Low temperature solution of the Sherrington-Kirkpatrick model

Sergey Pankov
National High Magnetic Field Laboratory, Florida State University, Tallahassee, FL 32306
(Dated: October 5, 2018)

We propose a simple scaling ansatz for the full replica symmetry breaking solution of the Sherrington-Kirkpatrick model in the low energy sector. This solution is shown to become exact in the limit \(x \to 0, \beta x \to \infty\) of the Parisi replica symmetry breaking scheme parameter \(x\). The distribution function \(P(x,y)\) of the frozen fields \(y\) has been known to develop a linear gap at zero temperature. We integrate the scaling equations to find an exact numerical value for the slope of the gap \(\partial P(x,y)/\partial y\big|_{y=0} = 0.3014046\ldots\). We also use the scaling solution to devise an inexpensive numerical procedure for computing finite timescale \((x = 1)\) quantities. The entropy, the zero field cooled susceptibility and the local field distribution function are computed in the low temperature limit with high precision, barely achievable by currently available methods.

The field of spin glass physics has been actively studied for the last thirty years. Much attention has been devoted to the Sherrington-Kirkpatrick (SK) model (a mean field analog of realistic spin glasses. The mean field treatment of this model, although exact, is highly nontrivial). In its static formulation, the mean field description of the glassy phase involves infinitely many steps of replica symmetry breaking (RSB), compactly written as a set of integro-differential equations for the spin glass order parameter \(x\) and some auxiliary functions \(m(x,y)\) and \(P(x,y)\). A dynamical formulation leads to similar results and interprets the variable \(x\) as a parametrization of (diverging) timescales of the model, with smaller values of \(x\) corresponding to the longer times. The functions \(m(x,y)\) and \(P(x,y)\) are interpreted, respectively, as a local magnetization in the presence of a frozen field \(y\) and a distribution function of the frozen fields, measured at a timescale \(x\). The full RSB equations cannot be solved analytically, and even solving them numerically is not a trivial task. There were attempts to conjecture certain scaling laws, which, if correct, would grant an exact solution of the problem. Those conjectures, however, turned out to be merely a good approximation.

The idea of scaling, nevertheless, deserves attention. The spin glass phase is characterized by marginal criticality, and critical systems generically exhibit some kind of universal (scaling) behavior. To be more specific we use insights gained from both the static and dynamical viewpoints of the problem. The space of spin glass states is ultrametric and can be visualized as a tree of states with the leaves representing quasi-equilibrium (pure) states. In the dynamical description the system can explore the tree, reaching the states with smaller and smaller overlap \(q\), connected to the initial state through larger and larger branches. The self-similarity of the tree, away from its leaves \((x \sim 1)\) and its root \((x \lesssim T)\), is an important ingredient which could be related to the scaling behavior of the functions \(m(x,y)\) and \(P(x,y)\) in the range \(T \ll x \ll 1\). Another condition for universality to occur is that the typical frozen fields should play no role in the problem. In other words, the scale of disorder should be irrelevant. This can only happen in the zero temperature limit, when any finite frozen field completely polarizes a spin, thus excluding self from contributing to the dynamical evolution of the system. Therefore, it is the limit of \(x \to 0\) and \(\beta x \to \infty\), where one should expect to find a scaling solution, which would be valid for \(y \ll 1\). In this paper we present such a solution and prove that it becomes exact in the above limit, by investigating the correction to scaling. We then use the scaling solution to solve the Parisi equations on the scale \(y \sim T\) and \(x > 0\), for the first time directly in the zero temperature limit. Various physical quantities, measurable on short time scales, are also computed.

Model. We consider the Sherrington-Kirkpatrick model—perhaps the most studied model of spin glass physics:

\[
H = \sum_{i<j} J_{ij} s_i s_j , \tag{1}
\]

where the spins are classical variables \(s_i = \pm 1\) and the bonds are randomly Gaussian quenched with a variance \(J\), which is set to unity through the rest of the paper. Upon cooling, the ergodicity breaks down at \(T = 1\) and the systems freezes in one of the multitude of quasiequilibrium states. This phase is referred to as the glassy phase and can be described with use of the Parisi RSB formalism. A spin on a site \(i\) experiences a local field \(\sum_j J_{ij} s_j\). The local fields can be measured with or without the site \(i\) present in the lattice. They are called, accordingly, the instantaneous or frozen fields. For \(T < 1\) a pseudogap in the distribution of fields opens, becoming (asymptotically) linear in the limit \(y \to 0, \beta y \to \infty\).

Scaling ansatz. The Parisi infinite RSB scheme equations, written in a usual form, read:

\[
\dot{m}(x,y) = -\frac{\bar{q}(x)}{2} \left[ m''(x,y) + 2\beta x m^2(x,y) m'(x,y) \right] , \tag{2}
\]

\[
\dot{P}(x,y) = -\frac{\bar{q}(x)}{2} \left\{ P''(x,y) - 2\beta x [ m(x,y) P(x,y) ]' \right\} , \tag{3}
\]

where the dot and prime are derivatives with respect to \(x\) and \(y\) variables, correspondingly. The differential
equations are supplemented with the initial conditions: \( m(1, y) = \tanh \beta y \) and \( P(0, y) = \delta(y) \). In principle, these equations can be solved iteratively. One can compute \( m \) and \( P \) for a given order parameter \( q \), which, in its turn, is computed from \( m \) and \( P \):

\[
q(x) = \int dy P(x, y)m^2(x, y).
\]

We introduce new notations: \( m(x, y) = \tilde{m}(x, z) \), \( P(x, y) = \frac{1}{\beta x}\tilde{p}(x, z) \), \( \dot{q}(x) = 2\beta x^{-3}c(x) \), where \( z = \beta xy \). It will become clear later that in the scaling regime the functions \( \tilde{m}, \tilde{p} \) and \( c \) lose their dependence on the variable \( x \). We recast Eqs. (2,3) using new definitions:

\[
\dot{\tilde{m}} = -c(\tilde{m}'' + 2\tilde{m}'') - z\tilde{m}',
\]

\[
\dot{\tilde{p}} = c(\tilde{p}'' - 2\tilde{p}''') - z\tilde{p}' + \tilde{p},
\]

where the dot and prime are now derivatives with respect to \( x \) and \( z \) variables, correspondingly. The functions' arguments are omitted for compactness. Our scaling ansatz states that \( \tilde{m} \) and \( \tilde{p} \) are functions of the scaling variable \( z \) only, and \( c \) is a constant, given by \( c = \int dz\tilde{p}(z)(1 - \tilde{m}^2(z)) \), as follows from Eq. (4). Hereafter we refer to Eqs. (2,3) with the left hand sides set to zero and \( c \) being a constant, as the scaling equations.

As expected, the ansatz does not respect the boundary conditions (where the tree of states is not self-similar), but we will demonstrate that it becomes asymptotically exact in the scaling regime described above. This is precisely where the slopes of the linear gap are formed. Thus the scaling ansatz allows us to compute the slope of \( P(1, y) \) in the low temperature limit. Because the scaling equations can be easily integrated (numerically), one can obtain the value of the slope with arbitrary precision. The initial conditions for the scaling equations follow from the definition of \( m \) and \( P \) and from the linearity of the gap at large \( z \):

\[
z = 0, \quad \tilde{m} = 0, \quad \tilde{p}' = 0; \quad |z| \gg 1, \quad \tilde{m} = \text{sign}(z), \quad \tilde{p} = \gamma(|z| + 2c).
\]

With a fixed constant \( c \) the function \( \tilde{p} \) enters the equations linearly, therefore one can set \( \gamma = 1 \) when solving for \( c \), and then compute the slope as \( \gamma = c\int dz\tilde{p}(z)(1 - \tilde{m}^2(z))^{-1} \). Up to ten digits of precision we found \( c = 0.4108020997, \gamma = 0.3010464715 \). The functions \( \tilde{m}(z) \) and \( \tilde{p}(z) \) are shown in Fig. 1.

**Stability of the scaling solution.** To substantiate our findings we have to prove the existence of the scaling regime. Equations (2,3) are free of singularities, and because the scaling solution differs from the initial conditions at \( x = 1 \), our scaling [coinciding with Parisi-Toulouse scaling for \( q(x) \)] cannot hold for all \( x \). The important question is if there is at all such an \( x \) where the proposed scaling solution becomes exact. One can answer this question by investigating the stability of the scaling solution, expanding around it to linear order. We look for a correction to scaling in the form:

\[
\delta \tilde{m}(x, z) = x^\lambda \delta \tilde{m}(z), \quad \delta \tilde{p}(x, z) = x^\lambda \delta \tilde{p}(z), \quad \delta c(x) = x^\lambda \delta c.
\]

This procedure leads to a set of inhomogeneous linear differential equations

\[
\lambda \delta \tilde{m} = -c(\delta \tilde{m}'' + 2\delta \tilde{m}''') - z\delta \tilde{m}' + z\tilde{m}''/c,
\]

\[
(\lambda - 1)\delta \tilde{p} = c(\delta \tilde{p}'' - 2(\delta \tilde{p}''') - 2(\delta \tilde{p}''))
\]

\[
- z\delta \tilde{p}' + (z\tilde{p}' - \tilde{p})/c,
\]

supplemented by two constraints:

\[
1 = \left(1 - \frac{\lambda}{2}\right) \int dz \left[\delta \tilde{p}(1 - \tilde{m}^2 - 2\tilde{p}\delta \tilde{m})\right],
\]

\[
0 = \int dz \left[\delta \tilde{m}^2 + 2\tilde{p}\delta \tilde{m}'\right],
\]

where \( \delta \tilde{m} = \delta \tilde{m}/\delta c, \delta \tilde{p} = \delta \tilde{p}/\delta c, \) and all functions depend on the scaling variable \( z \) only. The least stable solution corresponds to the smallest \( \lambda \). Integrating (numerically) the above equations we found \( \lambda \approx 5.41 \), a fairly large value, indicating that not only the scaling solution is locally stable in the limit \( x \to 0 \), but also that it serves...
terms exhibit quick decay with \( x \) linear stability analysis Eqs (9-12). Correction to scaling \( \tilde{c} \) as a good approximation at finite \( x \). The correction to scaling functions \( \delta \tilde{m}(z) \) and \( \delta \tilde{p}(z) \) are shown in Fig. 2.

**Numerical solution of the RSB equations.** To complete the proof we have to demonstrate that the exact solution of Parisi equations indeed flows to the fixed point, represented by our scaling solution, which we had shown to be attractive. A simple way to do this is to solve the exact RSB equations [Eqs. (5,6)] numerically on the interval \( x \in [x_0, 1] \), using the scaling solution \( \tilde{p} \) as the initial conditions at the point \( x_0 \rightarrow 0 \). If the function \( c(x) \) computed in that way is consistent with our scaling ansatz, that is \( c(x) \) is approaching, for small \( x \), the constant \( c = 0.410802 \) (and \( \tilde{m} \) is approaching its scaling counterpart), then one can claim that the scaling region does exist. A similar self-consistency check could be done on the interval \( x \in [0, x_0] \), though it is not as simple. In addition to proving our point, the outlined procedure will allow us to compute certain low energy quantities characterized by a finite time scale \( x = 1 \).

In practice, when integrating Eqs. (5,6) numerically at \( T = 0 \). It quickly approaches the constant value \( c = 0.410802 \ldots \) as \( x \rightarrow 0 \). The inset shows \( \ln[|c - c(x)|] \) plotted vs \( \ln x \) (circles) along with the linear analysis prediction 5.41 \( x + \text{const} \) (solid line). Notice that the function \( c - c(x) \) spans more than three orders of magnitude.

![FIG. 2: Normalized correction to scaling functions \( \delta \tilde{m}(z) = \delta \tilde{m}(x, z)/\delta c(x) \) and \( \delta \tilde{p}(z) = \delta \tilde{p}(x, z)/\delta c(x) \), obtained from the linear stability analysis Eqs (9-12). Correction to scaling terms exhibit quick decay with \( x \), as \( \sim x^{-0.14} \) - a manifestation of good quality of the Parisi-Toulouse scaling approximation.](image1)

![FIG. 3: Function \( c(x) \) obtained by solving the Parisi equations numerically at \( T = 0 \). It quickly approaches the constant value \( c = 0.410802 \ldots \) as \( x \rightarrow 0 \). The inset shows \( \ln[|c - c(x)|] \) plotted vs \( \ln x \) (circles) along with the linear analysis prediction 5.41 \( x + \text{const} \) (solid line). Notice that the function \( c - c(x) \) spans more than three orders of magnitude.](image2)
The distribution function $p(z)$ is the entropy and zero field cooled susceptibility. Asymptotic behavior of finite time scale quantities, such as $p$ at $T \to 0$, is experimentally measurable distribution of instantaneous fields \cite{12}, corresponding to the tunneling density of states in the context of charge glasses:

$$p(z) = \int \frac{dz'}{\sqrt{2\pi\beta\chi_0}} \tilde{p}(1, z') \frac{\cosh z}{\cosh z'} \exp \left[ -\frac{(z - z')^2}{2\beta\chi_0} - \frac{\beta\chi_0}{2} \right]. \quad (15)$$

The distribution function $p(z)$ is plotted in Fig. 4 together with $\tilde{p}(1, z)$. At large $z$ the distribution of instantaneous fields is shifted, relative to the distribution of frozen fields due to effect of the Onsager reaction term \cite{14,15} by $\gamma\beta\chi_0 \approx 0.479$.

**Conclusion.** To summarize, in this paper we presented a scaling ansatz for the zero temperature solution of the SK model. We assessed the stability of the scaling solution, both analytically and numerically, and found that it becomes exact in the limit of zero temperature and low energy, for $T \ll x \to 0$ and $y \ll 1$. Our ansatz enabled us to compute the slope of the gap in the local fields distribution function numerically exactly, that is up to an arbitrary requested precision. Using the scaling solution as the initial conditions for the RSB equations at $x \to 0$, we could compute, with high precision, the asymptotic behavior of finite time scale quantities, such as the entropy and zero field cooled susceptibility.

We would like to note that the idea of existence of a scaling solution of some sort for the SK model is not new. One of the recent attempts was reported in Ref. \cite{17}. In our paper however, for the first time, a proposed scaling ansatz is proven to become asymptotically exact in a certain limit. The presented method is not limited to the SK model only. It is expected to be applicable and very useful for a wide variety of spin glass models which admit full RSB scenario, such as the Ising $p$-spin model or a recently introduced mean field description of the Coulomb glass \cite{16,17,18}. The scaling ideas may also be promising for understanding quantum glasses, which are thought to be relevant to the most challenging problems of strongly correlated systems \cite{20}.

The author acknowledges extremely enlightening discussions with Vladimir Dobrosavljevic and Markus Mueller.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{distribution_function.png}
\caption{Universal distribution functions $p(z)$ (instantaneous fields, bottom graph) and $\tilde{p}(1, z)$ (frozen fields, top graph) obtained by numerically solving the Parisi equations at $T = 0$. The instantaneous fields are enhanced, relative to the frozen fields, by the value of the Onsager term $\beta\chi_0$ at large $z$.}
\end{figure}

[1] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792-1796 (1975)
[2] G. Parisi, Phys. Rev. Lett. 43, 1754-1756 (1979); Phys. Rev. Lett. 50, 1946-1948 (1983)
[3] G. Parisi, J. Phys. A 13, L115, 1101, 1887 (1980)
[4] J. R. L. de Almeida and E. J. S. Lage, J. Phys. C 16, 939 (1983)
[5] H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. 47, 359-362 (1981)
[6] H. Sompolinsky, Phys. Rev. Lett. 47, 935-938 (1981)
[7] H. J. Sommers and W. Dupont, J. Phys. C 17, 5785 (1984)
[8] G. Parisi and G. Toulouse, J. Phys. (Paris) Lett. 41, L361 (1980)
[9] A. Crisanti, T. Rizzo, and T. Temesvari, Eur. Phys. J. B 33, 203-207 (2003)
[10] F. Pázmáni, G. Zaránd, and G. T. Zimányi, Phys. Rev. Lett. 83, 1034-1037 (1999); A. A. Pastor and V. Dobrosavljevic, Phys. Rev. Lett. 83, 4642-4645 (1999)
[11] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. 52, 1156-1159 (1984)
[12] M. Thomsen, M. F. Thorpe, T. C. Choy, D. Sherrington, and H. J. Sommers, Phys. Rev. B 33, 1931-1947 (1986)
[13] A. Crisanti and T. Rizzo, Phys. Rev. E 65, 046137 (2002)
[14] A. Crisanti, L. Leuzzi, G. Parisi, and T. Rizzo, Phys. Rev. B 70, 064423 (2004)
[15] D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. 35, 593 (1977)
[16] M. Müller and L. B. Ioffe, Phys. Rev. Lett. 93, 256403 (2004)
[17] R. Oppermann and D. Sherrington, Phys. Rev. Lett. 95, 197203 (2005)
[18] S. Pankov and V. Dobrosavljevic, Phys. Rev. Lett. 94, 046402 (2005)
[19] M. Müller and S. Pankov, unpublished
[20] E. Dagotto, Science 309, 257 (2005); E. Miranda and V. Dobrosavljevic, Rep. Prog. Phys. 68, 2337 (2005)