Abstract
In this note we report the solution of the problem of defining the $S$-transform in Free Probability Theory in arbitrary dimensions. This is achieved by generalising the theory and embedding it into an algebraic-geometric framework. Finally, we classify the groups arising as distributions of $s$-tuples of non-commutative random variables.

1 Introduction
The addition and multiplication problem for pairs of free non-commutative random variables was solved by Voiculescu [8]. He introduced two series, the $S_V$ and the $R_V$ transform, cf. [9]. Subsequently, Speicher [7] and Nica and Speicher [6] extended the $R$-transform to arbitrary dimensions considering the free cumulants instead of the moment series. The essential tool to achieve this, is the insight that free probability theory can be described in terms of non-crossing partitions, as realised and developed by Speicher [7]. Further, a multiplicative operation, the so-called “boxed” convolution was introduced and studied. Among the many remarkable properties it has, one should mention that it defines a generally non-commutative group structure on the set of non-commutative power series with complex coefficients.

Earlier, Voiculescu described in the one-dimensional case the group structure on non-commutative probability laws as a projective limit of finite dimensional complex Lie groups [8].

Since its introduction in 1987 [8], however, the nature of the higher-dimensional $S$-transform has remained an unresolved problem. Additionally, the question of how to go, from the additive to the multiplicative problem, if possible at all, also stayed open, cf. [7].
Here, we give the answers to all the questions raised above. The main tool
to do so is by generalising the entire theory of free probability theory from
its usual $\mathbb{C}$-valued setting to arbitrary commutative unital rings, which by
its combinatorial nature is possible, and then to study it by using algebraic-
geometric methods. As it turns out, the theory becomes nice and pleasant.

This is a slightly revised version of a talk delivered at the Fields Institute
in July 2013. Therefore, it contains only the “bare bones” and some of the
major results. General references for standard results and common notations
of Free Probability Theory include $[6, 9]$. The detailed content, in particular
the proofs, and also further extensions can be found in the following (up-
coming) publications $[2, 3]$.

2 The one-dimensional case

We prove that in the one-dimensional non-commutative case, the situation
is analogous to classical probability theory.

**Theorem 2.1** ([$2$]). There exists a group isomorphism $\text{LOG} : (\Sigma^\times_1, \boxtimes) \rightarrow (\Sigma, \boxplus)$, with inverse $\text{EXP}$, defined by the diagram:

\[
\begin{array}{ccc}
\Lambda(\mathbb{C}) & \xrightarrow{S_V} & (\Sigma^\times_1, \boxtimes_V) \\
\downarrow \text{log} & \text{“ghost map”} & \downarrow \text{LOG} \\
\mathbb{C}[[z]] & \xrightarrow{R_V^{-1}} & (\Sigma, \boxplus_V)
\end{array}
\]

In fact, $\mathbb{C}[[z]]$ is a commutative unital ring, endowed with the Hadamard
multiplication:

\[
\left( \sum_{n=0}^{\infty} a_n z^n \right) \ast \left( \sum_{n=0}^{\infty} b_n z^n \right) := \sum_{n=0}^{\infty} a_n b_n z^n.
\]

3 Witt vectors and Free Probability

The algebraic structure of the distribution of non-commutative random vari-
ables is isomorphic to the abelian part of the Witt vectors, as we showed
in [$2$].
Theorem 3.1. Let $\Sigma_{1}^{x}$ of probability distributions with mean 1, carries the structure of a commutative unital ring, with binary operations $(\boxplus_{V}, \boxtimes)$, such that $(\Sigma_{1}^{x}, \boxplus_{V}, \boxtimes)$ is isomorphic to the ring $(\Lambda(\mathbb{C}), \cdot, *, 1, 1 - z)$. The multiplication $\boxtimes$ in $\Sigma_{1}^{x}$ satisfies

$$\mu_{1} \boxtimes \mu_{2} = S_{V}^{-1}(S_{V}(\mu_{1}) * S_{V}(\mu_{2})) \quad \text{for} \quad \mu_{1}, \mu_{2} \in \Sigma_{1}^{x}.$$ 

• There exists an unital ring structure on $\Sigma$, with binary operations $(\boxplus_{V}, \boxtimes)$, where $\boxtimes$ denotes the multiplication. This ring is isomorphic to the ring $(\mathbb{C}[[z]], +, *, 0, 1)$.

• The rings $(\Sigma_{1}^{x}, \boxplus_{V}, \boxtimes)$ and $(\Sigma, \boxplus_{V}, \boxtimes)$ are isomorphic, with the isomorphisms given by LOG and EXP, respectively.

4 Partial ring structure on one-dimensional probability measures

As a consequence we derive a partial ring structure for the one-dimensional probability distributions have a partial ring structure, i.e. certain types of probability measures behave also nicely under multiplication. For reference and background information on the distributions discussed, cf. [6, 9].

Let $\delta_{a}$ be a Dirac distribution supported at $a \in \mathbb{R}$. Its Cauchy transform is

$$G_{\delta_{a}}(z) = \int_{\mathbb{R}} \frac{1}{z - t} \delta_{a}(t)dt = \frac{1}{z - a},$$

and

$$\frac{1}{R_{V}(z) + (1/z) - a} = z \iff R_{V}(z) = a.$$

Proposition 4.1. Let $a, b \in \mathbb{R}$, and $\delta_{a}, \delta_{b}$ the corresponding point masses supported at $a$ and $b$, respectively. Then Dirac distributions are freely Hadamard multiplied by multiplying their supports, i.e.

$$\delta_{a} \boxtimes \delta_{b} = \delta_{ab}.$$ 

For $a, b \in \mathbb{R}$, $r, s > 0$ the semicircle law centred at $a$, of radius $r$, is defined as the distribution $\gamma_{a,r} : \mathbb{C}[z] \to \mathbb{C}$ given by

$$\gamma_{a,r}(z^{n}) := \frac{2}{\pi r^{2}} \int_{a-r}^{a+r} t^{n} \sqrt{r^{2} - (t - a)^{2}} dt \quad \forall n \in \mathbb{N}.$$
It is determined, as is the Gaussian law, by its first and second moment, i.e., by \( \gamma_{a,r}(z) = a \) and \( \gamma_{a,r}(z^2) = a^2 + r^2/4 \). Its \( R_V \)-transform equals
\[
R_{\gamma_{a,r}}(z) = a + \frac{r^2}{4}z.
\]

**Proposition 4.2.** The family of semicircular distributions is closed both under additive free \( \boxplus_V \), and point-wise (Hadamard) free multiplicative \( \boxdot \) convolution, i.e.
\[
\begin{align*}
\gamma_{a,r} \boxplus_V \gamma_{b,s} &= \gamma_{a+b,\sqrt{r^2+s^2}} \\
\gamma_{a,r} \boxdot \gamma_{b,s} &= \gamma_{ab,rs/2}
\end{align*}
\]

The free compound Poisson distribution with rate \( \lambda \geq 0 \) and jump size \( \alpha \in \mathbb{R} \) is the limit in distribution for \( N \to \infty \) of
\[
\nu_N := \nu_{N,\lambda,\alpha} := \left( \left( 1 - \frac{\lambda}{N} \right) \delta_0 + \frac{\lambda}{N} \delta_\alpha \right)^{\boxplus N}
\]
with \( \boxplus N \) being \( n \)-fold additive free self-convolution. The free cumulants \( \kappa_n \) are given by \( \kappa_n = \lambda \alpha^n \) for \( n \in \mathbb{N}^\times \).

The \( R_V \)-transform of the limit \( \nu_\infty \) is
\[
R_V(\nu_\infty)(z) = \lambda \alpha \frac{1}{1 - \alpha z} = \sum_{n=0}^{\infty} \lambda \alpha^{n+1}z^n = \lambda \alpha + \lambda \alpha^2 z + \lambda \alpha^3 z^2 + \cdots.
\]

**Proposition 4.3.** The large \( N \)-limit of the free Poisson distribution \( \nu_{\infty,\lambda,\alpha} \) with rate \( \lambda = 1 \) and jump size \( \alpha = 1 \), corresponds to the \textbf{unit} with respect to \( \boxdot \) multiplication in \( \mathbb{C}[[z]] \), i.e. to
\[
\sum_{n \in \mathbb{N}} z^n = 1 + z + z^2 + \ldots.
\]

## 5 Free \( k \)-probability theory

Let \( k \) be a commutative ring with unit \( 1_k \) and \( R \in \text{cAlg}_k \). For \( s \in \mathbb{N}^\times \) consider the alphabet \([s] := \{1, \ldots, s\}\) with words \( w \). The set of all finite words over \([s]\), including the empty word \( \emptyset \), is denoted by \([s]^\ast\), and set \([s]^\ast_+ := [s]^\ast \setminus \emptyset\).

**Definition 5.1.** A non-commutative \( R \)-valued \( k \)-probability space (non-commutative \( R-k \)-probability space) consists of a pair \((\mathcal{A}, \phi)\), with \( \mathcal{A} \in \text{Alg}_k \) and \( \phi \) a fixed \( k \)-linear functional with values in \( R \in \text{cAlg}_k \) such that \( \phi(1_A) = 1_R \).

The elements \( a \in \mathcal{A} \) are called \textbf{non-commutative random variables}. 

Definition 5.2. The \( s \)-dimensional distribution or law of \( \phi \), with values in \( R \), is the map

\[
\mu : \mathcal{A}^s \to \text{Hom}_{k,1}(k \langle z_1, \ldots, z_s \rangle, R),
\]

such that for any \( s \)-tuple \( \underline{a} := (a_1, \ldots, a_s) \) we have

\[
\mu_{(a_1, \ldots, a_s)} : k \langle z_1, \ldots, z_s \rangle \to R
\]

\[
z_w \mapsto \mu_{(a_1, \ldots, a_s)}(a_w) := \phi(a_w) \quad \forall w \in [s]^*.
\]

Then \( \mu_{(a_1, \ldots, a_n)} \) is called the \( R \)-distribution of the \( s \)-tupel \( (a_1, \ldots, a_s) \).

6 Combinatorial Freeness

The notion of freeness is modified in the general setting.

Definition 6.1. Let \((\mathcal{A}, \phi)\) be a non-commutative \( R - k \)-probability space. The subsets \( M_1, \ldots, M_n \subset \mathcal{A} \) are called combinatorially free if they have vanishing mixed free cumulants.

Proposition 6.1. 1. Let \( A_1, \ldots, A_n \subset \mathcal{A} \) be combinatorially free unital subalgebras in a non-commutative \( R - k \)-probability space \((\mathcal{A}, \phi)\). Then they are also classically free, i.e., for all \( m \geq 1 \) \( \phi(a_1 \cdots a_m) = 0 \) whenever \( a_1 \in A_{i_1}, \ldots, a_m \in A_{i_m} \) with \( i_1 \neq i_2, \ldots, i_{m-1} \neq i_m \) and \( \phi(a_i) = 0 \)

2. For any \( k \)-valued non-commutative \( k \)-probability space, i.e. \( \phi : \mathcal{A} \to k \), classical and combinatorial freeness are equivalent.

The essential tool in the proofs is the so-called “Centering trick”:

\[
a - \phi(a)1_{\mathcal{A}} \text{ which satisfy } \phi(a - \phi(a)1_{\mathcal{A}}) = \phi(a) - \phi(a)1_k = 0 \text{ as } \phi \text{ is } k \text{-linear and unital.}
\]

7 The combinatorial \( R \)-transform

Definition 7.1. Let \( \underline{a} = (a_1, \ldots, a_s) \in \mathcal{A}^s, s \geq 1 \). The \( R \)-transform with values in \( R \in \text{cAlg}_k \), is the map \( \mathcal{R} : \mathcal{A}^s \to R_+(\langle z_1, \ldots, z_s \rangle) \) which
assigns to every $a$ the formal power series $R(a)$ in $s$ non-commuting variables $\{z_1, \ldots, z_s\}$, given by

$$R(a) := R_a(z_1, \ldots, z_s) := \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1} s \sum_{1 \leq i_1, \ldots, i_n \leq s} \kappa_n(a_{i_1}, \ldots, a_{i_n}) z_{i_1} \cdots z_{i_n}$$

$$= \sum_{|w| \geq 1} \kappa_{|w|}(a_w) z_w,$$

where $\kappa_n$ denotes the free cumulants.

8 Addition and multiplication of combinatorially free random variables

For $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathcal{A}^n$ and $\lambda \in k$ we have

$$a + b := (a_1 + b_1, \ldots, a_n + b_n),$$

$$a \ast b := (a_1 \cdot b_1, \ldots, a_n \cdot b_n),$$

$$\lambda \cdot a := (\lambda \cdot a_1, \ldots, \lambda \cdot a_n).$$

In general, the moment map does not preserve any of the above algebraic structures, e.g. in the additive case we would have

$$\mathcal{M}(a + b) \neq \mathcal{M}(a) \ast \mathcal{M}(b),$$

and in the multiplicative case

$$\mathcal{M}(a \ast b) \neq \mathcal{M}(a) \ast \mathcal{M}(b).$$

**Proposition 8.1.** Let $(\mathcal{A}, \phi)$ be a non-commutative $R - k$-probability space, and let $a = (a_1, \ldots, a_s), b = (b_1, \ldots, b_s) \in \mathcal{A}^s$ be such that $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$ are combinatorially free. Then there exist two binary operations $\boxplus_V$ and $\boxtimes_V$ on $R_+\langle\langle z_1, \ldots, z_s\rangle\rangle$ such that the $\mathcal{M}$-transform defines algebraic morphisms

$$\mathcal{M} : \mathcal{A}^s \to R_+\langle\langle z_1, \ldots, z_s\rangle\rangle,$$

viz., we have

- in the additive case

$$\mathcal{M}(a + b) = \mathcal{M}(a) \boxplus_V \mathcal{M}(b)$$

6
• **in the multiplicative case**

\[
\mathcal{M}(a \star b) = \mathcal{M}(a) \boxtimes_{\mathcal{V}} \mathcal{M}(b)
\]

# 9 Boxed convolution

Let \( \pi \) denote a non-crossing partition and \( K(\pi) \) its Kreweras complement.

**Definition 9.1.** Let \( R \in \text{cAlg}_k \). The **boxed convolution** \( \boxtimes \), is a binary operation

\[
R_+ \langle x_1, \ldots, x_s \rangle \times R_+ \langle x_1, \ldots, x_s \rangle \rightarrow R_+ \langle x_1, \ldots, x_s \rangle,
\]

\[
(f, g) \mapsto f \boxtimes g,
\]

which for every word \( w = (i_1, \ldots, i_n) \) and integer \( n \geq 1 \) satisfies:

\[
X_w(f \boxtimes g) = \sum_{\pi \in NC(|w|)} X_{w, \pi}(f) \cdot R X_{w, K(\pi)}(g).
\]

**Remark 9.1.** The operation \( \boxtimes \) is completely described by **universal polynomials with** \( \mathbb{Z} \)-**coefficients**, determined by the underlying combinatorics. The calculation of a coefficient of degree \( |w| \) depends only on the coefficients of the monomials of degree \( \leq |w| \).

# 10 Moment-cumulant formula

For any \( \underline{a} \in \mathcal{A}^s \), we have: \( \mathcal{R}(\underline{a}) \boxtimes \text{Zeta} = \mathcal{M}(\underline{a}) \), i.e.,

\[
R_+ \langle \langle z_1, \ldots, z_s \rangle \rangle \xrightarrow{\mathcal{R}-\text{transform}} \mathcal{A}^s \xrightarrow{\mathcal{M}-\text{transform}} \mathcal{Zeta} \xrightarrow{\boxtimes} R_+ \langle \langle z_1, \ldots, z_s \rangle \rangle
\]

**Proposition 10.1.** The relation between \( + \) and \( \boxplus_{\mathcal{V}} \), and between \( \boxtimes \) and \( \boxtimes_{\mathcal{V}} \), respectively, is given by

\[
\boxplus_{\mathcal{V}} = (\bullet_1 \boxtimes \text{Moeb}_s + \bullet_2 \boxtimes \text{Moeb}_s) \boxtimes \text{Zeta}_s,
\]

and

\[
\boxtimes_{\mathcal{V}} = \bullet_1 \boxtimes \text{Moeb}_s \boxtimes \bullet_2.
\]

with \( \bullet_i, \ i = 1, 2 \), denoting the first and the second argument, respectively.
11 The $k$-functors

Let $s \in \mathbb{N}^\times$, $R \in c\text{Alg}_k$.

\[
\mathcal{M}^s(R) := \{ \sum_{|w| \geq 1} \alpha_w z_w | \alpha_w \in R, w \in [s]^* \} \cong R^{[s]^+},
\]
\[
\mathcal{G}^s(R) := \{ (r_1, \ldots, r_s, \ldots, r_w, \ldots) | r_i \in R^\times, i \in [s], r_w \in R, |w| \geq 2 \},
\]
\[
\mathcal{G}^s_+(R) := \{ (1_1, \ldots, 1_s, \ldots, r_w, \ldots) | r_w \in R, |w| \geq 2 \},
\]
and the finite-dimensional restrictions of $\mathcal{M}^s(R)$, $\mathcal{G}^s(R)$ and $\mathcal{G}^s_+(R)$, respectively, for $n \geq 1$

\[
(\mathcal{M}^s)_n(R) := \{ \sum_{1 \leq |w| \leq n} \alpha_w z_w \}.
\]
\[
(\mathcal{G}^s)_n(R) := \{ \sum_{1 \leq |w| \leq n} \alpha_w z_w | \alpha_i \in R^\times \text{ for } i = 1, \ldots, s \},
\]
\[
(\mathcal{G}^s_+)_n(R) := \{ \sum_{1 \leq |w| \leq n} \alpha_w z_w | \alpha_i = 1_R \text{ for } i = 1, \ldots, s \}.
\]

12 Affine group schemes

**Proposition 12.1.** For all $s \in \mathbb{N}^\times$, $R \in c\text{Alg}_k$ and the binary operation $\boxtimes$, the following hold:

1. $(\mathcal{M}^s(R), +)$, with component-wise addition, is an abelian group, and it has as additive neutral element, $0_{\mathcal{M}^s(R)}$, corresponding to the series with all coefficients being zero.

2. $(\mathcal{M}^s(R), \boxtimes)$ is an associative multiplicative monoid, with unit

\[
1_{\mathcal{M}^s(R)} = 1_R z_1 + \cdots + 1_R z_s.
\]

3. $(\mathcal{M}^s(R), +, \boxtimes)$, is an associative and **not** distributive unital ring, which for $s \geq 2$ is also non-commutative.

4. $\mathcal{G}^s(R)$ is a group with neutral element $1_{\mathcal{G}^s(R)} = 1_R z_1 + \cdots + 1_R z_s$.

For $s = 1$ it is abelian and for $s \geq 2$ non-abelian.
Proposition 12.2. For all $s \in \mathbb{N}^\times$, $R \in \text{cAlg}_k$, and the binary operation $\boxtimes$, the following holds:

1. $\mathfrak{G}^s(R)$ is a group, which is the semi-direct product of the normal subgroup $\mathfrak{G}^s_+(R)$ and the s-torus $(\mathfrak{G}^s)^1(R) \cong \mathfrak{G}^s_m(R)$, i.e.,

$$\mathfrak{G}^s(R) = \mathfrak{G}^s_1(R) \ltimes (\mathfrak{G}^s_+)(R),$$

or equivalently, we have the exact sequence

$$0 \xrightarrow{} \mathfrak{G}^s_m(R) \xrightarrow{i} \mathfrak{G}^s(R) \xrightarrow{p} \mathfrak{G}^s_+(R) \xrightarrow{} 1.$$ 

2. $\mathfrak{G}^s(R)$ and $\mathfrak{G}^s_+(R)$ are groups, filtered in ascending order by the subgroups $(\mathfrak{G}^s)_n(R)$ and $(\mathfrak{G}^s_+)_n(R)$, respectively.

3. $\mathfrak{G}^s(R)$ and $\mathfrak{G}^s_+(R)$ are projective limits of finite dimensional groups, i.e.

$$\mathfrak{G}^s(R) = \lim_{\leftarrow n \in \mathbb{N}} (\mathfrak{G}^s)_n(R) \quad \text{and} \quad \mathfrak{G}^s_+(R) = \lim_{\leftarrow n \in \mathbb{N}} (\mathfrak{G}^s_+)_n(R).$$

13 Lie Groups and Lie Algebras in characteristic 0

Theorem 13.1. Let $R$ be a $\mathbb{Q}$-algebra. Then the formal groups $\mathfrak{G}^N_u(R)$ and $(\mathfrak{G}^1_+(R), \boxtimes)$ are isomorphic.

Proof. 

- This follows from Lazard’s Theorem: “Over $\mathbb{Q}$ every commutative formal group law $F$ is isomorphic to the dim($F$)-dimensional additive group law”.

- $(\mathfrak{G}^1_+(R), \boxtimes)$ defines an infinite commutative formal group law.

\[\square\]

Remark 13.2. 

- Our LOG in Thm. 2.1 is the concrete form of the above isomorphism.

- The above theorem answers one of the question we started with.

- For $s \geq 2$, $(\mathfrak{G}^s_+, \boxtimes)$ is not isomorphic to the additive group, as it is not commutative.
14 The co-ordinate algebras

The \( k \)-functors are group valued and representable. By the Yoneda Lemma this gives rise to Hopf algebras, which we described next.

**Theorem 14.1.** Let \( s \in \mathbb{N}^\times \) and \( k \) be a ring. For the formal group schemes we have

- \( \mathfrak{G}^s \) is represented by
  \[
  k[X_1^{\pm 1}, \ldots, X_s^{\pm 1}, X_w : |w| \geq 2];
  \]
  and it is not connected.

- \( (\mathfrak{G}^s)_1 \) is represented by
  \[
  k[X_1, X_1^{-1}, \ldots, X_s, X_s^{-1}],
  \]
  consisting of group-like elements.

- \( \mathfrak{G}^*_s \) is represented by the graded and connected Hopf algebra
  \[
  k[\bar{X}_w : |w| \geq 2]
  \]
  which for \( s = 1 \) is co-commutative.

The co-structure is given by:

- The co-product: \( \Delta X_w(f, g) := X_w(f \boxtimes g) \),
  \[
  \Delta X_w = \sum_{\pi \in NC(|w|)} X_{w,\pi} \otimes X_{w,K(\pi)},
  \]
  where \( X_{w,\pi} \) and \( X_{w,K(\pi)} \) are as above.

- For the co-unit: \( \varepsilon(X_w) := X_w(1_{(\mathfrak{G}^s)_n}) \) we get
  \[
  \varepsilon(X_w) = \begin{cases} 
  1 & \text{for } w = (i), i \in \{1, \ldots, s\}, \\
  0 & \text{for } |w| \geq 2.
  \end{cases}
  \]
  (1)

This defines on \( R[X_w|1 \leq |w| \leq n] \) the structure of a bi-algebra.
Remark 14.2. The elements $X_i, i \in \{1, \ldots, s\}$, are group-like, i.e., they satisfy
\[
\Delta(X_i) = X_i \otimes X_i \quad \text{and} \quad \varepsilon(X_i) = 1.
\]

The full structure is obtained with Proposition 14.3 (Antipode of the boxed convolution).

Let $w = (i_1 \ldots i_n) \in [s]^*$ with $n \geq 2$. The antipode corresponding to the boxed convolution $\boxtimes$, is given by the recursive relation
\[
S(X_w) = S(X_{(i_1 \ldots i_n)}) = -(X_{i_1}^{-1} \cdots X_{i_n}^{-1}) \cdot \sum_{\pi \neq 0_n} X_{w, \pi} \cdot S(X_{w, K(\pi)}).
\] (2)

15 Some types of matrices

In the remaining sections we assume $k$ to be a field.

- The set of diagonal matrices
  \[
  T_n = \{ \text{diag}(t_1, \ldots, t_n) \mid t_i \in k^\times \}
  \]
in $GL_n(k)$ is called the torus,

- the set of upper-triangular matrices
  \[
  B_n(k) := \{ (a_{ij})_{i,j} \in GL_n(k) \mid a_{ij} = 0 \text{ for } i > j \}
  \]
known as the Borel subgroups.

Let
\[
B_\infty := \lim_{\leftarrow} B_n(k) \quad \text{and} \quad U_\infty := \lim_{\leftarrow} U_n(k)
\]
be the corresponding projective limits.

Definition 15.1. Let $A$ be a square matrix $A$ and $E$ the unit matrix. Then, $A$ is called unipotent if $A - E$ is nilpotent, i.e., there exists an integer $n \geq 1$ such that $(A - E)^n = 0$.

An element $r$ in a ring $R$ with unit $1_R$, is called unipotent, if $(r - 1_R)$ is nilpotent, i.e., $\exists n \geq 1, (r - 1_R)^n = 0_R$. 
The upper-triangular matrices with constant diagonal 1 are unipotent

\[ \mathbb{U}_n(k) := \begin{pmatrix}
1 & * & \ldots & * \\
0 & 1 & \ddots & \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & 1
\end{pmatrix} \]

with * denoting arbitrary entries from \( k \).

The \( n \)-dimensional additive group \( \mathbb{G}_a^n(k) \) is unipotent and is faithfully represented by

\[ \begin{pmatrix}
1 & x_1 & \ldots & x_n \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix} \]

with \( x_i \in k, i = 1, \ldots, n \).

16 Faithful representations

Proposition 16.1. The groups \(((\mathfrak{G}^s)_n, \boxtimes)\) and \(((\mathfrak{G}^s_+)_n, \boxtimes)\) are faithfully representable as closed subgroups of some \( \text{GL}_N \), \( N = N(n, s) \) and \((\mathfrak{G}^s, \boxtimes)\) and \((\mathfrak{G}^s_+, \boxtimes)\) are pro-algebraic groups, i.e. projective limits of algebraic groups.

Theorem 16.2. Let \( k \) be a field and \( s, n \in \mathbb{N}^\times \). There exist faithful representations \( \rho_{n,s} \), for

- \(((\mathfrak{G}^s)_n(k), \boxtimes)\) as closed subgroups of the Borel groups \( \mathbb{B}_N(k), N = \mathbb{N}(n, s) \),
- \(((\mathfrak{G}^s_+)_n(k), \boxtimes)\) as a closed subgroups of the unipotent groups \( \mathbb{U}_N(k), N = \mathbb{N}(n, s) \),
- \(((\mathfrak{G}^s)_1(k), \boxtimes)\) as a group of multiplicative type \( \mathcal{T}_s(k) \).

17 The general \( S \)-transform

Definition 17.1 (\( S \)-transform). The minimal faithful representation \( \rho_k : (G^s_+(k), \boxtimes) \rightarrow \mathbb{B}_\infty(k) \) is called the \( s \)-dimensional \( S \)-transform.
If \((\mathcal{A}, \phi)\) is a non-commutative \(k\)-probability space and \(a = (a_1, \ldots, a_s) \in \mathcal{A}^s\) then the \(S\)-transform of \(a\) is defined as

\[
S_k(a) := \rho_k(\mathcal{R}_k(a))
\]

(3)

**Theorem 17.1.** Let \((\mathcal{A}, \phi)\) be a non-commutative \(k\)-probability space and \(a = (a_1, \ldots, a_s), b = (b_1, \ldots, b_s) \in \mathcal{A}^s\) such that \(\{a_1, \ldots, a_s\}\) and \(\{b_1, \ldots, b_s\}\) are combinatorially free. Then the \(S\)-transform is a faithful morphism

\[
S_k : \mathcal{A}^s \to \lim_{\leftarrow} \mathcal{B}_N(k)
\]

which satisfies

\[
S_k(a \ast b) = S_k(a) \cdot S_k(b)
\]

(4)

**Remark 17.2.** An equivalent description can be given in terms of moment series, by using the moment-cumulant formula.

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