Disentangling perturbative and power corrections in precision tau decay analysis

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Hadronic tau decay precision data are analyzed with account of both perturbative and power corrections of high orders within QCD. It is found that contributions of high order power corrections are essential for extracting a numerical value for the strange quark mass from the data on Cabibbo suppressed tau decays. We show that with inclusion of new five-loop perturbative corrections in the analysis the convergence of perturbation theory remains acceptable only for few low order moments. We obtain $m_s(M_R) = 130 \pm 27$ MeV in agreement with previous estimates.

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Effects of strong interactions is a real stumbling block for investigating the electroweak sector of the Standard Model.\textsuperscript{4} While there remain still many principal problems of QCD as an underlying theory of strong interactions unresolved, an account of hadronic effects at the level of few percent is becoming a must for the high precision tests of the Standard Model and search for new physics.\textsuperscript{5} Although the phenomenon of confinement is still beyond a complete quantitative theoretical explanation there is a solid qualitative understanding of many features of QCD beyond perturbation theory that allows for a reliable use of perturbation theory (pQCD) in its applicability area for obtaining high precision predictions. The nonperturbative effects are accounted for through several phenomenological parameters.\textsuperscript{6} A high precision achieved for the hadronic $\tau$-lepton decays both theoretically and experimentally makes the $\tau$-system a unique testing ground of particle interactions.\textsuperscript{7} The analysis of $\tau$-decays provides useful information in a variety of ways for: (i) extracting QCD parameters with high precision – strong coupling constant, $s$-quark mass, vacuum condensates of local operators within the operator product expansion; (ii) understanding general properties of perturbation theory and its asymptotic behavior at high orders; (iii) evaluating the hadronic contributions necessary in the high precision tests of the Standard Model, e. g. the electromagnetic coupling $\alpha_{\text{EM}}(M_Z)$, muon $g - 2$, Higgs mass.

In this note we present a new analysis of the hadronic $\tau$-lepton decays with the main emphasize on the precision and reliability of the theoretical description within QCD.

The total $\tau$-lepton decay rate into tau neutrino and hadrons normalized to the corresponding pure leptonic decay $R_\tau = \Gamma(\tau \to h\nu) / \Gamma(\tau \to l\nu)$ splits into a sum of strange and non-strange channels $R_\tau = R_{\tau}^{S=0} + R_{\tau}^{S=1}$ with the experimental values $R_\tau = 3.642 \pm 0.012$ and $R_{\tau}^{S=1} = 0.1625 \pm 0.0066$\textsuperscript{14} \textsuperscript{15} \textsuperscript{16}. The numerical values for the decay rates $R_{\tau}^{S=0}$ and $R_{\tau}^{S=1}$ are plausibly understood since in the parton model approximation the non-strange and strange parts of the decay rate are associated with the $ud$ and $us$ decay channels $R_{\tau}^{S=0} \propto N_c |V_{ud}|^2$ and $R_{\tau}^{S=1} \propto N_c |V_{us}|^2$, respectively. The determination of the ratio $R_{\tau}^{S=1}/R_{\tau}^{S=0}$ is a key measurement for the determination of the strange quark mass $m_s$ in QCD. To this end, data on Cabibbo suppressed $\tau$-decays have been extracted from the experimental data. Theoretically, the moments of the differential decay rate of the $\tau$ lepton into hadrons

\begin{equation}
R_{\tau}^{kl} = \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds}
\end{equation}

have been extracted from the experimental data. Theoretically, the moments of the differential decay rate $R_{\tau}^{kl}$ are calculable in QCD perturbation theory (pQCD) within the operator product expansion for current correlators (OPE).\textsuperscript{13} The range of indices $(k,l)$ for the moments $R_{\tau}^{kl}$ in eq. (1) should be properly chosen in order to guarantee the applicability of QCD perturbation theory for evaluating the moments with a strict control over the theoretical precision.\textsuperscript{10} The analysis of the non-strange part of the decay rate results in an accurate determination of the strong interaction coupling constant $\alpha_s(\mu)$ directly in the low energy domain for $\mu \sim M_\tau$.\textsuperscript{20}\textsuperscript{21} The determination of the numerical value for the $s$-quark mass exploits the difference

\begin{equation}
\delta R_{\tau}^{kl} = R_{\tau}^{kl}_{S=0}/|V_{ud}|^2 - R_{\tau}^{kl}_{S=1}/|V_{us}|^2.
\end{equation}

This is a sensible setup as CKM is an external quantity to QCD and should be factor out. The analysis of the difference $\delta R_{\tau}^{kl}$ is more demanding theoretically and requires much care in the interpretation of perturbation theory calculations in order to retain the full control over the obtained precision.

The theoretical expression for the difference $\delta R_{\tau}^{kl}$ is usually written in the general form

\begin{equation}
\delta R_{\tau}^{kl} = 2N_c S_{EW} \sum_{n \geq 2} \delta_n^{kl}(m_s, \alpha_s)
\end{equation}

where $S_{EW} = 1.02$ is the electroweak correction.\textsuperscript{22} The quantities $\delta_n^{kl}(m_s, \alpha_s)$ with $n \geq 2$ give corrections emerging within OPE technique in QCD. They include mass corrections within pQCD for $n = 2$ and power corrections for $n > 2$. The determination of $m_s$ and $\alpha_s$ is particularly important in order to extend the precision tests of the Standard Model and search for new physical effects.\textsuperscript{23}
TABLE I: Perturbation theory moments of ud part

| (k, l) | (δP)kl^{LO} | (δP)kl^{NNLO} | (δP)kl^{NNL0} |
|-------|-------------|----------------|----------------|
| (0, 0) | 0.200 ± 0.005 | 0.200 ± 0.005 | 0.200 ± 0.005 |
| (1, 0) | 0.159 ± 0.004 | 0.167 ± 0.006 | 0.167 ± 0.007 |
| (2, 0) | 0.135 ± 0.004 | 0.145 ± 0.006 | 0.151 ± 0.008 |
| (3, 0) | 0.120 ± 0.004 | 0.137 ± 0.007 | 0.144 ± 0.009 |
| (4, 0) | 0.110 ± 0.004 | 0.125 ± 0.008 | 0.126 ± 0.010 |

The mass correction in the function \( \Pi_q(q^2) \) is very clean as they are computed through the two-point correlator of the weak charged currents \( j_\mu(x) \)

\[
i \int dx e^{ix\phi} \langle Tj_\mu(x)j_\nu(0) \rangle = \delta_{\mu\nu} \langle q^2 \rangle + g_{\mu\nu} \Pi_g(q^2) \tag{3}\]

with two scalar form factors \( \Pi_q(q^2) \) and \( \Pi_g(q^2) \). Such a decomposition of the general tensor correlator in eq. (3) into a sum of scalar form factors avoids kinematical singularities. The function \( \Pi_q(q^2) \) receives contributions from the states with the total angular momentum \( J = 1 \) only that gives an attractive opportunity to analyze the decay data with respect to the spin content of the particles in the final state [23].

The differential \( \tau \)-lepton decay rate is proportional to discontinuities \( R_{q,g}(s) \) of the functions \( \Pi_{q,g}(q^2) \) across the physical cut along the positive semi-axis \( q^2 = s > 0 \) in the complex \( q^2 \) plane

\[
\frac{dR_\tau}{ds} \propto \left( 1 - \frac{s}{M^2} \right)^2 \left( R_q(s) - \frac{2}{M^2} R_g(s) \right) \tag{4}\]

The numerical values for the moments of the ud part of the decay rate calculated in QCD are given in Table I. The experimental input for the calculation is a perturbative correction \( \delta_P \) to the total decay rate for the non-strange decays defined through

\[
R^{ud}_\tau = \frac{N_c S_{EW} |V_{ud}|^2 (1 + \delta_P + \delta_{NP} + \delta_{EW})}{s} \]

with \( \delta_P = 0.200 \pm 0.005 \). The power correction contribution \( \delta_{NP} = -0.003 \pm 0.004 \) is small and consistent with zero. The additive electroweak correction is also negligibly small, \( \delta_{EW} = 0.001 \). This fixes the numerical value for the moment \((k, l) = (0, 0)\) which is used as the input for the determination of the numerical value for the strong coupling constant \( \alpha_s(M_\tau) \) from \( \tau \)-decays. The perturbation theory results are obtained in the approximation of massless \( u, d \) quarks and are rather stable up to the next-to-next-to-leading order (NNLO), i.e. including two corrections to the leading order non-vanishing result. The inclusion of the fifth order contribution in the coupling constant (NNNLO or three perturbation theory corrections to the leading order non-vanishing result with the “estimated” numerical value for the last coefficient of perturbation theory expansion at \( n = 25 \) for details and discussion see [5]) is still reasonable, we find \( (\delta_P)^{NNNLO}_{10} = 0.170 \pm 0.007 \) and \( (\delta_P)^{NNNLO}_{30} = 0.147 \pm 0.013 \). Thus, the pattern of convergence for the perturbation theory correction to the decay rate is

\[
(\delta_P)^{NNNLO}_{10} = 0.170 = 0.159 + 0.008 + 0.000 + 0.003
\]

for the \((1, 0)\) moment and

\[
(\delta_P)^{NNNLO}_{30} = 0.147 = 0.120 + 0.017 + 0.007 + 0.003
\]

for the \((3, 0)\) moment. It is difficult to estimate the actual accuracy of the truncation of the asymptotic series in the coupling constant.

The leading power corrections within the operator product expansion for the correlator from eq. (3) are given by \( m_s^2 \) and quark condensate \( m_s \langle ss \rangle \)

\[
\Pi'^{a}(q^2) - \Pi^{ad}(q^2) = \frac{3}{4 \pi^2} \frac{m_s^2 + m_s \langle ss \rangle}{q^2} + O \left( \frac{1}{q^4} \right)
\]

\[
\Pi'^{g}(q^2) - \Pi^{gd}(q^2) = -\frac{3}{8 \pi^2} m_s^2 \ln \left( \frac{\mu^2}{q^2} \right) + O \left( \frac{1}{q^6} \right). \tag{5}\]

The correction to \( \Pi_g(q^2) \) is ultraviolet divergent with \( \mu \) being a subtraction point.

The \( m_s^2 \) contribution to \( \delta_P^{kl} \) reads

\[
\delta^{kl}_2(\alpha_s) = F^{kl} m_s^2 M^2 \delta^{KL}_{10} + \frac{(k + 2)!!}{(k + l + 3)!} \tag{6}\]

The mass correction in the function \( \Pi_q(q^2) \) from eq. (5) gives no contribution to the moments with \( l > 0 \) in the leading order of perturbation theory [23]. The leading order result for the \( m_s^2 \) contribution from eq. (6) gets strongly renormalized in higher orders of perturbation theory [26, 27, 28]. In the approach based on the finite order perturbation theory one finds

\[
F_{00} = \frac{4}{3} \left( 1 + 5.333a_s + 46.0a_s^2 + (283.6 + k_3^2)a_s^3 \right) \tag{7}\]

with \( a_s = \alpha_s(M_\tau)/\pi \). At the \( \alpha_s^3 \) order there is an unknown constant \( k_3^2 \) in the correlator \( \Pi_q(q^2) \) that contributes to the moments with \( l = 0 \). The use of the moments with \( l > 0 \) allows for pushing the theoretical accuracy to the \( \alpha_s^3 \) level but the experimental data for such moments is less precise. For the numerical value of the strong coupling constant \( \alpha_s(M_\tau) = 0.344 \pm 0.006 \) as extracted from the \( \tau \) lepton decay rate into non-strange hadrons the convergence of perturbation theory series is slow. The explicit convergence of higher moments within perturbation theory is worse, e.g.

\[
F_{20} = \frac{6}{5} \left( 1 + 6.456a_s + 62.25a_s^2 + (547.8 + k_3^2)a_s^3 \right). \tag{8}\]

The detailed analysis of convergence in the finite order perturbation theory for the moments of order \( m_s^2 \) is beyond the scope of this paper.
The calculational techniques developed for this evaluation (for details and further references see [34]) allows for computing the quantity \( k^2 \) as well.

The contribution of quark condensate

\[
\delta_{\bar{u}u}^{k_0} = -4\pi^2 \frac{m_s\langle \bar{s}s \rangle}{M_T^2} (k + 2) \tag{9}
\]

is linear in \( m_s \) that implies a potentially large numerical magnitude. It suffices to use the leading order approximation of the coefficient function as the operator \( m_s \bar{s}s \) is renormalization group invariant that makes the change of the coefficient function small with running. In the numerical analysis we use \( \langle \bar{s}s \rangle = (0.8 \pm 0.2)\langle \bar{u}u \rangle \) and \( \langle \bar{u}u \rangle = -(0.23 \text{ GeV})^3 \).\[33\].

Further power corrections are written as

\[
\Pi_1 = \sum_{n \geq 3} \frac{(O_{2n}^{\bar{u}u})}{(-q^2)^n}, \quad \Pi_2 = \sum_{n \geq 3} \frac{(O_{2n}^{\bar{s}s})}{(-q^2)^n(n-1)} \tag{10}
\]

that leads to \( n \geq 3 \) contributions

\[
\delta_{2n}^{k_0} = -4\pi^2 a_{2n} \frac{(k + 2)!}{M_T^{2n} (n-1)! (k + n + 3)!} \tag{11}
\]

with \( a_{2n} = \langle O_{2n}^{\bar{s}s} \rangle - 2 \langle O_{2n}^{\bar{u}u} \rangle / M_T^2 \) (see eq. [34]). Thus, the LO expression for \( \delta R_{k_0}^{00} \) reads

\[
\frac{1}{2N_c} \delta R_{k_0}^{00} = 4 \frac{m_s^2}{M_T^4} - 8\pi^2 \frac{m_s \langle \bar{s}s \rangle}{M_T^4} - 4\pi^2 \frac{a_6}{M_T^4}. \tag{12}
\]

The quantity \( a_6 = \langle O_6^\alpha \rangle / 2 \langle O_8^\alpha \rangle / M^2 \) contains a contribution from dimension eight operators \( O_8 \) appearing in the g-part of the correlator \[34\]. Note that this part receives only \( J = 1 \) contributions.

The perturbation theory coefficients \( F_{kl} \) of the \( m_s^2 \) contribution are given in Table IV up to NNLO. An account of new NNNLO results from ref. [24] shows that the convergence virtually disappears for \( k \geq 2 \). Thus, the \( (2,0) \) moment is indeed marginal for the perturbation theory calculations to be trusted. The quantity \( \delta R_{k_0}^{00} \) is sensitive to the power corrections up to \( a_{2n} \) which include the maximal dimension operator \( O_{2n+2}^{\bar{s}s} \).

Experimental results for the moments of the decay rate from ref. [13] are given in Table III. Recently published results are rather close \[36\]. Using these data and theoretical calculation at the next-to-leading order (NNLO) of perturbation theory we extract the numerical value for \( m_s \). With \( a_6 = 0.001 \text{ GeV}^6 \) which is obtained in the approximation \( \langle O_8^\alpha \rangle = 0 \) and neglecting the power corrections \( a_{2n} \) with \( n > 3 \) one finds the following results at NNLO:

\[
\begin{align*}
(0, 0) & : m_s(M_T^2) = 130 \pm 27_{\text{exp}} \pm x_{th} \text{ MeV} \\
(1, 0) & : m_s(M_T^2) = 110 \pm 13_{\text{exp}} \pm x_{th} \text{ MeV} \tag{13} \\
(2, 0) & : m_s(M_T^2) = 94 \pm 8_{\text{exp}} \pm x_{th} \text{ MeV}
\end{align*}
\]

Here \( x_{th} \) denotes a theoretical uncertainty of a given moment to be discussed in much detail later. Note that the error in \( m_s \) due to the experimental uncertainty of the measured moments is large. Thus the linear approximation (propagation of errors) is not really applicable for the error analysis of the extracted value of \( m_s \). Considering the leading order of perturbation theory for the illustration we obtain the following values for the extracted masses

\[
\begin{align*}
(0, 0) & : m_s(M_T^2)_{\text{LO}} = 171 \pm 37_{\text{exp}} \text{ MeV} \\
(1, 0) & : m_s(M_T^2)_{\text{LO}} = 155 \pm 19_{\text{exp}} \text{ MeV} \tag{14} \\
(2, 0) & : m_s(M_T^2)_{\text{LO}} = 141 \pm 13_{\text{exp}} \text{ MeV}
\end{align*}
\]

For the low value \( 0.394 - 0.137 = 0.257 \) of the moment \((0,0)\) we find \( m_s(M_T^2) = 171 - 37 = 134 \text{ MeV} \) while for the high value \( 0.394 + 0.137 = 0.531 \) we have \( m_s(M_T^2) = 171 + 31 = 202 \text{ MeV} \). In our analysis we take the biggest error as a conservative estimate for the uncertainty of the extracted numerical value for the quark mass.

It is instructive to analyze how the theoretical predictions for the moments are composed from the different QCD contributions. To see the relation between perturbation theory and nonperturbative contributions is most interesting. We obtain the following decomposition of
the numerical values for the experimental moments into the contributions of perturbation theory and power corrections at NNLO:

\[(0, 0) : m_s(M_2^2) = 0.394 = 0.334 + 0.060 \]
\[(1, 0) : m_s(M_2^2) = 0.383 = 0.306 + 0.077 \]  \hspace{1cm} (15)
\[(2, 0) : m_s(M_2^2) = 0.373 = 0.286 + 0.087 \]

The first term is a “trivial” perturbation theory power correction term proportional to the mass squared of the strange quark \(m_s^2\)-term.

The second term is given by the quark condensate and is linear in \(m_s\) that makes it potentially large numerically. This term is renormalization group invariant. Note also that by construction the ratios of quark condensate contributions in different perturbation theory orders are equal to the ratios of the corresponding values of quark masses. One sees that the contribution of the leading power correction is numerically significant. The relative magnitude of the quark condensate contribution increases for larger \((k, 0)\) moments from 18% to 30%. This calls for the analysis of contributions of higher order power corrections.

There are no general systematic techniques to estimate the theoretical errors due to truncation of the asymptotic series. This is really difficult problem that should be considered for any given case. Let us consider a pattern of convergence for the extracted masses in different orders of perturbation theory for coefficient functions:

\[(0, 0) : m_s(M_2^2) = 130 = 171 - 30 - 11 \text{ MeV} \]
\[(1, 0) : m_s(M_2^2) = 110 = 155 - 31 - 14 \text{ MeV} \]  \hspace{1cm} (16)
\[(2, 0) : m_s(M_2^2) = 94 = 141 - 32 - 15 \text{ MeV}. \]

Taking one half of the last term as the estimate of the truncation error we find the following results:

\[(0, 0) : m_s(M_2^2) = 130 \pm 6 \text{ MeV} \]
\[(1, 0) : m_s(M_2^2) = 110 \pm 7 \text{ MeV} \]  \hspace{1cm} (17)
\[(2, 0) : m_s(M_2^2) = 94 \pm 8 \text{ MeV}. \]

This uncertainty comes from the truncation of the perturbation theory series for the coefficients \(F_{kl}\). Combining them with the experimental errors we finally get:

\[(0, 0) : m_s(M_2^2) = 130 \pm 27_{\text{exp}} \pm 6_{\text{tr}} \text{ MeV} \]
\[(1, 0) : m_s(M_2^2) = 110 \pm 13_{\text{exp}} \pm 7_{\text{tr}} \text{ MeV} \]  \hspace{1cm} (18)
\[(2, 0) : m_s(M_2^2) = 94 \pm 8_{\text{exp}} \pm 8_{\text{tr}} \text{ MeV}. \]

For higher moments the relative weight of truncation error is larger.

Still, this is not a whole story and we have to estimate uncertainty due to higher power corrections. One possibility to perform such an estimate is to consider the “allowed” region for the \(m_s\) moment. For the \((0, 0)\) moment there is no contribution of higher power correction terms that leaves it untouched.

For the \((1, 0)\) moment there is a contribution of one higher power correction. Taking again one half of the last term we get the uncertainty \(0.077/2 \approx 0.04\) from eq. (15). This uncertainty leads to an additional error in the mass of 7 MeV (this number can be obtained either by a direct computation or from the results for the experimental error in the linear approximation, it should be around one half of it, \(13/2 \approx 7\)).

For the \((2, 0)\) moment there are at least two next power correction contributions. This fact can make the resulting error larger if these corrections are correlated. Still taking one half of the last term we get the uncertainty \(0.087/2 \approx 0.044\) from eq. (15). This uncertainty leads to the additional error in the extracted mass of 7 MeV. Having in mind the possibility of correlation of power corrections one could add 50% to the error (see also eq. (20)) and finally get 10 MeV.

This analysis leads to the following uncertainties due to power corrections that should be added to the perturbation theory uncertainties from eq. (17):

\[(0, 0) : m_s(M_2^2) = 130 \pm 6 \pm 0 \text{ MeV} \]
\[(1, 0) : m_s(M_2^2) = 110 \pm 7 \pm 7 \text{ MeV} \]  \hspace{1cm} (19)
\[(2, 0) : m_s(M_2^2) = 94 \pm 8 \pm 10 \text{ MeV}. \]

These uncertainties are only indicative as it is really difficult to make any reliable evaluation of the errors due to higher power corrections.

A remarkable feature of the obtained results for the numerical value of the strange quark mass is the strong dependence of the extracted numerical value for \(m_s(M_2^2)\) on the particular moment used for the analysis \([15,16,17,18]\). The theoretical errors are dominated by the uncertainty due to the truncation of the perturbation theory series.

The experimental and theoretical errors are independent and added in quadrature give the total uncertainty of 28, 16, 15 MeV for the \(m_s\) values extracted from the \((0, 0), (1, 0), (2, 0)\) moments. From perturbation theory point of view the moment \((0, 0)\) is the best one but its experimental accuracy is low. With neglecting the contributions due to higher power corrections the value of \(m_s\) extracted from the \((1, 0)\) moment has a smaller total uncertainty because the experimental accuracy for the \((1, 0)\) moment is much better than that of the \((0, 0)\) moment. This analysis makes clear that in order to improve upon the total precision one should reduce the experimental errors for the low moments.

Three estimates for the numerical value of \(m_s\) in eq. (15) are consistent within the error bars since the experimental errors are rather large. However, the situation is definitely unsatisfactory as there is a systematic decrease of the central value for \(m_s\) extracted from different moments. The reason for this systematic decrease is likely the neglect of higher power corrections.
tions in the analysis. The analysis of experimental data with the inclusion of higher power corrections as free parameters of the fit gives $m_s(M_0^2) = 130 \pm 27_{\exp} \pm 6_{\text{the}} \text{MeV}$, $a_8 = 0.05 \text{GeV}^8$, $a_{10} = -0.3 \text{GeV}^{10}$. The value for $m_s$ has not changed since the higher power corrections do not contribute to the $(0,0)$ moment. However the equations for moments $(0,1)$ and $(0,2)$ are satisfied with the same $m_s$ because of contributions due to higher power terms $a_{8,10}$. It is instructive to see how the moments are composed with an account of the contributions of power corrections (ordered by the dimensionality)

\[
(0,0) : 0.394 = 0.341|m_s|^2 + 0.061|m_s| - 0.008|a_8| (20)
(1,0) : 0.383 = 0.433|m_s|^2 + 0.092|m_s| - 0.024|a_8|
-0.118|a_{10}|
(2,0) : 0.373 = 0.552|m_s|^2 + 0.123|m_s| - 0.048|a_8|
-0.472|a_{10}| + 0.218|a_{10}|.
\]

There is a sizable contribution from higher power corrections to the high moments. The numerical values for the high dimensional condensates are extracted as fit parameters from the data. Can they be understood from the present knowledge of the numerical magnitude of power corrections? The structure of power corrections for the relevant correlator is $a_{2n} = C_{2n}m_s^{2n-1}$ with $\Lambda$ being a nonperturbative infrared scale of QCD and the factor $m_s$ appears as the power corrections to the difference of the correlators should vanish in the chiral limit. For $m_s = 130 \text{MeV}$ and $\Lambda \sim m_N = 0.77 \text{GeV}$ one finds $a_8 = 0.02C_8$ and $a_{10} = 0.012C_{10}$. One expects a fast growth of coefficients $C_{2n}$ with $n$, e.g. $25$. Assuming a factorial growth $C_{2n} \sim n!$ one obtains for the ratio $a_{10}/a_8 = 5\Lambda^2 = 3 \text{GeV}^2$ while the values extracted from the data is $6 \text{GeV}^2$. This is quite reasonable result given the simplicity of the estimate. Note that the value of $a_8$ seems to be very small within this logic. It is known however that there is a huge numerical cancellation between first several low dimensional power corrections in the vector and axial channels. Our method of estimation cannot catch such delicate features of the physical spectrum.

One can also estimate the magnitude of power corrections phenomenologically. Assuming for simplicity that the continuum contribution basically cancels in the difference of the resonance contributions. For the axial part it is given by the difference

\[
\Pi_a^q(q^2) - \Pi_q^a(q^2) \sim \frac{f_{2a}}{m_{a}^2 - q^2} - \frac{f_{2a}}{m_{a}^2 - q^2}.
\]

where $m_{a,ud}^2$ and $m_{a,us}^2$ are some characteristic scales in the corresponding channels. Taking $f_{2a}^2 = f_{2a}^2 = 0.6/4\pi^2 \text{GeV}^2$ and $m_{a,ud} = m_{a,us} = 1.24 \text{GeV}$, $m_{a,K} = 1.4 \text{GeV}$ one finds $|a_8| = 0.06 \text{GeV}^8$ and $|a_{10}| = 0.14 \text{GeV}^{10}$ which is in agreement by eye with the fit. The contribution of the vector channel to the difference is much smaller because the masses of corresponding resonances $\rho(770)$ and $K^*(890)$ are smaller.

It seems hopeless to estimate numerical values for high dimensional contributions from first principles. Even if the basis of relevant operators is identified and the coefficient functions are found one remains with the problem of numerical values for vacuum condensates. The factorization hypothesis becomes less trustworthy with increasing dimension of the condensates. Still our consideration shows that the fit to data is consistent with the estimates based on the structure of hadronic spectra. The emerging picture is fairly sensible from the physical point of view. The large $k$-moments give a high energy resolution of the spectrum as they are saturated with quite a limited part of the total spectrum; in fact, the large $k$-moments virtually become the exclusive observables. For such almost exclusive observables the influence of high order power corrections should be large.

Note that the precision of experimental data is not sufficiently good that allows for (marginal) fit of all the moments within the error bars with only one value of the strange quark mass and without including the higher power corrections. We do not discuss this possibility as power corrections are known to be definitely included as free parameters of the fit.

The obtained numerical value for the strange quark mass is compatible with results available in the literature. However the central value is larger than the lattice results. We obtain $m_s(2 \text{GeV}) = 125 \pm 28 \text{MeV}$.

To conclude, our current analysis of the $\tau$-decay data shows that the requirement of the applicability of perturbation theory strictly limits the allowed range of moments with the $(k,l) = (2,0)$ moment being almost marginal since for the higher moments the convergence of perturbation theory series is virtually absent. The contributions of power corrections are large for the high $k$-moments and have to be retained that stabilizes the value of the strange quark mass extracted from the data.

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