THE REEB FOLIATION ARISES AS A FAMILY OF LEGENDRIAN SUBMANIFOLDS AT THE END OF A DEFORMATION OF THE STANDARD $S^3$ IN $S^5$

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ABSTRACT. We realize the Reeb foliation of $S^3$ as a family of Legendrian submanifolds of the unit $S^3 \subset \mathbb{C}^3$. Moreover we construct a deformation of the standard contact $S^3$ in $S^5$, via a family of contact submanifolds, into this realization.

1. INTRODUCTION

The Reeb foliation is a codimension one smooth foliation of the 3-sphere $S^3$ obtained by gluing two Reeb components $S^1 \times D^2$ and $D^2 \times S^1$. Since the one-sided holonomies of the Reeb components along $\{1\} \times \partial D^2$ and $\partial D^2 \times \{1\}$ are trivial, the Reeb foliation is not analytic ("Haefliger’s remark").

On the other hand the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R}) \approx \mathbb{R}^{2n+1}$ for a function of $n$ variables carries the canonical contact structure. It is contactomorphic to the unit sphere $S^{2n+1}$ through the Legendrian torus $\alpha$ denotes the vertex of the solutions. $F$ is a family of Legendrian submanifolds of $S^{2n+1}$ such that

There exists a smooth family $\{M^3_t\}_{t \in [0,3/2]}$ of codimension-2 submanifolds of $S^5$ such that

(1) $M^3_0$ is the standard $S^3(\subset \mathbb{C}^2 \subset \mathbb{C}^3)$,
(2) $M^3_1$ is an embedded contact submanifold for $0 \leq t < 1$,
(3) $M^3_1$ admits a Reeb foliation by injectively immersed Legendrian submanifolds of $S^5$, and
(4) $-M^3_1$ is an embedded overtwisted contact submanifold for $1 < t < 3/2$.

The foliated submanifold $M^3_1$ is obtained by joining two great circles $\{r_1 = 1\}, \{r_2 = 1\} \subset S^5$ through the Legendrian torus $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3}, \theta_1 + \theta_2 + \theta_3 = 0\}$. The family $M^3_1$ is obtained as a byproduct in the process of isotoping $M_1 \subset S^5$ to the unknot. The author is seeking the converse approach, i.e., to find a foliated submanifold by using contact topology or open-books (see Remark 1 in §2).

2. PROOF AND REMARK

Proof. Let $\pi$ be the natural projection of $S^5$ to the 2-simplex $\Delta = \{(r_1^2, r_2^2, r_3^2) \mid r_1^2 + r_2^2 + r_3^2 = 1\} \subset \mathbb{R}^3$, which sends the Legendrian 2-torus $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3}, \theta_1 + \theta_2 + \theta_3 = 0\} \subset S^5$ to the barycenter $G$. The set $\Gamma = \pi^{-1}(\partial \Delta)$ contains the great circles $\pi^{-1}(\{V_1, V_2, V_3\})$ where $V_i$ denotes the vertex $r_i^2 = 1$. Except them $\pi|\Gamma$ is a $T^2$-fibration. On the other hand, $\pi|(S^5 \setminus \Gamma)$ is a

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We generate $M^3$ where $P \in \Delta$, i.e.,

\[
3r_1^2 = 1 + 2x - y(\geq 0), \quad 3r_2^2 = 1 - x + 2y(\geq 0), \quad \text{and} \quad 3r_3^2 = 1 - x - y(\geq 0).
\]

Let $M^3_0$ be the standard $S^3 = \pi^{-1}(V_1 \cup V_2)$. We deform $M^3_0$ with the help of a certain family of simple curves $C_t : x = x_t(s), y = y_t(s), -\delta \leq s \leq \delta$ depicted in Fig.[I] ($0 < \delta \ll 1, 0 \leq t \leq 3/2$). Note that $C_1$ has a break point $G$ while $x_1(s)$ and $y_1(s)$ are smooth on $(-\delta, \delta)$.

![Figure 1. The curve $C_t$ on $\Delta$ and its parametrization by $s$](image)

We generate $M^3_t \subset S^5$ by moving the intersection of the “wall” $W_t = \text{cl}\{\theta_1 + \theta_2 + \theta_3 = s\} \subset S^5$ with the fibre $\pi^{-1}(x_t(s), y_t(s))$ for $-\delta \leq s \leq \delta$. Then we can see that $M^3_t$ realizes the join of two large circles $\pi^{-1}(V_2)$ and $\pi^{-1}(V_1)$. Now we give a precise definition of the curve $C_t$. Put $\varphi_0(u) = \frac{1}{2}(1 + u)$ for $u \in [-1, 1]$, and take a smooth function $\varphi_1(u)$ and a smooth odd function $s(u)$ such that

\[
\varphi_1(u) = 0 \quad (-1 \leq u \leq 0), \quad \varphi_1'(u) > 0 \quad (0 < u \leq 1), \quad \varphi_1(u) = \varphi_0(u) \quad (1/2 \leq u \leq 1),
\]

\[
s'(u) > 0 \quad (-1 < u < 1), \quad s(1) = \delta, \quad s(-1) = -\delta, \quad \text{and} \quad s(u) \text{ is } C^\infty\text{-tangent to } \pm \delta.
\]

The inverse function $u(s)$ of $s(u)$ is defined on $[-\delta, \delta]$. It is smooth on $(-\delta, \delta)$ ($u'(\pm \delta) = +\infty$). We put $\varphi_t(u) = (1 - t)\varphi_0(u) + t\varphi_1(u)$, and take the curve $C_t : x = x_t(s) = \varphi_t(u(s)), y = y_t(s) = \varphi_t(u(-s)), -\delta \leq s \leq \delta$.

Next we show that $M^3_t$ is a smooth submanifold. By moving the 2-torus $(M^3_t \setminus \Gamma) \cap W_4$ for $-\delta < s < \delta$, we see that $M^3_t \setminus \Gamma$ is diffeomorphic to $T^2 \times (-\delta, \delta)$. Moreover $M^3_t$ is topologically the join $S^1 \ast S^3 \approx S^3$. Thus it only remains for us to examine the smoothness of $M^3_t$ along $M^3_t \cap \Gamma$. We restrict ourself to the connected component of $M^3_t \cap \Gamma$ corresponding to $s = +\delta$ and omit the other component. We put

\[
\tilde{M}^3_t : \left\{ \begin{array}{l}
\sqrt{r_1^2 + r_2^2} + r_3^2 = 1 \\
3r_2^2 = 1 - \frac{t}{1}(1 + u) + (1 - t)(1 - u) \\
3r_3^2 = 1 - \frac{t}{1}(1 + u) - \frac{1 - t}{2}(1 - u) = \frac{t}{3 - 2t} \cdot 3r_2^2 \\
\theta_1 + \theta_2 + \theta_3 = 1
\end{array} \right.
\]

where $u \in [1/2, 1]$ is a parameter to be eliminated. Then $\{\theta_1 = \text{const}\} \subset \tilde{M}^3_t$ is a smooth disk since it tangents to the real 2-plane $\left\{ z_1 = \exp \sqrt{-1}\theta_1, \quad z_3 = \frac{1}{\sqrt{3 - 2t}} \cdot \exp \{\sqrt{-1}(1 - \theta_1)\} \right\} \subset \mathbb{C}^3$ at $u = 1$. Since the function $s(u)$ smoothly tangents to $\delta$ at $u = 1$, $M^3_t$ is a smooth 3-sphere.

Next we consider the (non-)integrability of the restriction $\lambda_t = \alpha|M^3_t|$ of the standard contact form $\alpha = r_1^2d\theta_1 + r_2^2d\theta_2 + r_3^2d\theta_3|S^5$. Using $(\theta_1, \theta_2, s)$ as coordinates of $M^3_t \setminus \Gamma$, we can write

\[
\lambda_t = x_t(s)d\theta_1 + y_t(s)d\theta_2 + (1 - x_t(s) - y_t(s))ds.
\]

Here the sign of $\lambda_t \wedge d\lambda_t$ with respect to $d\theta_1 \wedge d\theta_2 \wedge ds > 0$ coincides with that of $x_t'(s)y_t(s) - x_t(s)y_t'(s)$, and that of $1 - t$. More generally, if a submanifold $M^3(t) \cong T^2 \times \mathbb{R} \subset S^5$ is presented by a simple curve $C : x = x(s), y = y(s)$ on int$\Delta$, the negative area velocity $x'(s)y(s) - x(s)y'(s)$
still presents the non-integrability of $\alpha|\mathcal{M}^3$. In the case where $t = 1$, the integrability means the vanishing of the area verocity. That is why the curve $C_1$ is broken into two rays to/from the origin $G$, and $M^3_1$ is non-analytic.

On the other hand, for cylindrical coordinates $(\theta_1, (r_2, \theta_2))$, $\mu_t = \alpha|\mathcal{M}^3_t$ and $\mu_t \wedge d\mu_t$ are written as

$$\mu_t = \left(1 - \frac{3}{3 - 2t}r_2^2\right)d\theta_1 + \frac{3(1 - t)}{3 - 2t}r_2d\theta_2 \quad \text{and} \quad \mu_t \wedge d\mu_t = \frac{6(1 - t)}{3 - 2t}d\theta_1 \wedge (r_2dr_2 \wedge d\theta_2).$$

This implies that the sign of $\lambda_1 \wedge d\lambda_1$ everywhere coincides with that of $1 - t$.

Now we show that the foliation of $M^3_1$ is a Reeb foliation. The definition of $M^3_1$ is

$$\begin{cases}
3r_1^2 = 1 + 2\varphi(u(s)) - \varphi(u(-s)) \\
3r_2^2 = 1 - \varphi(u(s)) + 2\varphi(u(-s)) \\
3r_3^2 = 1 - \varphi(u(s)) - \varphi(u(-s)) \\
\theta_1 + \theta_2 + \theta_3 = s
\end{cases}$$

where $s \in [-\delta, \delta]$ is a parameter to be eliminated. On the open solid torus $H = \{s > 0\} \subset M^3_1$, we have

$$\alpha|H = \varphi(u(s))d\theta_1 + \{1 - \varphi(u(s))\}ds.$$ 

Thus the surface of $\theta_2$-revolution of the graph of $\theta_1 = \int \frac{\varphi(u(s)) - 1}{\varphi(u(s))}ds$ is a leaf. Similarly, we can describe the foliation on $\{s < 0\}$. These foliations spiral into $T$ and form a transversely oriented Reeb foliation, to which the positive Hopf link $\{r_1 = 1\} \cup \{r_2 = 1\}$ is positively transverse.

Finally we see from $d(\theta_1 + \theta_2) \wedge d\lambda_1 = \{x'_i(s) - y'_i(s)\}d\theta_1 \wedge d\theta_2 \wedge ds > 0 \ (t \neq 1)$ that the positive Hopf band $\ker(d\theta_1 + d\theta_2)$ is a supporting open-book for $0 \leq t < 1$. On the other hand, the negative Hopf band $\ker(-d\theta_1 - d\theta_2)$ on $-M^3_t(\simeq S^3)$ is a supporting open-book for $1 < t < 3/2$. Thus $-M^3_t$ is overtwisted. Indeed it has the half-Lutz tube $\{x_i(s) \leq 0\}$. Moreover, we can reverse the orientation of $S^3$ by a diffeotopy, we obtain the following “negative stabilization” lemma. This ends the proof. \hfill $\square$

**Lemma 2.1.** The overtwisted contact submanifold $-M^3_{5/4} \subset S^5$ is diffeotopic to the standard $S^3 \subset S^5$. Particularly $-M^3_{5/4}$ is differential topologically unknotted, but contact topologically knotted.

**Remark 1.** Any closed oriented 3-manifold admits an open-book decomposition (Alexander [1]). We can associate to it a contact structure (Thurston-Winkelnkemper [3]) as well as a spinnable foliation (see [4]). Further any contact structure is supported by an open-book decomposition (Giroux [3]). Using this result, the author constructed a certain immersion of any contact 3-manifold into $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$ or $S^5$ ([5]). This construction was generalized to any dimension, i.e., $M^{2n+1} \rightarrow \mathcal{J}^1(\mathbb{R}^{2n}, \mathbb{R})$ or $S^{4n+1}$ by Martínez Torres ([1]). The author proved that any/some contact structure of $M^3$ can be deformed into some/any spinnable foliation ([5], see also [2]). He also proved that a certain higher dimensional contact structure can be deformed into a foliation ([6]). It is interesting to generalize the present result to these cases.

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