HERMITIAN METRICS, \((n-1,n-1)\) FORMS AND MONGE-AMPERÈ EQUATIONS∗

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Abstract. We show existence of unique smooth solutions to the Monge-Ampère equation for \((n-1)\)-plurisubharmonic functions on Hermitian manifolds, generalizing previous work of the authors. As a consequence we obtain Calabi-Yau theorems for Gauduchon and strongly Gauduchon metrics on a class of non-Kähler manifolds: those satisfying the Jost-Yau condition known as Astheno-Kähler.

Gauduchon conjectured in 1984 that a Calabi-Yau theorem for Gauduchon metrics holds on all compact complex manifolds. We discuss another Monge-Ampère equation, recently introduced by Popovici, and show that the full Gauduchon conjecture can be reduced to a second order estimate of Hou-Ma-Wu type.

1. Introduction

The complex Monge-Ampère equation
\[
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = e^{F} \omega^n, \quad \omega + \sqrt{-1} \partial \bar{\partial} u > 0,
\]
on a compact Kähler manifold \((M, \omega)\) was solved by Yau in the 1970’s [36] and has played an ubiquitous role in Kähler geometry ever since. Here \(F\) is a given smooth function on \(M\), normalized so that \(\int_M e^{F} \omega^n = \int_M \omega^n\). Yau’s Theorem, conjectured in the 1950’s by Calabi, is that \((1.1)\) has a unique solution \(u\) with \(\sup_M u = 0\).

For a general Hermitian metric \(\omega\), the complex Monge-Ampère equation was solved in full generality by the authors in [30] (see also [2, 13, 29]). In this case, there exists a unique pair \((u, b)\) with \(u\) a smooth function satisfying \(\sup_M u = 0\) and \(b\) a constant such that
\[
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = e^{F+b} \omega^n, \quad \omega + \sqrt{-1} \partial \bar{\partial} u > 0.
\]
This equation has applications to the study of cohomology classes and notions of positivity on complex manifolds [31, 3, 35].

Let \(M\) be a compact complex manifold of dimension \(n > 2\). We can consider metrics \(\omega\) satisfying conditions which are weaker than Kähler. There is a bijection from the space of positive definite \((1,1)\) forms to positive definite \((n-1,n-1)\) forms, given by
\[
\omega \mapsto \omega^{n-1}.
\]

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A key point is that closedness conditions on $\omega^{n-1}$ impose fewer equations than the same conditions on $\omega$. Indeed, Gauduchon [10] showed that every Hermitian metric is conformal to a metric $\omega$ satisfying
\[ \partial \overline{\partial} (\omega^{-1}) = 0, \]
a condition now known as Gauduchon. A stronger condition, introduced recently by Popovici [23],
\[ \partial (\omega^{n-1}) \text{ is } \partial\text{-exact}, \]
is known as strongly Gauduchon. 

There are natural Monge-Ampère equations associated to these conditions, obtained by replacing $(1,1)$ forms by $(n-1,n-1)$ forms. Our first result is about an equation which we call the Monge-Ampère equation for $(n-1)$-plurisubharmonic functions.

**Theorem 1.1.** Let $M$ be a compact complex manifold with two Hermitian metrics $\omega_0$ and $\omega$. Let $F$ be a smooth function. Then there exists a unique pair $(u,b)$ with $u$ a smooth function on $M$ and $b$ a constant such that
\[
\det (\omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2}) = e^{F+b} \det (\omega^{n-1}),
\]
with
\[
\omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} > 0, \quad \sup_M u = 0.
\]

This generalizes a recent result of the authors [33], where it was proved for $\omega$ Kähler, a conjecture of Fu-Xiao [8] (see also [6, 24]). The equation (1.4) for $\omega$ Kähler was first introduced by Fu-Wang-Wu [6, 7] who proved uniqueness and a number of other properties, including existence in the case when $\omega$ satisfies an assumption on curvature. The authors learned about equations of the type (1.4) from J.-P. Demailly in relation to questions about strongly Gauduchon metrics.

Note that a function $u$ satisfying
\[
\omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} > 0
\]
is called $(n-1)$-plurisubharmonic with respect to $\omega_0^{n-1}$ and $\omega$. The reason for this terminology is as follows. If $\omega$ is the Euclidean metric on $\mathbb{C}^n$ then the condition
\[
\sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} \geq 0,
\]
is the statement that $u$ is $(n-1)$-plurisubharmonic in the sense of Harvey-Lawson [14]. It is equivalent to the assertion that $u$ is subharmonic when restricted to every complex $(n-1)$-plane in $\mathbb{C}^n$. Thus equation (1.4) can be regarded as the Monge-Ampère equation for $(n-1)$-plurisubharmonic functions on Hermitian manifolds.

We now describe briefly a technical innovation which we use to prove Theorem 1.1. A key difficulty in the case of Hermitian metrics arises in the second order estimate of $u$ (Theorem 3.1 below). There are new terms of the form $T \ast D^3 u$, where $T$ is the torsion of $\omega$ and $D^3 u$ an expression involving
three derivatives of $u$. To control these terms, the idea is to add a small multiple of the quantity $g^{pqr} \eta_{ij} \eta_{pq}$, the “square of the tensor $\eta$ with respect to the metric $g$”, where $\eta_{ij}$ is the same tensor as defined in [33]. By doing so we obtain positive terms which can bound most of the $T^*D^3u$ terms. A fortunate cancellation of two remaining torsion terms allows the argument to work (cf. (3.37) and (3.60) below).

We now explain how equation (1.4) is related to Gauduchon and strongly Gauduchon metrics. In fact, Theorem 1.1 gives Calabi-Yau theorems for these metrics for a certain class of non-Kähler manifolds. Suppose that $\omega$ satisfies the condition

$$\partial \bar{\partial} (\omega^{n-2}) = 0.$$  

Then if $\omega_0$ is Gauduchon and $u$ a smooth function, a metric $\omega_u$ whose $(n-1)$th power is given by

$$\omega_u^{n-1} = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} > 0$$

is also Gauduchon. The same assertion holds if we replace “Gauduchon” by “strongly Gauduchon”.

A metric $\omega$ satisfying (1.5) is called Astheno-Kähler. Astheno-Kähler metrics were introduced by Jost-Yau in [17]. See also [5, 19, 20, 21] for more on these metrics, and for examples of Astheno-Kähler manifolds without Kähler metrics.

We obtain:

**Corollary 1.2.** Let $M$ be a compact complex manifold equipped with an Astheno-Kähler metric $\omega$. Let $\omega_0$ be a Gauduchon (resp. strongly Gauduchon) metric on $M$ and $F'$ a smooth function on $M$. Then there exists a unique constant $b'$ and a unique Gauduchon (resp. strongly Gauduchon) metric, which we write as $\omega_u$, with

$$\omega_u^{n-1} = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}$$

for some smooth function $u$, solving the Calabi-Yau equation

$$\omega_u^n = e^{F'+b'} \omega^n.$$  

(1.6)

This result was proved by the authors in [33] under the stronger assumption that $\omega$ is Kähler. In fact, when $\omega$ is Kähler then the result of [33] also gives a Calabi-Yau Theorem for balanced metrics [22] (see also [8, 9, 18, 27, 28]).

Gauduchon made the following conjecture in 1984 [11 IV.5]:

**Conjecture 1.3.** Let $M$ be a compact complex manifold. Let $\psi$ be a closed real $(1,1)$ form on $M$ with $[\psi] = c_1^{BC}(M)$ in $H_{1,1}^{BC}(M, \mathbb{R})$. Then there exists a Gauduchon metric $\tilde{\omega}$ on $M$ with

$$\text{Ric}(\tilde{\omega}) = \psi.$$  

(1.7)
Here $\text{Ric}(\tilde{\omega})$ is the Chern-Ricci curvature, given locally by
\[
\text{Ric}(\tilde{\omega}) = -\sqrt{-1} \partial \overline{\partial} \log \tilde{\omega}^n,
\]
and $H_{\text{BC}}^{1,1}(M, \mathbb{R})$ is the Bott-Chern cohomology group of $d$-closed real $(1,1)$ forms modulo $\partial \overline{\partial}$-exact ones. Gauduchon’s conjecture is a natural extension of the celebrated Calabi conjecture \cite{36} to compact complex manifolds. In complex dimension 2, Gauduchon’s conjecture follows from the result of Cherrier \cite{2} (see \cite{29} for an alternative proof) since in that case it follows from the solution of the complex Monge-Ampère equation \cite{12}.

As a consequence of Corollary \ref{corollary:gauduchon}, we can prove Gauduchon’s conjecture if $M$ admits an Astheno-Kähler metric.

**Corollary 1.4.** Let $M$ be a compact complex manifold equipped with an Astheno-Kähler metric $\omega$. Then Conjecture \ref{conjecture:gauduchon} holds on $M$.

In the special case when $(M, \omega)$ is Kähler then Yau’s solution of the Calabi conjecture already gives $\tilde{\omega}$ Kähler (and hence Gauduchon) satisfying \cite{17} in every Kähler class. The previous result of the authors in \cite{33} shows that in this case one can also find a Gauduchon metric $\tilde{\omega}$ satisfying \cite{17} where $\tilde{\omega}^{n-1}$ has the form $\tilde{\omega}^{n-1} = \omega_0^{n-1} + \sqrt{-1} \overline{\partial} u \wedge \omega_0^{n-2}$ for any given Gauduchon metric $\omega_0$.

Next, we discuss a different Monge-Ampère equation which is closely related to Gauduchon’s conjecture. This equation is a modification of \cite{14} and first appeared in the very recent preprint of Popovici \cite{25}. The solution of this equation would solve the full conjecture of Gauduchon.

Let $\omega_0$ be any Hermitian metric and $\omega$ a Gauduchon metric. To deal with the fact that $\partial \overline{\partial} \omega^{n-2} \neq 0$ in general, we consider the following $(n-1)$th root of $\Phi_u$ is that if $\omega_0$ is Gauduchon (strongly Gauduchon) and $\Phi_0 > 0$ then the $(n-1)$th root of $\Phi_u$ is also Gauduchon (strongly Gauduchon). Indeed, the $(n-1)$th root of $\Phi_u = \sqrt{-1} \overline{\partial} u \wedge \omega_0^{n-2} + \text{Re} \left( \sqrt{-1} \overline{\partial} u \wedge \overline{\partial} (\omega_0^{n-2}) \right)$ is $\partial \overline{\partial}$-closed. Moreover $\overline{\partial} \Phi_u$ is $\partial$-exact. In fact, $\beta_u$ is the $(n-1,n-1)$-part of the $d$-exact $(2n-2)$ form $d(d^c u \wedge \omega_0^{n-2})$.

Replacing $\omega_0^{n-1} + \sqrt{-1} \overline{\partial} u \wedge \omega_0^{n-2}$ by $\Phi_u$ gives a Monge-Ampère type equation. We conjecture that this equation can always be solved (Popovici posed the same statement as a question \cite{25}).

**Conjecture 1.5.** Let $M$ be a compact complex manifold with a Hermitian metric $\omega_0$ and a Gauduchon metric $\omega$. Let $F$ be a smooth function. Then there exists a unique pair $(u, b)$ with $u$ a smooth function on $M$ and $b$ a constant solving the equation
\[
\text{det}(\Phi_u) = e^{F+b} \text{det}(\omega^{n-1}),
\]

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The authors independently discovered the equation \cite{12} in June 2013 and completed Theorems \ref{theorem:main} and \ref{theorem:bott-chern} of this paper before \cite{25} was posted on the arXiv.
with
\[ \Phi_u = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right) > 0 \]
and \( \sup_M u = 0 \).

This conjecture would imply the conjecture of Gauduchon (see Section 4). Popovici [25] proved uniqueness for this equation and computed its linearization.

We now describe some progress towards Conjecture 1.5. First, we establish the following \textit{a priori} \( L^\infty \) estimate for (1.8).

\textbf{Theorem 1.6.} In the setting of Conjecture 1.5, let \( u \) be a smooth solution of (1.8). Then there exists a uniform constant \( C \) depending only on \( \omega, \omega_0 \) and \( \sup_M |F| \) such that
\[ \| u \|_{L^\infty} \leq C. \]

In the proof of this result we again make use of a cancellation between some bothersome torsion terms (see (5.11) below). As a consequence of this and suitable modifications of arguments of [33], it follows that Conjecture 1.5 can be reduced to an \textit{a priori} second order estimate.

\textbf{Theorem 1.7.} Let \( u \) solve (1.8) as above. If there exists a uniform constant \( C \), depending only on \( \omega, \omega_0 \) and bounds for \( F \) such that
\[ (1.9) \quad \Delta u \leq C (\sup_M |\nabla u|^2 + 1), \]
then Conjecture 1.5 (and hence also Conjecture 1.3) holds.

An estimate of the type (1.9) was proved by Hou-Ma-Wu [16] for the complex Hessian equations, and similar arguments are used for Monge-Ampère type equations in [33] and in the proof of Theorem 1.1. We expect that (1.9) does indeed hold for solutions of (1.8), but torsion terms arising from quantity \( \text{Re} (\sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2})) \) have so far thwarted our attempts to prove it.

The outline of the paper is as follows. In Sections 2 and 3 we prove zero and second order \textit{a priori} estimates for a solution \( u \) of the equation (1.4). From these estimates together with arguments adapted from [33], we deduce in Section 4 the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.4. In Sections 5 and 6 we discuss the Monge-Ampère equation (1.8) and prove Theorems 1.6 and 1.7 respectively.

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\section{2. Zero Order Estimate}

In this section we prove an \textit{a priori} zeroth order estimate for a solution of the Monge-Ampère equation (1.4). The argument we give is similar to the one when \( \omega \) is Kähler [33], except that we have to make use of arguments of [2, 29, 30] to deal with torsion terms that arise when integrating by parts.
First, we introduce some notation which will be used throughout the paper. Given a Hermitian metric \( g \) on \( M \), which we write in local holomorphic coordinates as \((g_{i\overline{j}}) > 0\), we define its associated \((1,1)\) form \( \omega \) to be

\[
\omega = \sqrt{-1} g_{i\overline{j}} dz^i \wedge d\overline{z}^j.
\]

Here we are summing in the repeated indices \( i \) and \( j \). We commonly refer to both \( \omega \) and \( g \) as “the Hermitian metric”.

Equation (1.4) is an equation of \((n-1,n-1)\) forms. Given a smooth function \( u \) on \( M \) we write \( \Psi_u \) for the positive definite \((n-1,n-1)\) form

\[
\Psi_u = \omega_{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} > 0.
\]

Recall that an \((n-1,n-1)\) form \( \Psi \) is positive definite if \( \Psi \wedge \sqrt{-1} \gamma \wedge \overline{\gamma} > 0 \), for all nonzero \((1,0)\) forms \( \gamma \). For \( \omega \) as above, we define the determinant of \( \omega_{n-1} \) to be

\[
\det(\omega^{n-1}) = (\det g)^{n-1},
\]

and this defines the determinant of a general positive definite \((n-1,n-1)\) form via (1.3).

As in [33], to solve (1.4) we will first apply the Hodge star operator \( * \) of \( \omega \) to rewrite it as an equation of \((1,1)\) forms. Define

\[
\tilde{\omega} := \frac{1}{(n-1)!} * \Psi_u, \quad \omega_h = \frac{1}{(n-1)!} * \omega_{n-1}.
\]

Then \( \tilde{\omega} \) and \( \omega_h \) are Hermitian metrics on \( M \). Write

\[
\tilde{\omega} = \sqrt{-1} \tilde{g}_{i\overline{j}} dz^i \wedge d\overline{z}^j, \quad \omega_h = \sqrt{-1} h_{i\overline{j}} dz^i \wedge d\overline{z}^j.
\]

We have (see Section 2 of [33])

\[
(2.1) \quad \tilde{\omega} = \omega_h + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \overline{\partial} u) > 0,
\]

and the equation (1.4) becomes

\[
(2.2) \quad \tilde{\omega}^n = e^{F+b} \omega^n,
\]

where we recall that we normalize \( u \) by \( \sup_M u = 0 \). Taking the trace of (2.1) we see that

\[
(2.3) \quad \text{tr}_\omega \tilde{\omega} = \text{tr}_\omega \omega_h + \Delta u,
\]

and therefore

\[
(2.4) \quad \sqrt{-1} \partial \overline{\partial} u = (n-1) \omega_h + (\text{tr}_\omega \tilde{\omega} - \text{tr}_\omega \omega_h) \omega - (n-1) \tilde{\omega}.
\]

Before we state the \( L^\infty \) estimate for \( u \), we first note that, by looking at the points where \( u \) achieves its maximum and its minimum, we immediately conclude that

\[
(2.5) \quad |b| \leq \sup_M |F| + C,
\]
for a uniform constant $C$ (which depends only on $(M, \omega_0)$ and $\omega$). Let $u$ solve (2.2) with $\sup_M u = 0$, where we recall that $\tilde{\omega}$ is given by (2.1). We will prove:

**Theorem 2.1.** There exists a constant $C$ which depends only on $\omega_0, \omega$ and $\sup_M |F|$, such that

$$\|u\|_{L^\infty} \leq C.$$

**Proof.** We make use of the following lemma from [33] (see Lemma 3.2 there):

**Lemma 2.2.** Assume that Hermitian metrics $\omega, \omega_h, \omega_0$ and $\tilde{\omega}$ are related by

$$\omega_h = \frac{1}{(n-1)!} * \omega_0^{n-1} \quad \text{and} \quad \tilde{\omega}^n = e^{\tilde{F}} \omega^n,$$

for $*$ the Hodge star operator of $\omega$. Define

$$\alpha = (n-1)\omega_h + (\text{tr}_\omega \tilde{\omega} - \text{tr}_\omega \omega_h)\omega - (n-1)\tilde{\omega}.$$

Then

$$\alpha \wedge (2\omega_0^{n-1} + \alpha \wedge \omega^{n-2}) \leq C \omega^n,$$

for a constant $C$ depending only on $\sup_M |\tilde{F}|$, $\omega$ and $\omega_0$ (in particular, $C$ is independent of $\tilde{\omega}$).

Note that the lemma is a pointwise statement about Hermitian metrics. Here, we have $\sqrt{-1} \partial \bar{\partial} u = \alpha$ as defined by (2.6) and $\tilde{F} = F + b$. Using (2.5) and this lemma, we conclude that

$$\sqrt{-1} \partial \bar{\partial} u \wedge (2\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}) \leq C \omega^n,$$

for a constant $C$ as in the statement of Theorem 2.1 (we will refer to such constants as uniform constants).

We make use of (2.7) in a Moser iteration argument. We compute for $p$ sufficiently large,

$$\int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge (2\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}) = p \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (2\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2})$$

$$+ \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \partial (2\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}) =: (A) + (B).$$

We use the inequality

$$\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} > 0,$$

to see that

$$\int_M e^{-pu} \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \bar{\partial} e^{-\frac{pu}{2}} \wedge \omega^{n-1} =: (A) \geq \frac{1}{Cp} \int_M \sqrt{-1} \partial \bar{\partial} e^{-\frac{pu}{2}} \wedge \bar{\partial} e^{-\frac{pu}{2}} \wedge \omega^{n-1}.$$
for a uniform $C > 0$. For (B), using $e^{-pu}\sqrt{-1}\partial u = -\frac{1}{p}\sqrt{-1}\partial e^{-pu}$, we have, for $p$ sufficiently large,

\begin{equation}
(B) = \frac{1}{p} \int_M e^{-pu}\sqrt{-1}\partial (2\omega_0^{n-1} + \sqrt{-1}\partial \omega \wedge \omega^{n-2}) \tag{2.10}
\end{equation}

\begin{align*}
&= \frac{1}{p} \int_M e^{-pu}\sqrt{-1}\partial u \wedge \sqrt{-1}\partial \omega \wedge \omega^{n-2} + \frac{2}{p} \int_M e^{-pu}\sqrt{-1}\partial \omega_0^{n-1} \\
&= \int_M e^{-pu}\sqrt{-1}\partial u \wedge \sqrt{-1}\partial \omega \wedge \omega^{n-2} + \frac{2}{p} \int_M e^{-pu}\sqrt{-1}\partial \omega_0^{n-1} \\
&= \frac{4}{p^2} \int_M \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \omega^{n-2} + \frac{2}{p} \int_M e^{-pu}\sqrt{-1}\partial \omega_0^{n-1} \\
&\geq -\frac{1}{2Cp} \int_M \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \omega^{n-1} - \frac{C'}{p} \int_M e^{-pu}\omega^n,
\end{align*}

where the constant $C$ is the same as the one in (2.9). On the other hand, multiplying (2.7) by $\omega$ and integrating, we have

\begin{equation}
\int_M e^{-pu}\sqrt{-1}\partial u \wedge (2\omega_0^{n-1} + \sqrt{-1}\partial \omega \wedge \omega^{n-2}) \leq C \int_M e^{-pu}\omega^n. \tag{2.11}
\end{equation}

Combining (2.8), (2.9), (2.10), (2.11) we conclude that for $p$ sufficiently large we have the “Cherrier-type” inequality (see [2] [30])

\begin{equation}
\int_M |\partial e^{-\frac{pu}{2}}|^2 \omega^n \leq Cp \int_M e^{-pu}\omega^n. \tag{2.12}
\end{equation}

In [30, Lemma 2.2] we proved that (2.12) implies that $|\{u \leq \inf_M u + C_0\}| \geq \delta > 0$, for some uniform $\delta, C_0$. We give two proofs of how to derive a $C^0$ bound on $u$ from this.

For the first proof, note that by Gauduchon’s Theorem [10], there exists a smooth function $\sigma$ on $M$ so that $g' = e^{\sigma} g$ is Gauduchon. Write $\Delta' = g'^{ij} \partial_i \partial_j$ for the Laplacian of $g'$ acting on functions. We use the existence of a Green’s function for $\Delta'$: since $\Delta'$ is an elliptic second order differential operator, with the kernel consisting of just constants, standard linear PDE theory (see e.g. [11 Appendix A]) shows that there exists a green function $G$ for $\Delta'$ which satisfies $G(x,y) \geq -C, \|G(x,\cdot)\|_{L^1} \leq C$, and

$$u(x) = \frac{1}{\int_M \omega^n} \int_M u \omega^n - \int_M \Delta' u(y) G(x,y) \omega^n(y),$$

for all $u$ and $x$. Since $\omega'$ is Gauduchon we have $\int_M \Delta' \omega'^n = 0$. Therefore, we are free to add a large constant to $G(x,y)$ to make it nonnegative, while preserving the same Green formula. Using $\Delta' u = e^{-\sigma} \Delta u \geq -e^{-\sigma} \text{tr}_\omega \omega_h \geq -C,$
and \( \sup_{M} u = 0 \), we immediately deduce that \( \int_{M} (-u) \omega^{n} \leq C \). But then we have

\[
- \delta \inf_{M} u \leq \int_{\{u \leq \inf_{M} u + C_0\}} (-u + C_0) \leq C.
\]

The second proof uses Moser iteration, in the spirit of [30]. Let \( v = u - \inf_{M} u \), so that \( \Delta' v \geq -C \). For any \( p \geq 1 \) compute

\[
\int_{M} |\partial v_{\frac{p+1}{2}}|^{2} \omega^{n} = \frac{n(p+1)^{2}}{4} \int_{M} \sqrt{-1} v_{p-1} \partial v \wedge \omega^{n-1} \\
= \frac{n(p+1)^{2}}{4p} \int_{M} \sqrt{-1} v_{p} \wedge \partial v \wedge \omega^{n-1} \\
= \frac{(p+1)^{2}}{4p} \int_{M} v^{p}(-\Delta' v) \omega^{n} + \frac{n(p+1)}{4p} \int_{M} \sqrt{-1} v_{p+1} \wedge \partial(\omega^{n-1}) \\
= \frac{(p+1)^{2}}{4p} \int_{M} v^{p}(-\Delta' v) \omega^{n} \\
\leq C \frac{(p+1)^{2}}{4p} \int_{M} v^{p} \omega^{n},
\]

using the Gauduchon condition \( \partial \bar{\partial}(\omega^{n-1}) = 0 \). Since \( \sigma \) is bounded we can switch from \( \omega \) to \( \omega' \) and obtain:

\[
\int_{M} |\partial v_{\frac{p+1}{2}}|^{2} \omega^{n} \leq C \frac{(p+1)^{2}}{4p} \int_{M} v^{p} \omega^{n}, \tag{2.14}
\]

A standard Moser iteration argument implies that

\[
- \inf_{M} u = \sup_{M} v \leq C \int_{M} v \omega^{n} + C.
\]

To bound \( \|v\|_{L^{1}} \), use the Poincaré inequality and (2.14) with \( p = 1 \) to get

\[
\|v - \overline{v}\|_{L^{2}} \leq C \left( \int_{M} |\partial v|^{2} \omega^{n} \right)^{\frac{1}{2}} \leq C' \|v\|_{L^{1}}^{\frac{1}{2}},
\]

where \( \overline{v} \) is the average of \( v \) with respect to \( \omega^{n} \). But also

\[
\frac{\delta}{\int_{M} \omega^{n}} \|v\|_{L^{1}} = \delta \overline{v} \leq \int_{\{v \leq C_0\}} \overline{v} \omega^{n} \leq \int_{\{v \leq C_0\}} (|v - \overline{v}| + C_0) \omega^{n} \leq \|v - \overline{v}\|_{L^{1}} + C,
\]

hence

\[
\|v\|_{L^{1}} \leq C(\|v - \overline{v}\|_{L^{1}} + 1) \leq C(\|v - \overline{v}\|_{L^{2}} + 1) \leq C(\|v\|_{L^{1}}^{\frac{1}{2}} + 1),
\]

which implies that \( \|v\|_{L^{1}} \leq C \), and we are done. \( \square \)
3. Second order estimate

In this section we continue the proof of Theorem 1.1 by establishing an estimate on the metric $\tilde{g}$ in terms of the gradient of $u$. The setup is the same as in the previous section, and in particular, $\tilde{g}$ is a Hermitian metric given by

$$\tilde{g}_{ij} = h_{ij} + \frac{1}{n-1} \left( (\Delta u) g_{ij} - u_{ij} \right),$$

which solves

$$\det \tilde{g} = e^{F + b} \det g.$$

We prove:

**Theorem 3.1.** There exists a uniform constant $C$ depending only on $(M, \omega_0), \omega$ and bounds for $F$ such that

$$\text{tr}_g \tilde{g} \leq C \left( \sup_M |\nabla u|_g^2 + 1 \right).$$

Noting that $\text{tr}_g \tilde{g} = \text{tr}_g h + \Delta u$, we define, as in [33],

$$\eta_{ij} = u_{ij} + (\text{tr}_g h) g_{ij} - (n-1) h_{ij} = (\text{tr}_g \tilde{g}) g_{ij} - (n-1) \tilde{g}_{ij}.$$

For $x \in M$ and $\xi$ a unit vector with respect to $g$, we define $H(x, \xi)$ by

$$H(x, \xi) = \log(\eta_{ij} \xi^i \xi^j) + c \log(g^{kp} \eta_{pq} \eta_{ij} \xi^i \xi^j) + \varphi(|\nabla u|_g^2) + \psi(u),$$

where $\varphi, \psi$ are given by

$$\varphi(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad \text{for } 0 \leq s \leq K - 1,$$

$$\psi(t) = -A \log \left( 1 + \frac{t}{2L} \right), \quad \text{for } -L + 1 \leq t \leq 0,$$

for

$$K := \sup_M |\nabla u|_g^2 + 1, \quad L := \sup_M |u| + 1, \quad A := 2L(C_1 + 1)$$

and $C_1$ is to be determined later. The constant $c > 0$ is a small constant, depending only on $n$, which will also be determined later. The quantities $\varphi$ and $\psi$ are identical to those in the computation of Hou-Ma-Wu [16].

Recall that we have normalized $u$ so that $\sup_M u = 0$. $L$ is uniformly bounded. The quantities $\varphi(|\nabla u|_g^2)$ and $\psi(u)$ are both uniformly bounded.

For later use note that, as in [16], we have

$$\frac{1}{2K} \geq \varphi' \geq \frac{1}{4K} > 0, \quad \varphi'' = 2(\varphi')^2 > 0$$

and

$$\frac{A}{L} \geq -\psi' \geq \frac{A}{2L} = C_1 + 1, \quad \psi'' \geq \frac{2\varepsilon}{1 - \varepsilon} (\psi')^2, \quad \text{for all } \varepsilon \leq \frac{1}{2A + 1},$$

whenever $\varphi$ and $\psi$ are evaluated at $|\nabla u|_g^2$ and $u$ respectively.
Note that the difference between $H$ here and the quantity considered in [33] is that we have added a small multiple of the quantity
\[
\log(g^{p\overline{q}} \eta_{p} \eta_{q} \xi^{i} \overline{\xi^{j}}).
\]
This is important in what follows, since after applying the linearized operator to our $H$ we obtain additional positive terms (see (3.17), below) which are needed to bound torsion terms that did not arise in [33].

We restrict the function $H$ to the compact set $W$ in the $g$-unit tangent bundle of $M$ where $\eta_{i} \xi^{i} \overline{\xi^{j}} \geq 0$, defining $H = -\infty$ whenever $\eta_{i} \xi^{i} \overline{\xi^{j}} = 0$. Note that $\eta_{i} \xi^{i} \overline{\xi^{j}} > 0$ implies $g^{p\overline{q}} \eta_{p} \eta_{q} \xi^{i} \overline{\xi^{j}} > 0$. This way, $H$ is upper semi-continuous on $W$ and hence $H$ achieves a maximum at some point $(x_0, \xi_0)$ where $\eta_i(x_0) \xi^i \overline{\xi^j} > 0$.

Choose a holomorphic coordinate system $z^1, \ldots, z^n$ centered at $x_0$, that at $x_0$,
\[
g_{ij} = \delta_{ij}, \quad \eta_{ij} = \delta_{ij} \eta_1, \quad \eta_{1} \geq \eta_{2} \geq \cdots \geq \eta_{n}.
\]
From (3.3), $(\tilde{g}_{i\overline{j}})$ is also diagonal at this point. Writing $\lambda_i = \tilde{g}_{i\overline{j}}$ we have at $x_0$,
\[
\eta_1 = \sum_{j=1}^{n} \lambda_j - (n-1) \lambda_1,
\]
and in particular $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The three quantities $\lambda_n$, $\eta_1$ and $\text{tr}_{\omega} \omega$ are uniformly equivalent:
\[
\frac{1}{n} \text{tr}_{\omega} \omega \leq \lambda_n \leq \eta_1 \leq (n-1) \lambda_n \leq (n-1) \text{tr}_{\omega} \omega.
\]
Observe that $\eta_n$ could be negative, and it is possible in general that $-\eta_n > \eta_1$. This only happens for $n \geq 4$, since it is easy to check that when $n = 3$ we have $\eta_1 > -\eta_n$. However, in any case we have
\[
|\eta_n| \leq (n-2) \eta_1.
\]
At $x_0$ we have
\[
\log(\eta_{i} \xi^{i} \overline{\xi^{j}}) + c \log(g^{p\overline{q}} \eta_{p} \eta_{q} \xi^{i} \overline{\xi^{j}}) = \log \left( \sum_{i} \eta_i |\xi^i|^2 \right) + c \log \left( \sum_{i} \eta_i^2 |\xi^i|^2 \right).
\]
Using (3.9), the reader can check that if $c > 0$ is chosen sufficiently small, depending only on $n$, then this quantity is maximized by $\xi_0 = \partial / \partial z^1$. In fact, one can take any positive $c$ with $c < 1/(n-3)$ for $n > 3$ and any $c > 0$ for $n = 3$.

We extend $\xi_0$ to a locally defined smooth unit vector field
\[
\xi_0 = g_{1\overline{1}}^{-1/2} \frac{\partial}{\partial z^1}.
\]
Define a new quantity, in a neighborhood of $x_0$, by
\[
Q(x) = H(x, \xi_0) = \log(g_{1\overline{1}}^{-1} \eta_1) + c \log(g_{1\overline{1}}^{-1} g^{1\overline{1}} \eta_1 \eta_1) + \varphi(|\nabla u|_g^2) + \psi(u),
\]
which achieves a maximum at $x_0$. Note that at $x_0$ we have
\[ g_{\mathbf{T}T}^{-1} g^{\mathbf{T}} \eta_{\mathbf{T}} \eta_{\mathbf{T}} = (\eta_{\mathbf{T}})^2. \]

Our goal is to prove that, at $x_0$,
\begin{equation}
|\eta_{\mathbf{T}}| \leq CK,
\end{equation}
for a uniform constant $C$. This will prove the theorem: at any given point $x \in M$, we choose coordinates are before, so that $\frac{1}{n} \text{tr}_\omega \tilde{\omega}(x)$ is bounded from above by $\eta_{\mathbf{T}}(x)$. But this is in turn bounded above by
\[
\sup_{\xi \in W} \left\{ (\eta_{\mathbf{K}} \xi^k \xi^\ell)^{1/(1+2c)} \left( g^{\mathbf{K} \ell} \eta_{\mathbf{K}} \eta_{\mathbf{J}} \xi^k \xi^j \right)^{c/(1+2c)} \right\},
\]
and so
\[
\sup_M \text{tr}_\omega \tilde{\omega} \leq n \sup_{\xi \in W} \left\{ (\eta_{\mathbf{K}} \xi^k \xi^\ell)^{1/(1+2c)} \left( g^{\mathbf{K} \ell} \eta_{\mathbf{K}} \eta_{\mathbf{J}} \xi^k \xi^j \right)^{c/(1+2c)} \right\}
\leq C e^{Q(x_0)/(1+2c)} \leq C' K = C' (\sup_M |\nabla u|^2 g + 1),
\]
as required. We may and do assume, without loss of generality, that $|\eta_{\mathbf{T}}| >> 1$ at $x_0$.

Define a linear operator $L$ on functions by $L(v) = \Theta \bar{\partial}_i \bar{\partial}_j v$ where
\[
\Theta \bar{\partial}_j = \frac{1}{n-1} ( (\text{tr}_g g) g^{i \bar{j}} - \bar{g}^{i \bar{j}} ) > 0.
\]
At $x_0$, $\Theta \bar{\partial}_j$ is diagonal, and we have
\begin{equation}
\sum_i \Theta \bar{\partial}_i = \text{tr}_g g.
\end{equation}

We will apply the maximum principle to $Q$ by computing $L(Q)$. We use covariant derivatives with respect to the Hermitian metric $g$, which we denote by subscripts. First, at $x_0$, we have from (3.10), dropping the zero subscript in $\xi_0$,
\begin{equation}
\xi_i^1 + \bar{\xi}_i = 0.
\end{equation}
Using this, compute
\begin{equation}
0 = Q_i = (1 + 2c) \eta_{\mathbf{T}T} + \varphi' \left( \sum_p u_p u_{\bar{p}} + \sum_p u_{\bar{p}} u_p \right) + \psi'u_i.
\end{equation}
Next

\begin{equation}
L(Q) = (1 + 2c) \sum_i \frac{\Theta^\tau |\eta_{1Ti}|^2}{\eta_{1T}} + c \sum_{i \neq 1} \sum_{p} \frac{\Theta^\tau |\eta_{pTi}|^2}{(\eta_{1T})^2} + c \sum_{i \neq 1} \sum_{p} \frac{\Theta^\tau |\eta_{Ti}|^2}{(\eta_{1T})^2} - (1 + 2c) \sum_i \frac{\Theta^\tau |\eta_{1Ti}|^2}{(\eta_{1T})^2} + \psi' \sum_i \Theta^\tau u_{i1} + \psi'' \sum_i \Theta^\tau |u_i|^2 + \varphi' \sum_{i,p} \Theta^\tau (u_{pi1}|u_p|^2 + u_{p1}u_p) + \varphi' \sum_{i,p} \Theta^\tau u_{p1}u_p + \sum_{p} \sum_{i} u_{p1}u_{pi} \bigg|\bigg| + \psi' \sum_{i,p} \Theta^\tau u_{p1}u_p + u_{p1}u_p,\end{equation}

where

\begin{equation}
(*) = 2(1 + c) \sum_{p \neq 1} \sum_i \frac{\Theta^\tau}{\eta_{1T}} \text{Re}(\eta_{pTi} \xi_p^i + \eta_{Ti} \xi_p^i) + 2c \sum_{p \neq 1} \sum_i \frac{\Theta^\tau}{(\eta_{1T})^2} \text{Re}(\eta_{pTi} \eta_{pTi} \xi_p^i + \eta_{Ti} \eta_{pTi} \xi_p^i) + (1 + c) \sum_i \Theta^\tau (\xi_{1i}^i + \xi_{1i}^i) + \sum_{p,i} \frac{\Theta^\tau}{\eta_{1T}} \text{Re}(\xi_p^i \xi_p^i) + c \sum_{p,i} \frac{\Theta^\tau}{(\eta_{1T})^2} \text{Re}(\xi_p^i \xi_p^i).
\end{equation}

We have used again the equation (3.13), which is why the first two \( p \) summations in (*) do not include \( p = 1 \). Observe that the second and third terms on the first line of the expression for \( L(Q) \) give new positive terms that did not appear in \( \Delta S \). We will use half of these terms to bound (*), and the other half later. We obtain:

\begin{equation}
L(Q) \geq (1 + 2c) \sum_i \frac{\Theta^\tau |\eta_{1Ti}|^2}{\eta_{1T}} + \frac{c}{2} \sum_{i \neq 1} \sum_{p} \frac{\Theta^\tau |\eta_{pTi}|^2}{(\eta_{1T})^2} + \frac{c}{2} \sum_{i \neq 1} \sum_{p} \frac{\Theta^\tau |\eta_{Ti}|^2}{(\eta_{1T})^2} - (1 + 2c) \sum_i \frac{\Theta^\tau |\eta_{1Ti}|^2}{(\eta_{1T})^2} + \psi' \sum_i \Theta^\tau u_{i1} + \psi'' \sum_i \Theta^\tau |u_i|^2 + \varphi' \sum_{i,p} \Theta^\tau (u_{pi1}|u_p|^2 + u_{p1}u_p) + \varphi' \sum_{i,p} \Theta^\tau u_{p1}u_p + \sum_{p} \sum_{i} u_{p1}u_{pi} \bigg|\bigg| + \psi' \sum_{i,p} \Theta^\tau u_{p1}u_p + u_{p1}u_p - \text{tr}_g.$
for a uniform $C$.

Differentiating (3.2), we obtain

$$\tag{3.18} \tilde{g}^j \nabla_k \tilde{g}_j = F_\ell, \quad \tilde{g}^j \nabla_m \nabla_k \tilde{g}_j - \tilde{g}^j \tilde{g}^{pj} \nabla_m \tilde{g}_{pj} \nabla_k \tilde{g}_j = F_{\ell m}.$$  

In particular, we obtain at $x_0$

$$\tag{3.19} \sum_i \Theta_i^\alpha u_{i1T} + \sum_i \tilde{g}^i \tilde{h}_{i1T} = - \frac{1}{(n-1)^2} \sum_{i,j} \tilde{g}^i \tilde{g}^j (g_j^i \sum_a u_{a\ell1} - u_{j1} + \hat{h}_{j1}) = F_{T1},$$

where we are writing $\hat{h}_{j1} = (n-1) h_{j1}$.

We have the following commutation formulae:

$$\tag{3.20} u_{i\ell} = u_i - u_p R_{\ell j}^{p,}, \quad u_{p\ell} = u_{pm} - \frac{T_{m j}^q u_{pq}}{T_{m q}^j u_{pq}}, \quad u_{\ell \ell} = u_{\ell} - \frac{1}{T_{m j}^q u_{pq}},$$

$$u_{i m} = u_{i m} + u_{p} R_{m j}^{p} - u_{m} R_{i j}^{p} - T_{m j}^q u_{pq} - \frac{T_{m q}^j u_{pq}}{T_{m j}^q u_{pq}}.$$  

Indeed, these follow from the standard formulae (using conventions of [32]):

$$\nabla_i a_\ell = \partial_i a_\ell - \Gamma_{i A}^p a_p, \quad \Gamma_{ij}^k = g^{k q} \partial_i g_{jq}, \quad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k, \quad R_{m q}^p = - \partial_m \Gamma_{i q}^p,$$

and

$$\tag{3.21} [\nabla_i, \nabla_j] a_\ell = - R_{\ell j}^{p} a_p, \quad [\nabla_i, \nabla_j] a_m = R_{ij}^{q} a_q.$$  

Then from (3.19) and (3.20),

$$\tag{3.22} \sum_i \Theta_i^\alpha u_{1T1} + \sum_i \tilde{g}^i \tilde{h}_{i1T} + \sum_i \Theta_i^\alpha (u_{a T 11}^a - u_{a T 11}^a)$$

$$+ \sum_i \Theta_i^\alpha (T_{11}^p u_{pT1} - T_{11}^p u_{1p} - T_{11}^p u_{a T}$$

$$- \frac{1}{(n-1)^2} \sum_{i,j} \tilde{g}^i \tilde{g}^j (g_j^i \sum_a u_{a\ell1} - u_{j1} + \hat{h}_{j1}) = F_{T1}.$$  

From (3.3),

$$\tag{3.23} u_{1T1} = \eta_{1T1} + \hat{h}_{1T1} - (\text{tr}_g h)^{\gamma}_{\gamma}.$$
From (3.17), (3.22) and (3.23) and using the fact that at $x_0$ we have $L(Q) \leq 0$, we obtain
(3.24)
\[
0 \geq (1 + 2c) \sum_{i,j} \tilde{g} \tilde{g} (g_{j} \sum_a u_{at} - u_{jt} + \hat{h}_{jt}) (g_{i} \sum_b u_{bt} - u_{it} + \hat{h}_{it})
\]
\[
+ \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^i \eta_{1pi}}{(\eta_{1i})^2} + \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^i \eta_{1pi}}{(\eta_{1i})^2} - (1 + 2c) \sum_i \frac{\Theta^i \eta_{1i}}{(\eta_{1i})^2}
\]
\[
-C \text{tr} \tilde{g} + \frac{1 + 2c}{\eta_{1i}} \left( F_{1i} - \sum_i \Theta^i (u_{at} R_{1i}^i - u_{at} R_{1i}^a) \right)
\]
\[
- \sum_i \tilde{g} \tilde{h}_{11} - \sum_i \Theta^i \left( \hat{h}_{11} - (\text{tr} \tilde{h}) \right)
\]
\[
+ \frac{1 + 2c}{\eta_{1i}} \left( \sum_i \Theta^i \left( T_{1i}^p u_{1pi} + T_{1i}^b u_{1pi} + T_{1i}^p T_{1i}^b u_{1pi} \right) \right)
\]
\[
+ \psi^i \sum_i \Theta^i u_{1i} + \psi'' \sum_i \Theta^i |u_i|^2 + \varphi'' \sum_i \Theta^i \sum_p u_{pi} u_{1pi} + \sum_p u_{pi} u_{1pi} \right|^2
\]
\[
+ \varphi^i \sum_i \Theta^i \left( |u_{1pi}|^2 + |u_{pi}|^2 \right) + \varphi \sum_i \Theta^i \left( u_{1pi} u_{1pi} + u_{1pi} u_{1pi} \right).
\]
=: (1) + (2) + (3) + (4) + (5) + (6) + (7),
where the numbers (1) - (7) refer to the lines in the expression above.

We now make the observation that we may and do assume that, at the point $x_0$,
(3.25)
\[ |u_{ij}^2| \leq 2|\eta_{1i}|, \quad \text{for all } i, j. \]
Indeed, since our goal is to prove (3.11) we may assume without loss of generality that $|\eta_{1i}|$ is large. Then (3.25) follows immediately from (3.3).

We will now deal with each line of (3.24) in turn, starting with the easiest.

Lines (3) and (4) of (3.24). From (3.12) and (3.25), we immediately have a lower bound for the third and fourth lines of (3.24):
(3.26)
\[ (3) + (4) \geq - C \text{tr} \tilde{g} - C. \]

Line (6) of (3.24). The second and third terms in this line are nonnegative (since $\psi'', \varphi'' > 0$) and we will make use of them later. Note that by taking trace with respect to $\tilde{g}$ of (3.11) we obtain
(3.27)
\[ \psi \sum_i \Theta^i u_{1i} = (-\psi') (\text{tr} \tilde{h} - n). \]
On the other hand recall that from (3.6),

\[ \frac{A}{L} \geq -\psi' \geq \frac{A}{2L} = C_1 + 1, \]

where \( C_1 \) is still to be determined. So we have

\[ \psi' \sum_i \Theta^{\tilde{\eta}} u_{\tilde{\eta}} \geq -2n(C_1 + 1) + (C_1 + 1)\text{tr}_{\tilde{g}}h \]

and hence

\[ (6) \geq -2n(C_1 + 1) + (C_1 + 1)\text{tr}_{\tilde{g}}h + \psi'' \sum_i \Theta^{\tilde{\eta}} |u_i|^2 \]

(3.28)

\[ + \varphi'' \sum_i \Theta^{\tilde{\eta}} \left| \sum_p u_{\tilde{\eta}p} u_p \right|^2. \]

(3.29)

**Line (7) of (3.24).** The first term of line (7) is nonnegative since \( \varphi' > 0 \). For the second term we argue as follows. From (3.18), we have at \( x_0 \),

\[ \sum_i \Theta^{\tilde{\eta}} u_{\tilde{\eta}p} = F_p - \sum_i \tilde{g}^{\tilde{\eta}} h_{\tilde{\eta}p}. \]

Using (3.20) we have at \( x_0 \),

\[ \psi' \sum_{i,p} \Theta^{\tilde{\eta}} (u_{\tilde{\eta}1} u_p + u_{\tilde{\eta}1} u_{\tilde{\eta}p}) = 2(1 + 2c) \eta_1 \sum_{i,p} \Theta^{\tilde{\eta}} \text{Re} (T_{i1} u_{1p}). \]

(3.30)

Making use of (3.5) we have the estimate

\[ \varphi' \sum_{i,p} \Theta^{\tilde{\eta}} (u_{\tilde{\eta}1} u_p + u_{\tilde{\eta}1} u_{\tilde{\eta}p}) \geq -C - C\text{tr}_{\tilde{g}}g - \frac{1}{10} \psi' \sum_{i,p} \Theta^{\tilde{\eta}} (|u_{\tilde{\eta}1}|^2 + |u_{\tilde{\eta}1}|^2). \]

(3.31)

Hence

\[ (7) \geq \frac{9}{10} \varphi' \sum_{i,p} \Theta^{\tilde{\eta}} (|u_{\tilde{\eta}1}|^2 + |u_{\tilde{\eta}1}|^2) - C - C\text{tr}_{\tilde{g}}g. \]

(3.32)

**Line (5) of (3.24).** Here is where torsion terms appear that need to be controlled using the first and second terms of line (2) of (3.24). We first bound

\[ \frac{1 + 2c}{\eta_1} \sum_{i,p} \Theta^{\tilde{\eta}} \left( T_{1i} u_{1p} + T_{1i} u_{1\tilde{\eta}p} \right) = \frac{2(1 + 2c)}{\eta_1} \sum_{i,p} \Theta^{\tilde{\eta}} \text{Re} \left( T_{1i} u_{1\tilde{\eta}p} \right). \]

We have

\[ u_{1\tilde{\eta}p} = \eta_{1\tilde{\eta}} - (\text{tr}_{\tilde{g}}h)_{\tilde{\eta}1\tilde{\eta}} + \tilde{h}_{1\tilde{\eta}p}. \]
Hence
\[
\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i,p} \Theta^\alpha \Re \left( \overline{T^p_{i1}} u_{i\mathbf{T}_{\alpha}} \right)
\]
(3.33)
\[
\geq \frac{2(1 + 2c)}{\eta_{1T}} \sum_{i,p} \Theta^\alpha \Re \left( \overline{T^p_{i1}} \eta_{i\mathbf{T}_{\alpha}} \right) - C \text{tr}_g g.
\]

Notice that if we consider only the summands \( p \neq 1 \), we have
\[
\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i} \frac{\Theta^\alpha \Re \left( \overline{T^p_{i1}} \eta_{i\mathbf{T}_{\alpha}} \right)}{\eta_{1T}} \geq - \frac{c}{4} \sum_{i} \frac{\Theta^\alpha |\eta_{i\mathbf{T}_{\alpha}}|^2}{(\eta_{1T})^2} - C \text{tr}_g g,
\]
and observe that the first term on the right hand side of this inequality can be controlled by the second term on line (2) of (3.24).

The term when \( p = 1 \) can be written as
\[
\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i} \Theta^\alpha \Re \left( \overline{T^1_{i1}} \eta_{i\mathbf{T}_{1}} \right) = \frac{2(1 + 2c)}{\eta_{1T}} \sum_{i \neq 1} \Theta^\alpha \Re \left( \overline{T^1_{i1}} \eta_{i\mathbf{T}_{1}} \right),
\]
using the skew-symmetry of torsion.

The third term of line (5) of (3.24) can be easily bounded by
\[
\left| \frac{1 + 2c}{\eta_{1T}} \sum_{i} \Theta^\alpha T^1_{i1} \overline{T^1_{i1}} u_{i\mathbf{a}1} \right| \leq C \text{tr}_g g.
\]
(3.36)

Combining (3.33), (3.34), (3.35) and (3.36), we obtain
\[
0 \geq (1 + 2c) \left( \sum_{i,j} g^\alpha_{ij} g^\beta_{ij} \sum_a u_{a\mathbf{a}1} - u_{j\mathbf{a}1} + h_{j\mathbf{a}1} \right) \left( \sum_b u_{b\mathbf{a}1} - u_{i\mathbf{a}1} + h_{i\mathbf{a}1} \right) \frac{(n - 1)^2}{(1 + 2c) \sum_{i} \Theta^\alpha |\eta_{i\mathbf{T}_{\alpha}}|^2 (\eta_{1T})^2}
\]
\[
+ \frac{c}{2} \sum_{i} \sum_{p} \frac{\Theta^\alpha |\eta_{i\mathbf{T}_{\alpha}}|^2}{(\eta_{1T})^2} + \frac{c}{4} \sum_{i} \sum_{p} \frac{\Theta^\alpha |\eta_{i\mathbf{T}_{\alpha}}|^2}{(\eta_{1T})^2} - (1 + 2c) \sum_{i} \frac{\Theta^\alpha |\eta_{i\mathbf{T}_{\alpha}}|^2}{(\eta_{1T})^2}
\]
\[
+ \psi'' \sum_{i} \Theta^\alpha |u_i|^2 + \varphi'' \sum_{i} \Theta^\alpha \left( \sum_p u_{p\mathbf{a}1} + \sum_p u_{p\mathbf{a}1} \right)^2
\]
\[
+ \frac{9}{10} \varphi'' \sum_{i,p} \Theta^\alpha |u_{p\mathbf{a}1}|^2 - C_0 \text{tr}_g g - C
\]
\[
+ \frac{2(1 + 2c)}{\eta_{1T}} \sum_{i \neq 1} \Theta^\alpha \Re \left( \overline{T^1_{i1}} \eta_{i\mathbf{T}_{1}} \right) + (C_1 + 1) \text{tr}_g h,
\]
(3.37)
where $C_0$ is a uniform constant (independent of the value of $C_1$). Let $C_2$ be a uniform constant such that

$$
\sup_M |T|^2 \lesssim C_2.
$$

We now pick $C_1$ sufficiently large (and uniformly bounded) so that

$$(C_1 + 1) \tr g h \geq (C_0 + 2) \tr g$$

and define

$$
\delta = \min \left\{ \frac{1}{1 + 4L(C_1 + 1)}, \frac{1}{16(1 + 2c)C_2} \right\},
$$

which is a small uniform constant.

We first consider:

**Case 1:** $\lambda_2 \leq (1 - \delta) \lambda_n$.

In this case, we simply throw away several nonnegative terms in (3.38) and get

$$
0 \geq - (1 + 2c) \sum_i \Theta_i^2 |\eta_i T_i|^2 + \varphi'' \sum_i \Theta_i^2 \left| \sum_p u_p \eta_i p + \sum_p u_p \eta_i p \right|^2 - C_0 \tr g - C
$$

from (3.14) and (3.12) we have

$$
(1 + 2c)^2 \sum_i \frac{\Theta_i^2 |\eta_i T_i|^2}{(\eta_i T_i)^2} + \varphi'' \sum_i \Theta_i^2 \left( \sum_p u_p \eta_i p + \sum_p u_p \eta_i p \right)^2
$$

From (3.14) and (3.12) we have

$$
(1 + 2c)^2 \sum_i \frac{\Theta_i^2 |\eta_i T_i|^2}{(\eta_i T_i)^2} = - \sum_i \Theta_i^2 \left( \sum_p u_p \eta_i p + \sum_p u_p \eta_i p \right)^2
$$

$$(\varphi')^2 \sum_i \Theta_i^2 \left( \sum_p u_p \eta_i p + \sum_p u_p \eta_i p \right)^2
$$

$$
\geq - 2(\varphi')^2 \sum_i \Theta_i^2 \left( \sum_p u_p \eta_i p + \sum_p u_p \eta_i p \right)^2 - 8(C_1 + 1)^2 K \tr g,
$$
where we have used \( |\psi'| \leq A/L = 2(C_1 + 1) \). Then, using (3.3) and (3.6), we have from (3.41),

\[
0 \geq \frac{1}{8K} \sum_i \Theta^2 u_i^2 - CK \text{tr} \tilde{g} - C \text{tr} \tilde{g},
\]

giving

\[
\frac{1}{8K} \Theta^{\alpha \alpha} u_{\alpha \alpha}^2 \leq CK \text{tr} \tilde{g}.
\]

From the definition of \( \Theta^2 \), we have \( \Theta^{\alpha \alpha} \geq \frac{1}{n} \sum_i \Theta^2 = \frac{1}{n} \text{tr} \tilde{g} \) and so

\[
\frac{1}{n} (\text{tr} \tilde{g}) u_{\alpha \alpha}^2 \leq CK^2 \text{tr} \tilde{g}.
\]

But the assumption \( \lambda_2 \leq (1 - \delta) \lambda_n \) together with (3.3), implies that

\[
u_{\alpha \alpha} \leq \sum_{i=1}^{n} \lambda_i - (n - 1) \lambda_n + C
\]
\[
\leq \lambda_1 + \lambda_2 + (n - 2) \lambda_n - (n - 1) \lambda_n + C
\]
\[
\leq \lambda_1 - \delta \lambda_n + C \leq -\frac{\delta}{2} \lambda_n,
\]

where we use the following: since \( \lambda_1 \leq \cdots \leq \lambda_n \) and the equation gives us \( \lambda_1 \cdots \lambda_n = e^\tilde{F} \) (recall that \( \tilde{F} = F + b \) is bounded), we may assume without loss of generality that \( \lambda_1 << 1 \), say (so in particular, \( \text{tr} \tilde{g} \geq 1 \)). Hence we have \( \delta^2 \lambda_n^2 \leq 4 u_{\alpha \alpha}^2 \) which from (3.45) gives the uniform bound

\[
\lambda_n \leq CK.
\]

By (3.8), this implies the estimate \( \eta_{\alpha \alpha} \leq C'K \). This completes the proof of Theorem 3.1 in Case 1.

**Case 2:** \( \lambda_2 \geq (1 - \delta) \lambda_n \).

From (3.3) we immediately get

\[
|u_{\alpha j}| \leq C \quad \text{for } i \neq j,
\]
\[
|u_{1 \alpha} - \eta_{\alpha \alpha}| \leq C.
\]
At this point we throw away two nonnegative terms in (3.38) to obtain

\[(3.47)\]

\[0 \geq (1 + 2c) \sum_{i,j} \bar{g}_{ij} \bar{g}_{ji} (g_{ji} \sum_a u_{a \alpha} - u_{j1} + \hat{h}_{j1}) (g_{ij} \sum_b u_{b \beta} - u_{i1} + \hat{h}_{i1}) (n - 1)^2 \eta_{1T}^{-1} \]

\[- (1 + 2c) \sum_i \frac{\Theta_1^i |\eta_{1T}|^2}{(\eta_{1T})^2} + \psi' \sum_i \Theta_1^i |u_i|^2 \]

\[+ \varphi'' \sum_i \Theta_1^i \left| \sum_p u_p u_{\alpha} + \sum_p u_{p1} u_{\beta} \right|^2 + \frac{9}{10} \varphi' \sum_{i,p} \Theta_1^i (|u_{p\alpha}|^2 + |u_{p\beta}|^2) \]

\[- C_0 \text{tr} g - C + 2(1 + 2c) \sum_{i=2}^n \Theta_1^i \text{Re} \left( \frac{\bar{T}_{1i} \eta_{1T}}{\eta_{1T}} \right) + (C_1 + 1) \text{tr} h,\]

We wish to deal with the bad term \(-(1 + 2c) \sum_i \frac{\Theta_1^i |\eta_{1T}|^2}{(\eta_{1T})^2}\) in (3.47). We first consider the summand when the index \(i\) is equal to 1. Compute, using (3.14),

\[-(1 + 2c)^2 \Theta_1^1 |\eta_{1T}|^2 \]

\[= - \Theta_1^1 \left| \varphi' \left( \sum_p u_p u_{\alpha} + \sum_p u_{p1} u_{\beta} \right) + \psi' u_1 \right|^2 \]

\[(3.48)\]

\[\geq - 2 \Theta_1^1 (\varphi')^2 \left| \sum_p u_p u_{\alpha} + \sum_p u_{p1} u_{\beta} \right|^2 - 2(\psi')^2 K \Theta_1^1 \]

\[\geq - \varphi'' \Theta_1^1 \left| \sum_p u_p u_{\alpha} + \sum_p u_{p1} u_{\beta} \right|^2 - C,\]

where we have used that (3.5) and (3.6), that \(\Theta_1^1 \approx (\eta_{1T})^{-1}\). Recall that we may assume without loss of generality that \(\eta_{1T} \gg K\).

Next we exploit again the good first term on the third line of (3.47) together with the as-yet unused good second term on the same line to kill a “small part” of the bad term

\[-(1 + 2c) \sum_{i=2}^n \frac{\Theta_1^i |\eta_{1T}|^2}{(\eta_{1T})^2}.\]
First, observe that applying again (3.14) and the fact that \( \varphi'' = 2(\varphi')^2 \),

\[
(3.49) \quad \varphi'' \sum_{i=2}^{n} \Theta^i \left| \sum_p u_p u_{\overline{p}} + \sum_p u_{pi} u_{\overline{p}} \right|^2 \\
= 2 \sum_{i=2}^{n} \Theta^i \left| (1 + 2c) \frac{\eta_{1i}}{\eta_{11}} + \psi' u_i \right|^2 \\
\geq 2(1 + 2c)^2 \delta \sum_{i=2}^{n} \frac{\Theta^i |\eta_{1i}|^2}{(\eta_{11})^2} - \frac{2\delta (\psi')^2}{1 - \delta} \sum_{i=2}^{n} \Theta^i |u_i|^2,
\]

using [16] Proposition 2.3. Recall that \( \delta \leq \frac{1}{1 + 2A} \) and so from (3.16),

\[
(3.50) \quad \frac{2\delta (\psi')^2}{1 - \delta} \sum_{i=2}^{n} \Theta^i |u_i|^2 \leq \psi'' \sum_{i=2}^{n} \Theta^i |u_i|^2 \leq \psi'' \sum_{i=2}^{n} \Theta^i |u_i|^2.
\]

Putting this together gives

\[
\varphi'' \sum_{i=2}^{n} \Theta^i \left| \sum_p u_p u_{\overline{p}} + \sum_p u_{pi} u_{\overline{p}} \right|^2 \\
\geq 2(1 + 2c)^2 \delta \sum_{i=2}^{n} \frac{\Theta^i |\eta_{1i}|^2}{(\eta_{11})^2} - \psi'' \sum_{i=1}^{n} \Theta^i |u_i|^2.
\]

Combining (3.48) and (3.51) we get

\[
(3.52) \quad -(1 + 2c) \sum_i \frac{\Theta^i |\eta_{1i}|^2}{(\eta_{11})^2} + \psi'' \sum_i \Theta^i |u_i|^2 + \varphi'' \sum_i \Theta^i \left| \sum_p u_p u_{\overline{p}} + \sum_p u_{pi} u_{\overline{p}} \right|^2 \\
\geq -(1 + 2c - 2(1 + 2c)^2 \delta) \sum_{i=2}^{n} \frac{\Theta^i |\eta_{1i}|^2}{(\eta_{11})^2} - C.
\]

We will now make use of the good first line of (3.47). In fact we will only need a part of this term, namely the summands with \( j = 1 \) and \( i = 2, \ldots, n \), which equal

\[
\frac{1 + 2c}{(n - 1)^2 \eta_{11}} \sum_{i=2}^{n} g^i g_{1i} |u_{1i} - \hat{h}_{1i}|^2.
\]

We have from (3.20),

\[
u_{1i} - \hat{h}_{1i} = \eta_{1i} + \mathcal{E}_i
\]

where

\[
(3.53) \quad \mathcal{E}_i = \sum_p T^p_{1i} u_{p1} - \hat{h}_{1i} - (\text{tr} h)_i + \hat{h}_{1i}.
\]
Hence
\[
\frac{1 + 2c}{(n - 1)^2 \eta_1 T} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |u_i T_1 - \hat{h}_i T_1|^2
\]
\[
= \frac{1 + 2c}{(n - 1)^2 \eta_1 T} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |\eta_1 T_i + \mathcal{E}_i|^2
\]
\[
(3.54)
\]
\[
= \frac{1 + 2c}{(n - 1)^2 \eta_1 T} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |\eta_1 T_i|^2 + \frac{1 + 2c}{(n - 1)^2 \eta_1 T} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |\mathcal{E}_i|^2
\]
\[
+ \frac{2(1 + 2c)}{(n - 1)^2 \eta_1 T} \text{Re} \left\{ \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger \eta_1 T_i \mathcal{E}_i \right\}.
\]

We claim that
\[
(3.55) \quad \frac{1 + 2c}{(n - 1)^2 \eta_1 T} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |\eta_1 T_i|^2 \geq (1 + 2c - (1 + 2c)^2 \delta) \sum_{i=2}^{n} \Theta_i |\eta_1 T_i|^2 / (\eta_1 T)^2.
\]

Indeed this follows from the stronger inequality
\[
(3.56) \quad (1 - (1 + 2c)\delta) \frac{\Theta_i^\dagger}{\eta_1 T} \leq \frac{1}{(n - 1)^2} \tilde{g} \tilde{g}^\dagger, \quad \text{for } i = 2, \ldots, n,
\]
which was proved in [33, (4.41)] with $3\delta/2$ instead of $(1 + 2c)\delta$. The argument is exactly the same and so we will not repeat it here.

The next term that we need to bound is the cross term of (3.54):
\[
\frac{2(1 + 2c)}{(n - 1)^2 \eta_1 T} \text{Re} \left\{ \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger \eta_1 T_i \mathcal{E}_i \right\}.
\]

From (3.53), we have for $i = 2, \ldots, n$,
\[
(3.57) \quad \mathcal{E}_i = \eta_1 T_i T^\dagger T_i + \tilde{\mathcal{E}}_i,
\]
where
\[
\tilde{\mathcal{E}}_i = O(1),
\]
thanks to (3.46). Here we are writing $T^\dagger = g_{k\ell} T^k_{\ell j}$.

Now using the facts that for $i = 2, \ldots, n$, the quantity $\tilde{g} \tilde{g}^\dagger$ is comparable to $(\eta_1 T)^{-1}$ and $\tilde{g} \tilde{g}^\dagger$ is comparable to $\Theta_i^\dagger$, we have
\[
\frac{2(1 + 2c)}{(n - 1)^2 \eta_1 T} \text{Re} \left\{ \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger \eta_1 T_i \mathcal{E}_i \right\}
\]
\[
\geq - \frac{C'}{(\eta_1 T)^2} \sum_{i=2}^{n} \tilde{g} \tilde{g}^\dagger |\eta_1 T_i|
\]
\[
\geq - \frac{\delta}{4} \sum_{i=2}^{n} \Theta_i |\eta_1 T_i|^2 / (\eta_1 T)^2 - \frac{C'}{\delta(\eta_1 T)^2} \text{tr} \tilde{g} \tilde{g}.
\]
Without loss of generality, we may assume that $(\eta_1)^2 \geq 2C'/\delta$, so that

\begin{equation}
(3.58) \quad \frac{2(1 + 2c)}{(n - 1)^2 \eta_1^T} \text{Re} \left\{ \sum_{i=2}^{n} \bar{g}^{\eta_1^T} \eta_1^T \bar{\eta}_i \right\} \geq -\frac{\delta}{4} \sum_{i=2}^{n} \Theta^{\eta_1^T} |\eta_1^T|^2 - \frac{1}{2} \text{tr} \tilde{g}.
\end{equation}

Next we make the observation that, assuming without loss of generality $\lambda_n \gg \lambda_1$, we have for $i = 2, \ldots, n$,

\begin{equation}
(3.59) \quad \left| \bar{g}^{\eta_1^T} - \frac{(n-1)}{\eta_1^T} \right| \leq \frac{2(n-1)\delta}{\eta_1^T}, \quad \text{for } i = 2, \ldots, n.
\end{equation}

Using (3.59) and the fact that $g^{\eta_1^T} \leq (n-1)\Theta^{\eta_1^T}$ for $i \geq 2$, we obtain

\begin{equation}
(3.60) \quad \frac{2(1 + 2c)}{(n - 1)^2 \eta_1^T} \text{Re} \left\{ \sum_{i=2}^{n} \bar{g}^{\eta_1^T} \eta_1^T \bar{\eta}_i \right\} \geq \frac{2(1 + 2c)\bar{g}^{\eta_1^T}}{(n - 1)\eta_1^T} \text{Re} \left\{ \sum_{i=2}^{n} \eta_1^T \bar{\eta}_i \right\} \geq \frac{2(1 + 2c)\bar{g}^{\eta_1^T}}{(n - 1)\eta_1^T} \text{Re} \left\{ \sum_{i=2}^{n} \eta_1^T \bar{\eta}_i \right\} - \frac{\delta}{4} \sum_{i=2}^{n} \Theta^{\eta_1^T} |\eta_1^T|^2 - 16(1 + 2c)^2 \delta \text{tr} \tilde{g}.
\end{equation}

Putting together (3.54), (3.55), (3.57), (3.58), (3.60) we obtain the lower bound

\begin{equation}
(3.61) \quad \frac{1 + 2c}{(n - 1)^2 \eta_1^T} \sum_{i=2}^{n} \bar{g}^{\eta_1^T} \eta_1^T |u_i \eta_1^T - \tilde{h}_i \eta_1^T|^2 \geq (1 + 2c - (1 + 2c)^2 \delta - \frac{\delta}{2} \sum_{i=2}^{n} \Theta^{\eta_1^T} |\eta_1^T|^2 \frac{\eta_1^T}{(\eta_1^T)^2} \right\} - 16(1 + 2c)^2 \delta \text{tr} \tilde{g} - \frac{1}{2} \text{tr} \tilde{g}.
\end{equation}
Combining (3.47), (3.52) and (3.61) we conclude that

\[ 0 \geq \frac{\delta}{2} \sum_{i=2}^{n} \Theta^i \frac{|\eta_{i,T}|^2}{(\eta_T)^2} - \frac{1}{2} \text{tr}_g g - 16(1 + 2c)^2 C_2 \delta \text{tr}_g g \]

\[ + (C_1 + 1) \text{tr}_g h + \frac{9}{10} \varphi' \sum_{i,p} \Theta^i (|u_{pi}|^2 + |u_{pi}'|^2) - C_0 \text{tr}_g g - C \]

\[ + \frac{2(1 + 2c)}{n-1} \eta_T \text{Re} \left( \sum_{i=2}^{n} \eta_{i,T} \overline{\eta_{i,T}} \right) \]

\[ + \frac{2(1 + 2c)}{n} \eta_T \text{Re} \left( \sum_{i=2}^{n} \eta_{i,T} \overline{\eta_{i,T}} \right) + \frac{2(1 + 2c)}{\eta_T} \sum_{i=2}^{n} \Theta^i \text{Re} \left( \frac{T_i}{\eta_{i,T}} \right) . \]

We now deal with the last two terms in this equation. Recall that

\[ \Theta^i = \frac{1}{n-1} g^T + O((\eta_T)^{-1}), \quad \text{for } i = 2, \ldots, n, \]

and in particular, \( \Theta^i \) is large when \( i \neq 1 \). Then the last term in (3.62) satisfies

\[ \frac{2(1 + 2c)}{n-1} \sum_{i=2}^{n} \Theta^i \text{Re} \left( \frac{T_i}{\eta_{i,T}} \right) \]

\[ \geq \frac{2(1 + 2c)}{n-1} \eta_T \sum_{i=2}^{n} g^T \text{Re} \left( \eta_{i,T} \overline{\eta_{i,T}} \right) - \frac{C}{(\eta_T)^2} \sum_{i=2}^{n} |\eta_{i,T}| \]

\[ \geq - \frac{2(1 + 2c)}{n-1} \sum_{i=2}^{n} g^T \text{Re} \left( \eta_{i,T} \overline{\eta_{i,T}} \right) - \frac{\delta}{2} \sum_{i=2}^{n} \Theta^i |\eta_{i,T}|^2 - \frac{1}{\delta(\eta_T)^2}, \]

and the first term on the right hand side precisely cancels the other torsion term in (3.62). We can also assume that \( (\eta_T)^2 \geq \frac{1}{\delta} \). Combining (3.62) and (3.63), we get

\[ 0 \geq - \frac{1}{2} \text{tr}_g g - 16(1 + 2c)^2 C_2 \delta \text{tr}_g g + (C_1 + 1) \text{tr}_g h \]

\[ + \frac{9}{10} \varphi' \sum_{i,p} \Theta^i (|u_{pi}|^2 + |u_{pi}'|^2) - C_0 \text{tr}_g g - C. \]

But recall that \( (C_1 + 1) \text{tr}_g h \geq (C_0 + 2) \text{tr}_g g \) and \( \varphi' > 0 \) so

\[ 0 \geq - 16(1 + 2c)^2 C_2 \delta \text{tr}_g g + \frac{3}{2} \text{tr}_g g - C. \]

We chose \( \delta \) so that \( 16(1 + 2c)^2 C_2 \delta \leq 1 \), and so we conclude that \( \text{tr}_g g \) is bounded from above at the maximum, and in particular,

\[ \frac{1}{\lambda_1} \leq C. \]

From the assumption \( \lambda_j \geq (1 - \delta) \lambda_n \) for \( j \geq 2 \), together with the Monge-Ampère equation, we conclude that

\[ \lambda_n^{n-1} \leq C \lambda_2 \ldots \lambda_n \leq C \frac{\lambda_2 \ldots \lambda_n}{\lambda_1 \ldots \lambda_n} = C \frac{1}{\lambda_1} \leq C. \]
Since $n \geq 3$, we conclude that $\lambda_n \leq C$ and so $\eta_{1T} \leq C$ and we are done.

4. **Proofs of Theorem 1.1 and its Corollaries**

**Proof of Theorem 1.1.** Assume that we are in the setting of Theorem 1.1. We claim that we have an *a priori* gradient estimate

\[
\sup_M |\nabla u|_g \leq C,
\]

where the constant $C$ depends only on $\|F\|_{C^2(M,g)}$ and the fixed data $(M, \omega_0)$, $\omega$. Indeed, thanks to the estimates from Theorems 2.1 and 3.1, the proof of (4.1) is identical to [33, Theorem 5.1], since the fact that $\omega$ was assumed to be Kähler there played no role in the proof. The basic ideas are as follows.

The estimate

\[
\text{tr}_g \tilde{g} \leq C \left( \sup_M |\nabla u|_g^2 + 1 \right),
\]

of Theorem 3.1 is compatible with a blow-up argument (cf. [16]) so we can apply the Liouville-type theorem [33, Theorem 5.2] that a $(n-1)$-PSH function on $C^n$ which is Lipschitz continuous and maximal with bounded $L^\infty$ and Lipschitz norms must be constant. The proof of the Liouville Theorem uses key ideas of Dinew-Kołodziej [4].

Given (4.1), we follow [33, Theorem 6.1] (see also [6, 7]) to derive higher order estimates

\[
\|u\|_{C^k(M,g)} \leq C_k,
\]

for $k = 0, 1, 2, \ldots$, and

\[
\tilde{g} \geq \frac{1}{C_0} g,
\]

where each $C_k$ is a positive constant which depends only on $k$ and the fixed data $(M, \omega_0)$, $\omega$ and $F$.

Indeed, combining the results of Theorems 2.1 and 3.1 with (4.1), we have the following estimates:

\[
\sup_M |u| + \sup_M |\partial u|_g + \sup_M |\partial \overline{u}|_g \leq C,
\]

and from the equation (1.4) and (2.5), the uniform upper bound on $\text{tr}_g \tilde{g}$ gives

\[
C^{-1} g \leq \tilde{g} \leq C g.
\]

By the standard linear elliptic theory, it suffices to obtain a $C^{2+\alpha}(M,g)$ bound for $u$ for some $\alpha > 0$. This can be done with the usual Evans-Krylov method, adapted to the complex setting (see [26, 34]), and the details are the same as in [33, Theorem 6.1] (see also Section 6 below).

To finish the proof of Theorem 1.1, it remains to set up a continuity method and establish “openness”, and prove the uniqueness of the solution. The proofs of these items are identical to the ones given in [33, Section 6], where the assumption that $\omega$ was Kähler was never used. □
Proof of Corollary 1.2. As we remarked in the introduction, if $\omega$ is Astheno-Kähler, $\omega_0$ is Gauduchon, and $u$ is a smooth function such that

$$\Psi_u = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} > 0,$$

then $\partial \bar{\partial} \Psi_u = 0$ too. A simple linear algebra argument (see [22]) shows that there exists a unique Hermitian metric $\omega_u$ such that $\omega_u^{n-1} = \Psi_u$. It follows that $\omega_u$ is Gauduchon. Similarly, if $\omega_0$ is strongly Gauduchon then so is $\omega_u$. Then it is easy to see (cf. [7] or [33, Section 2]) that (1.6) is equivalent to (1.4) if we set $F' = F/(n-1), b' = b/(n-1)$. Theorem 1.1 therefore implies Corollary 1.2. \qed

Proof of Corollary 1.4. By assumption, there is a smooth function $F$ such that $\text{Ric}(\omega) = \psi + \sqrt{-1} \partial \bar{\partial} F$.

We use Corollary 1.2 to solve the Calabi-Yau equation

$$\tilde{\omega}^n = e^{F+b} \omega^n,$$

for some constant $b$, with $\tilde{\omega}$ a Gauduchon metric. Taking $-\sqrt{-1} \partial \bar{\partial} \log$ of this equation gives

$$\text{Ric}(\tilde{\omega}) = \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} F = \psi,$$

as required. \qed

Proof that Conjecture 1.5 implies Conjecture 1.3. For $n=2$ this follows from the result of Cherrier [2]. For $n>2$, this is a combination of the arguments of the previous two proofs. Fix a Gauduchon metric $\omega$ on $M$. By assumption we have

$$\text{Ric}(\omega) = \psi + \frac{1}{n-1} \sqrt{-1} \partial \bar{\partial} F,$$

for a smooth function $F$. With this choice of $F$, let $\omega_0 = \omega$ and let $(u,b)$ solve (1.8) with $\Phi_u > 0$. Then for $\tilde{\omega}$ the $(n-1)$th root of $\Phi_u$, we have

$$\tilde{\omega}^n = e^{(F+b)/(n-1)} \omega^n,$$

and hence $\text{Ric}(\tilde{\omega}) = \psi$. Since $\Phi_u$ is $\partial \bar{\partial}$-closed, $\tilde{\omega}$ is Gauduchon. \qed

5. An $L^\infty$ estimate for (1.8).

In this section we prove Theorem 1.6. The arguments use the same basic ideas as in Section 2 except that we have to deal with the extra term $\text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right)$ in $\Phi_u$, and we make use of a crucial cancellation between certain torsion terms. Recall that

$$\Phi_u = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right) > 0,$$

with $u$ normalized by $\sup_M u = 0$. Set

$$E = \frac{1}{(n-1)!} \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right).$$
Define $\bar{\omega} := \frac{1}{(n-1)!} \ast \Phi_u$, for $\ast$ the Hodge star operator of $\omega$. This is a Hermitian metric, and can be written

\begin{equation}
\bar{\omega} = \omega_h + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \overline{\partial} u) + \ast E,
\end{equation}

where $\omega_h$ is the Hermitian metric defined by

\begin{equation}
\omega_h = \frac{1}{(n-1)!} \ast \omega^{n-1}.
\end{equation}

Let us also define $H = \text{tr}_\omega(\ast E)$, so that

\begin{equation}
n \ast E \wedge \omega^{n-1} = n! E \wedge \omega = H \omega^n.
\end{equation}

Then taking the trace of (5.1) we see that

\begin{equation}
\text{tr}_\omega \bar{\omega} = \text{tr}_\omega \omega_h + \Delta u + H,
\end{equation}

and therefore

\begin{equation}
\sqrt{-1} \partial \overline{\partial} u = (n-1) \omega_h + (\text{tr}_\omega \bar{\omega} - \text{tr}_\omega \omega_h - H) \omega - (n-1) \bar{\omega} + (n-1) \ast E.
\end{equation}

Since $\Phi_u$ satisfies $\det \Phi_u = e^{F+b} \det(\omega^{n-1})$ it follows that $\bar{\omega}$ satisfies the Monge-Ampère equation

\begin{equation}
\bar{\omega}^n = e^{F+b} \omega^n.
\end{equation}

As in Section 2, we have $|b| \leq \sup_M |F| + C$, for a uniform constant $C$ which depends only on $(M, \omega_0)$ and $\omega$. To prove the $L^\infty$ estimate of $u$, we first prove the following lemma, which is the analogue of (2.7).

**Lemma 5.1.** For a uniform $C$, we have

\begin{equation}
\sqrt{-1} \partial \overline{\partial} u \wedge (2 \omega_0^{n-1} + \omega^{n-2} + (n-1)! E) \leq C(1 + |\nabla u|^2) \omega^n - (n-1)! \sqrt{-1} \partial \overline{\partial} u \wedge E.
\end{equation}

**Proof.** Observe that

\begin{equation}
|H| \leq C |\nabla u|, \quad |E| \leq C |\nabla u|.
\end{equation}

Let us define

\begin{equation}
S = ((n-1) \omega_h + (\text{tr}_\omega \bar{\omega} - \text{tr}_\omega \omega_h) \omega - (n-1) \bar{\omega}) \wedge (2 \omega_0^{n-1} + ((n-1) \omega_h + (\text{tr}_\omega \bar{\omega} - \text{tr}_\omega \omega_h) \omega - (n-1) \bar{\omega}) \wedge \omega^{n-2}).
\end{equation}

Then using Lemma 2.2, we have

\begin{equation}
S \leq C \omega^n.
\end{equation}
Indeed, we can compute at a point where
\[
(5.10) \\
\sqrt{-1}d\bar{\omega}u \wedge (2\omega_0^{n-1} + \sqrt{-1}d\bar{\omega}u \wedge \omega^{n-2} + (n-1)!E) \\
= S + (n-1)!\sqrt{-1}d\bar{\omega}u \wedge E - (n-1)H\omega_h \wedge \omega^{n-1} + (n-1)^2 \ast E \wedge \omega_h \wedge \omega^{n-2} \\
- (\text{tr}_\omega \bar{\omega})H\omega^n + (n-1)(\text{tr}_\omega \bar{\omega}) \ast E \wedge \omega^{n-1} + (\text{tr}_\omega \omega_h)H\omega^n - (n-1)(\text{tr}_\omega \omega_h) \ast E \wedge \omega^{n-1} \\
- 2H\omega_0^{n-1} \wedge \omega - (n-1)H\omega_h \wedge \omega^{n-1} - H(\text{tr}_\omega \bar{\omega})\omega^n + H(\text{tr}_\omega \omega_h)\omega^n + H^2\omega^n \\
+ (n-1)H\bar{\omega} \wedge \omega^{n-1} - (n-1)H \ast E \wedge \omega^{n-1} + (n-1)H\bar{\omega} \wedge \omega^{n-1} \\
- (n-1)^2 \ast E \wedge \bar{\omega} \wedge \omega^{n-2} + 2(n-1) \ast E \wedge \omega_0^{n-1} + (n-1)^2 \ast E \wedge \omega_h \wedge \omega^{n-2} \\
+ (n-1)(\text{tr}_\omega \bar{\omega}) \ast E \wedge \omega^{n-1} - (n-1)(\text{tr}_\omega \omega_h) \ast E \wedge \omega^{n-1} - (n-1)H \ast E \wedge \omega^{n-1} \\
- (n-1)^2 \ast E \wedge \bar{\omega} \wedge \omega^{n-2} + (n-1)^2 \ast E \wedge \omega_h \wedge \omega^{n-2} \\
\leq (n-1)!\sqrt{-1}d\bar{\omega}u \wedge E + C(1 + |\nabla u|^2)\omega^n \\
+ \frac{2n-4}{n}(\text{tr}_\omega \bar{\omega})H\omega^n - 2(n-1)^2 \ast E \wedge \bar{\omega} \wedge \omega^{n-2}.
\]

A simple calculation shows that
\[
(5.10) \\
\ast(\bar{\omega} \wedge \omega^{n-2}) = (n-2)!((\text{tr}_\omega \bar{\omega})\omega - \bar{\omega}).
\]

Indeed, we can compute at a point where $\omega = \sum_i e_i$ and $\bar{\omega} = \sum_j \lambda_j e_j$, where $e_i = \sqrt{-1}dz^i \wedge d\bar{z}^i$, and then
\[
\bar{\omega} \wedge \omega^{n-2} = (n-2)! \left( \sum_j \lambda_j e_j \right) \wedge \sum_{k<\ell} e_k \ldots \hat{e}_i \ldots e_n \\
= (n-2)! \sum_k \left( \sum_{j \neq k} \lambda_j \right) e_1 \ldots \hat{e}_k \ldots e_n,
\]
from which (5.10) follows. Therefore
\[
-2(n-1)^2 \ast E \wedge \bar{\omega} \wedge \omega^{n-2} = -2(n-1)^2 E \wedge \ast(\bar{\omega} \wedge \omega^{n-2}) \\
= -2(n-1)(n-1)!((\text{tr}_\omega \bar{\omega})E \wedge \omega + 2(n-1)(n-1)!E \wedge \bar{\omega} \\
= -\frac{2(n-2)}{n}(\text{tr}_\omega \bar{\omega})H\omega^n + 2(n-1)(n-1)!E \wedge \bar{\omega},
\]
and, using again (5.2),
\begin{align*}
2n - 4 & (\text{tr}_\omega \bar{\omega}) H \omega^n - 2(n - 1)^2 \ast E \wedge \bar{\omega} \wedge \omega^{n-2} \\
&= -2(n - 1)(n - 1)! E \wedge \bar{\omega} \\
&= -2(n - 1)! \Delta u E \wedge \omega - 2(n - 1)! E \wedge \sqrt{-1} \partial \bar{\partial} u + 2(n - 1)(n - 1)! E \wedge \ast E \\
&= -2(n - 1)! E \wedge \sqrt{-1} \partial \bar{\partial} u + C(1 + |\nabla u|^2) \omega^n.
\end{align*}

Note that the two bad terms $-\frac{2}{n} H \Delta u \omega^n$ and $2(n - 1)! \Delta u E \wedge \omega$ exactly canceled out. Combining (5.9) and (5.11) concludes the proof of (5.6). □

Proof of Theorem 2.6. Following the method of Section 2, we use (5.6) in a Moser iteration argument. For $p$ sufficiently large, as in (2.8),
\begin{align*}
\int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge (2 \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + (n - 1)! E) \\
\geq 1 \frac{p}{C} \int_M \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \bar{\partial} e^{-\frac{pu}{2}} \wedge \omega^{n-1} - \frac{C}{p} \int_M e^{-pu} \omega^n,
\end{align*}
where we have used
\begin{align*}
\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + (n - 1)! E > 0,
\end{align*}
and
\begin{align*}
\partial \bar{\partial} \left( \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + (n - 1)! E \right) = 0.
\end{align*}
Using claim (5.6), we see that
\begin{align*}
\int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge (2 \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + (n - 1)! E) \\
\leq C \int_M e^{-pu} \omega^n + C \int_M e^{-pu} |\nabla u|^2 \omega^n - \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right).
\end{align*}
We have that
\begin{align*}
C \int_M e^{-pu} |\nabla u|^2 \omega^n \leq \frac{C}{p^2} \int_M \sqrt{-1} \partial e^{-\frac{pu}{2}} \wedge \bar{\partial} e^{-\frac{pu}{2}} \wedge \omega^{n-1}.
\end{align*}
and

\[
- \int_M e^{-p u} \sqrt{-1} \bar{\partial} \partial u \wedge \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right) \\
= - \text{Re} \int_M e^{-p u} \sqrt{-1} \bar{\partial} \partial u \wedge \sqrt{-1} \partial \bar{\partial} u \wedge \bar{\partial} (\omega^{n-2}) \\
= \frac{1}{p} \text{Re} \int_M \sqrt{-1} \partial (e^{-p u}) \wedge \sqrt{-1} \bar{\partial} \partial u \wedge \bar{\partial} (\omega^{n-2}) \\
= \frac{1}{p} \text{Re} \int_M \sqrt{-1} \partial (e^{-p u}) \wedge \bar{\partial} u \wedge \sqrt{-1} \bar{\partial} \bar{\partial} (\omega^{n-2}) \\
= - \frac{4}{p^2} \int_M \sqrt{-1} \partial e^{-\frac{p u}{2}} \wedge \bar{\partial} e^{-\frac{p u}{2}} \wedge \sqrt{-1} \bar{\partial} \bar{\partial} (\omega^{n-2}) \\
\leq C \frac{p}{p^2} \int_M \sqrt{-1} \partial e^{-\frac{p u}{2}} \wedge \bar{\partial} e^{-\frac{p u}{2}} \wedge \omega^{n-1}.
\]

(5.14)

Putting all these together, we conclude that for \( p \) sufficiently large we have

\[
\int_M \left| \partial e^{-\frac{p u}{2}} \right|^2 \omega^n \leq C p \int_M e^{-p u} \omega^n.
\]

From here we can conclude the proof exactly as in Theorem 2.1. Again, we have two different ways to complete the proof. We can modify the first proof there by replacing the operator \( \Delta' \) with the operator \( Lu = \Delta u + H(u) \), where as before

\[
H(u) = \frac{n \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \wedge \omega \right)}{\omega^n} = \frac{n(n-2)}{(n-1)} \cdot \frac{\text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-1}) \right)}{\omega^n}.
\]

The assumption that \( \omega \) is Gauduchon implies that \( \int_M Lu \omega^n = 0 \), and we have

\[
Lu \geq -\text{tr}_\omega \omega_h \geq -C.
\]

Therefore one can use the Green’s formula for \( L \) in exactly the same way as before.

We can also modify the second proof of Theorem 2.1 using Moser iteration. The function \( v = u - \inf_M u \) satisfies \( \Delta v \geq -C - H(v) \). Then, as in
for any $p \geq 1$ we compute

\[(5.15)\]

\[
\int_M |\partial v^{p+1}_2|^2_\omega \omega^n = \frac{(p+1)^2}{4p} \int_M v^p (-\Delta v) \omega^n + \frac{n(p+1)}{4p} \int_M \sqrt{-1} \partial v^{p+1} \wedge \partial (\omega^{n-1})
\]

\[
= \frac{(p+1)^2}{4p} \int_M v^p (-\Delta v) \omega^n
\]

\[
\leq C \frac{(p+1)^2}{4p} \int_M v^p \omega^n + \frac{n(n-2)(p+1)^2}{4p(n-1)} \int_M \sqrt{-1} \partial v \wedge \overline{\partial} (\omega^{n-1})
\]

\[
= \frac{(p+1)^2}{4p} \int_M v^p \omega^n + \frac{n(n-2)(p+1)}{4p(n-1)} \int_M \sqrt{-1} \partial v^{p+1} \wedge \overline{\partial} (\omega^{n-1})
\]

\[
= \frac{(p+1)^2}{4p} \int_M v^p \omega^n,
\]

using twice the Gauduchon condition $\partial \overline{\partial} (\omega^{n-1}) = 0$. From here we conclude exactly as in Theorem 2.1. \(\Box\)

6. Proof of Theorem 1.7

Finally, we give the proof of Theorem 1.7. Suppose that $u$ solves (1.8). Thanks to Theorem 1.6 we have $\|u\|_{L^\infty} \leq C$ for a uniform constant $C$ (which depends only on $\omega, \omega_0$ and $F$). Since we are assuming that (1.9) holds, we can use a blow-up argument as in [33] (which in turn uses the ideas of Dinew-Ko\l osiej [4]) to show that

\[(6.1)\]

\[
\sup_M |\nabla u|_g \leq C,
\]

for a uniform constant $C$.

Indeed the blow-up argument of [33] Theorem 5.1 can be applied with minor modifications to give (6.1). The only difference here is the presence of the term $*E(u)$. But this term is linear in $\partial u$ and hence converges to zero uniformly on compact subsets under the the rescaling procedure of [33]. The rest of the argument is identical to [33].

Combining $\|u\|_{L^\infty} \leq C$ with (6.1) and (1.9) we have

\[(6.2)\]

\[
\sup_M |u| + \sup_M |\partial u|_g + \sup_M |\partial \overline{\partial} u|_g \leq C,
\]

and (1.8) together with the bound on $|b|$ and the uniform upper bound on $\text{tr}_g \tilde{g}$ give

\[(6.3)\]

\[
C^{-1} g \leq \tilde{g} \leq CG.
\]

By the standard linear elliptic theory, it suffices to obtain a $C^{2+\alpha}(M,g)$ bound for $u$ for some $\alpha > 0$. To do this, we again follow the strategy of [33] using the Evans-Krylov theory (see also [7]). However, there are new difficulties that arise. Let us define a tensor

\[
Z = *E = \sqrt{-1} Z_{ij} dz^i \wedge \overline{dz}^j,
\]
so that
\[ \tilde{g}_{ij} = g_{ij} + \frac{1}{n-1} \left( (\Delta u) g_{ij} - u_{ij} \right) + Z_{ij}. \]
We have \( |Z|_g \leq C |\nabla u|_g \), and \( |\nabla Z|_g \leq C (|\nabla \nabla u|_g + |\nabla \nabla u|_g) \). We will also use the notation \( \hat{Z} = (n-1)Z \).

As in [33], we will work in a small open subset of \( \mathbb{C}^n \), containing a ball \( B_{2R} \) of radius \( 2R \). The equation is given by
\[ \log \frac{\det \tilde{g}}{\det g} = \tilde{F}, \]
where \( \tilde{F} = F + b \). Let \( \gamma = (\gamma^i) \) be a unit vector in \( \mathbb{C}^n \). The same calculation as in [3.19], using (3.20), gives
\[ \Theta^\gamma u_{\gamma ij} \geq G - C \sum_{p,q} |u_{\gamma pq}| + S - C |u_{ij}|_g, \]
with
\[ S = \frac{1}{(n-1)^2} \tilde{g}_{\gamma}^{\alpha \beta} \tilde{g}_{\gamma}^{\kappa \eta} \left( g_{\alpha \beta} g^{\gamma \eta} u_{\gamma \kappa \eta} - u_{\gamma \kappa \eta} + \hat{B}_{\gamma \kappa \eta} \right) \left( g_{ij} g^{\alpha \beta} u_{\alpha \beta \gamma} - u_{ij \gamma} + \hat{B}_{ij \gamma} \right), \]
and
\[ B_{ij \gamma} = \hat{h}_{ij \gamma} + \hat{Z}_{ij \gamma}, \]
and for \( G \) a uniformly bounded function (which depends on \( \gamma \)), which may change from line to line. Here we are writing \( |u_{ij}|_g^2 \) for \( |\nabla \nabla u|_g^2 \). We convert these covariant derivatives into partial derivatives and obtain
\[ \Theta^\gamma \partial_i \partial_j u_{\gamma \gamma} \geq G - C \sum_{p,q} |u_{\gamma pq}| + S - C |u_{ij}|_g. \]
For a uniform \( C' > 0 \),
\[ S \geq C'^{-1} \tilde{g}_{\gamma}^{\alpha \beta} \tilde{g}_{\gamma}^{\kappa \eta} \left( g_{\alpha \beta} g^{\gamma \eta} u_{\gamma \kappa \eta} - u_{\gamma \kappa \eta} + \hat{B}_{\gamma \kappa \eta} \right) \left( g_{ij} g^{\alpha \beta} u_{\alpha \beta \gamma} - u_{ij \gamma} + \hat{B}_{ij \gamma} \right) \]
\[ \geq C'^{-1} \tilde{g}_{\gamma}^{\alpha \beta} \tilde{g}_{\gamma}^{\kappa \eta} \left( g_{\alpha \beta} g^{\gamma \eta} u_{\gamma \kappa \eta} - u_{\gamma \kappa \eta} + \hat{B}_{\gamma \kappa \eta} \right) \left( g_{ij} g^{\alpha \beta} u_{\alpha \beta \gamma} - u_{ij \gamma} + \hat{B}_{ij \gamma} \right) - C' |B_{ij \gamma}|_g^2 \]
\[ = C'^{-1} \left( (n-2) \sum_{p,q} |u_{\gamma pq}|^2 + |u_{ij \gamma}|_g^2 \right) - C' \sum_{p,q} |u_{\gamma pq}| - C' |B_{ij \gamma}|_g^2 - C', \]
and
\[ |B_{ij \gamma}|_g^2 \leq C(1 + |u_{ij}|_g^2). \]
Hence, we conclude that
\[ \Theta^\gamma \partial_i \partial_j u_{\gamma \gamma} \geq G - C_0 (1 + |u_{ij}|_g^2), \]
where \( G \) and \( C_0 \) depend on \( \gamma \), and \( G \) is bounded. To deal with the bad term \( |u_{ij}|_g^2 \), we claim that we have the estimate
\[ \Theta^\gamma \partial_i \partial_j |\nabla u|_g^2 \geq C_1^{-1} |u_{ij}|_g^2 - C, \]
for a uniform \( C_1 > 0 \). To see this, we compute
\[
\Theta^{ij}\partial_i\partial_j|\nabla u|^2_g = \Theta^{ij}g^{PQ}(u_{Pj}u_{Qj} + u_{Pi}u_{Qi}) + \Theta^{ij}g^{PQ}(u_{Pi}u_{Qi} + u_{Qj}u_{Pj}) \\
\geq C'\left|u_{ij}\right|^2_g + \Theta^{ij}g^{PQ}(u_{Pi}u_{Qi} + u_{Qj}u_{Pj}).
\]

The analog of (3.18) for equation (1.8) is
\[
\Theta^{ij}u_{ij} = F_p - \tilde{g}^{ij}(h_{ij} + Z_{ij}).
\]

Switching covariant derivatives and using the bound
\[
\left|\nabla Z\right|_g \leq C(1 + \left|u_{ij}\right|_g),
\]
we obtain
\[
(6.8) \quad \left|\Theta^{ij}g^{PQ}(u_{Pi}u_{Qi} + u_{Qj}u_{Pj})\right| \leq (2C')^{-1}\left|u_{ij}\right|^2_g + C.
\]

Combining (6.7) and (6.8) proves (6.6). Then for \( A \) sufficiently large (depending on \( \gamma \)) we have
\[
\Theta^{ij}\partial_i\partial_j(u_{\gamma} + A|\nabla u|^2_g) \geq G.
\]

For the next step, we proceed as in [33] and consider the metric \( \hat{g}_{\gamma} \) on \( B_{2R} \) given by \( \hat{g}_{\gamma} = g_{\gamma}(0) \). Define
\[
\hat{\Theta}^{ij} = \frac{1}{n-1}\left((\text{tr}_{\hat{g}}\hat{g})\hat{g}^{ij} - \hat{g}^{ij}\right).
\]

By concavity of log det and arguing as in [33], we get
\[
(6.9) \quad \sum_{i,j} \hat{\Theta}^{ij}(y)\left(u_{\gamma}(y) - u_{\gamma}(x)\right) \leq C'R.
\]

As in [29] for example, we find a set of unit vectors \( \gamma_1, \ldots, \gamma_N \) of \( \mathbb{C}^n \), containing an orthonormal basis, with the property that
\[
\Theta^{ij}(y) = \sum_{\nu=1}^N \beta_\nu(y)(\gamma_\nu)^i(\gamma_\nu)^j,
\]
for \( \beta_\nu \) with \( 0 < C^{-1} \leq \beta_\nu \leq C \). Now define
\[
w_\nu = u_{\gamma_\nu} + A|\nabla u|^2_g.
\]

Since we have a uniform bound on \( u \) and \( \Delta u \) it follows that \( |\nabla u|^2_g \) is bounded in \( C^\alpha \) for any \( \alpha \) with \( 0 < \alpha < 1 \), which we fix once and for all. Then from (6.9), we have
\[
(6.10) \quad \sum_{\nu=1}^N \beta_\nu(w_\nu(y) - w_\nu(x)) \leq CR^\alpha.
\]

On the other hand, from (6.5), we have
\[
(6.11) \quad \Theta^{ij}\partial_i\partial_jw_\nu \geq -C.
\]
From (6.10) and (6.11) and the arguments of [29], we obtain
\[ \Omega(R) \leq \delta \Omega(2R) + CR^\alpha, \]
for \( \Omega = \sum_{\nu=1}^{N} \text{osc}_{B_R} w_{\nu} \), for a uniform \( 0 < \delta < 1 \) and for \( R \) sufficiently small. Applying [12, Lemma 8.23] we obtain
\[ \Omega(R) \leq CR^\kappa, \]
for a small but uniform \( \kappa > 0 \) (with \( \kappa < \alpha \)) and hence we obtain a Hölder estimate for the second derivatives \( u_{ij} \) of \( u \) (since we already have a \( C^\alpha \) estimate for \( |\nabla u|^2 g \)).

Combining all these estimates, we have proved (assuming (1.9) holds) that there are constants \( C_k, k = 0, 1, \ldots \), which depend only on \( k, \omega, \omega_0 \) and \( F \), such that if \( u \) solves (1.8), then
\[ \|u\|_{C^k(M, g)} \leq C_k, \]
and
\[ \tilde{g} \geq \frac{1}{C_0} g. \]

To complete the proof of Theorem 1.7, we need to set up a continuity method and establish “openness”, and prove the uniqueness of the solution. These follow from modifications of the arguments of [7] or [33], which we now briefly explain (uniqueness is also proved in [25], which furthermore contains a calculation of the linearized equation).

We consider the family of equations \( u_t, b_t \), for \( t \in [0, 1] \),
\[
(\omega_h + \frac{1}{n-1}((\Delta u_t) \omega - \sqrt{-1} \partial \bar{\partial} u_t) + *E(u_t))^n = e^{tF + b_t} \omega_h^n,
\]
with
\[
\omega_h + \frac{1}{n-1}((\Delta u_t) \omega - \sqrt{-1} \partial \bar{\partial} u_t) + *E(u_t) > 0, \quad \sup_M u_t = 0.
\]
Suppose that we have a solution of (6.12), (6.13) for \( t = \hat{t} \) and write
\[ \hat{\omega} = \omega_h + \frac{1}{n-1}((\Delta u_t) \omega - \sqrt{-1} \partial \bar{\partial} u_t) + *E(u_t). \]
Define
\[ \hat{g}^{ij} = \frac{1}{n-1}((\text{tr}_{g} g) g^{ij} - \hat{g}^{ij}). \]
Consider the linear differential operator
\[ L(v) = \hat{\Theta}^{ij} v_{ij} + \hat{g}^{ij} Z(v) g_{ij}. \]
It is elliptic and its kernel are the constants. Denote by \( L^* \) the adjoint of \( L \) with respect to the \( L^2 \) inner product with volume form \( \hat{\omega}^n \). We then argue as in [10]. The index of \( L \) is zero and hence the kernel of \( L^* \) is one-dimensional, spanned by a smooth function \( f \). The maximum principle implies that every nonzero function in the image of \( L \) must change sign, and
since $f$ is orthogonal to the image of $L$, it must have constant sign. We can assume that $f \geq 0$. The strong maximum principle then implies that $f > 0$, and so we can write $f = e^{\sigma}$ for a smooth function $\sigma$. We may and do assume that $\int_M e^{\sigma} \hat{\omega}^n = 1$.

Consider the operator
\[
\Upsilon(v) = \log \left( \hat{\omega} + \frac{1}{n-1}((\Delta v)\omega - \sqrt{-1} \partial \bar{\partial} v) + *E(v) \right)^n
\]
\[
- \log \left( \int_M e^{\sigma} \left( \hat{\omega} + \frac{1}{n-1}((\Delta v)\omega - \sqrt{-1} \partial \bar{\partial} v) + *E(v) \right)^n \right),
\]
which maps $C^{2+\alpha}$ functions with integral zero to the space of $C^\alpha$ functions $w$ satisfying $\int_M e^w e^{\sigma} \hat{\omega}^n = 1$. Note that the tangent space at 0 of the latter space consists of $C^\alpha$ functions orthogonal to the kernel of $L^*$. Since for any $C^{2+\alpha}$ function $\zeta$ we have
\[
\int_M e^\sigma \left( \Theta \zeta \zeta \bar{\zeta} + \bar{g} \bar{Z}(\zeta) \bar{\zeta} \right) \hat{\omega}^n = \int_M e^\sigma L(\zeta) \hat{\omega}^n = \int_M \zeta L^* (e^\sigma) \hat{\omega}^n = 0,
\]
it follows that the linearization of $\Upsilon$ at 0 is the operator $L$. From the Fredholm alternative, $L$ gives an isomorphism of the tangent spaces. By the Inverse Function Theorem, we obtain a solution of (6.12) for $t$ close to $\hat{t}$, as required.

Lastly, the uniqueness of the solution $(u, b)$ of (1.8) follows almost exactly as in [7] or [33]. The only difference is the additional term $*E$ in the equation. But one only needs to observe that the map $w \mapsto *E(w)$ is linear in $w$ and vanishes at a maximum or minimum of $w$. The rest of the argument is the same and we refer the reader to Section 6 of [33].

This completes the proof of Theorem 1.7.

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