BRAIDED QUANTUM SYMMETRIES OF GRAPH C$^*$-ALGEBRAS

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ABSTRACT. We prove the existence of a universal braided compact quantum group acting on a graph C$^*$-algebra in the category of T-C$^*$-algebras with a twisted monoidal structure, in the spirit of the seminal work of S. Wang. To achieve this, we construct a braided analogue of the free unitary quantum group and study its bosonization. As a concrete example, we compute this universal braided compact quantum group for the Cuntz algebra.

1. Introduction

Since the work of Manin in [Man88], the concept of quantum symmetries of a space, in both classical and noncommutative sense, has been (and still being) thoroughly investigated and clarified in a number of works. It is shown in [Man88] that SL$_q(2, \mathbb{C})$, one of the very first examples of a quantum group ([Dri87]) in Drinfeld’s sense, i.e., a noncommutative and noncocommutative Hopf algebra, appears as a quantum symmetry group of a noncommutative space. Woronowicz’s theory of compact matrix pseudogroups ([Wor87a]), describing a compact topological group (of matrices) in a purely C$^*$-algebraic language, marked a new beginning which is also better suited to Connes’ approach to noncommutative geometry ([Con94]).

A landmark example ([Wor87b]) constructed by Woronowicz is the compact quantum group SU$_q(2)$ for each $0 < q \leq 1$. Thus for each $q \in (0, 1]$, one has a unital C$^*$-algebra C(SU$_q(2)$) and a unital $^*$-homomorphism $\Delta_{SU_q(2)}$ from C(SU$_q(2)$) to C(SU$_q(2)$) $\otimes$ C(SU$_q(2)$) satisfying some natural conditions. Setting $q = 1$, one recovers the algebra of continuous functions on the compact group SU(2) and the morphism induced by the group-multiplication. The discovery of SU$_q(2)$ together with the dream of making contact with Connes’ enterprise, resulted, following Wang’s pioneering work on quantum symmetries of finite spaces ([Wan98]), in several thematically entwined constructions and insights. We mention, as a necessarily incomplete sampling,

1. Banica, Bichon and collaborators on quantum symmetries of discrete structures, see [Ban05a, Ban05b, Bic03];
2. Goswami, Bhowmick and collaborators on quantum isometries of spectral triples, see [BG10, Gos20, GJ18, BG19, BG09, BBG21];
3. Banica, Skalski and collaborators on quantum symmetries of C$^*$-algebras equipped with orthogonal filtrations, see [BS13, BMRS19, TdC14];
4. and more recently, Goswami and collaborators on quantum symmetries of subfactors, see [BCG22, HG21].

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The deformation parameter $q$ in the definition of $C(SU_q(2))$ can be relaxed so as to require it to be a nonzero real number $q \in \mathbb{R}^\times$. Letting $q$ to be any nonzero complex number $q \in \mathbb{C} \setminus \mathbb{R}$, other than the reals, unlocks a plethora of interesting phenomena; for starter, the comultiplication $\Delta_{SU_q(2)}$ does not take values in the minimal tensor product $C(SU_q(2)) \otimes C(SU_q(2))$ anymore. What the definition of a compact quantum group misses in this case is a fine structure of the underlying $C^*$-algebra $C(SU_q(2))$; it is the hidden $\mathbb{T}$-structure on $C(SU_q(2))$, plus a twisting by the nontrivial bicharacter on $\mathbb{Z}$ governed by the unit complex number $\zeta = q/\bar{q}$, which is 1, of course, when $q$ is real. It turns out that the receptacle of the morphism $\Delta_{SU_q(2)}$ in this case is the braided tensor product for the $\mathbb{T}$-structure mentioned just above, in the sense of [MRW14, MRW16, Roy22], which, as expected, becomes the minimal tensor product when $q$ is real.

Braided tensor product of two $C^*$-algebras were systematically studied by Meyer, the third author and Woronowicz in [MRW14, MRW16] (though there are predecessors, for instance [Vae05]) to accommodate several existing tensor products and crossed product-like constructions. For a quasitriangular quantum group $G$, which is a suitable generalization of the algebraic case, this braided tensor product is then used to introduce a monoidal structure $\boxtimes$ on the category of $G$-$C^*$-algebras. With this in hand, a braided compact quantum group is then defined ([MRW16]) exactly as an ordinary compact quantum group, the only difference being that the minimal tensor product $\otimes$ is replaced by the braided tensor product $\boxtimes$. $SU_q(2)$ for a complex deformation parameter $q$ is a braided compact quantum group ([KMRW16]) in this sense with the quasitriangular quantum group $G$ being the circle group $\mathbb{T}$ and so is the braided free orthogonal quantum group, constructed in [MR22].

The next natural step, that of viewing these braided compact quantum groups as symmetry objects of suitable spaces, is explored by the third author in [Roy21] for finite spaces, obtaining further examples of braided compact quantum groups (see also [Sol20]). Our aim in this paper is to continue along this line. The class of (noncommutative) spaces considered in this paper is that of graph $C^*$-algebras. The relatively recent study of quantum symmetries of this well-studied class of $C^*$-algebras were taken up by the second author and Mandal in [JM18, JM21]. A graph $C^*$-algebra carries a natural generalized gauge action and a canonical equivariant state which is KMS (see, for instance [BR97]), under some mild restrictions on the underlying graph. These two structures are crucial to our existence result, which is otherwise an extension of the one obtained in [JM21]. However, there are several technical and conceptual difficulties that are to be overcome. We defer introducing the relevant notations and definitions, but state the main result now.

**Theorem 1.1.** Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks such that the KMS state exists. Then there is a universal braided compact quantum group $Qaut(C^*(E))$ acting linearly, faithfully on $C^*(E)$ and preserving the KMS state.

In the ordinary compact quantum group case, the path followed in [JM21] to reach this result uses the existence of the free unitary quantum group $U^+(F)$, $(F \in \text{GL}(n, \mathbb{C}))$, a braided version of which is missing hitherto. We recall that the free unitary quantum group is defined by only demanding the fundamental representation to be unitary and its conjugate to be equivalent to a unitary representation. The first obstacle in defining a braided analogue of $U^+(F)$ is the definition of the conjugate representation. We achieve this in Definition 2.14. Then following [MR22] closely,
we are able to construct the required braided analogue of the free unitary quantum group.

**Theorem 1.2.** The braided free unitary quantum group, denoted $U^\pm_\zeta(F)$, exists for any $\zeta \in T$ and satisfies the natural universal property.

The bosonization construction, introduced in [MRW16], based on the algebraic Radford bosonization (which in turn is based on the semidirect product construction for groups) provides an equivalence between braided compact quantum groups and a class of ordinary quantum groups (those with an idempotent quantum group morphism). We obtain an explicit description of the bosonization of the braided free unitary quantum group, again following [MR22].

**Theorem 1.3.** The bosonization $U^\pm_\zeta(F) \rtimes T$ of $U^\pm_\zeta(F)$ is a compact quantum group such that $C(U^\pm_\zeta(F) \rtimes T)$ is the crossed product of $C(U^\pm_\zeta(F))$ with $\mathbb{Z}$, the $\mathbb{Z}$-action being given by the automorphism induced by $\zeta^{-1}$.

This theorem is in turn used in the main existence theorem stated above. We continue by explicitly computing $\text{Qaut}(C^*(E))$ when $C^*(E)$ is the Cuntz algebra $O_n$; in this case $\text{Qaut}(C^*(E))$ turns out to be $U^+_\zeta(I_n)$.

**Theorem 1.4.** The braided quantum symmetry group for the Cuntz algebra $O_n$ is the free unitary quantum group $U^+_\zeta(I_n)$, $I_n$ being the $n \times n$ identity matrix.

The demonstration of the above theorem is naturally divided into two parts; the first part shows that $U^+_\zeta(I_n)$ is indeed a candidate to be the braided quantum symmetry group (in technical terms it is an object of a suitable category) and the second part shows its universality. We remark that the proof of the first part (Proposition 4.22) is an improvement over the proof as given in [JM18, JM21] in the unbraided case.

We note that although the braided free unitary quantum group is constructed out of necessity, it is, nevertheless, interesting in its own right. To not interrupt the flow of the article, we have pushed to an Appendix a very brief discussion of the representation theory of the braided free unitary quantum group. More precisely, we have described the irreducible representations and the fusion rule of the bosonization $U^+_\zeta(F) \rtimes T$ for a diagonal $F$. As is known from [MR22], the representation category of $U^+_\zeta(F)$ is equivalent to that of the bosonization $U^+_\zeta(F) \rtimes T$, thus yielding complete knowledge of the representation category for such a diagonal $F$, and in particular, for $U^+_\zeta(I_n)$.

**Theorem 1.5.** Let $F \in \text{GL}(n, \mathbb{C})$ be diagonal and admissible. Then the braided free unitary quantum group $U^+_\zeta(F)$ has irreducible representations $r_{(x,x')}$, $x \in \mathbb{Z}$, $x' \in \mathbb{N} \times \mathbb{N}$ such that any irreducible representation is unitarily equivalent to exactly one of these and moreover they satisfy the fusion rule

$$r_{(x,x')} \boxtimes r_{(y,y')} \cong \bigoplus_{\{a,b \in \mathbb{N} | x'=ag, y'=\bar{gb}\}} r_{(x+y, ab)}.$$

We end this Introduction by describing the organization of this paper. In Section 2 we begin with recalling the definition of a braided quantum group. We then define the conjugate of a representation (Definition 2.14) and obtain the main results of this section, i.e., the construction of the free unitary quantum group (Definition 2.19). We then state its universal property (Theorem 2.21). We note that we have tried to
be as self-contained as possible in defining the braided tensor product and related constructions, partly owing to the length of this section. Section 3 describes the bosonization $U^+_\mathbb{T}(F) \rtimes \mathbb{T}$ of $U^+_\mathbb{T}(F)$ and proves that it is a compact matrix quantum group (Theorem 3.3 and Corollary 3.5 respectively). In Section 4 we prove the main existence result on the braided quantum symmetry group of the graph $C^*$-algebra $C^*(E)$ (Theorem 4.19) and compute it for the Cuntz algebra $O_n$ (Theorem 4.23). The Appendix A at the end describes the irreducible representations and the fusion rule of the bosonization $U^+_\mathbb{T}(F) \rtimes \mathbb{T}$ for a diagonal $F$.

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Notations. For two $C^*$-algebras $A$ and $B$, $A \otimes B$ denotes the minimal tensor product of $C^*$-algebras. For a $C^*$-algebra $A$ and two closed subspaces $X,Y \subseteq A$, $XY$ denotes the norm-closed linear span of the set of products $xy$, $x \in X$ and $y \in Y$. For an object $X$ in some category, $\text{id}_X$ denotes the identity morphism of $X$. For a unital $C^*$-algebra $A$, $1_A$ denotes the unit element in $A$, and $\mathcal{M}(A)$ denotes the multiplier algebra of $A$.

2. The Braided Free Unitary Quantum Group

In this section, we gather some preliminaries regarding braided compact quantum groups over $\mathbb{T}$ (see [MRW14,MRW16,KMRW16]), presented in a way that naturally leads us to the definition of the braided free unitary quantum group.

Let $\mathcal{C}^*$ be the category of $C^*$-algebras. For $A$ and $B$ in $\text{Obj}(\mathcal{C}^*)$, we write the set of morphisms as $\text{Mor}(A,B)$ which consists of $*$-homomorphisms $\pi : A \to \mathcal{M}(B)$ such that $\pi(A)B = B$, where $\mathcal{M}(B)$ is the multiplier algebra of $B$. Thus for unital $A$ and $B$, $\text{Mor}(A,B)$ consists of unital $*$-homomorphisms from $A$ to $B$. We recall that a compact quantum group $G$ consists of a pair $G = (C(G), \Delta_G)$ where $C(G)$ is a unital $C^*$-algebra and $\Delta_G : C(G) \to C(G) \otimes C(G)$ is a coassociative morphism satisfying a cancellation property. We also recall that the comultiplication $\Delta_T : C(\mathbb{T}) \to C(\mathbb{T}) \otimes C(\mathbb{T})$ of the compact quantum group $C(\mathbb{T})$ sends $z$ to $z \otimes z$.

Definition 2.1. We define the category $\mathcal{C}^*_\mathbb{T}$ of $\mathbb{T}$-$C^*$-algebras and $\mathbb{T}$-equivariant morphisms as follows. An object of $\mathcal{C}^*_\mathbb{T}$ is a pair $(X, \rho^X)$, where $X$ is a unital $C^*$-algebra and $\rho^X \in \text{Mor}(X, X \otimes C(\mathbb{T}))$ such that

1. $(\rho^X \otimes \text{id}_{C(\mathbb{T})}) \circ \rho^X = (\text{id}_X \otimes \Delta_T) \circ \rho^X$;
2. $\rho^X(X)(1_X \otimes C(\mathbb{T})) = X \otimes C(\mathbb{T})$.

Let $(X, \rho^X)$ and $(Y, \rho^Y)$ be two $\mathbb{T}$-$C^*$-algebras. A morphism $\phi : (X, \rho^X) \to (Y, \rho^Y)$ in $\mathcal{C}^*_\mathbb{T}$ (or equivalently, a $\mathbb{T}$-equivariant morphism) is by definition a $\phi \in \text{Mor}(X, Y)$ such that $\rho^Y \circ \phi = (\phi \otimes \text{id}_{C(\mathbb{T})}) \circ \rho^X$. We write $\text{Mor}^\mathbb{T}(X,Y)$ for the set of morphisms between $(X, \rho^X)$ and $(Y, \rho^Y)$ in $\mathcal{C}^*_\mathbb{T}$.
Remark 2.2. A T-C*-algebra $X$ (with $\rho^X$ understood) comes with an associated $\mathbb{Z}$-grading defined as follows. We call an element $x \in X$ homogeneous of degree $n \in \mathbb{Z}$ if $\rho^X(x) = x \otimes x^n$ and write $\deg(x) = n$. For each $n \in \mathbb{Z}$, we let $X(n)$ denote the set consisting of homogeneous elements of degree $n$: $X(n) = \{ x \in X \mid \deg(x) = n \}$. The collection $\{ X(n) \}_{n \in \mathbb{Z}}$ enjoys the following:

1. for each $n \in \mathbb{Z}$, $X(n)$ is a closed subspace of $X$;
2. for $m, n \in \mathbb{Z}$, $X(m)X(n) \subset X(m + n)$;
3. for each $n \in \mathbb{Z}$, $X(n)^* = X(-n)$;
4. the algebraic direct sum $\bigoplus_{n \in \mathbb{Z}} X(n)$ is norm-dense $X$.

Roughly speaking, a braided compact quantum group over $\mathbb{T}$ is a “compact quantum group” object in $C^*_\mathbb{T}$, which is endowed with a monoidal structure using a braided tensor product $\otimes_\zeta$ depending on a parameter $\zeta \in \mathbb{T}$. Perhaps the simplest way to define $\otimes_\zeta$ is to use the noncommutative torus as described in [KMRW16], which we follow closely.

Let $(X, \rho^X)$ and $(Y, \rho^Y)$ be two objects of $C^*_\mathbb{T}$. Let $\zeta \in \mathbb{T}$ and let $C(\mathbb{T}^2_\zeta)$ denote the unital C*-algebra generated by two unitaries $U$ and $V$, subject to the relation $VU = \zeta UV$. Fixing $\zeta \in \mathbb{T}$ amounts to fixing an R-matrix, in the sense of [MRW16].

Lemma 2.3. There exist unique morphisms $\iota_1, \iota_2 \in \text{Mor}(C(\mathbb{T}), C(\mathbb{T}^2_\zeta))$ such that $\iota_1(z) = U$ and $\iota_2(z) = V$.

Definition 2.4. [KMRW16] We define $j_1 \in \text{Mor}(X, X \otimes Y \otimes C(\mathbb{T}^2_\zeta))$ and $j_2 \in \text{Mor}(Y, X \otimes Y \otimes C(\mathbb{T}^2_\zeta))$ as the composites

$$X \xrightarrow{\rho^X} X \otimes C(\mathbb{T}) \xleftarrow{\text{embedding}} X \otimes Y \otimes C(\mathbb{T}) \xrightarrow{id_X \otimes id_Y \otimes \iota_1} X \otimes Y \otimes C(\mathbb{T}^2_\zeta),$$

and

$$Y \xrightarrow{\rho^Y} Y \otimes C(\mathbb{T}) \xleftarrow{\text{embedding}} X \otimes Y \otimes C(\mathbb{T}) \xrightarrow{id_X \otimes id_Y \otimes \iota_2} X \otimes Y \otimes C(\mathbb{T}^2_\zeta),$$

respectively.

Thus, for $x \in X(k)$, $j_1(x) = x \otimes 1_Y \otimes U^k$ and for $y \in Y(l)$, $j_2(y) = 1_X \otimes y \otimes V^l$. It follows that for $x \in X(k)$ and $y \in Y(l)$,

$$j_2(y)j_1(x) = \zeta^{lk}j_1(x)j_2(y)$$

and therefore $j_2(Y)j_1(X) = j_1(X)j_2(Y)$, implying that $j_1(X)j_2(Y)$ is a C*-algebra.

Definition 2.5. [KMRW16] The braided tensor product $X \otimes_\zeta Y$ is defined to be the C*-algebra $j_1(X)j_2(Y)$, i.e., $X \otimes_\zeta Y := j_1(X)j_2(Y)$.

By abuse of notation, we write $j_1 \in \text{Mor}(X, X \otimes_\zeta Y)$ and $j_2 \in \text{Mor}(Y, X \otimes_\zeta Y)$. We sometimes write $x \otimes_\zeta 1_Y$ for $j_1(x)$ (and similarly, $1_X \otimes_\zeta y = j_2(y)$), so that for homogeneous $x$ and $y$,

$$(x \otimes_\zeta 1_Y)(1_X \otimes_\zeta y) = j_1(x)j_2(y) = \zeta^{-\deg(x)\deg(y)}j_2(y)j_1(x)$$

$$= \zeta^{-\deg(x)\deg(y)}(1_X \otimes_\zeta y)(x \otimes_\zeta 1_Y).$$

Lemma 2.6. There is a unique coaction $\rho^{X \otimes_\zeta Y}$ of $C(\mathbb{T})$ on $X \otimes_\zeta Y$ such that $j_1 \in \text{Mor}(X, X \otimes_\zeta Y)$ and $j_2 \in \text{Mor}(Y, X \otimes_\zeta Y)$.
At the algebraic level, if \( x \in X(k) \) and \( y \in Y(l) \) then \( j_1(x)j_2(y) \in (X \boxtimes Y)(k+l) \). Throughout the paper, \( X \boxtimes Y \) is equipped with this \( C(T) \)-coaction and thus becomes an object of \( C^*_T \).

Finally, suppose we are given two \( T \)-equivariant morphisms \( \pi_1 \in \text{Mor}^T(X_1, Y_1) \) and \( \pi_2 \in \text{Mor}^T(X_2, Y_2) \).

**Lemma 2.7.** [KMRW16] There is a unique \( T \)-equivariant morphism \( \pi_1 \boxtimes \pi_2 \in \text{Mor}^T(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \) such that \((\pi_1 \boxtimes \pi_2)(j_1(x_1)j_2(x_2)) = j_1(\pi_1(x_1))j_2(\pi_2(x_2))\), for \( x_1 \in X_1 \) and \( x_2 \in X_2 \).

Now we can define a braided compact quantum group.

**Definition 2.8.** [MRW16] A braided compact quantum group (over \( T \)) \( G \) is a triple \( G = (C(G), \rho^{C(G)}, \Delta_G) \), where \( C(G) \) is a unital \( C^* \)-algebra, \( \rho^{C(G)} \) is a \( C(T) \)-coaction on \( C(G) \) so that \( (C(G), \rho^{C(G)}) \) is an object of \( C^*_T \), \( \Delta_G \) is a \( T \)-equivariant morphism \( \Delta_G \in \text{Mor}^T(C(G), C(G) \boxtimes C(G)) \) such that

1. \( (\Delta_G \boxtimes \text{id}_{C(G)}) \circ \Delta_G = (\text{id}_{C(G)} \boxtimes \Delta_G) \circ \Delta_G \) (coassociativity);
2. \( \Delta_G(\rho^{C(G)}(x)j_1(C(G))) = \Delta_G(\rho^{C(G)}(x))j_1(C(G)) = C(G) \boxtimes C(G) \) (bisetifiability).

Before defining a representation, we need one more notation. Let \((A, \rho^A) \in \text{Obj}(C^*_T)\) be a \( T \)-\( C^* \)-algebra. For \( a \in A \) and \( z \in T \), we write the value of the map \( \rho^A(a) \in \text{id}_C(T) \equiv C(T, A) \) at \( z \) as \( \rho^A_z(a) \), i.e., \( \rho^A_z(a) = \rho^A_z(a) \in A \).

**Definition 2.9.** Let \( H \) be a Hilbert space equipped with a (strongly continuous) unitary representation of \( T, U : T \to U(H) \). We define \( \rho^K_{\mathcal{H}} : K(H) \to K(H) \otimes C(T) \) as follows: for \( z \in T \) and \( x \in K(H), \rho^K_{\mathcal{H}}(x) = U(z)xU(z)^* \), making \((K(H), \rho^K_{\mathcal{H}})\) a \( T \)-\( C^* \)-algebra.

**Remark 2.10.** It is shown in [MRW16] that for a \( T \)-\( C^* \)-algebra \((A, \rho^A)\), the braided tensor product \( K(H) \boxtimes_A C \) can be identified with the minimal tensor product \( K(H) \otimes C \) and we identify them henceforth.

**Definition 2.11.** [MRW16] Let \( G = (C(G), \rho^{C(G)}, \Delta_G) \) be a braided compact quantum group. A (unitary) representation of \( G \) is a triple \((\mathcal{H}, U, \rho^U)\), where \( \mathcal{H} \) is a Hilbert space, \( U \) is a strongly continuous unitary representation of \( T \) on \( \mathcal{H} \) and \( u \in M(K(H) \otimes C(G)) \) is a (unitary) element such that

1. for each \( z \in T \), \( \rho^U(z \otimes \rho^{C(G)})(u) = u \), i.e., \( U \) is \( T \)-invariant;
2. \( \text{id}_H \otimes \Delta_G(u) = (\text{id}_H \otimes j_1)(u)(\text{id}_H \otimes j_2)(u) \).

Now we focus on the case when \( \mathcal{H} \) is finite dimensional. This is already described in [MR22] but we repeat the discussion for the sake of readability. So let \((\mathcal{H}, U, \rho^U)\) be a unitary representation of the braided compact quantum group \( G = (C(G), \rho^{C(G)}, \Delta_G) \) with \( \mathcal{H} \) finite dimensional. Let \( \{e_1, \ldots, e_n\} \) be an ordered basis of \( \mathcal{H} \) such that for each \( z \in T \), and for each \( i = 1, \ldots, n, U(z)(e_i) = z^{d_i}e_i \), where \( (d_1, \ldots, d_n) \in \mathbb{Z}^n \) with \( d_1 \leq \cdots \leq d_n \). We identify \( M(K(H) \otimes C(G)) \) with \( L(H) \otimes C(G) \). Let \( \{e_{ij}\}_{1 \leq i, j \leq n} \) be the “matrix units” of \( L(H) \), i.e., \( e_{ij}(e_k) = \delta_{jk}e_i \) for \( 1 \leq i, j, k \leq n \). Thus an element \( u \in M(K(H) \otimes C(G)) \) may be written as \( u = \sum_{1 \leq i, j \leq n} e_{ij} \otimes u_{ij} \), with \( u_{ij} \in C(G) \) or alternately as a matrix \( (u_{ij})_{1 \leq i, j \leq n} \). From this we also see that \( u \in M(K(H) \otimes C(G)) \) is a unitary if and only if the matrix \( (u_{ij})_{1 \leq i, j \leq n} \) is unitary.
in the ordinary sense, i.e.,
\[
(2.2) \quad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^{n} u_{ik} u_{jk}^*,
\]
for \(1 \leq i, j \leq n\). The induced coaction \(\rho^K(\mathcal{H})\) on \(\mathcal{K}(\mathcal{H}) = \mathcal{L}(\mathcal{H})\) is then given by \(\rho^K(\mathcal{H})(e_{ij}) = z^{d_i-d_j} e_{ij}, z \in \mathbb{T}\). Then \(u \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(G))\) is \(\mathbb{T}\)-invariant if and only if \(\rho^C(G)\) satisfies
\[
(2.3) \quad \rho^C(G)(u_{ij}) = z^{d_j-d_i} u_{ij},
\]
for \(z \in \mathbb{T}\), i.e., each \(u_{ij}\) is homogeneous of degree \(d_j - d_i\). And finally, \((\text{id}_H \otimes \Delta_G)(u) = (\text{id}_H \otimes j_1)(u)(\text{id}_H \otimes j_2)(u)\) translates into
\[
(2.4) \quad \Delta_G(u_{ij}) = \sum_{k=1}^{n} j_1(u_{ik}) j_2(u_{kj}).
\]

Summarizing, we record the following.

**Proposition 2.12.** Let \(G = (C(G), \rho^C(G), \Delta_G)\) be a braided compact quantum group and \((\mathcal{H}, U)\) be a pair consisting of a finite dimensional Hilbert space \(\mathcal{H}\) and a strongly continuous unitary representation \(U\) of \(\mathbb{T}\) on \(\mathcal{H}\). Then a matrix \(u = (u_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(G)) = \mathcal{L}(\mathcal{H}) \otimes C(G)\) is a unitary representation of \(G\) if and only if

1. \(\sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^{n} u_{ik} u_{jk}^*, \) for \(1 \leq i, j \leq n\);
2. \(\rho^C(G)(u_{ij}) = z^{d_j-d_i} u_{ij}, \) for \(1 \leq i, j \leq n\);
3. \(\Delta_G(u_{ij}) = \sum_{k=1}^{n} j_1(u_{ik}) j_2(u_{kj}), \) for \(1 \leq i, j \leq n\).

In the ordinary compact quantum group case, \(u = (u_{ij})_{1 \leq i, j \leq n}\) being a (unitary) representation automatically implies that the matrix conjugate \(\overline{u} = (u_{ij}^*)_{1 \leq i, j \leq n}\) is also (equivalent to) a (unitary) representation. However, in the braided case, this is not so, simply because simple tensors of the form \((a \otimes \zeta b)\) don’t commute:

\[
\Delta_G(u_{ij}^*) = \Delta_G(u_{ij})^* = \left(\sum_{k=1}^{n} j_1(u_{ik}) j_2(u_{kj})\right)^* = \sum_{k=1}^{n} j_2(u_{kj})^* j_1(u_{ik})^* = \sum_{k=1}^{n} \zeta^{(d_k-d_j)-(d_i-d_k)} j_1(u_{ik}^*) j_2(u_{kj}^*) = \sum_{k=1}^{n} j_1(u_{ik}^*) j_2(u_{kj}^*) \neq \sum_{k=1}^{n} j_1(u_{ik}^*) j_2(u_{kj}^*),
\]

unless \(\zeta = 1\); here we use that \(\deg(u_{ij}^*) = -\deg(u_{ij}) = d_i - d_j\). A moment’s thought reveals that

\[
\sum_{k=1}^{n} \zeta^{(d_k-d_j)-(d_i-d_k)} j_1(u_{ik}^*) j_2(u_{kj}^*) = \zeta^{-d_i(d_j-d_i)} \sum_{k=1}^{n} j_1(\zeta^{d_k(d_j-d_k)} u_{ik}) j_2(\zeta^{d_k(d_j-d_k)} u_{kj}),
\]
which implies

\[ \Delta_G(\zeta^{d_i(d_j-d_k)} u^*_{ij}) = \sum_{k=1}^{n} j_1(\zeta^{d_i(d_k-d_j)} u^*_{ik}) j_2(\zeta^{d_k(d_j-d_k)} u^*_{kj}), \]

yielding the following.

**Proposition 2.13.** Let \((\mathcal{H}, U, u)\) be a representation of the braided compact quantum group \(G\) on the Hilbert space \(\mathcal{H}\). Then the triple \((\mathcal{H}, U, u)\) is also a representation of \(G\), where \(\mathcal{H}\) is the conjugate Hilbert space, \(U\) is the conjugate \(T\)-representation on \(\mathcal{H}\) and \(u\) is the matrix \((\zeta^{d_i(d_j-d_k)} u^*_{ij})_{1 \leq i,j \leq n}\).

**Proof.** Only the \(T\)-invariance is left to be shown. For that, we need only observe that in the notation of the discussion following Definition 2.11 \{\(\pi_1, \ldots, \pi_m\}\} is an eigenbasis (for \(\mathcal{U}\)) of \(\mathcal{H}\) such that for each \(z \in T\) and for each \(i = 1, \ldots, n\), \(\mathcal{U}(z)(\pi_i) = \mathcal{U}(z)(e_i) = z^{d_i} e_i = z^{-d_i} \pi_i\). Thus the operator \(\pi^*_j\) that sends \(\pi_j\) to \(\pi_i\) and \(\pi_k\) to 0 for \(k \neq j\) has homogeneous degree \(d_j - d_i\). \(\square\)

**Definition 2.14.** Let \(G = \langle C(G), \rho C(G), \Delta_G \rangle\) be a braided compact quantum group and let \((\mathcal{H}, U, u)\) be a finite dimensional representation of \(G\). The conjugate representation corresponding to \(u\), denoted as \((\mathcal{H}, U, \pi_u)\) (as opposed to the standard \(u\), to emphasize the role of \(\zeta\)) is defined to be the matrix \(\pi_u = (\zeta^{d_i(d_j-d_k)} u^*_{ij})_{1 \leq i,j \leq n}\).

As in the ordinary compact quantum group case, forcing \(\pi_u\) to be equivalent to a unitary representation yields a braided analogue of the free unitary quantum group. To obtain the defining relations, let us still work with a general braided compact quantum group \(G = \langle C(G), \rho C(G), \Delta_G \rangle\) having a unitary representation \((\mathcal{H}, U, u)\) such that \((\mathcal{H}, U, \pi_u)\) is equivalent to a unitary representation, say \((\mathcal{H}', U', u')\). Let \(F \in \mathcal{L}(\mathcal{H}, \mathcal{H}')\) be an equivalence between \(\pi_u\) and \(u'\) of homogeneous degree \(d\) (for the underlying \(T\)-representations). Let \((e_1', \ldots, e_n')\) be an ordered basis of \(\mathcal{H}'\) such that for each \(z \in T\), and for each \(i = 1, \ldots, n\), \(U'(z)(e_i') = z^{d_i'} e_i'\), where \((d_1', \ldots, d_n') \in \mathbb{Z}^n\) with \(d_1' \leq \cdots \leq d_n'\) and assume \(F(\pi_u) = \sum_{i=1}^{n} F_{ij} e_i'\) for each \(z \in T\). Since \(F\) is homogeneous of degree \(d\), \(F(\pi_u)\) is homogeneous of degree \(-d_j + d\) which forces \(F_{ij}\) to be 0 unless \(-d_j + d = d_i'\). Similarly, letting \(F'\) denote the matrix entries for the inverse \(F^{-1}\), we get that \(-d_i' \neq -d + d_i'\) implies \(F'_{ij} = 0\). Now \(u'_{ij}\) is homogeneous of degree \(d_j' - d_i'\) and since \(u' = F \pi_u F^{-1}\), the \(ij\)-th entry \((F \pi_u F^{-1})_{ij}\) should be homogeneous of degree \(d_j' - d_i'\), which we now check. First,

\[ (F \pi_u F^{-1})_{ij} = \sum_{\alpha, \beta} F_{i \alpha}(\pi_u)_{\alpha \beta} F'_{\beta j} = \sum_{\alpha, \beta} F_{i \alpha} \zeta^{d_{\alpha}(d_{\beta}-d_{\alpha})} u^*_{\alpha \beta} F'_{\beta j}, \]

and since we may assume \(-d_\alpha + d = d_\beta - d\) and \(-d_\beta = -d + d_j'\), we have that the right-hand side has degree \(d_\alpha - d_\beta = d_j' - d_i'\), which was to be obtained.

We can now construct the braided free unitary quantum group \(U^*_\mathcal{C}(F)\), proceeding step by step. First we shall build the underlying \(C^*\)-algebra \(C(U^*_1(F))\).

**Definition 2.15.** An \(F \in \text{GL}(n, \mathbb{C})\) is called admissible if there exist \(n\)-tuples \(d = (d_1, \ldots, d_n), d' = (d_1', \ldots, d_n') \in \mathbb{Z}^n\) and \(d \in \mathbb{Z}\) such that

\[ F_{ij} = 0 = F^{ji} \text{ whenever } -d_j + d \neq d_i', \]

where \(F = (F_{ij})_{1 \leq i,j \leq n}\) with inverse \(F^{-1} = (F^{ij})_{1 \leq i,j \leq n}\).
Definition 2.16. For such an admissible \( F \in \text{GL}(n, \mathbb{C}) \), we define \( C(U^+_\varsigma(F)) \) to be
the universal unital \( C^* \)-algebra with generators \( u_{ij} \) for \( 1 \leq i, j \leq n \) subject to the relations that make \( u \) and \( F \bar{u}_\varsigma F^{-1} \) unitaries, where \( u = (u_{ij})_{1 \leq i, j \leq n} \) and \( \bar{u}_\varsigma \) as in Definition 2.14.

To construct \( C(U^+_\varsigma(F)) \), we first observe that \( \|u_{ij}\| \leq 1 \) and the relations are polynomials in \( u_{ij}, u_{ij}^* \) ensuring its existence. Now let \( \mathcal{A}(U^+_\varsigma(F)) \) be the universal unital \( * \)-algebra with generators and relations. Given a \( C^* \)-seminorm \( \|\cdot\| \) on \( \mathcal{A}(U^+_\varsigma(F)) \), we have \( \|u_{ij}\| \leq 1 \) for \( 1 \leq i, j \leq n \), hence there is a largest \( C^* \)-seminorm on \( \mathcal{A}(U^+_\varsigma(F)) \) and \( C(U^+_\varsigma(F)) \) is the completion of \( \mathcal{A}(U^+_\varsigma(F)) \) in this largest \( C^* \)-seminorm. For convenience, we let \( u' = F \bar{u}_\varsigma F^{-1} \), so that \( u'_{ij} = (F \bar{u}_\varsigma F^{-1})_{ij} = \sum_{\alpha, \beta} F_{i\alpha}(\bar{u}_\varsigma)_{\alpha\beta} F_{\beta j} = \sum_{\alpha, \beta} F_{i\alpha} \varsigma^{d_\beta - d_\alpha} u_{\alpha \beta}^* F_{\beta j} \).

Proposition 2.17. There is a unique unital \( * \)-homomorphism
\[
\rho^{C(U^+_\varsigma(F))} : C(U^+_\varsigma(F)) \to C(\mathbb{T}, (C(U^+_\varsigma(F)))
\]
such that \( \rho^{C(U^+_\varsigma(F))}(u_{ij}) = z^{d_j - d_i} u_{ij} \) for \( 1 \leq i, j \leq n \) and \( z \in \mathbb{T} \), satisfying the two conditions in Definition 2.1 making \( (C(U^+_\varsigma(F)), \rho^{C(U^+_\varsigma(F))}) \) a \( \mathbb{T} \)-\( C^* \)-algebra.

Proof. We begin by remarking that, defining \( \rho^{C(U^+_\varsigma(F))}(u_{ij}) = z^{d_j - d_i} u_{ij} \) yields a \( \mathbb{T} \)-action on the free unital \( * \)-algebra with generators \( u_{ij} \). It also follows (see the discussion prior to Definition 2.15) that \( \rho^{C(U^+_\varsigma(F))}(u'_{ij}) = z^{d_j' - d_i'} u'_{ij} \). To conclude the proof, we only need to show that the defining relations (i.e., \( u \) and \( u' \) are unitaries) are homogeneous. But this follows from the following observations (see also Eq. (2.2) following Definition 2.11):
\[
\rho^{C(U^+_\varsigma(F))}(u_{ik}^* u_{kj}) = z^{d_j - d_i} u_{ik}^* u_{kj}, \quad \rho^{C(U^+_\varsigma(F))}(u_{ik} u_{jk}^*) = z^{d_j - d_i} u_{ik} u_{jk}^*,
\]
which together with
\[
\rho^{C(U^+_\varsigma(F))}(u_{ik}^* u_{kj}^*) = z^{d_j' - d_i'} u_{ik}^* u_{kj}^*, \quad \rho^{C(U^+_\varsigma(F))}(u_{ik} u_{jk}) = z^{d_j' - d_i'} u_{ik} u_{jk},
\]
for \( 1 \leq i, j \leq n \). That \( \rho^{C(U^+_\varsigma(F))} \) satisfies the two conditions in Definition 2.1 is an easy check.

Proposition 2.18. There is a unique unital \( * \)-homomorphism
\[
\Delta_{U^+_\varsigma(F)} : C(U^+_\varsigma(F)) \to C(U^+_\varsigma(F)) \otimes C(U^+_\varsigma(F))
\]
such that \( \Delta_{U^+_\varsigma(F)}(u_{ij}) = \sum_{k=1}^n \bar{j}_1(u_{ik}) j_2(u_{kj}) \) for \( 1 \leq i, j \leq n \). Furthermore, \( \Delta_{U^+_\varsigma(F)} \) is \( \mathbb{T} \)-equivariant, coassociative and bisimplifiable (see Definition 2.8).

Proof. Let \( U_{ij} = \sum_{k=1}^n \bar{j}_1(u_{ik}) j_2(u_{kj}) \) for \( 1 \leq i, j \leq n \) and \( U = (U_{ij})_{1 \leq i, j \leq n} \). We remark that each \( U_{ij} \) is homogeneous of degree \( d_j - d_i \). Let \( U' = F U \bar{u}_\varsigma F^{-1} \) and let \( U'_{ij} = \sum_{\alpha, \beta} F_{i\alpha} \varsigma^{d_\beta - d_\alpha} u_{\alpha \beta}^* F_{\beta j} \). Now, on one hand
\[
U'_{ij} = \sum_{\alpha, \beta, \gamma = 1}^n F_{i\alpha} \varsigma^{d_\beta - d_\alpha} j_2(u_{\gamma \beta}^*) j_1(u_{\alpha \gamma}) F_{\beta j}
\]

\[
\Delta _{U^+_\zeta (F)}(u'_{ij}) = \sum_{\alpha, \beta = 1}^{n} F_{i\alpha} \zeta^{d_{\alpha} - d_{\alpha}} (d_{\alpha} - d_{\alpha}) j_{1}(u_{\alpha \gamma}^*) j_{2}(u_{\gamma \beta}^*) F^{\beta j}
\]
and on the other,
\[
\Delta _{U^+_\zeta (F)}(u'_{ij}) = \sum_{\alpha, \beta = 1}^{n} F_{i\alpha} \Delta _{U^+_\zeta (F)}((\bar{\eta}_{\zeta})_{\alpha \beta}) F^{\beta j}
\]
so that \(\Delta _{U^+_\zeta (F)}(u_{ij}') = U_{ij}'\), for a unital \(*\)-homomorphism \(\Delta _{U^+_\zeta (F)}\) that satisfies \(\Delta _{U^+_\zeta (F)}(u_{ij}) = U_{ij}\). Thus, by the universal property, a necessarily unique unital \(*\)-homomorphism \(\Delta _{U^+_\zeta (F)} : C(U^+_\zeta (F)) \to C(U^+_\zeta (F)) \otimes C(U^+_\zeta (F))\) satisfying \(\Delta _{U^+_\zeta (F)}(u_{ij}) = U_{ij}\) exists if and only if \(U\) and \(U'\) are unitaries. Now
\[
\sum_{k=1}^{n} U_{k1} U_{kj} = \sum_{k, \alpha, \beta = 1}^{n} j_{2}(u_{\alpha \gamma}^*) j_{1}(u_{\gamma \beta}^*) j_{1}(u_{k \alpha}) j_{1}(u_{k \beta}) j_{2}(u_{\beta j})
\]
That \( \Delta \) and \( U \) contains the monomials in \( U \) in force for the rest of the argument. To show that \( \Delta \) is unitary (using that \( U' \) is unitary). Therefore, we have constructed a unique unital \(*\)-homomorphism \( \Delta_{U^\xi}(F) : C(U^\xi_+(F)) \to C(U^\xi_+(F)) \boxtimes C(U^\xi_+(F)) \) satisfying \( \Delta_{U^\xi}(F)(u_{ij}) = U_{ij} \) for \( 1 \leq i, j \leq n \).

As remarked above, for \( 1 \leq i, j \leq n, U_{ij} \) is homogeneous of degree \( d_j - d_i \) and so \( \Delta_{U^\xi}(F) \) is \( T \)-equivariant. Also, since both \( (\Delta_{U^\xi}(F) \boxtimes \text{id}_A) \circ \Delta_{U^\xi}(F) \) and \( (\text{id}_A \boxtimes \Delta_{U^\xi}(F)) \circ \Delta_{U^\xi}(F) \) send \( u_{ij} \) to \( \sum_{k,l=1}^n j_1(u_{ik})j_2(u_{kl})j_3(u_{lj}) \), we see that \( \Delta_{U^\xi}(F) \) is coassociative. Here we have written \( A \) for \( C(U^\xi_+(F)) \), which will also be in force for the rest of the argument. To show that \( \Delta_{U^\xi}(F) \) is (right) simplifiable, we consider as usual, \( S = \{ x \in A \mid j_1(x) \in \Delta_{U^\xi}(F)(A)j_2(A) \} \). Next we observe that

\[
\sum_{j=1}^n \Delta_{U^\xi}(F)(u_{ij})j_2(u_{kj}) = \sum_{j,l=1}^n j_1(u_{il})j_2(u_{lj})j_2(u_{kj}) = \sum_{l=1}^n j_1(u_{il})\delta_{kl} = j_1(u_{ik}),
\]

which implies that for \( 1 \leq i, j \leq n, u_{ij} \) and \( u_{*ij} \) belong to \( S \). We now show that \( S \) contains the monomials in \( u_{ij} \) and \( u_{*ij} \) and for this, we reproduce the argument in [MR22]. Let \( x \) and \( y \) be in \( S \), both homogeneous, say of degree \( k \) and \( l \), respectively. Then, using \( j_1(x)j_2(A) = j_2(A)j_1(x) \) and \( j_2(y)j_1(A) = j_1(A)j_2(y) \), we see that

\[
j_1(xy) = j_1(x)j_1(y) \in \Delta_{U^\xi}(F)(A)j_2(A)j_1(y)
= \Delta_{U^\xi}(F)(A)j_1(y)j_2(A)
\subseteq \Delta_{U^\xi}(F)(A)\Delta_{U^\xi}(F)(A)j_2(A)
= \Delta_{U^\xi}(F)(A)j_2(A).
\]

Thus for \( x, y \in S, xy \) belong to \( S \) too, so that \( S \) contains all the monomials in \( u_{ij} \) and \( u_{*ij} \) and so is dense in \( A \). This finishes the argument for right simplifiability. That \( \Delta_{U^\xi}(F) \) is left simplifiable too can be shown in exactly similar manner and so we skip the argument.

**Definition 2.19.** We define the braided free unitary quantum group, denoted \( U^\xi_+(F) \), to be the braided compact quantum group \( (C(U^\xi_+(F)), \rho^{C(U^\xi_+(F))}, \Delta_{U^\xi}(F)) \), constructed above.

**Remarks 2.20.**

1. We remark that the identity matrix \( I_n \in \text{GL}(n, \mathbb{C}) \) (and by the same token, any invertible diagonal matrix) is admissible. For this, we take \( d \in \mathbb{Z}^n \) to be
any $n$-tuple, $d' = -d$ and $d = 0$. A quick observation then yields that $i \neq j$ is necessary for $-d_j \neq -d_i$ to hold. We write $U^n_j(n)$ instead of $U^n_j(I_n)$.

(2) For our purposes, it suffices to take $F$ to be diagonal but we remark that non-diagonal admissible $F$ indeed exists. In fact, a non-diagonal admissible $F$ is necessary to recover the braided free orthogonal quantum group as in \[MR22\]. As one would expect, the braided free orthogonal quantum group may be obtained as a quantum subgroup of $U^n_\omega(F)$, with the extra relation $F\varpi_\zeta F^{-1} = u$. Setting $\omega_{ij} = \zeta^{-dd_i} F_{ji}$, one can show that $F\varpi_\zeta F^{-1} = u$ is equivalent to Eq.(5) of \[MR22\]. As this is not directly related to this article, we omit the details.

The significance of the name is described in the following theorem.

**Theorem 2.21.** Let $G = (C(G),\rho^{C(G)},\Delta_G)$ be a braided compact quantum group and let $(H,V,v)$ be a finite dimensional unitary representation such that $(H,\overline{V},\overline{v})$ is equivalent to a unitary representation. Then there is a unique $T$-equivariant Hopf $*$-homomorphism $\phi : C(U^n_\omega(F)) \to C(G)$ sending $u$ to $v$, for some admissible $F$.

**Proof.** This is really just rephrasing the discussion prior to Definition 2.15. \qed

We end this section with recalling a few more definitions needed in the following sections.

**Definition 2.22.** \[Roy21\] Let $G = (C(G),\rho^{C(G)},\Delta_G)$ be a braided compact quantum group. An action of $G$ (equivalently, a $C(G)$-coaction) on a $\mathbb{T}$-$C^*$-algebra $(B,\rho_B)$ is a $\mathbb{T}$-equivariant morphism $\eta^B \in \text{Mor}^{\mathbb{T}}(B,B \boxtimes_\zeta C(G))$ such that

1. $(\text{id}_B \boxtimes_\zeta \Delta_G) \circ \eta^B = (\eta^B \boxtimes_\zeta \text{id}_{C(G)}) \circ \eta^B$ (coassciativity);
2. $\eta^B(B)(1_B \boxtimes_\zeta C(G)) = B \boxtimes_\zeta C(G)$ (Podleś condition).

**Definition 2.23.** \[Roy21\] Let $(B,\rho_B)$ be a $\mathbb{T}$-$C^*$-algebra equipped with a $G$-action $\eta^B \in \text{Mor}^{\mathbb{T}}(B,B \boxtimes_\zeta C(G))$, where $G = (C(G),\rho^{C(G)},\Delta_G)$ is a braided compact quantum group. A $\mathbb{T}$-equivariant state $f : B \to \mathbb{C}$ on $B$ is one that satisfies

$$(f \otimes \text{id}_{C(G)})\rho_B(b) = f(b)1_{C(G)} \text{ for all } b \in B.$$ 

Such an $f : B \to \mathbb{C}$ is said to be preserved under the $G$-action $\eta^B$ if

$$(f \boxtimes_\zeta \text{id}_{C(G)})\eta^B(b) = f(b)1_{C(G)} \text{ for all } b \in B.$$ 

In all honesty, we have not defined $f \boxtimes_\zeta \text{id}_{C(G)}$ for a non-homomorphism $f$. Nevertheless, one may define it as follows. We recall that $B \boxtimes_\zeta C(G)$ is defined as a sub-$C^*$-algebra of $B \otimes C(G) \otimes C(T_\zeta^2)$ and clearly $f \otimes \text{id}_{C(G)} \otimes \text{id}_{C(T_\zeta^2)}$ is defined. We let $f \boxtimes_\zeta \text{id}_{C(G)}$ to be the restriction of $f \otimes \text{id}_{C(G)} \otimes \text{id}_{C(T_\zeta^2)}$ to $B \boxtimes_\zeta C(G)$.

**Definition 2.24.** \[Roy21\] A $G$-action $\eta^B \in \text{Mor}^{\mathbb{T}}(B,B \boxtimes_\zeta C(G))$ of $G$ on a $\mathbb{T}$-$C^*$-algebra $(B,\rho_B)$ is said to be faithful if the $\ast$-algebra generated by $\{(f \boxtimes_\zeta \text{id}_{C(G)})\eta^B(B) \mid f : B \to \mathbb{C} \text{ a } \mathbb{T}$-equivariant state$\}$ is norm-dense in $C(G)$.

### 3. Bosonization of the Braided Free Unitary Quantum Group

This section describes the bosonization construction which gives an equivalence between the category of braided compact quantum groups and the category of ordinary compact quantum groups together with an idempotent quantum group homomorphism. We shall describe explicitly the bosonization of the braided free
unitary quantum group constructed above, to be used in the sequel. We begin with
recalling a few necessary preliminaries.

**Proposition 3.1.** [MRW16] Let \((X, \rho^X)\) and \((Y, \rho^Y)\) be two \(T\)-C*-algebras. Then there is a unique morphism

\[
\psi^{X,Y} \in \text{Mor}(C(T) \boxtimes_{\zeta} X \boxtimes_{\zeta} Y, (C(T) \boxtimes_{\zeta} X) \otimes (C(T) \boxtimes_{\zeta} Y))
\]

such that

\[
\psi^{X,Y}(j_1(x)) = (j_1 \otimes j_1)\Delta_T(x),
\]

\[
\psi^{X,Y}(j_2(a)) = (j_2 \otimes j_1)(\rho^X(a)),
\]

\[
\psi^{X,Y}(j_3(b)) = 1_{C(T) \boxtimes_{\zeta} X} \otimes j_2(b),
\]

for \(x \in C(T)\), \(a \in X\), and \(b \in Y\).

With \(\psi\) in hand, we can now recall the definition of the bosonization of \(G\).

**Proposition 3.2.** [MRW16] Let \(G = (C(G), \rho^{C(G)}, \Delta_G)\) be a braided compact quantum group. Then the pair \((C(T) \boxtimes_{\zeta} C(G), \psi^{C(G)}, \Delta_G) \circ (\text{id}_{C(T)} \boxtimes_{\zeta} \Delta_G)\) satisfies the axioms for a compact quantum group, called the bosonization of \(G\) and denoted by \(G \rtimes \mathbb{T} = (C(G \rtimes \mathbb{T}), \Delta_{G \rtimes \mathbb{T}})\).

The next theorem is the main theorem of this section and describes the bosonization of \(U^+_\zeta(F)\) explicitly.

**Theorem 3.3.** Let \(U^+_\zeta(F) \rtimes \mathbb{T} = (C(U^+_\zeta(F) \rtimes \mathbb{T}), \Delta_{U^+_\zeta(F) \rtimes \mathbb{T}})\) be the bosonization of the braided free unitary quantum group \(U^+_\zeta(F)\) for an admissible \(F\). Then \(C(U^+_\zeta(F) \rtimes \mathbb{T})\) is the universal unital C*-algebra generated by elements \(z\) and \(u_{ij}\) for \(1 \leq i, j \leq n\) subject to

1. the relation \(zz^*_z = z^*z = 1\),
2. the commutation relations \(zu_{ij} = \zeta^{d_j - d_i}u_{ij}z\), for \(1 \leq i, j \leq n\),
3. and the relations that make the two matrices \(u = (u_{ij})_{1 \leq i, j \leq n}\) and \(F_{U\zeta}F^{-1}\) unitaries, where \(F_{U\zeta} = (\zeta^{d_i - d_j})_{1 \leq i, j \leq n}\).

Furthermore, the comultiplication \(\Delta_{U^+_\zeta(F) \rtimes \mathbb{T}}\) is given by

\[
\Delta_{U^+_\zeta(F) \rtimes \mathbb{T}}(z) = z \otimes z, \quad \Delta_{U^+_\zeta(F) \rtimes \mathbb{T}}(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes z^{d_k - d_i}u_{kj},
\]

for \(1 \leq i, j \leq n\).

**Proof.** For the reader’s convenience, we recall (see Definition 2.3) that the C*-algebra \(C(U^+_\zeta(F) \rtimes \mathbb{T}) = C(T) \boxtimes_{\zeta} C(U^+_\zeta(F))\) is defined to be the subalgebra

\[
j_1(C(T))j_2(C(U^+_\zeta(F))) \subseteq C(T) \otimes C(U^+_\zeta(F)) \otimes C(T^2).
\]

Therefore, from the definition itself, it is clear that the C*-algebra \(C(U^+_\zeta(F) \rtimes \mathbb{T})\) admits the specified generators satisfying the first and the third relations. For the second relation, we appeal to the commutation relation described after Definition 2.4. Here \(z\) has homogeneous degree 1 and \(u_{ij}\) has homogeneous degree \(d_j - d_i\), \(1 \leq i, j \leq n\) and therefore the commutation Eq. 2.1 becomes

\[
j_2(u_{ij})j_1(z) = \zeta^{d_j - d_i}j_1(z)j_2(u_{ij}),
\]
or equivalently
\[ j_1(z)j_2(u_{ij}) = \zeta^{d_i-d_j}j_2(u_{ij})j_1(z), \]
which is what we were after.

Now, we proceed to obtain the comultiplication \( \Delta_{U^+_\zeta(F) \rtimes \mathbb{T}} \) on the generators \( j_1(z) \) and \( j_2(u_{ij}) \), \( 1 \leq i, j \leq n \). Let us write, to simplify notation, \( \psi \) instead of \( \psi_{C(U^+_\zeta(F) \rtimes \mathbb{T})} \). It follows from the definition (see Proposition 3.2) that
\[
\Delta_{U^+_\zeta(F) \rtimes \mathbb{T}}(j_1(z)) = \psi(\text{id}_{C(\mathbb{T})} \boxtimes \zeta \Delta_{U^+_\zeta(F)}(j_1(z)) = j_1(z) \otimes j_1(z),
\]
where we have used Lemma 2.7 and Proposition 3.2 to obtain the second and third equalities, respectively; also for \( 1 \leq i, j \leq n \),
\[
\Delta_{U^+_\zeta(F) \rtimes \mathbb{T}}(j_2(u_{ij})) = \psi(\text{id}_{C(\mathbb{T})} \boxtimes \zeta \Delta_{U^+_\zeta(F)}(j_2(u_{ij}))) = \psi(j_2(\Delta_{U^+_\zeta(F)}(u_{ij})))
\]
\[
= \sum_{k=1}^{n} \psi(j_2(u_{ik})j_3(u_{kj}))
\]
\[
= \sum_{k=1}^{n} \psi(j_2(u_{ik}))\psi(j_3(u_{kj}))
\]
\[
= \sum_{k=1}^{n} (j_2(u_{ik}) \otimes j_1(z^{d_k-d_i})) (1_{C(\mathbb{T})} \boxtimes \zeta \Delta_{U^+_\zeta(F)} \otimes j_2(u_{kj}))
\]
\[
= \sum_{k=1}^{n} j_2(u_{ik}) \otimes j_1(z^{d_k-d_i})j_2(u_{kj}),
\]
where we have used again Lemma 2.7 and Proposition 3.2 to obtain the second and fifth equalities, respectively.

Our next aim is to prove that the bosonization \( U^+_\zeta(F) \rtimes \mathbb{T} \) is a compact matrix quantum group. For that, we need to dig slightly into the representation theory of \( U^+_\zeta(F) \); by the techniques in [MRW16], it can be shown that representations of \( U^+_\zeta(F) \) are equivalent to representations of the bosonization \( U^+_\zeta(F) \rtimes \mathbb{T} \). We refer the reader to the Appendix A where an explicit description is sketched out. Recall the admissibility condition from Definition 2.15

**Proposition 3.4.** Let \( t_{ij} = j_1(z^{d_i})j_2(u_{ij}) \in C(U^+_\zeta(F) \rtimes \mathbb{T}) = C(\mathbb{T}) \boxtimes \zeta C(U^+_\zeta(F)) \) for \( 1 \leq i, j \leq n \). Then \( t = (t_{ij})_{1 \leq i, j \leq n} \in M_n(C(U^+_\zeta(F) \rtimes \mathbb{T})) \) defines a finite dimensional unitary representation of the compact quantum group \( U^+_\zeta(F) \rtimes \mathbb{T} \).

**Proof.** First we show that the matrix \( t \) is a unitary. So for \( 1 \leq i, j \leq n \),
\[
\sum_{k=1}^{n} t^*_{ik}t_{kj} = \sum_{k=1}^{n} j_2(u_{ik}^*)j_1(z^{-d_k})j_1(z^{d_k})j_2(u_{kj})
\]
\[= \sum_{k=1}^{n} j_2(u_{ik}^*)j_2(u_{kj}) = \delta_{ij}, \]

and
\[ \sum_{k=1}^{n} t_{ik} t_{jk}^* = \sum_{k=1}^{n} j_1(z^{d_i}) j_2(u_{ik}) j_2(u_{jk}^*) j_1(z^{-d_j}) = j_1(z^{d_i}) \delta_{ij} j_1(z^{-d_j}) = \delta_{ij}, \]
which is what we wanted. Next,
\[ \Delta_{U^\times_\zeta^+(F) \rtimes T}(t_{ij}) = \Delta_{U^\times_\zeta^+(F) \rtimes T}(j_1(z^{d_i}) j_2(u_{ij})) = \Delta_{U^\times_\zeta^+(F) \rtimes T}(j_1(z^{d_i})) \Delta_{U^\times_\zeta^+(F) \rtimes T}(j_2(u_{ij})) = (j_1(z^{d_i}) \otimes j_2(z^{d_i})) \sum_{k=1}^{n} j_2(u_{ik}) \otimes j_1(z^{d_k-d_i}) j_2(u_{kj}) \]
\[ = \sum_{k=1}^{n} j_1(z^{d_i}) j_2(u_{ik}) \otimes j_1(z^{d_k}) j_2(u_{kj}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \]
yielding that \( t \) is indeed a unitary representation of \( U^\times_\zeta^+(F) \rtimes T \).

\[ \square \]

**Corollary 3.5.** The pair \((C(U^\times_\zeta^+(F) \rtimes T), z \oplus t)\) is a compact matrix quantum group.

**Proof.** From Theorem 3.3 and Proposition 3.4 it follows that the C*-algebra \( C(U^\times_\zeta^+(F) \rtimes T) \) is generated by the matrix coefficients of the representation \( z \oplus t \). Next, \( z \) and \( t \) being unitaries, we observe that \( z \oplus t \) is invertible. To finish the proof, we need to show that \( \tau \oplus \bar{\tau} \) is invertible. The relation \( zu_{ij} = \zeta^{d_i-d_j} u_{ij} z \)
yields the relation \( z^{d_i} u_{ij} = \zeta^{d_i-d_j} u_{ij} z^{d_i} \), for \( 1 \leq i, j \leq n \). Thus, \( t_{ij}^* u_{ij} z^{-d_i} = z^{-d_i} \zeta^{d_i-d_j} u_{ij} z^{d_i} = z^{-d_i} (\zeta u^{d_i})_{ij} \), for \( 1 \leq i, j \leq n \), which implies, in matrix terms, \( \bar{\tau} = \text{diag}(z^{-d_1}, \ldots, z^{-d_n}) \bar{\mu}_C \). But \( \bar{\mu}_C \) is equivalent to a unitary, hence invertible, and so \( \bar{\tau} \) is invertible too. Finally, since \( z \) is a one-dimensional representation, \( \tau = z^* = z^{-1} \), we obtain that \( \tau \oplus \bar{\tau} \) is indeed invertible. \[ \square \]

### 4. Symmetries of graph C*-algebras

In this section, we come to the main theme of this paper, that of braided symmetries of graph C*-algebras, which rely on the results obtained in the previous sections. We emphasize, however, that almost all the results are modelled on previous works in the unbraided case, as described, for instance, in [JM21, JM18] (see also [SW18]) and we follow the presentations therein. We essentially adopt the techniques developed in the cited references to our braided setting; our contribution is the observation that all the constructs in [JM18] are \( T \)-equivariant under the generalized gauge action as described below. Having said so, we briefly recall the basic definitions, see [Rae05] for more details.

Let \( E = (E^0, E^1, r, s) \) be a directed graph. Explicitly, this means that we have the set of vertices \( E^0 \), the set of edges \( E^1 \), the range and source maps \( r, s : E^1 \to E^0 \), respectively. An edge \( e \in E^1 \) “goes from” its source \( s(e) \in E^0 \) to its range \( r(e) \in E^0 \). Usually, the sets \( E^1 \) and \( E^0 \) are taken to be countable. A vertex \( v \in E^0 \) is called a sink (respectively, regular) if \( s^{-1}(v) = \{ e \in E^1 \mid s(e) = v \} \) is empty (respectively, standard.
Definition 4.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. The graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by families of projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$ subject to the following relations:

1. for $v, w \in E^0$, with $v \neq w$, $P_v P_w = 0$;
2. for $e, f \in E^1$, with $e \neq f$, $S_e^* S_f = 0$;
3. for $e \in E^1$, $S_e^* S_e = P_{r(e)}$;
4. for $e \in E^1$, $S_e S_e^* \leq P_{s(e)}$;
5. for $v \in E^0$ regular, $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$.

Let $E = (E^0, E^1, r, s)$ be a directed graph and $C^*(E)$ be the corresponding graph $C^*$-algebra. $C^*(E)$ comes equipped with a natural $\mathbb{T}$-$C^*$-algebra structure $\rho_{\text{gauge}}: C^*(E) \to C^*(E) \otimes C(\mathbb{T})$, called the gauge action of $\mathbb{T}$ and defined as follows: for $z \in \mathbb{T}$, $v \in E^0$, and $e \in E^1$, $\rho_z^{\text{gauge}}(P_v) = P_v$ and $\rho_z^{\text{gauge}}(S_e) = z S_e$. For finite graphs $E$, we can slightly generalize the gauge action as follows.

Definition 4.2. Let $\#(E^0) = m$, $\#(E^1) = n$ and $(d_1, \ldots, d_n) \in \mathbb{Z}^n$. The generalized gauge action is given by $\rho_z^{C^*(E)}(P_v) = P_v$ and $\rho_z^{C^*(E)}(S_e) = z^{d_j} S_{e_j}$ for $z \in \mathbb{T}$, $v_1, \ldots, v_m \in E^0$ and $e_1, \ldots, e_l \in E^1$.

Remark 4.3. For the rest of the section, we only consider finite graphs $E$ without sinks, use the generalized gauge action to equip $C^*(E)$ with the structure of a $\mathbb{T}$-$C^*$-algebra and abuse notation to write $P_i$ instead of $P_{v_i}$, $S_j$ instead of $S_{e_j}$. Furthermore, for a path $\alpha = e_1 \ldots e_l$, we shall write $S_{\alpha}$ for $S_{e_1} \ldots S_{e_l}$. We will also use the fact repeatedly that $C^*(E)$ is the closed linear span of $S_{\alpha} S_{\beta}^*$, where $\alpha, \beta$ are paths in $E$.

We recall that a $C^*$-dynamical $\mathbb{R}$-system consists of a $C^*$-algebra $X$ and a strongly continuous homomorphism $\sigma: \mathbb{R} \to \text{Qaut}(X)$ of $\mathbb{R}$ in the group of automorphisms of $X$. A self-adjoint operator $x \in \mathcal{L}(H)$ on a finite dimensional Hilbert space $H$ defines a $C^*$-dynamical $\mathbb{R}$-system $(\mathcal{L}(H), \mathbb{R}, \sigma)$ which is given by $\sigma_t(a) = e^{it}ae^{-it}$, $a \in \mathcal{L}(H)$. For such a dynamical system, it is well-known that at any inverse temperature $\beta \in \mathbb{R}$, the unique thermal equilibrium state is given by the Gibbs state

$$\omega_{\beta}(a) = \frac{\text{Tr}(e^{-\beta x} a)}{\text{Tr}(e^{-\beta x})}, \quad a \in X.$$ 

For a general $C^*$-dynamical $\mathbb{R}$-system $(X, \mathbb{R}, \sigma)$, the generalization of the Gibbs states are the KMS (Kubo-Martin-Schwinger) states.

Definition 4.4. [BR97] A KMS state for a $C^*$-dynamical $\mathbb{R}$-system $(X, \mathbb{R}, \sigma)$ at an inverse temperature $\beta \in \mathbb{R}$ is a state $\tau$ on $X$ that satisfies the KMS condition given by

$$\tau(ab) = \tau(b \sigma_{i\beta}(a)), \quad \text{for } a, b \text{ in a norm-dense subalgebra } X_{an} \text{ of } X \text{ called the algebra of analytic elements of the dynamical system } (X, \mathbb{R}, \sigma).$$
In this paper, we shall only consider KMS states on graph $C^*$-algebras with respect to the canonical gauge action. To that end, let $E$ be a finite, directed graph without sinks. We denote the vertex matrix (the $ij$-th entry of which is the number of edges between the $i$-th and $j$-th vertices) by $D$ and the spectral radius of the vertex matrix by $\rho(D)$. This $\rho(D)$ is called the critical inverse temperature and we shall be content with the existence of KMS states at this critical inverse temperature (see also [KW13]).

**Proposition 4.5.** Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks. Then the graph $C^*$-algebra $C^*(E)$ admits a KMS$_{\log(\rho(D))}$ state, say $\tau_E$ if and only if $\rho(D)$ is an eigenvalue of $D$ such that there is an eigenvector with all entries nonnegative. Furthermore, whenever $\tau_E$ exists, we have the following.

1. The vector $(\tau_E(P_1), \ldots, \tau_E(P_m))$ is an eigenvector corresponding to the eigenvalue $\rho(D)$ of the vertex matrix. Here $P_i$ is the projection corresponding to the vertex $v_i$, $i = 1, \ldots, m$ (see Definition 4.2).
2. The KMS state $\tau_E$ is given by

$$\tau_E(S_\alpha S_\beta^*) = \delta_{\alpha\beta} \frac{1}{\rho(D)^{|\alpha|}} \tau_E(P_{r(\alpha)}).$$

Here $|\alpha|$ and $r(\alpha)$ are the length and the range of the path $\alpha$, respectively. The symbol $\delta_{\alpha\beta}$ has the obvious meaning: it vanishes when the two paths $\alpha$ and $\beta$ are different and takes value 1 when $\alpha$ coincides with $\beta$.

3. The KMS state $\tau_E$ is $T$-equivariant for the generalized gauge action.

**Proof.** The demonstrations for the first three conclusions may be found in [JM21].

For the last, we simply compute. Before that, let us write $d_\alpha$ for $d_1 + \cdots + d_l$, where $\alpha = e_1 \ldots e_l$ is a path. Then for $z \in T$, $\rho_z^{C^*(E)}(S_\alpha) = z^{d_\alpha} S_\alpha$, so that

$$\tau_E(\rho_z^{C^*(E)}(S_\alpha S_\beta^*)) = \tau_E(z^{d_\alpha - d_\beta} S_\alpha S_\beta^*) = z^{d_\alpha - d_\beta} \tau_E(S_\alpha S_\beta^*)$$

$$= z^{d_\alpha - d_\beta} \delta_{\alpha\beta} \frac{1}{\rho(D)^{|\alpha|}} \tau_E(P_{r(\alpha)})$$

$$= \delta_{\alpha\beta} \frac{1}{\rho(D)^{|\alpha|}} \tau_E(P_{r(\alpha)}) = \tau_E(S_\alpha S_\beta^*).$$

The fourth equality can be argued in the following way. If the two paths $\alpha$ and $\beta$ coincide, then $z^{d_\alpha - d_\beta}$ vanishes. And if $\alpha$ and $\beta$ are different, then the right-hand side of the third equality vanishes.

**Remark 4.6.** The class of graphs satisfying the condition of the Proposition 4.5 is quite big. For example, it contains all the regular graphs. Let us denote the following condition by a (†)

(†) $\rho(D)$ is an eigenvalue of $D$ such that there is an eigenvector with all entries nonnegative

and refer to the graphs satisfying it as graphs satisfying condition (†). Thus a KMS$_{\log(\rho(D))}$ state exists if and only if the graph satisfies (†).

**Definition 4.7.** Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks and let $G = (C(G), \rho^{C(G)}, \Delta_G)$ be a braided compact quantum group. A $G$-action $\eta \in \text{Mor}^\times(C^*(E), C^*(E) \boxtimes_G C(G))$ is called linear if there exists $q = (q_{ij})_{1 \leq i, j \leq n} \in M_n(C(G))$ such that for $1 \leq j \leq n$, we have $\eta(S_j) = \sum_{i=1}^n j_1(S_i)j_2(q_{ij})$. 

We then observe that, by the $\mathbb{T}$-equivariance of $\eta$ and the homogeneity of degree $d_i - d_j$ for $1 \leq i, j \leq n$. Furthermore, the coassociativity of $\eta$ yields that $\Delta_G(q_{ij}) = \sum_{k=1}^{n} j_1(q_{ik})j_2(q_{kj})$. We summarize these observations in a lemma.

**Lemma 4.8.** Let $\eta : C^*(E) \to C^*(E) \otimes_{\mathbb{C}} C(G), S_j \mapsto \sum_{i=1}^{n} j_1(S_i)j_2(q_{ij})$ ($1 \leq j \leq n$) be a linear action of a braided compact quantum group $G = (C(G), \rho^{C(G)}, \Delta_G)$ on $C^*(E)$. Then

1. for $1 \leq i, j \leq n$ and $z \in \mathbb{T}$, $\rho_z^{C^*(E)}(q_{ij}) = z^{d_j - d_i}q_{ij}$;
2. for $1 \leq i, j \leq n$, $\Delta_G(q_{ij}) = \sum_{k=1}^{n} j_1(q_{ik})j_2(q_{kj})$;
3. for $1 \leq j \leq n$, $\eta(S_j^*) = \sum_{i=1}^{n} j_1(S_i^*)j_2((\bar{q}_\zeta)_{ij})$,

where $\bar{q}_\zeta = (\zeta^{d_i - d_j}q_{ij})_{1 \leq i, j \leq n}$.

**Proof.** Only the last conclusion needs to be demonstrated. To that end, we observe that for each $1 \leq j \leq n$,

$$\eta(S_j^*) = \eta(S_j^*)^* = \sum_{i=1}^{n} j_2(q_{ij}^*)j_1(S_i^*)$$

$$= \sum_{i=1}^{n} \zeta^{-d_i(d_j - d_i)}j_1(S_i^*)j_2(q_{ij}^*)$$

$$= \sum_{i=1}^{n} \zeta^{-d_i(d_j - d_i)}j_1(S_i^*)j_2(q_{ij}^*)$$

$$= \sum_{i=1}^{n} j_1(S_i^*)j_2((\zeta^{d_i(d_j - d_i)}q_{ij})$$

$$= \sum_{i=1}^{n} j_1(S_i^*)j_2((\bar{q}_\zeta)_{ij})$$

which is what we wanted. \hfill \Box

**Definition 4.9.** Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks satisfying condition (†) (see Remark 4.6) and let $\tau_E$ be the KMS state on $C^*(E)$. We define the category $C(E, \tau_E)$ as follows.

1. An object of $C(E, \tau_E)$ is a pair $(G, \eta)$, where $G = (C(G), \rho^{C(G)}, \Delta_G)$ is a braided compact quantum group, and $\eta \in \text{Mor}^\tau(C^*(E), C^*(E) \otimes_{\mathbb{C}} C(G))$ is a $\tau_E$-preserving, linear faithful (see Definition 2.24) action of $G$ on $C^*(E)$.

2. Let $(G_1, \eta_1)$ and $(G_2, \eta_2)$ be two objects in $C(E, \tau_E)$. A morphism $\phi : (G_1, \eta_1) \to (G_2, \eta_2)$ in $C(E, \tau_E)$ is by definition a $\mathbb{T}$-equivariant Hopf $*$-homomorphism $\phi : C(G_2) \to C(G_1)$ such that $(\text{id}_{C^*(E)} \otimes_{\mathbb{C}} \phi) \circ \eta_2 = \eta_1$.

**Definition 4.10.** A terminal object in $C(E, \tau_E)$ is called the braided quantum symmetry group of the graph $C^*$-algebra $C^*(E)$ and denoted $(\text{Qaut}(C^*(E)), \eta^E)$.

A priori, it is not clear that $(\text{Qaut}(C^*(E)), \eta^E)$ exists but as we shall see below, it indeed does (for the class of graphs we consider, i.e., finite, directed, without sinks and satisfying the condition (†), so that a KMS state exists at the critical inverse temperature). This is the main theorem of this section and we shall step by step build up to its proof.
Keeping the same notations as above, we recall that by Proposition 4.5 for $1 \leq i, j \leq n$, $\tau_E(S_i S_j^*) = \delta_{ij} \frac{1}{d_j} \tau_E(P_{i\ell})$. Now as observed in [JM18], for each $1 \leq i \leq n$, $\tau_E(P_i) \neq 0$, so that we can and in fact do, normalize $S_i$ to obtain $\tau_E(S_i S_j^*) = \delta_{ij}$. Let $\tilde{F}$ be the matrix $(\tau_E(S_i S_j))_{1 \leq i, j \leq n}$. Being a KMS state implies that $\tilde{F}$ is an invertible diagonal matrix and hence by Remark 2.20 it is admissible.

**Proposition 4.11.** Let $\eta : C^*(E) \to C^*(E) \boxtimes \mathbb{C} G$, $S_j \mapsto \sum_{i=1}^n j_1(S_i) j_2(q_{ij})$ be a linear action of a braided compact quantum group $G = (C(G), \rho(C(G)), \Delta_G)$ on $C^*(E)$.

If the $G$-action $\eta$ preserves $\tau_E$ then for each $1 \leq i, j \leq n$, the following relations hold.

1. $\sum_{k=1}^n \zeta^{d_k(d_j-d_i)} q_{ki} \xi_{kj}^* = \delta_{ij}$;
2. $\sum_{k=1}^n q_{ki}^* \xi_{kk} \eta_{kj} = \tilde{F}_{ij}$.

**Proof.** With the normalization mentioned above, we observe that if the $G$-action $\eta$ preserves $\tau_E$ then $(\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i S_j^*) = \delta_{ij} \eta_1 C(G)$ holds for each $1 \leq i, j \leq n$. Therefore, we first write $(\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i S_j^*)$ in terms of the $q_{ij}$ ($1 \leq i, j \leq n$), for which we simply compute:

\[
\begin{align*}
& (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i S_j^*) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i) \eta(S_j^*) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \left( \sum_{k,l=1}^n j_1(S_k) j_2(q_{kl}) j_1(S_j^*) j_2(\zeta^{d_j(d_j-d_l)} q_{lj}^*) \right) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \left( \sum_{k,l=1}^n j_1(S_k S_l^*) j_2(\zeta^{d_j(d_j-d_l)-d_l(d_i-d_k)} q_{ki} q_{lj}^*) \right) \\
& = \sum_{k,l=1}^n \tau_E(S_k S_l^*) \zeta^{d_k(d_j-d_l)-d_l(d_i-d_k)} q_{ki} q_{lj}^* \\
& = \sum_{k,l=1}^n \delta_{kl} \zeta^{d_k(d_j-d_l)-d_l(d_i-d_k)} q_{ki} q_{lj}^* \\
& = \sum_{k=1}^n \zeta^{d_k(d_j-d_i)} q_{ki} \xi_{kj}^*,
\end{align*}
\]

and so we obtain $\sum_{k=1}^n \zeta^{d_k(d_j-d_i)} q_{ki} \xi_{kj}^* = \delta_{ij}$.

For the second conclusion, we observe that if the $G$-action preserves $\tau_E$ then $(\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i^* S_j) = \tau_E(S_i^* S_j) \eta_1 C(G)$ holds for $1 \leq i, j \leq n$. Now

\[
\begin{align*}
& (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i^* S_j) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \eta(S_i^*) \eta(S_j) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \left( \sum_{k,l=1}^n j_1(S_k^*) j_2(\zeta^{d_k(d_i-d_k)} q_{ki}^*) j_1(S_l) j_2(q_{lj}) \right) \\
& = (\tau_E \boxtimes \zeta \text{id}_{C^*(E)}) \left( \sum_{k,l=1}^n j_1(S_k^* S_l) j_2(\zeta^{d_k(d_i-d_k)+d_l(d_i-d_k)} q_{ki}^* q_{lj}) \right) \\
& = \sum_{k,l=1}^n \tau_E(S_k^* S_l) \zeta^{d_k(d_i-d_k)+d_l(d_i-d_k)} q_{ki}^* q_{lj}.
\end{align*}
\]
Then the \( C \) such that which is what we wanted. Next, we observe that each of the diagonal entries of \( \tilde{q} \) where Proposition 4.12. Let \((G, \eta) \in \text{Obj}(C(E, \tau_E))\) be an object in the category \(C(E, \tau_E)\). Then the \( C^* \)-algebra \( C(G) \) is a quotient of the \( C^* \)-algebra \( C(U^+_\zeta(F^{-1}))\), where \( F \) is such that \( F^* F = \tilde{F} \).

Proof. Let the linear action \( \eta \) of \( G \) be given on the generators \( S_j \) by \( \eta(S_j) = \sum_{i=1}^d j_i(S_i)_j(q_{ij}) \), for \( 1 \leq j \leq n \). We now observe that the first relation of Proposition 4.11 i.e., \( \sum_{k=1}^n \zeta_k^{d_k}q_{kij} = \delta_{ij} \) can be written as \( (\overline{q}_\zeta)^* \overline{q}_\zeta = I_n \), where \( q = (q_{ij})_{1 \leq i, j \leq n} \), \( \overline{q}_\zeta = (\zeta_k^{d_k}q_{kij})_{1 \leq i, j \leq n} \) and \( (\overline{q}_\zeta)^* \) denotes the usual bar-conjugate i.e., bar-transpose. Indeed,

\[
\sum_{k=1}^n (\overline{q}_\zeta)^k_k(\overline{q}_\zeta)_kj = \sum_{k=1}^n \zeta_k^{d_k}q_{kij} = \sum_{k=1}^n \zeta_k^{d_k}q_{kij} = \sum_{k=1}^n \zeta_k^{d_k}q_{kij},
\]

yielding the required relation. The second relation obtained in Proposition 4.11 \( \sum_{k=1}^n q_{ki}^* \tilde{F}_{kk}q_{kj} = \tilde{F}_{ij} \) too can be put in a more compact form by writing \( q^* \tilde{F}q = \tilde{F} \).

Indeed,

\[
(q^* \tilde{F}q)_{ij} = \sum_{k,l=1}^n (q^*)_{ik} \tilde{F}_{kl}q_{lj} = \sum_{k,l=1}^n q_{ki}^* \delta_{kl} \tilde{F}_{kk}q_{kj} = \sum_{k=1}^n q_{ki}^* \tilde{F}_{kk}q_{kj},
\]

which is what we wanted. Next, we observe that each of the diagonal entries of \( \tilde{F} \) is positive; thus there is indeed an \( F \) such that \( F^* F = \tilde{F} \). Now we consider \( FqF^{-1} \) and compute

\[
(FqF^{-1})^*(FqF^{-1}) = (F^*)^{-1}q^*F^*FqF^{-1} = (F^*)^{-1}q^* \tilde{F}qF^{-1},
\]

which says that the relation \( q^* \tilde{F}q = F \) is equivalent to \( (FqF^{-1})^*(FqF^{-1}) = I_n \).

Letting \( q' = FqF^{-1} \), we now find out what \( F^{-1}q'_\zeta F \) is.

\[
(F^{-1}q'_\zeta F)_{ij} = \sum_{k,l=1}^n F^{-1}_{kl}(q'_\zeta)_{kj}F_{lj} = F^{-1}_{ii}(q'_\zeta)_{jj}F_{jj}
\]

i.e., we have \( F^{-1}(q'_\zeta)F = q'_\zeta \) and therefore, the relation \( (q'_\zeta)^*q'_\zeta = I_n \) is equivalent to \( (F^{-1}(q'_\zeta)F)^*F^{-1}(q'_\zeta)F = I_n \). Taken together, we see that the matrix \( q \) satisfies the relations \( (F^{-1}(q'_\zeta)F)^*F^{-1}(q'_\zeta)F = I_n \) and \( q^*q = I_n \). These two relations still hold after passing to the bosonization \( G \rtimes T \) of \( G \). Since \( G \rtimes T \) is a compact quantum
group, we obtain the remaining two relations i.e., \( F^{-1}(q^\eta)F(F^{-1}(q^\eta)F)^* = I_n \) and \( q^\eta q'^\eta = I_n \). Taking the four sets of relations together, we have that \( q^\eta \) and \( F^{-1}(q^\eta)F \) are unitaries and the universal property of the \( C^\ast \)-algebra \( C(U^+_\zeta(F^{-1})) \) yields the conclusion.

\[ \square \]

**Definition 4.13.** We now define another category \( C'(E) \) as follows. An object of \( C'(E) \) is a triple \( (X, \rho^X, \eta^X) \) which consists of

1. a \( \mathbb{T}-C^\ast \)-algebra \( (X, \rho^X) \) generated by \( \{t_{ij}\}_{1 \leq i, j \leq n} \) such that the two matrices \( F_t F^{-1} = (F_{t_{ij}}F_{j_i}^{-1})_{1 \leq i, j \leq n} \) and \( \tilde{t}_\zeta = (\zeta^{d_i(d_j-d_i)}t_{ij}^\ast)_{1 \leq i, j \leq n} \) are unitaries.

2. a \( \mathbb{T} \)-equivariant morphism \( \eta^X \in \text{Mor}^\mathbb{T}(C^\ast(E), C^\ast(E) \boxtimes \zeta X) \) such that, for each \( 1 \leq j \leq n \), \( \eta^X(S_j) = \sum_{i=1}^n j_1(S_i)j_2(t_{ij}) \).

Let \( (X, \rho^X, \eta^X) \) and \( (Y, \rho^Y, \eta^Y) \) be two objects in the category \( C'(E) \). A morphism \( \phi : (X, \rho^X, \eta^X) \to (Y, \rho^Y, \eta^Y) \) in \( C'(E) \) is by definition a \( \mathbb{T} \)-equivariant morphism \( \phi : X \to Y \) such that \( (\text{id}_{C'(E)} \boxtimes \zeta \phi) \circ \eta^X = \eta^Y \).

**Remark 4.14.** We remark that by the \( \mathbb{T} \)-equivariance of \( \eta \) and the homogeneity of \( S_j \), \( t_{ij} \) is homogeneous of degree \( d_j - d_i \) for \( 1 \leq i, j \leq n \).

**Lemma 4.15.** An initial object in the category \( C'(E) \) exists.

**Proof.** We begin by remarking again that the proof is essentially the same as given in [JM18]. We recall that for the matrix \( F \), \( C(U^+_\zeta(F^{-1})) \) is the universal \( C^\ast \)-algebra with generators \( u_{ij} \) \( (1 \leq i, j \leq n) \) subject to the relations that make \( u \) and \( F^{-1} \pi_{\zeta} \) unitaries, where \( u = (u_{ij})_{1 \leq i, j \leq n} \) and \( \pi_\zeta = (\zeta^{d_i(d_j-d_i)}u_{ij}^\ast)_{1 \leq i, j \leq n} \). Thus for any object \( (X, \rho^X, \eta^X) \in \text{Obj}(C'(E)) \), by the universal property of \( C(U^+_\zeta(F^{-1})) \), there is a surjective \( * \)-homomorphism \( \pi_\chi : C(U^+_\zeta(F^{-1})) \to X \) sending \( u_{ij} \) to \( F_{t_{ij}}F_{j_i}^{-1} \).

As remarked above, for \( 1 \leq i, j \leq n \), \( t_{ij} \) is homogeneous of degree \( d_j - d_i \) and so \( \pi_\chi \) is \( \mathbb{T} \)-equivariant. We define \( I = \bigcap_\Lambda \ker(\pi_\chi) \), where \( \Lambda = \{X : (X, \rho^X, \eta^X) \in \text{Obj}(C'(E))\} \) and observe that \( I \) is a \( \mathbb{T} \)-invariant closed two-sided ideal of \( C(U^+_\zeta(F^{-1})) \) so that \( C(U^+_\zeta(F^{-1}))/I \) makes sense. We denote \( C(U^+_\zeta(F^{-1}))/I \) by \( \mathcal{U} \) and let \( \pi : C(U^+_\zeta(F^{-1})) \to \mathcal{U} \) be the quotient map.

The \( \mathbb{T} \)-invariance of \( I \) induces a \( \mathbb{T}(\mathcal{U}) \)-coaction on \( \mathcal{U} \) which can also be described explicitly as follows. We write \( [a] = \pi(a) \) for the class of \( a \in C(U^+_\zeta(F^{-1})) \) in \( \mathcal{U} \). Then for \( z \in \mathbb{T}, \rho^\mathcal{U}([u_{ij}]) = [z^\zeta u_{ij}] \). Thus \( (\mathcal{U}, \rho^\mathcal{U}) \) is indeed a \( \mathbb{T} \)-\( C^\ast \)-algebra. Once we show that there is a \( \mathbb{T} \)-equivariant morphism \( \eta^\mathcal{U} \) satisfying the requirements as in Definition 4.13, the conclusion that \( (\mathcal{U}, \rho^\mathcal{U}, \eta^\mathcal{U}) \) is an initial object in \( C'(E) \) is immediate. To that end, let us set \( \eta^\mathcal{U}(S_j) = \sum_{i=1}^n j_1(S_i)j_2(F_{t_{ij}}F_{j_i}) \). It is a tedious but straightforward check that \( \eta^\mathcal{U} \) is indeed a morphism from \( C'(E) \) to \( C^\ast(E) \boxtimes \zeta X \) and it is clearly \( \mathbb{T} \)-equivariant, thus implying that \( \eta^\mathcal{U} \in \text{Mor}^\mathbb{T}(C^\ast(E), C^\ast(E) \boxtimes \zeta X) \).

This finishes the proof. □

Let us keep the notations from the above proof; thus \( (\mathcal{U}, \rho^\mathcal{U}, \eta^\mathcal{U}) \) is the initial object of \( C'(E) \). We shall provide \( \mathcal{U} \) with more structures so as to make it a braided compact quantum group and prove that it is the braided quantum symmetry group \( (\text{Qaut}(C^\ast(E)), \eta^\mathcal{E}) \). We first observe that \( \mathcal{U} \boxtimes \zeta \mathcal{U} \) can be made into an object of the category \( C'(E) \).

**Lemma 4.16.** The braided tensor product \( \mathcal{U} \boxtimes \zeta \mathcal{U} \) of \( \mathcal{U} \) with itself can be made into an object \( (\mathcal{U} \boxtimes \zeta \mathcal{U}, \rho^\mathcal{U}, \eta^\mathcal{U}) \) of \( C'(E) \).
Proof. Indeed, letting \( t_{ij} = \sum_{k=1}^{n} j_1([u_{ik}])j_2([u_{kj}]) \) for \( 1 \leq i, j \leq n \), we see that the C*-algebra \( \mathcal{U} \boxtimes \mathcal{U} \) is generated by \( t_{ij} \) (\( 1 \leq i, j \leq n \)). A repetition of the proof of Proposition 2.18 then yields that \( t_{ij} \) satisfy the required relations as in Definition 4.13.

Finally, the morphism \( \eta^\mathcal{U}_\mathcal{U} = (\eta^\mathcal{U} \boxtimes \mathcal{U} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} \in \text{Mor}^\mathcal{U}(\mathcal{U}, \mathcal{U} \boxtimes \mathcal{U}) \) satisfies

\[
\eta^\mathcal{U}_\mathcal{U}(S_j) = \sum_{k,i=1}^{n} j_1(S_i) j_2(F^{-1}_{ii}[u_{ik}]F_{kk}) j_3(F^{-1}_{kk}[u_{kj}]F_{jj}) \\
= \sum_{i=1}^{n} j_1(S_i) j_2(F^{-1}_{ii} t_{ij} F_{jj})
\]

which implies that \( (\mathcal{U} \boxtimes \mathcal{U}, \rho^\mathcal{U}_\mathcal{U}, \eta^\mathcal{U}_\mathcal{U}) \) is an object of \( \mathcal{C}(E) \).

Corollary 4.17. There exists a unique \( \mathbb{T} \)-equivariant morphism \( \Delta_\mathcal{U} : \mathcal{U} \to \mathcal{U} \boxtimes \mathcal{U} \) such that \( \Delta_\mathcal{U}([u_{ij}]) = \sum_{k=1}^{n} j_1([u_{ik}])j_2([u_{kj}]) \). Furthermore, \( \Delta_\mathcal{U} \) is coassociative and bisimplifrible (see Definition 2.8).

Proof. Since \( \mathcal{U} \) is the initial object in the category \( \mathcal{C}(E) \), there is a unique morphism \( \Delta_\mathcal{U} \) from \( (\mathcal{U}, \rho^\mathcal{U}, \eta^\mathcal{U}) \) to \( (\mathcal{U} \boxtimes \mathcal{U}, \rho^\mathcal{U}_\mathcal{U}, \eta^\mathcal{U}_\mathcal{U}) \). Explicitly, this means that \( \Delta_\mathcal{U} \) is a \( \mathbb{T} \)-equivariant morphism \( \Delta_\mathcal{U} : \mathcal{U} \to \mathcal{U} \boxtimes \mathcal{U} \) such that \( (\text{id}_{\mathcal{C}(E) \boxtimes \mathcal{U}} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} = \eta^\mathcal{U}_\mathcal{U} \). The last equality when evaluated at \( S_j \) (\( 1 \leq j \leq n \)), together with their linear independence, yields \( \Delta_\mathcal{U}([u_{ij}]) = \sum_{k=1}^{n} j_1([u_{ik}])j_2([u_{kj}]) \). The arguments for coassociativity and bisimplifiability are exactly similar as in the proof of the Proposition 2.18.

Corollary 4.18. There exists a braided compact quantum group (over \( \mathbb{T} \)) \( G_E \) such that \( (\mathcal{C}(G_E), \rho^{\mathcal{C}(G_E)}, \Delta_{G_E}) = (\mathcal{U}, \rho^\mathcal{U}, \Delta_\mathcal{U}) \). Furthermore, \( G_E \) acts linearly, faithfully on \( \mathcal{C}(E) \) preserving \( \tau_E \) via \( \eta^\mathcal{U} \), denoted henceforth by \( \eta_E \).

Proof. The first statement is essentially renaming. For the second, we observe that \( \eta^\mathcal{U} \) is coassociative. Indeed, by definition \( \eta^\mathcal{U}_\mathcal{U} = (\eta^\mathcal{U} \boxtimes \mathcal{U} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} \) and we have \( (\text{id}_{\mathcal{C}(E) \boxtimes \mathcal{U}} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} = \eta^\mathcal{U}_\mathcal{U} = (\eta^\mathcal{U} \boxtimes \mathcal{U} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} \). The Podleś condition can be checked again along the same lines as in the proof of Proposition 2.18. Finally, \( \eta^\mathcal{U} \) is, once again by definition, linear, faithful and it preserves \( \tau_E \) by Proposition 1.11. This completes the proof.

Theorem 4.19. Let \( E = (E_0, E^1, r, s) \) be a finite, directed graph without sinks that satisfies the condition \( \langle \rangle \). Then \( (\text{Quat}(\mathcal{C}(E)), \eta^E) \) exists.

Proof. As in [JM18], we shall show that \( G_E \) is the terminal object in the category \( \mathcal{C}(E, \tau_E) \) and thus is isomorphic to \( (\text{Quat}(\mathcal{C}(E)), \eta^E) \). To that end, let \( (G, \eta) \in \text{Obj}(\mathcal{C}(E, \tau_E)) \) be an object in the category \( \mathcal{C}(E, \tau_E) \). By Proposition 1.12 the triple \( (\mathcal{C}(G), \rho^{\mathcal{C}(G)}, \eta^{\mathcal{C}(G)} = \eta) \) is an object of the category \( \mathcal{C}(E) \), hence there is a unique \( \phi \in \text{Mor}^\mathcal{U}(\mathcal{C}(G_E), \mathcal{C}(G)) \) such that \( (\text{id}_{\mathcal{C}(E) \boxtimes \mathcal{U}} \boxtimes \mathcal{U}) \circ \eta^\mathcal{U} = \eta^{\mathcal{C}(G)} \). This is equivalent to saying that \( G_E \) is indeed the terminal object in the category \( \mathcal{C}(E, \tau_E) \).

Having proved the existence of the braided quantum symmetry group, we now explicitly compute it for the Cuntz algebra \( \mathcal{O}_n \). We recall that the Cuntz algebra \( \mathcal{O}_n \) is the graph C*-algebra corresponding to the graph (denoted by \( E_{\mathcal{O}_n} \)) with a
single vertex and $n$-loops at it. Explicitly, $\mathcal{O}_n$ is the universal C$^*$-algebra generated by $S_i$ for $1 \leq i \leq n$ subject to the relations

\[ S_i^* S_j = \delta_{ij} \quad (1 \leq i, j \leq n), \quad \text{and} \quad S_1 S_1^* + \cdots + S_n S_n^* = 1. \]

$\mathcal{O}_n$ is equipped with the generalized gauge action $\rho_{\mathcal{O}_n} : \mathcal{O}_n \to \mathcal{O}_n \rtimes \mathbb{C}(\mathbb{T})$ given by $\rho_{\mathcal{O}_n}(S_i) = z^d S_i$, $1 \leq i \leq n$, $z \in \mathbb{T}$, and $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$. The next Proposition shows that the braided free unitary quantum group $U^+_\zeta(n)$ acts on $\mathcal{O}_n$.

**Proposition 4.20.** There is a unique unital \*$\text{-}$homomorphism $\eta_{\mathcal{O}_n} : \mathcal{O}_n \to \mathcal{O}_n \rtimes \zeta C(U^+_\zeta(n))$ such that $\eta_{\mathcal{O}_n}(S_j) = \sum_{i=1}^n j_1(S_i)j_2(u_{ij})$ for $1 \leq i, j \leq n$. Furthermore, $\eta_{\mathcal{O}_n}$ is $\mathbb{T}$-equivariant, coassociative and satisfies Podleś condition (see Definition 2.22).

**Proof.** Let $S_j^* = \sum_{i=1}^n j_1(s_i)j_2(u_{ij})$ for $1 \leq i, j \leq n$. We remark that each $S_j^*$ is homogeneous of degree $d_j$. We first observe that for each $1 \leq j \leq n$

\[
S_j^* = \sum_{i=1}^n j_2(u_{ij}^*)j_1(s_i^*)
\]

\[
= \sum_{i=1}^n \zeta^{-d_i(d_j-d_j)}j_1(s_i^*)j_2(u_{ij})
\]

\[
= \sum_{i=1}^n \zeta^{-d_i(d_j-d_j)}j_1(s_i^*)j_2(u_{ij})
\]

\[
= \sum_{i=1}^n j_1(s_i^*)j_2(\zeta^{d_i(d_j-d_j)}u_{ij}^*) = \sum_{i=1}^n j_1(s_i^*)j_2((\eta_{\mathcal{O}_n})_{ij}).
\]

Now, by the universal property, we see that a (necessarily unique) \*$\text{-}$homomorphism $\eta_{\mathcal{O}_n} : \mathcal{O}_n \to \mathcal{O}_n \rtimes \zeta C(U^+_\zeta(n))$ satisfying $\eta_{\mathcal{O}_n}(S_j) = S_j^*$ exists if and only if $S_j^*$ (for $1 \leq j \leq n$) satisfy Eq. (4.2). To see that $S_j^*$ for $1 \leq j \leq n$ indeed satisfy Eq. (4.2), we compute

\[
S_i^* S_j^* = \sum_{\alpha, \beta=1}^n j_1(S_{\alpha}^*)j_2(\zeta^{d_{\alpha}(d_j-d_j)}u_{\alpha i}^*)j_1(S_{\beta})j_2(u_{\beta j})
\]

\[
= \sum_{\alpha, \beta=1}^n j_1(S_{\alpha}^*)j_1(S_{\beta})\zeta^{d_{\beta}(d_{\alpha}-d_{\alpha})}j_2(u_{\alpha i}^*)j_2(u_{\beta j})
\]

\[
= \sum_{\alpha, \beta=1}^n j_1(S_{\alpha}^*)S_{\beta}\zeta^{d_{\beta}(d_{\alpha}-d_{\alpha})}j_2(u_{\alpha i}^*)j_2(u_{\beta j})
\]

\[
= \sum_{\alpha, \beta=1}^n \delta_{\alpha \beta}\zeta^{d_{\beta}(d_{\alpha}-d_{\alpha})}j_2(u_{\alpha i}^*)j_2(u_{\beta j})
\]

\[
= \sum_{\alpha=1}^n j_2(u_{\alpha i}^*)u_{\alpha j} = \delta_{ij},
\]

The fourth and fifth equalities use the relations of $\mathcal{O}_n$ and that $u$ is a unitary, respectively. Similarly,

\[
\sum_{j=1}^n S_j^* S_j^* = \sum_{j, \alpha, \beta=1}^n j_1(S_{\alpha})j_2(u_{\alpha j})j_1(S_{\beta}^*)j_2(\zeta^{d_{\beta}(d_j-d_j)}u_{\beta j}^*)
\]
We recall the following lemma from [JM21].

We first remark that it suffices to check the equivariance on Case 1.

Now there are two cases.

The statement “Eq. (4.3) holds for paths $\alpha$” for good.

$\eta$ $\tau$ (4.3) $(\alpha, \beta)$ paths, i.e., faithful. To finish the proof, we need to show that $\eta$.

By Proposition 4.20, we already have that the $U^{\delta}_{\alpha}$-algebra spanned by $S_{\alpha, \beta}$, respectively. Therefore, we have constructed a unique and unital $\eta$.

$\square$ As remarked above, for $1 \leq j \leq n$, $S_{j}^{\delta}$ is homogeneous of degree $d_{j}$ and so $\eta^{O_{n}}$ is $T$-equivariant. The coassociativity and the Podleś can be proved along the same lines as in the proof of Proposition 2.18.

Now for the Cuntz algebra $O_{n}$, the vertex matrix $D$ is just a scalar $n$ and so $\rho(D) = n$. And therefore, the underlying graph satisfies the condition (i) of Remark 4.6 implying the existence of the KMS state $\tau_{E_{O_{n}}}$, which we denote by $\tau_{n}$, to shorten the notation. By Proposition 4.5, $\tau_{n}(S_{i}^{\delta}_{j}) = \delta_{ij} \frac{1}{n}$. In fact, on the dense $*$-algebra spanned by $S_{\alpha, \beta}$, where $\alpha, \beta$ are paths, $\tau_{n}$ is given by $\tau_{n}(S_{\alpha, \beta}^{\delta}) = \delta_{\alpha, \beta} \frac{1}{n^{m_{\beta}}}$.

We recall the following lemma from [JM21].

**Lemma 4.21.** The KMS state $\tau_{n}$ satisfies $\tau_{n}(S_{\alpha}xS_{\beta}^{\delta}) = \delta_{\alpha, \beta} \frac{1}{n^{m_{\beta}}} \tau_{n}(x)$ for all $x \in O_{n}$ and paths $\alpha, \beta$ with $|\alpha| = |\beta|$.

**Proposition 4.22.** The pair $(U_{\xi}^{\delta}(n), \eta^{O_{n}})$ is an object of the category $C(E_{O_{n}}, \tau_{n})$.

**Proof.** By Proposition 4.20, we already have that the $U_{\xi}^{\delta}(n)$-action is linear and faithful. To finish the proof, we need to show that $\eta^{O_{n}}$ preserves $\tau_{n}$. To that end, we first remark that it suffices to check the equivariance on $S_{\alpha, \beta}^{\delta}$, where $\alpha, \beta$ are paths, i.e.,

\[(\tau_{n} \otimes \varepsilon_{\delta} \text{id}_{O_{n}})\eta^{O_{n}}(S_{\alpha}S_{\beta}^{\delta}) = \tau_{n}(S_{\alpha}S_{\beta}^{\delta})1_{C(U_{\xi}^{\delta}(n))}.\]

Now there are two cases.

**Case 1.** $|\alpha| \neq |\beta|$. In this case, the right-hand side of Eq. (4.3) is 0. For the left-hand side, we observe that $\eta^{O_{n}}(S_{\alpha}S_{\beta}^{\delta}) = \eta^{O_{n}}(S_{\alpha})\eta^{O_{n}}(S_{\beta}^{\delta})$. Using linearity of $\eta^{O_{n}}$ and the commutation relations of the imbeddings $j_{1}$ and $j_{2}$, we find a $u_{\alpha, \beta, \mu, \nu} \in C(U_{\xi}^{\delta}(n))$, such that $\eta^{O_{n}}(S_{\alpha}S_{\beta}^{\delta})$ is a finite sum of elements of the form $j_{1}(S_{\mu}S_{\nu}^{\delta})j_{2}(u_{\alpha, \beta, \mu, \nu})$, where $\mu$ and $\nu$ are multi-indices of the form $(i_{1}, \ldots, i_{|\alpha|})$ and $(j_{1}, \ldots, j_{|\beta|})$, respectively. Then $(\tau_{n} \otimes \varepsilon_{\delta} \text{id}_{O_{n}})\eta^{O_{n}}(S_{\alpha}S_{\beta}^{\delta})$ is a finite sum of elements of the form $\tau_{n}(S_{\mu}S_{\nu}^{\delta})u_{\alpha, \beta, \mu, \nu}$. Since $|\alpha| \neq |\beta|$, $\tau_{n}(S_{\mu}S_{\nu}^{\delta})$ vanishes too, settling this case for good.

**Case 2.** $|\alpha| = |\beta|$. For this case, we use induction on $|\alpha| = |\beta|$. Thus let $P(k)$ be the statement “Eq. (4.3) holds for paths $\alpha$ and $\beta$ of lengths $|\alpha| = |\beta| = k$.”
Step 1. We prove that $\mathbf{P}(1)$ holds. For this, we observe that length 1 paths
are the generators themselves, i.e., $S_\alpha = S_i$ and $S_{\beta} = S_j$
for some $1 \leq i, j \leq n$. Since $u$ and $\pi_c$ are unitaries, a repetition
of the proof of Proposition 4.11 settles this step.

Step 2. Now we assume that $\mathbf{P}(m)$ holds for all $m < m_0$, i.e.,
eq \tau_n(S_\alpha S_{\beta}^*) = 0$ if $S_\alpha S_{\beta}^*$ has nonzero homogeneous
degree, i.e., in the notation of the proof of Proposition 4.5 $d_1 + \cdots + d_{|\alpha|} = d_\alpha \neq d_\beta$
$= d_1 + \cdots + d_{|\beta|}$. Indeed, $d_\alpha \neq d_\beta$ implies $\alpha \neq \beta$ and so
$\tau_n(S_\alpha S_{\beta}^*) = \delta_{\alpha \beta} \frac{1}{n+1}$ vanishes.

Step 2a. We remark that $\tau_n(S_\alpha S_{\beta}^*) = 0$ if $S_\alpha S_{\beta}^*$ has nonzero homogeneous degree,
\[ \tau_n(S_\alpha S_{\beta}^*) = 0 \]
and $\tau_n(S_\alpha S_{\beta}^*)$ is nonzero homogeneous degree. Then $\tau_n(S_\alpha S_{\beta}^*)$ reduces
$\tau_n(S_\alpha S_{\beta}^*) = 0$. Indeed, $d_\alpha \neq d_\beta$ implies $\alpha \neq \beta$ and so
$\tau_n(S_\alpha S_{\beta}^*) = \delta_{\alpha \beta} \frac{1}{n+1}$ vanishes.

Step 2b. With Step 2a in hand, we take paths $\alpha$ and $\beta$ with $|\alpha| = |\beta| = m_0$. We
write $S_\alpha S_{\beta}^*$ as $S_i S_j^*$ with $x$ of the form $S_i S_j^*$, where $\alpha'$ and $\beta'$ are
paths of lengths $m_0 - 1$. We note that Lemma 4.21 can be applied and so
$\tau_n(S_\alpha S_{\beta}^*) = \tau_n(S_i S_j^*) = \delta_{ij} \frac{1}{n} \tau_n(x)$. Now the right-hand side of Eq. (4.3). when evaluated at $S_i S_j^*$, reduces to the following.

\[
\begin{align*}
(\tau_n \boxtimes \iota_{C_{O_n}}) \eta^{O_n}(S_i S_j^*)
&= (\tau_n \boxtimes \iota_{C_{O_n}}) \eta^{O_n}(S_i) \eta^{O_n}(x) \eta^{O_n}(S_j^*) \\
&= (\tau_n \boxtimes \iota_{C_{O_n}}) \left( \sum_{(x)} \sum_{k,l=1}^{n} j_1(S_k) j_2(u_{ki}) j_1(x_0) j_2(x_1) j_1(S_i^*) j_2(u_{kj}) \right) \\
&= (\tau_n \boxtimes \iota_{C_{O_n}}) \left( \sum_{(x)} \sum_{k,l=1}^{n} j_1(S_k x_0 S_i^*) j_2(u_{ki} x_1 u_{kj}^*) \right) \\
&= \sum_{(x)} \sum_{k,l=1}^{n} \tau_n(S_k x_0 S_i^*) u_{ki} x_1 u_{kj}^* \zeta^{d_1(d_1 - d_l) + \deg(x_0)(d_1 - d_k) - d_1(d_1 - d_k + \deg(x_1))} \\
&= \sum_{(x)} \sum_{k,l=1}^{n} \delta_{kl} \frac{1}{n} \tau_n(x_0) u_{ki} x_1 u_{kj}^* \zeta^{d_1(d_1 - d_l) + \deg(x_0)(d_1 - d_k) - d_1(d_1 - d_k + \deg(x_1))} \\
&= \sum_{(x)} \sum_{k,l=1}^{n} \frac{1}{n} \tau_n(x_0) u_{ki} x_1 u_{kj}^* \zeta^{d_1(d_1 - d_l) + \deg(x_0)(d_1 - d_k) - d_1(d_1 - d_k + \deg(x_1))}
\end{align*}
\]

The second equality uses the definition of $\eta^{O_n}$ (see Proposition 4.20), Sweedler
notation for $\eta^{O_n}$ and Lemma 4.3, the third equality uses commutation relations
for the imbeddings $j_1$ and $j_2$; the fifth equality uses Lemma 4.21. We recall that we have
assumed $\mathbf{P}(m)$ holds for $m < m_0$; thus $(\tau_n \boxtimes \iota_{C_{O_n}}) \eta^{O_n}(x) = \tau_n(x) 1_{C(U^+(\mathbb{C}^n))}$, which,
in Sweedler notation as used above, is equivalent to $\sum_{(x)} \tau_n(x_0 x_1) = \tau_n(x) 1_{C(U^+(\mathbb{C}^n))}$. We
now further divide into two more steps.

Step 2b'. We first assume that $x$ has nonzero homogeneous degree. Then (4) becomes

\[
\begin{align*}
\sum_{(x)} \sum_{k,l=1}^{n} \frac{1}{n} \tau_n(x_0) u_{ki} x_1 u_{kj}^* \zeta^{d_1(d_1 - d_l) + \deg(x_0)(d_1 - d_k) - d_1(d_1 - d_k + \deg(x_1))}
\end{align*}
\]
Remark 4.24. Theorem 4.23. which is isomorphic to the crossed product 

This completes the step, hence the induction, and the proof.

The fourth equality uses the fact that

Step 2b. We now assume that \( \text{deg}(x) = 0 \). Now in \((\dagger)\), we observe that under the sum \( \sum_{x} \), only those \( x_0 \) survive for which \( \text{deg}(x_0) = 0 \), by Step 2a. But \( \text{deg}(x) = \text{deg}(x_0) = 0 \) imply \( \text{deg}(x_1) = 0 \). Then \((\dagger)\) reduces to the following.

\[
\sum \sum_{x} \frac{1}{n} \tau_n(x_0) u_{ki} x_1 u_{kj}^* \delta_k (d_j - d_k) + \text{deg}(x_0)(d_i - d_k) - d_k (d_i - d_k + \text{deg}(x_1))
\]

\[
\sum \sum_{x} \frac{1}{n} \tau_n(x_0) u_{ki} x_1 u_{kj}^* \delta_k (d_j - d_k) - d_k (d_i - d_k)
\]

\[
= \sum \sum_{x} \frac{1}{n} \tau_n(x) u_{ki} x_1 u_{kj}^* \delta_k (d_j - d_k) - d_k (d_i - d_k)
\]

\[
= \frac{1}{n} \tau_n(x) \sum_{k=1}^{n} u_{ki} u_{kj}^* \delta_k (d_j - d_i)
\]

\[
= \frac{1}{n} \tau_n(x) \delta_{ij}.
\]

The fourth equality uses the fact that \( \overline{u}_\zeta \) is a unitary. Therefore, the right-hand side of Eq. (4.3) equals \( \frac{1}{n} \tau_n(x) \delta_{ij} \) which, by Lemma 4.21, is the left-hand side of Eq. (4.3). This completes the step, hence the induction, and the proof.

Theorem 4.23. The pair \((U_+^+(n), \eta^{\mathcal{O}_n})\) is the terminal object in the category \( \mathcal{C}(E\mathcal{O}_n, \tau_n) \), i.e., \((U_+^+(n), \eta^{\mathcal{O}_n}) \cong (\text{Qaut}(\mathcal{O}_n), \eta^{\mathcal{O}_n})\).

Proof. The relations Eq. (4.2) imply that the matrix \( \tilde{F} = \tau_n(S_i^* S_j)_{1 \leq i, j \leq n} \) defined in the proof of Proposition 4.11 is just the identity matrix \( I_n \). The result now follows from Propositions 4.11 and 4.20.

Remark 4.24. The bosonization \( U_+^+(n) \rtimes \mathbb{T} \) acts on the C*-algebra \( C(\mathbb{T}) \boxtimes \zeta \mathcal{O}_n \), which is isomorphic to the crossed product \( \mathcal{O}_n \rtimes \mathbb{Z} \mathcal{O}_n \rtimes \mathbb{Z} \rtimes \zeta \mathcal{O}_n \) with \( \mathbb{Z} \), the \( \mathbb{Z} \)-action being given by the automorphism induced by \( \zeta^{-1} \). In fact, we expect it to be the universal object in a suitable category but as the main focus of this article is the braided quantum group itself, as opposed to its bosonization, we refrain from going further in this direction.

Appendix A.

In this appendix, we discuss the representation theory of the braided free unitary quantum group \( U_+^+(F) \). As this is mostly similar to the results in [MR22] and is not directly related to our main theorem, our discussion will be brief, only highlighting the crucial points. We also restrict to the case where \( F \) is diagonal as this is the
We note that it is known from [MR22], that the representation category of $U_x(F)\rtimes T$ is the universal unital $C^*$-algebra generated by $z$ and $t_{ij}$ for $1 \leq i,j \leq n$, subject to the relations that make $z, t = (t_{ij})_{1 \leq i,j \leq n}$, and $\bar{t} = (t^*_{ij})_{1 \leq i,j \leq n}$ unitaries, together with the relations $zt_{ij} = \zeta^{d_i - d_j}t_{ij}z$ for $1 \leq i,j \leq n$. Indeed, the proof of Corollary 3.5 shows that $\pi_\zeta$ being equivalent to a unitary is equivalent to $\bar{t}$ being equivalent to a unitary and since $F$ is diagonal, we have the prescribed relations by Theorem 3.3. Furthermore, the relation $zt_{ij} = \zeta^{d_i - d_j}t_{ij}z$ and its adjoint, yield equivalences $z \bar{t} t \equiv t \bar{t} z$ and $z \bar{z} \equiv \bar{t} \bar{t} z$, respectively.

Now for the matrix $F$, we consider the function algebra $A_u(F)$ of the free unitary quantum group, the generators being denoted by $x_{ij}$, $1 \leq i,j \leq n$. Multiplying the generators $x_{ij}$ by $\zeta^{d_i - d_j}$, $1 \leq i,j \leq n$ yields an automorphism $\alpha$ of the $C^*$-algebra $A_u(F)$. Following again [MR22], there is a Hopf $*$-homomorphism $\phi : A_u(F) \to C(U_\zeta(F) \rtimes T)$ mapping $x_{ij}$ to $t_{ij}$. The proof of the following is similar to Proposition 4.2 of [MR22].

**Proposition A.1.** The Hopf $*$-homomorphism $\phi : A_u(F) \to C(U_\zeta(F) \rtimes T)$ extends to an isomorphism $A_u(F) \rtimes_u Z \cong C(U_\zeta(F) \rtimes T)$.

The Hopf $*$-homomorphism $\phi$ induces a fully faithful strict tensor functor $\phi_*$ from the representation category of $A_u(F)$ to that of $U_\zeta(F)\rtimes T$. The irreducible representations of the free unitary quantum group, as described by Banica [Ban97], are enumerated by $r_x$, $a \in \mathbb{N} \ast \mathbb{N}$, with $r_x = 1$, $r_e = x$, $r_\beta = \bar{x}$, where $\mathbb{N} \ast \mathbb{N}$ is the coproduct in the category of monoids of two copies of $\mathbb{N}$, generated respectively by $\alpha$ and $\beta$; $e$ denotes the unit of $\mathbb{N} \ast \mathbb{N}$; and $\bar{()}$ denotes the involution of $\mathbb{N} \ast \mathbb{N}$ given by $\bar{e} = e$, $\bar{a} = \beta$, $\bar{\beta} = \alpha$ and extended by antimultiplicativity. One furthermore has for any $x$ and $y$ in $\mathbb{N} \ast \mathbb{N}$, $r_x = r_y = r_x$ and the fusion rule,

$$r_x \bar{t} r_y \cong \bigoplus_{\{a,b,g \in \mathbb{N} \ast \mathbb{N}| x = ag,y = gb\}} r_{ab}.$$

A lemma analogous to Lemma 4.3 of [MR22] says that the representations $z^x \bar{t} r_x(r_y)$ with $x \in Z$ and $x' \in \mathbb{N} \ast \mathbb{N}$ are all irreducible and distinct; furthermore, any irreducible representation is of this form. And finally, the fusion rules then become

$$\zeta^x \bar{t} r_{x'} \cong \zeta^{x'} \bar{t} z \cong \zeta^{-x} \bar{t} r_{x'},$$

$$(z^x \bar{t} r_{x'}) \bar{t} (z^y \bar{t} r_y) \cong z^{x+y} \bar{t} \bigoplus_{\{a,b,g \in \mathbb{N} \ast \mathbb{N}| x' = ag,y' = gb\}} r_{ab}.$$

We note that it is known from [MR22], that the representation category of $U_\zeta(F)$ is equivalent to that of the bosonization $U_\zeta(F)\rtimes T$, thus yielding complete knowledge of the representation category for a diagonal $F$, and in particular, for $U_\zeta(n)$. Collecting these together, we end this article with the following theorem.

**Theorem A.2.** Let $F \in GL(n, \mathbb{C})$ be diagonal and admissible. Then the braided free unitary quantum group $U_\zeta(F)$ has irreducible representations $r_{(x,x')}$, $x \in Z$, $x' \in \mathbb{N} \ast \mathbb{N}$ such that any irreducible representation is unitarily equivalent to exactly one of these and moreover they satisfy the fusion rule

$$r_{(x,x')} \bar{t} r_{(y,y')} \cong \bigoplus_{\{a,b,g \in \mathbb{N} \ast \mathbb{N}| x' = ag,y' = gb\}} r_{(x+y,ab)}.$$
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