I. INTRODUCTION

Differential Geometry, in its modern fiber-bundle language, provides the mathematical background for the theories describing the known fundamental interactions. The bundle of frames stands behind General Relativity, while other principal bundles, built up with the respective gauge groups, give a clear picture of the kinematic setup backing electroweak and strong interactions (see Trautman, 1970; Wu and Yang, 1975; Daniel and Viallet, 1980). The picture is nowadays commonplace: geometry supplies the stage-set, on which Lagrangians of phenomenological origin rule over dynamics. Dynamics confers different characters to gravitation, whose Lagrangian is of first order in the curvature, and to the other interactions, whose Lagrangians are of second order in the curvature. But in all cases it is a curvature which appears, and curvature is a quantity derived from a connection. The metric keeps the central role in gravitation, but the basic fields in the other cases are gauge potentials, that is, connections. A splendid experimental record favors the existing theories and justifies the belief that much of their content is of perennial value.

There are, however, some cloudy spots in this sunny landscape. There are too many arbitrary constants and an obstinate lack of unity with respect to great general principles. Gravitation alone is universal, can be locally simulated by a moving frame, has a problematic energy and is power-counting non-renormalizable. Some of the mediating bosons are massless and have problematic charges. Other are massive and have well-defined charges. And there is the question of the meaning to be attributed to spontaneous symmetry breakdown. The presence of a remnant scalar field adds to the difficulties (Gaillard, Grannis and Sciulli, 1999). This blending of experimental success and theoretical bafflement suggests that, though the gauge principle is promised an important role in an eventual final theory, the simple, direct gauge prescription will not have the last saying as it stands. In the search of a more comprehensive framework, string theory, with its ultimate goal of explaining, in principle, “everything”, is the dominating trend.

We want to present here a few more steps of another proposal (Aldrovandi, 1995), which starts from gauge theories and looks for the minimal modifications necessary to enlighten at least some of these difficulties. It takes into account two initial clues. The first is supported by all the experimental evidence and is concerned with the peculiar behavior of the electroweak gauge potentials. The gauge potentials appearing in chromodynamics and isolated electromagnetism, as well as the Christoffel symbols in gravitation, behave strictly as connections, but the vector fields describing real particles in electroweak theory do not. The theory does start with a connection-behaving gauge potential, but then spontaneously breaks the symmetry by introducing an external field. The final combinations, representing the physical fields, do not transform as connections. This leads to the second clue, more mathematical in nature: when a gauge field ceases to behave like a connection, the whole geometric picture provided by the underlying bundle is blurred.

What happens to the bundle picture when a connection, or part of it, adopts an abnormal behavior?

The connection adjoint behavior is essential to the bundle picture. On the bundle tangent spaces, it is reflected in the commutators of the vector fields coming from the base manifold (external space, spacetime) and from the structure (internal, gauge space) group. Vector fields are derivatives, and a connection allows mixing internal and external vectors to produce more general, covariant derivatives, while preserving the bundle makeup. This preservation comes from the connection adjoint behavior. Any deviation from that behavior changes the whole picture, and the electroweak physical fields do deviate. Some encouraging results have been obtained years ago, in which an abnormal behavior of a gauge potential was shown to engender fields strongly suggestive of linear connections with
their curvatures and torsions (Aldrovandi 1, 1991), hinting thereby to a relationship with gravitation. We intend here to present some new results on the subject, valid when the gauge group has, as in the Weinberg–Salam theory, dimension 4.

We start (section II) with a formal compact on Lie algebra extensions. In this section we also examine the behavior of the Lie algebra of fields on a manifold under changes of basis. To alleviate notation we shall, as a rule, omit projections, their differentials and corresponding pull–backs. In section III we introduce an enlarged concept of change of basis in the principal fiber bundle and we apply them successively to the simplest conceivable geometric configuration and obtain 3 kinds of commutation relations: those of a gauge theory, those of an extended gauge theory and those of a gravitational model. Non-covariant derivatives, akin to those appearing in electroweak theory, turn up naturally in the extended formalism. In Section IV we begin discussing which aspects of the geometric picture can still be retained in the presence of anomalous connections. When the base manifold and the gauge group have the same dimension, as is the case involving spacetime and the electroweak theory, a tetrad–like field is naturally introduced to represent the isomorphism of the underlying vector spaces of the tangent field algebras. The object measuring the breaking of the bundle structure acquires the aspect of an external, linear connection, which preserves the metric defined by the tetrad and is endowed with curvature and torsion. Thus, the same objects of usual geometry are found and strongly suggest a relation to gravitation. Section V is devoted to show that such objects have the expected geometrical properties and lead to reasonable dynamical equations. Many results previously found for the gauge sector can be obtained by assuming the persistence of the duality symmetry and, for the gravity sector, by a procedure analogous to that used in General Relativity. The final section sums up the results and the many still unsolved problems.

II. EXTENSIONS OF TANGENT ALGEBRAS

We shall find it necessary to call attention to a certain number of elementary facts, and profit to introduce notation through an overview of well-known notions. A Lie algebra is a vector space on which a binary internal operation is defined which is antisymmetric and satisfies the Jacobi identity. The operation will be indicated by the commutator \([Y_\alpha, Y_\beta]_V = f^{\gamma}_{\alpha\beta}Y_\gamma\). The numbers \(f^{\gamma}_{\alpha\beta}\) are the structure constants of \(V\). We shall sometimes indicate the algebra by one of its basis, as in \(V' = \{Y_\alpha\}\).

In order to discuss the extension of a Lie algebra \(L\) by another Lie algebra \(V\), notice beforehand that the direct sum \(E = L \oplus V\) of the underlying vector spaces \(L\) and \(V\) is always defined. To extend \(L\) by \(V\) means, in general terms, to give an answer to the following question: when and how can we combine \(L\) and \(V\) to build another Lie algebra \(E'\) with underlying vector space \(L \oplus V\)? In the generic case, many answers are possible, provided \(L'\) has a representation acting on \(V\). Two main points should be specified: (i) the insertion of the algebras in the enlarged space \(E\) and (ii) the relationship between the algebras after the insertion.

We shall be interested in extensions involving the algebras of vector fields on differentiable manifolds. The pattern introduced below is closely related to that present on the total manifold \(P\) of a principal bundle (Kobayashi and Nomizu, 1963).

Let \(P\) be a differentiable manifold. It will have a tangent space \(T_pP\) at each point \(p \in P\). A vector field \(X\) is a differentiable choice of a vector \(X_\mu\) at each \(T_pP\). In general, making such a choice is only possible locally, that is, on an open neighborhood of each point \(p\). For that reason all the discussion which follows will be purely local in character. If the manifold is \(C^\infty\), \(X\) will act on a space \(\mathbb{R}(P)\) of infinitely differentiable real functions on \(P\).

The set of all vector fields on \(P\) constitutes an infinite Lie algebra \(\Xi(P)\). Consider a Lie group whose Lie algebra \(G\) has generators \(J_\mu\) satisfying the commutation rules

\[
[J_\mu, J_\nu] = f^\lambda_{\mu\nu}J_\lambda .
\]

When \(G\) acts on \(P\) as a transformation group, there is a representation \(\rho\) of its generators by fields on \(P\). This means (Aldrovandi and Pereira, 1995) that \(\rho\) chooses, for each \(J_\mu\), a representative field \(Y_\mu \in \Xi(P)\):

\[
\rho : G' \rightarrow \Xi(P)
\]

\[
\rho : J_\mu \rightarrow Y_\mu = \rho(J_\mu) .
\]
The representation \( \rho \) will be a linear representation when the representative fields have the same commutation rules as the fields they represent:

\[
[Y_\mu, Y_\nu]_{\Xi(P)} = f^\lambda_{\mu\nu} Y_\lambda .
\]  

(3)

Suppose that a first representative algebra \( L' = \{ Y_\mu \} \) is given around a point \( p \) on \( P \), with a number \( d < \text{dim } P \) of generators. Consider also a linear representation, also around \( p \), of another algebra \( V' \), locally given by a number \( n = \text{dim } P - d \) of fields \( X_a \) with commutations

\[
[X_a, X_b]_{\Xi(P)} = f^c_{ab} X_c .
\]  

(4)

If all the involved fields \( Y_\mu \) and \( X_a \) are linearly independent, the set \( \{ Y_\mu, X_a \} \) constitutes a local basis around \( p \). Notice that, once an algebra is represented by vectors at a point \( p \in P \), its structure constants can become point-dependent (structure coefficients) when these vectors are extended into vector fields around \( p \). As a last basic assumption, suppose the commutation table in that basis to have the form

\[
[Y_\mu, Y_\nu]_{\Xi(P)} = f^\lambda_{\mu\nu} Y_\lambda - \beta^a_{\mu\nu} X_a ;
\]  

(5)

\[
[Y_\mu, X_a]_{\Xi(P)} = C^b_{\mu a} X_b ;
\]

\[
[X_a, X_b]_{\Xi(P)} = f^c_{ab} X_c .
\]

\( \beta^a_{\mu\nu} \) is a 2-form with values in the \( V' \) sector. It characterizes the deviation from the linearity of the algebra \( \{ Y_\mu \} \), caused by its association with the algebra \( \{ X_a \} \). The latter, by the above relations, is unaffected: it is simply included in \( E' \), and its structure coefficients remain constant:

\[
[X_a, X_b]_{\Xi(P)} = [X_a, X_b]_V = f^c_{ab} X_c .
\]

The middle expression in (3) says that the result of any action of \( L' \) on \( V' \) stays in \( V' \). For each fixed \( \mu \), the field \( Y_\mu \) is represented on the \( X_a \)'s by the matrix \( C_\mu \) whose entries are the coefficients \( C^b_{\mu a} \). The algebra \( E' \) specified by (4) is an extension of the representative field algebra of \( L' \) by the representative field algebra of \( V' \).

An extension is trivial when there is no departure from linearity, that is, when \( \beta^a_{\mu\nu} = 0 \). The extension is a direct product when the fields \( Y_\mu \) act on the \( X_a \)'s by the null representation, that is, when \( C^b_{\mu a} = 0 \). This will be a necessary (but not sufficient) condition for the geometry of gauge theories.

The compound so obtained depends, thus, on the pair \( (C^b_{\mu a}, \beta^a_{\mu\nu}) \). The extended algebra should be a Lie algebra, so that we impose the Jacobi identities on the fields obeying (3). Three conditions come out, which must be respected by any pair \( (C^b_{\mu a}, \beta^a_{\mu\nu}) \):

\[
Y_\mu (\beta^a_{\nu\sigma} + Y_\nu (\beta^a_{\mu\nu}) + Y_\sigma (\beta^a_{\mu\sigma}) + C^a_{\nu c} \beta^c_{\sigma \mu} + C^a_{\sigma c} \beta^c_{\mu \nu} + C^a_{\mu c} \beta^c_{\nu \sigma}) + f^\rho_{\mu \nu} \beta^a_{\sigma \rho} + f^\rho_{\nu \mu} \beta^a_{\sigma \rho} + f^\rho_{\sigma \mu} \beta^a_{\rho \nu} = 0 ;
\]

(6)

\[
Y_\mu (C^a_{\nu b}) - Y_\nu (C^a_{\mu b}) + C^a_{\mu c} C^c_{\nu b} - C^a_{\nu c} C^c_{\mu b} - f^\rho_{\mu \nu} C^a_{\rho b} - X_b (\beta^a_{\mu\nu}) - \beta^c_{\mu\nu} f^a_{bc} = 0 ;
\]

(7)

\[
X_a (C^c_{\mu b}) - X_b (C^c_{\mu a}) - C^d_{\mu a} f^c_{bd} + C^c_{\mu d} f^d_{ba} + C^d_{\mu b} f^c_{ad} = 0 .
\]

(8)

An extension is central when \( \beta^a_{\mu\nu} X_c \) has all its elements in the center of the algebra \( V' \). In particular, it follows from (7) that every direct product \( (C^a_{\mu b} = 0) \) is a central extension. In effect, in that case

\[
[X_b, \beta^a_{\mu\nu} X_a]_{\Xi(P)} = \{ X_b (\beta^c_{\mu\nu}) + f^c_{ba} \beta^a_{\mu\nu} \} X_c = 0 .
\]

(9)

Let us examine what happens to the above commutation tables under a change of basis. Starting from the basis \( \{ Y_\mu, X_a \} \) on the whole manifold \( P \), we introduce the particular transformationss
\[ Y'_\mu = Y_\mu - \alpha^c_\mu(x)X_c, \] (10)

where the \( \alpha^{\alpha}_\mu \)'s are point-dependent objects on \( P \). In the applications we have in mind \( P \) will be the total space of a bundle with spacetime as base manifold. The fields \( Y_\mu \) will represent translations on spacetime, so that \( f^\rho_{\mu\nu} = 0 \) in (\ref{10}), (\ref{11}) and (\ref{12}). Notice that the non-linearity indicator \( \beta^{\alpha}_{\mu\nu} \) is not an anholonomy coefficient for \{\( Y_\mu \}\), as it points to other directions in the algebra. Notice that we consider (10) as a change of basis on the whole local algebra of vector fields on \( P \). The new set of commutation relations is (we also drop the index “ \( \Xi(P) \)” from now on):

\[ [Y'_\mu, Y'_\nu] = -\beta^{\alpha}_{\mu\nu}X_a; \] (11)
\[ [Y'_\mu, X_a] = C^{\alpha}_{\mu a}X_b; \] (12)
\[ [X_a, X_b] = f_c^{ab}X_c, \] (13)

with new coefficients given by

\[ C^{\alpha}_{\mu a} = C^b_{\mu a} - \alpha^c_\mu f^b_{ca} + X_a(\alpha^b_\mu), \] (14)
\[ \beta^{\alpha}_{\mu\nu} = \beta^{\alpha}_{\mu\nu} + K^a_{\mu\nu}, \] (15)

and

\[ K^a_{\mu\nu} = Y'_\mu \alpha^a_\nu - Y'_\nu \alpha^a_\mu + \alpha^b_\mu X_b(\alpha^a_\nu) - \alpha^b_\nu X_b(\alpha^a_\mu) \]
\[ + \alpha^b_\nu C^a_{\mu b} - \alpha^b_\mu C^a_{\nu b} + f^a_{bc} \alpha^b_\mu \alpha^c_\nu. \] (16)

Relations (14)-(16) are such that the forms of the Jacobi identities are preserved under (10). This is important, because as we shall see later, the field equations will come from Jacobi Identities.

### III. CHANGES OF BASIS

The simple scheme of basis transformation presented above can, if we start from a trivial initial algebra, engender 3 types of algebra: that of a gauge theory, the extension given above or the forthcoming extended algebra of section IV, and the algebra corresponding to a gravitational model. Assuming the validity of the duality prescription applied to the Bianchi identities, we can also obtain the corresponding dynamics of each theory.

We shall take as starting field configuration that appearing on a fiber bundle whose structure group \( G \) has Lie algebra \( G' = \{X_a\} \), and whose base manifold is spacetime represented by the trivial holonomic basis \{\( \partial_\mu \}\}. The set of commutation relations is

\[ [\partial_\mu, \partial_\nu] = 0; \]
\[ [X_a, \partial_\mu] = 0; \] (17)
\[ [X_a, X_b] = f_c^{ab}X_c. \]

It represents a trivial and direct-product extension of the translation algebra by \( G' \), or vice-versa. Physically, it corresponds to a theory without interaction.

Let us first make in (17) a change of basis

\[ X_\mu = \partial_\mu - \alpha^a_\mu X_a, \] (18)

imposing that it preserves the direct-product character. It leads to

\[ [X_\mu, X_\nu] = -\beta^{\alpha}_{\mu\nu}X_a. \]
\[ [X_a, X_\mu] = 0 ; \] (19)
\[ [X_a, X_b] = f^c_{ab} X_c . \] 

It follows from (14) that
\[ X_a (\alpha_{\mu}^b) = f^b_{ca} \alpha_{\mu}^c . \] (20)

This behavior characterizes \( \alpha \) as a connection, or an adjoint-behaved 1–form. It may seem that a derivative, vacuum term is missing, but we are working on the bundle and the vacuum term only comes out when the connection is pulled–back to spacetime by a section.

From (15) and (16) we obtain the expression for the non–linearity indicator:
\[ \beta_{\mu\nu} = \partial_\mu \alpha_{\nu}^a - \partial_\nu \alpha_{\mu}^a + f_{bc}^a \alpha_{\mu}^b \alpha_{\nu}^c . \] (21)

Since in a direct product the extension is central, we must have
\[ [X_a, \beta_{\mu\nu} X_a] = 0 , \] (22)
and consequently
\[ X_a (\beta_{\mu\nu}) = f^c_{ba} \beta_{\mu\nu} . \] (23)

This condition, which can be equally obtained from (7), says that also \( \beta \) belongs to the adjoint representation of the group whose generators are represented by \( X_a \).

The above algebraic configuration is just the structure appearing in a gauge theory, where \( \beta \) is the field strength of the gauge potential \( \alpha \). The change of basis (18) corresponds to the covariant derivative introduced in gauge theories by the minimal coupling prescription.

Gauge field dynamics can be obtained via the duality prescription: the sourceless field equations are written just as the Bianchi identities, but applied to the dual of the field strength. This dual depends on the metric. Recall that, of Maxwell’s equations, one pair is metric–insensitive (they are Bianchi identities) while the other is metric–dependent (they are the real dynamical equations). In principle, any metric which is preserved by the derivation will do, but different metrics lead to inequivalent equations. We simply assume the existence of such a metric. We obtain the field equations for \( \alpha \) by first finding the Jacobi identity for three fields \( X_\mu, X_\nu, X_\rho \) in algebra (19) – which gives a Bianchi identity – and then applying the duality prescription. The Yang–Mills equations come out:
\[ X_\mu \beta^{a\mu\nu} = 0 . \] (24)

From the point of view of the theory of algebra extensions, the next natural step would be to break the direct product in (19) by another change of basis,
\[ X'_\mu = X_\mu - \gamma_{\mu}^a X_a \] (25)
and investigate the kind of physical theory the resulting configuration can be associated to. Expression (25) leads to the following commutation relations
\[ [X'_\mu, X'_\nu] = - \beta_{\mu\nu}^a X_a ; \]
\[ [X'_\mu, X_a] = C_{\mu\nu}^c X_c ; \]
\[ [X_a, X_b] = f_{ab}^c X_c , \] (26)
which just corresponds to the extended theory of the previous section. Now, it follows from (14) that
\[ X_b (\gamma_{\mu}^a) = f_{cb}^a \gamma_{\mu}^c + C_{\mu\nu}^{ab} . \] (27)

Comparison with (20) shows that \( C_{\mu\nu}^{ab} \) measures the deviation from covariant behavior of the object \( \gamma_{\mu}^a \) appearing in (25). With the help of (18), we can express (25) as
\[ X'_\mu = \partial_\mu - \sigma_{\mu}^a X_a \] (28)
with \( \sigma_{\mu}^a \equiv (\alpha_{\mu}^a + \gamma_{\mu}^a) \). We shall call (28) a generalized derivative. In fact we shall, from now on, give that name to each derivative which is not the standard gauge–covariant derivative. The behavior of \( \sigma \) under the group action is
The new non-linearity indicator $\beta^{a\mu} \nu$ can be obtained from (13) using (16) and (22):

$$
\beta^{a\mu} \nu = \partial_\mu \sigma^a \nu - \partial_\nu \sigma^a \mu + f^{f}_{\alpha \beta} \sigma^\alpha \mu \sigma^\beta \nu - C^a_{\alpha \mu} \sigma^\alpha \nu + C^a_{\nu \mu} \sigma^\nu \mu
$$

(30) $C^a_{\nu \mu} = - C^a_{\mu \nu})$. This is the general expression for the deviation from linearity in the presence of an object with behavior given by (29). The behavior of $\beta^\nu$ under the group action is fixed by the Jacobi identity (8), replacing $Y_\mu$ by $X'_{\mu}$ and $C$ by $C'$.

Dynamics associated to algebra (26) is obtained by applying the duality prescription in a way analogous to that leading to the Yang–Mills equations. The Jacobi identity involving three fields $X'_{\mu}$ in (23) is

$$
X'_{\mu}(\beta^{a\mu} \nu) - C^a_{\nu \mu} \beta^{c\nu} \mu + X'_{\sigma}(\beta^{a\mu} \nu) - C^a_{\sigma \nu} \beta^{c\sigma} \mu
$$

$$
+ X'_{\nu}(\beta^{a\sigma} \mu) - C^a_{\nu \sigma} \beta^{c\nu} \sigma = 0 .
$$

(31) Applying this expression to the dual of $\beta^{a\mu} \nu$, the field equations turn out to be

$$
X'_{\mu} \beta^{a\mu} \nu - C^a_{\nu \mu} \beta^{d\mu} \nu = 0 .
$$

(32) These equations are, of course, linked to the choice of $C'$, which is constrained by the Jacobi identity (8).

The set of commutators (26) can be obtained directly from (17) by the basis change (28). The above two-step procedure is, however, appropriate to show how it can be attained from the algebraic scheme of a gauge theory. The 1-form $\sigma^a \mu$, appearing in the generalized derivative can be seen as a connection deformed by the addition of a non–covariant form (Aldrovandi 2, 1991).

We can infer using (30) in (32) that a mass term for $\sigma$ can appear. Thus, this second change of basis (or a change of basis in a gauge configuration) leads to a theory with massive vector fields which do not behave like connections. This is what happens in the Weinberg-Salam model.

Another change of basis may be introduced as follows. Going back to (31) we see that it has the form of a Bianchi identity for a still more general, enlarged derivative $X'^a_{\mu}$, which can be defined by its action on a indexed object $Z^c$ as

$$
X'^a_{\mu}(Z^c) = X'_{\mu}(Z^c) - C^{c\nu}_{\mu}(Z^a) .
$$

(33) To be acceptable as a derivative, $X'^a_{\mu}$ must obey the Leibniz rule, which leads to some interesting consequences. For example, for a scalar of type $Z^a Z_a$,

$$
X'^a_{\mu}(Z^a Z_a) = X'_{\mu}(Z^a Z_a) .
$$

(34) For a lower–indexed object,

$$
X'^a_{\mu}(Z^c) = X'_{\mu}(Z^c) + C^{c\nu}_{\mu}(Z^c) .
$$

(35) and for a mixed product,

$$
X'^a_{\mu}(Z^d J_c) = X'_{\mu}(Z^d J_c) + C^{c\nu}_{\mu}(Z^d J_c) - C^{c\nu}_{\mu}(Z^d J_c) .
$$

(36) Expression (33) leads to the commutators

$$
[X'^a_{\mu}, X'^a_{\nu}](Z^c) = - \beta^{a\mu} \nu X_a(Z^c) - R^{d\mu}_{a\nu} Z^a ;
$$

(37)

$$
[X'^a_{\mu}, X_a](Z^c) = X_a(C^{c\mu}_{d\nu})Z^d ,
$$

(38) where $\beta^{a\mu} \nu$ is given by (31) and

$$
R^{c\mu}_{a\nu} = X'_{\mu} C^{c\nu}_{a\nu} - X'_{\nu} C^{c\mu}_{a\nu} - C^{c\mu}_{a\mu} C^{d\nu}_{a\nu} + C^{c\mu}_{a\nu} C^{d\nu}_{a\mu} .
$$

(39) The relation between $C'$ and its algebraic derivative is given by Jacobi identity (8).

Besides the same non-linear coefficient $\beta^{a\mu} \nu$ appearing in (28), the extra non–linear term $R^{c\mu}_{a\nu}$ turns up. The relationship between these coefficients is provided by the Jacobi identity for the fields $X_a, X'_{\mu}, X'_{\nu}$:

$$
X_b(\beta^{a\mu} \nu) + f^{a}_{bc} \beta^{c\mu} \nu + R^{a\mu}_{b\nu} = 0 .
$$

(40) The dynamics corresponding to configuration (34) and (38) is examined in the next two sections.
The commutator (37) can then be rewritten as
\[ \text{commutator of (26), we obtain} \]
\[ C \]
which shows that
\[ \text{This is the correct expression of the curvature of a connection} \]
where now
\[ a \text{ tetrad, with} \]
\[ \beta^a_{\mu\nu} \text{ term in (37) should be a torsion, or an anholonomy, but is not: for that, it should have values along } X'^{\prime \mu}. \]
Under the assumption that the dimensions of the two algebras are the same, a solution to these problems comes by
\[ \text{the isomorphism in view would actually be between the tangent spaces, and should hold at each point of the manifold. Provided some reasonable differentiability conditions are met, the set } \{H^a_\mu\} \text{ will be similar to a tetrad field. We shall use for the inverse the usual tetrad notation, so that } H^a_\mu H_b^\mu = \delta^a_b \text{ and } H^a_\mu H^\nu_a = \delta^\nu_\mu. \]
Applying (41) to the second commutator in (26) we obtain
\[ X'^{\prime \mu} = H^a_\mu X_a. \]

The mapping described by \( H^a_\mu \) should be invertible. If we have spacetime in mind, the group should be itself 4-dimensional. The isomorphism in view would actually be between the tangent spaces, and should hold at each point of the manifold. Provided some reasonable differentiability conditions are met, the set \( \{H^a_\mu\} \) will be similar to a tetrad field. We shall use for the inverse the usual tetrad notation, so that \( H^a_\mu H_b^\mu = \delta^a_b \) and \( H^a_\mu H^\nu_a = \delta^\nu_\mu. \) Applying (41) to the second commutator in (26) we obtain
\[ X_a H^d_\mu = f^d_{\epsilon \alpha} H^c_\mu - C^{\epsilon \mu}_{\rho d}. \]

A brief calculation leads to
\[ [X'^{\prime \mu}, X'^{\prime \nu}] = -\beta^{\prime \mu}_{\rho \nu} X'^{\prime \rho}, \]
with
\[ \beta^{\prime \mu}_{\rho \nu} = \beta^a_{\mu \nu} H^a_\rho, \]
showing \((-\beta^{\prime \mu}_{\rho \nu})\) in the role of the non-holonomy coefficient for the basis \( \{X'_\mu\}. \)

Taking (41) into (34) we obtain the relation between \( X_a \) and \( X'^{\prime \mu}: \)
\[ X_a Z^c = H^\mu_a (X'^{\prime \mu} Z^c + C^{\mu c}_{\beta \mu} Z^\beta) . \]
The commutator (37) can then be rewritten as
\[ [X'^{\prime \mu}, X'^{\prime \nu}] (Z^c) = -\beta^{\prime \mu}_{\rho \nu} X'^{\prime \rho} Z^c - R^c_{\alpha \mu \nu} Z^\alpha, \]
where now
\[ R^c_{\alpha \mu \nu} = X'^{\prime \mu} C^{\alpha c}_{\alpha \nu} - X'^{\prime \nu} C^{\alpha c}_{\alpha \mu} - C^{\alpha c}_{\alpha \mu} - C^{\alpha c}_{\beta \mu} C^{\beta \nu}_{\nu \alpha} + C^{\alpha c}_{\beta \mu} C^{\beta \nu}_{\nu \alpha} + \beta^{\prime \rho}_{\mu \nu} C^{\alpha c}_{\alpha \rho} . \]
This is the correct expression of the curvature of a connection \( C' \) in basis \( \{X'_\mu\} \) (Nakahara, 1990). It can be shown that the commutator in (16), if applied to a mixed object with internal and external indices, only acts on those internal.

Let us examine some more properties of the candidate–connection \( C' \). Taking \( X_a = H^\rho_\alpha X'_\mu \) into the second commutator of (24), we obtain
\[ C^{\alpha \mu}_{\rho \nu} = H^\alpha_\lambda C^{\lambda \nu}_{\rho \mu} H^\nu_a + H^\nu_a X'_\mu (H^\rho_\nu), \]
which shows that \( C' \) behaves, under the the action of \( H^\rho_\alpha, \) as a connection of the linear group would behave under a tetrad, with \( C^{\alpha \mu}_{\nu \rho} = \beta^{\alpha \mu}_{\nu \rho}. \) A curious consequence is that
\[ X'^{\prime \lambda}_{\alpha \rho} \beta^{\prime \rho}_{\mu \nu} = X'^{\prime \lambda}_{\alpha \rho} \beta^{\alpha \mu}_{\nu \rho}. \]
The torsion tensor is \( T^\rho_{\mu \nu} = -\beta^{\prime \rho}_{\mu \nu}. \) From (42) and (18) it can be written as
\[ T_{\rho \mu \nu} = H_{a \rho}^a (X_{\rho \mu}^a H_{a \nu}^a - X_{\rho \nu}^a H_{a \mu}^a + f^{a \beta \gamma}_{b \nu} H_{a \mu}^b H_{b \rho}^\gamma). \]  

The deformed Yang-Mills field strength acquires the rank of a torsion. Due to the last term in (50), we would better call \( T_{\rho \mu \nu} \) a “generalized torsion tensor”. It reduces to usual torsion in the abelian case (Aldrovandi 2, 1991). One should remember that usual torsion is a 2-form with values in the algebra of translation generators. In the present case, torsion has values in the assumed gauge group algebra, which is non abelian. This is the origin of the extra term.

As with usual tetrads, the \( H_{a \mu}^a \)'s can be used to transmute indices from the gauge algebra to spacetime and vice-versa. However, due to the presence of non-covariant objects, the usual properties do not follow automatically — every one must be verified by direct calculation. For example, computation gives, for the curvature, just what we would expect from a tensorial object,

\[ R_{\rho \sigma \mu \nu} = X_{\mu}^\alpha C_{\sigma \nu}^\alpha - X_{\nu}^\alpha C_{\rho \mu}^\alpha + C_{\rho \alpha \mu}^\nu C_{\sigma \nu}^\alpha + \beta^\gamma_{\mu \nu} C_{\rho \sigma}^\gamma. \]  

As in (49), it happens that

\[ X_{\mu}^\alpha R_{\rho \sigma \mu \nu} = X_{\mu}^\alpha R_{\rho \sigma \mu \nu}. \]  

It is important to notice that the enlarged derivative, once acting on objects with the indices transmuted to spacetime indices, changes its form. As happens with covariant derivatives, it will take a different aspect when acting on objects with one or two indices. The simplest way to discover its form is to read it from the Bianchi identities, as we shall do below.

**V. APPROACHING A GRAVITATIONAL MODEL**

We have in the previous section succeeded in obtaining (i) an anholonomy or torsion term in the commutator and (ii) the correct expression for the curvature in a non-holonomic basis. We shall in what follows show that two other geometrical landmarks also hold: the two Bianchi identities for linear connections. Despite their purely geometrical character, Bianchi identities are, both in gauge theories and in General Relativity, intimately related to dynamics, so that we shall also comment on the field equations. The procedure adopted here parallels those theories. The field equations are obtained by applying the duality prescription to the sole Bianchi identity present in the gauge sector. In the gravity sector, a contracted Bianchi identity is used to recognize which expression is to be identified to the source current. Properties (49) and (52) hold in general when we derive objects with external indices only. Thus, the metric \( g_{\alpha \beta} \) used in (32) can be used in the following. We see from (41) that, if preserved by \( X_{\mu}^\alpha \), it will be also gauge invariant. Recognizing (33) in the field equation (32) and adding a source current, we arrive at

\[ X_{\mu}^\alpha \beta^\alpha_{\mu \nu} = J_{\mu \nu}. \]

As the deformed Yang-Mills field coincides with torsion, this equation fixes the dynamics for both. Applying \( X_{\nu}^\alpha \) to this equation, a rather surprising result turns up:

\[ X_{\nu}^\alpha J_{\mu \nu} = 0. \]

This “current conservation” shows that some invariance must still be at work, although its meaning is not clear. Notice that the commutation relations, the new covariant derivatives and the dynamics of \( \sigma^c_{\mu} \) have all been constructed or obtained in the respect of the Jacobi identities which are, for tangent vector fields, integrability conditions. Once also the duality symmetry is supposed to hold, some invariance is to be expected.

Which kind of gravitational model would turn up? Using (47), equation (40) can be written as

\[ X_{\alpha \beta}^c_{\mu \nu} + f^c_{a \beta \gamma} \beta^c_{\mu \nu} = - R_{\alpha \mu \nu}^c + H_{a \mu \nu}^b \beta^d_{\mu \nu} C^{c}_{a \rho}, \]

which presents \( R_{a \mu \nu}^b \) as an effect of \( \beta \)'s non-covariance. Applying \( H_{a \mu}^b H_{a \rho}^\gamma \), we find

\[ R_{\mu \nu}^\alpha + X_{\nu}^\alpha (\beta^\mu_{\alpha \mu}) = 0. \]

The Ricci tensor is not symmetric,

\[ R_{\sigma \nu} + X_{\sigma}^\nu (\beta^\mu_{\nu \alpha}) = 0, \]
which is to be expected in the presence of torsion. The gravitational sector would be close to an Einstein-Cartan model, but with dynamical torsion. Combined with (56), (57) leads to

\[ X'_a(\beta^\rho_{\nu\mu}) + X'_\mu(\beta^\rho_{\lambda\nu}) + X'_\nu(\beta^\rho_{\mu\lambda}) = - \beta^\alpha_{\mu\nu} \beta^\rho_{\lambda\alpha} - \beta^\alpha_{\lambda\mu} \beta^\rho_{\nu\alpha} - \beta^\alpha_{\nu\lambda} \beta^\rho_{\mu\alpha} . \] (58)

Let us now show that the two Bianchi identities have the same formal aspect they have in usual geometry. For that, we start by calculating of the Jacobi identity for \( X'_\mu, X'_\nu \) and \( X'_\lambda, \)

\[ [X'_\mu, [X'_\mu, X'_\nu]] (Z^c) + [X'_\nu, [X'_\nu, X'_\mu]] (Z^c) + [X'_\mu, [X'_\nu, X'_\lambda]] (Z^c) = 0 . \] (59)

We first obtain one of the three cyclic terms:

\[ [X'_{\nu\lambda}, [X'_{\nu\mu}, X'_{\nu\nu}]] (Z^c) = \beta_{\mu\lambda}^\alpha [X'_{\nu\lambda}, X'_{\nu\mu}] X'_{\alpha}(Z^c) \]

Let us now show that the two Bianchi identities have the same formal aspect they have in usual geometry. For that, we start by calculating of the Jacobi identity for \( X'_{\nu\lambda}, X'_{\nu\mu} \) and \( X'_{\nu\nu}, \)

\[ + H'_{\nu\lambda} \nu\nu [\mathcal{R}_{\rho\mu\nu} \beta^\rho_{\lambda\rho} - \mathcal{R}_{\rho\mu\nu} \beta^\rho_{\nu\rho} - \mathcal{R}_{\rho\mu\nu} \beta^\rho_{\sigma\rho\nu} - X'_{\nu\lambda} \mathcal{R}_{\rho\mu\nu} \beta^\rho_{\sigma\rho\nu}] Z^a . \]

Identity (58) becomes then

\[ X'_{\nu\alpha}(Z^c) [\beta^\rho_{\mu\nu} \beta^\rho_{\nu\rho} + \beta_{\rho\nu}^\alpha \beta^\rho_{\nu\rho} + \beta^\rho_{\nu\rho} \beta^\rho_{\nu\rho} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu}] = - H'_{\nu\lambda} \nu\nu [\mathcal{R}_{\rho\mu\nu} \beta^\rho_{\lambda\rho} - \mathcal{R}_{\rho\mu\nu} \beta^\rho_{\nu\rho} + \mathcal{R}_{\rho\mu\nu} \mathcal{R}_{\rho\mu\nu} - \mathcal{R}_{\rho\mu\nu} \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu}] \]

\[ + X'_{\nu\lambda} \mathcal{R}_{\rho\mu\nu} - C'_{\rho\nu} \mathcal{R}_{\rho\mu\nu} + C'_{\rho\nu} \mathcal{R}_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \mathcal{R}_{\rho\mu\nu} + X'_{\nu\mu} \mathcal{R}_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \mathcal{R}_{\rho\mu\nu} = 0 . \] (60)

With the term proportional to \( X'_{\nu\alpha}(Z^c) \) in view, we calculate

\[ \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} \beta_{\rho\mu\nu} + \beta^\rho_{\mu\nu} \beta^\rho_{\nu\rho} \beta_{\rho\mu\nu} + \beta^\rho_{\mu\nu} \beta_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} = \]

\[ 2 \left[ X'_{\nu} \mathcal{R}_{\rho\mu\nu} + X'_{\nu} \mathcal{R}_{\rho\mu\nu} + X'_{\nu} \mathcal{R}_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \beta_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \beta_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \beta_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \beta_{\rho\mu\nu} \right] . \]

The term inside the brackets vanishes by the Jacobi identity (51) with all the indices in spacetime. The left-hand side is the factor of \( X'_{\nu\alpha}(Z^c) \) in the first term of (58), which consequently vanishes too. The remaining content of (60) is the vanishing of the term proportional to \( Z^a \), whose meaning we examine in the following. Let us notice before that the above left-hand side has another interest: the fact that it is zero, combined with (58), results in

\[ \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} + \mathcal{R}_{\rho\mu\nu} = X'_{\nu}(\beta^\rho_{\nu\nu}) + X'_{\nu}(\beta^\rho_{\nu\nu}) + X'_{\nu}(\beta^\rho_{\nu\nu}) . \] (61)

As \( \beta^\rho_{\mu\nu} \) is the torsion, this is just the expression of the first Bianchi identity to which linear connections submit (Kobayashi and Nomizu, 1963).

To analyse the vanishing of the term proportional to \( Z^a \), it is convenient to read in (60) itself the form of the enlarged derivative in the non-holonomic basis \( \{ X'_\mu \} \) when applied to an object with two transmuted indices, like \( \mathcal{R}_{\rho\mu\nu} \). It has the same expression of the usual covariant derivative:

\[ D_{\nu} \mathcal{R}_{\rho\mu\nu} = X'_{\nu} \mathcal{R}_{\rho\mu\nu} - C'_{\rho\nu} \mathcal{R}_{\rho\mu\nu} + C'_{\rho\nu} \mathcal{R}_{\rho\mu\nu} - \beta^\rho_{\mu\nu} \mathcal{R}_{\rho\mu\nu} , \] (62)

where use has been made of (52). The identity then reduces to
\[ D_{\nu} \mathcal{R}^\alpha_{\sigma \lambda \mu} + D_{\lambda} \mathcal{R}^\alpha_{\sigma \mu \nu} + D_{\mu} \mathcal{R}^\alpha_{\sigma \nu \lambda} = 0, \]

which has the form of the second Bianchi identity.

We now follow a path which parallels that used in General Relativity to identify the geometrical object appearing in the field equation, analogous to the Einstein tensor. Contracting first \( \alpha \) with \( \lambda \) and then using the preserved metric \( g_{\alpha \beta} \) to contract the remaining indices, we get

\[ D_{\nu} \mathcal{R}^\mu_{\mu} + D_{\alpha} \mathcal{R}^{\alpha \mu}_{\mu \nu} + D_{\mu} \mathcal{R}^\mu_{\nu} = 0. \]

This contracted Bianchi identity takes the form

\[ D_{\alpha} G^{\alpha \sigma} = 0 \tag{64} \]

provided we define

\[ G^{\alpha \sigma} = \mathcal{R}^{\alpha \sigma} - g^{\alpha \sigma} \mathcal{R} - g^{\sigma \nu} \mathcal{R}^{\alpha \mu}_{\mu \nu}. \tag{65} \]

This expression would lead to an object quite similar to the Einstein tensor if \( \mathcal{R}^{\alpha \beta}_{\lambda \mu} \) were antisymmetric in the first two indices.

We have above obtained the two Bianchi identities of usual geometry with torsion. To recover all the features of a real geometry the only missing point is the direct product of the vector field algebras. We see in (38) the possibility of recovering the direct product by setting \( X_a (C'_{cd\mu}) Z^d = 0 \), which includes the invariance of \( C' \) under the group action,

\[ X_a (C'_{cd\mu}) = 0. \tag{66} \]

This would mean a constant \( C'_{a b \mu} \), but not a constant \( C'_{\rho \mu \nu} \), so that the curvature would keep its general form (51). It is important to notice that such a condition to establish the direct product could only be realized because we have made the change of basis (33). It could not be done inside the extended gauge theory, since we wanted to preserve the misbehaving elements.

The validity of the scheme is restricted to gauge groups with the dimension of spacetime. In consequence the extended gauge scheme, besides describing a theory with massive fields that do not behave as connections, also describes a theory for a group with the dimension of spacetime. If we take this dimension equal to four, a group that could be chosen is the \( SU(2) \otimes U(1) \) of the Weinberg Salam model. The condition in the dimension of the group guarantees the existence of the mapping \( H_{a}^{\nu} \) and its inversibility. The introduction of \( H \) allowed us to recover the usual geometric interpretation of curvature and torsion. Significantly enough, the behavior of \( C' \) is the same as that of an external connection. The gravitational sector would exhibit curvature and see the deformed gauge field as a torsion, with dynamics given respectively by (57) and (53).

VI. OPEN QUESTIONS AND FINAL REMARKS

We have seen how, when a gauge potential ceases to behave like a connection, the bundle picture of gauge theories becomes shaky. The arena appropriate to discuss the new situation is no more bundle theory, but the theory of Lie algebra extensions, which allows for the modified local commutators coming to the fore. The object measuring the breaking of the bundle scheme is reminiscent of a linear, external, spacetime connection. A new, non-covariant, generalized derivative emerges naturally which is analogous to that appearing in electroweak theory in the presence of a gravitational field. This suggests a link between electroweak interactions and gravitation. The suggestion is strengthened by a dimensional coincidence: spacetime and the Lie algebra of the electroweak group are both 4-dimensional and, as vector spaces, isomorphic. This isomorphism can be realized by a tetrad-like field \( H \) which, once introduced, reorganizes the whole picture. Objects corresponding to the curvature and the torsion of the candidate linear connection turn up at the right places in the commutation relations and obey formally the two Bianchi identities of Differential Geometry. The broken gauge field strength appears in the role of torsion. The dynamical equations obtained correspond formally to a broken gauge model on a spacetime endowed with curvature.

We are far from having solved all the questions raised by the approach. The crucial, obvious problem which remains unsolved is that of the necessary index transmutation. We do obtain quantities resembling a connection, a curvature and a torsion by their behavior. The connections related to gravitation are, however, related to the Lorentz group. This means that, instead of our internal indices, we should have indices related to some vector or tensor representation of the Lorentz group. This is clear in the case of real tetrad fields, which are Lorentz vectors. Our latin indices should
be somehow transformed into Lorentz indices before we can really speak of gravitation. This question is not easy to answer in a satisfactory way. What we can do by now is to speculate on possible origins for such a transmutation. A first point to look at is the definition of \( H \). We have assumed equal dimensions to avoid an ill-defined mapping and this has led to the transformation \( [15] \) of \( C' \), which takes an object with internal indices into an object exclusively “external”. But the fact remains that the original group has nothing to do with spacetime. When the gauge group is the group of spacetime translations \( T^{3,1} \), \( H \) reduces to the usual vierbeine fields, which appear quite naturally (Aldrovandi 2, 1991). In that case \( C' \) turns up as a true connection for the linear or Lorentz group, with a Riemann curvature and an additional torsion. However, translation generators are Lorentz vectors and, in a sense, “external” from the start. We mention in passing that the Lorentz group does not affect spacetime directly, but through a representation, the vector representation in the case. It could happen that the group supposed above – say, the group of the electroweak interactions – come to do the same, so that the relationship to spacetime come to be realized through an intermediate, “interface” representation. This will depend on the available representations of the gauge group. The group of electroweak interactions is at present under study.

A point worth mentioning concerns universality. It is true that gravitation is the only universal interaction. However, the electroweak interaction presents a large amount of universality. Though with different strengths, all particles (except possibly the gluons) couple to it.

For the time being, the only positive clue we have to the possibility of transmutation is the appearance of torsion in expressions like \( [16] \). Torsion is specifically external, an effect of soldering which is absent in purely internal gauge bundles. Even when it vanishes, it is responsible for the presence of two — instead of only one — Bianchi identities. Another point worth remembering is that our approach is, up to now, purely classical. It is possible that transmutation come as a quantum effect. Indeed, getting “spin from isospin” has been studied in the seventies (Jackiw and Rebbi, 1976; Hasenfratz and ‘t Hooft, 1976; Goldhaber, 1976) as an instanton–induced transmutation of exactly the required kind. What we have done here has been to leave this question aside and investigate the purely formal aspects of the approach, to see whether it presents points enticing enough to justify further study. We think the results are highly positive.

ACKNOWLEDGMENTS

The authors are most grateful to FAPESP (São Paulo, Brazil), for financial support.

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