SO(2, d − 1) Gauge Theory of Gravity in d Dimensional Spacetime and AdS\(_d\)/CFT\(_{d-1}\) Correspondence.

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Abstract

Gravity in d dimensions is formulated as the gauge theory of local SO(2,d-1) gauge group. The Chern-Pontryagin index \(P_{2n}\) plays a crucial role in both gravity and gauge theories. \(P_{2n}\)(gravity) gives the gravitational Lagrangian in 2n dimensions, having the vacuum solution AdS\(_{2n}\). The same but global symmetry is shared with the gauge theories and 0,1-cochains of the Chern-Simon index \(C_{2n}(gauge)\) take part of CFT\(_{2n-1}\) and CFT\(_{2n-2}\), respectively. Gravity in odd dimensions is quite analogously formulated to that in even dimensions. This gives new insights on AdS/CFT correspondence.

1 Introduction

Gauge theory of gravity has been old and long standing theme since the seminal work of Utiyama [1]. In his work, spin connection \(\omega_{\mu ab}\) was introduced as the gauge field of local Lorentz transformation, whereas the tetrad was treated as external field. The tetrad was incorporated into “gauge field” of the Poincare group by Kibble [2]. However, this is restricted rather in formal analogy and left unclear the question why the tetrad transform covariantly under the gauge transformation since the gauge fields \(A_\mu\) transforms

\[
A_\mu' = U^{-1}(x)A_\mu U(x) + U^{-1}\partial_\mu U(x) .
\]  (1.1)

If we hope to incorporate the metric tensor or tetrad (vielbein, in general) in the usual gauge formulation, we must take the peculiar property of gravity into consideration, with which the usual gauge fields do not share certainly\(^2\). One of the most important points is the soldering of internal space of gauge symmetry of gravity with the external space.

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\(^2\)The metric was introduced from invariance under general coordinate transformations [3]
This has no analogy in the other Abelian and non-Abelian gauge fields. So the gauge theory of gravity must reflects this peculiarity. As we will show in this letter, SO(2,3) gauge group satisfies these properties. We proposed the gauge theory of gravity in the symmetry breking chain of the conformal (SO(2,4)) \(\rightarrow\) SO(2,3) [4].

The conformal transformation,
\[
dx'_a dx'_a = (\Omega(x))^{-2} dx_a dx_a \quad (a = 1, \ldots, d),
\]
(1.2)
in flat (1,d-1) dimensions corresponds to the "enlarged" SO(2,d) symmetry which maps
\[
\sum_1^{d+2} Z_A^2 = 0
\]
(1.3)
into itself, where two \(Z_A\) components are timelike. That is,
\[
Z'_A = \Lambda_A^B Z_B \quad \text{with} \quad \Lambda_A^C \Lambda_C^B = k(Z) \delta_A^B.
\]
(1.4)
There appears no scale parameter and SO(2,d) is the maximal spacetime symmetry of massless fields in d dimensional spacetime. We asserted that gravity is formulated as the gauge theory of local SO(2,d-1), leaving
\[
\sum_1^{d+1} Z_A^2 = -l^2.
\]
(1.5)
invariant [4] [5] [6]. Real \(l\) measures the scale breaking from SO(2,d) with two timelike \(Z_A\). Its vacuum solution AdS\(_d\) has SO(2,d-1) symmetry, which is also global symmetry of \(CFT_{d-1}\).

This fact reminds us the AdS/CFT correspondence [8][9].

However, it is a little bit curious to us who has studied the gauge theory of gravity since old and not so popular era. For gravity is also one of the gauge theories as mentioned above and there arises an expectation that the gauge theoretical view point of gravity may shed new light on AdS/CFT and vice versa\(^3\).

This paper is organized as follows. In section 2, we review gravity of SO(2,3) gauge group [4] corresponding to \(d = 4\). Arguments are extended to general \(d\) in section 3. \(d = 2\) has peculiar property and separately discussed in section 4. In these constructions, the topological objects of the Chern-Pontryagin class and Chern-Simon class concerned with the symmetry SO(2,d) play crucial roles in both gravity and gauge theories. Section 5 is devoted to the discussions.

## 2 SO(2,3) Gauge Theory of Gravity

Before discussing SO(2,3) invariance we will very briefly review the relation between the conformal transformation (1.2) and SO(2,d) [10]. We will describe conformally flat (1,d-1) coordinates as \(x^a\). The infinitesimal conformal transfomation
\[
x'^a = x^a + v^a(x)
\]
(2.1)
\(^3\)We do not discuss on the supersymmetry in this letter.
must satisfy
\[ \partial_a v_b + \partial_b v_a = \frac{2}{d} \delta_{ab}. \]  \hspace{1cm} (2.2)

The general solution to this equation is
\[ v_a = p_a + \omega_{ab} x^a + \lambda x_a + f_a x^2 - 2x_a f_b x^b. \]  \hspace{1cm} (2.3)

This symmetry is described by the SO(2,d) invariant transformation acting on six components projective coordinates \( Z_A \)
\[ Z_A = (Z_a, Z_-, Z_+), \quad (a = 1, \ldots, d) \]  \hspace{1cm} (2.4)

with the constraints
\[ Z_A^2 = Z_a Z_b \eta^{ab} - 2Z^+ Z^- = 0, \]  \hspace{1cm} (2.5)

where \( Z_\pm = Z_{d+1} \pm Z_{d+2} \) for timelike \( Z_{d+1} \). Indeed if we define
\[ x^a = \frac{Z_a}{Z^+}, \]  \hspace{1cm} (2.6)

SO(2,d) generators \( G^{AB} = Z^{[A} p^{B]} \) produce (2.3). Gauge group of gravitation represents the symmetry of the spacetime and naively we might consider the conformal symmetry as the gauge group. We spontaneously break the conformal invariance by
\[ (Z_{d+1}, Z_{d+2}) = (l, l) \text{ or } (-l, -l). \]  \hspace{1cm} (2.7)

However, it needs a dimensional parameter \( l \), which breaks conformal symmetry explicitly. Thus we considered the gauge theory of gravity whose gauge group is reduced from SO(2,d) to SO(2,d-1) \footnote{The implication of conformal invariance in gravity is further discussed in section 5.}. In this section, hereafter we consider \( d = 4 \) and
\[ Z_1^2 + \ldots + Z_4^2 + Z_5^2 = -l^2. \]  \hspace{1cm} (2.8)

From gauge theoretical view point we may put SO(1,4) in place of SO(2,3) with \( l^2 \) in place of \(-l^2\). However, we consider SO(2,3) taking supergravity into consideration. (The extra dimension must be compactified and in total ten dimensions, cosmological constant must be zero.) Corresponding to SO(2,3), the covariant derivative is defined by
\[ D_\mu = \partial_\mu - i \omega_{\mu AB} S_{AB}/2 \quad (A, B = 1, \ldots, 4, 5) \]  \hspace{1cm} (2.9)

Here \( \omega_{\mu AB} \) are \( 4 \times 10 \) connection fields and \( S_{AB} \) are the generators of (anti) de Sitter group. The field strength is derived from the commutation relation
\[ i [D_\mu, D_\nu] = -R_{\mu\nu AB} S_{AB}/2. \]  \hspace{1cm} (2.10)

\[ R_{\mu\nu AB} = \partial_\mu \omega_{\nu AB} - \partial_\nu \omega_{\mu AB} - \omega_{\mu AC} \omega_{\nu CB} + \omega_{\nu AC} \omega_{\mu CB} \]  \hspace{1cm} (2.11)
The Einstein’s action is written as

\[ I = \int d^4x \epsilon^{ABCD} \epsilon^{\mu\nu\rho\sigma} (Z_A/l) \left[ R_{\mu\nu BC} R_{\rho\sigma DE} / (16g^2) + D_\mu Z_B D_\nu Z_C D_\rho Z_D D_\sigma Z_E \sigma(x) \{(Z_A^2/l^2) - 1\}^2 \right]. \]

(2.12)

Here \( \epsilon^{\mu\nu\lambda\sigma} \) and \( \epsilon^{ABCD} \) are fully antisymmetric tensors with \( \epsilon^{1234} = 1 \) and \( \epsilon^{12345} = 1 \), respectively (4, 5 are timelike components). It should be remarked that this action is a geometrical invariant and that we do not introduce metric ad hoc. Hamilton formulation of action (2.12) was given by [7]. After the gauge choice

\[ Z^A = (0, 0, 0, 0, l) \quad (2.13) \]

\[ D_\mu Z_A = \left( \partial_\mu \delta_{AB} - \omega_{\mu AB} \right) Z_B = \{ \omega_{\mu a} \equiv e_{\mu a} \text{ if } A = a \}, \quad \text{if } A = 5 \]. \quad (2.14)

It is important that \( e_{\mu a} \) transforms covariantly under the remaining 4-dim Lorentz rotation. Generalized Riemannian tensor \( R_{\mu\nu ab} \) is divided into two terms

\[ R_{\mu\nu ab} = \hat{R}_{\mu\nu ab} - e_{[\mu a} e_{\nu]b}/l^2. \]

(2.15)

Here \( \hat{R}_{\mu\nu ab} \) is the conventional Riemannian tensor defined by

\[ \hat{R}_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \omega_{[\mu ac} \omega_{\nu]cb} \]

(2.16)

and \( e_{[\mu a} e_{\nu]b} \equiv e_{\mu a} e_{\nu b} - e_{\nu a} e_{\mu b} \).

\( L_{grav} \) takes the form

\[ L_{grav} = \mathcal{P}_4(\text{gravity}) = \epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu ab} R_{\rho\sigma cd} / (16g^2) \]

\[ = \partial_\mu C^\mu_4 - e \left( \hat{R} - \frac{6}{l^2} \right) / (16\pi G), \]

(2.17)

where

\[ 16\pi G \equiv g^2 l^2 \]

\[ e = \det e_{\mu a}, \quad \hat{R}_\mu = e^{\nu b} \hat{R}_{\mu\nu ab}, \quad \hat{R} = e^{\mu a} \hat{R}_{\mu a}, \]

(2.18)

and use has been made of

\[ \epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\mu a} e_{\nu b} e_{\rho c} e_{\sigma d} = 4! e \]

\[ \epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\mu a} e_{\nu b} = 2 e e^{[\rho c} e^{\sigma]d} \text{ etc.} \]

(2.20)

Here \( e^{\mu a} e_{\mu b} = \delta_{ab} \), \( e^{\mu a} e_{\nu a} = \delta^\mu_\nu \). The quadratic term in \( \hat{R}_{\mu\nu ab} \) is total derivative \( \partial_\mu C^\mu_4 \) (the Gauss-Bonnet term). This gauge theoretical construction of gravity may shed new light on the AdS/CFT correspondence [8] [9] which states the correspondence between \( AdS_d \) gravity and (d-1)dimensional conformal field theory. Indeed, the Gauss-Bonnet term in (2.17) does not affect the equation motion but does the boundary like event horizon of Black Hole (BH). So this may change the scenario of near-horizon extreme BH like Reissner-Nordstrom and Kerr etc. [11]. This is indeed the case and the special combinations of (2.17) gives the conserved mass and angular momentum for Kerr-AdS BH [12].
As we will show in the next section, we have higher derivative terms other than the linear Einstein and cosmological terms. These also modify BH solution.

Three dimensional action of Yang-Mills gauge field $F_{\mu \nu}^a \equiv F_{\mu \nu}$,

$$S = \int d^3 x \frac{-1}{2g^2} \text{Tr} F_{\mu \nu} F^{\mu \nu}. \quad (2.21)$$

has dimensional coupling (mass dimension of $g$ is 1/2) and not SO(2,3) invariant. In the above arguments, $AdS_4$ has been derived with 4-dimensional Chern-Pontryagin index $P_4(\text{gravity})$. Unlike the SO(2,3) invariant gravity, the corresponding counterpart in gauge theory $P_4(\text{gauge})$ is total derivative, being related with 3 dimensional Chern-Simons term as follows.

$$L_{\text{gauge}} = P_4(\text{gauge}) = \frac{1}{8\pi^2} \text{Tr} \epsilon^{\mu \nu \rho \sigma} (A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma). \quad (2.22)$$

where

$$C_4^{(0)}(A) = \frac{1}{8\pi^2} \text{Tr} \epsilon^{\mu \nu \rho \sigma} (A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma). \quad (2.23)$$

$C_4^{(0)}(A)$ may be called a 0-cochain [13][14] and

$$C_4^{(0)}(A^U) = C_4^{(0)}(A) + \frac{1}{8\pi^2} \text{Tr} \epsilon^{\mu \nu \rho \sigma} \partial (a_\rho A_\sigma) + \frac{1}{24\pi^2} \text{Tr} \epsilon^{\mu \nu \rho \sigma} a_\mu a_\nu a_\sigma$$

$$\equiv C_4^{(0)}(A) + \Delta C_4^{(0)}, \quad (2.24)$$

where $a_\nu = \partial_\nu U U^{-1}$. $\Delta C^{(0)}$ is also total derivative ,

$$\Delta C_4^{(0)} = \partial_\mu C_4^{(1)\mu}(A, U) \quad (2.25)$$

and we obtain one-cochain

$$C_4^{(1)}(A, U) = \frac{1}{8\pi^2} \text{Tr} \epsilon^{\alpha \beta \gamma} a_\alpha A_\beta + \frac{d^{-1}}{24\pi^2} \text{Tr} \epsilon^{\alpha \beta \gamma} a_\alpha a_\beta a_\gamma. \quad (2.26)$$

Here $d^{-1}$ is simbolic of integral and the explicit form is given for the specific case of SU(2) gauge in [13]. Thus (2.24) is gauge invariant up to topological winding number. The dimensional descent is continued further:

$$\Delta C_4^{(1)} = C_4^{(1)}(A^U_1; U_2) - C_4^{(1)}(A; U_1) + C_4^{(1)}(A^U_1; U_2)$$

$$= \partial_\mu C_4^{(2)\mu}(A; U_1, U_2) \quad (2.27)$$

The explicit form of 2-cochain $C_4^{(2)\mu}(A; U_1, U_2)$ is given in [15].

Thus $P_4(\text{grav})$ and $C_4^{(0)}(\text{gauge})$ are correspondents of $AdS_4/CFT_3$. They are related with SO(2,3): In the former, it is local gauge group of gravity, and in the latter it is global
symmetry of gauge theory. \( \mathcal{C}_4^{(1)}(A, U) \) is related with \( CFT_2 \) as will be shown in the next section.

In the following sections, this is generalized to \( \mathcal{P}_{2n}(\text{grav}) \) and \( \mathcal{C}_{2n}(\text{gauge}) \) for \( d = 2n \). \( d = 2n + 1 \) cases are also discussed. Though gravitational part for \( d = 2n + 1 \) is not described as such topological object unlike that of \( d = 2n \), procedures are quite analogous to \( d = 2n \) cases.

### 3 SO(2,d-1) Gravity for \( d \neq 2 \)

We have started with \( d = 4 \) dimensional spacetime with SO(2,3) gauge group. This formulation is easily extended to \( d = 5 \), five dimensional spacetime. That is

\[
I = -\int d^5x \epsilon^{ABCDEF} \epsilon^{\mu\nu\rho\sigma\lambda} (Z_A/l) D_\mu Z_B \left[ R_{\nu\rho\sigma\lambda} / (48 g^2 l) \right] + D_\nu Z_C D_\rho Z_D D_\sigma Z_E D_\lambda Z_F \sigma(x) \sum_{A=1}^{6} \left( \frac{Z_A^2}{l^2} - 1 \right)^2.
\]

(3.1)

with

\[ Z_A = (0, 0, 0, 0, 0, l). \]

(3.2)

In this case

\[ D_\mu Z_A = (\partial_\mu \delta_{AB} - \omega_{\mu AB}) Z_B = \{ \begin{cases} \omega_{\mu a} l = e_{\mu a} & \text{if } A = a \\ 0 & \text{if } A = 6 \end{cases}, \]

(3.3)

Here \( \mu \) and \( a \) run over 1,...,5 in world and local Lorentz coordinates, respectively. Consequently (3.1) is reduced to

\[ L_{\text{grav}} = \epsilon^{abcdef} \epsilon^{\mu\nu\rho\sigma\lambda} e_{\mu a} R_{\nu\rho bc} R_{\sigma\lambda de} / (48 g^2 l) \]

\[ = \epsilon^{abcdef} \epsilon^{\mu\nu\rho\sigma\lambda} e_{\mu a} \hat{R}_{\nu\rho bc} \hat{R}_{\sigma\lambda de} / (48 g^2 l) - e \left( \hat{R} - \frac{10}{l^4} \right) / (16 \pi G_5) \]

(3.4)

with \( 16 \pi G_5 = g^{2l^3} \). Thus we obtain \( AdS_5 \) as the vacuum solution. In this case, however, higher derivative terms (the first term of (3.4)) are not total derivatives and change the equation of motion in high energy region and do therefore Black Hole solution and its near horizon property.

Corresponding to SO(2,4) in gauge theory is

\[ S_{\text{gauge}} = -\frac{1}{2g^2} \int d^4x \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + \int d^4x \mathcal{P}_4(\text{gauge}), \]

(3.5)

where \( \mathcal{P}_4(\text{gauge}) \) is defined in (2.22). (3.4) and (3.5) constitute the correspondents in \( AdS_5/CFT_4 \) at least from the invariance property.

In six dimensional spacetime, gauge group of gravity is SO(2,5), and gravity action is

\[ L_{\text{gravity}} = -\epsilon^{ABCDEF} \epsilon^{\alpha\beta\mu\nu\rho} \left[ (Z_A/l) R_{\alpha\beta BC} R_{\mu\nu DE} R_{\rho\sigma FG} \right]. \]

(3.6)

Further processes follow analogously to the preceding arguments. By the gauge choice \( Z_A = (0, 0, 0, 0, 0, l) \), (3.6) becomes

\[ L_{\text{grav}} = \partial_\mu C_6^\mu (\hat{R}) + \text{quadratic of } \hat{R}, -e \hat{R} + \text{cosmological const}. \]

(3.7)
Thus we obtain the Einstein equation with negative cosmological constant in low energy. However, it includes terms quadratic in Riemannian tensor and Pontrjagin $C_6(R)$ term. The corresponding counterpart of gauge action is

$$L_{gauge} = \frac{1}{384\pi^3}e^{\alpha\beta\mu\nu\rho\sigma}\text{Tr}F_{\alpha\beta}F_{\mu\nu}F_{\rho\sigma}$$

with

$$C_6^\sigma = \frac{1}{192\pi^3}e^{\alpha\beta\mu\nu\rho\sigma}\text{Tr}\left(F_{\alpha\beta}F_{\mu\nu}A_\rho - F_{\alpha\beta}A_\mu A_\nu A_\rho + \frac{2}{5}A_\alpha A_\beta A_\mu A_\nu A_\rho\right).$$

and

$$C_6^{(0)} = \frac{1}{192\pi^3}e^{\alpha\beta\mu\nu\rho\sigma}\text{Tr}\left(F_{\alpha\beta}F_{\mu\nu}A_\rho - F_{\alpha\beta}A_\mu A_\nu A_\rho + \frac{2}{5}A_\alpha A_\beta A_\mu A_\nu A_\rho\right).$$

Thus $P_6^{(gravity)}$ and $C_6^{(0)}(gauge)$ are related with SO(2,5). We may add kinetic terms using scalar $\phi$ of mass dimension 1 and $CFT_5$ is,

$$\phi(\partial\phi)^2 + \phi F_{\mu\nu}^2 + C_6^{(0)}(gauge).$$

$C_6^{(1)}(gauge)$ defined analogously to (2.28) may be added into (3.5).

For $d = 7$, $AdS_7$ is straightforward and omit to describe it. The counterpart of gauge theories are guided by renormalizability [16]

$$\Phi F_{\mu\nu}^2 + B \wedge F \wedge F + (\partial\Phi)^2 + (dB)^2 + ...,\quad (3.13)$$

where a scalar $\Phi$ and a two form $B_{\mu\nu}$ both of dimension 2. Unlike for $d \leq 5$, we have no idea to derive (3.13) from the dimensional descent of higher $C_d$ since gauge field $B_{\mu\nu}$ has 2 dimensional world volume peculiar to superstring or supergravity.

The same procedures are performed for $d = 3$ dimensional case, SO(2,2) gravity.

$$I = \int d^3x \epsilon^{ABCD} \epsilon^{\mu\nu\rho}(Z_A/l)D_\mu Z_B \left[R_{\nu\rho\sigma}/(2g^2l) + D_\nu Z_C D_\rho Z_D \sigma(x) \sum_A \{(Z_A/l^2) - 1\}^2\right].$$

(3.14)

After the gauge fix $Z_A = (0, 0, 0, l)$, (3.14) is reduced to

$$S_{grav} = -\int d^3x \epsilon^{abc} \epsilon^{\mu\rho\sigma} \epsilon_{\mu\alpha} R_{\nu\rho\sigma}/(2g^2l)$$

$$= -\int d^3x \epsilon \left(\hat{R} - \frac{6}{l^2}\right)/(16\pi G_3),$$

(3.15)

where $g^2l = 16\pi G_3$. The SO(2,2) invariant gauge theory is naively

$$I_{gauge} = -\frac{1}{2g^2} \int d^2x \epsilon^{\mu\nu} F_{\mu\nu} = \int d^2x P_2(gauge)$$

(3.16)

but it is topological invariant. The corresponding gauge counterpart is the WZW model [17]

$$I_{WZW} = \frac{1}{4\pi} \int d^2x T \text{Tr}(a_\nu a_{\overline{\nu}}) + C_4^{(1)}(A, U).$$

(3.17)

Here $a_\nu$ is defined at (2.19) with $z = x - it$. $C_4^{(1)}(A, U)$ is given by (2.27).

We have extended our formulation to $d = 3, 5, 6, 7$. $d = 2$ case is discussed in the next section.
4 \ SO(2,1) \ Gravity

Two dimensional gauge theory of gravity is special in the sense that its gauge group \( \text{SO}(2,1) \) has an infinite set of generators,

\[
L_m = \frac{T}{2} \int_0^\pi e^{-2im\sigma} T_- d\sigma \quad (m : \text{integer}) \\
\bar{L}_m = \frac{T}{2} \int_0^\pi e^{2im\sigma} T_+ d\sigma. 
\]

These generators satisfy the Virasoro algebra

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. 
\]

Usually this corresponds to the fact that

\[
S = -\int d^2x \sqrt{-g} (R - \Lambda) 
\]

is total derivative. The following non trivial action was proposed by Jackiw and Teitelboim [18].

\[
S = -\int d^2x \sqrt{-g} (R - \Lambda)N 
\]

with auxiliary field \( N \). This action was formulated as the \( \text{SO}(2,1) \) gauge theory of gravity by us [19]. Naively we might consider

\[
S = -\frac{1}{2} \int d^2x \epsilon^{ABC} \epsilon^{\mu
u} R_{\mu\nu AB} Z_C / l. 
\]

Unfortunately, this leads us to (4.3).

However, two dimensionality has the peculiar property that the scalar fields (we denote them as \( \phi_A \)) have the canonical dimensionality 0, which makes us possible to construct

\[
S = -\frac{1}{2} \int d^2x \epsilon^{ABC} \epsilon^{\mu\nu} R_{\mu\nu AB} \phi_C. 
\]

The equations of motion derived from (4.6) are

\[
R_{AB} = d\omega_{AB} - \omega^2_{AB} = 0, \\
D\phi_A = d\phi_A - \omega_{AB}\phi_B = 0, 
\]

where \( R_{AB} = \frac{1}{2} R_{\mu\nu AB} dx^{\mu} \wedge dx^{\nu}, \omega_{AB} = \omega_{\mu AB} dx^{\mu} \). By decomposing (4.7) into (0,1) and (a,2) components, we obtain

\[
0 = de_a - \omega_{ab}e_b \quad (a, b = 0, 1), \\
0 = d\omega_{01} - e_0 e_1 / l^2. 
\]

In the same way, (4.8) gives

\[
0 = d\phi_a - \omega_{ab}\phi_b + e_a\phi_2 / l, \\
0 = d\phi_2 + e_a\phi_a / l. 
\]
(4.12) is used to describe $\phi_a$ in terms of $\phi_2(\equiv N)$ and (4.11) becomes the equation of motion for $N$,

$$
\phi_a = -le^a_\mu \partial_\mu N, \quad (4.13)
$$

$$
0 = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)N + g_{\mu\nu} N/l^2. \quad (4.14)
$$

(4.10) and (4.14) are exactly the same ones derived from (4.4). The canonical form of (4.6) was also given by [19] and two generators satisfy the conformal algebra without central charge,

$$
\{ \mathcal{H}_1^f, \mathcal{H}_1^g \} = \{ \mathcal{H}_1^f, \mathcal{H}_1^g \} = \mathcal{H}_1^h, \quad \{ \mathcal{H}_1^f, \mathcal{H}_1^g \} = \mathcal{H}_1^h, \quad (4.15)
$$

where

$$
h = f \partial_1 g - g \partial_1 f,
$$

$$
\mathcal{H}_1^f = \int dx^1 f \mathcal{H}_1, \text{ etc.} \quad (4.16)
$$

The explicit forms of $\mathcal{H}_1, \mathcal{H}_1$ are given in [19] and represent two dimensional diffeomorphism generators after gauge fixing

$$
e_{\mu a} = e^\chi \delta_{\mu a}. \quad (4.17)
$$

The gauge part of this SO(2,1) is

$$
\mathcal{P}_2(\text{gauge}) = -\frac{1}{2\pi} \text{Tr} e^{\mu \nu} F_{\mu \nu}
$$

$$
= \partial_\mu C_2^\mu \quad (4.18)
$$

with $C_2^\mu = -\frac{1}{2\pi} \text{Tr} e^{\mu \nu} A_\nu$ and

$$
C_2^{(0)}(A) = \frac{1}{2\pi} \text{Tr} A. \quad (4.19)
$$

5 Discussions

We have argued that gravity in d dimensional spacetime is formulated as the gauge theory of SO(2,d-1). The same but global symmetry is shared with conformal field theory in d-1 dimensional flat spacetime. AdS/CFT correspondence has been discussed in the framework of nonsusy local field theory. In both gravity and CFT, the Chern-Pontryagin index $\mathcal{P}_2$, especially $\mathcal{P}_4$, play a crucial role. In $\mathcal{P}_4(\text{gravity})$, Einstein gravity, linear term in the Riemannian tensor, survives by virtue of scale violation. Whereas, the Lagrangian of gauge part appears as the dimensional descents 0, 1—cochains of $\mathcal{C}_4(\text{gauge}) +$ kinetic terms in three, two dimensions, respectively.

Gravitational Lagrangian has the surface term and higher derivative terms for $d \geq 4$, which may change the boundary condition of BH solution and affect Reisner-Nordstrom and Kerr/CFT correspondence [11]. More concretely speaking, Strominger et.al. identified
near event-horizon extreme Kerr with \( CFT_2 \) by the central charge \([11]\). If the surface term appeared in our theory modifies the central charge, this correspondence may be affected. These are the soliton solution of gravity.

Gauge theory of gravitation allows the other kind of soliton solution. Let us consider the case of \( d = 5 \) case. we adopted the gauge (3.2). However we may set the kink solution

\[
Z_A = (0, 0, 0, 0, 0, l) \text{ at } x^5 \subset (0, \infty) \\
Z_A = (0, 0, 0, 0, 0, -l) \text{ at } x^5 \subset (-\infty, 0)
\] (5.1)

In this case the solution has the kink of step function at \( x^5 = 0 \). This may be interpreted 3 dimensional brane. Of course this is too simplified and we will discuss the detail of this process in separate form.

Lastly we comment on the conformal invariance of the gravity. For gravity, conformal transformation takes the form

\[
v_{\mu;\nu} + v_{\nu;\mu} = \frac{2}{d} g_{\mu\nu}
\] (5.2)

in place of (2.2). Here \( v_{\mu;\nu} \) implies the covariant derivative. The invariance under (5.2) is recovered by the conformal (Weyl) tensor

\[
C^{\mu\rho\sigma} = \hat{R}^{\mu\rho\sigma} - \frac{1}{d-2} \left(g^{\mu\rho} S^{\sigma\mu} - g^{\mu\sigma} S^{\rho\nu} - g^{\nu\rho} S^{\sigma\mu} + g^{\nu\sigma} S^{\rho\mu}\right)
\] (5.3)

with

\[
S^{\mu\nu} = \hat{R}^{\mu\nu} - \frac{1}{2(d-1)} g^{\mu\nu} \hat{R}.
\] (5.4)

This tensor vanishes iff the spacetime is conformally flat. In \( d = 3 \), this tensor vanishes identically but not all three dimensional spacetime is conformally flat. So this tensor does not characterize conformal flatness in \( d = 3 \). The ”Weyl” tensor in \( d = 3 \) is given by \( C_4^{(0)} (A) \) of (2.23) with replacement of \( A \) by the Christoffel symbol,

\[
S_{CS(gravity)} = \int d^3 x C_4^{(0)} (\Gamma) = -\frac{1}{8\pi^2} \int d^3 x Tr e^{\nu\rho\sigma} \left( \Gamma^{\alpha}_\nu \partial_\rho \Gamma^{\beta}_\sigma + \frac{2}{3} \Gamma^{\alpha}_\nu \Gamma^{\beta}_\rho \Gamma^{\lambda}_\sigma \Gamma^{\lambda}_{\sigma\nu} \right)
\] (5.5)

The new Weyl tensor \( C^{\mu\nu} \) is given by the variation [20]

\[
\delta S_{CS} = \frac{1}{4\pi^2} \int d^3 x \sqrt{g} C^{\mu\nu} \delta g_{\mu\nu}
\] (5.6)

and

\[
C^{\mu\nu} = \frac{1}{2\sqrt{g}} \left( e^{\mu\rho\sigma} R^{\nu}_{\sigma\rho} + e^{\nu\rho\sigma} R^{\mu}_{\sigma\rho} \right).
\] (5.7)

Thus the conformally flat transformation in gauge theory and conformal transformation in gravitation had some formal correspondence in the same dimension. However, in the real world, the gravitation breaks the scale invariance and new kind of correspondence, \( AdS_d/CFT_{d-1} \), appears.
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