies that $\hat{u}$ is compactly supported. Therefore, there exists $m_1 \in Z$ such that $\hat{u}(n) = 0$ if $n_1 \geq m_1$. Using the equation $(-\Delta_{disc} + \hat{V} - \lambda)\hat{u} = 0$, which holds on the whole $Z^d$, we then have
$$\frac{1}{4} \hat{u}(m_1 - 1, n') = \left(-\Delta^{(d-1)}_{disc} + \hat{V}(m_1, n') - \lambda\right) \hat{u}(m_1, n') + \frac{1}{2} \hat{u}(m_1, n') - \frac{1}{4} \hat{u}(m_1 + 1, n') = 0,$$where $n' = (n_2, \ldots, n_d)$ and $\Delta^{(d-1)}_{disc}$ is the discrete Laplacian on $Z^{d-1}$. Repeating this procedure, we have $\hat{u}(n) = 0$ for all $n$, and completes the proof. \hfill $\square$

### 2.2. Unique continuation property.

The next problem we address is the unique continuation property. We begin with the explanation of the exterior problem. A subset $\Omega \subset Z^d$ is said to be connected if for any $m, n \in \Omega$, there exists a sequence $n^{(0)}, \ldots, n^{(k)} \in \Omega$ with $n^{(0)} = m$, $n^{(k)} = n$ such that for all $0 \leq \ell \leq k - 1$, $|n^{(\ell)} - n^{(\ell + 1)}| = 1$. For a connected subset $\Omega \subset Z^d$, we put

$$(2.1) \quad \deg_\Omega(n) = \# \{ m \in \Omega ; |m - n| = 1 \}, \quad n \in \Omega.$$The interior $\overset{\circ}{\Omega}$ and the boundary $\partial \Omega$ are defined by

$$(2.2) \quad \overset{\circ}{\Omega} = \{ n \in \Omega ; \deg_\Omega(n) = 2d \},$$

$$(2.3) \quad \partial \Omega = \{ n \in \Omega ; \deg_\Omega(n) < 2d \}.$$The normal derivative on $\partial \Omega$ is defined by

$$(2.4) \quad \partial_n \hat{u}(n) = \frac{1}{4} \sum_{m \in \hat{\Omega}, |m - n| = 1} (\hat{u}(n) - \hat{u}(m)), \quad n \in \partial \Omega.$$
Then, for a bounded connected subset $\Omega$, the following Green formula holds:

$$\sum_{n \in \Omega} \left( (\Delta_{\text{disc}} \hat{u})(n) \cdot \hat{v}(n) - \hat{u}(n) \cdot (\Delta_{\text{disc}} \hat{v})(n) \right)$$

(2.5)

$$= \sum_{n \in \partial \Omega} \left( (\partial_n \hat{u})(n) \cdot \hat{v}(n) - \hat{u}(n) \cdot (\partial_n \hat{v})(n) \right).$$

Indeed, the standard definition of Laplacian on graph is (see e.g. [4])

$$-(\Delta_{\text{disc}} \hat{u})(n) := \begin{cases} - (\Delta_{\text{disc}} \hat{u})(n), & (n \in \Omega), \\ (\partial_n \hat{u})(n), & (n \in \partial \Omega), \end{cases}$$

which yields

(2.6) $$\sum_{n \in \Omega} (\Delta_{\text{disc}} \hat{u})(n) \cdot \hat{v}(n) = \sum_{n \in \Omega} \hat{u}(n) \cdot (\Delta_{\text{disc}} \hat{v})(n), \quad \hat{u}, \hat{v} \in \ell^2(\Omega).$$

Splitting the sum (2.6) into two parts, the ones over $\Omega$ and over $\partial \Omega$, we have (2.5).

We take a connected exterior domain $\Omega_{\text{ext}}$, which means that there is a bounded set $\Omega_{\text{int}}$ such that $\Omega_{\text{ext}} = \mathbb{Z}^d \setminus \Omega_{\text{int}}$, and consider the Schrödinger operator

(2.7) $$\hat{H}_{\text{ext}} = -\Delta_{\text{disc}} + \hat{V}$$

without imposing the boundary condition, where $\hat{V}$ is a real-valued compactly supported potential.

Now suppose there exists a $\lambda \in (0, d) \setminus \mathbb{Z}$, and $\hat{u}$ satisfying (1.2) and

(2.8) $$\hat{H}_{\text{ext}} - \lambda)\hat{u} = 0, \quad \text{in} \quad \Omega_{\text{ext}}^c.$$

By Theorem 1.1 $\hat{u}$ vanishes near infinity. However, in the discrete case the unique continuation property of Laplacian does not hold in general. It depends on the shape of the domain. To guarantee it, we introduce the following cone condition.

For $1 \leq i \leq d$ and $n \in \mathbb{Z}^d$, let $C_{i, \pm}(n)$ be the cone defined by

(2.9) $$C_{i, \pm}(n) = \left\{ m \in \mathbb{Z}^d : \sum_{k \neq i} |m_k - n_k| \leq \pm (m_i - n_i) \right\}.$$

**Definition 2.2.** An exterior domain $\Omega_{\text{ext}}$ is said to satisfy a cone condition if for any $n \in \Omega_{\text{ext}}$, there is a cone $C_{i, +}(n)$ or $C_{i, -}(n)$ such that $C_{i, \pm}(n) \subset \Omega_{\text{ext}}$.

Examples of the domain satisfying this cone condition are

- $(\Omega_{\text{ext}})^c$ = a rectangular polyhedron $= \{ n \in \mathbb{Z}^d : |n_i| \leq a_i, \ i = 1, \cdots, d \}$,
- $(\Omega_{\text{ext}})^c$ = a rotated cube $= \{ n \in \mathbb{Z}^d : \sum_{i=1}^d |n_i| \leq C \}$,
- a domain with zigzag type boundary (see Figure 1).

**Theorem 2.3.** Let $\hat{H}_{\text{ext}}$ be a Schrödinger operator in an exterior domain $\Omega_{\text{ext}} \subset \mathbb{Z}^d$ with compactly supported potential. Suppose $\Omega_{\text{ext}}$ satisfies the cone condition. If there exist $\lambda \in (0, d) \setminus \mathbb{Z}$ and $\hat{u}$ satisfying (2.8) and (1.2), then $\hat{u} = 0$ on $\Omega_{\text{ext}}$. 
Proof. Take any $n \in \Omega_{ext}$. By the cone condition, there is a cone, say $C_{1,+}(n)$, such that $C_{1,+}(n) \subset \Omega_{ext}$. There is $k_1$ such that $\hat{u}(m) = 0$ for $m \in C_{1,+}(n)$, $k_1 < m_1$. Arguing as in the proof of Theorem 2.1, we have $\hat{u}(k_1, m') = 0$, $(k_1, m') \in C_{1,+}(n)$. Repeating this procedure, we arrive at $\hat{u}(n) = 0$.

An example of the domain which does not satisfy the cone condition is the one (in 2-dimension) whose boundary in the 4th quadrant has the form illustrated in Figure 2 and is rectangular in the other quadrants. In this case, $\hat{a}$ defined as in the figure satisfies

$$\left(\hat{H}_{ext} - \frac{1}{2}\right)\hat{u} = 0,$$

in $\Omega_{ext}$,

and $\hat{u} = 0$ on $\Omega_{ext}$, however $\hat{u} \neq 0$ on $\partial\Omega_{ext}$.

Figure 1. The zig zag type boundary.

Figure 2. A counter example for unique continuation property.
2.3. Exterior eigenvalue problem. Now let $\hat{H}^{(D)}_{ext}$ be $\hat{H}_{ext}$ subject to the Dirichlet boundary condition
\begin{equation}
\text{Dom} (\hat{H}^{(D)}_{ext}) = \{ \hat{u} \in \ell^2(\Omega_{ext}) ; \hat{u}(n) = 0, \forall n \in \partial \Omega_{ext} \},
\end{equation}
and $\hat{H}^{(R)}_{ext}$ the Robin boundary condition
\begin{equation}
\text{Dom} (\hat{H}^{(R)}_{ext}) = \{ \hat{u} \in \ell^2(\Omega_{ext}) ; \partial_v \hat{u}(n) + c(n) \hat{u}(n) = 0, \forall n \in \partial \Omega_{ext} \},
\end{equation}
c(n) being a bounded real function on $\partial D$. They are bounded self-adjoint operators, and their essential spectra are
$$\sigma_{ess}(H^{(D)}_{ext}) = \sigma_{ess}(H^{(R)}_{ext}) = [0, d].$$
Since this follows from the standard perturbation theory, we omit the proof.

Theorem 2.3 asserts the non-existence of embedded eigenvalues for these operators. In particular, it is clear from the proof that for the Dirichlet case, we have only to assume the cone condition for $\Omega_{ext}$.

Theorem 2.4. (1) Let $\hat{H}^{(R)}_{ext}$ be a Schrödinger operator in an exterior domain $\Omega_{ext}$ with compactly supported potential subordinate to the Robin boundary condition. Then $\sigma_p(\hat{H}^{(R)}_{ext}) \cap ((0, d) \setminus Z) = \emptyset$, if $\Omega_{ext}$ satisfies the cone condition.

(2) Let $\hat{H}^{(D)}_{ext}$ be a Schrödinger operator in an exterior domain $\Omega_{ext}$ with compactly supported potential subordinate to the Dirichlet boundary condition. Then $\sigma_p(\hat{H}^{(D)}_{ext}) \cap ((0, d) \setminus Z) = \emptyset$, if for any $n \in \Omega_{ext}$, there is a cone $C_i,+(n)$ (or $C_i,-(n)$) such that $C_i,+(n) \subset \Omega_{ext}$ (or $C_i,-(n) \subset \Omega_{ext}$).

3. Sobolev and Besov spaces on compact manifolds

The condition (1) is reformulated as a spectral condition for the Laplacian on the torus, which can further be rewritten by the Fourier transform. We do it on a compact Riemannian manifold in this section.

3.1. General case. Let $M$ be a compact Riemannian manifold of dimension $d$. One way to introduce the Sobolev and Besov spaces on $M$ is to use the Laplace-Beltrami operator $L = -\sum_{i,j=1}^{d-1} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right)$. For $s \in \mathbb{R}$, we define $\mathcal{H}^s$ to be the completion of $C^\infty(M)$ by the norm $\| (L)^{s/2} u \|$, where $\| \cdot \|$ is the norm of $L^2(M)$. We also define $\mathcal{B}^s$ to be the completion of $C^\infty(M)$ by the norm $\sup_{R>1} \frac{1}{\sqrt{R}} \| \chi_R(\sqrt{L}) u \|$, where $\chi_R(t) = 1$ for $t < R, \chi_R(t) = 0$ for $t > R$.

Another way is to use the Fourier transform. Let $\{ \chi_j \}_{j=1}^N$ be a partition of unity on $M$ such that on each support of $\chi_j$, we can take one coordinate patch. We define $H^s$ to be the completion of $C^\infty(M)$ by the norm $\sum_{j=1}^N \| (\xi)^s (\mathcal{F} \chi_j u)(\xi) \|$, where $\mathcal{F} v = \hat{v}$ denotes the Fourier transform of $v$, $\| \cdot \|$ is the norm of $L^2(\mathbb{R}^d)$. We define $B^s$ to be the completion of $C^\infty(M)$ by the norm $\sum_{j=1}^N \sup_{R>1} \frac{1}{\sqrt{R}} \| \chi_R(\xi)(\mathcal{F} \chi_j u)(\xi) \|$. 

The following inclusion relations hold for $s > 1/2$:

$$(3.1) \quad L^2 \subset H^{-1/2} \subset B^s \subset H^{-s}, \quad L^2 \subset H^{-1/2} \subset B^s \subset H^{-s}.$$  

These definitions of Sobolev and Besov spaces coincide. We show

**Lemma 3.1.** $H^s = H^s$ for any $s \in \mathbb{R}$, and $B^s = B^s$.

**Proof.** We prove $B^s = B^s$. It is well-known that $H^s = H^s$ for $s \in \mathbb{R}$, whose proof is similar to, actually easier than, that for $B^s = B^s$ given below.

First let us recall a formula from functional calculus. Let $\psi(x) \in C^\infty(\mathbb{R})$ be such that

$$(3.2) \quad |\psi^{(k)}(x)| \leq C_k(x)^{m-k}, \quad \forall k \geq 0,$$

for some $m \in \mathbb{R}$. One can construct $\Psi(z) \in C^\infty(\mathbb{C})$, called an almost analytic extension of $\psi$, having the following properties:

$$(3.3) \quad \left\{ \begin{array}{l}
\Psi(x) = \psi(x), \quad \forall x \in \mathbb{R}, \\
|\Psi(z)| \leq C(|z|)^m, \quad \forall z \in \mathbb{C}, \\
|\partial_z^n \Psi(z)| \leq C_n |\text{Im} \, z|^{m-n-1}, \quad \forall n \geq 1, \quad \forall z \in \mathbb{C}, \\
\text{supp} \, \Psi(z) \subset \{ z; \, |\text{Im} \, z| \leq 2 + 2|\text{Re} \, z| \}. \end{array} \right.$$  

In particular, if $\psi(x) \in C_0^\infty(\mathbb{R})$, one can take $\Psi(z) \in C^\infty_0(\mathbb{C})$. Then, if $m < 0$, for any self-adjoint operator $A$, we have the following formula

$$(3.4) \quad \psi(A) = \frac{1}{2\pi i} \int_{C} \overline{\partial_z \Psi(z)} (z - A)^{-1} \, dz \, d\bar{z},$$

which is called the formula of Helffer-Sjöstrand. See [3], [3], p. 390.

We use a semi-classical analysis employing $\hbar = 1/R$ as a small parameter (see e.g. [21]). We show that $\psi(\hbar^2 L)$ is equal to, modulo a lower order term, a pseudo-differential operator ($\Psi DO$) with symbol $\psi(\ell(x, \hbar \xi))$, where $\ell(x, \xi) = \sum_{i,j=1}^d g^{ij}(x) \xi_i \xi_j$. In fact, take $\chi(x)$, $\chi_0(x) \in C^\infty(M)$ with small support such that $\chi_0(x) = 1$ on $\text{supp} \, \chi$. Consider a $\Psi DO$ $P_h(z)$ with symbol $(\ell(x, \hbar \xi) - z)^{-1}$. Then

$$(h^2 L - z)\chi_0 P_h(z)\chi = \chi + Q_h(z)\chi,$$

where $Q_h(z)$ is a $\Psi DO$ with symbol

$$(3.5) \quad \sum_{i=1}^2 h^l \sum_{j=1}^3 \frac{q_{ij}(x, \hbar \xi)}{(\ell(x, \hbar \xi) - z)^2}, \quad q_{ij}(x, \xi) \in S^{2j-1},$$

$S^{m}$ being the standard Hörmander class of symbols (see [19], p. 65). This implies

$$(h^2 L - z)^{-1} \chi = \chi_0 P_h(z)\chi - (h^2 L - z)^{-1}Q_h(z)\chi.$$  

By the symbolic calculus, we have

$$(3.6) \quad \| (h^2 L)^{(r+1)/2} Q_h(z) \chi (h^2 L)^{-s/2} \| \leq h C_{s, d} |\text{Im} \, z|^{-N} |\langle z \rangle|^N,$$

where $N > 0$ is a constant depending on $s$ and $d$, and $C_{s, d}$ does not depend on $h$.

We take $\psi \in C_0^\infty(\mathbb{R})$ and apply (3.4). Then we have

$$(3.7) \quad \psi(h^2 L)\chi = \chi_0 \Psi_h \chi + \Psi Q \chi,$$
where $\Psi_h$ is a $\Psi DO$ with symbol $\psi(\ell(x, h\xi))$ and

$$
(3.8) \quad \Psi_{Q,h} = \frac{1}{2\pi i} \int_C \overline{\partial_z \Psi(z)} (h^2 L - z)^{-1} Q_h(z) dz d\overline{z}.
$$

We then have, letting $z = x + iy$,

$$
\| \langle h^2 L \rangle^{(s+3)/2} \Psi_{Q,h} \chi \langle h^2 L \rangle^{-s/2} \| 
\leq C \int_C |\overline{\partial_z \Psi(z)}| \| \langle h^2 L \rangle^{(s+3)/2} (h^2 L - z)^{-1} Q_h(z) \chi \langle h^2 L \rangle^{-s/2} \| dz d\overline{z}
\leq C \int_C |\overline{\partial_z \Psi(z)}| \| \langle h^2 L \rangle^{(s+3)/2} (h^2 L - z)^{-1} (h^2 L)^{-(s+1)/2} \| 
\times \| \langle h^2 L \rangle^{(s+1)/2} Q_h(z) \chi \langle h^2 L \rangle^{-s/2} \| dz d\overline{z}.
$$

Since

$$
\| \langle h^2 L \rangle^{(s+3)/2} (h^2 L - z)^{-1} (h^2 L)^{-(s+1)/2} \| \leq \sup_{\lambda \in \mathbb{R}} \frac{\langle \lambda \rangle}{|\lambda - z|} \leq |\text{Im } z|^{-1} \langle z \rangle,
$$

we obtain, using (3.6),

$$
\| \langle h^2 L \rangle^{(s+3)/2} \Psi_{Q,h} \chi \langle h^2 L \rangle^{-s/2} \| \leq h C_{s,d} \int_C |\overline{\partial_z \Psi(z)}| |\text{Im } z|^{-1-N} \langle z \rangle^{N+1} dz d\overline{z}
\leq h C_{s,d} \int_C |z|^{-m-1} dz d\overline{z},
$$

where we have used (3.3) with $m < -1, n = N + 1$. This estimate implies

$$
(3.10) \quad \sup_{\lambda < 1} h^{1/2} \| \Psi_{Q,h} \chi u \| < \infty, \quad \text{if } u \in H^{-s}, \forall s > 1/2.
$$

In fact, taking $0 \leq t \leq 3/2$, we have choosing $s = -3$ in (3.9),

$$
h^{1/2} \| \Psi_{Q,h} \chi u \| \leq C h^{3/2} \| \langle h^2 L \rangle^{-1} u \| \leq C h^{-2t+3/2} \| \langle L \rangle^{-t} u \|.
$$

The right-hand side is bounded if $1/4 < t \leq 3/4$.

Now, by the definition of $B^*$, we have the following equivalence

$$
(3.11) \quad u \in B^* \iff \begin{cases}
\forall s > 1/2, \\
\sup_{\lambda < 1} h^{1/2} \| \psi(h^2 L) u \| < \infty, \quad \forall \psi \in \mathcal{C}_c^\infty (\mathbb{R}).
\end{cases}
$$

In fact, the left-hand side is equivalent to the right-hand side for one fixed $\psi$ such that $\psi(t) = 1$ for $|t| < 1$, and $\psi(t) = 0$ for $|t| > 2$.

By virtue of (3.7) and (3.10), (3.11) is equivalent to

$$
(3.12) \quad \begin{cases}
\forall s > 1/2, \\
\sup_{\lambda < 1} h^{1/2} \| \chi_0 \Psi_h \chi_j u \| < \infty, \quad \forall j.
\end{cases}
$$

The symbol of $(\Psi_h)^* \chi_0(x)^2 \Psi_h$ is equal to

$$
\chi_0(x)^2 \psi(\ell(x, h\xi))^2 + O(h).
$$

Then by a suitable choice of $0 < c_1 < c_2$, we have

$$
\psi \left( \frac{h^2 |\xi|^2}{c_1} \right)^2 \leq \psi(\ell(x, h\xi))^2 \leq \psi \left( \frac{h^2 |\xi|^2}{c_2} \right)^2.
$$
Moreover, we can assume that there exists \( q(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) such that
\[
\begin{align*}
\psi(|\xi|^2/c_2)^2 - \psi(\ell(x, \xi))^2 &= q(x, \xi)^2, \\
\text{supp } q(x, \xi) &\subset \mathbb{R}^d \times \{a < |\xi| < b\},
\end{align*}
\]
for some \( 0 < a < b \). Since the symbol of \( \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \) is equal to
\[
\chi_0(x)^2 \psi\left(\frac{\hbar^2 |\xi|^2}{c_2}\right)^2 + O(\hbar),
\]
we see that the symbol of \( \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) - \Psi_h^* \chi_0(x)^2 \Psi_h \) is estimated as
\[
\chi_0(x)^2 \psi\left(\frac{\hbar^2 |\xi|^2}{c_2}\right)^2 - \chi_0(x)^2 \psi(\ell(x, \hbar^2 \xi))^2 + O(\hbar) = \chi_0(x)^2 q(x, \hbar \xi)^2 + O(\hbar).
\]
Then we have
\[
\begin{align*}
\psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) - (\Psi_h)^* \chi_0(x)^2 \Psi_h \\
= (Q_{0,h})^* \chi_0(x)^2 Q_{0,h} + Q_{1,h},
\end{align*}
\]
In the right-hand side, \( Q_{0,h} \) is a \( \Psi DO \) with symbol \( q(x, \hbar \xi) \), where \( q(x, \xi) \) is given by (3.13), and \( Q_{1,h} \) is a \( \Psi DO \) with symbol \( q_1(x, \xi; h) \) admitting the asymptotic expansion
\[
q_1(x, \xi; h) \sim \sum_{j \geq 1} h^j q_j(x, \hbar \xi),
\]
with \( q_j(x, \xi) \) having the same support property as in (3.13). Since the 1st term of the right-hand side of (3.14) is non-negative, we have proven
\[
\psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \geq (\Psi_h)^* \chi_0(x)^2 \Psi_h + Q_{1,h}.
\]
By a similar computation, we can prove
\[
(\Psi_h)^* \chi_0(x)^2 \Psi_h \geq \psi\left(\frac{-\hbar^2 \Delta}{c_1}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_1}\right) + Q'_{1,h}.
\]
By (3.16) and (3.17), we have
\[
\hbar \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_2}\right) - \hbar Q_{1,h} \geq \hbar (\Psi_h)^* \chi_0(x)^2 \Psi_h \\
\geq \hbar \psi\left(\frac{-\hbar^2 \Delta}{c_1}\right) \chi_0(x)^2 \psi\left(\frac{-\hbar^2 \Delta}{c_1}\right) + \hbar Q'_{1,h},
\]
where \( Q_{1,h} \) and \( Q'_{1,h} \) have the property (3.15). As \( u \in H^{-s} \), \( \forall s > 1/2 \), we have
\[
\sup_{h < 1} \hbar \|Q_{1,h} u, u\| + \sup_{h < 1} \hbar \|Q'_{1,h} u, u\| < \infty.
\]
Therefore, by (3.18), \( \sup_{h < 1} \hbar \|\chi_0 \Psi_h \Psi_j u\|^2 < \infty \) is equivalent to
\[
\sup_{R > 1} \frac{1}{R} \int_{\mathbb{R}^d} \left| \psi\left(\frac{|\xi|^2}{c_1 R^2}\right) \langle \Psi_j u(\xi)\rangle \right|^2 d\xi < \infty,
\]
for some $c > 0$, which is equivalent to $u \in B^*$. We have thus completed the proof of Lemma 3.1.

By the same argument, we can also prove the following lemma.

**Lemma 3.2.** If $u \in B^*$, we have the following equivalence

$$\frac{1}{\sqrt{R}} \| \psi \left( \frac{L}{R^2} \right) u \| \to 0 \iff \frac{1}{\sqrt{R}} \| \psi \left( \frac{|\xi|^2}{R^2} \right) (\mathcal{F} \chi_j)(\xi) \| \to 0,$$

for any $j$ and any $\psi \in C_0^\infty(\mathbb{R})$, where $\{\chi_j\}_{j=1}^N$ is the partition of unity on $M$.

### 3.2. Torus

We interpret the above results for the case of the torus $T^d$ defined by (1.3). Let $U$ be the unitary operator from $\ell^2(\mathbb{Z}^d)$ to $L^2(T^d)$ defined by

$$(3.19) \quad (U \hat{f})(x) = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} \hat{f}(n)e^{-i n \cdot x}.$$  

Letting $H_0 = U \hat{H_0} U^*$, $\hat{H_0} = -\Delta_{\text{disc}}$, we have

$$H_0 = h(x) = \frac{1}{2} \left( d - \sum_{j=1}^d \cos x_j \right) = \sum_{j=1}^d \sin^2 \left( \frac{x_j}{2} \right).$$

We define operators $\hat{N}_j$ and $N_j$ by

$$(\hat{N}_j \hat{f})(n) = n_j \hat{f}(n), \quad N_j = U \hat{N}_j U^* = i \partial_{x_j}.$$  

We put $N = (N_1, \ldots, N_d)$, and let $N^2$ be the self-adjoint operator defined by

$$N^2 = \sum_{j=1}^d N_j^2 = -\Delta, \quad \text{on} \quad T^d,$$

where $\Delta$ denotes the Laplacian on $T^d = [-\pi, \pi]^d$ with periodic boundary condition. We can then apply the results in the previous subsection to $L = -\Delta$. We put

$$|N| = \sqrt{N^2} = \sqrt{-\Delta}.$$  

For $s \in \mathbb{R}$, let $\mathcal{H}^s$ be the completion of $D(|N|^s)$ with respect to the norm $\|u\|_s = \|(N)^s u\|$ i.e.

$$\mathcal{H}^s = \{ u \in D'(T^d) ; \|u\|_s = \| (N)^s u \| < \infty \},$$

where $D'(T^d)$ denotes the space of distribution on $T^d$. Put $\mathcal{H} = L^0 = L^2(T^d)$.

For a self-adjoint operator $T$, let $\chi(a \leq T < b)$ denote the operator $\chi_I(T)$, where $\chi_I$ is the characteristic function of the interval $I = [a, b)$. The operators $\chi(T < a)$ and $\chi(T \geq b)$ are defined similarly. Using the series $\{r_j\}_{j=0}^\infty$ with $r_{-1} = 0$, $r_j = 2^j$ ($j \geq 0$), we define the Besov space $\mathcal{B}$ by

$$\mathcal{B} = \left\{ f \in \mathcal{H} ; \|f\|_{\mathcal{B}} = \sum_{j=0}^\infty r_j^{1/2} \| \chi(r_{j-1} \leq |N| < r_j) f \| < \infty \right\}.$$
Its dual space $B^*$ is the completion of $\mathcal{H}$ by the following norm

$$\|u\|_{B^*} = \sup_{j \geq 0} 2^{-j/2} \|\chi(r_{j-1} \leq |N| < r_j)u\|.$$  

The following Lemma 3.3 is proved in the same way as in \cite{2}.

**Lemma 3.3.** (1) There exists a constant $C > 0$ such that

$$C^{-1}\|u\|_{B^*} \leq \left(\sup_{R>1} \frac{1}{R} \|\chi(|N| < R)u\|^2\right)^{1/2} \leq C\|u\|_{B^*}.$$  

(2) For $s > 1/2$, the following inclusion relations hold:

$$\mathcal{H}^s \subset B \subset \mathcal{H}^{1/2} \subset \mathcal{H} \subset \mathcal{H}^{-1/2} \subset B^* \subset \mathcal{H}^{-s}.$$  

In view of the above lemma, in the following, we use

$$\|u\|_{B^*} = \left(\sup_{R>1} \frac{1}{R} \|\chi(|N| < R)u\|^2\right)^{1/2}$$

as a norm on $B^*$.

We also put $\hat{\mathcal{H}} = \ell^2(\mathbb{Z}^d)$, and define $\hat{\mathcal{H}}^s, \hat{B}, \hat{B}^*$ by replacing $N$ by $\hat{N}$. Note that $\hat{\mathcal{H}}^s = \mathcal{U}^*\hat{\mathcal{H}}^s$ and so on. In particular, Parseval’s formula implies that

$$\|u\|_{\hat{H}^s}^2 = \|\hat{u}\|_{\hat{H}^s}^2 = \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^s |\hat{u}(n)|^2,$$

$$\|u\|_{\hat{B}^s}^2 = \|\hat{u}\|_{\hat{B}^s}^2 = \sup_{R>1} \frac{1}{R} \sum_{|n| < R} |\hat{u}(n)|^2,$$

$\hat{u}(n)$ being the Fourier coefficient of $u(x)$.

4. **Proof of Theorem 1.1**

We extend $\hat{u}(n)$ to be zero for $|n| \leq R_0$ and denote it by $\hat{u}$ again. Then we have

$$\left(\hat{\mathcal{H}}_0 - \lambda\right)\hat{u} = \hat{f},$$

where $\hat{f}$ is compactly supported. In fact, letting $\hat{P}(k)$ be the projection onto the site $k$, it is written as $\hat{f} = \sum_{|k| \leq R_0 + 1} c_k \hat{P}(k)\hat{u}$. We first note the following Lemma.

**Lemma 4.1.** Let $\lambda \in (0, d) \setminus \mathbb{Z}$ and $\hat{u}$ satisfy (1.1) and (1.2). Then $u \in \mathcal{C}^\infty(\mathbb{T}^d)$ and $\hat{f}$ satisfies

$$\left(\mathcal{U}\hat{f}\right)(x) = f(x) = 0 \quad \text{on} \quad M_\lambda.$$  

Proof. Passing to the Fourier series, (1.2) implies

$$\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{T}^d} \chi(|N| < R)u(x)^2 dx = 0.$$  

By Lemma 3.2, this is equivalent to

$$\lim_{R \to \infty} \frac{1}{R} \int_{|\xi| < R} |\hat{\chi}(\xi)|^2 d\xi = 0, \quad \text{for any} \quad j$$
where \( \{\chi_j\} \) is the partition of unity on \( \mathbb{T}^d \), and \( \tilde{v} (\xi) \) is the Fourier transform of \( v \):

\[
\tilde{v} (\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} v(x) dx.
\]

By (4.1), \( u \) satisfies

\[
(4.5) \quad (h(x) - \lambda) u = f \quad \text{on} \quad \mathbb{T}^d,
\]

where \( f \) is a polynomial of \( e^{ix_j}, j = 1, \ldots, d \), since \( \hat{f} \) is compactly supported. Take a point \( x^{(0)} \in M_\lambda \), and let \( \chi \in C^\infty(\mathbb{T}^d) \) be such that \( \chi(x^{(0)}) = 1 \) and \( \text{supp} \chi \) is sufficiently small. Letting \( v(x) = \chi(x) u(x) \), \( g(x) = \chi(x) f(x) \) and making the change of variable \( x \mapsto y \) so that \( y_1 = h(x) - \lambda \), we have by passing to the Fourier transform, \( \frac{\partial}{\partial \eta_1} \tilde{v} (\eta) = -i \tilde{g} (\eta) \). Integrating this equation, we have

\[
\tilde{v} (\eta) = -i \int_0^{\eta_1} \tilde{g} (s, \eta') ds + \tilde{v} (0, \eta').
\]

Since \( \tilde{g} (\eta) \) is rapidly decreasing, we then see the existence of the limit

\[
\lim_{\eta_1 \to \infty} \tilde{v} (\eta) = -i \int_0^{\infty} \tilde{g} (s, \eta') ds + \tilde{v} (0, \eta').
\]

We show that this limit vanishes. Let \( D_R \) be the slab such that

\[
D_R = \left\{ \eta : |\eta'| < \delta R, \quad \frac{R}{3} < \eta_1 < \frac{2R}{3} \right\}.
\]

Then we have \( D_R \subset \{ |\eta| < R \} \) for a sufficiently small \( \delta > 0 \). We then see that

\[
\frac{1}{R} \int_{D_R} |\tilde{v} (\eta)|^2 d\eta = \frac{1}{R} \int_{|\eta'|<\delta R} \int_{R/3}^{2R/3} |\tilde{v} (\eta_1, \eta')|^2 d\eta_1 d\eta' \leq \frac{1}{R} \int_{|\eta|<R} |\tilde{v} (\eta)|^2 d\eta.
\]

As \( R \to \infty \), the right-hand side tends to zero by (4.4), hence so does the left-hand side, which proves that \( \lim_{\eta_1 \to \infty} \tilde{v} (\eta) = 0 \).

We have, therefore,

\[
\tilde{v} (\eta) = i \int_{\eta_1}^{\infty} \tilde{g} (s, \eta') ds.
\]

This shows that \( v = \chi u \in C^\infty(\mathbb{T}^d) \).

It is easy to see that \( u \) is smooth outside \( M_\lambda \). Then \( u \in C^\infty(\mathbb{T}^d) \). In particular, \( f|_{M_\lambda} = 0 \) by (4.3), which proves the lemma. \( \square \)

We use the function theory of several complex variables. Let \( \mathbb{T}^d_C = \mathbb{C}^d/(2\pi\mathbb{Z})^d \) be the complex torus and define

\[
M^C_\lambda = \{ z \in \mathbb{T}^d_C : h(z) = \lambda \}.
\]

Then \( M^C_\lambda \cap \mathbb{R}^d = M_\lambda \).

**Lemma 4.2.** For \( \lambda \in (0, d) \setminus \mathbb{Z} \), \( M^C_\lambda \) is a \( (d - 1) \)-dimensional, connected complex submanifold of \( \mathbb{T}^d_C \).
Proof. Since $\nabla h(z) \neq 0$ on $M^C_{\lambda}$, $M^C_{\lambda}$ is a $(d - 1)$-dimensional complex manifold without singularities. Using the change of variables and taking into account the Riemann surface of $\arcsin w$, connectivity of $M^C_{\lambda}$ can be proven from the fact that the manifold $w^2_1 + \cdots + w^2_d = \mu$, $\mu > 0$, is connected in $C^d$.

This lemma then implies

**Lemma 4.3.** Let $\lambda \in (0, d) \setminus Z$. If $f$ is analytic on $T^d_C$, and $f = 0$ on $M_{\lambda}$, then $f = 0$ on $M^C_{\lambda}$.

Let us return to the equation (4.7). Since $f$ is a polynomial of $e^{i\varepsilon_j}$, $f(x) = \sum c_{\alpha} e^{i\alpha \cdot x}$, it is analytic on $T^d_C$. By virtue of Lemma 4.2 and 4.3, we then have

(4.7) $f(z) = 0$ on $M^C_{\lambda}$.

Since $(\partial_{z_1} h(z), \cdots, \partial_{z_d} h(z)) \neq 0$ on $M^C_{\lambda}$, taking $h(z) - \lambda$ as a new variable and using the Taylor expansion, we see that there exists a holomorphic function $g$ such that $f(z) = (h(z) - \lambda)g(z)$, hence $f(z)/(h(z) - \lambda)$ is an entire function of $z \in C^d$.

Here we pass to the variables $w_j = e^{\varepsilon_j}$, $j = 1, \cdots, d$. Note that the map $T^d_C \ni z \mapsto w \in C^d \setminus \bigcup_{j=1}^d A_j$, $A_j = \{w \in C^d \mid w_j = 0\}$, is biholomorphic. Then there exist positive integers $\alpha_j$ such that

$$f(z) = \sum c_{\beta} w^\beta = F(w) \prod_{j=1}^d w_j^{-\alpha_j},$$

where $F(w)$ is a polynomial, $F(w) = \sum a_{\gamma} w^\gamma$, with the property that

$$a_\gamma = 0,$$

if one of $\gamma_j \leq 0$ in $\gamma = (\gamma_1, \cdots, \gamma_d)$.

We factorize $h(z) - \lambda$ as

$$h(z) - \lambda = \frac{d}{2} - \lambda - \frac{1}{4} \sum_{j=1}^d (w_j + w_j^{-1}) = H_\lambda(w) \prod_{j=1}^d w_j^{-1},$$

where

(4.8) $H_\lambda(w) = \left(\frac{d}{2} - \lambda\right) \prod_{j=1}^d w_j - \frac{1}{4} \sum_{j=1}^d w_j \prod_{j=1}^d w_j - \frac{1}{4} \sum_{i \neq j} \prod_{j=1}^d w_i$.

Then

(4.9) $\frac{f(z)}{h(z) - \lambda} = \frac{F(w)}{H_\lambda(w)} \prod_{j=1}^d w_j^{1-\alpha_j}$.

Since $f(z)/(h(z) - \lambda)$ is analytic, $F(w)/H_\lambda(w)$ is also analytic except possibly on hyperplanes $A_j$, $j = 1, \cdots, d$. However, due to the expression (4.8), we have

$$H_\lambda(w) \neq 0,$$

if $w \in \bigcup_{k=1}^d V_k$, $V_k = \{(w_1, \cdots, w_{k-1}, 0, w_{k+1}, \cdots, w_d) \mid w_i \neq 0, i \neq k\}$.
Hence $F(w)/H_{\lambda}(w)$ is analytic except only on some sets of complex dimension $d-2$ (the intersection of two hyperplanes). Therefore,

- $F(w)/H_{\lambda}(w)$ is an entire function.

See e.g. Corollary 7.3.2 of [15]. In particular,

- $F(w) = 0$ on the set $\{w \in \mathbb{C}^d : H_{\lambda}(w) = 0\}$.

Finally, we use the following fact, a corollary of the Hilbert Nullstellensatz (See e.g. Appendix 6 of [24]). Let $C[w_1, \cdots, w_d]$ be the ring of polynomials of variables $w_1, \cdots, w_d$.

**Lemma 4.4.** If $f, g \in C[w_1, \cdots, w_d]$, and suppose that $f$ is irreducible. If $g = 0$ on all zeros of $f$, there exists $h \in C[w_1, \cdots, w_d]$ such that $g = fh$.

**Proof of Theorem 1.1, the final step.** We factorize $H_{\lambda}(w)$ so that

$$H_{\lambda}(w) = H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(N)}(w),$$

where each $H_{\lambda}^{(j)}(w)$ is an irreducible polynomial. We prove inductively that

$$F(w)/H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(k)}(w)$$

is a polynomial for $1 \leq k \leq N$.

Note that, since we know already that $F(w)/H_{\lambda}(w)$ is entire,

- $F(w)/H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(k)}(w)$ is also entire,
- $F(w) = 0$ on the zeros of $H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(k)}(w)$.

Consider the case $k = 1$. Since $F(w) = 0$ on the zeros of $H_{\lambda}^{(1)}(w)$, Lemma 4.4 implies that $F(w)/H_{\lambda}^{(1)}(w)$ is a polynomial.

Assuming the case $k \leq \ell - 1$, we consider the case $k = \ell$. By the induction hypothesis, there exists a polynomial $P_{\ell-1}(w)$ such that

$$\frac{F(w)}{H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(\ell-1)}(w)} = P_{\ell-1}(w).$$

Then we have $F(w)/(H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(\ell)}(w)) = P_{\ell-1}(w)/H_{\lambda}^{(\ell)}(w)$. This is entire. Therefore, $P_{\ell-1}(w) = 0$ on the zeros of $H_{\lambda}^{(\ell)}(w)$. By Lemma 4.4 there exists a polynomial $Q_{\ell}(w)$ such that

$$\frac{P_{\ell-1}(w)}{H_{\lambda}^{(\ell)}(w)} = Q_{\ell}(w).$$

Therefore, $F(w)/H_{\lambda}^{(1)}(w) \cdots H_{\lambda}^{(k)}(w)$ is a polynomial for $1 \leq k \leq N$. Taking $k = N$, we have that $F(w)/H_{\lambda}(w)$ is a polynomial of $w$, hence $f(z)/(h(z) - \lambda)$ is a polynomial of $e^{iz}$ by [19]. This implies that $\tilde{u}(n)$ is compactly supported. We have thus completed the proof of Theorem 1.1.
References

[1] S. Agmon, *Lower bounds for solutions of Schrödinger equations*, J. d’Anal. Math., **23** (1970), 1-25.
[2] S. Agmon and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. d’Anal. Math., **30** (1976), 1-38.
[3] J. Dereziński and C. Gérard, *Scattering Theory of Classical and Quantum Mechanical N-Particle Systems*, Springer, Berlin-Heidelberg-New York (1997).
[4] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc., **284** (1984), 787-794.
[5] D. M. Eidus, *The principle of limiting absorption*, Math. Sb. (N.S.) **58** (100) (1962), 65-86.
[6] M. S. Eskina, *The direct and the inverse scattering problem for a partial difference equation*, Soviet Math. Doklady, **7** (1966), 193-197.
[7] C. Gérard and F. Nier, *The Mourre theory for analytically fibred operators*, J. Funct. Anal. **152** (1989), 202-219.
[8] B. Helffer and J. Sjöstrand, *Equation de Schrödinger avec champ magnétique et équation de Harper*, Lecture Notes in Phys. 345, Schrödinger Operators, pp.118-197, eds. H. Holden, A. Jensen, Springer, Berlin-Heidelberg-New York (1989).
[9] L. Hörmander, *Lower bounds at infinity for solutions of differential equations with constant coefficients*, Israel J. Math. **16** (1973), 103-116.
[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators III, Pseudo-Differential Operators*, Springer-Verlag, Berlin Heidelberg New York Tokyo (1994).
[11] T. Ikebe and Y. Saito, *Limiting absorption method and absolute continuity for the Schrödinger operator*, J. Math. Kyoto Univ., **12** (1972), 513-542.
[12] H. Isozaki and E. Korotyaev, *Inverse problems, trace formulae for discrete Schrödinger operators*, Ann. Henri Poincaré, **13** (2012), 751-788.
[13] H. Isozaki and H. Morioka, *Inverse scattering at a fixed energy for discrete Schrödinger operators on the square lattice*, preprint.
[14] T. Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, CPAM, **12** (1959), 403-425.
[15] S. G. Krantz, *"Function Theory of Several Complex Variables"*, John Wiley and Sons Inc., 1982.
[16] P. Kuchment and B. Vainberg, *On absence of embedded eigenvalues for Schrödinger operators with perturbed periodic potentials*, Comm. PDE, **25** (2000), 1809-1826.
[17] W. Littman, *Decay at infinity of solutions to partial differential equations with constant coefficients*, Trans. Amer. Math. Soc., **123** (1966), 449-459.
[18] W. Littman, *Decay at infinity of solutions to partial differential equations*, Israel J. Math., **8** (1970), 403-407.
[19] M. Murata, *Asymptotic behaviors at infinity of solutions to certain partial differential equations*, J. Fac. Sci. Univ. Tokyo Sec. IA, **23** (1976), 107-148.
[20] F. Rellich, *Über das asymptotische Verhalten der Lösungen von Δu + λu = 0 in unendlichen Gebieten*, Jahresber. Deutch. Math. Verein., **53** (1943), 57-65.
[21] D. Robert, *Autour d’approximation semi-classique*, Birkhäuser, (1987).
[22] S. N. Roze, *On the spectrum of a second-order elliptic operator*, Math. Sb., **80** (122), (1969), 195-209.
[23] W. Shaban and B. Vainberg, *Radiation conditions for the difference Schrödinger operators*, Applicable Analysis, **80** (2001), 525-556.
[24] I. R. Shafarevich, *"Basic Algebraic Geometry 1", 2nd edition*, Springer-Verlag, Heidelberg, 1977.
[25] F. Treves, *Differential polynomials and decay at infinity*, Bull. Amer. Math. Soc., **66** (1960), 184-186.

Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, JAPAN, isoza-kih@math.tsukuba.ac.jp, hmorioka@math.tsukuba.ac.jp