1 Introduction

In several papers[1-5], we have developed the Landé interpretation of quantum mechanics[6-9] to devise new methods of treating spin systems. By these methods, we have not only derived the matrix treatment of spin from probability amplitudes but we have also obtained new generalized forms of spin operators, their eigenvectors, and of spin states. These results have been for isolated spin systems. In this paper, we extend the formalism to the case of systems of compounded angular momentum. While addition of angular momentum might seem to be adequately treated by standard methods, we show that new results and insights are the reward of applying the new methods to this subject.

Our task in this paper is to derive explicit joint probability amplitudes for measurements of spin projections of subsystems compounded to give new systems, and then to use them to derive the matrix treatment of the compounded systems. We here present a new method for calculating the probability amplitudes and we present results which are more generalized than any we have seen to date for these probability amplitudes. The theory is applied to the cases of total spin 0 and 1 resulting from the addition of two spins of spin 1/2 each. For spin 0, we obtain one matrix representation of the system. For spin 1 however, there are two matrix representations.

Our considerations lead us to the conclusion that the Clebsch-Gordan coefficients can be generalized. We give these new generalized forms for the particular Clebsch-Gordan coefficients appearing in the cases treated here.

As an example of an application, the results we obtain for $S = 0$ and $S = 1$ are used to investigate joint measurements of the kind used to study entangled systems. We confirm standard results and find a reason for the correlations in the results for the $S = 0$ state.

This paper is organized according to the following plan. Section 2, which follows, is an exposition of basic theory. We there give the main equation from the work of Landé which our considerations are based upon. In Section 2.2, we explain the basic features of our approach. In Section 3, this approach is applied to the general problem of adding two angular momenta. After explaining the notation in Section 3.1, we give in Section 3.2 the basic formulas for the matrix treatment, as derived from probability amplitudes for the vectors and the operators.

In Section 4, we pose some of the questions which measurements on a
compounded-spin system are designed to answer. We begin our answer of these questions in Section 5. After explaining all the possibilities in Section 5.1, we give general formulas for probability amplitudes in Section 5.2. Section 5.3 lists the results of measurements on two uncoupled spin-1/2 systems. These are combined into the results of joint measurements on such systems in Section 5.4.

We derive the probability amplitudes for measurements on the singlet state in Section 5.5. Section 5.6 gives the probability amplitudes for the triplet state. Section 6 is devoted to the calculation of probabilities: thus, Section 6.1 contains the singlet-state probability amplitudes, and Section 6.2 the triplet-state probability amplitudes.

According to our treatment, the Clebsch-Gordan coefficients can be generalized. This is discussed in Section 7, and the generalized Clebsch-Gordan coefficients for both singlet and triplet states are given.

In Section 8, we shift to matrix mechanics. We derive the matrix-mechanics treatment of the singlet state in Section 8.2 and of the triplet state in Section 8.3.

Section 9 applies the new results to the calculation of the expectation values for joint measurements on the singlet states and triplet states. Section 10 presents the Discussion and Conclusion, which closes the paper.

2 Theory

2.1 Preliminary Results

We begin by reminding ourselves of some basic features of the Landé approach [6-9]. Let a quantum system have the observables \( A, B \) and \( C \). The eigenvalues of \( A \) are \( A_1, A_2, \ldots; \) the eigenvalues of \( B \) are \( B_1, B_2, \ldots; \) and the eigenvalues of \( C \) are \( C_1, C_2, \ldots. \) If the system is in the state corresponding to the eigenvalue \( A_i \) of \( A \), measurement of \( C \) yields any of the eigenvalues \( C_1, C_2, \ldots \) with probability amplitudes \( \psi(A_i; C_n) \). Measurement of \( B \) yields any of the eigenvalues \( B_1, B_2, \ldots \) with probability amplitudes \( \chi(A_i; B_n) \). Finally, measurement of \( C \) when the system is in the state corresponding to the eigenvalue \( B_i \) of \( B \) yields eigenvalues of \( C \) with probability amplitudes \( \phi(B_i; C_n) \). Then the probability amplitudes are connected by [6-9]
\( \psi(A_i; C_n) = \sum_q \chi(A_i; B_q) \phi(B_q; C_n), \) \hspace{1cm} (1)

which we shall call the Landé formula. The probability amplitudes satisfy the Hermiticity condition

\[ \psi(A_i; C_n) = \psi^*(C_n; A_i). \] \hspace{1cm} (2)

The expansions for the \( \phi \)'s and \( \chi \)'s are [1]

\[ \phi(B_l; C_k) = \sum_i \chi(B_l; A_i) \psi(A_i; C_k) \] \hspace{1cm} (3)

and

\[ \chi(A_i; B_m) = \sum_k \psi(A_i; C_k) \phi(C_k; B_m). \] \hspace{1cm} (4)

### 2.2 Further Results

The basis of our treatment is the expansion Eq. (1) and the interpretation of the wave function due to Landé. According to Landé, the wave function or the eigenfunction is to be interpreted as a probability amplitude which connects well-defined initial and final states. Thus the solution \( u_E(r) \) of the time-independent Schrödinger equation for a particular system is a probability amplitude that connects the initial state defined by the energy eigenvalue \( E \) to the final state corresponding to the position eigenvalue \( r \). Therefore \( |u_E(r)|^2 \, dr \) is the probability that if the system is initially in the state corresponding to the energy \( E \), a measurement of its position yields the value \( r \) in the volume element \( dr \). By the same token, the spherical harmonic \( Y_{lm}(\theta, \phi) \) is a probability amplitude connecting an initial state defined by the quantum numbers \( l \) and \( m \) to a final state characterized by the eigenvalues \((\theta, \varphi)\). Hence, \( |Y_{lm}(\theta, \varphi)|^2 \, d\Omega \) is the probability that if the square of the angular momentum is initially \( l(l+1)\hbar^2 \) and its \( z \) component is \( m\hbar \), a measurement of the angular position gives \((\theta, \varphi)\) in \( d\Omega \).

This approach can be very fruitfully applied to the treatment of spin, as we have shown[1-5]. We have utilized it to develop a way to derive generalized probability amplitudes, operators and vector states for spin and have illustrated the method by applying it to the cases of spin \( 1/2 \)[1,2,4,5] and
spin 1 \[3\]. We shall denote the spin probability amplitudes by the functions \(\chi(m(\hat{a}); m(\hat{b}))\). Here \(\hat{a}\) is a direction vector with respect to which the spin projection is initially known and \(\hat{b}\) is the direction vector with respect to which we subsequently measure it. Hence, \(m(\hat{a})\), for example, represents the projection \(m(\hat{a})\hbar\) in the direction \(\hat{a}\). Thus \(|\chi(+(\frac{1}{2})\hat{a}; +(\frac{1}{2})\hat{b})|^2\) is the probability that if the spin projection is initially up with respect to the unit vector \(\hat{a}\), a new measurement finds it up with respect to the vector \(\hat{b}\). Similarly, \(|\chi((-\frac{1}{2})\hat{a}; +(\frac{1}{2})\hat{b})|^2\) is the probability of finding the spin projection to be up with respect to \(\hat{b}\) if it was initially down with respect to \(\hat{a}\). For spin 1/2, there are two other probability amplitudes \(\chi((+\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{b})\) and \(\chi((-\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{b})\), whose interpretation is obvious.

By expanding the probability amplitudes using Eq. (1), we obtain the matrix treatment of spin. We have demonstrated this for spin 1/2 and spin 1. Thus, by means of this expansion, the treatment of spin is made analogous to that of orbital angular momentum, since the matrix treatment of orbital angular momentum is achieved with the aid of an expansion in terms of the spherical harmonics. The main difference between the two cases derives from the fact that spin is distinguished from most other dynamical variables by the circumstance that the eigenvalues corresponding to both initial and final states are discrete. In contrast, orbital angular momentum as described by the spherical harmonics is characterized by a discrete initial eigenvalue spectrum and a continuous final spectrum.

### 3 General Addition of Angular Momentum

#### 3.1 Re-interpretation and Notation

We now wish to generalize the new method of deriving probability amplitudes to systems compounded of other systems. We are specifically interested in the singlet and triplet states obtained by adding two spins of value 1/2. We want to derive the probability amplitudes pertaining to measurements on such systems and we want to show that the matrix treatment of such systems can be derived from first principles. To this end, we first look at the general problem of obtaining a matrix treatment of a compounded angular-momentum system.
Our approach to the problem is based on the re-interpretation of standard quantities and equations in terms of the Landé approach. Let the angular momenta of two subsystems 1 and 2 be $J_1$ and $J_2$. Let these angular momenta be added to give the total angular momentum $J$. Let the angular momenta be $J_1$ and $J_2$.

Let these angular momenta be added to give the total angular momentum $J$. Let the simultaneous eigenfunction of $J_2$, $J_z$, and $J_2$ be $\Psi_{J,M,j_1,j_2}(\theta_1, \varphi_1, \theta_2, \varphi_2)$.

Here $(\theta_1, \varphi_1)$ are the angular variables pertaining to $J_1$, while $(\theta_2, \varphi_2)$ are the variables for $J_2$. In the spirit of the Landé interpretation of quantum mechanics, this function is a probability amplitude which connects an initial state characterized by the eigenvalues $j(j+1)\hbar^2$, $M\hbar$, $j_1(j_1+1)\hbar^2$ and $j_2(j_2+1)\hbar^2$ with a final state characterized by the eigenvalues $\theta_1, \varphi_1, \theta_2$ and $\varphi_2$. For this reason, $|\Psi_{J,M,j_1,j_2}(\theta_1, \varphi_1, \theta_2, \varphi_2)|^2\ d\Omega_1d\Omega_2$ is the probability that starting from the initial state characterised by the eigenvalues $j(j+1)\hbar^2$, $M\hbar$, $j_1(j_1+1)\hbar^2$ and $j_2(j_2+1)\hbar^2$, a measurement of angular position gives $(\theta_1, \varphi_1, \theta_2, \varphi_2)$ in $d\Omega_1d\Omega_2$.

Let the eigenfunctions for isolated subsystem 1 be $\phi^{(1)}_{j_1m_1}(\theta_1, \varphi_1)$ and those for isolated subsystem 2 be $\phi^{(2)}_{j_2m_2}(\theta_2, \varphi_2)$. According to the new formalism, the wave function for subsystem 1 gives the probability amplitude that if the initial eigenstate of the system has the quantum numbers $j_1$ and $m_1$ a measurement of angular position gives $(\theta_1, \varphi_1)$. A corresponding interpretation holds for subsystem 2. To bring out this interpretation, we rewrite these probability amplitudes as

$$\phi_1(j_1, m_1; \theta_1, \varphi_1) = \phi^{(1)}_{j_1m_1}(\theta_1, \varphi_1)$$  \hfill (5)

and

$$\phi_2(j_2, m_2; \theta_2, \varphi_2) = \phi^{(2)}_{j_2m_2}(\theta_2, \varphi_2).$$  \hfill (6)

We denote their product by

$$\Phi(j_1, m_1, j_2, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2) = \phi_1(j_1, m_1; \theta_1, \varphi_1)\phi_2(j_2, m_2; \theta_2, \varphi_2).$$  \hfill (7)

The eigenfunction for the state obtained by adding the two angular momenta $J_1$ and $J_2$ is[10]

$$\Psi_{J,M,j_1,j_2}(\theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} C(j_1,j_2; m_1m_2M)\phi^{(1)}_{j_1m_1}(\theta_1, \varphi_1)\phi^{(2)}_{j_2m_2}(\theta_2, \varphi_2),$$  \hfill (8)
where we have used the notation in Rose[10] for the Clebsch-Gordan coefficients $C(j_1 j_2 j; m_1 m_2 M)$. In order to interpret this eigenfunction in terms of the Landé approach, we have to change the notation appropriately. Hence we rewrite Eq. (8) as

$$
\Psi(j, M, j_1, j_2; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} C(j_1 j_2 j; m_1 m_2 M) \\
\times \phi_1(j_1, m_1; \theta_1, \varphi_1) \phi_2(j_2, m_2; \theta_2, \varphi_2).
$$

(9)

This eigenfunction is the probability amplitude that if the initial state is characterized by the quantum numbers $(j, M, j_1, j_2)$, a measurement of angular position gives $(\theta_1, \varphi_1, \theta_2, \varphi_2)$.

Now if we compare the basic equation Eq. (1) and the expansion Eq. (9), the Clebsch-Gordan coefficients are immediately seen to be probability amplitudes. Thus $C(j_1 j_2 j; m_1 m_2 M)$ is the probability amplitude that if the compound system is in a state corresponding to the quantum numbers $(j, M, j_1, j_2)$, a measurement of the $z$ components of the spins of systems 1 and 2 gives $m_1 \hbar$ and $m_2 \hbar$ respectively, while a measurement of the squares of the spins gives $j_1(j_1 + 1)\hbar^2$ and $j_2(j_2 + 1)\hbar^2$ respectively. To emphasize the fact that the Clebsch-Gordan coefficients are just probability amplitudes, and in order to cast Eq. (9) in terms of the fundamental expansion Eq. (1), we set

$$
\eta(j_1, j_2, j, M; j_1, j_2, m_1, m_2) = C(j_1 j_2 j; m_1 m_2 M).
$$

(10)

Thus, with the aid of Eq. (7) we express Eq. (9) in the form

$$
\Psi(j, j_1, j_2, M; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} \eta(j_1, j_2, j, M; j_1, j_2, m_1, m_2) \\
\times \Phi(j_1, j_2, m_1, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2).
$$

(11)

We can suppress the indices $j_1$ and $j_2$ because for given subsystems 1 and 2, they are fixed. However, they may, of course, give different values of $j$ within the range $|j_1 - j_2| \leq j \leq j_1 + j_2$. In addition, we can suppress the index $m_2$ because of the constraint $m_1 + m_2 = M$. We then have

$$
\eta(j, M; m_1) = \eta(j_1, j_2, j, M; j_1, j_2, m_1, m_2)
$$

(12)

so that
\[ \Psi(j, M; \theta_1, \phi_1, \theta_2, \phi_2) = \sum_{m_1} \eta(j, M; m_1) \Phi(m_1; \theta_1, \phi_1, \theta_2, \phi_2). \] (13)

We see immediately that the standard expression for the wave function of a system of compounded angular momentum is of the form Eq. (1). This justifies the interpretation we have given to the Clebsch-Gordan coefficients.

In the expansion, \( \Psi \) and the \( \eta' \)s are not known, while the \( \Phi' \)s are known. If we are adding two orbital angular momenta, \( \Phi \) is the product of two spherical harmonics. When we are adding spins, \( \Phi \) is the product of two spin probability amplitudes. If orbital angular momentum and spin are being compounded, then \( \Phi \) is the product of a spherical harmonic and a spin probability amplitude.

### 3.2 General Expressions For Operators And Vectors

In this section, we show how to obtain the matrix treatment of added angular momentum. Thus, we obtain the general formulas for the vector states and operators. We achieve this by going through the definition of the expectation value. Consider the case where we are measuring values of the observable \( R(\theta_1, \varphi_1, \theta_2, \varphi_2) \). The expectation value of \( R \) is

\[ \langle R \rangle = \int |\Psi(j, M; \theta_1, \varphi_1, \theta_2, \varphi_2)|^2 R(\theta_1, \varphi_1, \theta_2, \varphi_2) d\Omega_1 d\Omega_2. \] (14)

Using

\[ \Psi(j, M; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} \eta(j, M; m_1) \Phi(m_1; \theta_1, \varphi_1, \theta_2, \varphi_2) \] (15)

and

\[ \Psi^*(j, M; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m'_1} \eta^*(j, M; m'_1) \Phi^*(m'_1; \theta_1, \varphi_1, \theta_2, \varphi_2), \] (16)

we obtain

\[ \langle R \rangle = \sum_{m_1} \sum_{m'_1} \eta^*(j, M; m'_1) R_{m'_1 m_1} \eta(j, M; m_1), \] (17)
where

\[ R_{m'_1 m_1} = \int \Phi^*(m'_1; \theta_1, \varphi_1, \theta_2, \varphi_2) R(\theta_1, \varphi_1, \theta_2, \varphi_2) \times \Phi(m_1; \theta_1, \varphi_1, \theta_2, \varphi_2) d\Omega_1 d\Omega_2. \]  

(18)

Hence

\[ \langle R \rangle = [\Psi]^\dagger [R] [\Psi], \]  

(19)

where

\[ [\Psi] = \begin{pmatrix} \eta(j, M; (m_1)_1) \\ \eta(j, M; (m_1)_2) \\ \vdots \\ \eta(j, M; (m_1)_N) \end{pmatrix} \]  

(20)

and

\[ [R] = \begin{pmatrix} R_{11} & R_{22} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{pmatrix}. \]  

(21)

Here \( N \) is the number of combinations of \( m_1 \) and \( m_2 \) such that \( m_1 + m_2 = M \). We note also that \( (m_1)_i \) are individual values of \( m_1 \), given labels from 1 to \( N \).

We see that the matrix representation of the state is a row vector whose elements are the Clebsch-Gordan coefficients.

Actually, the matrix representation presented here is not the only one possible. Others can be realized. They arise if we seek the generalized form of the probability amplitude Eq. (13).

The standard treatment of angular momentum addition assumes that the initial direction of projection of the total spin is the \( z \) direction. But in the generalized treatment we shall give here, this direction is arbitrary. In that case, Eq. (13) is replaced by

\[ \Psi(j, M^\parallel; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1, m_2} \eta(j, M^\parallel; m_1, m_2) \Phi(m_1, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2), \]  

(22)

where the generalized probability amplitude \( \Psi(j, M^\parallel; \theta_1, \varphi_1, \theta_2, \varphi_2) \) is such that the initial projection of the total spin is with respect to the direction
By the same token, \( \eta(j, M(\hat{a}); m_1, m_2) \) now contains the quantum number \( M(\hat{a}) \), which refers to the direction \( \hat{a} \) instead of the \( z \) direction. This quantity is no longer a Clebsch-Gordan coefficient, but as we shall see later, it can be expressed in terms of the Clebsch-Gordan coefficients, by means of the expansion Eq. (4). But this new expansion can be used to realize a different matrix representation. We observe that since \( \eta(j, M(\hat{a}); m_1, m_2) \) is no longer a Clebsch-Gordan coefficient, the condition \( m_1 + m_2 = M \) no longer necessarily holds, so that in principle the summation in Eq. (22) is over both \( m_1 \) and \( m_2 \).

4 Measurements on Systems of Compounded Spin

The theory we have outlined above is most easily applied to spin systems. We consider the case of two such systems compounded to give one system. The total spin of the compounded system is

\[
S = S_1 + S_2, \tag{23}
\]

where \( S_1 \) is the spin of subsystem 1 and \( S_2 \) is the spin of subsystem 2. We may ask the following questions regarding these systems.

(a) Suppose we focus attention on one of the subsystems. If the spin projection of that subsystem is initially known in a given direction, what is the probability of obtaining a given value of the spin projection in a new direction? The answer is easily obtained since the subsystem can be treated as if isolated. When the subsystem is of spin \( 1/2 \), or of spin 1, the probability amplitudes that answer this question for the general case are as given in Refs. [1,2,4,5] and [3] respectively. If the spin is not \( 1/2 \) or 1, we can derive the generalized probability amplitudes by the method illustrated in these references. If we are satisfied with a less general answer, we can use standard expressions [11] for the probability amplitudes. As shown in Refs. [1-5], the standard expressions can be obtained from the generalized expressions by setting the direction in which we initially know the spin projection to be the \( z \) direction.

(b) Suppose we focus attention on the compound system. If its spin projection is initially known along a given direction, what is the probability
of obtaining a given value of the projection along a new direction? The answer depends only on the total spin of the compound system. It is not necessary to know what individual spins have been combined to give the particular value of the compound spin. If the compound spin is 0, then of course there is only one projection and the question is trivial because the required probability is unity. If the compound spin is $1/2$ or 1, the probability amplitudes are the ones derived in Refs. [1,2,4,5] and in [3] respectively. If the compound spin is neither $1/2$ nor 1, it is necessary to use the method outlined in Refs. [1-5] to derive these probability amplitudes. Again, if we are satisfied with less generality, we can use the standard formulas for that value of spin.

(c) Finally, suppose we focus attention on the compound system. If the projection of the total spin is initially known along the direction $\hat{a}$, what is the probability that a measurement of the spin projection of subsystem 1 along the vector $\hat{c}_1$ gives a specified value while a simultaneous measurement of the spin projection of subsystem 2 along the direction $\hat{c}_2$ gives a certain value? This is the question we address here. We show that the probability amplitude represented by the functions for the compound spin are correspond to precisely this measurement.

5 Probability Amplitudes

5.1 Possible Results of Measurements on Subsystems

The compounded system has states characterized by simultaneous eigenvalues of $S^2$, $S_1^2$, $S_2^2$ and $S_z$. Since $s_1$ and $s_2$ are fixed, we omit them in the labelling of the states. The basic quantity we wish to obtain is the probability amplitude for obtaining a specified value of $S_1z$ along the vector $\hat{c}_1$ and a specified value of $S_2z$ along the vector $\hat{c}_2$ starting from a state characterised by the specified value $S_z$ along the vector $\hat{a}$. We shall make our investigation concrete by considering the case $s_1 = s_2 = 1/2$. The reason for this is that this is the simplest non-trivial case to which we may apply the new theory we shall develop here.

For $s_1 = s_2 = 1/2$, there are only two possible values of the spin projection for each subsystem. If the spin projection is found up (down) with respect to $\hat{c}_1$ or $\hat{c}_2$, we label this outcome by $(+\frac{1}{2})(\hat{c}_1)$ $((-\frac{1}{2})(\hat{c}_1))$ or $(+\frac{1}{2})(\hat{c}_2)$ $((-\frac{1}{2})(\hat{c}_2))$, respectively. Thus the possible combinations of the results of the
measurements are as follows:

1) spin 1 up with respect to \(\hat{c}_1\) and spin 2 up with respect to \(\hat{c}_2\) (denoted by \(((+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)})\));

2) spin 1 up with respect to \(\hat{c}_1\) and spin 2 down with respect to \(\hat{c}_2\) (denoted by \(((+\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)})\));

3) spin 1 down with respect to \(\hat{c}_1\) and spin 2 up with respect to \(\hat{c}_2\) (denoted by \((-\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)})\));

4) spin 1 down with respect to \(\hat{c}_1\) and spin 2 down with respect to \(\hat{c}_2\) (denoted by \((-\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)})\)).

We shall denote a particular such outcome by \(((m_1)^{(c_1)}, (m_2)^{(c_2)})\), with \(u,v = 1,2\). Thus,

\[
((m_1)^{(c_1)_1}, (m_2)^{(c_2)_1}) = ((+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}) ,
\]

\[
((m_1)^{(c_1)_2}, (m_2)^{(c_2)_2}) = ((+\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}) ,
\]

\[
((m_1)^{(c_1)_2}, (m_2)^{(c_2)_1}) = ((-\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}) ,
\]

\[
((m_1)^{(c_1)_1}, (m_2)^{(c_2)_2}) = ((-\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}) .
\]

Corresponding to each initial state characterised by the quantum number \(M_{(\hat{a})}\) for the spin projection along \(\hat{a}\) of the compound system, there are four possible probability amplitudes. These possibilities are denoted by 

\[
\Psi(s, M_{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}), \Psi(s, M_{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}), \Psi(s, M_{(\hat{a})}; (-\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)})
\]

and

\[
\Psi(s, M_{(\hat{a})}; (-\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}).
\]

When we mean to denote these probability amplitudes in a general way, we shall use the shorthand \(\Psi(s, M_{(\hat{a})}; (m_1)^{(c_1)}, (m_2)^{(c_2)})\).

### 5.2 General Formulas For Probability Amplitudes

We now seek the explicit forms of these probability amplitudes. The basic method is to use the Landé expansion, Eq. (41), to express the required probability amplitudes in terms of known probability amplitudes. Consider the probability amplitude \(\Psi(s, M_{(\hat{a})}; (m_1)^{(c_1)}, (m_2)^{(c_2)})\). Referring to Eq. (41), and assuming that the total spin \(s\) is fixed, we first identify the observable
with projection along the $z$ axis. The reason for this choice of intermediate states will become clear later. Thus, we have

$$
\Psi(s, M_{i}^\sigma; (m_{1})^\sigma_{u}, (m_{2})^\sigma_{v}) = \sum_{j} \chi(s, M_{j}^\delta; s, M_{j}^k)
\times \xi(s, M_{j}^{\hat{k}}; (m_{1})^{\hat{c}_{1}}, (m_{2})^{\hat{c}_{2}}),
$$

(28)

where $\chi(s, M_{i}^\sigma; s, M_{j}^k)$ is the probability amplitude for finding that the spin projection along the $z$ direction $\hat{k}$ is $M_{j}^k\hbar$ when initially, the spin projection is $M_{i}^\sigma\hbar$ along the direction $\hat{\sigma}$. Since $s_{1} = s_{2} = 1/2$, it follows that $s = 0, 1$. For this reason the probability amplitudes $\chi$ belong to $s = 0$ or $s = 1$ and are therefore known. If $s = 0$, then there is only one value $M_{i}^\sigma = 0$; hence $\chi(s, 0^\sigma; s, 0^k) = e^{i\delta}$, where $\delta$ is a real number which we shall set equal to zero. For $s = 1$, the probability amplitudes $\chi$ are given in Ref. [3].

The function $\xi(s, M_{j}^{\hat{k}}; (m_{1})^{\hat{c}_{1}}, (m_{2})^{\hat{c}_{2}})$ is the probability amplitude that if the state of the compound system is initially characterized by the eigenvalue $M_{j}^{\hat{k}}\hbar$, a measurement of the spin projections of subsystems 1 and 2 yields the projections $(m_{1})_{u}\hbar$ and $(m_{2})_{v}\hbar$ with respect to the direction vectors $\hat{c}_{1}$ and $\hat{c}_{2}$ respectively. However, the functions $\xi$ are not known. But they can be obtained by using the Landé expansion once more. Thus, we set

$$
\xi(s, M_{j}^{\hat{k}}; (m_{1})^{\hat{c}_{1}}, (m_{2})^{\hat{c}_{2}}) = \sum_{\alpha,\alpha'} \eta(s, M_{j}^{\hat{k}}; (m_{1})^{\hat{c}_{1}}, (m_{2})^{\hat{c}_{2}})
\times \psi((m_{1})^{\hat{k}}, (m_{2})^{\hat{k}}; (m_{1})^{\hat{c}_{1}}, (m_{2})^{\hat{c}_{2}}).
$$

(29)

In this expansion, the values $((m_{1})^{\hat{k}}, (m_{2})^{\hat{k}})$ are given by Eqs. (24) - (27) with $\hat{c}_{1} = \hat{c}_{2} = \hat{k}$. Thus for example, $\eta(s, M_{j}^{\hat{k}}; (+1/2)^{\hat{k}}, (+1/2)^{\hat{k}})$ is the probability amplitude for the spin projection of subsystem 1 to be found up with respect to the $z$ direction and for that of subsystem 2 to be found up with
respect to $z$ direction upon measurement, when the initial compound state is characterized by the spin projection being $M_j^{(\hat{k})}\hbar$ along the $z$ direction. Thus, according to the interpretation of Section 3.1, the $\eta(s, M_j^{(\hat{k})}; (m_1)_{\alpha}^{(\hat{k})}, (m_2)_{\alpha'}^{(\hat{k})})$ must be Clebsch-Gordan coefficients. Hence, when systems 1 and 2 are spin-1/2 systems, we have

$$\eta(s, M_j^{(\hat{k})}; (\pm \frac{1}{2})^{(\hat{k})}, (\pm \frac{1}{2})^{(\hat{k})}) = C\left(\frac{1}{2}, \frac{1}{2}; \pm \frac{1}{2}, M_j^{(\hat{k})}\right). \quad (30)$$

On the other hand, the function $\psi((m_1)_{\alpha}^{(\hat{k})}, (m_2)_{\alpha'}^{(\hat{k})}; (m_1)_{u}^{(\hat{c}_1)}, (m_2)_{v}^{(\hat{c}_2)})$ is the probability amplitude that if the spin projection of subsystem 1 is $(m_1)_{\alpha}\hbar$ along the $z$ axis, and that of subsystem 2 is $(m_2)_{\alpha'}\hbar$ along the $z$ axis, a measurement of the spin projection of subsystem 1 along $\hat{c}_1$ finds $(m_1)_{u}\hbar$ while a measurement of the spin projection of subsystem 2 along $\hat{c}_2$ finds $(m_2)_{v}\hbar$. Thus, it follows that

$$\psi((m_1)_{\alpha}^{(\hat{k})}, (m_2)_{\alpha'}^{(\hat{k})}; (m_1)_{u}^{(\hat{c}_1)}, (m_2)_{v}^{(\hat{c}_2)}) = \phi_1((m_1)_{\alpha}^{(\hat{k})}; (m_1)_{u}^{(\hat{c}_1)})$$
$$\times \phi_2((m_2)_{\alpha'}^{(\hat{k})}; (m_2)_{v}^{(\hat{c}_2)}), \quad (31)$$

where the $\phi_1$ are the probability amplitudes for the measurement of spin projections of subsystem 1 from one direction to another. By the same token, the $\phi_2$ are the probability amplitudes for the measurement of spin projections of subsystem 2 from one direction to another. The probability amplitudes $\phi_1$ and $\phi_2$ come in a variety of forms, depending on the choice of phase made when they are being derived[1,4].

### 5.3 Results of Measurements on Isolated Spin-1/2 Systems

The derivation of the probability amplitudes $\phi_1((m_1)_{\alpha}^{(\hat{k})}; (m_1)_{u}^{(\hat{c}_1)})$ or $\phi_2((m_2)_{\alpha'}^{(\hat{k})}; (m_2)_{v}^{(\hat{c}_2)})$ has already been done for two different choices of phase[1,4]. Consider either subsystem 1 or 2. Let the spin projection be initially in the direction of the vector $\hat{d}$, whose polar angles are $(\theta, \varphi)$. We subsequently measure it in the direction of the vector $\hat{e}$, whose polar angles are $(\theta', \varphi')$. In Ref. [1], we found the following probability amplitudes for the measurements:

$$\phi((\pm \frac{1}{2})^{(\hat{d})}; (\pm \frac{1}{2})^{(\hat{e})}) = \cos \theta/2 \cos \theta'/2 + e^{i(\varphi-\varphi')} \sin \theta/2 \sin \theta'/2, \quad (32)$$
\[ \phi\left(\left(\frac{1}{2}\right)\hat{d}; \left(-\frac{1}{2}\right)\hat{e}\right) = \cos \frac{\theta}{2} \sin \theta' \frac{1}{2} - e^{i(\varphi-\varphi')} \sin \frac{\theta}{2} \cos \theta' \frac{1}{2}, \]  
(33)

\[ \phi\left(\left(-\frac{1}{2}\right)\hat{d}; \left(\frac{1}{2}\right)\hat{e}\right) = \sin \frac{\theta}{2} \cos \theta' \frac{1}{2} - e^{i(\varphi-\varphi')} \cos \frac{\theta}{2} \sin \theta' \frac{1}{2} \]  
(34)

and

\[ \phi\left(\left(-\frac{1}{2}\right)\hat{d}; \left(-\frac{1}{2}\right)\hat{e}\right) = \sin \frac{\theta}{2} \sin \theta' \frac{1}{2} + e^{i(\varphi-\varphi')} \cos \frac{\theta}{2} \cos \theta' \frac{1}{2}. \]  
(35)

It turns out however that these probability amplitudes will not serve here, because they ultimately lead to incorrect results when employed. Since they differ from other sets of probability amplitudes for the same measurements only in phase, this means that in general we cannot use an arbitrary choice of phase to develop our theory. That the probability amplitudes Eq. (32) - (35) lead to the wrong results when used is deduced from the following circumstance. The probability amplitudes Eq. (28) must reduce to the Clebsch-Gordan coefficients to within a phase factor if we set \( \hat{a} = \hat{c}_1 = \hat{c}_2 = \hat{k} \). (The argument leading to this requirement is given in Section 7, where the Clebsch-Gordan coefficients are discussed). When we use Eqs. (32) - (35) in the treatment that follows, we produce probability amplitudes \( \Psi(s, M_i; (m_1)_{\hat{c}_1}, (m_2)_{\hat{c}_2}) \) that do not have this property. For this reason, it is necessary to try other choices of phase until this condition is satisfied. In fact the phase choice that gives the correct results corresponds to the following probability amplitudes:

\[ \phi\left(\left(\frac{1}{2}\right)\hat{d}; \left(\frac{1}{2}\right)\hat{e}\right) = \cos \frac{\theta}{2} \cos \theta' \frac{1}{2} + e^{i(\varphi-\varphi')} \sin \frac{\theta}{2} \sin \theta' \frac{1}{2}, \]  
(36)

\[ \phi\left(\left(\frac{1}{2}\right)\hat{d}; \left(-\frac{1}{2}\right)\hat{e}\right) = -\cos \frac{\theta}{2} \sin \theta' \frac{1}{2} + e^{i(\varphi-\varphi')} \sin \frac{\theta}{2} \cos \theta' \frac{1}{2}, \]  
(37)

\[ \phi\left(\left(-\frac{1}{2}\right)\hat{d}; \left(\frac{1}{2}\right)\hat{e}\right) = -\sin \frac{\theta}{2} \cos \theta' \frac{1}{2} + e^{i(\varphi-\varphi')} \cos \frac{\theta}{2} \sin \theta' \frac{1}{2} \]  
(38)
and

\[ \phi((-\frac{1}{2})\hat{d}; (-\frac{1}{2})\hat{e}) = \sin \frac{\theta}{2} \sin \frac{\theta'}{2} + e^{i(\varphi - \varphi')} \cos \frac{\theta}{2} \cos \frac{\theta'}{2}. \]  

(39)

Setting \( \theta = \varphi = 0 \), so that \( \hat{d} = \hat{k} \), and letting \( \theta' = \theta_1 \), \( \varphi' = \varphi_1 \) so that \( \hat{e} = \hat{c}_1 \) (whose polar angles are \( (\theta_1, \varphi_1) \)), we get

\[ \phi_1((+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1) = \cos \frac{\theta_1}{2}, \]  

(40)

\[ \phi_1((+\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1) = -\sin \frac{\theta_1}{2}, \]  

(41)

\[ \phi_1((-\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1) = e^{-i\varphi_1} \sin \frac{\theta_1}{2} \]  

(42)

and

\[ \phi_1((-\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1) = e^{-i\varphi_1} \cos \frac{\theta_1}{2}. \]  

(43)

The same formulas apply for the \( \phi_2 \) except that the index 1 is everywhere replaced by the index 2.

### 5.4 Results of Joint Measurements on the Uncoupled Subsystems

It is now a straightforward matter to obtain the \( \psi((m_1)_{\alpha}^{(\hat{k})}, (m_2)_{\alpha'}^{(\hat{k})}: (m_1)_{\alpha}^{(\hat{c}_1)}, (m_2)_{\alpha}^{(\hat{c}_2)}) \). Using Eq. (31), and Eqs. (40) - (43), we obtain

\[ \psi((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}, \]  

(44)

\[ \psi((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = -\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \]  

(45)

\[ \psi((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = -\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}, \]  

(46)

\[ \psi((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \]  

(47)
With these results, we are in a position to compute the probability am-

\[ \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2)) = \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2}, \]  

(48)

\[ \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2}, \]  

(49)

\[ \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = -\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2}, \]  

(50)

\[ \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = -\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2}, \]  

(51)

\[ \psi((-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1}, \]  

(52)

\[ \psi((-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = -\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}, \]  

(53)

\[ \psi((-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1}, \]  

(54)

\[ \psi((-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = -\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}, \]  

(55)

\[ \psi((-\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)}, \]  

(56)

\[ \psi((-\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)}, \]  

(57)

\[ \psi((-\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \]  

(58)

and

\[ \psi((-\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}; (-\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)}. \]  

(59)

With these results, we are in a position to compute the probability am-

It seems there's a missing text in line 16. Can you provide the correct content for line 16?
5.5 The Singlet-State Probability Amplitudes

We first deal with the case \( s = 0 \). Thus, there is only one projection, characterised by \( M^{(\bar{\alpha})} = 0 \). For this case therefore, there is only the summation in Eq. (29) to be carried out. The one in Eq. (28) is redundant. Since

\[
\chi(s = 0, M^{(\bar{\alpha})} = 0; s = 0, M^{(\bar{k})} = 0) = \chi(0, 0^{(\bar{\alpha})}; 0, 0^{(\bar{k})}) = 1,
\]

we have

\[
\Psi(0, 0^{(\bar{\alpha})}; (m_1)^u_{\alpha} (c_1), (m_2)^v_{\alpha'} (c_2)) = \xi(0, 0^{(\bar{\alpha})}; (m_1)^u_{\alpha} (c_1), (m_2)^v_{\alpha'} (c_2))
\]

\[
= \sum_{\alpha, \alpha'} \eta(0, 0^{(\bar{k})}; (m_1)^k_{\alpha} (\bar{c}_1), (m_2)^k_{\alpha'} (\bar{c}_2)) \times \psi((m_1)^k_{\alpha} (\bar{c}_1), (m_2)^k_{\alpha'} (\bar{c}_2)).
\]

Since the \( \eta \)'s are Clebsch-Gordan coefficients, they are

\[
\eta(0, 0^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2} 0; \frac{1}{2} \frac{1}{2} 0) = 0,
\]

\[
\eta(0, 0^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2} 0; \frac{1}{2} \frac{1}{2} 0) = \frac{1}{\sqrt{2}},
\]

\[
\eta(0, 0^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2} 0; -\frac{1}{2} \frac{1}{2} 0) = -\frac{1}{\sqrt{2}},
\]

and

\[
\eta(0, 0^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2} 0; -\frac{1}{2} -\frac{1}{2} 0) = 0.
\]

Here, we have obtained the values of the Clebsch-Gordan coefficients from Rose[10].

Using the expressions for the \( \psi \)'s given by Eqs. (44) - (59), we obtain the probability amplitudes

\[
\Psi(0, 0^{(\bar{\alpha})}; (+\frac{1}{2})^{(\bar{c}_1)}, (+\frac{1}{2})^{(\bar{c}_2)}) = \frac{1}{\sqrt{2}} \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} - \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} \right],
\]

\[
\Psi(0, 0^{(\bar{\alpha})}; (+\frac{1}{2})^{(\bar{c}_1)}, (-\frac{1}{2})^{(\bar{c}_2)}) = \frac{1}{\sqrt{2}} \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} \right],
\]

(66)
\[ \Psi(0, 0; (-\frac{1}{2}) \hat{c}_1, (+\frac{1}{2}) \hat{c}_2) = -\frac{1}{\sqrt{2}} \left[ \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} + \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} \right] \]  

(68)

and

\[ \Psi(0, 0; (-\frac{1}{2}) \hat{c}_1, (-\frac{1}{2}) \hat{c}_2) = \frac{1}{\sqrt{2}} \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\varphi_1} - \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} \right]. \]  

(69)

5.6 The Triplet State Probability Amplitudes

Owing to the presence of two summations, the triplet case is a bit more involved than the singlet case. The general formula for the probability amplitudes is given by Eq. (28). Each of the four probability amplitudes has the form

\[ \Psi(1, M; (\hat{a})_i, (\hat{c})_j) = \chi(1, M; (\hat{a})_i; 1, M; (\hat{k})_j) \xi(1, M; (\hat{a})_i, (\hat{c})_j) \]

(70)

The probability amplitudes \( \chi(1, M; (\hat{a})_i; 1, M; (\hat{k})_j) \) connect spin projections measurements such that the initial state corresponds to the vector \( \hat{a} \), while the result corresponds to the vector \( \hat{k} \). Since \( s = 1 \), there are three values of \( M^{(\hat{a})} \) and \( M^{(\hat{k})} \): these are \( M^{(\hat{a})} = 0, \pm 1 \). These probability amplitudes \( \chi \) are derived in Ref. [3].

5.6.1 The \( M^{(\hat{a})} = 1 \) State

For \( M^{(\hat{a})} = 1 \), the \( \chi(1, M^{(\hat{a})}; 1, M^{(\hat{k})}) \) are

\[ \chi(1, 1; 1, 1) = \cos^2 \frac{\theta}{2} e^{-i\varphi}, \]

(71)

\[ \chi(1, 1; 1, 0) = \frac{1}{\sqrt{2}} \sin \theta \]

(72)

and
\[ \chi(1, 1^{(\bar{a})}; 1, (-1)^{(\bar{k})}) = \sin^2 \frac{\theta}{2} e^{i\varphi}. \]  

(73)

The probability amplitudes \( \xi \) are given by Eq. (29) and are:

\[
\xi(1, M_j^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v) = \eta(1, M_j^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) \\
\times \psi((+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v) \\
+ \eta(1, M_j^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) \psi((+\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v) \\
+ \eta(1, M_j^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) \psi((-\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v) \\
+ \eta(1, M_j^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) \psi((-\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v).
\]

(74)

The \( \eta \)'s are Clebsch-Gordan coefficients; thus setting \( M^{(\bar{k})} = 1 \), we have

\[
\eta(1, 1^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; 1 \frac{1}{2} 1) = 1, \\
\eta(1, 1^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; 1 \frac{1}{2} -\frac{1}{2}) = 0, \\
\eta(1, 1^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; -\frac{1}{2} 1) = 0
\]

(75) 

(76) 

(77)

and

\[
\eta(1, 1^{(\bar{k})}; (-\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; -\frac{1}{2} -\frac{1}{2}) = 0.
\]

(78)

Therefore,

\[
\xi(1, 1^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v) = \psi((+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}; (m_1)^{(\bar{c}_1)}_u, (m_2)^{(\bar{c}_2)}_v).
\]

(79)

Setting \( M^{(\bar{k})} = 0 \), we find

\[
\eta(1, 0^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (+\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; 1 \frac{1}{2} 0) = 0, \\
\eta(1, 0^{(\bar{k})}; (+\frac{1}{2})^{(\bar{k})}, (-\frac{1}{2})^{(\bar{k})}) = C(\frac{1}{2} \frac{1}{2}; 1 \frac{1}{2} -\frac{1}{2}) = \frac{1}{\sqrt{2}}.
\]

(80) 

(81)
Thus, \( \eta(1, 0^{(\hat{k})}; (-\frac{1}{2})^{(\hat{k})}, (+\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; -\frac{1}{2}, 0) = \frac{1}{\sqrt{2}}, \) \hspace{1cm} (82)

and

\( \eta(1, 0^{(\hat{k})}; (-\frac{1}{2})^{(\hat{k})}, (-\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; -\frac{1}{2}, -\frac{1}{2}, 0) = 0. \) \hspace{1cm} (83)

Thus

\( \xi(1, 0^{(\hat{k})}; (m_1)^{(c_1)}, (m_2)^{(c_2)}) = \frac{1}{\sqrt{2}} \psi((+\frac{1}{2})^{(\hat{k})}, (-\frac{1}{2})^{(\hat{k})}, (m_1)^{(c_1)}, (m_2)^{(c_2)}) + \psi((-\frac{1}{2})^{(\hat{k})}, (+\frac{1}{2})^{(\hat{k})}, (m_1)^{(c_1)}, (m_2)^{(c_2)})]. \) \hspace{1cm} (84)

Finally, setting \( M^{(\hat{c})} = -1 \), we find

\( \eta(1, -1^{(\hat{k})}; (+\frac{1}{2})^{(\hat{k})}, (+\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; \frac{1}{2}, -1) = 0, \) \hspace{1cm} (85)

\( \eta(1, -1^{(\hat{k})}; (+\frac{1}{2})^{(\hat{k})}, (-\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; \frac{1}{2}, -1) = 0, \) \hspace{1cm} (86)

\( \eta(1, -1^{(\hat{k})}; (-\frac{1}{2})^{(\hat{k})}, (+\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; \frac{1}{2}, -1) = 0, \) \hspace{1cm} (87)

and

\( \eta(1, -1^{(\hat{k})}; (-\frac{1}{2})^{(\hat{k})}, (-\frac{1}{2})^{(\hat{k})}) = C(\frac{1}{2}, 1; -\frac{1}{2}, -\frac{1}{2}, -1) = 1. \) \hspace{1cm} (88)

Thus,

\( \xi(1, -1^{(\hat{k})}; (m_1)^{(c_1)}, (m_2)^{(c_2)}) = \psi((-\frac{1}{2})^{(\hat{k})}, (-\frac{1}{2})^{(\hat{k})}, (m_1)^{(c_1)}, (m_2)^{(c_2)}). \) \hspace{1cm} (89)

Using all these results, and recalling Eqs. (44) - (53) for the \( \psi \), we find that for \( M^{(\hat{a})} = 1, \)

\[ \Psi(1, 1^{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}) = \cos^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi} \]

\[ + \sin^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi - \varphi_1 - \varphi_2)} \]

\[ + \frac{1}{2} \sin \theta \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} \right]. \] \hspace{1cm} (90)
\[
\Psi(1, 1^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) = -\cos^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \\
+ \sin^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)} \\
+ \frac{1}{2} \sin \theta [\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}], 
\] (91)

\[
\Psi(1, 1^{(\hat{a})}; (-\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) = -\cos^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi} \\
+ \sin^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)} \\
+ \frac{1}{2} \sin \theta [\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2}] 
\] (92)

and

\[
\Psi(1, 1^{(\hat{a})}; (-\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) = \cos^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \\
+ \sin^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)} \\
- \frac{1}{2} \sin \theta [\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} + \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}], 
\] (93)

5.6.2 The \(M^{(\hat{a})} = 0\) State

For \(M^{(\hat{a})} = 0\), the \(\xi\)'s remain the same as for \(M^{(\hat{a})} = 1\). However, the \(\chi\)'s are[3]

\[
\chi(1, 0^{(\hat{a})}; 1, 1^{(\hat{k})}) = -\frac{1}{\sqrt{2}} \sin \theta e^{-i\varphi}, 
\] (94)

\[
\chi(1, 0^{(\hat{a})}; 1, 0^{(\hat{k})}) = \cos \theta 
\] (95)

and

\[
\chi(1, 0^{(\hat{a})}; 1, (-1)^{(\hat{k})}) = \frac{1}{\sqrt{2}} \sin \theta e^{i\varphi}. 
\] (96)
Therefore, the probability amplitudes are

\[
\Psi(1, \theta; (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}} \sin \theta [\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)} - \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi}]
\]

\[
+ \frac{1}{\sqrt{2}} \cos \theta [\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1}],
\]  

(97)

\[
\Psi(1, \theta; (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}} \sin \theta [\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)}
\]

\[
+ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi}
\]

\[
+ \frac{1}{\sqrt{2}} \cos \theta [\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}],
\]  

(98)

\[
\Psi(1, \theta; (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}} \sin \theta [\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)}
\]

\[
+ \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi}
\]

\[
+ \frac{1}{\sqrt{2}} \cos \theta [\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2}]
\]  

(99)

and

\[
\Psi(1, \theta; (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}} \sin \theta [\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi_1 - \varphi_2)}
\]

\[
- \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi}
\]

\[
- \frac{1}{\sqrt{2}} \cos \theta [\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} + \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1}].
\]  

(100)

5.6.3 The $M(\hat{a}) = -1$ State

For $M(\hat{a}) = -1$, again the $\xi$’s remain the same, while the $\chi$’s are [3]

\[
\chi(1, (-1)(\hat{a}); 1, 1(\hat{k})) = - \sin^2 \frac{\theta}{2} e^{-i\varphi},
\]  

(101)
\[ \chi(1, (-1)^{(\hat{a})} ; 1, 0^{(\hat{k})}) = \frac{1}{\sqrt{2}} \sin \theta \]  

and

\[ \chi(1, (-1)^{(\hat{a})} ; 1, (-1)^{(\hat{k})}) = -\cos^2 \frac{\theta}{2} e^{i\varphi}. \]  

Hence, the probability amplitudes are

\begin{align*}
\Psi(1, (-1)^{(\hat{a})} ; (\frac{1}{2})^{(\hat{c}_1)}, (\frac{1}{2})^{(\hat{c}_2)}) &= -\sin^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi} \\
&\quad - \cos^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi - \varphi_1 - \varphi_2)} \\
&\quad + \frac{1}{2} \sin \theta \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} \right]. \tag{104} 
\end{align*}

\begin{align*}
\Psi(1, (-1)^{(\hat{a})} ; (+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) &= \sin^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \\
&\quad - \cos^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi - \varphi_1 - \varphi_2)} \\
&\quad + \frac{1}{2} \sin \theta \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} \right]. \tag{105} 
\end{align*}

\begin{align*}
\Psi(1, (-1)^{(\hat{a})} ; (-\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) &= \sin^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi} \\
&\quad - \cos^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi - \varphi_1 - \varphi_2)} \\
&\quad + \frac{1}{2} \sin \theta \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} \right] \tag{106} 
\end{align*}

and

\begin{align*}
\Psi(1, (-1)^{(\hat{a})} ; (-\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) &= -\sin^2 \frac{\theta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \\
&\quad - \cos^2 \frac{\theta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\varphi - \varphi_1 - \varphi_2)} \\
&\quad - \frac{1}{2} \sin \theta \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} \right]. \tag{107} 
\end{align*}
6 Probabilities

6.1 Singlet-State Probabilities

The probabilities for the singlet state are

\[ P(0, 0; (\hat{a}), (\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = |\Psi(0, 0; (\hat{a}), (\hat{c}_1), (+\frac{1}{2})(\hat{c}_2))|^2 \]

\[ = \frac{1}{2} [\sin^2 \left(\frac{\theta_2 - \theta_1}{2}\right) + \sin \theta_1 \sin \theta_2 \sin^2 \left(\frac{\varphi_2 - \varphi_1}{2}\right)], \]  

(108)

\[ P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \frac{1}{2} \cos^2 \left(\frac{\theta_2 - \theta_1}{2}\right) \]

\[ - \sin \theta_1 \sin \theta_2 \sin^2 \left(\frac{\varphi_2 - \varphi_1}{2}\right), \]  

(109)

\[ P(0, 0; (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \]  

(110)

and

\[ P(0, 0; (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)). \]  

(111)

Now let \( P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1)) \) be the probability of finding the spin projection of subsystem 1 up with respect to the direction \( \hat{c}_1 \) irrespective of the projection found for subsystem 2. Then

\[ P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1)) = P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \]

\[ + P(0, 0; (\hat{a}), (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \]

\[ = \frac{1}{2}. \]  

(112)

Let \( P(0, 0; (\hat{a}), (-\frac{1}{2})(\hat{c}_1)) \) be the probability of finding the spin projection of subsystem 1 down with respect to the direction \( \hat{c}_1 \) irrespective of the projection found for subsystem 2. Then
\[ P(0, 0^{(\hat{a})}; (-\frac{1}{2})^{(c_1)}; (\frac{1}{2})^{(c_2)}) = P(0, 0^{(\hat{a})}; (-\frac{1}{2})^{(c_1)}, (\frac{1}{2})^{(c_2)}) \\
+ P(0, 0^{(\hat{a})}; (-\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}) \\
= \frac{1}{2}. \] (113)

Similarly,
\[ P(0, 0^{(\hat{a})}; (+\frac{1}{2})^{(c_2)}) = \frac{1}{2} \] (114)
and
\[ P(0, 0^{(\hat{a})}; (-\frac{1}{2})^{(c_2)}) = \frac{1}{2}. \] (115)

6.2 Triplet-State Probabilities

6.2.1 The \( M^{(\hat{a})} = 1 \) State

The probabilities corresponding to the amplitudes for the initial state characterized by \( s = 1, M^{(\hat{a})} = 1 \) are

\[ P(1, 1^{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}) = \left| \Psi(1, 1^{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (+\frac{1}{2})^{(c_2)}) \right|^2 \\
= \cos^4 \theta \frac{1}{2} \cos^2 \theta_1 \cos^2 \theta_2 + \sin^4 \theta \frac{1}{2} \sin^2 \theta_1 \sin^2 \theta_2 \\
+ \frac{1}{4} \sin^2 \theta \cos^2 \theta_1 \sin^2 \theta_2 + \frac{1}{4} \sin^2 \theta \sin^2 \theta_1 \cos^2 \theta_2 \\
+ \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
+ \frac{1}{2} \sin \theta \sin \theta_1 \left( \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} \right) \cos(\varphi_1 - \varphi) \\
+ \frac{1}{2} \sin \theta \sin \theta_2 \left( \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \right) \cos(\varphi_2 - \varphi), \] (116)

\[ P(1, 1^{(\hat{a})}; (+\frac{1}{2})^{(c_1)}, (-\frac{1}{2})^{(c_2)}) = \cos^4 \theta \frac{1}{2} \cos^2 \theta_1 \sin^2 \theta_2 \]
\[
\begin{align*}
+ \sin^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
- \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
+ \frac{1}{2} \sin \theta \sin \theta_1 (\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2}) \cos(\varphi_1 - \varphi) \\
- \frac{1}{2} \sin \theta \sin \theta_2 (\cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2}) \cos(\varphi_2 - \varphi),
\end{align*}
\]

(117)

\[
\begin{align*}
P(1, 1; (-\frac{1}{2})^{(\hat{a})_1}, (+\frac{1}{2})^{(\hat{c})_2}) &= \cos^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
- \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
- \frac{1}{2} \sin \theta \sin \theta_1 (\cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2}) \cos(\varphi_1 - \varphi) \\
+ \frac{1}{2} \sin \theta \sin \theta_2 (\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2}) \cos(\varphi_2 - \varphi),
\end{align*}
\]

(118)

\[
\begin{align*}
P(1, 1; (-\frac{1}{2})^{(\hat{a})_1}, (-\frac{1}{2})^{(\hat{c})_2}) &= \cos^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
- \frac{1}{2} \sin \theta \sin \theta_1 (\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2}) \cos(\varphi_1 - \varphi)
\end{align*}
\]

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\[-\frac{1}{2} \sin \theta \sin \theta_2 (\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2}) \cos (\varphi_2 - \varphi),\]

(119)

These probabilities add to unity. We also find

\[
P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_1) = P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_1, (1/2)\hat{(c)}_2) + P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_1, (-1/2)\hat{(c)}_2)
\]

\[
= \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \frac{1}{2} \sin \theta \sin \theta_1 \cos (\varphi_1 - \varphi),
\]

(120)

\[
P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_1) = P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_1, (1/2)\hat{(c)}_2) + P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_1, (-1/2)\hat{(c)}_2)
\]

\[
= \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} - \frac{1}{2} \sin \theta \sin \theta_1 \cos (\varphi_1 - \varphi),
\]

(121)

\[
P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_2) = P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_1, (1/2)\hat{(c)}_2) + P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_1, (1/2)\hat{(c)}_2)
\]

\[
= \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \frac{1}{2} \sin \theta \sin \theta_2 \cos (\varphi_2 - \varphi)
\]

(122)

and

\[
P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_2) = P(1, 1(\hat{a})_1; (1/2)\hat{(c)}_1, (-1/2)\hat{(c)}_2) + P(1, 1(\hat{a})_1; (-1/2)\hat{(c)}_1, (-1/2)\hat{(c)}_2)
\]

\[
= \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta \sin \theta_2 \cos (\varphi_2 - \varphi).
\]

(123)
The $M^{(\hat{a})} = 0$ State For this case, the probabilities are

\[
P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) = \frac{1}{2} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}
\]

\[
+ \frac{1}{2} \cos^2 \theta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \cos^2 \theta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}
\]

\[
- \frac{1}{2} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2)
\]

\[
+ \frac{1}{4} \sin \theta_1 \sin \theta_2 \cos(\varphi_2 - \varphi_1)
\]

\[
- \frac{1}{2} \sin \theta \cos \theta \sin \theta_1 \cos \theta_2 \cos(\varphi_1 - \varphi)
\]

\[
- \frac{1}{4} \sin \theta \cos \theta \sin \theta_2 \cos \theta_1 \cos(\varphi_2 - \varphi),
\]

(124)

\[
P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) = \frac{1}{2} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}
\]

\[
+ \frac{1}{2} \cos^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \frac{1}{2} \cos^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}
\]

\[
+ \frac{1}{2} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2)
\]

\[
- \frac{1}{4} \sin \theta_1 \sin \theta_2 \cos(\varphi_2 - \varphi_1)
\]

\[
+ \frac{1}{2} \sin \theta \cos \theta \sin \theta_1 \cos \theta_2 \cos(\varphi_1 - \varphi)
\]

\[
+ \frac{1}{2} \sin \theta \cos \theta \sin \theta_2 \cos \theta_1 \cos(\varphi_2 - \varphi),
\]

(125)

\[
P(1, 0^{(\hat{a})}; (-\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) = P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)})
\]

(126)

and
\[ P(1, 0^{(\hat{a})}; (\frac{1}{2})^{(\hat{c}_1)}, (\frac{1}{2})^{(\hat{c}_2)}) = P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}). \] (127)

These probabilities sum to unity. Also, we have

\[
\begin{align*}
P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}) &= P(1, 0^{(\hat{a})}; (-\frac{1}{2})^{(\hat{c}_1)}) \\
&= P(1, 0^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_2)}) \\
&= P(1, 0^{(\hat{a})}; (-\frac{1}{2})^{(\hat{c}_2)}) = \frac{1}{2}.
\end{align*}
\] (128)

### 6.2.2 The \( M^{(\hat{a})} = -1 \) State

For this case, the probabilities are

\[
P(1, (-1)^{(\hat{a})}; (+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) = \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \cos^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
- \frac{1}{2} \sin \theta \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi) \\
- \frac{1}{2} \sin \theta \sin \theta_1 \cos^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_2}{2} \sin^2 \frac{\theta_1}{2} \cos(\varphi_2 - \varphi).
\] (129)

\[
P(1, (-1)^{(\hat{a})}, (+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) = \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \cos^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
- \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2)
\]
−\frac{1}{2} \sin \theta \sin \theta_1 [\sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2}] \cos(\varphi_1 - \varphi) \\
+ \frac{1}{2} \sin \theta \sin \theta_2 [\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2}] \cos(\varphi_2 - \varphi), \\
(130)

P(1, (-1)^{(\tilde{a})}; (-\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)}) = \sin^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \cos^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
- \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
+ \frac{1}{2} \sin \theta \sin \theta_1 [\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2}] \cos(\varphi_1 - \varphi) \\
- \frac{1}{2} \sin \theta \sin \theta_2 [\sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2}] \cos(\varphi_2 - \varphi), \\
(131)

and

P(1, (-1)^{(\tilde{a})}; (-\frac{1}{2})^{(\tilde{c}_1)}, (-\frac{1}{2})^{(\tilde{c}_2)}) = \sin^4 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \cos^4 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \theta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
+ \frac{1}{4} \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_1) \cos(\varphi - \varphi_2) \\
+ \frac{1}{2} \sin \theta \sin \theta_1 [\sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2}] \cos(\varphi_1 - \varphi) \\
+ \frac{1}{2} \sin \theta \sin \theta_2 [\sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2}] \cos(\varphi_2 - \varphi). \\
(132)

These probabilities add up to unity. We also have
\[ P(1, (-1)^{\hat{a}}; (+\frac{1}{2})^{\hat{c}_1}) = \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} - \frac{1}{2} \sin \theta \sin \theta_1 \cos(\varphi_1 - \varphi), \quad (133) \]

\[ P(1, (-1)^{\hat{a}}; (-\frac{1}{2})^{\hat{c}_1}) = \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_1}{2} + \frac{1}{2} \sin \theta \sin \theta_1 \cos(\varphi_1 - \varphi), \quad (134) \]

\[ P(1, (-1)^{\hat{a}}; (+\frac{1}{2})^{\hat{c}_2}) = \sin^2 \frac{\theta^2}{2} \cos^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta \sin \theta_2 \cos(\varphi_2 - \varphi), \quad (135) \]

\[ P(1, (-1)^{\hat{a}}; (-\frac{1}{2})^{\hat{c}_2}) = \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_2}{2} + \frac{1}{2} \sin \theta \sin \theta_2 \cos(\varphi_2 - \varphi). \quad (136) \]

**7 Clebsch-Gordan Coefficients**

In this paper, we have assumed that the Clebsch-Gordan coefficients are probability amplitudes for obtaining, starting from a given state of the compound system, certain special combinations of spin projections for joint measurements on the subsystems. Such a measurement is a special case of the general measurement which we have treated in this paper. In the general case, we are interested in the probability amplitudes of joint measurements of the spin projections of the subsystems with respect to arbitrary directions. The special case that yields the Clebsch-Gordan coefficients arises in the following way.

The vectors \( \hat{a}, \hat{c}_1 \) and \( \hat{c}_2 \) define the situation obtaining. While \( \hat{a} \) gives the initial direction of the compound spin, \( \hat{c}_1 \) and \( \hat{c}_2 \) give the directions along
which we measure the spin projections of subsystems 1 and 2 respectively. The probability amplitudes for measurements of these projections are given by the Clebsch-Gordan coefficients if \( \hat{c}_1 \) defines the direction of spin \( S_1 \) and \( \hat{c}_2 \) that of spin \( S_2 \) in such a way that the spin addition condition

\[
S = S_1 + S_2
\]

is satisfied. But this is not all. In addition, \( \hat{a} \) must be oriented to point in the \( z \) direction.

This immediately suggests a generalization of the Clebsch-Gordan coefficients. If the direction \( \hat{a} \) is made arbitrary, but with the directions of \( \hat{c}_1 \) and \( \hat{c}_2 \) still such that the condition Eq. (137) holds, then we obtain joint probability amplitudes which are generalized forms of the Clebsch-Gordan coefficients.

For \( s = 0 \) or \( s = 1 \), as in this paper, the generalized Clebsch-Gordan coefficients can be deduced from the joint probability amplitudes if we set \( \hat{c}_1 = \hat{c}_2 = \hat{a} \). If in addition, we set \( \hat{a} = \hat{k} \), so that \( \theta = \varphi = 0 \), we obtain the standard Clebsch-Gordan coefficients. Though the systems in this paper are such that the vectors \( \hat{a}, \hat{c}_1 \) and \( \hat{c}_2 \) are collinear for the situation that yields the Clebsch-Gordan coefficients, this will clearly not be the case for an arbitrary system. The exact mutual orientations of the vectors will have to be deduced on a system-by-system basis. However, in all cases, the generalized Clebsch-Gordan coefficients can be obtained from the standard ones by a rotation that carries the vector \( \hat{k} \) into the vector \( \hat{a} \). Also, the probabilities resulting from generalized Clebsch-Gordan coefficients will always equal those resulting from the standard Clebsch-Gordan coefficients.

We denote the generalized Clebsch-Gordan coefficients by \( C(s_1 s_2 s; m_1 m_2 M)_{\text{gen}} \). We obtain them from Eqs. (66)-(69), Eqs. (90) - (93), Eqs. (97) - (100) and Eqs. (104) - (107) by setting \( \theta_1 = \theta_2 = \theta \) and \( \varphi_1 = \varphi_2 = \varphi \).

For the singlet state, they are

\[
C(\frac{1}{2} \frac{1}{2} 0; \frac{1}{2} \frac{1}{2} 0)_{\text{gen}} = \Psi(0, 0(\hat{a}); (+\frac{1}{2})(\hat{a}), (+\frac{1}{2})(\hat{a})) = 0, \quad (138)
\]

\[
C(\frac{1}{2} \frac{1}{2} 0; \frac{1}{2} \frac{1}{2} 0)_{\text{gen}} = \Psi(0, 0(\hat{a}); (+\frac{1}{2})(\hat{a}), (-\frac{1}{2})(\hat{a})) = \frac{1}{\sqrt{2}} e^{-i\varphi}, \quad (139)
\]

\[
C(\frac{1}{2} \frac{1}{2} 0; \frac{1}{2} \frac{1}{2} 0)_{\text{gen}} = \Psi(0, 0(\hat{a}); (-\frac{1}{2})(\hat{a}), (+\frac{1}{2})(\hat{a})) = -\frac{1}{\sqrt{2}} e^{-i\varphi} \quad (140)
\]
and
\[ C(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}, 0)_{\text{gen}} = \Psi(0, 0; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0. \] (141)

For the case \( s = 1 \), the generalized Clebsch-Gordan coefficients for \( M^a = 1 \) are
\[ C(\frac{1}{2}^a; \frac{1}{2}, 1)_{\text{gen}} = \Psi(1, 1; (\frac{1}{2})^a, (\frac{1}{2})^a) = e^{-i\phi}, \] (142)
\[ C(\frac{1}{2}^a; \frac{1}{2}, -\frac{1}{2})_{\text{gen}} = \Psi(1, 1; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0, \] (143)
\[ C(\frac{1}{2}^a; -\frac{1}{2}, 1)_{\text{gen}} = \Psi(1, 1; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0 \] (144)

and
\[ C(\frac{1}{2}^a; -\frac{1}{2}, -\frac{1}{2}, 1)_{\text{gen}} = \Psi(1, 1; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0. \] (145)

For \( s = 1, M^a = 0 \), they are
\[ C(\frac{1}{2}^a; \frac{1}{2}, \frac{1}{2}0)_{\text{gen}} = \Psi(1, 0^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = e^{-i\phi}, \] (146)
\[ C(\frac{1}{2}^a; \frac{1}{2}, -\frac{1}{2}0)_{\text{gen}} = \Psi(1, 0^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = \frac{1}{\sqrt{2}} e^{-i\phi}, \] (147)
\[ C(\frac{1}{2}^a; \frac{1}{2}, \frac{1}{2}0)_{\text{gen}} = \Psi(1, 0^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = \frac{1}{\sqrt{2}} e^{-i\phi} \] (148)

and
\[ C(\frac{1}{2}^a; -\frac{1}{2}, -\frac{1}{2}, 0)_{\text{gen}} = \Psi(1, 0^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0. \] (149)

Finally, for \( s = 1, M^a = -1 \), the generalized Clebsch-Gordan coefficients are
\[ C(\frac{1}{2}^a; \frac{1}{2}, \frac{1}{2}; -1)_{\text{gen}} = \Psi(1, -1^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0, \] (150)
\[ C(\frac{1}{2}^a; \frac{1}{2}, -\frac{1}{2}, -1)_{\text{gen}} = \Psi(1, -1^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0, \] (151)
\[ C(\frac{1}{2}^a; -\frac{1}{2}, -\frac{1}{2}, -1)_{\text{gen}} = \Psi(1, -1^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = 0 \] (152)

and
\[ C(\frac{1}{2}^a; -\frac{1}{2}, -\frac{1}{2}, -1)_{\text{gen}} = \Psi(1, -1^a; (\frac{1}{2})^a, (\frac{1}{2})^a) = -e^{-i\phi}. \] (153)

Thus, we see that we do indeed obtain quantities that we may justly term generalized Clebsch-Gordan coefficients. The simple appearance of these
expressions in this case is probably due to the fact that $S_1$, $S_2$ and $S$ always lie along one line in the cases dealt with here. More complicated expressions ought to arise from those cases where the three angular momenta actually form a triangle. But this needs to be investigated fully by actually obtaining the probability amplitudes for such cases.

8 Matrix Mechanics

8.1 Introductory Remarks

We obtain a matrix description from a probability-amplitude description by means of the expansion Eq. (1) of the probability amplitudes. In order to obtain the triplet-state probability amplitudes $Ψ(1, 1\hat{a}; (m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)})$, we have had to utilize two expansions, which means that we can have at least two matrix descriptions of these probability amplitudes and of the expectation values we can calculate through them. In order to obtain the probability amplitudes $Ψ(0, 0\hat{a}; (m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)})$ for the singlet state, we have employed only one expansion, which gives only one way of casting expectation values involving this state into matrix form. We start off with this case.

8.2 Vectors and Operators for the Singlet State

For this case, the expansion of the probability amplitudes connecting the initial state with the possible final states is

$$Ψ(0, 0\hat{a}; (m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)}) = \sum_{a,a'} \eta(0, 0\hat{k}; (m_1)_{a}^{(c_1)}, (m_2)_{a'}^{(c_2)})$$

$$\times ψ((m_1)_{a}^{(c_1)}, (m_2)_{a'}^{(c_2)}; (m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)}).$$

(154)

The expectation value of $R = R((m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)})$ is

$$\langle R((m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)}) \rangle = \sum_{u,v} |Ψ(0, 0\hat{a}; (m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)})|^2$$

$$\times R((m_1)_{u}^{(c_1)}, (m_2)_{v}^{(c_2)}).$$

(155)
In order to facilitate the transition to matrix mechanics, we define a new observable corresponding to combined values of \((m_1)_{\vec{k}}, (m_2)_{\vec{k}}\). This is denoted by \(B\). Hence \(B_\gamma = ((m_1)_{\vec{k}}, (m_2)_{\vec{k}})_{\gamma}^{\gamma}, \) where \(\gamma = 1, 2, 3, 4\). Hence we have

\[
B_1 = (m_1)_{1^{\vec{k}}}, (m_2)_{1^{\vec{k}}} = ((\frac{1}{2})_{\vec{k}}, (\frac{1}{2})_{\vec{k}}),
\]

\[
B_2 = (m_1)_{1^{\vec{k}}}, (m_2)_{2^{\vec{k}}} = ((\frac{1}{2})_{\vec{k}}, (\frac{1}{2})_{\vec{k}}),
\]

\[
B_3 = (m_1)_{2^{\vec{k}}}, (m_2)_{1^{\vec{k}}} = ((\frac{1}{2})_{\vec{k}}, (\frac{1}{2})_{\vec{k}}),
\]

\[
B_4 = (m_1)_{2^{\vec{k}}}, (m_2)_{2^{\vec{k}}} = ((\frac{1}{2})_{\vec{k}}, (\frac{1}{2})_{\vec{k}}).
\]

Thus, the probability amplitude Eq. \((154)\) becomes

\[
\Psi^\dagger(0,0; 0^{\vec{a}}, (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2}) = \sum_{\gamma} \eta^\dagger(0,0; B_\gamma) \psi^\dagger(B_\gamma; (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})
\]

and

\[
\Psi(0,0; 0^{\vec{a}}, (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2}) = \sum_{\gamma'} \eta(0,0; B_{\gamma'}) \psi(B_{\gamma'}; (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})
\]

The expectation value becomes

\[
\langle R((m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})) \rangle = \sum_{\gamma, \gamma'} \eta^\dagger(0,0; B_\gamma) R_{\gamma, \gamma'} \eta(0,0; B_{\gamma'}),
\]

where

\[
R_{\gamma, \gamma'} = \sum_{u,v=1}^{2} \psi^\dagger(B_\gamma; (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2}) R((m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})^\dagger \psi(B_{\gamma'}; (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2}).
\]

Therefore

\[
\langle R((m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})) \rangle = [\Psi(0,0; 0^{\vec{a}}, (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})]^\dagger [R] \times [\Psi(0,0; 0^{\vec{a}}, (m_1)_{\vec{c}_1}, (m_2)_{\vec{c}_2})],
\]

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where

\[
[\Psi(0, 0; (m_1)_{\hat{c}_1}, (m_2)_{\hat{c}_2})] = \begin{pmatrix}
\eta(0, 0; B_1) \\
\eta(0, 0; B_2) \\
\eta(0, 0; B_3) \\
\eta(0, 0; B_4)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\eta(0, 0; (+\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}) \\
\eta(0, 0; (+\frac{1}{2})_{\hat{k}}, (-\frac{1}{2})_{\hat{k}}) \\
\eta(0, 0; (-\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}) \\
\eta(0, 0; (-\frac{1}{2})_{\hat{k}}, (-\frac{1}{2})_{\hat{k}})
\end{pmatrix}
\]

and

\[
[R] = \begin{pmatrix}
R_{11} & R_{12} & R_{13} & R_{14} \\
R_{21} & R_{22} & R_{23} & R_{24} \\
R_{31} & R_{32} & R_{33} & R_{34} \\
R_{41} & R_{42} & R_{43} & R_{44}
\end{pmatrix}.
\]

As the \(\eta\)'s are the Clebsch-Gordan coefficients, Eqs. (62) - (65), we have

\[
[\Psi(0, 0; (m_1)_{\hat{c}_1}, (m_2)_{\hat{c}_2})] = \begin{pmatrix}
0 \\
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} \\
0
\end{pmatrix}
\]

(165)

The elements of \([R]\) are as follows.

\[
R_{11} = \left| \psi((+\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}; (m_1)_{\hat{c}_1}, (m_2)_{\hat{c}_2}) \right|^2 R((+\frac{1}{2})_{\hat{c}_1}, (+\frac{1}{2})_{\hat{c}_2})
\]

\[
+ \left| \psi((+\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}; (m_1)_{\hat{c}_1}, (-\frac{1}{2})_{\hat{c}_2}) \right|^2 R((+\frac{1}{2})_{\hat{c}_1}, (-\frac{1}{2})_{\hat{c}_2})
\]

\[
+ \left| \psi((+\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}; (-\frac{1}{2})_{\hat{c}_1}, (+\frac{1}{2})_{\hat{c}_2}) \right|^2 R((-\frac{1}{2})_{\hat{c}_1}, (+\frac{1}{2})_{\hat{c}_2})
\]

\[
+ \left| \psi((+\frac{1}{2})_{\hat{k}}, (+\frac{1}{2})_{\hat{k}}; (-\frac{1}{2})_{\hat{c}_1}, (-\frac{1}{2})_{\hat{c}_2}) \right|^2 R((-\frac{1}{2})_{\hat{c}_1}, (-\frac{1}{2})_{\hat{c}_2})
\]

\[
= \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})_{\hat{c}_1}, (+\frac{1}{2})_{\hat{c}_2})
\]
\[+ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2))
+ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2))
+ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)),\] (166)

By the same token,

\[R_{12} = \psi^*((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}; (+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2))
\times \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2))
+ \psi^*((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2))
\times \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2))
+ \psi^*((+\frac{1}{2})\hat{k}, (+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2))
\times \psi((+\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{k}, (-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)),\] (167)

Thus,

\[R_{12} = \frac{1}{2} \cos^2 \frac{\theta_1}{2} \sin \theta_2 e^{-i\varphi_2} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) - R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2))]
+ \frac{1}{2} \sin^2 \frac{\theta_1}{2} \sin \theta_2 e^{-i\varphi_2} [R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) - R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)),\] (168)

\[R_{13} = \frac{1}{2} \cos^2 \frac{\theta_2}{2} \sin \theta_1 e^{-i\varphi_1} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) - R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2))]
+ \frac{1}{2} \sin^2 \frac{\theta_2}{2} \sin \theta_1 e^{-i\varphi_1} [R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) - R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)),\] (169)
\begin{align*}
R_{14} &= \frac{1}{4} \sin \theta_1 \sin \theta_2 e^{-i(\varphi_1 + \varphi_2)} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) - R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)] - R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) + R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)], \\
R_{21} &= R_{12}^*, \quad (171) \\
R_{22} &= \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) \\
&+ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2) \\
&+ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) \\
&+ \cos^2 \frac{\theta_2}{2} \sin^2 \frac{\theta_1}{2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)], \quad (172) \\
R_{23} &= \frac{1}{4} \sin \theta_1 \sin \theta_2 e^{i(\varphi_2 - \varphi_1)} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) - R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)] \\
&- R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) + R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)], \quad (173) \\
R_{24} &= \frac{1}{2} \sin^2 \frac{\theta_2}{2} \sin \theta_1 e^{-i\varphi_1} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) - R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)] \\
&+ \frac{1}{2} \cos^2 \frac{\theta_2}{2} \sin \theta_1 e^{-i\varphi_1} [R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2) - R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)], \quad (174) \\
R_{31} &= R_{13}^*, \quad (175) \\
R_{32} &= R_{23}^*, \quad (176)
\end{align*}
\[ R_{33} = \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) \\
+ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) \\
+ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) \\
+ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) \]  

(177)

\[ R_{34} = \frac{1}{2} \sin^2 \frac{\theta_1}{2} \sin \theta_2 e^{-i\varphi_2} [R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) - R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) \\
+ \frac{1}{2} \cos^2 \frac{\theta_1}{2} \sin \theta_2 e^{-i\varphi_2} [R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) - R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) \]  

(178)

\[ R_{41} = R_{41}^*, \]  

(179)

\[ R_{42} = R_{24}^*, \]  

(180)

\[ R_{43} = R_{34}^* \]  

(181)

and

\[ R_{44} = \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) \\
+ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)) \\
+ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)) \\
+ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)). \]  

(182)

The formulas given here correspond to the general results given in Section 3.2.
8.3 The Triplet State

8.3.1 Vectors and Operators in Four-Dimensional Treatment

As observed above, the case $s = 1$ admits of two different matrix representations because the calculation of the amplitudes involves two different expansions. We shall give both.

The probability amplitudes Eqs. (28) can be written in the form

$$\Psi(1, M_i^{(a)}; (m_1)^{(c_1)}, (m_2)^{(c_2)}) = \sum_{\alpha, \alpha'} \beta(1, M_i^{(a)}; (m_1)^{(k)}, (m_2)^{(k)})$$

$$\times \psi((m_1)^{(k)}, (m_2)^{(k)}, (m_1)^{(c_1)}, (m_2)^{(c_2)}),$$

(183)

where

$$\beta(1, M_i^{(a)}; (m_1)^{(k)}, (m_2)^{(k)}) = \sum_s \chi(1, M_i^{(a)}; 1, M_s^{(k)})$$

$$\times \eta(1, M_s^{(k)}; (m_1)^{(k)}, (m_2)^{(k)}).$$

(184)

Alternatively the probability amplitudes can be written in the form

$$\Psi(1, M_i^{(a)}; (m_1)^{(c_1)}, (m_2)^{(c_2)}) = \sum_r \chi(1, M_i^{(a)}; 1, M_r^{(k)})$$

$$\times \xi(1, M_r^{(k)}; (m_1)^{(c_1)}, (m_2)^{(c_2)}),$$

(185)

where

$$\xi(1, M_r^{(k)}; (m_1)^{(c_1)}, (m_2)^{(c_2)}) = \sum_{\alpha, \alpha'} \eta(1, M_r^{(k)}; (m_1)^{(k)}, (m_2)^{(k)})$$

$$\times \psi((m_1)^{(k)}, (m_2)^{(k)}, (m_1)^{(c_1)}, (m_2)^{(c_2)}).$$

(186)

If we start off with the choice Eq. (183), then the operator is a $4 \times 4$ matrix and the vector is a $4 \times 1$ column. The expectation value of $R((m_1)^{(c_1)}, (m_2)^{(c_2)})$ is given by

$$\langle R((m_1)^{(c_1)}, (m_2)^{(c_2)}) \rangle = [\Psi(1, M_i^{(a)}; (m_1)^{(c_1)}, (m_2)^{(c_2)})]^\dagger [R]$$

$$\times [\Psi(1, M_i^{(a)}; (m_1)^{(c_1)}, (m_2)^{(c_2)})],$$

(187)
where

\[
[\Psi(1, M_i^a; (m_1)\hat{c}_1)(m_2)\hat{c}_2)] = \begin{pmatrix}
\beta(1, M_i^a; (+1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, M_i^a; (+1/2)(\hat{k}), (-1/2)(\hat{k})) \\
\beta(1, M_i^a; (-1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, M_i^a; (-1/2)(\hat{k}), (-1/2)(\hat{k}))
\end{pmatrix}. \tag{188}
\]

The operator is the same as was found for the singlet state and so its elements are given by Eqs. (166) - (182).

The elements of the vectors are known. We consider first the case \(M^a = 1\). We have from Eqs. (71) - (75) combined with Eqs. (75) - (78), Eqs. (80) - (83) and Eqs. (83) - (88)

\[
[\Psi(1, 1^a; (m_1)\hat{c}_1)(m_2)\hat{c}_2)] = \begin{pmatrix}
\beta(1, 1^a; (+1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, 1^a; (+1/2)(\hat{k}), (-1/2)(\hat{k})) \\
\beta(1, 1^a; (-1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, 1^a; (-1/2)(\hat{k}), (-1/2)(\hat{k}))
\end{pmatrix} = \begin{pmatrix}
\cos^2 \theta e^{-i\phi} \\
\frac{1}{2} \sin \theta \\
\frac{1}{2} \sin \theta \\
\sin^2 \theta e^{i\phi}
\end{pmatrix}. \tag{189}
\]

For \(M^a = 0\), we use Eqs. (74) - (76) together with Eqs. (75) - (78), Eqs. (80) - (83) and Eqs. (83) - (88) to obtain

\[
[\Psi(1, 0^a; (m_1)\hat{c}_1)(m_2)\hat{c}_2)] = \begin{pmatrix}
\beta(1, 0^a; (+1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, 0^a; (+1/2)(\hat{k}), (-1/2)(\hat{k})) \\
\beta(1, 0^a; (-1/2)(\hat{k}), (+1/2)(\hat{k})) \\
\beta(1, 0^a; (-1/2)(\hat{k}), (-1/2)(\hat{k}))
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{\sqrt{2}} \sin \theta e^{-i\phi} \\
\frac{1}{2} \cos \theta \\
\sqrt{2} \cos \theta \\
\frac{1}{\sqrt{2}} \sin \theta e^{-i\phi}
\end{pmatrix}. \tag{190}
\]
Finally for or $M(\hat{a}) = -1$, we have combined Eqs. (101) - (103) with Eqs. (75) - (78), Eqs. (80) - (83) and Eqs. (85) - (88) to get

$$[\Psi(1, (-1)^{\hat{a}}; (m_1)^{\hat{c}_1}(m_2)^{\hat{c}_2})] =$$

$$\begin{pmatrix}
\beta(1, (-1)^{\hat{a}}; (+\frac{1}{2})^{\hat{k}}, (+\frac{1}{2})^{\hat{k}}) \\
\beta(1, (-1)^{\hat{a}}; (+\frac{1}{2})^{\hat{k}}, (-\frac{1}{2})^{\hat{k}}) \\
\beta(1, (-1)^{\hat{a}}; (-\frac{1}{2})^{\hat{k}}, (+\frac{1}{2})^{\hat{k}}) \\
\beta(1, (-1)^{\hat{a}}; (-\frac{1}{2})^{\hat{k}}, (-\frac{1}{2})^{\hat{k}})
\end{pmatrix}
$$

$$= \begin{pmatrix}
-\sin^2 \frac{\theta}{2} e^{-i\phi} \\
\frac{1}{2} \sin \theta \\
\frac{1}{2} \sin \theta \\
-\cos^2 \frac{\theta}{2} e^{i\phi}
\end{pmatrix}.$$  \hspace{1cm} (191)

These then are the matrix forms of the various states. The three vectors are mutually orthogonal. As expected, each one is normalized to unity. We reiterate the basic result that the elements of the vectors are probability amplitudes. They are probability amplitudes for getting spin projections along the $z$ axis starting from states of specified projection of the total spin along the arbitrary vector $\hat{a}$.

### 8.3.2 Vectors and Operators in Three-Dimensional Treatment

If we choose to write the probability amplitude as in Eq. (185), the expectation value of $R((m_1)^{\hat{c}_1}, (m_2)^{\hat{c}_2})$ is

$$\langle R((m_1)^{\hat{c}_1}, (m_2)^{\hat{c}_2}) \rangle = [\Psi(1, M_i^{\hat{a}}; (m_1)^{\hat{c}_1}(m_2)^{\hat{c}_2})]^\dagger [R]$$

$$\times [\Psi(1, M_i^{\hat{a}}; (m_1)^{\hat{c}_1}(m_2)^{\hat{c}_2})],$$  \hspace{1cm} (192)

with the matrix states being

$$[\Psi(1, M_i^{\hat{a}}; (m_1)^{\hat{c}_1}(m_2)^{\hat{c}_2})] = \begin{pmatrix}
\chi(1, M_i^{\hat{a}}; 1, 1^{\hat{k}}) \\
\chi(1, M_i^{\hat{a}}; 1, 0^{\hat{k}}) \\
\chi(1, M_i^{\hat{a}}; 1, (-1)^{\hat{k}})
\end{pmatrix}. \hspace{1cm} (193)$$
and the operator \([R]\) being a 3 \times 3 matrix whose elements are given by

\[
R_{pp'} = \sum_{u,v} \xi^*(1, M_p^{(k)}; (m_1)_u^{(c_1)}, (m_2)_v^{(c_2)}) R((m_1)_u^{(c_1)}, (m_2)_v^{(c_2)})
\]

\[
\times \xi(1, M_{p'}^{(k)}; (m_1)_u^{(c_1)}, (m_2)_v^{(c_2)}).
\]  

(194)

The states \([\Psi(1, M_i^{(a)}; (m_1)^{(c_1)}(m_2)^{(c_2)})]\) are known because their elements are given by Eqs. (72) - (73), Eqs. (94) - (96) and Eqs. (101) - (103). They are:

\[
[\Psi(1, 1^{(a)}; (m_1)^{(c_1)}(m_2)^{(c_2)})] = \begin{pmatrix}
\cos^2 \frac{\theta}{2} e^{-i\varphi} \\
\frac{1}{\sqrt{2}} \sin \theta \\
\sin^2 \frac{\theta}{2} e^{i\varphi}
\end{pmatrix}
\]  

(195)

\[
[\Psi(1, 0^{(a)}; (m_1)^{(c_1)}(m_2)^{(c_2)})] = \begin{pmatrix}
-\frac{1}{\sqrt{2}} \sin \theta e^{-i\varphi} \\
\cos \theta \\
\frac{1}{\sqrt{2}} \sin \theta e^{i\varphi}
\end{pmatrix}
\]  

(196)

and

\[
[\Psi(1, (-1)^{\tilde{a}}; (m_1)^{(c_1)}(m_2)^{(c_2)})] = \begin{pmatrix}
-s\sin \frac{\theta}{2} e^{-i\varphi} \\
\sqrt{2} \sin \theta \\
-\cos^2 \frac{\theta}{2} e^{i\varphi}
\end{pmatrix}
\]  

(197)

Using Eq. (184) for the definition of \(\xi(1, 1, M_p^{(k)}; (m_1)_u^{(c_1)}, (m_2)_v^{(c_2)})\), we find that

\[
\xi(1, 1^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)})
\]

\[
= \eta(1, 1^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{k})}, (+\frac{1}{2})^{(\tilde{k})}) \psi((+\frac{1}{2})^{(\tilde{k})}, (+\frac{1}{2})^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)}) + \eta(1, 1^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{k})}, (-\frac{1}{2})^{(\tilde{k})}) \psi((+\frac{1}{2})^{(\tilde{k})}, (-\frac{1}{2})^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)}) + \eta(1, 1^{(\tilde{k})}; (-\frac{1}{2})^{(\tilde{k})}, (-\frac{1}{2})^{(\tilde{k})}) \psi((-\frac{1}{2})^{(\tilde{k})}, (-\frac{1}{2})^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)}) + \eta(1, 1^{(\tilde{k})}; (-\frac{1}{2})^{(\tilde{k})}, (+\frac{1}{2})^{(\tilde{k})}) \psi((-\frac{1}{2})^{(\tilde{k})}, (+\frac{1}{2})^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)})
\]

\[
= \psi((+\frac{1}{2})^{(\tilde{k})}, (+\frac{1}{2})^{(\tilde{k})}; (+\frac{1}{2})^{(\tilde{c}_1)}, (+\frac{1}{2})^{(\tilde{c}_2)}) = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}.
\]  

(198)
by virtue of the values of the Clebsch-Gordan coefficients $\eta$ given in Eqs. (75) - (78), and the expressions for the $\psi$'s given in Eqs. (14) - (53).

Similarly,

\[
\xi(1, \hat{k}, (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \psi((+\frac{1}{2})(\hat{k}), (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = -\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2},
\]  

(199)

\[
\xi(1, \hat{k}, (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = \psi((+\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = -\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2},
\]  

(200)

\[
\xi(1, \hat{k}, (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \psi((+\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2},
\]  

(201)

\[
\xi(1, \hat{k}, (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}}[\psi((+\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) + \psi((-\frac{1}{2})(\hat{k}), (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2))]
\]  

(202)

\[
\xi(1, \hat{k}, (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}}[\psi((+\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) + \psi((-\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2))]
\]  

(203)

\[
\xi(1, \hat{k}, (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) = \frac{1}{\sqrt{2}}[\psi((-\frac{1}{2})(\hat{k}), (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) + \psi((-\frac{1}{2})(\hat{k}), (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2))]
\]  

(204)
\[ \xi(1, 0(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})) = \frac{1}{\sqrt{2}} \left[ \psi\left((\frac{1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \right. \\
\left. + \psi\left((\frac{-1}{2})(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \right] \\
= -\frac{1}{\sqrt{2}} \left[ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} \right], \quad (205) \]

\[ \xi(1, (\frac{-1}{2})(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})) = \psi(-k, -k; (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})) \]

\[ = \frac{1}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \]

\[ \xi(1, (\frac{-1}{2})(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})) = \psi\left((\frac{-1}{2})(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})\right) \]

\[ = \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \]

(206)

\[ \xi(1, (\frac{-1}{2})(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})) = \psi\left((\frac{-1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \]

\[ = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \] 

and 

\[ \xi(1, (\frac{-1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})) = \psi\left((\frac{-1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \]

\[ = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \] 

(207)

\[ \xi(1, (\frac{-1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})) = \psi\left((\frac{-1}{2})(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \]

\[ = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\varphi_1 + \varphi_2)} \]

The elements \( R_{pp'} \) of \([R]\) are

\[ R_{11} = \left| \xi(1, 1(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{1}{2})) \right|^2 R\left((\frac{1}{2})(\hat{G}_1), (\frac{1}{2})\right) \]

\[ + \left| \xi(1, 1(\hat{k}); (\frac{1}{2})(\hat{G}_1), (\frac{-1}{2})) \right|^2 R\left((\frac{1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \]

\[ + \left| \xi(1, 1(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{1}{2})) \right|^2 R\left((\frac{-1}{2})(\hat{G}_1), (\frac{1}{2})\right) \]

\[ + \left| \xi(1, 1(\hat{k}); (\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})) \right|^2 R\left((\frac{-1}{2})(\hat{G}_1), (\frac{-1}{2})\right) \]

45
\[ R_{12} = \xi^*(1, 1(\hat{k}); (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \times \xi(1, 0(\hat{k}); (+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \] 

\[ R_{12} = \xi^*(1, 1(\hat{k}); (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \times \xi(1, 0(\hat{k}); (+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \] 

\[ R_{12} = \xi^*(1, 1(\hat{k}); (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \times \xi(1, 0(\hat{k}); (-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \] 

\[ R_{12} = \xi^*(1, 1(\hat{k}); (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \times \xi(1, 0(\hat{k}); (-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \] 

\[ R_{12} = \frac{1}{\sqrt{2}} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} [\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\phi_2} + \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\phi_2}] \times R((+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \]

\[ R_{12} = \frac{1}{\sqrt{2}} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} [\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\phi_1} - \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\phi_1}] \times R((+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \]

\[ R_{12} = \frac{1}{\sqrt{2}} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} [\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\phi_2} - \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\phi_2}] \times R((-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) \]

\[ R_{12} = \frac{1}{\sqrt{2}} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} [\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\phi_1} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\phi_1}] \times R((-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \]

Thus,

\[ R_{12} = \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) + \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)) \]

\[ + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((-\frac{1}{2})(\hat{c}_1), (+\frac{1}{2})(\hat{c}_2)) + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})(\hat{c}_1), (-\frac{1}{2})(\hat{c}_2)). \] (210)
\begin{equation}
\times R((-\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}),
\end{equation}

(212)

In the same way,

\begin{equation}
R_{13} = \frac{1}{4} \sin \theta_1 \sin \theta_2 e^{-i(\varphi_1 + \varphi_2)} [R((+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) - R((+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) - R((-\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) + R((-\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)})],
\end{equation}

(213)

\begin{equation}
R_{21} = R_{12}^*,
\end{equation}

(214)

\begin{align*}
R_{22} &= \frac{1}{2} \left[ \cos \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
&+ \frac{1}{2} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] R((+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) \\
&+ \frac{1}{2} \left[ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
&- \frac{1}{2} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] R((+\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}) \\
&+ \frac{1}{2} \left[ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
&- \frac{1}{2} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] R((-\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) \\
&+ \frac{1}{2} \left[ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\
&+ \frac{1}{2} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] R((-\frac{1}{2})^{(\hat{c}_1)}, (-\frac{1}{2})^{(\hat{c}_2)}),
\end{align*}

(215)

\begin{align*}
R_{23} &= \frac{1}{\sqrt{2}} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} \\
&+ \frac{1}{\sqrt{2}} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2}] R((+\frac{1}{2})^{(\hat{c}_1)}, (+\frac{1}{2})^{(\hat{c}_2)}) \\
&+ \frac{1}{\sqrt{2}} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1}
\end{align*}
\[-\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2)\]
\[+ \frac{1}{\sqrt{2}} \cos \theta_1 \cos \theta_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_2} \]
\[-\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_1} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)\]
\[-\frac{1}{\sqrt{2}} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i\varphi_1}\]
\[+ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2), (216)\]

\[R_{31} = R^*_{13}, \quad \text{(217)}\]
\[R_{32} = R^*_{23} \quad \text{(218)}\]

and

\[R_{33} = \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)\]
\[+ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2) \quad \text{(219)}\]
\[+ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2)\]
\[+ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2). \quad \text{(220)}\]

Thus, this form of the matrix representation of the triplet state consists of $3 \times 1$ columns for the state and $3 \times 3$ matrices for the operators.

9 Applications to Measurements on Entangled Systems

9.1 Singlet State

We have given the general form for the operator of any quantity which is a function of the final spin projections of subsystems 1 and 2. In this section,
we shall see how this general operator can be used to calculate expectation values for joint measurements on a system such as singlet or a triplet state. These measurements reveal correlations which play a very important role in the debate over the interpretation of quantum theory [12]. The most interesting such system is the singlet state.

To use our formulas to study the correlations in the results of measurements on the singlet state, we start by investigating the behaviour of the probability amplitudes. For one thing, we ought to find that for \( \hat{c}_1 = \hat{c}_2 \) the probability amplitudes for finding the two spin projections parallel should vanish. Indeed, when we set \( \theta_1 = \theta_2 \) and \( \varphi_1 = \varphi_2 \) so that \( \hat{c}_1 = \hat{c}_2 = \hat{c} \) in Eqs. (66) - (69), we find that

\[
P(0, 0; (+\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = P(0, 0; (-\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = 0 \tag{221}
\]

but

\[
P(0, 0; (+\frac{1}{2})\hat{c}_1, (-\frac{1}{2})\hat{c}_2) = P(0, 0; (-\frac{1}{2})\hat{c}_1, (+\frac{1}{2})\hat{c}_2) = \frac{1}{2}. \tag{222}
\]

Thus, whenever one subspin is found up, the other will be found down with certainty. The present approach gives a reason for this correlation. Since the angles defining the initial direction \( \hat{a} \) are absent from the expressions for the probability amplitudes, the singlet state is symmetric with respect to the coordinate system. Thus, whatever vector \( \hat{c} \) we choose in order to make our spin projection measurements takes on the role of the vector \( \hat{a} \) along which the two spin-1/2 projections are assumed to lie. To put it another way, the absence of the angles for \( \hat{a} \) from the probability amplitudes means that the state does not recognise the original direction along which lie the two spins added to give zero. When one measurement is made, the direction with respect to which it is done assumes the role of the vector along which the two spins lie anti-parallel. Hence a measurement of the other spin projection along the same vector finds it anti-parallel to the other. Thus, the correlation is a symmetry effect.

We expect similar results to occur whenever the spin of the composite system is zero. In such cases, the expansion of the probability amplitudes will involve only one summation, as in Eq. (61) owing to the fact that the summation over the spin projections of the compound spin vanishes. There will then be no correlation between the original orientations of the constituent subsystems and the directions of the vectors \( \hat{c} = \hat{c}_1 = \hat{c}_2 \).
In order to compute the expectation values for the entangled system, we consider the observable $R$ defined by assigning the value $+1$ ($-1$) when the spin projection of subsystem 1 or subsystem 2 is up (down) with respect to $\hat{c}_1$ or $\hat{c}_2$, respectively. If these values are then multiplied to give an observable describing joint measurements on the subsystems, the four possible outcomes are

$$R((+\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) = R((-\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2) = 1$$  \hspace{1cm} (223) \\

and

$$R((+\frac{1}{2})\hat{c}_1), (-\frac{1}{2})\hat{c}_2) = R((-\frac{1}{2})\hat{c}_1), (+\frac{1}{2})\hat{c}_2) = -1.$$  \hspace{1cm} (224) \\

Hence the operator $[R]$ has the elements

$$R_{11} = \cos \theta_1 \cos \theta_2,$$  \hspace{1cm} (225) \\

$$R_{12} = \cos \theta_1 \sin \theta_2 e^{-i\varphi_2},$$  \hspace{1cm} (226) \\

$$R_{13} = \sin \theta_1 \cos \theta_2 e^{-i\varphi_1},$$  \hspace{1cm} (227) \\

$$R_{14} = \sin \theta_1 \sin \theta_2 e^{-i(\varphi_1 + \varphi_2)},$$  \hspace{1cm} (228) \\

$$R_{21} = R_{12}^*,$$  \hspace{1cm} (229) \\

$$R_{22} = -\cos \theta_1 \cos \theta_2,$$  \hspace{1cm} (230) \\

$$R_{23} = \sin \theta_1 \sin \theta_2 e^{i(\varphi_2 - \varphi_1)},$$  \hspace{1cm} (231) \\

$$R_{24} = -\sin \theta_1 \cos \theta_2 e^{-i\varphi_1},$$  \hspace{1cm} (232) \\

$$R_{31} = R_{13}^*,$$  \hspace{1cm} (233) \\

$$R_{32} = R_{23}^*,$$  \hspace{1cm} (234) \\

$$R_{33} = -\cos \theta_1 \cos \theta_2,$$  \hspace{1cm} (235) \\

$$R_{34} = -\sin \theta_2 \cos \theta_1 e^{-i\varphi_2},$$  \hspace{1cm} (236) \\

$$R_{41} = R_{14}^*,$$  \hspace{1cm} (237)
\[ R_{42} = R_{24}^*, \quad R_{43} = R_{34}^* \] (238)

and

\[ R_{44} = \cos \theta_1 \cos \theta_2. \] (240)

Using this operator and the state, Eq. (165), the expectation value of \( R \) is found to be

\[
\langle R \rangle = -[\cos(\theta_2 - \theta_1) - 2 \sin \theta_1 \sin \theta_2 \sin^2 \left(\frac{\varphi_2 - \varphi_1}{2}\right)]
= -\mathbf{c}_1 \cdot \mathbf{c}_2.
\] (241)

### 9.2 Triplet State

As we have seen, there are two matrix representations for the triplet state. The four-dimensional representation has the same operator as the singlet-state case, but with the states being given by Eqs. (189), (190) and (191) for the cases \( M = 1, 0 \) and \(-1\) respectively.

For the three-dimensional treatment, the elements of the operator \( [R] \) are obtained by putting

\[
R\left(\left(+\frac{1}{2}\right)\mathbf{\hat{c}}_1, \left(+\frac{1}{2}\right)\mathbf{\hat{c}}_2\right) = R\left(\left(-\frac{1}{2}\right)\mathbf{\hat{c}}_1, \left(-\frac{1}{2}\right)\mathbf{\hat{c}}_2\right) = 1
\] (242)

and

\[
R\left(\left(+\frac{1}{2}\right)\mathbf{\hat{c}}_1, \left(-\frac{1}{2}\right)\mathbf{\hat{c}}_2\right) = R\left(\left(-\frac{1}{2}\right)\mathbf{\hat{c}}_1, \left(+\frac{1}{2}\right)\mathbf{\hat{c}}_2\right) = -1
\] (243)

in Eqs. (210) - (220), and are

\[ R_{11} = -\cos \theta_1 \cos \theta_2, \] (244)

\[ R_{12} = \frac{1}{\sqrt{2}}[\sin \theta_1 \cos \theta_2 e^{-i\varphi_1} + \cos \theta_1 \sin \theta_2 e^{-i\varphi_2}], \] (245)

\[ R_{13} = \sin \theta_1 \sin \theta_2 e^{-i(\varphi_1 + \varphi_2)}, \] (246)

\[ R_{21} = R_{12}^*, \] (247)

\[ R_{22} = -\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2), \] (248)
\[ R_{23} = -\frac{1}{\sqrt{2}} [\sin \theta_1 \cos \theta_2 e^{-i\varphi_1} + \cos \theta_1 \sin \theta_2 e^{-i\varphi_2}] \]  
(249)

\[ R_{31} = R_{13}^*, \]  
(250)

\[ R_{32} = R_{23}^* \]  
(251)

and

\[ R_{33} = \cos \theta_1 \cos \theta_2. \]  
(252)

We can calculate the expectation value by using the probability amplitudes directly, or by the matrix-mechanics approach, in which case we have a choice between the three- and the four-dimensional representations. Whatever method we use, the expectation values are

\[ \langle R \rangle_{M(\hat{a})=1} = \langle R \rangle_{M(\hat{a})=-1} = \cos^2 \theta \cos \theta_1 \cos \theta_2 \]

\[ + \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_2) \cos(\varphi - \varphi_1) \]

\[ + \sin \theta \sin \theta_1 \cos \theta \cos \theta_2 \cos(\varphi - \varphi_1) \]

\[ + \sin \theta \sin \theta_2 \cos \theta \cos \theta_1 \cos(\varphi - \varphi_2) \]  
(253)

and

\[ \langle R \rangle_{M(\hat{a})=0} = - \cos 2\theta \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_2 - \varphi_1) \]

\[ - 2 \sin^2 \theta \sin \theta_1 \sin \theta_2 \cos(\varphi - \varphi_2) \cos(\varphi - \varphi_1) \]

\[ - \sin 2\theta \sin \theta_1 \cos \theta_2 \cos(\varphi - \varphi_1) \]

\[ - \sin 2\theta \sin \theta_2 \cos \theta_1 \cos(\varphi - \varphi_2). \]  
(254)

The standard expectation values correspond to setting \( \theta = \varphi = 0 \), so that the compound spin is initially along the z axis. In this limit, we find

\[ \langle R \rangle_{M(\hat{a})=1} = \langle R \rangle_{M(\hat{a})=-1} = \cos \theta_1 \cos \theta_2 \]  
(255)

and

\[ \langle R \rangle_{M(\hat{a})=0} = - \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_2 - \varphi_1). \]  
(256)

If in addition \( \hat{c}_2 = \hat{c}_1 \), then

\[ \langle R \rangle_{M(\hat{a})=1} = \langle R \rangle_{M(\hat{a})=-1} = \cos^2 \theta_1 \]  
(257)
and
\[ \langle R \rangle_{M(\hat{a})=0} = -\cos 2\theta_1. \] (258)

Further applications are easily realized by plugging the appropriate values \( R((m_1)^{(C_1)}, (m_2)^{(C_2)}) \) into the formulas Eqs. (211) - (221) if we select the three-dimensional treatment or into Eqs. (166) - (182) if we prefer the four-dimensional treatment.

10 Discussion and Conclusion

In this paper, we have approached the treatment of systems of compounded angular momentum from a new direction. We have given a general theory for their treatment which can be used on all such systems. This theory has been specialized to the case of the addition of two spins of value \( 1/2 \) each to obtain total spin 0 and 1. Thus, for the singlet and triplet states thus resulting, we have derived the probability amplitudes for obtaining by measurement all possible combinations of the spin projections of the subsystems along arbitrary directions.

We have obtained new matrix treatments of these systems from first principles. Thus, we have given the matrix representations of the \( S = 0 \) and \( S = 1 \) states and given the general form of the matrix operator corresponding to measurements on these systems. For the \( S = 1 \) system, we have found two matrix representations. It is evident that in a general case where a probability amplitude is obtained indirectly through several expansions, it is possible to use any of those expansions to define a matrix representation. Thus, a general system has as many matrix representations as there are possible intermediate expansions of its probability amplitudes. In fact from this point of view, wave mechanics is merely matrix mechanics of \( N = 1 \) dimensions. We can see this from the wave-mechanics expectation value of the quantity \( R(C) \):
\[ \langle R \rangle = \sum_j |\Psi(A_i; C_j)|^2 R(C_j). \] (259)

This can be expressed as
$$\langle R \rangle = (1) \left( \sum_j |\Psi(A_i; C_j)|^2 R(C_j) \right)$$

$$= [\Psi(A_i)]^\dagger [R][\Psi(A_i)],$$

where

$$[\Psi(A_i)] = (1)$$

and

$$[R] = \left( \sum_j |\Psi(A_i, C_j)|^2 R(C_j) \right).$$

In our work on isolated spin-1/2 systems,[1,2,4], we saw that changing the phase of the probability amplitudes alters the forms of the spin operators and vectors. In the present work, we have seen that not all choices of phase for the spin-1/2 probability amplitudes are equivalent. It must be realized that there are several choices of phase possible for the probability amplitudes for the isolated spin-1 system as well; the choice that appears in Ref. [3], and which has been adopted for use here, was arbitrarily picked. In fact, the total number of phase choices for the spin-1 case is greater than for the spin-1/2 case. It is certain that the choice of spin-1 probability amplitude phase has a bearing on the choice of spin-1/2 phase to be combined with it in order to obtain the treatment of the compound system. If in our derivation of the spin quantities for spin 1 we had used a different choice of phase for the probability amplitudes, we would certainly have been obliged to use a different phase for the spin-1/2 probability amplitudes in order to obtain the results for the compound system. We then would have found different forms for the compound-system probability amplitudes, the matrix states and the matrix operators. Thus, we see that the results we have presented here are not unique, but are only one set in a family of such results. But as we have seen here, it is not true that we can combine any spin-1/2 probability amplitudes with just any spin-1 probability amplitudes. Some combinations will not lead to acceptable results. The rule for determining the correct combination of spin-1 and spin-1/2 phase choices is that in the appropriate limit, the compound-state probability amplitudes should reduce to the Clebsch-Gordan coefficients. The matter of just how to match phase choices

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for spin-1/2 and spin-1 is of considerable interest in its own right and deserves deeper investigation. It is possible that the phase choice combinations which are not acceptable in the present work will be found appropriate in other investigations where both spin-1/2 and spin-1 probability amplitudes both enter.

According to the present approach, the Clebsch-Gordan coefficients can be generalized and we have given the generalized forms for the singlet and triplet states. For these cases, the forms are not very different from the standard forms. However, the generalized Clebsch-Gordan coefficients are themselves functions of the phase combinations deriving from the spin-1/2 and spin-1 phase choices. It is possible that even for the singlet and triplet cases, a more involved dependence of the generalized Clebsch-Gordan coefficients on the angles results from a different combination of phases from that used here.

We have applied the results obtained to one practical problem - the issue of correlations in measurements on entangled systems. We have gained a useful insight into the systems with the observation that the correlations in the results for the singlet state are due to the probability amplitudes not being functions of the angles that define the initial direction of the compounded spin. Here this direction is the direction along which the subspins which add up to zero lie. We deduce that whenever we compound any number of spins in order to get a total spin of zero, the probability amplitudes we obtain will be independent of the directions of the spins which are added to yield zero. Therefore in all such cases, we should observe correlations in the results of spin-projection measurements on the systems.

The fact that we have been able to do all this proves the soundness of Landé’s ideas, which underlie our method. The method has the advantage of great generality and clarity and should prove useful in other departments of physics. It is probably possible to use this approach to obtain the matrix treatment from a probability amplitude basis for other systems which are currently described only by matrices.

The extension of the present ideas to more complex systems is straightforward, if more tedious. A system that suggests itself as being but one step removed in complexity from the one treated here involves adding the spins of spin-1 and spin-1/2 subsystems to obtain or a spin half or a spin 3/2 system. For the former case, we have all the tools we need because we already have the individual treatments for the spin-1 and spin-1/2 subsystems and
thereby also for the spin-1/2 compound system. But if we wish to compound
to get spin-3/2, then we need to apply the general method we have devised
in previous papers in order to obtain the probability amplitudes for this
case. Strictly speaking, we can get away with using the standard generalized
probability amplitudes. Whatever, both these cases promise to be interest-
ing because they ought to yield generalized Clebsch-Gordan coefficients with
a less simple dependence on the angles than the ones for the cases in the
present paper.

It is very striking that the matrix treatment we have derived here is com-
pletely different from the standard treatment which involves $2 \times 2$ operators
for each subsystem directly multiplied to give the operator for the compound
system. The transition from one treatment to the other deserves investiga-

11 References

1. Mweene H. V., "Derivation of Spin Vectors and Operators From First
   Principles", quant-ph/9905012
2. Mweene H. V., "Generalized Spin-1/2 Operators and Their Eigenvect-
   tors", quant-ph/9906002
3. Mweene H. V., "Vectors and Operators for Spin 1 Derived From First
   Principles", quant-ph/9906043
4. Mweene H. V., "Alternative Forms of Generalized Vectors and Ope-
   rators for Spin 1/2", quant-ph/9907031
5. Mweene H. V., "Spin Description and Calculations in the Landé Inp-
   erpretation of Quantum Mechanics", quant-ph/9907033
6. Landé A., "From Dualism To Unity in Quantum Physics", Cambridge
   University Press, 1960.
7. Landé A., "New Foundations of Quantum Mechanics", Cambridge
   University Press, 1965.
8. Landé A., "Foundations of Quantum Theory," Yale University Press,
   1955.
9. Landé A., "Quantum Mechanics in a New Key," Exposition Press,
   1973.
10. Rose M. E., "Elementary Theory of Angular Momentum", John Wiley
    and Sons, Inc. (New York), 1957

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11. Bransden and Joachain, "Introduction to Quantum Mechanics", Long-man Scientific & Technical, 1989.
12. See for example, Bell J. S., Physics 1 (1964), 195