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The New Wave Structures to the Fractional Ion Sound and Langmuir Waves Equation in Plasma Physics

Mahmoud A. E. Abdelrahman 1,2,*, S. Z. Hassan 3, R. A. Alomair 3 and D. M. Alsaleh 4

1 Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah 42353, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Department of Mathematics, College of Science and Humanities, Imam Abdulrahman Bin Faisal University, Jubail 35811, Saudi Arabia; szhassan@iau.edu.sa (S.Z.H.); ralomair@iau.edu.sa (R.A.A.)
4 Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, Dammam 34212, Saudi Arabia; dalsaleh@iau.edu.sa
* Correspondence: mahmoud.abdelrahman@mans.edu.eg

Abstract: In this paper, we consider the fractional ion sound and Langmuir waves (FISALWs) equation. We apply the unified solver technique in order to extract some new solutions for the FISALWs equation. The fractional derivative is defined in the sense of a conformable fractional derivative. The proposed solver is based on He’s semi-inverse method and gives beneficial solutions in explicit form. The recital of the method is trustworthy and useful and gives new, more general exact solutions. The constraint conditions for the existence of valid soliton solutions are reported. The enforcement of the presented solutions might be especially interesting in the applications of plasma physics such as bursty waves in cusp regions, Langmuir turbulence, and solar wind. Finally, the proposed solver can be extended to many other models in new physics and applied science.

Keywords: fractional ion sound and Langmuir waves equation; unified solver; conformable derivative; He’s variational method

1. Introduction

Most complex phenomena are modeled and interpreted by the nonlinear fractional or classical partial differential equations. Nonlinear fractional partial differential equations (NFPDEs) are generalizations of classical differential equations of integer order. NFPDEs are very important in many fields of applied science, such as superfluid, chemical engineering, plasma physics, deep water, fiber communications, optoelectronics, and many other fields [1–10]. Therefore, the study of the solutions of NFPDEs is of interest to many researchers. Accordingly, several techniques have been proposed to find analytical and numerical solutions of NFPDEs, such as the $(G'/G)$-expansion technique [11], tanh-sech technique [12], variational iteration technique [13] sub-equation technique [14], modified Kudryashov technique [15], exponential function technique [16], first integral technique [17], and fractional sub-equation technique [18].

There are several definitions for the fractional differential equations, such as Riemann–Liouville and Caputo’s fractional derivatives [19]. He’s fractional derivative [20,21], the local fractional derivative [22], and the Abel–Riemann fractional derivative [23]. Many vital complex phenomena in electrochemistry, electromagnetic, and material science are well described by differential equations of fractional order [24–27]. A physical interpretation of fractional calculus was presented in [28,29]. Khalil et al. [30] presented a vital conformable fractional derivative. This definition is very interesting due to its simplicity and efficiency. Thus, many researchers have introduced several works in light of this definition [31–33]. In the following, we present the basic notion definition of the conformable derivative [30].
Definition 1 ([30]). Consider a function $\Psi : (0, \infty) \to \mathbb{R}$, then the conformable fraction derivative of $\Psi$ of order $\delta$ is

$$T_\delta(\Psi)(t) = \lim_{\epsilon \to 0} \frac{\Psi(t + \epsilon t^{1-\delta}) - \Psi(t)}{\epsilon}, \quad t > 0, \ 0 < \delta \leq 1.$$ 

The conformable fractional derivative obeys:

(i) $T_\delta(a \Phi + b \Psi) = aT_\delta(\Phi) + bT_\delta(\Psi)$, $a, b \in \mathbb{R}$;
(ii) $T_\delta(t^m) = mt^{m-\delta}$, $m \in \mathbb{R}$;
(iii) $T_\delta(\Phi \Psi) = \Phi T_\delta(\Psi) + \Psi T_\delta(\Phi)$;
(vi) $T_\delta\left(\frac{\Phi}{\Psi}\right) = \frac{T_\delta(\Phi)\Phi - \Phi T_\delta(\Psi)}{\Psi^{\delta+1}}$;
(v) If $\Psi$ is differentiable, thus $T_\delta(\Psi)(t) = t^{1-\delta} \frac{d\Psi}{dt}$.

Theorem 1 ([30]). Let $\Phi, \Psi : (0, \infty) \to \mathbb{R}$ be differentiable and also $\delta$-differentiable; hence, the following rule satisfies:

$$T_\delta(\Phi \circ \Psi)(t) = t^{1-\delta} \Psi'(t)\Phi'(\Psi(t)).$$

This paper is concerned with the following FISALWs equation [34,35]:

$$iD_t^\delta E + \frac{1}{2}E_{xx} - \nu E = 0,$$

$$D_t^{2\delta} \nu + v_{xx} - 2(|E|^2)_{xx} = 0, \quad 0 < \delta \leq 1,$$  

$E(x,t)$ is the electrostatic field excitations, and $E_0 e^{-i\omega_0 t}$ denotes the normalized electric field of the Langmuir oscillation, where $\nu$ denotes the normalized density perturbation and $\omega_0$ is the plasma frequency, which depends on $v$. The second equation represents the speed of ion sound equation. Equation (2) investigates the nonlinear phenomenon called Langmuir collapsing wave that produces essential physical information for the strong Langmuir turbulences and understanding of undetermined subsonic and supersonic behavior limits in these turbulences and observations of Langmuir wave in solar radio bursts. The electron temperature of the plasma is computed by calculating the frequency of the hyperbolic and trigonometric function solutions. Alruwaili et al. studied Equation (2) via the Atangana–Baleanu fractional derivative, and they introduced multilwave, periodic cross-kink, rational, and interaction solutions.

The variational method is very important since it can reduce the order of the differential equation to make the model simpler and obtain the optimal solution by the stationary condition. Indeed, He’s variational method is a very powerful tool to construct the solitary wave solutions in different fields of applied science [39–42]. In this work, we introduce a new unified solver technique based on He’s variations technique to solve NFPDEs. This technique gives the complete wave structure of several kinds of NFPDEs. The proposed approach is direct, sturdy, efficacious, and adequate. Our work is motivated to grasp some new solutions of the FISALWs equation by using the unified solver, based on He’s semi-inverse technique [43–45]. These solutions may be applicable in space, bursty Langmuir waves observed in cusps, and solar wind [46,47]. Our results clarify that the proposed technique can be implemented for many other NFPDEs emerging in new physics and applied science.
The organization of the paper is as follows: Section 2 illustrates the unified solver technique. Section 3 constructs new solutions for the fractional FISALWs equation. Section 4 displays the physical interpretation for the presented results, and some 3D graphs are depicted. Finally, some conclusions are drawn in Section 5.

2. Unified Solver Technique

We present the unified solver for the widely used NFPDEs arising in the nonlinear sciences. For a given NFPDE with some physical fields $\Phi(x,t)$, consider the NFPDEs as follows:

$$F(\Phi, D_\delta^\Phi, D_\delta^2\Phi, D_\delta^3\Phi, D_\delta^4\Phi, ..., ) = 0, \quad 0 < \delta \leq 1. \tag{3}$$

Utilizing the wave transformation:

$$\Phi(x,t) = \Phi(\xi), \quad \xi = \frac{x^\delta}{\delta} - \gamma \frac{t^\delta}{\delta}, \tag{4}$$

converts Equation (3) into the following ODE:

$$G(\Phi, \Phi', \Phi'', \Phi''', ....) = 0. \tag{5}$$

Based on He’s semi-inverse technique [43–45], if possible, integrate (5) term by term one or more times. The variational model for the differential Equation (5) can be obtained by the semi-inverse method [48], which reads:

$$J(\Phi) = \int_0^\infty L d\xi, \tag{6}$$

$L$ is the Lagrangian function corresponding to the problem characterized by Equation (5) and given in the form:

$$L = \frac{1}{2} \Phi'^2 - V, \tag{7}$$

where $V$ is the potential function. By the Ritz technique, we can find several forms of solitary wave solutions, such as $\Phi(x,t) = \alpha \operatorname{sech}(\beta \xi)$, $\Phi(x,t) = \alpha \operatorname{csch}(\beta \xi)$, $\Phi(x,t) = \alpha \tanh(\beta \xi)$, and $\Phi(x,t) = \alpha \cotch(\beta \xi)$, and $\Phi(x,t) = \alpha \operatorname{sech}^2(\beta \xi)$, where $\alpha$ and $\beta$ are constants to be determined.

In this work, we look for solitary wave solutions in the forms

$$\Phi(x,t) = \alpha \operatorname{sech}(\beta \xi), \Phi(x,t) = \alpha \operatorname{sech}^2(\beta \xi), \Phi(x,t) = \alpha \tanh(\beta \xi). \tag{8}$$

There are so many models of NFPDEs in applied science and new physics that have been turned into the following nonlinear ordinary differential equation:

$$A_1 \Phi'' + A_2 \Phi^3 + A_3 \Phi = 0, \tag{9}$$

$A_i, i = 1, 2, 3$ are constants. Equation (9) represents a Hamiltonian system with many entertaining applications [49]. As a result of the vital role of Equation (9), we present a sturdy and efficient solver for various classes of the NFPDEs. This solver presents the closed-form solutions to the NFPDEs. Multiplying Equation (9) by $\Phi'$ and integrating with respect to $\Phi$, we obtain the equation:

$$\frac{1}{2} \Phi'^2 + \frac{A_2}{4A_1} \Phi^4 + \frac{A_3}{2A_1} \Phi^2 + A_0 = 0 \tag{10}$$

and $A_0$ is a constant of integration. Thus, Equation (9) can be written in the form:

$$\Phi'' = -\frac{\partial V}{\partial \Phi}, \quad V = -(\gamma_2 \Phi^4 + \gamma_1 \Phi^2 + \gamma_0), \tag{11}$$
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where:

\[ \gamma_2 = -\frac{A_2}{4A_1} \]
\[ \gamma_1 = -\frac{A_3}{2A_1} \]
\[ \gamma_0 = -A_0 \]

(12)

and \( V \) is the potential function (Sagdeev potential).

We apply He’s semi-inverse technique [43–45] to solve Equation (9), which constructs the following variational formulation from Equation (10):

\[ J = \int_0^\infty \left[ \frac{1}{2} \Phi'^2 + \gamma_2 \Phi^4 + \gamma_1 \Phi^2 + \gamma_0 \right] d\xi. \]

(13)

Using a Ritz-like technique, we search for a wave solution in the form given by Equation (8). Substituting Equation (8) into Equation (13) and making \( J \) stationary with respect to \( \alpha \) and \( \beta \) give:

\[ \frac{\partial J}{\partial \alpha} = 0; \quad \frac{\partial J}{\partial \beta} = 0. \]

(14)

Solving simultaneously the system of equations given by (14), we obtain \( \alpha \) and \( \beta \). Consequently, the solitary wave solution given by Equation (8) is well determined.

2.1. The First Family

The first family of solutions is:

\[ \Phi(\xi) = \alpha \sech \theta, \quad \theta = \beta \xi. \]

(15)

By substituting Equation (15) in Equation (13), we obtain:

\[ J = \frac{1}{\beta} \int_0^\infty \left[ \frac{1}{2} \alpha^2 \beta^2 \sech^2 \theta \tanh^2 \theta + \gamma_2 \alpha^4 \sech^4 \theta + \gamma_1 \alpha^2 \sech^2 \theta + \gamma_0 \right] d\theta. \]

Since \( \gamma_0 \) is a constant of integration, we can let \( \gamma_0 = 0 \), and consequently:

\[ J = \frac{\alpha}{12\beta} \left[ 2\alpha \beta^2 + 8\gamma_2 \alpha^3 + 12\gamma_1 \alpha \right]. \]

(16)

Making \( J \) stationary with respect to \( \alpha \) and \( \beta \) gives:

\[ \frac{\partial J}{\partial \alpha} = \frac{1}{12\beta} \left[ 32\gamma_2 \alpha^3 + 24\gamma_1 \alpha + 4\alpha \beta^2 \right] = 0, \]

(17)

\[ \frac{\partial J}{\partial \beta} = -\frac{\alpha}{12\beta^2} \left[ 8\gamma_2 \alpha^3 + 12\gamma_1 \alpha - 2\alpha \beta^2 \right] = 0. \]

(18)

By solving these equations and using Equation (15), the solutions of Equation (9) take the form:

\[ \Phi(\xi) = \pm \sqrt{-\frac{\gamma_1}{\gamma_2}} \sech \left( \pm \sqrt{2\gamma_1} \xi \right) \]

(19)

or

\[ \Phi(\xi) = \pm \sqrt{-\frac{2A_3}{A_2}} \sech \left( \pm \sqrt{-\frac{A_3}{A_1}} \xi \right). \]

(20)

2.2. The Second Family

The second family of solutions takes the form:

\[ \Phi(\xi) = \alpha \sech^2 \theta; \quad \theta = \beta \xi. \]

(21)
Substituting Equation (21) in Equation (13) leads to:

\[ J = \frac{\alpha}{\beta} \int_0^\infty \left[ 2\alpha\beta^2 \text{sech}^2 \theta \tanh^2 \theta + \gamma_2 \alpha^3 \text{sech}^4 \theta + \gamma_1 \alpha \text{sech}^4 \theta + \frac{\gamma_0}{\alpha} \right] d\theta. \]

Assuming that \( \gamma_0 = 0 \), we find that:

\[ J = \frac{2\alpha^2}{105\beta} \left[ 14\alpha^2 \beta^2 + 24\gamma_2 \alpha^3 + 35\gamma_1 \alpha \right]. \quad (22) \]

Making \( J \) stationary with respect to \( \alpha \) and \( \beta \) gives:

\[ 14\alpha^2 \beta^2 + 48\gamma_2 \alpha^3 + 35\gamma_1 \alpha = 0, \quad (23) \]
\[ 14\alpha^2 \beta^2 - 24\gamma_2 \alpha^3 - 35\gamma_1 \alpha = 0. \quad (24) \]

By solving these equations and using Equation (21), the solutions of Equation (9) take the form:

\[ \Phi(\xi) = \pm \sqrt{-\frac{35\gamma_1}{36\gamma_2}} \text{sech}^2 \left( \pm \sqrt{\frac{5}{6}\gamma_1 \xi} \right) \quad (25) \]

or

\[ \Phi(\xi) = \pm \sqrt{-\frac{35 A_3}{18 A_2}} \text{sech}^2 \left( \pm \sqrt{-\frac{5 A_3}{12 A_1}} \xi \right). \quad (26) \]

### 2.3. The Third Family

The third family of solutions takes the form:

\[ \Phi(\xi) = \alpha \tanh(\beta \xi). \quad (27) \]

Substituting the previous equation into Equation (13) implies:

\[ J = \frac{1}{\beta} \int_0^\infty \left[ \frac{1}{2} \alpha^2 \beta^2 \text{sech}^4 \theta + \gamma_2 \alpha^4 \tanh^4 \theta + \gamma_1 \alpha^2 \tanh^2 \theta + \gamma_0 \right] d\theta. \quad (28) \]

Under the condition \( \gamma_0 = -\alpha^2 (\gamma_2 \alpha^2 + \gamma_1) \), we find that:

\[ J = -\frac{\alpha^2}{3\beta} \left[ 4\gamma_2 \alpha^2 + 3\gamma_1 - \beta^2 \right]. \quad (29) \]

Making \( J \) stationary with respect to \( \alpha \) and \( \beta \) gives:

\[ \frac{\partial J}{\partial \alpha} = -\frac{\alpha}{3\beta} \left[ 16\gamma_2 \alpha^2 + 6\gamma_1 - 2\beta^2 \right] = 0, \quad (30) \]
\[ \frac{\partial J}{\partial \beta} = \frac{\alpha^2}{3\beta^2} \left[ 4\gamma_2 \alpha^2 + 3\gamma_1 + \beta^2 \right] = 0. \quad (31) \]

By solving these equations and using Equation (27), the solutions of Equation (9) take the form:

\[ \Phi(\xi) = \pm \sqrt{-\frac{\gamma_1}{2\gamma_2}} \tanh(\pm \sqrt{-\gamma_1} \xi) \quad (32) \]

or

\[ \Phi(\xi) = \pm \sqrt{-\frac{A_3}{A_2}} \tanh \left( \pm \sqrt{\frac{A_3}{2A_1}} \xi \right). \quad (33) \]

### 3. Mathematical Analysis

We applied the unified solver to introduce some vital solutions for the FISALWs equation.
Using the wave transformation [34]:

\[
E(x,t) = e^{i\omega} \phi(\zeta), \quad \eta(x,t) = \nu(\zeta), \quad \zeta = cx + \frac{t^{\delta}}{\delta}, \quad \omega = kx + \frac{\lambda t^\delta}{\delta}.
\]  

Substituting Equation (34) into Equation (2), yields

\[
i(\mu + kc)\phi' = 0, \quad (35)
\]

\[
c^2\phi'' - (2\lambda + k^2)\phi - 2\nu = 0, \quad (36)
\]

\[
(\mu^2 - c^2)\nu'' - 2c^2(\phi^2)'' = 0. \quad (37)
\]

Integrating Equation (37) two times with respect to \(\zeta\), taking the integration constant to be zero, and using Equation (35), we obtain

\[
\nu = \frac{2c^2}{\mu^2 - c^2} \varphi^2 = \frac{2}{k^2 - 1} \varphi^2, \quad (38)
\]

\[
\mu = -kc. \quad (39)
\]

Substituting Equation (38) in Equation (36) gives

\[
L\phi'' + M\phi^3 + N\phi = 0, \quad (40)
\]

where \(L = (k^2 - 1)c^2, M = -4,\) and \(N = -(k^2 - 1)(2\lambda + k^2)\).

### 3.1. Solutions of FISALWs Equation via the Unified Solver Method

In view of the unified solver approach introduced in Section 2, the solutions of Equation (40) are as follows.

#### 3.1.1. The First Family of Solutions

From Equations (20) and (38), we can find that:

\[
\phi_{1,2}(\zeta) = \pm \frac{1}{2} \sqrt{2(1 - k^2)(2\lambda + k^2)} \text{sech}\left( \pm \sqrt{-\frac{(2\lambda + k^2)}{c^2}} \zeta \right), \quad (41)
\]

\[
\nu_{1,2}(\zeta) = -(2\lambda + k^2) \text{sech}^2\left( \pm \sqrt{-\frac{(2\lambda + k^2)}{c^2}} \zeta \right), \quad \zeta = cx + \frac{t^\delta}{\delta}, \quad (42)
\]

where \((2\lambda + k^2) < 0\) and \(k^2 > 1\) for valid soliton solutions. Therefore, the first family of solutions is

\[
E_{1,2}(x,t) = \pm \frac{1}{2} \sqrt{2(1 - k^2)(2\lambda + k^2)} e^{i(kx + \lambda t^\delta)} \text{sech}\left( \pm \sqrt{-\frac{(2\lambda + k^2)}{c^2}} \zeta \right), \quad (43)
\]

\[
\eta_{1,2}(x,t) = -(2\lambda + k^2) \text{sech}^2\left( \pm \sqrt{-\frac{(2\lambda + k^2)}{c^2}} \zeta \right). \quad (44)
\]

#### 3.1.2. The Second Family of Solutions

Similarly, using Equations (26) and (38), one can write

\[
\phi_{3,4}(\zeta) = \pm \sqrt{-\frac{35 (k^2 - 1)(2\lambda + k^2)}{72}} \text{sech}^2\left( \pm \sqrt{\frac{5(2\lambda + k^2)}{12c^2}} \zeta \right). \quad (45)
\]
\[ \nu_{3,4}(\zeta) = \frac{-35(2\lambda+k^2)}{72}\text{sech}^4\left(\pm \sqrt{\frac{5(2\lambda+k^2)}{12c^2}} \zeta\right), \tag{46} \]

where \((2\lambda+k^2) > 0\) and \(k^2 < 1\) for valid soliton solutions. Therefore, the second family of solutions is

\[ E_{3,4}(x,y,t) = \pm \sqrt{-\frac{35(k^2-1)(2\lambda+k^2)}{72}} e^{ikx+\lambda t} \text{sech}^2\left(\pm \sqrt{\frac{5(2\lambda+k^2)}{12c^2}} \zeta\right), \tag{47} \]

\[ \eta_{3,4}(\zeta) = \frac{-35(2\lambda+k^2)}{72}\text{sech}^4\left(\pm \sqrt{\frac{5(2\lambda+k^2)}{12c^2}} \zeta\right). \tag{48} \]

### 3.1.3. The Third Family of Solutions

By Equation (33), we obtain:

\[ \phi_{5,6}(\zeta) = \pm \sqrt{-\frac{(k^2-1)(2\lambda+k^2)}{4}} \tanh\left(\pm \sqrt{-\frac{(2\lambda+k^2)}{2c^2}} \zeta\right), \tag{49} \]

\[ \nu_{5,6}(\zeta) = \frac{-2(2\lambda+k^2)}{2}\tanh^2\left(\pm \sqrt{-\frac{(2\lambda+k^2)}{2c^2}} \zeta\right), \tag{50} \]

where \((2\lambda+k^2) < 0\) and \(k^2 > 1\) for valid soliton solutions. Therefore, the third family of solutions is

\[ E_{5,6}(x,y,t) = \pm \sqrt{-\frac{(k^2-1)(2\lambda+k^2)}{4}} e^{ikx+\lambda t} \tanh\left(\pm \sqrt{-\frac{(2\lambda+k^2)}{2c^2}} \zeta\right), \tag{51} \]

\[ \eta_{5,6}(x,y,t) = \frac{-2(2\lambda+k^2)}{2}\tanh^2\left(\pm \sqrt{-\frac{(2\lambda+k^2)}{2c^2}} \zeta\right). \tag{52} \]

### 4. Results and Discussion

The dynamics of Langmuir and ion sound waves are very important types of waves in plasma; these waves gain this feature since they propagate (acoustic trends) on the surfaces depending on the wave frequency and wave oscillation [50,51]. The wave soliton solution is one of the most intensively studied objects in plasma theory due to its importance in understanding the dynamics of nonlinear waves. We implemented the unified solver approach to find some new soliton solutions for the FISALWs model. We also gave the certain constraint conditions for the existence of the valid soliton solutions. The presented solutions are hyperbolic function solutions. These solutions may be quite valuable in plasma physics, e.g., electromagnetic structures at higher-order plasma frequency harmonics [52], solar wind [52], Langmuir turbulence [53], and the spin-electron acoustic waves [54].

The proposed solver can be used as a box solver for engineers, physicists, and mathematicians. To the best of our knowledge, no previous study has been performed using the proposed solver for solving the FISALWs model. Indeed, this solver can be implemented for solving many equations of the NFPDEs; see, for example, [55–59]. Our results showed that the proposed solver avoids monotonous and complex computations and introduces vital solutions in an explicit form. Finally, these results illustrated the efficiency and reliability of the proposed solver for finding some new solutions for many complicated models in applied science and new physics.

### 5. Conclusions

We applied the unified solver approach to derive some new solutions for the FISALWs equation. The existence of the valid soliton solution depends on the certain constraint
conditions. The proposed solutions possess various potential applications in plasma, for example in bursty waves in cusp regions, Langmuir turbulence, and solar wind. The proposed technique for the FISALWs model is a powerful, effective, apposite, and sturdy method for considering other equations of NFPDEs in applied science.

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