OPTIMALITY OF CWIKEL-SOLOMYAK ESTIMATES IN ORLICZ SPACES

F. SUKOČHEV AND D. ZANIN

Dedicated to the memory of M.Z. Solomyak

Abstract. We discuss the optimality of Cwikel-Solomyak estimates for the uniform operator norm and establish optimality of M.Z Solomyak’s results [19] within the class of Orlicz spaces. Our methods are based on finding the optimal version of the Sobolev embedding theorem.

1. Introduction

In this text we establish three main results.

Firstly, we establish a distributional version of Sobolev inequality written in terms of Hardy-Littlewood submajorization (see Theorem 3). We demonstrate that our version of this inequality is optimal in the sense that it cannot be improved in terms of the distribution function (see Proposition 6).

Secondly, we show that our version of Sobolev inequality yields the optimal Sobolev embedding theorem (see Proposition 7 and Corollary 13). These results enhance earlier results due to Hansson [9], Brezis and Wainger [6], and Cwikel and Pustylnik [7].

Thirdly, we show that Solomyak’s version of Cwikel-Solomyak estimate given in [19] is optimal in the class of Orlicz spaces (see Corollary 22).

We explain the details of our contribution in Section 3 below after explaining the notations.

2. Preliminaries

2.1. Symmetric function spaces. For detailed information about decreasing rearrangements and symmetric spaces briefly discussed below, we refer the reader to the standard textbook [11] (see also [12, 13]). Let $(\Omega, \nu)$ be a measure space. Let $S(\Omega, \nu)$ be the collection of all $\nu$-measurable functions on $\Omega$ such that, for some $n \in \mathbb{N}$, the function $|f| \chi_{\{|f|>n\}}$ is supported on a set of finite measure. For every $f \in S(\Omega, \nu)$ one can define the notion of decreasing rearrangement of $f$ (denoted by $\mu(f)$). This is a positive decreasing function on $\mathbb{R}_+$ equimeasurable with $|f|$.

Let $E(\Omega, \nu) \subset S(\Omega, \nu)$ and let $\| \cdot \|_E$ be Banach norm on $E(\Omega, \nu)$ such that

1. if $f \in E(\Omega, \nu)$ and $g \in S(\Omega, \nu)$ be such $|g| \leq |f|$, then $g \in E(\Omega, \nu)$ and $\|g\|_E \leq \|f\|_E$;
2. if $f \in E(\Omega, \nu)$ and $g \in S(\Omega, \nu)$ be such $\mu(g) = \mu(f)$, then $g \in E(\Omega, \nu)$ and $\|g\|_E = \|f\|_E$;

We say that $(E(\Omega, \nu), \| \cdot \|_E)$ (or simply $E$) is a symmetric Banach function space (symmetric space, for brevity). The classical example of such spaces is given by Lebesgue $L_p$-spaces, $(L_p(\Omega, \nu), \| \cdot \|_p)$, $1 \leq p \leq \infty$. 

If $\Omega = \mathbb{R}_+$, then the function

$$t \rightarrow \|\chi_{(0,t)}\|_E, \quad t > 0,$$

is called the fundamental function of $E$. Similar definition is available when $\Omega$ is an interval or an arbitrary $\sigma$-finite measure space. The concrete examples of measure spaces $(\Omega, \nu)$ considered in this paper are $d$-dimensional torus $\mathbb{T}^d$ (equipped with Haar measure), $\mathbb{R}_+$, $\mathbb{R}^d$ (equipped with Lebesgue measure), their measurable subsets and compact $d$-dimensional Riemannian manifolds $(X, g)$.

Among concrete symmetric spaces used in this paper are $L_p$-spaces, Orlicz, Lorentz spaces and Marcinkiewicz spaces. Given a convex function $M$ on $\mathbb{R}_+$ (continuous at 0) such that $M(0) = 0$, Orlicz space $L_M(\Omega, \nu)$ is defined by setting

$$L_M(\Omega, \nu) = \left\{ f \in S(\Omega, \nu) : M\left(\frac{|f|}{\lambda}\right) \in L_1(\Omega, \nu) \text{ for some } \lambda > 0 \right\}.$$

We equip it with a Banach norm

$$\|f\|_{L_M} = \inf \left\{ \lambda > 0 : \left\| M\left(\frac{|f|}{\lambda}\right) \right\|_1 \leq 1 \right\}.$$

We refer the reader to [10, 11, 12] for further information about Orlicz spaces.

Given a concave increasing function $\psi$, Lorentz space $\Lambda_\psi$ is defined by setting

$$\Lambda_\psi(\Omega, \nu) = \left\{ f \in S(\Omega, \nu) : \int_0^\infty \mu(s, f) d\psi(s) < \infty \right\}.$$

Marcinkiewicz space $M_\psi$ is defined by setting

$$M_\psi(\Omega, \nu) = \left\{ f \in S(\Omega, \nu) : \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(s, f) ds < \infty \right\}.$$

Spaces $L_{2,1}$ and $L_{2,\infty}$ are Lorentz and Marcinkiewicz space with $\psi(t) = t^{\frac{1}{2}}$, $t > 0$ respectively (see e.g. [12]).

All mentioned examples are closed with respect to the Hardy-Littlewood sub-majorization. The latter is a pre-order defined on $L_1 + L_\infty$ by setting

$$\int_0^t \mu(s, y) ds \leq \int_0^t \mu(s, x) ds, \quad t > 0,$$

and is denoted by $y \prec\prec x$.

Let us fix throughout this paper concave functions

$$\psi(t) = \frac{1}{\log\left(\frac{e}{t}\right)}, \quad \phi(t) = t \log\left(\frac{e}{t}\right), \quad t \in (0, 1)$$

and consider Orlicz functions

$$M(t) = t \log(e + t), \quad G(t) = e^t - 1, \quad t > 0.$$

Throughout this text, the interplay between Lorentz spaces $\Lambda_\psi(0,1)$ and $\Lambda_\phi(0,1)$, Orlicz spaces $L_M(0,1)$ and $L_G(0,1)$, and Marcinkiewicz spaces $M_\psi(0,1)$ and $M_\phi(0,1)$ play a fundamental role. The space $M_\psi$ (respectively, the space $\Lambda_\phi$) is the largest (respectively, the smallest) symmetric Banach function space with the fundamental function $\phi$.

First of all, we observe that the spaces $M_\phi(0,1)$ and $L_G(0,1)$ coincide (the corresponding norms are equivalent). Indeed, the fundamental functions of those spaces coincide, hence $L_G(0,1) \subset M_\phi(0,1)$. On the other hand, $\phi' \in L_G(0,1)$ and, therefore, $M_\phi(0,1) \subset L_G(0,1)$. This fact was first observed in [4].
By invoking Köthe duality (see [11] and [12]), we infer that the spaces $\Lambda_\phi(0,1)$ and $L_M(0,1)$ also coincide and (the corresponding norms are equivalent).

In the statement of our main result, we use the notion of 2-convexification of the symmetric space. Given a symmetric space $E(\Omega, \nu)$, we set

$$E^{(2)}(\Omega, \nu) = \{ f \in S(\Omega, \nu) : |f|^2 \in E(\Omega, \nu) \},$$

$$\|f\|_{E^{(2)}} = \| |f|^2 \|_E^{\frac{1}{2}}, \quad f \in E^{(2)}(\Omega, \nu).$$

More details on the 2-convexification of a function space may be found in the book [12].

We also need a definition of dilation operator $\sigma_u$, $u > 0$, which acts on $S(\mathbb{R})$ by the formula

$$(\sigma_u f)(t) = f(tu), \quad f \in S(\mathbb{R}).$$

We frequently use the mapping $r_d : \mathbb{R}^d \to \mathbb{R}_+$ defined by the formula

$$r_d(t) = |t|^d, \quad t \in \mathbb{R}^d.$$  

3. Connection with earlier results

3.1. Connection with Cwikel-type estimates. Let $d \in \mathbb{N}$ and let $f$ be a measurable function on $d$-dimensional torus $\mathbb{T}^d$ and let $M_f$ be an (unbounded) multiplication operator by $f$ on $L_2(\mathbb{T}^d)$. Let $\Delta_{\mathbb{T}^d}$ be the torical Laplacian. In Proposition 18 we show that (symmetrized) Cwikel operator

$$(1 - \Delta_{\mathbb{T}^d})^{-\frac{d}{4}} M_f (1 - \Delta_{\mathbb{T}^d})^{-\frac{d}{4}}$$

is bounded for $f \in \mathcal{M}_\psi(\mathbb{T}^d)$.

Symmetrized Cwikel estimate in the weak-trace ideal $L_{1,\infty}$ as well as in the ideal $\mathcal{M}_{1,\infty}$ are important in Non-commutative Geometry and in Mathematical Physics. For the background concerning weak trace ideals $L_{p,\infty}$, $1 < p < \infty$ and Marcinkiewicz ideal $\mathcal{M}_{1,\infty}$ we refer to [13] (see also subsection 6.3 below).

Symmetrized Cwikel estimate in $L_{1,\infty}$ (on the torus) for even $d$ appeared in the foundational papers [18, 19]. The estimate there was given in terms of the Orlicz norm $\| \cdot \|_{L_M}$ (which is equivalent to $\| \cdot \|_{\Lambda_\phi}$). Recently, symmetrized Cwikel estimates in the ideal $\mathcal{M}_{1,\infty}$ were established (on the Euclidean space) in [13]. The estimate was given in terms of the Lorentz norm

$$f \to \| \mu(f) \chi_{(0,1)} \|_{\Lambda_\phi(0,1)} + \| f \|_{L_1(\mathbb{R}^d)}.$$  

The surprising fact that $\Lambda_\phi(0,1) = L_M(0,1)$ demonstrates the convergence of those totally unrelated approaches.

The results in this paper complement the results cited in the preceding paragraph. Indeed, Cwikel operator belongs to $L_\infty$ for $f \in \mathcal{M}_\psi$ and to $L_{1,\infty}$ for $f \in \Lambda_\phi$. This opens an avenue (using interpolation methods) to determine the least ideal to which the Cwikel operator belongs.

3.2. Connection with Sobolev-type estimates. The following Sobolev inequality is customarily credited to [9] and [6]. Actually, none of those papers contain a complete proof or even a clear-cut statement. The proof is available in [7]. For a notion of Sobolev space on $\Omega \subset \mathbb{R}^d$ we refer the reader to Chapter 7 in [2].
Theorem 1. Let $d$ be even and let $\Omega$ be a bounded domain in $\mathbb{R}^d$ (conditions apply). There exists a constant $c_{\Omega}$ such that

$$\|u\|_{\Lambda^2_\psi} \leq c_{\Omega} \|u\|_{W^{2,2}_d}, \quad u \in W^{2,2}_d(\Omega).$$

This result is optimal due to the following theorem (proved in [7]).

Theorem 2. Let $d$ be even and let $\Omega$ be a bounded domain in $\mathbb{R}^d$ (conditions apply). If $X$ be a Banach symmetric function space on $\Omega$ such that

$$\|u\|_X \leq c_{\Omega} \|u\|_{W^{2,2}_d}, \quad u \in W^{2,2}_d(\Omega)$$

for some constant $c_{\Omega}$, then $\Lambda^2_\psi \subset X$.

In the next sections, we prove the assertions which substantially strengthen theorems above and extend them to arbitrary dimensions $d \geq 1$. In the course of the proof, we also recast Sobolev inequality using distribution functions. Those results are very much inspired by [7] and our technique is a substantial improvement of that in [7].

4. DISTRIBUTIONAL SOBOLEV-TYPE INEQUALITY

In this section, we introduce Hardy-type operator $T$ (see e.g. [11]) and employ it as a technical tool for our distributional version of Sobolev inequality. In the second subsection, we show that our result is optimal.

4.1. Distributional Sobolev-type inequality. Let $T : L_2(0,1) \rightarrow L_2(0,1)$ be the operator defined by the formula

$$(Tx)(t) = t^{-\frac{1}{2}} \int_0^t x(s)ds + \int_1^t \frac{x(s)ds}{s^{\frac{1}{2}}}, \quad x \in L_2(0,1).$$

The boundedness of $T : L_2(0,1) \rightarrow L_2(0,1)$ is guaranteed by the fact that $T$ is actually a Hilbert-Schmidt operator. Indeed, its integral kernel is given by the formula

$$(t, s) \rightarrow t^{-\frac{1}{2}} \chi_{(s< t)} + s^{-\frac{1}{2}} \chi_{(t< s)}, \quad 0 < s, t < 1,$

which is, obviously, square-integrable.

The gist of (the critical case of the) Sobolev inequality on $\mathbb{R}^d$ may be informally understood as the boundedness of the operator $(1-\Delta)^{-\frac{d}{4}}$ from $L_2(\mathbb{R}^d)$ into a symmetric Banach function space on $\mathbb{R}^d$ which is sufficiently close to $L_{\infty}(\mathbb{R}^d)$. We suggest to view this result as a statement concerning distribution functions of elements $(1-\Delta)^{-\frac{d}{4}} x$ and $T(\mu(x) \chi_{(0,1)})$ where $x \in L_2(\mathbb{R}^d)$.

Our distributional version of Sobolev inequality is as follows:

Theorem 3. Let $d \in \mathbb{N}$. For every $x \in L_2(\mathbb{R}^d)$, we have

$$\mu((1-\Delta)^{-\frac{d}{4}} x) \chi_{(0,1)} \prec \prec c_d T(\mu(x) \chi_{(0,1)}).$$

Here, $\prec \prec$ denotes the Hardy-Littlewood submajorization.

We need the following lemma stated in terms of convolutions.

Lemma 4. Let $x \in L_2(\mathbb{R}^d)$ and $g \in (L_{2,\infty} \cap L_1)(\mathbb{R}^d)$. We have

$$\mu(x \ast g) \chi_{(0,1)} \prec \prec 4\|g\|_{L_{2,\infty} \cap L_1} T(\mu(x) \chi_{(0,1)}).$$
Proof. Lemma 1.5 in [15] states that
\[ \int_0^t \mu(s, x * g) ds \leq \int_0^t \mu(s, x) ds \cdot \int_0^t \mu(s, g) ds + t \int_0^\infty \mu(s, x) \mu(s, g) ds. \]
Obviously,
\[ \int_0^t \mu(s, x) ds \leq \|g\|_{L^2, \infty} \cdot \int_0^t s^{-\frac{1}{2}} ds = 2t^{\frac{1}{2}} \|g\|_{L^2, \infty} \leq 2 \|g\|_{L^2, \infty} \cap L^1 \cdot t^{\frac{1}{2}}. \]
For \( t \in (0, 1) \), we have
\[ \int_t^\infty \mu(s, x) \mu(s, g) ds = \int_t^1 \mu(s, x) \mu(s, g) ds + \int_1^\infty \mu(s, x) \mu(s, g) ds \leq \]
\[ \leq t \|g\|_{L^2, \infty} \int_t^1 \mu(s, x) \frac{ds}{s^{\frac{1}{2}}} + t \mu(1, x) \|g\|_1 \leq \]
\[ \leq \|g\|_{L^2, \infty} \cap L^1 \cdot \left( t \int_t^1 \mu(s, x) \frac{ds}{s^{\frac{1}{2}}} + t \mu(1, x) \right). \]
We, therefore, have
\[ (1) \quad \int_0^t \mu(s, x * g) ds \leq 2 \|g\|_{L^2, \infty} \cdot F(t), \quad t \in (0, 1), \]
where
\[ F(t) = t^{\frac{1}{2}} \int_0^t \mu(s, x) ds + \int_t^1 \mu(s, x) \frac{ds}{s^{\frac{1}{2}}} + t \mu(1, x), \quad t > 0. \]
Computing the derivative, we obtain
\[ F'(t) = \frac{1}{2} t^{-\frac{1}{2}} \int_0^t \mu(s, x) ds + \int_t^1 \mu(s, x) \frac{ds}{s^{\frac{1}{2}}} + \mu(1, x) \leq \]
\[ \leq (T \mu(x) \chi_{(0,1)})(t) + \mu(1, x). \]
Next,
\[ (T \mu(x) \chi_{(0,1)})(t) \geq t^{-\frac{1}{2}} \int_0^t \mu(1, x) ds + \int_t^1 \mu(1, x) ds = \]
\[ = \mu(1, x) \cdot (2 - t^{\frac{1}{2}}) \geq \mu(1, x), \quad t \in (0, 1). \]
Thus,
\[ F'(t) \leq 2(T \mu(x) \chi_{(0,1)})(t), \quad t \in (0, 1). \]
It is now immediate that
\[ F(t) \leq 2 \int_0^t \left( (T \mu(x) \chi_{(0,1)})(s) \right) ds, \quad t \in (0, 1). \]
Combining this equation with (1), we obtain
\[ \int_0^t \mu(s, x * g) ds \leq 4 \|g\|_{L^2, \infty} \cdot \int_0^t \left( (T \mu(x) \chi_{(0,1)})(s) \right) ds, \quad t \in (0, 1). \]
This is exactly the required assertion. \( \square \)
Proof of Theorem 3. We rewrite \( (1 - \Delta)^{-\frac{1}{d}} \) as a convolution operator. Namely, 
\[ (1 - \Delta)^{-\frac{1}{d}} x = x * g, \quad x \in L_2(\mathbb{R}^d), \]
where \( g \) is the Fourier transform of the function 
\[ t \rightarrow (1 + |t|^2)^{-\frac{1}{d}}, \quad t \in \mathbb{R}^d. \]
Precise expression for the function \( g \) involves Macdonald function \( K_{\frac{d}{2}} \) and is given in [3] (see formulae (2.7) and (2.10) there) as follows 
\[ g(t) = c_d |t|^{-\frac{d}{2}} K_{\frac{d}{2}}(|t|), \quad t \in \mathbb{R}^d. \]
Let \( \mathbb{B}^d \) be the unit ball in \( \mathbb{R}^d \). If \( d \neq 0 \mod 4 \), it follows from formulae (9.6.2) and (9.6.10) in [1] that \( g \chi_{\mathbb{B}^d} \in L_{2,\infty}(\mathbb{B}^d) \subset L_1(\mathbb{B}^d) \). If \( d = 0 \mod 4 \), formula (9.6.11) in [1] yields that \( g \chi_{\mathbb{B}^d} \in L_{2,\infty}(\mathbb{B}^d) \subset L_1(\mathbb{B}^d) \). Also, \( g \chi_{\mathbb{B}^d \setminus \mathbb{B}^d} \in (L_1 \cap L_\infty)(\mathbb{R}^d \setminus \mathbb{B}^d) \) by formula (9.7.2) in [1]. Thus, \( g \in (L_{2,\infty} \cap L_1)(\mathbb{R}^d) \). The assertion follows from Lemma 4.

4.2. Optimality of the distributional Sobolev-type inequality. We start with the following easy observation.

Lemma 5. For \( x \in S(0, \infty) \), we have 
\[ \mu(x \circ r_d) = \sigma_{\omega_d} \mu(x), \quad r_d(t) = |t|^d, \quad t \in \mathbb{R}^d, \]
where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \).
Proof. Indeed, for every interval \( (a, b) \), we have 
\[ m(r_d^{-1}(a, b)) = m(\{ t \in \mathbb{R}^d : a < |t|^d < b \}) = m(\{ t \in \mathbb{R}^d : |t| < b^{\frac{1}{d}} \}) - m(\{ t \in \mathbb{R}^d : |t| < a^{\frac{1}{d}} \}) = \omega_d \cdot (b - a). \]
Hence, for every set \( A \subset (0, \infty) \), we have 
\[ m(r_d^{-1}(A)) = \omega_d m(A). \]
In other words, the mapping \( r_d : \mathbb{R}^d \to \mathbb{R} \) preserves a measure (modulo a constant factor \( \omega_d \)). This suffices to conclude the argument.

The following proposition shows (with the help of Lemma 5) that Theorem 3 is optimal.

Proposition 6. There exists a strictly positive constant \( c_d' \) depending only on \( d \) such that 
\[ (1 - \Delta)^{-\frac{1}{d}} (x \circ r_d) \geq c_d' (T \circ r_d) \]
for every \( 0 \leq x \in L_2(0, \infty) \) supported in the interval \( (0, 1) \).
Proof. As in the proof of Theorem 3 above, we rewrite \( (1 - \Delta)^{-\frac{1}{d}} \) as a convolution operator. Namely, 
\[ (1 - \Delta)^{-\frac{1}{d}} z = z * g, \quad z \in L_2(\mathbb{R}^d) \]
where \( g \) is given by the formula 
\[ g(t) = c_d |t|^{-\frac{d}{2}} K_{\frac{d}{2}}(|t|), \quad t \in \mathbb{R}^d. \]
By formula (9.6.23) in [1], this is a strictly positive function.
Let $B^d$ be the unit ball in $\mathbb{R}^d$. Since $g$ is strictly positive, it follows from the formulae (9.6.2) and (9.6.10) in [1] (when $d \neq 0 \mod 4$) or from the formula (9.6.11) in [1] (when $d = 0 \mod 4$) that
\[
g(t) \geq c'_d |t|^{-\frac{d}{4}}, \quad 0 < |t| < 2.
\]

Since $x \geq 0$, it follows that
\[
((1 - \Delta)^{-\frac{d}{4}} (x \circ r_d))(t) = ((x \circ r_d) * g)(t) \geq c'_d \int_{|t-s|<2} |t-s|^{-\frac{4}{4}} x(|s|)ds.
\]

When $|t| < 1$, we have
\[
\{ s \in \mathbb{R}^d : |s| < 1 \} \subset \{ s \in \mathbb{R}^d : |t-s| < 2 \}.
\]
Thus,
\[
((1 - \Delta)^{-\frac{d}{4}} (x \circ r_d))(t) \geq c'_d \int_{|s|<1} |t-s|^{-\frac{4}{4}} x(|s|)ds, \quad |t| < 1.
\]

Obviously,
\[
|t-s| \leq |t| + |s| \leq 2 \max\{|t|, |s|\}
\]
and, therefore,
\[
|t-s|^{-\frac{d}{4}} \geq 2^{-\frac{d}{4}} \min\{|t|^{-\frac{d}{4}}, |s|^{-\frac{d}{4}}\}.
\]

It follows that
\[
((1 - \Delta)^{-\frac{d}{4}} (x \circ r_d))(t) \geq 2^{-\frac{d}{4}} c'_d \int_{|s|<1} \min\{|t|^{-\frac{d}{4}}, |s|^{-\frac{d}{4}}\} x(|s|)ds, \quad |t| < 1.
\]

Passing to spherical coordinates, we obtain
\[
\int_{|s|<1} \min\{|t|^{-\frac{d}{4}}, |s|^{-\frac{d}{4}}\} x(|s|)ds =
\frac{1}{d} \int_0^1 \min\{|t|^{-\frac{d}{4}}, r^{-\frac{d}{4}}\} r^{d-1} x(r)dr \cdot \int_{S^{d-1}} ds.
\]

Making the substitution $r^d = u$, we write
\[
\frac{1}{d} \int_0^1 \min\{|t|^{-\frac{d}{4}}, r^{-\frac{d}{4}}\} r^{d-1} x(r)dr = \frac{1}{d} \int_0^{|T_x|} \min\{|t|^{-\frac{d}{4}}, u^{-\frac{d}{4}}\} x(u)du =
\frac{1}{d} \left( \int_0^{|t|} |t|^{-\frac{d}{4}} x(u)du + \int_{|t|}^{|T_x|} u^{-\frac{d}{4}} x(u)du \right) = \frac{1}{d} (T_x)(|t|^d).
\]

Combining the last 2 equations, we complete the proof. \(\square\)

5. Sobolev embedding theorem in arbitrary dimension

In this section, we extend Theorems [1] (see Proposition [7] and [2] (see Corollary [13]) to Euclidean spaces of arbitrary dimension. We provide new proofs of both theorems in their original setting (for both even and odd $d$).

In what follows, we extend $\psi$ to a concave increasing function on $(0, \infty)$ by setting
\[
\psi(t) = \begin{cases} \frac{1}{\log(1/t)}, & t \in (0, 1) \\ t, & t \in [1, \infty). \end{cases}
\]
**Proposition 7.** Let \( d \in \mathbb{N} \). For every \( x \in L_2(\mathbb{R}^d) \), we have
\[
\| (1 - \Delta)^{-\frac{d}{4}} x \|_{\Lambda_\psi^{(2)}} \leq c_d \| x \|_2.
\]

With some effort, the crucial lemma below can be inferred from the proof of Theorem 5.1 in \([8]\). We provide a direct proof. For the definition of real interpolation method employed below we refer the reader to \([5]\).

**Lemma 8.** We have
\[
[M_{\phi}, L_\infty]_{\frac{1}{2}, 2} \subset \Lambda_\psi^{(2)}.
\]

**Proof.** Let \( t \in (0, 1) \) and set \( s = \psi^{-1}(t) \).
Let \( x \in M_{\phi} + L_\infty \) and let \( x = f + g \), where \( f \in M_{\phi} \) and where \( g \in L_\infty \). Set
\[
A = |x|^{-1}([\mu(s, x), \infty)), \quad B = |f|^{-1}([\frac{1}{2}\mu(s, x), \infty)), \quad C = |g|^{-1}([\frac{1}{2}\mu(s, x), \infty)).
\]

Note that \( m(A) \geq s \).

We have
\[
|f|(u) < \frac{1}{2} \mu(s, x), \quad |g|(u) < \frac{1}{2} \mu(s, x), \quad u \in B^c \cap C^c.
\]

Thus,
\[
|f|(u) + |g|(u) < \mu(s, x), \quad u \in B^c \cap C^c.
\]

So, \( u \in B^c \cap C^c \) implies that \( u \in A^c \). That is, \( B^c \cap C^c \subset A^c \) or, equivalently, \( A \subset B \cup C \).

If \( m(B) \geq \frac{1}{2} m(A) \), then
\[
\| f \|_{M_{\phi}} + t \| g \|_{L_\infty} \geq \| f \chi_B \|_{M_{\phi}} \geq \frac{1}{2} \mu(s, x) \| \chi_B \|_{M_{\phi}} =
\]
\[
= \frac{1}{2} \mu(s, x) \psi(m(B)) \geq \frac{1}{4} \mu(s, x) \phi(m(A)) \geq \frac{1}{4} t \mu(s, x).
\]

If \( m(C) \geq \frac{1}{2} m(A) \), then
\[
\| f \|_{M_{\phi}} + t \| g \|_{L_\infty} \geq t \| g \chi_C \|_{L_\infty} \geq \frac{1}{2} t \mu(s, x).
\]

In either case, we have
\[
K(t, x, M_{\phi}, L_\infty) \geq \frac{1}{4} t \mu(s, x).
\]

Thus,
\[
\| x \|_{[M_{\phi}, L_\infty]_{\frac{1}{2}, 2}} \leq \int_0^1 \left( \frac{1}{t} K(t, x, M_{\phi}, L_\infty) \right)^2 dt \geq
\]
\[
\geq \frac{1}{16} \int_0^1 \mu^2(s, x) dt \quad \text{for} \quad t = \psi(s) = \frac{1}{16} \int_0^1 \mu^2(s, x) ds.
\]

The following lemma shows that the receptacle of the operator \( T \) is strictly smaller than \( \exp(L_2) \) (the space suggested by the Moser-Trudinger inequality).

**Lemma 9.** We have \( T : L_2(0, 1) \to \Lambda_\psi^{(2)}(0, 1) \).
Proof. Let $x \in L_{2,1}(0,1)$. It is immediate that
\[ |(Tx)(t)| \leq t^{-\frac{1}{2}} \int_0^t |x(s)| ds + \int_t^1 |x(s)| \frac{ds}{s^{\frac{1}{2}}} \leq \]
\[ \leq t^{-\frac{1}{2}} \int_0^t \mu(s, x) ds + \int_0^1 |x(s)| \frac{ds}{s^{\frac{1}{2}}} \leq t^{-\frac{1}{2}} \int_0^t \mu(s, x) ds + \int_0^1 \mu(s, x) \frac{ds}{s^{\frac{1}{2}}} = \]
\[ = t^{-\frac{1}{2}} \int_0^t \mu(s, x) ds + 2 \|x\|_{2,1} \leq t^{-\frac{1}{2}} \|x\|_{2,\infty} \int_0^t \frac{ds}{s^{\frac{1}{2}}} + 2 \|x\|_{2,1} = \]
\[ = 2 \|x\|_{2,\infty} + 2 \|x\|_{2,1} \leq c_{abs} \|x\|_{2,1}. \]
Thus,
\[ \|T\|_{L_{2,1} \rightarrow L_{\infty}} \leq c_{abs}. \]
Let $x \in L_{2,\infty}(0,1)$. It is immediate that
\[ |(Tx)(t)| \leq \int_0^1 |x(s)| \min\{t^{-\frac{1}{2}}, s^{-\frac{1}{2}}\} ds \leq \]
\[ \leq \int_0^1 \mu(s, x) \min\{t^{-\frac{1}{2}}, s^{-\frac{1}{2}}\} ds \leq \|x\|_{2,\infty} \int_0^1 s^{-\frac{1}{2}} \min\{t^{-\frac{1}{2}}, s^{-\frac{1}{2}}\} ds. \]
Obviously,
\[ \int_0^1 s^{-\frac{1}{2}} \min\{t^{-\frac{1}{2}}, s^{-\frac{1}{2}}\} ds = t^{-\frac{1}{2}} \int_0^1 \frac{ds}{s^{\frac{1}{2}}} + \int_1^\infty \frac{ds}{s} = \log\left(\frac{t}{e}\right). \]
That is,
\[ |(Tx)(t)| \leq \|x\|_{2,\infty} \log\left(\frac{t}{e}\right), \quad t \in (0,1). \]
Since the mapping $t \rightarrow \log(\frac{t}{e})$, $t \in (0,1)$, falls into $\exp(L_1)(0,1)$, it follows that
\[ \|T\|_{L_{2,\infty} \rightarrow \exp(L_1)} \leq c_{abs}. \]
By real interpolation, we have
\[ T : [L_{2,\infty}, L_{2,1}]_{\frac{1}{2},2} \rightarrow [\exp(L_1), L_{\infty}]_{\frac{1}{2},2} \]
is a bounded mapping. By Lemma $\S$ we have
\[ [L_{2,\infty}, L_{2,1}]_{\frac{1}{2},2} = L_2, \quad [\exp(L_1), L_{\infty}]_{\frac{1}{2},2} \overset{L^\S}{\subset} \Lambda^{(2)}_{\psi}. \]
Thus, $T : L_2(0,1) \rightarrow \Lambda^{(2)}_{\psi}(0,1)$ is a bounded mapping. \hfill $\square$

Proof of Proposition $\S$ By Theorem $\S$ and Lemma $\S$ we have
\[ \left\| \mu\left( (1-\Delta)^{-\frac{1}{2}} x \right) \chi_{(0,1)} \right\|_{\Lambda^{(2)}_{\psi}} \leq c_d \|x\|_{2}. \]
In other words,
\[ \left\| (1-\Delta)^{-\frac{1}{2}} x \right\|_{\Lambda^{(2)}_{\psi} + L_{\infty}} \leq c_d \|x\|_{2}. \]
On the other hand, we have
\[ \left\| (1-\Delta)^{-\frac{1}{2}} x \right\|_2 \leq \|x\|_2. \]
Thus,
\[ \left\| (1-\Delta)^{-\frac{1}{2}} x \right\|_{(\Lambda^{(2)}_{\psi} + L_{\infty}) \cap L_2} \leq c_d \|x\|_{2}. \]
Obviously,
\[(\Lambda^{(2)}_\psi + L_\infty) \cap L_2 = \Lambda^{(2)}_\psi\]
and the assertion follows.

The next assertion should be compared with Theorem 4 in [7] (it is proved in
the companion paper [8] and constitutes the key part of the proof of Theorem 5.7
in that paper). Note that our \(T\) is different from that in [7]. This difference is the
reason why our proof is so much simpler.

**Theorem 10.** For every \(z = \mu(z) \in \Lambda^{(2)}_\psi(0,1)\), there exists \(x = \mu(x) \in L_2(0,1)\)
such that \(\mu(z) \leq Tx\) and \(\|x\|_2 \leq c_{\text{abs}} \|z\|_{\Lambda^{(2)}_\psi}\).

The technical part of the proof of Theorem 10 is concentrated in the next lemma.

**Lemma 11.** Let \(z \in \Lambda^{(2)}_\psi(0,1)\) and let
\[y(t) = \sup_{0 < s < t} \psi(s) \mu(s, z), \quad 0 < t < 1.\]
We have
\[\|y\|_{L_2((0,1), t)} \leq 3^{\frac{2}{\gamma}} \|z\|_{\Lambda^{(2)}_\psi}.\]

**Proof.** We claim that (here \(C\) is the classical Cesaro operator)
\[\int_0^1 \sup_{0 < s < t} \psi^2(s)(C\mu(w))(s) \frac{dt}{t} \leq 3\|w\|_{\Lambda_\psi}, \quad w \in \Lambda_\psi.\]
The functional on the left hand side is normal, subadditive and positively homoge-
neous. By Lemma II.5.2 in [11], it suffices to prove the inequality for the indicator
functions.

Let \(w = \chi_A\) and let \(m(A) = u\). Obviously, \(\mu(w) = \chi_{(0,u)}\) and
\[(C\mu(w))(s) = \frac{\min\{u, s\}}{s}, \quad 0 < s < 1.\]
Thus,
\[
\int_0^1 \sup_{0 < s < t} \psi^2(s)(C\mu(w))(s) \frac{dt}{t} = \int_0^u \sup_{0 < s < t} \psi^2(s) \frac{dt}{t} + \\
+ \int_u^1 \max \left\{ \sup_{0 < s < u} \psi^2(s), \sup_{u < s < t} \psi^2(s) \frac{u}{s} \right\} \frac{dt}{t} = \\
= \int_0^u \psi^2(t) \frac{dt}{t} + \int_u^1 \max \left\{ \psi^2(u), u \sup_{u < s < t} s^{-1}\psi^2(s) \right\} \frac{dt}{t}.
\]
The function \(s \to s^{-1}\psi^2(s), s \in (0,1)\), decreases on the interval (0, \(e^{-1}\)) and in-
creases on the interval (\(e^{-1}\), 1). Thus,
\[
\sup_{u < s < t} s^{-1}\psi^2(s) = \begin{cases} 
  u^{-1}\psi^2(u), & t \in (0, e^{-1}) \\
  t^{-1}\psi^2(t), & u \in (e^{-1}, 1) \\
  \frac{u}{t}, & u < e^{-1} < t
\end{cases}
\]
For \(u \in (e^{-1}, 1)\), we have
\[
\int_0^1 \sup_{0 < s < t} \psi^2(s)(C\mu(w))(s) \frac{dt}{t} = \int_0^u \psi^2(t) \frac{dt}{t} + \int_u^1 \max \left\{ \psi^2(u), ut^{-1}\psi^2(t) \right\} \frac{dt}{t} =
\]
\[\int_0^u \psi^2(t) \frac{dt}{t} + \int_u^1 \psi^2(u) \frac{dt}{t} = \psi(u) + \psi^2(u) \log \left( \frac{1}{u} \right) \leq 2\psi(u).\]

For \( u \in (0, e^{-1}) \), we have
\[
\int_0^1 \sup_{0 < s < t} \psi^2(s)(C\mu(w))(s) \frac{dt}{t} = \int_0^u \psi^2(t) \frac{dt}{t} + \int_u^{e^{-1}} \psi^2(u) \frac{dt}{t} + \int_{e^{-1}}^1 \max \left\{ \psi^2(u), \frac{e}{4} u \right\} \frac{dt}{t} = \psi(u) + \psi^2(u) \log \left( \frac{1}{e u} \right) + \max \left\{ \psi^2(u), \frac{e}{4} u \right\} \leq 3\psi(u).
\]

This yields the claim.

Let \( w = \mu^2(z) \in \Lambda_\psi \). We have
\[
\int_0^1 \sup_{0 < s < t} \psi^2(s)\mu(s, w) \frac{dt}{t} \leq \int_0^1 \sup_{0 < s < t} \psi^2(s)(C\mu(w))(s) \frac{dt}{t} \leq 3\|w\|_{\Lambda_\psi}.
\]

The assertion follows immediately. \(\square\)

The following lemma affords a very substantial simplification and streamlining of the arguments employed in [7].

**Lemma 12.** Let \( y \in L_2((0, 1), \frac{dt}{t}) \) and let \( x(t) = t^{-\frac{1}{2}} y(t), \ 0 < t < 1 \). If \( y \) is positive and increasing, then
\[
\psi(t) \cdot (Tx)(t) \geq \frac{1}{4e} y(t^2), \quad 0 < t < 1.
\]

**Proof.** Suppose \( t \in (0, e^{-1}) \). We have
\[
\psi(t) \cdot (Tx)(t) \geq \psi(t) \int_t^1 \frac{y(s)ds}{s} \geq y(t) \cdot \psi(t) \int_t^1 \frac{ds}{s} \geq \frac{1}{2} y(t) \geq \frac{1}{4e} y(t^2).
\]

Suppose \( t \in (e^{-1}, 1) \). We have
\[
\psi(t) \cdot (Tx)(t) \geq \frac{1}{2} (Tx)(t) \geq \frac{1}{2t^2} \int_0^t y(s)ds \geq \frac{1}{2} \int_0^t y(s)ds \geq \frac{1}{4e} y(t^2).
\]

\(\square\)

**Proof of Theorem 17.** Let \( y \) be as in Lemma 11 and let \( x \) be as in Lemma 12. It follows from Lemma 11 that \( \|x\|_2 \leq 3^\frac{1}{2} \|z\|_{\Lambda^2_\psi} \). Obviously, \( y \) is positive and increasing. It follows from Lemma 12 that
\[
\psi(t) \cdot (Tx)(t) \geq \frac{1}{4e} y(t^2) \geq \frac{1}{4e} \psi(t^2) \mu(t, z) \geq \frac{1}{8e} \psi(t) \mu(t, z), \quad 0 < t < 1.
\]

Thus,
\[
T\mu(x) \geq T x \geq \frac{1}{8e} \mu(z).
\]

\(\square\)

The next corollary shows that the result of Proposition 7 is optimal.

**Corollary 13.** Let \( E(\mathbb{R}^d) \) be a symmetric Banach function space on \( \mathbb{R}^d \). If
\[
(1 - \Delta)^{-\frac{1}{2}} : L_2(\mathbb{R}^d) \to E(\mathbb{R}^d),
\]
then \( \Lambda^2_\psi(\mathbb{R}^d) \subset (E + L_\infty)(\mathbb{R}^d) \).
6. Degenerate case of Cwikel estimate

In this case, we consider operators
\((1 - \Delta)^{-\frac{\theta}{4}} M_f(1 - \Delta)^{-\frac{\theta}{4}}\) and \((1 - \Delta_\varphi^d)^{-\frac{\theta}{4}} M_f(1 - \Delta_\varphi^d)^{-\frac{\theta}{4}}\)
which act, respectively, on \(L_2(\mathbb{R}^d)\) and \(L_2(\mathbb{T}^d)\) and evaluate their uniform norms.
We show that the maximal (symmetric Banach function) space \(E\) such that the operators above are bounded for every \(f \in E\) is the Marcinkiewicz space \(\mathcal{M}_\psi\). For Euclidean space, this follows from Propositions 14 and 15 below. For torus, this follows from Propositions 18 and 19 below.

6.1. Estimates for Euclidean space.

Proposition 14. Let \(d \in \mathbb{N}\). We have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_f(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty \leq c_d\|f\|_{\mathcal{M}_\psi}, \quad f \in \mathcal{M}_\psi(\mathbb{R}^d).
\]

Proof. Without loss of generality, \(f\) is real-valued and positive. We have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_f(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty = \sup_{\|x\|_2 \leq 1} \left|\langle (1 - \Delta)^{-\frac{\theta}{4}} M_f(1 - \Delta)^{-\frac{\theta}{4}} x, x \rangle\right| = \\
= \sup_{\|x\|_2 \leq 1} \left|\langle f \cdot (1 - \Delta)^{-\frac{\theta}{4}} x, (1 - \Delta)^{-\frac{\theta}{4}} x \rangle\right| = \\
= \sup_{\|x\|_2 \leq 1} \left\|f \cdot (1 - \Delta)^{-\frac{\theta}{4}} x\right\|^2 \leq \sup_{\|x\|_2 \leq 1} \|f\|_{\mathcal{M}_\psi} \left\|(1 - \Delta)^{-\frac{\theta}{4}} x\right\|^2_{\Lambda_{\theta}^2}.
\]
The assertion follows now from Proposition 7. □

The converse inequality follows from Proposition 6 and Theorem 10.

Proposition 15. Let \(d \in \mathbb{N}\). Let \(f = \mu(f) \in \mathcal{M}_\psi(0, \infty)\). We have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_{
abla r_d}(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty \geq c_d\|f\|_{\mathcal{M}_\psi},
\]

Proof. We have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_{
abla r_d}(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty = \\
= \sup_{\|\xi\|_2 \leq 1} \left|\langle (1 - \Delta)^{-\frac{\theta}{4}} M_{
abla r_d}(1 - \Delta)^{-\frac{\theta}{4}} \xi, \xi \rangle\right| = \\
= \sup_{\|\xi\|_2 \leq 1} \left|\langle (f \circ r_d) \cdot (1 - \Delta)^{-\frac{\theta}{4}} \xi, (1 - \Delta)^{-\frac{\theta}{4}} \xi \rangle\right| = \\
= \sup_{\|\xi\|_2 \leq 1} \left\|(f \circ r_d) \cdot (1 - \Delta)^{-\frac{\theta}{4}} \xi\right\|^2_1.
\]

Let us now restrict the supremum to the radial \(\xi\). That is, let \(\xi = x \circ r_d\), where \(x \in L_2(0, \infty)\) and \(\|x\|_2 \leq \omega_d^{-\frac{\theta}{2}}\). We have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_{\nabla r_d}(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty \geq \\
\geq \sup_{\|x\|_2 \leq \omega_d^{-\frac{\theta}{2}}} \left\|(f \circ r_d) \cdot (1 - \Delta)^{-\frac{\theta}{4}} (x \circ r_d)\right\|^2_1,
\]

Let us further assume that \(x = \mu(x)\) is supported on the interval \((0, 1)\). By Proposition 6 we have
\[
\left\|(1 - \Delta)^{-\frac{\theta}{4}} M_{\nabla r_d}(1 - \Delta)^{-\frac{\theta}{4}}\right\|_\infty \geq
\]
\[ \geq \sup_{x = \mu(x)} \left\| (f \circ r_d) \cdot \left( (1 - \Delta)^{-\frac{d}{4}} (x \circ r_d) \right)^2 \right\|_1 \]
\[ \geq c_d \cdot \sup_{x = \mu(x)} \left\| (f \circ r_d) \cdot |Tx \circ r_d|^2 \right\|_1 = c_d \omega_d \cdot \sup_{x = \mu(x)} \left\| f \cdot |Tx|^2 \right\|_1. \]

By Theorem [10] we have
\[ \left\| (1 - \Delta)^{-\frac{d}{4}} M_{\text{for}_d} (1 - \Delta)^{-\frac{d}{4}} \right\|_\infty \geq c_d \omega_d \cdot \sup_{z = \mu(z) \in \Lambda_{y_{\psi}}(0,1)} \left\| f \cdot z^2 \right\|_1 = c_d \| f \|_{\text{abs}} \cdot \sup_{z = \mu(z) \in \Lambda_{y_{\psi}}(0,1)} \left\| f \cdot z \right\|_1. \]

It is clear that
\[ \sup_{z = \mu(z) \in \Lambda_{y_{\psi}}(0,1)} \left\| f \cdot z \right\|_1 = \| f \chi_{(0,1)} \|_{y_{\psi}}. \]

Since \( f = \mu(f) \) and since \( \psi \) is linear on \((1, \infty)\), it follows that
\[ \| f \chi_{(0,1)} \|_{y_{\psi}} = \| \mu(f) \chi_{(0,1)} \|_{y_{\psi}} \approx_d \| f \|_{y_{\psi} + L_\infty} \approx_d \| f \|_{y_{\psi}}. \]

Combining three last equations, we complete the proof. \( \square \)

6.2. Estimates for the torus. The following lemma is taken from [20] (see Lemmas 4.5 and 4.6 there).

**Lemma 16.** Let \( h \) be a measurable function on \([-1, 1]^d\). We have
\[ M_h (1 - \Delta)^{-\frac{d}{4}} M_h \left|_{L_2([-1, 1]^d)} \right. = M_h a(\nabla_T d) M_h \left|_{L_2([-1, 1]^d)} \right. \]
where
\[ a(n) = (1 + |n|^2)^{-\frac{d}{4}} + b(n), \quad b(n) = O((1 + |n|^2)^{-\frac{d+1}{2}}), \quad n \in \mathbb{Z}^d. \]

The following lemma relies on the post-critical Sobolev inequality:
\[ \left\| (1 - \Delta_{T^d})^{-\frac{d+1}{4}} \right\|_{L_\infty(T^d) \to L_\infty(T^d)} \leq c_d. \]

The validity of this inequality follows immediately from the fact
\[ \left\{ (1 + |n|^2)^{-\frac{d+1}{2}} \right\}_{n \in \mathbb{Z}^d} \in l_1(\mathbb{Z}^d). \]

**Lemma 17.** Let \( h \in L_1(T^d) \). We have
\[ \left\| (1 - \Delta_{T^d})^{-\frac{d+1}{4}} M_h (1 - \Delta_{T^d})^{-\frac{d+1}{4}} \right\|_\infty \leq c_d \| h \|_1. \]

**Proof.** Without loss of generality, \( h \) is real-valued and positive. We have
\[ \left\| (1 - \Delta_{T^d})^{-\frac{d+1}{4}} M_h (1 - \Delta_{T^d})^{-\frac{d+1}{4}} \right\| \leq \sup_{\| x \|_2 \leq 1} \left\| (1 - \Delta_{T^d})^{-\frac{d+1}{4}} M_h (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x, x \right\| = \]
\[ = \sup_{\| x \|_2 \leq 1} \left\| h \cdot (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x, (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x \right\| = \]
\[ = \sup_{\| x \|_2 \leq 1} \left\| h \cdot (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x, (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x \right\| = \]
\[ = \sup_{\| x \|_2 \leq 1} \left\| h \cdot (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x, (1 - \Delta_{T^d})^{-\frac{d+1}{4}} x \right\| = \]
Let \( \parallel \)

\[ \text{Proposition 18.} \quad \text{The sharpness of the result will be demonstrated below in Proposition 19.} \]

Without loss of generality, \( \triangleq \)

\[ \text{The assertion follows from Proposition 14 and Lemma 17.} \]

Firstly, note that \( \parallel \)

\[ \text{The following proposition is a version of Proposition 15 for} \quad \parallel \]

\[ \text{It is of crucial importance that the estimate in the preceding lemma is given in terms of} \quad \parallel \]

\[ \text{The result below is the best possible criterion for the boundedness of the} \quad \parallel \]

\[ \text{The sharpness of the result will be demonstrated below in Proposition 19.} \]

**Proposition 18.** Let \( d \in \mathbb{N} \). Let \( f \in \mathfrak{H}_\psi(\mathbb{T}^d) \). We have

\[ \left\| (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_f (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} \right\|_{\infty} = \left\| M_{\frac{\psi}{2}} (1 - \Delta)^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{\infty} = \right. \]

\[ = \left\| M_{\frac{\psi}{2}} (1 - \Delta)^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{L^2([-1,1]^d)} \]

By Lemma 16 we have

\[ \left\| M_{\frac{\psi}{2}} (1 - \Delta)^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{L^2([-1,1]^d)} = \]

\[ = M_{\frac{\psi}{2}} (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \left|_{L^2([-1,1]^d)} + M_{\frac{\psi}{2}} b(\nabla_{\mathbb{T}^d}) M_{\frac{\psi}{2}} \right|_{L^2([-1,1]^d)} \]

By triangle inequality, we have

\[ \left\| M_{\frac{\psi}{2}} (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{L^2([-1,1]^d)} \]

\[ \leq \left\| M_{\frac{\psi}{2}} (1 - \Delta)^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{L^2([-1,1]^d)} + \left\| M_{\frac{\psi}{2}} b(\nabla_{\mathbb{T}^d}) M_{\frac{\psi}{2}} \right|_{L^2([-1,1]^d)} \]

\[ = \left\| M_{\frac{\psi}{2}} (1 - \Delta)^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{\infty} + \left\| M_{\frac{\psi}{2}} b(\nabla_{\mathbb{T}^d}) M_{\frac{\psi}{2}} \right\|_{\infty} \]

\[ \leq \left\| (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_f (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} \right\|_{\infty} + c_d \left\| M_{\frac{\psi}{2}} (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_{\frac{\psi}{2}} \right\|_{\infty}. \]

The assertion follows from Proposition 14 and Lemma 17.

The following proposition is a version of Proposition 18 for \( \mathbb{T}^d \).

**Proposition 19.** Let \( d \in \mathbb{N} \). Let \( f = \mu(f) \in \mathfrak{H}_\psi(0, 1) \). We have

\[ \left\| (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} M_{\text{for}_d} (1 - \Delta_{\mathbb{T}^d})^{-\frac{\psi}{2}} \right\|_{\infty} \geq c_d \left\| f \right\|_{\mathfrak{H}_\psi}. \]
Proof. Firstly, note that
\[ \left\| (1 - \Delta)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta)^{-\frac{d}{2}} \right\|_{\infty} = \left\| M_{f^{T\text{ord}}} (1 - \Delta)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \right\|_{\infty} = \left\| M_{f^{T\text{ord}}} (1 - \Delta)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \right\|_{L_2([-1,1]^d)} \]

By Lemma 10 we have
\[ M_{f^{T\text{ord}}} (1 - \Delta)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \]

By triangle inequality, we have
\[ \left\| M_{f^{T\text{ord}}} (1 - \Delta)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \right\|_{L_2([-1,1]^d)} \leq \left\| M_{f^{T\text{ord}}} (1 - \Delta T_a)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \right\|_{L_2([-1,1]^d)} + \left\| M_{f^{T\text{ord}}} b(T_a) M_{f^{T\text{ord}}} \right\|_{L_2([-1,1]^d)} \]

Recall that
\[ |b(n)| \leq c_d (1 + |n|^2)^{-\frac{d}{2}}, \quad n \in \mathbb{Z}^d. \]

Thus,
\[ \left\| (1 - \Delta)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta)^{-\frac{d}{2}} \right\|_{\infty} \leq \left\| M_{f^{T\text{ord}}} (1 - \Delta T_a)^{-\frac{d}{2}} M_{f^{T\text{ord}}} \right\|_{\infty} + \left\| M_{f^{T\text{ord}}} a(T_a) M_{f^{T\text{ord}}} \right\|_{\infty} \leq \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} + c_d \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} \]

By Proposition 15 and Lemma 17 we have
\[ c_d^f \|f\|_{\kappa_0} \leq \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} + c_d^f \|f\|_1. \]

In other words, we have
\[ (2) \quad \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} \geq c_d^f \|f\|_{\kappa_0} - c_d^f \|f\|_1. \]

On the other hand, we have
\[ \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} \geq \langle (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} 1, 1 \rangle = \langle (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} 1, 1 \rangle = \langle M_{\text{ord}} 1, (1 - \Delta T_a)^{-\frac{d}{2}} 1 \rangle = \langle M_{\text{ord}} 1, 1 \rangle. \]

Thus,
\[ (3) \quad \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} \geq \|f\|_1. \]

Combining (2) and (3), we obtain
\[ (1 + c_d^f) \left\| (1 - \Delta T_a)^{-\frac{d}{2}} M_{\text{ord}} (1 - \Delta T_a)^{-\frac{d}{2}} \right\|_{\infty} \geq c_d^f \|f\|_{\kappa_0}. \]

This completes the proof. \qed
6.3. Optimality of Cwikel-Solomyak estimates within the class of Orlicz spaces. The following material is standard; for more details we refer the reader to [3, 10]. Let $H$ be a complex separable infinite dimensional Hilbert space, and let $B(H)$ denote the set of all bounded operators on $H$, and let $K(H)$ denote the ideal of compact operators on $H$. Given $T \in K(H)$, the sequence of singular values $\mu(T) = \{\mu(k, T)\}_{k=0}^{\infty}$ is defined as:

$$\mu(k, T) = \inf\{\|T - R\|_\infty : \text{rank}(R) \leq k\}.$$ 

Let $p \in (0, \infty)$. The weak Schatten class $L_{p,\infty}$ is the set of operators $T$ such that $\mu(T)$ is in the weak $L_p$-space $l_{p,\infty}$, with quasi-norm:

$$\|T\|_{p,\infty} = \sup_{k \geq 0} (k+1)^{\frac{1}{p}} \mu(k, T) < \infty.$$ 

Obviously, $L_{p,\infty}$ is an ideal in $B(H)$.

Our next lemma follows from Proposition [19] and provides one of the two ingredients in the proof of our third main result.

**Lemma 20.** Let $E(\mathbb{T}^d)$ be a symmetric Banach function space on $\mathbb{T}^d$. Suppose that

$$\left\|\left(1 - \Delta_{\mathbb{T}^d}\right)^{-\frac{1}{2}} M_f \left(1 - \Delta_{\mathbb{T}^d}\right)^{-\frac{1}{2}}\right\|_\infty \leq c_{d,E} \|f\|_E,$$ 

for $f \in E(\mathbb{T}^d)$. It follows that $E \subset \mathcal{M}_\psi$.

**Proof.** Take $h = \mu(h) \in E(0, 1)$ and let $f = h \circ r_d$. We have

$$\left\|\left(1 - \Delta_{\mathbb{T}^d}\right)^{-\frac{1}{2}} M_{\nu \circ r_d} \left(1 - \Delta_{\mathbb{T}^d}\right)^{-\frac{1}{2}}\right\|_\infty \leq c_{d,E} \|h\|_E,$$ 

for $h = \mu(h) \in E(0, 1)$.

By Proposition [19] we have

$$c_d \|h\|_{\mathcal{M}_\psi} \leq c_{d,E} \|h\|_E,$$ 

for $h = \mu(h) \in E(0, 1)$.

This completes the proof. 

The next lemma demonstrates efficiency of general theory of symmetric function spaces in the study of Cwikel estimates.

**Lemma 21.** If $N$ is an Orlicz function such that $L_N(0, 1) \subset \mathcal{M}_\psi(0, 1)$, then $L_N(0, 1) \subset L_M(0, 1)$, where $M(t) = t \log(e + t)$, $t > 0$.

**Proof.** We have

$$\|\chi_{(0,t)}\|_{\mathcal{M}_\psi} \leq c_N \|\chi_{(0,t)}\|_{L_N}, \quad 0 < t < 1.$$ 

Thus,

$$t \log\left(\frac{e}{t}\right) \leq \frac{c_N}{N^{-1}(\frac{1}{t})}, \quad t \in (0, 1).$$

Setting $u = t^{-1}$, we write

$$u^{-1} \log(eu) \leq \frac{c_N}{N^{-1}(u)}, \quad u > 1.$$ 

Setting $v = N^{-1}(u)$, we write

$$N(v)^{-1} \log(eN(v)) \leq \frac{c_N}{v}, \quad v > N^{-1}(1).$$

Equivalently,

$$N(v) \geq c_{N^{-1}} v \log(eN(v)), \quad v > N^{-1}(1).$$
By convexity,
\[ N(v) \geq \frac{v}{N^{-1}(1)}, \quad v > N^{-1}(1). \]

Thus,
\[ N(v) \geq c_N^{-1}v \log \left( \frac{e^v}{N^{-1}(1)} \right), \quad v > N^{-1}(1). \]

Thus,
\[ N(v) \geq c_M' N(v), \quad v > c''_N. \]

The following corollary is our third main result. It demonstrates that Cwikel inequality proved by Solomyak (for even \( d \)) cannot be improved within the class of Orlicz spaces. Observe that a version of Solomyak inequality for an arbitrary \( d \) is established in [21].

It is interesting to compare the result of the following corollary with Theorem 9.4 in [17] which is proved under an artificial condition on Orlicz function \( N \). The result below holds for an arbitrary Orlicz function.

**Corollary 22.** If \( N \) is an Orlicz function such that
\[
\left\| (1 - \Delta_T^{-d})^{-\frac{d}{2}} M_f (1 - \Delta_T^{-d})^{-\frac{d}{2}} \right\|_{p,q} \leq c_{d,N} \|f\|_{L_N}, \quad f \in L_N(T^d),
\]
then \( L_N(0,1) \subset L_M(0,1) \).

**Proof.** By Lemma [20] \( L_N \subset M_\psi \). By Lemma [21] \( L_N \subset L_M \). \( \square \)

6.4. Cwikel-Solomyak estimate in Lorentz ideals. We have the uniform norm estimate for Cwikel operator. Solomyak proved the \( \| \cdot \|_{1,\infty} \)-quasi-norm estimate for Cwikel operator. The next natural step is to involve real interpolation to obtain Lorentz norm estimates for Cwikel operator.

**Corollary 23.** Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \). We have
\[
\left\| (1 - \Delta_T^{-d})^{-\frac{d}{2}} M_f (1 - \Delta_T^{-d})^{-\frac{d}{2}} \right\|_{p,q} \leq c_{p,q,d} \|f\|_{[\Lambda_\phi, M_\psi]_1 - \frac{1}{p},q}, \quad f \in [\Lambda_\phi, M_\psi]_1 - \frac{1}{p},q(T^d).
\]

**Proof.** Let \( A : M_\psi(T^d) \rightarrow \mathcal{L}_\infty \) be a bounded operator defined by the setting
\[
A : f \rightarrow (1 - \Delta_T^{-d})^{-\frac{d}{2}} M_f (1 - \Delta_T^{-d})^{-\frac{d}{2}}.
\]

We have that \( A : \Lambda_\psi(T^d) \rightarrow \mathcal{L}_{1,\infty} \) is bounded. By real interpolation, we have
\[
A : [\Lambda_\phi, M_\psi]_1 - \frac{1}{p},q(T^d) \rightarrow [\mathcal{L}_{1,\infty}, \mathcal{L}_\infty]_1 - \frac{1}{p},q.
\]

It is well-known that
\[
[\mathcal{L}_{1,\infty}, \mathcal{L}_\infty]_1 - \frac{1}{p},q = \mathcal{L}_{p,q}.
\]

This completes the proof. \( \square \)
References

[1] Abramowitz M., Stegun I. Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series, 55.

[2] Adams R. Sobolev spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[3] Aronszajn N., Smith K. Theory of Bessel potentials. I. Ann. Inst. Fourier 11 (1961), 385–475.

[4] Astashkin S., Sukochev F. Series of independent random variables in rearrangement invariant spaces: an operator approach. Israel J. Math. 145 (2005), 125–156.

[5] Bergh J., Lofström J. Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[6] Brezis H., Wainger S. A note on limiting cases of Sobolev embeddings and convolution inequalities. Comm. Partial Differential Equations 5 (1980), no. 7, 773–789.

[7] Cwikel M., Pustylnik E. Sobolev type embeddings in the limiting case. J. Fourier Anal. Appl. 4 (1998), no. 4-5, 433–446.

[8] Cwikel M., Pustylnik E. Weak type interpolation near "endpoint" spaces. J. Funct. Anal. 171 (2000), no. 2, 235–277.

[9] Hansson K. Imbedding theorems of Sobolev type in potential theory. Math. Scand. 45 (1979), no. 1, 77–102.

[10] Krasnoselskii M., Rutitskii Y. Convex functions and Orlicz spaces. P. Noordhoff Ltd., Groningen, 1961.

[11] Krein S., Petunin Yu., Semenov E. Interpolation of linear operators. Translations of Mathematical Monographs, 54. American Mathematical Society, Providence, R.I., 1982.

[12] Lindenstrauss J., Tzafriri L. Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 97, Springer-Verlag, Berlin-New York, 1979.

[13] Lord S., Sukochev F., Zanin D. Singular traces. Theory and applications. De Gruyter Studies in Mathematics, 46. De Gruyter, Berlin, 2013.

[14] Lord S., Sukochev F., Zanin D. A last theorem of Kalton and finiteness of Connes’ integral. Journal of Functional Analysis, https://doi.org/10.1016/j.jfa.2020.108664

[15] O’Neil R. Convolution operators and L(p, q) spaces. Duke Math. J. 30 (1963), 129–142.

[16] Simon B. Trace ideals and their applications. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.

[17] Shargorodsky E. On negative eigenvalues of two-dimensional Schrödinger operators. Proc. Lond. Math. Soc. (3) 108 (2014), no. 2, 441–483.

[18] Solomyak M. Piecewise-polynomial approximation of functions from H((0,1)^d), 2l = d, and applications to the spectral theory of the Schrödinger operator. Israel J. Math. 86 (1994), no. 1-3, 253–275.

[19] Solomyak M. Spectral problems related to the critical exponent in the Sobolev embedding theorem. Proc. London Math. Soc. (3) 71 (1995), no. 1, 53–75.

[20] Sukochev F., Zanin D. A C*-algebraic approach to the principal symbol. I. J. Operator Theory 80 (2018), no. 2, 481–522.

[21] Sukochev F., Zanin D. Cwikel-Solomyak estimates on tori and Euclidean spaces. submitted manuscript

School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, Australia
Email address: f.sukochev@unsw.edu.au

School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, Australia
Email address: d.zanin@unsw.edu.au