Bipartite Matchings with Group Fairness and Individual Fairness Constraints

Atasi Panda
atasipanda@iisc.ac.in
Indian Institute of Science

Anand Louis *
anandl@iisc.ac.in
Indian Institute of Science

Prajakta Nimbhorkar *
prajakta@cmi.ac.in
Chennai Mathematical Institute

August 24, 2022

We address group as well as individual fairness constraints in matchings in the context of assigning items to platforms. Each item belongs to certain groups and has a preference ordering over platforms. Each platform enforces group fairness by specifying an upper and a lower bound on the number of items that can be matched to it from each group. There could be multiple optimal solutions that satisfy the group fairness constraints. To achieve individual fairness, we introduce ‘probabilistic individual fairness’, where the goal is to compute a distribution over ‘group fair’ matchings such that every item has a reasonable probability of being matched to a platform among its top choices.

In the case where each item belongs to exactly one group, we provide a polynomial-time algorithm that computes a probabilistic individually fair distribution over group fair matchings. When an item can belong to multiple groups, and the group fairness constraints are specified as only upper bounds, we rehash the same algorithm to achieve three different polynomial-time approximation algorithms.

1 Introduction

Matchings is a central topic in theoretical computer science, well-studied over several years. Many real-world problems can be modeled as the problem of computing a maximum matching on a bipartite graph. A few examples are ad-auctions [Mehta, 2013, Mehta et al., 2007], resource allocation [Halabian et al., 2012], scheduling [McKeown et al., 1996], school choice [Abdulkadiroglu and Sonmez, 2003], and healthcare rationing [Aziz and Brandl, 2021]. We refer to the two bipartitions of the underlying bipartite graph as items and platforms. A matching is an allocation of items to platforms, where each item and platform can get multiple matches assigned to them.

In real-world instances, items may have various attributes, leading to the classification of items into different groups. Additional group fairness constraints are imposed to ensure that, among the items matched to any platform, no group is under-represented or over-represented. Moreover, items can have preferences over platforms. So a matching that satisfies group fairness constraints may not be fair to an individual item, thus making the introduction of individual fairness constraints necessary.

The importance of group-fairness constraints has been repeatedly stressed in literature [Luss, 1999, Devanur et al., 2013, Costello et al., 2016, Cels et al., 2017, Segal-Halevi and Suksompong, 2019, Ray et al., 2015, Bolukbasi et al., 2016]. For example, in the case of school choice, group-fairness constraints can be imposed to ensure diversity among students assigned to each school in terms of attributes like ethnicity and economic and social background. Such diversity constraints have been implemented in practice [Cowen Institute, 2011]. Similarly, while forming teams to work on projects, group-fairness constraints ensure that each team has experts from all the fields required for the project. Another example where group-fairness constraints are applicable is residency-matching, also known as the hospital-residents problem where medical interns are to be assigned to hospitals. Group-fairness constraints can be imposed so that each hospital gets interns from diverse specialties.

*These two authors contributed equally
In all the above examples, however, group-fairness constraints alone do not accommodate the preferences of individuals. Students have preferences over schools, people have preferences over projects, and medical interns have preferences over hospitals. A matching that satisfies group-fairness constraints without any consideration for the preferences of individuals may match many individuals to their less preferred choice. Moreover, any matching computed by a deterministic algorithm always ends up assigning top choices to some individuals and less preferred choices to some. Individual fairness constraints address this issue. Instead of outputting a single matching, the goal is to output a distribution on group-fair matchings so that the expected choice assigned to each individual satisfies the individual fairness constraints. In this paper, we address the problem of simultaneously achieving group fairness and individual fairness in bipartite matching, formally defined below:

**Our problem:** The input instance consists of a bipartite graph denoted as \( G = (A \cup P, E) \). Here \( A \) denotes the set of items and \( P \) is the set of platforms. There is an edge from an item \( a \) to a platform \( p \) if \( a \) can be assigned to \( p \). The items can be grouped into possibly non-disjoint subsets \( A_1, A_2, \ldots, A_\chi \) for some given integer \( \chi \geq 1 \) such that \( \bigcup_{h \in [\chi]} A_h = A \). Here \( \chi \) denotes the total number of groups. Each platform \( p \) has a collection of groups \( C_p \subseteq 2^{N(p)} \), where \( N(p) \) denotes the set of neighbors of \( p \) in \( G \). Each group \( C \subseteq C_p \) has upper and lower bounds \( u_{p,C} \) and \( l_{p,C} \) respectively, denoting the maximum and the minimum number of items from \( C \) that can be assigned to \( p \). These are referred to as *group fairness constraints* in this paper. For each \( C \subseteq C_p \), let \( E_{p,C} \) denote the set of edges \( \{(a,p) : a \in C\} \).

**Group fairness:** A matching \( M \subseteq E \) is said to be *group-fair* if and only if
\[
l_{p,C} \leq |E_{p,C} \cap M| \leq u_{p,C} \quad \forall p \in P, C \in C_p.
\]  

**Probabilistic individual fairness:** In addition to the group fairness constraints, the input also contains individual fairness parameters \( L_{a,S} \in [0,1] \) and \( U_{a,S} \in [0,1] \) for each item \( a \) and each subset \( S \) of \( N(a) \), where \( N(a) \) denotes the neighborhood of \( a \) in \( G \). A distribution \( D \) on group-fair matchings in \( G \) is *probabilistically individually fair* if and only if
\[
L_{a,S} \leq \mathbb{Pr}_{M \sim D}[\exists p \in S \text{ s.t. } (a,p) \in M] \leq U_{a,S} \quad \forall a \in A,
\]
\[
\forall S \subseteq N(a)
\]

Let \( I = (G, A_1, \ldots, A_\chi, \bar{u}, \bar{L}, \bar{U}) \) denote an instance of our problem. The goal is to compute a probabilistic individually fair distribution over a set of group-fair matchings such that the expected size is maximized when a matching is picked uniformly from this distribution.

1.1 Our Results

When each item belongs to exactly one group, we get a solution to our problem that optimally satisfies group-fairness and individual fairness constraints.

**Theorem 1.1.** Given an instance of our problem where each item belongs to exactly one group, we provide a polynomial-time algorithm that either computes a probabilistic individually fair distribution over a set of group-fair matchings or reports infeasibility if no such distribution exists.

We first construct a linear programming formulation of the problem and then prove that if a solution to the linear program exists, it can be written as a convex combination of group-fair matchings (as defined in our problem).

When each item can belong to multiple groups, the problem of finding a maximum size group-fair matching is NP-hard \cite{Ma2020} even when all the group lower bounds are zero, and there are no individual fairness constraints. We give an approximation algorithm in terms of \( \Delta \) which is the maximum number of groups an item belongs to. We also give two approximation algorithms in terms of \( g \), the maximum number of groups per platform.

**Theorem 1.2.** Given an instance of our problem where each item can belong to at most \( \Delta \) groups, any constant \( \epsilon > 0 \), and \( l_{p,C} = 0 \quad \forall p \in P, C \in C_p \), that is there are no lower bound constraints, we provide a polynomial-time algorithm that computes a distribution \( D \) over a set of group-fair matchings such that the expected size is at least \( \frac{1}{\Delta}(OPT + \epsilon) \) where \( f_c = \mathcal{O}(\Delta \log(n/\epsilon)) \) and \( n \) is the total number of items.
Table 1: Comparison of Approximation Algorithms.

| Approximation Type          | Theorem 1.2, Algorithm 4 | Theorem 1.3 | Theorem 1.4 |
|-----------------------------|--------------------------|-------------|-------------|
| Size-approximation          | \( \frac{1}{f_e} (OPT + \epsilon) \) | \( \frac{OPT}{2g} \) | \( \frac{OPT}{g} \) |
| Group Fairness Violation    | \( \frac{1}{f_e} \)-additive | None | \( \frac{1}{g} \)-multiplicative |
| Individual Fairness Violation| \( \frac{1}{f_e} \)-multiplicative, \( \frac{1}{f_e} \)-additive | \( \frac{1}{g} \)-multiplicative | \( \frac{1}{g} \)-additive |

Given the individual fairness parameters, \( L_{a,S} \in [0,1] \) and \( U_{a,S} \in [0,1] \), for each item, \( a \in A \), and subset, \( S \subseteq N(a) \),

\[
\frac{1}{f_e} (L_{a,S} - \epsilon) \leq \Pr_{M \sim D} [\tilde{M} \text{ matches } a \text{ to a platform in } S] \\
\leq \frac{1}{f_e} (U_{a,S} + \epsilon).
\]

Note that if we set \( \epsilon = \min_{a \in A, S \subseteq N(a)} \frac{L_{a,S}}{2} \) in Theorem 1.2 then \( \forall a \in A, S \subseteq N(a) \),

\[
\frac{L_{a,S}}{2f_e} \leq \frac{1}{f_e} (L_{a,S} - \epsilon) \leq \Pr_{M \sim D} [\tilde{M} \text{ matches } a \text{ to a platform in } S] \leq \frac{1}{f_e} (U_{a,S} + \epsilon) \leq \frac{3U_{a,S}}{2f_e}.
\]

Therefore, we only get a multiplicative violation of individual fairness for \( \epsilon = \min_{a \in A, S \subseteq N(a)} \frac{L_{a,S}}{2} \).

**Theorem 1.3.** Given an instance of our problem, where items are classified into at most \( g \) groups per platform, each item belongs to at most \( \Delta \) groups, \( l_{p,C} = 0 \) \( \forall p \in P, C \in C_p \), that is, there are no lower bound constraints, and \( u_{p,C} \geq g \) \( \forall p \in P, C \in C_p \), we provide a polynomial-time algorithm that computes a distribution, \( D \), over a set of group-fair matchings such that the expected size of a matching chosen uniformly from \( D \) is at least \( \frac{OPT}{2g} \). Given the individual fairness parameters \( L_{a,S} \in [0,1] \) and \( U_{a,S} \in [0,1] \) for each item, \( a \in A \) and subset \( S \subseteq N(a) \),

\[
\frac{L_{a,S}}{2g} \leq \Pr_{M \sim D} [\tilde{M} \text{ matches } a \text{ to a platform in } S] \leq \frac{U_{a,S}}{2g}.
\]

**Theorem 1.4.** Given an instance of our problem, where items are classified into at most \( g \) groups per platform, each item belongs to at most \( \Delta \) groups, \( l_{p,C} = 0 \) \( \forall p \in P, C \in C_p \), that is, there are no lower bound constraints, and \( u_{p,C} \geq g \) \( \forall p \in P, C \in C_p \), we provide a polynomial-time algorithm that computes a distribution \( D \) over a set of matchings such that the expected size of a matching chosen uniformly from \( D \) is at least \( \frac{OPT}{g} \) and each matching in the distribution violates group-fairness by an additive factor of at most \( \Delta \). Given the individual fairness parameters \( L_{a,S} \in [0,1] \) and \( U_{a,S} \in [0,1] \) for each item \( a \in A \) and subset \( S \subseteq N(a) \),

\[
\frac{L_{a,S}}{g} \leq \Pr_{M \sim D} [\tilde{M} \text{ matches } a \text{ to a platform in } S] \leq \frac{U_{a,S}}{g}.
\]

Table 1 shows a comparison of all the three approximation algorithms used to prove Theorem 1.2, Theorem 1.3 and Theorem 1.4.

### 1.2 Related Work

In [García-Soriano and Bonchi, 2020], the authors introduce the distributional maxmin fairness framework, which provides the strongest guarantee possible simultaneously for each individual in terms of being matched in the solution. Their distribution is only over maximum matchings, and we extend this idea to a distribution over maximum group-fair matchings, and a stronger notion of individual fairness.

Several allocation problems like resource allocation [Halabian et al., 2011], kidney exchange programs [Farnadi et al., 2021], school choice [Abdulkadiroglu and Sonmez, 2003], candidate selection [Bei et al., 2020]...
summer internship programs [Aziz et al., 2020], and matching residents to hospitals [Makino et al., 2022] are modeled as matching problems. Since the people/items to be matched may belong to different groups, bipartite matching under group fairness constraints has been studied by quite a few works recently. The fairness constraints are captured by upper and lower bounds [Huang, 2010], or in terms of proportion of the final matching [Bei et al., 2020], especially in cases where the final number of matched items is not fixed. In some applications, the items could belong to multiple groups as well. Sankar et al., 2021 give a polynomial-time algorithm achieving an approximation ratio of \( \frac{1}{\Delta+1} \) where each item belongs to at most \( \Delta \) laminar families of groups per platform, and [Nasre et al., 2019] show that without a laminar structure, the problem is NP-hard in general. While both the papers focus only on upper bounds, we would like to incorporate lower bounds and a notion of individual fairness.

Different works look at different notions of individual fairness. In [Farnadi et al., 2021], the authors enumerate all optimal solutions and use randomized policies to select a matching to promote patients’ equal opportunity to get a transplant. [Nasre et al., 2019] address matching based on rank-maximality and popularity, and [Garcia-Soriano and Bonchi, 2020] look at distributional maximin fairness, which is the strongest guarantee possible simultaneously for all individuals in terms of the probability of being matched in the solution. We want to develop an algorithm that provides a distributional individual fairness guarantee, but each matching in the support of the resulting distribution should satisfy group fairness constraints. [Garcia-Soriano and Bonchi, 2021] provide an algorithm to find maximin-fair distributions of general search problems under group fairness constraints and [Esmaili et al., 2022] look at the Rawlsian definition of individual and group fairness in bipartite matching. However, to the best of our knowledge, no work addresses both the individual and group fairness constraints in the same instance of bipartite matching.

Among other notions of fairness, the notion proposed by Sühr et al. [Sühr et al., 2019] for ride-hailing platforms involves distributing fairness over time, thereby requiring that, over a period of time, all drivers receive benefits proportional to the amount of time spent on the platform. Singh et al., 2021 study fairness in ranking in the presence of uncertainty and provide a fairness framework that extends the definition of fairness by modeling uncertainty and the lack of complete information explicitly. [Kletti et al., 2022] provide an efficient algorithm for computing a sequence of rankings that maximizes consumer-side utility while minimizing producer-side individual unfairness of exposure.

In the area of matchings under preferences, the term fair matchings is used for a different notion of fairness. In this notion, a fair matching is a maximum matching that minimizes the number of agents matched to the last choice, subject to that, the number of agents matched to the second last choice and so on [Huang et al., 2016]. In [Fleiner and Kamiyama, 2016], Fleiner and Kamiyama introduce matroid constraints on acceptable stable matchings, whereas in [Huang, 2010], Huang addresses the problem of computing stable matchings under group fairness constraints with lower bounds. Beyhaghi and Tardos [Beyhaghi and Eva Tardos, 2021] compute the expected size of a stable matching in a job market where candidates can apply to only a few positions, and the number of interviews that can be conducted for each job is limited, under uniform and independent preferences. Patro et al. [Patro et al., 2020] consider a two-sided market of producers and consumers on a platform and formulate a method to incrementally deploy the changes in platform algorithms to guarantee fairness to both parties at any point.

2 Approximation Algorithm for the case of disjoint groups

In this section, we will focus on computing a probabilistic individually fair distribution, as defined in our problem, over a set of integer group-fair matchings on a bipartite graph where each item belongs to exactly one group; that is, all the groups are disjoint. In most real-world applications, the lower and upper bounds enforced by the platforms would be integers. Hence we will assume the same. We will first prove that any basic feasible solution of the LP GFLP(Definition 2.2) is integral. Then, to prove Theorem 1.1 given any feasible solution of LP 2.1 say \( \bar{x} \), we will use Algorithm 3 and GFLP to compute a convex combination of integer matchings and prove that \( \bar{x} \) can be written as the same. Let us first look at the LPs 2.1 and GFLP.
The proof of Lemma 2.4 can be found in Appendix A.1. In the rest of the section we will show that Algorithm 1 is a polynomial-time algorithm that returns a distribution over a set of group-fair matchings, which proves the same for Algorithm 3. To achieve that, we will first prove the following Lemma.

**Lemma 2.5**. \( \bar{x}^{(i)} \) always lies within the polytope of \( GFLP \), where \( i+1 \) denotes an arbitrary iteration of the while loop in Algorithm 1.

**Proof.** We will prove this using induction. For the base case, \( i+1 = 1 \), \( \bar{x}^{(0)} = \bar{x} \). Since \( \bar{x} \) is an optimal solution of LP 2.1, the Lemma holds by Observation 2.3. Let us assume that the Lemma holds for \( \bar{x}^{(i-1)} \), where \( i \) denotes an arbitrary iteration of the while loop in Algorithm 1. Now, we will show that the Lemma also holds for \( \bar{x}^{(i)} \). If \( \bar{x}^{(i-1)} \) is non-zero, then there exists at least one \( p \in P, C \in C_p \), such that \( u_p^{(i)} \) and \( u_{p,C}^{(i)} \) values are at least one. Therefore, \( M^{(i)} \) is a non-empty matching on \( G^{(i)} \), since \( GFLP \) returns a maximum matching that satisfies the updated group fairness constraints. First let us look at constraints 8 and 9 for an arbitrary platform, \( p \in P \). Let \( m_p^{(i)} \) be the number of edges picked in \( M^{(i)} \) for platform \( p \). From steps 8 to 8 in Algorithm 1, we know that we can have one of the following cases:

1. \( \sum_{a \in N(p)} x_{ap}^{(i-1)} \) is an integer in which case \( l_p^{(i)} = u_p^{(i)} = \sum_{a \in N(p)} x_{ap}^{(i-1)} \). Since \( M^{(i)} \) is an integer matching by Lemma 2.4, \( m_p^{(i)} = l_p^{(i)} = u_p^{(i)} = \sum_{a \in N(p)} x_{ap}^{(i-1)} \). Therefore, for all values of \( \alpha^{(i)} \in (0, 1] \),

\[
\sum_{a \in N(p)} x_a^{(i)} = \sum_{a \in N(p)} x_{ap}^{(i-1)} - \alpha^{(i)} \cdot m_p^{(i)} = \sum_{a \in N(p)} x_{ap}^{(i-1)} = l_p^{(i)} = u_p^{(i)}
\]
Algorithm 1: Distribution-Calculator($\mathcal{I} = (G, A_1 \cdots A_g, \vec{\ell}, \vec{u}, \vec{L}, \vec{U}), \vec{x}, LP$)

Input : $\mathcal{I}, \vec{x}, LP$
Output : Distribution $\mathcal{D}$ over integer matchings

1. $G^{(0)} \leftarrow G$, $\vec{x}^{(0)} \leftarrow \vec{x}$, $\mathcal{D} \leftarrow \emptyset$
2. $\alpha^{(0)} \leftarrow 0$, $\Gamma^{(0)} \leftarrow 1$, $\beta^{(0)} \leftarrow 1$
3. while $\vec{x}^{(i)} \neq \vec{0}$ do
   4. $i \leftarrow i + 1$
   5. $G^{(i)} \leftarrow G^{(i-1)} - \{(a, p) \mid x_{ap}^{(i-1)} = 0\}$
   6. $l_{p}^{(i)} \leftarrow \lfloor \sum_{a \in N(p)} x_{ap}^{(i-1)} \rfloor$, $\forall p \in P$
   7. $u_{p}^{(i)} \leftarrow \lceil \sum_{C \in \mathcal{C}(p)} x_{ap}^{(i-1)} \rceil$, $\forall p \in P$
   8. $u_{p,C}^{(i)} \leftarrow \lceil \sum_{a \in C} x_{ap}^{(i-1)} \rceil$, $\forall p \in P$, $C \in C_p$
9. Solve LP with $l_{p}^{(i)}$, $u_{p}^{(i)}$, $u_{p,C}^{(i)}$ as the bounds and let the matching returned by LP on $G^{(i)}$
   be $\vec{M}^{(i)}$
10. $\alpha^{(i)} \leftarrow \text{Find-Coefficient}(G^{(i)}, A_1 \cdots A_g, \vec{x}^{(i-1)}, \vec{M}^{(i)})$
11. $\vec{x}^{(i)} \leftarrow \vec{x}^{(i-1)} - \alpha^{(i)} \vec{M}^{(i)}$
12. $\beta^{(i)} \leftarrow \Gamma^{(i-1)} \cdot \alpha^{(i)}$
13. $\mathcal{D}^{(i)} \leftarrow (\vec{M}^{(i)}, \beta^{(i)})$
14. $\Gamma^{(i)} \leftarrow \Gamma^{(i-1)} \cdot (1 - \alpha^{(i)})$
15. $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}^{(i)}$
16. end
17. if $\mathcal{D} = \emptyset$ then
   18. Return 'Infeasible'
19. end
20. Return $\mathcal{D}$

2. $\sum_{a \in N(p)} x_{ap}^{(i-1)}$ is fractional in which case $u_{p}^{(i)} - l_{p}^{(i)} = 1$. Since $\vec{M}^{(i)}$ is an integer matching by Lemma 2.3, $m_{p}^{(i)} = u_{p}^{(i)}$ or $m_{p}^{(i)} = l_{p}^{(i)}$. Therefore, we can have the following sub cases:
   (a) $m_{p}^{(i)} = l_{p}^{(i)}$: It is easy to see that for all values of $\alpha^{(i)} \in (0, 1]$, the lower bound is always satisfied. Based on step 7 of the Routine Find-Coefficient that is called in step 10 of Algorithm 1, we know that $\alpha^{(i)} \leq \sum_{a \in N(p)} x_{ap}^{(i-1)} - \sum_{a \in N(p)} x_{ap}^{(i-1)} = u_{p}^{(i)} - \sum_{a \in N(p)} x_{ap}^{(i-1)}$. Therefore,
      $\sum_{a \in N(p)} x_{ap}^{(i-1)} \leq u_{p}^{(i)} - \alpha^{(i)}(u_{p}^{(i)} - l_{p}^{(i)}) \implies \sum_{a \in N(p)} x_{ap}^{(i-1)} - \alpha^{(i)} \cdot l_{p}^{(i)} \leq u_{p}^{(i)} \implies \sum_{a \in N(p)} x^{(i)} \leq u_{p}^{(i)}$
   Hence constraint 9 is not violated.
   (b) $m_{p}^{(i)} = u_{p}^{(i)}$: It is easy to see that for all values of $\alpha^{(i)} \in (0, 1]$, the upper bound is always satisfied. Based on step 7 of the Routine Find-Coefficient, we know that $\alpha^{(i)} \leq \sum_{a \in N(p)} x_{ap}^{(i-1)} - \sum_{a \in N(p)} x_{ap}^{(i-1)} = u_{p}^{(i)} - l_{p}^{(i)}$. Following steps similar to the above sub case, we have
      $l_{p}^{(i)} \geq \sum_{a \in N(p)} x_{ap}^{(i-1)} - \alpha^{(i)} \cdot u_{p}^{(i)} \cdot u_{p}^{(i)} \sum_{a \in N(p)} x^{(i)}$
   Hence constraint 6 is not violated.
Similar arguments can be used to show that constraints 8 and 10 are also not violated. Constraint 10 is satisfied trivially. Therefore, $\vec{x}^{(i)}$ also satisfies all the constraints of GFLP and hence lies within the polytope of GFLP.

Claim 2.6. In an arbitrary iteration of the while loop in Algorithm 1, say $i^{th}$ iteration, if there is an edge, $(a, p)$, such that $\sum_{a \in C} x_{ap} = u_{p,C}^{(i)}$ or $\sum_{a \in C} x_{ap} = l_{p,C}^{(i)}$ for some $C \in C_p$, then $l_{p,C}^{(i)} = \sum_{a \in C} x_{ap}^{(i-1)} = \sum_{a \in C} x_{ap}$.
Lemma 2.7. Algorithm 3 terminates in polynomial time.
Proof. We will show that in each iteration, at least an edge is removed, or at least one constraint becomes tight. From Claim 7, we know that once a constraint becomes tight, it stays so in the rest of the rounds. Therefore, we get a polynomial bound on the total number of iterations since the total number of constraints and edges is polynomial. Let us look at an arbitrary iteration, say $i$th iteration. If $\alpha^{(i)} = \min_{e \in M^{(i)}} x_e^{(i-1)}$, then there is at least one edge, say $(a, p) \in E$, such that $x_p^{(i)} = 0$. Otherwise, there is some platform, say $p \in P$, such that $\alpha^{(i)} = \min \left( \sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)}, \gamma_p - \sum_{a \in N(p)} x_a^{(i-1)} \right)$, or there is some platform, say $p' \in P$, with some group, say $K \in C_{p'}$, such that $\alpha^{(i)} = \min \left( \sum_{a \in K} x_a^{(i-1)} - \ell^{(i)}_{p', K}, u^{(i)}_{p', K} - \sum_{a \in K} x_{a}^{(i-1)} \right)$. Let $\alpha^{(i)} = \sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)}$, then from step 4 of Routine 2 we know that $m_p^{(i)} = u_p^{(i)}$ where $m_p^{(i)}$ is the number of edges picked in $M^{(i)}$ for platform $p$. Therefore,

$$\sum_{a \in N(p)} x_a^{(i-1)} = \frac{\sum_{a \in N(p)} x_a^{(i-1)} - \alpha^{(i)} m_p^{(i)}}{1 - \alpha^{(i)}} = \frac{\sum_{a \in N(p)} x_a^{(i-1)} - (\sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)}) \cdot m_p^{(i)}}{1 - (\sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)})} \quad (11)$$

Note that $\sum_{a \in N(p)} x_a^{(i-1)}$ must be fractional if $\alpha^{(i)} = \sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)}$ based on step 9 of Routine 2. Therefore, $\ell_p^{(i)} = [\sum_{a \in N(p)} x_a^{(i-1)}] = [\sum_{a \in N(p)} x_a^{(i-1)}] - 1 = u_p^{(i)} - 1$. Hence, from Equation (11),

$$\sum_{a \in N(p)} x_a^{(i-1)} = \frac{\sum_{a \in N(p)} x_a^{(i-1)} - (1 + \sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)}) u_p^{(i)}}{1 - (1 + \sum_{a \in N(p)} x_a^{(i-1)} - \ell_p^{(i)})} = \frac{(u_p^{(i)} - 1)(u_p^{(i)} - \sum_{a \in N(p)} x_a^{(i-1)})}{(u_p^{(i)} - \sum_{a \in N(p)} x_a^{(i-1)})} = \ell_p^{(i)}$$

Similarly, we can show that if

1. $\alpha^{(i)} = u_p^{(i)} - \sum_{a \in N(p)} x_a^{(i-1)}$, then $\sum_{a \in N(p)} x_a^{(i)} = u_p^{(i)}$.
2. $\alpha^{(i)} = \sum_{a \in K} x_a^{(i-1)} - \ell_{p', K}^{(i)}$, then $\sum_{a \in K} x_a^{(i)} = \ell_{p', K}^{(i)}$.
3. $\alpha^{(i)} = u_p^{(i)} - \sum_{a \in K} x_a^{(i-1)}$, then $\sum_{a \in K} x_a^{(i)} = u_p^{(i)}$.

Therefore, if $\alpha^{(i)} \neq \min_{e \in M^{(i)}} x_e^{(i-1)}$, that is if an edge is not removed, then either constraint 7 or constraint 8 becomes tight in the $i$th round for some platform or there is some platform, say $p' \in P$, such that either constraint 8 or constraint 9 becomes tight in the $i$th round for some group, say $K \in C_{p'}$. Since the total number of constraints is $O(|V|)$, $|E| = O(|V|^3)$, and the routine Find-Coefficient runs in $O(|V|)$ time, we get $O(|V|^3)$ iterations for the loop in Algorithm 1. Since GFLP can be solved in polynomial time, Algorithm 1 Distribution-Calculator is a polynomial-time algorithm. LP 2.1 can be solved in polynomial time, therefore, Algorithm 3 also runs in polynomial time.

Claim 2.8. Let Algorithm 3 terminate in $k$ rounds and return a set of tuples, $D = \{(\vec{M}^{(i)}, \beta^{(i)})\}_{i \in [k]}$, then, $\forall i \in [k]$,

$$\vec{x}^{(i)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i} \beta^{(j)} \vec{M}^{(j)}}{\Gamma^{(i)}}$$

Proof. From steps 12 and 13 in Algorithm 3, we know that $\beta^{(i)} = \Gamma^{(i-1)} \alpha^{(i)}$, and $\Gamma^{(i)} = \Gamma^{(i-1)} (1 - \alpha^{(i)})$ respectively, and $\Gamma^{(0)} = 1$. For the base case, $i = 1$, we know from step 11 in Algorithm 4 that $\vec{x}^{(1)} = \frac{\vec{x}^{(0)} - \alpha^{(1)} \vec{M}^{(1)}}{1 - \alpha^{(1)}}$. It is easy to see that $\Gamma^{(1)} = (1 - \alpha^{(1)})$ and $\beta^{(1)} = \alpha^{(1)}$, therefore,

$$\vec{x}^{(1)} = \frac{\vec{x}^{(0)} - \beta^{(1)} \vec{M}^{(1)}}{\Gamma^{(1)}}$$

For the induction step, for some $i \in \mathbb{Z} \cap (1, k]$, let $\vec{x}^{(i-1)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i-1} \beta^{(j)} \vec{M}^{(j)}}{\Gamma^{(i-1)}}$. We know that $\vec{x}^{(i)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i} \beta^{(j)} \vec{M}^{(j)}}{1 - \alpha^{(i)}}$, therefore, by induction hypothesis,

$$\vec{x}^{(i)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i-1} \beta^{(j)} \vec{M}^{(j)}}{1 - \alpha^{(i)}} - \alpha^{(i)} \vec{M}^{(i)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i-1} \beta^{(j)} \vec{M}^{(j)} - \Gamma^{(i-1)} \alpha^{(i)} \vec{M}^{(i)}}{\Gamma^{(i-1)} (1 - \alpha^{(i)})},$$

hence,

$$\vec{x}^{(i)} = \frac{\vec{x}^{(0)} - \sum_{j=1}^{i} \beta^{(j)} \vec{M}^{(j)}}{\Gamma^{(i)}}$$

\qed
Lemma 2.9. Let Algorithm 2 terminate in $k$ rounds and return a set of tuples, $D = \{(\bar{M}^{(i)}, \beta^{(i)})\}_{i \in [k]}$, then, $\bar{x} = \sum_{i=1}^{k} \beta^{(i)} \bar{M}^{(i)}$, where $\bar{x}$ is computed in step 4 of Algorithm 2 and $\sum_{i=1}^{k} \beta^{(i)} = 1$.

Proof. From Claim 2.8, we know that $\forall i \in [k]$:

$$\bar{x}^{(i)} = \frac{\bar{x}^{(i)} - \sum_{j=1}^{k} \beta^{(j)} \bar{M}^{(j)}}{\Gamma^{(i)}}.$$

Since $\bar{x}^{(k)} = \bar{0}$, we have $\bar{x}^{(0)} = \bar{x} = \sum_{i=1}^{k} \beta^{(i)} \bar{M}^{(i)}$.

We will prove that $\sum_{i=1}^{k} \beta^{(i)} = 1$, using induction on $i$, backwards from $k$ to 0. For the base case, $i = k$, $\bar{x}^{(k)} = \bar{0}$, therefore, for any real values of $\alpha$, $\bar{x}^{(k)} = (1 - \alpha)\bar{M} + \alpha\bar{M}$, where $\bar{M}$ is an empty matching. Note that if $u_p, l_p, u_p, C, l_p, C$ are set to 0, $\forall p \in P, C \in C_p$, GFLP would compute an empty matching. Therefore, $\bar{x}^{(k)}$ can be written as a convex combination of integer matchings computed by GFLP. For the induction step, let us assume that $\bar{x}^{(i+1)} = \sum_{j} \gamma_j \bar{M}_j$, where $\bar{M}_j$ is an integer matching computed by GFLP for all values of $j$ and $\sum \gamma_j = 1$, for some $i \in [k-1]$. We know that $\bar{x}^{(i+1)} = \frac{\bar{x}^{(i)} - \alpha \bar{x}^{(i)} \bar{M}^{(i)}}{1 - \alpha \bar{x}^{(i)} \bar{M}^{(i)}}$.

Therefore, $\bar{x}^{(i)} = (1 - \alpha^{(i)}) \bar{x}^{(i+1)} + \alpha^{(i)} \bar{M}^{(i)} = (1 - \alpha^{(i)}) \sum_{j} \gamma_j \bar{M}_j + \alpha^{(i)} \bar{M}^{(i)}$

Since $\sum \gamma_j = 1$, by the induction hypothesis, $(1 - \alpha^{(i)}) \sum \gamma_j + \alpha^{(i)} = 1$, therefore, $\bar{x}^{(i)}$ is also a convex combination of integer matchings computed by GFLP. From Claim 2.8 we know that, $\forall i \in [k]$, $\bar{x}^{(i)} = \frac{\bar{x}^{(i)} - \sum_{j=1}^{k} \beta^{(j)} \bar{M}^{(j)}}{\Gamma^{(i)}}$, hence,

$$\bar{x}^{(0)} = \Gamma^{(i)} \bar{x}^{(i)} + \sum_{j=1}^{i} \beta^{(j)} \bar{M}^{(j)}.$$

Since, we have already shown that, $\bar{x}^{(i)}$ is a convex combination of integer matchings computed by GFLP using induction, we just need to show that $\Gamma^{(i)} + \sum_{j=1}^{i} \beta^{(j)} = 1$. Expanding $\Gamma^{(i)}$ and $\sum_{j=1}^{i} \beta^{(j)}$, we have

$$\Gamma^{(i)} + \sum_{j=1}^{i} \beta^{(j)} = \Pi_{j=1}^{i} (1 - \alpha^{(j)}) + \sum_{j=1}^{i} \Pi_{l=1}^{j-1} (1 - \alpha^{(l)}) \alpha^{(j)} = 1$$

Therefore, $\sum_{i=1}^{k} \beta^{(i)} = 1$. $\square$

Lemma 2.10. Let us consider a set of tuples, $D = \{(\bar{M}^{(i)}, \beta^{(i)})\}_{i \in [k]}$, where $\bar{M}^{(i)}$ is an integer matching and $\beta^{(i)}$ is a scalar, $\forall i \in [k]$, where $k \in \mathbb{Z}$. Let $\bar{x} = \sum_{i=1}^{k} \beta^{(i)} \bar{M}^{(i)}$ such that $\sum_{i=1}^{k} \beta^{(i)} = 1$, and $\|\bar{x} - \bar{x}_x\|_1 \leq \delta$ where $\bar{x}$ is any feasible solution of LP [2.7] $\delta \in [0, 1)$, and $t \geq 1$. The probability that an item, $a \in A$, is matched to a platform $p \in S$, where $S \subseteq N(a)$, in a matching sampled from the support of $D$ is

$$\frac{L_{a,S}}{t} - \delta \leq \Pr_{\bar{M} \sim D} \{\bar{M} \text{ matches } a \text{ to some platform in } S\} \leq \frac{L_{a,S}}{t} + \delta.$$

Proof. Given that $\|\bar{x} - \bar{x}_x\|_1 \leq \delta$, therefore,

$$\sum_{(a,p) \in E} \left| \bar{x}_{ap} - \frac{x_{ap}}{t} \right| \leq \delta \quad \text{(12)}$$

Let’s fix an arbitrary item $a \in A$, and let $S$ be an arbitrary subset of $N(a)$, then from Equation (12),

$$\sum_{p \in S} \left| \bar{x}_{ap} - \frac{x_{ap}}{t} \right| \leq \delta \implies \sum_{p \in S} \left| \bar{x}_{ap} - \frac{x_{ap}}{t} \right| \leq \delta$$
The last inequality holds due to triangle inequality. Therefore,
\[
\frac{1}{t} \sum_{p \in S} x_{ap} - \delta < \sum_{p \in S} \hat{x}_{ap} < \frac{1}{t} \sum_{p \in S} x_{ap} + \delta
\]  \tag{13}

Since \( \bar{x} = \sum_{i=1}^{k} \beta^{(i)} \bar{M}^{(i)} \), the probability that an item \( a \in A \) is matched to a platform \( p \in S \), where \( S \subseteq N(a) \), in a matching sampled from \( D \) is
\[
\Pr[a \text{ is matched to a platform in } S] = \sum_{i:M^{(i)} \text{ matches } a \text{ to } p} \beta^{(i)} = \sum_{p \in S} \sum_{i:M^{(i)} \text{ matches } a \text{ to } p} \beta^{(i)} = \sum_{p \in S} \hat{x}_{ap}
\]

Therefore, from constraints \( 4 \) and \( 5 \) and Equation \( 13 \), we have
\[
\frac{L_{a,S}}{t} - \delta \leq \Pr_{\bar{M} \sim \mathcal{D}} [\bar{M} \text{ matches } a \text{ to some platform in } S] \leq \frac{U_{a,S}}{t} + \delta.
\]

Proof of Theorem 1.1. Let Algorithm 3 terminate in \( k \) rounds and return a set of tuples, \( D = \{ (\bar{M}^{(i)}, \beta^{(i)}) \}_{i \in [k]} \).

Therefore, \( \bar{x} = \sum_{i=1}^{k} \beta^{(i)} \bar{M}^{(i)} \) and \( \sum_{i=1}^{k} \beta^{(i)} = 1 \) by Lemma 2.5. We know that after every iteration, we get another point inside the polytope of GFLP by Lemma 2.5.

In every iteration, the integer matching being computed in step \( 9 \) of Algorithm 3 satisfies group fairness constraints. Therefore, Algorithm 3 returns a distribution over group-fair integer matchings. By substituting \( \bar{x} = \bar{y} = 0 \) and \( t = 1 \) in Lemma 2.10, we get that the probability that an item \( a \in A \) is matched to a platform \( p \in S \), where \( S \subseteq N(a) \), in a matching sampled from \( D \) is
\[
\frac{L_{a,S}}{t} \leq \sum_{p \in S} x_{ap} \leq \frac{U_{a,S}}{t},
\]

\( \forall a \in A, S \subseteq N(a) \). Hence, \( D \) is a probabilistic individually fair distribution. The run time has been shown to be polynomial in \( \mathcal{L} 3.1 \) This proves the theorem.

In the next two sections, we will focus on computing a probabilistic individually fair distribution over a set of integer group-fair matchings on a bipartite graph with a maximum of \( g \) groups per platform, where each item can belong to at most \( \Delta \) groups.

3 \( O(\Delta \log n) \) bicriteria approximation algorithm

In this section, we will prove Theorem 1.2 using Algorithm 4 and LP 3.2, which is the dual of LP 3.1.

LP 3.1.
\[
\max \sum_{(a,p) \in E} x_{ap} \tag{14}
\]

such that
\[
\sum_{a \in C} x_{ap} \leq u_{p,C}, \quad \forall C \subseteq C_p, \forall p \in P \tag{15}
\]
\[
0 \leq x_{ap} \leq 1, \quad \forall (a,p) \in E \tag{16}
\]

LP 3.2.
\[
\min \sum_{p \in P} \sum_{C \subseteq C_p} u_{p,C} w_{p,C} + \sum_{(a,p) \in E} y_{ap} \tag{17}
\]

such that
\[
1 \leq \sum_{C \subseteq C_p} w_{p,C} + y_{ap}, \quad \forall (a,p) \in E \tag{18}
\]

We will first prove that any greedy maximal matching computed in step 10 of Algorithm 4 is a \((\Delta + 1)\)-approximation of any feasible solution of LP 3.1 using dual fitting analysis technique. \[Williamson and Shmoys, 2011\] in the following two lemmas.

Observation 3.3. Any feasible solution of LP 3.1 augmented with constraints 4 and 5 is also a feasible solution of LP 3.1.

Lemma 3.4. Let \( \bar{x} \) be a greedy maximal matching computed in step 10 of Algorithm 4. Let \( \bar{y} \) be a vector such that \( \forall (a,p) \in E, y_{ap} \) is set to 1 if \( x_{ap} = 1 \), and \( \bar{w} \) be a vector such that \( \forall p \in P, C \subseteq C_p, w_{p,C} \) is set to 1 if \( \sum_{a \in C} x_{ap} = u_{p,C} \). Then, \( \bar{y} \) and \( \bar{w} \) are a feasible solution of LP 3.2.
Algorithm 4: $O(\Delta \log n)$-BicriteriaApprox($\mathcal{I} = (G, A_1 \cdots A_{\Delta}, \bar{\mathcal{I}}, \bar{\mathcal{U}}, \bar{\mathcal{U}})$, $\epsilon$)

**Input**: $\mathcal{I}, \epsilon$

**Output**: Distribution over matchings satisfying the guarantees in Theorem \ref{thm:bicriteria}

1. Solve LP 3.3 augmented with constraints 4 and 5 on $G$ and store the result in $\hat{\mathcal{I}}$.
2. $i \leftarrow 0$
3. $\alpha^{(0)} \leftarrow 0$
4. $\mathcal{G}^{(0)} \leftarrow G$
5. $\hat{x}^{(0)} \leftarrow \hat{x}$
6. $\sum = 0$
7. while $\|\hat{x}^{(i)}\|_1 \geq \epsilon$ do
   8. $i \leftarrow i + 1$
   9. $\mathcal{G}^{(i)} \leftarrow \mathcal{G}^{(i-1)} - \{(a, p) \mid x_{ap}^{(i-1)} = 0\}$
   10. Greedily find a Maximal Matching $\hat{\mathcal{M}}^{(i)}$ in $\mathcal{G}^{(i)}$ such that constraints (15) are not violated.
   11. $\alpha^{(i)} \leftarrow \min_{(a, p) \in \hat{\mathcal{M}}^{(i)}} \{x_{ap}^{(i-1)}\}$
   12. $\sum \leftarrow \sum + \alpha^{(i)}$
   13. $\hat{x}^{(i)} \leftarrow \hat{x}^{(i-1)} - \alpha^{(i)} \cdot \hat{\mathcal{M}}^{(i)}$
   14. $\mathcal{D}^{(i)} \leftarrow (\hat{\mathcal{M}}^{(i)}, \alpha^{(i)})$
   15. $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}^{(i)}$
8. end
17. for $\mathcal{D}^{(i)} \in \mathcal{D}$ do
18. $\mathcal{D}^{(i)} \leftarrow (\hat{\mathcal{M}}^{(i)}, \frac{\alpha^{(i)}}{\sum})$
19. end
20. Return $\mathcal{D}$

**Proof.** Let us fix an arbitrary edge, say $(a, p) \in E$. If $x_{ap} = 1$, $y_{ap} = 1$ by definition, therefore constraint \textbf{[18]} is satisfied. Let $T_a$ denote a set of groups such that $a \in K$, $\forall K \in T_a$. If $x_{ap} = 0$, we will show that there exists at least one group, say $K_q \in C_p \cap T_a$, such that $w_{p,K_q} = 1$. Suppose $\forall K \in C_p \cap T_a$, $w_{p,K} = 0$. This implies that $\forall K \in C_p \cap T_a$, $\sum_{p \in K} x_{ap} < u_{p,K}$, by definition of $\bar{\mathcal{I}}$, therefore the edge, $(a, p)$ can be included in the matching without violating constraint \textbf{15}, which is a contradiction since $\bar{\mathcal{I}}$ is a maximal matching. Hence, there exists at least one group, say $K_q \in C_p$, such that $K_q \in T_a$ and $\sum_{a \in K_q} x_{ap} = u_{p,K_q}$; which in turn implies that there exists at least one group, $K_q \in C_p$, such that $K_q \in T_a$ and $w_{p,K_q} = 1$. Therefore, constraint \textbf{[18]} is not violated and $\bar{y}$ and $\bar{w}$ are a feasible solution to LP \textbf{5.2}.

**Lemma 3.5.** If $\bar{x}$ is a greedy maximal matching computed in step \textbf{[14]} of Algorithm \textbf{4}, then \[
\sum_{(a, p) \in E} \frac{\Psi_{ap}}{\Delta + 1}, \text{ where } \Psi \text{ is any feasible solution of LP } 3.3 \text{ augmented with constraints } 4 \text{ and } 7.
\]

**Proof.** Let $\tilde{y}$ be a vector such that $\forall (a, p) \in E$, $y_{ap}$ is set to 1 iff $x_{ap} = 1$, and $\tilde{w}$ be a vector such that $\forall p \in P, C \in C_p$, $w_{p,C}$ is set to 1 iff $\sum_{a \in C} x_{ap} = u_{p,C}$. From Lemma \textbf{3.3} we know that $\tilde{y}$ and $\tilde{w}$ are a feasible solution of LP \textbf{4.2}. Let $\tilde{\psi}$ be the dual objective function evaluated at $\tilde{y}$ and $\tilde{w}$ and $\psi$ be the primal objective function evaluated at $\bar{x}$. Note that by definition of $\tilde{y}$, $\sum_{(a, p) \in E} y_{ap}$ is equal to the number of edges in the maximal matching, which is nothing but $\psi$. Since $w_{p,C}$ is set to 1 iff $\sum_{a \in C} x_{ap} = u_{p,C}$, \[
\sum_{p \in P} \sum_{C \in C_p} u_{p,C} w_{p,C} = \sum_{p \in P} \sum_{C \in C_p} x_{ap}.
\]

Since any item, say $a \in A$, can belong to at most $\Delta$ groups, any edge, $(a, p)$, such that $x_{ap} = 1$, can contribute to at most $\Delta$ many tight upper bounds, therefore, \[
\sum_{p \in P} \sum_{C \in C_p} \sum_{a \in C} x_{ap} \leq \Delta \psi.
\]

Hence, \[
\hat{\psi} = \sum_{p \in P} \sum_{C \in C_p} u_{p,C} w_{p,C} + \psi \leq \Delta \psi + \psi = (\Delta + 1)\psi.
\]
Let $\psi^*$ and $\tilde{\psi}^*$ be the optimal objective costs of LP 3.1 and LP 3.2, respectively, since LP 3.1 is a maximization, we get

$$(\Delta + 1)\psi \geq \tilde{\psi} \geq \psi^*$$

\[\square\]

**Lemma 3.6.** Algorithm 4 runs in polynomial time.

**Proof.** Let $i$ denote an arbitrary iteration of the while loop in Algorithm 4. Since $\alpha^{(i)} = \min_{(a,p) \in M^{(i)}} x_{ap}^{(i-1)}$, as seen in step 11 at least one edge is removed from the support of the solution, in each iteration. Hence the norm can go to zero in $|E| = O(|V|^2)$ iterations, and since the algorithm exits once $\|\tilde{x}\| < \epsilon$, it runs for at most $|E|$ rounds, therefore, the while loop in Algorithm 4 terminates in $O(|V|^2)$ time. Since the LP in step 11 can be solved in polynomial time, Algorithm 4 runs in polynomial time.

\[\square\]

**Observation 3.7.** Let Algorithm 4 terminate in $k$ iteration, then, $\forall i \in \{0\} \cup [k]$, $x_{ap}^{(i)} \geq 0, \forall (a, p) \in E$.

**Proof.** We will use induction to show this. For the base case, $i = 0$, since $\tilde{x}$ is a feasible solution of LP 3.1 by Observation 3.3, $x_{ap}^{(0)} = x_{ap} \geq 0, \forall (a, p) \in E$ because of constraint 16. For the induction step let us assume that $x_{ap}^{(i-1)} \geq 0, \forall (a, p) \in E$. Since no edge, say $(a, p)$, such that $x_{ap}^{(i-1)} = 0$, will be picked in the maximal matching, $M^{(i)}$, $x_{ap}^{(i)} = x_{ap}^{(i-1)} - \min_{(a,p) \in M^{(i)}} \{x_{ap}^{(i-1)}\}$, if $x_{ap}^{(i-1)} \neq 0$. Therefore, $x_{ap}^{(i)} \geq 0, \forall (a, p) \in E$.

\[\square\]

**Claim 3.8.** Let Algorithm 4 terminate in $k$ iteration, then, $\forall i \in [k-1]$,

$$\|\sum_{j=1}^{k} \alpha^{(j)} \tilde{M}^{(j)}\|_1 \leq \|\tilde{x}^{(i-1)}\|_1$$

**Proof.** If Algorithm 4 terminates in $k$ rounds, from step 13 of Algorithm 4 we know that $\forall (a, p) \in E$,

$$x_{ap}^{(k)} = x_{ap}^{(k-1)} - \alpha^{(k)} \tilde{M}_{ap}^{(k)}$$

(19)

Let us consider an integer, $i \leq k - 1$, then, recursively replacing $x_{ap}^{(k-1)}$ on the RHS of Equation (19) until the index reaches $i - 1$, we have

$$x_{ap}^{(i)} = x_{ap}^{(i-1)} - \sum_{j=i}^{k} \alpha^{(j)} \tilde{M}_{ap}^{(j)}$$

Since $x_{ap}^{(i)} \geq 0, \forall (a, p) \in E$, from Observation 3.7 $\forall (a, p) \in E$, $\sum_{j=i}^{k} \alpha^{(j)} \tilde{M}_{ap}^{(j)} \leq x_{ap}^{(i-1)}$. Therefore, $\forall i \in [k-1]$,

$$\|\sum_{j=1}^{k} \alpha^{(j)} \tilde{M}^{(j)}\|_1 \leq \|\tilde{x}^{(i-1)}\|_1$$

\[\square\]

**Lemma 3.9.** Let $i_h$ denote the first iteration of the while loop in Algorithm 4 such that $\|\tilde{x} - \sum_{j=1}^{i_h} \alpha^{(j)} \cdot \tilde{M}^{(j)}\|_1 < \frac{\|\tilde{x}\|}{2}$, then $\sum_{j=1}^{i_h} \alpha^{(j)} \leq 2h(\Delta + 1)$.

**Proof.** Let Algorithm 4 terminate in $k$ rounds. Since $\forall i \in \{0\} \cup [k]$, $x_{ap}^{(i)} \geq 0 \forall (a, p) \in E$, from Observation 3.7 it is easy to see that $\alpha^{(i)} = \min_{(a,p) \in M^{(i)}} \{x_{ap}^{(i-1)}\} > 0, \forall i \in [k]$. Hence $\sum_{a \in C} x_{ap}^{(i)} \leq u_{p,C}$ $\forall C \in C_p, \forall p \in P$, and $\tilde{x}^{(i)}$ is a feasible solution of LP 3.1. Therefore, using Lemma 3.3 we have

$$\|\tilde{M}^{(j)}\|_1 \geq \frac{\|\tilde{x} - \sum_{j=1}^{i_h-1} \alpha^{(j)} \cdot \tilde{M}^{(j)}\|_1}{\Delta + 1}$$

(20)

Now we will prove the Lemma by induction on $h$. Let’s first look at the base case where $h = 1$. By definition,

$$\|\tilde{x} - \sum_{j=1}^{i_1} \alpha^{(j)} \cdot \tilde{M}^{(j)}\|_1 < \frac{\|\tilde{x}\|}{2}$$
Since, \( \forall j \leq i_1, \| \vec{x}^{(j)} \|_1 \geq \frac{\| \vec{x} \|_1}{2} \), from Equation (20) we have \( \| \vec{M}'^{(j)} \|_1 \geq \frac{\| \vec{x}^{(j)} \|_1}{\Delta + 1} \geq \frac{\| \vec{x} \|_1}{2(\Delta + 1)} \), \( \forall j \leq i_1 \). From Claim 3.8 we know that \( \| \sum_{j=1}^{k} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \leq \| \vec{x}^{(0)} \|_1 = \| \vec{x} \|_1 \). Since \( i_1 \leq k \), \( \| \sum_{j=1}^{i_1} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \leq \| \vec{x} \|_1 \). Therefore,

\[
\| \vec{x} \|_1 \geq \| \sum_{j=1}^{i_1} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \geq \sum_{j=1}^{i_1} \alpha^{(j)} \cdot \frac{\| \vec{x} \|_1}{2(\Delta + 1)}
\]

\[
\Rightarrow \sum_{j=1}^{i_1} \alpha^{(j)} \leq 2(\Delta + 1)
\]

(21)

For the induction step, let us assume that for some iteration \( i_{h-1} \),

\[
\sum_{j=1}^{i_{h-1}} \alpha^{(j)} \leq 2(h-1)(\Delta + 1)
\]

(22)

By definition, \( \| \vec{x} - \sum_{j=1}^{i_h} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 < \frac{\| \vec{x} \|_1}{2} \), therefore, \( \forall j \leq i_h \), \( \| \vec{x}^{(j)} \|_1 \geq \frac{\| \vec{x} \|_1}{2} \). From Equation (20), we get \( \forall j \leq i_h, \| \vec{M}^{(j)} \|_1 \geq \frac{\| \vec{x} \|_1}{2(\Delta + 1)} \). Therefore,

\[
\| \sum_{j=i_{h-1}+1}^{i_h} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \geq \sum_{j=i_{h-1}+1}^{i_h} \alpha^{(j)} \cdot \frac{\| \vec{x} \|_1}{2^h(\Delta + 1)}
\]

(23)

From Claim 3.8 we know that \( \| \sum_{j=i_h-1+1}^{i_h} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \leq \| \vec{x}^{(i_h-1)} \|_1 \). Since, \( i_h \leq k \),

\[
\| \sum_{j=i_{h-1}+1}^{i_h} \alpha^{(j)} \cdot \vec{M}^{(j)} \|_1 \leq \| \vec{x}^{(i_h-1)} \|_1 = \| \vec{x} - \sum_{j=1}^{i_{h-1}} \alpha^{(j)} \vec{M}^{(j)} \|_1 < \frac{\| \vec{x} \|_1}{2^{h-1}}
\]

Therefore, using Equation (23), we have

\[
\sum_{j=i_{h-1}+1}^{i_h} \alpha^{(j)} < 2(\Delta + 1)
\]

(24)

Combining Equation (22) and Equation (24),

\[
\sum_{j=1}^{i_h} \alpha^{(j)} = \sum_{j=1}^{i_{h-1}} \alpha^{(j)} + \sum_{j=i_{h-1}+1}^{i_h} \alpha^{(j)} < 2(h-1)(\Delta + 1) + 2(\Delta + 1) = 2h(\Delta + 1)
\]

\[
\square
\]

**Proof.** Proof of Theorem 1.2 We know that each matching in the distribution is group-fair because in each iteration, the matching being computed in step [10] is a group-fair maximal matching. Let \( \vec{x}^{(i)} \) be the state of \( \vec{x} \) after \( i \) rounds of the loop in Algorithm 4. Let \( \vec{M}^{(i)} \) and \( \alpha^{(i)} \) be the greedy maximal matching and it’s coefficient being calculated in the \( i \)th round of the loop in Algorithm 4. Let Algorithm 4 terminate after \( k \) iterations, then \( \vec{x}^{(k)} = \vec{x} - \sum_{(\vec{M}^{(i)}, \alpha^{(i)}) \in D} \alpha^{(i)} \vec{M}^{(i)} \). Therefore,

\[
\vec{x} - \vec{x}^{(k)} = \sum_{(\vec{M}^{(i)}, \alpha^{(i)}) \in D} \alpha^{(i)} \vec{M}^{(i)} \Rightarrow \vec{x} - \vec{x}^{(k)} = \sum_{i=1}^{k} \alpha^{(i)} \vec{M}^{(i)} = \sum_{i=1}^{k} \frac{\alpha^{(i)}}{\sum_{i=1}^{k} \alpha^{(i)}} \vec{M}^{(i)}
\]

In other words, \( \frac{\vec{x} - \vec{x}^{(k)}}{\sum_{i=1}^{k} \alpha^{(i)}} \) can be written as a convex combination of group-fair greedy maximal matchings.

We will first find an upper bound for \( \sum_{j=1}^{k} \alpha^{(j)} \). Let \( h' \) be such that \( \frac{\| \vec{x} \|_1}{2^h} \leq \epsilon \), by Lemma 3.9

\[
\sum_{j=1}^{h'} \alpha^{(j)} \leq 2h'(\Delta + 1).
\]
Algorithm 5: 2g-BicriteriaApprox(\(I = (G, A_1 \cdots A_\chi, \vec{t}, \vec{u}, \vec{L}, \vec{U})\))

**Input:** \(I\)

**Output:** Distribution over matchings satisfying the guarantees in Theorem 1.3

1. Solve LP 3.1 augmented with constraints 4 and 5 on \(G\) and store the result in \(\vec{x}\)

2. \(g = \max_{p \in P} |C_p|\)

3. For each item \(a \in A\), we remove it from every group other than \(C_a\) where \(C_a = \arg \min_{C \in C_p : a \in C} u_{p,C}\). Let the resulting graph be \(G'\).

4. \(I' = (G', A_1' \cdots A'_\chi, \vec{t}, \vec{u}, \vec{L}, \vec{U})\)

5. Return Distribution-Calculator\(I', \vec{x}_2^g, LP 4.5\)

Since \(\|x\|_1 \leq n\), setting \(\|x\|_1 = n\), we have \(\frac{n}{2^\chi} < \epsilon\). Setting \(h' = \log(n/\epsilon) + 1\), we have

\[
\sum_{i=1}^{h'} a(i) < 2(\Delta + 1)(\log(n/\epsilon) + 1).
\]

Let \(f_\epsilon = 2(\Delta + 1)(\log(n/\epsilon) + 1)\), then \(\vec{x}_2^g\) lies inside the polytope of LP 4.1. This proves the theorem.

### 4 \(O(g)\) Bicriteria Approximation Algorithms

In this section, we will continue to work with an instance of a bipartite graph, \(G(A,P,E)\), that has a maximum of \(g\) groups per platform, and any item, \(a \in A\), can belong to at most \(\Delta\) groups. We first reduce this instance to one where \(\Delta = 1\), then use GFLP with specific bounds in the form of LP 4.1 and Algorithm 5 to compute a distribution over matchings in Algorithm 5. Since Section 2 also addresses an instance where the groups are disjoint, we will use Lemmas from, and analysis technique similar to Section 2 in order to prove Theorem 1.3. Let us first look at LP 4.1.

**LP 4.1.**

\[
\text{max} \sum_{(a,p) \in E} x_{ap} \quad \text{such that} \quad \begin{align*}
\sum_{a \in C} x_{ap} &\leq \left\lfloor \frac{u_{p,C}}{g} \right\rfloor , & \forall C \in C_p , \forall p \in P \\
0 &\leq x_{ap} \leq 1 & \forall (a,p) \in E
\end{align*}
\]

**Observation 4.2.** Let \(\vec{x}\) be a feasible solution of LP 3.1 and \(u_{p,C} \geq g \forall p \in P,C \in C_p\), then \(\vec{x}_2^g\) lies inside the polytope of LP 4.1.

**Proof.** Let \(d\) be any positive real number, then we will consider the following two cases:

1. \(d \geq 2\): It is trivial to see that \(\frac{d}{2} \leq d - 1 \leq |d|\) in this case.

2. \(d \in [1,2)\): In this case, \(d = 1 + \delta\) where \(\delta \in [0,1)\). Therefore, \(\frac{d}{2} < \frac{1}{2}\), hence,

\[
\frac{d}{2} = \frac{1}{2} + \frac{\delta}{2} < \frac{1}{2} < 1 = |d|
\]

Therefore, for any positive real number \(d \geq 1\),

\[
\frac{d}{2} \leq |d|
\]
Since $\vec{x}$ is a feasible solution of LP 3.1, \( \sum_{a \in C} x_{ap} \leq u_{p,C} \) \( \forall p \in P, C \in C_p \) by constraint 26, therefore, \( \forall p \in P, C \in C_p \)
\[
\frac{\sum_{a \in C} x_{ap}}{2g} \leq \frac{u_{p,C}}{2g} \leq \left\lfloor \frac{u_{p,C}}{g} \right\rfloor
\]

The last inequality holds because of the assumption \( u_{p,C} \geq g \) \( \forall p \in P, C \in C_p \), which implies \( \frac{u_{p,C}}{g} \geq 1 \) \( \forall p \in P, C \in C_p \). Therefore, \( \frac{\vec{x}}{2g} \) satisfies constraint 29, constraint 27 is also satisfied because \( \forall (a, p) \in E \), \( 0 \leq x_{ap} \leq 1 \), therefore, \( 0 \leq \frac{\vec{x}}{2g} \leq 1 \).

**Lemma 4.3.** Let $\vec{x}$ be any optimal solution of LP 4.1 on the graph resulting after step 3 in Algorithm 1, then $\vec{x}$ is an integer matching on $G$ that satisfies the group fairness constraint 15.

**Proof.** Any vertex solution of LP 4.1 on $G'$ is integral if the groups are disjoint, by Lemma 2.4, therefore, $\vec{x}$ is a matching if the groups are disjoint. Let's fix an arbitrary platform, $p$, and number all the groups in $C_p$ in the ascending order of their upper bounds, breaking ties arbitrarily, that is, for any two groups say $K_i, K_j \in C_p$, $u_{p,K_i} > u_{p,K_j}$ if $j > i$. Let us consider an arbitrary group, $K_{q-1} \in C_p$, with upper bound $u_{p,K_{q-1}}$. Any item $a \in A$ that has been removed from this group in step 3 of Algorithm 5 could only be in one of the groups from $K_1$ to $K_{q-1}$. This is because an item stays in the group with the lowest upper bound. Let $m_i = \sum_{a \in K_i} x_{ap}$, then $m_i \leq \left\lfloor \frac{u_{p,K_i}}{g} \right\rfloor$ due to constraint 29. Therefore,
\[
\sum_{i=1}^{g} m_i \leq \sum_{i=1}^{g} \left\lfloor \frac{u_{p,K_i}}{g} \right\rfloor \leq \sum_{i=1}^{g} \frac{u_{p,K_i}}{g} \leq \sum_{i=1}^{g} \frac{u_{p,K_q}}{g} = u_{p,K_q}
\]

Therefore, $\sum_{a \in C} x_{ap} \leq u_{p,C} \forall C \in C_p$ after all the items are returned to all the groups they belonged to in the original graph. Therefore, $\vec{x}$ satisfies constraint 15.

**Lemma 4.4.** Given a bipartite graph $G(A, P, E)$ with possibly non-disjoint groups and an optimal solution of LP 4.1, Algorithm 5 returns a distribution over integer matchings in polynomial time, such that each matching satisfies group fairness constraints.

**Proof.** We start with an optimal solution of LP 3.1 augmented with constraints 4 and 5 which is a feasible solution of LP 4.1, therefore, \( \vec{x} \) is a feasible solution of LP 4.1 by Observation 1.2. Let $\vec{x}^{(i)}$ be the state of the optimal solution of LP 3.1 after the \( i \)th iteration of Algorithm 1. Note that LP 4.1 is GFLP (Definition 2.2) with specific upper and lower bounds, therefore, $\vec{x}^{(i)}$ always lies within the polytope of LP 4.1 by Lemma 2.7, \( \forall i \in [k - 1] \), where $k$ is the number of iterations after which Algorithm 1 terminates. Therefore, if $\vec{x}^{(i-1)}$ is non empty, a non empty integer matching is computed in step 6 of Algorithm 1 for $g$ rounds and by Lemma 1.3 we know that each such matching satisfies group fairness constraints. From Lemma 2.9 we know that $\vec{x}$ can be written as a convex combination of integer matchings computed by LP 4.1. Therefore, Algorithm 6 returns a distribution over group-fair integer matchings. The run time of Algorithm 1 has been shown to be polynomial in Lemma 2.7 since LP 3.1 can be solved in polynomial time. Algorithm 6 also runs in polynomial time.

**Proof of Theorem 1.3** Let $\vec{x}$ be any optimal solution of LP 3.1 augmented with 4 and 5, then, Algorithm 6 can be used to represent $\vec{x}$ as a convex combination of integer group-fair matchings in polynomial time by Lemma 1.4. By setting $\vec{x} = \vec{x}, \delta = 0$, and $t = 2g$ in Lemma 2.10 we have $\forall a \in A, S \subseteq N(a)$,
\[
\frac{L_{a,S}}{2g} \leq \Pr_{M \sim D} [\vec{M} \text{ matches } a \text{ to some platform in } S] \leq \frac{U_{a,S}}{2g}
\]
This proves the theorem.

### 4.1 Group Fairness Violation

In this section we will prove Theorem 1.3 and use LP 4.5 instead of LP 4.1 to reduce the problem to something similar to the problem we saw in Section 2 then use Algorithm 6 which is a slightly modified version of Algorithm 5 and Lemmas from Section 2 to prove Theorem 1.3.
LP 4.5.

\[
\max \sum_{(a,p) \in E} x_{ap} \tag{28}
\]
\[
\text{such that } \sum_{a \in C} x_{ap} \leq \left\lceil \frac{u_{p,C}}{g} \right\rceil, \quad \forall C \in C_p, \forall p \in P \tag{29}
\]
\[
0 \leq x_{ap} \leq 1 \quad \forall (a,p) \in E \tag{30}
\]

Observation 4.6. Let \(x\) be a feasible solution of LP 4.1, then \(\frac{\hat{x}}{g}\) lies inside the polytope of LP 4.5.

Lemma 4.7. The solution computed by LP 4.5 in Algorithm 6 is an integer matching that violates the group fairness constraint \(\frac{\hat{x}}{g}\) by an additive factor of at most \(\Delta\).

Proof. Any optimal solution of LP 4.5 on \(G'\) is integral if the groups are disjoint, by Lemma 2.4, therefore, \(x\) is an integer matching on \(G'\). Let’s fix an arbitrary platform, \(p\), and number all the groups in \(C_p\) in the ascending order of their upper bounds, that is, for any two groups say \(K_i, K_j \in C_p\), \(u_{p,K_j} \geq u_{p,K_i}\) if \(j \geq i\). Let us consider an arbitrary group, \(K_q \in C_p\), with upper bound \(u_{p,K_q}\). Any item \(a \in A\) that has been removed from this group in step 3 of Algorithm 6 could only be in one of the groups from \(K_1\) to \(K_{q-1}\). This is because an item stays in the group with the lowest upper bound. Let \(m_i = \sum_{a \in K_i} x_{ap}\), then \(m_i \leq \left\lceil \frac{u_{p,K_i}}{g} \right\rceil\) due to constraint 29. Therefore,

\[
\sum_{i=1}^{q} m_i \leq \sum_{i=1}^{q} \left\lceil \frac{u_{p,K_i}}{g} \right\rceil \leq \sum_{i=1}^{q} \left( \frac{u_{p,K_i}}{g} + 1 \right) \leq \sum_{i=1}^{q} \left( \frac{u_{p,K_q}}{g} + 1 \right) \leq \Delta \frac{u_{p,K_q}}{g} + \Delta \leq u_{p,K_q} + \Delta
\]

The last second inequality holds because any item can belong to at most \(\Delta\) groups. Let \(x_{ap} \in \{0, 1\}\) be the value assigned to an edge \((a,p) \in E\) in \(\hat{M}\) returned by LP 4.5 then \(\sum_{a \in C} x_{ap} \leq u_{p,C} + \Delta \forall C \in C_p\) after all the items are returned to all the groups they belonged to in the original graph. Therefore, \(\hat{M}\) violates constraint 14 by an additive factor of at most \(\Delta\).

Lemma 4.8. Given a bipartite graph \(G(A, P, E)\) with possibly non-disjoint groups and an optimal solution of LP 4.4, Algorithm 7 returns a distribution over integer matchings such that each matching violates group fairness constraints by an additive factor of at most \(\Delta\), in polynomial time.

Proof. The proof is similar to the proof of Lemma 4.7 with one key difference that in each iteration, the matching being computed in step 9 of Algorithm 1 does not satisfy group fairness constraints but violates group fairness constraints by an additive factor of at most \(\Delta\) by Lemma 4.7.

Proof of Theorem 1.3. Let \(x\) be any optimal solution of LP 4.1 augmented with 4 and 5, then, Algorithm 6 can be used to represent \(\frac{x}{g}\) as a convex combination of integer matchings that violate group fairness constraints by an additive factor of at most \(\Delta\), in polynomial time by Lemma 4.8. By setting \(\tilde{x} = x\), \(\delta = 0\), and \(t = g\) in Lemma 2.10, we have \(\forall a \in A, S \subseteq N(a), \frac{L_{a,S}}{g} \leq \Pr_\tilde{M} (\tilde{M}\text{ matches } a\text{ to some platform in } S) \leq \frac{U_{a,S}}{g}\).
The run time of Algorithm 1 has been shown to be polynomial in Lemma 2.7, since LP 3.1 can be solved in polynomial time. Algorithm 6 also runs in polynomial time. This proves the theorem.

5 Experiments

In this section, we apply our approximation algorithm from Theorem 1.2 on two real-world datasets. In our experiments on standard datasets, the algorithm performs much better than the guarantee of $2(\Delta + 1)(\log(n/\epsilon) + 1)$ approximation provided in the analysis of the algorithm. Here $n$ is the total number of items, $\Delta$ is the maximum number of groups an item can belong to and $\epsilon > 0$ is a small value.

5.1 Datasets

Employee Access data 1: This data is from Amazon, collected from 2010-2011, and published on the Kaggle platform. We use the testing set with 58921 samples for our experiments. Each row in the dataset represents an access request made by an employee for some resource within the company. In our model, the employees and the resources correspond to items and platforms, respectively, and each request represents an edge. We group the employees based on their role family. An employee can make multiple requests, each under a different role family. Therefore, each item can have edges to different platforms and belong to more than one group. We run our experiments on datasets of sizes 1000, 2000, 3000, and 5000 sampled from this dataset.

Grant Application Data 2: This data is from the University of Melbourne on grant applications collected between 2004 and 2008 and published on the Kaggle platform. We use the training set with 8,707 grant applications for our experiments. In our model, the applicants and the grants correspond to items and platforms, respectively, and each grant application represents an edge. We group the applicants based on their research fields. The same applicant, de-identified in the dataset, can apply to different grants under different research fields represented as RFCD code in the dataset. Therefore, each item can have edges to different platforms and belong to more than one group.

5.2 Experimental Setup and Results

We implement our algorithm in Python 3.7 using the libraries NumPy, scipy, and Pandas. All the experiments we run use Google colab notebook on a virtual machine with Intel(R) Xeon(R) CPU @ 2.20GHz and 13GB RAM. Both the datasets on which we run our algorithm, are taken from Kaggle. We run our experiments on one complete dataset and three different sample sizes on another dataset. The sample size denotes the total number of rows present in the unprocessed sample. The total number of edges can differ from the sample size after data cleaning like removing null values and dropping duplicate edges if any. For group fairness bounds, we set the same upper and lower bounds for each platform group pair. If $n$ is the number of items, $m$ is the number of platforms, and $g$ is the number of groups, the upper bound is $\frac{knmg}{g^2}$, where $k = \lceil \frac{mg}{n} \rceil$. All the lower bounds are set to 0. For individual fairness constraints, we first choose a random permutation of the platforms to create a ranking and then add constraints such that an item should have $\frac{r}{2}$% chance of being matched to a platform in the top $r$% in the ranking. For all the runs, $\epsilon = 0.0001$.

We use the solution obtained by solving LP 2.1 as an upper bound on OPT. We denote it by $UB$, and $SOL$ denotes the expected size of the solution given by Algorithm 4 on different samples. Let $2(\Delta + 1)(\log(n/\epsilon) + 1)$ be denoted by ‘approx’. In Table 2, we compare the actual approximation ratio, $UB/SOL$, with the theoretical approximation ratio, ‘approx’. As can be seen in Table 2, in our experiments on standard real datasets, the algorithm performs much better than the worst-case theoretical guarantee provided by Algorithm 4. We repeatedly apply our approximation algorithm from Theorem 1.2 on multiple datasets sampled from the Employee Access dataset under the same experimental setup except for the $\epsilon$-value which is now set to 0.001. We see that the algorithm continues to perform much better than the guarantee of $2(\Delta + 1)(\log(n/\epsilon) + 1)$ approximation provided in the analysis of the algorithm in Section 3. The results can be seen in Table 3, Table 4, Table 5, and Table 6.

1https://www.kaggle.com/datasets/lucamassaron/amazon-employee-access-challenge
2https://www.kaggle.com/competitions/unimelb/data
| Dataset               | Sample Size | \(\Delta\) | \(\frac{UB}{OPT}\) | approx. | no. of matchings | run-time (seconds) |
|----------------------|-------------|-------------|------------------|---------|------------------|-------------------|
| Employee Access data | 1000        | 3           | 5.43             | 191.2   | 892              | 11                |
| Employee Access data | 2000        | 3           | 7.24             | 196.8   | 1871             | 60                |
| Employee Access data | 3000        | 4           | 9.19             | 250     | 2786             | 180               |
| Employee Access data | 5000        | 4           | 15.98            | 254     | 4651             | 900               |
| Grant Application Data | 8707    | 12          | 7.92             | 652     | 3836             | 540               |

Table 2: Comparison of solution values on real world datasets.

6 Conclusion

Various notions of group fairness and individual fairness in matching have been considered. However, to the best of our knowledge, this is the first work addressing both the individual and group fairness constraints in the same instance. Our work leads to several interesting open questions like improving the \(O(\Delta \log n)\) approximation ratio in Theorem 1.2 and extending our approximation results to the setting with lower bounds.

7 Acknowledgements

We thank Shankar Ram for initial discussions. AL was supported in part by SERB Award ECR/2017/003296, a Pratiksha Trust Young Investigator Award, and an IUSSTF virtual center on “Polynomials as an Algorithmic Paradigm”.

References

[Abdulkadiroglu and Sönmez, 2003] Abdulkadiroglu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. The American Economic Review.

[Aziz et al., 2020] Aziz, H., Baychkov, A., and Biró, P. (2020). Summer internship matching with funding constraints. In Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’20, page 97–104, Richland, SC. International Foundation for Autonomous Agents and Multiagent Systems.

[Aziz and Brandl, 2021] Aziz, H. and Brandl, F. (2021). Efficient, Fair, and Incentive-Compatible Healthcare Rationing, page 103–104. Association for Computing Machinery, New York, NY, USA.

[Bei et al., 2020] Bei, X., Liu, S., Poon, C. K., and Wang, H. (2020). Candidate selections with proportional fairness constraints. In Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’20, page 150–158, Richland, SC. International Foundation for Autonomous Agents and Multiagent Systems.

[Beyhaghi and Éva Tardos, 2021] Beyhaghi, H. and Éva Tardos (2021). Randomness and Fairness in Two-Sided Matching with Limited Interviews. In Lee, J. R., editor, 12th Innovations in Theoretical Computer Science Conference (ITCS 2021), volume 185 of Leibniz International Proceedings in Informatics (LIPIcs), pages 74:1–74:18, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[Bolukbasi et al., 2016] Bolukbasi, T., Chang, K.-W., Zou, J., Saligrama, V., and Kalai, A. (2016). Man is to computer programmer as woman is to homemaker? debiasing word embeddings. In NIPS’16.

[Celis et al., 2017] Celis, L. E., Straszak, D., and Vishnoi, N. K. (2017). Ranking with fairness constraints. In ICALP.

[Costello et al., 2016] Costello, M., Hawdon, J., Ratliff, T., and Grantham, T. (2016). Who views online extremism? individual attributes leading to exposure. Comput. Hum. Behav.

[Cowen Institute, 2011] Cowen Institute (2011). Case studies of school choice and open enrollment in four cities. Technical report, Cowen Institute.
[Devanur et al., 2013] Devanur, N. R., Jain, K., and Kleinberg, R. D. (2013). Randomized primal-dual analysis of ranking for online bipartite matching. In SODA ’13.

[Esmaeili et al., 2022] Esmaeili, S. A., Duppala, S., Nanda, V., Srinivasan, A., and Dickerson, J. P. (2022). Rawlsian fairness in online bipartite matching: Two-sided, group, and individual.

[Farnadi et al., 2021] Farnadi, G., St-Arnaud, W., Babaki, B., and Carvalho, M. (2021). Individual fairness in kidney exchange programs. Proceedings of the AAAI Conference on Artificial Intelligence, 35(13):11496–11505.

[Fleiner and Kamiyama, 2016] Fleiner, T. and Kamiyama, N. (2016). A matroid approach to stable matchings with lower quotas. Math. Oper. Res., 41(2):734–744.

[García-Soriano and Bonchi, 2021] García-Soriano, D. and Bonchi, F. (2021). Maxmin-fair ranking: Individual fairness under group-fairness constraints.

[García-Soriano and Bonchi, 2020] García-Soriano, D. and Bonchi, F. (2020). Fair-by-design matching. In Data Min Knowl Disc 34, page 1291–1335.

[Halabian et al., 2011] Halabian, H., Lambadaris, I., and Lung, C.-H. (2011). Optimal server assignment in multi-server queueing systems with random connectivities. In IEEE International Conference on Communications (ICC).

[Halabian et al., 2012] Halabian, H., Lambadaris, I., and Lung, C.-H. (2012). Optimal server assignment in multi-server parallel queueing systems with random connectivities and random service failures. In IEEE International Conference on Communications (ICC).

[Huang, 2010] Huang, C. (2010). Classified stable matching. In Charikar, M., editor, Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010, pages 1235–1253. SIAM.

[Huang et al., 2016] Huang, C., Kavitha, T., Mehlhorn, K., and Michail, D. (2016). Fair matchings and related problems. Algorithmica, 74(3):1184–1203.

[Jain et al., 2003] Jain, K., Mahdian, M., Markakis, E., Saberi, A., and Vazirani, V. V. (2003). Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP. J. ACM, 50(6):795–824.

[Kay et al., 2015] Kay, M., Matuszek, C., and Munson, S. A. (2015). Unequal representation and gender stereotypes in image search results for occupations. In Proceedings of the 33rd Annual ACM Conference on Human Factors in Computing Systems, CHI ’15.

[Klettis et al., 2022] Klettis, T., Renders, J.-M., and Loiseau, P. (2022). Introducing the expohedron for efficient pareto-optimal fairness-utility amortizations in repeated rankings. In Proceedings of the Fifteenth ACM International Conference on Web Search and Data Mining. ACM.

[Luss, 1999] Luss, H. (1999). On equitable resource allocation problems: A lexicographic minimax approach. Operations Research.

[Ma et al., 2020] Ma, W., Xu, P., and Xu, Y. (2020). Group-level fairness maximization in online bipartite matching.

[Makino et al., 2022] Makino, K., Miyazaki, S., and Yokoi, Y. (2022). Maximally satisfying lower quotas in the hospitals/residents problem with ties and incomplete lists.

[McKeown et al., 1996] McKeown, N., Anantharam, V., and Walrand, J. (1996). Achieving 100% throughput in an input-queued switch. In INFOCOM’96.

[Mehta, 2013] Mehta, A. (2013). Online matching and ad allocation. Foundations and Trends® in Theoretical Computer Science.

[Mehta et al., 2007] Mehta, A., Saberi, A., Vazirani, U., and Vazirani, V. (2007). Adwords and generalized online matching. J. ACM.
Since $x_{a,p}$ is an integer, $\forall p \in P$, $P_l$, $l_{p,C}$ and $u_{p,C}$ values are integers, $\forall p \in P$, $\forall C \in C_p$.

Proof. Let $\bar{x}$ be a basic feasible solution of $GFLP$. For an arbitrary platform, $p' \in P$, let there be $r$ groups in $C_{p'}$, say $K_1, K_2 \ldots K_r$. Suppose $\sum_{a \in N(p') \atop i=1} x_{a,p'} = \sum_{a \in K_i} x_{a,p'}$ is not an integer. This implies that there exists at least one group, say $K_q \in C_{p'}$, such that $\sum_{a \in K_q} x_{a,p'}$ is fractional, which in turn implies that there is at least one item, say $b \in K_q$, such that $x_{b,p'}$ is fractional. Let

$$w = \min \left( x_{b,p'}, \left[ x_{b,p'} \right] - x_{b,p'}, \lfloor \sum_{a \in K_q} x_{a,p'} \rfloor - \sum_{a \in K_q} x_{a,p'} \right),$$

$$\left[ \sum_{a \in N(p')} x_{a,p'} \right] - \sum_{a \in K_q} x_{a,p'} - \sum_{a \in K_q} x_{a,p'} - \left[ \sum_{a \in K_q} x_{a,p'} \right],$$

$$\sum_{a \in N(p')} x_{a,p'} - \left[ \sum_{a \in N(p')} x_{a,p'} \right].$$

Since $x_{a,p'}$, $\sum_{a \in K_q} x_{a,p'}$, and $\sum_{a \in N(p')} x_{a,p'}$ are not integers by our assumption, $w \in (0, 1)$. Let us modify $\bar{x}$ by replacing $x_{b,p'}$ with $x_{b,p'} + w$, and let the resulting vector be $\bar{y}$. By the definition of $w$ and the
assumption that \( \forall p \in P, \forall C \in C_p \), the \( u_p \) and \( u_{p,C} \) values are integers, \( \bar{y} \) doesn’t violate the constraints 6 to 10. Similarly, since \( \forall p \in P, \forall C \in C_p \), the \( l_p \) and \( l_{p,C} \) values are assumed to be integers, if we modify \( \bar{x} \) by replacing \( x_{bp'} \) with \( x_{bp'} - w \), the resulting vector, say \( \bar{z} \), will also not violate the constraints 6 to 10. Hence \( \bar{y} \) and \( \bar{z} \) are feasible solutions of GFLP. Clearly,

\[
\bar{x} = \frac{1}{2} \bar{y} + \frac{1}{2} \bar{z},
\]

which is a contradiction since a basic feasible solution of any LP cannot be written as a convex combination of two other points in the polytope of the same LP.

\( \square \)

Proof of Lemma 2.4. Let \( \bar{x} \) be a basic feasible solution of GFLP. Let us suppose that \( \bar{x} \) is fractional. From Claim A.1 we know that \( \sum_{a \in N(p)} x_{ap} \) is an integer, \( \forall p \in P \), in any vertex solution of GFLP.

Therefore, for some arbitrary platform, say \( p' \in P \), if there is an edge, say \((b, p')\) where \( b \in A \), such that \( x_{bp'} \) is fractional, then there must be at least one other edge \((b', p')\) where \( b' \in A \), such that \( x_{b'p'} \) is also fractional. Let \( b \in K_1 \) and \( b' \in K_2 \) where \( K_1, K_2 \in C_p' \) and let \( x_{bp'} > x_{b'p'} \) without loss of generality.

\[
w = \min \left( 1 - x_{bp'}, x_{b'p'}, \sum_{a \in K_1} x_{ap'} - \left\lfloor \sum_{a \in K_1} x_{ap'} \right\rfloor, \right)
\]

\[
\sum_{a \in K_2} x_{ap'} - \left\lfloor \sum_{a \in K_1} x_{ap'} \right\rfloor - \sum_{a \in K_1} x_{ap'},
\]

\[
\left[ \sum_{a \in K_2} x_{ap'} - \sum_{a \in K_1} x_{ap'} \right]
\]

Let us modify \( \bar{x} \) by replacing \( x_{bp'} \) and \( x_{b'p'} \) with \( x_{bp'} + w \) and \( x_{b'p'} - w \), respectively, and let the resulting vector be \( \bar{y} \). By the definition of \( w \) and the assumption that \( \forall p \in P, \forall C \in C_p \), the \( u_{p,C} \) values are integers, \( \bar{y} \) doesn’t violate the constraints 6 to 10. Similarly, if we modify \( \bar{x} \) by replacing \( x_{bp'} \) and \( x_{b'p'} \) with \( x_{bp'} - w \) and \( x_{b'p'} + w \), respectively, the resulting vector, say \( \bar{z} \), will also not violate the constraints 6 to 10. It is easy to see that

\[
\sum_{a \in N(p')} y_{ap'} = \sum_{a \in N(p')} z_{ap'} = \sum_{a \in N(p')} x_{ap'}
\]

Hence \( \bar{y} \) and \( \bar{z} \) also satisfy constraints 6 and 7 and hence are feasible solutions of GFLP. Clearly,

\[
\bar{x} = \frac{1}{2} \bar{y} + \frac{1}{2} \bar{z},
\]

which is a contradiction since a basic feasible solution of any LP cannot be written as a convex combination of two other points in the polytope of the same LP.

\( \square \)
| $\Delta$ | $UB_{SOL}$ | approx | no. of matchings | run-time (seconds) |
|-------|-----------|--------|------------------|-------------------|
| 3     | 6.74      | 164.82 | 914              | 17.1              |
| 3     | 3.39      | 164.58 | 908              | 17.8              |
| 3     | 4.45      | 164.68 | 923              | 17.8              |
| 3     | 6.34      | 164.63 | 897              | 16.6              |
| 3     | 5.04      | 164.89 | 913              | 16.7              |
| 3     | 3.78      | 164.83 | 905              | 20.4              |
| 3     | 3.57      | 164.84 | 914              | 17.7              |
| 3     | 5.4       | 164.76 | 904              | 17                |
| 2     | 7.33      | 123.54 | 906              | 16.9              |
| 2     | 4.11      | 123.6  | 935              | 17.7              |

Table 3: Comparison of solution values with the theoretical bound for samples of size 1000.

| $\Delta$ | $UB_{SOL}$ | approx | no. of matchings | run-time (seconds) |
|-------|-----------|--------|------------------|-------------------|
| 3     | 8.54      | 170.56 | 1852             | 105.5             |
| 3     | 9.97      | 170.6  | 1863             | 117               |
| 4     | 9.94      | 213.36 | 1850             | 105.1             |
| 3     | 8.52      | 170.53 | 1879             | 108.7             |
| 3     | 6.53      | 170.76 | 1864             | 109.1             |
| 4     | 10.54     | 213.48 | 1868             | 113.3             |
| 4     | 9.75      | 213.13 | 1867             | 108.1             |
| 4     | 9.55      | 213.17 | 1865             | 105.5             |
| 4     | 7.62      | 213.28 | 1848             | 107               |
| 3     | 9.60      | 170.79 | 1846             | 103.1             |

Table 4: Comparison of solution values with the theoretical bound for samples of size 2000.

| $\Delta$ | $UB_{SOL}$ | approx | no. of matchings | run-time (seconds) |
|-------|-----------|--------|------------------|-------------------|
| 4     | 14.13     | 221.28 | 4671             | 1315.4            |
| 4     | 13.28     | 220.78 | 4606             | 1332.6            |
| 4     | 12.70     | 221.29 | 4670             | 1298.9            |
| 4     | 18.81     | 221.14 | 4661             | 1292.6            |
| 4     | 20.97     | 220.89 | 4617             | 1160.8            |
| 5     | 13.76     | 265.67 | 4626             | 1494.9            |
| 4     | 10.9      | 221.21 | 4646             | 1286.2            |
| 5     | 11.95     | 265.23 | 4648             | 1295.6            |
| 4     | 20.1      | 221.08 | 4609             | 1166.9            |
| 5     | 9.88      | 265.09 | 4658             | 1229.5            |

Table 5: Comparison of solution values with the theoretical bound for samples of size 3000.

| $\Delta$ | $UB_{SOL}$ | approx | no. of matchings | run-time (seconds) |
|-------|-----------|--------|------------------|-------------------|
| 4     | 14.13     | 221.28 | 4671             | 1315.4            |
| 4     | 13.28     | 220.78 | 4606             | 1332.6            |
| 4     | 12.70     | 221.29 | 4670             | 1298.9            |
| 4     | 18.81     | 221.14 | 4661             | 1292.6            |
| 4     | 20.97     | 220.89 | 4617             | 1160.8            |
| 5     | 13.76     | 265.67 | 4626             | 1494.9            |
| 4     | 10.9      | 221.21 | 4646             | 1286.2            |
| 5     | 11.95     | 265.23 | 4648             | 1295.6            |
| 4     | 20.1      | 221.08 | 4609             | 1166.9            |
| 5     | 9.88      | 265.09 | 4658             | 1229.5            |

Table 6: Comparison of solution values with the theoretical bound for samples of size 5000.