I give a short historical and a critical review of the determinations of light quark masses from QCD at dawn of the next millennium. QCD spectral sum rules combined with ChPT give, to order $\alpha_3^3$, the world average for the running masses:

- $m_s(2\text{ GeV}) = (118.9 \pm 12.2)\text{ MeV}$,
- $m_d(2\text{ GeV}) = (6.3 \pm 0.8)\text{ MeV}$,
- $m_u(2\text{ GeV}) = (3.5 \pm 0.4)\text{ MeV}$

and the corresponding values of the invariant masses given in Eq. 24. Lower and upper bounds derived from the positivity of spectral moments are presented in Tables 2 and 3. For a comparison, we critically review the recent lattice results (section 8 and Table 5) and attempt to deduce the present QCD grand average determination (Table 6):

- $m_s(2\text{ GeV}) = (110.9 \pm 8.8)\text{ MeV}$, to be used with a great care.

Then, we deduce the value: $B_{6/2} - 0.45(\text{resp. 0.32})B_{8/2} \approx 1.6 \pm 0.4$ (resp. $1.1 \pm 0.3$) and the lower bound $1.1 \pm 0.2$ (resp. $0.7 \pm 0.1$), for the combination of the penguin operators, governing the CP-violating parameters $\epsilon'/\epsilon$ without (resp. with) the inclusion of the final state interaction effects. The result signals a possible deviation from the leading $1/N$ prediction by about $1 \sim 3\sigma$, which should be tested using accurate non-perturbative calculations.

1. INTRODUCTION

One of the most important parameters of the standard model and chiral symmetry is the light quark masses. Indeed, they are useful for a much better understanding of the realizations of chiral symmetry breaking [1–3] and for some eventual explanation of the origin of quark masses in unified models of interactions [3]. Within some popular parametrizations of the hadronic matrix elements [1], the strange quark mass can also largely influence the Standard Model prediction of the $CP$ violating parameters $\epsilon'/\epsilon$, which have been measured recently [2]. However, contrary to the leptons, where the physical masses can be identified with the pole of the propagator [3], the quark masses are difficult to define because of confinement. Instead, they can be treated as coupling constants of the QCD Lagrangian, where the notion of the running and invariant masses, which are renormalization scheme and scale dependents, has been introduced [3]. In practice, these masses are conveniently defined within the standard $\overline{MS}$-scheme. In addition to the determination of the ratios of light quark masses (which are scale independent) from current algebra [1], and from chiral perturbation theory (ChPT), its modern version [4], a lot of effort reflected in the literature [5] has been put into extracting directly from the data the running quark masses using the SVZ QCD spectral sum rules (QSSR) [6], LEP experiments [7] and lattice data [8]. In this talk, I shall review the different determinations from these QCD approaches, by emphasizing the historical developments of the field.

2. RUNNING AND INVARIANT LIGHT QUARK MASSES IN QCD

It is convenient to introduce the dimensionless coupling $x_i(\nu) \equiv m_i(\nu)/\nu$, where $\nu$ is the renormalization scheme subtraction constant. The running quark mass is a solution of the differential equation:

$$\frac{dx_i}{dt} = (1 + \gamma(\alpha_s))\mathcal{F}_i(t) : \mathcal{F}_i(t = 0) = x_i(\nu).$$  (1)
In the $\overline{MS}$-scheme, its solution to order $a_s^3$ ($a_s = \alpha_s/\pi$) is:

$$\bar{m}_i(\nu) = \bar{m}_i(-\beta_1 a_s(\nu))^{-\gamma_1/\beta_1}\left\{1 + \frac{\beta_2}{\beta_1} \left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right) a_s(\nu) + \frac{1}{2} \left[\frac{\beta_2^2}{\beta_1^2} \left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right) - \frac{\beta_3}{\beta_1} \left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_3}{\beta_3}\right)\right] a_s^2(\nu) + 1.95168a_s^3\right\}, \quad (2)$$

where the $a_s^3$ term comes from $[15]$; $\gamma_i$ are the $O(a_s^i)$ coefficients of the quark-mass anomalous dimension, which read for three flavours:

$$\gamma_1 = 2, \quad \gamma_2 = 91/12, \quad \gamma_3 = 24.8404. \quad (3)$$

The invariant mass $\bar{m}_i$ has been introduced for the first time by $[8]$ in connection with the analysis of the breaking of the Weinberg sum rules by the quark mass terms in QCD.

3. RATIOS OF LIGHT QUARK MASSES

The ratios of light quark masses are well-determined from current algebra $[3]$, and ChPT $[4]$. In this approach, the meson masses are expressed using a systematic expansion in terms of the light quark masses:

$$M_{s^+}^2 = (m_u + m_d)B + O(m^2) + ...$$
$$M_{s^+}^2 = (m_u + m_s)B + O(m^2) + ...$$
$$M_{K^0}^2 = (m_d + m_s)B + O(m^2) + ... \quad (4)$$

where $B \equiv -\langle \bar{\psi}\psi \rangle/f^2$, from the Gell-Mann, Oakes, Renner relation $[10]$:

$$m_{s^+}^2 f_s^2 \simeq -(m_u + m_d)\langle \bar{\psi}\psi \rangle + O(m^2). \quad (5)$$

However, only the ratio, which is scale independent can be well determined. To leading order in

![Figure 1. $m_s/m_d$ versus $m_u/m_d$ from [17].](image)

$m \leq 17$:

$$\frac{m_u}{m_d} \approx \frac{M_{s^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{s^+}^2 + M_{K^0}^2 - M_{K^+}^2} \approx 0.66$$

$$\frac{m_s}{m_d} \approx \frac{-M_{s^+}^2 + M_{K^0}^2 + M_{K^+}^2}{M_{s^+}^2 + M_{K^0}^2 - M_{K^+}^2} \approx 20 \quad (6)$$

Including the next order + electromagnetic corrections, the ratios of masses are constrained on the ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 = 1 \quad (7)$$

where $Q^2 \simeq (m_s - \bar{m})^2/(m_d - m_u)$ using the value of the $\eta \rightarrow \pi^+\pi^-\pi^0$ from the PDG average $[10]$, though this value can well be in the range 22–26, to be compared with the Dashen’s formula $[19]$ of 24.2; $\bar{m} \equiv (1/2)(m_u + m_d)$. In Fig. 1, one shows the range spanned by $R \equiv (m_s - \bar{m})/(m_d - m_u)$ and the corrections to the GMO mass formula $\Delta_M$: $M^2 = (1/3)(4M^2 - m_s^2)(1 + \Delta_M)$. The Weinberg mass ratio $[4]$ is also shown

In Generalized ChPT, the contribution of the $m^2$-term can be as large as the $m$ one $[18]$, which modifies drastically these ratios.
which corresponds to the Dashen’s formula and $R \simeq 43$. At the intersection of different ranges, one deduces [20]:

$$\frac{m_u}{m_d} = 0.553 \pm 0.043, \quad \frac{m_s}{m_d} = 19.8 \pm 0.8,$$

$$\frac{2m_s}{(m_d + m_u)} = 24.4 \pm 1.5.$$  \hspace{1cm} (8)

The possibility to have a $m_u = 0$ advocated in [20] appears to be unlikely as it implies too strong flavour symmetry breaking and is not supported by the QSSR results from 2-point correlators of the divergences of the axial and vector currents, as will be shown in the next sections.

4. QCD SPECTRAL SUM RULES

4.1. Description of the method
Since its discovery in 79 [11], QSSR has proved to be a powerful method for understanding the hadronic properties in terms of the fundamental QCD parameters such as the QCD coupling $\alpha_s$, the (running) quark masses and the quark and/or gluon QCD vacuum condensates. The description of the method has been often discussed in the literature, where a pedagogical introduction can be, for instance, found in the book [13]. In practice (like also the lattice), one starts the analysis from the two-point correlator:

$$\psi_H(q^2) \equiv i \int d^4x e^{iqx} \langle 0 | \bar{T} J_H(x) (J_H(0)) | 0 \rangle,$$

built from the hadronic local currents $J_H(x)$, which select some specific quantum numbers. However, unlike the lattice which evaluates the correlator in the Minkowski space-time, one exploits, in the sum rule approaches, the analyticity property of the correlator which obeys the well-known Källen–Lehmann dispersion relation:

$$\psi_H(q^2) = \int_0^\infty dt \frac{dt}{t - q^2 - i\epsilon} \frac{1}{\pi} \text{Im} \psi_H(t) + ..., \quad (10)$$

where ... represent subtraction points, which are polynomials in the $q^2$-variable. In this way, the sum rule expresses in a clear way the duality between the integral involving the spectral function $\text{Im} \psi_H(t)$ (which can be measured experimentally), and the full correlator $\psi_H(q^2)$. The latter can be calculated directly in the QCD Euclidean space-time using perturbation theory (provided that $-q^2 + m^2$ (m being the quark mass) is much greater than $\Lambda^2$), and the Wilson expansion in terms of the increasing dimensions of the quark and/or gluon condensates which simulate the non-perturbative effects of QCD.

4.2. Beyond the usual SVZ expansion
Using the Operator Product Expansion (OPE) [13], the two-point correlator reads:

$$\psi_H(q^2) \simeq \sum_{D=0,2,...} \frac{1}{(q^2)^{D/2}} \sum_{\text{dim}O=D} C(q^2, \nu) \langle O(\nu) \rangle,$$

where $\nu$ is an arbitrary scale that separates the long- and short-distance dynamics; $C$ are the Wilson coefficients calculable in perturbative QCD by means of Feynman diagrams techniques; $\langle O(\nu) \rangle$ are the quark and/or gluon condensates of dimension $D$. In the massless quark limit, one may expect the absence of the terms of dimension 2 due to gauge invariance. However, it has been emphasized recently [21] that the resummation of the large order terms of the perturbative series, and the effects of the higher dimension condensates due e.g. to instantons, can be mimicked by the effect of a tachyonic gluon mass $\lambda$ which generates an extra $D = 2$ term not present in the original OPE. Its presence might be understood from the analogy with the short distance linear part of the QCD potential [4]. The strength of this short distance mass has been estimated from the $e^+e^-$ data to be [23,24]:

$$\frac{\alpha_s \lambda^2}{\pi} \simeq -0.06 \sim 0.07 \text{ GeV}^2,$$

which leads to the value of the square of the (short distance) string tension: $\sigma \simeq -\frac{2}{3} \alpha_s \lambda^2 \simeq [(400 \pm 20) \text{ MeV}]^2$ in an (unexpected) good agreement with the lattice result [24] of about $[(440 \pm 38) \text{ MeV}]^2$. In addition to Eq. 5, the strengths of the vacuum condensates having dimensions $D \leq 6$ are also under good control, namely:

- $\langle \bar{s}s \rangle/\langle \bar{d}d \rangle \simeq 0.7 \pm 0.2$ from the meson [12] and baryon systems [20];

Some evidence of this term is found from the lattice analysis of the static quark potential [24], though the extraction of the continuum result needs to be clarified.
Among the different sum rules discussed in the prior literature, two free external parameters (\(\alpha_s, \tau\)) insensitive to their variations. In some cases, the \(\tau\)-stability is not reached due to the too naive parametrization of the spectral function. One can either fix the \(\tau\)-values by the help of FESR (local duality) or improve the parametrization of the spectral function by introducing threshold effects fixed by chiral perturbation theory, ..., in order to restore the \(\tau\)-stability of the results. The results discussed below satisfy these stability criteria.

4.3. Spectral function

In the absence of the complete data, the spectral function is often parametrized using the “naïve” duality ansatz:

\[
\frac{1}{\pi} \text{Im}\psi_H(t) \approx 2M^2_H f_H^2 \delta(t - M^2_H) + \text{"QCD continuum"} \times \delta(t - t_c),
\]

which has been tested [12] using \(e^+e^-\) and \(\tau\) decay data, to give a good description of the spectral integral in the sum rule analysis; \(f_H\) (analogue to \(f_\pi\)) is the hadron’s coupling to the current; \(2\pi\) is the dimension of the correlator; while \(t_c\) is the QCD continuum’s threshold.

4.4. Form of the sum rules and optimization procedure

Among the different sum rules discussed in the literature [12], we shall be concerned with the:

- **Laplace sum rule (LSR)** [11,13,18,28]:

\[
\mathcal{L}_n(\tau) = \int_0^\infty dt \, t^n \exp(-\tau t) \frac{1}{\pi} \text{Im}\psi_H(t).
\]

The advantage of the Laplace sum rules with respect to the previous dispersion relation is the presence of the exponential weight factor, which enhances the contribution of the lowest resonance and low-energy region accessible experimentally.

For the QCD side, this procedure has eliminated the ambiguity carried by subtraction constants, arbitrary polynomial in \(q^2\), and has improved the convergence of the OPE by the presence of the factorial damping factor for each condensates of given dimensions. As one can notice, there are “a priori” two free external parameters (\(\tau, t_c\)) in the analysis. The optimized result will be (in principle) insensitive to their variations. In some cases, the \(t_c\)-stability is not reached due to the too naive parametrization of the spectral function. One can either fix the \(t_c\)-values by the help of FESR (local duality) or improve the parametrization of the spectral function by introducing threshold effects fixed by chiral perturbation theory, ..., in order to restore the \(t_c\)-stability of the results. The results discussed below satisfy these stability criteria.

- **Finite Energy Sum Rule (FESR)** [33–38]:

\[
R_n^m = \int_0^{M^2} dt \, t^n \left(1 - \frac{t}{M^2}\right)^m - \frac{1}{\pi} \text{Im}\psi_H(t),
\]

The advantage of the FESR is the separation (to leading order in \(\alpha_s\)) of the terms of given dimensions, which gives a set of local duality constraints. However, unlike the two formers, FESR is sensitive to the high-energy tails of the spectral integral and needs an accurate treatment of this region, in order that the optimal results are insensitive to the changes of \(t_c\).

5. UP AND DOWN RUNNING MASSES

5.1. Pseudoscalar sum rules

- **Values of \((\bar{m}_u + \bar{m}_d)\)** have been extracted for the first time in [23] using the sum rule of the 2-point correlator associated to the pseudoscalar current:

\[
\partial_\mu A^\mu(x) = (m_i + m_j) : \bar{u}(i\gamma_5)d :.
\]

The analysis has been improved (or disproved) later on by many groups [12, 13–50], by the inclusion of higher order terms or/and by a more involved parametrization of the spectral function (threshold effects, ChPT,...). However, this channel is quite peculiar due to the Goldstone nature of the pseudoscalar mesons.

- \(\bar{m}_{u,d}\) are the quark masses, and \(\bar{m}_{u,d}\) are the quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are the up and down quark masses, respectively.

- \(\bar{m}_{u,d}\) are the strange quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are strange quark masses.

- \(\bar{m}_{u,d}\) are the charm quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are charm quark masses.

- \(\bar{m}_{u,d}\) are the bottom quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are bottom quark masses.

- \(\bar{m}_{u,d}\) are the top quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are top quark masses.

- \(\bar{m}_{u,d}\) are the quark masses, and \(\bar{m}_{u,d}\) are the quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are quark masses.

- \(\bar{m}_{u,d}\) are the quark masses, and \(\bar{m}_{u,d}\) are the quark masses.

- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are quark masses.

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- \(\bar{m}_{u,d}\) and \(\bar{m}_{u,d}\) are quark masses.
Table 1

| (\overline{m}_u + \overline{m}_d)(1) | Sources | Authors |
|--------------------------------------|---------|---------|
| Best estimate                       |         |         |
| 12.8 ± 2.5 \(\pi + \text{ChPT} \) | BPR95 [46,47] |
| 12.1 ± 2.4 BPR+ tachyonic gluon     |         |         |
| 12.6 ± 3.2 \(\langle \psi\psi \rangle\) | DN98* [52] |
| Others                              |         |         |
| 13.2 ± 4.4 \(\pi + \text{moments} \) | Y97 [50] |
| 15.6 ± 3.4 LSR + \(\pi + \pi'\) NWA |         |         |
| Average                             |         |         |
| 13.1 ± 1.5(stat) ± 1.3(syst)        |         |         |

* to order \(\alpha_s\) and not included in the average.

of the pion, where the value of the sum rule scale (1/\(\tau\) for Laplace and \(t_c\) for FESR) is relatively large of about 2 GeV$^2$ compared with the pion mass, where the duality between QCD and the pion is lost. This implies an important role of the higher states (radial excitations or/and theoretical parametrizations of the spectral function above the 3\(\pi\) threshold) in the analysis, and then led to some controversial results, which hopefully can be cured by the presence of the new 1/\(q^2\) [21,24] due to the tachyonic gluon mass, which enlarges the duality region to lower scale and then minimizes the role of the higher states into the sum rule. The errors due to the QCD part of the sum rules, which is now known to order \(\alpha_s^3\), are much less than from the parametrization of the spectral function. Among the available results, we consider that the best estimates of \((\overline{m}_u + \overline{m}_d)\) from this channel come from [10] \((\pi + \text{ChPT parametrization of the 3\(\pi\) continuum}) and

from [24] (inclusion of the tachyonic gluon mass into the analysis of [46]). The result of [10] to order \(\alpha_s^3\) has been extended to order \(\alpha_s^4\) by [47]. Also the result of [10] updates the one in [15]. Ref. [50] uses the positivity of the higher state contributions plus the moment inequalities to order \(\alpha_s\). In [14], one treats the \(\pi'\) in a Narrow Width Approximation (NWA), while the QCD expression is to order \(\alpha_s^2\). These different determinations are quoted in Table 1, after including into these published results the perturbative contributions of the known \(\alpha_s^3\)-term [51]. The effect of this term is quite small as the PT series converges quite well at the sum rule working region of about 1.5 GeV. Indeed, the PT expression of e.g. the Laplace sum rule normalized to \((\overline{m}_u + \overline{m}_d)^2\) reads:

\[ \mathcal{L} \sim 1 + 4.82a_s + 21.98a_s^2 + 53.14a_s^3 + O(a_s^4). \quad (17) \]

One can notice the good consistency of the results from the different forms of the sum rules in the pion channel. We give in Table 1 the average of these updated determinations, where we have added an extra 10% error which takes into

| Table 2 |
|---------|
| Lower bounds on \(\overline{m}_{u,d,s}(2)\) in MeV to order \(\alpha_s\) |

| Observables | Sources | Authors |
|-------------|---------|---------|
| \(\overline{m}_u + \overline{m}_d\) | \(\pi, \sigma\) | LRT97 [49] |
| 8           |         | Y97 [50] |
| 7.3         | \(\pi\) |         |
| 7.          | \(\langle \psi\psi \rangle + \text{GMOR} \) | DN98* [52] |
| \(\overline{m}_d - \overline{m}_u\) | \(K\pi\) | Y97 [50] |
| 1.5         |         |         |
| \(\overline{m}_s\) | \(K\) | LRT97 [49] |
| 100         |         |         |
| 104         | \(K\) | Y97 [50] |
| 90          | \(\langle \psi\psi \rangle + \text{ChPT}\) | DN98* [52] |

[21,24]
account the systematics of the approach.

- **Lower bounds for** $\overline{m}_{u} + \overline{m}_{d}$ **based on** moments inequalities and the positivity of the spectral functions have been obtained, for the first time, in [42]. These bounds have been derived recently in [49,50] to order $\alpha_s^3$. Their optimal values quoted in Table 2 exclude the low value of about 6 MeV given by [48].

### 5.2. Scalar sum rules

- **Lower bounds on** $\overline{m}_{d} - \overline{m}_{u}$ **have been extracted for the first time in** [52] **using the sum rule of the 2-point correlator associated to the scalar current**:

  \[ \partial_\mu V^\mu(x) = (m_d - m_u) : \bar{d}(i) u : \, \]  

  which is sensitive to leading order to the quark-mass difference. The analysis has been extended later on by many authors [45,50]. However, the analysis relies heavily on the less controlled nature of the $a_0(980)$, where its $q\bar{q}$ nature appears to be favoured by the present data [53]. In the $I = 0$ channel, the situation of the $\pi\pi$ continuum is much more involved due to the possible gluonium nature of the low mass and wide $\sigma$ meson [30,55].

5.3. **Direct extraction of $\langle \bar{\psi}\psi \rangle$**

- **The chiral condensate** can be directly extracted from the sum rules (nucleon, $B^* - B$ splitting, vector form factor of $D^* \to K^*\ell\nu$), which are particularly sensitive to it and to the mixed condensate $\langle \bar{\psi}\sigma^{\mu\nu}(\lambda_\mu/2)G_{\mu\nu}\psi \rangle = M_0^2 \langle \bar{\psi}\psi \rangle$. A global fit from these different channels gives, to order $\alpha_s$, the running condensate value at 1 GeV [53]:

  \[ 0.6 \leq \langle \bar{\psi}\psi \rangle/[-225 \text{ MeV}]^3 \leq 1.5, \]  

  a result also recovered by the lattice [53].

- **Lower and upper bounds** on the light quark masses given in Tables 2 and 3, can be obtained by transforming this result using the PCAC relation in Eq. 5, and the positivity of the $\mathcal{O}(m^2)$ term. These results are independent on how chiral symmetry is realized (ChPT or generalized ChPT?).

### 5.4. $\overline{m}_{u,d,s}$ to order $\alpha_s^3$ from sum rules + ChPT

One should note from Table 1 the consistency of the results from the pion channel and the one from the direct extraction of $\langle \bar{\psi}\psi \rangle$, which is an a posteriori support of the validity of the OPE for the $\pi$-sum rule in the working region, and signals the absence of the large effects due to instantons, which may break the OPE. Using the ratios from ChPT in section 2, one can deduce in units of MeV, the value of the running masses at 2 GeV to order $\alpha_s^3$ given in Table 6. We have used the conversion factor:

\[ \overline{m}_i(1) \simeq (1.38 \pm 0.06)\overline{m}_i(2), \]  

for running, to order $\alpha_s^3$, the results from 1 to 2 GeV, which corresponds to the average value of the QCD scale $\Lambda_3 \simeq (375 \pm 50)$ MeV from PDG [10] and others [58]. I remind that the errors in these determinations already take into account the systematics of the method (see Table 1).

### 6. DIRECT EXTRACTIONS OF $\overline{m}_i$
6.2. Scalar sum rules
Following the pioneer’s analysis of \[54\], \(m_s\) has been obtained by different authors \[59\]–\[64\] by using the \(K\pi\) phase shift data for parametrizing the spectral function. The different values obtained from this channel to order \(\alpha_s^3\) is given in Table 4. Like in the case of the pseudoscalar channel, the errors are dominated by the uncertainties for parametrizing the spectral function.

6.3. \(m_s\) from \(e^+e^- + \tau\)-decay data
One can combine the \(e^+e^- \rightarrow I = 0, 1\) hadrons and the rotated recent \(\Delta S = 0\) component of the \(\tau\)-decay data in order to extract \(m_s\). Unlike previous sum rules, one has the advantage to have a complete measurement of the spectral function in the region covered by the analysis. We shall work with:

\[
R_{\tau,\phi} \equiv \frac{3|V_{ud}|^2}{2\pi\alpha^2} S_{EW} \int_0^{M^2} ds \left(1 - \frac{s}{M^2}\right)^2 \left(1 + \frac{2s}{M^2}\right) \frac{s}{M^2} \sigma_{e^+e^- \rightarrow \phi,\phi',...},
\]

(21)

and the \(SU(3)\)-breaking combinations \[37,53\]:

\[
\Delta_{10} \equiv R_{\tau,1} - R_{\tau,0}, \quad \Delta_{10} \equiv R_{\tau,1} - 3R_{\tau,0},
\]

(22)

which vanish in the \(SU(3)\) symmetry limit; \(\Delta_{10}\) involves the difference of the isoscalar \((R_{\tau,0})\) and isovector \((R_{\tau,1})\) sum rules à la Das-mathur-Okubo \[87\]. The PT series converges quite well at the optimization scale of about 1.6 GeV \[22\]. E.g., normalized to \(m_s^2\), one has:

\[
\Delta_{10} \sim 1 + \frac{13}{3} a_s + 30.4a_s^2 + (173.4 \pm 109.2)a_s^3.
\]

(23)

It has been argued in \[67\] that \(\Delta_{10}\) can be affected by large \(SU(2)\) breakings, but this claim has not been confirmed from the result based on the other sum rules not affected by these terms \[67\]. The largest range of values from different form of the sum rules is given in Table 4, which one can compare with the average of \((178 \pm 33)\) MeV given in \[53\]. An upper bound deduced from the positivity of \(R_{\tau,\phi}\) is given in Table 3.

6.4. \(m_s\) from the \(\Delta S = -1\) part of \(\tau\)-decay
One can also extract \(m_s\) from the Cabibbo suppressed channel of \(\tau\)-decay \[14,58,22\] using different \(\tau\)-like moments. Unlike the case of the

| Channels   | \(\overline{m}_s(1)\) | Sources   | Authors |
|------------|---------------------|-----------|---------|
| Kaon SR    | 165 ± 15            | SN89\[44\] |         |
|            | 155 ± 25            | DPS99\[59\] |         |
|            | 155 ± 25 \(\text{Largest Range}\) |          |         |
| Scalar SR  | 203 ± 20            | CPS97\[41\] |         |
|            | 143 ± 17            | CFNP97\[52\] |         |
|            | 160 ± 30            | J98\[63\] |         |
|            | 158 ± 11            | M99\[54\] |         |
|            | 175 ± 48 \(\text{Largest Range}\) |          |         |
| \((\tau\text{-like } \phi\text{ SR})\) | 173 ± 33            | R\[64\] |         |
| \(e^+e^-\text{ data}\) | 176 ± 31            | SN95,99\[37,65\] |         |
| \(+\tau\text{-decay}\) | 186 ± 31            | SN95,99\[37,65\] |         |
| \(\Delta S = -1\) \(\text{part of } 200 \pm 50\) | 200 ± 50            | CKP98\[58\] |         |
| \(\tau\text{-decay}\) | 164 ± 33            | PP99\[70\] |         |
| \(213 ± 82 \(\text{Largest Range}\) |          |         |
| \(
\text{Average of Largest Ranges } 166.7 ± 18.8
\) |          |         |

6.1. Pseudoscalar sum rules
In the strange quark channel, we quote in Table 4 the results from \[44\] and \[59\], and consider the largest range spanned by these previously quoted results. We consider that this conservative range already takes into account and may even overestimate the systematics of the method. One should notice here that, unlike the case of the pion, the result is less sensitive to the contribution of the higher states continuum due to the relatively higher value of \(M_K\), though the parametrization of the spectral function still gives larger errors than the QCD series.
neutral $\phi$-meson current, where the QCD series is more convergent, here the convergence is quite bad, such that one needs to select an appropriate combination (spin 1+0 pieces) for obtaining an acceptable result. Though a complete agreement has been obtained in the previous analysis of [69] with the two other determinations [73,78], a recent analysis in [70] is lower and more precise than the former, though still in agreement with the previous ones due to the generous errors given there. Ref. [70] argues that one should consider the previous results as an upper bound of [69] with the two other determinations [14,68], a recent analysis in [70] is lower and more precise, rather than taking their average quoted in Table 6. Combined with the ratios from ChPT in section 2, this value leads to the values of $m_{u,d}$ given in Table 6. As already discussed in previous sections, the quoted error already include the systematics of the methods. The size of the error is within the expected accuracy of the sum rule results. Using Eq. 2, it is trivial to extract the value of the invariant mass $\hat{m}_i$. One obtains in units of MeV:

\[
\hat{m}_u = 3.9 \pm 0.7, \quad \hat{m}_d = 7.1 \pm 0.8
\]

where the error is larger than the corresponding running mass due to $\Lambda$ in the evolution procedure.

### 8. COMPARISON WITH THE LATTICE

#### 8.1. Lattice approaches for/by non-experts
One usually starts from the QCD action and partition function:

\[
Z = \int D A_\mu \det M e^{\int d^4 x (-\frac{1}{4} G_{\mu\nu} G_{\mu\nu})} \tag{25}
\]

integrated over gauge field configurations. The fermion contributions are included into the non-local $\det M$ term. For the analysis, one works like in the sum rule approach, with the 2-point correlator defined in previous sections, which is saturated by the intermediate states $|n\rangle \langle n|$. In this way, the two-point correlator can be expressed as:

\[
\sum_x |0\rangle \langle J(x) J(0)\rangle \langle 0| = \sum_n |0\rangle \langle J(x) |n\rangle \langle n| \langle J(0)\rangle \langle 0| = \frac{e^{-E_n t}}{2E_n} \tag{26}
\]

where the zero momentum states $E_n$ tend to the masses $M_n$ of the resonances. In the (ideal) asymptotic limit $t \to \infty$, the exponential factor kills the effect of the different excitations, such that the lowest ground state contribution dominates. In practice, this approximation is expected to be realized when the splitting between

### 7. QSSR + ChPT FINAL RESULTS
We take the average of $m_s$ from $m_u + m_d$ (Table 1) + the ChPT ratio and from the direct determination in Table 4. Then, we obtain the final average from QSSR+ChPT to order $\alpha_s^3$ in Table 6. Combined with the ratios from ChPT in section 2, this value leads to the values of $m_{u,d}$ given in Table 6. As already discussed in previous sections, the quoted error already include the systematics of the methods. The size of the error is within the expected accuracy of the sum rule results. Using Eq. 2, it is trivial to extract the value of the invariant mass $\hat{m}_i$. One obtains in units of MeV:

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the ground state and the radial excitation is large enough.

8.2. Practical limits of the lattice

Besides the usual statistical and finite size (about 1% if the lattice size \( L \geq 3 \) fermi and \( m_s L \geq 6 \)), errors inherent to the lattice, which can be minimized using modern technology, there are still large uncertainties related to the uses of field theory on the lattice due to the finite values of the lattice spacing \( a \):

- The different operators mix at finite \( a \).
- The discretization errors specific to each actions, which are \( \mathcal{O}(a) \) for the Wilson (explicit breaking of chiral symmetry (\( \chi S \))) and Domain wall (extra 5th dimension in order to preserve \( \chi S \)) actions, \( \mathcal{O}(a^2) \) for the staggered (reduction of quark couplings with high-momenta gluons) and \( \mathcal{O}(a) \) for the Clover (inclusion of the mixed quark-gluon operator) actions. For typical values of \( 1/a \approx 2 \) GeV, the error is \( \approx 10-30\% \), which can be reduced by computing at different values of \( a \).
- The well-known quenched approximation (no inclusion of the fermion contribution \( \ln \text{Det} M \)), which implies a modification of \( \chi S \) with unphysical singularities for \( m_q = 0 \) or practically for \( m_q \leq m_s/3 \) (recall that in this approximation: \( M_q' \approx m_q = 0 \) (\( \equiv \text{large } N_c \)-limit)), which induces an error of about 20% that can be estimated from the deviation of the predictions from the observed meson masses and couplings or/from the choices of the mesons for setting the scale (string tension).
- The extrapolation of the results to light quark masses with the help of the meson mass dependence expected from ChPT, which for typical values \( 1/a \approx 2 \) GeV, and keeping \( m_s L \geq 6 \), one requires \( L/a \geq 90 \) in order to avoid finite volume effects. At present, \( L/a \approx 32 \) (quenched) and \( L/a \approx 24 \) (unquenched) far below this limit.
- The errors due to the matching of the lattice and the continuum at a typical lattice conversion scale of 2 GeV can be minimized using the non-perturbative renormalization.

8.3. Lattice results and estimated errors

From the previous discussions, we consider that:

- The conservative quenched lattice errors are about 20%.
- The extraction of \( m_{u,d} \) is less reliable than \( m_s \).

Therefore, we shall only consider the value of \( m_s \) obtained from the lattice which we shall compare with the one obtained in previous sections. Lattice results prior 98 have been already reviewed in [13]. The different results for 98 and 99 are given in Table 5 for different actions, where one can see a large spread of predictions, which with the given errors are inconsistent each others. We mainly attribute the source of this discrepancy to the underestimate of the systematic errors given there. Most of these results have been obtained using the non-perturbative renormalization [80], and Ward identities for the axial (AWI) and/or vec-

| Group     | \( m_s \) (2) | Sources          | Comments         |
|-----------|---------------|------------------|------------------|
| 98        |               |                  |                  |
| OHIO[71]  | 129±23        | Staggered        |                  |
| APE[72]   | 130±18        | K*               | NLO, AWI         |
|           | 121±13        | \( K, \phi \)    | NLO, Clover      |
| Wilson,   |               |                  | NPR+AWI          |
| 99        |               |                  |                  |
| CP-PACS[73]| 143±6        | \( \phi \)       | AWI+VWI          |
| JLQCD[74] | 115±2         | \( K \)          | \( q \text{ChPT} \) |
|           | 129±12        | \( \phi \)       | AWI+VWI          |
|           | 106±7         | \( K \)          | \( q \text{ChPT} \) |
| \( \alpha \)-UKQCD[75]| 97±4       | \( f_K \)        | NNLO             |
|           | 105±4         | \( K^* \)        | NNLO             |
| DESY[76]  |               |                  | AWI, I. Wilson, |
| BNL[77]   | 95±26         | \( f_\pi, K \)   | Domain           |
| BNL[78]   | 130±21        |                  | Walls            |
| APE[79]   | 114±9         | Q-Prop.          | NNLO             |
| Average   | 112.9±1.5     | (stat.)          | ±22.3 (syst.)    |
\[ \overline{m}_{s}(2) \simeq (69 - 181) \text{ MeV} \]  

where part of this range is already excluded by the bounds given in Tables 2 and 3. Instead, one can also quote (to be taken carefully) the naïve average given in Table 5 at NNLO \([79]\) where we have added our guessed 20\% estimate of the lattice systematic errors based on the previous comments. 

At this approximation, where a comparison with the previous results from QCD spectral sum rules is meaningful, one can notice a surprisingly good agreement.

Some attempts to put dynamical fermions have been done \([82]\), and more recently with 2 flavours in \([83]\). In \([83]\), some of the problems encountered in the quenched approximation (discrepancy between the \(\phi\) and \(K\) results,...) seems to be resolved. Though promising, the approach is not enough mature for the different systematics to be fully under control. We quote in Table 6 this result adopting a more conservative error than the original one.

\section*{9. SUMMARY}

We have reviewed the different determinations of light quark masses from ChPT, QCD spectral sum rules (QSSR), \(e^+e^-\) and \(\tau\)-decay data, and compared the one of the strange quark mass with the recent lattice results:

- The sum of light quark masses \(m_u + m_d\) to order \(\alpha_3^s\) from different QSSR analysis is given in Table 1 and the resulting average value.
- Lower (resp. upper) bounds based on the positivity and analyticity properties of the spectral functions are given in Table 2 (resp. Table 3).
- Different direct determinations of the strange quark mass to order \(\alpha_3^s\) are compiled in Table 4.
- Combined results from these four methods lead
to the final average given in Table 6 to order $\alpha_s^3$, where the errors are typically the 10% systematics of the QSSR approach. An eventual failure of this result should signal a new phenomena not accounted for in the QPE discussed in this paper.

- We have compared this final result with the recent (after 98) lattice determinations (Table 5) which belong in the range given by Eq. 27 and which lead to the average in Table 5.
- Within the present uncertainties of various approaches, we consider that there is a good agreement between the previous sum rule and lattice results. Attempting to give the final QCD value, we average the different results in Table 6, and deduce the QCD Grand Average given in this table, to be used carefully.
- However, we expect that future high precision measurement of the light quark masses will be difficult to reach due to the systematic errors inherent to each method, which, often, different authors do not include into their results!
- Finally, one should remind that, in the phenomenological analysis, one should use the value of $m_s$ into the expression of any hadronic matrix element or/and observables which are known at the same level of approximation. This consistency condition is not often respected in the literature.

10. APPLICATION TO $\epsilon'/\epsilon$

One of the most fashionable applications of the previous result is the one to the CP violating parameters $\epsilon'/\epsilon$, where $\epsilon'$ is related to $A[K_L \to (\pi\pi)_{I=2}]/A[K_S \to (\pi\pi)_{I=0}]$ and characterizes the (direct) CP-violation in the decay amplitude of $K \to \pi\pi$; $\epsilon = A[K_L \to (\pi\pi)_{I=0}]/A[K_S \to (\pi\pi)_{I=0}]$ is the (indirect) CP-violation from $K^0$-$\bar{K}^0$ mixing. It is known [3] that the dominant effects in the analysis of $K_L,S \to (\pi\pi)_{I=0,2}$ amplitudes are due to the QCD and electroweak penguin operators:

$$Q_6 \equiv (\bar{s}_d d_\beta)_{V-A} \sum_{u,d,s} (\bar{\psi}_a \psi_\beta)_{V+A} \approx B_6^{3/2}/m_s^2 + O(1/N)$$

$$Q_8 \equiv \frac{3}{2} (\bar{s}_d d_\beta)_{V-A} \sum_{u,d,s} (\bar{c}_d \bar{\psi}_a \psi_\beta)_{V+A} \approx B_8^{3/2}/m_s^2 + O(1/N).$$

For $M_t \simeq 165$ GeV and $\Lambda_{QCD} \simeq 340$ MeV, the simplified SM prediction without (resp. with) the inclusion of final-state interaction effects [5], is [8]:

$$\frac{\epsilon'}{\epsilon} \approx 9.75(\text{resp. } 15.34) \text{Im} \lambda_t \left[ \frac{110 \text{ MeV}}{m_s(2)} \right]^2 \times \left[ B_6^{1/2} - 0.54(\text{resp. } 0.32) B_8^{3/2} \right].$$

where $\lambda_t = V_{td} V_{ts}^*$ is expressed in terms of the CKM matrix elements. Using $\text{Im} \lambda_t \approx (1.34 \pm 0.30) \times 10^{-4}$ [85], and the measured value $\epsilon'/\epsilon_{\text{exp}} \approx (21.4 \pm 4.0) \times 10^{-4}$, one can deduce, from the average value and the lower bound of $m_s$ in Table 6 [5],

$$B_6^{1/2} - 0.54(0.32) B_8^{3/2} \approx 1.6 \pm 0.4 (1.1 \pm 0.3),$$

which signals a violation of about $1 \sim 3\sigma$ for the leading $1/N$ vacuum saturation prediction [3];

$$B_6^{1/2} \approx B_8^{3/2} \approx 1.$$ This result and the final-state interaction effects should be tested using more accurate non-perturbative calculations. It is only after performing these tests that one can make a sharper conclusion on the SM prediction of the CP-violation [9].

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QUESTIONS AND DISCUSSIONS

Lively general discussions, many questions and some comments have followed this talk, As they have been also addressed to the previous talks in this session, they have not been reported here.