I. INTRODUCTION

In the usual approach to interacting fermion systems, the starting point is the treatment of the ideal Fermi gas, followed by the development of diagrammatic perturbation theory. Finally, contact with the phenomenological Landau Fermi-liquid theory is made, with a discussion of Landau quasiparticles and collective modes. An exception to this paradigm is found in one-dimensional systems, where perturbation theory diverges, and the quasi-particle structure is in general destroyed by interactions. In this case, a different paradigm has developed from Tomonaga’s observation that the low-energy degrees of freedom of a 1D Fermi gas are completely collective, and the development of the “bosonization” technique. In these lectures, I will try to present a generalization of the “bosonization” description as a general treatment of Fermi surface dynamics in any dimension. This suggests some new interpretations of the Fermi surface as an “order parameter” for metals, and of its notional formation as $T \to 0$ as a type of critical phenomenon. One virtue of a treatment that starts with the collective description, arriving at quasiparticles (for $d > 1$) at the end, is that the special features that distinguish $d = 1$ and $d > 1$ occur at the end of the treatment, rather than right at the beginning. It also makes clear that the existence of a Fermi surface is not necessarily synonymous with the validity of the Landau quasiparticle description. It seems in principle possible that systems with a Fermi surface but which are non Landau Fermi-liquids may exist, and the bosonization methods seem promising tools for investigating such possibilities.

I will start by developing a one-dimensional interpretation of bosonization as Fermi-surface dynamics, then extend it to higher dimensions, review spin-charge separation and fractionalization of electrons into spin and charge degrees of freedom, and end with some intriguing new results on persistence of special features of the ideal gas in some solvable models with “spinons”.

II. LUTTINGER’S THEOREM

I will take the conceptual starting point for the bosonization of the Fermi surface to be the Luttinger theorem expressing the total particle number and momentum of the Fermi gas purely in terms of the Fermi surface geometry. I will initially describe the treatment of one-dimensional, spinless fermions, and eventually extend it to three-dimensional electrons with spin.

The particle and momentum density of a one-dimensional Fermi gas are given by

$$2\pi N/L = \int dk \, n(k)$$

$$2\pi P/\hbar L = \int dk \, kn(k)$$

where for free fermions at zero temperature

$$n(k) = \theta(E_F - \epsilon(k)).$$

We must now express this in terms of the Fermi surface geometry. In this case, the Fermi surface is described by a set of Fermi points $\{k_{Fi}\}$ at which there is a step discontinuity $\Delta\nu_i = n(k_{Fi} + \delta) - n(k_{Fi} - \delta)$ in $n(k)$, with $\Delta\nu_i = \ldots$
\[ \pm 1 \text{ and } \sum_i \Delta \nu_i = 0. \] We may then write

\[ 2\pi N/L = \sum_i \Delta \nu_i k_{Fi}, \]
\[ 2\pi P/\hbar L = \frac{1}{2} \sum_i \Delta \nu_i (k_{Fi})^2. \]

For free electrons this is a “trivial” result. However the deep result of Luttinger is that (with some reinterpretation) this result remains valid (at least in perturbation theory) even when there are interactions between the fermions. In this case \( \Delta \nu \) is no longer the value of a step discontinuity in \( n(k) \): \( k_{Fi} \) still marks a singularity in \( n(k) \), but (in one dimension) it is generally weaker than a step singularity. Instead, the absolute value of \( \Delta \nu \) is an index characterizing the nature of the Fermi surface singularity, and its sign characterizes the orientation of the surface, which in one dimension has an outward normal pointing either to the right or to the left. The usual value \(|\Delta \nu| = 1\) indicates that the singularity in \( n(k) \) arises from the Pauli principle, but other rational values such as \( 1/m \) can occur, principally in connection with the fractional quantum Hall effect, so I will develop the treatment for general \( \Delta \nu \).

Luttinger’s theorem is proved using methods of diagrammatic perturbation theory, which in any case fails to converge in one-dimensional systems. It follows from the fact that particle number and momentum are additively conserved quantities carried by the particles and conserved in total at each interaction vertex of a diagram. Since I am invoking Luttinger’s theorem outside the strict validity of its derivation from diagrammatic perturbation theory, I am in essence taking it as an axiom that is in principle justified by experimental fact. The aim of this treatment will be to show it can be taken as the starting point for the discussion of Fermi surface dynamics.

One further comment is in order. The Luttinger theorem for the total momentum assumes strict momentum conservation; on a lattice, momentum is only conserved modulo reciprocal lattice vectors. However, unless the Fermi surface geometry is commensurate with reciprocal lattice vectors, Umklapp processes are “frozen out” at low temperature, and the non-conservation of momentum on a lattice is technically an irrelevant perturbation to the low-energy fixed point.

To proceed, I now formulate the Luttinger theorem in a differential, local form. On lengthscales \( \xi \) where

\[ |k_{Fi} - k_{Fj}|\xi >> 1 \quad (i \neq j) \]

we can locally define the Fermi surface \( k_{Fi}(x) \). Low-energy, long-wavelength excited states will then be described purely in terms of local Fermi surface fluctuations about the “reference” (ground state) Fermi surface \( k^0_{Fi} \):

\[ k_{Fi}(x,t) = k^0_{Fi} + \delta k_{Fi}(x,t). \]

This is essentially a “semiclassical” treatment of the Fermi surface where momentum and position are simultaneously specified on a coarse-grained scale. The local charge density \( \rho(x) \) (relative to the uniform density ground state) is then given by

\[ 2\pi \rho(x) = \sum_i \delta \nu_i \delta k_{Fi}. \]

Similarly, the local momentum density \( \Pi(x) \) is

\[ 2\pi\hbar^{-1}\Pi(x) = \sum_i \delta \nu_i \left( k^0_{Fi} \delta k_{Fi}(x) + \frac{1}{2}(\delta k_{Fi}(x))^2 \right). \]

Thus the generators of continuous symmetries (particle conservation, or \( U(1) \) gauge invariance, and translations) are expressed purely in terms of the locations of the \( T = 0 \) singularities of \( n(k) \), now defined locally on large lengthscales.

The quantities \( \Delta \nu_i \) are “adiabatic invariants” that remain unchanged as the Hamiltonian is adiabatically varied (and the \( k^0_{Fi} \) in general change), provided the basic structure of the Fermi surface does not change, and are defined by the differential relation

\[ \delta \rho(x)/\delta k_{Fi}(x') = (2\pi)^{-1}\Delta \nu_i \delta(x - x'). \]

I again stress that the Pauli principle gives \( \Delta \nu = \pm 1 \), and while the unit step-discontinuity in the free-fermion \( n(k) \) is reduced to a step \( Z < 1 \) in Landau Fermi-liquid theory, and a weaker power-law discontinuity in the 1D Luttinger-liquid, the value of \( \Delta \nu \) which characterizes the Luttinger theorem remains fixed.
The case $\Delta \nu \neq \pm 1$ occurs in the application to edge states in the fractional quantum Hall effect (FQHE), which have been extensively described by Wen. As a thought experiment, put electrons in a strong uniform magnetic field in the $z$-direction and confine them to the $xy$-plane and the lowest Landau level. Add a substrate potential $V(y)$ that is translationally-invariant in the $x$-direction. The single-particle dispersion relation is

$$\epsilon(k_x) \approx \frac{1}{2} \hbar \omega_c + V(k_x \ell^2)$$

where $\ell = (\hbar c/eB)^{1/2}$ is the “magnetic length”, and $V(y)$ is assumed to vary slowly on this scale.

In this geometry the $N$-particle Laughlin state takes the form,

$$\Psi_m \propto \prod_{i<j}(z_i - z_j)^m \prod_i(z_i^j e^{-y_i^2/\ell^2})$$

where $z_j = \exp(i(x_j + iy_j)\ell)$. This state has non-zero $n(k)$ in the range

$$2\pi J/L = k_{F-} < k < k_{F+} = 2\pi(J + m(N - 1))/L,$$

and the mean occupation of states in this range is $1/m$. A caricature of this state is given by occupying $N$ orbitals in this range so that $m - 1$ empty orbitals separate successive occupied orbitals, giving the mean occupation $1/m$. For $m = 3$, the resulting occupation pattern is the binary string $\ldots 1001001001001 \ldots$

Such a state, interpreted as a Slater determinant, is the Tao-Thouless state, advanced as a rival model to the Laughlin state. As a thought experiment, put electrons in a strong uniform magnetic field in the $z$-direction and confine them to the $xy$-plane and the lowest Landau level. Add a substrate potential $V(y)$ that is translationally-invariant in the $x$-direction. The single-particle dispersion relation is

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$$2\pi J/L = k_{F-} < k < k_{F+} = 2\pi(J + m(N - 1))/L,$$
total charge $\Delta N$ is conserved. It is important to note that the Fermi surface (in the sense of a singularity in the $n(k)$ distribution) strictly only appears in the $T \to 0$ limit, and it is only in this limit that the Luttinger theorem becomes precise. The existence of the Fermi surface at $T = 0$ implies a separate charge conservation law ($U(1)$ gauge symmetry) at each point on the Fermi surface. In some sense, this is just a restatement of the Landau Fermi-liquid theory principle that the lifetime of a Fermi-liquid quasiparticle becomes infinite at the Fermi surface as $T \to 0$. From this viewpoint, the formation of the Fermi surface as $T \to 0$ is a critical phenomenon, and it is not surprising that new symmetries not present in the microscopic Hamiltonian appear at a critical point: in a renormalization-group sense, the symmetry-breaking terms present in the microscopic model will correspond to irrelevant perturbations of the fixed-point effective Hamiltonian.

We must now construct the most general effective Hamiltonian compatible with separate charge conservation at each Fermi point. For free electrons, with $\Delta \nu = \pm 1$, the Hamiltonian $\Delta H = \Delta(H - E_F N)$ is given by

$$\Delta H_0 = \frac{1}{2} \sum_i \int \frac{dx}{2\pi} v_{F_i}(\delta k_{F_i}(x))^2$$

where the Fermi velocity is $v_{F_i} \Delta \nu_i$. (If charge is added to change $k_{F_i}$ by $\delta k_{F_i}$, the mean momentum of the additional particles is $k_{F_i}^0 + \frac{1}{2} \delta k_{F_i}$. ) The most general possibility is essentially a Landau-type form

$$\Delta H_{eff} = \frac{1}{2} \sum_{ij} \int \frac{dx}{2\pi} \int \frac{dx'}{2\pi} \Gamma_{ij}(x-x') \delta k_{F_i} \delta k_{F_j}.$$ 

Here stability requires that

$$\Gamma_{ij}(q) = \int dx \Gamma_{ij}(x)e^{iqx}$$

is a real positive-definite symmetric matrix.

It is useful to review the conservation laws and gauge symmetries of (a) of free electrons, (b) of the microscopic Hamiltonian, (c) of this effective Hamiltonian, and (d) of the Landau Fermi liquid ($D \geq 2$). Free spinless fermions have a huge set of $U(1)$ gauge symmetries, one for each orbital with a conserved occupation number in the $D$-dimensional reciprocal space (i.e., there is a $D$-dimensional manifold of gauge symmetries).

In contrast, only the global $U(1)$ symmetry is present in the microscopic interacting model. The degrees of freedom in the effective Hamiltonian derived above correspond only to fermion orbitals close to the Fermi surface; its explicit symmetries correspond to gauge changes where all orbitals directly above and below a given point on the Fermi surface have the same phase change, corresponding to a $(D-1)$-dimensional manifold of gauge symmetries (one for each point on the Fermi surface). (In 1D this is a discrete set, one per Fermi point.)

Finally, because the interactions in a Landau Fermi liquid leave a low-energy spectrum of fermionic quasiparticles in one-to-one correspondence with bare electron states “near” the Fermi surface, it has the full $D$-dimensional manifold of gauge symmetries asymptotically close to the Fermi surface, with conserved quasiparticle occupation numbers.

From this we conclude that the existence of the Fermi Surface implies only the $(D-1)$-dimensional manifold of symmetries, not the full $D$-dimensional manifold that reappears in the Landau theory. This indicates that the Fermi surface can exist even when the Landau quasiparticle picture is not applicable, and non-Fermi-liquid systems can still have a Fermi surface obeying the Luttinger theorems. This is in fact the case in interacting 1D systems (which are not Landau Fermi liquids), and leaves open the possibility (discussed later) of such a possibility for $D > 1$.

The term “Luttinger liquid” has been used by Anderson to refer to a system with a Fermi surface that obeys the Luttinger theorem, but which is not a Landau Fermi-liquid. As a historical note, when I coined the term “Luttinger liquid” in the 1D context I was referring to the exactly solvable Luttinger model which in 1D played the role of the “zeroth order” model to which residual interactions are added, in analogy to Landau’s use of the free fermion model as the “zeroth-order” model for $D \geq 2$ Fermi liquids. However, it is serendipitous that Luttinger originated both the theorem and the model, and Anderson’s interpretation of the term “Luttinger liquid” is particularly appropriate.

### III. QUANTIZATION OF THE FERMI-SURFACE VARIABLES.

So far, I have derived expressions for $\Delta H$, $\Delta P$, and $\Delta N$ relative to the ground state as quadratic expressions in terms of the local Fermi-surface displacements $\delta k_{F_i}(x)$. To quantize these degrees of freedom, we need to find their dynamical algebra. This is found by the Tomonaga’s bosonization method[10].
The total electron density $\rho(x)$ has commuting Fourier components $[\rho_q, \rho_{q'}] = 0$, where
\[ \rho_q = \sum_{kk'} \delta_{k-k',q}(c^\dagger_k c_{k'}) - \langle 0|c^\dagger_k c_{k'}|0\rangle \]

The Tomonaga procedure is to decompose the total electron density into components associated with each Fermi point:
\[ \rho_q \approx \sum_i \rho_{qi}. \]

This is done by defining small non-overlapping domains of width $\Lambda$ in reciprocal space around each Fermi point:
\[ f_i(k) = \theta(\Lambda^2 - |k-k^{0}_{F_i}|^2) \]

Then
\[ \rho_{qi} = \sum_{kk'} f_i(k)f_i(k')\delta_{k-k',q}(c^\dagger_k c_{k'} - \langle 0|c^\dagger_k c_{k'}|0\rangle). \]

The commutation relations are
\[ [\rho_{qi}, \rho_{q'j}] = \delta_{ij} \left( \delta_{q+q',0} \sum_k f_i(k)f_i(k+q)|0\rangle|n_{k+q} - n_k|0\rangle + X_i(q, q') \right), \]

where
\[ X_i(q, q') = \sum_{kk'} f_i(k)f_i(k')|f_i(k'+q) - f_i(k'+q')|(c^\dagger_k c_{k'} - \langle 0|c^\dagger_k c_{k'}|0\rangle). \]

For $|q| < \Lambda$, the factor $f_i(k)f_i(k+q) \approx 1$ over almost all the range where $(0)|n_{k+q} - n_k|0\rangle$ is non-negligible. Similarly, the factor $f_i(k)f_i(k')|f_i(k'+q) - f_i(k'+q')|$ vanishes over most of the range of $k$ and $k'$. Up to corrections of order $|q|/\Lambda$, the commutator becomes
\[ [\rho_{qi}, \rho_{q'j}] = \delta_{ij} \delta_{q+q',0}(qL/2\pi)\Delta \nu_i, \]

where $\Delta \nu_i$ is the shift in $(0)|n_k|0\rangle$ in going from $k \approx k^{0}_{F_i} - \Lambda$ to $k \approx k^{0}_{F_i} + \Lambda$. With the identification $2\pi\rho_i(x) = \Delta \nu_i \delta k_{F_i}(x)$, we obtain the local form of the dynamical algebra of the Fermi-surface dispacements: taking $\Delta \nu_i$ to be the rational number $\xi_i p_i/q_i$ with $p_i$ and $q_i$ positive, and $\xi_i = \pm 1$,
\[ \delta k_{F_i}(x), \delta k_{F_j} = (2\pi i q \xi_i/p)\delta_{ij} \delta'(x-x'). \]

(Here $\delta'(x)$ is the derivative of the Dirac delta-function.) Since the RHS of this commutation relation is a c-number, and the Hamiltonian is quadratic, the effective Hamiltonian has been reduced to a harmonic oscillator problem.

The operator that creates an electron in a wave-packet of states near the $i$'th Fermi point has the form
\[ \Psi^\dagger_i(x) = A_i \exp \left( i \int x' \delta k_{F_i}(x) \right). \]

We may write this as
\[ \Psi^\dagger_i(x) = A_i e^{i\varphi_i(x)}. \]

It changes the total charge at the Fermi point by one unit:
\[ [\Delta N_i, \Psi^\dagger_i(x)] \delta_{ij} \Psi^\dagger_j(x). \]

From the Tomonaga commutation relations,
\[ [\varphi_i(x), \rho_j(x')] = i\delta(x-x') \]

so $\varphi_i(x)$ is the conjugate field to $\rho_i(x)$, and is subject to the chiral constraint
\[ \partial \varphi_i(x)/\partial x = k^0_{F_i} + \delta k_{F_i}(x) = k^0_{F_i} + 2\pi i (\xi_i q_i/p_i)\rho_i(x). \]
Thus $\rho_i(x)$ and $\varphi_i(x)$ are not independent canonical fields, and $\rho_i(x)$ is proportional to the derivative of its own conjugate field. The explicit representation of $\varphi_i(x)$ is

$$\varphi_i(x) = \theta_i + \int_0^x dx' \frac{\partial}{\partial x'} \varphi_i(x).$$  \hspace{1cm} (1)

The integration constant $\theta_i$ is the conjugate phase to the number operator $\Delta N_i$, and is not constructed from the harmonic fluctuation modes.

Integrating the commutation relations gives

$$[\varphi_i(x), \varphi_j(x')] = (i\pi \xi \epsilon_i / p_i) \delta_{ij} \text{sign}(x - x').$$

From this we get (for $x \neq x'$)

$$\Psi_i^\dagger(x) \Psi_i^\dagger(x') = e^{i\pi q_i / p_i \text{sign}(x') \Delta N}.$$

For $\Psi_i^\dagger(x)$ to be a fermion creation operator, we require that $p_i = 1$ and that $q_i$ is odd. Anticommutation of operators $\Psi_i^\dagger(x)$ and $\Psi_j^\dagger(x')$ when $i$ and $j$ are different requires that the Klein factor is

$$A_i = \exp \left( i\pi \sum_j \text{sign}(i - j) \Delta N_j \right)$$

where an arbitrary ordering of the Fermi points has been introduced.

IV. DIAGONALIZING THE ONE-DIMENSIONAL HAMILTONIAN

Let us first consider systems with no coupling between different Fermi points, so $\Gamma_{ij}(x - x') = \gamma_i \delta_{ij} \delta(x - x')$, with $\gamma_i > 0$. When $\Delta \nu = \pm 1$, this just describes free electrons, or Landau quasiparticles. I will extend the discussion to the Laughlin-state case $\Delta \nu = \pm 1/m$, following WEN [2], as this is relevant for fractional quantum Hall effect edge states.

The Hamiltonian is a sum of decoupled terms associated with each Fermi point, $\mathcal{H} = \sum_i \mathcal{H}_i$, where

$$\mathcal{H}_i = \epsilon F \Delta N_i + \frac{1}{2} \int dx \gamma_i : (\rho_i(x))^2 :$$  \hspace{1cm} (2)

Using the commutation relation $[\rho_i(x), \rho_i(x')] = i(2\pi m)^{-1} \xi \delta'(x - x')$, with $\xi = \pm 1$, we get

$$[\mathcal{H}, \rho_i(x)] = i\xi v_{Fi} \frac{\partial}{\partial x} \rho_i(x)$$  \hspace{1cm} (3)

where the Fermi velocity is given by $v_{Fi} = \gamma_i / (2\pi m)$. In terms of boson creation and destruction operators obtained by normalizing the Fourier components of $\rho_i(x)$, the Hamiltonian is given by

$$\mathcal{H}_i = \epsilon F \Delta N_i + v_{Fi} \left( \frac{2\pi m}{L} (\Delta N_i)^2 + \sum_q \theta(q\xi_i) q b_q^\dagger b_q \right).$$  \hspace{1cm} (4)

The $m$-dependence shows up only in the term involving the total charge at the Fermi point. It is straightforward to compute the Greens function using the bosonic representation of the fermion creation operators:

$$\langle \psi_i(x, t) \psi_i^\dagger(0, 0) \rangle \approx \frac{Z_i e^{i(k_{Fi} x - \epsilon_{Fi} t)}}{(x - v_{Fi} \xi_i t)^m}$$  \hspace{1cm} (5)

where $Z_i$ is an undetermined normalization. The Fourier transform gives the singular part of the occupation factor $n(k)$:

$$n(k) = n(k)_{\text{reg}} + Z_i |k - k_{Fi}|^{-1} (k - k_{Fi})^m. $$  \hspace{1cm} (6)
More generally, the electron creation operator will take the form

\[ \psi^\dagger(x) = \sum_{\{n_i\}} A(\{n_i\}) e^{i \sum_i n_i \chi_i(x)}, \]  

(7)

where \( A(\{n_i\}) = 0 \) unless \( \sum_i n_i = 1 \). The occupation factor \( n(k) \) will in general have singularities at \( k = \sum_i n_i k_{Fi} \).

I now turn to the full problem, when \( \Gamma_{ij}(x) \approx \Gamma_{ij}^\delta(x-x') \) is not diagonal, and the different Fermi points are coupled. The problem is to diagonalize

\[ H = \frac{2\pi}{2L} \sum_{ij} \Gamma_{ij} \rho_{qi} \rho_{q'j}. \]  

(8)

\[ [\rho_{qi}, \rho_{q'j}] = (qL/2\pi)\delta_{q+q',0}(\delta_{ij} \xi_i/m_i). \]  

(9)

The normal modes \( \rho_q^\lambda \) are obtained from a real non-symmetric matrix eigenproblem:

\[ \rho_q^\lambda = \sum_i \chi_i^\lambda \rho_{qi}; \quad \rho_{qi} = \sum_{\lambda} \psi_i^\lambda \rho_q^\lambda. \]  

(10)

where

\[ \sum_i \chi_i^\lambda_1 \psi_i^\lambda_2 = \delta^\lambda_{\lambda';} \quad \sum_{\lambda} \chi_i^\lambda \psi_j^\lambda = \delta_{ij}. \]  

(11)

and

\[ \psi_i^\lambda = (\Gamma^{-1})_{ij} \chi_j^\lambda; \quad (\xi_j/m_j) \chi_j^\lambda = v^\lambda \psi_j^\lambda. \]  

(12)

Here, the eigenvalues \( v_j \) is real, since \( \Gamma_{ij} \) is positive definite; these are the set of renormalized normal-mode velocities. To calculate the Greens’ function, for example, the expression for \( \rho_{qi} \) in terms of the normal modes \( \rho_q^\lambda \) must be substituted into \( \varphi_i(x) \). The result for the diagonal Greens function at Fermi point \( i \) is

\[ G_i(x, t) = Z_i \prod_{\lambda} (x - v^\lambda t)^{-\alpha_i^\lambda}, \]  

(13)

where \( \alpha_i^\lambda = (\chi_i^\lambda)^2/2\pi|v_\lambda^\lambda| \). Detailed examination shows that \( \sum_{\lambda} \alpha_i^\lambda \geq m_i \), so the Fermi surface singularity in \( n(k) \) is always weakened by coupling between different Fermi points. General expressions for correlation exponents when many different Fermi points interact have also been developed by PENC and SOLYOM[11].

I now come to what is one of the central ideas of the “Luttinger liquid theory”[7]. This is that (unless crossover to a non-Luttinger-liquid fixed point occurs), the Landau parameters \( \Gamma_{ij} \) can be determined by identifying the excitations of a finite interacting system with periodic boundary conditions, that are associated with changing the net charges at the different Fermi points. If we suppress the finite-wavelength harmonic oscillator modes, the residual charge terms in the excitation spectrum are:

\[ \Delta P = \sum_i k_{Fi}^0 I_N^i + \frac{\pi}{L} \xi_i m_i (\Delta N_i)^2, \]  

(14)

\[ \Delta H = \sum_i \epsilon F_i \Delta N_i + \frac{\pi}{L} \sum_{ij} \Gamma_{ij} \Delta N_i \Delta N_j. \]  

(15)

By fitting the low-energy excitations of a system studied by finite-size numerical diagonalization or the Bethe Ansatz to this form, the low-energy effective Hamiltonian is determined, and its asymptotic correlations, etc. can be calculated from them. The program was demonstrated in detail on the spinless fermion system equivalent to the XXZ spin chain in a magnetic field[12], where there are just two Fermi points, R and L. The correspondence between the parametrization of[12] and that used here is that \( \Gamma_{R} R = \Gamma_{RL} L = v_N + v_J \), and \( \Gamma_{RL} = \Gamma_{LR} = v_N - v_J \).

Finally, we must discuss what happens when we include electron spin. If a magnetic field is present, this is just an application of the “spinless” treatment with double the number of Fermi points. However, if the Fermi points have spin degeneracy, the full non-Abelian \( SU(2) \) symmetry at each Fermi point must be considered. In this case, coupling between the spin degrees of freedom at different Fermi points is incompatible with the existence of independent spin rotation symmetries at each Fermi point. The renormalization group treatment[13] shows that either the couplings scale to zero at the low-energy fixed point, or they scale to a strong coupling, non-Luttinger liquid fixed point.
V. GENERALIZATION TO HIGHER DIMENSIONS

In the one-dimensional systems, the low-energy degrees of freedom are described by an independent set of bosonic variables at each Fermi point, representing harmonic fluctuations of the Fermi surface. LUTHER made a pioneering attempt to describe higher-dimensional Fermi surface degrees of freedom by bosonization, but his “tomographic” construction restricts attention to particle-hole pairs carrying a net momentum strictly normal to the local Fermi vector. Recently, I found that the Tomonaga bosonization algebra could be formulated in a more explicitly higher-dimensional form; this formulation has been reviewed by HOUGHTON and MARSTON who use it to discuss vector. Recently, I found that the Tomonaga bosonization algebra could be formulated in a more explicitly higher-dimensional form; this formulation has been reviewed by HOUGHTON and MARSTON who use it to discuss

The basic idea is again to remark that the Luttinger theorem expresses the ground-state particle density and momentum density purely in terms of the \((d-1)\)-dimensional Fermi surface in the \(d\)-dimensional reciprocal space defined by the singularity in the ground-state occupation number distribution \(n(k)\). This of course is making the assumption that the ground state \(n(k)\) has such a feature, and that no BCS or density-wave instability occurs at low temperatures.

It is generally believed that in the absence of any other instability, a BCS instability in some channel will always occur below some critical temperature, and destroy the singularity in \(n(k)\), so we must in principal first exclude the BCS processes from the effective Hamiltonian, then restore the (presumably relevant) perturbations. The BCS terms can be recognized as deriving from the special shape of the Fermi surface, which in the presence of either time-reversal symmetry or spatial inversion symmetry, has inversion symmetry in reciprocal space. If, for the moment, we ignore or conceptually abolish this symmetry, it should in principle be possible to have higher-dimensional interacting systems with a stable Fermi-surface singularity in their ground state. From this viewpoint the BCS instability, like density-wave instabilities, is classified as a special feature associated with a particular class of Fermi surface shapes.

In the Fermi-liquid theory, the Fermi surface singularity is a step discontinuity across which \(n(k)\) decreases by an amount \(Z\), but I will make no \textit{a priori} assumption about the nature of the singularity, and merely use the property that it defines a surface satisfying the Luttinger theorem. It will become clear that, in dimensions greater than one, non-Fermi-liquid behavior (such as spin-charge separation) requires a sufficiently-strong singular forward-scattering term in the phenomenological Landau parameters; the possible existence of such singular terms, which ANDERSON has argued are generically present in two-dimensional fermion systems, is controversial, and currently a subject of active investigation, though to date, no microscopic treatment has clearly demonstrated the existence of such terms.

In general dimensions, the Fermi surface is described by a function \(k_F(s)\), where \(s\) is a \((d-1)\)-dimensional surface coordinate. On large length-scales, I again describe the system in terms of local fluctuations of the Fermi surface geometry:

\[
k_F(x, s) = k_{F0}(s) + \hat{n}(s)\hat{\kappa}_\parallel(x, s) + \hat{t}_\mu(s)\hat{\kappa}_\perp^\mu(x, s)
\]

where \(x\) now represents a \(d\)-dimensional spatial coordinate, \(\hat{n}(s)\) is the local direction of the Fermi velocity (the outward normal direction of the Fermi surface), and \(\{\hat{t}_\mu(s)\}\) are a basis of the \(d-1\) unit vectors tangent to the Fermi surface. A treatment that is quadratic in the normal and tangential fluctuations \(\kappa_\parallel(x, s), \kappa_\perp^\mu(x, s)\) will be developed, with the recognition that the transverse fluctuations are essentially gauge variables describing infinitesimal \textit{reparametrizations} of the (curvilinear) surface coordinates \(s = \{s_1, \ldots, s_{d-1}\}\), without change in the shape of the surface. The physical quantities such as the local fluctuation in total particle density and momentum density can be completely expressed in terms of the normal fluctuations \(\kappa_\parallel(x, s)\). Classically, the gauge condition \(\kappa_\perp^\mu(x, s) = 0\) could be imposed; however, since as a quantum operator \(\kappa_\perp^\mu(x, s)\) has non-trivial commutation relations, the gauge condition is the action \(\kappa_\perp^\mu(x, s)|\Psi\rangle = 0\) on physical states \(|\Psi\rangle\).

Thus

\[
\Delta N = \int d^dx \rho(x)
\]

where the local change \(\rho(x)\) in particle density relative to the ground state is given by

\[
\rho(x) = \int \frac{\omega(s)ds^{d-1}}{(2\pi)^d} \kappa_\parallel(x, s),
\]

where \(\omega(s)\) is the surface area measure. Similarly,

\[
\Delta \tilde{P} = \hbar \int d^dx \tilde{P}(x),
\]
where the local change in momentum density is
\[
\Pi(x) = \int \frac{\omega(s)d^{d-1}s}{(2\pi)^d} \left( \tilde{k}_F(s) \kappa(x, s) + \frac{1}{2} \tilde{n}(s) : (\kappa(x, s))^2 : \right). \tag{20}
\]
(Here the notation \((\kappa(x, s))^2 : \) anticipates the normal-ordering needed in the quantized formulation.) I note that the first and second functional derivatives of \(\Pi(x)\) with respect to \(\kappa(x, s)\) define the two fundamental geometric properties of the Fermi surface, its \textit{shape} \(\tilde{k}_F(s)\) and its \textit{orientation} \(\tilde{n}(s)\).

In the spirit of Landau theory, the effective Hamiltonian is a quadratic form where \(\Delta(H - \epsilon_F N)\) is given by
\[
\frac{1}{2} \int d^d x \int d^d x' \int \frac{\omega(s)d^{d-1}s}{(2\pi)^d} \int \frac{\omega(s')d^{d-1}s'}{(2\pi)^d} \Gamma(s, s'; x - x') : \kappa(x, s)\kappa(x', s') :, \tag{21}
\]
where
\[
\Gamma(s, s'; x) = \frac{(2\pi)^d}{\omega(s)} v_F(s) \delta^{d-1}(s - s') \delta^d(x - x') + f(\tilde{k}_F(s), \tilde{k}_F(s'); x). \tag{22}
\]
Note that the kinetic energy ("effective mass") term appears as a \((2d - 1)\)-dimensional delta-function term in \(\Gamma\). The conventional Landau \(f\)-function is given by
\[
f(\tilde{k}_F(s), \tilde{k}_F(s')) = \int d^d x f(\tilde{k}_F(s), \tilde{k}_F(s'); x) \tag{23}
\]
and \(\delta^{d-1}(s - s') = \omega(s)\delta^{d-1}(\tilde{k}_F(s) - \tilde{k}_F(s'))\). The stability of the Fermi surface against spontaneous shape deformations requires that
\[
\tilde{\Gamma}(s, s'; \tilde{q}) = \int d^d x e^{i\tilde{q}\cdot\tilde{x}} \Gamma(s, s'; x) \tag{24}
\]
is a positive-definite quadratic form in \(s, s'\) for all \(\tilde{q}\).

In metals, the Fermi surface will in general consist of a number of distinct manifolds in the primitive (or sometimes extended) Brillouin zone. The formal integral \(\int d^{d-1}s\) can be considered to implicitly include sums over such discrete band indices distinguishing distinct manifolds. In the discussion here, I will also assume that the Fermi surface sheets are smooth differentiable (orientable) manifolds with finite curvature at all points. If some microscopic parameter is varied through a critical point at which the Fermi-surface topology changes, at the critical point there will be a Van-Hove singularity on the Fermi surface at which the curvature is infinite, and the linearized treatment of fluctuations will fail. Systems at or close to such critical points may also be a place to look for non-Fermi-liquid behavior.

Having quadratic expressions for the various conserved quantities in terms of the fluctuation variables \(\kappa(x, s)\), we now need the \(d\)-dimensional version of the quantum algebra of the Fermi-surface displacements. I will first give the answer, then sketch its derivation using a generalization of Tomonaga’s method. The commutation relations are
\[
[\kappa(x, s), \kappa(x', s')] = (2\pi)^d D[\delta^d(x - x')\delta^{d-1}(\tilde{k}_F(s) - \tilde{k}_F(s'))], \tag{25}
\]
where
\[
D[f(x, s)] \equiv (\hat{n}(s) \cdot \hat{\nabla}) f(x, s) \tag{26}
\]
is a "covariant derivative". The fact that the RHS of the commutation relation is a \(c\)-number means that \(\kappa(x, s)\) can be expressed as a linear combination of harmonic oscillator variables.

As a side comment, I note that if a static magnetic field \(\tilde{B}(x)\) is present, the covariant derivative becomes
\[
D[f(x, s)] \equiv (\hat{n}(s) \cdot \hat{\nabla}) f(x, s) + 2\pi \Phi_0^{-1} \hat{n}(s) \times \tilde{B}(x) \cdot \hat{n} x \frac{\partial}{\partial s'} f(x, s). \tag{27}
\]
Here \(\Phi_0\) is the London flux quantum \(2\pi \hbar/e\). This derivative encodes the information that if a wave packet of states centered at real-space position \(x\) and Fermi-surface point \(s\) is made, the spatial coordinate evolves in the direction \(\hat{n}(s)\) and the Fermi-surface coordinate evolves in the direction \(\tilde{B}(x) \times \hat{n}(s)\). The rate at which this evolution takes place is however encoded in the effective Hamiltonian, rather than the Fermi-surface displacement algebra. In what follows, I will assume that no magnetic field is present.
We may also write \( \kappa_{\parallel}(x, s) = D[\varphi(x, s)] \), where
\[
[\varphi(x, s), \kappa_{\parallel}(x', s')] = (2\pi)^d i \delta^d(x - x') \delta^{d-1}(\vec{k}_F(s) - \vec{k}_F(s')),
\]
so \( \kappa_{\parallel} \) is the derivative of its own conjugate field. The scalar phase field \( \varphi(x, s) \) obeys the algebra
\[
[\varphi(x, s), \varphi(x', s')] = i\pi \delta^{d-1}(\vec{k}_F(s) - \vec{k}_F(s')) \delta^{d-1}(\dot{n}(s) \times (\vec{x} - \vec{x}')) \text{sgn}(\dot{n}(s) \cdot (\vec{x} - \vec{x}')).
\]
This allows the commutation relations of both the normal and tangential Fermi-surface fluctuations to be obtained from the identification
\[
\vec{k}(x, s) = \vec{\nabla} \varphi(x, s).
\]

The microscopic derivation of the commutation relations follows Tomonaga’s approach to the 1D system. First, all large momentum-transfer scattering processes are in principle integrated out, in a renormalization-group sense, leaving an effective Hamiltonian that keeps only electron states within a reciprocal space distance \( \Lambda \) from the Fermi surface. The Fermi surface is then broken up into “patches” of area about \( \lambda^{d-1} \), and reciprocal space near the Fermi surface is broken up into little domains centered on each patch. It is convenient to consider these domains as little spheres of radius \( \Lambda \) centered on a mesh of points representing a triangulation of the Fermi surface, but since we are seeking an effective long-wavelength theory, the detailed cutoff structure should not matter. In this case, the “patches” would be circular \((d-1)-\)spheres; since a space cannot be fully tiled with non-overlapping spheres, the radius \( \Lambda \) would have to be chosen so that areas of Fermi surface that are double-counted because circular patches overlap are exactly compensated by omitted areas between the patches, so the sum of all patch areas exactly equals the Fermi surface area. Other tiling schemes could be used. An important condition is that \( \Lambda \) must be small enough so that within each domain, the Fermi-surface is quasi-flat. This means that there must be some finite upper bound to the Fermi surface curvature, and excludes the possibility of a Van Hove singularity on the Fermi surface.

Now let \( \theta(x, \alpha(\vec{k})) = 1 \) if \( \vec{k} \) is inside the spherical domain centered on the patch with label \( \alpha \), and let it vanish otherwise. Then we define
\[
\kappa_\alpha(\vec{q}) = \sum_{\vec{k}} \theta(\vec{k} + \vec{q}) \theta(\vec{k}) \left( e^{\dagger}_{\vec{k} + \vec{q}} c_{\vec{k}} \delta_{\vec{q}, 0} (n_{\vec{k}})_{0} \right),
\]
and
\[
\Pi_\alpha(\vec{q}) = \sum_{\vec{k}} \theta(\vec{k} + \vec{q}) \theta(\vec{k}) \tilde{n}_\alpha \cdot (\vec{k} - \vec{k}_F) + \frac{1}{2} \vec{q} \left( e^{\dagger}_{\vec{k} + \vec{q}} c_{\vec{k}} - \delta_{\vec{q}, 0} (n_{\vec{k}})_{0} \right).
\]

We must now approximately evaluate the commutation relations, and drop “cutoff-dependent terms” in the spirit of Tomonaga’s treatment. Then (ignoring any overlap between patches)
\[
[\kappa_\alpha(\vec{q}), \kappa_{\alpha'}(\vec{q}')] = \delta_{\alpha \alpha'} (X_{\alpha}(\vec{q}, \vec{q}') + \delta_{\vec{q} + \vec{q}', 0} g_{\alpha}(\vec{q}))
\]
where
\[
X_{\alpha}(\vec{q}, \vec{q}') = \sum_{\vec{k}} \theta(\vec{k} + \vec{q} + \vec{q}') \left( \theta(\vec{k} + \vec{q}) - \theta(\vec{k} + \vec{q}') \right) \left( e^{\dagger}_{\vec{k} + \vec{q} + \vec{q}'} c_{\vec{k}} - \delta_{\vec{q} + \vec{q}', 0} (n_{\vec{k}})_{0} \right).
\]

This operator-valued term can be neglected for \( |\vec{q}|, |\vec{q}'| \ll \Lambda \), as the factor \( (\theta(\vec{k} + \vec{q}) - \theta(\vec{k} + \vec{q}')) \) vanishes except at the surface of the spherical domain, and is a “cutoff-dependent correction”. Because (in contrast to the original Tomonaga calculation in 1D) some of this “correction” involves states at the Fermi surface, this is perhaps not as innocuous an approximation in higher dimensions, but appears to be valid in the long-wavelength limit. The residual term in the RHS of commutation relation is the \( c \)-number term
\[
g_{\alpha}(\vec{q}) = \sum_{\vec{k}} \theta(\vec{k} + \vec{q}) \theta(\vec{k}) (n_{\vec{k} + \vec{q}} - n_{\vec{k}})_{0}.
\]

In this case, for \( |\vec{q}| \ll \Lambda \) the value of this is just the number of allowed \( k \)-space points inside the volume of reciprocal space swept out by displacing the patch of Fermi surface by \( \vec{q} \). Note that it is independent of the detailed structure of \( (n_{\vec{k}})_{0} \) near the Fermi surface and only involves the change in asymptotic values of the occupation factor from deep inside to far outside the Fermi surface. Thus the commutation relation becomes
\[
[\kappa_\alpha(\vec{q}), \kappa_\beta(\vec{q}')] = a V \delta_{\alpha \beta} \delta_{\vec{q} + \vec{q}', 0} \tilde{n}_\alpha \cdot \vec{q}.
\]
where \( a \) is the surface area of the patch on the Fermi surface. This a generalization of an Abelian \((U(1))\) Kac-Moody algebra. The other commutation relations are similarly evaluated at long wavelengths as

\[
[\Pi_\alpha(q), \Pi_\beta(q')] = n_\alpha \cdot \bar{q}_\alpha (q + q') \delta_{\alpha\beta}
\]

and

\[
[\Pi_\alpha(q), \Pi_\beta(q')] = n_\alpha \cdot (\bar{q} - q') \delta_{\alpha\beta} \Pi_\alpha (q + q') + \frac{1}{12} \delta_{\alpha\beta} aV \delta_{q+q} (\bar{n}_\alpha \cdot \bar{q})^3
\]

The full structure is the generalization of the Kac-Moody and associated Virasoro algebras to \( d > 1 \), where at each point on the Fermi surface, the spatial coordinates separate into one special normal direction \( \bar{n}_\alpha \) along which derivatives are taken, and \( d-1 \) transverse directions. This differs from the earlier “tomographic” bosonization proposed by Luther\(^{[14]}\) which only considers the case when \( \bar{q} \) is parallel to \( \bar{n}_\alpha \), and does not allow any natural coupling between Fermi surface points where the normals are not parallel or antiparallel. The thermodynamic limit may now be taken, and Kronecker delta-functions on the discrete mesh of reciprocal space points allowed by periodic boundary conditions become Dirac delta-functions.

The fermion phase field \( \varphi_\alpha(x) \) is formally given by

\[
\varphi_\alpha(x) = \theta_\alpha(x) + \int_0^x dx' (\bar{n}_\alpha \cdot \nabla) \kappa_{||\alpha}(x),
\]

where \( \theta_\alpha(x) \) is an integration constant. The operator \( \exp(i\theta_\alpha(x)) \) must be interpreted as the operator that adds charge on the Fermi surface patch \( \alpha \). This will in a wave packet that is completely delocalized along the direction in real space parallel to \( \bar{n}(s) \), but is localized in the transverse direction to within a distance \( \Lambda^{-1} \) of the transverse spatial coordinate \( x_\perp \). The electron creation operator will again be proportional to \( \exp(i\varphi_\alpha(x)) \); as in the 1D case, a Klein factor can be added to make electron operators defined in different Fermi-surface patches anticommute. Electron creation operators defined in the same patch, and sharing (to within \( \Lambda^{-1} \)) a common transverse spatial coordinate \( x_\perp \) will also automatically anticommute. Anticommutation of creation operators in the same patch, but at different transverse spatial coordinates must be imposed through the integration constant \( \theta_\alpha(x_\perp) \):

\[
e^{i\theta_\alpha(x_\perp)} e^{i\theta_\alpha(x_\perp')} e^{i\theta_\alpha(x_\perp)} = 0 \quad (|x_\perp - x_\perp'| \gg \Lambda^{-1}).
\]

If there are only one or two transverse dimensions, this can be represented with Jordan-Wigner or “anyon” gauge fields.

It is straightforward to include spin degrees of freedom in the preceding treatment, and define

\[
\kappa_{\alpha\uparrow}(q) + \kappa_{\alpha\downarrow}(q) = 2\kappa_{\alpha}(q); \quad \kappa_{\alpha\uparrow}(q) - \kappa_{\alpha\downarrow}(q) = 2\sigma^z_\alpha(q).
\]

Then

\[
[\kappa_{\alpha}(q), \kappa_{\beta}(q')] = \frac{1}{2} aV \delta_{\alpha\beta} \delta_{q+q',\alpha} \bar{n}_\alpha \cdot \bar{q},
\]

\[
[\sigma^a_\alpha(q), \sigma^b_\beta(q')] = \delta^{ab} \left( \frac{1}{2} aV \delta_{\alpha\beta} \delta_{q+q',\alpha} \bar{n}_\alpha \cdot \bar{q} + i\epsilon^{abc} \sigma^c(q + q') \right).
\]

The spin degrees of freedom now obey a \( d > 1 \) version of the non-Abelian \( SU(2) \) Kac-Moody algebra.

VI. DIAGONALIZATION OF THE HARMONIC OSCILLATORS

Within the approximation that keeps only the terms which are quadratic in the Fermi surface fluctuations, the problem of the interacting Fermi system reduces to a harmonic oscillator problem. In fact this is (of course) essentially just the zero-sound problem of Fermi liquid theory. It is convenient to first rescale the Fermi surface normal fluctuation operators, and write

\[
\tilde{\rho}_{qa} = \left( \frac{v_{Fa}}{aV} \right)^{1/2} \kappa_{\alpha}(q).
\]

Then the commutation relations become

\[
[\tilde{\rho}_{qa}, \tilde{\rho}_{q'a'}] = \delta_{\alpha\beta} \delta_{q+q',\alpha} \omega^2_{\alpha}(q).
\]
where $\omega^0_\alpha (\hat{q}) = v_{F\alpha} (\hat{n}_\alpha \cdot \hat{q})$. The Hamiltonian is now
\[ \frac{1}{2} \sum_{q} \sum_{\alpha \beta} \tilde{\Gamma}_{\alpha\beta}(q) \hat{\rho}_{qa} \hat{\rho}_{-q\beta}, \]
where $\tilde{\Gamma}_{\alpha\beta}$ is the positive definite matrix
\[ \tilde{\Gamma}_{\alpha\beta}(q) = \delta_{\alpha\beta} + \Lambda^{d-1} f(\tilde{k}_{F\alpha}, \tilde{k}_{F\beta}, \hat{q}) (v_{F\alpha} v_{F\beta})^{1/2}, \]
which is an even function of $\hat{q}$. We now see the reason why the 1D case (with no transverse degrees of freedom) is special. In the scaling limit $\lambda \to 0$, with $(d - 1) > 0$, $\tilde{\Gamma}_{\alpha\beta} \to \delta_{\alpha\beta}$ (free fermions), unless either (a) $f(\tilde{k}_{F\alpha}, \tilde{k}_{F\beta}) \to \infty$ or (b) $v_{F\alpha} v_{F\beta} \to 0$. Put another way, for $d > 1$ unless the effective Landau parameters are singular, the coupling between different patches on the Fermi surface contains a factor (patch area)/(Fermi surface area) = 1/$N_{\text{patch}}$, and the only modification of the collective excitation spectrum is that a finite number of zero-sound collective modes are pushed up above the continuum of modes with frequencies up to $v_F |q|$.

Formally, to diagonalize the harmonic oscillator problem, we must express its normal modes $\rho^\lambda_q$ in terms of the local modes $\hat{\rho}_{qa}$ defined on each patch:
\[ \rho^\lambda_q = \sum_\alpha \chi^\lambda_\alpha (\hat{q}) \hat{\rho}_{qa} \]
with the inverse relation
\[ \hat{\rho}_{qa} = \sum_\lambda \psi^\lambda_\alpha (\hat{q}) \rho^\lambda_q. \]
If this involved the 1D problem of the coupling of a single pair of Fermi points with opposite-direction normals, this problem would be simple to treat by expressing it in terms of canonically-normalized boson creation and annihilation operators, and carrying out a Bogoliubov transformation. In the general case, this is not so convenient; instead it can be recognized (of course) as the zero-sound problem, and regarded as a real non-symmetric eigenproblem where all the eigenvalues $\omega^\lambda$ are real because the matrix $\tilde{\Gamma}_{\alpha\beta}$ is positive definite. Then
\[ \sum_{\beta} \tilde{\Gamma}_{\alpha\beta} \omega^0_\beta \chi^\lambda_\beta = \omega^\lambda \chi^\lambda_\alpha \]
and
\[ \sum_{\beta} \omega^0_\alpha \tilde{\Gamma}_{\alpha\beta} \psi^\lambda_\beta = \omega^\lambda \psi^\lambda_\alpha \]
with the orthogonality relation
\[ \sum_{\alpha} \psi^\lambda_\alpha \chi^\lambda_\alpha = \delta^{\lambda\lambda'}. \]
The eigenvalues $\omega^\lambda(\hat{q})$ are real, with the symmetry $\omega^\lambda(-\hat{q}) = -\omega^\lambda(\hat{q})$. This formal solution is useful for carrying out calculations of correlation functions, etc.

If the Fermi surface is regarded as the analog of the “order parameter” of a metal, its shape fluctuations are its “Goldstone modes”. The cutoff $\Lambda$ means that only modes with $|q| < \Lambda$ should be counted as independent, and there is then one linear-dispersion “Goldstone mode” per patch. These modes have a spectrum of velocities that becomes continuous in the in the limit $\Lambda \to 0$. For $\hat{q}$ along some direction $\hat{\Omega}$, the density of mode velocities remains finite at zero frequency provided that some part of the Fermi surface is tangential to $\hat{\Omega}$. It is this feature that gives the universal $T$-linear specific heat of fermion liquids in this formalism, in contrast to the $T^d$ specific heat of systems where the Goldstone mode velocities remain finite. The $T$-linear heat capacity (or entropy) is extensive in not only the real-space volume $V$, but also in the Fermi-surface area, and derives from the $(2d-1)$-dimensional delta-function term proportional to $\delta^d(x)\delta^{d-1}(\tilde{k}_F (s) - \tilde{k}_F (s'))$ in $\Gamma (s, s'; x)$. Furthermore, this term controls the upper limit (the Fermi velocity) to the continuous spectrum of velocities of modes traveling in a given direction. In the absence of a contribution to this delta-function part of $\Gamma$ coming from singular terms in the Landau parameters, there is no renormalization of the Fermi velocity or the $T$-linear specific heat.
The collective zero-sound modes give $T^d$ corrections to the low-temperature specific heat that depend on the Landau parameters. In fact, in three dimensions, the leading corrections to the $T$-linear specific heat is the $T^3 \log T$ term elucidated in detail by PETHICK and CARNEIRO [17] within a standard Fermi-liquid approach: HOUGHTON and MARSTON [15] have recently reported that such terms can also be recovered in the bosonized approach discussed here.

Because of this “all-or-nothing” character of the contribution of the Landau $f(\vec{k}_F(s), \vec{k}_F(s'))$ couplings to a shift in the Fermi velocity and the $T$-linear specific heat, it is instructive the consider the case when $f(\vec{k}_F(s), \vec{k}_F(s'))$ is a smooth function with a strong anomaly in a narrow cone around the forward scattering direction, and allow some control parameter to continuously evolve this anomaly into a true delta-function. If the Fermi surface is spherical, as the cone of the anomaly becomes narrower, more terms in the spherical-harmonic expansion of the Landau parameters are needed to adequately represent it. Roughly speaking, there will be one extra zero-sound collective mode pushed out above the continuum of modes with $0 < \omega < v_F|q|$ for each additional spherical harmonic term that becomes significant. These modes will proliferate and become dense in the range $v_F|q| < \omega < (v_F + \delta v_F)|q|$ as the Landau parameters develop a delta-function singularity in the forward scattering direction. Similarly, when the anomaly becomes pronounced, the specific heat will develop a “pseudo-$T$-linear” regime characterized by what will become the renormalized Fermi velocity, but which crosses over to the true unrenormalized $T$-linear regime at lower temperatures; this crossover temperature vanishes as the singularity in the Landau parameters develops.

VII. SPIN-CHARGE SEPARATION

Spin-Charge separation is seen quite generally in one-dimensional systems. It is associated with forward scattering of particles at the same Fermi point, and is not directly related to the other characteristic one-dimensional phenomenon where the long-wavelength coupling of the low-energy degrees of freedom of different Fermi points renormalizes the correlation function exponents away from their free fermion values. In fact, an exactly solvable model, the “supersymmetric $t$–$J$ model” with inverse-square interactions [18] exists in which spin-charge separation exists without correlation function exponent renormalizations. In this case, the simple pole of the electronic Greens function splits into a branch cut terminated by inverse square-root singularities:

$$\text{Im}G(k_F + \delta k, E_F + \delta E) \propto \theta((\delta E - v_s \delta k)(v_s \delta k - \delta E)) (\delta E - v_c \delta k)(v_c \delta k - \delta E)^{-1/2}. \quad (53)$$

An electron injected into the system in a wavepacket of states near such a Fermi point, and localized in space will physically separate into spatially separated charge and spin components, moving with velocities $v_s$ and $v_c$ as the state evolves.

Could such a phenomenon occur in two dimensions, as proposed by Anderson [9]? We have seen that in dimensions greater than one, the Fermi velocity is defined by the upper limit of the continuum of velocities of the “Goldstone modes” (Fermi-surface shape fluctuation modes), and that this cannot be renormalized by non-singular Landau couplings. For free fermions, and Landau Fermi liquids, the spin and charge velocities are strictly equal, which as we shall see can be interpreted in terms of a “gauge symmetry”. (The equal spin and charge velocities, defined by the dispersion relation associated with the low-energy pole of the Landau Fermi-liquid single-particle Green’s function should not be confused with the propagation velocities of the various spin and charge fluctuation collective zero-sound excitations that are present in a Fermi liquid). To get spin-charge separation in higher dimensions, singular forward scattering terms that differ in the singlet and triplet scattering channels would be required, as proposed by ANDERSON in two dimensions. However, it should again be emphasized that his proposal remains controversial.

The phenomenological description outlined here treats the Landau parameters as an input, and cannot provide guidance about their microscopic origin or validity. It may again be useful to consider what would occur if there was, for example, a strong forward scattering in the triplet but not the singlet channel, but not a true singularity. In this case, at higher energy scales the spin and charge degrees of freedom would presumably separate over shorter lengthscales, but finally, at the longest lengthscales and lowest energies, the spin and charge quantum numbers of the electron would be confined together to form a Landau quasiparticle. A deconfinement transition would take place if the Landau parameters were “tuned” to become (sufficiently) singular.

While spin-charge separation in two or higher dimensions remains obscure, I will now examine it more closely in the one-dimensional context from a symmetry viewpoint.

VIII. HIDDEN SYMMETRIES IN SPIN-CHARGE SEPARATED SYSTEMS

As noted earlier, the ideal Fermi gas exhibits an infinite set of gauge symmetries, as the occupation numbers of each orbital are separately conserved. When spin degrees of freedom are included, there is a infinite set of non-
Abelian $SU(2)$ symmetries, one for each orbital. This means that the spin of each singly-occupied orbital can be independently rotated, the spin degeneracy of a state with $N$ singly-occupied orbitals is $2^N$, and it is a highly reducible representation of the global $SU(2)$ group. When interactions are “switched on” this non-generic structure of the ideal gas is lost, and the eigenstates will become irreducible representations of the spin rotation group (assuming no spin-orbit coupling). The essence of the Landau Fermi-liquid state is that, asymptotically at the low-energy fixed point, the extra symmetries of the ideal gas are restored, but in the form of the quasiparticle occupations.

I now pose the question, if forward scattering processes at a Fermi point are included, so as to induce spin-charge separation, but the other interactions that couple different Fermi points are omitted, is any remnant of the “hidden” quasiparticle gauge symmetries retained? There is some remarkable evidence from certain exactly solvable one-dimensional models that this is indeed the case. These models are perhaps the closest interacting models to the ideal gas, and seem to be the simplest non-trivial interacting models. They have scale-invariant inverse-square interactions, and ground state wavefunctions which can be considered as (full) Gutzwiller projections of free fermion states.

The simplest of these models is the $S = 1/2$ spin chain which I and Shastry introduced independently a few years ago:

$$H = J \sum_{i<j} d(i-j)^{-2} \vec{S}_i \cdot \vec{S}_j$$

Here $d(j) = j$, or $(N\pi)\sin(\pi k/N)$ if periodic boundary conditions on a chain of $N$ sites is used. This model only has spin degrees of freedom. But an extension of this to the “supersymmetric $t - J$ model” was introduced by KURAMOTO and YOKOYAMA:

$$H = J \sum_{i<j} d(i-j)^{-2} \left( -P_G(\sum_{a=1}^4 c_{ia}c_{ja} + h.c.)P_G + (\vec{S}_i \cdot \vec{S}_j - \frac{1}{4} n_i n_j) \right)$$

where $P_G$ is the full Gutzwiller projection operator that prevents multiple occupancy of any site. This model has both spin and charge degrees of freedom, and exhibits spin-charge separation without coupling of low-energy degrees of freedom at different Fermi points. The periodic versions of these models exhibit remarkable “supermultiplet” degeneracies meaning that their energy levels form highly reducible representations of $SU(2)$. This is analogous to the free fermion gas degeneracies, but with a much less straightforward structure, and is what I will interpret as the remnant of the orbital occupation number symmetries that survives spin-charge separation. Since these symmetries just involve the spin sector, it is convenient to consider just the spin chain.

To put the results into context, it is first useful to consider the conformal limit, where the low-energy spin degrees of freedom are described by the $k = 1$ Kac-Moody algebra (Wess-Zumino-Witten conformal field theory). In this language, one writes a (say, right-moving) spin density field $\sigma^a_{\alpha \sigma}$ associated with a single Fermi point as $J^a_m$, with $q = 2\pi m/L$, and $m = 0, \pm 1, \pm 2 \ldots$. Then the Kac-Moody algebra takes its standard form

$$[J^a_m, J^b_{m'}] = km/2\delta_{m+m',0} + i\epsilon^{abc} J^c_{m+m'}.$$  

and the Hamiltonian becomes $\mathcal{H}^{eff} = v_s P$, where the momentum $P$ is given by

$$P = \frac{2\pi}{L} L_0$$

where $L_0$ is the “zero mode” of the associated Virasoro algebra:

$$L_0 = \frac{1}{k+2} \left( J^a_0 J^a_0 + \sum_{m=1}^{\infty} J^a_{-m} J^a_m \right)$$

where $[L_0, J^a_m] = -mJ^a_m$, and $J^a_m|0\rangle = 0$ for $m > 0$. The Hamiltonian is very degenerate, since $L_0$ takes only values $n + h$, where $n = 0, 1, 2 \ldots$ and (for the $k = 1$ SU(2) algebra) $h = 0$ for integer total spin, and $h = \frac{1}{4}$ for half-integral total spin. The standard descriptions of this spectrum are through “Abelian bosonization” (which is essentially what has been described in this these lectures) or the “Verma module” basis (see, e.g. the book by KAKU for an introduction), but these do not describe the fractional-statistics particle-like $S = \frac{1}{4}$ excitations (“spinons”) that turn out to be the appropriate basis for describing the inverse-square perturbation of the conformal limit.

It is useful to introduce a short-distance “point-splitting” cutoff that regularizes the conformal field theory as follows:

$$\mathcal{H}^{eff} = \frac{1}{2} \int dx \int dx' j(x - x') \vec{\sigma}(x) \cdot \vec{\sigma}(x').$$
The conformal field theory is recovered in the limit

$$ j(x) \rightarrow \frac{v_s}{4\pi} \delta(x - x'), $$

(60)

or \( \tilde{j}(0) = v_s/4\pi \), where

$$ \tilde{j}(q) = \int dx j(x)e^{iqx}. $$

(61)

Usually one takes some point-splitting function \( j(x) \) that falls off exponentially for large separations, but consider the case when it has algebraic tails falling off as \( x^{-1+\alpha} \). Then as \( |q| \rightarrow 0 \),

$$ \tilde{j}(q) = (v_s/4\pi) + A|q|^{\alpha} + Bq^2. $$

(62)

For \( 0 < \alpha < 2 \), the non-analytic term is the leading correction to the conformal limit; the new term in the Hamiltonian can be written

$$ H^{(2)} = \sum_{m=1}^{\infty} m^2 J^a_m J^a_m. $$

(63)

Since this commutes with \( L_0 \), this term can be studied numerically by diagonalizing it with the finite-dimensional subspace of states with a given value of \( L_0 \). In the limit \( \alpha \rightarrow 0 \), \( H^{(2)} \) merely splits the states at a given \( L_0 \) into groups with the same total spin. However, for general \( \alpha > 0 \), the spectrum of \( H^{(2)} \) is completely broken up into distinct energy levels, each of which corresponds to an irreducible representation of \( SU(2) \) with no unexpected additional degeneracies. This represents the complete destruction of all the higher symmetries of the conformal field theory by the point-splitting cutoff. A striking exception to this is seen in the special case \( \alpha = 1 \), corresponding to the inverse-square fall-off of \( j(x) \); in this case the levels partially regroup into “supermultiplets” which a highly reducible representations of \( SU(2) \). No other “special” values of \( \alpha \) are detected by this calculation.

Clearly a large residual part of the symmetry of the conformal field theory survives in the presence of the inverse-square corrections to the conformal limit. This symmetry has recently been identified as a “quantum group” symmetry [23, 24, 25] called the Yangian \( Y(sl_2) \), generated by \( J_0^a \) and

$$ J^a = i\hbar e^{abc} \sum_{m=1}^{\infty} J^b_m J^c_m $$

(64)

where \( \hbar \) is here the “quantum deformation parameter” defined by the “non-co-commutative co-product”

$$ \Delta(J^a) = \mathbb{1} \otimes J^a + J^a \otimes \mathbb{1} + \frac{1}{2} i\hbar e^{abc} J^b_0 \otimes J^c_0. $$

(65)

It is perhaps out of place to describe the technical aspects of “quantum groups” (which are in fact algebras, not groups) in any detail here; suffice it to say that quantum groups are infinite-dimensional algebras that are “quantum deformations” of Lie algebras, with the feature that they have a tensor-product operation (the “co-product”) where (unlike Lie algebras) the result of a sequence of tensor products depends on the order in which they are made (analogous to the action of a sequence of operators in quantum mechanics). “Quantum groups” are intimately related to braiding and fractional statistics. A physical explanation of the appearance of quantum groups in connection with spin-charge separation is that if a spin-1/2 fermion is factorized into independent spin and charge factors, the two components are each semions, fractional-statistics entities half-way between fermions and bosons [24, 26]. The Yangian \( Y(sl_2) \) is the quantum group which has \( sl_2 \) (the Lie Algebra of \( SU(2) \) generators) as a subalgebra.

The discrete spin chain also has this “quantum group” symmetry, \( [H, J^a] \), with

$$ J^a = \frac{\hbar}{2} \sum_{i<j} \cot(\pi(i - j)/N) e^{abc} S^b_i S^c_j. $$

(66)

The energy levels are given by [23] the construction

$$ E = 2J(\pi/N)^2 \sum_{i=1}^M m_1(m_1 - N); \quad e^{iK} = \prod_i \exp(2\pi im_i/N), $$

(67)
where \( \{m_i\} \) are a set of distinct integers in the range \( 0 < m_i < m_{i+1} < N \), subject to the “generalized Pauli principle” that not only are they distinct, but also that they cannot be consecutive. These quantum numbers can be represented by a binary sequence of length \( N - 1 \), where a “1” represents a value in the set \( \{m_i\} \). This means that the ground state sequence is 1010101...1010101, and has no consecutive pairs of zeroes, which represent spinon excitations. Removing a “1” from the sequence thus creates a two-spinon state ...101000010101... , which can be rearranged to give states such as ...1010100101010010101... A sequence such as this represents a fourfold-degenerate state with \( SU(2) \) representation content \( (\frac{1}{2}) \otimes (\frac{1}{2}) = 0 \otimes 1 \).

A sequence such as “...1000001...” where there are four successive “00” combinations represents a “four-string” in CHARI and PRESSLEY’s \([25]\) representation theory of \( Y(sl_2) \), and hence contributes a \( S = 2 \) factor in the tensor product of \( SU(2) \) representations that makes up the representation of \( Y(sl_2) \). Physically, this was previously interpreted \([27]\) as “four spinons in the same orbital” with a selection rule that spinons “in the same orbital” could only be in a symmetric spin state \([28]\). This empirically-observed rule, discovered by detailed examination of the results from numerical diagonalization \([27]\), now is seen to precisely correspond to the \( Y(sl_2) \) representation theory \([25]\).

This example suggests that “quantum-group” techniques may turn out to have important applications in connection with fractional statistics, as a more algebraic formulation that makes contact with “occupation number” descriptions and the Pauli principle.

IX. CONCLUSION

In these lectures, I sketched out the logic of an approach to Fermi fluids based on the idea that the Fermi surface is an analog of an order parameter, and that the low-energy degrees of freedom can be fully treated in terms of “bosonized” variables describing local fluctuations of the shape of the Fermi surface. The Luttinger theorem relating the volume of the Fermi surface to the particle density is seen to be the key principle. While bosonization has been a key tool in treating the one dimensional systems, it clearly shows promise in higher dimensions too. Much remains to be done to make this method a real working tool for higher dimensions. On the one hand, it will be interesting to see how much of the standard Fermi liquid results can be reproduced using such methods. On the other hand, they seem to have potential for the study of possible non-Fermi-liquid states, since they are not based on a perturbative expansion about the non-interacting Fermi gas. I also considered spin-charge separation, primarily in one dimension, and described some recent hints that “quantum group” methods may be important in cases where fermion variables fractionalize into fractional-statistics objects.

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