QUANTUM COSMOLOGY
IN SCALAR-TENSOR THEORIES
WITH NON MINIMAL COUPLING

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Abstract
Quantization in the minisuperspace of non minimal scalar-tensor theories leads to a partial differential equation which is non separable. Through a conformal transformation we can recast the Wheeler-DeWitt equation in an integrable form, which corresponds to the minimal coupling case, whose general solution is known. Performing the inverse conformal transformation in the solution so found, we can construct the corresponding one in the original frame. This procedure can also be employed with the bohmian trajectories. In this way, we can study the classical limit of some solutions of this quantum model. While the classical limit of these solutions occurs for small scale factors in the Einstein’s frame, it happens for small values of the scalar field non minimally coupled to gravity in the Jordan’s frame, which includes large scale factors. PACS number(s): 04.20.Cv., 98.80.Hw

1 Introduction
The theories of unification of interactions, like Kaluza-Klein, supergravity or superstring theories, predict non minimal couplings between the geometry of spacetime and a scalar field. The covariant quantization of these theories in term of perturbative methods is incomplete and it is interesting to pursue non-perturbative methods or simplifications in order to obtain some qualitative features of the quantization of these theories. The most dramatic simplification is the minisuperspace quantization, where all but a finite number of degrees of freedom are frozen yielding, in general, a solvable quantum problem. The interest on these models appears because, in spite of all this simplification, there are some issues which still persist, like the issue of time, how to interpret the wave function of the Universe, the elimination of singularities by quantum effects, and the conditions for the classical limit.
In the present paper, we will study these problems in the context of minisuperspace quantization of scalar-tensor theories with non minimal coupling. These theories are derived from the effective action of string theories [1], and from Kaluza-Klein theories compactified to four dimensions [2, 3]. In both cases, the resulting model has a non minimal coupling between gravity and one or more scalar fields. A conformal transformation permits to write these effective models in the Einstein’s frame, where the scalar field is minimally coupled to the gravity sector. There are claims that these minimally coupled formulations correspond to the physical frame in such a way that the conformal transformation does not constitute a mathematical tool only [4, 5]; to our point of view this is not a closed subject, and the choice of a physical frame depends on the theoretical structure one has in mind, for example, if we intend to consider a constant or a variable gravitational coupling. In some cases, stability considerations may select a preferred frame [6].

In a recent paper [7], the minisuperspace quantization of scalar-tensor theories with minimal coupling was studied. The presence of singularities at the quantum level, the classical limit, the issue of time, and problems of interpretation were discussed. It was shown that the causal interpretation of quantum mechanics [8], one of the possible interpretations which can be used in quantum cosmology (for reviews on interpretations and classical limit in quantum cosmology, see Ref. [9]), where the issue of time is absent at the minisuperspace level [10], yields a classical limit which is in agreement with semiclassical considerations. The general solution of the Wheeler-DeWitt equation was found. For some particular exact solutions, it was shown that, contrary to common belief, the classical limit occurs for small scale factors, and hence the classical singularities are not avoided by quantum effects, which become important only for larger values of the scale factor. For the case of scalar-tensor theories with non minimal coupling, the corresponding Wheeler-DeWitt equation is non separable and more difficult to solve. However, we were able to show that not only the classical solutions but also the quantum theory of this model can be mapped to the minimal coupling case by a conformal transformation. The solutions of the Wheeler-DeWitt equation found in the Einstein’s frame can be mapped to the solutions of the more complicated Wheeler-DeWitt equation in the Jordan’s frame, and also the bohmian trajectories which describe the quantum evolution of the solutions.

At the classical level, the effective models considered here are labeled by a coupling constant $\omega$, which in principle may take values in the interval $-\frac{3}{2} < \omega < \infty$, as in the Brans-Dicke theory. The models that will be studied here differ from the usual Brans-Dicke theory by the presence of a second scalar field minimally coupled to gravity but with a non trivial coupling with the usual scalar field of the Brans-Dicke theory. This extra scalar field is usually present in the effective action quoted above. For string effective action $\omega = -1$, while for compactified Kaluza-Klein theories, $\omega = -\frac{d+2}{4}$, where $n = 4 + d$ is the dimension of space-time. In this paper, we will consider only spatial sections of the Friedmann’s model with positive curvature. For $\omega < 0$, the classical solutions represent universes with bounce (but not necessarily singularity-free), while for $\omega > 0$, the cosmological scenario is qualitatively like the traditional Friedmann’s cosmological scenarios. For $\omega = 0$, we have a remarkable oscillating Universe, where the scale factor never goes to zero, constituting a complete singularity-free model.
At the quantum level, the bohmian trajectories of the quantum solutions studied in this paper show the same behavior as the classical solutions whenever the scalar field non minimally coupled to gravity is small (with the exception of the very particular case $\omega = 0$), which coincides with small values of the scale factor $a$ when $\omega > 0$, and large values of $a$ when $-\frac{3}{2} > \omega > 0$. Hence, contrary to the minimal coupling case, the bohmian trajectories may coincide with the classical solutions for large values of the scale factor.

In the next section we exhibit the classical theory, with their corresponding classical solutions. In section 3 we show the conformal mapping of the Wheeler-DeWitt equation in the mini-superspace from Jordan’s to Einstein’s frames, from where we determine their solutions in the non minimal coupling case. In section 4, the conformal equivalence of the bohmian trajectories is also determined, and their classical limits are compared and discussed. The conclusions are presented in section 5.

2 Scalar-tensor theories

In absence of ordinary matter, the most general Lagrangian density we can write for scalar-tensor theories is,

$$L = \sqrt{-g} \left( f(\phi) R - \omega(\phi) \frac{\phi^2}{\phi} + V(\phi) \right),$$

where $f(\phi)$ and $\omega(\phi)$ are arbitrary functions of the scalar field, while $V(\phi)$ is a potential term. In general the potential term may be added by hand, although in some cases the choice can be dictated by microphysical considerations. The Brans-Dicke theory is represented by the special case where $f(\phi) = \phi$ and $\omega(\phi) = \text{constant}$; generally, in Brans-Dicke theory, the potential term is put equal to zero.

Scalar-tensor theories appear in the low energy limit of string theory, and in the reduction of Kaluza-Klein theories to four dimensions. The form of these effective Lagrangian depends on the way the compactification is made, and on the original multidimensional theories (as in supergravity theories). In cosmology, generally, we are interested only in the bosonic sector without gauge fields in four dimensions. A large class of these effective theories is represented by a Lagrangian of the type

$$L = \sqrt{-g} \left( \phi R - \omega \frac{\phi^2}{\phi} - \frac{\chi^2}{\phi} \right).$$

This Lagrangian is the same as the Brans-Dicke one, but it has an additional scalar field, which couples non trivially with the Brans-Dicke field $\phi$. Such effective Lagrangian appears in string cosmology with axion field ($\omega = -1$), and in multidimensional theories where gravity is coupled to external (non geometric) gauge fields in the higher dimensional space-time $n = 4 + d$ ($\omega = -(d-1)/d$).

We would like to emphasize that these comparisons and results were obtained for some particular exact solutions of the Wheeler-DeWitt equation in both frames, which can be written in term of elementary functions. For more complicate solutions, the classical limit may present other features. However, these examples are sufficient to show that the quantum theory in Einstein’s and Jordan’s frames have quite different properties.
Employing the variational principle, we find the field equations corresponding to this Lagrangian:

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\omega}{\phi^2} \left( \phi_{\mu\phi_{\nu}} - \frac{1}{2}g_{\mu\nu}\phi_{_{\phi}^2} \right) + \frac{1}{\phi} \left( \phi_{_{\phi\phi\mu\nu}} - g_{\mu\nu}\Box\phi \right) + \frac{1}{\phi^2} \left( \chi_{_{\phi\phi\mu\nu}} - \frac{1}{2}g_{\mu\nu}\chi_{_{\phi\phi}} \right) ;
\]

(3)

\[
\Box\phi + \frac{2}{3 + 2\omega} \frac{\chi_{_{\phi\phi}}}{\phi} = 0 ;
\]

(4)

\[
\Box\chi - \frac{\phi_{_{\phi\phi}}}{\phi} \chi_{_{\phi}} = 0 .
\]

(5)

We insert in the field equations (3,4,5) the Friedmann-Robertson-Walker metric

\[
ds^2 = N^2 dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right)
\]

(6)

where \(a\) is the scale factor of the Universe, and \(k = 0, 1, -1\) represents the constant curvature of the spatial section. The factor \(N\) is the lapse function, which we will set equal to one in this section. The resulting equations of motion are:

\[
3\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \frac{3\ddot{a}}{a} \frac{\phi'}{\phi} + \frac{1}{2} \left( \frac{\chi'}{\phi} \right)^2 ;
\]

(7)

\[
\ddot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} + \frac{2}{3 + 2\omega} \frac{\chi^2}{\phi} = 0 ;
\]

(8)

\[
\ddot{\chi} + 3\frac{\dot{a}}{a} \chi - \frac{\phi'}{\phi} \dot{\chi} = 0 .
\]

(9)

The dot means derivative with respect to the cosmic time \(t\). To solve the equations (7,8,9) it is easier to reparametrize the time coordinate as \(dt = a^3 d\theta\). In terms of \(\theta\), we obtain the equations of motion,

\[
3\left( \frac{a'}{a} \right)^2 + 3ka^4 = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - 3\frac{a'}{a} \frac{\phi'}{\phi} + \frac{1}{2} \left( \frac{\chi'}{\phi} \right)^2 ;
\]

(10)

\[
\ddot{\phi} + 3\frac{a'}{a} \dot{\phi} + \frac{2}{3 + 2\omega} \frac{\chi^2}{\phi} = 0 ;
\]

(11)

\[
\ddot{\chi} - \frac{\phi'}{\phi} \chi' = 0 .
\]

(12)

The prime means derivative with respect to \(\theta\).

The solution of equation (12) is direct and reads,

\[
\chi' = C\phi ,
\]

(13)
where $C$ is an integration constant. Inserting (13) into (11) we obtain an harmonic oscillator equation for $\phi$ with the solution,

$$\phi = D \sin(\lambda \theta),$$

(14)

$D$ being another integration constant and $\lambda = \sqrt{\frac{2}{3+2\omega}}C$. Now, we turn to equation (10). We insert in it the solutions (13,14). Redefining the scale factor $a = \phi^{-1/2}b$, we can integrate the resulting equation for $b$. This redefinition is the same that transforms the original Lagrangian written in the Jordan’s frame to the equivalent Lagrangian written in the Einstein’s frame, whose cosmological solutions for a closed spatial section has been solved previously [7]. The final solution for the scale factor is:

$$a = \sqrt{\frac{2Cr \tan^\alpha(\frac{\lambda\theta}{2})}{\sqrt{6 \sin(\lambda\theta)[1 + r^2 \tan^2(\frac{\lambda\theta}{2})]}}},$$

(15)

where $\alpha \equiv \sqrt{1 + \frac{2}{3}\omega}$, and $r$ is an integration constant. When $\omega > 0$, (15) represents an Universe that has an expanding phase, coming from a singularity, reaching a maximum value for $a$, and then collapsing again to a singularity. For $\omega < 0$, the scenario is completely different: the Universe has initially a contracting phase coming from $a \to \infty$, reaches a minimum, and then enter in an expansion phase until $a \to \infty$. This bouncing Universe is free of singularities when $-\frac{3}{2} < \omega < -\frac{4}{3}$, with deflationary and inflationary periods when $a \to \infty$ (in the case where $w = 4/3$ this inflation is exponential), otherwise there are singularities when $a \to \infty$. When $\omega > 0$ all energy conditions are satisfied, while in the second case, the strong energy condition is always violated and the weak energy conditions can be violated in some specific regions. In the limits $\theta \to 0$ and $\theta \to \pi/\lambda$, the solutions take the form,

$$a \propto t^{\frac{\alpha-1}{3\alpha-1}}, \quad \phi \propto t^{\frac{2}{3\alpha-1}}, \quad \chi \propto \text{const},$$

(16)

with $t \propto \theta^{(3\alpha-1)/2}$ when $\theta \to 0$ or $t \propto (\pi - \lambda \theta)^{(3\alpha-1)/2}$ when $\theta \to \pi/\lambda$. For $\omega > 0$ there is no inflationary phase. In the case $-\frac{3}{2} < \omega < -\frac{4}{3}$ ($0 < \alpha < \frac{1}{3}$), $\theta \to 0$ implies $t \to -\infty$, while in all other cases $\theta \to 0$ implies $t \to 0$.

The solutions with big-crunch following the big-bang ($\omega > 0$), and those with bounce ($\omega < 0$), are separated by the particular case where $\omega = 0$, for which the scale factor oscillates between a maximum and a minimum value, taking the form,

$$a = \sqrt{\frac{Cr}{\sqrt{6(\cos^2(\frac{\lambda\theta}{2}) + r^2 \sin^2(\frac{\lambda\theta}{2}))}}},$$

(17)

This is a non singular solution. In the particular case where $r = 1$, the scale factor is constant. It is an static Universe, even with the scalar fields evolving in time.

It is worth to remember the corresponding solutions for the minimal coupling case, which can be obtained by performing the transformation $a = \phi^{-1/2}b$, yielding

$$b = \sqrt{\lambda_1 \text{sech}(2\lambda_1 \epsilon)},$$

(18)
\[
\phi = D \text{sech} \left( \frac{2\lambda_1}{\alpha} \epsilon \right),
\]
\[
\chi = \sqrt{\frac{3}{2}} \alpha D \text{tanh} \left( \frac{2\lambda_1}{\alpha} \epsilon \right) + \chi_0,
\]
where \(d\theta = \phi d\epsilon\), \(\lambda_1 \equiv CD/\sqrt{6}\), and \(\chi_0\) is an integration constant. Near the singularities, the above solutions in the Einstein’s frame behave as \(b \propto \tau^{1/3}\), \(\phi \propto \tau^{2/3\alpha}\) and \(\chi \propto \text{const.}\), where \(\tau\) is the proper time in that frame, \(d\tau = \phi^{1/2}\,dt\).

3 Quantum solutions in the minisuperspace

We now insert the metric (3) in the Lagrangian (2). After integration by parts, we obtain the expression,
\[
L = \frac{1}{N} \left( 12a\dot{a}^2\phi + 12a^2\dot{a}\dot{\phi} - 2\omega \dot{\phi}^2 a^3 - 2\frac{\chi^2}{\phi} a^3 \right) - 12kNa\phi,
\]
The conjugate momenta are,
\[
\pi_a = \frac{\partial L}{\partial \dot{a}} = 12 a^2 \phi N \left( 2\frac{\dot{a}}{a} + \frac{\dot{\phi}}{\phi} \right),
\]
\[
\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = 4 a^3 \frac{\dot{a}}{N} \left( \frac{3\dot{a}}{a} - \omega \frac{\dot{\phi}}{\phi} \right),
\]
\[
\pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = -4 a^3 \frac{\dot{a}}{N\phi} \dot{\chi}.
\]
The lapse function \(N\) is a Lagrangian multiplier whose variation leads to a constraint equation. The Hamiltonian can be obtained in the canonical way, and it reads,
\[
H = N\mathcal{H},
\]
with
\[
\mathcal{H} = \frac{a^3}{\phi} \left( \omega \frac{\pi_a^2}{4(3 + 2\omega) a^4} + \phi \frac{\pi_\phi^2}{a^5} - \frac{1}{4(3 + 2\omega)} \frac{\phi^2}{a^6} \pi_\phi^2 - \frac{\phi^2}{8a^6} \pi_\chi^2 + 12 a^2 \phi \right).
\]
Since the function \(N\) is a Lagrange multiplier, we have the constraint, \(\mathcal{H} \approx 0\). Applying the Dirac quantization procedure, the quantum states must be annihilated by the operator version of \(\mathcal{H}\), yielding,
\[
\hat{\mathcal{H}}\Psi = 0
\]
where \(\Psi\) is the wavefunction of the Universe. Inserting the explicit operator version for \(\hat{\mathcal{H}}\) in the above equation by substituting \(\pi_a = -i\partial_a\), \(\pi_\phi = -i\partial_\phi\), \(\pi_\chi = -i\partial_\chi\), we obtain the differential equation for \(\Psi = \Psi(a, \phi, \chi)\):
\[
\left\{ \frac{\omega}{6(2\omega + 3)} a^2 \left( \partial_a^2 + \frac{p}{a} \partial_a \right) + \frac{1}{2\omega + 3} a \phi \partial_a \partial_\phi - \frac{1}{2\omega + 3} \phi^2 \left( \partial_\phi^2 + \frac{q}{\phi} \partial_\phi \right) - \frac{1}{2} \phi^2 \partial_\chi^2 \right\} \Psi = 48k a^4 \phi^2 \Psi,
\]
where \( p \) and \( q \) are factor ordering terms.

The equation (28) has mixed second order derivatives in the variables \( a \) and \( \phi \); at the same time, the potential in the right hand side mixes these same variables. In this form, the equation (28) can not be solved through the method of separation of variables. However, we can redefine \( a \) and \( \phi \), writing

\[
a = \phi^{-\frac{3}{2}}b, \quad \phi' = \phi', \quad \chi' = \chi.
\]  

(29)

This transformation is the minisuperspace version of the transformation that takes Lagrangian (4), expressed in the so-called Jordan’s frame, to a new Lagrangian where gravity is coupled minimally to two scalar fields. Inserting the transformation (29) in (28), we obtain a new equation for the dynamical variables \( b \) and \( \phi \):

\[
\left\{ \frac{b^2}{12} \left( \partial_b^2 + \frac{p'}{b} \partial_b \right) - \frac{\phi^2}{3 + 2\omega} \left( \partial_\phi^2 + \frac{q'}{\phi} \partial_\phi \right) - \frac{\phi^2}{2} \partial_\chi \right\} \Psi = 48b^4 \Psi .
\]  

(30)

The new factor order terms \( p' \) and \( q' \) are related to the older ones, \( p \) and \( q \), by the expressions,

\[
p' = \frac{12}{3 + 2\omega} \left( \frac{\omega}{6} p + \frac{3}{4} \frac{q}{2} \right), \quad q' = q .
\]  

(31)

When \( p = q = 1 \) then \( p' = q' = 1 \). From this transformation, we can deduce that the Wheeler-DeWitt equation for a minimal coupled gravity/scalar field is equivalent to a non minimal coupled version of this model. The conformal equivalence at the classical level is preserved at the quantum level. Moreover, the factor ordering is preserved only when \( p = q = 1 \) because in this case the differential operator in the Wheeler-DeWitt equation is the Laplacian one, which is covariant under general field redefinitions.

Equation (30) has already been studied in the literature [7]. The solutions were found using the separation of variables method. To obtain the solutions of Eq. (28) (which is in the Jordan’s frame), we have just to employ the transformation (29) in the solutions so found. Writing \( \Psi(b, \phi, \chi) = \rho(b)\beta(\phi)\gamma(\chi) \), they read

\[
\rho(b = a\phi^{1/2}) = a^{(1-\nu)/2} \left[ A_\beta I_n(12a^2\phi) + B_\beta K_n(12a^2\phi) \right] ,
\]  

(32)

\[
n = \frac{\sqrt{(\nu-1)^2 - 48k_2}}{4} ,
\]

\[
\beta(\phi) = \phi^{(1-\nu)/2} \left[ A_\beta I_m(2\sqrt{(3 + 2\omega)k_1}\phi) + B_\beta K_m(2\sqrt{(3 + 2\omega)k_1}\phi) \right] ,
\]  

(33)

\[
m = \frac{\sqrt{(\nu-1)^2 - 4(3 + 2\omega)k_2}}{2} ,
\]

\[
\gamma(\chi) = A_\gamma \exp(i\sqrt{8k_1}\chi) + B_\gamma \exp(-i\sqrt{8k_1}\chi) ,
\]  

(34)

where \( k_1 \) and \( k_2 \) are two constants of separation. The general solution can be written as

\[
\Psi = \int A(k_1, k_2)\rho_{k_1}(a, \phi)\beta_{k_2, k_1}(\phi)\gamma_{k_1}(\chi)dk_1dk_2 .
\]  

(35)
4 The quantum bohmian trajectories

In this section, we will apply the rules of the causal interpretation to the wave functions we have obtained in the previous section. We first summarize these rules for the case of homogeneous minisuperspace models. The general minisuperspace Wheeler-DeWitt equation is:

$$\mathcal{H}(\hat{p}^\alpha(t), \hat{q}_\alpha(t))\Psi(q) = 0. \tag{36}$$

where $p^\alpha(t)$ and $q_\alpha(t)$ represent the homogeneous degrees of freedom coming from the gravitational and matter degrees of freedom. Writing $\Psi = R \exp(iS/\hbar)$, and substituting it into (36), we obtain the following equation:

$$\frac{1}{2} f_{\alpha\beta}(q_\mu) \frac{\partial S}{\partial q_\alpha} \frac{\partial S}{\partial q_\beta} + U(q_\mu) + Q(q_\mu) = 0, \tag{37}$$

where

$$Q(q_\mu) = -\frac{1}{R} f_{\alpha\beta} \frac{\partial^2 R}{\partial q_\alpha \partial q_\beta}, \tag{38}$$

and $f_{\alpha\beta}(q_\mu)$ and $U(q_\mu)$ are the minisuperspace version of the DeWitt metric [12] and of the scalar curvature density of the spacelike hypersurfaces, respectively. The causal interpretation applied to quantum cosmology states that the trajectories $q_\alpha(t)$ are real, independently of any observations. Eq. (37) is the Hamilton-Jacobi equation for them, which is the classical one amended with a quantum potential term (38), responsible for the quantum effects. This suggests to define:

$$p^\alpha = \frac{\partial S}{\partial q_\alpha}, \tag{39}$$

where the momenta are related to the velocities in the usual way:

$$p^\alpha = f^{\alpha\beta} \frac{1}{N} \frac{\partial q_\beta}{\partial t}. \tag{40}$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations:

$$\frac{\partial S(q_\alpha)}{\partial q_\alpha} = f^{\alpha\beta} \frac{1}{N} \frac{\partial q_\beta}{\partial t}. \tag{41}$$

Eqs. (41) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same spacetime geometry for a given non-classical solution $q_\alpha(t)$. There is no problem of time in the causal interpretation of minisuperspace quantum cosmology. The classical limit is given by the region where $Q = 0$ and the Hamilton-Jacobi equation (37) reduces to the classical one.

Let us then apply this interpretation to our minisuperspace models and choose the gauge $N = 1$. The bohmian trajectories can be obtained by integrating the relations,

$$\pi_a = \partial_a S, \tag{42}$$

$$\pi_\phi = \partial_\phi S, \tag{43}$$

$$\pi_\chi = \partial_\chi S. \tag{44}$$
which lead to the differential equations,
\begin{align*}
24a\ddot{a}\phi + 12a^2 \dot{\phi} &= \partial_a S, \quad (45) \\
12a^2 \ddot{a} - 4\omega \dot{\phi} \dot{a}^3 &= \partial_\phi S, \quad (46) \\
-4a^3 \dot{\phi} &= \partial_\chi S. \quad (47)
\end{align*}

Employing the conformal transformations (29) and remembering that \( dt = \phi^{-1/2} d\tau \), where \( \tau \) is the proper time in Einstein’s frame, we obtain the differential equations
\begin{align*}
24bb' &= \partial_b S, \quad (48) \\
-2(3 + 2\omega)b^3 \frac{\phi'}{\phi^2} &= \partial_\phi S, \quad (49) \\
-4b^3 \frac{\phi^2 \chi'}{\phi^2} &= \partial_\chi S. \quad (50)
\end{align*}

The primes here mean derivatives with respect to the proper time in the Einstein frame. Relations (48,49,50) are the same that were found in the case of a minimal coupling between gravity and a scalar field.

Inserting in the Wheeler-DeWitt equation the expression \( \Psi = R \exp(iS/\bar{\hbar}) \), we find the corresponding Hamilton-Jacobi equation, which reads for the non minimal coupling case,
\begin{equation}
-\frac{\omega a^2}{6(2\omega + 3)} (\partial_a S)^2 - \frac{a\phi}{2\omega + 3} (\partial_a S)(\partial_\phi S) + \frac{\phi^2}{2\omega + 3} (\partial_\phi S)^2 + \frac{\phi^2}{2} (\partial_\chi S)^2 + Q + V_{cl} = 0, \quad (51)
\end{equation}
where
\begin{align*}
V_{cl} &= -48k a^4 \phi^2, \quad (52) \\
Q &= \frac{1}{R} \left[ \frac{\omega a^2}{6(2\omega + 3)} \left( \partial_a^2 R + \frac{p}{a} \partial_a R \right) + \frac{a\phi}{2\omega + 3} \partial_\phi^2 R \\
&\quad - \frac{\phi^2}{2\omega + 3} \left( \partial_\phi^2 R + \frac{q}{\phi} \partial_\phi R \right) - \frac{\phi^2}{2} \partial_\chi^2 R \right], \quad (53)
\end{align*}
where \( V_{cl} \) and \( Q \) are the classical and quantum potentials. The transformations (29) map equation (51) and the potentials (52,53) into the corresponding Hamilton-Jacobi equation, classical and quantum potentials of the minimal coupling case [7]. This complete the conformal equivalence (from the mathematical point of view) between minimal and non-minimal coupling at the quantum level in the causal interpretation.

In [7], we have obtained the bohmian trajectories by integrating Eqs. (48,49,50) for some exact solutions of the Wheeler-DeWitt equation, and we have found that the classical limit is recovered when the scale factor is small. With these results, we can obtain the bohmian trajectories by using the conformal transformations (29), and investigate the presence of singularities and classical limit of these solutions in the non minimal coupling case.
First, we remember the expressions connecting the scale factor and proper time in the minimal and non minimal coupling cases: \( a = b \phi^{-1/2} \) and \( t = \int \phi^{-1/2} d\tau \). The classical limit in the minimal case is

\[
    b \sim \tau^{1/3}, \quad \phi \sim \tau^{2/3}, \quad \chi = \text{const}.
\]

Using the conformal transformation, we obtain the classical limit for the non minimal coupling,

\[
    a \propto t^{\alpha - 1/3}, \quad \phi \propto t^{2/3}, \quad \chi = \text{const}.
\]

where \( \alpha = \sqrt{1 + \frac{2}{3} \omega} \), which were obtained directly from the classical solutions in Jordan’s frame and shown in Eq. (54). In the Einstein’s frame, the qualitative behavior of the scale factor is the same irrespective of the value of \( \omega \), while in the Jordan’s frame its behavior depends crucially on the value of \( \omega \), as explained in the section (2). In order to analyze all these possibilities at the quantum level, we will study four examples. The solutions obtained in cases (I,II,III) where extracted from Ref. [13].

Case I: \( \omega = 9/2 \) (\( \omega > 0 \)), which yields \( \alpha = 2 \).

We will choose \( A(k_1, k_2) = \frac{3}{2} \delta(k_1 - \exp(i \frac{\pi}{3})/48) \sinh(\pi \nu) \), with \( \nu \equiv \sqrt{3k_2} \), \( A_\rho = A_\beta = B_\gamma = 0 \), and \( p' = q' = 1 \) in Eq. (55).

1) The minimal coupling case.

In this case, the wavefunction takes the form,

\[
    \Psi = \frac{\pi^{3/2}}{2^{5/2} \sqrt{y}} \exp\left(-y - \frac{\phi^2}{16y} - \frac{\chi}{2 \sqrt{6}}\phi^2\right) \exp\left[i \left(\frac{\pi}{6} - \frac{\sqrt{3} \phi^2}{16y} + \frac{\chi}{2 \sqrt{2}}\right)\right].
\]

where \( y \equiv 12b^2 \). We obtain the following expressions for the bohmian trajectories:

\[
    b \propto \tau^{1/3} \quad \text{or} \quad y \propto \tau^{2/3},
\]

\[
    \phi \propto \tau^{1/3},
\]

\[
    \chi \propto \tau^{2/3} + \chi_0.
\]

They coincide with the classical trajectories for \( \tau, b, \phi \) very small (see Eq. (54)) but they differ from them when these conditions are not satisfied (see Eqs. (53,54)). The quantum potential is

\[
    Q = \frac{y^2}{3} - \frac{\phi^2}{16},
\]

while the classical potential is

\[
    V_{cl} = -\frac{y^2}{3}.
\]

The kinetic terms are

\[
    K_b = -\frac{b^2}{12} \left(\frac{\partial S}{\partial b}\right)^2 \propto \frac{\phi^4}{y^2},
\]
\[
K_\phi = \frac{\phi^2}{2} \left( \frac{\partial S}{\partial \phi} \right)^2 \propto \frac{\phi^4}{y^2}, \quad (63)
\]
\[
K_\chi = \frac{\phi^2}{12} \left( \frac{\partial S}{\partial \chi} \right)^2 \propto \phi^2. \quad (64)
\]

It can be seen from the above expressions that, for small \( \tau \), \( K_b \) and \( K_\phi \) dominates while, for large \( \tau \), \( Q \) and \( V_{cl} \) dominates (\( Q/V_{cl} \approx 1 \)). In other words, quantum effects are negligible for small \( b \) and \( \phi \) but become important otherwise.

2) The non minimal coupling case.

To obtain the new wavefunction we have just to make the substitution \( y = x\phi \) where \( x = 12a^2 \), yielding
\[
\Psi = \frac{\pi^{3/2}}{2^{5/2}} \sqrt{\frac{\phi}{x}} \exp \left( -x\phi - \frac{\phi}{16x} - \frac{\chi}{2\sqrt{6}} \right) \exp \left[ \frac{i}{2} \left( \frac{\pi}{6} - \frac{\sqrt{3}\phi}{16x} + \frac{\chi}{2\sqrt{2}} \right) \right]. \quad (65)
\]

The bohmian trajectories can be calculated either by solving equations (45,46,47) directly, or by making the conformal mapping \( x = y/\phi, \phi = \phi, \chi = \chi \) in the solutions of the minimal coupling case. We obtain the following expressions for the bohmian trajectories:
\[
a \propto t^{1/5} \quad \text{or} \quad x \propto t^{2/5}, \quad (66)
\]
\[
\phi \propto t^{2/5}, \quad (67)
\]
\[
\chi \propto t^{4/5} + \chi_0. \quad (68)
\]

They coincide with the classical trajectories for \( t, b, \phi \) very small (see Eq. (55)) but they differ from them when these conditions are not satisfied. The quantum potential is
\[
Q = \frac{x^2 \phi^2}{3} - \frac{\phi^2}{16}, \quad (69)
\]
while the classical potential is
\[
V_{cl} = -\frac{x^2 \phi^2}{3}. \quad (70)
\]

The kinetic terms are
\[
K_a = -\frac{a^2}{16} \left( \frac{\partial S}{\partial a} \right)^2 \propto \frac{\phi^2}{x^2}, \quad (71)
\]
\[
K_\phi = \frac{\phi^2}{12} \left( \frac{\partial S}{\partial \phi} \right)^2 \propto \frac{\phi^2}{x}, \quad (72)
\]
\[
K_{a\phi} = -\frac{a\phi^2}{12} \frac{\partial S}{\partial a} \frac{\partial S}{\partial \phi} \propto \frac{\phi^2}{x^2}, \quad (73)
\]
\[
K_\chi = \frac{\phi^2}{2} \left( \frac{\partial S}{\partial \chi} \right)^2 \propto \phi^2. \quad (74)
\]

It can be seen from the above expressions that, for small \( t \), \( K_a \) and \( K_{a\phi} \) dominates while, for large \( t \), \( Q \) and \( V_{cl} \) dominates (\( Q/V_{cl} \approx 1 \)). In other words, quantum effects are
negligible for small $a$ and $\phi$ but become important otherwise.

Case II: $w = 0$ which yields $\alpha = 1$.
We will choose $A(k_1, k_2) = \frac{3}{2}\delta(k_1 + \frac{1}{12})\tan(\pi\nu)$, with $\nu \equiv \sqrt{3k_2}$, $A_\rho = A_\beta = 0$, $A_\gamma = B_\gamma = 1$, and $p' = q' = 1$ in Eq. (35).

1) The minimal coupling case.
In this case, the wavefunction takes the form,
\[
\Psi = \frac{\pi}{2} \cosh\left(\sqrt{\frac{2}{3}}\chi\right)\sqrt{\frac{y\phi}{y^2 + \phi^2}} e^{-y} \exp\left[i\left(\frac{\pi}{4} - \phi - \arctan\left(\frac{\phi}{y}\right)\right)\right],
\]
(75)
The bohmian trajectories are (see Ref. [7]):
\begin{align*}
    b &= \frac{1}{\sqrt{12}} \left[ \ln\left(\frac{C}{\sqrt{1+4\eta^2}}\right)\right]^{\frac{1}{2}}, \\
    \phi &= \frac{1}{2\eta} \ln\left(\frac{C}{\sqrt{1+4\eta^2}}\right) = -\frac{a^2}{\eta}, \\
    \chi &= \text{const.},
\end{align*}
(76)
where $\eta = \int \frac{dx}{b}$ is the conformal time and $C$ is an integration constant. For small $b$, when $\eta$ approaches $\pm \sqrt{\frac{c^2 - 1}{2}}$, these functions tend to:
\begin{align*}
    b(\tau) &\propto \tau^{\frac{2}{3}}, \\
    \phi(\tau) &\propto \tau^{\frac{2}{3}} \propto b^2, \\
    \chi &= \text{const.},
\end{align*}
(77)
which is exactly the classical behavior for $\omega = 0$. When $b$ is not small, the trajectories are not classical. This can be explained by the behavior of the quantum potential
\[
Q = \frac{1}{3} \frac{(y^4 - 2\phi^2 y - \phi^4)}{y^2 + \phi^2}.
\]
(78)
When $b$ is small the kinetic terms dominate while for $b$ large the quantum and classical potential dominate (see Ref. [7] for details).

2) The non minimal coupling case.
A conformal transformation in the solution (75) yields
\[
\Psi = \frac{\pi}{2} \cosh\left(\frac{2}{3}\chi\right)\sqrt{\frac{x}{x^2 + 1}} e^{-x\phi} \exp\left[i\left(\frac{\pi}{4} - \phi - \arctan\left(\frac{1}{x}\right)\right)\right].
\]
(79)
The bohmian trajectories can be calculated either by solving equations (45,46,47) directly or by making the conformal mapping $x = y/\phi, \phi = \phi, \chi = \chi$ in the solutions of the minimal
coupling case. The new solutions are

\[ a = \left( -\frac{t}{4} \right)^{1/3}, \]
\[ \phi = -\frac{1}{3(2t)^{2/3}} \ln\left( \frac{C}{\sqrt{1+9(-2t)^{1/3}}} \right), \]
\[ \xi = \text{const.}, \]

(80)

They have an initial singularity, and they are completely different from the classical trajectories, which in this case are oscillatory and without singularities. This is an example where quantum effects can create a singularity. There is no classical limit. This can be seen by examining the behavior of the quantum potential

\[ Q = \frac{1}{3} \left( \phi^2 x^4 - 2\phi^2 x - \phi^2 \right) \]
\[ x^2 + 1 \]

(81)

when compared with the classical potential and kinetic terms

\[ V_{cl} = -\frac{x^2 \phi^2}{3}, \]
\[ K_a = 0, \]
\[ K_\phi = \frac{\phi^2}{3} \left( \frac{\partial S}{\partial \phi} \right)^2 \propto \phi^2, \]
\[ K_{a\phi} = -\frac{a \phi^2}{3} \frac{\partial S}{\partial a} \frac{\partial S}{\partial \phi} \propto \frac{x \phi}{1 + x^2}, \]
\[ K_\chi = \frac{\phi^2}{2} \left( \frac{\partial S}{\partial \chi} \right)^2 = 0. \]

(82)
(83)
(84)
(85)
(86)

When \( t \) is small, \( Q \) and \( K_\phi \) dominate while for \( t \) large \( Q \) and \( V_{cl} \) dominate.

Case III: \( \omega = -9/8 \) \((-4/3 < \omega < 0)\), which yields \( \alpha = 1/2 \).

We will choose \( A(k_1, k_2) = \frac{3}{2} \delta(k_1 - \frac{1}{3} \exp(i\frac{\pi}{2})) \sin(\frac{\pi \nu}{2}) \), with \( \nu \equiv \sqrt{3k_2} \), \( A_\rho = A_\beta = B_\gamma = 0 \), and \( p' = q' = 1 \) in Eq. (35).

1) The minimal coupling case.

In this case, the wavefunction takes the form,

\[ \Psi = \frac{\pi^{3/2}}{2^{1/2}} \frac{y}{\sqrt{\phi}} \exp \left[ -\sqrt{3} \left( \frac{\phi + y^2}{8\phi} - \frac{2\chi}{\sqrt{6}} \right) \exp \left[ i \left( -\frac{\pi}{12} - \frac{\phi}{2} + \frac{y^2}{16\phi} + 2\chi \right) \right] \right]. \]

(87)

We obtain the following expressions for the bohmian trajectories:

\[ b = \lambda_1 \sqrt{\text{sech}(\lambda_1^2 \epsilon)}, \]
\[ \phi = 6\lambda_1^2 \text{cosech}(2\lambda_1^2 \epsilon), \]
\[ \chi = 9\lambda_1^2 \text{cotanh}(2\lambda_1^2 \epsilon) + \chi_0. \]

(88)
(89)
(90)
where $d\tau = b^3 d\epsilon$. They coincide with the classical trajectories for $\epsilon$ very large and $b, \phi$ very small, where they present the behavior of Eq. (54), but they differ from them when these conditions are not satisfied. The quantum potential is

$$Q = \frac{y^2}{4} - \frac{4\phi^2}{3}, \quad (91)$$

while the classical potential, is as usual,

$$V_{cl} = -\frac{y^2}{3}. \quad (92)$$

The kinetic terms are

$$K_b = -\frac{b^2}{12} \left( \frac{\partial S}{\partial b} \right)^2 \propto \frac{y^4}{\phi^2}, \quad (93)$$

$$K_\phi = \frac{4\phi^2}{3} \left( \frac{\partial S}{\partial \phi} \right)^2 = -\frac{1}{3} \left( \phi^2 + \frac{y^2}{4} - \frac{y^4}{64\phi^2} \right), \quad (94)$$

$$K_\chi = \frac{\phi^2}{2} \left( \frac{\partial S}{\partial \chi} \right)^2 \propto \phi^2. \quad (95)$$

It can be seen from the above expressions that, for large $\epsilon$, $K_b$ and $K_\phi$ dominates while, for $\epsilon \ll 1$, $Q$, $K_\phi$ and $K_\chi$ dominates. Quantum effects are negligible for small $b$ and $\phi$ but become important otherwise.

2) The non minimal coupling case.

To obtain the new wavefunction we have just to make the substitution $y = x\phi$ where $x = 12a^2$, yielding

$$\Psi = \frac{\pi^{3/2}}{2^{1/2}} x^{\sqrt{6}} \phi^{\sqrt{\chi}} \exp \left[ -\frac{\sqrt{3}}{2} (\phi + \frac{x^2\phi}{8}) - \frac{2\chi}{\sqrt{6}} \right] \exp \left[ i \left( -\frac{\pi}{12} - \frac{\phi}{2} + \frac{x^2\phi}{16} + 2\chi \right) \right]. \quad (96)$$

We obtain the following expressions for the bohmian trajectories:

$$a = \sqrt{\frac{\sinh(\lambda_1^2\epsilon)}{3}}, \quad (97)$$

$$\phi = 6\lambda_1^2 \text{cosech}(2\lambda_1^2\epsilon), \quad (98)$$

$$\chi = 9\lambda_1^2 \text{cotanh}(2\lambda_1^2\epsilon) + \chi_0, \quad (99)$$

where $dt = \phi a^3 d\epsilon$ and we will define $t_\infty \equiv t(\epsilon = \infty)$. They coincide with the classical trajectories for $\epsilon, a$ very large, and $\phi$ very small. Their behaviors in the limit $t \to t_\infty$ are

$$a \propto (t_\infty - t)^{-1} \quad \text{or} \quad x \propto (t_\infty - t)^{-2}, \quad (100)$$

$$\phi \propto (t_\infty - t)^4, \quad (101)$$

$$\chi \propto \chi_0. \quad (102)$$
which coincide with Eq. (55) for $\alpha = 1/2$, but they differ from them when these conditions are not satisfied. For instance, when $\epsilon$, and $t$, are very small they are

\begin{align*}
a & \propto t^{1/3} \quad \text{or} \quad x \propto t^{2/3}, \\
\phi & \propto t^{-2/3}, \\
\chi & \propto t^{-2/3}.
\end{align*}

which is not the classical behavior in this case. Note that one of the singularities of the classical solution is changed to a big-bang or a big-crunch. The quantum potential is

\begin{equation}
Q = \frac{y^2 \phi^2}{4} - \frac{4 \phi^2}{3},
\end{equation}

while the classical potential is

\begin{equation}
V_{cl} = -\frac{y^2}{3}.
\end{equation}

The kinetic terms are

\begin{align*}
K_a & = -\frac{a^2}{4} \left( \frac{\partial S}{\partial a} \right)^2 \propto \phi x^3, \\
K_\phi & = \frac{4 \phi^2}{3} \left( \frac{\partial S}{\partial \phi} \right)^2 \propto \frac{\phi^2}{2} \left( 1 - \frac{x^2}{8} \right), \\
K_{a\phi} & = -\frac{a \phi^2}{9} \frac{\partial S}{\partial a} \frac{\partial S}{\partial \phi} \propto \frac{\phi^2 x^2}{8} \left( 1 - \frac{x^2}{8} \right), \\
K_\chi & = \frac{\phi^2}{2} \left( \frac{\partial S}{\partial \chi} \right)^2 \propto \phi^2.
\end{align*}

It can be seen from the above expressions that, for $t \to t_\infty$, $K_a$ dominates while, for small $t$, $Q$, $K_\phi$ and $K_\chi$ dominate. The quantum effects are negligible for large $a$ and small $\phi$ but become important otherwise.

Case IV: $-3/2 < \omega < -4/3$, which yields $\alpha \leq 1/3$.

In this case, we do not take superpositions, but the wave functions (32,33,34) themselves. We will choose $B_\rho = B_\beta = B_\gamma = 0$, and $p' = q' = 1$ yielding

\begin{equation}
\Psi = I_n(x\phi)I_m(2\sqrt{(3 + 2\omega)k_1\phi}) \exp(i\sqrt{8k_1\chi})
\end{equation}

with $n = i\nu$ and $m = i\sqrt{(3 + 2\omega)k_2}$. For small $\phi$ we can approximate the Bessel functions for

\begin{equation}
\Psi = C_0 \exp[i(\nu \ln(x\phi) + \mu \ln(2\sqrt{(3 + 2\omega)k_1\phi}) + \sqrt{8k_1\chi})]
\end{equation}

As $C_0$ is a constant, the quantum potential is null and the bohmian trajectories are the classical ones in this limit. One can verify this by inserting the phase of the above wave function into Eqs. (45,46,47), obtaining

\begin{align*}
a & \propto t^{\frac{\omega - 1}{2}}, \\
\phi & \propto t^{\frac{2}{\omega - 1}}.
\end{align*}
with $\alpha = \sqrt{1 + \frac{2}{3} \omega}$ as before. Note that even in the cases where $a$ becomes large, the product $x\phi \propto t^{\frac{2\alpha}{2\alpha - 1}}$ is always small, justifying our approximation. We can see that the classical limit happens for $\phi$ small, but it can happen for large $a$.

5 Conclusions

The possible avoidance of classical singularities due to quantum effects, and the prediction of the classical behaviour of the Universe are some of the most important issues in quantum cosmology. These questions have been studied by different methods in the literature. In Ref. [7], it has been proposed to use the bohmian trajectories to study these issues, and it was verified that, in the context of scalar-tensor theories coming from string and Kaluza-Klein theories, re-expressed in the Einstein’s frame, their conclusions agrees with usual semi-classical considerations. The results obtained for some exact solutions of the Wheeler-DeWitt equation indicated that the Universe is classical when the scale factor is small (comparable to Planck scale), and the singularities are not avoided. In the present work we have extended this analysis to non minimal coupled scalar-tensor theories. The conformal equivalence between Jordan’s and Einstein’s frame at classical and quantum level was established, even for the bohmian trajectories and quantum potentials. Having the particular exact solutions of the Wheeler-DeWitt equation in the Einstein’s frame, we were able to obtain the corresponding solutions in the Jordan’s frame. The bohmian trajectories in Jordan’s frame present classical behavior whenever the scalar field non minimally coupled with gravity is small, which coincides with small values of the scale factor $a$ when $\omega > 0$, and large values of $a$ when $-\frac{3}{2} < \omega < 0$). Quantum behavior appears when the scalar field is not small, which means for large values of $a$ when $\omega > 0$, and for small values of $a$ when $-\frac{3}{2} < \omega < 0$). Hence, contrary to the minimal coupling case, the bohmian trajectories may coincide with the classical solutions for large values of $a$. The exception is the very particular case $\omega = 0$. In this case, the classical solution in Jordan’s frame is non-singular and periodic. In an example studied in this paper, the corresponding bohmian trajectory has always quantum behavior, even when the scalar field becomes small. More interesting, this quantum solution presents an initial singularity showing that, depending on the quantum state of the system, quantum effects, rather than avoid, may create singularities where classically there is none.

These results show that the quantum properties of quantum minisuperspace solutions may be quite different in Einstein’s and Jordan’s frame, although they may be linked by a conformal transformation. Those frames have not the same physical content, and the equivalence is manifest only from the mathematical point of view. A simple example is the fact that a G-variable theory demands the Jordan’s frame, while a G-constant is a natural feature of the Einstein’s frame. Also, our results express the importance of the choice of boundary conditions for the Wheeler-DeWitt equation. Many solutions we have obtained here present unphysical features, like quantum behavior for large scale factors. It should be interesting to investigate other solutions of the Wheeler-DeWitt equation presented in this paper which may be not expressible in term of elementary functions, but which present more appealing features, like absence of singularities, classical limit for large $a$,
and inflation. One possibility is to study gaussian superpositions of the Bessel functions presented in section 3, and study them numerically. As gaussian superpositions mixes negative and positive indices of the Bessel functions (here we made superpositions with only one sign of the indices because these are the ones which can be expressed in term of elementary functions), and as these indices are connected with expansion and contraction, it may be possible to obtain non-singular quantum solutions in both frames. It should also be interesting to extend the present analysis to any value of the the curvature of the spatial sections. These are subjects of our future investigations.

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