A remark on inverse scattering for time dependent Hartree equations

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Abstract. We consider an inverse scattering problem of identifying a common potential and an interaction potential acting on particles from the scattering operator defined by the solution to the time dependent Hartree equation. In this paper we give a reconstruction procedure for them in the case of two-particle systems.

1. Introduction

Let $V(x)$ be a common potential acting on particles and $v_{jk}(x)$, $1 \leq j < k \leq N$ be interactions between $j$-th and $k$-th particles. Then the time dependent Hartree equation, which is derived from an $N$-body Schrödinger equation, is given in the form:

$$i \partial_t u_j = -\Delta u_j + V(x)u_j + \sum_{k \neq j}^N (v_{jk} * |u_k|^2)u_j, \quad j = 1, \cdots, N,$$

(1)

where $u_j = u_j(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ are unknown functions, $i = \sqrt{-1}$, $\Delta$ is the $n$-dimensional Laplacian and $*$ denotes the convolution. In this paper we study an inverse scattering problem of identifying a common potential $V(x)$ and interactions $v_{jk}(x)$, $1 \leq j < k \leq N$.

In [9], inverse scattering problem for a simple case of the Hartree equation (1) given by

$$i \partial_t u = -\Delta u + V(x)u + \lambda(|\cdot|^{-\sigma} * |u|^2)u$$

(2)

was studied. Moreover, it was given that a reconstruction procedure to identify $V(x)$ and $\lambda$ from the corresponding scattering operator. In case $V(x) \equiv 0$, reconstruction formulae for $\sigma$ and $\lambda$ were given in [10]. To derive the reconstruction formula for $\sigma$, it was essentially important that the identity

$$T[\phi_R] = R^{n-2n-2} T[\phi], \quad \text{for any } R \in \mathbb{R}$$

(3)

holds, where $T[\phi] = \lim_{\epsilon \to 0} \frac{1}{\pi} \langle [S - I](\epsilon \phi), \phi \rangle$, $S$ is the scattering operator for (2) in the case of $V(x) \equiv 0$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^n)$ (see [10, p. 1481]). In case $V(x) \not\equiv 0$, however, it is not expected that we have the simple identity like (3). So, the aim of this paper is to give a new method for the reconstruction of $\sigma$.

Throughout this paper, we denote the norm of $L^p = L^p(\mathbb{R}^n)$ and the usual Sobolev space $H^{k,p} = H^{k,p}(\mathbb{R}^n)$ of order $k$ in $L^p$ by $\| \cdot \|_p$ and $\| \cdot \|_{k,p}$, respectively; in particular, we abbreviate
Suppose that Assumption I and Assumption II are satisfied. Then there exists Theorem 2.1. proved by Yajima [19] (for the other results in the multidimensional case holds for all $\phi = (\phi_1, \cdots, \phi_N)$.

2. Problem and result
To state our problem and result, we first construct the scattering operator for (1). Let $H_0 = -\Delta$, $H = H_0 + V(x)$ and $D_{\alpha} = D_{\alpha_1} \cdots D_{\alpha_n}$ with $D_j = -i\partial/\partial x_j$.

**Assumption I**
We assume that the real valued function $V(x)$ satisfies,

(i) $H$ has no eigenvalues and no zero resonance.

(ii) For $\delta > (3n)/2 + 1$, $p_0 > n/2$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq 1 + l_0$,

\[
\sup_{x \in \mathbb{R}^n} (x')^\delta \left( \int_{|x-y| \leq 1} |D_{\alpha}^n V(y)|^{p_0} \, dy \right)^{1/p_0} < \infty,
\]

where $l_0 = 0$ if $n = 3$ and $l_0 = [(n - 1)/2]$ if $n \geq 4$.

**Assumption II**
We assume that the real valued function $v_{jk}(x)$ satisfies,

\[
|v_{jk}(x)| \leq \lambda |x|^{-\sigma}, \quad 2 \leq \sigma \leq 4 \text{ and } \sigma < n,
\]

where $\lambda$ is a positive constant.

We remark that under Assumption I, wave operators $W_{\pm} : = s-lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$ exist and are complete, where the limit is taken in the strong topology of $L^2$. Moreover they are bounded in $H^{1,p}$ for $1 \leq p \leq \infty$. Furthermore an $H^{1,p} - H^{1,p'}$ estimate on $e^{-itH}$:

\[
\|e^{-itH}f\|_{1,p} \leq c |t|^{-d} \|f\|_{1,p'}
\]

holds for all $t \neq 0$, $2 \leq p \leq \infty$ and $f \in L^2 \cap H^{1,p'}$, where $d = n(1/p' - 1/2)$. These results were proved by Yajima [19] (for the other results in the multidimensional case $n \geq 3$, see Yajima [17, 18]). By these results, we easily obtain the following theorem. The proof was given in [7] and [9]. We also refer to papers [1], [4], [5] and [8] on the Cauchy problem for the Hartree equations.

In what follows, we denote by $u(t)$ the function $u(t, \cdot)$.

**Theorem 2.1.** Suppose that Assumption I and Assumption II are satisfied. Then there exists $\rho > 0$ such that for any $\phi^{(-)}_j \in H^{1,2}$ with $\|\phi^{(-)}_j\|_{1,2} < \rho$, $j = 1, 2, \cdots N$, (1) admits a unique solution

\[
u_j \in L^\infty(\mathbb{R}; H^{1,2}) \cap L^3(\mathbb{R}; H^{1,q}), \quad \frac{1}{q} = \frac{1}{2} - \frac{2}{3n}, \quad j = 1, 2, \cdots, N
\]

satisfying

\[
\lim_{t \to -\infty} \|u_j(t) - e^{-itH_0} \phi^{(-)}_j\|_{1,2} = 0, \quad j = 1, 2, \cdots, N.
\]

Moreover there is a unique $\phi^{(+)}_j \in H^{1,2}$ such that

\[
\lim_{t \to -\infty} \|u_j(t) - e^{-itH_0} \phi^{(+)}_j\|_{1,2} = 0, \quad j = 1, 2, \cdots, N.
\]
The map $S: \phi^(-) \rightarrow \phi^(+)$ which relates asymptotic states at $t = -\infty$ and $t = +\infty$ is called the scattering operator and is defined by

$$ (S_j \phi)(x) = (W_+^* P_j W_- \phi)(x), \quad j = 1, 2, \cdots N, $$

(6)

where

$$ (P_j \phi)(x) = \phi_j(x) + \frac{1}{i} \int_{\mathbb{R}} e^{itH} F_j(u(t)) \, dt, $$

(7)

$$ F_j(u(t)) = \sum_{k \neq j} (v_{jk} * |u_k|^2)u_j, $$

and $u(t)$ is the solution to (1) constructed in theorem 2.1.

We now state our problem and result. Our inverse problem is: for given functions $\{(S_j \phi)(x), \phi_j(x)\}, j = 1, 2, \cdots N$, find $V(x)$ and $v_{jk}(x)$, $1 \leq j < k \leq N$.

Consider two-particle systems $(j = 1, 2)$ with the interaction of the form $v_{12}(x) = v(x) = \lambda |x|^{-\sigma}$, where $\lambda$ and $\sigma$ are constants such that

$$ \lambda \neq 0, \quad 2 \leq \sigma \leq 4 \quad \text{and} \quad \sigma < n. $$

(8)

Obviously, $n \geq 3$.

In this paper we prove the following:

**Theorem 2.2.** Suppose that Assumption I and Assumption II are satisfied. Then there is a reconstruction procedure to identify $\lambda$, $\sigma$ and $V(x)$ for any $x \in \mathbb{R}^n$ from functions $\{(S_j \phi)(x), \phi_j(x)\}, j = 1, 2$.

3. Proof of Theorem 2.2

In this section we give a reconstruction procedure for $V(x)$, $\sigma$ and $\lambda$, respectively.

**Reconstruction of $V(x)$**

Since wave operators $W_\pm$ exist and are complete under Assumption I, the scattering operator for linear Schrödinger operator $H = H_0 + V(x)$ is defined by $S_V := W_+^* W_-$. As was proved in [9], the identity

$$ \frac{d}{d\varepsilon} S_j(\varepsilon \phi) \bigg|_{\varepsilon = 0} = S_V \phi_j $$

holds, where the derivative exists in the strong topology of $H^{1,2}$. Hence, by the well known result that $S_V$ uniquely determine $V(x)$ (see, e.g., [2] or [3]), we can reconstruct $V(x)$ from $S_j \phi$.

**Reconstruction of $\sigma$**

Since $V(x)$ is already known, we can compute wave operators $W_\pm$. It follows from the definition of the scattering operator (6) that we find $P_j = W_+ S W_-^*$. As in [7, Lemma 3 p. 299], small amplitude limit $L_j[\phi] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \langle [P_j - I] (\varepsilon \phi), \phi_j \rangle$ exists for any $\phi_j \in H^{1,2}$, $j = 1, 2$ and is given by

$$ L_j[\phi] = \iint_{\mathbb{R} \times \mathbb{R}^n} e^{-itH} \phi_j F_j(e^{-itH} \phi) \, dx \, dt. $$
We note that in our case, $L_1$ and $L_2$ are given by

\[
L_1[\phi] = \lambda \int_{\mathbb{R} \times \mathbb{R}^n} (| \cdot | \sigma \ast |e^{-itH_0} \phi_2|^2)(x) |e^{-itH} \phi_1(x)|^2 \, dx \, dt,
\]

and

\[
L_2[\phi] = \lambda \int_{\mathbb{R} \times \mathbb{R}^n} (| \cdot | \sigma \ast |e^{-itH_0} \phi_1|^2)(x) |e^{-itH} \phi_2(x)|^2 \, dx \, dt,
\]

respectively. Thus by Fubini’s theorem, $L_1[\phi] = L_2[\phi]$. Define $\phi_R(x) = \phi(Rx)$. Then we have

\[
(e^{-itH} \phi_R)(x) = (e^{-iR^2tH_0} \phi)(Rx),
\]

where $H_R = H_0 + R^{-2}V(x/R)$. See [15, pp. 1232-1233] for the detail. Using this identity and (9), we find by a simple computation that

\[
R^{2+2n-\sigma} L_1[\phi_R] = \lambda \int_{\mathbb{R} \times \mathbb{R}^n} (| \cdot | \sigma \ast |e^{-itH_0} \phi_2|^2)(x) |e^{-itH_0} \phi_1(x)|^2 \, dx \, dt.
\]

Next we show that the right-hand side of (10) converges to

\[
\lambda \int_{\mathbb{R} \times \mathbb{R}^n} (| \cdot | \sigma \ast |e^{-itH_0} \phi_2|^2)(x) |e^{-itH_0} \phi_1(x)|^2 \, dx \, dt
\]

as $R \to \infty$. To prove this, write $v_R(t) = e^{-itH_R} \phi_1(x)$, $w_R(t) = e^{-itH_R} \phi_2(x)$, $v_0(t) = e^{-itH_0} \phi_1(x)$, and $w_0(t) = e^{-itH_0} \phi_2(x)$. Then

\[
\left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast |w_R(t)|^2)|v_R(t)|^2 \, dx \, dt - \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast |w_0(t)|^2)|v_0(t)|^2 \, dx \, dt \right|
\]

\[
\leq \left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast \{w_R(t) - w_0(t)\}|\overline{w}_R(t))|v_R(t)|^2 \, dx \, dt \right|
\]

\[
+ \left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast w_0(t)|\overline{w}_R(t)) \{v_R(t) - v_0(t)\}|\overline{v}_R(t)\} \, dx \, dt \right|
\]

\[
+ \left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast w_0(t)|\overline{w}_R(t) - \overline{w}_0(t)) \{v_0(t)\}|\overline{v}_R(t)\} \, dx \, dt \right|
\]

\[
+ \left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast |w_0(t)|^2) v_0(t) \{\overline{v}_R(t) - \overline{v}_0(t)\} \, dx \, dt \right|.
\]

It follows from the Hardy-Littlewood inequality and the Hölder inequality that

\[
\left| \int_{\mathbb{R} \times \mathbb{R}^n} (\cdot | \sigma \ast \{w_R(t) - w_0(t)\}|\overline{w}_R(t))|v_R(t)|^2 \, dx \, dt \right|
\]

\[
\leq \int_{\mathbb{R}} \left| \left< e^{itH_R} \left[ (\cdot | \sigma \ast \{w_R(t) - w_0(t)\}|\overline{w}_R(t)) \right] v_R(t) \right>, \phi_1 \right| \, dt
\]

\[
\leq \|\phi_1\| \int_{\mathbb{R}} \left\| \left[ (\cdot | \sigma \ast \{w_R(t) - w_0(t)\}|\overline{w}_R(t)) \right] v_R(t) \right\| \, dt
\]

\[
\leq C \|\phi_1\| \int_{\mathbb{R}} \|w_R(t) - w_0(t)\|_{2a} \|w_R(t)\|_{2b} \|v_R(t)\|_{2b} \, dt,
\]

where $a, b$ and $h$ satisfy

\[
\frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{h} - 1 \right) = 1 - \frac{\sigma}{n}, \quad \sigma < n.
\]
By means of $L^p$ ($2 \leq p \leq \infty$) strong convergence of $e^{-itH_R} \to e^{-itH_0}$ as $R \to \infty$ (see Weder [13, p. 3644] or Weder [15, pp. 1233-1234]), we find that the integrand of (12) vanishes as $R \to \infty$. The same argument shows that the other terms of the right-hand side of (12) vanish as $\lim_{R \to \infty} R^2 \leq \infty$. Thus we find that $\lim_{R \to \infty} R^2 \leq 1$. Hence, if $2 \leq \sigma \leq 4$ and $\sigma < n$, it follows from Sobolev imbedding theorem that

$$\|w_R(t) - w_0(t)\|_{2a} \leq C_1 \|w_0(t)\|_{2a} \leq 2C\|w_0(t)\|_{1,q} \|v_0(t)\|_{1,q},$$

where $1/q = 1/2 - 2/(3n)$. Since $w_0$, $v_0 \in L^2(\mathbb{R}; H^{1,q})$ (see Mochizuki [6, pp. 148-151]), it follows from the dominated convergence theorem that the right-hand side of (11) vanishes as $R \to \infty$. The same argument shows that the other terms of the right-hand side of (12) vanish as $R \to \infty$. Thus we find that $\lim_{R \to \infty} R^2 \leq 2 + 2n - \sigma$. Choosing any $\gamma \neq 0$, we compute the limit $R^\gamma L_1[\phi_R]$ as $R \to \infty$. If the limit does not exist, it means that $\gamma > 2 + 2n - \sigma$. Next we take $\gamma_1 < \gamma$. If $R^{\gamma_1} L_1[\phi_R] \to 0$ as $R \to \infty$, then we find that $\gamma_1 < 2 + 2n - \sigma$. We choose a constant $\gamma_2$ such that $\gamma_1 < \gamma_2 < \gamma$ and compute the limit $R^{\gamma_2} L_1[\phi_R]$ again. Thus, repeating this computation till the limit $R^{\gamma_2} L_1[\phi_R]$ tends to some non-zero constant, we can reconstruct $\sigma$ by $\sigma = 2 + 2n - \rho$.

**Reconstruction of $\lambda$**

Once $V(x)$ and $\sigma$ are determined, we can easily obtain the formula from (9) that

$$\lambda = \frac{L_1[\phi]}{\int_{\mathbb{R}^n} (|\cdot|^{-\sigma} * |e^{-itH_0}\phi_2|^2(x))e^{-itH_0}\phi_1(x)^2 \, dx \, dt},$$

if $\phi_j \neq 0$, $j = 1, 2$. We note that in the step of determining $\sigma$, the value of the limit $\lim_{R \to \infty} R^{\gamma} L_1[\phi_R]$ are already known. Hence, using the identity (10), we can derive another formula for $\lambda$:

$$\lambda = \lim_{R \to \infty} \frac{R^{\gamma} L_1[\phi_R]}{\int_{\mathbb{R}^n} (|\cdot|^{-\sigma} * |e^{-itH_0}\phi_2|^2(x))e^{-itH_0}\phi_1(x)^2 \, dx \, dt}.$$

The proof is complete.

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