LIOUVILLE’S EQUATION FOR CURVATURE AND SYSTOLIC DEFECT

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Abstract. We analyze the probabilistic variance of a solution of Liouville’s equation for curvature, given suitable bounds on the Gaussian curvature. The related systolic geometry was recently studied by Horowitz, Katz, and Katz in [12], where we obtained a strengthening of Loewner’s torus inequality containing a “defect term”, similar to Bonnesen’s strengthening of the isoperimetric inequality. Here the analogous isosystolic defect term depends on the metric and “measures” its deviation from being flat. Namely, the defect is the variance of the function \( f \) which appears as the conformal factor expressing the metric on the torus as \( f^2(x, y)(dx^2 + dy^2) \), in terms of the flat unit-area metric in its conformal class. A key tool turns out to be the computational formula for probabilistic variance, which is a kind of a sharpened version of the Cauchy-Schwartz inequality.

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1. Liouville’s equation for Gaussian curvature

Given a function \( f \) satisfying Liouville’s equation for curvature in a domain, we are interested in studying lower bounds for the probabilistic

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variance of \( f \), or more precisely, for the average square deviation of \( f \) from its mean, in the domain.

A geometric application we have in mind is obtaining lower bounds for the variance of the conformal factor \( f \) in a fundamental domain of a doubly-periodic metric \( f^2(x, y)(dx^2 + dy^2) \) in the plane, and hence for the isosystolic defect in Loewner’s inequality for the corresponding torus (see Section 2).

Liouville’s equation for curvature is usually stated in terms of the Laplace-Beltrami operator \( \Delta_{\text{LB}} \) on a surface with a metric \( f^2(x, y)ds^2 \), where \( ds^2 = dx^2 + dy^2 \). In isothermal coordinates \((x, y)\), the operator is given by \( \Delta_{\text{LB}} = \frac{1}{f^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \). We obtain the following form of Liouville’s equation, see [9], [13, p. 26] for details.

**Theorem 1.1 (Liouville’s equation).** The Gaussian curvature \( K = K(x, y) \) of the metric \( f^2(x, y)(dx^2 + dy^2) \) is minus the Laplace-Beltrami operator of the log of the conformal factor \( f \):

\[
K = -\Delta_{\text{LB}} \log f. \tag{1.1}
\]

In terms of the flat Laplacian \( \Delta_0(h) = \text{trace} \, \text{Hess}(h) \), the equation can be written as follows: \(-\Delta_0 \log f = Kf^2\). Setting \( h = \log f \), we obtain yet another equivalent form, \(-\Delta_0 h = Ke^{2h}\).

This equation in the case of constant curvature \( K \) is called Liouville’s equation in [9, p. 118], see also Rogers and Schief [18, p. 154] (where it appears as the first Gauss-Mainardi-Codazzi equation), and [14, 15, 21].

Liouville’s equation \( K = -\Delta_{\text{LB}} \log f \) can be written as follows in terms of partial derivatives:

\[
-f(f_{xx} + f_{yy}) + (f_x^2 + f_y^2) = Kf^4, \tag{1.2}
\]
in other words \(-f\Delta_0 f + |\nabla f|^2 = Kf^4\). The flat Laplacian can be written as follows in polar coordinates:

\[
\Delta_0 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_{S^1} f, \tag{1.3}
\]

where \( \Delta_{S^1} f \) is the Laplace-Beltrami operator on the circle.

2. A GEOMETRIC APPLICATION

In this section, we present a differential-geometric application of lower bounds for the variance of a solution of Liouville’s equation for curvature. The reader mostly interested in Liouville’s equation itself can skip to the next section.

The *systole* of a compact metric space \( X \) is a metric invariant of \( X \), defined to be the least length of a noncontractible loop in \( X \). We
will denote it Sys = Sys(X), cf. M. Gromov [10, 11]. When X is a graph, the invariant is usually referred to as the girth, ever since W. Tutte’s article [20]. C. Loewner proved his systolic inequality Area(g) − $\sqrt{3}2\text{Sys}(g)^2 \geq 0$ for the torus $(T^2, g)$ in 1949, as reported by Pu [16].

The classical Bonnesen inequality from 1921 is the strengthened isoperimetric inequality $L^2 − 4\pi A \geq \pi^2(R−r)^2$, see [8, p. 3], [6]. Here $A$ is the area of the region bounded by a closed Jordan curve of length (perimeter) $L$ in the plane, $R$ is the circumradius of the bounded region, and $r$ is its inradius. The error term $\pi^2(R−r)^2$ on the right hand side of Bonnesen’s inequality is traditionally called the isoperimetric defect.

Loewner’s torus inequality can be similarly strengthened, by introducing a “defect” term à la Bonnesen. If we use conformal representation to express the metric $g$ on the torus $T^2$ as $f^2(dx^2 + dy^2)$ with respect to a unit area flat metric on the torus $\mathbb{R}^2/L$ (see below), then the defect term in question is the variance of the conformal factor $f$ above. Then the inequality with the defect term looks as follows:

$$\text{Area} − \frac{\sqrt{3}}{2}\text{Sys}^2 \geq \text{Var}(f).$$

Is there a geometrically meaningful estimate for the systolic defect term in Loewner’s torus inequality? Thus, one could look for lower bounds for the variance of the conformal factor $f$ of the metric, in terms of some curvature conditions, say if the curvature is bounded away from zero on a region $D$ whose area is bounded below. Liouville’s equation for Gaussian curvature $K$ is $−\Delta \log f = Kf^2$. One is led to the following problem in connection with Liouville’s equation. The solutions in the constant curvature case are of the form $f = \frac{|a'(z)|}{1+|a(z)|^2}$ where $a(z)$ is a holomorphic function on a disk (here the curvature, assumed constant, is normalized to the value $+4$). One seeks lower bounds for the variance of $f$. Here a lower bound for the area of the region $D$ translates into a lower bound for the $L^2$ norm of $\frac{|a'(z)|}{1+|a(z)|^2}$.

Recall that the uniformisation theorem in the genus $1$ case can be formulated as follows: For every metric $g$ on the 2-torus $T^2$, there exists a lattice $L \subset \mathbb{R}^2$ and a positive $L$-periodic function $f(x, y)$ on $\mathbb{R}^2$ such that the torus $(T^2, g)$ is isometric to $(\mathbb{R}^2/L, f^2ds^2)$, where $ds^2 = dx^2 + dy^2$ is the standard flat metric of $\mathbb{R}^2$. Similarly to the isoperimetric inequality, Loewner’s torus inequality relates the total area, to a suitable 1-dimensional invariant, namely the systole, i.e., least length of a noncontractible loop on the torus $(T^2, g)$:

$$\text{Area}(g) − \frac{\sqrt{3}}{2}\text{Sys}(g)^2 \geq 0. \quad (2.1)$$
In analogy with Bonnesen’s inequality, there exists a following version of Loewner’s torus inequality with an error term:

\[ \text{Area}(g) - \frac{\sqrt{3}}{2} \text{Sys}(g)^2 \geq \text{Var}(f), \]  

(2.2)

see [12]. Here the error term, or isosystolic defect, is given by the variance \( \text{Var}(f) = \int_{T^2} (f - m)^2 \) of the conformal factor \( f \) of the metric \( g = f^2 g_0 \) on the torus, relative to the unit area flat metric \( g_0 \) in the same conformal class. Here \( m = \int_{T^2} f \) is the mean of \( f \). More concretely, if \( (T^2, g_0) = \mathbb{R}^2/L \), where \( L \) is a lattice of unit coarea, and \( D \) is a fundamental domain for the action of \( L \) on \( \mathbb{R}^2 \) by translations, then the mean can be written as \( m = \int_D f(x,y) dxdy \), where \( dxdy \) is the standard measure of \( \mathbb{R}^2 \).

**Question 2.1.** Unlike Bonnesen’s inequality where the error term has clear geometric significance, the error term in (2.2) is of an analytic nature. It would be interesting to obtain a lower bound whose geometric significance is more transparent. Can such a bound be expressed in terms of suitable curvature bounds?

The proof of inequalities with isosystolic defect relies upon the computational formula for the variance of a random variable in terms of expected values. Keeping our differential geometric application in mind, we will denote the random variable \( f \). Namely, we have the formula

\[ E_\mu(f^2) - (E_\mu(f))^2 = \text{Var}(f), \]  

(2.3)

where \( \mu \) is a probability measure. Here the variance is \( \text{Var}(f) = E_\mu((f - m)^2) \), where \( m = E_\mu(f) \) is the expected value (i.e., the mean).

Now consider a flat metric \( g_0 \) of unit area on the 2-torus \( T^2 \). Denote the associated measure by \( \mu \). Since \( \mu \) is a probability measure, we can apply formula (2.3) to it. Consider a metric \( g = f^2 g_0 \) conformal to the flat one, with conformal factor \( f > 0 \), and new measure \( f^2 \mu \). Then we have

\[ E_\mu(f^2) = \int_{T^2} f^2 \mu = \text{Area}(g). \]

Equation (2.3) therefore becomes

\[ \text{Area}(g) - (E_\mu(f))^2 = \text{Var}(f). \]

(2.4)

Next, one relates the expected value \( E_\mu(f) \) to the systole of the metric \( g \). Recall that the first successive minimum, \( \lambda_1(L) \) is the least length of a nonzero vector in \( L \). The lattice of the Eisenstein integers is the lattice in \( \mathbb{C} \) spanned by the elements 1 and the sixth root of unity. To visualize this lattice, start with an equilateral triangle in \( \mathbb{C} \) with vertices 0, 1, and \( \frac{1}{2} + i \frac{\sqrt{3}}{2} \), and construct a tiling of the plane by repeatedly reflecting in all sides. The Eisenstein integers are by definition the set of vertices of the resulting tiling. The maximal ratio \( \frac{\lambda_1^2}{\text{Area}} \) (the area is that of a fundamental domain) for a lattice in the plane
is $\gamma_2 = \frac{2}{\sqrt{3}} = 1.1547 \ldots$. The corresponding critical lattice is homothetic to the $\mathbb{Z}$-span of the cube roots of unity in $\mathbb{C}$, i.e., the Eisenstein integers. This result yields a proof of Loewner’s torus inequality for the metric $g = f^2 g_0$ using the computational formula for the variance. Let us analyze the expected value term $E_\mu(f) = \int_{T^2} f \mu$ in (2.4). Indeed, the lattice of deck transformations of the flat torus $g_0$ admits a $\mathbb{Z}$-basis similar to $\{\tau, 1\} \subset \mathbb{C}$, where $\tau$ belongs to the standard fundamental domain. In other words, the lattice is similar to $\mathbb{Z}\tau + \mathbb{Z}1 \subset \mathbb{C}$. Consider the imaginary part $\Im(\tau)$ and set $\sigma^2 := \Im(\tau) > 0$. From the geometry of the fundamental domain it follows that $\sigma^2 \geq \sqrt{3}/2$, with equality if and only if $\tau$ is the primitive cube or sixth root of unity. Since $g_0$ is assumed to be of unit area, the basis for its group of deck transformations can therefore be taken to be $\{\sigma^{-1}\tau, \sigma^{-1}\}$, where $\Im(\sigma^{-1}\tau) = \sigma$. With these normalisations, we see that the flat torus is ruled by a pencil of horizontal closed geodesics, denoted $\gamma_y = \gamma_y(x)$, each of length $\sigma^{-1}$, where the “width” of the pencil equals $\sigma$, i.e. the parameter $y$ ranges through the interval $[0, \sigma]$, with $\gamma_\sigma = \gamma_0$. By Fubini’s theorem, we obtain the following lower bound for the expected value: $E_\mu(f) = \int_0^\sigma \left(\int_{\gamma_y} f(x) dx\right) dy = \int_0^\sigma \text{length}(\gamma_y) dy \geq \sigma \text{Sys}(g)$, see [13, p. 41, 44] for details. Substituting into (2.4), we obtain the inequality

$$\text{Area}(g) - \sigma^2 \text{Sys}(g)^2 \geq \text{Var}(f), \quad (2.5)$$

where $f$ is the conformal factor of the metric $g$ with respect to the unit area flat metric $g_0$. Since $\sigma^2 \geq \sqrt{3}/2$, we obtain in particular Loewner’s torus inequality with isosystolic defect (2.2). Hence, a metric satisfying the boundary case of equality in Loewner’s torus inequality (2.1) is necessarily flat and homothetic to the quotient of $\mathbb{R}^2$ by the lattice of Eisenstein integers. Indeed, if a metric $f^2 ds^2$ satisfies the boundary case of equality in (2.1), then the variance of the conformal factor $f$ must vanish by (2.2). Hence $f$ is a constant function.

If $\tau$ is pure imaginary, i.e., the lattice $L$ is a rectangular lattice of coarea 1, then the metric $g = f^2 g_0$ satisfies the inequality

$$\text{Area}(g) - \text{Sys}(g)^2 \geq \text{Var}(f). \quad (2.6)$$

Indeed, if $\tau$ is pure imaginary then $\sigma \geq 1$, and the inequality follows from (2.5). In particular, every surface of revolution satisfies (2.6), since its lattice is rectangular.

Lower bounds for the variance of the conformal factor yield lower bounds for the isosystolic defect, see next section.
3. Rotationally invariant case

If \( f \) is rotationally invariant, then \( \Delta_0 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} \) from (1.3). Thinking of \( f \) as a single-variable function \( f(r, \theta) = f(r) \), we obtain

\[
\Delta_0 f = f''(r) + \frac{1}{r} f'(r). \tag{3.1}
\]

Liouville’s equation then becomes an ordinary differential equation

\[
-f \left( f''(r) + \frac{1}{r} f'(r) \right) + (f'(r))^2 = K f^4(r),
\]

cf. (1.2). The substitution \( \zeta = r^2 \) results in an equation in the variable \( \zeta \), namely

\[
-f \mathcal{T}(f) + 4\zeta (f'(\zeta))^2 = K f^4,
\]

where the relation between \( f(r) \) and \( f(\zeta) \) needs to be explained (two different \( f \)'s).

Alternatively, we can proceed as follows. Instead of the substitution above, we study the variance of the product \( rf(r) \), where \( f \) is the conformal factor. Thus we retain the variable \( r \). Is there a convenient form of the equation for this new function \( rf(r) \)? However, the natural \( L^2 \) normalisation is for \( \int f^2(r) r dr d\theta \), hence here we need to integrate \( rf^2(r) \). In other words we are integrating \( rf(r) \) with a weight function \( 1/r \). We consider again the operator (3.1), which we now denote \( \mathcal{T} \): \( \mathcal{T} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \), where \( r \geq 0 \). Applying the change of variable \( \zeta = r^2 \), we can write \( \mathcal{T} \) as \( \mathcal{T} = 4 \frac{d}{d\zeta} \zeta \frac{d}{d\zeta} \). Furthermore the substitution \( \zeta = e^t \) allows us to write the operator as

\[
\mathcal{T} = \frac{4}{\zeta} \frac{d^2}{dt^2}. \tag{3.2}
\]

Lower bounds for the second derivative lead easily to estimates for the variance, expressed by the following Lemma: If the curvature is positive, then the function \( f(r) \) is decreasing, while \( \log f \) is concave with respect to the variable \( t \). Indeed, rewriting the equation \( -\mathcal{T} \log f = K f^2 \) as \( -\frac{d^2 \log f}{dt^2} = \frac{4}{\zeta} K f^2 \) immediately implies the concavity of \( \log f \) with respect to \( t \). Integrating with respect to \( t \), we obtain \( -\frac{d \log f}{dt} = \int \frac{4}{\zeta} K f^2 > 0 \) if the curvature is positive. Hence \( \log f \) is decreasing, and therefore so is \( f \) itself. The significance of the variable \( \zeta \) stems from the following elementary fact.

**Lemma 3.1.** The variance of a rotationally invariant conformal factor \( f \) on a disk is proportional to the variance of the corresponding single variable function, with respect to the variable \( \zeta = r^2 \).
Indeed, the proof is immediate from the fact that \( \frac{1}{2} d\zeta = rdr \) is the line measure inherited from polar coordinates. Liouville’s equation for a rotationally invariant conformal factor \( f = f(\zeta(r)) \) can be rewritten as

\[
4 \frac{d}{d\zeta} \zeta \frac{d}{d\zeta} \log f = -Kf^2.
\]

In terms of the reciprocal function \( \phi = \frac{1}{f} \), this becomes

\[
4 \phi^2 \frac{d}{d\zeta} \zeta \frac{d}{d\zeta} \log \phi = K.
\]

Note that the linear function \( \phi(\zeta) = 1 + \frac{K}{4} \zeta \) solves this equation, as pointed out by B. Riemann, cf. (6.1).

**Question 3.2.** Can one translate the differential inequality

\[
4 \phi^2 \frac{d}{d\zeta} \zeta \frac{d}{d\zeta} \log \phi \geq K
\]

into a geometric condition involving a comparison of an arbitrary solution, with Riemann’s linear solution?

4. **Averaging \( h = \log f \) respects the differential inequality**

If the curvature satisfies a lower bound \( K(x, y) \geq \alpha \), Liouville’s equation yields a differential inequality

\[
-\Delta_0 h \geq \alpha e^{2h(r)}.
\]

Given a disk in \( \mathbb{R}^2 \) where \( h \) is defined, we can average \( h \) by the circle of rotations about the center of the disk to obtain a rotationally symmetric function \( h_{av}(r) = h_{av}(r, \theta) \) on the same disk, where \( \theta \) is the polar angle.

**Proposition 4.1.** If \( h \) satisfies the differential inequality (4.1) then its rotationally symmetric average \( h_{av}(r) \) satisfies the ordinary differential inequality

\[
-\left( h''_{av}(r) + \frac{1}{r} h'_{av}(r) \right) \geq \alpha e^{h_{av}},
\]

where \( \alpha \) is a lower bound for Gaussian curvature.

**Proof.** By linearity of the flat Laplacian, we obtain from (4.1)

\[
-\Delta_0 (h_{av}) = -\text{av} (\Delta_0 h) = \text{av} (K e^{2h}) \geq \alpha \text{av} (e^{2h}) \geq \alpha e^{2h_{av}},
\]

by Jensen’s inequality applied to the exponential function. Applying (3.1), we complete the proof of Proposition 4.1. \( \square \)

The logarithmic average \( f_{LA} \) is by definition the function \( f_{LA} = e^{\text{av}(\log f)} \). The above proposition can be restated in terms of the logarithmic average as follows.

**Corollary 4.2.** If the metric \( f^2 ds^2 \) admits a lower bound \( K \geq \alpha \) on a disk, then the logarithmic average \( f_{LA} \) satisfies the differential inequality

\[
-\Delta_{LB} \log f_{LA} \geq \alpha.
\]

Namely, at the level of the function \( h = \log f \), the differential inequality \( -\Delta_0 h \geq \alpha e^{2h} \) averages well by Jensen’s inequality (see above).
Remark 4.3. One can find lower bounds for the variance of the conformal factor \( f \), by relating the variances \( \text{Var}(f) \) and \( \text{Var}(f_{LA}) \), and developing variance estimates for a rotationally invariant function, and applying them to the logarithmic average \( f_{LA} \).

5. The effect of averaging on variance

Lemma 5.1. Rotational averaging does not increase the variance.

Indeed, we claim that the rotational average \( h_{av} \) of a function \( h \) on a disk \( D \) satisfies \( \text{Var}(h_{av}) \leq \text{Var}(h) \). More specifically, let \( h_{av}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(r, \theta) d\theta \). Note that \( \int_0^{2\pi} h_{av}(r)d\theta = \frac{3}{2\pi} \int_0^{2\pi} h(r, \theta)d\theta = \int_0^{2\pi} f(r, \theta)d\theta \). Assume for simplicity that \( D \) has unit area. Then

\[
\text{E}(h_{av}) = \int r dr \left( \int_0^{2\pi} h_{av}(r, \theta) d\theta \right) = \int r dr \left( \int_0^{2\pi} h(r, \theta) d\theta \right) = \text{E}(h).
\]

(5.1)

Now consider the term \( \text{E}(h^2) \). We have \( \int_0^{2\pi} h_{av}^2(r)d\theta = 2\pi h_{av}^2(r) = 2\pi \left( \frac{1}{2\pi} \int_0^{2\pi} h(r, \theta)d\theta \right)^2 \). Since the squaring function is a convex function, we can apply Jensen’s inequality to obtain \( \left( \frac{1}{2\pi} \int h(r, \theta)d\theta \right)^2 \leq \frac{1}{2\pi} \int h^2(r, \theta)d\theta \). Thus \( \int_0^{2\pi} h_{av}^2(r)d\theta \leq 2\pi \frac{1}{2\pi} \int h^2(r, \theta)d\theta \). Hence

\[
\text{E}(h_{av}^2) = \int \left( r dr \int_0^{2\pi} h_{av}^2(r)d\theta \right) \leq \int \left( r dr \int_0^{2\pi} h^2(r)d\theta \right) = \text{E}(f^2).
\]

(5.2)

Combining (5.1) and (5.2), we obtain \( \text{Var}(h_{av}) = \text{E}(h_{av}^2) - \text{E}(h_{av})^2 = \text{E}(h_{av}^2) - \text{E}(h)^2 \leq \text{E}(h^2) - \text{E}(h)^2 \), as required.

6. Some curvature estimates and Liouville’s equation

Assuming the existence of a region of positive Gaussian curvature \( K(x, y) \geq \alpha \) for the metric \( g \), we would like to compare the variance of an arbitrary solution of Liouville’s equation (1.1), to that of the standard rotationally invariant solution \( f_0 \) given by

\[
f_0 = \frac{1}{1 + \frac{\alpha}{4} r^2}
\]

(6.1)

already appearing in B. Riemann’s 1854 essay [17].

Given an metric \( g = f^2 ds^2 \) with an arbitrary conformal factor \( f > 0 \), we seek estimates of the following type.

We will denote by \( D(\rho) \) the disk of radius \( \rho > 0 \) for the background flat metric of unit area.
Question 6.1. Suppose the torus admits a region of positive Gaussian curvature $K \geq \alpha$, which is expressed by a conformal factor $f$ of normalized $L^2$ norm on $D(\rho)$. We seek a lower bound for the variance $\text{Var}(f) \geq N(\rho)$ where $N$ is an explicit function of $\rho$.

Applying inequalities 2.2 and 2.5, we can then obtain the following corollary: Let $\tau$ be the parameter of the underlying flat metric. Then $\text{Area}(g) - \Im(\tau)\text{Sys}(g)^2 \geq N(\rho) \ldots$. In other words, we seek an estimate only dependent on the lower curvature bound and the size of the disk, modulo a normalisation of the $L^2$-norm of $f$. Can such a bound be obtained by studying the reciprocal $\phi$ instead of $f$, and using the convexity of the reciprocal function? Is Riemann’s solution 6.1 optimal as far as the variance is concerned?

We will work with the hypothesis $K(x, y) \geq \alpha > 0$ on the metric $f^2 ds^2$ in a disk of radius $2\rho > 0$. We need a lower bound for the variance of $f = f(\zeta)$ with respect to the standard measure $d\zeta$, as already discussed above in Lemma 3.1.

Lemma 6.2. Assume the Gaussian curvature $K(x, y)$ satisfies a lower bound $K \geq \alpha > 0$. Then the substitution $\zeta = e^t$ results in a concave decreasing function $u(\zeta(t))$ in the variable $t$. The second derivative $\frac{d^2u(\zeta(t))}{dt^2}$ is bounded away from zero on the interval $\zeta \in [\rho, 2\rho]$ as follows: $-\frac{d^2u}{dt^2} \geq \frac{\alpha}{4} f^2$.

Indeed, since the function $u = \log f_{LA}$ is rotationally invariant, we can write $-T_u \geq \alpha f_{LA}^2$, or

$$-4 \zeta u''(t) \geq \alpha f_{LA}^2$$  \hspace{1cm} (6.2)

by (3.2). Monotonicity has already been checked in the Lemma above. Inequality (6.2) takes the form $-\frac{d^2u}{dt^2} \geq \frac{\zeta \alpha f^2}{4}$.

Can the concavity be used to obtain an estimate for the $t$-variance? Can one relate the $t$-variance and the $\zeta$-variance?

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