In this paper, the linearized field equations related to the quadratic curvature gravity theory have been obtained in the four-dimensional de Sitter (dS) space-time. The massless spin-2 field equations have been written in terms of the Casimir operators of dS group making use of the ambient space notations. By imposing some simple constraints, arisen from group theoretical interpretation of the field equations, a new four-dimensional Gauss-Bonnet (GB)-like action has been introduced with the related field equations transforming according to the unitary irreducible representations (UIR’s) of dS group. Since, the field equations transform according to the UIR’s of dS group, the GB-like action, we just obtained, is expected to be a successful model of modified gravity. For more clarifying, the gauge invariant field equations have been solved in terms of a gauge-fixing parameter $C$. It has been shown that the solution can be written as the multiplication of a symmetric rank-2 polarization tensor and a massless minimally coupled scalar field on dS space. The Gupta-Bleuler quantization method has been utilized and the covariant two-point function has been calculated in terms of the massless minimally coupled scalar two-point function, using the ambient space notations. It has been written in terms of dS intrinsic coordinates from the ambient space counterpart. The two-point functions are dS invariant and free of any theoretical problems. It means that the proposed model is a successful model of modified gravity and it can produce significant results in the contexts of classical theory of gravity and quantum gravity toy models.

Keywords: Modified theories of gravity; Classical theories of gravity; Linear gravity; de Sitter space-time.

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1 Introduction

Among the reasons it should be interesting to study the physics of dS and asymptotically dS spacetimes we may mention that: (1) Recent analysis of astronomical data indicates that there is a positive cosmological constant and we live in a universe which will look like dS spacetime in the future [1, 2, 3]. (2) The interesting proposal of defining, in a manner analogous to the AdS/CFT correspondence, the dS/CFT correspondence has been suggested recently. According to dS/CFT correspondence, there is a dual between quantum gravity on dS space and a Euclidean conformal field theory on the boundary of dS space [4]. Therefore, understanding of quantum field theory on dS space-time is of considerable interest.

Although the Einstein tensorial theory of gravitation, known today as general relativity, was more successful to pass observational tests and became the standard theory of gravitation, according to the recent cosmological observations it seems that this theory may be incomplete. The most important failures was the inability to describe accelerating expansion of the universe [5, 6, 7], cosmic microwave background anisotropies [8, 9], large scale structure formation [10, 11], baryon oscillations [12] and weak lensing [13], etc. One of the main approaches for explanation of these phenomena is to modify Einstein general relativity. In this regards, various modifications of Einstein gravity were proposed in the literatures. Among them Lovelock gravity [14, 15, 16], braneworld scenario [17, 18, 19], scalar-tensor theories [20, 21, 22], f(R) gravity [23]-[26] and, in particular, non-linear gravity theories or higher-order theories of gravity have provided interesting results [27]-[37]. Some of the models are based on gravitational actions which are non-linear in the Ricci scalar, the Riemann and Ricci invariants, known as quadratic curvature gravity theory, or contain terms involving combinations of derivatives of the Ricci scalar [38]-[41].

Nowadays, the quadratic curvature gravity theories and their applications have been the subject of many interesting works and a lot of papers have been appeared in which the usual theory of gravity is extended quadratically making use of the curvature invariants (for example see [42]-[47]). Quadratic curvature gravity is a natural generalization on Einstein’s gravity. That includes higher order derivatives of metric. But due to the presence of these higher order derivatives, massive ghosts appear. However, higher order models of modified gravity play an important role in high energy physics. These certain classes of higher order gravity theories are known as the quantum gravity toy models [48, 49].

We already know that the gravitational field equations in the linear approximation describe a massless spin-2 particle (the graviton, if it exists) which propagates on the background space-time. Following the Wigner’s theorem, a linear gravitational field should transform according to the UIR’s of the symmetric group of the background space-time. In the previous paper [50], making use of an action containing quadratic form of both the Ricci scalar and the Ricci tensor, we obtained the physical massless spin-2 field equations and showed that it can be associated with UIR’s of dS group with a suitable choice of constant coefficients. Also we obtained the massless rank-2 tensor field and the related two-point function as the solutions to the physical field equations.

The main purpose of this paper is to introduce a new extension of the Hilbert-Einstein action of the gravitational field. This action is composed by taking a linear combination of the squared Ricci scalar, the Ricci and Riemann curvature invariants. To this end, we consider the linearized form of the corresponding field equation around dS background. Recasting these linearized field equations in terms of Casimir operators dS group one is led to a new GB-like quadratic action in the four dimensions. Since the field equations, stem from the new action we just obtained, are correspond to the UIR’s of dS group it is a successful model of modified gravity and can produce reasonable results. For more clarifying, we solve the field equations in terms of the massless minimally coupled scalar field in dS space. Also we obtain the related two-point function and show that it can be written in terms of
the massless minimally coupled scalar two-pint function on dS space. The results of the calculations are free of any theoretical problems and confirm the validity and successfulness of the proposed model of modified gravity.

This paper is structured as follows. In section-2, the most general form of the quadratically-extended gravitational action, as the generalization of the Hilbert-Einstein action, has been introduced and corresponding linearized field equations have been obtained in terms of the intrinsic dS coordinates as the background. In section-3, after a brief review of the ambient space notations and Casimir operators of dS group, the full field equations have been written in terms of Casimir operators of dS group. By imposing some simple conditions a new GB-like action has been obtained with the field equations which are correspond to the UIR’s of dS group. In order to illustrate the validity and successfulness of the action, the related field equations have been solved in terms of the gauge-fixing parameter \( C \) making use of the ambient space notations. The solution is written as the multiplication of a symmetric generalized polarization rank-2 tensor and a massless minimally coupled scalar field on dS space. Also, making use of the Gupta-Bleuler quantization method, we calculated the corresponding two-point function in terms of the massless minimally coupled scalar two-point function using the ambient space formalism. The dS-invariant graviton two-point function has been written in terms of 4-dimensional intrinsic coordinates from its ambient space counterpart. The results are summarized and discussed in section-4. Some useful mathematical relations and details of derivations have been given in appendices.

2 Linear dS field equations

The gravitational action for the quadratic curvature gravity theory in the 4-dimensional dS space-time with the positive cosmological constant \( \Lambda = 3H^2 \) can be written as

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ a_0(\mathcal{R} - 2\Lambda) + a\mathcal{R}^2 + b\mathcal{R}^{ab}\mathcal{R}_{ab} + c\mathcal{R}^{abcd}\mathcal{R}_{abcd} \right],
\]

where \( \mathcal{R}_{abcd} \) is the Riemann tensor, \( \mathcal{R}_{ab} \) is the Ricci tensor and \( \mathcal{R} = g^{ab}\mathcal{R}_{ab} \) is the Ricci scalar of the space-time under consideration. \( a_0, a, b \) and \( c \) are constant coefficients. The coefficients \( a, b \) and \( c \) are positive having dimension of (Length)^2. The theory described by this action is referred to as fourth-order gravity since it leads to fourth order equations. Numerous papers have been devoted to the study of fourth-order gravity theories. Note that the Gauss-Bonnet action

\[
\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( \mathcal{R}^2 - 4\mathcal{R}^{ab}\mathcal{R}_{ab} + \mathcal{R}^{abcd}\mathcal{R}_{abcd} \right),
\]

is a total divergence. It does not contribute to the field equations of 4-dimensional space-times.

By varying the action (2.1) with respect to the metric tensor \( g_{ab} \) the modified gravitational field equations can be obtained as [47]

\[
E_{ab} = a_0\mathcal{E}^{(0)}_{ab} + a\mathcal{E}^{(1)}_{ab} + b\mathcal{E}^{(2)}_{ab} + c\mathcal{E}^{(3)}_{ab} = 0,
\]

where \( \mathcal{E}^{(0)}_{ab} = G_{ab} + \Lambda g_{ab} \) and \( G_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R} g_{ab} \) is the Einstein tensor and

\[
\mathcal{E}^{(1)}_{ab} = 2\mathcal{R}\mathcal{R}_{ab} - 2\nabla_a \nabla_b \mathcal{R} - \frac{1}{2}g_{ab}(\mathcal{R}^2 - 4\Box \mathcal{R}),
\]

\[
\mathcal{E}^{(2)}_{ab} = \Box \mathcal{R}_{ab} - \nabla_a \nabla_b \mathcal{R} + 2\mathcal{R}_{acbd}\mathcal{R}^{cd} - \frac{1}{2}g_{ab}(\mathcal{R}^{cd}\mathcal{R}_{cd} - \Box \mathcal{R}),
\]

\[
\mathcal{E}^{(3)}_{ab} = \Box \mathcal{R}_{ab} - \nabla_a \nabla_b \mathcal{R} + 2\mathcal{R}_{acbd}\mathcal{R}^{cd} - \frac{1}{2}g_{ab}(\mathcal{R}^{cd}\mathcal{R}_{cd} - \Box \mathcal{R}),
\]

\[
\mathcal{E}^{(4)}_{ab} = \Box \mathcal{R}_{ab} - \nabla_a \nabla_b \mathcal{R} + 2\mathcal{R}_{acbd}\mathcal{R}^{cd} - \frac{1}{2}g_{ab}(\mathcal{R}^{cd}\mathcal{R}_{cd} - \Box \mathcal{R}),
\]
\[
\mathcal{E}_{ab}^{(3)} = 4\Box R_{ab} - 2\nabla_a \nabla_b R - 4R_{ac} R^c_b + 4R_{acbd} R^{cd} + 2R_{acde} R^{cf} - \frac{1}{2} g_{ab} R_{cf} R^{cf},
\]
and \( \Box \equiv \nabla^a \nabla_a = g^{ab} \nabla_a \nabla_b \) is the d’Alembertian operator.

We use the background field method to obtain the linearized form of the field equations (2.3). According to the background field method, originally developed by Christian Fronsdal [51], one can assume \( g_{ab}^{(BG)} + h_{ab} \), in which \( g_{ab}^{(BG)} \) is the background metric and \( h_{ab} \) are its small fluctuations. Indices are raised and lowered by the background metric. We suppose that \( g_{ab}^{(BG)} = g_{ab}^{(ds)} \equiv \tilde{g}_{ab} \). So we have

\[
g_{ab} \simeq \tilde{g}_{ab} + h_{ab} \quad \text{and} \quad g^{ab} \simeq \tilde{g}^{ab} - h^{ab}. \tag{2.7}
\]

The metric \( \tilde{g}_{ab} \) is a solution to Einstein’s field equations with the positive cosmological constant \( \Lambda = 3H^2 \):

\[
\tilde{R}_{ab} - \frac{1}{2} \tilde{R} \tilde{g}_{ab} + 3H^2 \tilde{g}_{ab} = 0. \tag{2.8}
\]

Making use of the approximations given in Eq.(2.7), in Eq.(2.3), we have

\[
(E_{ab})^{(0)}_L = \frac{1}{2} (\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} - \Box h_{ab} - \nabla_a \nabla_b h^c + 2H^2 h_{ab}) + \frac{1}{2} \tilde{g}_{ab} (\Box h' - \nabla_c \nabla_d h^{cd} + H^2 h' + 2H h' + 2H h'), \tag{2.9}
\]

in which \( h' = h_{ab}^a \) is the trace of \( h_{ab} \) with respect to the background metric and \( \nabla^b \) is the background covariant derivative. Making use of the relations given in appendix-A it is easy to obtain the linearized field equations in dS space as

\[
(E_{ab})^{(1)}_L = +12^H \nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} - \Box h_{ab} - 2\nabla_a \nabla_b \left( \nabla_c \nabla_d h^{cd} - \Box h' + 3H^2 h' \right)
+ 24H^4 h_{ab} - 2\tilde{g}_{ab} \left( 3H^2 \nabla_c \nabla_d h^{cd} + 3H^4 h' - \Box \nabla_c \nabla_d h^{cd} + \Box^2 h' \right), \tag{2.10}
\]

\[
(E_{ab})^{(2)}_L = \frac{1}{2} \left[ \Box (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a) - 2H^2 \Box h_{ab} - \Box^2 h_{ab} + \nabla_a \nabla_b \Box h' \right]
+ 2H^2 (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a) - \nabla_a \nabla_b \nabla_c h^{cd} - 3H^2 \nabla_a \nabla_b h' + 4H^4 h_{ab}
+ \frac{1}{2} \tilde{g}_{ab} \left( 2H^2 \nabla_c \nabla_d h^{cd} - 2H^4 h' + 7H^2 \Box h' + \Box \nabla_c \nabla_d h^{cd} - \Box^2 h' \right), \tag{2.11}
\]

\[
(E_{ab})^{(3)}_L = 2\Box (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a) + 8H^2 \Box h_{ab} - 2\Box^2 h_{ab} - 8H^4 h_{ab} - 6H^2 \nabla_a \nabla_b h'
- 4H^2 (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a) - 2\nabla_a \nabla_c \nabla_d h^{cd} + 2H^2 \tilde{g}_{ab} \left( \nabla_c \nabla_d h^{cd} - \Box h' + H^2 h' \right). \tag{2.12}
\]

Now, making use of Eqs.(2.9)-(2.12) and Eq.(2.3) we have

\[
(E_{ab})_L = a_0 (E_{ab})^{(0)}_L + a (E_{ab})^{(1)}_L + b (E_{ab})^{(2)}_L + c (E_{ab})^{(3)}_L = 0. \tag{2.13}
\]

Eq.(2.13) is the linearized quadratically-extended gravitational field equations in dS background, which has been written in terms of the intrinsic coordinates \( X_a \) of the 4-dimensional dS space-time. The linear field equations (2.13) can be rewritten in the following explicit form
Making use of these relations in the field equations (2.14) we obtain
\[
\frac{1}{2} g_{ab} \left[ (-a_0 - 12a H^2 - 2b H^2 + 4c H^2) \nabla_c \nabla_d h^{cd} + \left( a_0 - b H^2 - 4c H^2 \right) \Box h' \right] \\
+ \left( a_0 - 12a H^2 - 2b H^2 + 4c H^2 \right) H^2 h' + \left( 4a + b \right) \left( \Box \nabla_c \nabla_d h^{cd} - \Box^2 h' \right) \\
- \left( 2a + b + 2c \right) \nabla_a \nabla_b \nabla_c \nabla_d h^{cd} = 0. 
\]  
(2.14)

It is easy to show that the massless spin-2 dS field equations (2.14) is invariant under the following gauge transformations
\[
h_{ab} \rightarrow h_{ab}^{(gt)} = h_{ab} + \nabla_a \Lambda_b + \nabla_b \Lambda_a, 
\]
where \( \Lambda_a \) is an arbitrary four-vector.

Letting \( a = c = \alpha_g \) and \( b = -4\alpha_g \), where \( \alpha_g \) is a constant coefficient with dimension of \((\text{Length})^2\), the proposed gravitational theory (2.1) reduces to the Einstein-Gauss-Bonnet theory as

\[
\frac{1}{16\pi G} \int d^4 \sqrt{-g} \left[ a_0 (\mathcal{R} - 2\Lambda) + \alpha_g \left( \mathcal{R}^2 - 4\mathcal{R}_{ab} \mathcal{R}_{ab} + \mathcal{R}^{abcd} \mathcal{R}_{abcd} \right) \right], 
\]
and the linearized field equations (2.14) reduces to \((\mathcal{E}_{ab})_L = a_0 (\mathcal{E}_{ab}^{(0)})_L \). It means that the Gauss-Bonnet action is a total divergence and does not contribute to the field equations even if in its linear approximation. It is evident that the theory with these constraints is not of interest here.

The Minkowskian correspondence of the theory can be obtained by taking the zero curvature limit (i.e. \( H \rightarrow 0 \)) of Eq.(2.14); it is,

\[
2(E_{ab})_{L}^{\text{Mink.}} = - (b + 4c) \Box^2 h_{ab} - a_0 \Box h_{ab} + \left( 4a + b \right) \partial_a \partial_b \Box h' - a_0 \partial_a \partial_b h' \\
+ \left( a_0 + b \Box + 4c \Box \right) \left( \partial_a \partial^b h_{bc} + \partial_b \partial^c h_{ac} \right) - 2(2a + b + 2c) \partial_a \partial_b \partial_c \partial_d h^{cd}_{ab} \\
\eta_{ab} \left[ a_0 \Box h' - a_0 \partial_c \partial_d h^{cd} + (4a + b) \left( \Box \partial_a \partial_d h^{cd} - \Box^2 h' \right) \right] = 0. 
\]
(2.16)

where \( \eta_{ab} \) is the metric and \( \Box = \eta_{ab} \partial^a \partial^b = \partial^a \partial_a \) is the wave operator in the flat space. The linearized Minkowskian field equations (2.16) is invariant under the following gauge transformations
\[
h_{ab} \rightarrow h_{ab}^{(gt)} = h_{ab} + \partial_a \lambda_b + \partial_b \lambda_a, 
\]
for the arbitrary four-vector \( \lambda_a \).

Now, we calculate the trace of the field equations (2.14). In terms of the dimensionless constants \( A = a H^2 \), \( B = b H^2 \) and \( C = c H^2 \), that is

\[
(E')_L = g^{ab} (E_{ab})_L = (a_0 - 18 A - 6 B - 6 C) \Box h' + 3 H^2 a_0 h' - 2 (3A + B + C) H^{-2} \Box^2 h' \\
- a_0 \nabla_c \nabla_d h^{cd} + 2 (3A + B + C) H^{-2} \Box \nabla_c \nabla_d h^{cd} = 0, 
\]
(2.17)

which immediately reads as
\[
a_0 = 0, \quad \text{and} \quad 3A + B + C = 0. 
\]
(2.18)

Making use of these relations in the field equations (2.14) we obtain
\[
-2(E_{ab})_L = (B + 4C) \left[ H^{-2} \Box^2 h_{ab} - H^{-2} \Box (\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac}) - 6 \Box h_{ab} + 8 H^2 h_{ab} \\
+ 4 (\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac}) + 2 \nabla_a \nabla_b h' + \frac{2}{3} H^{-2} \nabla_a \nabla_c \nabla_c h^{cd} + \frac{1}{3} H^{-2} \nabla_a \nabla_b \Box h' \right]. 
\]
\[-\tilde{g}_{ab} \left( 2 \nabla_c \nabla_d h^{cd} - \Box h' + 2H^2 h' - \frac{1}{3} H^{-2} \left( \Box \nabla_c \nabla_d h^{cd} - \Box^2 h' \right) \right) = 0. \tag{2.19} \]

Since \( B \neq -4C \), Eq. (2.19) can be rewritten as

\[
H^{-2} \Box^2 h_{ab} - H^{-2} \Box \left( \nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} \right) - 6 \Box h_{ab} + 8H^2 h_{ab} \\
+ 4 \left( \nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} \right) + 2 \nabla_a \nabla_b h' + \frac{2}{3} H^{-2} \nabla_a \nabla_b \nabla^c h_{cd} + \frac{1}{3} H^{-2} \nabla_a \nabla_b \Box h' \\
- \tilde{g}_{ab} \left[ 2 \nabla_c \nabla_d h^{cd} - \Box h' + 2H^2 h' - \frac{1}{3} H^{-2} \left( \Box \nabla_c \nabla_d h^{cd} - \Box^2 h' \right) \right] = 0. \tag{2.20} \]

## 3 The field equations in the ambient space

In this section we try to rewrite the linearized field equations (2.20) making use of the five-dimensional ambient space formalism. For this purpose, at first we review briefly the ambient space notations and remind the Casimir operators of dS group. We consider a symmetric and transverse (i.e. \( x \cdot K(x) = 0 \)) tensor field \( K_{\alpha\beta}(x) \) in ambient space notations. It is related to the “intrinsic” field \( h_{ab}(X) \) through the following tensorial transformation rule [52, 53, 54]

\[
h_{ab}(X) = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} K_{\alpha\beta}(x(X)). \tag{3.1} \]

The covariant derivative in the ambient space notations is defined as

\[
D_\beta T_{\alpha_1...\alpha_n} = \partial_\beta T_{\alpha_1...\alpha_n} - H^2 \sum_{i=1}^{n} x_{\alpha_i} T_{\alpha_1...\beta\alpha_{n}}, \tag{3.2} \]

where \( \partial \) is tangential (or transverse) derivative in dS space

\[
\partial_{\alpha} = \theta_{\alpha\beta} \partial^\beta = \partial_{\alpha} + H^2 x_{\alpha} \cdot \partial, \quad x \cdot \partial = 0, \tag{3.3} \]

and \( \theta_{\alpha\beta} \) can be written in terms of the ambient space metric \( \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1) \) as \( \theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_{\alpha} x_{\beta} \). It is easily shown that the dS metric \( \tilde{g}_{ab} \) corresponds to the transverse projector \( \theta_{\alpha\beta} \), that is,

\[
\tilde{g}_{ab}(X) = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \theta_{\alpha\beta}(x). \tag{3.4} \]

The two Casimir operators of dS group are labeled by \( Q_1^{(1)} \) and \( Q_2^{(2)} \) where the subscript \( s \) reminds the rank of tensors in consideration. The action of the Casimir operators \( Q_1^{(1)} \) and \( Q_2^{(1)} \) on \( K \) and \( \mathcal{K} \) respectively can be written in the more explicit form

\[
Q_1^{(1)} K(x) = \left( Q_0^{(1)} - 2 \right) K(x) + 2x \partial \cdot K(x) + 2H^2 x \cdot K(x) - 2 \partial \cdot K(x), \tag{3.5} \]

\[
Q_2^{(1)} \mathcal{K}(x) = \left( Q_0^{(1)} - 6 \right) \mathcal{K}(x) + 2\eta \mathcal{K}' + 2S x \partial \cdot \mathcal{K}(x) - 2S \partial x \cdot \mathcal{K}(x) + 2H^2 S x x \cdot \mathcal{K}(x), \tag{3.6} \]

where, \( Q_0^{(1)} = -H^{-2}(\partial)^2 \) is the scalar Casimir operator. The symmetrizer \( S \) is defined for two vectors \( \xi_\alpha \) and \( \omega_\beta \) by \( S(\xi_\alpha \omega_\beta) = \xi_\alpha \omega_\beta + \xi_\beta \omega_\alpha \). \( \mathcal{K}' = \eta^{\alpha\beta} \mathcal{K}_{\alpha\beta} \) is the trace of the tensor \( \mathcal{K} \). The readers are referred to [52] and references therein for more details.

Making use of the ambient space formalism, the field equations (2.20) can be written as (appendix-B)

\[
(Q_2^{(1)} + 4)(Q_2^{(1)} + 6) \mathcal{K} + (Q_2^{(1)} + 4)D_2(\partial_2 \mathcal{K}) = 0, \tag{3.7} \]
where $D_2$ is the generalized gradient operator defined by $D_2 K = S(D_1 - x) K$ with $D_{2a} = H^{-2} \delta_{a}^{x}$ and $\partial_{x}K = \partial_{x}K - H^2 x K' - \frac{1}{2} \partial K'$ [52].

Noting the following identities [54]

$$Q_2^{(1)} D_2 A = D_2 Q_1^{(1)} A, \quad (3.8)$$

$$\partial_{x} D_2 A = -(Q_1^{(1)} + 6) A, \quad (3.9)$$

one can show that Eq.(3.7) is invariant under the following gauge transformations

$$\mathcal{K} \rightarrow \mathcal{K}^{(gt)} = \mathcal{K} + D_2 A, \quad (3.10)$$

for an arbitrary five-vector $A$ in the ambient space. It means that one can rewrite the field equations (3.7) in the following form

$$(Q_2^{(1)} + 4)(Q_2^{(1)} + 6) \mathcal{K} + C(Q_2^{(1)} + 4)D_2(\partial_{x} \mathcal{K}) = 0, \quad (3.11)$$

where $C$ is a gauge-fixing parameter. It is notable that if one let $C = 0$ or if one impose the physical conditions $\partial. \mathcal{K} = 0 = \mathcal{K}'$ in Eq.(3.11), the physical sector of the theory obtains as

$$(Q_2^{(1)} + 4)(Q_2^{(1)} + 6) \mathcal{K} = 0, \quad (3.12)$$

which transforms according to two of the UIR’s of dS group denoted by $\Pi^{\pm}_{2, 2}$ and $\Pi^{\pm}_{2, 1}$ in discrete series. The physical graviton field equations (3.12) has been solved and the corresponding two-point function has been obtained in [50].

Now, we are in a position to introduce a new four-dimensional GB-like action in the form

$$I = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( a R^2 + b R^{ab} R_{ab} + c R^{abcd} R_{abcd} \right), \quad (3.13)$$

with the constraints $3a + b + c = 0$ and $b \neq -4c$. Since the gravitational field equations related to the GB-like action (3.13) transform according to the UIR’s of dS group, we claim that it can be a successful model of modified gravity and can produce realized and significant physical results in the context of classical theory of gravity such as classical black holes. For more clarifying, we would like to solve the field equations and calculate the related two-point functions in the following subsections.

It must be noted that if we set $c = 0, a = -\frac{1}{8}$ and $b = 1$, in the proposed gravitational theory given by action (3.13), it reduces to the well-known Weyl theory of gravity. As an immediate consequence of the calculations presented here the field equations related to the Weyl theory of gravity, in the physical state, transform according to two of the UIR’s of dS group labeled by $\Pi^{\pm}_{2, 2}$ and $\Pi^{\pm}_{2, 1}$ in discrete series. This result is not compatible with that of ref.[59].

### 3.1 Solution to dS massless spin-2 field equations

In this section we would like to solve the field equations (3.11) in the ambient space notations. The general solution of to the field equations can be constructed out by the suitable combination of a scalar field and two vector fields. Let us introduce a rank-2 tensor field $\mathcal{K}$ in terms of a five-dimensional constant vector $Z_1 = (Z_{1a})$ and a scalar field $\phi_1$ and two vector fields $K$ and $K_g$ by putting [50, 52], [54]-[58]

$$\mathcal{K} = \theta \phi_1 + S \bar{Z}_1 K + D_2 K_g, \quad (3.14)$$

where $\bar{Z}_{1a} = \theta_{a\beta} Z_{1}^{\beta}$. Making use of the ansatz (3.14) in the field equations and noting relations (3.8) and (3.9) with the help of the following identity [54]

$$Q_2^{(1)} S \bar{Z} K = S \bar{Z}(Q_1^{(1)} + 4)K - 2H^2 D_2(x.Z_1)K + 4\theta Z_1.K, \quad (3.15)$$
we arrived at the following equation

\[ \theta \left[ (Q_0^{(1)})^2 + 4(Q_0^{(1)}) + 6 \phi_1 + 8(Q_0^{(1)}) + 2)Z_1.K \right] + S \bar{Z}_1 \left[ Q_1^{(1)}(Q_1^{(1)} + 2)K \right] \\
+ D_2 \left[ (1 - C)(Q_1^{(1)}) + 4(Q_1^{(1)}) + 6K_2 - 4H^2 \left( (Q_1^{(1)}) + 5)(x.Z_1.K) + Z_1D_1.K - xZ_1.K \right) \\
- CH^2(Q_1^{(1)}) + 4 \left( D_1\phi_1 + D_1Z_1.K - H^2Z_1.\bar{\partial}K + x(Z_1.K) - H^2Z_1.\bar{\partial}K - 5(x.Z_1.K) \right) \right] = 0, \quad (3.16) \]

which immediately reads as the following system of coupled differential equations

\[ Q_1^{(1)}(Q_1^{(1)} + 2)K = 0, \quad \text{or} \quad Q_1^{(1)}Q_0^{(1)}K = 0, \quad \partial.K = 0 = x.K, \quad (3.17) \]

\[ (Q_0^{(1)} + 4)(Q_0^{(1)} + 6)\phi_1 + 8(Q_0^{(1)}) + 2)Z_1.K = 0, \quad (3.18) \]

\[ (Q_1^{(1)} + 4)(Q_1^{(1)} + 6)K_g^{(c)} = \frac{4H^2}{1 - C} \left[ (Q_1^{(1)}) + 5)(x.Z_1.K) + (Z_1.D_1)K - x(Z_1.K) \right] \\
+ \frac{CH^2}{1 - C} (Q_1^{(1)} + 4) \left[ D_1\phi_1 + D_1(Z_1.K) - (Z_1.D_1)K + x(Z_1.K) - 5(x.Z_1.K) \right], \quad C \neq 1. \quad (3.19) \]

The solution to the first two equations can be written as [50]

\[ \phi_1 = - \frac{2}{3}Z_1.K, \quad K = \frac{\sigma}{2} \left[ (\sigma + 2)\bar{Z}_2 + (\sigma^2 + 2\sigma - 2)\frac{x.Z_2.\bar{\xi}}{x.\bar{\xi}} \right] \phi_s, \quad (3.20) \]

where \( \phi_s \) is the massless minimally coupled scalar field in dS space such that

\[ Q_0^{(1)}\phi_s = 0, \quad \phi_s = (Hx.\bar{\xi})^{\sigma}, \quad \xi^2 = 0, \quad \sigma = 0, -3. \quad (3.21) \]

Regarding Eq.(3.20), Eq.(3.19) can be rewritten as

\[ K_g^{(c)} = \frac{4H^2}{1 - C}(Q_1^{(1)} + 6)^{-1}(Q_1^{(1)} + 4)^{-1} \left[ (Q_1^{(1)}) + 5)(x.Z_1.K) + (Z_1.D_1)K - x(Z_1.K) \right] \\
+ \frac{CH^2}{1 - C}(Q_1^{(1)} + 6)^{-1} \left[ \frac{1}{3}D_1(Z_1.K) - (Z_1.D_1)K + x(Z_1.K) - 5(x.Z_1.K) \right]. \quad (3.22) \]

Making use of the identities given in ref.[50], we can write the gauge dependent vector \( K_g^{(c)} \) as

\[ K_g^{(c)} = \frac{H^2}{3(1 - C)} \left[ (x.Z_1.K) + \frac{1}{9}D_1(Z_1.K) \right] \\
+ \frac{CH^2}{1 - C}(Q_1^{(1)} + 6)^{-1} \left[ \frac{1}{3}D_1(Z_1.K) - (Z_1.D_1)K + x(Z_1.K) - 5(x.Z_1.K) \right]. \quad (3.23) \]

Now, using the following identities

\[ (Q_1^{(1)} + 6) \left[ \frac{1}{9}D_1(Z_1.K) + (x.Z_1.K) \right] = 6(x.Z_1.K), \quad (3.24) \]

\[ (Q_1^{(1)} + 6)Z.\bar{\partial}K = 6Z.\bar{\partial}K + 2H^2D_1Z.K, \quad (3.25) \]

\[ (Q_1^{(1)} + 6)D_1Z.K = 6D_1Z.K, \quad (3.26) \]

\[ (Q_1^{(1)} + 6)xZ.K = 6xZ.K, \quad (3.27) \]
in Eq. (3.23) we have
\[
K^{(c)}_g = \frac{CH^2}{6(1-C)} \left[ \frac{2+C}{9C} D_1(Z_1.K) + x(Z_1.K) - (Z_1.D_1)K + \frac{2-5C}{C}(x.Z_1)K \right].
\] (3.28)

Note that \( K^{(c)}_g \) satisfies the conditions
\[
x.K^{(c)}_g = 0, \quad \text{and} \quad \partial.K^{(c)}_g = \frac{1}{3}H^2Z_1.K.
\] (3.29)

In addition, if one set \( C = 0 \) in Eq. (3.28), \( K^{(c)}_g \) reduces to its physical correspondence given in [50]. It is obvious from Eq. (3.28) that the simplest gage-fixed value is not \( C = 0 \). But it is \( C = \frac{2}{2+1} \), which obeys the relation \( C = \frac{2}{2+1} \) and \( s \) denotes spin of the particle under consideration (graviton, if it exists). The case corresponds to \( C = \frac{2}{2+1} \) in named conventionally as the “Minimal case” (see [60] for the massless spin-1 particles and [55] for massless spin-2 particles in the Einstein’s gravity). In the minimal case we have
\[
K^{(\xi)}_g = \frac{2H^2}{27} [2D_1(Z_1.K) + x(Z_1.K) - (Z_1.D_1)K].
\] (3.30)

Now, Eqs. (3.20) and (3.28) show that one can construct the tensor field \( K \) in terms of a “massless” minimally coupled scalar field on dS space. After some calculations we can show that
\[
k^{(c)}_{\alpha\beta}(x) = \mathcal{E}^{(c)}_{\alpha\beta}(x, \xi, Z_1, Z_2)\phi_s,
\] (3.31)

where \( \mathcal{E}^{(c)} \) is a generalized symmetric polarization tensor which can be explicitly written as
\[
\mathcal{E}^{(c)}_{\alpha\beta} = S [AZ_{1\alpha}Z_{2\beta} + BZ_{1\alpha}\xi_{\beta} + CZ_{2\alpha}\xi_{\beta} + D\xi_{\alpha}\xi_{\beta} + E\theta_{\alpha\beta}],
\] (3.32)

with
\[
A = -\frac{\sigma}{2} + \frac{\sigma C}{12(1-C)} \left[ 2(\sigma + 3) + \frac{2+C}{9C} + \frac{2-5C}{C} \right],
\]
\[
B = -\frac{\sigma}{2} + \frac{\sigma C}{12(1-C)} \left[ 2(\sigma + 3) + \frac{2+C}{9C} + (\sigma + 2) \frac{2-5C}{C} \right] \frac{x.Z_2}{x.\xi},
\]
\[
C = \frac{\sigma C}{6(1-C)} \left[ \sigma(\sigma + 3) + \frac{2+C}{9C} + (\sigma + 1) \frac{2-5C}{C} - \sigma(\sigma + 1) \right] \frac{x.Z_1}{x.\xi},
\]
\[
D = \frac{\sigma(\sigma - 1)C}{12(1-C)} \left[ H^{-2} \left( \sigma \frac{2+C}{9C} - 2 - \sigma \right) \frac{Z_1.Z_2}{(x.\xi)^2} + (\sigma + 3) \frac{2+C}{9C} \right] \frac{(x.Z_1)(x.Z_2)}{(x.\xi)^2},
\]
\[
E = \frac{\sigma}{6} \left[ \left( 1 + \frac{C}{2(1-C)} \right) \left[ \sigma \frac{2+C}{9C} - 2 \right] \right] \frac{Z_1.Z_2}{x.\xi},
\]
\[
+ (\sigma + 3) \left( 1 + \frac{1}{9(1-C)} \right) \left[ 2(\sigma + 2) \frac{2-5c}{c} - \sigma(\sigma + 2) \right] \frac{Z_1.Z_2}{x.\xi}.
\]

As mentioned before, the ansatz (3.14) is a solution to the field equations (3.11) with respect to \( x' \) too. If we reexamine (3.14) in Eq. (3.11) and treat \( x' \) as the variable instead of \( x \), the resulting equations are exactly the same as Eqs. (3.20) and (3.27) with replacing \( x \) by \( x' \) and operators act on
the variable $x'$ [52]. But the final expression for the tensor field $K^{(C)}_{\alpha\beta}(x)$ is not other than that given in Eq.(3.32).

We now return to the gauge-fixed value $C = 1$ in Eq.(3.11) and reexamine the solution given in Eq.(3.14), we find that Eqs.(3.17) and (3.18) remain unchanged and Eq.(3.19) modifies as

$$Q_1^{(1)} \left[ x(Z_1K) - (Z_1D_1K) - (x.Z_1K) + \frac{1}{3} D_1(Z_1K) \right] + \frac{4}{3} D_1(Z_1K) = 0. \quad (3.33)$$

Noting Eqs.(3.24)-(3.27), one can easily confirm the validity of Eq.(3.33). It means that $K^{(C)}_g$ can be regarded as an arbitrary vector field without any constraint.

### 3.2 The graviton two-point function

The graviton two-point function $W_{\alpha\beta\gamma\delta}(x,x')$, which is a solution of the wave equation with respect to $x$ or $x'$, can be found in terms of the scalar two-point function. Let us try the following possibility [50, 52], [54]-[58]

$$W(x,x') = \theta \theta' W_0(x, x') + SS' \theta \theta' W_1(x, x') + D_2 D_2' W_g(x, x'), \quad (3.34)$$

where $W$, $W_1$ and $W_g$ are transverse bi-vectors, $W_0$ is bi-scalar and $D_2 D_2' = D_2' D_2$. We now substitute the two-point function (3.34) in the field equations (3.11) as a solution with respect to $x$. It is easy to show that

$$Q_1^{(1)} (Q_0^{(1)} + 2) W_1 = 0, \quad \text{or} \quad Q_1^{(1)} Q_0^{(1)} W_1 = 0, \quad \partial \partial W_1 = 0 = x.W_1, \quad (3.35)$$

$$(Q_0^{(1)} + 4)(Q_0^{(1)} + 6) \theta' W_0 + 8(Q_0^{(1)} + 2) S' \theta' W_1 = 0, \quad (3.36)$$

$$(Q_1^{(1)} + 4)(Q_1^{(1)} + 6) D_2 W^{(C)}_g = \frac{4H^2}{1 - C} S' \left[ (Q_1^{(1)} + 5)(x.\theta') W_1 + \theta'.D_1 W_1 + x.\theta'.W_1 \right]$$

$$+ \frac{C H^2}{1 - C} (Q_1^{(1)} + 4) \left[ D_1 \theta' W_0 + S' \left( (D_1 \theta'.W_1) - (\theta'.D_1) W_1 + x(\theta'.W_1) - 5(x.\theta') W_1 \right) \right], \quad C \neq 1. \quad (3.37)$$

The solution to Eqs.(3.35) and (3.36) are as follows

$$\theta' W_0(x,x') = -\frac{2}{3} S' \theta' W_1(x, x'), \quad (3.38)$$

$$W_1 = \left[ \theta \theta' + \frac{1}{2} D_1 \left( H^2 x.\theta' Q_0^{(1)} - \theta'.\bar{\partial} - 2H^2 x.\theta' \right) \right] W_s, \quad (3.39)$$

where $W_s$ is the two-point function for dS massless minimally coupled scalar field, obtained from “Gupta-Bleuler vacuum” state, with the following explicit form [61]

$$W_s(x,x') = \frac{iH^2}{8\pi} \epsilon(x^0 - x'^0) [\delta(1 - Z(x,x')) + \theta(Z(x,x') - 1)], \quad (3.40)$$

with

$$Z = -H^2 x.x', \quad \text{and} \quad \epsilon(x^0 - x'^0) = \begin{cases} 
1 & x^0 > x'^0, \\
0 & x^0 = x'^0, \\
-1 & x^0 < x'^0.
\end{cases} \quad (3.41)$$

which preserves dS-invariance.
Combining Eqs. (3.37) and (3.38) we have
\[
D^2 g^{(C)} = \frac{H^2}{3(1 - C)} S' \left[ (x', \theta) W_1 + \frac{1}{9} D_1 \theta W_1 \right]
\]
\[+ \frac{C H^2}{1 - C} (Q_2^{(1)} + 6)^{-1} S' \left[ \frac{1}{3} D_1 (\theta W_1) - (\theta W_1) \frac{1}{9} D_1 + x (\theta W_1) - 5 (x, \theta) W_1 \right],
\]
from which we obtain
\[
D^2 g^{(C)}(x, x') = \frac{CH^2}{6(1 - C)} S' \left[ \frac{2}{3} D_1 \theta W_1 + 2 + 5 \theta W_1 + x \theta W_1 - H^{-2} \theta \partial W_1 \right].
\]
Making use of Eqs. (3.38), (3.39) and (3.43), after some relatively simple and straightforward calculations, it turns out that the tensor two-point function can be written in the form
\[
W_{\alpha \beta \alpha' \beta'}(x, x') = \frac{2Z}{27(1 - C)(1 - Z^2)^2} S S' \left[ P_1(C, Z) \theta_{\alpha \beta} \theta_{\alpha' \beta'} + P_2(C, Z) (\theta_{\alpha \alpha'})(\theta_{\beta \beta'}) + H^2 P_3(C, Z) \left( \theta_{\alpha \beta} (x, x') (x, x') + \theta_{\alpha' \beta'} (x, x') (x, x') \right) + P_4(C, Z) H^4 (x, x') (x, x') (x, x')
\]
\[+ P_5(C, Z) H^2 (\theta_{\alpha \beta})(x, x')(x, x') \frac{d}{dZ} W_s(Z),
\]
where
\[
P_1(C, Z) = (1 - Z^2) \left[ 2 + C + 3(C - 1)Z^2 \right],
\]
\[
P_2(C, Z) = (1 - Z^2) \left[ 17C - 11 + 9(1 - C)Z^2 \right],
\]
\[
P_3(C, Z) = 3 \left[ 7C - 1 + (C - 1)Z^2 \right],
\]
\[
P_4(C, Z) = - \frac{3}{(1 - Z^2)} \left[ 3(7 - 19C) - 2(1 + 5C)Z^2 - 3(1 - 1)Z^4 \right],
\]
\[
P_5(C, Z) = \frac{1}{Z} \left[ 10(4 - 7C) + 2(1 - 22C)Z^2 - 18(1 - 1)Z^4 \right].
\]
The Eq. (3.44) is the explicit form of the graviton two-point function, based on the quadratically extended gravitational theory, in the ambient space notations. It is clearly dS-invariant and free of any divergences. It is well-known that if we require the statement (3.34) to be the solution of Eq. (3.11), with respect to $x'$ as the variable, the final result is not other than (3.44).

Let’s translate the two-point function (3.44) to the dS intrinsic coordinates. Using the translation rules given in [52, 54, 57] we have
\[
W^{(C)}_{aba' b'}(X, X') = \frac{2Z}{27(1 - C)} S S' \left[ \frac{P_1}{(1 - Z^2)^2} g_{ab} g_{a'b'} + \frac{P_2}{(1 - Z^2)^2} g_{ab} g_{a'b'}
\]
\[+ \frac{P_3}{1 - Z^2} \left( g_{ab} n_{a} n_{b} + g_{a'b'} n_{a} n_{b} \right) + \left( \frac{2(Z - 1)P_2}{1 - Z^2} + P_4 \right) n_{a} n_{b} n_{a'} n_{b'}
\]
\[+ \left( \frac{P_2}{(1 + Z)^2} - \frac{P_5}{1 + Z} + P_4 \right) n_{a} n_{b} n_{a'} n_{b'} \right] \frac{d}{dZ} W_s(Z).
\]
If we put $C = 0$ the two-point functions (3.44) and (3.45) will reduce to the corresponding physical two-point functions. It is notable that $W^{(C)}_{aba' b'}(X, X')$ and $W^{(C)}_{\alpha \beta \alpha' \beta'}(x, x')$ are related through the following tensorial transformation rule
\[
W^{(C)}_{aba' b'}(X, X') = \frac{\partial x^a}{\partial X^s} \frac{\partial x^b}{\partial X^s} \frac{\partial x^{a'}}{\partial X^{a'}} \frac{\partial x^{b'}}{\partial X^{b'}} W^{(C)}_{\alpha \beta \alpha' \beta'}(x, x').
\]
4 Conclusion

This work considers an extension of the Einstein-Hilbert gravitational action, which is constructed out by the linear combination of Ricci scalar, Ricci invariant and Riemann invariant in dS space. Varying the proposed action with respect to metric tensor leads to the fourth order field equations, conventionally named as the quadratically-extended gravitational field equations (Eq.(2.3)). The background field method is utilized and the linearized field equations are obtained in terms of intrinsic coordinates in the four-dimensional dS space as the background (Eq.(2.14)). The linearized field equations is invariant under some gauge transformations. We obtained the Minkowskian correspondence of the theory by taking the zero curvature limit (Eq.(2.16)). It remains invariant under some gauge transformations too.

By imposing some simple conditions (Eq.(2.18)), the massless spin-2 field equations has been written in the ambient space notations with the Casimir operators appearing in it (Eq.(3.7)). It is also gauge invariant under some special gauge transformations. Because of this gauge freedom a gauge-fixing parameter $C$ is inserted in the field equations. The field equations, in the physical state, are correspond to two of the UIR’s of dS group denoted by $\Pi_{2,2}^\pm$ and $\Pi_{2,1}^\pm$ in discrete series. We believe that for a model of gravity theory to be valid and successful it is necessary for the field equations to transform according to the UIR’s of the symmetric group [50]. With this issue in mind we introduced a new four-dimensional GB-like action (Eq.(3.13)), which is expected to be a successful model of modified gravity theory and have some new and realized physical consequences in the context of quantum gravity toy models.

Next we tried to illustrate the validity and successfulness of the gravitational model we just obtained. For this purpose we solved the related field equations. The symmetric tensor field, as the solution of the field equations, has been obtained in terms of the gauge-fixing parameter $C$. It has been written as the multiplication of a symmetric rank-2 generalized polarization tensor and a massless minimally coupled scalar field in dS space (Eq.(3.31)). Also, we have calculated the relevant two-point function, making use of the ambient formalism. We showed that the two-point function can be written in terms of the massless minimally coupled scalar two-point function in dS space (Eq.(3.44)). It is dS-invariant and free of any theoretical problems. The graviton two-point function has been written in terms of dS intrinsic coordinates from its ambient space counterpart which is dS-invariant and free of any theoretical problems too (Eq.(3.45)). Therefore, we claim that the proposed model of modified gravitational theory (i.e. the four-dimensional GB-like action given in Eq.(3.13)) is a valid and successful model and can produce reasonable and interesting results in the contexts of classical theory of gravity and quantum gravity toy models. It must be stressed that the results of this work confirm that it is necessary for a theory of modified gravity to be successful if it transforms according to the UIR’s of the related symmetric group. It is compatible with the results of [50].

A Some useful mathematical relations

The following relations have been used in deriving the linearized field equations.

$$\tilde{R}_{abcd} = H^2(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc}),$$  \hspace{1cm} (A.1)

$$\tilde{R}_{ab} = 3H^2\tilde{g}_{ab},$$  \hspace{1cm} (A.2)

$$\tilde{R} = 12H^2,$$  \hspace{1cm} (A.3)

$$(\mathcal{R})_L = \nabla_c\nabla_b h^{cb} - \Box h' - 3H^2 h'.$$  \hspace{1cm} (A.4)

$$(\nabla_a \nabla_b \mathcal{R})_L = \nabla_a \nabla_b \left(\nabla_c \nabla_d h^{cd} - \Box h' - 3H^2 h'\right).$$  \hspace{1cm} (A.5)
\((\Box R)_L = \Box \left( \nabla_e \nabla_d h^{ed} - \Box h' - 3H^2 h' \right)\). 

\((R_{ab})_L = \frac{1}{2} \left( \nabla^c \nabla_a h^c_{bd} + \nabla_d \nabla_a h^{ac} + 8H^2 h^{cd}_d - 2H^2 h' \bar{g}^{cd} - \Box h_d - \nabla^c \nabla_d h' \right) - 3H^2 h^c_d.\) 

\((R^{bc})_L = \frac{1}{2} \left( \nabla^c \nabla_a h^{ab} + \nabla^b \nabla_a h^{ac} + 8H^2 h^{bc}_d - 2H^2 h' \bar{g}^{bc} - \Box h^{bc} - \nabla^c \nabla^b h' \right) - 6H^2 h^{bc}.\) 

\((R_{ab})_L = \frac{1}{2} \left( \nabla_a \nabla_b \bar{h}_d + \nabla_b \nabla_a h^{e}_{cd} + 8H^2 h_{ab} - 2H^2 h' \bar{g}_{ab} - \Box h_{ab} - \nabla_a \nabla_b h' \right).\) 

\((\nabla_a \nabla_b \nabla_{cd})_L = \frac{1}{2} \nabla_a \nabla_b \left( \nabla^c \nabla_d h^{e}_{ce} + \nabla^d \nabla_a h^{e}_c + 2H^2 h^{ed}_d - 2H^2 h' \bar{g}^{ed} - \Box h^{cd} - \nabla^c \nabla_d h' \right).\) 

\((\nabla_a \nabla_b \nabla^{cd})_L = \frac{1}{2} \left( \nabla^e \nabla_a h^{d}_{ed} + \nabla_d \nabla_a h^{e}_c + 2H^2 h^{ed}_c - 2H^2 h' \bar{g}^{cd} - \Box h^{cd} - \nabla^e \nabla_d h' \right).\) 

\((\Box R^{c}_{\text{dab}})_L = \frac{1}{2} \left( \nabla_a (\nabla^c \nabla^b h^d_{bd} + \nabla_b \nabla^c h^d_{ab} - \nabla^{c} h_{ab}) - \nabla_b (\nabla^d h^e_{ca} + \nabla_a h^e_{db} - \nabla^{c} h_{ad}) \right).\) 

\((\Box R^{acde})_L = \frac{1}{2} \left( \nabla^d (\nabla^c h^e_{ae} + \nabla_e h^{ae} - \nabla^c h_{ae}) - \nabla_e (\nabla^c h_{ad} + \nabla_d h^e_{ca} - \nabla_{a} h_{cd}) \right) + g_{ec} h_{ad} - g_{cd} h^{ae}.\) 

\((R_{b}^{cde})_L = \frac{1}{2} \left[ \nabla^d (\nabla^e h^f_{b} + \nabla^f h^e_{b} - \nabla^{e} h_{b}) - \nabla^f (\nabla^d h^{e}_{b} + \nabla^d h_{b} - \nabla^{d} h_{b}) \right] + 2 \left( g_{e}^{f} h^{d}_{b} - g_{d}^{e} h^{f}_{b} \right).\) 

\((R^{cdef})_L = \frac{1}{2} \left[ \nabla^e \left( \nabla^d h^{f}_{b} + \nabla^f h^{d}_{b} - \nabla^{d} h^{f}_{b} \right) - \nabla^f \left( \nabla^d h^{e}_{b} + \nabla^d h_{b} - \nabla^{d} h_{b} \right) \right] + 2 \left( g^{e} h^{f}_{b} - g^{f} h^{e}_{b} \right).\) 

### B Details of derivations of Eq.(3.7)

In order to write the field equations in terms of the five dimensional ambient space notations, we need the following identities

\[ D_{\alpha} D_{\beta} K_{\gamma} = \bar{\partial}_{\alpha}(\bar{\partial}_{\gamma} K_{\beta}) - H^2 K_{\beta} g_{\alpha \beta} - H^2 x_{\beta} \bar{\partial}_{\alpha} K' - H^2 x_{\beta} (\bar{\partial}_{\gamma} K_{\alpha}), \]  

\[ \Box A K_{\alpha \beta} = D_{\gamma} D_{\gamma} K_{\alpha \beta} = -H^2 Q_{\alpha \beta}^{(1)} - 2H^2 K_{\alpha \beta} + 2H^2 S_{\alpha \beta} (\bar{\partial}_{\gamma} K_{\beta}) + 2H^2 x_{\beta} x_{\beta} K'_{\alpha \beta}, \]  

\[ D_{\alpha} D_{\beta} (\Box A K') = D_{\alpha} D_{\beta} (D_{\gamma} D_{\gamma} K') = -H^2 \bar{\partial}_{\alpha} \bar{\partial}_{\beta} Q_{\alpha \beta}^{(1)} + H^2 x_{\beta} \bar{\partial}_{\alpha} Q_{\alpha \beta}^{(1)} K', \]  

\[ D_{\alpha} D_{\beta} (D_{\gamma} D_{\lambda} K^{\gamma \lambda}) = \bar{\partial}_{\alpha} \bar{\partial}_{\beta} (\bar{\partial}_{\gamma} K_{\beta}) - H^2 x_{\beta} \bar{\partial}_{\alpha} \bar{\partial}_{\beta} (\bar{\partial}_{\gamma} K_{\beta}) - 4H^2 \bar{\partial}_{\alpha} \bar{\partial}_{\beta} K_{\beta} + 4H^4 x_{\beta} \bar{\partial}_{\alpha} K'_{\beta}, \]  

\[ D_{\alpha} D_{\beta} K'_{\gamma} = \bar{\partial}_{\alpha} \bar{\partial}_{\beta} K'_{\gamma} - H^2 x_{\beta} \bar{\partial}_{\alpha} K', \]  

\[ \Box A K' = D_{\alpha} D_{\alpha} K' - H^2 Q_{\alpha}^{(1)} K', \]  

\[ \Box A K_{\alpha} = (D_{\alpha} D_{\alpha})^2 K_{\alpha} = H^4 (Q_{\alpha}^{(1)})^2 K', \]  

\[ D_{\alpha} D_{\beta} K_{\alpha \beta} = \bar{\partial}_{\alpha}(\bar{\partial}_{\beta} K) - 4H^2 K', \]  

\[ H^{-4} \Box A K_{\alpha \beta} = H^{-4} (D_{\gamma} D_{\gamma})^2 K_{\alpha \beta} = (Q_{\alpha}^{(1)})^2 K_{\alpha \beta} + 4Q_{\alpha}^{(1)} K_{\alpha \beta} + 4K_{\alpha \beta} \]
Regarding Eqs.(B.12) and (B.14), the field equations (B.11) reduces to

\[ + S \left[ 4x_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + 8x_\alpha (\bar{\partial} K)_\beta - 14H^2 x_\alpha x_\beta K' - 4H^{-2} \bar{\partial}_\alpha (\bar{\partial} K)_\beta \right. \]
\[ + 4x_\alpha x_\beta \bar{\partial}_\beta (\bar{\partial} K) - 2H^{-2} x_\alpha x_\beta Q_0^{(1)} K' + 4x_\alpha \bar{\partial}_\beta K' + 2\theta_{\alpha \beta} K' \left. \right], \tag{B.9} \]

\[ \Box_A S \left( D_\alpha D_\gamma K_{\gamma}^{(1)} \right) = -H^2 S \left[ \bar{\partial}_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + 6\bar{\partial}_\alpha (\bar{\partial} K)_\beta - H^2 x_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + 2x_\alpha \bar{\partial}_\beta \bar{\partial}_\beta (\bar{\partial} K) \right. \]
\[ - 2H^2 x_\alpha x_\beta \bar{\partial}_\beta (\bar{\partial} K) - 12H^2 x_\alpha \bar{\partial}_\beta K' - 6H^2 x_\alpha (\bar{\partial} K)_\beta + 10H^4 x_\alpha x_\beta K' \]
\[ - H^2 \eta_{\alpha \beta} Q_0^{(1)} K' - 2H^2 \theta_{\alpha \beta} K' - H^2 x_\alpha \bar{\partial}_\beta Q_0^{(1)} K' + 2\theta_{\alpha \beta} K' \right]. \tag{B.10} \]

In obtaining the above identities, Eq.(3.2) has been used.

Now substituting Eqs.(B.1)-(B.10) in Eq.(3.10) and adopting the metric signature (+,−,−,−) results in

\[ Q_0^{(1)} (Q_0^{(1)} - 2) K_{\alpha \beta} + S \left[ 3x_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + H^{-2} \bar{\partial}_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta - 6x_\alpha (\bar{\partial} K)_\beta + 2H^2 x_\alpha x_\beta K' \right. \]
\[ - 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K)_\beta + 2x_\alpha x_\beta \bar{\partial}_\beta (\bar{\partial} K) - 2H^2 x_\alpha x_\beta Q_0^{(1)} K' - 4x_\alpha \bar{\partial}_\beta K' + 2H^{-2} x_\alpha \bar{\partial}_\beta \bar{\partial}_\beta (\bar{\partial} K) \]
\[ - \eta_{\alpha \beta} Q_0^{(1)} K' + 2H^{-2} \bar{\partial}_\alpha \bar{\partial}_\beta K' - x_\beta \bar{\partial}_\alpha Q_0^{(1)} K' + 4\eta_{\alpha \beta} K' \]
\[ - \frac{1}{3} \left( H^{-2} \bar{\partial}_\alpha \bar{\partial}_\beta - x_\beta \bar{\partial}_\alpha \right) \left[ 14K' + Q_0^{(1)} K' - 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K) \right. \]
\[ + \theta_{\alpha \beta} \left[ 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K) - 10K' + \frac{7}{3} Q_0^{(1)} K' - \frac{4}{3} H^{-2} Q_0^{(1)} (\bar{\partial} K) - \frac{1}{3} \left( Q_0^{(1)} \right)^2 K' \right]. \tag{B.11} \]

On the other hand, making use of Eq.(3.6) we can show that

\[ (Q_2^{(1)} + 4)(Q_2^{(1)} + 6) K_{\alpha \beta} = Q_0^{(1)} \left( Q_0^{(1)} - 2 \right) K_{\alpha \beta} + 4Q_0^{(1)} K' \eta_{\alpha \beta} + 8K' \eta_{\alpha \beta} \]
\[ + 4S \left[ - H^2 x_\alpha x_\beta K' + H^2 x_\alpha x_\beta \bar{\partial}_\beta (\bar{\partial} K) + x_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + x_\alpha \bar{\partial}_\beta K' - x_\alpha (\bar{\partial} K)_\beta - H^{-2} \bar{\partial}_\alpha (\bar{\partial} K)_\beta \right]. \tag{B.12} \]

Also one can show that

\[ (D_2 (\partial_2 K))_{\alpha \beta} = H^{-2} S \left[ - H^2 x_\alpha \bar{\partial}_\beta K' - H^2 x_\alpha (\bar{\partial} K)_\beta + \bar{\partial}_\alpha (\bar{\partial} K)_\beta \right. \]
\[ - 2K' \eta_{\alpha \beta} - H^{-2} \bar{\partial}_\alpha \bar{\partial}_\beta K' + x_\beta \bar{\partial}_\alpha K', \tag{B.13} \]

and

\[ (Q_2^{(1)} + 4) (D_2 (\partial_2 K))_{\alpha \beta} = S \left[ - x_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta + H^{-2} \bar{\partial}_\alpha Q_0^{(1)} (\bar{\partial} K)_\beta - 2x_\alpha (\bar{\partial} K)_\beta + 6H^2 x_\alpha x_\beta K' \right. \]
\[ + 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K)_\beta - 2x_\alpha x_\beta \bar{\partial}_\beta (\bar{\partial} K) - 2H^2 x_\alpha x_\beta Q_0^{(1)} K' - 6x_\alpha \bar{\partial}_\beta K' + 2H^{-2} x_\alpha \bar{\partial}_\beta \bar{\partial}_\beta (\bar{\partial} K) \]
\[ - 3\eta_{\alpha \beta} Q_0^{(1)} K' - x_\alpha \bar{\partial}_\beta Q_0^{(1)} K' \right] - \theta_{\alpha \beta} \left[ 12K' - 4Q_0^{(1)} K' \right] + \left( - H^{-2} \bar{\partial}_\alpha \bar{\partial}_\beta + x_\beta \bar{\partial}_\alpha \right) Q_0^{(1)} K'. \tag{B.14} \]

Regarding Eqs.(B.12) and (B.14), the field equations (B.11) reduces to

\[ (Q_2^{(1)} + 4) \left[ (Q_2^{(1)} + 6) K_{\alpha \beta} + (D_2 (\partial_2 K))_{\alpha \beta} \right. \]
\[ - \frac{1}{3} \left( - H^{-2} \bar{\partial}_\alpha \bar{\partial}_\beta + x_\beta \bar{\partial}_\alpha \right) \left[ 2Q_0^{(1)} K' + 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K) - 2K' \right. \]
\[ + \theta_{\alpha \beta} \left[ 2H^{-2} \bar{\partial}_\alpha (\bar{\partial} K) + 2K' - \frac{5}{3} Q_0^{(1)} K' - \frac{1}{3} H^{-2} Q_0^{(1)} (\bar{\partial} K) - \frac{1}{3} \left( Q_0^{(1)} \right)^2 K' \right] = 0. \tag{B.15} \]
Now, Eq. (B.15) can be rewritten as
\[ T_{\alpha\beta} + T_{\alpha\beta} - \frac{1}{4} \theta_{\alpha\beta} (T' + T') = 0, \]  
(B.16)
where
\[ T_{\alpha\beta} = (Q_2^{(1)} + 4) \left[ (Q_2^{(1)} + 6) K_{\alpha\beta} + (D_2(\partial_2 K))_{\alpha\beta} \right], \]  
(B.17)
\[ T' = \text{trace} [T_{\alpha\beta}] = 2 \left( Q_0^{(1)} \right)^2 K' + 6Q_0^{(1)} K' - 8K' - 8H^{-2} \tilde{\partial}.(\tilde{\partial}.K) + 2H^{-2} Q_0^{(1)} \tilde{\partial}.(\tilde{\partial}.K), \]  
(B.18)
\[ \mathcal{T}_{\alpha\beta} = D_{\alpha\beta} T, \quad D_{\alpha\beta} = -1/3 \left( -H^{-2} \tilde{\partial}_a \tilde{\partial}_\beta + x_\beta \tilde{\partial}_a \right) \text{is a symmetric tensor operator,} \]  
(B.19)
\[ \mathcal{T} = \text{trace} \left[ (Q_2^{(1)} + 6) K_{\alpha\beta} + (D_2(\partial_2 K))_{\alpha\beta} \right] = 2Q_0^{(1)} K' - 2K' + 2H^{-2} \tilde{\partial}.(\tilde{\partial}.K), \]  
(B.20)
\[ \mathcal{T}' = \text{trace} [\mathcal{T}_{\alpha\beta}] = -\frac{2}{3} \left[ \left( Q_0^{(1)} \right)^2 K' - Q_0^{(1)} K' + H^{-2} Q_0^{(1)} \tilde{\partial}.(\tilde{\partial}.K) \right]. \]  
(B.21)

Now, the field equations (B.16) can be written as
\[ \mathcal{O} \left[ (Q_2^{(1)} + 4) (Q_2^{(1)} + 6) K_{\alpha\beta} + (Q_2^{(1)} + 4) (D_2(\partial_2 K))_{\alpha\beta} \right] = 0, \]  
(B.22)
where \( \mathcal{O} \) is an operator with the operation
\[ \mathcal{O} T_{\alpha\beta} = T_{\alpha\beta} + T_{\alpha\beta} - \frac{1}{4} \theta_{\alpha\beta} (T' + T'). \]  
(B.23)

Since \( \mathcal{O} \) has an inverse Eq.(B.22) is nothing but Eq.(3.7).

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