Nonuniformity of the $1/N$ Expansion for Two-Dimensional $O(N)$ Models

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Abstract

We point out that the $1/N$ expansion, which is widely invoked to infer properties of the 2D $O(N)$ models, is nonuniform in the temperature, i.e. with decreasing temperature the $1/N$ expansion truncated at a fixed order deviates more and more from the true answer. This fact precludes the use of the expansion to deduce low temperature properties such as asymptotic scaling for those models. By contrast, in the 1D $O(N)$ chains, there are no signs of such a nonuniformity.

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1 Introduction

The $1/N$ expansion [1] is a very popular tool to analyze $2D \mathbb{O}(N)$ models, especially in the critical dimension 2. But already the first rigorous paper by A. Kupiainen [2], which established the fact that the expansion is a valid asymptotic expansion in the thermodynamic limit at fixed $\tilde{\beta} = \beta/N$, contained hints that the expansion is really only good at high enough temperature: the main result stated that the expansion is valid provided $\tilde{\beta}$ is less than the critical value of the ‘spherical model’, i.e., the $N \to \infty$ limit of the model. Likewise the followup paper by Fröhlich et al [4] proved Borel summability of the expansion only for small $\tilde{\beta}$. Müller, Raddatz and Rühl [5] computed the $1/N$ corrections to the mass gap and the susceptibility for large $\tilde{\beta}$; their results are nonuniform in $\tilde{\beta}$ in the sense that the corrections grow beyond all bounds as $\tilde{\beta} \to \infty$. This fact, which is apparent from the equations in [5] was, however, not noted explicitly by the authors. Finally, U. Wolff computed the correlation length of the $\mathbb{O}(N)$ model for $N = 2, 3, 4, 8$. Comparing his results with the spherical model makes it quite obvious that the approach to the $N \to \infty$ limit slows down with increasing $\tilde{\beta}$ (see Tab.1).

In spite of all those facts, many authors (e.g.,[1, 6, 7]) continue to cite the $1/N$ expansion and the properties of the spherical model as evidence for the supposed asymptotic scaling and absence of a critical point of the $2D \mathbb{O}(N)$ model at finite $N$. In this note we report a few more calculations that show explicitly that the $O(1/N)$ corrections for a number of quantities grow linearly with $\tilde{\beta}$. All those quantities are of long range character, such as correlation lengths, susceptibilities or two-point functions at large distances. We did not see such effects in short range quantities such as the energy (= two point function at one lattice distance); this does not mean, however, that the $1/N$ expansion for short range quantities is uniform in $\tilde{\beta}$. Uniformity would require control of the deviation of the true result from its truncated $1/N$ expansion, something that is not analyzed here. We should, however, point out that our recent results on the problems of perturbation theory (PT) [8] in non-Abelian models suggest that in fact there is nonuniformity even in those short-distance quantities.

The situation is different in 1D: no indication of any non-uniformity is found. The underlying reason is that in 1D the necessary computations can be reduced to the analysis of simple 1 dimensional integrals, so that no infrared effects, which in our mind are responsible for the nonuniformity in 2D, can arise.

2 Notation and Definitions

Consider the lattice $\mathbb{Z}^D$ and let $\Lambda$ be some given finite subset of it. The non-linear $O(N)$ $\sigma$-model with standard nearest neighbor interaction (s.n.n.i.) at
inverse temperature $\beta = \tilde{\beta}N$ is defined by the generating function

$$Z_\Lambda(g) = \int \prod_{(ij)} \exp(\beta s_i \cdot s_j) \prod_i \left( \exp(g \cdot s_i) \delta(s_i^2 - 1) ds_i \right)$$

(1)

or equivalently the Gibbs measure

$$\frac{1}{Z_\Lambda(0)} \prod_{(ij)} \exp(\beta s_i \cdot s_j) \prod_i \left( \delta(s_i^2 - 1) ds_i \right)$$

(2)

Here $s$ is a $N$-vector, the integration is over the spins inside $\Lambda$ with some boundary conditions.

It is well known (see for instance [4]) how one can develop an asymptotic expansion in powers of $1/N$ by introducing a ‘Langrange multiplier field’ a dual to the $\delta$-function constraint and a standard saddle point expansion. In terms of the Lagrange multiplier field one obtains a dual representation of the model; the generating function is now given by

$$Z(g) = \int \prod_i da_i \exp(-N\hat{S}) \exp\left(\frac{1}{2\tilde{\beta}N}(g, (-\Delta + m^2 - \frac{2ia}{\sqrt{N}})^{-1} g)\right)$$

(3)

where the dual action is given by

$$\hat{S} = \frac{1}{2} \text{tr} \ln(-\Delta + m^2 - \frac{2ia}{\sqrt{N}}) - i\tilde{\beta} \sum_i a_i \sqrt{N}$$

(4)

The mass parameter $m$ appearing in this equation is fixed by the gap equation

$$\tilde{\beta} = (-\Delta + m^2)^{-1}$$

(5)

Kupiainen [2] showed, using the technique of cluster expansions, that the resulting asymptotic expansion is uniform in the volume $\Lambda$ and can be used to obtain a valid asymptotic $1/N$ expansion in the infinite volume, provided $\tilde{\beta} < \tilde{\beta}_{\text{sph}}$, where $\tilde{\beta}_{\text{sph}}$ is the critical point of the spherical model. It is well known how the $1/N$ expansion can be cast in the form of Feynman graphs involving two types of propagators $B$ and $C$ given by

$$C_{ij} = (-\Delta + m^2)^{-1}$$

(6)

$$B^{-1}_{ij} = \frac{1}{2}(C^2_{ij})$$

(7)

Those Feynman graphs can be easily evaluated numerically on finite lattices. In this paper we are considering the $1/N$ corrections to the correlation length $\xi$, the susceptibility $\chi$ and the following ratio of two-point functions

$$A \equiv \frac{G(\xi_\infty/2)}{G(\xi_\infty/4)}$$

(8)
where $\xi_\infty$ is the correlation length of the spherical model (up to a correction that is negligible for $\tilde{\beta} > .5$, $\xi_\infty = 1/m$ where $m$ is the solution of the gap equation) and

$$G(x) \equiv \langle s(0) \cdot s(x) \rangle$$  \hspace{1cm} (9)

The quantities $\xi$, $\chi$, $A$ as well as the energy $E$ are expanded as follows:

$$\xi = \xi_\infty (1 + \frac{1}{N} \xi_1 + O(\frac{1}{N^2}))$$  \hspace{1cm} (10)

$$\chi = \chi_\infty (1 + \frac{1}{N} \chi_1 + O(\frac{1}{N^2}));$$  \hspace{1cm} (11)

$$\frac{G(\frac{1}{2} \xi_\infty)}{G(\frac{1}{4} \xi_\infty)} = A_\infty (1 + \frac{1}{N} a_1 + O(\frac{1}{N^2}))$$  \hspace{1cm} (12)

$$E \equiv \langle s(0) \cdot s(1) \rangle = E_\infty (1 + \frac{1}{N} e_1 + O(\frac{1}{N^2}))$$  \hspace{1cm} (13)

In the following we will study the behavior of the coefficients $\xi_1$, $\chi_1$ and $a_1$ for large $\tilde{\beta}$.

### 3 What is the problem?

Let us consider a certain physical quantity $A(N, \tilde{\beta})$. According to the preceding discussion it will have an asymptotic expansion of the form

$$A(N, \tilde{\beta}) \sim \sum_{k=0}^\infty \frac{a_k(\tilde{\beta})}{N^k}$$  \hspace{1cm} (14)

The meaning of the symbol $\sim$ is that the series on the right hand side, if truncated at the term of order $N^{-k}$, will approximate the left hand side by up to a remainder term which is $o(N^{-k})$, and this is true for any $k \geq 0$. That means

$$r_k(N, \tilde{\beta}) \equiv A(N, \tilde{\beta}) - \sum_{k=0}^k \frac{a_k(\tilde{\beta})}{N^k}$$  \hspace{1cm} (15)

satisfies

$$\lim_{N \to \infty} |r_k(N, \tilde{\beta})|N^k = 0$$  \hspace{1cm} (16)

Uniform asymptoticity means the stronger property

$$|r_k(N, \tilde{\beta})|N^k \leq c_k(N)$$  \hspace{1cm} (17)

where $c_k(N)$ is independent of $\tilde{\beta}$ and satisfies

$$\lim_{N \to \infty} c_k = 0$$  \hspace{1cm} (18)
It is easy to see that uniform asymptoticity does not hold if any one of the expansion coefficients $a_k(\beta)$ is unbounded for large $\beta$, but it should be stressed that this is not a necessary condition. There are simple examples of functions where every one of the coefficients $c_k(\beta)$ is bounded and yet the expansion is not uniformly asymptotic. Consider for instance the function
\[ f(N\beta) = \beta \exp(-N/\beta) \] (19)

Its asymptotic expansion in powers of $1/N$ has all coefficients $a_k(\beta) = 0$ and the remainder term is always equal to $f$, which is unbounded.

## 4 One Dimension

In 1D the situation is particularly simple because the thermodynamic limit of the expectation values of local quantities can be obtained by computing in a finite box with free boundary conditions. For nearest neighbor quantities like the energy $E$ one only has to evaluate a 1D integral; long range quantities such as the correlation length $\xi$ or the susceptibility $\chi$ can be expressed in terms of $E$. It turns out that for this reason no nonuniformity in any of those quantities is found.

The energy of a 1D $O(N)$ chain can be obtained as follows:
\[ E \equiv \langle s(0) \cdot s(1) \rangle = \frac{d}{d\beta} \ln Z \] (20)

where $Z$ is the partition function for one link:
\[ Z = \int_{-1}^{1} dt e^{\beta t} (1 - t^2)^{N-3} \] (21)

Instead of using the general scheme outlined in the previous section, it is convenient to obtain the $1/N$ expansion directly from this expression. The two point function in the thermodynamic limit is simply given by
\[ \langle s(0) \cdot s(n) \rangle = E^n \] (22)

and hence the correlation length is
\[ \xi = -\ln E \] (23)

and the susceptibility
\[ \chi = \frac{1 + E}{1 - E} \] (24)

Introducing the rescaled inverse temperature
\[ \bar{\beta} = \frac{\beta}{N - 3} \]  
(25)

the partition function takes the form

\[ Z = \int_{-1}^{1} dt e^{(N-3)f(t)} \]  
(26)

with

\[ f(t) = \bar{\beta}t + \frac{1}{2} \ln(1 - t^2) \]  
(27)

It is now straightforward to produce a saddle point expansion in powers of \( 1/(N - 3) \): the saddle point value \( t_o \) is given by

\[ t_o(\bar{\beta}) = \sqrt{1 + \frac{1}{4\bar{\beta}^2} - \frac{1}{2\bar{\beta}}} \]  
(28)

and the logarithm of the partition function is expanded as

\[ \ln Z = (N - 3)f_{\bar{\beta}(t_o)} + \frac{1}{2} \ln \left( \frac{2\pi}{(N - 3)(-f''_{\bar{\beta}(t_o)})} \right) + O\left( \frac{1}{(N - 3)^2} \right) \]  
(29)

Since \( t_o \) is a function that increases monotonically from 0 to 1 as \( \bar{\beta} \) increases from 0 to \( \infty \), we may use \( t_o \) instead of \( \bar{\beta} \) as an independent variable. Using

\[ \frac{d}{d\bar{\beta}} = \frac{(1 - t_o^2)^2}{1 + t_o^2} \frac{d}{dt_o} \]  
(30)

the expansion for the energy becomes

\[ E = t_o - \frac{t_o}{N - 3} \frac{(1 - t_o^2)(3 + 2t_o^2)}{1 + t_o^2} + O\left( \frac{1}{(N - 3)^2} \right) \]  
(31)

It is already visible that the ration of the \( 1/(N - 3) \) correction to the leading term remains bounded as \( \bar{\beta} \to \infty \), i.e. \( t_o \to 1 \); in fact it is \( O(1/\bar{\beta}) \). But we want to expand in powers of \( 1/N \), not \( 1/(N - 3) \), hence we have to re-expand the leading term, using

\[ \tilde{\beta} = \frac{\bar{\beta}}{1 - 3/N} \]  
(32)

Again it is very easy to see that we just obtain another correction of order \( 1/\tilde{\beta} \) to the \( 1/N \) contribution \( e_1 \) (see eq.(13)). It is also easy to see that the result is in agreement with the one found by Hasenfratz for the infinite chain.

Using this information and the formulae above it is now trivial to conclude that
\[ \xi_1 = \frac{e_1}{\ln E} \] (33)

\[ \chi_1 = -\frac{2e_1E^2}{(1 + E)(1 - E)} \] (34)

\[ a_1 = -\frac{1}{2} e_1 \ln E \] (35)

are all \( O(1) \) for \( \tilde{\beta} \to \infty \) as claimed.

5 Two Dimensions

We computed the Feynman graphs for the \( 1/N \) correction to the two point function numerically on lattices satisfying \( L/\xi_\infty = 7 \). It is observed that this is sufficient to reach the thermodynamic limit with a precision of 1 to 2%.

The expansion for the two point function has the form

\[ G(i) = G_\infty(i) + \frac{1}{N} \sum_{j,l \in \Lambda} \frac{1}{L^4} G_\infty(i - j)\Sigma(j - l)G_\infty(l) \] (36)

where \( G_\infty(i) = C(i)/\tilde{\beta} \) and the ‘self-energy’ \( \Sigma \) is given by

\[ \Sigma(i) = -B(i)G_\infty(i) + \frac{1}{2} \sum_{j,l,k \in \Lambda} \frac{1}{L^6} B(i - j)B(l - k)G_\infty(l - k)G_\infty(l - j)G_\infty(k - j) \] (37)

and analogously in momentum space

\[ \hat{G}(p) = \hat{G}_\infty(p)^2 \hat{\Sigma}(p) \] (38)

where \( \hat{\Sigma} \) is given by

\[ \hat{\Sigma}(p) = -\sum_k \frac{1}{L^2} \hat{G}_\infty(k)B(p - k) + \frac{1}{2} B(0) \sum_{k,q} \frac{1}{L^4} \hat{G}_\infty(k)^2 \hat{G}_\infty(k + q)B(q) \] (39)

the momenta \( p, k, q \) range over \( 2\pi i/L, i = 1, 2, \ldots L \). From the momentum space two-point functions \( \hat{G}(0) \) and \( \hat{G}(2\pi/L) \) we obtain the susceptibility and also the (effective) correlation length using the formula

\[ \xi = \frac{1}{2 \sin \frac{\pi}{2L}} \sqrt{\frac{\hat{G}(0)}{\hat{G}(2\pi/L)}} - 1 \] (40)

which is expanded in powers of \( 1/N \) using the expansion for the two-point functions.
The results are given in Table 2. They show clearly that the $1/N$ corrections for all three quantities grow linearly with $\tilde{\beta}$, respectively $\ln m$, in agreement with the findings of [5].

6 Discussion

We have found by explicit calculation that the $1/N$ expansion is not uniformly asymptotic in $\tilde{\beta}$ for a variety of long range quantities in the $2D O(N)$ model. The nonuniformity appears in the most blatant form: the correction grows linearly with $\tilde{\beta}$. For short distance quantities like the energy we could not find this kind of phenomenon up to $O(1/N^3)$. This does not mean, however, that the $1/N$ expansion is uniform for those quantities; to show uniformity one would have to control the remainder terms.

This situation is in striking contrast with the one for the $1D$ chains: since all computations reduce to one-dimensional integrals, there is no nonuniformity occurring.

Some authors (for instance [6]) try to improve the quality of the $1/N$ expansion by introducing a certain ‘mass renormalization’. Since we are working on a lattice, there is first of all no need to do so since there are no divergences to be absorbed. Furthermore, the procedure corresponds to a a resummation of certain terms in the $1/N$ expansion and therefore is no longer a systematic asymptotic expansion in any parameter. While apparently improving the agreement with the Monte Carlo data in the $\tilde{\beta}$ region studied, the mathematical status of the procedure remains unclear and it can be expected to reveal the same problems as the systematic expansion at larger $\tilde{\beta}$.

In [8] we found that PT for the energy yields results that depend on the boundary conditions to order $1/\tilde{\beta}^2$; this dependence survives the limit $N \to \infty$. In our mind this is a strong indication that the $1/N$ expansion is nonuniform in $\tilde{\beta}$ even for short distance quantities, since the spherical limit of the model does not show this phenomenon.

But the main conclusion is that that the nonuniformity for the mass gap and similar quantities implies that it also holds for the Callan-Symanzik $\beta$-function. This means that the $1/N$ expansion cannot be used to draw any conclusions about the phase structure of the model for finite $N$, in particular the fact that the spherical model does not have a soft phase in $2D$ is in no way in conflict with the arguments advanced by us [9, 10] that strongly suggest the existence of such a phase for any finite $N$. 
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Table 1: Comparison of Monte Carlo results with the spherical limit; Monte Carlo data taken from \([11](O(2)), [12](O(3)), [13](O(4)) \) and \([14](O(4) \text{ and } O(8))\). The numbers represent \(\xi/\xi_\infty\).

| \(\hat{\beta}\) | \(.500\) | \(.5667\) | \(.575\) | \(.600\) | \(.650\) | \(.725\) |
|------------------|--------|--------|--------|--------|--------|--------|
| \(N = 8\)       | 1.307  | 1.494  | 1.711  | 1.983  |        |        |
| \(N = 4\)       | 1.866  | 2.857  | 3.301  | 4.425  |        |        |
| \(N = 3\)       | 2.644  | 5.479  | 8.348  | 15.629 |        |        |
| \(N = 2\)       | 9.691  | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) |

Table 2: The \(1/N\) corrections to various long range quantities. For the definition of the quantities given see eq. (10),(11),(12).

| \(\beta\) | \(.49349\) | \(.60600\) | \(.67105\) | \(.71704\) | \(.78173\) | \(.82758\) |
|-----------|------------|------------|------------|------------|------------|------------|
| \(L\)    | 28         | 56         | 84         | 112        | 168        | 224        |
| \(\xi_\infty\) | 4         | 8          | 12         | 16         | 24         | 32         |
| \(\xi_1\) | 1.764      | 2.927      | 3.640      | 4.152      | 4.879      | 5.397      |
| \(\chi_1\) | 2.860      | 4.932      | 6.229      | 7.171      | 8.518      | 9.487      |
| \(a_1\)  | \(-.486\)  | \(-.853\)  | \(-1.058\) | \(-1.205\) | \(-1.416\) | \(-1.567\) |