A PLATFORM PRESENTATION FOR SURFACE-LINKS

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Abstract. In this paper, we introduce a method, called a plat form, of describing a surface-link in the 4-space using a braided surface. We prove that every surface-link, which is not necessarily orientable, can be described in a plat form. The plat index is defined as a surface-link invariant, which is an analogy of the bridge index for a link in the 3-space. We classify surface-links with plat index 1 and show some examples of surface-links in plat forms.

1. Introduction

In knot theory we often use two methods of presenting links in the 3-space using braids: One is a closed braid form as in Figure 1 and the other is a plat form as in Figure 2.

Figure 1. A closed braid form. Figure 2. A plat form.

A surface-link is a closed surface embedded in \( \mathbb{R}^4 \), and a 2-knot is a 2-sphere embedded in \( \mathbb{R}^4 \). Two surface-links are considered to be equivalent if they are ambient isotopic in \( \mathbb{R}^4 \). It is known that every orientable surface-link is equivalent to a surface-link in a closed 2-dimensional braid form (cf. [9, 12, 21]). It is an analogy of a closed braid form for a link.

The purpose of this paper is to introduce a new method of presenting a surface-link, which we call a plat form, as an analogy of a plat form for a link.

Theorem 1.1. Every surface-link is equivalent to a surface-link in a plat form.

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We emphasize that our method works for every surface-link, while the closed 2-dimensional braid form works only for orientable ones. A genuine plat form is a special case of a plat form. Some surface-links can be presented in genuine plat forms.

**Theorem 1.2.** Every orientable surface-link is equivalent to a surface-link in a genuine plat form.

We show that the normal Euler number $e(F)$ of a surface-link $F$ in a genuine plat form is zero (Proposition 5.9). It is unknown to the author whether every surface-link with $e(F) = 0$ is equivalent to one in a genuine plat form.

We define two surface-link invariants, which are called the plat index and the genuine plat index, denoted by $\text{Plat}(F)$ and $g.\text{Plat}(F)$, respectively. These are analogies of the plat index, or the bridge index, of a link.

Using a theory of braided surfaces and 2-dimensional braids, we show that a surface-link $F$ with $\text{Plat}(F) = 1$ or with $g.\text{Plat}(F) = 1$ is trivial (Theorem 5.5) and that a 2-knot with $g.\text{Plat}(F) = 2$ is ribbon (Theorem 5.7). We also see an example of a 2-knot whose plat index and genuine plat index are different (Proposition 5.8). An example of a non-trivial surface-link in a plat form is shown in Figure 3 by using a motion picture (Proposition 5.8).

![Figure 3. The 2-twist spun trefoil in a (normal) plat form.](image)

This paper is organized as follows. In Section 2, we recall the notions of braids, surface-links, and braided surfaces. We also recall the definition of a plat form for a link. In Section 3, we define a (normal) plat form and a genuine plat form for a surface-link. In Section 4, we prove Theorems 1.1 and 1.2. In Section 5, we discuss the plat index and the genuine plat index of a surface-link, and show some examples.

We work in the PL or smooth category. Surfaces embedded in the 4-space are assumed to be locally flat in the PL category.

2. Preliminaries

2.1. A plat form presentation for a link. Let $n$ be a positive integer, $I = [0, 1]$ the interval, $D$ the square $I^2$ in $\mathbb{R}^2$, $\text{Int} D$ the interior of $D$, and $Q_n =$
the subset of \( n \) points in \( D \) such that \( q_k = (1/2, k/(n + 1)) \) for \( k = 1, 2, \ldots, n \).

An \( n \)-braid is a union of \( n \) intervals \( \beta \) embedded in \( D \times I \) such that each component
intersects with every open disk \( \text{Int} \, D \times \{ t \} \) \(( t \in I \)) transversely
at a single point, and \( \partial \beta = Q_n \times \{ 0, 1 \} \). The \( n \)-braid group \( B_n \) is the group
consisting of equivalence classes of \( n \)-braids in \( D \times I \). The braid group \( B_n \)
is identified with the fundamental group \( \pi_1(C_n, Q_n) \) of the configuration space \( C_n \) of \( n \)
points of \( \text{Int} \, D \). We denote by \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \) the standard
generators of \( B_n \) or their representatives due to Artin ([11]).

To define the plat closure of a braided surface in Section 3, we introduce the space of \( m \) wickets.

**Definition 2.1** ([3]). A wicket is a semicircle in \( D \times I \) that meets \( D \times \{ 0 \} \)
orthonally at its endpoints in \( \text{Int} \, D \times \{ 0 \} \). A configuration of \( m \) wickets
is a disjoint union of \( m \) wickets in \( D \times I \). The space consisting of all configurations of \( m \) wickets.

For a configuration \( w = w_1 \cup \cdots \cup w_m \) of \( m \) wickets, we denote by \( |\partial w| \) the
\( 2m \) points \( \partial w_1 \cup \cdots \cup \partial w_m \) in \( \text{Int} \, D \), which is identified with \( \text{Int} \, D \times \{ 0 \} \),
and by \( \partial w \) the \( 2m \) points \( |\partial w| \) equipped with the partition \( \{ \partial w_1, \ldots, \partial w_m \} \). Note
that if two configurations \( w \) and \( w' \) satisfy \( \partial w = \partial w' \), then \( w = w' \).

The set \( Q_{2m} \) equipped with the partition \( \{ q_1, q_2, \ldots, q_{2m} \} \) bounds
a unique configuration of \( m \) wickets, which we call the standard configuration of \( m \) wickets
and denote by \( w_0 \).

The fundamental group \( \pi_1(\{(W_m, w_0) \rightarrow (C_{2m}, Q_{2m}) \) be the continuous map sending \( w \) to \( |\partial w| \).
It induces a homomorphism \( |\partial| : \pi_1(\{(W_m, w_0) \rightarrow \pi_1(C_{2m}, Q_{2m}) = B_{2m}. \)

Hilden’s subgroup \( K_{2m} \) is the subgroup of \( B_{2m} \) generated by \( \sigma_1, \sigma_2 \sigma_1 \sigma_3 \sigma_2, \)
and \( \sigma_2 \sigma_1 \sigma_3 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \) for \( i = 1, \ldots, m - 1 \) ([7], cf. [2]).

**Proposition 2.2** ([3]). For each positive integer \( m \), the homomorphism \( |\partial| : \pi_1(\{(W_m, w_0) \rightarrow \pi_1(C_{2m}, Q_{2m}) = B_{2m} \) is injective and the image is Hilden’s
subgroup \( K_{2m} \). Namely, the wicket group \( \pi_1(\{(W_m, w) \) is isomorphic to Hilden’s
subgroup \( K_{2m} \).

The isomorphism from \( \pi_1(\{(W_m, w) \) to \( K_{2m} \) is restated as follows: Let \( f : (I, \partial I) \rightarrow (W_m, w_0) \) be a loop. Consider a 2m-braid \( \beta_f = \bigcup_{t \in I} |\partial f(t)| \times \{ t \} \subset D \times I \),
then the isomorphism sends \( [f] \in \pi_1(\{(W_m, w) \) to \( |\partial f| \in K_{2m}. \)

**Definition 2.3.** A loop \( g : (I, \partial I) \rightarrow (C_{2m}, Q_{2m}) \) is liftable if there exists a
loop \( f : (I, \partial I) \rightarrow (W_m, w_0) \) such that \( g = |\partial| \circ f. \)

**Definition 2.4.** A 2m-braid \( \beta \) in \( D \times I \) is adequate or wicket-adequate if the
associated loop \( g : (I, \partial I) \rightarrow (C_{2m}, Q_{2m}) \) is liftable, namely, there exists a
loop \( f : (I, \partial I) \rightarrow (W_m, w_0) \) such that \( \beta = \beta_f. \)
Note that Hilden’s subgroup $K_{2m}$ consists of the elements of $B_{2m}$ represented by some adequate $2m$-braids.

Let $\beta$ be a $2m$-braid in $D \times I \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. Attach a pair of the standard configurations of $m$ wickets to $\beta$ as in Figure 4, and we obtain a link which is called the \textit{plat closure} of $\beta$ and denoted by $\beta$. A link is said to be \textit{in a plat form} when it is the plat closure of a braid. Every link is equivalent to a link in a plat form.

$$\beta = \text{plat closure of a braid}, \quad \widetilde{\beta} = \text{plat closure of the trivial braid}.$$  

\textbf{Figure 4.} The plat closure of a braid.

In Section 3 we introduce a plat form of a surface-link in $\mathbb{R}^4$. We will also introduce a \textit{normal plat form}, which is a plat form satisfying a nice condition such that its motion picture is easy to describe.

To define a normal plat form of a surface-link in Section 3 we construct an isotopic deformation changing the plat closure of an adequate braid to the plat closure of the trivial braid as follows: Let $f : (I, \partial I) \rightarrow (W_m, w_0)$ be a loop. For each $t \in I$, let $\beta_t$ be $\bigcup_{s \in I} |\partial f((1-t)s)| \times \{s\}$ in $D \times I$, which is a union of $2m$ arcs. We denote by $L_t$ a link obtained from $\beta_t$ by attaching the configuration $f(t)$ of $m$ wickets to the side of $D \times \{1\}$ and the standard configuration $w_0$ to the side of $D \times \{0\}$ in $\mathbb{R}^3$. See Figure 5. Then, $\{L_t\}_{rel}$ is a 1-parameter family of links in $\mathbb{R}^3$ such that $L_0$ is $\beta_f$ and $L_1$ is the plat closure of the trivial $2m$-braid as in Figure 5. We call $\{L_t\}_{rel}$ the \textit{isotopic deformation changing $\beta_f$ to the plat closure of the trivial braid}.

As a corollary, the plat closure of an adequate $2m$-braid is an $m$-component trivial link.

\textbf{Figure 5.} The isotopic deformation changing $\widetilde{\beta_f}$ to the plat closure of the trivial braid.
2.2. **Surface-links.** A *surface-link* is a closed surface embedded in $\mathbb{R}^4$, and a *surface-knot* is a connected surface-link. A *2-knot* is a surface-knot homeomorphic to a 2-sphere. A *2-link* is a surface-link consisting of 2-spheres. Two surface-links $F$ and $F'$ are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^4$. We denote it by $F \simeq F'$ that $F$ and $F'$ are equivalent.

Let $h : \mathbb{R}^3 \times \mathbb{R}^1 \to \mathbb{R}^1$ be the projection onto the second factor. Set $F[t] = F \cap \mathbb{R}^3 \times \{t\}$ for $t \in \mathbb{R}$, which is called the *cross-section* of $F$ at $t$. A *motion picture* of $F$ is a 1-parameter family $\{F[t]\}_{t \in \mathbb{R}}$. We often describe surface-links using motion pictures.

A surface-knot is *trivial* if it is equivalent to a connected sum of standardly embedded 2-spheres, tori, and projective planes ([8]). Here standardly embedded projective planes $P^+$ and $P^-$ are illustrated in Figure 6.

![Figure 6. Motion pictures of $P_+$ and $P_-$.

2.3. **Braided surfaces and 2-dimensional braids.** A braided surface was introduced by Rudolph [19] and a *2-dimensional braid* was introduced by Viro (cf. [10, 11, 12]). Let $D_1$ and $D_2$ be the squares $I^2 \subset \mathbb{R}^2$ and $\text{pr}_i : D_1 \times D_2 \to D_i$ ($i = 1, 2$) the projection onto the $i$-th factor. Let $y_0 \in \partial D_2$ be a fixed base point.

**Definition 2.5** ([19], [21]). A *(pointed)* braided surface of degree $n$ is a surface $S$ embedded in $D_1 \times D_2$ satisfying the following conditions:

1. $\pi_S = \text{pr}_2|_S : S \to D_2$ is a simple branched covering map of degree $n$ (i.e., the preimage of each branch locus consists of $n - 1$ points).
2. $\partial S$ is the closure of an $n$-braid in the solid torus $D_1 \times \partial D_2$.
3. $\text{pr}_1(\pi_S^{-1}(y_0)) = Q_n$.

In particular, a 2-dimensional braid of degree $n$ is a braided surface $S$ of degree $n$ such that $\partial S$ is trivial, i.e., $\text{pr}_1(\pi_S^{-1}(y)) = Q_n$ for all $y \in \partial D_2$.

The degree of $S$ is denoted by $\deg S$. We say that two braided surfaces of the same degree are *equivalent* if they are ambient isotopic by an isotopy $\{h_t\}_{t \in I}$ of $D_1 \times D_2$ such that each $h_t$ ($s \in I$) is fiber-preserving when we regard $D_1 \times D_2$ as the trivial $D_1$-bundle over $D_2$, and the restriction of $h_t$ to
pr_{2}^{-1}(y_{0}) is the identity map. A braided surface is trivial if it is equivalent to $Q_{n} \times D_{2}$.

**Lemma 2.6 (cf. [12]).** A braided surface $S$ is trivial if and only if $S$ has no branch points.

We assume $D_{1} \times D_{2} \subset \mathbb{R}^{2} \times \mathbb{R}^{2} = \mathbb{R}^{4}$. Let $S$ be a 2-dimensional braid of degree $n$. The closure of $S$ is an orientable surface-link in $\mathbb{R}^{4}$ obtained from $S$ by attaching $n$ 2-disks trivially outside $D_{1} \times D_{2}$ in $\mathbb{R}^{4}$ along the boundary $\partial S$. It is described in Figure 7 when $n = 3$, where $\varepsilon$ is a positive number and $S_{[t]} = S \cap D_{1} \times (I \times \{t\})$ ($t \in I$).

**Proposition 2.7 ([11], [21]).** Every orientable surface-link is equivalent to the closure of a 2-dimensional braid.

![Figure 7. The closure $\overline{S}$ of a 2-dimensional braid $S$.](image)

For an orientable surface-link $F$, the braid index of $F$, denoted by $\text{Braid}(F)$, is the minimum degree of 2-dimensional braids whose closures are equivalent to $F$.

### 3. A Plattform Presentation for a Surface-Link

In this section, we introduce a Plattform for a surface-link.

We fix a loop $\mu : (I, \partial I) \to (\partial D_{2}, y_{0})$ which runs once on $\partial D_{2}$ counterclockwise. For a braided surface $S$ of degree $n$, let $g_{S} : (I, \partial I) \to (C_{n}, Q_{n})$ be a loop in the configuration space $C_{n}$ obtained by

$$g_{S}(t) = \text{pr}_{1}(\pi_{S}^{-1}(\mu(t)))$$

and $\beta_{S}$ an $n$-braid in $D_{1} \times I$ obtained by

$$\beta_{S} = \bigcup_{t \in I} \text{pr}_{1}(\pi_{S}^{-1}(\mu(t))) \times \{t\},$$
where \( \pi_S : S \to D_2 \) is the simple branched covering map appearing in the definition of a braided surface. Then \( \partial S \) is the closure of \( \beta_S \) in \( D_1 \times \partial D_2 \).

**Definition 3.1.** A braided surface \( S \) in \( D_1 \times D_2 \) is adequate if \( g_S \) is liftable or equivalently if \( \beta_S \) is adequate.

Note that the degree of an adequate braided surface is even. For an adequate braided surface \( S \) of degree \( 2m \), let \( f_S : (I, \partial I) \to (W_m, w_0) \) be the lift of \( g_S \), i.e., a loop in \( W_m \) with \( g_S = |\partial| \circ f_S \).

Let \( N \) be a regular neighborhood of \( \partial D_2 \) in \( \mathbb{R}^2 \setminus \text{Int} D_2 \). Since \( N \) is homeomorphic to an annulus \( I \times S^1 \), we identify them by a fixed identification map \( \phi : I \times S^1 \to N \) such that \( \phi(0, p(t)) = \mu(t) \in \partial D_2 \) for all \( t \in I \), where \( p : I \to S^1 = I/\partial I \) is the quotient map.

**Definition 3.2.** A properly embedded surface \( A \) in \( D_1 \times N \) is of wicket type if there exists a loop \( f : (I, \partial I) \to (W_m, w_0) \) such that
\[
A = \bigcup_{t \in I} f(t) \times \{ p(t) \} \subset (D_1 \times I) \times S^1 = D_1 \times N.
\]
In this case, we say that \( A \) is associated with \( f \) and denote it by \( A_f \).

We remark that a surface \( A \) of wicket type is a union of annuli or Möbius bands, and that \( \partial A = \partial A_f \) is expressed as
\[
\partial A = \bigcup_{t \in I} |\partial f(t)| \times \{ p(t) \} \subset D_1 \times S^1 = D_1 \times \partial D^2.
\]
Since two loops \( f \) and \( f' \) in \( (W_m, w_0) \) with \( |\partial| \circ f = |\partial| \circ f' \) are the same, we see that two surfaces \( A \) and \( A' \) of wicket type with \( \partial A = \partial A' \) are the same.

Let \( S \) be an adequate braided surface, and let \( f : (I, \partial I) \to (W_m, w_0) \) be a loop with \( g_S = |\partial| \circ f \). Then it holds that \( S \cap A_f = \partial S = \partial A_f \). We denote \( A_f \) by \( A_S \) and say that \( A_S \) is the surface of wicket type associated with \( S \).

**Definition 3.3.** Let \( S \) be an adequate braided surface and \( A_S \) the surface of wicket type associated with \( S \). The plat closure of \( S \), denoted by \( \overline{S} \), is the union of \( S \) and \( A_S \) in \( \mathbb{R}^4 \).

When \( \deg S = 2m \) and \( S \) has \( r \) branch points, the Euler characteristic \( \chi(S) \) of \( S \) is \( 2m - r \). Since \( \chi(A_S) = \chi(\partial A_S) = 0 \), we have \( \chi(\overline{S}) = 2m - r \).

**Definition 3.4.** A surface-link is said to be in a plat form if it is the plat closure of an adequate braided surface. Moreover, a surface-link is said to be in a genuine plat form if it is that of a 2-dimensional braid.

We introduce a normal plat form for a surface-link by using a motion picture as follows: Let \( \overline{S} \) be the plat closure of an adequate braided surface \( S \) of degree \( 2m \), and set \( S_{[0]} = \overline{S} \cap \mathbb{R}^3 \times \{ t \} (t \in \mathbb{R}) \) and \( S_{[1]} = S \cap D_1 \times (t \times \{ t \}) = S \cap \mathbb{R}^3 \times \{ t \} (t \in [0, 1]) \). Replacing \( S \) with an equivalent braided surface if
necessary, we may assume that $S$ satisfies the following conditions for some $t_0 \in [0, 1]$

1. $S$ has no branch points over $I \times [t_0, 1] \subset D_2$.
2. $\text{pr}_1(\pi_1^{-1}(y)) = Q_{2m}$ for every $y \in \partial D_2 \setminus ([1] \times [t_0, 1])$.
3. $S_{[t_0]} = S_{[t_0]} \times \{t_0\}$.

In particular, $S_{[0]}$ and $S_{[1]}$ are both the trivial braids. Furthermore, replacing $S$ with an equivalent braided surface if necessary, we may assume that the motion picture $[S_t]_{t \in [0, 1]}$ between $t = t_0$ and $t = 1$ is the isotopic deformation changing $e_\beta f$ to the plat closure of the trivial braid. (See Figure 5.) Finally, deforming $A_S$ by an ambient isotopy rel boundary, we have a surface-link $F$, equivalent to $\widetilde{S}$, described by a motion picture as in Figure 8. The surface-link $F$ in this form is said to be in a normal plat form.

4. Proofs of Theorems 1.1 and 1.2

In this section, we give proofs of Theorems 1.1 and 1.2. To prove them, we discuss a plat form for a link and a banded link presentation for a surface-link.

4.1. Stabilization and generalized stabilization for braids. For positive integers $n$ and $n'$ with $n \leq n'$, let $i_n' : B_n \to B_{n'}$ denote the natural inclusion map from $B_n$ to $B_{n'}$ sending each generator $\sigma_i \in B_n$ to $\sigma_i \in B_{n'}$.

A stabilization of a $2m$-braid $\beta$ is a replacement of $\beta$ with a $2m'$-braid $\beta'$ such that

$$\beta' = t_{2m'}^{2m} (\beta) \sigma_2 \sigma_{2(m+1)} \sigma_{2(m+2)} \cdots \sigma_{2(m'-1)},$$

where $m'$ is an integer with $m \leq m'$. We also call a stabilization an $l$-stabilization when $l = m' - m$.

It is obvious that if $\beta'$ is obtained from $\beta$ by an $l$-stabilization then the plat closure of $\beta'$ is equivalent to that of $\beta$ as links in $\mathbb{R}^3$. See Figure 9 for $l = 1, 2$. 

\begin{align*}
&\quad \hspace{1.5cm} t = -\varepsilon \quad \quad -\varepsilon < t < 0 \quad \quad 0 \leq t < t_0 \quad \quad t = t_0 \quad \quad t = t_0 \\
&\quad \hspace{1.5cm} \beta S \\
&\quad \hspace{1.5cm} S_{[t]} \\
&\quad \hspace{1.5cm} S_{[t_0]} \\
&\quad \hspace{1.5cm} \beta S \\
&\quad \hspace{1.5cm} t = t_0 \quad \quad t = 1 \quad \quad 1 \leq t < 1 + \varepsilon \quad \quad t = 1 + \varepsilon \\
&\quad \hspace{1.5cm} \beta S \\
\end{align*}

Figure 8. A surface-link in a normal plat form.
Proposition 4.1. Let $\beta_i$ $(i = 1, 2)$ be a $2m_i$-braid such that the plat closure $e\beta_i$ is a knot. Then $\beta_1$ is equivalent as knots in $\mathbb{R}^3$ to $e\beta_2$ if and only if there exists an integer $t \geq \max\{m_1, m_2\}$ such that for each $m \geq t$, the $2m$-braids $\beta'_i$ $(i = 1, 2)$ obtained from $\beta_i$ by stabilization belong to the same double coset of $B_{2m}$ modulo $K_{2m}$.

Proposition 4.1 is generalized into the case of links in $\mathbb{R}^3$ by using the notion of a generalized stabilization.

Let $\Lambda_m$ be the set of $m$-tuples of non-negative integers. For two elements $\lambda = (l_1, \ldots, l_m)$ and $\lambda' = (l'_1, \ldots, l'_m)$ of $\Lambda_m$, we write $\lambda \leq \lambda'$ if $l_i \leq l'_i$ for each $i = 1, \ldots, m$. Then $\preceq$ is a (directed) partial ordering on $\Lambda_m$. Put $|\lambda|_0 = m$, $|\lambda|_i = m + l_1 + \cdots + l_i$ $(i = 1, \ldots, m)$, and $|\lambda| = |\lambda|_m$. For a given $\lambda \in \Lambda_m$, we denote $\tau_i = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i} \in K_{2m}$ $(1 \leq i \leq |\lambda| - 1)$ and $T_{i, j} = \prod_{k=i}^{m-1} \tau_k \cdot \prod_{k=j}^{m-1} \tau_k^{-1} \in K_{2m}$ $(1 \leq i \leq m, m - 1 \leq j \leq |\lambda|)$, where the former or later product is assumed to be the identity element of the group if $i = m$ or $j = m - 1$, respectively, and we construct a $2|\lambda|$-braid $T(\lambda)$ as follows:

$$T(\lambda) = \prod_{i=1}^{m} T_{i, (|\lambda|_i - 1)} \sigma_{2|\lambda|} \sigma_{2|\lambda| - 1} \cdots \sigma_{2} \cdot T_{i, (|\lambda|_i - 1)}^{-1}.$$

For a $2m$-braid $\beta$ and $\lambda \in \Lambda_m$, we let $\beta^\lambda$ denote a $2|\lambda|$-braid such that

$$\beta^\lambda = \tau_{2m}^{2|\lambda|}(\beta) \cdot T(\lambda).$$

A generalized stabilization (with respect to $\lambda$) or $\lambda$-stabilization of $\beta$ is a replacement of $\beta$ with $\beta^\lambda$. A $\lambda$-stabilization is a composition of $l_i$-stabilization performed on the $2i$-th strand of $\beta$ for each $i = 1, \ldots, m$. A $l$-stabilization of $\beta$ is a $\lambda$-stabilization with $\lambda = (0, \ldots, 0, l) \in \Lambda_m$. Figure 10 depicts the plat closure of a 12-braid obtained from a 6-braid $\beta$ by a generalized stabilization with respect to $\lambda = (2, 0, 1) \in \Lambda_3$. 
The following proposition states that two braids of even degrees have equivalent plat closures as links in $\mathbb{R}^3$ if and only if, after applying a generalized stabilization suitably, they belong to the same double coset of $B_{2m}$ modulo $K_{2m}$.

**Proposition 4.2** (cf. [2]). Let $\beta_i$ ($i = 1, 2$) be a $2m_i$-braid. The plat closure $e_{\beta_i}$ is equivalent to $e_{\beta_2}$ as links in $\mathbb{R}^3$ if and only if there exists an element $\lambda \in \Lambda_m$ satisfying the following condition: For any $\lambda_1 \geq \lambda$, there exists $\lambda_2 \in \Lambda_{m_2}$ with $|\lambda_1| = |\lambda_2|$ such that $\beta_{1\lambda_1}$ and $\beta_{2\lambda_2}$ belong to the same double coset of $B_{2m}$ modulo $K_{2m}$, where $m = |\lambda_1| = |\lambda_2|$.

Proposition 4.2 is proved directly by applying the proof of Proposition 4.1 given in [2] for each component of a link.

**4.2. A banded link presentation for a surface-link.** A banded link in $\mathbb{R}^3$ means a pair $(L, B)$ of a link $L$ and a family $B$ of mutually disjoint bands attaching to $L$. We let $L_B$ denote the link obtained from $L$ by surgery along the bands belonging to $B$. A banded link $(L, B)$ is admissible if both $L$ and $L_B$ are trivial links.

Let $(L, B)$ be an admissible banded link in $\mathbb{R}^3$. Let $d$ and $D$ be unions of mutually disjoint 2-disks embedded in $\mathbb{R}^3$ bounded by $L$ and $L_B$, respectively. Consider a closed surface $F = F(L, B)$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ defined by

$$p(F \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} 
D & (t = 1), \\
L_B & (0 < t < 1), \\
L \cup |B| & (t = 0), \\
L & (-1 < t < 0), \\
d & (t = -1), \\
\emptyset & \text{otherwise},
\end{cases}$$

Figure 10. The plat closure of a $(2, 0, 1)$-stabilized braid.
where \(|B|\) is the union of the bands belonging to \(B\). We call \(F(L, B)\) a closed realizing surface of \((L, B)\). Although it depends on a choice of \(d\) and \(D\), the equivalence class as surface-links does not depend on them (cf. [13, 16]).

Let \(r\) be a real number, and let \(h : \mathbb{R}^3 \times (-\infty, r) \to (-\infty, r]\) be the projection onto the second factor, which we regard as a height function of \(\mathbb{R}^3 \times (-\infty, r]\).

**Lemma 4.3** (cf. [13, 16]). Let \(F\) and \(F'\) be compact surfaces properly embedded in \(\mathbb{R}^3 \times (-\infty, r]\) such that all critical points of \(F\) and \(F'\) are minimal points with respect to \(h\), and their boundaries are the same trivial link in \(\mathbb{R}^3 \times \{r\}\). Then, \(F\) and \(F'\) are ambient isotopic in \(\mathbb{R}^3 \times (-\infty, r]\) rel \(\mathbb{R}^3 \times \{r\}\).

**Lemma 4.4** ([16]). If two admissible banded links \((L, B)\) and \((L', B')\) are ambient isotopic in \(\mathbb{R}^3\), then their closed realizing surfaces \(F(L, B)\) and \(F(L', B')\) are equivalent.

**Lemma 4.5** ([16]). Any surface-link \(F\) is equivalent to a closed realizing surface \(F(L, B)\) of an admissible banded link \((L, B)\).

**Lemma 4.6.** By an isotopy of \(\mathbb{R}^3\), any banded link \((L, B)\) in \(\mathbb{R}^3\) is deformed to a banded link \((L_0, B_0)\) satisfying the following conditions:

1. There exists a disk \(D\) in \(\mathbb{R}^2\) and a 2\(m_0\)-braid \(\beta_0\) in \(D \times I\) (\(\subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3\)) for some \(m_0 \in \mathbb{N}\) such that \(\beta_0 = L_0 \cap D \times I\) and \(\overline{\beta_0} = L_0\).

2. There exist mutually disjoint \(n\) subcylinders \(U_i = d_i \times [s_i, t_i]\) (\(i = 1, \ldots, n\)) in \(D \times I\) such that each \(U_i\) contains a part of \(L_0\) as a pair of vertical line segments and a half-twisted band \(b_i \in B_0\) as in Figure 11, where \(n\) is the number of bands belonging to \(B_0\).

Furthermore, we may take subcylinders \(U_i = d_i \times [s_i, t_i]\) such that \(d_1, \ldots, d_n\) are mutually disjoint disks in \(\text{Int } D\) and \([s_i, t_i] = [2/5, 3/5]\) for \(i = 1, \ldots, n\).

**Figure 11.** A local model of \(L_0\) and \(b_i\) in \(U_i = d_i \times [s_i, t_i]\).

**Proof.** Let \(d_1, \ldots, d_n\) be mutually disjoint disks in \(\text{Int } D\) and let \(U_i = d_i \times [2/5, 3/5]\) for \(i = 1, \ldots, n\). By an isotopy of \(\mathbb{R}^3\), \((L, B)\) is deformed into \((L_1, B_0)\) such that for each \(i\), \(U_i\) intersects with \((L_1, B_0)\) as in Figure 11.

By an isotopy of \(\mathbb{R}^3\) keeping \(U_i\) (\(i = 1, \ldots, n\)) fixed pointwise, \((L_1, B_0)\) is deformed into \((L_2, B_0)\) such that all maximal points of \(L_2\) are in \(\mathbb{R}^2 \times \{1\}\).
Lemma 4.6. Let \( \beta \)

We prove the theorem by 3 steps. Let \( c \) along bands belonging to \( B \) respectively. Since \( U \) to the trivial braid \( 1 \) that obtained by generalized stabilization. Since \( \beta = L_0 \) is a trivial link of \( c \) components, the plat closure \( \beta' \) is equivalent to the plat closure \( \lambda = U \times I \), there exist a positive integer \( m \in \mathbb{Z} \), three elements \( \lambda \in \Lambda_{m0}, \lambda_1 \in \Lambda_{c_1}, \lambda_2 \in \Lambda_{c_2}, \) and four adequate \( 2m \)-braids \( \gamma, \gamma', \delta, \delta' \) in \( D \times I \) such that \( |\lambda| = |\lambda_1| = |\lambda_2| = m \) and

\[
\beta_1 = \gamma \alpha_1 \gamma', \quad \beta_2 = \delta \alpha_2 \delta'
\]

in \( B_{2m} \), where \( \beta_1 = \beta^{i_1}_{0}, \alpha_1 = 1^{i_1}_{2c_1}, \beta_2 = (\beta_0)^i_{B_0} \) and \( \alpha_2 = 1^{j_2}_{2c_2} \) are \( 2m \)-braids in \( D \times I \) obtained by generalized stabilization.

Since \( \beta_1 \) is a \( \lambda \)-stabilized \( \beta_0 \), there exists a subcylinder \( U \) of \( D \times I \) such that \( \beta_1 \cap U = \beta_0 \) under an identification of \( U \) and \( D \times I \). Let \( B_1 \) be the set of bands attaching to \( \beta_1 \) obtained from \( B_0 \) via the identification. Then, \( \beta_2 \) and \( (\beta_1)_{B_1} \) are the same braid. Note that \( (\beta_1, B_1) \) is ambient isotopic to \( (L_0, B_0) \).

Step 2: We construct a properly embedded compact surface \( S_0 \) in \( D_1 \times D_2 \) and a braided surface \( S \) of degree \( 2m \) in \( D_1 \times D_2 \). Let \( 0 = t_0 < t_1 < \cdots < t_6 < t_7 = 1 \) be a partition of \( I = [0, 1] \). We divide \( D_2 = I \times I \) into seven pieces \( E_0, \ldots, E_6 \) with \( E_i = I \times [t_i, t_{i+1}] \). Let \( \alpha_1^i \) and \( \alpha_2^i \) be \( 2m \)-braids in \( D_1 \times I \) given by

\[
\alpha_1^i = \prod_{i=1}^{m} T_i, (|\lambda_1|_{i-1})^{-1} T_i^{-1}, (|\lambda_1|_{i-1})^{-1}, \quad \alpha_2^i = \prod_{i=1}^{m} T_i, (|\lambda_2|_{i-1})^{-1} T_i^{-1}, (|\lambda_2|_{i-1})^{-1},
\]

which are obtained from \( \alpha_1 = 1^{i_1}_{2c_1} = T(\lambda_1) \) and \( \alpha_2 = 1^{j_2}_{2c_2} = T(\lambda_2) \) by removing the parts \( \sigma^{2|\lambda_1|_{i-1}} \sigma^{2|\lambda_1|_{i-1}+1} \cdots \sigma^{2|\lambda_1|_m} \) and \( \sigma^{2|\lambda_2|_{i-1}} \sigma^{2|\lambda_2|_{i-1}+1} \cdots \sigma^{2|\lambda_2|_m} \) (\( i = 1, \ldots, m \)), respectively (Figure 14). Note that \( \alpha_1^i \) and \( \alpha_2^i \) are equivalent to the trivial braid \( 1_{2m} = Q_{2m} \times I \) as braids in \( D_1 \times I \).
Let $p_1 : D_1 \times I \to D_1$ and $p_2 : D_1 \times I \to I$ be the projections onto the first and second factors, respectively. Let $p_i : D_1 \times D_2 \to D_i$ be the projections onto the $i$-th factors ($i = 1, 2$). For a braid $b$ in $D_1 \times I$ and $s \in I$, we denote by $b_{s,t}$ the image $p_1(b \cap p_2^{-1}(s))$ in $D_1$ of the intersection $b \cap p_2^{-1}(s)$.

Now, we define a properly embedded compact surface $S_0$ in $D_1 \times D_2$, step by step, as follows:

(0) First, we define $S_0 \cap D_1 \times \partial E_0$ by

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s,t)) = \begin{cases}
(\alpha_1^*[s]) & ((s, t) \in I \times \{t_1\}), \\
Q_{2m} & ((s, t) \in [0, 1] \times \{t_0, t_1\}), \\
Q_{2m} & ((s, t) \in I \times \{t_0\}).
\end{cases}
$$

See Figure 13. Since $\alpha_1^*$ is equivalent to the trivial $2m$-braid, we may define $S_0 \cap D_1 \times E_0$ as a braided surface of degree $2m$ without branch points in $D_1 \times E_0$, which is trivial by Lemma 2.6.

Figure 13. A blueprint for a surface $S_0$. Each braid is appeared as the section of $S_0$. 
(1) We define $S_0 \cap D_1 \times (E_1 \setminus I \times \{(t_1 + t_2)/2\})$ as follows:

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\alpha_1^\ast)[s] & ((s, t) \in I \times [t_1, (t_1 + t_2)/2]), \\
(\alpha_1)[s] & ((s, t) \in I \times ((t_1 + t_2)/2, t_2]].
\end{cases}$$

Then, we define $S_0 \cap D_1 \times (I \times \{(t_1 + t_2)/2\})$ as the $2m$-braid $\alpha_1^\ast$ with bands such that the surgery result of $\alpha_1^\ast$ is $\alpha_1$ (see Figure 14). We denote by $B_1^\ast$ the set of these bands.

\[\text{Figure 14. A motion picture of } S_0 \ (t_1 \leq t \leq t_2)\]

(2) We construct $S_0 \cap D_1 \times E_2$ similarly to the case (0). First, we define $S_0 \cap D_1 \times \partial E_2$ by

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\beta_1)[s] & ((s, t) \in I \times \{t_3\}), \\
(\gamma'_{[(t-t_2)/(t_3-t_2)]})[s] & ((s, t) \in \{1\} \times [t_2, t_3]), \\
(\gamma'_{[(t-t_3)/(t_2-t_3)]})[s] & ((s, t) \in I \times \{t_2\}), \\
(\alpha_1)[s] & ((s, t) \in \{0\} \times [t_2, t_3]).
\end{cases}$$

Since $\beta_1 = \gamma \alpha_1 \gamma'$, the closed braid $S_0 \cap D_1 \times \partial E_2$ is equivalent to the trivial closed braid in $D_1 \times \partial E_2$. Thus we may define $S_0 \cap D_1 \times E_2$ as a braided surface of degree $2m$ without branch points.

(3) We construct $S_0 \cap D_1 \times E_3$ similarly to the case (1). First, we define $S_0 \cap D_1 \times (E_3 \setminus I \times \{(t_3 + t_4)/2\})$ by

$$\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\beta_1)[s] & ((s, t) \in I \times [t_3, (t_3 + t_4)/2]), \\
(\beta_2)[s] & ((s, t) \in I \times ((t_3 + t_4)/2, t_4]).
\end{cases}$$

Then, we define $S_0 \cap D_1 \times (I \times \{(t_3 + t_4)/2\})$ as the $2m$-braid $\beta_1$ with bands belonging to $B_1$. 
(4) We construct $S_0 \cap D_1 \times E_4$ similarly to the case (2). We define $S_0 \cap D_1 \times \partial E_4$ by

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\beta_2)_{[s]} & ((s, t) \in I \times \{t_4\}), \\
\delta'_{[t-t_5]/(t_5-t_3)} & ((s, t) \in \{1\} \times [t_4, t_5]), \\
(\alpha_2)_{[s]} & ((s, t) \in I \times \{t_3\}), \\
\delta_{[t-t_4]/(t_4-t_3)} & ((s, t) \in \{0\} \times [t_4, t_5]).
\end{cases}
$$

Since $\beta_2 = \delta \alpha_2 \delta'$, we define $S_0 \cap D_1 \times E_4$ as a braided surface of degree $2m$ without branch points.

(5) We construct $S_0 \cap D_1 \times E_5$ similarly to the case (1). We define $S_0 \cap D_1 \times (E_5 \setminus \{(t_5 + t_6)/2\})$ by

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\alpha_2)_{[s]} & ((s, t) \in I \times [t_5, (t_5 + t_6)/2]), \\
(\alpha_2')_{[s]} & ((s, t) \in I \times ((t_5 + t_6)/2, t_6)).
\end{cases}
$$

Then, we define $S_0 \cap D_1 \times I \times \{(t_5 + t_6)/2\}$ as the $2m$-braid $\alpha_2^*$ with bands attaching to $\alpha_2^*$ as in the opposite direction of Figure 14 such that the surgery result of $\alpha_2^*$ is $\alpha_2$. We denote by $B^*_7$ the set of these bands.

(6) We construct $S_0 \cap D_1 \times E_6$ similarly to the case (0). First, we define $S_0 \cap D_1 \times \partial E_6$ by

$$
\text{pr}_1(S_0 \cap \text{pr}_2^{-1}(s, t)) = \begin{cases} 
(\alpha_2^*)_{[s]} & ((s, t) \in I \times \{t_7\}), \\
Q_{2m} & ((s, t) \in [0, 1] \times [t_6, t_7]), \\
Q_{2m} & ((s, t) \in I \times \{t_6\}).
\end{cases}
$$

Since $\alpha_2^*$ is equivalent to the trivial $2m$-braid, we may define $S_0 \cap D_1 \times E_6$ as a braided surface of degree $2m$ without branch points.

As a result, we have a properly embedded surface $S_0$ in $D_1 \times D_2$. We take a based point $y_0 = (0, 0) \in \partial D_2$. Then, $S_0$ is a braided surface of degree $2m$ except in neighborhoods of the bands appearing in (1), (3), and (5). By an ambient isotopy of a neighborhood of each band, we can change the band to a branch point as shown in Figure 15. Hence, we obtain a braided surface $S$ of degree $2m$ from $S_0$. The braided surface $S$ is adequate because the $2m$-braid $\beta_S$ is the composition of adequate $2m$-braids $\gamma^{-1}$, $\delta$, $\delta'$, and $\gamma'^{-1}$.

Step 3: Finally, we show that the surface-link $F$ is equivalent to the plat closure $\bar{S}$ of $S$.

Let $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ and $h : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be the projections onto the first and second factors, respectively. We regard $h$ as a height function of $\mathbb{R}^4$. Let $A$ be the surface of wicket type associated with $S$. Note that $\partial A = \partial S = \partial S_0$. Let $F_0 = S_0 \cup A$. Then $F_0$ is a surface-link equivalent to $\bar{S} = S \cup A$. Thus we show that $F_0$ and $F$ are equivalent.
By an ambient isotopy of $\mathbb{R}^4$ keeping $\mathbb{R}^3 \times (t_0, t_7)$ fixed pointwise, we deform $F_0$ to a surface-link $F_1$ such that

\[
p(F_1 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} 
D_1 & (t = t_7), \\
\overline{\alpha_2} & ((t_5 + t_6)/2 < t < t_7), \\
\alpha_2 \cup |B_1^+| & (t = (t_5 + t_6)/2), \\
p(F_0 \cap \mathbb{R}^3 \times \{t\}) & ((t_1 + t_2)/2 < t < (t_5 + t_6)/2), \\
\overline{\alpha_2} \cup |B_1^+| & (t = (t_1 + t_2)/2), \\
\alpha_2 & (t_0 < t < (t_1 + t_2)/2), \\
d_1 & (t = t_0), \\
\emptyset & \text{otherwise},
\end{cases}
\]

where $|B_1^-|$ (resp. $|B_1^+|$) is the union of the bands belonging to $B_1^-$ (resp. $B_1^+$), and $d_1$ (resp. $D_1$) is a union of mutually disjoint $m$ 2-disks in $\mathbb{R}^3$ bounded by $\alpha_1$ (resp. $\alpha_2$) such that $d_1$ (resp. $D_1$) is disjoint from $|B_1^-|$ (resp. $|B_1^+|$) as in the left of Figure 16 except for the attaching arcs of the bands, respectively.

Next, we define a surface-link $F_2$ in $\mathbb{R}^4$ by

\[
p(F_2 \cap \mathbb{R}^3 \times \{t\}) = \begin{cases} 
D_2 & (t = t_7), \\
\overline{\alpha_2} & ((t_5 + t_6)/2 \leq t < t_7), \\
p(F_0 \cap \mathbb{R}^3 \times \{t\}) & ((t_1 + t_2)/2 < t < (t_5 + t_6)/2), \\
\overline{\alpha_1} & (t_0 < t \leq (t_1 + t_2)/2), \\
d_2 & (t = t_0), \\
\emptyset & \text{otherwise},
\end{cases}
\]

where $d_2 = d_1 \cup |B_1^-|$ (resp. $D_2 = D_1 \cup |B_1^+|$) is the union of mutually disjoint $c_1$ (resp. $c_2$) 2-disks as in the right of Figure 16, respectively.

Then, $F_2$ is obtained from $F_1$ by cellular moves (cf. [18]) along 3-cells $|B_1^-| \times [t_0, (t_1 + t_2)/2] \cup |B_1^+| \times [(t_5 + t_6)/2, t_7]$. This implies that $F_1$ and $F_2$ are equivalent.
Figure 16. $d_2$ is the union of $d_1$ and $|B_1|$. 

Note that $F_2 \cap \mathbb{R}^2 \times \{t_0\} = d_2 \times \{t_0\}$ is the union of all minimal disks of $F$ with respect to the height function $h$, $F_2 \cap \mathbb{R}^2 \times \{t_1\} = D_2 \times \{t_1\}$ is the union of all maximal disks of $F$, and all saddle bands of $F$ appear at $t = (t_3 + t_4)/2$ as bands belonging to $B_1$. By Lemma 4.3, $F_2$ is equivalent to a closed realizing surface of the banded link $(\tilde{\beta}_1, B_1)$.

Since $(L_0, B_0)$ is ambient isotopic to $(\tilde{\beta}_1, B_1)$ and $F$ is equivalent to a closed realizing surface of $(L_0, B_0)$, we see that $F_2$ is equivalent to $F$. □

Next, we show Theorem 1.2. We define the $2m$-braid $\Delta_m$ by $\Delta_1 = Q_2 \times I$ and $\Delta_m = \prod_{k=1}^{m-1} (\sigma_2 \sigma_2 \cdots \sigma_2 \sigma_1)$ for $m \geq 2$. See Figure 17.

Figure 17. The $2m$-braids $\Delta_m$ ($m = 1, 2, 3$).

Note that the closure of an $m$-braid $b$ is equivalent to the plat closure of a $2m$-braid $\Delta_m \iota_m^2(b) \Delta_m^{-1}$. See Figures 18 and 19.

Figure 18. An isotopic deformation of $\Delta_m$ with the standard wicket configuration $w_0$ to a configuration of wickets appearing in a closed braid form (Figure 1).
Proof of Theorem 1.2. Let $F$ be an orientable surface-link. By Proposition 2.7, there exists a 2-dimensional braid $S$ in $D_1 \times D_2 = D_1 \times I \times I$ whose closure in $\mathbb{R}^4$ is equivalent to $F$. Let $m$ be the degree of $S$ and $S_{[t]}$ the cross-section $S \cap D_1 \times (I \times \{t\})$ for each $t \in I$. See Figure 7 when $m = 3$. Let $S_1$ be the 2-dimensional braid of degree $2m$ obtained from $S$ by adding trivial $m$ sheets.

Let $\varepsilon$ be a positive number and let $D'_2 = I \times [-\varepsilon, 1+\varepsilon]$. We consider a 2-dimensional braid $S_2$ of degree $2m$ in $D_1 \times D'_2 = D_1 \times (I \times [-\varepsilon, 1+\varepsilon])$ with a motion picture $(S_2)_{[t]}$ as in Figure 20. Here, the motion picture $(S_2)_{[t]}$ for $t \in [-\varepsilon, 0]$ (or $t \in [1, 1+\varepsilon]$) is the 1-parameter family of $2m$-braids changing $1_{2m}$ to $\Delta_{m} \Delta_{m}^{-1}$ (or $\Delta_{m} \Delta_{m}^{-1}$ to $1_{2m}$), respectively, and the motion picture $(S_2)_{[t]}$ for $t \in I = [0, 1]$ is the composition of $\Delta_{m}$, $(S_1)_{[t]}$ and $\Delta_{m}^{-1}$.

As a result, the plat closure of $S_2$ has the motion picture as in Figure 21. By comparing Figure 7 and Figure 21, we see that the closure of $S$ is equivalent to the plat closure of $S_2$. Hence, $F$ has a genuine plat form presentation. □

Remark 4.7. In Lemma 4.6, each subcylinder $U_i$ contains a part of a banded link as in the left of Figure 22. However, we may assume that for each $i$, 

\[ \begin{array}{c}
\text{Figure 19. A transformation from the closure of } b \text{ to the plat closure of } \Delta_m \epsilon_m^2(b) \Delta_m^{-1} (m = 3). \\
\end{array} \]
the band in $U_i$ is either as in the left or as in the right of Figure 22. Then we have another braided surface in the proof of Theorem 1.1, where the corresponding branch point changes the sign. (A branch point of a braided surface is positive (or negative) if the local monodromy is a conjugate of a standard generator (or its inverse), cf. [12, 13]).

**Figure 22.** Two types of half-twisted bands in a subcylinder $U_i = d_i \times [s_i, t_i]$.

5. THE PLAT INDEX OF A SURFACE-LINK AND EXAMPLES

In this section, we introduce two surface-link invariants called the plat index and the genuine plat index.

**Definition 5.1.** Let $F$ be a surface-link. The *plat index of $F$*, denoted by $\text{Plat}(F)$, is defined as the half of the minimum degree of all adequate braided surfaces whose plat closures are equivalent to $F$:

$$\text{Plat}(F) = \frac{1}{2} \min \{ \deg S/2 \mid S \text{ is a braided surface with } \tilde{S} \cong F \}.$$

**Definition 5.2.** Let $F$ be a surface-link. If $F$ admits a genuine plat form, the *genuine plat index of $F$*, denoted by $g\text{.Plat}(F)$, is defined as the half of the minimum degree of all 2-dimensional braids whose plat closures are
equivalent to \( F \). If \( F \) does not admit a genuine plat form, it is defined as infinity:

\[
g_{\text{Plat}}(F) = \begin{cases} \min \{ \deg S/2 \mid S \text{ is a 2-dimensional braid with } S \cong F \} , \\ \infty & \text{if } F \text{ admits no genuine plat forms.} \end{cases}
\]

By definition, it holds that \( \text{Plat}(F) \leq g_{\text{Plat}}(F) \) for every surface-link \( F \). Moreover, from the proof of Theorem \([1,2]\) we have the following proposition.

**Proposition 5.3.** The following inequalities hold for every orientable surface-link \( F \):

\[
\text{Plat}(F) \leq g_{\text{Plat}}(F) \leq \text{Braid}(F).
\]

In the rest of this section, we show some examples of surface-links in plat forms and discuss the plat index and the genuine plat index.

We first recall the notion of a braid system of a braided surface. Refer to \([12]\) for more details. Let \( \text{pr}_i : D_1 \times D_2 \to D_i \) \((i = 1, 2)\) be the projection and \( C_n \) the configuration space of \( n \) points of \( \text{Int} D_1 \). Let \( S \) be a braided surface of degree \( n \), and \( \Sigma(S) \) the branch locus of \( \pi_n = \text{pr}_2|_S : S \to D_2 \). Let \( y_0 \in \partial D_2 \) be a fixed base point.

The **braid monodromy** of \( S \) is a homomorphism \( \rho_S : \pi_1(D_2 \setminus \Sigma(S), y_0) \to \pi_1(C_n, Q_n) = B_n \) defined as follows: For a loop \( c : (I, \partial I) \to (D_2 \setminus \Sigma(S), y_0) \), define a loop \( \rho_S(c) : (I, \partial I) \to (C_n, Q_n) \) as \( \rho_S(c)(t) = \text{pr}_1(\pi_n^{-1}(c(t))) \). Then the braid monodromy of \( S \) is defined as a group homomorphism sending \( [c] \) to \( [\rho_S(c)] \in \pi_1(C_n, Q_n) \).

Let \( r \) be a positive integer. A **Hurwitz arc system** in \( D_2 \) (with the base point \( y_0 \)) is an \( r \)-tuple \( \mathcal{A} = (\alpha_1, \cdots, \alpha_r) \) of oriented simple arcs in \( D_2 \) such that

1. for each \( i \), \( \alpha_i \cap \partial D_2 = \partial \alpha_i \cap \partial D_2 = \{ y_0 \} \) and this is the terminal point of \( \alpha_i \),
2. for \( i \neq j \), \( \alpha_i \cap \alpha_j = \{ y_0 \} \), and
3. \( \alpha_1, \ldots, \alpha_r \) appear in this order around the base point \( y_0 \).

The set of initial points of \( \alpha_1, \ldots, \alpha_r \) is called the **starting point set** of \( \mathcal{A} \).

Let \( \mathcal{A} = (\alpha_1, \cdots, \alpha_r) \) be a Hurwitz arc system with the starting point set \( \Sigma(S) \). For each \( i \), let \( N_i \) be a (small) regular neighborhood of the starting point of \( \alpha_i \), \( \overline{\alpha_i} \) an oriented arc obtained from \( \alpha_i \) by restricting to \( D_2 \setminus \text{Int} N_i \), and \( \gamma_i \) a loop \( \overline{\alpha_i}^{-1} \cdot \partial N_i \cdot \overline{\alpha_i} \) in \( D_2 \setminus \Sigma(S) \) with base point \( y_0 \). Here, \( \partial N_i \) is oriented counter-clockwise. Then \( \pi_1(D_2 \setminus \Sigma(S), y_0) \) is generated by \( [\gamma_1], [\gamma_2], \ldots, [\gamma_r] \) and we have \( \overline{\partial D_2} = [\gamma_1] \cdots [\gamma_r] \). The **braid system** of \( S \) associated with \( \mathcal{A} \) is an \( r \)-tuple \( b_S \) of elements of \( B_n \) defined as

\[
b_S = (\rho_S([\gamma_1]), \ldots, \rho_S([\gamma_r])) \in (B_n)^r.
\]
It is known that $\rho_S(\gamma_i)$ is a conjugation of a standard generator or its inverse, $\sigma_1^\epsilon$ ($\epsilon \in \{\pm 1\}$), such that $\epsilon$ is the sign of the branch point which is the starting point of $\alpha_i$. The composition $\rho_S(\gamma_1)\rho_S(\gamma_2)\cdots\rho_S(\gamma_r)$ is equal to $\beta_S$ in $B_n$.

The slide action of the braid group $B_r$ on $(B_n)'$ is a left group action defined as

$$\text{slide}(\sigma_j)(\beta_1, \ldots, \beta_r) = (\beta_1, \ldots, \beta_{j-1}, \beta_j\beta_{j+1}\beta_j^{-1}, \beta_j, \beta_{j+2}, \ldots, \beta_r)$$

for $\sigma_j \in B_r$ and $(\beta_1, \ldots, \beta_r) \in (B_n)'$. Two elements of $(B_n)'$ are said to be Hurwitz equivalent if they are in the same orbit of the slide action of $B_r$.

**Lemma 5.4** (cf. [12, 17, 20]). Two braided surfaces in $D_1 \times D_2$ are equivalent if and only if their braid systems are Hurwitz equivalent.

Let $e(F)$ be the normal Euler number of a surface-knot $F$. The normal Euler number of any orientable surface-knot is 0, and the normal Euler number of a trivial non-orientable surface-knot, which is a connected sum of $p$ copies of $P_+$ and $q$ copies of $P_-$, is $2(p - q)$ (cf. [4, 8]).

**Theorem 5.5.** Let $F$ be a surface-link.

1. $\text{Plat}(F) = 1$ if and only if $F$ is either a trivial 2-knot or a trivial non-orientable surface-knot.
2. $g \cdot \text{Plat}(F) = 1$ if and only if $F$ is either a trivial 2-knot or a trivial non-orientable surface-knot with $e(F) = 0$.
3. If $F$ is a trivial orientable surface-knot with positive genus, then $\text{Plat}(F) = g \cdot \text{Plat}(F) = 2$.

**Proof.** (1) Let $F$ be a surface-link with $\text{Plat}(F) = 1$ and $S$ a braided surface of degree 2 with $\bar{S} \cong F$. Let $p$ and $q$ be the numbers of positive and negative branch points of $S$, respectively. Then a braid system for $S$ is Hurwitz equivalent to $(\sigma_1, \ldots, \sigma_1, \sigma_1^{-1}, \ldots, \sigma_1^{-1})$ consisting of $p$ $\sigma_1$'s and $q$ $\sigma_1^{-1}$'s. Hence, the equivalence class of $S$ is determined from $p$ and $q$. Figure 23 is a motion picture of the plat closure of a braided surface of degree 2 with $p$ positive branch points and $q$ negative branch points. The motion picture describes a trivial 2-knot if $p = q = 0$ holds, otherwise, it describes a connected sum of $p$ copies of $P_+$ and $q$ copies of $P_-$. Therefore, $F$ is either a trivial 2-knot or a trivial non-orientable surface-knot.

Conversely, if $F$ is a trivial 2-knot, then $\text{Plat}(F) = 1$. If $F$ is a trivial non-orientable surface-knot, then $F$ is equivalent to a surface-knot described in Figure 23 and hence $\text{Plat}(F) = 1$.

(2) Let $F$ be a surface-link with $g \cdot \text{Plat}(F) = 1$ and $S$ a 2-dimensional braid of degree 2 with $\bar{S} \cong F$. Since $S$ is a 2-dimensional braid, the number of positive branch points of $S$, denoted by $p$, is equal to the number of negative ones. Hence, the argument in the proof of (1) implies that $F$ is
equivalent to a trivial 2-knot, when \( p = 0 \), or a connected sum of \( p \) copies of \( P_+ \) and \( p \) copies of \( P_- \). In particular, it holds that \( e(F) = 0 \).

Conversely, if \( F \) is a trivial 2-knot, then \( g \cdot \text{Plat}(F) = 1 \). If \( F \) is a trivial non-orientable surface-knot with \( e(F) = 0 \), then \( F \) is a connected sum of \( p \) copies of \( P_+ \) and \( p \) copies of \( P_- \) for some \( p > 0 \), which is equivalent to a surface-knot described in Figure 23 with \( p = q \). Hence \( g \cdot \text{Plat}(F) = 1 \).

(3) Let \( F \) be a trivial orientable surface-knot with a positive genus. Since \( \text{Braid}(F) = 2 \) (cf. [9, 12]), by Proposition 5.3, we have \( \text{Plat}(F) \leq g \cdot \text{Plat}(F) \leq 2 \). On the other hand, by (1), it holds that \( \text{Plat}(F) \neq 1 \). Hence, we have \( \text{Plat}(F) = g \cdot \text{Plat}(F) = 2 \). (Figure 24 shows a motion picture of a genuine plat form of \( F \).)

\[ \begin{array}{c}
\text{Figure 23. A surface-knot in a (normal) plat form.} \\
\end{array} \]

Proposition 5.6. Let \( F \) be the 2-knot denoted by \( 2,2 \) in the table of [15], which is depicted in Figure 25. Then \( \text{Plat}(F) = g \cdot \text{Plat}(F) = 2 \).

Proof. Figure 26 shows a deformation of a banded link by an isotopy of \( \mathbb{R}^3 \). Using the isotopy, we see that \( F \) is equivalent to a surface-knot in a genuine plat form depicted in Figure 27. Hence, we have the inequality \( \text{Plat}(F) \leq g \cdot \text{Plat}(F) \leq 2 \). Since \( F \) is not a trivial 2-knot, we have \( \text{Plat}(F) = g \cdot \text{Plat}(F) = 2 \). □

The braid index of every non-trivial surface-knot is greater than 2 ([9]). Hence, \( 2,2 \) is an example such that the equality in \( g \cdot \text{Plat}(F) \leq \text{Braid}(F) \) in Proposition 5.3 does not hold.

\[ \begin{array}{c}
\text{Figure 24. A trivial orientable surface-knot with a positive genus in a genuine plat form.} \\
\end{array} \]
A surface-link is said to be \textit{ribbon} if it is obtained from a trivial 2-link by some 1-handle surgeries.

\textbf{Theorem 5.7.} Let $F$ be a 2-knot (or a surface-link with $\chi(F) = 2$) with $g.\text{Plat}(F) = 2$. Then, $F$ is ribbon.

\textit{Proof.} Let $S$ be a 2-dimensional braid of degree 4 with $\overline{S} \simeq F$, and $r$ the number of branch points of $S$. Since $\chi(F) = 2$, we see that $r = 2$ from $\chi(\overline{S}) = 4 - r$. Let $b_S = (\beta_1, \beta_2) \in (B_4)^2$ be a braid system of $S$. Since $S$ is a 2-dimensional braid, $\beta_S = \beta_1 \beta_2 = 1$ in $B_4$, i.e., $\beta_2 = \beta_1^{-1}$. A 2-dimensional braid with a symmetric braid system $(\beta_1, \beta_1^{-1})$ is known as a ribbon 2-dimensional braid (\cite{12}), which is equivalent to a 2-dimensional braid $S'$ in $D_1 \times D_2 = D_1 \times (I \times [0,1])$ such that $S'$ is symmetric with respect to $t = 1/2$. Then the plat closure of $S'$ is symmetric with respect to $t = 1/2$ and it is in a normal form in the sense of \cite{16}. Hence $\overline{S}'$ is ribbon (cf. Theorem 11.4 of \cite{12}). Since $F \simeq \overline{S}'$ and $\overline{S} \simeq \overline{S}'$, $F$ is ribbon. \qed
Proposition 5.8. Let $k(n)$ be the twist knot ($n \in \mathbb{Z}$) and $F(n)$ the 2-twist spin of $k(n)$ (23). Then $\text{Plat}(F(n)) = 2$ holds for $n \neq 0$.

Proof. The 2-knot $F(n)$ has a motion picture described in [14] as in Figure 28, where $m = 2n + 1$ and a box labeled by $m$ contains $m$ positive half-twists or $-m$ negative half-twists for $m < 0$. Since the trivial link depicted in (4) of Figure 28 is the plat closure of an adequate braid of degree 4, this motion picture gives us a (normal) plat form presentation for $F(n)$. On the other hand, it is known that $F(n)$ is a non-trivial 2-knot if $n \neq 0$. Hence, we have that $\text{Plat}(F(n)) = 2$. □

Figure 28. The 2-knot $F(n)$ in a plat form ($m = 2n + 1$).

Furthermore, it is known that $F(n)$ is not a ribbon 2-knot for $n \neq 0$ (6). By Theorem 5.7, the genuine plat index of $F(n)$ is greater than 2. Thus, Proposition 5.8 gives us examples of 2-knots such that the equality in $\text{Plat}(F) \leq g \cdot \text{Plat}(F)$ in Proposition 5.3 does not hold.

A $P^2$-link is a surface-link whose components are projective planes. Replacing $m = 2n + 1$ (or $-m = -2n - 1$) crossings in Figure 28 with $2n$ (or $-2n$) crossings, respectively, we have a 2-component $P^2$-link in a plat form. In particular, in the case of $n = 1$, the $P^2$-link is a $P^2$-link denoted by $8_{1, -1}$ in Yoshikawa’s table (22).

Proposition 5.9. Let $F$ be a surface-link in a genuine plat form. Each component of $F$ is a surface-knot whose normal Euler number is zero.

Proof. Each connected component of $F$ is regarded as a surface-knot in a genuine plat form by forgetting other components of $F$. Thus it is sufficient to show that $e(F) = 0$ for a surface-knot $F$ in a genuine plat form.

For a (broken surface) diagram $D$ of $F$ (cf. 15), let $b_+(D)$ (resp. $b_-(D)$) be the number of positive (resp. negative) branch points of $D$. Then, the normal Euler number $e(F)$ is equal to $b_+(D) - b_-(D)$.

When $F = S$ is in a genuine plat form, taking a diagram suitably, positive (resp. negative) branch points of $S$ (in the sense of a 2-dimensional braid)
correspond to positive (resp. negative) branch points of $D$, and vice versa. Since $S$ is a 2-dimensional braid, the number of positive branch points of $S$ and that of negative branch points of $S$ are the same. Thus we have $e(F) = b_+(D) - b_-(D) = 0$. □

It is unknown to the author whether every surface-link consisting of surface-knots whose normal Euler numbers are zero is equivalent to a surface-link in a genuine plat form.

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