A Boundary Schwarz Lemma for Pluriharmonic Mappings Between the Unit Polydiscs of Any Dimensions

Ziyan Huang, Di Zhao, Hongyi Li

Abstract. In this paper, we present a boundary Schwarz lemma for pluriharmonic mappings between the unit polydiscs of any dimensions, which extends the classical Schwarz lemma for bounded harmonic functions to higher dimensions.

1. Introduction

The Schwarz lemma is regarded as one of the most important results in complex analysis. Let \( f \) be a holomorphic self-mapping of the unit disk \( D \). The classical Schwarz lemma states that for holomorphic mapping \( f \) satisfying the condition \( f(0) = 0 \), the inequality \( |f(z)| \leq |z| \) is true for any \( z \in D \). This result is a potent tool to study several research fields in complex analysis. An increasing number of mathematicians thus focus attention on establishing various versions of the Schwarz lemma.

Schwarz lemma at the boundary is an active topic in complex analysis. Various interesting results associated with the boundary Schwarz lemma have been presented in recent years. For the convenience of representation, we introduce some notations and definitions.

Let \( \mathbb{C}^n \) be the complex space of dimension \( n \) with the norm given by \( ||z|| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2} \) for any \( z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n \). For any \( z = (z_1, z_2, \cdots, z_n)^T, \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \in \mathbb{C}^n \), the inner product on \( \mathbb{C}^n \) is defined by \( (z, \omega) = \sum_{i=1}^{n} z_i \bar{\omega}_i \), therefore \( (z, z)^{1/2} = ||z|| \) also represents the norm of \( z \). Let \( B^n = \{ z \in \mathbb{C}^n : ||z|| < 1 \} \) be the unit ball in \( \mathbb{C}^n \), and \( \partial B^n = \{ z \in \mathbb{C}^n : ||z|| = 1 \} \) be the unit sphere. Denote by \( D \) the unit disk with unit circle \( T \) in the complex plane \( \mathbb{C} \), then the unit polydisc can be represented as \( D^n = D \times \cdots \times D = \{ z \in \mathbb{C}^n : |z_i| < 1, 1 \leq i \leq n \} \) which belongs to the complex space \( \mathbb{C}^n \). Furthermore, denote \( ||z||_n = \max_{1 \leq i \leq n} |z_i| \), then we have \( \partial D^n = \{ z \in \mathbb{C}^n : ||z||_n = 1 \} \) and \( T^n = T \times \cdots \times T = \{ z \in \mathbb{C}^n : |z_i| = 1 \leq i \leq n \} \) which represent the topological boundary and the distinguished boundary of \( D^n \), respectively. If there are only \( r \) (\( 1 \leq r \leq n \)) components of \( z_0 \) whose modules equals to 1, then the set of all this kind of boundary points is denoted by \( E_r \). It is obvious that \( E_n = T^n \) and \( \bigcup_{1 \leq r \leq n} E_r = \partial D^n \) if taking all boundary points into consider.

Denote the set of all holomorphic mappings between the bounded domains of any dimensions as \( H(\Omega_1, \Omega_2) \) where \( \Omega_1 \subset \mathbb{C}^n \) and \( \Omega_2 \subset \mathbb{C}^n \). For any \( f = (f_1, f_2, \cdots, f_n)^T \in H(\Omega_1, \Omega_2) \), the Jacobian matrix of \( f \)
at \( z \in \Omega_1 \) is given by
\[
D f(z) = \left[ \frac{\partial f_i}{\partial z_j}(z) \right]_{N \times n}.
\]
Moreover, we use \( \overline{D} f(z) \) to represent the \( N \times n \) matrix \( \left( \frac{\partial f_i}{\partial z_j}(z) \right)_{N \times n} \). For the same function, denote by \( f'(z) \) the \( 2N \times 2n \) Jacobian matrix of \( f \) at \( z \) in terms of real coordinates. Let \( C^r(V) \) be the set of all functions \( f \) on the bounded domain \( V \) for which
\[
\sup \left\{ \frac{|f(z) - f(z')|}{|z - z'|^\alpha} : z, z' \in V \right\}
\]
is finite with \( 0 < \alpha < 1 \). Then we denote \( C^{k+\alpha}(V) \) as the set of all functions \( f \) on \( V \) whose \( k \)-th order partial derivatives exist and belong to \( C^\alpha(V) \) for an integer \( k \).

In [1], the classical boundary Schwarz lemma for holomorphic mappings is described as follows:

**Theorem 1.1.** [1] Let \( f \in H(D, D) \) be a holomorphic mapping. If \( f \) is holomorphic at \( z = 1 \) with \( f(0) = 0 \) and \( f(1) = 1 \), then \( f'(1) \geq 1 \). Moreover, the inequality is sharp.

If we remove the condition \( f(0) = 0 \) in the above theorem and take the holomorphic mapping
\[
g(z) = \frac{1 - \overline{f(0)} f(z) - f(0)}{1 - \overline{f(0)} f(z)},
\]
we have the following estimate instead:
\[
f'(1) \geq \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0.
\]

Chelst[2] and Osserman[3] further studied the Schwarz lemma at the boundary of the unit disk, respectively. Ornek[4] explored some new expressions of Schwarz inequality at the boundary of the unit disk and acquired the sharpness of these inequalities.

Moreover, in the case of several complex variables, Wu generalized the classical Schwarz lemma for holomorphic mappings to higher dimension [5]. Recently, Liu et al.[6] presented a version of the boundary Schwarz lemma for holomorphic mappings from the unit ball \( B^n \) to the unit ball \( B^N \), which is not restricted by the condition \( f(0) = 0 \).

**Theorem 1.2.** [6] Let \( f \in H(B^n, B^N) \) for \( n, N \geq 1 \). If \( f \) is \( C^{1+\alpha} \) at \( z_0 \in \partial B^n \) with \( f(z_0) = \omega_0 \in \partial B^N \), then there exists \( \lambda \in \mathbb{R} \) such that
\[
\overline{D} f(z_0)^T \omega_0 = \lambda \omega_0
\]
where \( \lambda = \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0 \), \( \omega = f(0) \).

Furthermore, in [7] Liu et al. presented the result of Schwarz lemma for holomorphic mappings from the unit polydisc \( D^n \) to the unit ball \( B^N \) at the boundary as follows.

**Theorem 1.3.** [7] Let \( f \in H(D^n, B^N) \) for \( n, N \geq 1 \). Given \( z_0 \in \partial D^n \). Assume \( z_0 \in E \), with the first \( r \) components at the boundary of \( D \) for some \( 1 \leq r \leq n \). If \( f \) is \( C^{1+\alpha} \) at \( z_0 \) with \( f(z_0) = \omega_0 \in \partial B^N \), then there exist a sequence of nonnegative real numbers \( \gamma_1, \gamma_2, \cdots, \gamma_r \), satisfying \( \sum_{j=1}^r \gamma_j \geq 1 \) and \( \lambda \in \mathbb{R} \) such that
\[
\overline{D} f(z_0)^T \omega_0 = \lambda \text{diag}(\gamma_1, \cdots, \gamma_r, 0, \cdots, 0) \omega_0
\]
where \( \lambda = \frac{|1 - \overline{f(0)}|^2}{1 - |f(0)|^2} > 0 \), \( \omega = f(0) \) and "\( \text{diag} \)" represents the diagonal matrix.
Harmonic mapping is a complex-valued harmonic function defined in the complex space, which is in touch with geometric functions and locally quasiconformal mappings. For the harmonic mappings, there are also some interesting analogues of the Schwarz lemma. For example, the Schwarz lemma for the harmonic self-mapping of the unit disk is stated as follows.

**Theorem 1.4.** [8] Let $f$ is a harmonic mapping of the unit disk $D$ on itself, and $f(0) = 0$, then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|, z \in D.$$ 

In [9], the boundary Schwarz lemma for the harmonic self-mapping of the unit disk is restated with a simple proof. Considering the several complex variables, Mateljević offered the boundary Schwarz lemma for harmonic mappings between the unit balls with any dimensions in [10].

Note that the pluriharmonic mapping can be considered as a generalization of the harmonic function. A continuous complex-valued function $f$ defined on a domain $\Omega \in \mathbb{C}^n$ is said to be pluriharmonic if for each fixed $z \in \Omega$ and $\theta \in \partial B^n$, the function $f(z + \theta \zeta)$ is harmonic in $\{ \zeta : ||\zeta|| < d\Omega(z) \}$, where $d\Omega(z)$ denotes the distance from $z$ to the boundary $\partial \Omega$ of $\Omega$. Therefore, it is a very natural task to obtain various versions of the Schwarz lemma for pluriharmonic mappings.

It is obtained in [11] that when $\Omega$ is a simply connected domain, then $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if $f$ could be represented by $f = \eta + \tilde{\zeta}$ where $\eta$ and $\zeta$ are holomorphic in $\Omega$. Hence, a holomorphic mapping can be regarded as a special pluriharmonic function. Furthermore, $f : \Omega \to \mathbb{C}^n$ is called a pluriharmonic mappings if all its components are pluriharmonic functions from $\Omega$ to $\mathbb{C}$. Similarly to $H(\Omega_1, \Omega_2)$, the set of pluriharmonic mappings between the bounded domains of any dimensions is denoted as $P(\Omega_1, \Omega_2)$ where $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^N$.

In [12], Mateljević introduced Kobayashi metrics and obtained the Kobayashi-Schwarz lemma for the holomorphic mappings on the bounded connected open subsets of complex Banach space. As an application of the lemma obtained, a boundary Schwarz lemma is established for pluriharmonic mappings defined on the unit ball $B^2$.

For the pluriharmonic mappings between unit balls with any dimensions, in [13], Liu et al. presented the boundary Schwarz lemma for pluriharmonic mappings defined on the unit ball.

**Theorem 1.5.** [13] Let $f \in P(B^n, B^N)$ for $n, N \geq 1$. If $f$ is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ and $f(z_0) = \omega_0 \in \partial B^N$, then there exists a positive $\lambda \in \mathbb{R}$ such that

$$Df(z_0)^T \omega'_0 = \lambda z'_0$$

where $z'_0$ and $\omega'_0$ are real versions of $z_0$ and $\omega_0$, and $\lambda \geq \frac{1-\|f(00)\|}{\frac{n}{2n+1}} > 0$.

In this paper, we extend the boundary Schwarz lemma for planar harmonic mappings to higher dimensions, and establish a novel boundary Schwarz lemma for pluriharmonic mappings between the unit polydiscs of any dimensions.

Inspired by [13], we consider the real version of this problem. For $z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n$ with $z_i = x_i + iy_i$, where $1 \leq i \leq n$, denote $z'$ as the real version of $z$ and $z' = (x_1, y_1, \cdots, x_n, y_n)^T \in \mathbb{R}^{2n}$ only containing real elements. Therefore, $D^n$ in $\mathbb{C}^n$ is equivalent to the unit polydisc $D^{2n} \subset \mathbb{R}^{2n}$.

We first combine the Harnack’s inequality with the minimum principle and establish a new inequality for the nonnegative harmonic function defined on the unit polydisc $D^{2n}$ (see Lemma 2.1). This lemma provides an important technique support for estimating the lower bound of the function in the proof of the main results. Furthermore, we also present the Schwarz lemma for the pluriharmonic mapping $f \in P(D^n, D^N)$ (see Lemma 2.2), which generalizes the corresponding results in Theorem 1.4 to higher dimensions and plays a significant role in the proof of Theorem 1.6.

Then we get the following boundary Schwarz lemma for pluriharmonic mappings in $P(D^n, D^N)$.

**Theorem 1.6.** Let $f \in P(D^n, D^N)$ with $f(0) = 0$ for $n, N \geq 1$. Given $z_0 = (z_1, \cdots, z_r, z_{r+1}, \cdots, z_n)^T \in E \subset \partial D^n$. If $f$ is $C^{1+\alpha}$ at $z_0$ and $f(z_0) = \omega_0 \in E_m \subset \partial D^N$, then there exist a sequence of nonnegative real numbers $\gamma_1, \gamma_2, \cdots, \gamma_r$ such that the following statements hold.
Proof. Let $f$ be a nonnegative function defined on the unit polydisc $D^{2n}$ in $\mathbb{R}^{2n}$. If $f$ is continuous on the unit polydisc and harmonic on its interior, then for any $z = (x_1, y_1, \cdots, x_n, y_n)^T \in D^{2n}$ satisfying $\sqrt{x_i^2 + y_i^2} = r_0 < 1$ for $1 \leq i < r < n$ and $z_i = y_i = 0$ for $r + 1 \leq i \leq n$, the following inequality holds:

$$f(z) \geq \frac{1 - r_0}{(1 + r_0)^{2n-1}} f(0).$$

Proof. Suppose that $f$ is a nonnegative function defined on the unit ball $B^{2n}$ in $\mathbb{R}^{2n}$. According to the description of [13], we know that if $f$ is continuous on the unit ball and harmonic on its interior, then for any $z_0 \in B^{2n}$ with $\|z_0\| = r_0 < 1$ we have the Harnack’s inequality

$$\frac{1 - r_0}{(1 + r_0)^{2n-1}} f(0) \leq f(z) \leq \frac{1 + r_0}{(1 - r_0)^{2n-1}} f(0).$$

Since the conditions that $z = (x_1, y_1, \cdots, x_n, y_n)^T \in D^{2n}$ satisfies $|z_i| = \sqrt{x_i^2 + y_i^2} = r_0 < 1$ for $1 \leq i < r < n$ and $z_i = y_i = 0$ for $r + 1 \leq i \leq n$, it is not difficult to derive that $\|z\| = (\sum_{i=1}^{n} |z_i|^2)^{\frac{1}{2}} = \sqrt{r_0 < \sqrt{r}}$. Then we have

$$\frac{1 - r_0}{(1 + r_0)^{2n-1}} f(0) \leq f(z) \leq \frac{1 + r_0}{(1 - r_0)^{2n-1}} f(0).$$

Therefore, it follows from the minimum principle for harmonic function that

$$f(z) \geq \frac{1 - r_0}{(1 + r_0)^{2n-1}} f(0).$$

This completes the proof. $\square$

Lemma 2.2. Let $f = (f_1, \cdots, f_n)^T \in P(D^n, D^n)$ with $n, N \geq 1$ and $f(0) = 0$, then

$$\|f(z)\|_{\infty} \leq \frac{4}{n} \arctan \|z\|_{\infty}.$$

Proof. For any fixed $z' \in D^n \setminus \{0\}$ and any $\zeta \in D$, let

$$F_i(\zeta) = \left( \frac{\zeta z'}{\|z'\|_{\infty}}, \frac{f_i(z')}{\|f(z')\|} \right)$$

for any $1 \leq i \leq n$. Applying Theorem 1.4 to the complex-valued harmonic mapping $F_i$, we have the inequality

$$|f_i(z')| = |F_i(\|z\|_{\infty})| \leq \frac{4}{n} \arctan \|z\|_{\infty}.$$
for any $1 \leq i \leq n$. Thus, the inequality

$$\|f(z)\|_\infty \leq \frac{4}{\pi} \arctan \|z\|_\infty$$

holds for any $z \in D^n$.

The proof of the lemma is complete. \hfill \Box

**Remark 2.3.** When $n = N = 1$, Lemma 2.2 reduces to Theorem 1.4, which extends the boundary Schwarz lemma to high dimensions.

### 3. Proof of Theorem 1.6

In the following, we will prove Theorem 1.6.

**Proof.** 1) The first proof is divided into six steps for reader’s convenience.

**Step 1.** Assume $z_0 \in \partial D^n$ and $f = (f_1, \ldots, f_n)^T$ is $C^{1+\alpha}$ in a neighborhood $V$ of $z_0$. Without loss of generality, we let $z_0 = \sum_{i=1}^n e_i^0$, where $e_i^0$ represents the $i$-th column of the identity matrix $I_n$. Since $f_j = u_j + i\omega_j$ for $1 \leq j \leq N$ is defined on the unit polydisc, it is obtained from $f$ that $u_k(\sum_{i=1}^n e_i^0) = 1$ for $1 \leq k \leq m$ and $N$. Moreover, $1 - u_k \geq 0$ is harmonic on the unit polydisc. By using Lemma 2.1 and considering $x = (x_1, 0, x_2, 0, \ldots, x_r, 0, 0, \ldots, 0)^T \in \mathbb{R}^{2n}$ for $x_i = r_0$ near $1 (1 \leq i \leq r \leq n)$, we have

$$1 - u_k(x) \geq \frac{1 - x_i}{(1 + x_i)^{2n-1}} (1 - u_k(0)),$$

which gives that

$$\frac{1 - u_k(x)}{1 - x_i} \geq \frac{1 - u_k(0)}{(1 + x_i)^{2n-1}}.$$

Letting $x_i \to 1^-$, we can derive

$$\frac{\partial u_k(\sum_{i=1}^n e_i^0)}{\partial x_i} = \lim_{x_i \to 1^-} \frac{(1 - u_k(x)) - (1 - u_k(\sum_{i=1}^n e_i^0))}{1 - x_i} \geq \frac{1 - u_k(0)}{2^{2n-1}}$$

for $1 \leq i \leq r$ and $1 \leq k \leq m$.

**Step 2.** Let $p = z_0, q_i = -\sum_{j=1}^r e_j^0 + ike_j^0$ for $1 \leq l \leq r$ and $k \in \mathbb{R}$. It is clear that $p + tq = (1-t)z_0 + ikt e_j^0$ for $t \in \mathbb{R}$, so we have $\|p + tq\|_\infty < 1 \Leftrightarrow |1 - t + ikt| < 1$ and $|1 - t| < 1 \Leftrightarrow 0 < t < \frac{1}{1 + k^2}$. The equivalence relationship means that for a given $k$ when $t \to 0^+$, $p + tq \in D^n \cap V$. For $t \to 0^+$, taking the Taylor expansion of $f_j(p + tq)$ at $t = 0$, we have

$$f_j(p + tq) = (\omega_0)_j + Df_j(z_0)q_it + \bar{T}f_j(z_0)\bar{q}t + O(t^{1+\alpha})$$

where $(\omega_0)_j$ denotes the $j$-th element of $\omega_0$. By Lemma 2.2,

$$\|f_j(p + tq)\|_\infty = \max_{i \in \{1, N\}} |f_j(p + tq)| \leq \frac{4}{\pi} \arctan \|p + tq\|_\infty.$$  \hfill (3)

Considering that

$$\|p + tq\|_\infty = |1 - t + ikt|\|1 - t|,$$

it is easy to derive

$$\arctan |1 - t| = \frac{\pi}{4} - \frac{1}{4}t - \frac{1}{4}t + O(t^{1+\alpha}) = \frac{\pi}{4} - \frac{1}{2}t + O(t^{1+\alpha})$$

for any $1 \leq i \leq n$. Thus, the inequality

$$\|f(z)\|_\infty \leq \frac{4}{\pi} \arctan \|z\|_\infty$$

holds for any $z \in D^n$.
and
\[
\arctan[1 + t(1 + ik)] = \frac{\pi}{4} + \frac{1}{4}(-1 + ik)t + \frac{1}{4}(-1 - ik)t + O(t^{1+\alpha}) = \frac{\pi}{4} - \frac{1}{2}t + O(t^{1+\alpha}).
\]
Thus (3) is equivalent to
\[
\max_{1 \leq j \leq N} \left| (\omega_0) + Df_j(z_0)q_j t + \overline{Df_j(z_0)} q_j t + O(t^{1+\alpha}) \right| \leq 1 - \frac{2}{\pi} t + O(t^{1+\alpha}).
\]
Substituting \(\omega_0 = \sum_{j=1}^{m} e_j^n\), we have
\[
\max_{1 \leq j \leq N} \left| 1 + 2 \text{Re}(Df_j(z_0)q_j + \overline{Df_j(z_0)} q_j t + O(t^{1+\alpha}) \right| \leq 1 - \frac{4}{\pi} t + O(t^{1+\alpha}).
\]
Letting \(t \to 0^+\), we deduce that
\[
\max_{1 \leq j \leq N} \left| \text{Re}(Df_j(z_0)q_j + \overline{Df_j(z_0)} q_j) \right| \leq \frac{2}{\pi}.
\]
(4)
Substituting \(q_j = -\sum e_i^n + ike_i^n\), we have
\[
\max_{1 \leq j \leq N} \left| \text{Re} \left( Df_j(z_0) \left(-\sum e_i^n + ike_i^n \right) + \overline{Df_j(z_0)} \left(-\sum e_i^n - ike_i^n \right) \right) \right| \leq \frac{2}{\pi},
\]
which equals to
\[
\max_{1 \leq j \leq N} \left| \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} + \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} \right) \right) \right| \leq \frac{2}{\pi}.
\]
Let \(\frac{g_i}{z_i} = \text{Re} \frac{g_i}{z_i} + \text{Im} \frac{g_i}{z_i}\) and \(\frac{g_i}{z_i} = \text{Re} \frac{g_i}{z_i} + \text{Im} \frac{g_i}{z_i}\). From the above inequality, we have
\[
\max_{1 \leq j \leq N} \left| \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} - k \text{Im} \frac{g_i}{z_i} \right) - \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} + k \text{Im} \frac{g_i}{z_i} \right) \right| \leq \frac{2}{\pi},
\]
i.e.
\[
\min_{1 \leq j \leq N} \left( \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} + \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} \right) \right) \right) \geq \frac{2}{\pi}.
\]
(5)
Since (5) is valid for any \(k \in \mathbb{R}\), so that
\[
\text{Im} \frac{g_i}{z_i} - \text{Im} \frac{g_i}{z_i} = 0, 1 \leq l \leq r,
\]
which gives
\[
\min_{1 \leq j \leq N} \left( \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} + \text{Re} \left( \sum_{i=1}^{r} \frac{g_i}{z_i} \right) \right) \right) \geq \frac{2}{\pi},
\]
(7)
and
\[
\text{Re} \left( \frac{g_i}{z_i} - \text{Re} \left( \frac{g_i}{z_i} \right) \right) = \frac{g_i}{z_i} - \frac{g_i}{z_i}, 1 \leq l \leq r.
\]
(8)
Step 3. Consider \( p = z_0, q_l = -\sum_{i=1}^{r} e_i^n + ke_i^n \) for \( 1 \leq l \leq r \) and \( k \leq 0 \). Then \( p + tq_l = (1-t)z_0 + kte_i^n \) for \( t \in \mathbb{R} \). Hence \( \|p + tq_l\|_{\infty} < 1 \Leftrightarrow |1 - t + kt| < 1 \) and \( |1 - t| < 1 \Leftrightarrow 0 < t < \frac{1}{1 + k} \). The equivalence relationship implies that for any given \( k \leq 0 \) there is \( t \to 0^+ \) such that \( p + tq_l \in D^n \cap V \). Taking the Taylor expansion of \( f_j(p + tq_l) \) at \( t = 0 \) and applying Lemma 2.2, we get

\[
\max_{1 \leq j \leq N} \left| (\omega_0)_j + Df_j(z_0)q_l t + \overline{Df_j(z_0)}\overline{q_l} t + O(t^{1+n}) \right| \leq \frac{4}{\pi} \arctan \|p + tq_l\|_{\infty}.
\]

Same to Step 2, it is not difficult to obtain

\[
\max_{1 \leq j \leq N} \left( 1 + 2Re(Df_j(z_0)q_l + \overline{Df_j(z_0)}\overline{q_l} t + O(t^{1+n}) \right) \leq 1 - \frac{4}{\pi} t + O(t^{1+n}).
\]

We also substitute \( q_l = -\sum_{i=1}^{r} e_i^n + ke_i^n \) and let \( t \to 0^+ \), then

\[
\max_{1 \leq j \leq N} \left| Re\left(Df_j(z_0) + \sum_{i=1}^{r} e_i^n + ke_i^n \right) + \overline{Df_j(z_0)}\left( -\sum_{i=1}^{r} e_i^n - ke_i^n \right) \right| \leq -\frac{2}{\pi}.
\]

A straightforward computation shows that

\[
\max_{1 \leq j \leq N} \left| Re\left( -\sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial z_i} + k \frac{\partial f_j(z_0)}{\partial z_i} - \sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial \overline{z_i}} - k \frac{\partial f_j(z_0)}{\partial \overline{z_i}} \right) \right| \leq -\frac{2}{\pi},
\]

which is equivalent to

\[
\min_{1 \leq j \leq N} \left| Re\left( \sum_{i=1}^{r} \frac{\partial f_j(z_0)}{\partial z_i} + k \frac{\partial f_j(z_0)}{\partial z_i} - k \left( Re\frac{\partial f_j(z_0)}{\partial z_i} - Re\frac{\partial f_j(z_0)}{\partial \overline{z_i}} \right) \right) \right| \geq \frac{2}{\pi}.
\]

Since (7) and (8) is valid for \( k \in \mathbb{R} \), we get

\[
\min_{1 \leq j \leq N} -k \left( \frac{\partial f_j(z_0)}{\partial z_i} - \frac{\partial f_j(z_0)}{\partial \overline{z_i}} \right) \geq 0
\]

for \( k \leq 0 \). We further derive

\[
\frac{\partial f_j(z_0)}{\partial z_i} \geq \frac{\partial f_j(z_0)}{\partial \overline{z_i}}, 1 \leq l \leq r, 1 \leq j \leq N
\]

since \( k \leq 0 \) is arbitrary.

Step 4. Let \( p = z_0, q_l = -\sum_{i=1}^{r} e_i^n + ike_i^n \) for \( r + 1 \leq l \leq n \) and \( k \neq 0 \in \mathbb{R} \). Then \( p + tq_l = (1-t)\sum_{i=1}^{r} e_i^n + ikte_i^n \) for \( t \in \mathbb{R} \). It is not difficult to verify that \( \|p + tq_l\|_{\infty} < 1 \Leftrightarrow |1 - t| < 1 \) and \( |ikt|^2 < 1 \Leftrightarrow 0 < t < \min\left(\frac{1}{|k|}, 2\right) \). Therefore, given a \( k \neq 0 \in \mathbb{R} \), there exists \( t \to 0^+ \) such that \( p + tq_l \in D^n \cap V \). Then taking the Taylor expansion of \( f_j(p + tq_l) \) at \( t = 0 \), we can derive

\[
\max_{1 \leq j \leq N} \left| (\omega_0)_j + Df_j(z_0)q_l t + \overline{Df_j(z_0)}\overline{q_l} t + O(t^{1+n}) \right| \leq \frac{4}{\pi} \arctan \|p + tq_l\|_{\infty},
\]

from which it is obvious that

\[
\max_{1 \leq j \leq N} \left( 1 + 2Re(Df_j(z_0)q_l + \overline{Df_j(z_0)}\overline{q_l} t + O(t^{1+n}) \right) \leq 1 - \frac{4}{\pi} t + O(t^{1+n}).
\]

Substituting \( q_l = -\sum_{i=1}^{r} e_i^n + ike_i^n \) and letting \( t \to 0^+ \), we get

\[
\max_{1 \leq j \leq N} \left| Re\left(Df_j(z_0) - \sum_{i=1}^{r} e_i^n + ike_i^n \right) + \overline{Df_j(z_0)}\left( -\sum_{i=1}^{r} e_i^n - ike_i^n \right) \right| \leq -\frac{2}{\pi},
\]
i.e.

$$\max_{1 \leq i \leq N} \left\{ \text{Re} \left( -\sum_{t=1}^{r} \frac{\partial f_j(z_0)}{\partial x_t} + i k \frac{\partial f_j(z_0)}{\partial y_t} - \sum_{t=1}^{r} \frac{\partial f_j(z_0)}{\partial z_t} - i k \frac{\partial f_j(z_0)}{\partial \bar{z}_t} \right) \right\} \leq \frac{2}{\pi}.$$ 

Reviewing that \( \frac{\partial f_j(z_0)}{\partial z_t} = \text{Re} \frac{\partial f_j(z_0)}{\partial z_t} + i \text{Im} \frac{\partial f_j(z_0)}{\partial z_t} \) and \( \frac{\partial f_j(z_0)}{\partial \bar{z}_t} = \text{Re} \frac{\partial f_j(z_0)}{\partial \bar{z}_t} + i \text{Im} \frac{\partial f_j(z_0)}{\partial \bar{z}_t} \), and exploiting (7), it is not difficult to obtain

$$\max_{1 \leq i \leq N} \left( \text{Im} \frac{\partial f_j(z_0)}{\partial z_t} - \text{Im} \frac{\partial f_j(z_0)}{\partial \bar{z}_t} \right) \leq 0, r + 1 \leq t \leq n.$$ 

Since the above equality is valid for \( k \neq 0 \in \mathbb{R} \), with the similar argument to Step 2, we have

$$\text{Im} \frac{\partial f_j(z_0)}{\partial z_t} = 0, r + 1 \leq t \leq n. \quad (10)$$

**Step 5.** Let \( p = z_0, q_t = -\sum_{i=1}^{r} e_i^t + k e_i^t \) for \( r + 1 \leq t \leq n \) and any \( k \neq 0 \in \mathbb{R} \). Then \( p + t q_t = (1 - t) \sum_{i=1}^{r} e_i^t + k t e_i^t \) for \( t \in \mathbb{R} \). It is not difficult to verify that \( \| p + t q_t \|_\infty < 1 \Leftrightarrow |1 - t| < 1 \) and \( |kt|^2 < 1 \Leftrightarrow 0 < t < \min \{\frac{1}{|k|}, 2\} \). Therefore, given a \( k \neq 0 \in \mathbb{R} \), there exists \( t \to 0^+ \) such that \( p + t q_t \in D^n \cap \mathbb{V} \). With the similar argument to Step 4, it is not difficult to obtain

$$\text{Re} \sum_{i=1}^{m} \frac{\partial f_j(z_0)}{\partial z_t} + \text{Re} \sum_{i=1}^{m} \frac{\partial f_j(z_0)}{\partial \bar{z}_t} = 0, r + 1 \leq t \leq n. \quad (11)$$

Review the formulas \( f_j = u_j + i v_j \) and \( z_t = x_t + i y_t \) for \( 1 \leq i \leq n, 1 \leq j \leq N \). Considering that \( \frac{\partial}{\partial z_t} = \frac{1}{2} \left( \frac{\partial}{\partial x_t} - i \frac{\partial}{\partial y_t} \right) \) and \( \frac{\partial}{\partial \bar{z}_t} = \frac{1}{2} \left( \frac{\partial}{\partial x_t} + i \frac{\partial}{\partial y_t} \right) \), we can derive the following results for any \( 1 \leq i \leq n, 1 \leq j \leq N \):

$$\frac{\partial f_j}{\partial z_t} = \frac{1}{2} \left( \frac{\partial}{\partial x_t} - i \frac{\partial}{\partial y_t} \right) \left( u_j + i v_j \right) = \frac{1}{2} \left[ \frac{\partial u_j}{\partial x_t} + \frac{\partial v_j}{\partial y_t} + i \left( \frac{\partial v_j}{\partial x_t} - \frac{\partial u_j}{\partial y_t} \right) \right],$$

$$\frac{\partial f_j}{\partial \bar{z}_t} = \frac{1}{2} \left( \frac{\partial}{\partial x_t} + i \frac{\partial}{\partial y_t} \right) \left( u_j + i v_j \right) = \frac{1}{2} \left[ \frac{\partial u_j}{\partial x_t} - \frac{\partial v_j}{\partial y_t} + i \left( \frac{\partial v_j}{\partial x_t} + \frac{\partial u_j}{\partial y_t} \right) \right].$$

In view of (2) and (6), it is obvious that for any \( 1 \leq j \leq m \) we have

$$\frac{\partial u_j}{\partial y_t} = 0, \frac{\partial u_j}{\partial x_t} = 1 - u_j(0) \frac{1}{2^{m-1}}, 1 \leq i \leq r.$$ 

Similarly, it follows from (10) and (11) that

$$\frac{\partial u_j}{\partial y_t} = 0, \frac{\partial u_j}{\partial x_t} = 0, r + 1 \leq i \leq n. \quad (13)$$

Rewrite \( z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n \) by \( z' = (x_1, y_1, \ldots, x_n, y_n)^T \in \mathbb{R}^{2n} \), then \( z_0' = (e_0^N)' + (e_3^N)' + \cdots + (e_{2m-1}^N)' = (1, 0, \cdots, 1, 0, \cdots, 0)^T \in \mathbb{R}^{2n} \) where \( (e_i^N)' \) represents the \( i \)-th column of identity matrix \( I_{2n} \). According to (12) and (13), it is concluded that

$$f_j(z_0') \omega_0' = \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z_0'$$

where \( \omega_0' = (e_0^N)' + (e_3^N)' + \cdots + (e_N^N)' \) and \( \lambda_i = \sum_{j=1}^{m} \frac{\partial u_j}{\partial x_t} \) with \( \frac{\partial u_j}{\partial x_t} \geq \frac{1 - u_j(0)}{2^{m-1}} \) for \( 1 \leq i \leq r \).
Step 6. Let \( z'_0 \) be any given point at \( \partial \mathbb{D}^{2n} \subset \mathbb{R}^{2n} \). That is, \( z'_0 \) is not necessary \((e^0_1)' + (e^0_2)' + \cdots + (e^0_{2m-1})' \). Then there exists a special kind of real-valued diagonal unitary matrix \( U_{z'_0} \) such that \( U_{z'_0}(z'_0) = (e^0_1)' + (e^0_2)' + \cdots + (e^0_{2m-1})' \) and \( \omega(z'_0) \), referring to [7]. Assume \( f' \) is the real version of \( f \), and \( f'(z'_0) = \omega'_0 \) where \( \omega'_0 \) is not necessary \((e^N_1)' + (e^N_2)' + \cdots + (e^N_{2m-1})' \) \( \omega \)' at \( \partial \mathbb{D}^{2n} \). In the same way, there exists a real-valued diagonal unitary matrix \( U_{\omega'_0} \) such that \( U_{\omega'_0}(\omega'_0) = (e^N_1)' + (e^N_2)' + \cdots + (e^N_{2m-1})' \) \( \omega'_0 \). Denote
\[
g'(z') = U_{\omega'_0} \circ f' \circ U^T_{\omega'_0}(z'), z' \in \mathbb{D}^{2n}
\]
and
\[
g(z) = U_{\omega'_0} \circ f \circ U^T_{\omega'_0}(z), z \in \mathbb{D}^{2n}
\]
where \( U_{z_0} \) and \( U_{\omega_0} \) represent complex unitary matrices corresponding to \( U_{z'_0} \) and \( U_{\omega'_0} \) such that \( U_{z_0}(z_0) = \sum_{i=1}^r e^i_0 \) and \( U_{\omega_0}(\omega_0) = \sum_{i=1}^m e^N_j \). It is easy to verify that \( g' \) is the real version of \( g \) and \( g(\sum_{i=1}^r e^i_0) = \sum_{i=1}^m e^N_j \).
Furthermore, the Jacobian matrix of \( g \) could be denoted as
\[
J_g(z') = U_{\omega'_0} (f'(U^T_{\omega'_0}(z'))) U^T_{\omega'_0}(z'), z' \in \mathbb{D}^{2n}.
\]
(14)

According to Step 5, we have
\[
J_g(z'_0)^T \omega'_0 = \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z'_0
\]
where \( \lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i} \) with \( \frac{\partial u_i}{\partial x_i} \leq \frac{1-n_i(0)}{2m} \) for \( 1 \leq i \leq r \). As a result, we obtain
\[
\left( U_{\omega'_0} (f'(U^T_{\omega'_0}(z'_0))) U^T_{\omega'_0}(z'_0) \right)^T \omega'_0 = \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z'_0
\]
which equals to
\[
U_{z'_0} f'(U^T_{\omega'_0}(z'_0)) U^T_{\omega'_0}(z'_0) \omega'_0 = \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z'_0
\]

Multiplying \( U^T_{\omega'_0} \) at both sides of the above equation gives
\[
J_f(z'_0) \omega'_0 = \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0)z'_0
\]
where \( \lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i} \) with \( \frac{\partial u_i}{\partial x_i} \geq \frac{1-n_i(0)}{2m} \) for \( 1 \leq i \leq r \).
2) According to (7), it is not difficult to obtain
\[
\sum_{i=1}^r \sum_{j=1}^m \frac{\partial u_j}{\partial x_i} \geq \frac{2}{\pi}
\]
Since \( \text{tr} \left( \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) \right) = \sum_{i=1}^r \lambda_i \) and \( \lambda_i = \sum_{j=1}^m \frac{\partial u_j}{\partial x_i} \), it is obvious that
\[
\text{tr} \left( \text{diag}(\lambda_1, 0, \cdots, \lambda_r, 0, \cdots, 0) \right) \geq \frac{2}{\pi}
\]
The proof of Theorem 1.6 is finished. \( \square \)
References

[1] J. B. Garnett, Bounded Analytic Functions, Academic Press, 1981.
[2] D. Chelst, A generalized Schwarz lemma at the boundary, Proceedings of the American Mathematical Society 129(11) (2000) 3275-3278.
[3] R. Osserman, A Sharp Schwarz Inequality on the Boundary, Proceedings of the American Mathematical Society 128(12) (1997) 3513-3518.
[4] B. N. Ormek, Sharpened forms of the Schwarz lemma on the boundary, Bulletin of the Korean Mathematical Society 50(6) (2013) 2053-2059.
[5] H. Wu, Normal families of holomorphic mappings, Acta Mathematica 119(1) (1967) 193-233.
[6] Y. Liu, Z. Chen, Y. Pan, A boundary Schwarz Lemma for holomorphic mappings between unit balls of different dimensions, Eprint Arxiv, (2014).
[7] Y. Liu, Z. Chen, Y. Pan, A boundary Schwarz lemma for holomorphic mappings on the polydisc, Chinese Annals of Mathematics, Series B 39(1) (2018) 9-16.
[8] E. Heinz, On one-to-one harmonic mappings, Pacific Journal of Mathematics 9(1) (1959) 101-105.
[9] M. Mateljević, Schwarz Lemma and Distortion for Harmonic Functions Via Length and Area, Potential Analysis (2020).
[10] M. Mateljević, Rigidity of holomorphic mappings, Schwarz and Jack lemma, DOI: 10.13140/RG.2.2.34140.90249.
[11] V. S. Vladimirov, Methods of the Theory of Functions of Many Complex Variables, MIT Press (1996).
[12] M. Mateljević, Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions, Journal of Mathematical Analysis and Applications 464 (2018) 78-100.
[13] Y. Liu, Z. Chen, Y. Pan, Boundary Schwarz lemma for pluriharmonic mappings between unit balls, Journal of Mathematical Analysis and Applications 422(1) (2016) 487-495.