The existence, the uniqueness and the functoriality of the perfect locality over a Frobenius $P$-category

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Abstract: Let $p$ be a prime, $P$ a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category. The question on the existence of a suitable category $\mathcal{L}^c$ extending the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ goes back to Dave Benson in 1994 [1]. In 2002 Carles Broto, Ran Levi and Bob Oliver formulate the existence and the uniqueness of the category $\mathcal{L}^c$ in terms of the annulation of an obstruction $3$-cohomology element and of the vanishing of a $2$-cohomology group, and they state a sufficient condition for the vanishing of these $n$-cohomology groups. Recently, Amy Chermak has proved the existence and the uniqueness of $\mathcal{L}^c$, moreover, in [11] we already show that $\mathcal{L}^c$ can be completed in a suitable category $\mathcal{L}$ extending $\mathcal{F}$ and here we prove some functoriality of this correspondence.

1. Introduction

1.1. Let $p$ be a prime, $P$ a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category [10]. The question on the existence of a suitable category $\mathcal{L}^c$ extending the full subcategory $\mathcal{F}^c$ of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ [10, §3] goes back to Dave Benson in 1994 [1]. Indeed, considering our suggestion of constructing a topological space from the family of classifying spaces of the $\mathcal{F}$-localizers — a family of finite groups indexed by the $\mathcal{F}$-selfcentralizing subgroups of $P$ we had just introduced at that time [8] — Benson, in his tentative construction, had foreseen the interest of this extension, actually as a generalization for Frobenius $P$-categories of our old $O$-locality for finite groups in [7].

1.2. In [2] Carles Broto, Ran Levi and Bob Oliver formulate the existence and the uniqueness of the category $\mathcal{L}^c$ in terms of the annulation of an obstruction $3$-cohomology element and of the vanishing of a $2$-cohomology group, respectively. They actually state a sufficient condition for the vanishing of the corresponding $n$-cohomology groups and moreover, assuming the existence of $\mathcal{L}^c$, they succeed on the construction of a classifying space.

1.3. As a matter of fact, if $G$ is a finite group and $P$ a Sylow $p$-subgroup of $G$, the corresponding Frobenius $P$-category $\mathcal{F}_G$ introduced in [7] admits an extension $\mathcal{L}_G$ defined over all the subgroups of $P$ where, for any pair of subgroups $Q$ and $R$ of $P$, the set of morphisms from $R$ to $Q$ is the following quotient set of the $G$-transporter

$$\mathcal{L}_G(Q, R) = T_G(R, Q)/\mathcal{O}^p(C_G(R))$$

1.3.1.
Analogously, in the general setting, if we are interested in some functoriality for our constructions, we need not only the existence of $\mathcal{L}^\pi$ but the existence of a suitable category $\mathcal{L}$ extending $\mathcal{F}$ and containing $\mathcal{L}^\pi$ as a full subcategory. Soon after [2], on the one hand we showed that the contravariant functor from $\mathcal{F}_G$ mapping $Q$ on $C_G(Q)/\mathcal{O}_P(C_G(Q))$ can be indeed generalized to a contravariant functor $\mathfrak{c}^b_F$ from any Frobenius $P$-category $\mathcal{F}$ (see 2.4 below); on the other hand, we already proved in [9] that the existence of the so-called perfect $\mathcal{F}^\pi$-locality $\mathcal{L}^\pi$ forces the existence of a unique extension $\mathcal{L}$ of $\mathcal{F}$ by $\mathfrak{c}^b_F$, called the perfect $\mathcal{F}$-locality.

1.4. Recently, Andrew Chermak [3] has proved the existence and the uniqueness of $\mathcal{L}^\pi$ via his objective partial groups, and Bob Oliver [6], following some of Chermak’s methods, has also proved for $n \geq 2$ the vanishing of the $n$-cohomology groups mentioned above. In reading their preprints, we were disappointed not only because their proofs depend on the so-called Classification of the finite simple groups (CFSG), but because in their arguments they need strong properties of the finite groups. Indeed, since [7] we are convinced that a previous classification of the so-called “local structures” will be the way to clarify CFSG in future versions; thus, our effort in creating the Frobenius $P$-categories was directed to provide a precise formal support to the vague notion of “local structures”, independently of “environmental” finite groups and of most of finite group properties.

1.5. Here we will show that till now our intuition was correct, namely that there is a direct proof of the existence and the uniqueness of $\mathcal{L}^\pi$; that is to say, a proof that can be qualified of inner or tautological in the sense that only pushes far enough the initial axioms of Frobenius $P$-categories. Moreover, as we mention above, the existence and the uniqueness of the perfect $\mathcal{F}^\pi$-locality $\mathcal{L}^\pi$ will guarantee the existence and the uniqueness of the perfect $\mathcal{F}$-locality $\mathcal{L}$ defined over all the subgroups of $P$ — which no longer can be described in terms of Chermak’s objective partial groups — and then it makes sense to discuss the functorial nature of the correspondence mapping $\mathcal{F}$ on $\mathcal{L}$.

1.6. Let us explain how our method works. In [11, Chap. 18] we introduced the $\mathcal{F}$-localizers mentioned above and, as a matter of fact, we already introduce the $\mathcal{F}$-localizer $L_F(Q)$ for any subgroup $Q$ of $P$ (see Theorem 2.10 below), which is indeed an extension of the group $\mathcal{F}(Q)$ of $\mathcal{F}$-automorphisms of $Q$ by the $p$-group $\mathfrak{c}^b_P(Q)$ (cf. 1.3). More generally, in [11, Chap. 17] we introduce the $\mathcal{F}$-localities as a wider framework where to look for the perfect $\mathcal{F}$-locality. Namely, considering the category $\mathcal{T}_P$ — where the objects are all the subgroups of $P$, the set of morphisms from $R$ to $Q$ is the $P$-transporter $T_P(R,Q)$, and the composition is induced by the product in $P$ — we call $\mathcal{F}$-locality any extension $\pi: \mathcal{L} \to \mathcal{F}$ of the category $\mathcal{F}$, endowed with a functor $\tau: \mathcal{T}_P \to \mathcal{L}$ such that the composition $\pi \circ \tau: \mathcal{T}_P \to \mathcal{F}$ is the canonical
functor defined by the conjugation in $P$; of course, we add some suitable conditions as divisibility and $p$-coherence (see 2.8 below). As a matter of fact, a perfect $\mathcal{F}$-locality is just a divisible $\mathcal{F}$-locality $\mathcal{L}$ where the group $\mathcal{L}(Q)$ of $\mathcal{L}$-automorphisms of any subgroup $Q$ of $P$ coincides with the $\mathcal{F}$-localizer of $Q$ (see 2.13 below).

1.7. It turns out that there are indeed other $\mathcal{F}$-localities — easier to construct — which deserve consideration; their construction depends on the existence of the $\mathcal{F}$-basic $P \times P$-sets $\Omega$ (see section 3 below) introduced in [11, Chap. 21], which allows the realization of $\mathcal{F}$ inside the symmetric group of $\Omega$ and then it allows the possibility of considering localities as defined in [7]. In [11, Chap. 22] we introduce the so-called basic $\mathcal{F}$-locality $\mathcal{L}^b$ which is canonically associated with $\mathcal{F}$ (see section 4 below). More precisely, in [11, Chap. 24] we show that the very structure of a perfect $\mathcal{F}^{sc}$-locality $\mathcal{L}^{sc}$ supplies particular $\mathcal{F}$-basic $P \times P$-sets.

1.8. From any of these $\mathcal{F}$-basic $P \times P$-sets $\Omega$ we can construct a particular $\mathcal{F}$-locality $\mathcal{L}^\Omega$ (see 4.7 below) in such a way that the full subcategory $\mathcal{L}^{\Omega,sc}$ of $\mathcal{L}^\Omega$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ admits a quotient $\mathcal{L}^{\Omega,sc}$ — independent of our choice — containing $\mathcal{L}^{sc}$ (see Corollary 5.20 below). The point is that these particular $\mathcal{F}$-basic $P \times P$-sets can be described directly, without assuming the existence of $\mathcal{L}^{sc}$ (see Proposition 3.4 below); hence, we can introduce the so-called natural $\mathcal{F}^{sc}$-locality $\mathcal{L}^{sc}$; then, the $\mathcal{F}^{sc}$-locality $\mathcal{L}^{sc}$ supplies a support for the proof of the existence and the uniqueness of the perfect $\mathcal{F}^{sc}$-locality. Moreover, the basic $\mathcal{F}$-locality $\mathcal{L}^b$ also admits a quotient $\mathcal{L}^b$ in such a way that the full subcategory $\mathcal{L}^{b,sc}$ of $\mathcal{L}^b$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ contains $\mathcal{L}^{b,sc}$ (see 4.13.3 below); this fact is the key point in order to discuss functoriality.

1.9. More explicitly, we replace the whole set of $\mathcal{F}$-selfcentralizing subgroups of $P$ by a nonempty set $\mathfrak{X}$ of them, containing any subgroup of $P$ which admits an $\mathcal{F}$-morphism from some subgroup in $\mathfrak{X}$, and replace the categories $\mathcal{F}^{sc}$ and $\mathcal{L}^{n,sc}$ by their respective full subcategories $\mathcal{F}^\mathfrak{X}$ and $\mathcal{L}^{n,\mathfrak{X}}$ over $\mathfrak{X}$; then, we prove the existence and the uniqueness of the perfect $\mathcal{F}^\mathfrak{X}$-sublocality $\mathcal{L}^\mathfrak{X}$ in $\mathcal{L}^{n,\mathfrak{X}}$ arguing by induction on $|\mathfrak{X}|$. The proof depends on the annulation of the cohomology classes of 2- and 1-cocycles, respectively; this annulation comes from the vanishing of the corresponding cohomology groups from an inductive argument† supported by a homotopically trivial complex. Since this homotopically trivial situation admits a wider framework, we develop it in [12].

† We thank Bob Oliver who pointing out a mistake in a previous argument.
1.10. At this point, we have the perfect $\mathcal{F}^\infty$-locality $\mathcal{L}^\infty$ as a $\mathcal{F}^\infty$-sublocality of $\dot{\mathcal{L}}^\infty$ and therefore, as a $\mathcal{F}^\infty$-sublocality of $\dot{\mathcal{L}}^\infty$ (cf. 1.8). But, as mentioned in 1.3 above, the existence of the perfect $\mathcal{F}^\infty$-locality $\mathcal{L}^\infty$ forces the existence of the perfect $\mathcal{F}$-locality $\mathcal{L}$ [11, Theorem 20.24]; more generally, for any $p$-coherent $\mathcal{F}$-locality $\dot{\mathcal{L}}$, denoting by $\dot{\mathcal{L}}^\infty$ the full subcategory of $\dot{\mathcal{L}}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $\mathcal{P}$, any $\mathcal{F}$-locality functor from $\dot{\mathcal{L}}^\infty$ to $\dot{\mathcal{L}}^\infty$ can be extended to a unique $\mathcal{F}$-locality functor from $\mathcal{L}$ to $\dot{\mathcal{L}}$ (see section 7 below, where we give a slightly different and more correct proof of the existence of $\mathcal{L}$). In particular, we get an $\mathcal{F}$-locality functor from the perfect $\mathcal{F}$-locality $\mathcal{L}$ to $\hat{\mathcal{L}}^b$; once again, a homotopically trivial situation exhibited in [12] allows us to show that this functor can be lifted to an essentially unique functor from $\mathcal{L}$ to the basic $\mathcal{F}$-locality $\hat{\mathcal{L}}$ (see Theorem 8.10 below).

1.11. Let $P'$ be another finite $p$-group, $\mathcal{F}'$ a Frobenius $P'$-category and $\alpha : P \rightarrow P'$ an $\mathcal{F}'$-$\mathcal{F}$-functorial group homomorphism [11, 12.1], so that it determines a so-called Frobenius functor

$$f_\alpha : \mathcal{F} \longrightarrow \mathcal{F}'$$

1.11.1; once we know the existence and the uniqueness of the respective perfect $\mathcal{F}$- and $\mathcal{F}'$-localities $\mathcal{L}$ and $\mathcal{L}'$, it is reasonable to ask for the existence and the uniqueness of a suitable isomorphism class of functors

$$l_\alpha : \mathcal{L} \longrightarrow \mathcal{L}'$$

1.11.2 lifting $f_\alpha$; but here we only get a positive answer for the quotients

$$\tilde{l}_\alpha : \tilde{\mathcal{L}} = \mathcal{L}/[e_f^0, e_f^0] \longrightarrow \tilde{\mathcal{L}}' = \mathcal{L}'/[e_f^0, e_f^0]$$

1.11.3; then, for a third finite $p$-group $P''$ together with a Frobenius $P''$-category $\mathcal{F}''$ and an $\mathcal{F}'$-$\mathcal{F}$-functorial group homomorphism $\alpha' : P' \rightarrow P''$, the functors $l_{\alpha'} \circ l_\alpha$ and $l_{\alpha' \circ \alpha}$ are naturally isomorphic (see section 9 below).

1.12. Actually, if $\alpha$ is surjective and $\mathcal{F}' = \mathcal{F}/\ker(\alpha)$ then the existence of $l_\alpha$ follows from [11, Theorem 17.18] where, assuming the existence of the perfect $\mathcal{F}$-locality $\mathcal{L}$, we exhibit a perfect $\mathcal{F}'$-locality $\mathcal{L}'$ and a functor $l_\alpha : \mathcal{L} \rightarrow \mathcal{L}'$ lifting $f_\alpha$. Thus, here we actually may assume that $\alpha$ is injective; in this case, we start by getting a relationship between the natural $\mathcal{F}^\infty$-locality $\dot{\mathcal{L}}^\infty$ and the basic $\mathcal{F}^\infty$-locality $\dot{\mathcal{L}}^\infty$; more explicitly, the converse image $\text{Res}_{\mathcal{F}^\infty}(\dot{\mathcal{L}}^\infty)$ of $\mathcal{F}^\infty$ in $\dot{\mathcal{L}}^\infty$ is clearly a $p$-coherent $\mathcal{F}^\infty$-locality and we will exhibit a $\mathcal{F}^\infty$-locality functor from $\dot{\mathcal{L}}^\infty$ to a suitable quotient $\mathcal{F}^\infty$-locality $\text{Res}_{\mathcal{F}^\infty}(\dot{\mathcal{L}}^\infty)$ (see Theorem 9.10 below). Finally, from the $\mathcal{F}$-locality functors

$$\dot{\mathcal{L}}^\infty \longrightarrow \dot{\mathcal{L}}^b \longrightarrow \text{Res}_{\mathcal{F}^\infty}(\dot{\mathcal{L}}^\infty) \leftarrow \text{Res}_{\mathcal{F}^\infty}(\dot{\mathcal{L}}^b) \leftarrow \text{Res}_{\mathcal{F}^\infty}(\mathcal{L})$$

1.12.1 we will obtain an $\mathcal{F}$-locality functor $\dot{\mathcal{L}}^\infty \longrightarrow \text{Res}_{\mathcal{F}^\infty}(\mathcal{L}')$ which can be extended to a $\mathcal{F}$-locality functor $\dot{\mathcal{L}} \longrightarrow \text{Res}_{\mathcal{F}}(\mathcal{L}')$ (see Theorem 9.15 below).

$\dagger$ Our argument in [11, 20.16] has been scratched.
2. Frobenius $P$-categories and coherent $\mathcal{F}$-localities

2.1. Denote by $i\mathfrak{Gr}$ the category formed by the finite groups and by the injective group homomorphisms. Recall that, for any category $\mathcal{C}$, we denote by $\mathcal{C}^o$ the opposite category and, for any $\mathcal{C}$-object $C$, by $\mathcal{C}_C$ (or $(\mathcal{C})_C$ to avoid confusion) the category of “$\mathcal{C}$-morphisms to $C$” [11, 1.7]. If any $\mathcal{C}$-object admits inner automorphisms then we denote by $\mathcal{C}$ the corresponding quotient and call it the exterior quotient of $\mathcal{C}$ [11, 1.3]. Let $p$ be a prime; for any finite $p$-group $P$ we denote by $\mathcal{F}_P$ the subcategory of $i\mathfrak{Gr}$ where the objects are all the subgroups of $P$ and where the morphisms are the group homomorphisms induced by conjugation by elements of $P$.

2.2. A Frobenius $P$-category $\mathcal{F}$ is a subcategory of $i\mathfrak{Gr}$ containing $\mathcal{F}_P$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [11, 2.8 and Proposition 2.11]

2.2.1 For any subgroup $Q$ of $P$, the inclusion functor $(\mathcal{F})_Q \to (i\mathfrak{Gr})_Q$ is full.

2.2.2 $\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.

2.2.3 Let $Q$ be a subgroup of $P$ fulfilling $\xi(C_P(Q)) = C_P(\xi(Q))$ for any $\mathcal{F}$-morphism $\xi: Q \cdot C_P(Q) \to P$, let $\varphi: Q \to P$ be an $\mathcal{F}$-morphism and let $R$ be a subgroup of $N_P(\varphi(Q))$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R(\varphi(Q))$ over $Q$ via $\varphi$. Then there is an $\mathcal{F}$-morphism $\zeta: R \to P$ fulfilling $\zeta(\varphi(u)) = u$ for any $u \in Q$.

As in [11, 1.2], for any pair of subgroups $Q$ and $R$ of $P$, we denote by $\mathcal{F}(Q, R)$ the set of $\mathcal{F}$-morphisms from $R$ to $Q$ and set $\mathcal{F}(Q) = \mathcal{F}(Q, Q)$. If $G$ is a finite subgroup admitting $P$ as a Sylow $p$-subgroup, we denote by $\mathcal{F}_G$ the Frobenius $P$-category where the morphisms are the group homomorphisms induced by the conjugation by elements of $G$.

2.3. Fix a Frobenius $P$-category $\mathcal{F}$; for any subgroup $Q$ of $P$ and any subgroup $K$ of the group $\text{Aut}(Q)$ of automorphisms of $Q$, we say that $Q$ is fully $K$-normalized in $\mathcal{F}$ if, for any $\mathcal{F}$-morphism $\xi: Q \cdot N^K_P(Q) \to P$, we have [11, 2.6]

$$\xi(N^K_P(Q)) = N^Q_P(\xi(Q))$$

2.3.1, where $N^K_P(Q)$ is the converse image of $K$ in $N_P(Q)$ via the canonical group homomorphism $N_P(Q) \to \text{Aut}(Q)$, and $\xi^K$ denotes the image in $\text{Aut}(\xi(Q))$ of $K$ via $\xi$. Recall that if $Q$ is fully $K$-normalized in $\mathcal{F}$ then we have a new Frobenius $N^K_P(Q)$-category $N^K_P(Q)$ [11, 2.14 and Proposition 2.16] where, for any pair of subgroups $R$ and $T$ of $N^K_P(Q)$, the set of morphisms $(N^K_P(Q))(R, T)$ is the set of group homomorphisms from $T$ to $R$ induced by the $\mathcal{F}$-morphisms $\psi: Q \cdot T \to Q \cdot R$ which stabilize $Q$ and induce on it an
element of $K$. Note that from [11, statement 2.13.2 and Corollary 5.14] it is not difficult to prove that if $T$ contains $Q\cdot P(Q)$ then the canonical map

$$(N_{\bar{F}}^F(Q))(R, T) \rightarrow T_{K\cap F(Q)}(F_T(Q), F_R(Q))$$

is surjective.

2.4. We denote by $H_{\bar{F}}$ the $F$-hyperfocal subgroup of $P$, which is the subgroup generated by the sets $\{u^{-1}\sigma(u)\}_{u \in Q}$ where $Q$ runs over the set of subgroups of $P$ and $\sigma$ over the set of $p'$-elements of $F(Q)$ [11, 13.2]. As above, for any subgroup $Q$ of $P$ fully centralized in $F$ — namely, fully $\{1\}$-normalized in $F$ — we have the Frobenius $C_P(Q)$-category $C_{\bar{F}}(Q) = N_{\bar{F}}^F(Q)$ and therefore we can consider the $C_{\bar{F}}(Q)$-hyperfocal subgroup $H_{C_{\bar{F}}}(Q)$ of $C_{\bar{F}}(Q)$; then, in [11, Proposition 13.14] we exhibit a unique contravariant functor

$$\text{c}_{\bar{F}}^h : F \rightarrow \text{Gr}$$

where $\text{Gr}$ denotes the exterior quotient of the category $\text{Gr}$ of finite groups (cf. 2.1), mapping any subgroup $Q$ of $P$ fully centralized in $F$ on the quotient $C_P(Q)/H_{C_{\bar{F}}(Q)}$ and any $F$-morphism $\varphi : R \rightarrow Q$ from a subgroup $R$ of $P$ fully centralized in $F$ on a $\text{Gr}$-morphism induced by an $F$-morphism

$$\varphi(R)\cdot C_P(Q) \rightarrow R\cdot C_P(R)$$

sending $\varphi(v)$ to $v$ for any $v \in R$ (cf. condition 2.2.3).

2.5. We say that a subgroup $U$ of $P$ is $F$-stable if we have $\varphi(Q \cap U) \subset U$ for any subgroup $Q$ of $P$ and any $F$-morphism $\varphi : Q \rightarrow P$; then, setting $\bar{P} = P/U$, there is a Frobenius $P$-category $\bar{F} = F/U$ such that the canonical homomorphism $\varphi : P \rightarrow \bar{P}$ is $(\mathcal{F}, \bar{F})$-functorial and that the corresponding Frobenius functor $f_{\bar{F}} : F \rightarrow \bar{F}$ is full over the subgroups of $P$ containing $U$ [11, Proposition 12.3]. In particular, if $Q$ is a subgroup of $P$ fully normalized in $F$, it follows from [11, Proposition 13.9] that $H_{C_{\bar{F}}(Q)}$ is an $N_{\bar{F}}(Q)$-stable subgroup of $N_P(Q)$ and therefore we can consider the quotients

$$\overline{N_P(Q)} = N_P(Q)/H_{C_{\bar{F}}(Q)} \quad \text{and} \quad \overline{N_{\bar{F}}(Q)} = N_{\bar{F}}(Q)/H_{C_{\bar{F}}(Q)}$$

2.6. We say that a subgroup $Q$ of $P$ is $F$-selfcentralizing if we have

$$C_P(\varphi(Q)) \subset \varphi(Q)$$

for any $\varphi \in F(P, Q)$; we denote by $\mathcal{F}^{sc}$ the full subcategory of $F$ over the set of $F$-selfcentralizing subgroups of $P$. More generally, as mentioned above we consider a nonempty set $\mathfrak{X}$ of subgroups of $P$ containing any subgroup of $P$ admitting an $F$-morphism from some subgroup in $\mathfrak{X}$, and then we denote by $\mathcal{F}^{\mathfrak{X}}$ the full subcategory of $F$ over the set $\mathfrak{X}$ of objects; in most situations, the subgroups in $\mathfrak{X}$ will be $F$-selfcentralizing and if $\mathfrak{X}$ is the set of all the $F$-selfcentralizing subgroups of $P$, we write $\text{sc}$ instead of $\mathfrak{X}$. }
2.7. Denote by $\mathcal{T}_P^X$ the full subcategory of $\mathcal{T}_P$ (cf. 1.6) over the set $\mathcal{X}$ and by $\kappa^x : \mathcal{T}_P^X \to \mathcal{F}^x$ the canonical functor determined by the conjugation. An $\mathcal{F}^x$-\textit{locality} $\mathcal{L}^x$ is a category, where $\mathcal{X}$ is the set of objects, endowed with two functors

$$\tau^x : \mathcal{T}_P^X \to \mathcal{L}^x \quad \text{and} \quad \pi^x : \mathcal{L}^x \to \mathcal{F}^x$$

which are the identity on the set of objects and fulfill $\pi^x \circ \tau^x = \kappa^x$, $\pi^x$ being full; as above, for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$, we denote by $\mathcal{L}^x(Q,R)$ the set of $\mathcal{L}^x$-morphisms from $R$ to $Q$ and by

$$\tau_{Q,R}^x : \mathcal{T}_P^x(Q,R) \to \mathcal{L}^x(Q,R) \quad \text{and} \quad \pi_{Q,R}^x : \mathcal{L}^x(Q,R) \to \mathcal{F}^x(Q,R)$$

the corresponding maps; we write $Q$ only once if $Q = R$.

2.8. We say that $\mathcal{L}^x$ is \textit{divisible} if, for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$, Ker($\pi^x_R$) acts regularly on the “fibers” of $\pi_{Q,R}^x$, and that $\mathcal{L}^x$ is \textit{coherent} if moreover for any $x \in \mathcal{L}^x(Q,R)$ and any $v \in R$ we have [11, 17.8 and 17.9]

$$x \cdot \tau_R^x(v) = \tau_Q^x\left( \left( \pi_{Q,R}^x(x) \right)(v) \right) \cdot x$$

more precisely, we say that $\mathcal{L}^x$ is $A$-\textit{coherent} if it is \textit{coherent} and Ker($\pi^x_Q$) is Abelian for any $Q \in \mathcal{X}$; in this case, the divisibility determines a functor

$$\ker(\pi^x) : \mathcal{F}^x \to \text{Ab}$$

sending any $Q \in \mathcal{X}$ to Ker($\pi^x_Q$). On the other hand, we say that $\mathcal{L}^x$ is $p$-\textit{coherent} if it is \textit{coherent} and, for any subgroup $Q$ in $\mathcal{X}$, the kernel Ker($\pi^x_Q$) is a $p$-group; in this case, it follows from [11, 17.13] that if $Q$ is fully centralized in $\mathcal{F}$ then we have

$$H_{C_{\mathcal{F}}(Q)} \subset \ker(\tau^x_Q)$$

we say that $\mathcal{L}^x$ is $P$-\textit{bounded} if it is \textit{coherent} and, for any subgroup $Q$ in $\mathcal{X}$ fully normalized in $\mathcal{F}$, we have Ker($\pi^x_Q$) $\subset$ $\tau^x_Q\left( N_P(Q) \right)$. Finally, we say that $\mathcal{L}^x$ is \textit{perfect} if it is $P$-\textit{bounded} and for any subgroup $Q$ in $\mathcal{X}$ fully centralized in $\mathcal{F}$ we have [11, 17.13]

$$H_{C_{\mathcal{F}}(Q)} = \ker(\tau^x_Q)$$

2.9. If $\mathcal{L}'^x$ is a second $\mathcal{F}^x$-\textit{locality} with structural functors $\tau'^x$ and $\pi'^x$, we call $\mathcal{F}^x$-\textit{locality functor} from $\mathcal{L}^x$ to $\mathcal{L}'^x$ any functor $\mathcal{L}^x \to \mathcal{L}'^x$ fulfilling

$$\tau'^x = \mathcal{L} \circ \tau^x \quad \text{and} \quad \pi'^x \circ \mathcal{L} = \pi^x$$

2.8.1;
the composition of two $\mathcal{F}^x$-locality functors is obviously an $\mathcal{F}^x$-locality functor; we say that two $\mathcal{F}^x$-locality functors $\Gamma^x$ and $\bar{\Gamma}^x$ from $\mathcal{L}^x$ to $\mathcal{L}'^x$ are naturally $\mathcal{F}^x$-isomorphic whenever we have a natural isomorphism $\lambda^x : \Gamma^x \cong \bar{\Gamma}^x$ such that $\pi^x * \lambda^x = \text{id}_{\pi^x}$; then, for any $Q \in \mathcal{X}$, $(\lambda^x)_Q$ belongs to $\text{Re}r(\pi'^x)$ and, since

$$\Gamma^x (\tau^x_{P,Q}(1)) = \tau^x_{\pi^x P,Q}(1) = \bar{\Gamma}^x (\tau^x_{\pi^x P,Q}(1))$$

2.9.2., if $\mathcal{L}^x$ is divisible then $\lambda^x$ is uniquely determined by $(\lambda^x)_P$; indeed, we have

$$(\lambda^x)_P \cdot \tau^x_{P,Q}(1) = \tau^x_{\pi^x P,Q}(1) \cdot (\lambda^x)_Q$$

2.9.3.

2.10. Once again, the composition of a natural $\mathcal{F}^x$-isomorphism between $\mathcal{F}^x$-locality functors with an $\mathcal{F}^x$-locality functor or with another such a natural $\mathcal{F}^x$-isomorphism is a natural $\mathcal{F}^x$-isomorphism between $\mathcal{F}^x$-locality functors. Note that if $\mathcal{L}^x$ and $\mathcal{L}'^x$ are $A$-coherent then any $\mathcal{F}^x$-locality functor $\Gamma^x : \mathcal{L}^x \rightarrow \mathcal{L}'^x$ determines a natural map

$$\nu^x : \text{Re}r(\pi^x) \longrightarrow \text{Re}r(\pi'^x)$$

2.10.1 which is clearly compatible with the restrictions to $\text{Re}r(\kappa^x)$ of $\tau^x$ and $\tau'^x$; in this case, it is quite clear that any subfunctor $\xi^x$ of $\text{Re}r(\pi^x)$ determines a quotient $\mathcal{F}$-locality $\mathcal{L}^x / \xi^x$ defined, for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$, by the quotient set

$$(\mathcal{L}^x / \xi^x)(Q,R) = \mathcal{L}^x(Q,R)/\xi^x(R)$$

2.10.2, and by the corresponding induced composition.

2.11. With the notation in 2.5.1, we are interested in the $\mathcal{N}_F(Q)$-locality $\mathcal{N}_F(Q)$, where the morphisms are the pairs formed by an $\mathcal{N}_F(Q)$-morphism and by an automorphism of $Q$, both determined by the same $F$-morphism [11, 18.3], and where the composition and the structural functors are the obvious ones. Similarly, if $L$ is a finite group acting on $Q$, we are interested in the $\mathcal{F}_L$-locality $\mathcal{F}_L(Q)$ where the morphisms are the pairs formed by an $\mathcal{F}_L$-morphism and by an automorphism of $Q$, both determined by the same element of $L$. We are ready to describe the $\mathcal{F}$-localizer of $Q$ [11, Theorem 18.6].

† We thank John Rognes who pointed out that the proof of [11, Lemma 18.8] only works whenever the normal subgroup $Q$ of $M$ is Abelian. To complete the proof of [11, Theorem 18.6], we replace the application of this lemma in page 342, by quoting [5, Proposition 4.9]; indeed, [5, condition 4.9.1] follows from [11, conditions 18.6.2 and 18.6.3], and [5, condition 4.9.2] follows from [11, condition 18.8.1].
Theorem 2.12. For any subgroup $Q$ of $P$ fully normalized in $F$ there is a triple formed by a finite group $L_F(Q)$ and by two group homomorphisms

$$
\tau_Q : N_P(Q) \longrightarrow L_F(Q) \quad \text{and} \quad \pi_Q : L_F(Q) \longrightarrow F(Q)
$$

2.12.1

such that $\pi_Q \circ \tau_Q$ is induced by the $N_P(Q)$-conjugation, that we have the exact sequence

$$
1 \longrightarrow H_{C_F(Q)} \longrightarrow C_P(Q) \xrightarrow{\tau_Q} L_F(Q) \xrightarrow{\pi_Q} F(Q) \longrightarrow 1
$$

2.12.2

and that $\pi_Q$ and $\tau_Q$ induce an equivalence of categories

$$
\overline{N_{F,Q}(Q)}^b \cong F_{L_F(Q),Q}
$$

2.12.3.

Moreover, for another such a triple $L'$, $\tau'_Q$ and $\pi'_Q$, there is a group isomorphism $\lambda : L_F(Q) \cong L'$, unique up to $C_F(Q)$-conjugation, fulfilling $\lambda \circ \tau_Q = \tau'_Q$ and $\pi'_Q \circ \lambda = \pi_Q$.

2.13. For any subgroup $Q$ of $P$ fully normalized in $F$, we call $F$-localizer of $Q$ any finite group $L$ endowed with two group homomorphisms as in 2.12.1 fulfilling the conditions 2.12.2 and 2.12.3. Note that, if $L^x$ is an $F^x$-locality then, for any $Q \in \mathfrak{X}$, the structural functors $\tau^x$ and $\pi^x$ determine two group homomorphisms (cf. 2.7.2)

$$
\tau^x_Q : N_P(Q) \longrightarrow L^x(Q) \quad \text{and} \quad \pi^x_Q : L^x(Q) \longrightarrow F(Q)
$$

2.13.1

and $\pi^x_Q$ is surjective; in particular, if $Q$ is fully normalized in $F$ then, since $F_P(Q)$ is a Sylow $p$-subgroup of $F(Q)$ [11, Proposition 2.11], $\tau^x_Q(N_P(Q))$ is a Sylow $p$-subgroup of $L^x(Q)$ if and only if it contains a Sylow $p$-subgroup of $\text{Ker}(\pi^x_Q)$. Hence, if $L^x$ is divisible and for any $Q \in \mathfrak{X}$ fully normalized in $F$ the group $L^x(Q)$ endowed with $\tau^x_Q$ and $\pi^x_Q$ is an $F$-localizer of $Q$, it is easily checked from [11, Proposition 17.10] that $L^x$ is $p$-coherent and therefore that it is a perfect $F^x$-locality (cf. 2.8). Actually, the converse statement is true and it is easily checked from [11, Proposition 18.4].

2.14. We also need the $F$-localizing functor $\text{loc}_F$— defined in [11, 18.12.1 and Proposition 18.19] — from the proper category of chains $\mathfrak{ch}^*(F)$ of $F$. Explicitly, for any $F$-chain $q : \Delta_n \rightarrow F$ [11, A2.8], we denote by $F(q)$ the subgroup of elements in $F(q(n))$ which can be lifted to an automorphism of $q$; then, if $q$ is fully normalized in $F$ [11, 2.18], we consider the $F$-localizer $L_F(q(n))$ of $q(n)$ and denote by $L_F(q)$ and by $N_F(q)$ the respective converse
images of $F(q)$ in $L_F(q)$ and in $N_P(q(n))$, endowed with the suitable group homomorphisms

$$
\tau_q : N_P(q) \longrightarrow L_F(q) \quad \text{and} \quad \pi_q : L_F(q) \longrightarrow F(q)
$$

2.14.1; actually, in the construction of $\text{loc}_F$ we only can consider the Abelian part of the extension $\pi_q$, namely the quotient

$$
\overline{\pi}_q : \bar{L}_F(q) = L_F(q) / \left[ \text{Ker}(\pi_q), \text{Ker}(\pi_q) \right] \longrightarrow F(q)
$$

2.14.2.

2.15. Moreover, we denote by $\mathcal{Loc}$ the category where the objects are the pairs $(L, Q)$ formed by a finite group $L$ and a normal $p$-subgroup $Q$ of $L$, and where the morphisms from $(L, Q)$ to $(L', Q')$ are the group homomorphisms $f : L \rightarrow L'$ fulfilling $f(Q) \subset Q'$ [11, 18.12]; this category has an obvious inner structure [11, 1.3] mapping any object $(L, Q)$ on the subgroup of the group of automorphisms of $L$ determined by the $Q$-conjugation, and we denote by $\overline{\mathcal{Loc}}$ the corresponding exterior quotient and by $\text{lv} : \overline{\mathcal{Loc}} \rightarrow \mathcal{Gr}$ the functor sending $(L, Q)$ to $L/Q$. Then, it follows from [11, Proposition 18.19] that there is a suitable functor

$$
\text{loc}_F : \text{ch}^* (F) \longrightarrow \overline{\mathcal{Loc}}
$$

2.15.1 mapping any $F$-chain $q : \Delta_n \rightarrow F$ fully normalized in $F$ [11, 2.18] on the pair $(L_F(q), \text{Ker}(\overline{\pi}_q))$.

2.16. But, for a $p$-coherent $F^X$-locality $L^X$ it follows from [11, Proposition A2.10] that we have a functor

$$
\text{aut}_{L^X} : \text{ch}^*(L^X) \longrightarrow \mathcal{Gr}
$$

2.16.1 mapping any $L^X$-chain $\hat{q} : \Delta_n \rightarrow L^X$ on the group $L^X(\hat{q})$ of all the automorphisms of $\hat{q}$; moreover, if we assume that $\text{Ker}(\pi^X_q)$ is Abelian for any $Q \in X$, then $\text{aut}_{L^X}$ induces an obvious functor

$$
\text{loc}_{L^X} : \text{ch}^*(F^X) \longrightarrow \overline{\mathcal{Loc}}
$$

2.16.2 mapping any $F^X$-chain $q : \Delta_n \rightarrow F^X$ on the pair $(L^X(\hat{q}), \text{Ker}(\pi^X_q))$ for a $L^X$-chain $\hat{q} : \Delta_n \rightarrow L^X$ lifting $q$; here, we are interested in the following $X$-relative version of [11, Proposition 18.21].

**Proposition 2.17.** If $L^X$ is a $p$-coherent $F^X$-locality such that $\text{Ker}(\pi^X_q)$ is Abelian for any $Q \in X$, then there is a unique natural map

$$
\lambda_{L^X} : \text{loc}_{F^X} \longrightarrow \text{loc}_{L^X}
$$

such that $\text{lv} * \lambda_{L^X} = \text{id}_{\text{aut}_{F^X}}$ and that, for any $F^X$-chain $q : \Delta_n \rightarrow F^X$ fully normalized in $F^X$, we have $(\lambda_{L^X})_q \circ \overline{\tau}^X_q = \overline{\tau}^X_q \circ \tau^X_q$. 
3. The natural $\mathcal{F}$-basic $P \times P$-sets

3.1. Recall that a basic $P \times P$-set [11, 21.4] is a finite nonempty $P \times P$-set $\Omega$ such that $\{1\} \times P$ acts freely on $\Omega$, that we have

$$\Omega^\circ \cong \Omega \quad \text{and} \quad |\Omega|/|P| \not\equiv 0 \mod p$$

where we denote by $\Omega^\circ$ the $P \times P$-set obtained by exchanging both factors, and that, for any subgroup $Q$ of $P$ and any group homomorphism $\varphi : Q \to P$ such that $\Omega$ contains a $P \times P$-subset isomorphic to $(P \times P)/\Delta_\varphi(Q)$, we have a $Q \times P$-set isomorphism

$$\text{Res}_{\varphi \times \text{id}_P}(\Omega) \cong \text{Res}_{i_Q^P \times \text{id}_P}(\Omega)$$

where, for any pair of group homomorphisms $\varphi$ and $\varphi'$ from $Q$ to $P$, we set

$$\Delta_{\varphi,\varphi'}(Q) = \{(\varphi(u), \varphi'(u)) \mid u \in Q\} \quad \text{and} \quad \Delta_{\varphi}(Q) = \Delta_{\varphi, i_Q^P}(Q)$$

and denote by $i_Q^P$ the corresponding inclusion map.

3.2. Then, for any pair of subgroups $Q$ and $R$ of $P$, denoting by $\mathcal{F}^{Q}(Q,R)$ the set of group homomorphisms $\varphi : R \to P$ such that $\varphi(R) \subset Q$ and $\varphi \in \mathcal{F}(P,R)$, it follows from [11, Proposition 21.9] that $\mathcal{F}$ is a Frobenius $P$-category. Moreover, if $\mathcal{F}$ is a Frobenius $P$-category, let us say that $\Omega$ is a $\mathcal{F}$-basic $P \times P$-set whenever $\mathcal{F}^{\Omega} = \mathcal{F}$; then, it follows from [11, Proposition 21.12] that any Frobenius $P$-category $\mathcal{F}$ admits an $\mathcal{F}$-basic $P \times P$-set.

3.3. From now on, we fix a Frobenius $P$-category $\mathcal{F}$ and a nonempty set $X$ of subgroups of $P$ as in 2.6 above; more generally, we say that a $P \times P$-set $\Omega^x$ is $\mathcal{F}^x$-basic if it fulfills condition 3.1.1 and the statement [11, 21.7]

3.3.1 The stabilizer of any element of $\Omega^x$ coincides with $\Delta_{\psi,\psi'}(R)$ for some $R \in X$ and suitable $\psi, \psi' \in \mathcal{F}(P,R)$, and we have

$$|(\Omega^x)^{\Delta_{\varphi,\varphi'}(Q)}| = |(\Omega^x)^{\Delta_{\varphi}(Q)}|$$

for any $Q \in X$ and any $\varphi, \varphi' \in \mathcal{F}(P,Q)$. Recall that, according to [11, Proposition 21.12], for any $\mathcal{F}^x$-basic $P \times P$-set $\Omega^x$ there is an $\mathcal{F}$-basic $P \times P$-set $\Omega$ containing $\Omega^x$ and fulfilling

$$\Omega^{\Delta_{\varphi}(Q)} = (\Omega^x)^{\Delta_{\varphi}(Q)}$$

for any $Q \in X$ and any $\varphi \in \mathcal{F}(P,Q)$. In order to describe the $\mathcal{F}$-basic $P \times P$-set $\Omega$ announced in 1.8 above, we need the notation of [11, Chap. 6] which we actually recall in section 5 below (cf. 5.3.1).
Proposition 3.4. Assume that any element of \(X\) is \(F\)-selfcentralizing. Then, the \(P \times P\)-set
\[
\Omega^X = \bigsqcup_Q \bigsqcup (P \times P) / \Delta_\varphi(Q)
\]
where \(Q\) runs over a set of representatives for the set of \(P\)-conjugacy classes in \(X\) and \(\tilde{\varphi}\) runs over a set of representatives for the set of \(\tilde{F}_P(Q)\)-orbits in \(\tilde{F}(P,Q)\tilde{\iota}_Q^P\), is an \(\tilde{F}^X\)-basic \(P \times P\)-set which fulfills \(|\Omega^X\Delta(Q)| = |Z(Q)|\) for any \(Q \in X\).

Proof: Since we clearly have
\[
(\Omega^X)^0 \cong \Omega^X \quad \text{and} \quad |\Omega^X / P| \equiv |\tilde{F}(P)| \mod p
\]
it suffices to check that, for any \(R \in X\) and any \(\psi \in \tilde{F}(P,R)\), we have
\[
|((\Omega^X)^\Delta(R))| = |Z(R)|
\]
but, for any subgroup \(Q\) of \(P\) and any \(\varphi \in \tilde{F}(P,R)\), \(\Delta_\varphi(R)\) fixes the class of \((u,v) \in P \times P\) in \((P \times P)/\Delta_\varphi(Q)\) if and only if it is contained in \(\Delta_\varphi(Q)^{(u,v)}\) or, equivalently, we have
\[
vRv^{-1} \subset Q \quad \text{and} \quad \varphi(vu\psi^{-1}) = u\varphi(w)u^{-1} \text{ for any } w \in R
\]
which amounts to saying that the following \(\tilde{F}\)-diagram is commutative
\[
\begin{array}{ccc}
P & \xrightarrow{\kappa} & P \\
\| & \kappa & \| \\
Q & \xleftarrow{\tilde{\psi}} & P \\
\tilde{\kappa}_{Q,R}(v) & \xrightarrow{\tilde{\iota}_R^P} & P \\
\tilde{\psi} & \xleftarrow{\kappa_{Q,R}(v)} & R \\
\end{array}
\]
where \(\kappa_{Q,R}(v): R \rightarrow Q\) is the group homomorphism determined by the conjugation by \(v\).

Since \(\tilde{\varphi}\) belongs to \(\tilde{F}(P,Q)\tilde{\iota}_Q^P\) (see [11, 6.4.1] or 5.1.1 below), it follows from [11, Proposition 6.7] that the pair \((\tilde{\psi}, \tilde{\iota}_R^P)\) determines the isomorphism class of the \((\tilde{F}^*)_{R}\)-object (cf. 2.1)
\[
\tilde{\kappa}_{Q,R}(v): R \rightarrow Q
\]
that is to say, if \((u',v') \in P \times P\) is another element such that \(\Delta_\varphi(Q)^{(u',v')}\) contains \(\Delta_\varphi(R)\), we have \(v' = sv\) for some \(s \in Q\) and therefore we get
\[
\psi(w) = u^{-1} \varphi(suv^{-1}w^{-1})u' = \varphi(vwv^{-1})\varphi(s)^{-1}u'
\]
for any \(w \in R\); at this point, it follows from [11, Proposition 4.6] that, for a suitable \(z \in Z(R)\), we have \(\varphi(s)^{-1}u' = uz\), which proves our claim.
3.5. If any element of $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing, we call $\Omega^\mathfrak{X}$ the natural $\mathcal{F}^\mathfrak{X}$-basic $P \times P$-set. Recall that we say that an $\mathcal{F}$-basic $P \times P$-set $\Omega$ is thick if the multiplicity of the indecomposable $P \times P$-set $(P \times P)/\Delta_{\varphi}(Q)$ is at least two for any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$ [11, 21.4]. Let us call natural any $\mathcal{F}$-basic $P \times P$-set $\Omega$ which fulfills

$$|\Omega^{\Delta_{\varphi}(Q)}| = |Z(Q)|$$

for any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$, and is thick outside of the set of $\mathcal{F}$-selfcentralizing subgroups of $P$, namely the multiplicity of $(P \times P)/\Delta_{\varphi}(R)$ is at least two if $R$ is not $\mathcal{F}$-selfcentralizing; the existence of natural $\mathcal{F}$-basic $P \times P$-sets follows from Proposition 3.4 together with [11, Proposition 21.12].

3.6. Let $\Omega$ be an $\mathcal{F}$-basic $P \times P$-set and $Q$ a subgroup of $P$; it follows from our definition in 3.2 that any $Q \times P$-orbit in $\text{Res}_{Q \times P}(\Omega)$ is isomorphic to the quotient set $(Q \times P)/\Delta_{\varphi}(T)$ (cf. 3.1.3) for some subgroup $T$ of $P$ and some $\eta \in \mathcal{F}(Q,T)$; note that the isomorphism class of this $Q \times P$-set $(Q \times P)/\Delta_{\varphi}(T)$ only depends on the conjugacy class of $T$ in $P$ and on the class $\bar{\eta}$ of $\eta$ in $\tilde{\mathcal{F}}(Q,T)$; moreover, it is quite clear that $\tilde{N}_{Q \times P}(\Delta_{\varphi}(T))$ acts regularly on $((Q \times P)/\Delta_{\varphi}(T))^{\Delta_{\varphi}(T)}$ and that we have a group isomorphism

$$\text{Aut}_{Q \times P}((Q \times P)/\Delta_{\varphi}(T)) \cong \tilde{N}_{Q \times P}(\Delta_{\varphi}(T))$$

### Proposition 3.7

Let $\Omega$ be a natural $\mathcal{F}$-basic $P \times P$-set, $Q$ and $T$ a pair of $\mathcal{F}$-selfcentralizing subgroups of $P$ and $\eta$ an element of $\mathcal{F}(Q,T)$. Then, the multiplicity of $(Q \times P)/\Delta_{\varphi}(T)$ in $\text{Res}_{Q \times P}(\Omega)$ is at most one, and it is one if and only if $\bar{\eta}$ belongs to $\tilde{\mathcal{F}}(Q,T)_{\bar{T}}$. Moreover, in this case we have

$$\text{Aut}_{Q \times P}((Q \times P)/\Delta_{\varphi}(T)) \cong Z(T)$$

and the multiplicity of $(Q \times P)/\Delta_{\varphi}(T)$ in any $\mathcal{F}$-basic $P \times P$-set $\Omega'$ is at least one.

**Proof:** According to our definition (cf. 3.5.1), we have

$$|\Omega^{\Delta_{\varphi}(T)}| = |Z(T)|$$

hence, if the multiplicity of $(Q \times P)/\Delta_{\varphi}(T)$ in $\text{Res}_{Q \times P}(\Omega)$ is not zero, then it is one and we have (cf. 3.6)

$$|\tilde{N}_{Q \times P}(\Delta_{\varphi}(T))| \leq |Z(T)|$$

which forces isomorphism 3.7.1. In this case, since $N_{Q \times P}(\Delta_{\varphi}(T))$ covers the intersection $\mathcal{F}_{Q}(\eta(T)) \cap \eta\mathcal{F}_{P}(T)$ where $\eta\mathcal{F}_{P}(T)$ is the image of $\mathcal{F}_{P}(T)$ in $\text{Aut}(\eta(T))$ via $\eta$ (cf. 2.3), it follows from [11, 6.5] that $\bar{\eta}$ belongs to $\tilde{\mathcal{F}}(P,T)_{\bar{T}}$. 

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Moreover, for any $\mathcal{F}$-basic $P \times P$-set $\Omega'$, denoting by $\mathfrak{T}$ the set of subgroups $R$ of $Q$ such that $\mathcal{F}(R, T) \neq \emptyset$ and $|R| \neq |T|$, and by $\Omega'^{\mathfrak{T}}$ the subset of $\omega' \in \Omega'$ such that the stabilizer $(Q \times P)_{\omega'}$ coincides with $\Delta_{\omega}(R)$ for some $R \in \mathfrak{T}$, it is quite clear that $\tilde{\eta} \in \tilde{\mathcal{F}}(Q, T)_{\omega'}$ forces $(\Omega'^{\mathfrak{T}})_{\Delta_{\omega}(T)} = \emptyset$; since $\Omega'^{\Delta(T)}$ is clearly not empty, this proves the last statement.

4. Construction of $\mathcal{F}$-localities from $\mathcal{F}$-basic $P \times P$-sets

4.1. Let $\Omega$ be an $\mathcal{F}$-basic $P \times P$-set and denote by $G$ the group of permutations of $\text{Res}_{\Omega} \times P(\Omega)$; it is clear that we have an injective map from $P \times \{1\}$ into $G$; we identify this image with the $p$-group $P$ so that, from now on, $P$ is contained in $G$ and acts freely on $\Omega$. Recall that, for any pair of subgroups $Q$ and $R$ of $P$, we have (cf. 3.2)

$$T_G(R, Q)/C_G(R) \cong \mathcal{F}(Q, R)$$

where $T_G(R, Q)$ is the $G$-transporter from $R$ to $Q$.

4.2. Let $Q$ be a subgroup of $P$; clearly, the centralizer $C_G(Q)$ coincides with the group of permutations of $\text{Res}_{Q \times P}(\Omega)$ and therefore, denoting by $\mathfrak{D}_{\Omega, Q}$ the set of isomorphism classes of $Q \times P$-orbits of $\Omega$, by $k_{\Omega}$ the number of $Q \times P$-orbits of isomorphism class $\tilde{O} \in \mathfrak{D}_{\Omega, Q}$ in $\Omega$, by $\mathfrak{S}_{k_{\Omega}}$ the corresponding $k_{\Omega}$-symmetric group and by $\text{Aut}(O)$ the group of $Q \times P$-set automorphisms of $O \in \tilde{O}$, it is easily checked that we have a canonical $\mathfrak{S}_{\text{tr}}$-isomorphism [11, 22.5.1]

$$\tilde{\omega}_Q : C_G(Q) \cong \prod_{\tilde{O} \in \mathfrak{D}_{\Omega, Q}} \text{Aut}(O) \cdot \mathfrak{S}_{k_{\Omega}}$$

More precisely, as in [11, Proposition 22.11], for any subgroup $R$ of $Q$ we have a commutative $\mathfrak{S}_{\text{tr}}$-diagram

$$\begin{array}{ccc}
C_G(Q) & \longrightarrow & C_G(R) \\
\uparrow & & \uparrow \\
\prod_{\tilde{O} \in \mathfrak{D}_{\Omega, Q}} \mathfrak{S}_{k_{\Omega}} & \longrightarrow & \prod_{\tilde{M} \in \mathfrak{D}_{\Omega, R}} \mathfrak{S}_{k_{\tilde{M}}} \\
\end{array}$$

4.3. As in [11, Proposition 22.7], let us denote by $\mathfrak{S}_\Omega(Q)$ the minimal normal subgroup of $C_G(Q)$ containing $(\omega_{\Omega})^{-1} \left( \prod_{\tilde{O} \in \mathfrak{D}_{\Omega, Q}} \mathfrak{S}_{k_{\Omega}} \right)$ for a representative $\omega_{\Omega}$ of $\tilde{\omega}_Q$; then, denoting by $\mathfrak{D}_{\Omega, Q}^{1}$ the subset of isomorphism classes $\tilde{O} \in \mathfrak{D}_{\Omega, Q}$ with multiplicity one in $\Omega$ and by $ab(\text{Aut}(O))$ the maximal Abelian quotient of $\text{Aut}(O)$, it follows from [11, Lemma 22.8] that

$$C_G(Q)/\mathfrak{S}_\Omega(Q) \cong \prod_{\tilde{O} \in \mathfrak{D}_{\Omega, Q}^{1}} \text{Aut}(O) \times \prod_{\tilde{O} \in \mathfrak{D}_{\Omega, Q} - \mathfrak{D}_{\Omega, Q}^{1}} ab(\text{Aut}(O))$$
let us denote by $S_Ω(Q)$ the converse image in $C_G(Q)$ of the commutator subgroup of this quotient, so that we have

$$C_G(Q)/S_Ω(Q) \cong \prod_{\tilde{O} \in D_{Ω,Q}} ab(Aut(O)) \quad 4.3.2.$$  

4.4. Although in [11, Chap. 22] we assume that $Ω$ is thick (cf. 3.5), it is easily checked that the elementary arguments in [11, Proposition 22.11] still prove that, for any subgroup $R$ of $Q$, we have

$$S_Ω(Q) \subset S_Ω(R) \quad 4.4.1$$

and therefore we still get

$$S_Ω^1(Q) \subset S_Ω^1(R) \quad 4.4.2;$$

as a matter of fact, these arguments do not depend on the conditions 3.1.1. Thus, as in [11, 22.13], we get a contravariant functor

$$\tilde{c}_Ω : \tilde{F} \rightarrow Ab \quad 4.4.3$$

mapping any subgroup $Q$ of $P$ on the Abelian group

$$\tilde{c}_Ω(Q) = C_G(Q)/S_Ω^1(Q) \cong \prod_{\tilde{O} \in D_{Ω,Q}} ab(Aut(O)) \quad 4.4.4$$

and any $\tilde{F}$-morphism $\tilde{\varphi} : R \rightarrow Q$ on the group homomorphism

$$\tilde{c}_Ω(\tilde{\varphi}) : C_G(Q)/S_Ω^1(Q) \rightarrow C_G(R)/S_Ω^1(R) \quad 4.4.5$$

induced by conjugation in $G$ by any element $x \in T_G(R, Q)$ lifting $\tilde{\varphi}$ (cf. 4.1.1).

4.5. More precisely, for any $\tilde{F}$-morphism $\tilde{\varphi} : R \rightarrow Q$ and any $\tilde{M} \in D_{Ω,R}$, we consider the (possibly empty) set of the injective $R \times P$-set homomorphisms

$$f : M \rightarrow Res_{\varphi \times id_P}(O) \quad 4.5.1,$$

for $M \in \tilde{M}$, $O \in \tilde{O}$ and $\varphi \in \tilde{\varphi}$; it is clear that $Aut(M) \times Aut(O)$ acts on this set by left- and right-hand composition and let us denote by $T^0_M(\tilde{\varphi})$ a set of representatives for the set of $Aut(M) \times Aut(O)$-orbits. Then, if $f$ is such an injective $R \times P$-set homomorphism, denoting by $Aut(O)_{f}$ the stabilizer of $f(M)$ in $Aut(O)$, we get an obvious group homomorphism

$$\delta_f : Aut(O)_{f} \rightarrow Aut(M) \quad 4.5.2$$

and we denote by $\varepsilon_f : Aut(O)_{f} \rightarrow Aut(O)$ the corresponding inclusion group homomorphism; we are interested in the maximal Abelian quotients of these groups; explicitly, we denote by

$$\text{ab}(Aut(O)) \quad \text{ab}(Aut(M)) \quad \text{ab}(Aut(O)_{f}) \quad 4.5.3.$$
the group homomorphisms respectively determined by the \textit{transfert} induced by $\varepsilon_f$, and by $\delta_f$. With all this notation, from \cite[Proposition 22.17]{11} we get the following description of $\tilde{c}^G(\tilde{\phi})$

**Proposition 4.6.** For any $\tilde{\mathcal{F}}$-morphism $\tilde{\phi}: R \to Q$, we have

$$
\tilde{c}^G(\tilde{\phi}) = \sum_{\tilde{O} \in \tilde{\mathcal{O}}_{G,Q}} \sum_{\tilde{M} \in \tilde{\mathcal{O}}_{G,R}} \sum_{f \in \tilde{\mathcal{I}}_{\tilde{M}}} \tilde{ab}(\delta_f) \circ \tilde{ab}(\varepsilon_f)
$$

4.7. Then, the correspondence sending any pair of subgroups $Q$ and $R$ of $P$ to the quotient set

$$
\mathcal{L}^G(Q,R) = T_G(R,Q)/\mathcal{S}^G(R)
$$

endowed with the canonical maps

$$
\tau^G_{Q,R} : \mathcal{T}_P(Q,R) \to \mathcal{L}^G(Q,R) \quad \text{and} \quad \pi^G_{Q,R} : \mathcal{L}^G(Q,R) \to \mathcal{F}(Q,R)
$$

defines a $p$-coherent $\mathcal{F}$-locality $(\tau^G, \mathcal{L}^G, \pi^G)$. Indeed, from inclusion 4.4.2 it is not difficult to check that, for any triple of subgroups $Q$, $R$ and $T$ of $P$, the product in $G$ induces a map

$$
\mathcal{L}^G(Q,R) \times \mathcal{L}^G(R,T) \to \mathcal{L}^G(Q,T)
$$

then, it is quite clear that these maps determine a \textit{composition} in the correspondence $\mathcal{L}^G$ defined above and that the canonical maps in 4.7.2 define structural functors

$$
\tau^G : \mathcal{T}_P \to \mathcal{L}^G \quad \text{and} \quad \pi^G : \mathcal{L}^G \to \mathcal{F}
$$

moreover, the \textit{divisibility} and the \textit{coherence} of $\mathcal{L}^G$ (cf. 2.8) are easy consequences of the fact that $G$ is a group, whereas the $p$-\textit{coherence} follows from isomorphisms 3.6.1 and 4.3.2.

4.8. Let us denote by $P\text{-}\mathcal{S}et$ the category of finite $P$-sets endowed with the \textit{disjoint union} and with the \textit{inner} direct product mapping any pair of finite $P$-sets $X$ and $Y$ on the $P$-set — still noted $X \times Y$ — obtained from the restriction of the $P \times P$-set $X \times Y$ through the \textit{diagonal} map $\Delta : P \to P \times P$; as in \cite[2.5]{13}, let us consider the functor

$$
\mathfrak{f}_\Omega : P\text{-}\mathcal{S}et \to P\text{-}\mathcal{S}et
$$

mapping any $P$-set $X$ on the $P$-set noted $\Omega \times_P X$ and, as in \cite[3.5]{13}, we consider the positive integer

$$
\ell(\mathfrak{f}_\Omega) = \frac{|\mathfrak{f}_\Omega(P)|}{|P|} = |\Omega/P|
$$

recall that if $\alpha : P' \to P$ is a group homomorphism then the \textit{restriction} defines a functor \cite[2.3]{13}

$$
\text{res}_\alpha : P\text{-}\mathcal{S}et \to P'\text{-}\mathcal{S}et
$$
As a matter of fact, for any pair of subgroups \( Q \) and \( R \) of \( P \), we have a canonical bijection [13, 2.5.2]
\[
T^c_G(R, Q) \cong \mathfrak{Nat}^* (\res_{\ell} \circ f, \res_{\ell} \circ g \circ f) \quad 4.8.4
\]
where \( T^c_G(R, Q) \) denotes the converse image of \( \varphi \in \mathcal{F}(Q, R) \) in \( T_G(R, Q) \) via the bijection 4.1.1 and, for any pair of functors \( f \) and \( g \) from \( R\text{-}\mathcal{S}et \) to \( R\text{-}\mathcal{S}et \), \( \mathfrak{Nat}^*(f, g) \) denotes the set of of natural isomorphisms from \( f \) to \( g \): for short, we set \( \mathfrak{Nat}^*(f) = \mathfrak{Nat}^*(f, f) \). In particular, we have
\[
C_G(R) \cong \mathfrak{Nat}^*(\res_{\ell} \circ f) \quad 4.8.5
\]
and the image of \( \mathfrak{S}_\Omega(R) \) is easy to describe.

4.9. Recall that, for any \( P\text{-}\mathcal{S}et \) \( X \), the correspondence sending any \( P\text{-}\mathcal{S}et \) \( Y \) to the \( P\text{-}\mathcal{S}et \) \( X \times Y \) and any \( P\text{-}\mathcal{S}et \) map \( g : Y \to Y' \) to the \( P\text{-}\mathcal{S}et \) map
\[
\id_X \times g : X \times Y \to X \times Y'
\]
defines a functor preserving disjoint unions [13, 2.9]
\[
m_X : P\text{-}\mathcal{S}et \to P\text{-}\mathcal{S}et
\]
recall that \( (m_X)^\circ = m_X \) [13, 5.3 and 6.1] and that \( \ell(m_X) = |X| \) [13, 5.3]; we say that \( X \) is \( \mathcal{F}\text{-}stable \) if we have
\[
\res_{\varphi}(X) = \res_{\varphi}(X)
\]
for any subgroup \( Q \) of \( P \) and any \( \varphi \in \mathcal{F}(P, Q) \).

**Proposition 4.10.** For any \( \mathcal{F}\text{-}stable \) \( P\text{-}\mathcal{S}et \) \( X \) such that \( p \) does not divide \( |X| \), there are an \( \mathcal{F}\text{-}basic \) \( P \times P\text{-}\mathcal{S}et \) \( \Omega' \) containing \( \Omega \) and fulfilling \( f_{\Omega'} = f_{\Omega} \circ m_X \), an \( \mathcal{F}\text{-}locality functor \( L^\alpha \to L^{\alpha'} \) and an injective natural map \( \ell : \ell^\alpha \to \ell^{\alpha'} \).

**Proof:** From the very definition of a \( \mathcal{F}\text{-}basic \) \( P \times P\text{-}\mathcal{S}et \), it is easily checked that \( f_{\Omega} \) is \( \mathcal{F}\text{-}stable \) [13, 6.3] and that \( p \) does not divide \( \ell(f_{\Omega}) \); hence, it follows from [13, Theorem 6.6] that \( f_{\Omega} \) and \( m_X \) centralizes each other and, with the notation there, that we have \( m_X = \ell + \varphi \) for some \( \ell \in \mathcal{K}_H \) and \( \varphi \in \mathcal{H}_E \); then, it follows from [13, 2.8 and Proposition 6.5] that we get
\[
f_{\Omega} \circ m_X = m_X \circ f_{\Omega} = \varphi \circ f_{\Omega} \quad \text{and} \quad (f_{\Omega} \circ m_X)^\circ = f_{\Omega} \circ m_X
\]
so that \( f_{\Omega} \circ m_X \) is also \( \mathcal{F}\text{-}stable \); moreover, since [13, 5.3]
\[
\ell(f_{\Omega} \circ m_X) = \ell(f_{\Omega})\ell(m_X)
\]
p does not divide \( \ell(f_{\Omega} \circ m_X) \). Consequently, if \( \Omega' \) is the unique \( \mathcal{F}\text{-}stable \) \( P \times P\text{-}\mathcal{S}et \) \( \Omega' \) fulfilling \( f_{\Omega'} = f_{\Omega} \circ m_X \) [13, 2.8.1 and 3.1.1] then it is easily checked that \( \Omega' \) fulfills conditions 3.1.1 and 3.1.2 above, so that it is a basic \( P \times P\text{-}set \); moreover, according to [13, Proposition 7.6], it is a \( \mathcal{F}\text{-}basic \) \( P \times P\text{-}set \).
Denote by $G'$ the group of automorphisms of $\text{Res}_{\{1\} \times P}(\Omega')$; now, for any pair of subgroups $Q$ and $R$ of $P$, and any $\varphi \in \mathcal{F}(Q, R)$, the composition with $m_X$ determines a map

$$\mathfrak{Mat}^*(\text{res}_{P}^\varphi \circ f_{\Omega}, \text{res}_{Q}^\varphi \circ f_{\Omega}) \longrightarrow \mathfrak{Mat}^*(\text{res}_{P} \circ f_{\Omega}', \text{res}_{Q}^\varphi \circ f_{\Omega})$$

4.10.3

sending any natural isomorphism

$$\nu : \text{res}_{P}^\varphi \circ f_{\Omega} \cong \text{res}_{Q}^\varphi \circ f_{\Omega}$$

4.10.4
to the natural isomorphism

$$\nu * m_X : \text{res}_{P} \circ f_{\Omega} \cong \text{res}_{Q} \circ f_{\Omega}$$

4.10.5;

consequently, from the canonical bijections 4.8.4 we get a canonical map

$$t^\psi_{R,Q} : T^\psi_{G}(R, Q) \longrightarrow T^\psi_{G}(R, Q)$$

4.10.6.

Moreover, for any third subgroup $T$ of $P$, any $\mathcal{F}$-morphism $\psi : T \rightarrow R$ and any natural isomorphism

$$\eta : \text{res}_{T}^\varphi \circ f_{\Omega} \cong \text{res}_{T}^\varphi \circ f_{\Omega}$$

4.10.7

we have the natural isomorphism

$$(\text{res}_{\psi} * \nu) \circ \eta : \text{res}_{T}^\varphi \circ f_{\Omega} \cong \text{res}_{T}^\varphi \circ f_{\Omega}$$

4.10.8

and thus we get

$$((\text{res}_{\psi} * \nu) \circ \eta) * m_X = (\text{res}_{\psi} * (\nu * m_X)) \circ (\eta * m_X)$$

4.10.9;

that is to say, we obtain $t^\psi_{R,Q} \circ t^\psi_{T,R}$. In particular, we get a group homomorphism $t^\text{id}_{R} : C_{G}(R) \rightarrow C_{G}(R)$ and it is easily checked that $\mathcal{G}(\Omega')(R)$ contains $t^\text{id}_{R}(\mathcal{G}(\Omega')(R))$, which forces

$$t^\text{id}_{R}(\mathcal{G}(\Omega')(R)) \subset \mathcal{G}(\Omega')(R)$$

4.10.10.

Consequently, since we have

$$T_{G}(R, Q) = \bigcup_{\varphi \in \mathcal{F}(Q, R)} T^\psi_{G}(R, Q)$$

4.10.11,

the family of maps $t^\psi_{R,Q}$ induces a canonical functor and a natural map

$$\lambda : \check{\mathcal{G}}' \longrightarrow \check{\mathcal{G}}$$

and

$$\lambda : \check{\mathcal{G}}' \longrightarrow \check{\mathcal{G}}$$

4.10.12;

it is easily checked that $\lambda$ is compatible with the structural functors $\tau^\Omega$ and $\pi^\Omega$, and with the structural functors $\pi^\Omega$ and $\pi^{\Omega'}$, so that it is an $\mathcal{F}$-locality functor. Moreover, since $p$ does not divide $|X|$, $X$ contains the trivial $P$-set with multiplicity $k$ prime to $p$ and therefore $\Omega'$ contains $\Omega$ [13, 3.1.1]; more precisely, we claim that the group homomorphism

$$\lambda_R : \check{\mathcal{G}}(R) \longrightarrow \check{\mathcal{G}}(R)$$

4.10.13
is injective. Indeed, otherwise choose a nontrivial element \( a = (a_\odot)_{\odot \in \Omega_{\odot, R}} \) in
\[
\ker(\lambda_R) \subset \prod_{\odot \in \Omega_{\odot, R}} ab(\operatorname{Aut}(O))
\]
and an element \( \hat{O}_\odot \in \Omega_{\odot, Q} \) with \(|\hat{O}_\odot|\) minimal in such a way that \( a_{\hat{O}_\odot} \neq 0 \); then, it is easily checked that the component of \( \lambda_R(a) \) in the factor \( \hat{O}_\odot \) of
\[
\hat{e}^{\odot'}(R) \cong \prod_{\odot \in \Omega_{\odot', R}} ab(\operatorname{Aut}(O))
\]
coincides with \( k \cdot a_{\hat{O}_\odot} \neq 0 \), a contradiction. We are done.

**Corollary 4.11.** If \( \Omega \) is thick and \( \Omega' \) is an \( F \)-basic \( P \times P \)-set then we have an \( F \)-locality functor \( L^{\Omega'} \to L^{\Omega} \) and an injective natural map \( \hat{e}^{\Omega'} \to \hat{e}^{\Omega} \). In particular, if \( \Omega' \) is thick then we have an \( F \)-locality isomorphism \( L^{\Omega'} \cong L^{\Omega} \).

**Proof:** Denote by \( X \) and by \( X' \) the respective images by \( f_\Omega \) and by \( f_{\Omega'} \) of the trivial \( P \)-set; since \( X \) and \( X' \) are \( F \)-stable [13, Proposition 6.5] and \( p \) does not divide
\[
|\Omega'/P| = |X| \quad \text{and} \quad |\Omega'/P| = |X'|
\]
it follows from Proposition 4.10 above that there are two \( F \)-basic \( P \times P \)-sets \( \Omega'' \supset \Omega \) and \( \Omega''' \supset \Omega' \) fulfilling
\[
f_{\Omega''} = f_\Omega \circ m_X, \quad \text{and} \quad f_{\Omega'''} = f_{\Omega'} \circ m_X
\]
two \( F \)-locality functors
\[
L^{\Omega} \to L^{\Omega''} \quad \text{and} \quad L^{\Omega'} \to L^{\Omega'''}
\]
and two injective natural maps
\[
\hat{e}^{\Omega} \to \hat{e}^{\Omega''} \quad \text{and} \quad \hat{e}^{\Omega'} \to \hat{e}^{\Omega'''}
\]
Now, we claim that \( f_{\Omega'} = f_{\Omega'''} \); indeed, it follows from [13, Theorem 6.6] that, since \( f_{\Omega''} \) and \( f_{\Omega''''} \) are \( F \)-stable, it suffices to prove that the images by \( f_{\Omega'} \) and by \( f_{\Omega'''} \) of the trivial \( P \)-set coincide with each other; but, we clearly have
\[
(f_\Omega \circ m_X')(1) = (m_X' \circ f_\Omega)(1) = X' \cdot X
\]
\[
= X' \cdot X' = (m_X \circ f_{\Omega})(1) = (f_{\Omega'} \circ m_X)(1)
\]
Moreover, if \( \Omega \) is thick then \( \Omega'' \) is also thick and isomorphism 4.4.4 implies that the left-hand natural map in 4.11.4 is a natural isomorphism and therefore that the left-hand \( F \)-locality functor is an isomorphism, so that we have the \( F \)-locality functors and the injective natural maps
\[
L^{\Omega'} \to L^{\Omega''} = L^{\Omega'''} \cong L^{\Omega} \quad \text{and} \quad \hat{e}^{\Omega'} \to \hat{e}^{\Omega''} = \hat{e}^{\Omega'''} \cong \hat{e}^{\Omega}
\]
We are done.

† The proof of the uniqueness of the basic \( F \)-locality in [12, Proposition 22.12] is not correct.
4.12. When $\Omega$ is thick we call $L^\Omega$ the basic $\mathcal{F}$-locality and set $L^\Omega = L^b$, $
abla^{\Omega} = \nabla^b$, $\pi^{\Omega} = \pi^b$ and $\hat{c}^{\Omega} = \hat{c}^b$; according to Corollary 4.11, $L^b$ and $\hat{c}^b$ do not depend on the choice of the thick $\mathcal{F}$-basic $P \times P$-set. Let us denote by $L^{b,sc}$ the full subcategory of $L^b$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ and by $\hat{c}^{b,sc}$ the restriction to $\tilde{\mathcal{F}}^{sc}$ of the natural map $\hat{c}^b$; in general, denote by $L^{\Omega,sc}$ the full subcategory of $L^\Omega$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ and by $\hat{c}^{\Omega,sc}$ the restriction to $\tilde{\mathcal{F}}^{sc}$ of the natural map $\hat{c}^{\Omega}$, so that we have an $\mathcal{F}$-locality functor and an injective natural map

$$L^{\Omega,sc} \to L^{b,sc} \quad \text{and} \quad \hat{c}^{\Omega,sc} \to \hat{c}^{b,sc} \quad 4.12.1.$$ 

4.13. Moreover, for any subgroup $Q$ of $P$, consider the set of isomorphism classes $D_Q^{nsc}$ of indecomposable $Q \times P$-sets $(Q \times P)/\Delta_\theta(U)$ where $U$ is a $\mathcal{F}$-nonselfcentralizing subgroup of $P$ and $\theta$ belongs to $\mathcal{F}(Q,U)$; according to our arguments in [11, 23.2], it is easily checked that the correspondence mapping any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ on

$$\hat{c}^{nsc}(Q) = \prod_{\hat{c} \in D_Q^{nsc}} \text{ab}(\text{Aut}(O)) \quad 4.13.1$$

defines a contravariant subfunctor $\hat{c}^{nsc}: \tilde{\mathcal{F}}^{sc} \to \mathfrak{Ab}$ of $\hat{c}^{b,sc}$. Then, let us consider the quotient $\mathcal{F}^\prime$-localities (cf. 2.10.2)

$$\tilde{\mathcal{L}}^{b,sc} = L^{b,sc}/\hat{c}^{nsc} \quad \text{and} \quad \tilde{\mathcal{L}}^{\Omega,sc} = L^{\Omega,sc}/(\hat{c}^{\Omega,sc} \cap \hat{c}^{nsc}) \quad 4.13.2,$$

we still have a faithful $\mathcal{F}^\prime$-locality functor and an injective natural map

$$\tilde{\mathcal{L}}^{\Omega,sc} \to L^{b,sc} \quad \text{and} \quad \hat{c}^{\Omega,sc}/(\hat{c}^{\Omega,sc} \cap \hat{c}^{nsc}) \to \hat{c}^{b,sc}/\hat{c}^{nsc} \quad 4.13.3.$$ 

4.14. If $\Omega$ is natural (cf. 3.5), we claim that $\tilde{\mathcal{L}}^{\Omega,sc}$ does not depend on the choice of $\Omega$ and set $\tilde{\mathcal{L}}^{\Omega,sc} = \tilde{\mathcal{L}}^{n,sc}$, $\pi^{\Omega,sc} = \pi^{n,sc}$, $\pi^{\Omega,sc} = \pi^{n,sc}$ and $\hat{c}^{\Omega,sc} = \hat{c}^{n,sc}$. Explicitly, it follows from Proposition 3.7 and from [11, Lemma 22.8] that for any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ we have

$$\hat{c}^{\Omega,sc}(Q) \equiv Z(Q \cap \tilde{\mathcal{F}}^{sc}P) \times \hat{c}^{nsc}(Q) \quad 4.14.1$$

where, with the notation in 5.7.2 below, we set

$$Z(Q \cap \tilde{\mathcal{F}}^{sc}P) = \left( \prod_T \prod_{\gamma \in \tilde{\mathcal{F}}(P,T)_Q} Z(T) \right)^{Q \times \mathcal{F}_Q(T)} \quad 4.14.2,$$

$T$ running over the set of $\mathcal{F}$-selfcentralizing subgroups of $Q$. 
4.15. In order to prove our claim, for any pair of \( \mathcal{F} \)-selfcentralizing subgroups \( Q \) and \( T \) of \( P \), and any \( \mathcal{F} \)-morphism \( \eta : T \to Q \), consider the \( Q \times P \)-set \( O = (Q \times P) / \Delta_\eta(T) \), set
\[
Z_O = Z(T) \quad \text{and} \quad A_O = \eta^* \bar{F}_Q(\eta(T)) \cap \bar{F}_P(Q)
\]
\[
\bar{Z}_O = Z_O / [A_O, Z_O] \quad \text{and} \quad \bar{A}_O = A_O / [A_O, A_O]
\]
where \( \eta^* \bar{F}_Q(\eta(T)) \) is the corresponding image of \( \bar{F}_Q(\eta(T)) \) in \( \bar{\text{Aut}}(Q) \), and recall that we have [11, 23.8.5]
\[
\text{Ab}(\text{Aut}(O)) \cong \bar{Z}_O \times \bar{A}_O
\]

Then, denoting by \( O^\text{sc}_Q \) the set of isomorphism classes of such indecomposable \( Q \times P \)-sets, in [11, Proposition 23.10] we exhibit two contravariant functors
\[
\hat{\mathcal{j}}^\text{sc} : \hat{\mathcal{F}}^{\text{sc}} \to \mathfrak{Ab} \quad \text{and} \quad \hat{\mathcal{a}}^\text{sc} : \hat{\mathcal{F}}^{\text{sc}} \to \mathfrak{Ab}
\]
respectively mapping any \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) on
\[
\hat{\mathcal{j}}^\text{sc}(Q) = \prod_{\hat{\delta} \in O^\text{sc}_Q} Z_{\hat{\delta}} \quad \text{and} \quad \hat{\mathcal{a}}^\text{sc}(Q) = \prod_{\hat{\delta} \in O^\text{sc}_Q} A_{\hat{\delta}}
\]
and fulfilling \( \bar{\mathcal{c}}_{b,sc} / \bar{\mathcal{c}}_{n,sc} \cong \hat{\mathcal{j}}^\text{sc} \times \hat{\mathcal{a}}^\text{sc} \).

4.16. In particular, with an obvious notation, it is quite clear that we get (cf. 2.9)
\[
\mathcal{L}^{b,sc} \cong (\mathcal{L}^{b,sc} / \hat{\mathcal{j}}^\text{sc}) \times \mathcal{F}^{\text{sc}} (\mathcal{L}^{b,sc} / \hat{\mathcal{a}}^\text{sc})
\]
and then we set \( \mathcal{L}^{c,sc} = \mathcal{L}^{b,sc} / \hat{\mathcal{a}}^\text{sc} \) [11, 23.10.2]; if \( \Omega \) is natural, it is easily checked that the \( \mathcal{F}^{\text{sc}} \)-locality functor \( \mathcal{L}^{\Omega,sc} \to \mathcal{L}^{b,sc} \) still determines a faithful \( \mathcal{F}^{\text{sc}} \)-locality functor \( \hat{\mathcal{L}}^{\Omega,sc} \to \mathcal{L}^{c,sc} \). Moreover, in [11, Corollaries 23.24 and 23.28] we successively exhibit a sequence of \( \mathcal{F}^{\text{sc}} \)-sublocalities
\[
\mathcal{L}^{\text{sc}} \subset \mathcal{L}^{d,sc} \subset \mathcal{L}^{c,sc}
\]
and it is not difficult to check that we may assume that \( \mathcal{L}^{c,sc} \) contains the image of \( \hat{\mathcal{L}}^{\Omega,sc} \); finally, it follows from equality [11, 23.28.1] and from isomorphism 4.14.1 above that the functors \( \mathfrak{Fer}(\pi^{r,sc}) \) and \( \mathfrak{c}^{\Omega,sc} / \mathfrak{c}^{n,sc} \) coincide with each other. Consequently, we get an \( \mathcal{F}^{\text{sc}} \)-locality isomorphism
\[
\hat{\mathcal{L}}^{\Omega,sc} \cong \mathcal{L}^{c,sc}
\]
which proves our claim.
5. Construction of an $\mathcal{F}^x$-basic $P \times P$-set from a perfect $\mathcal{F}^x$-locality

5.1. Let $\mathcal{F}$ be a Frobenius $P$-category and $\mathfrak{X}$ a nonempty set of $\mathcal{F}$-selfcentralizing subgroups of $P$ which contains any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathfrak{X}$; let us denote by $\mathcal{F}^x$ and by $(\tilde{\mathcal{L}}^{n,sc})^x$ the respective full subcategories of $\mathcal{F}$ and of $\tilde{\mathcal{L}}^{n,sc}$ over $\mathfrak{X}$; it is quite clear that the correspondence mapping any $Q \in \mathfrak{X}$ on

$$\tilde{\mathfrak{X}}(Q) = \left( \prod_T \prod_{\gamma \in \tilde{\mathcal{F}}(P:T)_{\mathcal{F}Q}} \mathbb{Z}(T) \right)^{Q \times \mathcal{F}Q(T)}$$

5.1.1,

where $T$ runs over the set of $\mathcal{F}$-selfcentralizing subgroups of $Q$ which does not belong to $\mathfrak{X}$, determines a subfunctor $\tilde{\mathfrak{X}}: \tilde{\mathcal{F}}^x \rightarrow \mathfrak{Ab}$ of $(\text{Ker}(\tilde{\mathcal{L}}^{n,sc}))^x_{\mathcal{F}X}$; let us consider the quotient $\mathcal{F}^x$-locality (cf. 2.9)

$$\tilde{\mathcal{L}}^{n,x} = (\tilde{\mathcal{L}}^{n,sc})^x / \tilde{\mathfrak{X}}$$

5.1.2,

where we denote the structural functors by

$$\tilde{\tau}^{n,x}: \mathcal{F}^x \rightarrow \tilde{\mathcal{L}}^{n,x}$$

and

$$\tilde{\pi}^{n,x}: \tilde{\mathcal{L}}^{n,x} \rightarrow \mathcal{F}^x$$

5.1.3.

In this section we show that any possible perfect $\mathcal{F}^x$-locality $(\tau^x, \mathcal{P}^x, \pi^x)$ is contained in $(\tilde{\tau}^{n,x}, \tilde{\mathcal{L}}^{n,x}, \tilde{\pi}^{n,x})$ — called the natural $\mathcal{F}^x$-locality (cf. 1.8). Actually, when $\mathfrak{X}$ is the set of all the $\mathcal{F}$-selfcentralizing subgroups of $P$, this is already proved in [11, Corollary 24.18]; but, although the arguments there still hold for $\mathfrak{X}$, we state below the main steps of the proof for $\mathfrak{X}$ since we explicitly need the present context for our inductive argument.

5.2. First of all, let us recall the distributive direct product in $\text{ac}(\tilde{\mathcal{F}}^x)$ [11, Proposition 6.14]. The additive cover $\text{ac}(\tilde{\mathcal{F}}^x)$ of $\tilde{\mathcal{F}}^x$ is the category where the objects are the finite sequences $\bigoplus_{i \in I} Q_i$ of subgroups $Q_i$ in $\mathfrak{X}$, and where a morphism from another object $R = \bigoplus_{j \in J} R_j$ to $Q = \bigoplus_{i \in I} Q_i$ is a pair $(\tilde{\alpha}, f)$ formed by a map $f: J \rightarrow I$ and by a family $\tilde{\alpha} = \{\tilde{\alpha}_j\}_{j \in J}$ of $\tilde{\mathcal{F}}^x$-morphisms $\tilde{\alpha}_j: R_j \rightarrow Q_{f(j)}$. The composition of $(\tilde{\alpha}, f)$ with another $\text{ac}(\tilde{\mathcal{F}}^x)$-morphism

$$(\tilde{\beta}, g): T = \bigoplus_{\ell \in L} T_{\ell} \rightarrow R = \bigoplus_{j \in J} R_j$$

5.2.1,

formed by a map $g: L \rightarrow J$ and by a family $\tilde{\beta} = \{\tilde{\beta}_\ell\}_{\ell \in L}$, is the pair formed by $f \circ g$ and by the family $\{\tilde{\alpha}_{g(\ell)} \circ \tilde{\beta}_\ell\}_{\ell \in L}$ of composed morphisms

$$\tilde{\alpha}_{g(\ell)} \circ \tilde{\beta}_\ell: T_{\ell} \rightarrow R_{g(\ell)} \rightarrow Q_{(f \circ g)(\ell)}$$

5.2.2.
5.3. It follows from [11, Corollary 4.9] that, for any triple of subgroups 
$Q$, $R$ and $T$ in $\mathbb{X}$, any $\mathcal{F}$-morphism $\tilde{\alpha} : Q \to R$ induces an injective map from 
$\mathcal{F}(T, R)$ to $\mathcal{F}(T, Q)$ and then we set 
\[ \mathcal{F}(T, Q)_{\tilde{\alpha}} = \mathcal{F}(T, Q) - \bigcup_{\tilde{\theta}'} \mathcal{F}(T, Q') \circ \tilde{\theta}' \]
where $\tilde{\theta}'$ runs over the set of $\mathcal{F}$-nonisomorphisms $\tilde{\theta} : Q \to Q'$ from $Q$ such that $\tilde{\alpha}' \circ \tilde{\theta}' = \tilde{\alpha}$ for some $\tilde{\alpha}' \in \mathcal{F}(R, Q')$; in this case, according to [11, Corollary 4.9], $\tilde{\alpha}'$ is uniquely determined, and we simply say that $\tilde{\theta}'$ divides $\tilde{\alpha}$ setting $\tilde{\alpha}' = \tilde{\alpha} / \tilde{\theta}'$. Note that we have $\mathcal{F}(T, Q)_{\tilde{\alpha}} = \mathcal{F}(T, Q)$ if and only if $\tilde{\alpha}$ is an isomorphism.

5.4. Actually, an element $\tilde{\beta} \in \mathcal{F}(T, Q)$ which can be extended to $Q'$ via $\tilde{\theta} : Q \to Q'$, a fortiori can be extended to $N_Q(\tilde{\theta}(Q))$ for a representative $\tilde{\theta}' \in \tilde{\theta}'$; hence, it follows from condition 2.2.3 above that $\tilde{\beta}$ belongs to $\mathcal{F}(T, Q)_{\tilde{\alpha}}$ if and only if for some representatives $\alpha \in \tilde{\alpha}$ and $\beta \in \tilde{\beta}$ we have 
\[ \alpha^* \mathcal{F}_R(\alpha(Q)) \cap \beta^* \mathcal{F}_T(\beta(Q)) = \mathcal{F}_Q(Q) \]
where $\alpha^* : \alpha(Q) \cong Q$ and $\beta^* : \beta(Q) \cong Q$ denote the inverse of the isomorphisms respectively induced by $\alpha$ and $\beta$; in particular, we get [11, 6.5.2 and 6.6.4]

5.4.2 $\tilde{\beta} \in \mathcal{F}(T, Q)_{\tilde{\alpha}}$ is equivalent to $\tilde{\alpha} \in \mathcal{F}(R, \bar{Q})_{\tilde{\beta}}$, and moreover $N_R(\alpha(Q))$ acts freely on $\mathcal{F}(T, Q)_{\tilde{\alpha}}$.

The next result follows from [11, Proposition 6.7].

**Proposition 5.5.** For any triple of subgroups $Q$, $R$ and $T$ in $\mathbb{X}$ and any $\tilde{\alpha} \in \mathcal{F}(R, Q)$, we have
\[ \mathcal{F}(T, Q) = \bigcup_{\tilde{\theta}'} \mathcal{F}(T, Q')_{\tilde{\alpha} / \tilde{\theta}', \tilde{\theta}'} \]
where $\tilde{\theta}' : Q \to Q'$ runs over a set of representatives for the isomorphism classes of $\mathcal{F}^+$-morphism from $Q$ dividing $\tilde{\alpha}$. In particular, $p$ does not divide $|\mathcal{F}(P, Q)|$.

5.6. At this point, Proposition 5.5 allows us to define a **distributive direct product** in $\text{ac}(\mathcal{F}^+)$ (see [11, Chap. 6] and also [12, Proposition 4.5]).

If $R$ and $T$ are two subgroups in $\mathbb{X}$, let us consider the set $\mathcal{F}_{R,T}^x$ of strict triples $(\tilde{\alpha}, Q, \tilde{\beta})$ where $Q$ is a subgroup in $\mathbb{X}$, $\tilde{\alpha}$ and $\tilde{\beta}$ respectively belong to $\mathcal{F}(R, Q)$ and to $\mathcal{F}(T, Q)$, and we have $\tilde{\alpha} \in \mathcal{F}(R, Q)_{\tilde{\beta}}$ or, equivalently, $\tilde{\beta} \in \mathcal{F}(T, Q)_{\tilde{\alpha}}$. We say that two strict triples $(\tilde{\alpha}, Q, \tilde{\beta})$ and $(\tilde{\alpha}', Q', \tilde{\beta}')$ are **equivalent** if there is an $\mathcal{F}$-isomorphism $\tilde{\theta} : Q \cong Q'$ fulfilling 
\[ \tilde{\alpha}' \circ \tilde{\theta} = \tilde{\alpha} \quad \text{and} \quad \tilde{\beta}' \circ \tilde{\theta} = \tilde{\beta} \]
then, \( \tilde{\theta} \) is unique since, assuming that the triples coincide each other and choosing \( \alpha \in \tilde{\alpha} \), \( \beta \in \tilde{\beta} \) and \( \theta \in \tilde{\theta} \), it is easily checked that \( \theta \) belongs to (cf. 5.4.1)

\[
\alpha^{*} \mathcal{F}_{R}(\alpha(Q)) \cap \beta^{*} \mathcal{F}_{T}(\beta(Q)) = \mathcal{F}_{Q}(Q)
\]

5.6.2.

5.7. Denoting by \( \tilde{T}_{R,T}^{x} \) a set of representatives for the set of equivalence classes in \( \tilde{T}_{R,T}^{x} \), we call \( \mathcal{F}_{x} \)-intersection of \( R \) and \( T \) the \( \mathcal{F}_{x} \)-object

\[
R \cap \tilde{T}_{R,T}^{x} T = \bigoplus_{(\tilde{\alpha}, Q, \tilde{\beta}) \in \tilde{T}_{R,T}^{x}} Q
\]

5.7.1; note that, if we choose another set of representatives, then the uniqueness of the isomorphism in 5.6.1 above yields a unique \( \mathcal{F}_{x} \)-isomorphism between both \( \mathcal{F}_{x} \)-objects; in particular, we have

\[
R \cap \tilde{T}_{R,T}^{x} T \cong \bigoplus_{Q} \bigoplus_{\tilde{\gamma}} Q
\]

5.7.2.

where \( Q \) runs over a set of representatives for the set of \( R \)-conjugacy classes of elements in \( X \) contained in \( R \) and, for such a \( Q \), \( \tilde{\gamma} \) runs over a set of representatives for the \( \mathcal{F}_{R}(Q) \)-classes in \( \tilde{F}(T, Q)_{\tilde{T}_{R,T}^{x}} \). Finally, for any pair of \( \mathcal{F}_{x} \)-objects \( R = \bigoplus_{j \in J} R_{j} \) and \( T = \bigoplus_{t \in L} T_{t} \), we define

\[
R \cap \tilde{T}_{R,T}^{x} T = \bigoplus_{(j,t) \in J \times L} R_{j} \cap \tilde{T}_{R,T}^{x} T_{t}
\]

5.7.3.

The argument in [11, Proposition 6.14] still shows that the \( \tilde{T}_{R,T}^{x} \)-intersection defines a distributive direct product in \( \mathcal{F}_{x} \) (see also [12, Proposition 4.5]).

5.8. Analogously, the existence of a perfect \( \mathcal{F}_{x} \)-locality \((\tau^{x}, \mathcal{P}^{x}, \pi^{x})\) actually determines a distributive direct product in the additive cover \( \mathcal{F}_{x} \) of \( \mathcal{P}^{x} \); as we show in [12, Proposition 4.5], this fact depends on Lemma 5.9 and on Proposition 5.11 below which admit the same proofs as the proofs of [11, Proposition 24.2] and [11, Proposition 24.4].

**Lemma 5.9.** Any \( \mathcal{P}^{x} \)-morphism \( x: R \rightarrow Q \) is a monomorphism and an epimorphism.

5.10. Thus, for any triple of subgroups \( Q, R \) and \( T \) in \( X \), as in 5.3 above any \( \mathcal{P}^{x} \)-morphism \( x \in \mathcal{P}^{x}(T, Q) \) induces an injective map from \( \mathcal{P}^{x}(T, R) \) to \( \mathcal{P}^{x}(T, Q) \) and then, as in 5.3.1, we set

\[
\mathcal{P}^{x}(T, Q)_{x} = \mathcal{P}^{x}(T, Q) - \bigcup_{z'} \mathcal{P}^{x}(T, Q') \cdot z'
\]

5.10.1

where \( z' \) runs over the set of \( \mathcal{P}^{x} \)-nonisomorphisms \( z': Q \rightarrow Q' \) from \( Q \) such
that \( x' \cdot z' = x \) for some \( x' \in \mathcal{P}^x(R, Q') \); then, \( x' \) is uniquely determined by this equality and we simply say that \( z' \) divides \( x \) setting \( x' = x/z' \). Note that the existence of \( x' \) for some \( z' \in \mathcal{P}^x(Q', Q) \) is equivalent to the existence of a subgroup of \( R \) which is \( \mathcal{F} \)-isomorphic to \( Q' \) and contains \( \{ \pi_{R,Q'}(x) \}(Q) \); thus, it is quite clear that

5.10.2 \( \mathcal{P}^x(T, Q)_x \) is the converse image of \( \tilde{\mathcal{F}}^x(T, Q) \xrightarrow{\pi_{R,Q'}(x)} \mathcal{P}^x(T, Q) \).

Then, Proposition 5.5 implies the following result.

**Proposition 5.11.** For any triple of elements \( Q, R \) and \( T \) in \( \mathfrak{X} \), and any \( x \in \mathcal{P}^x(R, Q) \), we have

\[
\mathcal{P}^x(T, Q) = \bigsqcup_{z'} \mathcal{P}^x(T, Q')_{x/z' \cdot z'}
\]

5.11.1

where \( z' : Q \to Q' \) runs over a set of representatives for the isomorphism classes of \( \mathcal{P}^x \)-morphisms from \( Q \) dividing \( x \).

5.12. As above, if \( R \) and \( T \) are two subgroups in \( \mathfrak{X} \), we consider the set \( \mathcal{X}^x_{R,T} \) of strict \( \mathcal{P}^x \)-triples \( (x, Q, y) \) where \( Q \) belongs to \( \mathfrak{X} \), \( x \) and \( y \) respectively belong to \( \mathcal{P}^x(R, Q) \) and to \( \mathcal{P}^x(T, Q) \), and we have \( x \in \mathcal{P}^x(R, Q)_y \) or, equivalently, \( y \in \mathcal{P}^x(T, Q)_x \). Note that, for any \( v \in R \) and any \( w \in T \), the \( \mathcal{P}^x \)-triple

\[
v \cdot (x, Q, y) \cdot w^{-1} = (\tau^x_R(v) \cdot x, Q, \tau^x_T(w) \cdot y)
\]

5.12.1

still belongs to \( \mathcal{X}^x_{R,T} \) and therefore the quotient set \( (R \times T) \setminus \mathcal{X}^x_{R,T} \) clearly coincides with \( \mathcal{X}^x_{R,T} \). Similarly, we say that two strict \( \mathcal{P}^x \)-triples \( (x, Q, y) \) and \( (x', Q', y') \) are equivalent if there exists a \( \mathcal{P}^x \)-isomorphism \( z : Q \cong Q' \) fulfilling

\[
x' \cdot z = x \quad \text{and} \quad y' \cdot z = y
\]

5.12.2

since \( \mathcal{P}^x \) is divisible, such a \( \mathcal{P}^x \)-isomorphism \( z \) is unique; in particular, in any equivalent class we may find a unique element fulfilling

\[
Q \subset R \quad \text{and} \quad x = \tau^x_{R,Q}(1)
\]

5.12.3.

5.13. Coherently, for any \( Q \in \mathfrak{X} \) denoting by \( \mathcal{S}^x_Q \) the set of subgroups of \( Q \) belonging to \( \mathfrak{X} \), we call \( \mathcal{P}^x \)-intersection of \( R \) and \( T \) the \( \mathfrak{ac}(\mathcal{P}^x)_x \)-object

\[
R \cap \mathcal{P}^x T = \bigoplus_{Q \in \mathcal{S}^x_Q} \bigoplus_{y \in \mathcal{P}^x(T, Q)_x, \tau^x_{R,Q}(1)} Q
\]

5.13.1

and we clearly have canonical \( \mathfrak{ac}(\mathcal{P}^x) \)-morphisms

\[
R \leftarrow R \cap \mathcal{P}^x T \rightarrow T
\]

5.13.2
respectively determined by \( \tau_{R,Q}^X(1) \) and \( y \). Note that, for any other choice of a set of representatives for the set of equivalence classes in \( \mathcal{T}^X_{R,T} \), we get an isomorphic object and a unique \( \text{ac}(\mathcal{P}^X) \)-isomorphism which is compatible with the canonical morphisms. Then, either the arguments in [11, Proposition 24.8] or [12, Proposition 4.5] prove the following.

**Proposition 5.14.** The category \( \text{ac}(\mathcal{P}^X) \) admits a distributive direct product mapping any pair of elements \( R \) and \( T \) of \( \mathfrak{X} \) on their \( \mathcal{P}^X \)-intersection \( R \cap \mathcal{P}^X T \).

5.15. Here, we are particularly interested in the \( \mathcal{P}^X \)-intersection of \( P \) with itself; more explicitly, denoting by \( \Omega^X \) the set of pairs \((Q,y)\) formed by \( Q \in \mathfrak{X} \) and by \( y \in \mathcal{P}^X(P,Q)_{\tau^X_{P,Q}(1)} \), we have

\[
P \cap \mathcal{P}^X P = \bigoplus_{(Q,y) \in \Omega^X} Q \quad \text{5.15.1;}
\]

moreover, since \( P \times P \) acts on the set \( \mathcal{S}^X_{P,P} \) (cf. 5.12.1) preserving the equivalence classes, this group acts on \( \Omega^X \) and it is easily checked that [11, 24.9]

5.15.2 \( (u, v) \in P \times P \) maps \((Q, y) \in \Omega^X \) on \((Q^{u^{-1}}, \tau^X_P(v) \cdot y \cdot \tau^X_{Q,Q^u}(1)^{-1}(u^{-1})) \).

In particular, \( \{1\} \times P \) acts freely on \( \Omega^X \). On the other hand, it is clear that the map sending a strict \( \mathcal{P}^X \)-triple \((x, Q, y) \in \mathcal{S}^X_{P,P} \) to \((y, Q, x)\) induces a \( P \times P \)-set isomorphism \( \Omega^X \cong (\Omega^X)^{\circ} \). The point is that from [11, Proposition 24.10 and Corollary 24.11] and from Proposition 3.4 above we get (cf. 3.5).

5.15.3 \( \Omega^X \) is the natural \( \mathcal{F}^X \)-basic \( P \times P \)-set.

5.16. Consequently, we may assume that \( \Omega^X \) is contained in a natural \( \mathcal{F}^X \?-\text{basic} \ P \times P \)-set \( \Omega \) (cf. 3.5) and our purpose is to show that the perfect \( \mathcal{F}^X \)-locality \( \mathcal{P}^X \) is contained in the natural \( \mathcal{F}^X \)-locality \( \mathcal{L}^{\mathcal{F}^X} \) (cf. 5.1.3). First of all, it follows from Proposition 5.14 that for any \( Q \in \mathfrak{X} \) the inclusion \( Q \subset P \) determines an \( \text{ac}(\mathcal{P}^X) \)-morphism

\[
\tau^X_{P,Q}(1) \cap \mathcal{P}^X \tau^X_{P}(1) : Q \cap \mathcal{P}^X P \to P \cap \mathcal{P}^X P \quad \text{5.16.1;}
\]

actually, according to 5.13.1 and denoting by \( \Omega^X_Q \) the set of pairs \((T, z)\) formed by a subgroup \( T \) in \( \mathfrak{X} \) contained in \( Q \) and by an element \( z \) of \( \mathcal{P}^X(P,T)_{\tau^X_{Q,T}(1)} \), we have

\[
Q \cap \mathcal{P}^X P = \bigoplus_{(T,z) \in \Omega^X_Q} T \quad \text{5.16.2},
\]
the group $Q \times P$ acts on $\Omega^x$, and the $\text{ac}(P^x)$-morphism 5.16.1 determines a $Q \times P$-set homomorphism

$$f^x_Q : \Omega^x \rightarrow \text{Res}_{Q \times P}(\Omega^x) \subset \text{Res}_{Q \times P}(\Omega)$$

From the arguments in [11, Proposition 24.15] we get the following result.

**Proposition 5.17.** For any $Q \in \mathcal{X}$, the map $f^x_Q : \Omega^x \rightarrow \Omega^x$ sends an element $(T, z) \in \Omega^x$ to $(R, y) \in \Omega^x$ if and only if we have $T = Q \cap R$ and $z = y \cdot \tau^x_{\eta,z}(1)$. In particular, this map is injective.

5.18. Thus, according to this proposition, the image of $\Omega^x_Q$ in the natural $\mathcal{F}$-basic $P \times P$-set $\Omega$ coincides with the union of all the $Q \times P$-orbits isomorphic to $(Q \times P)/\Delta_Q(T)$ for some $T \in \mathcal{X}$ and some $\tilde{\eta} \in \tilde{\mathcal{F}}(Q, T)_{\eta^P}$. On the other hand, for any $P^x$-isomorphism $x : Q \cong Q'$, it follows again from Proposition 5.14 that we have an $\text{ac}(P^x)$-isomorphism

$$x \cap P^x \tau^x_{\eta^x}(1) : Q \cap P^x \cong Q' \cap P^x$$

and therefore we get a bijection between the sets of indices $\Omega^x_Q$ and $\Omega^x_{Q'}$, which is compatible via $\pi^x_{Q', Q}(x)$ with the respective actions of $Q \times P$ and $Q' \times P$; that is to say, we get a $Q \times P$-set isomorphism

$$f^x_x : \Omega^x \cong \text{Res}_{\pi^x_{Q', Q}(x) \times \text{id}_P}(\Omega^x_{Q'})$$

As above, we set $G = \text{Aut}_{(1) \times P}(\Omega)$.

**Proposition 5.19.** For any $P^x$-isomorphism $x : Q \cong Q'$, the $Q \times P$-set isomorphism

$$f^x_x : \Omega^x \cong \text{Res}_{\pi^x_{Q', Q}(x) \times \text{id}_P}(\Omega^x_{Q'})$$

can be extended to an element $f^x_x$ of $T_G(Q, Q')$ and the image of $f^x_x$ in $E^{x,x}(Q', Q)$ is uniquely determined by $x$.

**Proof:** Since the $Q \times P$-sets $\text{Res}_{Q \times P}(\Omega)$ and $\text{Res}_{\pi^x_{Q', Q}(x) \times \text{id}_P}(\text{Res}_{Q \times P}(\Omega))$ are isomorphic (cf. 3.1.2), and the $Q \times P$- and $Q' \times P$-set homomorphisms

$$f^x_Q : \Omega^x \rightarrow \text{Res}_{Q \times P}(\Omega) \quad \text{and} \quad f^x_{Q'} : \Omega^x_{Q'} \rightarrow \text{Res}_{Q' \times P}(\Omega)$$

are injective (cf. Proposition 5.17), identifying $\Omega^x_Q$ and $\Omega^x_{Q'}$ with their images in $\Omega$, $f^x_x$ can be extended to a $Q \times P$-set isomorphism

$$f^x_x : \text{Res}_{Q \times P}(\Omega) \cong \text{Res}_{\pi^x_{Q', Q}(x) \times \text{id}_P}(\text{Res}_{Q' \times P}(\Omega))$$

that is to say, we get an element $f^x_x$ of $T_G(Q, Q')$ (cf. 3.1).
Then, we claim that the image of \( f_x \) in \( \mathcal{L}^{n,x}(Q',Q) \) is independent of our choices; indeed, for another choice \( g_x \in T_{G}(Q',Q) \) fulfilling the above conditions, the composed map \((f_x)^{-1} \circ g_x \) belongs to \( C_{G}(Q) \) and induces the identity on \( \Omega^{x}_{Q} \); since we know that (cf. 4.4.4 and 4.14.1)

\[
C_{G}(Q)/\mathcal{S}_{1}^{1}(Q) \cong Z(Q \cap \mathcal{F}^{x}P) \times \mathcal{S}_{0}(Q)
\]

and \((f_x)^{-1} \circ g_x \) induces the identity on \( \Omega^{x}_{Q} \), it is clear from 5.18 that the image in this quotient of \((f_x)^{-1} \circ g_x \) belongs to \( \mathcal{F}^{x}(Q) \) and therefore it has a trivial image in \( \mathcal{L}^{n,x}(Q) \), so that \( f_x \) and \( g_x \) have the same image in \( \mathcal{L}^{n,x}(Q',Q) \). We are done.

**Corollary 5.20.** There is a faithful \( \mathcal{F}^{x} \)-locality functor \( \lambda^{x}: \mathcal{P}^{x} \rightarrow \mathcal{L}^{n,x} \)

sending any \( \mathcal{P}^{x} \)-isomorphism \( x:Q \cong Q' \) to the image of \( f_x \) in \( \mathcal{L}^{n,x}(Q',Q) \).

Moreover, any \( \mathcal{F}^{x} \)-locality functor \( \mu^{x}: \mathcal{P}^{x} \rightarrow \mathcal{L}^{n,x} \) is naturally \( \mathcal{F}^{x} \)-isomorphic to \( \lambda^{x} \).

**Proof:** Let us denote by \( \lambda^{x}(x) \) the image of \( f_x \) in \( \mathcal{L}^{n,x}(Q',Q) \); first of all, let \( x':Q' \cong Q'' \) be a second \( \mathcal{P}^{x} \)-isomorphism; it is clear that the automorphism \( \text{Res}^{x}_{Q',Q}(x) \times \text{id}_{P} \) extends \( \text{Res}^{x}_{Q',Q}(x) \times \text{id}_{P} \) and \( f_x \) to \( \text{Res}^{x}_{Q',Q}(x) \times \text{id}_{P} \) : consequently, by the proposition above, we get

\[
\lambda^{x}(x') = \lambda^{x}(x) \cdot \lambda^{x}(x) \quad 5.20.1.
\]

On the other hand, by the divisibility of \( \mathcal{P}^{x} \), any \( \mathcal{P}^{x} \)-morphism \( z:T \rightarrow Q \) is the composition of \( \tau^{x}_{Q,T} \) with a \( \mathcal{P}^{x} \)-isomorphism \( z:Q \cong Q' \) where we set \( T' = (\tau^{x}_{Q,T}(z))(T) \); then, we simply define

\[
\lambda^{x}(z) = \tau^{x}_{Q,T}(z) \cdot \lambda^{x}(z) \quad 5.20.2.
\]

Now, in order to prove that this correspondence defines a functor, it suffices to show that, for any \( \mathcal{P}^{x} \)-isomorphism \( x:Q \cong Q' \) and any subgroup \( R \) of \( Q \), setting \( R' = (\tau^{x}_{Q',Q}(x))(R) \) and denoting by \( y:R \cong R' \) the \( \mathcal{P}^{x} \)-isomorphism induced by \( x \) (cf. 2.8), we still have

\[
\lambda^{x}(x) \cdot \tilde{\tau}^{n,x}(1) = \tilde{\tau}^{n,x}(1) \cdot \lambda^{x}(y) \quad 5.20.3.
\]

But, it is quite clear that the commutative \( \text{ac}(\mathcal{P}^{x}) \)-diagram (cf. Proposition 5.14)

\[
\begin{array}{ccc}
R \cap \mathcal{P}^{x}P & \xrightarrow{\tau^{x}_{Q,R}(1) \cap \mathcal{P}^{x}P(1)} & Q \cap \mathcal{P}^{x}P \\
\downarrow_{y \cap \mathcal{P}^{x}P} & & \downarrow_{x \cap \mathcal{P}^{x}P} \\
R' \cap \mathcal{P}^{x}P & \xrightarrow{\tau^{x}_{Q',R}(1) \cap \mathcal{P}^{x}P(1)} & Q' \cap \mathcal{P}^{x}P \\
\end{array}
\]

5.20.4.
determines a commutative diagram of $R \times P$-sets (cf. 5.16)

\[
\begin{array}{c}
\Omega^X_R \rightarrow \text{Res}^Q_{R\times P}(\Omega^X_Q) \\
f_y \downarrow \text{Res}^Q_{R\times P}(f_y) \\
\text{Res}_{\pi_q \times \text{id}_P}(\Omega^X_R) \rightarrow \text{Res}^Q_{R\times P}(\text{Res}_{\pi_q \times \text{id}_P}(\Omega^X_Q))
\end{array}
\]

5.20.5.

Consequently, the element $f_x$ of $T_G(Q, Q')$ extending $f_x^\#: \text{ker}$ also extends $f_y^\#$ and we can choose $f_y = f_x$. On the other hand, since $\tau_n^x$ is faithful, it is easily checked that $\lambda^x$ induces an injective group homomorphism $\mathcal{P}^x(Q) \rightarrow \mathcal{L}^{n,x}(Q)$ for any $Q \in \mathfrak{X}$ and therefore this functor is faithful too.

Moreover, if $\mu^x : \mathcal{P}^x \rightarrow \mathcal{L}^{n,x}$ is another $\mathcal{X}$-locality functor then, for any $\mathcal{P}^x$-morphism $x : R \rightarrow Q$, we have $\mu^x(x) = \lambda^x(x) \cdot c_x$ for some $c_x \in \text{ker}(\pi^x_R)$; since $\mu^x \circ \tau^n = \pi^{n,x} = \lambda^x \circ \tau^n$ (cf. 2.9), $c_x$ only depends on the class $\tilde{x}$ of $x$ in $\tilde{\mathcal{F}}^x(Q, R)$; then, for another $\mathcal{P}^x$-morphism $y : T \rightarrow R$, we get

\[
\lambda^x(x \cdot y) \cdot c_{xy} = \mu^x(x \cdot y) = \mu^x(x) \cdot \mu^y(y) = (\lambda^x(x) \cdot c_x)(\lambda^y(y) \cdot c_y)
\]

5.20.6; thus, employing additive notation in the Abelian group $(\text{Ret}(\pi^{n,x})(T), \cdot)$, we still get

\[
0 = (\text{Ret}(\pi^{n,x})(y))(c_x) - c_{xy} + c_y
\]

5.20.7.

That is to say, setting

\[
\mathcal{C}^n(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x})) = \prod_{\tilde{a} \in \tilde{\mathcal{F}}(\Delta_n, \tilde{\mathcal{F}}^x)} \text{ker}(\pi^{n,x}_{\tilde{a}(0)})
\]

5.20.8.

for any $n \in \mathbb{N}$, the family $c = (c_x)_{\tilde{x}}$ where $\tilde{x}$ runs over the set of $\tilde{\mathcal{F}}^x$-morphisms is an element of $\mathcal{C}^1(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x}))$ and equality 5.20.7 shows that this element belongs to the kernel of the usual differential map

\[
d_1^x : \mathcal{C}^1(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x})) \rightarrow \mathcal{C}^2(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x}))
\]

5.20.9.

But, according to [12, 4.2] and 5.7 above, the main result of [12, §4] can be applied to the category $\tilde{\mathcal{F}}^x$ and to the functor $\text{Ret}(\pi^{n,x})$ since for any $Q \in \mathfrak{X}$ we have (cf. 5.7.1)

\[
\text{Ret}(\pi^{n,x})(Q) = Z(Q \cap \tilde{\mathcal{F}}^x P)
\]

5.20.10.

Consequently, for any $n \geq 1$ we have

\[
\mathbb{H}^n(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x})) = \{0\}
\]

5.20.11.

and, in particular, we have $c = d^x(z)$ for a suitable element $z = (z_Q)_{Q \in \mathfrak{X}}$ in $\mathcal{C}^0(\tilde{\mathcal{F}}^x, \text{Ret}(\pi^{n,x}))$; hence, for any $\mathcal{P}^x$-morphism $x : R \rightarrow Q$, in additive notation we get

\[
c_x = \left(\text{Ret}(\pi^{n,x})(\tilde{x})\right)(z_Q) - z_R
\]

5.20.12.
and therefore we still get $\mu^x(x) \cdot z_R = z_Q \cdot \lambda^x(x)$, so that the family of $L^{n,x}$-isomorphisms $z_Q : Q \cong Q$ where $Q$ runs over $X$ defines a natural $\mathcal{F}^x$-isomorphism between $\lambda^x$ and $\mu^x$ (cf. 2.9).

**Corollary 5.21.** Let $\mathcal{P}^x$ and $\mathcal{P}'^x$ be perfect $\mathcal{F}^x$-localities and assume that they are $\mathcal{F}^x$-isomorphic. Then, there is an $\mathcal{F}^x$-isomorphism $\rho^x : \mathcal{P}^x \cong \mathcal{P}'^x$ such that we have the commutative diagram

$$\mathcal{P}^x \xrightarrow{\rho^x} \mathcal{P}'^x$$

$$\lambda^x \searrow \bigtriangleup $$

$$\mathcal{L}^{n,x} \swarrow \lambda'^x$$

5.21.1.

**Proof:** Considering the set $\Omega^x$ of pairs $(Q, y)$ formed by $Q \in X$ and by $y \in \mathcal{P}^x(P, Q)_{\tau^x(P)}$, it follows from 5.15.3 that the $P \times P$-sets $\Omega^x$ and $\Omega^x$ are mutually isomorphic; hence, up to suitable identifications, it follows from Corollary 5.20 that there is also a faithful $\mathcal{F}^x$-locality functor $\lambda^x : \mathcal{P}^x \rightarrow \mathcal{L}^{n,x}$ as above; thus, we have a new $\mathcal{F}^x$-locality functor

$$\lambda^x \circ \rho^x : \mathcal{P}^x \rightarrow \mathcal{L}^{n,x}$$

5.21.2.

and therefore, according to Corollary 5.20 and 2.9, it suffices to modify the identification between $\Omega^x$ and $\Omega^x$ with a suitable element of $C_G(P)$ to get

$$\lambda^x \circ \rho^x = \lambda^x$$

5.21.3.

6. The perfect $\mathcal{F}^x$-locality contained in the natural $\mathcal{F}^x$-locality

6.1. Let $\mathcal{F}$ be a Frobenius $P$-category and $X$ a nonempty set of $\mathcal{F}$-self-centralizing subgroups of $P$ which contains any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $X$; we keep our notation in 5.1 above. In this section we prove the existence and the uniqueness of a perfect $\mathcal{F}^x$-locality $(\tau^x, \mathcal{P}^x, \pi^x)$. The existence and the uniqueness of the localizer $(\tau_P, L_{\mathcal{F}}(P), \pi_P)$ (cf. Theorem 2.12) proves the existence and the uniqueness of the perfect $\mathcal{F}^x$-locality whenever $X = \{P\}$; thus, assume that $X \neq \{P\}$, choose a minimal element $U$ in $X$ fully normalized in $\mathcal{F}$ and set

$$\mathcal{G} = X - \{\theta(U) \mid \theta \in \mathcal{F}(P, U)\}$$

6.1.1.

then, arguing by induction on $|X|$ we may assume that there is a perfect $\mathcal{F}^x$-locality $(\tau^x, \mathcal{P}^x, \pi^x)$ which is unique up to $\mathcal{F}^x$-locality isomorphisms. At this point, according to Corollary 5.20, we may assume that $(\tau^x, \mathcal{P}^x, \pi^x)$ is an $\mathcal{F}^n$-sublocality of the natural $\mathcal{F}^n$-locality $(\tau_n^n, \mathcal{L}^{n,x^n}, \pi_n^n)$ (cf. 5.1);
then, denoting by \((\mathcal{L}^{n,x})^\mathfrak{g}\) the full subcategory of \(\mathcal{L}^{n,x}\) over \(\mathfrak{g}\), we clearly have an obvious functor \((\mathcal{L}^{n,x})^\mathfrak{g}\rightarrow\mathcal{L}^{n,\mathfrak{g}}\) and we look to the pull-back
\[
\begin{array}{c}
\mathcal{P}^\mathfrak{g} \subset \mathcal{L}^{n,\mathfrak{g}} \\
\uparrow \\
\mathcal{M}^\mathfrak{g} \subset (\mathcal{L}^{n,x})^\mathfrak{g}
\end{array}
\]
so that we get a \(p\)-coherent \(\mathcal{F}^\mathfrak{g}\)-locality \((\nu^\mathfrak{g},\mathcal{M}^\mathfrak{g},\rho^\mathfrak{g})\); more explicitly, it is easily checked from 5.1 that, for any \(Q \in \mathfrak{g}\), we have the exact sequence
\[
1 \rightarrow \prod_V \prod_{\tilde{\theta}} Z(V) \rightarrow \mathcal{M}^\mathfrak{g}(Q) \rightarrow \mathcal{P}^\mathfrak{g}(Q) \rightarrow 1
\]
where \(V\) runs over a set of representatives for the set of \(Q\)-conjugacy classes of elements of \(\mathfrak{X} - \mathfrak{g}\) contained in \(Q\), and \(\tilde{\theta}\) over a set of representatives for the set of \(\mathcal{F}_Q(V)\)-classes in \(\tilde{F}(P,V)_{\nu^\mathfrak{g}}\).

6.2. Then, let us consider the quotient \(\mathcal{F}^\mathfrak{g}\)-locality \((\bar{\nu}^\mathfrak{g},\bar{\mathcal{M}}^\mathfrak{g},\bar{\rho}^\mathfrak{g})\) of \(\mathcal{M}^\mathfrak{g}\) defined by
\[
\bar{\mathcal{M}}^\mathfrak{g}(Q,R) = \mathcal{M}^\mathfrak{g}(Q,R)/\nu^\mathfrak{g}_R(Z(R))
\]
together with the induced natural maps
\[
\bar{\nu}^\mathfrak{g}_{Q,R} : \mathcal{T}_P(Q,R) \rightarrow \bar{\mathcal{M}}^\mathfrak{g}(Q,R) \quad \text{and} \quad \bar{\rho}^\mathfrak{g}_{Q,R} : \bar{\mathcal{M}}^\mathfrak{g}(Q,R) \rightarrow \mathcal{F}(Q,R)
\]
for any \(Q,R \in \mathfrak{g}\); in order to show the existence of a perfect \(\mathcal{F}^x\)-locality \((\tau^x,\mathcal{P}^x,\pi^x)\) it suffices to prove that \(\bar{\rho}^\mathfrak{g}\) admits a functorial section
\[
\sigma^\mathfrak{g} : \mathcal{F}^\mathfrak{g} \rightarrow \bar{\mathcal{M}}^\mathfrak{g}
\]
such that the image of \(\sigma^\mathfrak{g}\) contains the image of \(\nu^\mathfrak{g}\). Note that the exterior quotients of \(\mathcal{M}^\mathfrak{g}\) and \(\bar{\mathcal{M}}^\mathfrak{g}\) coincide with each other.

6.3. Indeed, in this case it is clear that the converse image \(\mathcal{N}^\mathfrak{g}\) in \(\mathcal{M}^\mathfrak{g}\) of the “image” of \(\sigma^\mathfrak{g}\) in \(\bar{\mathcal{M}}^\mathfrak{g}\) is an \(\mathcal{F}^\mathfrak{g}\)-sublocality isomorphic to \(\mathcal{P}^\mathfrak{g}\), so that it is a perfect \(\mathcal{F}^x\)-locality; at this point, we consider the \(\mathcal{F}^x\)-sublocality \(\mathcal{N}^x\) of \(\mathcal{L}^{n,x}\) containing \(\mathcal{N}^\mathfrak{g}\) as a full subcategory over \(\mathfrak{g}\) and fulfilling
\[
\mathcal{N}^x(Q,V) = \mathcal{L}^{n,x}(Q,V)
\]
for any \(Q \in \mathfrak{X}\) and any \(V \in \mathfrak{X} - \mathfrak{g}\). On the other hand, by the very definition of \(\mathcal{L}^{n,x}\) (cf. 5.1), we have
\[
\text{Ker}(\bar{\pi}_{V}^{n,x}) = \xi^{n,x}(V)/\tilde{\xi}^{x}(V) = \prod_{\tilde{\theta} \in \tilde{F}(P,V)} Z(V)
\]
and therefore, since $p$ does not divide $|\tilde{F}(P,V)|$ (cf. Proposition 5.5), we have a surjective group homomorphism

$$\nabla^x_V : \text{Ker}(\tilde{\pi}^{n,x}_V) \to Z(V)$$

mapping $z = (z_{\tilde{\theta}})_{\tilde{\theta} \in \tilde{F}(P,V)}$ on

$$\nabla^x_V(z) = \frac{1}{|\tilde{F}(P,V)|} \sum_{\tilde{\theta} \in \tilde{F}(P,V)} z_{\tilde{\theta}}$$

which is a section of the restriction to $Z(V)$ and $\text{Ker}(\tilde{\pi}^{n,x}_V)$ of

$$\tilde{\tau}^{n,x}_V : N_P(V) \to \tilde{L}^{n,x}(V)$$

6.4. Finally, considering the \textit{contravariant Dirac functor} $d^x : \mathcal{N}^x \to \mathfrak{A}$ mapping any $Q \in \mathcal{Y}$ on $\{0\}$ and any $V \in \mathfrak{X} - \mathcal{Y}$ on $\text{Ker}(\nabla^x_V)$, the quotient $\mathcal{F}^x$-locality $\mathcal{P}^x = \mathcal{N}^x / d^x$ (cf. 2.10), where we denote the structural functors by

$$\tau^x : \mathcal{T}_P \to \mathcal{P}^x$$

and

$$\pi^x : \mathcal{P}^x \to \mathcal{F}$$

is actually a \textit{perfect} $\mathcal{F}^x$-locality; indeed, it is quite clear that the $\mathcal{F}^x$-locality $\mathcal{N}^x$ and therefore the $\mathcal{F}^x$-locality $\mathcal{P}^x$ are both $p$-coherent (cf. 2.8); moreover, for any $Q \in \mathcal{Y}$ fully normalized in $\mathcal{F}$, we already know that $\mathcal{P}^x(Q) = \mathcal{N}^x(Q)$ is an $\mathcal{F}$-\textit{localizer} of $Q$ and, for any $V \in \mathfrak{X} - \mathcal{Y}$, we have the exact sequence

$$1 \to Z(V) \to \mathcal{P}^x(V) \to \mathcal{F}(V) \to 1$$

which, together with the group homomorphisms

$$\tau^x_V : N_P(V) \to \mathcal{P}^x(V)$$

and

$$\pi^x_V : \mathcal{P}^x(V) \to \mathcal{F}(V)$$

is an $\mathcal{F}^x$-\textit{localizer} of $V$ whenever $V$ is fully normalized in $\mathcal{F}$; now, our claim follows from 2.13 above.

6.5. Similarly, in order to show the uniqueness of $(\tau^x, \mathcal{P}^x, \pi^x)$, in the general setting we have to consider the $\mathcal{F}^x$-sublocality $(u^x, \mathcal{M}^x, \rho^x)$ of $\tilde{\mathcal{L}}^{n,x}$ containing $\mathcal{M}^\mathcal{Y}$ as a \textit{full} subcategory over $\mathcal{Y}$ and fulfilling

$$\mathcal{M}^x(Q,V) = \tilde{L}^{n,x}(Q,V)$$

for any $Q \in \mathfrak{X}$ and any $V \in \mathfrak{X} - \mathcal{Y}$, and as above the quotient $\mathcal{F}^x$-locality $(\tilde{\nu}^x, \tilde{\mathcal{M}}^x, \tilde{\rho}^x)$ of $\mathcal{M}^x$ defined by

$$\tilde{\mathcal{M}}^x(Q,R) = \mathcal{M}^x(Q,R) / u^{n,x}_R(Z(R))$$

together with the induced natural maps

$$\tilde{\nu}^x_{Q,R} : \mathcal{T}_P(Q,R) \to \tilde{\mathcal{M}}^x(Q,R)$$

and

$$\tilde{\rho}^x_{Q,R} : \tilde{\mathcal{M}}^x(Q,R) \to \mathcal{F}(Q,R)$$
for any $Q, R \in \mathcal{X}$; thus, for any $Q \in \mathcal{Y}$, from the exact sequence 6.1.3 we obtain the exact sequence

$$1 \longrightarrow \prod_{W} \prod_{\tilde{\theta}} Z(W) \longrightarrow \mathcal{M}^x(Q) \longrightarrow \mathcal{F}^x(Q) \longrightarrow 1 \quad \text{6.5.4},$$

where $W$ runs over a set of representatives for the set of $Q$-conjugacy classes of elements of $\mathcal{X} - \mathcal{Y}$ contained in $Q$ and $\tilde{\theta}$ over a set of representatives for the $\mathcal{F}_Q(W)$-classes in $\tilde{\mathcal{F}}(P, W)_{i_Q}$, whereas for any $V \in \mathcal{X} - \mathcal{Y}$ it follows again from 5.1 that we have the exact sequence

$$1 \longrightarrow \text{Ker}(\nabla^x) \longrightarrow \tilde{\mathcal{M}}^x(V) \longrightarrow \mathcal{F}^x(V) \longrightarrow 1 \quad \text{6.5.5}.$$

6.6. At this point, if $(\tau^x, \mathcal{P}^x, \pi^x)$ is another perfect $\mathcal{F}^x$-locality, once again it follows from Corollary 5.20 that we may assume that $(\tau^x, \mathcal{P}^x, \pi^x)$ is a $\mathcal{F}^x$-sublocality of the natural $\mathcal{F}^x$-locality $(\tilde{\tau}^x, \tilde{\mathcal{P}}^x, \tilde{\pi}^x)$ (cf. 5.1); then, for the corresponding full subcategories over $\mathcal{Y}$, we have $(\mathcal{P}^x)^{\mathcal{Y}} \subset (\tilde{\mathcal{P}}^x)^{\mathcal{Y}}$; moreover, from the induction hypothesis and from Corollary 5.21, we may assume that the image $\mathcal{P}^{\mathcal{Y}}$ of $(\mathcal{P}^x)^{\mathcal{Y}}$ in $\tilde{\mathcal{P}}^{\mathcal{Y}}$ coincides with $\mathcal{P}^{\mathcal{Y}}$. Consequently, $(\mathcal{P}^x)^{\mathcal{Y}}$ is contained in $\mathcal{M}^{\mathcal{Y}}$ and therefore $\mathcal{P}^x$ is contained in $\mathcal{M}^x$; thus, the image of $\tilde{\mathcal{P}}^x$ in $\tilde{\mathcal{M}}^x$, determines a functorial section

$$\sigma^x : \mathcal{F}^x \longrightarrow \tilde{\mathcal{M}}^x \quad \text{6.6.1}$$

which induces a functorial section $\sigma^{\mathcal{Y}}$ as in 6.2.3 above and, according to Theorem 6.12 below, this functorial section is naturally $\mathcal{F}^{\mathcal{Y}}$-isomorphic to $\sigma^{\mathcal{Y}}$, so that we may assume that both coincide with each other (cf. 2.9); that is to say, we get an $\mathcal{F}^x$-locality isomorphism between $(\tau^x, \mathcal{P}^x, \pi^x)$ and the the quotient $\mathcal{F}^x$-locality $(\tau^x, \mathcal{P}^x, \pi^x)$ in 6.4 above. In conclusion, the existence and the uniqueness of the perfect $\mathcal{F}^x$-locality is a consequence of Theorem 6.12 below.

6.7. In order to prove this theorem, we apply the methods developed in [12, §4 and §5]; as in [12, 4.2], $\tilde{\mathcal{F}}^x$ can be considered as an $\tilde{\mathcal{F}}_{\mathcal{P}}^x$-category and clearly it fulfills condition [12, 4.2.1]; then, through the $\tilde{\mathcal{F}}^x$-intersection in 5.7 above, $\tilde{\mathcal{F}}^x$ becomes a multiplicative $\tilde{\mathcal{F}}_{\mathcal{P}}^x$-category and, as in [12, 4.6], we consider the functor and the natural map defined there

$$m_p^x : \tilde{\mathcal{F}}^x \longrightarrow \mathcal{ac}(\tilde{\mathcal{F}}^x) \quad \text{and} \quad \omega^x : m_p^x \longrightarrow j^x \quad \text{6.7.1}$$

where $j^x : \tilde{\mathcal{F}}^x \to \mathcal{ac}(\tilde{\mathcal{F}}^x)$ denotes the canonical functor. Also, as in [12, 4.8], we consider the representation $I$ of $\tilde{\mathcal{F}}^x$ (cf. 2.2) mapping any $Q \in \mathcal{X}$ on a set $I_Q = \tilde{\tau}^x_\mathcal{Q}_Q$, $\rho$ (cf. 5.7) of suitable representatives for the set of equivalence classes of the corresponding strict triples, and the functor defined in [12, 4.8.3]

$$n^x : I \times \tilde{\mathcal{F}}^x \longrightarrow \tilde{\mathcal{F}}_{\mathcal{P}}^x \subset \tilde{\mathcal{F}}^x \quad \text{6.7.2}.$$
6.8. In our situation, we have to consider the contravariant functor
\[ \delta_U^x : \tilde{F}^x \longrightarrow \mathbb{Ab} \]
6.8.1
mapping any \( Q \in \mathcal{Z} \) on \( \{0\} \) and any \( V \in \mathcal{X} - \mathcal{Z} \) on \( Z(V) \); as in [12, 4.6], it is clear that \( \delta_U^x \) determines an additive contravariant functor
\[ \delta_U^{x, ac} : ac(\tilde{F}^x) \longrightarrow \mathbb{Ab} \]
6.8.2.
and recall that for any \( n \in \mathbb{N} \) we have a canonical isomorphism [12, 4.10.7]
\[ C^n(\tilde{F}^x, \delta_U^{x, ac} \circ m^n_\theta) \cong C^n(I \times \tilde{F}^x, \delta_U^{x, ac} \circ n_\rho) \]
6.8.3.
Now, consider the contravariant functor
\[ \mathfrak{Re}(\bar{\rho}) : \tilde{F}^x \longrightarrow \mathbb{Ab} \]
6.8.4
which maps any \( Q \in \mathcal{X} \) on \( \text{Ker}(\bar{\rho}_Q^x) \) (cf. 2.8); then, from the functor and the natural map in 6.7.1, it is easily checked that we get the following exact sequence of contravariant functors
\[ \{0\} \longrightarrow \delta_U^{x, ac} \longrightarrow \delta_U^{x, ac} \circ m^n_\theta \longrightarrow \mathfrak{Re}(\bar{\rho}) \longrightarrow \{0\} \]
6.8.5.

6.9. At this point, it follows from [12, Proposition 5.14] that for any \( n \geq 1 \) we have
\[ H^n(\tilde{F}^x, \mathfrak{Re}(\bar{\rho}^x)) = \{0\} \]
6.9.1
provided we prove that, for any \( \tilde{F}^x \)-chain \( \tilde{q} : \Delta_n \rightarrow \tilde{F}^x \) such that \( \tilde{q}(0) \cong U \), \( p \) does not divide the cardinal of the set \( T_{\tilde{q}} \) of triples \( (\tilde{\alpha}, V, \tilde{\gamma}) \) such that \( \tilde{\alpha} : \tilde{q}(n) \rightarrow P \) is an \( \tilde{F}^x \)-morphism, the triple \( (\tilde{\gamma}, V, i^n_\rho) \) belongs to \( I_{\tilde{q}(0)} \) and any minimal \( \tilde{F}^x \)-morphism \( \tilde{\theta} : V \rightarrow Q \) dividing \( \tilde{\alpha} \circ \tilde{q}(0 \bullet n) \circ \tilde{\gamma} \) also divides \( i^n_\rho \).

In order to give another description of this set, we consider
\[ \tilde{F}(P, V)_\gamma = \bigcup_T \tilde{F}(P, V)_\gamma \]
6.9.2
where \( T \) runs over the set of subgroups of \( P \) which strictly contain \( V \); moreover, for any \( \tilde{F}^x \)-morphism \( \theta : V \rightarrow Q \), we denote by \( \theta_* : V \cong \theta(V) \) the isomorphism determined by \( \theta \).

Lemma 6.10. Let \( \tilde{q} : \Delta_n \rightarrow \tilde{F}^x \) be an \( \tilde{F}^x \)-chain such that \( \tilde{q}(0) \cong U \) and choose a representative \( \chi : q(0) \rightarrow q(n) \) of \( \tilde{q}(0 \bullet n) \). Then, we have a bijection between the complement in \( \tilde{F}(P, q(n)) \times \tilde{F}(P, q(0)) \) of the subset
\[ \{ (\tilde{\alpha}, \tilde{\delta} \circ (\alpha \circ \chi)^{-1}) \mid \tilde{\alpha} \in \tilde{F}(P, q(n)), \tilde{\delta} \in \tilde{F}(P, (\alpha \circ \chi)_*, q(0)) \} \]
6.10.1.
where \( \alpha \) is a representative of \( \tilde{\alpha} \), and the set \( T_{\tilde{q}} \), mapping \( (\tilde{\alpha}, \tilde{\delta} \circ (\alpha \circ \chi)^{-1}) \) on \( (\tilde{\alpha}, \delta_\gamma(q(0)), \delta^{-1}) \), where \( \delta \) is a representative of \( \tilde{\delta} \). In particular, \( p \) does not divide \( |T_{\tilde{q}}| \).
Proof: Since \( p \) does not divide \( |\tilde{F}(P, q(n)) \times \tilde{F}(P, q(0))| \) (cf. Proposition 5.5), the second statement follows from the first one provided we prove that \( p \) divides \( |\tilde{F}(P, V)| \); but, it is easily checked that if \( R \) is a subgroup of \( P \) containing \( T \) then \( \tilde{F}(P, V)_{\xi} \) is contained in \( \tilde{F}(P, V)_{\xi'} \); thus, we have
\[
\tilde{F}(P, V)_\gamma = \bigcup_{\xi} \tilde{F}(P, V)_{\xi'}
\]
where \( \bar{T} \) runs over the set of subgroups of order \( p \) of \( \bar{N}_P(V) \) and \( T \) is the converse image of \( \bar{T} \) in \( N_P(V) \). Since \( \bar{N}_P(V) \) acts on \( \tilde{F}(P, V)_\gamma \), in order to prove that \( p \) divides \( |\tilde{F}(P, V)| \), it suffices to show that this action has no fixed points. Arguing by contradiction, let \( \bar{\theta} \in \tilde{F}(P, V) \), be an \( \bar{N}_P(V) \)-fixed element; thus, there is a subgroup \( \bar{T} \) of order \( p \) of \( \bar{N}_P(V) \) such that \( \bar{\theta} \) belongs to \( \tilde{F}(P, V)_{\xi'} \) and then, according to [11, statement 6.6.4], the \( \bar{T} \)-orbit of \( \bar{\theta} \) is regular; since \( \bar{T} \) is contained in \( \bar{N}_P(V) \), we get a contradiction.

In order to prove the announced bijection, first of all note that, since \( q(0) \) is isomorphic to \( U \), we have
\[
\check{q}(0) \cap \tilde{F}^x = \bigoplus_{(\gamma, V, \xi') \in \check{V}(0)} V \cong \bigoplus_{\delta \in \tilde{F}(P, q(0))} \check{q}(0)
\]
and an \( \tilde{F}^x \)-morphism \( \bar{\theta} : V \to Q \) divides \( \tilde{\alpha} \circ \check{q}(0 \bullet n) \circ \tilde{\gamma} \) if and only if, for a representative \( \gamma \) of \( \tilde{\gamma} \), \( \tilde{\theta} \circ \tilde{\gamma}^{-1} \) divides \( \tilde{\alpha} \circ \tilde{q}(0 \bullet n) \); similarly, \( \bar{\theta} \) divides \( \xi' \) if and only if \( \bar{\theta} \circ \tilde{\gamma}_s^{-1} \) divides \( \xi' \circ \tilde{\gamma}_s^{-1} \). Moreover, setting \( W = (\alpha \circ \chi)_* (q(0)) \), the minimal \( \tilde{F}^x \)-morphisms from \( \check{q}(0) \) dividing \( \tilde{\alpha} \circ \check{q}(0 \bullet n) \) correspond bijectively with the subgroups of order \( p \) of \( \bar{N}_P(W) \); thus, the existence of a minimal \( \tilde{F}^x \)-morphism from \( \check{q}(0) \) dividing \( \tilde{\alpha} \circ \check{q}(0 \bullet n) \), which does not divide \( \check{\delta} \) in \( \tilde{F}(P, q(0)) \) is equivalent to the existence of a subgroup \( \bar{T} \) of order \( p \) of \( \bar{N}_P(W) \) such that \( \check{\delta} \) belongs to \( \tilde{F}(P, W)_{\xi'} \circ (\alpha \circ \chi)_* \). Consequently, with all this notation, \( (\tilde{\alpha}, V, \tilde{\gamma}) \) belongs to \( T_\check{q} \) if and only if \( \xi' \circ \tilde{\gamma}_s^{-1} \) does not belong to \( \tilde{F}(P, W)_{\xi} \circ (\alpha \circ \chi)_* \). We are done.

6.11. On the other hand, let us consider the functors (cf. 2.16)
\[
\text{aut}_x : ch^*(F^x) \to \mathcal{G} \subset \mathcal{G} \quad \text{and} \quad \text{loc}_{\mathcal{M}^x} : ch^*(F^x) \to \mathcal{G}
\]
respectively mapping any \( F^x \)-chain \( q : \Delta_n \to F^x \) on \( (F(q), \{1\}) \) and on \( (\tilde{\mathcal{M}}^x(\check{q}), \text{Ker}(\rho^\check{q})) \) where \( \check{q} : \Delta_n \to \tilde{\mathcal{M}}^x \) is a \( \mathcal{M}^x \)-chain lifting \( q \) and \( \rho^\check{q} \) denotes the restriction of \( (\rho^\check{q})_{\check{q}(n)} \) to \( \tilde{\mathcal{M}}^x(\check{q}) \); then, we have an obvious natural map
\[
\nu_{\mathcal{M}^x} : \text{loc}_{\mathcal{M}^x} \to \text{aut}_x
\]
determined by the structural functor \( \bar{\rho} : \tilde{\mathcal{M}}^x \to F^x \).
Theorem 6.12. With the notation above, the structural functor $\rho^x$ admits a unique natural $F^x$-isomorphism class of $F^x$-locality functorial sections

$$\sigma^x : F^x \rightarrow M^x$$

Proof: It follows from Proposition 2.17 that at least the natural map $\nu_{\tilde{M}^x}$ admits a unique natural section

$$\mu_{\tilde{M}^x} : \text{aut}_{F^x} \rightarrow \text{loc}_{\tilde{M}^x}$$

fulfilling the conditions there; that is to say, for any $\tilde{M}^x$-chain $\bar{q} : \Delta_n \rightarrow \tilde{M}^x$, the second structural group homomorphism $\tilde{M}^x(q) \rightarrow F(\tilde{\rho}^x \circ \bar{q})$ admits a section

$$\mu_{\bar{q}} : F(\tilde{\rho}^x \circ \bar{q}) \rightarrow \tilde{M}^x(\bar{q})$$

which is compatible with the first structural functor and is unique up to $\text{Ker}(\tilde{\rho}^x)$-conjugation; moreover, for another $\tilde{M}^x$-chain $\bar{r} : \Delta_m \rightarrow \tilde{M}^x$ and any $\chi^*(\tilde{M}^x)$-morphism $(\bar{\eta}, \bar{\delta}) : (\bar{r}, \Delta_m) \rightarrow (\bar{q}, \Delta_n)$, the diagram

$$\xymatrix{ F(\tilde{\rho}^x \circ \bar{q}) \ar[r]^{\mu_{\bar{q}}} \ar[d] & \tilde{M}^x(\bar{q}) \ar[d] \\ F(\tilde{\rho}^x \circ \bar{r}) \ar[r]^{\mu_{\bar{r}}} & \tilde{M}^x(\bar{r}) }$$

is commutative up to $\text{Ker}(\tilde{\rho}^x)$-conjugation.

In particular, assume that $n = 0$, $m = 1$ and $\delta = \delta_0^1$, and, setting $Q = \bar{r}(1)$, $R = \bar{r}(0)$, $x = \bar{r}(0 \cdot 1)$ and $\varphi = (\tilde{\rho}^x \circ \bar{r})(0 \bullet 1)$, assume that $\bar{q}(0) = R$, $\bar{\eta}_0 = (\tilde{\rho}^x)(1)$ and $\tilde{\rho}^x(\bar{\eta})_0 = \text{id}_R$; then, $F(\tilde{\rho}^x \circ \bar{r})$ coincides with the stabilizer $F^x(Q)_\varphi$ of $\varphi(R)$ in $F^x(Q)$, $\tilde{M}(\bar{r})$ coincides with the stabilizer $\tilde{M}^x(Q)_x$ of $\varphi(R)$ in $\tilde{M}^x(Q)$ and diagram 6.12.4 becomes

$$\xymatrix{ F^x(R) \ar[r]^{\mu_{\bar{r}}} \ar[d] & \tilde{M}^x(R) \ar[d] \\ F^x(Q)_\varphi \ar[r]^{\mu_{\bar{r}}} & \tilde{M}^x(Q)_x }$$

where $\kappa_x : \tilde{M}^x(Q)_x \rightarrow \tilde{M}^x(R)$ sends $a \in \tilde{M}^x(Q)_x$ to the unique $b \in \tilde{M}^x(R)$ fulfilling $x \cdot b = a \cdot x$; moreover, since $F^x(Q)_\varphi$ and $\tilde{M}^x(Q)_x$ are respectively contained in $F^x(Q)$ and in $\tilde{M}^x(Q)$, and since $\tilde{M}^x(Q)_x$ contains $\text{Ker}(\tilde{\rho}^x)_x$, we actually may assume that $\mu_x$ is just the restriction of $\mu_Q$; then note that, for some choice of $x$ lifting $\varphi$, diagram 6.12.5 becomes commutative.

Considering the action of $F(Q) \times F(R)$ on $\tilde{M}^x(Q, R)$ defined by the composition on the left- and on the right-hand via the chosen sections

$$\mu_Q : F(Q) \rightarrow \tilde{M}^x(Q) \quad \text{and} \quad \mu_R : F(R) \rightarrow \tilde{M}^x(R)$$

6.12.6,
if we choose a lifting \( x \) of \( \varphi \) such that diagram 6.12.5 becomes commutative then we have the equality of stabilizers
\[
(F(Q) \times F(R))_x = (F(Q) \times F(R))_{\varphi}
\] 6.12.7.
Indeed, since \( x \) lifts \( \varphi \), the inclusion of the left-hand member in the right-hand one is clear; conversely, for any pair \((\alpha, \beta) \in (F(Q) \times F(R))_{\varphi}\) we have \( \alpha \circ \varphi = \varphi \circ \beta \) and, in particular, \( \alpha \) belongs to \( F(Q)_{\varphi} \); then, the commutativity of diagram 6.12.5 above yields
\[
\kappa_x(\mu_Q(\alpha)) = \mu_R(\beta)
\] 6.12.8,
which amounts to saying that \( x \cdot \mu_R(\beta) = \mu_Q(\alpha) \cdot x \), so that \((\alpha, \beta)\) belongs to \( (F(Q) \times F(R))_x \).

This allows us to choose a family of liftings \( \{x_{\varphi}\}_\varphi \), where \( \varphi \) runs over the set of \( F^x \)-morphisms, which is compatible with \( F^x \)-isomorphisms; precisely, choose a set of representatives \( \mathcal{X} \) for the set of \( F \)-isomorphism classes in \( \mathcal{X} \), for any pair of subgroups \( Q \) and \( R \) in \( \mathcal{X} \) choose a set of representatives \( F_{Q,R} \) in \( F(Q,R) \) for the set of \( F(Q) \times F(R) \)-orbits, and for any \( \varphi \in F(Q,R) \) choose a lifting \( x_{\varphi} \in M^x(Q,R) \) such that equality 6.12.7 holds; thus, any \( Q \in \mathcal{X} \) determines a unique \( \hat{Q} \) in \( \mathcal{X} \), and moreover we choose an \( F \)-isomorphism \( \omega_Q : Q \cong \hat{Q} \) and a lifting \( x_Q \in M^x(\hat{Q},Q) \) of \( \omega_Q \) such that equality 6.12.7 holds. Consequently, any \( F^x \)-morphism \( \varphi : R \to Q \) determines \( \hat{Q}, \hat{R} \in \mathcal{X} \) and \( \hat{\varphi} \in F_{\hat{Q},\hat{R}} \) fulfilling
\[
\varphi = \omega_Q^{-1} \circ \hat{\alpha} \circ \hat{\varphi} \circ \hat{\beta} \circ \omega_R
\] 6.12.9
for suitable \( \hat{\alpha} \in F(\hat{Q}) \) and \( \hat{\beta} \in F(\hat{R}) \), and then we define
\[
x_{\varphi} = x_Q^{-1} \cdot \mu_Q(\hat{\alpha}) \cdot x_{\varphi} \cdot \mu_R(\hat{\beta}) \cdot x_R
\] 6.12.10.
At this point, it is routine to check that
6.12.11 For any \( F^x \)-isomorphisms \( \alpha \in F(Q',Q) \) and \( \beta \in F(R,R') \) we have
\[
x_{\alpha \circ \varphi \circ \beta} = x_{\alpha} \cdot x_{\varphi} \cdot x_{\beta}
\]
Note that, for any \( Q \in \mathcal{X} \) and any \( \alpha \in F(Q) \) we may assume that
\[
x_{\alpha} = \mu_Q(\alpha)
\] 6.12.12;
in particular, since the section \( \mu_Q \) is compatible with the first structural functor, for any \( \zeta \in F_P(Q) \) we have \( x_{\zeta} = \hat{\varphi}_Q(\zeta) \) for some \( u_{\zeta} \in N_P(Q) \) lifting \( \zeta \).

Then, for any triple of subgroups \( Q, R \) and \( T \) in \( \mathcal{X} \), and any pair of \( F \)-morphisms \( \psi : T \to R \) and \( \varphi : R \to Q \), since \( x_{\varphi} \cdot x_{\psi} \) and \( x_{\varphi \circ \psi} \) have the same image \( \varphi \circ \psi \in F(Q,T) \), the divisibility of \( M^x \) guarantees the existence and the uniqueness of \( k_{\varphi,\psi} \in \text{Ker}(\hat{\rho}_T^x) \) fulfilling
\[
x_{\varphi} \cdot x_{\psi} = x_{\varphi \circ \psi} \cdot k_{\varphi,\psi}
\] 6.12.13.
That is to say, we get a correspondence mapping any $\mathcal{F}^x$-chain $q : \Delta_2 \to \mathcal{F}^x$ on $k_{q(0 \bullet 1), q(1 \bullet 2)}$ and, considering the contravariant functor (cf. 2.8)

$$\text{Ref}(\hat{\rho}^x) : \hat{\mathcal{F}}^x \to \text{Ab}$$

6.12.14

and setting

$$C^n(\hat{\mathcal{F}}^x, \text{Ref}(\hat{\rho}^x)) = \prod_{\bar{q} \in \text{st}(\Delta_n, \hat{\mathcal{F}}^x)} \text{Ker}(\hat{\rho}^x_{\bar{q}(0)})$$

6.12.15

for any $n \in \mathbb{N}$, we claim that this correspondence determines a regular stable element $k$ of $C^2(\hat{\mathcal{F}}^x, \text{Ref}(\hat{\rho}^x))$ [11, A3.17 and A5.7].

Indeed, from 6.12.11 above, it is quite clear that $k_{\varphi, \psi} = 0$ if either $\varphi$ or $\psi$ is an isomorphism; moreover, for another isomorphic $\mathcal{F}^x$-chain $q' : \Delta_2 \to \mathcal{F}^x$ and a natural isomorphism $\nu : q \cong q'$, setting

$$\psi = q(0 \bullet 1), \quad \varphi = q(1 \bullet 2), \quad \psi' = q'(0 \bullet 1), \quad \varphi' = q'(1 \bullet 2)$$

$$\nu_0 = \gamma, \quad \nu_1 = \beta \quad \text{and} \quad \nu_2 = \alpha$$

6.12.16,

once again from 6.12.11 we have

$$x_{\varphi'} = x_{\alpha} \cdot x_{\beta} \cdot x_{\gamma}^{-1}, \quad x_{\psi'} = x_{\beta} \cdot x_{\gamma} \cdot x_{\delta}^{-1}$$

and therefore we get

$$x_{\varphi' \circ \psi'} = x_{\alpha} \cdot x_{\beta} \cdot x_{\gamma}^{-1} \cdot x_{\delta}^{-1} \cdot x_{\gamma} \cdot x_{\delta}^{-1} \cdot x_{\gamma}^{-1}$$

6.12.17

$$= x_{\alpha} \cdot x_{\beta} \cdot x_{\gamma}^{-1} \cdot x_{\delta}^{-1} \cdot x_{\gamma}^{-1} \cdot x_{\gamma}^{-1} \cdot x_{\delta}^{-1} \cdot x_{\gamma}^{-1}$$

6.12.18

$$= x_{\varphi' \circ \psi'} \cdot (\text{Ref}(\hat{\rho}^x)(\gamma^{-1}))(k_{\varphi, \psi})$$

so that $k_{\varphi', \psi'} = (\text{Ref}(\hat{\rho}^x)(\gamma^{-1}))(k_{\varphi, \psi})$; this proves that the correspondence $k$ sending $(\tilde{\varphi}, \tilde{\psi})$ to $k_{\varphi, \psi}$ is stable and, in particular, that $k_{\varphi, \psi}$ only depends on the corresponding $\hat{\mathcal{F}}^x$-morphisms $\tilde{\varphi}$ and $\tilde{\psi}$, so that we set $k_{\tilde{\varphi}, \tilde{\psi}} = k_{\varphi, \psi}$.

On the other hand, considering the usual differential map

$$d^x_2 : C^2(\hat{\mathcal{F}}^x, \text{Ref}(\hat{\rho}^x)) \to C^3(\hat{\mathcal{F}}^x, \text{Ref}(\hat{\rho}^x))$$

6.12.19,

we claim that $d^x_2(k) = 0$; indeed, for a third $\mathcal{F}^x$-morphism $\eta : W \to T$ we get

$$(x_{\varphi' \cdot \psi'}) \cdot x_{\eta} = (x_{\varphi' \circ \psi'}) \cdot x_{\eta} = (x_{\varphi' \circ \psi') \cdot (\text{Ref}(\hat{\rho}^x)(\overline{\eta}))(k_{\tilde{\varphi}, \tilde{\psi}})$$

6.12.20

$$x_{\varphi' \cdot \psi'} \cdot x_{\eta} = x_{\varphi' \cdot \psi'} \cdot x_{\eta} = x_{\varphi' \circ \psi'} \cdot x_{\eta} = x_{\varphi' \circ \psi'} \cdot x_{\eta}$$
and the divisibility of $\hat{M}^x$ forces
\[ k_{\tilde{x}_o\tilde{y},\tilde{y}} (\tilde{\text{Rec}}(\tilde{\rho}^x)(\tilde{\eta})) (k_{\tilde{x},\tilde{y}}) = k_{\tilde{x}_o\tilde{y},\tilde{y}} k_{\tilde{x},\tilde{y}} \]
6.12.21;
since $\text{Ker}(\tilde{\rho}^x)$ is Abelian, with the additive notation we obtain
\[ 0 = (\tilde{\text{Rec}}(\tilde{\rho}^x)(\tilde{\eta})) (k_{\tilde{x},\tilde{y}}) - k_{\tilde{x}_o\tilde{y},\tilde{y}} + k_{\tilde{x}_o\tilde{y},\tilde{y}} - k_{\tilde{x},\tilde{y}} \]
6.12.22,
proving our claim.

At this point, it follows from [12, Proposition 5.14] and from Lemma 6.10 above that we have $\Pi^2(\tilde{F}^x, \tilde{\text{Rec}}(\tilde{\rho}^x)) = \{0\}$ and therefore that $k = d_x(\ell)$ for some element $\ell = (\ell_1)_{\tilde{F}^x}^{\text{Rec}(\Delta_1,\tilde{F}^x)}$ in $C^1(\tilde{F}^x, \tilde{\text{Rec}}(\tilde{\rho}^x))$; that is to say, with the notation above we get
\[ k_{\tilde{x},\tilde{y}} = (\tilde{\text{Rec}}(\tilde{\rho}^x)(\tilde{\eta})) (\ell_{\tilde{x}} - (\ell_{\tilde{y}}))^{-1} \]
6.12.23
where we identify any $\tilde{F}^x$-morphism with the obvious $\tilde{F}^x$-chain $\Delta_1 \to \tilde{F}^x$; hence, from equality 6.12.13 we obtain
\[ (\ell_{\tilde{x}} - (\ell_{\tilde{y}}))^{-1} = (x_{\varphi} - x_{\psi}) \cdot \left( (\tilde{\text{Rec}}(\tilde{\rho}^x)(\tilde{\eta})) (\ell_{\tilde{x}} - (\ell_{\tilde{y}}))^{-1} \right) \]
6.12.24,
which amounts to saying that the correspondence $\sigma^x$ sending $\varphi \in F(Q,R)$ to $x_{\varphi} - x_{\psi} \in \hat{M}^x(Q,R)$ defines a functorial section of $\tilde{\rho}^x$. Note that in the case that $Q = R = T$ and that $\varphi$ and $\psi$ are both inner automorphisms then equality 6.12.23 forces $\ell_{\tilde{x}} = \ell_{\tilde{y}}$ and from 6.12.12 we get $\sigma^x(\varphi) = \psi(u_{\varphi})$ for some $u_{\varphi} \in NP(Q)$ lifting $\varphi$.

We can modify this correspondence in order to get an $F^x$-locality functorial section; indeed, for any $\tilde{F}^x$-morphism $\zeta: R \to Q$, choosing $u_{\zeta}$ in $TP(R,Q)$ lifting $\zeta$, the $\hat{M}^x$-morphisms $\sigma^x(\zeta)$ and $\psi_{Q,R}(u_{\zeta})$ both lift $\zeta$; once again, the divisibility of $\hat{M}^x$ guarantees the existence and the uniqueness of $m_{\zeta} \in \text{Ker}(\tilde{\rho}^x)$ fulfilling
\[ \psi_{Q,R}(u_{\zeta}) = \sigma^x(\zeta) m_{\zeta} \]
6.12.25
and the remark above proves that $m_{\zeta}$ only depends on $\zeta \in \tilde{F}^x_p(Q,R)$. Moreover, for a second $\tilde{F}^x$-morphism $\zeta: T \to R$, we get
\[ \sigma^x(\zeta \circ \xi) m_{\zeta \circ \xi} = \psi_{Q,T}(u_{\zeta \circ \xi}) = \psi_{Q,R}(u_{\zeta}) \psi_{R,T}(u_{\xi}) = \sigma^x(\zeta) m_{\zeta} \sigma^x(\xi) m_{\xi} \]
6.12.26.

Always the divisibility of $\tilde{M}^x$ forces

$$m_{\tilde{\xi}:\tilde{\xi}} = (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\xi}))(m_{\tilde{\xi}})$$  6.12.27

and, since $\text{Ker}(\tilde{\rho}_x)$ is Abelian, with the additive notation we obtain

$$0 = (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\xi}))(m_{\tilde{\xi}}) - m_{\tilde{\xi}:\tilde{\xi}} + m_{\tilde{\xi}}$$  6.12.28;

that is to say, denoting by $i^* : \tilde{F}_p^x \subset \tilde{F}_p^x$ the obvious inclusion functor, the correspondence $m$ sending any $\tilde{F}_p^x$-morphism $\tilde{\xi} : R \to Q$ to $m_{\tilde{\xi}}$ defines a $1$-cocycle in $\tilde{C}^1(\tilde{F}_p^x, \hat{\text{ker}}(\tilde{\rho})^x \circ i^*)$; but, since the category $\tilde{F}_p^x$ has a final object, we actually have [11, Corollary A4.8]

$$\tilde{H}^1(\tilde{F}_p^x, \hat{\text{ker}}(\tilde{\rho})^x \circ i^*) = \{0\}$$  6.12.29;

consequently, we obtain $m = d_0^x(z)$ for some element $z = (z_Q)_{Q \in \mathcal{X}}$ in

$$\tilde{C}^0(\tilde{F}_p^x, \hat{\text{ker}}(\tilde{\rho})^x \circ i^*) = \tilde{C}^0(\tilde{F}_p^x, \hat{\text{ker}}(\tilde{\rho})^x)$$  6.12.30.

In conclusion, equality 6.12.25 becomes

$$\tilde{\nu}_{Q,R}^x(u_\xi) = \sigma^x(\xi) \cdot (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\xi}))(z_Q) \cdot z_R^{-1} = z_Q \cdot \sigma^x(\xi) \cdot z_R^{-1}$$  6.12.31

and therefore the correspondence $\sigma^x : \mathcal{F}_x \to \tilde{M}^x$ is a second $\mathcal{F}_x$-locality functorial section of $\tilde{\rho}^x$.

Finally, assume that $\sigma''^x : \mathcal{F}_x \to \tilde{M}^x$ is a second $\mathcal{F}_x$-locality functorial section of $\tilde{\rho}^x$; for any $\mathcal{F}_x$-morphism $\varphi : R \to Q$, we set $x'_\varphi = \sigma''^x(\varphi)$ and $x''_\varphi = \sigma''^x(\varphi)$ for short; since these elements have the same image in $\mathcal{F}(Q, R)$, the divisibility of $\tilde{M}^x$ forces again the existence of a unique $\ell_\varphi \in \text{Ker}(\tilde{\rho}_x^\pi)$ fulfilling $x''_\varphi = x'_\varphi \cdot \ell_\varphi$; moreover, since $\sigma'$ and $\sigma''$ are $\mathcal{F}_x$-locality functors, it is easily checked that $\ell_\varphi$ only depends on $\tilde{\varphi} \in \mathcal{F}(Q, R)$. Now, denoting by $\ell$ the element of $\tilde{C}^1(\tilde{F}_p^x, \hat{\text{ker}}(\tilde{\rho})^x)$ defined by the correspondence sending $\tilde{\varphi}$ to $\ell_{\tilde{\varphi}}$, we claim that $d_1^x(\ell) = 0$; indeed, for a second $\mathcal{F}_x$-morphism $\tilde{\psi} : T \to R$ we get

$$x''_{\varphi\tilde{\psi}} = x''_{\varphi} \cdot x''_{\tilde{\psi}} = (x'_\varphi \cdot \ell_{\tilde{\varphi}}) \cdot (x'_\varphi \cdot \ell_{\tilde{\varphi}}) = x'_{\varphi \tilde{\psi}} \cdot (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\psi}))(\ell_{\tilde{\varphi}} \cdot \ell_{\tilde{\psi}})$$  6.12.32;

then, as above, the divisibility of $\tilde{M}^x$ forces

$$\ell_{\tilde{\varphi} \tilde{\psi}} = (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\psi}))(\ell_{\tilde{\varphi}} \cdot \ell_{\tilde{\psi}})$$  6.12.33

and, since $\text{Ker}(\tilde{\rho}_x^\pi)$ is Abelian, with the additive notation we obtain

$$0 = (\hat{\text{ker}}(\tilde{\rho})^x(\tilde{\psi}))(\ell_{\tilde{\varphi}}) - \ell_{\tilde{\varphi} \tilde{\psi}} + \ell_{\tilde{\psi}}$$  6.12.34.
At this point, it follows from [12, Proposition 5.14] and from Lemma 6.10 above that we have \( \mathbb{H}^1(\tilde{\mathcal{F}}^x, \mathfrak{Re}t(\tilde{\rho}^x)) = \{0\} \) and therefore that \( \ell = d^x_v(z) \) for some element \( z = (z_Q)_{Q \in \mathcal{X}} \) in \( \mathbb{C}^0(\tilde{\mathcal{F}}^x, \mathfrak{Re}t(\tilde{\rho}^x)) \); that is to say, with the notation above we get

\[
\ell = (\mathfrak{Re}t(\tilde{\rho}^x)(\tilde{\varphi}))(z_Q) - z_R^{-1}
\]

where we identify any \( \tilde{\mathcal{F}}^x \)-object with the obvious \( \tilde{\mathcal{F}}^x \)-chain \( \Delta_0 \to \tilde{\mathcal{F}}^x \); hence, we obtain

\[
\sigma''(\varphi) = x'' = x'_r(\mathfrak{Re}t(\tilde{\rho}^x)(\tilde{\varphi}))(z_Q) - z_R^{-1} = z_Q \sigma^x(\tilde{\varphi}) z_R^{-1} \]

which amounts to saying that the correspondence \( \nu \) sending \( Q \) to \( z_Q \) defines a natural \( \mathcal{F}^x \)-isomorphism between \( \sigma'' \) and \( \sigma'' \). We are done.

7. The perfect \( \mathcal{F} \)-locality extending the perfect \( \mathcal{F}^{\sigma} \)-locality

7.1 Let \( P \) be a finite \( p \)-group and \( \mathcal{F} \) a Frobenius \( P \)-category; from section 6 we already know the existence and the uniqueness of a perfect \( \mathcal{F}^{\sigma} \)-locality \( \mathcal{P}^{\sigma} \); as a matter of fact, in [11, Chap. 20]† we already have proved that any perfect \( \mathcal{F}^{\sigma} \)-locality \( \mathcal{P}^{\sigma} \) can be extended to a unique perfect \( \mathcal{F} \)-locality \( \mathcal{P} \) (cf. 2.8); in this section, we prove the following more precise result which actually shows the existence and the uniqueness of a perfect \( \mathcal{F} \)-locality.

Theorem 7.2. Any perfect \( \mathcal{F}^{\sigma} \)-locality \( \mathcal{P}^{\sigma} \) can be extended to a unique perfect \( \mathcal{F} \)-locality \( \mathcal{P} \). Moreover, for any \( p \)-coherent \( \mathcal{F} \)-locality \( L \), any \( \mathcal{F}^{\sigma} \)-locality functor \( h^{\sigma} \) from \( \mathcal{P}^{\sigma} \) to \( L^{\sigma} \) can be extended to a unique \( \mathcal{F} \)-locality functor

\[
h : \mathcal{P} \longrightarrow L
\]

7.3. Let us consider a set \( \mathcal{X} \) of subgroups of \( P \) containing the set of \( \mathcal{F} \)-selfcentralizing subgroups and any subgroup \( Q \) of \( P \) such that \( \mathcal{F}(Q, R) \neq \emptyset \) for some \( R \in \mathcal{X} \). Arguing by induction on \( |\mathcal{X}| \), we will construct the perfect \( \mathcal{F} \)-locality \( \mathcal{P} \) extending \( \mathcal{P}^{\sigma} \) and the \( \mathcal{F} \)-locality functor \( h : \mathcal{P} \to L^{\sigma} \) extending \( h^{\sigma} \). We may assume that \( \mathcal{X} \) contains subgroups of \( P \) which are not \( \mathcal{F} \)-selfcentralizing, choose a minimal one \( U \) and set

\[
\mathcal{Y} = \mathcal{X} - \{ \theta(U) \mid \theta \in \mathcal{F}(P, U) \}
\]

from now on, we assume that there exist a perfect \( \mathcal{F}^{\theta} \)-locality \( \mathcal{P}^{\theta} \) and a functor \( h^{\theta} : \mathcal{P}^{\theta} \to L^{\theta} \), and we denote by

\[
\tau^{\theta} : \mathcal{T}^{\theta} \longrightarrow \mathcal{P}^{\theta} \quad \text{and} \quad \pi^{\theta} : \mathcal{P}^{\theta} \longrightarrow \mathcal{F}^{\theta}
\]

\[
\bar{\tau}^{\theta} : \mathcal{T}^{\theta} \longrightarrow \mathcal{L}^{\theta} \quad \text{and} \quad \bar{\pi}^{\theta} : \mathcal{L}^{\theta} \longrightarrow \mathcal{F}^{\theta}
\]

† The argument in [11, 20.16] has been scratched; below we develop a correct argument.
the corresponding structural functors (cf. 2.7.1); for any pair of subgroups $Q$ and $R$ in $\mathfrak{Q}$ we set $\mathcal{P}_{}^\times(Q,R) = \mathcal{P}^\otimes_0(Q,R)$ and $h^\otimes_{Q,R} = h^\otimes_{Q,R}$, and if $R \subset Q$ then we set $i^Q_R = \tau^\otimes_{Q,R}(1)$ and $t^Q_R = \tau^\otimes_{Q,R}(1)$.

7.4. If $V \in \mathfrak{X} - \mathfrak{Q}$ is fully centralized in $\mathcal{F}$ then we consider $\hat{V} = V \cdot C_P(V)$ which is clearly $\mathcal{F}$-selfcentralizing; in particular, $\mathcal{P}_{}^\times(\hat{V})$ has been already defined above, and the structural functor $\tau^\otimes : \mathcal{P}_{}^\otimes \to \mathcal{P}^\otimes$ determines a group homomorphism $\tau^\otimes : N_P(\hat{V}) \to \mathcal{P}_{}^\times(\hat{V})$. Let $Q$ be a subgroup in $\mathfrak{Q}$ which contains and normalizes $V$; thus, $Q$ normalizes $\hat{V}$ and we set $\hat{Q} = Q \hat{V}$ which coincides with the converse image of $\mathcal{F}_Q(V)$ in $N_P(V)$; since $V$ is also fully $\mathcal{F}_Q(V)$-normalized in $\mathcal{F}$ [11, 2.10] and we have $N^\mathcal{F}_Q(V) = \hat{Q}$, we get the following Frobenius $Q$-category $\mathcal{F}^\times_\mathcal{Q}$ [11, Proposition 2.16] and the associated perfect $\mathcal{F}^\times_\mathcal{Q}$-locality [11, 17.4 and 17.5]

$$\mathcal{F}^\times_\mathcal{Q} = N^\mathcal{F}_Q(V)(V) \quad \text{and} \quad \mathcal{P}^\times_\mathcal{Q} = N^\mathcal{F}_Q(V)(V)$$

7.5. Since the hyperfocal subgroup $H_{\mathcal{F}^\times_\mathcal{Q}}$ (cf. 2.4) is a $\mathcal{F}^\times_\mathcal{Q}$-stable subgroup of $\hat{Q}$ [11, Lemma 13.3], it follows from [11, Theorem 17.18] that we have the quotient perfect $\mathcal{F}^\times_\mathcal{Q}/H_{\mathcal{F}^\times_\mathcal{Q}}$-locality $\mathcal{P}^\times_\mathcal{Q}/H_{\mathcal{F}^\times_\mathcal{Q}}$; but, it is easily checked from [11, Lemma 13.3] that $\mathcal{P}^\times_\mathcal{Q}/H_{\mathcal{F}^\times_\mathcal{Q}}$ can be identified to the full subcategory of $\mathcal{T}_{\mathcal{Q}_R/H} \mathcal{F}_R$ [11, 17.2] over the set of images in $\mathcal{Q}/H_{\mathcal{F}^\times_\mathcal{Q}}$ of the subgroups in $\mathfrak{Q}^\times_\mathcal{Q}$; hence, we have a canonical functor

$$\mathcal{T}^\times_\mathcal{Q} : \mathcal{P}^\times_\mathcal{Q} \longrightarrow \mathcal{T}_{\mathcal{Q}_R/H} \mathcal{F}_R$$

compatible with the structural functors; in particular, we have a group homomorphism $(\mathcal{T}^\times_\mathcal{Q})_{\otimes} : \mathcal{P}_{}^\times(\hat{V}) \to \mathcal{Q}/H_{\mathcal{F}^\times_\mathcal{Q}}$ and we consider its kernel

$$O(V) = \text{Ker}\left((\mathcal{T}^\times_\mathcal{Q})_{\otimes}\right) = \mathcal{O}^\otimes\left(C_{\mathcal{L}_{}^\times(\hat{V})}\left(\tau^\otimes_{\hat{V}}(V)\right)\right)\cdot \tau^\otimes_{\hat{V}}(H_{C_{\mathcal{F}_R}(V)})$$

7.6. If $V' \in \mathfrak{X} - \mathfrak{Q}$ is also fully centralized in $\mathcal{F}$, setting $\hat{V}' = V' \cdot C_P(V')$ it follows from [11, statement 2.10.1] that any $\varphi \in \mathcal{F}(V',\hat{V'})$ can be extended to a suitable $\hat{\varphi} \in \mathcal{F}(V',\hat{V})$; thus, the restriction map

$$\mathcal{F}(\hat{V}',\hat{V})_{V',V} : \mathcal{F}(V',\hat{V})_{V',V} \longrightarrow \mathcal{F}(V',V)$$
is surjective. In this case, \( \mathcal{P}^x(\hat{V}', \hat{V}) \) has been already defined above, and the groups \( \mathcal{P}^x(\hat{V}) \) and \( \mathcal{P}^x(\hat{V}') \) act on this set by composition on the right-hand and on the left-hand respectively; moreover, it is quite clear that the respective subgroups \( O(V) \) and \( O(V') \) stabilize \( \mathcal{P}^x(\hat{V}', \hat{V})_{V', V} \), and that the corresponding quotient sets \( \mathcal{P}^x(\hat{V}', \hat{V})_{V', V}/O(V) \) and \( O(V')\mathcal{P}^x(\hat{V}', \hat{V})_{V', V} \) coincide with each other. Thus, we can define

\[
\mathcal{P}^x(V', V) = \mathcal{P}^x(\hat{V}', \hat{V})_{V', V}/O(V) = O(V')\mathcal{P}^x(\hat{V}', \hat{V})_{V', V}
\]

and we denote by

\[
\varphi_{V', V} : \mathcal{P}^x(\hat{V}', \hat{V})_{V', V} \rightarrow \mathcal{P}^x(V', V)
\]

the canonical map; moreover, since \( \mathcal{L} \) is \( p \)-coherent, the image of \( \hat{O}(V) \) in \( \mathcal{L}(V) \) is trivial, and therefore \( \hat{h}^x_{V', V} \) induces a map

\[
\hat{h}^x_{V', V} : \mathcal{P}(V', V) \rightarrow \mathcal{L}(V', V)
\]

fulfilling \( \varphi_{V', V} \circ \hat{h}^x_{V', V} = \varphi_{V', V} \circ \hat{h}^x_{V', V} \) where

\[
\hat{g}^x_{V', V} : \mathcal{L}^x(\hat{V}', \hat{V})_{V', V} \rightarrow \mathcal{L}^x(V', V)
\]

denotes the map determined by the divisibility in \( \mathcal{L} \).

7.7. It is clear that there is a unique map

\[
\pi_{V', V} : \mathcal{P}^x(V', V) \rightarrow C(V', V)
\]

such that, for any \( \hat{x} \in \mathcal{P}^x(\hat{V}', \hat{V})_{V', V} \), we have (cf. 7.6.1)

\[
\pi_{V', V}(\varphi_{V', V}(\hat{x})) = \hat{f}^x_{V', V}(\pi_{V', V}(\hat{x}))
\]

Similarly, if \( u \in P \) belongs to \( \mathcal{T}_p(V', V) \) then it belongs to \( \mathcal{T}_p(\hat{V}', \hat{V}) \) too, and we consider the map \( \tau^x_{V', V} : \mathcal{T}_p(V', V) \rightarrow \mathcal{P}^x(V', V) \) defined by

\[
\tau^x_{V', V}(u) = \varphi_{V', V}(\tau^y_{V', V}(u))
\]

Moreover, it is easily checked that

\[
\hat{h}^x_{V', V} \circ \tau^x_{V', V} = \pi_{V', V} \quad \text{and} \quad \pi_{V', V} \circ \hat{h}^x_{V', V} = \pi_{V', V}
\]

7.8. Further, it is clear that the composition in \( \mathcal{P}^y \) defines a compatible composition between those morphisms; explicitly, if \( V'' \in \mathfrak{A} - \mathfrak{I} \) is fully centralized in \( \mathcal{F} \), setting \( \bar{V}'' = V'' \cdot C_p(V'') \) we get a composition map

\[
\mathcal{P}^x(V'', V') \times \mathcal{P}^x(V', V) \rightarrow \mathcal{P}^x(V'', V)
\]

such that, for any \( \hat{x} \in \mathcal{P}^x(\hat{V}', \hat{V})_{V', V} \) and any \( \hat{x}' \in \mathcal{P}^x(\hat{V}'', \hat{V})_{V'', V'} \), we have

\[
\varphi_{V'', V}'(\hat{x}', \hat{x}) = \varphi_{V'', V}'(\hat{x}') \circ \varphi_{V', V}(\hat{x})
\]
and the associativity of the composition in $\mathcal{P}^\chi$ forces the obvious associativity here. Once again, it is easily checked that for any $x \in \mathcal{P}^x(V', V)$ and any $x' \in \mathcal{P}^x(V'', V')$ we have
\[
h^x_{V', V}(x' \cdot x) = h^x_{V'', V}(x') \cdot h^x_{V', V}(x) \quad 7.8.3.
\]

**Proposition 7.9.** With the notation and the hypothesis above, let $Q'$ be a subgroup $\mathcal{F}$-isomorphic to $Q$ which contains and normalizes $V'$, and set $\hat{Q}' = \hat{Q}' \hat{V}'$. For any element $x \in \mathcal{P}^x(Q', Q)_{V', V}$ there is $\hat{a} \in \mathcal{P}^x(\hat{Q}', \hat{Q})_{\hat{V}', \hat{V}}$ such that
\[
\hat{a}^{-1} \cdot i_{\hat{Q}' \cdot x} \in \mathcal{P}^\chi_{\hat{Q}' \cdot x}(\hat{Q}, Q) \quad \text{and} \quad \hat{t} \cdot i_{\hat{Q}' \cdot x}(\hat{a}^{-1} \cdot i_{\hat{Q}' \cdot x}) = 1 \quad 7.9.1.
\]
Moreover, the correspondence sending $x$ to $g_{\hat{V}', \hat{V}}(\hat{Q}' \cdot \hat{Q} \cdot (\hat{a}))$ determines a map
\[
g_{\hat{V}', \hat{V}}: \mathcal{P}^x(Q', Q)_{V', V} \rightarrow \mathcal{P}^x(V', V) \quad 7.9.2.
\]

**Proof:** Denoting by $\chi \in \mathcal{F}(V', V)$ the element fulfilling $\pi^x_{\hat{Q}' \cdot Q}(x) \circ i_{\hat{Q}'} = i_{\hat{Q}} \circ \chi$, we clearly have $\mathcal{F}_Q(\hat{Q}) = \mathcal{F}_{Q'}(V')$; then, since $V$ is normal in $\hat{Q}$ and we have $\mathcal{F}_Q(\hat{Q}) = \mathcal{F}_Q(V)$, it follows from [11, statement 2.10.1] that $\chi$ can be extended to an $\mathcal{F}$-morphism $\alpha: \hat{Q} \rightarrow P$ and, since $Q'$ coincides with the inverse image of $\mathcal{F}_Q(V')$ in $N_P(V')$, we clearly have $\alpha(\hat{Q}) = \hat{Q}'$; in particular, there is $\hat{a} \in \mathcal{P}(\hat{Q}', \hat{Q})_{\hat{V}', \hat{V}}$ such that $\hat{a}^{-1} \cdot i_{\hat{Q}' \cdot x}$ belongs to $\mathcal{P}^\chi_{\hat{Q}', \hat{Q}}(\hat{Q}, Q)$, and actually we can modify our choice of $\hat{a}$ in such a way that
\[
\hat{t} \cdot i_{\hat{Q}' \cdot x}(\hat{a}^{-1} \cdot i_{\hat{Q}' \cdot x}) = 1 \quad 7.9.3.
\]
Moreover, if $\hat{a}' \in \mathcal{P}(\hat{Q}', \hat{Q})_{\hat{V}', \hat{V}}$ is another choice such that
\[
\hat{a}'^{-1} \cdot i_{\hat{Q}' \cdot x} \in \mathcal{P}^\chi_{\hat{Q}' \cdot x}(\hat{Q}, Q) \quad \text{and} \quad \hat{t} \cdot i_{\hat{Q}' \cdot x}(\hat{a}'^{-1} \cdot i_{\hat{Q}' \cdot x}) = 1 \quad 7.9.4.
\]
then the difference $\hat{a}^{-1} \cdot \hat{a}'$ belongs to $\mathcal{P}^\chi_{\hat{Q}' \cdot x}(\hat{Q})$ and we have $\hat{t} \cdot i_{\hat{Q}' \cdot x}(\hat{a}^{-1} \cdot \hat{a}') = 1$; but, as in 7.5.2 above, we have
\[
\text{Ker}((\hat{t} \cdot i_{\hat{Q}' \cdot x})_\hat{Q}) = \mathcal{O}^\hat{p} \left(C_P^\chi_\hat{Q}(\tau^\chi_\hat{Q}(V)) \cdot \tau^\chi_\hat{Q}(H_{C_\mathcal{F}(V)}) \right) \quad 7.9.5;
\]
consequently, since we have
\[
g_{\hat{V}', \hat{V}} \left(\mathcal{O}^\hat{p} \left(C_P^\chi_\hat{Q}(\tau^\chi_\hat{Q}(V)) \right) \right) \subset \mathcal{O}^\hat{p} \left(C_P^\chi_\hat{V}(\tau^\chi_\hat{V}(V)) \right) \quad 7.9.6.
\]
the element $g_{\hat{V}', \hat{V}}(\hat{a}^{-1} \cdot \hat{a}')$ belongs to $O(V)$. We are done.
7.10. Now, we claim that the family of maps \( g_{v', v}^{Q', Q} \) obtained in Proposition 7.9 is compatible with the composition maps defined in 7.8 and it fulfills the corresponding transitivity condition.

**Proposition 7.11.** With the notation and hypothesis above, let \( Q'' \) be a subgroup of \( P \) which is \( \mathcal{F} \)-isomorphic to \( Q \) and \( Q' \), and \( V'' \in \mathcal{X} - \mathcal{Y} \) a normal subgroup of \( Q'' \) fully centralized in \( \mathcal{F} \). Then, for any \( x' \in \mathcal{P}(Q'', Q')_{V'', V'} \) and any \( x \in \mathcal{P}(Q', Q)_{V', V} \), we have

\[
g_{v''_{v''}, v''}^{Q'', Q'}(x' \cdot x) = g_{v''_{v''}, v''}^{Q'', Q'}(x') \cdot g_{v, v}^{Q', Q}(x) \tag{7.11.1}
\]

**Proof:** With the notation above, set \( \hat{Q}'' = Q'' \cdot \hat{V}'' \) and choose an element \( \hat{a}' \) in \( \mathcal{P}(\hat{Q}'', \hat{Q}')_{V'', V'} \) such that

\[
\hat{a}''^{-1} \hat{Q}''_{v''} x' \in \mathcal{P}^{v', Q'}(\hat{Q}', Q') \quad \text{and} \quad t^{v', Q'}(\hat{a}''^{-1} \hat{Q}''_{v''} x') = 1 \tag{7.11.2}
\]

then, setting \( \hat{a}'' = \hat{a}'' \cdot \hat{a} \) and \( x'' = x' \cdot x \), we claim that

\[
\hat{a}''^{-1} \hat{Q}''_{v''} x'' \in \mathcal{P}^{v, Q}(\hat{Q}, Q) \quad \text{and} \quad t^{v, Q}(\hat{a}''^{-1} \hat{Q}''_{v''} x'') = 1 \tag{7.11.3}
\]

We argue by induction on the length \( \ell \) of \( \pi^{x}_{Q''_{v''}, Q} \left( \hat{a}''^{-1} \hat{Q}''_{v''} x' \right) \) as an \( \mathcal{F}^{v', Q'} \)-morphism [11, 5.15 and 20.8.2]; if \( \ell = 0 \) then we have

\[
\hat{a}''^{-1} \hat{Q}''_{v''} x' = n' \hat{Q}''_{v'} \tag{7.11.4}
\]

for a suitable \( n' \in \mathcal{P}^{v', Q'}(\hat{Q}') \) fulfilling \( t^{v', Q'}(n') = 1 \) since we already know that \( t^{v', Q'}(\hat{Q}'_{v'}) = 1 \), and therefore we easily get

\[
\hat{a}''^{-1} \hat{Q}''_{v''} x'' = (\hat{a}''^{-1} n' \hat{a}) \cdot (\hat{a}''^{-1} \hat{Q}''_{v''} x) \tag{7.11.5}
\]

since \( \hat{a}''^{-1} n' \hat{a} \) belongs to \( \mathcal{P}^{v, Q}(\hat{Q}) \) and we have \( t^{v, Q}(\hat{a}''^{-1} n' \hat{a}) = 1 \), in this case we are done.

If \( \ell \geq 1 \) then we have

\[
\hat{a}''^{-1} \hat{Q}''_{v''} x' = y' \cdot v' \tag{7.11.6}
\]

for some \( \mathcal{F}^{v', Q'} \)-essential subgroup \( E' \) [11, 5.18] of \( \hat{Q}' \), some \( p' \)-element \( y' \) of the converse image of \( X_{\mathcal{F}^{v', Q'}(E')} \) [11, Corollary 5.13] in \( \mathcal{P}^{v', Q'}(E') \) and some element \( v' \) of \( \mathcal{P}^{v', Q'}(E', Q') \) in such a way that \( \pi^{x}_{Q''_{v''}, Q''} \left( t_{E'}^{Q', Q'}(y' \cdot v') \right) \) has length \( \ell - 1 \) [11, 5.15]. Note that we have \( \pi^{x}_{Q''_{v''}, Q''}(V') = V' \) and that, setting

\[
Q''' = \pi^{x}_{Q''_{v''}, Q''}(Q') \subset E' \subset \hat{Q}' \tag{7.11.7}
\]
and denoting by \( v'_* \) and \( y'_* \) the respective elements of \( \mathcal{P}^{v', Q'}(Q''', Q') \) and \( \mathcal{P}^{v', Q'}(Q', Q''') \) determined by \( v' \) and \( y' \), we get
\[
i^{Q'}_{v'} \cdot v' = i^{Q'''}_{Q'''} \cdot v'_* \quad \text{and} \quad \hat{a}^{-1} \cdot i^{Q'}_{Q''}, x' = i^{Q'}_{Q'} \cdot y'_* \cdot v'_* \] 7.11.8;

moreover, since \( y' \) is a \( p' \)-element, we have \( t^{v', Q'}(y'_*) = 1 \) which implies that \( t^{v', Q'}(y'_*) = 1 \) and therefore, since \( t^{v', Q'}(\hat{a}^{-1} \cdot i^{Q''}, x') = 1 \), we successively obtain \( t^{v', Q'}(v'_*) = 1 \) and \( t^{v', Q'}(i^{Q'}_{v'} \cdot v'_*) = 1 \).

Thus, by the induction hypothesis, we already know that
\[
\hat{a}^{-1} \cdot (i^{Q'}_{E'} \cdot v') \cdot x \in \mathcal{P}^{v, Q}(\hat{Q}, Q) \quad \text{and} \quad t^{v, Q}(\hat{a}^{-1} \cdot (i^{Q'}_{E'} \cdot v') \cdot x) = 1 \] 7.11.9

and therefore, setting \( E = (\pi \frac{E}{Q})^{-1}(E') \) and denoting by \( b \in \mathcal{P}^{x}(E', E) \) the element such that \( \hat{a}^{-1} \cdot i^{Q'}_{E'} = i^{Q}_{E} \cdot b^{-1} \), the divisibility in \( \mathcal{P}^{v, Q} \) implies that \( b^{-1} \cdot v' \cdot x \) belongs to \( \mathcal{P}^{v, Q}(E, Q) \) and then we still have \( t^{v, Q}(b^{-1} \cdot v' \cdot x) = 1 \); consequently, we still get
\[
\hat{a}^{-1} \cdot (i^{Q'}_{E'} \cdot y') \cdot V \cdot x = \hat{a}^{-1} \cdot (i^{Q'}_{E'} \cdot y' \cdot b) \cdot (b^{-1} \cdot v' \cdot x) \] 7.11.10

and, since \( b^{-1} \cdot y' \cdot b \) is a \( p' \)-element of \( \mathcal{P}^{v, Q}(E) \), we have \( t^{v, Q}(b^{-1} \cdot y' \cdot b) = 1 \), which proves our claim.

Now, according to 7.8 and to Proposition 7.9, we have
\[
\mathcal{g}^{v, Q}(x'') = \mathcal{g}^{v', Q}(\mathcal{g}^{v''. Q'}(\mathcal{g}^{v'''. Q'''}(\mathcal{g}^{v'''. Q'''}(\mathcal{g}^{v''. Q'}(\mathcal{g}^{v'. Q'}(\mathcal{g}^{v'. Q'}(\mathcal{g}^{v. Q'}(\mathcal{g}^{v. Q'}(x)))))))) \quad 7.11.11.
\]

We are done.

**Corollary 7.12.** With the notation and the hypothesis above, we have a group homomorphism
\[
\mathcal{g}^{Q}_v : \mathcal{P}^\tau(Q)_V \longrightarrow \mathcal{P}^\tau(V) \] 7.12.1.

fulfilling \( \mathcal{O}^p(\mathcal{Ker}(\mathcal{g}^{Q}_v)) = \mathcal{O}^p(\mathcal{C}_{\mathcal{P}^\tau(Q)}(\tau^\tau_Q(V))) \), and moreover have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{P}^\tau(Q', Q)_V & \xrightarrow{\mathcal{g}^{Q', Q}_v} & \mathcal{P}^\tau(V', V) \\
\downarrow^{b^x_{Q', Q}} & & \downarrow^{b^x_{V', V}} \\
\mathcal{L}^\tau(Q', Q)_V & \xrightarrow{\mathcal{g}^{Q', Q}_v} & \mathcal{L}^\tau(V', V)
\end{array} \] 7.12.2.
Proof: The first statement is easily checked from proposition 7.11 and from the following exact sequence (cf. 2.12.2)

\[ 1 \rightarrow H_{C_p(V)} \rightarrow C_p(V) \rightarrow \mathcal{P}(V) \rightarrow \mathcal{F}(V) \rightarrow 1 \]

7.12.3.

Moreover, for any \( x \in \mathcal{P}(Q', Q)_{V, V} \) and any \( \hat{a} \in \mathcal{P}(\hat{Q}', Q')_{V, V} \) fulfilling condition 7.9.1 above, it follows from the very definition of the perfect \( \mathcal{F}_{V, Q} \)-locality \( \mathcal{P}_{V, Q} / H_{P_{V, Q}} \) that the equality \( \tilde{t}^{V, Q}(\hat{a}^{-1}, i_{Q, x}) = 1 \) implies that (cf. 7.5.1)

\[ \tilde{t}^{-1, i_{Q, x}} \in i_{Q, x} \text{Ker}(\tilde{t}^{V, Q}) \]

7.12.4

and, since \( \mathcal{L} \) is \( p \)-coherent, it is easily checked that

\[ \tilde{g}_{v}^{Q}\left(h_{x}^{v}(\text{Ker}(\tilde{t}^{V, Q}))\right) = \{ i_{v}^{v} \} \]

7.12.5

hence, we get

\[ \tilde{g}_{v}^{Q, Q}(h_{x}^{Q}(x)) = h_{x}^{Q}(i_{Q, Q}(x)) = h_{x}^{Q}(\hat{a}^{-1}, i_{Q, Q}(x)) \]

7.12.6

\[ = h_{x}^{Q}(\hat{a}) \cdot h_{x}^{Q}(\hat{a}^{-1}, i_{Q, Q}(x)) = h_{x}^{Q}(\hat{a}) \cdot i_{Q, Q} \]

\[ = \tilde{Q}, Q_{v} \cdot h_{x}^{Q}(\hat{a}) \]

and therefore from Proposition 7.9 we still get

\[ \tilde{g}_{v}^{Q, Q}(h_{x}^{Q}(x)) = \tilde{g}_{v}^{Q, Q}(h_{x}^{Q}(\hat{a})) = \tilde{g}_{v}^{Q, Q}(h_{x}^{Q}(\hat{a})) \]

7.12.7

We are done.

Proposition 7.13. With the notation and the hypothesis above, let \( R \) be a subgroup of \( Q \) which contains \( V \). Then, for any \( x \in \mathcal{P}(Q', Q)_{V, V} \), setting \( R' = \pi_{x}^{Q}(R) \) we have

\[ g_{v, v}^{Q, Q}(x) = g_{v, v}^{Q, Q}(x) \]

7.13.1

Proof: As above we set

\[ \hat{R} = R \cdot \hat{V} \subset \hat{Q} \quad \text{and} \quad \hat{R}' = R' \cdot \hat{V}' \subset \hat{Q}' \]

7.13.2

and, denoting by \( \chi \in \mathcal{F}(V', V) \) the element fulfilling \( \pi_{x}^{Q}(x) \circ i_{v}^{Q} = i_{v}^{Q} \circ \chi \), we clearly have \( \chi \mathcal{F}_{R}(V) = \mathcal{F}_{R}(V') \); as above, considering an \( \mathcal{F} \)-morphism
\[\alpha: \hat{Q} \to P\] extending \(\chi\), since \(\hat{R}\) coincides with the converse image of \(F_{\hat{R}}(V')\) in \(N_{P}(V')\), we clearly have \(\alpha(R) = \hat{R}'\). Moreover, up to suitable identifications, it is quite clear that (cf. 7.4.1)

\[F_{\hat{R}}^{V,R} \subset F_{\hat{V}}^{V,Q} \quad \text{and} \quad P_{\hat{R}}^{V,R} \subset P_{\hat{V}}^{V,Q} \quad 7.13.3.\]

Consequently, for a choice of \(\hat{a} \in P_{\hat{R}}^{\hat{V}',\hat{Q}'}(\hat{Q}', \hat{Q})\) in \(\hat{P}_{\hat{R}}^{\hat{V}',\hat{Q}'}\), we have

\[\hat{a}^{-1}\hat{\cdot}^{\hat{Q}'}_{\hat{Q}}, x \in P_{\hat{V}}^{V,Q}(\hat{Q}, \hat{Q}) \quad \text{and} \quad \hat{t}^{V,Q}(\hat{a}^{-1}\hat{\cdot}^{\hat{Q}'}_{\hat{Q}}, x) = 1 \quad 7.13.4,
\]

the element \(\mathfrak{g}_{\hat{R}',\hat{R}}^{\hat{V}',\hat{Q}'}(\hat{a})\) in \(P_{\hat{R}',\hat{R}}^{\hat{V}',\hat{Q}'}(\hat{R}, \hat{V}')\) fulfills the analogous conditions with respect to \(\mathfrak{g}_{\hat{R}',\hat{R}}^{\hat{V}',\hat{Q}'}(x)\), which proves the proposition.

7.14. We are ready to define the set \(\mathcal{P}_{\hat{V}}^{\hat{V}'}(V', V)\) for any pair of subgroups \(V\) and \(V'\) in \(X - \mathcal{Q}\); we clearly have \(N = N_{P}(V) \neq V\) and it follows from [11, Proposition 2.7] that there is an \(F\)-morphism \(\nu: N \to P\) such that \(\nu(V)\) is fully centralized in \(F\); moreover, we choose \(n \in \mathcal{P}_{\hat{V}}^{\hat{V}}(\nu(N), N)\) lifting \(\nu\). That is to say, we may assume that

\[7.14.1 \quad \text{There is a pair } (N, n) \text{ formed by a subgroup } N \text{ of } P \text{ which strictly contains and normalizes } V, \text{ and by an element } n \text{ in } \mathcal{P}_{\hat{V}}^{\hat{V}}(\nu(N), N) \text{ lifting } \nu \text{ for a suitable } F\text{-morphism } \nu: N \to P \text{ such that } \nu(V) \text{ is fully centralized in } F.\]

We denote by \(\mathfrak{M}(V)\) the set of such pairs and often we write \(n\) instead of \((N, n)\), setting \(nN = \nu(N)\) and \(\pi_{n}^{x} = \pi_{\nu(N), N}^{x}(n)\).

7.15. For another pair \((\hat{N}, \hat{n})\) in \(\mathfrak{M}(V)\), denoting by \(\hat{\nu}: \hat{N} \to P\) the \(F\)-morphism determined by \(\hat{n}\), setting \(M = (N, \hat{N})\) and considering a new \(F\)-morphism \(\mu: M \to P\) such that \(\mu(V)\) is fully centralized in \(F\), we can obtain a third pair \((M, m)\) in \(\mathfrak{M}(V)\); then, \(\mathfrak{g}_{m,N,N}^{m,M,M}(m)n^{-1}\) and \(\mathfrak{g}_{m,N,N}^{m,M,M}(m)\hat{n}^{-1}\) respectively belong to \(\mathcal{P}_{\hat{V}}^{\hat{V}}(nN, \hat{n}\hat{N})\) and to \(\mathcal{P}_{\hat{V}}^{\hat{V}}(mN, \hat{m}\hat{N})\); in particular, since \(nV, \hat{n}V\) and \(mV, \hat{m}V\) are fully centralized in \(F\), the sets \(\mathcal{P}_{\hat{V}}^{\hat{V}}(nV, \hat{n}V)\), \(\mathcal{P}_{\hat{V}}^{\hat{V}}(mV, \hat{m}V)\), and \(\mathcal{P}_{\hat{V}}^{\hat{V}}(nV, \hat{n}V)\) have been already defined above, and we consider the element

\[g_{\hat{n}, n} = \mathfrak{g}_{m,N,N}^{m,M,M}(m)\hat{n}^{-1} \quad 7.15.1\]

in \(\mathcal{P}_{\hat{V}}^{\hat{V}}(\hat{n}V, nV)\), which actually does not depend on the choice of \(m\).

7.16. Indeed, for another pair \((M, m')\) in \(\mathfrak{M}(V)\) we have

\[\mathfrak{g}_{m',M,M}^{m,M,M}(m') = \mathfrak{g}_{m',N,N}^{m,M,M}(m') \quad 7.16.1\]

and

\[\mathfrak{g}_{m',N,N}^{m,M,M}(m') = \mathfrak{g}_{m',N,N}^{m,M,M}(m') \quad 7.16.1\]
and therefore it follows from Proposition 7.13 that we get
\[
\begin{align*}
\varrho_{m',N,N}^{m',M,M} & (\varrho_{m',N,N}^{m',M,M} (m') \cdot n^{-1}) \\
& = \varrho_{m',N,N}^{m',M,M} (m' \cdot m^{-1}) \cdot \varrho_{m',N,N}^{m',M,M} (m) \cdot n^{-1}) \\
& = \varrho_{m',N,N}^{m',M,M} (m' \cdot m^{-1}) \cdot \varrho_{m',N,N}^{m',M,M} (m) \cdot n^{-1}) \\
& = \varrho_{m',N,N}^{m',M,M} (m' \cdot m^{-1}) \cdot \varrho_{m',N,N}^{m',M,M} (m) \cdot n^{-1})
\end{align*}
\] 7.16.2,

which proves our claim. Similarly, for any triple of pairs \((N, n), (\hat{N}, \hat{n})\) and \((\hat{N}, \hat{n})\) in \(\mathcal{R}(V)\), considering a pair \((\langle N, \hat{N}, \hat{n}, m \rangle)\) in \(\mathcal{R}(V)\), it follows from Propositions 7.11 and 7.13 that
\[
\varrho_{\hat{N},n} \varrho_{\hat{n},n} = \varrho_{\hat{n},n}
\] 7.16.3.

Note that if \(V\) is fully centralized in \(\mathcal{F}\) then \(N = N_{p}(V)\) is \(\mathcal{F}\)-selfcentralizing, so that it is fully centralized too, and therefore the pair \((N, n)\) belongs to \(\mathcal{R}(V)\).

7.17. Then, for any pair of subgroups \(V\) and \(V'\) in \(\mathfrak{X} - \mathfrak{Y}\), since for any \((N, n)\) in \(\mathcal{R}(V)\) and any \((N', n')\) in \(\mathcal{R}(V')\) the set \(\mathcal{P}^{x}(n'V', nV)\) is already defined, we denote by \(\mathcal{P}^{x}(V', V)\) the subset of the product
\[
\prod_{n' \in \mathcal{R}(V')} \prod_{n' \in \mathcal{R}(V')} \mathcal{P}^{x}(n'V', nV)
\] 7.17.1
formed by the families \(\{x_{n',n}\}_{n' \in \mathcal{R}(V'), n' \in \mathcal{R}(V')}\) fulfilling
\[
\varrho_{n',n'} x_{n',n} = x_{n',n} \varrho_{n',n}.
\] 7.17.2.

In other words, the set \(\mathcal{P}^{x}(V', V)\) is the inverse limit of the family formed by the sets \(\mathcal{P}^{x}(n'V', nV)\) and by the bijections between them induced by the \(\mathcal{P}^{y}\)-morphisms \(\varrho_{n,n'}\) and \(\varrho_{n',n'}\).

7.18. Note that, according to equalities 7.16.3, the projection map onto the factor labeled by the pair \(\langle (N, n), (N', n') \rangle\) induces a bijection
\[
n_{n',n} : \mathcal{P}^{x}(V', V) \cong \mathcal{P}^{x}(n'V', nV)
\] 7.18.1;
in particular, if \( V \) and \( V' \) are fully centralized in \( \mathcal{F} \), setting \( N = N_P(V) \) and \( N' = N_P(V') \), the pairs \((N, i^N_N)\) and \((N', i^{N'}_{N'})\) respectively belong to \( \mathfrak{R}(V) \) and to \( \mathfrak{R}(V') \), and therefore we have a canonical bijection

\[
\eta^{N',N}_{i^{N',N}} : \mathcal{P}^x(V',V) \cong \mathcal{P}^x(i^{N'}_N V', i^N_N V)
\]

so that our notation is coherent. At this point, since the map \( \eta^{N',N}_{i^{N',N}} \) is already defined, we can define a map

\[
The fo \text{ro} x \in \mathcal{P}^x(V',V)
\]

sending \( x \in \mathcal{P}^x(V',V) \) to

\[
\eta^x_{i^{N',N}}(x) = \tilde{g}_{n',n}^{-1} \eta^x_{n,n}(x) \tilde{g}_{n,n}(\tilde{n})
\]

where we are setting \( \tilde{n} = \eta^{N,N}_{i^{N,N}}(n) \) and \( \tilde{n}' = \eta^{N',N}_{i^{N',N}}(n') \). From Corollary 7.12 and Proposition 7.13 it is not difficult to check that this map does not depend on the choice of the pairs \((N,n)\) and \((N',n')\).

7.19. Moreover, we have a map

\[
\pi^x_{i^{N',N}} : \mathcal{P}^x(V',V) \rightarrow \mathcal{F}(V',V)
\]

sending \( x \in \mathcal{P}^x(V',V) \) to \((\text{cf. 7.7})\)

\[
\pi^x_{i^{N',N}}(x) = \eta^{N,N}_{i^{N,N}}(\pi^x_n) \circ \pi^x_{n,n}(x) \circ \eta^{N,N}_{i^{N',N}}(\pi^x_n)
\]

then, from condition 7.17.2, it is not difficult to prove that this map does not depend on the choice of the pairs \((N,n)\) and \((N',n')\). Similarly, if \( u \) belongs to \( \mathcal{T}_P(V',V) \) then we may assume that it belongs to \( \mathcal{T}_P(N',N) \), and we consider the map \( \tau^x_{i^{N',N}} : \mathcal{T}_P(V',V) \rightarrow \mathcal{P}^x(V',V) \) determined by

\[
\eta^x_{n,n}(\tau^x_{i^{N',N}}(u)) = \eta^x_{n,n}(\tau^x_{i^{N',N}}(u))^{-1}
\]

Now, it is easy to check that

\[
\eta^x_{i^{N',N}} \circ \tau^x_{i^{N',N}} = \tau^x_{i^{N',N}} \circ \eta^x_{i^{N',N}}
\]

7.20. On the other hand, for any \( V'' \in \mathfrak{X} - \mathfrak{Y} \), the composition map in 7.8 can be extended to a new composition map

\[
\mathcal{P}^x(V'',V') \times \mathcal{P}^x(V',V) \rightarrow \mathcal{P}^x(V'',V)
\]

sending \((x',x) \in \mathcal{P}^x(V'',V') \times \mathcal{P}^x(V',V)\) to

\[
x' \cdot x = (\eta^x_{n',n} - 1)(\eta^x_{n',n}(x') \cdot n_{n',n}(x))
\]
for a choice of \((N, n)\) in \(\mathfrak{M}(V)\), of \((N', n')\) in \(\mathfrak{M}(V')\) and of \((N'', n'')\) in \(\mathfrak{M}(V'')\). This composition map does not depend on our choice; indeed, for another choice of pairs \((\tilde{N}, \tilde{n}) \in \mathfrak{M}(V)\), \((N', \tilde{n}') \in \mathfrak{M}(V')\) and \((N'', \tilde{n}'') \in \mathfrak{M}(V'')\), we get (cf. 7.17.2)

\[
g_{n''', n'}(\pi_{n'''}_{n'}(x') \cdot n_{n', n}(x)) = n_{n''', n'}(x') \cdot g_{n', n}(n_{n', n}(x)) = n_{n''', n'}(x') \cdot g_{n, n}(n_{n', n}(x))
\]

7.20.3.

Moreover, it is compatible with the maps \(h^x_{\nu, \psi} \) defined in equality 7.18.4 above since, setting \(\bar{m} = \hat{\varphi}_{n, \nu, \psi}(n)\), \(\bar{m}' = \hat{\varphi}_{n, \nu, \psi}(n')\) and \(\bar{m}'' = \hat{\varphi}_{n, \nu, \psi}(n'')\), we get

\[
h^x_{\nu, \psi}(x') \cdot \bar{m} = \bar{m}' \cdot h^x_{\nu, \psi}(n_{n', n}(x)) \cdot \bar{m}
\]

7.20.4.

Finally, for any \(V''' \in \mathfrak{X} - \mathfrak{G}\) and any \(\psi'' \in \mathcal{P}^x(V'''', V''')\), it is quite clear that \((x'''' \cdot x') = x''' \cdot (x' \cdot x)\)

7.20.5.

7.21. We are ready to complete our construction of the announced perfect \(\mathcal{X}^\nu\)-locality \(\mathcal{P}^x\) and the \(\mathcal{X}^\nu\)-locality functor \(h^x : \mathcal{P}^x \to \mathcal{L}^\nu\); for any subgroups \(V\) in \(\mathfrak{X} - \mathfrak{G}\) and \(Q\) in \(\mathfrak{G}\) we define

\[
\mathcal{P}^x(V, Q) = \emptyset \quad \text{and} \quad \mathcal{P}^x(Q, V) = \bigsqcup_{V'} \mathcal{P}^x(V', V)
\]

7.21.1

where \(V'\) runs over the set of subgroups \(V' \in \mathfrak{X} - \mathfrak{G}\) contained in \(Q\), and the map

\[
h^x_{\psi, V} : \mathcal{P}^x(Q, V) \to \mathcal{L}^\psi(Q, V)
\]

7.21.2

sends \(x \in \mathcal{P}^x(V', V) \subset \mathcal{P}^x(Q, V)\) to \(\hat{i}^Q_{\psi, V}, h^x_{\psi, V}(x)\). In order to define the composition of two \(\mathcal{P}^x\)-morphisms \(x : R \to Q\) and \(y : T \to R\) we already may assume that \(T\) does not belong to \(\mathfrak{G}\); if \(Q\) does not belong to \(\mathfrak{G}\) then the composition \(x \cdot y\) is given by the map 7.20.1 which is compatible with the maps \(h^x_{\psi, V}\) defined above. If \(Q \in \mathfrak{G}\) but \(R\) does not belong to \(\mathfrak{G}\) then, setting \(R' = (\pi_{Q, R}^x(x))(R)\), \(x\) is actually an element of \(\mathcal{P}^x(R', R)\) and, according to definition 7.21.1, the element \(x \cdot y\) defined by the map 7.20.1 belongs to \(\mathcal{P}^x(R', T) \subset \mathcal{P}^x(Q, T)\), so that we still have

\[
h^x_{Q, T}(x \cdot y) = \hat{i}_{R', R}^Q(h^x_{R', R}(x)) \cdot h^x_{Q, R}(y) = h^x_{Q, R}(x) \cdot h^x_{R, S}(y)
\]

7.21.3.
7.22. Finally, assume that $R$ belongs to $\mathcal{Q}$ and consider the respective subgroups of $R$ and $Q$

$$T' = (\pi_x^{T'}(y))(T) \quad \text{and} \quad T'' = (\pi_x^{T''}(x))(T')$$

setting $N' = N_P(T')$ and $N'' = N_P(T'')$, and considering pairs $(N', n')$ in $\mathfrak{H}(T')$ and $(N'', n'')$ in $\mathfrak{H}(T'')$, the divisibility of $\mathcal{P}^x$ forces the existence of a unique $\mathcal{P}^x$-morphism $r : N' \to N''$ fulfilling $\tau^Q \cdot r = x \cdot i^R_{N'}$; then, we consider the element $s$ in $\mathcal{P}^x(T'', T')$ determined by the equality (cf. 7.18.1)

$$n_{n'', n'}(s) = \mathfrak{g}_{n'' N'' n', n'}(n'' \cdot r \cdot n''')$$

and, since $y \in \mathcal{P}^x(T'', T) \subset \mathcal{P}^x(R, T')$, we can define $x \cdot y = s \cdot y$.

7.23. Once again, we have $\mathfrak{h}_{Q, T'}^x(x, y) = i^Q_{T'} \cdot \mathfrak{h}_{T', T'}^x(s) \cdot \mathfrak{h}_{T', T'}^x(y)$; but, according to our definition, we have $\mathfrak{h}_{R, T'}^x(y) = i^R_{T'} \cdot \mathfrak{h}_{T', T'}^x(y)$ and from $i^Q_{N'} \cdot r = x \cdot i^R_{N'}$ we get

$$\mathfrak{b}_{Q, R}^x(x \cdot i^R_{T'}) = i^Q_{N''} \cdot \mathfrak{b}_{N'' N'' N'}^x(r)$$

so that we obtain

$$\mathfrak{b}_{Q, R}^x(x \cdot i^R_{T'}) = i^Q_{N''} \cdot \mathfrak{b}_{N'' N'' N'}^x(r)$$

thus, setting $m' = \mathfrak{g}_{n'' N'' n', n'}^x(\mathfrak{h}_{n'' N'' n', n'}^x(n'))$ and $m'' = \mathfrak{g}_{n'' N'' n', n'}^x(\mathfrak{h}_{n'' N'' n', n'}^x(n''))$, we get

$$\mathfrak{g}_{n'' N'' n', n'}^x(n'' N'' n', n') \cdot m' \cdot m''$$

consequently, we have

$$\mathfrak{b}_{Q, R}^x(x) \cdot \mathfrak{b}_{R, T'}^x(y) = \mathfrak{b}_{Q, R}^x(x) \cdot i^R_{T'} \cdot \mathfrak{b}_{T', T'}^x(y)$$

This completes the definition of the composition in $\mathcal{P}^x$ and the compatibility of this composition with $\mathfrak{h}^x$. 
7.24. This composition \( x \cdot y \) does not depend on our choice; indeed, for another choice of pairs \((N', \hat{n}') \in \mathfrak{N}(T')\) and \((N'', \hat{n}'') \in \mathfrak{N}(T'')\), it follows from Proposition 7.11 and from equality 7.17.2 that we have

\[
g_{n''N', n''N'}(\hat{n}' \cdot \hat{n}'') = g_{n''N', n''N'}( (\hat{n}' \cdot n'' \cdot \hat{n}'') \cdot (n'' \cdot \hat{n}'') \cdot (\hat{n}' \cdot n'' \cdot \hat{n}'') )
\]

\[
= g_{n''N', n''N'}( (\hat{n}' \cdot n'' \cdot \hat{n}'') \cdot (n'' \cdot \hat{n}'') \cdot (\hat{n}' \cdot n'' \cdot \hat{n}'')) \cdot n'' \cdot \hat{n}''
\]

\[
= n'' \cdot \hat{n}''(s)
\]

Then, the associativity follows from equality 7.20.5, the structural functors are easily defined from 7.19 and from the right-hand definition in 7.21.1, and the equalities 7.19.4 show that \( \mathfrak{h}^x \) is an \( F^x \)-locality functor. We are done.

8. Functor from the perfect \( F \)-locality to the basic \( F \)-locality \( L^b \)

8.1. Let \( F \) be a Frobenius \( P \)-category. From section 6 we already know the existence of a perfect \( F^{sc} \)-locality \( P^{sc} \) canonically contained in the natural \( F^{sc} \)-locality \( \bar{L}^{n,sc} \) and therefore \( P^{sc} \) is also contained in the corresponding quotient \( \bar{L}^{b,sc} \) (cf. 4.13.3) of the full \( F^{sc} \)-sublocality \( L^{b,sc} \) of the basic \( F \)-locality \( L^b \) (cf. Corollary 4.11); actually, \( \bar{L}^{b,sc} \) is also the full \( F^{sc} \)-sublocality of a quotient \( \bar{L}^b \) of the basic \( F \)-locality \( L^b \) (see 8.5 below) and therefore it follows from Theorem 7.2 that the inclusion \( P^{sc} \subset \bar{L}^{b,sc} \) can be extended to a unique \( F \)-locality functor \( \mathfrak{h} : P \to \bar{L}^b \); the main purpose of this section is to prove that this functor can be lifted to an \( F \)-locality functor \( \mathfrak{h} : P \to L^b \), in an essentially unique way.

8.2. From 4.4 and 4.12 we have a contravariant functor

\[
\mathcal{c}^b : \mathcal{F} \to \mathbb{Ab}
\]

mapping any subgroup \( Q \) of \( P \) on the Abelian group

\[
\mathcal{c}^b(Q) = \prod_{O \in \mathcal{O}_Q} \text{ab}(\text{Aut}(O))
\]

where we denote by \( \mathcal{O}_Q \) the set of isomorphism classes of indecomposable \( Q \times P \)-sets \( (Q \times P)/\Delta(U) \) where \( U \) is a subgroup of \( P \) and \( \theta : U \to Q \) an \( F \)-morphism, and mapping any \( \mathcal{F} \)-morphism \( \mathfrak{c} : R \to Q \) on the group homomorphism

\[
\mathcal{c}^b(\mathfrak{c}) : \prod_{O \in \mathcal{O}_Q} \text{ab}(\text{Aut}(O)) \to \prod_{O \in \mathcal{O}_R} \text{ab}(\text{Aut}(O))
\]

described in Proposition 4.6 above.
8.3. For any $\tilde{O} \in \mathfrak{O}_Q$, note that the homomorphism $\tilde{c}^b(\tilde{\varphi})$ sends an element of $\text{ab}(\text{Aut}(O))$ to a family of terms indexed by $R \times P$-orbits with "smaller" stabilizers. More precisely, consider a set $\mathfrak{N}$ of subgroups of $P$ such that any subgroup $U$ of $P$ fulfilling $\mathcal{F}(T, U) \neq \emptyset$ for some $T \in \mathfrak{N}$ belongs to $\mathfrak{N}$, and for any subgroup $Q$ of $P$ denote by $\mathfrak{O}_Q$ the subset of $\tilde{O} \in \mathfrak{O}_Q$ such that $(Q \times P)/\Delta_\eta(T)$ belongs to $\tilde{O}$ if and only if $T$ belongs to $\mathfrak{N}$.

**Corollary 8.4.** With the notation above, the correspondence sending any subgroup $Q$ of $P$ to

$$\tilde{c}^n(Q) = \prod_{O \in \mathfrak{O}_Q} \text{ab}(\text{Aut}(O))$$

defines a contravariant subfunctor $\tilde{c}^n : \tilde{\mathcal{F}} \to \text{Ab}$ of $\tilde{c}^b$.

**Proof:** Straightforward.

8.5. In particular, considering the set of subgroups of $P$ which are not $\mathcal{F}$-selfcentralizing and denoting by $\tilde{c}^b : \tilde{\mathcal{F}} \to \text{Ab}$ the corresponding subfunctor of $\tilde{c}^b$, we get the quotient $\tilde{L}^b = \mathcal{L}^b/\tilde{c}^{\text{sec}}$ of the basic $\mathcal{F}$-locality (cf. 2.9); as mentioned above, it follows from 4.13.3 and from section 6 that the perfect $\mathcal{F}^\text{sc}$-locality $\mathcal{P}^\text{sc}$ is contained in the full subcategory $\mathcal{L}^b/\tilde{c}^{\text{sec}}$ of $\mathcal{L}^b$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$, and then it follows from Theorem 7.2 that this inclusion can be extended to a unique $\mathcal{F}$-locality functor

$$\tilde{h} : \mathcal{P} \to \tilde{L}^b$$

where $\mathcal{P}$ denotes the perfect $\mathcal{F}$-locality. As a matter of fact, if $R$ is not $\mathcal{F}$-selfcentralizing then we have $\mathcal{L}^b(Q, R) = \mathcal{F}(Q, R)$ and, in this case, the existence of $\tilde{h}$ admits an easy direct proof.

8.6. More generally, for any set $\mathfrak{N}$ as in 8.3 above, we consider the corresponding quotient — denoted by $(\tilde{c}^n, \tilde{\pi}^n, \tilde{\mathcal{L}}^n)$ — of the basic $\mathcal{F}$-locality; if all the subgroups in $\mathfrak{N}$ are not $\mathcal{F}$-selfcentralizing, then we claim that $\tilde{h}$ can be lifted to a unique natural $\mathcal{F}$-isomorphism class of $\mathcal{F}$-locality functors

$$\tilde{h}^n : \mathcal{P} \to \tilde{\mathcal{L}}^n$$

The induction argument on the cardinal of the complement of $\mathfrak{N}$ in the set of all the subgroups of $P$ suggests the following general construction; assume that this complement is not empty, choose on it a minimal element $U$ fully normalized in $\mathcal{F}$ and set

$$\mathfrak{M} = \mathfrak{N} \cup \{\theta(U) \mid \theta \in \mathcal{F}(P, U)\}$$

then, we clearly have a canonical functor $\tilde{\mathcal{L}}^n : \tilde{\mathcal{L}}^n \to \tilde{\mathcal{L}}^n$ and, since we have $\tilde{\mathcal{P}} = \tilde{\mathcal{F}}$, it makes sense to consider the quotient contravariant functor

$$\tilde{i}^n = \text{Ret}(\tilde{c}^n) = \tilde{c}^n/\tilde{c}^n : \tilde{\mathcal{F}} \to \text{Ab}$$
actually, we prove below that this contravariant functor admits a compatible complement in the sense of [12, 5.7], and therefore for any \( n \geq 1 \) we have
\[
\mathbb{H}_n^p(\tilde{\mathcal{F}}, 1^U) = \{0\}
\]
this result will be also quoted in section 9.

8.7. Recall that, for any subgroup \( R \) of \( P \) and any element \( \eta \in \mathcal{F}(R, U) \), we have (cf. 3.6.1)
\[
\text{Aut}((R \times P)/\Delta_\eta(U)) \cong \tilde{N}_{R \times P}(\Delta_\eta(U))
\]
and, for any \( \mathcal{F} \)-morphism \( \varphi : R \to Q \), denote by
\[
\tilde{\varphi}_\eta : \tilde{N}_{R \times P}(\Delta_\eta(U)) \to \tilde{N}_{Q \times P}(\Delta_{\varphi \eta}(U))
\]
the group homomorphism induced by \( \varphi \times \text{id}_P \). Moreover, for any \( \tilde{\mathcal{F}} \)-morphisms \( \tilde{\varphi} : R \to Q \) and \( \tilde{\theta} : U \to Q \), the existence of an injective \( R \times P \)-set homomorphism
\[
f : (R \times P)/\Delta_\eta(U) \to \text{Res}_{\varphi \times \text{id}_P}((Q \times P)/\Delta_\theta(U))
\]
for some \( \varphi \in \tilde{\mathcal{F}} \) and some \( \theta \in \tilde{\mathcal{F}} \), is equivalent to the existence of \((v, u) \in Q \times P\) such that
\[
(\varphi \times \text{id}_P)(\Delta_\eta(U)) = \Delta_\theta(U)^{(v,u)}
\]
which implies that \( u \) belongs to \( N_P(U) \) and that \( \tilde{\varphi} \circ \tilde{\eta} = \tilde{\theta} \circ \tilde{\kappa}_v(u) \) (cf. 2.7).

8.8. More precisely, as in 4.5 above set
\[
M = (R \times P)/\Delta_\eta(U) \quad \text{and} \quad O = (Q \times P)/\Delta_\theta(U)
\]
and denote by \( \text{Inj}_{R \times P}(M, \text{Res}_{\varphi \times \text{id}_P}(O)) \) the corresponding set of injective \( R \times P \)-set homomorphisms; assuming that this set is not empty, in 8.7.3 let us denote by \((w_f, \hat{w}_f) \in Q \times P\) an element belonging to the class in \( O \) which is the image by \( f \) of the class of \((1, 1)\) in \( M \); from [11, Lemma 22.19] it follows that

8.8.2 The correspondence mapping \( f \) on the double class \( \varphi(R)w_f \theta(U) \) determines a bijection from the set of \( \text{Aut}(M) \)-orbits in \( \text{Inj}_{R \times P}(M, \text{Res}_{\varphi \times \text{id}_P}(O)) \) onto the set of double classes \( \varphi(R)v \theta(U) \) in \( \varphi(R)\backslash Q/\theta(U) \) admitting a representative \( v \in Q \) such that \((v, 0)\) normalizes \( \Delta_\theta(U) \) for some \( \hat{v} \in N_P(U) \).

In particular, since \((v, \hat{v})\) belongs to \( \tilde{N}_{Q \times P}(\Delta_\theta(U)) \cong \text{Aut}(O) \), in the present situation in 4.5 we have \(|T_M^O| = 1\), we may assume that \( \tilde{\varphi} \circ \tilde{\eta} = \tilde{\theta} \), \( \delta_f \) is an isomorphism and moreover we have
\[
ab(\delta_f) \circ ab^\theta(\varepsilon_f) = \ab^\varphi(\tilde{\varphi}_\eta)
\]
Proposition 4.6. That, for any \( \theta \) equality 8.9.1 follows from 3.6.1 up to suitable identifications.

It is clear that \( \tilde{\varphi} \) runs over \( \tilde{\mathcal{F}}(R,U)/\tilde{\mathcal{F}}P(U) \) having an element \( \tilde{\gamma} \) such that \( \tilde{\varphi} \circ \tilde{\gamma} = \tilde{\theta} \), and setting \( M_{\tilde{\gamma}} = (R \times P)/\Delta_{\tilde{\gamma}}(U) \) for some \( \gamma \in \tilde{\gamma} \), we actually have a bijection

\[
\bigcup_{\tilde{\gamma} \in \tilde{\Omega}_\theta^\gamma} M_{\tilde{\gamma}}/\text{Aut}(M_{\tilde{\gamma}}) \cong O/\text{Aut}(O)
\]

8.8.4.

**Proposition 8.9.** With the notation above, the contravariant functor \( \tilde{\mathcal{F}}^U \) maps any subgroup \( Q \) of \( P \) on

\[
\tilde{\mathcal{F}}^U(Q) = \left( \prod_{\tilde{\delta} \in \tilde{\mathcal{F}}(Q,U)} \text{ab}(\tilde{N}_{Q \times P}(\Delta_{\tilde{\varphi}(\tilde{\eta})(U)))) \right)^{\tilde{\mathcal{F}}P(U)}
\]

8.9.1 and any \( \tilde{\mathcal{F}} \)-morphism \( \tilde{\varphi}: R \to Q \) on the homomorphism induced by the sum of the group homomorphisms

\[
\text{ab}^\circ(\tilde{\varphi}) : \text{ab}(\tilde{N}_{Q \times P}(\Delta_{\varphi(\eta)}(U))) \longrightarrow \text{ab}(\tilde{N}_{R \times P}(\Delta_{\eta}(U)))
\]

8.9.2

where \( \tilde{\eta} \) runs over \( \tilde{\mathcal{F}}(R,U) \). In particular, \( \tilde{\mathcal{F}}^U \) admits a compatible complement mapping \( \tilde{\varphi}: R \to Q \) on the homomorphism \( \tilde{\mathcal{F}}^U(\tilde{\varphi})^\circ \) induced by the sum of the group homomorphisms

\[
\text{ab}(\tilde{\varphi}) : \text{ab}(\tilde{N}_{R \times P}(\Delta_{\eta}(U))) \longrightarrow \text{ab}(\tilde{N}_{Q \times P}(\Delta_{\varphi(\eta)}(U)))
\]

8.9.3

where \( \tilde{\eta} \) runs over \( \tilde{\mathcal{F}}(R,U) \).

**Proof:** It is clear that

\[
\tilde{\mathcal{F}}^U(Q) = \prod_{\tilde{\delta} \in \tilde{\Omega}_\theta^\gamma - \Omega_Q^\gamma} \text{ab}((\text{Aut}(O))
\]

8.9.4;

but, for any \( O \in \Omega_Q^\gamma - \Omega_Q^\gamma \), we necessarily have \( O \cong (Q \times P)/\Delta_{\theta}(U) \) for some \( \theta \in \mathcal{F}(Q,U) \); moreover, it is clear that we have a \( Q \times P \)-set isomorphism

\[
(Q \times P)/\Delta_{\theta}(U) \cong (Q \times P)/\Delta_{\theta'}(U)
\]

8.9.5

if and only if \( \Delta_{\theta}(U) \) and \( \Delta_{\theta'}(U) \) are \( Q \times P \)-conjugate to each other; now, equality 8.9.1 follows from 3.6.1 up to suitable identifications.

On the other hand, if \( \tilde{\varphi}: R \to Q \) is an \( \tilde{\mathcal{F}} \)-morphism then it follows from Proposition 4.6 that, for any \( \theta \in \mathcal{F}(Q,U) \) and any \( a \in \text{ab}(\tilde{N}_{Q \times P}(\Delta_{\theta}(U))) \), setting \( O = (Q \times P)/\Delta_{\theta}(U) \) we get

\[
(\tilde{\mathcal{F}}^U(\tilde{\varphi}))(a) = \sum_{\tilde{M} \in \tilde{\Omega}_Q^\gamma} \sum_{f \in T_{\tilde{\varphi}}(\tilde{M})} (\text{ab}(\delta_f) \circ \text{ab}^\circ(\varepsilon_f))(a)
\]

8.9.6,
so that, according to equality 8.9.4, we still get
\[
(\hat{t}^U(\hat{\varphi}))(a) = \sum_{M \in \mathcal{M}_R - \mathcal{M}_R} \sum_{f \in \mathcal{T}_M(\hat{\varphi})} (\mathbf{ab}(\delta_f) \circ \mathbf{ab}^\varphi(\varepsilon_f))(a) \quad 8.9.7;
\]
once again, for any \(M \in \mathcal{M}_R - \mathcal{M}_R\), we necessarily have \(M \cong (R \times P)/\Delta_\eta(U)\) for some \(\eta \in \mathcal{F}(R, U)\); consequently, from 8.8 we obtain
\[
\hat{\varphi} \circ \hat{\eta} = \hat{\theta} \quad \text{and} \quad (\hat{t}^U(\hat{\varphi}))(a) = \sum_{\bar{\eta} \in \mathcal{O}_\eta} (\mathbf{ab}^\varphi(\bar{\varphi}_\eta))(a) \quad 8.9.8.
\]

In order to prove the last statement in the proposition, with the notation above we have to compute the following element of \(\hat{t}^U(Q)\)
\[
\sum_{\bar{\eta} \in \mathcal{O}_\eta} (\mathbf{ab}(\bar{\varphi}_\eta))(\mathbf{ab}^\varphi(\bar{\varphi}_\eta))(a) = \sum_{\bar{\eta} \in \mathcal{O}_\eta} \frac{|N_{Q \times P}(\Delta_\eta(U))|}{|N_{R \times P}(\Delta_\eta(U))|} a \quad 8.9.9;
\]
but, according to the bijection 8.8.4, we get
\[
\sum_{\bar{\eta} \in \mathcal{O}_\eta} \frac{|R \times P|}{|N_{R \times P}(\Delta_\eta(U))|} = \frac{|Q \times P|}{|N_{Q \times P}(\Delta_\eta(U))|} \quad 8.9.10;
\]
consequently, we obtain \(\hat{t}^U(\hat{\varphi}) \circ \hat{t}^U(\hat{\varphi}) = |Q|/|R| \cdot \mathbf{id}_{\hat{t}^U(Q)}\).

Finally, consider a special \(\mathbf{ac}(\bar{\mathcal{F}})\)-square \([12, 5.1 \text{ and } 5.2]\)
\[
\begin{array}{c}
Q \\
R \\
S \\
T
\end{array}
\xrightarrow[\bar{\varphi}]{\kappa} \xrightarrow[\hat{\varphi}]{\hat{\varphi}}
\xrightarrow[\xi]{\xi}
\]
in order to prove that \(\hat{t}^U\) admits a compatible complement, we may assume that \(Q, R \text{ and } T\) are just subgroups of \(P\); more precisely, up to isomorphisms, we may assume that \(Q\) contains \(R \text{ and } T\), and that \(\hat{\varphi} = \hat{t}_R^Q\) and \(\hat{\psi} = \hat{t}_T^Q\); in this case, by the very definition of the special \(\mathbf{ac}(\bar{\mathcal{F}})\)-squares, we have
\[
S = \bigoplus_{w \in W} S_w \quad 8.9.12
\]
where \(W \subseteq Q\) is a set of representatives for the set of double classes \(R \setminus Q/T\) and, for any \(w \in W\), we set \(S_w = R^w \cap T\) and respectively denote by
\[
\hat{t}_w^R : S_w \to R \quad \text{and} \quad \hat{t}_w^T : S_w \to T \quad 8.9.13
\]
the \(\mathcal{F}\)-morphisms mapping \(s \in S_w\) on \(wsw^{-1}\) and on \(s\); moreover, the \(\mathbf{ac}(\bar{\mathcal{F}})\)-morphisms
\[
\hat{\zeta} : \bigoplus_{w \in W} S_w \to R \quad \text{and} \quad \hat{\xi} : \bigoplus_{w \in W} S_w \to T \quad 8.9.14
\]
are respectively determined by the families \(\{\hat{t}_w^R\}_{w \in W}\) and \(\{\hat{t}_w^T\}_{w \in W}\).
Now, it suffices to prove the commutativity of the following diagram

\[
\begin{array}{ccc}
\tilde{t}^U(Q) & \xrightarrow{\tilde{t}^U(\eta)} & \tilde{t}^U(T) \\
\tilde{t}^U(\theta) & \xrightarrow{\prod_{w \in W} \tilde{t}^U(S_w)} & \prod_{w \in W} \tilde{t}^U(s_w)
\end{array}
\]

that is to say, for any \( \eta \in F(R, U) \) and any \( a \in ab\left(\tilde{N}_R \times P(\Delta_{\eta}(U))\right) \), it suffices to prove that

\[
\tilde{t}^U(\eta) \circ \tilde{t}^U(\phi)(a) = \tilde{t}^U(\phi) \circ \tilde{t}^U(\eta)(a) \quad 8.9.16.
\]

According to our definition of \( \tilde{t}^U(\eta) \), we have

\[
\tilde{t}^U(\eta)(a) = (ab((i^Q_R)_\eta))(a) \quad 8.9.17
\]

and therefore, setting \( \theta = i^Q_R \circ \eta \), it follows from 8.9.8 that we get

\[
\tilde{t}^U(\eta)(\tilde{t}^U(\phi)(a)) = \sum_{\gamma \in \Delta_{\eta}} (ab^\circ((i^Q_R)_{\gamma}) \circ ab((i^Q_R)_\eta))(a) \quad 8.9.18.
\]

On the other hand, it follows from 8.9.8 that

\[
\tilde{t}^U(\phi)(\tilde{t}^U(\eta)(a)) = \sum_{w \in W} \sum_{\tilde{v} \in D^\phi_{w}} (ab((i^Q_R)_{\tilde{v}}) \circ ab^\circ(((i^Q_R)_w))(a) \quad 8.9.19
\]

and, according to our definition of \( \tilde{t}^U(\phi) \), we obtain

\[
\tilde{t}^U(\phi)(\tilde{t}^U(\phi)(a)) = \sum_{w \in W} \sum_{\tilde{v} \in D^\phi_{w}} (ab((i^Q_R)_{\tilde{v}}) \circ ab^\circ(((i^Q_R)_{\tilde{v}}))(a) \quad 8.9.20.
\]

Thus, it suffices to prove the commutativity of the following diagram

\[
\begin{array}{ccc}
ab\left(\tilde{N}_Q \times P(\Delta_{\eta}(U))\right) & \xrightarrow{ab\left(\tilde{N}_R \times P(\Delta_{\eta}(U))\right)} & \prod_{\tilde{v} \in D^\phi_{w}} ab\left(\tilde{N}_T \times P(\Delta_{\eta}(U))\right) \\
ab\left(\tilde{N}_Q \times P(\Delta_{\phi}(U))\right) & \xrightarrow{\sum_{\tilde{v} \in D^\phi_{w}} ab\left(((i^Q_R)_{\tilde{v}})\right)} & \prod_{\tilde{v} \in D^\phi_{w}} ab\left(\tilde{N}_R \times P(\Delta_{\phi}(U))\right) \\
\sum_{w \in W} \sum_{\tilde{v} \in D^\phi_{w}} ab^\circ(((i^Q_R)_{\tilde{v}})) & \xrightarrow{\sum_{w \in W} \sum_{\tilde{v} \in D^\phi_{w}} ab^\circ(((i^Q_R)_{\tilde{v}}))} & \prod_{\tilde{v} \in D^\phi_{w}} ab\left(\tilde{N}_S \times P(\Delta_{\phi}(U))\right)
\end{array}
\]

8.9.21;
but, denoting by $Q'$ the converse image in $P$ of $\theta^*_F(\theta(U)) \cap \bar{F}_P(U)$ where $\theta^*: \theta(U) \cong U$ is the inverse of the isomorphism induced by $\theta$, $N_{Q \times P}(\Delta_\theta(U))$ is clearly contained in $Q' \times P$ and therefore we have

$$ab\left(\bar{N}_{Q \times P}(\Delta_\theta(U))\right) = ab\left(\bar{N}_{Q' \times P}(\Delta_\theta(U))\right)$$

8.9.22;

explicitly, it follows from [11, statement 2.10.1] that the composition $i^T_{Q'} \circ \theta^*$ can be extended to an $F$-morphism $\theta^*: Q' \to P$ and then we actually have

$$N_{Q' \times P}(\Delta_\theta(U)) = (\{1\} \times C_P(U)) \times N_{\Delta_{\theta^*}(Q')}(\Delta_\theta(U))$$

8.9.23

where we set $\Delta_{\theta^*}(Q') = \Delta_{id_{Q'}, \theta^*}(Q')$.

According to our choice of $\theta$, we have $\Delta_{\eta}(U) = \Delta_{\eta}(U)$ and therefore, setting $R' = R \cap Q'$, we similarly have

$$ab\left(\bar{N}_{R' \times P}(\Delta_\theta(U))\right) = ab\left(\bar{N}_{R' \times P}(\Delta_\theta(U))\right)$$

8.9.24

and, denoting by $\eta^*: R' \to P$ the restriction of $\theta^*$, we still have

$$N_{R' \times P}(\Delta_\theta(U)) = (\{1\} \times C_P(U)) \times N_{\Delta_{\eta^*}(R')}(\Delta_\theta(U))$$

8.9.25.

On the other hand, for any $\tilde{\gamma} \in \mathcal{D}_{\Delta_{\theta}}^{\Delta_{\theta}}$, we have an injective $T \times P$-set homomorphism

$$(T \times P)/\Delta_{\gamma}(U) \longrightarrow \text{Res}_{\Delta_{\gamma} \times \text{id}_P}(\{(Q \times P)/\Delta_\theta(U)\})$$

8.9.26

and therefore, setting $T' = T \cap Q'$ and denoting by $\theta': U \to Q'$ the restriction of $\theta$, it follows from statement 8.8.2 that, up to a modification of our choice of the set of representatives in $\bar{F}(T, U)/\bar{F}_P(U)$ and our choice of $\gamma$ in $\tilde{\gamma}$, we also may assume that $\Delta_{\gamma}(U) = \Delta_{\gamma}(U)$ and then it is easily checked that we get an $\mathcal{F}$-morphism $\gamma': U \to T'$ such that $i^T_{T'} \circ \gamma'$ is a representative of $\tilde{\gamma}$ and that we have $\theta' = i^T_{T'} \circ \gamma'$; that is to say, there is $v' \in Q'$ such that we have $\theta'(u) = v'\gamma'(u)$ for any $u \in U$ and therefore we get (cf. 2.7)

$$\theta' \circ \kappa_{\mathcal{U}}(\theta^*(u')) = i^T_{T'} \circ \gamma'$$

8.9.27;

since $\theta^*(u')$ belongs to $N_P(U)$, we actually obtain $|\mathcal{D}_{\Delta_{\theta}}^{\Delta_{\theta}}| = 1$ (cf. 8.8). Moreover, denoting by $\gamma'^*: T' \to P$ the restriction of $\theta^*$, we still have

$$ab\left(\bar{N}_{T' \times P}(\Delta_\theta(U))\right) = ab\left(\bar{N}_{T' \times P}(\Delta_\theta(U))\right)$$

$$= \{1\} \times C_P(U) \times N_{\Delta_{\gamma'^*}(T')} \left(\Delta_\theta(U)\right)$$

8.9.28.
Finally, for any \( w \in W \) and any \( \tilde{w} \in \Omega^\phi_{t_w} \) we have an injective \( S_w \times P \)-set homomorphism

\[
(S_w \times P)/\Delta_w(U) \longrightarrow \text{Res}_{w \times \text{id}_P}((R \times P)/\Delta_R(U)) \tag{8.9.29}
\]

and therefore, since \( \Delta_R(U) = \Delta_\phi(U) \), setting \( t^Q_w = \iota^Q_w \circ t^R_w \) we also have an injective \( S_w \times P \)-set homomorphism

\[
(S_w \times P)/\Delta_w(U) \longrightarrow \text{Res}_{w \times \text{id}_P}((Q \times P)/\Delta_\phi(U)) \tag{8.9.30}
\]

once again, setting \( S'_w = S_w \cap Q' \), it follows from statement 8.8.2 that, up to a modification of our choice of the set of representatives in \( \tilde{F}(S_w, U)/\tilde{F}_P(U) \) and our choice of \( \nu \) in \( \tilde{\nu} \), we also assume that \( (w, 1) \Delta_w(U) = \Delta_\phi(U) \); in this case, up to a modification of our choice of \( W \), we may assume that \( w \) belongs to \( Q' \).

Thus, denoting by \( W' \) the set of \( w \in W \) such that \( \Omega^\phi_{t_w} \neq \emptyset \), we may assume that \( W' \) is contained in \( Q' \) and then it is actually a set of representatives for the set of double classes \( R' \setminus Q'/T' \). Moreover, the argument above also proves that, for any \( w' \in W' \), we still have \( |\Omega^\phi_{t_{w'}}| = 1 \) (cf. 8.8) and then we still obtain

\[
\text{ab}\left(\tilde{N}_{S'_{w'}} \times P(\Delta_{w'}(U))\right) = \text{ab}\left(\tilde{N}_{S'_{w'}} \times P(\Delta_w(U))\right)
\]

\[
N_{S'_{w'}} \times P(\Delta_{w'}(U)) = \left(\{1\} \times C_P(U)\right) \times \Delta_{S'_{w'}}(T')^{(\Delta_{w'}(U))}
\]

where \( \nu': U \to S'_{w'} \) and \( \nu'^* : S'_{w'} \to P \) are the respective restrictions of \( \theta^{w'} \) and of \( (\theta^*)^{w'} \).

In conclusion, respectively denoting by \( \tilde{\gamma} \) and by \( \tilde{\nu}_w \) the unique elements of \( \Omega^\phi_{t^\gamma_w} \) and of \( \Omega^\phi_{t^\nu_w} \) for any \( w \in W' \), diagram 8.9.21 becomes

\[
\text{ab}\left(\tilde{N}_{Q \times P}(\Delta_\phi(U))\right) \xrightarrow{\text{ab}((\iota^Q)^{\gamma})_{\nu}} \text{ab}\left(\tilde{N}_{R \times P}(\Delta_\phi(U))\right) \xrightarrow{\text{ab}((\iota^R)^{\gamma})_{\nu}} \text{ab}\left(\tilde{N}_{T \times P}(\Delta_\phi(U))\right)
\]

\[
\sum_{w \in W'} \text{ab}((\iota^Q)_{\nu_w}) \xrightarrow{\sum_{w \in W'} \text{ab}((\iota^R)_{\nu_w})} \prod_{w \in W'} \text{ab}\left(\tilde{N}_{S_w \times P}(\Delta_{w'}(U))\right)
\]

but, denoting by \( i\mathcal{G} \) the category of finite groups with injective group homomorphisms, the \textit{contravariant} functor determined by the \textit{transfert}

\[
\text{ab}^\circ : i\mathcal{G} \longrightarrow \text{Ab} \tag{8.9.33}
\]
admits the functor $ab : iGr \to \mathfrak{Ab}$ as a Mackey complement; moreover, the bijective image of $W'$ in $\tilde{N}_{Q \times P}(\Delta_{\theta}(U))$ is a set of representatives for the set of double classes

$$\tilde{N}_{Q \times P}(\Delta_{\eta}(U)) \setminus \tilde{N}_{Q \times P}(\Delta_{\phi}(U)) / \tilde{N}_{T \times P}(\Delta_{\gamma}(U))$$

consequently, the commutativity of diagram 8.9.32 follows from the Mackey formula applied to the pair $(ab^o, ab)$. We are done.

**Theorem 8.10.** The $F$-locality functor $\tilde{h} : \mathcal{P} \to \bar{\mathcal{L}}^b$ can be lifted to a unique natural $F$-isomorphism class of $F$-locality functors $\bar{h}_N : \mathcal{P} \to \bar{\mathcal{L}}_N^b$.

**Proof:** As above, consider a set $\mathfrak{N}$ of subgroups of $P$ in such a way that any subgroup $V$ of $P$ fulfilling $F(T, V) \neq \emptyset$ for some $T \in \mathfrak{N}$ belongs to $\mathfrak{N}$; assume that all the subgroups in $\mathfrak{N}$ are not $F$-selfcentralizing; arguing by induction on the cardinal $c$ of the complement of $\mathfrak{N}$ in this set, we will prove that $\tilde{h} : \mathcal{P} \to \bar{\mathcal{L}}^b$ can be lifted to a unique natural $F$-isomorphism class of $F$-locality functors $\tilde{h}^\mathfrak{N} : \mathcal{P} \to \bar{\mathcal{L}}^{\mathfrak{N}, b}$.

We may assume that $c \neq 0$ and then in the complement of $\mathfrak{N}$ we choose a minimal element $U$ fully normalized in $F$ and set

$$\mathfrak{M} = \mathfrak{N} \cup \{ \theta(U) \mid \theta \in F(P, U) \}$$

according to the induction hypothesis, we have a unique natural $F$-isomorphism class of $F$-locality functors $\tilde{h}^\mathfrak{M} : \mathcal{P} \to \bar{\mathcal{L}}^{\mathfrak{M}, b}$ lifting $\tilde{h}$; then, it suffices to prove that such a functor can be lifted to a unique natural $F$-isomorphism class of $F$-locality functors $\tilde{h}_{\mathfrak{M}}$ as in 8.10.1. As a matter of fact, our proof follows the same pattern as the proof of Theorem 6.12 but a common argument would not be handy to read.

As in 6.11 above, let us consider the functors $[11, 18.20.3]$

$$\text{loc}_{\mathcal{P}} : ch^*(\mathcal{F}) \to \bar{\text{loc}} \quad \text{and} \quad \text{loc}_{\mathcal{L}^{\mathfrak{M},b}} : ch^*(\mathcal{F}) \to \bar{\text{loc}}$$

respectively mapping any $\mathcal{F}$-chain $\mathfrak{q} : \Delta_n \to \mathcal{F}$ on $(\mathcal{P}(\hat{q}), \ker(\pi_{\hat{q}}))$ and on $(\mathcal{L}^{\mathfrak{M},b}(\hat{q}^\mathfrak{M}), \ker(\pi_{\hat{q}^\mathfrak{M}}))$ (cf. 2.15) where

$$\hat{q} : \Delta_n \to \mathcal{P} \quad \text{and} \quad \hat{q}^\mathfrak{M} : \Delta_n \to \mathcal{L}^{\mathfrak{M},b}$$

are respective $\mathcal{P}$- and $\mathcal{L}^{\mathfrak{M},b}$-chains lifting $\mathfrak{q}$, and consider the obvious natural map

$$\text{loc}_{\hat{q}^\mathfrak{M}} : \text{loc}_{\mathcal{P}} \to \text{loc}_{\mathcal{L}^{\mathfrak{M},b}}$$

determined by the $F$-locality functor $\tilde{h}^\mathfrak{M}$.?
Actually, from the uniqueness part of [11, Proposition 18.19], it is clear that the functor $\mathfrak{loc} \mathcal{F}$ coincides with the $\mathcal{F}$-localizing functor $\mathfrak{loc}_\mathcal{F}$ [11, 18.12.1] and, as in the proof of Theorem 6.12 above, the point is that, according to Proposition 2.17 above, the *natural map* $\mathfrak{loc}_\mathcal{F}$ already can be lifted to a unique *natural map*
\[ \lambda_{\mathcal{F}^{n,b}} : \mathfrak{loc}_{\mathcal{F}^{n,b}} \rightarrow \mathfrak{loc}_{\mathcal{F}^{n,b}} \]
fulfilling the conditions there. That is to say, for any $\mathcal{F}$-chain $q : \Delta_n \rightarrow \mathcal{F}$, we have a group homomorphism
\[ \lambda_q = (\lambda_{\mathcal{F}^{n,b}})_q : P(\hat{q}) \rightarrow L_{\mathcal{F}^{n,b}}(\hat{q}^n) \]
lifting $(\mathfrak{loc}_{\mathcal{F}^{n,b}})_q$ which is compatible with the corresponding structural functors and, according to definition 8.6.3, is unique up to $\hat{\nu}$-conjugation; analogously, for a second $\mathcal{F}$-chain $r : \Delta_m \rightarrow \mathcal{F}$ and any $\mathfrak{ch}(\mathcal{F})$-morphism $(\mu, \delta) : (r, \Delta_m) \rightarrow (q, \Delta_n)$ [11, A2.8], the diagram
\[ \begin{array}{ccc}
\mathcal{P}(\hat{q}) & \xrightarrow{\lambda_q} & L_{\mathcal{F}^{n,b}}(\hat{q}) \\
\uparrow & & \uparrow \\
\mathcal{P}(\hat{r}) & \xrightarrow{\lambda_q} & L_{\mathcal{F}^{n,b}}(\hat{r})
\end{array} \]
is commutative up to $\hat{\nu}$-conjugation (cf. 2.15).

In particular, assume that $n = 0$, $m = 1$ and $\delta = \delta_0^1$, and setting $Q = r(1)$, $R = r(0)$, $x = r^0(0 \cdot 1)$ and $\hat{\phi} = \hat{r}(0 \cdot 1)$, assume that $q(0) = R$ and that (cf. 2.15)
\[ \mathfrak{loc}_\mathcal{F}(\mu, \delta) = (r_\mathcal{F}(1), id_{\Delta_n}) \quad \text{and} \quad \mathfrak{loc}_{\mathcal{F}^{n,b}} = (r_{\mathcal{F}^{n,b}}(1), id_{\Delta_n}) \]
then, $\mathcal{P}(\hat{r})$ coincides with the stabilizer $\mathcal{P}(Q)_{\hat{\phi}}$ of $\varphi(R)$ in $\mathcal{P}(Q)$, $L_{\mathcal{F}^{n,b}}(\hat{r}^n)$ coincides with the stabilizer $L_{\mathcal{F}^{n,b}}(Q)_x$ of $\varphi(R)$ in $L_{\mathcal{F}^{n,b}}(Q)$ and diagram 8.10.8 becomes
\[ \begin{array}{ccc}
\mathcal{P}(R) & \xrightarrow{\lambda_R} & L_{\mathcal{F}^{n,b}}(R) \\
\uparrow & & \uparrow \\
\mathcal{P}(Q)_{\hat{\phi}} & \xrightarrow{\lambda_R} & L_{\mathcal{F}^{n,b}}(Q)_x
\end{array} \]
where $\mu_x : L_{\mathcal{F}^{n,b}}(Q)_x \rightarrow L_{\mathcal{F}^{n,b}}(R)$ sends $a \in L_{\mathcal{F}^{n,b}}(Q)_x$ to the unique $b \in L_{\mathcal{F}^{n,b}}(R)$ fulfilling $x \cdot b = a \cdot x$; moreover, since $\mathcal{P}(Q)_{\hat{\phi}}$ and $L_{\mathcal{F}^{n,b}}(Q)_x$ are respectively contained in $\mathcal{P}(Q)$ and $L_{\mathcal{F}^{n,b}}(Q)$, and since $L_{\mathcal{F}^{n,b}}(Q)_x$ contains $\text{Ker}(\pi_{\mathcal{F}^{n,b}})$, we actually may assume that $\lambda_q^n$ is just the restriction of $\lambda^q$; then note that, for some choice of $x$ lifting $h_{\mathcal{F}^{n,b}}(\hat{\phi})$, diagram 8.10.10 becomes commutative.
Consider the actions of $\mathcal{P}(Q) \times \mathcal{P}(R)$ on $\tilde{L}^{m,b}(Q, R)$ and on $\tilde{L}^{n,b}(Q, R)$ defined by the composition on the left- and the right-hand respectively via the functor $\tilde{h}^m$ and via the group homomorphisms

\[
\lambda_Q : \mathcal{P}(Q) \rightarrow \tilde{L}^{m,b}(Q) \quad \text{and} \quad \lambda_R : \mathcal{P}(R) \rightarrow \tilde{L}^{n,b}(R) \quad 8.10.11
\]

and, for any $\hat{\phi} \in \mathcal{P}(Q, R)$, choose a lifting $x_{\hat{\phi}} \in \tilde{L}^{m,b}(Q, R)$ of $\tilde{h}^m(\hat{\phi})$ such that the corresponding diagram $8.10.10$ is commutative; then, we have the equality $8.10.12$ holds; thus, any subgroup $\tilde{Q}$ of $\tilde{h}^m(\hat{\phi})$ lift of $\hat{\phi} \circ \pi : \hat{\mathcal{P}} \rightarrow \mathcal{F}$ the second structural functor, we get

\[
\hat{\phi} \circ \pi_{Q,R} = \pi_{Q,R} \circ \hat{\phi} \circ \pi_{R} \quad 8.10.13
\]

and therefore $\hat{\phi}$ belongs to $\mathcal{P}(Q)_{\hat{\phi}}$; then, since we assume that the corresponding diagram $8.10.10$ is commutative, we still get

\[
\mu_{x_{\hat{\phi}}} (\lambda_Q(\hat{\phi})) = \lambda_R(\hat{\phi}) \quad 8.10.14,
\]

which amounts to saying that $x_{\hat{\phi}} \cdot \lambda_R(\hat{\phi}) = \lambda_Q(\hat{\phi}) \cdot x_{\hat{\phi}}$, so that $(\hat{\alpha}, \hat{\beta})$ belongs to $(\mathcal{P}(Q) \times \mathcal{P}(R))_{\hat{\phi}}$.

This allows us to choose a family of liftings \{x_{\hat{\phi}}\}_{\hat{\phi}} where $\hat{\phi}$ runs over the set of $\mathcal{P}$-morphisms, which is compatible with $\mathcal{P}$-isomorphisms. Precisely, choose a set of representatives $\mathcal{X}$ for the set of $\mathcal{P}$-isomorphism classes of subgroups of $P_r$ for any pair of subgroups $Q$ and $R$ in $\mathcal{X}$ choose a set of representatives $\mathcal{P}_{Q,R}$ in $\mathcal{P}(Q, R)$ for the set of $\mathcal{P}(Q) \times \mathcal{P}(R)$-orbits, and for any $\hat{\phi} \in \mathcal{P}(Q, R)$ choose a lifting $x_{\hat{\phi}} \in \tilde{L}^{m,b}(Q, R)$ such that the corresponding equality $8.10.12$ holds; thus, any subgroup $Q$ of $P$ determines a unique $\hat{Q}$ in $\mathcal{X}$ and, moreover, we choose an $\mathcal{P}$-isomorphism $\hat{\omega}_Q : Q \cong \hat{Q}$ and a lifting $x_{\hat{\phi}} \in \tilde{L}^{m,b}(\hat{Q}, Q)$ of $\hat{\omega}_Q$ such that the corresponding equality $8.10.12$ holds. Hence, any $\mathcal{P}$-morphism $\hat{\phi} : R \rightarrow Q$ determines $Q, \hat{R} \in \mathcal{X}$ and $\hat{\phi} \in \mathcal{P}_{Q,\hat{R}}$ fulfilling

\[
\hat{\phi} = \hat{\omega}_Q^{-1} \cdot \hat{\alpha} \cdot \hat{\phi} \cdot \hat{\beta} \cdot \hat{\omega}_R \quad 8.10.15
\]

for suitable $\hat{\alpha} \in \mathcal{P}(\hat{Q})$ and $\hat{\beta} \in \mathcal{P}(\hat{R})$, and then we define

\[
x_{\hat{\phi}} = x_{\hat{Q}}^{-1} \cdot \lambda_Q(\hat{\phi}) \cdot x_{\hat{\phi}} \cdot \lambda_R(\hat{\beta}) \cdot x_R \quad 8.10.16.
\]
At this point, it is routine to check that

\[ 8.10.17 \quad \text{We have } x_{\hat{\alpha} \circ \hat{\beta} \circ \hat{\theta}} = x_{\alpha} \cdot x_{\varphi} \cdot x_{\beta} \text{ for any } \mathcal{P}-\text{isomorphisms } \hat{\alpha} \in \mathcal{P}(Q', Q) \text{ and } \hat{\beta} \in \mathcal{P}(R, R'). \]

Note that, for any subgroup \( Q \) of \( P \) and any \( \hat{\alpha} \in \mathcal{P}(Q) \), we may assume that

\[ x_{\hat{\alpha}} = \lambda^m_Q(\hat{\alpha}) \quad 8.10.18; \]

in particular, since the section \( \lambda^m_Q \) is compatible with the first structural functor \( \tau: \mathcal{P} \to \mathcal{F} \), for any \( u \in N_P(Q) \) we have \( x_{\tau_Q(u)} = \tilde{\tau}^m_Q(u) \).

Then, for any triple of subgroups \( Q, R \) and \( T \) of \( P \), and any pair of \( \mathcal{P} \)-morphisms \( \hat{\psi}: T \to R \) and \( \hat{\phi}: R \to Q \), since \( x_{\varphi} \cdot x_{\psi} \) and \( x_{\hat{\varphi} \cdot \hat{\psi}} \) have the same image \( \hat{h}^m(\hat{\varphi} \cdot \hat{\psi}) \) in \( \mathcal{L}^m(Q', T) \), the divisibility of \( \mathcal{L}^m(Q, T) \) guarantees the existence and the uniqueness of \( t_{\hat{\phi}, \hat{\psi}} \) fulfilling

\[ x_{\hat{\phi} \cdot x_{\hat{\psi}}} = x_{\hat{\phi}} \cdot x_{\hat{\psi}} \quad 8.10.19. \]

That is to say, we get a correspondence mapping any \( \mathcal{P} \)-chain \( q: \Delta_2 \to \mathcal{P} \) on \( t_{q(0 \cdot 1), q(1 \cdot 2)} \) and, considering the contravariant functor \( \mathcal{t}' \) (cf. 8.6.3) and setting

\[ \mathcal{C}^n(\hat{\mathcal{F}}, \mathcal{t}') = \prod_{\hat{q} \in \hat{\mathcal{G}}(\Delta_n, \hat{\mathcal{F}})} \mathcal{t}'(\hat{q}(0)) \quad 8.10.20 \]

for any \( n \in \mathbb{N} \), we claim that this correspondence determines a stable element \( t \) of \( \mathcal{C}^2(\hat{\mathcal{F}}, \mathcal{t}') \). [11, A3.18].

Indeed, for another isomorphic \( \mathcal{P} \)-chain \( q': \Delta_2 \to \mathcal{P} \) and a natural isomorphism \( \nu: q \equiv q' \), setting

\[ \hat{\psi} = q(0 \cdot 1), \quad \hat{\phi} = q(1 \cdot 2), \quad \hat{\psi}' = q'(0 \cdot 1), \quad \hat{\phi}' = q'(1 \cdot 2) \]

\[ \nu_0 = \hat{\gamma}, \quad \nu_1 = \hat{\beta} \quad \text{and} \quad \nu_2 = \hat{\alpha} \quad 8.10.21, \]

from statement 8.10.17 we have

\[ x_{\hat{\phi}'} = x_{\hat{\phi} \cdot x_{\hat{\phi}^{-1}}} \]

\[ x_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} = x_{\hat{\phi} \cdot x_{\hat{\psi} \cdot x_{\hat{\phi}^{-1}}}^{-1}} \text{ and } x_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} = x_{\hat{\phi} \cdot x_{\hat{\psi} \cdot x_{\hat{\phi}^{-1}}}^{-1}} \quad 8.10.22 \]

and therefore we get (cf. 8.6.3)

\[ x_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} \cdot \mathcal{t}_{\hat{\phi}, \hat{\psi}'} = \mathcal{t}_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} = (x_{\hat{\phi} \cdot x_{\hat{\psi} \cdot x_{\hat{\phi}^{-1}}}^{-1}}) \cdot (x_{\hat{\beta} \cdot x_{\hat{\phi}^{-1}}}) \]

\[ = x_{\hat{\alpha}} \cdot (x_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} \cdot x_{\hat{\phi}^{-1}}) = x_{\hat{\phi} \cdot \hat{\psi} \cdot \hat{\phi}'} (t' (\hat{\gamma}^{-1})) (t_{\hat{\phi}, \hat{\psi}}) \quad 8.10.23 \]

which proves that the correspondence \( t \) sending \((\hat{\phi}, \hat{\psi})\) to \( t_{\hat{\phi}, \hat{\psi}} \) is stable and, in particular, that \( t_{\hat{\phi}, \hat{\psi}} \) only depends on the corresponding \( \mathcal{F} \)-morphisms \( \hat{\phi} \) and \( \hat{\psi} \).
Moreover, considering the usual differential map
\[ d_2 : \mathbb{C}^2(\tilde{\mathcal{F}}, i_\tilde{t}) \rightarrow \mathbb{C}^3(\tilde{\mathcal{F}}, \check{t}^i) \]
we claim that \( d_2(t) = 0 \); indeed, for a third \( \mathcal{P} \)-morphism \( \check{\eta} : W \rightarrow T \) we get
\[
(x_{\check{\varphi}} \cdot x_{\check{\psi}}) \cdot x_{\check{\eta}} = (x_{\check{\varphi}} \cdot x_{\check{\psi}} \cdot t_{\check{\varphi}, \check{\psi}}) \cdot x_{\check{\eta}} = (x_{\check{\varphi}} \cdot x_{\check{\psi}} \cdot (i_\check{t}(\check{\eta}))) \cdot (t_{\check{\varphi}, \check{\psi}})
\]
\[ = x_{\check{\varphi}} \cdot x_{\check{\psi}} \cdot t_{\check{\varphi}, \check{\psi}, \check{\eta}} \cdot (i_\check{t}(\check{\eta}))(t_{\check{\varphi}, \check{\psi}}) \]
\[ x_{\check{\varphi}} \cdot (x_{\check{\psi}} \cdot x_{\check{\eta}}) = x_{\check{\varphi}} \cdot (x_{\check{\psi}} \cdot t_{\check{\varphi}, \check{\psi}, \check{\eta}}) = x_{\check{\varphi}} \cdot x_{\check{\psi}} \cdot t_{\check{\varphi}, \check{\psi}, \check{\eta}} \cdot t_{\check{\varphi}, \check{\psi}, \check{\eta}}
\]
and the divisibility of \( \hat{\mathcal{L}}^{m,b} \) forces
\[ t_{\check{\varphi}, \check{\psi}, \check{\eta}} \cdot (i_\check{t}(\check{\eta}))(t_{\check{\varphi}, \check{\psi}}) = t_{\check{\varphi}, \check{\psi}, \check{\eta}} \]
\[ t_{\check{\varphi}, \check{\psi}, \check{\eta}}(i_\check{t}(\check{\eta}))(t_{\check{\varphi}, \check{\psi}}) = t_{\check{\varphi}, \check{\psi}, \check{\eta}} \]
since \( i_\check{t}(W) \) is Abelian, with the additive notation we obtain
\[ 0 = (i_\check{t}(\check{\eta}))(t_{\check{\varphi}, \check{\psi}}) - t_{\check{\varphi}, \check{\psi}, \check{\eta}} + t_{\check{\varphi}, \check{\psi}, \check{\eta}} - t_{\check{\varphi}, \check{\psi}, \check{\eta}} \]
proving our claim.

At this point, since \( \mathbb{H}^2_{\mathcal{F}}(\tilde{\mathcal{F}}, i_\check{t}) = 0 \) (cf. 8.6.4), we have \( t = d_1(s) \) for some element \( s = (s_t)_{t \in \mathcal{P}}(\Delta_1, \mathcal{F}) \) in \( \mathbb{C}^0(\tilde{\mathcal{F}}, i_\check{t}) \); that is to say, with the notation above we get
\[ t_{\check{\varphi}, \check{\psi}} = (i_\check{t}(\check{\eta}))(s_{\check{\eta}})(s_{\check{\psi}})^{-1} \]
where we identify any \( \tilde{\mathcal{F}} \)-morphism with the obvious \( \tilde{\mathcal{F}} \)-chain \( \Delta_1 \rightarrow \tilde{\mathcal{F}} \); hence, from equality 8.10.19 we obtain
\[
(x_{\check{\varphi}}(s_{\check{\eta}})^{-1}) \cdot (x_{\check{\psi}}(s_{\check{\psi}})^{-1}) = (x_{\check{\varphi}} \cdot x_{\check{\psi}}) \cdot (i_\check{t}(\check{\eta}))(s_{\check{\eta}})(s_{\check{\psi}})^{-1}
\]
\[ x_{\check{\varphi}} \cdot x_{\check{\psi}}(s_{\check{\eta}})^{-1}
\]
which amounts to saying that the correspondence sending \( \check{\varphi} \in \mathcal{P}(Q,R) \) to \( x_{\check{\varphi}}(s_{\check{\eta}})^{-1} \in \hat{\mathcal{L}}^{m,b}(Q,R) \) defines a functor \( \check{\eta} : \mathcal{P} \rightarrow \hat{\mathcal{L}}^{m,b} \) lifting \( \eta^m \).

Note that in the case that \( Q = R = T \) and that \( \check{\varphi} \) and \( \check{\psi} \) are both inner \( \mathcal{P} \)-automorphisms then equality 8.10.28 forces \( s_{\check{\varphi}} = 1 \) and from 8.10.18 we get \( x_{\check{\varphi}} = \tau_{Q,R}^m(u) \) for some \( u \in N_P(Q) \).

We can modify this correspondence in order to get an \( \mathcal{F} \)-locality functor; indeed, for any \( u \in T_P(R, Q) \), the \( \hat{\mathcal{L}}^{m,b} \)-morphisms \( \check{\eta} = \tau_{Q,R}^m(u) \) and \( \hat{\eta} = \tau_{Q,R}^m(u) \) both lift \( \tau_{Q,R}(u) \); once again, the divisibility of \( \hat{\mathcal{L}}^{m,b} \) guarantees the existence and the uniqueness of \( \ell_u \in \text{Ker}(\hat{f}_{Q,R}) \) fulfilling
\[ \tau_{Q,R}^m(u) = \hat{\eta} \cdot \tau_{Q,R}(u) \cdot \ell_u \]
and the remark above proves that \( \ell_\theta \) only depends on \( \tilde{u} \in \tilde{\mathcal{F}}_p(Q, R) \). Moreover, for a second \( \tilde{\mathcal{F}}_p \)-morphism \( v : T \to R \), we get

\[
\tilde{h}(\tau_{Q,T}(uv)) \cdot \ell_{\tilde{\theta}} = \tilde{\tau}_{Q,T}(uv) = \tilde{\tau}_{Q,R}(u) \cdot \tilde{\tau}_{R,T}(v) = \tilde{h}(\tau_{Q,R}(u)) \cdot \ell_\theta \cdot \tilde{h}(\tau_{R,T}(v)) \cdot \ell_\bar{\theta} \]
\[
= \tilde{h}(\tau_{Q,T}(uv)) \cdot (\tilde{t}(\tilde{u}))((\ell_\theta) \cdot (\ell_\bar{\theta}))
\]

8.10.31.

Always the divisibility of \( \tilde{L}^{m,b} \) forces \( \ell_{\alpha \text{div}} = (\tilde{t}(\tilde{u}))((\ell_\alpha) \cdot (\ell_\bar{\alpha})) \) and, since \( \tilde{t}(T) \) is Abelian, with the additive notation we obtain

\[
0 = (\tilde{t}(\tilde{v}))((\ell_\alpha) - \ell_{\alpha \text{div}} + \ell_\bar{\alpha})
\]

8.10.32;

that is to say, denoting by \( i : \tilde{\mathcal{F}}_P \subset \tilde{\mathcal{F}} \) the obvious inclusion functor, the correspondence \( \ell \) sending any \( \tilde{\mathcal{F}}_p \)-morphism \( \tilde{u} : R \to Q \) to \( \ell_\theta \) defines a \( 1 \)-cocycle in \( C^1(\tilde{\mathcal{F}}_P, \tilde{t}(\tilde{u}) \circ i) \); but, since the category \( \tilde{\mathcal{F}}_P \) has a final object, we actually have [11, Corollary A4.8]

\[
\mathbb{H}^1(\tilde{\mathcal{F}}_P, \tilde{t}(\tilde{u}) \circ i) = \{0\}
\]

8.10.33;

consequently, we obtain \( \ell = d_0(z) \) for some element \( z = (z_Q)_Q \) in

\[
\mathbb{C}^0(\tilde{\mathcal{F}}_P, \tilde{t}(\tilde{u}) \circ i) = \mathbb{C}^0(\tilde{\mathcal{F}}, \tilde{t}(\tilde{u}))
\]

8.10.34.

In conclusion, equality 8.10.30 becomes

\[
\tilde{\tau}_{Q,R}(u) = \tilde{h}(\tau_{Q,R}(u)) \cdot (\tilde{t}(\tilde{u}))(z_Q) \cdot z_R^{-1} = z_Q \cdot \tilde{h}(\tau_{Q,R}(u)) \cdot z_R^{-1}
\]

8.10.35

and therefore the correspondence \( \sigma'x \) sending \( \varphi \in \mathcal{F}(Q, R) \) to \( z_Q \cdot \sigma'(\varphi) \cdot z_R^{-1} \) defines a \( \mathcal{F}^x \)-locality functorial section of \( \tilde{\rho}^x \).

Assume that there is a second \( \mathcal{F} \)-locality functor \( \tilde{h}^m : \mathcal{P} \to \tilde{L}^{m,b} \) lifting \( \tilde{h}^m \); the uniqueness of the natural map \( \lambda_{\mathcal{L}^{m,b}} \) in 8.10.6 already guarantees that, in order to prove that \( \tilde{h}^m \) is naturally \( \mathcal{F} \)-isomorphic to \( \tilde{h}^m \), we may assume that

\[
\tilde{h}^m(\tilde{\alpha}) = \lambda_Q(\tilde{\alpha}) = \tilde{h}^m(\tilde{\alpha})
\]

8.10.36

for any subgroup \( Q \) of \( P \) and any \( \tilde{\alpha} \in \mathcal{P}(Q) \); more precisely, we may assume that \( \tilde{h}^m(\alpha) = \tilde{h}^m(\alpha) \) for any \( \mathcal{F}^x \)-isomorphism \( \alpha \in \mathcal{F}(Q', Q) \). For any \( \mathcal{P} \)-morphism \( \tilde{\varphi} : R \to Q \), set \( x_{\tilde{\varphi}} = \tilde{h}^{m}(\tilde{\varphi}) \) and \( x'_{\tilde{\varphi}} = \tilde{h}^{m}(\tilde{\varphi}) \); for short; the divisibility of \( \tilde{L}^{m,b} \) forces again the existence of a unique \( s_{\tilde{\varphi}} \in \tilde{t}(R) \) fulfilling \( x'_{\tilde{\varphi}} = x_{\tilde{\varphi}} \cdot s_{\tilde{\varphi}} \); note that equality 8.10.36 forces \( s_{\tilde{\alpha}} = 1 \). That is to say,
we get a correspondence mapping any $\mathcal{P}$-chain $\tau: \Delta_1 \to \mathcal{P}$ on $s_\tau(0 \bullet 1)$ and we claim that this correspondence determines a stable element $s$ of $\mathbb{C}^1(\widetilde{\mathcal{F}}, \tilde{t}^u)$ [11, A3.18].

Indeed, for another isomorphic $\mathcal{P}$-chain $q': \Delta_1 \to \mathcal{P}$ and a natural isomorphism $\nu: q \cong q'$, setting

\[ \hat{\phi} = q(0 \bullet 1), \quad \hat{\phi}' = q'(0 \bullet 1), \quad \nu_0 = \hat{\beta} \quad \text{and} \quad \nu_1 = \hat{\alpha} \quad 8.10.37, \]

from our choice we have $s_{\hat{\alpha}} = 1$ and $s_{\hat{\beta}} = 1$ and therefore we get

\[
x'_{\hat{\phi}'} = x_{\hat{\phi}'} \cdot s_{\hat{\phi}'} = (x_{\hat{\alpha}} \cdot x_{\hat{\beta}} \cdot x_{\hat{\beta}}^{-1}) \cdot s_{\hat{\phi}'} = (x_{\hat{\alpha}} \cdot x_{\hat{\beta}}^{-1} \cdot x_{\hat{\beta}}^{-1}) \cdot s_{\hat{\phi}'}
\]

\[
= (x_{\hat{\alpha}} \cdot x_{\hat{\beta}}^{-1}) \cdot (\tilde{t}^u (\hat{\beta}^{-1})) (-1) \cdot s_{\hat{\phi}'}
\]

\[
= x_{\phi} \cdot (\tilde{t}^u (\hat{\beta}^{-1})) (-1) \cdot s_{\hat{\phi}'}
\]

which proves that the correspondence $s$ sending $\hat{\phi}$ to $s_{\hat{\phi}}$ is stable and, in particular, that $s_{\hat{\phi}}$ only depends on the corresponding $\tilde{F}$-morphism $\hat{\phi}$.

Moreover, we also claim that $d_1(s) = 0$; indeed, for a second $\mathcal{P}$-morphism $\hat{\psi}: T \to R$ we get

\[
x'_{\hat{\psi} \cdot \hat{\phi}} = x'_{\hat{\phi}} \cdot x'_{\hat{\psi}} = (x_{\hat{\phi}} \cdot x_{\hat{\psi}} \cdot s_{\hat{\phi}} \cdot s_{\hat{\psi}}) \cdot (\tilde{t}^u (\hat{\psi})) (-1) \cdot s_{\hat{\phi}}
\]

8.10.39

and the divisibility of $\mathcal{L}^{m,n}$ forces $s_{\hat{\psi} \cdot \hat{\phi}} = (\tilde{t}^u (\hat{\psi})) (-1) \cdot s_{\hat{\phi}}$; since $\tilde{t}^u(T)$ is Abelian, with the additive notation we obtain

\[
0 = (\tilde{t}^u (\hat{\psi})) (-1) \cdot s_{\hat{\phi}}
\]

8.10.40

proving our claim.

Finally, since $H^1_\mathbb{Z}(\tilde{\mathcal{F}}, \tilde{t}^u) = 0$ (cf. 8.6.4), we have $t = d_0(n)$ for some element $n = (n_Q)_{Q}$ in $\mathbb{C}^0(\tilde{\mathcal{F}}, \tilde{t}^u)$ where we identify any subgroup $Q$ of $P$ with the obvious $\tilde{F}$-chain $\Delta_0 \to \tilde{F}$; that is to say, with the notation above we get

\[
s_{\hat{\phi}} = (\tilde{t}^u (\hat{\phi})) (-1)
\]

8.10.41

hence, we obtain

\[
\mathcal{H}^m_\mathbb{Z} (\hat{\phi}) = x_{\hat{\phi}} \cdot (\tilde{t}^u (\hat{\phi})) (-1) = n_Q \cdot \mathcal{H}^m_\mathbb{Z} (\hat{\phi})
\]

8.10.42

which amounts to saying that the correspondence $\nu$ sending $Q$ to $n_Q$ defines a natural $\mathcal{F}$-isomorphism between $\mathcal{H}^m_\mathbb{Z}$ and $\mathcal{H}^m_\mathbb{Z}$. We are done.
9. Functoriality of the perfect $\mathcal{F}$-locality

9.1. It remains to discuss the functoriality of the perfect $\mathcal{F}$-locality $\mathcal{P}$; let $P'$ be a second finite $p$-group, $\mathcal{F}'$ a Frobenius $P'$-category and $\mathcal{P}'$ the corresponding perfect $\mathcal{F}'$-locality, and denote by

$$\tau' : \mathcal{T}_{P'} \longrightarrow \mathcal{P}'$$

and

$$\pi' : \mathcal{P}' \longrightarrow \mathcal{F}'$$

the structural functors; let $\alpha : P \rightarrow P'$ be an ($\mathcal{F}, \mathcal{F}'$)-functorial group homomorphism [11, Theorem 17.18]; recall that we have a so-called Frobenius functor $f_\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ [11, 12.1], and let us denote by $t_\alpha : \mathcal{T}_P \rightarrow \mathcal{T}_{P'}$ the functor induced by $\alpha$. In this section, replacing $\mathcal{F}$ and $\mathcal{P}$ by the quotients

$$\bar{\mathcal{P}} = \mathcal{P}/[c^b_{\bar{P}}, c^a_{\bar{P}}]$$

and

$$\bar{\mathcal{P}}' = \mathcal{P}'/[c^b_{\bar{P}'}, c^a_{\bar{P}'}]$$

we prove that there is a unique isomorphism class of functors $\bar{\mathcal{g}}_\alpha : \bar{\mathcal{P}} \rightarrow \bar{\mathcal{P}}'$ fulfilling

$$\bar{\pi}' \circ t_\alpha = \bar{\mathcal{g}}_\alpha \circ \tau$$

and

$$\pi' \circ \bar{\mathcal{g}}_\alpha = \bar{f}_\alpha \circ \pi$$

as a consequence, if $P''$ is a third finite $p$-group, $\mathcal{F}''$ a Frobenius $P''$-category, $\mathcal{P}''$ the perfect $\mathcal{F}''$-locality and $\alpha' : P' \rightarrow P''$ an ($\mathcal{F}', \mathcal{F}''$)-functorial group homomorphism, then the functors $\bar{\mathcal{g}}_{\alpha'} \circ \bar{\mathcal{g}}_\alpha$ and $\bar{\mathcal{g}}_{\alpha \circ \alpha'}$ from $\bar{\mathcal{P}}$ to $\bar{\mathcal{P}}''$ are naturally isomorphic.

9.2. As a matter of fact, assuming the existence of $\mathcal{P}$, we already have proved in [11, Theorem 17.18] the existence of all the possible perfect quotients $\bar{\mathcal{P}}$ of $\mathcal{P}$; presently, this simplifies our work. Let us recall the construction of $\bar{\mathcal{P}}$; let $U$ be an $\mathcal{F}$-stable subgroup of $P$ (cf. 2.5), set $\bar{P} = P/U$ and denote by $\bar{\mathcal{F}}$ the quotient Frobenius $\bar{P}$-category $\mathcal{F}/U$ [11, Proposition 12.3]; for any subgroup $Q$ of $P$, denote by $\bar{Q}$ the image of $Q$ in $\bar{P}$ and by $U_{\mathcal{F}}(Q)$ the kernel of the canonical group homomorphism $\mathcal{F}(Q) \rightarrow \bar{\mathcal{F}}(\bar{Q})$; moreover, if $Q$ is fully normalized in $\mathcal{F}$, for short we set

$$P^Q = N^U_{\mathcal{F}}(Q)$$

and

$$\mathcal{F}^Q = N^U_{\mathcal{F}}(Q)$$

so that $\mathcal{F}^Q$ is a Frobenius $P^Q$-category, and in the group $\mathcal{P}(Q)$ we put the definition (cf. 2.4)

$$U_\mathcal{P}(Q) = \Omega^p \left( \pi_\alpha^{-1}(U_{\mathcal{F}}(Q)) \right) \cdot \tau_Q \left( N_U(Q) \cdot H_{\mathcal{F}^Q} \right)$$

actually, via $\mathcal{P}$-isomorphisms we can extend the definition of $U_\mathcal{P}(Q)$ to any subgroup $Q$ of $P$. Then, $\bar{\mathcal{P}}$ is the perfect $\bar{\mathcal{F}}$-locality fulfilling [11, 17.15-17.17]

$$\bar{\mathcal{P}}(\bar{Q}, \bar{R}) = \mathcal{P}(Q, R)/U_\mathcal{P}(R)$$

for any pair of subgroups $Q$ and $R$ of $P$. 
9.3. In the general case, setting $U = \operatorname{Ker}(\alpha)$ and $\tilde{P} = P/U$, we have an injective group homomorphism $\tilde{\alpha} : \tilde{P} \to P'$ and from [11, Proposition 12.3] it is easily checked that we have a faithful Frobenius functor $\bar{f}_{\tilde{\alpha}} : \tilde{F} \to \tilde{F}'$; moreover, from the description above it is easily checked that any functor $\tilde{\alpha}_n : \tilde{P} \to \tilde{P}'$ fulfilling condition 9.1.3 factorizes throughout the quotient $\tilde{P}/[\tilde{h}^b, \tilde{c}_\bar{P}]$; consequently, in order to prove the existence and the uniqueness of $\tilde{\alpha}_n$, it suffices to prove the existence and the uniqueness of a suitable functor $\tilde{\alpha}_n : \tilde{P}/[\tilde{h}^b, \tilde{c}_\bar{P}] \to \tilde{P}'$. Thus, we may assume that $\alpha$ is injective and, in this case, let us identify $P$ and $\mathcal{F}$ with their respective images in $P'$ and $\mathcal{F}'$. In order to relate $P$ and $\mathcal{P}'$, we start by getting a relationship between the natural $\mathcal{F}'$-locality $\mathcal{L}^{\mathfrak{r}, sc}$ and the basic $\mathcal{F}'$-locality $\mathcal{L}^b$; more explicitly, the converse image $\operatorname{Res}_{\mathcal{F}^{sc}}(\mathcal{L}^b)$ of $\tilde{F}'$ in $\tilde{L}^b$ is clearly a $p$-coherent $\mathcal{F}'$-locality and we will exhibit a canonical $\mathcal{F}'^{sc}$-locality functor from $\tilde{L}^{\mathfrak{r}, sc}$ to a suitable quotient $\mathcal{F}'^{sc}$-locality of $\operatorname{Res}_{\mathcal{F}^{sc}}(\mathcal{L}^b)$.

9.4. Choose a natural $\mathcal{F}$-basic $P \times P$-set $\Omega$ (cf. 3.3 and 3.5), denote by $G$ the group of automorphisms of $\operatorname{Res}_{\{1\} \times P}(\Omega)$ and identify the $p$-group $P$ with the image of $P \times \{1\}$ in $G$, so that for any pair of $\mathcal{F}$-selfcentralizing subgroups $Q$ and $R$ of $P$ we have (cf. 4.12 and 4.14)

$$
\mathcal{L}^{\mathfrak{r}, sc}(Q, R) = T_G(R, Q)/\tilde{\Theta}_\Omega^1(R) \quad \text{and} \quad \mathcal{L}^{\mathfrak{r}, sc} = \mathcal{L}^{\mathfrak{r}, sc}/\tilde{\Theta}_\Omega^1 9.4.1;
$$

it is clear that $G$ acts faithfully on the $P \times P'$-set $\Omega \times P'$ centralizing the action of $\{1\} \times P'$. We claim that $9.4.2$. There is a thick $\mathcal{F}'$-basic $P' \times P'$-set $\Omega'$ such that $\operatorname{Res}_{P \times P'}(\Omega')$ contains $\Omega \times P'$.

Indeed, any $P \times P'$-orbit $O''$ of $\Omega \times P'$ is isomorphic to $(P \times P')/\Delta_\varphi(Q)$ for some subgroup $Q$ of $P$ and some $\varphi \in \mathcal{F}(P, Q)$, and therefore it is isomorphic to a $P \times P'$-orbit of $\Omega'$; hence, up to replacing $\Omega'$ by the disjoint union of $k$ copies of $\Omega'$ for a suitable $k$ prime to $p$, we may assume that $\operatorname{Res}_{P \times P'}(\Omega')$ contains $\Omega \times P'$. Similarly, denote by $G'$ the group of automorphisms of $\operatorname{Res}_{\{1\} \times P'}(\Omega')$ and identify the $p$-group $P'$ with the image of $P' \times \{1\}$ in $G'$; once again, for any pair of subgroups $Q'$ and $R'$ of $P'$ we have (cf. 4.14)

$$
\mathcal{L}^{\mathfrak{r}}(Q', R') = T_{G'}(R', Q')/\tilde{\Theta}_{\Omega'}(R') 9.4.3.
$$

In particular, for any subgroup $Q$ of $P$, $(\operatorname{Res}_{\mathcal{F}}(\mathcal{L}^{\mathfrak{r}}))(P, Q)$ is the set of classes $f' \tilde{\Theta}_{\Omega'}(Q)$ in $\tilde{L}^b(P, Q)$ where

$$
f' : \operatorname{Res}_{Q \times P'}(\Omega') \cong \operatorname{Res}_{\varphi \times \operatorname{id}_{P'}}(\operatorname{Res}_{P \times P'}(\Omega')) 9.4.4
$$

is a $Q \times P'$-set isomorphism for some $\varphi \in \mathcal{F}(P, Q)$. 
9.5. As in section 6 above, for induction purposes we have to consider a nonempty set \( X \) of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \) which contains any subgroup of \( P \) admitting an \( \mathcal{F} \)-morphism from some subgroup in \( X \); recall that \( \Omega \) contains \( \Omega^x \) (cf. 3.5) and, for any \( Q \in X \), denote by \( \Omega_Q^x \) the union of all the \( Q \times P \) -orbits of \( \Omega \) isomorphic to \( (Q \times P)/\Delta_{\eta}(T) \) for some \( T \in X \) and some \( \eta \in \mathcal{F}(Q,T) \) (cf. 5.18); then, for any \( \varphi \in \mathcal{F}(Q,P,Q) \) note that any \( Q \times P \) -set isomorphism

\[
\hat{f} : \text{Res}_{Q \times P}(\Omega) \cong \text{Res}_{\varphi \times \text{id}_P} (\text{Res}_{P \times P}(\Omega))
\]

maps \( \Omega_Q^x \) onto \( \Omega_{\varphi(Q)}^x \). Moreover, let us denote by \( \hat{\Omega}_{Q}^x \subset \text{Res}_{Q \times P}(\Omega') \) the union of all the \( Q \times P' \) -orbits of \( \Omega' \) which are isomorphic to a \( Q \times P' \) -orbit of \( \Omega_{Q}^x \).

**Proposition 9.6.** For any \( Q, R \in X \), any \( \varphi \in \mathcal{F}(Q,R) \) and any \( R \times P' \) -set isomorphism

\[
f' : \text{Res}_{R \times P'}(\Omega') \cong \text{Res}_{\varphi \times \text{id}_{P'}} (\text{Res}_{P \times P'}(\Omega'))
\]

we have \( f'(\Omega_{Q}^x) \subset \hat{\Omega}_{Q}^x \).

**Proof:** For any \( T, U \in X \), any \( \theta \in \mathcal{F}(R,U) \) and any \( \eta' \in \mathcal{F}'(Q,T) \) , it suffices to prove that, if there is an injective \( R \times P' \) -set homomorphism

\[
(R \times P')/\Delta_{\theta}(U) \longrightarrow \text{Res}_{\varphi \times \text{id}_{P'}} ((Q \times P')/\Delta_{\eta'}(T))
\]

then there is \( u' \in P' \) such that \( T^{u'} \) is contained in \( P \) and that \( \eta' \circ \kappa_{T,T^{u'}}(u') \) belongs to \( \mathcal{F}(Q,T^{u'}) \); we argue by induction on \( |P : R| \) and may assume that \( R \neq P \). Since we have an injective \( Q \times P' \) -set homomorphism

\[
(Q \times P')/\Delta_{\eta'}(T) \longrightarrow \text{Res}_{\varphi \times \text{id}_{P'}} ((P \times P')/\Delta_{\eta'\circ \eta'}(T))
\]

we may assume that \( Q = P \); since we still have an injective \( U \times P' \) -set homomorphism

\[
(U \times P')/\Delta_{\text{id}_P}(U) \longrightarrow \text{Res}_{\varphi \times \text{id}_{P'}} ((R \times P')/\Delta_{\theta}(U))
\]

we actually may assume that \( U = R \) and \( \theta = \text{id}_R \).

In this case, we also may assume that

\[
(\varphi \times \text{id}_{P'})(\Delta(R)) \subset \Delta_{\eta'}(T)
\]

so that \( R \) is contained in \( T \) and that \( \eta' \) extends \( \varphi \); in particular, we still may assume that \( R \neq T \) and we set \( \hat{R} = N_T(R) \) and \( \hat{P} = N_P(R) \); then, it follows from [11, 2.10] that \( \varphi \) can be extended to an \( \mathcal{F} \)-morphism \( \hat{\varphi} : \hat{R} \rightarrow P \) and that \( R \) is fully \( \mathcal{F}_P(R) \) -normalized in \( \mathcal{F} \).
Since \( \eta' \) and \( \hat{\eta} \) coincide over \( R \) and we have \( C_p(R) = Z(R) \), we still have \( \eta'(-) = \hat{\eta}(-) \); that is to say, there is \( \sigma' \in \mathcal{F}'(\hat{R}) \) such that the restriction of \( \eta' \) to \( \hat{R} \) coincides with \( \hat{\eta} \circ \sigma' \); thus, \( \sigma' \) acts trivially on \( R \) and therefore, denoting by \( \mathcal{F}'(\hat{R})_R \) the stabilizer of \( R \) in \( \mathcal{F}'(\hat{R}) \), \( \sigma' \) belongs to \( \mathcal{O}_p(\mathcal{F}'(\hat{R})_R) \) [4, Ch. 5, Theorem 3.4].

On the other hand, it follows from [11, Proposition 2.7] that there is \( \zeta' \in \mathcal{F}'(P', \hat{P}) \) such that \( \hat{R} = \zeta'(R) \) is fully normalized in \( \mathcal{F}' \) and, moreover, that \( \hat{R} = \zeta'(\hat{R}) \) is fully normalized in \( \mathcal{N}_{\mathcal{F}'}(\hat{R}) \); then \( \mathcal{F}_{p'}(\hat{R})_\hat{R} \) is a Sylow \( p \)-subgroup of \( \mathcal{F}'(\hat{R})_\hat{R} \) [11, Proposition 2.11] and therefore it contains \( \mathcal{O}_{p'}(\mathcal{F}'(\hat{R})_\hat{R}) \) : set \( \hat{P} = \zeta'(\hat{P}) \) and respectively denote by \( \hat{\phi} : \hat{R} \to \hat{P} \) and \( \hat{\sigma}' \in \mathcal{F}'(\hat{R}) \) the items determined by \( \hat{\phi} \) and \( \sigma' \) via the group isomorphism \( \zeta' : \hat{P} \cong \hat{P} \) induced by \( \zeta' \).

In particular, \( \hat{\sigma}' \) belongs to \( \mathcal{F}_{p'}(\hat{R}) \) and we have an injective \( \hat{R} \times P' \)-set homomorphism
\[
(\hat{R} \times P')/\Delta(\hat{R}) \longrightarrow \text{Res}_{\hat{\phi} \times \text{id}_{p'}}((\hat{P} \times P')/\Delta_{\hat{\phi} \circ \hat{\sigma}'}(\hat{R})) \quad 9.6.6;
\]
then, still denoting by \( \hat{\phi} : \hat{R} \to \hat{P} \) the \( \mathcal{F} \)-morphism determined by \( \hat{\phi} \) above, the inverse of the group isomorphism \( \zeta' \times \text{id}_{p'} \) determines an injective \( \hat{R} \times P' \)-set homomorphism
\[
(\hat{R} \times P')/\Delta(\hat{R}) \longrightarrow \text{Res}_{\hat{\phi} \times \text{id}_{p'}}((\hat{P} \times P')/\Delta_{\hat{\phi} \circ \hat{\sigma}'}(\hat{R})) \quad 9.6.7;
\]
moreover, since the restriction of \( \eta' \) to \( \hat{R} \) coincides with \( \hat{\phi} \circ \sigma' \), we have
\[
\Delta_{\hat{\phi} \circ \hat{\sigma}'}(\hat{R}) = (\hat{P} \times P') \cap \Delta_{\eta'}(\hat{R}) \quad 9.6.8
\]
and therefore we still have an injective \( \hat{R} \times P' \)-set homomorphism
\[
(\hat{R} \times P')/\Delta(\hat{R}) \longrightarrow \text{Res}_{\hat{\phi} \times \text{id}_{p'}}((P \times P')/\Delta_{\eta'}(T)) \quad 9.6.9;
\]
finally, it suffices to apply the induction hypothesis.

**Proposition 9.7.** For any \( Q \in \mathcal{X} \) the inclusion of the set of \( Q \times P' \)-orbits in \( \Omega^x_{Q \times P} P' \) in the set of \( Q \times P' \)-orbits in \( \hat{\Omega}^x_Q \) admits a section \( s^x_Q \) such that, for any \( Q \times P' \)-orbit \( \hat{O} \) in \( \hat{\Omega}^x_Q \), we have a \( Q \times P' \)-set isomorphism \( s^x_Q(\hat{O}) \cong \hat{O} \) and that, for any \( Q \times P \)-orbit \( O \) in \( \Omega^x_Q \), any \( \varphi \in \mathcal{F}(P, Q) \) and any \( Q \times P \)-set isomorphism
\[
f : \text{Res}_{Q \times P}(\Omega) \cong \text{Res}_{\varphi \times \text{id}_P}(\text{Res}_{P \times P}(\Omega)) \quad 9.7.1,
\]
setting \( \hat{Q} = \varphi(Q) \) and \( \hat{O} = f(O) \), we have
\[
|\langle s^x_Q \rangle^{-1}(\hat{O} \times P')| = |\langle s^x_Q \rangle^{-1}(O \times P')| \quad 9.7.2.
\]
Proof: Let $\hat{O}$ be a $Q \times P'$-orbit in $\hat{\Omega}_Q^x$ and $\hat{\omega}$ an element of $\hat{O}$; since $\hat{O}$ is isomorphic to $O \times P'$ for some $Q \times P$-orbit $O$ in $\hat{\Omega}_Q^x$, there is $u' \in P'$ such that the stabilizer of $\hat{\omega}u'v^{-1}$ in $Q \times P'$ coincides with the stabilizer $(Q \times P)_\omega$ in $Q \times P$ of some $\omega \in O$; but, we have $(Q \times P)_\omega = \Delta_\eta(T)$ for some $T \in \mathcal{X}$ and some $\eta \in \mathcal{F}(Q, T)$ and therefore, according to Proposition 5.17, this stabilizer determines the set $Z(T)_\omega$. Moreover, if $v'$ is another element of $P'$ such that the stabilizer of $\hat{\omega}u'v'^{-1}$ coincides with the stabilizer of some element of $O$ then, since this stabilizer coincides with the stabilizer of $\omega(u'v'^{-1})$, the element $u'v'^{-1}$ belongs to $P$ and thus, in $\Omega_Q^x \times P'$, we have $(\omega, u') = (\omega(u'v'^{-1}), v')$.

In conclusion, the correspondence mapping $\hat{O}$ on $O \times P'$ defines a map $s_Q^x$ from the set of $Q \times P'$-orbits in $\hat{\Omega}_Q^x$ to the set of $Q \times P'$-orbits in $\hat{\Omega}_Q^x \times P'$, and it is clear that $s_Q^x(O \times P') = O \times P'$. Finally, for any $\varphi \in \mathcal{F}(P, Q)$ and any $Q \times P$-set isomorphism

$$f : \text{Res}_Q \times P(\Omega) \cong \text{Res}_{\varphi \times id_P}(\text{Res}_P \times P(\Omega))$$  \hspace{1cm} 9.7.3,$$

setting $Q = \varphi(Q)$ and denoting by $\varphi_\times : Q \cong Q$ the isomorphism induced by $\varphi$, we have an obvious $Q \times P'$-set isomorphism

$$f \times_P \text{id}_{P'} : \text{Res}_Q \times P'(\Omega \times P') \cong \text{Res}_{\varphi_\times \times id_{P'}}(\text{Res}_P \times P'(\Omega \times P'))$$  \hspace{1cm} 9.7.4,$$

but, since $\mathcal{F}(P, Q) \subset \mathcal{F}'(P, Q)$, we also have a $Q \times P'$-set isomorphism

$$\text{Res}_Q \times P'(\Omega') \cong \text{Res}_{\varphi_\times \times id_{P'}}(\text{Res}_P \times P'(\Omega'))$$  \hspace{1cm} 9.7.5,$$

hence, we get a $Q \times P'$-set isomorphism

$$\text{Res}_Q \times P'(\Omega' - \Omega \times P') \cong \text{Res}_{\varphi_\times \times id_{P'}}(\text{Res}_P \times P'(\Omega' - \Omega \times P'))$$  \hspace{1cm} 9.7.6,$$

and therefore $f \times_P \text{id}_{P'}$ can be extended to a $Q \times P'$-set isomorphism

$$f' : \text{Res}_Q \times P'(\Omega') \cong \text{Res}_{\varphi_\times \times id_{P'}}(\text{Res}_P \times P'(\Omega'))$$  \hspace{1cm} 9.7.7.$$

At this point, it follows from Proposition 9.6 that $f'(\hat{\Omega}_Q^x) = \hat{\Omega}_Q^x$ and then it is quite clear that

$$s_Q^x(f'(\hat{O})) = f'(s_Q^x(\hat{O}))$$  \hspace{1cm} 9.7.8,$$

which proves equality 9.7.2.

9.8. Now, for any $Q \in \mathcal{X}$ and any $Q \times P'$-orbit $O'$ in $\Omega_Q^x \times P'$ we clearly can choose a subset $\mathcal{S}_{Q, O'}^x$ of $\mathcal{G}_{\Omega'}(Q)$ containing the trivial element and fulfilling

$$(s_Q^x)^{-1}(O') = \{s'(O')\}_{s' \in \mathcal{S}_{Q, O'}^x} \quad \text{and} \quad |(s_Q^x)^{-1}(O')| = |\mathcal{S}_{Q, O'}^x|$$  \hspace{1cm} 9.8.1;
thus, according to Proposition 9.7, for any $\varphi \in \mathcal{F}(P, Q)$ and any $Q \times P$-set isomorphism
\[ f : \text{Res}_{Q \times P}(\Omega) \cong \text{Res}_{\varphi \times \text{id}_P}(\text{Res}_{P \times P}(\Omega)) \]
9.8.2, setting $\bar{Q} = \varphi(Q)$ and $\bar{O}' = (f \times_P \text{id}_{P'})(O')$, we can choose a bijection
\[ \sigma^x_{\bar{Q}, \bar{O}', \varphi, f} : \mathcal{S}^x_{\bar{Q}, \bar{O}'} \cong \mathcal{S}^x_{\bar{Q}, \bar{O}'} \]
9.8.3 preserving the trivial element.

9.9. As we mention in 9.3 above, we have to consider a suitable quotient of the $\mathcal{F}^x$-locality $\text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)$; let us denote by $\tilde{\mathcal{F}}^b : \mathcal{F}^x \rightarrow \mathfrak{A}b$ the contravariant functor mapping $Q \in \mathcal{X}$ on (cf. Corollary 8.4)
\[ \tilde{\mathcal{F}}^b(Q) = \prod_O \text{ab(\text{Aut}(O))} \]
9.9.1, where $O$ runs over a set of representatives for the set of isomorphism classes of $Q \times P'$-sets $(Q \times P')/\Delta_f(T')$ where $T'$ is a subgroup of $P'$ such that any subgroup $U'$ of $P'$ fulfilling $\mathcal{F}'(T', U') \neq \emptyset$ does not belong to $\mathcal{X}$, and $\eta'$ belongs to $\mathcal{F}'(Q, T')$; actually, this definition still makes sense for any subgroup $Q$ of $P$, defining a contravariant functor $\tilde{\mathcal{F}} \rightarrow \mathfrak{A}b$ that we still denote by $\tilde{\mathcal{F}}^b$; then, the quotient $\mathcal{F}$-locality $\text{Res}_{\mathcal{F}}(\mathcal{L}^b)/\tilde{\mathcal{F}}^b$ (cf. 2.10) simply maps any pair of subgroups $Q$ and $R$ of $P$ such that $R \notin \mathcal{X}$ on $\mathcal{F}(Q, R)$; this remark will be useful in 9.11 below.

**Theorem 9.10.** With the notation and the choice above, there is a unique $\mathcal{F}^x$-locality functor
\[ \tilde{\mathcal{F}}^x : \mathcal{L}^c \rightarrow \text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)/\tilde{\mathcal{F}}^b \]
9.10.1 in such a way that, for any $Q, R \in \mathcal{X}$, any $\varphi \in \mathcal{F}(Q, R)$ and any $Q \times P$-set isomorphism
\[ f : \text{Res}_{Q \times P}(\Omega) \cong \text{Res}_{\varphi \times \text{id}_P}(\text{Res}_{Q \times P}(\Omega)) \]
9.10.2, $\tilde{\mathcal{F}}^x$ maps the image $\tilde{f}$ of $f$ in $\mathcal{L}^x(Q, R)$ on the class $\tilde{f}'$ in $\mathcal{L}^b(Q, R)/\tilde{\mathcal{F}}^b(R)$ of a $R \times P'$-set isomorphism
\[ f' : \text{Res}_{R \times P'}(\Omega') \cong \text{Res}_{\varphi \times \text{id}_{P'}}(\text{Res}_{Q \times P'}(\Omega')) \]
9.10.3 which, for any $R \times P'$-orbit $O'$ in $\Omega^x_{R \times P} \times_{P'} P'$, any $\omega' \in O'$ and any $s' \in \mathcal{S}^x_{R, O'}$, fulfills
\[ f'(s'(\omega')) = \tilde{s}'((f \times_P \text{id}_{P'})(\omega')) \]
9.10.4 where we set $\tilde{s}' = \sigma^x_{\bar{Q}, \bar{O}', \varphi, f}(s')$ for $\varphi^* = \tilde{\varphi}$.
Proof: Setting \( \hat{R} = \varphi(R) \) and denoting by \( \varphi_* : R \cong \hat{R} \) the isomorphism determined by \( \varphi \), we already know that \( f(\Omega^x_R) = \Omega^x_{\hat{R}} \) (cf. 9.5) and therefore \( f \) induces an \( R \times P' \)-set isomorphism

\[
f^x_R = f \times_p \text{id}_{P'} : \Omega^x_R \times P \cong \text{Res}_{\varphi_* \times \text{id}_{P'}}(\Omega^x_R \times P')
\]

then, it is quite clear that this \( R \times P' \)-set isomorphism can be uniquely extended to a \( R \times P' \)-set isomorphism

\[
f^x_R : \hat{\Omega}_R \cong \text{Res}_{\varphi_* \times \text{id}_{P'}}(\hat{\Omega}^x_R)
\]

fulfilling condition 9.10.4; once again, since we also have an \( R \)-acts trivially on \( \hat{\Omega}_R \)

Moreover, if \( g' : \text{Res}_{R \times P'}(\Omega') \cong \text{Res}_{\varphi_* \times \text{id}_{P'}}(\text{Res}_{\hat{R} \times P'}(\Omega')) \)

is another \( R \times P' \)-set isomorphism extending \( f^x_R \) then the composition

\[
f'^{-1} \circ g' : \text{Res}_{R \times P'}(\Omega') \cong \text{Res}_{R \times P'}(\Omega')
\]

acts trivially on \( \hat{\Omega}_R \) and in particular the image of \( f'^{-1} \circ g' \) in \( \mathcal{C}_{G'}(R) / \mathfrak{G}_{\Omega'}(R) \) belongs to \( \mathfrak{G}^{h,x}(R) \); that is to say, \( f \) determines a unique class \( \hat{f}' \) in the quotient set \( \mathcal{L}^{h,x}(Q, R) / \mathfrak{G}^{h,x}(R) \).

On the other hand, the class \( \hat{f}' \) does not depend on our choice of the sets \( \mathcal{S}^x_{R, \Omega'} \) and the bijections \( \sigma^x_{R, \Omega', \varphi_* P, f} \) in 9.8; indeed, another choice of them determines another \( R \times P' \)-set isomorphism

\[
g^x_R : \hat{\Omega}_R \cong \text{Res}_{\varphi_* \times \text{id}_{P'}}(\hat{\Omega}^x_R)
\]

extending \( f^x_R \) and it is easily checked that the extension to \( \Omega' \) by the identity on \( \Omega' - \hat{\Omega}_R \) of the difference \( (f^x_R)^{-1} \circ g^x_R \) belongs to \( \mathfrak{G}_{\Omega'}(R) \).

Now, we claim that the correspondence mapping \( f \in T_G(R, Q) \) on the class \( f' \in \mathcal{L}^{h,x}(Q, R) / \mathfrak{G}^{h,x}(R) \) is functorial; namely we claim that, for any \( T \in \mathfrak{X} \), any \( \psi \in \mathcal{F}(R, T) \), any \( T \times P \)-set isomorphism

\[
g : \text{Res}_{T \times P}(\Omega) \cong \text{Res}_{\psi \times \text{id}_{P'}}(\text{Res}_{R \times P}(\Omega))
\]

and any \( T \times P' \)-set isomorphism

\[
g' : \text{Res}_{T \times P'}(\Omega') \cong \text{Res}_{\psi \times \text{id}_{P'}}(\text{Res}_{R \times P'}(\Omega'))
\]
which for any $T \times P'$-orbit $M'$ in $\Omega^x_T \times_P P'$, any $\omega' \in M'$ and any $s' \in S^x_{T,M'}$, fulfills

$$g'(s'(\omega')) = \tilde{s}'((g \times_P \text{id}_{P'})(\omega'))$$

9.10.14

where $\tilde{s}' = \sigma^x_{T,M',\psi_P,g}(s')$, the above correspondence maps the composition $f \circ g$ on the composition of the classes

$$\tilde{g}' \in \mathcal{L}^h_x(T,T)/\mathcal{L}^h_x(T) \text{ and } \tilde{f}' \in \mathcal{L}^h_x(Q,R)/\mathcal{L}^h_x(R)$$

9.10.15.

Indeed, setting $\tilde{T} = \psi(T)$ and $\tilde{T} = \tilde{\varphi}(\tilde{T})$, and denoting by $\psi_* : T \cong \tilde{T}$ and $\tilde{\varphi}_* : \tilde{T} \cong \tilde{T}$ the respective isomorphisms induced by $\psi$ and $\tilde{\varphi}_*$, it is clear that $g'$ and $f'$ induce a $T \times P'$- and a $\tilde{T} \times P'$-set isomorphisms

$$g' : \text{Res}_{T \times P'}(\Omega') \cong \text{Res}_{\psi \times \text{id}_{P'}}(\text{Res}_{\tilde{T} \times P'}(\Omega'))$$

9.10.16

$$\tilde{f}' : \text{Res}_{\tilde{T} \times P'}(\Omega') \cong \text{Res}_{\tilde{\varphi}_* \times \text{id}_{P'}}(\text{Res}_{\tilde{T} \times P'}(\Omega'))$$

and, according to Proposition 9.6, we have $\tilde{f}'(\tilde{\Omega}^x_T) = \tilde{\Omega}^x_{\tilde{T}}$; moreover, since $f'$ extends $\tilde{f}'$, it is clear that $\tilde{f}'$ extends the corresponding $f^x_T$; in particular, denoting by $O'$ the $R \times P'$-orbit in $\Omega^x_O \times_P P'$ containing $M' = g'(M')$ and by $\tilde{O}'$ the $\tilde{R} \times P'$-orbit in $\Omega^x_{\tilde{R}} \times_P P'$ containing $\tilde{M}' = f'(M')$, it follows again from Proposition 5.17 that we can do our choice in 9.8 above in such a way that we have $S^x_{T,M'} \subset S^x_{O,O'}$ and $S^x_{\tilde{T},\tilde{M}'} \subset S^x_{\tilde{R},\tilde{O}'}$, that $\sigma^x_{\tilde{R},\tilde{O}',\psi_P,f}$ maps $S^x_{\tilde{T},\tilde{M}'}$ onto $S^x_{T,M'}$, and that $\sigma^x_{T,M',\psi_P,f}$ coincides with the restriction of $\sigma^x_{\tilde{R},\tilde{O}',\psi_P,f}$.

In this situation, considering the $T \times P'$-set isomorphism

$$\tilde{f}' \circ g' : \text{Res}_{T \times P'}(\Omega') \cong \text{Res}_{\tilde{\varphi}_* \circ \psi \times \text{id}_{P'}}(\text{Res}_{\tilde{T} \times P'}(\Omega'))$$

9.10.17,

for any $T \times P'$-orbit $M'$ in $\Omega^x_T \times_P P'$, any $\omega' \in M'$ and any $s' \in S^x_{T,M'}$, we get

$$(\tilde{f}' \circ g')(s'(\omega')) = \tilde{f}'\left(\tilde{s}'((g \times_P \text{id}_{P'})(\omega'))\right)$$

9.10.18

$$= \tilde{s}'\left(((f \circ g) \times_P \text{id}_{P'})(\omega')\right)$$

where $\tilde{s}' = \sigma^x_{T,M',\psi_P,g}(s')$ and $\tilde{s}' = \sigma^x_{\tilde{T},\tilde{M}',\psi_{\tilde{P}}\tilde{f}}(\tilde{s}')$; thus, the composition $f' \circ g'$ also fulfills the corresponding condition 9.10.4 and therefore our correspondence above maps $f \circ g$ on the class of $f' \circ g'$ in $\mathcal{L}^h_x(Q,T)/\mathcal{L}^h_x(T)$, proving our claim.

Finally, recall that (cf. 4.12)

$$\mathcal{L}^h_x(Q,R) = \mathcal{L}^\alpha_x(Q,R)/(\mathcal{h}_x(T,R) \times \mathcal{c}^\text{res}_n(R))$$

9.10.19

and that $\mathcal{L}^\alpha_x(Q,R) = T\mathcal{G}(R,Q)/\mathfrak{S}_\mathcal{H}^1(R)$; thus, if $Q = R$, $\varphi = \text{id}_R$ and $f$ belongs to the converse image in $T\mathcal{G}(R,Q)$ of $\mathcal{h}_x(R) \times \mathcal{c}^\text{res}_n(R)$, the action of $f$
on $\Omega^x_R$ coincides with the action of some element in $\mathfrak{S}^1_{\Omega}(R)$; then, it is easily checked that the action of the uniquely extended $R \times P'$-set isomorphism

$$\hat{f}_R : \hat{\Omega}_R^x \cong \text{Res}_{\text{id}_R \times \text{id}_{P'}}(\hat{\Omega}_R^x)$$

fulfilling condition 9.10.4 also coincides with the action of some element belonging to $\mathfrak{S}_{\Omega^P}(R)$; in this case, it is not difficult to check that the class in $\mathcal{L}^{b,x}(R)/\mathfrak{t}^{b,x}(R)$ of a $R \times P'$-set isomorphism $f'$ extending $\hat{f}_R$ is trivial. This proves the existence and the uniqueness of the functor $\hat{f}_R$ in 9.10.1; the compatibility with the corresponding structural functors is easily checked, proving that it is actually an $\mathcal{F}^x$-locality functor. We are done.

9.11. It follows from section 6 that the $\mathcal{F}^x$-locality $\mathcal{L}^{n,x}$ contains a perfect $\mathcal{F}^x$-locality $\mathcal{P}^x = \check{P}^x$ and therefore from Theorem 9.10 we get a $\mathcal{F}^x$-locality functor

$$\check{h}^x : \mathcal{P}^x \to \text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)/\mathfrak{t}^{b,x}$$

as a matter of fact, $\check{h}^x$ can be extended to a $\mathcal{F}^x$-locality functor (cf. 9.9)

$$\mathcal{P} \to \text{Res}_{\mathcal{F}}(\mathcal{L}^b)/\mathfrak{t}^{b,x}$$

that we still denote by $\check{h}^x$, mapping any pair of subgroups $Q$ and $R$ of $P$ such that $R \notin \mathcal{X}$ on the structural map $\mathcal{P}(Q,R) \to \mathcal{F}(Q,R)$; this remark will be useful in the proof of the theorem below.

**Theorem 9.12.** Any $\mathcal{F}^x$-locality functor from $\mathcal{P}^x$ to $\text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)/\mathfrak{t}^{b,x}$ is naturally $\mathcal{F}^x$-isomorphic to $\check{h}^x$.

**Proof:** We argue by induction on $|\mathcal{X}|$; if $\mathcal{X} = \{P\}$ then the statement follows from Proposition 2.17; thus, assume that $\mathcal{X} \neq \{P\}$, choose a minimal element $U$ in $\mathcal{X}$ fully normalized in $\mathcal{F}$ and set

$$\mathcal{Y} = \mathcal{X} - \{\theta(U) \mid \theta \in \mathcal{F}(P,U)\}$$

we may assume that the full subcategory of $\mathcal{P}^x$ over $\mathcal{Y}$ coincides with $\mathcal{P}^\mathcal{Y}$; on the other hand, we have a canonical functor

$$(\text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)/\mathfrak{t}^{b,x})^\mathcal{Y} \to \text{Res}_{\mathcal{F}^\mathcal{Y}}(\mathcal{L}^b)/\mathfrak{t}^{b,\mathcal{Y}}$$

which composed with the restriction of $\check{h}^x$ to $\mathcal{P}^\mathcal{Y}$ clearly coincides with $\check{h}^\mathcal{Y}$.

In particular, by the induction hypothesis, for any $\mathcal{F}^x$-locality functor

$$\check{f}^x : \mathcal{P}^x \to \text{Res}_{\mathcal{F}^x}(\mathcal{L}^b)/\mathfrak{t}^{b,x}$$

its restriction to $\mathcal{P}^\mathcal{Y}$ composed with the canonical functor above is naturally $\mathcal{F}^\mathcal{Y}$-isomorphic to $\check{h}^\mathcal{Y}$; thus, up to modifying $\check{f}^x$ by conjugation with a suitable element in $(\text{Res}_{\mathcal{F}^x}(\mathcal{L}^b))(P)/\mathfrak{t}^{b,x}(P)$ (cf. 2.9.3), we may assume that the
restitution of \( f^X \) to \( \mathcal{P}^\square \) composed with the canonical functor above coincides with \( h^\square \). In this situation, the converse image \( h^\square(\mathcal{P}^\square) \) in \((\text{Res}_\mathcal{F}(\mathcal{L}^\mathcal{P}))\slash \hat{\mathcal{h}}^{\mathcal{h},X}(\mathcal{P}^\square))\) via homomorphism 9.12.2 of the image \( h^\square(\mathcal{P}^\square) \) of \( h^\square \) by \( h^\square \) contains the image \( f^X(\mathcal{P}^\square) \) of \( \mathcal{P}^\square \); more explicitly, denoting by \( \hat{h}^{\mathcal{h},X}(\mathcal{P}^\square) \) the subcategory of \( \text{Res}_\mathcal{F}(\mathcal{L}^\mathcal{P}))\slash \hat{\mathcal{h}}^{\mathcal{h},X} \) (cf. 9.9) which coincides with \( h^\square(\mathcal{P}^\square) \) over \( \mathcal{Q} \) and maps any pair of subgroups \( Q \) and \( R \) of \( P \) such that \( R \not\in \mathcal{Q} \) on

\[
\hat{h}^{\mathcal{h},X}(\mathcal{P}^\square)(Q, R) = (\text{Res}_\mathcal{F}(\mathcal{L}^\mathcal{P}))\slash \hat{\mathcal{h}}^{\mathcal{h},X}(\mathcal{P}^\square)(Q, R) \tag{9.12.4}
\]

we get two \( \mathcal{F} \)-locality functors (cf. 9.11.2)

\[
f^X : \mathcal{P} \rightarrow \hat{h}^{\mathcal{h},X}(\mathcal{P}^\square) \quad \text{and} \quad h^X : \mathcal{P} \rightarrow \hat{h}^{\mathcal{h},X}(\mathcal{P}^\square) \tag{9.12.5}
\]

where we still denote by \( f^X \) the obvious extension of the functor 9.12.3 mapping any pair of subgroups \( Q \) and \( R \) of \( P \) such that \( R \not\in \mathcal{X} \) on the structural map \( \mathcal{P}(Q, R) \rightarrow \mathcal{F}(Q, R) \).

Set \( \mathcal{M}^X = \hat{h}^{\mathcal{h},X}(\mathcal{P}^\square) \) for short, and denote by \( \rho^X : \mathcal{M}^X \rightarrow \mathcal{F} \) the second structural functor; first of all note that, according to 2.16, we have functors

\[
\text{loc}_\mathcal{P} : \text{ch}^X(\mathcal{F}) \rightarrow \hat{\text{loc}} \quad \text{and} \quad \text{loc}_{\mathcal{M}^X} : \text{ch}^X(\mathcal{F}) \rightarrow \hat{\text{loc}} \tag{9.12.6}
\]

and it is clear that the functors \( f^X \) and \( h^X \) determine natural maps \( \text{loc}_\mathcal{F} \) and \( \text{loc}_{\mathcal{M}^X} \) from \( \text{loc}_{\mathcal{P}} \) to \( \text{loc}_{\mathcal{M}^X} \); moreover, we know that \( \text{loc}_{\mathcal{F}} = \text{loc}_\mathcal{F} \) (cf. 2.15) and, since \( f^X \) and \( h^X \) are \( \mathcal{F} \)-locality functors, it is easily checked that \( \text{loc}_{f^X} \) and \( \text{loc}_{h^X} \) fulfill the conditions in Proposition 2.17; consequently, it follows from this proposition that we actually have the equality \( \text{loc}_{f^X} = \text{loc}_{h^X} \). In particular, for any \( \mathcal{P} \)-chain \( q : \Delta_n \rightarrow \mathcal{P} \), denoting by \( \bar{q} : \Delta_n \rightarrow \mathcal{F} \) the corresponding \( \mathcal{F} \)-chain, we have (cf. 2.16)

\[
\text{loc}_{\mathcal{P}}(\bar{q}) = (\mathcal{P}(q), \text{Ker}(\pi_q)) \quad \text{and} \quad \text{loc}_{\mathcal{M}^X}(\bar{q}) = (\mathcal{M}^X(f^X \circ q), \text{Ker}(\rho_{f^X \circ q})) = (\mathcal{M}^X(h^X \circ q), \text{Ker}(\rho_{h^X \circ q})) \tag{9.12.7}
\]

and it is not difficult to check that, up to replacing \( f^X \) for a naturally \( \mathcal{F} \)-isomorphic functor, we may assume that we have

\[
\mathcal{M}^X(f^X \circ q) = \mathcal{M}^X(h^X \circ q) \tag{9.12.8}
\]

and that the group homomorphisms mapping any \( \mathcal{P} \)-automorphism \( \alpha : q \cong q \) on

\[
f^X \circ \alpha : f^X \circ q \cong f^X \circ q \quad \text{and} \quad h^X \circ \alpha : h^X \circ q \cong h^X \circ q \tag{9.12.9}
\]

coincide with each other.

On the other hand, since \( f^X \) and \( h^X \) coincide over \( \mathcal{P}^\square \) and they are \( \mathcal{F} \)-locality functors, for any \( \mathcal{P} \)-morphism \( \varphi : R \rightarrow Q \) we have

\[
f^X(\varphi) = h^X(\varphi) \cdot \ell_\varphi \tag{9.12.10}
\]
for some element $\ell_\varphi \in M^\chi (R)$ belonging either to the kernel of the canonical homomorphism (cf. 9.12.2)
\[
\mathcal{L}^h (R)/\mathcal{L}^{b, x} (R) \longrightarrow \mathcal{L}^h (R)/\mathcal{L}^{b, \nu} (R)
\]
whenever $R$ belongs to $\mathcal{P}$, or to Ker$(\rho^h_R)$ otherwise. Moreover, in the situation above, for any $\alpha \in \mathcal{P}(R)$ we get $\tilde{f}^x (\alpha) = h^x (\alpha)$, so that $\ell_\alpha = 1$; actually, up to replacing $\tilde{f}^x$ by a naturally $\mathcal{F}$-isomorphic functor, we may assume that we have $\tilde{f}^x (\alpha) = h^x (\alpha)$ for any $\mathcal{P}$-isomorphism $\alpha \in \mathcal{P}(Q', Q)$.

But, we already know that the kernel of the structural homomorphism $\pi^h_R: \mathcal{L}^h (R) \rightarrow \mathcal{F}'(R)$ is given by (cf. 4.3.2)
\[
\text{Ker}(\pi^h_R) = \prod_{\delta \in \mathcal{D}'_R} \text{ab}(\text{Aut}(O'))
\]
where we denote by $\mathcal{D}'_R$ the set of isomorphism classes of the $R \times P'$-sets $(R \times P')/\Delta_{\theta'} (T')$ for any subgroup $T'$ of $P'$ and any element $\theta \in \mathcal{F}'(R, T')$; consequently, if $R$ belongs to $\mathcal{X}$, the kernel above involves the isomorphism classes of the $R \times P'$-orbits in $\hat{\Omega}^x_R$, and therefore we can choose a set of representatives in $\Omega^x_R \times P P'$; explicitly, considering the contravariant functor $\tilde{t}^U: \tilde{\mathcal{F}} \rightarrow \mathfrak{ab}$ introduced in Proposition 8.9 above, since $\tilde{t}^U (R) = \{0\}$ whenever $R$ does not belong to $\mathcal{X}$, in all the cases it is easily checked that $\ell_\varphi$ above belongs to
\[
\tilde{t}^U (R) \cong \left( \prod_{\delta \in \tilde{\mathcal{F}}(R, U)} \text{ab} \left( \text{Aut} \left( (R \times P')/\Delta_{\phi} (U) \right) \right) \right)^{\mathcal{F}'(U)}
\]
\[
\cong \left( \prod_{\delta \in \tilde{\mathcal{F}}(R, U)} \text{ab} \left( \mathcal{N}_{R \times P'} (\Delta_{\phi} (U)) \right) \right)^{\mathcal{F}'(U)}
\]
That is to say, we have obtained a correspondence mapping any $\mathcal{P}$-chain $q: \Delta_1 \rightarrow \mathcal{P}$ on $\ell_q (0 \bullet 1) \in \tilde{t}^U (q(0))$ and, with our choice of this correspondence vanishing over the $\mathcal{P}$-isomorphisms, we claim that it determines a stable element $\ell$ of $\mathcal{C}(\tilde{\mathcal{F}}, \tilde{t}^U)$ [11, A3.17]; indeed, for another isomorphic $\mathcal{P}$-chain $q': \Delta_1 \rightarrow \mathcal{P}$ and a natural isomorphism $\nu: q \cong q'$, setting
\[
\varphi = q(0 \bullet 1), \quad \varphi' = q'(0 \bullet 1), \quad \nu_0 = \beta \quad \text{and} \quad \nu_1 = \alpha
\]
from our choice we have $\ell_\alpha = 1$ and $\ell_\beta = 1$ and therefore we get
\[
\tilde{f}^x (\varphi') = h^x (\varphi') \cdot \ell_{\varphi'} = (h^x (\alpha) \cdot h^x (\varphi) \cdot h^x (\beta)^{-1}) \cdot \ell_{\varphi'}
\]
\[
= (\tilde{f}^x (\alpha) \cdot \tilde{f}^x (\varphi) \cdot \ell_{\varphi'} \cdot h^x (\beta)^{-1}) \cdot \ell_{\varphi'}
\]
\[
= \tilde{f}^x (\varphi') \cdot (\tilde{t}^U (\beta^{-1})) (\ell_{\varphi'}) \cdot \ell_{\varphi'}
\]
which proves that the correspondence $\ell$ sending $\tilde{\varphi}$ to $\ell_{\varphi}$ is stable and, in particular, that $\ell_{\varphi}$ only depends on the corresponding $\mathcal{P}$-morphism $\tilde{\varphi}$. 

Moreover, we also claim that \( d_1(\ell) = 0 \); indeed, for a second \( P \)-morphism \( \psi : T \to R \) we get
\[
\int (\varphi \circ \psi) = \int (\varphi) \cdot \int (\psi) = (\mathfrak{h} (\varphi) \cdot \ell (\varphi)) \cdot (\mathfrak{h} (\psi) \cdot \ell (\psi)) = \mathfrak{h} (\varphi \circ \psi) \cdot (\ell (\varphi)) (\ell (\psi)) \cdot \ell (\psi)
\]
and the \textit{divisibility} of \( M^T \) forces
\[
\ell (\varphi \circ \psi) = (\tilde{t} (\tilde{\varphi})) (\ell (\varphi)) \cdot \ell (\psi)
\]
since \( \tilde{t} (T) \) is Abelian, with the additive notation we obtain
\[
0 = (\tilde{t} (\tilde{\varphi})) (\ell (\varphi)) - \ell (\varphi \circ \psi) + \ell (\psi)
\]
proving our claim.

Finally, since \( H_1^T (\hat{\mathcal{F}}, \tilde{\mathfrak{f}}) = \{0\} \) (cf. 8.6.4), we have \( \ell = d_0 (n) \) for some element \( n = (n_Q)_Q \) of \( C^0 (\hat{\mathcal{F}}, \tilde{\mathfrak{f}}) \); that is to say, with the notation above we get
\[
\ell (\varphi) = (\tilde{t} (\tilde{\varphi})) (n_Q) n_R^{-1}
\]
where we identify any \( \tilde{\mathcal{F}} \)-object with the obvious \( \tilde{\mathcal{F}} \)-chain \( \Delta_0 \to \tilde{\mathcal{F}} \); hence, we obtain
\[
\int (\varphi) = \mathfrak{h} (\varphi) \cdot (\tilde{t} (\tilde{\varphi})) (n_Q) n_R^{-1} = n_Q \mathfrak{h} (\varphi) n_R^{-1}
\]
which amounts to saying that the correspondence sending \( Q \) to \( n_Q \) defines a \textit{natural isomorphism} between \( \mathfrak{h} \) and \( \int \); the compatibility with the corresponding structural functors is easily checked, proving that \( n \) defines a \textit{natural} \( \mathcal{F} \)-isomorphism. We are done.

9.13. Recall that, according to Theorem 8.10 above, we have a canonical \( \mathcal{F}' \)-\textit{locality} functor
\[
\mathfrak{b}' : \mathcal{P}' \longrightarrow \mathcal{L}'^b
\]
thus, we also have an \( \mathcal{F} \)-\textit{locality} functor
\[
\text{Res}_{\mathcal{F}} (\mathfrak{b}') : \text{Res}_{\mathcal{F}} (\mathcal{P}') \longrightarrow \text{Res}_{\mathcal{F}} (\mathcal{L}'^b)
\]
and we consider the induced \( \mathcal{F}^{sc} \)-\textit{locality} functor (cf. 9.1.2)
\[
\text{Res}_{\mathcal{F}^{sc}} (\mathfrak{b}') : \text{Res}_{\mathcal{F}^{sc}} (\mathcal{P}') \longrightarrow \text{Res}_{\mathcal{F}^{sc}} (\mathcal{L}'^{b,sc})
\]
which is actually \textit{faithful} as it proves the following lemma; recall that the \textit{kernel} of the structural functor \( \mathcal{P}' \to \mathcal{F}' \) is given by the \textit{contravariant} functor
[11, Proposition 13.14]
\[
\mathfrak{c}_\mathcal{F}' = \mathfrak{c}_\mathcal{F}' / [\mathfrak{c}_\mathcal{F}' , \mathfrak{c}_\mathcal{F}']
\]
Lemma 9.14. For any subgroup \( Q \) in \( \mathfrak{X} \) the group homomorphism

\[
\epsilon_{\mathcal{F}^x}(Q) \longrightarrow \text{Ker}(\pi_Q^h)/\mathbb{t}^{h,x}(Q)
\]

determined by the \( \mathcal{F}^x \)-locality functor \( \text{Res}_{\mathcal{F}^x}(\mathfrak{h}') \) admits a section \( \sigma_Q^{\mathfrak{h}} \) which is stable through \( \mathcal{F} \)-isomorphisms.

**Proof:** Choose an \( \mathcal{F}' \)-morphism \( \varphi' : Q \rightarrow P' \) such that \( Q' = \varphi'(Q) \) is fully centralized in \( \mathcal{F}' \); then, we know that \( \epsilon_{\mathcal{F}'}(Q') \) is the direct limit of the canonical functor from the Frobenius \( C_{P'}(Q') \)-category \( C_{\mathcal{F}'}(Q') \) to \( \mathfrak{G} \) \cite[13.1 and Proposition 13.14]{} and therefore we have a canonical group homomorphism

\[
\rho_Q : \text{ab}(C_{P'}(Q')) \longrightarrow \epsilon_{\mathcal{F}'}(Q')
\]

On the other hand, we know that

\[
\text{Ker}(\pi_Q^h)/\mathbb{t}^{h,x}(Q) \cong \prod_{O'} \text{ab}(\text{Aut}(O))
\]

where \( O' \) runs over a set of representatives for the isomorphism classes of \( Q \times P'/\Delta_{Q'}(T') \) where \( T' \) is a subgroup of \( P' \) such that for some subgroup \( U \) in \( \mathfrak{X} \) of \( P \) we have \( \mathcal{F}'(T',U) \neq \emptyset \), and \( \eta' \) belongs to \( \mathcal{F}'(Q,T') \); in particular, for \( T' = Q' \) and \( \eta' = \varphi'^* = (\varphi'_* )^{-1} \), in the right-hand member of equality 9.14.3 we have the factor

\[
\text{ab} \left( \tilde{N}_{Q \times P'}(\Delta_{\mathcal{F}'}(Q')) \right) \cong \text{ab}(C_{P'}(Q'))
\]

At this point, we denote by \( \sigma_Q^{\mathfrak{h}} \) the composition

\[
\text{Ker}(\pi_Q^h)/\mathbb{t}^{h,x}(Q) \longrightarrow \text{ab}(C_{P'}(Q')) \overset{\rho_Q}{\longrightarrow} \epsilon_{\mathcal{F}'}(Q') \quad \epsilon_{\mathcal{F}'}(Q') \quad \epsilon_{\mathcal{F}'}(Q')
\]

which is clearly stable through \( \mathcal{F} \)-isomorphisms and, since the \( \mathcal{F}' \)-locality functor \( \tilde{h} : \mathcal{P}' \rightarrow \mathcal{L}'' \) maps \( \tilde{h}'(u') \) on \( \tau_{Q'}^h(u') \) for any \( u' \in C_{P'}(Q') \), it is easily checked that \( \sigma_Q^{\mathfrak{h}} \) is a section of the homomorphism 9.14.1 above.

**Theorem 9.15.** There is a unique natural \( \mathcal{F} \)-isomorphism class of \( \mathcal{F} \)-locality functors \( \tilde{g} : \mathcal{P} \rightarrow \text{Res}_\mathcal{F}(\mathcal{P}') \).

**Proof:** This statement follows from Theorem 7.2 applied to the \( \mu \)-coherent \( \mathcal{F} \)-locality \( \text{Res}_\mathcal{F}(\mathcal{P}') \) provided we prove that there is a unique natural \( \mathcal{F}^x \)-isomorphism class of \( \mathcal{F}^x \)-locality functors \( g^x \) from \( \mathcal{P}^x \) to \( \text{Res}_\mathcal{F}^x(\mathcal{P}') \); we actually prove that, for any set \( \mathfrak{X} \) as in 9.5 above, there is a unique natural \( \mathcal{F}^x \)-isomorphism class of \( \mathcal{F}^x \)-locality functors \( g^x \) from \( \mathcal{P}^x \) to \( \text{Res}_\mathcal{F}^x(\mathcal{P}') \).

Arguing by induction on \( |\mathfrak{X}| \), if \( \mathfrak{X} = \{ P \} \) then the statement follows from Proposition 2.17 above; thus, assume that \( \mathfrak{X} \neq \{ P \} \), choose a minimal element \( U \) in \( \mathfrak{X} \) fully normalized in \( \mathcal{F} \) and set

\[
\mathfrak{Y} = \mathfrak{X} - \{ \theta(U) \mid \theta \in \mathcal{F}(P,U) \}
\]
it follows from the induction hypothesis that there is a unique natural $F^\mathfrak{y}$-isomorphism class of $F^\mathfrak{y}$-locality functors $g^\mathfrak{y}: \mathcal{P}^\mathfrak{y} \to \text{Res}_{F^\mathfrak{y}}(\bar{P}')$; then, considering the composition $\text{Res}_{F^\mathfrak{y}}(\bar{h}') \circ g^\mathfrak{y}$, it follows from Theorem 9.12 that, for a suitable choice of a $F^\mathfrak{y}$-locality functor $h^\mathfrak{y}$, the image of this functor is contained in the image of $\text{Res}_{F^\mathfrak{y}}(\bar{h}')$ (cf. 2.9.3).

But, always according to this theorem, $h^\mathfrak{y}$ can be extended to an $F^x$-locality functor $h^x$ over $\mathcal{P}^x$; thus, the image of this functor is contained in
\[
M^x(Q, V) = (\text{Res}_F(L'^x))(Q, V)/\tilde{t}^{b:x}(V)
\]
in particular, $h^x$ and $\bar{h}'$ induce two functors
\[
\mathcal{P}^x \longrightarrow M^x \leftarrow \text{Res}_{F^x}(\bar{P}')
\]
and therefore we get a quotient $F^x$-locality $M^x/\mathfrak{d}^x$ (cf. 2.9). At this point, it is easily checked that the composition of the right-hand functor in 9.15.3 with the canonical functor $M^x \to M^x/\mathfrak{d}^x$ induces an $F^x$-locality isomorphism
\[
\text{Res}_{F^x}(\bar{P}') \cong M^x/\mathfrak{d}^x
\]
thus, the composition of the left-hand functor in 9.15.3 with the functor $M^x \to \text{Res}_{F^x}(\bar{P}')$ obtained from isomorphism 9.15.5 supplies the announced $F^x$-locality functor
\[
g^x: \mathcal{P}^x \longrightarrow \text{Res}_{F^x}(\bar{P}')
\]
Moreover, if we have another $F^x$-locality functor $\hat{g}^x: \mathcal{P}^x \to \text{Res}_{F^x}(\bar{P}')$ then, from the induction hypothesis, we may assume that the restriction of this functor to $\mathcal{P}^\mathfrak{y}$ coincides with $g^\mathfrak{y}$ and may choose the same $F^x$-locality functor $h^x$ over $\mathcal{P}^x$; now, the images of $h^x$ and of the composition of $\hat{g}^x$ with the right-hand functor in 9.15.3 are contained in $M^x$ and it is not difficult to check from Theorem 9.12 that the corresponding functors from $\mathcal{P}^x$ to $M^x$ still are naturally $F$-isomorphic. Finally, the compositions of these functors with the functor $M^x \to \text{Res}_{F^x}(\bar{P}')$ obtained from isomorphism 9.15.5 remain naturally $F$-isomorphic to each other and, on the other hand, respectively coincide with $g^x$ and with $\hat{g}^x$. We are done.
10. Appendix on a second proof of Theorem 6.12

10.1. Let $\mathcal{F}$ be a Frobenius $P$-category and $\mathfrak{X}$ a nonempty set of $\mathcal{F}$-self-centralizing subgroups of $P$ which contains any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathfrak{X}$; we keep our notation and hypothesis in 6.1 - 6.5 above. In particular, we consider the $p$-coherent $\mathcal{F}^\mathfrak{X}$-sublocality $(\nu^\mathfrak{X}, \mathcal{M}^\mathfrak{X}, \rho^\mathfrak{X})$ of $\bar{\mathcal{L}}^n_{\mathfrak{X}}$ containing the converse image $\mathcal{M}^\mathfrak{X}$ of $\mathcal{P}^\mathfrak{X}$ in $\bar{\mathcal{L}}^n_{\mathfrak{X}}$ as a full subcategory over $\mathfrak{Y}$ and fulfilling
\[ \mathcal{M}^\mathfrak{X}(Q,V) = \bar{\mathcal{L}}^n_{\mathfrak{X}}(Q,V) \]
for any $Q \in \mathfrak{X}$ and any $V \in \mathfrak{X} - \mathfrak{Y}$, and the quotient $\mathcal{F}^\mathfrak{X}$-locality $(\bar{\nu}^\mathfrak{X}, \bar{\mathcal{M}}^\mathfrak{X}, \bar{\rho}^\mathfrak{X})$ of $\bar{\mathcal{M}}^\mathfrak{X}$ defined by
\[ \bar{\mathcal{M}}^\mathfrak{X}(Q,R) = \mathcal{M}^\mathfrak{X}(Q,R)/\nu^\mathfrak{X}_n(Z(R)) \]
together with the induced natural maps
\[ \bar{\nu}^\mathfrak{X}_{Q,R} : \mathcal{T}_P(Q,R) \to \bar{\mathcal{M}}^\mathfrak{X}(Q,R) \quad \text{and} \quad \bar{\rho}^\mathfrak{X}_{Q,R} : \bar{\mathcal{M}}^\mathfrak{X}(Q,R) \to \mathcal{F}(Q,R) \]
for any $Q,R \in \mathfrak{X}$.

10.2. Thus, for any $Q \in \mathfrak{Y}$, from the exact sequence 6.1.3 we obtain the exact sequence
\[ 1 \to \prod W \prod \bar{\theta} Z(W) \to \bar{\mathcal{M}}^\mathfrak{X}(Q) \to \mathcal{F}^\mathfrak{X}(Q) \to 1 \]
where $W$ runs over a set of representatives for the set of $Q$-conjugacy classes of elements of $\mathfrak{X} - \mathfrak{Y}$ contained in $Q$ and $\bar{\theta}$ over a set of representatives for the $\mathcal{F}_Q(W)$-classes in $\bar{\mathcal{F}}(P,W)^{\mathfrak{X}}$, whereas for any $V \in \mathfrak{X} - \mathfrak{Y}$ it follows from 6.3 above that we have the exact sequence
\[ 1 \to \text{Ker}(\bar{\nu}^\mathfrak{X}_V) \to \bar{\mathcal{M}}^\mathfrak{X}(V) \to \mathcal{F}^\mathfrak{X}(V) \to 1 \]
in particular, for any $Q \in \mathfrak{X}$ in $\bar{\mathcal{M}}^\mathfrak{X}(Q)$ we have
\[ \text{Ker}(\bar{\rho}^\mathfrak{X}_Q) \cap \bar{\nu}^\mathfrak{X}_Q(Q) = \{1\} \]

10.3. As in 6.8 above, we consider the contravariant functor
\[ \bar{\mathfrak{J}}^\mathfrak{X}_U : \bar{\mathcal{F}}^\mathfrak{X} \to \text{Ab} \]
mapping any $Q \in \mathfrak{Y}$ on $\{0\}$ and any $V \in \mathfrak{X} - \mathfrak{Y}$ on $Z(V)$, and the corresponding additive contravariant functor
\[ \bar{\mathfrak{J}}^{\mathfrak{X},\mathfrak{ak}}_U : \text{ac}(\bar{\mathcal{F}}^\mathfrak{X}) \to \text{Ab} \]
recall that from the functor and the natural map in 6.7.1 we get the following exact sequence of contravariant functors
\[ 0 \to \bar{\mathfrak{J}}^{\mathfrak{X},\mathfrak{ak}}_U \to \bar{\mathfrak{J}}^{\mathfrak{X},\mathfrak{ak}}_U \circ \mathfrak{m}_P \to \text{Ker}(\bar{\rho}^\mathfrak{X}) \to 0 \]
note that, according to 6.3 above, the sequence obtained by restricting these functors to the subcategory of \( \tilde{\mathcal{F}}^x \) formed by all the \( \tilde{\mathcal{F}}^x \)-isomorphisms is split. Since \( U \) is fully normalized in \( \mathcal{F} \), we have the Frobenius \( N_P(U) \)-category \( N_F(U) \) and we can consider the analogous contravariant functors; more explicitly, we set \( N = N_P(U) \) and \( N = N_F(U) \), denote by \( \mathfrak{N} \) the set of \( Q \in \mathfrak{X} \) contained in \( N \), and consider corresponding contravariant functors

\[
\tilde{\mathfrak{f}}_U : \tilde{N}^n \to \mathfrak{N}_b \quad \text{and} \quad \tilde{\mathfrak{f}}_{U,ac} : \mathfrak{A}(\tilde{N}^n) \to \mathfrak{N}_b \quad 10.3.4
\]

we clearly have an inclusion functor \( i_U : \tilde{N}^n \to \tilde{\mathcal{F}}^x \), which induces a functor \( i_U : \tilde{N}^n \to \tilde{\mathcal{F}}^x \) and note that \( \tilde{\mathfrak{f}}_U = \tilde{\mathfrak{f}} \circ i_U \); then, the following reduction result is the key of this second proof.

**Proposition 10.4.** For any \( n \in \mathbb{N} \) the inclusion functor \( i_U : \tilde{N}^n \to \tilde{\mathcal{F}}^x \) induces a group isomorphism

\[
\mathbb{H}_n^U(\tilde{\mathcal{F}}^x, \tilde{\mathfrak{f}}_U) \cong \mathbb{H}_n^U(\tilde{N}^n, \tilde{\mathfrak{f}}_U) \quad 10.4.1
\]

**Proof:** For \( n = 0 \), the isomorphism is clear since both members are zero. Otherwise, consider the categories \( U^P \) and \( U^N \) where the respective objects are the subgroups of \( P \) or of \( N \) strictly containing \( U \), and the morphisms are just the inclusions between them; note that we have obvious functors

\[
i : U^N \to U^P \quad \text{and} \quad n : U^P \to U^N \quad 10.4.2
\]

respectively sending any \( U^N \)-object \( R \) to \( R \), and any \( U^P \)-object \( Q \) to \( N_Q(U) \); moreover, \( U^P \) and \( U^N \) can be identified to subcategories of \( \tilde{\mathcal{F}}^x \) and \( \tilde{N}^n \) respectively.

Then, still denoting by \( Z(U) \) the constant contravariant functors from \( U^P \) and \( U^N \) to \( \mathfrak{N}_b \) sending any object to \( Z(U) \) and any morphism to \( \text{id}_{Z(U)} \), for any \( n \in \mathbb{N} \) we have obvious group homomorphisms

\[
r : \mathbb{C}^{n+1}(\tilde{\mathcal{F}}^x, \tilde{\mathfrak{f}}_U) \to \mathbb{C}^n(U^P, Z(U))
\]

\[
r_N : \mathbb{C}^{n+1}(\tilde{N}^n, \tilde{\mathfrak{f}}_U) \to \mathbb{C}^n(U^N, Z(U))
\]

respectively mapping \( z = (z_q)_q \) where \( q \) runs either over \( \mathfrak{f}(\Delta_{n+1}, \tilde{\mathcal{F}}^x) \) or over \( \mathfrak{f}(\Delta_{n+1}, \tilde{N}^n) \), on \( (z_u)_u \) where \( u \) runs either over \( \mathfrak{f}(\Delta_n, U^P) \) or over \( \mathfrak{f}(\Delta_n, U^N) \) and \( \hat{u} \) is either the \( \tilde{\mathcal{F}}^x \)-chain or the \( \tilde{N}^n \)-chain such that

\[
\hat{u}(0) = U \quad , \quad \hat{u}(0 \cdot 1) = \hat{u}^{u(0)} \quad \text{and} \quad \hat{u} \circ \delta^u_{0,1} = u \quad 10.4.4
\]

Moreover, it follows from [11, Proposition 4.6 and 14.8] that, in both cases, any natural isomorphism \( \hat{\nu} : \hat{u} \cong \hat{u}' \) is actually determined by the element \( \hat{\nu}_0 \) of \( \tilde{\mathcal{F}}(U) \); thus, setting [11, A3.18]

\[
\mathbb{C}^n_{\tilde{\mathcal{F}}(U)}(U^P, Z(U)) = r(\mathbb{C}^{n+1}(\tilde{\mathcal{F}}^x, \tilde{\mathfrak{f}}_U))
\]

\[
\mathbb{C}^n_{\tilde{\mathcal{F}}(U)}(U^N, Z(U)) = r_N(\mathbb{C}^{n+1}(\tilde{N}^n, \tilde{\mathfrak{f}}_U))
\]

10.4.5,
homomorphisms 10.4.3 finally determine group isomorphisms
\[
\mathbb{C}^n_{\ast+1}(\tilde{F}^X, \tilde{\delta}_F^X) \cong \mathbb{C}^n_{\ast}(U^P, Z(U)) \quad \text{10.4.6}
\]
\[
\mathbb{C}^n_{\ast+1}(\tilde{N}^n, \tilde{\delta}_F^n) \cong \mathbb{C}^n_{\ast}(U^N, Z(U))
\]
which are compatible with the corresponding differential maps; furthermore, it is easily checked that the right-hand members are compatible with the functors i and n in 10.4.2 above.

Let us denote by \(\mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U))\) and by \(\mathbb{H}^n_{\tilde{F}(U)}(U^N, Z(U))\) the cohomology groups of the right-hand members in 10.4.6; thus, in order to prove isomorphism 10.4.1, it suffices to show that these groups are isomorphic; since \(n \circ i = \text{id}_{U^N}\), we already get that the corresponding composition

\[
\begin{array}{ccc}
\mathbb{H}^n_{\tilde{F}(U)}(U^N, Z(U)) & \xrightarrow{n} & \mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U)) \\
\mathbb{H}^n_{\tilde{F}(U)}(i, Z(U)) & \xrightarrow{i} & \mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U)) \\
\mathbb{H}^n_{\tilde{F}(U)}(n, Z(U)) & \xrightarrow{n} & \mathbb{H}^n_{\tilde{F}(U)}(U^N, Z(U))
\end{array}
\]

10.4.7

coincides with the identity, so that the homomorphism \(\mathbb{H}^n_{\tilde{F}(U)}(n, Z(U))\) is injective and the homomorphism \(\mathbb{H}^n_{\tilde{F}(U)}(i, Z(U))\) is surjective.

On the other hand, the inclusion induces a natural map \(i : i \circ n \to \text{id}_{U^P}\), and it is easily checked that, for any \(n \in \mathbb{N}\), the corresponding homotopy map [11, A4.4.6] sends \(\mathbb{C}^n_{\ast+1}(U^P, Z(U))\) to \(\mathbb{C}^n_{\ast}(U^P, Z(U))\); hence, it follows from equality [11, A4.5.1] that we have the commutative diagram [11, A4.5.2]

\[
\begin{array}{ccc}
\mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U)) & \xrightarrow{\mathbb{H}^n_{\tilde{F}(U)}(i, Z(U))} & \mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U)) \\
\mathbb{H}^n_{\tilde{F}(U)}(i, Z(U)) & \xrightarrow{\mathbb{H}^n_{\tilde{F}(U)}(n, Z(U))} & \mathbb{H}^n_{\tilde{F}(U)}(U^N, Z(U))
\end{array}
\]

10.4.8,

so that we get

\[
\mathbb{H}^n_{\tilde{F}(U)}(n, Z(U)) \circ \mathbb{H}^n_{\tilde{F}(U)}(i, Z(U)) = \text{id}_{\mathbb{H}^n_{\tilde{F}(U)}(U^P, Z(U))}
\]

10.4.9,

which shows that \(\mathbb{H}^n_{\tilde{F}(U)}(i, Z(U))\) is also injective and \(\mathbb{H}^n_{\tilde{F}(U)}(n, Z(U))\) is also surjective.
Remark 10.5. Actually, it follows from [2, Proposition 3.2] that the inclusion functor \(i_U: \tilde{N}^\pi \to \tilde{F}^x\) induces a group isomorphism

\[
\mathbb{H}^n(\tilde{F}^x; \delta_U^x) \cong \mathbb{H}^n(\tilde{N}^\pi; \delta_U^\pi)
\]

since, according to [2, Proposition 3.2], both members are canonically isomorphic to \(\mathbb{H}^n(\tilde{T}_{\tilde{F}(U)}; \delta_{(1)})\) where the contravariant functor \(\delta_{(1)}: \tilde{T}_{\tilde{F}(U)} \to \text{Ab}\) maps any nontrivial \(p\)-subgroup of \(\tilde{F}(U)\) on \(0\) and \(1\) on \(Z(U)\).

10.6. On the other hand, *mutatis mutandis* \(\tilde{N}^\pi\) can be considered as an \(\tilde{F}^\pi\)-category and it fulfills the conditions in [12, 2.1]; thus, we can consider the corresponding functor and the corresponding natural map defined in [12, 4.6]

\[
m_N^x: \tilde{N}^\pi \to \text{ac}(\tilde{N}^\pi) \quad \text{and} \quad \omega^\pi: m_N^x \to j^\pi
\]

where \(j^\pi: \tilde{N}^\pi \to \text{ac}(\tilde{N}^\pi)\) denotes the canonical functor, and it is quite clear that we have the commutative diagram

\[
\begin{array}{ccc}
\tilde{F}^x & \xrightarrow{i_X} & \text{ac}(\tilde{F}^x) \\
\tilde{i}_U \uparrow & & \uparrow \text{ac}(\tilde{i}_U) \\
\tilde{N}^\pi & \xrightarrow{j^\pi} & \text{ac}(\tilde{N}^\pi)
\end{array}
\]

Explicitly, for any \(Q \in \mathcal{R}\) we may assume that

\[
m_N^x(Q) = \bigoplus_{Q'} \bigoplus_{\hat{\delta}} Q'
\]

where \(Q'\) runs over a set of representatives for the set of \(Q\)-conjugacy classes of elements of \(\mathcal{R}\) contained in \(Q\) and then \(\hat{\delta}\) runs over a set of representatives in \(\tilde{N}(N, Q')_{\tilde{Q'}}\) for the set of \(\tilde{F}_Q(Q')\)-orbits; note that

\[
i_N^P \circ \tilde{N}(N, Q')_{\tilde{Q'}} = \tilde{F}(P, Q')_{\tilde{Q'}} \cap (i_N^P \circ \tilde{N}(N, Q'))
\]

indeed, for any \(\hat{\delta} \in \tilde{N}(N, Q')\) and any subgroup \(R\) of \(Q\) strictly containing \(Q'\), the existence of an \(\tilde{F}\)-morphism \(\tilde{\psi}: R \to P\) fulfilling \(\tilde{\psi} \circ i_Q^R = i_N^R \circ \hat{\delta}\) forces \(\tilde{\psi} = i_N^R \circ \tilde{\psi}'\) for some \(\tilde{\psi}' \in \tilde{N}(N, R)\).

10.7. In particular, we clearly have the *natural map* between the functors \(\text{ac}(i_U) \circ m_N^x\) and \(m_P^x \circ \tilde{i}_U\) from \(\tilde{N}^\pi\) to \(\text{ac}(\tilde{F}^x)\)

\[
\theta_U: \text{ac}(i_U) \circ m_N^x \to m_P^x \circ \tilde{i}_U
\]
sending $Q \in \mathfrak{N}$ to the $\text{ac}(\mathcal{F})$-morphism $\mathfrak{m}_{N}^{\mathfrak{n}}(Q) \to \mathfrak{m}_{\mathfrak{N}}^{\mathfrak{n}}(Q)$ determined by the identity on the subgroups $Q'$ of $Q$ and by the maps

\[ \mathcal{N}(N, Q')_{\mathfrak{N}} / \mathfrak{F}(Q) \longrightarrow \mathcal{F}(P, Q')_{\mathfrak{N}} / \mathfrak{F}(Q) \]

sending the class of $\tilde{\delta} \in \mathcal{N}(N, Q')_{\mathfrak{N}}$ to the class of $\tilde{i}_{N}^{Q'} \circ \tilde{\delta}$; consequently, we still have a natural map between the \textbf{Ab}-valued contravariant functors $(\mathfrak{m}_{U}^{\mathfrak{n}}) \circ \tilde{i}_{U}$ and $\mathfrak{m}_{N}^{\mathfrak{n}}$ from $\mathcal{N}_{\mathfrak{N}}$

\[ \mathfrak{m}_{U}^{\mathfrak{n}} \circ \tilde{i}_{U} \longrightarrow \mathfrak{m}_{N}^{\mathfrak{n}} \]

and then, from the exact sequence 10.3.3, it is easily checked that we get the commutative diagram of \textbf{Ab}-valued contravariant functors from $\mathcal{N}_{\mathfrak{N}}$

\[ \begin{array}{ccc}
0 & \longrightarrow & \mathfrak{m}_{U}^{\mathfrak{n}} \\
\lvert & & \lvert \\
0 & \longrightarrow & \mathfrak{m}_{N}^{\mathfrak{n}}
\end{array} \]

10.8. In conclusion, since the sequences obtained by restricting these functors to the subcategory of $\mathcal{N}_{\mathfrak{N}}$ formed by all the $\mathcal{N}_{\mathfrak{N}}$-isomorphisms are split (cf. 10.3), from the exact sequence 10.3.3 and the diagram above we obtain the commutative diagram of long exact sequences of stable cohomology

\[ \begin{array}{ccc}
\mathbb{H}^{n}_{*}(\mathcal{F}, \mathfrak{m}_{U}^{\mathfrak{n}}) & \longrightarrow & \mathbb{H}^{n}_{*}(\mathcal{F}, \mathfrak{m}_{U}^{\mathfrak{n}}) \\
\lvert & & \lvert \\
\mathbb{H}^{n}_{*}(\mathcal{N}, \mathfrak{m}_{U}^{\mathfrak{n}}) & \longrightarrow & \mathbb{H}^{n}_{*}(\mathcal{N}, \mathfrak{m}_{U}^{\mathfrak{n}}) \\
\lvert & & \lvert \\
\mathbb{H}^{n}_{*}(\mathcal{N}, \mathfrak{m}_{U}^{\mathfrak{n}}) & \longrightarrow & \mathbb{H}^{n}_{*}(\mathcal{N}, \mathfrak{m}_{U}^{\mathfrak{n}})
\end{array} \]

but, it follows from [12, Proposition 4.12] that for any $n \geq 1$ we have

\[ \mathbb{H}^{n}_{*}(\mathcal{F}, \mathfrak{m}_{U}^{\mathfrak{n}}) = \{0\} = \mathbb{H}^{n}_{*}(\mathcal{N}, \mathfrak{m}_{U}^{\mathfrak{n}}) \]

hence, from the commutative diagram above of long exact sequences we get the commutative diagram

\[ \begin{array}{ccc}
\mathbb{H}^{n}_{*}(\mathcal{F}, \tilde{\text{ac}}(\mathcal{F})) & \cong & \mathbb{H}^{n+1}_{*}(\mathcal{F}, \tilde{\mathfrak{m}}_{\mathfrak{N}}) \\
\lvert & & \lvert \\
\mathbb{H}^{n}_{*}(\tilde{i}_{U}, \tilde{\text{ac}}(\mathcal{F})) & \cong & \mathbb{H}^{n+1}_{*}(\tilde{i}_{U}, \tilde{\mathfrak{m}}_{\mathfrak{N}}) \\
\lvert & & \lvert \\
\mathbb{H}^{n}_{*}(\mathcal{N}, \tilde{\text{ac}}(\mathcal{F})) & \cong & \mathbb{H}^{n+1}_{*}(\mathcal{N}, \tilde{\mathfrak{m}}_{\mathfrak{N}}) \\
\lvert & & \lvert \\
\mathbb{H}^{n}_{*}(\tilde{i}_{U}, \tilde{\text{ac}}(\mathcal{F})) & \cong & \mathbb{H}^{n+1}_{*}(\tilde{i}_{U}, \tilde{\mathfrak{m}}_{\mathfrak{N}})
\end{array} \]
10.9. At this point, it follows from Proposition 10.4 that in order to show
that an $n$-cocycle in $C^*_n(\tilde{\mathcal{F}}, \tilde{\text{Ret}}(\tilde{\rho}^x))$ is an $n$-coboundary, it suffices to prove
that the image in $C^*_n(N \times \tilde{\mathcal{F}}, \tilde{\text{Ret}}(\tilde{\rho}^x))$ of its restriction to $C^*_n(N \times \tilde{\mathcal{F}}, \tilde{\text{Ret}}(\tilde{\rho}^x) \circ \tilde{i}_U)$ is an $n$-coboundary. From now on, with the notation in the proof of
Theorem 6.12, we will apply this remark to the regular stable 2-cocycle $k$ in
$C^*_2(\tilde{\mathcal{F}}, \tilde{\text{Ret}}(\tilde{\rho}^x))$ defined in 6.12.13.

10.10. Recall that the $\mathcal{F}^x$-locality $\tilde{\mathcal{L}}^{n,x}$ comes from a natural $\mathcal{F}$-basic
$P \times P$-set $\Omega$ (cf. 3.5); more explicitly, denoting by $G$ the group of permutations
of $\text{Res}_{\{1\}}(\{1\}(\Omega))$ and identifying $P$ with the image of $P \times \{1\}$ into $G$ (cf. 4.1),
$\tilde{\mathcal{L}}^{n,x}$ is a suitable quotient (cf. 4.13 and 4.14) of the $\mathcal{F}^x$-locality defined by
the $G$-transporter (cf. 4.1). Then, it follows from [11, Proposition 21.11] that
the subset
$$\Gamma = \bigcup_{\chi \in \mathcal{F}(U)} \Omega_{\chi}(U)$$ \hspace{1cm} 10.10.1
is actually an $\mathcal{N}$-basic $N \times N$-set; mutatis mutandis, denote by $H$ the group of permutations of $\text{Res}_{\{1\}}(\{1\}(\Gamma))$ and, to avoid confusion, denote by $\tilde{N}$ the image
of $N \times \{1\}$ into $H$; since $N_G(U)/C_G(U) \cong \mathcal{F}(U)$ (cf. 4.1.1), it is clear that
$N_G(U)$ stabilizes $\Gamma$ and therefore we have a canonical group homomorphism
from $N_G(U)$ to $H$; moreover, denote by
$$\iota_U : N \to N_H(\tilde{U}) \quad \text{and} \quad \kappa_U : N_H(\tilde{U}) \to N(U) = \mathcal{F}(U)$$ \hspace{1cm} 10.10.2
the obvious maps.

**Proposition 10.11.** With the notation above, for any pair of subgroups $Q$
and $R$ of $N$ containing $U$ and any $\psi \in \mathcal{N}(Q, R)$, there exists at most one
$Q \times N$-orbit in $\Gamma$ isomorphic to $(Q \times N)/\Delta_\psi(R)$. In particular, $C_H(Q)$ is an
Abelian $p$-group.

**Proof:** Since $\Gamma$ is an $\mathcal{N}$-basic $N \times N$-set, for any $\omega \in \Gamma$ we already know that
$(Q \times N)\omega = \Delta_\psi(R)$ for some subgroup $R$ of $P$ and some $\psi \in \mathcal{F}(P, R)$ (cf. 3.1)
and, by the very definition of $\Gamma$, $\Delta_\psi(R)$ contains $\Delta_\chi(U)$ for some $\chi \in \mathcal{F}(U)$;
that is to say, $R$ contains $U$ and $\psi$ extends $\chi$, so that $\omega$ determines $\chi$.

We claim that we still have $(Q \times P)\omega = \Delta_\psi(R)$; indeed, if $(v, w) \in Q \times P$
fixes $\omega$ then $\Delta_\chi(U)$ fixes $\omega \cdot w^{-1}$; that is to say, for any $u \in U$ we have
$$\omega \cdot w^{-1} = \chi^v(u) \cdot w^{-1} \cdot u^{-1} = \omega \cdot (\chi \circ \chi^v)(u) \cdot w^{-1} \cdot u^{-1}$$ \hspace{1cm} 10.11.1
which forces $w^{-1} = (\chi \circ \chi^v)(u) \cdot w^{-1} \cdot u^{-1}$, so that $u = (\chi \circ \chi^v)(u)$; hence, $w$
belongs to $N$ which proves our claim. Consequently, any $Q \times N$-orbit in $\Gamma$
isomorphic to $(Q \times N)/\Delta_\psi(R)$ is the intersection of $\Gamma$ with a $Q \times P$-orbit
in $\Omega$ isomorphic to $(Q \times P)/\Delta_\psi(R)$ and it follows from Proposition 3.7 that
such a $Q \times P$-orbit is unique.

In particular, the last statement follows from isomorphism 4.2.1 applied
to the $\mathcal{N}$-basic $N \times N$-set $\Gamma$ together with isomorphism 3.7.1. We are done.
10.12. Let \((\tau_U, L, \pi_U)\) be an \(F\)-localizer of \(U\) (cf. 2.13); thus, \(L\) is a finite group, we have the injective and the surjective group homomorphisms

\[
\tau_U : N \rightarrow L \quad \text{and} \quad \pi_U : L \rightarrow \mathcal{F}(U)
\]

\(\tau_U(N)\) is a Sylow \(p\)-subgroup of \(L\), and we also have the exact sequence

\[
1 \rightarrow Z(U) \xrightarrow{\tau_U} L \xrightarrow{\pi_U} \mathcal{F}(U) \rightarrow 1
\]

we need the following result, essentially proved in [11, Proposition 20.12].

**Proposition 10.13.** With the notation above, there is a unique \(C_H(\bar{U})\)-conjugacy class of group homomorphisms

\[
\lambda_U : L \rightarrow N_H(\bar{U})
\]

fulfilling \(\lambda_U \circ \tau_U = \iota_U\) and \(\kappa_U \circ \lambda_U = \pi_U\).

**Proof:** We will apply [11, Lemma 18.8] to the groups \(L\) and \(N_H(\bar{U})\), to the quotient \(\mathcal{F}(U)\) of both, and to the group homomorphism

\[
\eta : \tau_U(N) \rightarrow N_H(\bar{U})
\]

mapping \(\tau_U(u)\) on \(\bar{u}\) for any \(u \in N\), which makes sense since \(\tau_U\) is injective; we already know that \(\tau_U(N)\) is a Sylow \(p\)-subgroup of \(L\) and, according to Proposition 10.11, that \(C_H(\bar{U})\) is an Abelian \(p\)-group; moreover, it is clear that the restriction of \(\pi_U\) to \(\tau_U(N)\) coincides with \(\kappa_U \circ \eta\).

Let \(Q\) be a subgroup of \(N\) and \(x \in L\) such that \(\tau_U(Q) \subset \tau_U(N)^x\); since we still have

\[
\tau_U(U\cdot Q) \subset \tau_U(N)^x
\]

it follows from the very definition of the \(F\)-localizer [11, Theorem 18.6] that there is an \(F\)-morphism \(\varphi : U\cdot Q \rightarrow N\) such that we have

\[
\varphi(u) = (\pi_U(x))(u) \quad \text{and} \quad \tau_U(\varphi(v)) = x\tau_U(v)x^{-1}
\]

for any \(u \in U\) and any \(v \in U\cdot Q\); thus, \(\varphi\) is also an \(N\)-morphism from \(U\cdot Q\) to \(N\) and therefore there is \(\bar{x} \in T_H(\bar{U}\cdot \bar{Q}, \bar{N})\) fulfilling [11, Proposition 21.9]

\[
\varphi(u) = (\kappa_U(\bar{x}))(u) \quad \text{and} \quad \bar{\varphi}(v) = \bar{x}\bar{v}\bar{x}^{-1}
\]

for any \(u \in U\) and any \(v \in U\cdot Q\). Consequently, we get

\[
\kappa_U(\bar{x}) = \pi_U(x) \quad \text{and} \quad \eta(x\tau_U(v)x^{-1}) = \bar{x}\bar{v}\bar{x}^{-1} = \bar{x}\eta(\tau_U(v))\bar{x}^{-1}
\]

for any \(v \in U\cdot Q\); that is to say, the condition in [11, Lemma 18.8] is fulfilled and therefore, since \(C_H(\bar{U})\) is an Abelian \(p\)-group, it follows from [11, Lemma 18.8] that there is a unique \(C_H(\bar{U})\)-conjugacy class of group homomorphisms as announced.
10.14. Let us denote by \( T^\mathcal{N}_H \) the category over the set of objects \( \mathfrak{R} \) determined by the 
transporter\ in \( N_H(U) \); that is to say, for any pair of subgroups \( Q \) and \( R \) in \( \mathfrak{R} \), we set

\[
T^\mathcal{N}_H(Q, R) = T_{N_H(U)}(\bar{R}, \bar{Q})
\]

the composition in \( T^\mathcal{N}_H \) being defined by the product in \( H \); similarly, let us denote by \( T^\mathcal{N}_L \) the category over the set of objects \( \mathfrak{R} \) determined by the 
transporter\ in \( L \). Actually, it is quite clear that from these two categories we get two \( \mathcal{N} \)-localities and that a group homomorphism in 10.13.1 above determines an \( \mathcal{N} \)-locality functor

\[
l_U : T^\mathcal{N}_L \longrightarrow T^\mathcal{N}_H
\]

10.15. On the other hand, setting \( \mathcal{M}^x = N_{\mathcal{M}^x}(U) \) \([11, 17.5]\), it is easily
checked from Proposition 10.11 that the canonical group homomorphism from \( N_G(U) \) to \( N_H(U) \) induces a functor

\[
g : \mathcal{M}^x \longrightarrow T^\mathcal{N}_H
\]

and, considering the corresponding quotients \( \mathcal{M}^x \) and \( T^\mathcal{N}_H \) by the contra-
variant functor \( \mathfrak{s}^\mathcal{N}_U : \mathcal{N}^\mathcal{N} \rightarrow \mathfrak{A} \) (cf. 2.10 and 10.3), we still have a functor

\[
\mathfrak{g} : \mathcal{M}^x \longrightarrow \mathcal{N}^\mathcal{N}
\]

Similarly, we still have a functor

\[
\mathfrak{l}_U : \mathcal{T}^\mathcal{N}_L \longrightarrow \mathcal{T}^\mathcal{N}_H
\]

and, by the very definition of the \( \mathcal{F} \)-localizer, we actually have \( \mathcal{T}^\mathcal{N}_L \cong \mathcal{N}^\mathcal{N} \).

10.16. Once again, let us borrow the notation in the proof of Theorem 6.12. In particular, for any \( Q \in \mathfrak{R} \) the natural section

\[
\mu_{\mathcal{M}^x_U} : \text{aut}_{\mathcal{N}^\mathcal{N}} \longrightarrow \text{loc}_{\mathcal{M}^x_U}
\]

supplies a section of the second structural homomorphism of \( \mathcal{M}^x_U \) at \( Q \)

\[
(\mu_{\mathcal{M}^x_U})_Q : \mathcal{N}^\mathcal{N}(Q) \longrightarrow \mathcal{M}^x_U(Q)
\]

similarly, the functor \( \tilde{l}_U \) above supplies a section of the second structural homomorphism of \( \mathcal{T}^\mathcal{N}_H \) at \( Q \)

\[
(\tilde{l}_U)_Q : \mathcal{N}^\mathcal{N}(Q) \longrightarrow \mathcal{T}^\mathcal{N}_H(Q)
\]

hence, according to the uniqueness of the natural section \( \mu_{\mathcal{T}^\mathcal{N}_H} \), the diagram

\[
\begin{array}{c}
\mathcal{M}^x_U(Q) \\
\downarrow \theta_Q
\end{array}
\begin{array}{c}
\mathcal{T}^\mathcal{N}_H(Q)
\end{array}
\begin{array}{c}
\downarrow (\tilde{l}_U)_Q
\end{array}
\begin{array}{c}
\mathcal{N}^\mathcal{N}(Q)
\end{array}
\]

is commutative up to \( \tilde{C}_H(\bar{Q}) \)-conjugation.
10.17. Consequently, up to the replacement of \( i_u \) by a natural \( \mathcal{N} \)-isomorphic functor (cf. 2.10), we may assume that diagrams 10.16.4 above are commutative for any \( Q \in \mathfrak{Q} \). At this point, for any pair of subgroups \( Q \) and \( R \) in \( \mathfrak{Q} \), and any \( \mathcal{N}^m \)-morphism \( \varphi: R \to Q \), we may assume that \( x_\varphi: R \to Q \) is an \( \mathcal{M}^x_U \)-morphism and therefore, since \( g \) is an \( \mathcal{N} \)-locality functor, for a suitable \( \tilde{z}_R \in C_H(R) \) we get

\[
\tilde{g}(x_\varphi) = i_u(\varphi)\cdot\tilde{z}_R
\]

10.17.1;

moreover, it is easily checked that the commutativity of the diagrams 10.16.4 above forces \( \tilde{z}_\varphi = 1 \) whenever \( \varphi \) is an isomorphism. Then, for any triple of subgroups \( Q, R \) and \( T \) in \( \mathfrak{Q} \), and any pair of \( \mathcal{N} \)-morphisms \( \psi: T \to R \) and \( \varphi: R \to Q \), in \( \mathcal{F}^m \) we get

\[
\tilde{g}(x_\varphi)\cdot\tilde{g}(x_\psi) = \tilde{g}(x_{\varphi \circ \psi}) \cdot \tilde{g}(k_{\varphi,\psi}) = i_u(\varphi)\cdot\tilde{z}_\varphi \cdot i_u(\psi)\cdot\tilde{z}_\psi = i_u(\varphi \circ \psi)\cdot(\tilde{\text{Re}}(\tilde{\kappa}^m_U)(\psi))(\tilde{z}_\varphi)\cdot\tilde{z}_\psi
\]

10.17.2

where \( \tilde{\kappa}^m_U : \mathcal{F}^m \to \mathcal{N}^m \) denotes the second structural functor; thus, since \( \mathcal{F}^m \) is divisible and \( C_H(T) \) is Abelian, with additive notation we obtain

\[
\tilde{g}(k_{\varphi,\psi}) = (\tilde{\text{Re}}(\tilde{\kappa}^m_U)(\psi))(\tilde{z}_\varphi) - \tilde{z}_{\varphi \circ \psi} + \tilde{z}_\psi
\]

10.17.3.

10.18. But, with the notation in 10.7 and 10.8 above, it is easily checked that we have

\[
\tilde{\text{Re}}(\tilde{\kappa}^m_U) \circ \tilde{\text{Re}}(\tilde{\kappa}^m_U) = \tilde{\text{Re}}(\tilde{\kappa}^m_U)
\]

10.18.1

and that the map

\[
C^2_*(i_u, \tilde{\text{Re}}(\tilde{\kappa}^m_U)) : C^2_*(\mathcal{F}^m, \tilde{\text{Re}}(\tilde{\kappa}^m_U)) \to C^2_*(\mathcal{N}^m, \tilde{\text{Re}}(\tilde{\kappa}^m_U))
\]

10.18.2

sends the \( \tilde{\text{Re}}(\tilde{\kappa}^m_U) \)-valued 2-cocycle \( k \) to the \( \tilde{\text{Re}}(\tilde{\kappa}^m_U) \)-valued 2-cocycle give by the family \( \{ \tilde{g}(k_{\varphi,\psi}) \}_{\varphi,\psi} \) where \( \varphi: R \to Q \) and \( \psi: T \to R \) run over the set of \( \mathcal{N}^m \)-morphisms. Moreover, we already know that \( k_{\varphi,\psi} = 0 \) if either \( \varphi \) or \( \psi \) is an isomorphism and that \( \tilde{z}_\varphi = 0 \) if \( \varphi \) is an isomorphism; from this remark and from the equalities 10.17.3 is easily checked that the element \( \{ \tilde{z}_\varphi \} \) in \( \mathbb{C}^1(\mathcal{N}^m, \tilde{\text{Re}}(\tilde{\kappa}^m_U)) \) is stable. In conclusion, the equalities 10.17.3 show that \( C^2_*(i_u, \tilde{\text{Re}}(\tilde{\kappa}^m_U))(k) \) has a trivial image in \( \mathbb{H}^2_*(\mathcal{N}^m, \tilde{\text{Re}}(\tilde{\kappa}^m_U)) \) and therefore it follows from Proposition 10.4 and from the commutative diagram 10.8.3 that \( k \) also has a trivial image in \( \mathbb{H}^2_*(\mathcal{F}^m, \tilde{\text{Re}}(\tilde{\kappa}^m_U)) \). Consequently, as in the proof of Theorem 6.12, we get a functorial section of \( \tilde{\rho}^x \) and, as in this proof, we can modify this section in order to get an \( \mathcal{F}^x \)-locality functorial section \( \sigma^x \).
10.19. Finally, Proposition 10.4 and the commutative diagram 10.8.3 also provide a second proof of the uniqueness of \( \sigma^x \). Indeed, assume that \( \sigma^x : \mathcal{F}^x \to \mathcal{M}^x \) is a second \( \mathcal{F}^x \)-locality functorial section; the uniqueness of the natural section \( \mu_{\mathcal{M}^x} \) in 6.12.2 already guarantees that, in order to prove that \( \sigma^x \) is naturally \( \mathcal{F}^x \)-isomorphic to \( \sigma^x \), we may assume that (cf. 6.12.12)

\[
\sigma^x(\alpha) = \mu_Q(\alpha) = \sigma^x(\alpha)
\]

for any \( Q \in \mathfrak{X} \) and any \( \alpha \in \mathcal{F}(Q) \); more precisely, we may assume that \( \sigma^x(\alpha) = \sigma^x(\alpha) \) for any \( \mathcal{F}^x \)-isomorphism \( \alpha \in \mathcal{F}(Q', Q) \).

10.20. For any \( \mathcal{F}^x \)-morphism \( \varphi : R \to Q \), as usual we set \( x_\varphi = \sigma^x(\varphi) \) and \( x'_\varphi = \sigma^x(\varphi) \) for short; it is easily checked that these elements fulfill statement 6.12.11 above; since they have the same image in \( \mathcal{F}(Q, R) \), the divisibility of \( \mathcal{M}^\alpha \) forces the existence of a unique \( \ell_\varphi \in \text{Ker}(\mathfrak{p}_R^\alpha) \) fulfilling \( x'_\varphi = x_\varphi \cdot \ell_\varphi \); actually, since \( \sigma^x \) and \( \sigma'^x \) are \( \mathcal{F}^x \)-locality functors, it is easily checked that \( \ell_\varphi \) only depends on \( \bar{\varphi} \in \mathcal{F}(Q, R) \); then, denoting by \( \ell \) the element of \( \mathcal{C}^1(\bar{\mathcal{F}}^x, \mathfrak{ret}(\bar{\rho}^x)) \) defined by the correspondence sending \( \bar{\varphi} \) to \( \ell_\varphi \), we get \( d_1(\ell) = 0 \) as in 6.12.32 above. Moreover, we claim that this element \( \ell \) is \( \text{stable} \); indeed, for a \( \text{natural isomorphism} \ \nu : q \equiv q' \) of \( \mathcal{F}^x \)-chains, setting

\[
\varphi = q(0 \cdot 1) \quad \varphi' = q'(0 \cdot 1) \quad \nu_0 = \beta \quad \text{and} \quad \nu_1 = \alpha
\]

from our choice we have \( \ell_\alpha = 1 \) and \( \ell_\beta = 1 \) and therefore we get

\[
x'_\varphi = x_\varphi \cdot \ell_\varphi = (x_\alpha \cdot x'_\varphi x^{-1}_\beta) \cdot \ell_\varphi = (x_\alpha \cdot x'_\varphi x^{-1}_\beta \cdot x^{-1}_\beta) \cdot \ell_\varphi
\]

\[
= (x_\alpha \cdot x'_\varphi x^{-1}_\beta) \cdot \mathfrak{ret}(\bar{\rho}^x)(\bar{\beta}^{-1}) (\ell_\varphi)
\]

\[
= x'_\varphi \cdot (\mathfrak{ret}(\bar{\rho}^x)(\bar{\beta}^{-1}) (\ell_\varphi)).
\]

10.21. At this point, in order to show that \( \sigma'^x \) is isomorphic to \( \sigma^x \) it suffices to prove that the image of \( \ell \) in \( \mathbb{H}_1(\bar{\mathcal{F}}^x, \mathfrak{ret}(\bar{\rho}^x)) \) is equal to zero; indeed, in this case we have \( \ell = d_0^x(n) \) for some element \( n = (n_Q)_{Q \in \mathfrak{X}} \) in \( \mathbb{C}^0(\bar{\mathcal{F}}^x, \mathfrak{ret}(\bar{\rho}^x)) \); that is to say, with the notation above we get

\[
\ell_\varphi = (\mathfrak{ret}(\bar{\rho}^x)(\bar{\varphi}))(n_Q \cdot n^{-1}_R)
\]

where we identify any \( \bar{\mathcal{F}}^x \)-object with the obvious \( \bar{\mathcal{F}}^x \)-chain \( \Delta_0 \to \bar{\mathcal{F}}^x \); hence, we obtain

\[
\sigma'^x(\varphi) = x'_\varphi = x_\varphi \cdot (\mathfrak{ret}(\bar{\rho}^x)(\bar{\varphi}))(n_Q \cdot n^{-1}_R) = n_Q \sigma^x(\varphi) \cdot n^{-1}_R
\]

10.22, which amounts to saying that the correspondence \( \nu^x \) sending \( Q \) to \( n_Q \) defines a \( \text{natural} \ \mathcal{F}^x \)-isomorphism between \( \sigma^x \) and \( \sigma'^x \).
10.22. As in 10.9 above, it suffices to prove that $C^1_*(\tilde{\mathcal{U}}, \tilde{\mathcal{F}}^\ast_{\mathcal{U}})(T)$ has a trivial image in $H^1_*(\tilde{\mathcal{N}}^\ast_{\mathcal{U}}, \tilde{\mathcal{H}}(\tilde{m}^\ast_{\mathcal{U}}))$; thus, since we have

$$H^1_*(\tilde{\mathcal{N}}^\ast_{\mathcal{U}}, \tilde{\mathcal{F}}^\ast_{\mathcal{U}}) = \{0\}$$ 10.22.1,

it actually suffices to prove that

$$C^1_*(\tilde{\mathcal{U}}, \tilde{\mathcal{F}}^\ast_{\mathcal{U}})(T) = \{\tilde{\mathcal{F}}(\ell_\varphi)\}$$ 10.22.2

where $\varphi$ runs over the set of $\tilde{\mathcal{N}}^\ast_{\mathcal{U}}$-morphisms can be lifted to a 1-cocycle $m = \{m_\varphi\}$ in $C^1_*(\tilde{\mathcal{N}}^\ast_{\mathcal{U}}, \tilde{\mathcal{F}}^\ast_{\mathcal{U}})$.

10.23. Let us define $m$ for any pair of subgroups $Q$ and $R$ in $\mathcal{M} \cap \mathcal{Q}$, and any $\tilde{\mathcal{N}}$-morphism $\varphi : R \to Q$, we simply set $m_\varphi = \ell_\varphi$ which makes sense since we know that

$$\tilde{\mathcal{N}}^\ast_{\mathcal{U}}(\tilde{\mathcal{N}}^\ast_{\mathcal{U}} \circ m^\mathcal{N})((R) = \tilde{\mathcal{H}}(\tilde{m}^\mathcal{N})(R)$$ 10.23.1;

on the other hand, for any $\tilde{\mathcal{N}}$-morphism $\gamma : U \to Q$, we set

$$m_\gamma = ((\tilde{\mathcal{N}}^\ast_{\mathcal{U}} \circ m^\mathcal{N})(\gamma^\ast_N))((m^\mathcal{Q})$$ 10.23.2

where $\gamma : U \cong U$ is the automorphism induced by $\gamma$ and $m^\mathcal{Q}$ is the unique element of $\text{Ker}\tilde{\mathcal{N}}^\ast_{\mathcal{U}}$ lifting $\ell^\mathcal{Q}_U$ (cf. 6.3.3).

10.24. For any triple of subgroups $Q$, $R$ and $T$ in $\mathcal{M}$, and any pair of $\tilde{\mathcal{N}}$-morphisms $\psi : T \to R$ and $\varphi : R \to Q$, if $T$ belongs to $\mathcal{M}$ then $R$ and $Q$ also belong to $\mathcal{Q}$ and therefore we still have

$$0 = ((\tilde{\mathcal{N}}^\ast_{\mathcal{U}} \circ m^\mathcal{N})(\psi^\ast_N))((m_\varphi) - m_\varphi \psi + m_\psi$$ 10.24.1;

otherwise, we have $T = U$ and $\psi = \ell^R_U \circ \psi_*$, and therefore it follows from the stability of $\ell$ that

$$\ell_\varphi = (\tilde{\mathcal{H}}(\tilde{m}^\mathcal{N})(\psi^\ast_N))((\ell^R_U)$$ \ell_\psi = (\tilde{\mathcal{H}}(\tilde{m}^\mathcal{N})(\psi^\ast_N))((\ell^R_U)$$ 10.24.2,

so that the equality $d^1_*(\ell) = 0$ implies that

$$0 = (\tilde{\mathcal{H}}(\tilde{m}^\mathcal{N})(\ell^R_U))((\ell^R_U) - \ell_\varphi \psi + \ell_\psi$$ 10.24.3.

If $R = U$ then we get $0 = \ell^R_U$ which forces $0 = m^R_U$ and therefore we still get

$$0 = ((\tilde{\mathcal{N}}^\ast_{\mathcal{U}} \circ m^\mathcal{N})(\ell^R_U))((m_\varphi) - m_\varphi \psi + m^R_U$$ 10.24.4.
10.25. Otherwise, since \( \ell_{(\varphi \circ \bar{t}_U)} \in 1 \) and both sections are compatible with the first structural functor \( \bar{v}^m_U : T^m_N \to \bar{M}^\chi_U \), we obtain

\[
x'_{\varphi \circ \bar{t}_U} = x'_{\varphi} (\bar{v}^m_U)_{R,U} (1) = x_{\varphi} \cdot \ell_{\varphi} (\bar{v}^m_U)_{R,U} (1) = x_{\varphi} (\bar{v}^m_U)_{R,U} (1) \cdot (\mathfrak{Ref}(\bar{k}^m_U) (\bar{t}_U)) (\ell_{\varphi})
\]

and therefore, with the additive notation, we still get \( 0 = (\mathfrak{Ref}(\bar{k}^m_U) (\bar{t}_U)) (\ell_{\varphi}) \); once again, the equality \( d^1 (\ell) = 0 \) forces \( 0 = -\ell_{\varphi \circ \bar{t}_U} + \ell_{\bar{t}_U} \) and therefore, according to our choice, we also have \( 0 = -m_{\varphi \circ \bar{t}_U} + m_{\bar{t}_U} \). But, since the element \( ((\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{i}_U^R)) (m_{\bar{t}}) \) in \( (\mathfrak{d}^m_{U, e} \circ m_N^m) (U) \) lifts \( (\mathfrak{Ref}(\bar{k}^m_U) (\bar{t}_U)) (\ell_{\varphi}) = 0 \), it belongs to the diagonal (cf. 6.8.5)

\[
(\mathfrak{d}^m_{U, e} \circ m_N^m) (U) = \prod_{\bar{\gamma} \in \bar{N}(N,U)} Z(U)
\]

whereas, according to 6.5.4 applied to \( \bar{N} \), it belongs to

\[
\text{Im} ((\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{i}_U^R)) = \prod_{\bar{\gamma} \in \bar{N}(N,U)_{\bar{i}_U^R}} Z(U)
\]

consequently, since \( p \) divides \( |\bar{N}(N,U)_{\bar{i}_U^R}| \) and does not divide \( |\bar{N}(N,U)| \) [11, 6.7.4 and 6.7.2], we also get \( 0 = ((\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{i}_U^R)) (m_{\bar{t}}) \) and in this case we also obtain

\[
0 = ((\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{i}_U^R)) (m_{\bar{t}}) - m_{\varphi \circ \bar{t}_U} + m_{\bar{t}_U}^T
\]

In both cases, it follows from our choice in 10.23.2 that \( (\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{t}) \) applied to equalities 10.24.4 and 10.25.4 yields

\[
0 = ((\mathfrak{d}^m_{U, e} \circ m_N^m) (\bar{t})) (m_{\bar{t}}) - m_{\varphi \circ \bar{t}} + m_{\bar{t}}
\]

so that \( m = \{m_{\bar{t}}\}_{\bar{t}} \) is indeed a 1-cocycle in \( C^1_+ (\bar{N}, \mathfrak{d}^m_{U, e} \circ m_N^m) \). We are done.
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