Proof of Polyakov conjecture for general elliptic singularities

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Abstract

A proof is given of Polyakov conjecture about the accessory parameters of the $SU(1,1)$ Riemann-Hilbert problem for general elliptic singularities on the Riemann sphere. Its relevance to $2 + 1$ dimensional gravity is stressed.

1 Introduction

Polyakov made the following conjecture \[1\] on the accessory parameters $\beta_n$ which appear in the solution of the $SU(1,1)$ Riemann-Hilbert problem

$$-\frac{1}{2\pi}dS_P = \sum_n \beta_n dz_n + c.c. \quad (1)$$

where $S_P$ is the regularized Liouville action \[3\], $S_P = \lim_{\epsilon \to 0} S_\epsilon$ with \[3\]

$$S_\epsilon[\phi] = i \int_{X_\epsilon} (\partial_z \phi \partial_{\bar{z}} \phi + \frac{e^\phi}{2}) dz \wedge d\bar{z} + i \sum_n g_n \oint_{\gamma_n} \phi \left( \frac{d\bar{z}}{\bar{z} - \bar{z}_n} - \frac{dz}{z - z_n} \right)$$

$$+ \frac{i}{2} g_\infty \oint_{\gamma_\infty} \phi \left( \frac{d\bar{z}}{\bar{z} - \bar{z}} - \pi \sum_n g_n^2 \ln \epsilon^2 - \pi g_\infty^2 \ln \epsilon^2 \right) \quad \text{where} \quad dz \wedge d\bar{z} = -2i dx \wedge dy \quad (2)$$

and $X_\epsilon$ is the disk of radius $1/\epsilon$ in the complex plane from which disks of radius $\epsilon$ around all singularities have been removed; $\gamma_n$ are the boundaries of the small disks and $\gamma_\infty$ is the boundary of the large disk. In eq.(1) $S_P$ has to be computed on the solution of the inhomogeneous Liouville equation which arises from the minimization of the action i.e.

$$4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n g_n \delta^2(z - z_n) \quad (3)$$

\[2\]Our conventions are slightly different from those of ref.\[8\]: our field $\phi$ is related the the field $\psi$ of ref.\[8\] by $\phi = \psi + \ln 2$
with asymptotic behavior at infinity $\phi = -g_\infty \ln z\bar{z} + O(1)$. Such a conjecture plays an important role in the quantum Liouville theory [3] and in the ADM formulation of 2 + 1 dimensional gravity [4, 5]. The conjecture is interesting in itself as it gives a new meaning to the rather elusive accessory parameters [6, 7] of the Riemann-Hilbert problem. In particular it implies that the form $\omega = \sum_n \beta_n dz_n + c.c.$ is exact.

Zograf and Takhtajan [8] provided a proof of eq.(1) for parabolic singularities using the technique of mapping the quotient of the upper half-plane by a fuchsian group to the Riemann surface and exploiting certain properties of the harmonic Beltrami differentials. In addition they remark that the same technique can be applied when some of the singularities are elliptic of finite order. The case of only parabolic singularities is of importance in the quantum Liouville theory [3] as such singularities provide the sources from which to compute the correlation functions. On the other hand in 2 + 1 gravity one is faced with general elliptic singularities and here the mapping technique cannot be directly applied. (In the case of elliptic singularities with rational $g_n$ some progress in the mapping technique that are relevant to this problem were made in [14]). As a matter of fact we shall see that the case of elliptic singularities is more closely related to the theory of elliptic non linear differential equations (potential theory) than to the theory of fuchsian groups.

In a series of papers at the turn of the past century Picard [9] proved that eq.(3) for real $\phi$ with asymptotic behavior at infinity

$$
\phi(z) = -g_\infty \ln(z\bar{z}) + O(1)
$$

and $-1 < g_n, \; 1 < g_\infty$ (which excludes the case of punctures) and $\sum_n g_n + g_\infty < 0$ admits one and only one solution (see also [10]). Picard [4] achieved the solution of (3) through an iteration process exploiting Schwarz alternating procedure. The same problem has been considered recently with modern variational techniques by Troyanov [11], obtaining results which include Picard’s findings. The interest of such results is that they solve the following variant of the Riemann-Hilbert problem: at $z_1, \ldots z_n$ we are given not with the monodromies but with the class, characterized by $g_j$, of the elliptic monodromies with the further request that all such monodromies belong to the group $SU(1, 1)$. The last requirement is imposed
by the fact that the solution of eq.(3) has to be single valued. Eq.(3) is the type of equation one encounters in the ADM treatment \[4, 5\] of 2 + 1 gravity coupled with point particles in the maximally slicing gauge \[12\]. In this case \(z\) varies on the Riemann sphere, \(N\) of the \(z_j\) are the particle singularities with residue \(g_j = -1 + \mu_j\) and \(N - 2\) of them are the so called apparent singularities \(z_B\) with residues \(g_B = 1\). The inequalities on the values of \(g_m\) are satisfied in 2 + 1 dimensional gravity due to the restriction on the masses of the particles \(0 < \mu_n < 1\) (in rationalized Planck units) and to the fact that the total energy \(\mu\) must satisfy the bound \(\sum_n \mu_n < \mu < 1\). For this reason in this paper we shall confine ourselves to the Riemann sphere. After solving all the constraints, the hamiltonian nature of the particle equations of motion is a consequence of Polyakov conjecture; actually is the consequence of a somewhat weaker form of it \[3\] i.e. of the relation one obtains by taking the derivative of Polyakov conjecture with respect to the total energy.

From eq.(3) one can easily prove \[10, 13\] that the function \(Q(z)\) defined by
\[
e^{\frac{\phi}{2}} \partial_z^2 e^{-\frac{\phi}{2}} = -Q(z)
\]
is analytic i.e. as pointed out in \[13\] \(Q(z)\) is given by the analytic component of the energy momentum tensor of a Liouville theory. \(Q(z)\) is meromorphic with poles up to the second order \[3\] i.e. of the form
\[
Q(z) = \sum_n \frac{g_n(g_n + 2)}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)}. \tag{6}
\]
All solutions of eq.(3) can be put in the form
\[
e^\phi = \frac{8f'\bar{f}'}{(1 - ff)^2} = \frac{8|w_{12}|^2}{(y_2\bar{y}_2 - y_1\bar{y}_1)^2}, \quad f(z) = \frac{y_1}{y_2} \tag{7}
\]
being \(y_1, y_2\) two properly chosen, linearly independent solutions of the fuchsian equation
\[
y'' + Q(z)y = 0. \tag{8}
\]
\(w_{12}\) is the constant wronskian. In fact following \[10, 13\] as \(e^{-\phi/2}\) solves the fuchsian equation (8) it can be put in the form
\[
e^{-\frac{\phi}{2}} = \frac{1}{\sqrt{8}}[\psi_2(z)\bar{\chi}_2(z) - \psi_1(z)\bar{\chi}_1(z)] \tag{9}
\]
with $\psi_j(z)$ solutions of eq. (8) with wronskian 1 and $\chi_j(z)$ also solutions of eq. (8) with wronskian 1. The solution of eq. (8) ($\phi = \text{real}$) with the stated behavior at infinity is unique [9, 11]. Exploiting the reality of $e^\phi$ it is possible by an $SL(2C)$ transformation to reduce eq. (9) to the form eq. (7). In fact, being $\chi_j$ linear combinations of the $\psi_j$, the reality of $e^\phi$ imposes

$$\psi_2(z)\bar{\chi}_2(\bar{z}) - \psi_1(z)\bar{\chi}_1(\bar{z}) = \sum_{jk} \bar{\psi}_j H_{jk} \psi_k$$

(10)

with the $2 \times 2$ matrix $H_{jk}$ hermitean and det $H = -1$. By means of a unitary transformation, which belongs to $SL(2C)$ we can reduce $H$ to diagonal form diag($-\lambda, \lambda^{-1}$) and with a subsequent $SL(2C)$ transformation we can reduce it to the form diag($-1, 1$) i.e. to the form (8). Through eq. (8), $\phi$ contains the full information about the accessory parameters $\beta_n$ defined in eq. (6). It is important to notice that being all of our monodromies elliptic, we can by means of an $SU(1, 1)$ transformation, choose around a given singularity $z_m$ (not around all singularities simultaneously) $y_1$ and $y_2$ with the following canonical behavior

$$y_1(\zeta) = k_m \zeta^{\frac{gm-1}{2}} A(\zeta), \quad y_2(\zeta) = \zeta^{\frac{-gm}{2}} B(\zeta)$$

(11)

with $\zeta = z - z_m$ and $A$ and $B$ analytic functions of $\zeta$ in a neighborhood of 0 with $A(0) = B(0) = 1$.

2 The realization of fuchsian $SU(1,1)$ monodromies

The result of Picard assures us that given the position of the singularities $z_n$ and the classes of monodromies characterized by the real numbers $g_n$ there exists a unique fuchsian equation which realizes $SU(1, 1)$ monodromies of the prescribed classes. In particular the uniqueness of the solution of Picard’s equation tells us that the accessory parameters $\beta_i$ are single valued functions of the parameter $z_n$ and $g_n$. We shall examine in this section how such dependence arises from the viewpoint of the imposition of the $SU(1, 1)$ condition on the monodromies in order to understand the nature of the dependence of the $\beta_i$ on the $g_n$ and on the $z_n$. The
proof of the real analytic dependence of the accessory parameters on the $z_n$ in the case of rational $g_n$ has been given by Kra [14].

Starting from the singularity in $z_1$ we can consider the canonical pair of solutions around $z_1$ i.e. those solutions which behave as a single fractional power multiplied by an analytic function with coefficient one as given in eq. (11). We shall call such pair of solutions $(y_1^1, y_1^2)$ and let $(y_1, y_2)$ the solution which realize $SU(1, 1)$ around all singularities. Obviously all conjugations with any element of $SU(1, 1)$ is still an equivalent solution in the sense that they provide the same conformal factor $\phi$. The canonical pairs around different singularities are linearly related i.e. $(y_1^1, y_2^1) = (y_1^2, y_2^2) C_{21}$. We fix the conjugation class by setting

$$(y_1, y_2) = (y_1^1, y_2^1) K$$

with $K = \text{diag}(k, k^{-1})$ being the overall constant irrelevant in determining $\phi$. Moreover if the solution $(y_1, y_2)$ realizes $SU(1, 1)$ monodromies around all singularities also $(y_1, y_2) \times \text{diag}(e^{i\alpha}, e^{-i\alpha})$ accomplishes the same purpose being $\text{diag}(e^{i\alpha}, e^{-i\alpha})$ an element of $SU(1, 1)$. Thus the phase of the number $k$ is irrelevant and so we can consider it real and positive. This choice of the canonical pairs is always possible in our case. In fact the roots of the indicial equation are $-\frac{g_m}{2}$ and $\frac{2m}{2} + 1$ and thus the monodromy matrix has eigenvalues $e^{-i\pi g_m}$ and $e^{i\pi g_m}$ which are different when $g_m$ is not an integer. If $g_m$ is an integer in general in the solution of the fuchsian equation the less singular solution possesses a logarithmic term which however has to be absent in our case (no logarithm condition [7]) in order to have a single valued $\phi$. In this case the monodromy matrix is simply the identity or minus the identity. The monodromy around $z_1$ thus belongs to $SU(1, 1)$ for any choice of $K$. If $D_n$ denote the diagonal monodromy matrices around $z_n$, we have that the monodromy around $z_1$ is $D_1$ and the one around $z_2$ is

$$M_2 = K^{-1} C_{12} D_2 C_{21} K. \quad (13)$$

where with $C_{12}$ we have denoted the inverse of the $2 \times 2$ matrix $C_{21}$.

In the case of three singularities (one of them at infinity) the counting of the degrees of freedom is the following: by using the freedom on $K$ we can reduce $M_2$ to the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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with Re $b = \text{Re } c$, or if either $\text{Re } b$ or $\text{Re } c$ is zero we can obtain $\text{Im } b = -\text{Im } c$. Then we use the fact that $D_1 M_2 = C D_\infty C^{-1}$ and thus in addition to $a + d = \text{real}$ we have also $ae^{i\pi g_1} + de^{-i\pi g_1} = \text{real}$, which gives $d = \bar{a}$ and thus using $a\bar{a} - bc = 1$ we have $c = \bar{b}$. The fact that a real $k$ is sufficient to perform the described reduction of the matrix $M_2$ is assured by Picard’s result on the solubility of the problem and in this simple case also by the explicit solution in terms of hypergeometric functions [12, 4].

We give now a qualitative discussion of the case with four singularities and then give the analytic treatment of it. The case with more that four singularities is a trivial extension of the four singularity case. The following treatment relies heavily on Picard’s result about the existence and uniqueness of the solution of eq.(3). We recall that the accessory parameters $\beta_n$ are bound by two algebraic relations known as Fuchs relations [7]. Thus after choosing $M_1$ of the form $M_1 = D_1 K$, in imposing the $SU(1,1)$ nature of the remaining monodromies we have at our disposal three real parameters i.e. $k$, Re $\beta_3$ and Im $\beta_3$. It is sufficient to impose the $SU(1,1)$ nature of $M_2$ and $M_3$ as the $SU(1,1)$ nature of $M_\infty$ is a consequence of them. As the matrices $M_n = K^{-1} C_{1n} D_n C_{n1} K$ satisfy identically $\det M_n = 1$ and $\text{Tr} M_n = 2 \cos \pi g_n$ we need to impose generically on $M_2$ only two real conditions e.g. Re $b_2 = \text{Re } c_2$ and Im $b_2 = -\text{Im } c_2$. The same for $M_3$. Thus is appears that we need to satisfy four real relations when we can vary only three real parameters. The reason why we need only three and not four is that for any solution of the fuchsian problem the following relation among the monodromy matrices is identically satisfied

$$D_1 K M_2 M_3 M_\infty = 1.$$  \hspace{1cm} (14)

Rigorously the conditions for realizing $SU(1,1)$ monodromies are

$$\text{Re } a_i = \text{Re } d_i, \quad \text{Im } a_i = -\text{Im } d_i, \quad \text{Re } b_i = \text{Re } c_i, \quad \text{Im } b_3 = -\text{Im } c_i \quad (i = 2, 3)$$  \hspace{1cm} (15)

Satisfying the eight above equations is a sufficient (and necessary) condition to realize the $SU(1,1)$ monodromies. The fact that given a $z_n^0$ in a neighborhood of such a point there exists one and only one solution to the eight equation (14) means that at least three of them are not identically satisfied in such a neighborhood and that the remaining are satisfied as
a consequence of them. We shall denote such equations as

$$\Delta^{(1)} = 0, \quad \Delta^{(2)} = 0, \quad \Delta^{(3)} = 0.$$  \hspace{1cm} (16)

The matrices $A_n = C_n K$ which give the solution of the problem in terms of the canonical solutions around the singularities are completely determined by the two equations

$$(y_1, y_2) = (y_1^{(n)}, y_2^{(n)}) A_n; \quad (y'_1, y'_2) = (y_1^{(n)'}, y_2^{(n)'}) A_n$$  \hspace{1cm} (17)

due to the non vanishing of the wronskian of $y_1^{(n)}, y_2^{(n)}$. Being $(y_1, y_2)$ solutions of a fuchsian equation, $A_n$ depend analytically on $k, z_1, z_2, z_3, \beta_3$. Thus eqs.$(16)$ which determine implicitly $k, \text{Re} \beta_3$ and $\text{Im} \beta_3$ state the vanishing of the real part of three analytic functions, functions of $z_n, k$ and $\beta_3$. It follows that $\Delta^{(i)}$ are analytic functions of the real and imaginary parts of the variables or equivalently of the independent variables $k, \beta_3, \bar{\beta}_3, z_n, \bar{z}_n$.

In order to understand the dependence of $\beta_3$ and $\bar{\beta}_3$ on $z_n, \bar{z}_n$ we apply around a solution (which due to Picard we know to exist) of the three equations, Weierstrass preparation theorem [15]. It states that in a neighborhood of a solution $z_n^0, k^0, \beta_3^0, \Delta^{(i)}$ can be written as

$$\Delta^{(i)} = P^{(i)}(k) u^{(i)}(k, z_n, \bar{z}_n, \beta_3, \bar{\beta}_3)$$  \hspace{1cm} (18)

where $P(k)$ is a polynomial in $k$ with coefficients analytic functions of $z_n, \bar{z}_n, \beta_3, \bar{\beta}_3$, while $u(k, z_n, \bar{z}_n, \beta_3, \bar{\beta}_3)$ is a “unit” i.e. an analytic function of the arguments, which does not vanish in a neighborhood of $z_n^0, k^0, \beta_3^0$. Thus our problem is reduced to the search of the real common zeros of the three polynomial $P^{(i)}(k)$. By algebraic elimination of the variable $k$ we reach a system of two equations which depend analytically on the variables $z_n, \bar{z}_n, \beta_3, \bar{\beta}_3$ and reasoning as above by elimination of $\bar{\beta}_3$ we reach as condition to be satisfied by Picard’s solution

$$V(\beta_3 | c_k(z_n, \bar{z}_n)) = 0$$  \hspace{1cm} (19)

where $V$ is a polynomial in $\beta_3$ with coefficients analytic functions of $z_n$ and $\bar{z}_n$. The derivative of $\beta_3$ with respect to $z_n$ (and similarly with respect to $\bar{z}_n$) is then given by

$$V'(\beta_3 | c_k) \frac{\partial \beta_3}{\partial z_n} + V(\beta_3 | \frac{\partial c_k}{\partial z_n}) = 0$$  \hspace{1cm} (20)
If \( V' \) vanishes identically on the \( \beta_3(z_n) \) provided by Picard’s solution we can adopt \( V' \) as determining such a function. The procedure can be repeated until \( V' \) does not vanish identically on Picard’s solution and thus in a neighborhood of \( z_n^0 \) the derivative \( \frac{\partial \beta_3}{\partial z_n} \) exists except for a finite number of points. Actually \( \beta_3 \) is an analytic functions of \( z_n \) and \( \bar{z}_n \) for all points of such a neighborhood in which \( V' \) does not vanish [15]. The extension to five or more singularities proceeds along the same line.

As we already mentioned if some of the \( g_m \) is an integer we have the so called apparent singularities which have monodromy \( I \) if \( g_m \) is even and monodromy \( -I \) if \( g_m \) is odd. In this case we have to impose the so called no-logarithm conditions (see e.g. [16]) which result in a linear combination of the \( \beta_n \) with \( n \neq m \) to be equal to the square of \( \beta_m \). Thus we can eliminate one \( \beta_n \) in favor of \( \beta_m \) and we have the same matching in the degrees of freedom.

3 Proof of Polyakov conjecture

As already stated we shall limit ourselves to the case of the Riemann sphere with a finite number of conical singularities, one of them at infinity, subject to the restrictions given by Picard and described in sect.1. The technique to prove Polyakov conjecture will be to express the original action in terms of a field \( \phi_M \) which is less singular than the original conformal field \( \phi \). This procedure will give rise to an action \( S \) for the field \( \phi_M \) which does not involve the \( \epsilon \to 0 \) process. Despite that, computing the derivative of the new action \( S \) is not completely trivial because one cannot take directly the derivative operation under the integral sign. In fact such unwarranted procedure would give rise to an integrand which is not absolutely summable. In the global coordinate system \( z \) on \( C \) one writes \( \phi = \phi_M + \phi_0 + \phi_B \) where \( \phi_B \) is a background conformal factor which is regular and behaves at infinity like \( \phi_B = -2 \ln(z\bar{z}) + c_B + O(1/|z|) \) while \( \phi_0 \) is a solution of

\[
4\partial_z\partial_{\bar{z}}\phi_0 = 4\pi \sum_n g_n \delta^2(z - z_n) - 4\alpha \partial_z\partial_{\bar{z}}\phi_B
\]

(21)
with behavior at infinity $\phi_0 = (2 - g_\infty) \ln(z\bar{z}) + O(1)$. Such a behavior fixes the value of $\alpha$ to

$$4\pi\left(\sum_n g_n + g_\infty - 2\right) + 8\pi\alpha = 0 \quad (22)$$

(a possible choice for $\phi_B$ is the conformal factor of the sphere i.e. $\phi_B = -2\ln(1 + z\bar{z})$). The fields $\phi_0$ and $\phi_M$ transform under a change of chart like scalars while $e^{\phi_B}$ transforms as a $(1,1)$ density. This choice is also in agreement with the invariance of eq.(21,24,25). The expression of $\phi_0$ is

$$\phi_0 = \sum_n g_n \ln|z - z_n|^2 - \alpha\phi_B + c_0. \quad (23)$$

Then we have for $\phi_M$

$$4\partial_z\partial_{\bar{z}}\phi_M = e^{\phi_0+\phi_B}\phi_M + (\alpha - 1)4\partial_z\partial_{\bar{z}}\phi_B. \quad (24)$$

$\phi_M$ is a continuous function on the Riemann sphere. The action which generates the above equation is

$$S = \int d\mu [e^{-\phi_B}\partial_z\phi_M\partial_{\bar{z}}\phi_M + \frac{e^{\phi_0+\phi_M}}{2} + 2(\alpha - 1)e^{-\phi_B}\phi_M\partial_z\partial_{\bar{z}}\phi_B] = \int d\mu F \quad \text{with} \quad d\mu \equiv e^{\phi_B} \frac{idz \wedge d\bar{z}}{2}$$

where the splitting between the measure and the integrand has been introduced for later convenience. Due to the behavior of $\phi_M$ and $\phi_0$ at the singularities and at infinity the integral in eq.(25) converges absolutely. It is straightforward to prove that the action $S$ computed on the solution of eq.(24) is related to the original Polyakov action $S_P$ also computed on the solution of eq.(24) by

$$S_P = S - (\alpha - 1)^2 \int \phi_B\partial_z\partial_{\bar{z}}\phi_B \frac{idz \wedge d\bar{z}}{2} + 2\pi(\alpha - 1)^2c_B +$$

$$+ \pi \sum_m \sum_{n \neq m} g_m g_n \ln|z_m - z_n|^2 + 4\pi c_0(1 - \alpha). \quad (26)$$
The behavior of \( \phi_M \) around the singularities \( z_m \) can be deduced from eqs. (7,11). Thus in a finite neighborhood of \( z_m \) we can write

\[
\phi_M = \sum_{nLMN} c_{nLMN} [(z - z_n)(\bar{z} - \bar{z}_n)]^{L(g_n + 1)}(z - z_n)^M(\bar{z} - \bar{z}_n)^N \rho(|z - z_n|) + \phi_{Mr} \tag{27}
\]

where the finite sum extends to the terms such that \( 2L(g_n + 1) + M + N \leq 3 \) and \( \rho(|\zeta|) \) is chosen \( C^\infty \) with \( \rho = 1 \) in a finite neighborhood of 0 and zero for \( |\zeta| > 1 \). \( \phi_{Mr} \) is a continuous function \( O(|z - z_n|^3) \) around each \( z_n \). We saw in sect.2 how except for a finite number of points in a neighborhood of \( z_n \) there exists the derivative of the parameters \( k, \text{Re}\beta_i, \text{Im}\beta_i \) which determine the solutions of the fuchsian equation related by eq. (7) to the conformal factor \( \phi \). Actually as pointed out at the end of sect.(2) around the points where \( V' \) does not vanish such parameters are analytic functions of \( z_n \). On the other hand the solutions of the fuchsian equation and thus \( \phi_M \) depend analytically on such parameters \[16\].

The procedure to compute the derivative will be to prove that

\[
\frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X_\epsilon} d\mu \frac{\partial F}{\partial z_m} \tag{28}
\]

where \( F \) is given in eq.(23) and \( X_\epsilon \) has been defined after eq.(2).

This is achieved by writing \( F = (F - f) + f \) where \( F - f \) is sufficiently regular, i.e. continuous and absolutely integrable with \( \frac{\partial(F-f)}{\partial z_m} \) continuous and \( |\frac{\partial(F-f)}{\partial z_m}| < M \) for any \( z \) while \( z_m \) varies in a finite interval, so that

\[
\frac{\partial}{\partial z_m} \int (F - f) d\mu = \int \frac{\partial}{\partial z_m} (F - f) d\mu \equiv \lim_{\epsilon \to 0} \int_{X_\epsilon} \frac{\partial}{\partial z_m} (F - f) d\mu \tag{29}
\]

proving at the same time that

\[
\frac{\partial}{\partial z_m} \int f d\mu = \lim_{\epsilon \to 0} \int_{X_\epsilon} \frac{\partial}{\partial z_m} f d\mu. \tag{30}
\]

Then by summing eq.(29) and eq.(30) we obtain eq.(28).

Using the expansions eq.(27) we shall choose the function \( f \) as

\[
f d\mu = \sum_{nLMN} b_{nLMN} \left[ (z - z_n)(\bar{z} - \bar{z}_n) \right]^{L(g_n + 1)}(z - z_n)^M(\bar{z} - \bar{z}_n)^N \rho(|z - z_n|) \frac{idz \wedge d\bar{z}}{2} \tag{31}
\]
\[ \sum_{nLMN} b_{nLMN} G_{nLMN}(z - z_n) \frac{idz \wedge d\bar{z}}{2} \]

where the finite sum extend to all singularities of \( F \) and \( M, N \geq 0 \) and \( L \geq 1 \) such that \( 2L(g_m + 1) + M + N \leq 3 \). We notice that

\[ \frac{\partial}{\partial z_m} \int f d\mu = \frac{\partial}{\partial z_m} \sum_{nLMN} b_{nLMN} \int G_{nLMN}(z - z_n) \frac{idz \wedge d\bar{z}}{2} = \] (32)

\[ \sum_{nLMN} \frac{\partial b_{nLMN}}{\partial z_m} \int G_{nLMN}(z - z_n) \frac{idz \wedge d\bar{z}}{2} \]

the point being that each integral in the sum does not depend on \( z_n \) due to translational invariance and thus we have to take only the derivative of the coefficients \( b_{nLMN} \). Moreover

\[ \int_{X_{\epsilon}} \frac{\partial f}{\partial z_m} d\mu = \sum_{nLMN} \frac{\partial b_{nLMN}}{\partial z_m} \int_{X_{\epsilon}} G_{nLMN}(z - z_n) \frac{idz \wedge d\bar{z}}{2} - \sum_{LMN} b_{mLMN} \int_{X_{\epsilon}} \frac{\partial}{\partial z} G_{mLMN}(z - z_m) \frac{idz \wedge d\bar{z}}{2}. \] (33)

The last term is either zero due to the phase of the integrand or goes to zero for \( \epsilon \to 0 \) by power counting and thus we have the stated result eq.(30). We are left to prove that in eq.(29) we can take the derivative operation under the integral sign. To this purpose it is sufficient to prove that \( F - f \) and \( \frac{\partial (F - f)}{\partial z_n} \) are continuous on the product of the Riemann sphere and a closed disk of \( z_n \) having for center the values of \( z_n \) for which according to sect.2 the derivative of \( k, \beta_n, \bar{\beta}_n \) exists. In fact \( F - f \) and \( \frac{\partial (F - f)}{\partial z_n} \) are free of singularities both at the finite and at infinity. As the product of the Riemann sphere and a closed disk is a compact set the hypothesis above stated are satisfied and this allows the exchange of the derivative operation with the integral sign. Using now the equation of motion (24) we obtain

\[ \frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X_{\epsilon}} \left[ \partial_{\epsilon} \left( \frac{\partial \phi_M}{\partial z_m} \partial_{\epsilon} \phi_M \right) + \partial_{\epsilon} \left( \frac{\partial \phi_M}{\partial z_m} \partial_{\epsilon} \phi_M \right) + \frac{\partial \phi_0}{\partial z_m} \frac{e^\phi}{2} \right] \frac{idz \wedge d\bar{z}}{2}. \] (34)

It is easily checked that the only contribution which survives in the limit \( \epsilon \to 0 \) is

\[ \frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X_{\epsilon}} \frac{e^\phi}{2} \frac{\partial \phi_0}{\partial z_m} \frac{idz \wedge d\bar{z}}{2}. \] (35)
which can be computed by using eq.(24) and \( \frac{\partial \phi_0}{\partial z_m} = -\frac{g_m}{z - z_m} \) to obtain

\[
\frac{\partial S}{\partial z_m} = -i g_m \lim_{\epsilon \to 0} \oint_{\gamma} \frac{1}{z - z_m} \partial_z (\phi_M - (\alpha - 1)\phi_B) \, dz.
\]  

(36)

Using \( \phi_M - (\alpha - 1)\phi_B = \phi - \sum_n g_n \ln |z - z_n|^2 \) and the expansion of \( A = 1 + c_1 \zeta + \cdots \) and \( B = 1 + c_2 \zeta + \cdots \) which are obtained by substituting into the differential equation (8) to obtain

\[
c_1 = -\frac{\beta_m}{2(g_m + 2)} \quad \text{and} \quad c_2 = \frac{\beta_m}{2g_m}
\]  

(37)

finally we have

\[
\frac{\partial S}{\partial z_m} = -2\pi \beta_m - 2\pi \sum_{n,n \neq m} \frac{g_m g_n}{z_m - z_n}
\]  

(38)

equivalent to Polyakov conjecture eq.(1) due to the relation (26) between \( S \) and \( S_P \).

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