A stabilized finite element method for advection–diffusion equations on surfaces

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[Received on 1 August 2012; revised on 19 December 2012]

A recently developed Eulerian finite element method (FEM) is applied to solve advection–diffusion equations posed on hypersurfaces. When transport processes on a surface dominate over diffusion, FEMs tend to be unstable unless the mesh is sufficiently fine. The paper introduces a stabilized FE formulation based on the streamline upwind Petrov-Galerkin (SUPG) technique. An error analysis of the method is given. Results of numerical experiments are presented, which illustrate the performance of the stabilized method.

Keywords: surface PDE; finite element method; transport equations; advection–diffusion equation; SUPG stabilization.

1. Introduction

Mathematical models involving partial differential equations (PDEs) posed on hypersurfaces occur in many applications. Often surface equations are coupled with other equations that are formulated in a (fixed) domain which contains the surface. This happens, for example, in common models of multiphase fluid dynamics if one takes the so-called surface active agents into account (Gross & Reusken, 2011). The surface transport of such surfactants is typically driven by convection and surface diffusion and the relative strength of these two is measured by the dimensionless surface Peclet number \( Pe_s = UL/D_s \). Here \( U \) and \( L \) denote typical velocity and length scales, respectively, and \( D_s \) is the surface diffusion coefficient. Typical surfactants have surface diffusion coefficients in the range \( D_s \approx 10^{-3} - 10^{-5} \text{ cm}^2/\text{s} \) (Agrawal & Neumann, 1988), leading to (very) large surface Peclet numbers in many applications. Hence, such applications result in advection–diffusion equations on the surface with dominating advection terms. The surface may evolve in time and be available only implicitly (for example, as a zero level of a level-set function).

It is well known that finite element (FE) discretization methods for advection–diffusion problems need an additional stabilization mechanism, unless the mesh size is sufficiently small to resolve...
boundary and internal layers in the solution of the differential equation. For the planar case, this topic has been studied extensively in the literature and a variety of stabilization methods has been developed; see, for example, Roos et al. (2008). However, we are not aware of any studies of stable finite element methods (FEMs) for advection–diffusion equations posed on surfaces.

In the past decade the study of numerical methods for PDEs on surfaces has been a rapidly growing research area. The development of FEMs for solving elliptic equations on surfaces can be traced back to the paper Dziuk (1988), which considers a piecewise polygonal surface and uses an FE space on a triangulation of this discrete surface. This approach has further been analysed and extended in several directions; see, for example, Dziuk & Elliott (2011) and the references therein. Another approach was introduced in Deckelnick et al. (2010) and builds on the ideas of Bertalmio et al. (2001). The method in that paper applies to cases in which the surface is given implicitly by some level-set function and the key idea is to solve the PDE on a narrow band around the surface. Unfitted FE spaces on this narrow band are used for discretization. Another surface FEM based on an outer (bulk) mesh was introduced in Olshanskii et al. (2009) and further studied in Olshanskii & Reusken (2009) and Demlow & Olshanskii (2012). The main idea of this method is to use FE spaces that are induced by triangulations of an outer domain to discretize the PDE on the surface by considering traces of the bulk FE space on the surface, instead of extending the PDE off the surface, as in Bertalmio et al. (2001) and Deckelnick et al. (2010). The method is particularly suitable for problems in which the surface is given implicitly by a level set or a volume of fluid (VOF) function and in which there is a coupling with a differential equation in a fixed outer domain. If in such problems one uses FE techniques for the discretization of equations in the outer domain, this setting immediately results in an easy-to-implement discretization method for the surface equation. The approach does not require additional surface elements.

In this paper we reconsider the volume mesh FEM from Olshanskii et al. (2009) and study a new aspect, which has not been studied in the literature so far, namely the stabilization of advection-dominated problems. We restrict ourselves to the case of a stationary surface. To stabilize the discrete problem for the case of large mesh Peclet numbers, we introduce a surface variant of the SUPG method. For a class of stationary advection–diffusion equations, an error analysis is presented. Although the convergence of the method is studied using a SUPG norm similar to the planar case (Roos et al., 2008), the analysis is not standard and contains new ingredients: some new approximation properties for the traces of FE are needed and geometric errors require special control. The main theoretical result is given in Theorem 3.11. It yields an error estimate in the SUPG norm which is almost robust in the sense that the dependence on the Peclet number is mild. This dependence is due to some insufficiently controlled geometric errors, as will be explained in Section 3.7.

The remainder of the paper is organized as follows. In Section 2, we recall equations for transport–diffusion processes on surfaces and present the stabilized FEM. Section 3 contains the theoretical results of the paper concerning the approximation properties of the FE space and discretization error bounds for the FEM. Finally, in Section 4 results of numerical experiments are given for both stationary and time-dependent advection-dominated surface transport–diffusion equations, which show that the stabilization performs well and that numerical results are consistent with what is expected from the SUPG method in the planar case.

2. Advection–diffusion equations on surfaces

Let $\Omega$ be an open domain in $\mathbb{R}^3$ and $\Gamma$ be a connected $C^2$ compact hypersurface contained in $\Omega$. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$, the tangential derivative at $\Gamma$ is defined by

$$\nabla_\Gamma g = \nabla g - (\nabla g \cdot \mathbf{n}_\Gamma)\mathbf{n}_\Gamma,$$

(2.1)
where \( \mathbf{n}_\Gamma \) denotes the unit normal to \( \Gamma \). Denote by \( \Delta \Gamma \) the Laplace–Beltrami operator on \( \Gamma \). Let \( \mathbf{w} : \Omega \to \mathbb{R}^3 \) be a given divergence-free (\( \text{div} \mathbf{w} = 0 \)) velocity field in \( \Omega \). If the surface \( \Gamma \) evolves with a normal velocity of \( \mathbf{w} \cdot \mathbf{n}_\Gamma \), then the conservation of a scalar quantity \( u \) with a diffusive flux on \( \Gamma(t) \) leads to the surface PDE

\[
\dot{u} + (\text{div}_\Gamma \mathbf{w}) u - \varepsilon \Delta \Gamma u = 0 \quad \text{on} \ \Gamma(t),
\]

where \( \dot{u} \) denotes the advective material derivative and \( \varepsilon \) is the diffusion coefficient. In Dziuk & Elliott (2007) problem (2.2) was shown to be well posed in a suitable weak sense.

In this paper, we study an FEM for an advection-dominated problem on a steady surface. Therefore, we assume that \( \mathbf{w} \cdot \mathbf{n}_\Gamma = 0 \), that is, the advection velocity is everywhere tangential to the surface. This and \( \text{div} \mathbf{w} = 0 \) imply \( \text{div}_\Gamma \mathbf{w} = 0 \), and the surface advection–diffusion equation takes the form

\[
u_t + \mathbf{w} \cdot \nabla \Gamma u - \varepsilon \Delta \Gamma u = 0 \quad \text{on} \ \Gamma.
\]

Although the methodology and numerical examples of the paper are applied to equation (2.3), the error analysis will be presented for the stationary problem

\[
-\varepsilon \Delta \Gamma u + \mathbf{w} \cdot \nabla \Gamma u + c(x)u = f \quad \text{on} \ \Gamma,
\]

with \( f \in L^2(\Gamma) \) and \( c(x) \geq 0 \). To simplify the presentation we assume \( c(\cdot) \) to be constant, that is, \( c(x) = c \geq 0 \). The analysis, however, also applies to nonconstant \( c \); cf. Section 3.7. Note that (2.3) and (2.4) can be written in intrinsic surface quantities, since \( \mathbf{w} \cdot \nabla \Gamma u = \mathbf{w}_\Gamma \cdot \nabla \Gamma u \), with the tangential velocity \( \mathbf{w}_\Gamma = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n}_\Gamma) \mathbf{n}_\Gamma \). We assume \( \mathbf{w}_\Gamma \in H^{1,\infty}(\Gamma) \cap L^\infty(\Gamma) \) and scale the equation so that \( \|\mathbf{w}_\Gamma\|_{L^\infty(\Gamma)} = 1 \) holds. Furthermore, since we are interested in the advection-dominated case we take \( \varepsilon \in (0, 1] \). Introduce the bilinear form and the functional

\[
\begin{align*}
a(u, v) &:= \varepsilon \int_\Gamma \nabla \Gamma u \cdot \nabla \Gamma v \, ds + \int_\Gamma (\mathbf{w} \cdot \nabla \Gamma u)v \, ds + \int_\Gamma c uv \, ds, \\
f(v) &:= \int_\Gamma fv \, ds.
\end{align*}
\]

The weak formulation of (2.4) is as follows: find \( u \in V \) such that

\[
a(u, v) = f(v) \quad \text{for all} \ v \in V,
\]

with

\[
V = \begin{cases} 
\left\{ v \in H^1(\Gamma) \mid \int_\Gamma v \, ds = 0 \right\} & \text{if} \ c = 0, \\
H^1(\Gamma) & \text{if} \ c > 0.
\end{cases}
\]

Owing to the Lax–Milgram lemma, there exists a unique solution of (2.5). For the case \( c = 0 \), the following Friedrich’s inequality (Sobolev, 1991) holds:

\[
\|v\|_{L^2(\Gamma)}^2 \leq C_F \|\nabla \Gamma v\|_{L^2(\Gamma)}^2 \quad \text{for all} \ v \in V.
\]

2.1 The stabilized volume mesh FEM

In this section, we recall the volume mesh FEM introduced in Olshanskii et al. (2009) and describe its SUPG-type stabilization.
Let \{T_h\}_{h>0} be a family of tetrahedral triangulations of the domain \(\Omega\). These triangulations are assumed to be regular, consistent and stable. To simplify the presentation we assume that this family of triangulations is quasi-uniform. The latter assumption, however, is not essential for our analysis. We assume that for each \(T_h\) a polygonal approximation of \(\Gamma\), denoted by \(\Gamma_h\), is given: \(\Gamma_h\) is a \(C^{0,1}\) surface without boundary and \(\Gamma_h\) can be partitioned into planar triangular segments. It is important to note that \(\Gamma_h\) is not a ‘triangulation of \(\Gamma\)’ in the usual sense (an \(O(h^2)\) approximation of \(\Gamma\), consisting of regular triangles). Instead, we (only) assume that \(\Gamma_h\) is \textit{consistent with the outer triangulation} \(T_h\) in the following sense. For any tetrahedron \(S_T \in T_h\) such that \(\text{meas}_2(S_T \cap \Gamma_h) > 0\), define \(T = S_T \cap \Gamma_h\). We assume that every \(T \in \Gamma_h\) is a \textit{planar} segment and thus it is either a triangle or a quadrilateral. Each quadrilateral segment can be divided into two triangles, so we may assume that every \(T\) is a triangle. An illustration of such a triangulation is given in Fig. 1. The results shown in this figure are obtained by representing a sphere \(\Gamma\) implicitly by its signed distance function, constructing the piecewise linear nodal interpolation of this distance function on a uniform tetrahedral triangulation \(T_h\) of \(\Omega\) and then considering the zero level of this interpolant.

Let \(F_h\) be the set of all triangular segments \(T\); then \(\Gamma_h\) can be decomposed as

\[
\Gamma_h = \bigcup_{T \in F_h} T. \tag{2.7}
\]

Note that the triangulation \(F_h\) is not necessarily regular, that is, elements from \(T\) may have very small internal angles and the size of neighbouring triangles can vary strongly; cf. Fig. 1. In applications with level-set functions (that represent \(\Gamma\) implicitly), the approximation \(\Gamma_h\) can be obtained as the zero level of a piecewise linear FE approximation of the level-set function on the tetrahedral triangulation \(T_h\).

The surface FE space is \textit{the space of traces on} \(\Gamma_h\) of all piecewise linear continuous functions with respect to the outer triangulation \(T_h\). This can be formally defined as follows. We define a subdomain that contains \(\Gamma_h\):

\[
\omega_h = \bigcup_{T \in F_h} S_T, \tag{2.8}
\]

and a corresponding volume mesh FE space:

\[
V_h := \{v_h \in C(\omega_h) \mid v_h|_{S_T} \in P_1 \text{ for all } T \in F_h\}, \tag{2.9}
\]
where $P_1$ is the space of polynomials of degree one. $V_h$ induces the following space on $\Gamma_h$:  
\[ V_h^\Gamma := \{ \psi_h \in H^1(\Gamma_h) \mid \exists v_h \in V_h \text{ such that } \psi_h = v_h|\Gamma_h \}. \tag{2.10} \]

When $c = 0$, we require that any function $v_h$ from $V_h^\Gamma$ satisfies $\int_{\Gamma_h} v_h \, ds = 0$. Given the surface FE space $V_h^\Gamma$, the FE discretization of (2.5) is as follows: find $u_h \in V_h^\Gamma$ such that  
\[ \varepsilon \int_{\Gamma_h} \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds + \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_\Gamma u)v \, ds + \int_{\Gamma_h} cuv \, ds = \int_{\Gamma_h} f^e v \, ds \tag{2.11} \]

for all $v_h \in V_h^\Gamma$. Here $\mathbf{w}^e$ and $f^e$ are the extensions of $\mathbf{w}_T$ and $f$, respectively, along normals to $\Gamma$ (the precise definition is given in the next section). Similarly to the plain Galerkin FE for advection–diffusion equations, method (2.11) is unstable unless the mesh is sufficiently fine such that the mesh Peclet number is less than 1.

We introduce the following stabilized FEM based on the standard SUPG approach (cf. Roos et al., 2008): find $u_h \in V_h^\Gamma$ such that  
\[ a_h(u_h, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h^\Gamma, \tag{2.12} \]

with  
\[ a_h(u, v) := \varepsilon \int_{\Gamma_h} \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds + \int_{\Gamma_h} cuv \, ds \]
\[ + \frac{1}{2} \left[ \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_\Gamma u)v \, ds - \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_\Gamma v)u \, ds \right] \]
\[ + \sum_{T \in \mathcal{T}_h} \delta_T \int_T (-\varepsilon \Delta_\Gamma u + \mathbf{w}^e \cdot \nabla_\Gamma u + cu)\mathbf{w}^e \cdot \nabla_\Gamma v \, ds, \tag{2.13} \]
\[ f_h(v) := \int_{\Gamma_h} f^e v \, ds + \sum_{T \in \mathcal{T}_h} \delta_T \int_T f^e (\mathbf{w}^e \cdot \nabla_\Gamma v) \, ds. \tag{2.14} \]

The stabilization parameter $\delta_T$ depends on $T \subset S_T$. The diameter of the tetrahedron $S_T$ is denoted by $h_{S_T}$. Let $\text{Pe}_T := h_{S_T} \| \mathbf{w}^e \|_{L^\infty(T)}/2\varepsilon$ be the cell Peclet number. We take  
\[ \bar{\delta}_T = \begin{cases} \delta_0 h_{S_T} & \text{if } \text{Pe}_T > 1, \\ \delta_1 h_{S_T}^2 & \text{if } \text{Pe}_T \leq 1, \end{cases} \quad \delta_T = \min\{\bar{\delta}_T, c^{-1}\}, \tag{2.15} \]

with some given positive constants $\delta_0, \delta_1 \geq 0$.

Since $u_h \in V_h^\Gamma$ is linear on every $T$ we have $\Delta_\Gamma u_h = 0$ on $T$, and thus $a_h(u_h, v_h)$ simplifies to  
\[ a_h(u_h, v_h) = \varepsilon \int_{\Gamma_h} \nabla_\Gamma u_h \cdot \nabla_\Gamma v_h \, ds + \frac{1}{2} \left[ \int_{\Gamma_h} (\mathbf{w}^e \cdot \nabla_\Gamma u_h)v_h - (\mathbf{w}^e \cdot \nabla_\Gamma v_h)u_h \, ds \right] \]
\[ + \int_{\Gamma_h} cu_h(v_h + \delta(x)\mathbf{w}^e \cdot \nabla_\Gamma v_h) \, ds + \int_{\Gamma_h} \delta(x)(\mathbf{w}^e \cdot \nabla_\Gamma u_h)(\mathbf{w}^e \cdot \nabla_\Gamma v_h) \, ds, \tag{2.16} \]

where $\delta(x) = \delta_T$ for $x \in T$. 
3. Error analysis

The analysis in this section is organized as follows. First we collect some definitions and useful results in Section 3.1. In Section 3.2, we derive a coercivity result. In Section 3.3, we present interpolation error bounds. In Sections 3.4 and 3.5, continuity and consistency results are derived. Combining this analysis we obtain the FE error bound given in Section 3.6. In the error analysis we use the following mesh-dependent norm:

\[ \|u\|_* := \left( \varepsilon \int_{\Gamma_h} |\nabla_{\Gamma_h} u|^2 \, ds + \int_{\Gamma_h} \delta(x) |w^e \cdot \nabla_{\Gamma_h} u|^2 \, ds + \int_{\Gamma_h} c |u|^2 \, ds \right)^{1/2}. \]  

(3.1)

Here and in the remainder, \( |\cdot| \) denotes the Euclidean norm for vectors and the corresponding spectral norm for matrices.

3.1 Preliminaries

For the hypersurface \( \Gamma \), we define its \( h \)-neighbourhood, 

\[ U_h := \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \Gamma) < c_0 h \}, \]  

(3.2)

and assume that \( c_0 \) is sufficiently large such that \( \omega_h \subset U_h \), and \( h \) sufficiently small such that

\[ 5c_0 h < \left( \max_{i=1,2} \|\kappa_i\|_{L^\infty(\Gamma)} \right)^{-1} \]  

(3.3)

holds, with \( \kappa_i \) being the principal curvatures of \( \Gamma \). Here and in what follows, \( h \) denotes the maximum diameter for tetrahedra of the outer triangulation: \( h = \max_{S \in \omega_h} \text{diam}(S) \).

Let \( d : U_h \to \mathbb{R} \) be the signed distance function, \( |d(x)| = \text{dist}(x, \Gamma) \) for all \( x \in U_h \). Thus, \( \Gamma \) is the zero level set of \( d \). We assume \( d < 0 \) in the interior of \( \Gamma \) and \( d > 0 \) in the exterior and define \( \mathbf{n}(x) := \nabla d(x) \) for all \( x \in U_h \). Hence, \( \mathbf{n} = \mathbf{n}_\Gamma \) on \( \Gamma \) and \( |\mathbf{n}(x)| = 1 \) for all \( x \in U_h \). The Hessian of \( d \) is denoted by

\[ H(x) := \nabla^2 d(x) \in \mathbb{R}^{3 \times 3}, \quad x \in U_h. \]  

(3.4)

The eigenvalues of \( H(x) \) are denoted by \( \kappa_1(x), \kappa_2(x) \) and 0. For \( x \in \Gamma \) the eigenvalues \( \kappa_i, i = 1,2 \) are the principal curvatures.

For each \( x \in U_h \), define the projection \( p : U_h \to \Gamma \) by

\[ p(x) = x - d(x)\mathbf{n}(x). \]  

(3.5)

Owing to assumption (3.3), the decomposition \( x = p(x) + d(x)\mathbf{n}(x) \) is unique. We will need the orthogonal projector

\[ P(x) := I - \mathbf{n}(x)\mathbf{n}(x)^T \quad \text{for} \ x \in U_h. \]

Note that \( \mathbf{n}(x) = \mathbf{n}(p(x)) \) and \( P(x) = P(p(x)) \) for \( x \in U_h \) holds. The tangential derivative can be written as \( \nabla_{\Gamma} g(x) = P \nabla g(x) \) for \( x \in \Gamma \). One can verify that for this projection and for the Hessian \( H \) the relation
\( HP = PH = H \) holds. Similarly, define

\[
P_h(x) := I - n_{\Gamma_h}(x)n_{\Gamma_h}(x)^T \quad \text{for} \ x \in \Gamma_h, \ x \text{ is not on an edge},
\]

where \( n_{\Gamma_h} \) is the unit (outward-pointing) normal at \( x \in \Gamma_h \) (not on an edge). The tangential derivative along \( \Gamma_h \) is given by \( \nabla_{\Gamma_h} g(x) = P_h(x) \nabla g(x) \) (not on an edge).

**Assumption 3.1** In this paper, we assume that for all \( \varepsilon \),

\[
\Gamma \text{ characterizes a function } v.
\]

Assumptions (3.7) and (3.8) imply that \( \varepsilon \) is given by

\[
\nabla_{\Gamma_h} = \text{the unit (outward-pointing) normal at } x \in \Gamma_h \text{ (not on an edge)}.
\]

For \( x \in \Gamma_h \), define

\[
\mu_h(x) = (1 - d(x)\kappa_1(x))(1 - d(x)\kappa_2(x))n^T(x)n_h(x).
\]

The surface measures \( ds \) and \( ds_h \) on \( \Gamma \) and \( \Gamma_h \), respectively, are related by

\[
\mu_h(x) ds_h(x) = ds(p(x)), \quad x \in \Gamma_h.
\]

Assumptions (3.7) and (3.8) imply that

\[
\text{ess sup}_{x \in \Gamma_h}(1 - \mu_h) \lesssim h^2;
\]

cf. Olshanskii et al. (2009, (3.37)). The solution of (2.4) is defined on \( \Gamma \), while its FE approximation \( u_h \in V_h \) is defined on \( \Gamma_h \). We need a suitable extension of a function from \( \Gamma \) to its neighbourhood. For a function \( v \) on \( \Gamma \), we define

\[
v^\varepsilon(x) := v(p(x)) \quad \text{for all } x \in U_h.
\]

The following formulas for this lifting function are known (cf. Demlow & Dziuk, 2007, Section 2.3):

\[
\nabla u^\varepsilon(x) = (I - d(x)H)\nabla_{\Gamma_h} u(p(x)) \quad \text{a.e. on } U_h,
\]

\[
\nabla_{\Gamma_h} u^\varepsilon(x) = P_h(x)(I - d(x)H)\nabla_{\Gamma_h} u(p(x)) \quad \text{a.e. on } \Gamma_h,
\]

with \( H = H(x) \). By direct computation one derives the relation

\[
\nabla^2 u^\varepsilon(x) = (P - d(x)H)\nabla_{\Gamma_h}^2 u(p(x))(P - d(x)H) - n^T \nabla_{\Gamma_h} u(p(x))H
\]

\[
- (H\nabla_{\Gamma_h} u(p(x)))n^T - n(H\nabla_{\Gamma_h} u(p(x)))^T - d\nabla_{\Gamma_h} H : \nabla_{\Gamma_h} u(p(x)).
\]
For sufficiently smooth $u$ and $|\mu| \leq 2$, using this relation one obtains the estimate

\[
|D^\mu u^e(x)| \lesssim \left( \sum_{|\mu|=2} |D^\mu u(p(x))| + |\nabla u(p(x))| \right) \quad \text{a.e. on } U_h
\]  

(cf. Dziuk, 1988, Lemma 3). This further leads to (cf. Olshanskii et al., 2009, Lemma 3.2)

\[
\|D^\mu u^e\|_{L^2(U_h)} \lesssim \sqrt{h}\|u\|_{H^2(\Gamma)}, \quad |\mu| \leq 2. (3.16)
\]

The next lemma is needed for the analysis in the following section.

**Lemma 3.2** The following holds:

\[
\|\text{div}_{\Gamma_h} w^e\|_{L^\infty(\Gamma_h)} \lesssim h\|\nabla w\|_{L^\infty(\Gamma)}.
\]

*Proof.* We use the following representation for the tangential divergence:

\[
\text{div}_{\Gamma_h} w^e(x) = \text{tr}(\nabla_{\Gamma_h} w^e(x)) = \text{tr}(P\nabla w(x)).
\]  

Take $x \in \Gamma_h$, not lying on an edge. Using (3.12) we obtain

\[
\text{div}_{\Gamma_h} w^e(x) = \text{tr}(P_h \nabla w(x)) = \text{tr}(P_h(I - d(x)H)\nabla w(p(x)))
\]

\[
= \text{tr}(P\nabla w(p(x))) + \text{tr}((P_h - P)\nabla w(p(x))) - d(x)\text{tr}(P_h H\nabla w(p(x))).
\]

The first term vanishes due to $\text{tr}(P\nabla w(p(x))) = \text{div}_{\Gamma} w(p(x)) = 0$. The second and the third terms can be bounded using (3.7) and (3.8):

\[
|P_h - P| \lesssim h, \quad |d(x)P_h H| \lesssim h^2.
\]

This proves the lemma. \qed

### 3.2 Coercivity analysis

In the next lemma we present a coercivity result. We use the norm introduced in (3.1).

**Lemma 3.3** The following holds:

\[
a_h(v_h, v_h) \geq \frac{1}{2} \|v_h\|^2_a \quad \text{for all } v_h \in V^F_h.
\]  

*Proof.* For any $v_h \in V^F_h$, we have

\[
a_h(v_h, v_h) = \|v_h\|^2_a + \int_{\Gamma_h} c \delta(x)v_h(w^e \cdot \nabla_{\Gamma_h} v_h) \, ds.
\]  

The choice of $\delta_T$ (cf. (2.15)) implies $c \delta(x) \leq 1$. Hence the last term in (3.19) can be estimated as follows:

\[
\left| \int_{\Gamma_h} c \delta(x)v_h(w^e \cdot \nabla_{\Gamma_h} v_h) \, ds \right| \leq \frac{1}{2} \int_{\Gamma_h} c v_h^2 \, ds + \frac{1}{2} \int_{\Gamma_h} c \delta(x)^2(w^e \cdot \nabla_{\Gamma_h} v_h)^2 \, ds \leq \frac{1}{2} \|v_h\|^2_a.
\]

This yields (3.18). \qed
As a consequence of this result, we obtain the well-posedness of the discrete problem (2.12).

### 3.3 Interpolation error bounds

Let \( I_h : C(\bar{\omega}_h) \to V_h \) be the nodal interpolation operator. For any \( u \in H^2(\Gamma) \) the surface FE function \( (I_hu^e)|_{T_h} \in V^T_h \) is an interpolant of \( u^e \) in \( V^T \).

For any \( u \in H^2(\Gamma) \) the following estimates hold (Olshanskii et al., 2009):

\[
\|u^e - (I_hu^e)|_{T_h}\|_{L^2(\Gamma_h)} \lesssim h^2 \|u\|_{H^2(\Gamma)},
\]
\[
\|
abla I_h u^e - \nabla (I_hu^e)|_{T_h}\|_{L^2(\Gamma_h)} \lesssim h \|u\|_{H^2(\Gamma)}.
\]

Using these results we easily obtain an interpolation error estimate in the \( \| \cdot \|_a \) norm.

**Lemma 3.4** For any \( u \in H^2(\Gamma) \) the following holds:

\[
\|u^e - (I_hu^e)|_{T_h}\|_a \lesssim h(e^{1/2} + h^{1/2} + e^{1/2}h) \|u\|_{H^2(\Gamma)}.
\]

**Proof.** Define \( \varphi := u^e - (I_hu^e)|_{T_h} \in H^1(\Gamma_h) \). Using definition (2.15) of \( \delta(x) \), we obtain

\[
\int \delta(x)|w^e \cdot \nabla \varphi|^2 \, ds = \sum_{T \in \mathcal{F}_h} \int_T \delta_T |w^e \cdot \nabla \varphi|^2 \, ds 
\lesssim h \|\nabla \varphi\|^2_{L^2(\Gamma)} \lesssim h^3 \|u\|^2_{H^2(\Gamma)}.
\]

The remaining two terms in \( \|u^e - (I_hu^e)|_{T_h}\|_a \) are estimated in a straightforward way using (3.20) and (3.21). This and (3.23) imply the inequality (3.22).

The next lemma estimates the interpolation error on the edges of the surface triangulation. In the remainder, \( E_h \) denotes the set of all edges in the interface triangulation \( \mathcal{F}_h \).

**Lemma 3.5** For all \( u \in H^2(\Gamma) \) the following holds:

\[
\left( \sum_{E \in E_h} \int_E (u^e - I_hu^e)^2 \, ds \right)^{1/2} \lesssim h^{3/2} \|u\|_{H^2(\Gamma)}.
\]

**Proof.** Define \( \phi := u^e - I_hu^e \in H^1(\omega_h) \). Take \( E \in E_h \) and let \( T \in \mathcal{F}_h \) be a corresponding planar segment of which \( E \) is an edge. Let \( W \) be a side of the tetrahedron \( S_T \) such that \( E \subset W \). From Hansbo & Hansbo (2002, Lemma 3), we have

\[
\|\phi\|_{L^2(E)}^2 \lesssim h^{-1} \|\phi\|_{L^2(W)}^2 + h \|\phi\|_{H^1(W)}^2.
\]

From the standard trace inequality

\[
\|w\|_{L^2(\partial S_T)}^2 \lesssim h^{-1} \|w\|_{L^2(S_T)}^2 + h \|w\|_{H^1(S_T)}^2 \quad \text{for all } w \in H^1(S_T),
\]

applied to \( \phi \) and \( \partial_i \phi \), where \( i = 1, 2, 3 \), we obtain

\[
h^{-1} \|\phi\|_{L^2(W)}^2 \lesssim h^{-2} \|\phi\|_{L^2(S_T)}^2 + \|\phi\|_{H^1(S_T)}^2,
\]
\[
h \|\phi\|_{H^1(W)}^2 \lesssim \|\phi\|_{H^1(S_T)}^2 + h^2 \|u^e\|_{H^2(S_T)}^2.
\]
From standard error bounds for the nodal interpolation operator $I_h$ we obtain
\[
\|\phi\|_{L^2(E)}^2 \lesssim h^{-2} \|\phi\|_{L^2(S_T)}^2 + \|\phi\|_{H^1(S_T)}^2 + h^2 \|u^e\|_{H^2(S_T)}^2 \lesssim h^2 \|u^e\|_{H^2(S_T)}^2.
\]
Summing over $E \in E_h$ and using $\|u^e\|_{H^2(\Omega_h)} \lesssim h^{1/2} \|u\|_{H^2(\Gamma)}$, (cf. (3.16)) results in
\[
\sum_{E \in E_h} \|\phi\|_{L^2(E)}^2 \lesssim h^2 \|u^e\|_{H^2(\Omega_h)}^2 \lesssim h^3 \|u\|_{H^2(\Gamma)}^2,
\]
which completes the proof. \qed

3.4 Continuity estimates

In this section we derive a continuity estimate for the bilinear form $a_h(\cdot, \cdot)$. If one applies partial integration to the integrals that occur in $a_h(\cdot, \cdot)$ then jumps across the edges $E \in E_h$ occur. We start with a lemma that yields bounds for such jump terms. Related to these jump terms we introduce the following notation. For each $T \in F_h$, denote by $m_h^\pm|_E$ the outer normal to an edge $E$ in the plane which contains element $T$. Let $[m_h^\pm] = m_h^+ - m_h^-$ be the jump of the outer normals to the edge in two neighbouring elements; cf. Fig. 2.

**Lemma 3.6** The following holds:
\[
|P(x)(m_h^\pm)(x)| \lesssim h^2 \quad \text{a.e. } x \in E. \tag{3.25}
\]

**Proof.** Let $E$ be the common side of two elements $T_1$ and $T_2$ in $F_h$, and $n_h^+, n_h^-, m_h^+$ and $m_h^-$ are the unit normals as illustrated in Fig. 2. Denote by $s_h$ the unit (constant) vector along the common side $E$, which can be represented as $s_h = n_h^+ \times m_h^+ = m_h^- \times n_h^-$. The jump across $E$ is given by
\[
[m_h^\pm] = s_h \times (n_h^+ - n_h^-).
\]
For each $x \in E$ and $p(x) \in \Gamma$, let $n = n(p(x))$ be the unit normal to $\Gamma$ at $p(x)$ and $P = P(x) = I - nn^T$ the corresponding orthogonal projection. Using (3.8), we obtain
\[
|n_h^- - n_h^+| \leq |n_h^+ - n| + |n_h^- - n| \lesssim h_{S_{T_1}} + h_{S_{T_2}} \lesssim h.
\]
Since \(|n^+_h| = |n^-_h| = |n| = 1\), the above estimate implies
\[
n^+_h - n^-_h = ch^2 n + e_1, \quad e_1 \perp n, \quad |e_1| \lesssim h.
\]
We also have
\[
s_h = n^+_h \times m_h^+ = (n + (n^+_h - n)) \times m_h^+ = n \times m_h^+ + e_2, \quad |e_2| \lesssim h.
\]
We use the decomposition
\[
P[m_h] = P[(n \times m_h^+) + e_2] \times (ch^2 n + e_1)].
\]
Since \(e_1 \perp n\) we have \((n \times m_h^+) \times e_1 \parallel n\) and thus \(P[(n \times m_h^+) \times e_1] = 0\). Therefore, we obtain
\[
|P[m_h]| \lesssim h^2 + |e_1| |e_2| \lesssim h^2, \quad (3.26)
\]
that is, the result (3.25) holds.

In the analysis below, we need an inequality of the form \(\|v_h\|_{L^2(I_h)} \lesssim \|v_h\|_*\) for all \(v_h \in V_h^\Gamma\). This result can be obtained as follows. First we consider the case \(c = 0\). Then the functions \(v_h \in V_h^\Gamma\) satisfy
\[
\int_{\Omega_h} v_h \, ds = 0.
\]
We assume that in \(V_h^\Gamma\) a discrete analogy of Friedrich’s inequality (2.6) holds uniformly with respect to \(h\), that is, there exists a constant \(C_F\) independent of \(h\) such that
\[
\|v_h\|_{L^2(I_h)}^2 \leq C_F \|\nabla I_h v_h\|_{L^2(I_h)}^2 \quad \text{for all } v_h \in V_h^\Gamma. \quad (3.27)
\]
Now we reduce the parameter domain \(\varepsilon \in (0, 1), \quad c > 0\) as follows. For a given generic constant \(c_0\) with \(0 < c_0 < 1\), in the remainder we restrict our attention to the parameter set
\[
\varepsilon \in (0, 1), \quad c \in \{0\} \cup [c_0 \varepsilon, \infty). \quad (3.28)
\]
For \(c > 0\) we then have
\[
c_0 \|v_h\|_{L^2(I_h)}^2 \leq \frac{2c_0}{c_0 + 1} \frac{1}{c^{1/2}} \|v_h\|_{L^2(I_h)}^2 \leq \frac{2}{c c_0 + c} \|v_h\|_*^2 \leq \frac{2}{\varepsilon + c} \|v_h\|_*^2,
\]
and combining this with the result in (3.27) for the case \(c = 0\) we obtain
\[
\|v_h\|_{L^2(I_h)} \lesssim \frac{1}{\sqrt{\varepsilon + c}} \|v_h\|_* \quad \text{for all } v_h \in V_h^\Gamma. \quad (3.29)
\]
and arbitrary \(\varepsilon \in (0, 1), \quad c \in \{0\} \cup [c_0 \varepsilon, \infty)\).

In the proof of Theorem 3.9 we need a bound for \(\|v_h\|_{L^2(T_h)}\) in terms of \(\|v_h\|_*\). Such a result is derived with the help of the following lemma.

**Lemma 3.7** Assume that the outer tetrahedra mesh size satisfies \(h \lesssim h_0\), with some sufficiently small \(h_0 \simeq 1\), depending only on the constant \(c_2\) from (3.8). The following holds:
\[
\sum_{E \in \mathcal{E}_h} \int_E v_h^2 \, ds \lesssim h^{-1} \|v_h\|_{L^2(I_h)}^2 + h \|\nabla I_h v_h\|_{L^2(I_h)}^2 \quad \text{for all } v_h \in V_h^\Gamma. \quad (3.30)
\]
projection on $P_n$ outer triangulation. Consider the patch $\tilde{\omega}(S_T)$ of all $S \in T_h$ touching $S_T$. Denote $\omega(S_T) = \tilde{\omega}(S_T) \cap T_h$. Let $v_h$ be an arbitrary fixed function from $V_h^\Gamma$. We shall prove the bound

$$\int_E v_h^2 \, ds \lesssim h^{-1} \|v_h\|^2_{L^2(\omega(S_T))} + h \|\nabla v_h\|^2_{L^2(\omega(S_T))}. \quad (3.31)$$

Then summing over all $E \in E_h$ and using that $\omega(S_T)$ consists of a uniformly bounded number of tetrahedra (due to the regularity of the outer mesh), we obtain (3.30).

Let $\mathbb{P}$ be a plane containing $T$. We can define, for sufficiently small $h$, an injective mapping $\phi: \omega(S_T) \rightarrow \mathbb{P}$ such that $|\nabla \phi| \lesssim 1$ and $|\nabla (\phi^{-1})| \lesssim 1$. For example, $\phi$ can be built by the orthogonal projection on $\mathbb{P}$. Then $|\nabla \phi| \lesssim 1$ and $|\nabla (\phi^{-1})(x)| \lesssim (\sin \alpha)^{-1}$, where $\alpha$ is the angle between $\mathbb{P}$ and $n_h(\phi^{-1}(x))$. Because of assumption (3.8) we have $1 \lesssim \sin \alpha$ for sufficiently small $h$. If $\phi$ is the orthogonal projection on $\mathbb{P}$, then $\phi(E) = E$. Thus, we obtain

$$\begin{align*}
\int_E v_h^2 \, ds = & \int_{\phi(E)} (v_h \circ \phi^{-1})^2 \, ds, \\
\|v_h\|^2_{L^2(\omega(S_T))} \simeq & \|v_h \circ \phi^{-1}\|^2_{L^2(\phi(\omega(S_T)))}, \\
\|\nabla v_h\|^2_{L^2(\omega(S_T))} \simeq & \|\nabla_{\mathbb{P}} (v_h \circ \phi^{-1})\|^2_{L^2(\phi(\omega(S_T)))}.
\end{align*} \quad (3.32)$$

Owing to the shape regularity of $S \in \tilde{\omega}(S_T)$ we have

$$h \lesssim \text{dist}(E, \partial \tilde{\omega}(S_T)) \lesssim \text{dist}(E, \partial \omega(S_T)).$$

Hence, from $|\nabla \phi^{-1}| \lesssim 1$ it follows that $h \lesssim \text{dist}(\phi(E), \partial \phi(\omega(S_T)))$. Thus, we may consider a rectangle $Q \subset \phi(\omega(S_T))$ such that $E = \phi(E)$ is a side of $Q$ and $|Q| \simeq h|E|$. By the standard trace theorem and a scaling argument we obtain

$$\int_{\phi(E)} (v_h \circ \phi^{-1})^2 \, ds \lesssim h^{-1} \|v_h \circ \phi^{-1}\|^2_{L^2(Q)} + h \|\nabla_{\mathbb{P}} (v_h \circ \phi^{-1})\|^2_{L^2(Q)}.$$

This together with (3.32) and $Q \subset \phi(\omega(S_T))$ implies (3.31). \hfill $\Box$

An immediate consequence of the lemma and (3.29) is the following corollary.

**Corollary 3.8** The following estimate holds:

$$h \sum_{E \in E_h} \int_E v_h^2 \, ds \lesssim \left( \frac{1}{\varepsilon + c} + \frac{h^2}{\varepsilon} \right) \|v_h\|^2_{\infty} \quad \text{for all } v_h \in V_h^\Gamma. \quad (3.33)$$

We are now in a position to prove a continuity result for the surface FE bilinear form.

**Lemma 3.9** For any $u \in H^2(\Gamma')$ and $v_h \in V_h^\Gamma$, we have

$$|a_h(u^e - (I_h u^e)|_{\Gamma_h}, v_h) | \lesssim \left( e^{1/2} + h^{1/2} + c^{1/2}h + \frac{h^2}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) h \|u\|_{H^2(\Gamma')} \|v_h\|_{\infty}. \quad (3.34)$$
Proof. Define $\phi = u^e - (I_h u^e)|_{\Gamma_h}$, then

$$a_h(\phi, v_h) = \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} \phi \cdot \nabla_{\Gamma_h} v_h \, ds$$

$$+ \int_{\Gamma_h} \frac{1}{2} ((w^e \cdot \nabla_{\Gamma_h} \phi) v_h - (w^e \cdot \nabla_{\Gamma_h} v_h) \phi) + c \phi v_h \, ds$$

$$+ \sum_{T \in \mathcal{T}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} \phi + w^e \cdot \nabla_{\Gamma_h} \phi + c \phi) w^e \cdot \nabla_{\Gamma_h} v_h \, ds. \quad (3.35)$$

We estimate $a_h(\phi, v_h)$ term by term. Because of (3.20) and (3.21), for the first term on the right-hand side of (3.35) we obtain

$$\left| \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} \phi \cdot \nabla_{\Gamma_h} v_h \, ds \right| \lesssim \varepsilon h \|u\|_{H^2(\Gamma)} \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} \lesssim \varepsilon^{1/2} h \|u\|_{H^2(\Gamma)} \|v_h\|. \quad (3.36)$$

To the second term on the right-hand side of (3.35) we apply integration by parts:

$$\int_{\Gamma_h} \frac{1}{2} ((w^e \cdot \nabla_{\Gamma_h} \phi) v_h - (w^e \cdot \nabla_{\Gamma_h} v_h) \phi) + c \phi v_h \, ds$$

$$= \int_{\Gamma_h} c \phi v_h \, ds - \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} v_h) \phi \, ds$$

$$+ \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T} (w^e \cdot \mathbf{m}_h) \phi v_h \, ds - \frac{1}{2} \int_{\Gamma_h} (\text{div}_{\Gamma_h} w^e) \phi v_h \, ds$$

$$=: I_1 + I_2 + I_3 + I_4. \quad (3.37)$$

The term $I_1$ can be estimated by

$$|I_1| \lesssim c^{1/2} \|\phi\|_{L^2(\Gamma_h)} \|\sqrt{\varepsilon} v_h\|_{L^2(\Gamma_h)} \lesssim h^{1/2} c^{1/2} \|u\|_{H^2(\Gamma)} \|v_h\|. \quad (3.36)$$

To estimate $I_2$, we consider the advection-dominated case and the diffusion-dominated case separately. If $\text{Pe}_T > 1$, we have

$$\int_T (w^e \cdot \nabla_{\Gamma_h} v_h) \phi \, ds \lesssim \delta_T^{-1/2} \|\phi\|_{L^2(T)} \left( \int_T \delta_T (w^e \cdot \nabla_{\Gamma_h} v_h)^2 \, ds \right)^{1/2}$$

$$\lesssim \max(h^{-1/2}, c^{1/2}) \|\phi\|_{L^2(T)} \left( \int_T \delta_T (w^e \cdot \nabla_{\Gamma_h} v_h)^2 \, ds \right)^{1/2},$$

and if $\text{Pe}_T \leq 1$,

$$\int_T (w^e \cdot \nabla_{\Gamma_h} v_h) \phi \, ds \lesssim \|w^e\|_{L^\infty(T)} \|\nabla_{\Gamma_h} v_h\|_{L^2(T)} \|\phi\|_{L^2(T)}$$

$$\lesssim \varepsilon^{1/2} h^{-1} \|\varepsilon^{1/2} \nabla_{\Gamma_h} v_h\|_{L^2(T)} \|\phi\|_{L^2(T)}. \quad (3.37)$$
Summing over $T \in \mathcal{F}_h$ we obtain

$$|I_2| \lesssim (h^{-1/2} + \varepsilon^{1/2} h^{-1}) \| \phi \|_{L^2(\Gamma_h)} \| v_h \|_* \lesssim h (c^{1/2} h + h^{1/2} + \varepsilon^{1/2}) \| u \|_{H^2(\Gamma)} \| v_h \|_*.$$  

The term $I_3$ is estimated using $Pw^e = w^e$, Lemmas 3.5 and 3.6 and Corollary 3.8:

$$|I_3| \lesssim \left| \sum_{E \in \mathcal{E}_h} \int_E (w^e \cdot [m_h]) \phi v_h \, ds \right|$$

$$\lesssim \left( \sum_{E \in \mathcal{E}_h} \int_E |\phi|^2 \, ds \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h} \int_E |P[m_h]|^2 v_h^2 \, ds \right)^{1/2}$$

$$\lesssim h^3 \| u \|_{H^2(\Gamma)} \left( h \sum_{E \in \mathcal{E}_h} \int_E v_h^2 \, ds \right)^{1/2} \lesssim \left( \frac{h^3}{\sqrt{\varepsilon} + c} + \frac{h^4}{\sqrt{\varepsilon}} \right) \| u \|_{H^2(\Gamma)} \| v_h \|_*.$$  

(3.38)

The term $I_4$ in (3.37) can be bounded due to Lemma 3.2, the interpolation bounds and (3.29):

$$|I_4| \lesssim \frac{1}{2} \| \text{div}_{\Gamma_h} w^e \|_{L^\infty(\Gamma_h)} \| \phi \|_{L^2(\Gamma_h)} \| v_h \|_{L^2(\Gamma_h)} \lesssim h^3 \| u \|_{H^2(\Gamma)} \| v_h \|_{L^2(\Gamma_h)}$$

$$\lesssim \frac{h^3}{\sqrt{\varepsilon} + c} \| u \|_{H^2(\Gamma)} \| v_h \|_*.$$  

Finally, we treat the third term on the right-hand side of (3.35). Using $\delta_T \| w^e \|_{L^\infty(T)} \lesssim h, \delta_T \varepsilon \lesssim 1, \delta_T c \lesssim 1$ and the interpolation estimates (3.20) and (3.21) we obtain

$$\sum_{T \in \mathcal{F}_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} \phi + w^e \cdot \nabla_{\Gamma_h} \phi + c\phi) w^e \cdot \nabla_{\Gamma_h} v_h \, ds$$

$$\lesssim \left( \sum_{T \in \mathcal{F}_h} \delta_T (\varepsilon^2 \| \Delta_{\Gamma_h} u^e \|_{L^2(T)}^2 + \| w^e \cdot \nabla_{\Gamma_h} \phi \|_{L^2(T)}^2 + c^2 \| \phi \|_{L^2(T)}^2) \right)^{1/2} \| v_h \|_*$$

$$\lesssim (\varepsilon^{1/2} + h^{1/2} + c^{1/2} h) h \| u \|_{H^2(\Gamma)} \| v_h \|_*.$$  

(3.39)

Combining the inequalities (3.36–3.39) proves the result of the lemma. \qed

### 3.5 Consistency estimate

The consistency error of the FEM (2.12) is due to geometric errors resulting from the approximation of $\Gamma$ by $\Gamma_h$. To estimate these geometric errors we need a few additional definitions and results, which can be found in, for example, Demlow & Dziuk (2007). For $x \in \Gamma_h$ define $\tilde{P}_h(x) = I - n_h(x) n(x)^T / (n_h(x) \cdot n(x))$. One can represent the surface gradient of $u \in H^1(\Gamma)$ in terms of $\nabla_{\Gamma_h} u^e$ as follows:

$$\nabla_{\Gamma} u(p(x)) = (I - d(x) H(x))^{-1} \tilde{P}_h(x) \nabla_{\Gamma_h} u^e(x) \quad \text{a.e.} \ x \in \Gamma_h.$$  

(3.40)
Owing to (3.9), we obtain
\[ \int_{\Gamma} \nabla u \nabla v \, ds = \int_{\Gamma_h} A_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v^e \, ds \quad \text{for all } v \in H^1(\Gamma), \tag{3.41} \]
with \( A_h(x) = \mu_h(x) \bar{P}_h^T(x)(I - d(x)H(x))^{-1} \bar{P}_h(x). \) From \( w \cdot n = 0 \) on \( \Gamma \) and \( w^e(x) = w(p(x)), n(x) = n(p(x)) \) it follows that \( n(x) \cdot w^e(x) = 0 \) and thus \( w(p(x)) = \bar{P}_h(x)w^e(x) \) holds. Using this, we obtain the relation
\[ \int_{\Gamma} (w \cdot \nabla u) v \, ds = \int_{\Gamma_h} (B_h w^e \cdot \nabla_{\Gamma_h} u^e) v^e \, ds, \tag{3.42} \]
with \( B_h = \mu_h(x) \bar{P}_h^T(I - dH)^{-1} \bar{P}_h. \) In the proof we use the lifting procedure \( \Gamma_h \to \Gamma \) given by
\[ v^e_h(p(x)) := v_h(x) \quad \text{for } x \in \Gamma_h. \tag{3.43} \]
It is easy to see that \( v^e_h \in H^1(\Gamma). \)

The following lemma estimates the consistency error of the FEM (2.12).

**Lemma 3.10** Let \( u \in H^2(\Gamma) \) be the solution of (2.5); then we have
\[ \sup_{v_h \in V_h} \frac{|f_h(v_h) - a_h(u^e, v_h)|}{\|v_h\|_1} \lesssim \left( h^{1/2} + c^{1/2} h + \frac{h}{\sqrt{c + \varepsilon}} \right) h(\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}). \tag{3.44} \]

**Proof.** The residual is decomposed as
\[ f_h(v_h) - a_h(u^e, v_h) = f_h(v_h) - f(v^e_h) + a(u, v^e_h) - a_h(u^e, v_h). \tag{3.45} \]
The following holds:
\[
\begin{align*}
\int_{\Gamma} f(v^e_h) \, ds & = \int_{\Gamma_h} \mu_h f^e v_h \, ds, \\
\int_{\Gamma} a(u, v^e_h) \, ds & = \varepsilon \int_{\Gamma} \nabla u \nabla v^e_h \, ds + \int_{\Gamma} (w \cdot \nabla u) v^e_h \, ds + \int_{\Gamma} c v^e_h \, ds \\
& = \varepsilon \int_{\Gamma} \nabla u \nabla v^e_h \, ds + \frac{1}{2} \int_{\Gamma} (w \cdot \nabla u) v^e_h - (w \cdot \nabla v^e_h) u \, ds + \int_{\Gamma} c v^e_h \, ds \\
& = \varepsilon \int_{\Gamma_h} A_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v_h \, ds + \frac{1}{2} \int_{\Gamma_h} (B_h w^e \cdot \nabla_{\Gamma_h} u^e) v_h \, ds \\
& - \frac{1}{2} \int_{\Gamma_h} (B_h w^e \cdot \nabla_{\Gamma_h} v_h) u^e \, ds + \int_{\Gamma_h} \mu_h c u^e v_h \, ds.
\end{align*}
\]
Substituting these relations into (3.45) and using (2.13), (2.14) results in

\[
\begin{align*}
  f_h(v_h) - a_h(u^e, v_h) &= \int_{\Gamma_h} (1 - \mu_h) f^e v_h \, ds + \varepsilon \int_{\Gamma_h} (A_h - P_h) \nabla \Gamma_h u^e \cdot \nabla \Gamma_h v_h \, ds \\
  &\quad + \frac{1}{2} \int_{\Gamma_h} (\mathbf{B}_h - P_h) \mathbf{w}^e \cdot \nabla \Gamma_h u^e \, v_h \, ds - \frac{1}{2} \int_{\Gamma_h} ((\mathbf{B}_h - P_h) \mathbf{w}^e \cdot \nabla \Gamma_h v_h) u^e \, ds \\
  &\quad + \int_{\Gamma_h} (\mu_h - 1) \mathbf{c} u^e v_h \, ds + \sum_{T \in \mathcal{T}_h} \delta_T \int_T (f^e + \varepsilon \Delta \Gamma_h u^e - \mathbf{w}^e \cdot \nabla \Gamma_h u^e - \mathbf{c} u^e) \mathbf{w}^e \cdot \nabla \Gamma_h v_h \, ds \\
  =: I_1 + I_2 + I_3 + I_4 + I_5 + I_7.
\end{align*}
\]

(3.46)

We estimate the \( I_i \) terms separately. Applying (3.10) and (3.29) we obtain

\[
I_1 \lesssim h^2 \| f^e \|_{L^2(\Gamma_h)} \| v_h \|_{L^2(\Gamma_h)} \lesssim \frac{h^2}{\sqrt{\varepsilon + \varepsilon}} \| f \|_{L^2(\Gamma^e)} \| v_h \|_*,
\]

(3.47)

\[
I_5 \lesssim h^2 c^{1/2} \| u^e \|_{L^2(\Gamma_h)} \| \sqrt{\varepsilon} v_h \|_{L^2(\Gamma_h)} \lesssim h^2 c^{1/2} \| u \|_{H^2(\Gamma)} \| v_h \|_*.
\]

(3.48)

One can show (cf. Olshanskii et al., 2009, (3.43)) the bound

\[
|P_h - A_h| = |P_h(I - A_h)| \lesssim h^2.
\]

Using this we obtain

\[
I_2 \lesssim \varepsilon h^2 \| \nabla \Gamma_h u^e \|_{L^2(\Gamma_h)} \| \nabla \Gamma_h v_h \|_{L^2(\Gamma_h)} \lesssim \varepsilon^{1/2} h^2 \| u^e \|_{H^2(\Gamma)} \| v_h \|_*.
\]

(3.49)

Since \((I - d\mathbf{H})^{-1} = I + O(h^2)\), we also estimate

\[
|\mathbf{B}_h - P_h| \lesssim h^2 + |A_h - P_h| \lesssim h^2.
\]

This yields

\[
I_3 \lesssim h^2 \| \nabla \Gamma_h u^e \|_{L^2(\Gamma_h)} \| v_h \|_{L^2(\Gamma_h)} \lesssim \frac{h^2}{\sqrt{\varepsilon + \varepsilon}} \| u \|_{H^2(\Gamma)} \| v_h \|_*.
\]

(3.50)

To estimate \( I_4 \) we use definition (2.15) of \( \delta_T \). If \( \mathbf{P} \mathbf{e}_T \leq 1 \), then \( \varepsilon^{-1/2} \| \mathbf{w}^e \|_{L^\infty(\mathcal{T})} \leq \sqrt{2} \| \mathbf{w}^e \|_{L^2(\mathcal{T})} h_{S_T}^{-1/2} \) holds. If \( \mathbf{P} \mathbf{e}_T > 1 \), then \( \delta_T^{-1/2} \leq \max(c^{1/2}, \delta_0^{-1/2}) \| \mathbf{w}^e \|_{L^2(\mathcal{T})} h_{S_T}^{-1/2} \) holds. Using the assumption that the outer triangulation is quasi-uniform we obtain \( h_{S_T}^{-1} \lesssim h^{-1} \). Thus, we obtain

\[
\min\{\varepsilon^{-1/2} \| \mathbf{w}^e \|_{L^\infty(\mathcal{T})}, \delta_T^{-1/2}\} \lesssim \max(c^{1/2}, h^{-1/2}) \lesssim c^{1/2} + h^{-1/2}
\]
and

\[ I_4 \lesssim \max_{x \in \Gamma_h} |B_h - P_h| \sum_{T \in \mathcal{F}_h} (\varepsilon^{1/2} \| \nabla \Gamma_h v_h \|_{L^2(T)} + \delta_T^{1/2} \| \nabla \Gamma_h v_h \|_{L^2(T)}) \]

\[ \times \min \{ \varepsilon^{-1/2} \| \nabla \Gamma_h \|_{L^\infty(T)}, \delta_T^{-1/2} \} \| u^\varepsilon \|_{L^2(T)} \]

\[ \lesssim h_1 (c^{1/2} + h^{1/2}) \| v_h \|_{*} \| u \|_{L^2(T)}. \]

\( (3.51) \)

Now we estimate \( \Pi_1 \). Using the equation \(-\varepsilon \Delta_{\Gamma} u + w \cdot \nabla_{\Gamma} u + cu = f \) on \( \Gamma \) we obtain

\[ \Pi_1 = \sum_{T \in \mathcal{F}_h} \delta_T \int_T (f \circ p + \varepsilon \Delta_{\Gamma} u^\varepsilon - w \cdot \nabla \Gamma_h u^\varepsilon - cu^\varepsilon) w \cdot \nabla \Gamma_h v_h \, ds \]

\[ = \sum_{T \in \mathcal{F}_h} \delta_T \int_T (-\varepsilon (\Delta_{\Gamma} u) \circ p + \varepsilon \Delta_{\Gamma} u^\varepsilon) w \cdot \nabla \Gamma_h v_h \, ds \]

\[ + \sum_{T \in \mathcal{F}_h} \delta_T \int_T ((w \cdot \nabla_{\Gamma} u) \circ p - w \cdot \nabla \Gamma_h u^\varepsilon) w \cdot \nabla \Gamma_h v_h \, ds \]

\[ + \sum_{T \in \mathcal{F}_h} \delta_T \int_T (cu \circ p - cu^\varepsilon) w \cdot \nabla \Gamma_h v_h \, ds \]

\[ =: \Pi_1^1 + \Pi_1^2 + \Pi_1^3. \]

\( (3.52) \)

From \( u^\varepsilon = u \circ p \) it follows that \( \Pi_1^3 = 0 \) holds. For \( \Pi_1^2 \) we obtain, using (3.40) and \( |\mu_h^{-1} B_h - P_h| \lesssim |\mu_h - 1| |\mu_h|^{-1} |B_h| + |B_h - P_h| \lesssim h^2 \),

\[ \Pi_1^2 = \sum_{T \in \mathcal{F}_h} \delta_T \int_T ((\mu_h^{-1} B_h - P_h) w \cdot \nabla \Gamma_h u^\varepsilon) w \cdot \nabla \Gamma_h v_h \, ds \]

\[ \lesssim h^2 \left( \sum_{T \in \mathcal{F}_h} \delta_T \| w \|_{L^\infty(T)} \| \nabla \Gamma_h u^\varepsilon \|_{L^2(T)} \right)^{1/2} \left( \sum_{T \in \mathcal{F}_h} \delta_T \| w \cdot \nabla \Gamma_h v_h \|_{L^2(T)} \right)^{1/2} \]

\[ \lesssim h^{5/2} \| u \|_{H^1(\Gamma)} \| v_h \|_{*}. \]

\( (3.53) \)

Since \( \delta_T \varepsilon \lesssim h^2 \), we obtain

\[ \Pi_1^1 \lesssim \left( \sum_{T \in \mathcal{F}_h} \varepsilon^2 \delta_T \| (\Delta_{\Gamma} u) \circ p - \Delta_{\Gamma} u^\varepsilon \|_{L^2(T)} \right)^{1/2} \left( \sum_{T \in \mathcal{F}_h} \delta_T \| w \cdot \nabla \Gamma_h v_h \|_{L^2(T)} \right)^{1/2} \]

\[ \lesssim h \varepsilon \left( \sum_{T \in \mathcal{F}_h} \| (\Delta_{\Gamma} u) \circ p - \Delta_{\Gamma} u^\varepsilon \|_{L^2(T)} \right)^{1/2} \| v_h \|_{*}. \]

\( (3.54) \)

We finally consider the term between brackets in (3.54). Using the identity \( \text{div}_{\Gamma} f = \text{tr}(\nabla_{\Gamma} f) \) and \( n \cdot \nabla u^\varepsilon(p(x)) = 0 \) we obtain for \( x \in T \), with \( \nabla^2 := \nabla \nabla^T \),

\[ \Delta_{\Gamma} u(p(x)) = \text{div}_{\Gamma} \nabla_{\Gamma} u(p(x)) = \text{tr}(P \nabla \nabla u^\varepsilon(p(x))) = \text{tr}(P \nabla^2 u^\varepsilon(p(x)) P). \]

\( (3.55) \)
From the same arguments it follows that

\[ \Delta_{\Gamma_h} u^\varepsilon(x) = \text{tr}(P_h \nabla P_h \nabla u^\varepsilon(x)) = \text{tr}(P_h \nabla^2 u^\varepsilon(x) P_h) \] (3.56)

holds. Using (3.14) and \(|d(x)| \lesssim h^2\), \(|P - P_h| \lesssim h\), \(|H| \lesssim 1\), \(|\nabla H| \lesssim 1\) we obtain

\[ P_h \nabla^2 u^\varepsilon(x) P_h = P \nabla^2 u^\varepsilon(p(x)) P + R, \]

\[ |R| \lesssim h(\nabla^2 u^\varepsilon(p(x))) + |\nabla u^\varepsilon(p(x))|. \]

Thus, using (3.55) and (3.56), we obtain

\[ |\Delta_{\Gamma} u(p(x)) - \Delta_{\Gamma_h} u^\varepsilon(x)| \lesssim |\text{tr}(P \nabla^2 u^\varepsilon(p(x)) P - P_h \nabla^2 u^\varepsilon(x) P_h)| \]

\[ \lesssim h(\nabla^2 u^\varepsilon(p(x))) + |\nabla u^\varepsilon(p(x))|, \]

and combining this with (3.54) yields

\[ \Pi_1 \lesssim h^{1/2} \left( \sum_{T \in \mathcal{T}_h} \| (\Delta_{\Gamma} u \circ p - \Delta_{\Gamma_h} u^\varepsilon)^2 \|_{L^2(T)} \right)^{1/2} \| v_h \|_* \lesssim \varepsilon^{1/2} h^2 \|u\|_{H^2(\Gamma)} \| v_h \|_*. \]

Combining this with the results (3.46–3.51) and (3.53) proves the lemma. \( \square \)

### 3.6 Main theorem

Now we put together the results derived in the previous sections to prove the main result of the paper.

**Theorem 3.11** Let Assumption 3.1 be satisfied. We consider problem parameters \( \varepsilon \) and \( c \) as in (3.28). Assume that the solution \( u \) of (2.5) has regularity \( u \in H^2(\Gamma) \). Let \( u_h \) be the discrete solution of the SUPG FEM (2.12). Then the following holds:

\[ \| u^\varepsilon - u_h \|_* \lesssim h \left( h^{1/2} + \varepsilon^{1/2} + \varepsilon c h \frac{1}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) (\| u \|_{H^2(\Gamma)} + \| f \|_{L^2(\Gamma)}). \] (3.57)

**Proof.** The triangle inequality yields

\[ \| u^\varepsilon - u_h \|_* \leq \| u^\varepsilon - (I_h u^\varepsilon) \|_{I_h} + \| (I_h u^\varepsilon) \|_{I_h} - u_h \|_* \]. (3.58)

The second term in the upper bound can be estimated using Lemmas 3.3, 3.9 and 3.10:

\[ \frac{1}{2} \| (I_h u^\varepsilon) \|_{I_h}^2 \leq a_h((I_h u^\varepsilon) \|_{I_h} - u_h, (I_h u^\varepsilon) |_{I_h} - u_h) \]

\[ = a_h((I_h u^\varepsilon) |_{I_h} - u^\varepsilon, (I_h u^\varepsilon) |_{I_h} - u_h) + a_h(u^\varepsilon - u_h, (I_h u^\varepsilon) |_{I_h} - u_h) \]
\[
\lesssim h \left( h^{1/2} + \varepsilon^{1/2} + c^{1/2} h + \frac{h^2}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) \|u\|_{H^2(\Gamma)} \| (I_h u^e) |_{\Gamma_h} - u_h \|_* \\
+ |a_h(u^e, (I_h u^e) |_{\Gamma_h} - u_h) - f_h((I_h u^e) |_{\Gamma_h} - u_h)| \\
\lesssim h \left( h^{1/2} + \varepsilon^{1/2} + c^{1/2} h + \frac{h}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) \\
\times (\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}) \| (I_h u^e) |_{\Gamma_h} - u_h \|_*. 
\]

This results in
\[
\| (I_h u^e) |_{\Gamma_h} - u_h \|_* \lesssim h \left( h^{1/2} + \varepsilon^{1/2} + c^{1/2} h + \frac{h}{\sqrt{\varepsilon + c}} + \frac{h^3}{\sqrt{\varepsilon}} \right) (\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}). \tag{3.59}
\]

The error estimate (3.57) follows from (3.22) and (3.59).

\[\square\]

### 3.7 Further discussion

We comment on some of the aspects related to the main theorem. With regard to the analysis, we note that the norm \(\| \cdot \|_*\), which measures the error on the left-hand side of (3.57), is the standard SUPG norm as found in standard analyses of planar streamline-diffusion FEMs. The analysis in this paper contains new ingredients compared to the planar case. To control the geometric errors (approximation as found in standard analyses of planar streamline-diffusion FEMs. The analysis in this paper contains.

The error estimates in this paper are in terms of the maximum mesh size over tetrahedra in \(\omega_h\), denoted by \(h\). In practice, the stabilization parameter \(\delta_T\) is based on the local Peclet number and the stabilization is switched off or reduced in the regions, where the mesh is ‘sufficiently fine’. To prove error estimates accounting for local smoothness of the solution \(u\) and the local mesh size (as available for the planar SUPG FE method), one needs local interpolation properties of FE spaces, instead of (3.20) and (3.21). Since our FE space is based on traces of piecewise linear functions, such local estimates are not immediately available. The extension of our analysis to this nonquasi-uniform case is left for future research.
Finally, we remark on the case of a varying reaction term coefficient \( c \). If the coefficient \( c \) in the third term of (2.4) varies, the above analysis is valid with minor modifications. We briefly explain these modifications. The stabilization parameter \( \delta_T \) should be based on elementwise values \( c_T = \max_{x \in T} c(x) \).

For the well-posedness of (2.4), it is sufficient to assume \( c \) to be strictly positive on a subset of \( \Gamma \) with positive measure:

\[
\mathcal{A} := \text{meas}\{x \in \Gamma : c(x) \geq c_0\} > 0,
\]

with some \( c_0 > 0 \). If this is satisfied, the Friedrich's-type inequality (Sobolev, 1991; see, also Olshanskii et al., 2009, Lemma 3.1),

\[
\|v\|_{L^2(\Gamma)}^2 \leq C_F (\|\nabla v\|_{L^2(\Gamma)}^2 + \|\sqrt{c} v\|_{L^2(\Gamma)}^2)
\]

holds, with a constant \( C_F \) depending on \( c_0 \) and \( \mathcal{A} \). Using this, all arguments in the analysis can be generalized to the case of a varying \( c(x) \) with obvious modifications. With \( c_{\min} := \text{ess inf}_{x \in \Gamma} c(x) \) and \( c_{\max} := \text{ess sup}_{x \in \Gamma} c(x) \), the final error estimate takes the form

\[
\|u^e - u_h\|_s \lesssim h \left( h^{1/2} + \varepsilon^{1/2} + c_{\max}^{1/2} h^{1/2} + \frac{h}{\sqrt{\varepsilon} + c_{\min}} + \frac{h^3}{\sqrt{\varepsilon}} \right) \left(\|u\|_{H^2(\Gamma)} + \|f\|_{L^2(\Gamma)}\right).
\]

### 4. Numerical experiments

In this section we show the results of a few numerical experiments which illustrate the performance of the method.

**Example 4.1** The stationary problem (2.4) is solved on the unit sphere \( \Gamma \), with the velocity field

\[
\mathbf{w}(x) = \left(-x_2 \sqrt{1-x_3^2}, x_1 \sqrt{1-x_3^2}, 0\right)^T,
\]

which is tangential to the sphere. We set \( \varepsilon = 10^{-6} \), \( c \equiv 1 \) and consider the solution

\[
u(x) = \frac{x_1 x_2}{\pi} \arctan \left( \frac{x_3}{\sqrt{\varepsilon}} \right).
\]

Note that \( u \) has a sharp internal layer along the equator of the sphere. The corresponding right-hand side function \( f \) is given by

\[
f(x) = \frac{8\varepsilon^{3/2}(2 + \varepsilon + 2x_3^2)x_1 x_2 x_3}{\pi (\varepsilon + 4x_3^2)^2} + \frac{6\varepsilon x_1 x_2 + \sqrt{x_1^2 + x_2^2 (x_1^2 - x_2^2)}}{\pi} \arctan \left( \frac{x_3}{\sqrt{\varepsilon}} \right) + u.
\]

We consider the standard (unstabilized) FEM in (2.11) and the stabilized method (2.12). A sequence of meshes was obtained by the gradual refinement of the outer triangulation. The induced surface FE spaces have dimensions \( N = 448, 1864, 7552, 30412 \). The resulting algebraic systems are solved by a direct sparse solver. FE errors are computed outside the layer: the variation of the quantities \( \text{err}_{L^2} = \|u -
Fig. 3. Discretization errors for Example 4.1.

Fig. 4. Example 4.1: solutions using the stabilized method and the standard method (2.11).

\[ u_h \|_{L^2(D)}, \text{err}_{H^1} = \| u - u_h \|_{H^1(D)} \] and \[ \text{err}_{\text{inf}} = \| u - u_h \|_{L^\infty(D)} \], with \( D = \{ x \in \Gamma : |x_3| > 0.3 \} \), are shown in Fig. 3. The results clearly show that the stabilized method performs much better than the standard one. The results for the stabilized method indicate an \( O(h^2) \) convergence in the \( L^2(D) \) norm and \( L^\infty(D) \) norm. In the \( H^1(D) \) norm, first-order convergence is observed. Note that the analysis predicts (only) \( O(h^{3/2}) \) order of convergence in the (global) \( L^2 \) norm.

Figure 4 shows the computed solutions with the two methods. Since the layer is unresolved, the FEM (2.11) produces a globally oscillating solution. The stabilized method gives a much better approximation, although the layer is slightly smeared, as is typical for the SUPG method.

**Example 4.2** Now we consider the stationary problem (2.4) with \( c = 0 \). The problem is posed on the unit sphere \( \Gamma \), with the same velocity field \( w \) as in Example 4.1. We set \( \varepsilon = 10^{-6} \), and consider the solution

\[ u(x) = \frac{x_1 x_2}{\pi} \arctan \left( \frac{x_3}{\sqrt{\varepsilon}} \right). \]
The corresponding right-hand side function $f$ is now given by

\[ f(x) = \frac{8\epsilon^{3/2}(2 + \epsilon + 2x_3^2)x_1x_2x_3}{\pi(\epsilon + 4x_3^2)^2} + \frac{6\epsilon x_1x_2 + \sqrt{x_1^2 + x_2^2(x_1^2 - x_3^2)}}{\pi} \arctan\left(\frac{x_3}{\sqrt{\epsilon}}\right). \]

Note that for $c = 0$ one loses explicit control of the $L_2$ norm in $\|\cdot\|_\ast$. Thus, we consider the streamline diffusion error,

\[ \text{err}_{SD} = \|w \cdot \nabla (u - u_h)\|_{L^2(D)}. \]

Results for this error quantity and for $\text{err}_{inf} = \|u - u_h\|_{L^\infty(D)}$ are shown in Fig. 5. We observe an $\mathcal{O}(h)$ behaviour for the streamline diffusion error, which is consistent with our theoretical analysis. The $L^\infty$ norm of the error also shows first-order convergence.

**Example 4.3** We show how this stabilization can be applied to a time-dependent problem and illustrate its stabilizing effect. We consider a nonstationary problem (2.3) posed on the torus

\[ \Gamma = \left\{(x_1, x_2, x_3) \mid \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2 + x_3^2 = \frac{1}{16}\right\}. \]

We set $\epsilon = 10^{-6}$ and define the advection field,

\[ w(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}}(-x_2, x_1, 0)^T, \]

which is divergence free and satisfies $w \cdot n_\Gamma = 0$. The initial condition is

\[ u_0(x) = \frac{x_1x_2}{\pi} \arctan\left(\frac{x_3}{\sqrt{\epsilon}}\right). \]

The function $u_0$ possesses an internal layer, as shown in Fig. 6(a).
The stabilized spatial semidiscretization of (2.3) reads as follows: determine $u_h = u_h(t) \in V^\Gamma_h$ such that

$$m(\partial_t u_h, v_h) + \hat{a}_h(u_h, v_h) = 0 \quad \text{for all } v_h \in V^\Gamma_h$$

with

$$m(\partial_t u, v) := \int_{\Gamma_h} \partial_t uv \, ds + \sum_{T \in F_h} \delta_T \int_T \partial_t u(w^e \cdot \nabla_{\Gamma_h} v) \, ds,$$

$$\hat{a}_h(u, v) := \varepsilon \int_{\Gamma_h} \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v \, ds + \frac{1}{2} \left[ \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} u) v \, ds - \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} v) u \, ds \right]$$

$$+ \sum_{T \in F_h} \delta_T \int_T (-\varepsilon \Delta_{\Gamma_h} u + w^e \cdot \nabla_{\Gamma_h} u) w^e \cdot \nabla_{\Gamma_h} v \, ds.$$ 

Note that $\hat{a}_h(\cdot, \cdot)$ is the same as $a_h(\cdot, \cdot)$ in (2.13) with $c \equiv 0$. The resulting system of ordinary differential equations is discretized in time by the Crank–Nicolson scheme.

For $\varepsilon = 0$ the exact solution is the transport of $u_0(x)$ by a rotation around the $x_3$-axis. Thus, the inner layer remains the same for all $t > 0$. For $\varepsilon = 10^{-6}$, the exact solution is similar, unless $t$ is large enough for dissipation to play a noticeable role. The space $V^\Gamma_h$ is constructed in the same way as in the previous examples. The spatial discretization has 5638 degrees of freedom. The fully discrete problem is obtained.
by combining the SUPG method in (4.2) and the Crank–Nicolson method with time step $\delta t = 0.1$. The evolution of the solution is illustrated in Fig. 6 demonstrating a smoothly 'rotated' pattern.

We repeated this experiment with $\delta T = 0$ in the bilinear forms $m(\cdot, \cdot)$ and $\tilde{a}_h(\cdot, \cdot)$ in (4.2), that is, the method without stabilization. As expected, we obtain (on the same grid) much less smooth discrete solutions (Fig. 7).

**Remark 4.4** With respect to mass conservation of the scheme we note the following. For $v_h \equiv 1$ in (4.2) we obtain, with $M_h(t) := \int_{\Gamma_h} u_h(x, t) \, ds$,

$$
\frac{d}{dt} M_h(t) = \left| \int_{\Gamma_h} \frac{1}{2} w^e \cdot \nabla_{\Gamma_h} u_h \, ds \right|
$$

$$
= \left| -\frac{1}{2} \sum_{E \in E_h} \int_E w^e \cdot [m] u_h \, ds + \frac{1}{2} \int_{\Gamma_h} \text{div}_{\Gamma_h} w^e u_h \, ds \right|
$$

$$
\lesssim h^2 \sum_{E \in E_h} \int_E |u_h| \, ds + C h \int_{\Gamma_h} |u_h| \, ds .
$$

Here we used estimates from Lemmas 3.2 and 3.6. Using Lemma 3.7 we obtain

$$
\sum_{E \in E_h} \int_E |u_h| \, ds \lesssim h^{-1/2} \left( \sum_{E \in E_h} \int_E u^2_h \, ds \right)^{1/2}
$$

$$
\lesssim h^{-1} (\|u_h\|_{L^2(\Gamma_h)} + h \|\nabla_{\Gamma_h} u_h\|_{L^2(\Gamma_h)}) .
$$

Assume that for the discrete solution we have a bound $\|u_h\|_{L^2(\Gamma_h)} + h \|\nabla_{\Gamma_h} u_h\|_{L^2(\Gamma_h)} \leq c$ with $c$ independent of $h$. Then we obtain $|\frac{d}{dt} M(t)| \lesssim h$ and thus $|M_h(t) - M_h(0)| \leq c h$, with a constant $c$ independent of $h$ and $t$. Hence, with respect to mass conservation we have an error that is (only) first order in $h$. With regard to mass conservation, it would be better to use a discretization in which in the discrete bilinear
form in (2.13) one replaces
\[
\frac{1}{2} \left[ \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} u) v \, ds - \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} v) u \, ds \right] \text{ by } - \int_{\Gamma_h} (w^e \cdot \nabla_{\Gamma_h} v) u \, ds.
\]
(4.3)

This method results in optimal mass conservation: \((d/dt)M_h(t) = 0\). It turns out, however, that (with our approach) it is more difficult to analyse. In particular, it is not clear how to derive a satisfactory coercivity bound.

In numerical experiments, we observed that the behaviour of the two methods (that is, with the two variants given in (4.3)) is very similar. In particular, the mass conservation error bound \(|M(t) - M(0)| \leq cth\) for the first method seems to be too pessimistic (in many cases). To illustrate this, we show results for the problem described above, but with initial condition
\[
u_0(x) = 1 + \frac{1}{\pi} \arctan \left( \frac{x_3}{\sqrt{\varepsilon}} \right), \quad \int_{\Gamma} u_0 \, ds = \pi^2 \approx 9.8696.
\]
Figure 8 shows the quantity \(M_h(t)\) for several mesh sizes \(h\). For \(t = 0\) we have, due to interpolation of the initial condition \(u_0\), a difference between \(M_h(0)\) and \(\int_{\Gamma} u_0 \, ds\) that is of order \(h^2\). For \(t > 0\) we see, except for the very coarse mesh with \(h = \frac{1}{4}\), very good mass conservation.

**Example 4.5** As a final illustration we show results for the nonstationary problem (2.3), but now on a surface with a ‘less regular’ shape. We take the surface given in Dziuk (1988):
\[
\Gamma = \{(x_1, x_2, x_3) \mid (x_1 - x_3^2)^2 + x_2^2 + x_3^2 = 1\}.
\]
(4.4)

We set \(\varepsilon = 10^{-6}\) and define the advection field as the \(\Gamma\)-tangential part of \(\tilde{w} = (-1, 0, 0)^T\). This velocity field does not satisfy the divergence-free condition. The initial condition is taken as \(u_0(x) = 1\). We apply the same method as in Example 4.3. The mesh size is \(\frac{1}{8}\) and the time step is \(\delta t = 0.1\). Figure 9 shows
the solution for several $t$ values. We observe that as time evolves, mass is transported from the two poles on the right to the pole on the left just as expected. Our discretization yields, for this strongly convection-dominated transport problem, a qualitatively good discrete result, even with a (very) low grid resolution.

**Acknowledgement**

We thank J. Grande for his support with the implementation of the methods and C. Lehrenfeld for useful discussions.

**Funding**

This work was supported in part by the Deutsche Forschungsgemeinschaft through grant RE1461/4-1, by the Russian Foundation for Basic Research through grants 12-01-91330, 12-01-00283 and by the National Natural Science Foundation of China project 11001260. X.X. was supported by DFG and NSFC M.O. was supported by RFBR.

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