Exact solution for the degenerate ground-state manifold of a strongly interacting one-dimensional Bose-Fermi mixture

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We present the exact solution for the many-body wavefunction of a one-dimensional mixture of bosons and spin-polarized fermions with equal masses and infinitely strong repulsive interactions under external confinement. Such a model displays a large degeneracy of the ground state. Using a generalized Bose-Fermi mapping we find the solution for the whole set of ground-state wavefunctions of the degenerate manifold and we characterize them according to group-symmetry considerations. We find that the density profile and the momentum distribution depends on the symmetry of the solution. By combining the wavefunctions of the degenerate manifold with suitable symmetry and guided by the strong-coupling form of the Bethe-Ansatz solution for the homogeneous system we propose an analytic expression for the many-body wavefunction of the inhomogeneous system which well describes the ground state at finite, large and equal interactions strengths, as validated by numerical simulations.

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I. INTRODUCTION

Ultracold atomic gases provide a versatile and controlled system for the study of quantum correlations and fluctuations which are particularly strong in one dimension (1D). Experiments on two-dimensional optical lattices or on a chip trap have reached the strongly interacting Tonks-Girardeau regime. In such impenetrable boson limit, repulsive interactions play the role of the Pauli exclusion principle and the many-body wavefunction can be exactly obtained by mapping onto the one of noninteracting fermions. Experimental advances on trapping and cooling ultracold Bose-Fermi mixtures and the possibility of trapping both species in tight atomic waveguides have boosted a theoretical activity on 1D mixtures. At increasing boson-fermion repulsions, mean-field and Luttinger liquid analysis at weak coupling predict an instability towards phase separation (i.e. demixing) of the two components. For a highly symmetric model with equal masses and coupling constants, further progress can be made by use of exact solutions. For the homogeneous system, a Bethe-Ansatz solution is known and no demixing is found. The long-wavelength properties of its correlation functions have been studied using conformal field theory. Inhomogeneous systems, as in the case of experiments, bring about novel issues, such as the spatial structure of the ground state. At intermediate interaction strength a partial demixing of the two clouds has been found by a local density approximation on the Bethe-Ansatz solution both at zero and finite temperature. In the Tonks-Girardeau limit of infinitely strong boson-boson and boson-fermion repulsions a large ground state degeneracy is expected, and is associated to the freedom of fixing the sign of the many-body wavefunction under the exchange of a boson with a fermion. For an inhomogeneous system, one exact solution of the degenerate manifold has been proposed and analyzed in detail. The corresponding density profiles display no demixing among the two species. Till now no expression was known for the other wavefunctions of the manifold. In this work we solve several open theoretical issues. First of all we find an exact analytical solution for all the wavefunctions of the degenerate manifold in the one-dimensional Bose-Fermi mixture, thus generalizing the solution of [21]. Secondly, we characterize the solutions in terms of their symmetry properties according to group theory considerations. At difference from fermionic or bosonic spinor systems, where the state of the system can be labelled on the basis of the spin quantum number, in order to label the states of the Bose-Fermi mixture we introduce a suitable Casimir operator which reflects the mixed symmetry under particle exchange. Furthermore, we find that such symmetry considerations allow for the understanding of the shape of the momentum distribution, which depends on the choice of the wavefunction within the manifold. Finally, by linear combination of the basis wavefunctions of the degenerate manifold we individuate the wavefunction which corresponds to the ground state at finite, large and equal interactions strengths, and we confirm this prediction by comparing with numerical DMRG simulations. For the nontrivial case of the Bose-Fermi mixture with large degeneracy, this analysis allows for the first time to draw a link between the Bethe-Ansatz solution of the homogeneous system and the Tonks-Girardeau solution of the inhomoge-
neous system. Our solution sheds light onto the general properties of the ground state wavefunction of a fully quantum problem in the strongly interacting limit.

II. ORTHONORMAL BASIS SET FOR THE DEGENERATE GROUND-STATE MANIFOLD

A. General considerations

We consider the model of $N_B$ bosons and $N_F$ spin-polarized fermions of masses $m_B = m_F = m$, confined by the same external potential. The particles interact via the contact potentials, $v_{BB}(x) = g_{BB}(x)$, $v_{BF}(x) = g_{BF}(x)$, and we focus on the limit $g_{BB} = g_{BF} \rightarrow \infty$. The effect of contact interactions can be replaced by the boundary condition that the wavefunction vanishes at each BB or BF contact, i.e.,

$$\Psi(..., x_j, ..., x_\ell, ...) = 0 \text{ whenever } x_j = x_\ell.$$  (1)

We adopt the convention that \{x_1, ..., x_{N_B}\} are bosonic coordinates and \{x_{N_B+1}, ..., x_{N}\} are fermionic ones. The ground state has a large degeneracy $C_{N_B}^N = N!/N_B!N_F!$, which can be interpreted as choosing $N_B$ positions for the bosons out of $N = N_B + N_F$, and amounts to fixing in several possible ways the sign of the wavefunction under the exchange of bosons with fermions.

In order to determine an orthonormal basis set for the degenerate manifold we proceed as follows. Consider a fermionic Slater determinant made of the first total $N$ orbitals,

$$\Psi_F(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \text{det} \phi_j(x_i),$$  (2)

where $j, \ell = 1, ..., N$, and $\phi_j(x)$ are obtained by the solution of the single-particle Schroedinger equation in the potential $V(x)$. $\Psi_F$ displays the correct nodes at each BB and BF contact. In a given coordinate sector, $x_{P(1)} < x_{P(2)} < ... < x_{P(N)}$, with $P$ being a permutation among the $N$ particles, the required many-body wavefunction is proportional to $\Psi_F(x_1, ..., x_N)$. A useful set of orthonormal basis is given by the “snippets”

$$\langle x_1, ..., x_N | P \rangle = \sqrt{N!} \Psi_F(x_1, ..., x_N)$$  (3)

in the coordinate sector $x_{P(1)} < x_{P(2)} < ... < x_{P(N)}$, and zero otherwise. To build the required basis for the manifold, we now collect the snippets which correspond to exchanging only the positions of the bosons or of the fermions among themselves, using the Fermi or Bose statistics to fix the relative sign of the various terms. The number of groups of snippets subdivided in such a way is exactly $C_{N_B}^N$. This yields the required orthonormal basis set \{\Psi^\alpha\} for the degenerate manifold, since each snippet is orthogonal to another and is used only once.

The density profiles associated to each wavefunction $\Psi_\alpha$ are given by

$$n_B^\alpha(x) = N_B \int dx_2 ... dx_N |\Psi_\alpha(x, x_2 ... x_N)|^2,$$

$$n_F^\alpha(x) = N_F \int dx_1 ... dx_{N-1} |\Psi_\alpha(x_1, ..., x_{N-1}, x)|^2,$$  (4)

which is equivalent to computing

$$n_{B(F)}^\alpha(x) = \sum_{i=1}^N p_{i, B(F)}^\alpha \rho_i(x),$$  (5)

with $p_{i, B(F)}^\alpha = 1$ if a boson (fermion) is at position $i = 1, ..., N$ in the configuration $\alpha$ and zero otherwise, and

$$\rho_i(x) = \int_{x_1 < x_2 < ... < x_N} dx_2 ... dx_N |\Psi_F(x_1, ..., x_N)|^2 \delta(x - x_i).$$  (6)

B. An illustration with $N_B = N_F = 2$

We illustrate the idea in the case $N = 4$, $N_B = N_F = 2$, and take a harmonic confinement $V(x) = \frac{1}{2}m\omega^2x^2$ for simplicity, assuming the same trapping frequency $\omega$ for the two species. We denote the bosonic coordinates by $x_1, x_2$ and the fermionic ones by $x_3, x_4$, and we expect a six-fold degeneracy. We label the basis set using the positions of the particles, i.e. BBFF, BBBF, BFFB, FBFB, FBBF, FBFB. Let us call this basis the “BBFF” basis. According to the above prescription the first wavefunction is

$$\Psi_{BBFF} = \frac{1}{2} \left[ (x_1, x_2, x_3, x_4 | (e + (12))(e - (34))) \right],$$  (7)

where by $(j\ell)$ we denote the permutation between the particles $j$ and $\ell$, $e$ is the identity permutation, and we adopt the usual convention of product among permutations: the snippet basis satisfies $\langle x_1, ..., x_N | P + Q \rangle = \langle x_1, ..., x_N | P \rangle + \langle x_1, ..., x_N | Q \rangle$. Similarly, the second wavefunction is obtained by

$$\Psi_{BBBF} = \frac{1}{2} \left[ (x_1, x_3, x_2, x_4 | (e + (12))(e - (34))) \right].$$  (8)

The other wavefunctions are built in the same way, taking as initial coordinate sector the one where the bosonic and fermionic coordinates are each in ascending order.

We display the density profiles of the six basis states in Fig. 12. Each peak corresponds to the position of a particle in the BBFF sequence, hence the basis set recalls the one of distinguishable particles, as in the case of spinor bosons.

III. SYMMETRY CHARACTERIZATION

A. Casimir invariance of the manifold

We would like to label the basis vectors by some additional quantum number. We proceed by exploiting the
antisymmetric in its last permutation (from the column) is counted as +1 and an antisymmetric permutation (from the row) is counted as −1. The total density (black solid thin line) is included in the first frame for reference.

exchange symmetry between bosons or fermions among themselves. According to a group theoretical analysis, there are only two possible Young tableaus associated to a quantum mechanical system of mixed Bose-Fermi symmetry, i.e. symmetric in its first \( N_B \) coordinates and antisymmetric in its last \( N_F \):

\[
Y = \begin{pmatrix}
F_1 & B_1 & \cdots & B_{N_B} \\
\vdots & \ddots & \ddots & \vdots \\
F_{N_F}
\end{pmatrix}, \quad Y' = \begin{pmatrix}
B_1 & \cdots & B_{N_B} \\
F_1 & \vdots & \ddots \\
\vdots & \ddots & \ddots \\
F_{N_F}
\end{pmatrix}, \quad (9)
\]

each with a dimension of \( C_{N_B}^{N-1} \), \( C_{N_F}^{N-1} \) respectively.

To each Young tableau it is possible to associate a value of a Casimir invariant, obtained from the generators of the permutation group \( S_N \), and which commutes with all elements of the group. In this particular case we choose \( C = \sum_{i<j} (ij) \), the sum over all the transpositions [28]. Two different eigenvalues of the Casimir operator are associated to the two tableaus, namely

\[
C_Y = C_{2}^{N_B+1} - C_{2}^{N_F-1},
\]

\[
C_{Y'} = C_{2}^{N_B-1} - C_{2}^{N_F+1}, \quad (10)
\]

with degeneracy corresponding to the dimension of the tableau. We remark that the eigenvalue \( C_Y(Y') \) essentially counts the number of two-particle exchange allowed by the corresponding tableau, where a symmetric permutation (from the row) is counted as +1 and an antisymmetric permutation (from the column) is counted as −1.

We represent the Casimir operator on the BBFF orthonormal basis set (see appendix A for some examples). The eigenvectors of the Casimir operators, obtained by diagonalization, are expressed as linear combinations of the BBFF basis vectors, and provide an alternative (but not orthonormal) basis set \( \Psi_\mu(x_1, ..., x_N) \) characterized by the symmetry of the tableau.

FIG. 1. (Color online) Bosonic (magenta solid line) and fermionic (blue dashed line) density profiles (in units of \( a_{ho}^{-1} = \sqrt{m\omega/\hbar} \)) as a function of the spatial coordinate \( x \) (in units of \( a_{ho} \)) for each BBFF wavefunction of the \( N_B = N_F = 2 \) mixture. The total density (black solid thin line) is included in the first frame for reference.

The related bosonic and fermionic momentum distributions

\[
n_B^\mu(p) = \frac{1}{2\pi} \int dx \ dx' e^{-ip(x-x')} \rho_B^\mu(x, x') \quad (11)
\]

can be computed from the bosonic and fermionic one-body density matrix

\[
\rho_B^\mu(x, x') = N_B \int dx_2 ... dx_N \Psi_\mu^*(x, x_2, ..., x_N) \\
\times \Psi_\mu(x', x_2, ..., x_N),
\]

\[
\rho_F^\mu(x, x') = N_F \int dx_1 ... dx_{N-1} \Psi_\mu^*(x_1, ..., x_{N-1}, x) \\
\times \Psi_\mu(x_1, ..., x_{N-1}, x'). \quad (12)
\]

B. Density profiles and momentum distributions at a given symmetry

We return to the earlier example of two bosons and two fermions in a harmonic confinement. The density profiles of the eigenstates of \( Y \) and \( Y' \) are shown in Fig. 2. We notice that in this highly symmetric problem the role of bosons and fermions is simply exchanged among the two submanifolds.

We also display the corresponding momentum distributions in Fig. 3. It is worth mentioning that the momentum distributions associated to the \( Y \) symmetry display less peaks (with a number of peaks in \( n_F(p) \) equal to \( N_F \) as in [22]) and are narrower than those associated to the \( Y' \) symmetry, following the intuition that the symmetry of the latter tableau is more “Fermi-like” (i.e. extended in the vertical direction).
We test its symmetry by evaluating the average value $Y_{\text{max}}$ of the Casimir operator, which corresponds to its average value in all coordinate sectors and has the symmetry of a given tableau $[30, 31]$. In analogy with fermionic systems, the solution is expressed in terms of spatial coordinates and “pseudospin” integer coordinates $y_i$, which correspond to the relative positions of the bosons in the coordinate sector $x_{P(1)} < x_{P(2)} < \ldots < x_{P(N_B)}$, namely $\{y_1, \ldots, y_{N_B}\} = \{P^{-1}(1), \ldots, P^{-1}(N_B)\}$. Imambekov and Demler study the strongly interacting limit of the Bethe-Ansatz solution (BA) for $N_B, N_F$ odd. They notice that the wavefunction decouples as a product of an “orbital” part and a “spin” part, as

$$\Psi_{BA} \sim \det[e^{i\frac{\pi}{N}y_j}]\Psi_F(x_1, \ldots, x_N), \quad (15)$$

where the orbital part is a Slater determinant of the first $N$ orbitals, of course chosen for the homogeneous problem $\Psi_F(x_1, \ldots, x_N) = (1/\sqrt{N!})\det[e^{ik_jx}],$ and the “spin” part includes only the bosonic coordinates $y_1, \ldots, y_{N_B}$, with $\kappa = \{-\frac{1}{2}, \ldots, N/2\} \cup \{-\frac{1}{2}, \ldots, N/2\}$. For the ground state. Taking advantage of the decoupling among orbital and spin part of the wavefunction, we generalize such a solution to the inhomogeneous system, by replacing the orbital part of $\Psi_{BA}$ by its corresponding expression under confinement, Eq. (2). We check its symmetry by evaluating the average of the Casimir operator. For $N_B = N_F = 3$ an explicit calculation by expansion of $\langle C \rangle_{BA}$ on the BBFF basis yields the maximal value $\langle C \rangle_{BA}/\langle \Psi_{BA} \Psi_{BA} \rangle = 3$, implying that the generalized BA wavefunction has the $Y$ symmetry. By construction, the BA solution for the trapped case has the same form as the one obtained for arbitrary interactions in the homogeneous system [34]. Notice that although the GM and generalized BA wavefunctions are not simply proportional to each other, their density profiles coincide, displaying no demixing. We remark that Eq. (15) can generate a wavefunction with the $Y'$ symmetry by adopting a different choice of spin rapidities, e.g. for $\kappa = \{-\frac{1}{2}, \ldots, N/2\}$ we obtain an average Casimir operator equal to $-3$.

It is important to notice that the form of the solution for the wavefunction at finite large interactions depends from the way in which the limit $g_{BB} \to \infty$ and $g_{BF} \to \infty$ is approached. Consider for example the case where $g_{BB}$ tends to infinity and $g_{BF}$ is finite. The above symmetry classifications are not useful in this case. The bosonic component can be mapped onto a fermionic one, and the problem can be reduced to the one of spin 1/2 fermions as in [32]. Also in such a case a decoupling of spatial and spin degrees of freedom is predicted with a different form for the spin part with respect to Eq. (15).

### IV. GROUND STATE AT LARGE, FINITE INTERACTION STRENGTH

#### A. Analysis of special solutions

At finite interactions $g_{BB} = g_{BF}$, the ground state is expected to display the $Y$ symmetry $[15, 20]$, since the associated wavefunction has less nodes than the one with $Y'$ symmetry. From a continuity argument starting from the noninteracting solution, we also expect that it is nonvanishing in all permutation sectors. In Ref. [21] a special solution with the latter property was proposed,

$$\Psi_{GM} = ABA_{BF}\Psi_F(x_1, \ldots, x_N), \quad (13)$$

where the mapping functions are

$$A_{BB} = \Pi_{1 \leq j < \ell \leq N_B} \text{sgn}(x_j - x_\ell),$$

$$A_{BF} = \Pi_{1 \leq j \leq N_B, N_B+1 \leq \ell \leq N} \text{sgn}(x_j - x_\ell). \quad (14)$$

We test its symmetry by evaluating the average value of the Casimir operator $\langle C \rangle = \langle \Psi | C | \Psi \rangle/\langle \Psi | \Psi \rangle$. The expression of the (unnormalized) GM wavefunction on the BBFF basis $\Psi_{GM} = \sum_{\alpha=1}^{N!/N_B!N_F!} c_{\alpha} \Psi_\alpha$ is $c_{\alpha} = 1$ for any $\alpha$. From the representation of the Casimir operator (see appendix A) we obtain that for $\frac{N_B}{N_F} = 3$ the average value of the Casimir operator corresponds to its maximal eigenvalue $C_Y = 3$, hence the GM wavefunction has the symmetry of the $Y$ tableau. This is not the case for $\frac{N_B}{N_F} = 2$, where the wavefunction which is nonvanishing in all coordinate sectors and has the $Y'$ symmetry is $\Psi_{GM} = A_{BB}\Psi_F(x_1, \ldots, x_N)$. More generally, we have proven that the GM wavefunction with odd $\frac{N_B}{N_F}$ has always the symmetry of the $Y$ tableau (see appendix B for demonstration).

For the homogeneous system, the Bethe-Ansatz method provides a solution of the model of a Bose-Fermi mixture with equal bosonic and fermionic masses and finite but equal coupling strengths $g_{BB} = g_{BF}$ $[15, 17, 19]$. The solution for the many-body wavefunction is built with the symmetry of a given tableau $[30, 31]$. In analogy with fermionic systems, the solution is expressed in terms of spatial coordinates and “pseudospin” integer coordinates $y_i$, which correspond to the relative positions of the bosons in the coordinate sector $x_{P(1)} < x_{P(2)} < \ldots < x_{P(N_B)}$, namely $\{y_1, \ldots, y_{N_B}\} = \{P^{-1}(1), \ldots, P^{-1}(N_B)\}$. Imambekov and Demler study the strongly interacting limit of the Bethe-Ansatz solution (BA) for $N_B, N_F$ odd. They notice that the wavefunction decouples as a product of an “orbital” part and a “spin” part, as

$$\Psi_{BA} \sim \det[e^{i\frac{\pi}{N}y_j}]\Psi_F(x_1, \ldots, x_N), \quad (15)$$

where the orbital part is a Slater determinant of the first $N$ orbitals, of course chosen for the homogeneous problem $\Psi_F(x_1, \ldots, x_N) = (1/\sqrt{N!})\det[e^{ik_jx}],$ and the “spin” part includes only the bosonic coordinates $y_1, \ldots, y_{N_B}$, with $\kappa = \{-\frac{1}{2}, \ldots, N/2\} \cup \{-\frac{1}{2}, \ldots, N/2\}$. For the ground state. Taking advantage of the decoupling among orbital and spin part of the wavefunction, we generalize such a solution to the inhomogeneous system, by replacing the orbital part of $\Psi_{BA}$ by its corresponding expression under confinement, Eq. (2). We check its symmetry by evaluating the average of the Casimir operator. For $N_B = N_F = 3$ an explicit calculation by expansion of $\langle C \rangle_{BA}$ on the BBFF basis yields the maximal value $\langle \Psi_{BA} | C | \Psi_{BA} \rangle/\langle \Psi_{BA} | \Psi_{BA} \rangle = 3$, implying that the generalized BA wavefunction has the $Y$ symmetry. By construction, the BA solution for the trapped case has the same form as the one obtained for arbitrary interactions in the homogeneous system [34]. Notice that although the GM and generalized BA wavefunctions are not simply proportional to each other, their density profiles coincide, displaying no demixing. We remark that Eq. (15) can generate a wavefunction with the $Y'$ symmetry by adopting a different choice of spin rapidities, e.g. for $\kappa = \{-\frac{1}{2}, \ldots, N/2\}$ we obtain an average Casimir operator equal to $-3$.

### B. Numerical illustration

We have tested our predictions by comparing with numerical DMRG simulations [36]. The DMRG techniques provide an efficient numerical solution for the lattice model of an interacting Bose-Fermi mixture subjected
The harmonic trap strength is \( V/t = 7 \times 10^{-6} \), and the number of lattice sites used in the simulation is \( L = 128 \). In the last panel the analytical prediction from the generalized BA wavefunction (thin black solid line) is shown.

The momentum distributions for the bosonic and fermionic components are illustrated in Fig. 4 as compared with the predictions of the generalized BA wavefunction. At increasing the interaction strength the momentum distributions approach those obtained by the generalized BA wavefunction, showing that this solution accurately describes the trapped mixture at finite, large but equal interactions.

V. OUTLOOK AND PERSPECTIVES

In this work we have studied the exact solutions of a highly symmetric model for Bose-Fermi mixture in the strongly interacting limit. Our results are relevant for the ongoing experiment on ultracold mixtures of atomic gases in tight atomic waveguides, with particular attention of the case of \(^{173}\)Yb-\(^{174}\)Yb Bose-Fermi mixtures \( \text{[12]} \) where the fractional mass difference among the two isotopes is small. We have provided first an orthogonal basis set of solutions which span the degenerate manifold. Secondly, we have grouped such solutions on the basis of the Casimir invariant, associated to a given Young tableau. Finally, we have analyzed two special solutions of the problem and discussed their symmetry according to the average value of the Casimir operator. By comparing with DMRG simulations, we have found that the wavefunction obtained by generalizing the Bethe-Ansatz solution to trapped systems accurately describes the density
profile and momentum distribution of the mixture at finite, large but equal BB and BF coupling strengths. This wavefunction can be used to describe the Bose-Fermi mixture in arbitrary external potential. Splittings and mixing of the states with different symmetries are expected when different masses and different BB and BF coupling constants are chosen. Our solution serves as a guideline for further numerical studies. This work opens also the way to the study of the dynamical properties of the strongly interacting Bose-Fermi mixture. Signatures of strong correlations could be found in the collective excitation spectrum. It would also be interesting to investigate how particular states in the degenerate manifold can be prepared and addressed.

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Appendix A: Casimir operator on the BBFF basis

For $N_B = 2$, $N_F = 2$ the representation of the Casimir operator $C = \sum_{i<j}(ij)$ on the orthonormal BBFF configuration basis reads

$$
\begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
-1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0
1 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0
1 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0
-1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0
1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0
0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1
0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1
0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 1 & -1 & 0
-1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0
0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}
$$

We prove here below that for odd $N_B = N_F$ the GM wavefunction has the symmetry of the $Y$ tableau.

$$
\begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0
1 & 0 & 1 & 1 & 0 & 1
-1 & 1 & 0 & 0 & 1 & -1
1 & 1 & 0 & 0 & 1 & 1
-1 & 0 & 1 & 1 & 0 & 1
0 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}
$$

Appendix B: Demonstration of the symmetry of a special wavefunction

We prove here below that for odd $N_B = N_F$ the GM wavefunction has the symmetry of the $Y$ tableau.
Casimir operator on the BBFF basis has $N_B N_F$ non-vanishing entries with value 1 in $N_B (N_F + 1)/2$ cases and value -1 in $N_B (N_F - 1)/2$ cases. Each line of the matrix corresponds then equally to the value of the average of the Casimir operator, $\langle C \rangle = \langle \Psi | C | \Psi \rangle / \langle \Psi | \Psi \rangle$, and its value corresponds to the maximal eigenvalue $C_Y = N_B = N_F$. For other values of $N_B$, $N_F$ the representation of the Casimir operator is more complicated to predict and the above demonstration does not hold.

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