Brief paper

Random asynchronous iterations in distributed coordination algorithms

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ABSTRACT

Distributed coordination algorithms (DCA) carry out information processing processes among a group of networked agents without centralized information fusion. Though it is well known that DCA characterized by an SIA (stochastic, indecomposable, aperiodic) matrix generate consensus asymptotically via synchronous iterations, the dynamics of DCA with asynchronous iterations have not been studied extensively, especially when viewed as stochastic processes. This paper aims to show that for any given irreducible stochastic matrix, even non-SIA, the corresponding DCA lead to consensus successfully via random asynchronous iterations under a wide range of conditions on the transition probability. Particularly, the transition probability is neither required to be i.i.d, nor characterized by a Markov chain.

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1. Introduction

Distributed coordination algorithms (DCA) use local information of a group of networked agents to generate specific collective behaviours. Nowadays, DCA have been used not only to explain social or economic phenomena (DeGroot, 1974), but also to solve practical engineering problems (Nedić & Olshevsky, 2015). A typical DCA generates aligned collective motion, usually referred to as consensus (or synchronization, Meng, Yang, Li, Ren, and Wu (2018) and Wang, Ma, Zheng, and Chen (2017)).

When agents update their states using DCA, they may do so synchronously or asynchronously. More precisely, by synchronous updating we mean all the agents update their states at the same time repeatedly. This requires the availability of a global clock or a set of identical local clocks, which is a stringent requirement in practice (see, Dolev, Halpern, & Strong, 1986). By asynchronous updating we mean each agent has an independent local clock, according to which it updates its own state without paying attention to when the other agents update. Only in this way DCA can be constructed in a fully distributed manner (Cao, Morse, & Anderson, 2008a). It has been recently reported that the synchronous and asynchronous implementation of DCA may lead to dramatically different asymptotic collective behaviours; for example, some DCA may converge under synchronous updating but diverge under asynchronous updating (Xia & Cao, 2014; Xia, Liu, Cao, Johansson, & Basar, 2019).

The possible significant differences of DCA in deterministic and stochastic settings are reflected in the fundamental differences in their corresponding analytical tools: the determination of convergence in DCA, even for a pair of stochastic matrices, has been proved to be NP-hard in a deterministic setting (Blondel & Olshevsky, 2014); in sharp comparison, the convergence for DCA with finite Markovian random switching modes can be determined by using the classic LMI-based techniques (Boukas & Yang, 1995), which have been proved to obtain desired solutions with high efficiency. Therefore, the analysis in stochastic settings often generates less conservative sufficient conditions for consensus: In 2007, Porfiri and Stilwell gave some sufficient conditions for consensus over random weighted directed graphs generated by i.i.d random variables (Porfiri & Stilwell, 2007). In 2013, You et al. established some necessary and sufficient conditions for consensus based on the assumption of Markovian switching topologies (You, Li, & Xie, 2013); Matei et al. gave some sufficient conditions for linear consensus problem under Markovian random graphs (Matei, Baras, & Somarakis, 2013). Note that the network topologies in these works are either Markovian or
generated by i.i.d random variables. When asynchronous updating in DCA is not generated by a Markovian chain or independent random variables, the analysis becomes much more challenging due to the limitations of the existing analytical methods.

In the deterministic setting, the topological condition for consensus of DCA with asynchronous iteration is generally very restrictive. In 2014, Xia and Cao proved that for any given scrambling matrix, the corresponding DCA reach consensus for any asynchronous iteration (Xia & Cao, 2014). It should be noted that any pair of nodes in the graph of a scrambling matrix share a common neighbor, which makes such a graph densely connected. In this paper, we will investigate the asynchronous iterations of DCA in the stochastic setting with the aim of relaxing the restrictive constraint on topological structures. To realize this purpose, we will transform the consensus problem of random asynchronous DCA to the random walk problem along a labeled directed cycle, and propose a graphical method to analyse the convergence of random asynchronous DCA. Specifically, the contributions of this paper can be summarized as follows:

(a) The convergence of asynchronous iterations of DCA is investigated in the stochastic setting for the first time. The obtained results only require the graph of the given matrix of DCA to be connected. Compared with the related results in the deterministic setting, we do not need to require the matrix to be SIA or scrambling;

(b) The critical conditions for consensus in the traditional stochastic setting, such as i.i.d or Markovian switching, existence of self-loops in the communication topology are not needed in our main result any more.

Moreover, removal of self-loops in the communication topology changes the computational complexity of the consensus problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficult: as indicated in Blondel and Olshevsky (2014), the deciding of consensus is problem and makes the analysis extremely difficulty: as indicated

In order to regroup the nodes in a graph, we assign a positive integer to each node of a graph, and these integers are called labels of the nodes. A graph \( G = (V, \mathcal{E}) \) is said to be a labelled graph if each node \( i \in V \) is assigned with an integer label \( \text{label}(i) \). It should be noted that there may exist two nodes \( i, j \in V \) with identical labels, i.e., \( \text{label}(i) = \text{label}(j) \). The following proposition tells us the fact: any strongly connected graph can be mapped to a labelled directed cycle.

**Proposition 1.** Given any strongly connected graph \( G = (V, \mathcal{E}) \) with \( N \) nodes, it can be mapped to a labelled directed cycle \( \mathcal{E} \) with length \( l \leq N(N - 1) \) and contains all the nodes of \( V \).

**Proof.** Due to the strong connectivity of graph \( G \), there exists a path from \( k \) to \( k + 1 \) (\( 1 \leq k \leq N - 1 \)) with length no more than \( N - 1 \), and there also exists a path from node \( N \) to 1 with length no more than \( N - 1 \). Concatenate these \( N \) paths and one generates a cycle which contains all the nodes of \( V \). \( \square \)

As shown in Fig. 1, a strongly connected graph with 4 nodes can be transformed to a labelled directed cycle with 6 nodes, in which \( V = \{1, 2, 3, 4, 5, 6\} \) and the label of each node \( i \in V \) is given in the bracket. In particular, nodes 2 and 5 share the same label 2, nodes 6 and 3 share the same label 6.

A matrix \( A = (a_{ij})_{i,j=1}^N \) is called stochastic (or row stochastic) if \( a_{ij} \geq 0 \) \( (\forall i, j \in V) \) and \( \sum_{j=1}^N a_{ij} = 1 \) \( (\forall i \in V) \). The graph \( G(A) = (V, \mathcal{E}) \) corresponding to \( A \) is defined by: \( V = \{1, 2, \ldots, N\} \) and \( (i, j) \in \mathcal{E} \) if and only if \( a_{ij} > 0 \). Based on \( G(A) \) we define \( \xi(A) = \xi(G(A)) \) and \( \mathcal{N}(A, \nu) = \{k \mid a_{ik} > 0, \ k \in V\} \) as the neighbour of node \( \nu \) in \( G(A) \). The ergodic coefficient of a stochastic matrix \( A \) is defined by

\[
\lambda(A) = 1 - \min_{i,j} \sum_{k=1}^N min(a_{ik}, a_{jk}).
\]

A stochastic matrix \( A \) is called scrambling if \( \lambda(A) < 1 \), and is called SIA if \( \lim_{k \to \infty} A^k = \xi 11^T \) for some \( \xi \in \mathbb{R}^N \), where \( 1 \in \mathbb{R}^N \) is a vector with each entry being 1.

Given two stochastic matrices \( A = (a_{ij})_{i,j=1}^N, B = (b_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N} \), we say \( A \) and \( B \) are of the same type if \( \text{sgn}(a_{ij}) = \text{sgn}(b_{ij}) \) holds for any \( i, j = 1, 2, \ldots, N \), denoted by \( A \sim B \), where \( \text{sgn}(\cdot) \) is the sign function.

Given a vector \( x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N \), we define

\[
\Delta(x) = \max_{i=1}^N x_i - \min_{i=1}^N x_i
\]

as the maximal discrepancy of vector \( x \). Given a stochastic matrix \( A = Ax \) with \( y \in \mathbb{R}^N \), it holds that \( \Delta(y) \leq \lambda(A)\Delta(x) \). (Cao, Morse, and Anderson (2008b))

Specifically, the ergodic coefficient \( \lambda(\cdot) \) has the following important properties.

(a) Given any stochastic matrices \( A_1, A_2 \in \mathbb{R}^{N \times N} \), it holds \( \lambda(A_1A_2) \leq \lambda(A_1)\lambda(A_2) \). (Cao et al. (2008a))

(b) Given any stochastic matrix \( A \in \mathbb{R}^{N \times N} \), it holds \( \lambda(A) = \sup_{\Delta(x)=1} \Delta(Ax) \). (Seneta (1981))
A matrix $A = (a_{ij})_{i,j=1}^N$ is called column stochastic if $A^T$ is row stochastic. In this paper, we use a column stochastic matrix to denote a Markovian chain. The entry $a_{ij}$ in a column stochastic matrix represents the transition probability from node $j$ to node $i$. Without specific declaration, for any stochastic matrix in this paper, we mean it is row stochastic.

2.2. Problem formulation

DCA characterizes the evolution of states in a network of agents via local interaction. A typical one of DCA is the following linear averaging protocol

$$x_i(k + 1) = \sum_{j=1}^N a_{ij} x_j(k), \quad \text{if agent } i \text{ updates}$$

$$x_i(k), \quad \text{at time instant } k;$$

$$x_i(k), \quad \text{if agent } i \text{ does not update at time instant } k,$$

where $x_i(k)$ is the state of agent $i$ at time $k$, $a_{ij}$ denotes the coupling coefficient between agent $i$ and $j$, $a_{ij} \geq 0$ for any $i, j \in V$, and $\sum_{j=1}^N a_{ij} = 1$ for any $i \in V$. Define $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N}$ as the corresponding stochastic coupling matrix.

If each agent $i \in V$ updates at any time $k$, then the above DCA transforms to

$$x(k + 1) = Ax(k), \quad k = 1, 2, \ldots, \tag{1}$$

where

$$x(k) = (x_1(k), x_2(k), \ldots, x_N(k))^T$$

represents the state vector for all the individuals. We call DCA (1) the synchronous DCA.

However, synchronous DCA is difficult to be implemented in practice since the local clocks associated with all the nodes are generally nonidentical (see Dolev et al. (1986), the impossibility of clock synchronization). In the case of asynchronous iteration, only part of the agents $\sigma_k \in 2^V$ ($2^V$ is the set of all subsets of $V$) update their states at each time instant $k$, then one generates the following asynchronous DCA:

$$x(k + 1) = A_{\sigma_k} x(k), \tag{2}$$

where $A_{\sigma_k}$ is the asynchronous iteration matrix with the index set $\sigma_k \subseteq V$, whose definition is given by the following construction process: if $j \in \sigma_k$, then the $j$th row of $A_{\sigma_k}$ equals the $j$th row of $A$; if $j \notin \sigma_k$, then the $j$th row of $A_{\sigma_k}$ is the $j$th elementary vector $e_j$. Specifically, when only one agent updates at time $k$, $\sigma_k$ becomes a singleton and in this case we also use $\sigma_k$ to denote the agent $\sigma (\sigma \in V')$ that updates in the sequel. It follows that

$$A_{\sigma} \triangleq A_{\sigma_k} = (e_1, e_2, \ldots, e_{\sigma_k - 1}, a_{\sigma_k}, e_{\sigma_k + 1}, \ldots, e_N)^T.$$

where $a_{\sigma_k}$ is the $\sigma_k$th row of $A$, $e_j$ are elementary column vectors.

For example, given a stochastic matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0.7 & 0.3 \end{pmatrix},$$

then $A_{\sigma_0}$ defines

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{\{1,3\}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0.7 & 0.3 \end{pmatrix}.$$

Denote $(A_{\sigma_k})_{k=1}^\infty$ as an asynchronous iteration sequence of matrix $A$. $(A_{\sigma_k})_{k=1}^\infty$ is said to generate consensus in the deterministic setting if for any initial value $x(1) \in \mathbb{R}^N$, there exists $\xi \in \mathbb{R}$ such that

$$\lim_{k \to \infty} x(k + 1) = \xi$$

in DCA (2).

The convergence of an asynchronous iteration sequence of a stochastic matrix cannot be determined by the SIA property of this matrix: the asynchronous implementation of an SIA matrix may not generate consensus, but the asynchronous implementation of a non-SIA matrix may generate consensus (see, examples in Chen et al. (2018a) and Xia and Cao (2014)).

In 2014, Xia and Cao proposed the following sufficient condition for consensus of asynchronous DCA (2) in the deterministic setting.

**Proposition 2.** If $A$ is scrambling and there exists $q > 0$ such that $\left( \sum_{k=1}^{q-1} \sigma_k = V \right)$ for any $j \geq 1$, then the asynchronous iteration sequence $\{A_{\sigma_k}\}_{k=1}^\infty$ generates consensus.

However, when we consider asynchronous iteration in the stochastic setting, i.e., $\{\sigma_k\}_{k=1}^\infty$ are generated by random variables $\{\mathcal{Z}_k\}_{k=1}^\infty$, the dynamics of asynchronous DCA (2) will be quite different. A major challenge for the analysis is that the value of $\sigma_k$ and the corresponding transition probability may depend on its historic values $\sigma_{k-1}, \sigma_{k-2}, \ldots, \sigma_1$. In this paper, we use $(\Omega, \mathcal{F}, P)$ to represent the probability space, where $\Omega$ is the sample space of $\sigma_k$, $\mathcal{F}$ is the $\sigma$-field, and $P$ is the probability function.

When $\sigma_k \in 2^V$, the possible values of $A_{\sigma_k}$ is $\mathcal{A} = \{A_\sigma : \sigma \in 2^V\}$, where $|\mathcal{A}| = 2^N$. When $\sigma_k \in V$, the possible values of $A_{\sigma_k}$ is $\mathcal{A} = \{A_\sigma : \sigma \in V\}$, where $|\mathcal{A}| = N$. Since the diagonal entries of $A$ may contain zeros, the set $\mathcal{A}$ may also contain elements whose diagonal entries have some zeros. The existence of zero diagonal entries in the coupling matrices usually brings big challenges in analysing the products of stochastic matrices (see, Chen, Xiong, and Li (2016), Chen, Xia, Cao, and Lü (2018b), Touri and Nedić (2012), and Touri and Nedić (2014)).

In the stochastic setting, the asynchronous DCA (2) is said to realize consensus almost surely if

$$\lim_{k \to \infty} \mathbb{P} \left( \sum_{k=1}^{N} \|x(k) - \frac{1}{N} \sum_{i=1}^{N} x_i(k) \|^2 \geq \varepsilon \right) = 0 \tag{3}$$

for any $\varepsilon > 0$ and $x(1) \in \mathbb{R}^N$. If one defines the following projection matrix $P = I - \frac{1}{N} 1_1^T 1^T$, where $1$ is the identity matrix, then (3) can be equivalently rewritten as

$$\lim_{k \to \infty} \mathbb{P} \left( \|Px(k)\| \geq \varepsilon \right) = 0 \tag{4}$$

for any $\varepsilon > 0$ and $x(1) \in \mathbb{R}^N$. In the rest of this paper, for the simplicity of expression, we denote $k_1 : k_2 = \{k_1, k_1 - 1, \ldots, k_2\}$, $\sigma_{k_1 : k_2} = \{\sigma_{k_1}, \sigma_{k_1 - 1}, \ldots, \sigma_{k_2}\}$, $A_{\sigma_{k_1 : k_2}} = A_{\sigma_{k_1}} A_{\sigma_{k_1 - 1}} A_{\sigma_{k_1 - 2}} \cdots A_{\sigma_{k_2}}, A_{\sigma_{k_1}}$ for any $k_1 \geq k_2 \geq 1$. Particularly, it follows $\sigma_{\infty : k} = \{\sigma_k\}$ and in the case of $\sigma_k \in V$, we do not distinguish the expressions of $\sigma_k$ and $\{\sigma_k\}$.

In the next section, we will give several sufficient conditions which guarantee almost surely consensus of asynchronous DCA (2).

3. Conditions for random asynchronous consensus

We briefly summarize the sufficient conditions for random asynchronous consensus as follows.
The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

(a) $\mathcal{G}(A)$ is rooted and $\sigma_k \in 2^V$.

(b) There exists $\alpha > 0$ such that if $\mathbb{P}(\sigma_k \mid \sigma_{k-1}) \neq 0$, then $\mathbb{P}(\sigma_k \mid \sigma_{k-1}) \geq \alpha$.

(c) For any given past values of $\sigma_{k-1}$, the set

$$\mathcal{S}_{\sigma_{k-1}} = \{\sigma : \mathbb{P} (\sigma_k = \sigma \mid \sigma_{k-1}) \neq 0\}$$

only depends on $k$ but not the historic values $\sigma_{k-1}$, i.e., there exists $\mathcal{S}_k$ such that

$$\mathcal{S}_{\sigma_{k-1}} = \mathcal{S}_k, \quad \forall \sigma_{k-1} \in 2^V \times 2^V \cdots \times 2^V, \text{ \(k-1\) times}.$$  

(d) There exists $q > 0$ such that

$$\bigcup_{\sigma \in \mathcal{S}_k} \bigcup_{\tau=k}^{k+q-1} \sigma = V, \quad \forall k \geq 1.$$  

(e) There exists a strongly connected component $\chi$ of $\mathcal{G}(A)$ such that for any $j \in \chi$, it holds $\mathcal{S}_j \neq \emptyset$ and

$$\chi \cap \left( \bigcap_{\sigma \notin \mathcal{S}_k} \sigma \right) = \emptyset, \quad \forall k \geq 1,$$

where

$$\mathcal{S}_j = \{\sigma : \sigma \in \mathcal{S}_k \text{ and } j \in \sigma\}.$$  

The meaning of the 5 conditions in Theorem 1 is intuitive: condition (a) gives the topological condition on $\mathcal{G}(A)$; condition (b) requires a positive infimum on all the nonzero transition probabilities; condition (c) is called historical independence of nonzero probabilities, which requires the set $\mathcal{S}_{\sigma_{k-1}}$ to be independent of the historic values $\sigma_{k-1}$; condition (d) is called the joint coverage condition, which means each node of $V$ has a nonzero probability to be chosen to update in every consecutive $q$ steps; condition (e) is called the quasi-singleton property of nodes in $\chi$, which means once some node $j \in \chi$ has a nonzero probability to be chosen at time $k$, then the intersection of $\chi$ and all the possible values of $\sigma_k$ which contain $j$ is itself.

To illustrate the meanings of the conditions (c)-(e) in Theorem 1, we give the following example:

Example 1. Suppose $\{\sigma_k\}_{k=1}^{\infty}$ are generated by i.i.d random variables, $V = \{1, 2, 3, 4\}$, and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $\mathcal{G}(A) = \{1, 2, 3\}$. Consider a subset of $2^V$, such as

$$\mathcal{S}_k = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3\}\}, \forall k \geq 1.$$  

Since $\mathcal{S}_k$ does not rely on the historic values $\sigma_{k-1}$, condition (c) naturally holds. Note that

$$\mathcal{S}_1 = \{[1, 2, 4], [1, 3, 4], [2, 3]\}, \quad \mathcal{S}_2 = \{[2, 3], [1, 2, 4]\},$$

$$\mathcal{S}_3 = \{[1, 3, 4], [2, 3]\},$$

one can verify that

$$\bigcap_{\sigma \in \mathcal{S}_1} \sigma = \{1, 4\}, \quad \bigcap_{\sigma \in \mathcal{S}_2} \sigma = 2, \quad \bigcap_{\sigma \in \mathcal{S}_3} \sigma = 3.$$  

Since $\mathcal{G}(A) = \{1, 2, 3\}$ and letting $\chi = \mathcal{G}(A)$, condition (e) of Theorem 1 is satisfied. One further sets $q = 1$ and calculates that $\bigcup_{\sigma \in \mathcal{S}_1} \sigma = \{1, 2, 3\} = V$. Hence, condition (d) holds too.

The structure of the transition probability in conditions (c)-(e) of Theorem 1 can be described by a trellis graph (Touri & Nedić, 2012). The trellis graph of the random process $(\mathcal{S}_k)_{k\geq 1}$ is an infinite directed graph $\mathcal{G} = (V, E, (\sigma_k)_{k\geq 1})$, where $V$ is the infinite grid $2^V \times \mathbb{Z}^+$ and

$$E = \{(\sigma, k), (\sigma', k+1) \mid \sigma, \sigma' \in 2^V, k \geq 1\}.$$  

According to the above definition, the link in a trellis graph is pointed from time $k$ to time $k+1$ if the corresponding transition probability is nonzero.

As shown in Fig. 2, the given trellis graph satisfies the conditions of Theorem 1 for $V = \{1, 2, 3, 4\}, \mathcal{G}(A) = \chi = \{1, 2\}$, and $q = 2$. It should be noted that the weight of each edge in Fig. 2 is dependent on the historic values.

In Theorem 1, we do not require $(\mathcal{S}_k)_{k\geq 1}$ to be i.i.d or Markovian. Hence, some popular methods for linear stochastic systems, such as LMI (linear matrix inequality), cannot be easily applied. Even if $(\mathcal{S}_k)_{k\geq 1}$ are Markovian, the number of states in the Markovian chain is $|Q| = 2^N$ and hence it generates $2^N$ LMIs by using the method given in Boukas and Yang (1995). Unfortunately, LMs with such a huge dimension is very difficult to be solved.

The conditions of Theorem 1 imply the sample space of $A_n$ may contain matrices with zero diagonal entries. Traditionally, the positivity of the diagonal entries in the stochastic matrices plays a very important role in random or deterministic consensus, which can be seen in Porfiri & Stiwerl (2007), Shi & Johansson (2013), and Tahbaz-Salehi & Jadbabaie (2010). In Theorem 1, we do not require positivity of the diagonal entries in the sample space $\mathcal{G}$ to guarantee consensus of asynchronous DCA (2).

As simple corollaries of Theorem 1, we consider the following two practical cases of the random process $(\mathcal{S}_k)_{k\geq 1}$ which governs the asynchronous iteration:

(a) A global asynchronous clock: In this case, a global clock determines which node to update, and there is only one node updates at each time. Hence, $\sigma_k \in V$ is the unique node updates at time $k$.

(b) Independent asynchronous clocks: In this case, each node has a local clock, such a clock determines the update of the corresponding node independently. Hence, the updated nodes $\sigma_k$ at time $k$ is a set and it can be decomposed as $\sigma_k = \bigcup_{j=1}^N \theta_k^j$, where $\theta_k^j$ is an associated with the $j$th local clock. If $\theta_k^j = \{j\}$, then node $j$ updates at time $k$. If $\theta_k^j = \emptyset$, then node $j$ does not update at time $k$.

The sufficient conditions for almost surely consensus of asynchronous DCA (2) in the above two cases are given as follows:

Theorem 2. The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

1. The set $\mathcal{S}_k$ can be set as time-varying, but we only give a time-invariant example here for simplicity.
If the conditions of Theorem 3 are satisfied, then for any \( k \geq 1 \)
\[ \mathbb{P}(\sigma_k = j) \geq \alpha, \quad \forall j \in V. \]

**Theorem 3.** The asynchronous DCA (2) generates consensus almost surely if all the following conditions hold:

(a) \( g(A) \) is rooted.
(b) There exists \( \alpha > 0 \) such that for any \( k \geq 1 \)
\[ \mathbb{P}(\sigma_k = j) \geq \alpha, \quad \forall j \in V. \]
(c) There exists \( \beta > 0 \) such that for any \( k \geq 1 \)
\[ \mathbb{P}(\sigma_k = j) \leq \beta, \quad \forall j \in V. \]

4. **Technical skeleton**

Due to the limitation of space, we only present the technical skeleton of the proof of Theorem 1 as follows and the details can be found in Chen et al. (2018a).

First, we prove the equivalence between random asynchronous consensus and the convergence of the ergodic coefficient of products of stochastic matrices (see, Lemmas 1 and 2).

**Lemma 1.** If the asynchronous DCA (2) realizes consensus almost surely under the conditions of Theorem 1, then it realizes consensus uniformly with respect to the region of the initial value \( \mathcal{U} = \{ x : \Delta(x) \leq 1 \} \).

**Lemma 2.** The asynchronous DCA (2) realizes consensus almost surely if and only if \( \lim_{k \to \infty} \mathbb{P}(\lambda(A_{\sigma(k)})) \geq 0 \) holds for any \( \varepsilon > 0 \).

Second, we show that the transition probability among \( (\sigma_k)_{k \geq 1} \) has a special property: the backward probabilities in the trellis graph have a positive infinitum (see, Lemma 3).

**Lemma 3.** If the conditions of Theorem 1 are satisfied, then for any \( T > 0 \), there exists \( \gamma \in (0, 1) \) such that
\[ \mathbb{P}(\sigma_{k-1} | \sigma_k) \geq \gamma. \quad \forall T \geq k \geq 2, \]
when \( \mathbb{P}(\sigma_{k-1} | \sigma_k) \neq 0. \)

Third, based on the result of step 2, the asynchronous consensus problem will be mapped to the problem of random backward walk along a labelled directed cycle (see, Lemmas 4, 5), and the existence of such a cycle is guaranteed by Proposition 1.

**Lemma 4.** Given a sequence of column stochastic matrices \( \{P_k\}_{k=1}^\infty \), if \( P_k \geq W \) and \( P_k \sim W \) for each \( k \geq 1 \), and \( g(W) \) is rooted with node 1 as the unique root which contains a self-loop, then there exist \( c_0 > 0 \) and \( \beta \in (0, 1) \) such that \( \|P_k \cdots P_{k-1} - e_1 \| \leq c_0 \beta^k \), where \( e_1 \) is the elementary unit vector with the 1st entry being 1 and the other entries being 0.

**Lemma 5** (Random Walk Along a Labelled Directed Cycle). Given a labelled directed cycle \( \mathcal{C} \) with the set of nodes \( V = \{1, 2, \ldots, l\} \) and the corresponding set of labels \( V = \{ \text{label}(i) : i \in V \} \), we define the following random walk \( \mathcal{W} \) along this cycle: Given two nodes moving along the cycle \( \mathcal{C} \), the positions of them at time \( k \) are denoted by \( x_k \) and \( y_k \) with the corresponding historic values \( W_k = (x_k, y_k) \), the transition from \( (i_k, j_k) \) to \( (i_{k+1}, j_{k+1}) \) is described by:

- When label \( (i_k, j_k) \) \( \neq \) label \( (i_{k+1}, j_{k+1}) \), it holds
\[ \mathbb{P}(\sigma_{k+1} = j) \geq \gamma, \quad \forall j \in V, \]
- When label \( (i_k, j_k) \) \( = \) label \( (i_{k+1}, j_{k+1}) \), it holds
\[ \mathbb{P}(\sigma_{k+1} = j) \geq 1; \]

where \( g(A) \) is strongly connected. According to Seneta (1981), one can see \( A \) is not an SIA matrix. As shown in Fig. 3, the asynchronous DCA (2) cannot realize consensus with the above \( A \). However, when we implement the asynchronous DCA (2) with the same matrix \( A \), the dynamics becomes quite different:

Firstly, suppose DCA (2) has a global asynchronous clock. For any \( 1 \leq j \leq 6 \), one sees \( \mathbb{P}(\sigma_{k} = j) = 1/6 \) and the conditions of Theorem 2 are satisfied. As shown in Fig. 4, the agents realize consensus almost surely when the initial values are chosen randomly from \([-1, 1] \).

Next, suppose DCA (2) has independent local clocks. For any \( 1 \leq j \leq 6 \), one sees \( \mathbb{P}(\phi_{i_j} = j) = \frac{1}{6} \) and the conditions of Theorem 3 are satisfied. As shown in Fig. 5, the agents realize consensus almost surely when the initial values are chosen randomly from \([-1, 1] \).

6. **Conclusions**

This paper investigated the asynchronous iteration problem of distributed coordination algorithms in the stochastic setting. By asynchronous iteration we mean only a fraction of nodes update their states at a given time instant. We have found that...
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