Factor Models with Real Data:  
a Robust Estimation of the Number of Factors  

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Abstract

Factor models are a very efficient way to describe high dimensional vectors of data in terms of a small number of common relevant factors. This problem, which is of fundamental importance in many disciplines, is usually reformulated in mathematical terms as follows. We are given the covariance matrix \( \Sigma \) of the available data. \( \Sigma \) must be additively decomposed as the sum of two positive semidefinite matrices \( D \) and \( L \): \( D \) — that accounts for the idiosyncratic noise affecting the knowledge of each component of the available vector of data — must be diagonal and \( L \) must have the smallest possible rank in order to describe the available data in terms of the smallest possible number of independent factors.

In practice, however, the matrix \( \Sigma \) is never known and therefore it must be estimated from the data so that only an approximation of \( \Sigma \) is actually available. This paper discusses the issues that arise from this uncertainty and provides a strategy to deal with the problem of robustly estimating the number of factors.

Index Terms

factor analysis; nuclear norm; convex optimization; duality theory.

I. INTRODUCTION

Describing a large amount of data by means of a small number of factors carrying most of the information is an important problem in modern data analysis with applications ranging in all fields of science. One of the classical methods for this purpose is to resort to factor
models that were first developed at the beginning of the last century by Spearman \[45\] in the framework of the so-called mental tests as an attempt at “the procedure of eliciting verifiable facts” in determining psychical tendencies from the tests results. From this first seed a rich stream of literature was developed at the interface between psychology and mathematics with the main focus on the case of a single common factor underlying the available data: necessary and sufficient conditions for the data to be compatible with a single common factor were derived in \[12\], \[46\], see also \[7\] and references therein for a detailed historical reconstruction of the derivation of these conditions. The interest for this kind of model has grown rapidly also outside the psychology community and analysis of factor models, or factor analysis has become an important tool in statistics, econometrics, systems theory and many engineering fields \[29\], \[43\], \[7\], \[39\], \[23\], \[40\], \[38\], \[28\], \[16\], \[27\], \[2\], \[41\], \[22\], \[19\], \[48\], \[36\]; see also the more recent papers \[9\], \[54\], \[15\], \[20\], \[8\], \[17\] where many other references are listed. A detailed geometric description of this problem is presented in \[42\]. In the seminal paper \[4\] a maximum likelihood approach in a statistical testing framework is proposed.

In the original formulation the construction of a factor model is equivalent to the mathematical problem of additively decomposing a given positive definite matrix \(\Sigma\) — modeling the covariance of the data — as

\[
\Sigma = L + D \tag{1}
\]

where both \(L\) and \(D\) are positive semidefinite, and \(D\) — modeling the covariance of the idiosyncratic noise — is diagonal. The rank of \(L\) is the number of (latent or hidden) common factors that explain the available data. One of the key aspects of factor analysis is to determine the minimum number of latent factors or, equivalently, a decomposition (1) where the rank of \(L\) is minimal. This is therefore a particular case of a matrix additive decomposition problem that arises naturally in numerous frameworks and have therefore received a great deal of attention, see \[13\], \[1\], \[55\], \[52\] and references therein. We hasten to remark that the problem of minimizing the rank of \(L\) in the decomposition (1) is extremely hard so that, the convex relaxation is usually considered where, in place of the rank, the nuclear norm (i.e. the trace) of \(L\) is minimized. This is a very good approximation that most often returns, with reasonable computational burden, a solution \(L\) with minimum rank.

In \[47\], \[30\], \[33\] an upper bound \(r(n)\) — known as Ledermann bound — was proposed for
the minimal rank $r_m(\Sigma)$ of $L$ in terms of the dimension $n$ of the matrix $\Sigma$:

$$r_m(\Sigma) \leq r(n) := \left\lfloor \frac{2n + 1 - \sqrt{8n + 1}}{2} \right\rfloor.$$ 

This bound, however, is based on heuristics that have never been proven rigorously; a *pétale de rose* is the prize for a positive demonstration of this fact. Interestingly, almost half a century later in [44] a related result was established: the set of matrices $\Sigma$ for which $r_m(\Sigma) < r(n)$ has zero Lebesgue measure. As a consequence of this result we have the following observation that may be regarded as the basic premise of our effort. When $n$ is large the Ledermann bound $r(n)$ is not much smaller than $n$. Therefore, even if our data do come from a factor model with a small number $r$ of latent factors, only a set of zero measure of $\hat{\Sigma}$ in a neighbourhood of $\Sigma$ can be decomposed in such a way that the corresponding $L$ matrix in its decomposition (1) has rank $r$. Thus, unless we know $\Sigma$ with absolute precision, we cannot rely only on the decomposition (1) to recover such $r$. An example of this phenomenon is illustrated in Figure 1.

![Fig. 1](image)

*Fig. 1.* First 20 singular values of the matrices $L$ (on the left) and $\hat{L}$ (on the right) obtained by applying the minimum trace factor analysis decomposition algorithm to a “true” covariance matrix $\Sigma \in \mathbb{R}^{40 \times 40}$ of a model with $r = 4$ latent factors and to an estimate $\hat{\Sigma}$ of $\Sigma$ obtained by generating $N = 1000$ independent samples from a normal distribution $N(0, \Sigma)$ and computing the corresponding sample covariance, respectively. Notice that the recovered matrix $L$ using as input the true $\Sigma$ is, up to negligible numerical errors, equal to the true low rank matrix.

The problem of estimating $r$ from an estimate $\hat{\Sigma}$ of $\Sigma$ is therefore of crucial importance and has been addressed in [6] and [32] by means of statistical methods. A similar issue has been addressed also in [37] in the framework of the robustness of Frisch scheme. Here, we propose

\[\text{[Indeed, not only is a rigorous proof missing but a precise statement is also needed. In fact, some further assumptions must be added for the validity of this bound as counterexamples can, otherwise, be easily produced [24].]}\]
an alternative optimization-based approach which is based only on the estimate $\hat{\Sigma}$ and takes into account the uncertainty of this estimate. Hence, even if we can start from $N$ n-dimensional vectors (observations) the data of our problem are just the sample covariance $\hat{\Sigma}$ of these vectors and their number $N$. These two quantities summarize all the relevant information for our method in which we compute the matrix $\Sigma$ in such a way that the trace of $L$ in its additive decomposition $[1]$ is minimized under a constraint limiting the Kullback-Leibler divergence between $\Sigma$ and $\hat{\Sigma}$ to a prescribed tolerance that depends on the precision of our estimate $\hat{\Sigma}$ and hence may be reliably chosen on the basis of the data numerosity $N$.

The proposed problem is analyzed by resorting to duality theory. The dual analysis is delicate to carry over, but yields a problem whose solution can be efficiently computed by employing an alternating direction method of multipliers (ADMM) algorithm. Moreover, the dual problem provides a necessary and sufficient condition for the uniqueness of the solution of the original problem.

The paper is organized as follows. In the Section II we recall the classical approach to factor analysis and, from it, we derive the formulation of our factor analysis problem. In Section III we describe how to establish, for a desired tolerance, an upper bound on the aforementioned Kullback-Leibler divergence. In Section IV we derive a dual formulation of our problem. In Section V we prove existence and uniqueness of the solution for the dual problem. Then, in Section VI we show how to recover the solution of the primal problem. In Section VII we present the numerical algorithm for solving the dual problem, while in Section VIII the results of numerical simulations and an application to a real world example are presented. Finally, some conclusions are provided. The less instructive proofs that are essentially based on calculations are deferred to the Appendix.

Some of the results of this paper have been presented in preliminary form and mostly without proof in our conference paper [14].

**Notation:** Given a vector space $\mathcal{V}$ and a subspace $\mathcal{W} \subset \mathcal{V}$, we denote by $\mathcal{W}^\perp$ the orthogonal complement of $\mathcal{W}$ in $\mathcal{V}$. Given a matrix $M$, we denote its transpose by $M^T$; if $M$ is a square matrix $\text{tr}(M)$ denotes its trace, i.e. the sum of the elements in the main diagonal of $M$; moreover, $|M|$ denotes the determinant of $M$ and $\sigma(M)$ denotes the spectrum of $M$, i.e. the set of its eigenvalues. We denote the spectral norm of $M$ as $\|M\|_2$. We endow the space of square real matrices with the following inner product: for $A, B \in \mathbb{R}^{n \times n}$, $\langle A, B \rangle := \text{tr}(A^T B)$. The kernel of
a matrix (or of a linear operator) is denoted by $\ker(\cdot)$. The symbol $\mathbb{Q}_n$ denotes the vector space of real symmetric matrices of size $n$. If $X \in \mathbb{Q}_n$ is positive definite or positive semi-definite we write $X \succ 0$ or $X \succeq 0$, respectively. Moreover, we denote by $\mathbb{D}_n$ the vector space of diagonal matrices of size $n$; $\mathbb{D}_n$ is clearly a subspace of $\mathbb{Q}_n$ and we denote by $\mathbb{M}_n := \mathbb{D}_n^\perp$ the orthogonal complement of $\mathbb{D}_n$ in $\mathbb{Q}_n$ (with respect to the inner product just defined). It is easy to see that $\mathbb{M}_n$ is the vector space of symmetric matrices of size $n$ having all the elements on the main diagonal equal to zero. We denote by $\text{diag}(\cdot)$ both the operator mapping $n$ real elements $d_i, i = 1, \ldots, n$ into the diagonal matrix having the $d_i$’s as elements in its main diagonal and the operator mapping a matrix $M \in \mathbb{R}^{n \times n}$ into an $n$-dimensional vector containing the diagonal elements of $M$. Then $\text{diag} \text{diag}(\cdot)$, that we denote by $\text{diag}^2(\cdot)$, is the (orthogonal projection) operator mapping a square matrix $M$ into a diagonal matrix of the same size having the same main diagonal of $M$. We denote by $\text{ofd}(\cdot)$ the self-adjoint operator orthogonally projecting $\mathbb{Q}_n$ onto $\mathbb{M}_n$, i.e. if $M \in \mathbb{Q}_n$, $\text{ofd}(M)$ is the matrix of $\mathbb{M}_n$ in which each off-diagonal element is equal to the corresponding element of $M$ (and each diagonal element is clearly zero). Finally, we denote by $\otimes$ the Kronecker product between two matrices and by $\text{vec}(X)$ the vectorization of a matrix $X$ formed by stacking the columns of $X$ into a single column vector.

II. Problem Formulation

We consider a standard factor model in its static linear formulation

$$y = Ax + z$$

(2)

where $A \in \mathbb{R}^{n \times r}$, with $r << n$, is the factor loading matrix, $x$ represents the (independent) latent factors and $z$ is the idiosyncratic component. $x$ and $z$ are independent Gaussian random vectors with zero mean and covariance matrix equal to the identity matrix of dimension $r$ and $D \in \mathbb{D}_n$, respectively. Note that, $Ax$ represents the latent variable. Consequently, $y$ is a Gaussian random vector with zero mean; we denote by $\Sigma$ its covariance matrix. Since $x$ and $z$ are independent we get that $\Sigma$ may be additively decomposed as in (1), where $L := AA^\top$ and $D$ are the covariance matrices of $Ax$ and $z$, respectively. Thus, $L$ has rank equal to $r$, and $D$ is diagonal.

The objective of factor analysis consists in finding the most parsimonious “low-rank plus diagonal” decomposition of $\Sigma$, that is a decomposition (1) for which the rank of $L$ is minimal.
This amounts to solve the minimum rank problem

\[
\min_{L,D \in \mathbb{Q}_n} \quad \text{rank}(L)
\]

subject to

\[
L, D \succeq 0
\]

\[
D \in \mathbb{D}_n
\]

\[
\Sigma = L + D
\]

which is, however, a hard problem. A well-known and widely used heuristic is the convex relaxation of (3), i.e. the trace minimization problem

\[
\min_{L,D \in \mathbb{Q}_n} \quad \text{tr}(L)
\]

subject to

\[
L, D \succeq 0
\]

\[
D \in \mathbb{D}_n
\]

\[
\Sigma = L + D
\]

The substitution of the rank with the trace is justified by the fact that \(\text{tr}(L)\), i.e. the nuclear norm of \(L\), is the convex hull of \(\text{rank}(L)\) over the set \(S := \{L \in \mathbb{Q}_n \text{ s.t. } \|L\|_2 \leq 1\}\). \cite{21}. The relation between Problem (3) and Problem (4) has been first studied in \cite{18} and while these two problems are, in general, not equivalent, very often they have the same solution.

In practice, however, matrix \(\Sigma\) is not known and needs to be estimated from a \(N\)-length realization (i.e. a data record) \(y_1 \ldots y_N\) of \(y\). The typical choice is to take the sample covariance estimate

\[
\hat{\Sigma} := \frac{1}{N} \sum_{k=1}^{N} y_k y_k^T
\]

which is statistically consistent, i.e. the corresponding estimator almost surely converges to \(\Sigma\) as \(N\) tends to infinity. As discussed in the Introduction, by replacing \(\Sigma\) with \(\hat{\Sigma}\) the solution, in terms of minimum rank, will rapidly degrade. Indeed a delicate problem in factor analysis is the one of estimating the number of factors. Such a problem has been addressed by several important contributions, see the seminal works of Bai and Ng \cite{6} and of Lam and Yao \cite{32} and the references therein. Our objective is to address the same problem from a different perspective. In fact, we propose an optimization problem whose solution provides an estimate of the minimum number of factors by introducing an appropriate model for the error in the estimation of \(\Sigma\). This
model is based on an auxiliary Gaussian random vector \( \hat{y} \) with zero mean and covariance matrix \( \hat{\Sigma} \) that is regarded as a “model approximation” for \( y \). To account for the estimation uncertainty, we assume that the distribution of \( y \) (that is completely specified by its covariance matrix and hence is referred to by \( \Sigma \)) belongs to a “ball” centred in \( \hat{y} \)

\[
\mathcal{B} := \{ \Sigma \in \mathbb{Q}_n \text{ s.t. } \Sigma \succ 0, \ D_{KL}(\Sigma \| \hat{\Sigma}) \leq \frac{\delta}{2} \}
\]  

which is formed by placing a bound (i.e. tolerance) on the Kullback-Leibler divergence between \( y \) and \( \hat{y} \):

\[
D_{KL}(\Sigma \| \hat{\Sigma}) := \frac{1}{2} \left( -\log |\Sigma| + \log |\hat{\Sigma}| + \text{tr}(\Sigma \hat{\Sigma}^{-1}) - n \right).
\]

This way to deal with model uncertainty has been successfully applied in econometrics for model mispecification [26] and in robust filtering [35], [50], [34], [53], [49], [51]. Accordingly, in order to estimate the minimum number of factors, we propose the following “robustification” of the minimum trace problem:

\[
\begin{align*}
\min_{\Sigma, L, D \in \mathbb{Q}_n} & \quad \text{tr}(L) \\
\text{subject to} & \quad L, D \succeq 0 \\
& \quad D \in \mathbb{D}_n \\
& \quad \Sigma = L + D \\
& \quad \Sigma \in \mathcal{B}.
\end{align*}
\]  

(7)

Note that, in (7) we can eliminate variable \( D \), obtaining the equivalent problem

\[
\begin{align*}
\min_{L, \Sigma \in \mathbb{Q}_n} & \quad \text{tr}(L) \\
\text{subject to} & \quad L, \Sigma - L \succeq 0 \\
& \quad \text{ofd}(\Sigma - L) = 0 \\
& \quad \Sigma \succ 0 \\
& \quad 2D_{KL}(\Sigma \| \hat{\Sigma}) \leq \delta.
\end{align*}
\]  

(8)

It is worth noting that an alternative to Problem (8) is to consider \( D_{KL}(\Sigma \| \hat{\Sigma}) \) as a penalty term in the objective function rather than as a constraint. Such approach, however, would require a cross validation procedure to set the regularization parameter \( \lambda \), i.e. we would have to solve an optimization problem for many values of \( \lambda \). In contrast, the proposed problem is solved only once provided that \( \delta \) is chosen in a suitable way, see the next section.
III. THE CHOICE OF $\delta$

The tolerance $\delta$ may be chosen by taking into account the accuracy of the estimate $\hat{\Sigma}$ of $\Sigma$ which, in turn, depends on the numerosity of the available data. This can be done by choosing a probability $\alpha \in (0, 1)$ and a neighborhood of “radius” $\delta_\alpha$ (in the Kullback-Leibler topology) centered in $\hat{\Sigma}$ containing the “true” $\Sigma$ with probability $\alpha$. The Kullback-Leibler divergence in (8) is a function of the estimated sample covariance and as such its accuracy depends crucially on the numerosity of the available data. To assess this accuracy we propose an approach that hinges on the following scale-invariance property of the Kullback-Leibler divergence.

**Lemma 3.1:** Let $y_i \sim \mathcal{N}(0, \Sigma)$, $i = 1, ..., N$ be i.i.d. random variables taking values in $\mathbb{R}^n$ and define the sample covariance estimator

$$\hat{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} y_i y_i^T.$$  

The Kullback-Leibler divergence between $\Sigma$ and $\hat{\Sigma}$ is a random variable whose distribution depends only of the number $N$ of random variables and on the dimension $n$ of each random variable.

**Proof:** We have

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} y_i y_i^T = \frac{1}{N} \Sigma^{1/2} \sum_{i=1}^{N} \tilde{y}_i \tilde{y}_i^T \Sigma^{1/2} = \Sigma^{1/2} Q_N \Sigma^{1/2}$$

with $\tilde{y}_i = \Sigma^{-1/2} y_i \sim \mathcal{N}(0, I_n)$ and $Q_N := \frac{1}{N} \sum_{i=1}^{N} \tilde{y}_i \tilde{y}_i^T$, is a random matrix taking values in $Q_n$. Notice that at this point $\tilde{y}$ are normalized Gaussian random vectors and hence do not depend on the data nor on $\Sigma$. Thus, $Q_N$ is a random matrix whose distribution only depends on $N$ and $n$ (see Section III-A for more details). Hence, the Kullback-Leibler divergence between $\Sigma$ and the sample covariance estimator is

$$d := D_{KL}(\Sigma \parallel \hat{\Sigma}) = \frac{1}{2} \left( \log(|\Sigma \Sigma^{-1}|) + \text{tr}(\Sigma \Sigma^{-1}) - n \right)$$

$$= \frac{1}{2} \left( \log(|Q_N|) + \text{tr}(Q_{N}^{-1}) - n \right). \quad (9)$$

In view of this result we can easily approximate the distribution of the random variable $2d = 2D_{KL}(\Sigma \parallel \hat{\Sigma})$ by a standard Monte Carlo method. In particular, we can reliably estimate with arbitrary precision the value of $\delta$ for which $Pr(2D_{KL}(\Sigma \parallel \hat{\Sigma}) \leq \delta) = \alpha$. As an alternative
to this empiric approach for determining $\delta_\alpha$, we can also resort to an analytic one as discussed below.

A. **Gaussian Orthogonal Ensemble**

Let us focus on the random matrix $Q_N$ that we have defined as $Q_N := \frac{1}{N} \sum_{i=1}^{N} \tilde{y}_i \tilde{y}_i^T$ where $\tilde{y}_i \in \mathbb{R}^n$, $i = 1, ..., N$, are i.i.d. random variables distributed as $\mathcal{N}(0, I_n)$. We now introduce a new matrix $\tilde{Q}_N := \sqrt{N} (Q_N - I_n) = \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} C_i$, where $C_i := \tilde{y}_i \tilde{y}_i^T - I$ are i.i.d. symmetric random matrices with zero mean. It is immediate to check that for each $i$: any two distinct elements $[C_i]_{h,j}$ and $[C_i]_{k,l}$ of $C_i$ are uncorrelated as long as they do not occupy symmetric positions, i.e. whenever $(h, j) \neq (l, k)$, and $\text{Var} ([C_i]_{h,j}) = \begin{cases} 1, & \text{if } h \neq j \\ 2, & \text{if } h = j \end{cases}$. By the multivariate Central Limit Theorem, we have that $\tilde{Q}_N$ converges in distribution to the random matrix

$$X = \left( \begin{array}{cccc} \sqrt{2} \xi_{1,1} & \cdots & \cdots & \xi_{1,n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \xi_{1,n} & \cdots & \cdots & \sqrt{2} \xi_{n,n} \end{array} \right) \in \mathbb{Q}_n,$$

where $\{\xi_{i,j}\}$ are i.i.d. Gaussian random variables with mean 0 and variance 1. The set of these matrices is known as the **Gaussian Orthogonal Ensemble**, see [3]. It is well known that the joint distribution of the eigenvalues $\lambda_1(X) \leq ... \leq \lambda_n(X)$ of such matrices takes the following form:

$$p(\lambda_1, ..., \lambda_n) = \bar{C}_n |\Delta(\lambda)| \prod_{i=1}^{n} e^{-\lambda_i^2/4}$$

where $\lambda := (\lambda_1, ..., \lambda_n)$, $|\Delta(\lambda)|$ is the Vandermonde determinant associated with $\lambda$, which is given by:

$$|\Delta(\lambda)| = \prod_{i<j} (\lambda_j - \lambda_i)$$

and $\bar{C}_n$ is defined as:

$$\bar{C}_n = \left( \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} |\Delta(\lambda)| \prod_{i=1}^{n} e^{-\lambda_i^2/4} d\lambda_i \right)^{-1}.$$ 

It is not difficult to see that (9) can be rewritten as:

$$d = d(\lambda_1, ..., \lambda_n) = \sum_{i=1}^{n} \frac{1}{2} \left( \log \left( \frac{\lambda_i}{\sqrt{N}} + 1 \right) - \frac{\lambda_i}{\lambda_i + \sqrt{N}} \right)$$

(10)
where \( \lambda_i \in \sigma(\tilde{Q}_N) \). Then, for a desired \( \alpha \), we are interested in finding \( \delta_\alpha \) such that
\[
Pr(2d \leq \delta_\alpha) = \alpha.
\]
Such a value for \( \delta_\alpha \) is given by the cumulative distribution function \( F(\cdot) \):
\[
F(\delta_\alpha) = Pr(2d \leq \delta_\alpha) = \int_{I(\delta_\alpha)} 2d(\lambda_1, \ldots, \lambda_n)p(\lambda_1, \ldots, \lambda_n) d\lambda
\]
where \( p(\lambda_1, \ldots, \lambda_n) \) denotes the joint probability density function of the eigenvalues \( \lambda_1, \ldots, \lambda_n \) and \( I(\delta_\alpha) := \{(\lambda_1, \ldots, \lambda_n) : d(\lambda_1, \ldots, \lambda_n) \leq \delta_\alpha/2\} \). Given \( \alpha \) the integral in (11) can be solved numerically for \( \delta_\alpha \).

**B. An upper bound for \( \delta_\alpha \)**

If the chosen level \( \alpha \) is too large with respect to the sample size \( N \), the computed \( \delta_\alpha \) become excessively large so that there are diagonal matrices \( \Sigma_D \) such that \( 2D_{KL}(\Sigma_D\|\hat{\Sigma}) \leq \delta_\alpha \). In this case Problem (8) admits the trivial solution \( L = 0 \) and \( D = \Sigma_D \). In order to rule out this trivial situation we need to require that the maximum value for \( \delta \) in (8) is strictly less than a certain \( \delta_{\text{max}} \) that can be determined as follows: since the trivial solution \( L = 0 \) would imply a diagonal \( \Sigma \), that is \( \Sigma = \Sigma_D := \text{diag}(d_1, \ldots, d_n) > 0 \), \( \delta_{\text{max}} \) can be determined by solving the following minimization problem
\[
\delta_{\text{max}} := \min_{\Sigma_D \in \mathcal{D}_n} 2D_{KL}(\Sigma_D\|\hat{\Sigma}).
\]

The following Proposition, whose proof is in Appendix, shows how to solve this problem.

**Proposition 3.1:** Let \( \gamma_i \) denote the \( i \)-th element in the main diagonal of the inverse of the sample covariance \( \hat{\Sigma}^{-1} \). Then, the optimal \( \Sigma_D \) which solves the minimization problem in (12) is given by
\[
\Sigma_D = \text{diag}(\gamma_1^{-1}, \ldots, \gamma_n^{-1}).
\]
Moreover, \( \delta_{\text{max}} \) can be determined as
\[
\delta_{\text{max}} = 2D_{KL}(\Sigma_D\|\hat{\Sigma}) = \log |\text{diag}^2(\hat{\Sigma}^{-1})\hat{\Sigma}|.
\]

In what follows, we always assume that \( \delta \) in (8) strictly less than \( \delta_{\text{max}} \), so that the trivial solution \( L = 0 \) is ruled out.
IV. DUAL PROBLEM

By duality theory, we reformulate the constrained minimization problem in (8) as an unconstrained minimization problem. The associated Lagrangian is

\[ L(L, \Sigma, \lambda, \Lambda, \Gamma, \Theta) = \text{tr}(L) + \lambda(-\log |\Sigma| + \log |\hat{\Sigma}|) - \text{tr}(\hat{\Sigma}^{-1}\Sigma) - \delta \]

\[ - \text{tr}(\Lambda L) - \text{tr}(\Gamma(\Sigma - L)) + \text{tr}(\Theta \text{ofd}(\Sigma - L)) \]

\[ = \text{tr}(L) + \lambda(-\log |\Sigma| + \log |\hat{\Sigma}|) - n + \text{tr}(\hat{\Sigma}^{-1}\Sigma) - \delta \]

\[ - \text{tr}(\Lambda L) - \text{tr}(\Gamma(\Sigma - L)) + \text{tr}(\Theta \text{ofd}(\Sigma - L)) \]

\[ = \text{tr}(L) + \lambda(-\log |\Sigma| + \log |\hat{\Sigma}|) - n + \text{tr}(\hat{\Sigma}^{-1}\Sigma) - \delta \]

\[ - \text{tr}(\Lambda L) - \text{tr}(\Gamma(\Sigma - L)) + \text{tr}(\Theta \text{ofd}(\Sigma - L)) \]

(14)

with \( \lambda \in \mathbb{R}, \lambda \geq 0 \), and \( \Lambda, \Gamma, \Theta \in Q_n \) with \( \Lambda, \Gamma \succeq 0 \). In the last equality, we exploited the fact that the operator \( \text{ofd}(\cdot) \) is self-adjoint. Note that the Lagrangian (14) does not include the constraint \( \Sigma \succ 0 \): as we will see this condition is automatically met by the solution of the dual problem.

Notice also that in (14) we can recognize the fit term \( 2\lambda \mathcal{D}(\Sigma \parallel \hat{\Sigma}) = \lambda(-\log |\Sigma| + \log |\hat{\Sigma}| - n + \text{tr}(\hat{\Sigma}^{-1}\Sigma)) \) and the term \( \text{tr}((I - \Lambda)L + (\text{ofd}((\Theta) - \Gamma)(\Sigma - L)) \) accounting for the complexity in the class of models (1) which induces low-rank on matrix \( L \). Thus (14) can be interpreted as an alternative to the likelihood function with a complexity term.

The dual function is defined as the infimum of \( L(L, \Sigma, \lambda, \Lambda, \Gamma, \Theta) \) over \( L \) and \( \Sigma \).

Thanks to the convexity of the Lagrangian, we rely on standard variational methods to characterize the minimum.

The first variation of the Lagrangian (14) at \( \Sigma \) in direction \( \delta \Sigma \in Q_n \) is

\[ \delta L(\Sigma; \delta \Sigma) = \text{tr}(-\lambda \Sigma^{-1} \delta \Sigma + \lambda \hat{\Sigma}^{-1} \delta \Sigma - \Gamma \delta \Sigma + \text{ofd}(\Theta) \delta \Sigma). \]

We impose the optimality condition

\[ \delta L(\Sigma; \delta \Sigma) = 0, \quad \forall \delta \Sigma \in Q_n, \]

which is equivalent to require \( \text{tr}(-\lambda \Sigma^{-1} \delta \Sigma + \lambda \hat{\Sigma}^{-1} \delta \Sigma - \Gamma \delta \Sigma + \text{ofd}(\Theta) \delta \Sigma) = 0 \) for all \( \delta \Sigma \in Q_n \), obtaining

\[ \Sigma = \lambda(\lambda \hat{\Sigma}^{-1} - \Gamma + \text{ofd}(\Theta))^{-1} \]

(15)
provided that $\lambda \hat{\Sigma}^{-1} - \Gamma + \text{ofd}(\Theta) \succ 0$ and $\lambda > 0$, which is clearly equivalent to require that the optimal $\Sigma$ that minimizes the Lagrangian satisfies the constraint $\Sigma \succ 0$.

The first variation of the Lagrangian (14) at $L$ in direction $\delta L \in \mathbb{Q}_n$ is

$$\delta \mathcal{L}(L; \delta L) = \text{tr}(\delta L - \Lambda \delta L + \Gamma \delta L - \text{ofd}(\Theta) \delta L).$$

Again, we impose the optimality condition

$$\delta \mathcal{L}(L; \delta L) = 0, \quad \forall \delta L \in \mathbb{Q}_n,$$

which is equivalent to require $\text{tr}(\delta L - \Lambda \delta L + \Gamma \delta L - \text{ofd}(\Theta) \delta L) = 0$ for all $\delta L \in \mathbb{Q}_n$ and we get that

$$I - \Lambda + \Gamma - \text{ofd}(\Theta) = 0. \quad (16)$$

The following result, whose proof is in Appendix, provides a precise formulation of the dual problem.

**Proposition 4.1:** The dual problem of (8) is

$$\max_{(\lambda, \Gamma, \Theta) \in C_0} J(\lambda, \Gamma, \Theta) \quad (17)$$

where

$$J(\lambda, \Gamma, \Theta) := \lambda (\log |(\hat{\Sigma}^{-1} + \lambda^{-1} (\text{ofd}(\Theta) - \Gamma))|$$

$$+ \log |\hat{\Sigma}| - \delta)$$

and $C_0$ is defined as

$$C_0 := \{ (\lambda, \Gamma, \Theta) : \lambda > 0, \ I + \Gamma - \text{ofd}(\Theta) \succeq 0, \ \Gamma \succeq 0,$$

$$\hat{\Sigma}^{-1} + \lambda^{-1} (\text{ofd}(\Theta) - \Gamma) \succ 0 \}. \quad (18)$$

**V. Existence and Uniqueness of the Solution for the Dual Problem**

We reformulate the maximization problem in (17) as a minimization problem:

$$\min_{(\lambda, \Gamma, \Theta) \in C_0} \tilde{J}(\lambda, \Gamma, \Theta) \quad (19)$$

where

$$\tilde{J}(\lambda, \Gamma, \Theta) = \lambda (- \log |\hat{\Sigma}^{-1} + \lambda^{-1} (\text{ofd}(\Theta) - \Gamma)|$$

$$- \log |\hat{\Sigma}| + \delta).$$
A. Existence

As it is often the case, existence of the optimal solution is a very delicate issue. Our strategy in order to deal with that is to prove that the dual problem in (19) admits solution. In doing that we show that we can restrict our set $C_0$ to a compact set $C$ over which the minimization problem is equivalent to the one in (19). Since the objective function is continuous over $C_0$, and hence over $C$, by Weierstrass’s theorem $\tilde{J}$ admits a minimum.

First, we recall that the operator $\text{ofd}(\cdot)$ is self-adjoint. Moreover, we notice that $\text{ofd}(\cdot)$ is not injective on $\Theta$, thus we can restrict the domain of $\text{ofd}(\cdot)$ to those $\Theta$ such that $\text{ofd}(\cdot)$ is injective.

Since $\text{ofd}$ is self-adjoint we have that:

$$\ker(\text{ofd}) = [\text{range } (\text{ofd})]^\perp.$$  

Thus, by restricting $\Theta$ to $\text{range}(\text{ofd}) = [\ker(\text{ofd})]^\perp = M_n$, the map becomes injective. Therefore, without loss of generality, from now on we can safely assume that $\Theta \in M_n$ so that $\text{ofd}(\Theta) = \Theta$ and we restrict our set $C_0$ to $C_1$:

$$C_1 := \{ (\lambda, \Gamma, \Theta) \in C_0 : \Theta \in M_n \}$$

$$= \{ (\lambda, \Gamma, \Theta) : \lambda > 0, I + \Gamma - \Theta \succeq 0, \Gamma \succeq 0,$$

$$\Theta \in M_n, (\hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma)) \succ 0 \}.$$  

Moreover, since $\Theta$ and $\Gamma$ enter into the problem always through their difference they cannot be univocally determined individually. However, their difference does. This allows us to restrict $\Gamma$ to the space of the diagonal positive semi-definite matrices. Indeed, for any sequence $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}} \in C_1$ such that $\inf \tilde{J}(\lambda, \Gamma, \Theta) = \lim_{k \to \infty} \tilde{J}(\lambda_k, \Gamma_k, \Theta_k)$ we can always consider a different sequence $(\lambda_k, \tilde{\Gamma}_k, \tilde{\Theta}_k)_{k \in \mathbb{N}}$ with $\tilde{\Gamma}_k := \text{diag}^2(\Gamma_k)$ and $\tilde{\Theta}_k := \Theta_k - \text{ofd}(\Gamma_k)$. It is now immediate to check that the new sequence still belongs to $C_1$ and that we still have $\inf \tilde{J}(\lambda, \Gamma, \Theta) = \lim_{k \to \infty} \tilde{J}(\lambda_k, \tilde{\Gamma}_k, \tilde{\Theta}_k)$. For this reason, we can further restrict our set $C_1$ to $C_2$:

$$C_2 := \{ (\lambda, \Gamma, \Theta) : \lambda > 0, I + \Gamma - \Theta \succeq 0, \Gamma \succeq 0, \Gamma \in D_n,$$

$$\Theta \in M_n, (\hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma)) \succ 0 \}.$$  

**Lemma 5.1**: Let $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ be a sequence of elements in $C_2$ such that

$$\lim_{k \to \infty} \lambda_k = 0.$$
Then \((\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}\) is not an infimizing sequence for \(\tilde{J}\).

**Proof:** We consider two cases separately. Let us first analyze the case of sequences \((\lambda_k, \Gamma_k, \Theta_k)\) in which, beside \(\lambda_k \to 0\), we also have \(\|\lambda_k^{-1}(\Theta_k - \Gamma_k)\| \to \infty\) as \(k \to \infty\). This implies that the largest singular value of \(\lambda_k^{-1}(\Theta_k - \Gamma_k)\) tends to infinity and this, by symmetry, implies in turn that

\[
\lim_{k \to \infty} \max_{\alpha_k \in \sigma(\lambda_k^{-1}(\Theta_k - \Gamma_k))} |\alpha_k| = +\infty. \tag{20}
\]

We now show that this implies

\[
\lim_{k \to \infty} \min_{\alpha_k \in \sigma(\lambda_k^{-1}(\Theta_k - \Gamma_k))} \alpha_k = -\infty. \tag{21}
\]

To this end, we observe that from (20) it follows that at least one of the following statements is true:

(21) holds (and in this case we are done) or

\[
\lim_{k \to \infty} \max_{\alpha_k \in \sigma(\lambda_k^{-1}(\Theta_k - \Gamma_k))} \alpha_k = +\infty. \tag{22}
\]

In the latter case, we use the fact that \(\Gamma_k \succeq 0\) and \(\lambda_k > 0\), so that

\[
\max_{\alpha_k \in \sigma(\lambda_k^{-1}\Theta_k)} \alpha_k \geq \max_{\alpha_k \in \sigma(\lambda_k^{-1}(\Theta_k - \Gamma_k))} \alpha_k \tag{23}
\]

which, together with (22) gives

\[
\lim_{k \to \infty} \max_{\alpha_k \in \sigma(\lambda_k^{-1}\Theta_k)} \alpha_k = +\infty. \tag{24}
\]

Since \(\text{tr}(\lambda_k^{-1}\Theta_k) = 0\), (24) implies that

\[
\lim_{k \to \infty} \min_{\alpha_k \in \sigma(\lambda_k^{-1}\Theta_k)} \alpha_k = -\infty. \tag{25}
\]

Now we use again the fact that \(\Gamma_k \succeq 0\) and \(\lambda_k > 0\), so that

\[
\min_{\alpha_k \in \sigma(\lambda_k^{-1}(\Theta_k - \Gamma_k))} \alpha_k \leq \min_{\alpha_k \in \sigma(\lambda_k^{-1}\Theta_k)} \alpha_k \tag{26}
\]

which, together with (25) implies (21). In conclusion, for sequences \((\lambda_k, \Gamma_k, \Theta_k)\) of this type and for a sufficiently large \(k\), \(\hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k)\) is no longer positive definite and therefore these sequences does not belong to \(C_2\).

Second, we consider the case of sequences \((\lambda_k, \Gamma_k, \Theta_k)\) in which, beside \(\lambda_k \to 0\), we also have \(\|\lambda_k^{-1}(\Theta_k - \Gamma_k)\| \to c\) as \(k \to \infty\), where \(c < +\infty\) is a non-negative value.
In this case, it is not difficult to see that \( \forall \varepsilon > 0, \exists \bar{k} \) such that the dual functional satisfies
\[
J(\lambda_k, \Gamma_k, \Theta_k) > -\varepsilon, \forall k \geq \bar{k}.
\]
In fact, since \( \|\lambda_k^{-1}(\Theta_k - \Gamma_k)\| \) is bounded, there exists \( l_0 > 0 \) such that \( \lambda_k^{-1}(\Theta_k - \Gamma_k) \leq l_0 I \) for all \( k \). Therefore, there exists \( l_1 > 0 \) such that for all \( k \),
\[
\hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k) \leq l_1 I \quad \text{and hence there exists } l_2 > 0 \text{ such that for all } k, \left| \hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k) \right| \leq l_2.
\]
In turn, there exists \( l_3 \in \mathbb{R} \) such that for all \( k \),
\[
\log \left| \hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k) \right| \leq l_3 \quad \text{and} \quad -\log \left| \hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k) \right| \geq -l_3.
\]
Eventually, there exists a real constant \( l_4 := -l_3 - \log \left| \hat{\Sigma} \right| + \delta \) such that, for all \( k \), \( \tilde{J}(\lambda_k, \Gamma_k, \Theta_k) \geq \lambda k l_4 \). Since \( l_4 \) is constant, the the right-hand side of this inequality converges to zero so that, by definition \( \forall \varepsilon > 0, \exists \bar{k} \) such that \( \lambda_k l_4 > -\varepsilon \forall k \geq \bar{k} \). As a consequence, \( \tilde{J}(\lambda_k, \Gamma_k, \Theta_k) > -\varepsilon, \forall k \geq \bar{k} \). It is therefore sufficient to exhibit a triple \( (\bar{\lambda}, \bar{\Gamma}, \bar{\Theta}) \in C_2 \) for which the dual functional is negative to conclude that sequences \( (\lambda_k, \Gamma_k, \Theta_k) \) of this kind cannot be minimizing sequences. Let us consider \( (\bar{\lambda}, \bar{\Gamma}, \bar{\Theta}) \) such that \( \bar{\lambda} > 0, \bar{\Gamma} = 0 \) and
\[
\bar{\Theta} = -\bar{\lambda} \operatorname{ofd}(\hat{\Sigma}^{-1}).
\]
For \( \bar{\lambda} \) sufficiently large, but finite, it is immediate to check that this triple is in \( C_2 \). For this choice of the multipliers and taking into account \( (13) \) we have that
\[
\tilde{J}(\bar{\lambda}, \bar{\Gamma}, \bar{\Theta}) = -\bar{\lambda} \log \left| \hat{\Sigma}^{-1} + \bar{\lambda}^{-1}(\bar{\Theta} - \bar{\Gamma}) \right| - \bar{\lambda} \log \left| \hat{\Sigma} \right| + \bar{\lambda} \delta
\]
\[
= -\bar{\lambda} \log \left| (\hat{\Sigma}^{-1} + \bar{\lambda}^{-1}\Theta)\hat{\Sigma} \right| + \bar{\lambda} \delta
\]
\[
= -\bar{\lambda} \log \left| \operatorname{diag}(\hat{\Sigma}^{-1}) \hat{\Sigma} \right| + \bar{\lambda} \delta
\]
\[
= -\bar{\lambda} (\delta_{\max} - \delta) < 0.
\]
This is sufficient to conclude the proof. In fact, the only other possible case is the one in which \( \lim_{k \to \infty} \|\lambda_k^{-1}(\Theta_k - \Gamma_k)\| \) does not exist. In this case however, we can consider a sub-sequence \( (\lambda_{k_j}, \Gamma_{k_j}, \Theta_{k_j}) \) for which the corresponding limit does exist (finite or infinite) and we are thus reduced to one of the previous two cases. \( \square \)

As a consequence of the previous result we have that the minimization of the dual functional over the set \( C_2 \) is equivalent to minimization over the set:
\[
C_3 := \{ (\lambda, \Gamma, \Theta) : \lambda \geq \varepsilon, I + \Gamma - \Theta \geq 0, \Gamma \geq 0, \Gamma \in D_n, \Theta \in M_n, \hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma) \geq 0 \}
\]
for a certain $\varepsilon > 0$.

The next result provides an upper bound for $\lambda$.

**Lemma 5.2:** Let $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ be a sequence of elements in $C_3$ such that

$$\lim_{k \to \infty} \lambda_k = \infty. \quad (27)$$

Then $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ is not an infimizing sequence for $\tilde{J}$.

**Proof:** Let us consider a sequence $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ such that (27) holds.

It follows from the condition $\Theta_k - \Gamma_k \preceq I$ that

$$\lambda_k^{-1} (\Theta_k - \Gamma_k) \preceq \lambda_k^{-1} I$$

which implies that

$$\tilde{J}(\lambda_k, \Gamma_k, \Theta_k) = \lambda_k (\log |(\hat{\Sigma}^{-1} + \lambda_k^{-1} (\Theta_k - \Gamma_k))^{-1} \hat{\Sigma}^{-1}| + \delta) \geq \lambda_k (\log |((\hat{\Sigma}^{-1} + \lambda_k^{-1} I)^{-1} \hat{\Sigma}^{-1})| + \delta) \quad (28)$$

$$\rightarrow +\infty$$

so that $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ cannot be an infimizing sequence. \hfill \Box

As a consequence of the previous result, the set $C_3$ can be further restricted to the set:

$$C_4 := \{ (\lambda, \Gamma, \Theta) : \varepsilon \leq \lambda \leq M, I + \Gamma - \Theta \succeq 0, \Gamma \succeq 0, \Gamma \in D_n, \Theta \in M_n, \hat{\Sigma}^{-1} + \lambda^{-1} (\Theta - \Gamma) \succ 0 \}$$

for a certain $M < \infty$.

The next result provides an upper bound for $\Theta - \Gamma$.

**Lemma 5.3:** Let $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ be a sequence of elements in $C_4$ such that

$$\lim_{k \to \infty} \|\Theta_k - \Gamma_k\| = +\infty. \quad (29)$$

Then $(\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}$ is not an infimizing sequence for $\tilde{J}$.

**Proof:** From (29) it follows that the largest singular value of $(\Theta_k - \Gamma_k)$ tends to $+\infty$ as $k \to \infty$. This in turn implies that, as $k \to \infty$, at least one of the eigenvalues of $(\Theta_k - \Gamma_k)$ diverges, because $(\Theta_k - \Gamma_k)$ is symmetric so that its singular values are the absolute values of
its eigenvalues. As before, since \((\Theta_k - \Gamma_k) \preceq I\) holds, the diverging eigenvalues have to tend to \(-\infty\). This implies that also \(\hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k)\) has an eigenvalue which tends to \(-\infty\) as \(k \to \infty\). But, this cannot be the case, because we have the positive definiteness constraint on \(\hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k)\).

\[\square\]

It follows from the previous result that there exists \(\rho\) such that \(|\rho| < \infty\) and \(\Theta - \Gamma \succeq \rho I\).

Therefore, the set \(C_4\) can be further restricted to the set:

\[
C_5 := \{ (\lambda, \Gamma, \Theta) : \varepsilon \leq \lambda \leq M, \rho I \preceq \Theta - \Gamma \preceq I, \Gamma \succeq 0, \\
\Gamma \in D_n, \Theta \in M_n, \hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma) \succ 0 \}.
\]

Now observe that in \(C_5\) \(\Theta\) and \(\Gamma\) are orthogonal so that if \((\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}\) is a sequence of elements in \(C_5\) such that

\[
\lim_{k \to \infty} \|\Gamma_k\| = +\infty
\]

or

\[
\lim_{k \to \infty} \|\Theta_k\| = +\infty
\]

then \((29)\) holds. Then we have the following Corollary.

**Corollary 5.1:** Let \((\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}\) be a sequence of elements in \(C_5\) such that \((30)\) or \((31)\) holds. Then \((\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}\) is not an infimizing sequence for \(\tilde{J}\).

Thus minimizing over the set \(C_5\) is equivalent to minimize over:

\[
C_6 := \{ (\lambda, \Gamma, \Theta) : \varepsilon \leq \lambda \leq M, \rho I \preceq \Theta - \Gamma \preceq I, 0 \preceq \Gamma \preceq \alpha I, \\
\Gamma \in D_n, \Theta \in M_n, \hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma) \succ 0 \}
\]

for a certain \(\alpha\) such that \(0 < \alpha < +\infty\).

Finally, let us consider a sequence \((\lambda_k, \Gamma_k, \Theta_k)_{k \in \mathbb{N}}\) such that, as \(k \to \infty\), the minimum eigenvalue of \(\hat{\Sigma} + \lambda_k^{-1}(\Theta_k - \Gamma_k)\) tends to zero. This implies that \(|\hat{\Sigma}^{-1} + \lambda_k^{-1}(\Theta_k - \Gamma_k)| \to 0\) and
hence $\tilde{J} \to +\infty$. Thus, such sequence does not infimize the dual functional. Thus, the final set $C$ is

$$C := \{(\lambda, \Gamma, \Theta) : \varepsilon \leq \lambda \leq M, \rho I \preceq \Theta - \Gamma \preceq I, 0 \preceq \Gamma \preceq \alpha I, \Gamma \in \mathcal{D}_n, \Theta \in \mathcal{M}_n, \hat{\Sigma}^{-1} + \lambda^{-1}(\Theta - \Gamma) \succeq \beta I\}$$

for a suitable $\beta > 0$.

Summing up we have the following Theorem.

**Theorem 5.1:** Problem (19) is equivalent to

$$\min_{(\lambda, \Gamma, \Theta) \in C} \tilde{J}(\lambda, \Gamma, \Theta). \quad (32)$$

Both these problems admit solution.

**Proof:** Equivalence of the two problems has already been proven by the previous argument. Since $C$ is closed and bounded and, hence, compact, and $\tilde{J}$ is continuous over $C$, by the Weierstrass’s Theorem the minimum exists. \hfill \square

Before discussing the uniqueness of the solution to (19), it is convenient to further simplify the dual optimization problem: consider the function

$$F(\lambda, X) := -\lambda[\log(|\hat{\Sigma}^{-1} + \lambda^{-1}X|) + \log|\hat{\Sigma}| - \delta]$$

where $\lambda > 0$ and $X \in \mathbb{Q}_n$. Note that

$$F(\lambda, \Theta - \Gamma) = \tilde{J}(\lambda, \Gamma, \Theta).$$

Moreover, $\Theta$ and $\Gamma$ are orthogonal over $C$ so that minimizing $\tilde{J}$ over $C_0$ is equivalent to minimize $F$ over the corresponding set

$$C_F := \{(\lambda, X) : \lambda > 0, X \in \mathbb{Q}_n, X \preceq I, -\text{diag}^2(X) \succeq 0, \hat{\Sigma}^{-1} + \lambda^{-1}X \succ 0\}.$$ 

Therefore, from now on we can consider the following problem

$$\min_{(\lambda, X) \in C_F} F(\lambda, X). \quad (33)$$

Once obtained the optimal solution $(\lambda^*, X^*)$ we can recover the optimal values of the original multipliers simply by setting $\Theta^* = \text{ofd}(X^*)$ and $\Gamma^* = -\text{diag}^2(X^*)$. 

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B. Uniqueness of the solution of the dual problem

The aim of this Section is to show that Problem (33) (and, hence Problem (19)) admits a unique solution. Since \( \tilde{J} \) is the opposite of the dual objective function, \( \tilde{J} \) is convex over \( C \). It is then easy to check that \( F \) is also a convex function over the convex set \( C_F \). However, as we will see, \( F \) is not strictly convex. Accordingly, establishing the uniqueness of the minimum is not a trivial task.

The following Proposition, whose proof is in Appendix, characterizes the second variation of \( F \) in direction \((\delta \lambda, \delta X)\), i.e. \( \delta^2 F(\lambda, X; \delta \lambda, \delta X) \).

**Proposition 5.1:** Let \( x := \text{vec}(X), \delta x := \text{vec}(\delta X) \), and \( K := (\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1} \otimes (\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1} \). Let also

\[
H := \begin{bmatrix}
\lambda^{-3}xx^\top Kx & -\lambda^{-2}x^\top K \\
-\lambda^{-2}Kx & \lambda^{-1}K
\end{bmatrix} \in \mathbb{R}^{(1+n^2) \times (1+n^2)}.
\]

Then, we have

\[
\delta^2 F(\lambda, X; \delta \lambda, \delta X) = [\delta \lambda \quad \delta x^\top ] H \begin{bmatrix}
\delta \lambda \\
\delta x
\end{bmatrix}.
\]

Since in \( C_F \) we have that \( K \in \mathbb{Q}_n \) is positive definite and \( \lambda > 0 \), the matrix \( H \), which has clearly the meaning of the Hessian of \( F \), has at least rank equal to \( n^2 \). Moreover, \( Hw = 0 \) with \( w = [\lambda x^\top]^\top \). We conclude that \( H \) has rank equal to \( n^2 \).

This means that \( F \) is convex and there is exactly one direction along which \( F \) is not strictly convex. We now analyse this direction in the neighbourhood of the optimal solution.

**Lemma 5.4:** Any optimal solution \((\lambda^*, X^*)\) minimizing \( F \) over \( C_F \) lies on the boundary of \( C_F \) and, specifically, is such that \( I - X^* \) is singular.

**Proof:** Let \((\lambda^*, X^*)\) be an optimal solution and assume, by contradiction, that \((\lambda^*, X^*)\) does not belong to the boundary of the feasible set \( C_F \), so that, in particular, \( X^* \prec I \). Thus there exists \( \varepsilon > 0 \) such that \((1 + \varepsilon)X^* \prec I \) so that

\[
((1 + \varepsilon)\lambda^*, (1 + \varepsilon)X^*) \in C_F.
\]

Now a direct computation yields

\[
F((1 + \varepsilon)\lambda^*, (1 + \varepsilon)X^*) = (1 + \varepsilon)F(\lambda^*, X^*) < F(\lambda^*, X^*) \quad (34)
\]
where the last inequality is a consequence of the fact that, as we have already seen in the proof of Lemma 5.1, the optimal value of $\tilde{J}$ (and, hence, of $F$) is negative. This a contradiction as $F(\lambda^*, X^*)$ is assumed to be a minimum. \hfill $\Box$

Remark 1: Notice that for any $(\lambda_0, X_0) \in C_F$, the direction $(\varepsilon \lambda_0, \varepsilon X_0)$ (which, by the way, is the direction considered in Lemma 5.4 for the specific case of the optimal solution $(\lambda^*, X^*)$) is exactly the unique direction along which $F$ is not strictly convex. In fact, along this direction $F$ is clearly a linear function of $\lambda$. Notice also that $F$ is constant along this direction if and only if $F(\lambda_0, X_0) = 0$. Since at any optimal solution $(\lambda^*, X^*)$ $F$ is necessarily negative, $F$ is not constant along the direction $(\varepsilon \lambda^*, \varepsilon X^*)$ (which is the only direction along which $F$ is not strictly convex).

As a consequence of this observation, we have the following result.

Corollary 5.2: Let $(\lambda_0, X_0)$ be a given point in $C_F$. If $w := (\delta \lambda, \delta X)$ is any direction along which $F(\lambda_0, X_0)$ is constant, i.e. $F(\lambda_0, X_0) = F(\lambda_0 + \alpha \delta \lambda, X_0 + \alpha \delta X)$ for any $\alpha$ such that $|\alpha| > 0$ is sufficiently small, then $F(\lambda_0, X_0) = 0$.

We are now ready to prove our main result.

Theorem 5.2: The dual problem admits a unique solution.

Proof: Assume, by contradiction, that there are two optimal solutions $(\lambda^*_1, X^*_1)$ and $(\lambda^*_2, X^*_2)$. By the convexity of $C_F$, the whole segment $S$ connecting $(\lambda^*_1, X^*_1)$ to $(\lambda^*_2, X^*_2)$ belongs also to $C_F$. It follows by the convexity of $F(\cdot, \cdot)$ that all the points in $S$ are optimal solutions. Notice, in passing, that in view of Lemma 5.4, this implies that $S$ belongs to the boundary of $C_F$. Now, $F$ is clearly negative and constant along $S$ and this is a contradiction in view of Corollary 5.2. \hfill $\Box$

VI. RECOVERING THE SOLUTION OF THE PRIMAL PROBLEM

By the uniqueness of the solution of the dual problem we know that the duality gap between the primal and the dual problem is zero. This allows us to recover the solution of the primal problem.

First, the optimal $\Sigma$ can be easily recovered by substituting the optimal solution of the dual problem $(\lambda^*, \Theta^*, \Gamma^*)$ into (15). Recovering the optimal $L$ is slightly more involved; since the duality gap is zero, from the KKT conditions we have:
\[ \text{tr}(\Lambda L) = 0 \quad (35) \]
\[ \text{tr}(\Gamma(\Sigma - L)) = 0 \quad (36) \]
\[ \text{tr}(\Theta(\Sigma - L)) = 0. \quad (37) \]

We begin by considering (35). It follows from (16) that
\[ \Lambda = I + \Gamma - \Theta \]
where we now know that \( \Lambda \) has deficient rank. Thus, we consider the following reduced singular value decomposition
\[ \Lambda = USU^\top \quad (38) \]
with \( S \in Q_{n-r} \) positive definite, i.e. \( n - r \) is the rank of \( \Lambda \), and \( U \in \mathbb{R}^{n \times n-r} \) such that \( U^\top U = I_{n-r} \). We plug (38) in (35) and get
\[ 0 = \text{tr}[\Lambda L] = \text{tr}[USU^\top L] \Rightarrow U^\top LU = 0. \quad (39) \]

Then, by selecting a matrix \( \tilde{U} \in \mathbb{R}^{n \times r} \) whose columns form an orthonormal bases of \( [\text{im}(U)]^\perp \), we can express \( L \) as:
\[ L = \tilde{U}Q\tilde{U}^\top \quad (40) \]
with \( Q \in Q_r \). Note that, in view of the fact that the columns of \( \tilde{U} \) form the orthogonal complement of the image of \( U \), the relationship \( U^\top \tilde{U} = 0 \) holds.

By (37), we know that \( \Sigma - L \) is diagonal. Thus, we plug (40) into (37) and obtain a linear system of equations: \( \text{ofd}(\Sigma - \tilde{U}Q\tilde{U}^\top) = 0 \), or equivalently,
\[ \text{ofd}(\tilde{U}Q\tilde{U}^\top) = \text{ofd}(\Sigma). \quad (41) \]

In an analogous fashion, using (36) we obtain an additional system of linear equations. In virtue of the fact that both the dual and the primal problem admit solution the resulting system of equations always admits solution in \( Q \). Moreover, the solution of this system of equations is unique if and only if the solution of the primal problem is unique.
VII. NUMERICAL IMPLEMENTATION

We propose an algorithm for finding the numerical solution of Problem (33). First, recall that the optimal solution lies in the boundary characterized by constraints $-\text{diag}^2(X) \geq 0$ and $X \preceq I$. Finding a descending direction $(\lambda, X)$ for $F(\lambda, X)$ satisfying simultaneously these two constraints is not trivial. Then we resort to the Alternating Direction Method of Multipliers (ADMM) algorithm, [10], for decoupling such constraints. Then, the corresponding ADMM updates can be performed by using a projection gradient algorithm. To this end, we rewrite (33) by introducing the new variable $Y \in \mathbb{Q}_n$ defined as $Y := I - X$:

$$\min_{\lambda, X, Y} F(\lambda, X)$$

subject to $(\lambda, X) \in \mathcal{C}_{\lambda, X}^*, Y \in \mathcal{C}_Y^*$

$$Y = I - X$$

with $\mathcal{C}_{\lambda, X}^*$ and $\mathcal{C}_Y^*$ defined, respectively, as

$$\mathcal{C}_{\lambda, X}^* := \{ (\lambda, X) : \lambda > 0, X \in \mathbb{Q}_n, \hat{\Sigma}^{-1} + \lambda^{-1} X \succ 0,$$

$$-\text{diag}^2(X) \geq 0 \}$$

$$\mathcal{C}_Y^* := \{ Y : Y \in \mathbb{Q}_n, Y \succeq 0 \}.$$ 

The augmented Lagrangian (see [10]) for the problem is

$$\mathcal{L}_\rho(\lambda, X, Y, M) = F(\lambda, X) + \langle M, Y - I + X \rangle + \frac{\rho}{2} ||Y - I + X||_F^2$$

where $M \in \mathbb{Q}_n$. Accordingly, given the initial values $\lambda^0, X^0, Y^0$ and $M^0$, the ADMM updates are:

$$(\lambda^{(k+1)}, X^{(k+1)}) := \arg \min_{(\lambda, X) \in \mathcal{C}_{\lambda, X}^*} \mathcal{L}_\rho(\lambda, X, Y^{(k)}, M^{(k)})$$

$$Y^{(k+1)} := \arg \min_{Y \in \mathcal{C}_Y^*} \mathcal{L}_\rho(\lambda^{(k+1)}, X^{(k+1)}, Y, M^{(k)})$$

$$M^{(k+1)} := M^{(k)} + \rho(Y^{(k+1)} - I + X^{(k+1)})$$

where $\rho > 0$ is the penalty parameter. Here, we choose $\rho = 0.5$. 

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Problem (42) has not a closed form solution. Thus, the solution is approximated by a projective gradient step:

\[
\lambda^{(k+1)} := \lambda^{(k)} - t_k \nabla_\lambda \mathcal{L}_\rho(\lambda^{(k)}, X^{(k)}, Y^{(k)}, M^{(k)}) \\
X^{(k+1)} := \Pi_{C_X^*} \left( X^{(k)} - t_k \nabla_X \mathcal{L}_\rho(\lambda^{(k)}, X^{(k)}, Y^{(k)}, M^{(k)}) \right)
\]

where \( \Pi_{C_X^*} \) denotes the projector operator from \( \mathbb{Q}_n \) onto

\[
C_X^* := \{ X : X \in \mathbb{Q}_n, -\text{diag}^2(X) \succeq 0 \},
\]

and \( \nabla_X \mathcal{L}_\rho, \nabla_X \mathcal{L}_\rho \) denotes the gradient with respect to \( \lambda \) and \( X \), respectively:

\[
\nabla_X \mathcal{L}_\rho(\lambda, X, Y, M) := \\
- \log |\hat{\Sigma}^{-1} + \lambda^{-1} X| - \log |\hat{\Sigma}| + \delta \\
+ \lambda^{-1} \text{tr}\left((\hat{\Sigma}^{-1} + \lambda^{-1} X)^{-1} X\right)
\]

\[
\nabla_X \mathcal{L}_\rho(\lambda, X, Y, M) := \\
- (\hat{\Sigma}^{-1} + \lambda^{-1} X)^{-1} + M + \rho(Y - I - X).
\]

It is not difficult to see that

\[
[\Pi_{C_X^*}(A)]_{ij} = \begin{cases} 
0, & \text{if } i = j \text{ and } [A]_{ij} > 0 \\
[A]_{ij}, & \text{otherwise}
\end{cases}
\]

where \([A]_{ij}\) denotes the entry in position \((i, j)\) of matrix \(A \in \mathbb{Q}_n\). The step size \(t_k\) is determined at each step \(k\) in an iterative fashion: we start by setting \(t_k = 1\) and we decrease it progressively until the two conditions \(\lambda^{(k+1)} > 0\) and \(\hat{\Sigma}^{-1} + \lambda^{-1} X \succ 0\) are met and the so-called Armijo condition, [11], are satisfied.

Problem (43) can be rewritten as

\[
Y^{(k+1)} = \arg \min_{Y \in C_Y^*} \| Y - X^{(k+1)} - \frac{1}{\rho} M^{(k)} - Y \|_F.
\]

We introduce the projection operator \( \Pi_{C_Y^*} : \mathbb{Q}_n \rightarrow C_Y^* \) which is defined as

\[
\Pi_{C_Y^*}(W) := \arg \min_{Z \in C_Y^*} \| W - Z \|_F^2.
\]
It is not difficult to see that, if \( A = U D U^\top \) is the eigenvalue decomposition of the matrix \( A \in \mathbb{Q}_n \), then

\[
\Pi_{C^*_Y}(A) = U \text{diag}(f(d_1), \ldots, f(d_n)) U^\top
\]

where

\[
f(d_i) := \begin{cases} d_i, & \text{if } d_i \geq 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Then the solution of (43) becomes

\[
Y^{(k+1)} = \Pi_{C^*_Y} \left( I - X^{(k+1)} - \frac{1}{\rho} M^{(k)} \right).
\]

In order to set the stopping criteria for the algorithm we define the primal and dual residual matrices:

\[
R^{(k+1)} := Y^{(k+1)} - I + X^{(k+1)},
\]

\[
S^{(k+1)} := \rho_k (Y^{(k+1)} - Y^{(k)}).
\]

The algorithm reaches an acceptable solution when the following conditions are met, [10]:

\[
\|R^{(k+1)}\|_F \leq n\epsilon^{\text{abs}} + \epsilon^{\text{rel}} \max \{ \sqrt{n}, \|X^k\|_F, \|Y^k\|_F \},
\]

\[
\|S^{(k+1)}\| \leq n\epsilon^{\text{abs}} + \epsilon^{\text{rel}} \|M^{(k)}\|_F
\]

where \( \epsilon^{\text{rel}} = 10^{-4} \) and \( \epsilon^{\text{abs}} = 10^{-4} \) are the relative and the absolute tolerance, respectively.

**VIII. Numerical Examples**

**A. Synthetic data**

In this Section we consider Monte Carlo studies composed by 200 experiments whose structure is as follows. For each experiment:

- we consider a factor model having the structure of (2) with the cross sectional dimension is \( n = 40 \); \( L \) and \( D \) are randomly generated in such a way that \( L \) has rank equal to \( r \) (a priori fixed), \( D \) is diagonal and the signal-to-noise ratio (defined as \( \|L\|/\|D\| \) ) between the latent and the idiosyncratic components is equal to one;
- a data sequence of length \( N \) for \( y \) is generated;
- we compute the sample covariance matrix \( \hat{\Sigma} \) from this data;
• we compute the estimate $\delta_\alpha$ of $\delta$ using the empirical procedure of Section III with $\alpha = 0.5$;

• we compute the solution $(L_{OPT}, \Sigma_{OPT})$ of Problem (8) where we replace $\delta$ with $\delta_\alpha$. Let $\lambda_i$, $i = 1 \ldots n$, denote the singular values of $L_{OPT}$ and define $i_{max}$ as the first $i$ such that $\lambda_{i+1}/\lambda_1 < 0.05$. Then, we define the “numerical rank” of $L_{OPT}$ as:

$$
r_{OPT} := \max_{i \leq i_{max}} \lambda_i/\lambda_{i+1}
$$

(45)

• we compute the solution of the standard problem (4) (exact decomposition with trace heuristic) and, with the same procedure of the previous point, we compute the numerical rank, $r_{ED}$, of the corresponding low rank matrix;

• we compute the minimum number of factors from the data sequence for $y$ by applying the three methods proposed by Bai and Ng [6], namely: ICP1, ICP2 and ICP3. We denote the corresponding estimates by $r_{ICP1}$, $r_{ICP2}$ and $r_{ICP3}$, respectively;

• we compute the minimum number of factors from the data sequence for $y$ by applying the method proposed by Lam and Yao [32]. We denote by $r_{LY}$ the resulting estimate of the rank.

Finally, we compute the root mean squared error:

$$
e = \sqrt{\frac{1}{200} \sum_{i=1}^{200} (r_i^* - r)^2}
$$

(46)

for $r_i^* = \{r_{OPT}, r_{ED}, r_{ICP1}, r_{ICP2}, r_{ICP3}, r_{LY}\}$ and where $r$ is the true rank of the data generating process. Table I shows error (46) for three Monte Carlo studies where $r = 4$ and the sample size is $N = 200$, $N = 500$ and $N = 1000$, respectively.

Usually, the problem becomes more challenging when the rank $r$ of the data generating process increases (yet remaining below the Ledermann bound). For this reason we repeat the above Monte Carlo studies for the case $r = 10$ (considering again the three sample sizes $N = 200$, $N = 500$, $N = 1000$). The corresponding root mean squared errors (46) are reported in Table II.

As one can see, in all these six Monte Carlo studies the proposed method outperforms the others.

\[\text{\textsuperscript{2}}\text{The estimation procedure for this method requires to set a parameter $k_0$ for the selection which only general considerations are provided: we decided to select $k_0$ using an “oracle” procedure i.e. for each Monte Carlo run we choose the value of $k_0$ which yields the most favourable result.}\]
We now analyze how well the proposed method recovers the subspace of \( L \) by considering the following measure of discrepancy. Let \( L = AA^\top \) be the low rank matrix of the data generating process and consider the singular value decomposition of \( L_{\text{OPT}} \), that is \( L_{\text{OPT}} = USV^\top \). Let \( \tilde{U} := U_{[1:n,1:r_{\text{OPT}}]} \), \( \tilde{U} \in \mathbb{R}^{n \times r_{\text{OPT}}} \) be the matrix formed by the first \( r_{\text{OPT}} \) columns of \( U \) and \( \tilde{S} := S_{[1:r_{\text{OPT}},1:r_{\text{OPT}}]} \), \( \tilde{S} \in \mathbb{R}^{r_{\text{OPT}} \times r_{\text{OPT}}} \) be the top left \( r_{\text{OPT}} \times r_{\text{OPT}} \) sub-matrix of \( S \). We define the projector onto the subspace of \( A_{\text{OPT}} := \tilde{U}\tilde{S} \) as

\[
P := A_{\text{OPT}}(A_{\text{OPT}}^\top A_{\text{OPT}})^{-1}A_{\text{OPT}}^\top.
\]

Then, a measure of discrepancy between the subspace of \( A \) and the subspace of \( A_{\text{OPT}} \) is given by:

\[
s(A_{\text{OPT}}) := \frac{\text{tr}(A^\top PA)}{\text{tr}(A^\top A)} \quad (47)
\]

where \( s(A_{\text{OPT}}) \) takes value between 0 and 1. Note that, if \( s(A_{\text{OPT}}) = 1 \) then \( A_{\text{OPT}} \) recovers exactly the image of \( A \). Figure 2 (left hand side panel) shows the box-plots for error (47) in the three Monte Carlo studies.

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**TABLE I**

Average root mean squared error between the estimated numerical rank and the true rank \( r = 4 \).

| \( N=200 \) | Proposed method | Exact Decomposition | Bai & Ng | Lam & Yao |
|-----------|----------------|---------------------|---------|-----------|
|           | 0.500          | 3.752               | 3.589   | 7.506     | 5.529     |
| \( N=500 \) | 0.000          | 0.9618              | 2.271   | 4.236     | 5.347     |
| \( N=1000 \) | 0.000          | 0.6557              | 3.587   | 3.927     | 5.421     |

**TABLE II**

Average root mean squared error between the estimated numerical rank and the true rank \( r = 10 \).

| \( N=200 \) | Proposed method | Exact Decomposition | Bai & Ng | Lam & Yao |
|-----------|----------------|---------------------|---------|-----------|
|           | 2.170          | 7.218               | 5.888   | 8.254     | 6.943     |
| \( N=500 \) | 0.174          | 5.221               | 5.214   | 5.812     | 6.536     |
| \( N=1000 \) | 0             | 2.961               | 5.302   | 5.490     | 6.669     |
Finally, we consider the example illustrated in Figure 1 of the Introduction. By applying our method, we obtain the situation illustrated in Figure 3 showing that our approach provides a numerical rank equal to the true value of $r$.

Fig. 3. A sample of numerosity 1000 has been generated from a factor model with $n = 40$ and $r = 4$. The Figure displays the first twenty singular values of the true matrix $L$ (on the left) and of the matrices $L_{OPT}$ (in the middle) and $L_{ED}$ (on the right) estimated, respectively, with the proposed method and with the trace heuristic with exact decomposition.

B. Data analysis for investment decision

In this sub-section we consider a cross section of 9 financial indicators ($n = 9$) collected across 94 different sectors ($N = 94$) of the US economy (each data vector represents the average for that sector). The data are taken from http://www.stern.nyu.edu/~adamodar/pc/datasets/betas.xls (data downloaded on June 2017).
The considered financial indicators can be computed from the balance sheet of the companies and from stocks market development and are customarily used for investment decisions. These indicators are: the beta, that is the systemic risk arising from the exposure to general market movements, the debt/equity ratio, the tax rate, the unlevered beta, the cash/firm value ratio, the unlevered beta corrected for cash, the Hi-Lo risk, the standard deviation of equity and the standard deviation of operating income. For a proper description of these indicators we refer to the aforementioned web site while for a more general treatment see for example [31].

It is reasonable to expect that the variability of the listed indicators may be successfully explained by a smaller number of factors and motivated by this reason we estimate the sample covariance matrix and we apply the proposed approach (with $\alpha = 0.5$). Indeed, we obtain an estimate $\hat{r} = 3$ for the number of latent factors. These seems to be reasonable, since the common variability of these indicators may be explained by factors such as the general market trend, the different fiscal regime and the different optimal capital structure across sectors. In this case, the POET method proposed in [20] provides an estimate of 1 latent factor and the methods proposed in [6] and [32] provide an estimate of 8 latent factors: the latter number does not seem very reliable as it is larger than the upper bound of 7 latent factors provided by the method based on exact decomposition of the covariance matrix.$^3$

**IX. CONCLUSION AND FURTHER RESEARCH DIRECTIONS**

In this paper we have proposed a new method to estimate the number of factors for the realistic situation in which the covariance matrix of the data is estimated with an error that is not negligible.

A question which arises naturally concerns the statistical properties of the proposed estimator, and, in particular, its asymptotic properties as the sample size approaches infinity. This is a complex issue that certainly cannot be fully addressed in the context of the present paper. We present only some ideas in this direction and a heuristic road-map that can be followed. A primary issue is the rank consistency of the minimum trace estimator, note that a similar matter has been studied e.g. in [5] for the Lasso problem. Restricting to the cases in which the minimizers of

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$^3$The eighth and ninth eigenvalues of the corresponding $L$ matrix obtained with this method are only numerically non-zero as they are smaller than $10^{-14}\lambda_1$, with $\lambda_1$ being the largest eigenvalue of $L$. 
the minimum rank problem and of the minimum trace problem coincide, a possible argument may be the following. By the construction presented in Section III, it holds that

$$ Pr[2D_{KL}(\Sigma||\hat{\Sigma}) < \delta] = \alpha. $$

We can let the desired precision $\alpha$ be a function of the sample size $N$, and choose $\alpha(N)$ such that, as $N \to \infty$, it holds that $\alpha(N) \to 1$. Moreover, we let $\alpha(N) \to 1$ sufficiently slowly so that it is reasonable to expect that $\delta(\alpha(N)) \to 0$ because $\hat{\Sigma} \to \Sigma$ almost surely. Consequently, as $N \to \infty$, the neighbourhood of $\hat{\Sigma}$ in which we seek for the solution becomes smaller and smaller and it contains the “true” $\Sigma$ with probability tending to 1. Moreover, the minimum rank problem (3) is a lower semi-continuous function of $\Sigma$ (see [37], Proposition 1) and, being integer valued, it does not decrease in a sufficiently small neighborhood of $\Sigma$. Therefore, it seems reasonable to conclude that $\forall \epsilon > 0 \exists \bar{\epsilon} > 0$ such that $\forall N > \bar{N}$ it holds that

$$ Pr[r_{OPT} \neq r_{true}] < \epsilon. $$

A rigorous study of this heuristic argument will be subject of future investigation.

Another natural direction of research, which will be subject of future investigation, is the extension of the presented results to the dynamical case in the spirit of [54].

**APPENDIX**

**A. Proof of Proposition 3.1**

For $\Sigma_D \in D_n$, the Kullback-Leibler divergence can be rewritten as

$$ 2D_{KL}(\Sigma_D||\hat{\Sigma}) = \sum_{j=1}^{n} - \log d_j + \text{tr}(\text{diag}(d_1, ..., d_n)\hat{\Sigma}^{-1}) + \log |\hat{\Sigma}| - n $$

$$ = \left( \sum_{j=1}^{n} \min_{d_j} - \log d_j + d_j \gamma_j \right) + \log |\hat{\Sigma}| - n. $$

Thus the minimization problem in (12) is equivalent to

$$ \min_{\Sigma_D \in D_n} 2D_{KL}(\Sigma_D||\hat{\Sigma}) $$

$$ = \left( \sum_{j=1}^{n} \min_{d_j} - \log d_j + d_j \gamma_j \right) + \log |\hat{\Sigma}| - n. $$
Since $-\log d_j + d_j \gamma_j$ is convex with respect to $d_j$, by setting equal to zero the first derivative with respect to $d_j$, $j = 1, \ldots, n$, it easily follows that

$$\Sigma_{OPT}^D = \text{diag}(\gamma_1^{-1}, \ldots, \gamma_n^{-1}).$$

Then, $\delta_{max}$ can be determined as

$$\delta_{max} = 2 \mathcal{D}_{KL}(\Sigma_{OPT}^D \parallel \hat{\Sigma})$$

$$= \left( \sum_{j=1}^{n} - \log(\gamma_j^{-1}) + 1 \right) + \log |\hat{\Sigma}| - n$$

$$= - \log |\text{diag}(\gamma_1^{-1}, \ldots, \gamma_n^{-1})| + \log |\hat{\Sigma}|$$

$$= \log |\text{diag}^2(\hat{\Sigma}^{-1})\hat{\Sigma}|.$$

B. Proof of Proposition 4.1

By substituting the obtained optimal conditions (16) and (15) into (14), we get the following expression where we have defined $\Delta := \lambda(\lambda \hat{\Sigma}^{-1} + \text{ofd}(\Theta) - \Gamma)^{-1}$:

$$J(\lambda, \Gamma, \Theta)$$

$$= \text{tr}(L) + \lambda(- \log |\Delta| + \text{tr}(\hat{\Sigma}^{-1}\Delta) + \log |\hat{\Sigma}| - n - \delta)$$

$$- \text{tr}((I + \Gamma - \text{ofd}(\Theta))L) - \text{tr}(\Gamma \Delta)$$

$$+ \text{tr}(\text{ofd}(\Theta)\Delta) + \text{tr}(\Gamma L) - \text{tr}(\text{ofd}(\Theta)L).$$
That simplifies into:

\[ J(\lambda, \Gamma, \Theta) \]

\[ = \lambda(- \log |(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta))^{-1})| + \log |\hat{\Sigma}| 
- n - \delta + \text{tr}(\hat{\Sigma}^{-1}(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
- \text{tr}(\Gamma(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
+ \text{tr}(\text{ofd}(\Theta)(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
\]

\[ = \lambda(- \log |(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}| + \log |\hat{\Sigma}| 
- n - \delta + \text{tr}(\lambda \hat{\Sigma}^{-1}(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
- \text{tr}(\Gamma(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
+ \text{tr}(\text{ofd}(\Theta)(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}) 
\]

\[ = \lambda(- \log |(\hat{\Sigma}^{-1} + \lambda^{-1}(\text{ofd}(\Theta) - \Gamma))^{-1}| + \log |\hat{\Sigma}| 
- n - \delta) + n\lambda \]

Since \( J \) does not depend on \( \Lambda \), we can eliminate it and, in view of (16), condition \( \Lambda \succeq 0 \), is replaced by

\[ I + \Gamma - \text{ofd}(\Theta) \succeq 0. \quad (48) \]

**C. Proof of Proposition 5.1**

Consider the function

\[ \tilde{F}(\lambda, X) = -\lambda \log |\hat{\Sigma} + \lambda^{-1}X|. \]
Since $\tilde{F}(\lambda, X)$ differs from $F(\lambda, X)$ only by terms which are linear in $(\lambda, X)$ the second variations of the two functions are equivalent. Thus, in what follows we will focus on $\tilde{F}(\lambda, X)$.

The first variation of $\tilde{F}(\lambda, X)$ in direction $(\delta \lambda, \delta X)$ is

$$
\delta \tilde{F}(\lambda, X; \delta \lambda, \delta X) = - \log |\hat{\Sigma} + \lambda^{-1}X| \delta \lambda \\
+ \lambda^{-1} \text{tr}((\hat{\Sigma} + \lambda^{-1}X)^{-1}X) \delta \lambda - \text{tr}((\hat{\Sigma} + \lambda^{-1}X)^{-1} \delta X).
$$

The second variation of $\tilde{F}(\lambda, X)$ in direction $(\delta \lambda, \delta X)$ is

$$
\delta^2 \tilde{F}(\lambda, X; \delta \lambda, \delta X) = \\
\lambda^{-1} \text{tr}((\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1}X(\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1} \delta X) \\
- 2 \left[ \lambda^{-2} \text{tr}((\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1}X(\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1} \delta X) \right] \delta \lambda \\
+ \lambda^{-3} \text{tr}((\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1}X(\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1}X) \delta \lambda^2.
$$

Now, by using the Kronecker product and the vec operator and defining $x := \text{vec}(X)$, $\delta x := \text{vec}(\delta X)$, and $K := (\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1} \otimes (\hat{\Sigma}^{-1} + \lambda^{-1}X)^{-1}$ the Hessian in Proposition 5.1 immediately follows.

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