RIGID GEOMETRY ON PROJECTIVE VARIETIES

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Abstract. We prove rigidity of various types of holomorphic geometric structures on smooth complex projective varieties.

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1. Introduction

All manifolds and maps henceforth are assumed complex analytic, and all Lie groups, algebras, etc. are complex. We will prove a collection of global rigidity theorems for holomorphic geometric structures. As an example:

Theorem 1. Suppose that $M$ is a smooth connected complex projective variety of complex dimension 7, bearing a holomorphic quaternionic contact structure. Then
$M = C_3/P$, where $C_3 = \text{Sp}(6, \mathbb{C})$ and $P \subset C_3$ is a certain complex parabolic subgroup. The holomorphic quaternionic contact structure is the standard flat holomorphic quaternionic contact structure on $C_3/P$ (defined in section 12 on page 13).

2. The main theorem

2.1. Definitions required to state the main theorem.

2.1.1. Definition of Cartan geometries.

Definition 1. If $E \to M$ is a principal right $G$-bundle, we will write the right $G$-action as $r_e g = eg$, where $e \in E$ and $g \in G$.

Throughout we use the convention that principal bundles are right principal bundles.

Definition 2. Let $H \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A $G/H$-geometry, or Cartan geometry modelled on $G/H$, on a manifold $M$ is a choice of $C^\infty$ principal $H$-bundle $E \to M$, and smooth 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the Cartan connection, which satisfies all of the following conditions:

1. $r^*_h \omega = \text{Ad}^{-1}_h \omega$ for all $h \in H$.
2. $\omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
3. For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on $E$ by the equation $\vec{A} \omega = A$. Then the vector fields $\vec{A}$ for $A \in \mathfrak{h}$ generate the $H$-action on $E$.

Sharpe [40] gives an introduction to Cartan geometries.

Example 1. The principal $H$-bundle $G \to G/H$ is a Cartan geometry, with Cartan connection $\omega = g^{-1} dg$ the left invariant Maurer–Cartan 1-form on $G$; this geometry is called the model Cartan geometry.

Definition 3. An isomorphism of $G/H$-geometries $E_0 \to M_0$ and $E_1 \to M_1$ with Cartan connections $\omega_0$ and $\omega_1$ is an $H$-equivariant diffeomorphism $F : E_0 \to E_1$ so that $F^* \omega_1 = \omega_0$.

2.1.2. Definition of lift of Cartan geometries.

Definition 4. Suppose that $H \subset H' \subset G$ are two closed subgroups, and $E \to M$ is a $G/H'$-geometry. Let $M = E/H$. Clearly $E \to M$ is a principal $H$-bundle. We can equip $E$ with the Cartan connection of the original $G/H'$-geometry, and then clearly $E \to M$ is a $G/H$-geometry. Moreover $M \to M'$ is a fiber bundle with fiber $H'/H$. The geometry $E \to M$ is called the $G/H$-lift of $E \to M'$ (or simply the lift). Conversely, we will say that a given $G/H$-geometry drops to a certain $G/H'$-geometry if it is isomorphic to the lift of that $G/H'$-geometry.

A Cartan geometry which drops can be completely recovered (up to isomorphism) from anything it drops to. So dropping encapsulates the same geometry in a lower dimensional reformulation.

2.1.3. Definition of generalized flag varieties.

Definition 5 (Knapp [30]). A parabolic subgroup $P$ of a complex semisimple Lie group $G$ is a subgroup containing a maximal solvable subgroup.

Remark 1. Parabolic subgroups are closed connected complex Lie subgroups.
A generalized flag variety is a homogeneous space $G/P$ where $G$ is a complex semisimple Lie group and $P$ is a parabolic subgroup. Every generalized flag variety is compact and connected.

2.1.4. Semicanonical modules.

Definition 7. Suppose that $G/H$ is a complex homogeneous space. An $H$-submodule $I \subset (\mathfrak{g}/\mathfrak{h})^*$ is semicanonical if there are integers $p \geq 0$ and $q > 0$ so that $(\det I)^\otimes q = (\det (\mathfrak{g}/\mathfrak{h}))^\otimes (-p)$. An $H$-submodule $I \subset (\mathfrak{g}/\mathfrak{h})^*$ is nontrivial if $I \neq 0$ and $I \neq (\mathfrak{g}/\mathfrak{h})^*$.

2.2. The main theorem. In various examples, we will prove rigidity of various Cartan geometries. Among many other results, we will prove the following theorem:

Theorem 2. Suppose that $G$ is a complex Lie group and $H \subset G$ is a maximal complex subgroup. Suppose that $I \subset (\mathfrak{g}/\mathfrak{h})^*$ is a nontrivial semicanonical module.

Suppose that $M$ is a connected smooth complex projective variety bearing a holomorphic Cartan geometry $E \to M$ modelled on $G/H$. Let $\mathcal{I} = E \times_H I \subset T^*M$. If the holomorphic subbundle $\mathcal{I}^\perp \subset T\mathcal{M}$ is not everywhere bracket closed, then

1. $M = G/H$ and
2. the Cartan geometry on $M$ is the model holomorphic Cartan geometry on $G/H$ and
3. $G/H$ is a generalized flag variety.

Remark 2. We will prove more general theorems below, and prove a similar theorem for compact Kähler manifolds for a different class of Cartan geometries. We will apply our theorems to prove rigidity of various types of holomorphic geometric structures.

3. Pfaffian systems

Definition 8. A Pfaffian system on a complex manifold $M$ is a holomorphic vector subbundle of the holomorphic cotangent bundle $T^*M$.

Remark 3. If $\mathcal{I} \subset T^*M$ is a Pfaffian system, the reader may feel more comfortable working with $\mathcal{V} = \mathcal{I}^\perp$, which is a holomorphic plane field (a.k.a. distribution, a.k.a. subbundle of the tangent bundle). The convenience of working with $\mathcal{I}$ rather than $\mathcal{V}$ will become clear, and will more than overcome the initial discomfort.

Definition 9. A Pfaffian system $\mathcal{I} \subset T^*M$ is Frobenius if the ideal it generates in the sheaf $\Lambda^* (T^*M)$ of differential forms is $d$-closed.

Remark 4. Equivalently, $\mathcal{I}$ is Frobenius if $\mathcal{V} = \mathcal{I}^\perp$ is bracket closed. Synonyms for Frobenius include integrable, completely integrable and involutive.

4. Brackets in Cartan geometries

Lemma 1 (Sharpe [39] p. 188, theorem 3.15). If $\pi : E \to M$ is any Cartan geometry, say with model $G/H$, then the Cartan connection of $E$ maps

$$
\begin{array}{cccccc}
0 & \longrightarrow & \ker \pi'(e) & \longrightarrow & T_eE & \longrightarrow & T_mM & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} & \longrightarrow & 0
\end{array}
$$
for any points \( m \in M \) and \( e \in E_m \); thus
\[
TM = E \times_H (g/h) \quad \text{and} \quad T^*M = E \times_H (g/h)^*.
\]
Under this identification, vector fields on \( M \) are identified with \( H \)-equivariant functions \( E \to g/h \), and sections of the cotangent bundle with \( H \)-equivariant functions \( E \to (g/h)^* \).

Remark 5. If \( I \subset (g/h)^* \) is an \( H \)-submodule, then each local holomorphic section of \( E \times_H I \subset T^*M \) is identified with an \( H \)-equivariant holomorphic map from an open subset of \( E \) to \( I \).

5. PSEUDEFFECTIVE LINE BUNDLES

Definition 10. A line bundle \( L \) on a Kähler manifold is pseudoeffective if \( c_1(L) \) can be represented by a closed positive \((1,1)\)-current. (See Demailly [20] for more information.)

Remark 6. Zero is considered positive in this definition.

Definition 11. If \( V \) is a holomorphic vector bundle of rank \( N \) on a complex manifold \( M \), let \( \det V = \Lambda^N(V) \).

Lemma 2. Let \( M \) be a closed Kähler manifold and \( I \subset T^*M \) a holomorphic Pfaffian system which is not Frobenius. Suppose that the line bundle \( \det I \) on \( M \) is pseudoeffective. Then \( I \) is Frobenius.

Proof. Suppose that \( I \) has rank \( q \). Define a line-bundle-valued differential form \( \vartheta \in \Omega^q(M) \otimes \det(TM/I^\perp) \) by
\[
\vartheta(v_1, v_2, \ldots, v_q) = (v_1 + I^\perp) \wedge (v_2 + I^\perp) \wedge \cdots \wedge (v_q + I^\perp).
\]
By Demailly [21] p. 1, Main Theorem applied to \( \vartheta \), if \( \det I \) is pseudoeffective, then \( I \) is Frobenius. \( \Box \)

Definition 12. We write the canonical bundle of a complex manifold \( M \) as \( \kappa_M \).

Definition 13. A holomorphic vector bundle \( I \) on a complex manifold \( M \) is semicanonical if there are integers \( p \geq 0, q > 0 \) so that \( (\det I)^\otimes q \otimes \kappa_M^{-\otimes p} \) is pseudoeffective.

Proposition 1. Suppose that

1. \( M \) is a compact Kähler manifold and
2. \( I \subset T^*M \) is a Pfaffian system and
3. \( I \) is not Frobenius and
4. \( I \) is semicanonical.

Then the canonical bundle of \( M \) is not pseudoeffective.

Proof. By lemma 2, \( \det I \) is not pseudoeffective. If \( \kappa_M \) is pseudoeffective, then so is \( \kappa_M^{-\otimes p} \) for any integer \( p \geq 0 \). Therefore \( (\det I)^\otimes q \) is pseudoeffective for some integer \( q > 0 \), and so \( \det I \) is also pseudoeffective. \( \Box \)

Lemma 3. Suppose that \( I \subset T^*M \) is a holomorphic contact structure. Then \( I \) is semicanonical.
Proof. Pick a local section \( \vartheta \) of \( I \) for which \( \vartheta \wedge d\vartheta^n \neq 0 \). But \( \vartheta \wedge d\vartheta^n \) is a holomorphic volume form. The map \( \phi : \mathcal{I} \to \kappa_M \), given on each local section by \( \vartheta \mapsto \vartheta \wedge d\vartheta^n \), depends only on the value of \( \vartheta \) pointwise, and scales like \( \phi(f\vartheta) = f^{n+1}\phi(\vartheta) \), so \( \mathcal{I} \otimes (n+1) = \kappa_M \).

Example 2. Suppose that \( M = M_1 \times M_2 \) is a product. If \( \mathcal{I}_1 \subset T^*M_1 \) and \( \mathcal{I}_2 \subset T^*M_2 \) are semicanonical, then so is \( \mathcal{I}_1 \oplus \mathcal{I}_2 \subset T^*M \).

Example 3. Let \( V \) be a rank 2 holomorphic subbundle \( V \subset TM \) on a complex manifold \( M \) with \( \dim \mathbb{C}M = 5 \). Say that a pair \( X, Y \) of local holomorphic sections of \( V \) is nondegenerate if the vector fields

\[
X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]
\]

are linearly independent at every point where \( X \) and \( Y \) are defined. Then \( V \) is Cartan or nondegenerate if near each point of \( M \) there is a nondegenerate pair of local holomorphic sections.

Given any nondegenerate pair \( X \) and \( Y \), let

\[
\xi(X, Y) = X \wedge Y \wedge [X, Y] \wedge [X, [X, Y]] \wedge [Y, [X, Y]].
\]

So \( \xi \) takes a pair of sections to a section of the anticanonical bundle \( \kappa_M^* \). If \( X' \) and \( Y' \) are any two local sections of \( V \), we can write

\[
X' = aX + bY \\
Y' = cX + dY
\]

for some holomorphic matrix valued function

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

on the overlap where \( X', Y', X \) and \( Y \) are defined. Check that

\[
\xi(X', Y') = \det(g)^5 \xi(X, Y).
\]

Therefore if \( V \) is nondegenerate, then \( \det V = \Lambda^2(V) \) and \( \Lambda^2(V)^{\otimes 5} = \kappa_M^* \). Let \( \mathcal{I} = V^\perp \). Clearly \( \det \mathcal{I} = \kappa_M \otimes \det V \), so \( (\det \mathcal{I})^{\otimes 5} = \kappa_M^4 \), so \( \mathcal{I} \) is semicanonical. By Demailly’s theorem, since \( V \) is not bracket closed (i.e. \( \mathcal{I} \) is not Frobenius), \( \kappa_M \) is pseudoeffective.

Example 4. A holomorphic \( k \)-plane field \( V \subset TM \) on a complex manifold \( M \) of dimension \( n = \dim \mathbb{C}M = k(k+1)/2 \) is first order nondegenerate if near each point of \( M \) there are local holomorphic sections

\[
X_1, X_2, \ldots, X_k
\]

of \( V \) so that

\[
X_1, X_2, \ldots, X_k, [X_1, X_2], [X_1, X_3], \ldots, [X_{k-1}, X_k]
\]

are linearly independent. Clearly any first order nondegenerate holomorphic \( k \)-plane field \( V \) has associated Pfaffian system \( \mathcal{I} = V^\perp \) semicanonical, with the same argument as for contact structures.
6. PSEUDEFFECTIVITY AND PFAFFIAN SYSTEMS IN CARTAN GEOMETRIES

Remark 7. Clearly if $I$ is a semicanonical submodule then $I = G \times_H I$ is a semicanonical vector bundle.

Example 5. If $G/H$ has a $G$-invariant contact structure, say $G \times_H I \subset T^*(G/H)$, then $I \subset (g/h)^*$ is semicanonical.

Proposition 2. Suppose that

1. $M$ is a compact Kähler manifold and
2. $M$ bears a holomorphic Cartan geometry modelled on a complex homogeneous space $G/H$ and
3. $I$ is an $H$-submodule $I \subset (g/h)^*$ and
4. $I$ is semicanonical and
5. $E \times_H I \subset T^* M$ is not Frobenius.

Then $M$ does not have pseudoeffective canonical bundle.

Proof. Clear from proposition [1] on page 4.

Example 6. Suppose that $G/P$ is a generalized flag variety. Every $P$-submodule $I \subset (g/p)^*$ is a direct sum of root spaces, since $P$ contains the Cartan subgroup of $G$. Associate to $I$ the set $S = S_I$ of roots whose root space lies in $I$. Then $S$ is a set of noncompact positive roots. For example, $(g/p)^*$ is the direct sum of all of the root spaces of all of the noncompact positive roots. Pick any root $\alpha \in S$. Pick $\beta$ to be either a compact root or a noncompact positive root. Then either $\alpha + \beta \in S$ or $\alpha + \beta$ is not a root. Conversely, if $S$ is any set of roots with this property, let $I = I_S$ be the sum of the root spaces of the roots that lie in $S$. Then $I \subset (g/p)^*$ is a $P$-submodule. So we can draw $I$ by drawing the root lattice of $G$ and indicating somehow which roots lie in $S$.

Next we need to test when $I$ is semicanonical. Let $W_{G/P}$ be the subgroup of the Weyl group of $G$ preserving the noncompact positive roots of $G/P$. Baston and Eastwood [1] prove that we can identify the weights of $P$ with the $W_{G/P}$-invariant weights of $G$. The weight of $\det I$ is

$$\sum_{\alpha \in S} \alpha.$$ 

In particular, the weight $\omega$ of $\det (g/p)^*$ as a weight of $P$ is the sum of the noncompact positive roots, say

$$\omega = \sum_{\alpha \in \Delta^\text{noncompact}} \alpha.$$ 

We can then see that $I$ is semicanonical if and only if

$$\sum_{\alpha \in S} \alpha = \frac{p}{q} \omega$$

for some rational number $0 \leq \frac{p}{q} \leq 1$, and nontrivial if and only if $0 < \frac{p}{q} < 1$.

Example 7. Figure 1 on the next page shows the roots of $G_2$. The dots are the roots whose root spaces lie in $p$, and the crosses are the other roots. The circled dot is the origin, representing the Cartan subgroup. The compact roots lie on the
Figure 1. The parabolic subgroup $P_1 \subset G_2$

Figure 2. The parabolic subgroup $P_2 \subset G_2$ together with a rank 3 Pfaffian system

line through $\beta$ and the origin. The positive noncompact roots lie in the upper half plane above this line. The sum of the noncompact positive roots is

$$\omega = 10 \alpha + 5 \beta.$$  

Figure 2 shows stars (⋆) on the noncompact positive roots

$$3 \alpha + 2 \beta, 2 \alpha + \beta, 3 \alpha + \beta.$$  

Let $S$ be the set of these roots, and $I = I_S$. Then $I$ is a 3-dimensional submodule $I \subset (\mathfrak{g}/\mathfrak{p})^\ast$. The weight of det $I$ is

$$\sum_{\alpha \in S} \alpha = (3 \alpha + 2 \beta) + (2 \alpha + \beta) + (3 \alpha + \beta)$$

$$= 8 \alpha + 4 \beta.$$  

So $I$ is semicanonical.

Example 8. Inspect the root lattices of all simple Lie groups $G$ of rank 2, using the same approach as the previous example. You see that for all generalized flag varieties $G/P$ with $G$ of rank 2, all submodules $I \subset (\mathfrak{g}/\mathfrak{p})^\ast$ are semicanonical, except for a few counterexamples. These counterexamples only occur for those $G/P$ where $P = B$ is a Borel subgroup. Specifically $G/P = \text{SO}(5, \mathbb{C})/B$ and $G/P = G_2/B$ have no nontrivial semicanonical modules $I \subset (\mathfrak{g}/\mathfrak{p})^\ast$ (i.e. other than $I = 0$ and $I = (\mathfrak{g}/\mathfrak{p})^\ast$). On the other hand, $G/P = A_2/B = \text{SL}(3, \mathbb{C})/B$ has precisely one nontrivial semicanonical submodule (as we will see in example 10 on the next page), and various nonsemicanonical submodules.
Example 9. More generally, if $B \subset G$ is a Borel subgroup of a complex semisimple Lie group $G$, let $V = g_{-\alpha}$ be the root space of any simple root $\alpha$. Let $I = V^\perp \subset (g/p)^*$, i.e. $I$ is the sum of the root spaces of all positive noncompact roots other than $\alpha$. So the weight of $\det I$ is the sum of all positive noncompact roots other than $\alpha$. Clearly $I$ is a $B$-submodule of $(g/b)^*$. However, $I$ is not semicanonical unless $G = SL(2, \mathbb{C})$. So there are some counterexamples in arbitrary rank.

Proposition 3. Suppose that $G$ is a a complex simple Lie group and $P \subset G$ a maximal parabolic subgroup. Then every submodule $I \subset (g/p)^*$ is semicanonical.

Proof. The maximal semisimple subgroup $M \subset P$ from the Langlands decomposition (see Knapp [30]) has Dynkin diagram given by removing the crossed (i.e. noncompact) simple roots from the Dynkin diagram of $G/P$. Since $P$ is maximal, there is one noncompact simple root, so the root lattice of $M$ spans a hyperplane in the root lattice of $G$. All weights of 1-dimensional $P$-modules lie in the line in the root lattice of $G$ perpendicular to the root lattice of $M$, by invariance under the Weyl group of $M$. So if $I$ is a $P$-submodule of $(g/p)^*$, then $\det I$ has weight lying on this line. The weight $\omega$ of $\det (g/p)^*$ is the sum of the noncompact positive roots, so is a nonzero vector on this line. Therefore $\det I$ must have weight a multiple of $\kappa$. We need to show that this multiple is not negative. This is clear because the weight is a sum of positive noncompact roots. 

Example 10. The generalized flag variety $G/P$ with $G = A_n$ and Dynkin diagram $\cdot \cdots \cdot \cdots \cdot \cdots \cdot \cdot $ represents the space of pairs $(p, L)$ where $L$ is a projective line in $\mathbb{P}^n$ and $p \in L$ is a point of that line. Map $G/P \to \mathbb{P}^n$ by $(p, L) \mapsto p$. There is an obvious Frobenius Pfaffian system $I_{\mathrm{point}}$ on $G/P$ consisting of the 1-forms vanishing on the fibers of this map. Similarly there a map $G/P \to \mathbb{P}^n^{\ast}$, $(p, L) \mapsto L$, and an obvious Frobenius Pfaffian system $I_{\mathrm{line}}$ on $G/P$ consisting of the 1-forms vanishing on the fibers of this map. Let $I_0 = I_{\mathrm{point}} \cap I_{\mathrm{line}}$.

As usual, $A_n$ has roots $e_i - e_j \in \mathbb{R}^{n+1}$ for $i \neq j$. A basis of positive simple roots is $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq n$. The compact roots are $e_i - e_j$ for $i, j \geq 3$ with $i \neq j$. The noncompact positive roots are $e_1 - e_i$ for $i > 1$ and $e_2 - e_i$ for $i > 2$. Write $\alpha \leq \beta$ to mean that $\beta - \alpha$ is a sum of positive noncompact roots and compact roots.

For each positive root $\alpha$, let

$$ I_{\alpha} = \bigoplus_{\alpha \leq \beta} g_{\beta}. $$

Note that $I_{\alpha} \subset (g/p)^*$ is a $P$-submodule. There are precisely 5 distinct $P$-submodules of $(g/p)^*$:

1. $0$,
2. $I_{\mathrm{point}} = I_{\alpha_1}$, $\dim_{\mathbb{C}} I_{\mathrm{point}} = n$,
3. $I_{\mathrm{line}} = I_{\alpha_2}$, $\dim_{\mathbb{C}} I_{\mathrm{line}} = 2n - 2$,
4. $I_0 = I_{\alpha_1 + \alpha_2}$, $\dim_{\mathbb{C}} I_0 = n - 1$, and
5. $(g/p)^*$.

The associated vector bundles on $G/P$ are the Pfaffian systems defined above.

As above let $\omega$ be the sum of the positive noncompact roots,

$$ \omega = n \alpha_1 + 2(n - 1) \alpha_2 + 2(n - 2) \alpha_3 + \cdots + 2 \alpha_n, $$
while the weights for the various submodules are
\[
\det I_{\alpha_1} : n \alpha_1 + (n - 1) \alpha_2 + (n - 2) \alpha_3 + \cdots + \alpha_n,
\]
\[
\det I_{\alpha_2} : (n - 1) \alpha_1 + 2(n - 1) \alpha_2 + 2(n - 2) \alpha_3 + \cdots + 2 \alpha_n,
\]
\[
\det I_0 : (n - 1) \alpha_1 + (n - 1) \alpha_2 + (n - 2) \alpha_3 + \cdots + \alpha_n.
\]
So \( I_0 \) is semicanonical precisely when \( n = 2 \), while \( I_{\text{point}} \) and \( I_{\text{line}} \) are not semicanonical for any \( n \).

7. Rational curves on smooth complex projective varieties

**Definition 14.** A complex projective variety is uniruled if every point lies on a rational curve; see [33].

**Theorem 3** (Boucksom et. al.[5]). A smooth complex projective variety is uniruled just when the variety has nonpseudoeffective canonical bundle.

**Corollary 1.** Suppose that \( I \subset (g/h)^* \) is a semicanonical module. Suppose that \( M \) is a smooth complex projective variety with a holomorphic Cartan geometry \( E \rightarrow M \) modelled on \( G/H \). If \( E \times_H I \) is not Frobenius then \( M \) contains a rational curve.

8. Dropping

**Theorem 4** (Biswas, McKay [4]). Suppose that

1. \( G/H \) is a complex homogeneous space,
2. \( M \) is a connected compact Kähler manifold and
3. \( M \) bears a holomorphic \( G/H \)-geometry.

Then the geometry drops to a unique \( G/H' \)-geometry on a connected compact Kähler manifold \( M' \), so that

1. \( H' \subset G \) is a closed complex subgroup,
2. \( H'/H \) is a generalized flag variety,
3. \( M \rightarrow M' \) is a holomorphic \( H/H' \)-bundle, and
4. the manifold \( M' \) contains no rational curves.

Any other drop \( M \rightarrow M'' \) for which \( M'' \) contains no rational curves factors uniquely through holomorphic drops \( M \rightarrow M' \rightarrow M'' \).

**Theorem 5.** Suppose that \( I \subset (g/h)^* \) is a semicanonical module. Suppose that \( M \) is a smooth connected complex projective variety with a holomorphic Cartan geometry \( E \rightarrow M \) modelled on \( G/H \). Suppose that \( E \times_H I \) is not Frobenius.

Then the geometry drops to a unique \( G/H' \)-geometry on a connected smooth complex projective variety \( M' \), so that

1. \( H' \subset G \) is a closed complex subgroup,
2. \( H'/H \) is a generalized flag variety,
3. \( \dim \mathbb{C} H' > \dim \mathbb{C} H \), i.e. \( \dim \mathbb{C} M' < \dim \mathbb{C} M \),
4. \( M \rightarrow M' \) is a holomorphic \( H/H' \)-bundle, and
5. the manifold \( M' \) contains no rational curves.

Any other drop \( M \rightarrow M'' \) for which \( M'' \) contains no rational curves factors uniquely through holomorphic drops \( M \rightarrow M' \rightarrow M'' \).

In particular, if there is no closed proper complex Lie subgroup \( H' \subset G \) with \( H \subset H' \) and \( H'/H \) a rational homogeneous variety, then \( M = G/H \) with its standard flat Cartan geometry, and \( G/H \) is a rational homogeneous variety.
Proof. The manifold $M$ contains a rational curve, by corollary on the preceding page. By theorem on the previous page, the geometry drops. If $H$ is not contained in a closed complex Lie subgroup $H' \subset G$ for which $H'/H$ is a rational homogeneous variety, then the geometry can only drop to a geometry modelled on $G/G$, a point. The original geometry on $M$ must be isomorphic to the lift of $G/G$, i.e. must be isomorphic to $G/H$. □

Definition 15. A parabolic geometry is a holomorphic Cartan geometry modelled on a generalized flag variety.

Remark 8. Let's develop a general criterion to ensure that a Pfaffian system $E \times_P I$ in a parabolic geometry cannot be Frobenius. There is a well known notion of regularity of parabolic geometries (see Calderbank and Diemer [10], Čap [11]). Čap [11] (unnumbered proposition on page 9) proves that if a parabolic geometry $E \to M$ is regular at a point of $M$, and if $G \times_P I$ is not Frobenius on $G/P$, then $E \times_P I$ is also not Frobenius on $M$. We need to see when $G \times_P I$ is Frobenius. It is easy to see that if $I \subset (g/p)^*$ is nontrivial and semicanonical, then $G \times_P I$ is not Frobenius. Therefore if $I$ is nontrivial and semicanonical, and $E \to M$ is regular at a single point of $M$, then $E \times_P I$ is not Frobenius. We will not need to make use of this regularity criterion in our examples.

9. Example: adjoint varieties

Example 11. Suppose that $G$ is a complex semisimple Lie group. Pick a highest weight vector $x \in g$, for some choice of Cartan subalgebra of $G$ and basis of simple roots. The adjoint variety of $G$ is the orbit $X = G[x] \subset \mathbb{P}g$ of the line $[x]$ spanned by $x$ in $g$. The stabilizer of $[x]$ in $G$ is a parabolic subgroup, say $P \subset G$ and $X = G/P$. For example, if $G$ is simple, the adjoint varieties have Dynkin diagrams as in figure on the facing page.

If $G$ is simple, then its adjoint variety is a holomorphic contact manifold, and every homogeneous compact complex contact manifold occurs as an adjoint variety; see Landsberg [34]. There is precisely one holomorphic contact structure on any adjoint variety.

If $G$ is not simple, then up to a finite covering $G$ is a product of simple factors $G = G_1 \times G_2 \times \cdots \times G_s$, and correspondingly $P = P_1 \times P_2 \times \cdots \times P_s$, where $P_j = P \cap G_j$. For each $G_j$, we can then consider the one dimensional $P_j$-submodule $I_j \subset (g_j/p_j)^*$ which arises from the holomorphic contact structure on $X_j = G_j/P_j$. We can then let $I = \bigoplus_j I_j$, and again $I$ is semicanonical on $X = G/P$, though not a contact structure.

Theorem 6. Suppose that $G$ is a complex simple Lie group and that $G/P$ is an adjoint variety with holomorphic contact structure $G \times_P I$. Suppose that $E \to M$ is holomorphic parabolic geometry modelled on $G/P$, on a smooth connected complex projective variety $M$. Let $\mathcal{I} = E \times_P I \subset T^*M$. Either

(1) $M$ is foliated by smooth hypersurfaces on which $\mathcal{I} = 0$ or
(2) $M = G/P$ with its usual adjoint variety geometry or
(3) $G = A_n$, and the geometry on $M$ drops to a holomorphic projective connection on a smooth connected complex projective variety.

| Group | Variety | dim | Diagram |
|-------|---------|-----|---------|
| $A_n$ | $\mathbb{P}^n \times \mathbb{P}^n$ | $2n - 1$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $B_n$ | $\text{Gr}_{\text{null}}(2, 2n + 1)$ | $4n - 5$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $C_n$ | $\mathbb{P}^{2n-1}$ | $2n - 1$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $D_n$ | $\text{Gr}_{\text{null}}(2, 2n)$ | $4n - 7$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $E_6$ | $X^\text{ad}_{E_6}$ | $21$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $E_7$ | $X^\text{ad}_{E_7}$ | $33$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $E_8$ | $X^\text{ad}_{E_8}$ | $57$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $F_4$ | $X^\text{ad}_{F_4}$ | $15$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |
| $G_2$ | $\text{Gr}_{\text{null}}(2, \text{Im } \mathbb{O})$ | $5$ | $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$ |

**Figure 3.** The adjoint varieties of the complex simple Lie groups.

The adjoint variety of $C_n$ is $\mathbb{P}^{2n-1}$ under the Veronese embedding, i.e. the set of rank 1 quadratic forms up to rescaling. A 2-plane in the octave numbers $\mathbb{O}$ is null if the multiplication is zero on it. The adjoint variety of $G_2$ is the set of null 2-planes in the imaginary complexified octave numbers. We don’t know a geometric description of the adjoint varieties of $E_6$, $E_7$, $E_8$ and $F_4$.

**Proof.** Either $\mathcal{I}$ is Frobenius, or the parabolic geometry drops by theorem 5 on page 9.

The adjoint variety $X = G/P$ of $G = A_n$ is the variety of pairs of a hyperplane in $\mathbb{P}^n$ and a point on that hyperplane. There are only two parabolic subgroups of $A_n$ containing $P$: forget the point or the hyperplane, i.e. $G/P'$ is either projective space or the dual projective space. Projective space and its dual are isomorphic, so the same parabolic geometries are modelled on either one. Suppose that $M$ is a smooth complex projective variety with a holomorphic parabolic geometry modelled on the adjoint variety of $A_n$. Then $M$ drops to a smooth complex projective variety with holomorphic projective connection.

Consider the adjoint variety $X = G/P$ of any other simple complex Lie group $G$ (i.e. $G = B_n, C_n, D_n, E_6, E_7, E_8, F_4$ or $G_2$). Then $P \subset G$ is a maximal parabolic subgroup. So there is only one regular parabolic geometry on any smooth complex projective variety modelled on that $G/P$: the model $G/P$ with its standard flat $G/P$-geometry.

**Remark 9.** We will reconsider the $A_n$-adjoint geometries in section 15 on page 15.

**10. Example: Cartan’s theory of 2-plane fields on 5-manifolds**

In example 7 on page 9 we saw that $G_2/P_1$ bears a holomorphic rank 3 Pfaffian system. We can see from the root lattice in figure 1 on page 7 that $\dim_{\mathbb{C}} G_2/P_1 = 5$.
(i.e. 5 crosses representing the 5 dimensions of $g_2/p_1$). We can also see that the rank 3 Pfaffian system is not Frobenius, because there is a pair of noncompact positive roots not among those 3 which add up to a root among those 3. The dual plane field is associated to the $P_1$-module $V = I^\perp$, i.e. the sum of root spaces of the two roots $-\alpha, -\alpha - \beta$. We can even see that the 2-plane field is Cartan, in the sense of example 3 on page 5, by looking at the brackets of vector fields in $g_2/p_1$, i.e. looking at sums of the roots $-\alpha, -\alpha - \beta$. (We leave this claim to the reader to prove, since it is not essential to our arguments.)

**Theorem 7** (Cartan [13, 42, 44]). If $V$ is a Cartan 2-plane field on a 5-dimensional complex manifold $M$, then there is a holomorphic parabolic geometry $E \to M$ modelled on $G_2/P_1$, so that $V = E \times_P V \subset TM$, where $V \subset g_2/p_1$ is the $P_1$-submodule constructed in example 7 on page 6.

**Theorem 8.** The only holomorphic Cartan 2-plane field on any smooth connected complex projective variety is the standard one on $G_2/P_1$ described in example 7 on page 6.

**Proof.** Suppose that $M$ is a smooth connected complex projective variety of complex dimension 5, bearing a holomorphic Cartan 2-plane field. From example 3 on page 5, we have seen that a smooth complex projective variety with a Cartan 2-plane field must have nonpseudoeffective canonical bundle. By theorem 3 on page 9, the variety must then be uniruled.

By Cartan’s theorem, we can assume that the Cartan 2-plane field is $E \times_P V \subset TM$. Let $I = V^\perp$. Since $I$ is semicanonical, and the Pfaffian system $E \times_P I$ is not Frobenius, again we see that the variety $M$ must be uniruled. By theorem 4 on page 9, the parabolic geometry must drop to a parabolic geometry with a lower dimensional model. The parabolic geometry can only drop to a parabolic geometry modelled on a point, since $P$ is maximal, so drops just when the parabolic geometry is isomorphic to the model. □

11. Example: 3-plane fields on 6-manifolds

**Definition 16.** A rank 3 Pfaffian system $I \subset T^*M$ on a complex manifold $M$ of complex dimension 6 is nondegenerate if near each point of $M$ there are 3 sections of $I$ with linearly independent exterior derivatives.

**Example 12.** Let $G = B_3 = PO(7, \mathbb{C})$ and $G/P$ be the space of null 3-planes in $\mathbb{C}^6$ for some nondegenerate complex inner product. The Dynkin diagram of $G/P$ is . Write the simple roots of $G$ as $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_2 + e_3$ in terms of the standard basis $e_1, e_2, e_3 \in \mathbb{R}^3$. The root $\alpha_3$ will be the noncompact positive simple root. The noncompact positive roots are

$$\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3.$$ 

Let $S$ be the set of roots

$$\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3$$

(i.e. the roots with 2 $\alpha_3$ in them). Let $I = I_S \subset (g/p)^*$ be the sum of the root spaces of roots in $S$; $I$ has dimension 3. We can see that $G \times_P I$ is nondegenerate in Bryant’s sense, since we can write the 3 roots in $S$ each as a sum of distinct pairs of roots not in $S$. (We again leave the reader to figure out the yoga relating exterior derivatives to root sums, since we won’t use this fact.)
Theorem 9 (Bryant [7]). If \( \mathcal{I} \subset T^*M \) is a nondegenerate rank 3 Pfaffian system on a smooth complex projective variety \( M \) with \( \dim_{\mathbb{C}} M = 6 \), then there is a parabolic geometry \( E \to M \) so that \( \mathcal{I} = E \times_{\mathcal{P}} \mathcal{I} \).

Theorem 10. Suppose that \( M \) is a smooth connected complex projective variety of complex dimension 6, bearing a nondegenerate rank 3 Pfaffian system. Then \( M = B_3/\mathcal{P} \) is the model defined in example 12 on the preceding page.

Proof. Suppose that \( \mathcal{I} \subset T^*M \) is a nondegenerate rank 3 Pfaffian system on a 6-dimensional connected smooth complex projective variety \( M \). By Bryant’s theorem, we can assume that \( \mathcal{I} = E \times_{\mathcal{P}} \mathcal{I} \), for some parabolic geometry \( E \to M \). Apply theorem 5 on page 9 to prove that the geometry on \( M \) drops. The group \( \mathcal{P} \subset B_3 \) is a maximal parabolic subgroup. Therefore \( M \) must drop to a point, i.e. must be isomorphic to \( B_3/\mathcal{P} \). \( \square \)

12. Example: quaternionic contact structures

Definition 17. Suppose that \( M \) is a complex manifold, \( \dim_{\mathbb{C}} M = 7 \) and that \( \mathcal{I} \subset T M \) is a holomorphic Pfaffian system of rank 3. For any two local sections \( \vartheta_0, \vartheta_1 \) of \( \mathcal{I} \), let
\[
q(\vartheta_0, \vartheta_1) = d\vartheta_0 \wedge d\vartheta_1|_{\mathcal{I}^\perp}.
\]
It is easy to check that \( q \) is a global holomorphic section of \( \text{Sym}^2 (\mathcal{I})^* \otimes \Lambda^4 (\mathcal{I}^\perp) \).

Say that \( \mathcal{I} \) is nondegenerate if \( \vartheta_0 \cdot q = 0 \) precisely when \( \vartheta_0 = 0 \). A quaternionic contact structure is a nondegenerate holomorphic Pfaffian system of rank 3 on a 7-manifold.

Remark 10. Quaternionic contact structures are very clearly explained by Montgomery [36]. For discussion of real forms of quaternionic contact structures, see [2, 3, 23].

Example 13. Let \( X = C_3/P = \text{Sp}(6, \mathbb{C})/P \) the space of subLagrangian 2-planes in \( \mathbb{C}^6 \), where \( P \) is the stabilizer of a subLagrangian 2-plane. The Dynkin diagram of \( X \) is \( \bullet \times \cdots \times \bullet \). There is a \( C_3 \)-invariant quaternionic contact structure on \( X \) defined as follows.

We can write the roots of \( C_3 \) as vertices and the middles of edges of an octahedron, say as \( \pm e_i \pm e_j \) for \( 1 \leq i, j \leq 3 \). The positive simple roots are
\[
\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = 2e_3.
\]
The positive noncompact roots of \( X \) are
\[
\alpha_2, \quad \alpha_1 + \alpha_2, \quad \alpha_2 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3, \quad 2\alpha_2 + \alpha_3, \quad \alpha_1 + 2\alpha_2 + \alpha_3, \quad 2\alpha_1 + 2\alpha_2 + \alpha_3.
\]
Consider the 3 roots
\[
2\alpha_2 + \alpha_3, \quad \alpha_1 + 2\alpha_2 + \alpha_3, \quad 2\alpha_1 + 2\alpha_2 + \alpha_3,
\]
i.e. those with \( 2\alpha_2 \) in them. Let \( I \subset \mathfrak{p} \) be the sum of the root spaces of those 3 roots. Consider the Pfaffian system \( \mathcal{I} = C_3 \times_{\mathcal{P}} I \subset T^*(C_3/P) \). One can directly calculate using the structure equations of \( C_3 \) that \( \mathcal{I} \) is a quaternionic contact structure. (Once more we leave this local calculation to the reader, since we don’t need this result.)
Theorem 11 (Montgomery [36]). If $\mathcal{I}$ is a quaternionic contact structure on a complex manifold $M$, then there is a holomorphic parabolic geometry $E \to M$ modelled on $X = C_3/P$, so that $E \times_P I = \mathcal{I}$, for $I = (g/p)^*$ the semicanonical $P$-submodule defined in example 13 on the previous page.

We now prove theorem 1 on page 1.

Proof. Suppose that $\mathcal{I}$ is a quaternionic contact structure on $M$. By Montgomery’s theorem, we can assume that $\mathcal{I} = E \times_P I$ for some holomorphic parabolic geometry $E \to M$. Apply theorem 5 on page 9 to prove that the parabolic geometry on $M$ drops. Since $P \subset C_3$ is a maximal parabolic subgroup, $M$ must drop to a point, i.e. must be isomorphic to $X$. □

13. Example: Čap–Neusser Pfaffian systems

Example 14. For $n \geq 3$, let $G = B_n = PO(2n + 1, \mathbb{C})$, Let $X = G/P$ be the set of all $n$-dimensional null subspaces of the standard complex linear inner product on $\mathbb{C}^{2n+1}$. The Dynkin diagram of $X$ is

If we order the positive roots according to the coefficient of $\alpha_n$, there are precisely $n$ positive noncompact roots $\alpha$ with coefficient 1 and precisely $n(n-1)/2$ positive noncompact roots $\alpha$ with coefficient 2. Let $S$ be the set of noncompact positive roots of coefficient 2. Let $I = I_S$ be the sum of the root spaces of these roots, so $I \subset (g/p)^*$. Then $I$ turns out to be a first order nondegenerate Pfaffian system in the sense of example 4 on page 5. (Again we leave this statement for the reader to prove.)

Theorem 12 (Čap and Neusser [12]). Suppose that $n \geq 3$. Suppose that $M$ is a complex manifold with $\dim_{\mathbb{C}} M = n(n + 1)/2$. Suppose that $\mathcal{I} \subset T^* M$ is a holomorphic first order nondegenerate Pfaffian system. Then there is a holomorphic parabolic geometry $E \to M$ so that $E \times_P I = \mathcal{I}$, where $I$ is the semicanonical $P$-module defined in example 13.

Theorem 13. Suppose that $M$ is a smooth connected complex projective variety with $\dim_{\mathbb{C}} M \geq 6$ bearing a holomorphic first order nondegenerate Pfaffian system. Then $M = B_n/P$ with its standard first order nondegenerate Pfaffian system as defined in example 14.

Proof. Suppose that $\mathcal{I}$ is a first order nondegenerate Pfaffian system on $M$. By theorem 12 of Čap and Neusser, we can assume that $\mathcal{I} = E \times_P I$ for some holomorphic parabolic geometry $E \to M$ modelled on $B_n/P$. Apply theorem 5 on page 9 to prove that the geometry on $M$ drops. Since the model $P \subset B_n$ is a maximal parabolic subgroup, $M$ must drop to a point, i.e. must be isomorphic to $X$. □

Remark 11. Theorem 10 on the previous page is the special case of theorem 13 for $\dim_{\mathbb{C}} M = 6$.

14. Example: parabolic geometries modelled on products

Theorem 14. Suppose that $E \to M$ is a holomorphic parabolic geometry on a smooth complex projective variety $M$, modelled on a generalized flag variety $G/P$. Suppose that $G$ splits into a product of simple complex Lie groups,

$$G = G_1 \times G_2 \times \cdots \times G_s.$$
Let $P_j = P \cap G_j$ for each $j$. Suppose that $P_j \subset G_j$ is maximal for each $j$. Suppose that each $G_j/P_j$ has a nontrivial semicanonical $P_j$-submodule $I_j \subset (\mathfrak{g}_j/\mathfrak{p}_j)^*$. Let $I = I_1 \oplus I_2 \oplus \cdots \oplus I_s$. Suppose that every $E \times_P I_j$ is not Frobenius. Then $M = G/P$ with its standard flat parabolic geometry.

Proof. Theorem 5 ensures that the parabolic geometry drops, say to a geometry with some model $G/Q$. $P \subset Q \subset G$ on some complex manifold $M'$. Since each $P_j \subset G_j$ is maximal, the group $Q$ must be obtained by setting $Q_j = P_j$ or $Q_j = G_j$ for each value of $j$, and then $Q = Q_1 \times Q_2 \times \cdots \times Q_s$. If $P_j \neq Q_j$, then $E \times_Q I_j$ is not Frobenius, since its local sections pull back to local sections of $E \times_Q I_j$. Therefore $Q = G$, and therefore $M'$ is a point, and so $M$ must be isomorphic to the model, i.e. $M = G/P$. \hfill $\square$

15. **Example: Double Legendre foliations**

We arrive at our most complicated example. We will study parabolic geometries modelled on the adjoint variety of $A_n$, but we will not obtain a complete classification.

**Example 15.** Consider the adjoint variety of $A_n = \text{SL}(n+1, \mathbb{C})$, say $X = G/P$, $G = A_n$. Let’s first find all of the $P$-submodules $I \subset (\mathfrak{g}/\mathfrak{p})^*$. Write the positive roots of $A_n$ as $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i \leq j$. Let $S_1$ be the set of roots

$$\alpha_1, \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

Let $S_n$ be the set of roots

$$\alpha_1 + \alpha_3 + \cdots + \alpha_n, \alpha_2 + \alpha_4 + \cdots + \alpha_n, \ldots, \alpha_n.$$

Let $I_1 \subset (\mathfrak{g}/\mathfrak{p})^*$ be the sum of all root spaces of positive noncompact roots $\alpha$ for which $\alpha \in S_1$, and similarly let $I_n \subset (\mathfrak{g}/\mathfrak{p})^*$ be the sum of all root spaces of positive noncompact roots $\alpha$ for which $\alpha \in S_n$. It is clear that no two noncompact positive roots can add up to a root in $S_1$, and similarly for $S_n$. It turns out to follow that $G \times_P I_1$ and $G \times_P I_n$ are Frobenius. (We leave the reader to figure out the yoga relating exterior derivatives to root sums, since we will only make use of it in examples where the claims made are an elementary calculation using Cartan’s structure equations.) By a similar argument, if we let $I_{1n} = I_1 \cap I_n$, then $G \times_P I_{1n}$ is a contact structure. Indeed $I_{1n}$ corresponds to the set $S_{1n} = S_1 \cap S_n$, which is just

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

which is a sum of noncompact positive roots

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1}) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_n),$$

corresponding to the exterior derivative of the contact form being a sum of the wedge products of various 2-forms. The leaves of $G \times_P I_1$ are the fibers of $G/P \to \mathbb{P}^n$, while the leaves of $G \times_P I_n$ are the fibers of $G/P \to \mathbb{P}^{n*}$. In particular, $G/P$ has two foliations (indeed fiber bundle mappings), with leaves integral manifolds of the contact structure.

**Definition 18** (Tabachnikov [43]). A **double Legendre foliation** [43] of a complex manifold $M$ of complex dimension $2n - 1$ is a pair $F_0, F_1 \subset TM$ of holomorphic foliations so that $F_0 \oplus F_1 \subset TM$ is a holomorphic contact structure, and the leaves of $F_0$ and of $F_1$ are Legendre submanifolds.
Example 16. The $A_n$-adjoint variety has a holomorphic double Legendre foliation.

Theorem 15 (Tabachnikov [43]). Suppose that $F_0, F_1$ is a holomorphic double Legendre foliation of a complex manifold $M$ of complex dimension $2n - 1$. Then there is a holomorphic parabolic geometry $E \rightarrow M$ modelled on the adjoint variety of $A_n = \text{SL}(n+1, \mathbb{C})$ so that (in the notation of example 15 on the previous page) $F_0 = E \times_p I^+_{1}$ and $F_1 = E \times_p I^-_{n}$.

Example 17. We will construct a parabolic geometry modelled on the adjoint variety of $A_n$ by lifting a holomorphic projective connection.

Write points of $\mathbb{C}^{n+1}$ as columns, spanned by the standard basis $e_0, e_1, \ldots, e_n$. Clearly $\mathbb{P}^n = \text{PSL}(n + 1, \mathbb{C})/P_1$ where $P_1 \subset G = \text{PSL}(n + 1, \mathbb{C})$ is the subgroup of matrices of the form

$$
\begin{bmatrix}
p_0 & p_1 \\
0 & p_2
\end{bmatrix},
$$

$1 \leq i, j \leq n$. and we identify an element of $g/p_1$ with a column in $\mathbb{C}^n$ by writing out the entries

$$
\begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n
\end{pmatrix} \in \mathbb{C}^n.
$$

Let $P_n \subset G = \text{PSL}(n + 1, \mathbb{C})$ be the subgroup of matrices of the form

$$
\begin{bmatrix}
p_0 & p_1 & p_2 \\
p_0 & p_1 & p_2 \\
0 & 0 & p_n
\end{bmatrix},
$$

$1 \leq i, j \leq n - 1$. Let $P = P_1 \cap P_n \subset G = \text{PSL}(n + 1, \mathbb{C})$ be the subgroup of matrices of the form

$$
\begin{bmatrix}
p_0 & p_1
p_2 & p_3 & p_4 \\
0 & 0 & p_2
\end{bmatrix},
$$

$1 \leq i, j \leq n - 1$.

Suppose that $M'$ is a complex manifold of complex dimension $n$ bearing a holomorphic projective connection, i.e. a holomorphic parabolic geometry $\pi : E \rightarrow M'$ modelled on $\mathbb{P}^n$. We will construct the lift of $M'$ to a parabolic geometry modelled on the adjoint variety of $A_n$. Let $M = \mathbb{P}T^*M'$, with its usual holomorphic contact structure. Write points of $M$ as $m = (m', H)$ where $m' \in M'$ and $H \subset T_{m'}M'$ is a complex hyperplane. Map $E \rightarrow M$ by

$e \in E \mapsto m = (m', H) \in M$,

taking $H$ to be the hyperplane identified by the Cartan connection $\omega$ with the span of $e_1, e_2, \ldots, e_{n-1} \in \mathbb{C}^n = g/p_1$. Because $\omega$ transforms in the adjoint $P$-representation, i.e.

$$r^*_p\omega = \text{Ad}(p)^{-1}\omega,$$

we can easily check that $M = E/P$. Therefore $M = \mathbb{P}T^*M'$ is the lift of $M'$, and $E \rightarrow M$ is a parabolic geometry modelled on the adjoint variety of $A_n$.

Consider on $E$ the following two linear Pfaffian systems: let $I_0 \subset T^*E$ be the system

$$\omega + p_1 = 0,$$
and let $\mathcal{I}_1 \subset T^*E$ be the system

$$\omega + p_\pi = 0.$$ 

The fibers of $E \to M$ are Cauchy characteristics for each of these systems, and both systems are $P$-invariant. Therefore these Pfaffian systems are pulled back from Pfaffian systems, which we denote by the same names, on $M$; see [8]. Clearly on $M$, $\mathcal{I}_0 = E \times_P I_0$ and $\mathcal{I}_1 = E \times_P I_1$. Let $F_0 = \mathcal{I}_0^\perp$ and $F_1 = \mathcal{I}_1^\perp$.

This holomorphic vector subbundle $F_1 \subset TM$ might not be a foliation. Clearly $F_0$ is a foliation. But $F_1$ is a foliation if and only if the projective connection on $M'$ satisfies a certain complicated condition on its curvature, ensuring the existence of a suitably large family of totally geodesic hypersurfaces, a local calculation which I leave to the reader.

On the other hand, if the projective connection on $M'$ has “enough” totally geodesic hypersurfaces, so that $F_1$ is a foliation, then each leaf of $F_1$ projects to an immersed complex hypersurface in $M'$, so that for every linear hyperplane $H \in PT^*M' = M$, there is a unique such complex hypersurface with tangent space $H$.

**Remark 12.** Suppose that $M'$ is a complex manifold with holomorphic normal projective connection (see Kobayashi and Nagano [31] for the definition of normal). We leave the reader to check that a projective connection has “enough” totally geodesic hypersurfaces (i.e. one through each point with each possible tangent hyperplane, i.e. $F_1$ is a foliation) if and only if the projective connection is flat.

**Theorem 16.** Suppose that $M$ is a smooth complex projective variety bearing a holomorphic double Legendre foliation. Then $M = \mathbb{P}T^*M'$ for some smooth complex projective variety $M'$, and $M$ is the lift of a holomorphic projective connection on $M'$.

**Remark 13.** This theorem reduces the classification of double Legendre foliations on smooth complex projective varieties to that of holomorphic projective connections satisfying the required curvature condition to have “enough” totally geodesic hyperplanes.

**Proof.** By theorem [15 on the facing page] for any double Legendre foliation, say with contact structure $\mathcal{I} \subset T^*M$, there is a parabolic geometry $E \to M$ so that $\mathcal{I} = E \times_P I$ for some $P$-module $I \subset (\mathfrak{g}/\mathfrak{p})^\ast$. Since $P \subset G$ is a maximal parabolic subgroup, by proposition [3 on page 8] $I$ is a semicanonical $P$-module. Therefore the parabolic geometry on $M$ has a semicanonical module whose associated Pfaffian system is not Frobenius. Apply theorem [5 on page 9] to see that the geometry must drop. Unless $M$ is isomorphic to the model geometry, there is only one space $A_n/P'$ that it can drop to, since there is only one parabolic subgroup $P' \subset A_n$ containing $P$, so $A_n/P' = \mathbb{P}^n$ a projective connection on some $M'$. (To be precise, there are actually two such subgroups, but there is only one up to outer automorphism.) So $M$ is a lift of a projective connection on $M'$. The lift of any projective connection to a parabolic geometry modelled on the adjoint variety of $A_n$ is given in detail in example [17 on the preceding page] and must be $M = \mathbb{P}T^*M'$. Since the subbundle $F_1 \subset TM$ is a foliation, the projective connection must have “enough hypersurfaces”. \[\square\]
16. Circles in parabolic geometries

So far we have one method to force dropping of Cartan geometries: semicanonical modules. Next we will describe a different method to force dropping, instead of using semicanonical modules. The method of semicanonical modules works particularly well on parabolic geometries modelled on $G/P$ with $P \subset G$ a maximal parabolic subgroup. Our new method, the method of rational circles, will work only on the opposite extreme: geometries modelled on $G/B$. All parabolic geometries, with any model $G/P$, lift to geometries modelled on $G/B$, so it is natural to focus on the $G/B$-geometries.

Definition 19. Suppose that $\alpha$ is a root of a complex semisimple Lie group $G$. Let $\mathfrak{sl}(2, \mathbb{C})_\alpha$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root spaces of $\alpha$ and $-\alpha$.

Remark 14. Any root of any complex semisimple Lie algebra is reduced, so $\mathfrak{sl}(2, \mathbb{C})_\alpha$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ [ES].

Definition 20. Suppose that $E \to M$ is a holomorphic parabolic geometry, with Cartan connection $\omega$, modelled on a generalized flag variety $G/P$. Suppose that $\alpha$ is a positive simple root of $G/P$. Define a Pfaffian system on $E$ by the equation

$$\omega = 0 \pmod{\mathfrak{sl}(2, \mathbb{C})_\alpha}$$

on tangent vectors. This Pfaffian system has the fibers of $E \to E/B$ as Cauchy characteristics, and is $B$-invariant, and therefore descends to a unique Pfaffian system on $E/B$. Call the maximal integral Riemann surfaces of this system on $E/B$ $\alpha$-circles, or just circles.

Remark 15. There is some danger of confusion here, since $\omega$ is not actually defined on $E/B$, and since the $\alpha$-circles of $M$ are complex 1-dimensional submanifolds of $E/B$, not of $M$.

Remark 16. Suppose that $G/P$ is a generalized flag variety and that $G$ splits into a product

$$G = G_1 \times G_2 \times \cdots \times G_s$$

of simple complex Lie groups. Then the Borel subgroup $B \subset G$ has the form

$$B = B_1 \times B_2 \times \cdots \times B_s,$$

where $B_i = B \cap G_i$. Let

$$B'_i = P_1 \times P_2 \times P_{i-1} \times B_i \times P_{i+1} \times \cdots \times P_s.$$

If $\alpha$ is a positive simple root of $\mathfrak{g}_i$, then we can define the circles by the equation

$$\omega = 0 \pmod{\mathfrak{sl}(2, \mathbb{C})_\alpha}$$

on $E$ as above, but we find that in fact the fibers of $E \to E/B'_i$ are Cauchy characteristics for this linear Pfaffian system. So in fact, we can define the circles as Riemann surfaces on the various $E/B'_i$. For our purposes in this paper, this observation has no significance, but it should save computation in examples.

Lemma 4. In the model, $G/P$, all circles are rational.

Proof. In the model $G/P$, with the standard model $G/P$-geometry, the $\alpha$-circles are precisely the orbits in $G/B$ of the connected subgroup $\text{SL}(2, \mathbb{C})_\alpha \subset G$ whose Lie algebra is $\mathfrak{sl}(2, \mathbb{C})_\alpha$. Note that $\mathfrak{sl}(2, \mathbb{C})_\alpha \cap \mathfrak{b} \subset \mathfrak{sl}(2, \mathbb{C})_\alpha$ is a Borel subalgebra. So the associated connected Lie subgroup $\text{SL}(2, \mathbb{C})_\alpha$ (which is either isomorphic
to SL(2, C) or to PSL(2, C) must act on the orbit as a complex semisimple Lie group acting on a generalized flag variety. The orbit has one complex dimension. Therefore the orbit is a rational curve. So in the model, all circles are rational. □

**Example 18.** Take $G/P = \mathbb{P}^2$ and $G = \text{PSL}(3, \mathbb{C})$. There is one noncompact positive simple root $\alpha$. Note that $G/B = \mathbb{P}T\mathbb{P}^2$. In fact $\alpha$-circles are precisely the lifts of complex projective lines in $\mathbb{P}^2$ to the projectivized tangent bundle. We lift each line $L = \mathbb{P}^1 \subset \mathbb{P}^2$ by taking each point $p \in L$ to $T_pL \in \mathbb{P}T\mathbb{P}^2 = G/B$. Clearly $\mathbb{P}T\mathbb{P}^2$ is foliated by the lifts of lines.

**Remark 17.** For each positive simple root $\alpha$ of $G/P$, the $\alpha$-circles foliate $E/B$. They are not actually defined inside $M$, although each $\alpha$-circle projects via a local biholomorphism to a Riemann surface in $M$. Therefore it is natural to picture the $\alpha$-circles as (not necessarily compact) curves in $M$.

**Remark 18.** Another description of the $\alpha$-circles: they are the leaves of the foliation $E \times_B g_{-\alpha} \subset T(E/B) = E \times_B (g/b)$.

**Theorem 17** (Brunella [6]). Suppose that $M$ is a compact Kähler manifold. Suppose that $F \subset TM$ is a holomorphic foliation by (not necessarily compact) curves, i.e. a rank 1 subbundle. Either (1) all of the leaves of $F$ are rational curves and $F^*$ is not pseudoeffective or (2) none of the leaves of $F$ are rational and $F^*$ is pseudoeffective.

**Remark 19.** Brunella’s theorem concerns holomorphic foliations with singularities, but we will only consider nowhere singular foliations, so case (2) above follows from Brunella’s remarks [6] p. 55.

**Proposition 4.** Suppose that $M$ is a compact Kähler manifold bearing a holomorphic parabolic geometry modelled on a generalized flag variety $G/P$. Suppose that $B \subset P$ is a Borel subgroup.

Draw the Dynkin diagram of $P$, but then change any cross (say corresponding to some noncompact simple root $\alpha$) to a dot if the $\alpha$-circles are rational. In other words, change a cross to a dot just when the line bundle $E \times_B g_{-\alpha}$ on $E/B$ is not pseudoeffective. Let $Q$ be the parabolic subgroup of $G$ whose Dynkin diagram we have just drawn.

Suppose $P' \subset G$ is a parabolic subgroup containing $P$. Then $M$ drops $M \to M'$ to a holomorphic parabolic geometry modelled on $G/P'$ if and only if $Q \subset P'$.

**Proof.** By theorem 17, these line bundles are pseudoeffective if and only if the $\alpha$-circles are rational.

If $M$ drops to $M \to M'$, then the $\alpha$-circles lie inside the fibers of $M \to M'$, i.e. inside generalized flag varieties $P'/P$. In these generalized flag varieties, the induced $P'/P$-geometry is the model geometry, and the $\alpha$-circles are therefore rational curves.

Conversely if the $\alpha$-circles are rational curves, theorem 4 on page 9 ensures that there is a drop $M \to M'$ so that all of the $\alpha$-circles lie in the fibers of $M \to M'$. The fibers are $P'/P$, some parabolic subgroup $P' \subset G$. For $P'/P$ to contain all of the $\alpha$-circles, $P'$ must have $\alpha$ as a compact root. □

**Theorem 18.** Suppose that

(1) $G$ is a complex semisimple Lie group with Borel subgroup $B \subset G$ and
(2) $M$ is a compact Kähler manifold and
(3) $E \to M$ is a holomorphic parabolic geometry modelled on $G/B$. Then either (1) this parabolic geometry drops to some lower dimensional holomorphic parabolic geometry on a compact Kähler manifold or (2) for every $B$-submodule $I \subset (g/b)^*$, the associated Pfaffian system $E \times_B I \subset T^*M$ is Frobenius.

Proof. By proposition 4 on the preceding page, if the geometry does not drop, then for every positive simple root $\alpha$, the line bundle $E \times_B g_\alpha$ on $M$ is pseudoeffective. Every positive root $\alpha$ is a sum, with nonnegative integer coefficients, of positive simple roots. So for any positive root $\alpha$, not necessarily simple, the line bundle $E \times_B g_\alpha$ on $M$ is also pseudoeffective. Pick any $B$-submodule $I \subset (g/b)^*$. Then

$$\det I = \bigotimes_{\alpha} g_\alpha,$$

where the tensor product is over positive roots $\alpha$ for which $g_\alpha \subset I$. Therefore the line bundle $E \times_B \det I$ on $M$ is pseudoeffective. By lemma 2 on page 4, $E \times_B I \subset T^*M$ is Frobenius. □

17. Example: second order scalar ordinary differential equations

A path geometry is a geometric description of a system of 2nd order ordinary differential equations.

Example 19. Take a 2nd order scalar order differential equation,

$$\frac{d^2 y}{dx^2} = f \left( x, y, \frac{dy}{dx} \right).$$

Pick a variable $p$, and consider the associated foliation

$$dy = p \, dx,$$
$$dp = f \left( x, y, p \right) \, dx,$$

whose leaves correspond to the solutions of the equation. Also consider the foliation

$$dy = 0,$$
$$dx = 0,$$

whose leaves correspond to the points $(x, y)$ of the configuration space.

Definition 21. A path geometry on a complex manifold $M$ with $\dim_{\mathbb{C}} M = 3$ is a choice of 2 nowhere tangent holomorphic foliations on $M$ by (not necessarily compact) curves, called integral curves, and stalks respectively, with both foliations being tangent to a (necessarily uniquely determined) holomorphic contact plane field.

Remark 20. In other words, a path geometry is a double Legendre foliation of a 3-manifold.

Remark 21. It turns out that near any point of $M$ there are local coordinates $x, y, \dot{y}$ on $M$ and there is a holomorphic function $f \left( x, y, \dot{y} \right)$ for which the integral curves are the solutions of

$$dy = \dot{y} \, dx,$$
$$d\dot{y} = f \, dx,$$

while the stalks are the solutions of $dx = dy = 0$. Conversely, for any holomorphic function $f \left( x, y, \dot{y} \right)$, these two holomorphic foliations are a path geometry.
Remark 22. If we interchange the two foliations of a path geometry, we obtain the dual path geometry.

**Theorem 19** (Cartan [14]). A holomorphic path geometry on a complex manifold $M$ determines and is determined by a holomorphic parabolic geometry $E \to M$ modelled on the adjoint variety $B_2/B$. The holomorphic contact structure is $E \times_B I$ for a semicanonical $P$-submodule $I \subset (g/b)^*$.

**Remark 23.** See [9] for a detailed exposition.

**Remark 24.** We encountered this adjoint variety in example 11 on page 10. Suppose that $M$ is a complex manifold with path geometry, and that $E \to M$ is the induced regular parabolic geometry of Cartan’s theorem. Labelling roots and $B$-modules as in example 10 on page 8, we can see that $E \times_B I_{12}$ is a holomorphic contact structure. This is the contact structure of the path geometry. The integral curves are circles of the root $\alpha_1$, while the stalks are the circles of the root $\alpha_2$.

Using our method of semicanonical modules, we can classify holomorphic path geometries on smooth complex projective varieties. By the method of circles, we find the complete classification on compact Kähler manifolds.

**Theorem 20.** Suppose that $M^3$ is a connected compact Kähler manifold with a holomorphic path geometry. Then $M = \mathbb{P}TM'$, where $M'$ is a compact Kähler surface with a holomorphic projective connection. The stalks of $M \to M'$ are either the stalks or the integral curves of $M$. The manifold $M'$ is

1. $\mathbb{P}^2$ (and $M$ is the model $B_2/B$ with its standard flat path geometry) or
2. a complex surface with an unramified covering by the unit ball in $\mathbb{C}^2$ (and $M$ is a quotient of an open set in the model $B_2/B$, with its standard flat path geometry), or
3. a complex surface with an unramified holomorphic covering by a 2-torus (and the pullback projective connection on the torus is translation invariant).

All of these possibilities for $M'$ occur. The parabolic geometry on $M$ is the path geometry associated to the geodesic equation of the projective connection on $M'$.

**Proof.** We have proven in a more general setting in section 15 on page 15 that $M = \mathbb{P}TM'$, (keeping in mind that $\mathbb{P}TM' = \mathbb{P}T^*M'$). The compact complex surfaces which bear projective connections have been classified [32]: $M' = \mathbb{P}^2$ or $M'$ is an unramified ball quotient or an unramified torus quotient. The projective connections on these surfaces have also been classified [28, 29] under the hypothesis of local flatness. In case $M' = \mathbb{P}^2$, the presence of rational curves in $\mathbb{P}^2$ ensures, by theorem [4 on page 9] that the holomorphic projective connection on $M'$ is flat. In case $M'$ is covered by the ball, Klinger’s arguments in [29] actually go through without change, to prove local flatness. Finally, if $M'$ is covered by a torus, then any projective connection on $M'$ is translation invariant as shown in [35]. A local calculation (see Bryant, Griffiths and Hsu [9]) shows that the lift $M$ of a projective connection on any complex surface $M'$ has parabolic geometry given by the path geometry of the geodesic equation of the projective connection.

**Remark 25.** Another perspective: either the scalar second order ordinary differential equations that comprise the parabolic geometry on $M$ are the geodesic equations of a holomorphic projective connection on a complex surface, or else they are the dual equations of such equations.
Remark 26. The 2nd order ODE of the path geometry, in the case when $M' = \mathbb{P}^2$ or a ball quotient, is locally equivalent to
\[ \frac{d^2 y}{dx^2} = 0. \]
If $M'$ is a surface covered by a torus, then the 2nd order ODE is locally equivalent to
\[ \frac{d^2 y}{dx^2} = p \left( \frac{dy}{dx} \right), \]
for $p$ a polynomial of degree at most 3 with constant coefficients (in linear holomorphic coordinates on the torus); see Cartan [17] for proof. In either case, these equations are solvable by quadratures.

18. Example: third order scalar ordinary differential equations

Sato & Yoshikawa [37] have results on third order ordinary differential equations similar to Cartan's above on second order ordinary differential equations. For any third order ordinary differential equation for one function of one variable, say
\[ \frac{d^3 y}{dx^3} = f \left( x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2} \right), \]
consider the manifold $M^4$ whose coordinates are $x, y, p, q$, equipped with the exterior differential system
\[ dy = p\, dx, \quad dp = q\, dx, \quad dq = f(x, y, p, q)\, dx. \]
Sato and Yoshikawa put a parabolic geometry on $M$. Their parabolic geometry is invariant under “contact transformations”.

In this context, a contact transformation (in the sense of Lie, not the sense of contact topology) is any biholomorphism that preserves a certain complete flag of Pfaffian systems. Let
\[ \vartheta_1 = dy - p\, dx, \quad \vartheta_2 = dp - q\, dx, \quad \vartheta_3 = dq - f(x, y, p, q)\, dx \]
and let $\mathcal{I}_j$ ($j = 1, 2, 3$) be the Pfaffian system spanned locally by $\vartheta_1, \ldots, \vartheta_j$. Then a contact transformation in Lie’s sense is a local biholomorphism preserving all of the Pfaffian systems $\mathcal{I}_j$.

The parabolic geometry of Sato and Yoshikawa is modelled on $\text{Sp}(4, \mathbb{C})/B = C_2/B = B_2/B = \text{PO}(5, \mathbb{C})/B$, where $B$ is the Borel subgroup. The root lattice of $B_2/B$ is drawn in figure [4]. One can see the 4 positive roots, drawn as dots. The 3 Pfaffian systems are associated to the sets of positive roots
\[ \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2. \]

The Dynkin diagram of the model is $\times \| \| \times$. Let’s refer to a parabolic geometry with this model which arises locally from a third order ordinary differential equation, following the method of Sato and Yoshikawa, as a third order ODE geometry. The fibers of the bundle map
Example 20. Suppose that $M'$ is a complex manifold, of complex dimension 3, with holomorphic parabolic geometry $E \to M'$ modelled on the smooth quadric hypersurface $Q^3 = \text{PO}(5, \mathbb{C})/P = B_2/P$. For example, a holomorphic conformal structure on $M'$ will impose such a holomorphic parabolic geometry. Conversely, every parabolic geometry modelled on $B_2/B$ imposes a holomorphic conformal structure, since the group $P$ acts in the representation $\mathfrak{g}/\mathfrak{p}$ preserving a nondegenerate quadratic cone. Let $M$ be the set of all null lines in the tangent spaces of $M'$. We can easily see that $M = E/B$, $B \subset B_2$ the Borel subgroup. Therefore $M$ is the lift of $M'$ to a $B_2/B$-geometry. Moreover, if the parabolic geometry on $M'$ is a holomorphic conformal structure, then the parabolic geometry of the lift $M$ is precisely the equation of circles in $M'$. We leave the reader to check these (purely local and elementary) assertions of parabolic geometry.

Remark 27. The construction of a third order ordinary differential equation out of a conformal structure has been well known since work of Wünschmann (see Chern [18, 19], Dunajski & Tod [22], Frittelli, Newman & Nurowski [24], Sato & Yoshikawa [37], Silva-Ortigoza & García-Godínez [41], Wünschmann [45]). Identification of the local obstruction to dropping with the Chern invariant is a long but straightforward calculation (see Sato & Yoshikawa [37]). Hitchin [26] pointed out that a rational curve on a surface with appropriate topological constraint on its normal bundle must lie in a moduli space of rational curves constituting the integral curves of a unique third order ordinary differential equation with vanishing Chern invariant.

Theorem 21. Suppose that $M$ is a compact Kähler 4-fold with holomorphic parabolic geometry modelled on $B_2/B$. Then the geometry on $M$ drops to a holomorphic parabolic geometry modelled on

1. the smooth quadric hypersurface $Q^3 = B_2/P$ or
2. the projective space $\mathbb{P}^3$ of null 2-planes in $\mathbb{C}^5$.

on a compact Kähler 3-fold $M'$. In particular, if the parabolic geometry on $M$ is locally a third order ODE geometry, then this geometry is the equation of

1. circles of a holomorphic conformal structure on $M'$ or
2. circles of a holomorphic Legendre connection on $M'$.
Remark 28. See Sato and Yoshikawa \[37\] for the definition of Legendre connection.

Remark 29. It is not known which compact Kähler 3-folds admit conformal geometries, or admit Legendre connections.

Proof. The model for third order ODE geometries is $B_2/B$, a quotient by a Borel subgroup, so theorem \[18\] on page \[19\] applies, ensuring that the parabolic geometry drops to one modelled on either \(\times\) \(\times\) or \(\cdot\) \(\cdot\) \(\cdot\). We call these $B_2/P_1$ and $B_2/P_2$ respectively. The variety $B_2/P_1$ is the Dynkin diagram of the model of a conformal geometry. The variety $B_2/P_2$ is the Dynkin diagram of the model of the parabolic geometry of a Legendre connection (again see Sato and Yoshikawa \[37\]). We leave the reader to check by examination of the structure equations of Sato and Yoshikawa \[37\] p. 1000 that if we take a parabolic geometry with either of these two models on some complex manifold $M'$ and lift it, say to a complex manifold $M$, then the lifted parabolic geometry on $M$ is the parabolic geometry associated by Sato and Yoshikawa to the third order ODE of the circles. □

Definition 22. Recall that the Lie ball is the noncompact Hermitian symmetric space dual to the smooth quadric hypersurface.

Theorem 22. Suppose that $M$ is a smooth complex projective 4-fold with a holomorphic third order ODE geometry. Then $M$ is the set of null lines in the tangent spaces of a 3-fold $M'$ with holomorphic conformal geometry. The 3rd order ODE geometry on $M$ is the one associated to the circles of $M'$. The smooth complex projective 3-folds $M'$ which admit holomorphic conformal structures are precisely

1. the quadric $Q^3$, with its standard flat conformal geometry,
2. 3-folds with unramified covering by an abelian 3-fold, with any translation invariant conformal geometry,
3. 3-folds covered by the Lie ball with the standard flat conformal geometry.

Proof. Apply theorem \[21\] on the previous page to ensure that the parabolic geometry on $M$ drops to a parabolic geometry on some 3-fold $M'$. The parabolic geometry on $M'$ could be either a conformal structure or a Legendre connection. Legendre connections admit a holomorphic contact structure, of the form $E \times_{P_2} I$, as we see from the structure equations of Sato and Yoshikawa, \[37\] p. 1000. A contact structure is semicanonical and not Frobenius. Therefore any Legendre connection on any smooth complex projective variety must be isomorphic to the model $P^3$ and this forces $M$ to be isomorphic to its model, so drops to the model $Q^3$ of conformal geometry.

Therefore we can assume that the parabolic geometry on $M$ drops to a conformal geometry on a smooth connected complex projective 3-fold $M'$. The classification of smooth connected complex projective 3-folds admitting conformal geometries is due to Jahnke and Radloff \[27\]. □

19. Conclusion

We have demonstrated rigidity phenomena for a large class of holomorphic geometric structures and holomorphic exterior differential systems on smooth complex projective varieties. Our motivation is the following conjecture.

Conjecture 1. Suppose that $G$ is a complex simple Lie group and $P \subset G$ a maximal parabolic subgroup. Suppose that $G/P$ is not a compact Hermitian symmetric
space (or, if $G/P$ is a compact Hermitian symmetric space, then suppose that $G$ is a proper subgroup of the identity component of the biholomorphism group of $G/P$). Then up to isomorphism, the only holomorphic parabolic geometry modelled on $G/P$ on any compact Kähler manifold is the standard flat parabolic geometry on $G/P$.

More generally, one would like to construct explicitly all of the holomorphic Cartan geometries on all compact Kähler manifolds. The methods in this paper say nothing about the parabolic geometries modelled on compact Hermitian symmetric spaces, perhaps the most important type of parabolic geometry [25, 29].

It might be possible to classify the semicanonical modules of generalized flag varieties. This is a complicated combinatorial problem about root systems.

It is frustrating to have many results for smooth complex projective varieties but so few for compact Kähler manifolds. Any Cartan 2-plane field $\mathcal{V}$ on any 5-dimensional complex manifold $M$ has a holomorphic quartic symmetric form on the 2-plane as an invariant: [13]. That quartic form has a discriminant, which is a holomorphic section of a positive power of the canonical bundle. If the underlying 5-fold is compact Kähler, then the canonical bundle is not pseudoeffective, as explained above. Therefore no positive power of the canonical bundle has any nonzero sections. So the discriminant vanishes, i.e. the quartic has a multiple root at every point. The 2-plane field together with its brackets spans a 3-plane field. Similarly, Cartan defines a holomorphic quartic symmetric form on the 3-plane field, which restricts to the quartic on the two plane field. Again this quartic can’t have any nonvanishing invariants in classical invariant theory, since these all occur in positive powers of the canonical bundle. By geometric invariant theory, the projectivized zero locus of the quartic must therefore have a triple point or tacnode. One might be able to find similar information about other invariants and thereby prove vanishing of curvature to prove that $G_2/P_1$ is the only compact Kähler 5-fold bearing a holomorphic Cartan 2-plane field. It is already known that the holomorphic Cartan 2-plane field on $G_2/P_1$ discovered by Cartan is the only holomorphic Cartan 2-plane field on $G_2/P_1$ [4].

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