Singular chains on Lie groups and the Cartan relations II

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Abstract

Let $G$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and denote by $C_\ast(G)$ the DG Hopf algebra of smooth singular chains on $G$. In a companion paper it was shown that the category of sufficiently smooth modules over $C_\ast(G)$ is equivalent to the category of representations of $\mathbb{T}_\mathfrak{g}$, the DG Lie algebra which is universal for the Cartan relations. In this paper we show that, if $G$ is compact, this equivalence of categories can be extended to an $A_\infty$-quasi-equivalence of the corresponding DG categories. As an intermediate step we construct an $A_\infty$-quasi-isomorphism between the Bott-Shulman-Stasheff DG algebra associated to $G$ and the DG algebra of Hochschild cochains on $C_\ast(G)$. The main ingredients in the proof are the Van Est map and Gugenheim’s $A_\infty$ version of De Rhams theorem.

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1 Introduction

This paper is a continuation of previous work by the first named author [5], where an infinitesimal description of the category of modules over the algebra of singular chains on a Lie group is presented. Our main result is that in the compact case, the correspondence of [5] can be promoted to an $A_{\infty}$ quasi-equivalence of DG categories.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and denote by $C_\bullet(G)$ the space of smooth singular chains on $G$. This space carries the structure of a DG Hopf algebra, where the product is induced by the Eilenberg-Zilber map and the coproduct by the Alexander-Whitney map. We write $\text{Mod}(C_\bullet(G))$ for the category of sufficiently smooth modules over this DG Hopf algebra. We also denote by $T\mathfrak{g}$ the DG Lie algebra which is universal for the Cartan relations on $\mathfrak{g}$, and write $\text{Rep}(T\mathfrak{g})$ for the corresponding category of representations. The main result of [5] is the following.

Theorem. Suppose that $G$ is a simply connected Lie group. There exists a differentiation functor $\mathcal{D}: \text{Mod}(C_\bullet(G)) \to \text{Rep}(T\mathfrak{g})$ and an integration functor $\mathcal{I}: \text{Rep}(T\mathfrak{g}) \to \text{Mod}(C_\bullet(G))$ which are inverses to one another. In particular, the categories $\text{Mod}(C_\bullet(G))$ and $\text{Rep}(T\mathfrak{g})$ are equivalent as symmetric monoidal categories.

Let us now explain the content of the present work. The category $\text{Mod}(C_\bullet(G))$ admits an enhancement to a DG category, which we shall denote by $\text{DGMod}(C_\bullet(G))$, and whose spaces of morphisms are Hochschild complexes for $C_\bullet(G)$. Similarly, the category $\text{Rep}(T\mathfrak{g})$ may be enhanced to a DG category, which we denote by $\text{DGRep}(T\mathfrak{g})$, and whose spaces of morphisms are constructed in terms of the Weil algebra of $\mathfrak{g}$. Our main result is the following.

Theorem A. Suppose that $G$ is compact and simply connected. There exists a zig-zag of $A_{\infty}$-quasi-equivalences that connects $\text{DGRep}(T\mathfrak{g})$ to $\text{DGMod}(C_\bullet(G))$. In particular, the DG categories $\text{DGRep}(T\mathfrak{g})$ and $\text{DGMod}(C_\bullet(G))$ are $A_{\infty}$-quasi-equivalent.

For the proof we introduce an intermediate DG category $\text{BSS}(G)$ and an invariant version of it $\text{BSS}^G(G)$, whose morphism spaces are defined by twisting the Bott-Shulman-Stasheff DG algebra introduced in [10]. The following diagram, where each arrow represents an $A_{\infty}$ equivalence of DG categories, summarizes the structure of the paper:

\[
\begin{array}{ccc}
\text{DGRep}(T\mathfrak{g}) & \xrightarrow{\mathcal{D}} & \text{BSS}(G) \\
\text{BSS}^G(G) & \xleftarrow{\mathcal{I}} & \text{DGMod}(C_\bullet(G)).
\end{array}
\]

The second arrow is an inclusion of categories which is a quasi-equivalence when $G$ is compact.

The comparison between $\text{BSS}^G(G)$ and $\text{DGRep}(T\mathfrak{g})$ uses the Van Est map [26, 6, 19, 14, 27, 22] and the noncommutative Weil algebra of Alekseev and Meinrenken [1, 3]. We prove the following results.
Theorem B. Let $G$ be a compact and simply connected Lie group. There exists a DG functor

$$\forall \mathcal{E} : \text{BSS}^G(G) \to \text{DGRep}(\mathfrak{g})$$

which is a quasi-equivalence.

In order to compare $\text{BSS}(G)$ and $\text{DGMod}(C_\bullet(G))$, we construct an $A_\infty$ quasi-isomorphism between the Bott-Shulman-Stasheff algebra and the algebra of Hochschild cochains on singular chains on $G$.

Theorem C. Let $G$ be a Lie group. There is an $A_\infty$-morphism

$$\text{DR}^\Theta : \operatorname{Tot}(\Omega^\bullet(BG)) \to \text{HC}^\bullet(C_\bullet(G))$$

which is a quasi-isomorphism.

This construction uses Chen’s iterated integrals and combines a version of Gugenheim’s $A_\infty$ De Rham’s theorem for the classifying space $BG$ with the Eilenberg-Zilber map.

Theorem D. Let $G$ be a Lie group. There exists an $A_\infty$-functor

$$\mathcal{B}R : \text{BSS}(G) \to \text{DGMod}(C_\bullet(G)),$$

which is an $A_\infty$-quasi-equivalence of DG categories.

The $A_\infty$-quasi-equivalence of DG categories between $\text{DGRep}(\mathfrak{g})$ and $\text{DGMod}(C_\bullet(G))$ may be interpreted in terms of Chern-Weil theory for $\infty$-local systems on the classifying space $BG$, as studied in [9, 23, 17, 2, 8, 11]. Indeed, $\infty$-local systems on a topological space $X$ are described as objects of the DG category $\text{DGMod}(C_\bullet(\Omega X))$, where $\Omega X$ denotes the Moore based loop space of $X$. In case $X$ is $BG$, the monoid of Moore loops on $BG$ is $A_\infty$ equivalent to $G$. Thus, the DG category $\text{DGMod}(C_\bullet(G))$ can be thought of as that which parametrises $\infty$-local systems on $BG$. The $A_\infty$-quasi-equivalence between $\text{DGRep}(\mathfrak{g})$ and $\text{DGMod}(C_\bullet(G))$ is then an extension of the Chern-Weil computation of the cohomology of $BG$ for the trivial $\infty$-local system to the case of arbitrary $\infty$-local systems. The explicit construction of a Chern-Weil DG functor categorifying the Chern-Weil homomorphism is the subject of a forthcoming work [7].

The structure of the paper is as follows. In Section 2 we collect some preliminaries on DG categories, Hochschild complexes, Gugenheim’s $A_\infty$ De Rham theorem, the Alexander-Whitney and Eilenberg-Zilber maps, representations of the DG Lie algebra $\mathfrak{g}$, and the main result of [3] concerning the equivalence between $\text{Rep}(\mathfrak{g})$ and $\text{Mod}(C_\bullet(G))$. Section 3 is devoted to the study of the properties of the Van Est map that are used in the proof of our main results, and the construction of the $A_\infty$-quasi-isomorphism between the Bott-Shulman-Stasheff DG algebra $\Omega^\bullet(BG)$ and the DG algebra of Hochschild cochains $\text{HC}^\bullet(C_\bullet(G))$. We conclude in Section 4 with a discussion of the DG enhanced categories $\text{DGRep}(\mathfrak{g})$ and $\text{BSS}(G)$ and the proof of Theorems B and C, which together imply our main result, Theorem A. We also present some examples.

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**Notation and Conventions**

All vector spaces and algebras are defined over the field of real numbers $\mathbb{R}$. If $V = \bigoplus_{k \in \mathbb{Z}} V^k$ is a graded vector space, we denote by $sV$ its suspension, that is, the graded vector space with grading defined by

$$(sV)^k = V^{k+1},$$

and by $uV$ its unsuspension, that is, the graded vector space with grading defined by

$$(uV)^k = V^{k-1}.$$

All our complexes will be cochain complexes, meaning that the differentials increase the degree by one. For each $n \geq 1$, we write $\Delta_n$ for the standard $n$-simplex. The geometric realisation of $\Delta_n$ that we take is

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 1 \geq t_1 \geq \cdots \geq t_n \geq 0\}.$$

If $M$ is a smooth manifold, we respectively denote by $\Omega^\bullet(M)$, $C^\bullet(M)$ and $C_\cdot(M)$ the spaces of differential forms, smooth singular cochains and smooth singular chains defined on $M$.

## 2 Preliminaries

In this section, we review the basic definitions and results that will be needed in the sequel, in an attempt at making our paper as self-contained as possible. For a more detailed exposition on some of the topics covered in Sections 2.1, 2.2, 2.4 and 2.5 the reader may consult [25], [20] and [21].

### 2.1 Hochschild chain and cochain complexes

Let $A$ be a DG algebra and let $M$ be a DG bimodule over $A$. The **Hochschild chain complex** of $A$ with values in $M$ is the graded vector space

$$\text{HC}^\bullet(A, M) = \bigoplus_{n \geq 0} M \otimes (uA)^{\otimes n}, \quad (2.1)$$

equipped with a differential $b$ which is the sum of two components $b_1$ and $b_2$ defined by the formulas

$$b_1(m \otimes ua_1 \otimes \cdots \otimes ua_n) = d_M m \otimes ua_1 \otimes \cdots \otimes ua_n$$

$$+ \sum_{i=1}^{n} (-1)^{|m| + \sum_{j=1}^{i-1} |a_j| - i} m \otimes ua_1 \otimes \cdots \otimes ua_{i-1} \otimes uda_i \otimes ua_{i+1} \otimes \cdots \otimes ua_n, \quad (2.2)$$

4
and
\[
b_2(m \otimes ua_0 \otimes \cdots \otimes ua_n) = (-1)^{|m|+1} ma_1 \otimes ua_2 \otimes \cdots \otimes ua_n \\
+ \sum_{i=1}^{n-1} (-1)^{|m|+\sum_{j=1}^{i} |a_j|-i+1} m \otimes ua_1 \otimes \cdots \otimes ua_{i-1} \otimes u(a_ia_{i+1}) \otimes ua_{i+2} \otimes \cdots \otimes ua_n \qquad (2.3)
+ (-1)^{|m|+\sum_{j=1}^{n-1} |a_j|-n-1} a_n m \otimes ua_1 \otimes \cdots \otimes ua_{n-1},
\]
for homogeneous elements \( m \in M \) and \( a_1, \ldots, a_n \in A \). The resulting cohomology is called the *Hochschild homology of \( A \) with values in \( M \). In the special case where \( M = \mathbb{R} \) is the trivial bimodule, we shall write \( \text{HC}_*(A) \) instead of \( \text{HC}_*(A, \mathbb{R}) \).

The *Hochschild cochain complex* of \( A \) with values in \( M \) is the cochain complex
\[
\text{HC}^*(A, M) = \text{Hom}(\text{HC}_*(A), M) = \bigoplus_{n \geq 0} \text{Hom}((uA)^{\otimes n}, M), \tag{2.4}
\]
with differential \( b \) characterised by
\[
(b\varphi)(ua_1 \otimes \cdots \otimes ua_n) = d_M(\varphi(ua_1 \otimes \cdots \otimes ua_n)) - (-1)^{|\varphi|}\varphi(b(ua_1 \otimes \cdots \otimes ua_n)) \\
+ (-1)^{|\varphi|(|a_1|+1)} a_1 \varphi(ua_2 \otimes \cdots \otimes ua_n) - (-1)^{|\varphi|+\sum_{j=1}^{n-1} |a_j|+n-1} \varphi(ua_1 \otimes \cdots \otimes ua_{n-1})a_n, \tag{2.5}
\]
for homogeneous elements \( \varphi \in \text{Hom}((uA)^{\otimes n}, M) \) and \( a_1, \ldots, a_n \in A \). The resulting cohomology is called the *Hochschild cohomology of \( A \) with values in \( M \). In case \( M \) is the trivial module \( \mathbb{R} \), we will write \( \text{HC}^*(A) \) instead of \( \text{HC}^*(A, M) \).

A case of special interest arises in the following way. Let \( V, V' \) and \( V'' \) be DG modules over \( A \), so that the hom-complexes \( \text{Hom}(V, V') \) and \( \text{Hom}(V', V'') \) are naturally DG bimodules over \( A \). Then there is a cup product
\[
\cup : \text{HC}^*(A, \text{Hom}(V', V'')) \otimes \text{HC}^*(A, \text{Hom}(V, V')) \longrightarrow \text{HC}^*(A, \text{Hom}(V, V'')),
\]
which is defined by
\[
(\psi \cup \varphi)(ua_1 \otimes \cdots \otimes uam_{m+n}) = (-1)^{|\psi|(|\sum_{i=1}^{m} |a_i|-m|)\psi(ua_1 \otimes \cdots \otimes uam) \circ \varphi(uam_{m+1} \otimes \cdots \otimes uam_{m+n}), \tag{2.6}
\]
for homogeneous elements \( \varphi \in \text{HC}^*(A, \text{Hom}(V, V')) \), \( \psi \in \text{HC}^*(A, \text{Hom}(V', V'')) \) and \( a_1, \ldots, a_{m+n} \in A \). This cup product is compatible with the differential \( b \) in the sense that it satisfies the Leibniz rule. Given a DG algebra \( A \), one can form the DG category of DG modules over \( A \), which we denote by \( \text{DGMod}(A) \). Its objects are, of course, DG modules over \( A \). For any two such objects \( V \) and \( V' \), the space of morphisms is the complex \( \text{HC}^*(A, \text{Hom}(V, V')) \) and the composition law is the cup product.

### 2.2 DG categories, DG functors and \( A_\infty \)-functors

A *DG category* (where DG stands for “differential graded”) over a field \( K \) is a \( K \)-linear category \( \mathcal{C} \) such that for every two objects \( X \) and \( Y \) the space of arrows \( \text{Hom}_K(X, Y) \) is equipped with a structure of a cochain complex of \( K \)-vector spaces, and for every three objects \( X, Y \) and \( Z \) the
composition map $\text{Hom}_c(Y, Z) \otimes_K \text{Hom}_c(X, Y) \to \text{Hom}_c(X, Z)$ is a morphism of cochain complexes. Thus, by definition,

$$
\text{Hom}_c(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_c^n(X, Y)
$$

is a graded $K$-vector space with a differential $d$: $\text{Hom}_c^n(X, Y) \to \text{Hom}_c^{n+1}(X, Y)$. The elements $f \in \text{Hom}_c^n(X, Y)$ are called homogeneous of degree $n$, and we write $|f| = n$. We shall denote the set of objects of $\mathcal{C}$ by $\text{Ob}\mathcal{C}$.

The fundamental example of a DG category is the category of cochain complexes of $K$-vector spaces, which we denote by $\text{DGVect}_K$. Its objects are cochain complexes of $K$-vector spaces and the morphism spaces $\text{Hom}_{\text{DGVect}_K}(X, Y)$ are endowed with the differential defined as

$$
d(f) = d_Y \circ f - (-1)^{n} f \circ d_X,
$$

for any homogeneous element $f$ of degree $n$.

Given a DG category $\mathcal{C}$ one defines an ordinary category $\text{Ho}(\mathcal{C})$ by keeping the same set of objects and replacing each Hom complex by its 0th cohomology. We call $\text{Ho}(\mathcal{C})$ the homotopy category of $\mathcal{C}$. If $\mathcal{C}$ and $\mathcal{D}$ are DG categories, a DG functor $F: \mathcal{C} \to \mathcal{D}$ is a $K$-linear functor whose associated map for $X, Y \in \text{Ob}\mathcal{C}$,

$$
F_{X,Y}: \text{Hom}_c(X, Y) \to \text{Hom}_D(F(X), F(Y)),
$$

is a morphism of cochain complexes. Notice that any DG functor $F: \mathcal{C} \to \mathcal{D}$ induces an ordinary functor

$$
\text{Ho}(F): \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})
$$

between the corresponding homotopy categories. A DG functor $F: \mathcal{C} \to \mathcal{D}$ is said to be quasi fully faithful if for every pair of objects $X, Y \in \text{Ob}\mathcal{C}$ the morphism $F_{X,Y}$ is a quasi-isomorphism. Moreover, the DG functor $F$ is said to be quasi essentially surjective if $\text{Ho}(F)$ is essentially surjective. A DG functor which is both quasi fully faithful and quasi essentially surjective is called a quasi-equivalence.

There is a more general notion of functor between DG categories, that of an $A_{\infty}$-functor, where the composition is preserved only up to an infinite sequence of coherence conditions. It will be useful to introduce first the Hochschild chain complex of a DG category.

Let $\mathcal{C}$ be a small DG category. The Hochschild cochain complex of $\mathcal{C}$ is the complex

$$
\bigoplus_{X_0, \ldots, X_n} s\text{Hom}_c(X_{n-1}, X_n) \otimes_K \cdots \otimes_K s\text{Hom}_c(X_0, X_1),
$$

where $X_0, \ldots, X_n$ range through the objects of $\mathcal{C}$, and whose differential $b$ is the sum of two components $b_1$ and $b_2$ given by the formulas

$$
b_1(f_{n-1} \otimes \cdots \otimes f_0) = \sum_{i=0}^{n-1} (-1)^{\sum_{j=i+1}^{n-1} |f_j| + n - i - 1} f_{n-1} \otimes \cdots \otimes df_i \otimes \cdots \otimes f_0
$$

and

$$
b_2(f_{n-1} \otimes \cdots \otimes f_0) = \sum_{i=0}^{n-2} (-1)^{\sum_{j=i+2}^{n-1} |f_j| + n - i - 1} f_{n-1} \otimes \cdots \otimes (f_{i+1} \circ f_i) \otimes \cdots \otimes f_0
$$
for homogeneous elements \( f_0 \in \text{shom}_\mathcal{C}(X_0, X_1), \ldots, f_{n-1} \in \text{shom}_\mathcal{C}(X_{n-1}, X_n) \). Here \( d \) denotes indistinctly the differential in any of the spaces \( \text{Hom}_\mathcal{C}(X_i, X_{i+1}) \). It is easy to check that indeed \( b^2 = 0 \), by cancellation of terms with opposite signs.

With this in mind, the formal definition of an \( A_\infty \)-functor is given as follows. Let \( \mathcal{C} \) and \( \mathcal{D} \) be DG categories. An \( A_\infty \)-functor \( F: \mathcal{C} \to \mathcal{D} \) is the datum of a map of sets \( F_0: \text{Ob}\mathcal{C} \to \text{Ob}\mathcal{D} \) and a collection of \( K \)-linear maps of degree 0

\[
F_n: \text{shom}_\mathcal{C}(X_{n-1}, X_n) \otimes_K \cdots \otimes_K \text{shom}_\mathcal{C}(X_0, X_1) \to \text{Hom}_\mathcal{D}(F_0(X_0), F_0(X_n))
\]

for every collection \( X_0, \ldots, X_n \in \text{Ob}\mathcal{C} \), such that the relation

\[
b_1 \circ F_n + \sum_{i+j=n} b_2 \circ (F_i \otimes F_j) = \sum_{i+j=n} F_n \circ (id^{\otimes i} \otimes b_1 \otimes id^{\otimes j}) + \sum_{i+j+2=n} F_{n-1} \circ (id^{\otimes i} \otimes b_2 \otimes id^{\otimes j})
\]

is satisfied for any \( n \geq 1 \). One also requires that \( F_1(id_X) = id_{F_0(X)} \) for all objects \( A \) in \( \mathcal{C} \), as well as \( F_n(f_{n-2} \otimes \cdots \otimes f_1 \otimes id_X \otimes f_{i-1} \otimes \cdots \otimes f_0) = 0 \) for any \( n \geq 1 \), any \( 0 \leq i \leq n-2 \), and any chain of morphisms \( f_0 \in \text{shom}_\mathcal{C}(X_0, X_1), \ldots, f_{n-2} \in \text{shom}_\mathcal{C}(X_{n-2}, X_n) \).

The above relation when \( n = 1 \) implies that \( F_1 \) is a morphism of cochain complexes. On the other hand, for \( n = 2 \) we find that \( F_1 \) preserves the compositions on \( \mathcal{C} \) and \( \mathcal{D} \), up to a homotopy defined by \( F_2 \). In particular, a DG functor between \( \mathcal{C} \) and \( \mathcal{D} \) is the same as an \( A_\infty \)-functor such that \( F_n = 0 \) for \( n \geq 2 \). It also follows that \( F_1 \) induces and ordinary functor

\[
\text{Ho}(F_1): \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D}).
\]

An \( A_\infty \)-functor \( F: \mathcal{C} \to \mathcal{D} \) is called \( A_\infty \)-quasi fully faithfull if \( F_1 \) is a quasi-isomorphism, and it is called \( A_\infty \)-quasi essentially surjective if \( \text{Ho}(F_1) \) is essentially surjective. Finally, an \( A_\infty \)-functor \( F \) is called a \( A_\infty \)-quasi-equivalence if it is both quasi fully faithfull and quasi essentially surjective. We say that two DG categories are \( A_\infty \) equivalent if they can be connected by a zig-zag of \( A_\infty \) quasi-equivalences.

### 2.3 Gugenheim’s \( A_\infty \) De Rham theorem

The usual De Rham map, which sends a differential form to a singular cochain by integration, is not an algebra map. However, it induces an isomorphism of algebras in cohomology. A more complete explanation of this fact is due to Gugenheim. In [16], this author uses Chen’s iterated integrals [13] to extend the De Rham map to an \( A_\infty \)-quasi-isomorphism of DG algebras. Here we will review this construction, which will be needed later. We follow the presentation in [1].

Let us start with some background. For a smooth manifold \( M \) we denote by \( \mathcal{P}M \) the path space of \( M \), that is, the space of all smooth maps from \( I \) to \( M \) which we regard as a diffeological space. Given another manifold \( X \), one says that a map \( f: X \to \mathcal{P}M \) is smooth if the map \( \hat{f}: [0,1] \times X \to M \) defined for any \( t \in I \) and \( x \in X \) by

\[
\hat{f}(t,x) = f(x)(t),
\]

is smooth. With this in mind, we may define differential forms on \( \mathcal{P}M \) as follows. We first consider the category \( C^\infty(-,\mathcal{P}M) \) whose objects are pairs \((X,f)\) where \( X \) is a smooth manifold and \( f \)
is a smooth map from $X$ to $\mathcal{P}M$ and whose morphisms from one such pair $(X, f)$ to another $(Y, g)$ are smooth maps $h: X \to Y$ such that $f = g \circ h$. Next, if $\textbf{Vect}_R$ denotes the category of real vector spaces, we consider the functor $R(-)$ from $C^\infty(-, \mathcal{P}M)$ to $\textbf{Vect}_R$ which sends any object in $C^\infty(-, \mathcal{P}M)$ to $R$ and every morphism to the identity, along with the functor $\Omega^*(-)$ from $C^\infty(-, \mathcal{P}M)$ to $\textbf{Vect}_R$ sending an object $(X, f)$ to $\Omega^*(X)$ and a morphism $h$ to its pullback $h^*$. Then, a differential form on $\mathcal{P}M$ is a natural transformation from $R(-)$ to $\Omega^*(-)$. This definition simply means that we declare a differential form on $\mathcal{P}M$ to be determined by its pullback along smooth maps from a smooth manifold. We shall now explain Chen’s iterated integrals taking values on differential forms on the path space $\mathcal{P}M$. First we need the following piece of notation. If $\Delta_n$ denotes the $n$-simplex, we write $ev: \Delta_n \times \mathcal{P}M \to M^n$ for the evaluation map defined as

$$ev((t_1, \ldots, t_n), \gamma) = (\gamma(t_1), \ldots, \gamma(t_n)),$$

for $(t_1, \ldots, t_n) \in \Delta_n$ and $\gamma \in \mathcal{P}M$. Further, we let $p_i$ stand for the $i$-th projection from $M^n$ to $M$ for any $i = 1, \ldots, n$, and $\pi$ for the projection from $\Delta_n \times M$ to $M$. Then, Chen’s map $C: (\mathcal{S}\Omega^*(M))^{\otimes n} \to \Omega^*(\mathcal{P}M)$ is defined by setting

$$C(\omega_1 \otimes \cdots \otimes \omega_n) = (-1)^{\sum_{j=1}^n (n-j)} \pi_*(ev^*(p_1^*\omega_1 \wedge \cdots \wedge p_n^*\omega_n)),$$

for homogeneous elements $\omega_1, \ldots, \omega_n \in \mathcal{S}\Omega^*(M)$, where here $\pi_*: \Omega^*(\Delta_n \times M) \to \Omega^*(M)$ denotes the pushforward along the projection $\pi$. Besides Chen’s map, Gugenheim’s construction uses some combinatorial maps that we now describe. For each $n \geq 1$, let $\lambda_n: I^{n-1} \to \mathcal{P}I^n$ be the map that sends every element $(s_1, \ldots, s_{n-1})$ of $I^{n-1}$ to the piecewise linear path which goes backwards through the $n+1$ points

$$0 \leftarrow s_1 \leftarrow s_1 + s_2 \leftarrow \cdots \leftarrow s_1 + \cdots + s_{n-1} \leftarrow s_1 + \cdots + s_{n-1} + e_n,$$

with $e_1, \ldots, e_n$ being the standard basis of $\mathbb{R}^n$, and $\pi_n: I^n \to \Delta_n$ the map given by

$$\pi_n(s_1, \ldots, s_n) = (t_1, \ldots, t_n),$$

for $(s_1, \ldots, s_n) \in I^n$, with $t_i = \max\{s_i, \ldots, s_n\}$ for any $i = 1, \ldots, n$. We then obtain, for each $n \geq 1$, a map $\theta_n: I^{n-1} \to \mathcal{P}\Delta_n$ which is defined as the composition

$$I^{n-1} \xrightarrow{\lambda_n} \mathcal{P}I^n \xrightarrow{\mathcal{P}\pi_n} \mathcal{P}\Delta_n,$$

where $\mathcal{P}\pi_n$ is the map induced on path spaces by $\pi_n$. We also, by convention, set $\theta_0$ to be the map from a point to a point.

Using the above notation, we consider the map $S: \Omega^*(\mathcal{P}M) \to \mathcal{S}\Omega^*(M)$ from the de Rham complex of the path space $\mathcal{P}M$ to the unsuspension of $\Omega^*(M)$, obtained as the composition of the map $\Omega^*(\mathcal{P}M) \to C^*(\mathcal{P}M)$ given, for each $\varphi \in \Omega^*(\mathcal{P}M)$, by

$$(\sigma: \Delta_n \to M) \mapsto \int_{I^{n-1}} \theta_n^*\mathcal{P}\sigma^*\varphi \in \mathbb{R},$$

\footnote{In the case when $M$ is compact and oriented, the pushforward $\pi_*$ is characterized by the property that

$$\int_M \pi_*\omega = \int_{\Delta_n \times M} \omega,$$

for all $\omega \in \Omega^*(\Delta_n \times M)$.}
followed by the unsuspension \( s: C^*(M) \to sC^*(M) \). We then proceed to define, for \( n \geq 1 \), a sequence of linear maps \( DR_n : (s\Omega^*(M))^\otimes n \to sC^*(M) \) in the following way. For \( n = 1 \), we set

\[
DR_1(\omega)(\sigma) = \int_{\Delta_k} \sigma^* \omega, \tag{2.11}
\]

for \( \omega \in s\Omega^*(M) \) and \( \sigma \in C_k(M) \). For \( n > 1 \), we set

\[
DR_n(\omega_1 \otimes \cdots \otimes \omega_n) = (-1)^{\sum_{j=1}^n |\omega_j| + n}(S \circ C)(\omega_1 \otimes \cdots \otimes \omega_n), \tag{2.12}
\]

for homogeneous elements \( \omega_1, \ldots, \omega_n \in s\Omega^*(M) \). In term of these, we are now in position to state Gugenheim’s main result.

**Theorem 2.1.** For \( n \geq 1 \), the sequence of maps \( DR_n : (s\Omega^*(M))^\otimes n \to sC^*(M) \) determines an \( A_\infty \)-morphism from the de Rham complex of \( M \) to the singular cochain complex of \( M \), both viewed as DG algebras. Moreover, this \( A_\infty \)-morphism is an \( A_\infty \)-quasi-isomorphism which is natural with respect to pullbacks along smooth maps.

### 2.4 Alexander-Whitney and Eilenberg-Zilber maps

In this subsection we recall the definitions of the Alexander-Whitney and Eilenberg-Zilber maps. These will enable us to give the singular chain complex of a Lie group the structure of a DG Hopf algebra. We begin with the Alexander-Whitney map. For \( p \leq n \), the inclusions of the standard \( p \)-simplex as the front and the back \( p \)-th face of the standard \( n \)-simplex will be denoted respectively by \( f_p^n : \Delta_p \to \Delta_n \) and \( b_p^n : \Delta_p \to \Delta_n \). Explicitly,

\[
f_p^n(t_1, \ldots, t_p) = (t_1, \ldots, t_p, 0, \ldots, 0),
\]

\[
b_p^n(t_1, \ldots, t_p) = (1, \ldots, 1, t_1, \ldots, t_p). \tag{2.13}
\]

Also, for two fixed smooth manifolds \( X \) and \( Y \), we let \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \) denote the two natural projections. Then, the Alexander-Whitney map \( AW : C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \) is the chain map given, for each singular \( n \)-simplex \( \sigma : \Delta_n \to X \times Y \), by the formula

\[
AW(\sigma) = \sum_{p+q=n} (\sigma_1 \circ f_p^n) \otimes (\sigma_2 \circ b_q^n), \tag{2.14}
\]

where \( \sigma_1 = \pi_1 \circ \sigma \) and \( \sigma_2 = \pi_2 \circ \sigma \). For us, the most important property of this map is that it allows us to define a DG coalgebra structure on the space of singular chains \( C_*(X) \). The coproduct \( \Delta : C_*(X) \to C_*(X) \otimes C_*(X) \) is formed by composition of the map \( C_*(X) \to C_*(X \times X) \) induced by the diagonal \( X \to X \times X \) with the Alexander-Whitney map \( AW : C_*(X \times X) \to C_*(X) \otimes C_*(X) \). The counit \( \varepsilon : C_*(X) \to \mathbb{R} \) is induced by the projection map which collapses \( X \) to a point.

Now we turn to the Eilenberg-Zilber map. Such a map is based on the simple fact that a cube of dimension \( n \) is the union of \( n! \) simplices.
We fix again two smooth manifolds $X$ and $Y$. The Eilenberg-Zilber map $\text{EZ}: C_\ast(X) \otimes C_\ast(Y) \to C_\ast(X \times Y)$ is the chain map given, for each singular $m$-simplex $\sigma: \Delta_m \to X$ and each singular $n$-simplex $\tau: \Delta_n \to Y$, by the formula

$$\text{EZ}(\sigma \otimes \tau) = \sum_{\chi \in \mathcal{C}_{m,n}} (-1)^\chi (\sigma \times \tau) \circ \chi_s,$$

(2.15)

where, as the notation implies, the sum over $\chi$ is taken over all $(m,n)$ shuffles and $\chi_s: \Delta_{m+n} \to \Delta_m \times \Delta_n$ is the map defined by

$$\chi_s(t_1, \ldots, t_{m+n}) = ((t_{\chi(1)}, \ldots, t_{\chi(m)}), (t_{\chi(m+1)}, \ldots, t_{\chi(m+n)}))$$

(2.16)

We state without proof the key properties of the Eilenberg-Zilber map (see [20] and [15]).

**Proposition 2.2.** The Eilenberg-Zilber map $\text{EZ}: C_\ast(X) \otimes C_\ast(Y) \to C_\ast(X \times Y)$ satisfies:

1. It is associative, that is, given a third smooth manifold $Z$, the following diagram commutes

\[
\begin{array}{ccc}
C_\ast(X) \otimes C_\ast(Y) \otimes C_\ast(Z) & \xrightarrow{id \otimes \text{EZ}} & C_\ast(X \times Y) \otimes C_\ast(Z) \\
\downarrow \text{EZ} & & \downarrow \text{EZ} \\
C_\ast(X) \otimes C_\ast(Y \times Z) & \xrightarrow{\text{EZ}} & C_\ast(X \times Y \times Z); \\
\end{array}
\]

2. It is a map of DG coalgebras.

3. $\text{EZ}$ and $\text{AW}$ are inverses up to natural chain homotopies.

From the associativity property, it follows that if $X_1, \ldots, X_r$ are smooth manifolds, then there is an $r$-fold Eilenberg-Zilber map $\text{EZ}: C_\ast(X_1) \otimes \cdots \otimes C_\ast(X_r) \to C_\ast(X_1 \times \cdots \times X_r)$ which is obtained by applying the binary Eilenberg-Zilber maps $r - 1$ times. Explicitly, this map is defined as follows. Given simplices $\sigma_i: \Delta_{n_i} \to X_i$ with $i = 1, \ldots, r$, one has:

$$\text{EZ}(\sigma_1 \otimes \cdots \otimes \sigma_r) = \sum_{\chi \in \mathcal{C}_{n_1, \ldots, n_r}} (-1)^\chi (\sigma_1 \times \cdots \times \sigma_r) \circ \chi_s,$$

(2.17)

where the sum over $\chi$ is taken over all $(n_1, \ldots, n_r)$-shuffles and $\chi_s: \Delta_{n_1 + \cdots + n_r} \to \Delta_{n_1} \times \cdots \times \Delta_{n_r}$ now denotes the map defined by

$$\chi_s(t_1, \ldots, t_{n_1+\cdots+n_r}) = ((t_{\chi(1)}, \ldots, t_{\chi(n_1)}), \ldots, (t_{\chi(n_1+\cdots+n_{r-1}+1)}, \ldots, t_{\chi(n_1+\cdots+n_r)})).$$

(2.18)

We will now specialize the discussion by replacing $X$ with a Lie group $G$. In this case, the space of singular chains $C_\ast(G)$ acquires the structure of a DG Hopf algebra. The product $m: C_\ast(G) \otimes C_\ast(G) \to C_\ast(G)$ is formed by composition of the Eilenberg-Zilber map $\text{EZ}: C_\ast(G) \otimes C_\ast(G) \to C_\ast(G \times G)$ with the map $\mu_s: C_\ast(G \times G) \to C_\ast(G)$ induced by the multiplication map $\mu: G \times G \to G$. The unit $u: \mathbb{R} \to C_\ast(G)$ is induced by the inclusion of a point as the identity element of $G$, and the antipode $S: C_\ast(G) \to C_\ast(G)$ is induced by the inversion map $\iota: G \to G$. 

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2.5 Representations of the DG Lie algebra $T\mathfrak{g}$

For a Lie algebra $\mathfrak{g}$, consider the DG Lie algebra $T\mathfrak{g} = u\mathfrak{g} \oplus \mathfrak{g}$ with degree $-1$ generators $i(x) \in u\mathfrak{g}$ and degree $0$ generators $L(x) \in \mathfrak{g}$ for $x \in \mathfrak{g}$. The Lie bracket of $T\mathfrak{g}$ is induced by the Lie bracket of $\mathfrak{g}$, and the differential is defined by

\[
\begin{align*}
t(i(x)) &= L(x), \\
t(L(x)) &= 0.
\end{align*}
\]

The generators $i(x)$ and $L(x)$ satisfy the Cartan relations

\[
\begin{align*}
[i(x), i(y)] &= 0, \\
[L(x), L(y)] &= L([x, y]), \\
[L(x), i(y)] &= i([x, y]).
\end{align*}
\]

By a representation of $T\mathfrak{g}$ we mean a cochain complex $V$ together with a DG Lie algebra homomorphism $T\mathfrak{g} \to \text{End}(V)$. That is, it consists of a representation of $T\mathfrak{g}$ on $V$, where the operators $i_x \in \text{End}(V)^{-1}$ and $L_x \in \text{End}(V)^0$ corresponding to $i(x), L(x) \in T\mathfrak{g}$ satisfy the relations

\[
\begin{align*}
[i_x, \delta] &= L_x, \\
[L_x, \delta] &= 0, \\
[i_x, i_y] &= 0, \\
[L_x, L_y] &= L_{[x,y]}, \\
[L_x, i_y] &= i_{[x,y]},
\end{align*}
\]

with $\delta$ being the differential of $V$. The operators $i_x$ are called contractions and the operators $L_x$ are called Lie derivatives.

An important example of a representation of $T\mathfrak{g}$ is the Chevalley-Eilenberg complex $\text{CE}(\mathfrak{g})$. As a graded algebra is the exterior algebra $\Lambda^* \mathfrak{g}^*$, where $\mathfrak{g}^*$ has degree $1$. The differential $\delta_{\text{CE}}$ is the unique derivation such that for $\xi \in \Lambda^1 \mathfrak{g}^*$, $\delta_{\text{CE}} \xi$ is the element in $\Lambda^2 \mathfrak{g}^*$ defined by

\[
(\delta_{\text{CE}} \xi)(x, y) = -\xi([x, y]).
\]

It follows from the Jacobi identity that $\delta_{\text{CE}}$ defined in this manner is a differential. The contraction $i_x$ and Lie derivatives $L_x$ are the unique derivations such that for $\xi \in \Lambda^1 \mathfrak{g}^*$,

\[
\begin{align*}
i_x \xi &= \langle \xi, x \rangle, \\
L_x \xi &= \text{ad}_x^* \xi,
\end{align*}
\]

where $\text{ad}_x^*$ denotes the infinitesimal coadjoint action of the element $x$.

Explicit formulas for these various maps, which will be useful later on, are obtained by introducing a basis for $\mathfrak{g}^*$. Let $e_a$ be a basis for $\mathfrak{g}$ with dual basis $e^a$ and structure constants $f_{bc}^a = \langle e^a, [e_b, e_c] \rangle$, and write $i_a$ and $L_a$ for the contraction $i_{e_a}$ and the Lie derivative $L_{e_a}$ acting on $\text{CE}(\mathfrak{g})$. Then the explicit formulas for $\delta_{\text{CE}}, i_a$ and $L_a$ are the following:

\[
\begin{align*}
\delta_{\text{CE}} e^a &= -\frac{1}{2} f_{bc}^a e^b \wedge e^c, \\
i_b e^a &= \delta_b^a, \\
L_b e^a &= -f_{bc}^a e^c.
\end{align*}
\]
Here the convention that repeated indices are summed over is in place.

Another example of a representation of $\mathfrak{T}_g$ is the Weil algebra $W_g$. The underlying graded commutative algebra of $W_g$ is the tensor product

$$W_g = \Lambda^\ast g^\ast \otimes S^\ast g^\ast,$$

where $S^\ast g^\ast$ is the symmetric algebra of $g^\ast$ and where we associate to each $\xi \in g^\ast$ the degree 1 generators $t(\xi) \in \Lambda^1 g^\ast$ and the degree 2 generators $w(\xi) \in S^1 g^\ast$. The differential on $W_g$ is given by

$$d_W(t(\xi)) = w(\xi) + \delta_{CE}(t(\xi)),
$$

$$d_W(w(\xi)) = \delta_{CE}(w(\xi)),
$$

where $\delta_{CE}$ is the differential of the Chevalley-Eilenberg complex $CE(g)$. The operators $i_x$ and $L_x$ are the unique derivations such that

$$i_x(t(\xi)) = (t(\xi), x),
$$

$$i_x(w(\xi)) = 0,
$$

$$L_x(t(\xi)) = \text{ad}^* x(t(\xi)),
$$

$$L_x(w(\xi)) = \text{ad}^* x(w(\xi)).
$$

As for $CE(g)$, it will be useful to express the differential $d_W$ and the operator $i_x$ and $L_x$ in terms of a dual basis $e^a$ of $g^\ast$ and the structure constants $f^a_{bc}$ of $g$. If we write $t^a = t(e^a)$ and $w^a = w(e^a)$, they are as follows:

$$d_W t^a = w^a - \frac{1}{2} f^a_{\ bc} l^b t^c,
$$

$$d_W w^a = f^a_{\ bc} w^b t^c,
$$

$$i_b t^a = \delta^a_b,
$$

$$i_b w^a = 0,
$$

$$L_b t^a = - f^a_{\ bc} t^c,
$$

$$L_b w^a = - f^a_{\ bc} w^c.
$$

Clearly, the elements $t^a$ and $d_W t^a$ also generate $W_g$ freely, which implies that the Weil algebra is acyclic.

If $V$ and $W$ are representations of $\mathfrak{T}_g$, a homomorphism $f : V \to W$ is a morphism of cochain complexes commuting with the operators $i_x$ and $L_x$. It is clear that the identity map of a representation of $\mathfrak{T}_g$ onto itself is a homomorphism, and that the composition of two homomorphisms is again a homomorphism. Thus, representations of $\mathfrak{T}_g$ and their homomorphisms form a category which we denote by $\text{Rep}(\mathfrak{T}_g)$. For later purposes, we note that this category is symmetric monoidal with tensor product $V \otimes V'$ of two objects $V$ and $V'$ given by the tensor product of the underlying cochain complexes equipped with the actions of $i_x$ and $L_x$ defined by the formulas

$$i_x(v \otimes v') = i_x v \otimes v' + (-1)^{|v|} v \otimes i_x v',
$$

$$L_x(v \otimes v') = L_x v \otimes v' + v \otimes L_x v',
$$

for homogeneous elements $v \in V$ and $v' \in V'$. Clearly, the unit object is the trivial representation $\mathbb{R}$ viewed as a complex concentrated in degree zero.
2.6 Differentiation and integration functors between \( \text{Mod}(C_\ast(G)) \) and \( \text{Rep}(\mathbb{T}_g) \)

Let \( G \) be a simply connected Lie group with Lie algebra \( g \). The main result of [5] states the existence of differentiation and integration functors between the monoidal categories \( \text{Mod}(C_\ast(G)) \) and \( \text{Rep}(\mathbb{T}_g) \) which are inverses to one another. This extends the well-known correspondence between representations of \( G \) and representations of \( g \). Let us explain briefly the construction of these functors.

We begin with the differentiation functor, which we will write as \( \mathcal{D} \): \( \text{Mod}(C_\ast(G)) \to \text{Rep}(\mathbb{T}_g) \).

For an element \( x \) of \( g \), take the singular 1-simplex \( \sigma[x]: \Delta_1 \to G \) defined by
\[
\sigma[x](s) = \exp(sx),
\]
where \( \exp: g \to G \) is the exponential map of \( G \). Then, given an object \( V \) in \( \text{Mod}(C\ast(G)) \) with structure homomorphism \( \rho: C\ast(G) \to \text{End}(V) \), the corresponding object \( \mathcal{D}(V) \) in \( \text{Rep}(\mathbb{T}_g) \) is \( V \) with structure homomorphism \( \mathcal{D}(\rho): \mathbb{T}_g \to \text{End}(V) \) determined by
\[
\mathcal{D}(\rho)(i(x)) = \left. \frac{d}{dt} \right|_{t=0} \rho(\sigma[t x]),
\]
\[
\mathcal{D}(\rho)(L(x)) = \left. \frac{d}{dt} \right|_{t=0} \rho(\exp(t x)).
\]

In addition, the functor \( \mathcal{D} \) acts as the identity on morphisms. Under this definition, it is not difficult to verify that \( \mathcal{D} \) is indeed monoidal. For details, see Theorem 3.3 of [5].

Next we turn to the integration functor, which we write as \( \mathcal{I} \): \( \text{Rep}(\mathbb{T}_g) \to \text{Mod}(C_\ast(G)) \).

First we need to record an important preliminary notion. In the category \( \text{Rep}(\mathbb{T}_g) \) let us fix an object \( V \), and for each \( k \geq 0 \), let us call \( \Phi^{(k)}_V \in \Omega^k(G) \otimes \text{End}(V)^{-k} \) the unique left-equivariant form on \( G \) with values in \( \text{End}(V) \) such that
\[
\Phi^{(k)}_V(e)(x_1, \ldots, x_k) = i_{x_1} \circ \cdots \circ i_{x_k},
\]
for all \( x_1, \ldots, x_k \in g \). With this definition, it can be concluded that the forms \( \Phi^{(k)}_V \) satisfy the "descent equations"
\[
d\Phi^{(k)}_V = (-1)^k \delta \Phi^{(k+1)}_V,
\]
where, as before, we write \( \delta \) for the the differential of \( V \). Furthermore, if \( \mu: G \times G \to G \) denotes the multiplication map for \( G \) and \( \pi_1, \pi_2: G \times G \to G \) are the projection onto the first and second component, respectively, we get the relation
\[
\mu^* \Phi^{(k)}_V = \sum_{i+j=k} (-1)^{ij} \pi_1^* \Phi^{(i)}_V \wedge \pi_2^* \Phi^{(j)}_V.
\]
We note finally that \( \Phi^{(0)}_V \in \Omega^0(G) \otimes \text{End}(V)^0 \) is a representation of \( G \) on \( V \) and, in particular,
\[
\Phi^{(0)}_V(e) = \text{id}_V.
\]
In general, an element \( \Phi \) of \( \Omega^*(G) \otimes \text{End}(V) \) is refereed to as a left-equivariant representation form for \( V \) if it can be decomposed as
\[
\Phi = \sum_{k \geq 0} \Phi^{(k)},
\]

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where the forms $\Phi^{(k)} \in \Omega^k(G) \otimes \text{End}(V)^{-k}$ satisfy the conditions (2.33), (2.34) and (2.35). In this way we set up a bijective correspondence between objects of $\text{Rep}(Tg)$ and their associated left-equivariant representation forms (see Proposition 3.18 of [5]). We can now define the integration functor $\mathcal{I}: \text{Rep}(Tg) \to \text{Mod}(C^\bullet(G))$ as follows. Given an object $V$ in $\text{Rep}(Tg)$ with structure homomorphism $\rho: Tg \to \text{End}(V)$, the corresponding object $\mathcal{I}(V)$ in $\text{Mod}(C^\bullet(G))$ is $V$ with structure homomorphism $\mathcal{I}(\rho): C^\bullet(G) \to \text{End}(V)$ determined on a singular $k$-simplex $\sigma: \Delta_k \to G$ by

$$\mathcal{I}(\rho)(\sigma) = \int_{\Delta_k} \sigma^* \Phi_V,$$

where $\Phi_V = \sum_{k \geq 0} \Phi^{(k)}_V$. Moreover, the functor $\mathcal{I}$ acts as the identity on morphisms. Under this definition, it is not hard to see that $D$ is simultaneously left and right inverse to $\mathcal{I}$. All the details can be found in §3.3 of [5].

3 \textit{A}_\infty\text{-quasi-isomorphisms of DG algebras}

In this section, we prove several technical results concerning the Van Est map and the Hochschild-De Rham $A_\infty$-quasi-isomorphism in the context of classifying spaces. These results are key components in the proof of our main theorem. They may also be of independent interest. Throughout the discussion, $G$ denotes a simply connected Lie group with Lie algebra $g$.

3.1 The Van Est map

Here we consider the Van Est map from the Bott-Shulman-Stasheff algebra $\Omega^\bullet(G_\bullet)$ to the Weil algebra of $g$. We follow the conventions in [19]. The Bott-Shulman-Stasheff algebra computes the cohomology of BG while the Van Est map is contractible, so the Van Est map models the pull-back map of the universal bundle. Our goal here is to show that if $G$ is connected and compact, there is a natural subalgebra of $\Omega^\bullet(G_\bullet)$ such that the restriction of the Van Est map to it lands on the basic part of the Weil algebra and is a quasi-isomorphism.

Let us consider the universal $G$-bundle $\pi: EG_\bullet \to BG_\bullet$ as in [24]. Recall that $EG_\bullet$ is the simplicial manifold with $EG_p = G \times \cdots \times G$ ($p+1$ copies) where the face operators $\varepsilon_i: EG_p \to EG_{p-1}$ are given by

$$\varepsilon_i(g_0, \ldots, g_p) = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_p),$$

for $0 \leq i \leq p$. Similarly, $BG_\bullet$ is defined by $BG_p = G \times \cdots \times G$ ($p$ copies) but here the face operators $\varepsilon_i: BG_p \to BG_{p-1}$ are given by

$$\varepsilon_i(g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p) & \text{if } i = 0, \\ (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+2}, \ldots, g_p) & \text{if } 0 < i < p, \\ (g_1, \ldots, g_{p-1}) & \text{if } i = p. \end{cases}$$

Finally, view each $EG_p$ as a principal $G$-bundle over $BG_p$, with action the diagonal action of $G$ from the right, and quotient map $\pi: EG_p \to BG_p$ given by

$$\pi(g_0, \ldots, g_p) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{p-1} g_p^{-1}).$$
By definition, the total space of the universal $G$-bundle $EG$ is the thick geometric realisation of the simplicial manifold $EG_\bullet$. From this it is easy to see that the classifying space $BG$ may be identified with the thick geometric realisation of $BG_\bullet$. This means the cohomology of $BG$ may be computed as the “De Rham cohomology” of $BG_\bullet$, which is defined as the cohomology of the following double complex $\Omega^\bullet(BG_\bullet)$:

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\Omega^2(BG_0) \xrightarrow{\partial} \Omega^2(BG_1) \xrightarrow{\partial} \Omega^2(BG_2) \xrightarrow{\partial} \cdots \\
\vdots \\
\Omega^1(BG_0) \xrightarrow{\partial} \Omega^1(BG_1) \xrightarrow{\partial} \Omega^1(BG_2) \xrightarrow{\partial} \cdots \\
\vdots \\
\Omega^0(BG_0) \xrightarrow{\partial} \Omega^0(BG_1) \xrightarrow{\partial} \Omega^0(BG_2) \xrightarrow{\partial} \cdots \\
\end{array}
\]

Here the vertical differential $\bar{d}: \Omega^q(BG_p) \to \Omega^{q+1}(BG_p)$ is $(-1)^p$ times the usual de exterior differential $d$ and the horizontal differential $\partial: \Omega^q(BG_p) \to \Omega^q(BG_{p+1})$ is given by

\[
\partial = \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*. 
\] (3.4)

We note that $\Omega^\bullet(BG_\bullet)$ has a graded ring structure with respect to the cup product defined as follows. For any $\omega \in \Omega^q(BG_p)$ and $\omega' \in \Omega^{q'}(BG_{p'})$, let $\omega \cup \omega' \in \Omega^{q+q'}(BG_{p+p'})$ be the differential form given by

\[
\omega \cup \omega' = (-1)^{qp'} \text{pr}^* \omega \wedge \text{pr}'^* \omega', 
\] (3.5)

where $\text{pr}: BG_{p+p'} \to BG_p$ is the front face projection

\[
\text{pr}(g_1, \ldots, g_{p+p'}) = (g_1, \ldots, g_p), 
\] (3.6)

and $\text{pr}': BG_{p+p'} \to BG_{p'}$ is the back face projection

\[
\text{pr}'(g_1, \ldots, g_{p+p'}) = (g_{p+1}, \ldots, g_{p+p'}). 
\] (3.7)

Both the vertical and horizontal differentials $\bar{d}$ and $\partial$ are graded derivations relative to the cup product, and we regard Bott-Shulman-Stasheff complex $\Omega^\bullet(BG_\bullet)$ as a DG algebra with respect to the total differential.

Now we turn to a discussion of the Van Est map. For this purpose, let us consider the action $\gamma_i(g)$ of elements $g$ of $G$ on $BG_p$ defined by

\[
\gamma_i(g)(g_1, \ldots, g_p) = (g_1, \ldots, g_{i-1}, g_i g^{-1}, g g_{i+1}, g_{i+2}, \ldots, g_p), 
\] (3.8)

where $1 \leq i \leq p$. For each $x \in \mathfrak{g}$, we denote by $x^{i\sharp}$ the vector field on $BG_p$ generated by this action. We also regard the Weil algebra of $\mathfrak{g}$ as a bigraded algebra $W^{\bullet, \bullet}_\mathfrak{g}$ with

\[
W^{p,q}_\mathfrak{g} = A^{p-q} \mathfrak{g}^* \otimes S^q \mathfrak{g}^*. 
\] (3.9)
Notice that any $x \in \mathfrak{g}$ defines two kinds of contraction operators $i_\Lambda(x)$ and $i_S(x)$ on $W^{\bullet, \bullet}\mathfrak{g}$ of bidegrees $(-1, 0)$ and $(-1, -1)$, corresponding to the contractions on $\Lambda^\bullet\mathfrak{g}$ and $S^\bullet\mathfrak{g}$, respectively. For elements $\xi \in W^{p, q}\mathfrak{g}$ and $x_1, \ldots, x_p \in \mathfrak{g}$ we put

$$\xi(x_1, \ldots, x_q, x_{q+1}, \ldots, x_p) = i_\Lambda(x_p) \cdots i_\Lambda(x_{q+1})i_S(x_q) \cdots i_S(x_1)\xi. \quad (3.10)$$

With these definitions, the Van Est map $\text{VE}: \Omega^\bullet(BG_\bullet) \to W^{\bullet, \bullet}\mathfrak{g}$ is the map of DG algebras given by the following formula, for $\omega \in \Omega^\bullet(BG_p)$ and $x_1, \ldots, x_p \in \mathfrak{g}$,

$$\text{VE}(\omega)(x_1, \ldots, x_q, x_{q+1}, \ldots, x_p) = \sum_{\sigma \in S_p} \varepsilon(\sigma) \left( i_{x_{\sigma(1)}} \cdots i_{x_{\sigma(q+1)}} L_{x_{\sigma(q+2)}} \cdots L_{x_{\sigma(p)}} \omega \right)(e, \ldots, e). \quad (3.11)$$

Here $e$ is the identity element of $G$, and $\varepsilon(\sigma)$ is equal to $+1$ if the number of pairs $(i, j)$ with $q + 1 \leq i < j \leq p$ but $\sigma(i) > \sigma(j)$ is even, and equal to $-1$ if that number is odd.

The Van Est map does not take values on the basic elements of $W^{\bullet, \bullet}\mathfrak{g}$. However, there is a subalgebra of $\Omega^\bullet(G_\bullet)$ whose image under the Van Est map consists of basic elements. We set $G_p = G \times \cdots \times G$ ($p$ copies) and think of it as a Lie group. Moreover, we let $\mathfrak{g}_p = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ ($p$ copies) denote the corresponding Lie algebra. Since the actions of $G$ on $BG_p$ defined by (3.8) commute, we obtain an action $\gamma(g_1, \ldots, g_p)$ of elements $(g_1, \ldots, g_p)$ of $G_p$ on $BG_p$ by putting

$$\gamma(g_1, \ldots, g_p) = \gamma_1(g_1) \circ \cdots \circ \gamma_p(g_p). \quad (3.12)$$

It is straightforward to check this action is transitive and free. Let us denote by $\Omega^\bullet(BG_p)^{G_p}$ the subspace of $G_p$-invariant elements of $\Omega^\bullet(BG_p)$\footnote{We remark that $\Omega^\bullet(BG_\bullet)^{G_\bullet}$ is not a DG subalgebra of $\Omega^\bullet(BG_\bullet)$}. Then each element of $\Omega^\bullet(BG_p)^{G_p}$ is completely and freely determined by its evaluation at $(e, \ldots, e) \in BG_p$. Therefore, evaluation at $(e, \ldots, e)$ gives an isomorphism of graded vector spaces between $\Omega^\bullet(BG_p)^{G_p}$ and $\Lambda^\bullet\mathfrak{g}_p^\ast$. On the other hand, consider the residual action $\gamma_0(g)$ of elements $g$ of $G$ on $BG_p$ defined by

$$\gamma_0(g)(g_1, \ldots, g_p) = (gg_1, g_2, \ldots, g_p). \quad (3.13)$$

Since this action commutes with the one given by (3.12), we end up with an action $\zeta(g_0, g_1, \ldots, g_p)$ of elements $(g_0, g_1, \ldots, g_p)$ of $G_{p+1}$ on $BG_p$ by setting

$$\zeta(g_0, g_1, \ldots, g_p) = \gamma_0(g_0) \circ \gamma(g_1, \ldots, g_p). \quad (3.14)$$

We let $\Omega^\bullet(BG_p)^{G_{p+1}}$ denote the subspace of $G_{p+1}$-invariant elements of $\Omega^\bullet(BG_p)$.

**Lemma 3.1.** $\Omega^\bullet(BG_\bullet)^{G_{\bullet+1}}$ is a DG subalgebra of $\Omega^\bullet(BG_\bullet)$ and the inclusion

$$\Omega^\bullet(BG_\bullet)^{G_{\bullet+1}} \to \Omega^\bullet(BG_\bullet)$$

is a quasi-isomorphism.

**Proof.** First, let us verify that $\Omega^\bullet(BG_\bullet)^{G_{\bullet+1}}$ is a double subcomplex of $\Omega^\bullet(BG_\bullet)$. By definition, it is clear that $\bar{d}: \Omega^\bullet(BG_p) \to \Omega^{\bullet+1}(BG_p)$ preserves $G_{p+1}$-invariant elements, since the exterior differential (and hence $\bar{d}$) commutes with pullback. On the other hand, it is not hard to see that
\[\varepsilon^* \omega \in \Omega^2(BG_{p+1})^{G_{p+2}} \text{ for all } \omega \in \Omega^q(BG_p)^{G_{p+1}}. \text{ Thus, from (3.4), we conclude that } \partial : \Omega^q(BG_p) \to \Omega^{q+1}(BG_{p+1}) \text{ also preserves the } G_{p+1}\text{-invariant elements.} \]

Next, we need to verify that \(\Omega^*(BG_*)^{G_{p+1}}\) is closed with respect to the cup product (3.5). This turns out to be a direct consequence of the following two identities

\[\text{pr}(\zeta(g_0, \ldots, g_{p+p'})) = \zeta(g_0, \ldots, g_p), \quad \text{pr}'(\zeta(g_0, \ldots, g_{p+p'})) = \zeta(g_p, \ldots, g_{p+p'}),\]

which follow at once from the definitions (3.6), (3.7) and (3.14).

Finally, to prove the second statement, since \(G_{p+1}\) is compact and connected, a theorem of Cartan [12] asserts that the inclusion \(\Omega^*(BG_p)^{G_{p+1}} \to \Omega^*(BG_p)\) is a quasi-isomorphism. The result then follows from the convergence of the spectral sequences for \(\Omega^*(BG_*)^{G_{p+1}}\) and \(\Omega^*(BG_*)\), together with the fact that the inclusion \(\Omega^*(BG_*)^{G_{p+1}} \to \Omega^*(BG_*)\) induces an isomorphism of spectral sequences on the \(E_1\)-term.

For our next preparatory result, we let \(\text{Ad}_p\) be the adjoint action of elements \(g\) of \(G\) on \(g\) and denote by the same symbol its extension to \(g_p\).

**Lemma 3.2.** The following diagram commutes

\[
\begin{array}{ccc}
\Omega^q(BG_p)^{G_p} & \xrightarrow{\gamma_0(g)^*} & \Omega^q(BG_p)^{G_p} \\
\downarrow & & \downarrow \\
\Lambda^q g_p^* & \xrightarrow{\text{Ad}_p} & \Lambda^q g_p^*
\end{array}
\]

where the vertical arrows denote evaluation at the element \((e, \ldots, e)\).

**Proof.** Take \(\omega \in \Omega^q(BG_p)^{G_p}\) and \(v_1, \ldots, v_p \in g_p\). We compute directly, using the definitions:

\[
(\gamma_0(g)^* \omega)_{(e, \ldots, e)}(v_1, \ldots, v_p) = \omega(g, e, \ldots, e)(d\gamma_0(g)(e, \ldots, e)(v_1), \ldots, d\gamma_0(g)(e, \ldots, e)(v_p))
\]

\[
= \omega((g^{-1}, \ldots, g^{-1})(e, \ldots, e), (e, \ldots, e), v_1, \ldots, d\gamma_0(g)(e, \ldots, e)(v_p))
\]

\[
= (\gamma(g^{-1}, \ldots, g^{-1})\omega)_{(e, \ldots, e)}(d\gamma(g, \ldots, g)(e, \ldots, e), d\gamma_0(g)(e, \ldots, e)(v_1),
\]

\[
\ldots, d\gamma(g, \ldots, g)(e, \ldots, e), d\gamma_0(g)(e, \ldots, e)(v_p))
\]

\[
= \omega((e, \ldots, e), (d\gamma(g, \ldots, g) \circ \gamma_0(g))(e, \ldots, e)(v_1), \ldots, d\gamma(g, \ldots, g) \circ \gamma_0(g))(e, \ldots, e)(v_p))
\]

But

\[
\zeta(g, \ldots, g)(g_1, \ldots, g_p) = (g g_1g^{-1}, \ldots, g g_pg^{-1}),
\]

from which it follows that \(d\zeta(g, \ldots, g)(e, \ldots, e) = \text{Ad}_g\). Substitution gives the result claimed.

Next, we record the following observation.

**Lemma 3.3.** The restriction of the Van Est map \(\text{VE}\) to \(\Omega^q(BG_p)^{G_p}\) vanishes unless \(q = p\).
Proof. If $\omega \in \Omega^q(BG_p)^{G_p}$, then $L_{x_1,1}\omega = 0$ for all $x \in g$. This, together with formula (3.11), implies that $\text{VE}(\omega) = 0$ unless $q = p$. □

As a consequence of this, we see that the restriction of the Van Est map $\text{VE}$ to $\Omega^q(BG_p)^{G_p}$, which we keep on denoting by $\text{VE}$, is given by the following expression, for $\omega \in \Omega^q(BG_p)^{G_p}$ and $x_1, \ldots, x_p \in g$.

$$\text{VE}(\omega)(x_1, \ldots, x_p) = \sum_{\sigma \in S_p} \left( i_{x_1} \cdots i_{x_p} \omega \right)(e, \ldots, e).$$

(3.15)

We also note that this map has its image contained in $S^p g^*$.

Before we can go further, we need the following piece of notation. For each $x \in g$, we let $x^i$ be element of $g_p$ having its $i$th and $(i + 1)$th coordinates equal to $-x$ and $x$, respectively, and all others zero. Hence, by definition, $x^{i\sharp}(e, \ldots, e) = x^i$. Thus, if we let $\widetilde{\text{VE}}: \Lambda^p g_p^* \rightarrow S^p g^*$ be the map defined for $\xi \in \Lambda^p g_p^*$ and $x_1, \ldots, x_p \in g$ by

$$\widetilde{\text{VE}}(\xi)(x_1, \ldots, x_p) = \sum_{\sigma \in S_p} \xi(x_{\sigma(1)}, \ldots, x_{\sigma(p)}),$$

(3.16)

we obtain the commutative diagram

$$\begin{array}{ccc}
\Omega^q(BG_p)^{G_p} & \xrightarrow{\text{VE}} & \Lambda^p g_p^* \\
\downarrow & & \downarrow \text{VE} \\
\Lambda^p g_p^* & \xrightarrow{\text{Ad}_g^*} & S^p g^*
\end{array}$$

where, as before, the vertical arrow denotes evaluation at $(e, \ldots, e)$. We may now state and prove the following result.

**Proposition 3.4.** The restriction of the Van Est map $\text{VE}$ to $\Omega^q(BG_p)^{G_{q+1}}$ has image contained in $(S^p g^*)^G$.

Proof. By virtue of Lemma 3.2 and the previous remarks, it is enough to show that the following diagram commutes

$$\begin{array}{ccc}
\Lambda^p g_p^* & \xrightarrow{\text{Ad}_g^*} & \Lambda^p g_p^* \\
\downarrow \text{VE} & & \downarrow \text{VE} \\
S^p g^* & \xrightarrow{\text{Ad}_g^*} & S^p g^*
\end{array}$$

So let us take $\xi \in \Lambda^p g_p^*$ and $x_1, \ldots, x_p \in g$. Then, attending to the definition (3.16), we have

$$\widetilde{\text{VE}}(\text{Ad}_g^* \xi)(x_1, \ldots, x_p) = \sum_{\sigma \in S_p} (\text{Ad}_g^\ast \xi)(x_{\sigma(1)}^1, \ldots, x_{\sigma(p)}^p)$$

$$= \sum_{\sigma \in S_p} \xi \left( \text{Ad}_g x_{\sigma(1)}^1, \ldots, \text{Ad}_g x_{\sigma(p)}^p \right)$$

$$= \text{VE}(\xi) \left( \text{Ad}_g x_{\sigma(1)}^1, \ldots, \text{Ad}_g x_{\sigma(p)}^p \right)$$

$$= (\text{Ad}_g^* \text{VE}(\xi))(x_1, \ldots, x_p),$$

from which the result follows. □
Next, we will show that the restricted Van Est map $\mathbb{V}: \Gamma^G(BG^*)_{G+1} \to (S^*g^*)^G$ is a quasi-isomorphism. For this, we need a small digression outlining some of the results of [4].

To begin with, recall that a $g$-DG algebra $A$ is by definition an object of $\text{Rep}(\mathbb{T}g)$ endowed with the structure of a graded ring such that the action of $\mathbb{T}g$ is by derivations. Homomorphisms of $g$-DG algebras are morphisms in $\text{Rep}(\mathbb{T}g)$ which are also homomorphisms of graded rings. Given a $g$-DG algebra $A$, an algebraic connection is a linear map $\theta: g^* \to A^1$, which satisfy the relations

$$i_x(\theta(\xi)) = \langle \xi, x \rangle,$$

$$L_x(\theta(\xi)) = \theta(\text{ad}^*_x \xi),$$

for all $x \in g$ and $\xi \in g^*$. One important example of a commutative $g$-DG algebra is provided by the Weil algebra $W_g$. It is obvious that $W_g$ carries a “tautological” connection given by the map $\iota : g^* \to W^1g$. As a matter of fact, $W_g$ is universal among commutative $g$-DG algebras with connection. Thus, given a $g$-DG algebra $A$ with connection $\theta$, there exists a $g$-DG algebra homomorphism $c^\theta : W_g \to A$ such that $c^\theta \circ \iota = \theta$. Following the terminology of [4], one refers to $c^\theta$ as the characteristic homomorphism for the connection $\theta$.

Our interest here, however, is on the De Rham complex $\Omega^d(EG_*)$ of the simplicial manifold $EG_*$, which is defined by exactly the same prescription that defined the Bott-Shulman-Stasheff complex $\Omega^d(BG_*)$. This turns out to be a noncommutative $g$-DG algebra where the graded ring structure is again defined by the cup product, and, if we let $\rho$ denote the infinitesimal action of $g$ on $EG_*$, $i_x$ is the inner product of a form with $\rho(x)$, and $L_x$ is the Lie derivative of the form along $\rho(x)$. What is more, it carries a natural connection $\theta : g^* \to \Omega^1(EG_0)$ given by the left-invariant Maurer-Cartan form on $G$. We would like to define a characteristic homomorphism for this connection $\theta$. For this we need a universal object among noncommutative $g$-DG algebras with connection, the so called noncommutative Weil algebra $\tilde{W}_g$. Its definition is as follows.

Recall that the Weil algebra $W_g$ may be identified with the Koszul algebra of the graded vector space $uq^*$. Accordingly, as a DG algebra, $\tilde{W}_g$ is the noncommutative Koszul algebra of $uq^*$. Just as in Section 2.4, we associate to each $\xi \in g^*$ a degree 1 generator $t(\xi)$ and a degree 2 generator $w(\xi)$, so that $W_g$ is freely generated by $t(\xi)$ and $w(\xi)$, $d_{\tilde{W}}t(\xi) = w(\xi)$ and $d_{\tilde{W}}w(\xi) = 0$. The formulas for the contractions $i_x$ and Lie derivatives $L_x$ are given on these generators by

$$i_x(t(\xi)) = \langle t(\xi), x \rangle,$$

$$i_x(w(\xi)) = \text{ad}^*_x(t(\xi)),$$

$$L_x(t(\xi)) = \text{ad}^*_x(t(\xi)),$$

$$L_x(w(\xi)) = \text{ad}^*_x(w(\xi)).$$

And just as in the commutative case, $\tilde{W}_g$ carries a “tautological” connection determined by the map $\tilde{\iota} : g^* \to \tilde{W}^1g$. It can then be shown that, given an arbitrary $g$-DG algebra $A$ with connection $\theta$, there is a $g$-DG algebra homomorphism $\tilde{c}^\theta : \tilde{W}_g \to A$ such that $\tilde{c}^\theta \circ \tilde{\iota} = \theta$. We should also point out that the quotient map $\tilde{W}_g \to W_g$ is a morphism in $\text{Rep}(\mathbb{T}g)$ which is a quasi-isomorphism with homotopy inverse given by symmetrisation $\text{sym} : W_g \to \tilde{W}_g$.

In light of the preceding discussion it is now clear that there is a Chern-Weil map

$$c^\theta = \tilde{c}^\theta \circ \text{sym} : W_g \longrightarrow \Omega^d(EG_*),$$

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which is defined by symmetrisation. This in turn induces a morphism of cochain complexes on the
basic subspaces $c^\theta: (Wg)_{bas} \to \Omega^* (EG_\ast)_{bas}$. As $S^g\ast$ is precisely the set of elements in $Wg$ killed by
$i_x$ for $x \in g$, it follows that $(Wg)_{bas}$ coincides with the algebra of invariant polynomials $(S^g\ast)^G$. On
the target complex we have we have on the other hand that $\Omega^* (EG_\ast)_{bas}$ is canonically isomorphic
to the Bott-Shulman-Stasheff complex $\Omega^* (BG_\ast)$. This latter isomorphism is induced by pullback
along the right inverse $\iota: BG \rightarrow EG$ to the quotient map $\pi$ which is defined by the formula

$$\iota(g_1, \ldots, g_i) = (e, g_1^{-1}, \ldots, (g_1 \cdots g_p)^{-1}).$$

Therefore we clearly get a map

$$AM^\theta = \iota^* \circ c^\theta: (S^g\ast)^G \rightarrow \Omega^* (BG_\ast),$$

to which we refer to as the Alekseev-Meinrenken map. The image of an invariant polynomial of
degree $r$ under this map has non-vanishing components only in bidegree $p + q = 2r$ with $p \leq r$. It
also induces an algebra homomorphism in cohomology, and in fact an algebra isomorphism if $G$ is
compact and connected (see Proposition 9.1 and Theorem 9.2 of [4]).

To proceed further, let us consider the action $\gamma_i(g)$ of elements $g$ of $G$ on $EG_p$ defined by

$$\gamma_i(g)(g_0, \ldots, g_p) = (g_0, \ldots, g_{i-1}, gg_i, g_{i+1}, \ldots, g_p),$$

where $0 \leq i \leq p$. It is then a simple matter to verify that all of these actions provide lifts of the
actions of $G$ on $BG_p$ determined by (3.8) and (3.13). To be more precise, we have a commutative
diagram

$$\begin{array}{ccc}
EG_p & \xrightarrow{\gamma_i(g)} & EG_p \\
\pi \downarrow & & \downarrow \pi \\
BG_p & \xrightarrow{\gamma_i(g)} & BG_p,
\end{array}$$

for all $0 \leq i \leq p$. This implies that if, for each $x \in g$, we let $\overline{x}^{i,\#}$ denote the vector field on $EG_p$
generated by the action (3.20), then $\overline{x}^{i,\#}$ and $x^{i,\#}$ are $\pi$-related. In particular, we have

$$d\pi(e, \ldots, e)(\overline{x}^{i,\#}(e, \ldots, e)) = x^{i,\#}(e, \ldots, e).$$

It is also worth pointing out that we get an action $\zeta(g_0, \ldots, g_p)$ of elements $(g_0, \ldots, g_p)$ of $G_{p+1}$ on
$EG_p$ by simply putting

$$\zeta(g_0, \ldots, g_p) = \gamma_0(g_0) \circ \cdots \circ \gamma_p(g_p),$$

and that this action provides a lift of the action of $G_{p+1}$ on $BG_p$ defined by (3.14). Let $\Omega^* (EG_p)_{G_{p+1}}$
denote the subspace of $G_{p+1}$-invariant elements of $\Omega^* (EG_p)$. By precisely the same argument as
that used to prove Lemma 3.1 we have the following.

**Lemma 3.5.** $\Omega^* (EG_\ast)_{G_{p+1}}^G$ is a DG subalgebra of $\Omega^* (EG_\ast)$ and the inclusion

$$\Omega^* (EG_\ast)_{G_{p+1}}^G \hookrightarrow \Omega^* (EG_\ast)$$

is a quasi-isomorphism.
The discussion in the previous paragraphs also yield the following result.

**Proposition 3.6.** The Alekseev-Meinrenken map \( \hat{AM}^\theta \) has image contained in \( \Omega^*(BG_\bullet)^{G\ast+1} \).

**Proof.** Since the Maurer-Cartan form on \( G \) is left-invariant, the restriction of the Chern-Weil map \( c^\theta \) to \( (S^*g^*)^G \) has its image contained in \( \Omega^*(EG_\bullet)^{G\ast+1} \). The result is thus a direct consequence of the fact that the action of \( G_{p+1} \) on \( EG_p \) is a lifting of the action of \( G_{p+1} \) on \( BG_p \). \( \square \)

With all of the above ingredients in place, we now let \( \hat{c}^\theta \) be the Chen-Weil map \( c^\theta \) seen as a map taking values in \( \Omega^p(EG_\bullet)^{G\ast+1} \). We set accordingly \( \hat{AM}^\theta = \iota^* \circ \hat{c}^\theta \) and notice that \( \hat{AM}^\theta \) is nothing but the Alekseev-Meinrenken map \( AM^\theta \) seen as taking values in \( \Omega^*(BG_\bullet)^{G\ast+1} \).

**Theorem 3.7.** The map \( \hat{AM}^\theta : (S^*g^*)^G \to \Omega^*(BG_\bullet)^{G\ast+1} \) is a left inverse of the Van Est map \( \text{VE} : \Omega^*(BG_\bullet)^{G\ast+1} \to (S^*g^*)^G \).

**Proof.** We will first write down an explicit formula for the map \( \hat{AM}^\theta \). To that end, we fix a basis \( e_a \) of \( g \) with dual basis \( e^a \) and recall from Section 2.4 that we have set \( w^a = w(e^a) \), so that \( S^*g^* \) can be identified with the polynomial algebra in these variables. We also set \( \theta^a = \theta(e^a) \). Notice that \( \theta^a \) lives in bidegree \( (0,1) \), \( d\theta^a \) lives in bidegree \( (0,2) \) and \( \partial \theta^a \) lives in bidegree \( (1,1) \). It follows that the image of \( w^a \) under \( \hat{c}^\theta \) is \( \partial \theta^a \). Therefore, if we pick a monomial \( w^{a_1} \cdots w^{a_p} \) in \( S^*g^* \), we get

\[
\hat{c}^\theta(w^{a_1} \cdots w^{a_p}) = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \partial \theta^{a_{\sigma(1)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}}.
\]

and, consequently,

\[
\hat{AM}^\theta(w^{a_1} \cdots w^{a_p}) = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \iota^* (\partial \theta^{a_{\sigma(1)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}}).
\]

(3.23)

Next, let us determine \( \text{VE}(\hat{AM}^\theta(w^{a_1} \cdots w^{a_p})) \). To start, we fix \( x_1, \ldots, x_p \in g \). By the definition in \( [3.15] \), and recalling that \( x_{\sigma(i)}(e, \ldots, e) = x_{\sigma(i)}^i \) for all \( 1 \leq i \leq p \), we have

\[
\text{VE}(\hat{AM}^\theta(w^{a_1} \cdots w^{a_p}))(x_1, \ldots, x_p) = \sum_{\sigma' \in \mathcal{S}_p} \hat{AM}^\theta(w^{a_1} \cdots w^{a_p})(e, \ldots, e)(x_{\sigma'(1)}^1, \ldots, x_{\sigma'(p)}^p).
\]

(3.24)

Upon using \( [3.23] \), this becomes

\[
\text{VE}(\hat{AM}^\theta(w^{a_1} \cdots w^{a_p}))(x_1, \ldots, x_p)
= \sum_{\sigma' \in \mathcal{S}_p} \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} (\partial \theta^{a_{\sigma(1)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}})(e, \ldots, e)(d_{(e, \ldots, e)}(x_{\sigma'(1)}^1), \ldots, d_{(e, \ldots, e)}(x_{\sigma'(p)}^p)).
\]

(3.24)

Let us evaluate each of the terms inside the double sum. Firstly, attending to the definition of the cup product \( [3.5] \), one easily verifies that

\[
\partial \theta^{a_{\sigma(1)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}} = \pi_{1,2}^* \partial \theta^{a_{\sigma(1)}} \wedge \cdots \wedge \pi_{p,p+1}^* \partial \theta^{a_{\sigma(p)}},
\]

(3.25)

where \( \pi_{i,i+1} : EG_p \to EG_1 \) is the projection onto the \( i \)th and \((i+1)\)th factors with \( 1 \leq i \leq p \). Secondly, by virtue of \( [3.21] \),

\[
d_{(e, \ldots, e)}(x_{\sigma'(i)}^i) = x_{\sigma'(i)}^i(e, \ldots, e),
\]

(3.26)
for all $1 \leq i \leq p$. Putting together (3.25) and (3.25), we thus find
\[
(\partial \theta^{a_{\sigma(1)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}})_{(e, \ldots, e)}(d_{1}(e, \ldots, e)(x^{1}_{\sigma(1)}), \ldots, d_{l}(e, \ldots, e)(x^{p}_{\sigma(p)}))
= \sum_{\sigma'' \in \mathfrak{S}_{p}} \text{sgn}(\sigma'') \prod_{i=1}^{p} (\partial \theta^{a_{\sigma''(i)}})_{(e, e)} \left( (d_{\pi^{\sigma''(i)}}(e, \ldots, e)(x^{1}_{\sigma''(i)}), \ldots, d_{\pi^{\sigma''(i+1)}}(e, \ldots, e)(x^{p}_{\sigma''(i)})) \right),
\] (3.27)
where $\text{sgn}(\sigma'')$ denotes the sign of the permutation $\sigma''$. Next notice that $\mathfrak{g}_{p+1}^{\cdot}$ is the element of $\mathfrak{g}_{p+1}$ having its $i$th coordinate equal to $x_{\sigma''(i)}$ and all others zero. Consequently, the only non-zero contribution to the sum in (3.27) comes from the identity permutation. Also, it is straightforward to calculate that
\[
(\partial \theta^{a_{\sigma(i)}})_{(e, e)} \left( (d_{\pi_{i,j+1}}(e, \ldots, e)(x^{1}_{\sigma(i)}), \ldots, d_{\pi_{i,j+1}}(e, \ldots, e)(x^{p}_{\sigma(i)})) \right) = \theta^{a_{\sigma(i)}}(x_{\sigma(i)}) = \theta^{a_{\sigma(i)}}(x_{\sigma(i)}).
\]
In this way, (3.27) becomes
\[
(\partial \theta^{a_{\sigma(i)}} \cup \cdots \cup \partial \theta^{a_{\sigma(p)}})_{(e, \ldots, e)}(d_{1}(e, \ldots, e)(x^{1}_{\sigma(i)}), \ldots, d_{l}(e, \ldots, e)(x^{p}_{\sigma(p)})) = \prod_{i=1}^{p} \theta^{a_{\sigma(i)}}(x_{\sigma(i)}).
\]
Inserting this back in (3.24) gives
\[
\text{VE}(\mathcal{AM}^{\theta}(w^{a_{1}} \cdots w^{a_{p}}))(x_{1}, \ldots, x_{p}) = \sum_{\sigma \in \mathfrak{S}_{p}} \frac{1}{p!} \sum_{\sigma' \in \mathfrak{S}_{p}} \prod_{i=1}^{p} \theta^{a_{\sigma(i)}}(x_{\sigma(i)}) = \sum_{\sigma \in \mathfrak{S}_{p}} \prod_{i=1}^{p} \theta^{a_{\sigma(i)}}(x_{i}).
\]
This allows us to conclude that
\[
\text{VE}(\mathcal{AM}^{\theta}(w^{a_{1}} \cdots w^{a_{p}})) = w^{a_{1}} \cdots w^{a_{p}}.
\]
Since any element of $(S^{p}\mathfrak{g})^{G}$ is a linear combination of monomials $w^{a_{1}} \cdots w^{a_{p}}$ in $S^{p}\mathfrak{g}^{*}$, conclusion follows at once.

Combining the previous result with the above remarks immediately yields the following.

**Corollary 3.8.** The restricted Van Est map $\text{VE}: \Omega^{*}(BG_{*})^{G_{*+1}} \to (S^{*}\mathfrak{g}^{*})^{G}$ is a quasi-isomorphism.

### 3.2 The De Rham $A_{\infty}$-quasi-isomorphism for classifying spaces

In this subsection we establish a version of Gugenheim’s $A_{\infty}$ De Rham theorem for the classifying space $BG$. We shall start with some general considerations concerning the totalisation of semi-cosimplicial DG algebras.

For any positive integer $n$, let $[n]$ denote the set $\{0, 1, \ldots, n\}$. We then consider, for $p+q \leq n$, the map $l_{p,q}^{n}: [p] \to [n]$ defined by
\[
l_{p,q}^{n}(k) = k + q.
\] (3.28)
Notice that these maps satisfy the relations
\[
l_{p,q}^{n} \circ l_{p',q'}^{n} = l_{p+q,q'}^{n}.
\]
Let $A_* = \{A_p\}_{p \geq 0}$ be a semi-cosimplicial DG algebra with coface maps $\partial'_i: A_{p-1} \to A_p$ for $0 \leq i \leq p$. For $p \geq 0$ fixed, we write $A_p = \bigoplus_{q \geq 0} A^q_p$ for the underlying graded decomposition. Associated to $A_*$, there is a canonical DG algebra $\text{Tot}(A_*)$, constructed as follows. As a graded vector space, its $n$th degree summand is defined as

$$\text{Tot}(A_*)^n = \bigoplus_{p+q = n} A^q_p.$$ 

This becomes a cochain complex if we set $\partial = \partial' + \partial'': \text{Tot}(A_*)^n \to \text{Tot}(A_*)^{n+1}$, where the differential $\partial': A^q_p \to A^q_{p+1}$ is the alternating sum

$$\partial' = \sum_{i=0}^{p+1} (-1)^i \partial'_i,$$  

(3.29)

and the differential $\partial'': A^q_p \to A^{q+1}_p$ is $(-1)^p$ times the differential of $A_p$. To define the product on $\text{Tot}(A_*)$, we take we take the map induced on $A_*$ by the map $l^n_{p,q}$ given in (3.28), which by abuse of notation we also call $l^n_{p,q}$. We then have $l^n_{p,q}: A_p \to A_n$ for $p + q \leq n$. For any $a \in A^q_p$ and $a' \in A^q_{p'}$, we let $aa' \in A^{q+q'}_{p+p'}$ be the element defined as

$$aa' = (-1)^q p + p' (a) l^n_{p',p} (a').$$  

(3.30)

With these operations, one can verify that $\text{Tot}(A_*)$ is in fact a DG algebra. We omit the details, but comment that this depends upon the fact that the maps $l^n_{p,q}$ satisfy the following relations:

$$\partial'_i \circ l^n_{p,q} = \begin{cases} 
  m^n_{p+1} & \text{if } i > p + q, \\
  m^n_{p+1,q} \circ \partial'_{i-q} & \text{if } q < i \leq p + q, \\
  m^n_{p,q+1} & \text{if } i \leq q. 
\end{cases}$$  

(3.31)

More importantly for our purposes, the construction of $\text{Tot}(A_*)$ gives the following result.

**Proposition 3.9.** The assignment $A_* \mapsto \text{Tot}(A_*)$ defines a functor from the category of semi-cosimplicial DG algebras with semi-cosimplicial $A_\infty$-morphisms to the category of DG algebras with $A_\infty$-morphisms.

**Proof.** To start with, recall that a DG algebra $A$ with differential $\partial$ can be thought of as an $A_\infty$-algebra with $A_\infty$-operations $d: uA \to uA$ and $m: uA \otimes uA \to uA$ defined by declaring

$$d(ua) = u\partial a,$$

$$m(ua \otimes ua') = (-1)^{|a|+1} u(aa'),$$  

(3.32)

for homogeneous elements $a, a' \in A$. Next, let us introduce some notation to facilitate the presentation. Given a semi-cosimplicial DG algebra $A_* = \{A_p\}_{p \geq 0}$, when we write $a \in A^q_p$ we mean that $|a| = p + q$; on the other hand, if we write $\overline{a} \in A^q_{p'}$, we mean that $|\overline{a}| = q$. For $p \geq 0$ fixed, we also denote the $A_\infty$-operations associated to $A_p$ in accord to (3.32) by $d^n_p$ and $m_p$, respectively. With this notation, and the definitions (3.29) and (3.30), we can view $\text{Tot}(A_*)$ as an $A_\infty$-algebra.
by setting \( d = d' + d'' : \mathfrak{u} \text{Tot}(A_\ast) \to \mathfrak{u} \text{Tot}(A_\ast) \), where

\[
d'(ua) = \sum_{i=0}^{p+1} (-1)^i u \partial_i' \overline{a},
\]

\[
d''(ua) = (-1)^p d''_p(u \overline{a}),
\]

and \( m : \mathfrak{u} \text{Tot}(A_\ast) \otimes \mathfrak{u} \text{Tot}(A_\ast) \to \mathfrak{u} \text{Tot}(A_\ast) \) to be given by

\[
m(ua \otimes ua') = (-1)^{p'+q'} m_{p+p'}(u l_{p,0}^{p+p'}(\overline{a}), u l_{p',0}^{p+p'}(\overline{a}')),
\]

for any \( a \in A_p^q \) and \( a' \in A_{p'}^{q'} \).

We wish to show that the assignment \( A_\ast \mapsto \text{Tot}(A_\ast) \) is functorial with respect to semi-cosimplicial \( A_\infty \)-morphisms. So let \( \phi_\ast : A_\ast \to B_\ast \) be one such \( A_\infty \)-morphisms. This means that for each \( p \geq 0 \) we have an \( A_\infty \)-morphism \( \phi_p : A_p \to B_p \), and these commute with the coface maps of \( A_\ast \) and \( B_\ast \).

The map \( \text{Tot}(\phi_\ast) : \text{Tot}(A_\ast) \to \text{Tot}(B_\ast) \) is explicitly given as follows. Let \( a_1 \in A_{p_1}^{q_1}, \ldots, a_n \in A_{p_n}^{q_n} \), and put \( p = \sum_{j=1}^n p_j \) and \( r_i = \sum_{j=1}^{i-1} p_j \). Then

\[
\text{Tot}(\phi_\ast)(u a_1 \otimes \cdots \otimes u a_n) = (-1)^{i \leq i \leq n} p_{j(q+1)} \phi_{p,n}(u l_{p,1}^{p+1}(\overline{a}_i) \otimes \cdots \otimes u l_{p_n,r_n}^{p}(\overline{a}_n)).
\]

We claim that \( \text{Tot}(\phi_\ast) \) satisfies the required relations to be an \( A_\infty \)-morphism, which read

\[
d \circ \text{Tot}(\phi_\ast)_n + \sum_{i+j=n} m \circ (\text{Tot}(\phi_\ast)_i \otimes \text{Tot}(\phi_\ast)_j)
\]

\[
= \sum_{i+j+1=n} \text{Tot}(\phi_\ast)_i \circ (\text{id}^\otimes \otimes d \circ \text{id}^\otimes) + \sum_{i+j+2=n} \text{Tot}(\phi_\ast)_{n-1} \circ (\text{id}^\otimes \otimes m \circ \text{id}^\otimes).
\]

To verify the claim, we note that to prove (3.36) it is enough to prove that

\[
d' \circ \text{Tot}(\phi_\ast)_n = \sum_{i+j+1=n} \text{Tot}(\phi_\ast)_i \circ (\text{id}^\otimes \otimes d' \circ \text{id}^\otimes),
\]

and

\[
d'' \circ \text{Tot}(\phi_\ast)_n + \sum_{i+j=n} m \circ (\text{Tot}(\phi_\ast)_i \otimes \text{Tot}(\phi_\ast)_j)
\]

\[
= \sum_{i+j+1=n} \text{Tot}(\phi_\ast)_i \circ (\text{id}^\otimes \otimes d'' \circ \text{id}^\otimes) + \sum_{i+j+2=n} \text{Tot}(\phi_\ast)_{n-1} \circ (\text{id}^\otimes \otimes m \circ \text{id}^\otimes),
\]

separately.

We begin with (3.37). To that end, we write \( \tilde{\partial}'_i = u \circ \delta'_i \circ s \) and set \( s = \sum_{1 \leq i < j \leq n} p_j(q_i + 1) \).
From (3.29) and the fact that $\phi_p$ commutes with the $\tilde{\partial}_l$, it follows that

$$d'(\text{Tot}(\phi_\ast)_{\ast}(u_{a_1} \otimes \cdots \otimes u_{a_n})) = \sum_{i=0}^{p-1} (-1)^{s+i} \tilde{\partial}_i (\phi_{p,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$

$$= (-1)^s \tilde{\partial}_0 (\phi_{p,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$

$$+ (-1)^{s+p+1} \tilde{\partial}_{p+1} (\phi_{p,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$

$$+ \sum_{i=1}^{p_1} \sum_{j=1}^{p_1} (-1)^{s+r_i+j} \tilde{\partial}_{r_i+j} (\phi_{p,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$

$$= (-1)^s \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n))$$

$$+ (-1)^{s+p+1} \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n))$$

$$+ \sum_{i=1}^{p_1} \sum_{j=1}^{p_1} (-1)^{s+r_i+j} \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n))$$

Taking into account (3.31), the left-hand side of the last equality becomes

$$(-1)^s \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)) + (-1)^{s+p+1} \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n))$$

$$+ \sum_{i=1}^{p_1} \sum_{j=1}^{p_1} (-1)^{s+r_i+j} \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n))$$

$$\otimes u_{p_1+1, r_i+1}^p (\pi_{i+1}) \otimes \cdots \otimes u_{p_n, r_n+1}^p (\pi_n)) \).$$

A simple calculation shows that the first and second term of this expression cancel each other out. Thus, upon using (3.29) and (3.33) and putting all together,

$$d'(\text{Tot}(\phi_\ast)_{\ast}(u_{a_1} \otimes \cdots \otimes u_{a_n}))$$

$$= \sum_{i=1}^{n} (-1)^{s+r_i} \phi_{p+1,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)) \otimes u_{p_1+1, r_i+1}^p (\pi_{i+1}) \otimes \cdots \otimes u_{p_n, r_n+1}^p (\pi_n)$$

$$= \sum_{i+j+1=n} \text{Tot}(\phi_\ast)_{\ast}((\text{id}^{\otimes i} \otimes d' \otimes \text{id}^{\otimes j})(u_{a_1} \otimes \cdots \otimes u_{a_n}))$$

from which (3.37) follows.

Let us now tackle (3.38). By attending to (3.33) and (3.34), for the terms on the left-hand side of (3.38) we have

$$d''(\text{Tot}(\phi_\ast)_{\ast}(u_{a_1} \otimes \cdots \otimes u_{a_n})) = (-1)^{s+p} d''_p (\phi_{p,n}(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$

and

$$m((\text{Tot}(\phi_\ast)_{i} \otimes \text{Tot}(\phi_\ast)_{j})(u_{a_1} \otimes \cdots \otimes u_{a_n}))$$

$$= (-1)^{s+p} m_p ((\phi_{p,i} \otimes \phi_{p,j})(u_{l_{p_1, r_1}}^p (\pi_1) \otimes \cdots \otimes u_{l_{p_n, r_n}}^p (\pi_n)))$$.
Similarly, for the terms on the right-hand side \((3.38)\), one obtains

\[
\begin{align*}
\text{Tot}(\phi)_n((\id^{\otimes i} \otimes d'' \otimes \id^{\otimes j})\omega_{a_1} \cdots \omega_{a_n}) & = (-1)^{s+p} \phi_{p,n}((\id^{\otimes i} \otimes d'' \otimes \id^{\otimes j})\omega_{p_{1},r_{1}}(\overline{a}_1) \cdots \omega_{p_{n},r_{n}}(\overline{a}_n)), \\
\text{and} & \\
\text{Tot}(\phi)_{n-1}((\id^{\otimes i} \otimes m \otimes \id^{\otimes j})\omega_{a_1} \cdots \omega_{a_n}) & = (-1)^{s+p} \phi_{p,n-1}((\id^{\otimes i} \otimes m_p \otimes \id^{\otimes j})\omega_{p_{1},r_{1}}(\overline{a}_1) \cdots \omega_{p_{n},r_{n}}(\overline{a}_n)).
\end{align*}
\]

The desired conclusion is therefore a consequence of the fact that \(\phi_p\) is an \(A_\infty\)-morphism.

In order to finish the proof, one needs to check that \(A_* \mapsto \text{Tot}(A_*)\) preserves compositions. This is a straightforward verification which we omit. \(\square\)

The following result should also be noted.

**Lemma 3.10.** Let \(A_*\) and \(B_*\) be two semi-cosimplicial positively graded DG algebras. If \(\phi_* : A_* \to B_*\) is a semi-cosimplicial \(A_\infty\)-quasi-isomorphism then \(\text{Tot}(\phi)_* : \text{Tot}(A_*) \to \text{Tot}(B_*)\) is an \(A_\infty\)-quasi-isomorphism.

**Proof.** Since \(A_*\) and \(B_*\) are assumed to be positively graded, the map \(\text{Tot}(\phi)_1\) is a morphism of first-quadrant double complexes. Besides, by our hypothesis on \(\phi_*\), we see that \(\text{Tot}(\phi)_1\) induces an isomorphism on the vertical directions. We conclude therefore that the map of spectral sequences is an isomorphism at the first page, and hence \(\text{Tot}(\phi)_1\) induces an isomorphism in cohomology. \(\square\)

With these preliminaries out of the way, we may now formulate the version of Gugenheim’s \(A_\infty\) De Rham theorem for \(BG\) we are after. As in the previous section, consider the simplicial manifold \(BG\). Then, the Bott-Shulman-Stasheff complex gives us a semi-cosimplicial DG algebra \(\Omega^*(BG) = \{\Omega^*(BG_p)\}_{p \geq 0}\). Also, by taking singular cochains, we get a second semi-cosimplicial DG algebra \(\text{C}^*(BG) = \{\text{C}^*(BG_p)\}_{p \geq 0}\). Invoking Theorem \([2.1]\) for each \(p \geq 0\), there is an \(A_\infty\)-morphism \(\text{DR}_p : \Omega^*(BG_p) \to \text{C}^*(BG_p)\) induced by Gugenheim’s construction. Since the latter is natural with respect to the simplicial operations, this \(A_\infty\)-morphisms commutes with the coface maps of \(\Omega^*(BG)\) and \(\text{C}^*(BG)\). Thus, we actually get a semi-cosimplicial \(A_\infty\)-morphism \(\text{DR}_* : \Omega^*(BG) \to \text{C}^*(BG)\). One obtains the following.

**Theorem 3.11.** The induced \(A_\infty\)-morphism \(\text{Tot}(\text{DR}_*) : \text{Tot}(\Omega^*(BG_*)) \to \text{Tot}(\text{C}^*(BG_*))\) is an \(A_\infty\)-quasi-isomorphism.

**Proof.** It follows from Theorem \([2.1]\) that \(\text{DR}_* : \Omega^*(BG) \to \text{C}^*(BG)\) is a semi-cosimplicial \(A_\infty\)-quasi-isomorphism. Hence conclusion is a consequence of Lemma \([3.10]\). \(\square\)

### 3.3 The Hochschild-De Rham \(A_\infty\)-quasi-isomorphism

The goal of this subsection is to construct an \(A_\infty\)-quasi-isomorphism between \(\Omega^*(BG)\) and the DG algebra of Hochschild cochains on the space of singular chains \(C_*(G)\). We begin with some generalities.
Let $A$ be a DG Hopf algebra and let $\varepsilon: A \to \mathbb{R}$ be its counit. For each $p \geq 0$, we set $\Delta_p(A) = A^{\otimes p}$ and define the maps $\partial_i: \Delta_p(A) \to \Delta_{p-1}(A)$ by

$$\partial_i(a_1 \otimes \cdots \otimes a_p) = \begin{cases} 
\varepsilon(a_1) a_2 \otimes \cdots \otimes a_p & \text{if } i = 0, \\
a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_p & \text{if } 0 < i < p, \\
a_2 \otimes \cdots \otimes a_{p-1} \varepsilon(a_p) & \text{if } i = p.
\end{cases} \tag{3.39}$$

The following lemma is crucial.

**Lemma 3.12.** The collection $\Delta_\bullet(A) = \{\Delta_p(A)\}_{p \geq 0}$ is a semi-simplicial DG coalgebra with face maps $\partial_i$.

**Proof.** Since, by hypothesis, $A$ is a DG Hopf algebra, its product map is a morphism of DG coalgebras, which means that the maps $\partial_i$ for $0 < i < p$ are indeed morphisms of DG coalgebras. On the other hand, the fact that $\varepsilon: A \to \mathbb{R}$ is a morphism of DG Hopf algebras implies that both $\partial_0$ and $\partial_p$ are also morphisms of DG coalgebras. It remains only to check that $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. This is a routine calculation which we leave to the reader. \quad \Box

We next let $\Delta^\bullet(A)$ be the semi-cosimplicial DG algebra obtained by dualising $\Delta_\bullet(A)$. The following observation should be made.

**Lemma 3.13.** The product in $\text{Tot}(\Delta^\bullet(A))$ is given, for homogeneous elements $\varphi$ and $\psi$, by the formula

$$s^p \varphi \cdot s^q \psi = (-1)^{|\varphi|q} s^{p+q} (\varphi \cup \psi),$$

where $\cup$ designates the cup product in the Hochschild cochain complex $\text{HC}^\bullet(A)$.

**Proof.** As earlier, for $p + q \leq n$, let $t^{n}_{p,q}: \Delta^p(A) \to \Delta^n(A)$ denote the map induced by that given in (3.39). This induces a corresponding map $t^{n,*}_{p,q}: A^{\otimes n} \to A^{\otimes p}$. We claim that, if $\Delta$ denotes the coproduct in $A$,

$$\left(t^{p+q}_{p,0} \otimes t^{p+q}_{q,p}\right) \circ \Delta^{\otimes (p+q)} = \text{id}_{A^{\otimes (p+q)}}. \tag{3.40}$$

To substantiate our claim, we fix homogeneous elements $a_1, \ldots, a_{p+q} \in A$, and notice that

$$t^{p+q}_{p,0}(a_1 \otimes \cdots \otimes a_{p+q}) = \varepsilon(a_{p+1} \cdots a_{p+q}) a_1 \otimes \cdots \otimes a_p, \tag{3.41}$$

and

$$t^{p+q}_{q,p}(a_1 \otimes \cdots \otimes a_{p+q}) = \varepsilon(a_1 \cdots a_p) a_{p+1} \otimes \cdots \otimes a_{p+q}. \tag{3.42}$$

We also make use of Sweedler’s notation and write, for each $1 \leq i \leq p + q$,

$$\Delta(a_i) = \sum a_{i(1)} \otimes a_{i(2)}. \tag{3.43}$$
Using (3.41), (3.42) and (3.43), we find that
\[
\left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right) \Delta^{(p+q)} (a_1 \otimes \cdots \otimes a_{p+q})
\]
\[
= \sum (-1)^{|a_{(1)}|}|a_{(2)}| \left( \frac{p+q}{p,0} \otimes \frac{p+q}{p,q} \right) (a_{(1)} \otimes \cdots \otimes a_{p+q(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{p+q(2)})
\]
\[
= \sum (-1)^{|a_{(1)}|}|a_{(2)}| a_{(1)} \otimes \cdots \otimes a_{p+q(1)} \varepsilon (a_{p+1(1)} \cdots a_{p+q(1)} a_{1(2)} \cdots a_{p(2)}) a_{p+1(2)} \otimes \cdots \otimes a_{p+q(2)}
\]
\[
= \sum a_{(1)} \otimes \cdots \otimes a_{p+q(1)} \varepsilon (a_{p+1(1)} \cdots a_{p+q(1)} a_{1(2)} \cdots a_{p(2)}) a_{p+1(2)} \otimes \cdots \otimes a_{p+q(2)}
\]
\[
= \left( \sum a_{(1)} \varepsilon (a_{1(2)}) \right) \otimes \cdots \otimes \left( \sum a_{p+q(1)} \varepsilon (a_{p+q(2)}) \right)
\]
\[
= a_1 \otimes \cdots \otimes a_{p+q},
\]
where in the third equality we have used the fact that \( \varepsilon \) vanishes on elements of positive degree in order to set the signs to zero, and, in the last one, that \( \varepsilon \) is a counit for the coproduct. Thus (3.40) is true.

Next, let us denote by \( \mu^{p,q} \) the natural map from \( A^\otimes p^* \otimes A^\otimes q^* \) to \( (A^\otimes p \otimes A^\otimes q)^* \). It is a simple matter to verify that
\[
\mu^{p+q,p+q} \circ \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right) = \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right)^* \circ \mu^{p,q}.
\]

Moreover, we also have that
\[
\mu^{p,q}(\varphi \otimes \psi) = \varphi \cup \psi,
\]
for \( \varphi \in \Delta^p(A) \) and \( \psi \in \Delta^q(A) \). Now, the product in \( \text{Tot}(\Delta^*(A)) \) is defined, up to a sign, as the composition
\[
\Delta^\otimes (p+q)^* \circ \mu^{p+q,p+q} \circ \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right).
\]

More explicitly, for homogeneous elements \( \varphi \in \Delta^p(A) \) and \( \psi \in \Delta^q(A) \), we may write
\[
s^p \varphi \cdot s^q \psi = (-1)^{|s^q|} \varepsilon_{s^p+q} \Delta^\otimes (p+q)^* \left( \mu^{p+q,p+q} \left( \frac{p+q^*}{p,0} \varphi \otimes \frac{p+q^*}{p,q} \psi \right) \right).
\]

Using (3.44) and (3.45), the left hand side of (3.46) becomes
\[
(-1)^{|\varphi| s^q \varphi \Delta^\otimes (p+q)^* \left( \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right)^* \left( \mu^{p,q}(\varphi \otimes \psi) \right) \right)}
\]
\[
= (-1)^{|\varphi| s^q \varphi \Delta^\otimes (p+q)^* \left( \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right)^* (\varphi \cup \psi) \right)}
\]
\[
= (-1)^{|\varphi| s^q \varphi \Delta^\otimes (p+q)^* \left( \left( \frac{p+q^*}{p,0} \otimes \frac{p+q^*}{p,q} \right) \circ \Delta^\otimes (p+q)^* \right)^* (\varphi \cup \psi)}.
\]

Combining this with (3.40) gives
\[
s^p \varphi \cdot s^q \psi = (-1)^{|\varphi| s^q \varphi \Delta^\otimes (p+q)^* (\varphi \cup \psi)},
\]
as we wished to show.

\[\square\]

**Lemma 3.14.** There is an isomorphism of DG algebras \( \Theta: \text{Tot}(\Delta^*(A)) \rightarrow \text{HC}^*(A) \), which is explicitly given by
\[
\Theta(s^p(\varphi)) (ua_1 \otimes \cdots \otimes u a_p) = (-1)^{p|\varphi| + \frac{p(p-1)}{2}} \sum_{i=1}^{p-1} (-1)^{|a_i|} a_i, (p-i) \varphi (a_1 \otimes \cdots \otimes a_p),
\]
for homogeneous elements \( \varphi \in \Delta^p(A) \) and \( a_1, \ldots, a_p \in A \).
Proof. It is obvious that \( \Theta \) is a linear isomorphism. We have to show that it is also an algebra homomorphism. For this, let us fix homogeneous elements \( \varphi \in \Delta^p(A), \psi \in \Delta^q(A) \) and \( a_1, \ldots, a_{p+q} \in A \). Then, one the one hand, by virtue of Lemma 3.13

\[
\Theta(s^p \varphi \cdot s^q \psi)(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}).
\]

On the other hand,

\[
(\Theta(s^p \varphi) \cup \Theta(s^q \psi))(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|(p+q)}(ua_1 \otimes \cdots \otimes ua_{p+q})\Theta(s^q \psi)(ua_{p+1} \otimes \cdots \otimes ua_{p+p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}) \\
= (-1)^{|\varphi|+|\psi|+1}(ua_1 \otimes \cdots \otimes ua_{p+q}).
\]

By comparing the last two equalities above we find that

\[
\Theta(s^p \varphi \cdot s^q \psi) = \Theta(s^p \varphi) \cup \Theta(s^q \psi).
\]

We conclude that \( \Theta \) is indeed an algebra homomorphism. We leave it to the reader the task of checking that it also preserves the differentials. \( \square \)

Before moving forward, we need some definitions. An \( n \)-singular simplex \( \sigma: \Delta_n \to G \) is said to be degenerate if it can be factored as \( \sigma = \sigma' \circ \eta_i \), where \( \sigma': \Delta_{n-1} \to G \) is an \((n-1)\)-singular simplex and \( \eta_i: \Delta_{n-1} \to \Delta_n \) is a degeneracy map. It is not hard to see that the vector space \( \mathcal{I} \) generated by degenerate simplices on \( G \) is both a DG ideal and a DG coideal of \( C_\ast(G) \). Thus, taking the quotient vector space \( \overline{\mathcal{C}}_\ast(G) = C_\ast(G)/\mathcal{I} \), we obtain canonically a DG Hopf algebra structure on \( \overline{\mathcal{C}}_\ast(G) \) such that the projection \( q: C_\ast(G) \to \overline{\mathcal{C}}_\ast(G) \) is a morphism of DG Hopf algebras. It can be shown that \( q \) is in fact a quasi-isomorphism. We shall refer to the DG Hopf algebra \( \overline{\mathcal{C}}_\ast(G) \) as the algebra of normalized singular chains on \( G \).

Now let us again consider the simplicial manifold \( BG_\ast \) from Section 3.1. As the construction above is functorial in \( G \), it defines a semi-simplicial DG coalgebra \( \overline{\mathcal{C}}_\ast(BG_\ast) \) and a corresponding projection \( q_\ast: \mathcal{C}_\ast(BG_\ast) \to \overline{\mathcal{C}}_\ast(BG_\ast) \). For each \( p \geq 0 \), we let \( \overline{\mathcal{E}}_{p_\ast}: \mathcal{C}_\ast(G) \otimes^p \to \overline{\mathcal{C}}_\ast(BG_\ast) \) be the Eilenberg-Zilber map defined as reviewed in Section 2.4. We further let \( \overline{\mathcal{E}}_{p_\ast}: \mathcal{C}_\ast(G) \otimes^p \to \overline{\mathcal{C}}_\ast(BG_\ast) \) be defined as the composition

\[
\overline{\mathcal{E}}_{p_\ast} = q_\ast \circ \mathcal{E}_{p_\ast}.
\]

Owing to Proposition 2.2 and the preceding discussion, the map \( \overline{\mathcal{E}}_{p_\ast} \) is a quasi-isomorphism of DG coalgebras. Also, we have the following.
Lemma 3.15. The collection \( \{E\Sigma_p\}_{p \geq 0} \) determines a morphism of semi-simplicial DG coalgebras 
\( E\Sigma: \Delta_*(C_*(G)) \to \overline{C}_*(BG_*) \).

Proof. We need merely to show that \( E\Sigma_{p-1} \circ \partial_i = \partial_i \circ E\Sigma_p \) for \( 0 \leq i \leq p \). First, consider the case \( i = 0 \). For every collection \( \sigma_1, \ldots, \sigma_p \) of singular simplices on \( G \), we see from (3.39) that
\[
E\Sigma_{p-1}(\partial_0(\sigma_1 \otimes \cdots \otimes \sigma_p)) = \varepsilon(\sigma_1)E\Sigma_{p-1}(\sigma_2 \otimes \cdots \otimes \sigma_p)
\]
\[
= \begin{cases} 
0 & \text{if } |\sigma_1| > 0, \\
E\Sigma_{p-1}(\sigma_2 \otimes \cdots \otimes \sigma_p) & \text{if } |\sigma_1| = 0.
\end{cases}
\]

On the other hand, we can see from (2.17) that
\[
\partial_0(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)) = \partial_0 \left( \sum_{\chi \in \mathcal{E}_{|\sigma_1|} \cdots |\sigma_p|} (-1)^{\chi} q_p \circ (\sigma_1 \times \cdots \times \sigma_p) \circ \chi \right)
\]
\[
= \sum_{\chi \in \mathcal{E}_{|\sigma_1|} \cdots |\sigma_p|} (-1)^{\chi} q_{p-1} \circ \partial_0 (\sigma_1 \times \cdots \times \sigma_p) \circ \chi.
\]
If \( |\sigma_1| = 0 \), then \( \partial_0 (\sigma_1 \times \cdots \times \sigma_p) = \sigma_2 \times \cdots \times \sigma_p \) and therefore
\[
\partial_0(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)) = E\Sigma_{p-1}(\sigma_2 \otimes \cdots \otimes \sigma_p).
\]
If \( |\sigma_1| = 0 \), then \( q_{p-1} \circ \partial_0 (\sigma_1 \times \cdots \times \sigma_p) \circ \chi = 0 \), since \( \partial_0 (\sigma_1 \times \cdots \times \sigma_p) \circ \chi \) is degenerate, from which it follows that
\[
\partial_0(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)) = 0,
\]
and hence the result. The case \( i = p \) is completely analogous. So there only remains the case \( 0 < i < p \). On the one hand, using (3.39) gives
\[
E\Sigma_{p-1}(\partial_i(\sigma_1 \otimes \cdots \otimes \sigma_p)) = q_{p-1}(E\Sigma_{p-1}(\sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \sigma_i \sigma_{i+1} \sigma_{i+2} \otimes \cdots \otimes \sigma_p)).
\]
On the other hand, observing that, in the notation employed at the end Section 2.4, the face map
\( \partial_i: C_*(BG_p) \to C_*(BG_{p-1}) \) is given by \( \partial_i = (id^{\otimes (i-1)} \times \mu \times id^{\otimes (p-i)})_* \) and that, therefore,
\[
\partial_i \circ E\Sigma_p = E\Sigma_{p-1} \circ (id^{\otimes (i-1)} \otimes (\mu_* \circ E\Sigma_2) \otimes id^{\otimes (p-1)}),
\]
we find that
\[
\partial_i(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)) = \partial_i(q_p(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)))
\]
\[
= q_{p-1}(\partial_i(E\Sigma_p(\sigma_1 \otimes \cdots \otimes \sigma_p)))
\]
\[
= q_{p-1}(E\Sigma_{p-1}(\sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \sigma_i \sigma_{i+1} \otimes \sigma_{i+2} \otimes \cdots \otimes \sigma_p)),
\]
as wished. \( \square \)

Let us now write \( \overline{C}_*(BG_*) \) to denote the semi-cosimplicial DG algebra obtained by dualising \( C_*(BG_*) \). Also, let us denote by \( q_*: C_*(BG_*) \to \overline{C}_*(BG_*) \) the inclusion dual to the projection \( q_*: C_*(BG_*) \to C_*(BG_*) \). We make an observation to be applied in the subsequent argument.
Lemma 3.16. The semi-cosimplicial $A_{\infty}$-morphism $\text{DR}_\ast : \Omega^\ast(BG_\ast) \to C^\ast(BG_\ast)$ factors through $C^\ast(BG_\ast)$, that is, there is a semi-cosimplicial $A_{\infty}$-morphism $\text{DR}_\ast : \Omega^\ast(BG_\ast) \to C^\ast(BG_\ast)$ such that $\text{DR}_\ast = e_\ast \circ \overline{\text{DR}}_\ast$.

Proof. It suffices to show that $\text{DR}_\ast : \Omega^\ast(BG_\ast) \to C^\ast(BG_\ast)$ takes values in those singular cochains that vanish on degenerate singular simplices. But this holds by Proposition 3.26 of [1].

Next we consider the semi-cosimplicial DG algebra $\Delta^\ast(C_\ast(G))$ dual to $\Delta^\ast(C_\ast(G))$. By taking the dual of $\mathbb{E}Z_\ast : \Delta^\ast(C_\ast(G)) \to C_\ast(BG_\ast)$, we obtain a morphism of semi-cosimplicial DG algebras $\mathbb{E}Z^\ast : C^\ast(BG_\ast) \to \Delta^\ast(C_\ast(G))$. We let $\text{DR}_\ast^\Delta : \Omega^\ast(BG_\ast) \to \Delta^\ast(C_\ast(G))$ be the semi-cosimplicial $A_{\infty}$-morphism defined as the composition

$$\text{DR}_\ast^\Delta = \mathbb{E}Z^\ast \circ \text{DR}_\ast,$$

where $\text{DR}_\ast : \Omega^\ast(BG_\ast) \to C^\ast(BG_\ast)$ is the semi-cosimplicial $A_{\infty}$-morphism from Lemma 3.16. By Proposition 3.9, the latter induces an $A_{\infty}$-morphism $\text{Tot}(\text{DR}_\ast^\Delta) : \text{Tot}(\Omega^\ast(BG_\ast)) \to \text{Tot}(\Delta^\ast(C_\ast(G)))$. We then apply Lemma 3.14 to get and $A_{\infty}$-morphism $\text{DR}^\Theta : \text{Tot}(\Omega^\ast(BG_\ast)) \to HC^\ast(C_\ast(G))$ defined as the composition

$$\text{DR}^\Theta = \Theta \circ \text{Tot}(\text{DR}_\ast^\Delta).$$

These observations taken together with the preceding results yield the following.

Theorem D. The induced $A_{\infty}$-morphism $\text{DR}^\Theta : \text{Tot}(\Omega^\ast(BG_\ast)) \to HC^\ast(C_\ast(G))$ is an $A_{\infty}$-quasi-isomorphism.

Proof. Because of Lemma 3.14, it is enough to prove that $\text{Tot}(\text{DR}_\ast^\Delta)$ is an $A_{\infty}$-quasi-isomorphism. In order to do so, notice that both $\mathbb{E}Z_\ast$ and $\text{DR}_\ast$ are semi-cosimplicial $A_{\infty}$-quasi-isomorphism. Thus, taking note of the definition (3.48), we conclude that $\text{DR}_\ast^\Delta$ is also a semi-cosimplicial $A_{\infty}$-quasi-isomorphism. The desired assertion now follows from Lemma 3.10.

In the remaining part of this section, we will prove a vanishing result for the $A_{\infty}$-morphism $\text{DR}^\Theta$, which will be needed later. First a little terminology. We say that an $r$-singular simplex $\sigma : \Delta^r \to BG_p$ is decomposable if there is a collection of $r_i$-singular simplices $\sigma_i : \Delta^r_i \to G$ with $i = 1, \ldots, p$ and $r = \sum_{i=1}^p r_i$, together with an $(r_1, \ldots, r_p)$-shuffle $\chi$ such that

$$\sigma = (\sigma_1 \times \cdots \times \sigma_p) \circ \chi_s.$$

(3.50)

It is immediately apparent from (2.17) that, for each $p \geq 0$, the image of the Eilenberg-Zilber map $\mathbb{E}Z_p : C_\ast(G)^{\otimes p} \to C_\ast(BG_p)$ is generated by decomposable singular simplices.

Proposition 3.17. Let $n > 1$ and consider differential forms $\omega_1 \in \Omega^{r_1}(G), \ldots, \omega_n \in \Omega^{r_n}(G)$. Then

$$\text{DR}^\Theta_n(u\omega_1 \otimes \cdots \otimes u\omega_n) = 0.$$

Proof. We will in fact show that

$$\text{Tot}(\text{DR}_\ast^\Delta)_n(u\omega_1 \otimes \cdots \otimes u\omega_n) = 0,$$

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which clearly suffices thanks to Lemma 3.14. To begin with, by Proposition 3.9 we know that \( \text{Tot}(DR^*_{\mathbb{R}}) = \text{Tot}(EZ^*) \circ \text{Tot}(DR_*). \) Thus, in view of our previous remark, it will be enough to show that if \( \sigma : \Delta_r \to BG_n \) with \( r = \sum_{i=1}^n q_i - n + 1 \) is an decomposable singular simplex, then
\[
\text{Tot}(DR_*)_n(u_{\omega_1} \otimes \cdots \otimes u_{\omega_n})(\sigma) = 0.
\]

According to definition (3.35), this means that
\[
DR_{n,n}(u_{l_{1,0}^n}(\omega_1) \otimes \cdots \otimes u_{l_{n-1}^n}(\omega_n))(\sigma) = 0.
\]

Next, notice that if, for \( i = 1, \ldots, n \), we let \( pr_i : BG_n \to G \) be the projection onto the \( i \)th factor, then \( l_{i,i-1}^n = pr_i^* \). Thus (3.51) becomes
\[
DR_{n,n}(upr_1^* \omega_1 \otimes \cdots \otimes upr_n^* \omega_n)(\sigma) = 0. \tag{3.52}
\]

On the other hand, using the notation for the \( A_\infty \) version of De Rham map from 2.3, we have that
\[
DR_{n,n}(upr_1^* \omega_1 \otimes \cdots \otimes upr_n^* \omega_n)(\sigma) = \pm \int_{1-r} \theta^*(P\sigma)^* \int_{\Delta_n} ev^*(\pi_1^* pr_1^* \omega_1 \wedge \cdots \wedge \pi_n^* pr_n^* \omega_1)
\]
\[
= \pm \int_{1-r} \theta^*(P\sigma)^* \int_{\Delta_n} (id \times P\sigma)^* ev^*(\pi_1^* pr_1^* \omega_1 \wedge \cdots \wedge \pi_n^* pr_n^* \omega_1).
\]

Consequently, to show (3.52), it is sufficient to show that
\[
(id \times P\sigma)^* ev^*(\pi_1^* pr_1^* \omega_1 \wedge \cdots \wedge \pi_n^* pr_n^* \omega_1) = 0. \tag{3.53}
\]

Now let us use the fact that \( \sigma \) is decomposable. By definition, this means that
\[
\sigma = (\sigma_1 \times \cdots \times \sigma_n) \circ \chi,*
\]
for a collection of \( r_i \)-singular simplices \( \sigma_i : \Delta_{r_i} \to G \) with \( i = 1, \ldots, n \) and \( r = \sum_{i=1}^n r_i \), and for an \((r_1, \ldots, r_n)\)-shuffle \( \chi \). Therefore,
\[
id \times P\sigma = (id \times P(\sigma_1 \times \cdots \times \sigma_n)) \circ (id \times P\chi_*),
\]
and hence
\[
ev \circ (id \times P\sigma) = ev \circ (id \times P(\sigma_1 \times \cdots \times \sigma_n)) \circ (id \times P\chi_*) = (\sigma_1 \times \cdots \times \sigma_n)^{\times n} \circ ev \circ P\chi_*.
\]

Thus, to show (3.53), it will be enough to show that
\[
ev^*((\sigma_1 \times \cdots \times \sigma_n)^{\times n} \times (\pi_1^* pr_1^* \omega_1 \wedge \cdots \wedge \pi_n^* pr_n^* \omega_1) = 0. \tag{3.54}
\]

Now a simple calculation reveals that
\[
ev^*((\sigma_1 \times \cdots \times \sigma_n)^{\times n} \times (\pi_1^* pr_1^* \omega_1 \wedge \cdots \wedge \pi_n^* pr_n^* \omega_1)) = ev^*((\pi_1^* \omega_1 \wedge \cdots \wedge \sigma_n^* \omega_1)). \tag{3.55}
\]

On the other hand, since \( n > 1 \), we have that \( r = \sum_{i=1}^n q_i - n + 1 < \sum_{i=1}^n q_i \). But \( r = \sum_{i=1}^n r_i \), so there must exists a \( k \in \{1, \ldots, n\} \) such that \( r_k < q_k \). This implies that \( \sigma_k^* \omega_k = 0 \), and as a result (3.54) follows from (3.55). \(\square\)
4 $A_\infty$-quasi-equivalence of DG categories

In this section we prove the main result of the paper, which is the construction of a zig-zag of $A_\infty$-quasi-equivalences between the DG enhancements of the categories $\text{Rep}(\mathbb{T}g)$ and $\text{Mod}(C_*(G))$.

4.1 DG enhancement of the category $\text{Rep}(\mathbb{T}g)$

In this subsection we describe a DG enhancement of the category $\text{Rep}(\mathbb{T}g)$. Let $V$ be an object of $\text{Rep}(\mathbb{T}g)$. For $x \in \mathfrak{g}$, by a slight abuse of notation, we will indistinctly write $i_x$ and $L_x$ for the contraction and Lie derivative operators acting on $W_\mathfrak{g}$ or $V$. An element $\alpha \in W_\mathfrak{g} \otimes V$ will be called \textit{basic} if

\[
(i_x \otimes 1)\alpha = 0,
\]

\[
(L_x \otimes 1 + 1 \otimes L_x)\alpha = 0,
\]

for every $x \in \mathfrak{g}$. Since the operators $i_x \otimes 1$ and $L_x \otimes 1 + 1 \otimes L_x$ are derivations, the basic elements form a graded subspace of $W_\mathfrak{g} \otimes V$. It will be denoted by $(W_\mathfrak{g} \otimes V)_{bas}$.

Next, consider the DG algebra $W_\mathfrak{g} \otimes \text{End}(V)$ with multiplication induced by the composition operation $\text{End}(V)$ and the differential $d_W + \delta$. Fix a basis $e_a$ of $\mathfrak{g}$ with structure constant $f^a_{bc}$ and recall from Section 2.4 that $t^a$ stands for the degree 1 generators of $\Lambda^1 \mathfrak{g}$ and $w^a$ stands for the degree 2 generators of $S^2 \mathfrak{g}$.

\textbf{Lemma 4.1.} The element $t^a \otimes L_a - w^a \otimes i_a$ is a Maurer-Cartan element of $W_\mathfrak{g} \otimes \text{End}(V)$.

\textit{Proof.} On the one hand, according to (2.35) and the relations (2.21),

\[
d_W(t^a \otimes L_a - w^a \otimes i_a) = d_W t^a \otimes L_a - d_W w^a \otimes i_a
\]

\[
= w^a \otimes L_a - \frac{1}{2} f^a_{bc} t^c \otimes L_a - f^c_{bc} b^b w^c \otimes i_a,
\]

and

\[
\delta(t^a \otimes L_a - w^a \otimes i_a) = -t^a \otimes [\delta, L_a] - w^a \otimes [\delta, i_a] = -w^a \otimes L_a.
\]

Hence,

\[
(d_W + \delta)(t^a \otimes L_a - w^a \otimes i_a) = -f^c_{bc} b^b w^c \otimes i_a - \frac{1}{2} f^c_{bc} b^b t^c \otimes L_a
\]

On the other hand, again using the relations (2.21), we find that

\[
(t^b \otimes L_b - w^b \otimes i_b)(t^c \otimes L_c - w^c \otimes i_c)
\]

\[
= t^b t^c \otimes L_b L_c - t^b w^c \otimes L_b i_c + w^b t^c \otimes i_b L_c + w^b w^c \otimes i_b i_c
\]

\[
= \frac{1}{2} t^b t^c \otimes [L_b, L_c] - t^b w^c \otimes [L_b, i_c] - t^b t^c \otimes i_c L_b + w^b t^c \otimes i_b L_c + \frac{1}{2} w^b w^c \otimes [i_b, i_c]
\]

\[
= \frac{1}{2} f^a_{bc} t^c \otimes L_a - f^a_{bc} b^b w^c \otimes i_a
\]

\[
= \frac{1}{2} f^a_{bc} b^b t^c \otimes L_a + f^a_{bc} b^b t^c \otimes i_a.
\]

In conclusion, we obtain

\[
(d_W + \delta)(t^a \otimes L_a - w^a \otimes i_a) + (t^b \otimes L_b - w^b \otimes i_b)(t^c \otimes L_c - w^c \otimes i_c) = 0,
\]

as required. \qed
This result has the following important consequence.

**Corollary 4.2.** *The operator* $D$ *in* $W\mathfrak{g} \otimes V$ *given by*

$$D = d_W + \delta + t^a \otimes L_a - w^a \otimes i_a,$$

*is a derivation of homogenous degree 1 that satisfies* $D^2 = 0$.

Also, the following property holds true.

**Lemma 4.3.** *The differential* $D$ *preserves the graded subspace* $(W\mathfrak{g} \otimes V)_{bas}$.

**Proof.** It suffices to show that

$$[D, L_c \otimes 1 + 1 \otimes L_c] = 0,$$

and

$$[D, i_c \otimes 1] = L_c \otimes 1 + 1 \otimes L_c.$$ Fix an element of $W\mathfrak{g} \otimes V$ of the form $\xi \otimes v$. Then a straightforward computation gives

$$D((L_c \otimes 1 + 1 \otimes L_c)(\xi \otimes v)) = d_W(L_c \xi) \otimes v + (-1)^{|\xi|} L_c \xi \otimes \delta v$$

$$+ d_W \xi \otimes L_c v + (-1)^{|\xi|} \xi \otimes \delta(L_c v)$$

$$+ (t^a L_c \xi) \otimes L_a v - (-1)^{|\xi|}(w^a L_c \xi) \otimes i_a v$$

$$+ (t^a \xi) \otimes L_a(L_c v) - (-1)^{|\xi|}(w^a \xi) \otimes i_a(L_c v),$$

and

$$(L_c \otimes 1 + 1 \otimes L_c)(D(\xi \otimes v)) = L_c(d_W \xi) \otimes v + (-1)^{|\xi|} L_c \xi \otimes \delta v$$

$$+ d_W \xi \otimes L_c v + (-1)^{|\xi|} \xi \otimes L_c(\delta v)$$

$$- f_{cb}^a(t^b \xi) \otimes L_a v + (t^a L_c \xi) \otimes L_a v$$

$$+ (-1)^{|\xi|} f_{cb}^a(w^b \xi) \otimes i_a v - (-1)^{|\xi|}(w^a L_c \xi) \otimes i_a v$$

$$+ (t^a \xi) \otimes L_a(L_c v) - (-1)^{|\xi|}(w^a \xi) \otimes L_c(i_a v).$$

Therefore, from (2.21), it follows that

$$[D, L_c \otimes 1 + 1 \otimes L_c](\xi \otimes v) = [d_W, L_c]\xi \otimes v + (-1)^{|\xi|} \xi \otimes [\delta, L_c]v$$

$$+ f_{cb}^a(t^b \xi) \otimes L_a v - (-1)^{|\xi|} f_{cb}^a(w^b \xi) \otimes i_a v$$

$$+ (t^a \xi) \otimes [L_a, L_c] v + (-1)^{|\xi|}(w^a \xi) \otimes [L_c, i_a] v$$

$$= f_{cb}^a(t^b \xi) \otimes L_a v - (-1)^{|\xi|} f_{cb}^a(w^b \xi) \otimes i_a v$$

$$+ f_{b}^b(t^a \xi) \otimes L_b v + (-1)^{|\xi|} f_{ba}^b(w^a \xi) \otimes i_b v$$

$$= 0.$$ Thus the first identity is established. On the other hand, again by a direct computation,

$$D((i_c \otimes 1)(\xi \otimes v)) = d_W(i_c \xi) \otimes v - (-1)^{|\xi|} i_c \xi \otimes \delta v$$

$$+ (t^a i_c \xi) \otimes L_a v + (-1)^{|\xi|}(w^a i_c \xi) \otimes i_a v,$$
and

\[(i_c \otimes 1)(D(\xi \otimes v)) = i_c(d_W \xi) \otimes v + (-1)^{|\xi|} i_c \xi \otimes \delta v + \delta_c^a \xi L_a v - (t^a i_c) \otimes L_a v - (-1)^{|\xi|} (w^a i_c) \otimes i_a v.\]

Hence, using (2.21) again, this gives

\[[D, i_c \otimes 1](\xi \otimes v) = [d_W, i_c] \xi \otimes v + \delta_c^a \xi L_a v = L_c \xi \otimes v + \xi L_c v = (L_c \otimes 1 + 1 \otimes L_c)(\xi \otimes v),\]

and, consequently, the second identity also holds.

The preceding discussion allows us to define a DG category, which provides a DG enhancement of the category \(\text{Rep}(\mathbb{T}_g)\), by the following data. The objects of this DG category are the same as those of \(\text{Rep}(\mathbb{T}_g)\). For any two objects \(V\) and \(V'\), with corresponding differentials \(D\) and \(D'\), the space of morphisms is the graded vector space \((W_g \otimes \text{Hom}(V, V'))_{\text{bas}}\) with the differential \(\partial_{D, D'}\) acting according to the formula

\[\partial_{D, D'} \phi = D' \circ \phi - (-1)^k \phi \circ D,\]

for any homogeneous element \(\phi\) of degree \(k\). The DG category given by this data will be denoted by \(\text{DGRep}(\mathbb{T}_g)\).

### 4.2 The Bott-Shulman-Stasheff DG category

In this subsection we introduce a DG category canonically associated to the Lie group \(G\), which is based on the Bott-Shulman-Stasheff model discussed in §3.1. This DG category will play an essential intermediate role in the proof of our main result.

Let \(V\) be object of \(\text{Rep}(\mathbb{T}_g)\) and consider the DG algebra \(\Omega^*(BG) \otimes \text{End}(V)\) with multiplication induced by the composition operation on \(\text{End}(V)\) and the differential \(\partial + \bar{\delta}\), where \(\bar{\delta}\) here is defined as \((-1)^p\) times the differential \(\delta\) when acting on \(\Omega^*(BG_p) \otimes \text{End}(V)\). Let also \(\Phi_V\) be the left-equivariant representation form associated to \(V\). We note that \(\Phi_V\) may be thought of as an element of \(\Omega^*(BG) \otimes \text{End}(V)\) of homogeneous of degree 1 with respect to the total degree.

**Lemma 4.4.** The element \(\Phi_V - \text{id}_V\) is a Maurer-Cartan element of \(\Omega^*(BG) \otimes \text{End}(V)\).

**Proof.** We must show that

\[(-d + \partial - \bar{\delta})(\Phi_V - \text{id}_V) + (\Phi_V - \text{id}_V) \circ (\Phi_V - \text{id}_V) = 0.\]

To prove this, we first notice that \(d(\text{id}_V) = 0\) and \(\delta(\text{id}_V) = 0\). Moreover, by decomposing \(\Phi_V = \sum_{k \geq 0} \Phi_V^{(k)}\) and bringing to mind the “descent equations” (2.33), we obtain that

\[(d + \bar{\delta})\Phi_V = 0.\]
We are thus left to show that
\[ \partial(\Phi_V - \text{id}_V) + (\Phi_V - \text{id}_V) \cup (\Phi_V - \text{id}_V) = 0. \]

Toward this end, we notice that in the present situation \( \partial = \varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* \). Furthermore, according to (3.2), the face maps \( \varepsilon_0, \varepsilon_1 \) and \( \varepsilon_2 \) coincide with the projection onto the second component \( \pi_2 \), the multiplication map \( \mu \) and the projection onto the first component \( \pi_1 \), respectively. Therefore,
\[ \partial(\Phi_V - \text{id}_V) = \pi_2^* \Phi_V - \mu^* \Phi_V + \pi_1^* \Phi_V - \text{id}_V. \]

On the other hand, taking note of the condition (2.34), we have, for the first cup product term,
\[ \Phi_V \cup \Phi_V = \sum_{k \geq 2} \sum_{i+j=k} \Phi_V^{(i)} \cdot \Phi_V^{(j)} = \sum_{k \geq 2} \sum_{i+j=k} (-1)^{i+j} (-1)^i \pi_1^* \Phi_V^{(i)} \land \pi_2^* \Phi_V^{(j)} \]
\[ = \sum_{k \geq 2} \sum_{i+j=k} (-1)^{i} \pi_1^* \Phi_V^{(i)} \land \pi_2^* \Phi_V^{(j)} = \sum_{k \geq 2} \mu^* \Phi_V^{(k)} = \mu^* \Phi_V. \]

For the remaining cup product terms, we compute
\[ \Phi_V \cup \text{id}_V = \sum_{k \geq 2} \Phi_V^{(k)} \land \text{id}_V = \sum_{k \geq 2} (-1)^{-k} (-1)^k \pi_1^* \Phi_V^{(k)} \land \pi_2^* \text{id}_V = \sum_{k \geq 2} \pi_1^* \Phi_V^{(k)} = \pi_1^* \Phi_V, \]
\[ \text{id}_V \cup \Phi_V = \sum_{k \geq 2} \text{id}_V \land \Phi_V^{(k)} = \sum_{k \geq 2} \pi_1^* \text{id}_V \land \pi_2^* \Phi_V^{(k)} = \sum_{k \geq 2} \pi_2^* \Phi_V^{(k)} = \pi_2^* \Phi_V, \]
\[ \text{id}_V \cup \text{id}_V = \pi_1^* \text{id}_V \land \pi_2^* \text{id}_V = \text{id}_V. \]

Consequently,
\[ \partial(\Phi_V - \text{id}_V) + (\Phi_V - \text{id}_V) \cup (\Phi_V - \text{id}_V) \]
\[ = \partial(\Phi_V - \text{id}_V) + \Phi_V \cup \Phi_V - \Phi_V \cup \text{id}_V - \text{id}_V \cup \Phi_V + \text{id}_V \cup \text{id}_V \]
\[ = \pi_2^* \Phi_V - \mu^* \Phi_V + \pi_1^* \Phi_V - \text{id}_V + \mu^* \Phi_V^{(k)} - \pi_1^* \Phi_V - \pi_2^* \Phi_V + \text{id}_V \]
\[ = 0, \]

as we wished to show. \( \square \)

As in the previous section, we have the following direct consequence of this result.

**Corollary 4.5.** The operator \( D \) in \( \Omega^*(BG, \ast) \otimes V \) given by
\[ D = \bar{d} + \partial + \bar{\delta} + \Phi_V - \text{id}_V, \]
is a derivation of homogeneous degree 1 that satisfies \( D^2 = 0 \).

In light of the above discussion, we can define a DG category by the following data. The objects of this DG category are the same as those of \( \text{Rep}(\mathbb{T}_g) \). For any two objects \( V \) and \( V' \), with corresponding differentials \( D \) and \( D' \), the space of morphisms is the graded vector space \( \Omega^*(BG, \ast) \otimes \text{Hom}(V, V') \) with the differential \( \partial_{D, D'} \) given by the same formula as the one for \( \text{DGRep}(\mathbb{T}_g) \). This DG category will be called the **Bott-Shulman-Stasheff DG category** and will be denoted by \( \text{BSS}(G) \).
4.3 The invariant Bott-Shulman-Stasheff DG category

Our aim now is to consider an invariant version of the Bott-Shulman-Stasheff DG category we have just introduced. It is this DG category that is linked to the “infinitesimal” DG category $\text{DGRep}(T_g)$ discussed in §4.1. We start with some preliminary remarks. The notation is the same as in §4.1.

Let $V$ be an object of $\text{Rep}(T_g)$ with associated left-equivariant representation form $\Phi_V$. Recall that, with respect to the decomposition $\Phi_V = \sum_{k \geq 0} \Phi^{(k)}_V$, the zeroth component $\Phi^{(0)}_V$ is a representation of $G$ on $V$. With this understanding, let us consider the action $\hat{\gamma}_0(g)$ of elements $g$ of $G$ on $\Omega^q(BG_p) \otimes V$ defined by

$$\hat{\gamma}_0(g)(\omega \otimes v) = \gamma_0(g)^* \omega \otimes (\Phi^{(0)}_V(g)(v)), \quad (4.3)$$

for $\omega \in \Omega^q(BG_p)$ and $v \in V$. We should also consider the action $\tilde{\gamma}(g_1, \ldots, g_p)$ of elements $(g_1, \ldots, g_p)$ of $G_p$ on $\Omega^q(BG_p) \otimes V$ given by

$$\tilde{\gamma}(g_1, \ldots, g_p)(\omega \otimes v) = \gamma(g_1, \ldots, g_p)^* \omega \otimes v, \quad (4.4)$$

for $\omega \in \Omega^q(BG_p)$ and $v \in V$. Noting that these two actions commute, we obtain an action $\hat{\zeta}(g_0, g_1, \ldots, g_p)$ of elements $(g_0, g_1, \ldots, g_p)$ of $G_{p+1}$ on $\Omega^q(BG_p) \otimes V$ by simply putting

$$\hat{\zeta}(g_0, g_1, \ldots, g_p) = \hat{\gamma}_0(g_0) \circ \tilde{\gamma}(g_1, \ldots, g_p). \quad (4.5)$$

For what follows, we let $[\Omega^q(BG_p) \otimes V]^{G_{p+1}}$ denote the subspace of $G_{p+1}$-invariant elements of $\Omega^q(BG_p) \otimes V$.

**Lemma 4.6.** $[\Omega^q(BG_\ast) \otimes V]^{G_{p+1}}$ is a subcomplex of $\Omega^q(BG_\ast) \otimes V$ and the inclusion

$$[\Omega^q(BG_\ast) \otimes V]^{G_{p+1}} \hookrightarrow \Omega^q(BG_\ast) \otimes V$$

is a quasi-isomorphism.

**Proof.** In view of the second relation in (4.2), we deduce that $\bar{\delta}$ preserves $G_{p+1}$-invariant elements. Consequently, the result follows from the first part of the proof of Lemma 3.1. \hfill \Box

Consider next the derivation $D$ of $\Omega^q(BG_\ast) \otimes V$ as defined in Corollary 4.5. We have the following important observation.

**Lemma 4.7.** $D$ preserves the subcomplex $[\Omega^q(BG_\ast) \otimes V]^{G_{p+1}}$.

**Proof.** From the definition, it is clearly sufficient to show that the left-equivariant representation form $\Phi_V$ preserves $[\Omega^q(BG_\ast) \otimes V]^{G_{p+1}}$. To prepare for this, we first observe that the $k$th component $\Phi^{(k)}_V$ of $\Phi_V$ satisfies

$$L_g \Phi^{(k)}_V = \Phi^{(0)}_V(g) \circ \Phi^{(k)}_V, \quad (4.6)$$

where $L_g$ indicates the left translation determined by the group element $g$; see Lemma 3.15 of [3]. Using this, we claim that

$$R_g^* \Phi^{(k)}_V = \Phi^{(k)}_V \circ \Phi^{(0)}_V(g), \quad (4.7)$$
where $R_g$ indicates the right translation determined by the group element $g$. Indeed, let $x_1, \ldots, x_k \in \mathfrak{g}$. Then, attending to the definition of $\Phi^{(k)}_V$ in (2.32), we get

\[
(R^*_g \Phi^{(k)}_V)(e)(x_1, \ldots, x_k) = \Phi^{(k)}_V(g)\left((dR_g)e(x_1), \ldots, (dR_g)e(x_k)\right)
\]

\[
= (L^*_g \Phi^{(k)}_V)(e)\left((d(L_{g^{-1}} \circ R_g)x_1), \ldots, (d(L_{g^{-1}} \circ R_g)x_k)\right)
\]

\[
= (L^*_g \Phi^{(k)}_V)(e)(\text{Ad}_{g^{-1}}x_1, \ldots, \text{Ad}_{g^{-1}}x_k)
\]

\[
= \Phi^{(0)}_V(g)\left(\Phi^{(k)}_V(e)(\text{Ad}_{g^{-1}}x_1, \ldots, \text{Ad}_{g^{-1}}x_k)\right)
\]

\[
= \Phi^{(0)}_V(g)\left(\Phi^{(0)}_V(g^{-1}) \circ \Phi^{(k)}_V(e)(x_1, \ldots, x_k) \circ \Phi^{(0)}_V(g)\right)
\]

\[
= \Phi^{(k)}_V(e)(x_1, \ldots, x_k) \circ \Phi^{(0)}_V(g),
\]

as we wished. Next, let us take an invariant element $\eta \in [\Omega^p(BG_p) \otimes V]^{G_{p+1}}$. On account of (4.3), (4.4) and (4.5), this means that

\[
\zeta(g_0, \ldots, g_p)^* \eta = \Phi^{(0)}_V(g_0)(\eta).
\]

(4.8)

We need to show that $\Phi^{(k)}_V \cup \eta \in [\Omega^{k+q}(BG_{p+1}) \otimes V]^{G_{p+2}}$. On this purpose we notice firstly that

\[
\Phi^{(k)}_V \cup \eta = \pi_1^* \Phi^{(k)}_V \wedge \pi_{(p)}^* \eta,
\]

where $\pi_1 : G_{p+1} \to G$ is the projection onto the first factor and $\pi_{(p)} : G_{p+1} \to G_p$ is the projection onto the remaining $p$ factors. Notice, secondly, that

\[
\pi_1 \circ \zeta(g_0, \ldots, g_{p+1}) = L_{g_0} \circ R_{g^{-1}} \circ \pi_1,
\]

\[
\pi_{(p)} \circ \zeta(g_1, \ldots, g_{p+1}) = \zeta(g_1, \ldots, g_{p+1}) \circ \pi_{(p)}.
\]

(4.10)

By using (4.6), (4.7), (4.8), (4.9) and (4.10), we find

\[
\zeta(g_0, \ldots, g_{p+1})^* (\Phi^{(k)}_V \cup \eta) = \zeta(g_0, \ldots, g_{p+1})^* (\pi_1^* \Phi^{(k)}_V \wedge \pi_{(p)}^* \eta)
\]

\[
= (\pi_1 \circ \zeta(g_0, \ldots, g_{p+1}))^* \Phi^{(k)}_V \wedge (\pi_{(p)} \circ \zeta(g_0, \ldots, g_{p+1}))^* \eta
\]

\[
= (L_{g_0} \circ R_{g^{-1}} \circ \pi_1)^* \Phi^{(k)}_V \wedge (\zeta(g_1, \ldots, g_{p+1}) \circ \pi_{(p)}^* \eta
\]

\[
= \pi_1^* R_{g^{-1}}^* L_{g_0}^* \Phi^{(k)}_V \wedge \pi_{(p)}^* \zeta(g_1, \ldots, g_{p+1})^* \eta
\]

\[
= \pi_1^* (\Phi^{(0)}_V(g_0) \circ \Phi^{(k)}_V(g_{1}^{-1})) \wedge \pi_{(p)}^* (\Phi^{(0)}_V(g_{1})(\eta))
\]

\[
= \Phi^{(0)}_V(g_0)(\pi_1^* \Phi^{(k)}_V \wedge \pi_{(p)}^* \eta)
\]

\[
= \Phi^{(0)}_V(g_0)(\Phi^{(k)}_V \cup \eta),
\]

which implies what we want.

With this result in hand, we can now define the equivariant version of the Bott-Shulman-Stasheff DG category, which we denote by $\text{BSS}^G(G)$. Its objects are the same as those of $\text{BSS}(G)$, and, as such, they are just objects in the category $\text{Rep}(\mathfrak{T}G)$. For any two objects $V$ and $V'$, the space of morphisms is the graded vector space $[\Omega^* (BG_* \otimes \text{Hom}(V, V'))]^{G_{p+1}}$ with differential $\partial_{D,D'}$ given by exactly the same formula as that of $\text{BSS}(G)$. Note that Lemmas (4.6 and 4.7) ensure that this is well-defined. We would also like to highlight the following key result.
Proposition 4.8. The inclusion DG functor from $\text{BSS}^G(G)$ to $\text{BSS}(G)$ is a quasi-equivalence.

Proof. For any pair of objects $V$ and $V'$ in $\text{BSS}^G(G)$, since $G_{p+1}$ is compact and connected, we know that the inclusion $[\Omega^*(BG_p) \otimes \text{Hom}(V, V')]^{G_{p+1}} \rightarrow [\Omega^*(BG_p) \otimes \text{Hom}(V, V')]$ is a quasi-isomorphism. The result thus follows by an argument entirely similar to that of the proof of Lemma 3.1.

4.4 The Van Est DG functor

In this subsection we describe the construction of a DG functor between the equivariant Bott-Shulman-Stasheff DG category $\text{BSS}^G(G)$ and the DG enhanced category $\text{DGRep}(\mathcal{T}_g)$, which is a quasi-equivalence when $G$ is compact. We use freely the definitions and notation from §3.1.

Let $V$ be an object of $\text{Rep}(\mathcal{T}_g)$ and consider again the cochain complex $\Omega^\bullet(BG_p) \otimes V$. For fixed $p$ and $q$, we let $\tilde{\gamma}_0^q(g \otimes v)$ denote the subspace of $G_p$-invariant elements of $\Omega^q(BG_p) \otimes V$ with respect to the action (4.4). From the definition it is obvious that $\tilde{\gamma}_0^q(g \otimes v)$ coincides with $\Omega^q(BG_p) \otimes V$. Thus, evaluation at $(e, \ldots, e)$ induces an isomorphism of graded vector spaces from $\tilde{\gamma}_0^q(g \otimes v)$ onto $\Lambda^q g^* \otimes V$. On the latter, we consider the action $\hat{\gamma}_0^q(g)$ of elements $g$ of $G$ defined by

$$\hat{\gamma}_0^q(g)(\xi \otimes v) = \text{Ad}_g^\pi \xi \otimes \left(\Phi^{(0)}_V(g)(v)\right),$$

for $\xi \in \Lambda^q g^*$ and $v \in V$. The following result, which is a direct consequence of Lemma 3.2, will be needed below.

Lemma 4.9. The following diagram commutes

$$
\begin{array}{ccc}
[\Omega^q(BG_p) \otimes V]^{G_p} & \xrightarrow{\gamma_0^q(g)} & [\Omega^q(BG_p) \otimes V]^{G_p} \\
\Lambda^q g_p^* \otimes V & \xrightarrow{\tilde{\gamma}_0^q(g)} & \Lambda^q g_p^* \otimes V,
\end{array}
$$

where the vertical arrows denote evaluation at the element $(e, \ldots, e)$.

Next we consider the morphism of cochain complexes defined by

$$\mathcal{V}\mathcal{E}_V = \text{VE} \otimes \text{id}_V : \Omega^*(G_\ast) \otimes V \rightarrow W^{\bullet \bullet} g \otimes V,$$

where $\text{VE} : \Omega^*(G_\ast) \rightarrow W^{\bullet \bullet} g$ is the Van Est map. By virtue of Lemma 3.3, the restriction of $\mathcal{V}\mathcal{E}_V$ to $[\Omega^q(BG_p) \otimes V]^{G_p}$ vanishes unless $q = p$. From this it follows at once that this restriction, which we still denote by $\mathcal{V}\mathcal{E}_V$, has its image contained in $S^p g^* \otimes V$. It is also worth pointing out that, if we consider the morphism of graded vector spaces defined by

$$\tilde{\mathcal{V}}\mathcal{E}_V = \tilde{\text{VE}} \otimes \text{id}_V : \Lambda^p g_p^* \otimes V \rightarrow S^p g^* \otimes V,$$

we get a commutative diagram

$$
\begin{array}{ccc}
[\Omega^p(BG_p) \otimes V]^{G_p} & \xrightarrow{\mathcal{V}\mathcal{E}_V} & S^p g^* \otimes V, \\
\Lambda^p g_p^* \otimes V & \xrightarrow{\tilde{\mathcal{V}}\mathcal{E}_V} & S^p g^* \otimes V,
\end{array}
$$
with the vertical arrow being the evaluation at \((e,\ldots,e)\). This instructs us to introduce yet one more action \(\hat{\gamma}''_0(g)\) of elements \(g\) of \(G\) on \(S^p\mathfrak{g}^* \otimes V\) defined by

\[
\hat{\gamma}''_0(g)(\xi \otimes v) = \text{Ad}_g^* f \otimes (\Phi_V^{(0)}(g)(v)),
\]

(4.12)

for \(f \in S^p\mathfrak{g}^*\) and \(v \in V\). The corresponding subspace of \(G\)-invariants elements of \(S^p\mathfrak{g}^* \otimes V\) will be denoted by \((S^p\mathfrak{g}^* \otimes V)^G\). Then we have the following result.

**Proposition 4.10.** The restriction of the morphism \(\mathcal{V}E_V\) to \([\Omega^*(BG,\ast) \otimes V]^{G_{\ast+1}}\) has its image contained in \((W^{\ast\cdot}\mathfrak{g} \otimes V)_{\text{bas}}\).

**Proof.** The first thing to notice is that, owing to the definitions in (4.1) and (4.12), the graded subspace \((W^{\ast\cdot}\mathfrak{g} \otimes V)_{\text{bas}}\) coincides with \((S^*\mathfrak{g}^* \otimes V)^G\). Therefore, in light of Lemma 4.9 it will suffice to show that the following diagram commutes

\[
\begin{array}{ccc}
\Lambda^p\mathfrak{g}^* \otimes V & \xrightarrow{\gamma''_0(g)} & \Lambda^p\mathfrak{g}^* \otimes V \\
\mathcal{V}E_V & \downarrow & \mathcal{V}E_V \\
S^p\mathfrak{g}^* \otimes V & \xrightarrow{\gamma''_0(g)} & S^p\mathfrak{g}^* \otimes V.
\end{array}
\]

But this is an easy consequence of the commutativity of the diagram we established in the course of the proof of Proposition 3.4. \(\square\)

We also note the following result here.

**Proposition 4.11.** The Maurer-Cartan element \(\Phi_V \ast \text{id}_V\) of \(\Omega^*(BG,\ast) \otimes \text{End}(V)\) is sent by the morphism \(\mathcal{V}E_{\text{End}(V)}\) to the Maurer-Cartan element \(t^a \otimes L_a - w^a \otimes i_a\) of \(W^{\ast\cdot}\mathfrak{g} \otimes \text{End}(V)\).

**Proof.** Let us first write \(\Phi_V\) as a sum \(\sum_{k \geq 0} \Phi_V^{(k)}\). Next, let us observe that, by definition,

\[
\mathcal{V}E : \Omega^k(G) \longrightarrow W^{1,k}\mathfrak{g} = \Lambda^{1-k}\mathfrak{g}^* \otimes S^k\mathfrak{g}^* = \begin{cases} 
\Lambda^1\mathfrak{g}^* & \text{if } k = 0, \\
S^1\mathfrak{g}^* & \text{if } k = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

This implies that \(\mathcal{V}E_{\text{End}(V)}(\Phi_V^{(k)}) = 0\) for \(k \geq 2\). On the other hand, attending to the definitions, for each \(x \in \mathfrak{g}\), we have

\[
\mathcal{V}E_{\text{End}(V)}(\Phi_V^{(0)})(x) = (L_x \Phi_V^{(0)})(e) = L_x \circ \Phi_V^{(0)}(e) = L_x \circ \text{id}_V = L_x,
\]

and

\[
\mathcal{V}E_{\text{End}(V)}(\Phi_V^{(1)})(x) = (i_{-x^*} \Phi_V^{(1)})(e) = \Phi_V^{(1)}(e)(-x^*(e)) = -\Phi_V^{(1)}(e)(x) = -i_x.
\]

Moreover, \(\mathcal{V}E_{\text{End}(V)}(\text{id}_V) = 0\). Since for any \(x \in \mathfrak{g}\), \((t^a \otimes L_a)(x) = L_x\) and \((w^a \otimes i_a)(x) = i_x\), conclusion follows. \(\square\)

With this preparatory work completed, we come now to the definition that concerns us. Take two object \(V\) and \(V'\) of \(\text{Rep}(\mathfrak{T g})\) and let us write \(\mathcal{V}E_{V,V'}\) for \(\mathcal{V}E_{\text{Hom}(V,V')}\). Applying Proposition 4.10 yields that

\[
\mathcal{V}E_{V,V'} : [\Omega^*(BG,\ast) \otimes \text{Hom}(V,V')]^{G_{\ast+1}} \longrightarrow (W^{\ast\cdot}\mathfrak{g} \otimes \text{Hom}(V,V'))_{\text{bas}}.
\]
When combined with Proposition 4.11, this shows that the collection of morphisms \( \mathcal{V} \mathcal{E}_{V,V'} \) defines a DG functor \( \mathcal{V} \mathcal{E} : \text{BSS}^G(G) \to \text{DGRep}(\mathbb{T}_g) \) which is the identity on objects. We shall henceforth refer to this as the Van Est DG functor. The following result is our main finding in this subsection.

**Theorem B.** The Van Est DG functor \( \mathcal{V} \mathcal{E} : \text{BSS}^G(G) \to \text{DGRep}(\mathbb{T}_g) \) is a quasi-equivalence.

**Proof.** Following the discussion in the last part of §3.1, let us again consider the map \( \hat{\mathcal{M}}^\theta \). We notice that, by definition, the following diagram commutes

\[
\begin{array}{ccc}
S^p g^* & \xrightarrow{\hat{\mathcal{M}}^\theta} & S^p g^* \\
\downarrow \text{id} & & \downarrow \text{id} \\
\Omega^p (BG_p)^{G_r} & \xrightarrow{\gamma_0(g)^*} & \Omega^p (BG_p)^{G_r}.
\end{array}
\]

Next, for any two objects \( V \) and \( V' \) of \( \text{Rep}(\mathbb{T}_g) \), we set

\[
\mathcal{M}_{V,V'}^\theta = \hat{\mathcal{M}}^\theta \otimes \text{id}_{\text{Hom}(V,V')} : S^p g^* \otimes \text{Hom}(V,V') \to \Omega^p (BG_p)^{G_r} \otimes \text{Hom}(V,V').
\]

Then, from the definitions (4.3) and (4.12), it follows on use of the above that the following diagram commutes

\[
\begin{array}{ccc}
S^p g^* \otimes \text{Hom}(V,V') & \xrightarrow{\hat{\mathcal{M}}^\theta} & S^p g^* \otimes \text{Hom}(V,V') \\
\downarrow \gamma_0'(g) & & \downarrow \gamma_0'(g) \\
\Omega^p (BG_p)^{G_r} \otimes \text{Hom}(V,V') & \xrightarrow{\gamma_0(g)} & \Omega^p (BG_p)^{G_r} \otimes \text{Hom}(V,V').
\end{array}
\]

Recalling that \( (S^p g^* \otimes \text{Hom}(V,V'))^G \) coincides with \( (W^* g \otimes \text{Hom}(V,V'))_{\text{bas}} \) and \( [\Omega^*(BG_* \otimes \text{Hom}(V,V'))]^G \) coincides with \( [\Omega^*(BG_* \otimes \text{Hom}(V,V'))]^{G_{++1}} \), we thus get a morphism of cochain complexes

\[
\mathcal{M}_{V,V'}^\theta : (W^* g \otimes \text{Hom}(V,V'))_{\text{bas}} \to [\Omega^*(BG_* \otimes \text{Hom}(V,V'))]^{G_{++1}}.
\]

Furthermore, invoking Theorem 3.7, we infer that \( \mathcal{M}_{V,V'}^\theta \) is a left inverse of \( \mathcal{V} \mathcal{E}_{V,V'} \). This, clearly, yields the result. \( \square \)

**Theorem 4.12.** Let \( G \) be a compact simply connected Lie group. The categories \( \text{BSS}(G) \) and \( \text{DGRep}(\mathbb{T}_g) \) are \( A_\infty \) quasi-equivalent.

### 4.5 The De Rham-Hochschild \( A_\infty \)-functor

In this subsection we shall construct an \( A_\infty \)-quasi-equivalence which connects the Bott-Shulman-Stasheff DG category \( \text{BSS}(G) \) to the DG enhanced category \( \text{DGMod}(\text{C}_*(G)) \). We will use the De Rham-Hochschild \( A_\infty \)-quasi-isomorphism from §3.3.

Let \( V \) be an object of \( \text{Rep}(\mathbb{T}_g) \).

By tensoring the \( A_\infty \) map \( \text{DR}^\theta : \text{Tot}(\Omega^*(BG_*)) \to \text{HC}^*(\text{C}_*(G)) \) with the identity on \( \text{End}(V) \) one obtains a map:

\[
\mathcal{DR} : \text{Tot}(\Omega^*(BG_*)) \otimes \text{End}(V) \to \text{HC}^*(\text{C}_*(G)) \otimes \text{End}(V),
\]

which is an \( A_\infty \)-quasi-isomorphism.

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To proceed further, let us denote by $\rho: \mathbb{T}G \to \text{End}(V)$ the structure homomorphism associated with $V$. Following the discussion of §2.6 we will designate by $\mathcal{J}(\rho): C_*(G) \to \text{End}(V)$ the structure homomorphism associated with $V$ when viewed as object of $\text{Mod}(C_*(G))$. We will also write $\mathbb{1}_V: C_*(G) \to \text{End}(V)$ to denote the homomorphism which associates $\text{id}_V$ to every singular 0-simplex on $G$, that is, the trivial module.

**Lemma 4.13.** The element $\mathcal{J}(\rho) - \mathbb{1}_V$ is a Maurer-Cartan element of $\text{HC}^*(C_*(G) \otimes \text{End}(V))$.

**Proof.** Applying Proposition 3.17 we see that, for each element $\Phi 
 Putting this together with the definitions in (2.11) and (2.37), we conclude that the Maurer-Cartan result follows.

As usual, an immediate consequence of this is the following.

**Corollary 4.14.** The operator $D$ in $\text{HC}^*(C_*(G), \text{End}(V))$ given by

$$D = b + \delta + \mathcal{J}(\rho) - \mathbb{1}_V,$$

is a derivation of homogeneous degree 1 that satisfies $D^2 = 0$.

With this point in mind, let us now take $V'$ to be another object of $\text{Rep}(\mathbb{T}G)$ with associated structure homomorphism $\rho': \mathbb{T}G \to \text{End}(V')$ and differential $D'$, and consider the graded vector space $\text{HC}^*(C_*(G), \text{End}(V'))$ endowed with the differential $\partial_{D,D'}$ given by the same formula as (4.2) above. The following result demonstrates the basic link between the latter and the the Hochschild differential on $\text{HC}^*(C_*(G), \text{End}(V))$ obtained by the prescription (2.5).

**Proposition 4.15.** The differential $\partial_{D,D'}$ coincides with the differential on the Hochschild cochain complex $\text{HC}^*(C_*(G), \text{End}(V))$.

**Proof.** Explicitly, we may write $\partial_{D,D'}$ as

$$\partial_{D,D'} \varphi(u_1 \otimes \cdots \otimes u_n) = \delta(\varphi(u_1 \otimes \cdots \otimes u_n)) - (-1)^{|\varphi|} \varphi(b(u_1 \otimes \cdots \otimes u_n))$$

$$+ (\mathcal{J}(\rho') \cup \varphi)(u_1 \otimes \cdots \otimes u_n) - (-1)^{|\varphi|} (\varphi \cup \mathcal{J}(\rho))(u_1 \otimes \cdots \otimes u_n),$$

for homogeneous elements $\varphi \in \text{Hom}((uC_*(G))^n, \text{End}(V))$ and $c_1, \ldots, c_n \in C_*(G)$. On the other hand, from the definition of the cup product (2.6), we have

$$(\mathcal{J}(\rho') \cup \varphi)(u_1 \otimes \cdots \otimes u_n) = (-1)^{|\varphi|(|c_1|+1)} \mathcal{J}(\rho')(c_1) \circ \varphi(u_2 \otimes \cdots \otimes u_n),$$

and

$$(\varphi \cup \mathcal{J}(\rho))(u_1 \otimes \cdots \otimes u_n) = (-1)^{\sum_{j=1}^{n-1} |c_j|+n-1} \varphi(u_1 \otimes \cdots \otimes u_{n-1}) \circ \mathcal{J}(\rho)(c_n).$$

Using this together with (2.5) gives the desired conclusion. 

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Now we proceed to define the $A_\infty$-functor from $BSS(G)$ to $DGMod(C_*(G))$, which we also denote by $\mathcal{DR}$ and will refer to as the Hochschild-De Rham $A_\infty$-functor. On objects $\mathcal{DR}$ acts as the integration functor $I : \text{Rep}(T_g) \to \text{Mod}(C_*(G))$ which gives a module over singular chains given a representation of $T_g$. For each $n \geq 1$ and for every collection of objects $V_0, \ldots, V_n$ of $\text{Rep}(T_g)$, we let

$$\mathcal{DR}_n : u(\text{Tot}(\Omega^*(BG_*) \otimes \text{Hom}(V_{n-1}, V_n)) \otimes \cdots \otimes u(\text{Tot}(\Omega^*(BG_*) \otimes \text{Hom}(V_0, V_1))) \rightarrow HC^*(C_*(G), \text{Hom}(V_0, V_n))$$

be defined by

$$\mathcal{DR}_n(u(\omega_{n-1} \otimes f_{n-1}) \otimes \cdots \otimes u(\omega_0 \otimes f_0)) = u(sDR_n^\theta (u\omega_{n-1} \otimes \cdots \otimes u\omega_0) \otimes (f_{n-1} \circ \cdots \circ f_0)), \tag{4.13}$$

for homogeneous elements $\omega_0, \ldots, \omega_{n-1} \in \text{Tot}(\Omega^*(BG_*)$ and for a composable chain of homogeneous homomorphisms $f_0 \in \text{Hom}(V_0, V_1), \ldots, f_{n-1} \in \text{Hom}(V_{n-1}, V_n)$. A straightforward computation, which takes into account Proposition 4.15 shows that the sequence of maps $\mathcal{DR}_n$ indeed defines an $A_\infty$-functor $\mathcal{DR} : BSS(G) \to DGMod(C_*(G))$.

**Theorem C.** The Hochschild-De Rham $A_\infty$-functor $\mathcal{DR} : BSS(G) \to DGMod(C_*(G))$ is an $A_\infty$-quasi-equivalence.

**Proof.** Since the functor coincides with $\mathcal{I} : \text{Rep}(T_g) \to \text{Mod}(C_*(G))$ on objects, we know that it is essentially surjective. It is also quasi-faithful because the map

$$\mathcal{DR} : \text{Tot}(\Omega^*(BG_*)) \otimes \text{End}(V) \rightarrow HC^*(C_*(G)) \otimes \text{End}(V),$$

is a quasi-isomorphism. \qed

### 4.6 The main theorem

We are at last in a position to state and prove the principal result of the paper.

**Theorem A.** Suppose that $G$ is compact and simply connected. There exists a zig-zag of $A_\infty$ quasi-equivalences of DG categories connecting $DGRep(T_g)$ and $DGMod(C_*(G))$.

**Proof.** It follows directly from Proposition 4.8, Theorem B and Theorem C. \qed

We finish the paper with a couple of examples that illustrate the content of our main result.

**Example 4.16.** Let us write $\mathcal{R}$ for the trivial representation of $T_g$. Then, on the one hand,

$$H^*(\text{End}_{DGRep(T_g)}(\mathcal{R})) = H^*((W_0)_{\text{bas}}) \cong (S^*g^*)^G \cong H^*(BG).$$

On the other hand,

$$H^*(\text{End}_{DGMod(C_*(G))}(\mathcal{R})) \cong H^*(HC^*(C_*(G))) = \text{HH}^*(C_*(G)),$$

where “$\text{HH}^*$” stands for Hochschild cohomology. Invoking Theorem 4.6, we conclude that

$$H^*(BG) \cong (S^*g^*)^G \cong \text{HH}^*(C_*(G)).$$

This recovers two models for computing the cohomology of the classifying space $BG$ with coefficients in the trivial local system.
Example 4.17. Let us consider the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ viewed as a representation of $\mathbb{T} \mathfrak{g}$. Then, on the one hand,

$$H^\bullet(\text{Hom}_{DG\text{Rep}(\mathbb{T} \mathfrak{g})}(R, CE(\mathfrak{g}))) = H^\bullet((W \mathfrak{g} \otimes CE(\mathfrak{g}))_{bas}) \cong H^\bullet((S^* \mathfrak{g}^* \otimes CE(\mathfrak{g}))^G).$$

On the other hand,

$$H^\bullet(\text{Hom}_{DG\text{Mod}}(C_\bullet(G))(R, CE(\mathfrak{g}))) = H^\bullet(\text{HC}^\bullet(C_\bullet(G), CE(\mathfrak{g}))) = \text{HH}^\bullet(C_\bullet(G), CE(\mathfrak{g})).$$

Since the latter is known to be isomorphic to the cohomology of the free loop space $\mathcal{L}BG$ of $BG$, Theorem 4.6 tells us that

$$H^\bullet(\mathcal{L}BG) \cong H^\bullet((S^* \mathfrak{g}^* \otimes CE(\mathfrak{g}))^G).$$

On recovers the fact that the equivariant cohomology of $G$ acting on itself by conjugation is the cohomology of the loop space of $BG$. This means that the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ corresponds to the Gauss-Manin local system for the loop space fibration $\pi: \mathcal{L}BG \to BG$.

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