Abstract

We give an approximation algorithm for MaxCut and provide guarantees on the average fraction of edges cut on $d$-regular graphs of girth $\geq 2k$. For every $d \geq 3$ and $k \geq 4$, our approximation guarantees are better than those of all other classical and quantum algorithms known to the authors. Our algorithm constructs an explicit vector solution to the standard semidefinite relaxation of MaxCut and applies hyperplane rounding. It may be viewed as a simplification of the previously best known technique, which approximates Gaussian wave processes on the infinite $d$-regular tree.
1 Introduction

Finding the maximum cut in a graph is a canonical NP-hard discrete optimization problem, and one of the simplest constraint satisfaction problems. However, to date, MaxCut does not have optimal, efficient approximation algorithms on most families of graphs. Approximation algorithms are typically measured by one of two metrics: the performance (or cut fraction), which is proportional to the cut size; or the approximation ratio, which is the ratio of cut size and optimal cut size (or cut fraction and optimal cut fraction).

Arguably the most famous strategy to approximate MaxCut is to solve its semidefinite programming (SDP) relaxation and round the solution to get a cut (Goemans and Williamson, 1995). Under the Unique Games Conjecture (UGC), this algorithm has the optimal worst-case approximation ratio $\alpha > 0.878$ (Khot, 2002; Khot, Kindler, Mossel, and O’Donnell, 2007; O’Donnell and Wu, 2008). In other words, the cut given is of size at least $\alpha$ times the optimal cut size on every graph. However, many graphs, such as large random graphs, have an optimal cut fraction less than $1/(2\alpha)$ (Dembo, Montanari, and Sen, 2017). In such cases, as an $\alpha$-approximation, Goemans-Williamson only guarantees a cut fraction less than 1/2. In contrast, randomly assigning vertices to one of two partitions has an expected cut fraction of 1/2. This is why considering the cut fraction of an algorithm is relevant: for example, one can show that Goemans-Williamson has cut fraction 1/2, just as random assignment.

Even if UGC is true, we do not have approximation algorithms on regular graphs that have an optimal worst-case approximation ratio (Halperin, Livnat, and Zwick, 2004). Regular graphs also admit straightforward bounds on cut fraction. Many approximation algorithms (including ours) bound the maximum cut size as a function of basic graph parameters, including the graph degree and number of edges (Erdős, 1967; Shearer, 1992). One example is a local threshold algorithm by Hirvonen, Rybicki, Schmid, and Suomela (2017), which first randomly assigns vertices to one of two partitions; then, switches the label of each vertex if the number of cut edges adjacent to it does not meet a threshold. Although the algorithm is simple, analyzing it is not straightforward; the optimal threshold is not known a priori, so it must be numerically tuned for each degree value $d$ to maximize performance.

Quantum algorithms, especially the Quantum Approximate Optimization Algorithm (QAOA), may be competitive with classical algorithms on many optimization problems (Farhi, Goldstone, and Gutmann, 2014, 2015). The QAOA includes a parameter $p$ corresponding to its depth; its performance under optimal parameter settings, as well as its complexity, monotonically increases with $p$. Although hardware limitations of quantum computers prevent the use of the QAOA at high depth (Harrigan, Sung, Neeley, et al., 2021), the QAOA may be competitive even at lower depths. For example, when the QAOA was proposed, the depth-1 version had the best known cut fraction guarantee on triangle-free regular graphs (Wang, Hadfield, Jiang, and Rieffel, 2018; Ryan-Anderson, 2018). However, Hastings (2019) numerically showed that the local threshold algorithm (Hirvonen et al., 2017) outperforms the depth-1 QAOA. At higher $p$ and progressively higher girth, both algorithms perform better; however, a practitioner has to tune $O(p)$ parameters to perform optimally, and few provable performance bounds are known (Farhi et al., 2014; Marwaha, 2021).

Another style of approximation algorithm uses invariant Gaussian processes (Csóka, Gerencsér, Harangi, and Virág, 2015; Csóka, 2016; Lyons, 2017; de Boor, 2019; Barak and Marwaha, 2021). These algorithms are not local, but they generally perform better than the threshold and quantum
algorithms described previously. First, one constructs an invariant Gaussian process satisfying an eigenvector equation on an infinite $d$-regular tree (more precisely, a Bethe lattice). This Gaussian wave process can then be approximated by a factor of an i.i.d. process assuming an eigenvalue condition is met. One can use this to derive an independent set (Csóka et al., 2015) or a cut (Csóka, 2016; Lyons, 2017). As $d$ and the girth go to infinity, these algorithms approach a cut fraction of $1/2 + 2/(\pi \sqrt{d})$. However, the analysis is quite technical; it involves approximating eigenvectors of the adjacency matrix of the infinite $d$-regular tree. In contrast, our approach only requires the eigenvectors of the adjacency matrix of a finite path, which are well understood and explicitly known in certain cases. In addition, the algorithms of Csóka et al. (2015) and Lyons (2017) were designed to prove existential results, and there is no claim of computational efficiency. The algorithm in Barak and Marwaha (2021) has a simpler form, but its analysis is complicated at finite degree; it is only exactly bounded in the large-degree limit. The algorithm of de Boor (2019) is the closest in spirit to ours; however, ours offers better performance.

**Our contributions.** We design an algorithm that retains desirable features of the above approaches but is simple to implement and analyze. As with Lyons (2017), our algorithm’s performance bound is a function of the degree $d$ and girth $\geq 2k$, and we do not require either to be large. In addition to a simplified approach, for $d \geq 3$ and $k \geq 4$, our approximation guarantees are better than those of all algorithms we are aware of.

We employ hyperplane rounding on a feasible SDP solution; however, instead of solving the SDP, we construct an explicit vector solution. The vectors collectively depend on $k$ parameters, and we optimize the SDP objective subject to these parameters. This optimization problem is considerably simpler than solving the SDP and amounts to finding a minimum eigenvector of the adjacency matrix of a weighted path on $k$ vertices. Moreover, we give a closed-form approximate solution which also outperforms all previous algorithms.

Our approach is inspired by the work of Carlson, Kolla, Li, Mani, Sudakov, and Trevisan (2020), who construct and round explicit vector solutions to find large cuts in graphs with few triangles and in $K_4$-free graphs. Furthermore, it turns out that our algorithm may be viewed as an simplification of the Gaussian wave process, as discussed in Section 3. Its performance, as a function of $d$ when $k$ goes to infinity, matches that of other algorithms based on invariant Gaussian processes.

We motivate our discussion with a comparison to the Goemans-Williamson algorithm and then explicitly define our algorithm and establish a lower bound on its performance. We then connect our algorithm to the Gaussian wave process, and prove that our bound is higher than several other performance guarantees.

### 1.1 A motivating example

Given a graph $G = (V, E)$, where $n = |V|$ and $m = |E|$, one can describe the unweighted\(^1\) MaxCut problem by the following binary quadratic program:

$$\max \frac{1}{2} \sum_{(i,j) \in E} (1 - z_i \cdot z_j) \text{ such that } z_i \in \{-1, 1\}, \forall i \in V$$

\(^1\)Our results readily generalize to the nonnegatively weighted case, replacing $m$ with the sum of all edge weights.
This can be relaxed to the following semidefinite program (SDP), where $S^n$ is the unit sphere in $n$-dimensional space:

$$\max \frac{1}{2} \sum_{(i,j) \in E} (1 - \mathbf{v}_i \cdot \mathbf{v}_j) \text{ such that } \mathbf{v}_i \in S^n, \forall i \in V$$

Goemans and Williamson (1995) show that given any feasible solution to the SDP (i.e., any set of unit vectors, $\{\mathbf{v}_i\}_{i=1}^n$ where $\mathbf{v}_i \in S^n$) and a random standard Gaussian vector $\mathbf{r} \in S^n$, the cut given by $A = \{i \mid \mathbf{v}_i \cdot \mathbf{r} > 0\}$ and $\bar{A} = V/A$ has a lower bound on its expected cut size (denoted by $E[W]$):

$$E[W] = \frac{1}{\pi} \sum_{(i,j) \in E} \arccos(\mathbf{v}_i \cdot \mathbf{v}_j) \geq \alpha \cdot \frac{1}{2} \sum_{(i,j) \in E} (1 - \mathbf{v}_i \cdot \mathbf{v}_j)$$

This scheme is known as **hyperplane rounding**: the partition of vertex $i$ is set by the sign of $\mathbf{r} \cdot \mathbf{v}_i$.

Since the optimal solution to the relaxed program has a value at least the optimal cut size, this algorithm guarantees a cut with approximation ratio $\alpha$. The cut fraction is not as easy to bound because it depends on the values of the inner products $\mathbf{v}_i \cdot \mathbf{v}_j$.

It turns out that the SDP relaxation can be used to give provably large cuts when the graph is locally tree-like. We use the Goemans-Williamson rounding scheme; however, instead of actually solving the SDP, we explicitly construct a vector solution where $\mathbf{v}_i \cdot \mathbf{v}_j$ is same for all edges and then round to obtain a cut. Consider the following example from Carlson et al. (2020), which is the inspiration for this paper. On a $d$-regular, triangle-free graph, define for every $i, j \in V$:

$$v_{ij} = \begin{cases} \frac{1}{\sqrt{2}} & i = j \\ \frac{1}{\sqrt{2d}} & (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{v}_i = [v_{i1}, ..., v_{in}]^T$. Since $\mathbf{v}_i \in S^n$, the set $\{\mathbf{v}_i\}_{i=1}^n$ is a feasible solution to the above semidefinite program. Note that $\mathbf{v}_i \cdot \mathbf{v}_j$ is the same for all edges $(i, j) \in E$. Applying the Goemans-Williamson hyperplane rounding scheme gives a cut with expected size

$$E[W] = \frac{1}{\pi} \sum_{(i,j) \in E} \arccos(\mathbf{v}_i \cdot \mathbf{v}_j) = \frac{m}{\pi} \arccos\left(\frac{-1}{\sqrt{d}}\right).$$

This algorithm outperforms depth-1 QAOA at every value of $d$ (Wang et al., 2018). Although it is not as good as the threshold algorithm from Hirvonen et al. (2017), we can generalize the vector assignment shown above for graphs with progressively higher girth. On each of these graphs, the explicit vector solution has a performance guarantee above that of Lyons (2017), the depth-1 and depth-2 QAOA, and the threshold algorithm and its depth-2 extension in Marwaha (2021).

# 2 Vector assignments to the semidefinite program

Consider a $d$-regular graph, $G = (V, E)$, where $n = |V|$ and $m = |E|$, with graph girth at least $2k$. For each vertex, we define a vector $\mathbf{v}_i$ for all $i \in V$, where $\mathbf{v}_i = [v_{i1}, ..., v_{in}]^T$, and

$$v_{ij} = \begin{cases} \alpha_0 & i = j \\ \alpha_\ell & \text{dist}(i,j) = \ell < k \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{v}_i = [v_{i1}, ..., v_{in}]^T$. Since $\mathbf{v}_i \in S^n$, the set $\{\mathbf{v}_i\}_{i=1}^n$ is a feasible solution to the above semidefinite program. Note that $\mathbf{v}_i \cdot \mathbf{v}_j$ is the same for all edges $(i, j) \in E$. Applying the Goemans-Williamson hyperplane rounding scheme gives a cut with expected size

$$E[W] = \frac{1}{\pi} \sum_{(i,j) \in E} \arccos(\mathbf{v}_i \cdot \mathbf{v}_j) = \frac{m}{\pi} \arccos\left(\frac{-1}{\sqrt{d}}\right).$$

This algorithm outperforms depth-1 QAOA at every value of $d$ (Wang et al., 2018). Although it is not as good as the threshold algorithm from Hirvonen et al. (2017), we can generalize the vector assignment shown above for graphs with progressively higher girth. On each of these graphs, the explicit vector solution has a performance guarantee above that of Lyons (2017), the depth-1 and depth-2 QAOA, and the threshold algorithm and its depth-2 extension in Marwaha (2021).
Here, \( \text{dist}(i, j) \) is the length of the shortest path between \( i \) and \( j \). The algorithm is then to round the vectors \( \mathbf{v}_i \) using the Goemans-Williamson rounding scheme. We choose \( \alpha_\ell \) to optimize the cut fraction. Notice that \( \mathbf{v}_i \cdot \mathbf{v}_j \) is the same for all \((i, j) \in E\). Minimizing this value, where the vectors are restricted to being unit length, becomes the following optimization problem:

\[
\min 2\alpha_0 + 2(d-1)\alpha_1 + \cdots + 2(d-1)^{k-2}\alpha_{k-1} \\
\text{such that } \alpha_0^2 + d\alpha_1^2 + (d-1)\alpha_2^2 + \cdots + d(d-1)^{k-2}\alpha_{k-1}^2 = 1
\]

By scaling the \( \alpha_\ell \)'s, the problem becomes a minimization of a quadratic function over \( S^k \):

\[
\min_{\|\beta\|=1} \frac{2}{\sqrt{d}}\beta_0 + \frac{2\sqrt{d-1}}{d}\beta_1 + \cdots + \frac{2\sqrt{d-1}}{d}\beta_{k-2}\beta_{k-1} = \min_{\|\beta\|=1} \beta^T A_k \beta
\]

\[
\beta_\ell = \begin{cases} \\
0 & \ell = 0 \\
\alpha_\ell \sqrt{\frac{d(d-1)^{\ell-1}}{d} - 1} & \ell > 0
\end{cases}
\]

The minimum \( \mathbf{v}_i \cdot \mathbf{v}_j \) is then the minimum eigenvalue of the following \( k \times k \) matrix:

\[
A_k = \begin{bmatrix} \\
0 & a & 0 & 0 & 0 & \cdots \\
a & b & 0 & 0 & 0 & \cdots \\
0 & b & 0 & 0 & 0 & \cdots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & b & 0 & b \\
0 & 0 & \cdots & 0 & b & 0
\end{bmatrix}
\]

The optimal \( \beta_\ell \) are the entries of the corresponding eigenvector. The \( \alpha_\ell \) are then set from the \( \beta_\ell \).

We now state our main result:

**Theorem 1.** Given a finite \( d \)-regular graph \( G = (V, E) \) where \( m = |E| \) with girth at least \( 2k \), there is an edge cut of \( G \) with size

\[
E[W] \geq \frac{m}{\pi} \arccos(\sigma_{d,k}),
\]

such that

\[
\sigma_{d,k} \leq \frac{-2\sqrt{d-1}}{d} \left( \cos \frac{\pi}{k+1} + \left( \frac{d}{d-1} - 1 \right) \frac{2}{k+1} \sin \frac{\pi}{k+1} \sin \frac{2\pi}{k+1} \right) < \frac{-2\sqrt{d-1}}{d} \cos \frac{\pi}{k+1}.
\]

**Proof.** Choose parameters where \( \beta = w \), such that

\[
w_\ell = \sqrt{\frac{2}{k+1}} \sin \left( \frac{(\ell+1)k\pi}{k+1} \right) \quad w^T = [w_0, \ldots, w_{k-1}].
\]

Let’s calculate the inner product \( w^T A_k w \) in this case. Consider the following \( k \times k \) matrix:

\[
B_k = \begin{bmatrix} \\
0 & b & 0 & 0 & 0 & \cdots \\
b & 0 & b & 0 & 0 & \cdots \\
0 & b & 0 & b & 0 & \cdots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & b & 0 & b \\
0 & 0 & \cdots & 0 & b & 0
\end{bmatrix}
\]
Since $B_k$ is a hollow tridiagonal Toeplitz matrix, its eigenvectors and eigenvalues are fully known (Noschese, Pasquini, and Reichel, 2013). Here, $w$ is the eigenvector associated with the minimum eigenvalue of $B_k$:

$$w^T B_k w = \lambda_{\min}(B_k) = 2b \cos \frac{k\pi}{k+1}$$

Because of this, the following inequality holds:

$$\lambda_{\min}(A_k) \leq w^T A_k w = 2(a - b)w_0w_1 + w^T B_k w = 2(a - b)w_0w_1 + 2b \cos \frac{k\pi}{k+1}$$

Since $\arccos$ is decreasing, the cut fraction is lower bounded:

$$E[W] = \frac{1}{\pi} \sum_{(i,j) \in E} \arccos(v_i \cdot v_j) = \frac{m}{\pi} \arccos(\lambda_{\min}(A_k)) \geq \frac{m}{\pi} \arccos(w^T A_k w)$$

Algebraic manipulation of the expression for $w^T A_k w$ finishes the proof.\(\square\)

Assigning $\beta$ as the minimum eigenvector of $B_k$ does not require any additional calculation, because a closed-form expression for the eigenvector is known for all $k$. However, this algorithm has highest performance when assigning $\beta_\ell$ from the minimum eigenvector of $A_k$. At large $d$ or $k$, the difference between $A_k$ and $B_k$ tends to zero.

When $d \geq 3$ and $k \geq 4$, this is the best known cut fraction for $d$-regular graphs of girth $\geq 2k$. See Section 4 for a comparison of this bound with other algorithms’ performance guarantees.

### 3 Relationship to the Gaussian wave process

Consider a vector assignment given by our algorithm, $\{v_i\}_{i=1}^n$, and let $V = [v_1...v_n]$. Hyperplane rounding can now be thought of as sampling from a $n$-variable Gaussian distribution $x \sim \mathcal{N}(0^n, \Sigma)$ (where $\Sigma = V^T V$), and taking the sign of each variable. Any multivariate Gaussian distribution can be expressed as linear combinations of i.i.d. Gaussians. In other words and when applied to our setting, $x = Vz$, where $z$ encodes $n$ i.i.d Gaussian random variables. This yields

$$x_i = \sum_{j \in V} \alpha_{d(i,j)} z_j = \sum_{\ell=0}^{k-1} \sum_{j: \ell \cdot d(i,j) = \ell} \alpha_{\ell} z_j.$$  

Thus, our algorithm may be viewed as a linear factor of an i.i.d process, presented in Csóka et al. (2015) to approximate the Gaussian wave and used in Lyons (2017):

$$x_i = \sum_{j \in V} \alpha_{d(i,j)} z_j = \sum_{\ell=0}^{\infty} \sum_{j: \ell \cdot d(i,j) = \ell} \alpha_{\ell} z_j$$

The first equation may be recovered by setting $\alpha_{\ell} = 0$ for all $\ell \geq k$.

Csóka et al. (2015) construct a linear factor of an i.i.d. process on the infinite $d$-regular tree to approximate the Gaussian wave process. They observe that in this setting, one may assume that

\[^2\text{Note that because } w_0w_1 \leq 0 \text{ and } a \geq b, w^T A_k w \leq w^T B_k w = \lambda_{\min}(B_k).\]
only a finite number of the $\alpha_\ell$ are nonzero; however, their results are on graphs of sufficiently high girth, and they do not explicitly consider girth as a parameter.

The approach of Chapter 4.1 in de Boor (2019) is similar to ours. The main difference is that de Boor uses one free parameter $c$ to determine $\{\alpha_\ell\}_{\ell=0}^{k-1}$ rather than obtaining the optimal values as we do. In fact, de Boor’s optimal choice of $c$ exactly reproduces the block FIID in Theorem 4.2 of Lyons (2017), with the same bound on cut fraction.\footnote{de Boor makes a mathematical mistake; on page 20, his $\zeta^2$ should equal $1 + c^2dL$.} Our approach is guaranteed to perform as well any other approach based on a linear factor of an i.i.d. process using only $\{\alpha_\ell\}_{\ell=0}^{k-1}$.

4 Comparison of performance guarantees

The expected cut size given by our algorithm is larger than the guarantee given by Theorem 4.2 of Lyons (2017). We state this formally:

**Theorem 2.** Given a finite $d$-regular graph $G = (V, E)$ with girth at least $2k$, the explicit vector solution has a higher performance guarantee than that of the algorithm in Lyons (2017) for every $k \geq 3$ and $d \geq 3$.

**Proof.** First consider this form of the expected cut fraction:

$$E[W] = \frac{m}{\pi} \arccos(-2b \cdot \xi)$$

We compare $\xi$ for the explicit vector solution and the algorithm in Lyons (2017). Call $\xi$ the relative expectation. The explicit vector solution has relative expectation

$$\xi_{EV} \geq \frac{\lambda_{\min}(B_k)}{-2b} = \cos \frac{\pi}{k+1} \geq 1 - 0.5\left(\frac{\pi}{k+1}\right)^2.$$  

The algorithm in Lyons (2017) has relative expectation

$$\xi_L = \left(1 + \frac{d-1}{d(k-1)}\right)^{-1} = \frac{k-1}{k-1/d}.$$  

The condition $\xi_{EV} > \xi_L$ holds when

$$1 - 0.5\left(\frac{\pi}{k+1}\right)^2 > \frac{k-1}{k-1/d}$$  

$\iff$  

$$k - 1/d > \frac{k-1}{1 - 0.5\left(\frac{\pi}{k+1}\right)^2}$$  

$\iff$  

$$d > \left(k - \left(\frac{k-1}{1 - 0.5\left(\frac{\pi}{k+1}\right)^2}\right)\right)^{-1}.$$  

The righthand expression decreases with $k$, so if the condition holds at $k_*$, it holds at all $k \geq k_*$. Consider $k \in \{3, 4, 5\}$. The condition holds when

$$d > 9.26 \quad (k = 3)$$  

$$d > 3.82 \quad (k = 4)$$  

$$d > 2.75 \quad (k = 5).$$  

All that remains to show is that the relative expectation is higher for $(k = 3, 3 \leq d \leq 9)$ and $(k = 4, d = 3)$. A numerical calculation verifies this; see Table 1. This completes the proof.\qed

3 de Boor makes a mathematical mistake; on page 20, his $\zeta^2$ should equal $1 + c^2dL$. 

6
### Table 1: Comparison of relative expectation of our solution and the lower bound of a previous Gaussian wave algorithm.

The calculation uses the bound $w^T A_k w$ given in the proof of Theorem 1, where $w$ is the minimum eigenvector of $B_k$. All values are truncated.

The explicit vector solution also outperforms the QAOA and threshold algorithm, at depth-1 and depth-2, for all $k \geq 3$ and $d \geq 3$. Our solution has performance increasing with $k$, and the other algorithms have performance increasing with depth, so we compare the $k = 3$ solution to the depth-2 algorithms. Consider the cut fraction $1/2 + c_d/\sqrt{d}$, where $c_d$ is the normalized value. At large $d$, our algorithm has normalized value $c_\infty = \sqrt{2}/\pi > 0.45$, higher than that of the QAOA ($c_\infty \approx 0.41$) and the threshold algorithm ($c_\infty \approx 0.42$). The normalized value decreases with $d$ for $d \geq 3$; for example, $c_8 \approx 0.454$. Compare this with Figure 2 and Table 1 of Marwaha (2021); the explicit vector solution outperforms the QAOA and threshold algorithm at all $d \geq 3$.

The modified threshold algorithm of $d \in \{3, 4, 5\}$ was found with a computer search over a large class of algorithms (Marwaha, 2021). This outperforms our solution at $k = 3$ but not at $k = 4$.

### 5 Conclusions

We provide a procedure to find a large cut on a high-girth graph. For a given girth $g \geq 8$, this algorithm provides a cut fraction guarantee higher than that of all other classical and quantum algorithms to date. It gives a simple, explicit solution to the semidefinite program. Moreover, the process of using hyperplane rounding on this solution can be thought of as a simplification of the Gaussian wave process previously used.

Although our analysis requires $k$ upper bounded by the girth, in practice one can use $k$ equal to the diameter of the graph. Numerically, this change improves the cut fraction, but a new analysis is needed to prove it.

One can also use the explicit vector technique in other optimization problems. For example, it can be directly applied to the independent set algorithm in Csóka et al. (2015). Though this does not improve the performance, further work (for example a tighter bound) could lead to an improvement.

There is recent interest in the development of local improvement algorithms which start with an existing approximation (as opposed to random initialization). For example, Egger, Mareček, and Woerner (2021) implement “warm-start” QAOA, initializing the quantum computer with a solution produced by the Goemans-Williamson algorithm. This technique could also be applied here. Using the output of our algorithm, one could apply a local threshold procedure to improve
the cut. Although previous results are numerical, analytic bounds may be possible here because of
our solution’s simple form.

No current algorithm provably achieves the optimal cut fraction on locally tree-like graphs (Dembo et al., 2017). It is suspected that one may exist using approximate message-passing (Alaoui, Montanari, and Sellke, 2020; Montanari, 2021); but for now, this is an open question. This means that better algorithms than the one we propose may exist, including better quantum algorithms.

Acknowledgements

KM thanks Boaz Barak for personal guidance. OP and JT are affiliated with Sandia National Laboratories, which is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA-0003525. OP and JT were supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Accelerated Research in Quantum Computing and Quantum Algorithms Teams programs.

References

A. E. Alaoui, A. Montanari, and M. Sellke. Optimization of mean-field spin glasses, 2020. 2001.00904.
B. Barak and K. Marwaha. Classical algorithms and quantum limitations for maximum cut on high-girth graphs, 2021. 2106.05900.
C. Carlson, A. Kolla, R. Li, N. Mani, B. Sudakov, and L.Trevisan. Lower bounds for max-cut via semidefinite programming. In Y. Kohayakawa and F. K. Miyazawa, editors, LATIN 2020: Theoretical Informatics, pages 479–490, Cham, 2020. Springer International Publishing. ISBN 978-3-030-61792-9. 1810.10044.
E. Csóka, B. Gerencsér, V. Harangi, and B. Virág. Invariant gaussian processes and independent sets on regular graphs of large girth. Random Structures & Algorithms, 47(2):284–303, 2015. 1305.3977.
E. Csóka. Independent sets and cuts in large-girth regular graphs, 2016. 1602.02747.
C. de Boor. In search of degree-4 sum-of-squares lower bounds for maxcut. Technical Report CMU-CS-19-118, Carnegie Mellon University, 2019. URL http://reports-archive.adm.cs.cmu.edu/anon/2019/CMU-CS-19-118.pdf.
A. Dembo, A. Montanari, and S. Sen. Extremal cuts of sparse random graphs. The Annals of Probability, 45(2):1190–1217, 2017. 1503.03923.
D. J. Egger, J. Mareček, and S. Woerner. Warm-starting quantum optimization. Quantum, 5:479, Jun 2021. ISSN 2521-327X. doi: 10.22331/q-2021-06-17-479. 2009.10095.
P. Erdös. On even subgraphs of graphs. Mat. Lapok, 18(3, 4):283–288, 1967. URL http://real-j.mtak.hu/id/eprint/9365.
E. Farhi, J. Goldstone, and S. Gutmann. A quantum approximate optimization algorithm. arXiv preprint, 2014. 1411.4028.
E. Farhi, J. Goldstone, and S. Gutmann. A quantum approximate optimization algorithm applied to a bounded occurrence constraint problem, 2015. 1412.6062.
M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and
satisfiability problems using semidefinite programming. *Journal of the ACM (JACM)*, 42(6):1115–1145, 1995. URL [http://www-math.mit.edu/~goemans/PAPERS/maxcut-jacm.pdf](http://www-math.mit.edu/~goemans/PAPERS/maxcut-jacm.pdf). 1

E. Halperin, D. Livnat, and U. Zwick. Max cut in cubic graphs. *Journal of Algorithms*, 53(2):169–185, 2004. ISSN 0196-6774. doi: https://doi.org/10.1016/j.jalgor.2004.06.001. URL [https://www.sciencedirect.com/science/article/pii/S0196677404000902](https://www.sciencedirect.com/science/article/pii/S0196677404000902). 1

M. P. Harrigan, K. J. Sung, M. Neeley, et al. Quantum approximate optimization of non-planar graph problems on a planar superconducting processor. *Nature Physics*, 17(3):332–336, Mar 2021. ISSN 1745-2481. doi: 10.1038/s41567-020-01105-y. 2004.04197. 1

M. B. Hastings. Classical and quantum bounded depth approximation algorithms, 2019. 1905.07047. 1

J. Hirvonen, J. Rybicki, S. Schmid, and J. Suomela. Large cuts with local algorithms on triangle-free graphs. *The Electronic Journal of Combinatorics*, pages P4–21, 2017. 1402.2543. Preliminary version on arXiv, 2014. 1, 1, 3

S. Khot. On the power of unique 2-prover 1-round games. In *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, STOC ’02, page 767–775, New York, NY, USA, 2002. Association for Computing Machinery. ISBN 1581134959. doi: 10.1145/509907.510017. URL [http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.133.5651&rep=rep1&type=pdf](http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.133.5651&rep=rep1&type=pdf). 1

S. Khot, G. Kindler, E. Mossel, and R. O’Donnell. Optimal inapproximability results for MAX-CUT and other 2-variable csps? *SIAM J. Comput.*, 37(1):319–357, 2007. doi: 10.1137/S0097539705447372. URL [https://www.cs.cmu.edu/~odonnell/papers/maxcut.pdf](https://www.cs.cmu.edu/~odonnell/papers/maxcut.pdf). Preliminary version in FOCS 2004. 1

R. Lyons. Factors of iid on trees. *Combinatorics, Probability and Computing*, 26(2):285–300, 2017. 1401.4197. 1, 2, 2, 3, 5, 6, 6, 6, 6, 6

K. Marwaha. Local classical MAX-CUT algorithm outperforms $p = 2$ QAOA on high-girth regular graphs. *Quantum*, 5:437, Apr. 2021. ISSN 2521-327X. doi: 10.22331/q-2021-04-20-437. 2101.05513. 1, 3, 7, 7

A. Montanari. Optimization of the sherrington–kirkpatrick hamiltonian. *SIAM Journal on Computing*, 0(0):FOCS19–1, 2021. 1812.10897. Preliminary version in FOCS ’19. 8

S. Noschese, L. Pasquini, and L. Reichel. Tridiagonal toeplitz matrices: properties and novel applications. *Numerical linear algebra with applications*, 20(2):302–326, 2013. URL [http://www.math.kent.edu/~reichel/publications/toep3.pdf](http://www.math.kent.edu/~reichel/publications/toep3.pdf). 5

R. O’Donnell and Y. Wu. An optimal sdp algorithm for max-cut, and equally optimal long code tests. In *Proceedings of the Fortyeth Annual ACM Symposium on Theory of Computing*, STOC ’08, page 335–344, New York, NY, USA, 2008. Association for Computing Machinery. ISBN 9781605580470. doi: 10.1145/1374376.1374425. URL [https://doi.org/10.1145/1374376.1374425](https://doi.org/10.1145/1374376.1374425). 1

C. Ryan-Anderson. Quantum algorithms, architecture, and error correction, 2018. 1812.04735. 1

J. B. Shearer. A note on bipartite subgraphs of triangle-free graphs. *Random Structures & Algorithms*, 3(2):223–226, 1992. URL [https://onlinelibrary.wiley.com/doi/abs/10.1002/rsa.3240030211](https://onlinelibrary.wiley.com/doi/abs/10.1002/rsa.3240030211). 1

Z. Wang, S. Hadfield, Z. Jiang, and E. G. Rieffel. Quantum approximate optimization algorithm for maxcut: A fermionic view. *Phys. Rev. A*, 97:022304, Feb 2018. doi: 10.1103/PhysRevA.97.022304. 1706.02998. 1, 3