Two-loop renormalization group profile of the standard model and a new generation

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Abstract

The two-loop renormalization group global profile of the Standard Model (SM) in its full parameter space is investigated. Restrictions on the Higgs boson mass as a function of a cutoff scale are obtained from the stability of electroweak vacuum and from the perturbative validity both in the Higgs and Yukawa sectors. The cutoff equal to the Planck scale requires the Higgs mass to be \(M_H = (161.3 \pm 20.6)^{+10}_{-10}\) GeV and \(M_H \geq 140.7^{+10}_{-10}\) GeV, where the \(M_H\) corridor is the theoretical one and the errors are due to the top mass uncertainty. Modification of the two-loop global profile of the SM extended by one new chiral generation is studied, and bounds on the masses of the generation are derived. Under the precision experiment restriction \(M_H \leq 200\) GeV, the fourth chiral generation, taken alone, is excluded. Nevertheless a pair of the chiral generations constituting the vector-like one could exist.

The renormalization group (RG) study of a field theory (for the review see, e.g., Refs. [1, 2]) enables one to understand in grosso the structure of the theory as a function of a characteristic energy scale. Of special interest are the cases when self-consistency of the theory is under danger of violation. They may signal either the breakdown of the perturbative validity or/and the onset of a “new physics”.

There are two problems of the kind in the Standard Model (SM). First, it encounters when some of the running couplings tend to blow up at the finite scales violating thus the perturbativity. In the case of the \(\phi^4\) scalar theory, the problem is known for a long time as the triviality problem. Second,
the problem occurs when a running coupling leaves the physical region at some finite scale. In the SM, this happens when the Higgs quartic effective coupling becomes negative, indicating the absence of a ground state in the quantum theory. It is the so-called electroweak vacuum stability problem. It is a real problem of the quantum field theory because this phenomenon takes place in the realm of the perturbative validity. In the framework of the SM, the light Higgs bosons resulting in the unstable electroweak vacuum should be forbidden. On the other hand, if this happens some new physics beyond the SM will be required to stabilize the vacuum.

The SM self-consistency study in the framework of the one-loop RG and restrictions thereof on the SM heavy particles, the Higgs boson and the top quark, was undertaken in Refs. [3, 4, 5]. A generalization to the two-loop level was given in Refs. [6]–[8]. The one-loop RG restrictions on a new heavy chiral family were studied in Ref. [9], and that on the vector-like family were investigated in Ref. [10]. In Ref. [11] the RG study of the SM extended by the fourth chiral family was generalized to the two-loop level. This required a generalization of the SM two-loop $\beta$ functions to the massive neutrino case which was presented. As a by-product, the two-loop RG global profile of the SM in its parameter space at all conceivable scales was investigated. In particular the self-consistency restrictions on the Higgs mass were refined. More generally, the problem of what principally new the fourth heavy chiral family brings in the RG global profile of the SM was considered. A short summary of these results is given in the report.

1. Renormalization group profile of the SM

The SM two-loop $\beta$ functions in the $\overline{\text{MS}}$ renormalization scheme are well-known in the literature [12]–[14] (compact summaries can be found in Refs. [13, 14]). In what follows we put just the generic structure of the emerging one- and two-loop RG differential equations. So let $g_i, i = 1, 2, 3, y_f, \lambda, v$ and $\mu$ be the SM gauge couplings, the Yukawa couplings for fermions $f$, the Higgs self-interaction coupling, the vacuum expectation value (VEV) and the renormalization scale, respectively. We neglected for simplicity by the mixing of the Yukawa couplings and, thus, by the CP violating phase.

The following essential features of the SM RG system are readily ascertained. At one-loop order, one has a kind of the three-level up-down hierarchy among the SM couplings, so that the differential equations for $g_i, y_f$ and $\lambda$
disentangle. One can first find $g_i(\mu)$, then insert them into $\beta_{y_f}^{(1)}$ and find $y_f(\mu)$, and finally put $g_i(\mu)$, $y_f(\mu)$ into the equation for $\lambda$ and integrate it. The solution to the equation for $v$ is determined completely by those for the first three equations, both in one and two loops.

In two loops, the RG equations partially entangle with each other due to a down-up feedback to the neighbour level: from $\lambda$ to $y_f$ and from $y_f$ to $g_i$. But there is no direct influence of $\lambda$ on $g_i$. It emerges only in three loops. Hence to completely entangle the RG system one needs the three-loop SM $\beta$ functions, which are unknown at present. Thus we have to restrict ourselves to the two-loop order. On the other hand, the two- and higher-loop contributions to $\beta$ functions, even the sign including, are known to depend in a multi-coupling theory on the renormalization scheme [2]. Hence the physical meaning of the running couplings becomes ambiguous, and it is impossible to improve the perturbative RG analysis of the SM in the scheme-independent way beyond one loop.

We integrated the SM RG system numerically for $\mu \geq M_Z$ by the first-order Runge-Cutta method with the initial conditions at the scale $M_Z$ taken as $\alpha_1(M_Z) = 0.0102$, $\alpha_2(M_Z) = 0.0338$ and $\alpha_3(M_Z) = 0.123$ in accordance with $\alpha(M_Z) = 1/127.90$ and $\sin^2 \theta_W(M_Z) = 0.2315$ [17]. Our normalizations of the gauge couplings are as follows: $g_1 = (5/3)^{1/2}g'$, $g_2 = g$ and $g_3 = g_S$, with $g'$, $g$ and $g_S$ being the conventional SM couplings. We choose also the relations $m_f = y_f v$ and $m_H = \lambda^{1/2}v$ as the definition of normalization for the Higgs and Yukawa couplings, with $v = (\sqrt{2} G_F)^{-1/2} = 246.22$ GeV being the Higgs VEV. Because the evolution of $v(\mu)$ is gauge dependent we use in what follows only the gauge independent observable $v \equiv v(M_Z)$. Besides we used at $\mu = M_Z$ the one-loop matching condition for the physical and running fermion masses $M_f$ and $m_f(\mu) \equiv y_f(\mu) v$, respectively. We have got at $M_H = 150$ GeV: $m_{\tau}(M_Z) = 1.764$ GeV, $m_b(M_Z) = (4.47 \pm 0.50)$ GeV and $m_t(M_Z) = (171.8^{+4.0}_{-3.7})$ GeV. The last two values correspond in turn to the physical bottom and top masses $m_b = (4.5 \pm 0.5)$ GeV [17] and $m_t = (175 \pm 5)$ GeV [18], respectively. Only errors in the top mass are left as the main source of the subsequent uncertainties.

As a field theory, SM is legitimate to be pulled to its inner ultimate limits. This may help to understand better its structure in the physically reasonable region $\mu < M_{Pl}$, $M_{Pl} \simeq 10^{19}$ GeV, which is to be considered more seriously. So all the numerical results were obtained at all allowed $\mu$ with the exact two-
loop $\beta$ functions. Let us discuss the results in turn for the gauge, Yukawa and Higgs sectors of the SM.

(i) Gauge sector

Fig. 1: Running of the inverse gauge couplings squared $\alpha_i^{-1}$, $i = 1, 2, 3$. Number of generations is $n_g = 3$. Represented Higgs masses are those corresponding to the typical heavy Higgs and to the lower critical Higgs curve shown in bold in Fig. 3.

Fig. 1 shows the running with $\mu$ of the inverse gauge couplings squared. It is seen that the coupling $g_1$ develops a pole singularity at $\Lambda_{g_1}$, $\log \Lambda_{g_1} \simeq 41$. Validity of the perturbation theory in $g_1$ restricts $\alpha_1 \leq 4\pi$ and hence $\log \mu \leq 40$, which is in the logarithmic scale twice as large as the Planck scale. We should assume this restriction on the physical grounds in what follows. Nevertheless, taken at its face value the two-loop RG has the meaning by itself. So, to understand better the mathematical structure of its solutions we extend them up to the singularity point $\Lambda_{g_1}$. The actual influence of $y_f(\lambda(\mu))$ on $g_1$ in two loops is somewhat sizable only for the heavy Higgs. It diminishes the slope of $g_1(\mu)$ at $\mu$ beyond the Planck scale, where $y_f$ are large, and shifts the singularity position $\Lambda_{g_1}^{(2)}$ upwards to $\log \Lambda_{g_1}^{(2)} = 47$ for the heavy Higgs ($m_H(M_Z) = 450$ GeV). The critical value $m_H(M_Z) = 136.1$ GeV corresponds to the maximal lower bound of the vacuum stability.
(ii) Yukawa sector

Figure 2: Running of the third family Yukawa couplings \( n_g = 3 \). The falling down curves shown in bold correspond to the lower critical Higgs mass. The thin lines, close to the latter bold ones, correspond to the one-loop approximation.

Fig. 2 presents the evolution of the Yukawa couplings \( y_f \) for the third family SM fermions: \( t \), \( b \) quarks, and \( \tau \) lepton. In one loop, all the \( y_f \) are falling down with \( \mu \) and lie in the weak coupling regime. To be more precise, the one-loop trajectory for the \( \tau \) lepton is mildly convex, so that it intersects with the curve for the \( b \) quark near the GUT scale. But in two loops the behaviour changes drastically. An approximate UV stable fixed point appears at \( y_f^{(UV)} \approx 5.2 \). In the real world, this critical value is approached from below both for \( t \), \( b \) quarks and for \( \tau \) lepton, the faster the heavier Higgs boson is. (see Fig. 1). Hence for the sufficiently heavy Higgs, \( m_H \geq 200 \) GeV, all the third family fermions would fall into the strong coupling regime at sufficiently high \( \mu \). This would make the third family fermions much more alike at the high scales than at the electroweak one. In practice, prior to \( M_{Pl} \) the strong coupling develops only for \( t \) quark when Higgs is rather heavy, \( m_H \geq 450 \) GeV. Because from the combined LEP data on the precision experiments it
follows that $M_H \leq 200$ GeV at 95% C.L. [19], one may conclude that the Yukawa sector of the SM is weakly coupled along all the physically reasonable region of $\mu$, $\mu \leq M_{Pl}$.

Figure 3: The two-loop running of the Higgs quartic coupling ($n_g = 3$). The critical curves are shown in bold.

(iii) Higgs sector

Fig. 3 presents running of the Higgs quartic coupling. In two loops, there are three critical curves shown in bold. First of all, there appears an approximate UV stable fixed point at $\lambda_{UV}^{1/2} \approx 4.93$ produced by the compensation of the one- and two-loop terms: $\lambda^2$ and $\lambda^3$. It corresponds to boundary value of the Higgs mass $m_{H_{max}}^{(2)}(M_Z) = 1200$ GeV, at and above which the theory is definitely strongly coupled. The boundary Higgs mass for the vacuum instability in two loops is $m_{H_{min}}^{(2)}(M_Z) = 136.1$ GeV. The third critical value $m_{H_{inter}}^{(2)}(M_Z) \approx 156.7$ GeV borders the region with the potentially strongly coupled Higgs from the one with the weakly coupled Higgs. Note that theory with $m_{H_{min}}^{(2)} < m_{H}(M_Z) < m_{H_{inter}}^{(2)}$ is consistent in two loops up to the ultimate scale $\mu = \Lambda_{g_1}^{(2)}$. For completeness, we present in Fig. 4 the plot for
Figure 4: Running of the Higgs VEV ($v(\mu)$) in the 't Hooft–Landau gauge both in one and two loops. It is seen that the electroweak symmetry never restores prior to the Plank scale.

Finally, one can impose the requirement of the SM self-consistency up to some cutoff scale $\Lambda$. In other terms, the theory should be neither strongly coupled nor unstable at $\mu \leq \Lambda$. In one loop, this means that the $\lambda$ singularity position fulfills the requirement $\Lambda_\lambda^{(1)} \geq \Lambda$, and simultaneously one has $\mu |_{\lambda=0} \geq \Lambda$. In two loops, we should choose as a criterion for the onset of the strong coupling regime the requirements $\beta_\mu^{(2)} / \beta_\mu^{(1)}|_{\Lambda}$ and $\beta_\mu^{(3)} / \beta_\mu^{(2)}|_{\Lambda} < 1$ which could guarantee the perturbativity and the scheme independence. In neglect by all the couplings but $\lambda$ this would mean that $\lambda^{1/2} \leq 2$, in particular $m_H(M_Z) \leq 500$ GeV, the restriction we retain for the whole SM. Not knowing $\beta_t^{(3)}$ we restricted ourselves just by the requirement that $\beta_t^{(2)} / \beta_t^{(1)}|_{\Lambda} \ll 1$ which is definitely fulfilled at $y_t \leq 2 < y_t^{(UV)}$.

The resulting one- and two-loop restrictions are drawn in Fig. 5. Here the transition was made from the MS value $m_H(M_Z)$ to the physical Higgs mass $M_H$. The sensitivity of the allowed region of the Higgs mass to the uncertainty of the top quark mass is also indicated. Strictly speaking, allowed
Figure 5: The SM one- and two-loop self-consistency plot ($n_g = 3$): the allowed Higgs mass vs. the cutoff scale $\Lambda$.

is the region between the most upper and the most lower curves. This means that for $\Lambda = M_{Pl}$ the legitimate Higgs mass is $M_H = (161.3 \pm 20.6)^{+4}_{-10}$ GeV. One gets also the lower bound $M_H \geq 140.7 \pm 10$ GeV at such a cutoff. The allowed Higgs mass region is much wider for a lower cutoff scale $\Lambda$.

2. Restrictions on the fourth chiral family

We considered the minimum SM extension by means of the additional heavy fermion families. If alone, the fourth family should have with necessity the same chirality pattern as the three light families. This is to be required to avoid the potential problem of the large direct mass mixing for the fourth family with the light ones.

What concerns the fifth family, there are two possibilities: either it has the same chirality as the four previous families, or it is a mirror one (or to state it differently, it is charge conjugate with respect to the rest of the families). In the first case, the analysis repeats itself just with more parameters. In the second case, the large direct mass terms could be introduced for the pair of the fourth and fifth families, in addition to Yukawa couplings. This proliferates enormously the number of free parameters and makes the general analysis impossible. On the other hand, if one chooses a mass independent renormalization scheme, say $\overline{\text{MS}}$, the net influence of the direct mass terms on the evolution of the SM parameters will be just in the threshold effects. Barring them, this case, which may likewise be attributed to one vector-like family, is technically equivalent to the case with two chiral families.
For these reasons we restricted ourselves by one new chiral family. In order to conform with experimental value for the number of light neutrinos \( n_\nu = 3 \), we should also add the right-handed neutrinos \( \nu_R \) (at least for the fourth family) and the proper Yukawa couplings for them. The right-handed neutrinos may possess the explicit Majorana mass as well, so that the physical neutrino masses may be quite different from their Yukawa counterparts. Because in the mass independent renormalization the explicit mass terms are important only in the threshold effects, we disregard them in what follows.

We generalized the two-loop SM \( \beta \) functions of Ref. [12] to the case with the neutrino Yukawa couplings [11]. For practical calculations with the fourth family we neglected by the light neutrino Yukawa couplings.

At present, there are no theoretical hints on the existence (or v.v.) of the fourth (and the higher) family. Nevertheless, one can extract some restrictions on the corresponding fermion masses. They are twofold, direct and indirect ones, being in a sense complementary to each other. The first group gives bounds on the common mass scale of the fourth family, the second one restricts the splitting of the masses inside the family.

The existing direct experimental bounds on the masses of the fourth family quarks \( t_4 \) and \( b_4 \) depend somewhat on the assumptions about their decays. If the lightest of the quarks, say \( b_4 \), is stable enough to leave the detector, the limit on its mass is \( M_4 \geq 140 \text{ GeV} \) [20]. On the other hand, for unstable quarks, decaying inside the detector, the limit can be estimated from the CDF and D0 searches for the top quark [18] to be about \( M_t \). What concerns the neutral and charged leptons of the fourth family, \( \nu_4 \) and \( e_4 \), it follows from LEP searches that \( M_{\nu_4} \geq 59 \text{ GeV} \) and \( M_{e_4} \geq 90 \text{ GeV} \) at 95% C.L. [17, 21].

The indirect restrictions can be extracted from the precision electroweak data, and they are related to the absence of decoupling with respect to the heavy chiral fermions. This results in the quadratically growing dependence of the electroweak radiative corrections on the heavy fermion masses. To avoid such a large radiative corrections, as the precision data require, the members of a heavy fermion doublet should be highly degenerate. Namely, one should have for the quarks \( t_4 \) and \( b_4 \) that \( (M_{t_4}^2 - M_{b_4}^2)/M_Z^2 \leq 1 \), and similarly for the leptons \( \nu_4 \), \( e_4 \). One important peculiarity of the vector-like family is the decoupling with respect to the explicit mass term, at the Yukawa couplings being finite. Hence unlike the chiral family, there is no need here for the high degeneracy in the Yukawa couplings to suppress the large radiative corrections.
To reduce the number of free parameters we assume in what follows that $m_{t_4} = m_{b_4} = m_Q$ and $m_{\nu_4} = m_{\nu_4} = m_L$. As representative, we considered two cases: $m_L/m_Q = 1/2$ and 1, with the common mass $m_Q$ of the heavy quarks given by the fourth family scale $m_4$. It follows that both these typical cases do not contradict to the direct experimental bounds if $m_4 \geq 180$ GeV. Our results for the case $m_4 = 200$ GeV which we consider as more realistic and $m_L/m_Q = 1/2$ are presented in Figs. 6–10.

Figure 6: Running of the gauge couplings ($n_g = 4$). The fourth family mass scale is $m_4 = 200$ GeV and $m_L/m_Q = 1/2$.

Fig. 6 shows the evolution of $\alpha_i^{-1}$ with $\mu$. It may be noted that the GUT triangle shrinks and shifts towards somewhat lower scale but the conceivable gauge unification takes place beyond the region of perturbativity both in Yukawa and Higgs sectors. It is seen that the two-loop contributions manifest themselves at rather low scales, $\mu = (10^7 - 10^8)$ GeV. They are governed by the onset of the strong coupling regime in the Yukawa sector at such a $\mu$ (see Fig. 7). Accordingly, the perturbatively consistent region of $\mu$ in the Higgs sector shrinks to the same values (see Fig. 8).

Applying now the same criteria of self-consistency as in the case of the minimal SM we get the allowed values of $m_H(M_Z)$ depending on the cutoff scale $\Lambda$ (Fig. 9). The sensitivity to the shift in the mass $m_4$ is also indicated. Finally, Fig. 10 presents the two-loop allowed region in the $m_4-M_H$ plane.
Figure 7: Two-loop running of the Yukawa couplings \((n_g = 4)\) for the third and fourth families at \(m_4 = 200\) GeV and \(m_L/m_Q = 1/2\). The upper and lower curves correspond to the Higgs masses, respectively, for the upper and lower Higgs critical curves shown in bold in Fig. 8.

The dependence on \(\Delta M_t\) is faint, and it is not shown.

Let us summarize the differences in the RG global profiles of the SM with three and four generations. For three generations with the experimentally known masses, the Yukawa sector is weakly coupled in the one-loop approximation. Prior to the Planck scale, the strong coupling may appear in one loop only in the Higgs self-interactions for the sufficiently heavy Higgs. It drives strong coupling for the Yukawa sector as well, but only through two loops. As a result, this influence is reduced, and the Yukawa sector stays weakly coupled up to the Planck scale for all experimentally preferred values of the Higgs mass, \(M_H \leq 200\) GeV. Validity of the perturbative SM up to the Planck scale, the Yukawa sector including, as well as the vacuum stability require the Higgs mass to be \(M_H = (161.3 \pm 20.6)^{+4}_{-10}\) GeV and \(M_H \geq 140.7^{+10}_{-10}\) GeV. Here the \(M_H\) corridor is the theoretical one and the errors are produced by the top mass uncertainty. The allowed Higgs mass region is wider for a lower cutoff scale \(\Lambda\).

The inclusion of the fourth heavy chiral family qualitatively changes the mode of the SM realization. With the addition of the family, the strong
Figure 8: Two-loop running of the Higgs quartic coupling ($n_g = 4$) at $m_4 = 200$ GeV and $m_L/m_Q = 1/2$.

Figure 9: One- an two-loop self-consistency plots ($n_g = 4$): the allowed Higgs mass vs. the cutoff scale $\Lambda$ at $m_4 = 200$ GeV and $m_L/m_Q = 1/2$. 
coupling is driven in one loop by the Yukawa interactions. It transmits to the Higgs self-interactions at the one-loop order, too. Hence the strong coupling develops in both these sectors in parallel, and their coupling constants blow up at sufficiently low scales. As a result, the requirement of self-consistency of the perturbative SM as an underlying theory up to the Planck or GUT scale excludes the fourth chiral family. But as an effective theory, the SM allows the heavy chiral family with the mass up to 250 GeV depending on the Higgs mass and the cutoff scale. Under precision experiment restriction $M_H \leq 200$ GeV, the fourth chiral family, taken alone, is excluded. Though a pair of the chiral families constituting the vector-like one could still exist.

![Figure 10](image.png)

Figure 10: Two-loop self-consistency plot under the $y \leq 1.5$ restriction on the Yukawa couplings ($n_g = 4$).

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