Abstract

This paper considers multidimensional jump type stochastic differential equations with super linear and non-Lipschitz coefficients. After establishing a sufficient condition for nonexplosion, this paper presents sufficient local non-Lipschitz conditions for pathwise uniqueness. The non confluence property for solutions is investigated. Feller and strong Feller properties under local non-Lipschitz conditions are investigated via the coupling method. Sufficient conditions for irreducibility and exponential ergodicity are derived. As applications, this paper also studies multidimensional stochastic differential equations driven by Lévy processes and presents a Feynman-Kac formula for Lévy type operators.

Key Words and Phrases. Pathwise uniqueness, non-explosion, non confluence, Feller and strong Feller properties, irreducibility, exponential ergodicity, Lévy type operator, Feynman-Kac formula.

1 Introduction

Let $(U, \mathcal{U})$ be a measurable space and $\nu$ a $\sigma$-finite measure on $U$. Let $d \geq 2$ be a positive integer, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $c : \mathbb{R}^d \times U \to \mathbb{R}^d$ be Borel measurable functions. Consider the following stochastic differential equation (SDE)

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \int_U c(X(t-), u)\tilde{N}(dt, du),$$

(1.1)
where $W$ is a standard $d$-dimensional Brownian motion, and $N$ is a Poisson random measure on $[0, \infty) \times U$ with intensity $dt \nu(du)$ and compensated Poisson random measure $\tilde{N}$. It is well-known that if the coefficients $b, \sigma$ and $c$ of (1.1) satisfy the linear growth and local Lipschitz conditions, then (1.1) admits a non-explooding strong solution and the solution is pathwise unique; see, for example, (Ikeda and Watanabe, 1989, Theorem IV.9.1) for details.

The linear growth condition is a standard assumption in the literature; it guarantees that the solution $X$ to (1.1) does not explode in finite time with probability one. But such a condition is often too restrictive in practice. For example, in many mathematical ecological models (such as those in Khasminskii and Klebaner (2001), Mao et al. (2002), Zhu and Yin (2009)), the coefficients do not satisfy the linear growth condition; yet non-explosion is still guaranteed thanks to the special structures of the underlying SDEs in these papers. For general multidimensional SDEs without jumps, the relaxation of linear growth condition can be found in Fang and Zhang (2005) and Lan and Wu (2014). For jump type SDEs, can we relax the usual linear growth condition as well? In this paper, we provide a sufficient condition in Theorem 2.2 for non-explosion for solutions to (1.1) when the coefficients have super linear growth in a neighborhood of $\infty$.

Concerning the pathwise uniqueness, the usual argument is to use the (local) Lipschitz condition and Gronwall’s inequality to demonstrate that the $L_2$ distance $E[|\tilde{X}(t) - X(t)|^2]$ between two solutions $\tilde{X}, X$ vanishes if they have the same initial condition; see, for example, the proof of Ikeda and Watanabe (1989, Theorem IV.9.1). The paper Yamada and Watanabe (1971) relaxes the local Lipschitz condition to Hölder condition for one-dimensional SDEs without jumps. Since then, the problem of existence and pathwise uniqueness of solutions to SDEs with non-Lipschitz conditions has attracted growing attention. To name just a few, Bass (2003) presents a sharp condition for existence and pathwise uniqueness for a one-dimensional SDE with a symmetric stable driving noise; Fu and Li (2010) and Li and Mytnik (2011) provide sufficient conditions for existence and pathwise uniqueness for one-dimensional jump type SDEs with non-Lipschitz conditions; a crucial assumption in these two papers is that the kernel for the compensated Poisson integral term is non-decreasing. Such a nondecreasing kernel assumption was weakened in Fournier (2013) and Li and Pu (2012). It should be noted that pathwise uniqueness need not hold in general if the diffusion matrix is merely uniformly nondegenerate, bounded and continuous even in the one-dimensional case; see Bass et al. (2004) for such an example of one-dimensional SDE driven by a symmetric stable process in which pathwise uniqueness fails. See also the discussion in Tanaka et al. (1974), in which pathwise uniqueness fails for some one-dimensional SDEs driven by symmetric Lévy processes with Hölder continuous drift coefficients. All the aforementioned references focus on one-dimensional SDEs and less is known for the multi-dimensional case. Fang and Zhang (2005) establishes sufficient non-Lipschitz conditions for pathwise uniqueness for multidimensional SDEs without jumps. These conditions were further relaxed in Lan and Wu (2014) using Euler’s approximation method. Further studies on jump type SDEs with non-Lipschitz coefficients can be found in Priola (2012, 2015), Qiao (2014), Qiao and Zhang (2008), among others.

This paper aims to establish sufficient non-Lipschitz conditions for pathwise uniqueness for multidimensional SDEs with jumps. Two sets of sufficient non-Lipschitz conditions (Assumptions 2.3 and 2.5) for pathwise uniqueness are provided; both of them only require the
Dong (3.4) holds, we follow Yamada and Watanabe’s idea and construct a sequence of smooth functions to control the $L_1$ distance of two solutions $\tilde{X}, X$ up to an appropriately defined stopping time. Next we use a Bihari’s inequality type argument to show that such an $L_1$ distance vanishes if the two solutions start from the same initial conditions. Then we argue that $\tilde{X}(t) = X(t)$ a.s. for any $t \geq 0$, which, in turn, leads to the desired pathwise uniqueness. The details are spelled out in Theorem 2.4. When Assumption 2.5 is in force, we develop a quite different and more direct proof in Theorem 2.6. In lieu of a sequence of smooth functions, a single smooth function is used to estimate, roughly speaking, a “scaled” $L_2$ distance of two solutions to (1.1), which helps us to immediately obtain $\tilde{X}(t) = X(t)$ a.s. Example 2.10 is provided to demonstrate the utility of our results.

Now suppose (1.1) has a unique non-exploding strong solution for any initial condition. We say that the solution $X$ of (1.1) satisfies the non confluence property, if for all $x \neq y \in \mathbb{R}^d$,

$$\mathbb{P}\{X^x(t) \neq X^y(t), \text{ for all } t \geq 0\} = 1,$$

where $X^x$ and $X^y$ denote solutions to (1.1) with initial conditions $x$ and $y$, respectively. We refer to Fang and Zhang (2005) and Lan and Wu (2014) for sufficient conditions for non confluence for SDEs without jumps. The recent paper Dong (2018) contains some sufficient conditions for non confluence for jump SDEs. The key assumption in Dong (2018) is on the jumps: for each $u \in U$, the function $x \mapsto x + c(x,u)$ is homeomorphic and that its inverse satisfies the linear growth and Lipschitz conditions. Such conditions are quite strong and not easy to verify in practice. We aim to relax such conditions in this paper. First, as long as the function $x \mapsto x + c(x,u)$ is one-to-one for $\nu$-almost all $u \in U$, Theorem 3.1 proposes a set of sufficient conditions in terms of the existence of a certain Lyapunov function for non confluence for (1.1). Then in Corollary 3.3, we prove that under a slightly stronger condition on the function $x \mapsto x + c(x,u)$, the non confluence property holds if the coefficients of (1.1) is Lipschitz continuous. Remark 3.4 demonstrates that our condition is quite easy to verify in general.

This paper next considers Feller and strong Feller properties for solutions to (1.1) under non-Lipschitz conditions. Suppose (1.1) has a solution $X$ which is unique in the sense of probability law. For $f \in \mathfrak{B}_b(\mathbb{R}^d)$ (the set of bounded and measurable functions), set

$$P_t f(x) := \mathbb{E}_x[f(X(t))] = \mathbb{E}[f(X^x(t))], \quad t \geq 0, x \in \mathbb{R}^d. \quad (1.2)$$

The family of operators $\{P_t\}_{t \geq 0}$ forms a semigroup of bounded linear operators on $\mathfrak{B}_b(\mathbb{R}^d)$. We are interested in the continuous properties of the semigroup. The semigroup or the corresponding process is said to be Feller if $P_t$ maps $C_b(\mathbb{R}^d)$ (the set of bounded and continuous functions) into itself and strong Feller if it maps $\mathfrak{B}_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ for each $t > 0$. Most work on Feller and strong Feller properties assumes (local) Lipschitz conditions on the coefficients of the underlying processes; see, for example, Theorem 6.3.4 of Stroock and Varadhan (1979) for diffusion processes, Proposition 2.1 of Wang (2010) for jump diffusions and Theorems 4.5 and 5.6 of Xi (2009) for regime-switching jump diffusions. By contrast, this paper establishes these properties under non-Lipschitz conditions. Proposition 4.2 and Theorem 4.4...
deal with Feller property while Theorem 5.2 and Proposition 5.4 establish strong Feller property. In these results, we only require certain local modulus of continuity of the coefficients of (1.1) in a small neighborhood of the diagonal line. These results improve substantially over the related work in the literature, even for SDEs without jumps. See Remark 5.5 for more details. Our main tool in establishing these two theorems is the coupling method, which has been extensively applied in the literature to study various properties of many processes, see, for example, Chen and Li (1989), Lindvall (2002), Priola and Wang (2006), Wang (2010) and the references therein.

Next we take up the issue of exponential ergodicity for the process \( X \) of (1.1). Following the same approach as those in Priola et al. (2012), Qiao (2014), Zhang (2009), we first show that the process \( X \) of (1.1) is irreducible under Assumptions 2.1 and 2.5. The conditions for irreducibility in Qiao (2014) are somewhat relaxed here; see Remark 6.2 for more details. The irreducibility and strong Feller property together then imply the uniqueness of an invariant measure for the process \( X \). A Foster-Lyapunov type drift condition then leads to the existence of an invariant measure as well as the exponential ergodicity. The details are spelled out in Theorem 6.6.

As applications, we consider SDEs driven by multidimensional Lévy processes \( dX(t) = \psi(X(t-))dL(t) \), in which \( \psi : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d} \) is Borel measurable and non-Lipschitz, and \( L \) is a multidimensional Lévy process, e.g., a symmetric stable process of order \( \alpha \) with \( \alpha \in (0, 2) \). Under what conditions on \( \psi \) so that this SDE has a unique non-explooding strong solution? We aim to answer this question in Section 7.1. For another application, we consider a Cauchy problem related to a Lévy type operator (7.7). Our goal is to establish a non-standard Feynman-Kac formula for solutions to the Cauchy problem and therefore establish a connection between integral-differential equations and SDEs of the form (1.1). The details are spelled out in Section 7.2.

Upon the completion of the manuscript, we learned that the recent paper Dong (2018) also contains sufficient conditions for non-explosion, pathwise uniqueness and non confluence for jump type SDEs. These conditions are quite different from our corresponding conditions and they do not seem to imply one another. In addition, the methodologies in Dong (2018) and this paper have different flavors, even though certain technical aspects are similar.

The rest of the paper is organized as follows. Section 2 presents sufficient conditions for non-explosion and pathwise uniqueness. The non confluence property for solutions to (1.1) is investigated in Section 3. Section 4 is focused on Feller property under non-Lipschitz condition. Strong Feller property is treated in Section 5. Section 6 studies irreducibility and exponential ergodicity. Finally Section 7 studies SDEs driven by multidimensional Lévy processes and establishes a Feynman-Kac formula for Lévy type operators. Several technical proofs are arranged in Appendix A.

To facilitate the presentation, we introduce some notation that will be used often in later sections. Throughout the paper, we use \( \langle x, y \rangle \) or \( x \cdot y \) interchangeably to denote the inner product of the vectors \( x \) and \( y \) with compatible dimensions. If \( A \) is a vector or matrix, let \( A^T \) denote the transpose of \( A \) and set \( |A| := \sqrt{\text{tr}(AA^T)} \). For a sufficiently smooth function \( \phi : \mathbb{R}^d \rightarrow \mathbb{R}, D_{x_i} \phi = \frac{\partial \phi}{\partial x_i}, D_{x_ix_j} \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \) and we denote by \( D\phi = (D_{x_1} \phi, \ldots, D_{x_d} \phi)^T \in \mathbb{R}^d \) and \( D^2 \phi = (D_{x_ix_j} \phi) \in \mathbb{R}^{d \times d} \) the gradient and Hessian matrix of \( \phi \), respectively. For \( k \in \mathbb{N} \),
$C^k(\mathbb{R}^d)$ is the collection of functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ with continuous partial derivatives up to the $k$th order while $C^c_c(\mathbb{R}^d)$ denotes the space of $C^k$ functions with compact support. If $B$ is a set, we use $I_B$ to denote the indicator function of $B$. Throughout the paper, we adopt the conventions that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. Finally, we note that the infinitesimal generator $\mathcal{L}$ of (1.1) is given by

$$\mathcal{L}f(x) := \langle Df(x), b(x) \rangle + \frac{1}{2} \text{tr}(\sigma(x)\sigma(x)^T D^2 f(x))$$

$$+ \int_{U} [f(x + c(x, u)) - f(x) - \langle Df(x), c(x, u) \rangle] \nu(du), \quad f \in C^2_c(\mathbb{R}^d).$$

## 2 Nonexplosion and Pathwise Uniqueness

In this section, we consider nonexplosion and pathwise uniqueness for SDE (1.1). Assume throughout this paper that the functions $b(\cdot)$, $\sigma(\cdot)$, and $c(\cdot, u)$ (for each $u \in U$) are continuous and that $c(\cdot, \cdot)$ is Borel measurable such that the function $x \mapsto \int_{U} |c(x, u)|^2 \nu(du)$ is continuous. For the convenience of later presentations, let us recall several important notions from Ikeda and Watanabe (1989) (as well as the presentations in Situ (2005)). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. Let $W = \{W(t), t \geq 0\}$ be a standard $d$-dimensional $\{\mathcal{F}_t\}$-Brownian motion and let $p = \{p(t), t \geq 0\}$ be an $\{\mathcal{F}_t\}$-Poisson point processes on $U$ with characteristic measures $\nu(du)$, where as mentioned in the introduction, $(U, \mathcal{U})$ is a measurable space and $\nu$ a $\sigma$-finite measure on $U$. Suppose that $W$ and $p$ are independent. Let $N(ds, du)$ be the Poisson random measures associated with $p$ and let $\tilde{N}(ds, du)$ be the compensated Poisson random measure of $N(ds, du)$. By a weak solution up to an explosion time to (1.1), we mean an $\mathbb{R}^d$-valued càdlàg and $\{\mathcal{F}_t\}$-adapted process $X = \{X(t), t \geq 0\}$ such that the equation

$$X(t \wedge \tau_n) = X(0) + \int_0^{t \wedge \tau_n} b(X(s)) ds + \int_0^{t \wedge \tau_n} \sigma(X(s)) dW(s) + \int_0^{t \wedge \tau_n} \int_U c(X(s-), u) \tilde{N}(ds, du)$$

holds for all $n \in \mathbb{N}$ and $t \geq 0$ a.s., where the initial condition $X(0) \in \mathcal{F}_0$ and $\tau_n := \inf\{t \geq 0 : |X(t)| > n\}$ is the first exit time from the closed ball $B(n) := \{x \in \mathbb{R}^d : |x| \leq n\}$. Clearly the sequence $\{\tau_n, n \in \mathbb{N}\}$ is nondecreasing. The limit $\tau := \lim_{n \to \infty} \tau_n$, finite or infinite, is called the explosion time or lifetime for the process $X$. In particular, we say that $X$ is explosive if $\mathbb{P}\{\tau < \infty\} > 0$; otherwise, $X$ is said to be non-explosive. We say pathwise uniqueness holds for (1.1) if for any two solutions $X_1, X_2$ of the equation satisfying $\mathbb{P}\{X_1(0) = X_2(0)\} = 1$ we have $\mathbb{P}\{X_1(t) = X_2(t) \text{ for all } t \geq 0\} = 1$. Let $\{\mathcal{G}_t\}_{t \geq 0}$ be the augmented natural filtration generated by $W$ and $p$. A solution $X$ of (1.1) is called a strong solution if it is adapted with respect to $\{\mathcal{G}_t\}_{t \geq 0}$.

The classical results (e.g., Ikeda and Watanabe (1989)) indicate that if the coefficients satisfy the usual linear growth condition, then the solution to (1.1) is non-explosive. This section aims to relax the linear growth condition.

**Assumption 2.1.** There exists a nondecreasing function $\zeta : [0, \infty) \mapsto [1, \infty)$ that is continuously differentiable and satisfies

$$\int_0^\infty \frac{dr}{r^2 \zeta^2(r) + 1} = \infty,$$
such that for all $x \in \mathbb{R}^d$,

$$2\langle x, b(x) \rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(du) \leq \kappa [ |x|^2 \zeta(|x|^2) + 1],$$

(2.2)

where $\kappa$ is a positive constant.

Some common functions satisfying (2.1) include $\zeta(r) = 1$, $\zeta(r) = \log r$ and $\zeta(r) = \log r \log(\log r)$ for $r$ large.

**Theorem 2.2.** Under Assumption 2.1, any solution to (1.1) is non-explosive.

**Proof.** This proof is motivated by the proof of Theorem A in Fang and Zhang (2005). Consider the function $\phi(r) := \exp\left\{ \int_0^r \frac{dz}{z\zeta(z)+1} \right\}$ for $r > 0$. Then we have

$$\phi'(r) = \frac{\phi(r)}{r\zeta(r)+1} > 0, \quad \text{and} \quad \phi''(r) = \phi(r) \frac{1 - \zeta(r) - r\zeta'(r)}{(r\zeta(r)+1)^2}.$$  

Since $\zeta(r) \geq 1$ and $\zeta$ is nondecreasing, it follows that $\phi''(r) \leq 0$ and hence $\phi$ is a concave function. On the other hand, thanks to (2.1), we have $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Now consider the function $\Phi : \mathbb{R}^d \mapsto \mathbb{R}^+$ defined by $\Phi(x) = \phi(|x|^2)$. We have $\Phi(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Moreover, straightforward computations lead to $D\Phi(x) = 2\phi'(|x|^2)x$ and $D^2\Phi(x) = 2\phi'(|x|^2)I + 4\phi''(|x|^2)x x^T$. Since $\phi$ is concave, we have $\phi(r) \leq \phi(r_0) + \phi'(r_0)(r-r_0)$ for all $r, r_0 \in (0, \infty)$. Using this inequality with $r_0 = |x|^2$ and $r = |x + c(x, u)|^2$, we have

$$\phi(|x + c(x, u)|^2) - \phi(|x|^2) \leq \phi'(|x|^2)|x + c(x, u)|^2 - |x|^2 = \phi'(|x|^2)[2\langle x, c(x, u) \rangle + |c(x, u)|^2].$$

Then it follows that

$$\int_U \left[ \Phi(x + c(x, u)) - \Phi(x) - \langle D\Phi(x), c(x, u) \rangle \right] \nu(du)$$

$$= \int_U [\phi(|x + c(x, u)|^2) - \phi(|x|^2) - 2\phi'(|x|^2)\langle x, c(x, u) \rangle] \nu(du)$$

$$\leq \int_U \left[ \phi'(|x|^2)[2\langle x, c(x, u) \rangle + |c(x, u)|^2] - 2\phi'(|x|^2)\langle x, c(x, u) \rangle \right] \nu(du)$$

$$= \int_U \phi'(|x|^2)|c(x, u)|^2 \nu(du).$$

Consequently we can compute

$$L\Phi(x) = 2\phi'(|x|^2)\langle x, b(x) \rangle + \frac{1}{2} \text{tr} \left( \sigma(x)\sigma'(x) \left[ 2\phi'(|x|^2)I + 4\phi''(|x|^2)x x^T \right] \right)$$

$$+ \int_U [\Phi(x + c(x, u)) - \Phi(x) - \langle D\Phi(x), c(x, u) \rangle] \nu(du)$$

$$\leq \phi'(|x|^2) \left( 2\langle x, b(x) \rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(du) \right) + 2\phi''(|x|)\langle x, \sigma(x) \rangle |^2$$

$$\leq \frac{\phi(|x|^2)}{|x|^2 \zeta(|x|^2) + 1} \kappa |x|^2 \zeta(|x|^2) + 1 \leq \kappa \phi(|x|^2) = \kappa \Phi(x),$$

6
where we used (2.2) and the fact that $\phi''(r) \leq 0$ to derive the second inequality. The rest of the proof is quite standard: one can apply Itô’s formula and the optional sampling theorem to the process $\{e^{-rt}\Phi(X(t)), t \geq 0\}$ to argue that $\mathbb{P}\{\lim_{n \to \infty} \tau_n = \infty\} = 1$. Indeed similar arguments can be found in, e.g., the proofs of Theorem 2.1 of Meyn and Tweedie (1993), Theorem A of Fang and Zhang (2005), and Theorem 2.1 of Dong (2018). We shall omit the details here.

The rest of the section is focused on sufficient conditions for pathwise uniqueness for the stochastic differential equation (1.1). Let us first make the following assumption:

**Assumption 2.3.** There exist a positive constant $\delta_0$ and a nondecreasing and concave function $\rho : [0, \infty) \mapsto [0, \infty)$ satisfying $\rho(r) > 0$ for $r > 0$, and

$$
\int_{0+} \frac{dr}{\rho(r)} = \infty,
$$

(2.3)

such that for all $R > 0$ and $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$,

$$
2\langle z - x, b(z) - b(x) \rangle + |\sigma(z) - \sigma(x)|^2 \leq \kappa_R|z - x|\rho(|z - x|),
$$

(2.4)

$$
\int_U |c(z, u) - c(x, u)|\nu(du) \leq \kappa_R\rho(|z - x|),
$$

(2.5)

where $\kappa_R$ is a positive constant. In addition, assume $\int_U |c(0, u)|\nu(du) < \infty$.

**Theorem 2.4.** Under Assumptions 2.1 and 2.3, pathwise uniqueness holds for (1.1).

The proof of Theorem 2.4 is in the same spirit of Yamada and Watanabe’s argument for pathwise uniqueness in Yamada and Watanabe (1971) and Fu and Li (2010), Li and Mytnik (2011). The key idea is to construct a sequence of monotone $C^2$ functions $\{\psi_n\}$ satisfying certain conditions so that one can bound the growth of the $L_1$ distance $\mathbb{E}[|\tilde{X}(t \wedge S_{\delta_0}) - X(t \wedge S_{\delta_0})|]$ of two solutions $\tilde{X}, X$ with the same initial condition, where $S_{\delta_0}$ is a stopping time related to the solutions $\tilde{X}, X$. Next we use a Bihari’s inequality type argument to obtain $\mathbb{E}[|\tilde{X}(t \wedge S_{\delta_0}) - X(t \wedge S_{\delta_0})|] = 0$, from which we derive $\tilde{X}(t) = X(t)$ a.s. This, together with the right-continuity of solutions to (1.1), enables us to establish the pathwise uniqueness result. To preserve the flow of presentation, we relegate the proof of Theorem 2.4 to Appendix A.

Next we propose a different assumption than that of Assumption 2.3 for pathwise uniqueness.

**Assumption 2.5.** There exist a positive number $\delta_0$ and a nondecreasing and concave function $\varrho : [0, \infty) \mapsto [0, \infty)$ satisfying

$$
0 < \varrho(r) \leq (1 + r)^2 \varrho(r/(1 + r)) \text{ for all } r > 0, \quad \text{and} \quad \int_{0+} \frac{dr}{\varrho(r)} = \infty,
$$

(2.6)

such that for all $R > 0$ and $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$,

$$
2\langle x - z, b(x) - b(z) \rangle + |\sigma(x) - \sigma(z)|^2 + \int_U |c(x, u) - c(z, u)|^2\nu(du) \leq \kappa_R\varrho(|x - z|^2),
$$

(2.7)

where $\kappa_R$ is a positive constant.
Some common functions satisfying Assumptions 2.3 and 2.5 include $g(r) = r$ and concave and increasing functions such as $g(r) = r \log(1/r)$, $g(r) = r \log(\log(1/r))$, and $g(r) = r \log(1/r) \log(\log(1/r))$ for $r \in (0, \delta)$ with $\delta > 0$ small enough. It is worth pointing out that (2.4) and (2.5) in Assumption 2.3 and (2.7) in Assumption 2.5 only require the modulus continuity to hold in a small neighborhood of the diagonal line $x = z$ in $\mathbb{R}^d \otimes \mathbb{R}^d$ with $|x| \vee |z| \leq R$ for each $R > 0$. This is in contrast to those in Fu and Li (2010), Li and Mytnik (2011). Note, in particular, that the constant $\kappa_R$ in (2.4), (2.5) and (2.7) may depend on $R$. These conditions are very general but make our analysis very subtle; careful analysis are required to accommodate various stopping times. On the other hand, even in the case with $g(r) = r$, since $\nu(U)$ is not necessarily finite, Assumptions 2.3 and 2.5 in general cannot imply each other. Moreover, instead of using a sequence of $C^2$ functions $\{\psi_n\}$, we use a single $C^2$ function $h$ to obtain the desired pathwise uniqueness result in Theorem 2.6. Compared with the aforementioned references, the proof of Theorem 2.6 is simpler and more direct. Again, we arrange the proof of Theorem 2.6 to Appendix A.

**Theorem 2.6.** Under Assumptions 2.1 and 2.5, pathwise uniqueness holds for (1.1).

**Remark 2.7.** In case that the solution to (1.1) has a finite explosion time with positive probability, then pathwise uniqueness holds up to the explosion time under Assumptions 2.3 or 2.5.

**Theorem 2.8.** Suppose Assumption 2.1 and either Assumption 2.3 or Assumption 2.5 hold. Then for any $x \in \mathbb{R}^d$, (1.1) has a unique strong non-explosive solution $X = \{X(t), t \geq 0\}$ satisfying $X(0) = x$.

**Proof.** Suppose that Assumptions 2.1 and 2.3 hold; the proof for the case under Assumptions 2.1 and 2.5 is similar. Let us fix some $x \in \mathbb{R}^d$. For each $n \in \mathbb{N}$ with $|x| < n$, let $\psi_n : \mathbb{R}^d \to [0, 1]$ be a $C^\infty$ function such that $\psi_n(x) = 1$ for $|x| \leq n$ and $\psi_n(x) = 0$ for $|x| \geq n + 1$. Define $b_n := \psi_n b, \sigma_n := \psi_n \sigma$ and $c_n := \psi_n c$. Then

$$b_m(x) = b_n(x) = b(x), \quad \sigma_m(x) = \sigma_n(x) = \sigma(x), \quad \text{and} \quad c_m(x, u) = c_n(x, u) = c(x, u) \quad (2.8)$$

for all $\{(x, u) \in \mathbb{R}^d \times U : |x| \leq n\}$ and $m \geq n$. Obviously, for each $n \in \mathbb{N}$, $b_n(\cdot)$ and $\sigma_n(\cdot)$ are bounded and continuous and that $c_n(\cdot, \cdot)$ is measurable. Moreover, for any $x \in \mathbb{R}^d$, $M(x, B) := \nu\{u \in U : c_n(x, u) \in B\}, B \in \mathcal{B}(\mathbb{R}^d)$, is a $\sigma$-finite measure on $\mathcal{B}(\mathbb{R}^d)$ and satisfies

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} M(x, dy) = \int_U \frac{|c_n(x, u)|^2}{1 + |c_n(x, u)|^2} \nu(du) \leq \int_U \psi_n(x)^2 |c(x, u)|^2 \nu(du) < \infty.$$

Likewise, for any $\phi \in C_b(\mathbb{R}^d)$, the function

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi(y) M(x, dy) = \int_U \frac{|c_n(x, u)|^2}{1 + |c_n(x, u)|^2} \phi(c_n(x, u)) \nu(du) \leq \|\phi\|_\infty \int_U \psi_n(x)^2 |c(x, u)|^2 \nu(du) \leq \kappa \psi_n(x)^2 \|\phi\|_\infty |x|^2 \zeta(|x|^2) + 1]$$
is bounded and continuous, where the last inequality follows from (2.2) in Assumption 2.1. Now consider the the operator

$$
\mathcal{L}_n f(x) := \langle Df(x), b_n(x) \rangle + \frac{1}{2} \text{tr}(\sigma_n(x)\sigma_n(x)^T D^2f(x)) \\
+ \int_U [f(x + c_n(x, u)) - f(x) - \langle Df(x), c_n(x, u) \rangle] \nu(du) \\
= \langle Df(x), b_n(x) \rangle + \frac{1}{2} \text{tr}(\sigma_n(x)\sigma_n(x)^T D^2f(x)) \\
+ \int_{\mathbb{R}^d} [f(x + y) - f(x) - \langle Df(x), y \rangle] M(x, dy), \quad f \in C^2_c(\mathbb{R}^d).
$$

Thanks to Theorem 2.2 in Stroock (1975), the martingale problem for $\mathcal{L}_n$ has a solution. Then by virtue of Theorem 2.3 of Kurtz (2011), the stochastic differential equation

$$
X^{(n)}(t) = x + \int_0^t b_n(X^{(n)}(s))ds + \int_0^t \sigma_n(X^{(n)}(s))dW(s) + \int_0^t \int_U c_n(X^{(n)}(s-), u)\tilde{N}(ds, du) 
$$

has a weak solution $X^{(n)}$.

Apparently $b_n$ and $\sigma_n$ satisfy Assumption 2.3. On the other hand, for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$, we have from (2.5) that

$$
\int_U |c_n(x, u) - c_n(z, u)| \nu(du) \leq \int_U \left[ |\psi_n(x)||c(x, u) - c(z, u)| + |\psi_n(x) - \psi_n(z)||c(z, u)| \right] \nu(du) \\
\leq \int_U |c(x, u) - c(z, u)| \nu(du) + |\psi_n(x) - \psi_n(z)| \int_U c(z, u) \nu(du) \\
\leq \kappa_R \rho(|x - z|) + K_R |x - z|, \quad (2.10)
$$

where we used the facts that $\psi_n$ is locally Lipschitz and that the function $x \mapsto \int_U |c(x, u)| \nu(du)$ is locally bounded to obtain the last inequality. Furthermore, since $\rho(\cdot)$ is concave and $\rho(0) = 0$, it follows that $\rho(r) \geq \frac{\rho(\delta_0)}{\delta_0} r$ or $r \leq \frac{\delta_0}{\rho(\delta_0)} \rho(r)$ for all $r \in [0, \delta_0]$. Applying this observation in (2.10) leads to

$$
\int_U |c_n(x, u) - c_n(z, u)| \nu(du) \leq \tilde{\kappa}_R \rho(|x - z|),
$$

for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$, where $\tilde{\kappa}_R$ is a positive constant. Therefore $c_n$ also satisfies Assumption 2.3. Theorem 2.4 then implies that pathwise uniqueness holds. Now by Theorem 2 of Barczy et al. (2015), for each $n \in \mathbb{N}$, a unique strong solution $X^{(n)}$ to (2.9) exists. Let $\tau_n := \inf\{t \geq 0 : |X^{(n)}(t)| > n\}$ denote the first exit time of $X^{(n)}$ from $B(n)$.

Furthermore, for any $m \geq n$, again thanks to the pathwise uniqueness as well as (2.8), the processes $X^{(m)}$ and $X^{(n)}$ have the same first exit time $\tau_n$ from $B(n)$ and $X^{(m)}(t) = X^{(n)}(t)$ for all $t < \tau_n$. Now the process $X$ defined by $X(t) := X^{(n)}(t)$ for all $t < \tau_n$, $n \in \mathbb{N}$ is the unique strong solution to (1.1) with $X(0) = x$; Theorem 2.2 implies that $X$ has no finite explosion time. This completes the proof. \qed
Corollary 2.9. Let $U_0 \subset U$ so that $\nu(U \setminus U_0) < \infty$. Suppose Assumption 2.1 and either Assumption 2.3 or Assumption 2.5 (with $U$ replaced by $U_0$) hold. Then for any initial condition $x \in \mathbb{R}^d$, the stochastic differential equation

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s)$$

$$+ \int_0^t \int_{U_0} c(X(s-), u)\tilde{N}(ds, du) + \int_0^t \int_{U \setminus U_0} c(X(s-), u)N(ds, du)$$

(2.11)

has a unique strong non-explosive solution $X = \{X(t), t \geq 0\}$.

Proof. This corollary follows from the standard interlacing procedure as in the proof Theorem 6.2.9 of Applebaum (2009). Indeed, under Assumptions 2.1 and 2.3 or Assumptions 2.1 and 2.5 (with $U$ replaced by $U_0$), for any initial condition, Theorem 2.8 implies that the SDE

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dW(t) + \int_{U_0} c(Y(t-), u)\tilde{N}(dt, du)$$

has a unique strong non-explosive solution. Next we use the interlacing procedure as in the proof Theorem 6.2.9 of Applebaum (2009) to construct a solution to (2.11). The solution is unique thanks to Theorems 2.4 or 2.6 and the interlacing structure. □

Example 2.10. Let us consider the following SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \int_{U_0} c(X(t-), u)\tilde{N}(dt, du), X(0) = x \in \mathbb{R}^3,$$

(2.12)

where $W$ is a 3-dimensional standard Brownian motion, $\tilde{N}(dt, du)$ is a compensated Poisson random measure with compensator $dt \nu(du)$ on $[0, \infty) \times U$, in which $U = \{u \in \mathbb{R}^3 : 0 < |u| < 1\}$ and $\nu(du) := \frac{du}{|u|^{\alpha+1}}$ for some $\alpha \in (0, 2)$. The coefficients of (2.12) are given by

$$b(x) = \begin{pmatrix}
-x_1^{1/3} - x_1^3 \\
-x_2^{1/3} - x_2^3 \\
-x_3^{1/3} - x_3^3
\end{pmatrix}, \quad \sigma(x) = \begin{pmatrix}
x_2^2/\sqrt{2} + 1 & x_3^2/\sqrt{2} \\
x_2^2/\sqrt{2} & x_3^2/\sqrt{2} + 1 \\
x_2^2/\sqrt{2} & x_3^2/\sqrt{2}
\end{pmatrix}, \quad c(x, u) = \begin{pmatrix}
\gamma x_1^{2/3} |u| \\
\gamma x_2^{2/3} |u| \\
\gamma x_3^{2/3} |u|
\end{pmatrix},$$

in which $\gamma$ is a positive constant so that $\gamma^2 \int_U |u|^2 \nu(du) = \frac{1}{2}$.

Note that even without jumps, the coefficients of (2.12) do not satisfy conditions (H1) and (H2) in Fang and Zhang (2005) since $\sigma$ and $b$ grow very fast in the neighborhood of $\infty$ and they are Hölder continuous with orders $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Nevertheless, the coefficients of (2.12) still satisfy Assumptions 2.1 and 2.5 and hence a unique non-explosive strong solution of (2.12) exists. The verifications of these assumptions are as follows.

$$2 \langle x, b(x) \rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(du)$$

$$= 2 \sum_{j=1}^3 x_j (-x_j^{1/3} - x_j^3) + \sum_{j=1}^3 \left( \frac{1}{2} x_j^{4/3} + \frac{2}{9} x_j^4 + \sqrt{2} x_j^{2/3} + 1 \right) + \int_U \gamma^2 |u|^2 \sum_{j=1}^3 x_j^{4/3} \nu(du)$$
\[
= -\frac{16}{9} \sum_{j=1}^{3} x_j^4 - \sum_{j=1}^{3} x_j^{4/3} + \sqrt{2} \sum_{j=1}^{3} x_j^{2/3} + 3. \tag{2.13}
\]

This verifies Assumption 2.1. For the verification of Assumption 2.5, we compute

\[
2\langle x - y, b(x) - b(y) \rangle + |\sigma(x) - \sigma(y)|^2 + \int_U |c(x, u) - c(y, u)|^2 \nu(du)
= -2 \sum_{j=1}^{3} (x_j - y_j)(x_j^{1/3} - y_j^{1/3} + x_j^{3/3} - y_j^{3/3}) + \frac{1}{2} \sum_{j=1}^{3} (x_j^{2/3} - y_j^{2/3})^2
+ \frac{2}{9} \sum_{j=1}^{3} (x_j^2 - y_j^2)^2 + \int_U \sum_{j=1}^{3} \gamma^2(x_j^{2/3} - y_j^{2/3})^2 |u|^2 \nu(du)
= -\frac{16}{9} \sum_{j=1}^{3} (x_j - y_j)^2 \left[ (x_j + \frac{7}{16} y_j)^2 + \frac{207}{256} y_j^2 \right] - \sum_{j=1}^{3} (x_j^{1/3} - y_j^{1/3})^2 (x_j^{2/3} + y_j^{2/3}). \tag{2.14}
\]

Obviously this verifies Assumption 2.5.

3 Non Confluence Property

**Theorem 3.1.** Assume the conditions of Theorem 2.8. In addition, suppose

* for \( \nu \)-almost all \( u \), the function \( x \mapsto x + c(x, u) \) is one-to-one. \( \tag{3.1} \)

Moreover, assume that there exist a nondecreasing and concave function \( \psi : [0, \infty) \to [0, \infty) \) that vanishes only at \( r = 0 \), and a \( C^2 \) function \( V : (0, \infty) \mapsto (0, \infty) \) satisfying

(i) \( V \) is nonincreasing in a neighborhood of 0 and \( \lim_{r \to 0} V(r) = \infty \), and

(ii) for all \( |x - z| > 0 \),

\[
\psi(V(|x - z|)) \geq \frac{1}{2} \left( \frac{V''(|x - z|)}{|x - z|} - \frac{V'(|x - z|)}{|x - z|} \right) |\langle x - z, \sigma(x) - \sigma(z) \rangle|^2
+ \frac{V''(|x - z|)}{2|x - z|} \left( 2 \langle x - z, b(x) - b(z) \rangle + |\sigma(x) - \sigma(z)|^2 \right) \tag{3.2}
+ \int_U \left[ V(|x - z + c(x, u) - c(z, u)|) - V(|x - z|)
- \frac{V'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle \right] \nu(du).
\]

Then the non confluence property for (1.1) holds:

* If \( \bar{x} \neq x \), then \( \mathbb{P}\{X_{\bar{x}}(t) \neq X_x(t) \text{ for all } t \geq 0\} = 1 \), \( \tag{3.3} \)

where \( X_{\bar{x}} \) and \( X_x \) denote the solutions to (1.1) with initial conditions \( \bar{x} \) and \( x \), respectively.
Remark 3.2. Note that (3.1) prevents the process $X^\tilde{\pi}(t) - X^\pi(t)$ from jumping to 0 from a nonzero location. Also, by Itô’s formula, the right hand side of (3.2) is the extended generator $\tilde{\mathcal{L}}$ of the process $X^\tilde{\pi} - X^\pi$ applied to the function $(x - z) \mapsto V(|x - z|)$; see Meyn and Tweedie (1993) for the definition of the extended generator. We can also regard $\tilde{\mathcal{L}}$ as the basic coupling operator of $\mathcal{L}$ of (1.3); see Section 4 for more details.

Proof of Theorem 3.1. Let $\tilde{X}(t) = X^\tilde{\pi}(t)$, $X(t) = X^\pi(t)$ and denote $\Delta_t := \tilde{X}(t) - X(t)$ as in the proof of Theorem 2.4. In addition, assume that $|\Delta_0| = |\tilde{x} - x| > 0$. For all $N \ni n > \frac{1}{|\Delta_0|}$ and $R > |\tilde{x}| \lor |x|$, define

$$T_{1/n} := \inf\{t \geq 0 : |\Delta_t| \leq 1/n\}, \text{ and } \tau_R := \inf\{t \geq 0 : |\tilde{X}(t)| \lor |X(t)| > R\}.$$

Putting $T_0 := \inf\{t \geq 0 : |\Delta_t| = 0\}$. Then we have $T_0 = \lim_{n \to \infty} T_{1/n}$ and $\lim_{R \to \infty} \tau_R = \infty$ a.s. Applying Itô’s formula to the process $V(|\Delta_{\tau_R \wedge T_{1/n}}|)$ and using (3.2), we have

$$\mathbb{E}[V(|\Delta_{\tau_R \wedge T_{1/n}}|)] = V(|\Delta_0|) + \mathbb{E} \left[ \int_0^{\tau_R \wedge T_{1/n}} \tilde{\mathcal{L}}V(|\tilde{X}(s) - X(s)|) ds \right]$$

$$\leq V(|\Delta_0|) + \mathbb{E} \left[ \int_0^{\tau_R \wedge T_{1/n}} \psi(V(|\Delta_s|)) ds \right]$$

$$\leq V(|\Delta_0|) + \mathbb{E} \left[ \int_0^{\tau_R \wedge T_{1/n}} \psi(V(|\Delta_s|)) ds \right]$$

$$\leq V(|\Delta_0|) + \int_0^{\tau_R \wedge T_{1/n}} \psi(\mathbb{E}[V(|\Delta_s|)]) ds,$$

where we used the concavity of $\psi$ and Jensen’s inequality to obtain the last inequality. Denote $u(t) := \mathbb{E}[V(|\Delta_{\tau_R \wedge T_{1/n}}|)]$. Then $u$ satisfies $0 \leq u(t) \leq V(|\Delta_0|) + \int_0^t \psi(u(s)) ds$. We can use a similar argument as that in the end of the proof of Theorem 2.4 to show that

$$0 \leq u(t) = \mathbb{E}[V(|\Delta_{\tau_R \wedge T_{1/n}}|)] \leq G^{-1}(G(V(|\Delta_0|) + t),$$

in which $G(r) := \int_1^r \frac{ds}{\psi(s)}$, $r \in [0, \infty)$ and $G^{-1}(y) := \inf\{s \geq 0 : G(s) > y\}$ for $y \in \mathbb{R}$. Note that since $\psi$ is nonnegative, both $G$ and $G^{-1}$ are nondecreasing. In addition, since $\infty > V(|\Delta_0|) > 0$, we have $\infty > G(V(|\Delta_0|)) + t > -\infty$ and hence $G^{-1}(G(V(|\Delta_0|) + t)) \geq 0$ is finite. Now letting $R \to \infty$ in (3.4), we obtain from Fatou’s lemma that

$$0 \leq \mathbb{E}[V(|\Delta_{\tau_{1/n}}|)] \leq G^{-1}(G(V(|\Delta_0|)) + t).$$

Furthermore, on the set $\{T_{1/n} < t\}$, $|\Delta_{\tau_{1/n}}| \leq 1/n$. Thus it follows from condition (i) that

$$V(1/n) \mathbb{P}\{T_{1/n} < t\} \leq \mathbb{E}[V(|\Delta_{\tau_{1/n}}|)I_{(T_{1/n} < t)}] \leq \mathbb{E}[V(|\Delta_{\tau_{1/n}}|)] \leq G^{-1}(G(V(|\Delta_0|)) + t).$$

Rewrite the above inequality as

$$\mathbb{P}\{T_{1/n} < t\} \leq \frac{G^{-1}(G(V(|\Delta_0|)) + t)}{V(1/n)}.$$

Now passing to the limit as $n \to \infty$, we obtain from condition (i) that $\mathbb{P}\{T_0 < t\} = 0$. This is true for any $t \geq 0$ so letting $t \to \infty$, we obtain $\mathbb{P}\{T_0 < \infty\} = 0$. In other words, $|\Delta_t|$ is positive on the interval $[0, \infty)$ a.s. This completes the proof. \[\square\]
Theorem 3.1 presents sufficient condition for non confluence in terms of the existence of a certain Lyapunov function. Often, it is not an easy task to find such a Lyapunov function. The following corollary indicates that as long as the coefficients of (1.1) is Lipschitz, then the non confluence property holds.

**Corollary 3.3.** Suppose Assumption 2.1 and that there exists a $\delta > 0$ such that

$$\nu \left\{ u \in U : \text{there exist } x, z \in \mathbb{R}^d \text{ such that } x - z \neq 0 \right\} = 0 \quad \text{but } |x - z + c(x, u) - c(z, u)| \leq \delta |x - z| = 0. \quad (3.5)$$

Assume the coefficients of (1.1) satisfy for some positive constant $K$ that

$$2|\langle x - z, b(x) - b(z) \rangle| + |\sigma(x) - \sigma(z)|^2 + \int_{U} [ |c(x, u) - c(z, u)|^2 + |(x - z) \cdot (c(x, u) - c(z, u))|] \nu(du) \leq K|x - z|^2, \quad (3.6)$$

for all $x, z \in \mathbb{R}^d$. Then the non confluence property for (1.1) holds.

**Proof.** Apparently (3.6) verifies Assumption 2.5. This, together with Assumption 2.1, implies that (1.1) has a unique strong non-exploding solution $X^x$ for any initial condition $x \in \mathbb{R}^d$. Note also that (3.5) implies (3.1). The remaining proof is to find a smooth function $V$ satisfying the conditions of Theorem 3.1.

Consider the function $V(r) := r^{-2}$ for $r > 0$. Of course $V$ satisfies condition (i) of Theorem 3.1. It remains to verify condition (ii). To this end, let us first prove that for all $x, y \in \mathbb{R}^n$ with $x \neq 0$ and $|x + y| \geq \delta|x|$, where $\delta > 0$ is some constant, we have

$$V(|x + y|) - V(|x|) - DV(|x|) \cdot y = \frac{1}{|x + y|^2} - \frac{1}{|x|^2} + \frac{2x \cdot y}{|x|^4} \leq K \frac{|y|^2 \vee |x \cdot y|}{|x|^4}, \quad (3.7)$$

in which $K$ is a positive constant. Let us prove (3.7) in three cases:

**Case 1:** $x \cdot y \geq 0$. In this case, it is easy to verify that for any $\theta \in [0, 1]$, we have $|x + \theta y|^2 = |x|^2 + 2\theta x \cdot y + \theta^2|y|^2 \geq |x|^2$. Therefore we can use the Taylor expansion with integral reminder to compute

$$\frac{1}{|x + y|^2} - \frac{1}{|x|^2} + \frac{2x \cdot y}{|x|^4} = \int_0^1 \frac{1}{2} y \cdot D^2V(x + \theta y)y \, d\theta$$

$$= \int_0^1 \left[ -\frac{|y|^2}{|x + \theta y|^4} + 2\frac{y^T(x + \theta y)(x + \theta y)^Ty}{|x + \theta y|^6} \right] \, d\theta$$

$$\leq 2 \int_0^1 \frac{|y|^2}{|x + \theta y|^4} \, d\theta \leq 2 \int_0^1 \frac{|y|^2}{|x|^4} \, d\theta = \frac{2|y|^2}{|x|^4}.$$

**Case 2:** $x \cdot y < 0$ and $2x \cdot y + |y|^2 \geq 0$. In this case, we have $|x + y|^2 = |x|^2 + 2x \cdot y + |y|^2 \geq |x|^2$ and hence $|x + y|^2 - |x|^2 \leq 0$; which together with $x \cdot y \leq 0$ implies that $|x + y|^2 - |x|^2 + 2|x|^{-2}x \cdot y \leq 0$. 

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Case 3: $x \cdot y < 0$ and $2x \cdot y + |y|^2 < 0$. In this case, we use the bound $|x+y|^2 \geq \delta^2 |x|^2$ to compute

$$
\frac{1}{|x+y|^2} - \frac{1}{|x|^2} + \frac{2x \cdot y}{|x|^4} = \frac{|x|^2 - |x+y|^2}{|x|^2 |x+y|^2} + \frac{2x \cdot y}{|x|^4} - \frac{|y|^2}{|x|^2 |x+y|^2} + \frac{2x \cdot y}{|x|^4} \\
\leq -\frac{2x \cdot y}{|x|^2 |x+y|^2} \leq -\frac{2x \cdot y}{\delta^2 |x|^4}.
$$

Combining the three cases gives (3.7).

For all $x \neq z \in \mathbb{R}^d$, (3.5) implies that $\nu \{ u \in U : |x - z + c(x, u) - c(z, u)| \leq \delta |x - z| \} = 0$. Hence, with the notations $A(x, z), B(x, z)$ defined in (A.2), we can use (3.6) and (3.7) to compute

$$
\hat{V}(|x - z|) = \frac{1}{2} V''(|x - z|) A(x, z) + \frac{V'(|x - z|)}{2|x - z|} (2B(x, z) - A(x, z) + |\sigma(x) - \sigma(z)|^2) \\
+ \int_U \left[ V(|x - z + c(x, u) - c(z, u)|) - V(|x - z|) - \frac{V'(|x - z|)}{|x - z|} c(x, u) - c(z, u) \right] \nu(du) \\
\leq \frac{4}{|x - z|^4} \frac{|x - z|^2 |\sigma(x) - \sigma(z)|^2}{|x - z|^2} + \frac{1}{|x - z|^4} |x - z, b(x) - b(z)| \\
+ K \int_U \frac{|c(x, u) - c(z, u)|^2 \vee |x - z, c(x, u) - c(z, u)|}{|x - z|^4} \nu(du) \\
\leq K|x - z|^{-2},
$$

where $K$ is some positive constant. This verifies condition (ii) of Theorem 3.1 and hence finishes the proof of the corollary. □

Remark 3.4. Assume that either

$$
\langle x - z, c(x, u) - c(z, u) \rangle \geq 0,
$$

or

$$
\langle x - z, c(x, u) - c(z, u) \rangle < 0 \quad \text{and} \quad 2\langle x - z, c(x, u) - c(z, u) \rangle + |c(x, u) - c(z, u)|^2 \geq 0,
$$

for all $x, z \in \mathbb{R}^d$ and $u \in U$. Then (3.5) is automatically satisfied and moreover, the integrand of the integral term in (3.6) can be replaced by $|c(x, u) - c(z, u)|^2$. This is clear from Cases 1 and 2 for the proof of (3.7).

4 Feller Property

Assumption 4.1. For any initial condition $x \in \mathbb{R}^d$, the stochastic differential equation (1.1) has a non-exploding weak solution $X^x$ and the solution is unique in the sense of probability law.
Under Assumption 4.1, we can define the semigroup \( P_t f(x) := \mathbb{E}_x[f(X(t))] = \mathbb{E}[f(X^x(t))] \) for \( f \in \mathfrak{B}_b(\mathbb{R}^d) \) and \( t \geq 0 \), where \( X^x \) denotes the unique weak solution of (1.1) with initial condition \( X^x(0) = x \in \mathbb{R}^d \).

We have the following result:

**Proposition 4.2.** Suppose that Assumption 4.1 and either Assumption 2.3 or Assumption 2.5 hold, then the process \( X \) is Feller continuous.

**Proof.** Let Assumptions 4.1 and 2.3 hold and use the same notations as in the proof of Theorem 2.4. The end of the proof of Theorem 2.4 (cf. (A.7)) reveals that for any \( R > 0 \)

\[
\lim_{|\bar{x} - x| \to 0} \mathbb{E}[|\Delta_t \wedge S_{\delta_0} \wedge \tau_R|] = \lim_{|\bar{x} - x| \to 0} \mathbb{E}[|\bar{X}(t \wedge S_{\delta_0} \wedge \tau_R) - X(t \wedge S_{\delta_0} \wedge \tau_R)|] = 0 \quad \text{for all } t \geq 0. \tag{4.1}
\]

On the set \( \{ S_{\delta_0} \leq t \wedge \tau_R \} \), we have \( |\Delta_t \wedge S_{\delta_0} \wedge \tau_R| \geq \delta_0 \) and hence \( \delta_0 \mathbb{P}\{ S_{\delta_0} \leq t \wedge \tau_R \} \leq \mathbb{E}[|\Delta_t \wedge S_{\delta_0} \wedge \tau_R|] \). For any \( \epsilon > 0 \) and \( t \geq 0 \), we can choose an \( R > 0 \) sufficiently large so that \( \mathbb{P}(\tau_R < t) < \epsilon \). For any \( \epsilon > 0 \), we can compute

\[
\mathbb{P}\{|\Delta_t| > \epsilon \} = \mathbb{P}\{|\Delta_t| > \epsilon, \tau_R < t\} + \mathbb{P}\{|\Delta_t| > \epsilon, \tau_R \geq t, S_{\delta_0} > t\} + \mathbb{P}\{|\Delta_t| > \epsilon, \tau_R \geq t, S_{\delta_0} \leq t\} \\
\leq \epsilon + \mathbb{P}\{|\Delta_t \wedge S_{\delta_0} \wedge \tau_R| > \epsilon, \tau_R \geq t, S_{\delta_0} > t\} + \mathbb{P}\{ S_{\delta_0} \leq t \wedge \tau_R \} \\
\leq \epsilon + \mathbb{P}\{|\Delta_t \wedge S_{\delta_0} \wedge \tau_R| > \epsilon\} + \frac{\mathbb{E}[|\Delta_t \wedge S_{\delta_0} \wedge \tau_R|]}{\delta_0} \\
\leq \epsilon + \frac{\mathbb{E}[|\Delta_t \wedge S_{\delta_0} \wedge \tau_R|]}{\epsilon} + \frac{\mathbb{E}[|\Delta_t \wedge S_{\delta_0} \wedge \tau_R|]}{\delta_0} \\
\to \epsilon + 0,
\]

as \( \bar{x} - x \to 0 \), where we used (4.1) in the last step. Since \( \epsilon > 0 \) is arbitrary, it follows from that \( \Delta_t \) converges to 0 in probability as \( \bar{x} - x \to 0 \).

Recall that \( \Delta_t = \bar{X}(t) - X(t) \), in which \( \bar{X} \) and \( X \) denote the solutions to (1.1) with initial conditions \( \bar{x} \) and \( x \), respectively. Thus we see that \( \bar{X}(t) \) converges to \( X(t) \) in probability as \( \bar{x} \to x \). For any \( f \in C_b(\mathbb{R}^d) \), the mapping theorem (see, e.g., Theorem 2.7 of Billingsley (1999)) implies that \( f(\bar{X}(t)) \) converges weakly to \( f(X(t)) \) as \( \bar{x} \to x \). The bounded convergence theorem further implies that \( \mathbb{E}[f(\bar{X}(t))] \to \mathbb{E}[f(X(t))] \) as \( \bar{x} \to x \). The Feller continuity therefore follows.

Similar argument leads to the Feller continuity under Assumptions 4.1 and 2.5 as well. \( \square \)

Assumptions 2.3 and 2.5 impose continuity conditions on \( \int_U |c(x, u) - c(z, u)| \nu(du) \) and \( \int_U |c(x, u) - c(c(z, u))|^2 \nu(du) \), respectively. These conditions are sometimes restrictive for the function \( c \) and the Lévy measure \( \nu \). For example, suppose \( U = \mathbb{R}^d_0 \), \( \nu(du) = \frac{du}{|u|^{d+\alpha}} \), in which \( \alpha \in (1, 2) \), and \( c(x, u) = c(x)u \) with \( c(x) \in \mathbb{R}^{d \times d} \) being a non-constant matrix. In such a case, we have \( c(x, u) - c(z, u) = (c(x) - c(z))u \) and thus both \( \int_U |c(x, u) - c(z, u)| \nu(du) \) and \( \int_U |c(x, u) - c(z, u)|^2 \nu(du) \) may diverge to \( \infty \). Then neither Assumptions 2.3 nor 2.5 can be applied to derive the Feller continuity. We wish to relax such conditions and thus improve Proposition 4.2.
**Assumption 4.3.** There exist a positive constant $\delta_0$ and a nondecreasing and concave function $\rho : [0, \infty) \mapsto [0, \infty)$ satisfying (2.6) such that

$$
\begin{align*}
\int_{U} & \left[ |c(x, u) - c(z, u)|^2 \wedge (4|x - z| \cdot |c(x, u) - c(z, u)|) \right] \nu(du) \\
& + 2\langle x - z, b(x) - b(z) \rangle + |\sigma(x) - \sigma(z)|^2 \leq 2\kappa_R |x - z|\rho(|x - z|)
\end{align*}
$$

(4.2)

for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$, where $\kappa_R$ is a positive constant.

Apparently Assumption 4.3 relaxes the conditions on $c$ and $\nu$ over those in Assumptions 2.3 and 2.5. The main result of this section is:

**Theorem 4.4.** Suppose Assumptions 4.1 and 4.3 hold. Then the process $X$ is Feller continuous.

We will use the coupling method to prove Theorem 4.4. To this end, we recall the infinitesimal generator $\mathcal{L}$ of (1.1) defined in (1.3). To construct the basic coupling operator for $\mathcal{L}$, let us first introduce some notations. For $x, z \in \mathbb{R}^d$, we set

$$
a(x, z) = \begin{pmatrix} a(x) \\ \sigma(z)\sigma(x)^T \end{pmatrix}, \quad b(x, z) = \begin{pmatrix} b(x) \\ b(z) \end{pmatrix},
$$

where $a(x) = \sigma(x)\sigma(x)^T$ and $a(z)$ is similarly defined. Next we define the basic coupling operator (Chen (2004), Wang (2010)) for the operator $\mathcal{L}$ of (1.3)

$$
\tilde{\mathcal{L}} f(x, z) := \left[ \tilde{\Omega}_{\text{diffusion}} + \tilde{\Omega}_{\text{jump}} \right] f(x, z),
$$

(4.3)

where $f(x, z) \in C^2_c(\mathbb{R}^d \times \mathbb{R}^d)$, and

$$
\begin{align*}
\tilde{\Omega}_{\text{diffusion}} f(x, z) &= \frac{1}{2}\text{tr} \left( a(x, z) D^2 f(x, z) \right) + \langle b(x, z), D f(x, z) \rangle, \\
\tilde{\Omega}_{\text{jump}} f(x, z) &= \int_{U} \left[ f(x + c(x, u), z + c(z, u)) - f(x, z) ight. \\
& \quad \left. - \langle D_x f(x, z), c(x, u) \rangle - \langle D_z f(x, z), c(z, u) \rangle \right] \nu(du).
\end{align*}
$$

(4.4)

(4.5)

Here and below, $D f(x, z)$ represents the gradient of $f$ with respect to the variables $x$ and $z$, that is, $D f(x, z) = (D_x f(x, z), D_z f(x, z))^\prime$. Likewise, $D^2 f(x, z)$ denotes the Hessian of $f$ with respect to $x$ and $z$.

**Lemma 4.5.** Suppose Assumption 4.3 holds. Then

$$
\tilde{\mathcal{L}} F(|x - z|) \leq \kappa_R \rho(F(|x - z|))
$$

(4.6)

for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $0 < |x - z| \leq \delta_0$, where the function $F$ is defined by $F(r) := \frac{r}{1 + r}$, $r \geq 0$.

The proof of Lemma 4.5 involves straightforward but lengthy computations. To preserve the flow of the presentation, we arrange it in Appendix A.
Proof of Theorem 4.4. By virtue of Theorem 5.6 in Chen (2004), it suffices to prove that

\[ W_d(P(t, x, \cdot), P(t, z, \cdot)) \to 0 \text{ as } z \to x, \quad (4.7) \]

where \( \{P(t, x, \cdot) \colon t > 0, x \in \mathbb{R}^d\} \) is the transition probability family associated with the process \(X\) of (1.1) and \( W_d(\cdot, \cdot) \) denotes the Wasserstein metric between two probability measures:

\[ W_d(\mu, \nu) := \inf \left\{ \int d(x, y)\pi(dx, dy) : \pi \in \mathcal{C}(\mu, \nu) \right\}, \]

where \( \mathcal{C}(\mu, \nu) \) denotes the family of coupling measures of \( \mu \) and \( \nu \), and \( d(x, y) := \frac{|x-y|}{1+|x-y|} \) for \( x, y \in \mathbb{R}^d \).

Given \( x \neq z \) with \( \delta_0 > |x - z| > \frac{1}{n_0} \), where \( n_0 \in \mathbb{N} \), let \((\tilde{X}, \tilde{Z})\) be the coupling process corresponding to the operator \( \mathcal{L} \) of (4.3) with \((\tilde{X}(0), \tilde{Z}(0)) = (x, z)\). Denote by \( T \) the coupling time. For \( n \geq n_0 \) and \( R > |x| \vee |z| \), define

\[ T_n := \inf \left\{ t \geq 0 : |\tilde{X}(t) - \tilde{Z}(t)| < \frac{1}{n} \right\}, \quad \tau_R := \inf \{ t \geq 0 : |\tilde{X}(t)| \vee |\tilde{Z}(t)| > R \}, \quad (4.8) \]

and

\[ S_{\delta_0} := \inf \{ t \geq 0 : |\tilde{X}(t) - \tilde{Z}(t)| > \delta_0 \}. \quad (4.9) \]

We have \( \tau_R \to \infty \) and \( T_n \to T \) a.s. as \( R \to \infty \) and \( n \to \infty \), respectively. Moreover, by Itô’s formula and (4.6), we have

\[
\begin{align*}
\mathbb{E}[F(|\tilde{X}(\cdot \wedge T_n \wedge S_{\delta_0} \wedge \tau_R) - \tilde{Z}(\cdot \wedge T_n \wedge S_{\delta_0} \wedge \tau_R)|)] & = F(|x - z|) + \mathbb{E} \left[ \int_0^{T_n \wedge S_{\delta_0} \wedge \tau_R} \mathcal{L}F(|\tilde{X}(s) - \tilde{Z}(s)|)ds \right] \\
& \leq F(|x - z|) + \kappa_R \mathbb{E} \left[ \int_0^{T_n \wedge S_{\delta_0} \wedge \tau_R} \Theta(F(|\tilde{X}(s) - \tilde{Z}(s)|))ds \right].
\end{align*}
\]

Now, passing to the limit as \( n \to \infty \), it follows from the bounded and monotone convergence theorems that

\[
\begin{align*}
\mathbb{E}[F(|\tilde{X}(\cdot \wedge T \wedge S_{\delta_0} \wedge \tau_R) - \tilde{Z}(\cdot \wedge T \wedge S_{\delta_0} \wedge \tau_R)|)] & \leq F(|x - z|) + \kappa_R \mathbb{E} \left[ \int_0^{T \wedge S_{\delta_0} \wedge \tau_R} \Theta(F(|\tilde{X}(s) - \tilde{Z}(s)|))ds \right] \\
& \leq F(|x - z|) + \kappa_R \mathbb{E} \left[ \int_0^t \Theta(F(|\tilde{X}(s \wedge T \wedge S_{\delta_0} \wedge \tau_R) - \tilde{Z}(s \wedge T \wedge S_{\delta_0} \wedge \tau_R)|))ds \right] \\
& \leq F(|x - z|) + \kappa_R \int_0^t \Theta(\mathbb{E}[F(|\tilde{X}(s \wedge T \wedge S_{\delta_0} \wedge \tau_R) - \tilde{Z}(s \wedge T \wedge S_{\delta_0} \wedge \tau_R)|)])ds,
\end{align*}
\]

where we use the concavity of \( \Theta \) and Jensen’s inequality to obtain the last inequality. Then using Bihari’s inequality, we have

\[
\mathbb{E}[F(|\tilde{X}(\cdot \wedge T \wedge S_{\delta_0} \wedge \tau_R) - \tilde{Z}(\cdot \wedge T \wedge S_{\delta_0} \wedge \tau_R)|)] \leq G^{-1}(G \circ F(|x - z| + \kappa_R t)),
\]

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where the function $G(r) := \int_1^r \frac{ds}{\sigma(s)}$ is strictly increasing and satisfies $G(r) \to -\infty$ as $r \downarrow 0$. In addition, since the function $F$ is strictly increasing, we have

$$F(\delta_0)\mathbb{P}\{S_{\delta_0} < t \wedge T \wedge \tau_R\} \leq \mathbb{E}[F(|\bar{X}(t \wedge T \wedge S_{\delta_0} \wedge \tau_R)|)I_{\{S_{\delta_0} < t \wedge T \wedge \tau_R\}}]$$

$$\leq \mathbb{E}[F(|\bar{X}(t \wedge T \wedge S_{\delta_0} \wedge \tau_R)| - \tilde{Z}(t \wedge T \wedge S_{\delta_0} \wedge \tau_R)|)]$$

$$\leq G^{-1}(G \circ F(|x - z|) + \kappa_R t).$$

For any $t \geq 0$ and $\varepsilon > 0$, since $\lim_{R \to \infty} \tau_R = \infty$ a.s., we can choose some $R > 0$ sufficiently large so that $\mathbb{P}(t > \tau_R) < \varepsilon$. Then it follows that

$$\mathbb{E}[F(|\bar{X}(t) - \tilde{Z}(t)|)]$$

$$= \mathbb{E}[F(|\bar{X}(t \wedge \tau_R) - \tilde{Z}(t \wedge \tau_R)|)I_{\{t \leq \tau_R\}}] + \mathbb{E}[F(|\bar{X}(t) - \tilde{Z}(t)|)I_{\{t > \tau_R\}}]$$

$$\leq \mathbb{E}[F(|\bar{X}(t \wedge T \wedge \tau_R) - \tilde{Z}(t \wedge T \wedge \tau_R)|)I_{\{S_{\delta_0} \leq t \wedge T \wedge \tau_R\}}] + \mathbb{E}[F(|\bar{X}(t \wedge T \wedge \tau_R) - \tilde{Z}(t \wedge T \wedge \tau_R)|)I_{\{S_{\delta_0} \geq t \wedge T \wedge \tau_R\}}] + \varepsilon$$

$$\leq \mathbb{P}\{S_{\delta_0} < t \wedge T \wedge \tau_R\} + \mathbb{E}[F(|\bar{X}(t \wedge T \wedge \tau_R \wedge S_{\delta_0}) - \tilde{Z}(t \wedge T \wedge \tau_R \wedge S_{\delta_0})|)] + \varepsilon$$

$$\leq \frac{1 + 2\delta_0}{\delta_0}G^{-1}(G \circ F(|x - z|) + \kappa_R t) + \varepsilon.$$

Now passing to the limit, we obtain $\lim_{x \to z, R \to \infty} \mathbb{E}[F(|\bar{X}(t) - \tilde{Z}(t)|)] \leq 0 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\lim_{x \to z, R \to \infty} \mathbb{E}[F(|\bar{X}(t) - \tilde{Z}(t)|)] = 0$. This leads to (4.7) because by the definition of $W_d$, we have $W_d(P(t, x, \cdot), P(t, z, \cdot)) \leq \mathbb{E}[F(|\bar{X}(t) - \tilde{Z}(t)|)]$. This gives the Feller property as desired. \hfill \square

## 5 Strong Feller Property

**Assumption 5.1.** There exists a $\lambda_0 > 0$ such that $\langle \xi, a(x)\xi \rangle \geq \lambda_0|\xi|^2$ for all $x, \xi \in \mathbb{R}^d$, where $a(x) := \sigma(x)\sigma(x)^T$. Denote by $\sigma_{\lambda_0}$ the unique symmetric nonnegative definite matrix-valued function such that $\sigma_{\lambda_0}^2 = a - \lambda_0 I$. In addition, there exist positive constants $\delta_0, \kappa_0$ and a nonnegative function $\vartheta$ defined on $[0, \delta_0]$ satisfying $\lim_{r \to 0} \vartheta(r) = 0$ such that

$$\int_U \left[|c(x, u) - c(z, u)|^2 \wedge (4|x - z| \cdot |c(x, u) - c(z, u)|)\right] \nu(du)$$

$$+ 2\langle x - z, b(x) - b(z) \rangle + |\sigma_{\lambda_0}(x) - \sigma_{\lambda_0}(z)|^2 \leq 2\kappa_0|x - z|\vartheta(|x - z|),$$

for all $x, z \in \mathbb{R}^d$ with $|x - z| \leq \delta_0$.

The main result of this section is:

**Theorem 5.2.** Under Assumptions 4.1 and 5.1, for any $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, we have

$$\sup_{x \neq z} \frac{|P_t f(x) - P_t f(z)|}{|x - z|} \leq K\|f\|_{\infty},$$

where $K = K(t, \delta_0, \kappa_0)$ is a positive constant. In particular, it follows that the process $X$ of (1.1) is strong Feller continuous.
As in Chen and Li (1989), Priola and Wang (2006), Wang (2010), we construct the coupling by reflection operator \( \hat{\mathcal{L}} \) of \( \mathcal{L} \) as follows. For \( x, z \in \mathbb{R}^d \), put \( g(x, z) := -\lambda_0 I + \sigma_{\lambda_0}(x)\sigma_{\lambda_0}(z)^T \) and set
\[
\hat{a}(x, z) = \begin{pmatrix} a(x) & g(x, z) \\ g(x, z)^T & a(z) \end{pmatrix}, \quad b(x, z) = \begin{pmatrix} b(x) \\ b(z) \end{pmatrix}.
\]
We can verify directly that \( \hat{a}(x, z) \) is symmetric and nonnegative definite. Then we define
\[
\hat{\Omega}_{\mathrm{diffusion}} h(x, z) := \frac{1}{2} \text{tr} (\hat{a}(x, z) D^2 h(x, z)) + \langle b(x, z), Dh(x, z) \rangle,
\]
and
\[
\hat{\mathcal{L}} h(x, z) := \hat{\Omega}_{\mathrm{diffusion}} h(x, z) + \hat{\Omega}_{\mathrm{jump}} h(x, z), \tag{5.2}
\]
where \( h \in C^2_0(\mathbb{R}^d \times \mathbb{R}^d) \) and \( \hat{\Omega}_{\mathrm{jump}} \) is defined in (4.5). Let
\[
A(x, z) = a(x) + a(z) - 2g(x, z), \quad \overline{A}_{\lambda_0}(x, z) = \frac{1}{|x - z|^2} (x - z, A(x, z)(x - z)).
\]
Then straightforward computations lead to
\[
\text{tr}(A(x, z)) = |\sigma_{\lambda_0}(x) - \sigma_{\lambda_0}(z)|^2 + 4\lambda_0 \text{ and } \overline{A}_{\lambda_0}(x, z) \geq 4\lambda_0.
\]
We need the following lemma to prove Theorem 5.2:

**Lemma 5.3.** Under Assumption 5.1, there exist some positive constants \( \beta \) and \( \delta \) such that
\[
\hat{\mathcal{L}} F(|x - z|) \leq -\beta < 0 \tag{5.3}
\]
for all \( x, z \in \mathbb{R}^d \) with \( 0 < |x - z| \leq \delta \), where the function \( F \) is defined by \( F(r) := \frac{r}{1+r}, r \geq 0 \).

**Proof.** We have \( F'(r) = \frac{1}{(1+r)^2} > 0 \) and \( F''(r) = -\frac{2}{(1+r)^3} < 0 \) for all \( r \geq 0 \). Moreover we can verify directly that for all \( x, z \in \mathbb{R}^d \) with \( 0 < |x - z| \leq \delta_0 \),
\[
\hat{\Omega}_{\mathrm{diffusion}} F(|x - z|) = \frac{F''(|x - z|)}{2} \overline{A}_{\lambda_0}(x, z) + \frac{F'(|x - z|)}{2|x - z|} \left[ \text{tr}(A(x, z)) - \overline{A}_{\lambda_0}(x, z) + 2B(x, z) \right]
\]
\[
\leq 2\lambda_0 F''(|x - z|) + \frac{F'(|x - z|)}{2|x - z|} \left[ |\sigma_{\lambda_0}(x) - \sigma_{\lambda_0}(z)|^2 + 2B(x, z) \right]
\]
\[
\leq \frac{-4\lambda_0}{(1 + |x - z|)^2} \frac{\kappa_0}{(1 + |x - z|)^2} \vartheta(|x - z|), \tag{5.4}
\]
where the last inequality follows from (5.1).

Then it follows from (A.13) and (5.1) that for all \( x, z \in \mathbb{R}^d \) with \( 0 < |x - z| \leq \delta_0 \), we have
\[
\hat{\Omega}_{\mathrm{jump}} F(|x - z|) = \int_U \left[ F(|x + c(x, u) - z - c(z, u)|) - F(|x - z|) \right. \\
\left. - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle \right] \nu(du)
\]
\[
\leq \frac{\kappa_0}{(1 + |x - z|)^2} \vartheta(|x - z|). \tag{5.5}
\]
In particular, the desired strong Feller property follows.

Proof of Theorem 5.2. Let $\beta, \delta$ and $F$ be as in Lemma 5.3. Given $x \neq z$ with $\delta > |x-z| > \frac{1}{n_0}$, where $n_0 \in \mathbb{N}$, let $(\tilde{X}, \tilde{Z})$ be the coupling process corresponding to the operator $\tilde{L}$ of (5.2) with $(\tilde{X}(0), \tilde{Z}(0)) = (x, z)$. Denote by $T$ the coupling time. For $N \ni n \geq n_0$ and $R > 0$, define the stopping times $T_n$ and $\tau_R$ as in (4.8). Also define $S_\delta$ as in (4.9) (with $\delta_0$ replaced by $\delta$). We have

$$0 \leq F(\delta)P\{T_n \wedge \tau_R > S_\delta\} \leq E\left[F((\tilde{X}(T_n \wedge S_\delta \wedge \tau_R)) - \tilde{Z}(T_n \wedge S_\delta \wedge \tau_R))\right] \leq F(|x-z|) + E\left[\int_0^{T_n \wedge S_\delta \wedge \tau_R} \tilde{L}F((\tilde{X}(s) - \tilde{Z}(s))ds\right] \leq F(|x-z|) - \beta E[T_n \wedge S_\delta \wedge \tau_R].$$

Then it follows that

$$F(\delta)P\{T_n \wedge \tau_R > S_\delta\} + \beta E[T_n \wedge S_\delta \wedge \tau_R] \leq F(|x-z|). \quad (5.7)$$

Since $T_n \to T$ a.s. as $n \to \infty$ and $\tau_R \to \infty$ a.s. as $R \to \infty$, we have

$$F(\delta)P\{T > S_\delta\} + \beta E[T \wedge S_\delta] \leq F(|x-z|).$$

Then for any $t > 0$ and $0 < |x-z| < \delta$,

$$P\{T > t\} = P\{T > t, S_\delta > t\} + P\{T > t, S_\delta \leq t\} \leq P\{T \wedge S_\delta > t\} + P\{T > S_\delta\} \leq \frac{1}{t} E[T \wedge S_\delta] + P\{T > S_\delta\} \leq \left(\frac{1}{t\beta} + \frac{1}{F(\delta)}\right) F(|x-z|).$$

Finally, for any $f \in \mathfrak{B}_0(\mathbb{R}^d)$, $t > 0$, and $0 < |x-z| < \delta$, we can write

$$|P_tf(x) - P_tf(z)| = |E[f(\tilde{X}(t)) - f(\tilde{Z}(t))]| \leq 2\|f\|_\infty P\{T > t\} \leq 2\|f\|_\infty \frac{1}{t\beta} + \frac{1}{F(\delta)} F(|x-z|) = 2\|f\|_\infty \frac{1}{t\beta} + \frac{1}{\delta} \frac{1+\delta}{1+|x-z|} |x-z| \leq 2\|f\|_\infty \frac{1}{t\beta} + \frac{1}{\delta} |x-z|. $$

On the other hand, if $|x-z| \geq \delta$, then we can write

$$|P_tf(x) - P_tf(z)| \leq 2\|f\|_\infty \leq 2\|f\|_\infty \frac{|x-z|}{\delta}. $$

We can combine the above two displayed equations to obtain

$$\frac{|P_tf(x) - P_tf(z)|}{|x-z|} \leq 2\|f\|_\infty \left[\frac{1}{t\beta} + \frac{1+\delta}{\delta} \vee \frac{1}{\delta}\right] = 2\|f\|_\infty \left(\frac{1}{t\beta} + \frac{1+\delta}{\delta}\right).$$

In particular, the desired strong Feller property follows.
In view of Theorem 4.4, one may naturally ask whether the strong Feller property holds under a “localized” version of Assumption 5.1? The following result gives an affirmative answer:

**Proposition 5.4.** Let Assumption 4.1 hold. Suppose that for each $R > 0$, there exist positive constants $\lambda_R$ and $\kappa_R$ such that for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$, we have

$$
\langle \xi, a(x)\xi \rangle \geq \lambda_R|\xi|^2, \quad \forall \xi \in \mathbb{R},
$$

and

$$
\int_U \left[ |c(x, u) - c(z, u)|^2 \wedge (4|x - z| \cdot |c(x, u) - c(z, u)|) \right] \nu(du)
+ 2\langle x - z, b(x) - b(z) \rangle + |\sigma_{\lambda_R}(x) - \sigma_{\lambda_R}(z)|^2 \leq 2\kappa_R|x - z|\vartheta(|x - z|), \quad \forall |x - z| \leq \delta_0,
$$

where $\delta_0$ is a positive constant and $\vartheta$ is a function satisfying the conditions specified in Assumption 5.1, and $\sigma_{\lambda_R}$ the unique symmetric nonnegative definite matrix-valued function such that $\sigma_{\lambda_0}^2 = a - \lambda_R I$. then the process $X$ is strong Feller continuous.

**Proof.** The same computations as those in the proof of Lemma 5.3 reveal that for each $R > 0$ and all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $0 < |x - z| \leq \delta_R$, there exist positive constants $\delta_R$ and $\beta_R$ such that

$$
\hat{\mathcal{L}}F(|x - z|) \leq -\beta_R < 0.
$$

Use the same notations as those in the proof of Theorem 5.2. For every $\varepsilon > 0$ and $t > 0$, we choose some $R > 0$ sufficiently large so that $\mathbb{P}(t > \tau_R) < \varepsilon$. For this chosen $R$, (5.7), in which the constant $\beta$ is replaced by $\beta_R$ and the stopping time $S_\delta$ replaced by $S_{\delta_R}$, remains valid. Now passing to limit as $n \to \infty$ in (5.7) yields

$$
F(\delta)\mathbb{P}\{T \land \tau_R > S_{\delta_R}\} + \beta_R\mathbb{E}[T \land \tau_R \land S_{\delta_R}] \leq F(|x - z|).
$$

Then for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $0 < |x - z| \leq \delta_R$, we can compute

$$
\mathbb{P}\{T > t\} = \mathbb{P}\{T > t, \tau_R \geq t, S_{\delta_R} > t\} + \mathbb{P}\{T > t, \tau_R > t, S_{\delta_R} < t\} + \mathbb{P}\{T > t, \tau_R < t\}
\leq \mathbb{P}\{T \land \tau_R < S_{\delta_R}\} + \mathbb{P}\{T \land \tau_R > S_{\delta_R}\} + \varepsilon
\leq \left(\frac{1}{t\beta_R} + \frac{1}{F(\delta)}\right)F(|x - z|) + \varepsilon.
$$

Consequently for any $f \in \mathfrak{B}_b(\mathbb{R}^d)$ and all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $0 < |x - z| \leq \delta_R$, we have

$$
|P_t f(x) - P_t f(z)| \leq 2\|f\|_\infty \left(\frac{1}{t\beta_R} + \frac{1 + \delta}{\delta}\right)|x - z| + 2\varepsilon\|f\|_\infty.
$$

In particular, since $\varepsilon > 0$ is arbitrary, it follows that $\lim_{x - z \to 0} |P_t f(x) - P_t f(z)| = 0$; this gives the desired strong Feller property.

**Remark 5.5.** Note that Assumption 5.1 places very mild condition on the function $\vartheta$. For instance, when $c \equiv 0$, Theorem 5.2 and Proposition 5.4 allow us to derive strong Feller property as long as the function $b$ is locally uniformly continuous, and $\sigma_{\lambda_0}$ is locally Hölder
continuous with exponent $\delta_{a(x)} > \frac{1}{2}$. On the other hand, the uniform ellipticity condition for the diffusion matrix $a(x,k)$ in Assumption 5.1 is quite standard in the literature. Indeed, similar assumptions are used in Priola and Wang (2006), Qiao (2014), Wang (2010) to obtain the strong Feller property. Proposition 5.4 further relaxes this condition to a “local” one. In case that the diffusion matrix is degenerate, one needs to place certain conditions on the jumps to obtain strong Feller property; see Wang (2011) for related work.

6 Irreducibility and Exponential Ergodicity

The semigroup $P_t$ defined in (1.2) is said to be irreducible if for any $t > 0$ and $x \in \mathbb{R}^d$,

$$P_t(x, B) > 0 \text{ for all non-empty and open } B \subset \mathbb{R}^d.$$ 

A probability measure $\mu$ on $\mathbb{R}^d$ is said to be an invariant measure for the semigroup $P_t$ if $P_t^\ast \mu = \mu$ for all $t > 0$, where $P_t^\ast \mu(B) := \int_{\mathbb{R}^d} P_t(x, B)\mu(dx), B \in \mathcal{B}(\mathbb{R}^d)$.

The following result improves Proposition 2.4 of Qiao (2014):

**Lemma 6.1.** Suppose Assumption 2.1 (with $\zeta \equiv 1$) and Assumption 2.5 hold. Assume that there exists a constant $\lambda_0 > 0$ such that

$$\langle y, a(x)y \rangle \geq \lambda_0|y|^2, \quad \text{for all } x, y \in \mathbb{R}^d, \quad (6.1)$$

where $a(x) = \sigma(x)\sigma(x)^T$. Then the semigroup $P_t$ of (1.2) is irreducible.

**Remark 6.2.** Proposition 2.4 in Qiao (2014) assumes slightly stronger conditions than those in Lemma 6.1. In particular, Qiao (2014) assumes that

$$2\langle x - y, b(x) - b(y) \rangle + |\sigma(x) - \sigma(y)|^2 + \int_U |c(x,u) - c(y,u)|^2\nu(du) \leq K|x - y|^2\kappa(|x - y|),$$

for all $x, y \in \mathbb{R}^d$, where $K > 0$ and $\kappa$ is a positive and continuous function satisfying $\lim_{r \downarrow 0} \frac{\kappa(r)}{\log(r^{-1})} = \delta < \infty$. This condition excludes functions such as $r \mapsto \log(r^{-1})\log(\log(r^{-1}))$ for $r > 0$ small. By contrast, Assumption 2.5 allows the modulus of continuity of the coefficients of (1.1) to be of the form $r^2\log(r^{-2})\log(\log(r^{-2}))$ for $r > 0$ small. Thanks to this relaxation, the estimation techniques used in Qiao (2014) is not directly applicable in our analysis here. In addition, instead of requiring the modulus of continuity to hold for all $x, y \in \mathbb{R}^d$ as in Qiao (2014), Assumption 2.5 only requires it in a small neighborhood of the diagonal line $x = y$ in $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| \vee |y| \leq R\}$ for each $R > 0$. Also, we note that the condition $\int_U |c(x,u)|^2\nu(du) \leq K(1 + |x|^4)$ in Qiao (2014) is not necessary.

Even though we use essentially the same ideas of approximate controllability and Girsanov theorem as those in Qiao (2014) and Zhang (2009), the technical difficulties arising from the relaxed assumptions merit a sketch of proof of Lemma 6.1 in Appendix A.

**Corollary 6.3.** Under Assumptions 2.1 (with $\zeta \equiv 1$), 2.5, and 5.1, then the semigroup $P_t$ of (1.2) has at most one invariant measure.
Proof. It is well known (see, for example, Cerrai (2001)) that if a semigroup $P_t$ is irreducible and strong Feller, then it admits at most one invariant measure. Under the stated assumptions, the semigroup $P_t$ is irreducible (by Lemma 6.1) and strong Feller (by Theorem 5.2). Therefore the uniqueness of the invariant measure follows immediately. □

Lemma 6.4. Let Assumptions 2.1 and either 2.3 or 2.5 hold. Suppose there exist a positive constant $\alpha$, a compact $C \subset \mathbb{R}^d$, a measurable function $f : \mathbb{R}^d \mapsto [1, \infty)$, and twice continuously differentiable function $V : \mathbb{R}^d \mapsto \mathbb{R}_+$ satisfying

$$\mathcal{L}V(x) \leq -\alpha f(x) + I_C(x), \text{ for all } x \in \mathbb{R}^d. \quad (6.2)$$

Then the process $X$ of (1.1) has an invariant measure.

Proof. This lemma can be proved using exactly the same arguments as those in the proof of Theorem 3.3 in Xi (2004). For brevity, we shall omit the details here. □

A combination of Corollary 6.3 and Lemma 6.4 yields the following proposition:

Proposition 6.5. Under the assumptions of Corollary 6.3 and Lemma 6.4, the semigroup $P_t$ of (1.2) has a unique invariant measure.

For any positive function $f : \mathbb{R}^d \mapsto [1, \infty)$ and any signed measure $\nu$ defined on $\mathcal{B}(\mathbb{R}^d)$, we write

$$\|\nu\|_f := \sup\{\nu(g) : g \in \mathcal{B}(\mathbb{R}^d) \text{ satisfying } |g| \leq f\},$$

where $\nu(g) := \int_{\mathbb{R}^d} g(x) \nu(dx)$ is the integral of the function $g$ with respect to the measure $\nu$. Note that the usual total variation norm $\|\nu\|_{\text{Var}}$ is just $\|\nu\|_f$ in the special case when $f \equiv 1$. For a function $f : \mathbb{R}^d \mapsto [1, \infty)$, the process $X$ is said to be $f$-exponentially ergodic if there exists a probability measure $\pi(\cdot)$, a constant $\theta \in (0, 1)$ and a finite-valued function $\Theta(x)$ such that

$$\|P_t(x, \cdot) - \pi(\cdot)\|_f \leq \Theta(x)\theta^t \quad (6.3)$$

for all $t \geq 0$ and all $x \in \mathbb{R}^d$.

Theorem 6.6. Suppose Assumptions 2.1 (with $\zeta \equiv 1$), 2.5, and 5.1 hold. In addition, assume that there exist positive numbers $\alpha, \beta$ and a nonnegative function $V \in C^2(\mathbb{R}^d)$ satisfying

(i) $V(x) \to \infty$ as $|x| \to \infty$,

(ii) $\mathcal{L}V(x) \leq -\alpha V(x) + \beta$, $x \in \mathbb{R}^d$.

Then the process $X$ is $f$-exponentially ergodic with $f(x) = V(x) + 1$.

Proof. Apparently conditions (i) and (ii) in the statement of the theorem imply (6.2) and hence the existence and uniqueness of an invariant measure $\pi$ follows from Proposition 6.5. Next we can use the same argument as those in the proof of Theorem 6.3 in Xi (2009) to obtain the desired $f$-exponential ergodicity for the process $X$. □
Remark 6.7. Note that the condition
\[
2\langle x, b(x) \rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(du) \leq -\lambda_3 |x|^r + \lambda_4, \tag{6.4}
\]
in which \(\lambda_3 > 0, \lambda_4 \geq 0\) and \(r \geq 2\), in Theorem 1.3 of Qiao (2014) is a special case of the drift condition in Theorem 6.6. Indeed, the left hand side of (6.4) is just the infinitesimal generator \(L\) applied to the function \(V(x) = |x|^2\). And since \(r \geq 2\), we can find positive constants \(\alpha\) and \(\beta\) so that \(-\lambda_3 |x|^r + \lambda_4 \leq -\alpha |x|^2 + \beta = -\alpha V(x) + \beta\) for all \(x \in \mathbb{R}^d\). In other words, (6.4) implies the drift condition of Theorem 6.6.

Example 6.8. Let us consider the following SDE
\[
dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \int_U c(X(t-u), u)\tilde{N}(dt, du), X(0) = x \in \mathbb{R}^3, \tag{6.5}
\]
where \(W\) is a 3-dimensional standard Brownian motion, \(\tilde{N}(dt, du)\) is a compensated Poisson random measure with compensator \(dt \nu(du)\) on \([0, \infty) \times U\), in which \(U = \{u \in \mathbb{R}^3 : 0 < |u| < 1\}\) and \(\nu(du) := \frac{4u}{|u|^{1+\alpha}}\) for some \(\alpha \in (0, 2)\). The coefficients of (6.5) are given by
\[
b(x) = \begin{pmatrix} -x_1^{1/3} - x_3^3 \\ -x_2^{1/3} - x_3^3 \\ -x_3^{1/3} - x_3^3 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad c(x, u) = \begin{pmatrix} \gamma x_1^{2/3} |u| \\ \gamma x_2^{2/3} |u| \\ \gamma x_3^{2/3} |u| \end{pmatrix},
\]
in which \(\gamma\) is a positive constant so that \(\gamma^2 \int_U |u|^2 \nu(du) = \frac{1}{2}\).

We claim that all conditions in Theorem 6.6 are satisfied and hence the process \(X\) of (6.5) is exponentially ergodic. Indeed, detailed calculations similar to those in (2.13) and (2.14) help to verify Assumptions 2.1 (with \(\zeta \equiv 1\)) and 2.5. On the other hand, it is clear that the matrix \(a(x) = \sigma(x)\sigma(x)^T = \begin{pmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{pmatrix}\) is uniformly positive definite. Moreover, using similar calculations as those in (2.14), we can verify condition (5.1) and hence Assumption 5.1. Finally we turn to the drift condition stated in Theorem 6.6. To this end, we consider the function \(V(x) = |x|^2\), \(x \in \mathbb{R}^d\), which clearly satisfies condition (i) in the statement of Theorem 6.6. On the other hand, straightforward calculations lead to
\[
\mathcal{L}V(x) = 2 \langle x, b(x) \rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(du) = -\frac{3}{2} \sum_{j=1}^3 x_j^{4/3} - 2 \sum_{j=1}^3 x_j^4 + 42 
\]
\[
\leq -2 \sum_{j=1}^3 x_j^4 + 42 \leq -\alpha \sum_{j=1}^3 x_j^2 + \beta = -\alpha V(x) + \beta,
\]
for all \(x \in \mathbb{R}^d\) and some positive constants \(\alpha, \beta\). This gives condition (ii) in the statement of Theorem 6.6 and hence the claimed exponential ergodicity.
7 Applications

7.1 SDEs driven by Lévy processes

We consider the stochastic differential equation

$$dX(t) = \psi(X(t-))dL(t), \quad X(0) = x \in \mathbb{R}^d,$$  (7.1)

where the function $\psi : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is Borel measurable and $L \in \mathbb{R}^d$ is a Lévy process with triplet $(b, Q, \nu)$. That is, $b \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ is symmetric and nonnegative definite, and $\nu$ is a Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d_0} 1 \land |u|^2 \nu(du) < \infty$. It is well known that if $\psi$ is locally Lipschitz, then pathwise uniqueness holds for (7.1). Thus our focus in this section is to derive non-Lipschitz conditions under which pathwise uniqueness still holds for (7.1).

Thanks to the Lévy-Itô decomposition theorem (see, for example, Theorem 2.4.16 of Applebaum (2009)), we can write $L$ as:

$$L(t) = bt + \sigma W(t) + \int_{\mathbb{R}^d_0} uI_{\{|u| \leq 1\}} \tilde{N}(t, du) + \int_{\mathbb{R}^d_0} uI_{\{|u| > 1\}} N(t, du),$$

where $W \in \mathbb{R}^d$ is a standard Brownian motion, and $\sigma \in \mathbb{R}^{d \times d}$ satisfies $\sigma \sigma^T = Q$. Using this Lévy-Itô decomposition, we can rewrite (7.1) as

$$dX(t) = \psi(X(t-))bdt + \psi(X(t-))\sigma dW(t)$$

$$+ \int_{\{|u| \leq 1\}} \psi(X(t-))u \tilde{N}(dt, du) + \int_{\{|u| > 1\}} \psi(X(t-))u N(dt, du).$$  (7.2)

Proposition 7.1. The following assertions hold:

(i) Suppose there exist a positive constant $K$ and a nondecreasing and continuously differentiable function $\zeta : [0, \infty) \to [1, \infty)$ satisfying (2.1) such that

$$|\psi(x)|^2 \leq K(|x|^2 \zeta(|x|^2) + 1), \quad \text{for all } x \in \mathbb{R}^d.$$  (7.3)

Then the solution to (7.1) has no finite explosion time a.s.

(ii) Suppose that there exist positive constants $\delta_0, K$ and a nondecreasing, continuous and concave function $\varrho : [0, \infty) \to [0, \infty)$ satisfying (2.6) and $r \leq K \varrho(r)$ for all $r \in [0, \delta_0]$ such that

$$|\psi(x) - \psi(z)|^2 \leq K \varrho(|x - z|^2), \quad \text{for all } x, z \in \mathbb{R}^d \text{ with } |x - z| \leq \delta_0.$$  (7.4)

Then pathwise uniqueness holds for (7.1).

Some common functions satisfying the conditions of Proposition 7.1 (ii) include $\varrho(r) = r, r \log(\frac{r}{2}), r \log(\log(\frac{r}{2})), r \log(\frac{1}{2}) \log(\log(\frac{1}{2})), \ldots$ for $r$ in a small neighborhood $(0, \delta_0]$ of 0.

Proof. These assertions follow directly from applying Theorems 2.2, 2.6, and Corollary 2.9 to (7.2), respectively. For brevity, we shall omit the straightforward computations here. □
Next we consider sufficient conditions for Feller and strong Feller properties for the weak solution \( X \) to (7.1).

**Proposition 7.2.** Assume that the Lévy measure \( \nu \) also satisfies \( \int_{|u| \geq 1} |u| \nu(du) < \infty \) and that (7.1) has a unique non-exploding weak solution for every initial condition. Suppose also that there exist positive constants \( K, \delta_0 \) and a nondecreasing and concave function \( \varrho : [0, \infty) \to [0, \infty) \) satisfying (2.6) and \( r \leq K\theta(r) \) for all \( r \in [0, \delta_0] \) such that

\[
|\psi(x) - \psi(z)|^2 \leq K|x - z|\varrho(|x - z|), \quad \text{for all } x, z \in \mathbb{R}^d \text{ with } |x - z| \leq \delta_0,
\]

where \( \delta_0 > 0 \). Then the weak solution \( X \) to (7.1) is Feller continuous. In addition, suppose there exists a positive number \( \lambda_0 \) such that

\[
\langle \xi, \psi(x)Q\psi(x)^T\xi \rangle \geq \lambda_0|\xi|^2, \quad \text{for all } x, \xi \in \mathbb{R}^d.
\]

Then the weak solution \( X \) to (7.1) is strong Feller continuous.

**Proof.** For the proof of Feller property, it is enough to verify that the coefficients of (7.2) satisfy Assumption 4.3. Apparently (7.7) and the condition \( r \leq K\varrho(r) \) for all \( r \in [0, \delta_0] \) imply that \( |x - z||\psi(x) - \psi(z)| \leq K|x - z|\varrho(|x - z|) \) and hence

\[
\langle x - z, (\psi(x) - \psi(z))b \rangle + |(\psi(x) - \psi(z))\sigma|^2 \leq K|x - z|\varrho(|x - z|), \quad \text{for all } |x - z| \leq \delta_0.
\]

On the other hand,

\[
\int_{\mathbb{R}^d} |\psi(x)u - \psi(z)u|^2 \land (4|x - z||\psi(x)u - \psi(z)u|)\nu(du)
\leq |\psi(x) - \psi(z)|^2 \int_{\mathbb{R}^d} |u|^2I_{|u| \leq 1}\nu(du) + 4|x - z||\psi(x) - \psi(z)|\int_{\mathbb{R}^d} |u|I_{|u| > 1}\nu(du)
\leq K|x - z|\varrho(|x - z|).
\]

A combination of the above displayed equations gives (4.2) and hence verifies Assumption 4.3. Then we derive the Feller property for \( X \) by Theorem 4.4.

Concerning the strong Feller property, (7.6) and the calculations in the previous paragraph guarantee that Assumption 5.1 is satisfied and thus the desired strong Feller property holds true thanks to Theorem 5.2. \( \square \)

### 7.2 Lévy Type Operator and Feynman-Kac Formula

We consider the Lévy type operator

\[
\mathcal{L}f(x) = \frac{1}{2} \sum_{j,k=1}^d a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^d b_j(x) \frac{\partial}{\partial x_j} f(x)
+ \int_{\mathbb{R}^d} [f(x + y) - f(x) - y \cdot Df(x)]\nu(x, dy),
\]

(7.7)
in which \( a(x) = (a_{jk}(x)) \in \mathbb{R}^{d \times d} \) is measurable, symmetric and nonnegative definite for all \( x \in \mathbb{R}^d \), \( f \in C_c^2(\mathbb{R}^d) \) and \( \nu(x, dy) \) is a Lévy measure satisfying \( \int_{\mathbb{R}^d} |y| \wedge |y|^2 \nu(x, dy) < \infty \) for all \( x \in \mathbb{R}^d \). In addition, we assume that there exist a positive constant \( K \) and a nondecreasing function \( \zeta : [0, \infty) \mapsto [1, \infty) \) that is continuously differentiable and satisfies (2.1) so that

\[
2\langle x, b(x) \rangle + \text{tr}(a(x)) + \int_{\mathbb{R}^d} |y|^2 \nu(x, dy) \leq K(|x|^2 \zeta(|x|^2) + 1), \text{ for all } x \in \mathbb{R}^d. \tag{7.8}
\]

We wish to establish a Feynman-Kac formula for the solution to the Cauchy problem related to the Lévy type operator \( \mathcal{L} \) of (7.7):

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) + \mathcal{L} u(t, x) - \rho(t, x) u(t, x) &= g(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
u(t, x) &= f(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{7.9}
\]

where the functions \( \rho(\cdot, \cdot) \geq 0, g(\cdot, \cdot), \) and \( f(\cdot) \) are continuous, and \( \mathcal{L} u(t, x) \) is interpreted as the operator \( \mathcal{L} \) applied to the function \( x \mapsto u(t, x) \) and thus in particular, we require

\[
\int_{\mathbb{R}^d} |u(t, x + y) - u(t, x) - y \cdot D_x u(t, x)| \nu(x, dy) < \infty, \text{ for all } x \in \mathbb{R}^d.
\]

Let us first present the following lemma whose proof can be found in the Appendix A.

**Lemma 7.3.** There exist a measurable function \( c : \mathbb{R}^d \times U \mapsto \mathbb{R}^d \) and a \( \sigma \)-finite measure \( M \) on a measurable space \((U, \mathfrak{U})\) such that

\[
\nu(x, \Gamma) = \int_U I_\Gamma(c(x, u)) M(du), \tag{7.10}
\]

for all \( x \in \mathbb{R}^d \) and \( \Gamma \in \mathfrak{B}(\mathbb{R}^d) \). Consequently the operator \( \mathcal{L} \) of (7.7) can be rewritten as

\[
\mathcal{L} f(x) = \frac{1}{2} \sum_{j,k=1}^d a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^d b_j(x) \frac{\partial}{\partial x_j} f(x) \tag{7.11}
\]

\[
+ \int_U [f(x + c(x, u)) - f(x) - c(x, u) \cdot D f(x)] M(du).
\]

Lemma 7.3 now enables us to derive a stochastic differential equation corresponding to the Lévy type operator \( \mathcal{L} \) of (7.7). Indeed, let \( N \) be a Poisson random measure on \( U \times [0, \infty) \) with mean measure \( \nu(du) dt \) and denote its compensator measure by \( \tilde{N}(du, dt) = N(du, dt) - \nu(du) dt \). Let \( \sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d} \) be a measurable square root of \( a \) so that \( \sigma \sigma'(x) = a(x) \) for all \( x \in \mathbb{R}^d \times \mathcal{S} \). Consider the following stochastic differential equation

\[
X(s) = x + \int_t^s b(X(s)) ds + \int_t^s \sigma(X(s)) dW(s) + \int_t^s \int_U c(X(s-), u) \tilde{N}(du, ds), \quad s \geq t,
\tag{7.12}
\]

where \((t, x) \in [0, \infty) \times \mathbb{R}^d \) and \( W \) is a standard \( d \)-dimensional Brownian motion.
Assumption 7.4. For any \((t, x) \in [0, \infty) \times \mathbb{R}^d\), the SDE (7.12) has a unique weak solution \(((\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_s\}_{s \geq t}, (W, N), X)\), in which \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, \(\{\mathcal{F}_s\}_{s \geq t}\) is a filtration of \(\mathcal{F}\) satisfying the usual condition, \(W\) is an \(\{\mathcal{F}_s\}_{s \geq t}\)-adapted Brownian motion, \(N\) is an \(\{\mathcal{F}_s\}_{s \geq t}\)-adapted Poisson random measure, and \(X\) satisfies (7.12). For simplicity, we denote the weak solution by \(X = X^{t,x}\).

Note that Assumption 7.4 is equivalent to that the martingale problem for the infinitesimal generator \(\mathcal{L}\) of (7.7) is well-posed for any initial condition \((t, x) \in [0, \infty) \times \mathbb{R}^d\); see, for example, Theorem 2.3 of Kurtz (2011). We refer to Stroock (1973) and Komatsu (1973) for investigations of the well-posedness of martingale problems for Lévy type operators.

Theorem 7.5. Let Assumption 7.4 be satisfied. Let \(T > 0\). Suppose that \(u(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}\) is of class \(C^{1,2}([0, T] \times \mathbb{R}^d) \cap C_0([0, T] \times \mathbb{R}^d)\) and satisfies the Cauchy problem (7.9). Assume that the functions \(f, g\) are uniformly bounded. Then we have

\[
u(t) = \mathbb{E}_{t,x} \left[ e^{-\int_t^T \rho(r,X(r))dr} f(X(T)) - \int_t^T e^{-\int_t^s \rho(r,X(r))dr} g(s, X(s))ds \right], \quad 0 \leq t \leq T, x \in \mathbb{R}^d.
\]

(7.13)

Proof. Thanks to Lemma 7.3, we have

\[
\int_{\mathbb{R}^d} |y|^2 \nu(x, dy) = \int_{U} |c(x, u)|^2 M(du).
\]

(7.14)

Putting this observation into (7.8), we see that the coefficients of (7.12) satisfies Assumption 2.1. Therefore for any \((t, x) \in [0, \infty) \times \mathbb{R}^d\), Theorem 2.2 implies that the unique weak solution \(X = X^{t,x}\) of (7.12) has no finite explosion time with probability 1. We can then apply Itô’s formula to the process \(e^{-\int_t^T \rho(r,X(r))dr} u(s, X(s)), s \in [t, T]\) and use the first equation of (7.9) to see that

\[
\xi(s; t, x) := e^{-\int_t^s \rho(r,X(r))dr} u(s, X(s)) - u(t, x) - \int_t^s e^{-\int_t^r \rho(u,X(u))du} g(r, X(r))dr, \quad s \in [t, T]
\]

is a local martingale. The boundedness assumptions on \(u\) and \(g\) in fact implies that \(\xi\) is a bounded local martingale and hence a martingale. In particular, we have \(\mathbb{E}[\xi(T; t, x)] = 0\), which, together with the terminal condition of (7.9), leads to (7.13). This completes the proof. \(\square\)

Remark 7.6. Note that in the traditional setting for Feynman-Kac formula, one typically imposes linear growth condition or boundedness condition on the coefficients \(b, \sigma\) and \(c\); see, for example, Theorem 5.7.6 of Karatzas and Shreve (1991) for the diffusion case and Theorem 6.7.9 of Applebaum (2009) for the jump diffusion case. For our version of Feynman-Kac formula presented in Theorem 7.5, (7.8) allows the coefficients \(b, \sigma\) and \(c\) to grow super linearly. If we also know that \(X\) has certain moment estimates, say, \(\mathbb{E}[\sup_{0 \leq s \leq T} |X(s)|^2] < \infty\), then we can relax the boundedness assumption on \(u, f,\) and \(g\) to polynomial growth condition as in Theorem 3.2 of Zhu et al. (2015).
A Several Technical Proofs

Proof of Theorem 2.4. Thanks to the assumptions imposed on the function $\rho$, we can find a strictly decreasing sequence $\{a_n\} \subset (0, 1]$ with $a_0 = 1$, $\lim_{n \to \infty} a_n = 0$ and $\int_{a_n}^{a_{n-1}} \rho^{-1}(r)dr = n$ for every $n \geq 1$. For each $n \geq 1$, there exists a continuous function $\rho_n$ on $\mathbb{R}$ with support in $(a_n, a_{n-1})$ so that $0 \leq \rho_n(r) \leq 2n^{-1}\rho^{-1}(r)$ holds for every $r > 0$, and $\int_{a_n}^{a_{n-1}} \rho_n(r)dr = 1$.

Now consider the sequence of functions

$$\psi_n(r) := \int_0^{|r|} \int_0^{y} \rho_n(u)dudy, \quad r \in \mathbb{R}, n \geq 1.$$  \hfill (A.1)

We can immediately verify that $\psi_n$ is even and twice continuously differentiable, with $|\psi_n'(r)| \leq 1$ and $\lim_{n \to \infty} \psi_n(r) = |r|$ for $r \in \mathbb{R}$. Furthermore, for each $r > 0$, the sequence $\{\psi_n(r)\}_{n \geq 1}$ is nondecreasing. Note also that for each $n \in \mathbb{N}$, $\psi_n, \psi_n'$ and $\psi_n''$ all vanish on the interval $(-a_n, a_n)$. By direct computations, we have for $0 \neq x \in \mathbb{R}^d$

$$D\psi_n(|x|) = \psi_n'(|x|) \frac{x}{|x|}, \quad \text{and} \quad D^2\psi_n(|x|) = \psi_n''(|x|) \frac{xx^T}{|x|^2} + \psi_n'(|x|) \left[ \frac{I}{|x|} - \frac{xx^T}{|x|^3} \right].$$

Now suppose that $X$ and $\tilde{X}$ satisfy

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \int_U c(X(s), u)\tilde{N}(ds, du),$$

$$\tilde{X}(t) = \tilde{x} + \int_0^t b(\tilde{X}(s))ds + \int_0^t \sigma(\tilde{X}(s))dW(s) + \int_0^t \int_U c(\tilde{X}(s), u)\tilde{N}(ds, du),$$

for all $t \geq 0$, where $\tilde{x}, x \in \mathbb{R}^d$. Denote $\Delta_t := \tilde{X}(t) - X(t)$ for $t \geq 0$. Assume $|\Delta_0| = |\tilde{x} - x| < \delta_0$ and define $S_{\delta_0} := \inf\{t \geq 0 : |\Delta_t| \geq \delta_0\} = \inf\{t \geq 0 : |\tilde{X}(t) - X(t)| \geq \delta_0\}$.

For $R > 0$, let $\tau_R := \inf\{t \geq 0 : |\tilde{X}(t)| \vee |X(t)| > R\}$. By virtue of Theorem 2.2, we have $\tau_R \to \infty$ a.s. as $R \to \infty$.

Let us introduce the notations:

$$A(x, z) := \frac{|\langle x - z, \sigma(x) - \sigma(z) \rangle|^2}{|x - z|^2}, \quad B(x, z) := \langle x - z, b(x) - b(z) \rangle.$$  \hfill (A.2)

Applying Itô’s formula, we have

$$\mathbb{E}[\psi_n(|\Delta_{t \wedge \tau_R \wedge S_{\delta_0}}|)]$$

$$= \psi_n(|\Delta_0|) + \mathbb{E} \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0}} \mathbb{I}_{|\Delta_s| \neq 0} \left[ \frac{1}{2} \left( \psi_n''(|\Delta_s|) - \frac{\psi_n'(|\Delta_s|)}{|\Delta_s|} \right) A(\tilde{X}(s), X(s)) \right. \right.$$

$$\left. + \frac{\psi_n'(|\Delta_s|)}{2|\Delta_s|} \left( 2B(\tilde{X}(s), X(s)) + |\sigma(\tilde{X}(s)) - \sigma(X(s))|^2 \right) \right] ds.$$  \hfill (A.3)
\[+ \int_0^{t \wedge \tau \wedge S_{b_0}} \int_U \left[ \psi_n(|\Delta_s + c(\tilde{X}(s), u) - c(X(s), u)|) - \psi_n(|\Delta_s|) \right.\]
\[\left. - I_{\{\Delta_s \neq 0\}} \frac{\psi_n(|\Delta_s|)}{|\Delta_s|} \langle \Delta_s, c(\tilde{X}(s), u) - c(X(s), u) \rangle \right] \nu(du) ds\].

Recall that we have \(0 \leq \psi_n'(r) \leq 1\) for each \(r \geq 0\). Thus it follows from (2.4) that
\[
\mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} I_{\{\Delta_s \neq 0\}} \frac{\psi_n'(|\Delta_s|)}{2|\Delta_s|} \left(2B(\tilde{X}(s), X(s)) + |\sigma(\tilde{X}(s)) - \sigma(X(s))|^2\right) ds \right]
\leq \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} I_{\{\Delta_s \neq 0\}} \frac{\psi_n''(|\Delta_s|)}{2|\Delta_s|} \kappa R |\Delta_s| \rho(|\Delta_s|) ds \right]
\leq \frac{\kappa R}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \rho(|\Delta_s|) ds \right]. \quad (A.4)
\]

On the other hand, thanks to the construction of \(\psi_n\), we have for all \(r \geq 0\), \(\psi_n''(r) = \rho_n(r) \leq \frac{2}{n \rho'(r)} I_{(a_n, a_{n-1})}(r)\). Then it follows from (2.4) that
\[
\mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \frac{1}{2} I_{\{\Delta_s \neq 0\}} \left( \psi_n'(|\Delta_s|) - \frac{\psi_n'(|\Delta_s|)}{|\Delta_s|} \right) A(\tilde{X}(s), X(s)) ds \right]
\leq \frac{1}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} I_{\{\Delta_s \neq 0\}} \psi_n''(|\Delta_s|) \rho(|\Delta_s|) ds \right]
\leq \frac{1}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \frac{2}{n \rho(|\Delta_s|)} I_{(a_n, a_{n-1})}(|\Delta_s|) \left| \frac{\Delta_s^2}{|\Delta_s|^2} \sigma(\tilde{X}(s)) - \sigma(X(s))^2 \right| ds \right]
\leq \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \frac{\kappa R}{n \rho(|\Delta_s|)} I_{(a_n, a_{n-1})}(|\Delta_s|) \rho(|\Delta_s|) ds \right] \leq \frac{\kappa R l a_{n-1}}{n}. \quad (A.5)
\]

Using (2.5) and the fact that \(|\psi_n'(r)| \leq 1\), we can compute
\[
\mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \int_U \left( \psi_n(|\Delta_s + c(\tilde{X}(s), u) - c(X(s), u)|) - \psi_n(|\Delta_s|) \right.\]
\[\left. - I_{\{\Delta_s \neq 0\}} \frac{\psi_n'(|\Delta_s|)}{|\Delta_s|} \langle \Delta_s, c(\tilde{X}(s), u) - c(X(s), u) \rangle \right) \nu(du) ds \right]
\leq 2 \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \int_U |c(\tilde{X}(s), u) - c(X(s), u)| \nu(du) ds \right]
\leq 2 \kappa R \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \rho(|\Delta_s|) ds \right]. \quad (A.6)
\]

Plugging (A.4)–(A.6) into (A.3), we obtain
\[
\mathbb{E} [\psi_n(|\Delta_{t \wedge \tau \wedge S_{b_0}}|)] \leq \psi_n(|\Delta_0|) + \frac{\kappa R l a_{n-1}}{n} + \frac{5 \kappa R}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \rho(|\Delta_s|) ds \right].
\]

Letting \(n \to \infty\) yields
\[
\mathbb{E} [\Delta_{t \wedge \tau \wedge S_{b_0}}] \leq |\Delta_0| + \frac{5 \kappa R}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau \wedge S_{b_0}} \rho(|\Delta_s|) ds \right] \leq |\Delta_0| + \frac{5 \kappa R}{2} \mathbb{E} \left[ \int_0^t \rho(|\Delta_{s \wedge \tau \wedge S_{b_0}}|) ds \right].
\]

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\[
\leq |\Delta_0| + \frac{5K_R}{2} \int_0^t \rho(\mathbb{E}[|\Delta_s \wedge \tau_R \wedge S_{\delta_0}|]) ds,
\]
where we used the concavity of \( \rho \) and Jensen’s inequality to derive the last inequality. Let \( u(t) := \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|] \). Then \( u \) satisfies
\[
0 \leq u(t) \leq v(t) := |\Delta_0| + \frac{5K_R}{2} \int_0^t \rho(u(s)) ds.
\]

Define \( G(r) := \int_1^r \frac{ds}{\rho(s)} \) for \( r > 0 \). Then \( G \) is nondecreasing and satisfies \( G(r) > -\infty \) for \( r > 0 \) and \( \lim_{r \downarrow 0} G(r) = -\infty \) thanks to (2.3). In addition, we have
\[
G(u(t)) \leq G(v(t)) = G(|\Delta_0|) + \int_0^t G'(v(s)) v'(s) ds
= G(|\Delta_0|) + \frac{5K_R}{2} \int_0^t \frac{\rho(u(s))}{\rho(v(s))} ds \leq G(|\Delta_0|) + \frac{5K_R}{2} t,
\]
where the last inequality follows from the assumption that \( \rho \) is nondecreasing. Now sending \( |\Delta_0| = |\tilde{x} - x| \to 0 \), we see that the right-hand side of the above inequality converges to \(-\infty\) and so does the left-hand side. Hence
\[
\lim_{|\tilde{x} - x| \to 0} u(t) = \lim_{|\tilde{x} - x| \to 0} \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|] = 0. \tag{A.7}
\]
In particular, when \( \tilde{x} = x \), we have \( \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|] = 0 \). Recall that \( \lim_{R \to \infty} \tau_R = \infty \) a.s. Thus by Fatou’s lemma, we have \( 0 \leq \mathbb{E}[|\Delta_t \wedge S_{\delta_0}|] \leq \lim_{R \to \infty} \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|] = 0 \). This gives \( \mathbb{E}[|\Delta_t \wedge S_{\delta_0}|] = 0 \) and therefore \( \Delta_t \wedge S_{\delta_0} = 0 \) a.s.

On the set \( \{S_{\delta_0} \leq t\} \), we have \( |\Delta_t \wedge S_{\delta_0}| \geq \delta_0 \). Thus it follows that \( 0 = \mathbb{E}[|\Delta_t \wedge S_{\delta_0}|] \geq \delta_0 \mathbb{P}\{S_{\delta_0} \leq t\} \). Then, we have \( \mathbb{P}\{S_{\delta_0} \leq t\} = 0 \) and hence \( \Delta_t = 0 \) a.s. The desired pathwise uniqueness result then follows from the fact that \( \tilde{X} \) and \( X \) have right continuous sample paths.

**Proof of Theorem 2.6.** Let \( X(t), \tilde{X}(t), \Delta_t, S_{\delta_0} \), and \( \tau_R \) be defined as in the proof of Theorem 2.4. Consider the function \( H(r) := \frac{r^2}{1+r^2}, \ r \in \mathbb{R} \). We have \( H'(r) = \frac{2r}{(1+r^2)^2} \) and \( H''(r) = \frac{2}{(1+r^2)^2} - \frac{8r^2}{(1+r^2)^3} \). Note that \( H, H' \) and \( H'' \) are uniformly bounded. By direct computations, we have for all \( x \in \mathbb{R}^d \)
\[
DH(|x|) = \frac{2x}{(1+|x|^2)^2}, \text{ and } D^2H(|x|) = \frac{2I}{(1+|x|^2)^2} - \frac{8xx^T}{(1+|x|^2)^3}.
\]
Applying Itô’s formula to the process \( H(|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|) \), we have
\[
\mathbb{E}[H(|\Delta_t \wedge \tau_R \wedge S_{\delta_0}|)]
= H(|\Delta_0|) + \mathbb{E} \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0}} \left\{ \frac{2\Delta_s b(\tilde{X}(s)) - b(X(s))}{(1+|\Delta_s|^2)^2} \right\} ds \right.
+ \frac{1}{2} \text{tr} \left( \left( \sigma(\tilde{X}(s)) - \sigma(X(s)) \right) \left( \sigma(\tilde{X}(s)) - \sigma(X(s)) \right)^T \left( \frac{2I}{(1+|\Delta_s|^2)^2} - \frac{8\Delta_s \Delta_s^T}{(1+|\Delta_s|^2)^3} \right) \right) \Bigg] ds
\]

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Then we have
\[
\int_0^{t \wedge \tau_R \wedge S_0} \left[ H(|\Delta_s + c(\tilde{X}(s), u) - c(X(s), u)|) - H(|\Delta_s|) \right] - \frac{2}{(1 + |\Delta_s|^2)^2} \langle \Delta_s, c(\tilde{X}(s), u) - c(X(s), u) \rangle \nu(du) ds \leq H(|\Delta_0|) + \mathbb{E}\left[ \int_0^{t \wedge \tau_R \wedge S_0} \left( H(|\Delta_s + c(\tilde{X}(s), u) - c(X(s), u)|) - H(|\Delta_s|) \right) \right.
\]
\[
- \frac{2}{(1 + |\Delta_s|^2)^2} \langle \Delta_s, c(\tilde{X}(s), u) - c(X(s), u) \rangle \nu(du) ds \right].
\]

To simplify notations, for any \(x, z \in \mathbb{R}^d\) and \(u \in U\), let us denote \(w := w(x, z, u) = c(x, u) - c(z, u)\). Then
\[
H(|x + c(x, u) - z - c(z, u)|) - H(|x - z|) - \frac{2}{(1 + |x - z|^2)^2} \langle x - z, c(x, u) - c(z, u) \rangle
\]
\[
= H(|x - z + w|) - H(|x - z|) - \frac{H'(|x - z|)}{|x - z|} \langle x - z, w \rangle
\]
\[
= \frac{|x - z + w|^2}{1 + |x - z + w|^2} - \frac{|x - z|^2}{1 + |x - z|^2} - \frac{2\langle x - z, w \rangle}{(1 + |x - z|^2)^2}
\]
\[
= \frac{|x - z + w|^2}{(1 + |x - z + w|^2)(1 + |x - z|^2)} - \frac{|x - z|^2}{(1 + |x - z|^2)^2} - \frac{2\langle x - z, w \rangle}{(1 + |x - z|^2)^2}
\]
\[
= \frac{|x - z + w|^2 - |x - z|^2}{1 + |x - z|^2} \left[ \frac{1}{1 + |x - z + w|^2} - \frac{1}{1 + |x - z|^2} \right] + \frac{|w|^2}{(1 + |x - z|^2)^2}
\]
\[
\leq \frac{|w|^2}{(1 + |x - z|^2)^2}.
\]

Then we have
\[
\int_U \left( H(|\Delta_s + c(\tilde{X}(s), u) - c(X(s), u)|) - H(|\Delta_s|) \right)
\]
\[
- \frac{2\langle \Delta_s, c(\tilde{X}(s), u) - c(X(s), u) \rangle}{(1 + |\Delta_s|^2)^2} \nu(du)
\]
\[
\leq \int_U \frac{|c(\tilde{X}(s), u) - c(X(s), u)|^2}{(1 + |\Delta_s|^2)^2} \nu(du).
\]

Using this estimate in (A.8), we obtain
\[
\mathbb{E}[H(|\Delta_{t \wedge \tau_R \wedge S_0}|)] - H(|\Delta_0|)
\]
\[
\leq \mathbb{E}\left[ \int_0^{t \wedge \tau_R \wedge S_0} \frac{1}{(1 + |\Delta_s|^2)^2} \left( 2\langle \Delta_s, b(\tilde{X}(s)) - b(X(s)) \rangle + |\sigma(\tilde{X}(s)) - \sigma(X(s))|^2 \right) \nu(du) \right].
\]
bounded convergence theorem further implies that 
\[ \text{by 1, it follows that } 0 \leq (A.7) \] 
where we used the concavity of \( \rho \) and Jensen’s inequality to derive the last inequality. When \( \bar{x} = x \) or \( \Delta_0 = 0 \), the same argument as that in the end of the proof of Theorem 2.4 reveals that \( \mathbb{E}[H(\{\Delta_{t\wedge t^R}\wedge S_{\delta_0}\})] = 0 \). Since \( \lim_{R \to \infty} \tau_R = \infty \) a.s. and \( 0 \leq H(r) \leq 1 \) for all \( r \geq 0 \), the bounded convergence theorem further implies that \( \mathbb{E}[H(\{\Delta_{t\wedge t^R}\wedge S_{\delta_0}\})] = 0 \).

On the set \( \{S_{\delta_0} < t\}, |\Delta_{S_{\delta_0}}| \geq \delta_0 \). Since \( H \) is increasing on \((0, \infty)\) and bounded above by 1, it follows that \( 0 < H(\delta_0) \leq H(|\Delta_{S_{\delta_0}}|) \leq 1 \) and hence
\[
H(\delta_0)\mathbb{P}\{S_{\delta_0} < t\} \leq \mathbb{E}[H(\{\Delta_{t\wedge t^R}\wedge S_{\delta_0}\})I_{\{S_{\delta_0} < t\}}] \leq \mathbb{E}[H(|\Delta_{S_{\delta_0}}|)] = 0.
\]
Therefore it follows that \( \mathbb{P}\{S_{\delta_0} < t\} = 0 \). Then \( 0 \leq \mathbb{E}[H(|\Delta_{S_{\delta_0}}|)I_{\{S_{\delta_0} < t\}}] \leq \mathbb{E}[1 \cdot I_{\{S_{\delta_0} < t\}}] = 0 \) and thus
\[
0 = \mathbb{E}[H(|\Delta_{t\wedge t^R}\wedge S_{\delta_0}|)] = \mathbb{E}[H(|\Delta_t|)I_{\{t \leq S_{\delta_0}\}}] + \mathbb{E}[H(|\Delta_{S_{\delta_0}}|)I_{\{S_{\delta_0} < t\}}] = \mathbb{E}[H(|\Delta_t|)I_{\{t \leq S_{\delta_0}\}}].
\]
Next we observe that
\[
| \mathbb{E}[H(|\Delta_t|)] - \mathbb{E}[H(|\Delta_{t\wedge t^R}\wedge S_{\delta_0}|)] | = \left| \mathbb{E}[H(|\Delta_t|)] - \mathbb{E}[H(|\Delta_t|)I_{\{t \leq S_{\delta_0}\}}] \right|
\leq \mathbb{P}\{S_{\delta_0} < t\} = 0.
\]
Hence it holds that \( \mathbb{E}[H(|\Delta_t|)] = 0 \) and hence \( \Delta_t = 0 \) a.s. As observed in the end of the proof of Theorem 2.4, this gives the desired pathwise uniqueness result. \( \square \)

**Proof of Lemma 4.5.** We have \( F'(r) = \frac{1}{(1+r)^2} \) and \( F''(r) = -\frac{2}{(1+r)^3} \). Recall the notations \( \overline{A}(x, z) \) and \( B(x, z) \) defined in (A.2). Then as in the proof of Theorem 3.1 in Chen and Li (1989), straightforward calculations lead to
\[
\tilde{\Omega}_{	ext{diffusion}}F(|x-z|) = \frac{F''(|x-z|)}{2} \overline{A}(x, z) + \frac{F'(|x-z|)}{2|x-z|} [\sigma(x) - \sigma(z)]^2 - \overline{A}(x, z) + 2B(x, z)
\leq \frac{[\sigma(x) - \sigma(z)]^2 + 2B(x, z)}{2|x-z|(1 + |x-z|)^2}.
\]
Following the same arguments as those in the proof of Proposition 3.1 in Wang (2010), we can verify that

\[
F(|x + c(x, u) - z - c(z, u)|) - F(|x - z|) - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle \leq \frac{|c(x, u) - c(z, u)|^2}{2|x - z|(1 + |x - z|)^2}.
\]  

(A.10)

On the other hand, since the function \( F \) is concave, it follows that \( F(r) - F(r_0) \leq F'(r_0)(r - r_0) \) for all \( r, r_0 \geq 0 \). Using this inequality with \( r_0 = |x - z| \) and \( r = |x + c(x, u) - z - c(z, u)| \), and noting that \( F'(r_0) > 0 \), we can compute

\[
F(|x + c(x, u) - z - c(z, u)|) - F(|x - z|) - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle
\leq F'(|x - z|)(|x + c(x, u) - z - c(z, u)| - |x - z|) - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle
\leq F'(|x - z|)c(x, u) - c(z, u) + \frac{F'(|x - z|)}{|x - z|}|x - z| \cdot |c(x, u) - c(z, u)|
\leq 2F'(|x - z|)|c(x, u) - c(z, u)|
\leq \frac{2|x - z||c(x, u) - c(z, u)|}{|x - z|(1 + |x - z|)^2}.
\]

(A.11)

Combining (A.10) and (A.11) yields

\[
F(|x + c(x, u) - z - c(z, u)|) - F(|x - z|) - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle
\leq \frac{1}{2|x - z|(1 + |x - z|)^2} [ |c(x, u) - c(z, u)|^2 \wedge (4|x - z||c(x, u) - c(z, u)|) ].
\]  

(A.12)

Using (A.12) in \( \tilde{\Omega}_{jump} \) of (4.5), it follows that for all \( x \neq z \),

\[
\tilde{\Omega}_{jump} F(|x - z|)
= \int_U \left[ F(|x + c(x, u) - z - c(z, u)|) - F(|x - z|) - \frac{F'(|x - z|)}{|x - z|} \langle x - z, c(x, u) - c(z, u) \rangle \right] \nu(du)
\leq \frac{1}{2|x - z|(1 + |x - z|)^2} \int_U [ |c(x, u) - c(z, u)|^2 \wedge (4|x - z||c(x, u) - c(z, u)|) ] \nu(du).
\]  

(A.13)

Combining (A.9) and (A.13), and using condition (4.2), we obtain

\[
\tilde{\mathcal{L}} F(|x - z|) \leq \frac{\kappa_R \theta(|x - z|)}{(1 + |x - z|)^2} \leq \kappa_R \theta \left( \frac{|x - z|}{1 + |x - z|} \right) = \kappa_R \theta \left( \frac{|x - z|}{1 + |x - z|} \right),
\]

for all \( x, z \in \mathbb{R}^d \) with \( |x| \vee |z| \leq R \) and \( 0 < |x - z| \leq \delta_0 \), where we used (2.6) to derive the second inequality above. This establishes (4.6) and hence completes the proof of the lemma. □
Proof of Lemma 6.1. Let us fix $T > 0, r > 0$ and $x, a \in \mathbb{R}^d$. We need to show that $P_t(x, B(a, r)) := \mathbb{P}\{|X^x(T) - a| \leq r\} > 0$, or equivalently, $\mathbb{P}\{|X^x(T) - a| > r\} < 1$. To this end, we choose $t_0 \in (0, T)$, whose exact value will be specified later. Set for $n \in \mathbb{N}$, $X^n(t_0) := X(t_0)_I[\{X(t_0)\leq n\}]$. Then we have

$$\lim_{n \to \infty} \mathbb{E}[H(|X(t_0) - X^n(t_0)|)] = 0, \tag{A.14}$$

where the function $H(r) = \frac{1}{r^2}, r \geq 0$ is defined in the proof of Theorem 2.6.

For $t \in [t_0, T]$, we define

$$J^n(t) := \frac{T - t}{T - t_0}X^n(t_0) + \frac{t - t_0}{T - t_0}a, \text{ and } h^n(t) := \frac{a - X^n(t_0)}{T - t_0} - b(J^n(t)).$$

Then $J^n(t_0) = X^n(t_0), J^n(T) = a$, and $J^n$ satisfies the following SDE:

$$J^n(t) = X^n(t_0) + \int_{t_0}^t b(J^n(s))ds + \int_{t_0}^t h^n(s)ds, \quad t \in [t_0, T].$$

Let us also consider the SDE

$$Y(t) := X(t_0) + \int_{t_0}^t [b(Y(s)) + h^n(s)]ds + \int_{t_0}^t \sigma(Y(s))dW(s) + \int_{t_0}^t \int_U c(X(s, u))\tilde{N}(dsdu),$$

for $t \in [t_0, T]$. Also let $Y(t) := X(t)$ for $t \in [0, t_0]$. Denote $\Delta_t := Y(t) - J^n(t)$ for $t \in [t_0, T]$. Note that $\Delta_{t_0} = X(t_0) - X^n(t_0)$ and $\Delta_T = Y(T) - a$.

Define $\tau_R := \inf\{t \geq t_0 : |Y(t)| \vee |J^n(t)| > R\} \land T$ and $S_{\delta_0} := \inf\{t \geq t_0 : |Y(t) - J^n(t)| \geq \delta_0\} \land T$. Then detailed calculations as those in the proof of Theorem 2.6 reveal that

$$\mathbb{E}[H(|\Delta_{T \land \tau_R \land S_{\delta_0}}|)] - \mathbb{E}[H(|\Delta_{t_0}|)]$$

$$= \mathbb{E}\left[\int_{t_0}^{T \land \tau_R \land S_{\delta_0}} 2\langle \Delta_s, b(Y(s)) - b(J^n(s)) \rangle + |\sigma(Y(s))|^2 - 4|\langle \sigma(Y(s), \Delta_s \rangle|^2 ds \right]$$

$$+ \mathbb{E}\left[\int_{t_0}^{T \land \tau_R \land S_{\delta_0}} \int_U \left(H(|\Delta_s + c(Y(s, u))|) - H(|\Delta_s|) - \frac{2\langle \Delta_s, c(Y(s, u)) \rangle}{(1 + |\Delta_s|^2)^2}\right)\nu(du)ds \right]$$

$$\leq K_R \mathbb{E}\left[\int_{t_0}^{T \land \tau_R \land S_{\delta_0}} \left(g(H(|\Delta_s|)) + |\sigma(Y(s)|^2 + \int_U |c(Y(s, u)|^2\nu(du)ds \right) \right]$$

$$\leq K_R \mathbb{E}\left[\int_{t_0}^{T \land \tau_R \land S_{\delta_0}} \left(g(H(|\Delta_s|)) + 1 + |Y(s)|^2 \right)^2 ds \right]$$

$$\leq K_R \mathbb{E}\left[\int_{t_0}^{T \land \tau_R \land S_{\delta_0}} (g(H(|\Delta_{s \land \tau_R \land S_{\delta_0}}|)) + 1 + |Y(s \land \tau_R \land S_{\delta_0})|^2)^2 ds \right],$$

where the second last inequality follows from the linear growth condition given by Assumption 2.1, and $K_R$ is a positive constant. Also, throughout the proof, $K_R$ is generic positive constant whose exact value may change from line to line. Furthermore, by virtue of Zhu et al. (2015), we have $\mathbb{E}\sup_{t \in [0, T]} |Y(t)|^2 \leq K$, where $K$ is a positive constant independent of $t_0$ and $R$. Thus we have

$$\mathbb{E}[H(|\Delta_{T \land S_{\delta_0} \land \tau_R}|)] \leq \mathbb{E}[H(|\Delta_{t_0}|)] + K_R(T - t_0) + K_R \int_{t_0}^{T} g(\mathbb{E}[H(|\Delta_{s \land S_{\delta_0} \land \tau_R}|)])ds.$$
Note that we also used Jensen’s inequality to obtain the above inequality. Consequently as in the proof of Theorem 2.6, we have

$$\mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta})|] \leq G^{-1}(G(\mathbb{E}[H(|\Delta_{t_0}|)] + K_R(T - t_0)) + K_R(T - t_0)).$$  \hspace{1cm} (A.15)$$

where \(G(r) := \int_1^r \frac{dr}{\varphi(r)}\) and \(G^{-1}\) is the (left) inverse function of \(G: G^{-1}(x) := \inf\{y \geq 0 : G(y) \geq x\}, x \in \mathbb{R}\).

Next we observe that for the positive constant \(\frac{1}{H(\delta_0)} = 1 + \frac{1}{\delta_0}\), we have

$$\mathbb{E}[H(|\Delta_T|)] \leq \frac{1}{H(\delta_0)} \mathbb{E}[H(|\Delta_{T^{\land}s_\delta}|)].$$  \hspace{1cm} (A.16)$$

To see this, we notice that on the set \(\{s_\delta < T \land \tau_R\}\), we have \(\Delta_{T^{\land}\tau_R^{\land}s_\delta} \geq \delta_0\) and hence \(H(\delta_0) \leq H(\Delta_{T^{\land}\tau_R^{\land}s_\delta})\) since \(H\) is increasing. Therefore,

$$\mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta}|)] = \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)I_{T^{\land}\tau_R \leq s_\delta}] + \mathbb{E}[H(|\Delta_{s_\delta}|)I_{s_\delta < T^{\land}\tau_R}] \geq \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)I_{T^{\land}\tau_R \leq s_\delta}] + H(\delta_0)\mathbb{P}\{s_\delta < T \land \tau_R\}.$$

Then it follows that

$$\frac{\mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta}|)]}{H(\delta_0)} - \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)] \geq \frac{\mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)I_{T^{\land}\tau_R \leq s_\delta}] + H(\delta_0)\mathbb{P}\{s_\delta < T \land \tau_R\} - \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)]}{H(\delta_0)} \geq \mathbb{P}\{s_\delta < T \land \tau_R\} + \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)I_{T^{\land}\tau_R \leq s_\delta}] - \mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)] \geq \mathbb{P}\{s_\delta < T \land \tau_R\} - \mathbb{E}[1 \cdot I_{s_\delta < T \land \tau_R}] = 0.$$

Consequently \(\mathbb{E}[H(|\Delta_{T^{\land}\tau_R}|)] \leq \frac{\mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta}|)]}{H(\delta_0)}\) for each \(R > 0\). Thanks to Theorem 2.2, \(\lim_{R \to \infty} \tau_R = \infty\) a.s. Also note that \(\bar{H}\) is uniformly bounded. Thus, by the bounded convergence theorem, passing to the limit as \(R \to \infty\) establishes (A.16).

For any \(\varepsilon > 0\), we can choose some \(R_0 > 0\) sufficiently large so that \(\mathbb{P}\{\tau_{R_0} \leq T \land S_0\} \leq \mathbb{P}\{\tau_{R_0} \leq T\} < \varepsilon\). Then we have from (A.15) that

$$\mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta}|)] = \mathbb{E}[H(|\Delta_{T^{\land}s_\delta}|)I_{T^{\land}s_\delta \leq \tau_{R_0}}] + \mathbb{E}[H(|\Delta_{T^{\land}s_\delta}|)I_{T^{\land}s_\delta > \tau_{R_0}}] \leq \mathbb{E}[H(|\Delta_{T^{\land}\tau_R^{\land}s_\delta}|)I_{T^{\land}s_\delta \leq \tau_{R_0}}] + \mathbb{P}\{\tau_{R_0} \leq T \land S_0\} \leq G^{-1}(G(\mathbb{E}[H(|\Delta_{t_0}|)] + K_{R_0}(T - t_0)) + K_{R_0}(T - t_0)) + \varepsilon.$$

(A.17)

The rest of the proof is very similar to those in the proof of Proposition 2.4 of Qiao (2014). Note that \(Y\) satisfies the SDE

$$Y(t) := x + \int_0^t [b(Y(s)) + h^n(s)I_{s > t_0}]ds + \int_0^t \sigma(Y(s))dW(s) + \int_0^t \int_U c(X(s-), u)\tilde{N}(dsdu),$$

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for \( t \in [0, T] \). Put \( \tilde{H}(t) := I_{(t > t_0)} \sigma^{-1}(Y(t)) h^n(t) \) and

\[
M(t) := \exp \left\{ \int_0^t \langle \tilde{H}(s), dW(s) \rangle - \frac{1}{2} \int_0^t |\tilde{H}(s)|^2 ds \right\}, \quad t \in [0, T].
\]

As observed in Qiao (2014), \( M \) is an a.s. strictly positive martingale under \( \mathbb{P} \) with \( \mathbb{E}[M(T)] = 1 \), the measure \( \mathbb{Q} \) defined by \( \mathbb{Q}(A) = \mathbb{E}[M(T) I_A] \), \( A \in \mathcal{F}_T \) is probability measure equivalent to \( \mathbb{P} \) on \( \mathcal{F}_T \). \( \tilde{W}(t) := W(t) + \int_0^t \tilde{H}(s)ds \) is a \( \mathbb{Q} \)-Brownian motion, and \( \tilde{N}(dt, du) \) is a \( \mathbb{Q} \)-compensated Poisson random measure with compensator \( dt \nu(du) \). Moreover, under \( \mathbb{Q} \), \( Y \) solves the SDE

\[
Y(t) := x + \int_0^t b(Y(s)) ds + \int_0^t \sigma(Y(s)) d\tilde{W}(s) + \int_0^t \int_U c(Y(s), u) \tilde{N}(dsdu), \quad t \in [0, T].
\]

By the pathwise uniqueness result established in Theorem 2.6, it follows that \( \mathbb{P} \{ |X^x(T) - a| > r \} = \mathbb{Q} \{ |Y(T) - a| > r \} \). Furthermore, since \( \mathbb{P}, \mathbb{Q} \) are equivalent, the desired assertion \( \mathbb{P} \{ |X^x(T) - a| > r \} < 1 \) will follow if we can show that \( \mathbb{P} \{ |Y(T) - a| > r \} < 1 \). To this end, we deduce as follows. Since the function \( H \) is increasing, we can use (A.16) and (A.17) to derive

\[
\mathbb{P} \{ |Y(T) - a| > r \} \leq \mathbb{P} \{ H(|Y(T) - a|) > H(r) \} \leq \mathbb{E}[H(|Y(T) - a|)] I_{[0, r]} + \frac{\mathbb{E}[H(|\Delta_T|)]}{H(r)} \frac{H(r)}{H(r)}
\]

\[
\leq \frac{\mathbb{E}[H(|\Delta_{T \wedge S_{t_0}}|)]}{H(r) H(\delta_0)} \leq \frac{G^{-1}(G(\mathbb{E}[H(|\Delta_{t_0}|)] + K_{R_0}(T - t_0)) + K_{R_0}(T - t_0)) + \varepsilon}{H(r) H(\delta_0)}.
\]

Finally, in view of (A.14) and the asymptotic properties of \( G \) and \( G^{-1} \), we can make the value of the last fraction in the above equation arbitrarily small by choosing \( n \) sufficiently large and \( t_0 \) sufficiently close to \( T \). This completes the proof.

**Proof of Lemma 7.3.** We give a constructive proof motivated by Kurtz (2011). Since \( \nu(x, \cdot) \) is a \( \sigma \)-finite measure on \( \mathbb{R}_0^d \), we can find a measurable partition \( \{ A_n \}_{n = -\infty}^\infty \) of \( \mathbb{R}_0^d \) such that \( 0 < \nu(x, A_n) \leq 1 \) for each \( n \). Now let

\[
\nu_n(x, \cdot) := \nu(x, \cdot \cap A_n), \quad \mu_n(x, \cdot) := \frac{\nu_n(x, \cdot)}{\nu_n(x, \mathbb{R}_0^d)}, \quad n \in \mathbb{Z}.
\]

Obviously we have \( \nu(x, \Gamma) = \sum_{n = -\infty}^\infty \nu_n(x, \Gamma) \) for each \( \Gamma \in \mathfrak{B}(\mathbb{R}_0^d) \). Using the measurable selection theorem (see, e.g. Kuratowski and Ryll-Nardzewski (1965) or (Stroock and Varadhan, 1979, Chapter 12)), we can choose \( \nu_n(x, \cdot) \) so that \( \nu_n(\cdot, \Gamma) \) is measurable for each \( n \) and \( \Gamma \in \mathfrak{B}(\mathbb{R}_0^d) \). For any complete and separable metric space \( E \), denoting by \( \mathcal{P}(E) \) the set of probability measures on \( E \), there exists a Borel measurable function \( h : \mathcal{P}(E) \times [0, 1] \mapsto \mathbb{R}_0^d \) such that \( h(\mu, Z) \overset{d}= \mu \), where \( \mu \in \mathcal{P}(E) \) and \( Z \) is uniformly distributed on \( [0, 1] \).

Now define functions \( \gamma : \mathbb{R}_0^d \times \mathbb{R} \mapsto \mathbb{R}_0^d \) and \( \lambda : \mathbb{R}_0^d \times \mathbb{R} \mapsto \mathbb{R} \) by

\[
\gamma(x, \xi) := \sum_{k = -\infty}^\infty h(\mu_k(x, \cdot), \xi) I_{[k, k+1)}(\xi), \quad \text{and} \quad \lambda(x, \xi) := \sum_{k = -\infty}^\infty \nu_k(x, \mathbb{R}_0^d) I_{[k, k+1)}(\xi).
\]
Then it follows that for any $\Gamma \in \mathfrak{B}(\mathbb{R}_0^d)$, we have

$$\int_{\mathbb{R}} \lambda(x, \xi) I_\Gamma(\gamma(x, \xi))d\xi = \sum_{k=-\infty}^{\infty} \int_{k}^{k+1} \nu_k(x, \mathbb{R}_0^d) I_\Gamma(h(\mu_k(x, \cdot), \xi))d\xi = \sum_{k=-\infty}^{\infty} \nu_k(x, \mathbb{R}_0^d) \mu_k(x, \Gamma) = \sum_{k=-\infty}^{\infty} \nu_k(x, \Gamma) = \nu(x, \Gamma).$$

Since $0 \notin \Gamma$, we can write

$$\nu(x, \Gamma) = \int_{\mathbb{R}} \lambda(x, \xi) I_\Gamma(\gamma(x, \xi))d\xi = \int_{\mathbb{R}} \int_{0}^{1} I_{[0, \lambda(x, \xi)]}(\eta)d\eta I_\Gamma(\gamma(x, \xi))d\xi = \int_{\mathbb{R} \times [0,1]} I_\Gamma(\gamma(x, \xi)I_{[0, \lambda(x, \xi)]}(\eta))d\eta d\xi.$$

This gives (7.10) with $c(x, u) = \gamma(x, \xi)I_{[0, \lambda(x, \xi)]}(\eta)$, $(U, \mathcal{U}) = (\mathbb{R} \times [0,1], \mathfrak{B}(\mathbb{R} \times [0,1]))$, and $M(\cdot)$ being the Lebesgue measure on $\mathbb{R} \times [0,1]$. The lemma is therefore proved. \\vspace{10pt}

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