POLYNOMIALS WHOSE COEFFICIENTS COINCIDE WITH THEIR ZEROS

OKSANA BIHUN, AND DAMIANO FULGUESU

Abstract. In this paper we consider monic polynomials such that their coefficients coincide with their zeros. These polynomials were first considered by S. Ulam. We obtain estimates on the number of Ulam polynomials of degree $N$ using methods of algebraic geometry. We show that the only Ulam polynomial eigenfunctions of hypergeometric type differential operators are the polynomials $\{x^N\}_{N=0}^\infty$, which are eigenfunctions of the differential operator $\alpha x^2 \frac{d^2}{dx^2} - x \frac{d}{dx}$ with the corresponding eigenvalues $\{N(N - 1)\alpha - N\}_{N=0}^\infty$. We propose a family of solvable $N$-body problems such that their stable equilibria are the zeros of certain Ulam polynomials.

1. Introduction

Let $N$ be a positive integer. We can describe the set of monic polynomials in $\mathbb{C}[z]$ of degree $N$ as the space $\mathbb{C}^N$ representing the coefficients other than the leading coefficient.

We define the map

$$\psi^{(N)} : \mathbb{C}^N \rightarrow \mathbb{C}^N$$

$$(c_1, c_2, \ldots, c_N) \mapsto (-s_1^{(N)}, s_2^{(N)}, \ldots, (-1)^N s_N^{(N)})$$

where $s_j^{(N)}$ is the $j^{th}$ symmetric polynomial in the $N$ variables $c_i$:

$$s_j^{(N)} = s_j^{(N)}(c_1, \ldots, c_N) = \frac{1}{j!} \sum_{n_1, \ldots, n_j = 1}^N c_{m_1} \cdots c_{m_j}.$$ (1)

The basic idea is that the map $\psi^{(N)}$ sends a monic polynomial with coefficients $c_1, c_2, \ldots, c_N$ into another monic polynomial whose zeros are exactly $c_1, c_2, \ldots, c_N$. The map $\psi^{(N)}$ was proposed by Ulam (see [6], p.31), hence we will refer to it as the Ulam map. Ulam wrote that "Many of the statements about algebraic equations are translatable into the elementary properties of this mapping." One of the questions posed by Ulam is the identification of nontrivial fixed points of the map.
ψ^{(N)}(\gamma) \equiv \gamma, that is to say the monic polynomials such that their zeros coincide with their coefficients. In the following we will refer to such polynomials as Ulam polynomials. In [5] it is shown that for \( N \geq 5 \) the Ulam map \( \psi^{(N)} \) does not have a fixed point with the property that all its components are real and distinct from zero. In this paper, we focus on counting the number of the fixed points of \( \psi^{(N)} \) and on finding nontrivial complex fixed points of \( \psi^{(N)} \). Moreover, we show that the only Ulam polynomial eigenfunctions of hypergeometric type differential operators are the polynomials \( \{x^N\}_{N=0}^{\infty} \), which are eigenfunctions of the differential operator \( \alpha x^2 \frac{d^2}{dx^2} - x \frac{d}{dx} \) with the corresponding eigenvalues \( \{N(N-1)\alpha - N\}_{N=0}^{\infty} \). We propose a family of solvable \( N \)-body problems such that their stable equilibria are the zeros of certain Ulam polynomials.

Below we provide several useful definitions and theorems that will be used in the subsequent exposition.

**Definition 1.1.** The dimension of a ring \( R \), written \( \text{dim}(R) \), is the maximum integer \( N \) such that there is a strictly ascending chain of prime ideals

\[
P_0 \subset P_1 \subset \cdots \subset P_N.
\]

The dimension of an ideal \( I \subset R \), written \( \text{dim}(R/I) \) is the dimension of the quotient ring \( R/I \).

**Definition 1.2.** The radical of an ideal \( I \) in a ring \( R \) is the set

\[
\sqrt{I} := \{ f \in R | f^m \in I \text{ for some } m \in \mathbb{N} \}.
\]

An ideal \( I \) is said to be radical if \( \sqrt{I} = I \).

The following statement is a consequence of the general Bezout’s Theorem and it is a known result in algebraic geometry.

**Theorem 1.3.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_N \) be polynomials in \( \mathbb{C}[c_1, c_2, \ldots, c_N] \) respectively of degree \( d_1, d_2, \ldots, d_N \). Moreover, let \( I \) be the ideal generated by \( \alpha_1, \alpha_2, \ldots, \alpha_N \). Assume that \( \text{dim}(I) = 0 \), that is to say the variety associated to \( I \) is a set of points. Then the system \( \{\alpha_i = 0\}_{i=1}^{N} \)

has exactly \( \prod_{j=1}^{N} d_j \) solutions if and only if the following two conditions are satisfied:

(a) \( I \) is a radical ideal,
(b) the system \(\{\alpha_i = 0\}_{i=1}^{N}\) does not have solutions at infinity (see Remark 2.3 for an explanation of solutions at infinity).

2. Some Properties of the Fixed Points of the Ulam Maps

Recall that a point \(\gamma = (\gamma_1, \ldots, \gamma_N)\) is a fixed point of the Ulam map \(\psi^{(N)}\) if and only if the zeros of the monic polynomial \(p_N(x) = x^N + \sum_{m=1}^{N} \gamma_m x^{N-m}\) coincide with its coefficients

\[
p_N(x) = x^N + \sum_{m=1}^{N} \gamma_m x^{N-m} = \prod_{n=1}^{N} (x - \gamma_n)
\]

or, equivalently, these coefficients satisfy the \(N \times N\) system

\[
c_j = (-1)^j s_j^{(N)}(c_1, \ldots, c_N), \quad j = 1, 2, \ldots, N,
\]

where the symmetric polynomials \(s_j^{(N)}\) are given by (1).

On the other hand, every fixed point \(\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N, \tilde{\gamma}_{N+1})\) of the Ulam map \(\psi^{(N+1)}\) satisfies the \((N+1) \times (N+1)\) system

\[
\tilde{c}_j = (-1)^j \left[ s_j^{(N)}(\tilde{c}_1, \ldots, \tilde{c}_N) + \tilde{c}_{N+1} s_{j-1}^{(N)}(\tilde{c}_1, \ldots, \tilde{c}_N) \right], \quad j = 1, 2, \ldots, N, N + 1.
\]

The last equation in system (3) reads

\[
\tilde{\gamma}_{N+1} = (-1)^N \tilde{\gamma}_1 \cdots \tilde{\gamma}_N \tilde{\gamma}_{N+1}.
\]

Clearly, if \(\tilde{c}_{N+1} = 0\), system (3) reduces to system (2) and it is easy to conclude the following.

**Proposition 2.1.** Suppose that \((\gamma_1, \ldots, \gamma_N) \in \mathbb{C}^N\) is a fixed point of the map \(\psi^{(N)}\). Then the following are true.

(a) For every positive integer \(n\) the point \((\gamma_1, \ldots, \gamma_N, 0, \ldots, 0) \in \mathbb{C}^{N+n}\) is a fixed point of the map \(\psi^{(N+n)}\).

(b) If one of the components of the vector \((\gamma_1, \ldots, \gamma_N)\) vanishes, then all the subsequent components vanish as well. That is, if \(\gamma_j = 0\) for some \(j \in \{1, 2, \ldots, N-1\}\), then \(\gamma_{j+1} = \gamma_{j+2} = \ldots = \gamma_N = 0\).

Another way to illustrate statement (a) of Proposition 2.1 is to say that if the zeros of a monic polynomial \(p_N(x)\) coincide with its coefficients, then the zeros of the monic polynomial \(x^n p_N(x)\) also coincide with its coefficients.

In the following we make use of the polynomials \(\alpha_j\) defined by

\[
\alpha_j = \alpha_j(c_1, \ldots, c_N) = s_j^{(N)}(c_1, \ldots, c_N) - (-1)^j c_j, \quad j = 1, 2, \ldots, N.
\]
Let $I_N$ be the ideal in $\mathbb{C}[c_1, \ldots, c_N]$ generated by the set $\{\alpha_i\}_{i=1,2,\ldots,N}$. Let $V(I_N)$ be the algebraic variety generated by the ideal $I_N$, the set of common zeros of all polynomials in $I_N$. In other words, $V(I_N)$ is the set of solutions of the system $\{\alpha_i = 0\}_{i=1,2,\ldots,N}$.

In the following section, we will make use of Theorem 1.3, so as a preliminary step, we prove the following statement.

**Lemma 2.2.** For all $N$, the ideal $I_N$ generated by the polynomials $\{\alpha_i\}_{i=1,2,\ldots,N}$ defined by (4) has dimension zero, that is, $\dim(I_N) = 0$.

*Proof.* The statement has already been proved in [4]. We provide a slightly different proof in order to point out a crucial step that we will use in Corollary 2.3.

Consider the projective space $\mathbb{P}^N(\mathbb{C})$. We introduce homogeneous co-ordinates in $\mathbb{P}^N(\mathbb{C}) : [C_0 : C_1 : \cdots : C_N]$ such that $\mathbb{C}^N$ is the chart corresponding to coordinates $c_i = \frac{C_i}{C_0}$ for $i = 1, 2, \ldots, N$. Using standard results in algebraic geometry, we conclude the following. If $V(I_N)$ has a component with positive dimension, then its closure $\bar{V}(I_N)$ in $\mathbb{P}^N(\mathbb{C})$ must intersect the hyperplane $C_0 = 0$. The algebraic set $\bar{V}(I_N)$ is defined by equations:

\begin{equation}
\tag{5} s_j^{(N)}(C_1, \ldots, C_N) - (-1)^j C_j C_0^{N-1} = 0, \quad j = 1, 2, \ldots, N.
\end{equation}

By substituting $C_0 = 0$ into system (5), we obtain

\begin{align*}
  s_1^{(N)}(C_1, \ldots, C_N) &= -C_1, \\
  s_j^{(N)}(C_1, \ldots, C_N) &= 0, \quad j = 2, \ldots, N - 1, \\
  C_1 C_2 \ldots C_N &= 0.
\end{align*}

The last equation in system (6) implies that at least one among the $C_i$ for $i = 1 \ldots N$ must equal zero. By going backward through system (6), we obtain that all $C_i$ must equal zero, hence the system has no solutions in $\mathbb{P}^N(\mathbb{C})$. \qed

**Corollary 2.3.** System $\alpha_j = 0$ for $j = 1, \ldots, N$ does not have solutions at infinity, that is, the compactification $\bar{V}(I_N)$ does not intersect the hyperplane $C_0 = 0$ in the projective space $\mathbb{P}^N(\mathbb{C})$ defined above.

In conclusion, we have that the ideal $I_N$ satisfies all the hypotheses of Theorem 1.3 except for the radicality condition (a). In the next section we show that $I_N$ is not radical if $N \geq 4$. Instead, we make use of a modified ideal $\bar{I}_N$ which we verify to be radical up to $N = 5$. 

3. Number of Ulam polynomials of degree $N$

Let $U_N$ be the set of Ulam polynomials of degree $N$. In this section we derive some statements on the number $|U_N|$ for arbitrary values of $N$ and also compute $|U_N|$ for small values of $N$.

Recall that $I_N$ is the ideal generated by the polynomials $\{\alpha_j\}_{j=1}^{N}$ defined by (4). We define a new ideal $\tilde{I}_N := \langle \alpha_1, \ldots, \alpha_{N-1}, \tilde{\alpha}_N \rangle$ where $\tilde{\alpha}_N = s_{N-1}(c_1, \ldots, c_{N-1}) - (-1)^N$.

Since $\alpha_N = c_N \cdot \tilde{\alpha}_N$, we have $I_N \subset \tilde{I}_N$ and therefore $V(\tilde{I}_N) \subset V(I_N)$. In particular, $V(\tilde{I}_N)$ contains all the solutions of system (2) such that $c_N \neq 0$. However, the set $V(\tilde{I}_N)$ may still contain some solutions with $c_N = 0$. Let us also define the ideal $I_0^N := \langle I_N, c_N \rangle$.

From the inclusion-exclusion principle we have

$$|U_N| = |V(I_0^N)| + |V(\tilde{I}_N)| - |V(I_0^N) \cap V(\tilde{I}_N)|.$$  

(7)

In order to perform computations in some special cases, we need the following Lemmas.

**Lemma 3.1.** For all integers $N > 1$, we have the identity

$$|V(I_0^N)| = |U_{N-1}|.$$  

Proof. Let $U_0^N$ denote the set of Ulam polynomials of degree $N$ that have a root $x = 0$ (i.e. divisible by $x$). Consider the map $\phi : U_{N-1} \to U_0^N$ defined by $\phi(P(x)) = x \cdot P(x)$. It is straightforward to show that $\phi$ is a bijection.  

**Lemma 3.2.** If $\tilde{I}_N$ is radical, then $|V(\tilde{I}_N)| = (N-1) \cdot (N-1)!$ for all integers $N > 0$.

Proof. Since $V(\tilde{I}_N) \subset V(I_N)$, we know from Lemma 2.2 that $|V(\tilde{I}_N)|$ is finite. Moreover, by arguing as in the proof of Lemma 2.2 we know that the system $\alpha_1 = 0, \ldots, \alpha_{N-1} = 0, \tilde{\alpha}_N = 0$ does not have solutions at infinity. Therefore we can apply Theorem 1.3.

**Lemma 3.3.** The ideal $\tilde{I}_N$ is radical for $N = 1, 2, 3, 4, 5$.

Proof. From Bezout’s Theorem we know that $|V(I_N)| \leq n!$. By combining this result with formula (7) and Theorem 1.3 we have that if $|V(I_N)| = N!$, then $\tilde{I}_N$ must be radical. In Subsection 3.1 we show that $|V(I_N)| = N!$ for $N = 1, 2, 3$ and, moreover, provide details of our check of the radicality of $\tilde{I}_4$ and $\tilde{I}_5$, which we performed using the programming environment Maple.
3.1. **Calculations of $|U_N|$ for $N = 1, 2, 3, 4, 5$.**

- $|U_1| = 1$. It is straightforward to check that the only Ulam polynomial of degree 1 is $x$.

- $|U_2| = 2$. In this case, we directly solve the system
  \[
  \begin{align*}
  c_1 + c_2 &= -c_1, \\
  c_1c_2 &= c_2,
  \end{align*}
  \]
  and obtain $U_2 = \{x^2, x^2 + x - 2\}$ and $|U_2| = 2$.

- $|U_3| = 6$. We directly solve system (2) again, this time for $N = 3$, but now we refer to equation (7). We already know that $|V(I_0^3)| = |U_2| = 2$. Moreover, the ideal $\tilde{I}_3$ is generated by
  \[
  \begin{align*}
  2c_1 + c_2 + c_3, \\
  c_1c_2 + c_1c_3 + c_2c_3 - c_2, \\
  c_1c_2 - 1.
  \end{align*}
  \]
  By solving the associated system for $c_1$, we obtain that $c_1$ must be a zero of the polynomial $(x-1) \cdot (2x^3 + 2x^2 - 1)$. Therefore, we obtain four solutions given by
  \[
  (c_1, c_2, c_3) = (1, -1, -1),
  \]
  \[
  (c_1, c_2, c_3) = \left(\beta_i, -\frac{1}{\beta_i}, \frac{1}{\beta_i + 1}\right),
  \]
  where $\beta_1, \beta_2$ and $\beta_3$ are the three distinct zeros of $2x^3 + 2x^2 - 1$.
  In all of these four solutions we have $c_3 \neq 0$, therefore the set
  \[
  |V(I_0^3) \cap V(\tilde{I}_3)|
  \]
  is empty. In conclusion, we have $|U_3| = 2 + 4 = 6$.

- $|U_4| = 23$. We already know that $|V(I_0^4)| = |U_3| = 6$. We verified that the ideal $\tilde{I}_4$ is radical using Maple. Therefore, from Lemma 3.3 we have $|V(\tilde{I}_4)| = 18$. 
In order to determine $|V(I_0^0) \cap V(\tilde{I}_4)|$, we need to find the number of solutions of the system

\[
\begin{align*}
c_1 + c_2 + c_3 &= -c_1, \\
c_1c_2 + c_1c_3 + c_2c_3 &= c_2, \\
c_1c_2c_3 &= -c_3, \\
c_1c_2c_3 &= 1,
\end{align*}
\]

which is the number of solutions in the case $N = 3$ with the extra condition $c_1c_2c_3 = 1$. A simple check shows that exactly one of the solutions in the previous case satisfies this extra condition: $(c_1, c_2, c_3) = (1, -1, -1)$.

In conclusion, we have $|U_4| = 6 + 18 - 1 = 23$.

- $|U_5| = 119$. We already know that $|V(I_0^5)| = |U_4| = 23$. We verified that the ideal $\tilde{I}_5$ is radical using Maple. Therefore, from Lemma 3.3, we have $|V(\tilde{I}_5)| = 96$.

The number $|V(I_0^5) \cap V(\tilde{I}_5)|$ is equal to the number of solutions of the system

\[
\begin{align*}
c_1 + c_2 + c_3 + c_4 &= -c_1, \\
c_1c_2 + c_1c_3 + c_2c_3 + c_4(c_1 + c_2 + c_3) &= c_2, \\
c_1c_2c_3 + c_4(c_1c_2 + c_1c_3 + c_2c_3) &= -c_3, \\
c_1c_2c_3c_4 &= c_4 \\
c_1c_2c_3c_4 &= -1.
\end{align*}
\]

From the last two equations we obtain that $c_4 = -1$. Therefore, the system can be rewritten as follows:

\[
\begin{align*}
c_1 + c_2 + c_3 &= 1 - c_1, \\
c_1c_2 + c_1c_3 + c_2c_3 &= 1 - c_1 + c_2, \\
c_1c_2c_3 &= 1 - c_1 + c_2 - c_3, \\
c_1c_2c_3 &= 1.
\end{align*}
\]

From the last two equations we obtain

\[-c_1 + c_2 - c_3 = 0.\]
By combining this last equation with the first of the two equations, we obtain

\[ c_2 = \frac{1}{2}(1 - c_1), \]

(9)

\[ c_3 = \frac{1}{2}(1 - 3c_1). \]

We substitute the last two expressions for \( c_2 \) and \( c_3 \) into the equation \( c_1c_2 + c_1c_3 + c_2c_3 = 1 - c_1 + c_2 \) to obtain a polynomial of degree two in \( c_1 \) with the roots

\[ c_1 = \frac{3 \pm 4i}{5}. \]

Now, we use equations (9) in order to determine \( c_2 \) and \( c_3 \). It is straightforward to check that in both of the two cases we have \( c_1c_2c_3 \neq 1 \). Therefore, system (8) has no solutions and \( |V(I_0^5) \cap V(I_5^{|})| = 0. \)

In conclusion, we have \( |U_5| = 23 + 96 = 119. \)

**Theorem 3.4.** For all \( N \geq 4 \), the system \( \{\alpha_j = 0\}_{j=1,2,\ldots,N} \) has a solution of multiplicity larger than 1. In particular, \( |U_N| < N! \) for all \( N \geq 4. \)

**Proof.** Bezout’s Theorem tells us that there is a solution of multiplicity larger than 1 if and only if \( |U_N| < N! \). From formula (7) and Lemma 3.3 we have that \( |U_N| = N! \) if and only if the following three conditions are simultaneously satisfied:

(a) \( |U_{N-1}| = (N - 1)! \),
(b) \( \tilde{I}_N \) is radical,
(c) \( |V(I_N^0) \cap V(\tilde{I}_N)| = 0. \)

If for some \( N_0 \) any of the above conditions is false, then \( |U_N| < N! \) for all \( N \geq N_0. \) As we showed above, we have \( |U_4| = 23 < 4!. \) \( \square \)

**Remark 3.5.** It is evident that \( |U_N| \geq 1 \) because \( x^N \) is a (trivial) Ulam polynomial. Sharper lower bounds for \( U_N \) can be obtained by using Proposition 2.1. If \( p_m(x) \) is an Ulam polynomial of degree \( m \), then \( x^kp_m(x) \) is an Ulam polynomial of degree \( m + k \). In Subsection 3.1 we showed the existence of 8 nontrivial Ulam polynomials of degrees 2 and 3, thus there exist at least 8 nontrivial Ulam polynomials of degree \( N \), for all \( N \geq 4. \)
4. Ulam Polynomial Eigenfunctions of Hypergeometric Type Differential Operators

In this section we explore the following question: Are there sequences of Ulam polynomials that are eigenfunctions of hypergeometric type differential operators? This question is related to a more general question: Are there sequences of Ulam polynomials that are orthogonal with respect to some measure?

Let \( \{p_N(x)\}_{N=0}^{\infty} \) be a sequence of Ulam polynomials, in which the \( N \)-th term is given by \( p_N(x) = x^N + \sum_{n=1}^{N} \gamma_n^{(N)} x^{N-n} = \prod_{m=1}^{N} (x - \gamma_m^{(N)}) \). Of course, for each \( N \in \mathbb{N} \), the coefficients \( (\gamma_1^{(N)}, \ldots, \gamma_N^{(N)}) \) solve system (2). According to [2], a monic polynomial \( y_N(x) = x^N + \sum_{n=1}^{N} C_n^{(N)} x^{N-n} \) of degree \( N \) is an eigenfunction of the differential operator \( p(x) \frac{d^2}{dx^2} + q(x) \frac{d}{dx} \), that is

\[
(10) \quad p \ y''_N + q \ y'_N + \lambda_N \ y_N = 0,
\]

where \( p(x) = \alpha x^2 + \beta x \) and \( q(x) = -(x + a_1) \), if and only if \( \lambda_N = N - N(N-1)\alpha \) and the coefficients \( C_j^{(N)} \) satisfy the recurrence relations

\[
C_1^{(N)} = \frac{N a_1 - N(N-1)\beta}{\lambda_N - \lambda_{N-1}},
\]

\[
C_\ell^{(N)} = \frac{(N - \ell + 1)[a_1 - (n - \ell)\beta]}{\lambda_N - \lambda_{N-\ell}} C_{\ell-1}^{(N)} - \frac{(n - \ell + 2)(n - \ell + 1)}{\lambda_N - \lambda_{N-\ell}} C_{\ell-2}^{(N)},
\]

\[
C_0^{(N)} = 0,
\]

see [2] and note the misprint in the cited paper: \( p(x) = \alpha x^2 + \beta x \), not \( p(x) = \alpha x^2 + \beta x + c \).

Suppose that each polynomial \( p_N(x) \) in the sequence \( \{p_N(x)\}_{N=0}^{\infty} \) of Ulam polynomials solves differential equation (10). Then its coefficients \( (\gamma_1^{(N)}, \ldots, \gamma_N^{(N)}) \) not only solve system (2), but also satisfy recurrence relations (11).

For example, if \( N = 2 \), recurrence relations (11) give

\[
\gamma_1^{(2)} = \frac{2(a_1 - \beta)}{1 - 2\alpha},
\]

\[
\gamma_2^{(2)} = \frac{2a_1(a_1 - \beta)}{(1 - 2\alpha)^2}.
\]
Similarly, if $N = 3$, recurrence relations (11) give

\[
\gamma_1^{(3)} = \frac{3(a_1 - 2\beta)}{1 - 4\alpha},
\]
\[
\gamma_2^{(3)} = \frac{3(a_1 - 2\beta)(a_1 - 2\beta)}{(1 - 3\alpha)(1 - 4\alpha)},
\]
\[
\gamma_3^{(3)} = \frac{(a_1 - 2\beta)(a_1(a_1 - \beta) - 2 + 6\alpha)}{(1 - 2\alpha)(1 - 3\alpha)(1 - 4\alpha)}.
\]

By substituting the above expressions for the five coefficients $\gamma_1^{(2)}, \gamma_2^{(2)}, \gamma_1^{(3)}, \gamma_2^{(3)}, \gamma_3^{(3)}$ into the system

\[
2\gamma_1^{(2)} + \gamma_2^{(2)} = 0,
\]
\[
\gamma_2^{(2)} - \gamma_1^{(2)}\gamma_2^{(2)} = 0,
\]
\[
2\gamma_1^{(3)} + \gamma_2^{(3)} + \gamma_3^{(3)} = 0,
\]
\[
\gamma_2^{(3)} - \gamma_1^{(3)}\gamma_2^{(3)} - \gamma_2^{(3)}\gamma_3^{(3)} - \gamma_1^{(3)}\gamma_3^{(3)} = 0,
\]
\[
\gamma_3^{(3)} + \gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)} = 0,
\]
we obtain $a_1 = \beta = 0$. Note that the first two equations of the last system ensure that $p_2(x) = x^2 + \gamma_1^{(2)}x + \gamma_2^{(2)}$ is an Ulam polynomial, while the remaining three equations ensure that $p_3(x) = x^3 + \gamma_1^{(3)}x^2 + \gamma_2^{(3)}x + \gamma_3^{(3)}$ is also an Ulam polynomial. Therefore, by (12), (13), $\gamma_1^{(2)} = \gamma_2^{(2)} = \gamma_1^{(3)} = \gamma_2^{(3)} = \gamma_3^{(3)} = 0$. Moreover, from (11) we conclude that $\gamma_j^{(N)} = 0$ for all $j = 1, \ldots, N$, where $N \in \mathbb{N}$.

It is easy to verify that for each $N \in \mathbb{N}$, the Ulam polynomial $p_N(x) = x^N$ solves the differential equation (10) with $a_1 = \beta = 0$. Thus we have proved the following result.

**Theorem** If a solution $y_N$ of the differential equation (10) is an Ulam polynomial of degree $N$, then $a_1 = \beta = 0$ and $y_N = x^N$. In other words, the only Ulam polynomial eigenfunctions of hypergeometric type differential operators are the polynomials $\{x^N\}_N^{\infty}$, which are eigenfunctions of the differential operator $\alpha x^2 \frac{d^2}{dx^2} - x \frac{d}{dx}$ with the corresponding eigenvalues $\{N(N - 1)\alpha - N\}_N^{\infty}$.

5. **Zeros of Ulam Polynomials as Equilibria of Certain Dynamical Systems**

Let $\gamma = (\gamma_1, \ldots, \gamma_N)$ be the coefficients of an Ulam polynomial

\[
p_N(z) \equiv p_N(z, \gamma) = z^N + \sum_{m=1}^{N} \gamma_m z^{N-m} = \prod_{n=1}^{N} (z - \gamma_n)
\]
such that the components of $\gamma$ are all different among themselves. Consider the polynomial

$$q_N(z, t) \equiv q_N(z, t; a, b)$$

$$= e^t z^N + \sum_{m=1}^{N} [\gamma_m(e^t + a) + b_m] z^{N-m} = e^t \prod_{n=1}^{N} (z - \zeta_n(t))$$

(15)

with time-dependend coefficients, where $a$ is a constant and $b = (b_1, \ldots, b_N)$ is an $N$-vector of constants, while $\zeta_n(t)$ are the zeros of the polynomial $q_N(z, t)$. Upon differentiation of $q_N(z, t)$ defined by (15) with respect to $t$ followed by the substitution $z = \zeta_n(t)$, we obtain a system of nonlinear ODEs satisfied by the time-dependent zeros $\zeta_n(t)$ of $q_N(z, t)$:

$$\dot{\zeta}_n(t) = -\left[ \prod_{\ell=1, \ell \neq n}^{N} (\zeta_n - \zeta_\ell)^{-1} \right] \left[ (\zeta_n)^N + \sum_{m=1}^{N} \gamma_m(\zeta_n)^{N-m} \right].$$

(16)

System (16) is solvable in the sense that the process of finding its solutions can be reduced to the process of finding zeros of the polynomial $q_N(z, t)$. Clearly, the vector of coefficients (and the zeros) $\gamma$ of the Ulam polynomial $p_N(z; \gamma)$ is an equilibrium of system (16). The same is true for each of the distinct vectors $\gamma_\sigma$ obtained by permuting the components of $\gamma$, where $1 \leq \sigma \leq N!$.

Let us linearize system (16) about its equilibrium $\gamma$. For convenience, let us denote the right-hand side of the $n$-th equation in system (16) by $f_n(\zeta)$, where $\zeta = (\zeta_1, \ldots, \zeta_N)$ and consider the vector function

$$f(\zeta) = (f_1(\zeta), \ldots, f_N(\zeta))$$

so that system (16) is recast in the form

$$\frac{d\zeta}{dt} = f(\zeta).$$

(17)

Note that the function $f$ is of class $C^2$ in an open neighborhood of the point $\gamma$ because the components of $\gamma$ are all different among themselves. By Taylor’s Theorem, there exists a constant $\alpha > 0$ such that for every $\zeta$ in the open ball $B(\gamma, \alpha)$ centered at $\gamma$ and having the radius $\alpha$ we have

$$f(\zeta) = Df(\gamma)(\zeta - \gamma) + \tilde{g}(\zeta).$$

(18)

Moreover, there exist positive constants $\beta$ and $\kappa$ such that for every $\zeta \in B(\gamma, \beta)$ we have

$$|\tilde{g}(\zeta)| < \kappa |\zeta - \gamma|^2.$$
It is easy to verify that $Df(\gamma) = -I$ is the negative of the $N \times N$ identity matrix $I$. We thus recast system (16) or (17) as
\begin{equation}
\dot{\xi} = -\xi + g(\xi),
\end{equation}
where $\xi(t) = \zeta(t) - \gamma$ and $g(\xi) = \tilde{g}(\xi + \gamma)$ satisfies
\[|g(\xi)| < \kappa |\xi|^2\]
for all $\xi \in B(0, \beta)$. The fundamental matrix solution of the linearization
\begin{equation}
\dot{y} = -y,
\end{equation}
of system (19) is $e^{-It}$, where $y(t) = (y_1(t), \ldots, y_N(t))$. Therefore, by the variation of parameters formula [3], the solution $\xi(t)$ of system (19) with the initial condition $\xi(t_0) = \xi_0$ is given by
\begin{equation}
\xi(t) = e^{-I(t-t_0)}\xi_0 + \int_{t_0}^{t} e^{-I(t-s)}g(\xi(s)) \, ds.
\end{equation}
By a theorem about stability of equilibria of nonlinear dynamical systems [3], $0 \in \mathbb{R}^N$ is a stable equilibrium of system (19), hence $\gamma$ is a stable equilibrium of system (16).

6. Discussion and Outlook

The authors plan to improve the results reported in this paper by obtaining sharper estimates or exact formulas for the number of Ulam polynomials of degree $N$. Other possible investigations include discovery of differential equations satisfied by Ulam polynomials, existence or non-existence of measures with respect to which sequences of Ulam polynomials are orthogonal and further investigation of dynamical systems such that their equilibria are the zeros of Ulam polynomials. We also plan to study sequences of monic polynomials defined by
\[p_0(x) = x^N + \sum_{m=1}^{N} c^{(0)}_m x^{N-m},\]
\[p_1(x) = \prod_{m=1}^{N} (x - c^{(0)}_m) = x^N + \sum_{m=1}^{N} c^{(1)}_m x^{N-m},\]
\[p_n(x) = \prod_{m=1}^{N} (x - c^{(n-1)}_m) = x^N + \sum_{m=1}^{N} c^{(n)}_m x^{N-m},\]
in contrast with the hierarchies of monic polynomials introduced in [1]. In particular, in Summer 2015 the authors conceived the idea to study
periodic orbits of the operators $T(n) = p_n(x)$ defined in terms of such sequences.

7. History and Acknowledgements

The work on this paper began in Summer 2015 during O. Bihun’s visit to the “La Sapienza” University of Rome. The first electronic draft is dated Feb. 11, 2016. Some more work on the paper was done during O. Bihun’s visit to the “La Sapienza” University of Rome in June 2016 and D. Fulghesu’s visit to the Scuola Normale Superiore in Pisa in June-July 2016. Both authors are grateful for the hospitality of the respective institutions. D. Fulghesu’s research is supported in part by a Simons Foundation Grant “Collaboration Grants for Mathematicians”.

REFERENCES

[1] O. Bihun and F. Calogero, Generations of Monic Polynomials such that the Coefficients of the Polynomials of the Next Generation Coincide with the Zeros of the Polynomials of the Current Generation, and New Solvable Many-body Problems, *Lett. Math. Phys.*, Vol. 106, No. 7 (2016) 1011-1031, [arXiv:1510.05017](https://arxiv.org/abs/1510.05017) [math-ph].

[2] W.C. Brenke, On Polynomial Solutions of a Class of Linear Differential Equations of the Second Order, *Faculty Publications*, Department of Mathematics, University of Nebraska-Lincoln, Paper 13 (1930).

[3] C. Chicone, *Ordinary Differential Equations with Applications*, Springer (2006).

[4] A. J. Di Scala, Ó. Maciá, Finiteness of Ulam Polynomials, [arXiv:0904.0133](https://arxiv.org/abs/0904.0133) [math.AG] (2009).

[5] P.R. Stein, On Polynomial Equations with Coefficients Equal to Their Roots, *The American Mathematical Monthly*, Vol. 73, No. 3 (1966) 272-274.

[6] S. Ulam, *A collection of mathematical problems*, Interscience, New York (1960).