1. Introduction

The SU(1, 1) Lie algebra has many applications in quantum optics because it can characterize many kinds of quantum optical systems [1–4]. In order to study many problems in this field, it has recently been used by many researchers to investigate the nonclassical properties of light in quantum optical systems [5–7]. In recent years there has been much interest in applications and generalizations of the Barut-Girardello coherent states(BG-CS) [8]. The BG-CS were introduced [9] as eigenstates of the lowering Weyl operator \( K_− \) in the framework of SU(1, 1) Lie algebra. We employ the second-order correlation function to discuss some non-classical properties, and violations of Cauchy-Schwarz inequalities. The phenomenon of squeezing is examined, squeezing is clear and Q-functions support that. Finally the phase distribution in the framework of an appropriate Pegg and Barnett formalism is considered and discussed.

2. The SU(1, 1) pair coherent state

Let us have two independent systems where operators are described by the generators of SU(1, 1) group. These generators are \( \{ K^a_+, K^a_−, K^a_z \} \) where \( a = a, b \). Let us define new operators which are given by the following

\[
K_+^{ab} = K_a^+ K_b^+, \quad K_-^{ab} = K_a^- K_b^-
\]

where \( \{ K^a_+, K^a_−, K^a_z \} \) obey SU(1, 1) Lie algebra commutation relation [15]

\[
[K_+^{ij}, K_-^{kl}] = \pm K_-^{ij}, \quad [K_+^{ij}, K_z^{kl}] = 2 K_z^{ij}, \quad i = a, b
\]
while, 

\[ [K_{ab}^-, K_{+}^+] = 2[K_{a}^2(K_{b}^{b2} - (K_{b}^+)^2) + K_{b}^b(K_{a}^{a2} - (K_{a}^-)^2)]. \]

Let us introduce a two-mode basis \(|m, k_i; n, k_j\rangle = |m, k_i\rangle \otimes |n, k_j\rangle\) governed by SU(1, 1) group algebra in terms of eigenstates of two independent modes denoted by \(a, b\). The effect of generators of first mode \(a\) (second mode \(b\)) on \(|m, k_i; n, k_j\rangle\) is

\[
\begin{align*}
(K_{a})^2|m, k_i; n, k_j\rangle &= k_1(k_1 - 1)|m, k_i; n, k_j\rangle \\
K_{a}^a|m, k_i; n, k_j\rangle &= (m + 1)(m + 2k_2)|m + 1, k_i; n, k_j\rangle \\
K_{a}^b|m, k_i; n, k_j\rangle &= (m + 2k_1 - 1)|m - 1, k_i; n, k_j\rangle \\
K_{a}^b|m, k_i; n, k_j\rangle &= (m + k_1)|m, k_i; n, k_j\rangle 
\end{align*}
\]

(2.1)

where \((K_{a})^2\) is a Casimir operator of first mode \(a\) with similar relation, for the second mode \(b\).

The corresponding Hilbert space \(H = H_1 \otimes H_2\) is spanned by the complete orthonormal basis \(|m, k_i; n, k_j\rangle, (m, n = 1, 2, 3, \ldots)\) and the completeness relation is given by

\[
\sum_{n,m=0}^{\infty} |m, k_i; n, k_j\rangle \langle m, k_i; n, k_j| = I
\]

We define the new pair coherent state as an eigenstate of the lowering generator \(K_{a}^b\),

\[
K_{a}^b|\xi, q, k_i, k_j\rangle = \xi|\xi, q, k_i, k_j\rangle,
\]

where \(\xi\) is an arbitrary complex number and \(q\) is a real number. The state can be decomposed over the orthonormal two mode state basis. The action of the operators \(K_{a}^b\) and \((K_{a}^a - K_{a}^b)\) on states \(|m, k_i; n, k_j\rangle\) is

\[
\begin{align*}
K_{a}^b|m, k_i; n, k_j\rangle &= \sqrt{mn(m + 2k_2 - 1)(n + 2k_2 - 1)}|m - 1, k_i; n - 1, k_j\rangle \\
(K_{a}^a - K_{a}^b)|m, k_i; n, k_j\rangle &= (m + k_1 - n - k_2)|m, k_i; n, k_j\rangle
\end{align*}
\]

(2.3)

We assume that the eigenvalue \(q\) of the operator \((K_{a}^a - K_{a}^b)\) is positive where

\[
q = m + k_1 - n - k_2
\]

that is given through the condition of the state (2.2). The expansion of \(|\xi, q, k_i, k_j\rangle\) in the two-mode basis is composed of states of the form \(|n + q + k_2 - k_1, k_i; n, k_j\rangle\) and is given through the formula

\[
|\xi, q, k_i, k_j\rangle = \sum_{n=0}^{\infty} \xi^n C_n(q, k_i, k_j)|n + q + k_2 - k_1, k_i; n, k_j\rangle
\]

\[
C_n(q, k_i, k_j) = \frac{N}{\Delta_n} \left( \frac{\xi^n}{\Delta_n} \right)^2
\]

\[
\Delta_n = n!\Gamma(n + 2k_2)\Gamma(q + n + k_2 - k_1 + 1)\Gamma(q + n + k_2 + k_1)
\]

(2.4)

where \(\Gamma(x)\) is Euler’s Gamma function, \(N\) is normalization factor. Equation (2.4) represents SU(1, 1) quantum pair coherent state which can be considered as a generalization to the Barut-Girardello coherent state

### 3. Completeness of the states \(|\xi, q, k_i, k_j\rangle\)

Resolution of the identity (completeness) in terms of a certain set of states is very important because it allows the practical use of these states as bases in the Hilbert space. The problem here consists in finding a weight function \(\sigma(\xi)\) with \(\xi = re^{i\theta}\) such that

\[
\int d\sigma(\xi)|\xi, q, k_i, k_j\rangle \langle \xi, q, k_i, k_j| = 1
\]

(3.1)

Let \(\sigma(\xi) = N^{-1}r(|\xi|)d^2\xi\), with \(d^2\xi = r\,dr\,d\theta\) and \(|\xi| = r\) where \(N\) is defined in (2.4), \(0 < r < \infty\) and \(0 < \theta < 2\pi\). The integration in (3.1) is
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n + q - k_1, k_1; n, k_2\rangle \langle m + q - k_1, k_1|}{\sqrt{\Delta_n \Delta_m}} \times \int_{0}^{2\pi} e^{i(n-m)\mu(r)rr'^{m}dr} \\
= \sum_{n=0}^{\infty} \frac{|n + q - k_1, k_1; n, k_2\rangle \langle n + q - k_1, k_1|}{\Delta_n} \times 2\pi \int_{0}^{\infty} \mu(r)rr'^{2n}dr
\]

(3.2)

where \(\Delta_n\) defined in (2.4). Hence we must have

\[
\pi \int_{0}^{\infty} \mu(r)r^{2n}d(r^2) = \Delta_n
\]

(3.3)

Following to [16] the solution of this moment problem can be found as the general solution of this integral equation in terms of the Meijer’s G-function [17]

\[
\mu(r) = G_{0}^{1}(r^2)[0, 2k_2 - 1, q + k_2 - k_1, q + k_2 + k_1 - 1]
\]

Then the weight function \(\sigma(\xi)\) is given by

\[
\sigma(\xi) = N^{-2}C_{4b}^{10}(\xi^2)[0, 2k_2 - 1, q + k_2 - k_1, q + k_2 + k_1 - 1] \frac{d^2\xi}{\pi}
\]

This completes requirements for the resolution of the identity

4. Generation scheme

It is to be mentioned that \(su(1, 1)\) Lie algebra can be realized in terms of boson annihilation and creation operators, where we can define \(K_\pm^a\) and \(K_\pm^b\) where \(i = a, b\) as follows

\[
K_+^a = \frac{1}{2} a + \frac{1}{2}, \quad K_-^a = \frac{1}{2} a^\dagger + \frac{1}{2}, \quad K_+^b = \frac{1}{2} b + \frac{1}{2}, \quad K_-^b = \frac{1}{2} b^\dagger + \frac{1}{2}
\]

Here, we are going to study a generation scheme of the state which is the eigenstate to \(K_\pm^a\) within the framework of the motion of a trapped ion [18] in a two dimensional harmonic potential. Consider a single ion of mass \(M\) trapped in a two dimensional harmonic potential with frequencies \(\omega_1\) (in the x-direction), \(\omega_2\) (in the y-direction). In the rotating wave approximation the Hamiltonian of the system is written as

\[
\frac{H}{\hbar} = \omega_1 a^\dagger a + \omega_2 b^\dagger b + \frac{\omega_0}{2} \sigma_z + \mu [E_1 e^{i(k_1 x + k_2 y - \omega_1 t)} + E_2 e^{i(k_1 x + k_2 y - \omega_2 t)}] \sigma_x + h.c.
\]

(4.1)

The Hamiltonian (4.1) describes a two-level ion confined within a two dimensional trap that is approximated as harmonic oscillators of frequencies \(\omega_1\) and \(\omega_2\). The frequency \(\omega_0\) is the energy difference between the two levels of the atom. The \(\sigma_+\) (\(\sigma_-\)) and \(\sigma_z\) are the raising (lowering) and phase operator, and represent the Pauli operator of the electronic two-level ion. The parameter \(\mu\) is the dipole matrix element and \(k_i\) (\(k\)’s) is the wavevector of ith driving the laser field of amplitude \(E_1\) and \(E_2\). The position of the center-of-mass of the trapped ion is given by \((\hat{x}, \hat{y})\) quantized by the operators \(a\), \(a^\dagger\) and \(b\), \(b^\dagger\) which the annihilation and creation operators of the vibrational motion of the center-of-mass of the ion. The quantized centre-of-mass position \(\hat{x}\) and \(\hat{y}\) can be written as

\[
\hat{x} = \frac{\hbar}{\sqrt{2\mu_0}} (\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{y} = \frac{\hbar}{\sqrt{2\mu_1}} (\hat{b} + \hat{b}^\dagger)
\]

We may use a vibrational rotating wave approximation and neglect the terms with fast oscillations [19]. Thus the interactions Hamiltonian (4.1) is simplified to
\[ H_{IN} = \exp\left( -\frac{\eta_1^2 + \eta_2^2}{2}\right) \]

\[ \left[ \sigma_x \left( \Omega_0 \sum_{m,m_2} \frac{(i \eta_1)^2m(i \eta_2)^2m_2m_2m_{m_2}}{(m_1)^2(m_2)!^2}b^m b^m_{m_2} + \Omega_2 \sum_{m,m_2} \frac{(i \eta_1)^2m_2(i \eta_2)^2m_2m_2m_{m_2}}{m!(m + 2)!m!(m + 2)!)^2}b^m b^m_{m_2} + h.c \right) \right] \]  

(4.2)

\[ \Omega_0 = |\mu \cdot E_0| \text{ and } \Omega_2 = |\mu \cdot E_2| \text{ are the Rabi frequencies to the laser fields and } \eta_2 \text{ is the Lamb-Dicke parameter}, \]

where \( \eta_2 = k_2 \sqrt{\frac{\hbar}{2M_0}} \) and \( k_L \approx |k_L| \approx |k_L'| \). It should be noted that the operator \( K^a_4 - K^b_3 \) is a constant of motion for the Hamiltonian (4.2). For small Lamb-Dicke parameters \( \eta_2 \ll 1, \ i = 1, 2 \), one may consider lowest terms with \( m_1 = 0 = m_2 \) in the summation. Hence the Hamiltonian (4.2) can be approximated to

\[ H_{IN} = \sigma_+ \{ \Omega_0 + \frac{\Omega_2}{4} (i \eta_1)^2(i \eta_2)^2 \} + h.c \]  

(4.3)

the term between parentheses can be written as

\[ \hat{G} = \lambda \left( \frac{1}{4} \hat{a}^2 \hat{b}^2 - \zeta \right) \]  

(4.4)

where

\[ \lambda = \Omega_\nu \eta_1^2 \eta_2^2 \]

and

\[ \zeta = -\frac{\Omega_0}{\Omega_\nu \eta_1^2 \eta_2^2} \]

The master equation for the density matrix under spontaneous emission with energy dissipation rate \( \gamma \) is given by [19]

\[ \frac{\partial \rho}{\partial t} = -i [H_{IN}, \rho] + \frac{\gamma}{2} \left( \sigma_+ \rho \sigma_- - \sigma_- \sigma_+ - \rho \sigma_+ \sigma_- \right) \]  

(4.5)

The stationary solution \( \tilde{\rho} \) for this master equation is obtained by setting \( \frac{\partial \rho}{\partial t} = 0 \). A solution \( \tilde{\rho} \) can be given as

\[ \tilde{\rho} = \langle g | \rangle \langle \tilde{\zeta} | \rangle \]  

(4.6)

with \( \langle g | \rangle \) the electronic ground state \( (\sigma_- \langle g |) = 0 \), \( \langle g | \rangle \langle \tilde{\zeta} | \rangle \) is the vibration eigenstate that satisfies \( H_{IN} \langle \tilde{\zeta} | \rangle = 0 \). It is straightforward to show that \( \langle \tilde{\zeta} | \rangle \) belongs to the class of the SU(1, 1) pair coherent states,

\[ \hat{G} \langle \tilde{\zeta} | \rangle = 0 \Rightarrow \lambda \left( \frac{1}{4} \hat{a}^2 \hat{b}^2 - \zeta \right) \langle \tilde{\zeta} | \rangle = 0 \Rightarrow \frac{1}{4} \hat{a}^2 \hat{b}^2 \langle \tilde{\zeta} | \rangle = \zeta \langle \tilde{\zeta} | \rangle \]

\[ K^a_4 K^b_3 \langle \tilde{\zeta} | \rangle = \zeta \langle \tilde{\zeta} | \rangle \Rightarrow K^b_3 \langle \tilde{\zeta} | \rangle = \zeta \langle \tilde{\zeta} | \rangle \]

given by (2.2) and (2.4)

5. Probability distribution

A probability distribution \( P_n \) for any quantum state \( |\psi\rangle \) is defined as

\[ P(n) = |\langle n |\psi\rangle|^2, \quad n = 0, \ 1, \ 2, \ldots \]

must satisfy

\[ P(n) \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} P(n) = 1 \]

For the state (2.4)

\[ P(n) = |\langle n + q + k_2 - k_1, n; k_1, k_2 |\xi, k_1, k_2\rangle|^2, \quad n = 0, \ 1, \ 2, \ldots \]

\[ P(n) = |\xi \langle q, k_1, k_2 \rangle|^2 \]

Where \( C_n(q, k_1, k_2) \) is given in (2.4). To study the effect of both \( \xi, q, k_1 \) and \( k_2 \) on probability distribution function we plot \( P(n) \) against \( n \). At figure 1(a) \( q = 1, 5, 10, \xi = 20 \), \( k_1 = 1 \) and \( k_2 = 2 \). We find that when \( q \) increases the maximum value of the probability distribution curve moves towards lower value of \( n \). We not that at \( q = 0 \) the distributional behavior like a Gaussian distribution. In figure 1(b) \( \xi = 5, 10, 20 \) and \( k_1 = 1, k_2 = 2 \), \( q = 5 \) we find that when \( \xi \) increases the maximum value of the probability distribution curve moves towards higher values of \( n \). It is observed that at \( \xi = 5 \) the distributional behavior like a thermal distribution.
6. Nonclassical properties

6.1. Second order correlation

To study the quantum statistical properties of any quantum state, we must pay attention to the nonclassical behavior such as (sub-Poissonian behavior [20]). So that we introduce the second-order correlation function [21, 22], which leads to better understanding of the nonclassical behavior of quantum states [4, 23]. A state for which \( g^{(2)}_i < 1 \) has sub-Poissonian (nonclassical behavior), a state for which \( g^{(2)}_i > 1 \) is super-Poissonian (classical behavior), while the state is Poissonian when the function \( g^{(2)}_i = 1 \). Therefore, we devote the present section to discussing this correlation function. This can be introduced for the SU(1, 1) group generated as follows

\[
g^{(2)}_i(\xi) = \frac{\langle (K^+_i)^2(K^-_i)^2 \rangle}{\langle K^+_iK^-_i \rangle^2} \quad i = a, b
\]  

(6.1)

In order to discuss the behavior of the correlation function, we calculate the expectation values of the quantities \( (K^+_i)^2(K^-_i)^2 \), \( K^+_iK^-_i \) at \( i = a \) for the first mode, \( i = b \) for the second mode.

For the first mode \( i = a \), the second order correlation function is

\[
g^{(2)}_a(\xi) = \frac{\langle (K^+_a)^2(K^-_a)^2 \rangle}{\langle K^+_aK^-_a \rangle^2}
\]  

(6.2)
where

\[
\langle K^a_n K^b_n \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n + q + k_1 + k_2 - 1)
\]

(6.3)

\[
\langle (K^a_n)^2 (K^b_n)^2 \rangle = \sum_{n=0}^{\infty} (n + q + k_2 - k_1)(n + q + k_1 + k_2 - 1)(n + q + k_1 + k_2 - 2)|\xi^n C_n(q, k_1, k_2)|^2
\]

(6.4)

For second mod \(i = b\), the second order correlation function is

\[
g^{(2)}_b(\xi) = \frac{\langle (K^b_n)^2 (K^b_n)^2 \rangle}{\langle K^b_n K^b_n \rangle^2}
\]

(6.5)

where

\[
\langle K^b_n K^b_n \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n + 2k_2 - 1)
\]

(6.6)

\[
\langle (K^b_n)^2 (K^b_n)^2 \rangle = \sum_{n=0}^{\infty} |\xi^n C_n(q, k_1, k_2)|^2 (n + 2k_2 - 1)
\]

\[
\times (n - 1)(n + 2k_2 - 2)
\]

(6.7)

To show the behavior of the correlation function for the state under consideration, we plot \(g^{(2)}_b, \ i = a, \ b\). Figure 2(a) for first, figure 2(b) for second mode. We find that the state has nonclassical behavior at all values of \(q\) and \(\xi\), as it may be expected from the form of the \(C_n\) coefficient and their dependence on \(n\)

### 6.2. Cauchy-Schwarz inequality

We now consider violation of the Cauchy-Schwarz inequality between the single mode and cross-correlation second-order coherence functions. In the classical theory, this inequality can be expressed as

\[
|g^{(2)}_a(\xi)| |g^{(2)}_b(\xi)| \geq |g^{(2)}_{ab}(\xi)|^2
\]

In order to measure the deviation from the classical inequality, we define the quantity [4, 24]

\[
I_\theta = \frac{|\langle g^{(2)}_a(\xi) g^{(2)}_b(\xi) \rangle |}{g^{(2)}_{ab}(\xi)} - 1
\]

where

\[
g^{(2)}_{ab}(\xi) = \frac{\langle (K^a_n)^2 (K^b_n)^2 \rangle}{\langle K^a_n K^b_n \rangle^2} = 1
\]

As we can observe in figure 3 this function is always negative, which means that the inter-mode correlation is larger than the correlation between the same mode. The strongest violations of the Cauchy-Schwarz inequality occur at lower \(q\) for a fixed values of \(k_1 = 1, k_2 = 2\).
6.3. Squeezing effect

Squeezing fluctuations are important in quantum measurement and communication theories [25]. In the SU(1, 1) Lie group [7], one can define two hermitian operators $X$ and $P$ as follows:

\[
X = \frac{K_1 - K_2}{2}, \quad P = \frac{K_1 - K_2}{2i}
\]

which satisfies the commutation relation

\[ [X, P] = iC \]

The uncertainty relation for these operators takes the form

\[
\Delta X\Delta P \geq \frac{1}{2}|\langle C \rangle|
\]

where

\[
C = K_1^2(K_2^2 - (K_1^2))^2 + K_2^2(K_1^2 - (K_2^2))^2
\]

fluctuations in the $X$ (or $P$) component are squeezed if the following condition is satisfied

\[
(\Delta X)^2 < \frac{1}{2}|\langle C \rangle| \quad \text{or} \quad (\Delta P)^2 < \frac{1}{2}|\langle C \rangle|
\]

where

\[
\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}, \quad \Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2}
\]

To measure the degree of squeezing, we define the following squeezing parameters,

\[
S_X = \frac{(\Delta X)^2 - 0.5|\langle C \rangle|}{0.5|\langle C \rangle|} \quad \text{and} \quad S_P = \frac{(\Delta P)^2 - 0.5|\langle C \rangle|}{0.5|\langle C \rangle|}
\]

The squeezing condition can be expressed as $S_X < 0$ or $S_P < 0$.

In figure 4 we note that state (2.4) achieves squeezing phenomenon in $S_P$. It is to be observed that squeezing increases when $q$ decreases at the fixed values $k_1 = 1$, $k_2 = 2$ as shown in figure 4.

7. Q-function

It is well known that the quasiprobability distribution functions are important tools to give insight on the statistical description of quantum dynamics [26]. Therefore, we devote the present section to concentrate on one of these functions, that is the Husimi Q-function [27]. In fact the Q-function is not only a convenient tool to calculate expectation values of anti-normally ordered products of the operators, but also interpreted as a true phase space probability distribution. For the state (2.4) $|\xi, q_1, k_1, k_2\rangle$, we present the following definition for the Q-function of two modes as [28]

\[
Q(\alpha, \beta) = \frac{1}{\pi^2}|\langle \alpha, \beta; \hat{k}, \hat{k}[\xi, q_1, k_1, k_2] \rangle|^2
\]
where

Consequently a phase distribution function

This definition is generalized for two mode case as follows:

\[ |\theta, k\rangle = \lim_{s \to \infty} \frac{1}{\sqrt{5}} \sum_{m=0}^{r-1} \exp(i\theta K_3) |m, k\rangle \]

This definition is generalized for two mode case as follows:

\[ |\theta_1, k_1; \theta_2, k_2\rangle = \lim_{s_1, s_2 \to \infty} \frac{1}{\sqrt{s_1 s_2}} \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} e^{iK_3\theta_1} e^{iK_3\theta_2} |m, k_l; l, k_2\rangle \]

Consequently a phase distribution function
\[ P(\theta_1, \theta_2) \]
can be obtained from:

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]

where \( \theta = \theta_1 + \theta_2 \) and \( C_n(q, k, k_1, k_2) \) is defined in (2.4). To show behavior of \( P(\theta) \) we plot it for \(-\pi < \theta < \pi\) and different values of \( \xi \) and \( q \) for fixed \( k_1 = 1, k_2 = 2 \). In figure 6(a) we take \( q = 5 \) and plot \( P(\theta) \) against \( \theta, \xi \). We note that phase distribution function appears as a peak centered at \( \theta = 0 \). No information for \( \xi = 0 \) but as \( \xi \)

8. Phase distribution

To study the phase distribution of the state (2.4) we use the definition for the SU(1, 1) phase state [29]

\[ |\theta, k\rangle = \lim_{s \to \infty} \frac{1}{\sqrt{5}} \sum_{m=0}^{r-1} \exp(i\theta K_3) |m, k\rangle \]

This definition is generalized for two mode case as follows:

\[ |\theta_1, k_1; \theta_2, k_2\rangle = \lim_{s_1, s_2 \to \infty} \frac{1}{\sqrt{s_1 s_2}} \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} e^{iK_3\theta_1} e^{iK_3\theta_2} |m, k_l; l, k_2\rangle \]

Consequently a phase distribution function
\[ P(\theta_1, \theta_2) \]
can be obtained from:

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]

\[ P(\theta_1, \theta_2) = \left( \frac{e^{i\theta_2}}{2\pi} \right) \left| \sum_{m=0}^{r-1} \sum_{l=0}^{r-1} C_n(q, k, k_1) \xi^n e^{-i\theta_1} |m, k_l; l, k_2\rangle \right|^2 \]
increases $P(\theta)$ increase and information about phase starts to build up around $\theta = 0$. At figure 6(b) we take $\xi = 20$ and plot $P(\theta)$ against $\theta, q$ we note that phase distribution function appears as a peak centered at $\theta = 0$ when $q$ increases the pack of $P(\theta)$ decreases in height and information about phase decreases for the fixed values of $\xi$.

9. Conclusions

In this article we have introduced and examined some statistical properties of a new pair coherent quantum state of the SU(1, 1) algebra. A suggested generation scheme is presented based on the vibrational motion of the center of mass of a trapped ion in two-dimensional harmonic potential. The present scheme could be realized experimentally. We calculate and plot probability distribution function. Quantum statistical properties of these states have been studied in some detail. We have found interesting nonclassical features of these states. The sub-Poissonian distribution, Cauchy-Schwarz inequality valuation and squeezing phenomenon were displayed for these particular states for fixed parameter values. We studied the Q-function and showed its behavior for different parameters. Finally, we introduced phase.

Figure 5. Q-function at $k_1 = 1, k_2 = 2, \xi = 5, 10, 20, q = 5$ and $q = 0, 10, 15, \xi = 20$.
distribution function. We note that this state has non-classical properties for squeezing, phase distribution and these properties are sensitive to change in the parameters of the state.

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