analytic $SU(3)$ unitary matrix and application to a $\sigma$-model

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abstract: The $SU(2)$ unitary matrix $U$ has both exponential and analytic representations, related by $U = \exp [i \tau \cdot \hat{\pi} \theta] = \cos \theta + i \tau \cdot \hat{\pi} \sin \theta$, where $\tau$ are Pauli matrices and $\pi = (\pi_1, \pi_2, \pi_3)$. One extends this result to the $SU(3)$ unitary matrix by deriving an analytic expression which, for Gell-Mann matrices $\lambda$, yields

$$U = \exp \left[ i v \cdot \lambda \right] = \left[ (F + \frac{2}{3} G) I + \left( H \hat{v} + \frac{1}{\sqrt{3}} G \hat{b} \right) \cdot \lambda \right]$$

$$+ i \left[ (Y + \frac{2}{3} Z) I + \left( X \hat{v} + \frac{1}{\sqrt{3}} Z \hat{b} \right) \cdot \lambda \right],$$

with $v_i = [v_1, \ldots, v_8]$, $b_i = d_{ijk} v_j v_k$, and factors $F, \cdots Z$ written in terms of elementary functions depending on $v = |v|$ and $\eta = 2 d_{ijk} \hat{v}_i \hat{v}_j \hat{v}_k / 3$. This result does not depend on the particular meaning attached to the variable $v$. Unitarity constrains the factors $F, \cdots Z$ to the surface of a four-dimensional sphere and, in the classical limit, corresponding to $\eta \rightarrow 0$, one gets the cyclic structure

$$U = \left[ \frac{1}{3} (1 + 2 \cos v) I + \frac{1}{\sqrt{3}} (-1 + \cos v) \hat{b} \cdot \lambda \right] + i (\sin v) \hat{v} \cdot \lambda,$$

which gives rise to a tilted circumference with radius $\sqrt{2/3}$ in the space defined by $I$, $\hat{b} \cdot \lambda$, and $\hat{v} \cdot \lambda$. The analytic expression for $U$ is used to calculate explicitly the corresponding left and right forms.

With the purpose of applying analytic results to low-energy hadron physics, one associates $v$ with pseudoscalar meson fields $\phi$ and evaluates the axial transformations of the matrix $U$. They guide the construction of a non-linear $SU(3) \times SU(3)$ version of the $\sigma$-model based on a generalization of the classical $U$ matrix relying on 17 fields, instead of the usual 18 present in the linear case. Owing to the absence of the ninth pseudoscalar field, it cannot account for $\eta - \eta'$ mixing. Nevertheless, basic equations are similar in both approaches and small differences are associated with symmetry breaking terms. The same holds for the conservative predictions for scalar masses, which read $m_a = 950 \text{ MeV}$ and $m_\kappa = 789 \text{ MeV}$, and $m_8 = 564 \text{ MeV}$, when one identifies $m_0$ with the $f_0(980)$. The conceptual structure of the non-linear model suggests that scalar resonances correspond to quasi-bound states of two pseudoscalars.
I. MOTIVATION

The considerable progress in the description of low-energy hadronic properties achieved over the last sixty years is closely associated with chiral symmetry. Quantum chromodynamics (QCD), the present-day strong theory, involves gluons and quarks with six different flavors, including color. Direct applications to low-energy processes are very difficult owing to gluon–gluon interactions and one has to resort to either lattice methods[1] or effective descriptions. The latter depart from the symmetries of QCD, namely the continuous Poincaré group, discrete C, P and T inversions, together electric charge and baryon number conservation. The quark masses \( m_q \) are external parameters and the lightest ones, \( m_u, m_d, \) and \( m_s \), can be considered as small in the scale \( \Lambda \sim 1 \text{ GeV} \). This rationale underlies the idea of chiral symmetry, an approximate scheme that becomes exact only in the ideal limit \( m_q \to 0 \).

In this case, helicity is a good quantum number and the quark fields \( q \) are written as linear combinations of \( q_R \) and \( q_L \), with spins respectively parallel and anti-parallel to their momenta. As helicity is conserved in interactions, the fields \( q_R \) and \( q_L \) do not couple and the Lagrangian is symmetric under the chiral group \( U(N)_R \times U(N)_L \), where \( N \) is the number of flavors. However, owing to the the \( U(1)_A \) anomaly, the actual group to be considered is \( U(1)_V \times SU(3)_R \times SU(3)_L \). In effective descriptions, these symmetries of QCD are associated directly with hadronic degrees of freedom, bypassing quarks and gluons.

The incorporation of chiral symmetry into hadron physics precedes QCD and was already being discussed in 1960. A long-lasting contribution from that year is the idea that the strong vacuum is not empty, presented in a paper by Gell-Mann and Lévy[2] discussing both linear and non-linear \( \sigma \)-models, including nucleons and pions. The former also relied on the \( \sigma \), a scalar particle proposed earlier by Schwinger[3], and provided a unique tool for dealing with the strong vacuum. In the symmetric version, the model involves just two parameters, usually denoted by \( \mu \) and \( \lambda \), whose values determine whether the ground state of the theory is either empty or contains a classical component, associated with a condensate. Almost simultaneously, in 1961, Nambu and Jona-Lasinio[4] studied the strong vacuum employing an alternative chiral model inspired by superconductivity, which also involved a scalar-isoscalar state. Their model was based on fermionic fields, the pion being a collective state, and contained a vacuum phase transition, described by a gap-equation and controlled by a free parameter. A common feature of both models is the indication that chiral symmetry allows the ground state state of strong systems to be realized in two different modes, namely: (i) the Wigner–Weyl mode, in which states with opposite parities are degenerate and the vacuum is empty; (ii) the Nambu–Goldstone mode, in which the pion is a massless Goldstone boson, the scalar state is massive, and the vacuum contains a condensate. Also in 1961, Skyrme succeeded in describing baryons as topological solitons composed of chiral pions, carrying a well defined quantum number[5, 6]. He employed classical pion fields constrained by a nonlinear condition and assumed the proton to be a deformation of the strong vacuum, kept
stable for topological reasons. Nowadays, these states are known as skyrmions but, at the
time, they were criticized for not having spin and deserved little attention. However, about
two decades later, spins were incorporated into the model by Adkins, Nappi and Witten[7],
and its rich structure could be properly appreciated.

After QCD became established as the strong theory, applications of chiral symmetry were
aimed mostly at improving the precision of predictions and nowadays chiral perturbation
theory (ChPT) is employed to tackle low-energy hadronic processes. This research program
was outlined by Weinberg in 1979[8] and fully developed by Gasser and Leutwyler for the
SU(2) sector in 1984[9]. Low-energy interactions are strongly dominated by quarks $u$ and $d$
and their small masses are treated as perturbations into a massless $SU(2) \times SU(2)$ symmetric
Lagrangian involving effective pion fields. ChPT is a well-defined theory and allows the
systematic expansion of low-energy amplitudes in powers of a typical scale $q \sim M_\pi < 1\,\text{GeV}$. Nevertheless, while QCD is fully renormalizable, ChPT can only be renormalized
order by order[8]. The effective lagrangian consists of strings of terms possessing the most
general structure consistent with broken chiral symmetry and both its the form and the
number of low-energy constants (LECs) associated with renormalization, depend on the
order considered.

All approaches to strong interactions mentioned, namely the models produced by Gell-
Mann and Lévy, Nambu and Jona-Lasinio, and Skyrme, together with ChPT, did bring
important progress to the area. With hindsight, however, one realizes that all of them have
specific limitations and none has superseded completely the others. So, in spite of their
differences, they coexist and the relevance of each one depends on the particular problem
considered. A common feature of these competing strategies is that, in all cases, early works
were performed in the framework of $SU(2)$ for reasons of simplicity. The basic unitary $SU(2)$
matrix $U$ can be represented as

$$U = \exp \left[ i \tau \cdot \hat{\pi} \theta \right], \quad \rightarrow \text{exponential representation}$$  \hspace{1cm} (1)

where $\hat{\pi}$ is the direction of the pion field in isospin space. As it is well known, the series
implicit in the exponential can be summed and one gets the equivalent form

$$U = \cos \theta + i \tau \cdot \hat{\pi} \sin \theta, \quad \rightarrow \text{analytic representation}$$  \hspace{1cm} (2)

which one calls analytic, in the want of a better name. It is employed in the non-linear
$\sigma$-model and suited to comparisons with the linear version, based on the non-unitary matrix

$$M = \sigma + i \tau \cdot \pi.$$  \hspace{1cm} (3)

The simplicity of these mathematical structures facilitates comparisons among different
schemes and allows one to study the mathematical reasons backing their main main fea-
tures.
The various approaches have been generalized to $SU(3)$ and the new version of the $\sigma$-model[10] employs a matrix $M$ composed by nonets of pseudoscalar and scalar states, whereas the extended version of ChPT relies on the exponential form[11]. In the case of the Skyrme model, the $SU(3)$ group is employed just in the quantization of the soliton, which is carried out formally[12, 13]. The conceptual mobility among these generalizations to $SU(3)$ is more difficult than in the $SU(2)$ case, partly owing to the absence of a suitable analytic expression for the matrix $U$. Analytic results based on Euler angles already exist for this operator[14–16], and find applications in many areas of physics dealing with three state systems, such as color superconductivity[17], optics[18], geometric phases[19], and quantum entanglement in computation and communication[20]. Nevertheless, a possible inconvenience of using Euler angles in hadron physics is the need of a set of external axes. In sect.II of this work one derives an alternative representation for the operator $U$, written in terms of internal degrees of freedom, which corresponds to an extension of eq.(2). Besides other applications, this result may prove to be instrumental to study topological properties of $SU(3)$ for both flavor and color, in analogy to the case of the skyrmion.

The unitarity of $U$ in analytic matrix form is explored in sect.III, its classical limit discussed in sect.IV, the corresponding left and right forms are presented in sect.V and its chiral transformations given in sect.VI. As a possible application of these results, a pure $SU(3) \times SU(3)$ version of the $\sigma$-model is discussed in sect.VII. Conclusions are summarized in sect.VIII, whereas technical matters are presented in four appendices.

II. ANALYTIC FORM

The exponential form of the unitary operator $U$ is written in terms of the Gell-Mann matrices $\mathbf{\lambda} = [\lambda_1, \cdots \lambda_8]$ and a generic $SU(3)$ octet $\mathbf{v} = [v_1, \cdots v_8]$ as

$$U = \exp \left[i \mathbf{v} \cdot \mathbf{\lambda}\right]$$

$$= \left[1 - \frac{v^2}{2!} [\hat{v} \cdot \mathbf{\lambda}]^2 + \cdots \right] + i \left[\frac{v}{1!} [\hat{v} \cdot \mathbf{\lambda}] - \frac{v^3}{3!} [\hat{v} \cdot \mathbf{\lambda}]^3 + \cdots \right]$$

(4)

with $\mathbf{v} \cdot \mathbf{\lambda} = v_i \lambda_i$, $v = \sqrt{v_i v_i}$, and $\hat{v} = \mathbf{v}/v$.

One uses two auxiliary variables in the derivation of the analytic form. One of them is the bilinear construct $\mathbf{b} = [b_1, \cdots b_8]$, 

$$b_i = d_{ijk} v_j v_k ,$$

(5)

with $b = \sqrt{b_i b_i} = v^2/\sqrt{3}$ and $\hat{b} = \mathbf{b}/b$. The other is 

$$\eta = \frac{2}{3 v^3} D = \frac{2}{3 \sqrt{3}} \hat{v} \cdot \hat{b} ,$$

(6)
where \( D = \mathbf{v} \cdot \mathbf{b} = d_{ijk} v_i v_j v_k \). The quantity \( \mathbf{b} \) is even under \( \mathbf{v} \to -\mathbf{v} \), \( \eta \) is odd, and the latter is a measure of the overlap between \( \mathbf{b} \) and \( \mathbf{v} \). The explicit forms of \( b_i \) and \( D \) are given in App.A and shown to satisfy the conditions

\[
\begin{align*}
  f_{ijk} v_j b_k &= 0 , \\
  d_{ijk} v_j b_k &= \frac{1}{3} v^2 v_i , \\
  d_{ijk} b_j b_k &= \eta v^3 v_i - \frac{1}{3} v^2 b_i ,
\end{align*}
\]

which allow one to write

\[
\begin{align*}
  [ \hat{\mathbf{v}} \cdot \mathbf{\lambda} ] [ \hat{\mathbf{v}} \cdot \mathbf{\lambda} ] &= \frac{2}{3} + \frac{1}{\sqrt{3}} \hat{\mathbf{b}} \cdot \mathbf{\lambda} , \\
  [ \hat{\mathbf{v}} \cdot \mathbf{\lambda} ] [ \hat{\mathbf{v}} \cdot \mathbf{\lambda} ] [ \hat{\mathbf{v}} \cdot \mathbf{\lambda} ] &= \hat{\mathbf{v}} \cdot \mathbf{\lambda} + \eta .
\end{align*}
\]

In order to simplify the notation, one defines

\[
\begin{align*}
  A &= \hat{\mathbf{v}} \cdot \mathbf{\lambda} , \\
  B &= \frac{2}{3} + \frac{1}{\sqrt{3}} \hat{\mathbf{b}} \cdot \mathbf{\lambda} ,
\end{align*}
\]

so that

\[
\begin{align*}
  A^2 &= AA = B , \\
  A^3 &= AB = A + \eta , \\
  A^4 &= AAB = BB = B + \eta A .
\end{align*}
\]

Thus, \( A^5 = A (B + \eta A) = A + \eta + \eta B \), and so on. These results mean that, in matrix space, the operator \( U \) is a linear combinations of the identity \( I \), \( A \) and \( B \). The matrices \( I \) and \( B \) are even under \( \mathbf{v} \to -\mathbf{v} \), whereas \( A \) is odd.

The real and imaginary parts of \( U \) have different structures, one writes

\[
U = \sum_{n=0}^{\infty} \frac{i^n}{n!} A^n = U_{\text{re}} + i U_{\text{im}}
\]

and has

\[
\begin{align*}
  \frac{\partial U_{\text{re}}}{\partial v} &= -A U_{\text{im}} , \\
  \frac{\partial U_{\text{im}}}{\partial v} &= A U_{\text{re}} .
\end{align*}
\]
In matrix space, one writes

\[ U_{\text{re}} = \sum_{n=0}^{\infty} (i)^{2n} \frac{v^{2n}}{(2n)!} [f_{2n} I + g_{2n} B + h_{2n} A] \]

\[ = F(v, \eta) I + G(v, \eta) B + H(v, \eta) A , \]  

(20)

\[ U_{\text{im}} = \sum_{n=0}^{\infty} (i)^{2n} \frac{v^{2n+1}}{(2n+1)!} [x_{2n+1} A + y_{2n+1} I + z_{2n+1} B] \]

\[ = X(v, \eta) A + Y(v, \eta) I + Z(v, \eta) B , \]  

(21)

where the functions \( F, G, H, X, Y, \) and \( Z \) are determined in the sequence. In tables I and II, one displays a few partial contributions to these series and it is possible to note that the dependences on \( v \) and \( \eta \) do not mix. Real and imaginary components are related by the action of the operator \( A \), which yields

\[ A [f_{2n} I + g_{2n} B + h_{2n} A] = [(f_{2n} + g_{2n}) A + \eta g_{2n} I + h_{2n} B] \]

\[ = [x_{2n+1} A + y_{2n+1} I + z_{2n+1} B] , \]  

(22)

\[ A [x_{2n+1} A + y_{2n+1} I + z_{2n+1} B] = [\eta z_{2n+1} I + x_{2n+1} B + (y_{2n+1} + z_{2n+1}) A] \]

\[ = [f_{2n+2} I + g_{2n+2} B + h_{2n+2} A] . \]  

(23)

| n   | \((i v)^n/n!\) | \(f_n \times I\) | \(g_n \times B\) | \(h_n \times A\) |
|-----|----------------|----------------|----------------|----------------|
| 0   | 1              | 1              |                |                |
| 2   | \(-v^2/2!\)    | 1              |                |                |
| 4   | \(v^4/4!\)     | 1              | \(\eta\)       |                |
| 6   | \(-v^6/6!\)    | \(\eta^2\)    | 1              | \(2\eta\)      |
| 8   | \(v^8/8!\)     | 2 \(\eta^2\)  | 1 + \(\eta^2\) | \(3\eta\)      |
| 10  | \(-v^{10}/10!\)| 3 \(\eta^2\)  | 1 + 3 \(\eta^2\)| 4 \(\eta + \eta^3\) |
| 12  | \(v^{12}/12!\) | 4 \(\eta^2 + \eta^4\)| 1 + 6 \(\eta^2\) | 5 \(\eta + 4 \eta^3\) |
| 14  | \(-v^{14}/14!\)| 5 \(\eta^2 + 4 \eta^4\)| 1 + 10 \(\eta^2 + \eta^4\)| 6 \(\eta + 10 \eta^3\) |
| 16  | \(v^{16}/16!\) | 6 \(\eta^2 + 10 \eta^4\)| 1 + 15 \(\eta^2 + 5 \eta^4\)| 7 \(\eta + 20 \eta^3 + \eta^5\) |

**TABLE I**: Structure of \( U_{\text{re}} \), eq.(20).
Using results (18)-(21), one writes
\[
\frac{\partial}{\partial v} [F I + G B + H A] = -[\eta Z I + X B + (Y + Z)A] ,
\]
\[
\frac{\partial}{\partial v} [X A + Y I + Z B] = [(F + G) A + \eta I + H B] .
\]
and obtains a set of first order differential equations, which couple real and imaginary components
\[
\frac{\partial F}{\partial v} = -\eta Z , \quad \frac{\partial G}{\partial v} = -X , \quad \frac{\partial H}{\partial v} = -Y - Z ,
\]
\[
\frac{\partial X}{\partial v} = F + G , \quad \frac{\partial Y}{\partial v} = \eta G , \quad \frac{\partial Z}{\partial v} = H .
\]
A further derivation decouples real and imaginary sectors, yielding
\[
\frac{\partial^2 F}{\partial v^2} = -\eta H , \quad \frac{\partial^2 G}{\partial v^2} = -F - G , \quad \frac{\partial^2 H}{\partial v^2} = -\eta G - H ,
\]
\[
\frac{\partial^2 X}{\partial v^2} = -\eta Z - X , \quad \frac{\partial^2 Y}{\partial v^2} = -\eta X , \quad \frac{\partial^2 Z}{\partial v^2} = -Y - Z .
\]
In order to get an uncoupled differential equation for \( F \), one increases the number of derivatives and finds
\[
\frac{\partial^6 F}{\partial v^6} + 2 \frac{\partial^4 F}{\partial v^4} + \frac{\partial^2 F}{\partial v^2} + \eta^2 F = 0 .
\]
Its general solution is discussed in App.B and given by

\[ F = \beta_1 \cos(k_1 v) + \beta_2 \cos(k_2 v) + \beta_3 \cos(k_3 v) \]  \hspace{1cm} (31)

where \( \beta_i \) are constants,

\[ k_i = \frac{2}{\sqrt{3}} \sin(\theta/6 + \delta_i \pi/3) \]  \hspace{1cm} (32)

\[ \theta = \tan^{-1} \left[ \frac{3\sqrt{3} \eta \sqrt{1 - 27 \eta^2/4}}{1 - 27 \eta^2/2} \right] \]  \hspace{1cm} (33)

and \( \delta_1 = 0, \delta_2 = 1, \delta_3 = -1 \).

The constants \( \beta_i \) are fixed by expanding \( \cos(k_i v) \) in series and, expressing results in terms of the roots \( \alpha_i = -k_i^2 \) of the cubic equation \( \alpha_i^3 + 2 \alpha_i^2 + \alpha_i + \eta^2 = 0 \), eq.(B3), one has

\[ F = (\beta_1 + \beta_2 + \beta_3) + \frac{v^2}{2!} (\beta_1 \alpha_1 + \beta_2 \alpha_2 + \beta_3 \alpha_3) + \frac{v^4}{4!} (\beta_1 \alpha_1^2 + \beta_2 \alpha_2^2 + \beta_3 \alpha_3^2) \]
\[ + \frac{v^6}{6!} \sum \beta_i \alpha_i^3 + \frac{v^8}{8!} \sum \beta_i \alpha_i^4 + \frac{v^{10}}{10!} \sum \beta_i \alpha_i^5 + \frac{v^{12}}{12!} \sum \beta_i \alpha_i^6 + \cdots \]  \hspace{1cm} (34)

Comparing results for \( v^0, v^2 \) and \( v^4 \) with those of table I, one learns that

\[ \beta_1 + \beta_2 + \beta_3 = 1 \] ,  \hspace{1cm} (35)
\[ \beta_1 \alpha_1 + \beta_2 \alpha_2 + \beta_3 \alpha_3 = 0 \] ,  \hspace{1cm} (36)
\[ \beta_1 \alpha_1^2 + \beta_2 \alpha_2^2 + \beta_3 \alpha_3^2 = 0 \] .  \hspace{1cm} (37)

Terms proportional to powers of \( v \geq 6 \) are evaluated using combinations of eqs.(35)-(37) and (B3). Thus, for instance

\[ \sum \beta_i \alpha_i^3 = \sum \beta_i [-2 \alpha_i^2 - \alpha_i - \eta^2] = -\eta^2 \]  \hspace{1cm} (38)
\[ \sum \beta_i \alpha_i^4 = \sum \beta_i \alpha_i [-2 \alpha_i^2 - \alpha_i - \eta^2] = 2 \eta^2 \]  \hspace{1cm} (39)
\[ \sum \beta_i \alpha_i^5 = \sum \beta_i \alpha_i^2 [-2 \alpha_i^2 - \alpha_i - \eta^2] = -3 \eta^2 \]  \hspace{1cm} (40)
\[ \sum \beta_i \alpha_i^6 = \sum \beta_i [-2 \alpha_i^2 - \alpha_i - \eta^2]^2 = 4 \eta^2 + \eta^4 \]  \hspace{1cm} (41)

Using these results into eq.(31), one has

\[ F = -\frac{v^6}{6!} \eta^2 + \frac{v^8}{8!} 2 \eta^2 - \frac{v^{10}}{10!} 3 \eta^2 + \frac{v^{12}}{12!} (4 \eta^2 + \eta^4) + \cdots \]  \hspace{1cm} (42)

and the entry in table I is reproduced.

Expressions (35)-(37) yield directly

\[ \beta_i = -\frac{\alpha_j \alpha_k}{(\alpha_i - \alpha_j)(\alpha_k - \alpha_i)} , \]  \hspace{1cm} (43)
with \([i, j, k] \rightarrow \text{cyclic permutations of } [1, 2, 3]\). Alternative versions are useful in calculations and, employing condition (B3), one has

\[
\beta_i = \frac{\eta^2}{\alpha_i (\alpha_i - \alpha_j) (\alpha_k - \alpha_i)}. \tag{44}
\]

The denominator can be simplified using results of App.B and one finds the set of alternatives

\[
\beta_i = \frac{\eta^2}{2 (\alpha_i^2 + \alpha_i) + 3 \eta^2}, \tag{45}
\]

\[
\beta_i = -\frac{\eta^2}{(\alpha_i^2 + \alpha_i) (3 \alpha_i + 1)}, \tag{46}
\]

\[
\beta_i = \frac{\alpha_i + 1}{3 \alpha_i + 1}. \tag{47}
\]

The last result determines the condition

\[
\sum_i \frac{1}{(3 \alpha_i + 1)} = 0. \tag{48}
\]

Result (31) for \(F\) and eqs.(26)-(29) determine the set of functions \(G, H, X, Y,\) and \(Z\). Choosing form (47) for the \(\beta_i\), one has

\[
F = \frac{(\alpha_1 + 1)}{(3 \alpha_1 + 1)} \cos(k_1 v) + [1 \rightarrow 2, 3], \tag{49}
\]

\[
G = -\frac{1}{(3 \alpha_1 + 1)} \cos(k_1 v) + [1 \rightarrow 2, 3], \tag{50}
\]

\[
H = \frac{\eta}{(\alpha_1 + 1) (3 \alpha_1 + 1)} \cos(k_1 v) + [1 \rightarrow 2, 3], \tag{51}
\]

\[
X = -\frac{1}{(3 \alpha_1 + 1)} k_1 \sin(k_1 v) + [1 \rightarrow 2, 3], \tag{52}
\]

\[
Y = \frac{\eta}{\alpha_1 (3 \alpha_1 + 1)} k_1 \sin(k_1 v) + [1 \rightarrow 2, 3], \tag{53}
\]

\[
Z = -\frac{\eta}{\alpha_1 (\alpha_1 + 1) (3 \alpha_1 + 1)} k_1 \sin(k_1 v) + [1 \rightarrow 2, 3]. \tag{54}
\]

The parity of these functions under \(v \rightarrow -v\) is determined by \(\eta\) and therefore \(F, G,\) and \(X\) are even, whereas \(H, Y,\) and \(Z\) are odd.

The analytic form of the operator \(U\), derived from eqs.(20) and (21), reads

\[
U = (FI + GB + HA) + i(YI + ZB + XA), \tag{55}
\]

\[
= \left[ (F + \frac{2}{3} G) I + \left( H \hat{v} + \frac{1}{\sqrt{3}} G \hat{b} \right) \cdot \lambda \right]
\]

\[
+ i \left[ (Y + \frac{2}{3} Z) I + \left( X \hat{v} + \frac{1}{\sqrt{3}} Z \hat{b} \right) \cdot \lambda \right]. \tag{56}
\]
As there is an overlap between $\hat{v}$ and $\hat{b}$, one might consider replacing the latter by the unit vector $\hat{u}$ given by

$$
\hat{b} = \frac{3\sqrt{3}}{2} \eta \hat{v} + \sqrt{1 - \frac{27\eta^2}{4}} \hat{u},
$$

such that $\hat{v} \cdot \hat{u} = 0$. However, this is not especially useful.

In order to deal with a more compact expression, one defines the quantities

$$
S = (F + \frac{2}{3} G), \quad Q_i = \sqrt{\frac{2}{3}} \left( H \hat{v}_i + \frac{1}{\sqrt{3}} G \hat{b}_i \right),
$$

$$
W = (Y + \frac{2}{3} Z), \quad P_i = \sqrt{\frac{2}{3}} \left( X \hat{v}_i + \frac{1}{\sqrt{3}} Z \hat{b}_i \right),
$$

and has

$$
U = \left[ SI + \sqrt{\frac{2}{3}} Q \cdot \lambda \right] + i \left[ WI + \sqrt{\frac{2}{3}} P \cdot \lambda \right].
$$

### III. UNITARITY

The matrix $U$ given by eqs. (55), (56) and (60) is unitary. Explicit multiplication using form (55), together with eqs. (14)-(16), yields

$$
U U^\dagger = C_I I + C_B B + C_A A,
$$

with

$$
C_I = F^2 + 2\eta GH + Y^2 + 2\eta XZ,
$$

$$
C_B = G^2 + H^2 + 2FG + X^2 + Z^2 + 2YZ,
$$

$$
C_A = \eta G^2 + 2FH + 2GH + \eta Z^2 + 2XY + 2XZ,
$$

and, in App.C, one shows that

$$
C_I = 1, \quad C_B = 0, \quad C_A = 0.
$$

Alternatively, form (60) gives rise to

$$
U U^\dagger = S^2 + Q^2 + W^2 + P^2 + \left[ \sqrt{6} SQ_k + \frac{2}{3} Q_i Q_j d_{ijk} + \sqrt{6} WP_k + \frac{3}{2} P_i P_j d_{ijk} + \frac{3}{2} Q_i P_j f_{ijk} \right] \lambda_k.
$$

Definitions (58), (59), with results (5), (8) and (9), allow one to show that the term within square brackets is

$$
[\cdots] = C_A \hat{v}_k + \frac{1}{\sqrt{3}} C_B \hat{b}_k = 0.
$$
Thus one finds

\[ U U^\dagger = S^2 + Q^2 + W^2 + P^2 = 1 \]  

(68)

and writing

\[ Q^2 = \frac{2}{3} G^2 + \frac{2}{3} H^2 + 2 \eta GH \quad , \quad P^2 = \frac{2}{3} X^2 + \frac{2}{3} Z^2 + 2 \eta XZ \]  

(69)

one has

\[ U U^\dagger = F^2 + \frac{4}{3} FG + \frac{2}{3} G^2 + \frac{2}{3} H^2 + 2 \eta GH + Y^2 + \frac{4}{3} YZ + \frac{2}{3} Z^2 + \frac{2}{3} X^2 + 2 \eta XZ \]

\[ = C_I + \frac{2}{3} C_B = 1 . \]  

(70)

The \( SU(3) \) unitarity condition in form (68) indicates that the variables \( S, Q, W, P \) are constrained to the surface of a four-dimensional sphere, irrespective of the values of the free parameters \( v \) and \( \eta \). The dependence of the functions \( S^2 \), \( Q^2 \), \( W^2 \), and \( P^2 \) on \( v \) are given in fig.1, where full and dashed curves correspond to \( \eta = 0 \) and \( \eta = 0.1361 \), respectively. As expected from the explicit results for \( F, \cdots Z \) in eqs.(49)-(54), just the case \( \eta = 0 \) corresponds to cyclic structures. It is worth noting that only for \( \eta \neq 0 \), the odd scalar term \( W \) is non-vanishing.

![FIG. 1: \( S^2, Q^2, W^2 \) and \( P^2 \) as functions of \( v \) for \( \eta = 0 \) (continuous curves) and \( \eta = 0.1361 \) (dashed curves).](image)

The situation in \( SU(3) \) contrasts with the \( SU(2) \) case, where the variation of the chiral angle \( \theta \) gives rise to oscillations of scalar and pseudoscalar variables, constrained to a circle. In fig.2 one shows the behaviour of the components \( U_{\text{re}}^2 = S^2 + Q^2 \) and \( U_{\text{im}}^2 = P^2 \) as functions of \( v \) for the case \( \eta = 0 \), in which \( W = 0 \). Although these functions oscillate, their values are restricted to the intervals \( 1 \geq U_{\text{re}}^2 \geq 1/3 \) and \( 2/3 \geq U_{\text{im}}^2 \geq 0 \). The individual scalar
contributions $S^2$ and $Q^2$ do vanish at specific points, but their sum does not. This interplay between $S$ and $Q$ within the real sector of sector $U$ is a distinctive feature of the $SU(3)$ case. For the sake of completeness, in fig.2 we also display the curves corresponding to the case $\eta = 0.1361$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{$U^2_{\text{re}}$ and $U^2_{\text{im}}$ as functions of $v$ for $\eta = 0$ (continuous curves) and $\eta = 0.1361$ (dashed curves).}
\end{figure}

\section*{IV. CLASSICAL LIMIT}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Classical $S$, $Q$ and $P$ as functions of $v$.}
\end{figure}

The variable $v$ is the analogous of the chiral angle $\theta$ in $SU(3)$ and, in some problems, it may be assumed to be classical. As the same does not apply for $\eta$, the classical case is formally obtained by making $\eta \to 0$. In this limit, one has

\begin{align}
  k_1 & \to \eta \quad \to \quad a_1 \to -\eta^2, \\
  k_{2,3} & \to \frac{1 \pm \eta}{2} \quad \to \quad a_{2,3} \to \mp (1 + \eta),
\end{align}

\section*{REFERENCES}
and finds

\[ F \rightarrow 1 , \quad (73) \]
\[ G \rightarrow -1 + \cos v , \quad (74) \]
\[ X \rightarrow \sin v , \quad (75) \]

whereas \( H, Y, Z \rightarrow \mathcal{O}(\eta) \).

![Diagram](image)

**FIG. 4:** Projections of the classical circle over planes (a) \( PQ \), (b) \( QS \), and (c) \( SP \); figure (b) shows the profile of the circle over the plane representing \( U_{re} \), whereas figures (a) and (c) are obtained by rotating it by \( \pi/2 \) along axes \( Q \) and \( S \) respectively; in all figures, the axis not shown points out of the page.

The behavior of the functions

\[ S \rightarrow \frac{1}{3} (1 + 2 \cos v) , \quad (76) \]
\[ Q \rightarrow \frac{\sqrt{2}}{3} (-1 + \cos v) , \quad (77) \]
\[ P \rightarrow \frac{\sqrt{2}}{3} \sin v , \quad (78) \]

is shown in fig.3 and the matrix \( U \) becomes

\[ U = \left[ \frac{1}{3} (1 + 2 \cos v) \mathbf{I} + \frac{1}{\sqrt{3}} (-1 + \cos v) \hat{b} \cdot \mathbf{\lambda} \right] + i (\sin v) \hat{v} \cdot \mathbf{\lambda} . \quad (79) \]

The unitary condition (68) constrains \( S, Q \) and \( P \) to the surface of a sphere because

\[ U U^\dagger \rightarrow S^2 + Q^2 + P^2 = 1 , \quad (80) \]

and the variation of \( v \) gives rise to a circumference, with projections over planes \( PQ, QS, \) and \( SP \) shown in fig.4. Figure (b), depicting the two components of \( U_{re} \), is particularly interesting, for it shows the profile of a circle as a straight line, since eqs.(76) and (77) yield

\[ S = 1 + \frac{1}{\sqrt{2}} Q . \quad (81) \]
Thus, the path determined by \( v \) is tilted circumference, defined by the intersection of the unit sphere with a plane orthogonal to the axes \( Q \) and \( S \), inclined by an angle \( \epsilon = \tan^{-1} \sqrt{2} \), which amounts to \( \sin \epsilon = \sqrt{2/3} \), \( \cos \epsilon = \sqrt{1/3} \), and \( \epsilon \sim 54.76^\circ \). Performing a rotation around the \( P \) axis, as in fig.5, one has

\[
S' = \sqrt{\frac{1}{3}} S - \sqrt{\frac{2}{3}} Q, \quad Q' = \sqrt{\frac{2}{3}} S + \sqrt{\frac{1}{3}} Q, \tag{82}
\]

and the equation of the plane containing the circle is \( S' = \sqrt{1/3} \). Its edge is determined by condition (80), which now reads \( Q'^2 + P^2 = 1 - S'^2 = 2/3 \), corresponding to a radius of \( \sqrt{2/3} \) and to \( Q' = \sqrt{2/3} \cos v \).

![Figure 5](image)

**FIG. 5**: Projections of the classical circle over planes (a) \( Q'S' \) and (b) \( Q'P \); in all figures, the axis not shown points out of the page.

V. LEFT AND RIGHT FORMS

The analytic result for \( U \), eq.(60), allows one to derive the left and right forms \( L^\mu \) and \( R^\mu \), defined by

\[
L^\mu = U^\dagger \frac{\partial U}{\partial x^\mu}, \quad R^\mu = U \frac{\partial U^\dagger}{\partial x^\mu}. \tag{83}
\]

They are related to the vector and axial currents \( V^\mu \) and \( A^\mu \) by

\[
L^\mu = i \left( V^\mu - A^\mu \right), \quad R^\mu = i \left( V^\mu + A^\mu \right) \tag{84}
\]

and, owing to the unitarity condition \( U U^\dagger = 1 \), one has

\[
[L^\mu]^\dagger = -L^\mu, \quad [R^\mu]^\dagger = -R^\mu. \tag{85}
\]
The left form is evaluated in App.D and reads

\[ L^\mu = i \left\{ \frac{3}{2} \left[ (Q_i \partial^\mu Q_j + P_i \partial^\mu P_j) f_{ijk} \right] + \sqrt{\frac{2}{3}} (S \partial^\mu P_k - P_k \partial^\mu S - W \partial^\mu Q_k + Q_k \partial^\mu W) + \frac{3}{2} (Q_i \partial^\mu P_j - P_i \partial^\mu Q_j) d_{ijk} \right\} \lambda_k. \]  

(86)

Writing \( L^\mu = i (V_k^\mu - A_k^\mu) \lambda_k \), eqs.(D38) and (D40) allow one to express this result in terms of the basic functions \( F, \ldots, Z \) as

\[ V_k^\mu = \left[ (H^2 + X^2) \hat{v}_i \partial^\mu \hat{v}_j + \frac{1}{\sqrt{3}} (G H + X Z) \left( \hat{v}_i \partial^\mu \hat{b}_j + \hat{b}_i \partial^\mu \hat{v}_j \right) \right. \]

\[ \left. + \frac{1}{3} (G^2 + Z^2) \hat{b}_i \partial^\mu \hat{b}_j \right] f_{ijk}, \]  

(87)

\[ A_k^\mu = - \left\{ [1] \hat{v}_k \partial^\mu v \right. \]

\[ + \frac{1}{(1 - \frac{2}{3} \eta^2)} \left[ (G Y - F Z) + \frac{2}{3} \eta (F X - H Y) \right. \]

\[ + \frac{3}{4} \eta (H Z - G X) - \frac{2}{3} v \eta \right] \hat{v}_k \partial^\mu \eta \]

\[ + \frac{1}{(1 - \frac{2}{3} \eta^2)} \frac{1}{\sqrt{3}} \left[ \frac{3}{2} (H Y - F X) + \frac{1}{2} (G X - H Z) \right. \]

\[ + \frac{2}{3} \eta (F Z - G Y) + \frac{3}{2} v \right] \hat{b}_k \partial^\mu \eta \]

\[ + \left[ (F + \frac{2}{3} G) X - H (Y + \frac{2}{3} Z) \right] \partial^\mu \hat{v}_k \]

\[ + \frac{1}{\sqrt{3}} \left[ (F + \frac{2}{3} G) Z - G (Y + \frac{2}{3} Z) \right] \partial^\mu \hat{b}_k \]

\[ + \frac{1}{\sqrt{3}} (H Z - G X) \left( \hat{v}_i \partial^\mu \hat{b}_j - \hat{b}_i \partial^\mu \hat{v}_j \right) d_{ijk} \right\}. \]  

(88)

VI. CHIRAL SYMMETRY

Results presented hitherto are generic and hold for unspecified \( SU(3) \) octet-vectors \( v \). One now concentrates on the case of pseudoscalar mesons \( \phi \) and, making \( v \rightarrow \phi \), discusses the chiral transformations of the matrix \( U(\phi) \) given by eq.(60). Its vector transformations are associated with changes in the directions of \( \hat{\phi} \) and \( \hat{b} \) and need not be written explicitly. Concerning axial transformations \( \delta A^A \phi \), the most general non-linear form has been discussed by Weinberg[21] and is given by

\[ \delta A^A \phi_a = f^A(\phi^2) \beta_a + g^A(\phi^2) \beta_i \phi_i \phi_a, \]  

(89)
where $\beta_i$ are free parameters $\beta_i$ and $f^A$ is an arbitrary function, whereas

$$g^A = \frac{2f^A f'^A + 1}{f^A - 2\phi^2 f'^A},$$  \hspace{1cm} (90)

with $f'^A = df^A/d\phi^2$. The axial transformation of a generic function $\psi(v, \eta)$ is

$$\delta^A \psi = \frac{d\psi}{d\phi} \delta^A \phi_a = \left[ \frac{\partial \psi}{\partial \phi} \hat{\phi}_a + \frac{\partial \psi}{\partial \eta} \frac{1}{v} \left( -3\eta \hat{\phi}_a + \frac{2}{\sqrt{3}} \hat{b}_a \right) \right] \delta^A \phi_a,$$  \hspace{1cm} (91)

using eq.(D30). Evaluating the derivatives of $(F, \cdots Z)$ with the help of eqs.(26), (27) and (D11)-(D16), one has

$$\delta^A F = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( 9\eta^2 G - 3\eta H \right) + \left( \frac{9}{2} \eta^2 X - 3\eta Y - \eta Z \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( -3\eta G + H \right) + \left( -\frac{3}{2} \eta X + Y + \frac{9}{4} \eta^2 Z \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (92)

$$\delta^A G = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( -\frac{27}{4} \eta^2 G + \frac{9}{2} \eta H \right) + \left( -X + \frac{9}{2} \eta Y + \frac{3}{2} \eta Z \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( \frac{\eta G - \frac{3}{2} H} + \left( \frac{9}{4} \eta X - \frac{3}{2} \eta Y - \frac{1}{2} \eta Z \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (93)

$$\delta^A H = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( 3\eta G - \frac{27}{4} \eta^2 H \right) + \left( \frac{3}{2} \eta X - Y - Z + \frac{9}{4} \eta^2 Z \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( -G + \frac{9}{4} \eta H \right) + \left( -\frac{1}{2} X + \frac{9}{4} \eta Y + \frac{3}{4} \eta Z \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (94)

$$\delta^A X = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( -\frac{27}{4} \eta^2 X + 3\eta Z \right) + \left( F + G - \frac{9}{2} \eta^2 G - \frac{3}{2} \eta H \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( \frac{9}{4} \eta X - Z \right) + \left( -\frac{9}{4} F - \frac{3}{4} \eta G + \frac{1}{2} H \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (95)

$$\delta^A Y = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( -3\eta X + 9\eta^2 Z \right) + \left( 3\eta F + \eta G - \frac{9}{2} \eta^2 H \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( X - 3\eta Z \right) + \left( -F + \frac{3}{4} \eta G + \frac{3}{2} \eta H \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (96)

$$\delta^A Z = \frac{1}{1 - \frac{27}{4} \eta^2} \left\{ \left[ \frac{1}{v} \left( \frac{9}{2} \eta X - \frac{27}{2} \eta^2 Z \right) + \left( -\frac{9}{2} \eta F - \frac{3}{2} \eta G + H \right) \right] \hat{\phi}_a ight. $$

$$+ \frac{2}{\sqrt{3}} \left[ \frac{1}{v} \left( -\frac{3}{2} X + \frac{9}{2} \eta Z \right) + \left( \frac{3}{2} F + \frac{1}{2} G - \frac{9}{4} \eta H \right) \right] \hat{b}_a \right\} \delta^A \phi_a,$$  \hspace{1cm} (97)
whereas the two directions transform as

\[
\delta^A \hat{\phi}_i = \frac{1}{\phi} \left( \delta_{ia} - \hat{\phi}_i \hat{\phi}_a \right) \delta^A \phi_a ,
\]

\[
(98)
\]

\[
\delta^A \hat{b}_i = \frac{2\sqrt{3}}{\phi} \left( d_{ija} \hat{\phi}_j - \frac{1}{\sqrt{3}} \hat{b}_i \hat{\phi}_a \right) \delta^A \phi_a .
\]

\[
(99)
\]

One notes that, as expected, axial transformations change the parities of the functions \(F, \cdots, Z\), and the directions \(\hat{\phi}\) and \(\hat{b}\), under the operation \(\phi \rightarrow -\phi\).

Using results (92)-(97) one can, for instance, show that the functions \(C_I, C_B\) and \(C_A\) given by eqs.(62)-(63) are invariant under axial transformations by means of explicit calculations. In the case of classical fields, these transformations become much simpler and read

\[
\delta^A F = 0 , \quad \rightarrow \quad \delta^A 1 = 0 ,
\]

\[
(100)
\]

\[
\delta^A G = -X \hat{\phi}_a \delta^A \phi_a , \quad \rightarrow \quad \delta^A (-1 + \cos \phi) = -\sin \phi \delta^A \phi ,
\]

\[
(101)
\]

\[
\delta^A X = (F + G) \hat{\phi}_a \delta^A \phi_a , \quad \rightarrow \quad \delta^A \sin \phi = \cos \phi \delta^A \phi .
\]

\[
(102)
\]

where one has used \(\hat{\phi}_a \delta^A \phi_a = \delta^A \phi\). Thus, the axial transformation implements a rotation along the tilted circumference discussed in sect.IV.

**VII. SIGMA-MODEL**

Sigma models are formulated in terms of effective degrees of freedom. Its \(SU(2)\) version was proposed before quarks were known[2], but its \(SU(3)\) extension was already based on a quark model and employed nonets of scalar and pseudoscalar fields, which change into each other under axial transformations[10]. This strategy for implementing chiral symmetry is therefore different from the adopted in the \(SU(2)\) case, where axial charges transform back and forth tree pions into a single \(\sigma\). The use of 18 fields accomodated into two nonets indicates that one is in fact dealing with a \(U(3) \times U(3)\) symmetry. From the very beginning, the standard dynamical lagrangian also included a trilinear term which is chiral invariant "for transformations that do not involve the ninth meson"[22], also needed to circumvent problems associated with \(U(1)_A\) anomaly[23–25]. So, in fact, the \(SU(3)\) linear \(\sigma\)-model is based on the group \(U_V(1) \times SU(3)_R \times SU(3)_L\). Attempts to move beyond this framework already exist in the literature[26].

An important feature of \(\sigma\)-models is their ability to describe the strong vacuum as either an empty state or quark condensates. The latter picture applies to the QDC ground state, which contains \(u\bar{u}, d\bar{d}\), and \(s\bar{s}\) condensates and is close to the Nambu-Goldstone realization of \(SU(3) \times SU(3)\), differences being ascribed to the small quark massses. The massless ground state is the point of departure for the formulation of chiral perturbation theory (ChPT)[8, 9, 11, 27], based on hadronic effective lagrangians in which pseudoscalar
mesons are the lowest energy excitations. The theory replicates carefully the symmetries of QCD, whereas quark masses are treated as small symmetry breaking terms, introduced by means of perturbative calculations including loops and renormalization counter-terms. ChPT is precise, accounting for a wide range of phenomena. Although originally developed for low-energy processes involving just pseudoscalars, it has been reliably extended to include resonances[28], baryons[29–32], and heavy mesons[33]. However, as resonances correspond to nonperturbative states, the applicability of ChPT is restricted to energies below the \( \rho(770) \) mass. Above that point, one needs to resort to further extensions, which may be performed by means of dispersion relations[34], dynamical models[35] or employing unitarization techniques[36]. The properties of the QCD vacuum lead to a mathematical formulation of ChPT[11] based on \( SU(3) \times SU(3) \) symmetry and involving just an octet of pseudoscalar mesons, the \( \eta' \) being considered as an external source. This is in sharp contrast with the \( U(3) \times U(3) \) approach based on nonets.

In the sequence, an alternative \( SU(3) \times SU(3) \) non-linear \( \sigma \)-model is presented, with the purpose of providing an application of results discussed in the previous sections. The basic lagrangian has the standard form introduced by Lévy[10] (see also ref.[22]) and is written as

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu M \partial^\mu M^\dagger) + \frac{1}{2} \mu^2 (M M^\dagger) + c \left[ \text{det}(M) + \text{det}(M^\dagger) \right] \\
- \lambda \langle M M^\dagger M M^\dagger \rangle - \lambda' \langle M M^\dagger \rangle + \mathcal{L}_{SB},
\]

(103)

where \( \langle \cdots \rangle \) indicates the trace and \( \mathcal{L}_{SB} \) is a symmetry breaking term. The the usual \( U(3) \times U(3) \) linear version is based on the matrix

\[
\sqrt{2} M_L = \frac{\sqrt{2}}{\sqrt{3}} (\sigma_0 + i \pi_0) \lambda_0 + (\sigma_i + i \pi_i) \lambda_i,
\]

(104)

with \( \lambda_0 = \sqrt{\frac{2}{3}} I \) and \( i = 1, \cdots 8 \), whereas the non-linear version employs a matrix \( M \) with 17 fields, constructed by generalizing eq.(56) in the classical limit. One writes

\[
U = (F + \frac{2}{3} G) I + \left( \frac{1}{\sqrt{3}} G \hat{b}_i + i X \hat{\phi}_i \right) \lambda_i \\
\rightarrow \sqrt{2} M = (\Phi + \frac{2}{3} R) I + \left( \frac{1}{\sqrt{3}} R \hat{b}_i + i \pi \hat{\phi}_i \right) \lambda_i,
\]

(105)

using \( \bar{f} F \rightarrow \Phi, \bar{f} G \hat{b}_i \rightarrow R \hat{b}_i = R_i, \) and \( \bar{f} X \hat{\phi}_i \rightarrow \pi \hat{\phi}_i = \pi_i \), where \( \bar{f} \) is a quantity with dimension of mass, needed because \( F, G \) and \( X \) are dimensionless. The quantum numbers of these new functions ensure that \( \Phi \) is a \( SU(3) \) scalar, \( \pi_i \) is pseudoscalar, and \( R_i \) can be associated with a scalar resonance. Their axial transformations are given by eqs.(98)-(99), together with

\[
\delta^A \Phi = 0, \quad \delta^A R = -\pi \hat{\pi}_a \delta^A \phi_a, \quad \delta^A \pi = (\Phi + R) \hat{\pi}_a \delta^A \phi_a,
\]

(106)

induced from eqs.(100)-(102).
In the absence of $\mathcal{L}_{SB}$, the lagrangian (103) is chiral invariant. Indeed, explicit calculations yield
\[
\langle 2 M M^\dagger \rangle = 3 \Phi^2 + 2 (2\Phi R + R^2 + \pi^2) ,
\]
\[
[\det(M) + \det(M^\dagger)] = 2 \Phi (\Phi^2 + 2 \Phi R + R^2 + \pi^2) ,
\]
\[
\langle 4 M M^\dagger M M^\dagger \rangle = 3 \Phi^4 + 2 (2\Phi^2 + 2 \Phi R + R^2 + \pi^2) (2\Phi R + R^2 + \pi^2) .
\]
Using (106), one has $\delta^4 (2\Phi R + R^2 + \pi^2) = 2 [(\Phi + R) \delta^4 R + \pi \delta^4 \pi] = 0$ and concludes that the axial transformations of these quantities also vanish. The symmetry breaking term is given by
\[
\mathcal{L}_{SB} = \frac{1}{4} \langle \chi^\dagger M M + M^\dagger M^\dagger \chi \rangle ,
\]
with $\chi = \epsilon_0 + \epsilon_8 \lambda$. One remarks that this function is written in terms of non-linear fields and not suited to be directly compared with its linear counterpart. A specific transformation of the linear $\sigma$-model lagrangian into a non-linear one is presented in ref.[37] and there it is possible to follow the change in form of the symmetry breaking term.

In calculations, it is convenient to employ a new variable, defined by
\[
\sqrt{\frac{2}{3}} \Sigma = (\Phi + \frac{2}{3} R)
\]
and the lagrangian in Cartesian coordinates becomes
\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \Sigma \partial^\mu \Sigma + \frac{1}{3} \partial_\mu R \partial^\mu R + \partial_\mu \pi \partial^\mu \pi) + \frac{1}{2} \mu^2 (\Sigma^2 + \frac{1}{3} R^2 + \pi^2)
\]
\[
+ c 2 \sqrt{\frac{2}{3}} \left[ \frac{2}{3} \Sigma^2 + \Sigma (-\frac{1}{3} R^2 + \pi^2) + \sqrt{2} R_i \left( \frac{1}{3} R_j R_k - \pi_j \pi_k \right) d_{ijk} \right]
\]
\[
- \left( \frac{1}{3} \lambda + \lambda' \right) (\Sigma^2 + \frac{1}{3} R^2 + \pi^2)^2
- \frac{1}{2} \lambda \left[ \frac{8}{3} \Sigma^2 R^2 + \frac{4 \sqrt{2}}{3} \Sigma R_i \left( \frac{1}{3} R_j R_k + \pi_j \pi_k \right) + \left( \frac{1}{3} R_i R_j + \frac{1}{2} \pi_i \pi_k \right) d_{ij} + \frac{1}{3} R_i \pi_j \pi_k f i j s k l s \right] + \epsilon_0 \frac{1}{2} \left( \Sigma^2 + \frac{1}{3} R^2 - \pi^2 \right) + \epsilon_8 \frac{1}{2} \left[ \frac{2 \sqrt{2}}{3} \Sigma R_i + \left( \frac{1}{3} R_i R_j - \pi_j \pi_j \right) d_{ij} \right] .
\]
Fluctuations are disentangled from the classical components $\sigma$ and $\sigma_8$ by writing
\[
\Sigma = \sigma + f ,
\]
\[
R_i = \delta_{i8} \sigma_8 + \sqrt{3} r_i ,
\]
where the factor $\sqrt{3}$ ensures the proper normalization of the $r_i$. This decomposition yields a vacuum energy
\[
V_{vac} = -\frac{1}{2} \mu^2 \left[ \sigma^2 + \frac{1}{3} \sigma_8^2 \right] - c \sqrt{\frac{2}{3}} \left[ \sigma^2 - \frac{1}{2} \sigma_8^2 - \frac{1}{3 \sqrt{3}} \sigma_8^2 \right] + \left( \lambda' + \frac{1}{3} \lambda \right) \left( \sigma^2 + \frac{1}{3} \sigma_8^2 \right)^2
\]
\[
+ \lambda \left( \frac{2}{3} \sigma^2 \sigma_8 - \frac{1}{\sqrt{6}} \sigma^2 \sigma_8^3 + \frac{1}{24} s_8^4 \right) - \epsilon_0 \left( \sigma^2 + \frac{1}{3} \sigma_8^2 \right) - \epsilon_8 \frac{\sqrt{2}}{3} \left( \sigma - \frac{1}{2 \sqrt{6}} \sigma_8 \right) \sigma_8
\]
and two minimization conditions, given by
\[
\frac{\delta V_{\text{vac}}}{\delta \sigma} = -\mu^2 - c 4 \sqrt{\frac{2}{3}} (\sigma^2 - \frac{1}{6} \sigma_s^2) + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2) \sigma + \lambda \frac{8}{9} \left(s - \frac{1}{2 \sqrt{6}} \sigma_s\right) \sigma_s^2
\]
\[-\epsilon_0 \sigma - \epsilon_s \frac{\sqrt{2}}{3} \sigma_s = 0 , \tag{116}\]
\[
3 \frac{\delta V_{\text{vac}}}{\delta \sigma_s} = -\mu^2 + c 4 \sqrt{\frac{2}{3}} (\sigma + \frac{1}{6} \sigma_s) \sigma_s + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2) \sigma
\]
\[+ \lambda \frac{1}{3} \left(s^2 - \frac{1}{2 \sqrt{6}} \sigma_s \sigma_s^2 + \frac{1}{12} s_s^2\right) \sigma_s^2 - \epsilon_0 \sigma_s - \epsilon_s \sqrt{2} \left(\sigma - \frac{1}{\sqrt{6}} \sigma_s\right) = 0 . \tag{117}\]
The pseudoscalar masses have the generic form[38]
\[
M_{ij}^2 = \left[-\mu^2 - c 4 \sqrt{\frac{2}{3}} \sigma + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2) + \epsilon_0\right] \delta_{ij}
\[+ \left[-c \frac{8}{\sqrt{3}} \sigma_s + \lambda \frac{4 \sqrt{2}}{3} \left(\sigma - \frac{1}{2 \sqrt{6}} \sigma_s\right) \sigma_s + \epsilon_s\right] d_{ij} + \lambda \frac{4}{3} \sigma_s^2 f_{sij} f_{sij} \tag{118}\]
and correspond to
\[
M_{\pi}^2 = -\mu^2 - c 4 \sqrt{\frac{2}{3}} \left(\sigma - \frac{1}{\sqrt{3}} \sigma_s\right) + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2)
\[+ \lambda \frac{4 \sqrt{2}}{3} \left(\sigma - \frac{5}{\sqrt{6}} \sigma_s\right) \sigma_s + \epsilon_0 + \frac{1}{\sqrt{3}} \epsilon_s \right) , \tag{119}\]
\[
M_{K}^2 = -\mu^2 - c 4 \sqrt{\frac{2}{3}} \left(\sigma + \frac{1}{\sqrt{3}} \sigma_s\right) + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2)
\[+ \frac{1}{\sqrt{3}} \epsilon_s \right) , \tag{120}\]
\[
M_{s}^2 = -\mu^2 - c 4 \sqrt{\frac{2}{3}} \left(\sigma + \frac{2}{\sqrt{3}} \sigma_s\right) + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2)
\[+ \frac{1}{\sqrt{3}} \epsilon_s \right) , \tag{121}\]
The general expression for the masses of the scalar resonances reads
\[
m^2 \times \text{fields}^2 = \left[-\mu^2 - c 8 \sqrt{\frac{2}{3}} \sigma_0 + (\lambda' + \frac{1}{3} \lambda) 8 \sigma^2 + \lambda \frac{8}{9} \sigma_s^2 - \epsilon_0\right] f^2
\[+ \left[-\mu^2 + c 4 \sqrt{\frac{2}{3}} \sigma_0 + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \frac{1}{3} \sigma_s^2) - \epsilon_0\right] r^2 + (\lambda' + \frac{1}{3} \lambda) \sigma_s^2 r_s^2
\[+ \left[-c \frac{8}{\sqrt{3}} \sigma_s + \lambda 4 \sqrt{2} \left(s_0 s_s - \frac{1}{6 \sqrt{6}} \sigma_s^2\right) - \epsilon_s\right] r_i r_j d_{ij} + \lambda \frac{4}{3} r_i r_j d_{is} d_{sj} \tag{122}\]
\[+ \left[c \frac{8}{\sqrt{3}} \sigma_s + (\lambda' + \frac{1}{3} \lambda) \frac{16}{\sqrt{3}} \sigma_s \sigma_s + \lambda \frac{32}{3 \sqrt{3}} \left(s_0 s_s - \frac{1}{4} \sqrt{2} \sigma_s^2\right) - 2 \frac{2}{3} \epsilon_s\right] f r_s \]
and yields

\[ m_0^2 = -\mu^2 + c 4 \sqrt{\frac{2}{3}} \left( \sigma - \sqrt{\frac{2}{3}} s_8 \right) + (\lambda' + \frac{1}{3} \lambda) 4 \left( \sigma^2 + \frac{1}{3} \sigma_8^2 \right) \]

\[ + \lambda 4 \sqrt{2} \left( \sigma + \frac{1}{\sqrt{3}} \sigma_8 \right) \sigma_8 - \epsilon_0 - \frac{1}{\sqrt{3}} \epsilon_8 \] (123)

\[ m_\kappa^2 = -\mu^2 + c 4 \sqrt{\frac{2}{3}} \left( \sigma + \frac{1}{\sqrt{3}} s_8 \right) + (\lambda' + \frac{1}{3} \lambda) 4 \left( \sigma^2 + \frac{1}{3} \sigma_8^2 \right) \]

\[ - \lambda 2 \sqrt{2} \left( \sigma - \frac{1}{3 \sqrt{6}} \sigma_8 \right) \sigma_8 - \epsilon_0 + \frac{1}{2 \sqrt{6}} \epsilon_8 \] (124)

\[ m_{88}^2 = -\mu^2 + c 4 \sqrt{\frac{2}{3}} \left( \sigma + \sqrt{\frac{2}{3}} \sigma_8 \right) + (\lambda' + \frac{1}{3} \lambda) 4 \left( \sigma^2 + \frac{1}{3} \sigma_8^2 \right) \]

\[ - \lambda 4 \sqrt{2} \left( \sigma - \frac{1}{3 \sqrt{6}} \sigma_8 \right) \sigma_8 + (\lambda' + \frac{1}{3} \lambda) \frac{8}{3} \sigma_8^2 - \epsilon_0 + \frac{1}{\sqrt{3}} \epsilon_8 \] (125)

\[ m_{00}^2 = -\mu^2 - c 8 \sqrt{\frac{2}{3}} \sigma + (\lambda' + \frac{1}{3} \lambda) 8 \sigma^2 + \lambda \frac{8}{3} \sigma_8^2 - \epsilon_0 \] (126)

\[ m_{08}^2 = c \frac{8}{\sqrt{3}} \sigma_8 + (\lambda' + \frac{1}{3} \lambda) \frac{16}{\sqrt{3}} \sigma \sigma_8 + \lambda \frac{32}{3 \sqrt{3}} \left( \sigma \sigma_8 - \frac{1}{3 \sqrt{2}} \sigma_8^2 \right) - 2 \sqrt{\frac{2}{3}} \epsilon_8 \] (127)

The 0-8 sector is diagonalized by writing

\[ m_0^2 = m_{00}^2 \cos^2 \theta + m_{88}^2 \sin^2 \theta - m_{08}^2 \sin \theta \cos \theta \],

\[ m_8^2 = m_{00}^2 \sin^2 \theta + m_{88}^2 \cos^2 \theta + m_{08}^2 \sin \theta \cos \theta \],

with

\[ \theta = \frac{1}{2} \tan^{-1} \left[ \frac{m_{08}^2}{m_{88}^2 - m_{00}^2} \right] \] (130)

In order to fix the parameters of the model, one notes that eq.(86) gives rise to the classical axial current

\[ A_k^\mu = -(F + \frac{2}{3} G) \partial^\mu (X \hat{v}_k) + (X \hat{v}_k) \partial^\mu (F + \frac{2}{3} G) \]

\[-\frac{1}{\sqrt{3}} \left[ (G b_i) \partial^\mu (X \hat{v}_j) - (X \hat{v}_j) \partial^\mu (G b_i) \right] d_{ijk} \]

\[ \rightarrow - \sqrt{\frac{2}{3}} \Sigma \partial^\mu \pi_k + \sqrt{\frac{2}{3}} \partial^\mu \Sigma \pi_k - \frac{1}{\sqrt{3}} (R_i \partial^\mu \pi_j - \pi_i \partial^\mu R_j) d_{ijk} \]

\[ = - \left[ \sqrt{\frac{2}{3}} \sigma \delta_{jk} + \frac{1}{\sqrt{3}} \sigma_8 d_{8jk} \right] \partial^\mu \pi_j + \text{fluctuations} \] (131)

This result is a precursor of PCAC and already allows one to identify the pseudoscalar decay constants as

\[ f_s = \sqrt{\frac{2}{3}} \sigma + \frac{1}{3} \sigma_8 \], \[ f_K = \sqrt{\frac{2}{3}} \sigma - \frac{1}{6} \sigma_8 \], \[ f_s = \sqrt{\frac{2}{3}} \sigma - \frac{1}{3} \sigma_8 \]. \[ (132) \]
Adopting $f_K = 1.22 f_\pi$, one has $f_8 = f_\eta = 1.3 f_\pi$, in agreement with the prediction of ChPT\cite{11}. As eq.\((111)\) indicates, the chiral invariant field $\Phi$ also has a classical component $\Phi_{\text{cl}}$, given by

$$\Phi_{\text{cl}} = \sqrt{\frac{2}{3}} \sigma - \frac{2}{3} \sigma_8$$

(133)

and, numerically, $\sigma = 1.40 f_\pi$, $\sigma_8 = -0.44 f_\pi$, $\Phi_{\text{cl}} = 1.44 f_\pi$. Internal consistency requires that the symmetry breaking lagrangian (110) gives rise to PCAC and its expansion produces

$$\mathcal{L}_{\text{SB}} = \frac{1}{2} \left( \epsilon_0 \pi^2 + \epsilon_8 \pi_i \pi_j d_{ij8} \right) + \text{scalar contributions} .$$

(134)

The axial transformation of the fields $\pi_i$ are given by eq.\((89)\) and one has $\delta A_i \pi_i = f_A(0) \beta_i + \text{higher order terms}$, where $f_A(0)$ is a constant to be determined. Thus

$$\partial_\mu A_\mu^i = - \frac{\delta A \mathcal{L}_{\text{SB}}}{\delta \beta_i} = f^A(0) \left( \epsilon_0 \delta_{ij} + \epsilon_8 d_{ij8} \right) \pi_j + \text{higher order terms} ,$$

(135)

and the PCAC condition yields

$$f^A(0) \left( \epsilon_0 + \frac{1}{\sqrt{3}} \epsilon_8 \right) = f_\pi M^2_\pi ,$$

(136)

$$f^A(0) \left( \epsilon_0 - \frac{1}{\sqrt{3}} \epsilon_8 \right) = f_K M^2_K ,$$

(137)

$$f^A(0) \left( \epsilon_0 - \frac{1}{\sqrt{3}} \epsilon_8 \right) = f_8 M^2_8 .$$

(138)

Using eqs.\((136)\), \((137)\), and \((132)\), one writes

$$f^A(0) \epsilon_0 = \frac{1}{3} \sqrt{\frac{2}{3}} \sigma \left( M^2_\pi + 2M^2_K \right) + \frac{1}{3} \sigma_8 \left( M^2_\pi - M^2_K \right) ,$$

(139)

$$f^A(0) \epsilon_8 = \frac{2 \sqrt{2} \lambda'}{3} \sigma \left( M^2_\pi - M^2_K \right) + \frac{1}{3 \sqrt{3}} \sigma_8 \left( 2M^2_\pi + M^2_K \right) ,$$

(140)

and finds a corrected Gell-Mann-Okubo mass relation

$$M^2_8 = \frac{1}{3} \left( 4M^2_K - 2M^2_\pi \right) - \frac{1}{3} \sqrt{\frac{2}{3}} \frac{\sigma_8}{\sigma - \frac{1}{\sqrt{6}} \sigma_8} \left( M^2_\pi - M^2_K \right) .$$

(141)

Employing the charge average masses $M_\pi = 138.0 \text{ MeV}$ and $M_K = 495.6 \text{ MeV}$, one finds $M_8 = 551.4 \text{ MeV}$, closer to the observed $M_\eta = 547.9 \text{ MeV}$ than the uncorrected value $M^\text{GMO}_8 = 566.8 \text{ MeV}$. One notes that this result depends just on the classical axial current and PCAC.

The non-linear $\sigma$-model depends on six parameters, namely $\mu^2$, $c$, $\lambda$, $\lambda'$, $\epsilon_0$ and $\epsilon_8$, but just the combination $[-\mu^2 + (\lambda' + \frac{1}{3} \lambda) 4 (\sigma^2 + \sigma_8^2)]$ occurs in uncoupled masses. One checks the ability of the model to reproduce basic data by fixing $M^2_\pi$ and $M^2_K$ to their charge average values, enforcing conditions \((116)\) and \((117)\), and considering two different scenarios,
presented in table III. In scenario 1, the symmetry breaking parameters \( \epsilon_8 \) and \( \epsilon_8 \) are kept correlated by eqs.(139) and (140), one finds \( f^A(0) = 242.7 \text{ MeV} \) and obtains the scalar masses \( m_a = 950 \text{ MeV} \) and \( m_\kappa = 789 \text{ MeV} \). The masses in the 0-8 sector depend on \( \mu^2 \) and, as in ref.[39], one identifies \( m_0 \) with the \( f_0(980) \), finding \( \mu = 804 \text{ MeV}, \theta = 41.2^\circ \), and \( m_8 = 564.3 \text{ MeV} \). These conservative results are not far from the masses of the states \( a_0(980), K_0^*(800) \) and \( f_0(500) \) found in the PDG[40]. In scenario 2, one explores the fact that the expression for \( M_{K^*}^2 \), eq.(120), receives a contribution from the term proportional to \( f_{ij}s f_{kls} \) in eq.(118) and is not bound to \( M_\pi^2 \) and \( M_8^2 \) by the Gell-Mann-Okubo relation[38]. One then forces \( M_8^2 \) to be identical with that given by eq.(141), relaxing the constraint between \( \epsilon_0 \) and \( \epsilon_8 \). In this case, the scalar masses decrease considerably, being around 500 MeV.

| parameter | scenario 1 | scenario 2 |
|-----------|------------|------------|
| \( \mu (\text{GeV}) \) | 0.804 | 0.798 |
| \( c (\text{GeV}) \) | 0.824 | 0.353 |
| \( \lambda' \) | 15.955 | 9.807 |
| \( \lambda \) | -1.438 | 7.674 |
| \( \epsilon_0 (\text{GeV})^2 \) | 0.0790 | 0.0778 |
| \( \epsilon_8 (\text{GeV})^2 \) | -0.1242 | -0.0977 |

| results | scenario 1 | scenario 2 |
|---------|------------|------------|
| \( M_\pi (\text{GeV}) \) | input (0.1380) | input (0.1380) |
| \( M_K (\text{GeV}) \) | input (0.4956) | input (0.4956) |
| \( M_8 (\text{GeV}) \) | 0.5696 | input (0.5514) |
| \( m_a (\text{GeV}) \) | 0.9500 | 0.5210 |
| \( m_\kappa (\text{GeV}) \) | 0.7890 | 0.5743 |
| \( m_8 (\text{GeV}) \) | 0.5643 | 0.3637 |
| \( m_{f_0} (\text{GeV}) \) | input (0.980) | input (0.980) |
| \( \theta \) | 41.22^\circ | 35.39^\circ |

TABLE III: Parameters and predictions for two different scenarios.

The difference \( M_8^2 - M_\eta^2 \) is usually ascribed to a mixture between \( \phi_8 \) and \( \phi_0 \). The linear version of the \( SU(3) \) \( \sigma \)-model does include the \( \phi_0 \) and hence gives rise to a prediction for the 8-0 pseudoscalar mixing angle. On the other hand, in ChPT[11], the field \( \phi_0 \) is treated as an external source and the 8-0 mixing is described with the help of a free parameter.
As the non-linear version of the $\sigma$-model does not include the $\phi_0$, it is interesting to study its predictions for $M_8$, which depend on both the axial current and the specific lagrangian adopted. The former yields results (132), whereas PCAC, which depends on the form of $L_{SB}$, gives rise to expressions (136)-(138). In association, they produce the value $M_8 = 551.4$ MeV, given by eq.(141). In scenario 1, the ratio $\varepsilon_8/\varepsilon_0 = -1.572$ predicted by eqs.(139) and (140) is satisfied, but the result for $M_8^2$ is 7% too large, whereas in scenario 2, one has $\varepsilon_8/\varepsilon_0 = -1.256$, and eqs.(136)-(138) are violated by about 20%, indicating that the model still has has problems with fine tuning. This may be associated with the present form of the symmetry breaking lagrangian adopted and alternatives will be discussed elsewhere.

Apart from $\eta - \eta'$ mixing, which is absent in the non-linear model, results for scenario 1 are compatible with those from the linear version[39, 41], based on different fitting strategies. This is not surprising, since the basic equations produced by both models are quite similar, the main differences being symmetry breaking contributions. One should also mention that the factor $c[\det(M) + \det(M^\dagger)]$ in the lagrangian (103) is decisive in the determination of results and its $SU(3) \times SU(3)$ structure already excludes the $\phi_0$ in the linear version. The non-linear prediction for the scalar mixing angle is larger than the value $\theta = 19^\circ \pm 5^\circ$ quoted in ref.[42].

The associations $\bar{f}X \hat{\phi}_i \rightarrow \pi_i$ and $\bar{f}G \hat{b}_i \rightarrow R_i$ employed in the non-linear model already hint at the structure of scalar resonances as quasi-bound states of two pseudoscalars. As the functions $\bar{f}X$ and $\bar{f}G$ have even parity, the pseudoscalar or scalar nature of the fields is determined by the factors $\hat{\phi}_i$ and $\hat{b}_i$. The latter is the bilinear combination $\hat{b}_i = \sqrt{3} d_{ijk} \hat{\pi}_j \hat{\pi}_k$, which suggests a two-pseudoscalar content. This interpretation is supported by the analysis of poles of $T$-matrices produced in ref.[42] and a particular instance illustrating the presence of a $\kappa$-pole in the $K\pi$ amplitude presented in ref.[43]. A comprehensive discussion of the assignment and content of scalar resonances can be found in the review section *Scalar Mesons below 2 GeV* of the PDG[40], which includes relevant references. As mentioned in the introduction, the main purpose of this discussion of the $\sigma$-model is to show that knowledge of the analytic matrix $U$ allows it to be formulated in the framework of $SU(3) \times SU(3)$, instead of the usual $U(3) \times U(3)$. The non-linear model allows the pseudoscalar octet to coexist with a scalar nonet, but this requires the introduction of a chiral invariant field $\Phi$, which might be associated with a glueball. The implications of this possibility will be discussed elsewhere.

VIII. SUMMARY

This work contains two complementary parts. First, (a) one presents an analytic expression for the $SU(3)$ unitary matrix which, although motivated by low-energy hadron physics, has general validity. In the sequence, (b) one produces an application to a non-linear $\sigma$-model. Results are summarized below.
The structure of the $SU(2)$ unitary matrix $U$ is well known to have two equivalent representations, given by

$$U = \exp [i \tau \cdot \hat{\pi} \theta],$$

$$= \cos \theta + i \tau \cdot \hat{\pi} \sin \theta,$$

where $\tau$ are Pauli matrices and $\pi = (\pi_1, \pi_2, \pi_3)$. In sect.II one extends this result to the $SU(3)$ case and, for Gell-Mann matrices $\lambda$, the identity holds

$$U = \exp [i v \cdot \lambda]$$

$$= \left[ SI + \sqrt{2} Q \cdot \lambda \right] + i \left[ WI + \sqrt{2} P \cdot \lambda \right]$$

with

$$S = (F + \frac{2}{3} G), \quad Q_i = \sqrt{2} \left( H \hat{v}_i + \frac{1}{\sqrt{3}} G \hat{b}_i \right),$$

$$W = (Y + \frac{2}{3} Z), \quad P_i = \sqrt{2} \left( X \hat{v}_i + \frac{1}{\sqrt{3}} Z \hat{b}_i \right),$$

$v_i = [v_1, \ldots v_8], b_i = d_{ijk} v_j v_k$, and functions $F, \ldots, Z$ given by eqs.(49)-(54), depending on $v = |v|$ and $\eta = 2 d_{ijk} \hat{v}_i \hat{v}_j \hat{v}_k/3$.

Unitarity constrains the functions $S, Q, W, P$ to the surface of a four-sphere, since

$$U U^\dagger = S^2 + Q^2 + W^2 + P^2 = 1,$$

for all values of $v$ and $\eta$.

In the classical limit, corresponding to $\eta \to 0$, one has $W \to 0$ and

$$U = \left[ \frac{1}{3} (1 + 2 \cos v) I + \frac{1}{\sqrt{3}} (-1 + \cos v) \hat{b} \cdot \lambda \right] + i (\sin v) \hat{v} \cdot \lambda.$$

The operator $U = U_{re} + i U_{im}$ becomes a cyclic function of $v$ and oscillates, but its components remain restricted to the intervals $1 \geq U_{re}^2 \geq 1/3$ and $2/3 \geq U_{im}^2 \geq 0$. The variation of $v$ defines a tilted circumference with radius $\sqrt{2}/3$, illustrated in fig.4. In terms of the variable $Q' = 2 S/\sqrt{3} + Q/\sqrt{3}$, its edge is given by the condition $Q'^2 + P^2 = 2/3$.

The analytic result for $U$ allows one to evaluate the left form, which is given by

$$L^\mu = i \left\{ \frac{3}{2} \left[ (Q_i \partial^\mu Q_j + P_i \partial^\mu P_j) f_{ijk} \right] \right. $$

$$+ \left[ \sqrt{\frac{3}{2}} \left( S \partial^\mu P_k - P_k \partial^\mu S - W \partial^\mu Q_k + Q_k \partial^\mu W \right) \right]$$

$$+ \frac{3}{2} (Q_i \partial^\mu P_j - P_i \partial^\mu Q_j) d_{ijk} \right\} \lambda_k.$$
It gives rise to the right form as well as to vector and axial currents and, in sect.V, one presents expressions in terms of the functions $F, \cdots Z$.

a5. Results presented in sects.II-V are generic and not committed to a particular interpretation of the variable $v$. Hence, they may prove to be useful in problems involving three degrees of freedom or three state systems. In QCD, one has the lightest flavors $u, d, s$ and the basic colors, where it might be instrumental to study color superconductivity. It also allows one to investigate topological properties of the classical solution, as in the Skyrme model. Nevertheless, the interest of the analytic form of $U$ is not restricted to hadron physics, since it may also be applied in other areas, such as optics, geometric phases, and quantum computation and communication.

a6. In sect.VI, the generic analytic expression for $U$ is adapted to low-energy flavor $SU(3)$ by associating the $v_i$ with pseudoscalar fields $\phi_i$ and one displays its axial transformation properties, involving both the functions $F, \cdots Z$ and the directions $\hat{\phi}$ and $\hat{b}$. In the classical limit, one has

$$
\delta^A \cos \phi = -\sin \phi \delta^A \phi, \quad \text{and} \quad \delta^A \sin \phi = \cos \phi \delta^A \phi,
$$

with $\phi = |\phi|$, indicating that the axial transformation corresponds to a rotation along the tilted circumference.

b1. The $\sigma$-model lagrangian has the standard form and is written in terms of a matrix $M$. Instead of the usual $U(3) \times U(3)$ version based on 18 fields

$$
\sqrt{2} M_L = \frac{\sqrt{3}}{\sqrt{2}} (\sigma_0 + i \pi_0) \lambda_0 + (\sigma_i + i \pi_i) \lambda_i,
$$

one employs a generalization of the classical $U$ matrix with 17 fields, given by

$$
U = (F + \frac{2}{3} G)I + \left( \frac{1}{\sqrt{3}} G \hat{b}_i + i X \hat{\phi}_i \right) \lambda_i,
$$

$$
\rightarrow \sqrt{2} M = (\Phi + \frac{2}{3} R)I + \left( \frac{1}{\sqrt{3}} R_i + i \pi_i \right) \lambda_i,
$$

with $\bar{f}F \rightarrow \Phi$, $\bar{f}G \hat{b}_i \rightarrow R_i$, and $\bar{f}X \hat{\phi}_i \rightarrow \pi_i$, where $\bar{f}$ a quantity with dimension of mass.

b2. The axial transformations of the fields $\Phi, R_i, \pi_i$ are taken from sect.VI and one shows that, excluding $L_{SB}$, the lagrangian based on $M$ is chiral invariant. In actual calculations on employs th variable $\sqrt{2} \Sigma/\sqrt{3} = (\Phi + 2R/3)$, whereas fluctuations are distinguished from the classical components $\sigma$ and $\sigma_8$ by writing $\Sigma = \sigma + f$ and $R_i = \delta_{i8} \sigma_8 + \sqrt{3} r_i$. The usual procedure gives rise to a vacuum energy $V_{\text{vac}}$, to minimization conditions $\delta V_{\text{vac}}/\delta \sigma = \delta V_{\text{vac}}/\delta \sigma_8 = 0$, to 8 pseudoscalar and 9 scalar masses, besides interaction
contributions, all expressed in terms of the parameters of the model.

**b3.** Free parameters are fixed in two steps. First, one employs the axial current derived in sect.V to relate $\sigma$ and $\sigma_8$ with the pseudoscalar decay constants $f_i$. Adopting $f_K = 1.22 f_\pi$, one finds $f_8 = 1.3 f_\pi$, in agreement with a prediction from ChPT. The symmetry breaking lagrangian gives rise to PCAC relations for the fields $\pi_i$. When results derived from the axial current and its divergence are taken together, one obtains the corrected Gell-Mann-Okubo mass relation

$$M_8^2 = \frac{1}{3} (4M_K^2 - M_\pi^2) - \frac{1}{3} \sqrt{\frac{2}{3}} \frac{\sigma_8}{\sqrt{\sigma}} (M_\pi^2 - M_K^2) .$$

The charge average masses $M_\pi = 138.0$ MeV and $M_K = 495.6$ MeV, yield $M_8 = 551.4$ MeV, closer to the observed $M_\eta = 547.9$ MeV than the uncorrected value $M_8^{\text{GMO}} = 566.8$ MeV.

**b4.** The remaining parameters determine masses and interaction vertices. The uncoupled masses are obtained by fixing $M_\pi^2$ and $M_K^2$ to their charge average values, and one finds the conservative values $m_\pi = 950$ MeV and $m_\kappa = 789$ MeV. Regarding the 8-0 scalar sector, one identifies $m_0$ with the $f_0(980)$ and has $m_8 = 564.3$ MeV, with a mixing angle $\theta = 41.2^\circ$. These results are not far from the masses of the states $a_0(980)$, $K_0^*(800)$ and $f_0(500)$ found in the PDG. Basic equations are compatible with those from the linear version, apart from differences associated with symmetry breaking terms, and the same holds for predictions. This feature may be associated with the presence in both models of the important $SU(3) \times SU(3)$ structure $c \left[ \det(M) + \det(M^\dagger) \right]$.  

**b5.** The non-linear version of the $\sigma$-model does not include the ninth pseudoscalar field $\phi_0$ and cannot account for $\eta - \eta'$ mixing. However, similarly to the case of ChPT, the field $\phi_0$ can be treated as an external source and 8-0 mixing in the pseudoscalar sector can be described by means of a free parameter. In this framework, the prediction $M_8 = 570$ MeV might seem acceptable, but it does not agree with the value arising from the corrected Gell-Mann-Okubo expression. Exploring an extra freedom in the expression for $M_K^2$ associated with a contribution proportional to $f_{ijs} f_{klt}$, one forced consistency between the two values of $M_8$ and found scalar masses around 500 MeV, which are unacceptable. This indicates that the symmetry breaking lagrangian employed in the model needs to be refined.

**b6.** The associations $\bar{f} X \hat{\phi}_i \rightarrow \pi_i$ and $\bar{f} G \hat{b}_i \rightarrow R_i$ incorporated into the model already suggest that scalar resonances correspond to quasi-bound states of two pseudoscalars. The parities of these fields are determined by $\hat{\phi}_i$ and $\hat{b}_i$, the latter representing the bilinear combination $\hat{b}_i = \sqrt{3} d_{ijk} \hat{\pi}_j \hat{\pi}_k$ of two pseudoscalars.
b7. In the non-linear model, the pseudoscalar octet coexists with a scalar nonet, but this requires a chiral invariant field $\Phi$, possibly associated with a glueball.

Appendix A: auxiliary functions

The explicit components of the vector $\mathbf{b}$ are given by

\begin{align}
\mathbf{b}_1 &= \frac{2}{\sqrt{3}} v_1 v_8 + v_4 v_6 + v_5 v_7, \\
\mathbf{b}_2 &= \frac{2}{\sqrt{3}} v_2 v_8 - v_4 v_7 + v_5 v_6, \\
\mathbf{b}_3 &= \frac{2}{\sqrt{3}} v_3 v_8 + \frac{1}{2} \left( v_4^2 + v_5^2 - v_6^2 - v_7^2 \right), \\
\mathbf{b}_4 &= v_1 v_6 - v_2 v_7 + v_3 v_4 - \frac{1}{\sqrt{3}} v_4 v_8, \\
\mathbf{b}_5 &= v_1 v_7 + v_2 v_6 + v_3 v_5 - \frac{1}{\sqrt{3}} v_5 v_8, \\
\mathbf{b}_6 &= v_1 v_4 + v_2 v_5 - v_3 v_6 - \frac{1}{\sqrt{3}} v_6 v_8, \\
\mathbf{b}_7 &= v_1 v_5 - v_2 v_4 - v_3 v_7 - \frac{1}{\sqrt{3}} v_7 v_8, \\
\mathbf{b}_8 &= \frac{1}{\sqrt{3}} \left[ v_1^2 + v_2^2 + v_3^2 - \frac{1}{2} \left( v_4^2 - v_5^2 - v_6^2 - v_7^2 \right) \right] v_8, \\
\end{align}

whereas the function $D$ reads

\begin{align}
D &= d_{ijk} v_i v_j v_k \\
&= \sqrt{3} \left[ v_1^2 + v_2^2 + v_3^2 - \frac{1}{2} \left( v_4^2 - v_5^2 - v_6^2 - v_7^2 \right) \right] v_8 \\
&+ 3 v_1 \left( v_4 v_6 + v_5 v_7 \right) + 3 v_2 \left( -v_4 v_7 + v_5 v_6 \right) + \frac{3}{2} v_3 \left( v_4^2 + v_5^2 - v_6^2 - v_7^2 \right). \\
\end{align}

The components of $\mathbf{b}$ satisfy the conditions

\begin{align}
f_{ijk} v_j b_k &= 0, \\
d_{jks} v_j b_k &= \frac{1}{3} v^2 v_i. \\
\end{align}

It is straightforward to prove these results by using directly eqs.(A1)-(A8). Multiplying eq.(A11) by $b_i$, one finds $b^2 = v^4/3$. Also, using $B B = B + \eta A$, eq.(16), one has $(\hat{b} \cdot \lambda) (\hat{b} \cdot \lambda) = 2/3 + 3 \eta \hat{v} \cdot \lambda - \hat{b} \cdot \lambda / \sqrt{3}$, which yields

\begin{align}
d_{ijk} b_j b_k &= \eta v^3 v_i - \frac{1}{3} v^2 b_i. \\
\end{align}

Appendix B: differential equation

One considers the differential equation (30), that reads

\begin{align}
\frac{\partial^6 F}{\partial v^6} + 2 \frac{\partial^4 F}{\partial v^4} + \frac{\partial^2 F}{\partial v^2} + \eta^2 F = 0.
\end{align}
Its solution has the general form $F = \exp(q v)$, where $q$ satisfies the algebraic equation

$$q^6 + 2q^4 + q^2 + \eta^2 = 0 \ . \tag{B2}$$

Defining $\alpha = q^2$, one has the cubic equation

$$\alpha^3 + 2\alpha^2 + \alpha + \eta^2 = 0 \ , \tag{B3}$$

which has the solutions

$$\alpha_1 = q_1^2 = -\frac{2}{3} \left[ 1 - \cos(\theta/3) \right] \ , \tag{B4}$$
$$\alpha_2 = q_2^2 = -\frac{2}{3} \left[ 1 - \cos(\theta/3 + 2\pi/3) \right] \ , \tag{B5}$$
$$\alpha_3 = q_3^2 = -\frac{2}{3} \left[ 1 - \cos(\theta/3 - 2\pi/3) \right] \ , \tag{B6}$$

with

$$\cos(\theta) = 1 - 27 \eta^2/2 \ , \tag{B7}$$
$$\sin(\theta) = 3\sqrt{3}\eta \sqrt{1 - 27 \eta^2/4} \ . \tag{B8}$$

As $\alpha_i < 0$, one defines $q_i = i k_i$, and has

$$k_1 = \frac{2}{\sqrt{3}} \sin(\theta/6) \ , \tag{B9}$$
$$k_2 = \frac{2}{\sqrt{3}} \sin(\theta/6 + \pi/3) = \cos(\theta/6) + \frac{1}{\sqrt{3}} \sin(\theta/6) \ , \tag{B10}$$
$$k_3 = \frac{2}{\sqrt{3}} \sin(\theta/6 - \pi/3) = -\cos(\theta/6) + \frac{1}{\sqrt{3}} \sin(\theta/6) \ . \tag{B11}$$

The function $F$ is real and its most general form reads

$$F = \beta_1 \cos(k_1 v) + \beta_2 \cos(k_2 v) + \beta_3 \cos(k_3 v) \ , \tag{B12}$$

where the $\beta_i$ are constants.

The $k_i$ satisfy the constraints

$$k_1 = k_2 + k_3 \ , \tag{B13}$$
$$k_1^2 + k_2^2 + k_3^2 = 2 \ , \tag{B14}$$
$$k_1 k_2 k_3 = -\eta \ , \tag{B15}$$

whereas, for the roots of the cubic equation (B3) one has the usual conditions

$$\alpha_1 + \alpha_2 + \alpha_3 = -2 \ , \tag{B16}$$
$$\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 = 1 \ , \tag{B17}$$
$$\alpha_1 \alpha_2 \alpha_3 = -\eta^2 \ , \tag{B18}$$
Combining (B16) and (B17), one finds the useful result
\[ \alpha_i^2 + 2\alpha_i + \alpha_j^2 + 2\alpha_j + \alpha_i\alpha_j + 1 = 0, \quad (B19) \]
that can also be rewritten as
\[ (\alpha_i^2 + \alpha_i) + (\alpha_j^2 + \alpha_j) = -(\alpha_i + 1)(\alpha_j + 1). \quad (B20) \]
Multiplying it by \(\alpha_i\) and using (B3), one gets
\[ \alpha_i^2\alpha_j + 2\alpha_i\alpha_i + \alpha_i\alpha_j^2 - \eta^2 = \alpha_i\alpha_j (2 + \alpha_i + \alpha_j) - \eta^2 = 0 \quad (B21) \]
and, using (B19) and (B21), one also shows that
\[ (\alpha_i^2 + \alpha_i)(\alpha_j^2 + \alpha_j) = \eta^2(1 + \alpha_i + \alpha_j). \quad (B22) \]

Appendix C: unitarity - proof

The unitarity of the operator \(U\) is indicated in eq.(61) and here one proves the validity of conditions (65). Using the shorthands \(c_i = \cos(k_i v)\) and \(s_i = \sin(k_i v)\) in eqs.(49)-(54) and results from App.B, one has

\[ F^2 = \left\{ \beta_1^2 c_1^2 [1] + 2\beta_1\beta_2 c_1 c_2 [1] + \cdots \right\}, \quad (C1) \]
\[ G^2 = \frac{1}{\eta^2} \left\{ \beta_1^2 c_1^2 [-\alpha_1] + 2\beta_1\beta_2 c_1 c_2 [1 + \alpha_1 + \alpha_2] + \cdots \right\}, \quad (C2) \]
\[ H^2 = \frac{1}{\eta^2} \left\{ \beta_1^2 c_1^2 [\alpha_1^2] + 2\beta_1\beta_2 c_1 c_2 [\alpha_1 \alpha_2] + \cdots \right\}, \quad (C3) \]
\[ 2FG = \frac{1}{\eta^2} \left\{ \beta_1^2 c_1^2 [2(\alpha_i^2 + \alpha_1)] + 2\beta_1\beta_2 c_1 c_2 [-(\alpha_1 + 1)(\alpha_2 + 1)] + \cdots \right\}, \quad (C4) \]
\[ 2FH = \frac{1}{\eta} \left\{ \beta_1^2 c_1 [2 - 2\alpha_1] + 2\beta_1\beta_2 c_1 c_2 [-(\alpha_1 + \alpha_2)] + \cdots \right\}, \quad (C5) \]
\[ 2GH = \frac{1}{\eta} \left\{ \beta_1^2 c_1 \left[ \frac{2\alpha_1}{(\alpha_1 + 1)} \right] + 2\beta_1\beta_2 c_1 c_2 [-1] + \cdots \right\}, \quad (C6) \]
\[ X^2 = \frac{1}{\eta^2} \left\{ \beta_1^2 s_1^2 [\alpha_1^2] + 2\beta_1\beta_2 k_1 k_2 s_1 s_2 [1 + \alpha_1 + \alpha_2] + \cdots \right\}, \quad (C7) \]
\[ Y^2 = \left\{ \beta_1^2 s_1^2 [1] + 2\beta_1\beta_2 k_1 k_2 s_1 s_2 \left[ \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{\eta^2} \right] + \cdots \right\}, \quad (C8) \]
\[ Z^2 = \frac{1}{\eta^2} \left\{ \beta_1^2 s_1^2 [-\alpha_1] + 2\beta_1\beta_2 k_1 k_2 s_1 s_2 [1 + \cdots] \right\}, \quad (C9) \]
Explicit calculations together with eqs. (35)-(37) and (47) yield

\[ C_I = F^2 + 2\eta GH + Y^2 + 2\eta XZ = \sum \beta_i \left( \frac{3\alpha_i + 1}{\alpha_i + 1} \right) = \sum \beta_i = 1, \]

\[ C_B = G^2 + H^2 + 2FG + X^2 + Z^2 YZ = \frac{1}{\eta^2} \sum \beta_i \left( 3\alpha_i^2 + \alpha_i \right) = \frac{1}{\eta^2} \sum \beta_i \left( \alpha_i^2 + \alpha_i \right) = 0, \]

\[ C_A = \eta G^2 + 2FH + 2GH + \eta Z^2 + 2XY + 2XZ = \frac{1}{\eta} \sum \beta_i \left( -\frac{3\alpha_i^2 + \alpha_i}{\alpha_i + 1} \right) = -\frac{1}{\eta} \sum \beta_i \alpha_i = 0. \]

Appendix D: left form

Direct evaluation of \( L^\mu \) by means of eqs. (60), (83), and condition (85), yields

\[
L^\mu = i \left[ S \partial^\mu W - W \partial^\mu S + Q_i \partial^\mu P_i - P_i \partial^\mu Q_i \right] + i \left\{ \frac{3}{2} \left[ (Q_i \partial^\mu Q_j + P_i \partial^\mu P_j) f_{ijk} \right] \right. \\
+ \left[ \sqrt{\frac{3}{2}} \left( S \partial^\mu P_k - P_k \partial^\mu S - W \partial^\mu Q_k + Q_k \partial^\mu W \right) \\
+ \frac{3}{2} (Q_i \partial^\mu P_j - P_i \partial^\mu Q_j) d_{ijk} \right\} \lambda_k.
\]

In the sequence, one shows that the first term of this expression vanishes and evaluates the other ones in terms of the functions \( F, \cdots, Z \). This requires a set of auxiliary results, presented below.

1. derivatives with respect to \( \eta \)

For any function \( \psi(v, \eta) \), one has

\[
\frac{\partial \psi}{\partial x^\mu} = \frac{\partial \psi}{\partial v_a} \frac{\partial v_a}{\partial x^\mu} = \left( \frac{\partial \psi}{\partial v} \frac{\partial v}{\partial v_a} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial v_a} \right) \frac{\partial v_a}{\partial x^\mu}.
\]
Derivatives with respect to \( v \) are given by eqs.(26)-(27), whereas for \( \partial \psi / \partial \eta \) one uses
\[
\frac{d\alpha_i}{d\eta} = -\frac{2\eta}{(\alpha_i + 1) (3 \alpha_i + 1)} ,
\]
\[
\frac{d\kappa_i}{d\eta} = -\kappa_i \frac{\eta}{\alpha_i (\alpha_i + 1) (3 \alpha_i + 1)} ,
\]
together with \( c_i = \cos(k_i v), \) \( s_i = \sin(k_i v), \) and obtains
\[
\frac{\partial F}{\partial \eta} = \frac{\eta}{(3 \alpha_1 + 1)^3} \left[ \frac{4}{(\alpha_1 + 1)} \right] c_1 + \frac{v \eta}{(3 \alpha_1 + 1)^2 \alpha_1} k_1 s_1 + (1 \to 2, 3) ,
\]
\[
\frac{\partial G}{\partial \eta} = -\frac{\eta}{(3 \alpha_1 + 1)^3} \left[ \frac{6}{(\alpha_1 + 1)} \right] c_1 - \frac{v \eta}{(3 \alpha_1 + 1)^2 \alpha_1 (\alpha_1 + 1)} k_1 s_1 + (1 \to 2, 3) ,
\]
\[
\frac{\partial H}{\partial \eta} = -\frac{1}{(3 \alpha_1 + 1)^3} [(3 \alpha_1 - 1)] c_1 - \frac{v}{(3 \alpha_1 + 1)^2} k_1 s_1 + (1 \to 2, 3) ,
\]
\[
\frac{\partial X}{\partial \eta} = -\frac{\eta}{(3 \alpha_1 + 1)^3} \left[ \frac{(3 \alpha_1 - 1)}{\alpha_1 (\alpha_1 + 1)} \right] k_1 s_1 - \frac{v \eta}{(3 \alpha_1 + 1)^2 (\alpha_1 + 1)} c_1 + (1 \to 2, 3) ,
\]
\[
\frac{\partial Y}{\partial \eta} = -\frac{1}{(3 \alpha_1 + 1)^3} [4] k_1 s_1 - \frac{v (\alpha_1 + 1)}{(3 \alpha_1 + 1)^2} c_1 + (1 \to 2, 3) ,
\]
\[
\frac{\partial Z}{\partial \eta} = \frac{1}{(3 \alpha_1 + 1)^3} [6] k_1 s_1 + \frac{v}{(3 \alpha_1 + 1)^2} c_1 + (1 \to 2, 3) .
\]
Employing eqs.(49)-(54), one reexpresses these results as
\[
\frac{\partial F}{\partial \eta} = \left[ (-3 \eta G + H) + v \left( -\frac{3}{2} \eta X + Y + \frac{9}{4} \eta^2 Z \right) \right] / (1 - \frac{27}{4} \eta^2) ,
\]
\[
\frac{\partial G}{\partial \eta} = \left[ \left( \frac{9}{4} \eta G - \frac{3}{2} H \right) + v \left( \frac{9}{4} \eta X - \frac{3}{2} Y - \frac{1}{2} Z \right) \right] / (1 - \frac{27}{4} \eta^2) ,
\]
\[
\frac{\partial H}{\partial \eta} = \left[ (-G + \frac{9}{4} \eta H) + v \left( -\frac{1}{2} X + \frac{9}{4} \eta Y + \frac{3}{4} \eta Z \right) \right] / (1 - \frac{27}{4} \eta^2) ,
\]
\[
\frac{\partial X}{\partial \eta} = \left[ \left( \frac{9}{4} \eta X - Z \right) + v \left( -\frac{9}{4} \eta F - \frac{3}{4} \eta G + \frac{1}{2} H \right) \right] / (1 - \frac{27}{4} \eta^2) ,
\]
\[
\frac{\partial Y}{\partial \eta} = \left[ (X - 3 \eta Z) + v \left( -F - \frac{9}{4} \eta^2 G + \frac{3}{2} \eta H \right) \right] / (1 - \frac{27}{4} \eta^2) ,
\]
\[
\frac{\partial Z}{\partial \eta} = \left[ \left( -\frac{3}{2} X + \frac{9}{4} \eta Z \right) + v \left( \frac{3}{2} F + \frac{1}{2} G - \frac{9}{4} \eta H \right) \right] / (1 - \frac{27}{4} \eta^2) .
\]
2. derivatives of vectors

Various combinations of the unit vectors \( \hat{v} \) and \( \hat{b} \) are also needed and they are listed below for convenience. Results from App.A yield

\[
\hat{v}_i \, \hat{v}_j \, d_{ijk} = \frac{1}{\sqrt{3}} \hat{b}_k , \quad (D17)
\]
\[
\hat{v}_i \, \hat{b}_j \, d_{ijk} = \frac{1}{\sqrt{3}} \hat{v}_k , \quad (D18)
\]
\[
\hat{b}_i \, \hat{b}_j \, d_{ijk} = 3 \eta \, \hat{v}_k - \frac{1}{\sqrt{3}} \hat{b}_k . \quad (D19)
\]

For terms involving derivatives, one uses \( \partial v/\partial v_i = \hat{v}_i \) and finds

\[
\frac{\partial \hat{v}_s}{\partial v_a} = \frac{1}{v} \left( \delta_{as} - \hat{v}_a \, \hat{v}_s \right) , \quad (D20)
\]
\[
\frac{\partial \hat{b}_s}{\partial v_a} = \frac{2\sqrt{3}}{v} \left( \hat{v}_j \, d_{jas} - \frac{1}{\sqrt{3}} \hat{v}_a \, \hat{b}_s \right) , \quad (D21)
\]
\[
\hat{v}_s \, \frac{\partial \hat{v}_s}{\partial v_a} = 0 , \quad (D22)
\]
\[
\hat{b}_s \, \frac{\partial \hat{v}_s}{\partial v_a} = 0 \left( -\hat{v} \cdot \hat{b} \, \hat{v}_a + \hat{b}_a \right) , \quad (D23)
\]
\[
\hat{v}_s \, \frac{\partial \hat{b}_s}{\partial v_a} = \frac{2}{v} \left( -\hat{v} \cdot \hat{b} \, \hat{v}_a + \hat{b}_a \right) , \quad (D24)
\]
\[
\hat{b}_s \, \frac{\partial \hat{b}_s}{\partial v_a} = 0 , \quad (D25)
\]
\[
\hat{v}_j \, \frac{\partial \hat{v}_s}{\partial v_a} d_{jsk} = \frac{1}{v} \left( \hat{v}_j \, d_{ajk} - \frac{1}{\sqrt{3}} \, \hat{v}_a \, \hat{b}_k \right) , \quad (D26)
\]
\[
\hat{b}_j \, \frac{\partial \hat{v}_s}{\partial v_a} d_{jsk} = \frac{1}{v} \left( -\frac{1}{\sqrt{3}} \, \hat{v}_a \, \hat{v}_j + \hat{b}_j \, d_{ajk} \right) , \quad (D27)
\]
\[
\hat{v}_j \, \frac{\partial \hat{b}_s}{\partial v_a} d_{jsk} = \frac{1}{v} \left( \frac{1}{\sqrt{3}} \, \delta_{ak} - \hat{b}_j \, d_{ajk} \right) , \quad (D28)
\]
\[
\hat{b}_j \, \frac{\partial \hat{b}_s}{\partial v_a} d_{jsk} = \frac{1}{v} \left( \frac{3}{2} \eta \, \delta_{ak} - 6 \eta \, \hat{v}_a \, \hat{v}_k - \hat{v}_j \, d_{ajk} + \frac{1}{\sqrt{3}} \, \hat{v}_a \, \hat{b}_k + \frac{3}{\sqrt{3}} \, \hat{b}_a \, \hat{v}_k \right) . \quad (D29)
\]

This allows one to write

\[
\frac{\partial \eta}{\partial v_a} = \frac{2}{3 \sqrt{3}} \left( \frac{\partial (\hat{v} \cdot \hat{b})}{\partial v_a} \right) = \frac{1}{v} \left( -3 \eta \, \hat{v}_a + \frac{2}{\sqrt{3}} \, \hat{b}_a \right) , \quad (D30)
\]
\[
\hat{b}_s \, \frac{\partial \hat{v}_s}{\partial v_a} = \frac{\sqrt{3}}{2} \frac{\partial \eta}{\partial v_a} , \quad (D31)
\]
\[
\hat{v}_s \, \frac{\partial \hat{b}_s}{\partial v_a} = \frac{\sqrt{3}}{2} \frac{\partial \eta}{\partial v_a} , \quad (D32)
\]
\[ \partial^\mu v_a = (\partial^\mu v) \dot{v}_a + v \partial^\mu \dot{v}_a , \quad (D33) \]
\[ \frac{\partial \dot{v}_s}{\partial v_a} \partial^\mu v_a = \partial^\mu \dot{v}_s , \quad (D34) \]
\[ \frac{\partial \dot{b}_s}{\partial v_a} \partial^\mu v_a = \partial^\mu \dot{b}_s \quad (D35) \]
\[ \frac{\partial \eta}{\partial v_a} \partial^\mu v_a = \partial^\mu \eta \quad (D36) \]

### 3. Results

Recalling eqs.(58), (59), and using \( \partial/\partial v_a \to \partial_a \), \( \partial/\partial v \to \partial_v \) and \( \partial/\partial \eta \to \partial_\eta \), one writes

\[
\left[ S \partial^\mu W - W \partial^\mu S + Q_1 \partial^\mu P_i - P_i \partial^\mu Q_1 \right] \\
= \left[ F \partial_a \left( Y + \frac{2}{3}Z \right) + G \partial_a \left( \frac{2}{3}Y + \frac{2}{3}Z + \eta X \right) + H \partial_a \left( \frac{2}{3}X + \eta Z \right) \\
- Y \partial_a \left( F + \frac{2}{3}G \right) - Z \partial_a \left( \frac{2}{3}F + \frac{2}{3}G + \eta H \right) - X \partial_a \left( \frac{2}{3}H + \eta G \right) \right] \dot{v}_a \\
+ \frac{2}{3\sqrt{3}} (HZ - GX) \left( \dot{v}_s \partial_\eta \dot{b}_s - \dot{b}_s \partial_a \dot{v}_s \right) \\
= \left[ F \partial_v \left( Y + \frac{2}{3}Z \right) + G \partial_v \left( \frac{2}{3}Y + \frac{2}{3}Z + \eta X \right) + H \partial_v \left( \frac{2}{3}X + \eta Z \right) \\
- Y \partial_v \left( F + \frac{2}{3}G \right) - Z \partial_v \left( \frac{2}{3}F + \frac{2}{3}G + \eta H \right) - X \partial_v \left( \frac{2}{3}H + \eta G \right) \right] \dot{v}_a \\
+ \frac{1}{3} (HZ - GX) \partial_\eta \eta \\
= \left[ \eta C_B + \frac{2}{3} C_A \right] \dot{v}_a + \left[ \frac{1}{3} v C_B \right] \partial_a \eta = 0 , \quad (D37) \]

after transforming terms involving derivatives into binomials of the basic functions by means of eqs.(26), (27), (D11)-(D16), and using results from App.C.

The vector contribution is obtained by a straightforward calculation and reads

\[
i \frac{3}{2} \left( Q_i \partial^\mu Q_j + P_i \partial^\mu P_j \right) f_{ijk} = i \left[ (H^2 + X^2) \dot{v}_i \partial^\mu \dot{v}_j \\
+ \frac{1}{\sqrt{3}} (GH + XZ) \left( \dot{v}_i \partial^\mu \dot{b}_j + \dot{b}_i \partial^\mu \dot{v}_j \right) + \frac{1}{3} (G^2 + Z^2) \dot{b}_i \partial^\mu \dot{b}_j \right] f_{ijk} . \quad (D38)\]
The axial term is

\[
    i \left[ \sqrt{\frac{3}{2}} (S \partial^\mu P_k - P_k \partial^\mu S - W \partial^\mu Q_k + Q_k \partial^\mu W) + \frac{3}{2} (Q_i \partial^\mu P_j - P_i \partial^\mu Q_j) d_{ijk} \right]
\]

= \left\{ \left[ [(F + G) \partial_a X + H \partial_a Y + (H + \eta G) \partial_a Z - (Y + Z) \partial_a H - X \partial_a F - (X + \eta Z) \partial_a G] \hat{v}_k \\
+ \frac{1}{\sqrt{3}} [(F + G) \partial_a Z + G \partial_a Y + H \partial_a X - (Y + Z) \partial_a G - Z \partial_a F - X \partial_a H] \hat{b}_k \\
+ \left[ (F + \frac{2}{3} G) X - (Y + \frac{2}{3} Z) H \right] \partial_a \hat{\nu}_k + \frac{1}{\sqrt{3}} \left[ (F + \frac{2}{3} G) Z - (Y + \frac{2}{3} Z) G \right] \partial_a \hat{b}_k \\
+ \frac{1}{\sqrt{3}} (HZ - GX) \left( \hat{v}_i \partial_a \hat{b}_j - \hat{b}_i \partial_a \hat{v}_j \right) d_{ijk} \right\} \partial^\mu v^a
\]

Reexpressing the derivatives by means of (D2) and employing results from App.C, one has

\[
    i \left[ \sqrt{\frac{3}{2}} (S \partial^\mu P_k - P_k \partial^\mu S - W \partial^\mu Q_k + Q_k \partial^\mu W) + \frac{3}{2} (Q_i \partial^\mu P_j - P_i \partial^\mu Q_j) d_{ijk} \right]
\]

= \left\{ \left[ [(F + G) \partial_a X + H \partial_a Y + (H + \eta G) \partial_a Z - (Y + Z) \partial_a H - X \partial_a F - (X + \eta Z) \partial_a G] \hat{v}_k \\
+ \frac{1}{(1 - 2\eta^2)} \left[ (FY - FZ) + \frac{3}{2} \eta (FX - HY) + \frac{3}{2} \eta (HZ - GX) - \frac{9}{4} v \eta \right] \hat{v}_k \partial^\mu \eta \\
+ \frac{1}{(1 - 2\eta^2)} \frac{1}{\sqrt{3}} \left[ \left( \frac{3}{2} (HY - FX) + \frac{1}{2} (GX - HZ) + \frac{9}{2} \eta (FZ - GY) + \frac{9}{2} v \right] \hat{b}_k \partial^\mu \eta \\
+ \left[ (F + \frac{2}{3} G) X - H \left( Y + \frac{2}{3} Z \right) \right] \partial^\mu \hat{\nu}_k + \frac{1}{\sqrt{3}} \left[ (F + \frac{2}{3} G) Z - G \left( Y + \frac{2}{3} Z \right) \right] \partial^\mu \hat{b}_k \\
+ \frac{1}{\sqrt{3}} (HZ - GX) \left( \hat{v}_i \partial^\mu \hat{b}_j - \hat{b}_i \partial^\mu \hat{v}_j \right) d_{ijk} \right\}
\]
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