Cavity method for force transmission in jammed disordered packings of hard particles

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The force distribution of jammed disordered packings has always been considered a central object in the physics of granular materials. However, many of its features are poorly understood. In particular, analytic relations to other key macroscopic properties of jammed matter, such as the contact network and its coordination number, are still lacking. Here we develop a mean-field approach to this problem, based on the consideration of the contact network as a random graph where the force transmission becomes a constraint optimization problem. We can thus use the cavity method developed in the last decades within the statistical physics of spin glasses and hard computer science problems. This method allows us to compute the force distribution $P(f)$ for random packings of hard particles of any shape, with or without friction. We find a new signature of jamming in the small force behavior $P(f) \sim f^\theta$, whose exponent has attracted recent active interest. We find a finite value for $P(f=0)$, along with $\theta=0$ over an unprecedented six decades of force data, which agrees with experimental measurements on emulsion droplets. Furthermore, we relate the force distribution to a lower bound of the average coordination number $\langle \xi_{\text{min}} \rangle$ of jammed packings of frictional spheres with coefficient $\mu$. This bridges the gap between the two known isostatic limits $\langle \xi \rangle (\mu=0) = 2D$ (in dimension $D$) and $\langle \xi \rangle (\mu \to \infty) = D+1$ by extending the naive Maxwell’s counting argument to frictional spheres. The framework describes different types of systems, such as non-spherical objects and dimensions, providing a common mean-field scenario to investigate force transmission, contact networks and coordination numbers of jammed disordered packings.

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I. INTRODUCTION

Mechanically stable packings of granular media are important to a wide variety of technical processes [1]. One approach to characterize jammed granular packings is via the interparticle contact force network. In turn, this network determines the probability density of inter-particle contact forces $P(f)$ and the average coordination number $\bar{\xi}$. While the force network has been studied for years, there is yet no unified theoretical framework to explain the common observations in granular packings, ranging from frictional to frictionless systems, from spherical to non-spherical particles and in any dimensions.

Experimental force measurements [2–8] and simulations [9–12] have shown that the interparticle forces are inhomogeneously distributed with common features: $P(f)$ near the jamming transition has a peak at small forces and an exponential tail in the limit of large $f$. It is argued that the development of a peak is a signature of the jamming transition [10]. Some parts of the qualitative behavior of $P(f)$ are correctly captured by simplified mean-field models, such as the q-model [3, 13] and Edwards’ model [7, 8]. Both of them describe the exponential decay at large force and a power law behavior $P(f) \sim f^\theta$ for small forces [7, 8, 14]. However, the exponent obtained by the q-model (an integer $\theta \geq 1$ [3, 14]) is larger than the experimental value $\theta \approx 0$ [2] or the one obtained by recent numerical simulations accessing the low force limit with increasing accuracy $0 \leq \theta \leq 0.5$ [15, 16], while the exponent obtained by Edwards’ model $\theta = 1/(\bar{\xi} - 2)$ depends strongly on the dimension (through $\bar{\xi}$), when simulations show it does not [15]. Other approaches based on entropy maximization similar to Edwards’ statistical mechanics [17] also recover the large force exponential decay [18–22]. Some of those works however predict a Gaussian tail [23], and so does a mean-field theory based on replica theory of spin glasses [15, 24, 25].

The average coordination number per particle $\bar{\xi}$ is another key signature of jamming. Close to jamming, many observables (pressure, volume fraction, shear modulus or viscosity [26] to name a few) scale with the distance $\bar{\xi} \approx \bar{\xi}_c$ to the average coordination number $\bar{\xi}_c$ at the transition. Understanding the value of $\bar{\xi}_c$ is thus of primary importance. The case of frictionless spheres, for which, $\bar{\xi}_c = 2D$ where $D$ is the dimension, has been rationalized based on counting arguments leading to the isostatic conjecture [27–29]: a lower bound $\bar{\xi}_c \geq 2D$ is provided by Maxwell’s stability argument [30], and an upper bound $\bar{\xi}_c \leq 2D$ is given by the geometric constraint of having the particles exactly at contact, without overlap. The problem, however, turns out to be more complicated when friction is considered, and no method is known so far to predict $\bar{\xi}_c$ in such a case. The (naive) generalization of the counting arguments gives the bounds $D+1 \leq \bar{\xi}_c \leq 2D$ in frictional packings, independently of the value of the interparticle friction coefficient $\mu$. Indeed, there is a range of $\bar{\xi}_c$ obtained numerically and experimentally [22, 29, 31–38]. However, for small $\mu$, this range never extends to the predicted lower bound, as packings with low friction coefficient lie close to $\bar{\xi}_c = 2D$.

Here we present a theoretical framework at a mean-field level to consider force transmission as a constraint optimization problem on random graphs, and study
this problem with standard tools, namely the cavity method [39]. We first obtain the force distribution for spheres, in frictionless and frictional cases, both in two and three-dimensions. Besides showing the experimentally known exponential fall-off at large forces, these distributions bring a new insight in the much less known small force regime. In particular, we find for frictionless spheres in both 2 and 3 dimensions a finite value for \( P(f) \) in \( f = 0 \), leading to an exponent \( \theta = 0 \) for small forces that extends over four decades, beyond currently performed simulated data [15, 16].

We also show how the frictional coefficient \( \mu \) affects the average number of contacting neighbors \( z_c \) at the jamming transition, and we find a lower bound \( z_c^{\text{min}}(\mu) \) for this number (which is also a lower bound on \( z \), since \( z \geq z_c \) in a packing). We achieve this by generalizing in a careful way the naive Maxwell counting arguments, considering the satisfiability of force and torque balance equations. Linking \( z_c \) to the behavior of force and torque balances is not a new idea, as it was already suggested by Silbert et al. [32]. Furthermore the generalized isostaticity picture [34] gives a bound on the number of fully mobilized forces (ie the number of tangential forces which are taking the maximally value allowed by the Coulomb law) based on the value of \( z_c \). However, none of these works derive a bound for \( z_c \) itself, at a given \( \mu \). The bound \( z_c^{\text{min}}(\mu) \) that we obtain interpolates smoothly between the two isostatic limits at \( \mu = 0 \) and \( \mu \to \infty \).

II. METHODS

A. Force balance as a satisfiability problem

Similarly to the global rigidity condition, force and torque balance are entirely constrained by the contact network of the packing. If we define \( \hat{r}_{ia} \) as the vector joining the center of particle \( a \) and the contact \( i \) on it, the contact network is uniquely defined by the complete set \( \{\hat{r}_{ia}\} \). Calling \( \vec{f}_{ia} \), the force acting on particle \( a \) from contact \( i \), force and torque balances read:

\[
\sum_{i \in \partial a} \hat{r}_{ia} \cdot \vec{f}_{ia} = 0, \quad \vec{r}_{ia} \cdot \vec{f}_{ia} < 0, \\
\sum_{i \in \partial a} \vec{r}_{ia} \times \vec{f}_{ia} = 0, \quad |\vec{f}_{ia}| \leq \mu |f_{ia}|, \quad \forall a \tag{1}
\]

where the notation \( \partial a \) denotes the set of contacts of particle \( a \). We explicitly take into account friction by decomposing the force into normal and tangential parts \( \vec{f}_{ia} = -f_{ia}^n \hat{n}_{ia} + f_{ia}^t \hat{t}_{ia} \), where \( \hat{n}_{ia} \) and \( \hat{t}_{ia} \) are normal and tangential unit vectors to the contact, respectively. The inequality \( \vec{r}_{ia} \cdot \vec{f}_{ia} < 0 \) ensures the repulsive nature of the normal force. The last inequality ensures Coulomb’s law with friction coefficient \( \mu \).

The constraints induced by force and torque balances on the forces \( f_i \) are not always satisfiable. In the case of frictionless spheres, we can recover the known \( z \geq 2D \) directly from force balance alone. The naive Maxwell counting argument [29] applied to frictionless spheres reduces Eq. (1) to a set of linear equations by taking into account only force balance and neglecting the repulsive nature of the forces. Maxwell argument considers the minimal number of forces needed to satisfy Eq. (1) which gives, per sphere, \( D \) equations and \( z/2 \) variables (forces), implying \( z \geq 2D \) to have a solution. Below this threshold, there is generically no solution to Eq. (1). To accurately extend this counting argument to more general conditions (frictional, repulsive, and/or non-spherical particles), one must take into account all the constraints in Eq. (1), including the repulsive nature of the forces and the Coulomb condition for frictional packings. Indeed, the naive Maxwell argument, neglecting those constraints, concludes \( z > z_c^{\text{min}} = D+1 \) for any frictional packing of spheres, ignoring the dependence on the friction coefficient [40]. On the other hand, below we show that including the above mentioned constraints we obtain an accurate lower bound \( z_c^{\text{min}}(\mu) \), explicitly depending on \( \mu \).

We tackle the problem of satisfiability of force and torque balances Eq. (1) by looking at the contact network in an amorphous packing as an instance of random graph. As depicted in Fig. 1A, starting from a packing of \( N \) particles, we explicitly construct a so-called factor graph [41], considering the \( M = zN/2 \) contacts as ‘sites’, and the \( N \) particles as ‘interaction nodes’. Each site \( i \) bears two vectors \( \vec{r}_{ia}^a \) and \( \vec{r}_{ib} \) and two opposite forces \( \vec{f}_{ia} = -\vec{f}_{ib} \) (one per particle involved in the contact). Note that \( \hat{n}_{ia} \) is uniquely determined by the contact network \( \{\vec{r}_{ia}\} \) and represents the ‘quenched’ disorder in the system, whereas \( \hat{t}_{ia} \) is free to rotate in the plane tangent to contact. On each interaction node (particle) \( a \), we enforce force balance, torque balance, repulsive interactions and Coulomb friction conditions on its \( z_a \) neighboring sites by an interaction function,

\[
\chi_a\{f^n, f^t, \hat{n}_{ia}, \hat{t}_{ia}\}_{\partial a} = \delta\left(\sum_{i \in \partial a} \vec{f}_{ia}^n \right) \delta\left(\sum_{i \in \partial a} \vec{r}_{ia}^a \times \vec{f}_{ia}^t \right) \times \prod_{i \in \partial a} \Theta(f_{ia}^n)\Theta(\mu f_{ia}^t - f_{ia}^t). \tag{2}
\]

We define the partition function \( Z \) and entropy \( S \) for the problem of satisfiability of force and torque balances for a fixed realization of the quenched disorder \( \{\hat{n}_{ia}\} \) as (see SI for more details):

\[
e^{-S} = Z = \int \prod_{i=1}^{M} df_{ia}^n df_{ia}^t d\hat{n}_{ia} d\hat{t}_{ia} \delta(\sum_i f_{ia}^n - Mp) \prod_{a=1}^{N} \chi_a\{f^n, f^t, \hat{n}_{ia}, \hat{t}_{ia}\}_{\partial a}. \tag{3}
\]

Without loss of generality, we work in the constant pressure \( p \) ensemble: if we find a solution to the force and torque balances problem having a pressure \( p' \), we can always find one solution with pressure \( p \) by multiplying all forces by \( p/p' \). If the entropy is finite, there exists a solution to force and torque balances. The satisfiab-
From this point of view, we stand on the same ground as tingly used in the context of spin-glasses for example [39]. This approximation is rou-
itly updated by the uncorrelated force probability $Q^{a \rightarrow j}$ on neighboring edges with the force/torque balance constraint $\chi_a(\{f^n, f^i, \hat{n}^a, \hat{f}^a\})$ on that particle $a$, verifying the cavity equation Eq. (6). The marginal probability $Q^i(f^n, f^i)$ on site $i$ is calculated as a product of $Q^{a \rightarrow i}$ and $Q^{b \rightarrow i}$ as in Eq. (5).

As the simplest case, we restrict the description of this section to packings of spheres, with obvious generalization to non-spherical objects. For a given disordered packing, each particle $a$ has unique surroundings, different from its neighbors or other particles in the packing. These surroundings are defined by the contact number $z_a$, contact vectors $\{\vec{r}^a\}$. If the system is underdetermined, several sets of forces in the system satisfy force and torque balance, and each contact force has a certain probability distribution $Q^i(f^n, f^i)$.

The local disorder makes each contact unique, and the probability distributions of forces $Q^i(f^n, f^i)$ are different from contact to contact. We define the overall force distribution in a packing $P(f^n, f^i)$ as an average over the probability distributions of forces over the contacts:

$$P(f^n, f^i) \equiv \langle Q^i(f^n, f^i) \rangle = \frac{1}{M} \sum_i Q^i(f^n, f^i).$$

(4)

Next, we show that on a random graph, we can access the distributions $Q^i(f^n, f^i)$ with a self-consistent set of local equations using the cavity method [41]. In this description, we work at fixed pressure $p$, i.e. we consider any two solutions differing only by an overall rescaling of the pressure to be only one genuine solution (see SI for the detailed implementation in the following cavity formalism). Each contact is linked to two particles, $a$ and $b$. We denote $Q^{a \rightarrow i}(f^n, f^i)$ the probability distribution of the force $f^i$ of a site (contact) $i$, if $i$ is connected only to the interaction node (particle) $a$, that is, if we remove (dig a cavity) particle $b$ from the packing. The main assumption of the cavity method is to consider that $Q^{a \rightarrow i}(f^n, f^i)$ and $Q^{b \rightarrow i}(f^n, f^i)$ are uncorrelated. Therefore, we can write the probability of forces at contact $i$ as:

$$Q^i(f^n, f^i) = \frac{1}{Z^i} Q^{a \rightarrow i}(f^n, f^i) Q^{b \rightarrow i}(f^n, f^i), \quad \{a, b\} = \partial i$$

(5)

with $Z^i$ the normalization.

Under the mean-field assumption a set of local equations (called cavity equations) relates the $Q^i$’s, as depicted in Fig. 1B:

$$Q^{a \rightarrow i}(f^n, f^j) = \frac{1}{Z^{a \rightarrow i}} \int \frac{d\vec{r}^a_i}{\partial a \rightarrow i} \prod_{j \in \partial a \rightarrow i} df^n_i df^j_i df^a_i \times \chi_a(\{f^n, f^i, \hat{n}^a, \hat{f}^a\}) \prod_{c=\partial j \rightarrow a} Q^{c \rightarrow j}(f^n_i, f^j_i)$$

(6)

where the notation $\partial x - y$ stands for the set of neighbors of $x$ on the graph except $y$, and $Z^{a \rightarrow i}$ is the normalization. Crucially, we do not average over the contact directions $\{\hat{n}^a\}_{\partial a}$ at this stage (whereas the q-model [3, 13] and Edwards’ model [7, 8] do). This implies that every
link $a \rightarrow i$ has a different distribution, due to the local ‘quenched’ disorder provided by the contact network \{\hat{n}_a\}_{a} and contact number $z_a$. Hence, finding a set of $Q^-$ that are solutions of Eq. (6) allows to get the distribution of forces on every contact individually, through the use of Eq. (5).

Looking for a solution of Eq. (6) for a given instance of the contact directions (meaning for one given packing) is possible. These equations are commonly encountered as ‘cavity equations’ in the context of spin glasses or optimization problems defined on random graphs [41], and they can be solved by message passing algorithms like Belief Propagation. Here we follow another route, since we are interested in $P(f^n, f^t)$ for not only one packing but over the ensemble of all random packings. Thus, we study the solutions of the cavity equations in the thermodynamic limit to provide typical solutions for large packings. As in statistical mechanics, the partition function will be dominated by the relevant typical configurations which we expect will be realized in experiments.

In the thermodynamic limit, the set of $Q^-$’s that are solutions of Eq. (6) are distributed according to the probability $Q(Q^-)$:

$$Q(Q^-) = \frac{1}{Z} \sum_{n \rightarrow i} \delta [Q^- - Q^\rightarrow]$$

(7)

In this case, we can replace the sum over $a \rightarrow i$ by a continuum description of the $Q^-$’s based on their distribution $Q(Q^-)$. The probability that a given $Q^-$ is set by a cavity equation Eq. (6) involving $z - 1$ contact is proportional to $z \Omega(\hat{n}_i, \{\hat{n}_j\})$. Thus, averaging over the ensemble of random graphs, Eq. (7) becomes a self-consistent equation [39, 41]:

$$Q(Q^-) = \frac{1}{Z} \sum_{z} z R(z) \int \Omega(\hat{n}, \{\hat{n}_j\}) \prod_{j=1}^{z-1} d\hat{n}_j \times DQ^\rightarrow J Q(Q^\rightarrow) \delta [Q^- - F^-] Q(\{Q^\rightarrow\}).$$

(8)

where $Z$ is the normalization. Note that the value of the integral does not depend on the choice of $\hat{n}$. Once a solution to Eq. (8) is known, we deduce the force distribution $P(f^n, f^t)$ in the overall packing as the average of all these probability distributions and contacts:

$$P(f^n, f^t) = (Q(f^n, f^t)) = \frac{1}{Z_P} \left[ \int DQ^- Q(Q^-) Q^- (f^n, f^t) \right]^2$$

(9)

where $Z_P$ is the normalization to ensure $\int P(f^n, f^t) = 1$.

Equation (8) stands out as the main and crucial difference with previous approaches, in particular the q-model [3, 13] and Edwards’ description [7, 8]. Although these approaches also neglect correlations, our work does not reduce to those models due to a fundamentally different way of treating the disorder in the packing. Here, we consider a site-dependent $Q'(f^n_i, f^t_i)$, where the Edwards’ model and q-model create all sites equal. Thus in our method the average over the packing configurations is not done at the same level as the average over forces. That is, we perform a quenched average over the disorder of the graph. As random packings in two or three dimensions have a rather small connectivity, the fluctuations in the environment of one particle are large: no particle stands in a ‘typical’ surrounding. Hence, the average over the ‘quenched’ disorder (the packing configurations) must be done with care. Averaging directly Eq. (6) (a so-called ‘annealed’ average in spin-glass terminology), as the previously cited approaches do [3, 7, 8, 13], amounts to neglect the site to site fluctuations. Performing a ‘quenched’ average as in Eq. (8), however, allows to take into account these fluctuations correctly [39], and leads to a force distribution which is the average force distribution over the ensemble of possible packings, as opposed to the force distribution of an averaged packing.

This issue becomes also crucial to the study of the satisfiability transition of force and torque balances. For example, the q-model describes a force distribution in a packing of frictionless spheres (ie it finds solution to force balance), even in cases where we know there is no solution, such as when $\hat{z} < 6$ in 3-D frictionless systems. This can be understood by looking at the entropy defined in Eq. (3). The annealed average over disorder done in the q-model amounts to compute the averaged partition function $Z$, and get the entropy through $S_{an} = \ln Z$, with $Z$ defined in Eq. (3). But one can show that $Z$ is always finite for $\hat{z} \geq 2$. Indeed, for $\hat{z} = 2$, an infinite row of perfectly aligned spheres will satisfy force and torque balances and will contribute to the partition function. A straightforward generalization of this example shows that for $\hat{z} \geq 2$, the annealed entropy is finite. On the contrary, the quenched average amounts to compute the averaged entropy $S_{qu} = \ln Z$. Now, for our frictionless sphere example, typical configurations with $\hat{z} < 6$ cannot satisfy force balance, and their diverging negative entropy will dominate the average $S_{qu}$. Hence, the quenched average correctly captures the satisfiability/unsatisfiability transition at $\hat{z} = 6$ while the annealed average does not.

Equation (8) is typically hard to solve, since it is a self-consistent equation for a distribution of distributions $Q(Q^-)$. For this purpose, we use a Population Dynamics algorithm (details in SI), familiar to optimization problems [41]. This method consists to describe the distribution $Q$ via a discrete sampling (a ‘population’) made of a large number of distributions $Q^-$. Applying iteratively Eq. (8), we find, if it exists, a fixed-point of the distribution $Q(Q^-)$.

It is interesting to discuss the different types of solutions expected from Eq. (8). For a given contact network, if the system is satisfiable and underdetermined, hence admits an infinite set of solutions for force and torque balances, the distributions $Q'(f^n_i, f^t_i)$ should be broad, allowing each force to take values in a non-vanishing range. This means that on each contact, $Q^\rightarrow_i (f^n_i, f^t_i)$ and $Q^-_i (f^n_i, f^t_i)$ should be broad and overlapping. If the system is neither under- nor overdetermined (i.e. iso-
static), there is only one solution to force and torque balances for every site \(i\), and each \(Q' (f^n, f')\) is a Dirac \(\delta\)-function centered on the solution. If the system is overdetermined or unsatisfiable, there is typically no solution to Eq. (8), meaning that an algorithm designed to solve it would not converge. In practice, since we perform a population dynamics algorithm to average over all possible packings, if one start with a set of broad \(\{Q_a^{n-\gamma} (f^n, f')\}\) as a guess for the solution, both isostatic and overdetermined ensembles show that all \(\{Q_a^{n-\gamma} (f^n, f')\}\) shrink to \(\delta\)-distributions, while underdetermined ensemble always gives broad (not vanishing) probabilities. Therefore the threshold \(z_{c\text{min}}(\mu)\) of the satisfiability/unsatisfiability transition for force transmission can be located by measuring the width of the force distributions \(\{Q_a^{n-\gamma} (f^n, f')\}\).

The location of this transition, in turn, constitutes a lower bound for the possible coordination number \(z_{c\text{min}}(\mu)\), which extends Maxwell’s counting argument for \(\mu = 0\) to any friction. An additional quantity available is the force distribution itself, as a function of \(z_{c\text{min}}(\mu)\). Therefore, our approach explicitly relates two essential properties of the jamming transition: the average coordination number and the force distribution.

### III. RESULTS

#### A. Force distribution for frictionless sphere packings

We start by computing the force distribution \(P (f = f^n)\) for two and three-dimensional frictionless sphere packings, when we fix the average contact number \(\bar{z} = \bar{z}_c = 4\) and 6 respectively, by solving SI-Eq. (7). Results in Fig. 2 reproduce the exponential tail of the force distribution at large forces, as seen in numerical simulations [9–12, 32, 33, 43, 44] and experiments [3–8, 12]. The force distribution we obtain can be well fitted with

\[
\bar{z} = 4
\]

\[
\bar{z} = 6
\]

P(x) = \[(7.84x^2 + 0.86 - 0.75/(1 + 4.10x))e^{-2.67x}\] for 2-D (Fig. 2A) and \(P(x) = [7.45x^2 + 1.20 - 1.06/(1 + 2.33x)]e^{-2.65x}\) for 3-D (Fig. 2B), with \(x = f/f\). Both fitting functions are close to the empirical fit \(P(x) = [3.43x^2 + 1.45 - 1.18/(1 + 4.71x)]e^{-2.25x}\) to the force distribution of dense amorphous packings generated by Lubachevsky-Stillinger algorithm in 3-D by Donve et al. [43]. More interestingly, our method allows to access the small force region with unprecedented definition. We gather data down to \(10^{-6}\) times the peak force (which is of the order of the pressure). This range is way below what is accessible with the art simulations of packings [9–11, 15, 16, 32, 33, 44]. The reason for this is that we avoid two problems: (i) as we have no real packing to generate, we can easily a huge number of forces (we used \(10^6\) to compute our \(P(f)\), compare with few tens of thousands that can be generated in simulated packings [9–11, 32]), and (ii) we can work exactly at the jamming transition point, as we set \(\bar{z} = 2D\), which contrasts with actual numerical or experimental studies where the limit of vanishing pressure with \(\bar{z} = 2D\) is very challenging.

The behavior of \(P(f)\) at small forces has recently attracted attention [15, 16, 33, 44]. Wyart [16] pointed out a relation between the small force scaling \(P(f) \sim f^\theta\) and the distribution function of the gaps \(h\) between particles close to contact \(g(h) \sim h^{-\gamma}\) via the inequality \(\gamma \geq 1/(2 + \theta)\). Few empirical data exist so far for the \(\theta\) exponent, even if some recent efforts greatly improved available values [15, 44].

Here we find for frictionless spheres a distribution of contact forces having a finite value for \(f = 0\). This translates as an exponent \(\theta = 0\) over four decades of data in both 2-D and 3-D packings (Fig. 2). Such a finite value for the force distribution was observed in experiments on concentrated emulsion droplets designed to capture their small deformation using confocal microscopy of fluores-
cent dyes to highlight the contact network [2], with which our prediction agrees. On the theoretical side, several values of $\theta$ have been predicted: the replica theory also predicts a finite value for $P(f = 0)$ [15, 24, 25] (it predicts a Gaussian for $P(f)$); the q-model can give several values for $\theta$, depending on the underlying assumptions, but in any case predicts $\theta \geq 1$ [3, 13]; and Edwards’ model predicts $\theta = 1/(z - 2)$ [7, 8]. The differences between Edwards’ and q-model approaches, which mostly stem from the treatment of local disorder in the contact normal directions, indicate that the way to deal with this disorder is crucial for a correct description of the small force behavior. According to the inequality $\gamma \geq 1/(2 + \theta)$ [16], our prediction of $\theta = 0$ gives $\gamma \geq 1/2$, which would mean that the commonly reported value of $\gamma = 1/2$ [31] actually saturates this bound.

Interestingly, we find an intermediate regime for $f$ slightly smaller than its average value $\langle f \rangle$, for which $P(f) \sim f^{0.52}$ (Fig. 2). This regime, extending from $f \approx 10^{-2}\langle f \rangle$ to $f \approx \langle f \rangle$ is the one probed by most experiments and simulations. We suggest that the discrepancy between the several values of $\theta$ reported so far might stem from the existence of this regime and the crossover to the $\theta = 0$ regime at smaller $f$.

B. Calculation of $\bar{z}_{c}^{\text{min}}(\mu)$ for frictional sphere packings

We turn to the determination of the force distribution for arbitrary friction coefficient $\mu$ and a lower bound on $z_{c}$ for the existence of random packings of spheres at a given $\mu$. This threshold corresponds to the point where solutions of the cavity equations Eq. (6) and Eq. (8) no longer exist.

We search for the existence of a solution by applying iteratively Eq. (6) to a population of force distributions $\{Q^{-}(f^{n}, f^{t})\}$, according to the Population Dynamics algorithm (SI). A solution exists if this process leads to a converged solution $Q(\bar{Q}^{-})$. The convergence of $Q(\bar{Q}^{-})$ is hard to get numerically, as it requires a very large population of $\{Q^{-}\}$ to describe the $Q^{-}$ space precisely enough. However, for our purpose, we do not need to describe $Q(\bar{Q}^{-})$ in detail, since we just need to know if it exists. We thus adopt a simpler criterion to test this existence. It is based on the convergence of the average width of the distribution $\{Q^{-}\}$. If this width converges to a finite value, a solution to the cavity equations exists (satisfiability), whereas if it vanishes, no solution exists (unsatisfiability), as described in the previous section. This convergence can be studied with a smaller population, and it occurs more rapidly than the convergence of $Q(\bar{Q}^{-})$. Still, the computational cost is high, and we here apply the method to 2D packings only.

We define the ‘width’ of $Q^{-}(f^{n}, f^{t})$ on $f^{n}$ and $f^{t}$ by $W_{n}$ and $W_{t}$ respectively as the difference of two extreme values of $f^{n}$ and $f^{t}$ at which $Q^{-}(f^{n}, f^{t})$ is equal to $10^{-3}$ (see SI-Fig. 1B). For frictionless packings, we calculate the average width over the sites as $\langle W_{n} \rangle$, whereas in frictional case, we calculate $\langle W_{n} \rangle$ and $\langle W_{t} \rangle$, the average width on variable $f^{n}$ and $f^{t}$, respectively. Fig. 3A shows the evolution of the average width of distributions at different $\bar{z}$ versus time steps in population dynamics in 2-D frictionless packing. $\bar{z}_{c}^{\text{min}}(\mu = 0)$ is found at 4.0, as expected from counting argument. (B) Linear-Log plot of $\bar{z}_{c}^{\text{min}}(\mu)$ vs. $\mu$ for various friction coefficients in 2-D disks packings. $\bar{z}_{c}^{\text{min}}(\mu)$ shows a monotonic decrease with increasing $\mu$ from $\bar{z}_{c}^{\text{min}}(\mu = 0) = 2D = 4$ to $\bar{z}_{c}^{\text{min}}(\mu = \infty) \geq D + 1 = 3$. Error bar indicates the range from the largest $\bar{z}_{c}^{\text{min}}(\mu)$ having no solution to the smallest $\bar{z}_{c}^{\text{min}}(\mu)$ having solution. In the inset, two power law scaling relations $\bar{z}_{c}^{\text{min}}(0) - \bar{z}_{c}^{\text{min}}(\mu) \sim \mu^{\alpha}$, $\bar{z}_{c}^{\text{min}}(\mu) - \bar{z}_{c}^{\text{min}}(\infty) \sim \mu^{-\beta}$ are found with $\alpha = 0.67$, $\beta = 0.35$, respectively.
goes below data discretization (the interval of force to integrate the force distribution), leading to no solution of $Q^f(f^n, f^i)$ according to Eq. (5). In this case the system is overdetermined in the sense that $z$ contacts per particle are not enough to stabilize the whole packing. In contrast, when $z$ is increased to $z = 4.5 > z_{c\min}^\infty(\mu)$, the distributions in $\{Q^{-}\}$ become broad (see SI-Fig. 1B).

The average of width converges to a finite value as seen in Fig. 3A, allowing a non-vanishing force distribution $Q^f(f^n, f^i)$ at each contact. In this case a set of solutions to force and torque balances exists. The point where the average widths of the distributions $(\langle W_n \rangle)$ and $(\langle W_i \rangle)$ vanish is $z_{c\min}^\infty(\mu)$.

Frictionless packings show a transition point $z_{c\min}^\infty(\mu = 0) = 2D$, in $D = 2$ and $D = 3$, as expected. The full curve $z_{c\min}^\infty(\mu)$ is shown in Fig. 3B for $D = 2$. We observe a monotonic decrease with increasing $\mu$ from $2D = 4$ at $\mu = 0$, a well-known behavior of frictional packings, previously found in numerous studies, both experimentally and numerically [22, 29, 31–38]. Notice that the critical contact number we obtain at infinite friction is slightly above the Maxwell argument $D + 1$; also a typical feature [31, 34, 36]. This is interesting, as it means that the naive counting argument, ignoring the repulsive nature of the forces, fails to reproduce the correct bound for such a simple case as sphere packings with $\mu \to \infty$, where neither Coulomb condition nor non-trivial geometrical features complexify the picture. The fact that the naive Maxwell counting argument still gives the correct bound for frictionless sphere packing can therefore be seen as a quite fortunate isolated prediction. In the inset of Fig. 3B, two power law scaling relations $z_{c\min}^\infty(0) - z_{c\min}^\infty(\mu) \sim \mu^{\alpha}$, $z_{c\min}^\infty(\mu) - z_{c\min}^\infty(\infty) \sim \mu^{-\beta}$ are found with $\alpha = 0.67$, $\beta = 0.35$ respectively. Our result $\alpha$ agrees well with previous simulation of 2-D polydisperse packings $\alpha = 0.70$ [34], and close to the prediction of 2-D monodisperse packing $\alpha = 1$ by Wang et al [36], while $\beta$ is much smaller than their predicted value $\beta = 2$ and the result of $\beta = 1.86$ obtained from simulation [36].

### C. Frictional sphere packings: joint force distribution

Similar to the frictionless case, the cavity method can generate the joint force distribution $P_\mu(f^n, f^i)$ for frictional sphere packings with friction coefficient $\mu$.

The simplest case of infinite friction, $P_\mu=\infty(f^n, f^i)$ for 2-D and 3-D sphere packings is shown in Fig. 4A and Fig. 4B respectively, and follow similar behavior as measured in previous numerical studies [36]. In particular, we recover the non-trivial qualitative change shown in [36] between 2-D and 3-D: while in 3-D, $P_\mu=\infty(f^n, f^i) \sim P_\mu=\infty(f^i, f^n)$, in 2-D, this symmetry is clearly broken. This is a consequence of the more symmetrical role tangential and normal forces play in 3-D with as many torque balance as force balance equations, whereas in 2-D, there are twice less torque balance than force balance equations.

Furthermore, we obtain the distributions of the normal and tangential components as plotted in Fig. 4C and Fig. 4D. The normal force distribution have slight peaks around the mean and approximate exponential tails at large forces. Below the mean, the normal force distribution for infinite friction has a nonzero probability at zero force whereas it shows a dip towards zero for $\mu = 0.2$ (see SI-Fig. 2 for results of $\mu = 0.2$ in 2-D). The tangential force distribution also has an exponential tail, however, it decreases monotonically without an obvious rise at small forces. Our results of the probability distribution of normal forces and tangential forces agree with previous experimental measurements in 2-D frictional discs packing [45].

When the friction coefficient is finite (see SI-Fig. 2A), the pattern inside the Coulomb cone looks similar to the one obtained at infinite friction. Quite interestingly, we do not observe any excess of forces at the Coulomb threshold $f^i = \mu f^n$, implying that there are no sliding contacts. This offers a theoretical explanation to a singular fact already observed in simulations: control parameters (essentially volume fraction and friction coefficient) being equal, the percentage of plastic contacts in a packing depends on the preparation protocol [33, 46]. Our formalism takes into account those different packings (hence protocols) by performing a statistical average over possible packings, and the outcome shows that packings without plastic contacts are dominant. In this regards, the fragility associated with the large number of plastic contacts in many experimentally or numerically generated packings could be mostly attributed to the preparation protocol, rather than to an inherent property of random packings of frictional spheres.

### IV. DISCUSSION

In conclusion, we develop a theoretical framework by using the cavity method, introduced initially for the study of spin-glasses and optimization problems, to obtain a statistical physics mean-field description of the force and torque balances constraints in a random packing. This allows us to get the force distribution and the lower bound on the average coordination number in frictional and frictionless spheres packings.

We find a signature of jamming in the finite value $P(f = 0)$ of the force distribution at small force, which agrees with experimental measurements on emulsion droplets [2]. We also notice that there is a power law rise $P(f) \sim f^{0.52}$ in the intermediate region of $P(f)$. Thus it is likely that one obtains an exponent $0 < \theta < 0.52$ if the simulations or experiments can not achieve data down to low enough forces. For frictional packings, we can access the complete joint distribution $P_\mu(f^n, f^i)$.

Concerning the average coordination number, we describe its lower bound $z_{c\min}^\infty(\mu)$, which interpolates smoothly between the isostatic frictionless case...
FIG. 4: The joint force distribution $P_\infty(f^n, f^t)$ at $\mu = \infty$ in (A) 2-D disks packing and (B) 3-D spheres packing. Plots of the probability distribution of normalized normal forces and frictional forces at $\mu = \infty$ in (C) 2-D disks packing and (D) 3-D spheres packing.

$z_c^{\text{min}}(0) = 2D$, and a large $\mu$ limit $z_c^{\text{min}}(\infty)$ slightly above $D + 1$. This confirms that there is no discontinuity at $\mu = 0$. We predict two scalings for small and large friction coefficients as $z_c^{\text{min}}(0) - z_c^{\text{min}}(\mu) \sim \mu^{0.67}$ and $z_c^{\text{min}}(\mu) - z_c^{\text{min}}(\infty) \sim \mu^{-0.35}$.

The statistical mechanics point of view on force and torque balances for random packings thus proves fruitful. Many features of these systems can be inferred from those simple considerations. The use of the cavity technique enables us to tackle this problem with a correct treatment of the disorder, leading to several new results. This formalism will be extended to packings of more general shapes and could be used to predict other properties of disordered packings, like the yield stress for instance. Granular materials are not the only systems subject to force and torque balance, and this constraint is seen in all overdamped systems, among which suspensions at low Reynolds number constitute an important example, both conceptually and practically. We hope that our results will motivate investigations in this direction.

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