Research Article

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On applications of bipartite graph associated with algebraic structures

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Abstract: The latest developments in algebra and graph theory allow us to ask a natural question, what is the application in real world of this graph associated with some mathematical system? Groups can be used to construct new non-associative algebraic structures, loops. Graph theory plays an important role in various fields through edge labeling. In this paper, we shall discuss some applications of bipartite graphs, related with Latin squares of Wilson loops, such as metabolic pathways, chemical reaction networks, routing and wavelength assignment problem, missile guidance, astronomy and x-ray crystallography.

Keywords: Wilson loop, bipartite graph, edge labeling, nucleus

MSC 2010: 05Exx, 05E40, 05E45, 05E15

1 Introduction

Ruth Moufang, German geometer, introduced Quasigroup to associate with non-desarguesian plane significantly. Naturally, this mathematical structure is the generalization of frequently studied algebraic system, group. After the origination, mathematicians discussed it with combinatorial analysis, projective plane, experimental design, algebra, topology, etc. All algebraic nets are the examples of Quasigroups. People worked on different algebraic structures, initiated from magma or groupoid, in the interval 1900 to 1970 and all these developments culminated after the appearance of Moufang loops and Bol loops. Loop theory has not only history of 70 years but also moving in the direction of well-known research areas with modernity.

Let \( E \) be a non-empty set such that with a binary operation \( \circ \), \((E, \circ)\) is a groupoid that is \( \forall \alpha, \beta \in E \) we have \( \alpha \circ \beta \in E \). If the system of equations \( p \circ \alpha = q \) and \( \beta \circ p = q \) have unique solutions for \( \alpha \) and \( \beta \) then \((E, \circ)\) is known as Quasigroup. Furthermore, if there exists a unique identity element \( e \in E \), then \((E, \circ)\) is said to be a loop. For each \( \alpha \in E \), the elements \( \alpha^l, \alpha^r \in E \) such that \( \alpha^l \circ \alpha = \alpha \circ \alpha^r = e \) are called left and right inverses of \( \alpha \) respectively. \( E \) is known as Wilson loop (WL) if and only if it obeys the Wilson Identity (WI);

\[
\alpha \circ (\alpha \circ \beta)^r = (\alpha \circ \gamma)(\alpha \circ (\beta \circ \gamma))^r, \quad \forall \alpha, \beta, \gamma \in E
\]
equivalently;

\((\beta \circ \alpha)^{\ell} \circ \alpha = ((\gamma \circ \beta) \circ \alpha)^{\ell} \circ (\gamma \circ \alpha), \ \forall \alpha, \beta, \gamma \in \Xi\)

Any loop \(\Xi\) satisfying \(a \circ (\beta \circ \alpha) = (a \circ \beta) \circ \alpha\) is flexible loop \(\forall \alpha, \beta \in \Xi\). Sets \(\aleph_\ell = \{p \in \Xi; \ p \circ (a \circ \beta) = (p \circ a) \circ \beta \ \forall \alpha, \beta \in \Xi\}\), \(\aleph_\chi = \{p \in \Xi; \ a \circ (p \circ \beta) = (a \circ p) \circ \beta \ \forall \alpha, \beta \in \Xi\}\) and \(\aleph_r = \{p \in \Xi; \ (a \circ \beta) \circ p = \alpha \circ (\beta \circ p) \ \forall \alpha, \beta \in \Xi\}\) are said to be left, middle and right nucleus, respectively. The set \(\aleph = \aleph_\ell \cap \aleph_\chi \cap \aleph_r\) consists of all elements that associate with any other two elements and is called the nucleus of \(\Xi\). For Wilson loop we have \(\aleph = \aleph_\ell = \aleph_\chi = \aleph_r\). \(\Xi\) is weak inverse property loop if and only if \((a \circ \beta) \circ \gamma = e\) implies \(a \circ (\beta \circ \gamma) = e\ \forall \alpha, \beta, \gamma \in \Xi\). \(\Xi\) is called conjugacy closed loop if the sets of left and right translations are closed under conjugation.

As worked by Goodaire and Robinson [1, Theorem 1], a loop \(\Xi\) is a Wilson loop if and only if is weak inverse property loop [2, p. 295][3, p. 132] and conjugacy closed loop [4, p. 843]. Originally Wilson loop is introduced by E. L. Wilson in [5, Theorem 5] where it is also given that a Moufang loop [6, p. 42][7, p. 194] is Wilson loop if and only if \(a^2 \in \aleph \ \forall a \in \Xi\). The developments of loop theory remained eclipsed under the fast moving research horizon of the theory of groups. After the completion of the list of simple groups, the research environment is getting more suitability for the structures of non-associative models like loops and Quasigroups. In the literature of loop theory, the groups are being used to derive new families of loops.

In the recent time researchers are using computers rapidly for mostly used applications and the second approach is graph theory. We can understand many real world applications by associating with several graphs. Graph theory is the extensively used branch of mathematics. In 1735, Koinisber bridge’s problem gave the origin of graph theory and later on researchers did work on Eulerian graph, complete graph and bipartite graph comprehensively. After Leonhard Euler’s work, Cauchy and L’Huilier played an important role to initiate a new branch, topology, of mathematics tremendously. Arthur Cayley was first mathematician who used trees for chemical composition in theoretical chemistry. Sylvester used term "graph" first time in his work and Frank Harary wrote an eminent book on graph theory in 1969 to connect mathematicians, biologists, computer experts, chemists, engineers and social scientists see Figure 1.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A module of a protein interaction graph.}
\end{figure}
Graph $\Gamma = (\Sigma, Y)$ is known as a simple graph if it does not contain loops and multiple edges where $\Sigma$ and $Y$ are respectively sets of vertices and edges of $\Gamma$. A simple graph $\Gamma = (\Sigma, Y)$ is said to be complete if there is an edge between any pair of distinct vertices. Secondly, $\Gamma = (\Sigma_1, \Sigma_2, E)$ is bipartite (or 2-mode network or bigraph) if $\forall e \in Y$ has one end in $\Sigma_1$ and the other in $\Sigma_2$ where the sets $\Sigma_1, \Sigma_2$ are disjoint [8, p. 2][9, p. 225]. Equivalently, $\Gamma$ is bipartite if it does not contain any odd length cycle. For instance, $K_{n,n}$ is a complete bipartite graph with cardinality of both $\Sigma_1, \Sigma_2$ is $n$. Graph, $\Gamma = (\Sigma_1, \Sigma_2, Y)$ is balanced bipartite graph if $|\Sigma_1| = |\Sigma_2|$. Bipartite graphs can be used broadly to consider bioentities, signal transduction, gene regulation, evolutionary relationships, metabolic pathways, gene expression etc. as vertices and their correlation as edges within a network.

Now biologists can understand more about yeast-two-hybrid [10, p. 246], protein-protein interactions (PPIs) for particular organisms [11, p. 822][12, p. 4570][13, p. 4880][14, p. 212][15, p. 624]. Microarrays and RNA-seq [16, p. 57][17, p. 201] with the help of bipartite graphs. Graph theory is a companionable and prolific tool to handle chemical reaction networks (CRNs) [18, p. 2309]. Absolutely, it has become an important structure to study in different fields specially computer science and chemistry.

In the modern world, it seems impossible to discuss properties of classical random graphs associated with algebraic structures see [22-27]. Without any restriction, algebraic operation, we can assign a Wilson Latin square to subcubic graphs (or 2-modenetwork or bigraph) $\Gamma$, where $\Sigma_1, \Sigma_2$ and $\Sigma_3, \Sigma_4$ with the help of bipartite graphs. Document/Word Graphs are the bipartite graphs where (say) $\Sigma_1$ and $\Sigma_2$ respectively consists of documents and words, $e = (v_1, v_2) \in Y$ represents word $v_2$ is in the document $v_1$. Edge labeling of a simple graph $\Gamma = (\Sigma, Y)$ is a mapping, $\Theta : Y \rightarrow \mathbb{A}$, from $Y$ to $\mathbb{A}$, set of integers or symbols. And with this $\Theta$ the graph $\Gamma$ is called edge-labeled graph. For an healthier understanding of graph labeling, its consequences and algebraic properties see [22-27]. Without any restriction, algebraic operation, we can assign a Wilson Latin square to a complete bipartite graph through edge labeling. In Figure 2, we label an element $(-1, 1)$ as an edge with respect to any two arbitrary vertices $A$ and $B$ so $K_{4,4}$ is desired bipartite graph for table 1 with Figure 2.

**Table 1: Wilson loop of order 4.**

| (1,0) | (1,1) | (-1,0) | (-1,1) |
|-------|-------|--------|--------|
| (1,1) | (1,0) | (-1,1) | (-1,0) |
| (-1,0) | (-1,1) | (1,1) | (1,0) |
| (-1,1) | (-1,0) | (1,0) | (1,1) |

**Figure 2: Edge (-1, 1).**

A path from $u$ to $v$ in the simple graph $\Gamma$ is a sequence of edges $(\zeta_0, \zeta_1), (\zeta_1, \zeta_2), (\zeta_2, \zeta_3), \ldots, (\zeta_{m-1}, \zeta_m)$ in $\Gamma$, where $m$ is a nonnegative integer, and $\zeta_0 = u$ and $\zeta_m = v$. It can be denoted by $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{m-1}, \zeta_m$ and has length $m$. In case of directed graphs, we say a path is increasing if the sequence of its edge labels is non-decreasing. Good edge-labeling is an edge-labeling in which for any two distinct vertices $u, v$ we have at most one increasing $(u, v)$-path. Subcubic $(C_3, K_{2,3})$-free graphs, planar graphs of girth at least 6, $C_3$-free outerplanar graphs, forests are the examples of graphs which admit the good edge-labeling and help us to
overcome RWA (Routing and Wavelength Assignment) problem for UPP-DAG [25, 28-30]. Graph labeling plays a vital role in a number of applications like data base management, communication network addressing, circuit design, x-ray crystallography, astronomy, radar and missile guidance. For further information see [31-33].

2 Main results

Let $\Psi_1$ and $\Psi_2$ be respectively groups under multiplication and addition. Moreover $\Psi_2$ is abelian group. The function $\triangleright: \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ with:

$$(1, F) \triangleright (F, 1) = 0, \quad \forall F \in \Psi_1,$$

is called a factor set. Binary operation $\circ$ on $\Psi_1 \times \Psi_2$ can be defined, with the help of $\triangleright$, as follows:

$$(F_1, v_1) \circ (F_2, v_2) = (F_1 F_2, v_1 + v_2 + (F_1, F_2)\triangleright) \quad \forall F_1, F_2 \in \Psi_1 \quad \forall v_1, v_2 \in \Psi_2.$$

Clearly the resulting groupoid is a loop denoted by $(\Psi_1, \Psi_2, \circ)$ with neutral element $(1, 0)$. Note that $(F, v)^{-1} = (F^{-1}, -v - (F, F^{-1})\triangleright)$ is the inverse of $(F, v)$ in $(\Psi_1, \Psi_2, \circ)$. The following theorem provides construction of the Wilson loops.

**Theorem 1.** Let $\triangleright: \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ be a factor set. Then $(\Psi_1, \Psi_2, \circ)$ is Wilson loop if and only if

$$(F_1, F_3)\triangleright + (F_1, F_2)\triangleright + (F_1 F_2, (F_1 F_2)^{-1})\triangleright + (F_1 F_3, (F_1 \cdot F_2 F_3)^{-1})\triangleright = (F_2, F_3)\triangleright + (F_1, F_2 F_3)\triangleright + (F_1 \cdot F_2 F_3, (F_1 \cdot F_2 F_3)^{-1})\triangleright + (F_1, (F_1 F_2)^{-1})\triangleright.$$

**Proof.** Let $(\Psi_1, \Psi_2, \circ)$ be the Wilson loop so it satisfies the identity

$$(F_1, v_1) \circ ((F_1, v_1) \circ (F_2, v_2))' = ((F_1, v_1) \circ (F_3, v_3)) \circ ((F_2, v_2) \circ (F_3, v_3))'$$

for all $(F_1, v_1), (F_2, v_2), (F_3, v_3) \in (\Psi_1, \Psi_2, \circ)$.

Now

$$(F_1, v_1) \circ ((F_1, v_1) \circ (F_2, v_2))' = (F_1, v_1) \circ ((F_1 F_2, v_1 + v_2 + (F_1, F_2)\triangleright)\circ)$$

$$= (F_1, v_1) \circ ((F_1 F_2)^{-1}, -v_1 - v_2 - (F_1, F_2)\triangleright - (F_1 F_2, (F_1 F_2)^{-1})\triangleright)$$

$$= (F_1, v_1) \circ ((F_1, F_2)^{-1}, v_1 - v_2 - (F_1, F_2)\triangleright - (F_1 F_2, F_1 F_2^{-1})\triangleright)$$

and

$$(F_1, v_1) \circ ((F_3, v_3) \circ ((F_1, v_1) \circ (F_2, v_2) \circ (F_3, v_3)))'$$

$$= (F_1, F_3, v_1 + v_3 + (F_1, F_3)\triangleright) \circ ((F_3, v_3) \circ (F_2 F_3, v_2 + v_3 + (F_2, F_3)\triangleright)\circ)$$

$$= (F_1, F_3, v_1 + v_3 + (F_1, F_3)\triangleright) \circ ((F_1 F_2 F_3, v_1 + v_2 + v_3 + (F_2, F_3)\triangleright + (F_1, F_2 F_3)\circ)$$

$$= (F_1, F_3, v_1 + v_3 + (F_1, F_3)\circ) \circ ((F_1 F_2 F_3)^{-1}, -v_1 - v_2 - v_3 - (F_2, F_3)\circ)$$

$$= (F_1 F_3, F_1^{-1} F_2^{-1}, F_1^{-1} F_2^{-1} F_1^{-1})\circ$$

and

$$(F_1, F_3)(F_3^{-1} F_2^{-1}, F_1^{-1})\circ, v_1 + v_3 + (F_1, F_3)\circ - v_1 - v_2 - v_3 - (F_2, F_3)\circ - (F_1, F_2 F_3)\circ$$

$$= (F_1, F_2 F_3, F_3^{-1} F_2^{-1} F_1^{-1})\circ$$

$$(F_1, F_3)(F_3^{-1} F_2^{-1}, F_1^{-1})\circ, v_1 + v_3 + (F_1, F_3)\circ - v_1 - v_2 - v_3 - (F_2, F_3)\circ - (F_1, F_2 F_3)\circ - (F_1, F_2 F_3)\circ$$

$$= (F_1, F_2 F_3, F_3^{-1} F_2^{-1} F_1^{-1})\circ$$

$$= (F_1 \cdot F_2 F_3, F_3^{-1} F_2^{-1} F_1^{-1})\circ$$

Nu
Using both results in the above Wilson identity

\[(F_1, F_3)b + (F_1, F_2)b + (F_1 F_2, (F_1 F_2)^{-1})b + (F_1 F_3, (F_1 F_2 F_3)^{-1})b\]
\[= (F_2, F_3)b + (F_1, F_2 F_3)b + (F_1 F_2 F_3, (F_1 F_2 F_3)^{-1})b + (F_1, (F_1 F_2)^{-1})b\]

Which is the required identity. The converse is easy to verify.

### 2.1 Wilson factor set

A factor set \( b : \Psi_1 \times \Psi_1 \rightarrow \Psi_2 \) with equation (1) is called Wilson factor-set. If \(|\Psi_1| = 2^s\) where \( s \) is the whole number then equation (1) reduces to

\[(F_1, F_3)b + (F_1, F_2)b + (F_1 F_2, F_1 F_3)b + (F_1 F_3, (F_1 F_2 F_3)^{-1})b\]
\[= (F_2, F_3)b + (F_1, F_2 F_3)b + (F_1 F_2 F_3, (F_1 F_2 F_3)^{-1})b + (F_1, (F_1 F_2)^{-1})b.\]

This Wilson-factor set is very helpful in construction of Wilson loops by the following manner.

**Proposition 1.** Let \( \Psi_2 \) be an additive abelian group with cardinality \( k \), positive integer greater than 1, and \( 0 \neq p \in \Psi_2 \). Let \( \Psi_1 = \{1, \square\} \) be the multiplicative group where \( \square = \cos\pi + i\sin\pi \) We define function \( b : \Psi_1 \times \Psi_1 \rightarrow \Psi_2 \) by

\[(F_1, F_2)b = \begin{cases} p, & \text{if } (F_1, F_2) = (\square, \square); \\ 0, & \text{if } (F_1, F_2) = (1, 1), (1, \square), (\square, 1). \end{cases}\]

Then \( (\Psi_1, \Psi_2, b) \) is a flexible, non-associative Wilson loop with nucleus \( \mathbb{N} = (1, v) \forall v \in \Psi_2 \) and \( \forall F_1, F_2 \in \Psi_1 \).

Let \( \Psi_1 = \{1, \square\} \), multiplicative group, and \( \Psi_2 = \{0, 1, 2, 3, \ldots, n - 1\} \), additive abelian group of modulo \( n \), table 2 shows a pattern of Wilson loops of even orders.

**Proposition 2.** Let \( \Psi_2 \) be an additive abelian group with \(|\Psi_2| > 2\), and order of \( p \) is greater than 2 where \( 0 \neq p \in \Psi_2 \). Let \( \Psi_1 = \{1, \varphi_1, \varphi_2, \varphi_3\} \) be the Klein group. Define \( b : \Psi_1 \times \Psi_1 \rightarrow \Psi_2 \) by

\[(F_1, F_2)b = \begin{cases} p, & \text{if } (F_1, F_2) = (\varphi_1, \varphi_3), (\varphi_3, \varphi_2), (\varphi_2, \varphi_1); \\ -p, & \text{if } (F_1, F_2) = (\varphi_1, \varphi_2), (\varphi_2, \varphi_3), (\varphi_3, \varphi_1); \\ 0, & \text{otherwise}. \end{cases}\]

Then \( (\Psi_1, \Psi_2, b) \) is a non-flexible (implies non-associative) Wilson loop with nucleus \( \mathbb{N} = (1, v) \forall v \in \Psi_2 \) and \( \forall F_1, F_2 \in \Psi_1 \).

**Proof.** Following table shows that function \( b \) is obviously Wilson-factor set.

| \( b \) | 1 | \( \varphi_1 \) | \( \varphi_2 \) | \( \varphi_3 \) |
|--------|---|---------------|---------------|---------------|
| 1      | 0 | 0             | 0             | 0             |
| \( \varphi_1 \) | 0 | 0             | -p            | p             |
| \( \varphi_2 \) | 0 | p             | 0             | -p            |
| \( \varphi_3 \) | 0 | -p            | p             | 0             |

To show that \( (\Psi_1, \Psi_2, b) \) is Wilson loop we verify equation (2). Since \( b \) is factor set, there is nothing to prove when \( F_3 = 1 \);

\[(F_1, F_2)b + (F_1 F_2, F_1 F_2)b + (F_1 F_2 F_3, (F_1 F_2 F_3)^{-1})b + (F_1, (F_1 F_2)^{-1})b.\]

When \( F_2 = 1 \);
Similarly it can be proved for $f_1 = 1$.

When $f_3 = \wp_1$:

\[(f_1, f_3)\ast (f_1, f_3) + (f_1, f_3) + (f_1, f_3) = (f_1, f_3) + (f_1, f_3) + (f_1, f_3).\]

Putting $f_1 = \wp_2, f_2 = \wp_3$ in the last identity, we have

\[(f_2, x) + (f_1, x) + (f_1, x) + (f_1, x) + (f_1, x) = (f_1, x) + (f_1, x) + (f_1, x) + (f_1, x) + (f_1, x).\]

Similarly we can check other cases when $f_3 = \wp_1$. By using same procedure for $f_2, f_1$ we can verify (2). Thus $(\Psi_1, \Psi_2, \ast)$ is Wilson loop. $(\Psi_1, \Psi_2, \ast)$ is non-commutative, non-associative Wilson loop. As let $\forall \nu \in \Psi_2$ and $0 \neq p, p \neq -p$

\[([(\wp_1, \nu) \circ (\wp_2, \nu)] \circ (\wp_1, \nu) = (\wp_1, \wp_2, \nu + \nu) + (\wp_1, \wp_2, \nu) \circ (\wp_1, \nu)\]
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Figure 3: Complete bipartite graph $K_{20,20}$.

\[
\begin{align*}
(\varphi_1, v) &\circ ((\varphi_2, v) \circ (\varphi_1, v)) = (\varphi_1, v) \circ (\varphi_2 \varphi_1, v + v + (\varphi_2, \varphi_1) v) \\
&= (\varphi_1, v) \circ (\varphi_1, 2v + p) \\
&= (\varphi_2, 3v + p + (\varphi_1, \varphi_3) v) \\
&= (\varphi_2, 3v + 2p)
\end{align*}
\]

from (3) and (4)

\[
((\varphi_1, v) \circ (\varphi_2, v)) \circ (\varphi_1, v) \neq (\varphi_1, v) \circ ((\varphi_2, v) \circ (\varphi_1, v)).
\]

It implies that $(\Psi_1, \Psi_2, v)$ is not flexible and $(\varphi_1, v), (\varphi_2, v)$ are not in $\mathbb{N}$. Similarly, $((\varphi_1, v) \circ (\varphi_3, v)) \circ (\varphi_1, v)$ gives $(\varphi_3, v)$ also not in $\mathbb{N}$. Finally $\forall F_1, F_2 \in \Psi_1$ and $\forall v_2, v_3 \in \Psi_2$

\[
(1, v) \circ (F_1, v_2) \circ (F_2, v_3) = (F_1 F_2, v + v_2 + v_3 + (F_1, F_2) v) = (1, v) \circ ((F_1, v_2) \circ (F_2, v_3))
\]

shows that $(1, v) \in \mathbb{N}$ represents star graph through above mentioned edge labeling.

**Example:** If $\Psi_1 = \{1, \varphi_1, \varphi_2, \varphi_3\}$ and $\Psi_2 = \{0, 1, 2, 3, 4\}$, with modulo 5, then $K_{20,20}$ is the associated graph see Figure 3. Let $\Psi_1 = \{1, \varphi_1, \varphi_2, \varphi_3\}$, Klein four group, and $\Psi_2 = \{0, 1, 2, 3, \ldots, n - 1\}$, additive abelian group of modulo $n$, table 3 also shows a pattern of Wilson loops.

We can recapitulate all the above discussion in the table 4.

3 Conclusion

This article deals with the application of graph theory in the pure mathematics. In particular the aim is to discover those algebraic structures and quasigroups which are closely associated with bipartite graphs. We have shown that graph labeling is a powerful tool to understand algebraic object namely the Wilson loop. The field is quite open in the sense, one can discover more connections between these two areas.

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Table 3: Wilson loop of order $4n$.

| $n$ | $(1,0)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | $(1,8)$ | $(1,9)$ | $(1,10)$ | $(1,11)$ | $(1,12)$ | $(1,13)$ | $(1,14)$ | $(1,15)$ | $(1,16)$ | $(1,17)$ | $(1,18)$ | $(1,19)$ | $(1,20)$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0   | $(1,0)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | $(1,8)$ | $(1,9)$ | $(1,10)$ | $(1,11)$ | $(1,12)$ | $(1,13)$ | $(1,14)$ | $(1,15)$ | $(1,16)$ | $(1,17)$ | $(1,18)$ | $(1,19)$ | $(1,20)$ |
| 1   | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | $(1,8)$ | $(1,9)$ | $(1,10)$ | $(1,11)$ | $(1,12)$ | $(1,13)$ | $(1,14)$ | $(1,15)$ | $(1,16)$ | $(1,17)$ | $(1,18)$ | $(1,19)$ | $(1,20)$ |
| 2   | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | $(1,8)$ | $(1,9)$ | $(1,10)$ | $(1,11)$ | $(1,12)$ | $(1,13)$ | $(1,14)$ | $(1,15)$ | $(1,16)$ | $(1,17)$ | $(1,18)$ | $(1,19)$ | $(1,20)$ |
| 3   | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ | $(1,8)$ | $(1,9)$ | $(1,10)$ | $(1,11)$ | $(1,12)$ | $(1,13)$ | $(1,14)$ | $(1,15)$ | $(1,16)$ | $(1,17)$ | $(1,18)$ | $(1,19)$ | $(1,20)$ |

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Table 4: Complete bipartite graphs associated with loops.

| Multiplicative Group | Additive group | Loop | p | −p | Bipartite graph | Star graph |
|----------------------|---------------|------|---|----|-----------------|-----------|
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 1 | 1 | $K_{4,6}$ | $K_{1,2}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 2 | 1 | $K_{6,6}$ | $K_{1,3}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2, 3\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 3 | 1 | $K_{8,8}$ | $K_{1,4}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2, 3, \ldots, n\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | n-1 | 1 | $K_{2n,2n}$ | $K_{1,n}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 2 | 1 | $K_{12,12}$ | $K_{1,3}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2, 3\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 3 | 1 | $K_{16,16}$ | $K_{1,4}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2, 3, 4\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | 4 | 1 | $K_{20,20}$ | $K_{1,5}$ |
| $\Psi_1 = \{1, \mathbb{Z}\}$ | $\Psi_2 = \{0, 1, 2, 3, \ldots, n\}$ | $(\Psi_1, \Psi_2, \mathbb{Z})$ | n-1 | 1 | $K_{\alpha n,\alpha n}$ | $K_{1,n}$ |

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