EQUIVALENCES OF GRADED FUSION CATEGORIES

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ABSTRACT. We use the techniques developed by Etingof, Nikshych, and Ostrik in [3] to classify the graded equivalences between two $G$-graded fusion categories with the same trivial piece. As a warm up for our main theorem we reprove the classification of graded extensions of a fusion category, making extensive use of graphical calculus. We provide several applications of our classification of graded equivalences.

1. INTRODUCTION

For any fusion category $C$, there is an important invariant $\text{BrPic}(C)$, the Brauer-Picard 3-group of $C$. This invariant is defined as the 3-group of invertible bimodules, bimodule equivalences, and bimodule functor natural isomorphisms. This invariant is important in many different areas of mathematics.

If $C$ is unitary, then the truncation $\text{BrPic}(C)$ exactly corresponds to finite index, finite depth subfactors whose even and dual even parts are both equivalent to $C$. For the classification of small index subfactors this correspondence has proved extremely useful. For example in [5] it is shown that the even part of the Haagerup-Izumi subfactor for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has Brauer-Picard group of order 360. In [1] the author classifies all subfactors whose even and dual even parts are equivalent to an even part of one of the $ADE$ subfactors, which revealed the existence of a new subfactor of small index. This classification is achieved by abstractly computing the Brauer-Picard group via the Drinfeld centre.

The application of the Brauer-Picard 3-group we focus on for this paper is the classification of $G$-graded extensions of $C$. In [3] it is shown that $G$-graded extensions of $C$ are classified by a 3-homomorphism $G \rightarrow \text{BrPic}(C)$, or equivalently by triples consisting of a homomorphism $G \rightarrow \text{BrPic}(C)$, a collection of certain bimodule equivalences, and a collection of certain bimodule natural isomorphisms, such that explicit obstructions vanish. More detail can be found in the mentioned paper, or in Section 4 of this paper. This theory has proved very rich for constructing new interesting examples of fusion categories. The only non-$ADE$ example of a 3-supertransitive fusion category is constructed in [6] as a $\mathbb{Z}/2\mathbb{Z}$-extension of the even part of the Asaeda-Haagerup subfactor. The graded extension theory may also prove useful in the classification of categories generated by an object of small dimension. An unpublished theorem of Morrison and Snyder shows that every pivotal fusion category generated by a object $X$ of dimension less than 2, satisfying the normality condition $X \otimes X^* \cong X^* \otimes X$, is a cyclic extension of the even part of one of the $ADE$ subfactors. This paper provides an improvement on the classification of graded categories, making it suitable for such classification problems.

The issue with using graded extension theory for classification results is that extensions are only considered up to equivalences that are the identity on the trivially graded piece of the extension. Thus the classification over counts extensions in certain examples. An example of this over counting occurs when classifying cyclic extensions of the even parts of the $ADE$ subfactors, which was the authors main motivation to write this paper. With the authors computation of the Brauer-Picard
group of $\frac{1}{2}D_{10}$, we can use the extension theory to compute that there are exactly 12 extensions of $\frac{1}{2}D_{10}$ with $D_{10}$ fusion rules. However, as a Corollary of the classification of subfactors of index less than 2, there are exactly 4 categories with $D_{10}$ fusion rules. In fact in Section 3 we show that whenever $C$ has a non-trivial monoidal auto-equivalence, the graded extensions of $C$ are over counted.

To fix this over counting we classify monoidal graded equivalences between two graded categories with the same trivial piece, a stronger condition than equivalences that are trivial on the trivial piece, but weaker than general monoidal equivalence (except for when both categories are graded by their universal grading groups). An easy example of these conditions can be seen through the auto-equivalences of $\text{Vec}(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$, thought of as a $\mathbb{Z}/3\mathbb{Z}$-graded category with trivial piece $\text{Vec}(\mathbb{Z}/3\mathbb{Z})$. For this example the group of all monoidal equivalences is $GL_2(3)$, the graded equivalences are the lower triangular matrices, and the equivalences that are trivial on the trivial piece are the matrices whose upper right entry is 0, and lower right entry is 1.

The main result of this paper is the proof of the following Theorem.

**Theorem 1.1.** Let $(c, M^C, A^C)$ and $(d, M^D, A^D)$ be two $G$-graded extensions of $B$, then graded monoidal equivalences between these two categories are classified by quadruples $(F_e, \phi, \hat{F}, \hat{\tau})$, where

- $F_e$ is an auto-equivalence of $B$,
- $\phi$ is an automorphism of $G$ such that $d = \text{inn}(B_{F_e}) \circ c \circ \phi$ and an explicit element $o_2(F_e, \phi) \in H^2(G, \text{Inv}(Z(B)))$ vanishes,
- $\hat{F}$ is an element of a certain $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$ torsor, such that an explicit element $o_3(F_e, \phi, \hat{F}) \in H^3(G, \mathbb{C}^*)$ vanishes,
- $\hat{\tau}$ is an element of a certain torsor over $H^2(G, \mathbb{C}^*)$.

Here $\hat{F}$ is a collection of bimodule equivalences

$$\hat{F}_g : B_{F_e} \boxtimes c_{\phi(g)} \boxtimes d_B B \to d_g,$$

the group $D^1(G, \text{Inv}(Z(B)))$ is the group of 1-cocycles $\rho$ such that $\rho_g \boxtimes \text{Id}_{\hat{F}_g}$ is isomorphic as a plain functor to $\text{Id}_{\hat{F}_g}$, and $\hat{\tau}$ is a collection of bimodule natural isomorphisms

$$\hat{\tau}_{g,h} : (\text{Id}_{F_e} B \boxtimes M^D_{\phi(g), \phi(h)} \boxtimes B_{F_e}) \circ (\text{Id}_{F_e} B \boxtimes \text{Id}_{d_{\phi(g)}} \boxtimes \text{ev}_{B_{F_e}} \boxtimes \text{Id}_{d_{\phi(h)}} \boxtimes \text{Id}_{B_{F_e}}) \circ (\hat{F}_g \boxtimes \hat{F}_h).$$

Graphical descriptions of $\hat{F}$ and $\hat{\tau}$ can be found in Section 6.
The explicit element $o_2(F_e, \phi) \in H^2(G, \text{Inv}(Z(B)))$ is the cohomology class of the 2-cocycle

$$T(F_e, \phi, \hat{F}) = \cdots$$

which is independent of choice of $\hat{F}$.

The explicit element $o_3(F_e, \phi, \hat{F}) \in H^3(G, \mathbb{C}^\times)$ is the cohomology class of the 3-cocycle given in Figure 1, which is independent of choice of $\hat{\tau}$.

We structure our paper in an unorthodox manner, beginning with the applications of our main Theorem in Section 3. We give a simple rederivation of the auto-equivalence group of a pointed fusion category. We solve our motivating problem of there being too many extensions of $\frac{1}{2}D_{10}$ by providing a Theorem that shows when two extensions of the same category are equivalent as graded monoidal categories. Finally we show that any modular category with distinguished boson or fermion has a (possibly trivial) auto-equivalence. We use this fact to construct a $\mathbb{Z}/2\mathbb{Z}$-graded extension of any super-modular category.

Our main theorem shares many similarities with the classification of graded extensions from [3]. In fact our theorem is very much inspired by their result, and the proof uses many of the same techniques. Due to the similarities of these two theorems we start the paper by rederiving the classification of graded extensions in Section 4. While the high-level arguments in our proof are exactly the same as in the original, we include more details, and we make extensive use of graphical calculus when possible. We hope these calculations will be useful to readers who like graphical proofs.

In Section 5 we define the abstract nonsense of twisted bimodule functors. This is a straightforward generalisation of bimodule functors, where now we require the functor to preserve the bimodule action up to some action of a fixed tensor auto-equivalence. The motivation behind such a definition is that the restriction of a graded equivalence to a graded piece gives exactly a twisted bimodule equivalence. Unfortunately twisted bimodule functors are in practice difficult to work with, as they have no nice graphical calculus. To fix this issue we prove Theorem 5.6, which shows that we can find all the information we need for twisted bimodule equivalences (and their natural isomorphisms), within the Brauer-Picard 3-group. The proof of this theorem is somewhat long and technical. We hide the proof in Appendix A, which the reader can view at their own risk.
Figure 1. Graphical description of $v(F_e, \phi, \widehat{F}_g, \widehat{\tau}_{g,h})$
In Section 6 we deconstruct a graded monoidal equivalence of extensions. We show that from a graded monoidal equivalence one can extract the four pieces of data as in Theorem 1.1. Conversely, given these four pieces of data we can reconstruct the graded equivalence. This proves the initial part of our main theorem, showing that graded monoidal equivalences are classified by such quadruples.

In Section 7 we classify all triples \((F_e, \phi, \hat{F})\) such that there exists some collection \(\hat{\tau}\) of bimodule equivalences (but not necessarily giving rise to a graded monoidal equivalence). This is equivalent to classifying graded quasi-monoidal equivalences between the graded categories. We found it very surprising that \(\hat{F}\) belongs to a \(Z_1(G, \text{Inv}(Z(B)))/D_1(G, \text{Inv}(Z(B)))\), rather than a \(H_1(G, \text{Inv}(Z(B)))\) torsor, which seems contrary with other results regarding graded extensions. To convince the reader that these two groups are different, and that \(Z_1(G, \text{Inv}(Z(B)))/D_1(G, \text{Inv}(Z(B)))\) is indeed the correct choice, we give a concrete computation at the end of this Section.

In Section 8 we complete the proof of our main Theorem by classifying collections \(\hat{\tau}\) such that the quadruple \((F_e, \phi, \hat{F}, \hat{\tau})\) gives rise to a graded monoidal equivalence.

This paper overlaps with the results of Ian Marshall’s thesis [7]. In this thesis a classification of the kernel and image of the restriction map 

\[
\text{Aut}(\bigoplus C_g) \to \text{Aut}(C_e)
\]

is given. In the language of our main theorem, computing the kernel corresponds to classifying all quadruples such that \(F_e = \text{Id}_{C_g}\), and computing the image corresponds to characterizing when, for a fixed \(F_e \in \text{Aut}(C_e)\), there exists a quadruple \((F_e, \phi, \hat{F}, \hat{\tau})\). Our result is more general as it classifies all quadruples, and it applies to graded equivalences between different categories, not just auto-equivalences.

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2. Preliminaries

We refer the reader to [2] for the basics of fusion categories, and bimodule categories.

\(G\)-graded fusion categories and functors. Let \(C\) be a fusion category, and \(G\) a finite group. We say \(C\) is \(G\)-graded if

\[
C = \bigoplus C_g
\]

for \(C_g\) abelian subcategories of \(C\), such that the tensor product restricted to \(C_g \times C_h\) has image in \(C_{gh}\).

A graded monoidal functor \(F : \bigoplus C_g \to \bigoplus D_g\) is a monoidal functor satisfying

\[
X, Y \in C_g \implies F(X), F(Y) \in D_h \text{ for some } h \in G.
\]

Quasi-monoidal categories, functors, and natural transformations. Crucial to our main classification result will be the intermediate classification of graded quasi-monoidal equivalences. We briefly define the basic details of quasi-monoidal fusion categories, functors, and natural transformations.
A quasi-monoidal fusion category is defined exactly the same as a fusion category, except there is no data of an associator, we only require the existence (but not choice) of a natural isomorphism \((X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\).

A quasi-monoidal functor \(F\) between two quasi-monoidal categories is a functor that admits a natural isomorphism \(F(X) \otimes F(Y) \to F(X \otimes Y)\). Note that the information of quasi-monoidal functor is just the functor itself, and not the choice of this natural isomorphism.

A quasi-monoidal natural transformation between quasi-monoidal functors is simply a natural transformation with no extra conditions.

Clearly every monoidal category, functor and natural transformation is also quasi, via forgetting.

The 3-category of invertible bimodules over a fusion category. This subsection recalls the information of \(\text{BrPic}(C)\), the 3-category of invertible bimodules over a fusion category \(C\). More details can be found in the papers \([4, 3]\).

We have in \(\text{BrPic}(C)\) that

- 0-morphism: The category \(C\)
- 1-morphisms \(C \to C\): Invertible \(C\)-\(C\) bimodules \(M\)
- 2-morphisms \(M \to N\): Bimodule equivalences \(F\)
- 3-morphisms \(F \to G\): Natural isomorphisms of bimodule functors \(\mu\)

Composition at the 2-level is the composition \(\circ\) of bimodule functors, and composition at the 3-level is the vertical composition \(\cdot\) of natural transformations. Composition at the 1-level is more complicated. We first define the category \(\text{Fun}^{\text{bal}}(M_1 \times M_2 \to N)\), for bimodules \(M_1, M_2, N\), as done in \([4]\). For this paper \(\times\) is the Deligne product of bimodule categories.

The objects of \(\text{Fun}^{\text{bal}}(M_1 \times M_2 \to N)\) are bimodule functors \(F: M_1 \times M_2 \to N\), along with a balancing isomorphism \(b^F_{m_1,c,m_2}: F(m_1 \triangleleft c \triangleleft m_2) \to F(m_1 \triangleleft c \triangleright m_2)\), satisfying the consistency condition:

We call the object \((F, b^F)\) a \(C\)-balanced bimodule functor.

The morphisms \((F, b^F) \to (G, b^G)\) of \(\text{Fun}^{\text{bal}}(M_1 \times M_2 \to N)\) are natural transformations of bimodule functors \(\mu: F \to G\) that commute with the balancing morphisms in the following sense:

We call the morphism \(\mu\) a \(C\)-balanced natural transformation.

With the category \(\text{Fun}^{\text{bal}}(M_1 \times M_2 \to N)\) in hand, we can now define the composition of bimodules in the 3-category \(\text{BrPic}(C)\). This composition of bimodules is called the relative tensor product.
Definition 2.1. Let $M_1$ and $M_2$ be bimodule categories. The relative tensor product of $M_1$ and $M_2$, $M_1 \boxtimes M_2$, is the unique bimodule category inducing, for every bimodule category $N$, an equivalence of categories:

$$\text{Fun}^{\text{bal}}(M_1 \times M_2 \to N) \simeq \text{Fun}(M_1 \boxtimes M_2 \to N).$$

From the above definition it isn’t clear that such a category $M_1 \boxtimes M_2$ exists. In [3] the authors show that $M_1 \boxtimes M_2$ can be realised as $Z_C(M_1 \times M_2)$, the relative Drinfeld centre of $M_1 \times M_2$.

Generalising the above definition we can define $\text{Fun}^{\text{bal}}(M_1 \times \ldots \times M_n \to N)$, the category of bimodule functors, balanced in each position, in a similar fashion. Here we now have objects a bimodule functor, along with $n-1$ balancing isomorphisms that are consistent between each other. We require natural transformations in this category to commute with each balancing isomorphism.

The same way as we defined the relative tensor product $M_1 \boxtimes M_2$, we can define the n-fold relative tensor product as the unique bimodule category $M_1 \boxtimes \ldots \boxtimes M_n$ inducing an equivalence:

$$(2.1) \quad \text{Fun}^{\text{bal}}(M_1 \times \ldots \times M_n \to N) \simeq \text{Fun}(M_1 \boxtimes \ldots \boxtimes M_n \to N).$$

The 3-category $\text{BrPic}(C)$ is non-empty due to the existence of $C$ as a $C$-$C$-bimodule over itself. Here the bimodule structure maps are given by associator of $C$, which we assume to be strict.

Given a monoidal auto-equivalence of $(F, \tau)$ of $C$, we can construct an invertible bimodule $C_F$, which as an abelian category is $C$. The left action is given by tensoring in $C$, and the right action is given by applying $F$ and tensoring in $C$. The left and central bimodule structure maps are trivial. The right bimodule structure map is given by $\tau$. Similarly we can define $F_C$ by swapping left and right in the above construction. It is shown in [9, Proposition 3.1.] that $C \simeq C_F$ if and only if $F$ is an inner auto-equivalence of $C$. While for this paper we will assume our bimodules our strict, we will not assume $C_F$ and $F_C$ are. This so we can have the bimodules $C$, $C_F$ and $F_C$ all share the same object set.

Graphical calculus for $\text{BrPic}(C)$. As $\text{BrPic}(C)$ is a 3-category, we have a 3-dimensional graphical calculus. We pick a coordinate system:
We can truncate $\text{BrPic}(C)$ to a 2-category $\text{BrPic}(C)$ by collapsing natural isomorphisms to identities. Thus we also have a 2-dimensional calculus for $\text{BrPic}(C)$. We pick a coordinate system:

$$\begin{array}{c}
\circ \\
\downarrow \\
\otimes
\end{array}$$

We will make heavy use of these graphical calculuses in this paper. We present some important examples of this graphical calculus.

Let $M$ be an invertible bimodule, in [3] the authors construct a bimodule $M^{\text{op}}$ along with a bimodule equivalence:

$$\text{ev}_M : M^{\text{op}} \boxtimes M \rightarrow C$$

Graphically we draw this equivalence as:

$$\begin{array}{c}
\bigcirc \\
M^{\text{op}} \\
\downarrow \\
M
\end{array}$$

and its inverse as:

$$\begin{array}{c}
\bigcirc \\
M^{\text{op}} \\
\downarrow \\
M
\end{array}$$

These bimodule equivalences satisfy the circle popping, and string recoupling relations:

$$\begin{array}{c}
M^{\text{op}} \\
\bigcirc \\
M \\
\cong \text{Id}_C
\end{array} \quad \text{and} \quad 
\begin{array}{c}
\bigcirc \\
M^{\text{op}} \\
\downarrow \\
M
\end{array}$$

Note that the bimodule $(M^{\text{op}})^{\text{op}}$ is strictly equal to $M$.

As an example of the usefulness of these string recoupling and circle popping relations, we prove the following Lemma which will be useful later in this paper.

**Lemma 2.2.** Let $M, N$ invertible bimodules, and let $F : M \rightarrow N$ and $H : N \rightarrow N$ be bimodule functors, then there exists $Z_H \in \text{Aut}(B)$ such that

$$H \circ F \cong Z_H \boxtimes F.$$
Proof. Using the graphical calculus we compute

\[
\begin{align*}
F & \quad H \\ N & \quad \rightsquigarrow \\
F & \quad M
\end{align*}
\]

\[
\begin{align*}
F & \quad H \\ N & \quad \rightsquigarrow \\
F & \quad M
\end{align*}
\]

\[
\begin{align*}
F & \quad H \\ N & \quad \rightsquigarrow \\
F & \quad M
\end{align*}
\]

\[
\begin{align*}
F & \quad H \\ N & \quad \rightsquigarrow \\
F & \quad M
\end{align*}
\]

Thus

\[
Z_H := \quad H
\]

satisfies the hypothesis.

\[
\square
\]

For every invertible bimodule functor \( F : M_1 \boxtimes M_2 \to N \), there exists by definition \( F^{-1} : N \to M_1 \boxtimes M_2 \) along with natural isomorphisms:

\[
ev_F : F^{-1} \circ F \to \text{Id}_M \quad \text{and} \quad ev_{F^{-1}} : F \circ F^{-1} \to \text{Id}_N.
\]

Graphically we draw these natural isomorphisms as:

\[
\begin{align*}
\text{and}
\end{align*}
\]

Of particular importance for this paper, when \( F = ev_M \) we get the bubble popping, and sheet recoupling relations:

\[
\begin{align*}
\text{id}_{\text{Id}_C} \quad \text{and} \quad \text{id}_{\text{Id}_N}
\end{align*}
\]
An action of BrPic($\mathbb{C}$) on Inv($Z(\mathbb{C})$). It is shown in [3] that the group of invertible elements of the centre of $\mathbb{C}$, Inv($Z(\mathbb{C})$), and the group of bimodule auto-equivalences of $\mathbb{C}$, Aut($\mathbb{C}$), are isomorphic. The isomorphism sends $(X, \gamma_{X,-})$ to the auto-equivalence $X \triangleright ?$. The left bimodule structure map is given by $\gamma_{X,-}$, and the right bimodule structure map is trivial. We will implicitly use this isomorphism throughout this paper.

Given an invertible bimodule $M$, and $z$ a bimodule auto-equivalence of the trivial bimodule $\mathbb{C}$, we can define

$$z^M := \text{ev}_M \circ (\text{id}_M \boxtimes z \boxtimes \text{id}_{M^{\text{op}}}) \circ \text{ev}_{M^{\text{op}}}$$

Graphically we draw $z^M$ as:

It isn’t hard to check using the string recoupling and bubble popping relations that this determines an action of BrPic($\mathbb{C}$) on Inv($Z(\mathbb{C})$). In fact this is exactly the action defined in [3], which they prove is independent of the choice of ev$_M$. The following lemma about this action is repeatedly used throughout this paper.

Lemma 2.3. There exists an isomorphism of bimodule functors:

$$z \cong z^M$$

Proof. We directly compute using the string recoupling and bubble popping relations:

3. Examples

Auto-equivalences of Vec$^\omega(G)$. Consider the category Vec$^\omega(G)$ for a 3-cocyle $\omega$. This category is a $G$-graded extension of Vec with extension data $c(g) = \text{Vec}$, $M_{g,h}$ the map induced by the Vec balanced functor $V \times W \mapsto V \otimes W$, and $\omega$ as above. We aim to recover the computation of Aut(Vec$^\omega(G)$). It is easy to see that any auto-equivalence of Vec$^\omega(G)$ is graded, thus we can use our main theorem to achieve the computation.
The fusion category $\text{Vec}$ has no non-trivial auto-equivalences, therefore our only choice of $F_e$ is $\text{Id}_{\text{Vec}}$. Let $\phi$ be any automorphism of $G$, then $\text{inn}(\text{Vec}) \circ c \circ \phi = c \circ \phi = c$. Thus we can choose any automorphism $\phi$.

There are obvious choices for the bimodule equivalences $C_{\phi(g)} \to D_g$, as both bimodule categories are the trivial $\text{Vec}$ bimodule category over itself. That is we choose each $F_g = \text{Id}_{\text{Vec}}$. The obstruction $T$ in $Z^2(G, \text{Inv}(\text{Z(Vec)}))$ vanishes as $\text{Inv}(\text{Z(Vec)}) = \{e\}$. For the same reason $Z^1(G, \text{Inv}(\text{Z(Vec)}))$ vanishes and so our chosen $F$ is the unique choice, up to isomorphism.

We need to choose a collection $\tau_{g,h}$ of natural isomorphisms $F_{gh} \circ M_{g,h} \to M_{\phi(g),\phi(h)} \circ (F_g \boxtimes F_h)$. For our case this is just a map $M_{g,h} \to M_{\phi(g),\phi(h)}$, so we can choose the identity natural isomorphism. It is straightforward to check that the obstruction $\nu(\text{Id}_{\text{Vec}},\phi,\text{Id}_{\text{Vec}},\text{id})$ in $Z^3(G, \mathbb{C}^*)$ vanishes if and only if $\omega_{\phi(f),\phi(g),\phi(h)} = \omega_{f,g,h}$ for all $f, g, h \in G$. We can scale $\text{id}$ by an element of $H^2(G, \mathbb{C}^*)$ to get all other possible choices of $\tau$.

Thus our main theorem implies that auto-equivalences of $\text{Vec}^\omega(G)$ are parameterised by quadruples $(\text{Id}_{\text{Vec}},\phi,\text{Id}_{\text{Vec}},\psi \text{id})$ for $\phi \in \text{Aut}(G)$ satisfying $\omega_{\phi(f),\phi(g),\phi(h)} = \omega_{f,g,h}$, and $\psi \in H^2(G, \mathbb{C}^*)$. Hence we recover the result that $\text{Aut}(\text{Vec}^\omega(G)) = \text{Aut}^\omega(G) \rtimes H^2(G, \mathbb{C}^*)$.

**Equivalences of graded extensions induced by an equivalence of the trivial piece.** Let $(c, M, A)$ be a $G$-graded extension of $B$, and $F$ a monoidal auto-equivalence of $B$. Using $F$ we can construct a new $G$-graded extension of $B$. We define

$$c^F := \text{inn}(B_F) \circ c,$$
$$M^F_{g,h} := (\text{Id}_{B_F} \boxtimes M_{g,h} \boxtimes \text{Id}_{F_B}) \circ (\text{Id}_{B_F} \boxtimes \text{Id}_{c_g} \boxtimes \text{ev}_{F_B} \boxtimes \text{Id}_{c_h} \boxtimes \text{Id}_{F_B}),$$
$$A^F_{f,g,h} := [\text{id}_{B_B} \boxtimes A_{f,g,h} \boxtimes \text{id}_{F_B}] [\text{id}_{B_B} \boxtimes \text{id}_{c_f} \boxtimes \text{id}_{c_B} \boxtimes \text{id}_{c_g} \boxtimes \text{id}_{c_{e,B}} \boxtimes \text{id}_{c_{B,B}} \boxtimes \text{id}_{F_B}].$$

We draw $M^F_{g,h}$ as:

![Diagram of $M^F_{g,h}$](image_url)

and $A^F_{f,g,h}$ as:

![Diagram of $A^F_{f,g,h}$](image_url)
For this triple to determine a $G$-graded extension we need the obstructions $T(c^F, M^F)$ and $v(c^F, M^F, A^F)$ to vanish. These obstructions vanish as $T(c, M)$ and $v(c, M, A)$ vanish, along with some string and sheet recoupling.

**Theorem 3.1.** Let $B$ a fusion category, $(c, M, A)$ a $G$-graded extension of $B$, and $F$ an auto-equivalence of $B$. Then there exists a graded equivalence $(c^F, M^F, A^F) \to (c, M, A)$.

**Proof.** We will apply our main theorem and produce a quadruple with vanishing obstructions. We choose

$$
F_e := F \\
\phi := \text{Id}_G \\
\hat{F}_g := \text{Id}_{B^F} \boxtimes \text{Id}_{c_g} \boxtimes \text{Id}_{F^B} \\
\hat{\tau}_{f,g} := (\text{id}_{\text{Id}_{B^F}} \boxtimes \text{id}_{M_{g,h}} \boxtimes \text{id}_{\text{Id}_{F^B}}) \circ (\text{id}_{\text{Id}_{B^F}} \boxtimes \text{id}_{c_f} \boxtimes \text{ev}_{\text{coev}_{F^B}} \boxtimes \text{id}_{c_g} \boxtimes \text{id}_{\text{Id}_{F^B}})
$$

Graphically we can draw $\hat{F}_g$ as:

```
B^F  c_g  F^B
```

and $\hat{\tau}_{f,g}$ as:

```
B^F  c_h  c_g  F^B
```

The condition that $c^F = \text{inn}(B^F_e) \circ \text{co} \circ \phi$ is satisfied by contruction. The obstruction $T(F_e, \phi, \hat{F})_{g,h}$ vanishes via a popping of a $F^B$ circle, and the cancellation of $M$ and its inverse. The obstruction $v(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$ vanishes via the cancellation of the $A$ and its inverse, and the popping of two $F^B$ bubbles. \qed

Recall that the motivating problem behind this paper was the seeming existence of too many categories with $D_{10}$ fusion rules via the extension theory of [3]. The above theorem fixes this problem, showing that for each $\text{Ad}(D_{10})$ fusion category, there are exactly 4 extensions with $D_{10}$ fusion rules. This is exactly the number we expect from the classification of small index subfactors.
We will describe this computation in detail when we compute the cyclic extensions of the even parts of the \textit{ADE} subfactors in later work.

**Auto-equivalences of bosionic and fermionic modular categories.** Let \((C, f)\) be a modular category \(C\) along with a distinguished boson or fermion \(f\). The category \(C\) has a \(\mathbb{Z}/2\mathbb{Z}\) grading, with
\[
C_0 := \{X \in C : \text{br}_{X,f} = \text{br}_{f,X}^{-1}\}
\]
\[
C_1 := \{X \in C : \text{br}_{X,f} = - \text{br}_{f,X}^{-1}\}.
\]
The modularity of \(C\) ensures that this grading is faithful.

The distinguished object \(f\) lives in \(C_0\), and lifts to \(Z(C_0)\) with the half-braiding \(\text{br}_{f,-}\). We aim to construct a graded auto-equivalence of \(C\) via \(f\) using our main theorem. We take \(F_e\) the trivial auto-equivalence of \(C_0\), and \(\phi\) the trivial automorphism of \(G\). Let \(F_1\) be the bimodule auto-equivalence of \(C_1\) defined by \(f \triangleright\?\). As \(F\) has to be an element of a torsor over \(H^1(\mathbb{Z}/2\mathbb{Z}, \text{Inv}(Z(C_0)))\), we need to verify that the function \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \text{Inv}(Z(C_0)) : 1 \times 1 \mapsto f\) is a cocycle.

**Lemma 3.2.** The action of the bimodule \(C_1\) on \(f\) is trivial.

**Proof.** This statement is equivalence to finding a natural isomorphism of bimodule functors \(? \odot f \to f \triangleright\?). Such a natural isomorphism is given by \(\text{br}_{f,-}\). \(\square\)

Thus the above function is a cocycle as \(f \otimes f \cong 1\).

As \((F_e, \phi, F)\) is the identity quasi-monoidal equivalence, acted on by a 2-cocycle, we have that \(T(F_e, \phi, F)\) vanishes.

We now have to choose \(\tau_{1,1} : \odot \circ (F_1 \boxtimes F_1) \to \odot\). We define \(\tau_{1,1}\) as the map induced by the balanced natural transformation
\[
\eta_{m \times n} := (\text{ev}_f \otimes \text{id}_m \otimes \text{id}_n)(\text{id}_z \otimes \text{br}_{m,f} \otimes \text{id}_n) : F_1(m) \otimes F_1(n) \to m \otimes n.
\]
For \(v(F_e, \phi, F, \tau)\) to vanish we have to verify the following braids are equal for \(m, n, p \in C_1\):

These two braids are equal as \(f\) is either a boson or fermion, so we can recouple the \(f\) string.

**Theorem 3.3.** Let \((C, f)\) be a modular category \(C\) along with a distinguished boson or fermion \(f\). There is a graded auto-equivalence \(F_z\) of \(C\) defined by
\[
F_z|_{C_0} := \text{Id}_{C_0},
\]
\[
F_z|_{C_1} := f \triangleright\?,
\]
with tensorator
\[
\tau_{m,n} := \begin{cases} 
\text{id}_m \otimes \text{id}_n & \text{if } m, n \in C_0, \\
\text{id}_f \otimes \text{id}_m \otimes \text{id}_n & \text{if } m \in C_1 \text{ and } n \in C_0, \\
\text{br}_{m,f} \otimes \text{id}_n & \text{if } m \in C_0 \text{ and } n \in C_1, \\
(\text{ev}_f \otimes \text{id}_m \otimes \text{id}_n)(\text{id}_z \otimes \text{br}_{m,f} \otimes \text{id}_n) & \text{if } m \in C_1 \text{ and } n \in C_1.
\end{cases}
\]

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Furthermore this auto-equivalence is braided if and only if $f$ is a fermion.

Proof. The only part left to prove is that $F_z$ is braided if and only if $f$ is a fermion. We directly compute for $m, n \in C_1$,

$$\tau_{n,m} \text{br}_{F(m),F(n)} = \begin{cases} 
-1 & f 
\end{cases} = \begin{cases} 
-t_f & f
\end{cases} = -t_f \text{br}_{F(m),n} \tau_{m,n}.$$

Thus $F$ is braided if and only if $t_f = -1$, i.e. $f$ is a fermion. \qed

Example 3.4. When $k$ is an even integer, the category $SU(2)_k$ is modular. This modular category always has a non-trivial invertible object $(k)$, which is a boson if $k \equiv 0 \pmod{4}$, and a fermion if $k \equiv 2 \pmod{4}$. Therefore an application of Theorem 3.3 gives auto-equivalences of these modular categories, which are non-trivial when $k > 2$. A simple fusion rule argument shows that these are the only auto-equivalences of $SU(2)_k$ for any $k$. Thus we get for $k > 2$:

| $k \pmod{4}$ | $\text{Aut}(SU(2)_k)$ | $\text{Aut}^{\text{br}}(SU(2)_k)$ |
|-------------|------------------------|------------------------|
| 0           | $\mathbb{Z}/2\mathbb{Z}$ | $\{e\}$               |
| 1           | $\{e\}$               | $\{e\}$               |
| 2           | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$|
| 3           | $\{e\}$               | $\{e\}$               |

This recovers the authors result from [1], with a much more elegant and simpler proof.

Given any super-modular category $C$, the Drinfeld centre $Z(C)$ is a modular category with distinguished fermion. Thus Lemma 3.3 gives a braided auto-equivalence of $Z(C)$, and thus an invertible bimodule $M$ of order 2 over $C$. There are canonical choices for the multiplication map $M \otimes M \to C$ such that the obstruction $T$ vanishes, and $H^4(Z/2\mathbb{Z}, C^\times)$ is trivial, so abstractly there exists an associator for this multiplication map. Therefore there exists a $Z/2\mathbb{Z}$-graded category $C \oplus M$.

Example 3.5. Consider the super-modular categories $SO(3)_{2\mathbb{N}+1}$. We can identify the $\mathbb{Z}/2\mathbb{Z}$ extensions for this family. They are the categories $(SO(3)_{2\mathbb{N}+1} \boxtimes \text{Ising})//\text{Rep}(\mathbb{Z}/2\mathbb{Z})$.

Question 3.6. Is the fusion category $C \oplus M$ always equivalent to $(C \boxtimes \text{Ising})//\text{Rep}(\mathbb{Z}/2\mathbb{Z})$?

Unfortunately this question seems difficult to answer, as we don’t have an explicit description of the bimodule $M$ in the general case.

4. Classification of graded extensions

As a warm up to the proof of our main theorem, we rederive the classification of graded extensions of a fusion category. Recall from [3] that $G$-graded extensions of $B$ are classified by

- a group homomorphism $c : G \to \text{BrPic}(B)$, such that a certain element $o_3(c) \in H^3(G, \text{Inv}(Z(B)))$ is trivial,
- an element $M$ of an $H^2(G, \text{Inv}(Z(B)))-$torsor, such that a certain element $o_4(c, M) \in H^3(G, \text{Inv}(Z(C)))$ is trivial,
- an element $A$ of an $H^3(G, C^\times)-$torsor.

The proof of our main theorem makes use of the techniques developed in the proof of this classification. To warm the reader up to these techniques, we quickly reprove this classification of Etingof, Nikshych, and Ostrik, putting an emphasis on string diagrams.
Let $c : G \to \text{BrPic}(B)$ be a group homomorphism. Then we can form the category

$$C := \bigoplus c_g.$$ 

We wish to make $C$ a graded quasi-monoidal category. Recall this is a category with a tensor product satisfying $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ via an unspecified isomorphism. As $c$ is a homomorphism there exists a collection of bimodule equivalences

$$M := \{ M_{g,h} : c_g \boxtimes c_h \to c_{gh} \} = \left\{ M_{g,h} \right\}.$$ 

Via Equation (2.1) the equivalences $M_{g,h}$ induce $B$-balanced bimodule functors $m_{g,h} : c_g \times c_h \to c_{gh}$. We define a tensor product on $C$ by:

$$X_g \otimes Y_h := m_{g,h}(X_g, Y_h).$$ 

For the category $C$ with tensor product coming from the collection $M$ to be quasi-monoidal we exactly need $B$-balanced isomorphisms:

$$m_{f,g,h} \circ (m_{f,g} \times \text{id}_{c_h}) \cong m_{f,gh} \circ (\text{id}_{c_f} \times m_{g,h}),$$

which via Equation (2.1) is equivalent to the existence of bimodule natural isomorphisms:

$$M_{f,g,h} \circ (M_{f,g} \boxtimes \text{id}_{c_h}) \cong M_{f,gh} \circ (\text{id}_{c_f} \boxtimes M_{g,h}).$$

Let

$$T(c, M)_{f,g,h} := \text{ev}_{c_{fg}^{op}} \circ ([M_{f,g,h} \circ (M_{f,g} \boxtimes \text{id}_{c_h}) \circ (\text{id}_{c_f} \boxtimes M_{g,h}^{-1}) \circ M_{f,gh}^{-1}] \boxtimes \text{id}_{c_{gh}^{op}}) \circ \text{ev}_{c_{gh}^{op}} : G \times G \times G \to \text{Inv}(Z(B)).$$

The category $C$ with tensor product coming from $M$ is quasi-monoidal if and only if $T(c, M)_{f,g,h} \cong \text{Id}_B$ for all $f, g, h \in G$. We call a collection of bimodule equivalences $M$ such that $T(c, M)$ vanishes a system of products for $c$. We consider systems of products for $c$ up to quasi-equivalence of the corresponding quasi-monoidal categories.

Graphically we draw:
We directly verify that $T(c, M)$ is a 3-cocycle using graphical calculus. At each stage in this calculation, we indicate with a red box the region we will modify. Each step consists of local applications of Lemma 2.3, and cancellation of terms.
To simplify notation and proofs for the remainder of the Section we introduce the following notation.

**Definition 4.1.** Given an element \( \rho \in C^2(G, \text{Inv}(Z(B))) \), and \( M \) a collection of bimodule equivalences. We define

\[
M^\rho := \{ \rho_{g,h} \boxtimes M_{g,h} : c_g \boxtimes c_h \to c_{gh} \} = \begin{cases} 
M_{g,h} & \rho_{g,h} \\
\rho_{g,h} & M_{g,h} \end{cases}
\]

The following Lemma gives a relation between \( T(c, M^\rho) \) and \( T(c, M) \) that we will repeatedly make use of.

**Lemma 4.2.** There exists an isomorphism of bimodule functors

\[
T(c, M^\rho) \cong \partial^3(\rho)_{f,g,h} T(c, M)_{f,g,h}.
\]

**Proof.** We compute using Lemma 2.3 that

\[
T(c, M^\rho)_{f,g,h} = \rho_{f,g,h} \rho_{f,g} \rho_{g,h}^{-1} c_f \rho_{f,g}^{-1} T(c, M).
\]
Suppose we have two collections of bimodule equivalences $M$ and $M'$. Then each $M'_{g,h}$ differs from $M_{g,h}$ by a bimodule auto-equivalence of $c_{gh}$, and thus by Lemma 2.2 we have $M'_{g,h} = L_{g,h}M_{g,h}$ for some $L_{g,h} \in \text{Inv}(Z(B))$. In the notation of Definition 4.1 we have $M' = M^L$. Applying Lemma 4.2 gives

$$T(c, M') = \partial^3(L)T(c, M).$$

Hence $T(c, M')$ differs from $T(c, M)$ by a coboundary in $Z^3(G, \text{Inv}(Z(B)))$. Thus the cohomology class of $T(c, M)$ is independent of the choice of $M$. This implies that

$$o_3(c) := T(c, -)$$

is an element of $H^3(G, \text{Inv}(Z(B)))$ that depends only on $c$.

**Lemma 4.3.** There exists a system of products for $c$ if and only if $o_3(c)$ is the trivial element of $H^3(G, \text{Inv}(Z(B)))$.

**Proof.** Choose $M$ an arbitrary collection of bimodule equivalences $\{M_{g,h} : c_g \boxtimes c_h \rightarrow c_{gh}\}$, which we can do as $c$ is a group homomorphism. As $o_3(c)$ is trivial in $H^3(G, \text{Inv}(Z(B)))$ we have that $T(c, M) = \partial^3(\rho)$ for some $\rho \in C^2(G, \text{Inv}(Z(B)))$. We choose a new collection of bimodule equivalences $M^{\rho^{-1}}$. An application of Lemma 4.2 gives

$$T(c, M^{\rho^{-1}}) = \partial^3(\rho^{-1})T(c, M) = \partial^3(\rho^{-1})\partial^3(\rho) = \text{Id}.$$ 

Thus $M^{\rho^{-1}}$ is a system of products for $c$.

Conversely, suppose that there exists a system of products $M$ for $c$. Then by definition $T(c, M)$ vanishes in $Z^3(G, \text{Inv}(Z(B)))$, and so $o_3(c)$ is trivial in $H^3(G, \text{Inv}(Z(B)))$. □

We now wish to classify all systems of products for $c$.

**Lemma 4.4.** Suppose $o_3(c)$ is trivial, then systems of products for $c$ form a torsor over $H^2(G, \text{Inv}(Z(B)))$.

**Proof.** Let $M$ be a system of products for $c$, and $\psi \in Z^2(G, \text{Inv}(Z(B)))$. Then $\psi$ acts on $M$ as in Definition 4.1. An application of Lemma 4.2 shows that

$$T(c, M^\psi) = \partial^3(\psi)T(c, M).$$

As $\psi$ is a 2-cocycle, $\partial^3(\psi)$ is trivial, and as $M$ is a system of products for $c$, $T(c, M)$ vanishes. Thus $T(c, M^\psi)$ also vanishes, and so $M^\psi$ is also a system of products for $c$. This shows that $Z^2(G, \text{Inv}(Z(B)))$ acts on system of products for $c$.

Let $M$ and $M'$ be two system of products for $c$. As explained in earlier discussion, each $M'_{g,h} = L_{g,h}M_{g,h}$ for some $L_{g,h} \in \text{Inv}(Z(B))$. Lemma 1.2 shows that

$$T(c, M') = \partial^3(L)T(c, M).$$

As both $T(c, M)$ and $T(c, M')$ vanish, we must have that $\partial^3(L)$ is trivial, therefore $L \in Z^2(G, \text{Inv}(Z(B)))$. This shows that the action of $Z^2(G, \text{Inv}(Z(B)))$ is transitive.

Let $M$ a system of products for $c$, and let $\psi \in Z^2(G, \text{Inv}(Z(B)))$ a coboundary. Then $\psi_{g,h}\rho_{g,h} = \rho_{g,h}c_{gh}$ for some $\rho \in C^1(G, \text{Inv}(Z(B)))$. We define an equivalence $F_\rho := \bigoplus \rho_{g} \text{Id}_{c_{gh}}$ between the
quasi-monoidal categories associated to \((c, M^\psi)\) and 
\((c, M)\). We compute

\[
M^\psi
\]

\[
\cong
\]

\[
\cong
\]

which implies \(F^\rho(X_g) \otimes F^\rho(Y_h) \cong F^\rho(X_g \otimes Y_h)\), and thus \(F^\rho\) is a quasi-monoidal equivalence. This shows the action of \(Z^2(G, \text{Inv}(Z(B)))\) descends to a well defined action of \(H^2(G, \text{Inv}(Z(B)))\).

Finally suppose there is a quasi-monoidal equivalence \(F\) from \(C\) with tensor product defined by \(M^\psi\), to \(C\) with tensor product defined by \(M\). Then \(F\) restricted to \(c_g\) gives an auto-equivalence of the bimodule \(c_g\), and thus by Lemma 2.2 \(F|_{c_g} = \rho_g \text{Id}_{c_g}\) for \(\rho_g \in \text{Inv}(Z(B))\). As \(F\) is quasi-monoidal, we have that

\[
M^\psi
\]

\[
\cong
\]

\[
\cong
\]

An application of Lemma 2.3 shows that \(\psi_{g,h} \rho_{g,h} = \rho_g \rho_h^{c_g}\). Thus \(\psi\) is a coboundary. This implies that the action of \(H^2(G, \text{Inv}(Z(B)))\) is free.

As \(\alpha_3(c)\) is trivial, we know from Lemma 4.3 that the set of system of products for \(c\) is non-empty. Therefore putting everything together we have that systems of products for \(c\) form a torsor over \(H^2(G, \text{Inv}(Z(B)))\).

Now that we have classified pairs \((c, M)\) giving quasi-monoidal categories, we wish to classify all associators for these quasi-monoidal categories. As \(M\) is a system of products for \(c\), by definition there exists a collection of isomorphisms:

\[
A := \{A_{f,g,h} : M_{f,g,h} \circ (M_{f,g} \boxtimes \text{id}_{c_h}) \cong M_{f,gh} \circ (\text{id}_{c_f} \boxtimes M_{g,h})\} = \begin{cases} C_f & \text{if } f \neq g, h \neq f, g \neq h \neq g, h \neq c, \text{otherwise} \end{cases}
\]

Applying Equation (2.1) to \(A\) we get \(B\)-balanced isomorphisms:

\[
a_{f,g,h} : m_{f,g,h} \circ (m_{f,g} \times \text{id}_{c_h}) \cong m_{f,gh} \circ (\text{id}_{c_f} \times m_{g,h}).
\]

We define an associator on the category with tensor product coming from \((c, M)\) by:

\[
\alpha_{X_f, Y_g, Z_h} := (a_{f,g,h})_{X_f, Y_g, Z_h}.
\]
For this associator to satisfying the pentagon axiom we exactly need the equality of $B$-balanced isomorphisms:
\[
[id_{mfghk} \circ (id_{Id_{f}} \times a_{gh,h,k})][a_{fg,h,k} \circ (id_{Id_{f}} \times id_{mg,h} \times id_{Id_{k}})][id_{mfgh,k} \circ (a_{fg,h} \times id_{Id_{k}})] =
[a_{fg,h,k} \circ (id_{Id_{f}} \times id_{Id_{g}} \times id_{mh,b})][a_{fg,h,k} \circ (id_{mg,b} \times id_{Id_{h}} \times id_{Id_{k}})].
\]

Via Equation (2.1) and a rearrangement, the above equality is equivalent to showing that the following bimodule functor natural automorphism:
\[
v(c, M, A)_{f,g,h,k} := [A_{fgh,k}^{-1} \circ (id_{M_{f}} \otimes id_{Id_{g}} \otimes id_{Id_{h,k}})](A_{fgh,k}^{-1} \circ id_{Id_{f}} \otimes id_{Id_{g}} \otimes id_{Id_{h,k}})]
\]
\[
(id_{M_{f}} \circ (id_{Id_{f}} \otimes A_{g,h,k})) [A_{fg,h,k} \circ (id_{Id_{f}} \otimes id_{M_{g,h}} \otimes id_{Id_{k}})] [id_{M_{fgh,k}} \circ (A_{fg,h} \otimes id_{Id_{k}})]
\]
is equal to the identity on the functor $H_{fgh,h,k} := M_{fgh,k} \circ (M_{fgh,h} \otimes id_{Id_{h}}) \circ (M_{f,g} \otimes id_{Id_{h}} \otimes id_{Id_{k}})$. A graphical description of $v(c, M, A)_{f,g,h,k}$ is given in Figure 2. We can collect a family of isomorphisms $A$ such that $v(c, M, A)$ vanishes, a system of associators for $(c, M)$. We consider system of associators for $(c, M)$ up to monoidal equivalence of the corresponding monoidal categories.

As $v(c, M, A)_{f,g,h,k}$ is a bimodule functor natural automorphism, we can identify it with a non-zero complex number. Thus $v(c, M, A)$ is a function $G \times G \times G \rightarrow \mathbb{C}^\times$. We can also identify $v(c, M, A)$ with a natural automorphism of $id_B$ in the following way:

\[
V(c, M, A)_{f,g,h,k} := ev^{-1}_{c_{fg,h,k}} [id_{ev^{-1}_{c_{fg,h,k}}} \circ (ev^{-1}_{c_{fg,h,k}} \otimes id_{Id_{f}} \otimes id_{Id_{g}} \otimes id_{Id_{h,k}})]
\]
\[
[id_{ev^{-1}_{c_{fg,h,k}}} \circ (ev^{-1}_{c_{fg,h,k}} \otimes id_{Id_{f}} \otimes id_{Id_{g}} \otimes id_{Id_{h,k}})] \circ (v(c, M, A)_{f,g,h,k} \otimes id_{ev^{-1}_{c_{fg,h,k}}}) \circ id_{ev^{-1}_{c_{fg,h,k}}}
\]
\[
[id_{ev^{-1}_{c_{fg,h,k}}} \circ (ev^{-1}_{c_{fg,h,k}} \otimes id_{Id_{f}} \otimes id_{Id_{g}} \otimes id_{Id_{h,k}})] \circ id_{ev^{-1}_{c_{fg,h,k}}} \circ ev^{-1}_{c_{fg,h,k}}.
\]

Graphically we can think of $V(c, M, A)_{f,g,h,k}$ as a compactification of the foam describing $v(c, M, A)_{f,g,h,k}$ in Figure 2. As a function $G \times G \times G \rightarrow \mathbb{C}^\times$, $V(c, M, A)$ is equal to $v(c, M, A)$.

We claim that $V(c, M, A)$ is a 4-cocycle. Showing that $V(c, M, A)_{g,h,k,i} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k}$ follows somewhat similar to the calculation showing $T(c, M)$ is a 3-cocycle, except now the proof is in 3 dimensions, so we need to recouple sheets instead of strings. The major difference in this calculation is that $V(c, M, A)_{g,h,k,i}$ needs to be wrapped in a $cf$ bubble to merge

\[
V(c, M, A)_{g,h,k,i} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k}
\]
into a single component via sheet recoupling. However this is easy to fix as $V(c, M, A)_{g,h,k,i}$ is a scalar, so we can pull $V(c, M, A)_{g,h,k,i}$ out of a $cf$ bubble using linearity, and then pop the bubble to see that $V(c, M, A)_{g,h,k,i}$ is equal to itself wrapped in a $cf$ bubble. The calculation now follows by merging both

\[
V(c, M, A)_{g,h,k,i} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k}
\]
and

\[
V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k} V(c, M, A)_{f,g,h,k}
\]
into single components, and rearranging to show they are equal. We omit the exact details of this calculation.
Figure 2. Graphical interpretation of \( v(c, M, A)_{f,g,h,k} \)
Suppose we choose some different collection of bimodule isomorphisms
\[ A'_{f,g,h} : M_{f,g,h} \circ (M_{f,g} \boxtimes \text{id}_{c}) \cong M_{f,g,h} \circ (\text{id}_c \boxtimes M_{g,h}). \]
Then each \( A'_{f,g,h} \) differs from \( A_{f,g,h} \) by a natural automorphism of \( c_{fg,h} \), and thus a non-zero complex number. Therefore \( A'_{f,g,h} = \lambda_{f,g,h} A_{f,g,h} \) for some \( \lambda_{f,g,h} \in \mathbb{C}^\times \). Using linearity it is easy to see that
\[ V(c, M, A')_{f,g,h,k} = \lambda_{g,h,k} \lambda_{f,g,h,k}^{-1} \lambda_{f,g,h,k}^{-1} V(c, M, A)_{f,g,h,k}. \]
Hence \( V(c, M, A') \) differs from \( V(c, M, A) \) by a coboundary in \( Z^4(G, \mathbb{C}^\times) \). Thus the cohomology class of \( V(c, M, A) \) is independent of the choice of \( A \). This implies that
\[ o_4(c, M) := V(c, M, -) \]
is an element \( H^4(G, \mathbb{C}^\times) \) that depends only on \( c \) and \( M \).

**Lemma 4.5.** There exists a system of associators for \((c, M)\) if and only if \( o_4(c, M) \) is the trivial element in \( H^4(G, \mathbb{C}^\times) \).

**Proof.** Suppose \( o_4(c, M) \) is trivial. As \( M \) is a system of products for \( c \), there exists some collection of bimodule isomorphisms \( A = \{ A_{f,g,h} : M_{f,g,h} \circ (M_{f,g} \boxtimes \text{id}_{c}) \cong M_{f,g,h} \circ (\text{id}_c \boxtimes M_{g,h}) \} \). As \( o_4(c, M) \) is trivial we know that \( v(c, M, A) = d^3(\rho) \) for some \( \rho \in C^3(G, \mathbb{C}^\times) \). We define a new collection of bimodule isomorphisms
\[ A_{f,g,h}^{-1} := \rho_{f,g,h}^{-1} A_{f,g,h}. \]
Using linearity it is easy to see that
\[ V(c, M, A^{-1})_{f,g,h,k} = \rho_{g,h,k}^{-1} \rho_{f,g,h,k}^{-1} \rho_{f,g,h,k} \rho_{f,g,h,k} V(c, M, A) = \text{id}_{\text{id}_B}. \]
Thus \( A^{-1} \) is a system of associators for \((c, M)\).

Conversely, suppose there exists a system of associators for \((c, M)\). Then by definition \( v(c, M, A) \) vanishes in \( Z^4(G, \mathbb{C}^\times) \), and so \( o_4(c, M) \) is trivial in \( H^4(G, \mathbb{C}^\times) \).

We now wish to classify all systems of associators for \((c, M)\).

**Lemma 4.6.** Suppose \( o_4(c, M) \) is trivial, then systems of associators for \((c, M)\) form a torsor over \( H^3(G, \mathbb{C}^\times) \).

**Proof.** Let \( A \) be such a system of associators for \((c, M)\), and \( \psi \in Z^3(G, \mathbb{C}^\times) \). We define new bimodule isomorphisms by:
\[ A^\psi_{f,g,h} := \psi_{f,g,h} A_{f,g,h}. \]
We compute using linearity that
\[ V(c, M, A^\psi)_{f,g,h,k} = \psi_{g,h,k} \psi_{f,g,h,k} \psi_{f,g,h,k}^{-1} \psi_{f,g,h,k}^{-1} V(c, M, A) = d^4(\psi)_{f,g,h,k} V(c, M, A) \]
which vanishes as \( A_{f,g,h} \) vanishes, and \( \psi \) is a 3-cocycle. Thus \( A^\psi \) is a system of associators for \((c, M)\). This shows that \( Z^3(G, \mathbb{C}^\times) \) acts on systems of associators for \((c, M)\).

Let \( A \) and \( A' \) be two systems of associators for \((c, M)\). We have that each \( A'_{f,g,h} = \psi_{f,g,h} A_{f,g,h} \) for some \( \psi_{f,g,h} \in \mathbb{C}^\times \), therefore \( A' = A^\psi \). The Equation (4.1), along with the fact that \( V(c, M, A) \) and \( V(c, M, A') \) vanish, implies that \( \psi \) is a coboundary. This shows that the action of \( Z^3(G, \mathbb{C}^\times) \) is transitive.

Let \( \psi \) be a coboundary in \( Z^3(G, \mathbb{C}^\times) \). Then \( \psi = \partial^3(\rho) \) for some \( \rho \in C^2(G, \mathbb{C}^\times) \). We define a monoidal equivalence \( F_\rho \) between the monoidal categories associated to \((c, M, A^\rho)\) and \((c, M, A)\).
As a functor $F_\rho$ is the identity and the tensorator is given by $\tau_{X_g,Y_h} := \rho_{g,h} \id_{X_g \otimes Y_h}$. To verify that $F_\rho$ is monoidal we compute

$$\alpha_{X_f,Y_g,Z_h} \tau_{f,g,h}[\tau_{X_f,Y_g} \otimes \id_{Z_h}] = (a_{f,g,h})_{X_f,Y_g,Z_h}[\rho_{f,g,h} \rho_{f,g} \id_{X_f \otimes Y_g \otimes Z_h}]$$

$$= (a_{f,g,h})_{X_f,Y_g,Z_h}[\rho_{f,g,h} \rho_{f,g} \id_{X_f \otimes Y_g \otimes Z_h}]$$

$$= [\rho_{g,h} \rho_{f,g} \id_{X_f \otimes Y_g \otimes Z_h}] \cdot \psi_{f,g,h}(a_{f,g,h})_{X_f,Y_g,Z_h}$$

$$= [\id_{X_f \otimes \tau_{g,h}}] \tau_{f,g,h} \alpha_{X_f,Y_g,Z_h}^{\psi}.$$

This shows that the action of $Z^3(G, \mathbb{C}^\times)$ descends to a well defined action of $H^3(G, \mathbb{C}^\times)$.

Finally suppose that there exists a monoidal equivalence $(F, \tau)$ from the graded monoidal category corresponding to $(c, M, A)$, to the graded monoidal category corresponding to $(c, M, A^\psi)$. The restriction of $\tau$ to $C_g \times C_h$ gives a bimodule automorphism of $\mathcal{M}_{g,h}$, and thus a non-zero complex number. As $(F, \tau)$ is monoidal we have

$$\tau|_{c_f,c_{gh}} \tau|_{c_g,c_h} \alpha_{f,g,h} = \alpha_{f,g,h} \tau|_{c_f,c_{gh}} \tau|_{c_f,c_g}.$$

Thus $\psi = \partial^3(\tau|_{-})$, so $\psi$ is a coboundary. This shows that the action of $H^3(G, \mathbb{C}^\times)$ is free.

As $\alpha_3(c, M)$ is trivial, we know from Lemma 4.5 that the set of system of associators for $(c, M)$ is non-empty. Therefore putting everything together we have that systems of associators for $(c, M)$ form a torsor over $H^3(G, \mathbb{C}^\times)$.\hfill \Box

We can now prove the classification of $G$-graded extensions of $B$. Given a $G$-graded extension of $B$, we can extract a group homomorphism $c : G \to \text{BrPic}(B)$, a system of products $M$ for $c$, and a system of associators $A$ for $(c, M)$. Conversely given this triple of data, we can reconstruct the $G$-extension as described earlier in this Section. Thus to classify $G$-extensions of $B$, we have to classify triples $(c, M, A)$.

Let $c$ be any group homomorphism $G \to \text{BrPic}(B)$, then Lemma 4.3 says we can pick a system of products for $c$ if and only if $\alpha_3(c)$ vanishes. As $\alpha_3(c)$ vanishes, Lemma 4.4 says that systems of products for $c$ form a torsor over $H^2(G, \text{Inv}(Z(B)))$. Let $M$ be a system of products for $c$, then Lemma 4.5 says we can pick a system of associators for $(c, M)$ if and only if $\alpha_4(c, M)$ vanishes. As $\alpha_4(c, M)$ vanishes, Lemma 4.6 says that systems of associators for $(c, M)$ form a torsor over $H^3(G, \mathbb{C}^\times)$.

This completes the classification of $G$-graded extensions of $B$, considered up to equivalence of extensions.

5. Twisted bimodule functors

In this section we introduce the abstract nonsense for twisted bimodule functors, and natural transformations between them. If the reader desires motivation for such abstract nonsense then we encourage them to skip to the next section, where the motivating example can be found.

**Definition 5.1.** Let $M$ and $N$ be bimodule categories over a fusion category $C$, and $(F, \tau)$ a monoidal auto-equivalence of $C$. A $F$-twisted bimodule functor $H : M \to N$ is an abelian functor, along with natural isomorphisms $L_{X,M} : F(X) \triangleright H(M) \to H(X) \triangleright M$ and $R_{M,X} : H(M) \rhd F(X) \to H(M \rhd X)$ satisfying:
Let $H : M \to N$ and $H' : M' \to N'$ be $F$-twisted bimodule functors. There is a functor $M \times M' \to N \boxtimes N'$ given by

$$B_{N,N'} \circ (H \times H').$$

This functor is $C$-balanced by the natural isomorphism

$$B_{N_1,N_2}(\text{id}_{H_1(m_1)} \times L_{X,m_2}^{-1}) \cdot b_{H_1(m_1),F(X),H_2(m_2)}^{B_{N_1,N_2}} \cdot B_{N_1,N_2}(R_{m_1,X}^H \times \text{id}_{H_2(m_2)}).$$

Thus via Equation (2.1), the above functor induces a functor $H_1 \boxtimes H_2 : M \boxtimes M' \to N \boxtimes N'$.

One can also compose twisted bimodule functors.
**Definition 5.3.** Let $H_1 : N \to M$, and $H_2 : M \to P$ be $F_1$ and $F_2$-twisted bimodule functors respectively. We can compose $H_1$ and $H_2$ as follows:

$$H_2 \circ H_1(m) := H_2(H_1(m)),$$

$$L_{X,m}^{H_2 \circ H_1} := H_2(L_{X,m}^{H_1}) \cdot L_{F_1(X),H_1(m)}^{H_2},$$

$$R_{m,X}^{H_2 \circ H_1} := H_2(R_{m,X}^{H_1}) \cdot R_{H_1(m),F_1(X)}^{H_2},$$

to obtain a $(F_2 \circ F_1)$-twisted bimodule functor.

We now define natural transformations between $F$-twisted bimodule functors. Again this is a straightforward generalisation of the untwisted case.

**Definition 5.4.** Let $H_1$ and $H_2$ be $F$-twisted bimodule functors with the same source and target. An $F$-twisted natural transformation $\mu : H_1 \to H_2$ is a natural transformation satisfying:

$$F(X) \triangleright H_1(m) \xrightarrow{id_{F(X)} \circ \mu m} F(X) \triangleright H_2(m) \quad H_1(m) \triangleleft F(X) \xrightarrow{\mu m \circ id_{F(X)}} H_2(m) \triangleleft F(X)$$

These $F$-twisted natural transformations can be composed with other $F$-twisted natural transformations, and with other $F'$-twisted natural transformations, given $F' \circ F$ is defined. The details of these operations are straightforward so we omit the details. The tensor product of two $F$-twisted natural transformations is more difficult so we include the details.

**Definition 5.5.** Let $H_1, H'_1 : M_1 \to N_1$, and $H_2, H'_2 : M_2 \to N_2$ be $F$-twisted bimodule functors, and $\mu_1 : H_1 \to H'_1$ and $\mu_2 : H_2 \to H'_2$ be $F$-twisted natural transformations. Consider the natural isomorphism

$$id_{B_{H'_1,H'_2}} \circ (\mu_1 \times \mu_2) : B_{N_1,N_2} \circ (H_1 \times H_2) \to B_{N_1,N_2} \circ (H'_1 \times H'_2).$$

It can be checked that this natural isomorphism is balanced, and thus via Equation (2.1) induces a natural transformation

$$\mu_1 \boxtimes \mu_2 : H_1 \boxtimes H_2 \to H'_1 \boxtimes H'_2.$$

When $F$ is the identity functor on $C$ we can assemble the above information into the 3-category $\text{BrPic}(C)$. When $F$ is non-trivial one wonders what higher categorical structure the above information fits into. Unfortunately we haven’t been able to provide a satisfactory answer to this question. The best we can guess is some sort of higher bimodule category over $\text{BrPic}(C)$. We haven’t put so much thought into this question due to the following theorem which shows that all the above information for twisted bimodule functors is actually contained in $\text{BrPic}(C)$, though in a not so obvious manner. The following Theorem begs to be a functor of 3-categories, with $\Omega^\boxtimes$ and $\Omega^\circ$ playing the role of the higher tensorators. However without an adequate description of the '3-category of twisted bimodule functors', it is impossible to state the theorem as such. We are forced to present the Theorem in its current messy state, which is nevertheless adequate for our uses later in this paper.

**Theorem 5.6.** Let $C$ a fusion category, and $F$ a monoidal auto-equivalence of $C$. Then

1. $F$-twisted bimodule functors $H : M \to N$ are in bijection (up to natural isomorphism) with untwisted bimodule functors $$\tilde{H} : C_F \boxtimes M \boxtimes F C \to N.$$
(2) Suppose \( H_1 : M_1 \to N_1 \) and \( H_2 : M_2 \to N_2 \) are \( F \)-twisted bimodule functors, then there exists a natural isomorphism of bimodule functors:

\[
\Omega^\otimes_{H_1,H_2} : H_1 \boxtimes H_2 \circ (\text{Id}_{C_F} \boxtimes \text{Id}_{N_1} \boxtimes \text{ev}_{C_F} \boxtimes \text{Id}_{N_2} \boxtimes \text{Id}_{F_C}) \to (\hat{H}_1 \boxtimes \hat{H}_2).
\]

(3) Suppose \( H_1 : M \to N \) is a \( F_1 \)-twisted bimodule functor and \( H_2 : N \to P \) is a \( F_2 \)-twisted bimodule functor, then there exists a natural isomorphism of bimodule functors:

\[
\Omega^\otimes_{H_1,H_2} : H_2 \circ \hat{H}_1 \circ (S_{F_2,F_1} \boxtimes \text{Id}_{M} \boxtimes S^*_{F_1,F_2}) \simeq \hat{H}_2 \circ (\text{Id}_{C_F} \boxtimes \hat{H}_1 \boxtimes \text{Id}_{F_C}),
\]

where \( S_{F_2,F_1} : C_{F_2} \boxtimes C_{F_1} \to C_{F_2 \circ F_1} \) and \( S^*_{F_1,F_2} : F_1 \boxtimes F_2 \to F_{2 \circ F_1} \) are certain bimodule functors.

(4) Natural isomorphisms of \( F \)-twisted bimodule functors \( \mu : H_1 \to H_2 \) are in bijection with natural isomorphisms of untwisted bimodule functors \( \hat{\mu} : \hat{H}_1 \to \hat{H}_2 \).

(5) If \( \mu_1 : H_1 \to H_1', \mu_2 : H_2 \to H_2' \) are natural isomorphisms of \( F \)-twisted bimodule functors \( H_1, H_1' : M_1 \to N_1 \) and \( H_2, H_2' : M_2 \to N_2 \) then

\[
\Omega^\otimes_{H_1,H_2} [\mu_1, \mu_2 ] (\text{Id}_{id_{C_F}} \boxtimes \text{Id}_{id_{N_1}} \boxtimes \text{id}_{ev_{C_F}} \boxtimes \text{Id}_{id_{N_2}} \boxtimes \text{Id}_{id_{F_C}}) = [\hat{\mu}_1, \hat{\mu}_2] \Omega^\otimes_{\hat{H}_1,\hat{H}_2}.
\]

(6) Let \( H_1, H_1' : M \to N \) be \( F_1 \)-twisted bimodule functors, and \( H_2, H_2' : N \to P \) be \( F_2 \)-twisted bimodule functors. Let \( \mu_1 : H_1 \to H_1' \) be a \( F_1 \)-twisted natural transformation, and \( \mu_2 : H_2 \to H_2' \) be a \( F_2 \)-twisted natural transformation. Then

\[
\Omega^\otimes_{H_1,H_2} [\mu_2, \mu_1 ] (\text{Id}_{S_{F_2,F_1}} \boxtimes \text{Id}_{M} \boxtimes \text{Id}_{S^*_{F_1,F_2}}) = [\hat{\mu}_2, \hat{\mu}_1] (\text{Id}_{id_{C_F}} \boxtimes \hat{\mu}_1 \boxtimes \text{Id}_{id_{F_C}}) \Omega^\otimes_{\hat{H}_1,\hat{H}_2}.
\]

(7) If \( \mu_1 : H \to H' \) and \( \mu_2 : H' \to H'' \) are natural isomorphisms of \( F \)-twisted bimodule functors then

\[
\mu_2 \hat{\mu}_1 = \hat{\mu}_2 \mu_1.
\]

Proof. The proof of this Theorem can be found in Appendix \[A\]. \( \square \)

6. DATA DETERMINED BY A GRADED EQUIVALENCE OF GRADED CATEGORIES

Let \((c, M^C, A^C)\) and \((d, M^D, A^D)\) be \(G\)-graded extensions of \(B\), as described in \[3\] or Section \[4\]. Let \((F, \mu)\) be a graded monoidal equivalence between these two categories. Let us see what data we can extract from \((F, \mu)\).

The first piece of data we find is an auto-equivalence of the trivial piece \(B\). This comes from the restriction of \((F, \mu)\) to \(B\). As we have required \((F, \mu)\) to be a graded equivalence, this restriction is a monoidal auto-equivalence of \(B\). We call this tensor auto-equivalence \(F_e\).

The second piece of data is an automorphism of the group \(G\). Let \(X_g\) be your favourite simple object in \(C_g\). The image of this object under \(F\) is simple and lives in some graded component of \(D\). As \(F\) is graded the resulting component was independent of choice of your favourite object. Thus \(F\) determines a map from the set of \\(\{C_g\}\) to the set of \\(\{D_g\}\), and hence a map \(\phi : G \to G\) satisfying \(C_{\phi(g)} \simeq D_g\). It can be checked that this map has the structure of a group automorphism.

The third piece of data is a collection of untwisted bimodule functors \(\hat{F}\). The restriction of \(F\) to \(C_{\phi(g)}\) gives a functor

\[
F_g : C_{\phi(g)} \xrightarrow{\sim} D_g.
\]

(6.1)
The structure map $\mu$ equips $F_g$ with the structure of an $F_e$-twisted bimodule equivalence. An application of Theorem 5.6 then gives a collection of untwisted bimodule functors:

$$
\hat{F} := \{ \hat{F}_g : B_{F_e} \boxtimes C_{\phi(g)} \boxtimes F_e \to D_g \}.
$$

The final piece of data is a collection of untwisted bimodule natural isomorphisms $\hat{\tau}$. The restriction of the natural isomorphism $\mu$ onto the graded pieces $C_g$ and $C_h$ gives isomorphisms of $F_e$-twisted bimodule functors:

$$
\mu_{g,h} : M_{g,h}^D \circ (F_g \boxtimes F_h) \xrightarrow{\sim} F_{gh} \circ M_{\phi(g),\phi(h)}^C.
$$

An application of Theorem 5.6 then gives a collection of untwisted bimodule natural isomorphisms:

$$
\hat{\tau} := \{ \hat{\tau}_{g,h} \} = \{ \}
$$

The tensorator axiom implies the following equality of $F_e$-twisted natural isomorphisms

$$
[\mu_{f,gh} \circ (\id_{\boxtimes F_{\phi(f)}} \boxtimes \id_{M_{\phi(g),\phi(h)}^C})][\id_{M_{f,g,h}} \circ (\id_{\boxtimes F_{\phi(g)}} \boxtimes \id_{M_{\phi(h),\phi(h)}^C})][A_{f,g,h}^D \circ (\id_{\boxtimes F_{\phi(h)}} \boxtimes \id_{M_{\phi(g),\phi(h)}^C})] = [A_{\phi(f),\phi(g),\phi(h)}^C][\id_{M_{\phi(f),\phi(g)}} \boxtimes \id_{M_{\phi(h),\phi(h)}^C}] \circ \mu_{f,g,h} \circ (\id_{\boxtimes F_{\phi(h)}} \boxtimes \id_{M_{\phi(g),\phi(h)}^C}),
$$

which via a rearrangement, and an application of Theorem 5.6 is equivalent to the bimodule natural automorphism $v(F_e, \phi, \hat{F}_g, \hat{\tau}_{g,h})$ defined in Figure 1 being the trivial automorphism of

$$
H_{f,g,h} := \{ \}
$$

for all $f, g, h \in G$.

In summary the auto-equivalence $(F, \mu)$ determines the following quadruple of data:

1. A monoidal auto-equivalence $F_e : B \to B,$
2. a group automorphism $\phi : G \to G,$
(3) a collection \( \hat{F} \) of bimodule functors,
(4) a collection \( \hat{\tau} \) of bimodule natural isomorphisms, such that \( v(F_e, \phi, \hat{F}, \hat{\tau}) \) is trivial.

Conversely, given any quadruple \((F_e, \phi, \hat{F}, \hat{\tau})\) of the above data, we can do the above construction in reverse to construct a graded monoidal equivalence. Thus graded monoidal equivalences between the monoidal graded categories corresponding to \((c, M^C, A^C)\) and \((d, M^D, A^D)\) exactly correspond to such quadruples. Our goal for the remainder of the paper is to classify such quadruples.

**Remark 6.1.** Graded monoidal equivalences also correspond to quadruples \((F_e, \phi, F, \mu)\), where \(F\) is a collection of \(F_e\)-twisted bimodule functors as in Equation (6.1), and \(\mu\) is a collection of \(F_e\)-twisted bimodule natural isomorphisms as in Equation (6.3), satisfying Equation (6.5). However these quadruples are much harder to classify, so we restrict our attention to the untwisted quadruples for this paper.

**Composition of quadruples.** Suppose we have two quadruples \((F_e, \phi, \hat{F}, \hat{\tau})\) and \((F'_e, \phi', \hat{F}', \hat{\tau}')\) corresponding to graded equivalences \(\bigoplus c_g \to \bigoplus d_g\) and \(\bigoplus d_g \to \bigoplus e_g\) respectively. Clearly we can compose the equivalences to obtain an equivalence \(\bigoplus c_g \to \bigoplus e_g\). The natural question to ask is what does the quadruple look like for this resulting equivalence. By reconstructing the equivalences from the quadruples, composing, then extracting the resulting quadruple we can arrive at an answer. However the resulting quadruple is extremely complicated so we omit the details. For twisted quadruples as in Remark 6.1 the answer is simpler. Let \((F_e, \phi, F, \mu)\) and \((F'_e, \phi', F', \mu')\) be two such twisted quadruples, with appropriate sources and targets, then the resulting twisted quadruple of their composition is

\[
(F'_e, \phi', F'_g, \mu'_{g,h}) \circ (F_e, \phi, F_g, \mu_{g,h}) = (F'_e \circ F_e, \phi' \circ \phi, F'_g \circ F_g, [\mu'_{g,h} \circ (\text{id}_{F_g} \boxtimes \text{id}_{F_e})]]([\text{id}_{F'_e} \boxtimes \text{id}_{F'_g}])].
\]

7. **Existence and classification of graded quasi-monoidal equivalences**

The goal for this section is the intermediate step of classifying graded quasi-monoidal equivalences between our two graded monoidal categories \((c, M^C, A^C)\) and \((d, M^D, A^D)\). Recall a quasi-monoidal equivalence \(J\) is an equivalence that admits an (unspecified) isomorphism \(J(X) \otimes J(Y) \to J(X \otimes Y)\) not satisfying any additional conditions. From the discussion of Section 6 this is equivalent to classifying triples \((F_e, \phi, \hat{F})\) such that there exists a collection \(\hat{\tau}\) not satisfying any additional conditions. For a fixed \(F_e\) and \(\phi\) we call such a collection \(\hat{F}\) a system of equivalences for \((F_e, \phi)\). We consider systems of equivalences for \((F_e, \phi)\) up to quasi-monoidal isomorphism of the corresponding quasi-monoidal functors. Explicitly this means that two systems of equivalences \(\hat{F}\) and \(\hat{F}'\) for \((F_e, \phi)\) are the same if and only if there exists a collection of natural isomorphisms of plain functors \(\{\mu_g : \hat{F}_g \to \hat{F}'_g\}\).

Let \(\rho \in C^1(G, \text{Inv}(Z(B)))\), and \(\hat{F}\) a collection of bimodule equivalences as in Equation (6.2). We define

\[
(7.1) \quad \hat{F}^\rho := \{\rho_g \boxtimes \hat{F}_g\} = \left\{ \begin{array}{c} \hat{F}_g \\ \rho_g \\ \delta_g \end{array}, \begin{array}{c} B_{F_e} \to \hat{F}_g, C_{\phi(g)} \to \rho_g \end{array} \right\}.
\]
Let

\[ T(F_e, \phi, \hat{F})_{g,h} := ev_{D_{gh}^{op}} \circ [M_{g,h}^D \circ (\hat{F}_g \Box \hat{F}_h) \circ (\text{Id}_{B_{F_e}} \Box \text{Id}_{C_{\phi(g)}} \Box ev_{B_{F_e}} \Box \text{Id}_{C_{\phi(h)}} \Box \text{Id}_{F_e B}) \]
\[ \circ (\text{Id}_{B_{F_e}} \Box M_{\phi(g), \phi(h)}^{-1} \Box \text{Id}_{F_e B}) \circ \hat{F}_{gh}^{-1} \Box \text{Id}_{D_{gh}^{op}}] \circ ev_{D_{gh}^{op}}^{-1} : G \times G \to \text{Inv}(Z(B)). \]

The collection \( \hat{F} \) is a system of equivalences for \((F_e, \phi)\) if and only if \( T(F_e, \phi, \hat{F})_{g,h} \) vanishes for all \( g, h \in G \). The function \( T(F_e, \phi, \hat{F}) \) only depends on the choice of \( F_e, \phi, \) and \( \hat{F} \), and not the choice of evaluation maps. Graphically we draw:

![Diagram](image)

We can directly verify that \( T(F_e, \phi, \hat{F}) \) is a 2-cocycle. At each stage in this calculation, we indicate with a red box the region we will modify. Each step consists of local applications of Lemma 2.3 and cancellation of terms. At one step we apply the associativity of \( M^C \) and \( M^D \).
\[ T(F_e, \phi, \widehat{F})_{g,h}^{d_f} T(F_e, \phi, \widehat{F})_{f,gh} = \]

\[ \sim \]

\[ c F_g c F_h \]

\[ M C \phi (g); \phi (h) - 1 \]

\[ M D g; h \]

\[ d F - 1 \]

\[ g h c F_f d F_{gh} \]

\[ M C \phi (f); \phi (gh) - 1 \]

\[ M D f; gh \]

\[ d F f g h - 1 \]

\[ \sim \]

\[ c F_f M C \phi (fg); \phi (h) - 1 \]

\[ M D fg; h \]

\[ c F_g c F_h \]

\[ M C \phi (f); \phi (g) - 1 \]

\[ B F_e c \phi (fg) F_e B \]

\[ \sim \]

\[ c F_f M D fg; h c F_g c F_h \]

\[ M C \phi (f); \phi (g) - 1 \]

\[ B F_e c \phi (fg) F_e B \]

\[ \sim \]
Suppose we have two collections of bimodule equivalences \( \hat{F} \) and \( \hat{F}' \). Then each \( \hat{F}'_g \) differs from \( \hat{F}_g \) by post-composition by an bimodule auto-equivalence of \( D_g \). Applying Lemma 2.2 gives
\[
\hat{F}'_g \cong L_g \hat{F}_g \quad \text{for some} \ L_g \in \text{Inv}(Z(B)).
\]
Thus we have \( \hat{F}' = \hat{F}^L \).

**Lemma 7.1.** For any \( \rho \in C^1(G, \text{Inv}(Z(B))) \), there exists an isomorphism of bimodule functors
\[
T(F_e, \phi, \hat{F}_g^\rho)_{g,h} \cong \partial^2(\rho)T(F_e, \phi, \hat{F})_{g,h}.
\]

**Proof.** We compute using Lemma 2.3.
Applying Lemma 7.1 shows $T(F_e, \phi, \widehat{F}')$ and $T(F_e, \phi, \widehat{F})$ differ by a coboundary. Hence $o_2(F_e, \phi) := T(F_e, \phi, -)$ is an element of $H^2(G, \text{Inv}(Z(B)))$ that depends only on the choice of $F_e$ and $\phi$.

**Lemma 7.2.** For fixed $F_e$ and $\phi$, there exists a system of equivalences for $(F_e, \phi)$ if and only if $\text{inn}(B_{F_e}) \circ c \circ \phi = d$ and $o_2(F_e, \phi)$ is the trivial element of $H^2(G, \text{Inv}(Z(B)))$.

**Proof.** Suppose there exists a system of equivalences $\widehat{F}$ for $(F_e, \phi)$. By definition $T(F_e, \phi, \widehat{F})$ is trivial in $Z^2(G, \text{Inv}(Z(B)))$, and therefore $o_2(F_e, \phi)$ is the trivial element of $H^2(G, \text{Inv}(Z(B)))$. As each $\widehat{F}_g$ is a bimodule equivalence $B_{F_e} \boxtimes c_{\phi(g)} \boxtimes F_e B \rightarrow d_g$ we have that $\text{inn}(B_{F_e}) \circ c \circ \phi = d$.

Conversely, suppose $\text{inn}(B_{F_e}) \circ c \circ \phi = d$, then we can make an arbitrary choice of bimodule equivalences $\widehat{F}_g : C_{\phi(g)} \rightarrow D_g$. As $o_2(F_e, \phi)$ is trivial in $H^2(G, \text{Inv}(Z(B)))$ we have that $T(F_e, \phi, \widehat{F})$ is a coboundary $\partial^2(L)$. Using Lemma 7.1 we have

$$T(F_e, \phi, \widehat{F}^{L^{-1}}) \cong \partial^2(L^{-1}) \partial^2(L) = \text{Id}.$$ 

Thus $\widehat{F}^{L^{-1}}$ is a system of equivalences for $(F_e, \phi)$. □

Now that we have shown exactly when we can choose a system of equivalences for $(F_e, \phi)$, we aim to classify all system of equivalences for $(F_e, \phi)$. To achieve this we introduce the group

$$D^1(G, \text{Inv}(Z(B))) := \{ \rho \in Z^1(G, \text{Inv}(Z(B))) : \rho_g \boxtimes \text{Id}_{d_g} \cong \text{Id}_{d_g} \} \text{ as plain functors}\}.$$ 

**Lemma 7.3.** For fixed $F_e$ and $\phi$, such that $\text{inn}(B_{F_e}) \circ c \circ \phi = d$ and $o_2(F_e, \phi)$ is trivial, systems of equivalences for $(F_e, \phi)$ form a $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$ torsor.

**Proof.** Let $\rho \in Z^1(G, \text{Inv}(Z(B)))$, and $\widehat{F}$ a system of equivalences for $(F_e, \phi)$. Then $\rho$ acts on $\widehat{F}$ as in Equation (7.1). As $\rho$ is a 1-cocycle, $\partial^2(\rho)$ is trivial, and thus Lemma 7.1 implies that $T(F_e, \phi, \widehat{F}^\rho) = T(F_e, \phi, \widehat{F})$. As $\widehat{F}$ is a system of equivalences for $(F_e, \phi)$ we know that $T(F_e, \phi, \widehat{F})$ vanishes, and thus $T(F_e, \phi, \widehat{F}^\rho)$ also vanishes. This shows that $Z^1(G, \text{Inv}(Z(B)))$ acts on systems of equivalences for $(F_e, \phi)$.

Let $\widehat{F}$ and $\widehat{F}'$ be two systems of equivalences for $(F_e, \phi)$. As explained earlier, $\widehat{F}' = L_g \widehat{F}_g$ for some $L_g \in \text{Inv}(Z(B))$, therefore $\widehat{F}' = \widehat{F}^L$. Using Lemma 2.3 we get that $T(F_e, \phi, \widehat{F}') = \partial^2(L)T(F_e, \phi, \widehat{F})$. As $\widehat{F}$ and $\widehat{F}'$ are both systems of equivalences for $(F_e, \phi)$, we have by definition that $T(F_e, \phi, \widehat{F}')$ and $T(F_e, \phi, \widehat{F})$ vanish. Thus $\partial^2(L)$ is trivial, and so $L$ is a 1-cocycle. This shows the action of $Z^1(G, \text{Inv}(Z(B)))$ is transitive.

Let $\widehat{F}$ be a system of equivalences for $(F_e, \phi)$, and let $\rho \in Z^1(G, \text{Inv}(Z(C)))$ such that $\rho_g \boxtimes \text{Id}_{d_g} \cong \text{Id}_{d_g}$ as plain functors. Let $\nu_g$ be the natural isomorphism $\rho_g \boxtimes \text{Id}_{d_g} \cong \text{Id}_{d_g}$, then $\nu_g \circ \text{id}_{\widehat{F}_g}$ is a natural isomorphism of plain functors $\widehat{F}_g^\rho \rightarrow \widehat{F}_g$. Thus the system of equivalences $\widehat{F}$ and $\widehat{F}^\rho$ are equivalent. This shows the action of $Z^1(G, \text{Inv}(Z(B)))$ descends to well defined action of $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$.

Let $\widehat{F}$ and $\widehat{F}'$ be equivalent systems of equivalences for $(F_e, \phi)$. By definition there exist natural isomorphisms of plain functors $\mu_g : \widehat{F}_g^\rho \rightarrow \widehat{F}_g$. Then $\mu_g \circ \text{id}_{\widehat{F}_g^\rho}$ gives a system of equivalences $\widehat{F}_g^\rho \rightarrow \widehat{F}_g$. This shows that action of $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$ is free.

Putting the four above statements together we have shown that systems of equivalences for $(F_e, \phi)$ form a $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$ torsor. □
It is very unexpected that system of equivalences form a $Z^1(G, \text{Inv}(Z(B)))/D^1(G, \text{Inv}(Z(B)))$ torsor rather than a $H^1(G, \text{Inv}(Z(B)))$ torsor. To convince the reader that these groups are different, we present the following example.

**Example 7.4.** Consider the pointed category $\text{Vec}(D_{2,3})$ with trivial associator. Recall $D_{2,3}$ is the semi-direct product of $Z/3Z$ by $Z/2Z$. Thus the category $\text{Vec}(D_{2,3})$ is a $Z/2Z$-extension of $\text{Vec}(Z/3Z)$, with the non-trivially graded piece the bimodule $\text{Vec}(Z/3Z)$ twisted by the order two auto-equivalence of $\text{Vec}(Z/3Z)$. Using [8 Proposition 2.3] we can compute that there are exactly 3 graded monoidal auto-equivalences of $\text{Vec}(D_{2,3})$ that are trivial on the trivial piece of this grading. All three of these equivalences are different as fusion ring automorphisms, and thus as quasi-monoidal auto-equivalences of $\text{Vec}(Z/3Z)$. Therefore for $F_e := \text{Id}_{\text{Vec}(Z/3Z)}$ and $\hat{\phi} := \text{id}_{Z/2Z}$, there are 3 systems of equivalences.

The category $\text{Vec}(Z/3Z)$ admits a modular braiding [10], therefore the centre of $\text{Vec}(Z/3Z)$ is $\text{Vec}(Z/3Z) \boxtimes \text{Vec}(Z/3Z)^{bop}$, and so the group of invertible elements of $Z(\text{Vec}(Z/3Z))$ is $Z/3Z \times Z/3Z$. The braided auto-equivalence of $\text{Vec}(Z/3Z) \boxtimes \text{Vec}(Z/3Z)^{bop}$ corresponding to the non-trivially graded piece of $\text{Vec}(D_{2,3})$ is the order 2 auto-equivalence of $\text{Vec}(Z/3Z)$ acting on both summands of the Deligne product. Thus the action of $Z/2Z$ on $Z/3Z \times Z/3Z$ is the map that sends each element to its inverse. With this information we compute:

$$Z^1(Z/2Z, \text{Inv}(Z(\text{Vec}(Z/3Z)))) = \{(g, h) \in Z/2Z \times Z/3Z : (g, h)^M = (1, 1)\} = Z/2Z \times Z/3Z,$$

$$B^1(Z/2Z, \text{Inv}(Z(\text{Vec}(Z/3Z)))) = \{(g, h)^2 \text{ for } (g, h) \in Z/2Z \times Z/3Z\} = Z/2Z \times Z/3Z,$$

$$D^1(Z/2Z, \text{Inv}(Z(\text{Vec}(Z/3Z)))) = \{(g, h) \in Z/3Z \times Z/3Z : gh = 1\} = Z/3Z.$$

This example shows that $H^1(G, \text{Inv}(Z(B)))$ is too small to be the torsor group for system of equivalences, and hopefully helps convince the reader that given torsor group is the correct choice.

8. **Existence and classification of tensorators for graded quasi-monoidal equivalences**

Now that we have classified triples $(F_e, \phi, \hat{F})$ admitting a collection of bimodule natural isomorphisms $\hat{\tau}$ as in Equation (6.4), we aim in this section to show when such a collection satisfies Equation (1) and thus gives a graded monoidal equivalence. We call such a collection a **system of tensorators** for $(F_e, \phi, \hat{F})$. We consider system of tensorators up to monoidal natural isomorphism of the corresponding monoidal functors. Explicitly this means that we consider two systems of tensorors $\hat{\tau}$ and $\hat{\tau}'$ the same if and only if there exists a collection $\mu_g \in \mathbb{C}^\times$, such that $\mu_{g,h} \hat{\tau}_{g,h} = \mu_g \mu_h \hat{\tau}'_{g,h}$.

As $v(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$ is a natural automorphism of $H_{f,g,h}$ we can identify it with a non-zero complex number. Thus $v(F_e, \phi, \hat{F}, \hat{\tau})$ is a function $G \times G \times G \to \mathbb{C}^\times$. We can also identify $v(F_e, \phi, \hat{F}, \hat{\tau}_{g,h})_{f,g,h}$ with a natural automorphism of $\text{Id}_B$ in the following way:

$$V(F_e, \phi, \hat{F}, \hat{\tau}_{g,h})_{f,g,h} := \text{ev}_{\text{ev}^{-1}_D} [\text{id}_{\text{ev}^{-1}_D} \circ (\text{id}_{H_{f,g,h}^{-1}} \boxtimes \text{id}_{\text{Id}^{-1}_D}) \circ \text{id}_{\text{ev}^{-1}_D}]_{\text{D}_{fgh}}$$

$$\text{ev}_{\text{ev}^{-1}_D} [\text{id}_{\text{ev}^{-1}_D} \circ (\text{id}_{H_{f,g,h}^{-1}} \boxtimes \text{id}_{\text{Id}^{-1}_D}) \circ (v(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h} \boxtimes \text{id}_{\text{Id}^{-1}_D}) \circ \text{id}_{\text{ev}^{-1}_D}]_{\text{D}_{fgh}}$$

$$\text{ev}_{\text{ev}^{-1}_D} [\text{id}_{\text{ev}^{-1}_D} \circ (\text{id}_{H_{f,g,h}^{-1}} \boxtimes \text{id}_{\text{Id}^{-1}_D}) \circ \text{id}_{\text{ev}^{-1}_D}]_{\text{D}_{fgh}} \circ \text{ev}_{\text{ev}^{-1}_D}. $$

Graphically we can think of $V(F_e, \phi, \hat{F}, \hat{\tau}_{g,h})_{f,g,h}$ as a compactification of the foam describing $v(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$ in Figure 1. As functions $G \times G \times G \to \mathbb{C}^\times$, $v(F_e, \phi, \hat{F}, \hat{\tau})$ and $V(F_e, \phi, \hat{F}, \hat{\tau})$ are equal.
We claim that $V(F_e, \phi, \hat{F}, \hat{\tau})$ is a 3-cocycle. Showing that

$$V(F_e, \phi, \hat{F}, \hat{\tau})_{g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h} = V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$$

follows somewhat similar to the calculation showing $T(F_e, \phi, \hat{F})$ is a 2-cocycle, except now the proof is in 3 dimensions, so we need to recouple sheets instead of strings. The major difference in this calculation is that $V(F_e, \phi, \hat{F}, \hat{\tau})_{g,h,k}$ needs to be wrapped in a $d_f$ bubble to merge

$$V(F_e, \phi, \hat{F}, \hat{\tau})_{g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$$

into a single component via sheet recoupling. However this is easy to fix as $V(F_e, \phi, \hat{F}, \hat{\tau}_{g,h,k})$ is a scalar, so we can pull $V(F_e, \phi, \hat{F}, \hat{\tau})_{g,h,k}$ out of a $d_f$ bubble using linearity, and then pop the bubble to see that $V(F_e, \phi, \hat{F}, \hat{\tau}_{g,h,k})$ is equal to itself wrapped in a $d_f$ bubble. The calculation now follows by merging both

$$V(F_e, \phi, \hat{F}, \hat{\tau})_{g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}$$

and

$$V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h,k}V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h,k}$$

into single components, cancelling terms, and applying the pentagon relations of $A^C$ and $A^D$ to show they are equal. We omit the exact details of this calculation.

Suppose we have two collections of bimodule isomorphisms $\hat{\tau}$ and $\hat{\tau}'$ as in Equation (6.4). Then each $\hat{\tau}_{g,h}' = \lambda_{g,h}^f \hat{\tau}_{g,h}$ for some $\lambda_{g,h} \in \mathbb{C}^\times$. Using linearity it is easy to see that

$$V(F_e, \phi, \hat{F}, \hat{\tau}')_{f,g,h} = \lambda_{g,h}^f \lambda_{g,h}^f \lambda_{f,g,h}^{-1} \lambda_{f,g,h}^{-1} V(F_e, \phi, \hat{F}, \hat{\tau})_{f,g,h}.$$ 

Thus $V(F_e, \phi, \hat{F}, \hat{\tau}')$ differs from $V(F_e, \phi, \hat{F}, \hat{\tau})$ by a coboundary in $Z^3(G, \mathbb{C}^\times)$. Therefore the cohomology class of $V(F_e, \phi, \hat{F}, \hat{\tau})$ is independent of the choice of $\hat{\tau}$. This implies that

$$o_3(F_e, \phi, \hat{F}) := V(F_e, \phi, \hat{F}, -)$$

is an element $H^3(G, \mathbb{C}^\times)$ that depends only on $F_e$, $\phi$, and $\hat{F}$.

**Lemma 8.1.** There exists a system of tensorors for $(F_e, \phi, \hat{F})$ if and only if $o_3(F_e, \phi, \hat{F})$ is the trivial element of $H^3(G, \mathbb{C}^\times)$.

**Proof.** Suppose there exists a system of tensorors $\hat{\tau}$ for $(F_e, \phi, \hat{F})$. Then by definition $v(F_e, \phi, \hat{F}, \hat{\tau})$ is the trivial element of $Z^3(G, \mathbb{C}^\times)$, and thus $o_3(F_e, \phi, \hat{F})$ is trivial in $H^3(G, \mathbb{C}^\times)$.

Conversely suppose $o_3(F_e, \phi, \hat{F})$ is trivial in $H^3(G, \mathbb{C}^\times)$. Let $\hat{\tau}$ be an arbitrary collection of bimodule natural isomorphisms as in Equation (6.4). As $o_3(F_e, \phi, \hat{F})$ is trivial, we have that $v(F_e, \phi, \hat{F}, \hat{\tau}) = d^3(\lambda)$ for some $\lambda \in C^2(G, \mathbb{C}^\times)$. We define a new collection of bimodule natural isomorphisms

$$\hat{\tau}_{g,h}^{\lambda^{-1}} := \lambda_{g,h}^{-1} \hat{\tau}_{g,h}.$$ 

Using linearity is is straightforward to see that $v(F_e, \phi, \hat{F}, \hat{\tau}^{\lambda^{-1}})$ is trivial, therefore $\hat{\tau}^{\lambda^{-1}}$ is a system of tensorors for $(F_e, \phi, \hat{F})$. $\square$

We now classify all systems of tensorors for $(F_e, \phi, \hat{F})$.

**Lemma 8.2.** Suppose $o_3(F_e, \phi, \hat{F})$ is trivial, then systems of tensorors for $(F_e, \phi, \hat{F})$ form a torsor over $H^2(G, \mathbb{C}^\times)$.
Proof. Let \( \lambda \in \mathbb{Z}^2(G, \mathbb{C}^\times) \), and \( \hat{F} \) a system of tensorators for \( (F_e, \phi, \hat{F}) \). We define an action:
\[
\hat{\tau}_{g,h}^\lambda := \lambda_{g,h} \hat{\tau}_{g,h}.
\]

Using linearity we compute
\[
v(F_e, \phi, \hat{F}, \hat{\tau}^\lambda) = d^3(\lambda)v(F_e, \phi, \hat{F}, \hat{\tau}),
\]
which is trivial as \( \lambda \) is a 2-cocycle, and \( \hat{\tau} \) is a system of tensorators for \( (F_e, \phi, \hat{F}) \). This shows that \( \mathbb{Z}^2(G, \mathbb{C}^\times) \) acts on systems of tensorators for \( (F_e, \phi, \hat{F}) \). Let \( \hat{\tau} \) and \( \hat{\tau}' \) be two systems of tensorators for \( (F_e, \phi, \hat{F}) \). As explained in the earlier discussion, each \( \hat{\tau}'_{g,h} = \lambda_{g,h} \hat{\tau}_{g,h} \) for some \( \lambda_{g,h} \in \mathbb{C}^\times \). Using linearity we compute
\[
v(F_e, \phi, \hat{F}, \hat{\tau}') = d^3(\lambda)v(F_e, \phi, \hat{F}, \hat{\tau}).
\]

As \( \hat{\tau} \) and \( \hat{\tau}' \) be two systems of tensorators for \( (F_e, \phi, \hat{F}) \), both \( v(F_e, \phi, \hat{F}, \hat{\tau}) \) and \( v(F_e, \phi, \hat{F}, \hat{\tau}') \) are trivial. Thus \( \lambda \) is a 2-cocyle. This shows that the action of \( \mathbb{Z}^2(G, \mathbb{C}^\times) \) is transitive.

Let \( \hat{\tau} \) be a system of tensorators for \( (F_e, \phi, \hat{F}) \) and suppose \( \lambda \) was a coboundary in \( \mathbb{Z}^2(G, \mathbb{C}^\times) \). Then \( \lambda = d^2(\mu) \) for some \( \mu \in C^1(G, \mathbb{C}^\times) \), and so
\[
\hat{\tau}_{g,h}^\lambda = \lambda_{g,h} \hat{\tau}_{g,h} = \mu_h \mu_{g,h}^{-1} \mu_g \hat{\tau}_{g,h}.
\]

Therefore the systems of tensorators \( \hat{\tau} \) and \( \hat{\tau}^\lambda \) are equivalent. This shows that the action of \( \mathbb{Z}^2(G, \mathbb{C}^\times) \) descends to a well defined action of \( H^2(G, \mathbb{C}^\times) \).

Finally, suppose the systems of tensorators \( \hat{\tau} \) and \( \hat{\tau}^\lambda \) are equivalent. Then there exists \( \mu \in C^1(G, \mathbb{C}^\times) \) such that
\[
\mu_{g,h} \hat{\tau}_{g,h} = \mu_g \mu_h \lambda_{g,h} \hat{\tau}_{g,h}.
\]

Therefore \( \lambda_{g,h} = \mu_g \mu_{g,h}^{-1} \mu_h \), so \( \lambda \) is a coboundary. This shows the action of \( H^2(G, \mathbb{C}^\times) \) is free.

Putting the four above statements together we have shown that systems of tensorators for \( (F_e, \phi, \hat{F}) \) form a torsor over \( H^2(G, \mathbb{C}^\times) \).

\[\blacksquare\]

APPENDIX A. PROOF OF THEOREM 5.6

Remark A.1. The following proof contains lengthy equations involving twisted bimodule functors and twisted natural transformations. To assist with readability we highlight in blue the relevant parts of each equation we change at each step of the equation.

Proof. [1]

We construct a functor \( \overline{H} : C_F \times M \times_F C \to N \) by
\[
X \times m \times Y \mapsto X \triangleright H(m) \triangleleft Y.
\]

This functor is \( C \)-balanced in the first and second position respectively via the maps:
\[
\begin{align*}
\overline{H}^{1}_{X,Z,m,Y} := \text{id}_X \triangleright L^H_{Z,m} \triangleleft \text{id}_Y : X \triangleright F(Z) \triangleright H(m) \triangleleft Y & \to X \triangleright H(Z \triangleright m) \triangleleft Y \\
\overline{H}^{2}_{X,m,Z,Y} := \text{id}_X \triangleright R^H_{m,Z} \triangleleft \text{id}_Y : X \triangleright H(m) \triangleleft F(Z) \triangleleft Y & \to X \triangleright H(m \triangleleft Z) \triangleleft Y.
\end{align*}
\]

We directly verify that the above maps satisfies the condition to be balanced in the first position, but leave the other conditions to be checked by the reader.

\[
\overline{H}^{1}_{X,Z_1 \otimes Z_2,m,Y}(\text{id}_X \triangleright \tau^F_{Z_1,Z_2} \otimes \text{id}_m \otimes \text{id}_Y) = (\text{id}_X \triangleright F^H_{Z_1 \otimes Z_2,m} \otimes \text{id}_Y)(\text{id}_X \triangleright \tau^F_{Z_1,Z_2} \otimes \text{id}_m \otimes \text{id}_Y) = (\text{id}_X \triangleright \text{id}_F(Z_1) \triangleright L^H_{Z_2,m} \otimes \text{id}_Y)(\text{id}_X \triangleright L^H_{Z_1,Z_2,m} \otimes \text{id}_Y)
\]

[1]
\[
= b^H_{X \angle Z_1, Z_2, m, Y} b^H_{X, Z_1, Z_2 \angle m, Y}.
\]

Via the equivalence \[2.1\] the functor \( \overline{H} \) gives a bimodule functor \( C_F \boxtimes M \boxtimes_F C \to N \) which we call \( \hat{H} \).

To show bijectivity consider an arbitrary bimodule functor \( J : C_F \boxtimes M \boxtimes_F C \to N \). The functor \( J \) induces a \( C \)-balanced functor \( j : C_F \times M \times_F C \to N \) via \[2.1\] We define a functor \( \hat{j} : M \to N \) given by

\[
j(m) := j(1, m, 1).
\]

The functor \( \hat{j} \) has the structure of a \( F \)-twisted bimodule functor via the following maps:

\[
L_{Z,m}^\hat{j} := b^j_{1,Z,m,1} L_{F(Z),1,m,1}^j \quad \text{and} \quad R_{m,Z}^\hat{j} := b^j_{1,m,Z,1} R_{1,m,1,F(Z)}^j.
\]

We leave it to the reader to verify that \( L^\hat{j} \) and \( R^\hat{j} \) satisfy the conditions for \( \hat{j} \) to be a \( F \)-twisted bimodule functor.

We finally show bijectivity. Let \( H : M \to N \) a \( F \)-twisted bimodule functor, then

\[
(\overline{H})(m) = \overline{H}(1, m, 1) = 1 \triangleright m \triangleleft 1 = m.
\]

Thus \( \overline{H} \) is strictly equal to \( H \).

Let \( \overline{J} : C_F \times M \times_F C \to N \) a \( C \)-balanced functor, then

\[
R_{X \times m \times 1,Y}^\hat{j}[L_{X \times 1 \times m \times 1,Y}^j \triangleleft \text{id}_Y] : (\overline{J})(X, m, Y) = X \triangleright \hat{j}(m) \triangleleft Y = X \triangleright j(1 \times m \times 1) \triangleleft Y \to J(X, m, Y).
\]

gives a natural isomorphism \( \overline{J} \to J \). We directly check that this natural isomorphism is balanced (only in the first position).

\[
\begin{align*}
&b^j_{X,Z,m,Y} R_{X \otimes F(Z) \times m \times 1,Y}^j [L_{X \otimes F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y] \\
&= b^j_{X,Z,m,Y} R_{X \otimes F(Z) \times m \times 1,Y}^j [L_{X,F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y][\text{id}_X \triangleright L_{F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y] \\
&= R_{X \times Z \otimes m \times 1,Y}^j [b^j_{X,Z,m,1} \triangleleft \text{id}_Y][L_{X,F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y][\text{id}_X \triangleright b^j_{1,Z,m,1} \triangleleft \text{id}_Y][\text{id}_X \triangleright L_{F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y] \\
&= R_{X \times Z \otimes m \times 1,Y}^j [L_{X,1 \times Z \otimes m \times 1,Y}^j \triangleleft \text{id}_Y][\text{id}_X \triangleright b^j_{1,Z,m,1} \triangleleft \text{id}_Y][\text{id}_X \triangleright L_{F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y] \\
&= R_{X \times Z \otimes m \times 1,Y}^j [L_{X,1 \times Z \otimes m \times 1,Y}^j \triangleleft \text{id}_Y][\text{id}_X \triangleright b^j_{1,Z,m,1} \triangleleft \text{id}_Y][\text{id}_X \triangleright L_{F(Z),1 \times m \times 1,Y}^j \triangleleft \text{id}_Y].
\end{align*}
\]

Thus \( \overline{J} \) is naturally isomorphic to \( J \).

Here we make a choice of the evaluation map \( F C \boxtimes C_F \to C \). We make the canonical choice, the functor induced by the balanced functor \( X \times Y \to F^{-1}(X \otimes Y) \), with identity balancing maps. The statement we are trying to prove is equivalent by \[2.1\] to finding a balanced natural isomorphism:

\[
\omega_{H_1,H_2}^{\otimes} : B_{N_1,N_2}(H_1 \times H_2) \triangleright (\text{Id}_{C_F} \times \text{Id}_{M_1} \times \text{ev}_{C_F} \times \text{Id}_{M_2} \times \text{Id}_{F_C}) \to B_{N_1,N_2}(\overline{H_1}, \overline{H_2}).
\]

We define the components of such a natural isomorphism as follows (We write \( B \) instead of \( B_{N_1,N_2} \) for readability):

\[
\omega_{X \times m_1 \times Y \times V \times m_2 \times W}^{\otimes} : b^B_{X \triangleright H_1(m_1),Y \triangleright H_2(m_2)\triangleright W} \triangleright -1 \text{id}_X \triangleright \text{id}_{H_1(m_1)} \times L_{F^{-1}(Y \otimes V),m_2}^{H_2} \triangleright \text{id}_W \\
R^B_{X \triangleright H_1(m_1) \times H_2(F^{-1}(Y \otimes V) \triangleright m_2),W} \triangleright -1 \text{id}_X \triangleright b^B_{X \triangleright H_1(m_1),Y \triangleright H_2(F^{-1}(Y \otimes V) \triangleright m_2)} \triangleright W \to B(X \triangleright H_1(m_1) \triangleleft Y \times V \triangleright H_2(m_2) \triangleleft W).
\]

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We directly verify that \( \omega_{H_1,H_2}^{\bigotimes} \) is balanced in the first position, and leave the other four to the reader.

\[
\omega_{H_1,H_2}^{\bigotimes} = (\omega_{H_1,H_2}^{\bigotimes})^{-1}
\]

The maps \( S_{F_2,F_1} : C_{F_2} \otimes C_{F_1} \to C_{F_2 \circ F_1} \) and \( S_{F_2,F_1}^{-1} : C_{F_2} \otimes C_{F_1} \to C_{F_2 \circ F_1} \) are induced from 2.1 via the balanced functors:

\[
s_{F_2,F_1} : X \otimes Y \to X \otimes F_2(Y) \quad \text{and} \quad s_{F_2,F_1}^{-1} : X \otimes Y \to F_2(X) \otimes Y
\]

respectively. The balancing morphisms are given by

\[
\text{id}_X \otimes \tau_{F_2}^{F_2} : X \otimes F_2(Z) \otimes F_2(Y) \to X \otimes F_2(Z \otimes Y), \quad \text{and} \quad \\
\tau_{F_2}^{F_2} \otimes \text{id}_Y : F_2(X \otimes Z) \otimes Y \to F_2(X) \otimes F_2(Z) \otimes Y.
\]

To prove the statement we have to give a balanced natural isomorphism:

\[
\omega_{H_1,H_2}^{\bigotimes} : \mathcal{H}_2 \circ \mathcal{H}_1 \circ (s_{F_2,F_1} \times \text{Id}_{M} \times s_{F_2,F_1}^{-1}) \to \mathcal{H}_2 \circ \text{Id}_{C_{F_2}} \times \mathcal{H}_1 \times \text{Id}_{F_2C})
\]
We make the following canonical choice (again we drop the \( H_1,H_2 \) subscript for readability):

\[
\omega_{X \otimes m \times V \times W}^\circ := [\id_X \triangleright R^H_{m,H_1(m),V} \triangleleft \id_W][\id_X \triangleright L^H_{Y,H_1(m)} \triangleleft \id_{F(V)} \triangleleft \id_W] \\
: (X \otimes F_2(Y)) \triangleright H_2(H_1(m)) \triangleleft (F_2(V) \otimes W) \to X \triangleright H_2(Y \triangleright H_1(m) \triangleleft V) \triangleleft W.
\]

Again we only verify that \( \omega_{H_1,H_2}^\circ \) is balanced in the first position and leave the rest to the reader.

\[
\omega_{X \otimes Y \otimes m,V,W}^\circ \circ \circ [(S_{F_2,F_1 \times \id_M} \times S_{F_1,F_2})] \\
= [\id_X \triangleright R^H_{Z \otimes Y,H_1(m),V} \triangleleft \id_W][\id_X \triangleright L^H_{Z,Y,H_1(m)} \triangleleft \id_{F_2(V)} \triangleleft \id_W][\id_X \triangleright \id_{F_2} \triangleright L^H_{Y,H_1(m)} \triangleleft \id_{F_2(V)} \triangleleft \id_W] \\
= [\id_X \triangleright R^H_{Z \otimes Y,H_1(m),V} \triangleleft \id_W][\id_X \triangleright L^H_{Z,Y,H_1(m)} \triangleleft \id_{F_2(V)} \triangleleft \id_W][\id_X \triangleright \id_{F_2} \triangleright L^H_{Y,H_1(m)} \triangleleft \id_{F_2(V)} \triangleleft \id_W] \\
= [\id_X \triangleright L^H_{Z,Y,H_1(m),V} \triangleleft \id_W][\id_X \triangleright \id_{F_2} \triangleright L^H_{Y,H_1(m)} \triangleleft \id_{F_2(V)} \triangleleft \id_W] \\
= b^H_{X,Z,Y,m,V,W} \circ \circ \circ [(S_{F_2,F_1 \times \id_M} \times S_{F_1,F_2})] \cdot \omega_{X \otimes F_2(Z),Y,m,V,W}^\circ.
\]

[4]

Given a natural isomorphism of \( F \)-twisted bimodule functors \( \mu : H_1 \to H_2 \), we need to construct a balanced natural isomorphism \( \overline{\mu} : \overline{H}_1 \to \overline{H}_2 \). We claim that

\[
\overline{\mu}_{X,m,Y} = \id_X \triangleright \mu_m \triangleleft \id_Y,
\]

is balanced. To see this we directly compute (only in the first position) that

\[
b_{X,Z,Y,m}^H \circ \overline{\mu}_{X \otimes m,Y} = [\id_X \triangleright L^H_{Z,M} \triangleleft \id_Y][\id_X \otimes Z \triangleright \mu_m \triangleleft \id_Y] \\
= \id_X \triangleright [L^H_{Z,M} \triangleright \id_{F(Z)} \triangleright \mu_m] \triangleleft \id_Y \\
= \id_X \triangleright [\mu_{Z \otimes m} \triangleright L^H_{Z,m}] \triangleleft \id_Y \\
= \overline{\mu}_{X,Z,m}^H \circ b_{X,Z,m,Y}^H.
\]

Thus via the equivalence [2.1] \( \mu \) induces a natural isomorphism \( \hat{\mu} : \hat{H}_1 \to \hat{H}_2 \).

Let \( \nu : \overline{H}_1 \to \overline{H}_2 \) a balanced natural isomorphism. We define a natural transformation \( H_1 \to H_2 \) by

\[
\nu_{X,m,Y} := \nu_{X,m,\id}.
\]

The fact that \( \nu \) is a natural isomorphism of \( F \)-twisted functors follows directly from the fact that \( \nu \) is itself a natural isomorphism of bimodule functors. It also follows directly that \( \overline{\nu}_{m}^H = \mu_m \) and \( \overline{(\nu)}_{X,m,Y} = \nu_{X,m} \). Thus via [2.1] the map \( \sim \) is bijective.

To prove this claim we have to verify that the balanced natural isomorphisms

\[
\omega_{H_1,H_2}^\circ \circ \circ [\mu_1 \times \mu_2 \circ (\id_{Id_{C_F} \times Id_{Id_{N_1} \times Id_{v_{C_F} \times Id_{Id_{N_2} \times Id_{Id_{F}}}}})]
\]

and

\[
[\overline{\mu_1} \times \overline{\mu_2}] \omega_{H_1,H_2}^O
\]

are equal. Let \( X \times m_1 \times Y \times V \times m_2 \times W \in C_F \times M_1 \times F \times C \times F \times M_2 \times F \times C \), then

\[
\omega_{H_1,H_2}^\circ \circ \circ [\mu_1 \times \mu_2 \circ (\id_{Id_{C_F} \times Id_{Id_{N_1} \times Id_{v_{C_F} \times Id_{Id_{N_2} \times Id_{Id_{F}}}}})][X \times m_1 \times Y \times V \times m_2 \times W] \\
= b_{X \otimes H_1(m_1),Y \otimes H_2(m_2) \otimes W}^{H_1}B(\id_X \triangleright id_{H_1(m_1)} \times L^{H_2}_{F^{-1}(Y \otimes V),m_2} \otimes \id_W)[R^{H_1 \times H_2} \otimes H_2(F^{-1}(Y \otimes V) \otimes m_2),W] \\
[L^{H_2}_{X,H_1(m_1) \times H_2(F^{-1}(Y \otimes V) \otimes m_2)} \otimes \id_W][\id_X \triangleright B(\mu_{m_1} \times \mu_{F^{-1}(Y \otimes V) \otimes m_2}) \triangleleft \id_W]
\]

\[39\]
= b^B_{X; H'_1(m_1), Y; V; H'_2(m_2) \otimes W}^{-1} B(id_X \triangleright id_{H'_1(m_1)} \times L_{F^{-1}(Y \otimes V), m_2}^{H'_2} \triangleright id_W) B(id_X \triangleright \mu_{1,m_1} \times \mu_{2,F^{-1}(Y \otimes V), m_2} \triangleright id_W)

R^B_{X; H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2), W}[L^B_{X, H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2) \otimes W} \triangleright id_W]

= b^B_{X; H'_1(m_1), Y; V; H'_2(m_2) \otimes W}^{-1} B(id_W \triangleright \mu_{1,m_1} \times id_Y \triangleright id_V \triangleright \mu_{2,m_2} \triangleright id_W) B(id_X \triangleright id_{H'_1(m_1)} \times L_{F^{-1}(Y \otimes V), m_2}^{H'_2} \triangleright id_W)

R^B_{X; H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2), W}[L^B_{X, H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2) \otimes W} \triangleright id_W]

= B(id_X \triangleright \mu_{1,m_1} \triangleright id_Y \triangleright id_V \triangleright \mu_{2,m_2} \triangleright id_W)b^B_{X; H'_1(m_1), Y; V; H'_2(m_2) \otimes W}^{-1} B(id_X \triangleright id_{H'_1(m_1)} \times L_{F^{-1}(Y \otimes V), m_2}^{H'_2} \triangleright id_W)

R^B_{X; H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2), W}[L^B_{X, H_1(m_1) \times H_2(F^{-1}(Y \otimes V), m_2) \otimes W} \triangleright id_W]

= [(\overline{\rho}_1 \times \overline{\rho}_2) \circ \omega^\omega_{H'_1, H'_2} X \times m_1 \times Y \times m_2 \times V \times W].

[6]

To prove this claim we have to verify that the balanced natural isomorphisms

$$\omega^\omega_{H'_1, H'_2} [\overline{\rho}_2 \circ \overline{\mu}_1 \circ (id_{S_{F_2, F_1} \times id_{I_{M \times F_1} \times C_{F_2}}}]$$

and

$$[\overline{\rho}_2 \circ (id_{I_{C_{F_2}} \times \overline{\rho}_1 \times id_{I_{F_2}}} \circ \omega_{H'_1, H'_2} [\overline{\rho}_2 \circ \overline{\mu}_1 \circ (id_{S_{F_2, F_1} \times id_{I_{M \times F_1} \times C_{F_2}}])$$

are equal. Let $X \times Y \times m \times V \times W \in C_{F_2} \times C_{F_1} \times M \times F_1 \times C_{F_2}$, then

$$\omega^\omega_{H'_1, H'_2} [\overline{\rho}_2 \circ \overline{\mu}_1 \circ (id_{S_{F_2, F_1} \times id_{I_{M \times F_1} \times C_{F_2}}}] X \times Y \times m \times V \times W

= [id_X \triangleright R^{H'_2}_{Y; Y'; H'_1(m), V} \triangleleft id_W][id_X \triangleright L^{H'_2}_{Y; Y'; H'_1(m)} \triangleleft id_W][id_X \triangleright id_{F(Y)} \triangleright \mu_2 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright id_{F(Y)} \triangleright \mu_1 \triangleleft id_{F(V)} \triangleleft id_W][id_X \triangleright id_{F(Y)} \triangleright \mu_1 \triangleleft id_{F(V)} \triangleleft id_W]

= [id_X \triangleright id_{F(Y)} \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright R^{H'_2}_{Y; Y'; H'_1(m), V} \triangleleft id_W][id_X \triangleright L^{H'_2}_{Y; Y'; H'_1(m)} \triangleleft id_W][id_X \triangleright \mu_2 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W]

= [id_X \triangleright id_{F(Y)} \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright R^{H'_2}_{Y; Y'; H'_1(m), V} \triangleleft id_W][id_X \triangleright \mu_2 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W]

= [id_X \triangleright R^{H'_2}_{Y; Y'; H'_1(m), V} \triangleleft id_W][id_X \triangleright \mu_2 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W]

= [id_X \triangleright \mu_2 \triangleright id_{F(V)} \triangleleft id_W][id_X \triangleright \mu_1 \triangleright id_{F(V)} \triangleleft id_W]

= [\overline{\rho}_2 \circ (id_{I_{C_{F_2}} \times \overline{\rho}_1 \times id_{I_{F_2}}} \circ \omega_{H'_1, H'_2} X \times Y \times m \times V \times W].

[7]

We have to show that the natural transformations $\overline{\rho}_2 \overline{\rho}_1$ and $\overline{\rho}_1 \overline{\rho}_2$ are equal. We compute

$$\overline{\rho}_2 \overline{\rho}_1 = id_X \triangleright \mu_2 \mu_1 \triangleleft id_Y

= [id_X \triangleright \mu_2 \triangleleft id_Y]id_X \triangleright \mu_1 \triangleleft id_Y

= \overline{\rho}_{X \times Y \times m \times V \times W}.

\[\square\]

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