Families index for
Boutet de Monvel operators

Severino T. Melo, Thomas Schick, and Elmar Schrohe

Abstract. We define the analytical and the topological indices for continuous families of operators in the C∗-closure of the Boutet de Monvel algebra. Using techniques of C∗-algebra K-theory and the Atiyah-Singer theorem for families of elliptic operators on a closed manifold, we prove that these two indices coincide.

2010 Mathematics Subject Classification: 19K56, 46L80, 58J32

INTRODUCTION

Boutet de Monvel’s calculus [5] provides a pseudodifferential framework which encompasses the classical differential boundary value problems. In an extension of the concept of Lopatinski and Shapiro, it associates to each operator two symbols: a pseudodifferential principal symbol, which is a bundle homomorphism, and an operator-valued boundary symbol. Ellipticity requires the invertibility of both. In this case, the calculus allows the construction of a parametrix. If the underlying manifold is compact, elliptic elements define Fredholm operators, and the parametrices are Fredholm inverses. Boutet de Monvel showed how then the index can be computed in topological terms. The crucial observation is that elliptic operators can be mapped to compactly supported K-theory classes on the cotangent bundle over the interior of the manifold. The topological index map, applied to this class, then furnishes an integer which is equal to the index of the operator.

For the construction of the above map, Boutet de Monvel combined operator homotopies and classical (vector bundle) K-theory in a very refined way. It therefore came as a surprise that this map – which is neither obvious nor trivial – can also be obtained as a composition of various standard maps in K-theory for C∗-algebras – which was not yet available when [5] was written. In fact, it turns out to be basically sufficient to have a precise understanding of the short exact sequence induced by the boundary symbol map, [17], see also [16].

In the spirit of the classical result of Atiyah and Singer [3], we introduce and consider in this article families of operators in Boutet de Monvel’s calculus, an issue that has not been addressed in [5].

More specifically, we consider a compact manifold \( X \) with boundary and then a fiber bundle \( Z \to Y \) with fiber \( X \) over a compact Hausdorff space \( Y \). We are then studying fiberwise (elliptic) Boutet de Monvel operators, depending continuously on \( y \in Y \). In order to be able to use the powerful tools of C∗-algebra K-theory we define such an operator family \( A \) over \( Y \) as a continuous section of a bundle of C∗-algebras over \( Y \), a concept which is slightly more general than that of Atiyah and Singer, who equip the set of operators with a Fréchet-space topology. In fact, restricted to the case without boundary, our algebra of continuous families \( \mathfrak{A} \) contains that of [3] as a dense subalgebra.

While the analytic index \( \text{ind}_a(A) \) of such an elliptic family \( A \) as an element of \( K(Y) \) is easily defined following Atiyah [2] and Jänich [11], cf. Definition [15] below, it is less obvious how to obtain the topological description. Similar to Boutet de Monvel’s approach, the essential step is the construction of a map which associates to an elliptic family an element of the compactly supported K-theory of the total space of the bundle of cotangent spaces over the interior of the underlying manifolds. We regard this map as a homomorphism defined on \( K_1(\mathfrak{A}/\mathfrak{R}) \), where \( \mathfrak{R} \) denotes the ideal of continuous families which have values in compact operators. In its definition, we use a fact which builds upon an observation of Boutet de Monvel: There exists a natural subalgebra \( \mathfrak{A}^\dag \) of \( \mathfrak{A} \) for which \( K_*(\mathfrak{A}^\dag/\mathfrak{R}) \cong K_*(\mathfrak{A}/\mathfrak{R}) \) so that each elliptic family \( A \) in \( \mathfrak{A} \) can be represented by a class \( a \in K_1(\mathfrak{A}^\dag/\mathfrak{R}) \). Moreover, \( \mathfrak{A}^\dag/\mathfrak{R} \) is commutative which allows us to make the connection to classical (vector bundle) K-theory. Then \( \text{ind}_t(A) \) is defined by applying the classical construction of the topological index to \( a \), compare Definition [16].

Our main result is then that these two indices are equal. To prove this, we reduce to the classical families index theorem of Atiyah and Singer [3]. We assign in a canonical way to \( A \) an index problem on
a bundle of closed manifolds, namely the double of our original bundle of manifolds with boundary. We then show that this associated family has the same analytic as well as topological index as $A$. In this step we make once more use of the isomorphism $K_1(\mathcal{A}/\mathcal{R}) \cong K_1(\mathcal{A}^1/\mathcal{R})$.

It is perhaps worth stressing that our index theorem does not use the Boutet de Monvel index theorem for boundary value problems, which can actually be obtained from ours by taking $Y$ equal to one point. Taking the families index theorem for granted, Albin and Melrose derived a more refined formula for the Chern character of the index bundle in terms of symbolic data [1, Theorem 3.8].

The paper is structured as follows: Section 1 starts with a review of the Boutet de Monvel calculus for a single manifold. We introduce the C*-algebra $A$ of Boutet de Monvel operators of order and class zero and the boundary symbol map $\gamma$. Section 2 gives the technical introduction of operator families in Boutet de Monvel’s calculus over a compact Hausdorff space $Y$. We define them as the continuous sections into a bundle of operator algebras whose typical fiber is the C*-algebra $A$. In order to keep the exposition simple, we first treat the case where $E$ is trivial one-dimensional and $F = 0$. We introduce $\gamma$ as the fiberwise symbol map and extend the results on the kernel and image of $\gamma$ to the family situation.

While in the single operator case this was sufficient to compute the K-theory of $A/\mathcal{K}$, the situation is more complicated in the families case. In fact, an important ingredient in [17] is that fact that whenever $X$ is connected and $\partial X \neq \emptyset$ there exists a continuous section of $S^*X$. This is no longer true in the families case. Instead, we prove in Theorem 14 the fact alluded to above: For $F = 0$ we define $\mathcal{A}^1$ as the C*-algebra generated by all sections whose pseudodifferential part is independent of the co-variable at the boundary and whose singular Green part vanishes. Then $\mathcal{A}^1/\mathcal{R}$ is commutative. Moreover, we use a Mayer-Vietoris argument to show that the inclusion map induces an isomorphism

$$K_*(\mathcal{A}^1/\mathcal{R}) \cong K_*(\mathcal{A}/\mathcal{R}).$$

In Section 3 we study the index problem. Again, we confine ourselves first to the case of trivial one-dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that the analytic and the topological index are equal. To achieve this, we reduce with the help of a doubling dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that $\gamma$ induces the

$$1.$$ $K_*(\mathcal{A}^1/\mathcal{R}) \cong K_*(\mathcal{A}/\mathcal{R}).$

In Section 4 we study the index problem. Again, we confine ourselves first to the case of trivial one-dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that the analytic and the topological index are equal. To achieve this, we reduce with the help of a doubling dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that $\gamma$ induces the

$$1.$$ $K_*(\mathcal{A}^1/\mathcal{R}) \cong K_*(\mathcal{A}/\mathcal{R}).$

In Section 4 we study the index problem. Again, we confine ourselves first to the case of trivial one-dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that the analytic and the topological index are equal. To achieve this, we reduce with the help of a doubling dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that $\gamma$ induces the

$$1.$$ $K_*(\mathcal{A}^1/\mathcal{R}) \cong K_*(\mathcal{A}/\mathcal{R}).$

In Section 4 we study the index problem. Again, we confine ourselves first to the case of trivial one-dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that the analytic and the topological index are equal. To achieve this, we reduce with the help of a doubling dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that $\gamma$ induces the

$$1.$$ $K_*(\mathcal{A}^1/\mathcal{R}) \cong K_*(\mathcal{A}/\mathcal{R}).$
Operators in Boutet de Monvel’s calculus have an order and a class or type. There are invertible elements in the calculus which allow us to reduce both, order and class, to zero. The operators then form a $*$-subalgebra of the bounded operators on the Hilbert space $H := L^2(X, E) \oplus L^2(\partial X, F)$.

**Definition 1.** Let $\mathcal{A}^0(E, F)$ denote the algebra of the (polyhomogeneous) Boutet de Monvel operators of order and class zero on $H = L^2(X, E) \oplus L^2(\partial X, F)$, endowed with its natural Fréchet topology, and $A$ its $C^*$-closure in the algebra of all bounded operators on $H$. We write $\mathcal{A}^0$ and $A$ if $E = X \times \mathbb{C}$ is trivial one-dimensional and $F = 0$.

Let $A \in \mathcal{A}^0(E, F)$ be given as in [2]. For each entry $P, S, G, T, K$ we have a symbol. This is the usual one for $P$ and $S$, while $G, T$, and $K$ can be considered as operator-valued pseudodifferential operators on $\partial X$ with classical symbols in the sense of Schulte [23].

These are defined as follows, see [22]: The principal pseudodifferential symbol $\sigma(A)$ of $A$ is the restriction of the principal symbol of $P$ to the cosphere bundle over $X$. In order to define the boundary principal symbol $\gamma(A)$ we first denote by $p^0$, $g^0$, $t^0$, $k^0$, and $s^0$ the principal symbols of $P$, $G$, $T$, $K$, and $S$, respectively. We let $E^{0}_{x', \xi'}$ be the pullback of $E_{\{x_n = 0\}}$ to the normal bundle of $X$, lifted to $(x', \xi') \in S^* \partial X$. For fixed $(x', \xi') \in S^* \partial X$, $\xi_n \mapsto p^0(x', 0, \xi', \xi_n)$ is a function on the conormal line in $(x', \xi')$, acting on $E^{0}_{x', \xi'}$. It induces a truncated pseudodifferential operator

$$ p^0(x', 0, \xi', D_n)_+ = r^+ p^0(x', 0, \xi', D_n) e^+: L^2(\mathbb{R}_{\geq 0}, E^{0}_{x', \xi'}) \rightarrow L^2(\mathbb{R}_{\geq 0}, E^{0}_{x', \xi'}). $$

In local coordinates near the boundary we then define the boundary principal symbol $\gamma(A)(x', \xi') : L^2(\mathbb{R}_{\geq 0}, E^{0}_{x', \xi'}) \oplus F_{x', \xi'} \rightarrow L^2(\mathbb{R}_{\geq 0}, E^{0}_{x', \xi'}) \oplus F_{x', \xi'}$ by

$$ \gamma(A)(x', \xi') := \begin{pmatrix} p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi', D_n) & k^0(x', \xi', D_n) \\ t^0(x', \xi', D_n) & s^0(x', \xi') \end{pmatrix}, $$

with $D_n$ indicating that we let the symbol act as an operator with respect to the variable $x_n$ only. Note that the operator $p^0(x', \xi', D_n)$ is compact and that $k^0(x', \xi', D_n)$, $t^0(x', \xi', D_n)$ and $s^0(x', \xi')$ even have finite rank. The operator $p^0(x', 0, \xi', D_n)_+$ on the other hand is a Toeplitz type operator; it will not be compact unless $p^0 = 0$.

Denoting by $K = K(H)$ the ideal of compact operators on $L(H)$, one has the following important estimate based on work by Gohberg [7], Seeley [24] and Grubb-Geymonat [9], see [20, 2.3.4.4, Theorem 1] for a proof:

$$ \inf_{\mathcal{K} \in \mathcal{K}} \|A + \mathcal{K}\| = \max\{\|\sigma(A)\|_{\sup}, \|\gamma(A)\|_{\sup}\}, $$

where the sup-norms on the right hand side are over the cosphere bundles in $X$ and $\partial X$, respectively. This estimate implies, in particular, that both symbols extend continuously to $C^*$-algebra homomorphisms defined on $\mathcal{A}(E, F)$. For fixed $(x', \xi')$ the range $\{\gamma(A)(x', \xi') \mid A \in \mathcal{A}\}$ forms an algebra of Wiener-Hopf type operators.

It also follows from this estimate that $\gamma$ vanishes on $\mathcal{K}$. Since the entries of $\gamma(A)(x', \xi')$ induced by $g^0$, $k^0$, $t^0$ and $s^0$ are (pointwise) compact while that induced by $p^0$ is not (unless $p^0 = 0$), we conclude that a Boutet de Monvel operator $A$ belongs to $\ker \gamma$ if and only if $\sigma(A)$ vanishes at the boundary. Based on this observation (see [10 Section 2] for details) one can show that $\sigma$ induces an isomorphism

$$ \ker \gamma / \mathcal{K} \cong C_0(S^* X^0). $$

The $K$-theory of the range of $\gamma$ was described in [10 Section 3]. Let $b : C(\partial X) \rightarrow \text{Im} \gamma$ denote the $C^*$-homomorphism that maps $g$ to $\gamma(m(f))$, where $m(f)$ is the operator of multiplication by a function $f \in C(\partial X)$ whose restriction to $\partial X$ equals $g$. Then $b$ induces a $K$-theory isomorphism.

**2. $K$-Theory of the families $C^*$-algebra**

To simplify the exposition, we shall assume in this section that $E = X \times \mathbb{C}$ is the trivial one-dimensional line bundle and $F = 0$.

Let $\text{Diff}(X)$ denote the group of diffeomorphisms of $X$, equipped with its usual Fréchet topology. Recall that $\delta : U \rightarrow \partial X \times [0, 1)$ is the collar fixed at the beginning of Section 1. Let $G$ denote the subgroup of $\text{Diff}(X)$ consisting of those $\phi$ such that $\delta \circ \phi \circ \delta^{-1} : \partial X \times [0, 1/2) \rightarrow \partial X \times [0, 1)$ is of the form $(x', x_n) \mapsto (\phi(x'), x_n)$ for some diffeomorphism $\phi : \partial X \rightarrow \partial X$. We are going to use two properties that each $\phi \in G$ satisfies: the boundary defining function is preserved $(x_n \circ \phi = x_n)$ for $0 \leq x_n \leq 1/2$, and the canonical map $2\phi : 2X \rightarrow 2X$, defined by $2\phi \circ i_{\pm} = i_{\pm} \circ \phi$, where $i_{\pm} : X^\pm \rightarrow 2X$ are the two canonical embeddings of $X$ in $2X$, is a diffeomorphism of $2X$. 

Münster Journal of Mathematics Vol. 1 (2008), 99999–99999
Throughout this paper, $\pi: Z \to Y$ will denote a fiber bundle over the compact Hausdorff space $Y$ with fiber $X$ and structure group $G$. Note, however, that this choice of structure group is just for convenience and can always be (essentially uniquely) arranged for a general bundle with typical fiber $X$, see the Appendix A for details.

We denote $Z_y := \pi^{-1}(y)$. Each $Z_y$ is a smooth manifold with boundary, non-canonically diffeomorphic to $X$. The restriction of $\pi$ to $\partial Z = \cup_y \partial Z_y$ is a fiber bundle $\pi_\partial: \partial Z \to Y$ with fiber $\partial X$ and structure group Diff($\partial X$).

Next we define a bundle of Hilbert spaces, and later a C*-algebra which will act on its space of sections. This is a bit delicate, as it depends on some further choices; therefore we give the details. We choose a continuous family of Riemannian metrics $(g_y)_{y \in Y}$ with corresponding measures $\mu_y$ on $Z_y$ and define $H_y := L^2(Z_y, \mu_y)$. Recall that such a family $(g_y)$ exists: we can patch them together using trivializations of the bundle and a partition of unity on $Y$, as the space of Riemannian metrics on $X$ is convex.

The union $\mathcal{H} = \bigcup_{y \in Y} H_y$ is a fiber bundle of topological vector spaces over $Y$, canonically associated to $\pi: Z \to Y$, with trivializations induced from the trivializations of $\pi$ in the obvious way. The structure group is the group of invertible bounded operators on $H$, equipped with the strong topology.

Remark 2. That we obtain here the strong topology and not the norm topology comes from the fact that the changes of trivialization are implemented by pullback with the diffeomorphisms of $G$.

Moreover, the choice $(g_y)_{y \in Y}$ gives rise to a continuous family of inner products on $\mathcal{H}$ inducing the given topology of the fibers $H_y$.

Let $\mathcal{A}_y$ be the Boutet de Monvel algebra of order and class zero on $L^2(Z_y)$. We want to define the bundle of Boutet de Monvel algebras $\mathcal{H} = \bigcup_{y \in Y} \mathcal{A}_y$ as locally trivial bundle with structure group the automorphism group of the C*-algebra $\mathcal{A}$ with the norm topology, associated to $Z \to Y$.

To achieve this, we need the diffeomorphism invariance of the Boutet de Monvel algebra in a precise form.

Definition 3. Given $\phi \in G$, let $T_\phi$ denote the bounded operator on $L^2(X)$ defined by $f \mapsto f \circ \phi^{-1}$.

Proposition 4. We have a well defined continuous action (for the Fréchet topology on $G$ and the norm topology on $\mathcal{A}$).

$$G \times \mathcal{A} \ni (\phi, A) \mapsto T_\phi AT_\phi^{-1} \in \mathcal{A}.$$ 

Moreover, by restriction we get an action $G \times \mathcal{A}^o \to \mathcal{A}^o$.

Proof. This corresponds to [3 Proposition 1.3]. In fact, even if $X$ is closed, Atiyah and Singer consider a slightly different situation in that they close $\mathcal{A}$ with respect to the operator norm of the action on all Sobolev spaces, while we only use the operator norm on $L^2$. Their argument still applies verbatim, since they treat the action on each Sobolev space separately.

Indeed, the proof of [3 Proposition 1.3] uses only a number of formal properties of the algebra of pseudodifferential operators which are also satisfied by the Boutet de Monvel algebra, and therefore applies in the same way to our general situation. To be more specific, let us list these properties:

- (1) the Boutet de Monvel algebra $\mathcal{A}^o$ is diffeomorphism invariant, i.e. in particular $T_\phi AT_\phi^{-1} \in \mathcal{A}^o$ for $A \in \mathcal{A}^o$ and $\phi \in G$.
- (2) Each $T_\phi$ is a bounded operator on $L^2(X)$ and the map $G \to \mathcal{L}(L^2(X))$ is strongly continuous. Moreover, for a sufficiently small open neighborhood of 1, the image has uniformly bounded norm. The proof of this fact as given in [3] works for compact manifolds with boundary exactly the same way as for closed manifolds.
- (3) Let $\mathcal{V}_G$ denote the space of vector fields on $X$ which, in the collar, pull back from vector fields on $\partial X$. The exponential map, defined with the help of Riemannian metrics which respect the collar structure, gives a local diffeomorphism (of Fréchet manifolds) between $\mathcal{V}_G$ and $G$.
- (4) If $V \in \mathcal{V}_G$ and $A \in \mathcal{A}^o$ then the commutator $[A, V]$ belongs to $\mathcal{A}^o$ by the rules of the calculus, cf. [8 Theorem 2.7.6].

All these properties are either well known or easy to establish. □

Corollary 5. We obtain the bundle $\mathcal{H} = \bigcup_{y \in Y} \mathcal{A}_y$ of topological algebras with bundle of subalgebras $\mathcal{H}^o = \bigcup_{y \in Y} \mathcal{A}_y^o$, modeled on $(\mathcal{A}, \mathcal{A}^o)$ with structure group the automorphism group of $\mathcal{A}$ with its norm topology and the automorphism group of $\mathcal{A}^o$ its Fréchet topology. The local trivializations are induced
by the local trivializations of $\pi: Z \to Y$, where a diffeomorphisms $\alpha_y: Z_y \to X$ obtained from the trivialization map $A_y$ in $A$ by conjugation with $T_{\alpha_y}$.

Moreover, the choice of metrics $(g_y)_{y \in Y}$ induces a continuous family of norms on the fibers of $N$ inducing the topology. With these norms the bundle becomes a bundle of $C^*$-algebras.

Proof. The statement about the bundle of topological algebras follows immediately from Proposition IV. Moreover, it is well known that each $A_y$ is closed under taking adjoints in $L^2(Z_y)$.

We now check that with this structure, we obtain a locally trivial bundle of $C^*$-algebras. Fix a local trivialization with diffeomorphisms $\alpha_y: Z_y \to X$. If we pull back the inner products on $H_y$ to $H = L^2(X)$ with the induced maps, then the corresponding Gram operator $G_y$, expressing this pullback inner product in terms of the original one on $L^2(X)$, is the multiplication with a smooth positive function $\mu_y$ which depends continuously on $y$: the density of $\alpha_y^*\mu_y$ with respect to a chosen measure $\mu$ on $X$. Note that $G_y$ belongs to $A$ and its norm, which is just the supremum, depends continuously on $y$. Now compose the original trivialization of $A_y$ with conjugation by $\sqrt{G_y}$ and the resulting trivialization will respect the $C^*$-algebra structures, but inherit the norm continuity of transition maps. To summarize: with a canonical modification (given in terms of the inner products) we have obtained trivializations of our bundle $N$ as a bundle of $C^*$-algebras, as claimed.

Definition 6. We denote by $\mathfrak{A}$ the set of continuous sections of the bundle $N$ of $C^*$-algebras. With the pointwise operations and the supremum norm, this becomes a $C^*$-algebra. The underlying topological algebra is canonically associated to $\pi: Z \to Y$, the norm and the $*$-operation depend on the choice of the family of metrics $(g_y)_{y \in Y}$.

The principal symbol and the boundary principal symbol extend continuously to two families of $C^*$-algebra homomorphisms

$$\sigma_y: A_y \to C(S^*Z_y) \text{ and } \gamma_y: A_y \to C(S^*\partial Z_y, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))),$$

where $S^*$ denotes cosphere bundle and $\mathcal{L}$ bounded operators. Here $\gamma_y$ is well defined, since the structure group of the bundle $\pi: Z \to Y$ leaves the boundary defining function invariant, see [3] Theorem 2.4.11.

Let us denote by $S^*Z$ the disjoint union of all $S^*Z_y$. This can canonically be viewed as the total space of a fiber bundle over $Y$ with structure group $G$. One analogously defines $S^*\partial Z = \cup_y S^*\partial Z_y$ and $S^*Z^c = \cup S^*Z_y$.

Definition 7. Given $A \in \mathfrak{A}$, let $\sigma_A$ be the function on $S^*Z$ defined by piecing together all the $\sigma_y$'s. Then $A \mapsto \sigma_A$ defines a $C^*$-algebra homomorphism

$$\sigma: \mathfrak{A} \to C(S^*Z).$$

One also gets, analogously,

$$\gamma: \mathfrak{A} \to C(S^*\partial Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))).$$

Let $\mathfrak{A}$ denote the subalgebra of $\mathfrak{A}$ consisting of the sections $(A_y)_{y \in Y}$ such that $A_y$ is compact for every $y \in Y$. It follows immediately from the corresponding statement for a single manifold that $\ker \sigma \cap \ker \gamma = \mathfrak{R}$. It is also straightforward to generalize the description of $\ker \gamma$ for a single manifold [4]:

Theorem 8. The principal symbol restricted to $\ker \gamma$ induces a $C^*$-algebra isomorphism

$$(6) \quad \ker \gamma/\mathfrak{R} \simeq C_0(S^*Z^c).$$

Here $C_0(S^*Z^c)$ consists of the elements of $C(S^*Z)$ which, for every $y \in Y$, vanish on all points of $S^*Z_y$ with base point belonging to $\partial Z_y$.

Regarding each $f \in C(Z)$ as a family of multiplication operators on $(H_y)_{y \in Y}$, furnishes an embedding of $C(Z)$ in $\mathfrak{A}$, which we denote $m: C(Z) \to \mathfrak{A}$. Mapping a $g \in C(\partial Z)$ to the boundary principal symbol of $m(f)$, where $f \in C(Z)$ is such that its restriction to $\partial Z$ is $g$, defines the $C^*$-algebra homomorphism $b: C(\partial Z) \to \text{Im} \gamma$.

Theorem 9. The homomorphisms $b_i: K_i(C(\partial Z)) \to K_i(\text{Im} \gamma)$, $i = 0, 1$, induced by $b$ are isomorphisms.

Proof. Given an open set $U \subseteq Y$, let us denote by $\pi_U: Z_U = \pi^{-1}(U) \to U$ the restriction of $\pi$ to $U$, by $\mathfrak{A}_U$ the algebra of sections in $\mathfrak{A}$ which vanish outside $U$ and by $\gamma_U$ the restriction of $\gamma$ to $\mathfrak{A}_U$. Moreover we let

$$C_0(\partial Z_U) = \{f \in C(\partial Z) : \supp f \subseteq \pi_0^{-1}(U)\}$$

and write $b_U$ for the restriction of $b$ to $C_0(\partial Z_U)$. If the bundle $\pi$ is trivial over $U$, then $\mathfrak{A}_U$ is isomorphic to $C_0(U, A)$ and, with respect to this isomorphism, $b_U$ corresponds to the tensor product of the identity on $C_0(U)$ with the corresponding map for a single manifold, also denoted by $b$ on $[14,17]$. It is the content
of [16] Corollary 8] that \( b \) induces a K-theory isomorphism onto the image of \( \gamma \). It then follows from the Künneth formula for C*-algebras [21] that \( b \) induces isomorphisms \( b \circ K_1(C_0(\partial Z_U)) \to K_1(\text{Im} \, \gamma_U) \), \( i = 0, 1 \), see Proposition 21 in Appendix [3].

Now let \( (\text{Im} \, \gamma_U) \) denote the subset of \( \text{Im} \, \gamma \) consisting of those functions which vanish outside \( \cup_{y \in U} S^* \partial Z_y \). It is obvious that \( \text{Im} \, \gamma_U \subseteq (\text{Im} \, \gamma)_U \). Since both \( \text{Im} \, \gamma_U \) and \( (\text{Im} \, \gamma)_U \) are closed in \( C(S^* \partial Z, \mathbb{L}(L^2(\mathbb{R}^2_+))) \), to show that they are equal it suffices to show that the former is dense in the latter. This follows from the fact that multiplication by a complex continuous function with support contained in \( U \) maps \( (\text{Im} \, \gamma)_U \) to \( \text{Im} \, \gamma_U \). This simple observation implies that, for open sets \( U \) and \( V \), we have a canonical C*-algebra isomorphism

\[
\text{Im} \, \gamma_U \cong \{ (f,g) \in \text{Im} \, \gamma_U \oplus \text{Im} \, \gamma_V ; f = g \}.
\]

Now suppose that we have shown \( b \circ \) to be an isomorphism for some open \( U \) and that \( V \) is open and \( \pi \) trivial over \( U \cap V \), and so in particular also over \( U \cap V \). We then consider the two — thanks to (7) — diagrams

\[
\begin{array}{ccc}
C_0(\partial Z_{U \cap V}) & \to & C_0(\partial Z_U) \\
\downarrow & & \downarrow \\
C_0(\partial Z_V) & \to & C_0(\partial Z_{U \cap V})
\end{array}
\begin{array}{ccc}
\text{Im} \, \gamma_{U \cap V} & \to & \text{Im} \, \gamma_U \\
& & \downarrow \\
\text{Im} \, \gamma_V & \to & \text{Im} \, \gamma_{U \cap V}
\end{array}
\]

Because they are cartesian, we may extract from both diagrams cyclic exact Mayer-Vietoris sequences (see [17, 21.2.2] or [15, 7.2.1]), and we may use the K-theory maps induced by \( b_U \) and \( b_{U \cap V} \) and \( b_{U \cap V} \) to map the first cyclic sequence to the second. By assumption and the case of trivial bundles, the maps induced by \( b_U \), \( b_V \) and \( b_{U \cap V} \) are isomorphisms. It then follows from the five-lemma that also \( b_{U \cap V} \) induces a K-theory isomorphism.

Since \( Y \) has a finite cover by open sets over which \( \pi \) is trivial, induction shows that \( b \) induces K-theory isomorphisms.

Using Theorem 8, we obtain the following commutative diagram of C*-algebra homomorphisms, whose horizontal lines are exact:

\[
\begin{array}{ccc}
0 & \to & C_0(S^*Z^\circ) \\
\downarrow m^\circ & & \downarrow m \\
0 & \to & C_0(Z^\circ)
\end{array}
\begin{array}{ccc}
\gamma & \to & \text{Im} \, \gamma \\
\downarrow & & \downarrow b \\
C(\partial Z) & \to & 0
\end{array}
\]

We have denoted by \( r \) the map that pieces together all restrictions \( r_y : C(Z_y) \to C(\partial Z_y) \), \( y \in Y \), and by \( Z^\circ \) the union \( \cup_y Z_y^\circ \). Since the isomorphism [13] is induced by the principal symbol, and the principal symbol of an operator of multiplication by a function is the function itself, the map \( m^\circ \) in the diagram above is actually the map of composition with the canonical projection \( S^*Z^\circ \to Z^\circ \). We may apply the cone-mapping functor [17] Lemma 9] to the above diagram and get (using the same arguments that prove (11) in [17]) the following commutative diagram of cyclic exact sequences

\[
\begin{array}{ccc}
K_0(C_0(Z^\circ)) & \to & K_0(C(\partial Z)) \\
\downarrow m^\circ & & \downarrow m \\
K_0(C_0(S^*Z^\circ)) & \to & K_0(\mathfrak{A}/\mathfrak{R}) \\
\downarrow & & \downarrow \\
K_1(C^\circ) & \to & K_1(Cm) \\
\downarrow \cong & & \downarrow \\
K_1(C_0(Z^\circ)) & \to & K_1(C(\partial Z)) \\
\downarrow m^\circ & & \downarrow m^\circ \\
K_1(C_0(S^*Z^\circ)) & \to & K_1(\mathfrak{A}/\mathfrak{R}) \\
\downarrow & & \downarrow \\
K_0(Cm^\circ) & \to & K_0(Cm) \\
\downarrow \cong & & \downarrow \\
K_0(C_0(Z^\circ)) & \to & K_0(C(\partial Z))
\end{array}
\]

where \( \cong \) denotes isomorphism.

Up to this point, everything goes exactly as in the case of a single manifold, but here comes a difference: The homomorphism \( m_0 \) does not necessarily have a left inverse (in the case of a single manifold \( X \), such a left inverse is defined by composition with a section of \( S^*X \)), and hence the cyclic exact sequences above do not have to split into short exact ones.

To proceed we now introduce the subalgebra \( \mathfrak{A}^\dagger \) of \( \mathfrak{A} \) and an associated subalgebra \( B \) of \( C(S^*Z) \) with the properties outlined in the introduction: For each \( y \in Y \), let \( B_y \) denote the subalgebra of \( C(S^*Z_y) \) consisting of the functions which do not depend on the co-variable over the boundary, that is,
an $f \in C(S^*Z_y)$ belongs to $B_y$ if and only if the restriction of $f$ to the points of $S^*Z_y$ over $\partial Z_y$ equals $g \circ p_y$, for some $g \in C(\partial Z_y)$, where $p_y : S^*Z_y \to Z_y$ is the canonical projection. We then define $A_y$ as the $C^*$-subalgebra of $A_y$ generated by $\{P_\gamma; \gamma \text{ is a pseudodifferential operator with } \text{the transmission property and } \sigma_0(P_\gamma) \in B_y\}$.

**Definition 10.** Let $B$ denote the subalgebra of $C(S^*Z)$ consisting of the functions whose restriction to each $S^*Z_y$ belongs to $B_y$. We let then $A^\dagger$ be the $C^*$-subalgebra of $A$ consisting of the sections $(A_y)_{y \in Y}$ such that $A_y \in A^\dagger_y$ for every $y \in Y$.

**Proposition 11.** The $C^*$-algebra $A^\dagger/\overline{\mathcal{R}}$ is commutative, and the map

$$A^\dagger/\overline{\mathcal{R}} \ni [A] \xrightarrow{\sigma} \sigma(A) \in B$$

is a $C^*$-algebra isomorphism.

**Proof.** Let $P = (P_y)$ be a family of pseudodifferential operators with symbol independent of the covariable over the boundary, i.e. a generator of $A^\dagger$. According to [3], $\gamma(P)$ can be considered as a function on $\partial Z$, acting for $z \in \partial Z$ on $L^2(\mathbb{R}_{\geq 0})$ by multiplication with $\gamma(P)(z)$. Moreover, for $z \in \partial Z$ we have $\gamma(z) = \sigma(z)$ independent of the covariable by assumption. It follows that the composed algebra homomorphism

$$\sigma : A^\dagger \xrightarrow{\sigma \circ \gamma} C(S^*Z) \oplus C(S^*Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))) \xrightarrow{\rho} C(S^*Z)$$

has the same kernel as $\sigma \circ \gamma$, namely $\overline{\mathcal{R}}$ and so the map we consider is injective and in particular $A^\dagger/\overline{\mathcal{R}}$ is commutative. By the very definition of $A^\dagger$, $\sigma : A^\dagger \to B$ has dense image, as a morphism of $C^*$-algebras it is therefore also surjective.

This allows us to describe the K-theory of $\mathcal{A}/\overline{\mathcal{R}}$:

**Theorem 12.** The composition

$$K_i(\mathcal{A}/\overline{\mathcal{R}}) \xrightarrow{\tau_{i-1}} K_i(\mathcal{A}^\dagger/\overline{\mathcal{R}}) \xrightarrow{\sigma} K_i(B)$$

is an isomorphism, $i = 0, 1$.

The proof makes use of the following proposition, which is easily established by a diagram chase, compare [10] Exercise 38 of Section 2.2:

**Proposition 13.** Let there be given a commutative diagram of abelian groups with exact rows,

$$\cdots \to A_i^{f_i} \xrightarrow{g_i} B_i \xrightarrow{b_i} C_i \xrightarrow{c_i} A_i^{f_i+1} \to \cdots$$

where each $c_i$ is an isomorphism. Then the sequence

$$\cdots \to A_i \xrightarrow{(a_i, -b_i)} A_i^{f_i} \oplus B_i \xrightarrow{(f_i, b_i)} B_i \xrightarrow{b_i \circ c_i^{-1} \circ g_i} A_{i+1} \to \cdots$$

is exact, where $(f_i, b_i)$ is the map defined by $(f_i, b_i)(\alpha, \beta) = f_i(\alpha) + b_i(\beta)$.

We are now ready to prove Theorem 12. Applying Proposition 13 to the diagram 8, we get the exact sequence

$$K_0(C_0(Z^\circ)) \to K_0(C(Z)) \oplus K_0(C_0(S^*Z^\circ)) \to K_0(\mathcal{A}/\overline{\mathcal{R}})$$

(9)

$$K_1(\mathcal{A}/\overline{\mathcal{R}}) \leftarrow K_1(C(Z)) \oplus K_1(C_0(S^*Z^\circ)) \leftarrow K_1(C_0(Z^\circ)).$$

We next consider the following diagram of commutative $C^*$-algebras

$$\begin{array}{ccc}
C_0(Z^\circ) & \xrightarrow{m_1} & C_0(S^*Z^\circ) \\
\downarrow & & \downarrow P_2 \\
C(Z) & \xrightarrow{p_1} & B
\end{array}$$

(10)

As $C_0(Z^\circ)$ is canonically isomorphic to

$$\{(f, g) \in C(Z) \oplus C_0(S^*Z^\circ); \ p_1(f) = p_2(g)\},$$

the Mayer-Vietoris exact sequence associated to (10) is the exact sequence

$$K_0(C_0(Z^\circ)) \to K_0(C(Z)) \oplus K_0(C_0(S^*Z^\circ)) \to K_0(B)$$

(11)

$$K_1(B) \leftarrow K_1(C(Z)) \oplus K_1(C_0(S^*Z^\circ)) \leftarrow K_1(C_0(Z^\circ)).$$
The map \( \iota : B \cong \mathfrak{A}/\mathcal{R} \to \mathfrak{A}/\mathcal{R} \) and the identity on the other K-theory groups furnish morphisms from the cyclic sequence (11) to the cyclic sequence (9). The five lemma then shows that the induced maps in K-theory are isomorphisms. Together with Proposition 14 we obtain the assertion. \( \square \)

3. The Boutet de Monvel family index theorem

The index of a continuous function with values in Fredholm operators was defined by Jänich [11] and Atiyah [2]. Using the following Proposition 14, their definition can be extended to sections of our \( \mathfrak{A} \).

**Proposition 14.** Let \( \mathfrak{A} \) and \( \mathfrak{A} \) be as above, \( k \in \mathbb{N} \) and let \( (A_y)_{y \in Y} \in M_k(\mathfrak{A}) \) be such that, for each \( y \), \( A_y \) is a Fredholm operator, where we interpret \( M_k(\mathfrak{A}) \) as the sections of the bundle with fiber \( M_k(A_y) \). Then there are continuous sections \( s_1, \ldots, s_g \) of \( \mathfrak{A} \) such that the maps

\[
\tilde{\mathbb{A}}_y : H_y^k \oplus \mathbb{C}^q \to H_y^k \oplus \mathbb{C}^q = (\tilde{\mathbb{A}}_y y + \sum_{j=1}^q \lambda_j s_j(y), 0)
\]

have image equal to \( H_y^k \oplus 0 \) for all \( y \in Y \) and \( (\ker \tilde{\mathbb{A}}_y)_{y \in Y} \) is a (finite dimensional) vector bundle over \( Y \).

**Proof:** Similar to [3] Proposition (2.2) and to [2] Proposition A5. \( \square \)

**Definition 15.** Given \( A = (A_y)_{y \in Y} \in \mathfrak{A} \) as in Proposition 14, we denote by \( \ker \tilde{\mathbb{A}} \) the bundle \( (\ker \tilde{\mathbb{A}}_y)_{y \in Y} \) and define

\[
\text{ind}_a(A) = [\ker \tilde{\mathbb{A}}] - [Y \times \mathbb{C}^q] \in K(Y).
\]

This is independent of the choices of \( q \) and of \( s_1, \ldots, s_g \) and we call it the analytical index of \( A \).

If \( A = (A_y)_{y \in Y} \in M_k(\mathfrak{A}) \) is a section such that each \( A_y \) is a Fredholm operator on \( H_y^k \) then the projection to \( M_k(\mathfrak{A}/\mathcal{R}) \) is invertible and hence defines an element of \( K_1(\mathfrak{A}/\mathcal{R}) \). Since \( \text{ind}_a(A) \) is invariant under stabilization, homotopies and perturbations by compact operator valued sections, we get a homomorphism

\[
(12) \quad \text{ind}_a : K_1(\mathfrak{A}/\mathcal{R}) \to K(Y).
\]

Next we define the topological index, also as a homomorphism

\[
\text{ind}_t : K_1(\mathfrak{A}/\mathcal{R}) \to K(Y).
\]

Let \( T^*Z \) denote the union of all \( T^*Z_y \) and \( B^*Z \) the union of all \( B^*Z_y \), equipped with their canonical topologies, where \( B^*Z_y \) denotes the bundle of closed unit balls of \( T^*Z_y \). One may regard \( B^*Z \) as a compactification of \( T^*Z \) and identify the “points at infinity” with \( S^*Z \).

Let \( \sim \) denote the equivalence relation that identifies, for each \( y \in Y \), all points of each ball of \( B^*Z_y \) which lies over a point of \( \partial Z_y \). The \( C^* \)-algebra \( B \) of Theorem 12 is isomorphic to the algebra of continuous functions on the quotient space \( S^*Z/\sim \). Let \( \beta : K_1(C(S^*Z/\sim)) \to K_0(C_0(T^*Z^2)) \) denote the index map associated to the short exact sequence

\[
0 \to C_0(T^*Z^2) \to C(B^*Z/\sim) \to C(S^*Z/\sim) \to 0,
\]

where \( T^*Z^2 \) is the union over \( y \in Y \) of all points of \( T^*Z_y \) which lie over interior points of \( Z_y \), and the map from \( C(B^*Z/\sim) \) to \( C(S^*Z/\sim) \) is induced by restriction.

Let \( 2Z \) denote the union \( \cup_y 2Z_y \), where each \( 2Z_y \) is the double of \( Z_y \), and \( \pi_d : 2Z \to Y \) the canonical projection. This can be given the structure of a Diff(\( 2X \))-bundle, with trivializations obtained by “doubling” (as explained at the beginning of Section 2) the trivializations of the bundle \( \pi : Z \to Y \). Each fiber \( 2Z_y \) is then equipped with the smooth structure induced by the trivializations of \( \pi_d : 2Z \to Y \) and we can form the bundles \( T^*2Z \) and \( S^*2Z \) as the unions, respectively, of all cotangent bundles \( T^*Z_y \) and of all cosphere bundles \( S^*(2Z_y) \), \( y \in Y \). We denote by \( \text{AS-ind}_t : K_0(C_0(T^*2Z)) \to K(Y) \) the composition of Atiyah and Singer’s [3] topological families-index for the bundle of closed manifolds \( 2Z \) with the canonical isomorphism \( K(T^*2Z) \cong K_0(C_0(T^*2Z)) \). Theorem 12 allows us to define the topological index:

**Definition 16.** The topological index \( \text{ind}_t \) is the following composition of maps

\[
\text{ind}_t : K_1(\mathfrak{A}/\mathcal{R}) \xrightarrow{\sigma \circ \iota^{-1}} K_1(C(S^*Z/\sim)) \xrightarrow{\beta} K_0(C_0(T^*Z^2)) \xrightarrow{\epsilon} K_0(C_0(T^*2Z)) \xrightarrow{\downarrow \text{AS-ind}_t} K(Y),
\]

where \( \epsilon : C_0(T^*Z^2) \to C_0(T^*2Z) \) denotes the map which extends by zero.
If $A = (A_y)_{y \in Y} \in \mathfrak{A}$ is a family of Fredholm operators we denote by $\text{ind}_s(A)$ the topological index evaluated at the element of $K_1(\mathfrak{A}/\mathfrak{R})$ that $A$ defines.

**Theorem 17.** Let $A = (A_y)_{y \in Y} \in \mathfrak{A}$ be a continuous family of Fredholm operators in the closure of the Boutet de Monvel algebra for each $y$. Then

\begin{equation}
\text{ind}_s(A) = \text{ind}_i(A).
\end{equation}

**Proof.** Our strategy is to derive the equality of the indices from the classical Atiyah-Singer index theorem for families [12, Theorem (3.1)]. To this end we define an operator family $\tilde{A}$ acting on a vector bundle over the double of $Z$ by a gluing technique involving the principal symbol family of $A$. We proceed in several steps. Step 1 consists of a few preliminary remarks on the choice of the representative of the $\mathcal{K}$-theory class of $A$. In Step 2 we describe the construction of the bundle. We then define the operator family $\tilde{A}$ over $2\mathcal{Z}$ in Step 3. Its topological index coincides with that of $A$ as we shall see in Step 4. The equality of the analytic indices of $A$ and $\tilde{A}$ is the content of Step 5.

**Step 1.** We need to prove that $\text{ind}_s$ and $\text{ind}_i$ coincide on $K_1(\mathfrak{A}/\mathfrak{R})$. Using that $K_1(\mathfrak{A}/\mathfrak{R}) = K_1(\mathfrak{A}^1/\mathfrak{R})$ by Theorem [12, an arbitrary element of $K_1(\mathfrak{A}/\mathfrak{R})$ is a class $[[A]]_1$ (the inner brackets denoting a class in the quotient by the compact). For each symbol family $A = (A_y)_{y \in Y} \in M_k(\mathfrak{A})$, $k \in \mathbb{N}$, such that, for each $y$, $A_y: H^k_y \to H^k_y$ is Fredholm operator with symbol in $\mathcal{B}$. It will be convenient to pick a representative with symmetrical properties. We denote by $C^\infty(S^*X/\sim)$ the subset of $C^\infty(S^*X)$ of functions which factor through $S^*X/\sim$, i.e., are independent of the co-variable at the boundary. The algebraic tensor product $C_0(U) \otimes C^\infty(S^*X/\sim)$ is dense in $C(U \times S^*X/\sim)$ for every open subset $U$ of $Y$. Furthermore, the inclusion of the space of all elements in $C^\infty(S^*X)$ which are independent of the co-variable in a neighborhood of $\partial Z$ into $C^\infty(S^*X/\sim)$ is a homotopy equivalence. We can therefore assume that the symbol family $(\sigma_y(A_y))_{y \in Y}$ is given as a finite sum of elements supported in open subsets $U$ of $Y$ over which $Z$ is trivial, and each of these is a pure tensor in $C_0(U) \otimes C^\infty(S^*X)$ which is independent of the co-variable near the boundary. Hence it suffices to prove equality for such an $A$.

**Step 2.** For each $y \in Y$, let $Z^+_y$ and $Z^-_y$ denote the two copies of $Z_y$ which are glued together at $\partial Z_y$ to form $2Z_y$. The map $i_2: \partial Z^+_y \to \partial Z^-_y$ identifies the two copies of $\partial Z_y$. We define $E_y$ as the quotient of the disjoint union $Z^+_y \times \mathbb{C}^k \cup Z^-_y \times \mathbb{C}^k$ by the equivalence relation that identifies the pairs $(x, v)$ and $(x', w)$ if and only if they are equal or $x' = i_y(x)$, $x \in \partial Z^+_y$, and $w = \sigma_y(A_y)(x)v$ (remembering that at points of $\partial Z_y$ over $\partial Z_y$, $\sigma_y(A_y)$ is independent of the co-vector variable). This set $E_y$ naturally becomes a smooth vector bundle over $Z_y$. Let $E$ denote the union of all $E_y$, which in the same way becomes a vector bundle over $Y$.

When defining families of smooth manifolds with smooth vector bundles, Atiyah and Singer make the technical assumption that the fiberwise vector bundles are isomorphic to a fixed vector bundle on the typical fiber. If $Y$ is not connected, this is not necessarily satisfied. However, the isomorphism type of $E_y$ depends only on the homotopy type of the map $\sigma_y$, in particular only on the component of the space of all continuous maps from $\partial Z_y$ to $M_k(\mathbb{C})$ in which it lies. By the compactness of $Y$, the latter decomposes into finitely many open and closed subsets over each of which the isomorphism type of $E_y$ is constant. As the $\mathcal{K}$-theory of $Y$ as well as $\mathfrak{A}/\mathfrak{R}$ split as direct sums under such disjoint union decompositions of $Y$, and as $\text{ind}_s$, $\text{ind}_i$ respect this, we can restrict to one such subset of $Y$. Then we are canonically in the situation of [12, Definition 1.2], i.e., $E$ is a smooth vector bundle over the family of smooth manifolds $2Z$.

**Step 3.** Let $\pi_z: S^*2Z \to 2Z$ denote the canonical projection and $S^*Z^+$ and $S^*Z^-$, respectively, the union of all $S^*Z^+_y$ and $S^*Z^-_y$, $y \in Y$. The bundle $\pi_z^*E$ can be seen as the disjoint union of $S^*Z^+ \times \mathbb{C}^k$ and $S^*Z^- \times \mathbb{C}^k$ quotiented by the equivalence relation that identifies a boundary point $(s, v)$ in $S^*Z^+ \times \mathbb{C}^k$ with $(s, \sigma_A(s) \cdot v)$ in $S^*Z^- \times \mathbb{C}^k$. Similarly, the bundle $S^*Z \times \mathbb{C}^k$ can be seen as the disjoint union of $S^*Z^+ \times \mathbb{C}^k$ and $S^*Z^- \times \mathbb{C}^k$ quotiented by the equivalence relation that identifies a boundary point $(s, v)$ in $S^*Z^+ \times \mathbb{C}^k$ with $(s, v)$ in $S^*Z^- \times \mathbb{C}^k$. We then define $\tilde{\sigma}(s, v)$ by

\begin{equation}
\tilde{\sigma}(s, v) = \begin{cases} 
\sigma_A(s) \cdot v, & \text{if } (s, v) \in S^*Z^+ \times \mathbb{C}^k, \\
v, & \text{if } (s, v) \in S^*Z^- \times \mathbb{C}^k.
\end{cases}
\end{equation}

We want to show that $\tilde{\sigma}$ is the symbol of a continuous family of pseudodifferential operators. As any element of $\text{Hom}(\pi_z^*E, S^*2Z \times \mathbb{C}^k)$, our $\tilde{\sigma}$ can be regarded as a family $(\tilde{\sigma}_y)_{y \in Y}$, $\tilde{\sigma}_y \in \text{Hom}(\pi_z^*E_y, S^*2Z_y \times \mathbb{C}^k)$. It is easily checked that our definition of $\tilde{\sigma}$ indeed mends continuously at boundary points. But more is true. Since $\sigma_y(A_y)$ is smooth and independent of the co-variable near the boundary, each $\tilde{\sigma}_y$ is...
smooth. Moreover, since we assumed in Step 1 that $a$ is a finite sum of local elementary tensors, we see that $\hat{a}$ is the symbol of an Atiyah-Singer family of pseudodifferential operators on $2\mathbb{Z}$.

Step 4. Let $\iota: K_0(C_0(T^*Z)) \to K(B^*Z, S^*2Z) \simeq K(T^*Z)$ denote the canonical isomorphism (we refer to [4] and mainly [2] for topological K-theory definitions and notation). By Definition [16] it is enough to show that $\iota(e_\sigma(\beta([\sigma_A])))$ is equal to the element of $K(B^*Z, S^*2Z)$ defined by the triple $(\pi_E^*E, B^*2Z \times \mathbb{C}^\delta, \hat{a})$, where $\pi_E: B^*2Z \to 2Z$ denotes the canonical projection.

The main step here is to understand $\beta([\sigma_A])$. Now, $\sigma_A$ can and will be considered as a function on $S^*2Z/\sim$ with values in $GL_2(\mathbb{C})$, representing an element in $K_1(C(S^*2Z/\sim))$ and at the same time the support of each element of the topological K-theory $K^1(S^*2Z/\sim)$. Recall from [12] 2.32 that for the pair of compact topological spaces $S^*Z/\sim \subset B^*Z/\sim$, the boundary map in topological K-theory assigns to $\sigma_A$ the relative K-class $((B^*Z/\sim) \times \mathbb{C}^\delta, (B^*Z/\sim) \times \mathbb{C}^\delta, \sigma_A)$, corresponding under the excision isomorphism $K((B^*Z/\sim), (S^*Z/\sim)) \cong K(B^*Z, S^*Z)$ to $(B^*Z \times \mathbb{C}^\delta, B^*Z \times \mathbb{C}^\delta, \sigma_A)$, compare [12] 2.35. Moreover, this corresponds to $\beta$ under the isomorphism with $\mathbb{C}$-algebra K-theory. We next have to compute the map $e_{\text{top}}: K(B^*Z, S^*Z) \to K(B^*2Z, S^*2Z)$ in topological K-theory, representing $e_\sigma: K_0(C_0(T^*Z)) \to K_0(C_0(T^*2Z))$. Recall, however, that $e_{\text{top}}(V,W,\tau)$ is given by any extension $\hat{V}$ of $\hat{V}$, $\hat{W}$ of $\hat{W}$ and an extension of $\tau$ to an isomorphism $\hat{\tau}$ between $\hat{V}$ and $\hat{W}$ on all of $(B^*2Z \setminus B^*Z) \cup S^*Z$, $\hat{\tau}$ finally restricted to $S^*Z$. Finally, observe that $(\pi_E^*E, B^*2Z \times \mathbb{C}^\delta, \hat{a})$ provides exactly such an extension (as $\hat{a}$ extends as $\text{id}$ over all of $B^*2Z \setminus B^*Z$) and therefore represents $e_{\text{top}}(\beta([\sigma_A]))$, as we had to prove.

Step 5. In order to show that the analytic indices coincide, we will introduce yet another operator family. Since $\sigma(A)$ is independent of the co-variable near the boundary, there is an open set $U \subset 2Z$ containing $Z^{-} = \cup_y Z^{-}_y$ and a bundle isomorphism

$$\Phi: E_{|U} \longrightarrow U \times \mathbb{C}^k$$

such that the restriction of $\hat{a}$ to $\pi^{-1}(U)$ is equal to the pullback of $\Phi$ by $\pi_x$. Let $(\chi^+_y)_{y \in Y}$ and $(\chi^+_x)_{x \in Y}$ be continuous families of smooth functions on $2Z$ with $0 \leq \chi^+_y \leq 1$, $(\chi^+_y)^2 + (\chi^+_x)^2 = 1$. Moreover, let the support of each $\chi^+_y$ be contained in the interior of $Z^+_y$ and $\chi^+_y = 1$ outside a neighborhood of $\partial Z^+_y$ in $U$. Then

$$\hat{B}_y = \chi^+_y \hat{A}_y \chi^+_y + \chi^-_y \Phi_y \chi^-_y,$$

defines a family of pseudodifferential operators in the sense of Atiyah and Singer which has the same principal symbol – and hence the same analytic index – as $\hat{A}$.

For each $y \in Y$, we canonically identify the space $L^2(E_y)$ of $L^2$-sections of $E_y$ with the direct sum $L^2(Z^+_y; \mathbb{C}^k) \oplus L^2(Z^-_y; \mathbb{C}^k)$ and denote by $e_\beta^y$ and $r_\beta^y$ the maps of extension by zero and restriction,

$$e_\beta^y : L^2(Z^+_y; \mathbb{C}^k) \to L^2(E_y), \quad r_\beta^y : L^2(Z^-_y; \mathbb{C}^k) \to L^2(Z^+_y; \mathbb{C}^k).$$

Then $B_y = r_\beta^y \hat{B}_y e_\beta^y$ defines a continuous family $B = (B_y)_{y \in Y}$ in $M_k(\mathbb{A})$. As $\sigma(A) = \sigma(B)$ (and hence $\gamma(A) = \gamma(B)$), it suffices to prove that the analytic indices of $B$ and $\hat{B}$ are equal.

Proposition 2.2 of [3], applied to the family $\hat{B}$ provides us with sections $s^y_\tau \in C^\infty(2Z^+_y; \mathbb{C}^k)$, $y \in Y$, $1 \leq j \leq q$, such that

$$\hat{Q}_y : C^\infty(2Z^+_y; \mathbb{C}^k) \times \mathbb{C}^q \longrightarrow C^\infty(2Z^+_y; \mathbb{C}^k) \times \mathbb{C}^q, \quad (u; \lambda_1, \cdots, \lambda_q) \longmapsto \hat{B}_y(u) + \sum_{j=1}^{q} \lambda_j s^y_j$$
is onto, ker $\hat{Q} = (\ker \hat{Q})_{y \in Y}$ is a vector bundle and the analytic index of $\hat{B}$ is equal to $[\ker \hat{Q}] - [Y \times \mathbb{C}^q]$. Now let $t^y_\tau = r^y_\beta s^y_\tau \in C^\infty(Z^+_y; \mathbb{C}^k)$. The continuity with respect to $y$ that we get from [3] Proposition 2.2] is enough to ensure that $(t^y_\tau)_{y \in Y}$ is a continuous section of our bundle of Hilbert spaces $\bigcup_{y \in Y} L^2(Z^+_y; \mathbb{C}^k)$.

We then define

$$Q_y : L^2(Z^+_y; \mathbb{C}^k) \times \mathbb{C}^q \longrightarrow L^2(Z^+_y; \mathbb{C}^k) \times \mathbb{C}^q, \quad (u; \lambda_1, \cdots, \lambda_q) \longmapsto B_y(u) + \sum_{j=1}^{q} \lambda_j t^y_j$$

Since $B_y$ is elliptic, ker $Q_y \subset C^\infty(Z^+_y; \mathbb{C}^k)$. Using that $\Phi_y$ is local, it is straightforward to check that

$$\hat{B}_y = e^+_y r^+_y B_y e^+_y r^+_y + e^-_y r^-_y \hat{B}_y e^-_y r^-_y = e^+_y B_y r^+_y + e^-_y r^-_y \Phi_y e^-_y r^-_y$$

and, hence, ker $Q_y$ and ker $\hat{Q}_y$ are isomorphic for each $y$ (because $\Phi$ is an isomorphism). Moreover, $Q_y$ is also surjective: Given $v \in L^2(Z^+_y; \mathbb{C}^k)$, if $u \in L^2(2Z^+_y; E_y)$ is a preimage of $e^+_y v$ under $\hat{Q}_y$, then $r^+_y u$ is

1Recall that they use a slightly stricter definition of operator families: While we here require continuity of the family with respect to the $L^2(X)$-operator norm, they take into account the norms on the whole range of Sobolev spaces.
a preimage of \( v \) under \( Q_y \). Hence the analytic index of \( B \) is given by \( [\ker Q] - [Y \times \mathbb{C}^n] \). The bundles \( \ker Q = (\ker Q_y)_{y \in Y} \) and \( \ker \hat{Q} \) are isomorphic and then

\[
\text{ind}_a(B) = [\ker Q] - [Y \times \mathbb{C}^n] = [\ker \hat{Q}] - [Y \times \mathbb{C}^n] = \text{ind}_a(\hat{B}),
\]
as we wanted. \( \Box \)

4. Nontrivial bundles

In this section we discuss families of Boutet de Monvel operators acting between vector bundles. The case considered in the first two sections correspond to the case of trivial bundles over the manifolds and the zero bundle over the boundary.

In addition to the data assumed up to this point (a bundle of manifolds \( \pi: Z \to Y \) with fiber \( X \)), we take smooth vector bundles \( E \) and \( F \) over \( X \) and \( \partial X \), respectively. Let \( \text{Diff}(\partial X, F) \) denote the group of diffeomorphisms of \( F \) which map fibers to fibers linearly, and let \( G_E \) denote the group of diffeomorphisms of \( E \) which map fibers to fibers linearly and whose restrictions to the base belong to the group \( G \) defined on page 100001. We equip \( \text{Diff}(\partial X, F) \) with its canonical topology \([3\), page 123\]) and do a similar construction for \( G_E \). Note that there are homomorphisms “forget the action in the fiber” \( h_\partial: \text{Diff}(\partial X, F) \to \text{Diff}(\partial Z) \) and \( h: G_E \to G \). Define the fiber product group

\( G_r := \{ (\phi, \psi) \in \text{Diff}(\partial X, F) \times G_E \mid h_\partial(\phi) = h(\psi) \} \).

Let \( (p: \tilde{E} \to Z; q: \tilde{F} \to \partial Z) \) be maps such that \( (\pi \circ p: \tilde{E} \to Y; \pi \circ q: \tilde{F} \to Y) \) are bundles with, respectively, fibers \( E \) and \( F \) and structure group \( G_r \). It follows that, for each pair of local trivializations \((\alpha, \beta)\) of \((\pi \circ p: \tilde{E} \to Y; \tilde{F} \to Y)\), there is a local trivialization \( \alpha_0 \) of \( \pi: Z \to Y \) and \( \beta_0 \) of \( \partial Z \to Y \) such that the diagram

\[
\begin{array}{ccc}
(p \circ \pi)^{-1}(U) & \xrightarrow{\alpha_0} & U \times E \\
\downarrow p & & \downarrow \\
\pi^{-1}(U) & \xrightarrow{\alpha} & U \times X
\end{array}
\]

commutes, where the right vertical arrow is the identity on \( U \) times the bundle projection on \( E \). This defines a vector bundle structure for \( p: \tilde{E} \to Z \). Moreover, for each \( y \in Y \), the restriction of \( p \) to \( \tilde{E}_y = (p \circ \pi)^{-1}(y) \) defines a smooth vector bundle \( p_y: \tilde{E}_y \to Z_y \), isomorphic to \( E \to X \). We obtain the corresponding result for the the map \( q \) and get a vector bundle \( q: \tilde{F} \to \partial Z \) and, for each \( y \in Y \), a smooth vector bundle \( q_y: \tilde{F}_y \to \partial Z_y \) isomorphic to \( F \to \partial X \).

Choose now, in addition to the family of Riemannian metrics \((g_y)_{y \in Y}\) families of Hermitean metrics on \( E_y \) and \( F_y \) which depend continuously on \( y \in Y \). Using them, we get families of Hilbert spaces \( H_y := L^2(Z_y; E_y) \oplus L^2(\partial Z_y; F_y) \) which patch together to a bundle of Hilbert spaces. Let \( \mathcal{A}(E, F)_y \) denote the \( C^*-\)subalgebra of the algebra of all bounded operators on \( H_y \) generated by the polyhomogeneous Boutet de Monvel operators of order and class zero.

Exactly as \([3\), Proposition 1.3\]) our Proposition \(4\) generalizes to the case of non-trivial bundles and their diffeomorphisms and is the basis for the generalization of Corollary \(5\) to the case of non-trivial bundles: the \( \mathcal{A}(E, F)_y \) form in a canonical way a continuous bundle of \( C^*-\)algebras, which we continue to call \( \mathcal{A} \) by abuse of notation.

Let \( \mathcal{A} \) denote the set of continuous sections of the bundle \( \mathcal{A} \), forming again a \( C^*-\)algebra with pointwise operations and supremum norm. The \( K\)-theory results of Section \(5\) can be extended to this more general setting using arguments similar to those used in \([17\]. In particular, the analytic and topological index given in Section \(5\) can also be defined as maps \( K_1(\mathcal{A}) \to K(Y) \). Theorem \(17\) then extends to this more general setting.

Remark 18. Variants of Theorem \(17\) the family index theorem for the Boutet de Monvel algebra for real \( K\)-theory or for equivariant \( K\)-theory should hold as well, and one should be able to derive them along the lines used in the present article.

Appendix A. Reduction of the structure group

Let, as in the main body of the text, \( X \) be a compact smooth manifold with boundary \( \partial X \), and fix a collar diffeomorphism \( \delta: U \to \partial X \times [0, 1) \) with collar coordinate \( x_n \). Recall that \( G \) was defined as the subgroup of the diffeomorphism group \( \text{Diff}(X) \) of those diffeomorphisms which respect the product structure and collar coordinate for \( x_n \in [0, 1/2) \). For convenience, in the text we were working with bundles of manifolds modelled on \( X \) and with structure group \( G \), i.e. with a canonically defined collar of the boundary in each fiber of the bundle.
In this appendix, we prove that, for any bundle (over a paracompact space) with structure group Diff(X) we have a unique (up to isomorphism) reduction to the structure group G. In other words, the functor from bundles (over a given paracompact base) with structure group G to bundles with structure group Diff(X) which “forgets the collar” is an equivalence of categories. [This is similar to the (unique up to isomorphism) choice of a Riemannian metric on a given finite dimensional vector bundle: reduction of the structure group from GL(n) to O(n).]

It is well known that we get this unique reduction of structure group if the inclusion G → Diff(X) is a homotopy equivalence, compare [5] for a rather refined version of this fact. We therefore show

**Theorem 19.** The inclusion G → Diff(X) (and therefore the corresponding map BG → BDiff(X)) are homotopy equivalences.

**Proof.** Observe first that G and Diff(X) as well as BG and BDiff(X) are paracompact Fréchet manifolds by [14] Sections 41, 42, 44.21] (the reference is for Diff(X), but the proofs easily generalize to G). Therefore it suffices by [19] Theorem 15) to show that G → Diff(X) is a weak homotopy equivalence and it follows automatically that it is a homotopy equivalence.

To show that the map is a weak homotopy equivalence, we have for a continuous map f: K → Diff(X), where K is a compact CW-complex, to construct a homotopy fτ, from f0 = f to an f1 which takes values in G. Moreover, the homotopy should be constant on every CW-subcomplex K0 of K where f already maps to G. Note that K0 is a deformation retract of a neighborhood U, i.e. there is a homotopy h: K × [0, 1] → K from the identity to h1 such that h1(U) = K0 and such that h0 is the identity on K0.

Be precomposing with h1 we can therefore assume that f maps the neighbourhood U of K0 to G.

Let us now construct the family fτ. Choose η ∈ (0, 1] such that f(ε) = δ ◦ f(k) ◦ δ−1 maps ∂X × [0, η] to ∂X × [0, η) for all k ∈ K and write f(k)(x, t) = (ϕ(x, t; k), τ(x, t)).

In two steps we shall now first deform τ to a function ˆτ which equals t for small t and then φ to a function which depends only on x′ for small t.

Observe that, as f(k) is a diffeomorphism of a manifold with boundary, δ function boundary, and therefore, by the compactness of K, if we choose η small enough, C > δ/η > c > 0 for some C > c > 0 on all of K × ∂X × [0, η).

Pick a smooth function a: [0, η) → (0, 1] such that a(t) ≡ 0 for t close to zero, a(t) ≡ 1 for t close to η and such that

\[ \hat{\tau}(x', t; k) = (1 - a(t))t + a(t)\tau(x', t; k), \quad (x', t) \in \partial X \times [0, \eta], \]

satisfies \( \partial \hat{\tau}(x', t; k)/\partial t \geq c/2 \) for every \( x' \in \partial X \) and \( k \in K \). To construct such an a, we use the uniform growth of \( \tau \): Choose, for some given \( \varepsilon > 0 \), the function a so that \( (1 - a(t)) \) is monotonically increasing on \( [0, 4\varepsilon] \) with \( (1 - a(t)) = t \) on \( [0, \varepsilon] \) and \( (1 - a(t)) = 2\varepsilon \) on \( [3\varepsilon, 4\varepsilon] \). Then a is necessarily increasing with \( a \equiv 0 \) near 0 and \( a(4\varepsilon) = 1/2 \). Moreover, \( \hat{\tau} \) is strictly increasing as \( \tau \) is.

Finally choose a on \( [4\varepsilon, \eta) \) such that \( (1 - a(t)) \) monotonically decreases to 0 and equals zero on \( [\eta - \varepsilon, \eta] \). Moreover, we arrange for the derivative \( \partial \hat{\tau}/\partial t \) to be always, \( a = -2\varepsilon/\eta \). Again, a is necessarily increasing with \( a \equiv 1 \) near \( \eta \). The derivative \( \partial \hat{\tau}/\partial t \) can therefore be estimated from below by \( c/2 \).

For \( \varepsilon \) sufficiently small, we will therefore have \( 2\varepsilon/\eta < c \) and thus \( \partial \hat{\tau}(x', t; k) > 0 \) for all \( x', t, k \). Note that then \( \hat{\tau}(x', t; k) = t \) for t close to zero, and \( \hat{\tau}(x', t; k) = \tau(x', t; k) \) for \( t \) close to \( \eta \), uniformly in \( k \). We then let

\[ \tau_s = s\hat{\tau} + (1 - s)\tau, \quad 0 \leq s \leq 1. \]

Then \( \partial \tau_s/\partial t \geq c/2 \) on \( K \times \partial X \times [0, \eta] \).

For the second step fix a smooth function \( \rho: [0, 1] \rightarrow [0, 1] \) with \( \rho(t) = 0 \) for \( t < \varepsilon \) and \( \rho(t) = t \) for \( t > 1 - \varepsilon \). Next choose a smooth family of smooth functions \( \rho_s, 0 \leq s \leq 1 \) such that \( \rho_0 \) is the identity and \( \rho_1 = \rho \). By compactness, we have a uniform bound \( |d\rho_s(t)/dt| \leq R \). For a given \( \eta > 0 \), define \( \rho^2_s([0, \eta]) \rightarrow [0, \eta]: t \mapsto \eta \rho_s(\eta^{-1}t) \). Then still \( |d\rho^2_s/dt| \leq R \), even independently of \( \eta \).

Let \( \varphi_s(x', t) := \varphi(x', \rho^2_s(t)) \) and \( \tilde{f}_s(k)(x', t) = (\varphi_s(x', t), \tau_s(t)) \). Then \( \tilde{f}_s \) equals the given \( \tilde{f} \) for \( t \) close to \( \eta \). Therefore \( \tilde{f}_s = \delta^{-1} \circ \tilde{f} \circ \delta \) extends (independently of \( s \)) to a self-map of \( X \). Moreover, \( |d\tilde{f}_s/dt| \leq |d\tilde{f}/dt| \) for all \( s \). And for \( t = 0 \) we have \( \partial \varphi_s/\partial t = 0 \). On the other hand, \( \partial \varphi_s/\partial t(x', t) = \partial \varphi/\partial t(x', \rho_s(t)) \) is, for \( s \) small enough, invertible on \( [0, \eta] \) with uniform bound on the norm of the inverse (and with better bounds if we choose \( \eta \) smaller), and \( |d\rho/\partial t(x', t)| = |d\rho/\partial t(x', \rho_s(t))| \cdot |d\rho/\partial t(\rho_s(t))| \) which is uniformly bounded, independent of \( \eta \).

By choosing \( \eta \) small enough, therefore \( \partial \tau_s \) will be linearly independent from \( \partial \varphi(x', \rho_s(t)) \) and so \( f_s(k) \) is a submersion for all \( s, k \). We check that actually constructed diffeomorphisms. We made our construction such that all the maps \( f_s(k) \) are submersions which map the boundary to itself, therefore the image is an open subset of \( X \). As \( X \) is compact, the image is also closed, and the map being a local
diffeomorphism, is a covering map. Because it is homotopic to the diffeomorphism \( f(k) \), it is a trivial covering map and therefore a diffeomorphism.

It is obvious that \( f_0 = f \) and \( f_1(k) \) lies in the variant of \( G \) where \( 1/2 \) is replaced by \( \eta - \delta \).

Next, we compose with a family of reparametrizations of the collar \([0, 1]\) which stretches \([0, \eta - \delta]\) to \([0, 1/2]\) such that in the end we really map to \( G \). Note that our construction is carried out in such a way that for \( k \in U \), where \( f(k) \) was already in \( G \), \( f_s(k) \in G \) for all \( s \), although, because of the last reparametrization step, not necessarily \( f_s(k) = f(k) \).

Therefore, finally, we choose a function \( \beta: K \to [0, 1] \) which is 1 outside \( U \) and 0 on \( K_0 \) and replace the homotopy \( f_s(k) \) with \( f_{\beta(k)s}(k) \).

This yields the desired homotopy from \( f_0 = f \) to an \( f_1 \) taking values in \( G \). Moreover, the mapping is constant on \( K_0 \).

□

APPENDIX B. THE KRÜNETH FORMULA

By the ”Küneth formula”, we mean the following theorem of Schochet [21]:

Theorem 20. Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) in the smallest subcategory of the category of separable nuclear \( C^* \)-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed product by \( \mathbb{Z} \) and by \( \mathbb{R} \). Then there is a natural \( \mathbb{Z}/2 \)-graded exact sequence

\[
0 \to K_* (A) \otimes K_* (B) \to K_* (A \otimes B) \to \text{Tor}(K_* (A), K_* (B)) \to 0.
\]

We use this Theorem to prove a statement made in the proof of Theorem 16.

Proposition 21. \( b_U : K_i (C_0 (\partial D^2)) \to K_i (\text{Im} \gamma_U ) \) is an isomorphism, \( i = 0, 1 \).

Proof. Let \( A = C_0 (U) \) and \( B = C(\partial X) \). Then \( \text{Im} \gamma_U \) is equal to \( A \otimes C \), where \( C \) is the image of the boundary principal symbol for the single manifold \( X \). As explained in the Introduction of [16], \( C \) can be regarded as a \( C^* \)-subalgebra of \( C(\mathbb{S}^* \partial X) \otimes T \), where \( T \) denotes the Toeplitz algebra. Since \( T \) belongs to the category defined in the statement of Theorem 20 (see Examples 5.6.4 and 6.5.1 in [18]), we may apply Schochet’s theorem for \( A \otimes B \) and for \( A \otimes C \).

Now let \( b : C(\partial X) \to C(\partial X) \) be the map analogous to the map \( b \) defined right before the statement of Theorem 16. In [16] Section 3], it is proven that \( b \) induces a \( K \)-theory isomorphism (\( b \) was denoted \( b \) in [16] [17]). Using that the exact sequence of Theorem 20 is natural, we can map (16) to the corresponding sequence obtained by replacing \( B \) with \( C \). Since the maps induced by \( b \) are isomorphisms, it follows from the five-lemma that the maps induced by \( b_U = \text{id}_A \otimes b \) are also isomorphisms.

□

Acknowledgements

We greatly benefited from numerous discussions with our friends Johannes Aastrup and Daniel Tausk. We thank them for their generosity and for the great time we had talking Math to them. We are also grateful to Jochen Ditsche for pointing out Proposition 13 to us. Severino Melo was partially supported by a grant from the Brazilian agency CNPq (Processo 304783/2009-9). Thomas Schick was partially supported by the Courant Center “Higher order structures of mathematics” within the Excellence initiative’s Institutional strategy of Georg-August-Universität Göttingen.

References

[1] P. Albin & R. Melrose. Relative Chern character, boundaries and index formulas. J. Topol. Anal. 1 (2009), no. 3, 207–250.
[2] M. F. Atiyah. K-Theory, Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.
[3] M. F. Atiyah & I. M. Singer. The index of elliptic operators IV. Ann. of Math. (2) 93 (1971), 119–138.
[4] B. Blackadar. K-Theory for Operator Algebras. Cambridge University Press, Cambridge, 1998.
[5] L. Boutet de Monvel. Boundary problems for pseudo-differential operators. Acta Math. 126 (1971), no. 1-2, 11–51.
[6] A. Dold. Partitions of unity in the theory of fibrations. Ann. of Math. (2) 78 (1963), 223–255.
[7] I. C. Gohberg. On the theory of multidimensional singular integral operators. Dokl. Akad. Nauk SSSR 133 (1960), 1279–1282.
[8] G. Grubb. Functional Calculus of Pseudodifferential Boundary Problems, Second Edition. Birkhäuser, Boston, 1996.
[9] G. Grubb & G. Geymonat. The essential spectrum of elliptic systems of mixed order. Math. Ann. 227 (1977), 247–276.
[10] A. Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002.
[11] K. Jänich. Vektorraumbündel und der Raum der Fredholm-Operatoren. Math. Ann. 161 (1965), 129–142.
[12] M. Karoubi. K-theory. An introduction. Grundlehren der Mathematischen Wissenschaften, Band 226. Springer-Verlag, Berlin-New York, 1978.

Münster Journal of Mathematics Vol. 1 (2008), 99999–99999
S. Melo, T. Schick, E. Schrohe

1. J. J. Kohn & L. Nirenberg. *An algebra of pseudo-differential operators.* Comm. Pure Appl. Math. 18 (1965), 269–305.
2. A. Kriegl & P. W. Michor. *The convenient setting of global analysis.* Mathematical Surveys and Monographs 53. American Mathematical Society, Providence, RI, 1997.
3. R. Matthes & W. Szymanski. *Lecture Notes on the K-Theory of Operator Algebras.* www.impan.gov.pl/Manuals/K_theory.pdf, 2007.
4. S. T. Melo, R. Nest, E. Schrohe. *C*-structure and K-theory of Boutet de Monvel’s algebra.* J. reine angew. Math. 561 (2003), 145–175.
5. S. T. Melo, T. Schick, E. Schrohe. *A K-theoretic proof of Boutet de Monvel’s index theorem for boundary value problems.* J. reine angew. Math. 599 (2006), 217–233.
6. G. J. Murphy. *C*-Algebras and Operator Theory.* Academic Press, Boston, 1990.
7. R. S. Palais. *Homotopy theory of infinite dimensional manifolds.* Topology 5 (1966), 1–16.
8. S. Rempel & B.-W. Schulze. *Index Theory of Elliptic Boundary Problems.* Akademie Verlag, Berlin, 1982.
9. C. Schochet. *Topological methods for C*-algebras II: geometric resolutions and the Künneth formula.* Pacific J. Math. 98 (1982), no. 2, 443–458.
10. E. Schrohe. *A short introduction to Boutet de Monvel’s calculus.* Approaches to Singular Analysis (Berlin, 1999). Oper. Theory Adv. Appl. 125, 85-116, Birkhäuser, Basel, 2001.
11. B.-W. Schulze. *Pseudo-Differential Operators on Manifolds with Singularities.* North Holland, Amsterdam, 1991.
12. R. T. Seeley. *Integro-differential operators on vector bundles.* Trans. Amer. Math. Soc. 117 (1965), 167–204.

Severino T. Melo
Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão 1010, 05508-090 São Paulo, Brazil—
E-mail: toscano@ime.usp.br
URL: http://www.ime.usp.br/~toscano

Thomas Schick
Mathematisches Institut, Georg-August-Universität Göttingen
Bunsenstr. 3-5, 37073 Göttingen, Germany
E-mail: schick@uni-math.gwdg.de
URL: http://www.uni-math.gwdg.de/schick

Elmar Schrohe
Institut für Analysis, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
E-mail: schrohe@math.uni-hannover.de
URL: http://www.analysis.uni-hannover.de/~schrohe/

Münster Journal of Mathematics Vol. 1 (2008), 99999–99999