An algorithm combining coordinate descent and split Bregman iterative for fractional image denoising model

Donghong Zhao, Yonghua Fan, Haoyu Liu and Yafeng Yang

Abstract
The split Bregman algorithm and the coordinate descent method are efficient tools for solving optimization problems, which have been proven to be effective for the total variation model. We propose an algorithm for fractional total variation model in this paper, and employ the coordinate descent method to decompose the fractional-order minimization problem into scalar sub-problems, then solve the sub-problem by using split Bregman algorithm. Numerical results are presented in the end to demonstrate the superiority of the proposed algorithm.

Keywords
Image denoising, coordinate descent method, fractional total variation, split Bregman algorithm

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Introduction
Image denoising is one of the image processing techniques, whose aim is to restore the degraded image \( g \) contaminated by noise \( n \) to the original image \( u \). An effective denoising technique can remove noise while preserving important image features such as image edge and texture.

\[ g = u + n. \]  

Image denoising is an inverse problem. The first denoising model was ill-posed and had no unique stable solution. To solve the ill-posed model, Tikhonov proposed a regularization technique and introduced it into image restoration problem. There are many methods of image restoration at present. Among them, the total variation (TV) model proposed by Rudin et al.\(^1\) has attracted much attention from scholars. Since the bounded variation functions space allows discontinuities, the edge-preserving ability of TV model is very good.\(^1\)–\(^3\) However, the staircase effect occurs in the image smooth region. The TV model is formulated as:

\[ \min_u \int_{\Omega} |\nabla u| + \frac{\lambda}{2} \| u - g \|^2. \]  

The above model has its advantages and its own temporary insurmountable disadvantages in image processing. In order to make the image processing technology more advanced, many researchers have been trying to study better algorithms. It introduced a novel mesh-free approach using TV regularization and a radial basis function for the numerical approximation of TV-based model to remove the additive noise from the measurements. The author proposed the anisotropic TV model for recovering image corrupted by impulse noise. The L2 fidelity term in formula (2) is replaced by L1. A hybrid regularization model was proposed; the related simulations indicated that the combination scheme performed better than the pure second-order or fourth-order partial differential equations (PDE) models. Engineering researches\(^4\)–\(^12\) have confirmed that the first-order derivative will increase the high-frequency components such as edge information. For the medium–low-frequency components such as texture details, the fractional derivative can keep the edge better. The theoretical analysis shows that the fractional-order denoising model has good texture retention ability. It extends the integer-order denoising model to the fractional-order and the proposed

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fractional-order TV model, the numerical simulation results proved that the fractional-order TV model has its superiority.

\[
\min_u \int_\Omega |\nabla u| + \frac{\lambda}{2} \| u - g \|^2. \tag{3}
\]

There are many solutions for minimizing the energy functional problem. Duli Chen et al. \cite{13} transformed the minimization problem (3) into a series of simple proximities operators and proposed an effective proximity algorithm. The dual algorithm proposed by Chambolle \cite{14} and proposed an effective proximity algorithm. The dual algorithm proposed by Qianshun Chang in the paper, three algorithms are adopted. Since the regularization terms of (3) are non-differential, the author proposed a projection algorithm for image denoising, transformed the original problem into a new problem through the sub-gradient and Fenchel conjugate properties.

Tian et al. \cite{19} proposed a primal–dual algorithm for TV model. This algorithm is more and more advantageous in the CPU time and the number of iterations. The Bregman iterative algorithm proposed in the 1960s for the general pursuit problem. Osher and Cai proposed the linear Bregman algorithm for the compressive sensing problem, Goldstein and Osher \cite{21,22} proposed the split Bregman algorithm for image restoration problem, whose convergence speed is fast.

In this paper, based on the previous work, the coordinate descent method is used to solve the fractional-order TV model and use the split Bregman algorithm for its sub-problem. The rest of paper is organized as follows. In section “Preliminary,” the split Bregman algorithm and the coordinate descent method are introduced; in section “Proposed algorithm,” the proposed algorithm for the fractional-order TV model is proposed. In section “Numerical experiments,” the numerical experiments and results to demonstrate the effectiveness of the proposed method are given. Finally, this paper is concluded in section “Conclusions.”

\section*{Preliminary}

Split Bregman algorithm and coordinate descent algorithm (CoD) are efficient for solving optimization problems, especially for L1 regulation problem. In this section, two algorithms are briefly introduced.

\section*{Split Bregman algorithm}

For the general minimization problem

\[
\min_u J(u) + H(u), \tag{4}
\]

where \( J(u) \) and \( H(u) \) are convex functions and \( H(u) \) is differentiable. Bregman defined the Bregman distance of function \( J \) at \( u \) and \( v \),

\[
D^J_J(u, v) = J(u) - J(v) - \langle p, u - v \rangle
\]

where \( u, v \in \text{dom}(E) \), \( p \) is the sub-gradient of \( J \) at \( v \). Bregman distance is not the distance in general because it does not satisfy the symmetry. Actually, it measures the closeness of \( u \) and \( v \). Bregman iterative formula is as follows:

\[
u^{k+1} := \min_u D^J_J(u, u^k) + H(u)\tag{5}
\]

\[
= \min_u J(u) - J(u^k) - \langle p^k, u - u^k \rangle + H(u),
\]

\[
p^{k+1} = p^k - \nabla H(u^{k+1}). \tag{6}
\]

\( \nabla H \) represents the derivative of \( H(u) \).

The energy functional regularization model based on image denoising can be reduced to the following minimization problem:

\[
\min_u |J(u)| + H(u), \tag{7}
\]

where \( | \cdot | \) is \( L_1 \) norm, \( J(u) \) and \( H(u) \) are convex functions. Goldstein and Osher \cite{21,22} proposed split Bregman algorithm based on Bregman iterative algorithm. By introducing variable \( d \), the unconstrained (7) equal to the following constrained problem:

\[
\min_u |d| + H(u), \text{ such that } d = J(u). \tag{8}
\]

Convert (8) into an equivalent unconstrained problem.

\[
\min_u |d| + H(u) + \mu \frac{1}{2} \| d - J(u) \|^2. \tag{9}
\]

The \( L_1 \) and \( L_2 \) portions of (7) are “de-couple” by split method. Next, let \( E(u, d) = |d| + H(u) \), using the Bregman iterative to solve (9), there would be:

\[
(u^{k+1}, d^{k+1}) = \min_{u,d} D^J_J(u, u^k, d, d^k) + \mu \frac{1}{2} \| d - J(u) \|^2
\]

\[
= J(u) + J(d) - J(u^k) - J(d^k) - \langle p^k, u - u^k \rangle
\]

\[
- \langle p^k, d - d^k \rangle + \mu \frac{1}{2} \| d - J(u) \|^2. \tag{10}
\]

\[
p^{k+1}_u = p^k - \mu \nabla J(u^{k+1}) - \mu d^{k+1}, \tag{11}
\]

\[
p^{k+1}_d = p^k - \mu (d^{k+1} - J(u^{k+1})). \tag{12}
\]

The above split Bregman iteration formulas can be reduced to the following equivalent formulas:

\[
(u^{k+1}, d^{k+1}) = \min_{u,d} |d| + H(u)
\]

\[
+ \frac{\mu}{2} \| d - J(u) - b^k \|^2, \tag{13}
\]

\[
b^{k+1} = b^k + \frac{\mu}{2} (J(u^{k+1}) - d^{k+1}). \tag{14}
\]
The key of split Bregman algorithm is to solve \((u^{k+1}, d^{k+1})\). Similar to the augmented Lagrangian algorithm, the split Bregman algorithm can be split into two subproblems of \(u\) and \(d\).

Step 1: \[ u^{k+1} = \min_u H(u) + \frac{\mu}{2} \|d^k - J(u) - b^k\|^2, \]  
(15)

Step 2: \[ d^{k+1} = \min_d |d| + \frac{\mu}{2} \|d - J(u) - b^k\|^2. \]  
(16)

Equation (15) can be solved by Gauss Seidel iteration or Fourier transform, and the solution of equation (16) can be expressed with the shrink operator.

**Coordinate descent algorithm**

The CoD was proposed in 1998 to solve the linear fitting problem. CoD-based denoising (CoDenoise) algorithm converges fast.\(^{23-25}\) Its basic idea is to decompose the problem about vectors or matrices into one-dimensional scalar sub-problems.

The following formulation is considered as:

\[ \min_u f(x) + h(x). \]  
(17)

where \(x \in \mathbb{R}^n\), \(f(x)\) and \(h(x)\) are convex functions and \(h(x)\) is smooth. Assuming \(f(x)\) is separable.

\[ f(x) = \sum_{i=1}^n f_i(x_i). \]  
(18)

Then, the problem (17) can be reduced to solve a series of scalar optimization sub-problem:\(^{26}\)

\[ \min_u f_i(x_i) + h_i(x_i). \]  
(19)

The coordinate selection mode, also calls sweeping path-wise, which is the emphasis of CoD method. Applying the effective sweeping path-wise into denoising models can solve problems faster.

**Proposed algorithm**

**The discrete model**

Let \(X = \mathbb{R}^{n^2}\), the size of image \(u\) is \(n \times n\) pixels, the discrete form of the fractional-order TV model is as follows:

\[
D_{\beta}^f(u_{ij}, u_{i,0}^j, d_{i,j}, d_{i,j-1}, d_{i,j+1}, d_{i,j+2}, d_{i,j-2}) = |d_{i,j}| + |d_{i,j+2}| + \frac{\lambda}{2} (u_{ij} - g_{ij})^2 - |d_{i,j+1}| - \frac{\lambda}{2} (u_{ij}^k - g_{ij})^2 - \left\{p_{d_{i,j}}^k, u_{ij} - u_{ij}^k\right\} - \left\{p_{d_{i,j+2}}^k, d_{i,j+2} - d_{i,j+2}^k\right\} - \left\{p_{d_{i,j-2}}^k, d_{i,j-2} - d_{i,j-2}^k\right\}
\]

**Solving sub-problem**

For every anisotropic fractional variable sub-problem, we use the split Bregman algorithm to solve it

\[ \min_{u_{ij}} |\nabla_x u_{ij}| + |\nabla_y u_{ij}| + \frac{\lambda}{2} (u_{ij} - g_{ij})^2. \]  
(22)

Introduce variables \(d_{i,j+1}\) and \(d_{i,j-1}\), (22) is equivalent to

\[ \min_{u_{ij}} |d_{i,j+1}| + |d_{i,j-1}| + \frac{\lambda}{2} (u_{ij} - g_{ij})^2 \quad \text{s.t.} \quad d_{i,j+1} = \nabla_x u_{ij}, \quad d_{i,j-1} = \nabla_y u_{ij}. \]  
(23)

The equivalent non-constrained problem is considered:

\[ \min_{u_{ij}} |d_{i,j}| + |d_{i,j+2}| + \frac{\lambda}{2} (u_{ij} - g_{ij})^2 + \frac{\mu}{2} (d_{i,j} - \nabla_x u_{ij})^2 + \frac{\mu}{2} (d_{i,j+2} - \nabla_y u_{ij})^2 \]  
(24)

Let \(E(u_{ij}, d_{i,j+1}, d_{i,j-1}) = |d_{i,j}| + |d_{i,j+2}| + \lambda (u_{ij} - g_{ij})^2\), then the Bregman distance of \(E\) at \((u_{ij}^k, d_{i,j+1}^k, d_{i,j-1}^k)\) is
Applying iterative split Bregman, there would be:

\[(u_j^{k+1}, d_j^{k+1}, d_{j2}^{k+1}) = \arg \min_{u_j, d_j, d_{j2}} D^k_u(u_j, u_{j1}, d_{j1}, d_{j2}, d_{j2}^k)\]

\[= \arg \min_{u_j, d_j, d_{j2}} \left( D^k_u(u_j, u_{j1}, d_{j1}, d_{j2}, d_{j2}^k) + \frac{\mu}{2} (d_{j1} - \nabla u_j)^2 + \frac{\mu}{2} (d_{j2} - \nabla u_{j2})^2 \right).\]  

(25)

\[d_{ij}^{k+1} = \text{shrink} \left( \sum_{k=0}^{i-1} (-1)^k C_k u_{i-k} + b_{ij}^{k-1}, \frac{1}{\mu} \right) \]  

(34)

\[d_{ij}^{k+1} = \text{shrink} \left( \sum_{k=0}^{i-1} (-1)^k C_k u_{i-k} + b_{ij}^{k-1}, \frac{1}{\mu} \right). \]  

(35)

We summarize the algorithm for the fractional variation model as follows; the algorithm has a sequential updating mode:

**Algorithm**

1. **Initialization.** Pick \(u_0^0\) and \(b_0^1 = 0, b_0^2 = 0, d_0^1 = 0, d_0^2 = 0\), choose feasible \(\hat{\lambda}\) and \(\mu\), set \(k = 0\).

2. **Iteration.** For every \(i, j = 1, 2, \ldots, n\), Compute \(d_j^{k+1}, d_{j2}^{k+1}, d_{j1}^{k+1}\) and \(b_{ij}^{k+1}\) as (33), (34), (35), (28) and (29).

3. If \(u^n\) satisfies the stopping criterion, then we terminate the iteration and output; otherwise, go to step 2.

In this paper, we utilize the fairly naive solution approach which is to discretize in time using the forward Euler finite-difference method and apply second-order finite-difference estimates to approximate the differential operators found:

\[\text{Set } u^0 := f \text{ and for } n = 1, 2 \ldots \text{ solve: }\]

\[u^{n+1} = u^n + \Delta t(A(u^n) + \lambda(f - u^n)),\]

where

\[A(u) := \frac{(u_x^2 + e^2)u_{xx} + (u_y^2 + e^2)u_{yy} - 2u_xu_yu_{xy}}{(u_x^2 + u_y^2 + e^3)\sqrt{2}}.\]  

(37)

We implement each of the differential operators as follows. The first-order derivatives are discretized by the symmetric difference quotient, the second-order derivatives are discretized by the usual second-order difference quotients, while the mixed derivative term is discretized by successively applying the first-order symmetric difference quotients.

**Numerical experiments**

In this part, we will use some experiment results to show the superiority of the method we proposed.

\[u_{ij}^{k+1} = \frac{\lambda g_{ij} + \mu \left( d_{ij}^{k+1} - \sum_{k=0}^{i-1} (-1)^k C_k u_{i-k}^{k+1} + b_{ij}^{k-1} + d_{ij}^{k} - \sum_{k=1}^{i-1} (-1)^k C_k u_{j-k} - b_{ij}^{k} \right)}{\lambda + 2\mu}\]  

(33)
Figure 1. Line graph of experiment data.

Figure 2. Performance testing. (a) Original image of “Actress”; (b) restoration image for $\sigma = 20$; (c) restoration image for $\sigma = 15$; (d) restoration image for $\sigma = 10$.

Figure 3. Image restoration effect of different orders. (a) Original image; (b) noisy image; (c) $\alpha = 1.1$; (d) $\alpha = 1.3$; (e) $\alpha = 1.5$; (f) $\alpha = 1.7$; (g) $\alpha = 1.9$.

Figure 4. Corresponding broken line statistical graph.

Figure 5. The restoration effect of different images. (a) Original image of “Actress”; (b) original image of “Building”; (c) original image of “Pepper”; (d) original image of “Dog”; (e) noisy image of “Actress”; (f) noisy image of “Building”; (g) noisy image of “Pepper”; (h) noisy image of “Dog”; (i) restoration image of “Actress”; (j) restoration image of “Building”; (k) restoration image of “Pepper”; (l) restoration image of “Dog.”

Table 1. Peak signal-to-noise ratio of restoration image for different iteration numbers.

| $\alpha$ | 1.1  | 1.2  | 1.3  | 1.4  | 1.5  | 1.6  | 1.7  | 1.8  | 1.9  |
|----------|------|------|------|------|------|------|------|------|------|
| 20       | 27.5189 | 27.7137 | 27.7360 | 27.6786 | 27.5706 | 27.4138 | 27.2072 | 26.9564 | 26.6706 |
| 30       | 28.2015 | 28.3539 | 28.2867 | 28.1710 | 28.0312 | 27.8423 | 27.5809 | 27.2459 | 26.8597 |
| 42       | 28.6025 | 28.6000 | 28.3994 | 28.2387 | 28.1155 | 27.9619 | 27.7171 | 27.3612 | 26.9145 |
| 50       | 28.7258 | 28.5930 | 28.3127 | 28.1364 | 28.0395 | 27.9291 | 27.7221 | 27.3740 | 26.9104 |
PSNR, the ratio of the peak signal energy to the average noise energy, is always used to quantify the denoised image, it can be calculated by the following formula:

\[
PSNR = 10 \log_{10} \frac{\text{MaxValue}^2}{\text{MSE}} \text{(dB)}
\]

\[
= 10 \log_{10} \frac{255^2}{\text{MSE}} \text{(dB)},
\]

where \(\text{MSE}\) is the mean variance measure, \(M \times N\) is the size of image, \(u, u_0\) are the restored image and the original image, respectively. The larger the value of PSNR, the better the image restoration.

Iteration of our model is terminated when the following condition is satisfied:

\[
\frac{\|u^{n+1} - u^n\|^2}{\|u^{n+1}\|^2} \leq \epsilon,
\]

where \(\epsilon\) is the maximum permissible error, in our experiments we set \(\epsilon = 0.0001\).

The used images in these experiments are “Actress” of size 128×128, “Building” of size 128×128, “Pepper” of size 128×128 and “Dog” of size 128×128. In addition, the numerical calculations are all accomplished by using MATLAB R2017b.

**Experiment 1:** In this experiment, we try to find the influence of parameters \(\lambda\) and \(\mu\) on the denoising effect. We add white Gaussian noise with standard variance \(\sigma = 15\) to “Actress” image. Many experiments show that the model is more effective when \(\lambda\) and \(\mu\) satisfied 100 to 50 times. Here we show only partial data results, and as is shown in Figure 1, the PSNR is bigger when the \(\lambda = 0.1\) or so.

Lambda is abbreviated to lam for brevity in the diagrams involved below.

**Experiment 2:** In this experiment, we set \(\lambda = 0.1\) and \(\mu = 0.001\). The processing effect of the proposed model after 42 iterations of three different noise values \(\sigma = 20, \sigma = 15, \sigma = 10\) is compared in Figure 2. Figure 2 shows that our presented model performing well for different variance value, we can get the restoration images which are close to the original image.

**Experiment 3:** In order to find out the influence of different fractional orders on denoising, in this experiment we add white Gaussian noise with standard variance \(\sigma = 15\) to Table 2.

| Image   | Actress | Building | Pepper | Dog   |
|---------|---------|----------|--------|-------|
| PSNR    | 26.3492 | 26.9380  | 28.2756| 28.4858|

![Figure 6. Line graph of experiment data.](image)

| Table 2. Peak signal-to-noise ratio (PSNR) of restoration images. |
|-----------------------|----------------|----------------|----------------|
| Image   | Actress | Building | Pepper | Dog   |
| PSNR    | 26.3492 | 26.9380  | 28.2756| 28.4858|

![Figure 7. Image restoration effect of different orders. (a) Original image; (b) noisy image; (c) \(\alpha = 1.9\); (d) \(\alpha = 1.5\); (e) \(\alpha = 1.1\).](image)

![Table 3. Peak signal-to-noise ratio (PSNR) of restoration image for different orders.](image)
“Dog” and implement noise elimination for $\lambda = 0.1$ and $\mu = 0.001$. As is shown in Figure 3, Figure 3(c) and (d) are clearer than Figure 3(f) and (g), that is to say, with other variables unchanged, the effect of image restoration seems to get worse and worse as the fractional order increases, we clearly make sure that the better denoising result, the smaller the order. To make the analysis more careful, the results of different fractional orders ($\alpha = 1.1$, $\alpha = 1.2$, $\alpha = 1.3$, $\alpha = 1.4$, $\alpha = 1.5$, $\alpha = 1.6$, $\alpha = 1.7$, $\alpha = 1.8$, $\alpha = 1.9$) under different iterations (20, 30, 42, 50) are shown in Table 1 (PSNR of restoration image for different iteration numbers) and Figure 4. We find that the PSNR values of this method are almost stable when the image is restored after 30 iterations, these show that the proposed model can achieve a high degree of noise removal very quickly.

Experiment 4: In the previous set of experiments, we find that the algorithm has a good denoising effect when the order is taken as 1.1, and we want this method to process some other images to verify the universal applicability of

Figure 8. Edge detection analysis. (a) Original image of “Dog”; (b) Canny edge detection image of (8a); (c) Laplacian of Gaussian edge detection image of (8a); (d) Denoised image of “Dog”; (e) Canny edge detection image of (8d); (f) Laplacian of Gaussian edge detection image of (8d); (g) Original image of “Building”; (h) Canny edge detection image of (8g); (i) Laplacian of Gaussian edge detection image of (8g); (j) Denoised image of “Building”; (k) Canny edge detection image of (8j); (l) Laplacian of Gaussian edge detection image of (8l).

Figure 9. Gray value histogram comparison. (a) Gray value histogram of the original image of “Dog”; (b) Gray value histogram of the original image of “Building”; (c) Gray value histogram of the degraded image of “Dog”; (d) Gray value histogram of the degraded image of “Building”; (e) Gray value histogram of the restored image of “Dog”; (f) Gray value histogram of the restored image of “Building.”
this method, so we added other images (“Building” of size 128 × 128, “Pepper” of size 128 × 128) in this experiment. All images with noise variance \( \sigma = 15 \) are processed with \( \alpha = 1.1, \lambda = 0.07, \) and \( \mu = 0.0011. \) After 42 iterations the restoration effect of different images are exhibited in Figure 5 and the PSNR of every image is shown in Table 2. The results show that the model can achieve good recovery effect.

In addition, during the experiment, we have carried out image denoising experiment on picture “Building” in detail; we find that the image “Building” has better restoration effect when the parameters were changed. The conclusion can be achieved more intuitively from Figure 6. Obviously, it is better with \( \lambda = 0.11. \) After that, we set \( \lambda = 0.11 \) and \( \mu = 0.0011, \) the image “Building” was been recovered for different orders. In the initial experiment, we made the selection about the range of \( \alpha, \) we found that the selection of 1–2 were better than those from other data processing, so we chose \( \alpha \) as 1–2 in the text for the experiment. What has been shown in Figure 7 and Table 3 is that the results of image restoration seem to get better and better as the fractional order decreases, which is consistent with the deduction of Experiment 3.

**Experiment 5:** To further analyze the denoising effect of the proposed method, we have made the original image and restored image edge detection analysis, all the experiment results are shown in Figure 8. In order to make the experiment more sufficient, Canny edge detection and the Laplacian of Gaussian edge detection were used to make the edge detection of the original image and the restored image, respectively. By conserving these images we can see that edge detection of the denoised images is very close to edge detection of the original clear images. The conclusion we get is that the method we proposed can well protect the edge of the image when image denoising.

Furthermore, we have made the histogram of gray value of the original image, degraded image, and restoration image in Figure 9. It is clear that the picture gap between Figure 9(a) and (c) is larger than that between Figure 9(a) and (e), the picture gap between Figure 9(b) and (d) is larger than that between Figure 9(b) and (f). This demonstrates that the method we proposed can restore the image perfectly.

In the fifth experiment, we add different levels of mixed noise to the three images and apply the L1L2/TV model and the coordinate descent algorithm to process the noisy images. The PSNR values of the denoised images are shown in Table 4. The first line in the table shows the PSNR result of the noisy images with only Gaussian noise. The second line shows the PSNR result of the noisy images with only salt and pepper noise. The third line

#### Table 4. Peak signal-to-noise ratio results of the noisy images with different noise level.

| Gaussian noise level | Actress | Pepper | Dog  |
|----------------------|---------|--------|------|
| \( s = 15 \)        | 27.9230 | 30.8067| 32.7217|
|                     | 27.8851 | 30.5762| 32.9227|
|                     | 27.6941 | 29.8738| 31.5648|
| \( s = 20 \)        | 26.8459 | 28.3364| 29.2075|
|                     | 26.4295 | 27.8235| 29.2027|
|                     | 26.1125 | 27.3502| 28.1795|

#### Table 5. Peak signal-to-noise ratio (PSNR) results of the noisy images between the two methods.

| TOYS   | Times | PSNR  | Time (s) |
|--------|-------|-------|----------|
| Times  | 43    | 21.8810| 28.712811|
| PSNR   | 52    | 22.6923| 15.173902|

**Figure 10.** The restoration effect of different decent methods. (a) Original image of “Toys”; (b) image of Gaussian noises; (c) denoised image with the decent method; (d) denoised image with the coordinate decent method.
shows the PSNR result of the noisy images with the mixed noise of Gaussian noise and salt and pepper noise. From the table, we observe that the PSNR values of images with different low levels of salt and pepper noise is almost the same at Gaussian noise $s = 15$. Similarly, the same experimental result at Gaussian noise $s = 20$ is obtained. The experiments have also shown that the L1L2/TV model and algorithm can effectively remove the mixed noise of Gaussian noise and salt and pepper noise.

**Experiment 6:** PSNR comparisons between the coordinate descent method and the common gradient descent can show the advantages of the coordinate decent method. In the following experiment, we set $\lambda = 0.2$ and $\mu = 0.002$. The processing effect of the proposed model after 52 iterations of noise values $\sigma = 20$ is compared in Figure 10. Table 5 shows that our presented model performing better between the coordinate decent method and the common gradient descent, we can get the restoration images which are close to the original image.

**Conclusions**

In this paper, we propose an algorithm for fractional TV model, and employ the coordinate descent method to decompose the fractional-order minimization problem into scalar sub-problems, and then solve it by using split Bregman algorithm. The split Bregman algorithm and the coordinate descent method are combined. The proposed algorithm divides the problem into sub-problems to solve, which reduces the difficulty of the problem. The final experimental results show that the proposed algorithm is effective for image denoising, but there is still space for improvement in discretization and programming, we also believe that this method will show more and more superiority based on continuous improvement.

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