A new discretization of classical and quantum general relativity

Ola Boström, Mark Miller and Lee Smolin

Abstract
We propose a new discrete approximation to the Einstein equations, based on the Capovilla-Dell-Jacobson form of the action for the Ashtekar variables. This formulation is analogous to the Regge calculus in that the curvature has support on sets of measure zero. Both a Lagrangian and Hamiltonian formulation are proposed and we report partial results about the constraint algebra of the Hamiltonian formulation. We find that the discrete versions of the diffeomorphism constraints do not commute with each other or with the Hamiltonian constraint.
1 Introduction

Einstein’s equations are beautiful because they capture the phenomena of gravitation in a simple geometrical statement: the vanishing of the Ricci curvature in empty regions of spacetime. However, when we use them to solve a practical problem in gravitation we discover they have another side—they are complicated nonlinear partial differential equations. In special cases, most typically when symmetries have been imposed, one sometimes can make use of the geometry to help discover the solution to a physical problem. But, when one is studying the generic problem of constructing solutions to, or evolving, the Einstein equations, little of the geometric beauty comes out in the techniques we use to try to solve the theory.

This is especially the case for numerical approximation methods. Such methods are crucial for making progress with important astrophysical questions such as gravitational wave production by realistic sources. By the time all of the elements necessary for making the numerical problem well defined are in place, including gauge fixing and finite differencing schemes, very little of the geometrical beauty of the equations remains.

For over thirty years there has been available an alternative to the finite differencing approaches to the Einstein equations, which is the Regge calculus[1]. In this approach a large and generic set of solutions is constructed by limiting solutions to manifolds in which the curvature is restricted to have support on sets of measure zero. Typically, the manifold is broken up into simplices, and the curvature is restricted to lie on the boundaries at which simplices are joined.

The idea of Regge calculus is that such simplicial manifolds could approximate a given smooth manifold arbitrarily well. Unfortunately, at least up until this time, Regge calculus has not been developed into a powerful tool for use in realistic calculations. Or, at least, one should say that this is the case in the classical theory, as recently a version of Regge calculus has been shown to yield a very effective method for calculating the path integrals in quantum gravity in two[2, 3, 4, 5, 6], three[7] and even four dimensions[8, 9, 10].

In this paper we introduce a new discretization of Einstein equations that results from applying the exact Einstein’s equations to a special restricted class of geometries. This new formulation has two basic features. First, it is based on the Ashtekar formulation of general relativity[16, 17, 18], in which the dynamical variables are frame fields and self-dual connections. Second, the restricted class of geometries we study are those in which all the fields are distributional.

These are connected in the following way. As the Ashtekar formalism is polynomial at both the Lagrangian and Hamiltonian level, the field equations allow a wider class of solutions than the conventional form of the Einstein equations. These include solutions in which the determinant of the metric vanishes, 

\footnote{For other work on four dimensional quantum gravity using Regge calculus, see [11, 12, 13]}. 

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for which the usual relations for the Christoffel symbols in terms of the metric components would not be defined. This is possible because one of the basic fields of the formalism is a three dimensional frame field, which is chosen to have density weight one. (Actually, such solutions are allowed for all first order formulations of the Einstein’s equation, as has been emphasized recently by Horowitz [19].)

Among such degenerate solutions are those for which the densitized frame field is actually distributional. Because of the Yang-Mills like gauge invariance of the theory, it turns out that there are many solutions in which the frame fields have support on one dimensional curves in the Hamiltonian formalisms. This is because the Gauss’s law constraint is solved by frame fields which are covariantly divergence free. These configurations may be taken to be of the form

\[ \tilde{E}_a^i(x) = a^2 \int ds \delta^3(x, \alpha(s)) \dot{\alpha}^a(s) e_i^a \]  

(1)

Here \( \alpha \) is a closed or open curve in the spatial three manifold, which we will denote \( \Sigma \) and \( e^i \) is an element of the Lie algebra of \( SU(2) \) which is associated with that curve. \( a \) is a constant with dimensions of length, which is necessary if, as is natural, the Lie algebra element \( e^i \) is dimensionless, so that the frame field can also be dimensionless.

In the Ashtekar formalism it is always important to keep track of the density weights. Thus, note that the right hand side of (1) is naturally a vector density, as is required.

We can consider distributional geometries of the form of (1) for complicated graphs or lattices. For example, let \( \Gamma \) be some graph in \( \Sigma \), with edges \( \gamma_I \) where \( I = 1, \ldots, N \). Then we consider distributional configurations of the form

\[ \tilde{E}_I^{\gamma I}(x) = a^2 \sum_I \int ds \delta^3(x, \gamma_I(s)) \dot{\gamma}_I^a(s) e_I^a \]  

(2)

Associated with the graph \( \Gamma \) is then a subspace of the configuration space of the theory which is then given by the \( N \) lie algebra elements \( e_I^a \).

Now, such a distributional frame field may not seem very physical. Indeed, that was the first impression when expressions of this form first arose in the quantum theory in [20, 21]. However, we have recently understood [22, 23, 24] that such distributional frame fields can actually approximate arbitrarily well any smooth metric, \( q_{ab} \), as long as the scale of the curvature of the metric is large compared to the spacing between the links of the graph, where both distances are measured in terms of \( q_{ab} \).

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\(^5\)We use throughout index conventions of reference [18]. Early latin indices (\( a, b, c, \ldots \)) are three dimensional spatial indices, middle latin indices (\( i, j, k, \ldots \)) are three dimensional frame indices, tilde’s indicate density weight one and capital middle latin indices \( I, J \), \( < \ldots \) refer to individual loops in a set of loops. Greek letters are generally used to denote loops.
Let us sketch here briefly the main ideas behind this, for more details the reader is referred to [22, 23, 24]. Certainly, many observables on the configuration space of general relativity cannot be defined on configurations of the form of (2). Included in these is the metric at a point. Since the metric is defined through the expression
\[
\tilde{q}^{ab}(x) = \tilde{E}^{ai}(x)\tilde{E}^{bj}(x)
\]
(3)
it is not, at least naively, well defined for configurations of the form of (2) as the product of the distributions is not defined. Moreover, it can be shown that the situation cannot be improved by any regularization or renormalization procedure of the type that is usually used in quantum field theory. The problem is that any such procedure either introduces spurious dependence on the regularization procedure or changes the density character of the observable[24, 23].

However, neither in classical nor in quantum physics do we ever observe the metric at a point of space or spacetime. Leaving aside the difficult issue of diffeomorphism invariant observables in general relativity, it is obvious that what is really observed in any real experiment in general relativity are averages of the metric, smeared over some regions of space or spacetime.

The question is then whether there are observables which measure the metric smeared over regions of space or spacetime which can be defined on distributional configurations of the form of (2). It turns out that there are a number of such observables. Among them are the following three [22, 24]:

1) The area of any given two dimensional surface, \( S \), in \( \Sigma \). It is denoted \( \hat{A}[S] \).

2) The volume of any three dimensional region \( R \) in \( \Sigma \). It is denoted \( \hat{V}[R] \).

3) An observable that measures the integrated norm of any one form \( \omega \) on \( \Sigma \). Written in terms of the classical three metric \( q_{ab} \) it is
\[
Q[\omega] = \int_{\Sigma} \sqrt{\text{det}(q)} q^{ab} \omega_a \omega_b.
\]
(4)

Note that the square root is a density and is thus integrable.

The key point is that in spite of the distributional character of the frame fields (2) the areas and volumes associated with them are finite when they are nonvanishing. For example, the area that a surface \( S \) acquires from the distributional configuration \( \tilde{E}^{ai} \) is nonvanishing as long as the surface intersects the graph at least once. In that case the area is
\[
\mathcal{A}[S] = a^2 \sum_I I^+[S, \gamma_I]|e_I|
\]
(5)
where \( |e_I| \) is the norm of the \( SU(2) \) Lie algebra and \( I^+[S, \gamma_I] \) is the unoriented intersection number, that simply counts positively the number of intersections of the surface and the curve. The area observable will be described in detail in the appendix.
Thus, if we are measuring, not the metric at a point, but the areas of surfaces, it is possible to arrange the graph and choose the $e_I$ such that the areas the surfaces have in the distributional configuration (2) approximate arbitrarily well the areas they have given a smooth metric $q_{ab}$. For example, one way to do this is the following. Fix $|e_I| = 1$. Then space the lines in the graph so that, on the average one line crosses every surface once per $a^2$ units of area of the surface, as it is measured by $q_{ab}$. This can be done consistently for any set of surfaces, as long as their radii of curvature are small compared to $a$, where again, the radii of curvatures are measured with respect to $q_{ab}$. Then, consider any such surface whose area from $q_{ab}$ is large compared to $a^2$. Its area according to the distributional geometry (2), from (5), will then be equal to the value from the smooth metric, up to errors of order $a^2$ divided by that area.

Similar statements can be made concerning the second and third observables. We may conclude that as long as we only measure such smeared observables, any smooth three metric may be approximated arbitrarily well by distributional metrics of the form of (2).

Having established this correspondence between distributional and smooth three metrics, we may go on to ask four additional questions. First, can we define a corresponding set of distributional connections such that the constraints of general relativity can be extended to the case of such distributional connections and frame fields? Second, can we find solutions to the constraints among these distributional initial data, yielding an extension of the physical phase space of general relativity to such distributional solutions. Third, can we define Poisson brackets on the space of distributional frame and connection fields that, together with the extension of the constraints to these fields, defines a constrained Hamiltonian system? Fourth, can we find corresponding distributional solutions to the full set of Einstein’s equations, either by evolving the distributional solutions to the constraints in time or by directly solving the Einstein equations.

The purpose of this paper is to show that the answer to the first three questions is yes. We find that the equations of general relativity can be consistently reduced to a set of equations that govern the evolution of distributional frame and connection fields. The starting point for this formulation of dynamics is to begin with a form of the action first written down by Capovilla, Dell and Jacobson. This allows us to make a consistent truncation of the constraint equations onto a finite, but arbitrarily large, dimensional phase space, that can approximate arbitrarily well smooth solutions to the constraints of general relativity. As in full general relativity, the Hamiltonian is a linear combination of constraints, and those constraints are in correspondence with the constraints of the continuum theory so that there are discrete analogues of the Hamiltonian, diffeomorphism and $SU(2)$ gauge constraints.

The key to these results is to invent connection fields that are associated with the faces of the graph $\Gamma$. If we label the faces by an index $\alpha$, the connection will also be specified by associating a Lie algebra $a_\alpha$ to each face. Associated to every graph $\Gamma$ we will then have a phase space $P_\Gamma$, in which $SU(2)$ Lie algebra
elements corresponding to frame fields of the form of (2) are associated to each link and the conjugate fields are Lie algebra elements associated to each face.

Given a formulation of a discrete approximation to general relativity as a constrained Hamiltonian system, we would like to use its evolution equations to construct solutions that approximate smooth solutions to general relativity. In order to do this, the key question which must be answered is that of the algebra of the constraints of the discrete theory. One’s first expectation is that the constraint algebra is most likely second class, as a result of the fact that the diffeomorphism invariance of the continuum theory appears to be broken by the reduction to the discrete theory. This is known to occur in several attempts to construct Hamiltonian formulations of the Regge calculus \[26, 27, 28, 29\].

We have studied the constraint algebra in the present model and we report here partial results concerning its form. While we have not yet been able to calculate the full algebra, we have been able to compute the algebra in a certain limit, in which the $SU(2)$ internal gauge symmetry is reduced to an abelian algebra which is $U(1)^3$. In the continuum theory this limit has been studied, and corresponds to the limit in which Newton’s constant $G$ is taken to zero in the Ashtekar formalism\[30\]. It corresponds to a chirally asymmetric theory which includes the full self-dual sector of the theory but only the linearization of the antiself-dual sector. What we find is that in this limit the algebra of the constraints analogous to the gauge and spatial diffeomorphism constraints is not first class in this discrete approximation.

Finally, it may also be interesting to construct discrete solutions to general relativity which approximate smooth solutions by restricting the four dimensional equations of motions of the theory to distributional fields so that time, as well as space, becomes discrete. We show in section 4 below that there is such a formulation, which is based as well on the Capovilla-Dell-Jacobson form of the action.

This paper is organized as follows. In the next section we derive the discrete Hamiltonian formulation by reducing the Capovilla-Dell-Jacobson form of the action for general relativity to a form appropriate for fields that are distributional on three surfaces, but continuous in time. In section 3 we study the algebra of constraints of the resulting Hamiltonian formulation, while section 4 is devoted to the construction of a four dimensional lagrangian in which the fields are distributional on the spacetime manifold. Our conclusions and suggestions for further work are in section 5. The appendix contains technical details about the extension of the area observable to distributional fields.

\section{The Hamiltonian formulation}

The best way to insure that a constrained Hamiltonian formulation is consistent is to derive it from an action principle. In this paper we will thus describe two closely related formulations of an action principle for distributional fields
in general relativity. In the first, spacetime is discretized, so that the full four dimensional Einstein equations are replaced by a finite set of difference equations. This is the subject of section 4 below. In the second, space is discretized, but time is kept continuous. This is necessary if we are to derive a discrete approximation to the Hamiltonian theory, and is the subject of the present section.

Both types of Lagrangians are constructed by using a form of the Lagrangian for general relativity first written down by Capovilla, Dell and Jacobson. It will be helpful to first explain this form, as it is the starting point for all the developments of this paper. On spacetime, which will be denoted \( M \), we consider two independent fields. These are an \( SU(2) \) connection one form, denoted \( A^i \) and a three by three matrix of scalar fields, denoted \( \phi^i_j \). We may then consider the action

\[
S(A, \phi) = \int F^i \wedge F^j \left[ \phi^{-1} \right]_{ij}
\]

In this form of the variational principle, \( \phi^i_j \) is to be varied respecting two constraints. These are a symmetry condition,

\[
\phi_{ij} = \phi_{ji}
\]

and a trace condition

\[
\sum_i \phi_{ii} = -6\Lambda
\]

where \( \Lambda \) is the cosmological constant (which may be zero).

Let us assume that \( M \) has the topology of \( \Sigma \times \mathbb{R} \), for some three manifold \( \Sigma \), and consider a \( 3 + 1 \) decomposition of \( M \). We may then split the spacetime coordinates, \( x^\alpha \) into a time coordinate \( t \) and spatial coordinates \( x^a \) on \( \Sigma \). It is then easy to show that if one defines the corresponding Hamiltonian theory for evolution in \( t \) the Ashtekar form of the Hamiltonian theory is found.

Let us sketch this, as we will shortly be following the same procedure in the distributional case. We may define the frame fields \( \tilde{E}^{ai} \) as the conjugate momenta to \( A^i_a \), according to the usual

\[
\tilde{E}^{ai} = \frac{\delta S}{\delta \dot{A}^i_a} = 2[\phi^{-1}]^i_j \tilde{B}^{aj}
\]

where \( \tilde{B}^{ai} \equiv \frac{1}{2} e^{abc} F_{bc}^i \). It is then one line to show that the action may be written as

\[
S = \int dt \int_{\Sigma} \left[ \tilde{E}^{ai} \dot{A}^i_a - A^i_a \dot{G}^i \right]
\]

\(^{6}\) A preliminary version of the formulation of this section was presented in [24]. We may note that several of the equations of that presentation have been corrected here, including the relationship between the connection and curvatures.
where $G^i$ is the Gauss’s law constraint defined by

$$G^i(x) = \mathcal{D}_a \tilde{E}^{ai}(x) = 0$$

where $\mathcal{D}_a$ is the $SU(2)$ gauge covariant derivative. The Poisson brackets may be read off from (10), they are,

$$\{A_{bj}(x), \tilde{E}^{ai}(y)\} = \delta^3(x,y) \delta^a_b \delta^i_j \tag{12}$$

However, because of the conditions (7) and (8) we put on $\phi_{ij}$, not all the $\tilde{E}^{ai}(x)$ defined by (9) are independent. Instead there are primary constraints. It is not hard to show that there are exactly the momentum and Hamiltonian constraints of the Ashtekar formalism:

$$C_a = F_{ab}^{ai} \tilde{E}^{bi} = 0 \tag{13}$$

$$C = \epsilon_{ijk} F_{ab}^{ai} \tilde{E}^{bj} \tilde{E}^{ck} + \Lambda \epsilon_{ijk} \epsilon_{abc} \tilde{E}^{ai} \tilde{E}^{bj} \tilde{E}^{ck} = 0 \tag{14}$$

where $F_{ab}^{ai}$ is the Yang-Mills curvature associated with $A_{i}^{a}$.

It is intriguing that in this formulation the dynamical constraints of general relativity are recovered as primary constraints. This will turn out to be a great help when constructing the discretization.

Now, we are going to construct a discretization by restricting this form of the action to distributional fields. The key question to be answered is what kind of distributional connections and curvatures will be employed to do this. In order to answer this question, let us first note that by (9) the support of $\tilde{E}^{ai}(x)$ is likely to coincide with the support of $B^{ai} = \frac{1}{2} \epsilon^{abc} F_{bc}^{ai}$. Thus, returning for the moment to the three dimensional formalism, we will expect to have curvatures of the form,

$$\tilde{B}_{ai}^\alpha(x) = \frac{1}{G} \sum_{I} \int ds \delta^3(x, \gamma_I(s)) \hat{\gamma}_{i}^\alpha(s) \hat{b}_{I}^a.$$  

Here, the $G$ is put in for dimensions. We would like the free factors $b_{n_a}^i$ to be dimensionless. In Ashtekar’s formalism it is $GF_{ab}^{ai}$ that has the dimensions of curvature, which is inverse length squared.

Thus, in the Hamiltonian theory the spacetime self-dual curvature will be represented by one $SU(2)$ Lie algebra element, $\hat{b}_{I}^a$, associated to each line of the graph.

The question we must now ask is what form of a distributional connection gives such a curvature? It turns out that the right answer is that the connection has support on the faces of the graph. In order to make the combinatorics simple and explicit, for the rest of this paper we will restrict attention to cubic lattices.

The vertices will be labeled by three integers $n$ and the links by the pair $(\vec{n}, \hat{a})$. Thus, $\gamma_{n\hat{a}}(s)$, with $s \in (0, 1)$ will be taken to refer to the link leaving the vertex $n$ in the positive $\hat{a}$ direction. From
now on, \(\hat{a}, \hat{b}, \hat{c}\) will be one of three positive directions in a right-handed system, see fig(1). The vertices themselves are located at points \(x_\vec{n} = \gamma_\vec{n}(0)\).

We will then take the frame field and curvature to be of the forms (2) and (15), with the index \(I\) over the links now labeled by \((\vec{n}, \hat{a})\).

Let us also label the faces by \((\vec{n}, \hat{a}, \hat{b})\) with \(\hat{a} > \hat{b}\). Then, we may consider distributional connections of the form,

\[
A^i_a(x) = \frac{1}{G} \sum_{\vec{n}\hat{a}, \hat{a}>\hat{b}} \int d^2 S_{\vec{n}\hat{ab}}(\sigma) \epsilon_{abc} \delta^i_3(x, S_{\vec{n}\hat{ab}}(\sigma)) a^i_{\vec{n}\hat{c}}
\]  

where \(\hat{c}\) is the positive normal of the face spanned by \(\hat{a}\) and \(\hat{b}\), see fig(1), and \(\sigma\) stands for an arbitrary pair of coordinates on the two dimensional surface. We must first compute the associated curvatures and show that they are of the form of (15), so that the \(b^i_{\vec{n}\hat{a}}\) can be expressed in terms of the \(a^i_{\vec{n}\hat{a}}\). To compute the first, derivative term of (15) smear the distribution with an appropriate one form \(f_{c\alpha}\) and compute,

\[
\int d^3 x f_{c\alpha} \epsilon^{abc} \partial_a A^i_b = \frac{1}{G} \sum_{\vec{n}\hat{a}} \left[ \sum_{\hat{b}, \hat{c}} \epsilon^{\hat{a}\hat{b}\hat{c}} \int ds \gamma_{\vec{n}\hat{a}}(s) f_{c\alpha}(\gamma_{\vec{n}\hat{a}}(s)) (a^i_{\vec{n}\hat{c}} - a^i_{\vec{n}(\hat{a}\hat{b}\hat{c})}) \right]
\]  

The second, non-abelian term is also found to have support only on the links of the lattice. The result is that the curvature of the connection (16) is of the form of (15), with

\[
b^i_{\vec{n}\hat{a}} = \sum_{\hat{b}, \hat{c} \neq \hat{a}} \epsilon^{\hat{a}\hat{b}\hat{c}} \left[ (a^i_{\vec{n}\hat{c}} - a^i_{\vec{n}(\hat{b}\hat{c})}) + \frac{1}{8} \epsilon^{ijk} (a^j_{\vec{n}\hat{b}} + a^j_{\vec{n}(\hat{c}\hat{b})}) (a^k_{\vec{n}\hat{c}} + a^k_{\vec{n}(\hat{b}\hat{c})}) \right]
\]  

see also fig(3).

In the continuum, the uncontracted Bianchi identity gives a relation between
the curvature $B^{ai}$ and the connection $A^{i}_{\hat{a}}$, namely

$$D_{[a}(\epsilon_{bc]d}\tilde{B}^{di}) = 0 \tag{19}$$

It is easy to verify this equation directly by substituting

$$\tilde{B}^{ai} = \epsilon^{abc}F_{bc} = \epsilon^{abc}\left(\partial_{a}A_{b}^{i} - \partial_{c}A_{b}^{i} + \epsilon^{ijk}A_{j}^{i}A_{k}^{b}\right) \tag{20}$$

into the left-hand side of (19). Substituting the distributional fields (15) and (16) into the left-hand side of (19) yields

$$D_{[a}(\epsilon_{bc]d}\tilde{B}^{di}) = \sum_{\hat{a}} \left[ b_{\hat{n}\hat{a}}^{i} - b_{(\hat{n}-\hat{a})\hat{a}}^{i} \right] + \frac{1}{8}\epsilon^{ijk} \left( a_{\hat{n}\hat{a}}^{j} + \sum_{b,\hat{c}\neq\hat{a}} b_{\hat{c}\hat{a}}^{j} \right) \left( b_{\hat{n}\hat{a}}^{k} + b_{(\hat{n}-\hat{a})\hat{a}}^{k} \right) \tag{21}$$

One might expect that substituting (18) into (21) would make the expression identically zero. This turns out not to be the case. The right hand side of (21), expressed purely in terms of $a_{\hat{n}\hat{a}}^{i}$ via (18), has terms that are linear, quadratic, and cubic in $a_{\hat{n}\hat{a}}^{i}$. Only the linear and quadratic terms cancel, leaving an expression cubic in $a_{\hat{n}\hat{a}}^{i}$ that, in general, is not zero.

To understand this, we remember that the Bianchi identity can be interpreted geometrically as a manifestation of the "boundary of a boundary = 0" principle[31]. Taking a cube in the dual lattice with lattice vertex $\hat{n}$ at the center, we parallel transport a test spinor around each of its faces in such a way that each link gets traversed twice, once in each direction. The resulting change in the test spinor is, to lowest non-zero order in $b_{\hat{n}\hat{a}}^{i}$, exactly given by the linear and quadratic (in $a_{\hat{n}\hat{a}}^{i}$) part of the right hand side of (21). Thus, the Bianchi
identity for our distributional frame fields is
\[
\sum_{\hat{a}} b^i_{\hat{a} \hat{a}} - b^i_{(\vec{n} - \hat{\alpha})\hat{a}}
\]

\[\left(\alpha^j_{\hat{a} \hat{a}} + \sum_{\hat{b}, \hat{c} \neq \hat{a}} 2 \frac{1}{2} \alpha^j_{(\vec{n} - (\hat{b} + \hat{c}))\hat{a}} + \alpha^j_{(\vec{n} - \hat{\alpha})\hat{a}}\right) \left(\beta^k_{\hat{a} \hat{a}} + \beta^k_{(\vec{n} - \hat{\alpha})\hat{a}}\right) = 0 \tag{22}\]

where
\[
\beta^i_{\hat{a} \hat{a}} = \sum_{\hat{b}, \hat{c} \neq \hat{a}} \epsilon^{\hat{a} \hat{b} \hat{c}} (a^i_{\hat{c} \hat{c}} - a^i_{(\vec{n} - \hat{\alpha})\hat{c}}) \tag{23}\]

That is, \(\beta^i_{\hat{a} \hat{a}}\) is equal to the linear terms of \(b^i_{\hat{a} \hat{a}}\) in (18).

This completes our specification of the kinematical structure of the discretization. The next question to be addressed is how to extend the constraint equations to distributional fields of the form of (2) and (16). This is not a completely trivial problem, as the computations involve products of distributions. For this reason, the simplest approach is to use the Lagrangian approach to define the dynamics.

In the continuum theory, the action can be written
\[
S(A_a, A_0, \phi) = 2 \int dt \int_{\Sigma} \left[ A^i_a - D_a A^i_a \right] \phi^{-1} \tag{24}\]

We now make the coefficients of the distributional fields, \(e^i_{\hat{a} \hat{a}}\) and \(a^i_{\hat{a} \hat{a}}\), time dependent and plug these fields into the above action. When we do this, we notice that in order to perform the integrations, some kind of restriction has to be put on \(\phi_{ij}\). To see what restriction we should choose, notice that in the Ashtekar formalism there are 18 configuration variables at each point in \(\Sigma\); \(E^{ai}\) and \(A^i_a\) (we do not count \(A^0_0\) as a configuration variable, since it plays the role of a Lagrange multiplier). We would therefore expect to have 18 configuration variables associated with each vertex of the lattice. This turns out to be the case if we restrict \(\phi_{ij}(x)\) to be constant on the three links associated with each vertex. That is, \(\phi_{ij}(x)\) has the same value, which we will denote as \(\phi_{ij}(x_{\vec{n}})\), on the three links obtained by starting at the vertex \(\vec{n}\) and moving in the three positive directions. Note that \(\phi_{ij}(x)\) is discontinuous at each vertex. We now have 18 configuration variables associated with each vertex (9 a’s and 9 φ’s).

If we make this restriction, keeping \(A^i_0(x)\) unrestricted, then the discrete version of the action is
\[
S = \int dt \mathcal{L}[a, A_0, \phi] \tag{25}\]

where
\[
\mathcal{L}[a, A_0, \phi] = \frac{2}{8G^2} \sum_{\vec{n}, \vec{\epsilon}} a^i_{\vec{n} \vec{\epsilon}} \times \left( b^i_{\vec{n} \vec{\epsilon}} \phi^{-1}_{ij} (x_{\vec{n}}) + b^i_{(\vec{n} - \vec{\epsilon}) \vec{\epsilon}} \phi^{-1}_{ij} (x_{\vec{n} - \vec{\epsilon}}) \right) \tag{26}\]
The Lagrangian is now
\[
G_{\bar{\alpha}} = \sum_{\hat{a}, b \neq \hat{c}} \frac{1}{2} (\bar{n} \bar{\alpha} + \bar{b}) e^{\phi_{ij}^{-1}}(x_{\bar{n} \bar{\alpha} + \bar{b}}) + b_i^{(\bar{n} \bar{\alpha} + \bar{a})} e^{\phi_{ij}^{-1}}(x_{\bar{n} \bar{\alpha} + \bar{a}}) + \\
+ \frac{1}{2} b_i^{(\bar{n} \bar{\alpha} + \bar{a} + \bar{b})} e^{\phi_{ij}^{-1}}(x_{\bar{n} \bar{\alpha} + \bar{a} + \bar{b}}) + b_i^{(\bar{n} \bar{\alpha} + \bar{a} + \bar{b})} e^{\phi_{ij}^{-1}}(x_{\bar{n} \bar{\alpha} + \bar{a} + \bar{b}}) + \\
+ \frac{1}{G} \sum_{\bar{n}} A_i^{(\bar{n})} G_{\bar{\alpha}}^i
\]

Here, \( G_{\bar{\alpha}}^i \) is the discrete version of the Gauss law constraint; it is given by,
\[
G_{\bar{\alpha}}^i = 2 \sum_{\alpha} \left( b_j^{(\bar{\alpha} \bar{a})} \phi_{ij}^{-1}(x_{\bar{n} \bar{\alpha} + \bar{b}}) - b_j^{(\bar{\alpha} \bar{a})} \phi_{ij}^{-1}(x_{\bar{n} \bar{\alpha} - \bar{a}}) \right) + \frac{1}{8} e^{ijkl} \left( b_j^{(\bar{\alpha} \bar{a})} \phi_{ij}^{-1}(x_{\bar{n} \bar{\alpha} + \bar{b}}) + b_j^{(\bar{\alpha} \bar{a})} \phi_{ij}^{-1}(x_{\bar{n} \bar{\alpha} - \bar{a}}) \right) \times \left( \alpha_k^{(\bar{n} \bar{\alpha} + \bar{b} + \bar{c})} + \sum_{b, \hat{b} \neq \hat{c}} \left[ \frac{1}{2} \alpha_k^{(\bar{n} \bar{\alpha} + \bar{b} + \bar{c})} + \alpha_k^{(\bar{n} \bar{\alpha} - \bar{a})} \right] \right)
\]

Following the continuum theory, let us define the coefficients of the frame field to be
\[
e_{\bar{\alpha} \bar{a}}^i = 2 \phi_{ij}^{-1}(x_{\bar{n} \bar{\alpha} + \bar{b}}) b_i^{\bar{a}} \tag{28}
\]
The Lagrangian is now
\[
\mathcal{L}[\alpha, A_0, e] = \frac{1}{8 G^2} \sum_{\bar{n}, \hat{c}} a_{\bar{n} \hat{c}}^i \times \left( e_{\bar{n} \hat{c}}^i + e_{(\bar{n} - \hat{a}) \hat{c}}^i \right) + \sum_{\hat{a}, b \neq \hat{c}} \left[ \frac{1}{2} e_{(\bar{n} \bar{\alpha} + \bar{b}) \hat{c}}^i + e_{(\bar{n} \bar{\alpha} + \bar{a}) \hat{c}}^i \right] + \frac{1}{2} \left[ (\bar{n} \bar{\alpha} + \bar{a} + \bar{b}) \hat{c} + e_{(\bar{n} \bar{\alpha} + \bar{a} + \bar{b}) \hat{c}}^i \right] + \\
+ \frac{1}{G} \sum_{\bar{n}} A_i^{(\bar{n})} G_{\bar{\alpha}}^i
\]

Now, the Gauss constraint is
\[
G_{\bar{\alpha}}^i = \sum_{\bar{a}} \left( e_{\bar{n} \bar{a}}^i - e_{(\bar{n} - \bar{a}) \bar{a}}^i \right) + \frac{1}{8} e^{ijk} \left( e_{\bar{n} \bar{a}}^k + e_{(\bar{n} - \bar{a}) \bar{a}}^k \right) \times \left( \alpha_k^{\bar{a}} + \sum_{\hat{b}, \hat{c} \neq \hat{a}} \left[ \frac{1}{2} \alpha_k^{(\bar{n} \bar{a} + \bar{b} + \bar{c})} + \alpha_k^{(\bar{n} - \bar{a} + \bar{b})} \right] \right)
\]

Notice that, at each vertex, we have replaced the five independent variables \( \phi_{ij}(x_{\bar{n} \bar{\alpha}}) \) with the nine variables \( e_{\bar{n} \bar{a}}^i \). Due to conditions (7) and (8) on \( \phi_{ij} \), we
have 4 primary constraints on $e^i_{\vec{n}\dot{\alpha}}$ at each vertex:

$$C_{\vec{n}\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}} e^i_{\vec{n}\dot{\beta}} b^i_{\vec{n}\dot{\gamma}}$$  \hspace{1cm} (31)

$$C_{\vec{n}} = \epsilon_{ijk} e^i_{\vec{n}\dot{a}} e^j_{\vec{n}\dot{b}} b^k_{\vec{n}\dot{c}}$$  \hspace{1cm} (32)

These are the lattice vector and scalar constraints, respectively.

In the continuum, the abelian term of the vector constraint is equal to the abelian term of the Gauss constraint times the connection. The physical interpretation of this is that the diffeomorphism constraint vanishes in the abelian limit when the $G_{Newton}$ constant goes to infinity \[32\]. This is not the case for the discretized constraints (31).

3 The constraint algebra

We would now like to find the Poisson bracket algebra of the constraints. Notice that the definition of the momenta conjugate to both $e^i_{\vec{n}a}$ and $a^i_{\vec{n}a}$ are actually primary constraints, since they do not involve time derivatives of $e^i_{\vec{n}a}$ and $a^i_{\vec{n}a}$:

$$\Pi_{a_{\vec{n}\dot{a}}} = \frac{d\mathcal{L}}{da^i_{\vec{n}\dot{a}}} = \frac{1}{8} \left\{ e^i_{\vec{n}\dot{a}} + e^i_{(n-\dot{a})\dot{c}} \right\}$$

$$\Pi_{e^i_{\vec{n}a}} = \frac{d\mathcal{L}}{de^i_{\vec{n}a}} = 0$$  \hspace{1cm} (34)

Let us label these constraints by

$$\chi_I = 0$$  \hspace{1cm} (35)

where $I$ is an index labeling the above constraints. Let us now treat a $L \times M \times N$ cubic lattice, with opposite points on the boundary identified, so that the topology is that of a 3-torus. This means that there are $(N-1)(M-1)(L-1)$ independent vertices. We also require $L$, $M$, and $N$ to be even numbers. It is easy to see that for any particular constraint $\chi_I$, there exists another constraint $\chi_J$ such that $\{\chi_I, \chi_J\}$ does not weakly vanish. Therefore, all constraints $\chi_I$ are second class constraints. We now follow the procedure first set forth by Dirac in dealing with second class constraints. We define the Dirac bracket of two functions on the phase space, $\xi$ and $\psi$, as

$$\{\xi, \psi\}_{db} = \{\xi, \psi\}_{pb} - \{\xi, \chi_I\}_{pb} \Omega^{IJ} \{\chi_J, \psi\}_{pb}$$  \hspace{1cm} (36)
where $\Omega^{IJ}$ is defined by
\[
\{\chi_I, \chi_J\} pb \Omega^{JK} = \delta^K_I \tag{37}
\]

Therefore, the Dirac bracket of any function on the phase space with any $\chi_I$ vanishes by construction.

It is not difficult to see that the resulting Dirac brackets are non-local, in the sense that an $a^i$ on one face of the lattice can have a nonvanishing Dirac bracket with an $e^i$ on a link arbitrarily far from it. This happens because to find the Dirac brackets we have to invert the matrix $\{\chi_I, \chi_J\}$. This may seem unphysical, but it is actually necessary if the Dirac algebra of the $a^i$’s and their conjugate momenta (defined by 33) is to be local. The problem is that the relation between the $e^i$’s and the conjugate momenta are nonlocal. (Actually, a natural rearrangement of the configuration variables makes the Dirac bracket local; see below.)

The straightforward way to proceed is to write a computer program to invert the relation (37) and find the Dirac bracket for any given lattice, which we did. However, a small trick gives us an easy form of $\Omega$. We rearrange the constraints
\[
0 = X_{a^i} = \Pi_{a^i} - \frac{d\mathcal{L}}{da_{i\hat{c}}} \tag{38}
\]
in such a way that all new constraints contain only one $e^i_{\hat{a}\hat{c}}$, and the rest in the same manner.

Using the above rearrangement of constraints, (37) is simple enough to solve analytically. The Dirac bracket is
\[
\{a^j_{\hat{m}\hat{d}}, e^i_{\hat{n}\hat{a}}\}_d = \begin{array}{c}
(-1)^{k+l+m-1}(\text{sign}(k) - \delta_{k,0})(\text{sign}(l) + \delta_{l,0}) \times \\
\times (\text{sign}(m) + \delta_{m,0})\delta_{\hat{a},\hat{d}}\delta_{i,j} \\
\hat{m} - \hat{n} = k\hat{a} + l\hat{b} + m\hat{c}; \quad \hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = \hat{b} \cdot \hat{c} = 0
\end{array} \tag{39}
\]
where
\[
\text{sign}(l) = 0 \quad \text{for} \quad l = 0, \quad +1 \quad \text{for} \quad l > 0, \quad -1 \quad \text{for} \quad l < 0 \tag{40}
\]
This expression was checked by directly performing the inversion required to solve (37) on a computer using a 6x6x6 cubic lattice.

This expression seems nonlocal at first glance. However, if we rearrange the configuration variables as
\[
e^{ri}_{\hat{n}\hat{a}} = e^{r}_{\hat{n} - \hat{a}, \hat{a}} + e^{i}_{\hat{n}\hat{a}} \tag{41}
\]
\[
a^{ri}_{\hat{n}\hat{a}} = a^{i}_{\hat{n}\hat{a}} + \sum_{\substack{\hat{b}, \hat{c} \neq \hat{a} \\hat{b} \neq \hat{c}}} \frac{1}{2} a^{j}_{\hat{n} - (\hat{b} + \hat{c})\hat{a}} + a^{i}_{\hat{n} - \hat{b}\hat{a}} \tag{42}
\]
\begin{align*}
\{a_{\hat{n}_{ab}}^i, e_{\hat{n}_{\hat{a}}}^i\}_{db} &= \delta_{\hat{a},\hat{b}}\delta_{i,j}\delta_{n,m} \quad (43)
\end{align*}

We now turn to the problem of finding the algebra of the constraints. To simplify the job of finding the Dirac bracket algebra of the constraints, we considered first an approximation in which we neglect all non-abelian terms in the constraints, so that

\begin{align*}
C_{\hat{n}_{\hat{a}}} &= \epsilon_{\hat{a}\hat{b}\hat{c}} e_{\hat{n}_{\hat{a}}}^i b_{\hat{n}_{\hat{b}}}^i \\
C_{\hat{n}} &= \epsilon_{ijk} \epsilon_{\hat{a}\hat{b}\hat{c}} e_{\hat{n}_{\hat{a}}}^i e_{\hat{n}_{\hat{b}}}^j b_{\hat{n}_{\hat{c}}}^k \\
G_{\hat{n}}^i &= \sum_{\hat{a}} (e_{\hat{n}_{\hat{a}}}^i - e_{\hat{n}-\hat{a}}^i)
\end{align*}  \quad (44, 45, 46)

are the vector, scalar, and Gauss constraints with

\begin{align*}
b_{\hat{n}_{\hat{a}}}^i &= \sum_{\hat{b}, \hat{c}\neq \hat{a}} \epsilon_{\hat{a}\hat{b}\hat{c}} \left(a_{\hat{n}_{\hat{c}}}^i - a_{\hat{n}-\hat{b}}^i\right)
\end{align*}  \quad (47)

As mentioned in the introduction, this approximation corresponds, in the continuum theory, to the limit in which Newton’s constant is taken to zero, as described in [30]. Note that both Bianchi identities (19) and (23) are now fulfilled, as there are no cubic terms at all.

The results of the Dirac bracket algebra is now as follows: The algebra of a generic lattice (as specified above) does not close. Almost all dirac brackets between the vector and scalar constraints can’t be expressed as a linear combination of constraints. This is obvious when computing a particular bracket and has been checked in a linearization program in the case of a $6 \times 6 \times 6$ lattice. Treated as (secondary) constraints, the non vanishing dirac brackets actually are first class. The algebra must eventually close, as there are a finite number of combinations of $a_{\hat{n}_{\hat{a}}}^i$’s and $e_{\hat{n}_{\hat{a}}}^i$’s. The full algebra including the brackets of the
new first class constraints is quite complicated for a generic lattice. Therefore, we present the full algebra for a simple but not trivial lattice and only a part for the generic lattice.

Let us start with the simple case. For a $2 \times 2 \times N$ lattice, meaning $N-1$ independent vertices on a line, much is simplified. Let the line be along the $x$-axes. Then, there are only $N-1$ constraints left (as $b_{\vec{n}x} = 0$ which means that $e_{\vec{n}x} = 0$), namely

$$C_{\vec{n}x} \equiv D_{\vec{n}} = e^{i}_{\vec{n}y} b^{i}_{\vec{n}z} - e^{i}_{\vec{n}z} b^{i}_{\vec{n}y} \quad \text{for } n = 1, 2, \ldots, N$$  \hspace{1cm} (48)

Now,

$$\{D_{\vec{n}}, D_{\vec{n} + \vec{m}}\} = (-1)^{m-1} \text{sign}(m)(e^{i}_{\vec{n}y} b^{i}_{\vec{m}z} - e^{i}_{\vec{n}z} b^{i}_{\vec{m}y} + e^{i}_{\vec{m}y} b^{i}_{\vec{n}z} - e^{i}_{\vec{m}z} b^{i}_{\vec{n}y})$$  \hspace{1cm} (49)

, where $\text{sign}(l)$ is defined in (11). The full algebra then becomes

$$\{D_{(\vec{n}+\vec{l},\vec{n}+\vec{m})}, D_{(\vec{n}+\vec{m},\vec{n}+\vec{q})}\} = (-1)^{l-p-1} \text{sign}(l-p)D_{(\vec{n}+\vec{l},\vec{n}+\vec{p})} + (-1)^{l-q-1} \text{sign}(l-q)D_{(\vec{n}+\vec{l},\vec{n}+\vec{q})} + (-1)^{m-p-1} \text{sign}(m-p)D_{(\vec{n}+\vec{p},\vec{n}+\vec{q})} + (-1)^{m-q-1} \text{sign}(m-q)D_{(\vec{n}+\vec{m},\vec{n}+\vec{q})}$$

where

$$2D_{\vec{n}} \equiv D_{(\vec{n},\vec{n})}$$

Among the $\left(\frac{1}{2}\right)$ number of $D_{(\vec{n},\vec{m})}$’s there are only one relation,

$$\sum_{\vec{n},\vec{m}} D_{(\vec{n},\vec{m})} = 0$$  \hspace{1cm} (55)

In case of a generic $(even) \times (even) \times (even)$ lattice with opposite sides identified, we give the following sample of the full algebra

$$\{C_{(\vec{n},\vec{n})}, C_{(\vec{n}+\vec{l},\vec{n})}\} = (-1)^{l-1} \text{sign}(l)e^{i}_{\vec{a} \vec{b}} b^{i}_{\vec{a} \vec{b}} \equiv (-1)^{l-1} \text{sign}(l)C_{(\vec{n},\vec{n}+\vec{l})}$$  \hspace{1cm} (56)

where parentheses denotes symmetrization.

It is possible to interpret the non-vanishing dirac brackets (56) if we treat them as constraints. They are then, as we have seen, first class and we may write the $C_{(\vec{n},\vec{n})}$’s using (28) as

$$C_{(\vec{n},\vec{m})} = (\phi_{ij}(x_{\vec{n}}) - \phi_{ji}(x_{\vec{m}}))e^{i}_{\vec{a} \vec{b}} b^{i}_{\vec{a} \vec{b}}$$

we realize that $\phi_{ij}(x_{\vec{n}})$ must be the same for all $\vec{n}$’s if the $C_{(\vec{n},\vec{m})}$’s are to vanish,

$$\phi_{ij}(x_{\vec{n}}) = \phi_{ij}(x_{\vec{m}}) \quad \text{for all } \vec{n} \text{ and } \vec{m}$$

Hence, all vertices are identical, there is only one independent vertex. That is, if we consider the "non-closedness of the original algebra" as constraints.
A four dimensional formulation

In the past two sections we worked out some details of a Hamiltonian formulation of a discretization of general relativity based on distributional fields of the form of (2). In this section we would like to sketch out a fully four dimensional lagrangian formulation which differs from the one we have just developed in that time, as well as space, will be treated discretely.

To do this we introduce into a four dimensional manifold $\mathcal{M}$, representing spacetime, an arbitrary hypercubic lattice. By extending the definitions (2) and (15) we will require that the self-dual curvature has support on the faces of the lattice, and the self-dual connection has support on the three dimensional surfaces of the lattice. We will see that the result of these assumptions is a natural reduction of the Lagrangian theory derived from the Capovilla-Dell-Jacobson form (6) to a Lagrangian form of a theory with a countable set of degrees of freedom.

We will follow what we did in the three dimensional case and label the sites of the four dimensional lattice by $x_\hat{n}$, the links by $\gamma_\hat{n}\hat{a}$, the faces by $S_{\hat{n}\hat{a}\hat{b}}$ and the three dimensional surfaces by $R_{\hat{n}\hat{a}\hat{b}\hat{c}}$. All of the indices are now four dimensional and refer to the object that is gotten by moving in the indicated positive directions from the site $\hat{n}$. Also, the labels of the two and three dimensional surfaces are assumed to be antisymmetric in the indices. Assuming a hypercubic structure, it is easy to label the faces making up the boundaries of the three dimensional surfaces, they are

$$\partial R_{\hat{n}\hat{a}\hat{b}\hat{c}} = S_{\hat{n}\hat{a}\hat{b}} \cup S_{\hat{n}+\hat{c}\hat{a}\hat{b}} \cup S_{\hat{n}+\hat{b}\hat{a}\hat{c}} \cup S_{\hat{n}+\hat{a}\hat{b}\hat{c}}$$

(59)

With these labellings, we then introduce a Lie algebra valued one form that has support on the three dimensional surfaces of the lattice. It is parametrized by a Lie algebra element $a^i_{\hat{n}\hat{a}\hat{b}\hat{c}}$ attached to each three dimensional surface, $R_{\hat{n}\hat{a}\hat{b}\hat{c}}$, and it is written as,

$$A^i_n(x) = \frac{1}{G} \sum_{\hat{n}\hat{a}\hat{b}\hat{c}} \int d^3R_{\hat{n}\hat{a}\hat{b}\hat{c}}(\rho) \epsilon_{abcd} \delta^4(x, R_{\hat{n}\hat{a}\hat{b}\hat{c}}(\rho)) a^i_{\hat{n}\hat{a}\hat{b}\hat{c}}.$$  

(60)

Here, $a, b, c, d$ are four dimensional spacetime indices and $\rho$ are three coordinates on the surface.

Following the derivation of (18) it is straightforward to show that the curvature associated with this connection is well defined and distributional, and has support on the two dimensional faces of the lattice. It is written as,

$$F^i_{ab}(x) = \frac{1}{G} \sum_{\hat{n}\hat{a}\hat{b}} \int d^2S_{\hat{n}\hat{a}\hat{b}}(\sigma) \epsilon_{abcd} \delta^4(x, S_{\hat{n}\hat{a}\hat{b}}(\sigma)) b^i_{\hat{n}\hat{a}\hat{b}},$$  

(61)

where the Lie algebra elements $b^i_{\hat{n}\hat{a}\hat{b}}$ associated to each face of the lattice are
given by

\[
b_{\tilde{n}\tilde{a}\tilde{b}}^{i} = \sum_{\tilde{c} \neq \tilde{a}, \tilde{b}} \left( a_{\tilde{n}\tilde{a}\tilde{b}}^{i} - a_{\tilde{n}-\tilde{c},\tilde{a}\tilde{b}}^{i} \right) + \frac{G}{16} \sum_{\tilde{d}} \epsilon_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \left[ a_{\tilde{n}\tilde{a}\tilde{b}}^{i} + a_{\tilde{n}-\tilde{c},\tilde{a}\tilde{b}}^{i} \right] \left[ a_{\tilde{n}\tilde{a}\tilde{b}}^{k} + a_{\tilde{n}-\tilde{d},\tilde{a}\tilde{b}}^{k} \right].
\]

(62)

It is now straightforward to plug (61) into the CDJ form of the action (6) and find that the reduced action is

\[
S[a, \phi] = \sum_{\tilde{n}\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \epsilon_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \bar{b}_{\tilde{n}\tilde{a}\tilde{b}}^{i} \bar{b}_{\tilde{n}\tilde{c}\tilde{d}}^{j} \phi_{\tilde{n}\tilde{a}\tilde{b}}^{-1} \phi_{\tilde{n}\tilde{c}\tilde{d}}^{-1}. 
\]

(63)

Here \( \phi_{\tilde{n}}^{ij} = \phi(x_{\tilde{n}})^{ij} \) is the matrix of scalar fields that satisfies (7) and (8) and \( \bar{b}_{\tilde{n}\tilde{a}\tilde{b}}^{i} \) is the average of the \( b_{\tilde{n}\tilde{a}\tilde{b}}^{i} \)'s on the four surfaces in the \( \tilde{a}\tilde{b} \) plane that touch the site \( \tilde{n} \),

\[
\bar{b}_{\tilde{n}\tilde{a}\tilde{b}}^{i} = \frac{1}{4} \left[ b_{\tilde{n}\tilde{a}\tilde{b}}^{i} + b_{\tilde{n}-\tilde{a},\tilde{a}\tilde{b}}^{i} + b_{\tilde{n}-\tilde{b},\tilde{a}\tilde{b}}^{i} + b_{\tilde{n}-\tilde{a}-\tilde{b},\tilde{a}\tilde{b}}^{i} \right].
\]

(64)

5 Conclusions

In this paper we have introduced two new approximation schemes for general relativity which are based on reductions to spaces of solutions based on finite dimensional spaces of solutions. We believe that it is likely that either the Lagrangian or the Hamiltonian formulation proposed here could be used to provide arbitrary good approximations to solutions to Einstein’s equations. However, there are a number of things that need to be checked in order to insure that either leads to a useful approximation scheme for either classical or quantum gravity. We would like to close by listing what we think remains to be done to establish the usefulness of the formulations introduced here.

1) The study of the algebra of the full constraints must be completed. It is not necessary that the algebra be first class for a reduction such as the one given here to provide a useful approximation method; in the case that the algebra is second class the reduction must be understood as constituting a partial gauge fixing. However, to go ahead, it is important to know what the algebra is.

2) The reality conditions must be formulated for the reduced theory, in both the Hamiltonian and Lagrangian case.

3) It may be more convenient, for some purposes, to use a simplicial rather than a cubic or hypercubic lattice. In this case it may be useful to introduce a dual lattice, whose faces are to be in one to one correspondence with the lines of the lattice (in the three dimensional case.) The faces of this dual lattice may then be assigned areas which, according to the results described in section 4 are given by \( |e_{I}| \), where \( I \) labels the faces (and the lines of the original lattice.) The simplices of this dual lattice can then be taken to constitute a piecewise flat Regge manifold whose edge lengths are determined from the areas of the
faces. In this way one can reformulate the system described here as a canonical formulation of the Regge calculus in which the three metric variables are treated conventionally, but the conjugate variables are the distributional self-dual curvature and connections, which live on a lattice dual to the simplicial lattice.

Furthermore, the Lagrangian formulation for simplicial lattices then appears to be a generalization of the four dimensional topological quantum field theories of Ooguri [33] and Crane and Yetter [34]. This is a connection that deserves further exploration.

4) It is interesting to speculate the extent to which this formulation could be useful for some approach to quantum gravity. It does seem unlikely that a direct quantization of the Hamiltonian system described in section 2 could be useful, given that direct imposition of the diffeomorphism constraints in the continuum already reduces quantum gravity to something very much like a finite system. On the other hand, it may be that the Lagrangian formulation described in section 4 could be the basis for a path integral quantization. Especially in the light of the recent impressive progress in four dimensional Monte Carlo simulations involving simplicial approximations, it would be interesting to investigate the duality suggested by the previous remarks between the present formulation and a Regge calculus formulation. It may also be that a direct attack on the path integral, given the action (63) may yield interesting results. As in any path integral formulation, the key problem that must be solved before reliable results can be extracted from the path integral is to find the correct measure of integration.

It is clear that a great deal of work remains to be done to establish whether or not the formulations proposed here may be useful for practical calculations in classical or quantum gravity. We hope that the results established here suffice to justify this further work.

Appendix. Observables and the meaning of distributional geometries

In this appendix we would like to return to the question of how distributional geometries of the kind we have been discussing can be used to approximate arbitrarily well smooth geometries. The idea of using such distributional geometries to approximate smooth geometries arose during work on the loop representation approach to quantum gravity [2], when it was realized that the classical limits of loop states could be associated with distributional geometries of the form (2). It was then discovered that certain classical observables, which are functions of the classical fields, could be extended unambiguously so that they were defined on an extension of the phase space of general relativity that includes distributional fields of the form of (2) and (16). Now, this extension cannot be made for
all observables, indeed, local observables, such as the metric at a point, cannot be given an unambiguous meaning for the distributional geometries. However, there are other observables, which are non-local which can be defined in such a way that they are meaningful when evaluated on the distributional geometries.

In references [22, 24, 18] one can find discussions of the quantum operators corresponding to these quantities. Very similar considerations apply to the extension of the observables to classical distributional geometries, as these results have never appeared we describe here the details for one observable, which is the area observable.

The main idea is to use a distributional geometry to approximate a smooth geometry by mimicking the weave construction of the quantum theory. For example, if one wants to approximate a metric, $q_{ab}$, which is slowly varying on a scale $L$, up to an accuracy of $a << L$, one can do the following. We require a set of curves $\alpha$, such that the distributional geometry $\tilde{E}_\alpha^a$, defined by (1), satisfies the following requirement:

For every surface $S$, whose area, given the metric $q_{ab}$, which we will call $A[S, q]$ is larger than $L^2$, and whose extrinsic curvature is bounded by $1/L^2$, we require that

$$|A[S, E_\alpha] - A[S, q]| < a^2 A[S, q]$$

If this is the case then we say that the distributional geometry $\tilde{E}_\alpha^a$ area-approximates the metric $q_{ab}$ on the scale $a$.

We may note that the problem of approximating a smooth geometry by a distributional one is closely connected to the problem of taking the classical limit of a loop state of the quantum theory and showing that it approximates a classical metric [23, 24]. These two problems are closely related because the loop states are eigenstates of certain operators that measure the three metric, such that the eigenvalues involve the distributional frame field associated to a loop by (1). Thus, the problem of extending an observable from smooth to distributional geometries is closely related to the problem of constructing good operators for those same quantities in the quantum theory through a regularization procedure.

We would thus like, in the remainder of this section, to show how the area operator can be extended to the distributional geometries. From the results it will be clear that it is easy to solve the problem of constructing distributional geometries that area-approximate any smooth geometry arbitrary well. Analogous results hold for other observables, including those involving the self-dual connections and frame fields, but we do not give them explicitly here.

Let us consider, then, an arbitrary two dimensional surface in $M$, which we will denote $S$. The coordinates of $S$ are given by $S^a(\sigma_\alpha)$, where the two $\sigma_\alpha$, $\alpha = 1, 2$ are coordinates on the surface. Let us consider the observable $A[S]$, which assigns to each surface $S$ its area induced from the three metric on $M$. One would not normally think that this observable could be extended to distributional geometries. However, we will now show that because our distributional
frame fields are also densities, $\mathcal{A}[\mathcal{S}]$ can be defined by a certain procedure, which allows it to extend unambiguously to distributional loop geometries. The result of this will be that, despite the fact that distributional geometries of the form of (2), are defined only on a set of measure zero, the observables $\mathcal{A}[\mathcal{S}]$ are, when nonzero, finite. They assign finite areas to surfaces that cross the loop $\alpha$.

We begin by writing the usual expression for $\mathcal{A}[\mathcal{S}]$ in the case of a smooth, nondegenerate geometry,

$$\mathcal{A}[\mathcal{S}] = \int_{\mathcal{S}} \sqrt{h},$$

(66)

where $h$ is the determinant of the induced two metric, $h_{ab} = q_{ab} - n_a n_b$, where $n^a$ is the unit normal. A simple calculation show that $h = \tilde{q}_{ab} n_a n_b$. Now, it is not hard to show that $\tilde{q}$ cannot itself be extended to distributional loop geometries. This can be demonstrated by a direct extension of the argument for the non-existence of a unambigous renormalized operator for $\tilde{q}$ in the loop representation [22, 23, 24]. As a result, we must construct the area (66) through a limit that does not need this extension. To do this, let us divide the surface up into $N$ disjoint regions $\mathcal{S}_i$, such that $\mathcal{S} = \bigcup_i \mathcal{S}_i$. We then have

$$\mathcal{A}[\mathcal{S}] = \sum_i \mathcal{A}[\mathcal{S}_i]$$

(67)

We will proceed by introducing an approximation for the square of $\mathcal{A}[\mathcal{S}_i]$ which becomes exact in the limit of infinitesimal surfaces. This is,

$$\mathcal{A}_{\text{approx}}^2[\mathcal{S}_i] \equiv \int d^2 S_{ab}^{\mathcal{S}_i} \int d^2 S_{cd}^{\mathcal{S}_i} T^{**}(S, S')_{ab cd}$$

(68)

where $T^{**}(x, y)_{ab cd} \equiv \epsilon_{abc} \epsilon_{cde} T^{ef}(x, y)$. To show that this approximates the area of the surface element for small surfaces, we use the facts that in the limit $T^{ab}(S, S') \approx \tilde{q}_{ab}$. We may invert the relation $h = \tilde{q}_{ab} n_a n_b$ to find that $\tilde{q} = h n^a n^b - \gamma^{ab}$ where $\gamma^{ab} n_b = 0$. An infinitesimal element of area is given by $dA = d^2 S_{ab} \sqrt{hn^a \epsilon_{abc}}$, from which it follows that,

$$dA^2 = d^2 S_{ab} d^2 S_{cd} \epsilon_{abc} \epsilon_{cde} \tilde{q}^{ef}$$

(69)

For smooth fields, this is then equal to (66) in the limit of small areas. We may then consider the limit in which we divide the surface up into smaller and smaller elements, so that $N \to \infty$. It then follows that,

$$\mathcal{A}[\mathcal{S}] = \lim_{N \to \infty} \sum_{i=1}^N \sqrt{\mathcal{A}_{\text{approx}}^2[\mathcal{S}]}.$$
We then evaluate (69) for a distributional loop geometry given by (1). We find,
\[ A_2^{\text{approx}}[\mathcal{S}] = a^4 \int d^2 S^a_{i} \oint d\alpha \delta^3 (S, \alpha) \epsilon_{abc} \int d^2 S^d_{i} \oint d\alpha (t) \delta^3 (S', \alpha) \epsilon_{def} \times \text{Tr} \left[ w(s) U_{\gamma_{S,S'}}(0, \pi) w(t) U_{\gamma_{S',S}}(\pi, 2\pi) \right] \]  
(71)

The expression
\[ \int d^2 S^a_{i} \oint d\alpha \delta^3 (S, \alpha) \epsilon_{abc} = I[S, \alpha] \]  
(72)

is the intersection number of the curve with the surface. It is zero unless they intersect, in which case it is equal to ±1 depending on the orientations. Now, as we take the limit of \( N \to \infty \) we will pass a point at which the absolute value of the intersection number of \( \alpha \) with each surface element is at most one. At that point we have
\[ A_2^{\text{approx}}[\mathcal{S}] = (a^2 |w(s^*_i)| I[S, \alpha])^2 \]  
(73)

where \( s^*_i \) is the intersection point of the curve with the \( i \)’th surface element. At this point further subdivisions do not change the value of the sum in (70), so the limit is equal to
\[ A[\mathcal{S}] = a^2 \sum_i |w(s^*_i)| \]  
(74)

where the sum is over the intersection points of the curve with the whole surface.

Given this result, it is clear how to construct a curve \( \alpha \) such that the distributional geometry \( E_\alpha^{\text{area}} \) area-approximates a given smooth metric \( q_{ab} \). Indeed, the construction mimicks the one that has been already given in the quantum case \([22, 23, 24]\); the main idea is to arrange the loops so that the sum of the contributions in (70) is equal to the area of each surface.

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