Existence of Solutions for Neutral Functional Integrodifferential Evolution Equations with Non Local Conditions

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Abstract

We study the existence of mild and strong solutions for nonlinear neutral functional integrodifferential evolution equations with nonlocal conditions in Banach spaces. The results are obtained by using the fractional powers of operators and Sadovskii’s fixed point theorem.

Keywords: Mild and Strong Solution, Neutral Equations, Nonlocal Condition, Semigroup

1. Introduction

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. The literature related to ordinary neutral functional differential equations is very extensive. The work in partial neutral functional differential equations with infinite delay was initiated by Hernandez and Henriquez. First-order partial neutral functional differential equations have been studied by different authors. The reader can consult Adimy¹, Hale¹³,¹⁴ and Wu¹⁵ for systems with finite delay and Hernandez Henriquez¹⁷,¹⁸ and Hernandez¹⁹ for the unbounded delay case. Hernandez¹⁵ established the existence results for partial neutral functional differential equations with nonlocal conditions modelled as

\[
\frac{d}{dt}(u(t)) + F(t, u(t)) = Au(t) + G(t, u(t)), \quad 0 \leq t \leq T
\]

\[
u_0 = \phi + q(u_{\tau_1}, u_{\tau_2}, \ldots, u_{\tau_s}) \in \Omega
\]

Bahuguna and Agarwal² studied the approximation of solution to a partial neutral functional differential equation with nonlocal history condition

\[
\frac{d}{dt}(u(t)) + g(t, u(t - \tau_1), \ldots, u(t - \tau_s)) + Au(t) = f(t, u(t), u(t - \tau_1)), \quad t > 0,
\]

\[
h(u) = \phi, \quad \text{on } [-\tau, 0]
\]

in a separable Hilbert space, where \( \tau = \max\{\tau_1, \tau_2\}, \tau_1, \tau_2 > 0 \). An extensive theory for ordinary neutral functional differential equations which includes qualitative behavior of classes of such equations and applications to biological and engineering processes. Several authors have studied the existence of solutions of neutral functional differential equations in Banach space²,³,⁴,⁶,¹¹,¹²,¹³,¹⁵,¹⁷,¹⁸,²³. The nonlocal Cauchy problem for semilinear evolution equations in Banach space was studied first by Byeszekswi⁷,⁸,⁹ where he established the existence and uniqueness of mild and classical solutions. The nonlocal conditions were motivated by physical problems and their importance is discussed in⁷,⁸. Balachandran et al¹⁰,²¹ studied the nonlocal Cauchy
problem for various type of nonlinear integrodiffrential equations. In addition, our result can also be regarded as an extension of the corresponding results on classical problem in $^\text{20,22}.$

In this paper, we study the following neutral functional integrodiffrential equation with nonlocal condition

$$\frac{d}{dt}[x(t)] + F(t, x(t), x(b(t)), \ldots, x(b_n(t)))] + A(t)x(t)
= G(t, x(t), x(a_1(t)), \ldots, x(a_m(t)))
+ K\left(t, x(t), \int_0^t k(t, s, x(s))ds\right).$$

$$x(0) + g(x) = x_0$$

(1.1)

2. Preliminaries

Let $-A$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $U(t, s)$ defined in the Banach space $X.$ Let $0 \leq \rho(A),$ then define the fractional power $A^\alpha,$ for $0 \leq \alpha \leq 1,$ as a closed linear operator on its domain $DA^\alpha(t)$ which is dense in $X.$ Further $DA^\alpha(t)$ is a Banach space under the norm

$$\|x\|_\alpha = \|A^\alpha x\|, x \in D(A^\alpha(t))$$

which we denote by $X_\alpha.$ Then for each $0 \leq \alpha \leq 1, X_\alpha \rightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ and the embedding is compact whenever the resolvent operator of $A$ is compact. We assume that

(a) there is a $M \geq 1$ such that $\|U(t, s)\| \leq M,$ for all $0 \leq t \leq a.$

(b) for any $a > 0,$ there exists a positive constant $C_a$ such that

$$\|A^\alpha U(t, s)\| \leq \frac{C_a}{t^\alpha}, 0 < t \leq T.$$

Now we represent the basic assumptions on equation (1.1):

(H$_1$) $F: [0, T] \times X^{n+1} \rightarrow X$ is a continuous function, $F([0, T] \times X^{n+1}) \subset DA^\alpha(t)$ with $n$ a positive integer, and there exists constants $L_1 > 0$ such that the function $A(t)F(t, x_0, x_1, \ldots, x_n)$ satisfies the Lipschitz condition:

$$\|A(t)F(t, x_0, x_1, \ldots, x_n) - A(t)F(t, y_0, y_1, \ldots, y_n)\| \leq L_1 \max_{i=0,1,\ldots,n} \|x_i - y_i\|$$

(2.2)

for every $0 \leq s_1, s_2 \leq T, x_i, \overline{x}_i \in X, i = 0, 1, \ldots, n,$ and the inequality

$$\|A(t)F(t, x_0, x_1, \ldots, x_n)\| \leq L_2 (\max \{\|x_i\| : i = 0, 1, \ldots, n\} + 1)$$

holds for any $(t, x_0, x_1, \ldots, x_n) \in [0, T] \times X^{n+1}.$

(H$_2$) The function $G : [0, T] \times X^{n+1} \rightarrow X$ satisfies the following condition:

(i) for each $t \in [0, T],$ the function $G(t, \cdot) : X^{n+1} \rightarrow X$ is continuous, and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1},$ the function $G(\cdot, x_0, x_1, \ldots, x_n) : [0, T] \rightarrow X$ is strongly measurable;

(ii) for each positive constant $k \in N,$ there is a positive function $g_k \in L^r([0, T])$ such that

$$\sup \|x_0, \ldots, x_m\| \leq k \|G(\cdot, x_0, x_1, \ldots, x_m)\| \leq g_k(t)$$

and

$$\lim_{k \rightarrow +\infty} \inf \int_0^T g_k(s)ds = \gamma < +\infty$$

(H$_3$) The function $K : [0, T] \times X \times X \rightarrow X$ satisfies the following condition:

(i) for each $t \in [0, T],$ the function $K(\cdot, \cdot, \cdot) : X \times X \rightarrow X$ and for each $x, y \in X, K(\cdot, x, y) : [0, T] \rightarrow X$ is strongly measurable.

(ii) For each positive number $r \in N,$ there is a positive function $\mu_r \in L^r([0, T])$ such that

$$\sup_{\|x\| \leq r} \left\|K(s, x(s), \int_0^T k(s, \tau, x(\tau))d\tau)\right\| \leq \mu_r(s)$$

and

$$\lim_{r \rightarrow +\infty} \inf \int_0^T \mu_r(s)ds = \gamma_1 < +\infty$$

(H$_4$) $a_i, b_j \in C([0, T]; [0, T]), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, g \in C(H; X)$ is completely continuous, where $H = C([0, T]; X),$ and there exists a constant $L_2 > 0$ such that $\|g(x)\| \leq L_2 \|x\|$ for each $x \in H.$

Theorem 2.1 (Sadovskii’s fixed point theorem, cf.$^\text{24}.$)

Let $P$ be a condensing operator on a Banach space $X,$ i.e., $P$ is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \leq \alpha(B)$ for every bounded set $B$ of $x$ with $\alpha(B) > 0.$ If $P(H) \subset H$ for convex, closed and bounded set $H$ of $X,$ then $P$ has a bounded point in $H$ (where $\alpha(.)$ denotes the Kuratowski’s measures of non-compactness).
3. Existence of Mild Solutions

DEFINITION 3.1
A continuous function \( x(.) : [0, T] \rightarrow X \) is said to be a mild solution of the nonlocal Cauchy problem (1.1), if the function \( U(t, s)F(s, x(b_1(s)), ..., x(b_n(s))) \), \( s \in (0, t) \) in integrable on \([0, t]\) and the following integral equation is verified:

\[
x(t) = U(t, 0)[x_0 + F(0, x(0), x(b_1(0)), ..., x(b_n(0))) - g(x)] - F(t, x(b_1(t)), ..., x(b_n(t)))
+ \int_0^t U(t, s)A(s)F(s, x(s), x(b_1(s)), ..., x(b_n(s)))\, ds
+ \int_0^t U(t, s)G(s, x(s), x(a_1(s)), ..., x(a_m(s)))\, ds
+ \int_0^t U(t, s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))\, d\tau\right)\, ds.
\]

THEOREM 3.1
If the assumption \((H_1) \sim (H_4)\) are satisfied and \( x \in X \), then the nonlocal Cauchy problem (1.1) has a mild solution provided that

\[
L_0 : L_0[(M + 1)M_0, MT] < 1
\]

and \( M_0L_1 + (L_2 + \gamma + \gamma_1 + M_0L_1 + L_1T)M < 1 \),

(3.4)

where \( M \) is from property \((f)\), \( M_0 = \sup\|A^{-1}(t)\| \).

PROOF.
For the sake of brevity, we rewrite \((t, x(t), x(b_1(t)), ..., x(b_n(t))) = (t, u(t))\) and \((t, x(t), x(a_1(t)), ..., x(a_m(t))) = (t, v(t))\). Define the operator \( P \) on \( C([0, T]; X) \) by the formula

\[
(Px)(t) = U(t, 0)[x_0 + F(0, v(0)) - g(x)] - F(t, v(t))
+ \int_0^t U(t, s)A(s)F(s, u(s))\, ds
+ \int_0^t U(t, s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))\, d\tau\right)\, ds, \quad 0 \leq t \leq T.
\]

for each positive number \( k \), let \( B_k = \{ x \in C([0, T]; X) : \|x(t)\| \leq k, \, 0 \leq t \leq T \} \), then for each \( k \), \( B_k \) is clearly a nonempty bounded closed convex set in \( X([0, T]; X) \), since the following relation holds

\[
\|U(t, s)A(s)F(s, v(s))\| \leq \|U(t, s)\|\|A(s)F(s, v(s))\| \leq ML_1(k + 1),
\]

then from Bouchner’s theorem it follows that \( U(t, s)A(s)F(s, v(s)) \) is integrable on \([0, t]\) since it is obviously strongly measurable, so \( P \) is well defined on \( B_k \). We claim that there exists a positive number \( k \) such that \( P(B_k) \subseteq B_k \). It is not true, then for each positive number \( k \), there is a function \( x_1(.) \in B_k \), but \( Px_1 \notin B_k \), that is \( \|Px_1(t)\| > k \) for some \( t(k) \in [0, T] \). However, on the other hand, we have

\[
k < \|Px_1(t)\| = \left\| U(t, 0)[x_0 + F(0, v(0)) - g(x)] - F(t, v(t))
+ \int_0^t U(t, s)A(s)F(s, u(s))\, ds
+ \int_0^t U(t, s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)\, ds \right\|
\]

\[
\leq \|U(t, 0)[x_0 - g(x)] - F(t, v(t))\|
\]

\[
+ \|A(t)A^{-1}(t)F(t, v(t))\|
\]

\[
+ \|A(t)A^{-1}(t)F(t, v(t))\| \leq M\|x_0 - g(x)\| + L_1 + M_0L_1 + M_1L_1 + M_1L_1 + ML_1T + M\int_0^T gk(s)\, ds + M\int_0^T \mu(s)\, ds.
\]

Dividing on both sides by \( k \) and taking the lower limit \( k \rightarrow +\infty \), we get

\[
M_0L_1 + (L_2 + M_0L_1 + L_1T + \gamma + \gamma_1)M \geq 1.
\]

This is contradicts (7). Hence some positive \( k \), \( PB_k \subseteq B_k \).

We will show that the operator \( P \) has a fixed point on \( B_k \) which implies that equation (1.1) has a mild solution. To this end, we decompose \( P \) into \( P = P_1 + P_2 \), where the operator \( P_1, P_2 \) are defined on \( B_k \) respectively by

\[
(P_1x)(t) = U(t, 0)[x_0 + F(0, v(0)) - g(x)]
+ \int_0^t U(t, s)A(s)F(s, v(s))\, ds
\]

and

\[
(P_2x)(t) = U(t, 0)[x_0 - g(x)]
+ \int_0^t U(t, s)G(s, u(s))\, ds
+ \int_0^t U(t, s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)\, ds,
\]

\( 0 \leq t \leq T \), and will verify that \( P_2 \) is contraction which \( P_2 \) is compact operator.

To prove \( P_2 \) is a contraction, we take \( x_1, x_2 \in B_k \), then for each \( t \in [0, T] \) and by condition \((H_1)\) and (6), we have

\[
\|P_2x_1(t) - P_2x_2(t)\| \leq \|U(t, 0)[x_0 - g(x)]\|
+ \|\int_0^t U(t, s)G(s, u(s))\, ds\|
+ \|\int_0^t U(t, s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)\, ds\|,
\]

\( 0 \leq t \leq T \), and will verify that \( P_1 \) is contraction which \( P_1 \) is compact operator.
\[ (P_2 x_n)(t) - (P_2 x_n)(t) \leq \|U(t, 0)[F(0, v_1(0)) - F(0, v_2(0))]\| + \|F(t, v_1(t)) - F(t, v_2(t))\| + \left\| \int_0^t U(t, s)A[F(s, v_1(s)) - F(s, v_2(s))]ds \right\| \leq (M+1)M_2L \sup_{0 \leq t \leq T} \|x_1(t) - x_2(t)\| \]

which shows that \( P_1 \) is contraction.

To prove that \( P_2 \) is compact, firstly we prove that \( P_2 \) is continuous on \( B_t \), Let \( \{x_n\} \subseteq B_t \) with \( x_n \rightarrow x \) is \( B_t \), then by \( (H) \), we have

\[ G(s, u_n(s)) \rightarrow G(s, u(s)), n \rightarrow \infty \]

\[ K(t, x_n(t), \int_0^t k(t, s, x_n(s))ds) \rightarrow K(t, x(t), \int_0^t k(t, s, x(s))ds) \]

as \( n \rightarrow \infty \). Since \( \|G(s, u_n(s)) - G(s, u(s))\| \leq 2g_1(s) \),

\[ \left\| K(t, x_n(t), \int_0^t k(t, s, x_n(s))ds) - K(t, x(t), \int_0^t k(t, s, x(s))ds) \right\| \leq 2\mu_1(s), \]

then by dominated convergence theorem we have,

\[ \|P_2 x_n - P_2 x\| = \sup_{0 \leq t \leq T} \|U(t, 0)[x_n(0) - x(0)] + \int_0^t U(t, s)G(s, u_n(s)) - G(s, u(s))ds \|
\]

\[ + \sup_{0 \leq t \leq T} \left\| \int_0^t U(t, s)K(s, x_n(s), \int_0^s k(s, \tau, x_n(\tau))d\tau)ds \right\| \rightarrow 0 \text{ as } n \rightarrow \infty. \]

That is \( P_2 \) is continuous.

We prove that the family \( \{P_2 x : x \in B_t\} \) is family of equicontinuous functions. To do this, let \( 0 \leq t_1 \leq t_2 \leq T, 0 < \epsilon < t_3 \), then

\[ \| (P_2 x)(t_2) - (P_2 x)(t_1) \| \leq \|U(t_2, 0) - U(t_1, 0)\| \|x(0)\| + \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\| \|G(s, u(s))\| ds \]

\[ + \int_{t_1}^{t_2} \|U(t_2, s)\| \|G(s, u(s))\| ds \]

Therefore, there are relatively compact sets arbitrarily close to the set \( V(t) \). Hence the set \( V(t) \) is also relatively compact in \( X \).

Thus by Arzela-Ascoli theorem \( P_1 \) is compact operator.

These arguments above enable us to conclude that \( P = P_1 + P_2 \) is condense mapping on \( B_t \), and by Theorem 2.1 there exists a fixed point \( z(\cdot) \) for \( P \) on \( B_t \), therefore the
nonlocal Cauchy problem (1.1) has mild solution. Then the proof is completed.

4. Existence of Strong Solutions

In this section, we provide conditions which allow the differentiation of the mild solutions obtained in section 3, i.e., these derivatives are shown to satisfy the differential equations of the form (1.1).

**Definition 4.2**

A function $X(.) : [0, T] \rightarrow X$ is said to be a strong solution of the nonlocal Cauchy problem (1.1), if (1) $x$ is continuous on $[0, T]$ and differentiable on $[0, T]$ (2) $x$ satisfies

$$
\begin{align*}
\frac{d}{dt} [x(t) + F(t, x(t), x(b_1(t)), \ldots, x(b_n(t)))] + A(t)x(t) & = (t, x(t), x(a_1(t)), \ldots, x(a_m(t))) \\
& + K \left( t, x(t), \int_0^t k(t, s, x(s)) ds \right) x(0) = g(x) = x_0
\end{align*}
$$

**Theorem 4.1**

Suppose that condition $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ are satisfied, and additionally the following conditions holds:

$(H'_7)$ For any function $y \in C([0, T]; X)$, the mapping $t \mapsto F(t, x(t), x(b_1(t)), \ldots, x(b_n(t)))$ is Holder continuous on $[0, T]$;

$(H_7)$ $G(., .)$ is Holder continuous i.e. there exists a constant $L_3 > 0$ such that

$$
\| G(s, x_0, \ldots, x_m) - G(\bar{s}, \bar{x}_0, \ldots, \bar{x}_m) \| \\
\leq L_3 \left[ |s - \bar{s}|^\theta + \max_{i = 0, 1, \ldots, m} |x_i - \bar{x}_i|^\theta \right]
$$

For $(s, x_0, \ldots, x_m, (\bar{s}, \bar{x}_0, \ldots, \bar{x}_m)) \in [0, T] \times X^{m+1}$.

$(H_8)$ There exists a constants $L_4, L_5, L_6 > 0$ such that

$$
\| K(t_1, x_1, y_1) - K(t_2, x_2, y_2) \| \leq L_4 \| t_1 - t_2 \|^\theta \\
+ L_5 \left( \| x_1 - x_2 \|^\theta + \| y_1 - y_2 \|^\theta \right)
$$

and $\| k(t, s, x) - k(t, s, y) \| \leq L_6 \| t - \tau \|^\theta$ for any $t, \tau \in [0, T]$ and $x \in X$.

$(H_9)$ There are constants $l_1, l_2 > 0$ such that $\| b_j(s) - b_j(\bar{s}) \| \leq l_1 |s - \bar{s}|$ for any $s, \bar{s} \in [0, T], j = 1, 2, \ldots, n$.

$(H_{10})$ $x_0 \in D(A)$, $g(x) \in D(A)$ for each $x \in H$, then the nonlocal Cauchy problem (1.1) has a strong solution on $[0, T]$, provided that (6) and (7).

**Proof.**

By Theorem 3.1. We see that equation (1.1) has a mild solution $x(t)$ on $[0, T]$ and we consider the differentiability of $x(t)$.

Let $f(t) = F(t, x(t), x(b_1(t)), \ldots, x(b_n(t)))$, $o(t) = U(t, 0)[x_0 + F(0, x(0), x(b_1(0)), \ldots, x(b_n(0)))] \in [0, x(0), x(b_0(0)), \ldots, x(b_n(0))]]$

$\| p(t) \| = \int_0^t U(s, t)A(s)F(s, x(s), x(b_1(s)), \ldots, x(b_n(s))) ds$

$\| q(t) \| = \int_0^t U(s, t)G(s, x(s), a_1(s)), \ldots, x(a_m(s))) ds$

Then from property (g) and (h) (see section 2) it follows immediately that $o(t), p(t), q(t)$ and $r(t)$ are Holder continuous on $[\varepsilon, T]$ for any $0 < \varepsilon < T$. Therefore, by condition $(H'_7)$ we obtain that $x(.)$ is Holder continuous on $[\varepsilon, T]$. We claim that the Lipschitz continuity of $A(., .)$ condition $H(1)$ implies that $A(., .)$ is locally Holder continuous. In fact, by assumption $(B_2)$ of $[A(t) : 0 \leq t \leq T]$, we have

$$
\begin{align*}
\| A(s)F(s, v) - A(\bar{s})F(\bar{s}, \bar{v}) \| & \leq \| A(s)F(s, v) - A(s)F(\bar{s}, \bar{v}) \| \\
& + \| A(s)F(s, v) - A(\bar{s})F(\bar{s}, \bar{v}) \| \\
& \leq L_4 |s - \bar{s}| + \max_{i = 0, 1, \ldots, m} |x_i - \bar{x}_i |
\end{align*}
$$

which shows that $A(., .)$ is locally Holder continuous. Hence condition $(H_7)$, $(H_8)$, $(H_9)$ assure that

$s \rightarrow A(s)F(s, x(s), x(b_1(s)), \ldots, x(b_n(s)))$

$s \rightarrow G(s, x(s), a_1(s)), \ldots, x(a_m(s))) ds$

and

$s \rightarrow K(s, x(s), \int_0^t k(s, t, x(t)) ds)
$$

are both Holder continuous on $[\varepsilon, T]$. Thus, from the proof of Theorem 5.7.1 of19 it is not difficult to see that $p(t) \in D(A)$, $q(t) \in D(A)$, $r(t) \in D(A)$ and

$$
\begin{align*}
p'(t) & = A(t)F(t, x(t), x(b_1(t)), \ldots, x(b_n(t))) \\
& - A(t) \int_0^t U(t, s)A(s)F(s, x(s), x(b_1(s)), \ldots, x(b_n(s))) ds
\end{align*}
$$
\[ q'(t) = G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))) - A(t) \int_0^t U(t, s)G(s, x(s), x(a_1(s)), \cdots, x(a_n(s))) \, ds \]

\[ r'(t) = K \left( t, x(t), \int_0^t k(t, s, x(s)) \, ds \right) - A(t) \left( \int_0^t U(t, s) \left[ K \left( s, x(s), \int_0^s k(s, \tau, x(\tau)) \, d\tau \right) \right] \right) \]

so we have that \( x' \) satisfies a.e. that

\[ \frac{d}{dt} \left[ x(t), F(t, x(t), x(b_1(t)), \cdots, x(b_n(t))) \right] = -A(t) \int_0^t U(t, s) \left[ G(s, x(s), x(a_1(s)), \cdots, x(a_n(s))) - g(x) \right] \]

\[ + \rho'(t) + q'(t) + r'(t) = A(t) \int_0^t U(t, s) \left[ F(s, x(s), x(b_1(s)), \cdots, x(b_n(s))) - g(x) \right] \]

\[ + A(t)F(t, x(t), x(b_1(t)), \cdots, x(b_n(t))) - A(t) \rho(t) \]

\[ + G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))) - A(t) q(t) \]

\[ + K(t, x(t), \int_0^t k(t, s, x(s)) \, ds) - A(t) r(t) \]

\[ = -A(t)x(t) + G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))) \]

\[ + K(t, x(t), \int_0^t k(t, s, x(s)) \, ds) \]

This shows that \( x(\cdot) \) is the strong solution of the nonlocal Cauchy problem (1.1). Thus the proof is completed.

5. References

1. Adimy M, Ezzinbi K. A class of linear partial neutral functional-differential equations with nondense domain. J. Differential Equations. 1998; 147:285–332.
2. Bahuguna D, Agarwal S. Approximations of solutions to neutral functional differential equations with nondense local history conditions. J Math Anal Appl. 2006; 317:583–602.
3. Balachandran K, Chandrasekaran M. Existence of solutions of a delay differential equation with nondense condition. Indian J Pure Appl Math. 1996; 27:443–9.
4. Balachandran K, Samuel FP. Existence of solutions for quasilinear delay integrodifferential equations with nondense conditions. Electronic J Differential Eqns. 2009; 6:1–7.
5. Balachandran K, Sakthivel R. Existence of solutions of neutral functional integrodifferential equation in Banach space. Proc Indian Acad Sci Math Sci. 1999; 109:325–32.
6. Benchohra M, Ntouyas S K. Nonlocal cauchy problems for neutral functional differential and integrodifferential inclusions. J Math Anal Appl. 2001; 258:573–90.
7. Byszewski L, Akca H. Existence of solutions of semilinear functional differential evolution nonlocal problem. Nonl Anal. 1998; 34:65–72.
8. Byszewski L. Theorem about existence and uniqueness of a solution of a semi linear evolution nonlocal Cauchy problem. J Math Anal Appl. 1991; 162:496–505.
9. L. Byszewski, Uniqueness criterion for solution to abstract nonlocal Cauchy problem. J Appl Math Stoch Anal. 1993; 6:49–54.
10. Fridman A. Partial Differential Equations. New York, Holt, Rienhat and Winston; 1969.
11. Fu X. On solutions of neutral nonlocal evolution equations with nondense domain. J Math Anal Appl. 2004; 299:392–410.
12. Fiu K, Ezzinbi K. Existence of solution for neutral functional evolution equations with nonlocal conditions. Nonl Anal. 2003; 54:215–27.
13. Hale JK. Partial neutral functional-differential equations. Rev Roumaine Math Pures Appl. 1994; 39:339–44.
14. Hale JK, Sjoerd MVL. Introduction to functional differential equations. Applied Math Sci. 99. New York, Springer-Verlag; 1993.
15. Hernandez E. Existence results for partial neutral functional differential equations with nonlocal conditions. Cadenos De Mathematica. 2001; 02:239–50.
16. Hernandez E. Existence results for partial neutral integrodifferential equations with unbounded delay. J Math Anal Appl. 2004; 292:194–210.
17. Hernandez E, Henriquez HR. Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay. J Math Anal Appl. 1998; 221(2):499–522.
18. Hernandez E, Henriquez HR. Existence results for partial neutral functional-differential equations with unbounded delay. J Math Anal Appl. 1998; 221:452–75.
19. Lin Y, Liu H. Semilinear integrodifferential equations with nonlocal Cauchy problem. Nonl Anal. 1996; 26:1023–33.
20. Marle CM. Measure et Probabilities. Paris; 1974.
21. Samuel FP, Balachandran K. Existence results for impulsive quasilinear integrodifferential equations in Banach spaces. Vietnam Journal of Math. 2010; 38:305–21.
22. Pazy A. Semigroup of linear operators and applications to partial differential equations. New York, Springer-Verlag; 1983.
23. Rankin SM. Existence and asymptotic behavior of a functional differential equations in Banach space. J Math Anal Appl. 1982; 88:531–42.
24. Sadovskii BN. On a fixed point principle. Funct Anal Appl. 1967; 1:74–6.
25. Wu J, Xia H. Self-sustained oscillations in a ring array of coupled lossless transmission lines. J Differential Equations. 1996; 124:247–78.