Logarithmic decay of hyperbolic equations with arbitrary boundary damping

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Abstract

In this paper, we study the logarithmic stability for the hyperbolic equations by arbitrary boundary observation. Based on Carleman estimate, we first prove an estimate of the resolvent operator of such equation. Then we prove the logarithmic stability estimate for the hyperbolic equations without any assumption on an observation sub-boundary.

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1 Introduction and main result

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial \Omega$ of class $C^2$. Denote by $\nu = (\nu_1, \cdots, \nu_n)$ the unit outward normal field along the boundary $\partial \Omega$, and $\overline{\Omega}$ the closure of $\Omega$. For simplicity, in the sequel, we use the notation $u_j = \partial u / \partial x_j$, where $x_j$ is the $j$-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use the notation $w_j, v_j$, etc. for the partial derivatives of $w$ and $v$ with respect to $x_j$. By $\overline{\gamma}$ we denote the complex conjugate of $c \in \mathbb{C}$. Throughout this paper, we will use $C$ to denote a generic positive constant which may vary from line to line (unless otherwise stated).

Let $a^{jk}(\cdot) \in C^2(\overline{\Omega}; \mathbb{R})$ be fixed satisfying

$$a^{jk} = a^{jk}(x) = a^{kj}(x), \quad \forall \, x \in \overline{\Omega}, \, j, k = 1, 2, \cdots, n,$$  

(1.1)

and for some constant $\beta > 0$,

$$\sum_{j, k=1}^n a^{jk}(x)\xi_j \xi_k^* \geq \beta |\xi|^2, \quad \forall \, (x, \xi) \in \overline{\Omega} \times \mathbb{C}^n,$$

(1.2)

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where \( \xi = (\xi^1,\cdots,\xi^n) \). Define a formal differential operator \( \mathcal{P} \) (associated with the matrix \( (a^{jk}(\cdot))_{n \times n} \)) as follows:

\[
\mathcal{P} \triangleq \sum_{j,k=1}^{n} \partial_j(a^{jk} \partial_j). \tag{1.3}
\]

Fix a real valued function \( a(\cdot) \in C^1(\partial \Omega; \mathbb{R}^+) \). In what follows, we assume that

\[
\Gamma_0 \triangleq \{ x \in \partial \Omega; \ a(x) > 0 \} \neq \emptyset. \tag{1.4}
\]

The main purpose of this article is to study the logarithmic decay of the following hyperbolic equations with a boundary damping term \( a(x)u_t \):

\[
\begin{aligned}
&u_{tt} - \mathcal{P}u = 0 \quad \text{in} \ \mathbb{R}^+ \times \Omega, \\
&\sum_{j,k=1}^{n} a^{jk} u_j \nu_k = 0 \quad \text{on} \ \mathbb{R}^+ \times \partial \Omega \setminus \Gamma_0, \\
&\sum_{j,k=1}^{n} a^{jk} u_j \nu_k + a(x) u_t = 0 \quad \text{on} \ \mathbb{R}^+ \times \Gamma_0, \\
&(u(0), u_t(0)) = (u^0, u^1) \quad \text{in} \ \Omega.
\end{aligned} \tag{1.5}
\]

Very interesting logarithmic decay results were given in [4, 11] for the above system under the regularity assumption that \( a^{jk}(\cdot), a(\cdot) \) and \( \partial \Omega \) are \( C^\infty \)-smooth ([11] considered the special case \( (a^{jk})_{n \times n} = I \), the identity matrix). Note that, since the sub-boundary \( \Gamma_0 \) in which the damping \( a(x)u_t \) is (uniformly) effective may be very “small”, the “geometric optics condition” introduced in [3] is not guaranteed for system (1.5), and therefore, in general, one can not expect exponential stability of this system. On the other hand, as pointed in [4, 11], for some special case of system (1.5), logarithmic stability is the best decay rate.

Put \( H \triangleq H^1(\Omega) \times L^2(\Omega) \). Define an unbounded operator \( A : H \to H \) by (Recall that \( u^0_j = \frac{\partial u^0}{\partial x_j} \))

\[
\begin{aligned}
&A \triangleq \begin{pmatrix} 0 & I \\ \mathcal{P} & 0 \end{pmatrix}, \\
&D(A) \triangleq \{ u = (u^0, u^1) \in H; \ Au \in H, \\
&\sum_{j,k=1}^{n} a^{jk} u^0_j \nu_k \big|_{\partial \Omega \setminus \Gamma_0} = 0, \left( \sum_{j,k=1}^{n} a^{jk} u^0_j \nu_k + a u^1 \right) \big|_{\Gamma_0} = 0 \}. \tag{1.6}
\end{aligned}
\]

It is easy to show that \( A \) generates a group \( \{ e^{tA} \}_{t \in \mathbb{R}} \) on \( H \).

The main result of this paper is stated as follows:

**Theorem 1.1** Let \( a^{jk}(\cdot) \in C^2(\overline{\Omega}; \mathbb{R}) \) satisfy (1.1)–(1.2) and \( a(\cdot) \in C^1(\partial \Omega; \mathbb{R}^+) \) satisfy (1.4). Then solutions \( e^{tA}(u^0, u^1) \equiv (u, u_t) \in C(\mathbb{R}; D(A)) \cap C^1(\mathbb{R}; H) \) of system (1.5) satisfy

\[
\| e^{tA}(u^0, u^1) \|_H \leq \frac{C}{\ln(2 + t)} \| (u^0, u^1) \|_{D(A)}, \quad \forall (u^0, u^1) \in D(A), \ \forall t > 0. \tag{1.7}
\]
Following [1] (see also [4]), Theorem 1.1 is a consequence of the following resolvent estimate for operator $A$:

**Theorem 1.2** Under the assumptions in Theorem 1.1, there exists a constant $C > 0$ such that

i) if $\lambda \in \text{Sp}(A) \setminus \{0\}$, then

$$\Re \lambda < - \frac{e^{-C|\text{Im}\lambda|}}{C};$$

ii) if

$$\Re \lambda \in \left[ - \frac{e^{-C|\text{Im}\lambda|}}{C}, 0 \right],$$

then

$$\| (A - \lambda I)^{-1} \|_{\mathcal{L}(H)} \leq Ce^{C|\text{Im}\lambda|}, \quad \text{for } |\lambda| > 1.$$ 

We shall develop an approach based on global Carleman estimate to prove Theorem 1.2, which is the main novelty of this paper. Our approach, stimulated by [10] (see also [6, 8, 17, 18]), is different from that in [1], which instead employed the classical local Carleman estimate and therefore needs $C^\infty$-regularity for the data.

It would be quite interesting to establish better decay rate (than logarithmic decay) for system (1.5) under further conditions (without geometric optics condition). There are some impressive results in this respect, say [2, 12, 13, 14] for polynomial decay of system (1.5) with special geometries. However, to the best of the author’s knowledge, the full picture of this problem is still unclear. We refer to [5, 15, 19] for related works.

The rest of this paper is organized as follows. In section 2, we collect some useful preliminary results which will be useful later. Another key preliminary, global Carleman estimate for elliptic equations without inhomogeneous boundary condition, is established in section 3. Sections 4–5 are addressed to the proof of our main results.

## 2 Some preliminaries

In this section, we collect some preliminaries which will be used in the sequel.

To begin with, we recall the following result (which is an easy consequence of known result in [9, 16], for example).

**Lemma 2.1** There exists a function $\hat{\psi} \in C^2(\overline{\Omega})$ such that

$$\begin{cases}
\hat{\psi} > 0 & \text{in } \Omega, \\
|\nabla \hat{\psi}| > 0 & \text{in } \overline{\Omega}, \\
\sum_{j,k=1}^n a^{jk} \hat{\psi}_j \nu_k \leq 0 & \text{on } \partial\Omega \setminus \Gamma_0.
\end{cases} \tag{2.1}$$

Next, for $n \in \mathbb{N}$, we denote by $O(\mu^n)$ a function of order $\mu^n$ for large $\mu$ (which is independent of $\lambda$); by $O_\mu(\lambda^n)$ a function of order $\lambda^n$ for fixed $\mu$ and for large $\lambda$. We now show the following pointwise estimate, which is a consequence of [8, Theorem 2.1] (see also [7]).
Lemma 2.2 Let $a^{jk} \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$ satisfying (1.1). Assume $z \in C^2(\mathbb{R}^{1+n}; C)$, $\Psi \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$ and $\ell \in C^4(\mathbb{R}^{1+n}; \mathbb{R})$. Set
\[
\theta = e^\ell, \quad v = \theta z, \quad \Psi = -2\ell_{ss} - 2 \sum_{j,k=1}^n (a^{jk}\ell_j)_k. \tag{2.2}
\]
Then
\[
\theta^2|z_{ss} + \sum_{j,k=1}^n (a^{jk}z_j)_k|^2 + M_s + \text{div } V
\geq 2 \left( 3\ell_{ss} + \sum_{j,k=1}^n (a^{jk}\ell_j)_k \right) |v_s|^2 + 4 \sum_{j,k=1}^n a^{jk}\ell_j(s) (v_k\nabla v_s + \nabla_k v_s)
+ \sum_{j,k=1}^n c^{jk}(v_k\nabla_j + \nabla_k v_j) + B|v|^2,
\tag{2.3}
\]
where
\[
A = \ell^2_s + \sum_{j,k=1}^n a^{jk}\ell_j \ell_k - \ell_{ss} - \sum_{j,k=1}^n (a^{jk}\ell_j)_k - \Psi,
\]
\[
M = 2\ell_s(|v_s|^2 - \sum_{j,k=1}^n a^{jk}\nabla_j v_k) + 2 \sum_{j=1}^n a^{jk}\ell_j(\nabla_s v_j + v_s \nabla_j)
- \Psi(\nabla_s v + v_s \nabla) + (2A\ell_s + \Psi_s)|v|^2,
\]
\[
V = [V_1, \ldots, V_k, \ldots, V_n],
\]
\[
V_k = \sum_{j,j',k'=1}^n \left\{ -2a^{jk}\ell_j |v_s|^2 + 2a^{jk}\ell_s(\nabla_j v_s + v_j \nabla_s) - \Psi a^{jk}(v_j \nabla + \nabla_j v)
+ (2a^{jk'}a^{j'k} - a^{jk}a^{j'k'}) \ell_j(v_j \nabla_k + \nabla_j v_k) + a^{jk}(2A\ell_j + \Psi_j - 2a\ell_j \ell_t) |v|^2 \right\},
\]
\[
c^{jk} = \sum_{j',k'=1}^n \left[ 2(a^{jk'}\ell_{j'})_k a^{j'k'} - a^{jk}a^{j'k'} \ell_{j'} + a^{jk}(a^{j'k'} \ell_{j'})_k \right] + a^{jk} \ell_{ss},
\]
\[
B = \sum_{j,k=1}^n (a^{jk}\Psi_k)_j + 2(A\ell_s)_s + 2 \sum_{j,k=1}^n (Aa^{jk}\ell_j)_k + 2A\Psi.
\tag{2.4}
\]

In particular, for any function $\psi \in C^4(\mathbb{R}^{1+n}; \mathbb{R})$ satisfying $\psi_{sj} = 0$ ($j=1, \ldots, n$), and any $\lambda, \mu > 1$, choosing the function $\ell(s,x)$ to be
\[
\ell = \lambda \phi, \quad \phi = e^{\mu \psi}, \tag{2.5}
\]
then
\[
\text{Left hand side of (2.3)} \geq 2 \left[ \lambda \mu^2 \phi \sum_{j,k=1}^n a^{jk}\psi_j \psi_k + \lambda \phi O(\mu) \right] (|v_s|^2 + \sum_{j,k=1}^n a^{jk}v_j \nabla_k)
+ 2 \left[ \lambda^2 \mu^3 \phi^3 \right] \sum_{j,k=1}^n a^{jk}\psi_j \psi_k |v|^2 + \lambda^3 \phi^3 O(\lambda^3) + O(\mu(\lambda^2)) |v|^2. \tag{2.6}
\]
Proof. Using Theorem 2.1 in [8] with \( m = 1 + n \), and
\[
t = s, \quad (a^{jk})_{m \times m} = \begin{pmatrix} 1 & 0 \\ 0 & (a^{jk})_{n \times n} \end{pmatrix}.
\]
By a direct calculation, we obtain (2.3).

On the other hand, by (2.5) and note that \( \psi_{aj} = 0 \) (\( j = 1, \ldots, n \)), it is easy to check that
\[
\begin{align*}
\ell_s &= \lambda \mu \phi \psi_s, \\
\ell_j &= \lambda \mu \phi \psi_j, \\
\ell_{ss} &= \lambda \mu^2 \phi \psi_s^2 + \lambda \mu \phi \psi_{ss}, \\
\ell_{jk} &= \lambda \mu^2 \phi \psi_j \psi_k + \lambda \mu \phi \psi_{jk}, \\
\ell_{js} &= \lambda \mu^2 \phi \psi_s \psi_j.
\end{align*}
\]
(2.7)

Next, recalling the definition of \( c^{jk} \) in (2.4), by (2.7) and note that \( a^{jk} \) satisfies (1.1), we have
\[
2\left(3\ell_{ss} + \sum_{j,k=1}^{n} (a^{jk} \ell_j)_{k}\right) s^2 + 4 \sum_{j,k=1}^{n} a^{jk} \ell_{js}(v_k \bar{v}_s + v_k v_s) + \sum_{j,k=1}^{n} c^{jk}(v_k \bar{v}_j + \bar{v}_k v_j)
\]
\[
= 2\{ \lambda \mu^2 \phi \left[3|\psi_s|^2 + \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right] + \lambda \phi \mathcal{O}(\mu) \} s^2 + 8\lambda \mu^2 \phi \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_j v_k \bar{v}_s
\]
\[
+ 4\lambda \mu^2 \left[ \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right]^2 + 2\{ \lambda \mu^2 \phi \left[ \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k + |\psi_s|^2 \right] + \lambda \phi \mathcal{O}(\mu) \} \sum_{j,k=1}^{n} a^{jk} v_k \bar{v}_j
\]
\[
= 4\lambda \mu^2 \phi s^2 v_s + \sum_{j,k=1}^{n} a^{jk} \psi_j v_k \left[ |\psi_j|^2 + 2\lambda \mu^2 \left[ \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right]^2 + 2\{ \lambda \mu^2 \phi \left[ \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k + |\psi_s|^2 \right] + \lambda \phi \mathcal{O}(\mu) \} \left| v_s \right|^2 + \sum_{j,k=1}^{n} a^{jk} v_j \bar{v}_k \right]
\]
\[
\geq 2\{ \lambda \mu^2 \phi \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k + \lambda \phi \mathcal{O}(\mu) \} \left| v_s \right|^2 + \sum_{j,k=1}^{n} a^{jk} v_j \bar{v}_k \right).
\]
Further, by (2.7) and recalling (2.4) and (2.2) for the definition of \( A \) and \( \Psi \), respectively, we have
\[
\begin{align*}
\Psi &= 2\lambda \mu^2 \phi \left[ |\psi_s|^2 + \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right] + \lambda \phi \mathcal{O}(\mu), \\
A &= (\lambda^2 \mu^2 \phi^2 + \lambda \mu^2 \phi) \left[ |\psi_s|^2 + \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right] + \lambda \phi \mathcal{O}(\mu).
\end{align*}
\]
(2.9)

Therefore, by (2.4), and note that \( a^{jk} \) satisfies (1.2), we have
\[
\begin{align*}
B &= 2\lambda^3 \mu^4 \phi^3 \left| \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k + |\psi_s|^2 \right|^2 + \lambda^3 \phi^3 \mathcal{O}(\mu^3) + \mathcal{O}_\mu(\lambda^2)
\]
\[
\geq 2\lambda^3 \mu^4 \phi^3 \left| \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right|^2 + \lambda^3 \phi^3 \mathcal{O}(\mu^3) + \mathcal{O}_\mu(\lambda^2).
\]
(2.10)

Combining (2.3), (2.8) and (2.10), we arrive at the desired result (2.6).

Finally, similar to [17, Lemma 3.3], we have the following result. \( \square \)
Lemma 2.3 Let \( a^{jk} \in C^1(\Omega) \) satisfy (1.1), and \( g \triangleq (g^1, \cdots, g^n) : \mathbb{R}_t \times \mathbb{R}^n_x \rightarrow \mathbb{R}^n \) be a vector field of class \( C^1 \). Then for any \( w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x; \mathbb{C}) \), we have

\[
- \sum_{k=1}^n \left[ (g \cdot \nabla w)^n \sum_{j=1}^n a^{jk} w_j + (g \cdot w)^n \sum_{i,l=1}^n a^{jl} w_j w_l \right]_k \\
= - \left[ w_{ss} + \sum_{j,k=1}^n (a^{jk} w_j)_k \right] g \cdot \nabla w - \left( w_{ss} + \sum_{j,k=1}^n (a^{jk} w_j)_k \right) g \cdot \nabla w \\
+ (w_s g \cdot \nabla w + w_s g \cdot w) - (w_s g_s \cdot \nabla w + w_s g \cdot w) \\
+ (\nabla \cdot g) |w_s|^2 - 2 \sum_{j,k,l=1}^n a^{jk} w_j \frac{\partial g^l}{\partial x_k} + \sum_{j,k=1}^n w_j w_k \nabla \cdot (a^{jk} g). 
\]  

(2.11)

Proof. On the one hand, we have

\[
w_{ss} g \cdot \nabla w + \overline{w_{ss}} g \cdot \nabla w \\
= (w_s g \cdot \nabla w + \overline{w_s g} \cdot \nabla w)_s - (w_s g_s \cdot \nabla w + \overline{w_s g} \cdot \nabla w) \\
- \sum_{j=1}^n (g^j |w_s|^2)_j + (\nabla \cdot g) |w_s|^2. 
\]  

(2.12)

On the other hand, by (1.1), we have

\[
\sum_{j,k=1}^n (a^{jk} w_j)_k g \cdot \nabla w + \sum_{j,k=1}^n (a^{jk} w_j)_k g \cdot \nabla w \\
= \sum_{j,k=1}^n \left[ a^{jk} w_j g \cdot \nabla w + a^{jk} \overline{w_j g} \cdot \nabla w \right]_k - 2 \sum_{j,k,l=1}^n a^{jk} w_j \frac{\partial g^l}{\partial x_k} \\
- \sum_{j,k,l=1}^n (a^{jk} g^l w_j w_k)_l + \sum_{j,k=1}^n w_j w_k \nabla \cdot (a^{jk} g). 
\]  

(2.13)

Combining (2.12)–(2.13), we get the desired result.

\[ \blacksquare \]

3 Global Carleman estimate for elliptic equations without inhomogeneous boundary condition

In this section, we shall derive a global Carleman estimate for elliptic equations with non-homogeneous and complex Neumann-like boundary condition.

Denote

\[
X = (-2,2) \times \Omega, \quad \Sigma = (-2,2) \times \partial \Omega, \quad Y = (-1,1) \times \Omega, \quad Z = (-2,2) \times \Gamma_0.
\]

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Let us consider the following elliptic equation:

\[
\begin{align*}
    z_{ss} + & \sum_{j,k=1}^{n} (a^{jk} z_j)_k = z^0 & \quad & \text{in } (-2, 2) \times \Omega, \\
    \sum_{j,k=1}^{n} a^{jk} z_j \nu_k = & \quad 0 & \quad & \text{on } (-2, 2) \times \partial\Omega \setminus \Gamma_0, \\
    \sum_{j,k=1}^{n} a^{jk} z_j \nu_k - ia(x)z_s = & \quad a(x)z^1 & \quad & \text{on } (-2, 2) \times \Gamma_0.
\end{align*}
\]

(3.1)

We now show the following Carleman estimate.

**Theorem 3.1** Under the assumptions in Theorem 1.1, there exists a constant \( C > 0 \) such that, for any \( \varepsilon > 0 \), any solution \( z \in C((-2, 2); H^1(\Omega)) \cap C^1((-2, 2); L^2(\Omega)) \) of system (3.1) satisfies

\[
||z||_{H^1(\Omega)} \leq C e^{C \varepsilon} \left[ ||z^0||_{L^2(\Omega)} + ||z^1||_{L^2(\Omega)} + ||z_s||_{L^2(\Omega)} + ||z_s||_{L^2(\Omega)} \right] + C e^{-2\varepsilon} ||z||_{H^1(\Omega)}.
\]

(3.2)

**Remark 3.1** For the general case of \( t \in (T_1, T_2) \) with \( T_1, T_2 \in \mathbb{R} \). By setting \( s = t - \frac{T_2 + T_1}{2} \), one deduces that

\[
s \in (-\alpha, \alpha), \quad \alpha \triangleq \frac{T_2 - T_1}{2}.
\]

Then by scaling, one need consider only the case of (3.1).

**Proof.** We divide the proof into several steps.

**Step 1.** Note that there is no boundary condition for \( z \) at \( s = \pm 2 \). Therefore, we need to introduce a cut-off function \( \varphi = \varphi(s) \in C_0^\infty(-b, b) \subset C_0^\infty(\mathbb{R}) \) such that

\[
\begin{align*}
0 \leq \varphi(s) \leq 1 & \quad |s| < b, \\
\varphi(s) = 1 & \quad |s| \leq b_0,
\end{align*}
\]

(3.3)

where \( b_0 \) and \( b \) (satisfying \( 1 < b_0 < b < 2 \)) will be given later. Put

\[
\hat{z} = \varphi z.
\]

(3.4)

Then, noting that \( \varphi \) does not depend on \( x \), by (3.1), it follows

\[
\begin{align*}
\hat{z}_{ss} + & \sum_{j,k=1}^{n} (a^{jk} \hat{z}_j)_k = \varphi_{ss}z + 2\varphi_s z_s + \varphi z^0 & \quad & \text{in } (-2, 2) \times \Omega, \\
\sum_{j,k=1}^{n} a^{jk} \hat{z}_j \nu_k = & \quad 0 & \quad & \text{on } (-2, 2) \times \partial\Omega \setminus \Gamma_0, \\
\sum_{j,k=1}^{n} a^{jk} \hat{z}_j \nu_k - ia(x)\hat{z}_s = & \quad -ia(x)\varphi_s z + a(x)\varphi z^1 & \quad & \text{on } (-2, 2) \times \Gamma_0.
\end{align*}
\]

(3.5)
Step 2. Put

\[ b \triangleq 1 + \frac{1}{\mu} \ln \left[ \frac{(2 + e\mu)e^{||\phi(x)||_{L^\infty(\Omega)}}}{\mu^\phi(x)} \right], \quad b_0 \triangleq 1 + \frac{1}{\mu} \ln \left[ \frac{(1 + e\mu)e^{||\phi(x)||_{L^\infty(\Omega)}}}{\mu^\phi(x)} \right], \quad (3.6) \]

where \( \mu > \ln 2, \hat{\psi}(x) \in C^2(\Omega) \) is given by Lemma 2.1. It is easy to see that

\[ 1 < b_0 < b \leq 2. \quad (3.7) \]

Put

\[ \psi = \psi(s, x) \triangleq -\frac{\hat{\psi}(x)}{||\hat{\psi}||_{L^\infty(\Omega)}} + b^2 - s^2. \quad (3.8) \]

It is easy to check that

\[ \begin{aligned}
\phi(s, \cdot) &\geq 2 + e\mu, \quad \text{for any } s \text{ satisfying } |s| \leq 1, \\
\phi(s, \cdot) &\leq 1 + e\mu, \quad \text{for any } s \text{ satisfying } b_0 \leq |s| \leq b.
\end{aligned} \quad (3.9) \]

On the other hand, by (3.8) and Lemma 2.1, we find

\[ h \triangleq |\nabla \psi| = \frac{1}{||\hat{\psi}||_{L^\infty(\Omega)}} |\nabla \hat{\psi}(x)| > 0, \quad \text{in } \Omega. \quad (3.10) \]

Next, recalling that \( a^kj \) satisfying (1.2) and by (3.10), we conclude that there exists a \( \mu_0 > 1 \), for any \( \mu \geq \mu_0 \), there exists \( \lambda_0(\mu) > 1 \) such that for any \( \lambda \geq \lambda_1 \), it holds

The right hand side of (2.6)

\[ \geq \lambda \mu^2 \phi \sum_{j,k=1}^n a^kj \psi j \psi k \left( |v_s|^2 + \sum_{j,k=1}^n a^kj v_j \nabla k \right) + \lambda^3 \mu^4 \phi^3 \left( \sum_{j,k=1}^n a^kj \psi j \psi k \right)^2 |v|^2 \]

\[ \geq \lambda \mu^2 \beta h^2 \phi \left( |v_s|^2 + \sum_{j,k=1}^n a^kj v_j \nabla k \right) + \lambda^3 \mu^4 \beta^2 h^4 \phi^3 |v|^2. \]

Now, integrating inequality (2.6) (with \( u \) replaced by \( \hat{\psi} \)) in \((-b, b) \times \Omega\), recalling that \( \varphi \)

vanishes near \( s = \pm b \), and by (3.5) and (3.11), one arrives at

\[ \lambda \mu^2 \int_{-b}^b \int_{\Omega} \phi(|\nabla v|^2 + |v_s|^2) dx ds + \lambda^3 \mu^4 \int_{-b}^b \int_{\Omega} \phi^3 |v|^2 dx ds \leq C \left\{ \int_{-b}^b \int_{\Omega} \theta^2 |\varphi_{ss} z + 2 \varphi_z \hat{z}_{ss} + \varphi(0)^2 |dx ds + \int_{-b}^b \int_{\partial \Omega} V \cdot \nu dx ds \right\}. \quad (3.12) \]

Recalling that \( v = \theta \hat{z} \), by (2.7), we get

\[ \frac{1}{C} \theta^2 (|\nabla \hat{z}|^2 + \lambda^2 \mu^2 \phi^2 |\hat{z}|^2) \leq |\nabla v|^2 + \lambda^2 \mu^2 \phi^2 |v|^2 \leq C \theta^2 (|\nabla \hat{z}|^2 + \lambda^2 \mu^2 \phi^2 |\hat{z}|^2). \quad (3.13) \]

Therefore, by (3.12) and (3.13), we end up with

\[ \lambda \mu^2 \int_{-b}^b \int_{\Omega} \phi^2 (|\nabla \hat{z}|^2 + |\hat{z}|^2) dx ds + \lambda^3 \mu^4 \int_{-b}^b \int_{\Omega} \phi^3 |\hat{z}|^2 dx ds \]

\[ \leq C \left\{ \int_{-b}^b \int_{\Omega} \theta^2 |\varphi_{ss} z + 2 \varphi_z \hat{z}_{ss} + \varphi(0)^2 |dx ds + \int_{-b}^b \int_{\partial \Omega} V \cdot \nu dx ds \right\}. \quad (3.14) \]
Step 3. We now estimate \( \int_{-b}^{b} \int_{\partial \Omega} V \cdot \nu dxds \). By (2.4) and nothing that \( v = \theta \hat{z} \), it follows

\[
\int_{-b}^{b} \int_{\partial \Omega} V \cdot \nu dxds = \sum_{k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} V_k \nu_k dxds
\]

\[
= \sum_{j,k,j',k'=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \left\{ -2\alpha^{jk} \ell_j \nu_k |v_s|^2 + 2\ell_s \alpha^{jk} \nu_k (\overline{v}_j v_s + v_j \overline{v}_s) - \Psi \alpha^{jk} \nu_k (v_j \overline{v} + \overline{v}_j v) + (2\alpha^{jk'} \alpha^{j'k} - \alpha^{jk} \alpha^{j'k'}) \ell_j (v_j \overline{\nu}_k + \overline{v}_j v_k) + \alpha^{jk} \nu_k (2A \ell_j + \Psi - 2a \ell_j \ell_s) |v|^2 \right\} dxds.
\]

Note that, by (2.1) and (2.7), we know that

\[
\sum_{j,k=1}^{n} a^{jk} \ell_j \nu_k = \lambda \mu \phi \sum_{j,k=1}^{n} a^{jk} \psi_j \nu_k = -\frac{\lambda \mu \phi}{||\psi||_{L^\infty(\Omega)}} \sum_{j,k=1}^{n} a^{jk} \psi_j \nu_k \geq 0, \quad \text{on } \partial \Omega \setminus \Gamma_0.
\]

Hence, recalling that \( v = \theta \hat{z} \), we have

\[
- \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} a^{jk} \ell_j \nu_k |v_s|^2 dxds \leq C\lambda \mu \int_{-b}^{b} \int_{\Gamma_0} \phi |v_s|^2 dxdt
\]

\[
\leq C e^{C_1} \int_{-b}^{b} \int_{\Gamma_0} (|\hat{z}_s|^2 + |\hat{z}|^2) dxds.
\]

Next, using \( v = \theta \hat{z} \) again, noting that \( \hat{z} \) vanishes near \( s = \pm b \), by (1.4) and (3.5), we have

\[
\sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \ell_s a^{jk} \nu_k (\overline{v}_j v_s + v_j \overline{v}_s) dxds - \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \Psi a^{jk} \nu_k (v_j \overline{v} + \overline{v}_j v) dxds
\]

\[
= \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \theta^2 \ell_s a^{jk} \nu_k (\overline{\hat{z}}_j \hat{z}_s + \hat{z}_j \overline{\hat{z}}_s) dxds
\]

\[
+ \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \theta^2 (\ell_s^2 - \Psi) a^{jk} \nu_k (\overline{\hat{z}}_j \hat{z} + \hat{z}_j \overline{\hat{z}}) dxds
\]

\[
+ \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} a^{jk} \ell_j \nu_k (\overline{\hat{z}} \hat{z} + \hat{z} \overline{\hat{z}}) dxds
\]

\[
+ 2 \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \theta^2 (\ell_s^2 - \Psi) a^{jk} \ell_j \nu_k |\hat{z}|^2 dxds
\]

\[
= \int_{-b}^{b} \int_{\Gamma_0} a(x) \theta^2 \ell_s [i \varphi_s (\hat{z}_s \overline{\hat{z}} - \overline{\hat{z}}_s \hat{z}) + \varphi (\overline{\hat{z}}_s \overline{\hat{z}} + \hat{z}_s \hat{z})] dxds
\]

\[
+ \int_{-b}^{b} \int_{\Gamma_0} a(x) \theta^2 (\ell_s^2 - \Psi) [i (\hat{z}_s \overline{\hat{z}} - \overline{\hat{z}}_s \hat{z}) + \varphi (\overline{\hat{z}}_s \overline{\hat{z}} + z_s \hat{z})] dxds
\]

\[
+ \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} (\theta^2 \ell_s a^{jk} \ell_j \nu_k |\hat{z}|^2) dxds - \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} \theta^2 (\ell_{ss} + 2 \Psi) a^{jk} \ell_j \nu_k |\hat{z}|^2 dxds
\]

\[
\leq C e^{C_1} \left[ \int_{-b}^{b} \int_{\Gamma_0} (|\hat{z}_s|^2 + |\varphi_s z|^2 + |\varphi z|^2) dxds + \int_{-b}^{b} \int_{\partial \Omega} |\hat{z}|^2 dxds \right].
\]
Further, by (3.16), and noting that \( v = \theta \hat{z} \), we get
\[
\sum_{j,k,j',k'=1}^n \int_{\partial \Omega} \left( 2a^{j'k'}a^{j''k'} - a^{jk'}a^{j'k'} \right) \ell_j(v_j-\nabla v_{k'}) \ell_j(v_{j'} \nabla v'_{k'}) \nu_k ds \]
\[
= \sum_{j,k,j',k'=1}^n \int_{\partial \Omega} a^{jk'} \ell_j(v_{j'} \nabla v'_{k'}) ds
\]
\[
\leq Ce^{C_{\lambda}} \sum_{j',k'=1}^n \int_{\partial \Omega} a^{j'k'}(v_{j'} \nabla v'_{k'}) ds
\]
\[
\leq Ce^{C_{\lambda}} \sum_{j',k'=1}^n \int_{\partial \Omega} \left[ a^{jk'} \hat{z}_{j'k} + a^{jk'} \ell_j \hat{|\hat{z}|}^2 \right] ds.
\]

Combining (3.15), (3.17)–(3.19), we obtain
\[
\int_{-b}^b \int_{\partial \Omega} V \cdot \nu ds \leq Ce^{C_{\lambda}} \left[ \int_{-b}^b \int_{\Gamma_0} (|\hat{z}_s|^2 + |\varphi_s z|^2 + |\varphi z|^2) ds \right]
\]
\[
+ \int_{-b}^b \int_{\partial \Omega \setminus \Gamma_0} |\nabla \hat{z}|^2 ds + \int_{-b}^b \int_{\partial \Omega} |\hat{z}|^2 ds.
\]

**Step 4.** Let us estimate \( \int_{-b}^b \int_{\partial \Omega} |\hat{z}|^2 ds \) and \( \int_{-b}^b \int_{\partial \Omega \setminus \Gamma_0} |\nabla \hat{z}|^2 ds \).

Firstly, by trace theory and Poincaré inequality, noting that \( \hat{z} \) vanishes near \( s = \pm b \), we have
\[
\int_{-b}^b \int_{\partial \Omega} |\hat{z}|^2 ds \leq C \int_{-b}^b \int_{\Omega} (|\hat{z}|^2 + |\nabla \hat{z}|^2) ds \leq C \int_{-b}^b \int_{\Omega} (|\hat{z}_s|^2 + |\nabla \hat{z}|^2) ds.
\]

Next, we choose a \( g \in C^1(\overline{\Omega}; \mathbb{R}) \) such that \( g = \nu \) on \( \partial \Omega \). Integrating (2.11) (in Lemma 2.3) in \((-b, b) \times \Omega\), with \( w \) replaced by \( \hat{z} \), using integrating by parts, and noting \( \hat{z}(-b) = \hat{z}(b) = 0 \), by (3.5) and using Poincaré inequality, we have
\[
- \sum_{k=1}^n \int_{-b}^b \int_{\partial \Omega} \left[ (g \cdot \nabla \hat{z}) \sum_{j=1}^n a^{jk} \hat{z}_j \nu_k + (g \cdot \nabla \hat{z}) \sum_{j=1}^n a^{jk} \hat{z}_j v_k \right] dx ds
\]
\[
+ \int_{-b}^b \int_{\partial \Omega} \left( |\hat{z}_s|^2 + \sum_{j,l=1}^n a^{jl} \hat{z}_j \hat{z}_l \right) ds
\]
\[
= - \int_{-b}^b \int_{\Omega} \left[ \left( \hat{z}_{ss} + \sum_{j,k=1}^n (a^{jk} \hat{z}_j)_{k} \right) g \cdot \nabla \hat{z} + \left( \hat{z}_{ss} + \sum_{j,k=1}^n (a^{jk} \hat{z}_j)_{k} \right) g \cdot \nabla \hat{z} \right] dx ds
\]
\[
- \int_{-b}^b \int_{\Omega} \left( \hat{z}_s g_s \cdot \nabla \hat{z} + \hat{z}_s g \cdot \nabla \hat{z} \right) dx ds
\]
\[
+ \int_{-b}^b \int_{\Omega} \left[ (\nabla \cdot g)|\hat{z}|^2 - 2 \sum_{j,k,l=1}^n a^{jk} \hat{z}_j \hat{z}_l \frac{\partial g}{\partial x_k} + \sum_{j,k=1}^n \hat{z}_j \hat{z}_k \nabla \cdot (a^{jk} g) \right] dx ds
\]
\[
\leq C \int_{-b}^b \int_{\Omega} \left[ |\varphi_{ss} z + 2\varphi_s \hat{z}_s + \varphi z|^2 + |\hat{z}_s|^2 + |\nabla \hat{z}|^2 \right] dx ds
\]
\[
\leq C \int_{-b}^b \int_{\Omega} (|\hat{z}|^2 + |\hat{z}_s|^2 + |\nabla \hat{z}|^2) dx ds.
\]
By (1.2), (3.5) and (3.22), we have

\[ \int_{-b}^{b} \int_{\partial \Omega} (|\dot{z}_a|^2 + \beta |\nabla \dot{z}|^2) dx ds \leq \int_{-b}^{b} \int_{\partial \Omega} \left( |\dot{z}_a|^2 + \sum_{j,l=1}^{n} a^{jl} \bar{z}_j z_l \right) dx ds \]

\[ \leq C \int_{-b}^{b} \int_{\Omega} (|\dot{z}_a|^2 + |\nabla \dot{z}|^2 + |z^0|^2) dx ds \]

\[ + \int_{-b}^{b} \int_{\Gamma_0} a(x) \left[ (g \cdot \nabla \dot{z})(i \dot{z}_a - i \varphi_s z + \varphi z^1) + (g \cdot \nabla \dot{z})(-i \bar{z}_a + i \varphi_s \bar{z} + \varphi \bar{z}^1) \right] dx ds \]

\[ \leq C \int_{-b}^{b} \int_{\Omega} (|\dot{z}_a|^2 + |\nabla \dot{z}|^2 + |z^0|^2) dx ds \]

\[ + \delta \int_{-b}^{b} \int_{\Gamma_0} |\nabla \dot{z}|^2 dx ds + C(\delta) \int_{-b}^{b} \int_{\Gamma_0} (|\varphi_s z|^2 + |\dot{z}_a|^2 + |\varphi z^1|^2) dx ds \]

where \( 0 < \delta < \beta \) is small, then

\[ \int_{-b}^{b} \int_{\partial \Omega \setminus \Gamma_0} |\nabla \dot{z}|^2 dx ds \]

\[ \leq C \left[ \int_{-b}^{b} \int_{\Omega} (|\dot{z}_a|^2 + |\nabla \dot{z}|^2 + |z^0|^2) dx ds + \int_{-b}^{b} \int_{\Omega} (|\dot{z}|^2 + |\dot{z}_a|^2 + |z|^2) dx ds \right]. \]  

(3.24)

Finally, by multiplying \( \bar{z} \) and \( \dot{z} \) on the first equation of (3.5), respectively, using integrating by parts, by (1.2) and using Poincaré inequality, we get

\[ 2 \int_{-b}^{b} \int_{\Omega} (|\dot{z}_a|^2 + \beta |\nabla \dot{z}|^2) dx ds \]

\[ \leq \int_{-b}^{b} \int_{\Omega} \left( 2|\dot{z}_a|^2 + \sum_{j,k=1}^{n} a^{jk} (\bar{z}_j \dot{z}_k + \dot{z}_j \bar{z}_k) \right) dx ds \]

\[ = \sum_{j,k=1}^{n} \int_{-b}^{b} \int_{\partial \Omega} (z a^{jk} \dot{z}_k \nu_j + \dot{z} a^{jk} \bar{z}_k \nu_j) dx ds - \int_{-b}^{b} \int_{\Omega} \varphi (z^0 \bar{z} + \bar{z}^0 \dot{z}) dx ds \]

\[ \leq C \left[ \int_{-b}^{b} \int_{\Gamma_0} (|\dot{z}|^2 + |\dot{z}_a|^2 + |z|^2) dx ds \right] + \frac{1}{\varepsilon^*} \int_{-b}^{b} \int_{\Omega} |z^0|^2 dx ds + \varepsilon^* \int_{-b}^{b} \int_{\Omega} |\dot{z}|^2 dx ds \]

\[ \leq C \left[ \int_{-b}^{b} \int_{\Gamma_0} (|\dot{z}|^2 + |\dot{z}_a|^2 + |z|^2) dx ds \right] \]

\[ + \frac{1}{\varepsilon^*} \int_{-b}^{b} \int_{\Omega} |z^0|^2 dx ds + C \varepsilon^* \int_{-b}^{b} \int_{\Omega} |\dot{z}_a|^2 dx ds \]

(3.25)

Taking \( \varepsilon^* = \frac{1}{C} \) small enough, and combining (3.21), (3.24) and (3.25), we get

\[ \int_{-b}^{b} \int_{\partial \Omega} |\dot{z}|^2 dx ds + \int_{b}^{b} \int_{\partial \Omega \setminus \Gamma_0} |\nabla \dot{z}|^2 dx ds \]

\[ \leq C \left[ \int_{-b}^{b} \int_{\Gamma_0} (|\dot{z}|^2 + |\dot{z}_a|^2 + |z|^2) dx ds + \int_{-b}^{b} \int_{\Omega} |z^0|^2 dx ds \right]. \]
By (3.20) and (3.26), and noting that \( \hat{\varepsilon} = \varphi z \), we obtain

\[
\int_{-b}^{b} \int_{\partial \Omega} V \cdot \nu dx ds \\
\leq Ce^{CA} \left[ \int_{-b}^{b} \int_{\Omega} |z|^{2} dx ds + \int_{-b}^{b} \int_{\Gamma_{0}} (|z|^{2} + |z_{s}|^{2} + |z_{1}|^{2}) dx ds \right]. \tag{3.27}
\]

**Step 5.** Combining (3.14), (3.24) and (3.27), we end up with

\[
\lambda \mu^{2} \int_{-b}^{b} \int_{\Omega} \theta^{2} \phi |(\nabla z|^{2} + |z_{s}|^{2}) dx ds + \lambda^{3} \mu^{4} \int_{-b}^{b} \int_{\Omega} \theta^{2} \phi^{3} |z|^{2} dx ds \\
\leq C \int_{-b}^{b} \int_{\Omega} \theta^{2} \phi^{3} |\varphi_{ss} z + 2 \varphi_{s} z_{s} + \varphi z_{0}|^{2} dx ds \\
+ Ce^{CA} \left[ \int_{-b}^{b} \int_{\Omega} |z|^{2} dx dt + \int_{-b}^{b} \int_{\Gamma_{0}} (|z|^{2} + |z_{s}|^{2} + |z_{1}|^{2}) dx ds \right]. \tag{3.28}
\]

Denote \( c_{0} = 2 + e^{\mu} > 1 \), and recall (3.6) for \( b_{0} \in (1, b) \). Fixing the parameter \( \mu \) in (3.28), using (3.3) and (3.9), one finds

\[
\lambda e^{2 \lambda c_{0}} \int_{-1}^{1} \int_{\Omega} (|\nabla z|^{2} + |z_{s}|^{2} + |z|^{2}) dx ds \\
\leq Ce^{CA} \left\{ \int_{-2}^{2} \int_{\Omega} |z|^{2} dx ds + \int_{-2}^{2} \int_{\partial \Omega} |z_{1}|^{2} dx ds + \int_{-2}^{2} \int_{\Gamma_{0}} (|z|^{2} + |z_{s}|^{2}) dx ds \right\} \tag{3.29}
+ Ce^{2 \lambda (c_{0} - 1)} \int_{(-b_{0}, b_{0})} \int_{\Omega} (|z|^{2} + |z_{s}|^{2}) dx ds.
\]

From (3.29), one concludes that there exists an \( \varepsilon_{2} > 0 \) such that the desired inequality (3.2) holds for \( \varepsilon \in (0, \varepsilon_{2}] \), which, in turn, implies that it holds for any \( \varepsilon > 0 \). This completes the proof of Theorem 3.1.

\[\square\]

### 4 Proof of Theorem 1.2

In this section, we will prove the existence and the estimate of the norm of the resolvent \((A - \lambda I)^{-1}\) when \( \text{Re} \lambda \in \left[ -e^{-C|\text{Im} \lambda|}/C, 0 \right] \).

**Proof.** We divide the proof into two steps.

**Step 1.** First, let \( f = (f^{0}, f^{1}) \in H \), and \( u = (u^{0}, u^{1}) \in D(A) \) with the boundary condition \( \sum_{j,k=1}^{n} a_{jk}^{0} u_{j}^{0} \nu_{k} \big|_{\partial \Omega \setminus \Gamma_{0}} = 0, \quad \left( \sum_{j,k=1}^{n} a_{jk}^{0} u_{j}^{0} \nu_{k} + au^{1} \right) \big|_{\Gamma_{0}} = 0. \)

Then, the following equation

\[
(A - \lambda I)u = f \tag{4.1}
\]

is equivalent to

\[
\begin{cases}
-\lambda u^{0} + u^{1} = f^{0}, \\
\sum_{j,k=1}^{n} (a_{jk}^{0} u_{j}^{0})_{k} - \lambda u^{1} = f^{1}.
\end{cases} \tag{4.2}
\]

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Hence, by substituting $u^1$ by $u^0$ in the second equation of (4.2) and with the boundary condition, we have

\[
\begin{align*}
\sum_{j,k=1}^{n} (a^{jk} u^0_j)_k - \lambda^2 u^0 = \lambda f^0 + f^1 & \quad \text{in } \Omega, \\
\sum_{j,k=1}^{n} a^{jk} u^0_j \nu_k = 0 & \quad \text{on } \partial \Omega \setminus \Gamma_0, \\
\sum_{j,k=1}^{n} a^{jk} u^0_j \nu_k + a \lambda u^0 = -af^0 & \quad \text{on } \Gamma_0, \\
u^1 = f^0 + \lambda u^0 & \quad \text{in } \Omega.
\end{align*}
\]

Put

\[v = e^{i\lambda s} u^0.\]  

(4.4)

It is easy check that $v$ satisfying the following equation:

\[
\begin{align*}
v_{ss} + \sum_{j,k=1}^{n} (a^{jk} v_j)_k = (\lambda f^0 + f^1) e^{i\lambda s} & \quad \text{in } \mathbb{R} \times \Omega, \\
\sum_{j,k=1}^{n} a^{jk} v_j \nu_k = 0 & \quad \text{on } \mathbb{R} \times \partial \Omega \setminus \Gamma_0, \\
\sum_{j,k=1}^{n} a^{jk} v_j \nu_k - i a v_s = -af^0 e^{i\lambda s} & \quad \text{on } \mathbb{R} \times \Gamma_0.
\end{align*}
\]

(4.5)

**Step 2.** By (4.4) and Remark 3.1, we have the following estimates.

\[
\begin{align*}
|u^0|_{H^1(\Omega)} & \leq Ce^{C|\text{Im } \lambda|} |v|_{H^1(\Gamma)}, \\
|v|_{H^1(\mathbb{R})} & \leq C(|\lambda| + 1) e^{C|\text{Im } \lambda|} |u^0|_{H^1(\Omega)}, \\
|v|_{L^2(\mathbb{R})} & \leq Ce^{C|\text{Im } \lambda|} |u^0|_{L^2(\mathbb{R})},
\end{align*}
\]

(4.6)

Now, we apply $v$ to Theorem 3.1, and combining (4.6), we have

\[
|u^0|_{H^2(\Omega)} \leq Ce^{C|\text{Im } \lambda|} \left[ |f^0|_{H^1(\Omega)} + |f^1|_{L^2(\Omega)} + |u^0|_{L^2(\Gamma_0)} \right].
\]

(4.7)

On the other hand, we multiplier (4.2) by $\bar{v}^0$, integrate it on $\Omega$, we get

\[
\int_{\Omega} \left( - \sum_{j,k=1}^{n} (a^{jk} u^0_j)_k + \lambda^2 u^0 \right) \cdot \bar{v}^0 \, dx
= \lambda^2 |u^0|^2_{L^2(\Omega)} + \sum_{j,k=1}^{n} \int_{\Omega} a^{jk} u^0_j \nu_k \bar{v}^0 \, dx - \sum_{j,k=1}^{n} \int_{\partial \Omega} a^{jk} u^0_j \nu_k \bar{v}^0 \, dx
= \lambda^2 |u^0|^2_{L^2(\Omega)} + \sum_{j,k=1}^{n} \int_{\Omega} a^{jk} u^0_j \nu_k \bar{v}^0 \, dx + \int_{\partial \Omega} (a \lambda u^0 + af^0) \bar{v}^0 \, dx.
\]
By taking the imaginary part, we find,
\[
|\text{Im } \lambda| \int_{\partial \Omega} a |u|^2 \, dx
\leq \left| - \sum_{j,k=1}^{n} (a_{jk} u^0_j) k + \lambda^2 u^0_j \right|_{L^2(\Omega)} |u^0|_{L^2(\Omega)}
+ 2 |\text{Im } \lambda||\text{Re } \lambda||u^0|_{L^2(\Omega)} + C|f^0|_{L^2(\partial \Omega)} |\sqrt{a} u^0|_{L^2(\partial \Omega)}
\leq C \left[ (\lambda f^0 + f^1)|u^0|_{L^2(\Omega)} + |\text{Im } \lambda||\text{Re } \lambda||u^0|_{L^2(\Omega)} + |f^0|_{H^1(\Omega)} |u^0|_{H^1(\Omega)} \right]
\]
Hence, combining (4.7) and (4.9), we have
\[
|u^0|_{H^1(\Omega)} \leq C e^{C |\text{Im } \lambda|} \left[ |f^0|_{H^1(\Omega)} + |f^1|_{L^2(\Omega)} + |\text{Im } \lambda||\text{Re } \lambda||u^0|_{H^1(\Omega)} \right].
\] (4.10)

Therefore, we take
\[
C e^{C |\text{Im } \lambda|} |\text{Im } \lambda||\text{Re } \lambda| \leq \frac{1}{2},
\]
which holds, as soon as $|\text{Re } \lambda| \leq - e^{C_0 |\text{Im } \lambda|}/C_0$ for some $C_0 > 0$. Then, we have
\[
|u^0|_{H^1(\Omega)} \leq C e^{C |\text{Im } \lambda|} (|f^0|_{H^1(\Omega)} + |f^1|_{L^2(\Omega)}).
\] (4.11)

Recalling that $u^1 = f^0 + \lambda u^0$, we have
\[
|u^1|_{L^2(\Omega)} \leq |f^0|_{L^2(\Omega)} + |\lambda||u^0|_{L^2(\Omega)} \leq C e^{C |\text{Im } \lambda|} (|f^0|_{H^1(\Omega)} + |f^1|_{L^2(\Omega)}).
\] (4.12)

By (4.11)–(4.12), we know that $A - \lambda I$ is injective. Thus $A - \lambda I$ is bi-injective from $D(A)$ to $H$. And moreover,
\[
||(A - \lambda I)^{-1}||_{L(H,H)} \leq C e^{C |\text{Im } \lambda|}, \quad \text{Re } \lambda \in (-e^{-C |\text{Im } \lambda|}/C, 0), \quad |\lambda| \geq 1.
\]
This completes the proof of Theorem 1.2.

\[\blacklozenge\]

5 Proof of Theorem 1.1

In this section, we adapt the proof of [1, Théorème 3] (and also the proof of [4, Theorem 3] on semigroups).

Proof of Theorem 1.1. By taking $\chi_1 = \chi_2 = I$, $A = iB$ and $k = 2$ in [1, Théorème 3], we have
\[
\left\| e^{tA} u \right\|_{L^2(H)} \leq \left( \frac{C}{\ln(2 + t)} \right)^2 \left\| u \right\|_{L^2(H)},
\] (5.1)
that is
\[
\left\| e^{tA} u \right\|_H \leq \left( \frac{C}{\ln(2 + t)} \right)^2 \left\| u \right\|_{D(A^2)}.
\] (5.2)

By definition, $D(A)$ is the interpolate space between $D(A^0) = H$ and $D(A^2)$. Since
\[
\left\| e^{tA} u \right\|_H \leq C \left\| u \right\|_H.
\] (5.3)
Then, combining (5.2)–(5.3), by applying interpolation theorem, we get the desired result.

\[\blacklozenge\]
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