Near Gaussian Multiple Access Channel Capacity Detection and Decoding

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Abstract—We consider the design of low-density parity-check (LDPC) codes with close-to-capacity performance for non-orthogonal interleave division multiple access (IDMA). The degree profile of the LDPC code is optimized using extrinsic information transfer (EXIT) charts to match an IDMA low complexity multi-user detector. Analytical EXIT functions for the IDMA system are derived while verifying that the Gaussian approximation stays valid for a sufficiently large number of users. The resulting LDPC codes are of rather low-rate (e.g., \( R_c = 0.03 \) for 32 users) and thus, the practical design of such codes is quite challenging and may, due to the low code-rate, not even yield a performance close to the Gaussian multiple access channel (GMAC) capacity. Adding a serial concatenation with a simple repetition code allows to design matching LDPC codes with a higher rate and, this way, achieving closer to GMAC capacity. Moreover, the designed IDMA system can be flexibly reconfigured to support a wide range of different numbers of users by simply varying the repetition code rate, i.e., without the need for redesigning the LDPC code itself, making it an attractive non-orthogonal multiple access (NOMA) scheme for future wireless communication systems.

Index Terms—Interleave division multiple access, EXIT chart, non-orthogonal multiple access, LDPC code.

I. INTRODUCTION

With the rapid development of information and coding theory, the fundamental capacity limit of information transmission between one transmitter and one receiver over an additive white Gaussian noise (AWGN) channel, laid down by Shannon, has virtually been achieved. In fact, there exist several powerful and practical codes, e.g., Turbo [1], LDPC [2] and Polar codes [3], that can operate close to capacity with both moderate codeword length as well as feasible decoding complexity. However, the situation changes when considering multi-user setups, such as the multiple access channel (MAC), i.e., multiple transmitters (or users) and one single receiver. The MAC capacity (region) has been known since the 1970s [4] and can be determined by a tuple of rates \( R_k, 1 \leq k \leq N \) of the \( N \) individual users. In this paper, we assume that no co-operation is allowed among users, e.g., no power control and/or cooperative encoding at transmitter. We refer interested readers to [5], [6], [7] for detailed discussion of capacity regions and different cooperative schemes in multiuser settings. Often, the capacity region is dominantly bounded by the sum capacity constraint of all users. These rate-tuples, resulting in maximum sum-rate, constitute edge or facet in vector space with dimension of two or higher, respectively. It is referred to as dominant face of the MAC capacity region, which is of particular practical interest.

To achieve arbitrary points of the dominant face (illustrated for the 2-user case in Fig. 1), joint multi-user decoding or successive interference cancellation (SIC) with time sharing between corner points is required. Joint decoding is not feasible in terms of complexity even for a small number of users. While SIC can achieve certain corner points of the capacity region, it suffers from high latency and potential error propagation. Other orthogonal methods such as time division multiple access (TDMA), frequency division multiple access (FDMA) and code division multiple access (CDMA) avoid interference by providing orthogonal channels so that each user can detect and decode its signal independently. However, these schemes may require, on the one hand, complex synchronization and tracking algorithms to maintain orthogonality in time/frequency/code domain, while on the other hand, achieve solely one point of the capacity region where optimum resource allocation in terms of bandwidth, power and time is performed (see green dot in Fig. 1).

A. Gaussian MAC capacity review: two-user example

It is instructive to review the Gaussian MAC capacity (Fig. 1). For the simple two-user case with transmit powers \( P_1 \) and \( P_2 \), the capacity region is determined by the rate pairs

\[
R_1 \leq \log_2 \left( 1 + \frac{P_1}{N_0} \right)
\]

\[
R_2 \leq \log_2 \left( 1 + \frac{P_2}{N_0} \right)
\]
where $B_1$ and $B_2$ with $B_1 + B_2 = 1$ denote the resource that is allocated to users 1 and 2, respectively. The maximum sum-rate is attained when

$$B_1 = \frac{P_1}{P_1 + P_2},$$

$$B_2 = \frac{P_2}{P_1 + P_2}.$$ 

Thus, it achieves solely one point of the dominant face when optimum resource allocation is performed.

For CDMA, the rates are given by

$$R_{1}^{\text{CDMA}} = \frac{1}{N} \log_2 \left(1 + \frac{NP}{N_0}\right),$$

$$R_{2}^{\text{CDMA}} = \frac{1}{N} \log_2 \left(1 + \frac{NP_2}{N_0}\right),$$

with equality achieved for $P_1 = P_2$. Thus, CDMA achieves solely one point of the dominant face in equal power settings.

Interleave division multiple access (IDMA) [9], [10] as a special form of superposition coding is theoretically optimal [11]. The initial (“iteration 0”) rates are given by

$$R_{1}^{\text{IDMA,0}} = \log_2 \left(1 + \frac{P_1}{P_2 + N_0}\right),$$

$$R_{2}^{\text{IDMA,0}} = \log_2 \left(1 + \frac{P_2}{P_1 + N_0}\right).$$

By exploiting soft interference cancellation and iterative detection and decoding, it can achieve any point of the dominant face with properly designed codes. The equations for general number of users $N$ follow straightforwardly.

### B. Problem formulation and related work

As the next generation wireless network is expected to accommodate a tremendous number of devices, non-orthogonal multiple access (NOMA) has become an emerging technology. In NOMA, the multiple access interference (MAI) rather than the noise is the bottleneck. Treating MAI as noise is strictly sub-optimal because interference shall not be considered the ultimate limitation of capacity in view of information theory, e.g., achieving the “it. 0” point in Fig. 1.

The SIC has been known for a long time for achieving the so-called dominant face corner points (marked as diamonds in Fig. 1) of the two operational (corner) points, the dominant face can be achieved.

For TDMA/FDMA, we can write

$$R_{1}^{\text{TF}} + R_{2}^{\text{TF}} = B_1 \log_2 \left(1 + \frac{P_1}{N_0 B_1}\right) + B_2 \log_2 \left(1 + \frac{P_2}{N_0 B_2}\right),$$

where $B_1$ and $B_2$ with $B_1 + B_2 = 1$ denote the resource that is allocated to users 1 and 2, respectively. The maximum sum-rate is attained when

$$B_1 = \frac{P_1}{P_1 + P_2},$$

$$B_2 = \frac{P_2}{P_1 + P_2}.$$ 

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that interleave division multiple access (IDMA) is an attractive method for approaching the MAC capacity with low-rate codes and low complexity receivers. In IDMA, the same error correction code may apply to all users. A low complexity parallel interference cancellation (PIC)-based multi-user detector (MUD) is serially concatenated in a “turbo-like” fashion (as an inner code component) to iteratively improve the achievable rate toward the MAC capacity. Unlike SIC-based schemes, IDMA requires merely one single code design that can be applied to all users and it can be operated fully asynchronously [12] in the time domain.

For close-to-optimal performance of the low-complexity PIC-based MUD Gaussian-distributed interference is required, an assumption that only holds for a large number of users when a finite alphabet, e.g., binary phase shift keying (BPSK), is used. In this case, the single user decoder is seriously challenged by MAI at the beginning of the turbo iterations due to the large number of users, leading to a very low initial effective SNR. For this reason, low-rate codes appear to be inevitable in IDMA systems.

As LDPC codes are capacity-approaching [2] and there exist several useful tools for its optimization, even taking into account a detector front-end [13] in the context of point-to-point (P2P) communications, we use LDPC codes for channel coding in the IDMA framework. However, such a low-rate code design and its decoding in the range of, e.g., code rates $R_c \leq \frac{1}{10}$ becomes a challenging task [14]. Besides the necessity of low-degree nodes [14], the Gaussian approximation (GA) of messages at the check nodes (CN) becomes inaccurate and, further, the decoding graph becomes relatively dense. Furthermore, the maximum node degrees should not exceed a certain limit to maintain feasible finite length code constructions. This renders code design for IDMA systems to be a challenging and interesting research direction.

Previous works [10, 9] show that PIC-based MUD exhibits low complexity and can achieve close to MAC capacity with low-rate Turbo-Hadamard codes. In [15], [16] extrinsic information transfer (EXIT) charts [17] are introduced to analyze the performance of iterative MUD and channel decoding. Therein, the main focus is on different multi-user detectors; a power allocation scheme among users is proposed to close the gap to the MAC capacity. Refs [18], [19] consider the coding-spreading trade-off and a hybrid MUD to improve convergence, yet only based on a few selected codes and a few number of users. A sophisticated and systematic design and optimization of the joint decoding and detection in the MAC still remains an open problem.

The most recent works in [20], [21], [22], [23] address the LDPC design with two users using optimal joint detection and a Gaussian-mixture model to approximate the message probability density function (PDF) at the check nodes. In [23], the LDPC codes are optimized for IDMA systems with a relatively large number of users, (6 to 10) again with an optimum joint MUD. This joint MUD eliminates one of the most attractive features of IDMA, namely, its low-complexity. Furthermore, for a moderate to large number of users, an optimum joint MUD is computationally prohibitive and, therefore, the code design for such large systems has not yet been adequately addressed, although the design of binary LDPC codes for iterative detection and decoding is well-understood in the EXIT-chart framework [13], [24], [25].

Even though the challenge of designing a practical low-rate code with close-to-capacity performance can be met, the flexibility is highly limited since the user load can vary, which, in turn, imposes a loss of information rate when the same code is applied, or requires a relatively complicated re-optimization of the code. To the best of the authors’ knowledge, a simple and flexible scheme within the IDMA framework for supporting a different number of active users has not been extensively studied thus far.

C. Contributions

We consider the design of channel codes for an arbitrary number of users, which shall be matched to the low complexity PIC-based MUD characteristics according to the EXIT-chart analysis. We will start with a large number of users (e.g., $N = 32$) thus Gaussian approximation can be assumed, and show later the code design also works for other number of users. We note that the PIC-based MUD is asymptotically optimum when the number of users approaches infinity. The performance loss incurred by the sub-optimum MUD is larger when the number of users is small.

In particular, we split the channel code into two parts, namely, a repetition code (REP) and an LDPC code. The intuition behind is that the repetition code is mainly concerned with the MAI while the LDPC code is predominantly engaged in combating noise and residual MAI. We start with the important case of equal power/rate/modulation for all users and derive a novel expression for the EXIT-curve capturing the PIC-based MUD. Subsequently, we concatenate the EXIT-curves of the MUD, the repetition code and the variable nodes (VN) of the LDPC code. The degree profiles of the VN and CN of the LDPC code are then optimized such that convergence of belief propagation (BP) decoding can be achieved with the highest possible code rate. As it turns out, the serial concatenation with a simple repetition code is surprisingly effective in terms of canceling out MAI. To achieve closer to GMAC capacity, a joint design of the repetition code and the LDPC code is further studied.
Once the LDPC degree profile along with its optimum repetition factor \( d_r \) is optimized to a certain number of users \( N \), any variation of user load in the system imposes a loss of information rate. To make the system design more universal to varying user load \( N \), we propose to use the simple repetition code as a “rate equalizer” to “absorb” the user load, thus avoiding complicated redesign of the LDPC code. Simulation results show that we can achieve close to GMAC capacity with 1.28 dB \( \frac{E_b}{N_0} \) loss at a sum-rate of 0.9375 bpcu supporting 30 users each with BPSK modulation.

This performance can be achieved over a wide range of number of users \( N \) by simply varying the repetition code rate while fixing the LDPC code.

### II. IDMA SYSTEM MODEL

Fig. 2 shows the IDMA system model with \( N \) uncooperative users. Each user encodes and decodes its data separately using an LDPC encoder of code rate \( R_c \) and a common serially concatenated repetition code of rate of \( R_r = \frac{1}{R} \). Note that the code parameters, e.g., degree profile and parity check \( \mathbf{H} \) matrix, are the same among all the \( N \) users. The total code-rate is thus \( R_{\text{tot}} = R_c R_r \). The interleaver is, on the contrary, user-specific to allow efficient user separation at the receiver. After interleaving, the coded bits are mapped to symbols, e.g., using binary phase shift keying (BPSK), and transmitted over a channel.

This paper considers the additive white Gaussian noise (AWGN) channel but extensions to other channels are straightforward. Hence, the \( m \)th received signal (i.e., the \( m \)th element of \( \mathbf{y} \) in Fig. 2) of all users can be written as

\[
y(m) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i(m) \cdot e^{j\varphi_i(m)} + n(m) \tag{1}
\]

where \( m \) is the discrete-time index, \( n \) is circularly symmetric (complex-valued) AWGN with zero mean and variance \( \sigma_n^2 \), and \( \varphi_i \) is a pseudo random phase scrambling to avoid ambiguity of the super-constellation (Cartesian product of all users constellation). This random phase shift could also be the consequence of, e.g., the channel and/or explicit “scrambling” and we include this into each user’s mapper. Throughout this paper, the phases \( \varphi_i \) are independently and uniformly distributed in \([0, \pi]\). The output of the mapper of the \( i \)th user with BPSK modulation at the \( m \)th time instant is \( \hat{x}_i(m) \in \{ \pm e^{j\varphi_i(m)} \} \).

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1. The target sum-rate is chosen such that the interference and noise have roughly the same impact on the performance. For low SNRs, the performance is mainly noise-limited and single-user coding suffices. For high SNRs, the performance is mainly interference-limited and orthogonal schemes or repetition coding suffice(s).
The received signal is first processed by a multi-user detector (MUD). An optimum MUD is to compute the maximum a posteriori (MAP) probability of each bit. This requires a complexity of $O(M^N)$ where $M$ denotes the number of constellation symbols per user. The exponentially increasing complexity with the number of users $N$ prohibits its practical implementation for a large number of users. Therefore, a sub-optimal soft interference cancellation (SoIC) based low complexity MUD was proposed in [9]. The sub-optimal MUD first cancels out other users’ signals; for instance, the $i$th user’s signal is estimated by

$$\hat{x}_i = \tanh \left( \frac{L_{M,i}^A}{2} \right) \cdot e^{j\phi_i}$$

based on, e.g., the a posteriori knowledge of the channel decoder $L_{M,i}^A$ (the LDPC decoder output) is then re-encoded by REP and re-interleaved with the outputs denoted by $\hat{L}_{M,i}$ and $L_{M,i}$, respectively. For an arbitrary user $j$ (the symbol index $m$ is dropped for brevity), the output of the MUD after the SoIC can be written as

$$y_j = \frac{1}{\sqrt{N}} \bar{x} + \frac{1}{\sqrt{N}} \sum_{i=1,i\neq j}^{N} (\hat{x}_i - \bar{x}) + n.$$ 

Then, each user starts its single user detection and decoding in parallel. The (soft) demapper computes the log-likelihood-ratio (LLR) of each bit while treating the residual interference as noise. For BPSK, an approximation of the true a posteriori LLR can be computed according to

$$L_{M,j}^E(m) = 2 \frac{\text{Re} \left\{ y_j(m) \cdot e^{-j\varphi_j(m)} \right\}}{\sigma_{L,j}^2 + \frac{N}{2} \sigma_n^2}$$

where the noise variance $\sigma_n^2$ and the random phase shifts $\varphi_j$ are assumed to be known to the receiver. The interference power can be estimated by

$$\sigma_{L,j}^2 = E \left[ \text{Re} \left\{ e^{-j\varphi_j} \sum_{i\neq j} (\hat{x}_i - \bar{x}_i) \right\} \right]^2$$

$$= \sum_{i\neq j} \cos^2 (\varphi_1 - \varphi_j) \left( 1 - E \left[ \tanh \left( \frac{L_{M,i}^A}{2} \right) \right]^2 \right)$$

where the interference term is assumed to be Gaussian distributed, provided that the number of users $N$ is large and the transmitted symbols are independent among users (central limit theorem).

Then, the LLRs are deinterleaved (denoted by $\hat{L}_{M,j}^E = L_{R\rightarrow M,j}^E$) which means the extrinsic message from MUD corresponds to the a priori knowledge of REP obtained by MUD) and sent to a repetition decoder. The extrinsic message from the repetition code to the LDPC decoder is given by

$$L_{D,j}^A(k) = L_{R\rightarrow D,j}^E(k) = \sum_{m=kd}^{(k+1)kd-1} \hat{L}_{M,j}^E(m).$$

Subsequently, belief propagation (BP) decoding can be performed by iterating between VN and CN of the LDPC code. The detailed decoding procedure will be discussed in Sec. III. The backwards message passing, i.e., from LDPC decoder to REP to MUD, is carried out similarly. We remark that only extrinsic messages shall be fed back for iterative processing.

The SoIC-based MUD is quite simple and requires only few multiplications and additions for variance estimation, interference subtraction and LLR computation (see [10]). The complexity becomes $O(MN)$ and many computations can be carried out in parallel when compared to SIC. Particularly, when all the users apply an LDPC code with the same parity check $H$-matrix, the coordination and code design among users can be dramatically reduced.

### III. EXIT Analysis

In order to optimize the LDPC degree profile, we elaborate on the EXIT-chart approach in the context of IDMA by separately and/or jointly considering the components of the multistage iterative receiver (MUD, repetition decoder and LDPC decoder). For this, we illustrate the graph representation of the belief propagation (BP) based receiver in Fig. 3. Furthermore, since we consider a system with a large number of users, the messages being exchanged over the graph (LLRs) can be assumed to be Gaussian distributed.

As common practice, it suffices to track the mean (in GA-based density evolution) or average mutual information (in EXIT-analysis) of the messages. In this paper, the mean will be tracked in the following as, when using the conventional mutual information-based EXIT chart analysis for the low-rate ($R_c < \frac{1}{10}$) code optimization, we observed too optimistic thresholds, leading to non-favorable code designs. By tracking the message mean $\mu$, our results turn out to be more accurate (i.e., closer to the predicted decoding threshold). We conjecture two reasons for this: firstly, the $J$-function which maps the mean $\mu$ to a mutual information $I = J(\mu)$ is more difficult to approximate, particularly when $\mu$ becomes very small; secondly, it relies on a more strict Gaussian approximation in terms of PDF (MI is obtained by solving the integral involving a Gaussian PDF) while directly tracking the mean $\mu$ does not strictly require Gaussian PDFs, but only the first order statistic (summation/product of mean assumes only independency).
A. SoIC-MUD nodes

As indicated in Fig. 3 each received symbol \( y(m), m \in [0,N_{\text{msg}} - 1] \) is represented by a MUD node which is connected to \( d_u = N \) repetition nodes, each REP node belonging to another user, and one channel observation. Consider an arbitrary MUD node, the message mean being passed from repetition node to channel observation. Upon convergence of the iterative processing between the SoIC-MUD and the repetition nodes, we obtain

\[
\begin{align*}
\mu_{M \rightarrow R} &= d_u \sigma_n^2 + (d_u - 1) \cdot \phi(\mu_{R \rightarrow M}) \\
\end{align*}
\]

(4)

where the interference power can be derived with (2) as

\[
\sigma_{I,j}^2 = \frac{d_u - 1}{2} \cdot \phi(\mu_{R \rightarrow M,j}).
\]

(3)

The detailed proof is given in Appendix A, and \( \phi(\mu) \) is given by

\[
\phi(\mu) = 1 - \int_{-\infty}^{\infty} e^{-\frac{y^2}{\sigma_n^2}} \tanh \left( \frac{\mu}{2} - \sqrt{\frac{\mu}{2} y} \right) dy,
\]

which denotes the minimum mean square error (MMSE) [26] of the SoIC with BPSK signaling. Thus, the updated message mean \( \mu_{M \rightarrow R} \) can be expressed as

\[
\begin{align*}
\mu_{M \rightarrow R} &= d_u \sigma_n^2 + (d_u - 1) \cdot \phi(\mu_{R \rightarrow M}) \\
\end{align*}
\]

(4)

B. Repetition nodes (REP)

The repetition node has \( d_r \) edges to MUD nodes and one edge to the LDPC variable node. Let \( \mu_{V \rightarrow R} \) denote the message mean of LLRs of the VN decoder (VND), we obtain the message mean from repetition node to MUD and VND as

\[
\begin{align*}
\mu_{R \rightarrow M} &= (d_r - 1) \cdot \mu_{M \rightarrow R} + \mu_{V \rightarrow R} \\
\mu_{R \rightarrow V} &= d_r \cdot \mu_{M \rightarrow R}
\end{align*}
\]

(5)

(6)

Upon convergence of the iterative processing between the SoIC-MUD and the repetition nodes, we obtain

\[
\mu_{M \rightarrow R} = d_u \sigma_n^2 + (d_u - 1) \cdot \phi(\mu_{R \rightarrow M}).
\]

If no LDPC code is available, i.e., \( \mu_{V \rightarrow R} = 0 \), the “uncoded” IDMA performance can be evaluated by the above message passing mechanism. In Fig. 4 the EXIT-chart is shown for a purely repetition coded IDMA system with \( d_u = 32 \) users at a multi-user SNR of \( \gamma_{s,\text{mu}} = 10 \log_{10} \frac{1}{\sigma_n^2} = 40 \text{ dB} \) (every single user decoder “sees” the equivalent single user SNR of \( \gamma_{s,\text{su}} = \frac{1}{\sigma_n^2} \) even in the absence of interference). Hence, the relation between single user and multi-user SNR is given by

\[
\gamma_{s,\text{su}} = d_u \cdot \gamma_{s,\text{mu}}.
\]

All the EXIT-curves are based on directly evaluating (4) and (5). For instance, the SoIC-MUD EXIT-curve is computed by

\[
I_{E,MUD} = J \left( d_u \sigma_n^2 + (d_u - 1) \cdot \phi(J^{-1}(I_{A,MUD})) \right)
\]

where the \( J(\mu) \)-function is given by

\[
J(\mu) = 1 - \int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{\sigma_n^2}} \log_2 \left( 1 + e^{-y} \right) dy
\]

and its inverse \( J^{-1} \) can be approximated as in [27]. The trajectories for the repetition rate \( R_r = \frac{4}{3} \) and \( \frac{7}{4} \) obtained by numerical simulations match very well with the EXIT-curve predictions. Small deviations are still
present in the high a priori knowledge region, i.e., large $J (\mu_{R \rightarrow M})$, due to the GA.

At such high SNR $\gamma_{s, \mu} = 40$ dB, the performance is mainly interference-limited. Interestingly, the required minimal repetition code rate is $1/2$ while a spreading factor of 16 would be necessary in an orthogonal CDMA scheme. This indicates that repetition codes are very efficient for interference cancellation. In a more realistic SNR region, the noise sets the upper limit on the MUD, i.e., $I_E = J \left( \frac{1}{\alpha_e \sigma_n^2} \right)$. Therefore, an LDPC code is further required to mitigate the noise disturbance, as described next.

C. VND and CND nodes

The message exchange and update between VN decoder (VND) and CN decoder (CND) is well known. For completeness, we review and tailor it for IDMA systems. Denote $\lambda_i$ and $\rho_j$ as the fraction of having VNDs and CNDs with degrees $d_{v,i} = i$ and $d_{c,j} = j$, respectively. The message means $\mu^{i}_{V \rightarrow C}$ and $\mu^{j}_{C \rightarrow V}$ (i.e., mean of message from CN/VN to VN/CN of degree $d_{v,i} \times d_{c,j}$, respectively) are given by

$$\mu^{i}_{V \rightarrow C} = \mu_{R \rightarrow V} + (i - 1) \cdot \sum_{j} \rho_{j} \mu^{j}_{C \rightarrow V}$$

and

$$\mu^{j}_{C \rightarrow V} = \phi^{-1} \left( 1 - \left( 1 - \sum_{i=2}^{\max} \lambda_{i} \phi \left( \mu^{i}_{V \rightarrow C} \right) \right)^{j-1} \right)$$

respectively, where the channel observation of the VND is now provided by the REP. Furthermore, we consider check-regular LDPC codes, i.e., $\rho_j = 1$ for a given $d_{c,j}$. The VND also passes a message to the connected repetition node, given by

$$\mu^{i}_{V \rightarrow R} = i \cdot \sum_{j} \rho_{j} \mu^{j}_{C \rightarrow V}.$$ 

IV. Degree Profile Optimization

A. LDPC Code Only

The LDPC degree profile, i.e., the coefficients $\lambda_i$'s and $\rho_j$'s, can be optimized to “match” (curve fitting) the EXIT-curve of the inner SoIC-MUD for achieving convergence at highest code rate. The optimization approach is summarized in Algorithm 1 where the repetition code is firstly not present, i.e., $d_r = 1$ and all redundancy is devoted to the LDPC code. In this case, the MUD, repetition and VND nodes can be merged into one effective MUD-VND node, and message passing is performed solely between CND and MUD-VND nodes. As shown in Appendix B, the stability condition in (9) is derived. The LDPC code shall be optimized and used to mitigate the MAI and noise disturbance. Note that the initial SINR (no iteration) can be very small. For instance, the SINRs are $-18$ dB and $-14.7$ dB for $\gamma_{s, \mu} = 0$ dB and $40$ dB (all with $d_u = 32$ users), respectively.

In Fig. 5, we show the EXIT-chart of the optimized LDPC code of code-rate $R_c = 0.1068$ at $\gamma_{s, \mu} = 40$ dB and $d_u = 32$. We limit the maximum variable/check node degrees to $d_{v, \max} = 320$ and $d_{c, \max} = 64$ to enable feasible finite length code construction. In the simulation, the LDPC codeword length is set to $N_{CW} = 10^4$. Throughout the paper, the parity check $H$ matrices are randomly generated and we select the matrix with the largest average girth for simulation. We use all-zero codewords with “phase scrambling” in the simulation to avoid the rather complicated LDPC encoding procedure for the random $H$ matrix, but still the users’ signals are kept independent.

It turns out that convergence is achieved with $d_u = 28$ users when fixing the multi-user SNR $\gamma_{s, \mu}$ to $40$ dB. We treat the SoIC-MUD nodes and the VND of the LDPC as one component, and the CND as another component. The analytical EXIT-functions are depicted by the solid blue and dashed red curves, respectively. For each iteration, the mutual information of the exchanged messages between the two components is simulated. The simulated average trajectory (denoted by green staircase) is shown in Fig. 5 for 171 iterations. It can be seen that the trajectory matches quite well with the analytical EXIT-functions despite a small deviation in the range.

\textsuperscript{3}We use this rather complicated notation to be consistent with other IDMA decoding components, i.e., REP and SoIC-MUD.

\textsuperscript{4}It turns out that not all degrees, particularly the high degrees, are needed. We select the relatively large numbers for more degree of freedoms.
of $\mu \in [0.07, 0.4]$, which verifies the viability of the algorithm.

The optimized LDPC code at such high SNR is not as efficient as the simple repetition code (comparing Fig. 5 and 6) both in terms of decoding complexity and achievable sum-rate ($R_{\text{sum}}^{\text{LDPC}} = d_u R_c = 2.99$ bpcu and $R_{\text{sum}}^{\text{REP}} = d_v R_c = 3.56$ bpcu). However, such a high SNR is usually not of practical interest. We simulate this SNR for the purpose of validating the Gaussian approximation of the MAI. As the results show, the GA is relatively good but the simple repetition is more efficient than the LDPC coding for the interference cancellation.

In Fig. 5, the BER of the optimized LDPC code at the target sum-rate of $R_{\text{sum}} = d_u R_c = 1$ bpcu is shown. The interleaver depth is set to $d_v N_{\text{CW}}$ bits in the simulation to mitigate short cycles due to the superposition of multiple users. It turns out that the performance depends, to a great extend, on the number of users $d_u$. In the simulation, we set the number of users to $d_u = 30$ although the code is optimized for $d_u = 32$ due to the fact that our parity check matrix exhibits relatively small girth. The corresponding GMAC capacity is at $-0.1$ dB, and the optimized code is more than 6 dB away from the GMAC capacity. The reasons are manifold; firstly, it is well known that low-rate LDPC code design does not yield satisfactory performance [14]; secondly, the BP-graph is essentially rather dense than sparse so that convergence is compromised. We remark that the performance can be improved by increasing the interleaver depth and/or constructing $H$ matrix using more sophisticated design methods.

B. Repetition code and LDPC

The repetition code seems to be an effective means for supporting interference cancellation. Then, we extend the LDPC degree profile optimization to a serially concatenated repetition code, i.e., $d_r > 1$. The procedure for the optimization of the degree profile involving a repetition code is summarized in Algorithm 1. Hereby, the MUD, repetition and VND nodes are merged into one MUD-REP-VND node. Within this node, a sufficient number of iterations is carried out until the message between MUD and repetition node convergences to a fixed value. The converged message mean, denoted by $\mu_{M \rightarrow R} (d_v, i \cdot \mu)$, depends on the incoming message from the connected VND $\mu_{V \rightarrow R} = d_v, i \mu$ and can be obtained by solving (2).

In Fig. 6, we present the $\frac{E_b}{N_0}$-gap to the Shannon limit of the joint repetition and LDPC code optimization for various repetition factors $d_r$ and a few target sum-rates, where the gap is calculated as

\[ \Delta \gamma = \frac{\zeta_t}{2 R_{\text{sum}} - 1} \]

where $\zeta_t$ denotes the decoding threshold (required SNR) and $R_{\text{sum}}$ denotes the achieved sum rate of all users. The maximum VN degree and CN degree are set to $d_{v, \text{max}} = 320$ and $d_{c, \text{max}} = 64$, respectively. As can be observed, there exists an optimum repetition code rate $d_r^* \frac{E_b}{N_0}$ for the outer LDPC code rate $R_c$ which maximizes spectral efficiency. The optimum repetition factor $d_r$ is marked in Fig. 6. This also indicates that repetition codes are efficient for interference cancellation, since it is well known that repetition code has no coding gain in the single user case. We remark that the SoIC-MUD based on GA is strictly sub-optimal for finite modulation order. The loss incurred by the sub-optimality becomes significant when SNR increases. Thus, the $\frac{E_b}{N_0}$-gap is larger when the target sum-rate increases, since it essentially requires higher SNR.

We select the target sum rate of $R_{\text{sum}} = 1$ bpcu whose GMAC capacity is achieved at $\frac{E_b}{N_0} = \gamma_{s,\text{sum}} = 0$ dB. In this case, the MAI and the noise have approximately the same power such that the interference cancellation and error correction are of the same importance. The

\[
\text{Algorithm 1 LDPC code design based on EXIT-Chart with and without repetition code.}
\]

\begin{itemize}
  \item \textbf{Input}: $d_u$, $d_{v, \text{max}}$, $d_r$, $\sigma_n^2$
  \item \textbf{Output}: $\lambda_{\text{opt}}$, $\rho_{\text{opt}}$, $d_r^\text{opt}$, $R_{\text{max}}$
  \item for $d_r = [d_r, \cdots, d_r]$ for $d_r = [d_r, \cdots, d_r, \text{max}]
  \text{Solve LP:}
\end{itemize}

\[
\max_{\lambda_{\text{opt}}} \sum_{i=2}^{\lambda_{\text{max}}} \frac{\lambda_i}{k}
\text{ s.t. } \lambda_i \geq 0
\sum_{i=2}^{\lambda_{\text{max}}} \lambda_i = 1
1 - \sum_{i=2}^{\lambda_{\text{max}}} \lambda_i \phi (d_r \cdot \mu_{M \rightarrow R} (d_v, i \cdot \mu) + (d_v, i - 1) \cdot \mu)
> d_r \sqrt{1 - \phi (\mu)} \quad (8)
\lambda_2 \leq e^{\frac{\sigma_n^2}{\sqrt{\lambda_1}}}
d_r - 1 \quad (9)
\text{Compute } R = \frac{1}{d_r} \left( 1 - \frac{1}{d_c} \sum_{i=2}^{\lambda_{\text{max}}} \lambda_i \right)
\end{itemize}

end
Select the maximum $R_{\text{sum}}$ and output the corresponding $d_r^\text{opt}$, $\lambda_{\text{opt}}$, and $\rho_{\text{opt}}$. 
interference-to-noise-ratio (INR) can be calculated by
\[
\gamma_{\text{INR}} = \frac{N-1}{N \cdot \sigma_n^2} \frac{d_u - 1}{d_u} \cdot \gamma_{s,\text{mu}}
\]
and \(\gamma_{\text{INR}} = -0.1379\) dB with \(d_u = 32\). The optimized LDPC code parameters\(^5\) are given in Tab. I.

In Fig. 7, the EXIT-chart based on \(\mu\) is depicted, where the EXIT-curves are analytically computed for the

\(^5\)The parity check \(H\)-matrices used for the simulation in this paper can be found in [28].
and the empirical approximation of the “channel input” to the LDPC decoder is expressed as

\[ \mu_{R \rightarrow V} (d_u, d_r, \sigma^2_n) \approx \mu_{R \rightarrow V} (\gamma_{RUR}, \sigma^2_n), \]

(10)

and we observe that the approximation holds for any \( \mu_{V \rightarrow R} \neq 0 \) through the EXIT analysis in (7).

This feature is particularly important in terms of adaptiveness to user load \( d_u \). For instance, if an LDPC code along with the repetition factor \( d_r \) is optimized for a \( d_u \) user IDMA system operating at the SNR of 0 dBi; If the number of active users \( d_u \) decreases or increases, the LDPC decoder “sees” a better or worse channel respectively, which drifts the system operating point away from capacity\(^*\) or the system collapses\(\text{7} \). If the approximation in (10) holds, we may only need to vary the \( d_r \) accordingly to maintain the constant RUR \( \gamma_{RUR} \) so that the “channel input” to the LDPC decoder remains unchanged and subsequently the coding part is decoupled from the multi-user front-end via the repetition code component.

Fig. 9 shows the gap-to-GMAC capacity where the LDPC code rate \( R_c = 0.0975 \) (optimized for \( d_u = 32, d_r = 4 \) and \( \gamma_{s,mu} = 0 \) dBi) is kept invariant. The number of users \( d_u \) and the repetition factor \( d_r \) are varied. Density evolution based on GA is used to estimate the decoding threshold for each combination of \( d_r \) and \( d_u \). Obviously, the spectral efficiency is quite invariant if

\(^6\)In contrast, the EXIT-analysis assumes convergence between MUD and REP, i.e., many iterations between MUD and REP before the message is passed to VND.

\(^7\)When the channel becomes better, the code rate of the LDPC code can be higher.

\(^8\)The optimized LDPC code at certain SNR can not operate when the real SNR becomes smaller.

\( \text{Fig. 9. GA-density evolution-based } \frac{P_{\text{ex}}}{N_0} \text{-gap to GMAC capacity for varying } \gamma_{RUR} \text{ with a same LDPC code optimized for } d_r = 4, d_u = 32 \text{ users and } \gamma_{s,mu} = 0 \text{ dB; Note that the same H-matrix is used in the simulation.} \)
Fig. 10. BER comparison of IDMA system with different repetition factor \( d_r \) and number of users \( d_u \); the LDPC code optimized for \( d_r = 4, d_u = 32 \) and \( \gamma_{s,\mu u} = 0 \) dB remains fixed; dashed black curve denotes \( \gamma_{\text{Hub}} = \frac{3}{4} \).

\( \gamma_{\text{Hub}} = \frac{3}{4} \) is kept constant. We note that this adjustment of repetition code does not incur any significant loss of spectral efficiency and can be easily reconfigured at transmitter and receiver due to its simple structure and decoding procedure. Furthermore, the simulated sum-rate and BER are shown in Fig. 10 and 11 for \( 1 \leq d_r \leq 8 \) and various \( d_u \), respectively. The sum-rate is obtained by summing up all users’ achievable mutual information after decoding. The results consolidate our empirical expression in (10) and suggest that the designed LDPC code for specific parameters can be easily generalized and reused in systems with arbitrarily varying user load by simply adjusting the repetition factor.

\[ y = \sqrt{P_{\text{st}}} \sum_{i=1}^{d_u/d_{\mu u}+1} \tilde{x}_i + \sqrt{P_{\text{weak}}} \sum_{j=d_{\mu u}+1}^{d_u} \tilde{x}_j + n \]

where half of the users, i.e., \( \frac{d_u}{2} \), have higher power than the other half number of users, i.e., \( P_{\text{st}} > P_{\text{weak}} \). The total power is normalized to one, i.e.,

\[ P_{\text{st}} + P_{\text{weak}} = \frac{2}{d_u} \]

and we define the “power asymmetry” as \( P_A = \frac{P_{\text{st}}}{P_{\text{weak}}} \).

In Fig. 12 we show the BER performance of an unequal-power IDMA system with the power asymmetry between strong users and weak users of \( P_A = 3 \) dB. However, the LDPC code remains unchanged (\( R_c = 0.125 \) and \( d_r = 4 \)). As the LDPC code is optimized

**V. BEYOND EQUAL POWER AND RATE**

Different users’ signals arrive at the base-station (BS) with different power levels since the path loss between the BS and individual users may be quite different. This fact indicates that each user’s optimum rate shall be adapted to its power level (or the BS regulates each user’s transmit power, thus equal power is achieved at the BS). We assume that each user transmits at full power for higher data rates and propose in the following two approaches to allow for multi-rate transmission, yet using the same channel code that is optimized for equal-power case.

1) **Layer aggregation.** The higher power users may divide their signals into several data layers or “virtual users”, each with a smaller power and the user with weakest power can only transmit one data layer. This approach is straightforward, thus not further discussed.

2) **Repetition code adaptation.** The serially concatenated repetition code can be further applied as a “rate equalizer” in unequal power cases. Users with higher power can reduce the repetition factor \( d_r \) and users with lower power shall increase the repetition factor.

We consider the following unequal power IDMA system
for equal power cases, the average performance becomes worse since the weak users require higher SNRs. If the repetition code adaptation is applied where the strong users use the repetition factor \( d_r = 3 \) and the weak users use the new repetition factor to \( d_r = 4 \), similar performance can be achieved compared to the equal power case as the adapted repetition is acting as a “rate/power equalizer”. An intuition for the performance gain is due to the effective codeword length is now different among users, but still using the interleaver depth of the length of the codeword length. It helps to break some loops that may be present due to the multiuser frontend (see Fig. 12). We remark that if the interleaver depth is set to the same for the equal power and unequal power with repetition code adaptation cases, very similar performance can be achieved. The extension to other power asymmetry cases is straightforward.

VI. CONCLUSION

We have studied LDPC code optimization for a wide range number of users in non-orthogonal multi-user IDMA systems. We provide a novel analytical EXIT analysis for various component decoders of the low complexity IDMA receiver. Furthermore, the difficulty of designing very low-rate LDPC codes is mitigated by a serial concatenation with a repetition code. The repetition code rate along with the LDPC degree profile is optimized based on EXIT-charts. As a result, we can achieve the GMAC capacity with different number of users within 1.3 dB SNR using an optimized LDPC code rate of 0.125 with a finite codeword length of \( 10^4 \) bit and a repetition code rate of \( 1/4 \) based on numerical simulation. Simulation results show that the design with Gaussian approximation-based EXIT-charts is accurate over a wide range of different code rates and number of users. We show that repetition codes allow to flexibly adapt the system to a varying user load and power levels without the need of re-designing the outer LDPC code while still operating close to the capacity limit, making it an attractive non-orthogonal multiple access scheme for future wireless communication systems.

APPENDIX A

PROOF OF (5)

We derive the expected interference power for an arbitrary user \( j \) with given incoming message mean \( \mu_{R \rightarrow M, i} \) from all other users. The residual interference after the SoIC can be expressed as

\[
\sigma_r^2 = \mathbb{E}_x \left[ \tilde{x}_i - \hat{x}_i \right]^2 = \phi(\mu_{R \rightarrow M, i})
\]

for BPSK [26]. With (2), we obtain

\[
\sigma_r^2 = \sum_{i \neq j} \mathbb{E}_x \left[ \cos^2 (\phi_i - \phi_j) \right] \phi(\mu_{R \rightarrow M, i})
\]

\[
\sigma_r^2 = \frac{d_u - 1}{2} \phi(\mu_{R \rightarrow M})
\]

since all users transmit with the same power, the incoming messages are of the same mean values, i.e., \( \mu_{R \rightarrow M, i} = \mu_{R \rightarrow M, j} = \mu_{R \rightarrow M} \), \( \forall i, j \).

APPENDIX B

PROOF OF (9)

The stability condition requires that the message \( \mu \) approaches infinity as the number of iterations goes to infinity. Assume that a very large \( \mu_l \) is reached after a finite number of iterations and let \( \tilde{\mu}_l \) denote

\[
\tilde{\mu}_l = \mu_l + \frac{4}{d_u \sigma_n^2 + (d_u - 1) \cdot \phi(2 \cdot \mu_l)}.
\]

The left term of (9) is lower-bounded by Taylor-series by

\[
\text{left term of (9)} \geq 1 - \lambda_2 \phi' \left( \tilde{\mu}_l \right) \Delta \mu
\]

\[
+ \frac{8 \lambda_2 \phi' \left( 2 \mu_l \right)}{(d_u \sigma_n^2 + (d_u - 1) \cdot \phi(2 \cdot \mu_l))^2} \Delta \mu
\]

and when \( \mu_l \to \infty \), we have \( \tilde{\mu}_l \to \mu_l + \frac{4}{d_u \sigma_n^2} \) and the left term can be written as

\[
1 - \lim_{\mu_l \to \infty} \lambda_2 \phi' \left( \mu_l + \frac{4}{d_u \sigma_n^2} \right) \cdot \Delta \mu.
\]
Similarly, the right term of (9) is approximated by
\[
1 - \frac{1}{d_c - 1} \lim_{\mu_i \to \infty} \phi'(\mu_i) \cdot \Delta \mu.
\]
Therefore, we obtain
\[
\lambda_2 \leq \frac{1}{d_c - 1} \cdot \lim_{\mu_i \to \infty} \phi'(\mu_i) = \frac{\mu_{ch}}{d_c - 1}
\]
We note that \( \phi(\mu) \approx \sqrt{\frac{e}{\mu}}\) is a tight approximation when \( \mu \) is large.

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