Quantum Mechanical Effects from Deformation Theory

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Abstract

We consider deformations of quantum mechanical operators by using the novel construction tool of warped convolutions. The deformation enables us to obtain several quantum mechanical effects where electromagnetic and gravitomagnetic fields play a role. Furthermore, a quantum plane can be defined by using the deformation techniques. This in turn gives an experimentally verifiable effect.

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1 Introduction

Deformation theory is an interesting subject of research, both from a mathematical and a physical point of view. Nowadays, many fundamental theories are reconsidered as deformations of more subtle theories. A fundamental example of deformation theory in a physical context is the deformation of classical mechanics to quantum physics, where the deformation parameter in that case is Planck’s constant $\hbar$. In this context the Poincaré group can also be considered as a deformation of the Galilei group, where the parameter characterizing the deformation is given by the speed of light, i.e. $1/c^2$. The opposite of a deformation in a group theoretical context is a contraction. It is induced by taking the limit of the deformation parameter to zero. In the example
of the Poincaré group, this would mean that we take the limit \(1/c^2 \to 0\). This limit is often taken by physicists as a consistency check and rarely recognized as a contraction. Another interesting example is the deformation of the Poincaré group to the Anti-de Sitter group by using the cosmological constant \(\Lambda\) as a deformation parameter.

Thus, from a physical point of view, deformation theory enters the game by the physical dimensionality of the deformation parameter. In this work, we emphasize the importance of choosing the deformation constant, in order to obtain physical effects.

One justified critique usually spoken out in the context of deformation theory is that the rightful deformation is only guessed after the physical theory has been formulated. Thus, to consider such deformations as fundamental, is often put into the category of wishful thinking of theoretical physicists. Therefore, the main aim of the current work is to understand a variety of physical effects, in a quantum mechanical context, by a deformation of the free theory. Furthermore, we propose an effect coming from deformation considerations.

The method that is used, in the current work, for deformation is known under the name of warped convolutions, \([GL07, BS, BLS11]\). Usually, this method is used in the realm of quantum field theory to deform free quantum fields and to construct non-trivial interacting fields which was done in \([Ala, GL07, GL08, Lec12, LST13, MM11, Alb12]\). It was also used in quantum measurement theory \([And13]\). One of the major advantages of this method is its easy accessibility to a physical regimen.

By using this novel tool in a quantum mechanical context, we recast many fundamental physical effects involving electromagnetism. This is done by the adjustment of the deformation parameter. Moreover, we are able to produce gravitomagnetic effects and interaction between magnetic and gravitomagnetic fields by this deformation procedure.

This paper is organized as follows: In Section 2 we give a brief introduction of the method of warped convolutions and introduce the basic notations for deformation in a quantum mechanical context. The free Hamiltonian is deformed in Section 3. We are obliged to show that the warped convolutions formula, originally formulated for a subset of bounded operators, is well-defined in the case of the deformation of unbounded operators. Section 4 is devoted to the emergence of physical effects from the deformation procedure.

## 2 Warped convolutions in QM

Since we constantly use warped convolutions we lay out the novel deformation procedure in this section and present the most important definitions, lemmas and propositions for the current paper. For proofs of the lemmas and propositions we refer the reader to the original works.

We start by assuming the existence of a strongly continuous unitary group \(U\) that is a representation of the additive group \(\mathbb{R}^n\), on some separable Hilbert space \(\mathcal{H}\). Let \(D\) be the dense domain of vectors in \(\mathcal{H}\) which transform smoothly under the adjoint action of \(U\). Then, the warped convolutions for operators \(F \in C^\infty\), where \(C^\infty\) is the \(*\)-algebra of smooth elements with respect to the adjoint action of \(U\), are given by the following definition.
**Definition 2.1.** Let $B$ be a real skew-symmetric matrix on $\mathbb{R}^n$, let $F \in C^\infty$ and let $E$ be the spectral resolution of the unitary operator $U$. Then, the corresponding warped convolution $F_B$ of $F$ is defined on the domain $D$ according to

$$F_B := \int \alpha_B x dE(x),$$

(2.1)

where $\alpha$ denotes the adjoint action of $U$ given by $\alpha_k(F) = U(k) F U(k)^{-1}$.

The restriction in the choice of operators is owed to the fact that the deformation is performed with operator valued integrals. Furthermore, one can represent the warped convolution of $A \in C^\infty$ by

$$\int \alpha_B x dE(x) \Phi = (2\pi)^{-n} \lim_{\epsilon \to 0} \int d^n x d^n y \chi(\epsilon x, \epsilon y) e^{-ixy} U(y) \alpha_B x (A) \Phi,$$

where $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. This representation makes calculations and proofs concerning the existence of integrals easier. In this work we use both representations.

In the following lemma we introduce the deformed product, also known as the Rieffel product [Rie93] by using warped convolutions. The two deformations are interrelated since warped convolutions supply isometric representations of Rieffel’s strict deformations of $C^*$-dynamical systems with actions of $\mathbb{R}^n$.

**Lemma 2.1.** Let $B$ be a real skew-symmetric matrix on $\mathbb{R}^n$ and let $A, E \in C^\infty$. Then

$$A_B E_B \Phi = (A \times_B E)_B \Phi, \quad \Phi \in D.$$

where $\times_B$ is known as the Rieffel product on $C^\infty$ and is given by,

$$(A \times_B E) \Phi = (2\pi)^{-n} \lim_{\epsilon \to 0} \int d^n x d^n y \chi(\epsilon x, \epsilon y) e^{-ixy} \alpha_B x (A) \alpha_y (E) \Phi.$$

(2.2)

Another proposition that seems a matter of technicality in the original work but has great physical significance is the following.

**Proposition 2.1.** Let $B_1, B_2$ be skew symmetric matrices. Then

$$(A_{B_1} B_2) = A_{B_1 + B_2}, \quad A \in C^\infty.$$

(2.3)

Next, we adopt Formula (2.1) to define the warped convolutions for an unbounded operator, with a real vector-valued function of the coordinate operator. To apply the definition of warped convolutions, we need self-adjoint operators that commute along their components. For this purpose let us give the following theorem, [RS75, Theorem VIII.6].

**Theorem 2.1.** Let $Q(\cdot)$ be an unbounded real vector-valued Borel function on $\mathbb{R}^n$ and let the dense domain $D_Q$ be given as,

$$D_Q = \{ \phi | \int_{-\infty}^{\infty} |Q_j(x)|^2 d\phi, P_x \phi \langle \infty, \quad j = 1, \ldots, n \}.$$
In this paper we consider unbounded real vector-valued functions of the coordinate operator and therefore we give the following definition.

**Definition 2.2.** Let $B$ be a real skew-symmetric matrix on $\mathbb{R}^n$ and let $\chi \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. Moreover, let $Q(X)$ be given as in Theorem 2.1. Then, the warped convolutions of an operator $A$ with operator $Q$, denoted as $A_{B,Q}$, are defined, in the same manner as in [BLS11], namely

$$A_{B,Q} := (2\pi)^{-n} \lim_{\epsilon \to 0} \int d^n y d^n k e^{-iyk} \chi(\epsilon y, \epsilon k) V(k) \alpha_{B\theta}(A). \quad (2.4)$$

The automorphisms $\alpha$ are implemented by the adjoint action of the strongly continuous unitary representation $V(y) = e^{iyQ^y}$ of $\mathbb{R}^n$ given by

$$\alpha_y(A) = V(y) A V(y)^{-1}, \quad y \in \mathbb{R}^n.$$ 

Since we deform unbounded operators we are obliged to prove that the deformation formula, given as an oscillatory integral, is well-defined. This is the subject of the next section.

### 3 Deforming Unbounded Operators

At first, we study deformations of the simplest Hamiltonian of quantum mechanics, that of a free particle. Further on we explore the physical consequences of the deformation and introduce to the reader how one can obtain a variety of physical effects using this method. For a deformation of the Hamiltonian we choose to work in the standard realization of quantum mechanics, the so called Schrödinger representation, [BEH08, RS75, Tes01]. In this representation the pair of operators $(P_i, X_j)$, satisfying the canonical commutation relations (CCR)

$$[X_i, P_k] = -i\hbar \delta_{ik}, \quad (3.1)$$

are represented as essentially self-adjoint operators on the dense domain $\mathcal{C}(\mathbb{R}^n)$. Here $X_i$ and $P_k$ are the closures of $x_i$ and multiplication by $i\partial/\partial x^k$ on $\mathcal{C}(\mathbb{R}^n)$ respectively.

In quantum mechanics the Hamiltonian of a free particle is given as follows

$$H_0 = -\frac{P_j P^j}{2m}. \quad (3.2)$$

This operator describes a non-relativistic and non-interacting particle. For the following considerations, we restrict the deformation to three space dimensions. This restriction is obvious due to its physical relevance. Let us start this section with a theorem concerning the domain of self-adjointness and the spectrum of the free undeformed Hamiltonian $H_0$, [Tes01].

**Theorem 3.1.** The free Schrödinger operator $H_0$ is self-adjoint on the domain $\mathcal{D}(H_0)$ given as

$$\mathcal{D}(H_0) = H^2(\mathbb{R}^3) = \{ \phi \in L^2(\mathbb{R}^3) | ||P \phi|| \in L^2(\mathbb{R}^3) \},$$

and its spectrum is characterized by $\sigma(H_0) = [0, \infty)$. 

Before proceeding with the deformation, one problem arises at this point of our work. The deformation formula given by warped convolutions is only well-defined in the strong operator topology for a subset of bounded operators that are smooth w.r.t. the unitary representation $U$ of $\mathbb{R}^n$. In view of the fact that we deal with unbounded operators, we are obliged to investigate the validity of the deformation Formula (2.4) for $H_0$. For this purpose we need a dense domain $\mathcal{E} \subseteq \mathcal{S}(\mathbb{R}^3)$ that fulfills additional requirements.

**Lemma 3.1.** Consider the self-adjoint operator

$$Q(X) = X/|X|^n, \quad n \in \mathbb{R}. \quad (3.3)$$

Then, for all $n \in \mathbb{R}$ there exists a dense domain $\mathcal{E} \subseteq \mathcal{S}(\mathbb{R}^3)$ such that

$$\|\{P_j, [Q, P_j]\}\Phi\| < \infty, \quad \|[Q, P_j][Q, P_j]\| < \infty, \quad \Phi \in \mathcal{E}. \quad (3.4)$$

**Proof.** From Theorem 2.1 it follows that all operators of the form $X/|X|^n$ are self-adjoint on their respective domains. Further we show the existence of a dense domain, satisfying Inequalities (3.4). To simplify calculations let us give general formulas for the commutators

$$[P_j, |X|^{n-1}] = i n X_j |X|^{-(n+2)}, \quad (3.5)$$

$$[P_j, X_j/|X|^n] = i (\eta_{jk} + n X_k X_j/|X|^2) |X|^{-n}. \quad (3.6)$$

Thus, for an arbitrary $n \in \mathbb{R}$ and $Q = X/|X|^n$ the anti-commutator in Inequality (3.4) is calculated as follows

$$\{P_j, [P_j, Q_k]\} = [P_j, [P_j, Q_k]] + 2[P_j, Q_k] P_j$$

$$= i \left( \eta_{jk} + n X_j X_k/|X|^2 \right) |X|^{-(n+2)} + 2i \left( \eta_{jk} + n X_j X_k/|X|^2 \right) |X|^{-n} P_j$$

$$= (n^2 - 3n) X_k |X|^{-(n+2)} + 2i \left( \eta_{jk} + n X_j X_k/|X|^2 \right) |X|^{-n} P_j,$$

where in the last lines we used the CCR, Equations (3.5) and (3.6). The norm of the anti-commutator is given by

$$\|\{P_j, [Q, P_j]\}\| = \|\eta_{jk} a(n) X_k |X|^{-(n+2)} + 2i \left( \eta_{jk} + n X_j X_k/|X|^2 \right) |X|^{-n} P_j\|$$

$$\leq \|\eta_{jk} a(n) X_k |X|^{-(n+2)} \Phi\| + \|2i \left( \eta_{jk} + n X_j X_k/|X|^2 \right) |X|^{-n} P_j\|$$

$$\leq \|a(n) |X|^{-(n+1)} \Phi\| + \|2 |X|^{-n} P\Phi\| + \|2n |X|^{-(n+1)} X^j P_j \Phi\|.$$

The term in the second inequality in (3.4) is given by

$$\|[Q, P_j][Q, P_j]\| = \|\left( \eta_{jl} + n X_l X_j/|X|^2 \right) \left( \eta_{jl} + n X_j X^j/|X|^2 \right) |X|^{-2n} \Phi\|$$

$$= \|\left( n^2 - 2n + 3 \right) |X|^{-2n} \Phi\|.$$

It is clear that if $n \in \mathbb{R}_0^+$ Inequalities (3.4) are satisfied for vectors in the dense domain $\mathcal{S}(\mathbb{R}^3)$, since the expressions in the norm are positive polynomial functions of the coordinate operator. For $n \in \mathbb{R}^+$ we consider the domain $\mathcal{E}$ which denotes the linear hull of the dense vectors $[\text{Thi81, Theorem 3.2.5}]$

$$\Phi(x) = e^{x_1^k x_2 k x_3}, \quad k = 0, 1, 2, \ldots.$$
Since the dense domain $E$ remains invariant under the action of positive functions of the coordinate and momentum operator (see proof of [Thi81, Theorem 3.2.5]), the remaining task is to show the finiteness of
\[ \|X^{(\lambda-1)}\Phi\|^2 = \int d^3x \|x\|^{2(\lambda-1)} e^{-|x|^2}, \quad \lambda \in \mathbb{R}^- \]
\[ = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \int_0^\infty r^{2\lambda} e^{-r^2} dr \]
\[ = 2\pi \int_0^\infty x^{2\lambda} e^{-x^2} dx. \]

This integral exists for all $\lambda$ and it is easily seen to be an analytic function in $\lambda$, [GS68, Chapter 1, Section 3.6]. Note that we choose the polynomial functions of components of $x$ to be equal to one. This choice is for the sake of argument, since positive polynomial functions improve the behavior of the integral. 

By using the former lemma, we show in the next proposition that the scalar product of the deformed free Hamiltonian, i.e.
\[ \langle \Psi, (H_0)_B, Q \Phi \rangle = (2\pi)^{-3} \lim_{\varepsilon \to 0} \int \int d^3y d^3k e^{-i\varepsilon k^l} \chi(\varepsilon y, \varepsilon k) \langle \Psi, V(k)\alpha_{By}(H_0)\Phi \rangle, \]
is bounded for $\forall \Psi \in \mathcal{H}$ and $\Phi \in E \subseteq \mathcal{S}(\mathbb{R}^3)$.

**Proposition 3.1.** Let $Q(X)$ be a self-adjoint operator of the form
\[ Q(X) = \frac{X}{|X|^n}, \quad n \in \mathbb{R}, \]
and let $(H_0)_B, Q$ denote the deformed free Hamiltonian (see Formula (2.4)). Then, the scalar product $\langle \Psi, (H_0)_B, Q \Phi \rangle$ is bounded by a finite constant $C_B$ as follows,
\[ |\langle \Psi, (H_0)_B, Q \Phi \rangle| \leq C_B \|\Psi\|, \quad \forall \Psi \in \mathcal{H}, \Phi \in E \subseteq \mathcal{S}(\mathbb{R}^3). \]

Therefore, the deformation formula for the unbounded operator $H_0$, given as an oscillatory integral, is well-defined and the explicit result of the deformation is
\[ (H_0)_B, Q \Phi = -\frac{1}{2m} \left( P_j + i(BQ)^k [Q_k, P_j] \right) \left( P^j + i(BQ)^r [Q_r, P^j] \right) \Phi. \quad (3.7) \]

**Proof.** To prove the boundedness of the scalar product $\langle \Psi, V(k)\alpha_{By}(H_0)\Phi \rangle$, we first derive the adjoint action of $V(By)$ on $H_0$ given by,
\[ \alpha_{By}(H_0) = -\frac{1}{2m} V(By)P_j P^j V(-By) \]
\[ = -\frac{1}{2m} V(By)P_j V(-By)V(By)P^j V(-By). \]

To solve this expression we first calculate the adjoint action of $V(By)$ on the momentum operator $P_j$ by using the Baker-Campbell-Hausdorff formula,
\[ V(By) P_j V(-By) = P_j + i(By)^k [Q_k, P_j] + \frac{i^2}{2}(By)^j (By)^k [Q_l, [Q_k, P_j]] + \ldots, \quad (3.8) \]
where in the last lines we used the CCR given in (3.1), and the commutativity of the coordinate operator, i.e. \([X_i, X_j] = 0\). Thus, the adjoint action w.r.t. \(V(By)\) on \(H_0\) is
\[
\alpha_{By}(H_0) = -\frac{1}{2m}(P_j + (By)^s X_{sj})(P_j + (By)_r X_{rj})
\]
\[
= H_0 - (By)^s \frac{1}{2m} \left( P_j X_{sj} + X_{sj} P_j \right) - (By)_r (By)^s \frac{1}{2m} X_{sj} X_{rj}
\]
\[
= H_0 - (By)^s N_s - (By)_r (By)^s R^s_r.
\]

Moreover, without loss of generality, one can choose the skew-symmetric matrix \(B\) to have the form \(B_{ij} = \varepsilon_{ijk} B^k\), where \(\varepsilon_{ijk}\) is the three dimensional epsilon-tensor. Then, we are able to derive the following inequality,
\[
||\langle By \rangle, e^t || \leq \sqrt{2} |B| |y|.
\]

This is easily seen by using Cauchy-Schwarz and the inequality \(|a| - |b| \leq |a| + |b|\).

\[
|\langle By \rangle, e^t |^2 = (-B_{ij} y^i B^{is} y_s)
\]
\[
= -\varepsilon_{ijk} B^k y^j \varepsilon^{rs} B_r y_s
\]
\[
= -\left( \delta^s_k \delta^r_j - \delta^r_k \delta^s_j \right) B_r B^k y^j y_s
\]
\[
= (B_r y^r B_k y^k - B_k B^k y^r)
\]
\[
\leq 2 |B|^2 |y|^2
\]

Thus, by using the adjoint action of \(V(By)\) on the free Hamiltonian and for \(B_{ij} = \varepsilon_{ijk} B^k\) we have the following inequality
\[
|\langle \Psi, (V(k)\alpha_{By}(H_0)\Phi) \rangle | \leq \|\Psi\| \|\langle H_0 - (By)^s N_s - (By)_r (By)^s R^s_r \rangle \| \Phi \|
\]
\[
\leq \|\Psi\| \left( \|H_0 \Phi\| + 2|y| \|B\| \|N \Phi\| + |y|^2 2 |B|^2 \|R \Phi\| \right)
\]
\[
\leq C_B \|\Psi\| (1 + |y|)^2.
\]

A finite constant \(C_B\) obeying the inequality exists, since \(C_2, C_3\) and \(C_4\) are finite for \(\Phi \in \mathcal{E}\), where \(\mathcal{E}\) is a dense set of vectors specified in Lemma 3.1. Therefore, the scalar product is polynomially bounded to the second order in \(y\), i.e.

\[
\frac{|\langle \Psi, (H_0)_{By} Q \Phi \rangle |}{C_B \|\Psi\|} \leq (2\pi)^{-3} \lim_{\varepsilon_1 \to 0} \int d^3y d^3k e^{-ik \varepsilon_1 y} \chi(\varepsilon_2 y, \varepsilon k) (1 + |y|)^2
\]
\[
= (2\pi)^{-3} \lim_{\varepsilon_1 \to 0} \left( \int d^3y \lim_{\varepsilon_2 \to 0} \left( \int d^3k e^{-ik y} \chi_2(\varepsilon_2 k) \right) \chi_1(\varepsilon_1 y) (1 + |y|)^2 \right)
\]
\[
= \lim_{\varepsilon_1 \to 0} \left( \int d^3y \delta(y) \chi(\varepsilon_1 y) (1 + |y|)^2 \right)
\]
\[
= 1.
\]

Here we used the fact that the oscillatory integral does not depend on the cut-off function chosen. As in [Rie93], we chose \(\chi(\varepsilon_2 k, \varepsilon y) = \chi_2(\varepsilon_2 k) \chi_1(\varepsilon_1 y)\) with \(\chi_1 \in \mathcal{Y}(\mathbb{R}^3 \times \mathbb{R}^3)\) and \(\chi(0, 0) = 1\), \(l = 1, 2\), and obtained the delta distribution \(\delta(y - q)\) in the limit \(\varepsilon_2 \to 0\), [Hör04, Section 7.8, Equation 7.8.5]. Since the former inequality proves the convergence of the oscillatory integral, we conclude that the deformation of the unbounded operator is well-defined.
Next, we turn to the actual result of the deformation. To simplify calculations it is easier to work in the spectral measure representation (see Equation 2.1). This can be done, since the two representations, one in terms of the spectral measure and the other as the limit of oscillatory integrals, are equal and we have proven that the deformation is well-defined.

\[
(H_0)_{B,Q} \Phi = \int dE(y) \alpha_{By} (H_0) \Phi \\
= -\frac{1}{2m} \int dE(y) \left( (P_j + i(By)_s [Q^*, P_j]) (P_j^* + i(By)^* [Q^*, P_j]) \right) \Phi \\
= -\frac{1}{2m} (P_j + i(BQ)_s [Q^*, P_j]) (P_j^* + i(BQ)^* [Q^*, P_j]) \Phi.
\]

The essential point of the proposition is that the deformation with the coordinate operator amounts to a non-constant shift in the momentum. In physics this is usually referred to as \textbf{minimal substitution}. Such a minimal substitution is in QM based on Galilei invariance and then implemented accordingly by an external electromagnetic field (see [JM67]). In our approach we obtain such a substitution by deformation. The connection between deformation and an external electromagnetic field is explored in the next sections.

For the next proposition we deform the momentum operator. Since the momentum operator is unbounded, we are as before obliged to show that deformation Formula (2.4) is given as a well-defined oscillatory integral.

**Proposition 3.2.** Let \( Q(X) \) be a self-adjoint operator of the form

\[
Q(X) = \frac{X}{|X|^n}, \quad n \in \mathbb{R},
\]

and let \( P_{B,Q} \) denote the deformed momentum operator (see Formula 2.4). Then, the scalar product \( \langle \Psi, P_{B,Q} \Phi \rangle \) is bounded by a finite constant \( C_D \) as follows,

\[
|\langle \Psi, P_{B,Q} \Phi \rangle| \leq C_D \| \Psi \|, \quad \forall \Psi \in \mathcal{H}, \quad \Phi \in E \subseteq \mathcal{S}(\mathbb{R}^3).
\]

Therefore, the deformation of the unbounded momentum operator, given as an oscillatory integral, is well-defined. Moreover, the explicit result of the deformation is given as

\[
P_{B,Q} \Phi = (P_j^* + i(BQ)^* [Q_j, P_j]) \Phi. \tag{3.11}
\]

\textbf{Proof.} As in the proof of the former proposition we show that \( |\langle \Psi, V(k)\alpha_{B,y}(P) \Phi \rangle| \), is polynomially bounded. To do so, we use the adjoint action of the unitary operator \( V(By) \) on the momentum operator (see Equation (3.8)) and the Cauchy-Schwarz inequality,

\[
|\langle \Psi, V(k)\alpha_{B,y}(P) \Phi \rangle| \leq \| \Psi \| \left\| |(P + i(By)^* [Q^*, P_j]) \Phi| \right\|
\leq \| \Psi \| \left( \left\| P \Phi \right\| + |y| \sqrt{2} |B| \| [Q, P] \Phi \| \right)
\leq C_D \| \Psi \| (1 + |y|),
\]

where we used Inequality (3.9) and the fact that a finite constant \( C_D \) obeying the inequality exists, since \( C_5 \) and \( C_6 \) are finite for \( \Phi \in \mathcal{E} \) (see Lemma 3.1). Therefore,
the whole expression is polynomially bounded to first order in $y$, i.e.
\[
\frac{\langle \psi, B, P^j \rangle_{\mathcal{B}}}{C_D \| \psi \|} \leq (2\pi)^{-3} \lim_{\epsilon \to 0} \int d^3y \int d^3k e^{-i\epsilon yk} \chi(\epsilon y, \epsilon k) (1 + |y|)
\]
\[
= (2\pi)^{-3} \lim_{\epsilon \to 0} \left( \int d^3y \lim_{\epsilon_2 \to 0} \left( \int d^3k e^{-i\epsilon yk} \chi(\epsilon 2k) \right) \chi(\epsilon 1y) (1 + |y|) \right)
\]
\[
= \lim_{\epsilon_1 \to 0} \left( \int d^3y \delta(y) \chi(\epsilon 1y) (1 + |y|) \right)
\]
\[
= 1.
\]

As before, we argue that due to the convergence of the integral the deformation of the momentum operator for all $\psi \in \mathcal{H}$ and $\phi \in \mathcal{E}$ is well-defined.

Next, we turn to the actual result of the deformation and again for simplicity we use the spectral measure for deformation,
\[
P^j_{B, \dot{Q}} \psi = \int dE y \alpha_B y (P^j) \psi
\]
\[
= \int dE y \left( P^j + i(BQ)_s[Q^s, P^j] \right) \psi
\]
\[
= \left( P^j + i(BQ)_s[Q^s, P^j] \right) \psi.
\]

Since the deformed Hamiltonian could be defined as the scalar product of the deformed momentum operators, we need to investigate the possible outcome. The investigation of the arbitrariness in the definition of the deformed free Hamiltonian is subject of the following theorem.

**Theorem 3.2.** The scalar product of the deformed momentum vectors is equal to the deformed free Hamiltonian (see Equation 3.7), i.e.
\[
(H_0)_{B, \dot{Q}} \psi = -\frac{1}{2m} P^j_{B, \dot{Q}} P^j_{B, \dot{Q}} \psi, \quad \psi \in \mathcal{E} \subseteq \mathcal{F}(\mathbb{R}^3).
\]

**Proof.** For the proof we calculate the Rieffel product, defined with the operator-valued vector $Q(X)$, of the deformed momentum vectors, i.e.
\[
(P_k \times_{B, \dot{Q}} P_j) \psi = (2\pi)^{-3} \lim_{\epsilon \to 0} \int d^3x d^3y \chi(\epsilon x, \epsilon y) e^{-ixy} \alpha_B x (P_k) \alpha_B y (P_j) \psi
\]
\[
= (P_k P_j - iB_{yx} \partial_x Q^l \partial_y Q^l) \psi,
\]
where in the last lines we used the CCR and the fact that the only terms that do not vanish are those of equal odd order in $x$ and $y$, (see proof of Lemma 5.3 in [Alb12]).

Now by summing over all components we obtain
\[
(P_k \times_{B, \dot{Q}} P^k) \psi = \left( P_k P^k - i B_{yx} \partial_x Q^l \partial_y Q^l \right) \psi = P_k P^k \psi,
\]
where we used the skew-symmetry of $B$ and commutativity of the coordinate operator. Thus, by using the last equation and Lemma 2.1 the following equality is given
\[
(H_0)_{B,Q} \Psi = -\frac{1}{2m} (P_k P_k)_{B,Q} \Psi \\
= -\frac{1}{2m} (P_k \times B_{,Q} P_k^k)_{B,Q} \Psi \\
= -\frac{1}{2m} P_{k,B,Q} P_k^k \Psi, \quad \Psi \in \mathcal{S}(\mathbb{R}^3).
\]

This is an important result resolving the question of arbitrariness of the deformation. Moreover, it is a group theoretical circumstance, since the deformation of a free Hamiltonian can be understood as the deformation of generators of the central extended Galilei (CEG) group (for CEG see for example [Bal98]). The deformation with the coordinate operator leaves all generators of the group invariant except for the momentum and the Hamiltonian. Since, the Hamiltonian is a function of the momentum it follows from the former proposition that the deformation respects the structure of the group. This fact is owed to the deformed product. Also note that the deformed momentum operator does not commute along its components.

As already mentioned in Section 2, for some arguments we deform the coordinate operator by using the momentum operator. Before doing so, we show in the next proposition that the deformation formula is well-defined even though the coordinate operator is unbounded.

**Proposition 3.3.** The scalar product \( \langle \Psi, X_{\theta, P} \Phi \rangle \) is bounded by a finite constant \( C_E \) as follows,

\[
|\langle \Psi, X_{\theta, P} \Phi \rangle| \leq C_E \| \Psi \|, \quad \forall \Psi \in \mathcal{H}, \ \Phi \in \mathcal{S}(\mathbb{R}^3).
\]

Therefore, the deformation of the unbounded coordinate operator, given as an oscillatory integral, is well-defined. Moreover, the explicit result of the deformation is given as

\[
X_{\theta, P}^j \Psi = (X^j - (\theta P)^j) \Psi, \quad \Psi \in \mathcal{S}(\mathbb{R}^3). \tag{3.12}
\]

**Proof.** Similar to the proofs of the former propositions we show that the scalar product \( |\langle \Psi, V(k)\alpha_{\theta y}(X)\Phi \rangle| \) is polynomially bounded,

\[
|\langle \Psi, V(k)\alpha_{\theta y}(X)\Phi \rangle| \leq \| \Psi \| \| (X + i(\theta y)^j P_j, X) \| \Phi \|
\]

\[
\leq \| \Psi \| \left( \| X \Phi \| + |y| \sqrt{2|\theta|} \| \Phi \| \right)_{=C_7}^{=C_8}
\]

\[
\leq C_E \| \Psi \| (1 + |y|),
\]

where in the last lines we used Inequality (3.9), and the fact that a finite constant \( C_E \) obeying the inequality exists, since \( C_7 \) and \( C_8 \) are finite for \( \Phi \in \mathcal{S}(\mathbb{R}^3) \). Therefore, the whole expression is polynomially bounded to the first order in \( y \), i.e.

\[
|\langle \Psi, X_{\theta, P} \Phi \rangle| \leq C_E \| \Psi \|,
\]

where we used the same arguments made in Proposition 3.2.

Next, we turn to the result of the deformation and again for simplicity we use the spectral measure for the deformation,
\[ X_{\delta,1}^j \Psi = \int dE(y) \alpha_{\theta y} (X^j) \Psi \]
\[ = \int dE(y) \left( X^j + i(\theta y)_{[s} [P^s, X^j] \right) \Psi \]
\[ = (X^j - (\theta P)^j) \Psi. \]

After this rather more technical part we turn in the next section to the physical implications of the deformation technique.

4 Physical models from deformation

One of the most important aspects of the interplay between mathematics and physics lies in the physical dimensionality of the physical constants. The main motivation of this work is the search for the physical meaning of the deformation parameter. Quantum mechanical deformations give us a variety of interesting answers and they are presented in this section.

4.1 Landau quantization

An example of a dynamical system interacting with a magnetic field in a quantum mechanical setting, is given by the Landau effect. It is also an important example of the appearance of quantum space in a physical context. The Landau effect describes the dynamics of a system of non-relativistic electrons confined to a plane, for example the \( y-z \) plane (\( \vec{A} = (0, y, z) \)), in the presence of a homogeneous magnetic field \( \vec{B} = B(1, 0, 0) \). In the symmetric gauge the Hamiltonian of the Landau effect is given by, [Eza08, Equation 9.2.1]

\[ H_L = -\frac{1}{2m} (P_i + e A_i) \left( P^i + e A^i \right), \]

where the gauge field is given as

\[ A_i = -\frac{1}{2} \varepsilon_{ijk} B^k X^j. \]

(4.1)

Next, we show that the deformed Hamiltonian \( (H_0)_{B,X} \) reproduces the Landau model after setting the parameters of the deformation matrix equal to a constant with physical dimension. This is the result of the following lemma.

**Lemma 4.1.** Let the deformation matrix \( B_{ij} \) be given as,

\[ B_{ij} = -(e/2) \varepsilon_{ijk} B^k, \]

where \( B^k \) characterizes a constant homogeneous magnetic field and \( e \) is the electric charge. Then, the deformed free Hamiltonian \( (H_0)_{B,X} \) becomes the Hamiltonian \( H_L \) of the Landau problem, i.e.,

\[ (H_0)_{B,X} \Psi = H_L \Psi, \quad \Psi \in \mathcal{E}. \]

**Proof.** For the proof we consider the free deformed Hamiltonian \( (H_0)_{B,X} \), given in Equation (3.7), with \( Q_j = X_j \)

\[ (H_0)_{B,X} \Psi = -\frac{1}{2m} (P_j + i(BX)^k [X_k, P_j]) \left( P^j + i(BQ)^r [X_r, P^j] \right) \Psi \]
\[ = -\frac{1}{2m} (P_j + B_{jk} X^k)(P^j + B^{jr} X_r) \Psi, \quad \Psi \in \mathcal{E}, \]  

(4.2)
where in the last line we used the CCR. By setting the deformation matrix equal to 
$B_{ij} = -(e/2) \varepsilon_{ijk} B^k$, where $B^k$ is a homogeneous magnetic field in the $x$-direction ($B^k = B(1,0,0)$), we obtain the Landau quantization.

This is an interesting result. We started with the free Hamiltonian and deformed it with warped convolutions using the coordinate operator. By simply taking the deformation parameters of the matrix $B_{ij}$ to be equal to certain physical quantities we obtain the Landau problem. Therefore, the quantization with the coordinate operator is physically of great importance. Note that our model is formulated in a general manner, and just for the specific choice of the deformation parameters we obtained the Landau effect.

A remark is in order about the current result. It is well-known that the non-commutative coordinates of the Landau quantization can be generated by minimally shifting the ordinary coordinate operator by a skew-symmetric matrix times the momentum operator. This rather ad hoc but remarkably insightful result is well-known as the Bopp-shift. In the context of deformation theory, we were able to give a systematic derivation of the Landau quantization, rather than postulating ad hoc a substitution. This derivation can be further applied to a variety of quantum mechanical effects involving gauge fields.

4.2 Zeeman effect

The Hamiltonian of the hydrogen atom is given as follows, [Thi81, Equation 4.1.1]

$$H^A = -\frac{P_j P_j}{2m} + \frac{e^2 |X|}{2}.$$ 

By solving the stationary Schrödinger equation $H^A \psi = E \psi$ one obtains the energy spectrum of a hydrogen atom, the so called Balmer series, [Str02]. In the presence of a constant magnetic field, an interesting physical effect occurs to the spectral lines of the hydrogen atom. The spectral lines split into further spectral lines depending on the presence of a homogeneous magnetic field $B_k$. This phenomenon is called the Zeeman effect and the Hamiltonian of this effect is given as follows, [Thi81, Equation 4.2.1]

$$H^{AZ} = -\frac{1}{2m} (P_j - (e/2) \varepsilon_{ijk} B^k X^i)(P^j - (e/2) \varepsilon^{jnl} B_l X_n) + \frac{e^2 |X|}{2}.$$ 

We recognized in the last section that the deformation with the coordinate operator induces a gauge field. Due to this lesson we preform a deformation on the Hamiltonian of the hydrogen atom to obtain the Hamiltonian of the Zeeman effect.

**Lemma 4.2.** Let the deformation matrix $B_{ij}$ be given as,

$$B_{ij} = -(e/2) \varepsilon_{ijk} B^k,$$

where $B^k$ characterizes a constant homogeneous magnetic field. Then, the deformed Hamiltonian of the hydrogen atom, denoted by $(H^A)_{B,X}$, becomes the Hamiltonian of the Zeeman effect $H^{AZ}$, i.e.

$$(H^A)_{B,X} \psi = H^{AZ} \psi, \quad \psi \in \mathcal{E}.$$ 

**Proof.** Due to the fact that the coordinate operator commutes with itself the only part of the Hamiltonian $H^A$ which is affected is the free part and therefore we obtain

$$(H^A)_{B,X} \psi = \left( -\frac{1}{2m} (P_j + B_{jk} X^k)(P^j + B^{jr} X_r) + \frac{e^2 |X|}{2} \right) \psi,$$ 

(4.4)
the Hamiltonian of the Zeeman effect for a homogeneous magnetic field in the \( x \)-direction, i.e. \((H^A)_{B,X} = H^{AZ}\).

As in the case of Landau quantization, the deformation parameter plays the role of the magnetic field which leads to this wonderful physical effect.

### 4.3 Aharonov-Bohm effect

In the last sections we recognized the consequence of a deformation with the coordinate operator. Warped convolutions with the coordinate operator induce a **gauge field**. Now since we work in a quantum mechanical setting we want to reproduce other physical effects where magnetic fields play a significant role. One of the most striking ones is the **Aharonov-Bohm (AB) effect**. It takes place in a system in which the gauge field influences the dynamics of a charged particle even in regions where the magnetic field vanishes, [Ber00, Eza08]. The gauge field of the magnetic AB effect, for a homogeneous magnetic field in \( x \)-direction, takes the following form

\[
A_i = \frac{\phi_M}{2\pi(X_2^2 + X_3^2)}\epsilon_{ijk}e^kX^j,
\]

where \( \phi_M \) is the **magnetic flux** and \( e^k \) is the unit vector in \( x \)-direction. Moreover, from quantum mechanical considerations it follows that the **interference pattern** is the same for two values of fluxes \( \phi_1 \) and \( \phi_2 \) if only if

\[
e(\phi_1 - \phi_2) = 2\pi n, \quad n \in \mathbb{Z}.
\]

In this section we take the free Hamiltonian and deform it with a vector-valued function of the coordinate operator. As before, after setting the deformation parameter equal to a physical constant, namely that of a magnetic flux, we obtain the AB effect.

**Proposition 4.1.** Let the deformation matrix \( B_{ij} \) and the operator \( Q_j(X) \) be given as

\[
B_{ij} = -\frac{e\phi_M}{2\pi}\epsilon_{ijk}e^k, \quad Q_j(X) := X_j/(\sum_{s=2}^{3} X_s X^s)^{1/2},
\]

where \( \phi_M \) characterizes the magnetic flux. Then, the deformed Hamiltonian \((H_0)_{B,Q}\) is equal to Hamiltonian of the Aharonov-Bohm, i.e.

\[
(H_0)_{B,Q}\Psi = \frac{1}{2m} (P - eA)^2 \Psi,
\]

where \( A \) is the gauge field of the Aharonov-Bohm effect (see Equation 4.5). Furthermore, if the deformation parameters of the matrices \( B_1 \) and \( B_2 \) fulfill Equation (4.6), the physical systems described by the Hamiltonians \( H_{B_1,F} \) and \( H_{B_2,F} \) have the same interference pattern.

**Proof.** For the deformation of \( H_0 \) we use Proposition 3.1, with \( Q_j(X) \) as given in Equation (4.7),

\[
(H_0)_{B,Q}\Psi = -\frac{1}{2m} \left( P_j + i(BQ)^k[Q_k,P_j] \right) \left( P^j + i(BQ)^r[Q_r,P^j] \right) \Psi
\]

\[
= -\frac{1}{2m} \left( P_j + (BX)_j/(-\sum_{s=2}^{3} X_s X^s) \right) \left( P^j + (BX)^i/(-\sum_{r=2}^{3} X_r X^r) \right) \Psi,
\]

where in the last lines we used the skew-symmetry of \( B \) and commutator relation 3.6. Thus, by setting the deformation matrix \( B_{ij} = -{(e\phi_M/2\pi)}\epsilon_{ijk}e^k \), the gauge field \( A_i(x) \) induced by deformation is the gauge field of the AB effect for a homogeneous magnetic field in \( x \)-direction.
This is an interesting result. We were able to induce the AB-gauge field by deforming the free Hamiltonian with a vector-valued function of the coordinate operator. In this case the deformation parameter corresponds to the magnetic flux rather, as in the previous cases, to the magnetic field.

There are two ways to interpret these results. The first one lies in understanding deformation, in the case of QM, as the rightful minimal substitution. Thus the procedure sheds new light on quantum mechanical effects involving magnetic fields. The fields can be understood as the outcome of a deformation with vector-valued functions of the coordinate operator. The other way of understanding the result is the following. The coupling of an external magnetic field in QM is well understood and studied for various physical applications and models. Deformation on the other hand is a mathematical tool, rather than a procedure that generates physical effects. Hence, in these examples deformation of a QM system can be understood as the coupling of an external field. Thus, if the deformation goes hand in hand with Moyal-type spaces one sees in these examples that Moyal spaces correspond to ordinary spaces in the presence of an external field. By having this observation in mind it does not seem far fetched that certain deformations of spacetime correspond to gravitation. Let us describe in the next sections how Moyal-Weyl spaces arise in this context.

4.4 Physical Moyal-Weyl plane

To describe the circular motion of an electron in the lowest Landau level we define the so called guiding center coordinates \( Q \), [Eza08, Sza04]

\[
Q_i := X_i + \frac{1}{2} (B^{-1})_{ik} P^k,
\]

with matrix \( B_{ij} = -(e/2) \varepsilon_{ijk} B^k \). Note that the inverse corresponds to the non-degenerate sub-matrix of \( B_{ij} \). By using the CCR it becomes apparent that the guiding center coordinates span a three dimensional Moyal-Weyl plane, i.e.

\[
[Q_i, Q_j] = i (B^{-1})_{ij}.
\]

Thus, the Landau-effect is an example of a physical noncommutative space. Now can we generate these noncommuting coordinates by the deformation procedure warped convolutions? Yes we can!

**Lemma 4.3.** The deformed coordinate operators \( X^j_{\theta, P} \) given as (see Equation 3.12)

\[
X^j_{\theta, P} = X^j - \theta^{jr} P_r,
\]

satisfy the commutation relations of the Moyal-Weyl plane \( \mathbb{R}^3_{-2\theta} \),

\[
[X^i_{\theta, P}, X^j_{\theta, P}] = -2i \theta^{ij}.
\]

Moreover, let \(-2\theta^{ij}\) be \((B^{-1})^{ij}\) then the deformed coordinate operators \( X^i_{\theta, P} \) are equal to the guiding center coordinates given in Equation (4.8).

**Proof.** The commutator of the deformed coordinate operator is calculated by using the canonical commutation relations and the skew-symmetry of the deformation matrix \( \theta_{jk} \).

\[
[X^j_{\theta, P}, X^k_{\theta, P}] = [X^j - \theta^{jr} P_r, X^k - \theta^{kl} P_l]
\]

\[
= -\theta^{kl} [X^j, P_l] + \theta^{lj} [X^k, P_j]
\]

\[
= -2i \theta^{jk}.
\]

\( \square \)
Lemma 4.3 gives a well defined path to obtain an effective quantum plane by the deformation using warped convolutions. As we showed, the lemma follows from well understood physical models and ideas, which are in circulation in condensed matter field theory, for quite some time. In the example of the Landau problem one defines guiding center coordinates, which satisfy the commutator relations of the Moyal-Weyl Plane. The reader is cautioned to notice that the effective quantum plane obtained by the Landau problem is not merely an abstract construct but has the precise meaning, that the space coordinates can not be measured simultaneously. A more precise mathematical way to obtain this Moyal-Weyl plane is introduced in this work. We obtain the Landau problem by deforming the Hamiltonian of a free non-relativistic particle with the coordinate operator and by setting the deformation parameter equal to a magnetic field. Furthermore, we show that the noncommuting coordinates referred to as the guiding coordinates are obtained by deforming the coordinate operator, using the momentum operator. In our opinion, this method can be further used in the quantum field theoretical (QFT) approach to define an effective quantum plane.

4.5 Gravitomagnetism in QM

The emergence of gravitomagnetism in QM from deformation theory is one of the centerpieces of this work. Before we prove the emergence of these effects let us introduce some basic notations. We consider slowly varying weak gravitational fields with energy momentum tensor of ordinary matter (dust-like). In this description the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h$ is the small perturbation from the flat spacetime and the energy momentum tensor can be written as

$$T_{\mu\nu} = \rho u_\mu u_\nu,$$

where $u$ is the 4-velocity and $\rho$ the scalar density. For slowly varying fields, the linearized Einstein field equations can be described by Maxwell-like field equations given by

$$\Delta \phi = 4\pi G \rho, \quad \Delta h^j = -16\pi G \rho v^j, \quad 4\dot{\phi} - \nabla \cdot h^j = 0,$$

with the definitions of the potentials

$$\phi := h_{00}/2, \quad h^j := h^{0j},$$

where we used the Lorentz condition and the fact that the fields considered are slowly varying, i.e. $\dddot{\phi}, \dddot{h}_k$ and $\dddot{h}_k$ can be neglected, (see for example [AC10], [Wei72]). Analogously to the electromagnetic case the gravitoelectric field $g$ and the gravitomagnetic field $\Omega$ are both defined by the potentials as

$$g = -\nabla \phi, \quad \Omega = \nabla \times h.$$

There are a few important examples that can be considered in the context gravitomagnetism. One of them is example of the vector potential $h$ inside a hollow spinning sphere with radius $r_{hs}$ and spin $\omega$ that is given by, [Wei72, Eq. 9.4.35]

$$h(x) = x \times \Omega,$$  \hspace{1cm} (4.11)

where $\Omega = 2MG/r_{hs}$ is the constant gravitomagnetic field inside the hollow sphere.

Now in [AC10] the authors derived the non-relativistic Schrödinger equation for a particle that is minimally coupled to an external electromagnetic and gravitoelectromagnetic field. The equation is given by [AC10, Eq. 5.1],
4.5 Gravitomagnetism in QM

\[ H_{GM} \Psi = -\frac{1}{2m} (P - eA)_i (P - eA)_i \Psi - h^i (P - eA)_i \Psi, \]  
(4.12)

where we set the potential \( V, \phi \) equal to zero and neglected the term of second derivative in \( \Psi \). This can be done since the term is just a relativistic correction which for slowly moving bodies can be neglected.

By using the former definitions and results we are able to reproduce the case of a constant gravitomagnetic field by deformation.

**Lemma 4.4.** Let the deformation matrix \( B_{ij} \) be given as

\[ B_{ij} = m \varepsilon_{ijk} \Omega^k, \]  
(4.13)

where \( \Omega^k = (2GM/r_{hs}) \omega^k \) is a constant gravitomagnetic field for a hollow spinning sphere. Then, the deformed free Hamiltonian \( (H_0)_{B,x} \Psi \), becomes the Hamiltonian of a quantum mechanical particle minimally coupled to a constant gravitomagnetic field, i.e.

\[ (H_0)_{B,x} \Psi(x) = -\frac{1}{2m} (P_j + m h_j)(P^j + m h^j)\Psi \]

where the vector \( h_j = \varepsilon_{jkl} x^k \Omega^l \) represents the gravitomagnetic vector potential for a hollow spinning sphere (see Equation (4.11)).

**Proof.** The free deformed Hamiltonian \( (H_0)_{B,x} \Psi \) is given by

\[ (H_0)_{B,x} \Psi = -\frac{1}{2m} (P_j + B_{jk} X^k)(P^j + B^{jr} X_r)\Psi \]

where in the last line we set the deformation matrix \( B_{ij} = m \varepsilon_{ijk} (2GM/r_{hs}) \omega^k \) and we neglected second order terms since we work in the linear approximation.

This an important result, since this means that we obtain gravitational effects from a well-defined deformation procedure by simply adjusting the deformation constants accordingly. Thus, gravitomagnetism can be understood as the outcome of a deformation procedure. Moreover, the physical constant used as deformation parameter in the gravitomagnetic case is the gravitational constant \( G \). Since by setting the gravitational constant to zero, i.e. neglecting gravitational effects, the deformed Hamiltonian describing gravitomagnetic effects becomes the free Hamiltonian.

Next we use Proposition 2.1 of the deformation technique to obtain the electromagnetic and gravitomagnetic coupling.

**Proposition 4.2.** Let the deformation matrix \( B_{ij}^1 \) be given as

\[ B_{ij}^1 = m \varepsilon_{ijk} \Omega^k, \]

where \( \Omega^k = (2GM/r_{hs}) \omega^k \) is the constant gravitomagnetic field for a hollow spinning sphere and let the deformation matrix \( B_{ij}^2 \) be given as

\[ B_{ij}^2 = -(e/2) \varepsilon_{ijk} B^k, \]
where $B^k$ is a homogeneous magnetic field.

Then, the deformed free Hamiltonian $\left( (H_0)_{B^1, B^2, \mathbf{x}} \right)$ becomes the Hamiltonian $H_{GEM}$ (see Equation 4.12) of a quantum mechanical particle minimally coupled to a constant external magnetic and gravitomagnetic field, i.e.

$$ (H_0)_{B^1, B^2, \mathbf{x}} \Psi = H_{GEM} \Psi, \quad \Psi \in \mathcal{E}. $$

Proof. First of all by the virtue of Proposition 2.1 the deformed Hamiltonian satisfies

$$ (H_0)_{B^1, B^2, \mathbf{x}} \Psi = H_{GEM} \Psi, \quad \Psi \in \mathcal{E}. $$

Next we consider the free deformed Hamiltonian $(H_0)_{B^1, B^2, \mathbf{x}}$, (see Equation (4.2)).

$$ (H_0)_{B^1, B^2, \mathbf{x}} \Psi = \frac{1}{2m} \left( P + \left( (B^1 + B^2) \mathbf{x} \right) \right)_j \left( P + \left( (B^1 + B^2) \mathbf{x} \right) \right)^j \Psi $$

$$ = \frac{1}{2m} \left( P_j - eA_j + mh_j \right) \left( P^j - eA^j + mh^j \right) \Psi $$

$$ = \frac{1}{2m} \left( P - eA \right)_j \left( P - eA \right)^j \Psi - h_j \left( P - eA \right)^j \Psi + O(h^2), $$

where we set the deformation matrix $B^1_{ij} = m\epsilon_{ijk}(2GM/r^3)\omega^k$ and $B^2_{ij} = -\epsilon^2\epsilon_{ijk}B^k$ and $h_j$ is the gravitomagnetic vector potential given in Equation (4.11) for a hollow spinning sphere and $-A^j$ the magnetic vector potential given in the Landau quantization, (see Equation (4.1)).

From this result it becomes clear that in a quantum mechanical setting one can obtain electromagnetic and gravitomagnetic effects by a deformation procedure. In the framework of deformation these effects simply correspond to certain deformation parameters that in turn are given by physical constants. One should also note that we obtained the Hamiltonian of a quantum mechanical system that is coupled to an external magnetic and gravitomagnetic field by deformation, rather than by advanced calculations and considerations as done in [AC10].

4.6 Lense-Thirring Precession

Another important gravitomagnetic effect is known under the name of Lense–Thirring precession. The effect is a general-relativistic correction to the precession of a gyroscope outside a massive stationary spinning sphere. The vector potential for such gravitomagnetic field is given as

$$ h = -(2GI/r^3)\mathbf{x} \times \omega, $$

(4.14)

where $I$ is the moment of inertia of the sphere and $r = |\mathbf{x}|$ the radius.

As for the constant gravitomagnetic field, we are also able to produce the vector potential of the Lense–Thirring effect.

**Proposition 4.3.** Let the deformation matrix $B_{ij}$ and the operator $Q_j$ be given as

$$ B_{ij} = m\epsilon_{ijk}\Omega^k, \quad Q_j(\mathbf{x}) = X_j/|\mathbf{x}|^{3/2}. $$

(4.15)

where $\Omega^k = (2GI)\omega^k$ and $I$ is the moment of inertia of a spinning sphere. Then, the deformed free Hamiltonian $(H_0)_{B, Q}$ becomes the Hamiltonian of a quantum mechanical particle minimally coupled to the gravitomagnetic field of the Lense-Thirring
effect, i.e.
\[
(H_0)_{B,Q} \Psi = -\frac{1}{2m} \left( P_j + (BX)_j / |X|^3 \right) \left( P^j + (BX)^j / |X|^3 \right) \Psi, \quad \Psi \in \mathcal{E}
\]
\[= H_0 \Psi - h_j P^j \Psi + \mathcal{O}(\hbar^2),\]
where the vector potential induced by deformation is the gauge field of the Lense-Thirring effect, i.e. $h_j = m \varepsilon_{ijk}(2GI) \omega^k X^k / |X|^3$.

**Proof.** To prove this proposition we use the spectral measure representation. The deformation of $H_0$ is then given as follows,
\[
(H_0)_{B,Q} \Psi = -\frac{1}{2m} \left( P_j + (BQ)_j \right)(Q^j + (BQ)^j) \Psi
\]
\[= -\frac{1}{2m} \left( P_j + (BX)_j / |X|^3 \right) \left( P^j + (BX)^j / |X|^3 \right) \Psi
\]
\[= -\frac{1}{2m} \left( P_j + m h_j \right) \left( P^j + m h^j \right) \Psi
\]
\[= H_0 \Psi - h_j P^j \Psi + \mathcal{O}(\hbar^2),\]
where in the last lines we used the skew-symmetry of $B$ and the commutator relation 3.6.

Next, we use the deformation technique to obtain the electromagnetic and gravitomagnetic coupling in the case of the Lense-Thirring effect. The effects emerge by a double deformation where once we use the coordinate operator and after that the operator-valued vector $Q_j(X)$. Note that the order of the deformation is irrelevant, since the two operators commute.

**Remark 4.1.** The proof that the deformation with two different operators is well-defined, is equivalent to proving Proposition 3.1, where one replaces the free Hamiltonian in Inequality (3.10) with $(H_0)_{B^2,X}$. It then follows that for $\Phi \in \mathcal{S}(\mathbb{R}^3)$, the expression $||(H_0)_{B^2,X}\Phi||$ is finite.

**Proposition 4.4.** Let the deformation matrix $B^1_{ij}$ be given as
\[
B^1_{ij} = m \varepsilon_{ijk} \Omega^k,
\]
where $\Omega^k = (2GI) \omega^k$ and let the deformation matrix $B^2_{ij}$ be given as
\[
B^2_{ij} = -(e/2) \varepsilon_{ijk} B^k,
\]
where $B^k$ is a homogeneous magnetic field. Moreover, let the operator $Q_j(X)$ be given by
\[
Q_j(X) = X_j / |X|^{3/2}.
\]
Then, the deformed free Hamiltonian $((H_0)_{B^2,X})_{B^1,Q}$ becomes the Hamiltonian $H_{GEM}$ of a quantum mechanical particle minimally coupled to a constant external magnetic and the gravitomagnetic field of the Lense-Thirring effect, i.e.
\[
((H_0)_{B^2,X})_{B^1,Q} \Psi = H_{GEM} \Psi, \quad \Psi \in \mathcal{E}.
\]

**Proof.** The free deformed Hamiltonian $(H_0)_{B^2,X}$ is given by, (see Equation (4.2)).
\[
(H_0)_{B^2,X} \Psi = -\frac{1}{2m} \left( P + (B^2 X) \right) \left( P + (B^2 X) \right)^j \Psi
\]
Due to the commutativity of the coordinate operators, deformations with $Q_j(X)$ do not influence the gauge field $(B^2 X)$ and vice versa, i.e. $((H_0)_{B^2,X})_{B^1,Q} = \ldots$
4.7 Gravitomagnetic Zeemaneffect

Thus, after choosing the deformation parameters as stated in Equations (4.16) and (4.17) we obtain for the deformed free Hamiltonian,

\[ ((H_0)_{B_1}, Q_{B_2})_{B_2 \cdot X} \]

\[ = -\frac{1}{2m} (P - eA + m h_j) (P - eA + m h_j) \Psi \]

\[ = -\frac{1}{2m} (P - eA) (P - eA) \Psi - h_j (P - eA) \Psi + O(h^2), \]

where \( h_j \) is the gravitomagnetic gauge field of the Lense-Thirring effect (see Equation (4.14)) and \(-A^j\) the magnetic vector potential given in the Landau quantization, (see Equation (4.1)).

4.7 Gravitomagnetic Zeemaneffect

Similar to the magnetic case, where the Zeemaneffect emerged by deforming the hydrogen Hamiltonian with the same deformation matrix used in the Landau quantization, we proceed in the gravitomagnetic case. Thus, we are able to predict a gravitomagnetic Zeemaneffect by deforming the hydrogen atom and using the constant gravitomagnetic deformation matrix.

**Lemma 4.5.** Let the deformation matrix \( B_{ij} \) be given as,

\[ B_{ij} = m \epsilon_{ijk} \Omega^k, \]

where \( \Omega^k = (2GM/r_{hs}) \omega^k \) is the constant gravitomagnetic field for a hollow spinning sphere. Then, the deformed Hamiltonian of the hydrogen atom, denoted by \((H^A)_{B_1 \cdot X}\), becomes the Hamiltonian of the gravitomagnetic Zeemaneffect, i.e.

\[ (H^A)_{B_1 \cdot X} \Psi = -\frac{1}{2m} (P_j + m \epsilon_{jkl} \Omega^l X^k)(P^j + m \epsilon^{jrs} \Omega_s X_r) \Psi + \frac{e^2}{|X|} \Psi, \quad \Psi \in \mathcal{E} \]

\[ = -\frac{1}{2m} (P_j + m h_j)(P^j + m h^j) \Psi + \frac{e^2}{|X|} \Psi \]

\[ = H_0 \Psi - h_j P^j \Psi + \frac{e^2}{|X|} \Psi + O(h^2). \]

**Proof.** The only difference to the proof of Lemma 4.2 consists in the choice of the deformation matrix, i.e. the proof is equivalent.

Analogously to the magnetic case, the presence of a constant gravitomagnetic field will split the spectral lines of the hydrogen atom. In this case the splitting depends on the strength of the gravitomagnetic field. This phenomenon is the **gravitomagnetic Zeemaneffect**, [Mas00]. Note that the linear approximation works just fine, since the quadratic terms of the gauge field are already neglected in the magnetic Zeemaneffect, [Str02].

In the next proposition we couple the two constant forces by a double deformation.

**Proposition 4.5.** Let the deformation matrix \( B^1_{ij} \) be given as

\[ B^1_{ij} = m \epsilon_{ijk} \Omega^k, \]

where \( \Omega^k = (2GM/r_{hs}) \omega^k \) is the constant gravitomagnetic field for a hollow spinning sphere and let the deformation matrix \( B^2_{ij} \) be given as

\[ B^2_{ij} = -(e/2) \epsilon_{ijk} B^k, \]
where \( B^k \) is a homogeneous magnetic field.

Then, the deformed Hamiltonian of the hydrogen atom, \( (H^A)_{B^1, X} \), becomes the Hamiltonian of the Zeeman effect generated by a constant external magnetic and gravitomagnetic field, i.e.

\[
(H^A)_{B^1 + B^2, X} \Psi = H_{\text{GEM}} \Psi, \quad \Psi \in \mathcal{E}.
\]

**Proof.** Since the deformation with the coordinate operator commutes with the potential term of the hydrogen atom, the proof is analogous to the proof of Proposition 4.4.

### 4.8 Arbitrary static gauge field

By only assuming the principle of Galilei-invariance the author in [Jau64] succeeded in deriving the minimally coupled Hamiltonian plus a potential. Thus by demanding that our deformed Hamiltonian respects the Galilei-invariance, we have to add a potential. This is justified since we showed that the deformation of a free Hamiltonian induces electromagnetism and gravitomagnetism. Moreover, in [Jau64] it was shown that the gauge field and the potential can only depend on the coordinates. Therefore, our deformation covers the whole range of abelian gauge fields, since we can induce such fields by choosing functions of the coordinate operator to obtain an arbitrary gauge field. This fact is used in the next sections to induce a variety of physical effects.

In the next proposition we show the importance of adding a potential to the Hamiltonian and for this purpose we need the four-momentum given as

\[
P_\mu = (H_0, P_i) = \left( -P_k P^k / (2m) + g \phi(X), P_i \right),
\]

where \( \phi(X) \) is the electromagnetic potential \( \phi_E \) or the gravitoelectromagnetic potential \( -\phi_G \). Moreover, \( g \) is a coupling constant given by \( e \) in the electromagnetic case and by \( -m \) in the gravitoelectromagnetic case.

**Proposition 4.6.** Let the gauge field induced by deformation of the Hamiltonian \( (H_0)_{B, Q} \) (see Proposition 3.1) be defined as

\[
-A_r(X) := (BQ(X))_k \partial_r Q^k(X),
\]

where \( A \) is the electromagnetic or the gravitoelectromagnetic vector potential. Then, the commutator of the deformed momentum vectors gives the spatial part of the field strength tensor \( F_{ij} \),

\[
[p^B_{i, Q}, p^B_{j, Q}] = -ig F_{ij}(X).
\]

Furthermore, the commutator of the deformed Hamiltonian with the deformed momentum gives the Lorentz force \( F^L \), i.e.

\[
[p^B_{0, Q}, p^B_{i, Q}] = -g [\phi(X), P_j] - i \frac{g}{m} F_{jk}(X) P^k_{B, Q} = iF^L_{ij}.
\]

Moreover, by using the Jacobi identities for the commutator relations between the deformed momentum and Hamiltonian operators the homogeneous Maxwell-equations emerge.

**Proof.** According to Proposition 3.1, the deformation of \( H_0 \) by an operator \( Q \) is given as follows

\[
(H_0)_{B, Q} \Psi = -\frac{1}{2m} \left( P_j + i(BQ)_s [Q^s, P_j] \right) \left( P^j + i(BQ)_r [Q^r, P^j] \right) \Psi + g \phi(X) \Psi
\]

\[
= \frac{1}{2m} \left( P - g A(X) \right)^2 \Psi + g \phi(X) \Psi,
\]
where we used the fact that the potential can only be a function of the coordinate operator and thus remains invariant under deformation. Since in the former propositions we identified deformations induced by the coordinate operator with the gauge field $-g A$, the induced term in this deformation can be identified with a general static gauge field. Next, we calculate Commutator (4.19). The deformed momentum operator is given in Proposition 3.11 as

$$P^j_{B,Q} \Psi = (P^j + i(BQ)_j/[Q^a,P^j]) \Psi$$

$$= (P^j - g A^j(X)) \Psi,$$

where in the last lines we identified the gauge field by Equation (4.18). Now by using the commutator relations $[X_i,X_j] = 0$ and the fact that the commutator $[Q^a,P^j]$ is again only a function of the coordinate operator, we obtain the following commutator relations for the deformed momentum operator

$$[P^k_{B,Q},P^l_{B,Q}] = -ig [P^k_{B,Q} F^{ij}_{B}(X)\leftrightarrow k l] = -ig F^{kl}(X).$$

Now we can also rewrite the deformed Hamiltonian $(H_0)_{B,Q}$ as

$$(H_0)_{B,Q} = -(1/2m) \left( P^j_{B,Q} P^j_{B,Q} \right) + g \phi(X).$$

This form of the deformed Hamiltonian simplifies the calculation of Commutator (4.20).

$$[(H_0)_{B,Q},P^k_{B,Q}] = -(1/2m) \left( P^j_{B,Q} P^j_{B,Q} \right), P^k_{B,Q} + g [\phi(X),P^k]$$

$$= i \frac{g}{m} P^j_{B,Q} F^{ij}_{B}(X) - ig \partial^k \phi(X)$$

$$= ig \left( E^k(X) - \varepsilon^{kjl} V^j_{B,Q} B_l(X) \right),$$

In the last lines we used the commutator relation $[P^j_{B,Q},F^{ij}_{B}(X)] = 0$ and the Heisenberg-equation to identify the velocity operator with the deformed momentum. Moreover, the fields $E$ and $B$ are the electromagnetic or the gravitoelectromagnetic fields, depending on the considered case. It is not surprising that the Lorentz force is obtained by calculating the commutator between the deformed Hamiltonian and the momentum, since it gives the equations of motion for the deformed system. This in turn is properly identified with a particle coupled to an electromagnetic or gravitoelectromagnetic force.

Next, we use the Jacobi identities for the commutators of the deformed momentum operator $F^i_j$ to obtain the homogeneous Maxwell-equations. From the Jacobi identity for the spatial part we have

$$[P^k_{B,Q},[P^j_{B,Q},P^j_{B,Q}]] + \text{cyclic} = -ig [P^k_{B,Q},F_{ij}(X)] + \text{cyclic}$$

$$= g \partial_k F_{ij}(X) + \text{cyclic}$$

$$= 0.$$

The last equation is the relativistic expression for the spatial part of the homogeneous Maxwell-equations. To obtain the homogeneous Maxwell-equations involving $F_{0j} = E_j$ we look at the other Jacobi identity of the deformed momentum, i.e.

$$[P^k_{B,Q},[P^j_{B,Q},P^j_{B,Q}]] + [P^j_{B,Q},P^k_{B,Q}] - i \leftrightarrow j = 0. \quad (4.21)$$
Let us take a look at the first term,

\[
[P^B_0, Q^i] [P^B_0, Q^j] = -ig\partial_0 F_{ij}(X) = -g\partial_0 F_{ij}(X) - \frac{g}{m} f^B_{ik} \partial_k F_{ij}(X),
\]

where we used the Heisenberg-equation. The other two terms in Equation (4.21) give

\[
-i[P^B_i, F^L_j] \leftrightarrow j = -g (\partial_i E_j(X) - \partial_j E_i(X)) - \frac{g}{m} P^B_k (\partial_i F_{jk}(X) - \partial_j F_{ik}(X)).
\]

By summing the two terms and using the spatial part of the homogeneous Maxwell-equations we obtain

\[
[P^B_0, Q^i, P^B_j] + \text{cyclic} = -g (\partial_i E_j(X) - \partial_j E_i(X)) - g\partial_0 F_{ij}(X).
\]

ometers and commutators of the velocity operators, one obtains the electromagnetic force. Furthermore, by using the Jacobi identities the Maxwell equations follow. Therefore, in a sense the deformations with warped convolutions reproduce the result of Feynman in a more sophisticated language, and moreover, the technique gives concrete representations of the operators that generate the electromagnetic and gravitoelectromagnetic fields. Thus, by the virtue of the deformation technique we have a deeper understanding of the surprising result of Feynman.

A crucial point is implied in the last proposition. The linear field equations of general relativity or the Maxwell equations emerge from a well-defined deformation procedure. The emergence is owed to the Jacobi identities. This observation gives an insightful hint how to receive substantial field equations from a deformation procedure.

### 4.9 Gravitomagnetic Moyal-Weyl plane

By using the deformations techniques we are able to understand how to generate the Landau quantization. Thus, by using the same procedures and by setting the deformation parameter equal to a constant gravitomagnetic field (see Lemma 4.4) we also obtain a Landau quantization in the gravitomagnetic case.

Analogously to the Landau quantization in the magnetic case, we can solve the eigenvalue equation of the deformed Hamiltonian \((H_0)_{B,X}\) with deformation matrix \(B_{ij} = m \varepsilon_{ijk} \Omega e^k\), i.e. with a constant homogeneous gravitomagnetic field in \(x\)-direction. The eigenvalue problem can be solved by diagonalizing the Hamiltonian and we obtain the following energy eigenvalues

\[
E_{B,n} = \frac{p_1^2}{2m} + \left( n + \frac{1}{2} \right) \omega_B, \quad p_1 \in \mathbb{R}, \quad n \in \mathbb{N},
\]

where the frequency of this harmonic oscillator is given by \(\omega_B = 2\Omega\). Therefore, quantum mechanical particles in a constant gravitomagnetic field can only occupy orbits with discrete energy values. This effect is the gravitomagnetic analogue of the Landau quantization in the magnetic case.
In Section 4.4 we identified the noncommutative coordinates of the Landau-quantization with the deformed coordinate operator. In the same manner we can obtain a physical Moyal-Weyl plane for the constant gravitomagnetic case. For this purpose, we set the deformation matrix of $X^A P$ equal to the inverse of a constant gravitomagnetic field times the coupling constant $m$. The deformed coordinate operators $X^{B^{-1}}_i P$ are then equal to the guiding center coordinates of an electron in the lowest Landau level and are given by

$$X^{B^{-1}}_i P = X_i + \frac{1}{2}(B^{-1})_{ik} P^k,$$

with commutator relations

$$[X^{B^{-1}}_i P, X^{B^{-1}}_j P] = i(B^{-1})_{ij},$$

and deformation matrix $B_{ij} = m \epsilon_{ijk} \Omega^k$. From the commutator relations we have the following uncertainty relations

$$\langle \Delta X^{B^{-1}}_2 P \rangle \langle \Delta X^{B^{-1}}_3 P \rangle \geq \hbar / (m \Omega).$$

These uncertainty relations have the precise meaning that coordinates of an electron can not be measured more accurately than the area $2\pi \hbar / (m \Omega)$. This is a physical effect that we predicted from a deformation procedure and may be experimentally verified.

## 5 Conclusion and Outlook

We obtained a variety of physical effects, in a QM context, containing electromagnetism and gravitomagnetism. These effects were generated by the deformation procedure warped convolutions. Thus, in this sense those two fundamental forces can be understood as deformations of free theories. The fundamental deformation parameters, for those forces, are given by the elementary electric charge $e$ and by the gravitational constant $G$. Therefore, not only $\hbar$ and $c$ can be used to deform the classical case (Galilei group) but also $e$ and $G$ play the role of deformation parameters responsible for electromagnetism and gravitomagnetism.

The deformation also shed a new light on the dynamics of a quantum mechanical particle in the presence of electromagnetic and gravitomagnetic forces. Namely, it gives a systematic derivation of a non-relativistic Hamiltonian in the presence of electromagnetic and gravitomagnetic effects.

Another striking implication of the deformation procedure is the deduction of a physical Moyal-Weyl plane. This plane is generated from the gravitomagnetic field times the mass and thus the strength of noncommutativity of the coordinates, in the lowest Landau level, is given by the inverse of the constant $m \Omega / \hbar$. This effect was purely deduced from deformation and could be one of the first effects that is theoretically predicted by deformation and verified experimentally. This would be a major physical breakthrough for deformation theory.

To obtain electric and gravitoelectric fields in the framework of deformation, we would have to extend the definition of warped convolutions. For example, it is not possible to obtain the Stark effect from deformation of the free Hamiltonian with warped convolutions. The reason herein lies in the fact, that the deformation leaves the spectrum of the operator invariant. Since the free Hamiltonian has a positive spectrum and the Hamiltonian of the Stark effect has the whole real line as spectrum,
a deformation respecting spectrum conditions can not reproduce such an effect.

Another line of work involves the extension of warped convolutions to a non-abelian setting. If this succeeds, we would be able to reproduce the weak and strong interaction as deformations. In this context, it seems intuitive to lift such deformations to a QFT setting. Thus, recasting the fundamental forces as deformations and most likely simplifying calculations involving interactions.

These and many other lines of work, in deformation theory, are to this date open and provide a broad, interesting and exciting area of research.

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References

[AC10] Ronald J. Adler and Pisin Chen. Gravitomagnetism in Quantum Mechanics. Phys. Rev., D82:025004, 2010.

[Ala] Sabina Alazzawi. Deformations of Fermionic Quantum Field Theories and Integrable Models. Lett. Math. Phys. 103 (2013) 37-58.

[Alb12] Much Albert. Wedge-local quantum fields on a nonconstant noncommutative spacetime. Journal of Mathematical Physics, 53(8):082303, August 2012.

[And13] Andreas Andersson. Operator deformations in quantum measurement theory. Letters in Mathematical Physics, pages 1–16, November 2013.

[Bal98] L.E. Ballentine. Quantum mechanics: a modern development. World Scientific Publishing Company Incorporated, 1998.

[BEH08] J. Blank, P. Exner, and M. Havlíček. Hilbert Space Operators in Quantum Physics. Springer, 2008.

[Ber00] R.A. Bertlmann. Anomalies in quantum field theory. International Series of Monographs on Physics. Oxford University Press, 2000.

[BLS11] Detlev Buchholz, Gandalf Lechner, and Stephen J. Summers. Warped Convolutions, Rieffel Deformations and the Construction of Quantum Field Theories. Commun.Math.Phys., 304:95–123, 2011.

[BS] Detlev Buchholz and Stephen J. Summers. Warped Convolutions: A Novel Tool in the Construction of Quantum Field Theories. Quantum Field Theory and Beyond, pp. 107–121. World Scientific, Singapore.

[Dys90] F. J. Dyson. Feynman’s proof of the Maxwell equations. Am. J. Phys., 58:209–211, 1990.

[Eza08] Z.F. Ezawa. Quantum Hall Effects: Field Theoretical Approach and Related Topics. World Scientific Publishing Company Incorporated, 2008.

[GL07] Harald Grosse and Gandalf Lechner. Wedge-Local Quantum Fields and Noncommutative Minkowski Space. JHEP, 0711:012, 2007.
[GL08] Harald Grosse and Gandalf Lechner. Noncommutative Deformations of Wightman Quantum Field Theories. JHEP, 0809:131, 2008.

[GS68] I.M. Gel’fand and G.E. Shilov. Generalized functions. Generalized Functions. Academic Press, 1968.

[Hör04] L. Hörmander. The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients. Springer, 2004.

[Jau64] J.M. Jauch. Gauge invariance as a consequence of Galilei-invariance for elementary particles. Hel. Phys. Acta, 37:284–292, 1964.

[JM67] Lévy-Leblond Jean-Marc. Nonrelativistic Particles and Wave Equations. Commu. math. Phys., 6:286–311, 1967.

[Lec12] Gandalf Lechner. Deformations of quantum field theories and integrable models. Commun.Math.Phys., 312:265–302, 2012.

[LST13] Gandalf Lechner, Jan Schlemmer, and Yoh Tanimoto. On the equivalence of two deformation schemes in quantum field theory. Lett.Math.Phys., 103:421–437, 2013.

[Mas00] Bahram Mashhoon. Gravitational couplings of intrinsic spin. Class. Quant. Grav., 17:2399–2410, 2000.

[MM11] Eric Morfa-Morales. Deformations of quantum field theories on de Sitter spacetime. J.Math.Phys., 52:102304, 2011.

[Rie93] M.A. Rieffel. Deformation quantization for actions of $\mathbb{R}^d$. Memoirs A.M.S., 506, 1993.

[RS75] M. Reed and B. Simon. Methods of Modern Mathematical Physics. 1. Functional Analysis. Gulf Professional Publishing, 1975.

[Str02] N. Straumann. Quantenmechanik: ein Grundkurs über nichtrelativistische Quantentheorie ; mit 2 Tabellen. Springer-Lehrbuch. Springer Verlag, 2002.

[Sza04] Richard J. Szabo. Magnetic backgrounds and noncommutative field theory. Int.J.Mod.Phys., A19:1837–1862, 2004.

[Tes01] G. Teschl. Mathematical Methods in Quantum Mechanics. Amer. Math. Soc., 99, 2001.

[Thi81] W.E. Thirring. Quantum mechanics of atoms and molecules. A Course in Mathematical Physics. Springer-Verlag, 1981.

[Wei72] S. Weinberg. Gravitation and cosmology: principles and applications of the general theory of relativity. Wiley, 1972.