NEW SUFFICIENT GLOBAL OPTIMALITY CONDITIONS FOR LINEARLY CONSTRAINED BIVALENT QUADRATIC OPTIMIZATION PROBLEMS

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Abstract. In this article, we obtain new sufficient global optimality conditions for bivalent quadratic optimization problems with linearly (equivalent and inequivalent) constraints, by exploring the local optimality condition. The global optimality condition can be further simplified when applied to special cases such as the p-dispersion-sum problem and the quadratic assignment problem.

1. Introduction. We begin with the following unconstrained bivalent quadratic optimization problem (UBQP)

$$\begin{align*}
\min & \quad f(x) = \frac{1}{2} x^T Q x + b^T x \\
\text{s.t.} & \quad x \in \{-1, 1\}^n,
\end{align*}$$

where $Q$ is an $n \times n$ real symmetric matrix, $b \in \mathbb{R}^n$. It is a generalization of the maximum cut problem, which is strongly NP-hard [6]. The UBQP has been investigated in numerous papers and has many applications, see [2, 5] and the references therein.

Throughout this paper we will use the following notation. For a vector $a$ and a matrix $A$, Diag$(a)$ and Diag$(A)$ are diagonal matrices with diagonal components $a_i$ and $A_{ii}$, respectively. vec$(A)$ gives the column vector obtained by stacking the columns of $A$ in order. The Kronecker product of matrices $A$ and $B$ is denoted by $A \otimes B$. $e$ denotes the vector with all components equal to one. $A \succeq 0$ means that $A$ is a positive semidefinite matrix. $\lambda_{\min}(A)$ denotes the minimal eigenvalue of $A$. For $n$-dimensional vectors $a$ and $b$, $a \leq b$ denotes $a_i \leq b_i$, $i = 1, \cdots, n$ and min$(a, b)$ is a vector with $i$-th component equal to min$(a_i, b_i)$. $[x]$ denotes the largest integer smaller than or equal to $x$.

Beck and Teboulle [1] derived a sufficient global optimality condition based on convex duality. Recently, it was improved in [15]. They are stated as the following Theorem 1.1 and Theorem 1.2, respectively.

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Theorem 1.1 ([1]). Let \( x \) be a feasible point for Problem (1)-(2). If
\[
\text{Diag}(x) (Qx + b) \leq \lambda_{\text{min}}(Q)e,
\]
then \( x \) is a globally optimal solution for Problem (1)-(2).

Theorem 1.2 ([15]). Let \( x \) be a feasible point for Problem (1)-(2). If
\[
\text{Diag}(x) (Qx + b) \leq \left( \min_{q \in \{1, \ldots, n\}} \frac{\alpha_q}{q} \right) e,
\]
where \( \alpha_q \) is any lower bound on the optimal value of the following problem
\[
\min y^T Qy,
\]
\[
s.t. \quad y^T y = q,
\]
\[
y \in \{-1, 0, 1\}^n,
\]
then \( x \) is a globally optimal solution for Problem (1)-(2).

Theorem 1.1 was later extended by Pinar [13] (see Theorem 1.3) to the following bivalent quadratic optimization problem with linear equivalent constraints:
\[
\min f(x) = \frac{1}{2} x^T Qx + b^T x
\]
\[
s.t. \quad Ax = c,
\]
\[
x \in \{-1, 1\}^n,
\]
where \( Q \) is an \( n \times n \) real symmetric matrix, \( b \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} \) has full row rank and \( c \in \mathbb{R}^m \). Problem (7)-(9) has many applications, e.g., the \( p \)-dispersion-sum problem (PDSP) [14] and the quadratic assignment problem (QAP) [4, 8, 11].

Theorem 1.3 ([13]). Let \( x \) be a feasible point for Problem (7)-(9). If there exists \( z \in \mathbb{R}^m \) which solves
\[
\text{Diag}(x) \left( Qx + b + A^T z \right) \leq \lambda_{\text{min}}(Q)e,
\]
then \( x \) is a global minimizer for Problem (7)-(9).

The above condition (10) can be further strengthened to the linear matrix inequality condition [13]:
\[
\text{Diag}(x) \text{Diag} \left( Qx + b + A^T z \right) \preceq Q.
\]

In this article, we first extend Theorem 1.2 to Problem (7)-(9), which improves Theorem 1.3. We give an example in which our new sufficient optimality condition is satisfied while even the condition (11) is not. For special structured cases such as the \( p \)-dispersion-sum problem and the quadratic assignment problem, we show that the sufficient global optimality conditions can be further simplified. Finally, we establish the corresponding sufficient global conditions for the binary quadratic programming problem with linear inequivalent constraints:
\[
\min f(x) = \frac{1}{2} x^T Qx + b^T x
\]
\[
s.t. \quad Gx \leq f,
\]
\[
x \in \{-1, 1\}^n.
\]
A special case of Problem (12)-(14) is the well-known quadratic knapsack, see [3] and references therein.

The rest of the article is organized as follows. Section 2 presents sufficient global optimality conditions for the bivalent quadratic programming problem with linear
equivalent constraints. Section 3 discusses how the conditions can be simplified for special cases. Section 4 presents sufficient global optimality conditions for the bivalent quadratic programming problem with linear inequivalent constraints. Conclusions are made in section 5.

2. Sufficient optimality conditions for BQP with linear equivalent constraints. First we introduce a kind of locally optimal solutions to Problem (7)-(9).

Definition 2.1. \( x' \in D := \{-1, 1\}^n \cap \{ x : Ax = c \} \) is a \( q \)-distance \((1 \leq q \leq n)\) ring solution of Problem (7)-(9) if it holds that

\[
\begin{align*}
    f(x') \leq & \min_{x} f(x) \\
    \text{s.t.} & \|x - x'\|_0 = q, \\
    x & \in D,
\end{align*}
\]

where \( \|x - x'\|_0 \) is the Hamming distance, i.e., the number of different components between \( x \) and \( x' \). Here, if the optimization problem on the right-hand side is infeasible, we set the minimal objective value to be \(+\infty\).

Lemma 2.2. \( x' \) is a \( q \)-distance ring solution of Problem (7)-(9) if and only if for any \( N' \subseteq N = \{1, 2, \cdots, n\} \) with \( q \) elements satisfying \( \sum_{j \in N'} A_{ij} x'_j = 0 \) \((i = 1, \cdots, m)\), it holds that

\[
\sum_{i \in N'} x'_i \left( \sum_{j \notin N'} Q_{ij} x'_j \right) + \sum_{i \in N'} b_i x'_i \leq 0. \tag{15}
\]

Proof. \( x' \) is a \( q \)-distance ring solution if and only if for any \( N' \subseteq N \) with \( q \) elements that \( \sum_{j \in N'} A_{ij} x'_j = 0 \) \((i = 1, \cdots, m)\),

\[
f(x') = \frac{1}{2} x'^T Q x' + b^T x' \]

\[
= \frac{1}{2} \sum_{i \notin N'} \sum_{j \notin N'} Q_{ij} x'_i x'_j + \frac{1}{2} \sum_{i \in N'} \sum_{j \notin N'} Q_{ij} x'_i x'_j + \sum_{i \in N'} x'_i \left( \sum_{j \notin N'} Q_{ij} x'_j \right) \\
+ \sum_{i \in N'} b_i x'_i + \sum_{i \in N'} b_i x'_i \\
\leq \frac{1}{2} \sum_{i \notin N'} \sum_{j \notin N'} Q_{ij} x'_i x'_j + \frac{1}{2} \sum_{i \in N'} \sum_{j \notin N'} Q_{ij} x'_i x'_j - \sum_{i \in N'} x'_i \left( \sum_{j \notin N'} Q_{ij} x'_j \right) \\
- \sum_{i \in N'} b_i x'_i + \sum_{i \in N'} b_i x'_i \\
= f(x''),
\]

where \( x'' \) with \( (x'')_i = \begin{cases} -x'_i & i \in N' \\ x'_i & i \notin N' \end{cases} \) is a \( q \)-distance neighbor of \( x' \).

From the above computation the following lemma readily follows

Lemma 2.3. Consider Problem (7)-(9). Let \( x \in D \). If

\[
\text{Diag}(x)(Qx + b) \leq \frac{\beta_q e}{q}, \tag{16}
\]

\]
where $\beta_q$ is any lower bound on the optimal value of the following problem

\[
\min y^T Q y, \quad (17)
\]

\[
s.t. \quad y^T y = q, \quad (18)
\]

\[
y \in T := \{ y \in \{-1, 0, 1\}^n : Ay = 0 \}, \quad (19)
\]

($\beta_q = +\infty$ if the (relaxed) constraints (18)-(19) are infeasible), then $x$ is a $q$-distance ring solution of Problem (7)-(9).

Proof. Inequalities (16) imply

\[
\sum_{i \in N'} x'_i (\sum_{j \notin N'} Q_{ij} x'_j) + \sum_{i \in N'} b_i x'_i \leq \beta_q - \sum_{i \in N'} x'_i (\sum_{j \notin N'} Q_{ij} x'_j) \leq \beta_q - \min_{y \in T, y^T y = q} \sum_{i=1}^n \sum_{j=1}^n Q_{ij} y_i y_j \leq 0.
\]

The proof is completed due to Lemma 2.2. \qed

Notice that $x$ is globally optimal if and only if it is a $q$-distance ring solution for all $q = 1, \cdots, n$. Therefore, Lemma 2.3 implies the following new sufficient global optimality condition.

**Theorem 2.4.** Consider Problem (7)-(9). Let $x \in D$ and $\beta_q$ be a lower bound on the optimal value of Problem (17)-(19). If

\[
\text{Diag}(x)(Q x + b) \leq \left( \min_{q \in \{1, \cdots, n\}} \frac{\beta_q}{q} \right) e, \quad (20)
\]

then $x$ is a globally optimal solution for Problem (7)-(9).

Applying Theorem 2.4 to the following equivalent formulation of Problem (7)-(9)

\[
\min \frac{1}{2} x^T Q x + b^T x + z^T (Ax - c) \quad (21)
\]

\[
s.t. \quad Ax = c, \quad (22)
\]

\[
x \in \{-1, 1\}^n, \quad (23)
\]

for any given $z \in \mathbb{R}^m$, we obtain

**Theorem 2.5.** Consider Problem (7)-(9). Let $x \in D$ and $\beta_q$ be a lower bound on the optimal value of Problem (17)-(19). If there exists $z_q \in \mathbb{R}^m$ ($q = 1, \cdots, n$) that

\[
\text{Diag}(x)(Q x + b + A^T z_q) \leq \left( \min_{q \in \{1, \cdots, n\}} \frac{\beta_q}{q} \right) e, \quad (24)
\]

then $x$ is a globally optimal solution for Problem (7)-(9).

Furthermore, Theorem 2.5 can be easily strengthened.

**Theorem 2.6.** Consider Problem (7)-(9). Let $x \in D$ and $\beta_q$ be a lower bound on the optimal value of Problem (17)-(19). If there exists $z_q \in \mathbb{R}^m$ ($q = 1, \cdots, n$) that

\[
\text{Diag}(x)(Q x + b + A^T z_q) \leq \frac{\beta_q}{q} e, \quad q = 1, \cdots, n, \quad (25)
\]

then $x$ is a globally optimal solution for Problem (7)-(9).
We show that Theorem 2.5 implies Theorem 1.3. Consider the following homogeneous trust-region type relaxation of Problem (17)-(19):

\[
\beta_q = \min_{y} y^T Q y \quad \text{s.t.} \quad y^T y = q, \tag{26}
\]

whose optimal solution is the eigenvector corresponding to the minimal eigenvalue of \(Q\), i.e.,

\[
\beta_q = q \cdot \lambda_{\min}(Q). \tag{28}
\]

Therefore, we may choose

\[
\min_{q \in \{1, \ldots, n\}} \frac{\beta_q}{q} = \lambda_{\min}(Q). \tag{29}
\]

To improve the one in (28), we focus on tighter relaxations of problem (17)-(19). We need only consider the nontrivial case when \(q\) is large, since \(\beta_q\) can be exactly calculated by enumeration for small \(q\), for example, \(q = 1, 2\). Again, if the relaxed constraints (18)-(19) are infeasible, set \(\beta_q = +\infty\).

The first is a minimal eigenvalue of the projected Hessian. Consider the following relaxation.

\[
\beta^1_q = \min_{y} y^T Q y \quad \text{s.t.} \quad y^T y = q, \tag{30}
\]

\[
\text{Ay} = 0. \tag{31}
\]

\[
\text{Ay} = 0 \quad \text{if and only if} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} y_2 \quad \text{and Problem (30)-(32) is equivalent to}
\]

\[
\beta^1_q = \min_{y_2} y_2^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} y_2 \quad \text{s.t.} \quad y_2^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} y_2 = q. \tag{33}
\]

Let \(B = B^T \in \mathbb{R}^{(n-m) \times (n-m)}\) satisfy \(B B^T = \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} \) and \(\hat{Q}\) denote the reduced Hessian \(B^{-1} \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} B^{-T}\). Introduce \(z = B^T y_2\) and then Problem (17)-(18) is reduced to

\[
\beta^1_q = \min_{z} z^T \hat{Q} z \quad \text{s.t.} \quad z^T z = q, \tag{35}
\]

whose optimal value is

\[
\beta^1_q = q \cdot \lambda_{\min}(\hat{Q}) = q \cdot \lambda_{\min} \left( B^{-1} \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} -A_1^{-1} A_2 \\ I_{n-m} \end{pmatrix} B^{-T} \right). \tag{37}
\]

It is easy to verify that

\[
\beta^1_q \geq q \cdot \lambda_{\min}(Q). \tag{38}
\]
The second is a further branching strategy. Let \( u \) be the eigenvector corresponding to the minimal eigenvalue of \( \hat{Q} \). Then \( \sqrt{\frac{q}{u^T u}}B^{-T}u \) is the optimal solution to Problem (33)-(34). Suppose there is an index \( i_0 \) such that \((B^{-T}u)_{i_0} \neq 0 \) and \( \frac{q}{u^T u}(B^{-T}u)_{i_0}^2 \neq 1 \), otherwise \( i_0 \in \emptyset \). We obtain a new lower bound:

\[
\beta_q^2 = \min_{z} z^T \hat{Q} z \quad \text{s.t.} \quad z^T z = q, \quad (B^{-T}z)_{i_0} \in \{-1, 0, 1\}. \tag{39}
\]

Observe that if \( z \) is feasible to this problem, then so is \(-z\). For the sake of convenience, let \( b^T_0 z \) denote the \( i_0 \)-th row of \( B^{-T} \), i.e., \((B^{-T}z)_{i_0} = b^T_0 z\). Then \( \beta_q^2 \) ((39)-(41)) is reduced to

\[
\beta_q^2 = \min \{ \beta_{q}^{21}, \beta_{q}^{22} \}, \tag{42}
\]

where

\[
\beta_{q}^{21} = \min_{z} z^T \hat{Q} z \quad \text{s.t.} \quad z^T z = q, \quad b^T_0 z = 1,
\]

\[
\beta_{q}^{22} = \min_{z} z^T \hat{Q} z \quad \text{s.t.} \quad z^T z = q, \quad b^T_0 z = 0.
\]

Let \( U \in \mathbb{R}^{n \times (n-1)} \) satisfy

\[
U^T b_0 = 0, \quad U^T U = I_{n-1}.
\]

Then the equations \( b^T_0 z = 1 \) and \( b^T_0 z = 0 \) are equivalent to \( z = \frac{1}{b_0^2} b^T_0 b_0 + U w \) and \( z = U w \) for \( w \in \mathbb{R}^{n-1} \), respectively. Furthermore, we have

\[
\left( \frac{1}{b_0^2} b_0 + U w \right)^T \left( \frac{1}{b_0^2} b_0 + U w \right) = 1 + w^T w \tag{43}
\]

\[
(U w)^T (U w) = w^T w. \tag{44}
\]

Therefore, \( \beta_{q}^{21} \) and \( \beta_{q}^{22} \) are reduced to the following trust-region subproblems:

\[
\beta_{q}^{21} = \min_{w} \left( \frac{1}{b_0^2} b_0 + U w \right)^T \hat{Q} \left( \frac{1}{b_0^2} b_0 + U w \right) \quad \text{s.t.} \quad w^T w = q - 1,
\]

\[
\beta_{q}^{22} = \min_{w} w^T U^T \hat{Q} U w \quad \text{s.t.} \quad w^T w = q,
\]

respectively. They can be efficiently solved, see [7, 9]. Regarding the tightness, due to the additional constraint (41), we have

\[
\beta_q^2 \geq \beta_q^{21}. \tag{45}
\]

The third approach is requiring the semidefinite programming (SDP) [10, 12] relaxation:

\[
\beta_q^3 = \min_{Z} \text{trace} (QZ) \quad \text{s.t.} \quad Z = q, \quad Z_{ii} \leq 1, \quad i = 1, \cdots , n, \quad \text{trace} (A^T AZ) = 0, \tag{46}
\]

\[
Z \succeq 0, \quad Z = Z^T \in \mathbb{R}^{n \times n}, \tag{47}
\]

where 'trace' denotes the sum of all diagonal elements, \( Z \) is actually a relaxation of \( yy^T, Ay = 0 \) is equivalent to \( y^T A^T Ay = 0 \), and then \( \text{trace}(A^T Ayy^T) = 0 \). It is easy to verify that

\[
\beta_q^3 \geq \beta_q^{21},
\]

where
since the latter is equal to the optimal objective function value of this SDP without
the constraints (48). But \( \beta_3^1 = \beta_1^1 \) since the additional constraints (48) are implied
by the constraints (47) and (50) when \( q = 1 \). This is the other reason that we do
not solve the tight relaxations to obtain \( \beta_q^q \) for small \( q \).

The following is an example in which our new sufficient optimality condition (3)
is satisfied while both the condition (10) and the stronger condition (11) are not.

Example 1. Consider Problem (7)-(9) with the data

\[
Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -1 \\ -4 \\ -1.1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = 1.
\]

whose optimal solution is \( x^* = (1, 1, 1, 1)^T \). It is not difficult to verify the following
equalities:

\[
\lambda_{\min}(Q) = 1 - \sqrt{2}, \quad (53)
\]

\[
\beta_1 = \beta_2 = 0, \quad \beta_3^3 = 1, \quad \beta_4^4 = +\infty. \quad (54)
\]

Then, for any \( z \in \mathbb{R} \), we have

\[
\text{Diag}(x^*)(Qx^* + b + A^T z) = \begin{pmatrix} z \\ 0 \\ -1 \\ -0.1 \end{pmatrix} \not\leq \lambda_{\min}(Q) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (55)
\]

\[
Q - \text{Diag}(x^*)(Qx^* + b + A^T z) = \begin{pmatrix} 1 - z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0.1 \end{pmatrix} \not\succeq 0, \quad (56)
\]

while

\[
\text{Diag}(x^*)(Qx^* + b + A^T 0) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -0.1 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \min \left( \beta_1, \beta_2, \beta_3^3, \beta_4^4 \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (57)
\]

3. Special cases. In this section, we show that Theorem 2.5 can be further simpli-
fi ed for some special structured cases such as the \( p \)-dispersion-sum problem (PDSP)
[14] and the quadratic assignment problem (QAP) [4, 8, 11].

First, we consider the \( p \)-dispersion-sum problem (PDSP) with many applications
in telecommunication and warehouse location. It is formulated as follows:

PDSP : \( \min \ x^T M x \)

\( s.t. \ e^T x = p, \)

\( x \in \{0, 1\}^n. \)

Define \( z = 2x - e \in \{-1, 1\}^n \). PDSP (58)-(60) is equivalent to

PDSP : \( \min \ \frac{1}{4} z^T M z + \frac{1}{2} e^T M z + \frac{1}{4} e^T M e \)

\( s.t. \ e^T z = 2p - n, \)

\( z \in \{-1, 1\}^n. \)
Theorem 3.1. Consider PDSP (58)-(60). Let \( x \) satisfy (59)-(60). If there exists \( z \in \mathbb{R} \) which solves
\[
\text{Diag}(2x - e)(Mx + ze) \leq \left( \min_{q=2,4,\ldots,2\left\lceil \frac{\beta_q}{q} \right\rceil} \frac{\beta_q}{q} \right) e,
\]
where \( \beta_q \) is any lower bound on the optimal value of the following problem
\[
\min y^T M y,
\]
\[
\text{s.t. } y^T y = q,
\]
\[
e^T y = 0,
\]
\[
y \in \{-1,0,1\}^n,
\]
then \( x \) is a globally optimal solution for PDSP (58)-(60).

Proof. Following Theorem 2.5, we need only explain the right hand side of (64). The constraints (67)-(68) imply that any feasible solution \( y \) must have \( 2k \) nonzero elements (1 or \(-1\)) for some integer \( k \), i.e., \( p = 2k \).

Next, consider the quadratic assignment problem (QAP), which is one of the great challenges in combinatorial optimization. For comprehensive surveys of QAPs, we refer to [4, 8, 11]. The general formulation for QAP is
\[
\text{QAP : } \min vec(X)^T Q vec(X) + b^T vec(X) \quad (69)
\]
\[
\text{s.t. } X e = e, \quad (70)
\]
\[
X^T e = e, \quad (71)
\]
\[
X \in \{-1,0,1\}^{n \times n}. \quad (72)
\]
Define \( Z = 2X - ee^T \in \{-1,1\}^{n \times n} \). QAP (69)-(72) is equivalent to
\[
\text{QAP : } \min \frac{1}{4} vec(Z)^T Q vec(Z) + b^T vec(Z) + \frac{1}{4} e^T Q e \quad (73)
\]
\[
\text{s.t. } \left( e_n^T \otimes I_n \right) \left( I_n \otimes e_n^T \right) vec(Z) = (2 - n)e_{2n}, \quad (74)
\]
\[
Z \in \{-1,1\}^{n \times n}. \quad (75)
\]
Notice that the rank of the coefficient matrix \( \left( e_n^T \otimes I_n \right) \left( I_n \otimes e_n^T \right) \) is \( 2n - 1 \). Let \( A \) denote the first \( (2n - 1) \times n^2 \) block, i.e., \( A = \left( e_n^T \otimes I_n \right) \left( I_{(n-1) \times n} \otimes e_n^T \right) \).

Now we establish a sufficient global optimality condition for QAP.

Theorem 3.2. Consider QAP (69)-(72). Let \( X \) satisfy (70)-(72). If there exists \( z \in \mathbb{R}^{2n-1} \) which solves
\[
\text{Diag}(2vec(X) - e) \left( Q vec(X) + b + A^T z \right) \leq \left( \min_{q=4,8,\ldots,2\left\lceil \frac{\beta_q}{q} \right\rceil} \frac{\beta_q}{q} \right) e, \quad (76)
\]
where \( \beta_q \) is any lower bound on the optimal value of the following problem
\[
\min vec(Y)^T Q vec(Y), \quad (77)
\]
\[
\text{s.t. } \text{trace}(Y^T Y) = q, \quad (78)
\]
\[
Ye = Y^T e = 0, \quad (79)
\]
\[
Y \in \{-1,0,1\}^{n \times n}, \quad (80)
\]
then $X$ is a globally optimal solution for QAP (69)-(72).

Proof. Following Theorem 2.5, we need only explain the right hand side of (76). The constraints (79)-(80) imply that any feasible solution $Y$ must have even number of nonzero elements (1 or $-1$) in each row and each column, respectively. Therefore, $Y$ has $4k$ nonzero elements for some integer $k$, i.e., $p = 4k$.

Corollary 1. Consider QAP (69)-(72). Let $X$ satisfy (70)-(72). If there exists $z \in \mathbb{R}^{2n-1}$ which solves

$$\text{Diag}(2\text{vec}(X) - e) (Q\text{vec}(X) + b + A^T z) \leq \lambda_{\min} ((V^T \otimes V^T)Q(V \otimes V)) e,$$

(81)

where

$$V = \begin{pmatrix}
-1/\sqrt{n} & -1/\sqrt{n} & \cdots & -1/\sqrt{n} \\
1 - 1/(n + \sqrt{n}) & -1/(n + \sqrt{n}) & \cdots & -1/(n + \sqrt{n}) \\
-1/(n + \sqrt{n}) & 1 - 1/(n + \sqrt{n}) & \cdots & -1/(n + \sqrt{n}) \\
\vdots & \ddots & \ddots & \vdots \\
-1/(n + \sqrt{n}) & -1/(n + \sqrt{n}) & \cdots & 1 - 1/(n + \sqrt{n})
\end{pmatrix}_{n \times (n-1)},$$

(82)

then $X$ is a globally optimal solution for QAP (69)-(72).

Proof. $Ye = Y^Te = 0$ implies that $Y$ has an eigenvalue equal to 0 and the corresponding eigenvector is $e$, i.e.,

$$Y = \begin{pmatrix} e & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} e^T \\ V^T \end{pmatrix} = VZV^T,$$

(83)

where $Z \in \mathbb{R}^{(n-1) \times (n-1)}$ and $V$ is an $n \times (n-1)$ matrix such that $V^Te = 0$, $V^TV = I_{n-1}$, as exemplified in (82). Then it follows that

$$\text{trace}(YY^T) = \text{trace}(VZV^TVZV^T) = \text{trace}(ZZ^TV^TV) = \text{trace}(ZZ^T).$$

We can choose $\beta_q$ as

$$\min \quad \text{vec}(Y)^T Q\text{vec}(Y),$$

s.t. \quad $\text{trace}(Y^TY) = q,$

$$Ye = Y^Te = 0,$$

(84-86)

which is equivalent to

$$\min \quad \text{vec}(Z)^T (V^T \otimes V^T)Q(V \otimes V)\text{vec}(Z),$$

s.t. \quad $\text{trace}(Z^TZ) = q,$

(87-88)

whose optimal objective function value is clearly $\lambda_{\min} ((V^T \otimes V^T)Q(V \otimes V))$.

4. Sufficient optimality conditions for BQP with linearly inequivalent constraints. In this section, we extend the sufficient optimality conditions to bivalent quadratic optimization problems with linearly inequivalent constraints, i.e., Problem (12)-(14). Since the extension is trivial, we only present the corresponding results without proofs.
Definition 4.1. \( x' \in E := \{-1,1\}^n \cap \{ x : Gx \leq f \} \) is a \( q \)-distance \((1 \leq q \leq n) \) ring solution of Problem (12)-(14) if it holds that
\[
f(x') \leq \min_{x \in E} f(x) \\
\text{s.t. } \|x - x'\|_0 = q,
\]
Again, if the optimization problem on the right-hand side is infeasible, we set the minimal objective value to be \(+\infty\).

Lemma 4.2. \( x' \) is a \( q \)-distance ring solution of Problem (12)-(14) if and only if for any \( N' \subseteq N = \{1,2,\cdots,n\} \) with \( q \) elements satisfying
\[
\sum_{j \in N'} G_{ij} x'_j \geq \sum_{j \not\in N'} G_{ij} x'_j - f_i \quad (i = 1, \ldots, m),
\]

\[
\sum_{i \in N'} x'_i (\sum_{j \not\in N'} Q_{ij} x'_j) + \sum_{i \in N'} b_i x'_i \leq 0. \quad (89)
\]

Lemma 4.3. Consider Problem (7)-(9). Let \( x \in E \). If
\[
\text{Diag}(x)(Qx + b) \leq \frac{\gamma_q(x)}{q} e, \quad (90)
\]
where \( \gamma_q(x) \) is any lower bound on the optimal value of the following problem
\[
\text{min } y^T Qy, \\
\text{s.t. } y^T y = q, \\
y \in T_x := \{ y \in \{-1,0,1\}^n : 2Gy \geq Gx - f \}, \quad (93)
\]
\((\gamma_q(x) = +\infty \text{ if the (relaxed) constraints (92)-(93) are infeasible)}, \) then \( x \) is a \( q \)-distance ring solution of Problem (12)-(14).

Theorem 4.4. Consider Problem (12)-(14). Let \( x \in E \) and \( \gamma_q(x) \) be a lower bound on the optimal value of Problem (91)-(93). If
\[
\text{Diag}(x)(Qx + b) \leq \left( \min_{q \in \{1,\ldots,n\}} \frac{\gamma_q(x)}{q} \right) e, \quad (94)
\]
then \( x \) is a globally optimal solution for Problem (12)-(14).

Applying Theorem 4.4 to the following equivalent formulation of Problem (12)-(14)
\[
\text{min } \frac{1}{2} x^T Qx + b^T x + z^T (Gx - f) \\
\text{s.t. } Gx \leq f, \\
x \in \{-1,1\}^n, 
\]
for any given \( z \in \mathbb{R}^m \) with nonnegative entries, we obtain

Theorem 4.5. Consider Problem (12)-(14). Let \( x \in E \) and \( \gamma_q(x) \) be a lower bound on the optimal value of Problem (91)-(93). If there exists \( z \in \mathbb{R}^m \) and \( z \geq 0 \) which solves
\[
\text{Diag}(x)(Qx + b + G^T z) \leq \left( \min_{q \in \{1,\ldots,n\}} \frac{\gamma_q(x)}{q} \right) e, \quad (98)
\]
then \( x \) is a globally optimal solution for Problem (12)-(14).
Theorem 4.6. Consider Problem (12)-(14). Let $x \in E$ and $\gamma_q(x)$ be a lower bound on the optimal value of Problem (91)-(93). If there exists $z_q \in \mathbb{R}^m$ and $z_q \geq 0$ ($q = 1, \cdots, n$) such that
\[
\text{Diag}(x)(Qx + b + G^Tz_q) \leq \frac{\gamma_q(x)}{q}e, \quad q = 1, \cdots, n, \tag{99}
\]
then $x$ is a globally optimal solution for Problem (12)-(14).

Finally, we discuss the computation of $\gamma_q(x)$. When $q$ is small (for example, $q = 1, 2$), $\gamma_q(x)$ can be exactly calculated by enumeration. Besides, if the relaxed constraints (92)-(93) are infeasible, we set $\gamma_q(x) = +\infty$. Below we need only consider the nontrivial case that $q$ is large and the constraints (92)-(93) are feasible. Applying the semidefinite programming relaxation, we have
\[
\gamma_q(x) = \min \text{ trace } (QZ) \tag{100}
\]
\[
s.t. \quad \text{trace } Z = q, \tag{101}
\]
\[
Z_{ii} \leq 1, \quad i = 1, \cdots, n, \tag{102}
\]
\[
2Gy \geq Gx - f, \tag{103}
\]
\[
\begin{pmatrix} 1 & y \\ y & Z \end{pmatrix} \succeq 0, \tag{104}
\]
\[
Z = Z^T \in \mathbb{R}^{n \times n}. \tag{105}
\]

Each time when checking the optimality of the current candidate solution $x$, we have to calculate $\gamma_q(x)$ ($q = 1, \cdots, n$) firstly, which is quite time-consuming. Next we show that the constraint (103) can be further relaxed to be independent of $x$:
\[
2Gy \geq h - f, \tag{106}
\]
where the $i$-th element of $h \in \mathbb{R}^m$ is obtained in advance by solving the following linear programming problem:
\[
h_i = \min \sum_{j=1}^{n} G_{ij}x_j \tag{107}
\]
\[
s.t. \quad Gx \leq f, \tag{108}
\]
\[
-e \leq x \leq e. \tag{109}
\]

5. Conclusion. In this article, we first consider the bivalent quadratic optimization problem with linear equivalent constraints. We establish new sufficient global optimality conditions, which imply the existing condition. Our approach is to first study the local optimality condition. We show that the quality of the conditions relies on the tightness of the relaxation of a quadratic integer programming. And the existing sufficient optimality condition exactly corresponds to the simple trust-region type relaxation, which is further improved by three strategies developed in this paper. It is illustrated by an example to show that the improvement can be significant. When applied to special cases such as the $p$-dispersion-sum problem and the quadratic assignment problem, the corresponding sufficient optimality conditions are further simplified. Finally, we extend the sufficient optimality conditions to the bivalent quadratic programming problems with linear inequivalent constraints.

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