On the properness of the moduli space of stable surfaces

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Abstract

The moduli spaces of stable surfaces serve as compactifications of the moduli spaces of canonical models of smooth surfaces in the same way the moduli spaces of stable curves compactify the moduli spaces of smooth curves. However, the natural definition of the moduli functor of stable surfaces allows in extra components parameterizing surfaces which do not smooth. This article verifies that these components are also proper.

1 Introduction

The theory of moduli spaces of stable curves has clearly shown the usefulness of a compactification of a moduli space of algebraic varieties which is itself a moduli space. By allowing curves with the mildest possible singularities into the moduli problem, one obtains a tractable projective variety which compactifies the space of all smooth curves of genus \( g \).

A similar space exists which provides a projective compactification of the components of the moduli space of canonically polarized surfaces (projective surfaces with ample canonical class and at most rational double points). The varieties parameterized by the boundary of this space have mild (in some sense) singularities, but singularities which are more complex than mere normal crossings points. The compactification is called the moduli space of stable surfaces.

Projectivity of this space follows from properness due to a result of Kollár ([9], Theorem 4.12). Properness follows from the boundedness results of Alexeev ([1], Section 7) and verification of the valuative criterion of properness, which in turn follows from the existence of semi-stable canonical models (cf. Chapter 7 of [8]).

After allowing limits of canonically polarized surfaces into the moduli problem, their deformations must also be admitted. In the case of curves, nodal curves all have unobstructed deformations, so this adds no new components to the moduli space. On the other hand, there are degenerations of smooth surfaces having deformations to singular surfaces which admit no smoothings. If these extra components of the moduli space are excluded somehow from the moduli functor, it is not clear how to define a scheme structure on the moduli space. What is the tangent space at these bad points if we artificially throw away components? Note that in his thesis [2], Hacking gives a functorial way (smoothable families) of removing these components in the case of log surfaces occurring as degenerations of the plane marked with a smooth curve.
One solution to this problem is not to throw away these components. In this case, a problem lingers: are these new components projective? The aforementioned results of Kollár and Alexeev reduce this question to the valuative criterion of properness. Checking this criterion is the subject of this article. This question was suggested by J. Kollár at the AIM workshop “Compact moduli spaces and birational geometry”. I am very grateful to Brendan Hassett for pointing out some errors in an older version.

1.1 Sketch of the argument

The basic method of finding “canonical” limits of families of smooth varieties is to apply the semistable reduction theorem (a good statement is Theorem 7.17 of [8]) to obtain (possibly after a finite base change) a family of reduced simple normal crossing varieties completing the original family. The ambiguity of choices of the special fibers is then removed by taking the relative canonical model of this family. For curves, this amounts to blowing down rational curves in the fibers with self-intersection -1 or -2 (that is, rational curves in a fiber meeting the rest of the fiber in one or two points). For surfaces, the process is significantly more complex, and uses essentially all the operations of the minimal model program.

When the general member of the family is singular, the total space may be non-normal, so the machinery of semistable reduction and canonical models does not directly apply. One could try to extend this machinery to the case of varieties with at worst normal crossings singularities in codimension one, but there are difficulties. It is better to normalize the original family, work on the components one by one, and attempt to reassemble the desired result.

For curves this is straightforward: pull the irreducible components of the total space apart, marking them with their (horizontal) conductors. Take a semistable resolution of the component pieces which also desingularizes (hence separates components of) the conductors. Now take the relative log canonical model of the pair of the total space of the family marked with its conductors. The conductors of the new family are nonsingular curves birational to the original conductors, so everything glues back together in the end. The separated conductors do not meet in the relative canonical model, since a relative canonical model is an lc morphism. Of course, since all nodal curves are smoothable, there is no need to check the valuative criterion on families of curves whose members are all singular: the nonsingular curves are an open dense set in the moduli space.

There are a few problems in extending this argument to higher dimensions. First, taking relative canonical models does not recover the fibers of the original family in general. For example, a surface which is smooth except for a simple elliptic singularity and which has ample canonical class is a stable surface whose minimal desingularization is also stable. Each of these surfaces is a log canonical model. Starting with a family of such singular surfaces and applying the algorithm above results in a family whose general member is the minimal desingularization of the general member of the original family.

The second problem is that in higher dimensions, it is not obvious that the conductors will glue back together, especially after making the additional modifications necessary to overcome the first problem. However, the fact that they do can be seen as
a consequence of the separatedness of the moduli space of stable varieties one dimension lower. Also, in general, components of the conductor may meet after taking the relative canonical model, but the conductors may still be glued together to admissible varieties (in the surface case, this leads to degenerate cusps).

1.2 Notation

I will use the notations and basic definitions from higher dimensional geometry following [8] and [10]. To keep notation simpler, I will use some conventions. First, if \((X, D)\) is a pair obtained from \((Y, B)\) by some birational morphism from \(X\) to \(Y\) or from \(Y\) to \(X\), \(D\) will be the birational ("strict" or "proper") transform of \(B\) on \(X\) (obtained by pushing forward \(B\) by the birational morphism in question or its inverse). I will write a morphism \(\pi : (X, D) \to C\) with a pair as domain to emphasize that canonical models to be taken are log canonical models. Consequently, I will frequently drop the adjectives "relative" and "log" and speak simply of canonical models. If \(C^0\) is an open set of \(C\), then \(X^0\) is the part of \(X\) lying over this open set, and similarly for \(D^0\). All restrictions of a morphism \(\pi\) to smaller sets will remain denoted \(\pi\). Finally, all equivalences are equivalences of \(\mathbb{Q}\)-divisors.

2 Preliminaries

The most useful definition of semi-log canonical for the purposes of this article is:

**Definition 2.1.** A variety \(X\) has semi-log canonical (slc) singularities if

1. \(X\) is \(S_2\);
2. the singularities of \(X\) in codimension one are (double) normal crossings;
3. \(X\) is \(\mathbb{Q}\)-Gorenstein, i.e. \(\omega_X^N\) (the reflexive hull of the \(N\)th tensor power of \(\omega_X\)) is locally free for some \(N\);
4. the pair \((X^\nu, D)\) consisting of the normalization of \(X\) marked with its conductor is log canonical (lc).

A stable variety is a projective variety with slc singularities such that \(\omega_X^N\) is an ample invertible sheaf for large and divisible \(N\).

We will use this definition in both directions: normalizing an slc variety produces a collection of lc pairs, and gluing a collection of lc pairs along their boundaries yields an slc variety.

Families of stable varieties are not represented by a separated space, so an additional condition is necessary.

**Definition 2.2.** Let \(\pi : X \to B\) be a flat projective morphism whose fibers are stable varieties.

1. \(\pi\) is weakly \(\mathbb{Q}\)-Gorenstein if \(X\) is \(\mathbb{Q}\)-Gorenstein.
2. $\pi$ is $\mathbb{Q}$-Gorenstein if $X$ is $\mathbb{Q}$-Gorenstein and the reflexive powers $\omega_{X/B}^{[n]}$ of the relative dualizing sheaf commute with arbitrary base change.

The second definition is better, since it leads to a natural deformation theory, but a fundamental remaining question in the theory is whether these notions differ. For the purposes of this article they do not:

**Proposition 2.3.** Notation as in the definition. If the base $B$ is a smooth curve and the fibers are curves or surfaces, then weakly $\mathbb{Q}$-Gorenstein implies strongly $\mathbb{Q}$-Gorenstein.

**Proof.** This is proved in [2], Proposition 10.14 if the general fiber is canonical. The main point in the proof is to show a certain divisor is $S_2$. If the general fiber is canonical, then the total space is canonical, which allows Hacking to conclude that the divisor in question is actually Cohen-Macaulay. In general, families with smooth base and slc fibers have slc total spaces (assuming Inversion of Adjunction), which allows us to conclude in the same manner as Hacking that his divisor $Z$ is $S_2$.

Therefore, I will prove the result for the weaker condition, and drop the adverb “weakly”.

The following is Definition 7.1 in *loc. cit.*:

**Definition 2.4.** A nonconstant morphism $f : (X, D) \to C$ to a smooth curve with $X$ normal and $D$ an effective $\mathbb{Q}$-divisor is called lc if for every closed point $c \in C$, $(X, D + f^{-1}(c))$ is an lc pair.

**Proposition 2.5.** If $\pi : (X, D) \to C$ is an lc morphism and $c \in C$ is a closed point, then $(f^{-1}(c), D|_{f^{-1}(c)})$ is slc if $D$ is $S_2$.

**Proof.** Cf. Lemma 7.4(2) of [3]. The boundaries of lc pairs are normal crossings in codimension one.

**Remark 2.6.** The final clause of the proposition is a real one. If $D$ is the boundary divisor in a family of stable varieties, there is no guarantee that $D$ should be $S_2$. For families of log surfaces this is true by the results of Hassett’s article [4].

Due to the last remark, I restrict attention to families of surfaces below. The results will be more generally valid if Hassett’s result mentioned in the remark can be extended.

## 3 Normal generic fibers

First I will show that $\mathbb{Q}$-Gorenstein families of stable lc pairs can be completed to families of stable pairs. For the remainder of this section, fix the following notation: $\pi : (X, D) \to C$ is a flat, projective morphism to (the germ of) a smooth curve, and there is a nonempty open subset $C^0 \subset C$ such that:

1. $K_{x_0} + D^0$ is $\mathbb{Q}$-Cartier and $\pi$-ample;
2. the fibers of $\pi$ over $C^0$ are lc pairs;
Theorem 3.1. There exists a finite and surjective base change \( C' \to C \) such that the pullback of \((X^0, D^0) \to C^0\) extends to an lc morphism \((X', D') \to C'\) with \( K_{X'} + D' \) relatively ample.

Proof. A key word in the statement of this theorem is extends.

Suppressing the base change in notation, assume \((X, D) \to C\) admits semistable resolution, and without loss of generality, \(C \setminus C^0\) is a single (closed) point \(0 \in C\).

Suppose \( f_i : (X_i, D_i) \to (X, D) \) for \( i = 1, 2 \) are semistable resolutions of \((X, D)\), and write:

\[
K_{X^0} + D^0_i = f_i^*(K_X^0 + D^0) + \sum a_j^{(i)} E_j^{(i)}
\]

where without loss of generality all \( E \) dominate \( C^0 \). Denote the closure of the \( E \) in \( X_i \) by the same symbol. Denote by \((X_{c i}, D_{c i})\) the relative canonical models (over \((X, D)\)) of \((X_i, D_i - \sum b_k^{(i)} a_k^{(i)} E_k^{(i)})\) and the corresponding morphisms to \((X, D)\) by \( g_i \). I make two claims:

1. \((X_{c i}, D_{c i})\) is independent of \( i \);
2. the fibers of \( \pi \circ g_i \) over \( C^0 \) agree with the fibers of \( \pi \).

Proof of Claim 1. Let \((\tilde{X}, \tilde{D})\) be a semistable resolution dominating \(X_1\) and \(X_2\) by morphisms \( h_1 \) and \( h_2 \). Write

\[
K_{\tilde{X}} + \tilde{D} = h_i^*(K_{X_i} + D_i) + \sum c_i G_i.
\]

Then since \(X_i\) is smooth, the \( c_i \) are all positive. Therefore applying Corollary 3.53 from \[8\], we conclude that the relative log canonical model of

\((\tilde{X}, \tilde{D} - \text{exc. divisors with negative discrepancy})\)

agrees with that of

\((X_i, D_i - \text{exc. divisors with negative discrepancy})\)

\(\square\)

Now denote by \((X', D')\) the common relative canonical model \((X_{c i}, D_{c i})\).

Proof of Claim 2. Let \( c \in C^0 \). The morphism \( g : (X'_c, D'_c) \to (X_c, D_c) \) is a morphism onto a canonical model. Write

\[
K_{X'_c} + D'_c = g^*(K_{X_c} + D_c) + \sum a_k E_k
\]

following the form above.

\[
K_{X'_c} + D'_c = \sum_{a_k < 0} a_k E_k - g^*(K_{X_c} + D_c)
\]

is effective and exceptional, so the canonical model of \((X'_c, D'_c)\) coincides with \((X_c, D_c)\) (ibid.).

\(\square\)
4 General families

In this section, I will first cover the case of families of surfaces, where the conjectures necessary for the existence of the moduli space are theorems, and then I will discuss the general case.

The following theorem proves the properness of the irreducible components of the moduli of stable surfaces.

Theorem 4.1 (Main Theorem). Suppose \( \pi: X \to C \) is a flat projective morphism to the germ of a smooth curve and that \( C^0 \) is a nonempty open subset of \( C \) such that

1. \( X \) is \( \mathbb{Q} \)-Gorenstein;
2. the fibers of \( \pi \) over \( C^0 \) are slc surfaces;
3. \( K_{X^0} \) is \( \pi \)-ample.

Then there exists a finite base change \( C' \to C \) and a \( \mathbb{Q} \)-Gorenstein family \( \tilde{X} \) of stable surfaces extending the pullback of \( X \) to \( C' \).

Proof. Denote by \( X' = \tilde{X}(X_i, D_i) \) the normalization of \( X \) marked by its horizontal conductors. I will ignore any vertical conductors (throwing them away makes \( (X_i, D_i) \) “even more” lc) and write the prime decomposition \( D_i = \sum_j D_{ij} \). Take a base change such that all the pairs \( (X_i, D_i) \) admit semistable resolution. Apply Theorem 3.1 to these pairs. If some \( D_{ij} \sim_{\text{birat}} D_{kl} \) as a result of the way \( X \) is glued together, making the base change does not affect this isomorphism. Furthermore, over \( C^0 \) (base change suppressed in notation), the birational transforms of \( D_{ij} \) and \( D_{kl} \) are isomorphic (by Claim 2 in the proof of 3.1), and over all of \( C \), these birational transforms are families of nodal curves, since the boundaries of log canonical surface pairs are nodal curves.

Denote a semistable resolution of the pair \( (X_i, D_i) \) by \( (\tilde{X}_i, \tilde{D}_i) \). Denote the relative log canonical model of \( (\tilde{X}_i, \tilde{D}_i) \) with respect to \( K_{\tilde{X}_i} + \tilde{D}_i \) by \( (\bar{X}_i, \bar{D}_i) \) and the relative canonical model with respect to

\[
K_{\bar{X}_i} + \bar{D}_i = \sum a_i E_i
\]

(as in the proof of 3.1) by \( (\hat{X}_i, \hat{D}_i) \). Since the fibers of \( (\hat{X}_i, \hat{D}_i) \) as well as those of \( (\tilde{X}_i, \tilde{D}_i) \) are lc pairs, \( \bar{D}_i \) and \( \hat{D}_i \) are families of nodal curves. Since \( \bar{X}_i \) and \( \hat{X}_i \) are smooth, by adjunction \( K_{\tilde{D}_i} = K_{\bar{X}_i} + \bar{D}_i \). Therefore, \( \hat{D}_i \) is the relative canonical model of a family of nodal curves (with \( K_{\hat{D}_i} \) big), so it is a family of stable curves (cf. Proposition 3.3 of [3]). \( \hat{D}_i \) is a family of nodal curves dominated by the family \( \bar{D}_i \), so it too is a family of stable curves. Therefore, the limiting curve \( \bar{D}_{i,0} \) is uniquely determined. At this point

The morphisms to each of these log canonical models may only contract some or all of the \( E_i \), or exceptional divisors mapping to \( 0 \in \tilde{C} \). Restricting to a fiber, the \( E \) restrict to curves which are not components of any \( D \). The collapsing of \( E \) may result in components of the conductor coming together. If this is so, then two families of stable curves are glued together, resulting in a family of stable curves, and the “mates” of these two components of the conductor must come together over the general fiber (again, by Claim 2 of [3]). Two components of the conductor cannot meet as a result
of collapsing divisors in the central fiber, since then the pair \((\bar{X}, \bar{D} + X_0)\) would not
be lc, so the relative canonical model would not be an lc morphism. Therefore, the
families \(\bar{D}_i\) and \(\hat{D}_i\) coincide (possibly another base change is needed to ensure this,
since two families of stable curves can have the same fibers and not be the same family),
if we renumber the \(D_i\) to refer to connected components of the conductor rather than irreducible components.

Therefore, the identifications among various conductors are preserved through the
process of taking base change, semistable resolution, and relative canonical models.
We may glue the various \((\hat{X}_i, \hat{D}_i)\) together to obtain a new family which is the \(\bar{X}\) in the
statement of the theorem.

Since all of the components \(\hat{X}_i\) and the identified loci \(\hat{D}_i\) are uniquely determined
by the original family, the limit is unique.

As noted above, this theorem together with the work of Alexeev and Kollár proves
the projectivity of the connected components of the moduli spaces of stable surfaces.

In general, what is needed for the existence of the moduli spaces and their projectivity is:

1. Local closedness of the moduli functor - this would follow from results of Has-
ssett and Kovacs [5] for the weakly \(Q\)-Gorenstein functor and from results of
Hacking [2] or unpublished results of Abramovich and Hassett for the strongly
\(Q\)-Gorenstein functor, if we knew that having semi-log canonical singularities is
an open condition. This in turn follows from the minimal model program: see
[6], Lemma 2.6. A recent preprint [7] of Kawakita proves the necessary Inver-
sion of Adjunction-type result without recourse to the minimal model program.

2. Boundedness of the moduli functor - what is needed here is a generalization of
Alexeev’s result. Specifically, one needs to know that the set of all log canonical
pairs \((X, D)\) such that \(K_X + D\) is ample and with fixed \((K_X + D)^n\) is bounded.
For smoothable stable surfaces, Karu (loc. cit., Theorem 1.1) shows that the
boundedness follows from the MMP (in one dimension higher than the moduli
problem under consideration), but this is not true in general.

3. Existence of relative canonical models of semistable resolutions - the proofs in
this article make it clear why this is necessary.

4. Cohen-Macaulayness of limits of Cohen-Macaulay varieties. See the remark in
the first section.

With all of these results in place, the argument at the end of [3,4] that the “extra” ex-
ceptional divisors to be blown down meet the conductor in points is not valid in higher
dimensions, but this should not matter. The family \(\hat{D}\) of stable varieties dominates the
family \(\bar{D}\) of slc varieties, hence \(\hat{D}\) is also a family of stable varieties and has a unique
limit.

One should also consider, following Alexeev, moduli spaces of stable pairs \((X, D)\).
It is essential in higher dimensions that \(D\) be reduced (or at least that some condition
be imposed). In this case, the arguments given here go through.
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