THE GEOMETRY OF THE POISSON BRACKET INvariant ON SURFACES

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Abstract. We study the Poisson bracket invariant $\mathcal{pb}$, which measures the level of Poisson noncommutativity of a smooth partition of unity, on closed symplectic surfaces. Building on preliminary work of Buhovsky and Tanny [BT], we prove that for any smooth partition of unity subordinated to an open cover consisting in discs of area at most $c$, if moreover the open cover satisfies some localization condition when the surface is a sphere, then the product of this invariant with $c$ is bounded from below by a universal constant. This result, which could be understood as a symplectic version of the mean value theorem, thereby answers a question of L. Polterovich [P3] for closed surfaces of genus $g \geq 1$. Polterovich’s question was also independently answered by Buhovsky-Logunov-Tanny for all closed surfaces via different methods [BLT]. However, whereas Buhovsky-Logunov-Tanny’s arguments are of a global nature, our arguments are more local; incidentally, the range of applicability of the two methods are not the same, and whenever both methods apply, ours appear to be sharper. This justifies a discussion of the sharpness of our own results. We also included a short survey on the work around Polterovich’s question at the end of the paper.

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1. Introduction

In his investigation of the function theory on symplectic manifolds, L. Polterovich introduced in [P2] the so-called Poisson bracket invariant as a quantitative measure of the level of Poisson
non-commutativity of functions forming a partition of unity on a symplectic manifold. In the same article, he also explained the relation of this symplectic invariant to operational quantum mechanics, more precisely describing how the invariant appears as a lower bound on the statistical noise of measurements of particular collections of quantum observables whose values in the (symplectic) phase space form an open cover of the latter. Consequently, it seems of some practical importance to establish lower bounds on the Poisson bracket invariant itself. The purpose in the present paper is to improve upon the known lower bounds on this invariant and to extend the range of applicability of these lower bounds when the symplectic manifolds are closed surfaces.

1.1. Preliminary notions. A symplectic manifold is a pair \((M, \omega)\) where \(M\) is a smooth manifold and \(\omega\) is a closed nondegenerate differential 2-form on \(M\), i.e. \(d\omega = 0\) and the bundle map \(\omega_y : TM \to T^*M : x \mapsto X_\omega \) is an isomorphism. This last property implies that \(M\) is necessarily even-dimensional, say \(\dim(M) = 2n\). In this paper, all symplectic manifolds are assumed to be closed, i.e. compact without boundary, with explicit mention otherwise.

The symplectic gradient of a smooth function \(H\) on \(M\) is the vector field \(X_H\) on \(M\) defined via the equation \(-dH = X_H \wedge \omega\), that is \(X_H := (\omega)^{-1}(-dH)\). The Poisson bracket associated to \((M, \omega)\) is the bilinear map

\[
\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : (G, H) \mapsto \{G, H\} := -\omega(X_G, X_H).
\]

A diffeomorphism \(\phi : M \to M\) is Hamiltonian if there exists a smooth time-dependent function \(H_t\) on \(M\), defined for \(t \in (-\epsilon, 1 + \epsilon)\) for some \(\epsilon > 0\), such that \(\phi = \phi^1_H\), where \(\phi^1_H\) solves the Cauchy problem \(\phi_0^H = I_d\) and \(\frac{d}{dt}\phi^1_H(x) = X_H \circ \phi^1_H(x)\). The set of Hamiltonian diffeomorphisms is denoted \(\text{Ham}(M, \omega)\); it turns out to be a group under composition (see for instance [Pi] [PR]). A symplectomorphism of \((M, \omega)\) is a diffeomorphism of \(\phi : M \to M\) such that \(\phi^*\omega = \omega\). Observe that symplectomorphisms preserve the volume form \(\wedge^n\omega/n!\). An important point is that every Hamiltonian diffeomorphisms are symplectomorphisms.

A subset \(X \subset (M, \omega)\) is (Hamiltonianly) displaceable if there exists \(\phi \in \text{Ham}(M, \omega)\) which disjoins \(X\) from itself its closure \(\overline{X}\), i.e. \(\phi(X) \cap \overline{X} = \emptyset\). Since Hamiltonian diffeomorphisms preserve volume, there is a volume constraint to displaceability: given a displaceable set \(X \subset (M, \omega)\) and \(\phi \in \text{Ham}(M, \omega)\) disjoining \(X\) from its closure,

\[
\text{Vol} M \geq \text{Vol} (\phi(X) \cup \overline{X}) \geq \text{Vol} \phi(X) + \text{Vol} \overline{X} - \text{Vol} (\phi(X) \cap \overline{X}) \geq 2 \text{Vol} X,
\]

where the volume is computed with respect to \(\wedge^n\omega/n!\). More quantitatively, we can associate to a set \(X \subset (M, \omega)\) its (Hofer’s) displacement energy, denoted \(e_H(X)\). Namely, first define the Hofer’s norm or energy of a Hamiltonian diffeomorphism \(\phi \in \text{Ham}(M, \omega)\) as

\[
\|\phi\|_H := \inf \left\{ \int_0^1 \left( \max_{t \in M} h_t(x) - \min_{x \in M} h_t(x) \right) dt : h \in C^\infty(M \times [0, 1]) \text{ such that } \phi = \phi^1_H \right\},
\]

then set

\[
e_H(X) := \inf \left\{ \|\phi\|_H : \phi \in \text{Ham}(M, \omega), \phi(X) \cap \overline{X} = \emptyset \right\},
\]

with the convention that \(e_H(X) = +\infty\) if \(X\) is nondisplaceable. The Hofer’s norm introduces a Finsler-like metric on the group \(\text{Ham}(M, \omega)\), leading to the so-called Hofer’s geometry, while the displacement energy introduces a sort of geometry on the underlying symplectic manifold.

Given a smooth function \(f : M \to \mathbb{R}\), its support is the closure of the set of points where \(f\) is nonzero, \(\text{supp}(f) := \text{Closure}\{x \in M \mid f(x) \neq 0\}\). We say that \(f\) is supported in an open set \(U\) if \(\text{supp}(f) \subseteq U\). A (smooth) positive collection is a finite collection \(F = \{f_i\}_{i=1}^N\) such that each \(f_i \geq 0\) and

\[
S_F(x) := \sum_{i=1}^N f_i(x) \geq 1 \quad \text{for all } x \in M.
\]
A (smooth) partition of unity is positive collection $\mathcal{F}$ such that $S_{\mathcal{F}} = 1$. A positive collection $\mathcal{F}$ is said to be subordinated to an open cover $\mathcal{U} = \{U_i\}_{i=1}^N$, what we denote $\mathcal{F} \prec \mathcal{U}$, if for each $i \in \{1, \ldots, N\}$, $f_i$ is supported in $U_i$. We denote by $\| \cdot \| : C^0(M; \mathbb{R}) \to [0, +\infty)$ the supremum norm on real-valued continuous functions on $M$.

1.2. Poisson bracket invariant. Following Polterovich [P2], define the Poisson bracket invariant of a positive collection $\mathcal{F}$ as

$$pb(\mathcal{F}) := \max_{a,b \in [-1,1]^N} \| \{a \cdot \mathcal{F}, b \cdot \mathcal{F}\}\| = \max_{a,b \in [-1,1]^N} \left\| \left\{ \sum_{i=1}^N a_i f_i, \sum_{j=1}^N b_j f_j \right\} \right\|,$$

and define the Poisson bracket invariant of an open cover $\mathcal{U}$ as

$$pb(\mathcal{U}) := \inf_{\mathcal{F} \prec \mathcal{U}} pb(\mathcal{F}).$$

These quantities are symplectic invariants to the extent that given a symplectomorphism $\phi$ of $(M, \omega)$, denoting $\phi_* \mathcal{F} = \{(\phi^{-1})^* f_i\}_{i=1}^N$ and $\phi_* \mathcal{U} = \{\phi(U_i)\}_{i=1}^N$, we have $pb(\phi_* \mathcal{F}) = pb(\mathcal{F})$ and $pb(\phi_* \mathcal{U}) = pb(\mathcal{U})$.

In this paper, we shall mainly consider a version of the invariant introduced by Buhovsky and Tanny [BT]. Define the Poisson bracket function of a positive collection $\mathcal{F}$ as

$$P_{\mathcal{F}} : M \to [0, \infty) : x \mapsto \sum_{i,j=1}^N |\{f_i, f_j\}(x)|.$$ 

It has been established that there exists a constant $0 < c(n) \leq 1$ (depending only on the dimension $\text{dim} M = 2n$) such that for any positive collection,

$$c(n) \|P_{\mathcal{F}}\| \leq pb(\mathcal{F}) \leq \|P_{\mathcal{F}}\|.$$ 

The upper bound is a straightforward application of the triangle inequality. The lower bound was first established in dimension 2 in [BT], and the full result is established in [BLT]. In order to find lower bounds on $pb(\mathcal{F})$, it is therefore sufficient (and necessary) to find lower bounds on $\|P_{\mathcal{F}}\|$, which would in turn follow from lower bounds on the $L^1$-norm of $P_{\mathcal{F}}$.

1.3. Polterovich’s Poisson bracket conjecture. Building upon previous work (e.g. [EP], [EPZ], [P2], Polterovich [P3] asked whether the following statement could be true:

**Conjecture 1.3.1** (Poisson bracket conjecture). There exists a constant $C > 0$ depending only on $(M, \omega)$ such that for any open cover $\mathcal{U}$ constituted of displaceable sets,

$$pb(\mathcal{U}) e_{H}(\mathcal{U}) \geq C.$$ 

A related and *a priori* simpler conjecture is:

**Conjecture 1.3.3** (Weak Poisson bracket conjecture). There exists a constant $C > 0$ depending only on $(M, \omega)$ such that for any open cover $\mathcal{U}$ constituted of displaceable sets,

$$pb(\mathcal{U}) \text{Vol}(M, \omega)^{1/n} \geq C.$$ 

The significance of this second conjecture is to claim that $pb$ is bounded away from 0 on covers made of displaceable sets (by a constant independent of the cover).

These two conjectures have been more or less explicitly discussed in [BT], [BLT], [EP], [EPZ], [LPa], [Pa], [P2], [P3], [P4], [PR], [SS], [SL]. Albeit we shall give a more detailed description of these work in section 5, we give here a brief motivation for these conjectures. Note that $pb(\mathcal{U})$ could vanish if $\mathcal{U}$ does not consist in displaceable sets: think of a partition of unity subordinated to a cover consisting in only two open sets. The displaceability assumption is therefore not superfluous.
On the other hand, Polterovich [P3] has established that whenever the open cover is formed by displaceable sets, then the above inequalities hold with a positive constant $C$ dependent on the cover $\mathcal{U}$.

We emphasize that both $pb$, $e_H$ and $\text{Vol}^{1/n}$ depend on the symplectic form $\omega$: namely, under the rescaling $\omega \mapsto \lambda \omega$ with $\lambda > 0$, for any given $\mathcal{U}$ we have $pb(\mathcal{U}) \mapsto \lambda^{-1}pb(\mathcal{U})$, $e_H(\mathcal{U}) \mapsto \lambda e_H(\mathcal{U})$ and $\text{Vol}(M,\omega)^{1/n} \mapsto \lambda \text{Vol}(M,\omega)^{1/n}$. Hence the quantities $pb(\mathcal{U})e_H(\mathcal{U})$ and $pb(\mathcal{U})\text{Vol}(M,\omega)^{1/n}$ depend only on the class $[\omega] \in \mathbb{P}(H^2_{dR}(M;\mathbb{R}))$, and so would the constant $C$ above. This invariance of the product $pb(\mathcal{U})e_H(\mathcal{U})$ under rescaling suggests its geometrical importance. In fact, the Poisson bracket conjecture could be interpreted as a symplectic version of the mean value theorem: for comparison, let $(M,g)$ be a Riemannian manifold, $U \subset M$ be an open subset with injectivity radius $\rho(U)$ and $f \in C^\infty(M;\mathbb{R})$ be supported in $U$. Let’s assume that $f(p) = 1$ at some point $p \in U$ so as to mimic the use of partitions of unity in the strong Poisson bracket conjecture. The mean value inequality implies

$$\|\nabla f\|_{L^\infty} \rho(U) \geq \|f\|_{L^\infty} \geq 1.$$  

Hence, the supremum norm of the first derivative of one or several smooth function(s) times the size of the set(s) which support the function(s) is bounded from below by a constant which is independent from both the function(s) and the set(s). This analogy served as an important guiding principle for the methods of this paper, as we shall look for geometric operations which preserve the above products.

The denomination "weak" is justified by the following consideration: the Poisson bracket conjecture can be formulated as

$$pb(\mathcal{U})\text{Vol}(M,\omega)^{1/n} \geq C \frac{\text{Vol}(M,\omega)^{1/n}}{e_H(\mathcal{U})},$$

which virtually implies the weak form of the conjecture whenever $e_H(\mathcal{U})$ is small enough. We admittedly do not know if on any given symplectic manifold the displacement energy of a displaceable open set is bounded from above, a statement whose validity would guarantee that the weak conjecture follows from the first conjecture. However this statement is true when $M$ is a surface, as in that case $e_H(U) \leq (1/2)\text{Area}(M,\omega)$ for any displaceable set $U \subset M$. Conversely, the weak Poisson bracket conjecture is not completely clueless about the stronger form of the conjecture, as the weak version can be written

$$pb(\mathcal{U})e_H(\mathcal{U}) \geq C \frac{e_H(\mathcal{U})}{\text{Vol}(M,\omega)^{1/n}},$$

hence implying, if it were true, the stronger version whenever $e_H(\mathcal{U})$ is large enough.

1.4. Statement of the main results. In the present work, we establish the Poisson bracket conjectures for every closed symplectic surfaces of genus $g \geq 1$, and also for $S^2$ under some conditions on the open covers, c.f. lemma 1.4.3 and theorem 1.4.4 below. The core of our approach is to study the way open covers by discs and the function $P_F$ behave when lifted along a symplectic covering map $\pi : (M',\omega') \to (M,\omega)$. This allows to reduce the problem to situations where an inequality obtained by Buhovsky–Tanny [BT] can be applied, c.f. theorem 1.4.8. In order to state the precise results, we need a few definitions.

**Definition 1.4.1.** Let $M$ be a smooth manifold. A locally finite open cover $\mathcal{U} = \{U_1, \ldots, U_N\}$ on $M$ is said to be in general position if the sets $U_i$ have smooth boundaries, if the boundaries intersect transversally i.e. $\partial U_i \cap \partial U_j$ for all $i \neq j$, and if the boundaries intersect at worst in double points i.e. $\partial U_i \cap \partial U_j \cap \partial U_k = \emptyset$ for every triple $(i,j,k)$ of different indices.
Definition 1.4.2. Let \((M, \omega)\) be a closed symplectic surface and \(\mathcal{U} = \{U_1, \ldots, U_N\}\) be an open cover. The capacity of \(\mathcal{U}\) is
\[
 c(\mathcal{U}) := \max_{1 \leq i \leq N} \text{Area}(U_i, \omega).
\]

Our method relies on the following fundamental fact:

**Lemma 1.4.3.** On closed symplectic surfaces, any positive collection \(\mathcal{F}\) subordinated to an open cover \(\mathcal{U}\) by displaceable sets is also subordinated to an open cover \(\mathcal{U}'\) by displaceable discs in general position whose displacement energy is arbitrarily close to that of \(\mathcal{U}\).

This shows that it suffices to restrict attention to open cover consisting in open discs in general position. This fact is a consequence of the apparently classical characterization of displaceable sets in dimension two, characterization which also implies that \(e_H(\mathcal{U}) = c(\mathcal{U})\) when \(\mathcal{U}\) consists in displaceable discs with smooth boundaries. For completeness, we prove the characterization in section 2.3 and deduce the above fact in section 2.4.

**Definition 1.4.4.** Let \((M, \omega)\) be a symplectic manifold and \(\mathcal{U} = \{U_1, \ldots, U_N\}\) be an open cover. Given \(x \in M\), denote
\[
 \mathcal{U}_x := \{ U_i \in \mathcal{U} : x \in U_i \}
\]
and let the *star of \(x\) (with respect to \(\mathcal{U}\)) to be the region
\[
 \text{St}(x) = \text{St}(x; \mathcal{U}) := \bigcup_{U_i \in \mathcal{U}_x} U_i \subseteq M.
\]

**Definition 1.4.5.** Let \(M\) be a smooth manifold, let \(\mathcal{U}\) be a finite open cover on \(M\) and \(x \in M\). The star \(\text{St}(x)\) is *confined (with respect to \(\mathcal{U}\)) if \(\partial \text{St}(x) \neq \emptyset\) and if some connected component of \(\partial \text{St}(x)\) is not contained in any single \(U \in \mathcal{U}\).

**Remark 1.4.6.** Morally, \(\text{St}(x)\) is confined if it does not spread in \(M\) so much as for its boundary components to be included in single sets of \(\mathcal{U}\); it is "confined" to stay "near" \(x\).

The next notion is borrowed from [BT, BLT].

**Definition 1.4.7.** Let \((M, \omega)\) be a symplectic manifold and \(\mathcal{U} = \{U_1, \ldots, U_N\}\) be a cover by open disc. A disc \(U \in \mathcal{U}\) is *essential (to \(\mathcal{U}\)) if \(\mathcal{U} \setminus \{U\}\) is not a cover of \(M\). Equivalently, there exists \(x \in U\) such that \(\mathcal{U}_x = \{U\}\), i.e. such that \(\text{St}(x) = U\).

The starting point of this paper is the following inequality established in [BT, BLT], which we state here in a slightly more general form.

**Theorem 1.4.8.** Let \((M, \omega)\) be a closed symplectic surface, \(\mathcal{U} = \{U_1, \ldots, U_N\}\) be open discs in general position and \(\mathcal{F} = \{f_1, \ldots, f_N\} \prec \mathcal{U}\) be a collection of smooth positive functions such that \(S_F \geq 1\). Assume that the set of confined essential discs \(\mathcal{J}_c(\mathcal{U}) \subseteq \mathcal{U}\) is nonempty. Then
\[
 \int_{U_j} \sum_{i=1}^N |\{f_i, f_j\}| \omega \geq 1 \quad \text{for all } U_j \in \mathcal{J}_c(\mathcal{U}).
\]
Corollary 1.4.9. Under the same assumption as in theorem 1.4.8, we have
\[
\max_{U_i \in \mathcal{U}} \int_{U_i} P_F \omega \geq \min_{U_i \in \mathcal{J}(\mathcal{U})} \int_{U_i} P_F \omega \geq 1 \quad \text{and} \quad \int_M P_F \omega \geq |\mathcal{J}(\mathcal{U})|.
\]

Proof. The first string of inequalities follows from \( P_F \geq \sum_{i=1}^N |\{f_i, f_j\}| \) and theorem 1.4.8. The second inequality follows similarly, observing that \(|\{f_i, f_j\}| = 0\) outside \(U_j\), so that
\[
\int_M P_F \omega \geq \int_M \sum_{i=1}^N \sum_{j : U_j \in \mathcal{J}(\mathcal{U})} |\{f_i, f_j\}| \omega \geq \sum_{j : U_j \in \mathcal{J}(\mathcal{U})} \int_U \sum_{i=1}^N |\{f_i, f_j\}| \omega \geq \sum_{j : U_j \in \mathcal{J}(\mathcal{U})} 1 = |\mathcal{J}(\mathcal{U})|.
\]

\[\square\]

Definition 1.4.10. Let \(M\) be a smooth manifold and let \(\mathcal{U}\) be a finite open cover on \(M\). We say that \(\mathcal{U}\) is **localized at points** \(x_1, \ldots, x_m\) if each \(U \in \mathcal{U}\) contains at most one of these points. For \(m \in \mathbb{N}\), we say that \(\mathcal{U}\) is **localized in \(m\) points** or **\(m\)-localized** if there are \(m\) points \(x_1, \ldots, x_m \in M\) at which \(\mathcal{U}\) is localized.

Remark 1.4.11. Observe that any open cover is \(1\)-localized. An open cover \(\mathcal{U}\) is localized at the points \(x_1, \ldots, x_m\) if and only if for all \(1 \leq i, j \leq m\), \(x_i \in St(x_j)\) implies \(x_i = x_j\).

As we mentioned, the central idea to our approach is to lift data along symplectic covering maps, and notably to understand how this procedure simplifies the topology of the stars \(St(x)\) to the point that they become confined. This leads to a proof of the "star" inequality that we state next, an inequality which somewhat locates the "Poisson non-commutativity" on the surface and which could also be understood as a generalization of the "essential" estimates.

Theorem 1.4.12 ("Star" inequality). Let \((M, \omega)\) be a closed symplectic surface equipped with an open cover \(\mathcal{U} = \{U_1, \ldots, U_N\}\) constituted of open discs in general position, \(\mathcal{F} \prec \mathcal{U}\) be a positive collection with \(S_F \geq 1\).

- If \(M\) has genus \(g \geq 1\), then for all \(x \in M\),
  \[
  \int_{St(x)} \sum_{i=1}^N \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}| \omega \geq 1 ;
  \]

- If \(M = S^2\) and \(x \in S^2\) is such that \(St(x)\) is confined, then the previous equality holds.
- If \(M = S^2\) and \(\mathcal{U}\) is \(m\)-localized with \(m \geq 3\), say at points \(x_1, x_2, x_3, \ldots, x_m\), then
  \[
  \int_{St(x)} \sum_{i=1}^N \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}| \omega \geq C(x)
  \]
  with \(C(x) = 1/4\) for any \(x \in S^2\) and with \(C(x) = 1\) if \(x \in \{x_1, \ldots, x_m\}\).

In particular, the weak Poisson bracket conjecture holds in those cases.

Remark 1.4.13. When \(M = S^2\), the assumptions of confinement on \(St(x)\) or of \(3\)-localization on \(\mathcal{U}\) are not superfluous. Indeed, consider \(S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}\) with the open cover \(\mathcal{U} = \{U_n, U_s\}\), where \(U_n = \{z > -1/2\}\) and \(U_s = \{z < 1/2\}\). It is \(2\)-localized and the possible star sets are \(\{U_n, U_s, S^2\}\) and are therefore never confined. However, for any partition of unity \(\mathcal{F} \prec \mathcal{U}\), one has \(P_F \equiv 0\).
Theorem 1.4.14 (Poisson bracket theorem). Let \((M, \omega)\) be a closed symplectic surface equipped with an open cover \(U = \{U_1, \ldots, U_N\}\) constituted of open discs in general position, and \(F \prec U\) be a positive collection with \(S_F \geq 1\). Suppose \(U\) is localized at the points \(x_1, \ldots, x_m\). If \(M = S^2\), further assume that \(m \geq 3\). Then for any probability measure \(\mu\) on \(M\),

\[
\int_M P_F \omega \geq \min_{1 \leq i \leq N} \frac{1}{\mu(U_i)}.
\]

In particular, the Poisson bracket conjecture holds in those cases, since

\[
\int_M P_F \omega \geq \max \left\{ \frac{\text{Area}(M, \omega)}{c(U)}, m \right\}.
\]

Theorem 1.4.14 is obtained by straightforward averaging of the star inequality over \(x \in M\) with respect to appropriate measures. In comparison, Buhovsky–Logunov–Tanny [BLT] obtained the lower bound

\[
(1.4.15) \quad \int_M P_F \omega \geq \frac{\text{Area}(M, \omega)}{2c(U)}
\]

under the sole condition that \(F\) is subordinated to an open cover \(U\) made of displaceable sets, thereby proving the full Poisson bracket conjecture on closed surfaces. For surface of genus \(g \geq 1\), by lifting the data along a double covering map as in section 3, it is easily seen that Buhovsky–Logunov–Tanny result holds verbatim for every open covers by discs, thereby recovering part of theorem 1.4.14 albeit with a smaller lower bound than what we achieved.

1.5. Structure of the paper. In section 2 we state some preliminary facts. We begin by considering in subsections 2.1 and 2.2 the behaviour of the quantities involved in the Poisson bracket conjectures under pullbacks along symplectic maps, as the essence of our methods is to lift the data \((U, F)\) on a symplectic surface \((M, \omega)\) to a symplectic covering space of sufficiently high degree for which, it turns out, we are able to apply theorem 1.4.12. We prove lemma 1.4.3 in subsection 2.4. This requires to appeal to the characterization of displaceable (closed) sets in two dimensions; this appears to be a well-known result to the experts, but as we were unable to find trace of it in the literature, we include a proof of this characterization in subsection 2.3.

The proofs of the main results are gathered in section 3. We first deduce theorem 1.4.14 from theorem 1.4.12 in 3.1 using a simple averaging technique. In the case of the sphere, this will yield theorem 1.4.14 with smaller lower bounds than what is claimed in this theorem; the full claim is proved only towards the end of subsection 3.3 after some more notations and facts have been established. Buhovsky–Tanny’s "confined disc inequality", theorem 1.4.8 is proved in subsection 3.2. We proceed in subsection 3.3 on proving theorem 1.4.12 for surfaces of genus \(g \geq 1\) by lifting the data \((U, F)\) along a symplectic covering map of sufficiently high degree. Subsection 3.4 is devoted to the proof of theorem 1.4.12 in the case of the sphere. The essence of the proof is the same as that of the previous results, but we now need to introduce, roughly speaking, "symplectic ramified covering maps" in order to lift the data on the sphere to data on the torus for which we will be able to apply theorem 1.4.14.

As we already mentioned, the general lower bounds we obtained are better than those obtained by Buhovsky–Logunov–Tanny. This motivates a discussion in section 4 of the sharpness of our own results. Subsection 4.1 shows that the consideration of the \(L^1\)-norm of the function

\[1\]

Our proof was motivated by the success of an "averaging technique" we used to prove directly a corollary of the Poisson bracket conjecture. We had started looking for an appropriate way to average the star inequality just a few hours before we became aware of the preprint [BLT]. Despite the great simplicity of the average process we found, we refrain from underestimating the extent to which Buhovsky–Logunov–Tanny’s specific lower bound helped us get on the right track. Accordingly, all the credit for proving the Poisson bracket conjecture should go to them.
Let \( P_F \) instead of its \( L^\infty \)-norm appearing in the statement of the Poisson bracket conjectures leads to no real lost in genrality if, as we do in this paper, we allow for open cover by non-displaceable discs. We prove in subsection 4.2 that the inequality in theorem 1.4.8 is sharp. However, we observe in subsection 4.3 that the lower bounds in theorem 1.4.14 are most probably not sharp.

Finally, section 5 is a short account of our reading of the history of the Poisson bracket invariant and of the Poisson bracket conjectures. It thereby includes some complementary informations on the Poisson bracket conjectures in dimension 2 established in the work [BLT, SL]. We also use this last section to briefly mention how the ideas of the present paper could be used to approach the Poisson bracket conjectures in higher dimensions.

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2. General considerations

2.1. Operations on the Poisson bracket function. Given a finite collection of smooth functions \( \mathcal{F} = \{ f_1, \ldots, f_N \} \) on a symplectic manifold \( (M, \omega) \), consider the functions

\[
S_\mathcal{F}(x) := \sum_{i=1}^N f_i(x) \quad \text{and} \quad P_\mathcal{F}(x) := \sum_{i=1}^N \{ f_i, f_j \}.
\]

We study here the behaviour of this function under some operations on the collection \( \mathcal{F} \).

Condensation. Given an integer \( M < N \) and a (surjective) map \( c : \{ 1, \ldots, N \} \to \{ 1, \ldots, M \} \), consider the new collection \( \mathcal{F}' = \{ f'_1, \ldots, f'_M \} \) obtained by setting

\[
f'_j := \sum_{i \in c^{-1}(j)} f_i, \quad \forall j \in \{ 1, \ldots, M \}.
\]

Clearly, \( S_{\mathcal{F}'} = S_\mathcal{F} \). Linearity of the Poisson bracket and the triangle inequality easily imply \( P_{\mathcal{F}'} \leq P_\mathcal{F} \).

Fragmentation. Suppose \( f_i \in \mathcal{F} \) has disconnected support. Write \( \text{supp}(f_i) = A \sqcup B \), where \( A \) and \( B \) are both reunions of connected components of \( \text{supp}(f_i) \), and set \( f_A := f_i|_A \) and \( f_B := f_i|_B \). Consider the new collection \( \mathcal{F}' = \{ f_1, \ldots, f_A, f_B, \ldots, f_N \} \). Since \( f_i = f_A + f_B \) and \( \{ f_A, f_B \} = 0 \), we easily get \( S_{\mathcal{F}'} = S_\mathcal{F} \) and \( P_{\mathcal{F}'} = P_\mathcal{F} \). We can evidently iterate this operation on \( A \) and \( B \) and apply it for each \( i \). Allowing ourselves to consider locally finite collections \( \mathcal{F}' = \{ f'_i \}_{i \in \mathbb{N}} \), i.e. for all \( x \in M \), only finitely many \( f'_i \in \mathcal{F}' \) do not vanish at \( x \), fragmentation leads to a locally finite collection \( \mathcal{F}' \) such that each \( f'_i \in \mathcal{F}' \) has connected support, \( S_{\mathcal{F}'} = S_\mathcal{F} \) and \( P_{\mathcal{F}'} = P_\mathcal{F} \).

Lift. Suppose \( p : (M', \omega') \to (M, \omega) \) is a symplectic covering map, i.e. a covering map such that \( p^* \omega = \omega' \). Consider the collection \( p^* \mathcal{F} := \mathcal{F}' = \{ f'_1, \ldots, f'_N \} \) on \( M' \) given by \( f'_i = p^* f_i \). Note that this collection is locally finite even when \( p \) has infinite degree. Clearly, \( S_{\mathcal{F}'} = p^* S_\mathcal{F} \). Since
\( p \) is a symplectic local diffeomorphism, we have \( p^* \{ g, h \} = \{ p^* g, p^* h \} \) for all \( g, h \in C^\infty(M) \); in particular, we deduce \( P_{\mathcal{F}'} = p^* P_{\mathcal{F}} \).

**Remark 2.1.1.** If \( \mathcal{F} \) is subordinated to an open cover \( \mathcal{U} \):

- a condensation \( \mathcal{F}' \) of \( \mathcal{F} \) is subordinated to a corresponding reunion \( \mathcal{U}' \) of sets in \( \mathcal{U} \), namely \( \text{supp}(f_j') \subset \bigcup_{i \in c^{-1}(j)} U_i \);
- a fragmentation \( \mathcal{F}' \) of \( \mathcal{F} \) is still subordinated to \( \mathcal{U} \), in the sense that for all \( f'_j \in \mathcal{F}' \), there exists \( U_i \in \mathcal{U} \) such that \( \text{supp}(f'_j) \subset U_i \);
- a lift \( p^* \mathcal{F} \) is subordinated to the open set \( p^* \mathcal{U} = \{ p^{-1} U_i \}_{i=1}^N \) of \( M' \).

**2.2. Symplectic lift and displaceability.** Suppose \( p : (M', \omega') \to (M, \omega) \) is a symplectic covering map. Note that given a Hamiltonian isotopy \( \phi_t \in \text{Ham}(M, \omega) \) generated by a time-dependent Hamiltonian function \( H_t \in C^\infty(M; \mathbb{R}) \), the lifted time-dependent function \( H'_t := p^* H_t \in C^\infty(M'; \mathbb{R}) \) generates a Hamiltonian isotopy \( \phi'_t \in \text{Ham}(M', \omega') \) which lifts \( \phi_t \), i.e. such that \( p \circ \phi'_t = \phi_t \circ p \). Since \( \max H'_t - \min H'_t = \max H_t - \min H_t \) at each time \( t \), we have \( \| \phi'_t \| H = \| \phi_t \| \). Moreover, if \( \phi_t \) displaces a set \( X \subset M \), then \( \phi'_t \) displaces \( X' := p^{-1}(X) \subset M' \). Consequently, we deduce \( e_H(X') \leq e_H(X) \); in particular, displaceable sets in \( M \) lift to displaceable sets under symplectic covering maps.

**2.3. Displaceability in dimension two.** An important fact which makes the Poisson bracket conjectures tractable in dimension two is the following characterization of displaceability for closed sets. Recall that in dimension two, a symplectic form if precisely an area form.

**Proposition 2.3.1.** Let \( (M, \omega) \) be a closed symplectic surface and \( X \subset M \) be a closed connected subset.

1. \( X \) is displaceable if and only if \( X \) is contained in a smoothly embedded closed disc \( D \subset M \) of \( \omega \)-area less than half that of \( M \).
2. If \( X \) is displaceable, its displacement energy is
   \[
   e_H(X) = \inf \left\{ \int_D \omega : X \subset D \subset M \text{ is a smoothly embedded closed disc} \right\}.
   \]

This appears to be a result well known to the experts; for instance, it is simply stated as remarks in \([BT], [BLT]\), although it is of central importance in their arguments too. As we were not able to locate this characterization in the literature, we sketch a proof of it in this subsection. We first consider a particular case.

**Lemma 2.3.2.** A smoothly embedded closed disc \( D \subset (M, \omega) \) is displaceable if and only if its \( \omega \)-area is less than half the \( \omega \)-area of \( M \), in which case its displacement energy equals its \( \omega \)-area.

**Proof.** For simplicity, set \( c = \int_D \omega \) and \( A = \int_M \omega \).

Assume \( D \) is displaceable. Since \( D \) is a closed subset of the Hausdorff compact space \( M \), a small open neighbourhood of \( D \) is also displaceable. By the area constraint, the area of this neighbourhood is at most \( A/2 \), hence \( c < A/2 \). That \( e_H(D) \geq c \) follows from Usher’s general and sharp energy-capacity inequality \([U]\).

Assume \( c < A/2 \); we shall show that \( D \) is displaceable and that \( e_H(D) \leq c \). Somewhat abusing notations, the smooth embedding \( D \hookrightarrow M \) can be extended to a smooth embedding \( D' \hookrightarrow M \) of a closed disc of area \( c' \in (2c, A) \). As a consequence of Moser’s trick, there are symplectic diffeomorphisms which are arbitrary close to mapping the pair \((D', D)\) to a pair of rectangles \((R', R)\) in \( \mathbb{R}^2 \) of corresponding areas, where \( R \) lies inside a half of \( R' \). It is well known
(see for instance the example in section 2.4 of [P1]) that (small neighbourhoods of) $R$ can be displaced within $R'$ via a Hamiltonian isotopy of energy $c + \epsilon$ for any $\epsilon > 0$. By invariance of the energy under symplectomorphisms, we get $e_H(D) < c + \epsilon$ for every $\epsilon > 0$.

\[\square\]

**Remark 2.3.3.** We take the opportunity to reflect upon the proof of $e_H(D) \geq c$. Let $\phi_t : M \times [0,1] \to M$ be a Hamiltonian isotopy displacing $D$ and having Hofer energy $e$. If $M = \Sigma_g$ with $g \geq 1$, the natural lift of $\phi_t$ to the universal cover $\mathbb{R}^2$ of $\Sigma_g$ is a Hamiltonian isotopy $\phi'_t$ with the same Hofer energy $e$ which displaces any given lift of $D$. It follows that $e$ is at least the Hofer displacement energy of a disc of area $c$ inside $\mathbb{R}^2$, denote it $e_H(c; \mathbb{R}^2)$. However, Hofer’s energy-capacity inequality [H] states that $e_H(c; \mathbb{R}^2) \geq c$; this is also established in [LM] via Gromov’s nonsqueezing theorem. As this is true for any possible $e$, we have $e_H(D) \geq c$.

If $M = S^2$, things are more complicated. If the whole isotopy $\phi_t$ would fix a particular point $p \in S^2$, the displacement of $D$ would effectively occur in an open disc $S \subset \mathbb{R}^2$ of area $A$ and essentially the same argument as before would apply. Otherwise, one could cook up a new Hamiltonian isotopy displacing $D$ and fixing a specific point at every times, but it is rather unclear to what extent the Hofer energy could increase in this way. Note that all of the above proofs of the energy-capacity energy $e_H(D) \geq c$ use hard symplectic topology results, even though we are working only on surfaces; this can be traced to the fact that we need also consider the time-dependent Hamiltonian isotopies, making the problem implicitly higher dimensional.

In order to prove proposition 2.3.1, we shall need the following lemma. We give two demonstrations: the first uses Lagrangian Floer theory, while the second uses more elementary and soft results. We thank Dominique Rathel-Fournier for discussions regarding both approaches.

**Lemma 2.3.4.** Let $(M, \omega)$ be a closed symplectic surface and $C \subset M$ a displaceable smoothly embedded circle. Then $C$ is contractible.

The case $M = S^2$ is trivial, so we shall assume $M \neq S^2$ below.

**Proof 1.** This proof uses Lagrangian Floer homology [F]. Since $M \neq S^2$, $\pi_2(M) = 0$. We argue by contradiction: assume $C$ is noncontractible. Since this loop bounds no disc, $\pi_2(M, C) = 0$. Theorem 1 in [F] states that for all $\phi \in \text{Ham}(M, \omega)$, $|C \cap \phi(C)| \geq c_{Z_2}(C) = 2$ where $c_{Z_2}(X)$ denotes the $Z_2$-cuplength of a topological space $X$, defined as the maximal integer $k$ such that there exist $k - 1$ nonzero degree cohomology classes $\alpha_j \in H^*(X; \mathbb{Z}_2)$ satisfying $\alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0$. Hence $C$ is not displaceable.

\[\square\]

**Proof 2.** Assume on the contrary that $C$ is noncontractible. Consider $\{\phi_t\}_{t \in [0,1]} \in \widehat{\text{Ham}}(M, \omega)$ displacing $C$ and set $C' = \phi_1(C)$. It is a classical fact due to Banyaga [Ba] that Hamiltonian isotopies lie in the kernel of the flux morphism (see also [P1])

$$\text{Flux} : \widehat{\text{Symp}}_0(M, \omega) \to H^1_{dR}(M; \mathbb{R}) : \{\psi_t\}_{t \in [0,1]} \mapsto \int_0^1 [X_t, \omega] dt,$$

from which we deduce that the isotopy $\{\phi_t\}_{t \in [0,1]}$ generates a (usually degenerate) cylinder $R : S^1 \times [0,1] \to M$ with boundary $-C + C'$ and area $\omega(R) = \int_{S^1 \times [0,1]} R^* \omega = 0$. Moreover, $C$ and $C'$ being embedded, isotopic, disjoint and noncontractible, it follows from Lemma 2.4 in [EP] that there exists an embedded cylinder $R' : S^1 \times [0,1] \hookrightarrow M$ such that $\partial R' = -C + C'$. Being embedded, its area $\omega(R') := \int_{S^1 \times [0,1]} (R')^* \omega$ satisfies $0 < \omega(R') < \omega(M)$. Since $R$ and $R'$ have the same boundary, these two 2-chains differ by a 2-cycle in $M$, hence the cohomology class $c = [R'] - [R] \in H_2(M; \mathbb{Z})$ has area $\omega(R') = \omega(c) \in \omega(M)\mathbb{Z}$, which is a contradiction.
Proof of proposition 2.3.4. Clearly, X is displaceable whenever X is contained in a displaceable set; in view of lemma 2.3.2 this is the case when X is contained in a smoothly embedded closed disc D with \( \int_D \omega < (1/2) \int_M \omega \). In this case, the lemma also implies that

\[
e_H(X) \leq \inf \left\{ \int_D \omega : X \subseteq D \subset M \text{ is a smoothly embedded closed disc} \right\}.
\]

Now suppose that X is displaceable. For every \( \epsilon > 0 \), there exists a Hamiltonian isotopy \( \phi_t \) with Hofer energy less than \( e_H(X) + \epsilon \) disjoining X from itself. Again, since X is a closed subset of the compact Hausdorff space M, the Hamiltonian isotopy displaces (the closure of) a small open neighbourhood U of X. Let \( \rho : M \to [0, 1] \) be a smooth function such that \( \rho^{-1}(0) = X \) and \( \rho^{-1}(1) = M \setminus U \). By Sard’s theorem, for almost every \( s \in (0, 1) \), namely the regular values of \( f \), the closed set \( \rho^{-1}([0, s]) \) has smooth boundaries; let \( X_s \) denote the connected component containing X. Since M is compact, \( \partial X_s \subseteq \rho^{-1}(s) \) and s is a regular, \( \partial X_s \) consists in a finite number of smoothly embedded circles. From lemma 2.3.4 each of those boundary circles are contractible. By the Jordan-Schoenflies theorem, we deduce that each of those circles separates M into two connected components, at least one of which is a disc; moreover, since \( X_s \) is connected, it is contained in one of these two components. If two discs bound a given boundary circle, one of the two has area less than \( (1/2) \int_M \omega \), since \( \phi_t \) disjoins the circle and preserves area. For each boundary circle of \( X_s \), consider the smallest disc bounding that circle. We claim that \( X_s \) belongs to one of these discs to finish the proof of the lemma when X is connected; suppose on the contrary that \( X_s \) is contained in none of these smallest discs. This implies that \( X_s \) is the complement of the reunion of these discs. Since \( X_s \) is connected, \( \phi_t \) sends \( X_s \) inside one of these smallest discs. Hence the diffeomorphism \( \phi_t \) sends either this disc or its (connected!) complement into this disc; both options contradict the area-preservation property of \( \phi_t \). Hence \( X_s \) indeed belongs to one of these smallest discs; this disc is disjoined by \( \phi_t \), for if it was not, then neither would be its boundary, which is however disjoined by \( \phi_t \) by construction. Incidentally, by the energy-capacity inequality, the energy of \( \phi_t \) is greater than the \( \omega \)-area of this disc. We have thus proved that

\[
\forall \epsilon > 0, \quad e_H(X) + \epsilon \geq \inf \left\{ \int_D \omega : X \subseteq D \subset M \text{ is a smoothly embedded closed disc} \right\}.
\]

\[\square\]

2.4. Reduction to the case of covers by discs. Let \( \mathcal{U} = \{U_1, \ldots, U_N\} \) be a finite open cover of a surface \((M, \omega)\) constituted of displaceable sets \( U_i \) and \( \mathcal{F} = \{f_1, \ldots, f_N\} \prec \mathcal{U} \) be a collection of functions subordinated to \( \mathcal{U} \), i.e. \( \text{supp}(f_i) \subset U_i \) for each i. Consequently, the (closed) support of \( f_i \) is displaceable, with displacement energy at most \( e_H(\mathcal{U}) \).

Arguing as in the proof of proposition 2.3.1 using a smooth function \( \rho_i : M \to [0, 1] \) such that \( \rho_i^{-1}(0) = \text{supp}(f_i) \) and \( \rho_i^{-1}(1) = M \setminus U_i \), we deduce that for every regular value \( s \) of \( \rho_i \), the closed set \( X_i(s) := \rho_i^{-1}([0, s]) \) has boundary \( \partial X_i(s) \) included in the compact regular level-set \( \rho_i^{-1}(s) \). Consequently, \( \partial X_i(s) \) and \( X_i(s) \) have finitely many connected components. The connected components of \( X_i(s) \) determine a (finite) fragmentation of the function \( f_i \); as a result, without lost of generality, we can assume from now on that each \( X_i(s) \) is connected. The proof of proposition 2.3.1 then implies that there exists a smoothly embedded closed disc \( D_i \supseteq X_i(s) \) such that \( e_H(D_i) \leq e_H(U_i) \leq e_H(\mathcal{U}) \). In view of lemma 2.3.2 for each \( \epsilon > 0 \), it is possible to slightly enlarge each \( D_i \) into an open disc \( U_i' \) (with smooth boundary) in such a way that \( e_H(U_i') < e_H(\mathcal{U}) + \epsilon \). The collection \( \mathcal{F} \) is therefore subordinated to an open cover \( \mathcal{U}' = \{U_1', \ldots, U_N'\} \) consisting in open discs (with smooth boundaries) such that \( e_H(\mathcal{U}') < e_H(\mathcal{U}) + \epsilon \).
3. Proofs of the main inequalities

3.1. Poisson bracket theorem. We present here the averaging method which allows to deduce theorem 1.4.14 from the "star" inequality (theorem 1.4.12), with the following caveat: when \( g = 0 \), the lower bound we shall obtain will not be as good as what we claimed; we postpone the proof of this lower bound to after the proof of theorem 1.4.12.

Proof of theorem 1.4.14 assuming theorem 1.4.12. Under the assumptions of theorem 1.4.14 there exists a (measurable) function \( C : M \to [1/4, \infty) \) such that for all \( x \in M \)

\[
\int_{\text{St}(x)} \sum_{i=1}^{N} \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}(y)| \omega_y \, d\mu_x
\]

Let \( \mu \) be a finite measure on the closed surface \( M \). We shall write \( d\mu_x \) to denote the density of \( \mu \) at \( x \in M \), and similarly write \( \omega_y \) to denote the value of the 2-form \( \omega \) at \( y \in M \). For a measurable function \( f : M \to \mathbb{R} \), let \( \mu(f) := \int_M f(x) \, d\mu_x \). Given a measurable set \( S \subset M \), we denote by \( \chi_S : M \to \{0, 1\} \) its characteristic function; set \( \mu(U) := \max_{U \in \mathcal{U}} \mu(\chi_U) \). Recall that if \( y \not\in \text{St}(x) \), then for all \( 1 \leq i \leq N \) and for all \( j \) such that \( U_j \in \mathcal{U}_x \), we have \( \{f_i, f_j\}(y) = 0 \).

We compute

\[
\mu(C) = \int_{x \in M} C(x) \, d\mu_x \leq \int_{x \in M} \int_{y \in \text{St}(x)} \sum_{i=1}^{N} \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}(y)| \omega_y \, d\mu_x
\]

\[
= \int_{x \in M} \int_{y \in M} \sum_{i=1}^{N} \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}(y)| \omega_y \, d\mu_x
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{x \in M} \chi_{U_j}(x) \left( \int_{y \in M} |\{f_i, f_j\}(y)| \omega_y \right) \, d\mu_x
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \mu(U_j) \int_{y \in M} |\{f_i, f_j\}(y)| \omega_y
\]

\[
\leq \mu(\mathcal{U}) \int_{y \in M} \sum_{i=1}^{N} \sum_{j=1}^{N} |\{f_i, f_j\}(y)| \omega_y
\]

which can be rewritten as

\[
\frac{\mu(C)}{\mu(\mathcal{U})} \leq \int_M P_F \omega.
\]

Let \( \mu \) be equal to the measure determined by the symplectic form \( \omega \). Since we can take \( C \equiv 1 \) when \( g \geq 1 \), we obtain in that case \( \mu(C) = \text{Area}(M, \omega) \) and \( \mu(\mathcal{U}) = c(\mathcal{U}) \), and thus

\[
\frac{\text{Area}(M, \omega)}{c(\mathcal{U})} \leq \int_M P_F \omega.
\]

When \( g = 0 \), we can take \( C \equiv 1/4 \), so that \( \mu(C) = (1/4)\text{Area}(M, \omega) \) and \( \mu(\mathcal{U}) = c(\mathcal{U}) \), hence

\[
\frac{1}{4} \frac{\text{Area}(M, \omega)}{c(\mathcal{U})} \leq \int_M P_F \omega.
\]

When \( \mathcal{U} \) is localized at the (distinct) points \( x_1, \ldots, x_m \), let \( \mu = \sum_{i=1}^{m} \delta_{x_i} \) where \( \delta_{x_i} \) denotes the Dirac measure supported at \( x_i \). We obtain \( \mu(C) = m \) since \( C(x_i) = 1 \) for all \( i \), and \( \mu(\mathcal{U}) = 1 \).
since each $U_j \in \mathcal{U}$ contains at most one $x_i$. Hence

$$m \leq \int_M P_\mathcal{F} \omega.$$ 

3.2. "Confined essential disc" inequality. We now prove theorem 1.4.8. Although the statement is more general and the result stronger than the corresponding results from [BT, BLT], the proof is essentially the same.

Remark 3.2.1. As the proof will make evident, the conclusion of theorem 1.4.8 remains valid for locally finite open covers by discs on any symplectic surfaces.

Proof of theorem 1.4.8. Without lost of generality, suppose $U_1 \in \mathcal{F}_c(\mathcal{U})$. The idea of the proof consists in using the flow generated by the Hamiltonian vector field associated to $f_1$ in order to dynamically parametrize (a portion of) $U_1$ via "energy-time" coordinates, to use Fubini theorem to express the above double integrals as iterated integrals in "time" and "energy", and finally to use the identity $|\{f_i, f_1\}| = |df_i(X_{f_1})|$ to understand the "time integrals" as measures of the total oscillation of $f_i$ along each integral curves of $X_{f_1}$.

Let's fix a Riemannian metric on $M$, hence allowing to take the gradient vector field of the smooth function $f_1$. Since $U_1$ is essential, there exists $x \in U_1$ such that no other $U_i \in \mathcal{U}$ contains $x$; incidentally, $f_1(x) = S_\mathcal{F}(x) \geq 1$. Also, $f_1(y) = 0$ for $y \in M \setminus U_1$, hence $(0, 1) \subset f_1(M)$ by the intermediate value theorem. By Sard’s theorem, the set of regular values $I := (0, 1) \setminus \text{crit}(f_1)$ has full measure 1, where $\text{crit}(f_1)$ denotes the set of critical values of $f_1$. Moreover, since $f_1$ is continuous and $M$ is compact, $I$ is open; it is therefore a reunion of disjoint open intervals $I = \bigcup_{\alpha \in A} I_\alpha$. For each $\alpha \in A$, choose $s_\alpha \in I_\alpha$; we wish to pick $p_\alpha \in f_1^{-1}(s_\alpha)$ appropriately, which necessitates a detour.

Let $s$ be any regular value of $f_1$ in $(0, 1)$. The regular level-set $f_1^{-1}(s)$ therefore consists in finitely many disjoint smoothly embedded circles $\{C_k\}_{k=1}^K$ included in the open disc $U_1$. By the Jordan-Schoenflies theorem, $U_1 \setminus C_k$ consists in an open disc $D_k$ and in an open annulus $A_k$ such that $\partial A_k = C_k \cup \partial U_1$. We claim that as least one $D_k$ contains $x$. Otherwise, $x$ would be contained in the intersection $W := \cap_{k=1}^K A_k$. This intersection is connected: indeed, observe that $C_l \subset D_k$ implies $D_l \subset D_k$. Inclusions of the discs $D_k$ into one another give a partial order on the discs; so there are finitely many maximal discs, each disjoint from one another, and the complement of their reunion is the intersection $W$. Contracting each maximal disc to a point within itself, we conclude that $W$ is homotopy equivalent to $U_1$ with a finite number of punctures, which is indeed connected. Consequently, there exists a continuous path $\gamma$ in $W$ from $x$ to $\partial U_1$; by the intermediate value theorem applied to $f_1 \circ \gamma$, we deduce that there exists $y \in W$ such that $f_1(y) = s$, in contradiction with the construction of $W$. Now pick any $D_k \ni x$; we claim that $C_k = \partial D_k$ intersects the complement of any disc $U_i$ in the open cover other than $U_1$. Otherwise, we would have $C_k \subset U_i$, hence $C_k$ would be contractible in $U_i$. By definition of $x$, we have $x \notin U_i$, which would imply that $C_k$ is contractible in $M \setminus \{x\}$, forcing $M$ to be $S^2$. In this case, the contraction of $C_k$ in $U_i$ and the disc $D_k$ could together be used to define a degree one (hence surjective) map from $S^2$ to $M$, so that $M \subset D_k \cup U_i \subset U_1 \cup U_2$, contradicting the assumption that $U_1$ is confined.

For each $\alpha \in A$, pick $p_\alpha$ in a circle of $f_1^{-1}(s_\alpha)$ winding around $x$. Following the gradient flow line of $f_1$ through $p_\alpha$, we get an embedding $\gamma_\alpha : I_\alpha \to U_1$ such that $\gamma_\alpha(s_\alpha) = p_\alpha$ and $f_1(\gamma_\alpha(s)) = s$. We define $\gamma : I \to U_1$ to be the embedding defined by $\gamma|_{I_\alpha} = \gamma_\alpha$. For each $s \in I$, let $C(s)$ denote the circle in $f_1^{-1}(s)$ containing $\gamma(s)$. Note that $C(s)$ winds around $x$ and, according to the previous paragraph, intersects the complement of each disc $U_i$, $i \neq 1$. Therefore, for each $i \neq 1$, $f_i$ vanishes somewhere on $C(s)$. Since $s \in I$ is a regular value of $f_1$,
each \( C(s) \) is an integral curve of the Hamiltonian vector field \( X_{f_1} \); denote by \( T : I \to (0, \infty) \) the (smooth) function giving the period of the integral curve of \( X_{f_1} \) along \( C(s) \).

Let’s consider the subset of the "energy-time space"

\[
R := \{(s, t) \in \mathbb{R}^2 : s \in I, t \in (0, T(s)) \}.
\]

The map \( \Phi : R \to f_1^{-1}(I) : (s, t) \mapsto \phi_s f_1(\gamma(s)) \) is a diffeomorphism onto its image. We observe that \( (T\Phi)(\partial_t) = X_{f_1} \), that \( \Phi^* f_1 = s \) and that \( -ds = (\partial_s) \cup (ds \wedge dt) \), whence \( \Phi^* \omega = ds \wedge dt \).

The rest of the proof is a computation:

\[
\int_{U_1} \sum_{i=1}^N |\{f_i, f_1\}| \omega = \sum_{i=2}^N \int_{U_1} |df_i(X_{f_1})| \omega \geq \sum_{i=2}^N \int_{\Phi(R)} |df_i(X_{f_1})| \omega = \sum_{i=2}^N \int_{s \in I} \int_{t=0}^{T(s)} \left| \frac{d(\Phi^* f_i)}{dt} \right| dt ds \\
\geq \sum_{i=2}^N \int_{s \in I} 2 \left( \max_{t \in (0, T(s))} \Phi^* f_i - \min_{t \in (0, T(s))} \Phi^* f_i \right) ds \\
\geq 2 \int_{s \in I} \max_{i=2}^N f_i(\gamma(s)) ds = 2 \int_{s \in I} (S_{\gamma}(\gamma(s)) - f_1(\gamma(s))) ds \\
\geq 2 \int_{s \in I} (1 - s) ds = 1.
\]

**Remark 3.2.2.** Note that the factor 2 appearing in the course of the calculation is due to the fact that each map \( t \mapsto |d(\Phi^* f_i)/dt| \) goes back in forth between its extremal values at least twice. In section 3.4 we shall see situations where this factor will be higher.

### 3.3. "Star" inequality in genus \( g \geq 1 \)

We now prove theorem \ref{thm:1.4.12} when \( g \geq 1 \). We shall be using the operations introduced in section 2.1; Figure 3.3.1 helps understanding the general effect of these operations.

![Figure 3.3.1](image.png)

**Figure 3.3.1.** Example on the torus of the lift of an open cover along a covering map of degree 4. Note that \( St(x') \) is a confined disc, but not \( St(x) \).

**Proof of theorem 1.4.12** Let \( p : (M', \omega') \to (M, \omega) \) be a symplectic covering map between closed symplectic surfaces, say of finite degree \( d \), \( U = \{U_1, \ldots, U_N\} \) be an open cover by discs
in general position and $\mathcal{F} \preceq \mathcal{U}$ be a positive collection. We shall more specifically choose $p$ towards the end of the proof.

**Step 1 - Lift and fragmentation.** Write for simplicity $N \times d = \{1, \ldots, N\} \times \{1, \ldots, d\}$. Since discs are contractible and covering maps have the unique homotopy lifting property, observe that the lift $p^{-1}(U_i)$ of each $U_i \in \mathcal{U}$ along $p$ consists in the disjoint union of $d$ discs $U'_i$, each symplectomorphic via $p$ to $U_i$, what we write $p^{-1}(U_i) = \bigcup_{k=1}^{d} U'_i$. In other words, the open cover $\mathcal{U}' = \{U'_i\}_{(k,l) \in N \times d}$ is the fragmentation (into connected discs) of the lift of $\mathcal{U}$ to $M'$. Similarly for each $f_i \in \mathcal{F}$, the function $p^* f_i$ fragment into $d$ functions $f'_{i,k}$ respectively supported in $U'_i$; in other words, the positive collection $\mathcal{F}' = \{f'_{i,k}\}_{(k,l) \in N \times d}$ is the fragmentation of the lift of $\mathcal{F}$ to $M'$, and $\mathcal{F}' \preceq \mathcal{U}'$.

**Step 2 - Equality of corresponding "star" integrals.** Let $x \in M$ and pick $x' \in p^{-1}(x)$. Denote simply $St(x) = St(x; \mathcal{U})$ and $St(x') = St(x'; \mathcal{U}')$. We claim that

$$\int_{St(x)} \sum_{i=1}^{N} \sum_{j : U_j \in \mathcal{U}_x} |\{f_i, f_j\}| \omega = \int_{St(x')} \sum_{i, (i,k) : U'_i \in \mathcal{U}'_x} \sum_{j, (j,l) : U'_j \in \mathcal{U}'_x} |\{f'_{i,k}, f'_{j,l}\}| \omega'. $$

Indeed, $U_j \in \mathcal{U}_x$ if and only if there exists (a necessarily unique) $1 \leq l \leq d$ such that $U_{j,l} \in \mathcal{U}_x'$; we denote $l(j)$ the unique such value (if it exists). The restriction $p|_{St(x')} : U'_{j,l(j)} \to U_j$ is then a symplectic diffeomorphism. We also note that $\{f'_{i,k}, f'_{j,l}\}$ and $\{f_{i,h}, f_{j,g}\}$ have disjoint support, since $f'_{i,k}$ and $f'_{i,h}$ have disjoint support. As a result, we get that

$$\sum_{(i,k), (j,l) : U'_{i,k} \in \mathcal{U}'_x} \int_{St(x')} |\{f'_{i,k}, f'_{j,l}\}| \omega' = \sum_{(i,k), (j,l) : U'_i \in \mathcal{U}'_x} \int_{U'_{j,l(j)}} |\{f'_{i,k}, f'_{j,l(j)}\}| \omega'

= \sum_{i} \sum_{j : U_j \in \mathcal{U}_x} \int_{U_j} |\bigcup_{k} \{f'_{i,k}, f'_{j,l(j)}\}| \omega'

= \sum_{i} \sum_{j : U_j \in \mathcal{U}_x} \int_{U_j} |\{f_i, f_j\}| \omega'

= \sum_{i} \sum_{j : U_j \in \mathcal{U}_x} \int_{St(x)} |\{f_i, f_j\}| \omega.$$
condensation, we have
\[
\int_{D''} \sum_{f_i' \in F'} \sum_{f_j' \in U_i \cap U_j} |\{f_i', f_j\}|\omega' \geq \int_{D''} \sum_{f_i' \in F'} \left\{ \sum_{f_j' \in U_i \cap U_j} f_j' \right\} \omega',
\]
while the right-hand side is at least 1 by theorem 1.4.8 and remark 3.2.1. Since the integrand of the left-hand side integral is supported in \(St(x') \subset D''\), we conclude from step 2.

\[\square\]

3.4. "Star" inequality in genus \(g = 0\). We finally prove theorem 1.4.12 when \(M = S^2\).

**Proof of theorem 1.4.12 when \(St(x)\) is confined.** Since \(St(x)\) is confined, \(\partial St(x)\) is nonempty and consists in disjoint piecewise smooth embedded circles, at least one of which is contained in no single \(U_i \in U\). Let \(C\) be such a circle. By the Jordan-Schoenflies theorem for \(S^2\), \(S^2 \setminus C\) consists in two discs. Since \(St(x)\) is connected, precisely one of these two discs contains \(St(x)\); denote it \(D\).

Let \(D'\) be an open disc with smooth boundary which contains \(D\) and which is contained in an arbitrarily small neighbourhood of \(D\). We form the new cover \(U' = (U \setminus U_z) \cup \{D'\}\) and the new positive collection \(F' = (F \setminus \{f_i : U_i \subset U_z\}) \cup \{\sum_{f_j \in U_i \cap U_j} f_j\}\). Note that \(F' \approx U'\) and that \(D'\) is a confined essential set for \(U'\). By property of condensation, we have
\[
\int_{D'} \sum_{f_i' \in F'} \sum_{f_j' \in U_i \cap U_j} |\{f_i', f_j\}|\omega \geq \int_{D'} \sum_{f_i' \in F'} \left\{ \sum_{f_j' \in U_i \cap U_j} f_j' \right\} \omega',
\]
and the right-hand side is at least 1 by theorem 1.4.8. Since the integrand of the left-hand side integral is supported in \(St(x') \subset D'\), we get the result.

\[\square\]

It remains to consider the case when \(U\) is 3-localized. The idea is again to used a – this time, ramified – covering map \(p : T^2 \rightarrow S^2\) in order to lift the data on \(S^2\) to appropriate data on \(T^2\), for which we will be able to evoke our results in genus \(g = 1\). We need some preparation.

**Observation 1.** Given two distinct points \(x_1, x_2\) on \(S^2\), there exists a degree 2 ramified covering map \(p_1 : S^2 \rightarrow S^2\) which is ramified precisely over these two points. Since the group of symplectic diffeomorphisms of \(S^2\) act transitively on pairs of distinct points, it suffices to prove this assertion for specific choices of \(x_1, x_2\). Identify \(S^2\) with \(\mathbb{C}P^1\), equipped with homogeneous coordinates \([z_0 : z_1]\). The map \([z_0 : z_1] \mapsto [z_0^2 : z_1^2]\) is a degree 2 ramified covering map, which is ramified precisely at the two points \([1 : 0]\) and \([0 : 1]\).

**Observation 2.** Given four distinct points \(x_1', x_2', x_3', x_4'\) on \(S^2\), there exists a degree 2 ramified covering map \(p_2 : T^2 \rightarrow S^2\) which is ramified precisely over these four points. Since the groups of symplectic diffeomorphisms of \(S^2\) and \(T^2\) act transitively on quadruples of distinct points, it suffices to prove this assertion for specific choices of \(x_1', \ldots, x_4'\). An example of such a ramified cover is given by some Weierstrass' elliptic \(\wp\)-function, but we shall rather give a topological construction. For \(S^2 = \mathbb{C}P^1\), set
\[
x_1' = [1 : 0], \quad x_2' = [0 : 1], \quad x_3' = [1 : i] \quad \text{and} \quad x_4' = [1 : -i]
\]
and consider the two disjoint branch cuts
\[
C_1 = \{ [1 - t : t] | t \in [0, 1] \} \quad \text{and} \quad C_2 = \{ [1 : e^{i\theta}] | \theta \in [\pi/2, 3\pi/2] \}.
\]
Lifting along the map \(p_1\) from observation 1, we obtain the six points
\[
\begin{align*}
x_1'' &= x_1', \quad x_2'' = x_2', \quad x_3'' = [1 : e^{i\pi/4}], \quad x_4'' = [1 : e^{-i\pi/4}], \quad x_5'' = [1 : e^{-3\pi/4}] \quad \text{and} \quad x_6'' = [1 : e^{i3\pi/4}],
\end{align*}
\]
which satisfy $p_1(x''_1) = x'_1$, $p_1(x''_2) = x'_2$, $p_1(x''_3) = p_1(x''_4) = x'_3$ and $p_1(x''_5) = p_1(x''_6) = x'_4$. The branch cut $C_2$ lifts to two branch cuts, namely
\[ C'_2 = \{(1 : e^{i\theta}) \mid \theta \in [\pi/4, 3\pi/4]\} \text{ and } C'_3 = \{(1 : e^{-i\theta}) \mid \theta \in [\pi/4, 3\pi/4]\}. \]
We can cut these two arcs open, thereby obtaining two "mouths", and we can identify the "upper lip" of one mouth with the "lower lip" of the other. More precisely, with $r$ denoting a real number, we make the identifications
\[ \forall \theta \in [\pi/4, 3\pi/4], \quad \lim_{r \rightarrow 1^+} [1 : re^{i\theta}] \sim \lim_{r \rightarrow 1^-} [1 : -re^{i\theta}] \text{ and } \lim_{r \rightarrow 1^+} [1 : re^{i\theta}] \sim \lim_{r \rightarrow 1^-} [1 : -re^{i\theta}]. \]
The resulting quotient space $Q$ is a closed oriented surface; the map $p_1$ passes to the quotient to yield a degree 2 ramified covering map $p_2 : Q \rightarrow S^2$ which is ramified precisely over the points $x'_1, \ldots, x'_4$; Riemann-Hurwitz formula implies that $Q$ has genus 1, hence $Q$ is a torus by the classification of closed surfaces.

**Observation 3.** Let $p : \Sigma \rightarrow S^2$ be a degree 2 ramified covering map, and $D \subset S^2$ be an embedded closed disc. If $D$ does not contain any branch point $x \in S^2$ of $p$, then $p^{-1}(D)$ consists in two discs, $D'_1$ and $D'_2$, such that $p|_{D'_i} : D'_i \rightarrow D$ is a diffeomorphism. If $D$ contains precisely one branch point $x \in S^2$ of $p$, then $D' := p^{-1}(D)$ is an embedded closed disc such that $p|_{D'} : D' \rightarrow D$ is a degree 2 ramified covering which is only ramified over $x$. We note that the $\mathbb{Z}/2\mathbb{Z}$-action of the nontrivial deck transformation of $p$ restricts to an action on $D'$.

**Proof of theorem 4.12 when $U$ is 3-localized.** Let $U$ be localized at $x_1, x_2, x_3, \ldots, x_m \in S^2$. Essentially, we plan to lift $U$ and $\mathcal{F}$ along a ramified covering map $p : T^2 \rightarrow S^2$ which is branched at $x_1, x_2, x_3$; the map $p$ will be a composition of the form $p_2 \circ p_1$.

**Step 1 - Perturbation of $\mathcal{F}$.** For all $\epsilon > 0$, it is possible to perturb the functions $f_j \in \mathcal{F}$ into smooth functions $f'_j$ such that:

1. the resulting collection $\mathcal{F}'$ is positive;
2. every $f'_j$ are constant over very small disjoint closed embedded discs $x_i \in D_i \subset \text{St}(x_i)$, $i \in \{1, 2, 3\}$, which contain none of the other points $x_k$;
3. $P_{\mathcal{F}'} < P_{\mathcal{F}} + \epsilon$ throughout $S^2$.

Step 2 - First lift. Using observation 1, let $p_1 : S^2 \rightarrow S^2$ denote a degree 2 ramified covering map that is branched at $x_1, x_2$. According to observation 3, and using fragmentation, $U$ lifts to an open cover $U'$ by discs. We observe that $U'$ is localized at $2m - 2$ points $x'_1, \ldots, x'_{2m-2}$, namely the preimages of the points $x_1, \ldots, x_m$:
\[ p_1(x'_1) = x_1, \quad p_1(x'_2) = x_2 \text{ and } p_1(x'_{2j+1}) = p_1(x'_{2j+2}) = x_{j+2} \text{ for } j \in \{1, \ldots, m-2\}. \]
The discs $D_i$ lift to discs $D'_1, D'_2, D'_3$ and $D'_4$; lifting and fragmenting the positive collection $\mathcal{F}$ yield a positive collection $\mathcal{F}'$ on $S^2$ all of whose functions $f'_j$ vanishes identically on the discs $D'_i$. The 2-form $p'_{1}\omega'$ is symplectic outside the points $x'_1, x'_2$; consider a very small non-negative smooth differential 2-form $\eta'$ on $S^2$ which is supported in $D'_1 \cup D'_2$, which is positive at the points $x'_1, x'_2$ and such that $\omega' := p'_{1}\omega + \eta'$ is a symplectic form of total area
\[ 2 \text{Area}(S^2, \omega) = \text{Area}(S^2, p'_{1}\omega) < \text{Area}(S^2, \omega') < \text{Area}(S^2, p'_{1}\omega) + \epsilon = 2 \text{Area}(S^2, \omega) + \epsilon. \]
Note that $p_1 : (S^2 \setminus (D'_1 \cup D'_2), \omega') \rightarrow (S^2 \setminus (D_1 \cup D_2), \omega)$ is symplectic. Consequently, if we compute $P_{\mathcal{F}'}$ with respect to $\omega'$, we obtain $P_{\mathcal{F}'} = p_{1}^{*} P_{\mathcal{F}}$ since both coincide outside $D'_1 \cup D'_2$ and both vanish on this reunion.

Step 3 - Second lift. Using observation 2, let $p_2 : T^2 \rightarrow S^2$ denote a degree 2 ramified covering map that is branched at $x'_1, x'_2, x'_3, x'_4$. According to observation 3, and using fragmentation, $U'$
lifts to an open cover $\mathcal{U}''$ by discs. We observe that $\mathcal{U}''$ is localized at $4m - 8$ points $x''_1, \ldots, x''_{4m-8}$, namely the preimages of the points $x'_1, \ldots, x'_{2m-2}$:

$$p_2(x''_i) = x'_i$$ for $i \in \{1, \ldots, 4\}$ and $p(x''_{j+3}) = p(x''_{j+4}) = x'_{j+4}$ for $j \in \{1, \ldots, 2m - 6\}$.

In a complete similar way as in step 2, the discs $D'_i$, $i \in \{1, \ldots, 4\}$, lift to discs $D''_i$; $p_2$ restricts to each $D''_i$ as a ramified covering map onto $D'_i$ branched only at $x'_i$; lifting and fragmenting $\mathcal{F}'$ yield a positive collection $\mathcal{F}'' \prec \mathcal{U}''$ all of whose functions vanish identically on the discs $D''_i$.

We can similarly define a symplectic form $\omega'' = p_2^\ast \omega' + \eta''$ on $T^2$ such that

$$4 \text{Area}(S^2, \omega) = \text{Area}(T^2, p_2^\ast \omega') \text{Area}(T^2, \omega'') < \text{Area}(T^2, p_2^\ast \omega') \epsilon + \epsilon = 4 \text{Area}(S^2, \omega) + 2 \epsilon,$$

and $p_2 : (T^2 \setminus \bigcup_{i=1}^4 D''_i, \omega'') \to (S^2 \setminus \bigcup_{i=1}^4 D'_i, \omega')$ is symplectic, so that $P_{\mathcal{F}''} = p_2^\ast P_{\mathcal{F}'}$.

**Step 4 - Star inequality on $T^2$.** We define $p : (T^2, \omega'') \to (S^2, \omega)$ to be $p = p_2 \circ p_1$. It is not a symplectic map, but since it is outside a set of arbitrarily small quantitative impact, we shall say that $p$ is "symplectic". We can therefore apply theorem 1.4.12 for $\mathcal{F}''$ on $T^2$: given $x'' \in T^2$, we have

$$\int_{S(t(x''))} \sum_{f_i'' \in \mathcal{F}''} \sum_{j'' \in U''_{i,j''}} |\{f''_i, f''_j\}| \omega'' \geq 1.$$

By construction, for each $U''_j \in \mathcal{U}''$, the map $p|_{U''_j} : U''_j \to p(U''_j)$ is a ramified "symplectic" map of degree $d(U''_j) \in \{1, 2, 4\}$ and $p(U''_j)$ is a disc in $\mathcal{U}$, where the value $d(U''_j)$ is 4 if $p(U''_j)$ contains either $x_1$ or $x_2$, 2 if $p(U''_j)$ contains $x_3$ and 1 otherwise. Denoting $p_*(f''_j) \in \mathcal{F}$ the function subordinated to $p(U''_j)$, we have

$$\int_{U''_j} \sum_{f_i'' \in \mathcal{F}''} |\{f_i'', f_j''\}| \omega'' = d(U''_j) \int_{p(U''_j)} \sum_{f_i \in \mathcal{F}} |\{f_i, p_*(f_j)\}| \omega.$$

Consequently, denoting $x = p(x'')$,

$$\int_{S(t(x))} \sum_{f_i \in \mathcal{F}} \sum_{U_j \in \mathcal{U}_x} |\{f_i, f_j\}| \omega \geq \frac{1}{4} \int_{S(t(x''))} \sum_{f_i'' \in \mathcal{F}''} \sum_{j'' \in U''_{i,j''}} |\{f''_i, f''_j\}| \omega'' \geq \frac{1}{4}.$$

When $x''$ is one of the $4m - 8$ points $x''_i$, the lower bound in the previous inequality can be increased to 1. Indeed, since $\mathcal{U}''$ is localized at those points, $d(U''_j)$ is the same for each $U''_j \in \mathcal{U}''$, namely 4 if $x'' \in \{x''_1, x''_2\}$, 2 if $x'' \in \{x''_3, x''_4\}$ and 1 otherwise. Denote this common degree $d(x''_i)$. Incidentally, $S(t(x''_1))$ and $S(t(x''_2))$ have a $\mathbb{Z}/2\mathbb{Z}$-symmetry and $S(t(x''_3))$ and $S(t(x''_4))$ have a $\mathbb{Z}/4\mathbb{Z}$-symmetry. Recall that the star inequality in genus $g = 1$ was deduced from theorem 1.4.8 going back to the proof of this result and considering remark 3.2.2, we deduce the stronger star inequalities

$$\int_{S(t(p(x'')))} \sum_{f_i \in \mathcal{F}} \sum_{U_j \in \mathcal{U}_{x''}} |\{f_i, p_*(f_j)\}| \omega = \frac{1}{d(x''_i)} \int_{S(t(x''_i))} \sum_{f_i'' \in \mathcal{F}''} \sum_{j'' \in U''_{i,j''}} |\{f''_i, f''_j\}| \omega'' \geq 1.$$ 

End of the proof of theorem 1.4.14 when $g = 0$. In the notations of the above proof, for any $\epsilon > 0$, we constructed a degree 4 "symplectic" ramified map $p : (T^2, \omega''(\epsilon)) \to (S^2, \omega)$ and an open cover $\mathcal{U}''$ by discs in general position such that $4 \text{Area}(S^2, \omega) < \text{Area}(T^2, \omega'')$ and $c(\mathcal{U}'') < 4c(\mathcal{U}) + 2 \epsilon$. Moreover, we constructed a positive collection $\mathcal{F}'' \prec \mathcal{U}''$ such that $P_{\mathcal{F}''} = p^\ast P_{\mathcal{F}}$ vanishes on the support of $\omega''(\epsilon) - p^\ast \omega$. Hence, we have on the one hand,

$$4 \int_{S^2} P_{\mathcal{F}} \omega = \int_{T^2} p^\ast P_{\mathcal{F}} p^\ast \omega = \int_{T^2} P_{\mathcal{F}''} \omega''(\epsilon),$$

while on the other hand, by theorem 1.4.14 in genus $g = 1$ applied to $\mathcal{F}''$

$$\int_{T^2} P_{\mathcal{F}''} \omega''(\epsilon) \geq \frac{\text{Area}(S^2, \omega)}{c(\mathcal{U}) + 2 \epsilon}.$$
Letting $\epsilon \to 0$, we obtain the sought-after inequality.

Sketch of proof of the claim in Step 1. Let $x \in \{x_1, \ldots, x_m\}$ be fixed, but arbitrary. For $\delta > 0$ sufficiently small, there is a Darboux chart $\Phi : (V, \omega) \to (B^2(\delta), \omega_0)$ sending $x \in V$ to $0 \in B^2(\delta) \subset \mathbb{R}^2$, where $\omega_0$ denotes the standard symplectic form.

Letting $0 < \sigma < \delta$ be small and $\rho : [\sigma, \delta) \to [0, \delta)$ be a surjective increasing diffeomorphism such that $1 \leq |\rho'(r)| < 1 + 2\sigma$ and $\rho(r) = r$ near $r = \delta$. In polar coordinates $(r, \theta)$ on $B^2(\delta)$, consider the map

$$\Phi : B^2(\delta) \setminus B^2(\sigma) \to B^2(\delta) : (r, \theta) \mapsto (\rho(r), \theta).$$

Given any smooth function $f : B^2(\delta) \to \mathbb{R}^2$, the pullback $\Phi^* f : B^2(\delta) \setminus B^2(\sigma) \to \mathbb{R}$ extends continuously to $B^2(\delta)$ in such a way that it is constant on $B^2(\sigma)$. Convoluting $\Phi^* f$ with a fixed sufficiently localized bump function, we obtain a smooth function $f'$ which is constant near $0$ and such that $f' = f$ near $\partial B^2(\delta)$.

Applying this to each $f_j \in \mathcal{F}$ (for $\delta$ small enough), the resulting functions $f'_j$ still form a positive collection subordinated to $\mathcal{U}$. Given any $\epsilon > 0$, since the functions $f_j$ are $C^1$-smooth, a careful study of $\{f'_j, f_j\}$ allows to prove that it can be made to satisfy $|\{f'_j, f_j\}| < |\{f_i, f_j\}| + \epsilon$ by picking $\delta$ and $\sigma$ small enough with respect to the given $\mathcal{F}$. We leave the fastidious details to the reader.

\[\square\]

4. Sharpness of the results

4.1. $L^1$-norm versus $L^\infty$-norm. In this article, we obtained lower bounds on the $L^\infty$-norm $\|P_\mathcal{F}\|$ by proving lower bounds on the $L^1$-norm $\int_M P_\mathcal{F} \omega$. Perhaps surprisingly, there was no essential lost in precision in doing so.

**Lemma 4.1.1.** Let $(M, \omega)$ be a surface, $\mathcal{U}$ a finite open cover on $M$ constituted of discs in general position and $\mathcal{F} \subset \mathcal{U}$ of positive collection. For any $\epsilon > 0$, there is a diffeomorphism $\phi : M \to M$ such that the positive collection $\mathcal{F}' := \phi^* \mathcal{F}$ subordinated to the cover $\mathcal{U}' := \phi^* \mathcal{U}$ together satisfy

$$P_{\mathcal{F}'} < \frac{\int_M P_{\mathcal{F}} \omega}{\int_M \omega} + \epsilon$$

and $c(\mathcal{U}') < \max_{i \in \{1, N\}} \frac{\int_{U_i} P_{\mathcal{F}} \omega}{\int_M P_{\mathcal{F}} \omega} \int_M \omega + \epsilon$.

Consequently, for all $\eta > 0$ we can find $\phi$, $\mathcal{U}'$ and $\mathcal{F}'$ as above such that

$$pb(\mathcal{F}') \text{Area}(M, \omega) < \int_M P_{\mathcal{F}} \omega + \eta \quad \text{and} \quad pb(\mathcal{F}')c(\mathcal{U}') < \max_{i \in \{1, N\}} \int_{U_i} P_{\mathcal{F}} \omega + \eta.$$

**Proof.** Without loss of generality, we can assume $\int_M P_\mathcal{F} \omega > 0$. For $\delta > 0$, Whitney’s approximation theorem allows us to find a smooth function $P : M \to \mathbb{R}$ satisfying $P_{\mathcal{F}} < P < P_{\mathcal{F}} + \delta$. In particular, $P$ is strictly positive everywhere on $M$. We shall set

$$\delta = \epsilon \cdot \min \left\{1, \frac{\int_M P_{\mathcal{F}} \omega}{\int_M \omega} \right\}.$$

Let’s consider the two differential forms

$$\omega_0 = \frac{\int_M \omega}{\int_M \omega} P \omega \quad \text{and} \quad \omega_1 = \omega.$$

The two-form $\omega_0$ is well-defined and nondegenerate as $P > 0$, and it is closed as we work on a surface. We observe that $\omega_0$ and $\omega_1$ give the same area to $M$, so that $[\omega_0] = [\omega_1] \in H^2_{dR}(M; \mathbb{R})$. For $t \in [0, 1]$ we set $\omega_t = (1 - t)\omega_0 + t\omega_1$, which is path of symplectic forms in the same cohomology class. Moser’s argument then yields a diffeomorphism $\phi$ of $M$ such that $\phi^*\omega_0 = \omega_1$.\[\square\]
Let \( \pi_t \) be the Poisson bivector associated to the symplectic form \( \omega_t \), that is if we denote \( \omega_t^\sharp \) the inverse of the isomorphism \( \omega_t^\sharp : TM \to T^*M : v \mapsto v \cdot \omega_t \) then \( \pi_t = -\omega_t \circ (\omega_t^\sharp \otimes \omega_t^\sharp) \). Incidentally, \( \{\alpha, \beta\}_\omega = \pi_t(da, db) \). It turns out that

\[
\pi_1 = \pi, \quad (\phi^{-1})_* \pi_0 = \pi_1 \quad \text{and that} \quad \pi_0 = \frac{\int_M P \omega}{\int_M \omega} \pi.
\]

As a result, for any \( \alpha, \beta \in C^\infty(M; \mathbb{R}) \) we compute at \( p \in M \)

\[
|\{\phi^* \alpha, \phi^* \beta\}_\omega(p)| = |\pi (d(\phi^* \alpha), d(\phi^* \beta))| = |\pi_1 (d\alpha, d\beta)| = \frac{\int_M P \omega}{\int_M \omega} \cdot \frac{\{\alpha, \beta\}_\omega(\phi(p))}{P(\phi(p))}.
\]

Taking \( \alpha = f_i \) and \( \beta = f_j \) and summing over all \( i \) and \( j \), this clearly implies \( P_{\mathcal{F}'} \leq \int_M P \omega / \int_M \omega \), which itself implies the first inequality claimed in the lemma by our choice of \( \delta \). We also compute

\[
\text{Area}(\phi^{-1} U_i, \omega) = \int_{\phi^{-1} U_i} \omega = \int_{U_i} (\phi^{-1})^* \omega_1 = \int_{U_i} \omega_0 = \frac{\int_{U_i} P_F \omega}{\int_M P_F \omega} \int_M \omega \geq \frac{\int_{U_i} P_F \omega}{\int_M P_F \omega} \int_M \omega + \delta \frac{(\int_M \omega)^2}{\int_M P_F \omega}.
\]

The second inequality claimed in the lemma easily follows. The last claim is then obvious.

\( \square \)

**Remark 4.1.2.** By choosing \( \delta \) also smaller than both

\[
\epsilon \rho_b(\mathcal{F}) \cdot \frac{\int_M \omega}{\int_M P_F \omega} \quad \text{and} \quad \epsilon \frac{\int_M P_F \omega}{\int_M \omega},
\]

we can similarly prove that, respectively,

\[
\rho_b(\mathcal{F}') > \frac{\int_M P_F \omega}{\int_M \omega} - \epsilon \quad \text{and} \quad c(\mathcal{U}') > \max_{i \in \{1, N\}} \frac{\int_{U_i} P_F \omega}{\int_M P_F \omega} \int_M \omega - \epsilon.
\]

Therefore, for all \( \eta > 0 \) we can find \( \phi, \mathcal{U}' \) and \( \mathcal{F}' \) as above such that

\[
\rho_b(\mathcal{F}') \text{Area}(M, \omega) \in \int_M P_F \omega + (-\eta, \eta) \quad \text{and} \quad \rho_b(\mathcal{F}') c(\mathcal{U}') \in \max_{i \in \{1, N\}} \int_{U_i} P_F \omega + (-\eta, \eta).
\]

### 4.2. Sharpness of the "confined essential disc" inequality.

In view of the proof of theorem 4.1.8 it seems feasible to come up with an example of a positive collection for which the inequalities encountered in the course of the proof are nearly sharp. One inequality however appears to more difficult to turn into an equality than the others: in the notations of the proof, we would need to find a curve \( \gamma \) such that for each function \( f_i, i \neq 1 \), attains its maximum on \( C(s) \) at \( \gamma(s) \), and that for every \( s \in I \). Notice however that the choice of \( \gamma(s) \in C(s) \) was arbitrary; averaging over this choice of point for each \( s \in I \), we obtain that we would need to have for each \( s \in I \) and each \( i \neq 1 \)

\[
\max C(s) f_i \min C(s) f_i \approx \frac{1}{T(s)} \int_0^{T(s)} (\Phi^* f_i)(t, s) \, dt.
\]

The next example uses this observation to show that theorem 4.1.8 is sharp.

**Example 4.2.1.** Consider the round sphere \( S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \) equipped with the usual area form \( \omega \). Let \( d \geq 2 \) be an integer, \( h : [-1, 1] \to [0, 1] \) be a smooth increasing function with \( h(0) = 0 \) near \( u = 0 \) and thus for \( u < 0 \) and \( h(1) = 1 \) near \( u = 1 \), and \( w : \mathbb{R}/2\pi \mathbb{Z} \to [0, 1] \) be a smooth function such that \( w(t) = 0 \) near \( t = 0 \) and \( w(t) = 1 \) for
Remark 4.3.2. Consider the positive collection $\mathcal{F} = \{f_+, f_-, f_0, \ldots, f_d\}$ given in cylindrical coordinates $(\theta, z) \in \mathbb{R}/2\pi \mathbb{Z} \times [-1, 1]$ as follows:

$$f_+(\theta, z) = h(z), \quad f_- (\theta, z) = h(-z) \quad \text{and} \quad f_j (\theta, z) = \frac{1}{d} (1 - h(z)) w \left( \theta + \frac{2\pi j}{d+1} \right) \forall j = 0, \ldots, d.$$  

The open cover is formed by discs which are slight enlargements of the support of these functions; the north pole is covered by a unique disc, which is therefore essential. We observe that $S_F(\theta, z) \in [1, 1 + 1/d]$. Using the general identity $|\{F, G\}|\omega = |dF \wedge dG|$ and the fact that $w(\theta + 2\pi j/(d+1))$ equals 1 on the support of the derivative of $w(\theta + 2\pi k/(d+1))$ whenever $j \neq k$, a straightforward computation yields

$$\int_M \sum_{f \in \mathcal{F}} |\{f_+, f\}| \omega = \int_M \sum_{j=0}^d |\{f_+, f_j\}| \omega = 1 + \frac{1}{d}.$$  

Letting $d \to \infty$ proves that theorem 1.4.8 is sharp. A slightly more involved calculation yields $\int_M P_F \omega = 8$ for all $d$.

4.3. Improving the lower bound in the Poisson bracket theorem. The lower bound we obtained in theorem 1.4.14 is already an improvement over the lower bound obtained by Buhovsky–Logunov–Tanny, c.f. eq. (1.4.15). The following proposition is a first indication that the lower bounds in theorem 1.4.14 might not be sharp.

Proposition 4.3.1. In the context of theorem 1.4.8, suppose that $\mathcal{F}$ is a partition of unity. Then

$$\max_{U_i \in \mathcal{U}} \int_{U_i} P_F \omega \geq \min_{U_i \in \mathcal{J}(\mathcal{U})} \int_{U_i} P_F \omega \geq 2.$$  

Moreover, if there are $J \geq 1$ disjoint confined essential sets, then

$$\int_M P_F \omega \geq 2J.$$  

Proof. Let $U_k \in \mathcal{J}(\mathcal{U})$. Since $\{f, 1\} = 0$ for all $f$, we have

$$\sum_{i,j=1}^N |f_i, f_j| = \sum_{i=1}^N |f_i, f_k| + \sum_{j \neq k} \sum_{i=1}^N |f_i, f_j| \geq \sum_{i=1}^N |f_i, f_k| + \sum_{i=1}^N \left| \left\{ f_i, \sum_{j \neq k} f_j \right\} \right| = \sum_{i=1}^N |f_i, f_k| + \sum_{i=1}^N |f_i, 1 - f_k| = 2 \sum_{i=1}^N |f_i, f_k|.$$  

The first inequality in the proposition therefore follows from theorem 1.4.8. If there are $J \geq 1$ disjoint confined essential sets, say $U_1, \ldots, U_J$, then $\cup_{i=1}^J U_i = \cup_{i=1}^J U_i$ and thus

$$\int_M P_F \omega \geq \int_{\cup_{i=1}^J U_i} P_F \omega = \sum_{i=1}^J \int_{U_i} P_F \omega \geq \sum_{i=1}^J 2 = 2J.$$  

Remark 4.3.2. We note that the proof of the last result cannot be readily adapted to the more general case when $S_F \geq 1$, although some intuition coming from the mean value theorem suggests that the result should also be valid.
In view of the previous proposition, of the remark following it and of the general spirit of the methods in the paper, we can suspect that under the assumptions of theorem \[1.4.14\]

\[
\int_M P_F \omega \geq 2.
\]

This is indeed the case, as follows from Shi–Lu’s argument \cite{SL} (c.f. section 5). More generally, it might be that every lower bounds in theorem \[1.4.14\] could be multiplied by a factor 2. Since theorem \[1.4.8\] is a sharp result, improving the lower bounds in theorem \[1.4.14\] seems a difficult task using our methods: a new ingredient is needed. It would be interesting to see if our techniques could be mixed with those of \cite{BLT, SL} to give a better and more complete understanding of the \(pb\) invariant on surfaces.

5. Short survey of the conjecture

We now describe in somewhat more details how the Poisson bracket invariants and the Poisson bracket conjectures were introduced and some of the progresses made on these conjectures.

Observe that \(pb(\mathcal{F})\), while nonnegative, might vanish: given a smooth function \(h : M \to \mathbb{R}\) and a smooth partition of unity \(G = \{g_i\}_{i=1}^N\) on \(h(M)\) subordinated to some open cover \(\mathcal{V} = \{V_i\}_{i=1}^N\) of \(h(M)\), then \(\mathcal{F} := \{f_i := g_i \circ h\}_{i=1}^N\) is a smooth partition of unity on \(M\) subordinated to the open cover \(\mathcal{U} := \{U_i := h^{-1}(V_i)\}_{i=1}^N\) such that \(pb(\mathcal{F}) = 0\) and hence \(pb(\mathcal{U}) = 0\).

In comparison, when \(\mathcal{U}\) is constituted of displaceable open sets, the invariant \(pb(\mathcal{U})\) cannot vanish if it is realized, i.e. if there exists \(\mathcal{F} \prec \mathcal{U}\) such that \(pb(\mathcal{F}) = pb(\mathcal{U})\). This follows from (the contrapositive of) the nondisplaceable fiber theorem \[EP\], which states that if a function \(\mathcal{F} : M \to \mathbb{R}^N : x \mapsto (f_1(x), \ldots, f_N(x))\) has components which all pairwise Poisson commute, then some preimage of \(\mathcal{F}\) is nondisplaceable in \((M, \omega)\), hence any open set \(U_i\) containing this fiber is also nondisplaceable.

In \[EPZ\], Entov, Polterovich and Zapolsky generalized the nondisplaceable fiber theorem in a more quantitative way by considering partitions of unity subordinated to open covers; this result was reformulated in \[P2\] in terms of the \(pb\) invariant. We state here this last formulation in a way closer to the formulation of our previous results and which can be deduced from the material in \[PR\]: whenever \(\mathcal{U}\) is constituted of displaceable open sets,

\[
(5.0.1) \quad + \infty > pb(\mathcal{U}) e_H(\mathcal{U}) \geq \frac{1}{8N^2}
\]

where \(N\) is the cardinality of the cover \(\mathcal{U}\) and where we defined \(e_H(\mathcal{U}) := \max_{i \in \{1, \ldots, N\}} e_H(U_i)\). In particular, \(pb(\mathcal{U})\) cannot vanish if \(\mathcal{U}\) consists in (finitely many) displaceable sets. The proofs of the aforementioned results are sophisticated, relying on the functional analytic apparatus of (symplectic) quasi-states on the function space \(C^\infty(M)\) (equipped with the Poisson bracket) constructed from spectral invariants obtained using the Hamiltonian Floer homology and the quantum cohomology of the symplectic manifold \((M, \omega)\).

In \[P3\], Polterovich established that if the cover \(\mathcal{U}\) is further assumed to be "regular" and "fine", morally meaning that each open set \(U_i\) can be displaced within a sufficiently "localized" neighbourhood of it with the aid of a Hamiltonian diffeomorphism of energy smaller than some prescribed value \(\mathcal{E}\), then there exists a constant \(C > 0\) depending on what is considered "sufficient" above, but not on the cardinality \(N\) of the cover, such that \(pb(\mathcal{U}) \mathcal{E} \geq C\). Notice that \(e_H(\mathcal{U})\) is smaller than \(\mathcal{E}\), possibly much smaller. Nevertheless, based on this result and on his intuition that "irregular" covers tend to have a higher \(pb\) invariant, Polterovich asked whether the conjecture \[1.3.1\] could be true.
To our knowledge, (5.0.1) is still the best result valid without further assumption on \((M, \omega)\) or on \(U\); it can however be refined in some circumstances. Motivated by the aforementioned "local" result of Polterovich, Seyfaddini [Se] and Ishikawa [I] studied more closely how the values of spectral invariants depend on localised data; under some monotonicity assumptions on \((M, \omega)\) and restricting \(U\) to consist either in images of symplectic embeddings of balls or of convex domains, respectively, they proved inequalities of the form \(pb(U)e_H(U) \geq C/D^2\) where \(C > 0\) is a universal constant and

\[
D = D(U) = \max_{1 \leq i \leq N} \sharp \{ j \in \{1, N\} : \bar{U}_i \cap \bar{U}_j \neq \emptyset \}
\]

is what they call the degree of \(U\). In fact, much like we do in this paper, Seyfaddini and Ishikawa establish somewhat stronger inequalities involving the capacities of the open sets rather than their (greater) displacement energy, thereby deducing that the displacement assumption in Polterovich conjecture is at best a sufficient condition for the nonvanishing of the \(pb\) invariant.

By a clever use of the lower semicontinuity of the \(C^0\)-norm of Poisson bracket on pairs of functions (see for instance [PR]), Polterovich [P2] and Buhovsky and Tanny [BT] established a close variant of the strong conjecture for a large class of covers \(U\): namely, given a Riemannian metric \(g\) on \(M\) compatible with \(\omega\), there exists \(\epsilon_0, C > 0\) depending only on \((M, \omega, g)\) such that for any \(0 < \epsilon < \epsilon_0\), if \(U\) consist in open sets \(U_i\) with diameter less than \(\epsilon\), then for any \(F < U\), \(pb(F)e^2 > C\). We note that \(e_H(U) \leq c\epsilon^2\) for some constant \(0 < c = c(M, g, \omega)\). Buhovsky–Tanny moreover proved in the same paper that the Poisson bracket conjecture is sharp, in the sense that it is possible to exhibit a family of such covers \(\{U_j\}_{j \in \mathbb{N}}\) and a sequence of positive numbers \(\{\epsilon_j\}_{j \in \mathbb{N}}\) such that each \(U_j\) consists in sets of diameters at most \(\epsilon_j\) and \(\lim_{j \to \infty} \epsilon_j^2 = 0\), but \(pb(U_j)e_j^2 < C'\) for some \(C' < +\infty\) independent of \(j \in \mathbb{N}\).

The situation on surfaces is more tractable than for general symplectic manifolds. On the one hand, there is an easy and well-known characterization of displaceability in dimension 2 which we prove in the appendix for completeness: an closed set \(X \subset M\) is displaceable if and only if it is contained in a closed smoothly embedded disc of area at most half that of \(M\). Combined with the behavior of \(pb\) with respect to refinements of open covers, the validity of a Poisson bracket conjecture on surfaces is essentially reduced to its validity on open covers by displaceable discs. On the other hand, by studying the \(L^1\)-norm of the Poisson bracket functions \(P_F\), Buhovsky–Tanny [BT] obtained several better lower bounds on \(pb\) valid uniformly on all surfaces; their results come into two sets of estimates, which we respectively dub "degree" estimates (which involve the degree of a cover) and "essential" estimates (which involve the existence of so-called essential sets to the cover). Explicitly, they proved that there exists a constant \(C > 0\) such that for any closed symplectic surface \((M, \omega)\) and any open cover \(U\) of \(M\) made of displaceable open discs\(^2\)

\[
(5.0.2)\quad pb(U)e_H(U) \geq C \max \{ \chi(J), \bar{D}^{-2} \} , \quad pb(U)\text{Area}(M, \omega) \geq C \max \{ |J|, (\log \bar{D})^{-1} \} .
\]

In the above, \(\bar{D} = \bar{D}(U) := \max_{1 \leq i \leq N} \sharp \{ j \in \{1, N\} : U_i \cap U_j \neq \emptyset \}\) is what Buhovsky and Tanny call the degree of \(U\), \(J = J(U) \subseteq U\) is the subset of essential sets of \(U\), where \(U_i \in U\) is essential if \(U \setminus \{U_i\}\) is not a cover of \(M\), \(|J|\) is the cardinality of \(J\), and \(\chi(J) = 1\) if \(J \neq \emptyset\) and 0 otherwise. These estimates follow from elementary, yet clever (and for the "degree" estimates, at times intricate) arguments with a strong geometric flavour.

In an updated version of [BT], Buhovsky, Logunov and Tanny [BLT] proved the Poisson bracket conjecture for every closed symplectic surfaces and for a universal constant \(C\) i.e.

\(^2\)In fact, the paper [BT] does not use the same definition of displaceability for open sets as the one we wrote above. It turns out that this does not affect the results, since the support of the functions in a partition of unity are assumed to be (strictly) contained in the open sets of the cover.
independent from \((M,\omega)\). They in fact accomplished more: given two partitions of unity \(\mathcal{F} = \{f_1, \ldots, f_N\}\) and \(\mathcal{G} = \{g_1, \ldots, g_L\}\) on \(M\), the authors considered the function
\[
P_{\mathcal{F},\mathcal{G}} : M \to [0,\infty) : x \mapsto \sum_{i=1}^{N} \sum_{j=1}^{L} \left| \{f_i, g_j\}(x) \right|.
\]

The quantity \(\|P_{\mathcal{F},\mathcal{G}}\|\) then generalizes \(pb(\mathcal{F})\), since \(P_{\mathcal{F}} = P_{\mathcal{F},\mathcal{F}}\). This sort of invariant (an instance of which was already considered in \([P3]\)) could be interpreted as a measure of the level of "Poisson noncommutativity" or of "Poisson interaction" of the two partitions of unity, so that \(pb(\mathcal{F}) \simeq pb(\mathcal{F},\mathcal{F})\) becomes a measure of "Poisson self-interaction". Buhovsky–Logunov–Tanny proved that for partitions of unity \(\mathcal{F}\) and \(\mathcal{G}\) respectively subordinated to open covers \(\mathcal{U} = \{U_1, \ldots, U_N\}\) and \(\mathcal{V} = \{V_1, \ldots, V_L\}\) of \((M,\omega)\) constituted of displaceable open sets,
\[
\int_M P_{\mathcal{F},\mathcal{G}} \omega \geq \frac{\text{Area}(M,\omega)}{2\max\{e_H(\mathcal{U}), e_H(\mathcal{V})\}},
\]
which readily implies
\[
\|P_{\mathcal{F},\mathcal{G}}\| \max\{e_H(\mathcal{U}), e_H(\mathcal{V})\} \geq 1/2.
\]
Loosely speaking, they achieved this by noticing that it is possible to bound \(\int_M P_{\mathcal{F},\mathcal{G}} \omega\) from below in terms of the numbers of intersection points of the level sets of the functions from \(\mathcal{F}\) and \(\mathcal{G}\), and that these numbers are themselves universally bounded from below. For comparison with the methods of the present paper, it is worth mentioning that their proof of the inequality in the case of only one open cover \(\mathcal{U}\) also requires to establish estimates on pairs of open covers.

More recently, Shi and Lu \([SL]\) adapted the arguments in \([BLT]\) to find a sufficient and necessary condition for an open cover by (not necessarily displaceable) discs in general position \(\mathcal{U}\) on any closed symplectic surface to have nonvanishing \(pb\) invariant: namely, no two discs from \(\mathcal{U}\) should suffice to cover \(M\). In other terms, any minimal subcover \(\mathcal{U}' \subset \mathcal{U}\) should have confined stars (the stars begin, by minimality, essential discs to the subcover)\(^3\). They moreover prove that, when this condition is satisfied, the weak Poisson bracket conjecture is valid, with
\[
\int_M P_{\mathcal{F}} \omega \geq 1.
\]

In fact, a refinement on their count of intersection points of so-called "\(A\)-divisions" allows to obtain the lower bound \(2\) in the above inequality. Although they do not discuss the case of two open covers \(\mathcal{U}\) and \(\mathcal{V}\), their methods implicitly imply the following more general fact: the infimum of \(P_{\mathcal{F},\mathcal{G}}\) over positive collections \(\mathcal{F} \prec \mathcal{U}\) and \(\mathcal{G} \prec \mathcal{V}\) is positive if and only if \(M \not\subset U \cup V\) for every \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\), in which case \(\int_M P_{\mathcal{F},\mathcal{G}} \omega \geq 2\).

The results of the present paper were obtained around the same time as those of \([BLT]\). They largely rely on the way Poisson brackets, displacement energies and areas behave under pullbacks along symplectic covering maps. This suggests that the right way of thinking about the Poisson bracket conjectures should be in terms of Poisson morphisms between symplectic manifolds (called symplectic submersions in \([Pa]\)). This idea has been explored in the author’s PhD thesis \([Pa]\) as a way to approach the Poisson bracket conjectures in higher dimensions and encountered some success, e.g. it yielded another proof of Polterovich–Buhovsky–Tanny’s result on the \(pb\) invariant of metrically small open covers. The line of attack in that work is, under some assumptions, to reduce the problem in higher dimensions to the two-dimensional situation by projecting the data \((\mathcal{U},\mathcal{F})\) defined on a symplectic manifold \((M,\omega)\) along a Poisson

\(^3\)We pointed out on multiple occasions that this condition is necessary. In the case of surfaces of genus \(g \geq 1\), the condition is automatically satisfied, since such surfaces have Lusternik–Schnirelmann category 3; theorem \(4.1\) therefore implicitly establishes the sufficiency of this condition when \(g \geq 1\). The sufficiency of the condition is thus most interesting in the case of the sphere, for which it nicely generalizes and simplifies our 3-localization assumption.
map to obtain appropriate data \((U', \mathcal{F}')\) on a symplectic surface. As far as we were able to figure things out, this reduction step requires one to consider non-displaceable sets and to evoke the star inequalities proved in the present paper. Explaining these results in further details shall be the object of a forthcoming paper.

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