On existence of an $x$-integral for a semi-discrete chain of hyperbolic type

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Abstract. A class of semi-discrete chains of the form $t_{1x} = f(x, t, t_x, t_1)$ is considered. For the given chains easily verifiable conditions for existence of $x$-integral of minimal order 4 are obtained.

1. Introduction

In the present paper we consider the integrable differential-difference chains of hyperbolic type

$$t_{1x} = f(x, t, t_1, t_x),$$

(1)

where the function $t(n, x)$ depends on discrete variable $n$ and continuous variable $x$. We use the following notations $t_x = \frac{\partial}{\partial x} t$ and $t_1 = t(n + 1, x)$. It is also convenient to denote $t[k] = \frac{\partial^k}{\partial x^k} t$, $k \in \mathbb{N}$ and $t_m = t(n + m, x)$, $m \in \mathbb{Z}$.

The integrability of the chain (1) is understood as Darboux integrability that is existence of so called $x$- and $n$-integrals [1, 4]. Let us give the necessary definitions.

**Definition 1** Function $F(x, t, t_1, \ldots, t_k)$ is called an $x$-integral of the equation (1) if

$$D_x F(x, t, t_1, \ldots, t_k) = 0$$

for all solutions of (1). The operator $D_x$ is the total derivative with respect to $x$.

**Definition 2** Function $G(x, t, t_x, \ldots, t[m])$ is called an $n$-integral of the equation (1) if

$$DG(x, t, t_x, \ldots, t[m]) = G(x, t, t_x, \ldots, t[m])$$

for all solutions of (1). The operator $D$ is a shift operator.

To show the existence of $x$- and $n$-integrals we can use the notion of characteristic ring. The notion of characteristic ring was introduced by Shabat to study hyperbolic systems of exponential type (see [11]). This approach turns out to be very convenient to study and classify the integrable equations of hyperbolic type (see [12] and references there in).

For difference and differential-difference chains the notion of characteristic ring was developed by Habibullin (see [3]-[8]). In particular, in [4] the following theorem was proved.
Theorem 3 (see [4]). A chain (1) admits a non-trivial \( x \)-integral if and only if its characteristic ring is of finite dimension.

A chain (1) admits a non-trivial \( n \)-integral if and only if its characteristic \( n \)-ring is of finite dimension.

For known examples of integrable chains the dimension of the characteristic ring is small. The differential-difference chains with three dimensional characteristic \( x \)-ring were considered in [6].

We consider chains with four dimensional characteristic \( x \)-ring, such chains admit \( x \)-integral of minimal order four. That is we obtain necessary and sufficient conditions for a chain to have a four dimensional characteristic \( x \)-ring. This conditions can be easily checked by direct calculations.

Note that if a chain (1) admits a nontrivial \( x \)-integral \( F(x, t, t, \ldots, t_k) \) and a non trivial \( n \)-integral \( G(x, t, t, \ldots, t_{[m]}) \) its solutions satisfy two ordinary equations

\[
F(x, t, t_1, \ldots, t_k) = a(n),
\]

\[
G(x, t, t, \ldots, t_{[m]}) = b(x)
\]

for some functions \( a(n) \) and \( b(x) \). This allows to solve (1) (see [9]).

The paper is organized as follows. In Section 2 we derive necessary and sufficient conditions on function \( f(x, t, t_1, t_x) \) so that the chain (1) has four dimensional characteristic ring and in Section 3 we consider some applications of the derived conditions.

2. Chains admitting four dimensional \( x \)-algebra.

Suppose \( F \) is an \( x \)-integral of the chain (1) then its positive shifts and negative shifts \( D^k F, \ k \in \mathbb{Z} \), are also \( x \)-integrals. So, looking for an \( x \)-integral it is convenient to assume that it depends on positive and negative shifts of \( t \).

To express \( x \) derivatives of negative shifts we can apply \( D^{-1} \) to the chain (1) and obtain

\[
t_x = f(x, t, t, t_x).
\]

Solving the above equation for \( t_{-1x} \) we get

\[
t_{-1x} = g(x, t, t, t_x).
\]

Let \( F(x, t, t_1, t_{-1}, \ldots) \) be an \( x \)-integral of the chain (1). Then on solutions of (1) we have

\[
D_x F = \frac{\partial F}{\partial x} + t_x \frac{\partial F}{\partial t} + t_{1x} \frac{\partial F}{\partial t_1} + t_{-1x} \frac{\partial F}{\partial t_{-1}} + t_{2x} \frac{\partial F}{\partial t_2} + t_{-2x} \frac{\partial F}{\partial t_{-2}} + \cdots = 0
\]

or

\[
D_x F = \frac{\partial F}{\partial x} + t_x \frac{\partial F}{\partial t} + f \frac{\partial F}{\partial t_1} + g \frac{\partial F}{\partial t_{-1}} + D_f \frac{\partial F}{\partial t_2} + D^{-1} g \frac{\partial F}{\partial t_{-2}} + \cdots = 0.
\]

Define a vector field

\[
K = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + D_f \frac{\partial}{\partial t_2} + D^{-1} g \frac{\partial}{\partial t_{-2}} + \cdots,
\]

then

\[
D_x F = K F.
\]

Note that \( F \) does not depend on \( t_x \) but the coefficients of \( K \) do depend on \( t_x \). So we introduce a vector field

\[
X = \frac{\partial}{\partial t_x}
\]
The vector fields $K$ and $X$ generate the characteristic $x$-ring $L_x$.

Let us introduce some other vector fields from $L_x$.

\[ C_1 = [X, K] \quad \text{and} \quad C_n = [X, C_{n-1}] \quad n = 2, 3, \ldots \]  \hfill (4)

and

\[ Z_1 = [K, C_1] \quad \text{and} \quad Z_n = [K, Z_{n-1}] \quad n = 2, 3, \ldots \]  \hfill (5)

Thus

\[ C_1 = \frac{\partial}{\partial t} + f_{tt} \frac{\partial}{\partial t_1} + g_{tx} \frac{\partial}{\partial t_1} + \ldots \]

\[ C_2 = f_{tx} t_x \frac{\partial}{\partial t_1} + g_{tx} t_x \frac{\partial}{\partial t_1} + \ldots \]

\[ Z_1 = \left( f_{tx} x f_{tx} t_x + f f_{tx} t_x + f f_{tx} t_x + f f_{tx} t_x \right) \frac{\partial}{\partial t_1} + \left( g_{tx} t_x + t_x g_{tx} t_x + g g_{tx} t_x - g t_x g_{tx} \right) \frac{\partial}{\partial t_1} + \ldots \]

and so on.

It is easy to see that if $f_{tx} t_x \neq 0$ then the vector fields $X$, $K$, $C_1$ and $C_2$ are linearly independent and must form a basis of $L_x$ provided $\dim L_x = 4$. By Lemma 3.6 in [6], if $f_{tx} t_x = 0$ and $(f_{tx} x + t_x f_{tx} + f f_{tx} t_x - f - f_{tx} f_{tx}) = 0$ then $\dim L_x = 3$. So in the case $f_{tx} t_x = 0$ we may assume $(f_{tx} x + t_x f_{tx} + f f_{tx} t_x - f - f_{tx} f_{tx}) \neq 0$. Then the vector fields $X$, $K$, $C_1$ and $Z_1$ are linearly independent and must form a basis of $L_x$ provided $\dim L_x = 4$. We consider this two cases separately.

In the rest of the paper we assume that the characteristic ring $L_x$ is four dimensional.

**Remark 4** It is convenient to check equalities between vector fields using the automorphism $D(\cdot)D^{-1}$. Direct calculations show that

\[ DXD^{-1} = \frac{1}{f_x} X, \]

\[ DKD^{-1} = K - \frac{f_x + t_x f_t + f f_{tx} X}{f_{tx}}. \]

The images of other vector fields under this automorphism can be obtained by commuting $DXD^{-1}$ and $DKD^{-1}$.

2.1. $f(x, t, t_1, t_x)$ is non linear with respect to $t_x$.

Let $f(x, t, t_1, t_x)$ be non linear with respect to $t_x$, $f_{tx} t_x \neq 0$. Then the vector fields $X$, $K$, $C_1$ and $C_2$ form a basis of $L_x$. For the algebra $L_x$ to be spanned by $X$, $K$, $C_1$ and $C_2$ it is enough that $C_3$ and $Z_1$ are linear combinations of $X$, $K$, $C_1$ and $C_2$. From the form of the vector fields it follows that we must have

\[ C_3 = \lambda C_2 \quad \text{and} \quad Z_1 = \mu C_2 \]

for some functions $\mu$ and $\lambda$. The conditions for the above equalities to hold are given by the following theorem.

**Theorem 5** The chain (1) with $f_{tx} t_x \neq 0$ has characteristic ring $L_x$ of dimension four if and only if the following conditions hold

\[ D \left( \frac{f_{tx} t_x}{f_{tx} t_x} \right) = \frac{f_{tx} t_x f_{tx} - 3 f_{tx}^2}{f_{tx} f_{tx}^2}. \]  \hfill (6)
The characteristic ring is generated by the vector fields $X, K, C_1, C_2$.

**Proof.** By Remark 4 we have

$$DC_2D^{-1} = \frac{1}{f_t^2} C_2 - \frac{f_{txt}f_t}{f_t^3} C_1 + \frac{f_{txt}f_t}{f_t^4} X$$

$$DC_3D^{-1} = \frac{1}{f_t^3} C_2 - \frac{3f_{txt}f_t}{f_t^4} C_2 - \frac{f_{txt}f_t f_t}{f_t^5} - \frac{3f_{txt}^2}{f_t^6} C_1 + \frac{f_{txt}f_t f_t}{f_t^7} - \frac{3f_{txt}^2}{f_t^8} X$$

$$DZ_1D^{-1} = \frac{1}{f_t} Z_1 - \left( \frac{mf_{txt} + p}{f_t^2} \right) \left( C_1 - \frac{f_t}{f_t^2} X \right),$$

where $p = \frac{f_x + t_x f_t + f f_{t}}{f_t}$ and $m = -\frac{(f_{tx} + t_x f_{tx} + f f_{t,t}) + f_t + f_{tx} f_t}{f_t}.

The equality $C_3 = \lambda C_2$ implies that

$$DC_3D^{-1} = (D\lambda) DC_2D^{-1}. \quad (8)$$

Substituting expressions for $DC_2D^{-1}$ and $DC_3D^{-1}$ into (8) and comparing coefficients of $C_1$, $C_2$ and $X$ we obtain that $\lambda$ satisfies

$$\lambda = f_{tx}(D\lambda) + \frac{3f_{txt}f_t}{f_t}$$

$$\lambda = \frac{f_{txt}f_t f_t - 3f_{txt}^2}{f_{txt} f_t^{2}}.$$

We can find $\lambda$ and $D\lambda$ independently and condition that $D\lambda$ is a shift of $\lambda$ leads to (6).

The equality $Z_1 = \mu C_2$ implies that

$$DZ_1D^{-1} = (D\mu) DC_2D^{-1}. \quad (9)$$

Substituting expressions for $DC_2D^{-1}$ and $DC_3D^{-1}$ into (9) and comparing coefficients of $C_1$, $C_2$ and $X$ we obtain that $\mu$ satisfies

$$\mu - \frac{f_x + t_x f_t + f f_{t}}{f_t} = \frac{(D\mu)}{f_t}$$

and

$$-(f_{tx} + t_x f_{tx} + f f_{t,tx} - f_t - f_{tx} f_{t}) + \frac{f_x + t_x f_t + f f_{t}}{f_t} f_{tx} f_{tx} = -\frac{f_{tx}(D\mu)}{f_t}.$$

We can find $\mu$ and $D\mu$ independently and condition that $D\mu$ is a shift of $\mu$ leads to (7). □

**Remark 6** Let $\dim L_x = 4$ and $f_{txx} \neq 0$. Then the characteristic ring $L_x$ have the following multiplication table
where \( \rho = \lambda \mu + X(\lambda) \) and \( \eta = X(\rho) - K(\mu) \).

**Example 7** Consider the following chain

\[
t_{1x} = \frac{tt_x - \sqrt{t^2_x - M^2(t_1 + t)}}{t_1}
\]

introduced by Habibullin and Zheltukhina [10]. We can easily check that the function

\[
f(t, t_1, t_x) = \frac{tt_x - \sqrt{t^2_x - M^2(t_1 + t)}}{t_1}
\]

satisfies the conditions of Theorem 5. Hence the corresponding \( x \)-algebra is four dimensional. The chain has the following \( x \)-integral

\[
F = \frac{(t^2_1 - t^2)(t^2_1 - t^2_2)}{t^2_1}.
\]

2.2. \( f(x, t, t_1, t_x) \) is linear with respect to \( t_x \).

Let \( f(x, t, t_1, t_x) \) be linear with respect to \( t_x \). Then vector fields \( X, K, C_1 \) and \( Z_1 \) form a basis of \( L_x \). The condition \( f_{t, t_x} = 0 \) also implies that the vector field \( C_2 = 0 \), see [6]. For the algebra \( L_x \) to be spanned by \( X, K, C_1 \) and \( Z \) it is enough that \( Z_2 \) is a linear combination of \( X, K, C_1 \) and \( Z_1 \). From the form of the vector fields it follows that we must have

\[
Z_2 = \alpha Z_1
\]

for some function \( \alpha \). The conditions for the above equality to hold given by the following theorem.

**Theorem 8** The chain (1) with \( f_{t, t_x} = 0 \) has the characteristic ring \( L_x \) of dimension four if and only if the following condition hold

\[
D \left( \frac{K(m)}{m} - m + \frac{f_t}{f_{t_x}} \right) = \frac{K(m)}{m} + m - f_{t_1}, \tag{10}
\]

where \( m = \frac{-f_{tx} + f_{ttx} + f_{ttxt} + f_x + f_{tx}f_{t_1}}{f_{tx}} \). The characteristic ring is generated by the vector fields \( X, K, C_1, Z_1 \).

**Proof.** By Remark 4 we have

\[
DZ_1 D^{-1} = \frac{1}{f_{tx}} Z_1 - \left( \frac{mf_{tx} + p}{f_{tx}^2} \right) \left( C_1 - \frac{f_t}{f_{tx}} X \right),
\]

and

\[
DZ_2 D^{-1} = \left( K \left( \frac{1}{f_{tx}} \right) + \frac{\alpha + m}{f_{tx}} \right) Z_1 + \left( K \left( \frac{m}{f_{tx}} \right) + \frac{mf_t}{f_{tx}^2} - pX \left( \frac{m}{f_{tx}} \right) \right) \left( C_1 - \frac{f_t}{f_{tx}} X \right).
\]
The equality $Z_2 = \alpha Z_1$ implies that

$$DZ_2D^{-1} = (D\alpha) DZ_1D^{-1}. $$

Substituting expressions for $DZ_1D^{-1}$ and $DZ_2D^{-1}$ into (11) and comparing coefficients of $C_1$, $Z_1$ and $X$ we obtain that $\alpha$ and $D(\alpha)$ satisfy

$$K \left( \frac{1}{ft_x} \right) + \frac{m}{ft_x} + \frac{\alpha}{ft_x} = \frac{D(\alpha)}{ft_x} \tag{11}$$

$$K \left( \frac{m}{ft_x} \right) + \frac{mf_t}{f^2_{tx}} = \frac{mD(\alpha)}{ft_x}$$

We can find $\alpha$ and $D(\alpha)$ independently and condition that $D(\alpha)$ is a shift of $\alpha$ leads to (10). □

**Remark 9** Let $\dim L_x = 4$ and $f_{txx} = 0$. Then the characteristic ring $L_x$ have the following multiplication table

|     | $X$ | $K$ | $C_1$ | $Z_1$ |
|-----|-----|-----|-------|-------|
| $X$ | 0   | 0   | $C_1$ | 0     |
| $K$ | $-C_1$ | 0   | $Z_1$ | $\alpha Z_1$ |
| $C_1$ | 0   | $-Z_1$ | 0     | $X(\alpha)Z_1$ |
| $Z_1$ | 0   | $-\alpha Z_1$ | $-X(\alpha)Z_1$ | 0     |

**Example 10** Consider the following chain

$$t_{1x} = t_x + e^{\frac{t_{1x}}{2}}$$

introduced by Dodd and Bullough [2]. We can easily check that the function

$$f(t, t_1, t_x) = t_x + e^{\frac{t_{1x}}{2}}$$

satisfies the conditions of Theorem 8. Hence the corresponding $x$-algebra is four dimensional.

The chain has the following $x$-integral

$$F = e^{\frac{t_{1x}}{2}} + e^{\frac{t_{1x}}{2}}$$

**3. Applications**

The conditions derived in the previous section can be used to determine some restrictions on the form of the function $f(x, t, t_1, t_x)$ in (1).

**Lemma 11** Let the chain (1) have four dimensional characteristic $x$-ring. Then

$$f = M(x, t, t_x)A(x, t, t_1) + t_xB(x, t, t_1) + C(x, t, t_1), $$

where $M$, $A$, $B$ and $C$ are some functions.

**Proof.** Let $f_{t_x t_x} \neq 0$ (if $f_{t_x t_x} = 0$ then $f$ obviously has the above form). Since characteristic $x$-ring has dimension four the condition (6) holds. It is easy to see that (6) implies that $\frac{f_{t_x t_x t_x}}{f_{t_x t_x}}$ does not depend on $t_1$. Hence

$$X(\ln |f_{t_x t_x}|) = M_1(x, t, t_x) \quad \text{and} \quad \ln |f_{t_x t_x}| = M_2(x, t, t_x) + A_1(x, t, t_1).$$

The last equality implies (12). □

We can also put some restrictions on the shifts of the function $f(x, t, t_1, t_x)$ in (1).
Lemma 12 Let the chain (1) have four dimensional characteristic $x$-ring and $f_{t_x t_x} \neq 0$. Then

$$Df = -H_1(x, t, t_1, t_2)t_x + H_2(x, t, t_1, t_2)f + H_3(x, t, t_1, t_2),$$

(13)

where $H_1$, $H_2$ and $H_3$ are some functions.

Proof. Note that the shift operator $D$ and the vector field $X$ satisfy

$$DX = \frac{1}{f_{t_x}} XD.$$  

(14)

The condition (6) can be written as

$$DX(\ln |f_{t_x t_x}|) = \frac{1}{f_{t_x}} X(\ln |f_{t_x t_x}| - \ln |f_{t_x}|^3)$$

Using (14) we get

$$\frac{1}{f_{t_x}} XD(\ln |f_{t_x t_x}|) = \frac{1}{f_{t_x}} X(\ln |f_{t_x t_x}| - \ln |f_{t_x}|^3)$$

which implies that

$$X\left(\ln |f_{t_x}^3 D f_{t_x t_x} t_x| \right) = 0 \quad \text{or} \quad X\left(\frac{f_{t_x}^3 D f_{t_x t_x}}{f_{t_x}} t_x\right) = 0.$$  

Thus $D f_{t_x t_x} = H_1(x, t, t_1, t_2)\frac{f_{t_x t_x}}{f_{t_x}^3}$. Since $D f_{t_x t_x} = DX(f_{t_x})$ and $\frac{f_{t_x t_x}}{f_{t_x}} = -\frac{1}{f_{t_x}} X(\frac{1}{f_{t_x}})$ we can rewrite previous equality using (14) as

$$X\left(D f_{t_x} + H_1(x, t, t_1, t_2)\frac{1}{f_{t_x}}\right) = 0$$

which implies

$$D f_{t_x} = -H_1(x, t, t_1, t_2)\frac{1}{f_{t_x}} + H_2(x, t, t_1, t_2).$$

Writing

$$DX(f) = -H_1(x, t, t_1, t_2)\frac{1}{f_{t_x}} + H_2(x, t, t_1, t_2)\frac{f_{t_x}}{f_{t_x}}$$

and applying (14) as before we get

$$X(Df + H_1(x, t, t_1, t_2)t_x - H_2(x, t, t_1, t_2)f) = 0.$$  

The last equality gives (13). \Box

Note that the equality (13) can be written as

$$t_{2x} = H_2(x, t, t_1, t_2)t_{1x} - H_1(x, t, t_1, t_2)t_x + H_3(x, t, t_1, t_2).$$
References
[1] Darboux G 1915 *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal* 2 (Paris: Gautier Villas)
[2] R K Dodd and R K Bullough 1976 *Proc. R. Soc. London* Ser. A 351 499
[3] Habibullin I T 2005 *SIGMA Symmetry Integrability Geom.: Methods Appl.* 1
[4] Habibullin I T and Pekcan A 2007 *Theoret. and Math. Phys.* 151 781790
[5] Habibullin I 2007 *Characteristic algebras of discrete equations. Difference equations, special functions and orthogonal polynomials* (Hackensack NJ World Sci. Publ.) 249-257
[6] Habibullin I, Zheltukhina N and Pekcan A 2008 *Turkish J. Math.* 32 277-292
[7] Habibullin I, Zheltukhina N and Pekcan A 2008 *J. Math. Phys.* 49 102702
[8] Habibullin I, Zheltukhina N and Pekcan A 2009 *J. Math. Phys.* 50 102710
[9] Habibullin I, Zheltukhina N and Sakieva A 2010 *J. Phys.* A 43 434017
[10] Habibullin I and Zheltukhina N 2014 *Discretization of Liouville type nonautonomous equations* Preprint nlin.SI:1402.3692v1
[11] Shabat A B and Yamilov R I 1981 *Exponential systems of type I and Cartan matrices* Preprint BBAS USSR Ufa
[12] Zhiber A B, Murtazina R D, Habibullin I T and Shabat A B 2012 *Ufa Math. J.* 4 17-85