Thermodynamic implications of non-reciprocity

Sarah A. M. Loos* and Sabine H. L. Klapp

Technische Universität Berlin Institut für Theoretische Physik
Hardenbergstr. 36, 10623 Berlin, Germany

Abstract

We study the thermodynamic properties induced by non-reciprocal interactions between stochastic degrees of freedom in time- and space-continuous systems. We show that, under fairly general conditions, non-reciprocal coupling alone implies a steady energy flow through the system, i.e., non-equilibrium. Projecting out the non-reciprocally coupled degrees of freedom renders non-Markovian, one-variable Langevin descriptions with complex types of memory, for which we find a generalized second law involving information flow. We demonstrate that non-reciprocal linear interactions can be used to engineer non-monotonic memory, which is typical for, e.g., time-delayed feedback control, and is automatically accompanied with a nonzero information flow through the system. Furthermore, already a single non-reciprocally coupled degree of freedom can extract energy from a single heat bath (at isothermal conditions), and can thus be viewed as a minimal version of a time-continuous, autonomous “Maxwell demon”. At the same time, the non-reciprocal system has characteristic features of active matter, such as a positive energy input on the level of the fluctuating trajectories, without global particle transport.

1 Introduction

Fundamental physical interactions between mutually coupled particles, such as atoms or molecules, are typically reciprocal. They are derivable from a Hamiltonian (i.e., conservative) and thus fulfill, automatically, Newton’s third law, actio = reactio. In the absence of driving forces or (temperature) gradients, systems with reciprocal interactions equilibrate and are well described by traditional thermodynamics. This holds even on the mesoscale, that is, when instead of the full microscopic dynamics, only few representative (stochastic) variables are considered by integrating out all other degrees of freedom (d.o.f.). This is the key idea of the celebrated Mori-Zwanzig approach [105] yielding a generalized Langevin equation, which involves noise and a memory kernel satisfying a fluctuation-dissipation relation (FDR), and may stochastically describe the motion of a colloid in a complex environment (e.g., a viscoelastic fluid [23, 57, 85, 104]).

However, the idea of reciprocal couplings and its thermodynamic implications breaks down in many living and artificial complex systems, where more general interactions, in particular, non-reciprocal couplings between mesoscopic subsystems, or (stochastic) d.o.f., naturally emerge [1, 19, 36, 40, 87]; as, e.g., in pedestrian dynamics [29, 37, 68], in complex plasmas [6, 12, 19, 67, 84], or in bio-chemical systems [9, 28, 42]. Moreover, state-of-the-art experimental techniques enable the realization of almost arbitrary interactions between colloidal particles [25, 38], including non-reciprocal ones [46]. Tuning the interactions opens up the possibility to experimentally explore fundamental principles, and to manufacture artificial systems on the fluctuating scale, like Brownian molecules [25, 38]. Recently, also in quantum systems it was demonstrated that the implementation of non-reciprocal couplings can be used to build new types of devices, e.g., directional amplifiers [20, 58, 59, 65, 93]. Further, non-reciprocal couplings between (effective) variables are present in various models for active matter. For example, to describe active self-propelled motion [22, 61, 77, 78, 95], the temporal evolution of the particle’s position is assumed to be affected by the orientation (due to the flagella or asymmetric flow field), but there is no backcoupling.

While some models which involve non-reciprocal interactions have already been studied from a thermodynamic perspective [3, 11, 15, 31, 62, 66, 72, 92], the general thermodynamic and information-theoretical implications of non-reciprocity itself have, to our knowledge, not been discussed so far. This is the first major goal of this paper. To this end, we will review and reinterpret some results from the literature (for systems with two d.o.f.), and derive new formulae for larger systems. In particular, we consider Markovian systems of n + 1 non-reciprocally coupled subsystems X_0,1,...,n with white noise. Each subsystem can represent, e.g., the position of a colloid in an experiment. By considering different thermodynamic quantities, we investigate the following questions: Can non-reciprocal systems reach a state of thermal equilibrium? Is there a crucial difference between nonequilibrium states induced by non-reciprocity, vs. external drivings? Indeed, we show here that, except for some specific cases, non-reciprocal systems are inherently out of equilibrium, even in the absence of external forces or (temperature) gradients. In order to discuss the fundamental consequences of non-reciprocity on a purely analytical basis, we will consider linear models. However, as we will discuss, several conclusions take over to non-linear models.

* sarahloos@itp.tu-berlin.de
Figure 1: Overview of various systems describable by the generic model \((1)\) with \(n = 1\). Left: for unidirectional coupling \((a_{01} = 0)\), \(X_1\) corresponds to a cellular sensor \([28]\). Center: for reciprocal coupling \((a_{01} = a_{10})\), \(X_{0,1}\) correspond to the angles of two mechanically coupled vanes \([91]\). Right: For unidirectional coupling \((a_{10} = 0)\), \(X_0\) corresponds to the position of a microswimmer within the AOUP model \([10, 11, 15, 60, 92]\). In the intermediate cases with bidirectional non-reciprocal coupling, \(X_1\) corresponds to the controller acting on a colloid at position \(X_0\).

The second main goal is to show under which conditions a setup with non-reciprocal linear couplings can be used to build a “microswimmer”, a “feedback controller” \((X_{j>0}\) representing a feedback controller acting on \(X_0)\), or a “Maxwell demon”. For microswimmers, thermodynamic notions are already a huge topic \([3, 11, 15, 21, 22, 48, 62, 66, 72, 92]\). Here, we calculate the information and energy flow between the particle (here \(X_0)\) and its propulsion mechanism (here represented by at least one subsystem \(X_1)\), confirming general expectations, e.g., the active swimmer heats up its environment but never cools it down. In contrast, in the context of time- and space-continuous feedback \([55, 69, 84, 85, 97]\), the connection to non-reciprocal coupling is rather uncommon and new. Therefore, we dedicate a more detailed analysis to this point. We show that linear non-reciprocal couplings can be used to construct a time-delayed feedback loop, and clarify under which conditions a non-reciprocal coupled d.o.f. can extract energy from a single heat bath, making it a “Maxwell demon”. We further find conditions under which thermal fluctuation suppression (or enhancement), i.e., “isothermal compression or expansion” of a single-particle gas are possible.

While some of the questions and connections discussed here may seem to be intuitively clear, almost representing “common wisdom”, there are only few studies where these issues are formally addressed. Moreover, we also detect counter-intuitive phenomena. For example, non-Markovian processes can exhibit a nonequilibrium steady state (NESS) without dissipation, where the entropy is exported purely in the form of information, implying that information and entropy are transported without accompanying energy flow (while in total sustaining this process relies on external energy supply). Furthermore, we show that, under certain conditions, a system of two isothermal subsystems with non-reciprocal coupling can be mapped onto a reciprocal system with a temperature gradient, building a bridge to other active matter models \([48, 71, 82]\). In addition, we provide a detailed derivation of the relevant information flows, which is, so far, a quantity that is not well-established for time- and space-continuous systems.

From a conceptual viewpoint it is important to also think about situations, where a portion of the d.o.f. might not be invisible to a (“marginal”) observer. Even more, in some theoretical models, a portion of the d.o.f. has no direct physical interpretation. Then, the dynamics can be equivalently formulated as a non-Markovian, one-variable equation (for \(X_0)\) with a memory kernel and colored noise, upon projecting out \(X_{j>0}\). In such a situation, the interpretation of thermodynamic quantities must be treated with care, and is indeed subject of a recent debate \([11, 15, 84, 85, 92]\). To account for this fact, we will pay special attention to the different measures of (non)equilibrium on the levels of the Markovian and non-Markovian description, and also explicitly consider the entropy balance of an individual subsystem. We will further comment on the connection to so-called “effective thermodynamic” descriptions \([30, 73]\).

We close this introduction with a brief outline. After introducing the model in Sec. II, we will investigate under which conditions detailed balance and the fluctuation-dissipation relation are satisfied (Sec. III). Then, we will calculate the total entropy production of the entire system and the dissipation of an individual subsystem in Sec. IV. Thereafter we will consider the entropy balance of an individual sub-system and derive explicit expressions for the information flows through the system (Sec. V). In Sec. VI, we show that, under certain conditions, a non-reciprocal system can be mapped onto a reciprocal one, and finally conclude in the Sec. VII.
with the vector $\mathbf{X} = (X_0, X_1, ..., X_n)^T \in \mathbb{R}^{n+1}$ involving $n + 1$ stochastic d.o.f. We will discuss thermodynamic properties of both, the entire system $\{X_0, X_1, ..., X_n\}$, and of the individual $X_j$. To set the focus, we will occasionally call $\{X_0, X_1, ..., X_n\}$ the “super-system”, while an individual $X_j$ will be called a “sub-system”. Further, $\xi_j$ denote zero-mean, Gaussian white noises with $\langle \xi_i(t)\xi_j(t') \rangle = 2k_B T \gamma_{ij} \delta(t - t')$ at temperatures $T_j \geq 0$, $j \in \{0, 1, ..., n\}$, with $k_B$, $\gamma_j$ being the Boltzmann and friction constants that also appear in the diagonal friction matrix $\gamma$ with $\gamma_{jj} = \gamma_j$. $f_0$ is an, in general, nonlinear force. The topology matrix $a$ defines the strength of the couplings $a_{ij}$, and gives the timescale $\gamma_{ij}/a_{ij}$ of the exponential relaxation dynamics of each d.o.f., due to the restoring forces $a_{ij} \dot{X}_j$.

At this point, we may already note one apparent difference between reciprocal system ($a_{ij} = a_{ji} \forall i, j$) and those that involve non-reciprocal couplings ($a_{ij} \neq a_{ji}$), that is, only the purely reciprocal coupled equations can be expressed as derivatives of a Hamiltonian, plus noise terms (and, if present, plus non-conservative forces $f_0$). In that case, (1) can be written as $\gamma_j \dot{X}_j = -\partial H/\partial X_j + \xi_j$, with the Hamiltonian

$$H = \sum_{j=0}^{n} [V_j(X_j) + \sum_{i>j} H_{int}(|X_i - X_j|)] = \sum_{j=0}^{n} [\frac{a_{jj}}{2} X_j^2 + \frac{a_{ji}}{2} \sum_{i>j}(X_i - X_j)^2],$$

(2)

where the last term in (2) represents the interaction part, $H_{int}$. In contrast, non-reciprocal couplings appear as a non-conservative force (like $f_0$). In that case, (1) corresponds to $\gamma_j \dot{X}_j = -\partial H/\partial X_j + \sum_{i\neq j} a_{ij}X_j + \xi_j$.

Equivalently to (1), one can describe the dynamics of one d.o.f., say $X_0$, by a one-variable LE

$$\gamma_0 \dot{X}_0(t) = a_{00} X_0(t) + \int_0^t K(t-t') X_0(t') dt' + f_0 + \nu(t) + \xi_0(t),$$

(3)

which can be derived by projecting the $X_{j>0}$ onto $X_0$, as described in [51, 105] and in Appendices A and B. Generally (unless the time-scales of $X_0$ and $X_{j>0}$ are well-separated), (3) is a non-Markovian LE, i.e., it comprises memory. In particular, it involves a time-nonlocal force depending on the past trajectory, weighted with a memory kernel $K$, and $\nu$ is a zero-mean, Gaussian colored noise (both depend on the topology of the coupling matrix, concrete examples are given below). For $T_{j>0} \equiv 0$, there is no colored noise in (3). We aim to emphasize that the dynamics of $X_0$ is identical to (1). Using (3) instead of (1) can be regarded as a coarse-graining or marginalization, because the dynamics of $\dot{X}_j$ is not explicitly considered. However, it does not imply loss of information about, or approximation of, $X_0$. One should note that, in reverse, for a non-Markovian process (3), a corresponding Markovian representation (1) is not unique. Thus, a specific memory can be realized by different Markovian networks [this can be seen, e.g., from Eq. (1) by the fact that $a_{00}$ and $a_{10}$ only arise as product, $a_{01}$ and $a_{10}$].

For the sake of generality, we deliberately do not focus on a specific model, and rather offer different interpretations for the involved d.o.f.; explicit examples will be given below. However, a situation of special interest is that the observer only sees parts of the system (say only $X_0$), while the other d.o.f. are “hidden.”
Even more, in some cases, only certain d.o.f. (say only $X_0$), represent actual, physical d.o.f. (such as the position of a colloid), whereas the others (say $X_{j>0}$) are effective (or auxiliary) variables representing those parts of the complex environment which generate a feedback loop or active motion. In such a situation, a non-Markovian description, which only involves $X_0$, may be the more fundamental one. We will discuss both situations, only $X_0$ or all $X_j$ being observed, in this paper.

Before we start with investigating the thermodynamic consequences of non-reciprocity, we first aim to discuss the relationship between non-reciprocal coupling in (1) and resulting memory in (3) and then give some examples for systems that can be modeled by (1) and (3).

## 2.1 Memory induced by non-reciprocally coupled systems

We begin by considering the smallest version of (1) with $n = 1$. While various aspects of this case have been studied previously [10, 11, 14, 15, 60, 75, 92], the full implications of non-reciprocity have so far, to the best of our knowledge, not been discussed. For $n = 1$, the memory kernel $K$ and the noise correlations $C_\nu(T) := \langle \nu(t)\nu(t+T) \rangle$ are both found to decay exponentially for reciprocal as well as non-reciprocal coupling, and read

$$K(T) = (a_{01}a_{10}/\gamma_1)e^{a_{11}T/\gamma_1},$$

$$C_\nu(T) = k_0 T_1 (a_{01}^2/a_{11}) e^{a_{11}T/\gamma_1}. \quad (4)$$

An exemplary plot of both functions is given in Fig. 2(a).

Let us now investigate the effect of adding more sub-systems $X_j$ to the super-system (1), such that there may be an interplay of multiple non-reciprocal interactions. Most importantly in the present context, this leads to complex types of memory beyond the single exponential decay. To illustrate this, let us consider a ring of three d.o.f., where all (counter-)clockwise couplings are set to $-p$, $-a_{jj} = p + \kappa$, as sketched in Fig 2(a). This super-system generates the memory kernel

$$K(t-t') = e^{-(p^2/2 + \kappa^2/2)} e^{-(p^2/2 - \kappa^2/2)} e^{2p\kappa t}, \quad (5)$$

(see Appendix A for a derivation). For reciprocal, i.e., conservative couplings, $\kappa = p$, (5) simplifies to an exponential decay $K(T = |t-t'|) = 2\kappa e^{-\kappa T}$. In contrast, if the coupling is non-reciprocal, we find that the super-system (1) generates a non-monotonic memory kernel, despite the linearity of all couplings. In the present example, the memory kernel (5) has a maximum at a finite time difference. In the limit of unidirectional coupling $p \to 0$, the memory kernel (5) converges to a Gamma-distribution $K(T) = \kappa^3 e^{-\kappa T}$, which has a pronounced maximum near $\kappa/3$, see Fig. 2(b). Noteworthy, in this limit, the kernel vanishes at $T = 0$, i.e., the instantaneous position does not contribute to the integral $\int X_0(t') K(t-t')dt'$ in (3) [while the integral is dominated by the instantaneous position for reciprocal coupling]. In Appendix A we discuss the general case where all couplings are different, yielding very cumbersome expressions while the overall characteristics are the same.

Playing around with different coupling topologies and system sizes, we generally find that non-reciprocal coupling is a crucial ingredient to generate non-monotonic memory, while reciprocal couplings always yield monotonic kernels. With an appropriate coupling topology, it is also possible to generate memory kernels with multiple maxima. We observe that a kernel with $n$ extrema can be represented via (at least) $n$ d.o.f.. On the other hand, we observe that the colored noise produced by linearly coupled d.o.f. is always monotonically decreasing (see, e.g., Fig. 2). A systematic study of the connections between coupling topology, the generated memory, and the resulting correlation functions will be presented in [18].

## 2.2 Examples

Let us now consider exemplary systems of type (1) with non-reciprocal interactions. We start with a brief summary of models known from the literature and then introduce our new models with feedback. Figure 1 provides an overview for the case $n = 1$.

For reciprocal coupling, the dynamics of the two d.o.f. $X_0$ and $X_1$, corresponds to the angles of two vanes that rotate in two different heat baths at $T_0$ and $T_1$, and are coupled by a torsion spring with spring constant $a_{01} = a_{10}$. At $T_0 \neq T_1$ this setup was considered as a minimal model for heat conducting through mechanical motion, as discussed in [91] (see p. 154). For unidirectional coupling ($a_{01} = 0$, $a_{10} > 0$), $X_0$ may correspond to the position of a microswimmer within the active Ornstein-Uhlenbeck particle (AOUP) model [10, 11, 15, 60, 92]. Then, $X_1$ represents the effect of the flagella of a bacterium, or the asymmetric flow field around a Janus colloid, pushing $X_0$ away from it and in this way creating the propulsion; while the memory (4), which in this case only appears in the form of noise color ($K \equiv 0$ since $a_{10} = 0$), yields the persistence of the motion. Furthermore,
the very same super-system with reversed unidirectional coupling (i.e., $a_{01} \equiv a_{10}$), was recently suggested as a model for a cellular sensor [9, 23]. Furthermore, the model for a cellular sensor with memory [23], corresponds to the case $n = 2$. Then, $X_0$ with $X_1$, and $X_1$ with $X_2$ are coupled unidirectionally, and there is no direct link between $X_0$ and $X_2$.

As we will show in this paper, the generic system \( [1] \) with non-reciprocal couplings also includes cases where a portion of the d.o.f., $X_J$, can be regarded as a feedback controller continuously operating on a system $X_0$ (where the noise terms $\xi_{J>0}$ represent errors of the controller). For example, the case $n = 1$ with bidirectional non-reciprocal coupling (i.e., $0 \neq a_{10} \neq a_{01} \neq 0$) can be interpreted as a minimal realization of such a controller.

A characteristic aspect of feedback control is the occurrence of a time delay between “measurement” and “control action”. In experimental setups, this delay either emerges naturally due to finite signal transmission or information processing times (e.g., think of optical feedback with the help of videomicroscopy [4, 17, 55, 100]), or may be implemented intentionally (e.g., in Pyragas control [76, 89]), because it is known to induce interesting dynamical and thermodynamical behavior, such as particle oscillations [7, 52, 88], transport [52], or a reversed heat flow [55]. The controller model with $n = 1$ yields an exponentially distributed delay with maximum at $t - t' = 0$. In contrast, the feedback loop often has a typical finite duration, i.e., the control action depends on $X_0(t - \tau)$, with a distinct characteristic delay time $\tau > 0$, implying that the equation of the controlled system (here $X_0$) involves a memory kernel with a maximum around $\tau$. It now becomes clear that a unidirectional ring with $n = 2$ can describe such a controller with preferred delay time. Specifically, setting $a_{10} = a_{21} = -a_{11} = -a_{22} = \gamma_1/\tau$, $T_1 = T_2$, $\gamma_1 = \gamma_2$, $k = a_{01}$ yields a kernel

\[
K(T) = (k/\tau^2) T e^{-T/\tau},
\]

\[
C_r(T) = (k_B T_1/(2\gamma_1)) k^2 (3\tau + T) e^{-T/\tau},
\]

with a pronounced maximum at $\tau$. The feedback force is $k \int X_0(t') K(t - t') dt'$, or, $kX_n$, in the non-Markovian or Markovian description, respectively. Note that, due to this setting, the only remaining free controller parameters are the time delay $\tau$ and the feedback gain $k$. To better compare the controllers with $n = 1$ and $n = 2$, we analogously set $a_{10} = -a_{11} = \gamma_1/\tau$, and $a_{01} = k$ in the case with $n = 1$, obtaining from \( [1] \),

\[
K(T) = (k/\tau) e^{-T/\tau},
\]

\[
C_r(T) = (k_B T_1/\gamma_1) k^2 T e^{-T/\tau}.
\]

In this paper, we focus on the cases $n = 1, 2$, a generalization towards higher $n$ will be discussed in [53]. We note that the limit $n \to \infty$ yields a $\delta$-distributed memory kernel around $\tau$ [51, 53], i.e., $K \propto \delta(T - \tau)$. Such stochastic delay differential equations are infinite-dimensional, which makes their treatment very involved, especially when it comes to thermodynamics [50, 53, 55]. In comparison, the model proposed here has in total three d.o.f. and is thus, quite handy.

3 Intrinsic non-equilibrium

Now we turn to the thermodynamic properties induced by the occurrence of non-reciprocal interactions, focusing on the long-time behavior $t \to \infty$, when transient dynamics due to the initial conditions have decayed and the system has approached a steady state.

We start by clarifying whether thermal equilibrium can exist despite non-reciprocity. As mentioned before, non-reciprocal interactions are non-conservative. One might therefore guess that a system with non-reciprocal interactions cannot reach thermal equilibrium. To investigate this question, we check the detailed balance (DB) condition on the level of the Markovian representation \( [1] \). Since the latter is only meaningful when all d.o.f. have a physical interpretation, we also discuss the fluctuation-dissipation relation (FDR) on the level of the non-Markovian description \( [3] \).

Since we are interested in analytical solutions, we will focus on the linear case, i.e., $f_0 = 0$. We stress, however, that the framework is readily adaptable to cases where a nonlinear force act on $X_0$, then requiring numerical solutions.

3.1 Detailed Balance

To investigate whether the super-system \( [1] \) can approach thermal equilibrium, we check the detailed balance condition. To this end, we consider the flow of the $(n + 1)$-point joint probability density function (pdf), $\rho_{n+1}(\mathbf{x},t)$, of $\mathbf{x} = (x_0, ..., x_n)^T$. To access this quantity, we utilize the closed, multivariate Fokker-Planck equation (FPE) [51] corresponding to \( [1] \), which reads

\[
\partial_t \rho_{n+1}(\mathbf{x}) = -\nabla \left[ \frac{1}{T_0} \mathbf{x} - D \nabla \rho_{n+1}(\mathbf{x}) \right],
\]

\[
= \frac{1}{T_0}
\]

\[
\rho_{n+1}(\mathbf{x})
\]

\[
\rho_{n+1}(\mathbf{x})
\]

\[
\rho_{n+1}(\mathbf{x})
\]
with the probability current $\mathbf{J}$ and diagonal diffusion matrix $D_{ij} = k_B T_j / \gamma_j$. We note that $\mathbf{J}$ is generally constant in steady states, and zero in equilibrium. Using the identity $\partial_t \rho = [\partial_x \ln(\rho)]_\rho$, we rewrite (8) as

$$\mathbf{v} = x^{-1} \mathbf{a} x - D^{-1} \nabla \ln \rho_{n+1}(x),$$

(9)

which is connected to the probability current by $\mathbf{J} = \nabla \rho_{n+1}$. DB means that all probability currents vanish, hence, $v_j = 0, \forall j$. From (9) we obtain the condition $D^{-1} x^{-1} \mathbf{a} x = \nabla \ln \rho_{n+1}$, which implies that the vector $D^{-1} x^{-1} \mathbf{a} x$ is the gradient of a scalar function. This, in turn, is true if and only if $\nabla \times (D^{-1} x^{-1} \mathbf{a} x) = 0$. Noting that $\gamma$ and $D$ are diagonal, this brings us to

$$a_{ij} T_j = a_{ji} T_i,$$

(10)

for all pairwise coupling constants between every two mutually coupled sub-systems. We stress that this condition is irrespective of the coupling topology, or system size. Remarkably, (10) shows that non-reciprocal systems that fulfill DB do exist, as long as $a_{ij} a_{ji} > 0$. However, unidirectional super-systems are by construction pure nonequilibrium models, including the (AOUP) microswimmer, or the controller with non-monotonic memory $(n = 2)$, see Eq. (6).

Condition (10) further implies that non-reciprocal systems can reach equilibrium despite $T_i \neq T_j$. This is in sharp contrast to reciprocally coupled (or “passive”) systems, which generally never equilibrate in the presence of temperature gradients.

3.2 Fluctuation-Dissipation Relation

Let us now turn to the corresponding non-Markovian process in $x_0$-space, which is more appropriate for models where $X_j > 0$ have no direct physical interpretations or if a marginal observer only sees $X_0$. On this level of description, the definition of a probability current is less clear, as there is, in general, no corresponding closed FPE [53]. However, from the non-Markovian LE [3] at $f_0 = 0$ alone, we can immediately deduce that the probability current in this marginalized space must vanish by a simple symmetry argument: On an ensemble-averaged “global” level, the equation is completely symmetric w.r.t. a coordinate inversion $x_0 \rightarrow -x_0$, consequently, the probability current cannot have any direction. Thus, naïvely repeating the analysis from Sec. 3.1, the system would always appear to be in equilibrium. This is, however, not true, as we see by instead considering the FDR [43]

$$\langle \mu(t) \mu(s) \rangle = k_B T_0 \gamma(\frac{t - s}{\tau}),$$

(11)

which describes a balance between the friction kernel $\gamma$ and thermal noise $\mu$. As well known for, e.g., viscoelastic fluids, the validity of a FDR implies that the system equilibrates in the absence of external driving [43, 57].

To check (11) for the present model, we rewrite (3) in the form of a generalized LE by converting $\dot{K}$ via partial integration into a friction kernel, which yields

$$\int_0^t \gamma(\frac{t - s}{\tau}) \dot{X}_0(s) ds = a_{00} X_0 + \tilde{K}(0) X_0(t) + \mu(t),$$

(12)

involving $\mu(t) = \xi_0(t) + \nu(t)$ and $\gamma(t - s) = 2 \gamma_0 \delta(t - s) + \tilde{K}(t - s)$. For the case $n = 1$, the integrated kernel reads $\tilde{K}(T) = (a_{01} a_{10} / a_{11}) e^{a_{11} T / \gamma_1}$. It can easily be verified [using (4) for the noise correlations] that the FDR holds if

$$a_{01} T_1 = a_{10} T_0,$$

(13)

which agrees with the DB condition (10). Thus, the non-Markovian process is out of equilibrium unless (13) holds, which is, for example never the case for the active microswimmer (where $a_{10} = 0$). For our $n = 2$ controller (6), $\tilde{K}(T) = k (1 + T / \tau) e^{-T / \tau}$, and FDR thus amounts to

$$T_0 k \left(1 + T / \tau \right) e^{-T / \tau} = T_1 k \left(r k / 2 \gamma_1 \right) \left(3 + T / \tau \right) e^{-T / \tau}.$$

(14)

There is no pair of $k$, $\tau$ that simultaneously obeys $T_0 2 \gamma_1 = T_1 3 r k$ and $T_0 2 \gamma_1 = T_1 r k$, which would be necessary to fulfill FDR. Thus, in this case, FDR (and DB) are never fulfilled (except for the trivial cases, where $k$ or $\tau$ nullify, or tend towards $\infty$).

For other coupling schemes and $n > 1$, we observe that a non-reciprocal system may fulfill FDR, but violate DB. We will present a detailed investigation, which is beyond the scope of this paper, in [18].

In this section, we have seen that non-reciprocity implies an intriguing property of the corresponding non-Markovian stochastic process, i.e., the existence of nonequilibrium steady states with zero probability currents.
Figure 3: *Left:* Steady-state mean heat flow $\hat{Q}_0$ \cite{21} for \cite{4} at $n = 1$. Along the diagonal $a_{01} = a_{10}$, the super-system is reciprocally coupled, and $X_{0,1}$ may model the angles of vanes coupled by a spring \cite{91}. For unidirectional coupling $a_{01} = 0$, $X_1$ correspond to a cellular sensor \cite{25}. Along the other unidirectional coupling line $a_{10} = 0$, $X_0$ corresponds to the position of an active swimmer in the AOP model \cite{10} \cite{11} \cite{15} \cite{60} \cite{92}. For bidirectional non-reciprocal coupling, $X_1$ may model a feedback controller acting on a colloid at $X_0$ \cite{the model \cite{7}} with $k = a_{01}$, $t = 1$ lies on the line $a_{10} = 1$. Here and in the following plots, $a_{11} = a_{00} = -1$, $k_B$ and all other parameters are set to unity. *Center:* Information flow to $X_0$. *Right:* Thermal fluctuations of $X_0$ measured by the second moment compared to the uncoupled case ($a_{01} = 0$), $\langle X_0^2 \rangle - \langle X_0^2 \rangle_{a_{01}=0}$. The grey areas indicate unstable regions (where $\langle X_0^2 \rangle \rightarrow \infty$).

This, in turn, also implies the absence of global particle transport, thus, intrinsic nonequilibrium. Such states occur in feedback-controlled systems \cite{52}, but are also common in active systems, see, e.g., \cite{104}. The reason is that, in both cases, the “driving” occurs directly on the level of the stochastic trajectories, yielding, e.g., persistence, but it does not come in the form of a global gradient, i.e., there is no global symmetry breaking (using the language of control theory, one might say that the driving is in a “closed-loop” form \cite{21} \cite{26} \cite{52}). In particular, the driving is hidden in the coupling forces. To further investigate this, we will next reconsider the system from an energetic perspective.

## 4 Energy & Entropy

To further unravel the nature of the intrinsic non-equilibrium, we consider the energy flows. Sekimoto’s framework \cite{91} tells us that the fluctuating heat exchange between each $X_j$ and its heat bath along a stochastic trajectory of length $dt$ is given by

$$\delta q_j(t) = (\gamma_j \dot{X}_j(t) - \xi_j(t)) \circ dX_j(t),$$  \hspace{1cm} (15)$$

yielding for the entire super-system a total dissipation of $\delta q = \sum_{j=0}^n \delta q_j$. Here, $\circ$ indicates Stratonovich calculus. Using the LE \cite{1}, we can write the ensemble average of the heat rate, denoting $\dot{Q} = \langle \delta q_j / dt \rangle$, $\dot{Q}_j = \langle \delta q_j / dt \rangle$, as

$$\dot{Q}(t) = \sum_{j=0}^n \dot{Q}_j(t) = \sum_{j=0}^n (\gamma_j \dot{X}_j(t) - \xi_j(t)) \circ \dot{X}_j(t) \sum_{j=0}^n a_{jj} \langle X_j(t) \dot{X}_j(t) \rangle + \sum_{i \neq j} a_{ij} \langle X_i(t) \dot{X}_j(t) \rangle$$ \hspace{1cm} (16)$$

[recall that $f_0 = 0$]. Now we utilize the steady-state identity $\langle X_k \dot{X}_l \rangle = -\langle X_l \dot{X}_k \rangle \forall k,l$, which readily follows from the fact that the correlations $\langle X_k(t) X_l(t) \rangle$ are time-independent and thus $\frac{d}{dt} \langle X_k(t) X_l(t) \rangle = \langle \dot{X}_k(t) X_l(t) \rangle + \langle X_k(t) \dot{X}_l(t) \rangle = 0$, and immediately obtain

$$\dot{Q} = \sum_{j=0}^{n-1} \sum_{i > j} (a_{ii} - a_{ij}) \langle X_i \dot{X}_j \rangle \geq 0.$$ \hspace{1cm} (17)$$

Accordingly, if all couplings are reciprocal, the total dissipation $\dot{Q}$ is zero, as expected. Equation (17) further reveals that, in contrast, a non-reciprocal interaction $a_{ij} = a_{ji}$ leads to a net dissipation. Let us discuss this in more depth.
First, we realize that $\dot{Q}$ is nonnegative, as follows from the connection to the total entropy production rate (EP) \[90\]

$$S_{\text{tot}} = \sum_{j=0}^{n} \dot{Q}_j / T_j + \dot{S}_{\text{sh}} \geq 0,$$  \hspace{1cm} (18)

with $S_{\text{sh}}$ being the ensemble average of the fluctuating multivariate (joint) Shannon entropy $s_{\text{sh}} = -k_B \ln[p_{n+1}(x)]$, and $\dot{S}_{\text{sh}} \equiv 0$ in steady states. Noteworthy, $[18]$ describes the actual total thermodynamic EP only when all d.o.f. have a physical interpretation. In other cases its meaning is debatable. However, in any case, the second law $S_{\text{tot}} \geq 0$ holds [as formally shown below in (29)], where $S_{\text{tot}} = 0$ in thermal equilibrium.

Second, according to the first law of thermodynamics, $\delta q = \delta w + du$, the net dissipation associated with each non-reciprocal interaction $[17]$, must result from work ($\delta w$) applied to the system, while the internal energy is conserved in steady states, $(du) = 0$. In other words, the total dissipation is due to a positive energy input at rate $\dot{W} = \dot{Q} \geq 0$ \[17\] into the system. Where does this energy come from? Because fundamental physical interactions are generally reciprocal, in order to realize a non-reciprocal coupling some (external) mechanism is necessary, which is here not explicitly modeled but “hidden” in the equations within the non-reciprocity. The positive energy input $\dot{W} = \dot{Q} \geq 0$ \[17\] gives the minimal energy needed (by this mechanism) to sustain the non-reciprocal coupling. We also note that a positive energy input on the level of the fluctuating trajectories is considered a defining property of active systems \[16\] \[23\] \[70\] \[77\] \[78\]. As we see here, it can be introduced in the form of a non-reciprocal interaction.

Next, we take a closer look at the individual heat flow between $X_0$ and its bath. We focus on $\dot{Q}_0$, as it is a characteristic thermodynamic quantity and it is independent of whether all d.o.f. have a clear physical interpretation, or not, and independent of the employed description (Markovian or non-Markovian). To calculate the steady-state ensemble average, we again utilize $\langle X_j X_j \rangle = 0$ and $\langle X_i \xi_j \rangle = 0$ for all $j \neq l$, and therewith find from \[15\] directly

$$\dot{Q}_0 = \sum_{j>0}^{n} a_{0j} \langle X_j \tilde{X}_0 \rangle = \sum_{i=0}^{n} \sum_{j>0}^{n} a_{0j} a_{0i} \langle X_j X_i \rangle = \dot{W}_0.$$

(19)

Likewise, one can calculate the heat flows of the other d.o.f. $\dot{Q}_l = \sum_{i=0}^{n} \sum_{j \neq l}^{n} a_{li} a_{0j} \langle X_j X_i \rangle$. It should be noted that by writing down this expression for the dissipation of $X_{j>0}$ and the total EP \[18\], we implicitly assume that all $X_j$ are even under time-reversal, that means, position-like variables. In contrast, odd variables would not contribute to the total EP, see \[92\].

Together with the (cross-)correlations $\langle X_i X_j \rangle$ that are derived in Appendix C, Eqs. \[18\] \[19\] represent analytical expressions for heat flow and entropy production for any $n$. Let us first consider the result for the case
n = 1 (which was also discussed in [14]), where the expression significantly simplifies and reads
\[ \dot{S}_{\text{tot}} = k_B \frac{(a_{10}T_0 - a_{01}T_1)^2}{T_0T_1(-a_{00}/\gamma_0 - a_{11}/\gamma_1)} \geq 0, \quad (20) \]
\[ \dot{Q}_0 = \frac{a_{01}(a_{10}T_0 - a_{01}T_1)}{(a_{00} + a_{11}/\gamma_1)} \cdot (21) \]

From (20) one immediately sees that the EP vanishes if, and only if, DB and the FDR (13) are fulfilled (as expected). Thus, all three notions of equilibrium are consistent.

Let us now consider the heat flow (21) for different coupling schemes, shown in Fig. 3 for \( T_0 = T_1, a_{00} = a_{11} \) and \( n = 1 \). Note that these isothermal conditions allow to better investigate the effect of non-reciprocity and, at the same time, are most realistic in regard to experimental realizations. For example, this could represent a system of two colloidal particles trapped in a harmonic potential of stiffness \( a_{00} = a_{11} \) and coupled with each other with the help of an external setup similar to [38, 46]. When the system is reciprocally coupled (along the dotted diagonal), it equilibrates and the heat flow nullifies. Then, the EP (20) is zero as well. The heat flow also vanishes in the trivial case \( a_{01} = 0 \), i.e., when \( X_0 \) does not “see” \( X_1 \) (dashed horizontal line), as is the case when \( X_1 \) corresponds to a sensor [28]. As one would expect, being measured does not bring \( X_0 \) out of equilibrium. If the unidirectional coupling is reversed (\( a_{10} = 0 \)), the heat flow is strictly nonnegative (dashed vertical line). This suits to the idea that \( X_0 \) is an active swimmer: the swimmer eventually heats up the surrounding fluid, but never has a net cooling effect. Remarkably, for cases with bidirectional non-reciprocal coupling, we observe that, \( \dot{Q}_0 \) can also become negative. Then, heat is constantly flowing out of the bath (on average), although the other subsystem is not colder, which would be a trivial case of heat extraction. We further remind the reader that a steady-state heat flow induced by a non-conservative external force acting on a passive, Markovian system is strictly nonnegative, as dictated by the second law, \( \dot{Q}_0/T_0 = \dot{S}_{\text{tot}} \geq 0 \). Here we find that, in contrast, the non-conservative interaction force \( a_{01}X_1 \) can induce a reversed heat flow \( \dot{Q}_0 < 0 \) (enabled by the usage of extracted information, see Sec. 3). The negative sign of \( \dot{Q}_0 \) implies a steady extraction of energy from the bath, which is converted into work \( W_0 \), i.e., a (potentially useful) form of energy. It is, of course, well-known that such an energy extraction can be realized by “Maxwell-demon”-type of devices [41, 64]. Here we see that the non-reciprocally coupled d.o.f. represents a minimal, time-continuous version of such a device, where the control action is automatically encoded in the non-reciprocity of the coupling. Note that the total EP, which is proportional to the sum over both heat flows, \( \dot{Q}_0 + \dot{Q}_1 \), is strictly positive also in this case, i.e., the isothermal “Maxwell demon” \( X_1 \) must heat up its own environment.

4.1 Conditions for reversed heat flow

To find out under which conditions the reversed heat flow occurs for \( n = 1 \) and 2 [with the parameter setting from [38, 47]], we vary the two important parameters, the feedback gain \( k \) and delay time \( \tau \). Figure 4 reveals that the heat flow \( \dot{Q}_0 \) is qualitatively and quantitatively similar for \( n = 1 \) and 2. The similarity of the two cases is indeed striking, given the differences between both systems. In particular, we here compare systems with monotonic memory kernel \( K(t = t') \), vs. non-monotonic \( K(t = t') \) which nullifies at \( t = t' \) (for \( n = 2 \)). At \( n = 1 \), the feedback force \( k \int K(t - t')X_0(t')dt' \) mostly depends on the instantaneous position \( X_0(t) \), while at \( n = 2 \) it is independent of the latter, and mostly depends on \( t - \tau \). Further, in regard to the Markovian super-system, there is a direct coupling from \( X_1 \) to \( X_0 \) in the case \( n = 1 \), while this coupling is only indirect (via a third sub-system) in the case \( n = 2 \). Nevertheless, the (blue) area of reversed heat flow lies in the same region of the \((\tau, k)\)-plane and is of similar size. Also, in both cases, it only occurs if \( k > 0 \).
In the context of control theory, it is common to characterize feedback loops as positive or negative feedback, according to the question whether the force points towards, or away from the delayed state (or, more generally, from the desired state for non-delayed feedback schemes). Thus, in both models, only negative feedback may induce a reversed heat flow.

Besides the trivial case, \( k = 0 \), there is, for both \( n \), a second line in Fig. 4 along which the heat flow vanishes. For \( n = 1 \), this line corresponds to parameters where DB and the FDR are fulfilled (dashed line), i.e., the system is in equilibrium. For \( n = 2 \), DB and the FDR are generally broken for all \((\tau, k)\). This second line hence reveals another interesting property of non-Markovian systems: They may be out of equilibrium without exhibiting showing dissipation (zero heat flow), in sharp contrast to reciprocal systems. In our system, such a state is found for \( n > 1 \) and non-reciprocal coupling only. From the viewpoint of the non-Markovian process \( X_0 \) this is indeed a bit puzzling. If \( X_0 \) is in a true nonequilibrium steady-state, there must be an associated entropy production. However, the zero heat flow indicates zero medium entropy production. Thus, where does the entropy go? To answer this question, we shall consider the entropy balance of the individual subsystem \( X_0 \), as we will do the next section.

We note that a NESS with zero heat flow and regimes of reversed heat flow may also occur in systems with distribution memory, which are moreover, nonlinear, as we have reported in [55].

5 Information

Now we turn to an information-theoretical investigation of non-reciprocal coupling. The motivation of this is two-fold. First, it will help us better understand the previous observations, for example: Why is heat extraction as we will do the next section.

In steady states, the first term naturally vanishes. To calculate the ensemble average of the sum, we use \( \langle \delta_x \rho_{n+1} \rangle = \int J_i A (x_j, t) dx_j \) [79, 90], with the probability currents \( J_i \). We consider natural boundary conditions \( \lim_{x \to \pm \infty} \rho (x) = 0 \), and denote improper integrals \( \lim_{x \to \infty} \int_{-x}^{x} \) simply as \( \int \). With these tools, we find the ensemble average of each summand of (23)

\[
\int \frac{\langle \delta_x \rho_{n+1} \rangle}{\rho_{n+1}} J_i dx_i = - \int \frac{\langle \delta_x \rho_{n+1} \rangle}{\rho_{n+1}} J_i dx_i = - \int \frac{[\ln(\rho_{n+1})J_i]_{-\infty}^{\infty}}{\rho_{n+1}} dx_i + \int \ln(\rho_{n+1}) \delta_x J_i dx_i
\]

where we have introduced the information flow \( \dot{I}_{i\to j} \) to \( X_i \). We stress that the involved information flow is from all other d.o.f. \( \{ X_j \neq i \} \) to \( X_i \). (Even if not directly coupled with each other, two d.o.f. can exchange information through a third d.o.f.) Furthermore, we recall that thermal equilibrium is characterized by vanishing probability current. Thus, from the definition (24), one can see that in equilibrium all individual information flows are necessarily zero.
To further proceed, we utilize the closed, multivariate Fokker-Planck equation [8], and find
\[ ... = -\dot{I}_{\rightarrow j} + \int \ln \rho_1(x_j) \left\{ -\partial_t \rho_{n+1} + \sum_{i \neq j} \partial x_i J_i \right\} \, dx - \dot{I}_{\rightarrow j} + \frac{\dot{S}_{sh}^j}{k_B} - \sum_{i \neq j} \int \ln \rho_1(x_j) \frac{[J_i]_j^\infty}{-\frac{\partial x_i}{\partial x_j}} \, dx_{j}, \] (25)
where we have introduced the change of the Shannon entropy of the marginal pdf \( \rho_1(x_j) \)
\[ \dot{S}_{sh}^j = -k_B \int \ln \rho_1(x_j) \partial_t \rho_1(x_j) \, dx_j. \] (26)
In sum, we have shown that
\[ \dot{S}_{sh} = -k_B \sum_{j=0}^{n} \dot{I}_{\rightarrow j} + \dot{S}_{sh}^j. \] (27)
The involved information flows are closely connected to the mutual information between all d.o.f., [22] via \( \dot{I} = \sum_{j=0}^{n} \dot{I}_{\rightarrow j} \), as we show in Appendix E. Since \( \dot{I} = 0 \), the information flows among all d.o.f. \( X_j \) in total cancel each other out (thus, from an information-theoretical point of view, the super-system as a whole is “closed”). However, they constitute an important contribution to the entropy balance, when an individual subsystem is considered.

To see this, we reconsider the summands of (23), and rewrite them using the FPE (8) as
\[ \frac{-\left( \partial x_j \rho_{n+1} \right)}{k_B \rho_{n+1}} \dot{X}_j = \gamma_j J_j(x_j, t) \dot{X}_j - \frac{\dot{Q}_j}{T_j} \]
\[ = \gamma_j J_j(x_j, t) \dot{X}_j - \frac{\dot{Q}_j}{T_j} \]
\[ := k_B S_{tot}^j. \] (28)
Combining (23, 24, 25), we obtain the entropy balance of each subsystem
\[ \dot{S}_{tot}^j = \dot{S}_{sh}^j - k_B \dot{I}_{\rightarrow j} + \dot{Q}_j \]
\[ \dot{S}_{tot}^{\text{in}} \sum_{j=0}^{n} \frac{\dot{Q}_j}{T_j} + \dot{S}_{sh} \sum_{j=0}^{n} \dot{S}_{tot}^j \geq 0. \] (29)
(30)
With (30), we have recovered the mean total EP (18).

Further, Eq. (29) may be seen as a generalized second law for each d.o.f., giving the entropy balance of an individual sub-system. In steady states, where \( \dot{S}_{sh} = 0 \), it implies
\[ \dot{Q}_j \geq k_B T_j \dot{I}_{\rightarrow j}, \] (31)
consistent with [2, 33].

Equation (31) states that a reversed steady heat flow, \( \dot{Q}_h < 0 \), is only possible, if \( \dot{I}_{\rightarrow 0} < 0 \), i.e., information is flowing from the \( X_0 \) to the rest of the system. The more information about \( X_0 \) is gathered by the other \( X_{j>0} \) (the controller d.o.f.), the more heat can be extracted from the bath. Figure 5 shows (for \( n = 1 \)) the information and heat flows, as well as the total EP, which are all connected via (29, 30). It also illustrates that in the reciprocal case, there is no “entropic cost” (zero EP), but, at the same time, no net information extraction is achieved, nor is a heat flow induced.

Due to the linearity of the model, we can calculate the information flows analytically. The steady-state pdfs are multivariate Gaussians with zero mean and with the covariance matrix \( \langle \Sigma \rangle_{i,j} = \langle X_i X_j \rangle \), which are described in Appendix E. To derive explicit expressions for the steady-state information flows, it turns out to be most convenient to start with (24). Using the general property of normal distributions, \( \partial x_j \rho_{n+1}(\xi) = [\text{and recalling } \dot{S}_{sh}^j = 0] \), we find
\[ \dot{I}_{\rightarrow j} = \left( \frac{\partial x_j \rho_{n+1}}{\rho_{n+1}} \dot{X}_j \right) = -\langle (\Sigma^{-1} X_j) \dot{X}_j \rangle. \] (32)
Inserting the Langevin equations (1), utilizing \( 2 \langle X_i \dot{X}_i \rangle = d(X_i^2)/dt = 0 \) and \( \langle X_i \xi_j \rangle = 0 \) for \( j \neq l \), we obtain the general formula
\[ \dot{I}_{\rightarrow j} = -\sum_{l=0}^{n} \gamma_j \langle \xi_l \dot{X}_j \rangle - \sum_{i=0}^{n} \sum_{l \neq j} \frac{\alpha_{ji}}{\gamma_j} \langle \Sigma^{-1} \rangle_{ij} \langle \xi_l \rangle_{li} = -\frac{\alpha_{ij}}{\gamma_j} + \frac{1}{\gamma_j} \langle \Sigma^{-1} \rangle_{jj} \langle \xi_j \rangle_{jj}. \] (33)
Equation (33) represents in combination with (60), an analytic expression for the steady-state information flow to any sub-system in (super-)systems of arbitrary sizes.
5.1 A single non-reciprocal interaction, n=1

We are now in the position to clarify the information-thermodynamic implications of non-reciprocal coupling. First we start with n = 1, where we find from (33)

\[
\dot{I}_{\rightarrow 0} = -\frac{a_{00}}{\gamma_0} - \frac{(X_0^2)(a_{00}(X_0^2) + a_{01}(X_0X_1))}{\gamma_0((X_0X_1)^2 - (X_0^2)(X_1^2))} - \frac{[a_{01}T_1 - a_{10}T_0][a_{00}a_{01}/T_0 + a_{11}a_{10}/\gamma(T_1\gamma_1)]}{T_0(\alpha_{00}\gamma_1 + a_{11}\gamma_0)^2 + \alpha_{01}^2/\gamma_1 - 2T_0a_{01}a_{10} + a_{00}^2/\gamma_1/\gamma_0}. \tag{34}
\]

Equation (34) explicitly shows that the information flow vanishes in thermal equilibrium when DB holds, \(T_0a_{01} = T_0a_{12}\), as already follows from its definition (24). Furthermore, it trivially vanishes if the cross-correlations nullify. If \(a_{01} \neq 0\), the information flow can be expressed as

\[
\dot{I}_{\rightarrow 0} = \frac{(X_0^2)}{\gamma_0} \frac{Q_0}{a_{01}T_0} = \frac{-\langle X_1X_0 \rangle}{\langle (X_1^2)(X_0^2) \rangle} a_{01} T_0, \tag{35}
\]

revealing that the information flow out of and into \(X_0\) necessarily nullifies, if the heat flow is zero (if \(a_{01} \neq 0\)).

The information flow is shown in Fig. 5 together with the heat flow. Along the unidirectional coupling axis \(a_{01} = 0\), there is net information flow from \(X_0\) to \(X_1\), but no net work applied to \(X_0\) (\(Q_0 = W_0 = 0\)). Thus, it is indeed sensible to consider \(X_1\) as “sensor” and the coupling a “sensing interaction”. If the unidirectional coupling is reversed (\(a_{01} = 0\)), the heat flow is always positive, \(Q_0 > 0\), i.e., an active swimmer eventually heats up its surrounding. In this case, there is as well a nonzero information flow, which is directed from the source of propulsion (e.g., the flagella) to the particle. This is also reasonable, as the propulsion force “carries” information: one could, on average, reconstruct the position of the flagella by only monitoring \(X_0\) whether the “sensing”, or the “active force” is stronger.

For non-reciprocal, bidirectional coupling, the information flow can be positive or negative, depending on whether the “sensing”, or the “active force” is stronger. It seems intuitive to consider \(X_0\) a feedback-controlled system, only if the net information flow out of \(X_0\) is positive, i.e., the controller “knows” more about \(X_0\) than vice versa. According to this definition, the control regime is given if \(|a_{10}| > |a_{01}|\) (blue regions in the middle panel of Fig. 5). This is exactly the regime where we have detected the reversed heat flow, i.e., here the controller may extract energy from a single heat bath (under isothermal conditions). Note that this observation is consistent with the generalized second law (31) which does not predict, but allow for a reversed heat flow in this very regime only.

Interestingly, we find that another intriguing phenomenon may occur (only) when the information flow is negative, namely, the suppression of thermal fluctuations. The latter can be measured by a reduced second moment \(\langle X_0^2 \rangle < \langle X_0^2 \rangle_{a_{01}=0}\), which we have displayed in Fig. 5 (right panel). In the blue areas, the second moment is reduced, thus, the feedback has the same effect as stiffening the trap. This resembles the situation in a recent experiment involving colloids in an optical trap 100, where time-delayed feedback was used to effectively stiffen a trap. Thermal fluctuation suppression can further be viewed as “isothermal compression” of a single-molecule gas, which represents, for example, an important step in the cycle of a (colloidal) heat engine 8,63. It also implies noise-reduction, which is desired in various experimental setups, and indeed one of the main applications of feedback control 13,96,99. Interestingly, by only varying \(a_{01}\) (which does not explicitly appear in the equation for \(X_0\), one can vary between fluctuation enhancement (isothermal expansion), and fluctuation suppression (isothermal compression). The suppression of thermal fluctuations is limited to the area where one direction of the coupling is attractive (\(a_{ij} < 0\)) while the reverse direction is repulsive (\(a_{ij} > 0\)). We find it quite remarkable that whenever \(\dot{I}_{\rightarrow 0} < 0\), such that \(X_1\) can be viewed as a controller, it either yields a suppression of the fluctuations of \(X_0\) (reduction of Shannon entropy), or a heat flow from the bath to \(X_0\) (reduction of medium entropy).

Lastly, we detect a further counter-intuitive property appearing exclusively in non-reciprocal super-systems: there are nonequilibrium steady states, where all information flows nullify (note that \(I_{\rightarrow 1} = -I_{\rightarrow 0}\) for \(n = 1\)). Thus, the subsystems may be driven out of equilibrium just due to their interaction (as signaled by finite dissipation), but without exchanging any information with each other.

5.2 Two non-reciprocal interactions, n=2

For higher \(n\), the explicit expressions for the information flow are quite cumbersome. For example, for \(n = 2\),

\[
\dot{I}_{\rightarrow 0} = -\frac{a_{00}}{\gamma_0} + \frac{\gamma_0^{-1}(a_{00}(X_0^2) + a_{01}(X_0X_1) + a_{02}(X_0X_2)(\langle X_1X_2 \rangle^2 - \langle X_1^2 \rangle^2) + \langle X_0^2 \rangle^2 + \langle X_1^2 \rangle^2 + \langle X_2^2 \rangle^2}{\langle X_0^2 \rangle^2 + \langle X_1^2 \rangle^2 + \langle X_2^2 \rangle^2 - 2\gamma_0\langle X_0X_2 \rangle^2 - 2\gamma_0\langle X_0X_2 \rangle - \langle X_0^2 \rangle + \langle X_1^2 \rangle + \langle X_2^2 \rangle}. \tag{36}
\]

Again, Eq. (36) reflects that the existence of a nonzero information flow necessarily implies that the d.o.f. are cross-correlated among each other. However, different from the case \(n = 1\), there is no proportionality between heat and information flow. In contrast, we find that for \(n > 1\), the relationship between those quantities becomes...
more complicated. To better understand their relationship, let us consider the cases \( n = 1, 2 \) again with the parameter setting from [29], shown in Fig. 3. Remarkably, despite the different nature of the super-system and the different type of memory, the information flow maps look almost identical for \( n = 1 \) and 2. This indicates that the information flow is almost exclusively affected by the direct coupling (here from \( X_0 \) to \( X_1 \)), which is, in principle, the same in both cases [given by the force \(- (1/\tau) X_1\)]. Thus, different from the energy flows, the information exchange is not affected by the additional indirect coupling though a third d.o.f. in the case \( n = 2 \). Furthermore, we again find that the areas of reversed heat flow [blue region in Fig. 4(a)] appear in the control regime of \( I_{\to 0} < 0 \) [blue region in Fig. 4(b)]. We note that, as in the case \( n = 1 \), the regime of thermal fluctuation suppression (not shown here) is limited to the area of negative information flow, i.e., to the control regime.

Apart from these similarities, we observe a phenomenon which only occurs for \( n > 1 \) and non-reciprocal coupling, that is, the existence of NESS where \( X_0 \) is out of equilibrium with broken FDR and \( I_{\to 0} < 0 \), but \( Q_0 = 0 \). Considering the entropy balance [29], the entropy produced in \( X_0 \) due to the non-reciprocal coupling force, is transported only in the form of information. This state corresponds to the aforementioned non-Markovian NESS with zero dissipation (see Sec. 4.1).

6 Mapping non-reciprocity onto temperature gradients

In the course of this paper, we have demonstrated that non-reciprocal coupling introduces “activity”, or more generally, intrinsic nonequilibrium. In contrast, there are several other recent publications which discuss (hidden) temperature gradients between reciprocally coupled stochastic d.o.f. as possible mechanisms that fuel active motion, see, e.g., [33, 71, 82]. In this last section we show that, in some cases, non-reciprocal coupled systems can indeed be mapped onto a reciprocally coupled system with an internal temperature gradient.

Consider the non-reciprocal system with \( n = 1 \) and \( a_{01} a_{10} \neq 0 \)

\[
\begin{align*}
\gamma_0 \dot{X}_0 &= a_{00} X_0 + a_{01} X_1 + \xi_0 \\
\gamma_1 \dot{X}_1 &= a_{10} X_0 + a_{11} X_1 + \xi_1.
\end{align*}
\] (37)

We now introduce new variables \( \tilde{X}_0 = \sqrt{|a_{10}|} X_0, \tilde{X}_1 = \sqrt{|a_{01}|} X_1 \), and \( \tilde{T}_0 = |a_{10}| T_0, \tilde{T}_1 = |a_{01}| T_1 \). We note that if the \( X_j \) are position-like d.o.f., their scaling should indeed be accompanied by scaling of the temperatures due to the connection between temperatures and the time-derivative of the positions. In this way, we find

\[
\begin{align*}
\gamma_0 \tilde{X}_0 &= a_{00} \tilde{X}_0 + \text{sgn}(a_{01}) \sqrt{|a_{10}| a_{01}} \tilde{X}_1 + \tilde{\xi}_0 \\
\gamma_1 \tilde{X}_1 &= \text{sgn}(a_{10}) \sqrt{|a_{01}| a_{10}} \tilde{X}_0 + a_{11} \tilde{X}_1 + \tilde{\xi}_1,
\end{align*}
\] (38)

with \( \tilde{\xi}_j(t) \sim \tilde{\xi}_j(t) \). If \( a_{01} a_{10} > 0 \), this system has reciprocal coupling. Further, even if \( T_0 = T_1 \), it involves a temperature gradient. The symmetric system [35] could, for example, model the angles of two vases in different heat baths, coupled by a torsion spring [91]. As well-known, such a reciprocally coupled system equilibrates if, and only if, \( T_1 = T_0 \Leftrightarrow |a_{01}| T_1 = |a_{10}| T_0 \). This is identical to the equilibrium condition [13]. In Appendix D we give an example for a non-reciprocal system with \( n = 2 \) that can also be mapped onto a reciprocally coupled one, if \( a_{ij} a_{ji} > 0 \), \( \forall i, j \in \{0, 1, 2\} \). Again, this mapping yields the same equilibrium conditions as found from DB.

Now we turn to the impact of this scaling on the thermodynamic quantities. For the energy flows, we find the relations

\[
\begin{align*}
\delta \dot{w}_0 &= \text{sgn}(a_{01}) \sqrt{|a_{10}| a_{01}} \tilde{X}_1 \circ \text{d} \tilde{X}_0 = a_{01} |a_{10}| X_1 \circ \text{d} X_0 = |a_{10}| \delta w_0, \\
\delta \dot{q}_0 &= (\gamma_0 \tilde{X}_0 - \xi_0) \circ \text{d} \tilde{X}_0 = |a_{10}| \delta q_0.
\end{align*}
\] (39)

Thus, the energy flows to and out of \( X_0 \) are both scaled with \( |a_{10}| \). Likewise, \( \delta \dot{w}_1 = |a_{01}| \delta w_1 \) and \( \delta \dot{q}_1 = |a_{01}| \delta q_1 \). This further means

\[
\Delta s_{\text{tot}} = \Delta s_{\text{sh}} + \frac{\delta \dot{q}_0}{|a_{10}| T_0} + \frac{\delta \dot{q}_1}{|a_{01}| T_1} = \Delta s_{\text{sh}} + \frac{\delta \dot{q}_0}{T_0} + \frac{\delta \dot{q}_1}{T_1} = \Delta s_{\text{tot}},
\] (40)

i.e., the EP in the scaled model is identical to the EP in the original model, while the energy flows in general differ.

We conclude that the two “driving mechanisms”, that is, non-reciprocal coupling (with \( a_{ij} a_{ji} > 0 \)), or a temperature gradient, can formally not be distinguished on the level of EP. This mapping also builds a bridge to active matter models where temperature gradients between reciprocally coupled stochastic d.o.f. fuel the active motion [38, 71, 82]. It should be emphasized, however, that a scaling as employed here cannot be found if \( a_{ij} a_{ji} \leq 0 \) (which, interestingly, includes unidirectional coupling, e.g., the AOUP model). This suggests that non-reciprocal coupling is the more general way to introduce intrinsic non-equilibrium.
7 Conclusion

This paper addresses the thermodynamic implications of non-reciprocal coupling between stochastic d.o.f., which is a form of non-conservative interaction appearing in various artificial or natural complex systems across the fields. The most important result is that the occurrence of a non-reciprocal coupling alone implies nonequilibrium as indicated by a broken detailed balance and fluctuation-dissipation relation, and is automatically associated with a net energy and information flow. We have shown that this is generally true except for some special cases, which are limited to systems with $a_{ij} a_{ji} > 0$ and $T_i / T_i = a_{ij} / a_{ji}$. Then, the non-reciprocal system with internal temperature gradient can be mapped onto a reciprocal one at isothermal conditions, giving a formal explanation for the observed exceptions. Another key result is that a non-reciprocal coupling between isothermal d.o.f. may induce, for one of the two d.o.f., a negative heat flow (while the total dissipation is always positive), meaning that energy is extracted from the bath. This shows a crucial difference between the thermodynamic implications of a non-conservative (non-reciprocal) interaction vs. a non-conservative external force, which could only induce a positive heat flow (as dictated by the second law).

We have considered active matter or feedback-controlled systems as different representatives of non-reciprocal systems. While a single unidirectional coupling makes $X_1$ a “propulsion mechanism” and $X_0$ an “active swimmer”, a single non-reciprocal bidirectional coupling may make $X_1$ a “feedback controller” that operates on $X_0$. Moreover, when the controller knows more about the controlled system than vice versa (indicated by an information flow to the controller), some major goals of feedback control can be achieved, including thermal fluctuation suppression, and energy extraction of the heat bath (i.e., a reversed heat flow) making $X_1$ a minimal version of a continuously operating “Maxwell demon”. The latter can only be achieved if (i) the information flow is directed from the system to the controller and (ii) the controller applies negative feedback, i.e., a feedback force pointing away from the delayed position of $X_0$.

Whereas one non-reciprocal coupling ($n = 1$) only induces exponentially decaying memory in the corresponding non-Markovian equation for the single d.o.f. (e.g., $X_0$), the interplay of multiple linear non-reciprocal interactions ($n > 1$) allows to generate non-monotonic memory, which, in turn, is typical for time-delayed feedback control. From a thermodynamic point of view, the cases $n = 1$ and $n = 2$ share the main characteristics. However, there is indeed a crucial difference, that is, the heat and information flows are not proportional to each other, if $n > 1$. Thus, one can find for $n = 2$ some interesting nonequilibrium steady states which only occur for $n > 1$ and non-reciprocal coupling. On the one hand, mutually coupled systems can be driven out of equilibrium due to their interaction, without at the same time exchanging any information. On the other hand, for a different non-reciprocal coupling topology, one can also find a state where one of these subsystems is in a NESS where it exports the entropy exclusively in the form of information without displaying a heat flow (no entropy is exported to the bath).

A major focus of recent research is the search of meaningful thermodynamic descriptions for active systems. This is indeed not the topic of the present work, and we have here merely scratched the surface of this issue. For example, it is generally not possible to access the full dissipation of a complex living system, as long as not all underlying bio-chemical processes are fully known, understood, and also observable. The last point, i.e., the observability is related to another main problem in this context, that is, the thermodynamic treatment of auxiliary, or effective variables, which lack of a clear physical interpretation, as it is the case for the variable $X_1$ in the AOU model. As we have pointed out several times throughout the paper, in such a situation the meaning of, e.g., the total EP is questionable. To account for this fact, we have discussed the different measures of (non)equilibrium on the Markovian and non-Markovian level of description. However, the detailed balance condition or the fluctuation-dissipation relation only yield a binary classification (equilibrium or not), but cannot quantify the distance from equilibrium. Finding out an appropriate way to do this is discussed, e.g., in [48]. An interesting line of research in the context of observability and auxiliary variables, is the search of “effective thermodynamic” descriptions [30, 73]. For a similar underdamped model with $n = 1$, different ways to obtain an “effective thermodynamic” description, where recently compared in [30]. A generalization towards higher $n$ (and overdamped models) represents a nontrivial but certainly worthwhile direction for future research. It would also be interesting to investigate the here observed special types of NESS, e.g., with zero dissipation but nonzero information flow, from this perspective.

Also, regarding the different measures for (non)equilibrium, our preliminary observations indicate that for $n > 1$, there are non-reciprocal systems that fulfill FDR but violate DB, i.e., are nonequilibrium models with fluctuation-dissipation relations. It might be interesting to study the corresponding information flows for these cases.

In this paper, we have analyzed the thermodynamic properties of small stochastic systems of few colloids with non-reciprocal couplings. As a next step, one could think about the implications of our findings for larger systems with numerous non-reciprocal couplings, which are, as a matter of fact, already realized in recent experiments [60]. Indeed, the non-reciprocity is found to yield intriguing clustering collective behavior. At this point, we also aim to note that in non-linear dynamics and network science, studying the effects of symmetry-broken coupling on the collective behavior is already a well-established research field [51]. For example, the
existence of chimera states, a special type of clustering, was linked to symmetry-broken coupling [74], and shown to persist in the presence of discrete delay [103] and Gamma-distributed memory [44]. Lastly, the unidirectionally coupled ring system studied here is very similar to the reservoir computers investigated in [45, 47]. A reservoir computer of this type may be experimentally realized by a laser network [50, 81], or by coupled RC circuits [39, 34]. Another link to machine learning is the similarity between the unidirectional ring and recurrent neural networks [102], used for example for reinforcement learning. In these contexts, the connection between non-reciprocal coupling and information flow discussed here might be of particular importance. Noteworthy, the architecture of the unidirectional ring considered here also resembles the architecture of a Brownian clock [5], which, in contrast, has discrete dynamics.

Acknowledgements

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 163436311 - SFB 910.

A APPENDIX: Memory kernel for up to three coupled systems

Here we derive the memory kernel for the case \( n = 1 \) and \( n = 2 \) by projecting the equations for \( X_{j>0} \) onto \( X_0 \). To this end, we solve the equations for \( X_{j\in\{1,2\}} \) in frequency space, making use of their linearity (we want to emphasize that their linearity is irrespective of the question whether the equation of \( X_0 \) is linear, thus, is result also applies to cases with nonlinear \( f_0 \)). First, we apply the Laplace transformation \( \mathcal{L}[X_j(t)](s) = \int_0^\infty X_j(t)e^{-st}dt \) to the LE \( \gamma_j \dot{X}_j = \sum_{l=0}^1 a_{jl} X_l + \xi_j \), which yields

\[
-X_j(0)\gamma_j + s\gamma_j \dot{X}_j(s) = \sum_{l=0}^1 a_{jl} \dot{X}_l(s) + \dot{\xi}_j(s). \tag{41}
\]

Since we are interested in steady-state dynamics in this paper, we can safely set \( X_j(0) \equiv 0 \) without loss of generality. We therewith obtain for all \( j \),

\[
\dot{X}_j(s) = \sum_{l\neq j} a_{jl} \dot{X}_l(s) + \frac{1}{(s\gamma_j - a_{jj})} \dot{\xi}_j(s). \tag{42}
\]

Let us first consider the case \( n = 1 \). We plug \((42)\) for \( j = 1 \) into the equation \((41)\) for \( X_0 \) and immediately find

\[
\dot{X}_0 = a_{00} \dot{X}_0 + a_{01} a_{10} (\dot{X}_1 + \dot{\xi}_1) + \frac{a_{01}}{(s\gamma_1 - a_{11})} \dot{\xi}_1 + \frac{a_{01}}{(s\gamma_1 - a_{11})} \dot{\xi}_1. \tag{43}
\]

Now we make use of the convolution theorem and the linearity of the Laplace transformation to transform back to real space, obtaining the non-Markovian process \((3)\) with a memory kernel given by the inverse Laplace transformation of \( \frac{a_{01} a_{10}}{(s\gamma_1 - a_{11})} \), as explicitly given in \([4]\). Analogously, one finds the Gaussian colored noise in \([3]\).

\[
\nu(t) = \int_0^t a_{01} \gamma_1 e^{a_{11}(t-t')/\gamma_1} \xi_1(t') dt', \tag{44}
\]

with correlation \( C_\nu(\Delta t) = \langle \nu(t)\nu(t+\Delta t) \rangle \)

\[
C_\nu(\Delta t) = a_{01}^2 \gamma_1 \int_0^t \int_0^{t+\Delta t} e^{a_{11}(t-t')\gamma_1} \xi_1(t') \xi_1(t'') dt' dt'' = \frac{k_B T_1 a_{01}^2}{a_{11}} e^{a_{11}\Delta t/\gamma_1} [1 - e^{a_{11}2t/\gamma_1}]. \tag{45}
\]

In the steady state \((t \to \infty)\), the second term vanishes (if \( a_{11} < 0 \)), yielding the correlation from \([4]\).

Next, we derive the memory kernel for the case \( n = 2 \). yields

\[
\dot{X}_1 = \frac{a_{10} \dot{X}_0}{(s\gamma_1 - a_{11})} + \frac{a_{12} a_{20} \dot{X}_0 + a_{21} \dot{X}_1}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11})} + \mathcal{O}(\dot{\xi}_1) + \mathcal{O}(\dot{\xi}_2), \tag{46}
\]

\[
\dot{X}_2 = \frac{a_{20} \dot{X}_0}{(s\gamma_2 - a_{22})} + \frac{a_{21} a_{10} \dot{X}_0 + a_{12} \dot{X}_2}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11})} + \mathcal{O}(\dot{\xi}_2) + \mathcal{O}(\dot{\xi}_2). \tag{47}
\]
Note that because \( \langle \xi_1 \xi_2 \rangle = 0 \), the terms \( \mathcal{O}(\xi_1 \xi_2) \) will not contribute in the end [see, e.g., Eq. \( 50 \)] and can thus be neglected. We can further simplify the expressions to

\[
\dot{X}_1 = \left[ \frac{a_{11}(s\gamma_2 - a_{22}) + a_{12}a_{20}}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11}) - a_{12}a_{21}} \right] \dot{X}_0 + \mathcal{O}(\xi_1),
\]

\[
\dot{X}_2 = \left[ \frac{a_{20}(s\gamma_1 - a_{11}) + a_{21}a_{10}}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11}) - a_{21}a_{12}} \right] \dot{X}_0 + \mathcal{O}(\xi_2).
\]

Substituting \( 48 \) in \( 41 \) for \( j = 0 \), one obtains

\[
\gamma_0 s \dot{X}_0 = a_{00} \dot{X}_0 + \left[ \frac{a_{01}a_{10}(s\gamma_2 - a_{22}) + a_{01}a_{12}a_{20}}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11}) - a_{12}a_{21}} + \frac{a_{02}a_{20}(s\gamma_1 - a_{11}) + a_{02}a_{21}a_{10}}{(s\gamma_2 - a_{22})(s\gamma_1 - a_{11}) - a_{21}a_{12}} \right] \dot{X}_0 + \mathcal{O}(\xi_j).
\]

Finally, transforming back to real space yields the non-Markovian process \( 3 \) with a memory kernel given by the inverse Laplace transformation of \( K(s) \).

Specifically, for the unidirectionally coupled ring system with \( n = 2 \) illustrated in Fig. 2 (a), which is described by the set of equations \( 3 \) with \( a_{jj} = -(p + \kappa) \), \( a_{jj+1} = p \), \( a_{j+1} = \kappa \), and \( \gamma_j = 1 \) for \( j \in \{0, 1, 2\} \), Eq. \( 50 \) simplifies to

\[
s \dot{X}_0 = \left[ \frac{p^3 + \kappa^3 + 2p\kappa(s + p + \kappa)}{(s + p + \kappa)^2 - p\kappa} \right] \dot{X}_0 - \kappa \dot{X}_0 - p \dot{X}_0 + \mathcal{O}(\xi_j).
\]

In real space, this memory kernel (given in square brackets) reads \( 5 \).

**B APPENDIX: Memory kernel and noise correlations**

We derive the memory kernel and colored noise in the model \( 6 \), i.e., a unidirectional ring with \( n = 2 \). Analogously to the derivation in Appendix \( A \), we first apply the Laplace transformation to the LE \( \dot{X}_j(t) = (1/\tau)(X_{j-1}(t) - X_j(t)) + \xi_j(t)/\gamma_1 \) for \( j \in \{1, 2\} \), and set \( X_j(0) \equiv 0 \), obtaining

\[
\dot{X}_j(s) = \tau^{-1} \dot{X}_{j-1}(s) + s^{-1} \xi_j(s).
\]

Iteratively substituting the solution \( 52 \) for \( j = 1 \) into \( 52 \) for \( j = 2 \), yields

\[
\dot{X}_2(s) = \frac{\tau^{-2}}{(s + \tau^{-1})^2} \dot{X}_0 + \frac{\tau^{-1}}{(s + \tau^{-1})^2} \xi_1(s) + \frac{1}{s + \tau^{-1}} \xi_2(s).
\]

Now we transform back to the real space via inverse Laplace transformation. In \( 53 \) we identify the Laplace-transform of the Gamma-distribution \( L[K_j(t)](s) = \frac{\tau^{-j}}{(s + \tau^{-1})^j} \) with the Gamma-distributed kernels \( K_j(t) = \frac{t^{j-1}}{\tau(j-1)!} e^{-t/\tau} \), and find

\[
X_2(t) = \int_0^t K_2(t - t')X_0(t') \, dt' + \nu_2(t)
\]

with the Gaussian colored noise

\[
\nu_2(t) = \int_0^t \frac{\tau}{\gamma_1} K_2(t - t') \xi_1(t') + \frac{\tau}{\gamma_1} K_1(t - t') \xi_2(t') \, dt'.
\]

Replacing \( X_2(t) \) from \( 54 \) in the Markovian LE \( \gamma \dot{X}_0(t) = a_{00}X_0(t) + kX_2(t) + \xi_0(t) \), yields the non-Markovian LE

\[
\gamma \dot{X}_0(t) = a_{00}X_0 + \int_0^t K(t - t')X_0(t') \, dt' + \xi_0(t) + \nu(t)
\]

with the memory kernel \( K(T) = kK_2(T) = \frac{\tau}{\gamma_1} T e^{-T/\tau} \) as given in \( 40 \) and the colored noise \( \nu(t) = k\nu_2(t) \).

The noise correlations \( C_\nu(\Delta t) = \langle \nu(t)\nu(t + \Delta t) \rangle \) can be calculated exactly for an arbitrary \( \Delta t > 0 \), as we will show in the following. First, we use the properties of the white noise, e.g., \( \langle \xi_1(t)\xi_2(t') \rangle = 0 \), \( \langle \xi_j(t)\xi_j(t') \rangle = 2k_B T_1 \gamma_1 \delta(t - t') \), and integrate out the delta distributions, yielding

\[
C_\nu(\Delta t) = \frac{\tau^2k^2k_B T_1}{\gamma_1} \int_0^t \sum_{j=1,2} K_j(t - t')K_j(t - t' + \Delta t) \, dt'.
\]
Plugging in, \( K_1(T) = \tau^{-1} e^{-T/\tau} \) and \( K_2(T) = \tau^{-2} T e^{-T/\tau} \), this can be further be simplified to

\[
C_v(\Delta t) = \frac{k^2 T_1}{\gamma_1} e^{-\Delta t/\tau} \int_0^T [1 + \tau^{-2}(t-t')(t-t' + \Delta t)] e^{-2(t-t')/\tau} \, dt'
\]

\[
= \frac{\tau^2 k^2 T_1}{\gamma_1} e^{-\Delta t/\tau} \int_0^{2T/\tau} [1 + (u/2) (\Delta t/\tau) + u^2/4] e^{-u} \, du.
\]

(57)

As we are interested in steady states, we now take the limit \( t \to \infty \) and then perform the integration using \( \int_0^\infty x^p e^{-x} \, dx = p! \), which readily yields the noise correlation given in (4). We note that the transient correlation could be calculated similarly by instead using the incomplete Gamma function.

### C APPENDIX: Analytical solutions

As indicated by \([19, 27, 18, 33]\), various (thermo-)dynamic quantities can be calculated on the basis of (cross-)correlations \( \langle X_i X_j \rangle \) [see (55)]. For example, the steady-state pdf \( \rho_{n+1} \) is, due to the linearity of the model, a Gaussian-distribution with zero mean and the covariance matrix \( (\Sigma)_{ij} = \langle X_i X_j \rangle \). Thus, it is fully determined by all the correlations \( \langle X_i X_j \rangle \).

Here we sketch how analytical expressions for these correlations can be obtained for arbitrary system sizes \( n \). To this end, we transform Eqs. (1) via the Fourier transformation \( \hat{X}(\omega) = \int_0^\infty X_j(t) e^{-i\omega t} \, dt \), which readily yields

\[
i\omega \gamma \hat{X}(\omega) = \alpha \hat{X}(\omega) + \hat{\xi}(\omega) \Rightarrow \hat{X}(\omega) = \left( \frac{i\omega - \alpha}{\lambda(\omega)} \right)^{-1} \hat{\xi}(\omega),
\]

(58)

with the Green’s function in Fourier-space \( \lambda(\omega) \), determined by the inverse of the topology matrix \( \alpha \). Using the well-known relationship between spatial correlations and the Green’s function from linear response theory \([27]\)

\[
C(\Delta t) = \frac{D_0}{\pi} \int_{-\infty}^{\infty} \lambda(-\omega) \lambda(-\omega) e^{-i\omega \Delta t} \, d\omega,
\]

(59)

one readily finds

\[
\langle X_j^2 \rangle = \sum_{p=0}^{n} \frac{k_B T_F \gamma_p}{\pi} \int_{-\infty}^{\infty} \hat{\lambda}_{jp}(\omega) \hat{\lambda}_{jp}(-\omega) \, d\omega,
\]

(60)

\[
\langle X_j X_i \rangle = \sum_{p=0}^{n} \frac{k_B T_F \gamma_p}{\pi} \int_{-\infty}^{\infty} \hat{\lambda}_{jp}(\omega) \hat{\lambda}_{ip}(-\omega) \, d\omega.
\]

(61)

These are analytical expressions for all correlations for arbitrary system sizes \( n \).

While this strategy in principle yields analytical expressions for various (linear) systems (which can, e.g., be numerically integrated), explicit closed-form solutions are only available for specific cases, where the inverse Fourier transformation is known (see \([25, 56]\) for some explicit results). For example, the correlations for \( n = 1 \) read \([11]\)

\[
(\Sigma) = \begin{pmatrix}
\langle X_0^2 \rangle & \langle X_0 X_1 \rangle \\
\langle X_0 X_1 \rangle & \langle X_1^2 \rangle
\end{pmatrix} = \begin{pmatrix}
-\frac{T_0 a_0 a_1}{a_0 a_1} - \frac{T_0 a_0 a_1}{a_0 a_1} & \frac{T_0 a_0 a_1}{a_0 a_1} - \frac{T_0 a_0 a_1}{a_0 a_1} \\
\frac{T_0 a_0 a_1}{a_0 a_1} - \frac{T_0 a_0 a_1}{a_0 a_1} & \frac{T_0 a_0 a_1}{a_0 a_1} - \frac{T_0 a_0 a_1}{a_0 a_1}
\end{pmatrix}.
\]

(62)

We could not find general closed-form solutions for the problem with \( n > 1 \).

The matrix inversion is indeed possible up to very large system sizes, if the coupling is sparse (e.g., for unidirectionally coupled ring systems). To evaluate the integrals, the residue theorem can be used. However, this requires finding the roots of a polynomial of order \( n + 1 \). Using computer algebra systems, this can be done reasonably fast up to about \( n = 10 \). We also note, for the case \( T_{F > 0} \), solutions up to \( n \sim 10^4 \) can be found in this way.

### D APPENDIX: Mapping onto a reciprocal super-system

In Sec. \([6]\) we discuss the mapping of an non-reciprocal coupled system onto a reciprocal system (with different temperatures) for systems with \( n = 1 \). In this Appendix, we generalize this idea to larger system sizes.
A specific type of non-reciprocal coupling topology, for which we could find a mapping, is
\[
\begin{pmatrix}
\gamma_0 \dot{X}_0 \\
\gamma_1 \dot{X}_1 \\
\gamma_2 \dot{X}_2
\end{pmatrix} = \begin{pmatrix}
a_{00} & r & v \\
p & a_{11} & v \\
p & r & a_{22}
\end{pmatrix} \begin{pmatrix}
\dot{\xi}_0 \\
\dot{\xi}_1 \\
\dot{\xi}_2
\end{pmatrix},
\]
i.e., the two outward connections of each sub-system are identical (e.g., the coupling from \(X_0\) to \(X_1\) and from \(X_0\) to \(X_2\)). Networks of type (63) can be mapped onto a reciprocally coupled system via the coordinate transformation \(\dot{\xi}_0 = \sqrt{|p|} \dot{X}_0, \dot{\xi}_1 = \sqrt{|p|} \dot{X}_1, \dot{\xi}_2 = \sqrt{|p|} \dot{X}_2, \) and \(\dot{\xi}_0 = |p| \dot{T}_0, \dot{\xi}_1 = |r| \dot{T}_1, \dot{\xi}_2 = |v| \dot{T}_2.\) The corresponding reciprocal super-system reads
\[
\begin{pmatrix}
\gamma_0 \dot{X}_0 \\
\gamma_1 \dot{X}_1 \\
\gamma_2 \dot{X}_2
\end{pmatrix} = \begin{pmatrix}
a_{00} & sgn(r)\sqrt{|p|} & sgn(v)\sqrt{|p|} \\
sgn(r)\sqrt{|p|} & a_{11} & sgn(v)\sqrt{|p|} \\
sgn(v)\sqrt{|p|} & sgn(r)\sqrt{|v|} & a_{22}
\end{pmatrix} \begin{pmatrix}
\dot{\xi}_0 \\
\dot{\xi}_1 \\
\dot{\xi}_2
\end{pmatrix},
\]
with \(\langle \tilde{\xi}_i(t)\tilde{\xi}_j(t') \rangle = 2k_B T \gamma_{ij} \delta(t-t')\). As in the case \(n = 1\), we cannot find such a mapping for general cases, but only under certain conditions, specifically: \(a_{ij}a_{ji} > 0, \forall i,j \in \{0,1,2\}\). As in the case \(n = 1\), we use the following argument: a reciprocally (i.e., “mechanical”) system equilibrates in the absence of temperature gradients, i.e., \(\dot{T}_0 = \dot{T}_1 = \dot{T}_2\), which in the original coordinates gives the same condition (10) as we found from DB. The mapping presented in this Appendix can straightforwardly be generalized to arbitrary \(n \in \mathbb{N}\).

**E APPENDIX: Mutual information**

Here we discuss the relation between the information flow considered in Sec. 5 and the mutual information (22) in steady states. We start with considering the total derivative of \(I\) from Eq. (22), that is,
\[
\dot{I} = \int \frac{\partial I}{\partial \rho_{n+1}(x)} \ln \frac{\rho_{n+1}(x)}{\rho_0(x_0)\cdots \rho_n(x_n)} \, dx + \int \frac{\partial I}{\partial \rho_{n+1}(x)} \left\{ - \frac{\partial \rho_{n+1}(x)}{\rho_{n+1}(x)} - \frac{\partial [\rho_0(x_0)\rho_1(x_1)\cdots \rho_n(x_n)]}{\rho_0(x_0)\rho_1(x_1)\cdots \rho_n(x_n)} \right\} \, dx.
\]
We substitute (*) by utilizing the multivariate FPE (8) \(\partial_t \rho_{n+1} = -\sum_{j=0}^n \partial_{x_j} J_j\), and find
\[
\dot{I} = \sum_{j=0}^n \int \partial_{x_j} J_j(x) \ln \frac{\rho_0(x_0)\cdots \rho_n(x_n)}{\rho_{n+1}(x)} \, dx - \int \frac{\partial \rho_{n+1}(x)}{\rho_{n+1}(x)} \, dx - \int \frac{\partial [\rho_0(x_0)\rho_1(x_1)\cdots \rho_n(x_n)]}{\rho_0(x_0)\rho_1(x_1)\cdots \rho_n(x_n)} \, dx.
\]
Let us now consider the individual summands. By application of basic properties of the logarithm and the natural boundary conditions, we find
\[
\int \partial_{x_j} J_j(x) \ln \frac{\rho_0(x_0)\rho_1(x_1)\cdots \rho_n(x_n)}{\rho_{n+1}(x)} \, dx = \int \int \ln \frac{\rho_0(x_0)\rho_1(x_{j\neq j})}{\rho_{n+1}(x)} \, dx_{j\neq j} \, dx_j - \int \int \ln \frac{\rho_0(x_0)}{\rho_{n+1}(x)} \, dx_{j\neq j} \, dx_j = \dot{I}_{\rightarrow j}
\]
Thus, the change of mutual information is given by the sum over all information flows, \(\sum_{j=0}^n \dot{I}_{\rightarrow j} = \dot{I}\). (As was shown in [2], the information flow \(\dot{I}_{\rightarrow j}\) is actually the “time-shifted mutual information” with the time shift applied to \(X_j\).)

**References**

[1] J. Agudo-Canalejo and R. Golestanian. Active Phase Separation in Mixtures of Chemically Interacting Particles. *Phys. Rev. Lett.*, 123:018101, 2019.
[2] A. E. Allahverdyan, D. Janzing, and G. Mahler. Thermodynamic efficiency of information and heat flow. *JSTAT*, 2009(09):P09011, 2009.
[3] A. Argun, A.-R. Moradi, E. Pince, G. B. Bagci, A. Imparato, and G. Volpe. Non-Boltzmann stationary distributions and nonequilibrium relations in active baths. *Phys. Rev. E*, 94(6):062150, 2016.
A. Balijepalli, J. J. Gorman, S. K. Gupta, and T. W. LeBrun. Significantly improved trapping lifetime of nanoparticles in an optical trap using feedback control. Nano Lett., 12(5):2347–2351, 2012.

A. C. Barato and U. Seifert. Cost and precision of Brownian clocks. Phys. Rev. X, 6(4):041053, 2016.

J. Bartnick, A. Kaiser, H. Löwen, and A. V. Ivlev. Emerging activity in bilayered dispersions with wake-mediated interactions. The Journal of Chemical Physics, 144:224901, 2016.

J. Bechhoefer. Feedback for physicists: A tutorial essay on control. Rev. Mod. Phys., 77:783, 2005.

V. Blickle and C. Bechinger. Realization of a micrometre-sized stochastic heat engine. Nat. Phys., 8(2):143, 2012.

L. Bonilla. Active Ornstein-Uhlenbeck particles. Phys. Rev. E, 100(2):022601, 2019.

L. Caprini, U. M. B. Marconi, A. Puglisi, and A. Vulpiani. The entropy production of Ornstein–Uhlenbeck active particles: a path integral method for correlations. J. Stat. Mech. Theor. Exp., 2019(5):053203, 2019.

M. Chaudhuri, A. V. Ivlev, S. A. Khrapak, T. H. M., and G. E. Morfill. Complex plasma—the plasma state of soft matter. Soft Matter, 7(4):1287–1298, 2011.

P.-F. Cohadon, A. Heidmann, and M. Pinard. Cooling of a mirror by radiation pressure. Phys. Rev. Lett., 83(16):3174, 1999.

A. Crisanti, A. Puglisi, and D. Villamaina. Nonequilibrium and information: The role of cross correlations. Phys. Rev. E, 65(2):061127, 2012.

L. Dabelow, S. Bo, and R. Eichhorn. Irreversibility in active matter systems: Fluctuation theorem and mutual information. Phys. Rev. X, 9(2):021009, 2019.

M. Debiossac, D. Grass, J. J. Alonso, E. Lutz, and N. Kiesel. Thermodynamics of continuous non-Markovian feedback control. ArXiv:1904.04889, 2019.

T. Doerries, S. A. M. Loos, and S. H. L. Klapp. Analytical expressions of correlation functions of non-Markovian systems beyond the case of single-exponential memory. to be submitted, 2020.

T. Doerries, S. A. M. Loos, and S. H. L. Klapp. Analytical expressions of correlation functions of non-Markovian systems beyond the case of single-exponential memory. to be submitted, 2020.

M. Durve, A. Saha, and A. Sayeed. Active particle condensation by non-reciprocal and time-delayed interactions. Eur. Phys. J. E, 41(4):49, 2018.

K. Fang, J. Luo, A. Metelmann, M. H. Matheny, F. Marquardt, A. A. Clerk, and O. Painter. Generalized non-reciprocity in an optomechanical circuit via synthetic magnetism and reservoir engineering. Nat. Phys., 13(5):465, 2017.

É. Fodor and M. C. Marchetti. The statistical physics of active matter: From self-catalytic colloids to living cells. Physica A, 504:106–120, 2018.

É. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland. How far from equilibrium is active matter? Physical review letters, 117(3):038103, 2016.

É. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland. How far from equilibrium is active matter? Phys. Rev. Lett., 117(3):038103, 2016.

T. Franosch, M. Grimm, M. Belushkin, F. M. Mor, G. Foffi, L. Forró, and S. Jeney. Resonances arising from hydrodynamic memory in Brownian motion. Nature, 478(7367):85, 2011.

D. Geiss, K. Kroy, and V. Holubec. Brownian molecules formed by delayed harmonic interactions. New J. Phys., 21:093014, 2019.

R. Gernert, S. A. M. Loos, K. Lichtner, and S. H. L. Klapp. Feedback control of colloidal transport. In E. Schöll, S. H. L. Klapp, and P. Hövel, editors, Control of Self-Organizing Nonlinear Systems, pages 375–392. Springer, 2016.

P. Hänggi and H. Thomas. Stochastic processes: Time evolution, symmetries and linear response. Phys. Rep., 88(4):207–319, 1982.

D. Hartich, A. C. Barato, and U. Seifert. Sensory capacity: An information theoretical measure of the performance of a sensor. Phys. Rev. E, 93(2):022116, 2016.

D. Helbing and P. Molnár. Social force model for pedestrian dynamics. Phys. Rev. E, 51(5):4282–4286, 1995.

T. Herpich, K. Shayanfard, and M. Esposito. Effective thermodynamics of two interacting underdamped brownian particles. Physical Review E, 101(2):022116, 2020.

H. Hinrichsen, T. Höfteld, M. Hirth, and P. Tran-Gia. Entropy production in stationary social networks. In Complex networks IV, pages 47–58. Springer, 2013.

J. M. Horowitz and M. Esposito. Thermodynamics with continuous information flow. Phys. Rev. X, 4(3):031015, 2014.

J. M. Horowitz and H. Sandberg. Second-law-like inequalities with information and their interpretations. New J. Phys., 16(12):125007, 2014.

J. M. Horowitz and S. Vaikuntanathan. Nonequilibrium detailed fluctuation theorem for repeated discrete feedback. Phys. Rev. E, 82(6):061120, 2010.
[35] S. Ito and T. Sagawa. Maxwell’s demon in biochemical signal transduction with feedback loop. Nat. Commun., 6:7498, 2015.

[36] A. V. Ivlev, J. Bartnick, M. Heinen, C.-R. Du, V. Nosenko, and H. Löwen. Statistical mechanics where Newton’s third law is broken. Phys. Rev. X, 5(1):011035, 2015.

[37] I. Karamouzas, B. Skinner, and S. J. Guy. Universal power law governing pedestrian interactions. Phys. Rev. Lett., 113(23):238701, 2014.

[38] U. Khadka, V. Holubec, H. Yang, and F. Cichos. Active Particles Bound by Information flows. Nat. Commun., 9:3864, 2018.

[39] L. B. Kish and C.-G. Granqvist. Electrical Maxwell demon and Szilard engine utilizing Johnson noise, measurement, logic and control. PloS one, 7(10):e46800, 2012.

[40] R. Kompaneets, S. Vladimirov, A. Ivlev, and G. Morfill. Reciprocal interparticle attraction in complex plasmas with cold ion flows. New J. Phys., 10(6):063018, 2008.

[41] J. V. Koski, V. F. Maisi, T. Sagawa, and J. P. Pekola. Experimental Observation of the Role of Mutual Information in the Nonequilibrium Dynamics of a Maxwell Demon. Phys. Rev. Lett., 113(3):030601, 2014.

[42] H. J. Kronzucker, M. W. Szczerba, L. M. Schulze, and D. T. Britto. Non-reciprocal interactions between k+ and na+ ions in barley (hordeum vulgare l.). Journal of Experimental Botany, 59(10):2793–2801, 2008.

[43] R. Kubo. The fluctuation-dissipation theorem. Rep. Prog. Phys, 29(1):255, 1966.

[44] Y. Kyrychko, K. Blyuss, and E. Schöll. Amplitude and phase dynamics in oscillators with distributed-delay coupling. Philos. Trans. Royal Soc. A, 371(1999):20120466, 2013.

[45] L. Larger, A. Baylón-Fuentes, R. Martinenghi, V. S. Udaltsov, Y. K. Chenmo, and M. Jacquot. High-speed photonic reservoir computing using a time-delay-based architecture: Million words per second classification. Phys. Rev. X, 7(1):011015, 2017.

[46] F. A. Lavergne, H. Wondehenne, T. Bäuerle, and C. Bechinger. Group formation and cohesion of active particles with visual perception–dependent motility. Science, 364(6435):70–74, 2019.

[47] J. Li, K. Bai, L. Liu, and Y. Yi. A deep learning based approach for analog hardware implementation of delayed feedback reservoir computing system. In 2018 19th International Symposium on Quality Electronic Design (ISQED), pages 308–313. IEEE, 2018.

[48] J. Li, J. M. Horowitz, T. R. Gingrich, and N. Fakhri. Quantifying dissipation using fluctuating currents. Nature communications, 10(1):1–9, 2019.

[49] I. I. Lisina and O. S. Vaulina. Formation of layered structures of particles with anisotropic pair interaction. Europhysics Letters, 103:55002, 2013.

[50] S. A. M. Loos. Stochastic systems with time delay. 2020.

[51] S. A. M. Loos, J. C. Claussen, E. Schöll, and A. Zakharova. Chimera patterns under the impact of noise. Phys. Rev. E, 93(1):012209, 2016.

[52] S. A. M. Loos, R. Gernert, and S. H. L. Klapp. Delay-induced transport in a rocking ratchet under feedback control. Phys. Rev. E, 89(5):052136, 2014.

[53] S. A. M. Loos, S. M. Herrmann, and S. H. L. Klapp. A markovian, nonreciprocal model for a time-delayed feedback controller. to be submitted, 2020.

[54] S. A. M. Loos and S. H. L. Klapp. Fokker-Planck equations for time-delayed systems via Markovian embedding. J. Stat. Phys., 177:95–118, 2019.

[55] S. A. M. Loos and S. H. L. Klapp. Heat flow due to time-delayed feedback. Sci. Rep., 9:2491, 2019.

[56] M. C. Marchetti, J.-F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha. Hydrodynamics of soft active matter. Rev. Mod. Phys., 85(3):1143, 2013.

[57] U. M. B. Marconi, A. Puglisi, and C. Maggi. Heat, temperature and clausius inequality in a model for active Brownian particles. Sci. Rep., 7:4696, 2017.

[58] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, J. M. Parrondo, and R. A. Rica. Brownian carnot engine. Nature physics, 12(1):67, 2016.

[59] J. C. Maxwell. Theory of heat. Dover Publications, inc., 1871.

[60] A. Metelmann and A. A. Clerk. Nonreciprocal photon transmission and amplification via reservoir engineering. Phys. Rev. X, 5(2):021025, 2015.
[66] G. Micali and R. G. Endres. Bacterial chemotaxis: information processing, thermodynamics, and behavior. Curr. Opin. Microbiol., 30:8–15, 2016.

[67] G. E. Morfill and A. V. Ivlev. Complex plasmas: An interdisciplinary research field. Rev. Mod. Phys., 81(4):1353–1404, 2009.

[68] M. Moussaid, D. Helbing, and G. Theraulaz. How simple rules determine pedestrian behavior and crowd disasters. Proceedings of the National Academy of Sciences, 108:6884–6888, 2011.

[69] T. Munakata and M. L. Rosinberg. Entropy production and fluctuation theorems for Langevin processes under continuous non-Markovian feedback control. Phys. Rev. Lett., 112(18):180601, 2014.

[70] C. Nardini, É. Fodor, E. Tjhung, F. Van Wijland, J. Tailleur, and M. E. Cates. Entropy production in field theories without time-reversal symmetry: quantifying the non-equilibrium character of active matter. Phys. Rev. X, 7(2):021007, 2017.

[71] R. R. Netz. Fluctuation-dissipation relation and stationary distribution of an exactly solvable many-particle model for active biomatter far from equilibrium. J. Chem. Phys., 148(18):185101, 2018.

[72] P. Pietzonka and U. Seifert. Entropy production of active particles and for particles in active baths. J. Phys. A: Math. and Theor., 51(1):01LT01, 2017.

[73] M. Polettini and M. Esposito. Effective thermodynamics for a marginal observer. Phys. Rev. Lett., 119(24):240601, 2017.

[74] K. Premalatha, V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan. Impact of symmetry breaking in networks of globally coupled oscillators. Phys. Rev. E, 91(5):052915, 2015.

[75] A. Pugiši and D. Villamaina. Irreversible effects of memory. EPL, 88(3):30004, 2009.

[76] K. Pyragas. Continuous control of chaos by self-controlling feedback. Phys. Lett. A, 170(6):421–428, 1992.

[77] S. Ramaswamy. The mechanics and statistics of active matter. Annu. Rev. Condens. Matter Phys., 1(1):323–345, 2010.

[78] S. Ramaswamy. Active matter. Annu. Rev. Condens. Matter Phys., 5(1):331–358, 2014.

[79] P. Reimann. Brownian motors noisy transport far from equilibrium. Phys. Rep., 361, 2002.

[80] A. Röhm, L. Jaurigue, and K. Lüdge. Reservoir Computing using Laser Networks. IEEE J. Sel. Top. Quant., 26(1):1–8, 2019.

[81] A. Röhm and K. Lüdge. Multiplexed networks: reservoir computing with virtual and real nodes. J. Phys. Commun., 2(8):085007, 2018.

[82] É. Roldán, J. Barral, P. Martin, J. M. Parrondo, and F. Jülicher. Arrow of time in Active fluctuations. ArXiv:1803.04744, 2018.

[83] M. Rosinberg, G. Tarjus, and T. Munakata. Influence of time delay on information exchanges between coupled linear stochastic systems. Phys. Rev. E, 98(3):032130, 2018.

[84] M. L. Rosinberg, T. Munakata, and G. Tarjus. Stochastic thermodynamics of Langevin systems under time-delayed feedback control: Second-law-like inequalities. Phys. Rev. E, 91:042114, 2015.

[85] M. L. Rosinberg, G. Tarjus, and T. Munakata. Stochastic thermodynamics of Langevin systems under time-delayed feedback control. II. Nonequilibrium steady-state fluctuations. Phys. Rev. E, 95(2):022123, 2017.

[86] P. E. Rouse Jr. A theory of the linear viscoelastic properties of dilute solutions of coiling polymers. The Journal of Chemical Physics, 21(7):1272–1280, 1953.

[87] S. Saha, J. Agudo-Canalejo, and R. Golestanian. Scalar active mixtures: The non-reciprocal cahn-hilliard particle model for active biomatter far from equilibrium. Philos. Trans. Royal Soc. A, 371(1999):20120472, 2013.

[88] I. Schneider. Delayed feedback control of three diffusively coupled Stuart–Landau oscillators: a case study in equivariant hopf bifurcation. Philos. Trans. Royal Soc. A, 371(1999):20120472, 2013.

[89] E. Schöll and H. G. Schuster, editors. Handbook of chaos control. John Wiley & Sons, 2008.

[90] U. Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. Rep. Prog. Phys., 75(12):126001, 2012.

[91] K. Sekimoto. Stochastic energetics, volume 799. Springer, 2010.

[92] S. Shankar and M. C. Marchetti. Hidden entropy production and work fluctuations in an ideal active gas. Phys. Rev. E, 98(2):020201(R), 2018.

[93] Z. Shen, Y.-L. Zhang, Y. Chen, F.-W. Sun, X.-B. Zou, G.-C. Guo, C.-L. Zou, and C.-H. Dong. Reconfigurable optomechanical circulator and directional amplifier. Nature communications, 7(2):021007, 2017.

[94] Z. Shen, Y.-L. Zhang, Y. Chen, F.-W. Sun, X.-B. Zou, G.-C. Guo, C.-L. Zou, and C.-H. Dong. Reconfigurable optomechanical circulator and directional amplifier. Nature communications, 7(2):021007, 2017.

[95] T. Speck. Thermodynamic approach to the self-diffusiophoresis of colloidal Janus particles. Phys. Rev. E, 99(6):060602(R), 2019.

[96] D. A. Steck, K. Jacobs, H. Mabuchi, S. Habib, and T. Bhattacharya. Feedback cooling of atomic motion in cavity QED. Phys. Rev. A, 74(1):012322, 2006.

[97] T. Van Vu and Y. Hasegawa. Uncertainty relations for time-delayed Langevin systems. Phys. Rev. E, 100:012134, 2019.
[98] O. S. Vaulina, I. I. Lisina, and E. A. Lisin. Kinetic energy in a system of particles with a nonreciprocal interaction. *Europhysics Letters*, 111:50003, 2015.

[99] A. Vinante, M. Bignotto, M. Bonaldi, M. Cerdonio, L. Conti, P. Falferi, N. Liguori, S. Longo, R. Mezzena, A. Ortolan, et al. Feedback cooling of the normal modes of a massive electromechanical system to submillikelvin temperature. *Phys. Rev. Lett.*, 101(3):033601, 2008.

[100] A. E. Wallin, H. Ojala, E. Hæggström, and R. Tuma. Stiffer optical tweezers through real-time feedback control. *Appl. Phys. Lett.*, 92(22):224104, 2008.

[101] J. B. Weiss. Coordinate invariance in stochastic dynamical systems. *Tellus A: Dynamic Meteorology and Oceanography*, 55(3):208–218, 2003.

[102] X. Xu, Y. Lu, and Y. Liang. *Time-delay recurrent neural networks for dynamic systems control*. Springer, 2004.

[103] A. Zakharova, S. A. M. Loos, J. Siebert, A. Gjurchinovski, J. C. Claussen, and E. Schöll. *Controlling chimera patterns in networks: interplay of structure, noise, and delay*. Springer, 2016.

[104] M. J. Zuckermann, C. N. Angstmann, R. Schmitt, G. A. Blab, E. H. Bromley, N. R. Forde, H. Linke, and P. M. Curmi. Motor properties from persistence: a linear molecular walker lacking spatial and temporal asymmetry. *New Journal of Physics*, 17(5):055017, 2015.

[105] R. Zwanzig. Nonlinear generalized Langevin equations. *J. Stat. Phys.*, 9(3):215–220, 1973.

[106] R. Zwanzig. *Nonequilibrium statistical mechanics*. Oxford University Press, 2001.