INVERSE PROBLEMS FOR EVOLUTION EQUATIONS WITH TIME DEPENDENT OPERATOR-COEFFICIENTS

MOHAMMED AL HORANI
Department of Mathematics, The University of Jordan (Sabbatical Year)
Amman, Jordan
Current Address: Department of Mathematics, Faculty of Science
University of Hail, Saudi Arabia

ANGELO FAVINI
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato, 5 – 40126 Bologna, Italy

HIROKI TANABE
Takarazuka, Hirai Sanso 12-13, 665-0817, Japan

Abstract. In this paper we study an inverse problem with time dependent operator-coefficients. We indicate sufficient conditions for the existence and the uniqueness of a solution to this problem. A number of concrete applications to partial differential equations is described.

1. Introduction. In the last years, many researches were devoted to identification problems for linear and nonlinear equations in Banach spaces, see [2]-[7], [10]-[18] and [22]. In particular, these identification problems, were concerned with degenerate differential equations in Banach spaces, of the type

\[ \frac{d}{dt}(My(t)) = Ly(t) + f(t)z + h(t), \quad 0 \leq t \leq \tau, \]

\[ (My)(0) = My_0, \]

\[ \Phi[My(t)] = g(t), \quad 0 \leq t \leq \tau, \]

where \( L, M \) are closed linear operators in the Banach space \( X \), \( h \in C([0, \tau]; X) \), \( z \in X \), \( y_0 \in D(M) \), \( g \in C([0, \tau]; \mathbb{C}) \), \( \Phi \in X^* \) are given and the unknowns are \( y \in C([0, \tau]; D(L)) \), \( f \in C([0, \tau]; \mathbb{C}) \) such that \( My \in C^1([0, \tau]; X) \). Note that \( M \) may have no bounded inverse. Such a problem was faced in both parabolic and hyperbolic cases, see [5]-[6] and [10]-[18]. Very recently, the authors in [15] have discussed the existence and regularity of solutions to the following time dependent coefficients problem in \( X = L^p(\Omega) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \geq 1 \), \( 1 < p < \infty \), with a sufficiently smooth boundary \( \partial \Omega \),

\[ \frac{d}{dt}(M(t)y(t)) = \alpha(t)L(t)y(t) + f(t)z + h(t), \quad 0 \leq t \leq \tau, \]

\[ M(0)y(0) = x_0 \]

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with \( \alpha(t) \) possibly non-negative, but such general hypotheses do not guarantee, in particular, that \( y \) is an \( X \)-valued continuous solution on \([0, \tau]\). To handle the related inverse problems, more regular solutions seem necessary.

In this paper, we are concerned to give a first treatment to the inverse problem with time dependent coefficients

\[
\frac{d}{dt} y(t) = A(t) y(t) + f(t) z + h(t), \quad 0 \leq t \leq \tau, \\
y(0) = y_0, \\
\Phi[y(t)] = g(t), \quad 0 \leq t \leq \tau.
\] (1.1)

It is well known that problems like (1.1)-(1.3) can be handled directly via fixed point argument. Recently in [14], the time independent coefficients case was faced by reducing the problem to a direct problem with perturbed operators \( A + B \). Clearly, one can tempt to extend this strategy to the general case, but the problem is to introduce assumptions guaranteeing that \( A(t) + B(t) \) has the required properties which govern a well posed evolution equation.

In order to solve (1.1)-(1.3) according to this strategy, we apply \( \Phi \in \mathcal{X}^* \) to both sides of equation (1.1) to obtain, by using (1.3), under the additional assumption \( g \in C^1([0, \tau]; \mathbb{C}) \)

\[
g'(t) = \Phi[A(t)y(t)] + f(t) \Phi[z] + \Phi[h(t)].
\] (1.4)

Suppose \( \Phi[z] \neq 0 \); then necessarily

\[
f(t) = \frac{g'(t) - \Phi[A(t)y(t)] - \Phi[h(t)]}{\Phi[z]}.
\] (1.5)

Substituting (1.5) in (1.1), we obtain the following new direct problem to be solved in the variable \( y(\cdot) \)

\[
\frac{d}{dt} y(t) = A(t) y(t) - \Phi[A(t)y(t)] z - \Phi[h(t)] z + \frac{g'(t)}{\Phi[z]} z + h(t), \quad 0 \leq t \leq \tau, \\
y(0) = y_0.
\]

Introduce operator \( B(t) : D(B(t)) = D(A(t)) \to \mathcal{X} \)

\[
B(t)y = -\frac{\Phi[A(t)y]}{\Phi[z]} z, \quad y \in D(B(t));
\]

all is reduced to existence, uniqueness and regularity of a solution \( y(\cdot) \) to the problem

\[
\frac{d}{dt} y(t) = (A(t) + B(t)) y(t) + g'(t) - \Phi[h(t)] z + h(t), \quad 0 \leq t \leq \tau, \\
y(0) = y_0.
\]

The essential role of our results is to guarantee that the perturbed operator \( A(t) + B(t) \) has a well appropriate behavior. To this point, the results from [9] will play an essential role in the hyperbolic case. To motivate this choice of using perturbation methods, for sake of simplicity, we confine to the parabolic case, cfr. [23]. Then for a given continuous function \( f : [0, \tau] \to \mathbb{C} \), the strict solution to the problem

\[
\frac{d}{dt} y(t) = A(t) y(t) + f(t) z, \quad 0 \leq t \leq \tau, \\
y(0) = y_0,
\]
necessarily has the form
\[ y(t) = U(t,0)y_0 + \int_0^t U(t,s)zf(s)ds, \]
where \( U(t,s), \ 0 \leq s \leq t \leq \tau \) is the evolution operator generated by \( A(t) \). Notice that under suitable regularity assumptions on the data, at least formally,
\[ \frac{d}{dt}y(t) = A(t)U(t,0)y_0 + f(t)z + \int_0^t A(t)U(t,s)zf(s)ds \]
so that
\[ \Phi[y'(t)] = \Phi[A(t)U(t,0)y_0] + f(t)\Phi[z] + \int_0^t \Phi[A(t)U(t,s)z]f(s)ds. \]
Then all is reduced to find a continuous function \( f(t) \) on \([0, \tau]\) such that, assuming \( g \in C^1([0, \tau]; \mathbb{C}) \),
\[ g'(t) = \Phi[z]f(t) + \Phi[A(t)U(t,0)y_0] + \int_0^t \Phi[A(t)U(t,s)z]f(s)ds \quad (1.6) \]
and thus the problem becomes the one to solve the integral equation (1.6) globally. This is not an obvious task, as it can be seen from the monograph [8]. Therefore, it seems that the proposed perturbation argument yields more quickly the desired results.

As previously remarked, solutions to the integro-differential equations of parabolic type and possibly degenerate of the form
\[ \frac{d}{dt}M(t)y(t) = L(t)y(t) + \int_0^t K(t,s)y(s)ds + f(t), \quad 0 \leq t \leq \tau, \]
have been, very recently, discussed in the paper [15], where \( L(t), K(t,s) \) are second order differential operators, \( M(t) \) is the multiplication operator by \( m(t,x) \geq 0 \) on the ambient space \( L^p(\Omega), \ 0 \leq t \leq \tau \). Theorem 9.1 in [15] on existence and uniqueness of solutions implies only that \( L(\cdot)y(\cdot) \in C((0, \tau]; L^p(\Omega)) \) and thus the related treatment of the inverse problem seems to be not an easy consequence, because more regularity to the solution must be assumed.

The contents of the paper are as follows. In Section 2 we recall some results of perturbation that we need. Section 3 contains the main results. In Section 4, some applications of partial differential equations are given to illustrate our abstract results.

2. Preliminaries. We start this section by recalling some definitions

**Definition 2.1.** (Stability) A family \( \{A(t)\}_{t \in [0,T]} \) of generators of \( C_0 \)–semigroups on the Banach space \( X \), with stability constants \( M \geq 0, \omega \in \mathbb{R} \) is stable in \( X \) if
(i) \( \rho(A(t)) \supseteq (\omega, \infty) \) for each \( t \in [0,T] \)
(ii) for each \( \lambda > \omega \) and for each finite sequence \( 0 \leq t_1 \leq ... \leq t_k \leq T, \ k = 1, 2, ... \) we have
\[ \left\| \prod_{i=1}^k (\lambda I - A(t_i))^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k} \]
where
\[ \prod_{i=1}^k (\lambda I - A(t_i))^{-1} = (\lambda I - A(t_k))^{-1} ...(\lambda I - A(t_1))^{-1}. \]
Hypothesis (D) (Constant Domain) \((D; \| \cdot \|_D)\) is a Banach space, densely and continuously embedded in \(X\) for all \(0 \leq t \leq T\), i.e., \(D(A(t)) = D\) for all \(t \in [0, T]\).

Of course, similarly, one can introduce stability of the restriction of \(A(t)\) to a subspace \(D\) of \(X\). The Favard class \(F\) of a generator of a \(C_0\)-semigroup in \(X\) is the interpolation space \((X, D)_1, \infty\). It is known that if \(X\) is reflexive, then the Favard class reduces to \(D\).

**Definition 2.2.** (CD-Systems) Let \(X\) be a separable Banach space. Then the triple
\[
\{\{A(t)\}_{t \in [0, T]}, X, D\}
\]
is a CD-system if \(\{A(t)\}_{t \in [0, T]}\) is a family of generators of \(C_0\)-semigroups in \(X\) and
(a) Hypothesis (D) is satisfied;
(b) the family \(\{A(t)\}_{t \in [0, T]}\) is stable with stability constants \(M\) and \(\omega\);
(c) the family \(\{A(t)\}_{t \in [0, T]}\) is Lipschitz continuous in the sense that \(A(t)\) is strongly Lipschitz continuous from \([0, T]\) into \(\mathcal{L}(D; X)\).

Theorem 4 in [9], p. 1197, affirms the result as follows

**Proposition 2.3.** Let \(\{A(t)\}_{t \in [0, T]}\) be a stable family of \(C_0\)-semigroups on \(X\) with stability constants \(M\), \(\omega\) and let (D) holds. Assume
\(\exists \mu \in \mathbb{R}\), such that for all \(t \in [0, T]\), we have \(\mu \in \rho(A(t))\) and
\[
\sup_{t \in [0, T]} \| (\mu I - A(t))^{-1} \|_{\mathcal{L}(X, D)} < \infty,
\]
\(b) t \mapsto A(t) \in BV([0, T], \mathcal{L}(D, X))\), the space of functions of bounded variation,
\(c) \sup_{t \in [0, T]} \| B(t) \|_{\mathcal{L}(D, F)} < \infty\) and \(t \mapsto B(t) \in BV([0, T], \mathcal{L}(D, X))\).

Then \(\{A(t) + B(t)\}_{t \in [0, T]}\) is a stable family of generators of \(C_0\)-semigroups on \(X\).

In particular, see [9], Proposition 6,

**Proposition 2.4.** (CD-Systems) Let \(X\) be a separable Banach space and let \(\{(A(t))_{t \in [0, T]}, X, D\}\) be a CD-system. If \(B(t)\) belongs to \(L^\infty([0, T], \mathcal{L}(D, F)) \cap \text{Lip}([0, T], \mathcal{L}(D, X))\), then \(\{(A(t) + B(t))_{t \in [0, T]}, X, D\}\) is a CD-system.

To solve the problem at all, we need the following essential well known result, precisely, Theorem 5.3, p. 147 in [21], related to the Cauchy problem in a Banach space \(X\) affirms that if \(\{A(t)\}_{t \in [0, T]}\) is a stable family of generators, with a constant domain \(D\), such that \(\forall y \in D\), \(A(t)y\) is continuously differentiable, then for all \(y_0 \in D\) and \(f \in C^1([0, T]; X)\), the Cauchy problem
\[
\begin{align*}
\frac{d}{dt} y(t) &= A(t)y(t) + f(t), & 0 \leq t \leq T, \\
\Phi(y(0)) &= y_0,
\end{align*}
\]
admits a unique strict solution \(y(\cdot)\).

3. **Main results.** As we noted in the introduction, the inverse problem to find \(y \in C([0, \tau]; D), f \in C([0, \tau]; \mathbb{C})\) such that
\[
\begin{align*}
\frac{d}{dt} y(t) &= A(t)y(t) + f(t)z + h(t), & 0 \leq t \leq \tau, \\
y(0) &= y_0, \\
\Phi[y(t)] &= g(t) \in C([0, \tau]; \mathbb{C}), & 0 \leq t \leq \tau
\end{align*}
\]
under the assumption that \( \{A(t)\}_{t \in [0,T]} \) is a stable family of generators of \( C_0 \)-semigroups with constant domain \( D \), is reduced to the direct problem of finding a strict solution \( y \) to the initial value problem

\[
\frac{d}{dt} y(t) = (A(t) + B(t)) y(t) + \frac{g'(t)}{\Phi[z]} z - \frac{\Phi[h(t)]}{\Phi[z]} z + h(t), \quad 0 \leq t \leq \tau, \tag{3.4}
\]

\[
y(0) = y_0, \tag{3.5}
\]

where

\[
B(t)x = -\frac{\Phi[A(t)x]}{\Phi[z]} z, \quad x \in D, \quad 0 \leq t \leq \tau. \tag{3.6}
\]

Let us assume that for all \( x \in D, A(t)x \) is continuously differentiable. Take \( F = D \) and assume a) in Proposition 2.3. We have

**Theorem 3.1.** Under the previous assumptions on \( A(t) \), problem (3.4)-(3.6) admits a unique strict solution \( y(\cdot) \), provided that \( y_0 \in D, g \in C^2([0,\tau];\mathbb{C}), z \in D, h \in C^1([0,\tau];X), \Phi[z] \neq 0 \).

**Proof.** The proof follows easily from Proposition 2.3 and the recalled existence and uniqueness result from Pazy [21]. \( \square \)

If \( D \) is stable for \( A(t) \), too, Theorem 5.2, p. 146 in [21] allows one to establish the following result on space regularity condition

**Corollary 3.2.** Under the assumptions in Theorem 3.1, if in addition, \( D \) is stable for \( A(t) \), then for any \( y_0 \in D, g \in C^1([0,\tau];\mathbb{C}), \Phi[z] \neq 0, h \in C([0,\tau];D), \) then problem (3.4)-(3.6) admits a unique strict \( D \)-valued solution.

Analogous results are obtained for CD-Systems in a separable Banach space. We omit these related results.

In the sequel, we shall devote ourselves to the parabolic case. Suppose that \( A(t) \) generates an analytic semigroup in the Banach space \( X, D(A(t)) = D \) is independent of \( t, B(t) \in \mathcal{L}(D, X) \) but range \( (B(t)) \in (X, D)_{\rho,\infty} \).

Let \( U(t,s) \) be the fundamental solution to the problem (3.4)-(3.5). Set \( \tilde{A}(t) = A(t) + B(t) \). Suppose that \( z \in (X, D)_{\rho,\infty} \) for some \( \rho \in (0,1), g \in C^1([0,\tau];\mathbb{C}) \) and \( h \in C^\rho([0,\tau];X) \) for some \( \rho \in (0,1) \). Then,

\[
\|U(t,s)z\| \leq C_0(t - s)^{\rho - 1} \|z\|_{(X, D)_{\rho,\infty}}
\]

for some constant \( C_0 \), and the function \( \int_0^t U(t,s)h(s)ds \) is differentiable in \( t, \)

\[
\int_0^t U(t,s)h(s)ds \in D(A(t)) \text{ for each } t \in [0,\tau] \text{ and}
\]

\[
\frac{d}{dt} \int_0^t U(t,s)h(s)ds = h(t) + \tilde{A}(t) \int_0^t U(t,s)h(s)ds.
\]

Let

\[
y(t) = U(t,0)y_0 + \frac{1}{\Phi[z]} \int_0^t g'(s)U(t,s)zds + \int_0^t U(t,s)h(s)ds - \frac{1}{\Phi[z]} \int_0^t \Phi[h(s)]U(t,s)zds.
\]
Then,
\[ y'(t) = \tilde{A}(t)U(t,0)y_0 + \frac{g'(t)}{\Phi[z]}z + \frac{1}{\Phi[z]} \int_0^t g'(s)\tilde{A}(t)U(t,s)zsds \]
\[ + h(t) + \int_0^t U(t,s)h(s)ds - \frac{1}{\Phi[z]} \int_0^t \Phi[h(t)]z - \frac{1}{\Phi[z]} \int_0^t \Phi[h(s)]\tilde{A}(t)U(t,s)zsds. \]

On the other hand
\[ \tilde{A}(t)y(t) = \tilde{A}(t)U(t,0)y_0 + \frac{1}{\Phi[z]} \int_0^t g'(s)\tilde{A}(t)U(t,s)zsds \]
\[ + \tilde{A}(t) \int_0^t U(t,s)h(s)ds - \frac{1}{\Phi[z]} \int_0^t \Phi[h(s)]\tilde{A}(t)U(t,s)zsds. \]

Hence it follows that (3.4) holds. Obviously \( y \) satisfies (3.5). Let \( f \) be the function defined by (1.5). Then \( (y, f) \) satisfies (1.1)-(1.3). If \( y_0 \in D \), then \( f \in C([0, \tau]; \mathbb{C}) \). If \( y_0 \in (X, D)_{\theta, \infty} \) for some \( \theta \in (0, 1) \), \( f \in C([0, \tau]; \mathbb{C}) \) and \( |f(t)| \leq Ct^{-\theta} \) for some constant \( C > 0 \).

We will also discuss the maximal regularity of such solutions. Repeating the arguments in [20] or [15], one can recognize that
\[ \left\| (\lambda - A(t) - B(t))^{-1} \right\|_{\mathcal{L}(X)} \leq C|\lambda|^{-1}, \quad \text{Re} \lambda > 0. \]

Indeed,
\[ \lambda - A(t) - B(t) = (\lambda - A(t)) \left( 1 - (\lambda - A(t))^{-1} B(t) \right) \]
implies, by using Favini and Yagi [19], p. 39, that
\[ (\lambda - A(t) - B(t))^{-1} = (1 - (\lambda - A(t))^{-1} B(t))^{-1} (\lambda - A(t))^{-1} \]
and
\[ \left\| (\lambda - A(t) - B(t))^{-1} \right\|_{\mathcal{L}(X)} \leq C|\lambda|^{-1}, \quad \text{for sufficiently large } |\lambda|. \]

Suppose \( A(\cdot) \in C^\delta([0, \tau], \mathcal{L}(D, X)), g \in C^{1+\delta}([0, \tau]; \mathbb{C}), h \in C^\delta([0, \tau]; X), z \in (X, D)_{\theta, \infty} \).

Then \( B(\cdot) \in C^\delta([0, \tau], \mathcal{L}(D, (X, D)_{\theta, \infty})). \) Hypothesis II of Acquistapace-Terreni, see [1], is satisfied by \( A(t) + B(t) - c \) with \( k = 1, \alpha_1 = \delta, \beta_i = 0 \) for sufficiently large \( c > 0 \). Hence in view of Theorem 6.1 of the above paper for \( \sigma \in (0, \delta], \) \( y_0 \in D \), problem (3.4)-(3.5) has a solution \( y \) such that \( y \in C^{1+\sigma}([0, \tau]; X), Ay \in C^\sigma([0, \tau]; X) \) provided that
\[ A(0)y_0 - \frac{\Phi[A(0)y_0]}{\Phi[z]}z + \frac{g'(0)}{\Phi[z]}z - \frac{\Phi[h(0)]}{\Phi[z]}z + h(0) \in (X, D)_{\sigma, \infty}. \]

We have

**Theorem 3.3.** Under previous assumptions, \( A(\cdot) \in C^\delta([0, \tau], \mathcal{L}(D, X)), g \in C^{1+\delta}([0, \tau]; \mathbb{C}), h \in C^\delta([0, \tau]; X), z \in (X, D)_{\theta, \infty}. \) If \( \sigma \in (0, \delta], y_0 \in D \), problem (3.1)-(3.3) has a solution \((y, f)\) such that \( y \in C^{1+\sigma}([0, \tau]; X), Ay \in C^\sigma([0, \tau]; X), f \in C^\sigma([0, \tau]; C) \) provided that
\[ A(0)y_0 - \frac{\Phi[A(0)y_0]}{\Phi[z]}z + \frac{g'(0)}{\Phi[z]}z - \frac{\Phi[h(0)]}{\Phi[z]}z + h(0) \in (X, D)_{\sigma, \infty}. \]
4. Applications.

Example 4.1. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with smooth boundary and
$$D(A(t)) = H^2(\Omega) \cap H^1_0(\Omega),$$
$$A(t) = \sqrt{-1} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{i,j}(x,t) \frac{\partial}{\partial x_i} \right),$$
where $\{a_{i,j}(x,t)\}$ is a positive definite symmetric matrix for each $(x,t) \in \overline{\Omega} \times [0,T]$.

Then, if $a_{i,j}, \frac{\partial a_{i,j}}{\partial x_j}, \frac{\partial a_{i,j}}{\partial t}, \frac{\partial^2 a_{i,j}}{\partial x_j \partial t} \in C(\overline{\Omega} \times [0,T]), i,j = 1, \ldots, n$,

Theorem 3.1 is applicable with $X = L^2(\Omega), D = H^2(\Omega) \cap H^1_0(\Omega)$.

Example 4.2. We consider a parabolic system as described in Tanabe [23], pp. 278-284. The space $X$ reduces to $L^p(\Omega), 1 < p < \infty$ where $\Omega$ is a bounded set in $\mathbb{R}^n$ of class $C^2$ and $L(x,t,D_x) = \sum_{\alpha \leq m} a_{\alpha}(x,t)D_x^\alpha$

with the coefficients are regular enough so that $L(x,t,D_x)$ is uniformly strongly elliptic in $\Omega \times [0,T]$. The boundary operators
$$B_j(x,D_x) = \sum_{\beta \leq m_j} b_{j\beta}(x)D_x^\beta, \quad j = 1, \ldots, m/2$$

are introduced, see Tanabe [23], p. 279. Let $A(t)$ be the operator described by
$$A(t)u = L(x,t,D_x)u, \quad \text{in } \Omega, \quad B_j(x,D_x)u = 0, \quad \text{on } \partial \Omega, \quad j = 1, \ldots, m/2.$$

Then $A(t)$ generates an analytic semigroup in $L^p(\Omega)$ satisfying the indicated properties.

We can handle the inverse problem to recover $y$ and $f$ such that
$$\frac{dy}{dt}(t) = A(t)y + f(t)z + h(t), \quad 0 \leq t \leq T,$$
$$y(x,0) = y_0(x),$$
$$\Phi[y(t)] = g(t), \quad 0 \leq t \leq T, \quad \Phi \in (L^p(\Omega))^*$$

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E-mail address: horani@ju.edu.jo, m.alhorani@uoh.edu.sa
E-mail address: favini@dm.unibo.it
E-mail address: bacbx403@jttk.zaq.ne.jp