Recovering two constants in a linear parabolic equation

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Abstract. We recover two real constants in a parabolic linear equation endowed with Robin boundary conditions. More exactly, we prove some theorems concerning existence or uniqueness when two pieces of additional information are given for large times.

1. Introduction

In this paper we are concerned with the problem of determining two constants in a linear parabolic equation related to a (smooth) $n$-dimensional bounded domain, using tools from Semi-group Theory and Functional Analysis.

We note that, to our knowledge, this approach seems not to have been used in literature. Moreover, concerning the determination of a single parameter we cite, e.g., the papers [1, 7, 13, 14], while concerning the determination of several parameters we cite, e.g., [3, 4, 5, 8, 10, 11], where, however, the parameters are the values of a piecewise constant coefficient.

We state now our problem. Let $L$ be the linear differential operator related to a bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary of class $C^3$ and defined by

$$L = \sum_{i,j=1}^{n} D_{x_i}(a_{i,j}(x)D_{x_j}),$$

where the coefficients satisfy

$$a_{i,j} \in C^1(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n, \quad \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \rho|\xi|^2, \quad \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^n,$$

for some positive constant $\rho$.

We are concerned with the problem consisting in determining the positive constant $a$ and the (non-zero) real constant $b$ entering the initial and boundary value problem

$$D_t u - Lu + au = bf, \quad \text{a.e. in } (0, T) \times \Omega,$$  \hfill (1.3)

$$u(0, \cdot) = u_0, \quad \text{a.e. on } \Omega,$$  \hfill (1.4)

$$D_{\nu_t} u + \beta u = 0, \quad \text{a.e. on } (0, T) \times \partial\Omega,$$  \hfill (1.5)
where \( T \in (0, +\infty) \) and \( \nu_L = ((\nu_L)_1, \ldots, (\nu_L)_n) \) stands for the conormal vector related to \( L \), i.e. \( (\nu_L)_j(x) = \sum_{i,j=1}^n \nu_i(x)a_{i,j}(x) \), \( j = 1, \ldots, n \), \( \nu(x) \) denoting the outward normal unit vector at \( x \in \partial \Omega \).

Our assumptions on the data are the following, where \( p \in (1, +\infty) \):

\[
\begin{align*}
  f &\in W^{\theta+2\alpha,p}(\Omega), \quad \alpha \in (0, 1/(2p)), \quad \theta \in (1/p, 1], \quad f \geq 0 \quad \text{a.e. in } \Omega, \quad f > 0, \quad \text{a.e. on } \Gamma, \quad (1.6) \\
  u_0 &\in W^2_p(\Omega), \quad u_0 \geq 0 \quad \text{a.e. in } \Omega, \quad \beta \in C^1(\partial \Omega), \quad \beta > 0, \quad \text{a.e. on } \partial \Omega, \quad (1.7)
\end{align*}
\]

where \( \Gamma \) is an open subset of \( \partial \Omega \) and

\[
W^2_p(\Omega) = \{ z \in W^2_p(\Omega) : D\nu_L z + \beta z = 0 \quad \text{on } \partial \Omega \}, \quad L_R u = Lu, \quad u \in W^2_p(\Omega). \quad (1.8)
\]

To recover \( a \) and \( b \) we are given the two following pieces of additional information:

\[
\int_{\partial \Omega} \psi_j(x)u(T, x)d\mu(x) = \gamma_j, \quad j = 1, 2, \quad (1.9)
\]

where \( \mu \) denotes the surface Lebesgue measure and

\[
\begin{align*}
  \gamma_j &\neq 0, \quad j = 1, 2, \quad \psi_j \in L^{p'}(\partial \Omega), \quad 1/p + 1/p' = 1, \quad \psi_j \geq 0 \quad \text{a.e. on } \partial \Omega, \quad (1.10) \\
  \psi_j &> 0 \quad \text{a.e. on } \Gamma, \quad \psi_1 \text{ and } \psi_2 \text{ are linearly independent.} \quad (1.11)
\end{align*}
\]

2. Representing the solution to the direct problem (1.1)-(1.3)

We will denote by \( u(a, b) \) the solution to problem (1.3)-(1.5), depending on the pair of parameters \( (a, b) \). We will show that \( u(a, b) \in C^1([0, +\infty); L^p(\Omega)) \cap C([0, +\infty); W^2_p(\Omega)) \).

For this purpose observe that the function

\[
v(a, b)(t) = e^{at}u(a, b)(t) \quad (2.1)
\]

solves the problem

\[
\begin{align*}
  D_t v - Lv &= be^{at}f, \quad \text{a.e. in } (0, +\infty) \times \Omega, \quad (2.2) \\
  v(0, \cdot) &= u_0, \quad \text{a.e. on } \Omega, \quad (2.3) \\
  D_t v + \beta v &= 0, \quad \text{a.e. on } (0, +\infty) \times \partial \Omega. \quad (2.4)
\end{align*}
\]

We note now that, according to the results in [2], we can conclude that the spectrum \( \sigma_p(L_R) \) of \( L_R \) (cf. (1.8)) is invariant with respect to \( p \in (1, +\infty) \). So, we have

\[
\sigma_p(L_R) = \sigma_2(L_R) = C \setminus \bigcup_{n=1}^{+\infty} \{ \lambda_n \},
\]

where \( \{ \lambda_n \}_{n=1}^{+\infty} \) is the decreasing sequence of (negative) eigenvalues of \( L_R \) diverging to \( -\infty \). This is implied by our assumptions \( \beta > 0 \) on \( \partial \Omega \). In particular, \( L_R \) admits a continuous inverse \( L_R^{-1} \).

Observe now that from Theorem 3.1.3 and Proposition 2.1.11 in [9] we deduce that there exist \( \mu_0 > \lambda_1 \) and \( \theta_0 \in (\pi/2, \pi) \) such that

\[
\| (\lambda - L_R)^{-1} \|_{L^p(\Omega)} \leq c_0|\lambda - \mu_0|^{-1}, \quad \text{if } \lambda = \mu_0 + \Sigma \theta_0, \quad (2.5)
\]

where \( \Sigma \theta_0 = \{ \mu \in C \setminus \{ 0 \} : |\arg \mu| \leq \theta_0 < \pi \} \).
Fix now $\mu_1 \in (\lambda_1, \mu_0)$ and a compact rectangle with sides parallel to the real and imaginary axes whose interior contains $\mu_0$ and the intersection of $\mu_0 + \Sigma_{\theta_0}$ with the straight line $\text{Re} \lambda = \mu_1$. Since the resolvent $(\lambda - L_R)^{-1}$ is an analytical function on $E$, it is, of course, bounded on $E$. Then some standard reasonings lead to the estimate

$$\|(\lambda - L_R)^{-1}\|_{L(\Omega)} \leq c_1|\lambda - \mu_1|^{-1}, \quad \text{if} \ \text{Re} \lambda \geq \mu_1.$$  

Applying again Proposition 2.1.11 in [9], we can extend estimate (2.5) to the following

$$\|(\lambda - L_R)^{-1}\|_{L(\Omega)} \leq c_2|\lambda - \mu_1|^{-1}, \quad \text{if} \ \lambda = \mu_1 + \Sigma_{\theta_1},$$  

(2.6)

for some $\theta_1 \in (\theta_0, \pi)$.

Introduce now the analytic semigroup of linear bounded operators $\{S(t)\}_{t \geq 0}$ generated - via the Dunford integral (cf. [9, Chapter 2]) - by operator $L_R$ endowed with homogeneous Robin boundary conditions.

Taking advantage of Proposition 2.3.1 and 2.2.7 in [9] we conclude that $S$ satisfies the following estimates for all $t \in \mathbb{R}_+$, $j \in \mathbb{N}$ and $p \in (1, +\infty)$:

$$\|D_j S(t)\|_{L^p(\Omega)} \leq C_j t^{-j} e^{-\omega t},$$  

(2.7)

$$\|\nabla_R S(t)\|_{L^p(\Omega)} \leq \tilde{C}_1 t^{-j/2} e^{-\omega t},$$  

(2.8)

$$\|L^j S(t)\|_{L^p(\Omega)} \leq \tilde{C}_1 t^{-j} e^{-\omega t},$$  

(2.9)

for suitable positive constants $C_j$, $\tilde{C}_1$ and $\omega \in (0, |\lambda_1|)$.

Using Theorem 4.3.1 in [9], it easily follows that the solution $v = v(a, b)$ to problem (2.2)–(2.4) belongs to $C^1([0, +\infty); L^p(\Omega)) \cap C([0, +\infty); W^{2,p}(\Omega))$. Consequently, $u(a, b) = e^{-at}v(a, b)$ belongs to the same space.

According to well-known results (cf. e.g. [9, Chapter 4]) $v(a, b)$ admits the following representation

$$v(a, b)(t) = S(t)u_0 + b \int_0^t e^{a(t-\tau)} S(\tau) f d\tau.$$  

(2.10)

From (2.1) and (2.10) we deduce

$$u(a, b)(t) = e^{-at}S(t)u_0 + b \int_0^t e^{-a\tau} S(\tau) f d\tau.$$  

(2.11)

3. Solving problem (1.3)-(1.5), (1.9) when $T = +\infty$

First we recall the fundamental formula (cf. [9, Lemma 2.1.6])

$$\int_0^{+\infty} e^{-a\tau} S(\tau) f d\tau = (a - L_R)^{-1} f, \quad a \in \mathbb{R}_+,$$  

(3.1)

Taking the limit at $t \to +\infty$ in (2.11), we get the formula

$$u(a, b)(+\infty) = b \int_0^{+\infty} e^{-a\tau} S(\tau) f d\tau = b(a - L_R)^{-1} f.$$  

(3.2)

Due to properties (1.6) and (1.7) and the minimum principle, we easily deduce that $u(a, b)$ is non-negative if $b > 0$ (for the reader’s convenience a proof is reported in Appendix). In particular, $(a - L_R)^{-1} f$ is non-negative.
Then from (3.2) and (1.9), with $T = +\infty$, we derive the following system for $(a, b)$:

$$bg_j(a) = \gamma_j, \quad j = 1, 2,$$

where $g_j : \mathbb{R}_+ \to [0, +\infty)$ is defined by

$$g_j(a) = \int_{\partial \Omega} \psi_j(x)(a - L_R)^{-1} f(x) \, d\mu(x), \quad j = 1, 2.$$  \hspace{1cm} (3.4)

According to assumption (1.10) from (3.3) we deduce that $b \neq 0$.

Observe now that $g_j$ is (real) analytic and nonincreasing in $\mathbb{R}_+$, since, owing to (1.10) and (1.6) we get

$$g_j'(a) = -\int_{\partial \Omega} \psi_j(x)(a - L_R)^{-2} f(x) \, d\mu(x) \leq 0, \quad a \in \mathbb{R}_+, j = 1, 2.$$  \hspace{1cm} (3.5)

To determine the behaviour of $g_j$, $j = 1, 2$, at $+\infty$, we begin by observing that from the identity

$$a(a - L_R)^{-1} f = f + L_R(a - L_R)^{-1} f, \quad \forall f \in L^p(\Omega),$$

from the generation estimate (2.6) and assumption (1.6), we get the relations

$$\|a(a - L_R)^{-1} f - f\|_{W^{1,1}(\Omega)} \leq C(\alpha, p) a^{-\alpha} \|f\|_{W^{1,2\alpha}(\Omega)}, \quad \forall a \in \mathbb{R}_+, \, \alpha \in (0, 1/(2p)).$$  \hspace{1cm} (3.7)

To show (3.7) we introduce in $W^2_{2\alpha}(\Omega)$ the norm $\|L_Ru\|_{L^p(\Omega)}$ that is equivalent to the usual one, since $0 \in \rho(L_R)$ (cf. (2.6)). Moreover, we note that the following interpolation formulae hold true (cf. [6, Theorem 7.5]):

$$W^2_{2\alpha}(\Omega) = (W^2_{1\alpha}(\Omega); L^p(\Omega))_{\theta, \alpha}, \quad \theta \in (0, (p + 1)/(2p)), \hspace{1cm} (3.8)$$

$$W^{\theta+2\alpha}(\Omega) = (W^2_{1\alpha}(\Omega); W^{2\alpha}(\Omega))_{\theta/2, \alpha}, \quad \theta \in (1/p, 1], \hspace{1cm} (3.9)$$

$W^{2\alpha}(\Omega)$ being defined by (1.8) with $W^{2\alpha}(\Omega)$ being replaced by $W^{2\alpha}(\Omega)$.

Introduce then the linear operator $L(a)$ depending on the parameter $a \in \mathbb{R}_+$ defined by

$$D(L(a)) = L^p(\Omega), \quad L(a) = L_R(a - L_R)^{-1}.$$  \hspace{1cm} (3.10)

Then $L(a) \in \mathcal{L}(L^p(\Omega))$ and satisfies the estimates

$$\|L(a)f\|_{L^p(\Omega)} \leq (\tilde{C}_1 + 1) \|f\|_{L^p(\Omega)},$$

$$\|L(a)f\|_{W^2_{2\alpha}(\Omega)} = \|(a - L_R)^{-1} L_R f\|_{L^p(\Omega)} \leq C_0 a^{-\alpha} \|f\|_{W^2_{1\alpha}(\Omega)}.$$  \hspace{1cm} (3.11)

We have thus shown that

$$L(a) \in \mathcal{L}(L^p(\Omega)) \cap \mathcal{L}(W^{2\alpha}(\Omega); L^p(\Omega)).$$

Then, by interpolation, we easily get

$$L(a) \in \mathcal{L}(W^{2\alpha}(\Omega); L^p(\Omega)), \quad \|L(a)\|_{\mathcal{L}(W^{2\alpha}(\Omega); L^p(\Omega))} \leq C_0(\alpha, p)^{1-\alpha} \tilde{C}_1(\alpha, p) a^{-\alpha},$$

Consider then the following estimate, where $f \in W^{2\alpha}(\Omega)$:

$$\|L(a)f\|_{W^{2\alpha}(\Omega)} = \|L_R(a - L_R)^{-1} L_R f\|_{L^p(\Omega)} \leq C(\alpha, p) a^{-\alpha} \|L_Rf\|_{W^{2\alpha}(\Omega)} \leq C_1(\alpha, p) a^{-\alpha} \|f\|_{W^{2\alpha}(\Omega)}.$$  \hspace{1cm} (3.12)
We have thus shown that
\[ L(a) \in \mathcal{L}(W^{2\alpha,p}_R(\Omega); L^p(\Omega)) \cap \mathcal{L}(W^{2+2\alpha,p}_R(\Omega); W^{2,p}_R(\Omega)). \]

Then, by interpolation, from (3.9) we get
\[ L(a) \in \mathcal{L}(W^{\theta+2\alpha,p}(\Omega); W^{\theta,p}(\Omega)) \]
\[ \|L(a)\|_{\mathcal{L}(W^{\theta+2\alpha,p}(\Omega); W^{\theta,p}(\Omega))} \leq C_0(\alpha,p)^{(1-\alpha)(1-\theta/2)} C_1(\alpha,p)^{\theta/2} a^{-\alpha}, \]
proving (3.7).

Since the traces on \( \partial \Omega \) of functions in \( W^{\theta,p}(\Omega) \), with \( \theta \in (1/p, 1] \), belong to \( L^p(\partial \Omega) \), from (3.7) we immediately deduce the relation
\[ a \int_{\partial \Omega} \psi_j(x)(a - L_R)^{-1} f(x) d\mu(x) \to \int_{\partial \Omega} \psi_j(x) f(x) d\mu(x), \quad \text{as } a \to +\infty, \ j = 1, 2. \quad (3.10) \]

Then, owing to (1.6) and (1.10), we easily conclude that
\[ \int_{\partial \Omega} \psi_j(x) f(x) d\mu(x) > 0, \quad j = 1, 2. \quad (3.11) \]

Consequently, we deduce that \( g_j, \ j = 1, 2, \) are strictly positive. Therefore, as far as solvability is concerned, system (3.3) is equivalent to the equation
\[ g(a) = 0, \quad a \in \mathbb{R}_+, \quad g(a) = \gamma_2 g_1(a) - \gamma_1 g_2(a). \]

Observe first that
\[ ag(a) \to \int_{\partial \Omega} [\gamma_2 \psi_1(x) - \gamma_1 \psi_2(x)] f(x) d\mu(x), \quad \text{as } a \to +\infty. \]

Observe now that from (3.4), (3.5), (3.11) we deduce
\[ g_j(0+) = -\int_{\partial \Omega} \psi_j(x) L_R^{-1} f(x) d\mu(x) > 0, \quad j = 1, 2. \quad (3.12) \]

**Remark 3.1.** From (3.5) and (3.12) we deduce
\[ 0 < g_j(a) \leq -\int_{\partial \Omega} \psi_j(x) L_R^{-1} f(x) d\mu(x), \quad \forall a \in (0, +\infty), \ j = 1, 2. \quad (3.13) \]

Moreover, there exists a positive functional \( K \) such that
\[ g_j(a) \geq \frac{K(\psi_j, f)}{1 + a}, \quad \forall a \in (0, +\infty), \ j = 1, 2. \quad (3.14) \]

Then from (3.12) we get
\[ g(0+) = \int_{\partial \Omega} [\gamma_1 \psi_2(x) - \gamma_2 \psi_1(x)] L_R^{-1} f(x) d\mu(x). \]

Assume now that the quintuplet \( (f, \psi_1, \psi_2, \gamma_1, \gamma_2) \) satisfies the condition
\[ \int_{\partial \Omega} [\gamma_1 \psi_2(x) - \gamma_2 \psi_1(x)] L_R^{-1} f(x) d\mu(x) \int_{\partial \Omega} [\gamma_1 \psi_2(x) - \gamma_2 \psi_1(x)] f(x) d\mu(x) > 0. \quad (3.15) \]
Then the analytic function $a \to g(a)$ changes its sign in the interval $(0, +\infty)$: therefore there exists (at least an) $a_0 \in (0, +\infty)$ such that $g(a_0) = 0$. Consequently, if condition (3.15) is satisfied, when $T = +\infty$ our problem admits at least the solution $(a, b) = (a_0, \gamma_2[g_2(a_0)]^{-1})$.

We note that, due to the analyticity of $g$, such a function admits at most countable many zeros $a_0 \in (0, +\infty)$.

We can summarize the results proved in this section in a theorem.

**Theorem 3.1.** Let conditions (1.2), (1.6)-(1.8), (1.10), (1.11) and (3.15) be satisfied and let $T = +\infty$. Then problem (1.3)-(1.5), (1.9) admits at least a solution $(u, a, b) \in [C^1([0, +\infty); L^p(\Omega)) \cap C([0, +\infty); W^2_{p, \Omega})] \times \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$.

We conclude this section with some discussion concerning condition (3.15).

Recall that

$$\int_{\partial \Omega} \psi_i(x)L_R^{-1}f(x) \,d\mu(x) \leq 0, \quad \int_{\partial \Omega} \psi_i(x)f(x) \,d\mu(x) > 0, \quad i = 1, 2. \tag{3.16}$$

Assume now

$$\int_{\partial \Omega} \psi_2(x)L_R^{-1}f(x) \,d\mu(x) < 0. \tag{3.17}$$

Then (3.15), considered as an inequality in $\gamma_1$, yields

$$\left\{ \begin{array}{ll} \gamma_2 \min \{I_1(f), I_2(f)\} < \gamma_1 < \gamma_2 \max \{I_1(f), I_2(f)\}, & \text{if } \gamma_2 > 0, \\
\gamma_2 \max \{I_1(f), I_2(f)\} < \gamma_1 < \gamma_2 \min \{I_1(f), I_2(f)\}, & \text{if } \gamma_2 < 0,
\end{array} \right.$$  

where

$$I_1(f) = \frac{\int_{\partial \Omega} \psi_1(x)L_R^{-1}f(x) \,d\mu(x)}{\int_{\partial \Omega} \psi_2(x)L_R^{-1}f(x) \,d\mu(x)}, \quad I_2(f) = \frac{\int_{\partial \Omega} \psi_1(x)f(x) \,d\mu(x)}{\int_{\partial \Omega} \psi_2(x)f(x) \,d\mu(x)}. \tag{3.18}$$

The previous conditions implies that $f$ must satisfy the relation

$$\frac{\int_{\partial \Omega} \psi_1(x)L_R^{-1}f(x) \,d\mu(x)}{\int_{\partial \Omega} \psi_2(x)L_R^{-1}f(x) \,d\mu(x)} \neq \frac{\int_{\partial \Omega} \psi_1(x)f(x) \,d\mu(x)}{\int_{\partial \Omega} \psi_2(x)f(x) \,d\mu(x)}. \tag{3.19}$$

4. A uniqueness result when $T = +\infty$

Assume that $u(a_j, b_j)$ is a solution to the direct problem (1.3)-(1.5) corresponding to the pair $(a_j, b_j)$, $j = 1, 2$, and satisfying the additional conditions (1.9). Introduce then the auxiliary functions

$$u = u_2 - u_1, \quad a = a_2 - a_1, \quad b = b_2 - b_1. \tag{4.1}$$

Then $u$ solves the problem

$$\begin{align*}
D_1u - Lu + a_2u &= bf - au_1(a_1, b_1), \quad \text{in } (0, +\infty) \times \Omega, \tag{4.2} \\
u(0, \cdot) &= 0, \quad \text{on } \Omega, \tag{4.3} \\
D_\nu u + \beta u &= 0, \quad \text{on } (0, +\infty) \times \partial \Omega, \tag{4.4} \\
\int_{\partial \Omega} \psi_j(x)u(+\infty, x) \,d\mu(x) &= 0, \quad j = 1, 2. \tag{4.5}
\end{align*}$$
Then observe that from [12, Chapter 4, Theorem 4.4] and (3.2), with \((a, b) = (a_1, b_1)\), we easily deduce the formula

\[
u(\pm \infty) = \lim_{t \to \pm \infty} \int_0^t e^{-a_2(t-\tau)} S(t-\tau)[b f - au_1(\tau)] \, d\tau = (a_2 - L_R)^{-1}[bf - au_1(a_1, b_1)(\pm \infty)]
\]

Following the same procedure as in Section 3, with \(bf\) being replaced with \(bf - ab_1(a_1 - L_R)^{-1}f\), we easily deduce that the pair \((a, b)\) must solve the linear system

\[-b_1\alpha_j(a_1, a_2)a + g_j(a_2)b = 0, \quad j = 1, 2, \quad (4.6)\]

where \(g_j\) is defined by (3.4) and, for all \(f \in L^p(\Omega)\), we have set

\[
\alpha_j(a_1, a_2) = \int_{\partial \Omega} \psi_j(x)(a_2 - L_R)^{-1}(a_1 - L_R)^{-1}f(x) \, d\mu(x), \quad j = 1, 2. \quad (4.7)
\]

If

\[
\alpha_1(a_1, a_2)g_2(a_2) - \alpha_2(a_1, a_2)g_1(a_2) \neq 0, \quad (4.8)
\]

we immediately deduce \(a_2 = a_1\) and \(b_2 = b_1\), implying \(u_2 = u_1\). When

\[
\alpha_1(a_1, a_2)g_2(a_2) - \alpha_2(a_1, a_2)g_1(a_2) = 0, \quad (4.9)
\]

the treatment is more difficult. However, we can show that the solution is locally unique. For this purpose observe that functions \(\alpha_j\) are holomorphic functions of \((a_1, a_2)\) and fix the pair \((a_1, b_1) \in \mathbb{R}_+ \times \mathbb{R}\). If inequality (4.8) holds with \(a_1 = a_2\), we can find a neighbourhood \(U(a_1) \subset \mathbb{R}_+\) of \(a_1\) such that (4.8) holds with \(a_2 \in U(a_1)\). In this case we have \(a_1 = a_2\) and \(b_1 = b_2\), implying the uniqueness of the solution of our identification problem. Observe now that the positive zeros of the holomorphic function \(a_1 \rightarrow \alpha_{1,1}(a_1, a_1)g_2(a_2) - \alpha_{2,1}(a_1, a_1)g_1(a_2)\) are at most countable whenever such a function is not identically zero. To exclude the latter case consider first the following identities holding for all \(f \in W^{2,p}_R(\Omega)\):

\[
a(a - L_R)^{-1}f = f + a^{-1}L_Rf + a^{-1}L_R(a - L_R)^{-1}L_Rf, \quad (4.10)
\]

\[
a^2(a - L_R)^{-2}f = f + 2a^{-1}L_Rf + 3(a - L_R)^{-1}L_R(a - L_R)^{-1}L_Rf - 2a^{-1}L_R^2(a - L_R)^{-2}L_Rf. \quad (4.11)
\]

For the sake of simplicity we limit ourselves to showing (4.11). For this purpose, let \(B : D(B) \subset X \to X\) be a closed linear operator such that \(1\) belongs to its resolvent set. Then we have the chain of identities

\[
(1 - B)^{-2}f - f - 2Bf = [1 - (1 + 2B)(1 - B)^2](1 - B)^{-2}f
\]

\[
= (3 - 2B)B^2(1 - B)^{-2}f = B^2(1 - B)^{-2}(3 - 2B)f, \quad \forall f \in D(B). \quad (4.12)
\]

Consequently, (4.11) follows from (4.12) with \(B = a^{-1}L_R\).
Observe now that, according to (3.7), the following estimates hold for all $f \in W^{2,p}_{\Omega}(\Omega)$, with $L_R f \in W^{2+2\alpha,p}_{\Omega}(\Omega)$, $\theta \in (1/p, 1]$ and $\alpha \in (0, 1/(2p))$:

$$
\|a(a-L_R)^{-1}f - f - a^{-1}L_R f\|_{W^{\theta,p}_{\Omega}(\Omega)} \\
\leq a^{-1}\|L_R(a-L_R)^{-1}L_R f\|_{W^{\theta,p}_{\Omega}(\Omega)} \\
\leq C a^{-1-\alpha}\|L_R f\|_{W^{2+2\alpha,p}_{\Omega}(\Omega)}, \\
\|a^2(a-L_R)^{-2}f - 2a^{-1}L_R f\|_{W^{\theta,p}_{\Omega}(\Omega)} \\
\leq C a^{-1}\|L_R(a-L_R)^{-1}L_R f\|_{W^{p,\theta}_{\Omega}(\Omega)} + 2a^{-1}\|L_R^2(a-L_R)^{-2}L_R f\|_{W^{p,\theta}_{\Omega}(\Omega)} \\
\leq C a^{-1-\alpha}\|L_R f\|_{W^{2+2\alpha,p}_{\Omega}(\Omega)}.
$$

Consequently, after some easy computations, we deduce the following asymptotic relations for all $f \in W^{2,p}_{\Omega}(\Omega)$, with $L_R f \in W^{2+2\alpha,p}_{\Omega}(\Omega)$, $\alpha \in (0, 1/(2p))$, $\theta \in (1/p, 1]$ and $j = 1, 2$:

$$
a_1^2[a_1(a_1,a_1)g_2(a_2) - a_2(a_1,a_1)g_1(a_1)] \\
= \int_{\Omega} \psi_1(x)[f(x) + 2a_1^{-1}L_R f(x)] \, d\mu(x) \int_{\Omega} \psi_2(x)[f(x) + a_1^{-1}L_R f(x)] \, d\mu(x) \\
- \int_{\Omega} \psi_2(x)[f(x) + 2a_1^{-1}L_R f(x)] \, d\mu(x) \int_{\Omega} \psi_1(x)[f(x) + a_1^{-1}L_R f(x)] \, d\mu(x) + O(a_1^{-2-2\alpha}) \\
\sim a_1^{-1}J(f, \psi_1, \psi_2) + O(a_1^{-2}), \text{ as } a_1 \to +\infty,
$$

where we have set (cf. also (3.19))

$$
J(f, \psi_1, \psi_2) = \int_{\Omega} \psi_1(x)L_R f(x) \, d\mu(x) \int_{\Omega} \psi_2(x)f(x) \, d\mu(x) \\
- \int_{\Omega} \psi_2(x)L_R f(x) \, d\mu(x) \int_{\Omega} \psi_1(x)f(x) \, d\mu(x)
$$

and we have assumed

$$
J(f, \psi_1, \psi_2) \neq 0. \tag{4.13}
$$

Under such a condition function $a_1 \to a_1(a_1,a_1)g_2(a_2) - a_2(a_1,a_1)g_1(a_1)$ is not the null function. This concludes our treatment of the local conditional uniqueness. Here conditional means that the local uniqueness holds except for at most countable many pairs $(a_n, b_n) \in R_+ \times (R \setminus \{0\})$.

We can summarize the results proved in this section in a theorem.

**Theorem 4.1.** Let $T = +\infty$ and $f \in W^{2,p}_{\Omega}(\Omega)$, $L_R f \in W^{\theta+2\alpha,p}_{\Omega}(\Omega)$, $\alpha \in (0, 1/(2p))$ $\theta \in (1/p, 1]$. Then, under conditions (1.2), (1.6)-(1.8), (1.10), (1.11) and (4.13), problem (1.3)-(1.5), (1.9) admits a locally unique solution solution $(u, a, b) \in [C^1([0, +\infty); L^p(\Omega))] \cap C([0, +\infty); W^{2,p}_{\Omega}(\Omega))] \times R_+ \times (R \setminus \{0\})$, except for at most countable many pairs $(a_n, b_n) \in R_+ \times (R \setminus \{0\})$.

We conclude this section by a remark on the local continuous dependence of $(u, a, b)$ on $(f, \gamma_1, \gamma_2)$. For this purpose assume that $(a', a', b')$ is a solution to problem (1.3)-(1.5), (1.9) corresponding to the data $(f', \gamma_1', \gamma_2')$, $i = 1, 2$, satisfying (1.6)-(1.8), (1.10), (1.11) and (1.15) with $f_i \in W^{2,p}_{\Omega}(\Omega)$, $L_R f_i \in W^{1+2\alpha,p}_{\Omega}(\Omega)$, $i = 1, 2$, $\alpha \in (0, 1/(2p))$.

Reasoning as at the beginning of this section and using the same notation, we easily deduce that system (4.6) changes to

$$
-b_1 \alpha_j(a_1,a_2)a + g_j(a_2)b = \gamma_j - b_2 \int_{\Omega} \psi_j(x)f(x) \, d\mu(x), \quad j = 1, 2, \tag{4.14}
$$
where $\gamma_j = \gamma_j^2 - \gamma_j^1$ and $f = f^2 - f^1$. Assume, as above, that the pair $(a_1, b_1)$ is fixed and (4.8) holds with $a_2 = a_1$. Then, system (4.14) ensures that $(a_2, b_2)$ tends to $(a_1, b_1)$ as $(f^2, \gamma_1^2, \gamma_2^2) \to (f^1, \gamma_1^1, \gamma_2^1)$ in $W^2_p(\Omega) \times \mathbb{R} \times \mathbb{R}$ with $L_f f^2 \to L_R f^1$ in $W^{p+2\alpha, p}(\Omega)$. Finally, well-known results for the direct problem (1.3)-(1.5) show that $u(a_2, b_2)$ tends to $u(a_1, b_1)$ in $C^1([0, +\infty); L^p(\Omega)) \cap C([0, +\infty); W^{2, p}(\Omega))$.

This proves that the continuous local dependence on the data of the solution $(u, a, b)$ to problem (1.3)-(1.5), (1.9) holds true, except for at most countable many pairs $(a_n, b_n) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$.

5. Solving problem (1.3)-(1.5), (1.9) for large $T$

From the fundamental formula (3.1) we now easily deduce

$$\int_0^T e^{-a\tau} S(\tau)f d\tau = (a - L_R)^{-1} f - \int_T^{+\infty} e^{-a\tau} S(\tau)f d\tau, \quad a \in \mathbb{R}_+. \quad (5.1)$$

Setting $t = T$ in (2.11), we get the formula

$$u(a, b)(T) = e^{-aT} S(T)u_0 + b \left\{ (a - L_R)^{-1} f - \int_T^{+\infty} e^{-a\tau} S(\tau)f d\tau \right\}. \quad (5.2)$$

Then from (5.2) we derive the following system for $(a, b)$:

$$b[g_{j,1}(a) - g_{j,2}(a, T)] = \gamma_j - h_j(a, T), \quad j = 1, 2, \quad (5.3)$$

where

$$g_{j,1}(a) = \int_{\partial \Omega} \psi_j(x)(a - L_R)^{-1} f(x) d\mu(x),$$

$$g_{j,2}(a, T) = \int_{\partial \Omega} \psi_j(x) d\mu(x) \int_T^{+\infty} e^{-a\tau} S(\tau)f d\tau,$$

$$h_j(a, T) = \int_{\partial \Omega} e^{-aT} \psi_j(x) S(T)u_0(x) d\mu(x).$$

We recall that $g_{j,1}$, $j = 1, 2$, are positive and nonincreasing functions (cf. (3.5) and (3.13)). We observe that problem (5.3) is equivalent to the following equation

$$0 = g(a, T) : = g(a) + \gamma_1 g_{2,2}(a) - \gamma_2 g_{1,2}(a, T) - k(a, T), \quad (5.4)$$

where

$$k(a, T) = h_2(a, T)[g_{1,1}(a) - g_{2,1}(a, T)] - h_1(a, T)[g_{2,1}(a) - g_{2,2}(a, T)], \quad (5.5)$$

if the following conditions are satisfied (cf. (3.14)):

$$|h_j(a, T)| \leq \frac{1}{2} |\gamma_j|, \quad |g_j(a, T)| \leq \frac{K(\psi_j, f)}{2(1 + a)} \leq \frac{1}{2} g_j(a) \quad \text{for some } j \in \{1, 2\}. \quad (5.6)$$

In this case we get

$$b = \frac{\gamma_j - h_j(a, T)}{g_{j,1}(a) - g_{j,2}(a, T)}. \quad (5.7)$$
To solve equation (5.4) first we compute $\lim_{a \to 0^+} g(a, T) = g(0+, T)$. For this purpose first we estimate $g_{j,2}$ and $h_{j,2}$, $j = 1, 2$:

\[
\| \int_T^{+\infty} e^{-\alpha t} S(\tau) f d\tau \|_{W^{1, p}(\Omega)} \leq \| f \|_{L^p(\Omega)} \int_T^{+\infty} e^{-\alpha t} \| S(\tau) \|_{L^1(W^{1, p}(\Omega); L^p(\Omega))} d\tau
\]

\[
\leq c_1 \| f \|_{L^p(\Omega)} \int_T^{+\infty} \tau^{-i/2} e^{-(a+\omega)\tau} d\tau = c_1 T^{-i/2}(a + \omega)^{-1} e^{-(a+\omega)T} \| f \|_{L^p(\Omega)}, \quad i = 0, 1, \quad (5.8)
\]

\[
\| \int_T^{+\infty} e^{-\alpha t} S(\tau) f d\tau \|_{L^p(\Omega)} \leq \int_T^{+\infty} e^{-\alpha t} \| L R S(\tau) f \|_{L^p(\Omega)} d\tau
\]

\[
\leq \tilde{C}_1 \| f \|_{L^p(\Omega)} \int_T^{+\infty} \tau^{-1} e^{-(a+\omega)\tau} d\tau \leq \tilde{C}_1 T^{-1}(a + \omega)^{-1} e^{-(a+\omega)T} \| f \|_{L^p(\Omega)}, \quad (5.9)
\]

\[
e^{-\alpha T} \| S(T) u_0 \|_{W^{1, p}(\Omega)} \leq C T^{-1/2} e^{-(a+\omega)T} \| u_0 \|_{L^p(\Omega)}, \quad i = 0, 1, \quad (5.10)
\]

where $c_0 = C_0$ and $c_1 = \tilde{C}_1$.

From (5.5)-(5.10) we deduce the estimates

\[
|g_{j,2}(a, T)| \leq C(p)c_1(a + \omega)^{-1} e^{-(a+\omega)T} T^{-1/2} \| \psi_j \|_{L^p(\partial \Omega)} \| f \|_{L^p(\Omega)}, \quad (5.11)
\]

\[
|h_{j}(a, T)| \leq C_0 C(p) T^{-1/2} e^{-(a+\omega)T} \| \psi_j \|_{L^p(\partial \Omega)} \| u_0 \|_{L^p(\Omega)}. \quad (5.12)
\]

Therefore we get

\[
|g_{j,2}(0) \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)}, \quad |g_{j,2}(0+, T) \|_{L^p(\Omega)} \leq C T^{-1/2} e^{-\alpha T} \| f \|_{L^p(\Omega)} \| \psi_j \|_{L^p(\partial \Omega)}, \quad (5.13)
\]

\[
|h_{j,2}(0+, T) \|_{L^p(\Omega)} \leq C T^{-1/2} e^{-\alpha T} \| \psi_j \|_{L^p(\partial \Omega)} \| u_0 \|_{L^p(\Omega)}, \quad (5.14)
\]

\[
|k(0+, T) \|_{L^p(\Omega)} \leq C e^{-\omega T}(1 + T^{-1/2} e^{-\alpha T})J_1(f, u_0, \psi_1, \psi_2), \quad (5.15)
\]

where

\[
J_1(f, u_0, \psi_1, \psi_2) = \| u_0 \|_{L^p(\Omega)} \| f \|_{L^p(\Omega)} \sum_{j=1}^{2} \| \psi_j \|_{L^p(\partial \Omega)} \left( 1 + \sum_{j=1}^{2} \| \psi_j \|_{L^p(\partial \Omega)} \right). \quad (5.16)
\]

Assume now

\[
I_1 = \int_{\partial \Omega} [\gamma_1 \psi_2(x) - \gamma_2 \psi_1(x)] L^{-1}_R f(x) d\mu(x) > 0, \quad (5.17)
\]

\[
I_2 = \int_{\partial \Omega} [\gamma_1 \psi_2(x) - \gamma_2 \psi_1(x)] f(x) d\mu(x) > 0. \quad (5.18)
\]

Consequently, from (5.13)-(5.18) we deduce

\[
g(0+, T) \geq I_1 - C e^{-\omega T} \left[ T^{-1/2} \| f \|_{L^p(\Omega)} \| \psi_j \|_{L^p(\partial \Omega)} + (1 + T^{-1/2} e^{-\alpha T})J_1(f, u_0, \psi_1, \psi_2) \right].
\]

Choose now $T \in [T_1, +\infty)$ so as to satisfy

\[
C e^{-\omega T} \left[ T^{-1/2} \| f \|_{L^p(\Omega)} \| \psi_j \|_{L^p(\partial \Omega)} + (1 + T^{-1/2} e^{-\alpha T})J_1(f, u_0, \psi_1, \psi_2) \right] \leq \frac{1}{2} I_1.
\]
This implies
\[ g(0+, T) \geq \frac{1}{2} I_1 > 0. \]
Reasoning as in Section 3 (cf. (3.6)–(3.10)), from (5.11) and (5.12) we easily deduce the relation
\[
ag(a, T) \leq -I_2 + C_{a}^{-\alpha} \|f\|_{W^{1+2\omega,p}(\Omega)} + C_{a} e^{-\omega T}T^{-1/2} \|f\|_{L^p(\Omega)} \sum_{j=1}^{2} \|\psi_j\|_{L^{p'}(\partial\Omega)} + (1 + T^{-1/2}e^{-\omega T})J_1(f, u_0, \psi_1, \psi_2).
\]
We now choose \(a_2 > 0\) and \(T_2 > 0\) so as to satisfy the following inequalities for all \(a > a_2\) and \(T > T_2\):
\[
C_{a}^{-\alpha} \|f\|_{W^{1+2\omega,p}(\Omega)} \leq \frac{1}{4} I_2
\]
\[
C_{a} e^{-\omega T}T^{-1/2} \|f\|_{L^p(\Omega)} \sum_{j=1}^{2} \|\psi_j\|_{L^{p'}(\partial\Omega)} + (1 + T^{-1/2}e^{-\omega T})J_1(f, u_0, \psi_1, \psi_2) \leq \frac{1}{4} I_2.
\]
Consequently, we deduce
\[
ag(a, T) \leq -\frac{1}{2} I_2 < 0, \quad \forall a \in [a_2, +\infty), \quad \forall T \in [T_2, +\infty).
\]
Finally, for each fixed \(T \in [\max (T_1, T_2), +\infty)\), we can conclude that the continuous function \(a \to g(a, T)\) changes its sign in \((0, +\infty)\). Therefore equation \(g(a, T) = 0\) is solvable for \(a\) for any such fixed \(T\).
As far as the unknown \(b\) is concerned, it suffices to note that inequalities (5.6) are satisfied for large \(T\) uniformly with respect to \(a \in (0, +\infty)\). Indeed, we have the following uniform inequalities:
\[
|h_j(a, T)| \leq CT^{-1/2}e^{-\omega T}\|\psi_j\|_{L^{p'}(\partial\Omega)}\|u_0\|_{L^p(\Omega)}
\]
\[
|g_{j,2}(a, T)| \leq C(a + 1)^{-1}e^{-\omega T}T^{-1/2}\|\psi_j\|_{L^{p'}(\partial\Omega)}\|f\|_{L^p(\Omega)},
\]
where we have used the inequality \((a + \omega)^{-1}e^{-\omega T} \leq C(\omega)T^{-1} \leq C(\omega)T^{-1} \leq C(\omega)\) for all \(a \in [0, +\infty)\).
Consider then the system of inequalities
\[
\begin{cases}
C_{a} e^{-\omega T}\|\psi_j\|_{L^{p'}(\partial\Omega)}\|u_0\|_{L^p(\Omega)} \leq \frac{1}{2} |\gamma_j| \\
C_{a} e^{-\omega T}T^{-1/2}\|\psi_j\|_{L^{p'}(\partial\Omega)}\|f\|_{L^p(\Omega)} \leq \frac{1}{2} K(\psi_j, f).
\end{cases}
\]
Consequently, there exists \(T_3 \in [\max (T_1, T_2), +\infty)\) such that inequalities (5.6) are satisfied for any \(a \in (0, +\infty)\) and \(T \geq T_3\).

**Remark 5.1.** Observe that, for each fixed \(t \in [T_3, +\infty)\) and any fixed \(a \in (0, +\infty)\) solving (5.4), equation (5.3) admits a unique solution \(b \in R \setminus \{0\}\) if and only if the following condition is satisfied
\[
[g_{1,1}(a) - g_{1,2}(a, T)]^2 + [g_{2,1}(a) - g_{2,2}(a, T)]^2 > 0.
\]
According to the latter inequality in (5.19) we conclude that \(g_{1,1}(a) - g_{1,2}(a, T) > 0\) for any \(a \in (0, +\infty)\) and \(T_3 \in [\max (T_1, T_2), +\infty)\). Therefore we have a unique solution \(b\).
We can summarize the results proved in this section in a theorem.

**Theorem 5.1.** Let conditions (1.2), (1.6)–(1.8), (1.10), (1.11), (5.17), (5.18) be satisfied and let \(T\) be large enough. Then problem (1.3)–(1.5), (1.9) admits at least a solution \((u, a, b) \in [C^1([0, T]; L^p(\Omega)) \cap C([0, T]; W^{2,p}_{R}(\Omega))] \times R_+ \times (R \setminus \{0\})\).
6. A uniqueness result for large $T$

Assume that $u(a_j, b_j)$ is a solution to the direct problem (1.3)-(1.5) corresponding to the pair $(a_j, b_j)$, $j = 1, 2$, and satisfying the additional condition (1.9). Introduce then the auxiliary functions

$$u = u_2 - u_1, \quad a = a_2 - a_1, \quad b = b_2 - b_1.$$ \hfill (6.1)

Then $u$ solves the problem

$$D_1 u - Lu + a_2 u = bf - au(a_1, b_1), \quad \text{in } (0, T) \times \Omega,$$ \hfill (6.2)

$$u(0, \cdot) = 0, \quad \text{on } \Omega,$$ \hfill (6.3)

$$D_\nu u + \beta u = 0, \quad \text{on } (0, T) \times \partial \Omega,$$ \hfill (6.4)

Then observe that from (6.1)-(6.3), (2.11), (5.1) and the semigroup property we easily deduce the formulae

$$u(T) = b \int_0^T e^{-a_2 \tau} S(\tau) f d\tau - a \int_0^T e^{-a_2 (T-\tau)} S(T-\tau) u(a_1, b_1)(\tau) d\tau,$$ \hfill (6.5)

$$\int_0^T e^{-a_2 \tau} S(\tau) f d\tau = (a_2 - L_R)^{-1} f - \int_T^{+\infty} e^{-a_2 \tau} S(\tau) f d\tau,$$ \hfill (6.6)

$$\int_0^T e^{-a_2 (T-\tau)} S(T-\tau) e^{-a_1 \tau} S(\tau) u_0 d\tau = \frac{1}{a_2 - a_1} (e^{-a_1 T} - e^{-a_2 T}) S(T) u_0,$$ \hfill (6.7)

$$\int_0^T e^{-a_2 (T-\tau)} S(T-\tau) d\tau \int_0^T e^{-a_1 \sigma} S(\sigma) f d\sigma = \int_0^T e^{-a_1 \sigma} d\sigma \int_0^T e^{-a_2 (T-\tau)} S(T-\tau + \sigma) f d\tau$$

$$= \int_0^T e^{-a_1 \sigma} d\sigma \int_0^T e^{-a_2 (\rho - \sigma)} S(\rho) f d\rho = \frac{1}{a_2 - a_1} \left[ \int_0^T e^{-a_2 \rho} S(\rho) f d\rho - \int_0^T e^{-a_1 \rho} S(\rho) f d\rho \right]$$

$$= -\frac{1}{a_2 - a_1} [(a_2 - L_R)^{-1} - (a_1 - L_R)^{-1}] f + \frac{1}{a_2 - a_1} \int_T^{+\infty} (e^{-a_2 \rho} - e^{-a_1 \rho}) S(\rho) f d\rho$$

$$= (a_2 - L_R)^{-1} (a_1 - L_R)^{-1} f + \frac{1}{a_2 - a_1} \int_T^{+\infty} (e^{-a_2 \rho} - e^{-a_1 \rho}) S(\rho) f d\rho.$$ \hfill (6.8)

Summing up, from (2.11), with $(a, b) = (a_1, b_1)$, and (6.5)–(6.8) we get the formula

$$u(T) = b \left[ (a_2 - L_R)^{-1} f - \int_T^{+\infty} e^{-a_2 \tau} S(\tau) f d\tau \right]$$

$$- a \left[ \frac{1}{a_2 - a_1} (e^{-a_1 T} - e^{-a_2 T}) S(T) u_0 + b_1 (a_2 - L_R)^{-1} (a_1 - L_R)^{-1} f \right]$$

$$+ \frac{b_1}{a_2 - a_1} \int_T^{+\infty} (e^{-a_2 \rho} - e^{-a_1 \rho}) S(\rho) f d\rho.$$ \hfill (6.9)

Following the same procedure as in Section 4, we easily deduce that the pair $(a, b)$ must solve the linear system

$$-[\alpha_j(a_1, a_2) + \alpha_{j,1}(a_1, a_2, T)] a + [g_j(a_2) - g_{j,1}(a_2, T)] b = 0, \quad j = 1, 2.$$
where \( g_j \) and \( \alpha_j \) are defined by (3.4) and (4.7), while, for all \( f \in W^{2,p}_R(\Omega), \) with \( Lf \in W^{1+2\alpha,p}(\Omega), \) and \( j = 1, 2, \) we have set

\[
\alpha_{j,1}(a_1, a_2, T) = \frac{1}{a_2 - a_1} (e^{-a_1T} - e^{-a_2T}) \int_{\partial \Omega} \psi_j(x) S(T) u_0(x) \, d\mu(x) + \frac{b_2}{a_2 - a_1} \int_{\partial \Omega} \psi_j(x) \, d\mu(x) \int_0^T (e^{-a_2 \rho} - e^{-a_1 \rho}) S(\rho) f(x) \, d\rho,
\]

\[
g_{j,1}(a_2, T) = \int_{\partial \Omega} \psi_j(x) \, d\mu(x) \int_0^T e^{-a_2 \tau} S(\tau) f \, d\tau.
\]

Since \( \alpha_{j,1}(a_1, a_2, T) \) and \( g_{j,1}(a_2, T) \) tend to 0 as \( T \to +\infty, \) uniformly with respect to the pair \((a_1, a_2) \in \mathbb{R}_+^2 \) (cf. (2.7)), for large enough \( T \) we immediately deduce \( a_2 = a_1 \) and \( b_2 = b_1 \) implying \( u_2 = u_1 \) if

\[
\alpha_1(a_1, a_2) g_2(a_2) - \alpha_2(a_1, a_2) g_1(a_2) \neq 0. \tag{6.9}
\]

When

\[
\alpha_1(a_1, a_2) g_2(a_2) - \alpha_2(a_1, a_2) g_1(a_2) = 0,
\]

the treatment is more difficult. However, we show that the solution is *locally* unique. For this purpose observe that functions \( \alpha_j, \ j = 1, 2, \) are holomorphic functions of \((a_1, a_2)\) and fix the pair \((a_1, b_1) \in \mathbb{R}_+ \times \mathbb{R}.\) Assume for the time being that inequality (6.9) holds with \( a_1 = a_2, \) i.e.

\[
\alpha_1(a_1, a_1) g_2(a_1) - \alpha_2(a_1, a_1) g_1(a_1) \neq 0.
\]

Then we can find a neighborhood \( U(a_1) \subset \mathbb{R}_+ \) of \( a_1 \) such that (6.9) holds with \( a_2 \in U(a_1). \) In this case for large enough \( T's, \) depending on the pair \((a_1, U(a_1)),\) also the following inequality holds

\[
[\alpha_1(a_1, a_2) + \alpha_{1,1}(a_1, a_2, T)] [g_2(a_2) + g_{2,1}(a_2, T)]
\]

\[-[g_1(a_2) + g_{1,2}(a_2, T)] [\alpha_2(a_1, a_2) + \alpha_{2,1}(a_1, a_2, T)] \neq 0.
\]

Hence, from (6.9) we get \( a_1 = a_2, \) implying the uniqueness of the solution of our identification problem. Observe now that, reasoning as at the end of Section 4, we can show that the holomorphic function \( a_1 \to \alpha_1(a_1, a_1) g_2(a_1) - g_1(a_1) \alpha_2(a_1, a_1) \) is not *identically* zero and its positive zeros are at most countable whenever condition (4.13) is fulfilled.

This concludes our treatment of the local conditional uniqueness. Here *conditional* means that the local uniqueness holds except for countable many pairs \((a_n, b_n) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}).\)

We can summarize the results proved in this section in a theorem.

**Theorem 6.1.** Let \( f \in W^{2,p}_R(\Omega), \) \( L_R f \in W^{\theta+2\alpha,p}(\Omega), \) \( \alpha \in (0,1/(2p)) \) and \( \theta \in (1/p, 1). \) Then, under conditions (1.2), (1.6)-(1.8), (1.10), (1.11), (4.13), problem (1.3)-(1.5), (1.9) admits, for large enough \( T, \) a locally unique solution solution \((u,a,b) \in [C^1([0,T];L^p(\Omega)) \cap C([0,T];W^{2,p}_R(\Omega))] \times \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})\), except for at most countable many pairs \((a_n, b_n) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})\).

We conclude this section by observing that, as at the end of Section 4, we can show the local continuous dependence of the solution \((u,a,b)\) to problem (1.3)-(1.5), (1.9) on \((f, \gamma_1, \gamma_2)\) for \( T \) large enough except for at most countable many pairs \((a_n, b_n) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}).\)

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Theorem A.1. Let \((f, u_0, \beta)\) satisfy properties (1.6), (1.7). Then the solution \(v \in \mathcal{C}^1([0, +\infty); L^p(\Omega)) \cap \mathcal{C}([0, +\infty); W^{2,p}_R(\Omega)), p \in (1, +\infty), \) to problem (2.2)–(2.4), with \(b = 1,\) is non-negative.

Proof. Denote the by \(v^+\) and \(v^-\) the positive and negative parts of \(v:\ v = v^+ - v^-\). Then \(v^+ \in W^{1,\infty}((0, +\infty); L^p(\Omega)) \cap \mathcal{C}([0, +\infty); W^{1,p}(\Omega))\). Moreover, the following formulae hold true for a.e. \(x \in \Omega,\) for a.e. fixed \(t \in (0, T)\):

\[
v(t, x)|v^-(t, x)|^{p-1} = -|v^-(t, x)|^p,
\]

\[
[v^-(t, x)]^{p-1}D_t v(t, x) = -[v^-(t, x)]^{p-1}D_t (v^-)(t, x) = -p^{-1}D_t [v^-(t, x)]^p,
\]

\[
D_{x_i} v(t, x)D_{x_j} (v^-)(t, x) = -D_{x_i} (v^-)(t, x)D_{x_j} (v^-)(t, x),
\]

and for a.e. fixed \(t \in (0, T)\) and any \(\delta \in (0, +\infty)\):

\[
\int_\Omega L_R v(t, x)[v^-(t, x)]^{p-1} dx = \lim_{\delta \to 0^+} \int_\Omega \sum_{i,j=1}^n D_{x_i}(a_{i,j}D_{x_j}v)(t, x)[v^-(t, x) + \delta]^{p-1} dx
\]

\[
= \lim_{\delta \to 0^+} \left\{ \int_\Omega [v^-(t, x) + \delta]^{p-1}D_{x_i}v(t, x) d\mu(x) \right\}
\]

\[
= -(p - 1) \int_\Omega [v^-(t, x) + \delta]^{p-2} \sum_{i,j=1}^n a_{i,j}(x)D_{x_j}v(t, x)D_{x_i}v^-(t, x) dx
\]

\[
= - \int_\Omega \beta(x) v(t, x)[v^-(t, x)]^{p-2} d\mu(x)
\]

\[
+ \int_\Omega [v^-(t, x) + \delta]^{p-2} \sum_{i,j=1}^n a_{i,j}(x)D_{x_j}(v^-)(t, x)D_{x_i}(v^-)(t, x) dx
\]

\[
\geq \int_{\partial \Omega} \beta(x)[v^-(t, x)]^{p-1} d\mu(x).
\]
Multiply then both sides in equation (2.2) by \(-[v^-(t,x)]^{p-1}\) and integrate over \((0, \tau) \times \Omega, \tau \in (0,T)\). Taking advantage of the previous formulae and assumptions, implying 

\[(f,v^-)_{L^2((0,\tau)\times\Omega)} = 0\] 

and \((u_0,v^-(0))_{L^2(\Omega)} = 0\), we easily get the inequalities:

\[
0 \geq p^{-1} \int_{\Omega} [v^-(\tau,x)]^p \, dx + \int_{(0,\tau)\times\partial\Omega} \beta(x)[v^-(t,x)]^{p-1} \, dtd\mu(x) \geq 0, \quad \text{for a.e. } \tau \in (0,T).
\]

Owing to (1.2), whence we immediately deduce \(v^-(\tau) = 0\) a.e. in \(\Omega\) for a.e. \(\tau \in (0,T)\), i.e. \(v(\tau, \cdot) = v^+(\tau, \cdot) \geq 0\) a.e. in \(\Omega\), for all \(\tau \in [0,T]\), since \(v \in C([0, +\infty); W^{1,p}(\Omega))\).

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