The minimum mean square estimator for a sublinear operator

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Abstract

In this paper, we study the minimum mean square estimator for a sublinear operator. Under some mild assumptions, we prove the existence and uniqueness of the minimum mean square estimator. Several characterizations of the minimum mean square estimator are obtained. We also explore the relationship between the minimum mean square estimator and the conditional coherent risk measure and conditional $g$-expectation.

Keywords: minimum mean square estimator; conditional nonlinear expectation; sublinear operator; coherent risk measure; $g$-expectation

1 Introduction

In the classical probability theory, the conditional expectation of a random variable $\xi$ in $L^2_\mathcal{F}(\Omega)$ is just the minimum mean square estimator. In more details, let $\mathcal{C}$ be a sub $\sigma$-algebra of $\mathcal{F}$. Then this minimum mean square estimator is the projection of $\xi$ from $L^2_\mathcal{F}(\Omega)$ to $L^2_\mathcal{C}(\Omega)$. Therefore, the minimum mean square estimator can be used as an alternative definition of conditional expectation.

In recent decades, nonlinear risk measures and nonlinear expectations have been proposed and developed rapidly. Various definitions of conditional nonlinear expectations (risk measures) are proposed. For example, Artzner et al. [2] introduced coherent risk measure theory in which the conditional expectation (conditional risk measure) is defined as

$$\bar{\Phi}_t[\xi] := \text{ess sup}_{P \in \mathcal{P}} E_P[\xi|\mathcal{F}_t],$$

where $\xi \in \mathcal{F}_T$, $\mathcal{F}_t$ is a sub $\sigma$-algebra of $\mathcal{F}_T$ and $\mathcal{P}$ is a family of probability measures. They have proved that if $\mathcal{P}$ is ‘stable’, then the conditional expectation defined above is time consistent. It is also well known that Peng studied $g$-expectation in [8] and defined the conditional $g$-expectation $E_g[\xi|\mathcal{F}_t]$ as the solution of a backward stochastic differential equation with the generator $g$ and the terminal value $\xi$ at time $t$. $g$-expectation has many good properties including time consistency, i.e., $\forall 0 \leq s \leq t \leq T$, we have $E_g[E_g[\xi|\mathcal{F}_t]|\mathcal{F}_s] = E_g[\xi|\mathcal{F}_s]$.

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So it is interesting to explore whether conditional nonlinear expectations still coincide with the minimum mean square estimators. Note that many interesting nonlinear expectations (risk measures) are sublinear. So the purpose of this paper is to study the minimum mean square estimator for a sublinear operator. We will show that for this minimum mean square estimator, generally speaking, the time consistency property does not hold. In accordance with this result, both of the above conditional nonlinear expectations (Artzner et al.’s and Peng’s) fail to be the minimum mean square estimators.

The paper is organized as follows. In section 2, we formulate our problem. Under some mild assumptions, we prove the existence and uniqueness of the minimum mean square estimator in section 3. In section 4, we obtain several characterizations of the minimum mean square estimator. At last section, we first give the basic properties of the minimum mean square estimator and the conditional coherent risk measure and conditional $g$-expectation.

2 Problem formulation

2.1 Preliminary

For a given measurable space $(\Omega, \mathcal{F})$, we denote all the bounded $\mathcal{F}$-measurable functions by $\mathbb{F}$.

**Definition 2.1** A sublinear operator $\rho$ is a functional $\rho : \mathbb{F} \mapsto \mathbb{R}$ satisfying

(i) **Monotonicity**: $\rho(\xi_1) \geq \rho(\xi_2)$ if $\xi_1 \geq \xi_2$;

(ii) **Constant preserving**: $\rho(c) = c$ for $c \in \mathbb{R}$;

(iii) **Sub-additivity**: For each $\xi_1, \xi_2 \in \mathbb{F}$, $\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2)$;

(iv) **Positive homogeneity**: $\rho(\lambda \xi) = \lambda \rho(\xi)$ for $\lambda \geq 0$.

Note that $\mathbb{F}$ is a Banach space endowed with the supremum norm. Denote the dual space of $\mathbb{F}$ by $\mathbb{F}^*$. It is well known that there is a one-to-one correspondence between $\mathbb{F}^*$ and the class of additive set functions. Then we denote an element in $\mathbb{F}^*$ by $E_P$ where $P$ is an additive set function. Sometimes we also use $P$ instead of $E_P$.

**Proposition 2.2** Suppose that the sublinear operator $\rho$ can be represented by a family of probability measures $\mathcal{P}$, i.e., $\rho(\xi) = \sup_{P \in \mathcal{P}} E_P[\xi]$. For a sequence $\{\xi_n\}_{n \in \mathbb{N}}$, if there exists a $M \in \mathbb{R}$ such that $\xi_n \geq M$ for all $n$, then we have

$$\rho(\lim \inf_n \xi_n) \leq \lim \inf_n \rho(\xi_n).$$

**Proof.** Set $\zeta_n = \inf_{k \geq n} \xi_k$. Then $\zeta_n \leq \xi_n$ and $\{\zeta_n\}_{n \in \mathbb{N}}$ is an increasing sequence. It is easy to see that

$$\rho(\lim \inf_n \xi_n) = \rho(\lim \zeta_n) = \lim \rho(\zeta_n) \leq \lim \inf_n \rho(\xi_n).$$

This completes the proof. ■

**Definition 2.3** We call a sublinear operator $\rho$ continuous from above on $\mathcal{F}$ if for each sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset \mathbb{F}$ satisfying $\xi_n \downarrow 0$, we have

$$\rho(\xi_n) \downarrow 0.$$
Lemma 2.4 If a sublinear operator $\rho$ is continuous from above on $\mathcal{F}$, then for any linear operator $E_P$ dominated by $\rho$, $P$ is a probability measure.

**Proof.** For any $A_n \downarrow \phi$, we have $\rho(I_{A_n}) \downarrow 0$. If a linear operator $E_P$ is dominated by $\rho$, then $P(A_n) \downarrow 0$. It is easy to see that $P(\Omega) = 1$. Thus, $P$ is a probability measure. ■

Proposition 2.5 A sublinear operator $\rho$ is continuous from above on $\mathcal{F}$ if and only if there exists a probability measure $P_0$ and a family of probability measures $P$ such that

i) $\rho(X) = \sup_{P \in P} E_P[X]$ for all $X \in \mathcal{F}$;

ii) any element in $P$ is absolutely continuous with respect to $P_0$;

iii) the set $\{\frac{dP}{dP_0}; P \in P\}$ is $\sigma(L^1(P_0), L^\infty(P_0))$-compact,

where $\sigma(L^1(P_0), L^\infty(P_0))$ denotes the weak topology defined on $L^1(P_0)$.

**Proof.** $\Rightarrow$ By Theorem A.1 in Appendix A, $\rho$ can be represented by the family of linear operators dominated by $\rho$. We denote by $P$ all the linear operators dominated by $\rho$. Since $\rho$ is continuous from above on $\mathcal{F}$, by Lemma 2.4 every element in $P$ is a probability measure. By Theorem A.2 $\mathcal{P}$ is $\sigma(\mathcal{F}^*, \mathcal{F})$-compact, where $\sigma(\mathcal{F}^*, \mathcal{F})$ denotes the weak* topology defined on $\mathcal{F}^*$. By Theorem A.3 there exists a $P_0 \in \mathcal{F}^*_c$ such that all the elements in $\mathcal{P}$ are absolutely continuous with respect to $P_0$, where $\mathcal{F}^*_c$ denotes all the countably additive measures in $\mathcal{F}^*$. Since $\rho$ is continuous from above on $\mathcal{F}$ and the dual space of $L^1(P_0)$ is $L^\infty(P_0)$, by Corollary 4.35 in [5], the set $\{\frac{dP}{dP_0}; P \in \mathcal{P}\}$ is $\sigma(L^1(P_0), L^\infty(P_0))$-compact, where $L^1(P_0)$ is the space of integrable random variables and $L^\infty(P_0)$ is the space of all equivalence classes of bounded real valued random variables.

$\Leftarrow$ We directly deduce this result by Dini’s theorem. ■

In the following, we will denote by $\mathcal{P}$ all the linear operators dominated by $\rho$.

**Definition 2.6** We call a sublinear operator $\rho$ proper if all the elements in $\mathcal{P}$ are equivalent to $P_0$, where $P_0$ is the probability measure in Proposition 2.5.

2.2 Minimum mean square estimator

Let $\mathcal{C}$ be a sub $\sigma$-algebra of $\mathcal{F}$ and $\mathcal{C}$ be the set of all the bounded $\mathcal{C}$-measurable functions. For a given $\xi \in \mathcal{F}$, our problem is to solve its minimum mean square estimator for the sublinear operator $\rho$ when we only know ”the information” $\mathcal{C}$. In more details, we want to solve the following optimization problem.

**Problem 2.7** Find a $\hat{\eta} \in \mathcal{C}$ such that

$$\rho(\xi - \hat{\eta})^2 = \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2.$$ (2.1)

The optimal solution $\hat{\eta}$ of (2.1) is called the minimum mean square estimator. It is also regarded as a minimax estimator in statistical decision theory.

If $\rho$ degenerates to a linear operator, then $\mathcal{P}$ contains only one probability measure and $\rho$ is the mathematical expectation under this probability measure. In this case, it is well known that the minimum mean square estimator $\hat{\eta}$ is just the conditional expectation $E[\xi | \mathcal{C}]$. 

3
3 Existence and uniqueness results

In this section, we study the existence and uniqueness of the minimum mean square estimator.

For a given $\xi \in \mathcal{F}$, we always suppose that $\sup |\xi| \leq M$ where $M$ is a positive constant.

3.1 Existence

Lemma 3.1 Suppose that $\xi \in \mathcal{F}$. Then we have

$$\inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2 = \inf_{\eta \in \mathcal{G}} \rho(\xi - \eta)^2,$$

where $\mathcal{G}$ is all the $\mathcal{C}$-measurable functions bounded by $M$.

Proof. For any $\eta \in \mathcal{C}$, let $\tilde{\eta} := \eta I_{-M \leq \eta \leq M} + MI_{\eta > M} - MI_{\eta < -M}$. Then $\tilde{\eta} \in \mathcal{G}$. For any $P \in \mathcal{P}$, we have

$$E_P[(\xi - \eta)^2] - E_P[(\xi - \tilde{\eta})^2] = E_P[(\tilde{\eta} - \eta)(2\xi - \eta - \tilde{\eta})] \geq E_P[(M - \eta)(2\xi - 2M)I_{\eta > M}] + E_P[(-M - \eta)(2\xi + 2M)I_{\eta < -M}].$$

Since $-M \leq \xi \leq M$, we have

$$E_P[(\xi - \eta)^2] \geq E_P[(\xi - \tilde{\eta})^2].$$

It yields that

$$\rho(\xi - \eta)^2 \geq \rho(\xi - \tilde{\eta})^2$$

and

$$\inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2 \geq \inf_{\eta \in \mathcal{G}} \rho(\xi - \eta)^2.$$

On the other hand, since $\mathcal{G} \subset \mathcal{C}$, the inverse inequality is obviously true.

Theorem 3.2 If the sublinear operator $\rho$ is continuous from above on $\mathcal{F}$, then there exists an optimal solution $\hat{\eta} \in \mathcal{G}$ for problem 2.7

Proof. By Lemma 3.1 there exists a sequence $\{\eta_n; n \in \mathbb{N}\} \subset \mathcal{G}$ such that

$$\rho(\xi - \eta_n)^2 < \alpha + \frac{1}{2n},$$

where $\alpha := \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2$. By Komlós theorem, there exists a subsequence $\{\eta_{n_k}\}_{k \geq 1}$ and a random variable $\hat{\eta}$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \eta_{n_k} = \hat{\eta}. \quad P_0 - a.s.$$

Since $\{\eta_n\}_{n \geq 1}$ is bounded by $M$, then $\hat{\eta} \in \mathcal{G}$. By Proposition 2.2 we have

$$\rho(\xi - \hat{\eta})^2 \leq \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \rho(\xi - \eta_{n_k})^2 \leq \lim_{k \to \infty} \left(\alpha + \frac{1}{k}\right) = \alpha.$$

This completes the proof.
Remark 3.3 If we only assume that there exists a probability measure $P_0$ such that the $\rho$ is a sublinear operator generated by a family of probability measures which are all absolutely continuous with respect to $P_0$, then Theorem 3.2 still holds.

3.2 Uniqueness

In this subsection, we prove that there exists a unique optimal solution of problem 2.7.

Lemma 3.4 If the sublinear operator $\rho$ is continuous from above on $\mathcal{F}$, then for a given $\xi \in \mathcal{F}$, we have

$$\sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{G}} E_P[|\xi - \eta|^2] = \max_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{G}} E_P[|\xi - \eta|^2].$$

Proof. By Proposition 2.5, there exists a probability measure $P_0$ such that $P \ll P_0$ for all $P \in \mathcal{P}$. Let $f_P := \frac{dP}{dP_0}$ and

$$\beta := \sup_{P \in \mathcal{P}, \eta \in \mathcal{G}} \inf E_P[|\xi - \eta|^2] = \sup_{P \in \mathcal{P}, \eta \in \mathcal{G}} \inf E_P[f_P(\xi - \eta)^2].$$

Then take a sequence $\{f_{P_n}; P_n \in \mathcal{P}\}_{n \geq 1}$ such that

$$\inf_{\eta \in \mathcal{G}} E_{P_0}[f_{P_n}(\xi - \eta)^2] \geq \beta - \frac{1}{2^n}.$$ 

By Komlós theorem, there exists a subsequence $\{f_{P_{n_k}}\}_{k \geq 1}$ of $\{f_{P_n}\}_{n \geq 1}$ and a random variable $f_P \in L^1(\Omega, \mathcal{F}, P_0)$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f_{P_{n_k}} = f_P \text{ P}_0 - a.s.$$ 

Set $g_k := \frac{1}{k} \sum_{i=1}^{k} f_{P_{n_i}}$. Then $g_k \in \{f_P; P \in \mathcal{P}\}$. By Proposition 2.5, we know $\{f_P; P \in \mathcal{P}\}$ is $\sigma(L^1(P_0), L^\infty(P_0))$-compact. By Dunford-Pettis theorem, it is uniformly integrable. Thus $\{g_k\}_{k \geq 1}$ is also uniformly integrable and $\|g_k - f_P\|_{L^1(P_0)} \to 0$. This shows $f_P \in \mathcal{P}$.

On the other hand, for any $\eta \in \mathcal{G}$ and $k \in \mathbb{N}$, we have

$$E_{P_0}[g_k(\xi - \eta)^2] \geq \inf_{\eta \in \mathcal{G}} E_{P_0}[g_k(\xi - \tilde{\eta})^2].$$

Then for any $\eta \in \mathcal{G}$, we have

$$\lim_{k \to \infty} E_{P_0}[g_k(\xi - \eta)^2] \geq \sup_{k \to \infty} \inf_{\eta \in \mathcal{G}} E_{P_0}[g_k(\xi - \tilde{\eta})^2].$$

Thus

$$\inf_{\eta \in \mathcal{G}} \lim_{k \to \infty} E_{P_0}[g_k(\xi - \eta)^2] \geq \sup_{k \to \infty} \inf_{\eta \in \mathcal{G}} E_{P_0}[g_k(\xi - \tilde{\eta})^2].$$

Since $\{g_k\}_{k \geq 1}$ is uniformly integrable and $\|(|\xi - \eta|^2)\|_{L^\infty} \leq 4M^2$, we have

$$\inf_{\eta \in \mathcal{G}} E_{P_0}[f_P(\xi - \eta)^2] = \inf_{\eta \in \mathcal{G}} E_{P_0}[\lim_{k \to \infty} g_k(\xi - \eta)^2] = \inf_{\eta \in \mathcal{G}} \lim_{k \to \infty} E_{P_0}[g_k(\xi - \eta)^2].$$

Then

$$\inf_{\eta \in \mathcal{G}} E_{P_0}[f_P(\xi - \eta)^2] \geq \sup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \inf_{\eta \in \mathcal{G}} E_{P_0}[f_{P_{n_i}}(\xi - \eta)^2] \geq \beta.$$ 

Since $f_P \in \mathcal{P}$, we have

$$\inf_{\eta \in \mathcal{G}} E_{P_0}[|\xi - \eta|^2] = \sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{G}} E_P[|\xi - \eta|^2].$$ 

This completes the proof. ■
Corollary 3.5 If the sublinear operator $\rho$ is continuous from above on $\mathcal{F}$, then for a given $\xi \in \mathcal{F}$, we have
\[
\sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2] = \max \inf_{P \in \mathcal{P}} E_P[(\xi - \eta)^2].
\]

Proof. Choose $\hat{P}$ as in Lemma 3.4 By Lemma 3.1 and Lemma 3.4 we have
\[
\sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2] \leq \sup_{P \in \mathcal{P}} E_P[(\xi - \eta)^2] = \inf_{\eta \in \mathcal{C}} \sup_{P \in \mathcal{P}} E_P[(\xi - \eta)^2].
\]

On the other hand, the inverse inequality is obvious. Then
\[
\inf_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2] = \sup_{P \in \mathcal{P}} E_P[(\xi - \eta)^2].
\]

Since $\hat{P} \in \mathcal{P}$, we have
\[
\sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2] = \max \inf_{P \in \mathcal{P}} E_P[(\xi - \eta)^2].
\]

This completes the proof. ■

Theorem 3.6 If the sublinear operator $\rho$ is continuous from above on $\mathcal{F}$ and proper, then for any given $\xi \in \mathcal{F}$, there exists a unique optimal solution of problem 2.7.

Proof. The existence result is proved in Theorem 3.2 Now we prove the optimal solution is unique. Since $\mathcal{P}$ is $\sigma(L^1(P_0), L^\infty(P_0))$-compact, by Minimax theorem, we have
\[
\inf_{\eta \in \mathcal{C}} \sup_{P \in \mathcal{P}} E_P[(\xi - \eta)^2] = \sup_{P \in \mathcal{P}} \inf_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2].
\]

Since the optimal solution exists, by Corollary 3.5, we have
\[
\min \sup_{\eta \in \mathcal{C}} E_P[(\xi - \eta)^2] = \max \inf_{P \in \mathcal{P}} E_P[(\xi - \eta)^2].
\]

Let $\tilde{\eta}$ be an optimal solution and $\hat{P}$ as in Corollary 3.5 By Minimax theorem, $(\tilde{\eta}, \hat{P})$ is an saddle point, i.e.,
\[
E_P[(\xi - \tilde{\eta})^2] \leq E_P[(\xi - \hat{\eta})^2] \leq E_P[(\xi - \eta)^2], \quad \forall P \in \mathcal{P}, \eta \in \mathcal{C}.
\]

This result shows that if $\tilde{\eta}$ is an optimal solution, then there exists a $\hat{P} \in \mathcal{P}$ such that $\hat{\eta} = E_{\hat{P}}[\xi|\mathcal{C}]$.

Suppose that there exist two optimal solutions $\eta_1$ and $\eta_2$. Denote the accompanying probabilities by $\hat{P}_1$ and $\hat{P}_2$ respectively. Then $\tilde{\eta}_1 = E_{\hat{P}_1}[\xi|\mathcal{C}]$ and $\tilde{\eta}_2 = E_{\hat{P}_2}[\xi|\mathcal{C}]$. Set $P^\lambda := \lambda \hat{P}_1 + (1 - \lambda) \hat{P}_2$, $\lambda \in (0, 1)$. Denote $\lambda E_{P^\lambda}[d_{\hat{P}_1}]$ by $\lambda_{\hat{P}_1}$ and $(1 - \lambda) E_{P^\lambda}[d_{\hat{P}_2}]$ by $\lambda_{\hat{P}_2}$. It is easy to see that $\lambda_{\hat{P}_1} + \lambda_{\hat{P}_2} = 1$.

\[
E_{P^\lambda}[(\xi - \lambda_{\hat{P}_1} \tilde{\eta}_1 - \lambda_{\hat{P}_2} \tilde{\eta}_2)^2] = E_{P^\lambda}[(\lambda_{\hat{P}_1} (\xi - \tilde{\eta}_1) + \lambda_{\hat{P}_2} (\xi - \tilde{\eta}_2))^2]
\]
\[
= E_{P^\lambda} [\lambda_{\hat{P}_1}^2 (\xi - \tilde{\eta}_1)^2 + \lambda_{\hat{P}_2}^2 (\xi - \tilde{\eta}_2)^2 + 2\lambda_{\hat{P}_1} \lambda_{\hat{P}_2} (\xi - \tilde{\eta}_1)(\xi - \tilde{\eta}_2)]
\]
\[
= E_{P^\lambda} [\lambda_{\hat{P}_1} (\xi - \tilde{\eta}_1)^2 + \lambda_{\hat{P}_2} (\xi - \tilde{\eta}_2)^2 - \lambda_{\hat{P}_1} \lambda_{\hat{P}_2} (\tilde{\eta}_1 - \tilde{\eta}_2)^2]
\]
\[
= \lambda E_{\hat{P}_1} [(\xi - \tilde{\eta}_1)^2] + (1 - \lambda) E_{\hat{P}_2} [(\xi - \tilde{\eta}_2)^2]
\]
\[
+ \lambda E_{\hat{P}_1} [\lambda_{\hat{P}_1}^2 (\tilde{\eta}_1 - \tilde{\eta}_2)^2] + (1 - \lambda) E_{\hat{P}_2} [\lambda_{\hat{P}_2}^2 (\tilde{\eta}_1 - \tilde{\eta}_2)^2]
\]
\[
\geq \alpha,
\]
where \( \alpha := \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2 \).

Since \( \rho \) is proper, we have that \( E_{P_\lambda}[(\xi - \lambda P_1 \hat{\eta}_1 - \lambda P_2 \hat{\eta}_2)^2] = \alpha \) if and only if \( \hat{\eta}_1 = \hat{\eta}_2 \) \( P_0 \)-a.s.

On the other hand, since \( \hat{\eta}_1 = \mathbb{E} \hat{P}_1[\xi | \mathcal{C}] = \mathbb{E} P_\lambda[\xi | \mathcal{C}] \frac{d \hat{P}_1}{d P_\lambda} | \mathcal{C} \) and \( \hat{\eta}_2 = \mathbb{E} \hat{P}_2[\xi | \mathcal{C}] = \mathbb{E} P_\lambda[\xi | \mathcal{C}] \frac{d \hat{P}_2}{d P_\lambda} | \mathcal{C} \)

we deduce that \( \lambda P_1 \hat{\eta}_1 + \lambda P_2 \hat{\eta}_2 = E_{P_\lambda}[\xi] \). Since \((\hat{\eta}_1, \hat{P}_1)\) is a saddle point, we have

\[
E_{P_\lambda}[(\xi - \lambda P_1 \hat{\eta}_1)^2] \leq E_{P_\lambda}[(\xi - \hat{\eta}_1)^2] \leq E_{P_\lambda}[(\xi - \hat{\eta}_1)^2] = \alpha.
\]

It yields that \( E_{P_\lambda}[(\xi - E_{P_\lambda}[\xi | \mathcal{C}])^2] = \alpha \). Thus, \( \hat{\eta}_1 = \hat{\eta}_2 \) \( P_0 \)-a.s. \[\blacksquare\]

4 Characterizations of the minimum mean square estimator

In this section, we obtain several characterizations of the minimum mean square estimator.

4.1 The orthogonal projection

If \( \mathcal{P} \) contains only one probability \( P \), then by probability theory, the minimum mean square estimator \( \hat{\eta} \) is just the conditional expectation \( \mathbb{E} P[\xi | \mathcal{C}] \). It is well known that a conditional expectation is an orthogonal projection. In more details, for any \( \eta \in \mathcal{C} \),

\[
\mathbb{E} P[\xi - \mathbb{E} P[\xi | \mathcal{C}]] = 0.
\]

Does the above property still hold when \( \rho \) is a sublinear operator? Note that for any \( \eta \in \mathcal{C} \),

\[
\rho([\xi - \hat{\eta}]\eta) = \sup_{P \in \mathcal{P}} \mathbb{E} P[(\xi - \hat{\eta})\eta] = \sup_{P \in \mathcal{P}} \mathbb{E} P[(\xi - \mathbb{E} P[\xi | \mathcal{C}])\eta] \geq 0.
\]

Thus, in this case, \( \hat{\eta} \) is not the orthogonal projection for \( \rho \). But we notice that \( \inf_{\eta \in \mathcal{C}} \rho([\xi - \hat{\eta}]\eta) = 0 \). This motivates us to introduce the following definition. For any given \( \xi \in \mathcal{F} \), define \( f: \mathcal{C} \mapsto \mathbb{R} \) by

\[
f(\bar{\eta}) = \inf_{\eta \in \mathcal{C}} \rho([\xi - \hat{\eta}]\eta).
\]

Denote the kernel of \( f \) by

\[
\ker(f) := \{ \eta \in \mathcal{C} | f(\eta) = 0 \}.
\]

In the previous section, we prove that the minimum mean square estimator is one element of the set \( \{ \mathbb{E} P[\xi | \mathcal{C}] | P \in \mathcal{P} \} \). In the following, we show that this set can be described by the kernel of \( f \).
Lemma 4.1 If \( \rho \) is a sublinear operator continuous from above on \( \mathcal{F} \), for any given \( \xi \in \mathcal{F} \),

\[
\ker(f) = \{ E_P[\xi|C]; P \in \mathcal{P} \}.
\]

Proof. For any \( P \in \mathcal{P} \) and \( \eta \in \mathbb{C} \), we have

\[
\rho([\xi - E_P[\xi|C]]\eta) \geq E_P([\xi - E_P[\xi|C]]\eta) = 0.
\]

Then

\[
\inf_{\eta \in \mathbb{C}} \rho([\xi - E_P[\xi|C]]\eta) \geq 0.
\]

It is obvious that

\[
\inf_{\eta \in \mathbb{C}} \rho([\xi - E_P[\xi|C]]\eta) \leq \rho([\xi - E_P[\xi|C]]0) = 0,
\]

which leads to \( \inf_{\eta \in \mathbb{C}} \rho([\xi - E_P[\xi|C]]\eta) = 0 \) for any \( P \in \mathcal{P} \). Thus, \( \{ E_P[\xi|C]; P \in \mathcal{P} \} \subset \ker(f) \).

On the other hand, \( \forall \tilde{\eta} \in \ker(f) \), since \( C \) is a convex set and \( \rho \) is a sublinear operator continuous from above, by Mazur-Orlicz theorem, there exists a probability \( \tilde{P} \in \mathcal{P} \) such that

\[
\inf_{\eta \in \mathbb{C}} E_{\tilde{P}}([\xi - \tilde{\eta}]\eta) = \inf_{\eta \in \mathbb{C}} \rho([\xi - \tilde{\eta}]\eta) = 0.
\]

If \( \tilde{\eta} \neq E_{\tilde{P}}[\xi|C] \), then it is easy to find a \( \eta' \in \mathbb{C} \) such that \( E_{\tilde{P}}([\xi - \tilde{\eta}]\eta'] < 0 \). Thus, \( \tilde{\eta}_0 = E_{\tilde{P}}[\xi|C] \) and \( \ker(f) \subset \{ E_P[\xi|C]; P \in \mathcal{P} \} \).

When \( \mathcal{P} \) satisfies stable property which was introduced in [1] (refer to [B.4] in Appendix B), we obtain \( \ker(f) \) of this case in the following theorem.

Theorem 4.2 If \( \rho \) is a sublinear operator continuous from above on \( \mathcal{F} \) and the corresponding \( \mathcal{P} \) is stable, then for given \( \xi \in \mathbb{F} \), \( \ker(f) \) is just the set

\[
\mathbb{B} := \{ \tilde{\eta} \in \mathbb{C} \mid \inf_{P \in \mathcal{P}} E_P[\xi|C] \leq \tilde{\eta} \leq \sup_{P \in \mathcal{P}} E_P[\xi|C] \}. 
\]

Proof. By Lemma 4.1 \( \ker(f) \) is a subset of \( \mathbb{B} \). So we only need to prove \( \mathbb{B} \subset \ker(f) \).

Since \( \mathcal{P} \) is ‘stable’, for any \( \eta \in \mathbb{C} \), we have

\[
\rho([\xi - \tilde{\eta}]\eta) = \rho(\sup_{P \in \mathcal{P}} [\xi - \tilde{\eta}]\eta|C] = \rho(\eta^+ (\sup_{P \in \mathcal{P}} E_P[\xi|C] - \tilde{\eta}) - \eta^- (\inf_{P \in \mathcal{P}} E_P[\xi|C] - \tilde{\eta})),
\]

where \( \eta^+ := \eta \wedge 0 \) and \( \eta^- := - (\eta \wedge 0) \).

It yields that for any \( \tilde{\eta} \in \mathbb{B} \) and \( \eta \in \mathbb{C} \),

\[
\rho([\xi - \tilde{\eta}]\eta) \geq 0.
\]

Then for any \( \tilde{\eta} \in \mathbb{B} \),

\[
\inf_{\eta \in \mathbb{L}} \rho([\xi - \tilde{\eta}]\eta) \geq 0.
\]

It is easy to see that

\[
\inf_{\eta \in \mathbb{L}} \rho([\xi - \tilde{\eta}]\eta) \leq \rho([\xi - \tilde{\eta}]0) = 0.
\]

Thus, \( \inf_{\eta \in \mathbb{C}} \rho([\xi - \tilde{\eta}]\eta) = 0 \) for any \( \tilde{\eta} \in \mathbb{B} \) which implies that \( \mathbb{B} \subset \ker(f) \). This completes the proof. ■
4.2 A sufficient and necessary condition

We give a sufficient and necessary condition for the minimum mean square estimator in this subsection. Especially, we do not need to assume that $\rho$ is continuous from above on $\mathcal{F}$.

**Lemma 4.3** For a given $\xi \in \mathbb{F}$, if $\hat{\eta}$ is an optimal solution of Problem 2.7, then

$$\rho(\hat{\eta}) \geq \rho(\xi - \eta)^2, \quad \forall \eta \in \mathbb{C}.$$ 

**Proof.** For any $\eta \in \mathbb{C}$, define $f: [0,1] \to \mathbb{R}$ by

$$f(\lambda) = \lambda^2 \rho(\xi - \eta)^2 + (1 - \lambda)^2 \rho(\xi - \hat{\eta})^2 + 2\lambda(1 - \lambda)\rho(\xi - \eta)(\xi - \hat{\eta}).$$

It is easy to check that

$$f(\lambda) \geq \rho(\xi - (\lambda\eta + (1 - \lambda)\hat{\eta}))^2 \geq \rho(\xi - \hat{\eta})^2.$$

This implies that $f(\lambda)$ attains the minimum on $[0,1]$ when $\lambda = 0$. Then for any $\eta \in \mathbb{C}$,

$$f'(\lambda)|_{\lambda=0} = -2\rho(\xi - \hat{\eta})^2 + 2\rho(\xi - \eta)(\xi - \hat{\eta}) \geq 0,$$

i.e.,

$$\rho(\xi - \eta)(\xi - \hat{\eta}) \geq \rho(\xi - \hat{\eta})^2, \quad \forall \eta \in \mathbb{C}.$$ 

**Lemma 4.4** For any $\xi_1, \xi_2 \in \mathbb{F}$ such that $\rho(|\xi_1|^p) > 0$ and $\rho(|\xi_2|^q) > 0$, we have

$$\rho(|\xi_1\xi_2|) \leq (\rho(|\xi_1|^p))^{\frac{\frac{1}{p}}{\frac{1}{p} + \frac{1}{q}}} (\rho(|\xi_2|^q))^{\frac{\frac{1}{q}}{\frac{1}{p} + \frac{1}{q}}},$$

where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** Set

$$X = \frac{\xi_1}{(\rho(|\xi_1|^p))^{\frac{1}{p}}}, \quad Y = \frac{\xi_2}{(\rho(|\xi_2|^q))^{\frac{1}{q}}}.$$ 

Since $|XY| \leq \frac{|X|^p}{p} + \frac{|Y|^q}{q}$, then

$$\rho(|XY|) \leq \rho\left(\frac{|X|^p}{p} + \frac{|Y|^q}{q}\right) \leq \rho\left(\frac{|X|^p}{p}\right) + \rho\left(\frac{|Y|^q}{q}\right) = 1,$$

i.e.,

$$\rho(|\xi_1\xi_2|) \leq (\rho(|\xi_1|^p))^{\frac{\frac{1}{p}}{\frac{1}{p} + \frac{1}{q}}} (\rho(|\xi_2|^q))^{\frac{\frac{1}{q}}{\frac{1}{p} + \frac{1}{q}}}.$$

**Remark 4.5** If $\rho$ can be represented by a family of probability measures, then the condition $\rho(|\xi_1|^p) > 0$ and $\rho(|\xi_2|^q) > 0$ can be abandoned.

**Theorem 4.6** Suppose $\inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)^2 > 0$. For a given $\xi \in \mathbb{F}$, $\hat{\eta}$ is the optimal solution of Problem 2.7 if and only if it is the bounded $\mathcal{C}$-measurable solution of the following equation

$$\inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)(\xi - \eta) = \rho(\xi - \hat{\eta})^2. \tag{4.1}$$
Proof. ⇒ Since \( \hat{\eta} \) is the optimal solution of Problem 2.7, by Lemma 4.3,

\[
\inf_{\eta \in \mathbb{C}} \rho(\xi - \hat{\eta})(\xi - \eta) \geq \rho(\xi - \hat{\eta})^2.
\]

It is obvious that

\[
\inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)(\xi - \eta) \leq \rho(\xi - \hat{\eta})^2.
\]

Then \( \hat{\eta} \) is the solution of (4.1).

⇐ If \( \hat{\eta} \in \mathbb{C} \) satisfying equation (4.1), by Lemma 4.4, we have

\[
\rho(\xi - \hat{\eta})^2 = \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)(\xi - \eta)
\]

\[
\leq \inf_{\eta \in \mathbb{C}} (\rho(\xi - \eta)^2)^{\frac{1}{2}} (\rho(\xi - \eta)^2)^{\frac{1}{2}}
\]

\[
= (\rho(\xi - \hat{\eta})^2)^{\frac{1}{2}} [\inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)^2]^{\frac{1}{2}}.
\]

Then \( \rho(\xi - \hat{\eta})^2 \leq \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)^2 \). This completes the proof. ■

Remark 4.7 If \( \rho \) is a linear operator generated by probability measure \( P \), then

\[
E_P[E_P(\xi|\mathcal{C})\eta] = E_P[\xi\eta], \quad \forall \eta \in \mathbb{C}.
\]

This means \( E_P(\xi|\mathcal{C}) \) not only satisfies (4.1) but also satisfies the following equation

\[
\sup_{\eta \in \mathbb{C}} E_P[(\xi - \hat{\eta})(\xi - \eta)] = E_P(\xi - \hat{\eta})^2.
\]

Remark 4.8 If \( \rho \) can be represented by a family of probability measures, then the condition \( \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta)^2 > 0 \) in Theorem 4.6 can be abandoned since Lemma 4.4 still holds for either \( \rho(||\xi_1||_p) = 0 \) or \( \rho(||\xi_2||_q) = 0 \).

5 Properties of the minimum mean square estimator

In this section, we will first give the basic properties of the minimum mean square estimator. Then we explore the relationship between the minimum mean square estimator and the conditional coherent risk measure and conditional \( g \)-expectation.

For a given \( \xi \in \mathbb{F} \), we will denote the minimum mean square estimator with respect to \( \mathcal{C} \) by \( \rho(\xi|\mathcal{C}) \). Then \( \rho(\xi|\mathcal{C}) \) satisfies the following properties.

Proposition 5.1 If the sublinear operator \( \rho \) is continuous from above on \( \mathbb{F} \) and proper, then for any \( \xi \in \mathbb{F} \), we have:

i) If \( C_1 \leq \xi(\omega) \leq C_2 \) for two constants \( C_1 \) and \( C_2 \), then \( C_1 \leq \rho(\xi|\mathcal{C}) \leq C_2 \).

ii) \( \rho(\lambda\xi|\mathcal{C}) = \lambda \rho(\xi|\mathcal{C}) \) for \( \lambda \in \mathbb{R} \).

iii) For each \( \eta_0 \in \mathbb{C} \), \( \rho(\xi + \eta_0|\mathcal{C}) = \rho(\xi|\mathcal{C}) + \eta_0 \).

iv) If under each \( P \in \mathcal{P} \), \( \xi \) is independent of the sub \( \sigma \)-algebra \( \mathcal{C} \), then \( \rho(\xi|\mathcal{C}) \) is a constant.
Proof. i) If $C_1 \leq \xi(\omega) \leq C_2$, then $C_1 \leq E_P[\xi|\mathcal{C}] \leq C_2$ for any $P \in \mathcal{P}$. Since $\rho(\xi|\mathcal{C}) \in \{E_P[\xi|\mathcal{C}]; P \in \mathcal{P}\}$ (refer to the proof of Theorem 3.6), it is easy to see that $\rho(\xi|\mathcal{C})$ lies in $[C_1, C_2]$.

ii) When $\lambda = 0$, the statement is obvious. When $\lambda \neq 0$, we have

$$\lambda^2 \rho(\xi - \frac{\rho(\xi|\mathcal{C})}{\lambda})^2 = \rho(\lambda \xi - \rho(\lambda \xi|\mathcal{C}))^2 = \inf_{\eta \in \mathcal{C}} \rho(\lambda \xi - \eta)^2 = \lambda^2 \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2.$$  

It yields that

$$\rho(\xi - \frac{\rho(\xi|\mathcal{C})}{\lambda})^2 = \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2.$$  

Thus, $\frac{\rho(\xi|\mathcal{C})}{\lambda} = \rho(\xi|\mathcal{C})$ due to the uniqueness result in section 3.

iii) Note that

$$\rho(\xi + \eta_0 - (\eta_0 + \rho(\xi|\mathcal{C})))^2 = \rho(\xi - \rho(\xi|\mathcal{C}))^2 = \inf_{\eta \in \mathcal{C}} \rho(\xi - \eta)^2 = \inf_{\eta \in \mathcal{C}} \rho(\xi + \eta_0 - \eta)^2.$$  

By the uniqueness of the minimum mean square estimator, we have $\rho(\xi + \eta_0|\mathcal{C}) = \eta_0 + \rho(\xi|\mathcal{C})$.

iv) If under each $P \in \mathcal{P}$, $\xi$ is independent of the sub-$\sigma$-algebra $\mathcal{C}$, then $E_P[\xi|\mathcal{C}]$ is a constant for each $P \in \mathcal{P}$. Since $\rho(\xi|\mathcal{C}) \in \{E_P[\xi|\mathcal{C}]; P \in \mathcal{P}\}$, $\rho(\xi|\mathcal{C})$ is also a constant. 

The coherent risk measures were introduced by Artzner et al. [1] and the $g$-expectations were introduced by Peng [3]. The conditional coherent risk measure and some special conditional $g$-expectations can be defined by $\text{ess sup} E_P[\xi|\mathcal{C}]$. In the next two examples, we will show the minimum mean square estimator is different from the conditional coherent risk measure and the conditional $g$-expectation.

Example 5.2 Let $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \Omega\}$ and $\mathcal{C} = \{\emptyset, \Omega\}$. Set $P_1 = \frac{1}{4} \mathbf{1}_{\{\omega_1\}} + \frac{3}{4} \mathbf{1}_{\{\omega_2\}}$, $P_2 = \frac{3}{4} \mathbf{1}_{\{\omega_1\}} + \frac{1}{4} \mathbf{1}_{\{\omega_2\}}$ and $\mathcal{P} = \{\lambda P_1 + (1 - \lambda)P_2; \lambda \in [0, 1]\}$. For each $\xi \in \mathbb{F}$, define

$$\rho(\xi) = \sup_{P \in \mathcal{P}} E_P[\xi].$$  

Set $\xi = 2\mathbf{1}_{\{\omega_1\}} + 8\mathbf{1}_{\{\omega_2\}}$. Then it is easy to check

$$\sup_{P \in \mathcal{P}} E_P[\xi] = \frac{1}{2}$$  

and $\rho(\xi|\mathcal{C}) = E_P[\xi|\mathcal{C}] = 5$, where $\hat{P} = \frac{1}{4} \mathbf{1}_{\{\omega_1\}} + \frac{3}{4} \mathbf{1}_{\{\omega_2\}}$.

Example 5.3 Let $W(\cdot)$ be a standard 1-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P_0)$. The information structure is given by a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, which is generated by $W(\cdot)$ and augmented by all the $P$-null sets. $M^2(0, T; \mathbb{R})$ denotes the space of all $\mathcal{F}_t$-progressively measurable processes $y_t$ such that $E_P \int_0^T |y_t|^2 dt < \infty$. Let us consider the $g$-expectation defined by the following BSDE:

$$y_t = \xi + \int_t^T |z_s| ds - \int_t^T z_s dW(s). \quad (5.1)$$  

where $\xi$ is a bounded $\mathcal{F}_T$-measurable function. Here $g(y, z) = |z|$. By the result in [4], there exists a unique adapted pair $(y, z)$ solves (5.1). We call the solution $y_t$ the conditional $g$-expectation with respect to $\mathcal{F}_t$ and denote it by $E_{[\xi]}(\xi|\mathcal{F}_t)$. 

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Consider the following linear case:

\[ y_t = \xi + \int_t^T \mu_s z_s \, ds - \int_t^T z_s \, dW(s), \quad (5.2) \]

where \(|\mu_s| \leq 1 \, P_0\text{-a.s.}\). By Girsanov transform, there exists a probability \(P^\mu\) such that \(\{y_t\}_{0 \leq t \leq T}\) of (5.2) is a martingale under \(P^\mu\). Set \(\mathcal{P} := \{P^\mu \mid |\mu_s| \leq 1 \, P_0\text{-a.s.}\}\). By Theorem 2.1 in \([6]\),

\[ \mathcal{E}_{\xi|z}(\xi) = \sup_{P^\mu \in \mathcal{P}} E_{P^\mu}[\xi], \quad \forall \xi \in \mathcal{F}_T \]

and

\[ \mathcal{E}_{\xi|z}(\xi|\mathcal{F}_t) = \operatorname{ess}
\sup_{P^\mu \in \mathcal{P}} E_{P^\mu}[\xi|\mathcal{F}_t]. \]

It is easy to see that \(\mathcal{E}_{\xi|z}(\cdot)\) is a sublinear operator. Denote the corresponding minimum mean square estimator by \(\rho_{\xi|z}(\xi|\mathcal{F}_t)\). We claim that the minimum mean square estimator \(\rho_{\xi|z}(\xi|\mathcal{F}_t)\) does not coincide with \(\mathcal{E}_{\xi|z}(\xi|\mathcal{F}_t)\) for all bounded \(\xi \in \mathcal{F}_T\). Otherwise, if for all bounded \(\xi \in \mathcal{F}_T\), \(\rho_{\xi|z}(\xi|\mathcal{F}_t) = \mathcal{E}_{\xi|z}(\xi|\mathcal{F}_t)\), then by the property ii) in Corollary 5.1, we have

\[ \operatorname{ess}
\sup_{P^\mu \in \mathcal{P}} E_{P^\mu}[\xi|\mathcal{F}_t] = \rho_{\xi|z}(\xi|\mathcal{F}_t) = -\rho_{\xi|z}(-\xi|\mathcal{F}_t), \]

Since the set \(\mathcal{P}\) contains more than one probability measure, the above equation can not be true for all bounded \(\xi \in \mathcal{F}_T\). Thus, our claim holds.

In the following, for simplicity, we denote \(\operatorname{ess}
\sup_{P^\mu \in \mathcal{P}} E_{P^\mu}[\xi|\mathcal{C}]\) by \(\eta_{\xi,\text{ess}}\). We first prove that \(\eta_{\xi,\text{ess}}\) is the optimal solution of a constrained mean square optimization problem.

**Definition 5.4** A sublinear operator \(\rho\) is called strictly comparable if for \(\xi_1, \xi_2 \in \mathcal{F}\) satisfying \(\xi_1 > \xi_2\) \(P_0\text{-a.s.}\), we have \(\rho(\xi_1) > \rho(\xi_2)\).

**Proposition 5.5** Suppose that a sublinear operator \(\rho\) is continuous from above, strictly comparable and the representation set \(\mathcal{P}\) of \(\rho\) is stable. Then for a given \(\xi \in \mathcal{F}\), \(\eta_{\xi,\text{ess}}\) is the unique solution of the following optimal problem:

\[ \inf_{\eta \in \mathcal{C}} \sup_{\tilde{\eta} \in \mathcal{C}^+} \rho[\xi - \eta]^2 + \tilde{\eta}[\xi - \eta], \quad (5.3) \]

where \(\mathcal{C}^+\) denotes all the nonnegative elements in \(\mathcal{C}\).

**Proof.** We first show if the optimal solution of (5.3) exists, the optimal solution \(\hat{\eta}\) of (5.3) lies in \(\mathcal{B}\), where \(\mathcal{B} := \{\eta \in \mathcal{C}; \eta \geq \eta_{\xi,\text{ess}}, P_0\text{-a.s.}\}\).

Since

\[
\inf_{\eta \in \mathcal{C}} \sup_{\tilde{\eta} \in \mathcal{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] \\
\leq \sup_{\tilde{\eta} \in \mathcal{C}^+} \rho[(\xi - \eta_{\text{ess}})^2 + \tilde{\eta}(\xi - \eta_{\text{ess}})] \\
\leq \rho[(\xi - \eta_{\text{ess}})^2] + \sup_{\tilde{\eta} \in \mathcal{C}^+} \rho[\tilde{\eta}(\xi - \eta_{\text{ess}})] \\
= \rho[(\xi - \eta_{\text{ess}})^2] < \infty,
\]

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then the value of our problem is finite.

On the other hand, we have

\[ \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[\tilde{\eta}(\xi - \tilde{\eta})] \leq \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \tilde{\eta})^2 + \tilde{\eta}(\xi - \tilde{\eta})]. \]

Since the representation set \( \mathcal{P} \) is ‘stable’, then

\[ \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[\tilde{\eta}(\xi - \tilde{\eta})] = \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[\tilde{\eta}(\eta_{\text{ess}} - \tilde{\eta})]. \]

If \( A := \{ \omega; \tilde{\eta} < \eta_{\text{ess}} \} \) is not a \( \mathcal{P} \)-null set, as \( \rho \) is strictly comparable, we can choose \( \tilde{\eta} \) to let \( \rho[\tilde{\eta}(\xi - \tilde{\eta})] \) larger than any real number. Then \( P_{\mathcal{P}}(A) = 0 \) and \( \hat{\eta} \geq \eta_{\text{ess}} \) \( \mathcal{P} \)-a.s..

For any \( \eta \in \mathbb{B} \) and \( \tilde{\eta} \in \mathbb{C}^+ \), we have

\[ \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] - \rho[(\xi - \eta)^2] \leq \rho[\tilde{\eta}(\xi - \eta)] \leq 0. \]

Then for any \( \eta \in \mathbb{B} \),

\[ \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] \leq \rho[(\xi - \eta)^2]. \]

On the other hand,

\[ \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] \geq \rho[(\xi - \eta)^2 + 0(\xi - \eta)] = \rho[(\xi - \eta)^2]. \]

Then for any \( \eta \in \mathbb{B} \),

\[ \rho[(\xi - \eta)^2] = \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)]. \]

We have

\[ \inf_{\eta \in B} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] = \inf_{\eta \in B} \rho[(\xi - \eta)^2]. \]

For any \( \eta \in B \),

\[ \rho[(\xi - \eta)^2] = \rho[(\xi - \eta_{\text{ess}})^2 + (\eta_{\text{ess}} - \eta)^2 - 2(\eta - \eta_{\text{ess}})(\xi - \eta_{\text{ess}})] \]

\[ \geq \rho[(\xi - \eta_{\text{ess}})^2 + (\eta_{\text{ess}} - \eta)^2] - 2\rho[(\eta - \eta_{\text{ess}})(\xi - \eta_{\text{ess}})] \geq \rho[(\xi - \eta_{\text{ess}})^2], \]

which means \( \eta_{\text{ess}} \) is the best mean-square estimate among \( B \).

On the other hand, for any \( \eta \notin B \), \( \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] \) is equal to \( \infty \). Then \( \eta_{\text{ess}} \) is also the optimal solution of (5.3). The uniqueness is from \( \rho \) is strictly comparable. \( \blacksquare \)

Proposition 5.5 is just equivalent to say \( \eta_{\text{ess}} \) is the unique solution of the following problem:

\[ \inf_{\eta \in \mathbb{C}} \rho[(\xi - \eta)^2] \]

subject to \( \rho[(\eta_{\text{ess}} - \eta)^+] = 0. \)

Using the same method as in Proposition 5.5, we can get \( \text{ess inf}_{\rho \in \mathcal{P}} E_{\rho \in \mathcal{P}}[\xi|\mathcal{C}] \) is the solution of the following problem:

\[ \inf_{\eta \in \mathbb{C}} \rho[(\xi - \eta)^2] \]

subject to \( \rho[(\eta - \text{ess inf}_{\rho \in \mathcal{P}} E_{\rho \in \mathcal{P}}[\xi|\mathcal{C})^+] = 0. \)

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We obtain the following necessary and sufficient condition for $\eta_{ess}$ being the minimum mean square estimator.

**Theorem 5.6** Under the assumptions in Proposition 5.5, for a given $\xi \in \mathbb{F}$, $\eta_{ess}$ is the optimal solution of Problem 2.7 if and only if

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)).
\]

**Proof.** If $\eta_{ess}$ is the optimal solution of Problem 2.7 then

\[
\sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) \geq \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2) = \rho((\xi - \eta_{ess})^2).
\]

On the other hand, by Proposition 5.3

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta_{ess}^2 + \tilde{\eta}(\xi - \eta_{ess}))
\]

\[
\leq \rho((\xi - \eta_{ess})^2) + \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\tilde{\eta}(\xi - \eta_{ess}))
\]

\[
= \rho((\xi - \eta_{ess})^2).
\]

Then

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) \leq \inf_{\eta \in \mathbb{C}} \rho([\xi - \eta_{ess}]^2).
\]

Since

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) \geq \sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta))
\]

is obvious, we have

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)).
\]

Conversely, for any $\tilde{\eta} \in \mathbb{C}^+$,

\[
\inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \inf_{\eta \in \mathbb{C}} \rho(\xi + \frac{\tilde{\eta}}{2} - \eta^2 - \frac{\tilde{\eta}^2}{4}) \leq \inf_{\eta \in \mathbb{C}} \rho(\xi + \frac{\tilde{\eta}}{2} - \eta^2) = \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2).
\]

Then

\[
\sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) \leq \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2).
\]

Since

\[
\sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) \geq \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2 + 0(\xi - \eta)) = \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2)
\]

is obvious, we have

\[
\sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2).
\]

This shows $\sup_{\tilde{\eta} \in \mathbb{C}^+} \inf_{\eta \in \mathbb{C}} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta))$ attains its supremum when $\tilde{\eta} = 0$. By Proposition 5.3, we also know $\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta))$ attains its infimum when $\eta = \eta_{ess}$. Since

\[
\inf_{\eta \in \mathbb{C}} \sup_{\tilde{\eta} \in \mathbb{C}^+} \rho(\xi - \eta^2 + \tilde{\eta}(\xi - \eta)) = \inf_{\eta \in \mathbb{C}} \rho((\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)),
\]

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then
\[ \min_{\eta \in C} \sup_{\tilde{\eta} \in C^+} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)] = \max_{\tilde{\eta} \in C^+} \inf_{\eta \in C} \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)]. \]

By Minimax theorem, \((\eta_{ess}, 0)\) is the saddle point, i.e., for any \(\eta \in C\) and \(\tilde{\eta} \in C^+\), we have
\[ \rho[(\xi - \eta_{ess})^2 + \tilde{\eta}(\xi - \eta_{ess})] \leq \rho[(\xi - \eta)^2 + \tilde{\eta}(\xi - \eta)]. \]

The second inequality means \(\eta_{ess}\) is the optimal solution of Problem 2.7.

### A Some basic results

In this section, some results are given which are used in our paper.

**Theorem A.1** If \(\rho\) is a sublinear operator and \(\mathcal{P}\) is the family of all linear operators dominated by \(\rho\), then
\[ \rho(\xi) = \max_{P \in \mathcal{P}} E_P[\xi], \forall \xi \in \mathbb{F}. \]

**Theorem A.2** Let \(\mathbb{F}\) be a normed space and \(\rho\) be a sublinear operator from \(\mathbb{F}\) to \(\mathbb{R}\) dominated by some scalar multiple of the norm of \(\mathbb{F}\). Then
\[ \{x^* \in \mathbb{F}^*: x^* \leq \rho \text{ on } \mathbb{F}\} \text{ is } \sigma(\mathbb{F}^*, \mathbb{F}) - \text{compact}. \]

We denote by \(\mathbb{F}^*_c\) the set of all countably additive measures.

**Theorem A.3** If \(\mathcal{P}\) is a subset of \(\mathbb{F}^*_c\) which is \(\sigma(\mathbb{F}^*, \mathbb{F}) - \text{compact}\), then there exists a nonnegative \(P_0 \in \mathbb{F}^*_c\) such that the measures in \(\mathcal{P}\) are uniformly \(P_0\)-continuous. i.e., if \(P_0(A) = 0\), then \(\sup_{P \in \mathcal{P}} P(A) = 0\).

### B Some results about coherent risk measure

In this section, we give some basic definitions and results about coherent risk measure which are used in our paper. Reader can refer [1], [2] and [3] for more details. Note that in order to ensure our statements of the entire paper is consistent, the operator we used in our paper is sublinear which is different from the coherent risk measure defined in [1], [2] or [3], in which it is superlinear. Thus it is represented as \(\sup_{P \in \mathcal{P}} E_P\) instead of \(\inf_{P \in \mathcal{P}} E_P\) and the conditional expectation is taken as \(\text{ess sup}_{P \in \mathcal{P}} E_P[\cdot | \mathcal{C}]\) instead of \(\text{ess inf}_{P \in \mathcal{P}} E_P[\cdot | \mathcal{C}]\). Though the definition is different, the methods and results are not affected.

For a given probability set \((\Omega, \mathcal{F}, P_0)\) by the filtration \(\{\mathcal{F}_n\}_{n \geq 1}\) such that \(\mathcal{F} := \bigvee_{n=1}^\infty \mathcal{F}_n\).

**Definition B.1** A map \(\pi : L^\infty(P_0) \rightarrow \mathbb{R}\) is called a coherent risk measure, if it satisfies the following properties:

i) Monotonicity: for all \(X\) and \(Y\), if \(X \geq Y\) then \(\pi(X) \geq \pi(Y)\),

ii) Translation invariance: if \(\lambda\) is a constant then for all \(X\), \(\pi(X + \lambda) = \lambda + \pi(X)\),

iii) Positive homogeneity: if \(\lambda \geq 0\), then for all \(X\), \(\pi(\lambda X) = \lambda \pi(X)\),

iv) Superadditivity: for all \(X\) and \(Y\), \(\pi(X + Y) \geq \pi(X) + \pi(Y)\).
**Definition B.2** The Fatou property for a risk-adjusted value $\pi$ is defined as: for any sequence functions $(X_n)_{n \geq 1}$ such that $\|X_n\|_{L^\infty} \leq 1$ and converging to $X$ in probability, then $\pi(X) \geq \limsup \pi(X_n)$.

**Lemma B.3** For any coherent risk-adjusted value $\pi$ having the Fatou property, there exists a convex $L^1(P_0)$-closed set $P$ of $P_0$-absolutely continuous probabilities on $(\Omega, \mathcal{F})$ called test probabilities, such that $\pi(X) = \inf_{P \in P} E_P[X]$.

**Definition B.4** (Stability) We say that the set $P$ of test probabilities is stable if for elements $Q, Q_0 \in P^e$ with associated martingales $Z^n_0, Z_n$, where $P^e$ denotes the elements in $P$ which is equivalent to $P_0$ and $Z^n_0 := E_{P_0}[\frac{dQ}{dP_0}|\mathcal{F}_n]$. For each stopping time $\tau$ the martingale $L$ defined as $L_n = Z^n_0$ for $n \leq \tau$ and $L_n = Z^n_0 Z_{\tau}$ for $n \geq \tau$ defines an element of $P$.

**Proposition B.5** A bounded $\mathcal{F}_\infty$-random variable $f$ is called a “final value”. We consider $\Phi_\tau(f) = \essinf_{Q \in P^e} E_Q[f|\mathcal{F}_\tau]$, then the following is equivalent:

i) Stability of the set $P$,

ii) Recursivity: for each final value $f$, the family $\{\phi_\nu(f)|\nu$ is a stopping time$\}$ satisfies: for every two stopping times $\sigma \leq \tau$, we have $\Phi_\sigma(f) = \Phi_\sigma(\Phi_\tau(f))$.

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