EQUILIBRIUM MEASURES AND PARTIAL BALAYAGE

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Abstract. We consider the equilibrium problem for an external background potential in weighted potential theory, and show that for a large class of background potentials there is a complementarity relationship between the measure solving the weighted equilibrium problem—the weighted equilibrium measure—and a certain partial balayage measure.

1. Introduction

Solving the equilibrium problem for a given external background potential, tantamount to finding the so-called equilibrium measure of the background potential, is a problem that in recent time has turned out to have various connections to many different fields, two of which are locating the zeros of orthogonal polynomials and finding the eigenvalues of random normal matrices; we refer to [13, 12, 11, 1, 7] for more information. For sake of studying the type of equilibrium measures that can arise in these problems, Balogh and Harnad introduced in [1] the class of superharmonically perturbed Gaussian background potentials of the form

\[ Q(z) = \alpha |z|^2 + U_\nu(z), \]

where \( \alpha > 0 \) and \( \nu \) is a finite positive Borel measure with compact support. Based on some of their results it is clear that there is a strong connection between the weighted equilibrium measure arising from such a background potential and a certain partial balayage operation, a connection which has, to the best of our knowledge, not yet been treated in detail in the literature. It turns out that the measure resulting from the partial balayage operation and the weighted equilibrium measure are in a sense complementary to each other, as is shown below in Theorem 6.4.

This paper is organized as follows: in section 2 we review the fundamental notions and results from weighted potential theory that will be used in the remainder of the article, in section 3 we define partial balayage in full generality and state some of the important properties, such as translational invariance, and in section 4 we discuss one sample background potential treated in detail in [1] that inspired the current work, and give a brief explanation as to from where the idea of the complementarity relationship between partial balayage and the weighted equilibrium measure came. The subsequent two sections are then devoted to proving this complementarity relationship in a bit more general setting: section 5 treats existence and certain properties of the particular partial balayage operation that we shall
employ, while section 6 combines the results from weighted potential theory with those for the partial balayage operation discussed in section 5 to prove the stated complementarity relationship, Theorem 6.4. In the final part of the paper, section 7, we treat the sample case of superharmonically perturbed Gaussian background potentials in full detail as an illustration of how Theorem 6.4 can be employed.

We will in this paper use the following notation:

\( \hat{C} \)  
Riemann sphere

\( D(a,r) \)  
Open disk centered at \( a \in \mathbb{C} \) with radius \( r \)

\( \text{diam}(S) \)  
Diameter of the set \( S \): \( \text{diam}(S) = \sup_{x,y \in S} |x-y| \)

\( m, s \)  
(Planar) Lebesgue measure, arc length measure

\( \delta_p \)  
Dirac measure at a point \( p \in \mathbb{C} \)

\( \mu[S] \)  
Restriction of the (possibly signed) measure \( \mu \) to the set \( S \), extended by zero outside \( S \)

\( f|_S \)  
Restriction of the function \( f \) to the set \( S \)

\( U^\mu \)  
Logarithmic potential of a (signed Radon) measure \( \mu \):

\[ U^\mu(z) = \int \log |z - \zeta|^{-1} \, d\mu(\zeta) \]

\( U^S \)  
Logarithmic potential of the Lebesgue measure restricted to the set \( S \):

\[ U^S = U^m|_S \]

\( \Delta \)  
Laplace operator; \( \Delta = \partial_x^2 + \partial_y^2 \) with \( z = x + iy \in \mathbb{C} \)

\( I(\mu) \)  
Logarithmic energy of \( \mu \):

\[ I(\mu) = \iint \log |z - \zeta|^{-1} \, d\mu(\zeta) \, d\mu(z) = \int U^\mu \, d\mu \]

\( \text{cap}(S) \)  
Logarithmic capacity of the set \( S \)

\( Q \)  
Background potential/external field

\( \mathcal{M}_t(E) \)  
Set of Borel measures \( \mu \) such that \( \mu(\mathbb{C}) = t \)

and \( \text{supp} \mu \subseteq E \)

\( I_{Q,t}(\mu) \)  
Weighted energy of a measure \( \mu \in \mathcal{M}_t(E) \)

\( \mu_{Q,t} \)  
Equilibrium measure with total mass \( t \) of a \((t\text{-admissible})\) background potential \( Q \)

\( F_{Q,t} \)  
Modified Robin constant corresponding to \( \mu_{Q,t} \)

\( \mathcal{S} \)  
Set of subharmonic functions in \( \mathbb{C} \)

\( \mathcal{S}_t \)  
Set of functions \( V(z) \in \mathcal{S} \) that are harmonic near infinity and \( V(z) \leq t \log |z| + O(1) \) for large \( |z| \) (with \( t > 0 \))

\( \text{Bal}(\mu, \lambda) \)  
Partial balayage of \( \mu \) to \( \lambda \)

Whenever we refer to a measure in this article, we always mean a positive measure. Every measure considered here will at the very least be a Borel measure, and most of them will be Radon measures (where, in our setting, Radon simply means that the measure is finite on every compact set; note that a Radon measure with compact support by necessity is a finite measure). We shall often utilize signed Radon measures, which simply are differences of two positive Radon measures. For any such signed Radon measure \( \sigma \) we can decompose it as \( \sigma = \sigma_+ - \sigma_- \), where we take \((\sigma_+, \sigma_-)\) to be the Jordan decomposition of \( \sigma \), i.e. the minimal decomposition into positive measures such that \( \sigma_+ \perp \sigma_- \). Relations between objects of not necessarily the same type (for instance an inequality between a function and a measure) will implicitly always be in the sense of distributions; the same applies to the Laplace operator, which, when applied to functions not necessarily twice differentiable, also always will be taken in the sense of distributions.
2. Weighted Potential Theory and Equilibrium Measures

We closely follow [9]. Let \( E \subseteq \mathbb{C} \) be a subset of the plane; throughout we always assume that \( E \) is a closed set. We denote by \( \mathcal{M}_t(E) \) the set of (positive) Borel measures with support in \( E \) and total mass \( t > 0 \); as a minor remark note that every measure in \( \mathcal{M}_t(E) \) by necessity is a Radon measure.

Remark 2.1. The weighted potential theory as described in [9] is stated in terms of probability measures, i.e. measures with total mass one. As we soon shall see, it will for our purposes be more convenient to allow these measures to have an arbitrary (but finite) total mass \( t \). One can easily translate between the two formulations of the theory, since if we let \( \mu_{Q,t} \) be the equilibrium measure of total mass \( t \) for a \( t \)-admissible background potential \( Q \) (as defined in Definition 2.2 and Proposition 2.5) and \( \hat{\mu}_Q \) be the equilibrium measure of total mass one for the background potential \( \hat{Q} := Q/t \) (as defined in [9]), then \( \mu_{Q,t} = t\hat{\mu}_Q \).

Definition 2.2. A function \( Q : E \to (-\infty, \infty] \) is called a \( t \)-admissible background potential (on \( E \)) for \( t > 0 \) if the following holds:

(i) \( Q \) is lower semicontinuous,
(ii) \( \text{cap}\{z \in E : Q(z) < \infty\} > 0 \), and
(iii) \( Q(z) - t \log |z| \to \infty \) as \( |z| \to \infty \), \( z \in E \) (if \( E \) is unbounded).

Remark 2.3. If the set \( E \) in the above definition is bounded (i.e. if \( E \) is compact) we see that requirement (iii) is empty and that the parameter \( t \) has no influence whatsoever on the admissibility of a background potential \( Q \). Whenever the domain of \( Q \) is compact we therefore for ease of notation simply say that \( Q \) is an admissible background potential if it satisfies properties (i) and (ii) in Definition 2.2. Moreover, if \( Q \) is a \( t \)-admissible background potential for every \( t > 0 \) we say that \( Q \) is a fully admissible background potential.

Definition 2.4. Given a \( t \)-admissible background potential \( Q \) on \( E \) we let \( w(z) := e^{-Q(z)} \) and define the weighted energy \( I_{Q,t}(\mu) \) of \( \mu \in \mathcal{M}_t(E) \) by

\[
I_{Q,t}(\mu) := \frac{1}{t} \int \int \frac{1}{|z - \zeta|^t} w(z) w(\zeta) d\mu(z) d\mu(\zeta)
= \int U \mu \ d\mu + 2 \int Q \ d\mu.
\]

The following existence property, essentially due to Frostman [2], for a solution to the equilibrium problem of an external background potential is an important part of weighted potential theory (see e.g. [8] or [9, Theorem I.1.3] for proofs of the case \( t = 1 \), the relation described in Remark 2.1 easily yields the proposition for arbitrary values of \( t \)):

Proposition 2.5. Let \( Q \) be a \( t \)-admissible background potential on \( E \) for some \( t > 0 \), and define

\[
V_{Q,t} := \inf_{\mu \in \mathcal{M}_t(E)} I_{Q,t}(\mu).
\]
Then $V_{Q,t}$ is finite, and there is a unique measure $Q_{Q,t} \in \mathcal{M}_t(E)$ such that $I_{Q,t}(Q_{Q,t}) = V_{Q,t}$. The measure $Q_{Q,t}$ has compact support, and is called the equilibrium measure of the background potential $Q$.

Determining the equilibrium measure $Q_{Q,t}$ is in applications often an important problem but also one that might be difficult to solve; the following result is quite useful:

**Proposition 2.6.** Let $Q$ be a t-admissible background potential on $E$ for some $t > 0$, and assume that $\mu \in \mathcal{M}_t(E)$ has finite logarithmic energy (i.e. $|I(\mu)| < \infty$) and that $\text{supp} \mu$ is compact. If there is a constant $F$ such that

\[
U^{\mu} + Q \geq F \quad \text{q.e. on } E,
\]

\[
U^{\mu} + Q = F \quad \text{q.e. on } \text{supp} \mu,
\]

then $\mu = Q_{Q,t}$ and $F = F_{Q,t}$, where $F_{Q,t}$ is the modified Robin constant

\[
F_{Q,t} := \frac{1}{t} \left( V_{Q,t} - \int Q \, d\mu_{Q,t} \right).
\]

We refer to [9, Theorem I.3.3] for a proof of the above. For future use we note the following: if we are given a t-admissible background potential $Q$ on some set $E$ and for some reason have prior knowledge that the support of the equilibrium measure $Q_{Q,t}$ is fully contained in some subset $\hat{E} \subset E$, then we can just as well switch to studying the equilibrium problem for the restriction of $Q$ to the set $\hat{E}$ instead; it is readily verified that $\mu|_{\hat{E},t} = Q_{Q,t}$.

In section 6 we are going to utilize the formulation of the problem of determining the potential of the equilibrium measure as a classical obstacle problem, for the sake of showing a relationship between the equilibrium measure and a partial balayage measure. For this purpose the following result, a slight reformulation of Theorem I.4.1 in [9] better suited for our purposes, will be useful:

**Proposition 2.7.** Let $t > 0$ be fixed, and define $\mathcal{S}_t$ to be the set of all subharmonic functions $V(z)$ on $C$ that for large $|z|$ are both harmonic and such that $V(z) \leq t \log|z| + O(1)$, i.e. $V(z) - t \log|z|$ is bounded from above near $\infty$. Then, for a t-admissible background potential $Q$ on a set $E$, the function

\[
F_{Q,t} - U^{\mu_{Q,t}}(z)
\]

is the upper envelope of the functions $V$ in $\mathcal{S}_t$ satisfying $V(z) \leq Q(z)$ for quasi-every $z \in E$.

We are also going to need the so called Principle of Domination (we refer to Theorem II.3.2 in [9] for a proof):

**Proposition 2.8.** Let $\mu$ and $\nu$ be two positive finite Borel measures with compact support on $C$, and suppose that the total mass of $\nu$ does not exceed that of $\mu$. Assume further that $\mu$ has finite logarithmic energy. If, for some constant $c$, the inequality

\[
U^{\mu}(z) \leq U^{\nu}(z) + c
\]

holds $\mu$-almost everywhere, then it holds for all $z \in \mathbb{C}$.
3. Partial Balayage

The concept of \textit{balayage} is well-known in classical potential theory, and is essentially the process of clearing the density of a measure in a prescribed region and altering the measure on the boundary of the region in such a way that the potential of the measure remains unchanged in the exterior of the region in question. \textit{Partial balayage} is a related concept, in the sense that we allow some density to remain after the process, while still demanding that the potential is unchanged in some region. Partial balayage can be defined in a multitude of (albeit often equivalent) ways \cite{5}, and we will in this paper use the following rather broad definition (see also \cite{6, 10, 4, 3}):

\textbf{Definition 3.1.} Let $\mu$ and $\lambda$ be (signed) Radon measures with compact supports. Set

$$\mathcal{F}_\mu^\lambda := \{ V \in D'(C) : V \leq U^\mu \text{ in } C, -\frac{1}{2\pi} \Delta V \leq \lambda \text{ in } C \},$$

where $D'(C)$ denotes the set of distributions on $C$. The set $\mathcal{F}_\mu^\lambda$ is of course highly dependent on the nature of $\mu$ and $\lambda$, but if it is nonempty and contains a largest element $V^\mu \equiv V^\mu_\lambda := \sup \mathcal{F}_\mu^\lambda$ then we define the \textit{partial balayage} of $\mu$ to $\lambda$ to be the signed Radon measure

$$\text{Bal}(\mu, \lambda) := -\frac{1}{2\pi} \Delta V^\mu.$$

\textbf{Remark 3.2.} Let us just note a sometimes very useful sort of translation invariance property of the partial balayage measure. As a consequence of the definition it is immediate that if $\mu$, $\lambda$ and $\sigma$ are suitable measures for which either $\mathcal{F}_\mu^\lambda$ or $\mathcal{F}_{\mu+\sigma}^{\lambda+\sigma}$ is nonempty then by necessity $\mathcal{F}_{\mu+\sigma}^{\lambda+\sigma} = \mathcal{F}_\mu^\lambda + U^\sigma$, and so it follows directly that

$$\text{Bal}(\mu + \sigma, \lambda + \sigma) = \text{Bal}(\mu, \lambda) + \sigma. \quad (3)$$

Existence of $\text{Bal}(\mu, \lambda)$ hence implies, through (3), the existence of any partial balayage measure of the form $\text{Bal}(\mu + \sigma, \lambda + \sigma)$, and vice versa.

In this paper we will exclusively work with partial balayage to the zero measure, \textit{i.e.} we will use partial balayage measures of the form $\text{Bal}(\sigma, 0)$; although somewhat similar, note that such a partial balayage measure in general will \textit{not} be the same as a classical balayage measure. Under certain assumptions on the signed Radon measure $\sigma$ partial balayage measures of this form are quite well-behaved, as we shall see in section 5.

4. Examples: Part I

Having discussed the basics of weighted potential theory and partial balayage let us consider an example before we move on to the main results. As mentioned in the introduction, the inspiration for this paper comes from \cite{1}, which studies a class of superharmonically perturbed Gaussian background potentials of the form

$$Q(z) = Q_{\alpha, \nu}(z) = \alpha |z|^2 + U^\nu(z), \quad (4)$$

defined on $C$, where $\alpha > 0$ is a parameter and $\nu$ is a compactly supported finite positive Borel measure. Any background potential of this form is readily seen to be fully admissible (\cite[Proposition 2.4]{1} shows this when...
\( t = 1 \), and the argument is easily generalized to arbitrary \( t \) using the relation described in Remark 2.1. The perhaps simplest non-trivial background potential of the form (4) is obtained by taking \( \nu = \beta \delta_a \), where \( \beta > 0 \) and \( a \in \mathbb{C} \), i.e. \( Q \) becomes
\[
Q(z) = \alpha |z|^2 + \beta \log \frac{1}{|z - a|}.
\]
This background potential is treated in detail in [1], and its equilibrium measure (for \( t = 1 \)) is in [1, Proposition 3.3] determined completely by the following (cf. Figure 1):

**Proposition 4.1.** Let the two radii \( R \) and \( r \) be defined by
\[
R := \sqrt{\frac{1 + \beta}{2 \alpha}} \quad \text{and} \quad r := \sqrt{\frac{\beta}{2 \alpha}}.
\]
The equilibrium measure \( \mu_Q \) of the background potential (5) is absolutely continuous with respect to the Lebesgue measure with density
\[
\frac{2 \alpha}{\pi m(S_Q)},
\]
where the support \( S_Q = \text{supp} \mu_Q \) depends on the geometric arrangement of the disks \( D(a,r) \) and \( D(0,R) \) in the following way:

(i) If \( D(a,r) \subset D(0,R) \) then \( S_Q = D(0,R) \setminus D(a,r) \).

(ii) If \( D(a,r) \not\subset D(0,R) \) then \( \mathbb{C} \setminus S_Q \) is given by a rational exterior conformal mapping of the form
\[
f : \mathbb{C} \setminus \{ \zeta : |\zeta| \leq 1 \} \to \mathbb{C} \setminus S_Q, \quad f(\zeta) = \rho \zeta + u + \frac{\nu}{\zeta - A},
\]
where the coefficients \( \rho \in \mathbb{R}^+, 0 < |A| < 1 \) and \( u, v \in \mathbb{C} \) of the mapping \( f(\zeta) \) are uniquely determined by the parameters \( \alpha, \beta \) and \( a \) of the potential \( Q(z) \).

One interpretation of the above result that becomes particularly clear when we look at Figure 1 is that the equilibrium measure \( \mu_Q \) effectively is the measure that is “removed” from the measure \( \frac{2 \alpha}{\pi m(D(0,R))} \) when we try to “sweep” the point charge \( \beta \) at \( a \in \mathbb{C} \) onto the disk \( D(0,R) \) (while factoring in the constant density \( 2\alpha/\pi \)). In terms of the partial balayage defined in the previous section we can interpret this as
\[
\text{Bal} \left( \beta \delta_a, \frac{2 \alpha}{\pi} m(D(0,R)) \right) = \frac{2 \alpha}{\pi} m(D(0,R) \setminus S_Q)
\]
\[
= \frac{2 \alpha}{\pi} \left( m(D(0,R)) - m(S_Q) \right)
\]
As noted in Remark 3.2 we can rewrite this as
\[
\text{Bal} \left( \beta \delta_a - \frac{2 \alpha}{\pi} m(D(0,R),0) \right) = -\frac{2 \alpha}{\pi} m(S_Q) = -\mu_Q,
\]
or, equivalently,
\[ \mu_Q + \text{Bal}(\sigma, 0) = 0 \]  
where
\[ \sigma = \beta \delta_a - \frac{2 \alpha}{\pi} m_{\overline{D}(0, R)}. \]

Of course, to fully justify the above we need to be careful and in detail verify each step. We omit the precise calculations, and in the following two sections instead formulate a more general theory of when relation (6) holds. As an illustration of how to use the results we obtain we will then in section 7 apply the theory to the entire class of background potentials of the form (4), hence in particular also to the background potential (5) used above.

5. Properties of certain Partial Balayage Measures

As previously mentioned we will exclusively consider partial balayage of a signed Radon measure to the zero measure, in particular partial balayage of the form \( \text{Bal}(\sigma, 0) \) for some signed Radon measure \( \sigma \). For later use we need to determine some of the properties of this type of partial balayage measure, given certain assumptions on the signed measure on which we perform the balayage operation. We require the following:

**Theorem 5.1.** Let \( \sigma = \sigma_+ - \sigma_- \) be a signed Radon measure with compact support for which \( \sigma(\mathbb{C}) < 0 \) and such that \( U^{\sigma_-} \) is continuous on \( \mathbb{C} \). Then \( \text{Bal}(\sigma, 0) \) exists, has the same total mass as \( \sigma \), satisfies
\[ \text{supp } \text{Bal}(\sigma, 0) \subseteq \text{supp } \sigma_- , \]  
and has finite logarithmic energy. Moreover, with \( V^\sigma \) as in Definition 3.1 for \( \nu := \text{Bal}(\sigma, 0) \), define the sets \( \omega := \{ z \in \mathbb{C} : V^\sigma(z) < U^\sigma(z) \} \) and \( \Omega := \mathbb{C} \setminus \text{supp } \nu \). Then \( \omega \) is an open set and \( \omega \subseteq \Omega \), so for every \( z \in \text{supp } \nu \) we have \( V^\sigma(z) = U^\sigma(z) \). Furthermore, there exists a constant \( c_0 \) such that \( V^\sigma = U^\nu + c_0 \).
Proof. Let us first show existence of the partial balayage measure. Since
\( U^\sigma = U^{\sigma^+} - U^{\sigma^-} \) and the potential of a positive measure with our definition
is superharmonic, hence lower semicontinuous, it clearly follows from the
assumption of continuity of \( U^{\sigma^-} \) that the potential \( U^\sigma \) is lower semicontinuous
on the whole of \( \mathbb{C} \). We assume that \( \sigma \) has compact support, but since
lower semicontinuous functions are bounded from below on any compact set
we can conclude, if we also use the fact that as \( |z| \to \infty \) we get
\[
U^\sigma(z) = \sigma(\mathbb{C}) \log \frac{1}{|z|} + O\left(\frac{1}{|z|}\right) \sim \frac{-\sigma(\mathbb{C})}{|z|} \to \infty,
\]
that there exists \( M \in \mathbb{R} \) such that \( M \leq U^\sigma(z) \) for all \( z \in \mathbb{C} \). If we now
let \( V(z) := M \) on \( \mathbb{C} \) we see that both \( V \leq U^\sigma \) and \( \Delta V \geq 0 \) hold on the
whole of \( \mathbb{C} \) (since \( \Delta V = 0 \)). The function \( V \) is thus a competing element
for the function \( V^\sigma \equiv V_0^\sigma := \sup \mathcal{F}^\sigma_0 \) as in Definition 3.1, \textit{i.e.} the set \( \mathcal{F}^\sigma_0 \) is
nonempty. If we can show that \( V^\sigma \in \mathcal{F}^\sigma_0 \) then the existence of \( \text{Bal}(\sigma,0) \)
follows. That \( V^\sigma \) indeed belongs to \( \mathcal{F}^\sigma_0 \) is nontrivial, but can be shown using
the same argument as in Lemma 1 in [4]: under the given assumptions it
is relatively easily seen that \( \mathcal{F}^\sigma_0 \) is locally uniformly bounded above, so the
upper semicontinuous regularization \( V^\sigma_0 \) of \( V^\sigma \) satisfies \( \Delta V^\sigma_0 \geq 0 \) and is equal
to \( V^\sigma \) almost everywhere (with respect to the Lebesgue measure), hence
everywhere by taking means over arbitrary small balls. Since \( V^\sigma = \sup \mathcal{F}^\sigma_0 \)
clearly satisfies \( V^\sigma \leq U^\sigma \) it now follows that \( V^\sigma = \sup \mathcal{F}^\sigma_0 = V^\sigma_0 \in \mathcal{F}^\sigma_0 \), and
we can conclude that the partial balayage measure \( \text{Bal}(\sigma,0) \) indeed exists.

The proposition for the location of the support of the partial balayage
measure under the given assumptions on \( \sigma \), \textit{i.e.} property (7), is clearly a
corollary from the fact that we have the inequalities
\[
-\sigma_+ \leq \text{Bal}(\sigma,0) \leq 0. \tag{8}
\]
It is clear by the definition of \( \text{Bal}(\sigma,0) \) that \( \text{Bal}(\sigma,0) \leq 0 \) holds everywhere.
The leftmost inequality in (8) follows from the fact that we can use (3) to
write
\[
\text{Bal}(\sigma,0) = \text{Bal}(\sigma_+-\sigma_-,\sigma_-) = \text{Bal}(\sigma_+,\sigma_-) - \sigma_-.
\]
Although [6] uses a slightly less general definition of partial balayage than
the one used in this paper, applying the same argument as that used in the
proof of part (b) of Theorem 2.1 in [6] shows that \( \text{Bal}(\sigma_+,\sigma_-) \geq 0 \) holds
under the given assumptions, and (8) follows.

As for \( \text{Bal}(\sigma,0) \) having finite logarithmic energy, we start by making the
observation that if \( d := \text{diam}(\text{supp } \sigma_-) \), then \( d > 0 \). Indeed, assume this
is not the case: then either \( \text{supp } \sigma_- = \emptyset \) or \( \text{supp } \sigma_- = \{a\} \) holds, for some
\( a \in \mathbb{C} \). The first case implies \( \sigma_- \equiv 0 \), which contradicts \( \sigma(\mathbb{C}) < 0 \), and the
second case implies that \( \sigma_- \) is a Dirac point mass, which contradicts the
assumed continuity of \( U^{\sigma^-} \) on the whole of \( \mathbb{C} \). It follows that, indeed, \( d > 0 \).

Let, for sake of simplicity, \( \mu := -\text{Bal}(\sigma,0) \); evidently \( \text{Bal}(\sigma,0) \) has finite
logarithmic energy if and only if \( \mu \) has finite logarithmic energy. We now
look at $I(\mu)$ in detail. By definition we get
\begin{align*}
I(\mu) &= \int\int \log \frac{1}{|z - \zeta|} \, d\mu(\zeta) d\mu(z) \\
&= \int\int \log \left| \frac{d}{z - \zeta} \right| \, d\mu(\zeta) d\mu(z) - \mu(\mathbb{C})^2 \log d. \quad (9)
\end{align*}

For any $z, \zeta \in \text{supp } \mu$ it follows from (7) that $|z - \zeta| \leq d$, so in particular we have $\log |d/(z - \zeta)| \geq 0$ for all $z, \zeta \in \text{supp } \mu$; note that this immediately implies $I(\mu) > -\infty$. We see that (8) yields $\mu \leq \sigma$, so (9) implies that
\begin{align*}
I(\mu) &\leq \int\int \log \left| \frac{d}{z - \zeta} \right| \, d\sigma_-(\zeta) d\sigma_-(z) - \mu(\mathbb{C})^2 \log d \\
&= \int\int \log \frac{1}{|z - \zeta|} \, d\sigma_-(\zeta) d\sigma_-(z) + (\sigma_-(\mathbb{C})^2 - \mu(\mathbb{C})^2) \log d \\
&= I(\sigma_-) + (\sigma_-(\mathbb{C})^2 - \mu(\mathbb{C})^2) \log d.
\end{align*}

The proposed finite logarithmic energy now clearly follows if we can show that $I(\sigma_-)$ is finite. However, this is nearly trivial: we assume that $U^{\sigma_-}$ is continuous on $\mathbb{C}$ and that $\text{supp } \sigma_-$ is compact, from which we obtain
\begin{align*}
|I(\sigma_-)| &= \left| \int U^{\sigma_-} d\sigma_- \right| \leq \max_{z \in \text{supp } \sigma_-} |U^{\sigma_-}(z)| \cdot \sigma_-(\mathbb{C}) < \infty.
\end{align*}

Having established the existence of the partial balayage measure, the location of its support and it having finite logarithmic energy, let us finally turn to proving the last part of the theorem, and, at the same time, show that the total mass of $\text{Bal}(\sigma, 0)$ is precisely $\sigma(\mathbb{C})$, as stated. That $\omega$ is an open set follows from the fact that the auxiliary function $u := U^{\sigma} - V^{\sigma}$ is a lower semicontinuous function with our assumptions on $\sigma$, and that $\omega = u^{-1}((0, \infty])$. To show that $\omega \subseteq \Omega$ it is enough to show that $\Delta V^{\sigma} = 0$ whenever $V^{\sigma} < U^{\sigma}$, since $\Delta V^{\sigma} = 0$ by the definition of $\text{Bal}(\sigma, 0)$ would imply $\nu = 0$ in $\omega$. For sake of argument let us simply assume the contrary, i.e. there is some point where $V^{\sigma} < U^{\sigma}$ while $\Delta V^{\sigma} > 0$ holds true. Then there would exist some ball $B$ contained in $\omega$ in which both $V^{\sigma} < U^{\sigma}$ and $\Delta V^{\sigma} > 0$ hold. In that case we could define a new function $\tilde{V}^{\sigma}$ by
\begin{align*}
\tilde{V}^{\sigma}(z) := \begin{cases}
V^{\sigma}(z) & \text{on } \mathbb{C} \setminus B, \\
p(z) & \text{on } B,
\end{cases}
\end{align*}

where $p$ is the Poisson integral on $B$ with boundary value $V^{\sigma}$ on $\partial B$. This function would then be at least equal to $V^{\sigma}$ everywhere and in fact be larger than $V^{\sigma}$ in $B$ while still satisfying $\tilde{V}^{\sigma} \leq U^{\sigma}$ and $\Delta \tilde{V}^{\sigma} \geq 0$, hence we would have $\tilde{V}^{\sigma} \in \mathcal{F}^{\sigma}_0$, contradicting the assumed maximality of $V^{\sigma} = \sup \mathcal{F}^{\sigma}_0$.

By the definition of $V^{\sigma}$ it immediately follows that the auxiliary function $u = U^{\sigma} - V^{\sigma}$ itself can be considered as the smallest non-negative function satisfying
\begin{align*}
\frac{1}{2\pi} \Delta u \geq \sigma \quad (10)
\end{align*}
in $\mathbb{C}$. Let $R > 0$ be large enough so that $\text{supp } \sigma \subset D(0, R)$; there evidently exists such an $R$ since the support of $\sigma$ is assumed compact. We claim that $u$ then must be harmonic in the set $\Theta := \mathbb{C} \setminus \overline{D}(0, R)$. Indeed, (10) implies
that $\Delta u \leq 0$ on $\Theta$, so the minimum principle for superharmonic functions implies that $u$ must be constantly equal to zero on the whole of $\Theta$ if it is zero at any point. (We in fact utilize that $u$ is assumed to be non-negative everywhere; a point in $\Theta$ where $u$ is zero would thus be a global minimum for the restriction of $u$ to any bounded neighborhood contained in $\Theta$, so that $u$ must be zero throughout that neighborhood, and we can cover $\Theta$ by such neighborhoods.) If, on the other hand, there is some point $z' \in \Theta$ where $u(z') > 0$, then $z' \in \omega = u^{-1}((0, \infty])$; the previous part in fact evidently implies that every point in $\Theta$ must be contained in $\omega$, i.e. $\Theta \subseteq \omega$. Since we already know that $\Delta V^\sigma = 0$ in $\omega$, and moreover also that $\Delta U^\sigma = 0$ outside the support of $\sigma$ (hence in $\Theta$), we obtain

$$\Delta u = \Delta U^\sigma - \Delta V^\sigma = 0$$

as desired.

Having established that the difference $u$ between $U^\sigma$ and $V^\sigma$ must be harmonic outside the support of $\sigma$, let us now investigate in detail its behavior near infinity. On the one hand, we already know that $U^\sigma$ has the expansion

$$U^\sigma(z) = t \log |z| + \mathcal{O}\left(\frac{1}{|z|}\right)$$

as $|z| \to \infty$, where $t := -\sigma(\mathbb{C}) > 0$. On the other hand, by once more applying that $U^\sigma$ is bounded from below in $\mathbb{C}$, we can find a constant $\tilde{M}$ such that the function $\tilde{V}$ defined by

$$\tilde{V}(z) := t \log_+ |z| + \tilde{M}$$

is both subharmonic and satisfies $\tilde{V} \leq U^\sigma$ everywhere (we here use the notation $\log_+ x = \max(\log x, 0)$). It follows that $\tilde{V} \in \mathcal{F}_0^\sigma$, hence $\tilde{V} \leq V^\sigma$, and combined with $V^\sigma \leq U^\sigma$ we obtain

$$0 \leq u(z) \leq U^\sigma(z) - t \log_+ |z| - \tilde{M};$$

as $|z| \to \infty$ this becomes

$$0 \leq u(z) \leq \mathcal{O}(1).$$

We can from this infer that the minimizing function $u$ must be both harmonic and bounded on $\Theta$, in particular that $V^\sigma$ must have the behavior

$$V^\sigma(z) = t \log |z| + \mathcal{O}(1)$$

as $|z| \to \infty$; as $\nu = \text{Bal}(\sigma, 0)$ is defined through

$$\nu = \text{Bal}(\sigma, 0) = -\frac{1}{2\pi} \Delta V^\sigma,$$

it follows that the logarithmic term in the expansion of $U^\nu(z)$ for large $|z|$ must be precisely $t \log |z|$, i.e. $\nu(\mathbb{C}) = -t = \sigma(\mathbb{C})$. Lastly, by Weyl’s Lemma [8, Lemma 3.7.10] it is straightforward to see that the difference between $U^\nu$ and $V^\sigma$ is a harmonic function on $\mathbb{C}$ that, due to the behaviors of both $U^\nu$ and $V^\sigma$, must be $\mathcal{O}(1)$ near infinity, hence bounded, and hence constant, by Liouville’s theorem. We thus conclude that, indeed, $V^\sigma = U^\nu + c_0$ for some constant $c_0$, as stated. □
6. Equilibrium Measures through Partial Balayage

As a motivational example of what will follow, let us combine a few results from weighted potential theory regarding the potential of an equilibrium measure, and consider the weighted equilibrium problem from the point of view as an obstacle problem. Assume \( Q : E \to (\mathbb{R}^+, \infty) \) is a \( t \)-admissible background potential on a closed set \( E \subseteq \mathbb{C} \) for some \( t > 0 \). From Proposition 2.6, in particular part (2), we know that \( Q = Q_{t} - U_{t}^{\mu_{Q,t}} \) holds q.e. on \( S_{Q,t} := \text{supp} \mu_{Q,t} \), but let us for sake of simplicity for the moment assume that \( Q \) is such that the polar exceptional set in this equality is empty, so that we in fact have \( Q(z) = Q_{t} - U_{t}^{\mu_{Q,t}}(z) \) for all \( z \in S_{Q,t} \) (this is not an unnatural assumption on \( Q \); all of the background potentials treated in section 7 all satisfy this property). We briefly note that an empty such polar exceptional set by Theorem I.4.4 in [9] in fact implies that \( U_{t}^{\mu_{Q,t}} \) must be continuous on \( \mathbb{C} \).

It is easily seen that if we let \( \hat{Q} \) be the restriction of \( Q \) to \( S_{Q,t} \) extended by positive infinity outside \( S_{Q,t} \), i.e. we let \( \hat{Q} : \mathbb{C} \rightarrow (\mathbb{R}^+, \infty) \) be defined by

\[
\hat{Q}(z) := \begin{cases} 
Q(z) & z \in S_{Q,t}, \\
\infty & z \notin S_{Q,t},
\end{cases}
\]

then \( \hat{Q} \) is also a \( t \)-admissible background potential, and, more importantly, \( \mu_{Q,t} = \mu_{\hat{Q},t} \). We now look at the equilibrium problem for \( \hat{Q} \) (which, as just mentioned, is equivalent to that of \( Q \)) but now from an obstacle problem point of view, and for this purpose we let \( \mathcal{H} \) denote the set of subharmonic functions in \( \mathbb{C} \), and define \( \mathcal{H}_{t} \) as the set of all functions \( V(z) \in \mathcal{H} \) that are harmonic for large \( |z| \) and such that \( V(z) - t \log |z| \) is bounded from above as \( |z| \to \infty \) (cf. Proposition 2.7); the obstacle problem for \( \hat{Q} \) is then to find the largest function \( \hat{V}_{\hat{Q}} \) in the class \( \mathcal{H}_{t} \) that is majorized by \( \hat{Q} \) quasi everywhere, i.e.

\[
\hat{V}_{\hat{Q}} = \sup \{ V \in \mathcal{H}_{t} : V(z) \leq \hat{Q}(z) \text{ for q.e. } z \in \mathbb{C} \}.
\]

The reason for studying the equilibrium problem as an obstacle problem is in part because of the fact that we want to establish a relationship between partial balayage and the weighted equilibrium measure, and the definition of partial balayage used in this paper is essentially as an obstacle problem, but also in part because of the fact that the obstacle problem formulation allows for some flexibility in the types of obstacles allowed that does not appear in the usual setting of weighted potential theory. In particular, we shall consider the obstacle problem with the obstacle \( \tilde{Q} := Q_{t} - U_{t}^{\mu_{Q,t}} \), now defined on the whole complex plane instead of simply a subset of it: this will result in a solution \( \tilde{V}_{\tilde{Q}} \) defined analogously to (12), and, as we will soon see, it will turn out that the two obstacle problems for \( \hat{Q} \) and \( \tilde{Q} \) in fact have the same solutions, i.e. \( \tilde{V}_{\tilde{Q}} = \hat{V}_{\hat{Q}} \). However, to specifically give an example of the flexibility in the obstacle problem formulation, note that \( \hat{Q} \) is a \( t \)-admissible background potential while \( \tilde{Q} \) is not: as \( |z| \) tends to infinity we have \( \hat{Q} = t \log |z| + \mathcal{O}(1) \), which violates admissibility requirement (iii) in Definition 2.2.
From our point of view it is the property that \( \hat{Q} \) and \( \tilde{Q} \) as obstacles have the same solution that is interesting. From the definitions above it is easily seen that \( \hat{Q} \) satisfies
\[
\hat{Q}(z) = \hat{Q}(z) \text{ for all } z \in S_{Q,t}, \\
\tilde{Q}(z) \leq \hat{Q}(z) \text{ for all } z \in \mathbb{C}.
\]
Combined with the previously mentioned behavior of \( \hat{Q} \) near infinity, as well as the above mentioned property that \( U^{\mu_{Q,t}} \) is continuous on \( \mathbb{C} \), and the (trivial) fact that \( \text{supp} \mu_{Q,t} \subseteq S_{Q,t} \), is it relatively easily seen that \( V_{\tilde{Q}} = V_{\hat{Q}} \) then must hold. In fact, we can in the above in general replace the set \( S_{Q,t} \) with some compact set \( V \), the constant \( F_{Q,t} \) with some constant \( c \), and the measure \( \mu_{Q,t} \) with some signed and compactly supported measure \( \sigma = \sigma_+ - \sigma_- \) as long as \( \sigma(\mathbb{C}) = -t \), \( U^{\sigma_-} \) is continuous on \( \mathbb{C} \), and \( \text{supp} \sigma_- \subseteq E' \); if it then holds that the function \( \hat{Q} := c + U^\sigma \) satisfies
\[
\tilde{Q}(z) = \hat{Q}(z) \text{ for q.e. } z \in E', \\
\tilde{Q}(z) \leq \hat{Q}(z) \text{ for q.e. } z \in \mathbb{C},
\]
then, as we soon shall see, we will still obtain \( V_{\tilde{Q}} = V_{\hat{Q}} \). With this in mind, we introduce the following:

**Definition 6.1.** Let \( Q \) be a \( t \)-admissible background potential on \( E \subseteq \mathbb{C} \) for some \( t > 0 \). If \( \mathbf{G} = (E', \sigma, c) \) is a triple with \( E' \subseteq E \) a compact set, \( \sigma \) a signed and compactly supported Radon measure with \( \sigma(\mathbb{C}) = -t \) and \( \text{supp} \sigma_- \subseteq E' \), and \( c \in \mathbb{R} \) a constant, such that the function \( \hat{Q} \) defined by \( \hat{Q}(z) := c + U^\sigma(z) \) satisfies
\[
\hat{Q}(z) = Q(z) \text{ for q.e. } z \in E', \\
\hat{Q}(z) \leq Q(z) \text{ for q.e. } z \in E,
\]
then we say that \( \mathbf{G} \) defines a \( t \)-extension of \( Q \) (relative to \( E' \)). Whenever it is clear from context precisely which \( \mathbf{G} \) we are working with, we simply refer to the generated function \( \hat{Q} = c + U^\sigma \) as a \( t \)-extension of \( Q \).

**Remark 6.2.** From the above definition it is natural to ask questions of existence and uniqueness of a \( t \)-defining extension \( \mathbf{G} \) for a given \( t \)-admissible background potential \( Q \). As for existence, it is evident from the motivational example given above that \( (\text{supp} \mu_{Q,t}, -\mu_{Q,t}, F_{Q,t}) \) always defines a \( t \)-extension of \( Q \). However, extensions for which \( \sigma = \mu_{Q,t} \) (and \( c = F_{Q,t} \) by necessity) are in some sense not particularly interesting: the \( t \)-extension \( \hat{Q} = F_{Q,t} - U^{\mu_{Q,t}} \) resulting from such a triple is, by Proposition 2.7, in fact itself the solution to the obstacle problem with \( \hat{Q} \) as an obstacle, i.e. \( V_{\hat{Q}} = \hat{Q} \).

In order to actually have something to work with we are more interested in trying to find a \( t \)-extension \( \tilde{Q} \) defined by some \( \mathbf{G} \) which can be constructed a priori and for which the solution \( V_{\tilde{Q}} \) to the obstacle problem is not necessarily equal to \( \hat{Q} \) everywhere. In other words, the interesting case for our purposes is precisely when we can find a \( t \)-extension \( \tilde{Q} = c + U^\sigma \) where \( \sigma \) is different from \( \mu_{Q,t} \). For an arbitrary \( Q \) we cannot in general guarantee that there exists a \( t \)-extension for which \( \sigma \neq \mu_{Q,t} \), and even when such an...
extension exists it may for some background potentials be the case that $\sigma$ is very difficult to calculate explicitly. On the other hand, in some settings it is relatively easy to find such a measure, and in the next section we will show one method that for instance works for the class of superharmonically perturbed Gaussian background potentials. However, that method will be based on us essentially attempting to create such a measure $\sigma$ from applying the Laplace operator (in the distributional sense) to $Q$. Applying this on a more general background potential need not always be fruitful, since $\Delta Q$ need not always be a measure; for instance, the background potential $Q$ could have jump discontinuities, so that $\Delta Q$ becomes a distribution of non-zero order.

Remark 6.3. Just like we extended the background potential by infinity in our motivational example in (11), we can always perform the same sort of extension on any general background potential $Q$. Thus, we may just as well think of the set $E \subseteq \mathbb{C}$ on which $Q$ is defined to be the entire complex plane, by simply letting $Q(z) := \infty$ for any $z \notin E$.

Having defined the notion of a $t$-extension, we are now ready to state a complementarity theorem between weighted equilibrium measures and partial balayage.

Theorem 6.4. Let $Q$ be a $t$-admissible background potential on $E \subseteq \mathbb{C}$, and assume $G = (E', \sigma, c)$ defines a $t$-extension $\tilde{Q} = c + U^\sigma$ of $Q$ relative to $E'$. If $U^\sigma-$ is continuous on $\mathbb{C}$, then

$$\mu_{Q,t} + \text{Bal}(\sigma, 0) = 0.$$  \hspace{1cm} (13)

Proof. As mentioned in Remark 6.3, we will, without loss of generality, consider $Q$ to be defined on the entire complex plane, i.e. $E = \mathbb{C}$. Under the given assumptions on $\sigma$, it follows from Theorem 5.1 that $\text{Bal}(\sigma, 0)$ exists.

Let, as before, $\mathcal{H}$ be the set of subharmonic functions in $\mathbb{C}$, and define $\mathcal{H}_t$ to be the set of all functions $V \in \mathcal{H}$ that are harmonic in a neighborhood of infinity and such that $V(z) - t \log |z|$ is bounded from above as $|z| \to \infty$, i.e. $V(z) \leq t \log |z| + O(1)$ for large $|z|$. We now consider the two obstacle problems

$$V_Q := \sup \{ V \in \mathcal{H}_t : V(z) \leq Q(z) \text{ for q.e. } z \in \mathbb{C} \}$$

and

$$V_{\tilde{Q}} := \sup \{ V \in \mathcal{H}_t : V(z) \leq \tilde{Q}(z) \text{ for q.e. } z \in \mathbb{C} \}$$

$$= \sup \{ V \in \mathcal{H}_t : V(z) \leq c + U^\sigma(z) \text{ for all } z \in \mathbb{C} \}.$$  

The last equality above is motivated in part by that quasi everywhere implies everywhere for this particular obstacle: the inequality is equivalent with $h_1(z) + h_2(z) \geq 0$ q.e. for $h_1(z) = U^\sigma(z) - V(z)$, $h_2(z) = c - U^\sigma-(z)$, and under the given assumptions we get that $h_1$ is superharmonic and $h_2$ is continuous everywhere, so a standard potential theoretic argument using mollifiers can be used. With this property established, it is evident that the solution $V_{\tilde{Q}}$ to the second obstacle problem in fact must be precisely $V_{\tilde{Q}} = V^\sigma + c$, where $V^\sigma \equiv V^\sigma_0 = \sup \mathcal{H}_0^\sigma$ is as in the definition of $\text{Bal}(\sigma, 0)$, since we from Theorem 5.1 can conclude that we must have $V^\sigma + c \in \mathcal{H}_t$. 


By the same theorem we obtain $V^\sigma = c_0 + U^\text{Bal}(\sigma,0)$ for some $c_0$, and hence, if we for sake of simplicity let $\nu$ be the positive measure $\mu := -\text{Bal}(\sigma,0)$, we can conclude that
\[
V_{\tilde{Q}} = \tilde{c} - U^\mu
\]
(14)
for some constant $\tilde{c} = c + c_0$. As for the solution to the first obstacle problem, we simply apply Proposition 2.7 to immediately obtain that we must have
\[
V_Q = F_{Q,t} - U^\mu_{Q,t}.
\]
(15)

We now claim that $V_Q = V_{\tilde{Q}}$. In other words, as earlier mentioned, from an obstacle problem point of view one could replace the obstacle $Q$, used in determining the potential of the weighted equilibrium measure, with the $t$-extension $\tilde{Q}$ and still get the same solution. Since the solution to the obstacle problem for the obstacle $\tilde{Q}$ evidently can be expressed in terms of the partial balayage measure $\text{Bal}(\sigma,0)$, this will establish the proposed complementarity relationship (13).

As for proving the equality between $V_Q$ and $V_{\tilde{Q}}$, we show that either one must be less than or equal to the other. Since we by definition of our $t$-extension $\tilde{Q}$ have $\tilde{Q} \leq Q$ quasi everywhere, it is clear that $V_{\tilde{Q}} \leq V_Q$ holds, and so the main difficulty is showing the we also have the reverse inequality, i.e. that $V_Q \leq V_{\tilde{Q}}$. Utilizing (14) and (15) it is evident that showing $V_Q \leq V_{\tilde{Q}}$ is equivalent with showing that $U^\mu \leq U^\mu_{Q,t} + (\tilde{c} - F_{Q,t})$ holds everywhere. Using the Principle of Domination, Proposition 2.8, we get that it in fact is enough to show that $V_Q \leq V_{\tilde{Q}}$ holds on the support of $\mu = -\text{Bal}(\sigma,0)$. However, this is easy: from Theorem 5.1 it follows that $V^\sigma = U^\sigma$ on $\text{supp Bal}(\sigma,0)$, hence
\[
V_{\tilde{Q}} = c + V^\sigma = c + U^\sigma = \tilde{Q}
\]
holds there. But from the same theorem and one of our starting assumptions we moreover also know that $\text{supp Bal}(\sigma,0) \subseteq \text{supp } \sigma_- \subseteq E'$. Since the $t$-extension $\tilde{Q}$ is defined to satisfy $\tilde{Q} = Q$ q.e. on the set $E'$, it now follows that $V_{\tilde{Q}} = Q$ holds on $\text{supp Bal}(\sigma,0)$. From this it is evident that $V_Q \leq V_{\tilde{Q}}$ holds on $\text{supp Bal}(\sigma,0)$, and we can therefore finally conclude that we indeed have $V_Q \leq V_{\tilde{Q}}$ everywhere, and hence
\[
F_{Q,t} - U^\mu_{Q,t} = V_Q = V_{\tilde{Q}} = \tilde{c} + U^{\text{Bal}(\sigma,0)}
\]
holds everywhere. Applying the Laplace operator on the leftmost and rightmost sides of this last equality to recover the measures yields (13). □

There are a few corollaries to Theorem 6.4 that we should mention. One of the main difficulties in determining the weighted equilibrium measure $\mu_{Q,t}$ is to locate its support, and for this reason the following result may be interesting:

**Corollary 6.5.** Let $Q$ and $G = (E',\sigma,c)$ be as in Theorem 6.4, with $\text{supp } \sigma_- \subseteq E'$ and $U^\sigma$- assumed continuous on $\mathbb{C}$. Then $\text{supp } \mu_{Q,t} \subseteq E'$. 

Proof. This follows immediately from applying the theorem, along with
\[ \text{supp } \mu_{Q,t} = \text{supp } \text{Bal}(\sigma,0) \subseteq \text{supp } \sigma_- \subseteq E'. \]
\[ \square \]

Also, if the set \( E \) on which \( Q \) is assumed to be defined is already a compact set, then Theorem 6.4 can be stated in a slightly simplified way:

**Corollary 6.6.** Let \( E \subset \mathbb{C} \) be a compact set, and let \( Q \) be an admissible background potential on \( E \). Assume there exists a signed Radon measure \( \sigma = \sigma_+ - \sigma_- \) with compact support such that \( Q = U^\sigma |_E + c \) for some constant \( c \in \mathbb{R} \) and such that \( \sigma \) satisfies \( t := -\sigma(\mathbb{C}) > 0 \), \( \text{supp } \sigma_- \subseteq E \) and \( U^\sigma_- \) is continuous on \( \mathbb{C} \). Then \( \mu_{Q,t} + \text{Bal}(\sigma,0) = 0 \).

Proof. Simply apply Theorem 6.4 using \( G = (E,\sigma,c) \).

\[ \square \]

7. **Examples: Part II**

Let us now once more turn our focus to the sample potential
\[ Q(z) = \alpha|z|^2 + \beta \log \frac{1}{|z-a|}, \]
with \( \alpha, \beta > 0 \) and \( a \in \mathbb{C} \) arbitrary, and attempt to find the equilibrium measure \( \mu_{Q,t} \) using the complementarity relationship to partial balayage given in Theorem 6.4, i.e. we essentially want to verify the results in section 4 using partial balayage. In fact, the precise technique we will use will work for any so-called Gaussian background potential with a superharmonic perturbation (as defined in [1]): we will in this section from hereon hence assume that our background potential \( Q \) has the form
\[ Q(z) = \alpha|z|^2 + U^\nu(z), \]
where \( \alpha > 0 \) and \( \nu \) is a finite positive Borel measure with compact support (note that our sample potential corresponds to \( \nu = \beta \delta_a \) with \( \beta > 0 \) and \( a \in \mathbb{C} \)). Throughout this section we will moreover assume that \( t > 0 \) is an arbitrary but fixed positive real number.

The method we will use will of course be that of finding a suitable \( t \)-extension \( \tilde{Q} \) of the background potential \( Q \), and apply the theorem given in the previous section. Finding such a \( t \)-extension is not a trivial task in general, but for \( Q \) of the form (16) it turns out that the following is suitable: let \( \rho > 0 \) be a constant, soon to be determined, let \( \tilde{Q} = c + U^\nu \) denote our desired \( t \)-extension relative to some compact set \( E' \) (i.e. \( G = (E',\sigma,c) \)), and in particular take \( E' := D(0,\rho) \); we thus need to find a signed Radon measure \( \sigma \) of compact support and a constant \( c \) such that the function \( \tilde{Q} = c + U^\nu \) satisfies \( \tilde{Q}(z) = Q(z) \) if \( |z| \leq \rho \) and \( \tilde{Q}(z) \leq Q(z) \) if \( |z| > \rho \) (see Figure 2 for an illustration of the case where \( \nu = \beta \delta_a \)). The way we will do this is to simply try to construct \( \tilde{Q} \) so that it gets the desired behavior on \( \mathbb{C} \), and then simply let \( \sigma \) be defined by
\[ \sigma := -\frac{1}{2\pi} \Delta \tilde{Q}; \]
Figure 2. A plot through the line containing the origin and the point $a$ of $Q(z) = \alpha |z|^2 + \beta \log \frac{1}{|z-a|}$, the $t$-extension $\tilde{Q}$, equal to $Q$ on $E' = D(0, \rho)$ and $\tilde{Q}(z) \sim t \log |z| + \mathcal{O}(1)$ near infinity, as well as the solution $V_Q (= V_{\tilde{Q}})$ to the obstacle problem with obstacle $Q$; as can be seen in the figure $\rho$ is assumed to satisfy $\rho > R = \sqrt{1 + \beta^2 \alpha}$. Note the discontinuity arising in the radial derivative of $\tilde{Q}$ illustrated by the arrows at $|z| = \rho$; the obtained $t$-extension $\tilde{Q}$ is $C^1$ at $|z| = \rho$ if and only if $\rho = R$.

It will soon become clear that there then exists some constant $c$ such that we indeed have $Q = c + U^\sigma$.

Since our background potential $Q$ is the sum of a Gaussian term $\alpha |z|^2$ with another term that is already a potential, let us attempt to find our $t$-extension $\tilde{Q}$ by simply modifying the part of $Q$ that is not already the potential of some measure, i.e. we will focus on the term $\alpha |z|^2$. For the relation $\tilde{Q} = c + U^\sigma$ to hold we are going to require that

$$\tilde{Q}(z) = \sigma(\mathbb{C}) \log \frac{1}{|z|} + \mathcal{O}(1) = (t + \nu(\mathbb{C})) \log |z| + \nu(\mathbb{C}) \log \frac{1}{|z|} + \mathcal{O}(1) \quad (17)$$

holds near infinity. As the second term on the rightmost side corresponds to the behavior of $U^\nu$ near infinity, let us simply base our ansatz for $\tilde{Q}$ directly on (17), and therefore see if we can find a constant $c \in \mathbb{R}$ such that

$$\tilde{Q}(z) = (t + \nu(\mathbb{C})) \log |z| + U^\nu(z) + c \quad (18)$$
holds for all $|z| > \rho$, with $\tilde{Q}$ still satisfying the required properties discussed above. The value of $c$ is easily determined: we obviously need that the resulting function $\tilde{Q}$ is a.e. continuous at the points on the set $\{ z : |z| = \rho \}$, and to achieve this while demanding that $\tilde{Q}(z) = Q(z)$ for $|z| < \rho$ and $\tilde{Q}$ given by (18) for $|z| > \rho$, it is clear that we need

$$c = \alpha \rho^2 - (t + \nu(C)) \log \rho.$$ 

As for the requirement that $\tilde{Q} \leq Q$ must hold everywhere, we see that this is clearly the case if we have

$$(t + \nu(C)) \log |z| + c \leq \alpha |z|^2$$

for all $|z| \geq \rho$. For sake of obtaining this property, let $f : [\rho, \infty) \to \mathbb{R}$ be defined by

$$f(x) := \alpha x^2 - (t + \nu(C)) \log x - c;$$

the inequality (19) is evidently equivalent with $f$ being a non-negative function on $[\rho, \infty)$. On the one hand it is clear from how we defined $c$ that $f(\rho) = 0$. Moreover, we have

$$f'(x) = 2\alpha x - \frac{t + \nu(C)}{x},$$

and so non-negativity of $f$ follows if we have $f'(x) \geq 0$ for all $x \geq \rho$. It is evident that $f'$ is monotonically increasing, so we only need to demand that $f'(\rho) \geq 0$; hence we require that

$$2\alpha \rho - \frac{t + \nu(C)}{\rho} \geq 0 \iff \rho \geq \sqrt{\frac{t + \nu(C)}{2\alpha}} =: R.$$ 

(20)

As earlier suggested we now define $\sigma$ through the relation $-\Delta \tilde{Q} = 2\pi \sigma$. An easy calculation, taking into account the possible measure arising on the set $\{ z : |z| = \rho \}$ due to the first derivative of $\tilde{Q}$ in the radial direction possibly being discontinuous there, yields that

$$\sigma = -\frac{1}{2\pi} \Delta \tilde{Q} = -\frac{2\alpha}{\pi} m[D(0, \rho)] + \nu + \frac{1}{2\pi} \left( 2\alpha \rho - \frac{t + \nu(C)}{\rho} \right) s,$$

where $s$ is the arc length measure on $\{ z : |z| = \rho \}$ of total mass $2\pi \rho$. Using the standard results from classical potential theory that the potential $U^{D(0,\rho)}$ of the Lebesgue measure restricted to the disk $D(0, \rho)$ is given by

$$U^{D(0,\rho)}(z) = \begin{cases} 
-\frac{\pi}{2} |z|^2 + \frac{\pi \rho^2}{2} \left( \log \frac{1}{\rho^2} + 1 \right), & |z| \leq \rho, \\
\pi \rho^2 \log \frac{1}{|z|}, & |z| > \rho,
\end{cases}$$

(21)

and that the potential $U^s$ of the arc length measure $s$ on $\{ z : |z| = \rho \}$ is given by

$$U^s(z) = \begin{cases} 
2\pi \rho \log \frac{1}{\rho} & |z| \leq \rho, \\
2\pi \rho \log \frac{1}{|z|} & |z| > \rho,
\end{cases}$$
one easily sees that for \(|z| \leq \rho\) we obtain
\[
U^\sigma(z) = -\frac{2\alpha}{\pi} U^D(0, \rho) + U^\nu(z) + \frac{1}{2\pi} \left(2\alpha \rho - \frac{t + \nu(C)}{\rho}\right) U^\nu(z)
\]
\[
= \alpha |z|^2 + \alpha \rho^2 \log \rho^2 - \alpha \rho^2 + U^\nu(z) - 2\alpha \rho^2 \log \rho + (t + \nu(C)) \log \rho
\]
\[
= \alpha |z|^2 + U^\nu(z) - (\alpha \rho^2 - (t + \nu(C)) \log \rho)
\]
\[
= \tilde{Q}(z) - c.
\]
For \(|z| > \rho\) we in a similar way instead get
\[
U^\sigma(z) = 2\alpha \rho^2 \log |z| + U^\nu(z) - 2\alpha \rho^2 \log |z| + (t + \nu(C)) \log |z|
\]
\[
= (t + \nu(C)) \log |z| + U^\nu(z).
\]
We can summarize these results as
\[
U^\sigma(z) + c = \begin{cases} 
Q(z) & |z| \leq \rho, \\
(t + \nu(C)) \log |z| + U^\nu(z) + c & |z| > \rho,
\end{cases}
\]
which shows that we indeed obtain \(\tilde{Q} = c + U^\sigma\) everywhere for our choice of \(\tilde{Q}\). Since we have already seen that this \(\tilde{Q}\) satisfies \(\tilde{Q} \leq Q\) everywhere, it follows that \(G := (F, \sigma, c)\) defines a \(t\)-extension of \(Q\) if we have \(\sigma(C) = -t\); this property of course follows immediately from that we originally defined \(\sigma\) so that the potential of it should behave like \(t \log |z| + O(1)\) near infinity, but we can just as well also calculate it explicitly:
\[
\sigma(C) = -\frac{2\alpha}{\pi} \cdot \pi \rho^2 + \nu(C) + \frac{1}{2\pi} \left(2\alpha \rho - \frac{t + \nu(C)}{\rho}\right) \cdot 2\pi \rho
\]
\[
= -2\alpha \rho^2 + \nu(C) + 2\alpha \rho^2 - (t + \nu(C)) = -t.
\]
Finally, from (20) we see that we have \(\sigma = \sigma_+ - \sigma_-\) with
\[
\sigma_+ = \nu + \frac{1}{2\pi} \left(2\alpha \rho - \frac{t + \nu(C)}{\rho}\right) s,
\]
\[
\sigma_- = \frac{2\alpha}{\pi} m [D(0, \rho)].
\]
Since it then is evident that \(\text{supp} \, \sigma_- = \overline{D(0, \rho)} = E'\) and that the potential \(U^{\sigma_-}\) is continuous on the complex plane (cf. (21)), it finally follows, by an application of Theorem 6.4, that we indeed have \(\mu_{Q,t} + \text{Bal}(\sigma, 0) = 0\). The support of \(\mu_{Q,t}\) must especially hence be contained in \(D(0, \rho)\) for every choice of \(\rho\) satisfying
\[
\rho \geq \sqrt{\frac{t + \nu(C)}{2\alpha}},
\]
so by taking the smallest such \(\rho\) we can conclude the following proposition, which in a similar form was conjectured in [1] but not proven there:

**Proposition 7.1.** Let \(Q(z) = \alpha |z|^2 + U^\nu(z)\) be a Gaussian background potential with a superharmonic perturbation, i.e. we assume that \(\alpha > 0\) and \(\nu\) is a finite positive Borel measure with compact support. Then
\[
\text{supp} \, \mu_{Q,t} \subseteq \overline{D} \left(0, \sqrt{\frac{t + \nu(C)}{2\alpha}} \right).
\]
As a final remark we note that
\[
\rho = \sqrt{\frac{t + \nu(\mathbb{C})}{2\alpha}} \quad \Rightarrow \quad 2\alpha\rho - \frac{t + \nu(\mathbb{C})}{\rho} = 0,
\]
i.e. the smallest possible value for \(\rho\) that still makes \(G = (E', \sigma, c)\) a \(t\)-extension is precisely the one that makes the arc measure supported on \(\{ z : |z| = \rho \}\) in \(\sigma\) vanish; this is precisely the radius required to make the radial derivative of the obtained \(t\)-extension \(\tilde{Q}\) continuous in a neighborhood of \(\{|z| = \rho\}\) (cf. Figure 2).

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