Algebraic Properties of Qualitative Spatio-Temporal Calculi

Frank Dylla¹, Till Mossakowski¹², Thomas Schneider³, and Diedrich Wolter¹

¹ Collaborative Research Center on Spatial Cognition (SFB/TR 8), Univ. of Bremen
{dylla,till,dwolter}@informatik.uni-bremen.de
² DFKI GmbH, Bremen Till.Mossakowski@dfki.de
³ Department of Mathematics and Computer Science, University of Bremen
tschneider@informatik.uni-bremen.de

Abstract Qualitative spatial and temporal reasoning is based on so-called qualitative calculi. Algebraic properties of these calculi have several implications on reasoning algorithms. But what exactly is a qualitative calculus? And to which extent do the qualitative calculi proposed meet these demands? The literature provides various answers to the first question but only few facts about the second. In this paper we identify the minimal requirements to binary spatio-temporal calculi and we discuss the relevance of the according axioms for representation and reasoning. We also analyze existing qualitative calculi and provide a classification involving different notions of a relation algebra.

1 Introduction

Qualitative spatial and temporal reasoning is a subfield of knowledge representation involved with representations of spatial and temporal domains that are based on finite sets of so-called qualitative relations. Qualitative relations serve to explicate knowledge relevant for a task at hand while at the same time they abstract from irrelevant knowledge. Often, these relations aim to relate to cognitive concepts. Qualitative spatial and temporal reasoning thus link cognitive approaches to knowledge representation with formal methods. Computationally, qualitative spatial and temporal reasoning is largely involved with constraint satisfaction problems over infinite domains where qualitative relations serve as constraints. Typical domains include, on the temporal side, points and intervals and, on the spatial side, regions or oriented points in the Euclidean plane.

In the past decades, a vast number of qualitative representations have been developed that are commonly referred to as qualitative calculi (see [18] for a recent overview). Yet the literature provides us with several definitions of what a qualitative calculus exactly is. Nebel and Scivos [29] have introduced the rather general and weak notion of a constraint algebra, which is a set of relations closed under converse, finite intersection, and composition. Ligozat and Renz [19] focus

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on so-called non-associative algebras, which are relation algebras without associativity axioms, and which have a much richer structure. Both approaches assume that the converse operation is strong, which is not the case for calculi like the Cardinal Direction (Relations) Calculus (CDR) \cite{37} or its recently introduced rectangular variant (RDR) \cite{28}.

The goal of this paper is to relate the existing definitions and to identify the essential representation-theoretic properties that characterize a qualitative calculus. It is achieved by the following contributions.

- We propose a definition of a qualitative calculus that includes existing spatio-temporal calculi by weakening the conditions usually imposed on the converse and composition relation (Section 2).
- We generalize the notions of constraint algebra and non-associative algebra to cover calculi with weak converse (Section 3.1).
- We discuss the role of algebraic properties of calculi for spatial reasoning, especially in connection with general-purpose reasoning tools like SparQ \cite{42, 41} and GQR \cite{10} (Section 3.2).
- We experimentally evaluate the algebraic properties of calculi and derive a reasoning procedure that is sensitive to these properties (Section 4).
- We examine information preservation properties of calculi during reasoning, i.e., how general relations become after several compositions (Section 5).

2 Qualitative Representations

In this section, we formulate minimal requirements to a qualitative calculus, discuss their relevance to spatio-temporal representation and reasoning, and list existing calculi. We restrict ourselves to calculi with binary relations because we want to examine their algebraic properties using the notion of a relation algebra, which is best understood for binary relations.

2.1 Requirements to Qualitative Representations

We start with minimal requirements used in the literature. Let us first fix some notation. Let $r, s, t$ range over binary relations over a non-empty universe $U$, i.e., $r \subseteq U \times U$. We use $\cup, \cap, \complement, \circ$ and $\circ$ to denote the union, intersection, complement, converse, and composition of relations, as well as the identity and universal relations $\text{id} = \{(u, u) | u \in U\}$ and $U = U \times U$. A relation $r \subseteq U \times U$ is called serial if, for every $u \in U$, there is some $v \in U$ such that $(u, v) \in r$.

Ligozat and Renz \cite{19} note that most spatial and temporal calculi are based on a set of JEPD (jointly exhaustive and pairwise disjoint) relations. The following definition is standard in the QSR literature \cite{19, 4}.

**Definition 1.** Let $U$ be a non-empty universe and $R$ a set of non-empty binary relations over $U$. $R$ is called a set of JEPD relations over $U$ if the relations in $R$ are pairwise disjoint and $U \times U = \bigcup_{r \in R} r$. 
An abstract partition scheme is a pair \((U, R)\) where \(R\) is a set of JEPD relations over \(U\). \((U, R)\) is called a partition scheme \([19]\) if \(R\) contains the identity relation \(\text{id}\) and, for every \(r \in R\), there is some \(s \in R\) such that \(r \circ s = s\).

The universe \(U\) represents the set of all spatial or temporal entities, and \(R\) being a set of JEPD relations ensures that each two entities are in exactly one relation from \(R\). Incomplete information about two entities is modelled by taking the union of base relations, with the universal relation (the union of all base relations) representing that no information is available. Disjointness of the base relations ensures that there is a unique way to represent an arbitrary relation, and exhaustiveness ensures that the empty relation can never occur in a consistent set of constraints (which are defined in Section 2.2).

Ligozat and Renz \([19]\) base their definition of a qualitative calculus on the notion of a partition scheme. This excludes calculi like CDR and RDR which do not have strong converses. Hence, we take a more general approach based on the notion of an abstract partition scheme. This accommodates existing calculi with these weaker properties: some existing spatio-temporal representations do not require an identity relation, and some representations are deliberately kept coarse and thus do not guarantee that the converse of a base relation is again a (base) relation. Furthermore, the computation of the converse operation may be easier when weaker properties are postulated. The same rationale applies to the composition operation. Thus, the following definition of a spatial calculus, based on abstract partition schemes, contains minimal requirements.

**Definition 2.** A qualitative calculus with binary relations is a tuple \((\text{Rel}, \text{Int}, \circ, \odot)\) with the following properties.

- \(\text{Rel}\) is a finite, non-empty set of base relations. The subsets of \(\text{Rel}\) are called relations. We use \(r, s, t\) to denote base relations and \(R, S, T\) to denote relations.
- \(\text{Int} = (U, \varphi)\) is an interpretation with a non-empty universe \(U\) and a map \(\varphi : \text{Rel} \to 2^{U \times U}\) with \((U, \{\varphi(r) \mid r \in \text{Rel}\})\) being a weak partition scheme. The map \(\varphi\) is extended to arbitrary relations by setting \(\varphi(R) = \bigcup_{r \in R} \varphi(r)\) for every \(R \subseteq \text{Rel}\).
- The converse operation \(\circ\) is a map \(\circ : \text{Rel} \to 2^{\text{Rel}}\) that satisfies
  \[
  \varphi(r) \supseteq \varphi(r) \circ \varphi(r) \quad (1)
  \]
  for every \(r \in \text{Rel}\). The operation \(\circ\) is extended to arbitrary relations by setting \(R\circ = \bigcup_{r \in R} R\circ\) for every \(R \subseteq \text{Rel}\).
- The composition operation \(\odot\) is a map \(\odot : \text{Rel} \times \text{Rel} \to 2^{\text{Rel}}\) that satisfies
  \[
  \varphi(r \circ s) \supseteq \varphi(r) \odot \varphi(s) \quad (2)
  \]
  for all \(r, s \in \text{Rel}\). The operation \(\odot\) is extended to arbitrary relations by setting \(R \circ S = \bigcup_{r \in R} \bigcup_{s \in S} r \circ s\) for every \(R, S \subseteq \text{Rel}\).

We call Properties (1) and (2) abstract converse and abstract composition, following Ligozat’s naming \([17]\). Our notion of a qualitative calculus makes weaker
requirements on the converse operation than Ligozat and Renz’s notions of a weak representation \[17, 19\]. We have already discussed a rationale behind choosing these “weaker than weak” variants and will name another in Section 2.2. On the other hand, our notion makes stronger requirements on the converse than Nebel and Scivos’s notion of a constraint algebra \[29\]. In Section 2.2, we will discuss a reason for adopting the minimal requirements we pose to abstract converse and composition. The following definition gives the stronger variants of converse and composition existing in the literature.

**Definition 3.** Let \( C = (\text{Rel}, \text{Int}, \circ, \diamond) \) be a qualitative calculus. C has weak converse if, for all \( r \in \text{Rel}: \)

\[
r^\circ = \bigcap \{ S \subseteq \text{Rel} | \varphi(S) \supseteq \varphi(r)^\circ \} \tag{3}
\]

C has strong converse if, for all \( r \in \text{Rel}: \)

\[
\varphi(r^\circ) = \varphi(r)^\circ \tag{4}
\]

C has weak composition if, for all \( r, s \in \text{Rel}: \)

\[
r \circ s = \bigcap \{ T \subseteq \text{Rel} | \varphi(T) \supseteq \varphi(r) \circ \varphi(s) \} \tag{5}
\]

C has strong composition if, for all \( r, s \in \text{Rel}: \)

\[
\varphi(r \circ s) = \varphi(r) \circ \varphi(s) \tag{6}
\]

The following fact captures that Properties (1)–(6) immediately carry over to arbitrary relations; the straightforward proof is given in the appendix A. It has consequences for efficient spatio-temporal reasoning, which are explained in Section 2.2.

**Fact 4** Given a qualitative calculus \((\text{Rel}, \text{Int}, \circ, \diamond)\) and relations \(R, S \subseteq \text{Rel}, \) the following hold:

\[
\varphi(R^\circ) \supseteq \varphi(R)^\circ \tag{7}
\]

\[
\varphi(R \circ S) \supseteq \varphi(R) \circ \varphi(S) \tag{8}
\]

If \(C\) has weak converse, then, for all \(R \subseteq \text{Rel}: \)

\[
R^\circ = \bigcap \{ S \subseteq \text{Rel} | \varphi(S) \supseteq \varphi(R)^\circ \} \tag{9}
\]

If \(C\) has strong converse, then, for all \(R \subseteq \text{Rel}: \)

\[
\varphi(R^\circ) = \varphi(R)^\circ \tag{10}
\]

If \(C\) has weak composition, then, for all \(R, S \subseteq \text{Rel}: \)

\[
R \circ S = \bigcap \{ T \subseteq \text{Rel} | \varphi(T) \supseteq \varphi(R) \circ \varphi(S) \} \tag{11}
\]

If \(C\) has strong composition, then, for all \(R, S \subseteq \text{Rel}: \)

\[
\varphi(R \circ S) = \varphi(R) \circ \varphi(S) \tag{12}
\]
Comparing Definitions [18] with the basic notions of a qualitative calculus in [19], a \textit{weak representation}, is a calculus with identity relation, strong converse and abstract composition, and a \textit{representation} is a weak representation with strong composition and an injective map $\varphi$. In contrast, our basic notion of a qualitative calculus does not require an identity relation, and it only requires abstract converse and composition.

\subsection{Spatio-Temporal Reasoning}

The most important flavor of spatio-temporal reasoning is constraint-based reasoning. Like with a classical constraint satisfaction problem (CSP), we are given a set of variables and constraints. The task of constraint satisfaction is to decide whether there exists a valuation of all variables that satisfies the constraints. In calculi for spatio-temporal reasoning, variables range over the specific spatial or temporal domain of a qualitative representation. The relations defined by the calculus serve as constraint relations. More formally, we have:

\begin{definition}[QCSP] Let $(\mathcal{Rel}, \mathcal{Int}, \mathcal{˘}, \circledast)$ be a binary qualitative calculus with $\mathcal{Int} = (\mathcal{U}, \varphi)$ and let $X$ be a set of variables ranging over $\mathcal{U}$. A \textit{qualitative constraint} is a formula $x_i R_j x_k$ with variables $x_i, x_k \in X$ and relation $R_j \in \mathcal{Rel}$. We say that a valuation $\psi : X \rightarrow \mathcal{U}$ satisfies $x_i R_j x_k$ if $(\psi(x_i), \psi(x_k)) \in \varphi(R_j)$ holds.

A \textit{qualitative constraint satisfaction problem} (QCSP) is the task to decide whether there is a valuation $\psi$ for a set of variables satisfying a set of constraints.
\end{definition}

In the following we use $X$ to refer to the set of variables and $r_{x,y}$ stands for the constraint relation between variables $x$ and $y$. For simplicity and wlog, it is assumed that for every pair of variables exactly one constraint relation is given.

Several techniques originally developed for finite domain CSP can be adapted to spatio-temporal QCSPs. Since deciding CSP instances is already NP-complete for search problems with finite domains, heuristics are important. One particular valuable technique is constraint propagation which aims at making implicit constraints explicit in order to identify variable assignments that would violate some constraint. By pruning away these variable assignments, a consistent valuation can be searched more efficiently. A common approach is to enforce $k$-consistency.

\begin{definition} A CSP is $k$-consistent if for all subsets of variables $X' \subset X$ with $|X'| = k - 1$ we can extend any valuation of $X'$ that satisfies the constraints to a valuation of $X' \cup \{x\}$ also satisfying the constraints, where $x \in X \setminus X'$ is any additional variable.
\end{definition}

QCSPs are naturally 1-consistent as the domains are infinite and there are no unary constraints. A QCSP is 2-consistent if $r_{x,y} = r_{y,x}$ and $r_{x,y} \neq \emptyset$ as relations are typically serial. A 3-consistent QCSP is also called \textit{path-consistent} and Definition 6 can be rewritten using compositions as

\begin{equation} r_{x,y} \subseteq \bigcap_{z \in X} r_{x,z} \circledast r_{z,y} \end{equation}
and we can enforce the 3-consistency by iterating the refinement operation

\[ r_{x,y} \leftarrow r_{x,y} \cap r_{x,z} \circ r_{z,y} \quad (14) \]

for all variables \( x, y, z \in X \) until a fix point is reached. This procedure is known as the path-consistency algorithm \[5\]. For finite constraint networks the algorithm always terminates since the refinement operation is monotone and there are only finitely many relations.

If a qualitative calculus does not provide strong composition, iterating Equation (14) is not possible as it would lead to relations not contained in \( \text{Rel} \). It is however straightforward to weaken Equation (14) using weak composition.

\[ r_{x,y} \leftarrow r_{x,y} \cap r_{x,z} \diamond r_{z,y} \quad (15) \]

This procedure is called enforcing \textit{algebraic closure} or \textit{a-closure} for short. The reason why, in Definition \[2\] we require composition to be at least abstract is that the underlying inequality guarantees that reasoning via a-closure is sound.

Enforcing \( k \)-consistency or algebraic closure does not change the solutions of a CSP as only impossible valuations are removed. If during application of Equation (15) an empty relation occurs, the QCSP is thus known to be inconsistent. In contrast, an algebraically closed QCSP may not be consistent though. However, for several qualitative calculi (or at least sub-algebras thereof) algebraic closure and consistency coincide.

Though we speak about composition in the following two paragraphs, the same statements hold for converse.

Fact \[3\] has the consequence that the composition operation of a calculus is uniquely determined if the composition of each pair of base relations is given. This information is usually stored in a table, the \textit{composition table}. Then, computing the composition of two arbitrary relations is just a matter of table look-ups which allows algebraic closure to be enforced efficiently. Speaking in terms of composition tables, abstract composition implies that each cell corresponding to \( r \circ s \) contains \textit{at least} those base relations \( t \) whose interpretation intersects with \( \varphi(r) \circ \varphi(s) \). In addition, weak composition implies that each cell contains \textit{exactly} those \( t \). If composition is strong, then \( \text{Rel} \) and \( \varphi \) even have to ensure that whenever \( \varphi(t) \) intersects with \( \varphi(r) \circ \varphi(s) \), it is contained in \( \varphi(r) \circ \varphi(s) \) – i.e., the composition of the interpretation of any two base relations has to be the union of interpretations of certain base relations.

### 2.3 Existing Qualitative Spatio-Temporal Representations

This paper is concerned with properties of binary spatio-temporal calculi that are described in the literature and implemented in the spatial representation and reasoning tool \textit{SparQ} \[42, 41\]. Table \[3\] lists these calculi.
| Name                                              | Ref. | Domain                  | #BR | RM       |
|---------------------------------------------------|------|-------------------------|-----|----------|
| 9-Intersection                                    | 8    | simple 2D regions       | 8   | I        |
| Allen’s interval relations                        | 4    | intervals (order)       | 13  | A        |
| Block Algebra                                     | 2    | n-dimensional blocks    | 13  | A        |
| Cardinal Dir. Calculus CDC                        | 9, 10| directions (point abstr.)| 9   | A        |
| Cardinal Dir. Relations CDR                       | 30   | regions                 | 218 | P        |
| CycOrd, binary CYC6                               | 13   | oriented lines          | 4   | U        |
| Dependency Calculus                               | 31   | points (partial order)  | 5   | A        |
| Dipole Calculus* DRAf                            | 24, 23| directions from line segm.| 72 | I        |
| DRA-connectivity                                  | 43   | connectivity of line segm.| 7  | U        |
| Geometric Orientation                             | 7    | relative orientation    | 4   | U        |
| INDU                                              | 30   | intervals (order, rel. dur.n) | 25 | P        |
| OPRAm, m = 1, . . . , 8                           | 22, 27| oriented points         | 4m · (4m + 1) |         |
| (Oriented Point Rel. Algebra)                     |      |                         |     | I        |
| Point Calculus                                    | 40   | points (total order)    | 3   | A        |
| Qualitat. Traject. Calc. QTCB11                   | 35, 39| moving point obj.s in 1D| 9   | U        |
| QTCB12                                           |      |                         |     |          |
| QTCB21                                           |      |                         |     |          |
| QTCB22                                           |      |                         |     |          |
| QTCc12                                           |      |                         |     |          |
| QTCc22                                           |      |                         |     |          |
| Region Connection Calc. RCC-5                     | 32   | regions                 | 5   | A        |
| RCC-8                                             | 32   | regions                 | 8   | A        |
| Rectangular Cardinal Rel.s RDR                    | 25   | regions                 | 36  | A        |
| Star Algebra STAR                                 | 34   | directions from a point  | 9   | P        |

*Variant DRAc is not based on a weak partition scheme – JEPD is violated [23].

#BR: number of base relations
RM: reasoning method used to decide consistency of CSPs with base relns only:

- **A**-closure; **P**olynomial: reducible to linear programming;
- **I**ntractable (assuming P ≠ NP); **U**nknown

Table 1. Overview of the binary calculi tested.

3 Relation Algebras

3.1 Definition

If we focus our attention on spatio-temporal calculi with binary relations, it is reasonable to ask whether they are relation algebras (RAs). If a calculus is a RA, it is guaranteed to have properties that allow several optimizations in constraint reasoners. For example, associativity of the composition operation \( \circ \) ensures that, if the reasoner encounters a path \( ArBsCtD \) of length 3, then the relation between \( A \) and \( D \) can be computed “from left to right”. Without associativity, \( (r \circ s) \circ t \) as well as \( r \circ (s \circ t) \) would have to be computed. RAs have been considered in the literature for spatio-temporal calculi [19, 6, 25].

An (abstract) RA is defined in [21]; here we use the symbols \( \cup, \circ, \), and \( id \) instead of +, ;, and 1. Let \( A \) be a set containing \( id \) and 1, and let \( \cup, \circ, \) be binary and \( `-`, \( `\` \) unary operations on \( A \). The relevant axioms (R1–R10, WA, SA, and PL)
are given in Table 2. All axioms except PL can be weakened to only one of two inclusions, which we denote by a superscript \( \supseteq \) or \( \subseteq \). For example, \( R_7^\supseteq \) denotes \((r^\supseteq)\supseteq r\). Likewise, we use \( PL^{\supseteq} \) and \( PL^{\subseteq} \). Then, \( \mathfrak{A} = (A, \cup, \cap, \circ, \bar{\circ}, \circlearrowright) \) is a

- non-associative relation algebra (NA) if it satisfies Axioms \( R_1-R_3, R_5-R_{10} \);
- semi-associative relation algebra (SA) if it is an NA and satisfies Axiom SA;
- weakly associative relation algebra (WA) if it is an NA and satisfies WA,
- relation algebra (RA) if it satisfies \( R_1-R_{10} \).

for all \( r, s, t \in A \). Every RA is a WA; every WA is an SA; every SA is an NA.

In the literature, a different axiomatization is sometimes used, for example in \[19\]. The most prominent difference is that \( R_{10} \) is replaced by PL, “a more intuitive and useful form, known as the Peircean law or De Morgan’s Theorem K” \[12\]. It is shown in \[12, Section 3.3.2\] that, given \( R_1-R_3, R_5, R_7-R_9 \), the axioms \( R_{10} \) and PL are equivalent. The implication \( PL \Rightarrow R_{10} \) does not need \( R_5 \) and \( R_8 \).

Furthermore, Table 2 contains the redundant axiom \( R_{6} \) because it may be satisfied when some of the other axioms are violated. It is straightforward to establish that \( R_4 \) and \( R_6 \) are equivalent given \( R_7 \) and \( R_9 \), see the appendix.

Due to our minimal requirements to a qualitative calculus given in Def. 2 certain axioms are always satisfied; see the appendix for a proof of the following

**Fact 7** Every qualitative calculus satisfies \( R_1-R_3, R_5, R_7^\supseteq, R_8, WA^{\supseteq}, SA^{\supseteq} \) for all (base and complex) relations. This axiom set is maximal: each of the remaining axioms in Table 2 is not satisfied by some qualitative calculus.

### 3.2 Discussion of the Axioms

We will now discuss the relevance of the above axioms for spatio-temporal representation and reasoning. Due to Fact 7 we only need to consider axioms \( R_4, R_6, R_7, R_9, R_{10} \) (or PL) and their weakenings \( R_{6}, SA, WA \).
Axiom $R_4$ is helpful for modelling. It allows for writing chains of compositions without parentheses, which have an unambiguous meaning. For example, consider the following statement in natural language about the relative length and location of two intervals $A$ and $D$.

Interval $A$ is before some equally long interval that is contained in some longer interval that meets the shorter $D$. This statement is just a conjunction of relations between $A$, the unnamed intermediary intervals $B, C$, and $D$. When we evaluate it, it intuitively does not matter whether we give priority to the composition of the relations between $A, B$ and $B, C$ or to the composition of the relations between $B, C$ and $C, D$.

However, INDU does not satisfy Axiom $R_4$ and, therefore, here the two ways of parenthesizing the above statement lead to different relations between $A$ and $D$. This behavior is sometimes attributed to the absence of strong composition, which we will refute in Section 4. Conversely, strong composition implies $R_4$ since composition of binary relations over $U$ is associative.

Fact 8 Let $C = (\text{Rel}, \text{Int}, \cdot, \cdot)$ be a qualitative calculus with strong composition. Then $C$ satisfies $R_4$.

Note that INDU still satisfies the weakenings SA and WA of $R_4$, and we already know from Fact 7 that the inequalities $SA^2$ and $WA^2$ are always satisfied.

Furthermore, Axiom $R_4$ is useful for optimizing reasoning algorithms: suppose a scenario that contains the constraints \{\(W \triangleright X, X \triangleleft Y, Y \triangleleft Z, W \triangleright Z\)\} with variables $W, X, Y, Z$ needs to be checked for consistency. If one of the inclusions $R_4^\leq$ is satisfied – say, $r \circ (s \circ t) \subseteq (r \circ s) \circ t$ – then it suffices to compute the “finer” composition result $r \circ (s \circ t)$ and check whether it contains $r'$. Otherwise, both results have to be computed and checked for containment of $r'$.

In case neither $R_4^\leq$ nor $R_4^\geq$ is satisfied, it is possible to establish associativity, at the cost of coarsening the composition operation and thereby reducing the “information content” inherent in the composition table, but with the benefit of providing a sound approximation with the advantages discussed so far. This can be done as follows. If there is some triple $(r, s, t)$ of base relations that violates associativity, for example, $r \circ (s \circ t) \subsetneq (r \circ s) \circ t$, then enrich the table entries that are used to compute $r \circ (s \circ t)$ with the base relations that are missing to reach $r \circ (s \circ t) = (r \circ s) \circ t$. Repeat this step until no more violations are found. This procedure terminates – in the worst case, it runs until all entries contain the universal relation. The resulting table represents an abstract composition operation that satisfies associativity, but its “information content” is lower than that of the original table because, by adding relations to certain entries, the disjunctions represented by these entries have been enlarged.

$R_6$ and $R_9$. Axioms $R_6$ and $R_9$ do not seem to play a significant role in (optimizing) satisfiability checking, but the presence of an id relation is needed for the standard reduction from the correspondence problem to satisfiability: to test whether a constraint system admits the equality of two variables $x, y$, one can add an id-constraint between $x, y$ and test the extended system for satisfiability.

Furthermore, the absence of an id relation may lead to an earlier loss of precision. For example, assume two variants of the 1D Point Calculus [40]: $PC_w$
with the relations less than ($<$), equal ($=$), and greater than ($>$), interpreted as
the natural relations $<$, $=$, $>$ over the domain of the reals, and its approximation
$PC\approx$ with the relations less than ($<$), approximately equal ($\approx$), and greater than
($>$), where $\approx$ is interpreted as the set of pairs of points whose distance is below
a certain threshold. Then, $=$ is the id-relation of $PC=\approx$ and $\approx \circ \approx$ results in $\{=\}$,
whereas $PC\approx$ has no id-relation and $\approx \circ \approx$ results in the universal relation.

$R_7$ and $R_9$. These axioms allow for certain optimizations in decision procedures
for satisfiability based on algebraic operations like algebraic closure. If $R_7$ holds,
the reasoning system does not need to store both constraints $ArB$ and $B r' B$,
as constraint relations can be reconstructed if needed. Similarly, $R_9$ grants that,
when enforcing algebraic closure by using Equation (15) to refine constraints
between variable $A$ and $B$, it is sufficient to compute composition once and, after
applying converse, reuse it to refine the constraint between $B$ and $A$ too.

Current reasoning algorithms and their implementations use the described
 optimizations; they produce incorrect results for calculi violating $R_7$ or $R_9$.

$R_{10}$ and $PL$. These axioms reflect that the relation symbols of a calculus indeed
represent binary relations, i.e., pairs of elements of a universe. This can be
explained from two different points of view.

1. If binary relations are considered as sets, $R_{10}$ is equivalent to $r \circ r \subseteq \bar{s}$. If
   we further assume the usual set-theoretic interpretation of the composition of
two relations, the above inclusion reads as: For any $X,Y$, if $Z r X$ for some
$Z$ and, $Z r U$ implies not $U s Y$ for any $U$, then not $X s Y$. This is certainly
true for binary relations over any universe because $x$ is one such $u$.
2. Under the same assumptions, each side of $PL$ says (in a different order) that
   there can be no triangle $X r Y,Y s Z,Z t X$. The equality then means that
   the “reading direction” does not matter, see also [6]. This allows for reducing
   nondeterminism in the a-closure procedure, as well as for efficient refinement
   and enumeration of consistent scenarios.

3.3 Prerequisites for Being a Relation Algebra

The following correspondence between properties of a calculus and notions of a
relation algebra is due to Ligozat and Renz [19].

**Proposition 9.** Every calculus $C$ based on a partition scheme is an NA. If, in
addition, the interpretations of the base relations are serial, then $C$ is an SA.

Furthermore, $R_7$ is equivalent to the requirement that a calculus has strong
converse. This is captured by the following lemma.

**Lemma 10.** Let $C = (\text{Rel, Int, } \circ)$ be a qualitative calculus. Then the following
properties are equivalent.

1. $C$ has strong converse.
2. Axiom $R_7$ is satisfied for all base relations $r \in \text{Rel}$.
3. Axiom $R_7$ is satisfied for all relations $R \subseteq \text{Rel}$.
Proof. Items (2) and (3) are equivalent due to distributivity of \( \sim \) over \( \cup \), which is introduced with the cases for non-base relations in Definition 2.

For “(1) \( \Rightarrow \) (2)”, the following chain of equalities, for any \( r \in \text{Rel} \), is due to \( C \) having strong converse: 
\[
\varphi(r \sim) = \varphi(r\sim) = \varphi(r) = \varphi(r).
\]
Since \( \text{Rel} \) is based on JEPD relations and \( \varphi \) is injective, this implies that \( r \sim = r \).

For “(2) \( \Rightarrow \) (1)”, we show the contrapositive. Assume that \( C \) does not have strong converse. Then \( \varphi(r) \supseteq \varphi(r\sim) \), for some \( r \in \text{Rel} \); hence \( \varphi(r\sim) \supseteq \varphi(r) \sim \). We can now modify the above chain of equalities replacing the first two equalities with inequalities, the first of which is due to Requirement (1) in the definition of the converse (Def. 2): 
\[
\varphi(r \sim) \supseteq \varphi(r\sim) \supseteq \varphi(r) = \varphi(r).
\]
Since \( \varphi(r \sim) \neq \varphi(r) \), we have that \( r \sim \neq r \).

\( \square \)

4 Algebraic Properties of Existing Calculi

In this section, we report on tests for algebraic properties we have performed on spatio-temporal calculi. We want to answer the following questions. (1) Which existing calculi correspond to relation algebras? (2) Which weaker notions of relation algebras correspond to calculi that do not fall under (1)?

We examined the corpus of the 31 calculi\(^4\) listed in Table 1. This selection is due to the restriction to calculi with (a) binary relations – because the notion of a relation algebra is best understood for binary relations – and (b) an existing implementation in \( \text{SparQ} \).

To answer Questions (1) and (2), we use the axioms for relation algebras listed in Table 2 using both the heterogeneous tool set \( \text{HETS} \) [26] and \( \text{SparQ} \). Due to Fact 7, it suffices to test Axioms \( R_4, R_6, R_7, R_9, R_{10} \) (or PL) and, if necessary, the weakenings \( SA, WA \), and \( R_{6l} \). The weakenings are relevant to capture weaker notions such as semi-associative or weakly associative algebras, or algebras that violate either \( R_6 \) or some of the axioms that imply the equivalence of \( R_6 \) and \( R_{6l} \). Because all axioms except \( R_{10} \) contain only operations that distribute over the union \( \cup \), it suffices to test them for base relations only. Therefore, we have written a CASL specification of \( R_4, R_6, R_7, R_9, \text{PL}, \text{SA}, \text{WA} \), and \( R_{6l} \), and used a \( \text{HETS} \) parser that reads the definitions of the above listed calculi in \( \text{SparQ} \) to test them against our CASL specification. In addition, we have tested all definitions against \( R_4, R_6, R_7, \text{PL}, \text{SA}, \text{WA} \), and \( R_{6l} \) using \( \text{SparQ} \)'s built-in function \text{analyze-calculus}. A part of the calculi have already been tested by Florian Mossakowski [25]; he comprehensively reports on the outcome of these tests, and on repairs made to the composition table where possible.

The results of our and Mossakowski’s tests are summarized in Table 3; details are listed in the appendix. With the exceptions of QTC, Cardinal Direction Relations (CDR) and Rectangular Direction Relations (RDR), all tested calculi are at least semi-associative relation algebras; most of them are even relation algebras. Hence, these calculi enjoy the advantages for representation and reasoning optimizations discussed in Section 3.2.

\(^4\) For the parameterized calculi DRA, OPRA, QTC, we count every variant separately.
| Calculus                          | Tests  | $R_4$ | SA | WA | $R_5$ | $R_7$ | $R_9$ | PL | $R_{10}$ |
|---------------------------------|--------|-------|----|----|-------|-------|-------|----|----------|
| Allen                           | MHS    | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| Block Algebra                   | HS     | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| Cardinal Direction Calculus     | MHS    | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| CYC$_{h}$, Geometric Orientation| HS     | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| DRA$_{fp}$, DRA-conn.           | HS     | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| Point Calculus                  | HS     | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| RCC-5, Dependency Calc.         | MHS    | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| RCC-8, 9-Intersection           | MHS    | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| STAR$_4$                        | HS     | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| DRA$_f$                         | MHS    | 19    | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| INDU                            | MHS    | 12    | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| OPRA$_n$, $n \leq 8$            | MHS    | 21–91 | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| QTC$_{max}$                     | MHS    | ✓     | ✓  | ✓  | ✓     | ✓     | ✓     | ✓  | ✓        |
| QTC$_{C21}$                     | HS     | 55    | ✓  | ✓  | 99    | 99    | ✓     | ✓  | <1, 1   |
| QTC$_{C22}$                     | HS     | 79    | ✓  | ✓  | 99    | 99    | ✓     | ✓  | <1, 1   |
| Rectang. Direction Relations    | HS     | ✓     | ✓  | ✓  | 97    | 92    | 89    | 66 | 7, 52    |
| Cardinal Direction Relations    | HS     | 28    | ✓  | ✓  | 99    | 99    | 98    | 12 | <1, 88   |

*calculus was tested by: M = HETS, H = HETS, S = SparQ

$b$21%, 69%, 78%, 83%, 86%, 88%, 90%, 91% for OPRA$_n$, $n = 1, \ldots, 8$

| Table 3. Overview of calculi tested and their properties. The symbol “✓” means that the axiom is satisfied; otherwise the percentage of counterexamples (relations, pairs or triples violating the axiom) is given. |

procedures, which already implement the optimizations described for $R_7$ and $R_9$, yield correct results for these calculi, and they could be optimized further by implementing the optimizations described for $R_4$, $R_{10}$, and PL.

The three groups of calculi that are SAs but not RAs are the Dipole Calculus variants DRA$_f$ (variants DRA$_{fp}$ and DRA-connectivity are even RAs!), as well as INDU and OPRA$_m$ for $m = 1, \ldots, 8$. These calculi do not even satisfy one of the inclusions $R_4 \supseteq R_7$ and $R_7 \subseteq R_4$, which implies that the reasoning optimizations described in Section 3.2 for Axiom $R_4$ cannot be applied, but this is the only disadvantage of these calculi over the others. Our observations suggest that the meaning of the letter combination “RA” in the abbreviations “DRA” and “OPRA” should stand for “Reasoning Algebra”, not for “Relation Algebra”.

In principle, it cannot be completely ruled out that associativity is reported to be violated due to errors in either the implementation of the respective calculus or the experimental setup. This even applies to non-violations, although it is much more likely that errors cause sporadic violations than systematic non-violations. In the case of DRA$_f$, INDU and OPRA$_m$, $m = 1, \ldots, 8$, the relatively high percentage of violations make implementation errors seem unlikely to be the cause. However, to obtain certainty that these calculi indeed violate $R_4$, one has to find concrete counterexamples and verify them using the original definition of the respective calculus. For DRA$_f$ and INDU, this has been done in the literature [25, 3]. Interestingly, the violation of associativity has been attributed to the absence of strong converse and strong composition, respectively. We remark,
however, that the latter cannot be responsible because, for example, DRA\textsubscript{fp} has an associative, but only weak, composition operation. While DRA\textsubscript{fp} has been proven to be associative due to strong composition in \cite{23}, for OPRA\textsubscript{m}, a general proof that \textit{none} of the variants for any \textit{m} are associative appears to be non-trivial due to the parameter \textit{m}, and is beyond the scope of this paper.

The \textit{B}-variants of QTC violate only the identity law \textit{R}_6 and \textit{R}_{6l}. As observed in \cite{25}, it is possible to equip them with a new \textit{id} relation, modify the interpretation of the other relations such that they become \textit{JEPD}, and adapt the converse and composition table accordingly. The thus modified calculi are then relation algebras.

The \textit{C}-variants of QTC additionally violate \textit{R}_4, \textit{R}_9, \textit{R}_{10}, and \textit{PL}. The corresponding algebra notion might be called a semi-associative Boolean algebra with converse-involution. As a consequence, most of the reasoning optimizations described in Section 3.2 cannot be applied to the \textit{C}-variants of QTC; hence, reasoning with these calculi is expected to be less efficient than with the calculi described so far. It is possible that the noticeably few violations of \textit{R}_9, \textit{R}_{10}, and \textit{PL} are due to errors in the composition table; the non-trivial verification is part of future work.

Cardinal Direction \textit{Relations} and Rectangular Direction Relations are the only calculi with weak converse that we have tested. The latter satisfies only \textit{WA} in addition to the axioms that are always satisfied by a Boolean algebra with distributivity. The corresponding algebra notion can be called a weakly associative Boolean algebra. Hence, this calculus enjoys none of the advantages for representation and reasoning discussed in Section 3.2. Similarly to the \textit{C}-variants of QTC, the relatively small number of violations of \textit{PL} may be due to errors in the implementation. Rectangular Direction Relations additionally satisfies \textit{R}_4 and therefore corresponds to what can be called an associative Boolean algebra. Since both calculi satisfy neither \textit{R}_7 nor \textit{R}_9, current reasoning algorithms and their implementations yield incorrect results for them, as seen in Section 3.2.

An overview of the algebra notions identified is given in Figure 1.

When making use of algebraic closure as inference mechanism it is essential to acknowledge that some axiom violations require special procedures in order to compute algebraic closure. Our analysis reveals that there indeed exist calculi that do not meet axioms that have been taken for granted. For example, the current version of GQR can fail to compute algebraic closure correctly for calculi that violate \textit{R}_9. In Algorithm 1 we present a universal algorithm to compute algebraic closure. For clarity and brevity of the presentation we stick to the well-known but simple control structure of PC-1. A real implementation would use an advanced control structure to avoid unnecessary invocations of the \texttt{revise} function, i.e., to use at least PC-2 \cite{20}. Conformance with \textit{R}_7 allows CSP storage to be restricted (flag \textit{s} in the algorithm), while violation of \textit{R}_9 requires two computations for the refinement operation Eq. 15 namely \((C_{i,j} \circ C_{j,k})\) and \((C_{j,k}^{-} \circ C_{i,j}^{-})^{-}\) (lines 10 [17]. \textit{R}_4 and \textit{R}_{10} are not used by the algorithm, since this would complicate the algorithm unduely.

\begin{figure}[h]
\centering
\caption{An overview of the algebra notions identified.}
\end{figure}
Algorithm 1 Universal algebraic closure algorithm

1: function lookup((C, i, j, s))
2:   if s \land (i < j) then
3:     return C_{i, j}^s
4:   else
5:     return C_{j, i}^s
6: end function

7: function revise((C, i, j, k, s))
8:   u ← false ▷ update flag to signal whether relation was updated
9:   r ← C_{i, j} \cap \text{lookup}(C, i, k, s) \odot \text{lookup}(C, k, j, s)
10:   if s \lor R_7 does not hold then
11:     r′ ← \text{lookup}(C, j, i, s) \cap (\text{lookup}(C, j, k, s) \odot \text{lookup}(C, k, i, s))
12:     r ← r \cap r'
13:     if r' \neq C_{j, i} then
14:       assert r' \neq \emptyset ▷ stop if inconsistency is detected
15:       u ← true
16:       C_{j, i} ← r'
17:     if r \neq C_{i, j} then
18:       assert r \neq \emptyset ▷ stop if inconsistency is detected
19:       u ← true
20:       C_{i, j} ← r
21: end function

22: return (C, u)

23: end function

24: function a-closure((n, \{x_1 r_1 y_1, \ldots, x_m r_m y_m\}))
25: if R_7 does not hold then
26:   s ← true ▷ without R_7 we must store converse relations
27:   C_{i, j} ← U, i = 1, \ldots, n, j = 1, \ldots, n
28: else
29:   s ← false ▷ for small calculi/CSPs storing converses may be more efficient
30:   C_{i, j} ← U, i = 1, \ldots, n, j = i + 1, \ldots, n ▷ use triangular matrix storage
31: end function

32: for i = 1, \ldots, m do
33:   x ← x_i, r ← r_i, y ← y_i ▷ process constraint x_i r_i y_i
34:   if \neg s \land (x > y) then
35:     (x, y) ← (y, x), r ← r' ▷ only write into upper half of matrix
36:     C_{x, y} ← C_{x, y} \cap r
37:     assert (x = y) \rightarrow (\text{id} \in C_{x, y})
38: end for
39: update ← true
40: while update do
41:   update ← false
42:   for i = 1, \ldots, n, j = i + 1, \ldots, n, k = 1, \ldots, n, k \neq i, k \neq j do
43:     (u, C) ← \text{revise}(C, i, j, k, s)
44:     update ← update \lor u
45:   end for
46: return C ▷ fix point reached
47: end function
5 A Quantitative Account of Qualitative Calculi

In this section, we report on computational properties of specific calculi which are beyond the computational complexity of constraint-based reasoning. For example, one might be interested to know how many relations are typically sufficient to describe a scene of \( n \) objects unequivocally or with a specific residual uncertainty. To this end, we developed two empirical measures that characterize certain aspects of qualitative calculi that are arguably relevant to applications.

We want to answer two questions: (1) How well do calculi with many relations make use of the usually higher information content? (2) Does information content differ significantly between the six classes of calculi established in Section 4?

The first measure we consider is information content of the composition operation. Our motivation is to estimate how much additional information can be gained by applying a composition operation. This allows us to estimate whether, for example, having observed relations \( r(A,B) \) and \( r'(B,C) \) in a scene, it is worthwhile to observe \( r''(A,C) \) too, as it may be improbable to derive \( r'' \) by composition \( (r \circ r') \). To obtain more general results we consider sequences of compositions \( r \circ r' \circ r'' \circ \ldots \) for several lengths. We define the information content \( I \) of a relation \( R \subseteq \text{Rel} \) to be

\[
I(R) = 1 - \frac{|R|}{|\text{Rel}|}
\]  

where \( |\text{Rel}| \) denotes the number of base relations of the calculus, and \( |R| \) the number of base relations \( R \) consists of. In case of the universal relation this results in \( I(U) = 0 \) as nothing is known, \( I(r) = 1 \) for all base relations \( r \in \text{Rel} \), and \( I(\emptyset) = 1 \). Obviously, the more base relations a calculus involves, the higher the information content can be for base relations. Therefore, we define

\[
I_C^k = \frac{\sum_{R \subseteq \text{Rel}} I(R)}{|\text{Rel}|^{k+1}}
\]
for a calculus $C$ with $r^k = \{r^0 \circ \ldots \circ r^k | r^0, \ldots, r^k \in \text{Rel}\}$ to be the average information content after $k$ composition operations, i.e., how restrictive relation are on average after information propagation with composition. In particular, $I_C^k$ is 1 minus the average proportion of base relations in any cell in the composition table. For example, for QTC$^{C22}$ ($|\text{Rel}| = 209$) or the Cardinal Direction Relations (CDR) ($|\text{Rel}| = 218$) $I_C^0 \approx 1$, whereas for the Point Calculus with three base relations $I_C^0 \approx 0.67$. We apply an iterative method to derive the values of $I_C^k$ that constructs $r^k$ for $k = 0, 1, \ldots$ rather than looping across combinations of base relations. Despite the potentially exponential size of $r^k$, the calculation remains feasible in many cases. Only for OPRA$_m$ with $m \geq 3$ and some QTC variants we were not able to derive values for higher $k$ in reasonable time. For the other calculi, computation was terminated after 14 compositions or if $I_C^k$ drops below 0.5.

As a second measure we determine the average degree of overlap that occurs after $k$ steps of composition for selected calculi. The degree of overlapping $O(R_i, R_j)$ is determined by counting the number of atomic relations shared by two relations, normalized by the total number of base relations:

$$O(R_i, R_j) = \frac{|R_i \cap R_j|}{|\text{Rel}|} \hspace{1cm} (18)$$

For example, if two relations in a calculus with eight base relations share four base relations, the overlap is 0.5. This value indicates how the information content differs between dealing with base relations only versus dealing with arbitrary relations (and thus how the results on information content generalize to arbitrary relations). Similar to $I(R)$ and $I_C^k$, we define $O_C^k$ to be the average overlap over all composition chains of length $k$.

$$O_C^k = \frac{\sum_{R_i, R_j \in r^k} O(R_i, R_j)}{|\text{Rel}|^{k+1}} \hspace{1cm} (19)$$

The results of the two measures are summarized in Figure 2 and Table 4 showing information content versus length $k$ of composition chains.

Figure 2 shows that the average information content for the Point Calculus after 1 step is $\approx 0.52$ and additionally, the overlap of $\approx 0.33$ is already quite high after a single composition. Therefore, in order to obtain detailed information it is reasonable to also observe $r_{AC}$ between objects $A$ and $C$ even if $r_{AB}$ and $r_{BC}$ are already known. By contrast, the INDU calculus has a very high information content ($\approx 0.87$) and a much smaller overlap. Therefore, it is not so informative to observe $r_{AC}$ as a lot of information is preserved after a composition. It is clear that the $O_C^k$ grows for increasing $k$ as composition results become coarser step by step. Nevertheless, information loss for PC is much higher than compared to Allen and INDU calculus: $I_{\text{PC}}^5$ and $I_{\text{Allen}}^5$ are close to $I_{\text{PC}}^2$ ($O_C^2$ respectively).

Our results show that there is no evidence for a relation between the information content of a calculus and its classification as per Figure 1. The only exceptions are some of the QTC calculi as $I_C^k$ starts to increase after some $k$ with increasing $k$. 
Figure 2. Information content and overlap after $k$ compositions for selected calculi

| Calculus       | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|---------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Allen         | 92.3 | 81.4 | 66.8 | 52.8 | 41.1 | 31.8 | 24.5 | 18.9 | 14.5 | 11.2 | 8.6 | 6.6 | 5.1 | 3.9 | 3.0 |
| Block Algebra | 99.4 | 96.5 | 89.0 | 77.7 | 65.3 | 53.4 | 43.0 | 34.1 | 27.0 | 21.1 | 16.4 | 12.8 | 9.9 | 7.7 | 5.9 |
| CDC           | 88.9 | 76.8 | 60.4 | 44.5 | 31.6 | 21.9 | 14.9 | 10.1 | 6.8  | 4.6  | 3.1  | 2.0  | 1.4  | 0.9 | 0.6 |
| CYC$_b$       | 75.0 | 62.5 | 46.9 | 32.8 | 21.9 | 14.1 | 8.8  | 5.4  | 3.2  | 1.9  | 1.1  | 0.6  | 0.4 |
| DRA$_{fp}$    | 98.8 | 89.9 | 69.0 | 45.0 | 25.8 | 13.4 | 6.5  | 3.0  | 1.3  | 0.6  | 0.2  |
| DRA-con       | 85.7 | 74.6 | 59.0 | 43.4 | 30.4 | 20.5 | 13.5 | 8.7  | 5.6  | 3.5  | 2.2  | 1.3  | 0.8  | 0.5 | 0.3 |
| Point Calculus| 66.7 | 51.9 | 37.0 | 25.5 | 17.3 | 11.6 | 7.8  | 5.2  | 3.5  | 2.3  | 1.5  | 1.0  | 0.7  | 0.5 |
| RCC-5         | 80.0 | 56.8 | 34.9 | 19.7 | 10.6 | 5.5  | 2.7  | 1.3  | 0.6  | 0.3  |
| RCC-8         | 87.5 | 62.3 | 38.0 | 21.1 | 11.0 | 5.5  | 2.6  | 1.2  | 0.6  | 0.3  |
| STAR$_4$      | 88.9 | 66.9 | 45.0 | 28.5 | 17.4 | 10.3 | 6.0  | 3.5  | 2.0  | 1.1  | 0.6  | 0.4 |
| DRA$_f$       | 98.6 | 90.6 | 70.4 | 46.3 | 26.7 | 13.9 | 6.7  | 3.0  | 1.3  | 0.6  | 0.2  |
| INDU          | 96.0 | 86.9 | 72.5 | 57.5 | 44.1 | 33.2 | 24.7 | 18.2 | 13.4 | 9.9  | 7.2  | 5.3  | 4.0  | 2.9 | 2.1 |
| OPRA$_1$      | 95.0 | 82.0 | 55.8 | 30.8 | 14.5 | 6.2  | 2.4  | 0.9  | 0.3  |
| OPRA$_2$      | 98.6 | 90.3 | 64.1 | 32.9 | 13.0 | 4.3  | 1.3  | 0.3  |
| OPRA$_3$      | 99.4 | 93.1 | 71.4 | 40.2 | 16.7 | 5.6  |
| OPRA$_4$      | 99.6 | 94.6 | 76.7 | 48.0 |
| QTC$_{B1}$    | 88.9 | 90.0 | 93.2 | 95.8 | 97.5 | 98.6 | 99.1 | 99.5 | 99.7 |
| QTC$_{B12}$   | 94.1 | 91.2 | 90.5 | 91.3 | 92.8 | 94.2 | 95.6 |
| QTC$_{B21}$   | 88.9 | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  |
| QTC$_{B22}$   | 96.3 | 51.9 | 37.0 | 25.5 | 17.3 | 11.6 | 7.8  | 5.2  | 3.5  | 2.3  | 1.5  | 1.0  | 0.7  | 0.5 | 0.3 |
| QTC$_{C21}$   | 98.8 | 92.5 | 76.6 | 68.6 | 69.5 | 73.0 | 76.5 | 79.4 | 81.8 | 83.7 | 85.2 | 86.4 | 87.4 |
| QTC$_{C22}$   | 99.5 | 95.1 | 78.0 | 69.3 | 51.2 |
| RDR           | 97.2 | 82.6 | 63.2 | 45.7 | 32.0 | 22.0 | 15.0 | 10.1 | 6.8  | 4.6  | 3.1  | 2.0  | 1.4  | 0.9 | 0.6 |
| CDR           | 99.5 | 78.8 | 60.9 | 48.9 | 39.6 | 32.1 | 26.1 | 21.2 | 17.2 | 14.0 | 11.4 | 9.3  | 7.6  | 6.2 | 5.1 |

Table 4. Information content $I_C^k$ for calculi in %
Although the calculi start with quite different values for $I^0$, most calculi have an information content less than 0.1 after six steps. The most notable exception is the Block Algebra where $I_{BA}^6 \approx 0.43$ and even after ten compositions it remains over 16%. Only Allen, INDU and CDR are somehow comparable. Concerning the classes we derived in Section 4 there is no uniform behavior observable. Thus, from a perspective of 'expression power' of calculi, there is no argument against applying calculi with weaker notions than RA. We have to note that the comparison of the values for calculi where it is known that a-closure decides consistency and those where it does not (or is unknown) is problematic. The latter ones may contain relations which are not physically realizable and thus, reduce the value of information content.

There are some interesting observations wrt. the various QTC variants. The QTC$_{B1x}$, QTC$_{B21}$ and QTC$_{C21}$ calculi behave differently from other calculi, whereas QTC$_{B23}$ behaves 'normal', i.e., $I^k$ increases, although it is very closely related to the other QTC variants. Interestingly, QTC$_{B1x}$ and QTC$_{C21}$ are the only calculi where $I^k$ increases with growing $k$. From our perspective, the reason lies in the multimodal structure of the calculus. As it combines points with line segments, the composition table (CT) contains empty relations as an object cannot be interpreted as a point and a line segment at the same time. Additionally, the CT contains only fairly small relations, i.e., small $|R|$. For example, the CT of QTC$_{B12}$ contains 29% empty relations, 29% atomic relations, and 42% other relations which have a maximal size of $|R| \leq 3$. The results for QTC$_{B21}$ are not surprising as the composition table only contains the universal relation and thus, for all $k$, $I^k = 0.0$ and $O^k = 100.0$. For QTC$_{C21}$ we observe that $I^k$ decreases to 69.5% at step 4, but starts to increase for $k \geq 5$. We assume that this is also the case for QTC$_{C22}$ as it is a refinement of QTC$_{C21}$, but we were not able to calculate necessary values due to the high complexity. So far, we have no explanation for this decrease.

An additional observation is that PC and QTC$_{B22}$ are similar with respect to information content, i.e., $I^k_{PC} \approx I^k_{QTC_{B22}}$ for $k \leq 14$. This congruence is interesting as the overlap values vary, the underlying partition scheme is different and the difference in base relations is significant (three for PC vs. 27 for QTC$_{B22}$). We leave the question for connections between these two calculi for future research.

6 Conclusion

We have looked at spatio-temporal representation and reasoning from an algebraic perspective, examining the implications of algebraic properties on modeling and reasoning algorithms, and testing these properties for a representative corpus of existing calculi. The resulting classification shows that calculi which have been described early in the literature tend to reside in the upper part of Figure 1; that is, they tend to have a rich algebraic structure. Few more recently developed calculi are based on generalizations and have a weaker structure. We have been able to conclude that common reasoning procedures are incorrect for the latter class of calculi, and have proposed a corrected universal a-closure algorithm that
makes use of reasoning optimizations where they are allowed. Furthermore, we found that algebraic properties do not necessarily relate to how much information is preserved in successive reasoning steps.

An interesting and significant line of future work is to extend this study to ternary calculi, which requires an extension of binary relation algebras to ternary relations, see also [35].

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A Additional proofs from Section 2.1

A.1 Proof of Fact 4

Given a qualitative calculus \((\text{Rel}, \text{Int}, \mathcal{\cdot}, \circ)\) and relations \(R, S \subseteq \text{Rel}\), the following hold:

\[
\varphi(R^\mathcal{\cdot}) \supseteq \varphi(R)\mathcal{\cdot} \quad (20)
\]

\[
\varphi(R \circ S) \supseteq \varphi(R) \circ \varphi(S) \quad (21)
\]

If \(C\) has weak converse, then, for all \(R \subseteq \text{Rel}\):

\[
R^\mathcal{\cdot} = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \varphi(R)\mathcal{\cdot}\} \quad (22)
\]

If \(C\) has strong converse, then, for all \(R \subseteq \text{Rel}\):

\[
\varphi(R^\mathcal{\cdot}) = \varphi(R)\mathcal{\cdot} \quad (23)
\]

If \(C\) has weak composition, then, for all \(R, S \subseteq \text{Rel}\):

\[
R \circ S = \bigcap \{T \subseteq \text{Rel} \mid \varphi(T) \supseteq \varphi(R) \circ \varphi(S)\} \quad (24)
\]

If \(C\) has strong composition, then, for all \(R, S \subseteq \text{Rel}\):

\[
\varphi(R \circ S) = \varphi(R) \circ \varphi(S) \quad (25)
\]

**Proof.** For (20), consider

\[
\varphi(R^\mathcal{\cdot}) = \bigcup_{r \in R} \varphi(r^\mathcal{\cdot}) \quad \text{definition of } \varphi(R^\mathcal{\cdot})
\]

\[
\supseteq \bigcup_{r \in R} \varphi(r)^\mathcal{\cdot} \quad \text{property } [1]
\]

\[
= \left( \bigcup_{r \in R} \varphi(r) \right)^\mathcal{\cdot} \quad \text{distributivity in set theory}
\]

\[
= \varphi(R^\mathcal{\cdot}) \quad \text{definition of } \varphi(R).
\]

For (21), consider

\[
\varphi(R \circ S) = \bigcup_{r \in R, s \in S} \varphi(r \circ s) \quad \text{definition of } \varphi(R \circ S)
\]

\[
\supseteq \bigcup_{r \in R} \bigcup_{s \in S} \varphi(r) \circ \varphi(s) \quad \text{property } [2]
\]

\[
= \left( \bigcup_{r \in R} \varphi(r) \right) \circ \left( \bigcup_{s \in S} \varphi(s) \right) \quad \text{distributivity in set theory}
\]

\[
= \varphi(R) \circ \varphi(S) \quad \text{definition of } \varphi(R) \text{ and } \varphi(S).
\]
Properties $23$ and $25$ are proven using $24$ and $26$ in the same way as we have just proven $20$ and $21$ using $1$ and $2$.

For $22$, we have $R = \{r_1, \ldots, r_n\}$ for some $n \geq 1$ and $r_1, \ldots, r_n \in \text{Rel}$. Due to Definition $2$, we have that

$$r_i^\ast = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \varphi(r_i)\}$$

for every $i = 1, \ldots, n$. Let $S_{i1}, \ldots, S_{im}$ be the $S$ over which the above intersection ranges, i.e.,

$$r_i^\ast = \bigcap_{j=1}^m S_{ij}.$$  

Due to Definition $2$, we have that

$$R^\ast = \bigcup_{i=1}^n r_i^\ast = \bigcup_{i=1}^n \bigcap_{j=1}^m S_{ij} = \bigcap_{j=1}^m \bigcup_{i=1}^n S_{ij},$$

where the last equality is due to the distributivity of intersection over union. Now $22$ follows if we show that, for every $S \in \text{Rel}$, the following are equivalent.

1. $\varphi(S) \supseteq \varphi(R)^\ast$
2. there exist $S_1, \ldots, S_n$ with $S = S_1 \cup \cdots \cup S_n$ and $\varphi(S_i) \supseteq \varphi(r_i)^\ast$ for every $i = 1, \ldots, n$.

For “$1 \Rightarrow 2$”, assume $\varphi(S) \supseteq \varphi(R)^\ast$, i.e., $\varphi(S) \supseteq \bigcup_{i=1}^n \varphi(r_i)^\ast$ (Definition $2$). If we further assume that $S = \{s_1, \ldots, s_t\}$, which implies that $\varphi(S) \supseteq \bigcup_{j=1}^t \varphi(s_j)$ (Definition $2$), then we can choose $S_i = \{s_j \mid \varphi(s_j) \cup \varphi(r_i)^\ast = \emptyset\}$ for every $i = 1, \ldots, n$. Because $C$ is based on JEPD relations, we have that $\varphi(S_i) \supseteq \varphi(r_i)^\ast$.

For “$2 \Rightarrow 1$”, let $S = S_1 \cup \cdots \cup S_n$ and $\varphi(S_i) \supseteq \varphi(r_i)^\ast$ for every $i = 1, \ldots, n$. Due to Definition $2$ and because $C$ is based on JEPD relations, we have that $\varphi(S) = \bigcup_{i=1}^n \varphi(S_i)$. Hence, $\varphi(S) \supseteq \bigcup_{i=1}^n \varphi(r_i)^\ast$ due to the assumption, and $\varphi(S) \supseteq \varphi(R)^\ast$ due to Definition $2$.

$24$ is proven analogously.

\[\square\]

A.2 $R_5$ and $R_{5i}$ from Table 2 are equivalent given $R_1$, $R_4$ and $R_6$–$R_9$.

We only show that $R_5$ implies $R_{5i}$; the converse direction is analogous.

\[
r \circ (s \cup t) = (r^\ast)^\ast \circ ((s \cup t)^\ast)^\ast \tag{R_7}
\]

\[
= ((s \cup t)^\ast)^\ast \circ r^\ast \tag{R_9}
\]

\[
= ((t^\ast \cup s^\ast)^\ast \circ r^\ast)^\ast \tag{R_6}
\]

\[
= ((t^\ast \circ r^\ast)^\ast \cup (s^\ast \circ r^\ast)^\ast)^\ast \tag{R_5}
\]

\[
= ((r \circ t)^\ast \cup (r \circ s)^\ast)^\ast \tag{R_9}
\]

\[
= ((r \circ t)^\ast)^\ast \cup ((r \circ s)^\ast)^\ast \tag{R_6}
\]

\[
= (r \circ t) \cup (r \circ s) \tag{R_7}
\]

\[
=r \circ (s \cup t) \tag{R_1}
\]

\[\square\]
A.3 \( R_6 \) and \( R_{6l} \) from Table 2 are equivalent given \( R_7 \) and \( R_9 \)

We only show that \( R_6 \) implies \( R_{6l} \); the converse direction is analogous. We first establish that \( \text{id}^\circ = \text{id} \).

\[
\begin{align*}
\text{id}^\circ &= \text{id}^\circ \circ \text{id} & (R_6) \\
&= \text{id}^\circ \circ (\text{id}^\circ)^\circ & (R_7) \\
&= (\text{id}^\circ \circ \text{id})^\circ & (R_9) \\
&= (\text{id}^\circ)^\circ & (R_6) \\
&= \text{id} & (R_7)
\end{align*}
\]

Now we use this lemma to establish \( R_{6l} \).

\[
\begin{align*}
\text{id} \circ r &= (\text{id}^\circ)^\circ \circ (r^\circ)^\circ & (R_7) \\
&= (r^\circ \circ \text{id}^\circ)^\circ & (R_9) \\
&= (r^\circ \circ \text{id})^\circ & (\text{Lemma}) \\
&= (r^\circ)^\circ & (R_6) \\
&= r & (R_7)
\end{align*}
\]

\( \square \)

B Additional proofs from Section 3

B.1 Proof of Fact 7

Axioms \( R_1 \)–\( R_3 \) are always satisfied because they are a characterization of a Boolean algebra; and the set operations on the relations form a Boolean algebra because \( \varphi \) maps base relations to a set of JEPD relations and complex relations to sets of interpretations of base relations.

The definition of the converse and composition operations for non-base relations in Definition 2 ensures that Axioms \( R_5 \) and \( R_8 \) hold.

Axiom \( R_7^\circ \) always holds due to JEPD and weak converse: For every \( r \in \text{Rel} \), we have that

\[
\varphi(r^\circ) \supseteq \varphi(r^\circ)^\circ \supseteq \varphi(r) = \varphi(r),
\]

where the first inclusion is due to Fact 4 (7) with \( R = r^\circ \), the second inclusion is due to Definition 2 (1) for \( r \), and the equation is due to the properties of binary relations over the universe \( \mathcal{U} \). Since the \( \varphi(r) \) are a set of JEPD relations, \( r^\circ \supseteq r \) follows. This reasoning carries over to arbitrary relations.

Axioms \( WA^\circ \) and \( SA^\circ \) always hold due to JEPD and weak composition: For every \( r \in \text{Rel} \), we have that

\[
\varphi((r \circ 1) \circ 1) \supseteq \varphi(r \circ 1) \circ \varphi(1) = \varphi(r \circ 1) \circ (\mathcal{U} \times \mathcal{U}) \supseteq \varphi(r \circ 1),
\]

where the first inclusion is due to to Fact 4 (8) with \( R = r \circ 1 \) and \( S = 1 \), and the last inclusion is due to the fact that \( R \circ (\mathcal{U} \times \mathcal{U}) \supseteq R \) for any binary relation
Since the $\varphi(r)$ are a set of JEPD relations, $(r \circ 1) \circ 1 \supseteq r \circ 1$ follows. Again, this reasoning carries over to arbitrary relations.

Axioms $R_0^c$, $R_{\delta_1}^c$, $R_0^c$ are violated by the following calculus. Let $\text{Rel} = \{r_1, r_2\}$, $\mathcal{U} = \{0, 1\}$, $\text{id} = r_1$, $1 = \{r_1, r_2\}$ with:

$$
\begin{align*}
\varphi(r_1) &= \{(0,0), (0,1)\} & r_1^\sim &= 1 & r_1 \circ r_1 &= 1 \\
\varphi(r_2) &= \{(1,0), (1,1)\} & r_2^\sim &= 1 & r_1 \circ r_2 &= r_1 \\
& & r_2 \circ r_1 &= 1 \\
& & r_2 \circ r_2 &= r_2 \\
\end{align*}
$$

This calculus satisfies the conditions from Definition 2 but violates Axioms $R_0^c$, $R_{\delta_1}^c$, $R_0^c$:

$$
\begin{align*}
R_0^c & \quad r_1 \circ \text{id} = 1 \not\supseteq r_1 \\
R_{\delta_1}^c & \quad \text{id} \circ r_1 = 1 \not\supseteq r_1 \\
R_0^c & \quad r_1^\sim = 1 \not\supseteq r_1 \\
\end{align*}
$$

Axioms $WA^c$, $SA^c$, $R_0^c$, $R_4^c$, $R_6^c$, $R_9^c$, $R_0^c$, $R_{\delta_1}^c$, $R_0^c$, $PL^{\neq}$, $PL^{=}$ are violated by the following calculus. Let $\text{Rel} = \{r_1, r_2, r_3, r_4\}$, $\mathcal{U} = \{0, 1\}$, $\text{id} = r_1$, $1 = \{r_1, r_2\}$ with:

$$
\begin{align*}
\varphi(r_1) &= \{(0,0)\} & r_1^\sim &= r_1 \\
\varphi(r_2) &= \{(1,1)\} & r_2^\sim &= r_2 \\
\varphi(r_3) &= \{(0,1)\} & r_3^\sim &= r_4 \\
\varphi(r_4) &= \{(1,0)\} & r_4^\sim &= r_3 \\
\end{align*}
$$

This calculus satisfies the conditions from Definition 2 but violates Axioms $WA^c$, $SA^c$, $R_0^c$, $R_4^c$, $R_6^c$, $R_9^c$, $R_0^c$, $R_{\delta_1}^c$, $R_0^c$, $PL^{\neq}$, $PL^{=}$:

$$
\begin{align*}
WA^c, SA^c &: \quad (r_1 \circ 1) \circ 1 = 1 \not\supseteq \{r_1, r_3, r_4\} = r_1 \circ 1 \\
R_0^c &: \quad (r_1 \circ r_3) \circ r_4 = r_3 \circ r_4 = \{r_1, r_4\} \not\supseteq \{r_1, r_3\} = r_1 \circ (r_3 \circ r_4) \\
R_4^c &: \quad (r_4 \circ r_3) \circ r_4 = r_2 \circ r_3 = r_2 \not\supseteq \{r_1, r_4\} = r_4 \circ (r_3 \circ r_4) \\
R_0^c &: \quad r_2 \circ \text{id} = r_2 \circ r_1 = 0 \not\supseteq r_2 \\
R_4^c &: \quad \text{id} \circ r_2 = r_1 \circ r_2 = 0 \not\supseteq r_2 \\
R_0^c, R_4^c &: \quad (r_3 \circ r_4)^\sim = \{r_1, r_4\} \not\supseteq \{r_1, r_3\} = r_3 \circ r_4 = r_4^\sim \circ r_3^\sim \\
R_{\delta_1}^c, R_0^c &: \quad r_3^\sim \circ r_2 \circ r_1 = r_4 \circ 0 = r_4 \circ 1 = \{r_1, r_2, r_4\} \not\supseteq \{r_2, r_3, r_4\} = \pi \\
PL^{\neq} &: \quad (r_1 \circ r_4) \cap r_1^\sim = \emptyset \cap r_1 = \emptyset \quad \text{but} \quad (r_4 \circ r_1) \cap r_1^\sim = \{r_4, r_1\} \cap r_1 = r_4 \neq \emptyset \\
PL^{=} &: \quad (r_4 \circ r_1) \cap r_1^\sim = \{r_4, r_1\} \cap r_1 = r_4 \neq \emptyset \quad \text{but} \quad (r_4 \circ r_1) \cap r_3^\sim = \emptyset \\
\end{align*}
$$

$\square$
Remark 11. Of course, there are calculi that satisfy only the weak conditions from Definition 2 but are a relation algebra, for example the following. Let \( \text{Rel} = \{r_0, r_1\} \), \( \mathcal{U} = \{0, 1\} \), \( \text{id} = r_1 \), \( 1 = \{r_1, r_2\} \) with:

\[
\begin{align*}
\varphi(r_1) &= \{(0, 0), (0, 1)\} & r_1^{-} &= r_2 & r_1 \circ r_1 &= r_1 \\
\varphi(r_2) &= \{(1, 0), (1, 1)\} & r_2^{-} &= r_1 & r_1 \circ r_2 &= 1 \\
& & r_2 \circ r_1 &= 1 & r_2 \circ r_2 &= r_2
\end{align*}
\]

C Detailed description of the test results in Section 4 by calculus

9-Intersection

- HETS: all axioms are satisfied.
- SparQ: all 6 tests are passed.

Allen’s interval relations

- \cite{25}: this calculus is a relation algebra.
- HETS: all axioms are satisfied.
- SparQ: all 6 tests are passed.

Block Algebra

- HETS: all axioms are satisfied.
- SparQ: all 6 tests are passed.

Cardinal Direction Calculus

- \cite{25}: this calculus is a relation algebra, as already reported in \cite{16}.
- HETS: all axioms are satisfied.
- SparQ: all 6 tests are passed.

Cardinal Direction Relations

- HETS: \( R_6 \) is violated for all base relations but one, \( R_{6l} \) for only 209 base relations, \( R_7 \) for 214 base relations, \( R_9 \) for 5,607 pairs, \( PL \) for 22,976 triples, \( R_4 \) for 2,936,946 triples, and \( SA \) for 38 base relations. WA is satisfied. CDC is therefore just a Boolean algebra with distributivity, weak associativity and weak involution.
- SparQ: the version with 218 base relations fails all 6 tests: \( R_6 \) and \( R_{6l} \) for 217 base relations, \( R_7 \) for 214 base relations, \( R_9 \) for 45,939 pairs (97%), \( PL \) for 22,976 triples (<1%), and \( R_4 \) for 2,936,946 triples (28%).

\footnote{This number is too large because SparQ uses a variant of \( R_6 \).}
Counterexamples:

First 5 counterexamples for PL

\[(W-NW-N-NE \circ NW-N-NE) \cap B-S^* = \{\}\]
\[\neq \{B\} = (NW-NE \circ B-S) \cap W-NW-N-NE-E^*\]

\[(W-NW-N-NE \circ NW-N-NE) \cap B^* = \{\}\]
\[\neq \{B\} = (NW-N \circ B-S) \cap W-NW-N-NE-E^*\]

\[(W-NW-N-NE \circ NW-N) \cap S-E-SE^* = \{\}\]
\[\neq \{B\} = (NW-N \circ S-E-SE) \cap W-NW-N-NE-E^*\]

\[(W-NW-N-NE \circ NW-N) \cap B-S-SE^* = \{\}\]
\[\neq \{B\} = (NW-N \circ B-S-SE) \cap W-NW-N-NE-E^*\]

First 10 counterexamples for $R_9$

\[(S \circ S)^* \neq S^* \circ S^*\]
\[(S \circ SW)^* \neq SW^* \circ S^*\]
\[(S \circ W)^* \neq W^* \circ S^*\]
\[(S \circ NW)^* \neq NW^* \circ S^*\]
\[(S \circ NE)^* \neq NE^* \circ S^*\]
\[(S \circ E)^* \neq E^* \circ S^*\]
\[(S \circ SE)^* \neq SE^* \circ S^*\]
\[(S \circ B)^* \neq B^* \circ S^*\]
\[(S \circ SW)^* \neq SW^* \circ S^*\]

First 10 counterexamples for $R_{10}$

\[S \circ S \circ S \not\subseteq S\]
\[S \circ S \circ SW \not\subseteq SW\]
\[S \circ S \circ W \not\subseteq W\]
\[S \circ S \circ NW \not\subseteq NW\]
\[S \circ S \circ N \not\subseteq N\]
\[S \circ S \circ NE \not\subseteq NE\]
\[S \circ S \circ E \not\subseteq E\]
\[S \circ S \circ SE \not\subseteq SE\]
\[S \circ S \circ B \not\subseteq B\]
\[S \circ S \circ SW \not\subseteq SW\]

First 10 counterexamples for SA

\[(B-S-W-NW \circ 1) \circ 1 \neq B-S-W-NW \circ 1\]
\[(B-W-N-NE \circ 1) \circ 1 \neq B-W-N-NE \circ 1\]
\[(B-N-E-SE \circ 1) \circ 1 \neq B-N-E-SE \circ 1\]
\[(B-S-SW-E \circ 1) \circ 1 \neq B-S-SW-E \circ 1\]
\[(B-S-NE-E \circ 1) \circ 1 \neq B-S-NE-E \circ 1\]
\[(B-S-W-SE \circ 1) \circ 1 \neq B-S-W-SE \circ 1\]
\[(B-SW-W-N \circ 1) \circ 1 \neq B-SW-W-N \circ 1\]
\[(B-NW-N-E \circ 1) \circ 1 \neq B-NW-N-E \circ 1\]
\[(B-S-W-NW-SE \circ 1) \circ 1 \neq B-S-W-NW-SE \circ 1\]
\[(B-SW-W-N-NE \circ 1) \circ 1 \neq B-SW-W-N-NE \circ 1\]
First 5 counterexamples for $R_4$

\[ W\text{-NW-N-NE-E-SE} \circ (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E}) \]
\[ \neq (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E-SE}) \circ W\text{-NW-N-NE-E} \]

\[ W\text{-NW-N-NE-E-SE} \circ (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE}) \]
\[ \neq (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E-SE}) \circ W\text{-NW-N-NE} \]

\[ W\text{-NW-N-NE-E-SE} \circ (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N}) \]
\[ \neq (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E-SE}) \circ W\text{-NW-N} \]

\[ W\text{-NW-N-NE-E-SE} \circ (W\text{-NW-N-NE-E-SE} \circ S\text{-SW-W-SE}) \]
\[ \neq (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E-SE}) \circ S\text{-SW-W-SE} \]

\[ W\text{-NW-N-NE-E-SE} \circ (W\text{-NW-N-NE-E-SE} \circ S\text{-SW-W-SE}) \]
\[ \neq (W\text{-NW-N-NE-E-SE} \circ W\text{-NW-N-NE-E-SE}) \circ S\text{-SW-W-SE} \]

\text{CYC}_b

- \text{HETS}: all axioms are satisfied.
- \text{SparQ}: all 6 tests are passed.

\text{Dependency Calculus}

- [25]: this calculus is a relation algebra (and homomorphically embeddable into RCC-5).
- \text{HETS}: all axioms are satisfied.
- \text{SparQ}: All 5 tests are passed.

\text{Dipole Calculus}

- \text{HETS}: and DRA$_f$ violates $R_4$ for 71,424 triples of base relations. For example, we have that
\[ rrl \circ (rrl \circ lrl) \neq (rrl \circ rrl) \circ lrl. \]

However, DRA$_f$ satisfies the weaker WA and SA. The violation of associativity attributed in [23] to the converse operation being weak, illustrated by the example $bfi \circ lllb = llll$ for DRA$_f$. DRA$_fp$ and DRA-connectivity satisfy all axioms.
- \text{SparQ}: $R_4$ fails for 71,424 triples in DRA$_f$ (about 19%). The other 4 tests are passed. DRA$_fp$ and DRA-connectivity pass all 6 tests.

\text{Geometric Orientation}

- \text{HETS}: all axioms are satisfied.
- \text{SparQ}: all 6 tests are passed.

\text{INDU}

- \text{HETS}: 1,880 triples violate $R_4$; the violation of associativity has already been reported and attributed to the absence of strong composition in [3]; for example,
\[ bi^* \circ (mi^* \circ m^*) \neq (bi^* \circ mi^*) \circ m^*. \]

All other axioms are satisfied, including SA and WA. Therefore, the Interval-Duration Calculus is a semi-associative relation algebra.
– **SparQ**: All tests except $R_4$ are passed; $R_4$ fails on 1880 triples (12 %).

**OPRA**$_n$
– [25]: for $n = 1, 2, 3, 4$, either the calculus is not associative, or there is an error in the composition table.
– **HETS**: $R_4$ is violated by 1,664 triples for $OPRA_1$,
257,024 triples for $OPRA_2$,
2,963,952 triples for $OPRA_3$,
16,711,680 triples for $OPRA_4$,
63,840,000 triples for $OPRA_5$,
190,771,200 triples for $OPRA_6$,
481,275,648 triples for $OPRA_7$,
1,072,693,248 triples for $OPRA_8$.

All other axioms are satisfied, including SA and WA. Therefore, OPRA$_n$, $n \leq 8$, are semi-associative relation algebras.

– **SparQ**: All tests except $R_4$ are passed. In $OPRA_1$, 1664 triples (21%) do not agree with $R_4$; in $OPRA_2$, there are 257,024 violations (69%); in $OPRA_3$, 2,963,952 (78%); in $OPRA_4$, 16,711,680 (83%); in $OPRA_5$, 63,840,000 (86%); and in $OPRA_6$, 190,771,200 (88%).

**Point Calculus**
– **HETS**: all axioms are satisfied.
– **SparQ**: all 6 tests are passed.

**QTC**
– [25]: id is not a neutral element of composition in QTC$_{B11}$ and QTC$_{B12}$.

After introducing a new id relation and making the relations JEPD, these two calculi pass all tests.
– **HETS**: QTC$_{B11}$ and QTC$_{B12}$ violate $R_6$ and $R_6l$ for all base relations that are not id. QTC$_{B21}$ (9 base relations) and QTC$_{B22}$ (27 base relations) violate $R_6$ and $R_6l$ for all base relations. QTC$_{C21}$ (81 base relations) does not satisfy $R_6$ and $R_6l$ for all base relations but one, $R_4$ for 292,424 triples, $R_9$ for 160 pairs, $R_{10}$ for 80 pairs,
and PL for 1056 triples\(^7\) \(QTC_{C22}\) (209 base relations) does not satisfy \(R_6\) and \(R_{6l}\) for 208 relations, \(R_9\) for 1248 pairs, \(R_{10}\) for 624 pairs, PL for 12,768 triples, and \(R_4\) for 7,201,800 triples, see also footnote \(^6\).

- **SparQ**: \(QTC_{B11}\) and \(QTC_{B12}\) are reported to have no identity relation, and they both fail the test for \(R_6\) and \(R_{6l}\) with all base relations but one. \(QTC_{B21}\) and \(QTC_{B22}\) are also reported to have no identity relation and fail the test for \(R_6\) and \(R_{6l}\) with all base relations.

\(QTC_{C21}\) (81 base relations) fails the test for \(R_4\) (292,424 triples, 55\%), \(R_6\) and \(R_{6l}\) (80 out of 81 relations), \(R_9\) (160 pairs, 2\%), and PL (1056 triples, <1\%).

\(QTC_{C22}\) (209 base relations) shows a similar result: it fails the test for \(R_4\) (7,201,800 triples, 79\%), \(R_6\) and \(R_{6l}\) (208 out of 209 relations), \(R_9\) (1248 pairs, 3\%), and PL (12,768 triples, <1\%).

**RCC**

- **[25]**: after certain repairs to the composition table, RCC-5 and RCC-8 are found to be relation algebras.

- **HETS**: all axioms are satisfied.

- **SparQ**: after one repair (removing EC from the entry \(NTPPi \circ NTPP\)), RCC-5 and RCC-8 pass all 6 tests.

**Rectangular Direction Relations**

- **HETS**: \(R_5\) is violated for all base relations but one, \(R_{6l}\) for only 33 base relations, \(R_7\) for 32 base relations, \(R_9\) for 855 pairs, \(R_{10}\) for 671 pairs, and PL for 3424 triples. \(R_4\), SA, and WA are satisfied. CDC is therefore just a Boolean algebra with distributivity, associativity and weak involution.

- **SparQ**: the calculus fails the following tests: \(R_6\) and \(R_{6l}\) for 35 out of 36 base relations, \(R_7\) for 32 base relations, \(R_9\) for 855 pairs (66\%), and PL for 3424 triples (7\%).

\(^6\) Note that, for calculi that violate \(R_9\), the equivalence between PL and \(R_{10}\) is no longer ensured, hence the mentioning of both of them. Furthermore, \(R_{10}\) is the only axiom that should be tested for all relations, but we have only tested it for all base relations. Therefore, there could be more violations than the four listed. The same cautions apply to \(QTC_{C22}\).
Counterexamples:
First 10 counterexamples for PL

\[(B \circ N) \cap B:W = \emptyset \Leftrightarrow (N \circ B:W) \cap B = \emptyset\]

\[(B \circ N) \cap B = \emptyset \Leftrightarrow (N \circ B) \cap B = \emptyset\]

\[(B \circ S) \cap B:W:N:NE:E = \emptyset \Leftrightarrow (S \circ B:W:N:NE:E) \cap B = \emptyset\]

\[(B \circ S) \cap B:N = \emptyset \Leftrightarrow (S \circ B:N) \cap B = \emptyset\]

\[(B \circ S) \cap B:E = \emptyset \Leftrightarrow (S \circ B:E) \cap B = \emptyset\]

\[(B \circ S) \cap B:W = \emptyset \Leftrightarrow (S \circ B:W) \cap B = \emptyset\]

First 10 counterexamples for R₉

\[(B \circ S:SW) = \emptyset \Leftrightarrow S:SW \cap B = \emptyset\]

\[(B \circ SW) = \emptyset \Leftrightarrow SW \cap B = \emptyset\]

\[(B \circ SE) = \emptyset \Leftrightarrow SE \cap B = \emptyset\]

\[(B \circ NW) = \emptyset \Leftrightarrow NW \cap B = \emptyset\]

\[(B \circ NE) = \emptyset \Leftrightarrow NE \cap B = \emptyset\]

\[(B \circ W) = \emptyset \Leftrightarrow W \cap B = \emptyset\]

\[(B \circ E) = \emptyset \Leftrightarrow E \cap B = \emptyset\]

\[(B \circ N) = \emptyset \Leftrightarrow N \cap B = \emptyset\]

\[(B \circ S) = \emptyset \Leftrightarrow S \cap B = \emptyset\]

\[(B \circ B) = \emptyset \Leftrightarrow B \cap B = \emptyset\]

First 10 counterexamples for R₁₀

\[B \circ B \circ S:SW \notin S:SW\]

\[B \circ B \circ SW \notin SW\]

\[B \circ B \circ SE \notin SE\]

\[B \circ B \circ NW \notin NW\]

\[B \circ B \circ NE \notin NE\]

\[B \circ B \circ W \notin W\]

\[B \circ B \circ E \notin E\]

\[B \circ B \circ N \notin N\]

\[B \circ B \circ S \notin S\]

\[B \circ B \circ B \notin B\]

STAR₄

- **HETS:** all axioms are satisfied.
- **SparQ:** all 6 tests are passed.