Abstract—In this work, we present a novel linear iterative solution to an electromagnetic inverse scattering problem with phaseless data (PD-ISPs) for strongly scattering, lossy media. It is based on an extended Rytov approximation (RA) that significantly widens the validity range of the conventional RA. By modifying this extension and including it in a distorted wave iterative formulation, accurate reconstruction results for both the real and imaginary components of permittivity can be obtained. We denote the technique as the distorted wave extended phaseless Rytov iterative method (DxPRIM) and we present its derivation and numerical formulation. Using simulation and experimental examples, we demonstrate that DxPRIM can reconstruct scatterers with phaseless measurement data and that it outperforms state-of-the-art phaseless techniques.

Index Terms—Born approximation (BA), inverse scattering, Rytov approximation (RA), wave scattering.

I. INTRODUCTION

Inverse scattering problems (ISPs) based on the time-independent form of the wave equation (Helmholtz equation) are challenging to solve under strong scattering conditions due to their nonlinearity and ill-posedness [1], [2], [3], [4]. The solution of these ISPs have a wide range of important applications including microwave imaging, remote sensing, medical imaging, and nondestructive evaluation [1], [3], [5], [6], [7], [8], [9], [10], [11], [12].

Most conventional inverse scattering techniques require full wave data consisting of both magnitude and phase measurements of the scattered field and are referred to as full data ISPs (FD-ISPs) here. However, for many applications, it is often not practical to accurately collect phase information. Measuring phase requires a high-precision measurement system with accurate synchronization between multiple receive and transmit nodes. The complexity and cost of these measurement systems increases as the frequency of the probing wave increases. Furthermore, they require accurate calibration. Hence, FD-ISPs have found limited applications in practical scenarios such as large-scale microwave and indoor imaging [8], [11], [12], [13], [14], [15], [16]. This has given rise to the development of techniques to solve ISPs with phaseless data (denoted as PD-ISPs hereafter) [1], [12], [13], [14], [15].

PD-ISPs are shown to be more ill-posed and nonlinear than FD-ISPs and hence are considered more difficult to handle [12], [14]. Even state-of-the-art nonlinear methods for solving PD-ISPs [1], [3], [7], [12], [17], such as phaseless data subspace optimization-based methods (known as PD-SOMs), fail to reconstruct strong scatterers with intricate profiles. These methods can also be more prone to experimental errors and noise due to the nonlinear formulation [1], [14]. Recent deep learning-based solutions to PD-ISPs [12] can moderately extend the validity range but require large datasets for training the deep learning networks. However, existing linear methods for solving PD-ISPs such as the phaseless Rytov approximation (RA) and line-of-sight radio tomography methods are computationally less expensive and robust to noise and experimental errors [8], [11], [13], [16], [18], [19], [20], [21]. However, there is a reduction in the validity range due to the linear approximations involved. Therefore, there is a trade-off between practicality (handling noise, experimental errors, and scarcity of measurements) and validity range while using linear versus nonlinear models.

In our recent work, we proposed corrections to the conventional RA to significantly increase its validity range in strongly scattering, lossy media [13], [18], [22]. We denoted the formulation as the extended phaseless RA in lossy media (xPRA-LM). In terms of shape estimation, it was demonstrated to successfully reconstruct objects up to extremely high permittivity and size, which far exceeds any known techniques. Because it is a linear phaseless model, it is also robust to experimental errors and noise. However, xPRA-LM cannot directly estimate the real and imaginary parts of relative permittivity and works only for homogeneous or piecewise homogeneous scatterers.

In this work, we modify xPRA-LM and incorporate it into the distorted wave iterative framework so that we can estimate shape as well as the complex-valued permittivity of strongly scattering, lossy scatterers. The distorted wave iterative framework is a well-known iterative approach to increase the range of applicability of existing approximations including the distorted wave Born iterative method (DBIM) and distorted wave Rytov iterative method (DRIM) and other approaches derived from these [1], [16], [20], [21]. However, only DRIM can be extended to handle phaseless data (which

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Amartansh Dubey is with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology (HKUST), Hong Kong (e-mail: adubey@connect.ust.hk).

Ross Murch is with the Department of Electronic and Computer Engineering and the Institute of Advanced Study, The Hong Kong University of Science and Technology (HKUST), Hong Kong.

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we denote here as PD-DRIM) [16]. PD-DRIM is shown to be useful for strong scattering conditions, but requires incident waves at multiple frequencies over a range of a few Gigahertz (requires relative bandwidth of over 100%) in order to achieve convergence. Overall, all existing linear, nonlinear, and iterative PD-ISP techniques have a limited validity range. Our key contributions in this work can be summarized as follows.

1) We formulate the distorted wave method using a modification of xPRA-LM so that it is valid in strongly scattering, lossy media. We denote the technique as the distorted wave extended phaseless Rytov method (DxPRM). It linearly relates change in an arbitrary complex-valued refractive index background profile to changes in the magnitude of the total received signal power.

2) We extend DxPRM to an iterative framework which we denote as the distorted wave extended phaseless Rytov iterative method (DxPRIM) for strongly scattering, lossy media. DxPRIM iteratively solves the forward scattering problem exactly, while the ISP is handled using the approximation DxPRM.

3) Using numerical and experimental results, we show that DxPRIM achieves a wide validity range and outperforms state-of-the-art PD-ISP methods in terms of accuracy when compared to benchmark scatterer profiles (such as the “Austria” profile).

Organization and Notation: Section II provides the formulations for the forward problem and corresponding inverse problems. The proposed DxPRM method is formulated in Section III along with a brief overview of xPRA-LM [13]. The DxPRIM algorithm is proposed in Section IV. Simulation and experimental results are provided in Section V. We use $\hat{X}$ and $\hat{x}$ to respectively denote the matrix and vector forms of a discretized parameter $X$. Lower case bold letters represent position vectors and italic letters are used to represent scalar parameters.

II. PROBLEM FORMULATION

Consider the setup shown in Fig. 1 which consists of a 2-D domain of interest, $D$, in which there are nonmagnetic scattering objects $S$. The domain of interest, $D$, is characterized by complex-valued relative permittivity $\varepsilon_r(r) = \varepsilon_R(r) + j\varepsilon_I(r)$ or equivalently by refractive index $n(r) = n_R(r) + jn_I(r) = (\varepsilon_r(r))^{1/2}$. To probe $D$, an array of electromagnetic sources (or transmitters) are placed along the circular boundary $T$. The total number of transmitters are $M_t$ and the location of each transmitter is given by $r_m \in T$ where, $m_t = 1, 2, \ldots, M_t$. The radiation from the transmitters is monochromatic, time harmonic ($e^{-j\omega t}$), and vertically polarized which is often referred to as transverse magnetic (TM) in inverse scattering contexts. The scattered waves from $D$ are collected by an array of receivers placed along the measurement boundary $R$. The total number of receivers is $M_r$ and the location of each receiver is given by $r_m \in R$, $m_r = 1, 2, \ldots, M_r$.

Let the free-space incident field at any point inside $D$ due to a transmitter (at $r_m$) be $E_{i,m}^i(r)$. The resultant total field inside $D$ [due to illumination by $E_{i,m}^i(r)$] can be written as the sum of the incident and scattered field, i.e., $E_{s,m}(r) = E_{i,m}^i(r) + E_{m}^i(r)$. The incident field $E_{i,m}^i(r)$ and total field $E_{s,m}(r)$ satisfy the homogeneous and inhomogeneous Helmholtz wave equation given by (1a) and (1b), respectively.

$$\nabla^2 E_{s,m}(r) + k^2_0 E_{s,m}(r) = 0, \quad r \in D \quad (1a)$$

$$\nabla^2 E_{i,m}^i(r) + k^2_0 E_{i,m}^i(r) = 0, \quad r \in D \quad (1b)$$

where $k_0 = 2\pi/\lambda_0$ is the free-space wavenumber, $\lambda_0$ is the free-space wavelength, and $v^2 = \varepsilon_r$. We would also like to point out that in the formulation (1) we only consider dielectric objects and do not consider objects with perfect electric conductors (PEC). This is a limitation of our work (it is also a very common assumption in previous work [1], [12]) and is an area for future investigation.

A. Forward Scattering Problem

The goal of the forward problem is to estimate the total field (or scattered field) at the receiver, given the permittivity profile $\varepsilon_r(r)$ inside $D$ and incident field $E_{i,m}^i$. Using (1a) and (1b), the forward problem can be formulated in the form of two equations. The first equation models the wave-matter interaction inside $D$ to estimate the total field inside $D$ (due to source at $r_m$)

$$E_{s,m}(r) = E_{i,m}^i(r) + k^2_0 \int_D g^0(r, r')\chi_{s}(r')E_{m}(r')d\hat{r}' \quad (2)$$

where $r, r' \in D$, and $\chi_s(r) = \varepsilon_s(r') - 1$ is denoted here as the contrast profile. $g^0$ is the 2-D free space homogeneous Green’s function defined as

$$g^0(r, r') = \frac{j}{4} H^{(1)}_0(k_0 |r - r'|). \quad (3)$$

Equation (2) is also known as the Lippmann–Schwinger equation [1], [23] and by solving it we can estimate the total field $E_{s,m}(r)$ for $r \in D$.

Using the estimate of the total field inside $D$, we can then estimate the scattered field $E_{m}^s(r) = E_{s,m}(r) - E_{i,m}^i(r)$ at any given receiver (at $r_m \in R$) using

$$E_{m}^s(r_m) = E_{s,m}(r_m) - E_{i,m}^i(r_m)$$

$$= k^2_0 \int_D g^0(r_m, r')\chi_{s}(r')E_{m}(r')d\hat{r}' \quad (4)$$

This equation describes the scattered field at the receiver as a radiadation of the induced contrast current $J_{m} = \chi_s \cdot E_{m}^i$. 

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Together, (2) and (4) describe the forward problem where (2) is known as the forward state equation and (4) is known as the forward data equation [1]. The forward state equation can be solved using the method of moments (MoMs) where we divide \( D \) into \( N \) discrete grids, with the position vector being the center of each grid \( r_n, n = 1, 2, \ldots, N \). This discrete equation can then be formulated as a system of linear equations for all \( M_t \) transmitters and \( M_r \) receivers. In particular we can rewrite (2) in matrix-vector form to estimate the total field (inside all \( N \) grids) as

\[
\vec{E}_D^i = \vec{D}(\vec{e}_r, \vec{g}^0) \cdot \vec{E}_D
\]

(5)

where \( \vec{E}_D \in \mathbb{C}^{N \times M_t} \) and \( \vec{E}_D^i \in \mathbb{C}^{N \times M_t} \) are complex-valued matrices and their \( m \)-th column represents the total field and incident field inside \( N \) grids due to illumination by a transmitter at \( r_m \). Matrix \( \vec{D} \in \mathbb{C}^{N \times N} \) is known as the forward state matrix which is a function of permittivity profile \( \epsilon \) and free-space Green’s function \( g^0(\vec{r}_m, \vec{r}_n) \) and it can be derived directly by applying MoM to (2) (see detailed derivation of \( \vec{D} \) in [1] and [23], [24]).

The forward data equation in (4) can also be written in matrix-vector form using the discretization used for (2) as

\[
\vec{E}_R = \vec{E}_R^i + \vec{S}(\vec{e}_r, \vec{g}^0) \cdot \vec{E}_D
\]

(6)

where \( \vec{E}_R, \vec{E}_R^i \in \mathbb{C}^{M_r \times M_t} \) are the total field and incident field matrix, respectively. Their elements \( [\vec{E}_R]_{m,t} \) and \( [\vec{E}_R^i]_{m,t} \) represent the total field and incident field at receiver \( r_m \), when \( D \) is illuminated by a transmitter at \( r_m \). Matrix \( \vec{S}(\vec{e}_r, \vec{g}^0) \in \mathbb{C}^{M_r \times N} \) is the forward data problem kernel matrix which is the product of the contrast profile \( \chi_\epsilon(\vec{r}) = (\epsilon_\epsilon(\vec{r}) - 1) \) and free space Green’s function \( g^0(\vec{r}_m, \vec{r}_n) \), scaled by the area of one grid. Matrix \( \vec{E}_D \in \mathbb{C}^{N \times M_t} \) represents the total field inside \( D \) as estimated by solving (5).

Overall, solving the forward problem (given \( \epsilon_\epsilon(\vec{r}), E^i(\vec{r}) \)) involves a two-step process. First estimating the total field inside \( D \) using (5) and then substituting it in (6) to estimate the total field at the receiver. The receiver field estimation using (5) and (6) provides numerically exact results as these equations are derived from the wave equation without any approximation.

**B. Inverse Scattering Problem**

Data equation (4) [or (6)] can also be posed as an inverse problem. That is, estimation of \( \chi_\epsilon \) in \( D \), given the total field \( E_{m_t}(\vec{r}) \) at the receivers \( \vec{r} \in \vec{R} \). However, it is nonlinear and ill-posed because along with \( \chi_\epsilon \), total field \( E_{m_t}(\vec{r}) \) inside \( D \) is also unknown in (4), which itself is a function of unknown \( \chi_\epsilon \). With phaseless data, there is even less information in the data and hence (4) becomes even more severely nonlinear and ill-posed.

Due to the difficulty in solving (4), approximate linear models have been proposed [1], [25], [26]. The most common include Born approximation (BA) and RA. These linear models have low computational complexity, and more importantly, due to a linear relation in measurements and contrast, they are less sensitive to experimental noise and errors. However, approximating a nonlinear problem using linearization, results in a limited validity range.

Despite their limited validity range, linear models have been shown to be useful when used with distorted wave iterative frameworks [1], [16], [20], where approximate inverse models are combined with exact forward models in an iterative framework. In distorted wave iterative frameworks (such as DBIM or DRIM), the algorithm alternates between incrementally correcting the contrast profile estimate and improving the estimate of the total field and inhomogeneous Green’s function inside \( D \). It updates the contrast profile and then estimates an improved estimate of the total field and inhomogeneous Green’s function. This process is repeated iteratively until convergence is achieved with the best contrast profile estimate.

To generate an estimate of the contrast profile at any iteration, linear models such as BA or RA are used in their distorted waveform.

Of the two approximations, BA and RA, only RA can be adapted for phaseless data. The Rytov transformation normalizes the total field \( E_{m_t}(\vec{r}) \) by the incident field \( E^i_{m_t}(\vec{r}) \) to express the scattering using a complex phase \( \phi_{m_t}(\vec{r}) \)

\[
\frac{E_{m_t}(\vec{r})}{E^i_{m_t}(\vec{r})} = e^{\phi_{m_t}(\vec{r})},
\]

(7)

Using (7), (1b), and (1a), we can obtain a nonlinear differential equation (Riccation equation in \( E^i_{m_t} \cdot \phi_{m_t} \)) [26], which can be written in integral form to obtain an expression for the total field at the receiver \( (\vec{r}_m) \) as

\[
E_{m_t}(\vec{r}_m) = E^i_{m_t}(\vec{r}_m) \cdot e^{\frac{k_0^2}{2 \epsilon_{m_t}(\vec{r}_m)^2} \int_D g^0(\vec{r}_m, \vec{r}) \chi_{RI}(\vec{r}) E^i_{m_t}(\vec{r}) d\vec{r}^2}
\]

(8)

where

\[
\chi_{RI}(\vec{r}) = \epsilon_{\epsilon}(\vec{r}) - 1 + \frac{\nabla \phi_{m_t}(\vec{r}) \cdot \nabla \phi_{m_t}^*(\vec{r})}{k_0^2}.
\]

(9)

We refer to (8) as the Rytov integral (RI) in the remainder of this article. Also note that \( \chi_{RI} \) is the contrast function of RI and is different from the conventional definition of contrast profile \( \chi_\epsilon = \epsilon_\epsilon - 1 \) used in the forward problem.

The term \( \nabla \phi_{m_t} \cdot \nabla \phi_{m_t}^* \) is neglected under a weak scattering assumption to arrive at the conventional RA formulation as

\[
E_{m_t}(\vec{r}_m) = E^i_{m_t}(\vec{r}_m) \cdot e^{\frac{k_0^2}{2 \epsilon_{m_t}(\vec{r}_m)^2} \int_D g^0(\vec{r}_m, \vec{r}) \chi_{RI}(\vec{r}) E^i_{m_t}(\vec{r}) d\vec{r}^2}.
\]

(10)

RA in (10) is transformed into phaseless form by multiplying (10) with its conjugate (which is not possible with other approximate models such as BA)

\[
|E_{m_t}(\vec{r}_m)|^2 = |E^i_{m_t}(\vec{r}_m)|^2 \cdot e^{\frac{2k_0^2}{\epsilon_{m_t}(\vec{r}_m)^2} \int_D g^0(\vec{r}_m, \vec{r}) \chi_{RI}(\vec{r}) E^i_{m_t}(\vec{r}) d\vec{r}^2},
\]

where Re is the real part operator. RA is useful only for weak scattering (\( \epsilon_R \approx 1 \)) because \( \nabla \phi_{m_t} \cdot \nabla \phi_{m_t}^* \) is not negligible when scattering is strong (even though RA does not impose a restriction on the size of the scatterer, unlike
BA [26], it fails if permittivity contrast is large). Estimation of \( \nabla \phi_{m_1} \cdot \nabla \phi_{m_1} \) is difficult as it requires solving the intractable nonlinear equation (8) [27], [28] which has not been performed for strongly scattering, lossy media. In our recent work [13], we corrected conventional RA to increase its validity range by approximately estimating \( \nabla \phi_{m_1} \cdot \nabla \phi_{m_1} \) using the high-frequency theory of inhomogeneous waves in strongly scattering, lossy media. This improved RA is denoted as xPRA-LM. It is briefly introduced in Section III, after which we modify it and obtain its distorted waveform for distorted iterative framework.

III. DISTORTED WAVE EXTENDED PHASELESS RYTOV METHOD

In this section, we first provide the background to xPRA-LM [13] and then propose a modification so that it can be formulated into a distorted waveform.

A. xPRA-LM AND ITS SIMPLIFICATION

xPRA-LM is obtained by approximately estimating the term \( \nabla \phi_{m_1} \cdot \nabla \phi_{m_1} \) to correct conventional RA. To perform this, we consider a plane wave in air/vacuum (in unit vector direction \( \hat{k}_i \)) incident on a lossy scatterer in the upper half-space as shown in Fig. 2.

In Fig. 2, an incident plane wave traveling in the direction given by the unit vector \( \hat{k}_i \) passes into the lossy scatterer becoming an inhomogeneous plane wave (IPW) in unit vector direction \( \hat{k}_t \). The high-frequency approximation [18], [29], [30] to IPW inside the scatterer can be written as

\[
E_{m_1}(r) = A_t \exp \left( jk_0 \int_{\hat{k}_t} (V_R(r) \hat{k}_t + jV_I(r) \hat{k}_a) dr \right) \tag{11}
\]

where \( A_t \) is transmission amplitude and \( V = V_R(r) \hat{k}_t + jV_I(r) \hat{k}_a \) is denoted as the effective refractive index [13], [29], [30]. The unit vectors \( \hat{k}_t \) and \( \hat{k}_a \) are normal to the constant phase planes and constant amplitude planes, respectively. The concept of effective refractive index is used to mathematically decompose an IPW into a linear combination of vectors along \( \hat{k}_t \) and \( \hat{k}_a \). Equation (11) represents a ray along path \( dr = dr \hat{k}_t \). An interesting feature of IPW is that the planes of constant amplitude are parallel to the media interface (or scatterer boundary), in other words, vector \( \hat{k}_a \) is always normal to the media interface [13], [29], [30]. This implies that \( \hat{k}_a \cdot \hat{k}_t = \cos \theta_t \) and \( \hat{k}_a \cdot \hat{k}_a = \cos \theta_a \), where \( \theta_t \) and \( \theta_a \) are the angles of incidence and refraction, respectively (see [13], [18] for more details).

Equating the total field in (11) to (7) and substituting the incident field as \( E_{m_1}'(r) = A_0 e^{j k_0 \hat{k}_i \cdot r} \) gives

\[
\nabla \phi_{m_1}'(r) \cdot \nabla \phi_{m_1}'(r) = k_0^2 \left[ V_I^2 - V_R^2 - 1 + 2V_R(\hat{k}_t \cdot \hat{k}_t) - 2j(V_R \hat{k}_t - \hat{k}_I) \cdot V_I \hat{k}_a \right] + (\nabla \ln T \cdot \nabla \ln T) + 2k_0(\nabla \ln T) \left[ j(V_R \hat{k}_t - \hat{k}_I) - V_I \hat{k}_a \right] \tag{12}
\]

where \( T = A_t/A_0 \) is the transmission coefficient. \( V_R, V_I \) are functions of \( r \) but for brevity we do not show the dependence.

Under the high-frequency assumption, since \( k_0 \) is large, terms in (12) with coefficient \( k_0^2 \) dominate. Furthermore, the gradient of \( \ln T \) is nonzero only along the boundaries of a scatterer. This makes terms containing \( \nabla \ln T \) almost zero inside and outside the scatterer. Hence, we can approximate \( \nabla \phi_{m_1}' \cdot \nabla \phi_{m_1}' \) by the dominant first term as

\[
\nabla \phi_{m_1}'(r) \cdot \nabla \phi_{m_1}'(r) \approx k_0^2 \left[ V_I^2 - V_R^2 - 1 + 2V_R(\hat{k}_t \cdot \hat{k}_t) - 2j(V_R \hat{k}_t - \hat{k}_I) \cdot V_I \hat{k}_a \right] \tag{13}
\]

Substituting \( \nabla \phi_{m_1}' \cdot \nabla \phi_{m_1}' \) from (13) into (9) and using (14) provides a new approximate expression for contrast profile \( \chi_{RI} \) as

\[
\chi_{RI}(r) \approx 2(V_R \cos \theta_t - 1) + j \frac{2V_RV_I}{\sqrt{V_R^2 - \sin^2 \theta_t}} \cos \theta_I
\]

\[
= 2(\sqrt{\epsilon_R} \cos \theta_t - 1) + j \frac{\epsilon_I}{\sqrt{\epsilon_R - \sin^2 \theta_t}} \cos \theta_I \tag{15}
\]

where \( \cos \theta_t = \hat{k}_t \cdot \hat{k}_t \) is the scattering angle (angle between incident and refRACTed rays) and the refractive index \( V = v_R + jv_I \) is related to relative permittivity \( \epsilon_r = \epsilon_R + j \epsilon_I \) as \( v_R = v_R + j v_I = \sqrt{\epsilon_R + j(\epsilon_I/2 \sqrt{\epsilon_R})} \) under the low-loss assumption. Note that there is a fundamental difference between the real and imaginary parts of the contrast derived in (15) compared to conventional RA \( \chi_{RA} = \epsilon_r(r) - 1 = v^2(r) - 1 \). Unlike in RA where contrast profile is a linear function of permittivity, now the new approximate contrast profile is a nonlinear function of permittivity. Also, the proposed contrast in (15) does not impose any assumptions on the scattering strength of the medium and hence it is also valid under strong scattering conditions. This happens because the term \( \nabla \phi_{m_1} \cdot \nabla \phi_{m_1} \) is not neglected while deriving (15) (unlike in RA, where \( \nabla \phi_{m_1} \cdot \nabla \phi_{m_1} \) is completely neglected).

To use the approximate contrast profile (15) in a distorted waveform (then later in a iterative framework), it needs to be further simplified. The most straightforward simplification is
to remove $\theta_i$ and $\theta_t$ dependence by approximating them as $\theta_i = 0$ and $\theta_t = 0$. This approximation will be valid if the waves are predominantly close to normal incidence. This gives our approximation as

$$\chi_{RI}(r) \approx 2(v_R - 1) + 2v_I$$

$$= 2(v_I - 1), \quad \text{(using } \epsilon_r = \epsilon_R + j \epsilon_I = v^2)$$

$$= \chi_{xRA}$$

where the approximation to $\chi_{RI}$ is denoted $\chi_{xRA}$. For normal incidence, the approximation in (16) is accurate. However, the assumption of normal incidence is not always true in the tomographic setup as waves can illuminate the object of interest from different incident angles. When waves are not incident at normal direction and the refractive index of the object is high, the multiple scattering effect is strong and the newly derived contrast in (15) cannot be simplified to obtain (16), and hence (16) becomes less accurate. However, as we shall show later, using the distorted wave iterative framework allows us to image the contrast in an incremental manner such that for each iteration, we estimate a small part of the contrast and hence instead of estimating the multiple scattering effect all at once, we incrementally correct it. Therefore, the error due to (16) can be handled using the distorted wave iterative framework.

It can be seen that (16) is linearly proportional to the refractive index. This agrees with Fermat’s principle where the incremental phase change of a ray is directly related to the product of the path length along the ray and refractive index contrast ($v(r) - 1$). In other words, the incremental phase change of a ray per wavelength should be proportional to $k_0(v - 1)$. For conventional RA, it is known (using asymptotic techniques) [25], [31] that the incremental phase change per wavelength is $(1/2)k_0(v^2 - 1)$ which does not match the expected phase change as per Fermat’s principal. Therefore, xPRA-LM with modified contrast in (16) also appears to better satisfy the underlying physics of the problem.

We can substitute (16) in RI (8) to obtain a phaseless form [similar to (11)] by multiplying (8) with its conjugate. In the resultant phaseless equation, we transform to received signal strength (RSS) in dB and new contrast [in (16)] to obtain modified xPRA-LM as

$$P_{m_i}(r_m)[\text{dB}] - P_{m_i}^{\text{i}}(r_m)[\text{dB}] = 20 \log_{10} \left| \frac{E_{m_i}(r_m)}{E_{m_i}^{\text{i}}(r_m)} \right|$$

$$= 20 \log_{10} e \cdot P_{m_i} \cdot P_{m_i}^{\text{i}} \quad \text{total power and free space incident power in dB.}$$

\section{DxPRM Formulation}

Using the contrast derived in (16), we can now formulate its use in a distorted waveform DxPRM. We estimate perturbation $\Delta \chi_{xRA}$ given that the refractive index and relative permittivity distribution before the perturbation are $\nu^b(r)$ and $\epsilon_r^b(r)$, respectively. The background incident field $E_{m_i}$, incident power $P_{m_i}^b$, and background Green’s function $g^b$ for this inhomogeneous background are given [or estimated by solving the forward problem for $\epsilon^b(r)$].

Letting the refractive index, relative permittivity, and the total received power after the perturbation be $\nu(r)$, $\epsilon_r(r) = v^2(r)$, and $P_{m_i}$, respectively, DxPRM can be written as

$$P_{m_i}(r_m)[\text{dB}] - P_{m_i}^{\text{i}}(r_m)[\text{dB}] = C_0 \cdot \text{Re} \left( \frac{k_0^2}{E_{m_i}^b(r_m)} \int_D g^b(r_{m_i}, r') \Delta \chi_{xRA}(r') E_{m_i}^b(r') dr'^2 \right)$$

$$\Delta \chi_{xRA} = 2(v - v^b) = 2\Delta \nu$$

is the change in contrast and is linearly proportional to the change in refractive index. The relative permittivity profile after perturbation can be given as

$$\epsilon_r = \left( \frac{\Delta \chi_{xRA}}{2} + \sqrt{\epsilon_r^b} \right)^2$$

Equation (18) can be solved using MoM, where we divide $D$ into $N$ discrete grids and then extend the discretized equation for all $M = M_i \cdot M_t$ transmitter-receiver links, resulting in a system of linear equations

$$\tilde{P} = \tilde{P}^b + \text{Re} \left( \tilde{H}^b(g^b, E^b) \cdot \Delta \tilde{\chi}_{xRA} \right)$$

Equation (20) can be written in compact form by substituting $\tilde{H}^b = [-\text{Re}(\tilde{H}^b) - \text{Im}(\tilde{H}^b)]$ where $\Delta \tilde{\chi}_{xRA}$ is the area of one grid. The operator $\text{Re}$ can be removed by rewriting (20) as

$$\tilde{P} - \tilde{P}^b = \text{Re}(\Delta \tilde{H}^b) - \text{Im}(\Delta \tilde{H}^b) \cdot \text{Re}(\Delta \tilde{\chi}_{xRA})$$

which can be written in compact form by substituting $\tilde{H}^b = [\text{Re}(\tilde{H}^b) - \text{Im}(\tilde{H}^b)] \in \mathbb{M}^{M \times 2N}$ as

$$\Delta \tilde{P}^b = \tilde{H}^b \cdot E^b \cdot \frac{\Delta \tilde{\chi}_{xRA}}{\Delta \tilde{\chi}_{xRA}}$$

where $\Delta \tilde{P}^b = \tilde{P} - \tilde{P}^b$ is change in RSS value (in dB) due to perturbation in the given background medium [with permittivity $\epsilon^b(r)$]. The vectors $\Delta \tilde{\chi}_{xRA}$ and $\Delta \tilde{\chi}_{xRA}$ represent $\text{Re}(\Delta \tilde{\chi}_{xRA})$ and $\text{Im}(\Delta \tilde{\chi}_{xRA})$, respectively.

When the background is free-space ($\epsilon_r = 1, \Delta \chi_{xRA} = \chi_{xRA}, E^b = E^i, \tilde{P}^b = P^i, g^b = g^0$), DxPRM in (23) reduces to the discrete form of modified xPRA-LM (17) to provide a linear system of equations

$$\Delta \tilde{P} = \tilde{H}^0 \cdot E^i \cdot \frac{\Delta \tilde{\chi}_{xRA}}{\Delta \tilde{\chi}_{xRA}}.$$
The inverse problem utilizing (23) and (24) is ill-posed as there are usually significantly fewer measurements ($M = M_1 \cdot M_r$) compared to the number of grids ($N$) with unknown permittivity, i.e., $M \ll N$. Furthermore, due to the removal of Re operator, we need to estimate $M$ which is a $2N \times 1$ vector, making the problem more severely ill-posed. Hence, $\tilde{\mathbf{H}}$ or its pseudo inverse is very poorly conditioned. To tackle this, regularization is needed. We use Tikhonov regularization to rewrite the optimization objective (23) as

$$\min_{\Delta \tilde{\mathbf{X}}_R, \Delta \tilde{\mathbf{X}}_l} \left\| \Delta \tilde{\mathbf{P}}_b - \mathbf{H}^{T} \Delta \tilde{\mathbf{X}}_R \mathbf{I}_2 + \beta \mathbf{H}^{T} \mathbf{I}_2 \right\|_2^2 + \beta \left\| \mathbf{I}_2 \Delta \tilde{\mathbf{X}}_l \right\|_2^2$$

(25)

where $\Gamma \in \mathbb{C}^{2N \times 2N}$ is the Tikhonov matrix which imposes the required prior on the optimization variable. The straightforward choice for $\Gamma$ is the identity matrix which reduces (25) to $l_2$ regularization (or Ridge regression). $\Gamma$ can also be the finite difference matrix to impose a smoothness prior. The smoothness prior is known to be better than Ridge in removing noise. However, Ridge requires less computation because the smoothness prior requires two finite difference matrices to be used (to impose smoothness in horizontal and vertical direction). In this work, we use Ridge for reconstructing homogeneous scatterers, whereas we can use the smoothness prior for more intricate inhomogeneous scatterers (note both can achieve the same final results and only the number of iterations required to converge to the optimum solution differs).

It is important to note that using the straightforward Tikhonov prior (with identity Tikhonov matrix) instead of an advanced sparsity and structure promoting prior such as TVAL3 [32] can make sure that sparsity and distortion removal is being learned by the proposed DxPRIM framework instead of advanced regularization techniques.

The analytical solution for DxPRIM [with Tikhonov regularization (25)] can be written as

$$\begin{bmatrix} \Delta \tilde{\mathbf{X}}_R \\ \Delta \tilde{\mathbf{X}}_l \end{bmatrix} = \left( \mathbf{H}^{T} \mathbf{H} + \beta \mathbf{I}_2 \mathbf{I}_2 \right)^{-1} \mathbf{H}^{T} \Delta \tilde{\mathbf{P}}_b$$

(26)

where $(\cdot)^T$ is the transpose of matrix (or Hermitian of matrix for complex-valued matrix). Similarly, the analytical solution when background is free-space can be written as

$$\begin{bmatrix} \Delta \tilde{\mathbf{X}}_R \\ \Delta \tilde{\mathbf{X}}_l \end{bmatrix} = \left( \mathbf{H}^{T}_{0} \mathbf{H}^{T}_{0} + \beta \mathbf{I}_2 \mathbf{I}_2 \right)^{-1} \mathbf{H}^{T}_{0} \Delta \tilde{\mathbf{P}}_b$$

(27)

and we refer to this as the modified xPRA-LM.

IV. DISTORTED WAVE EXTENDED PHASELESS
Rytov ITERATIVE METHOD

DxPRIM uses DxPR (26) iteratively for updating the contrast profile. The forward problem equations (5) and (6) are used for updating the total field (inside $\mathcal{D}$) and at receivers) using an estimate of the contrast profile given by DxPR (26) [or (27)]. The inhomogeneous Green’s function also needs to be updated iteratively. It can be estimated as a solution of the forward state problem [(2) or (5)]. Let $g^b$ be the inhomogeneous Green’s function for any given background profile with relative permittivity profile $\epsilon^b$. We can use the Lippmann–Schwinger equation (2) to estimate $g^b$ at a point $r$ inside $\mathcal{D}$ due to a fictitious source at the receiver location $r_m$ as

$$g^b(r_m, r) = E(r_m, r) + k_0^2 \int_{\mathcal{D}} g^0(r, r') (E^b(r') - 1) \times g^b(r', r_m) dr'^2$$

(28)

where $E(r_m, r)$ is the free-space Green’s function $g^0(r_m, r)$ such that we are estimating the incident field at point $r$ due to a source at $r_m$. Equation (28) can be solved using MoM [similar to solving (2) using MoM to obtain (5)]

$$\tilde{g}^0 = \tilde{D}(\epsilon^b, g^0) \cdot \tilde{g}^b$$

(29)

where $\tilde{g}^0 \in \mathbb{C}^{N \times M_r}$ is the free-space homogeneous Green’s function and $\tilde{g}^b \in \mathbb{C}^{N \times M_r}$ is the inhomogeneous Green’s function for the background with permittivity profile $\epsilon^b$. Matrix $\tilde{D} \in \mathbb{C}^{N \times N}$ is the forward state matrix (5).

Finally, using the three forward problem equations (5), (6), and (29) and two inverse problem equations (26) and (27), we can now formulate DxPRIM. Fig. 3 provides the flowchart for the DxPRIM algorithm which can be explained step-by-step as follows.

1) Initialize Relative Permittivity (Step $q = 0$): The initial guess for contrast profile $[\Delta \tilde{\mathbf{X}}_R, \Delta \tilde{\mathbf{X}}_l]^T \in \mathbb{R}^{2N \times 1}$ is obtained from (27). This is the solution to the modified xPRA-LM [in (17)]. The initial complex-valued relative permittivity profile $\tilde{\epsilon}^q \in \mathbb{C}^{N \times 1}$ is then obtained as

$$\tilde{\epsilon}^q = \left( \tilde{\epsilon}^q_{0} + j \tilde{\mathbf{K}}^0_{0} \right)^2$$

(30)

2) Update Total Field and Inhomogeneous Green’s Function: The estimate of the permittivity profile $\tilde{\epsilon}^q$ is used as the inhomogeneous background. Using this estimate of permittivity profile, the forward problem (5) is solved in order to update the total field $\tilde{E}^q_D$ inside $\mathcal{D}$

$$\tilde{E}^q_D = \tilde{D}(\tilde{\epsilon}^q_{r}, g^0) g^0$$

(31)

Similarly, the inhomogeneous Green’s function is estimated inside $\mathcal{D}$ using the permittivity profile estimate as

$$\tilde{g}^q = \tilde{D}(\tilde{\epsilon}^q_{r}, g^0) g^0$$

(32)

3) Update Measurements and Inverse Model Matrix: Using the estimate of the total field $\tilde{E}^q_D$ inside $\mathcal{D}$, the total field at the receiver is estimated as

$$\tilde{E}^q_{R} = \tilde{E}^q_D + \tilde{S}(\tilde{\epsilon}^q_{r}, g^0) \cdot \tilde{E}^q_D$$

(33)

In addition, update the change in RSS vector by using the current estimate of permittivity profile $\tilde{\epsilon}^q$ as background as

$$\Delta \tilde{P}^q = 20 \log \left| \frac{\tilde{E}}{\tilde{E}_R^q} \right| - 20 \log \left| \frac{\tilde{E}_R}{\tilde{E}_R^q} \right|$$

$$\Delta \tilde{P} = 20 \log \left| \frac{\tilde{E}}{\tilde{E}_R^q} \right| - 20 \log \left| \frac{\tilde{E}_R}{\tilde{E}_R^q} \right|$$

(34)
Note that in \((34)\), \(\Delta \vec{\rho} = 20 \log |\vec{E}/\bar{\vec{E}}q_R|\) is obtained from measurements and it represents differences in measured total power in the presence of the scatterer [with ground truth profile \(\epsilon_r(q)\)] and free-space incident power. However, \(\Delta \bar{\vec{\rho}}q\) represents the difference in the measured total power in the presence of the scatterer [with ground truth profile \(\epsilon_r(q)\)] and the background total power estimated using \(\bar{\epsilon}_q r\) as the inhomogeneous background instead of free space. Then, use the total field and the inhomogeneous Green’s function estimated in the previous step to update the DxPRM model matrix \(\bar{\vec{H}}q\) [as defined in (21)].

4) **Update Contrast and Relative Permittivity**: Obtain the change in contrast profile using DxPRM with the total field, model matrix, and change in RSS measurements obtained from the previous step as

\[
\begin{bmatrix}
\Delta \bar{\chi}^q R \\
\Delta \bar{\chi}^q I
\end{bmatrix}
= \left( \left( \bar{\vec{H}}q \right)^T \bar{\vec{H}}q + \beta \Gamma^T \Gamma \right)^{-1} \left( \bar{\vec{H}}q \right)^T \Delta \bar{\rho}q
\]

where \(\Delta \bar{\rho}q \in \mathbb{R}^{M \times 1}\) is the unrolled form of matrix \(\Delta \bar{\rho}q \in \mathbb{R}^{M \times M_t}\) which is estimated in the previous step. Use this change in contrast estimated from (35) to update the contrast for next iteration as

\[
\bar{\chi}^q R + j \bar{\chi}^q I + 1 = \left( \frac{\bar{\chi}^q R + j \bar{\chi}^q I + 1}{2} \right)^2
\]

which can be used to obtain an updated relative permittivity profile for the next iteration as

\[
\bar{\epsilon}^q R + 1 = \left( \frac{\bar{\chi}^q R + j \bar{\chi}^q I + 1}{2} \right)^2
\]

5) **Terminate or Continue to Next Iteration**: Using the current estimate of relative permittivity \(\bar{\epsilon}_q\), repeat steps 2–4 until the estimate of relative permittivity \(\bar{\epsilon}_q\) converges. Since the ground truth permittivity profile \(\epsilon_r\) is unknown, the decision of continuing or stopping can be based on checking if the current estimate of the magnitude of the total field \(|\vec{E}q R|\) at the receivers is close to the actual measured magnitude of total field \(|\vec{E} R|\) at the receivers. We define the relative error between the measured and estimated magnitude of total field as

\[
\text{RE} = \frac{|\vec{E} - \vec{E}q R|}{|\vec{E} R|}
\]

where \(|\cdot|_1\) is a norm 1 operator and \(\vec{E}\) and \(\vec{E}q R \in \mathbb{C}^{M \times 1}\) are unrolled form of matrices \(\vec{E}\) and \(\vec{E}q R \in \mathbb{C}^{M_r \times M_t}\), respectively. We can define a threshold \(\delta\) and if \(\text{RE} < \delta\), the iterative algorithm is terminated. The value of hyperparameter \(\delta\) depends on the type of the profile and has to be empirically selected depending on the type of scatterer. We observe that if the profile is smaller...
than a wavelength, then \( \delta \) can be extremely small (we use \( \delta = 0.01 \)), else it can be slightly larger (we use \( \delta = 0.1 \)).

V. SIMULATION RESULTS

In this section, we provide numerical results to demonstrate the performance of the proposed DxPRIM technique.

For performance benchmarking, we compare DxPRIM to the state-of-the-art phaseless inverse scattering techniques, namely the PD-SOM and a recent version of the phaseless data contrast source inversion (PD-CSI) [1], [12], [14]. PD-SOM formulates an optimization problem [1], [12], [14] that is solved using spectrum analysis (based on singular value decomposition) to divide the induced currents (contrast source) into two orthogonally complementary portions (deterministic and ambiguous/noisy portions). The PD-CSI approach used in this work is the recent implementation as described in [12]. This approach uses a similar procedure as the original PD-CSI work [17] with the key difference being the use of Fourier-based expansions for the induced current which can control the number of Fourier coefficients to determine the components of the induced currents. Recently, the validity range of these two techniques has been further improved using deep learning framework [12]; however, we only provide comparisons without deep learning frameworks so that the comparisons can be made without learning features from large datasets.

The implementation details for PD-SOM and PD-CSI techniques used in this work can be found in [1] and [12], respectively. The benchmark results for PD-SOM were obtained by using a modified version of the code used in [12]. The benchmark results for PD-CSI were provided directly by Prof. Kuiven Xu’s research group at Hangzhou Dianzi University and, therefore, provide an independent benchmarking of our work [1]. The stopping criteria for PD-SOM and PD-CSI were also as utilized in the original implementations of those approaches [1], [12]. We consider two scatterer profiles for performance analysis. These are profiles from the state-of-the-art PD-ISP literature reported in [1] and [12]. These profiles are defined as

1) The well-known “Austria” profile as shown in Figs. 4(a) and 5(a) [1], [2], [3], [4], [12], [14]. The Austria profile is widely used for benchmarking inverse scattering techniques. It is a particularly challenging profile to reconstruct with or without the phase data due to its intricate structure [1], [2], [3], [4], [12], [14].

2) Three eccentric circular solid scatterers with different complex-valued permittivity values and size as shown in Fig. 6(a). It is a homogeneous lossy scatterer profile utilized previously in [12]. It helps to evaluate the reconstruction techniques for an electrically large scatterer profile.

To quantify the performance of the techniques, we use the peak signal-to-noise ratio (PSNR) metric to compare the ground truth image to the reconstructed image. The higher the PSNR value, the better the reconstruction.

We use the same measurement setup as shown in Fig. 1 with domain of interest, \( \mathcal{D} \). To probe \( \mathcal{D} \), an array of 2 GHz electromagnetic sources (or transmitters) are placed around \( \mathcal{D} \) along a circular boundary of radius 5\( \lambda_0 \). The total number of transmitters is \( M_t \). An array of \( M_r \) receivers are placed along the circular boundary of radius 6\( \lambda_0 \). For reconstruction, \( \mathcal{D} \) is divided into 40 \( \times \) 40 discrete grids. Hence, we need to estimate \( 2N = 3200 \) unknowns using \( M_t \times M_r \) measurements. To generate measurement data (magnitude of total field, \( |E| \)), the forward problem is solved using MoM (as outlined in Section II). For the forward problem, \( \mathcal{D} \) is divided into 120 \( \times \) 120 grids. We also add 5% AWGN noise \( \pi \) to the simulated magnitude of the total field \( |\mathcal{E}| + \pi \), where \( |\mathcal{E}| \in \mathbb{R}^{M_r \times 1} \), and the level of noise is defined as \( |\pi|/||\mathcal{E}|\| \), where \( ||\cdot|| \) is the Euclidean norm. We added the noise directly to the total field (instead of the scattered field) because the inversion in each iteration of DxPRIM only needs the measured total field. We have not considered noise higher than 5\% in these examples as the reconstruction process then fails for all three techniques. In realistic conditions, noise may be higher than 5\%, and, therefore, enhancing the noise resilience of DxPRIM is a possible area for future research.

The first scatterer profile considered is the Austria profile as shown in Figs. 4(a) and 5(a). “Austria” is a benchmark
Fig. 5. Reconstructions for Austria profile with $\epsilon_r = 2 + 0.3j$ with $M_t = 20$ and $M_r = 20$. (a) Ground truth scatterer profile. Reconstruction using (b) DxPRIM, (c) PD-CSI, and (d) PD-SOM. $M_F = 7$ for PD-CSI (see [12]). Note that the colorbar scales for the subfigures are slightly different to clearly show the shape reconstructions. $x$- and $y$-axis are in m.

scatterer profile used in the inverse scattering community [1], [2], [3], [4], [12], [14] and is also used in state-of-the-art PD-ISP work [1], [12]. It is well-known that the “Austria” profile is a challenging profile for reconstruction [1], [2], [3], [4], [12], [14]. It is even more difficult to reconstruct with phaseless data as evident from [12], where deep learning has to be used along with nonlinear techniques such as PD-SOM and PD-CSI to obtain the reconstructions. The physical dimensions of the “Austria” profile in Figs. 4(a) and 5(a) are defined with respect to 2 GHz ($\lambda_0 = 0.15$ m). It consists of two disks and one ring. Both of the disks have a radius of $0.27\lambda_0$ m and the ring has an inner radius of $0.44\lambda_0$ m and an exterior radius of $0.8\lambda_0$ m. We also add a loss component to it so that the complex-valued relative permittivity becomes $\epsilon_r = \epsilon_R + j\epsilon_I$.

Fig. 4 provides reconstructions of the Austria profile with $\epsilon_r = 1.8 + 0.3j$ using the DxPRIM, PD-CSI, and PD-SOM techniques. The ground truth Austria profile is shown in Fig. 4(a) with the real part permittivity profile shown on the left and the imaginary part of the permittivity profile shown on the right. Fig. 4(b)–(d) show reconstructions using DxPRIM, PD-CSI, and PD-SOM, respectively. For this numerical example, we used $M_t = 20$ transmitters and $M_r = 40$ receivers which implies that we have an underdetermined problem with $2N = 3200$ unknowns and $M = M_t \times M_r = 800$ measurements ($M << 2N$). It can be clearly seen that DxPRIM outperforms PD-CSI and PD-SOM. It is also reported in [12] that to obtain this Austria profile reconstruction, both PD-CSI and PD-SOM need to be improved with a deep learning framework that needs large training data. However, as evident from Fig. 4(b), the proposed DxPRIM technique can reconstruct this profile directly without the need of any deep learning framework or large training data.

Next, in Fig. 5, we increase the relative permittivity of the Austria profile to $\epsilon_r = 2 + 0.3j$. We also make the problem more difficult by reducing the number of receivers to $M_r = 20$ (from $M_r = 40$ in the previous example). Therefore, the problem becomes further underdetermined as we have $M = M_t \times M_r = 400$ measurements to estimate $2N = 3200$ unknowns. It can be seen from Fig. 5(c) and (d) that PD-CSI and PD-SOM fail to reconstruct this profile. However, Fig. 5(b) shows that DxPRIM is able to reconstruct the profile with good accuracy.
Next, we select a homogeneous lossy scatterer profile used in recent deep learning-based PD-SOM work [12]. This profile consists of three circular scatterers with different permittivity values and sizes as shown in Fig. 6(a). Similar to the last example, we use \(M_r = 20\) transmitters and \(M_t = 20\) receivers. The reconstructions using DxPRIM, PD-CSI, and PD-SOM at 2 GHz are shown in Fig. 6(b)–(d), respectively. Similar to previous results, it can be seen that PD-CSI and PD-SOM fail to reconstruct the profile, whereas DxPRIM provides a high-accuracy reconstruction. The reason behind this is due to the larger size of the scatterers in this profile (the combined diameter of three scatterers is two times larger than the incident wavelength) as compared to the Austria profile.

To further test the validity range of DxPRIM, we increase the real part of permittivity \(\text{Re}(\varepsilon_r)\) of the smallest scatterer at top-left side in Fig. 6, while keeping the permittivity of the other two scatterers the same (the imaginary part of permittivity is left untouched). The two new resultant \(\text{Re}(\varepsilon_r)\) profiles are shown in Fig. 7(a) and (b) along with their reconstructions obtained using DxPRIM. In Fig. 7(a), the smallest scatterer on the top-left side has \(\text{Re}(\varepsilon_r) = 2\), whereas in Fig. 7(b), its permittivity is further increased to \(\text{Re}(\varepsilon_r) = 5\). We did not provide PD-CSI and PD-SOM reconstructions because both of the techniques failed to obtain reconstructions for smaller permittivity as shown in Fig. 6. It can be seen that even for the large relative permittivity of \(\text{Re}(\varepsilon_r) = 2\) and \(\text{Re}(\varepsilon_r) = 5\), DxPRIM is able to achieve reconstructions with acceptable accuracy.

Finally, we test the performance of the proposed DxPRIM method when both relative permittivity and the electrical size of the scatterer are large (larger than the incident wavelength). For this, we use the same profile as Fig. 6 but now instead of increasing the \(\text{Re}(\varepsilon_r)\) of the smallest scatterer, we increase \(\text{Re}(\varepsilon_r)\) for the large scatterer on the top-right corner (while keeping the permittivity of other two scatterers fixed). The two new resultant \(\text{Re}(\varepsilon_r)\) profiles are shown in Fig. 8(a) and (b) along with their reconstructions obtained using DxPRIM. In Fig. 8(a), the scatterer on the top-right side has \(\text{Re}(\varepsilon_r) = 2\), whereas in Fig. 8(a), the permittivity is further increased to \(\text{Re}(\varepsilon_r) = 5\). It can be seen that for \(\text{Re}(\varepsilon_r) = 2\), the proposed DxPRIM is still able to provide good reconstruction accuracy. For a higher value of permittivity \(\text{Re}(\varepsilon_r) = 5\), DxPRIM provides good shape reconstruction but fails to provide the permittivity value estimation.

To summarize, the proposed DxPRIM is shown to achieve accurate reconstructions for a variety of objects as can be seen in Figs. 4–6. Furthermore, DxPRIM also works for large values of relative permittivity \(\text{Re}(\varepsilon_r) \leq 5\) as can be seen from Fig. 7. Finally, when both permittivity and the electrical size of the scatterer is large, DxPRIM is still able to provide good shape reconstruction but fails to provide an accurate permittivity value estimation.

It is important to note that DxPRIM not only achieves a better validity range but also, as opposed to both PD-SOM and PD-CSI, does not need magnitude and phase measurements of the free-space incident field and relies on the change in the total field magnitude to image the change in contrast. This makes DxPRIM useful for imaging applications such as indoor RF imaging [15], [18], where it is not often feasible to collect free-space incident field data without the background clutter present in the imaging region (such as walls, ceilings, floors, and other objects).
A. Experimental Results

As a final verification of DxPRIM, we also provide reconstruction results using experimental data provided by the Fresnel Institute, France, where scattering data are collected in an anechoic chamber. This verification is important as it utilizes a dataset that is completely independent of our research group while also providing experimental results. The details of the experimental configurations and calibration process have been previously described [33]. We select the “FoamTwinDielTM” dataset which uses an inhomogeneous scatterer profile as shown in Fig. 9(a). It has overall size \( 15 \times 15 \) cm\(^2\) and consists of three cylindrical scatterers. The small red cylinder is of plastic material with \( \epsilon_r = 3 \pm 0.3 \). The large blue cylinder is made up of foam with \( \epsilon_f = 1.45 \pm 0.15 \). This large blue cylinder also contains an embedded small red cylinder with \( \epsilon_r = 3 \pm 0.3 \). The scatterers considered are lossless and the incident frequency is selected to be 2 GHz.

Fig. 9(b) provides reconstruction results for “FoamTwinDielTM” profile using DxPRIM. It can be seen that DxPRIM is able to recover the scatterer information. We have not provided the reconstruction of \( \text{Im}(\sigma) \) as it is predominantly sparse with maximum pixel value smaller than 0.04. The result is as expected as the profile “FoamTwinDielTM” is electrically smaller and with lower permittivity than the various simulation results provided. Nevertheless it is an important verification as it provides results from an independent experimental dataset.

VI. CONCLUSION

In this article, we presented a new DxPRIM to solve PD-ISPs. It is based on a correction to the conventional RA in strongly scattering, lossy media. The basis of the approach is to represent the ratio of the total to incident wavefield as IPWs. The wavefront deviation between the total and incident field can then be related to the incremental phase change of a ray passing through the scatterer similar to Fermat’s principle. We can then obtain a useful correction to the RA which we then formulate into a distorted waveform. This distorted waveform is utilized in an iterative framework to achieve high-quality reconstructions of strongly scattering, lossy objects. Using numerical results for benchmark profiles (such as Austria profile), we show that our proposed technique, DxPRIM, outperforms a state-of-the-art PD-CSI and PD-SOM techniques by a significant margin.

Currently our proposed method and existing techniques perform well for relative permittivity values \( \epsilon_r \leq 5 \) which are not sufficient for practical applications such as indoor imaging [18], where objects can have a much larger permittivity value. Future work can focus on further increasing the validity range of the proposed technique to handle stronger scatterers to make it viable for practical applications. Also, extensions that include some of the possible 3-D propagation effects would also be worth considering. The proposed DxPRIM can also be improved using deep learning techniques to further extend its validity range. Finally, the proposed model can be extended to reconstruct PECs.

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Amartansh Dubey (Graduate Student Member, IEEE) received the bachelor’s degree in electronics and communication engineering from the Visvesvaraya National Institute of Technology, Nagpur, India, in 2016, and the M.Phil. degree in electronic and computer engineering and the Ph.D. degree from The Hong Kong University of Science and Technology (HKUST), Hong Kong, in 2018 and 2022, respectively.

His research interests include inverse scattering problems, indoor imaging using Wi-Fi signals, statistical signal processing, machine learning, and computational electromagnetics.

Ross Murch (Fellow, IEEE) received the bachelor’s and Ph.D. degrees in electrical and electronic engineering from the University of Canterbury, Christchurch, New Zealand, in 1987 and 1990, respectively.

He was the Department Head with The Hong Kong University of Science and Technology (HKUST), Hong Kong, from 2009 to 2015. He is currently the Chair Professor with the Department of Electronic and Computer Engineering and a Senior Fellow with the Institute of Advanced Study, HKUST. His research contributions include more than 200 publications and 20 patents on wireless communication systems and antennas and these have attracted over 19,000 citations. His research interests include the Internet-of-Things, RF imaging, ambient RF systems, multiport antenna systems, reconfigurable intelligent surfaces, and acoustics. His unique expertise lies in his combination of knowledge from both wireless communication systems and electromagnetic areas.

Dr. Murch was a fellow of the Evaluation Committee. He has won several awards including the Computer Simulation Technology (CST) University Publication Award in 2015. He has served IEEE in various positions, including an Area Editor, the Technical Program Chair, and a Distinguished Lecturer.

He enjoys teaching and has won two teaching awards.