POINTEWISE WAVE BEHAVIOR OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE NONLINEAR DAMPED WAVE EQUATION IN $\mathbb{R}_+^n$

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Abstract. In this paper, the asymptotic wave behavior of the solution for the nonlinear damped wave equation in $\mathbb{R}_+^n$ is investigated. We describe the double mechanism of the hyperbolic effect and the parabolic effect using the explicit functions. With the absorbing and radiative boundary condition, we show that the Green’s function for the half space linear problem can be described in terms of the fundamental solution for the Cauchy problem and the reflected fundamental solution coupled with a boundary operator. Using the Duhamel’s principle, we see that due to the fast decay property of the Green’s function and the high nonlinearity, the pointwise decaying rate for the nonlinear solution and extra time decaying rate for its first order derivative are obtained.

1. Introduction. In this paper, we study the pointwise wave behavior of the solution for the nonlinear damped wave equation in the half space $\mathbb{R}_+^n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$
\begin{align*}
\partial_{tt} u - c^2 \Delta u + \nu \partial_t u &= f(u), \quad x \in \mathbb{R}_+^n, \quad t > 0, \\
|u|_{t=0} &= u_0(x), \\
|u_t|_{t=0} &= u_1(x),
\end{align*}
$$

with absorbing and radiative boundary condition

$$
\begin{align*}
(a_1 \partial_{x_1} u + a_2 u)(x_1 = 0, x', t) &= 0.
\end{align*}
$$

Here $x^T = (x_1, x'^T)$, with the normal spatial variable $x_1 \in \mathbb{R}_+ = (0, \infty)$, the tangential spatial variable $x'^T = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. $\nu$ is the viscosity, $a_1$ and $a_2$ are constants. The Laplacian $\Delta = \sum_{j=1}^n \partial^2_{x_j}$, $f(u)$ is the smooth nonlinear term with $f(u) = O(|u|^k)$ when $|u| \leq 1$, $k > 1 + \frac{2}{n}$.

The Cauchy problem for the damped wave equation has been investigated by many authors and the solution has the diffusive structure for the long time. The critical exponent $k$ for the global existence of the weak solution is proved to be the

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Fujita exponent, [16, 7]. To study the dissipative structure further more, the multi-dimensional Cauchy problem for the linear and nonlinear damped wave equation were studied by [16, 13, 12, 15, 14], etc.

For the half space problem, there are many results on the asymptotic stability of the stationary waves in half space, for example [6, 17, 18, 19]. The main approach used in these papers was the delicate energy estimate method in the physical space. [18] showed \((1 + t)^{-\frac{\alpha}{2}}\) convergence rate to the stationary wave for the degenerate damped wave equation in the one dimensional half space. For the non-degenerate case, [6] proved the \(L^p\) convergence rate \((1 + t)^{-\frac{n}{4} - \frac{1}{2} \left(\frac{1}{2} - \frac{1}{p}\right)}\). It seems that the energy estimates are too much global to get the sharper decaying rates for the solutions, one needs to find other ways to understand the mechanism of the boundary effect and get the satisfying rates. [19] indicated that it was useful to apply the energy method in the partial Fourier space to show sharper convergence rates. In their paper, they got the \(L^2\) estimates of the convergence rate \((1 + t)^{-\frac{n+1}{4}}\) for the planar wave.

In this paper, we consider the multi-dimensional nonlinear damped wave equation both for the Cauchy problem and the half space problem with mixed boundary condition, aim to give the asymptotic wave behavior of the solution. For the Cauchy problem, we explain the double mechanism of the hyperbolic effect \(u_1^2\) and the parabolic effect induced by the damping term \(u_t\) with the help of the exact functions. Our results show that the hyperbolic effect is dominant at first and the initial wave is propagating in the sound speed but with an exponentially decaying rate in time. Then after a period of time, the wave for the damped equation behaves like the heat kernel.

To study the initial-boundary value problem, we derive the expression of the Green’s function in the half space problem in the Fourier-Laplace space, and invert the Fourier-Laplace transform to get the pointwise estimate in the physical space. The accurate expression of the Green’s function and the Duhamel’s principle help us to get the wave behavior of the nonlinear problem. The difficulty of our problem lies in the complexity induced by the high dimension and the mixed boundary condition.

For the results about describing the Green’s function for the initial-boundary value problem in high dimensional case, one can refer [1] for the linear wave equation and [8] for the three dimensional Navier-Stokes equations. In [1], the authors showed how does the boundary condition affect the linear problem. In [8], the authors used the \(L^p\) estimates of solution for the linearized problem to get the \(L^p(2 \leq p \leq \infty)\) convergence rate for the nonlinear solution and only \(L^2\) convergence rate for its first derivative. In our paper, due to the accurate expression of Green’s function for the linear half space problem, we get the pointwise estimate for the nonlinear solution and the optimal \(L^p(1 < p \leq \infty)\) decaying rate. The carefully computations of nonlinear wave couplings show that the boundary effect is weaker than nonlinear effect so that we can get the optimal decaying rates for the derivatives, especially in the normal spatial direction. We only treat the case \(a_1a_2 \leq 0\). For the boundary condition of Dirichlet type \((a_1 = 0)\), we have

\[
\mathcal{G}(x_1, x', t; y_1) = G(x_1 - y_1, x', t) - G(x_1 + y_1, x', t),
\]

and Neumann type \((a_2 = 0)\)

\[
\mathcal{G}(x_1, x', t; y_1) = G(x_1 - y_1, x', t) + G(x_1 + y_1, x', t).
\]
For the mixed boundary case $a_1a_2 < 0$, extra surface operator is involved:
\[
G(x_1, x'_2; t) = G(x_1 - y_1, x'_2; t) + S(x'_2; t) \ast G(x_1 + y_1, x'_2; t).
\]
For the case of $a_1a_2 > 0$, the linearized problem is unstable, explained in Section 2. There have been other progress in developing the Green’s function method when studying the long time behavior of the solution for some initial-boundary value problems mainly in one dimensional case, such as [2, 4, 5, 9, 10] and the references therein.

The main results of our paper are given as follows:

**Theorem 1.1.** Assume the initial data $u_0(x), u_1(x) \in H^l(x)$, $l \geq \lceil \frac{n}{2} \rceil + 2$, and satisfy
\[
|\partial_x^\alpha u_0, \partial_x^\alpha u_1| \leq O(1)\varepsilon(1 + |x|^2)^{-r}, \quad r > \frac{n}{2}, \quad |\alpha| \leq 1, \tag{3}
\]
for $\varepsilon$ sufficiently small, then there exists a unique global classical solution to the problem (1) with the mixed boundary condition (2) while $a_1a_2 \leq 0$. The solution has the following pointwise estimates:
\[
|\partial_x^\alpha u(x, t)| \leq O(1)(1 + t)^{-|\alpha|/2} \left(1 + t + |x|^2 \right)^{-\frac{\varepsilon}{2}}.
\]
Moreover, we get the following optimal $L^p(\mathbb{R}_+^n)$ estimates of the solution
\[
\|\partial_x^\alpha u(\cdot, t)\|_{L^p(\mathbb{R}_+^n)} \leq O(1)(1 + t)^{-\frac{\varepsilon}{2}(1 - \frac{1}{p}) - |\alpha|/2}, \quad p \in (1, \infty).
\]

The estimates are true for all spatial dimension $n \geq 1$ since we can give the approximated spectra eigenvalues $\sigma_\pm$ up to the order of $(1 + |\xi|)^{a+1}$ for sufficient large $|\xi|$ in Section 2. In the rest of our paper, in order to illustrate the problem conveniently, we assume that the spatial dimension $n \leq 3$.

**Notations.** Let $C$, $E$ and $O(1)$ be denoted as generic positive constants. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial_x^\alpha = \partial_x^{\alpha_1} \cdots \partial_x^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$. Let $L^p$ denote the usual $L^p$ space on $x \in \mathbb{R}_+^n$. $[a] = \max\{b, b \text{ is an integer, } b \leq a\}$. For nonnegative integer $l$, we denote by $W^{l,p}(1 \leq p < \infty)$ the usual $L^p-$ Soblev space of order $l$: $W^{l,p}(l \geq 1), W^{0,p} = L^p$. The norm is denoted by $\| \cdot \|_{W^{l,p}} = \| \cdot \|_{W^{l,p}} = \sum_{|\alpha| \leq l} \| \partial_x^\alpha u \|_p$. When $p = 2$, we define $W^{l,2} = H^l$ for all $l \geq 0$ by $H^l = \{ u \in L^2, \| u \|_{H^l} < \infty \}$.

For the function $f(x, t)$, we denote its Fourier transform and Laplace transform by
\[
f(\xi, t) := \mathcal{F}[f](\xi, t) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x, t) dx, \quad f(\xi, s) := \mathcal{L}[f](\xi, s) = \int_0^\infty e^{-st} f(x, t) dt.
\]

The rest of paper is arranged as follows: in Section 2, we construct the fundamental solution for the Cauchy problem, and describe the double mechanism of the hyperbolic effect and the parabolic effect with the help of the exact functions. The exact expression of the fundamental solution is given through the complex analysis and the asymptotic expansion tools. In Section 3, the Green’s function for the linear initial-boundary value problem is constructed in the tangential-Fourier Laplace transformed spatial-time space. From the symbols in the transformed space, we get the relationship between the fundamental solution and the Green’s function. Therefore, one can describe the Green’s function in terms of the fundamental solution and the boundary effect. Finally, based on the Duhamel’s principle, the asymptotic wave behavior of the nonlinear solution is studied in Section 4.
2. Fundamental solution for the linear Cauchy problem. In this section, we
give the expression of the fundamental solution for the linear damped wave equation.
It is constructed by singularity removal, long wave- short wave decomposition in the
finite Mach number region and weighted energy estimates outside the finite Mach
number region.

One denotes $D_{\delta} := \{ \xi \in \mathbb{C}^n \mid |\text{Im}(\xi)| \leq \delta, i = 1, 2, \cdots, n \}$. 

Lemma 2.1. Suppose a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\mathcal{F}[f](\xi)$, $\xi = (\xi_1, \cdots, \xi_n)$ is analytic in $D_{\delta}$ and satisfies

$$|\mathcal{F}[f](\xi)| \leq \frac{E}{(1 + |\xi|)^{n+1}}, \quad \text{for } |\text{Im}(\xi_i)| \leq \delta, \quad \text{and } i = 1, 2, \cdots, n.$$ 

Then, the function $f(x)$ satisfies $|f(x)| \leq Ee^{-\delta|x|/C}$, for any positive constant $C > 1$.

The fundamental solution for the linear damped wave equation is defined

$$\left\{ \begin{array}{l}
\partial_t^2 G - c^2 \Delta G + \nu \partial_t G = 0, x \in \mathbb{R}^n, t > 0, \\
G(x, 0) = 0, G_t(x, 0) = \delta(x).
\end{array} \right. \quad (4)$$

It can be estimated by studying the Fourier transform $G(\xi, t)$ using the longwave-
shortwave decomposition method in the finite Mach number region $|x| \leq Mt$ for $M$
sufficiently large:

$$G(\xi, t) = \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-}, \quad \sigma_{\pm} = \frac{1}{2}(-\nu \pm \sqrt{\nu^2 - 4|\xi|^2c^2}).$$

We have the following

Lemma 2.2. The eigenvalues $\sigma_{\pm}$ are analytic functions of $\xi$ for $|\xi| \ll \varepsilon_0$, $\varepsilon_0$ sufficiently small, and satisfy the following asymptotic:

$$\left\{ \begin{array}{l}
\sigma_+ = -\frac{1}{\nu} |\xi|^2 + O(c^4\xi^4), \\
\sigma_- = -\nu + \frac{1}{\nu} |\xi|^2 + O(c^4\xi^4).
\end{array} \right.$$ 

Hence in the finite Mach number region $|x| \leq Mt$, we have the following estimates for the long wave component:

$$|\partial_t^2 G_l(x, t)| \leq C \frac{e^{-\frac{|\xi|^2}{c^2(t+1)}}}{(t+1)^{\frac{n+2}{2}}}.$$ 

The proof of this lemma is similar to that in [11], it relies on the complex contour
integral, and we omit it here.

For the short wave component, we adopt the local analysis in [3] to give a de-
scription about all types of singular functions.

When $|\xi| \to \infty$, we have the following Taylor expansion for $\sigma_{\pm}$:

$$\left\{ \begin{array}{l}
\sigma_+ = -\frac{\nu}{\nu} + i|\xi| - \frac{\nu^2}{8|\xi|^2} + O(1)|\xi|^{-5}, \\
\sigma_- = -\sigma_+ - \nu.
\end{array} \right.$$ 

This non-decaying property results in the singularities of the fundamental solution
$G$ in spatial variable. To investigate the singularities, we approximate the spectra $\sigma_{\pm}$ by $\sigma_{\pm}^{\infty}$:

$$\left\{ \begin{array}{l}
\sigma_+^{\infty} = -\frac{\nu}{\nu} + i \left( c|\xi| - \frac{\nu^2}{8(1 + |\xi|^2)} - \frac{\nu^2}{8(1 + |\xi|^2)^2} - \frac{\nu^2/8 + \nu^4/128}{(1 + |\xi|^2)^3} - \frac{\nu^4}{(1 + |\xi|^2)^4} \right), \\
\sigma_-^{\infty} = -\sigma_+^{\infty} - \nu.
\end{array} \right.$$
Here $J_0 > 0$ is chosen sufficiently large such that
\[ \inf_{\xi \in \mathbb{D}_v/c} |\sigma_+(\xi) - \sigma_-(\xi)| > 0, \quad \sup_{\xi \in \mathbb{D}_v/c} |\xi|^4 |\sigma_+(\xi) - \sigma_-(\xi)| < \infty \text{ as } |\xi| \to \infty. \]

Therefore, the approximated spectra $\sigma_{\pm}$ given above satisfy
\[ \left| \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*} \right| \leq O(1)(1 + |\xi|)^{-4}. \]

By Lemma 2.1, we have for $n \leq 3$,
\[ \left| \mathcal{F}^{-1} \left[ \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*} \right](\cdot, t) \right|_{L^\infty(\mathbb{R}^n)} = O(1), \]
which asserts that all singularities are contained in $\frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-}$. Moreover, one can also prove that the error of this approximation decays exponentially in time and space.

Set
\[ h(\xi) = c|\xi| - \frac{\nu^2}{8(1 + c|\xi|)} - \frac{\nu^2}{8(1 + c|\xi|)^2} - \frac{\nu^2/8 + \nu^4/128}{(1 + c|\xi|)^3} - \frac{J_0}{(1 + c|\xi|)^4}, \]
and one has
\[ \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*} = e^{-\frac{\nu}{2} t} \frac{\sin h(\xi) t}{h(\xi)}. \]

Straightforward computations give
\[ \frac{\sin h(\xi) t}{h(\xi)} - \frac{\sin c|\xi| t}{c|\xi|} = O(1) \cdot \frac{1}{1 + |\xi|^2}. \]
Denote
\[ Y_n(x) = \mathcal{F}^{-1} \left[ \frac{1}{1 + |\xi|^2} \right], \quad S_n(x) = \mathcal{F}^{-1} \left[ \frac{\sin c|\xi| t}{c|\xi|} \right]. \]

The $n$ dimension space Yukawa potential $Y_n$ and kernel function of the sine transform $S_n$ can be represented as
\[
Y_1(x) = \frac{e^{-|x|}}{2\sqrt{2\pi}}, \quad Y_2(x) = \frac{1}{2\pi} BesselK_0(|x|), \quad Y_3(x) = \frac{e^{-|x|}}{4\pi|x|};
\]
\[
S_1(x, t) = \frac{H(x + ct) - H(x - ct)}{2c}, \quad S_2(x, t) = \frac{H(c^2t^2 - x^2)}{2\pi c \sqrt{c^2t^2 - x^2}}, \quad S_3(x, t) = \delta(|x| - ct) / 4\pi c^2t.
\]
Here $BesselK_0(|x|)$ is the modified Bessel function of the second kind with degree 0, and the Heaviside function $H(x)$ is defined by
\[
H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}
\]

We conclude

**Lemma 2.3.** The short wave component has the following estimates in the finite Mach number region $|x| \geq M t$
\[
|G^S(x, t)| \leq O(1) \left( e^{-\frac{\nu}{2} S_n(x, t)} + e^{-\frac{\nu}{2} Y_n(x)} + e^{-c(|x| + t)/C} \right).
\]
Outside the finite Mach number region $|x| \geq Mt$, choosing the weighted function to be $W(x, t) = e^{i(|x|-cKt)}$ for $K$ sufficiently large, one can get the exponentially decaying rates in time and space using the standard weighted energy estimate method.

Summarizing, we get the explicit expression of the fundamental solution

**Lemma 2.4.** The fundamental solution has the following estimates for all $x \in \mathbb{R}^n$, $n \leq 3$, $|\alpha| \geq 0$,

$$\left| \partial^\alpha_x \left( G(x, t) - O(1)e^{-\frac{|x|^2}{4t}}(S_n(x, t) + Y_n(x)) \right) \right| \leq O(1) \frac{e^{-\frac{|x|^2}{4(t+1)}}}{(t+1)^{\alpha+1/2}} + O(1)e^{-\frac{1}{t}}.$$

3. The Green’s function for the initial-boundary value problem. In this section, we construct the Green’s function in the Fourier-Laplace space since the fundamental solution and the Green’s function are closely related in the transformed variables.

The Green’s function for the linear initial-boundary value problem is defined to satisfy the following system

$$\begin{cases} 
\partial_t^2 \mathcal{G} - c^2 \Delta \mathcal{G} + \nu \partial_t \mathcal{G} = 0, & x_1 > 0, y_1 > 0, x' \in \mathbb{R}^{n-1}, t > 0, \\
\mathcal{G}(x_1, x', 0; y_1) = 0, & \mathcal{G}_t(x_1, x', 0; y_1) = \delta(x_1 - y_1)\delta(x'), \\
\partial_1 \mathcal{G}(0, x', t; y_1) + a_2 \mathcal{G}(0, x', t; y_1) = 0.
\end{cases}
$$

(5)

Firstly, we represent the fundamental solution in Fourier-Laplace space. Taking Laplace transform in $t$ and Fourier transform in $x$, we have the transformed fundamental function in the whole space $\mathbb{R}^n$:

$$G(\xi, s) = \frac{1}{s^2 + c^2|\xi|^2 + \nu s}. $$

**Lemma 3.1.**

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{s^2 + c^2|\xi|^2 + \nu s} d\xi = \frac{e^{-\lambda|\xi_1|}}{2c^2 \lambda}, \quad \text{where} \quad \lambda = \lambda(\xi', s) = \frac{\sqrt{c^2|\xi'|^2 + s^2 + \nu s}}{c}.$$

**Proof.** We prove it by using the contour integral and the residue theorem. Note that

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{s^2 + c^2|\xi|^2 + \nu s} d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{\xi_1^2 + |\xi'|^2 + \frac{s^2 + \nu s}{c^2}} d\xi_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{(\xi_1 - \lambda i)(\xi_1 + \lambda i)} d\xi_1.$$

Define a closed path $\gamma$ containing $\Gamma := [-R, R]$ while $R$ is a positive constant, $\Omega = \gamma - \Gamma = \{|z| = Re^{\theta} \}$. If $x_1 > 0$, set $0 \leq \theta \leq \pi$, $R$ is chosen to be sufficiently large such that $\lambda i$ is contained in the domain surrounded by $\gamma$. Consider the contour integral over path $\gamma$. The contribution of the integration over $\Omega$ approaches to 0 when $R \to \infty$, therefore by the residue theorem, we have for $x_1 > 0$,

$$\frac{1}{2\pi c^2} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{(\xi_1 - \lambda i)(\xi_1 + \lambda i)} d\xi_1 = \frac{1}{2\pi c^2} 2\pi i \text{Res} \left( \frac{e^{i\xi_1 x_1}}{(\xi_1 - \lambda i)(\xi_1 + \lambda i)} \bigg|_{\xi_1 = \lambda i} \right) = \frac{e^{-\lambda x_1}}{2c^2 \lambda}.$$
The computation for the case $x_1 < 0$ is similar. Set $\pi \leq \theta \leq 2\pi$,
\[
\frac{1}{2\pi c^2} \int_{\mathbb{R}} \frac{e^{\xi_1 x_1}}{(\xi_1 - \lambda i)(\xi_1 + \lambda i)} \, d\xi_1 = -\frac{1}{2\pi c^2} 2\pi i \text{Res} \left( \frac{e^{\xi_1 x_1}}{(\xi_1 - \lambda i)(\xi_1 + \lambda i)} \right|_{\xi_1 = -\lambda i} \right) = \frac{\lambda x_1}{2c^2}.
\]
Hence we prove this lemma. \( \Box \)

Taking inverse Fourier transform of the fundamental solution only to the normal component $\xi_1$ and using Lemma 3.1, we get
\[
G(x_1, \xi', s) = \frac{e^{-\lambda|x_1|}}{2c^2}.
\]
In particular, when $\bar{x}_1 > 0$,
\[
G(-\bar{x}_1, \xi', s) = \frac{e^{-\lambda \bar{x}_1}}{2c^2}.
\]

We now construct the Green's function in the half space $x \in \mathbb{R}^n_+$. The first step is to make the initial value zero by considering the function $E(x_1, x', t; y_1) = \mathcal{G}(x_1, x', t; y_1) - G(x_1 - y_1, x', t)$:
\[
\begin{aligned}
\begin{cases}
\partial_t^2 E - c^2 \Delta E + \nu \partial_t E = 0, & x \in \mathbb{R}^n_+, y_1 > 0, t > 0, \\
E|_{t=0} = 0, & t \geq 0, \\
(\partial_1 \partial_{x_1} E + a_2 E)(x_1 = 0, x', t; y_1) = - (a_1 \partial_{x_1} + a_2) G(x_1 - y_1, x', t)|_{x_1 = 0}.
\end{cases}
\end{aligned}
\]

Taking Fourier transform in tangential spatial variable $x'$, Laplace transform in $t$, one has the following ODE problem
\[
\begin{aligned}
\begin{cases}
s^2 E - c^2 E_{x_1}^2 + (c^2 |\xi'|^2 + \nu s) E = 0, \\
(\partial_{x_1} E + a_2 E)(x_1 = 0, \xi', s; y_1) = -(a_1 \lambda + a_2) G(-y_1, \xi', s).
\end{cases}
\end{aligned}
\]

Solving it and dropping out the divergent mode as $x_1 \to +\infty$, using the boundary relationship, we have
\[
E(x_1, \xi', s; y_1) = \frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} e^{-\lambda x_1} G(-y_1, \xi', s) = \frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G(x_1 + y_1, \xi', s),
\]
where $\lambda$ is defined as before.

Therefore the transformed Green’s function $\mathcal{G}(x_1, \xi', s; y_1)$ is
\[
\mathcal{G}(x_1, \xi', s; y_1) = G(x_1 - y_1, \xi', s) - \frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G(x_1 + y_1, \xi', s) = G(x_1 - y_1, \xi', s) + G(x_1 + y_1, \xi', s) - \frac{2a_2}{a_2 - a_1 \lambda} G(x_1 + y_1, \xi', s),
\]
which reveals the connection between the fundamental solution and the Green’s function.

Hence,
\[
\mathcal{G}(x_1, x', t; y_1) = G(x_1 - y_1, x', t) + G(x_1 + y_1, x', t)
- \mathcal{F}^{-1}_{\xi \to x'} \mathcal{L}^{-1}_{s \to t} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \right]_{x', t} G(x_1 + y_1, x', t).
\]

The study of the Green’s function reduces to be the estimate of the boundary operator $\mathcal{F}^{-1}_{\xi \to x'} \mathcal{L}^{-1}_{s \to t} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \right]_{x', t}$. The function $\frac{1}{c a_2 - a_1 \sqrt{c^2 |\xi'|^2 + s^2 + \nu s}}$ has the poles in
the right half time space if $a_1a_2 > 0$, which suggests that the boundary term will grow exponentially in time. In the following we only consider the case $a_1a_2 < 0$.

Instead of inverting the boundary symbol, we follow the “differential equation” method initiated by [1]. Note that

\[ F_{\xi' \rightarrow \xi} L_{s \rightarrow t} \left[ \frac{2a_2}{a_2 - a_1\lambda} G(x_1 + y_1, \xi', s) \right] = \frac{a_2}{a_1\partial_{x_1} + a_2} G(x_1 + y_1, x', t), \]

set

\[ g(x_1, x', t) \equiv \frac{a_2}{a_1\partial_{x_1} + a_2} G(x_1, x', t), \]

then it satisfies

\[ (a_2 + a_1\partial_{x_1}) g = 2a_2 G(x_1, x', t). \]

Solving this ODE gives

\[ g(x_1, x', t) = 2a_2 \int_{x_1}^{\infty} e^{-\gamma(z-x_1)} G(z, x', t) dz = 2a_2 \int_{0}^{\infty} e^{-\gamma z} G(x_1 + z, x', t) dz, \]

while $\gamma \equiv -\frac{a_2}{a_1} > 0$.

Summarizing previous results we obtain

**Theorem 3.2.** The Green’s function $G(x_1, x', t; y_1)$ of the linear initial-boundary value problem (5) can be represented as

\[ G(x_1, x', t; y_1) = G(x_1 - y_1, x', t) + G(x_1 + y_1, x', t) - g(x_1 + y_1, x', t), \]

while $G(x_1, x', t)$ is the fundamental solution for (4).

4. **Time asymptotic behavior of the nonlinear problem.** The study of boundary operator in the last section suggests that we can only consider the case $a_1a_2 < 0$ for the nonlinear stability. The Green’s function $G(x_1, x', t; y_1)$ together with the Duhamel’s principle gives the representation of the solution $u(x, t)$:

\[ \partial_x u(x, t) = \partial_x g \int_{\mathbb{R}_+^n} G(x_1, x'-y', t; y_1) (u_0(y)+u_1(y)) + \partial_t G(x_1, x'-y', t; y_1) u_0(y) dy \]

\[ + \int_0^t \int_{\mathbb{R}_+^n} \partial_x G(x_1, x'-y', t-\tau; y_1) f(u(y, \tau)) dy d\tau \]

\[ = \partial_x^u T(x, t) + \partial_x^u N(x, t). \tag{6} \]

The energy estimates here are similar to those in [2]. Here we omit the computations and just state the result for the convenience of the readers. In the following we estimate the solution and its derivatives under a priori assumption:

\[ N(T) = \sup_{0 < t < T} \{ \|u(\cdot, t)\|_{H^l} \} \leq \delta_0, \quad 0 < \delta_0 \ll 1, l \geq 3. \]

Then the Sobolev inequality implies

\[ \sum_{0 \leq |\alpha| \leq l-2} \sup_{0 < t < T} |\partial_{x^\alpha} u| \leq C\delta_0. \tag{7} \]

Integrate the production of $u_t + \lambda u (0 < \lambda \ll 1)$ and the equation of (1) over $(0, t) \times \mathbb{R}_+^n$,
0 = \int_0^t \int_{\mathbb{R}_+^n} (u_t + \lambda u_t)(u_t - c^2 \Delta u + \nu u_t - f(u)) dx \, d\tau \\
= \int_{\mathbb{R}_+^n} (\lambda u^2 + \nu u_t + \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2) dx \bigg|_{t=0}^t + (\nu - \lambda) \int_0^t \int_{\mathbb{R}_+^n} u_t^2 dx d\tau + \lambda c^2 \int_0^t \int_{\mathbb{R}_+^n} |\nabla u|^2 dx d\tau \\
+ \int_0^t \int_{\mathbb{R}_+^n} (c^2 u_t u_{xt} + \lambda c^2 u u_{x(t, \tau)} dx' d\tau + \int_0^t \int_{\mathbb{R}_+^n} (u_t + \lambda u) f(u) dx d\tau.

Applying the mixed boundary condition (2) and the priori assumption (7) for nonlinear terms, we have
\[
\int_{\mathbb{R}_+^n} (\lambda u^2 + u_t^2 + |\nabla u|^2)(x, t) dx + \frac{a_2}{a_1} \int_{\mathbb{R}_+^n} u^2(0, x', t) dx' + (\nu - \lambda) \int_0^t \int_{\mathbb{R}_+^n} u_t^2 dx d\tau \\
+ \frac{a_2}{a_1} \lambda c^2 \int_0^t \int_{\mathbb{R}_+^n} u^2(0, x', t) dx' d\tau + \lambda c^2 \int_0^t \int_{\mathbb{R}_+^n} |\nabla u|^2 dx d\tau \\
\leq C \left( \int_{\mathbb{R}_+^n} (\lambda u^2 + u_t^2 + \frac{1}{2} |\nabla u|^2)(x, 0) dx + \int_{\mathbb{R}_+^n} u^2(0, x', 0) dx \right),
\]
for \((0 < \lambda \ll 1)\). The higher derivative estimates are similar as before. Thus, one can obtain the priori assumption.

Now we prove the pointwise nonlinear stability. Firstly, we give some computational lemmas for wave couplings.

**Lemma 4.1.** We have the following estimates for \(|\alpha| \leq 1,\)
\[
I_1 := \int_{\mathbb{R}^n} \frac{e^{-(x-y)^2/(\alpha + 1)}}{(1 + t)^{\frac{n+1}{2}}} (1 + |y|^2)^{-r} dy \\
\leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} \left( \frac{e^{-\frac{x^2}{4(t+1)^2}}}{(t+1)^{\frac{n}{2}}} (1 + t + |x|^2)^{-r} \right), \text{ for } r > \frac{n}{2}.
\]

**Proof.** Consider the following two cases.

When \(|x|^2 \leq 4(1 + t),\)
\[
I_1 \leq O(1) \frac{1}{(1 + t)^{\frac{n+1}{2} + |\alpha|}} \leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} (1 + t + x^2)^{-\frac{n}{2}}.
\]

When \(|x|^2 \geq 4(1 + t),\) we break integration into two parts
\[
I_1 \leq (1 + t)^{-\frac{|\alpha|}{2}} \left( \int_{|y| \leq \frac{|x|}{2}} \frac{e^{-(x-y)^2/(\alpha + 1)}}{(t+1)^{\frac{n}{2}}} (1 + |y|^2)^{-r} dy + \int_{|y| \geq \frac{|x|}{2}} \frac{e^{-(x-y)^2/(\alpha + 1)}}{(t+1)^{\frac{n}{2}}} (1 + |y|^2)^{-r} dy \right) \\
\leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} \left( \int_{|y| \leq \frac{|x|}{2}} \frac{e^{-\frac{x^2}{4(t+1)^2}}}{(t+1)^{\frac{n}{2}}} (1 + |y|^2)^{-r} dy + \int_{|y| \geq \frac{|x|}{2}} \frac{e^{-\frac{x^2}{4(t+1)^2}}}{(t+1)^{\frac{n}{2}}} (1 + |y|^2)^{-r} dy \right) \\
\leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} \left( \frac{e^{-\frac{x^2}{4(t+1)^2}}}{(t+1)^{\frac{n}{2}}} (1 + t + |x|^2)^{-r} \right).
\]
This proves the estimates. □
Lemma 4.2. In the finite Mach number region $|x| \leq Mt$, we have the following estimates for $|\alpha| \leq 1$:

$$
\int_{\mathbb{R}^n} \partial_{\alpha} G(x-y,t)(1+|y|^2)^{-\nu} dy \leq O(1)e^{-((|x|+t)/C)}, \text{ for } r > \frac{n}{2}.
$$

When $|\alpha|=0$, the proof of this lemma is obvious since the integrand is integrable with respect to $y$, and the exponential decaying rate in time can give the spatial decaying rate in the finite Mach number region. When $|\alpha|=1$, using the integration by parts and the condition (3), one can also get the estimates.

Lemma 4.3. For $x \in \mathbb{R}$, we have the following estimates in the finite Mach number region $|x| \leq Mt$:

$$
\int_0^t \int_{\mathbb{R}} e^{-\nu(\tau-x)} H(x-y+c(t-\tau)) - H(x-y-c(t-\tau)) \frac{1}{2c} (1+\tau+y^2)^{-k} dy d\tau
\leq O(1) (1+t+x^2)^{-k}.
$$

Lemma 4.4. For $x \in \mathbb{R}^2$, we have the following estimates in the finite Mach number region $|x| \leq Mt$,

$$
\int_0^t \int_{|x-y| \leq c(t-\tau)} e^{-\nu(\tau-x)} \frac{1}{2c\sqrt{c^2(t-\tau)^2 - (x-y)^2}} (1+\tau+|y|^2)^{-k} dy d\tau
\leq O(1) (1+t)^{-1} (1+t+x^2)^{-1}.
$$

Proof. When $0 < \tau \leq \frac{t}{2}$, we can get the exponent decay in time and space since the integrand is integrable with respect to $y$ and $\tau$.

When $\frac{t}{2} \leq \tau \leq t$,

$$
\int_{t/2}^t \int_{|x-y| \leq c(t-\tau)} e^{-\nu(\tau-x)} \frac{1}{2c\sqrt{c^2(t-\tau)^2 - (x-y)^2}} (1+\tau+|y|^2)^{-k} dy d\tau
\leq O(1) (1+t)^{-k} \int_{t/2}^t \int_{|x-y| \leq c(t-\tau)} e^{-\nu(\tau-x)} \frac{1}{\sqrt{c^2(t-\tau)^2 - (x-y)^2}} dy d\tau
\leq O(1) (1+t)^{-k} \int_{t/2}^t \int_{0<\tau \leq c(t-\tau)} e^{-\nu(\tau-x)} \frac{r}{\sqrt{c^2(t-\tau)^2 - \tau^2}} dr d\tau
\leq O(1) (1+t)^{-k} \int_{t/2}^t e^{-\nu(\tau-x)} (t-\tau) d\tau \leq O(1)(1+t)^{-1} (1+t+x^2)^{-1},
$$

since $k > 2$.

Lemma 4.5. For $x \in \mathbb{R}^3$, we have

$$
\int_0^t \int_{\mathbb{R}^3} e^{-\nu(\tau-x)} \delta(|x-y| \leq c(t-\tau)) \frac{1}{4\pi c^2(t-\tau)} (1+\tau+|y|^2)^{-\frac{3k}{2}} dy d\tau
\leq O(1)(1+t+x^2)^{-3k/2}.
$$

Lemma 4.6. For $n \leq 3$, $x \in \mathbb{R}^n$, we have

$$
\int_0^t \int_{\mathbb{R}^n} e^{-\nu(\tau-x)} Y_n(x-y)(1+\tau)^{-\frac{|\alpha|}{2}} (1+\tau+|y|^2)^{-\frac{n\beta}{2}} dy d\tau
\leq O(1)(1+t)^{-\frac{|\alpha|}{2}} (1+t+x^2)^{-nk/2},
$$

$$
\int_0^t \int_{\mathbb{R}^n} e^{-(\nu(\tau-x)^2) / 2} (1+\tau+|y|^2)^{-\frac{n\beta}{2}} dy d\tau
\leq O(1)(1+t)^{-\frac{|\alpha|}{2}} (1+t+|x|^2)^{-\frac{n}{2}}.
$$
Proof. The Yukawa potential $Y_n$ decays exponentially and the proof of the first estimate is obvious. We only compute the second one.

Case $|x|^2 \leq 1 + t$,
\[
\left( \int_0^{t/2} + \int_{t/2}^t \right) \int_{\mathbb{R}^n} \frac{e^{-\frac{(x-y)^2}{(1+t-\tau)^2 + |y|^2}}}{(1 + \tau + |y|^2)^{-\frac{n}{2}}} dy d\tau \leq O(1) (1 + t)^{-\left(\frac{\alpha}{2} + \frac{|\alpha|}{2}\right)} + O(1) (1 + t)^{-\left(\frac{n}{2} - 1\right)} \leq O(1) (1 + t)^{-\left(\frac{n}{2} + \frac{|\alpha|}{2}\right)} (1 + t + |x|^2)^{-\frac{\alpha}{2}}.
\]

Case $|x|^2 \geq 1 + t$, we consider
\[
\begin{cases}
|x - y| \geq \frac{\delta}{2}, & e^{-\frac{(x-y)^2}{(1+t-\tau)^2 + |y|^2}} \leq e^{-\frac{|x-y|^2}{(1+t-\tau)^2 + |y|^2}}; \\
|x - y| \leq \frac{\delta}{2}, & (1 + \tau + |y|^2)^{-\frac{n}{2}} \leq (1 + \tau + |x|^2)^{-\frac{n}{2}} \leq (1 + \tau + |x|^2 + t)^{-\frac{n}{2}},
\end{cases}
\]
and get the following estimates
\[
\int_0^{t/2} \int_{\mathbb{R}^n} \frac{e^{-\frac{(x-y)^2}{(1+t-\tau)^2 + |y|^2}}}{(1 + \tau + |y|^2)^{-\frac{n}{2}}} dy d\tau 
\leq O(1) e^{-\frac{|\alpha|}{2}} \int_{0}^{t/2} \int_{|x-y| \geq \frac{\delta}{2}} (1 + \tau + |y|^2)^{-\frac{n}{2}} dy d\tau 
\leq O(1) (1 + t + |x|^2)^{-\frac{n}{2}} (1 + t + |x|^2)^{-\frac{\alpha}{2}}.
\]

Now we estimate the each term in (6). According to Theorem 3.2, we denote
\[
\begin{align*}
\mathcal{G}^L(x_1, x', t; y_1) &= O(1)(G^L(x_1 - y_1, x', t) + G^L(x_1 + y_1, x', t)), \\
\mathcal{G}^S(x_1, x', t; y_1) &= O(1)(G^S(x_1 - y_1, x', t) + G^S(x_1 + y_1, x', t)).
\end{align*}
\]
By Lemma 4.1 and Lemma 4.2, the first part has the following estimates in the finite Mach number region $|x| \leq Mt$,
\[
|\mathcal{I}(x, t)| 
\leq \int_{\mathbb{R}^n} \left( \mathcal{G}^L(x_1, x' - y', t; y_1) (u_0(y) + u_1(y)) + \partial_t \mathcal{G}^L(x_1, x' - y', t; y_1) u_0(y) \right) dy.
\]
where

Here we use the integration by parts to estimate the short wave component part.

\[
\begin{align*}
+ & \int_{\mathbb{R}_+^n} \left( G^S (x_1, x' - y; t; y_1) (u_0(y) + u_1(y)) + \partial_t G^S (x_1, x' - y; t; y_1) u_0(y) \right) dy \\
+ & \int_{\mathbb{R}_+^n} \left( e^{-(x_1-y_1)^2/(t+1)^2} e^{-(x_1+y_1)^2/(t+1)^2} (t+1)^{-\frac{1}{2}} \right) (1 + y^2)^{-\frac{3}{2}} dy \\
+ & O(1) \left( e^{-(x_1-y_1)^2/(t+1)^2} \int_{\mathbb{R}_+^n} (Y_n(x_1 + y_1, x' - y') + Y_n(x_1 - y_1, x' - y')) \\
+ S_n(x_1 + y_1, x' - y', t) + S_n(x_1 - y_1, x' - y', t) \right) (1 + y^2)^{-\frac{3}{2}} dy \\
\leq & O(1) \left( e^{-(x_1-y_1)^2/(t+1)^2} \int_{\mathbb{R}_+^n} (Y_n(x - y) + S_n(x - y, t)) (1 + y^2)^{-\frac{3}{2}} dy \\
\leq & O(1) \left( \frac{e^{-(x_1-y_1)^2/(t+1)^2}}{(1+t)\sqrt{t}} + (1 + t + x^2)^{-\frac{3}{2}} \right). \quad (8)
\end{align*}
\]

For |\alpha| = 1,

\[
|{\partial_x}^\alpha I_n(x, t)| \\
\leq \left| \partial_x^\alpha \int_{\mathbb{R}_+^n} (G^L (x_1, x' - y; t; y_1) (u_0(y) + u_1(y)) + \partial_t G^L (x_1, x' - y; t; y_1) u_0(y)) dy \\
+ \int_{\mathbb{R}_+^n} (G^S (x_1, x' - y; t; y_1) (u_0(y) + u_1(y)) + \partial_t G^S (x_1, x' - y; t; y_1) u_0(y)) dy \right| \\
\leq O(1) \left( \frac{e^{-(x_1-y_1)^2/(t+1)^2}}{(t+1)\sqrt{t}} + \frac{e^{-(x_1+y_1)^2/(t+1)^2}}{(t+1)\sqrt{t}} \right) (1 + y^2)^{-\frac{3}{2}} dy \\
+ O(1) \left( e^{-(x_1+y_1)^2/(t+1)^2} \int_{\mathbb{R}_+^n} (Y_n(x_1 + y_1, x' - y') + Y_n(x_1 - y_1, x' - y')) \\
+ S_n(x_1 + y_1, x' - y', t) + S_n(x_1 - y_1, x' - y', t) \right) (1 + y^2)^{-\frac{3}{2}} dy' |_{y_1=0} \\
+ O(1) \left( e^{-(x_1+y_1)^2/(t+1)^2} \int_{\mathbb{R}_+^n} (Y_n(x_1 + y_1, x' - y') + Y_n(x_1 - y_1, x' - y')) \\
+ S_n(x_1 + y_1, x' - y', t) + S_n(x_1 - y_1, x' - y', t) \right) (1 + y^2)^{-\frac{3}{2}} dy \\
\leq & O(1) \left( \frac{e^{-(x_1-y_1)^2/(t+1)^2}}{(1+t)\sqrt{t}} + \frac{e^{-(x_1+y_1)^2/(t+1)^2}}{(1+t)\sqrt{t}} \right) (1 + t + |x|^2)^{-\frac{3}{2}} + O(1) e^{-(|x|+t)/C}, \quad (9)
\end{align*}
\]

where

\[
1(\partial_x^\alpha = \partial_{x_1}) = \begin{cases} 
1, & \text{if } \partial_x^\alpha = \partial_{x_1}, \\
0, & \text{otherwise}.
\end{cases}
\]

Here we use the integration by parts to estimate the short wave component part.
Outside the finite Mach number region, we have for $|\alpha| \leq 1$,

$$|\partial_\xi^\alpha \mathcal{I}(x, t)| \leq O(1) \varepsilon^{-\nu t} \int_{\mathbb{R}^n} e^{-|\xi-x|/\varepsilon^t} (1 + |y|^2)^{-r} \, dy \leq O(1) \varepsilon^{-\nu t} (1 + |x|^2)^{-\tau}. \quad (10)$$

Based on the estimates of (8)-(10), the ansatz is posed for the solution as follows:

$$|\partial_\xi^\alpha u(x, t)| \leq O(1) \varepsilon (1 + t)^{-\frac{|\alpha|}{2}} (1 + t + |x|^2)^{-\frac{\tau}{2}}.$$

Straightforward computations show that

$$|f(u)(x, t)| \leq O(1) \varepsilon^k (1 + t + |x|^2)^{-\frac{n}{2}}. \quad (11)$$

Now we justify the ansatz for the nonlinear term. For $\mathcal{N}(x, t)$, we have

$$|\mathcal{N}(x, t)| = \left| \int_0^t \int_{\mathbb{R}^n_+} G(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau \right|$$

$$\leq \left| \int_0^t \int_{\mathbb{R}^n_+} G^L(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau \right|$$

$$+ \left| \int_0^t \int_{\mathbb{R}^n_+} G^S(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau \right|$$

$$= N_1 + N_2.$$

Using Lemma 4.4-4.6, one gets

$$N_1 \leq O(1) \varepsilon^k \left| \int_0^t \int_{\mathbb{R}^n_-} \left( e^{-(x_1-y_1)^2 c(t-\tau_1)/(t-\tau_1)^{\frac{\nu}{2}}} + e^{-(x_1+y_1)^2 c(t-\tau_1)/(t-\tau_1)^{\frac{\nu}{2}}} \right) \left( \tau + 1 + |y|^2 \right)^{-\frac{n}{2}} \, dy \, d\tau \right|$$

$$\leq O(1) \varepsilon^k \left| \int_0^t \int_{\mathbb{R}^n} e^{-(x_1-y_1)^2 c(t-\tau_1)/(t-\tau_1)^{\frac{\nu}{2}}} \left( \tau + 1 + |y|^2 \right)^{-\frac{n}{2}} \, dy \, d\tau \right|$$

$$\leq O(1) \varepsilon^k (1 + t + |x|^2)^{-\frac{n}{2}}.$$

$$N_2 = \left| \int_0^t \int_{\mathbb{R}^n_+} G^S(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau \right|$$

$$\leq O(1) \varepsilon^k \left| \int_0^t \int_{\mathbb{R}^n_+} e^{-\frac{|x_1-y_1|^2}{c(t-\tau_1)^{\frac{\nu}{2}}}} \left( S_n(x_1, x' - y', t - \tau; y_1) + Y_n(x_1, x' - y'; y_1) \right) \left( \tau + 1 + |y|^2 \right)^{-\frac{n}{2}} \, dy \, d\tau \right|$$

$$\leq O(1) \varepsilon^k (1 + t + |x|^2)^{-\frac{n}{2}}.$$

In the following, we compute the estimate of $\partial_\xi^\alpha \mathcal{N}$ when $|\alpha| = 1$.

$$|\partial_\xi^\alpha \mathcal{N}(x, t)| = \left| \partial_\xi^\alpha \int_0^t \int_{\mathbb{R}^n_+} G(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau \right|$$

$$\leq \int_0^t \int_{\mathbb{R}^n_+} \partial_\xi^\alpha G^L(x_1, x' - y', t - \tau; y_1) f(u)(y, \tau) dy \, d\tau.$$
Hence the boundary term in (12) has the following estimates:

\[
\partial^a_\mathcal{N}_1 + \partial^c_\mathcal{N}_2.
\]

By Lemma 4.6 when \(|\alpha| = 1\), we have

\[
\partial^a_\mathcal{N}_1 = \left| (1) e^k \int_0^t \int_{\mathbb{R}^{n-1}} \left( e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{4c(t-\tau+\tau)}} + e^{-\frac{(x_1+y_1)^2+(x'-y')^2}{4c(t-\tau+\tau)}} \right) (\tau + 1 + y^2)^{-\frac{nk}{2}} dy dy_1 d\tau \right| 
\leq O(1) e^k \int_0^t \int_{\mathbb{R}^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{4c(t-\tau+\tau)}} (\tau + 1 + y^2)^{-\frac{nk}{2}} dy dy_1 d\tau 
\leq O(1) e^k (1 + t)^{-\frac{|n|}{2}} (1 + t + x^2)^{-\frac{1}{2}}.
\]

\[
\partial^c_\mathcal{N}_2 = \left| \int_0^t \int_{\mathbb{R}^{n-1}} \partial^c_\mu \mathcal{G}^S (x_1, x'-y', t-\tau; y_1) \, f(u)(y, \tau) \, dy \, d\tau \right| 
= \left| \int_0^t \int_{\mathbb{R}^{n-1}} \mathcal{G}^S (x_1, x'-y', t-\tau; y_1) \, f(u)(y, \tau) \, dy \big|_{y_1=0} \, d\tau \right| 
+ \left| 1_{\{a^n = \partial_{x_1}\}} \left( \int_0^t \int_{\mathbb{R}^{n-1}} \mathcal{G}^S (x_1, x'-y', t-\tau; y_1) \, \partial_\nu^c f(u)(y, \tau) \, dy \, d\tau \right) \right|. \tag{12}
\]

Note that

\[
\int_0^t e^{-\frac{\nu(t-\tau)}{4}} H(x + c(t-\tau)) - H(x - c(t-\tau)) (1 + \tau)^{-\frac{nk}{2}} \, d\tau 
\leq O(1) (1 + t)^{-\frac{1}{2}} (1 + t + x^2)^{-\frac{1}{2}}, \text{ for } n = 1;
\]

\[
\int_0^t \int_{|x-y| \leq c(t-\tau)} \frac{2\pi c \sqrt{c^2(t-\tau)^2 - (x-y)^2}}{4\pi c^2 (t-\tau)} (1 + \tau + |y|^2)^{-\frac{nk}{2}} \, dy \big|_{y_1=0} \, d\tau 
\leq O(1) (1 + t)^{-\frac{1}{2}} (1 + t + x^2)^{-1}, \text{ for } n = 2;
\]

\[
\int_0^t \int_{\mathbb{R}^2} e^{-\frac{\nu(t-\tau)}{4}} \delta((x-y) - c(t-\tau)) \frac{\nu(t-\tau)}{4\pi c^2 (t-\tau)} (1 + \tau + |y|^2)^{-\frac{nk}{2}} \, dy \big|_{y_1=0} \, d\tau 
\leq O(1) (1 + t)^{-\frac{1}{2}} (1 + t + x^2)^{-\frac{1}{2}}, \text{ for } n = 3.
\]

Hence the boundary term in (12) has the following estimates:

\[
\left| \int_0^t \int_{\mathbb{R}^{n-1}} \mathcal{G}^S (x_1, x'-y', t-\tau; y_1) \, f(u)(y, \tau) \, dy \big|_{y_1=0} \, d\tau \right| 
\leq O(1) e^k (1 + t)^{-\frac{|n|}{2}} (1 + t + x^2)^{-\frac{1}{2}}.
\]
The second term in (12) satisfies
\[
|O(1)\varepsilon^k \int_0^t \int_{\mathbb{R}^n_+} e^{-\frac{\nu(t-\tau)}{2}} (S_n(x_1, x' - y', t - \tau; y_1) + Y_n(x_1, x' - y'; y_1))

(1 + \tau)^{-\frac{\alpha}{2}} (\tau + 1 + y^2)^{-\frac{n+2}{2}} dy d\tau| \\
\leq O(1)\varepsilon^k (1 + t)^{-\frac{\alpha}{2}} (1 + t + x^2)^{-\frac{n}{2}}.
\]
Therefore one has the following estimates for the nonlinear term when $|\alpha| \leq 1$
\[
|\partial_x^\alpha N(x, t)| \leq O(1)\varepsilon^k (1 + t)^{-\frac{\alpha}{2}} (1 + t + x^2)^{-\frac{n}{2}}.
\]
Outside the finite Mach number region,
\[
|\partial_x^\alpha N(x, t)| \leq O(1)\varepsilon^k \left| \int_0^t \int_{\mathbb{R}^n} e^{-(\nu-(1+\tau))} \frac{x-y}{2} \left( \tau + 1 + y^2 \right)^{-\frac{n+2}{2}} dy d\tau \right| \\
\leq O(1)\varepsilon^k \left( t + 1 + x^2 \right)^{-\frac{n}{2}}.
\]
Thus, we verify the ansatz and finish the prove of Theorem 1.1.

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