Recently there has been a great deal of interest in geometric bounds on small eigenvalues of the Laplace operator on a Riemann surface [S.W.Y, D.P.R.S.]. Here we determine the precise asymptotic behaviour of these small eigenvalues. Let $S_\delta$ be a compact Riemann surface of genus $g \geq 2$ whose first $k$ nonzero eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ are small, i.e., $\lambda_k \leq \delta$ and $\lambda_{k+1} \geq c_1$. Then by [S.W.Y.] there exists a constant $a = a(g) > 0$ such that the closed geodesics $\gamma_1 \cdots \gamma_r$ of length less than $a \cdot \delta$ separate $S_\delta$ into $k + 1$ pieces $S_1, \ldots, S_{k+1}$ and all other closed geodesics of $S_\delta$ have length greater than $a(g)$. Let $\Lambda$ be the graph whose vertices are the pieces $S_i$. Suppose vertex $S_i$ has mass $v_i = \text{vol}(S_i)$ and the length $L_{ij}$ of an edge joining $S_i$ to $S_j$ is the total length of the geodesics contained in $S_i \cap S_j$. Furthermore, let $0 < \lambda_1(\Lambda) \leq \cdots \leq \lambda_k(\Lambda)$ be the spectrum of the quadratic form $\sum(F(S_i) - F(S_j))^2 L_{ij}$ with respect to the norm $\sum F(S_i)^2 v_i$. Then one has

**Theorem 1.**

$$\lim_{\delta \to 0} \frac{\lambda_j(S_\delta)}{\lambda_j(\Lambda)} = \frac{1}{\pi} \quad \text{for all } 1 \leq j \leq k.$$  

This convergence is uniform for all surfaces $S_\delta$ with $\lambda_{k+1}(S_\delta) \geq c_1$ and fixed genus.

**Remark.** The fact that $\limsup_{\delta \to 0} \lambda_j(S_\delta)/\lambda_j(\Lambda) \leq 1/\pi$ follows easily from [C.CdV]. This paper also shows the convergence of this ratio in the case that the lengths $l(\gamma_i)$ all have the same behaviour near zero, i.e. $l(\gamma_i) = d_i \varepsilon$ for $\varepsilon \to 0$, and fixed $d_i$.

**Sketch of Proof.** Complete $\gamma_1 \cdots \gamma_r$ to a set of geodesics $\gamma_1 \cdots \gamma_{3g-3}$, giving a decomposition of $S$ into $Y$-pieces with length $l(\gamma_i) \leq L_g$, a constant depending only on $g$ (see [Bu2, §13]). Then using a modified version of an argument of [B1] we show that $\lambda_j \cdot (1 + o(\sqrt{\delta})) \geq \pi^{-1} \lambda_j(\Gamma)$, where $\Gamma$ is the graph of the $Y$-pieces, and the length of an edge corresponding to a small geodesic is $l(\gamma)$. The proof of this also uses the asymptotic of the first nonzero eigenvalue of $Y_1 \cup Y_2$ for the Neumann problem, where $Y_1, Y_2$ are $Y$-pieces, $Y_1 \cap Y_2 = \gamma$ and $l(\gamma)$ is small. This can be deduced from [C.CdV], because there is only one small geodesic separating $Y_1 \cup Y_2$. To finish the proof we have then to compare $\lambda_j(\Gamma)$ with $\lambda_j(\Lambda)$. To do this we consider $\Lambda$ to be the graph of the connected components of $\Gamma$ after removing the small edges of $\Gamma$.

Received by the editors February 10, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 53C22, 58G25; Secondary 30F99.

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0273-0979/88 $1.00 + $.25 per page

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If $f$ is an eigenfunction corresponding to $\lambda_j(\Gamma)$ we write $f = h + g$, where, on each component, $h$ is constant and $g$ of mean zero: Applying the quadratic form to $h + g$ we estimate the resulting terms.

If one is interested in lower bounds on $\lambda_j$ in terms of $\lambda_j(\Lambda)$ it is more convenient to relate $\lambda_j$ to a graph associated to a geodesic triangulation of $S$ as considered in [Bu1] and used in [B2, §4]. As in the proof of Theorem 1, one relates then the spectrum of this graph to $\lambda_j(\Lambda)$.

**Theorem 2.** Let $S$ be a compact Riemann surface of genus $g \geq 2$ and $l(\gamma_1) \leq \cdots \leq l(\gamma_r) \leq l(\gamma_{r+1}) \cdots$ be the length of the closed geodesics of length smaller than $2 \ln 2$. Then there exists a universal constant $c > 0$ s.t. if $l(\gamma_r) \leq l(\gamma_{r+1})/g^2$ then $\lambda_j(S) \geq c\lambda_j(\Lambda)$ for $1 \leq j \leq k$. And $\Lambda$ is the graph associated to the components of $S$ after removing the geodesics $\gamma_1, \ldots, \gamma_r$.

**Remarks.** (a) It is easy to generalize Theorems 1 and 2 to the case of geometrically finite surfaces.

(b) In [D.P.R.S.] the inequality of Theorem 2 is stated with a constant $c(g)$ depending on the genus of $S$. But for fixed $g$ there are only finitely many graph structures which can occur if one forgets the length of the edges, thus a constant depending on $g$ “destroys” the combinatorial information contained in $\lambda_j(\Lambda)$.

(c) If there are no small geodesics or if the small geodesics don’t disconnect the surface then $\lambda_1(S) \geq c/g^2$, $c > 0$ being a universal constant.

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