Scheduling Algorithms for Minimizing Age of Information in Wireless Broadcast Networks with Random Arrivals

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Abstract

Age of information is a newly proposed metric that captures packet delay to end-users from an application layer perspective. The age measures the amount of time that elapsed since the latest information was generated at a source. In this context, we study an age minimization problem over a broadcast network to keep many users updated on timely information, where only one user can be served at a time. We formulate a Markov decision process (MDP) to find dynamic scheduling algorithms, with the purpose of minimizing the long-run average age. We show that an optimal scheduling algorithm for the MDP is a simple stationary switch type. However, the optimal scheduling algorithm is not easy to implement due to the MDP’s infinite state-space. Using a sequence of finite-state approximate MDPs, we successfully develop both optimal off-line and on-line scheduling algorithms. We validate the algorithms via numerical studies, and surprisingly show that the performance of no-buffer networks is very close to that of networks with buffers.

I. Introduction

Traditional networks’ designs have been focused on network throughput or delay. In addition to those performance metrics, in recent years there has been a growing interest in an age of information. The age of information is motivated by a variety of network applications requiring

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timely information. Examples of the applications range from information updates for network users, e.g., traffic, transportation, air quality, and weather, to status updates for cyber-physical systems or Internet of things, e.g., smart home systems, smart transportation systems, and smart grid systems.

Illustrated in Fig. 1 is a network whose network users $u_1, \cdots, u_N$ are running some applications that monitor time-varying information (e.g., user $u_1$ is interested in traffic and transportation information for planning the best route), while at some epochs, snapshots of the information are generated at the sources and sent to the applications in the form of packets over wire or wireless networks. In other words, the applications are being updated and keep the latest information only. Once a packet is generated at its source, it starts to age as time elapses. The age is therefore defined to capture the freshness of the information in the applications; more precisely, it measures the time that elapsed since the generation of those packets, and our goal is to minimize the long-run average age. In addition to the applications of the timely information for the network users, the cyber-physical systems or Internet of things also needs timely information (e.g., locations and velocities in smart vehicular networks, or renewable energy in smart grid networks) to accomplish some tasks (e.g., collision-free smart transportation systems or smart grid networks). As such, the age of information is a good metric to characterize performance of these age-sensitive applications.

While the packet delay is usually referred to as the elapsed time from the generation to its delivery, the age includes not only the packet delay but also the inter-delivery time because the
age of information increases until the information is updated at the end-users. We hence need to jointly consider the two parameters so as to design an age-optimal network. Moreover, while traditional relays (i.e., intermediate nodes) need to keep all packets that are not served yet, the relays in the networks of Fig. 1 for timely information only store the latest information and discard out-of-date packets. That is, a new arrival always replaces the old packet in a buffer.

In resource-constrained networks, scheduling is a critical issue to optimize network performance. In this paper, we consider a wireless broadcast network consisting of a base-station (BS) and many network users, associated with time-varying packet arrivals at the BS. Due to transmission capacity, we assume that the BS can serve at most one user for each transmission opportunity. We note that transmission scheduling schemes for maximizing throughput have been extensively studied using the Lyapunov theory (e.g., [2]). However, these Lyapunov-based algorithms might result in poor delay [3]. As the scheduling design in this paper is driven by the age of information, which includes the packet delay, our problem is challenging and unique.

We will design our dynamic scheduling algorithms based on Markov decision processes (MDPs) and reinforcement learning. We start with formulating our problem in the language of the MDPs, with the objective of determining an optimal decision for each state for minimizing the long-run average total age. Although there have been several classic works (e.g., [4–7]) on generic MDPs’ problems, there is no clear methodology to find optimal decisions for the problems that possess the properties of infinite horizon, average cost optimization, and have a countably infinite state-space. In fact, [5] concludes that such problems are difficult to analyze and obtain optimal algorithms.

A. Contribution and outlines

We start with a noiseless broadcast network without buffers at the BS in Section II, i.e., all arriving packets that are not served in the current slot are discarded. By formulating a MDP, in Section III we show that an optimal dynamic scheduling algorithm is stationary and deterministic. In particular, it is a simple switch type, i.e., given the ages of other users, an optimal decision for a user is a threshold type, where the BS updates the user if its age is larger than the threshold.

We then propose a sequence of finite-state approximations and rigorously show its convergence in Section IV-A since no practical algorithm can work on infinite-state MDPs like ours. In Section IV-B we proposed an optimal off-line scheduling algorithm based on the finite-state
approximate MDPs, and in Section IV-C we propose an *optimal on-line* scheduling algorithm by applying reinforcement learning techniques.

We then successfully extend our results and algorithms in Section V to the case where buffers are available at the BS to store the latest information for each user. Finally, in Section VI, we compare these scheduling algorithms via numerical studies, and surprisingly find that the buffers only improve performance marginally.

**B. Related works**

The general idea of *age* was proposed in [8] to study how to refresh a local copy of an autonomous information source to maintain the copy up-to-date. This paper considers a *pull-based* replication (i.e., the local copy is updated on request), and assumes no communication time between the data source and the local caches. On the contrary, the *age of information* recently proposed in [9] considers communication time while focusing on the *push-based* operation, i.e., it considers not only a network to deliver updates but a communication system over there. Moreover, the age defined in [8] is associated with *discrete* events at the information source, where the age is zero until the source is updated. Differently, the age of information in [9] measures the age of a sample of *continuous* events; therefore, the sample immediately becomes old after generated. Our work will contribute to the theory of the age of information.

The previous works [9–17] study the age of information for a single link. The papers [9–13] consider queues to store all unserved packets (i.e., out-of-date packets are also stored) and analyze the long-run average age, based on various queueing models. They show that the optimal sampling rate for the average age is not consistent with the throughput optimum or delay optimum. The paper [14] considers a *smart* update and shows the *always update* scheme cannot minimize the average age. Moreover, [15, 16] develop power-efficient updating algorithms for minimizing the average age. The model in [17] uses a small buffer to store the latest information.

Of the most relevant works on scheduling multiple users are [18–22]. The works [18–20] consider queues at a BS to store all unsent packets, different from ours. The paper [21] considers a buffer to store the latest information with periodic arrivals, while information updates in [22] can be generated at will. Our work is the first to consider random arrivals and to develop optimal off-line and on-line scheduling algorithms for both the no-buffer network and the buffer-network.
II. SYSTEM OVERVIEW

We consider a network consisting of a base-station (BS), and $N$ wireless users $u_1, \cdots, u_N$ as showed in Fig. 2 where each user $u_i$ is interested in information generated by an associated source $s_i$. The information is transmitted through the BS in the form of packets. The network is a fundamental model to investigate centralized scheduling schemes. Our focus is on the noiseless channels, where all transmissions from the BS to each user are considered to be successful with a sufficiently high probability, i.e., we ignore transmission errors. Please see Appendix H for an extension to unreliable channels.

We consider a discrete-time system with slots $t = 0, 1, \cdots$. The packets from the sources (if any) arrive at the BS at the beginning of each slot. We consider that the BS can transmit at most one packet during each slot, i.e., the BS can update at most one user in each slot. We assume that the arrivals at the BS for different users are independent of each other and also independent and identically distributed (i.i.d.) over slots, following a Bernoulli distribution. Precisely, by $\Lambda_i(t)$, we indicate if a packet from source $s_i$ arrives at the BS in slot $t$, in which $\Lambda_i(t) = 1$ if there is a packet; otherwise, $\Lambda_i(t) = 0$, where $P[\Lambda_i(t) = 1] = p_i$.

Moreover, the BS can buffer at most one packet for each user, i.e., an arriving packet always replaces the old one in the buffer. We start with the scenario without any buffer, where the arrivals are discarded if not transmitted in the current slot. The no-buffer network is easy to

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1 If many users have the same target information, we can group them as a super user.
implement for practical systems, and it was argued to achieve good performance in a single link (see [17]). In Section VI we will extend our results by considering the buffers. It turns out that the main results hold for the buffer-network, and the performance of the no-buffer network is very close to the buffer-network (see Sections VI and VII).

A. Age of information model

The age of information tells the freshness of the information at the users. We initialize the ages of the packets to be zero when reaching the BS. On receiving a packet, the age of information for the user is reset to be one due to one slot duration of the transmission time.

Let $A_i(t)$ be the age for user $u_i$ in slot $t$ before the BS makes a decision. Considering a linearly increasing age over slots, the age of user $u_i$ in slot $t$ is $A_i(t) = t - r_i(t)$, where $r_i(t)$ is the slot that the latest packet received by user $u_i$ was generated. Take Fig. 3 for example, where the arrivals in slots $t = 2, 5$ for user $u_i$ are delivered in slots $t = 3, 6$ and hence the dots represent the ages of information for user $u_i$ for each slot.

We remark that since the BS can update at most one user for each slot, $A_i(t) \geq 1$ for all $i$, $A_i(t) \neq A_j(t)$ for all $i \neq j$, and $\sum_{i=1}^{N} A_i(t) \geq 1 + 2 + \cdots + N$ for all $t$.

B. Markov decision process model

We use a Markov decision process (MDP) to develop scheduling algorithms for the BS to adaptively update a user at each transmission opportunity. According to [4], we describe the components of our MDP in detail below, followed by Example 1.
• **Decisions and decision epochs**: For each slot, the BS makes a decision immediately after receiving packets. By $D(t) \in \{0, 1, \cdots N\}$ we denote a decision of the MDP in slot $t$, where $D(t) = 0$ if the BS does not transmit any packet and $D(t) = i$ for $i = 1, \cdots, N$ if user $u_i$ is scheduled to be updated in slot $t$. Considering the no-buffer network, for each slot the BS will update a user given there is a corresponding arrival at the BS, i.e., $D(t) \in \{i : \Lambda_i(t) = 1\}$. Note that the decision-space is finite.

• **States**: We define the state $S(t)$ of the system in slot $t$ as $S(t) = (A_1(t), \cdots, A_N(t), \Lambda_1(t), \cdots, \Lambda_N(t))$. By $S$ we define the state-space including all possible states. Note that $S$ is a **countable infinite** set because the ages are possibly unbounded.

• **Transition probabilities**: Under decision $D(t) = d$, as the transmission time is one slot, the age of information in the next slot is

$$A_i(t + 1) = \begin{cases} 1 & \text{if } i = d; \\ A_i(t) + 1 & \text{else.} \end{cases}$$

Let $P_{s,s'}(d)$ be the transition probability from state $s = (a_1, \cdots, a_N, \lambda_1, \cdots, \lambda_N)$ to state $s' = (a'_1, \cdots, a'_N, \lambda'_1, \cdots, \lambda'_N)$ under decision $D(t) = d$, with the probability law as follows.

$$P_{s,s'}(d) = \begin{cases} \prod_{i : \lambda'_i = 1} p_i \prod_{i : \lambda'_i = 0} (1 - p_i) & \text{if } a'_d = 1 \text{ and } a'_i = a_i + 1 \text{ for } i \neq d; \\ 0 & \text{else.} \end{cases}$$

• **Cost**: Let $C(S(t), D(t) = d)$ be the **immediate cost** if decision $D(t) = d$ is taken in slot $t$ under system state $S(t)$. We consider the **total age** of the system after making decision in slot $t$:

$$C(S(t), D(t) = d) \triangleq \sum_{i=1}^{N} A_i(t + 1) = \sum_{i=1}^{N} (A_i(t) + 1) - A_d(t) \cdot \Lambda_d(t), \quad (1)$$

where we define $A_0(t) = 0$ and $\Lambda_0(t) = 0$ for all $t$ (for the case of $d = 0$), while the last term indicates that user $u_d$ is updated in slot $t$. We remark that we focus on the total age for the purpose of delivering clean results; whereas our analysis and design can also work perfectly for the **weighted sum** of the ages.

**Example 1.** In Fig. 4, we illustrate the age evolution (in the sub-figures) in terms of the arrivals and the decisions (in the table), where the initial ages are $A_1(0) = 2$ and $A_2(0) = 1$ for the two
users. We also show the immediate cost for each slot in the table. As $D(t) = 1$ at time $t = 0$, the resulting age in $t = 1$ for $u_1$ and $u_2$ are $A_1(1) = 1$ and $A_2(1) = 2$, respectively. As a result, the cost associated with the decision is $A_1(1) + A_2(1) = 3$. Similarly, we can calculate the cost for each slot in the table.

C. Average-optimal scheduling algorithm design

A scheduling algorithm $\theta = \{D(0), D(1), \cdots\}$ specifies the decisions at all decision epochs. An algorithm is history dependent if $D(t)$ depends on $D(0), \cdots, D(t - 1)$ and $S(0) \cdots, S(t)$, while it is Markov if $D(t)$ only depends on $S(t)$. An algorithm is stationary if $D(t_1) = D(t_2)$ when $S(t_1) = S(t_2)$ for any $t_1, t_2$. Moreover, a randomized algorithm specifies a probability distribution on the set of decisions for each decision epoch, while a deterministic algorithm makes a decision with certainty. In general, an algorithm belongs to one of the following sets [4]:

- $\Pi_{HR}$: a set of randomized history dependent algorithms;
- $\Pi_{MR}$: a set of randomized Markov algorithms;
- $\Pi_{SR}$: a set of randomized stationary algorithms;
- $\Pi_{SD}$: a set of deterministic stationary algorithms.
The *long-run average cost* for some algorithm $\theta \in \Pi^{HR}$ in general is given by

$$V(\theta) = \limsup_{T \to \infty} \frac{1}{T+1} E_\theta \left[ \sum_{t=0}^{T} C(S(t), D(t)) | S(0) \right],$$

where $E_\theta$ represents the conditional expectation, given that the algorithm $\theta$ is employed.

**Definition 2.** A scheduling algorithm $\theta$ (that belongs to $\Pi^{HR}$) is *average-optimal* if it minimizes the long-run average cost $V(\theta)$.

Our goal is to characterize and obtain an *average-optimal scheduling algorithm*. However, we note (as shown in [4]) that

$$\Pi^{SD} \subset \Pi^{SR} \subset \Pi^{MR} \subset \Pi^{HR}.$$  

The complexity of the algorithms increases from left to right above. According to [4], there may not exist a $\Pi^{SR}$ or $\Pi^{SD}$ algorithm that is average-optimal. Hence, we target at characterizing a regime such that an average-optimal scheduling lies in $\Pi^{SD}$ and developing a simple associated scheduling algorithm.

### III. Characterization of the Average-Optimality

We start with stating a preliminary result, as follows.

**Proposition 3 ([4], Theorem 5.5.3).** If there exists a scheduling algorithm in $\Pi^{HR}$ that is average-optimal, then there exists a scheduling algorithm in $\Pi^{MR}$ that is average-optimal.

Accordingly, the average-optimality of $\Pi^{MR}$ was established, and our focus is to investigate the average-optimality of $\Pi^{SR}$ or $\Pi^{SD}$. To that end, we introduce an infinite horizon $\alpha$-discounted cost case, where $0 < \alpha < 1$; we then connect to the average cost case because structures of an average-optimal algorithm is usually associated with the discounted cost case.

**A. $\alpha$-discounted MDP model**

Given initial state $S(0) = s$, the *total expected discounted cost* under a scheduling $\theta \in \Pi^{HR}$ is

$$V_\alpha(s; \theta) = \lim_{T \to \infty} E_\theta \left[ \sum_{t=0}^{T} \alpha^t C(S(t), D(t)) | S(0) = s \right].$$
Definition 4. A scheduling algorithm $\theta$ is $\alpha$-optimal if it minimizes the total expected discounted cost $V_\alpha(s; \theta)$. In particular, we define

$$V_\alpha(s) = \min_\theta V_\alpha(s; \theta).$$

Moreover, by $h_\alpha(s) = V_\alpha(s) - V_\alpha(0)$ we define the relative cost function, which is the difference of the discounted cost between state $s$ and a reference state $0$. We can arbitrarily choose the reference state, e.g., $0 = (1, 2, \cdots, N, 1, \cdots, 1)$. We first introduce the discounted cost optimality equation of $V_\alpha(s)$ below.

**Proposition 5.** The optimal expected discounted cost $V_\alpha(s)$, for state $s$, satisfies the following discounted cost optimality equation:

$$V_\alpha(s) = \min_{d \in \{0, 1, \cdots, N\}} C(s, d) + \alpha E[V_\alpha(s')],$$

where the expectation is taken over all possible next state $s'$ reachable from the state $s$. A deterministic stationary algorithm that realizes the minimum of right hand side (RHS) of the discounted cost optimality equation in Eq. (3) will be an $\alpha$-optimal algorithm. Moreover, we define $V_{\alpha,n}(s)$ by $V_{\alpha,0}(s) = 0$ and for any $n \geq 0$,

$$V_{\alpha,n+1}(s) = \min_{d \in \{0, 1, \cdots, N\}} C(s, d) + \alpha E[V_{\alpha,n}(s')].$$

(4)

Then, $V_{\alpha,n}(s) \to V_\alpha(s)$ as $n \to \infty$ for every $s$, and $\alpha$.

**Proof:** Please see Appendix A.

The value iteration in Eq. (4) will be helpful for identifying properties of $V_\alpha(s)$, e.g., to prove that $V_\alpha(s)$ is a non-decreasing function in the following.

**Proposition 6.** $V_\alpha(a_i, a_{-i}, \lambda)$ is a non-decreasing function in $a_i$ given $a_{-i} = \{a_1, \cdots, a_N\} - \{a_i\}$ and $\lambda = (\lambda_1, \cdots, \lambda_N)$.

**Proof:** Please see Appendix B.

B. Average-optimality of a deterministic stationary algorithm

Using the properties of $\alpha$-discounted MDP (Propositions 5 and 6), we show that the MDP defined in this paper has an average-optimal algorithm that is deterministic stationary.
Lemma 7. There exists a scheduling algorithm in $\Pi^{SD}$ that is average-optimal. Moreover, there exists a finite constant $V^* = \lim_{\alpha \to 1} (1 - \alpha) V_\alpha(s)$ for every state $s$ such that the average-optimal cost is $V^*$, independent of the initial state $S(0)$.

Proof: Please see Appendix C.

We want to elaborate further on Lemma 7. First, we notice that there is no condition for the existence of an average-optimal scheduling algorithm in $\Pi^{SD}$. In general, we need some conditions to guarantee the reduced Markov chain from applying a stationary scheduling algorithm is positive recurrent. We can think of the age of our problem as an age-queue, where the age-queueing system consists of a age-queue, input to the queue, and a server. The input rate is one per slot since the age increases by one for each slot, while the server can serve infinite number of packet for each service opportunity. As such, we always can find a scheduling algorithm such that the average arrival rate is less than the service rate and the reduced Markov chain is positive recurrent. Second, we remark that even though the average-optimality of a deterministic stationary algorithm is shown in Lemma 7, we might not arrive at the average cost optimaility equation like Eq. (3) or the value iteration like Eq. (4) (see [7, 23] for details).

C. Structural results

In addition to the average-optimality of a deterministic stationary algorithm, we show that an optimal scheduling algorithm has a nice structure facilitating the scheduling algorithm design in the next section.

Definition 8. A switch-type scheduling algorithm is a special deterministic stationary scheduling algorithm: for every user $u_i$, if the decision of the algorithm for state $s = (a_i, a_{-i}, \lambda)$ is $d_s = i$, then the decision for state $s' = (a_i + 1, a_{-i}, \lambda)$ is $d_{s'} = i$ as well.

Theorem 9. An average-optimal scheduling algorithm is the switch type.

Proof: (Sketch) We first prove that an $\alpha$-optimal scheduling algorithm is the switch type by applying the value iteration in Eq. (4). Then, we show that the structure holds for the average-optimum by letting $\alpha \to 1$. Please see Appendix D for details.

In particular, when the arrival rates of all information sources are the same, we can obtain a nice index algorithm as follows.
Corollary 10. If the arrival rates of all information sources are the same, i.e., \( p_i = p_j \) for all \( i \neq j \), then an optimal scheduling algorithm transmits a coming packet with the largest age of information, i.e., \( D(t) = \arg \max_i A_i(t) \Lambda_i(t) \) for each time \( t \).

**Proof:** Please see Appendix [E].

We also notice that the index algorithm is indeed an on-line algorithm, without the knowledge of the arrival statistics in advance. For general asymmetric arrivals, an average-optimal scheduling algorithm depends on both the arrival statistics and the current ages. However, it is not easy to get an average-optimal scheduling for the asymmetric arrivals. This key challenge hence motivates us to investigate both off-line and on-line scheduling algorithms in the next section.

IV. SCHEDULING ALGORITHM DESIGN

We start with proposing finite-state approximations to the original MDP as in practice we can only work on a finite-state MDP to avoid formidably high computational complexity. We will rigorously show the convergence of the proposed truncation as in general a MDP truncation might not converge to the original MDP according to [7].

Based on the approximate finite-state MDPs, we first develop a structural value iteration algorithm in Section [IV-B] to pre-compute an optimal decision for each state, whose complexity would be lower than the conventional value iteration algorithm by leveraging the switch-type structure. Moreover, we develop an on-line algorithm using reinforcement learning techniques [24, 25] and stochastic approximations [26] in Section [IV-C].

A. Finite-state MDP approximations

Let \( \Delta \) be the Markov decision process defined in Section [II-B]. By \( \{ \Delta_m \} \) we define a sequence of approximate MDPs for \( \Delta \) whose state-space is \( S_m = \{ s \in S : a_i \leq m \} \) with the bounded virtual ages, while the decision-space and cost definition (see Eq. (1)) are the same as \( \Delta \).

Let \( A_i^{(m)}(t) \) be the age of information for user \( u_i \) in slot \( t \) for \( \Delta_m \). Different from \( \Delta \), under decision \( D(t) = d \) the age of the next slot for \( \Delta_m \) is

\[
A_i^{(m)}(t + 1) = \begin{cases} 
1 & \text{if } i = d; \\
\left[ A_i^{(m)}(t) + 1 \right]^+_m & \text{else,}
\end{cases}
\]
Now, we are ready to propose scheduling algorithms based on $\Delta_m$. In other words, our algorithms make decisions according to the virtual age $A_i(t)$ on $S_m$, instead of the real age $A_i(t)$ on $S$. The real age can increase beyond $m$ but the virtual age will be smaller than $m + 1$.

![Example of the age evolution for the finite-state approximation $\Delta_3$.](image)

| $t$ | $A_1(t)$ | $A_2(t)$ | $D(t)$ | Cost(t) |
|-----|----------|----------|--------|--------|
| 0   | 1        | 1        | 1      | 3      |
| 1   | 0        | 0        | 0      | 5      |
| 2   | 0        | 1        | 2      | 4      |
| 3   | 1        | 1        | 2      | 4      |
| 4   | 1        | 0        | 1      | 3      |

where we define the notation $[x]_m^+$ by $[x]_m^+ = x$ if $x \leq m$ and $[x]_m^+ = m$ if $x > m$.

**Example 11.** In Fig. 5 we illustrate the age evolution for the finite-state approximation $\Delta_3$. Different from $\Delta$ in Fig. 4 in slot 4 the age of the user $u_1$ keeps three.

Let $P_{s,s'}^{(m)}(d)$ be the transition probability for $\Delta_m$. Then, $P_{s,s'}^{(m)}(d) = P_{s,s'}(d)$ if $s' \in S_m$; otherwise,

$$P_{s,s'}^{(m)}(d) = P_{s,s'}(d) + \sum_{r \in S - S_m} P_{s,r}(d),$$

(5)

for some excess probabilities on state $r \in S - S_m$.

Next, we will show that the proposed finite-state approximation will be asymptotically average-optimal.

**Theorem 12.** Let $V^*$ and $V_m^*$ be the average-optimal cost to $\Delta$ and $\Delta_m$, respectively. Then, $V_m^* \to V^*$ as $m \to \infty$.

**Proof:** Please see Appendix E. □

Now, we are ready to propose scheduling algorithms based on $\Delta_m$. In other words, our algorithms make decisions according to the virtual age $A_i^{(m)}(t)$ on $S_m$, instead of the real age $A_i(t)$ on $S$. The real age can increase beyond $m$ but the virtual age will be smaller than $m + 1$. 
(see Figs. 4 and 5 for example).

B. Structural off-line scheduling algorithm

The traditional relative value iteration algorithm (RVIA), as follows, can be applied to get an optimal deterministic stationary algorithm on $\Delta_m$:

$$V_{n+1}(s) = \min_{d \in \{0, 1, \ldots, N\}} C(s, d) + E[V_n(s')] - V_n(0),$$

for all $s \in S_m$. For each iteration $n$, we need to update decisions for all virtual states by minimizing the RHS of Eq. (6) as well as update $V(s)$ for all $s \in S_m$. The complexity is very high due to many users, i.e., curse of dimensionality [27]. Therefore, we propose in Alg. 1 a structural off-line algorithm based on the RVIA along with the switch-type structure.

In Alg. 1, we seek an optimal decision $d^*_s$ for each virtual state $s \in S_m$ by iteration. For each iteration, we update both the optimal decision $d^*_s$ and $V(s)$ for all virtual states. If the switch property holds, we can determine an optimal decision immediately in Line 5; otherwise we find an optimal decision according to Line 7. By $V_{tmp}(s)$ in Line 9 we temporarily keep the updated value, which will replace $V(s)$ in Line 11. Using the switch structure to prevent from the minimum operations on all virtual states in the RVIA, we can reduce the computational complexity greatly.

**Theorem 13.** The limit point of $d^*_s$ in Alg. 1 is an average-optimal decision for every virtual state $s \in S_M$. In particular, Alg. 1 converges in a finite number of iterations.

**Proof:** (Sketch) According to [4, Theorem 8.6.6], we only need to verify that the truncated MDPs are unichain. Please see Appendix G for details.

C. On-line scheduling algorithm

We note that Alg. 1 needs the arrival statistics to pre-compute an optimal decision for each virtual state. We will develop an on-line scheduling algorithm if the statistics is unavailable. Instead of updating $V(s)$ for all virtual states in each iteration, we update $V(s)$ following a

$^2$As the size of the state-space is $O(m^N)$, the complexity to update all states in each iteration of Eq. (5) is bigger than $O(m^N)$. 
Algorithm 1: Structural off-line scheduling algorithm

1 \( V(s) \leftarrow 0 \) for all states \( s \in S_m \);
2 \textbf{while} 1 \textbf{do}
3 \hspace{1em} \textbf{forall} \( s \in S_m \) \textbf{do}
4 \hspace{2em} \textbf{if} there exists \( \zeta > 0 \) and \( i \in \{1, \cdots, N\} \) such that \( d^*_a(\omega_1, \cdots, \omega_m) = i \) \textbf{then}
5 \hspace{3em} \( d^*_s \leftarrow i; \)
6 \hspace{2em} \textbf{else}
7 \hspace{3em} \( d^*_s \leftarrow \arg \min_{d \in \{0, 1, \cdots, N\}} C(s, d) + E[V(s')]; \)
8 \hspace{2em} \textbf{end}
9 \hspace{1em} \( V_{\text{tmp}}(s) \leftarrow C(s, d^*_s) + E[V(s')] - V(0); \)
10 \textbf{end}
11 \( V(s) \leftarrow V_{\text{tmp}}(s) \) for all \( s \in S_m \).
12 \textbf{end}

sample path, which is a set of outcomes of the arrivals over slots. It turns out that the sample-path updates will converge to the average-optimal solution.

To that end, we need a stochastic version of the RVIA. However, the RVIA in Eq. (6) is not suitable because the expectation is inside the minimization (as in [27]). While minimizing the RHS of Eq. (6) for a given current state, we would need the transition probabilities to calculate the expectation. To tackle these challenges, we design post-decision states [27, 28] for our problem.

We define the post-decision state \( \tilde{s} \) as the ages and the arrivals after a decision. The state we used before is referred to as the pre-decision state. If \( s = (a_1, \cdots, a_N, \lambda_1, \cdots, \lambda_N) \in S_m \) is a virtual state of the system, then the virtual post-decision state after decision \( d \) is \( \tilde{s} = (\tilde{a}_1, \cdots, \tilde{a}_N, \tilde{\lambda}_1, \cdots, \tilde{\lambda}_N) \) with

\[
\tilde{a}_i = \begin{cases} 
1 & \text{if } i = d \text{ and } \lambda_i = 1; \\
[a_i + 1]^+_m & \text{else},
\end{cases}
\]

and \( \tilde{\lambda}_i = \lambda_i \) for all \( i \).

Let \( \tilde{V}(\tilde{s}) \) be the value function based on the post-decision states defined by

\[
\tilde{V}(\tilde{s}) = E_s[V(s)],
\]

where the expectation \( E_s \) is taken over all possible the pre-decision states reachable from the
post-decision state. We can then write down the post-decision average cost optimality equation [27] for the virtual post-decision age $\tilde{s} = (\tilde{a}_1, \ldots, \tilde{a}_N, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N)$ with $\tilde{s} \in S_m$:

$$V(\tilde{s}) + V^* = E\left[\min_{d \in \{0,1,\ldots,N\}} C\left( (\tilde{a}, \tilde{\lambda}'), d \right) + V([\tilde{a} + 1 - \tilde{a}^+_d]_{jm}, \tilde{\lambda}') \right],$$

where $\tilde{a}_i = (0, \ldots, \tilde{a}_i, \ldots, 0)$ is the zero vector except for the $i$-th entry being substituted by $\tilde{a}_i$, and $1 = (1, \ldots, 1)$ is the unit vector. Moreover, $\tilde{\lambda}'$ summarizes the possible next arrivals.

We also recall that $V^*$ here is the constant as in Lemma 7.

From above optimality equation, the RVIA is as follows:

$${\tilde{V}}_{n+1}(\tilde{s}) = E\left[\min_{d \in \{0,1,\ldots,N\}} C\left( (\tilde{a}, \tilde{\lambda}'), d \right) + {\tilde{V}}_n([\tilde{a} + 1 - \tilde{a}^+_d]_{jm}, \tilde{\lambda}') \right] - {\tilde{V}}_n(0). \tag{7}$$

Subsequently, we propose the on-line algorithm in Alg. 2 based on the stochastic version of the RVIA. In Lines 1-3, we initialize $\tilde{V}(\tilde{s})$ of all virtual post-decision states and start from the reference point. Moreover, by $v$ we record $\tilde{V}(\tilde{s})$ of the current virtual post-decision state.

By observing the current arrivals $\Lambda(t)$ and plugging in Eq. (7), in Line 5 we optimally update a user by minimizing Eq. (8); as such, the expectation in Eq. (7) is removed. Then, we update $\tilde{V}(\tilde{s})$ of the current virtual post-decision state in Line 7 where $\gamma(t)$ is the stochastic step-size (see [27]) in slot $t$ and hence $\tilde{V}(\tilde{s})$ is balanced between the previous $\tilde{V}(\tilde{s})$ and the current value $v$. Finally, the next virtual post-decision state is updated in Lines 8 and 9.

Theorem 14. If $\sum_t \gamma(t) = \infty$ and $\sum_t \gamma^2(t) < \infty$, then Alg. 2 converges to the average-optimal value.

Proof: According to [24, 25], we only need to verify that the truncated MDPs are unichain, which has been completed in Appendix G.

In the above theorem, $\sum_t \gamma(t) = \infty$ implies that Alg. 2 needs infinite number of iterations to learn the average-optimal solution, while Alg. 1 can converge to the optimal solution in a finite number of iterations. Moreover, $\sum_t \gamma^2(t) < \infty$ means that the noise from measuring $\tilde{V}(\tilde{s})$ can be controlled.

Finally, we want to emphasize that the proposed Algs. 1 and 2 have been proven to be asymptotically optimal, i.e., they converge to the optimal solution when the finite state-space $m$ and the slot $t$ goes to infinity. In Section VI, we will also numerically study the performance of
Algorithm 2: On-line scheduling algorithm

/* Initialization */
1 \( \tilde{V}(\tilde{s}) \leftarrow 0 \) for all states \( \tilde{s} \in S_m \);
2 \( \tilde{s} \leftarrow 0 \);
3 \( v \leftarrow 0 \);
4 while 1 do
   /* Decision in slot t */
5   We optimally make a decision \( D^*(t) \) in slot \( t \) according to the current arrivals
6   \( \Lambda(t) = (\Lambda_1(t), \cdots, \Lambda_N(t)) \) in slot \( t \):
7       \[ D^*(t) = \arg \min_{d \in \{0, 1, \cdots, N\}} C((\tilde{a}, \Lambda(t)), d) + \tilde{V}([\tilde{a} + 1 - \tilde{a}^*_d]_M, \Lambda(t)); \]  \( (8) \)
   /* Value update */
6   \( v \leftarrow C((\tilde{a}, \Lambda(t)), D^*(t)) + \tilde{V}([\tilde{a} + 1 - \tilde{a}^*_D(t)]_M, \Lambda(t)) - \tilde{V}(0); \)
7   \( \tilde{V}(\tilde{s}) \leftarrow (1 - \gamma(t))\tilde{V}(\tilde{s}) + \gamma(t)v; \)
   /* Post-decision state update */
8   \( \tilde{a} \leftarrow [\tilde{a} + 1 - \tilde{a}^*_D(t)]_M; \)
9   \( \tilde{\lambda} \leftarrow \Lambda(t). \)
10 end

our algorithms over finite slots.

V. SCHEDULING FOR THE BUFFER-NETWORK

Thus far, our design and analysis of scheduling focus on the no-buffer network. Intuitively, by storing the latest information, the buffers at the BS can reduce the average age in case of no arrivals in the next slot. To what extent can the buffers improve the performance at the expense of the deployed buffers?

Hence, we want to answer the question by considering the buffer-network, where for each user there is a buffer at the BS to store the latest information. That is, a new arrival will replace the old packet (if any) in a buffer.

The MDP of the system will be similar to the no-buffer system. We emphasize the differences as follows.

• **States:** Due to the buffers at the BS, we need to redefine the states. In addition to the age \( A_i(t) \) of information in user \( u_i \), by \( I_i(t) \) we define the **initial age of the information** at the
buffer for user \( u_i \); precisely,
\[
I_i(t) = \begin{cases} 
0 & \text{if } \Lambda(t) = 1; \\
I_i(t-1) + 1 & \text{else}.
\end{cases}
\]

We note that the initial age at the BS keeps increasing even after the packet has been delivered. Then, we define the state of the system as \( S(t) = \{A_1(t), \ldots, A_N(t), I_1(t), \ldots, I_N(t)\} \).

- **Transition probabilities**: Under decision \( D(t) = d \), the ages are updated according to
\[
A_i(t+1) = \begin{cases} 
I_d(t) + 1 & \text{if } i = d; \\
A_i(t) + 1 & \text{else},
\end{cases}
\]
where the user \( u_d \) is updated on the packet in the buffer; as such, its age becomes \( I_d(t) \) plus one slot of the transmission. Similarly, we can write down the transition probability \( P_{s,s'}(d) \) from state \( s = (a_1, \ldots, a_N, I_1, \ldots, I_N) \) to state \( s' = (a'_1, \ldots, a'_N, I'_1, \ldots, I'_N) \) as follows.
\[
P_{s,s'}(d) = \begin{cases} 
\prod_{i: \lambda_i' = 1} p_i \prod_{i: \lambda_i' = 0} (1 - p_i) & \text{if } I'_i = 0 \text{ for } \lambda'_i = 1, \\
I'_i = I_i + 1 & \text{for } \lambda'_i = 0, \\
A'_d = I_d + 1, \\
A'_i = a_i + 1 & \text{for } i \neq d; \\
0 & \text{else}.
\end{cases}
\]

- **Cost**: The immediate cost is redefined as
\[
C(S(t), D(t) = d) = I_d(t) + 1 + \sum_{i \neq d} (A_i(t) + 1),
\]
where the first term of RHS indicates the age of user \( u_d \) who is updated to the initial age and we also add one slot for transmission.

**Example 15.** We investigate the advantage of the buffers through Fig. 6. Here, we illustrate the age evolution in Fig. 4 after deploying the buffers. We note that in slot 1, the user \( u_2 \) can be updated using the packet in the buffer. By means of the buffers, the BS can update a user even without any arrival at present.

We then consider the same long-run average cost as before and intend to find average-optimal
scheduling algorithms for the buffer-network. Using the same definition of the discounted cost, we can similarly show the average-optimality of a deterministic stationary algorithm.

**Theorem 16.** For the buffer-network, there exists a deterministic stationary algorithm that is average-optimal. Moreover, there exists a finite constant $V^* = \lim_{\alpha \to 1} (1 - \alpha)V_\alpha(s)$ for every state $s$ such that the average-optimal cost is $V^*$.

Let $I = (I_1, \cdots, I_N)$ be the vector including all initial ages. We then characterize a structure of an average-optimal algorithm as follows.

**Theorem 17.** An average-optimal scheduling algorithm for the buffer-network is a switch type. For every user $u_i$, if an average-optimal decision at state $s = (a_i, a_{-i}, I)$ is $d^*_i(a_i, a_{-i}, I) = i$, then $d^*_{i+1, a_{-i}, I} = i$.
A. Finite-state MDP approximations

Similar to Section IV-A, we define a sequence of finite-state approximations \{\Delta_m\} to the original MDP \(\Delta\) of the buffer-network, where the finite state-space for \(\Delta_m\) is \(S_m = \{s \in S : a_i \leq m, I_i \leq m\}\), while the decision-space and cost definition are the same as \(\Delta\).

Similarly, under decision \(D(t) = d\) the virtual age in the next slot for \(\Delta_m\) is

\[
A_i^{(m)}(t+1) = \begin{cases} 
\left[ I_d^{(m)}(t) + 1 \right]^+_m & \text{if } i = d; \\
\left[ A_i^{(m)}(t) + 1 \right]^+_m & \text{else.}
\end{cases}
\]

In addition, the virtual initial age is

\[
I_i^{(m)}(t) = \begin{cases} 
\left[ I_i^{(m)}(t-1) + 1 \right]^+_m & \text{if } \Lambda_i(t) = 0; \\
0 & \text{else.}
\end{cases}
\]

Similar to Theorem 12, we can show that the proposed finite-state approximation of the buffer-network will be asymptotically average-optimal, as follows.

**Theorem 18.** Let \(V^*\) and \(V_m^*\) be the average-optimal cost to \(\Delta\) and \(\Delta_m\) of the buffer-network, respectively. Then, \(V_m^* \to V^*\) as \(m \to \infty\).

B. Structural off-line scheduling algorithm

Similar to Alg. 1, we propose the structural off-line scheduling algorithm for the buffer-network in Alg. 3 to pre-compute an optimal decision for each virtual state. Note that in Line 5 the switch structure for the buffer-network is applied. Similar to Theorem 13, we can get the following result.

**Theorem 19.** The limit point of \(d_s^*\) in Alg. 3 is an average-optimal decision for every state \(s \in S_M\) of the buffer-network. In particular, Alg. 3 converges in a finite number of iterations.

C. On-line scheduling algorithm

Similar to Alg. 2, we propose the on-line scheduling algorithm for the buffer-network in Alg. 4. In Line 5, we update the virtual pre-decision state \(I\) by observing the current arrivals, while making a decision according to the current virtual initial age in Line 6, where \(I_i = (0, \cdots, I_i, \cdots, 0)\) is the zero vector except for the \(i\)-th item being replaced by \(I_i\).
**Algorithm 3:** Structural off-line scheduling algorithm for the buffer-network

1. $V(s) \leftarrow 0$ for all states $s \in S_m$;
2. while 1 do
3.   forall $s \in S_m$ do
4.     if there exists $\zeta > 0$ and $i \in \{1, \cdots, N\}$ such that $d_{(a_i-\zeta, a_i+\zeta)} = i$ then
5.       $d_s^* \leftarrow i$;
6.     else
7.       $d_s^* \leftarrow \arg \min_{d \in \{0,1,\cdots,N\}} C(s,d) + E[V(s')]$;
8.     end
9.   $V_{tmp}(s) \leftarrow C(s, d_s^*) + E[V(s')] - V(0)$;
10. end
11. $V(s) \leftarrow V_{tmp}(s)$ for all $s \in S_m$.
12. end

**Theorem 20.** For the buffer-network, if $\sum_t \gamma(t) = \infty$ and $\sum_t \gamma^2(t) < \infty$, then Alg. 4 converges to the average-optimal cost.

Similarly, the algorithms for the buffer-network are asymptotically optimal, while their performance over finite slots will be numerically studied in the next section.

**VI. Numerical results**

In this section, we conduct extensive simulations of the proposed scheduling algorithms. We will demonstrate the switch-type structure in Section VI-A. In Section VI-B we study the proposed off-line scheduling algorithms and compare the performance between the no-buffer network and buffer-network. While we have shown that the proposed on-line scheduling algorithms will converge to the optimal solution as the slot approaches infinity, in Section VI-C we show the performance of the on-line algorithms over finite slots.

**A. Switch-type structure of Alg. 1**

Figs. 7-(a) and 7-(b) show the switch-type structure of an average-optimal scheduling algorithm for two users in the no-buffer network. The experiment setting is as follows. We run Alg. 1 with the boundary $m = 10$ over 100,000 slots to search an optimal decision for each virtual state. Moreover, we consider various arrival rates with $(p_1, p_2)$ being $(0.9, 0.9)$ and $(0.9, 0.5)$, respectively, in Figs. 7-(a) and 7-(b), where the *dots* represent $D(t) = 1$ and the *stars* mean
Algorithm 4: On-line scheduling algorithm for the buffer-network

/* Initialization */
1 \( \tilde{V}(\tilde{s}) \leftarrow 0 \) for all states \( \tilde{s} \in S_m \);
2 \( \tilde{s} \leftarrow 0 \);
3 \( v \leftarrow 0 \);
4 while 1 do

/* Decision in slot \( t \) */
5 Update the initial ages according to the current arrivals \( \Lambda(t) = (\Lambda_1(t), \cdots, \Lambda_N(t)) \) in slot \( t \):
6 \( I_i \leftarrow \tilde{I}_i \) if \( \Lambda_i(t) = 0 \); otherwise, \( I_i \leftarrow 0 \);
7 We optimally make a decision \( D^*(t) \) in slot \( t \):
8 \( D^*(t) = \arg \min_{d \in \{0, 1, \cdots, N\}} C((\tilde{a}, I), d) + \tilde{V}([\tilde{a} + 1 - (\tilde{a}_d - I_d)]_m^+, [I + 1)_m^+) \);

/* Value update */
9 \( v \leftarrow C((\tilde{a}, I), D^*(t)) + \tilde{V}([\tilde{a} + 1 - (\tilde{a}_{D^*(t)} - I_{D^*(t)})]_m^+, [I + 1)_m^+) - \tilde{V}(0) \);
10 \( \tilde{V}(\tilde{s}) \leftarrow (1 - \gamma(t))\tilde{V}(\tilde{s}) + \gamma(t)v \);
11 /* Post-decision state update */
12 \( \tilde{a} \leftarrow [\tilde{a} + 1 - (\tilde{a}_{D^*(t)} - I_{D^*(t)})]_m^+ \);
13 \( \tilde{I} \leftarrow [I + 1]_m^+ \);
14 end

Fig. 7. (a) Switch structure for \( p_1 = p_2 = 0.9 \); (b) switch structure for \( p_1 = 0.9 \) and \( p_2 = 0.5 \).

\( D(t) = 2 \). We observe the switch structure in the figures, while Fig. 7(a) is consistent with the index algorithm by simply comparing the ages of the two users, as stated in Corollary 10. Moreover, after fixing the arrival rate \( p_1 = 0.9 \) for the first user, the BS will give a higher priority to the second user as \( p_2 \) decreases. That is because the second user takes more time to wait for the next arrival and becomes the bottleneck.
B. Off-line algorithms with and without buffers: Algs. 1 and 3

Thus far, we obtain an optimal scheduling decision for each state, like Section VI-A. Now, we are ready to investigate the time-average age by employing the obtained scheduling decisions in the last subsection. The experiment setting is as follows. We consider the truncated MDP with the boundary $m = 30$ and generate arrivals for each user according to the Bernoulli ($p$) distribution, i.e., $p_1 = p_2 = p$. The BS then decides to update a user according to the arrivals and the age of information for each user. After averaging the age over 100,000 slots, we obtain the average total age in Fig. 8 (blue curve with the circle markers) for various $p$. Moreover, we also show the average total age for the buffer-network in Fig. 8 (green curve with the square markers), where the BS updates a user based on the optimal decisions from Alg. 3 according to the arrivals, initial ages at the buffers, and ages of the users.

Intuitively, we can improve the average total age by exploiting the buffers to store unsent packets when both packets arrive in the same slot. However, it is quite interesting that the no-buffer network and the buffer-network result in the similar performance in Fig. 8. Let us discuss the following three cases when both users have the arrivals in some slot:

- **When both $p_1$ and $p_2$ are high**: That means the user who is not updated currently has a new arrival in the next slot with a high probability; as such, the old packet in the buffer seems not that effective.
- **When both $p_1$ and $p_2$ are low**: Then, the possibility of the two arrivals in the same slot is very low. Hence, this would be a trivial case.
• When one of $p_1$ and $p_2$ are high and the other is low: In this case, the BS will give the user with the lower arrival rate a higher update priority, while a packet for the other user will arrive shortly.

Moreover, since we aim at minimizing the average total age, the BS would prefer users with new arrivals to those with some old packets in the buffer. According to the above discussions, we observe that the buffers seem not that effective as expected; whereas, the no-buffer network is not only simple for practical implementation but also works pretty well.

We also compare the average age per user for different numbers of users in Fig. 9 by running the index algorithm (in Corollary 10), where the identical arrival rates are $p = 0.5$ (red line), 0.7 (green line), and 0.9 (blue line), respectively.

C. On-line algorithms with and without buffers: Algs. 2 and 4

Thus far, we analyze the average age of the optimal off-line algorithms. In this subsection, we will examine the proposed on-line algorithms, with the step size $\gamma(t) = 1/t$.

First, we show the average total age of the index algorithm and on-line scheduling algorithms in Fig. 10 where we consider two users with $p_1 = p_2 = p$. The BS then decides to update a user according to the current ages and arrivals by running Algs. 2 and 4 for the no-buffer network and buffer-network, respectively. The solid curves indicate the on-line algorithms for the no-buffer

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3The choice of the step size is similar to [28] and works well for the no-buffer network. How to choose the best step size is an interesting practical issue, but is not within the scope of this paper.
Fig. 10. Average total age for two users by running the on-line or off-line algorithms.

Fig. 11. Average total age for three users by running the on-line or off-line algorithms.

Fig. 12. Average total age for four users by running the on-line or off-line algorithms.
network and the buffer-network, whose performance is slightly worse than the index algorithm in the dashed curve.

Intuitively, the on-line algorithms learn the optimal value for each state by updating the value of the current post-decision state during the exploration \[27\], and then the on-line algorithms can use an optimal decision by exploiting the optimal value for each state after the exploration. However, if the state space is huge, it is quite easy to become stuck in a local solution because we have poor estimates of the values of some states or we never visit some states during the simulation (see \[27\] for more discussions). Hence, we can theoretically guarantee the performance of the on-line algorithm for infinite horizon; whereas, the optimal decisions are not available for problems with a large state-space.

Furthermore, we show the average total age for three and four users in Figs. 11 and 12, respectively. We find that the on-line algorithm for the no-buffer network even outperforms that for the buffer-network. That is because the state-space of the buffer-network is much larger than the no-buffer network. We note that for four users the state-space of the on-line algorithm for the buffer-network is \(m^8\), where \(m\) is the boundary of the truncated MDP, and it is \(30^8\) if \(m = 30\). During the simulation time of 100,000 slots, at least \(30^8 - 10^5\) states are not visited; as such, it is much harder for the on-line algorithm to learn optimal decisions in the buffer-network.

To conclude, the optimal average age of the no-buffer network is close to that of the buffer-network (see Fig. 8). Given the option to deploy buffers for the network, we would also suggest to implement the on-line algorithm for the simple no-buffer network due to the smaller state-space.

VII. Conclusion

In this paper, we consider a broadcast network, where many users are interested in different information that should be delivered by a base-station. We theoretically investigate the average age of information by designing and analyzing optimal scheduling algorithms. We show that an optimal scheduling algorithm is a simple stationary switch type. To tackle the infinite state-space Markov decision process (MDP), we propose a sequence of finite-state approximate MDPs. Based on the approximate MDPs, we propose both optimal off-line and on-line scheduling algorithms. The algorithms are further studied and compared by numerical results. It turns out that no-buffer networks are not only simple for practical implementation but have good performance close to buffer-networks.
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Appendix A

Proof of Proposition 5

According to [6, 23], it suffices to show that $V_\alpha(s) < \infty$ for every initial state $s$ and discount factor $\alpha$.

Let $f$ be the stationary algorithm of always doing nothing (i.e., $D(t) = 0$ for all $t$) for each
By the definition of the optimality, if \( V_\alpha(s; f) < \infty \), then \( V_\alpha(s) < \infty \). Note that

\[
V_\alpha(s; f) = \lim_{T \to \infty} E_f \left[ \sum_{t=0}^{T} \alpha^t C(S(t), D(t)) | S(0) = s \right] \\
= \sum_{t=0}^{\infty} \alpha^t (a_1 + t) + \cdots + (a_N + t) \\
= \frac{a_1 + \cdots + a_N}{1 - \alpha} + \frac{\alpha N}{(1 - \alpha)^2} < \infty.
\]

**Appendix B**

**Proof of Proposition 6**

The proof is based on induction on \( n \) in Eq. (4). The result clearly holds for \( V_{\alpha,0}(s) \).

Now, we assume that \( V_{\alpha,n}(s) \) is non-decreasing in \( a_i \). First, we note that the immediate cost \( C(s, d) = 1 + \sum_{j \neq d} (a_j + 1) \) is a non-decreasing function in \( a_i \). Next, \( E[V_{\alpha,n}(s')] \) is also a non-decreasing function in \( a_i \) according to the induction assumption. Since the minimum operator (in Eq. (4)) holds the non-decreasing property, we conclude that \( V_{\alpha,n+1}(s) \) is a non-decreasing function as well.

**Appendix C**

**Proof of Lemma 7**

According to [23], we need to prove that the following three conditions are satisfied.

1) \( V_\alpha(s) \) is finite for all \( s \) and \( \alpha \): It has been shown in Appendix A.

2) There exists a deterministic stationary algorithm \( f \) of our MDP such that the average total age is finite: We can view the age of information for each user as an age-queue, where the age-queueing system consists of the age-queue, inputs to the queue, and a server. The input rate of the age-queue is one per slot because the age increases by one per slot, while the service rate is infinity as the age is set to be one once updated. It is then easy to establish the algorithm \( f \) (e.g., to update the user with the largest age for each time) such that the average size of the age-queue is finite/stable [2], since the input rate is less than the service rate.

3) There exists a nonnegative \( L \) such that the relative cost function \( h_\alpha(s) \geq -L \) for all \( s \) and \( \alpha \): Let \( C_{s,s'}(\theta) \) be the expected cost of the first passage from state \( s \) to state \( s' \) using a
scheduling algorithm $\theta$. Then, using the deterministic stationary algorithm $f$ in the second condition, we have $C_{s,s'}(f) < \infty$ (see [23, Proposition 4]) and $h_\alpha(s) \geq -C_{0,s}$ (see [23, proof of Proposition 5] or [29, proof of Proposition 4.2]). Moreover, as $V_\alpha(s)$ is a non-decreasing function in $a_i$ (Proposition 6), only the states $s = (a_1, \ldots, a_N, 1, \ldots, 1)$ with $a_i \leq N$ can probably result in a lower value of $V_\alpha(s)$ than $V_\alpha(0)$. We hence can choose $L = \max_{s \in S: a_i \leq N} C_{0,s}$.

By verifying the three conditions, the result immediately follows from [23].

**Appendix D**

**Proof of Theorem 9**

First, we show that an $\alpha$-optimal scheduling algorithm is the switch type. Let $\nu_\alpha(s; d) = C(s, d) + \alpha E[V_\alpha(s')]$. Then, $V_\alpha(s) = \min_{d \in \{0, \ldots, N\}} \nu_\alpha(s; d)$. Without loss of generality, we suppose that an $\alpha$-optimal decision at state $s = (a_1, a_{-1}, \lambda)$ is to update the user $u_1$ with $\lambda_1 = 1$. Then, according to the $\alpha$-optimality of $d^*_{(a_1, a_{-1}, \lambda)} = 1$,

$$\nu_\alpha(a_1, a_{-1}, \lambda; 1) - \nu_\alpha(a_1, a_{-1}, \lambda; j) \leq 0,$$

for all $j \neq 1$.

Let $1 = (1, \ldots, 1)$ be the vector with all entries being one. Let $a_i = (0, \ldots, a_i, \ldots, 0)$ be the zero vector except for the $i$-th entry being replaced by $a_i$. To demonstrate the switch-type structure, we consider the two cases as follows.

1) For any other user $u_j$ with $\lambda_j = 1$: Since $V_\alpha(a_1, a_{-1}, \lambda)$ is a non-decreasing function in $a_1$ (Proposition 6), we get

$$\nu_\alpha(a_1 + 1, a_{-1}, \lambda; 1) - \nu_\alpha(a_1 + 1, a_{-1}, \lambda; j)$$

$$=a_j - (a_1 + 1) + \alpha E[V_\alpha(1, a_{-1} + 1, \lambda') - V_\alpha(a_1 + 2, a_{-1} + 1 - a_j, \lambda')]$$

$$\leq a_j + \alpha E[V_\alpha(1, a_{-1} + 1, \lambda') - V_\alpha(a_1 + 1, a_{-1} + 1 - a_j, \lambda')]$$

$$=\nu_\alpha(a_1, a_{-1}, \lambda; 1) - \nu_\alpha(a_1, a_{-1}, \lambda; j) \leq 0,$$

where $\lambda'$ summarizes the possible next arrivals.
2) For any other user $u_j$ with $\lambda_j = 0$: Similarly, we have

$$\nu_\alpha(a_1 + 1, a_{-1}, \lambda; 1) - \nu_\alpha(a_1 + 1, a_{-1}, \lambda; j)$$

$$= - (a_1 + 1) + \alpha E[V_\alpha(1, a_{-1} + 1, \lambda') - V_\alpha(a_1 + 2, a_{-1} + 1, \lambda')] \leq 0.$$ 

Considering the two cases, an $\alpha$-optimal decision for state $(a_1 + 1, a_{-1}, \lambda)$ is still to update $u_1$, yielding the switch-type structure.

Then, let $\{\alpha_n\}$ be a sequence of the discount factors. According to [23], there exists a subsequence $\{\beta_n\}$ such that an average-optimal algorithm is the limit point of the $\beta_n$-optimal algorithms. Similar to the argument in [30, Theorem 18], an average-optimal algorithm holds such the switch-type structure.

**APPENDIX E**

**PROOF OF COROLLARY 10**

Without loss of generality (according to Appendix D), we focus on an $\alpha$-optimal algorithm and assume that $a_1 \geq \max(a_2, \cdots a_N)$.

Let $a_{ij} = (0, \cdots, a_j, \cdots, 0)$ be the zero vector except for the $i$-the entry being replaced by $a_j$. By the symmetry of the users, swap of the initial ages of two user achieves the same expected discounted cost, i.e.,

$$E[V_\alpha(a_1, a_{-1}, \lambda)] = E[V_\alpha(a_j, a_{-1} - a_j + a_{j1}, \lambda)],$$

for all $j \neq 1$. Similar to Appendix D here we focus on the case that $\lambda_1 = 1$ and $\lambda_j = 1$ and the result follows from the non-decreasing function of $V_\alpha(a_1, a_{-1}, \lambda)$ and $a_1 \geq a_j$ for all $j \neq 1$:

$$\nu_\alpha(a_1, a_{-1}, \lambda; 1) - \nu_\alpha(a_1, a_{-1}, \lambda; j)$$

$$= a_j - a_1 + \alpha E[V_\alpha(1, a_{-1} + 1, \lambda') - V_\alpha(a_1 + 1, a_{-1} + 1 - a_j, \lambda')]$$

$$= a_j - a_1 + \alpha E[V_\alpha(a_j + 1, a_{-1} + 1 - a_j, \lambda') - V_\alpha(a_1 + 1, a_{-1} + 1 - a_j, \lambda')] \leq 0.$$
APPENDIX F
PROOF OF THEOREM 12

Let $V_{α}^{(m)}(s)$ and $h_{α}^{(m)}(s)$ be the $α$-optimal cost and the relative cost function for $Δ_m$, respectively. According to [29], we need to prove the following two conditions are satisfied.

1) **There exists a nonnegative $L$, a nonnegative finite function $F(.)$ on $S$, constants $m_0$, and $α_0 ∈ [0, 1)$ such that $-L ≤ h_{α}^{(m)}(s) ≤ F(s)$ for $s ∈ S_m$, $m ≥ m_0$, and $α ∈ (α_0, 1)$:** We consider a randomized stationary algorithm $f$ that updates each user (with packet arrival) with the equal probability for each slot. Similar to the proof of Lemma 7, let $C_{s,0}(f)$ and $C_{s,0}^{(m)}(f)$ be the expected cost from state $s ∈ S_m$ to the reference state 0 by applying the algorithm $f$ to $Δ$ and $Δ_m$, respectively. Then, $h_{α}^{(m)}(s) ≤ C_{s,0}^{(m)}(f)$ and $C_{s,0}(f) < ∞$ similar to the argument in the proof of Lemma 7. In the following, we will show that $C_{s,0}^{(m)}(f) ≤ C_{s,0}(f)$ and then we can choose the function $F(s) = C_{s,0}(f)$.

We note that the algorithm $f$ is independent of the age and then $C_{r,0}(f) ≥ C_{j,0}(f)$ for $r ≥ j$; as such, we have

$$
\sum_{s' ∈ S_m} P_{s,s'}(d)C_{s',0}(f)
\leq \sum_{s' ∈ S_m} P_{s,s'}(d)C_{s',0}(f) + \sum_{r ∈ S_m} P_{s,r}(d)C_{r,0}(f)
= \sum_{s' ∈ S} P_{s,s'}(d)C_{s',0}(f).
$$

Using the above inequality, we then can conclude $C_{s,0}(f) ≥ C_{s,0}^{(m)}(f)$ because

$$
C_{s,0}(f) = E_f[C(s,d) + \sum_{s' ∈ S} P_{s,s'}(d)C_{s',0}(f)]
\geq E_f[C(s,d) + \sum_{s' ∈ S_m} P_{s,s'}^{(m)}(d)C_{s',0}(f)],
$$

along with the fact that $C_{s,0}^{(m)}(f)$ is the solution to the equation:

$$
C_{s,0}^{(m)}(f) = E_f[C(s,d) + \sum_{s' ∈ S_m} P_{s,s'}^{(m)}(d)C_{s',0}^{(m)}(f)].
$$

On the other hand, we can choose $L = \max_{s ∈ S_0} C_{0,s}(f)$, since $h_{α}^{(m)}(s) ≥ -C_{0,s}^{(m)}(f)$ similar to the third condition in the proof of Lemma 7 and $-C_{0,s}^{(m)}(f) ≥ -C_{0,s}(f)$ similar
to above.

2) Let \( \limsup_{m \to \infty} V_m^* = V^*_\infty \) and then \( V^*_\infty \leq V^* \): We claim that \( V^{(m)}_{\alpha}(s) \leq V^\alpha(s) \) for all \( m \), and then the condition holds as

\[
V_m^* = \limsup_{\alpha \to 1} (1 - \alpha) V^{(m)}_{\alpha}(s) \\
\leq \limsup_{\alpha \to 1} (1 - \alpha) V^\alpha(s) = V^*.
\]

To verify this claim, we first note that since \( V^\alpha(s) \) is a non-decreasing function in \( a_i \), and hence we have

\[
\sum_{s' \in S_m} P^{(m)}_{s,s'}(d) V^\alpha(s') \leq \sum_{s' \in S} P_{s,s'}(d) V^\alpha(s'). \tag{9}
\]

We now prove the claim by induction on \( n \) in Eq. (4). It is obvious when \( n = 0 \). Suppose that \( V^{(m)}_{\alpha,n}(s) \leq V^\alpha_{\alpha,n}(s) \), and then

\[
V^{(m)}_{\alpha,n+1}(s) = \min_{d \in \{0,1,\ldots,N\}} C(s,d) + \alpha \sum_{s' \in S_m} P^{(m)}_{s,s'}(d) V^{(m)}_{\alpha,n}(s') \\
\leq \min_{d \in \{0,1,\ldots,N\}} C(s,d) + \alpha \sum_{s' \in S_m} P_{s,s'}(d) V^\alpha_{\alpha,n}(s') \\
\leq \min_{d \in \{0,1,\ldots,N\}} C(s,d) + \alpha \sum_{s' \in S} P_{s,s'}(d) V^\alpha_{\alpha,n}(s') \\
= V^\alpha_{\alpha,n+1}(s),
\]

where (a) results from the induction assumption, and (b) is according to Eq. (9).

**APPENDIX G**

**PROOF OF THEOREM 13**

According to [4, Theorem 8.6.6], the RVIA in Eq. (6) converges to the optimal solution in finite iterations if the truncated MDPs are unichain, i.e., the Markov chain corresponding to every deterministic stationary algorithm consists of a single recurrent class plus a possibly empty set of transient states. We notice that for every truncated MDP, there is only one recurrent class as the state \((m, \ldots, m, 0, \ldots, 0)\) is reachable (e.g., there is no arrival in the next \( m \) slots) from all
other states \[^{31}\] (where remember that \(m\) is the boundary of the truncated MDP). Hence, the truncated MDPs are unichain and the theorem follows immediately.

**APPENDIX H**

**UNRELIABLE CHANNELS**

Here, we discuss the extension of the proposed algorithms to fit unreliable networks. We focus on the ON-OFF channels, where by \(C_i(t) \in \{0, 1\}\) we indicate if the channel between the BS and the user \(u_i\) is ON or OFF in slot \(t\).

**A. The no-buffer network**

For the no-buffer network, the state in slot \(t\) can be revised as \(S(t) = (A_1(t), \ldots, A_N(t), \Lambda_1(t)C_i(t), \ldots, \Lambda_N(t)C_N(t))\), where we can regard \(\Lambda_i(t)C_i(t)\) as a virtual arrival, i.e., there is an effective arrival for user \(u_i\) in slot \(t\) when both \(\Lambda_i(t) = 1\) and \(C_i(t) = 1\). Then, the proposed off-line and on-line algorithms can be applied directly.

**B. The buffer-network**

By \(S(t) = (A_1(t), \ldots, A_N(t), I_1(t), \ldots, I_N(t), C_1(t), \ldots, C_N(t))\) we redefine the state for the buffer-network. Since the channel states are finite, the proposed finite-state approximations in Section \[^{\Box}\] can work perfectly; meanwhile, we can use the proposed optimal off-line and on-line scheduling algorithms in Alg. \(^{3}\) and Alg. \(^{4}\) associated with the revised state.