ON THE DISTRIBUTION OF A COTANGENT SUM

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Abstract. Maier and Rassias computed the moments and proved a distribution result for the cotangent sum $c_0(a/q) := -\sum_{m<q} \frac{m}{q} \cot\left(\frac{\pi ma}{q}\right)$ on average over $1/2 < A_0 \leq a/q < A_1 < 1$, as $q \to \infty$. We give a simple argument that recovers their results (with stronger error terms) and extends them to the full range $1 \leq a < q$. Moreover, we give a density result for $c_0$ and answer a question posed by Maier and Rassias on the growth of the moments of $c_0$.

1. Introduction

In this note, we consider the cotangent sum

$$c_0(a/q) := -\sum_{m=1}^{q-1} \frac{m}{q} \cot\left(\frac{\pi ma}{q}\right), \quad (a,q) = 1, q \geq 1,$$

which is related to the Nyman-Beurling criterion for the Riemann hypothesis (see, for example, [Bag], [BC]) and was recently studied in [BC] and [MR]. In [BC], Conrey and the author proved that $c_0$ satisfies the reciprocity formula

$$c_0(a/q) + q/a c_0(q/a) - (\pi q)^{-1} = \psi(a/q), \quad (1.1)$$

where $\psi(x)$ is an analytic function in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and satisfies several nice properties. In particular, it satisfies the three term relation $\psi(x) = \psi(x + 1) + (x + 1)^{-1} \psi(x/(x + 1))$ and has an asymptotic expansion for $x \to 0$, starting by

$$\psi(x) = -\frac{\log(2\pi x) - \gamma}{\pi x} + O(\log x). \quad (1.2)$$

Ishibashi [Ish] observed that $c_0$ is also related to the value at $s = 0$, or $s = 1$ by the functional equation, of the ("imaginary part" of the) Estermann function:

$$c_0(a/q) = \frac{1}{2} D_{\sin}(0, a/q) = 2q\pi^{-2} D_{\sin}(1, \bar{a}/q), \quad (1.3)$$

where for $x \in \mathbb{R}$, $\Re(s) > 1$,

$$D_{\sin}(s, x) := \sum_{n=1}^{\infty} \frac{d(n) \sin(2\pi nx)}{n^s}$$

with $d(n)$ indicating the divisor function. If $x \in \mathbb{Q}$, then $D_{\sin}(s, x)$ can be extended to an entire function of $s$ satisfying the functional equation

$$\Lambda_{\sin}(s, a/q) := \Gamma\left(\frac{1+s}{2}\right)^2 (q/\pi)^s D_{\sin}(s, a/q) = \Lambda_{\sin}(1 - s, \pi/q), \quad (1.4)$$
where $\pi$ denotes the inverse of $a$ modulo the denominator $q$.

If $x \in \mathbb{R} \setminus \mathbb{Q}$, then de la Bretèche and Tenenbaum [dBT] showed that the convergence of the series at $s = 1$ is equivalent to the convergence of $\sum_{n \geq 1}(-1)^n \log v_{n+1}/v_n$, where $u_n/v_n$ denotes the $n$-th partial quotient of $x$. Moreover, they also showed that for $x \notin \mathbb{Q}$

$$D_{\sin}(1, x) := \sum_{n=1}^{\infty} \frac{d(n) \sin(2\pi nx)}{n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} \{nx\},$$

whenever one of the two series converges, and where $\{x\}$ denotes the fractional part of $x$.

Recently, in a difficult paper Maier and Rassias [MR] computed the moments of $c_0(a/q)$ proving that

$$(1.5) \quad \frac{1}{\varphi(q)} \sum_{(a,q)=1, A_0 < a/q < A_1} c_0(a/q)^k = H_k q^k (1 + o(1))$$

as $q \to \infty$, for certain constants $H_k$ and any fixed $\frac{1}{2} < A_0 < A_1 < 1$ and where $\varphi(q)$ is Euler’s function. They also computed the distribution of $\frac{1}{q} c_0(a/q)$ and proved that

$$(1.6) \quad \frac{1}{(A_1 - A_0) \varphi(q)} \sum_{(a,q)=1, A_0 < a/q < A_1} f\left(\frac{1}{q} c_0(a/q)\right) = (1 + o(1)) \int_{\mathbb{R}} f(x) dF(x),$$

as $q \to \infty$ for any continuous function of compact support $f(x)$ and where $F(x)$ is the continuous (as it is shown in the same paper) function defined by

$$F(x) := \text{meas}\left\{z \in [0, 1] \mid 2\pi^{-2}D(1, z) \leq x\right\}.$$

**Figure 1.** The histogram of $2\pi^{-2}D(1, x)$ and the graph of $F(x)$ obtained by sampling $D(1, x)$ (truncated at $n \leq 10^5$) at $10^5$ points chosen uniformly in $[0, 1]$.

In this note we extend the results of [MR] to the full range $A_0 = 0, A_1 = 1$. Moreover, since we are averaging over the full range, our results immediately give moments and distribution also for $D(1, a/q)$ or, equivalently, for the Vasyunin sum $V(a, q) := -c_0(\pi/q)$.
Our method, with the orthogonality for additive characters replaced by Weil’s bound (as in Lemma 8 of [DFI]), gives also the case when \(A_0 \neq 0, A_1 \neq 1\), thus providing a new simpler proof of the results of Maier and Rassias with stronger bounds for the error terms. In particular, we obtain (1.5) with the error term \(o(1)\) replaced by \(O_\varepsilon(q^{k-\frac{1}{2}+\varepsilon}(Ak\log q)^{2k})\), allowing us to handle the case of short intervals \([A_0, A_1]\) with \(A_1 - A_0 \gg q^{-\frac{1}{2}+\delta}\) for some \(\delta > 0\). However, since the details are identical, we limit ourselves to the full range \(A_0 = 0, A_1 = 1\).

**Theorem 1.** Let \(q \geq 1\) and \(k \geq 0\). Then,

\[
\frac{1}{\varphi(q)} \sum_{a=1, (a,q)=1}^{q} c_0(a/q)^k = H_k q^k + O_\varepsilon(q^{k-1+\varepsilon}(Ak\log q)^{2k}),
\]

for some absolute constant \(A > 0\) and any \(\varepsilon > 0\), where

\[
H_k := (i\pi^2)^{-k} \sum_{(n_1,\ldots,n_k) \in (\mathbb{Z} \setminus 0)^k, n_1 + \cdots + n_k = 0} d(|n_1|) \cdots d(|n_k|) n_1 \cdots n_k.
\]

**Remark 1.** If \(k\) is odd, then both \(H_k\) and the left hand side of (1.7) are identically zero.

In the same paper Maier and Rassias ask whether \(\sum_{k=0}^{\infty} H_k t^k\) has a positive radius of convergence. The following Theorem answers their question in the affirmative.

**Theorem 2.** We have \(H_k \ll A^k k!\) for some \(A > 0\).

Since \(\sum_{k=0}^{\infty} H_k t^k/k!\) has a positive radius of convergence, we immediately obtain the distribution of \(c_0\) over the full range.

**Corollary 1.** Let \(q \geq 1\) and let \(f(x)\) be a continuous function with compact support. Then, as \(q \to \infty\) we have

\[
\frac{1}{\varphi(q)} \sum_{a=1, (a,q)=1}^{q} f\left(\frac{1}{q} c_0\left(\frac{a}{q}\right)\right) = (1 + o(1)) \int_{\mathbb{R}} f(x) dF(x).
\]

Using (1.1), we can also give an alternative expression for \(D_{\sin}(1, x)\) in terms of the denominators of the partial quotient of \(x\).

**Proposition 1.** Let \(<a_0; a_1, a_2, \ldots>\) be the continued fraction expansion of \(x \in \mathbb{R}\). Moreover, let \(u_r/v_r\) be the \(r\)-th partial quotient of \(x\). Then

\[
D_{\sin}(1, x) := \sum_{n=1}^{\infty} \frac{d(n) \sin(2\pi nx)}{n} = -\frac{\pi^2}{2} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{v_{\ell}^2} \left(\frac{1}{\pi v_{\ell}} + \psi\left(\frac{v_{\ell-1}}{v_{\ell}}\right)\right),
\]

whenever either of the two series is convergent.

If \(x = <a_0; a_1, \ldots, a_r>\) is a rational number, then the range of summation of the series on the right is to be interpreted to be \(1 \leq \ell \leq r\).
Remark 2. If $x \in \mathbb{Q}$ then one can write two different continued fraction expansion for $x$, but (1.8) holds regardless of the chosen expansion.

Proposition 1, which constitutes a refinement of the aforementioned work of de la Bretèche and Tenenbaum, can be interpreted as an extension of the reciprocity formula (1.1) to $x \notin \mathbb{Q}$. We also remark that Proposition 1 is of independent interest as $D(1, a/q) = -\frac{\pi^2}{2q}V(a, q)$ is exactly the sum appearing in the Nyman-Beurling criterion for the Riemann hypothesis (c.f. [BC]).

The exact formula (1.8) allows us to prove the following corollary which, combined with (1.6) and the periodicity modulo 1 of $c_0$, implies that $\{(a/q, \frac{1}{q}c_0(a/q)) \mid (a, q) = 1, q \geq 1\}$ is dense in $\mathbb{R}^2$.

Corollary 2. The function $F(x)$ is strictly increasing.

Corollary 3. We have that $\{(a/q, \frac{1}{q}c_0(a/q)) \mid (a, q) = 1, q \geq 1\}$ is dense in $\mathbb{R}^2$.

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After sending them a preprint of this paper, the author was informed by Maier and Rassias that they have also obtained independently Theorem 2.1 using a somewhat similar argument.

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2. Proof of Theorem

Both in this and in the following section, we will consider $D(1, a/q)$ rather than $c_0(a/q)$. The stated result then follows by (1.3). Moreover, we assume that $k$ is even, as the result is trivial otherwise.

First, we observe that we can have

\[(2.1) D_{\sin}(1, a/q) = \sum_{n \leq 2X} g_X(n) \frac{d(n) \sin(2\pi na/q)}{n} + O(q^{1+2\varepsilon_1}/X),\]

for $\varepsilon_1 = 0.1$ and where $g_X(x)$ is a bounded function supported in $[0, 2X]$ and identically 1 for $1 \leq x \leq X$. This can be seen by taking a smooth partition of unity satisfying

\[1 = \sum'_{M} F(x/M) \quad \forall x \in [1/2, \infty), \quad \sum'_{Y_1 \leq M \leq Y_2} 1 \ll \log(2 + Y_2/Y_1) \quad 1 \leq Y_1 \leq Y_2,\]

where $F(x)$ is smooth, supported in $[1/2, 1]$, and satisfying $F^{(j)}(x) \ll_j 1$ for all $j \geq 0$ (so that the Mellin transform $\hat{F}(s)$ of $F(x)$ is entire and decays rapidly on vertical strips).

Then, writing $F$ in terms of its Mellin transform $\hat{F}(s)$, we have

\[\sum_{n \geq 1} F\left(\frac{n}{M}\right) \frac{d(n) \sin(2\pi na/q)}{n} = \int_{(1)} \hat{F}(s) D(1 + s, \frac{a}{q}) M^s ds = \int_{(-1-\varepsilon_1)} \hat{F}(s) D(1 + s, \frac{a}{q}) M^s ds\]

\[\ll q^{1+2\varepsilon_1} M^{-1-\varepsilon_1}\]

as can be seen by using (1.4) and bounding trivially. Thus, (2.1) follows by taking $g_X(x) := \sum_{M \leq 2X} F(x/M)$. 

By Euler’s formula, when $X \geq q^{1+2\varepsilon}$ (2.1) gives

$$D_{\sin}(1, a/q)^k = (2i)^{-k} \sum_{n_1, \ldots, n_k \in B_{2X}^r} e\left((n_1 + \cdots + n_k) \frac{a}{q}\right) \tilde{d}(n_1, \ldots, n_k) + O\left(\frac{(A\log X)^{2k} q^{1+2\varepsilon}}{X}\right),$$

where $B_X^r := [-X, X] \cap \mathbb{Z}_{\neq 0}$,

$$\tilde{d}(n_1, \ldots, n_k) := d(|n_1|) \cdots d(|n_k|) g(|n_1|/X) \cdots g(|n_k|/X)$$

and $A$ denotes an absolute positive constant that might change from line to line. Thus, by Möbius inversion formula and the orthogonality of additive characters we have

$$\frac{1}{\varphi(q)} \sum_{\ell | q, (a, q) = 1} \mu(q/\ell) \sum_{a=1} \ell D_{\sin}(1, a/\ell)^k$$

$$= (2i)^{-k} \sum_{\ell | q} \mu(q/\ell) \ell \sum_{n_1, \ldots, n_k \in B_{2X}^r, n_1 + \cdots + n_k \equiv 0 (\text{mod } \ell)} \frac{\tilde{d}(n_1, \ldots, n_k)}{|n_1 \cdots n_k|} + O\left(\frac{(A\log X)^{2k} q^{1+3\varepsilon}}{X}\right).$$

The contribution of the terms with $n_1 + \cdots + n_k \neq 0$ is bounded by

$$\sum_{\ell | q} \ell \sum_{n_1, \ldots, n_k \in B_{2X}^r, 0 \neq n_1 + \cdots + n_k \equiv 0 (\text{mod } \ell)} \frac{|\tilde{d}(n_1, \ldots, n_k)|}{|n_1 \cdots n_k|} \ll \varepsilon \sum_{\ell | q} k \ell \sum_{n_1, \ldots, n_k \in B_{2X}^r, |n_1| \geq \ell/k, 0 \neq n_1 + \cdots + n_k \equiv 0 (\text{mod } q)} \frac{A^k X^\varepsilon d(|n_2|) \cdots d(|n_k|)}{|n_1 \cdots n_k|} \ll \varepsilon q^{-1+\varepsilon} A^k X^\varepsilon (\log X)^{2k},$$

since

$$\sum_{\ell/k \leq n \leq 2X, n \equiv 0 (\text{mod } \ell)} \frac{1}{n} \ll \frac{k}{\ell} + \frac{\log X}{\ell}.$$
3. Proof of Proposition [1] and Theorem [2]

The following lemmas give Proposition [1] in the cases of $x \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ respectively.

**Lemma 4.** Let $(a, q) = 1$, $q > 0$ and let $v_0, \ldots, v_r$ be the partial denominators of the continued fraction expansion $a/q = \langle a_0; a_1, \ldots, a_r \rangle$. Then

$$D_{\sin}(1, a/q) = -\frac{\pi^2}{2} \sum_{\ell=1}^{r} \left( -\frac{1}{v_\ell} \left( \frac{1}{\pi v_\ell} + \psi \left( \frac{v_{\ell-1}}{v_\ell} \right) \right) \right).$$

**Proof.** Write $b/q := (-1)^{r+1} \frac{a}{q}$. Then, one has the continued fraction expansion $b/q = \langle 0; b_1, \ldots, a_r \rangle = \langle 0; a_r, \ldots, a_1 \rangle$. Moreover, the Euclid algorithm for $b/q$ gives

$$y_1 = q, \quad y_2 = b,$$

$$y_{n-1} = b_{n-1} y_n + y_{n+1}, \quad n = 1, \ldots, r + 1,$$

with $y_{r+1-\ell} = v_\ell$, where $v_\ell$ is the $\ell$-th partial quotient of $a/q$ (as usual we put $v_{-1} := 0$). Thus, applying repeatedly the reciprocity formula (1.1) and using that $c_0(1) = 0$, we obtain

$$\frac{1}{q} c_0(b/q) = -\sum_{m=1}^{r} \frac{(-1)^m}{\pi y_m^2} - \sum_{m=1}^{r} \frac{(-1)^m}{y_m} \psi \left( \frac{y_{m+1}}{y_m} \right) = \sum_{\ell=1}^{r} \frac{(-1)^{\ell+r}}{v_\ell} \left( \frac{1}{\pi v_\ell} + \psi \left( \frac{v_{\ell-1}}{v_\ell} \right) \right)$$

and the Lemma follows by [1.3] since $D(s, x) = -D(s, -x)$. \hfill \Box

**Lemma 5.** Let $x \in \mathbb{R}\setminus\mathbb{Q}$ and assume $x$ has continued fraction expansion $x = \langle a_0; a_1, a_2, \ldots \rangle$ with partial quotients $v_0, v_1, v_2, \ldots$. Then

$$(3.1) \quad D_{\sin}(1, a/q) = -\frac{\pi^2}{2} \sum_{\ell=1}^{\infty} \left( -\frac{1}{v_\ell} \left( \frac{1}{\pi v_\ell} + \psi \left( \frac{v_{\ell-1}}{v_\ell} \right) \right) \right),$$

whenever $D_{\sin}(1, a/q)$ is defined. Moreover, writing

$$D_X(1, x) := \sum_{n \leq X} d(n) \frac{\sin(2\pi n x)}{n}, \quad S(x) := \sum_{n=1}^{\infty} \frac{\log v_{n+1}}{v_n},$$

we have $D_X(1, x) \ll S(x)$ and $D(1, x) \ll S(x)$, uniformly in $x \in [0, 1] \setminus \mathbb{Q}$, $X \geq 2$.

**Proof.** For a large positive constant $B \geq 5$, let $\xi_r = v_r (\log v_r)^B$ and let $R$ be the minimum integer such that $\xi_R \leq X$. We can split $D_X(1, x)$ into

$$(3.2) \quad D_X(1, x) = \sum_{n \leq \xi_R} d(n) \frac{\sin(2\pi n x)}{n} + \sum_{\xi_R \leq n \leq X} d(n) \frac{\sin(2\pi n x)}{n}.$$

The second addend can be treated using the work of de la Bretèche and Tenenbaum [dBT]. Indeed, by partial summation, if $B$ is sufficiently large we have

$$\sum_{\xi_R \leq n \leq X} d(n) \frac{\sin(2\pi n x)}{n} = \sum_{\xi_R \leq n \leq X} d(n) \frac{\sin(2\pi n x)}{X} + \int_{\xi_R}^{X} \sum_{\xi_R \leq n \leq t} d(n) \frac{\sin(2\pi n x)}{t^2} dt$$

$$= O \left( \frac{\log(v_{R+1})}{v_R} + \frac{1}{\log(v_R)} \right).$$
by (11.1) and (11.4) of [dBT]. For the first addend of the right hand side of (3.2), we first observe that

\[ \sum_{n \leq \xi R} d(n) \frac{\sin(2\pi nx)}{n} = \sum_{n \leq \xi R} d(n) \frac{\sin(\frac{2\pi nx}{v_R})}{n} + O\left(\frac{(\log v_R)^{1+B}}{v_{R+1}}\right) \]

since \( |x-u_R/v_R| \leq (v_R v_{R+1})^{-1} \). Moreover, we observe that by Mellin’s formula we have

\[ \sum_{n \leq \xi R} d(n) \frac{\sin(2\pi nx)}{n} = D \sin\left(\frac{1}{\xi R} \frac{u_R}{v_R}\right) + \frac{1}{2\pi i} \int_C D \sin(1+s, u_R/v_R) \xi_s ds + O((\log v_R)^2/T) \]

\[ = D \sin\left(\frac{1}{\xi R} \frac{u_R}{v_R}\right) + O(\log v_R)^2/\xi R + (\log v_R)^2/T \]

where \( C \) denotes the line from \( s = (-1 - \frac{1}{\log x}) - iT \) to \( s = (-1 - \frac{1}{\log x}) + iT \) with \( T = (\log v_R)^4 \) and where to bound the integral we used the functional equation (1.4) and a trivial bound.

Finally, by Lemma 4 we have

\[ D \sin\left(\frac{1}{\xi R} \frac{u_R}{v_R}\right) = -\frac{\pi^2}{2} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{v_\ell} \left( \frac{1}{\pi v_\ell} + \log\left(\frac{v_{\ell-1}}{v_\ell}\right) \right) \]

and thus

\[ D_X(1,x) = -\frac{\pi^2}{2} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{v_\ell} \left( \frac{1}{\pi v_\ell} + \log\left(\frac{v_{\ell-1}}{v_\ell}\right) \right) + O\left(\frac{(\log v_R)^{1+B}}{v_R} \right) \]

As \( X \to \infty \), we have \( v_R \to \infty \) and, by Theorem 4.4 of [dBT], \( \log(v_{R+1})/v_R \to 0 \) if (and only if) the series defining \( S(1,x) \) converges and thus we obtain (3.1). The second assertion of the Lemma then follows by (1.2).

We need two results from Khinchin’s book on continued fractions [Khi].

**Lemma 6.** For all \( x \in \mathbb{R} \setminus \mathbb{Q} \) and all \( n \geq 1 \) we have \( v_n \geq 2^{\frac{n-3}{2}} \).

**Proof.** This is Theorem 12 of [Khi].

The following lemma is a minor refinement of Theorem 31 of [Khi].

**Lemma 7.** Let \( K \geq 1 \). Then for all \( \varepsilon > 0 \), there exists \( B_\varepsilon > 0 \) such that

\[ E(K) := \text{meas}\left\{ x \in [0,1] \mid v_r(x) \geq K e^{B_r} \quad \exists r \geq 1 \right\} \ll \varepsilon K^{-1+\varepsilon} \]

**Proof.** Proceeding as in the proof of Theorem 31 of [Khi], we see that for all \( n, B \geq 1 \) we have

\[ \text{meas}(E_n(K)) \ll \frac{2^n}{K e^{B_n}} \sum_{\ell=0}^{n-1} \frac{(\log(K e^{B_n}))^\ell}{\ell!} \]
where \( E_n(K) := \{ x \in [0,1] \mid v_n(x) \geq Ke^{Bn} \} \) (this is the first equation on page 68 of [Khi], with \( g = Ke^n \)). Now, if \( K \leq e^{Bn} \) and \( B \) is large enough, then

\[
\text{meas}(E_n(K)) \ll \frac{2^n}{Ke^{Bn}} \sum_{\ell=0}^{n-1} \frac{(\log Ke^{Bn})^\ell}{\ell!} \ll \frac{2^n}{K} \frac{(2Bn)^\ell}{\ell!} \ll \frac{n}{K} \frac{(4ne^{-B})^n}{n!} \ll \frac{e^{-Bn/2}}{K}.
\]

where we used \( C^\ell/\ell! \ll C^n/n! \), valid for \( 0 \leq \ell \leq n \leq C \), and Stirling’s formula. In the same way, if \( K > e^{Bn} \) and \( B \) is large enough, then

\[
\text{meas}(E_n(K)) \ll \frac{n(4\log K)^n}{Kn!e^{Bn}} \ll \frac{(e^{-B/2}\log K)^n}{Kn!}.
\]

Thus, we have

\[
E(K) \leq \sum_{n=1}^{\infty} \text{meas}(E_n(K)) \ll \varepsilon K^{-1+e^{-B/2}}
\]

and the Lemma follows. \( \Box \)

**Corollary 8.** For \( K \geq 1 \), we have

\[
\text{meas}\{ x \in [0,1] \mid |S(x)| > K \} = O(e^{-\delta K})
\]

for some \( \delta > 0 \).

**Proof.** By Lemma 6 and 7 if \( x \in [0,1] \setminus E(e^K) \) we have

\[
S(x) \ll \sum_{n=1}^{\infty} \frac{B_\varepsilon n + K}{2^{n/2}} \ll \varepsilon 1 + K
\]

and (3.3) follows. \( \Box \)

We can now prove Theorem 2 and Corollary 1.

**Proof of Theorem 2 and Corollary 1**

Expressing the linear constraint in the definition of \( H_k \) as an integral, we see that

\[
H_k = (i\pi)^{-k} \lim_{X \to \infty} \int_0^1 \sum_{n_1, \ldots, n_k \neq 0} \frac{e((n_1 + \cdots + n_k)x) d([n_1]) \cdots d([n_k])}{n_1 \cdots n_k} dx
\]

\[
= \lim_{X \to \infty} \frac{2^k}{\pi^{2k}} \int_0^1 D_X(1, x)^k dx = \frac{2^k}{\pi^{2k}} \int_0^1 D(1, x)^k dx
\]

where the exchange of order of summation and integration is justified by the dominated convergence theorem, since \( D_X(1, x) \ll S(x) \) by Lemma 5 and, by (3.3),

\[
\int_0^1 S(x)^k dx \leq \sum_{L=1}^{\infty} \int_0^1 \chi_L(x) L^k dx \ll \sum_{L=1}^{\infty} L^k e^{-(L-1)^\delta} \ll A^k k!,
\]

for some \( \delta, A > 0 \) and where \( \chi_L \) is the characteristic function of the set \( \{ x \mid L - 1 \leq S(x) \leq L \} \).

Since we also have \( D(1, x) \ll S(x) \), the above computation also proves Theorem 2 and Corollary 1 then follows since \( \sum_{k=1}^{\infty} H_k t^k / k! \) has a positive radius of convergence. \( \Box \)
Finally, we prove Corollary 2.

**Proof of Corollary 2.** By Lemma 7 we can find some absolute constants $K, B$ such that

$$S(x_1, x_2, \kappa) := \{x = (0; 1, \ldots, 1, x_1, x_2, a_{\kappa+3}, a_{\kappa+4}, \ldots) \mid 1 \leq a_\ell \leq Ke^{B\ell}, \forall \ell \geq \kappa + 3\}$$

has positive measure for any $x_1, x_2, \kappa \in \mathbb{Z}_{>0}$. Thus, to prove the corollary it is enough to show that for any $z \in \mathbb{R}$, $\varepsilon > 0$ there exist integers $x_1, x_2, \kappa \geq 1$ such that $z < D(1, x) \leq z + \varepsilon$ for all $x \in S(x_1, x_2, \kappa)$.

Now, if $x \in S$ then by (1.8) we have

$$D_s(x, x) = A + B + C,$$

where

$$A = -\frac{\pi^2}{2} \sum_{\ell=1}^{\kappa} (-1)^\ell \left( \frac{1}{\pi v_\ell} + \psi\left( \frac{v_{\ell-1}}{v_\ell} \right) \right),$$

$$B = -\frac{\pi^2}{2} \sum_{\ell=\kappa+1}^{\kappa+2} (-1)^\ell \left( \frac{1}{\pi v_\ell} + \psi\left( \frac{v_{\ell-1}}{v_\ell} \right) \right),$$

and, by (1.2) and Lemma 6

$$C \ll \sum_{\ell=\kappa+3}^{\infty} \frac{\log v_\ell}{v_{\ell-1}} \ll \sum_{\ell=\kappa+3}^{\infty} \frac{\log K + B\ell}{2^\ell} \leq \varepsilon/10,$$

provided that $\kappa = \kappa_\varepsilon$ is large enough. Now, from the relation $v_n = a_nv_{n-1} + v_{n-2}$, we see that $v_{\kappa+1} = x_1v_\kappa + v_{\kappa-1}$ and $v_{\kappa+2} = x_2(x_1v_\kappa + v_{\kappa-1}) + v_\kappa$. Thus, by (1.2) we have

$$B = \alpha_\kappa \log(x_1) - \beta_\kappa \frac{\log(x_2)}{x_1 + \gamma_\kappa} + o(1)$$

$$= \alpha_\kappa \log(x_1) - \beta_\kappa \frac{\log(x_2)}{x_1} + o(1) + O\left( \frac{\log(x_2)}{x_1^2} \right),$$

as $x_1, x_2 \to \infty$ (and $\kappa$ fixed), for some $\alpha_\kappa, \beta_\kappa, \gamma_\kappa \neq 0$. Now, if we pick

$$x_2 := \lfloor \exp\left( \beta_\kappa^{-1}x_1(\alpha_\kappa \log(x_1) + A - z - \varepsilon/2) \right) \rfloor,$$

then $B = z - A + \varepsilon/2 + o(1)$. Thus, if $x_1$ is large enough we have

$$z < D(1, x) \leq z + \varepsilon$$

and the corollary follows. \hfill \Box

**Remark 3.** We remark that a modification of this proof in the spirit of [Hic] would have given Corollary 3 directly.

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