GL(3, R) gauge theory of gravity coupled with an electromagnetic field

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Consistency of GL(3, R) gauge theory of gravity coupled with an external electromagnetic field, is studied. It is shown that possible restrictions on Maxwell field can be avoided through introduction of auxiliary fields.

1) Introduction

Among several gauge formulations for gravity, we consider a Yang-Mills type with the Lie group GL(3, R) as a gauge group$^{1,2}$, where a covariant coupling scheme is known with material fields in a space-time with non null torsion provided$^2$. Here, we focus our attention on consistency between Einstein-Hilbert Theory (EHT) and a GL(3, R) gauge theory of gravity coupled with an external electromagnetic field, in a 2 + 1 dimensional space-time, not necessarily with null torsion. There, the standard definition of Maxwell field must depend on vector potential and torsion, so the source of gravitation, given by the electromagnetic energy-momentum tensor depends on equivalence class of connection. This fact conduces to an extension of model discussed in reference$^2$ and constitutes a particular type of a lagrangian formulation with four order self-interaction terms on connection.

This note is organized as follows. In section 2 we review a gauge formulation of gravity based on a frame bundle. Next, the coupling between gravity, Maxwell and auxiliary fields is performed, confirming consistency with EHT at the torsionless limit. We conclude with some remarks.

2) GL(3, R) gauge formulation for gravity

Let be $\mathcal{M}$ a GL(3, R) frame bundle on a 2+1 space-time as a base space with a metric $g_{\mu\nu}$ and coordinates $x^\mu$ provided. The connection 1-form shall be considered

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as an independent object from metric and it is introduced through $(\mathbb{A}_\lambda)^{\mu}{}_{\nu} \equiv \Gamma^{\mu}{}_{\lambda \nu}$, where $\Gamma^{\mu}{}_{\lambda \nu}$ is the affine connection which allows the construction of the covariant derivative, $\nabla_\mu$, the torsion tensor, $T^{\mu}{}_{\lambda \nu} = (\mathbb{A}_\lambda)^{\mu}{}_{\nu} - (\mathbb{A}_\nu)^{\mu}{}_{\lambda}$ and the curvature, $R^\sigma{}_{\alpha \mu \nu} \equiv (\mathbb{F}_{\mu \nu})^\sigma{}_{\alpha} = (\partial_\nu \mathbb{A}_\mu - \partial_\mu \mathbb{A}_\nu + [\mathbb{A}_\mu, \mathbb{A}_\nu])^\sigma{}_{\alpha}$. So, the $GL(3, \mathbb{R})$ gauge invariant action for the free model at the torsionless limit, is

$$S_0 = \kappa \int_{\mathcal{M}} d^3 x \sqrt{-g} \left( - \frac{1}{4} \text{tr} \mathbb{F}_{\alpha \beta} \mathbb{F}^{\alpha \beta} + C_{\alpha \beta} \varepsilon^{\beta \lambda \sigma} (\mathbb{A}_\lambda)^{\alpha}{}_{\sigma} \right),$$

where $\kappa$ has a length dimension and $C_{\alpha \beta}$ are the lagrange multipliers related to torsion constraint. This model could be cosmologically extended with the introduction of a lagrangian term $\lambda^2$ (see reference [2]), where $\lambda$ is the cosmological constant. Variations on connection and metric provide the field equations, $\nabla_\alpha R^{\lambda \sigma} - \nabla_\sigma R^{\lambda \alpha} = 0$ and $2R^{\sigma \mu} R^\sigma{}_{\nu} - 2RR^{\mu \nu} - g^{\mu \nu} R^{\rho \lambda} R_{\rho \lambda}^{\alpha} + \frac{3}{4} g^{\mu \nu} R^2 = 0$, which conduce to a flat solution ($\mathbb{F}_{\alpha \beta} = 0$), in consistency with free Einstein theory in 2+1 dimensions.

3) Coupling with Maxwell and auxiliary fields

It is well known that electromagnetic field couples with torsion because, if $a_\mu$ is the potential trivector, the Maxwell tensor is

$$f_{\mu \nu} = f_{\mu \nu}^{(o)} + a_\lambda \varepsilon^{\rho \alpha \beta} \varepsilon_{\rho \mu \nu} (\mathbb{A}_\alpha)^{\lambda}{}_{\beta},$$

where $f_{\mu \nu}^{(o)} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $\varepsilon^{\rho \alpha \beta}$ is the Levi-Civita (pseudo) tensor. Using (2), the Maxwell lagrangian density ($\mathcal{L}_M$) and symmetric momentum-energy tensor ($T^M_{\mu \nu}$) can be defined and they are given by

$$\mathcal{L}_M = -\frac{1}{4} \psi,$$

$$T^M_{\mu \nu} = \psi_{\mu \nu} - \frac{g_{\mu \nu}}{4} \psi,$$

where notation means $\psi_{\mu \nu} \equiv f_{\mu \sigma} f^{\nu \sigma}$ and $\psi \equiv f_{\mu \sigma} f^{\mu \sigma}$.

Following reference [2] and using (1) we introduce the non minimal coupling between
gravity, electromagnetism and auxiliary connection ($W_\alpha$), at the torsionless limit

$$S = S_0 + \kappa \int_M d^3x \sqrt{-g} \ell(g, f) - 8\pi G \text{tr} M^\alpha (g, f) \ast F_\alpha + \text{tr} J_\alpha (A_\alpha - W_\alpha)$$

$$+ \text{tr} H^{\alpha\beta} (A_\alpha - W_\alpha) (A_\beta - W_\beta),$$

(5)

where $\ell(g, f)$ is the modified Maxwell lagrangian density with second order correction given by parameters $b_1$ and $b_2$

$$\ell(g, f) \equiv L_M + b_1 (T^M)^2 + b_2 T^M T^{M\mu\nu}. \quad (6)$$

The components of the coupling tensor, $M^\alpha$ have linear dependence on $T^{\mu\nu}$, and they are

$$(M^\alpha)^\mu_\nu = [c_1 \varepsilon^{\alpha\mu \nu} g^{\sigma\rho} + c_2 \varepsilon^{\alpha\rho \nu} g^{\sigma\mu} + c_3 \varepsilon^{\alpha\mu\rho} \delta^\sigma_\nu] T^M_\sigma\rho + a \varepsilon^{\alpha\mu \nu}, \quad (7)$$

where $c_1$, $c_2$, $c_3$ and $a$ are free parameters, and $\ast F_\alpha$ in (5) is the Poincaré dual of curvature (i.e., $\ast F_\alpha \equiv \frac{1}{2} \varepsilon_{\alpha \mu \nu} F^{\mu \nu}$).

The coupling tensors for auxiliary fields are

$$(J_\beta)^\mu_\nu \equiv (d_1 + d_2 \psi) \varepsilon_\beta_{\mu\nu}, \quad (8)$$

$$(H^{\alpha\beta})^{\mu\nu} \equiv a_1 g^{\alpha\beta} g^{\mu\nu} + a_2 g^{\alpha\mu} g^{\beta\nu} + a_3 g^{\alpha\nu} g^{\beta\mu}, \quad (9)$$

where $d_1$, $d_2$, $a_1$, $a_2$ and $a_3$ are free parameters and $\psi \equiv f_{\mu\sigma} f^{\mu\sigma}$.

We underline some aspects of action (5). The lagrangian density $\ell(g, f)$ encloses Maxwell density and second order contributions on momentum-energy tensor, this means second order in $\psi_{\mu\nu}$. On the other hand, the lagrangian term ”$8\pi G \text{tr} M^\alpha \ast F_\alpha$” is covariantly defined and provide a minimal coupling contribution plus non minimal ”Proca” density. This fact suggests the shape of coupling with auxiliary fields $W_\alpha$.

The field equation for $W_\alpha$ is $J^{\beta} + H^{\alpha\beta} (A_\alpha - W_\alpha) + (A_\alpha - W_\alpha) H^{\beta\alpha} = 0$, so an ansatz for $W_\alpha$ is chosen, $(A_\alpha - W_\alpha)^\mu_\nu = (\theta_1 + \theta_2 \psi) \varepsilon_{\alpha\mu\nu}$ with $d_n = 2a_2 \theta_n$ for $n = 1, 2$.
and \(a_{21} \equiv a_2 - a_1\). Then, the field equation for \(GL(3, R)\) connection can be written like

\[
\nabla_\nu \left( R^\alpha_{\mu} - 8\pi G c_1 g^{\alpha \mu} T^M - 8\pi G c_2 T^M_{\alpha \mu} \right) - \nabla^\nu \left( R^\alpha_{\nu} - 8\pi G c_1 \delta^\alpha_\nu T^M - 8\pi G c_3 T^M_{\alpha \nu} \right) \\
+ 8\pi G \nabla_\beta \left( c_2 \delta^\alpha_\nu T^{M \alpha \beta} - c_3 g^{\alpha \mu} T^{M \beta}_{\mu \nu} \right) + \kappa^{-3} \varepsilon^{\lambda \alpha \mu} \tilde{C}_{\lambda \nu} = 0 \quad (10)
\]

where the lagrange multipliers have been rewritten as \(\tilde{C}_{\lambda \nu} \equiv C_{\lambda \nu} + (1 - 2(b_1 - b_2)) T^M - 88\pi GR a_\nu \ast f_\lambda - 8(b_2 T^M_{\rho \mu} + 8\pi GR^\rho_{\mu}) \varepsilon^\lambda_\rho a_\nu f^{\mu \sigma} \).

Demanding consistency between (10) and field equation of EHT (i.e., \(R_{\alpha \beta} - \frac{a_{21}^2}{2} R = -8\pi G T^M_{\alpha \beta}\)), some parameters are fixed, \(c_1 = -c_2 = -c_3 = 1\) and \(T^M_{\alpha \beta}\) must be a covariantly conserved tensor.

Variations on metric of (5) provide an equation whose evaluation on EHT conduce to

\[
\left( -8b_1 - 24b_2 + 24(8\pi G)^2 - 768a_{21} \theta_1^2 \right) \psi^{\rho \sigma} \\
+ \left( b_1 + 3b_2 - 17(8\pi G)^2 + 32a_{21} \theta_2^2 \right) \psi^2 g^{\rho \sigma} \\
+ \left( 16 - 32(8\pi G)^2 a - 768a_{21} \theta_1 \theta_2 \right) \psi^{\rho \sigma} \\
+ \left( -4 + 8(8\pi G)^2 a + 192a_{21} \theta_1 \theta_2 \right) \psi g^{\rho \sigma} \\
+ 96a_{21} \theta_2^2 g^{\rho \sigma} = 0 \quad , (11)
\]

and for all \(\psi_{\mu \nu} \equiv f_{\mu \sigma} f^{\nu}_{\sigma}\), the remaining free parameters can be fixed

\[
b_1 + 3b_2 = 3(8\pi G)^2 \quad , \quad (12)
\]

\[
a_{21} \theta_2^2 = 14(8\pi G)^2 \quad , \quad (13)
\]

\[
\theta_1 = 0 \quad , \quad (14)
\]

\[
2(8\pi G)^2 a = 1 \quad , \quad (15)
\]
so equation (11) is satisfied identically.

4) Concluding remark

The non minimal coupling scheme for the Yang-Mills-like gauge formulation of gravity provides the necessary terms which guarantee the fixing of free parameters in order to obtain consistency between $GL(3, R)$ connection and EHT. This fact says that a minimal coupling is not sufficient. However, metric field equation demands the presence of auxiliary connection, on the contrary, quadratic polynomial restrictions over $\psi_{\mu\nu}$ would appear. It is possible to choose parameters $b_1$ and $b_2$ in the way that $\ell(g, f) \rightarrow L_M$, when gravity decouples ($G \rightarrow 0$), so one can recover the free Maxwell theory in a flat space-time.

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