Transport Theorem and Continuity Equations: a Stochastic Approach

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Abstract
We give a stochastic generalization of transport theorem on smooth manifold. Furthermore, we deduce a system of continuity equation and present some application on torus.

Key words: stochastic flows, transport theorem, continuity equation.

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1 Introduction

The integration on smooth manifold is essential on the development of Mathematics and Science. One can integrate a form on a smooth manifold over a smooth flow. This integration yields the theory of fluid dynamics on smooth manifolds. For example, equations of motion, Euler’s equation and Navier-Stokes’s equation are deduced from this kind of integration (see for instance [4] and [1]). Moreover, in Mechanics there are many results yielded from integration on smooth manifolds (see for example [2]).

It is possible to make an integration over a stochastic flow on smooth manifolds, see Bismut [3] and Kunita [8]. Recently, Lázaro-Camí and Ortega used this kind of integral in order to study the mechanic flows, see [9] and [10].

The subject of this work is to give a stochastic transport theorem on smooth manifolds. In fact, we consider $X^0, X^1, \ldots, X^m$ time-dependent smooth vector fields, $\theta$ a time-dependent $p$-form with compact support, $\sigma_p$ a $p$-simplex and $(B^1_t, \ldots, B^m_t)$ a Brownian motion in $\mathbb{R}^m$. Let $\phi_t$ be the flow generated by the Stratonovich stochastic differential equation,

$$dx = X^0(t, x)dt + X^i(t, x) \circ dB^i_t,$$

$$x(0) = x.$$  

Then we have the Itô’s formula (see Theorem 4.2 [8] or Theorem 3.7 in [3, ch.IV]),

$$\int_{\phi_t(\sigma_p)} \theta = \int_{\sigma_p} \theta + \int_0^t \left( \int_{\phi_s(\sigma_p)} \frac{\partial \theta}{\partial t} \right) ds + \sum_{k=0}^m \int_0^t \left( \int_{\phi_s(\sigma_p)} L_{X^k_s} \theta \right) \circ dB^k_s.$$  

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If the smooth manifold has a volume form $\mu$, we prove a stochastic generalization of transport theorem

$$d\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\phi_t(\sigma_n)} \left( \frac{\partial f_t}{\partial t} + \sum_{k=0}^m \text{div}_{\mu}(f_t X_k^t) \circ B_t^k \right) \mu,$$

where $f$ is a smooth function.

In the case of a compact oriented Riemannian manifold we obtain a system of continuity equations for the mass density $\rho_t$,

$$\frac{\partial \rho_t}{\partial t} + \text{div}_{\mu}(\rho_t X_0(t)) = 0$$

$$\text{div}_{\mu}(\rho_t X_k(t)) = 0.$$

About the structure of this paper, in section 2 we proceed with study of the stochastic flows acting on integral, more specifically, we develop the necessary tools to show the Itô’s formula above. Then in section 4 we prove our main result, a stochastic generalization of transport theorem and we give some applications.

2 Stochastic flows acting on Integral

In this section, our mean is to construct the integral over forms upon a stochastic flow. We begin by introducing some notations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a probability space which satisfies the usual hypotheses (see for instance [8, ch.I]).

Let $M$ be a n-dimensional smooth manifold, $X_0, X^1, \ldots, X^m$ time-dependent smooth vector fields on $M$ and $B_t = (B^1_t, \ldots, B^m_t)$ a $m$-dimensional Brownian motion in $\mathbb{R}^m$. We consider the Stratonovich stochastic differential equation on $M$ given by

$$dx = X^0(t, x)dt + X^k(t, x) \circ dB^k_t$$

$$x(0) = x, x \in M.$$

It is well known that there exists an unique solution of this equation with a maximal time $T(x)$. A complete study about SDE (1) is founded in [8]. We denote the solution of SDE (1) by $\phi_t(\omega, x)$ or, simply, $\phi_t$.

Let $\theta$ be a time-dependent p-form on $M$ with compact support and $\sigma_p$ a p-simplex in $M$, we denote by $\int_{\phi_t(\sigma_p)} \theta$ the real semimartingale $\int_{\sigma_p} \phi_t^* \theta$.

We recall that the push-forward for a smooth vector field $X$ on $M$ by a diffeomorphism $\phi$ is given by

$$(\phi_* X)_y = \phi_{\phi^{-1}(y)}^* X(\phi^{-1}(y)).$$

We can now rephrase Theorem 4.3 in [8] ch.III] on time-dependent forms as follows.
**Theorem 2.1** Let \( M \) be a \( n \)-dimensional smooth manifold and \( \phi_t \) the flow in \( M \) given by SDE (1). Then for a time-dependent \( p \)-form \( \theta \) with compact support we have

\[
\phi_t^* \theta - \theta = \int_0^t \phi_s^* \frac{\partial \theta}{\partial t} ds + \sum_{k=0}^m \int_0^t (\phi_s^* L_{X_k^s} \theta) \circ dB_s^k.
\]

In order to write Itô’s formulas to real semimartingales \( \int_{\phi_t(\sigma_p)} \theta \), our next step is to show a Fubini’s Theorem.

**Proposition 2.2** Let \( M \) be \( n \)-dimensional smooth manifold, \( \theta \) a time-dependent \( p \)-form on \( M \) with compact support and \( B_t \) a real Brownian motion. Then

\[
\int_{\sigma_p} \left( \int_0^t \phi_s^* \theta \circ dB_s \right) = \int_0^t \left( \int_{\sigma_p} \phi_s^* \theta \right) \circ dB_s.
\]

**Proof:** Let \( \Delta = \{0 = t_0 < \ldots < t_n = T\} \) be a partition of the interval \([0,T]\), \( t_\star = \frac{t_{j+1} + t_j}{2} \) and \( \Delta B(\omega) = (B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) \). It is well known that, in \( L^2([0,T] \times \Omega) \),

\[
\lim_{j \to \infty} \mathbb{E} \left( \left| \int_0^t \left( \int_{\sigma_p} \phi_s^* \theta \right) \circ dB_s - \sum_j \left( \int_{\sigma_p} \phi_s^* \theta \right) \Delta B(\omega) \right|^2 \right) = 0.
\]

From uniqueness of limit we get

\[
\int_{\sigma_p} \left( \int_0^t \phi_s^* \theta \circ dB_s \right) = \int_0^t \left( \int_{\sigma_p} \phi_s^* \theta \right) \circ dB_s.
\]

\[\square\]

**Corollary 2.3** Let \( M \) be a \( n \)-dimensional smooth manifold and \( \phi_t \) the flow in \( M \) given by SDE (1). Then for a time-dependent \( p \)-form \( \theta \) with compact support we have

1. \[
\int_{\phi_t(\sigma_p)} \theta = \int_{\sigma_p} \theta + \int_0^t \left( \int_{\phi_s(\sigma_p)} \frac{\partial \theta}{\partial t} \right) ds + \sum_{k=0}^m \int_0^t \left( \int_{\phi_s(\sigma_p)} L_{X_k^s} \theta \right) \circ dB_s^k. \tag{2}
\]

2. \[
\int_{\phi_t(\sigma_p)} \theta = \int_{\sigma_p} \theta + \int_0^t \left( \int_{\phi_s(\sigma_p)} \left( \frac{\partial \theta}{\partial t} + \frac{1}{2} \sum_{k=1}^m L_{X_k^s}^2 \right) + \sum_{k=1}^m \int_0^t \left( \int_{\phi_s(\sigma_p)} L_{X_k^s} \theta \right) dB_s^k \right) ds \tag{3}
\]

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Proof: 1. From Theorem 2.1 we obtain
\[
\int_{\sigma_p} \theta = \int_0^t \theta + \int_0^t \int_0^t \phi_s \frac{\partial \theta}{\partial t} ds + \int_0^t \sum_{k=0}^m \int_0^t \phi_s^* L_{X_s^k} \theta \circ dB_s^k.
\]
Applying Proposition 2.2 in the right side we see that
\[
\int_0^t \sum_{k=0}^m \int_0^t \phi_s^* L_{X_s^k} \theta \circ dB_s^k = \sum_{k=0}^m \int_0^t \left( \int_0^t \phi_s^* L_{X_s^k} \theta \right) \circ dB_s^k. \tag{4}
\]
Analogously, we can see that
\[
\int_0^t \phi_s \frac{\partial \theta}{\partial t} ds = \int_0^t \int_0^t \frac{\partial \theta}{\partial t} ds. \tag{5}
\]
Therefore the formula (2) follows from (4) and (5).

2. We first apply the Stratonovich-Itô conversion formula in stochastic integrals of the formula (2). It follows that
\[
\int_0^t \left( \int_{\sigma_p} L_{X_s^k} \theta \right) \circ dB_s^k = \int_0^t \left( \int_{\sigma_p} L_{X_s^k} \theta \right) dB_s^k + \frac{1}{2} \left[ \int_{\sigma_p} L_{X_s^k} \theta \right], B_s^k].
\]
Using the formula (2) for \( \int_{\sigma_p} L_{X_s^k} \theta, k = 1, \ldots, n, \) we obtain
\[
\frac{1}{2} \left[ \int_{\sigma_p} L_{X_s^k} \theta \right], B_s^k] = \frac{1}{2} \sum_{l=0}^m \int_0^t \left( \int_{\sigma_p} L_{X_s^l} L_{X_s^k} \theta \right) \circ dB_s^l, B_s^k]
\]
\[
= \frac{1}{2} \int_0^t \left( \int_{\sigma_p} L_{X_s^k} L_{X_s^k} \theta \right) ds.
\]
Therefore we conclude that
\[
\int_0^t \left( \int_{\sigma_p} L_{X_s^k} \theta \right) \circ dB_s^k = \int_0^t \left( \int_{\sigma_p} L_{X_s^k} \theta \right) dB_s^k + \frac{1}{2} \int_0^t \left( \int_{\sigma_p} L_{X_s^k}^2 \theta \right) ds.
\]
Thus substituting the equality above in the formula (2) yields the formula (5). \( \square \)

This Corollary is reformulation of Theorem 3.7 in [3, ch.IV] in terms of time-dependent forms.

A direct consequence of the formula (3) is that if a time-dependent p-form \( \theta \) with compact support satisfies the differential equation
\[
\begin{cases}
\frac{\partial \theta}{\partial t} &= - \left( \frac{1}{2} \sum_{k=1}^m L_{X_s^k}^2 + L_{X_s} \right) \theta, \\
\theta(0,x) &= \theta_0(x)
\end{cases}
\]
then
\[
\int_{\sigma_p} \theta = \int_{\sigma_p} \theta_0 + \sum_{k=1}^m \int_0^t \left( \int_{\sigma_p} L_{X_s^k} \theta \right) dB_s^k.
\]
is a real martingale. Thus taking expectation we conclude that
\[
\int_{\sigma_p} \theta_0 = \mathbb{E} \left( \int_{\phi_i(\sigma_p)} \theta \right).
\]

3 Transport Theorem and Continuity Equations

In this section we develop some results in stochastic fluid mechanics. We follow close the presentation of Abraham, Ratiu and Marsden [1]. Our first result is a stochastic generalization of transport theorem.

**Theorem 3.1** Let \( M \) be a smooth manifold, \( \mu \) a volume form and \( \phi_t \) the flow generated by SDE (1). For a smooth function \( f : [0, \infty) \times M \to \mathbb{R} \), we have
\[
\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\sigma_n} f_0 \mu + \int_0^t \left( \int_{\phi_s(\sigma_n)} \frac{\partial f_s}{\partial t} \mu \right) ds + \sum_{k=0}^{m} \int_0^t \left( \int_{\phi_s(\sigma_n)} \text{div}_s(f_s X^k_s) \mu \right) dB^k_s
\]
for any \( n \)-simplex \( \sigma_n \).

**Remark 1** Being \( \mu \) a volume form on \( M \), we can generated a measure \( m \mu \) associated to it. Therefore the functions \( f \) adopted in Theorem above are smooth functions in \( L^1(M, \mu_M) \), as required in [1].

**Proof:** We first apply the Corollary 2.3 to obtain
\[
\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\sigma_n} f_0 \mu + \int_0^t \left( \int_{\phi_s(\sigma_n)} \frac{\partial f_s}{\partial t} \mu \right) ds + \sum_{k=0}^{m} \int_0^t \left( \int_{\phi_s(\sigma_n)} \text{div}_s(f_s X^k_s) \mu \right) dB^k_s
\]
As \( L_{X^k_t}(f_t \mu) = \text{div}_t(f_t X^k_t) \mu \) we have
\[
\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\sigma_n} f_0 \mu + \int_0^t \left( \int_{\phi_s(\sigma_n)} \frac{\partial f_s}{\partial t} \mu \right) ds + \sum_{k=0}^{m} \int_0^t \left( \int_{\phi_s(\sigma_n)} \text{div}_s(f_s X^k_s) \mu \right) dB^k_s,
\]
which proves the theorem. \( \square \)

**Theorem 3.2** Let \( M \) be a smooth manifold, \( \mu \) a volume form and \( \phi_t \) the flow generated by SDE [1]. For a smooth function \( f : [0, \infty) \times M \to \mathbb{R} \), we have
\[
d\mathbb{E} \left( \int_{\phi_t(\sigma_n)} f_t \mu \right) = \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + f_s(\text{div}_s(X^0_s)) + \frac{1}{2} \sum_{k=1}^{m} \text{div}_s(X^k_s)^2 \right) \mu \right) + \frac{1}{2} \sum_{k=1}^{m} \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( 2 X^k_s(f_s) \text{div}_s(X^k_s) + L_{X^k_s}(f_s \text{div}_s(X^k_s)) \right) \mu \right).
\]
for any \( n \)-simplex \( \sigma_n \), where \( L_s = X^0_s + \frac{1}{2} \sum_{k=1}^{m} X^k_s \) is the infinitesimal generator of sde [1].
Proof: We first apply the Corollary 2.3 to obtain

\[
\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\sigma_n} f_0 \mu + \int_0^t \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + \frac{1}{2} \sum_{k=1}^m L_{X^k_s}^2 + L_{X^0_s}(f_s \mu) \right) ds \right) + \sum_{k=1}^m \int_0^t \left( \int_{\phi_s(\sigma_n)} L_{X^k_s}(f_s \mu) \right) dB^k_s.
\]

As \( L_{X^k_s}(f_s \mu) = \text{div}_\mu(f_s X^k_s) \mu \) we have

\[
\int_{\phi_t(\sigma_n)} f_t \mu = \int_{\sigma_n} f_0 \mu + \int_0^t \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + \text{div}_\mu(f_s X^0_s) + \frac{1}{2} \sum_{k=1}^m \text{div}_\mu(\text{div}_\mu(f_s X^k_s)X^k_s) \right) \right) ds + \sum_{k=1}^m \int_0^t \left( \int_{\phi_s(\sigma_n)} \text{div}_\mu(f_s X^k_s) \right) dB^k_s.
\]

Taking the expectation we have

\[
\mathbb{E} \left( \int_{\phi_t(\sigma_n)} f_t \mu \right) = \int_{\sigma_n} f_0 \mu + \int_0^t \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + \text{div}_\mu(f_s X^0_s) + \frac{1}{2} \sum_{k=1}^m \text{div}_\mu(\text{div}_\mu(f_s X^k_s)X^k_s) \right) \right) ds.
\]

Differentiating we obtain

\[
d\mathbb{E} \left( \int_{\phi_t(\sigma_n)} f_t \mu \right) = \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + \text{div}_\mu(f_s X^0_s) + \frac{1}{2} \sum_{k=1}^m \text{div}_\mu(\text{div}_\mu(f_s X^k_s)X^k_s) \right) \right).
\]

Now applying the properties of \( \text{div}_\mu \) we can deduce that

\[
d\mathbb{E} \left( \int_{\phi_t(\sigma_n)} f_t \mu \right) = \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + f_s(\text{div}_\mu(X^0_s) + \frac{1}{2} \sum_{k=1}^m f_s(\text{div}_\mu(X^k_s)^2) \left( \mu \right) \right) \right) + \frac{1}{2} \sum_{k=1}^m \mathbb{E} \left( \int_{\phi_s(\sigma_n)} (2X^k_s(f_s)\text{div}_\mu(X^k_s) + L_{X^k_s}(f_s \mu)) \right) + \mathbb{E} \left( \int_{\phi_s(\sigma_n)} (X^0_s(f_s) + \frac{1}{2} \sum_{k=1}^m X^k_s(f_s)) \right).
\]

Corollary 3.3 Under hypothesis of Theorem we furthermore, if the vector fields \( X_0, \ldots, X_m \) in sde (7) are divergence free, then

\[
d\mathbb{E} \left( \int_{\phi_t(\sigma_n)} f_t \mu \right) = \mathbb{E} \left( \int_{\phi_s(\sigma_n)} \left( \frac{\partial f_s}{\partial t} + \mathcal{L}_s(f_s) \right) \right).
for any n-simplex $\sigma_n$, where $L_s = X^0_s + \frac{1}{2} \sum_{k=1}^{m} X^k_s$ is the infinitesimal generator of sde (1).

We use Theorem 3.1 to give a stochastic version of continuity equation. Let $M$ be a compact oriented Riemannian manifold and $\mu$ the Riemannian volume form. We observe that the mechanical interpretation of the right side of SDE (1)

$$dx = X_0(t,x)dt + X^k(t,x) \odot dB^k_t$$

is the velocity field of the fluid.

For each time $t$, we shall assume that the fluid has a well-defined mass density $\rho_t(x) = \rho(t,x)$. Taking any open set $U$ in $M$ we assume that the mass of fluid in $U$ at time $t$ is given by

$$m(t,U) = \int_U \rho_t \mu.$$

Assuming that mass is neither created nor destroyed. The meaning of this assumption to the open set $U$ is

$$\int_{\phi_s(U)} \rho_t \mu = \int_U \rho_0 \mu.$$

Applying Theorem 3.1 we obtain

$$\int_0^t \left( \int_{\phi_s(U)} \frac{\partial \rho}{\partial t} \right) ds + \sum_{k=0}^{m} \int_0^t \left( \int_{\phi_s(U)} \text{div}_\mu(\rho_s X^k_s) \mu \right) \odot dB^k_s = 0$$

We now apply the quadratic variation with respect to $B^l_t, l = 1, \ldots, n$, to get

$$\left[ \int_0^t \left( \int_{\phi_s(U)} \frac{\partial \rho}{\partial t} \right) ds + \sum_{k=0}^{m} \int_0^t \left( \int_{\phi_s(U)} \text{div}_\mu(\rho_s X^k_s) \mu \right) \odot dB^k_s, B^l_t \right] = 0.$$

Thus we conclude, for $k=1, \ldots, m$, that

$$\int_{\phi_s(U)} \text{div}_\mu(\rho_s X^k_s) \mu = 0.$$

Being $U$ an arbitrary open, the continuity equations are given by

$$\frac{\partial \rho}{\partial t} + \text{div}_\mu(\rho_t X^0_t) = 0$$

$$\text{div}_\mu(\rho_t X^k_t) = 0. \quad (6)$$

To end this work we apply the continuity equations on torus. The stochastic flow on torus has been studied in many works, we cite for example [5] and [6]. In our case we use a stochastic flow with divergence free vector fields.

Let $k = (k_1, k_2) \in \mathbb{Z}^2$ and $\theta = (\theta_1, \theta_2) \in \mathbb{T}$. Define the following vector fields on torus $\mathbb{T}$

$$A_k = k_2 \cos(k \cdot \theta) \partial_1 - k_1 \cos(k \cdot \theta) \partial_2$$

$$B_k = k_2 \sin(k \cdot \theta) \partial_1 - k_1 \sin(k \cdot \theta) \partial_2, \quad (7)$$
where $k \cdot \theta = k_1 \theta_1 + k_2 \theta_2$ and $\partial_1, \partial_2$ are the coordinate vector field on $\mathbb{T}$. In [5] is showed that $A_k$ and $B_k$ are divergence free.

Let $(B_1^t, B_2^t)$ be a $\mathbb{R}^2$-valued standard Brownian motion in $\mathbb{R}^2$. Now, consider a divergence free vector field $u(t) \in \mathbb{T}$ and the following Stratonovich stochastic differential equation

$$
dg^k(t) = u_t \, dt + A_k dB_1^t + B_k dB_2^t.
$$

We want to study the continuity equations (6) for the flow above and a volume form $\mu$ on torus. In fact, the continuity equations are given by

$$
\frac{\partial \rho^k_t}{\partial t} + \text{div}_\mu(\rho^k_t u_t) = 0
$$

$$
\text{div}_\mu(\rho^k_t A_k) = 0
$$

$$
\text{div}_\mu(\rho^k_t B_k) = 0.
$$

From the divergence free property we obtain

$$
\rho^k_t = \rho^k_0 + \int_0^t u_s(\rho^k_s)ds
$$

and

$$
< \nabla \rho^k_t, A_k > = 0 \quad \text{and} \quad < \nabla \rho^k_t, B_k > = 0.
$$

Applying (7) in the equalities above we obtain

$$
\cos(k \cdot \theta) < (k_2 \partial_1, -k_1 \partial_2), \nabla \rho^k_t > = 0
$$

$$
\sin(k \cdot \theta) < (k_2 \partial_1, -k_1 \partial_2), \nabla \rho^k_t > = 0
$$

and, consequently, $< (k_2 \partial_1, -k_1 \partial_2), \nabla \rho^k_t > = 0$.

Being $\theta$ arbitrary, we conclude that $\nabla \rho^k_t = (0, 0)$. It implies that $\partial_1 \rho^k_t = 0$ and $\partial_2 \rho^k_t = 0$. Since torus is a connected manifold, $\rho^k_t(\theta_1, \theta_2) = \rho^k_0(0, 0)$. Therefore applying $u_t$ in $\rho^k_t$ give us that $u_t(\rho^k_t) = 0$. We conclude from the equation (8) that

$$
\rho^k_t(\theta_1, \theta_2) = \rho^k_0(0, 0).
$$

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