Hadronic Light-by-Light Contribution
to the Muon $g - 2$

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Abstract

We present a calculation of the hadronic light-by-light contributions to the muon $g - 2$ in the $1/N_c$ expansion. We have used an Extended NJL model and introduced an explicit cut-off for the high energy region. We have then critically studied the relative size of the high energy contributions. Although we find them large we can give a conservative estimate of the light-by-light contribution to $a_\mu$ which is $-(11 \pm 5) \cdot 10^{-10}$. This is between two and three sigmas the expected experimental uncertainty at the forthcoming BNL experiment.

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The high precision measurement of the anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$, combined with the LEP results is expected to provide valuable information on the electroweak sector of the Standard Model and maybe unravel new physics. See [1] for reviews of both theoretical and experimental status. For carrying out this program, a new measurement of the muon anomalous magnetic moment at Brookhaven National Laboratory [2] aims to reduce the experimental uncertainty to $\sim 4 \cdot 10^{-10}$. On the theoretical side the error is dominated by the hadronic contributions and in particular by the hadronic vacuum polarization contribution. For a recent determination of this contribution and earlier references see [3]. The progress expected in measuring the total cross section $\sigma_{\text{total}}(e^+e^- \to \text{hadrons})$ will get the theoretical uncertainty from this contribution down to the order of the experimental uncertainty quoted above [3].

There is another source of hadronic uncertainty in the theoretical calculation of $a_\mu$ which has raised recently some discussion about its reliable calculation [4, 5, 6]. It is the hadronic light-by-light scattering contribution where a full four-point function made out of four vector quark currents is attached to a muon line with three of its legs coupling to photons in all possible ways and the fourth vector leg coupled to an on-shell external photon (see Figure 1). The difficulty here is that this contribution cannot be expressed in terms of experimental observables and thus one has to rely on our present knowledge in treating the strong interactions. There have been several attempts to calculate this contribution in the past [7, 8] and more recently in [9]. In this Letter we mainly address the calculation of the hadronic light-by-light contributions to $a_\mu$ in the large $N_c$ limit ($N_c =$ number of colors). These are $\mathcal{O}(N_c)$ in the $1/N_c$ expansion. An estimate of the effects of the $U(1)_A$ anomaly will be also included. We will use an Extended Nambu–Jona-Lasinio (ENJL) model and introduce an explicit cut-off. We defer a complete discussion and a more detailed presentation of our results to a forthcoming publication [10].

The momenta flowing through the three vector legs of the four-point function attached to the muon line run from zero up to infinity then covering both the non-perturbative and perturbative regimes of QCD. These two different regimes are naturally separated by the scale of the spontaneous symmetry breaking ($\Lambda_{\chi} \simeq 1$ GeV). Above this scale the strong interaction contributions have to match the perturbative QCD predictions in terms of quarks and gluons. At this point we would like also to check the current common wisdom statement that the bulk of the hadronic light-by-light contributions to $a_\mu$ is determined by the physics around the muon mass. In fact this was assumed in all previous calculations. If this was correct we could attempt to make a pure low energy calculation that would saturate this contribution. Were the contributions not negligible at some high scale we would need a more sophisticated model to calculate the vector four-point function. Indeed this appears to be the case from our results. We shall only be able to give a conservative estimate.

At very low energies (typically below the kaon mass), the framework to study
the strong interactions is Chiral Perturbation Theory (CHPT) and the relevant degrees of freedom are the lightest pseudoscalar mesons ($\pi$, $K$ and $\eta$). However, as emphasized in [6], there appear counterterms in the calculation of $a_\mu$ which are not determined by symmetry arguments alone. Instead we will use a low energy model. We will use an ENJL model because it possesses the following features: it encodes all the chiral constraints and therefore satisfies all the QCD Ward identities (both anomalous and non-anomalous); it has spontaneous symmetry breaking; both an $1/N_c$ expansion and a chiral expansion are possible; it reproduces the low energy phenomenology and the success of Vector Meson Dominance (VMD) models. To a large extent, as shown by the Weinberg Sum Rules, it also has the correct matching with the high energy QCD behaviour. Models to introduce vector fields like the Hidden Gauge Symmetry (HGS) model used in [9] do not always have this good behaviour. These characteristics were emphasized for $a_\mu$ in [6]. See [12] for more details, definitions and technical points about the specific model we are using.

Its major drawback is the lack of confinement. This can be smeared out by calculating with constituent quarks far off-shell and color singlet observables. This model has three parameters plus the current light quark masses. These are fixed as to reproduce the experimental pion and kaon masses. The three parameters can be chosen to be the couplings of the spin zero $G_S$ and spin one $G_V$ four-quark terms in the ENJL model (see [12] for details) and, since it is a non-renormalizable model, the cut-off $\Lambda$ of the regularization which we chose to be proper-time. Although this regulator breaks in general the Ward identities we impose them by adding the necessary counterterms including those for the anomaly [11]. The values of the parameters we use are the ones obtained in the first reference in [12] from a fit to low energy data: $G_S = 1.216$, $G_V = 1.263$ and $\Lambda = 1.16$ GeV. Then the constituent quark masses solution of the gap equation are $M_u = M_d = 275$ MeV and $M_s = 427$ MeV. The hadronic vacuum polarization contribution was estimated within the ENJL model in [6] using a procedure similar to the one below. The result agreed within about 15% with the one in [3].

Let us proceed to the calculation itself. We calculate to all orders in the chiral expansion. Notice that this is needed since we are integrating over three of the vector momenta. In previous calculations the lowest order CHPT result was convoluted with a naive VMD propagator. It is not clear how this procedure preserves the QCD Ward identities and the CHPT expansion itself.

At leading order in $1/N_c$ there are two classes of hadronic light-by-light diagrams contributing to $a_\mu$. The first one is in Figure 2a. This is a pure full four-point function with a constituent quark loop and the three vector legs attaching to the muon line dressed by full two-point functions. These full two-point functions are the sum of strings with one, two, · · · , $\infty$ constituent quark loops and can be found in [12]. There are six possible permutations for each quark flavor. The leading order in the CHPT expansion of this contribution is $O(p^8)$.
and thus potentially sensitive to the high energy region. The other class is in Figure 2b. Here we have two one-loop three-point functions with two vector legs each one and glued with a full two-point function that can be either pseudoscalar, scalar, mixed pseudoscalar–axial-vector, or axial-vector. The vector one does not contribute. The three vector legs attaching to the muon line are dressed with full two-point functions. The leading order in the CHPT expansion is $O(p^6)$ for the pseudoscalar exchange and $O(p^8)$ for the others. There are twelve possible permutations for each quark flavor.

Although the sum of the contributing terms is UV finite, each of them can be logarithmically divergent and one has to rely on potentially dangerous numerical cancellations. Instead, we used the method proposed in [13] to construct individually UV safe quantities. This is achieved by making use of the gauge invariance in the on-shell photon leg. We then construct the quantity in (2.9) of [13]. Momentum integrals are performed numerically in Euclidean space. This allows us to impose physically relevant cut-offs on the photons’ momenta.

The contribution of the first class of diagrams in Figure 2a can be written as a seven dimensional integral which we have evaluated using the Monte Carlo routine VEGAS. As a check we have reproduced the constituent quark and muon loops results in [8] and the electron loop results in [14]. Since we are dealing with a low-energy model we want to study the dependence on a high-energy cut-off $\mu$ on the vector legs’ momenta. The result only stabilizes at a rather high value of $\mu$. For a bare constituent quark with a mass of 300 MeV, the change between a cut-off of 2 GeV to a cut-off of 4 GeV is still around 20%. The change from 0.7 GeV to 2 GeV is typically a factor of 1.8. This invalidates the use of any low energy model to calculate the complete hadronic light-by-light contribution to $a_\mu$. The bulk of these contributions does not come from the dynamics at scales around the muon mass as it is often stated. This also explains the rather high sensitivity to the damping provided by the vector two-point functions observed in [8, 9]. Although the only rigorous result is for scales smaller than $(0.6 \sim 1)$ GeV, one still obtains an estimate. Mimicking the high energy behaviour of QCD by a bare constituent quark loop with a mass of about 1.5 GeV gives only an addition of $\sim 0.2 \cdot 10^{-10}$ and if there is any VMD suppression it will be even smaller. This we will take then as the uncertainty due to the high energy region contribution and the ENJL result where it stabilizes as our estimate.

Let us now turn to the second type of contributions in Figure 2b. This contribution can be factorized into a five dimensional integral which we have evaluated using the Monte Carlo routine VEGAS times two two-dimensional integrals and one one-dimensional integral that we have evaluated using Gaussian integration. Here, again we have followed the prescription in [13] to calculate the contribution to $a_\mu$. We have used two different approaches to calculate the quantity in (2.9) of [13]. The first one is using the Ward identities for four-point functions and the second one is using the Ward identities for three-point functions. Both agree exactly. We have done the same study of the cut-off dependence as
for the four-point function contribution. The contribution of the pseudoscalar exchange is more than one order of magnitude larger than the others. The reason that makes this contribution so different can be traced back both in the presence of two flavour anomaly vertices and the CHPT counting. It therefore deserves more attention. In fact, the pseudoscalar exchange has important next-to-leading corrections from the effects of the $U(1)_A$ anomaly that will leave the $\pi^0$ exchange as the dominant contribution to the muon $g-2$. We have taken into account the effects of the $U(1)_A$ anomaly by using the physical $\pi^0$, $\eta$ and $\eta'$ mass eigenstates as propagating states. We see less stability at high values of the cut-off $\mu$ than for the quark-loop contribution. Although the change from 0.7 GeV to 2 GeV is also around 1.8, the stability is worse for cut-off values above 4 GeV. Notice also that the error from integration routine VEGAS is larger for these values of the cut-off. The poor stability in the pseudoscalar exchange is mainly due to the subtraction terms we need to obtain the correct $SU(3)$ flavour anomaly. We shall make a detailed analysis of the intermediate and high energy region contributions to this pseudoscalar exchange in a more detailed publication [10].

Both scalar and axial-vector contributions are very suppressed and much smaller than our final error. In Table [1] we have listed the $\mathcal{O}(N_c)$ hadronic light-by-light two leading contributions to $a_\mu$ for the up and down quarks as functions of the cut-off together with the errors quoted by VEGAS. We have also included in the fourth column the estimation of the pseudoscalar exchange in the presence of the $U(1)_A$ anomaly. In the fifth column is the sum of the quark loop and the $\pi^0$, $\eta$ and $\eta'$ exchanges in the fourth column. Since the integrand is rather irregular, this error estimate is somewhat on the small side (see also [15]) and will be largely superseded by the error in our final result. For the quark loop contribution we used nonet symmetry. The contribution from the strange quark to the quark loop is in the range of the quoted errors in Table [1]. The charm quark contribution we calculate with a bare quark loop damped with $c\bar{c}$ meson dominance propagators in the photon legs. This contribution is very small. Both scalar and axial-vector exchange contributions are again in the range of the quoted errors in Table [1]. We therefore take as an estimate of the leading $\mathcal{O}(N_c)$ hadronic light-by-light contributions to $a_\mu$ including the effects of the $U(1)_A$ anomaly, the result in the fifth column of Table [1] plus the scalar and axial-vector exchange contribution, $0.75 \cdot 10^{-10}$, and the strange and charm quarks contributions, $0.05 \cdot 10^{-10}$:

$$
\left( a_\mu^{\text{light-by-light}} \right)_{\mathcal{O}(N_c)} = -(10.5 \pm 5.0) \cdot 10^{-10} .
$$

The error is mainly induced by the uncertainty from the intermediate and high energy contributions to the pseudoscalar exchange.

In addition to the leading $\mathcal{O}(N_c)$ result above there are the contributions from pion and kaon loops. These are $\mathcal{O}(1)$ in the $1/N_c$ expansion and have to be added to the $\mathcal{O}(N_c)$ result in [1]. We have seen that the lowest order CHPT result is damped by roughly the same factor in both the constituent quark loop
and the pseudoscalar meson exchange contributions and that the high energy region contributes significantly. This can be used to estimate that the result in [9] for the pion and kaon loops is in the right ball-park when vector mesons are included. As a first estimate we take the number and error from [9]

\[
\left( a_{\mu}^{\text{light-by-light}} \right)_{\mathcal{O}(1)} = (-0.45 \pm 0.80) \cdot 10^{-10} .
\]

We will return to this contribution in [10]. Adding the above \( \mathcal{O}(N_c) \) and \( \mathcal{O}(1) \) results we get our final estimate

\[
a_{\mu}^{\text{light-by-light}} = -(11 \pm 5) \cdot 10^{-10} .
\]

A more general comment is that although a HGS model can be derived from the ENJL model, this is only true after a series of approximations. In HGS models the consistency between the parameters in the anomalous and non-anomalous sectors is not obvious. In the ENJL model we are using the same parameters appear in both sectors. This is particularly important for the flavour anomaly contribution to the light-by-light scattering. For instance, the calculation in [9] assumes complete VMD for the anomalous sector. It was shown in [11] that complete VMD breaks the anomalous Ward identities. A prescription to include vector and axial-vector couplings was given there and was used in the present work. We find the pseudoscalar-exchange contribution to be negative and although the central value is three times larger it is compatible within one sigma with the values quoted in [9].

Our calculation establishes that the contribution to \( a_{\mu} \) from light-by-light scattering is negative and relatively large. It is of the same order as the one-loop electroweak corrections [16]. This result is between two and three sigmas the aimed experimental uncertainty at BNL. Our result has a large uncertainty due to intermediate and high energy contributions. Although we believe our estimate is conservative it has an unsatisfactory uncertainty that will be difficult to reduce. We will address this issue in [10]. Despite the uncertainty, the estimate in (3) is still an important theoretical result for the interpretation of the muon \( g - 2 \) measurement at the planned BNL experiment.

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**TABLE CAPTION**

Table 1 Results for the two dominant hadronic light-by-light contributions to $a_{\mu}$ in the ENJL model.

**FIGURE CAPTIONS**

Figure 1 Hadronic light-by-light contribution to $a_{\mu}$. The bottom line is the muon line. The wavy lines are photons and the cross-hatched circle depicts the hadronic part. The circled crossed vertex is an external vector source.

Figure 2 The two classes of hadronic light-by-light contributions to $a_{\mu}$ at leading $O(N_c)$. (a) The four-point functions class. (b) The product of two three-point functions class. The dots are ENJL vertices. The circled crossed vertices are where photons connect. The cross-hatched loops are full two-point functions and the lines are constituent quark propagators.
| Cut-off (GeV) | $a_\mu \times 10^{16}$ from Constituent Quark in Figure (2a) | $a_\mu \times 10^{16}$ from Pseudoscalar Exchange $\mathcal{O}(N_c)$ in Figure (2b) | $a_\mu \times 10^{16}$ from $\pi^0$, $\eta$ and $\eta'$ Exchanges $\mathcal{O}(N_c) + U(1)_A$ | Sum     |
|------------|-------------------------------------------------|-------------------------------------------------|----------------------------------------|--------|
| 0.7        | 1.14 ± 0.02                                      | -19.4 ± 0.1                                     | -7.2 ± 0.1                             | -6.1   |
| 1.0        | 1.44 ± 0.03                                      | -24.2 ± 0.2                                     | -9.4 ± 0.1                             | -8.0   |
| 2.0        | 1.78 ± 0.04                                      | -33.0 ± 0.2                                     | -13.2 ± 0.2                            | -11.4  |
| 4.0        | 1.98 ± 0.05                                      | -39.6 ± 0.6                                     | -15.9 ± 0.2                            | -13.9  |
| 8.0        | 2.00 ± 0.08                                      | -46.3 ± 1.5                                     | -18.6 ± 0.4                            | -16.6  |

Table 1:
Figure 2: