Resolution of singularities in foliated spaces

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2012

Abstract

Let $M$ be an analytic manifold over $\mathbb{R}$ or $\mathbb{C}$, $\theta$ an involutive singular distribution and $\mathcal{I}$ a coherent ideal sheaf defined on $M$. The aim of the work is to study the existence of a resolution of singularities of $\mathcal{I}$ in which nice properties of $\theta$ are preserved. This problem has a strong connection with resolution for families of ideal sheafs, equiresolution and monomialization of maps. We introduce a notion of admissible center, called $\theta$-admissible center, which is well-adapted to the singular distribution. We prove two Theorems of resolution of singularities under $\theta$-admissible centers: the first when $\mathcal{I}$ is invariant by $\theta$; the second without restrictions on $\mathcal{I}$ but assuming the leaf dimension of $\theta$ equals to one.

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1 Introduction

1.1 Motivation

Resolution of singularities is a powerful tool for studying singular varieties. It fulfills the desire to give a geometrical local description of a variety in the vicinity of its singularities. In this sense, reduction of singularities is the analogous tool for singular foliations. But, we remark that the existence of a reduction of singularities is still unknown for ambient spaces with dimension bigger than 3. We refer to [Hi, Sc, Ben] for landmarks in these subjects and [Ko, Ca, P] for modern expositions and results.

In applications, we may need to combine both topics. More precisely, suppose that we have an ambient space containing a singular variety and a singular foliation, and that the object of study is the “interaction” between them. It would be important to understand the local properties of each object and how they relate. As a first step into this study we may try to resolve the singular variety in a way that do not make the foliation “worse”. Provided that the starting foliation was already locally well-understood, this process would be sufficient. This leads to the following (informal) problem:

**Problem:** Can we obtain a resolution of singularities for a variety that preserves good conditions of an ambient foliation?

We formalize this problem in the next subsection. It is closely related with equiresolution, monomialization of maps and resolution on families of ideal sheafs. We briefly discuss applications to these subjects in subsection 1.4. We remark that an application to a dynamical system problem proposed by J-F Mattei can be found in chapter 8 of the thesis [Bel].

1.2 Results and Main ideas

A *foliated analytic manifold* is a triple $(M, \theta, E)$:
• $M$ is a smooth analytic manifold of dimension $n$ over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$);

• $E$ is an ordered collection $E = (E^{(1)}, \ldots, E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on $M$ such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings;

• $\theta$ is an involutive singular distribution defined over $M$ and everywhere tangent to $E$.

We recall the basic notions of singular distributions (we follow closely [BB]). Let $\text{Der}_M$ denote the sheaf of analytic vector fields over $M$, i.e. the sheaf of analytic sections of $TM$. An involutive singular distribution is a coherent sub-sheaf $\theta$ of $\text{Der}_M$ such that for each point $p$ in $M$, the stalk $\theta_p := \theta.\mathcal{O}_p$ is closed under the Lie bracket operation.

Consider the quotient sheaf $Q = \text{Der}_M/\theta$. The singular set of $\theta$ is defined by the closed analytic subset $S = \{p \in M : Q_p$ is not a free $\mathcal{O}_p$ module\}. A singular distribution $\theta$ is called regular if $S = \emptyset$. On $M \setminus S$ there exists an unique analytic subbundle $L$ of $TM|_{M \setminus S}$ such that $\theta$ is the sheaf of analytic sections of $L$. We assume that the dimension of the $\mathbb{K}$ vector space $L_p$ is the same for all points $p$ in $M \setminus S$ (this always holds if $M$ is connected). It will be called the leaf dimension of $\theta$ and denoted by $d$. In this case $\theta$ is called an involutive $d$-singular distribution and $(M, \theta, E)$ a $d$-foliated manifold.

A coherent set of generators of $\theta_p$ is a set $\{X_1, \ldots, X_{d_p}\}$ of $d_p \geq d$ vector fields germs with representatives defined in a neighborhood $U_p$ of $p$ such that $\{X_1, \ldots, X_{d_p}\}.\mathcal{O}_q$ generates $\theta_q$ for every point $q$ in $U_p$.

We recall that a blowing-up $\sigma : (M', E') \rightarrow (M, E)$ is admissible if the center $\mathcal{C}$ is a closed and regular sub-manifold of $M$ that has simple normal crossings with $E$ (see pages 137-138 of [Ko] for details).

We introduce a natural transform of $\theta$ under admissible blowing-up called adapted analytic strict transform. It is an involutive singular distribution $\theta'$, everywhere tangent to $E'$, obtained as a suitable extension of the pull-back of $\theta$ from $M \setminus \mathcal{C}$ to $M' \setminus \sigma^{-1}(\mathcal{C})$. The precise definition is given in subsection 2.2; we stress that, in general, it is neither the strict nor the
total transform of \( \theta \). We denote an admissible blowing-up by:

\[
\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)
\]

A foliated ideal sheaf is a quadruple \((M, \theta, I, E)\) where:

- \((M, \theta, E)\) is a foliated manifold;
- \(I\) is a coherent and everywhere non-zero ideal sheaf of \(M\).

The support of \(I\) is the subset:

\[
V(I) := \{p \in M; I.\mathcal{O}_p \subset m_p\}
\]

where \(m_p\) is the maximal ideal of the structural ring \(\mathcal{O}_p\).

An ideal sheaf \(I\) is invariant by a singular distribution \(\theta\) if \(\theta[I] \subset I\), where \(\theta\) is regarded as a set of derivations taking action over \(I\). An analytic sub-manifold \(N\) is invariant by a singular distribution \(\theta\) if the reduced ideal sheaf \(I_N\) that generates \(N\) (i.e. \(V(I_N) = N\)) is invariant by \(\theta\).

We say that an admissible blowing-up \(\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)\) is of order one on \((M, \theta, I, E)\) if the center \(C\) is contained in the variety \(V(I)\) (see definition 3.65 of \([Ko]\) for details). In this case, the controlled transform of the ideal sheaf \(I\) is the coherent and everywhere non-zero ideal sheaf \(I^c := \mathcal{O}(-F)(I.\mathcal{O}_{M'})\), where \(F\) stands for the exceptional divisor of the blowing-up (see subsection 3.58 of \([Ko]\)) and the total transform of of the ideal sheaf \(I\) is the coherent and everywhere non-zero ideal sheaf \(I^* := I.\mathcal{O}_{M'}\). Finally, an admissible blowing-up of order one of the foliated ideal sheaf is the mapping:

\[
\sigma : (M', \theta', I', E') \longrightarrow (M, \theta, I, E)
\]

where the ideal sheaf \(I'\) is the controlled transform of \(I\).
A resolution of a foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) is a sequence of admissible blowing-ups of order one:

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)
\]

such that \(\mathcal{I}_r = \mathcal{O}_{M_r}\) (i.e. each blowing-up \(\sigma_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \to (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1})\) is admissible of order one). In particular, \(\mathcal{I}.\mathcal{O}_{M_r}\) is the ideal sheaf of a SNC divisor on \(M_r\) with support contained in \(E_r\).

Our main objective is to find a resolution algorithm that preserves as much as possible “good” properties that the singular distribution \(\theta\) might posses. For example, one could ask if, assuming that the singular distribution \(\theta\) is regular (i.e. \(S(\theta) = \emptyset\)), there exists a resolution of the foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) such that the singular distribution \(\theta_r\) is regular. Unfortunately, it is easy to get examples of foliated ideal sheafs whose resolution necessarily breaks the regularity of a regular distribution:

**Example:** Let \((M, \theta, \mathcal{I}, E) = (\mathbb{C}^2, \frac{\partial}{\partial x}, (x,y), \emptyset)\): the only possible strategy for a resolution is to blow-up the origin, which breaks the regularity of the distribution.

The next best thing is a (locally) monomial singular distribution: given a ring \(R\) such that \(\mathbb{Z} \subset R \subset \mathbb{K}\), a \(d\)-singular distribution \(\theta\) is \(R\)-monomial at \(p \in M\) if there exists a local coordinate system \(x = (x_1, \ldots, x_n)\) and a coherent set of generators \(\{X_1, \ldots, X_d\}\) of \(\theta_p\) such that:

- Either \(X_i = \frac{\partial}{\partial x_i}\), or;
- \(X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}\) with \(\alpha_{i,j} \in R\).

A singular distribution is \(R\)-monomial if it is \(R\)-monomial at every point \(p \in M\) (see section 2.1 for details).

The main problem of this work can now be enunciated rigorously:
Problem: Given a foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) such that the singular distribution \(\theta\) is \(R\)-monomial, is there a resolution of \((M, \theta, \mathcal{I}, E)\):

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)
\]
such that the singular distribution \(\theta_r\) is also \(R\)-monomial?

Remark 1.2.1. A positive answer to this problem would lead to new ways of attacking some well-known problems, such as resolution in families, equiresolution and monomialization of maps (see subsection 1.4).

We will basically prove the following:

- If the ideal sheaf \(\mathcal{I}\) is invariant by the singular distribution \(\theta\) (i.e. \(\theta[\mathcal{I}] \subset \mathcal{I}\)), then there exists a resolution that preserves regularity and \(R\)-monomiality (see Theorem 4.1.1);

- If the leaf dimension of the singular distribution \(\theta\) is one, then there exists a resolution that preserves \(R\)-monomiality (see Theorem 5.1.1).

In order to be precise, we define the notion of local foliated ideal sheafs as quintuples \((M, M_0, \theta, \mathcal{I}, E)\):

- \((M, \theta, \mathcal{I}, E)\) is a foliated ideal sheaf;

- \(M_0\) is an open relatively compact subset of \(M\).

A resolution of a local foliated ideal sheaf \((M, M_0, \theta, \mathcal{I}, E)\) is a resolution of the foliated ideal sheaf \((M_0, \mathcal{I}_0, \theta_0, E_0) := (M_0, \mathcal{I}.\mathcal{O}_{M_0}, \theta.\mathcal{O}_{M_0}, E \cap M_0)\). With this notation, we present the main Theorems of this work in their simplest forms:

**Theorem 1.2.2.** Let \((M, M_0, \theta, \mathcal{I}, E)\) be a local \(d\)-foliated ideal sheaf and suppose that \(\mathcal{I}_0\) is \(\theta_0\)-invariant. There exists a resolution of \((M, M_0, \theta, \mathcal{I}, E)\):

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]
such that:
i ) If $\theta_0$ is $R$-monomial, then so is $\theta_r$; 

ii ) If $\theta_0$ is regular, then so is $\theta_r$.

**Theorem 1.2.3.** Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local $d$-foliated ideal sheaf and suppose that $\theta$ has leaf dimension equal to 1. There exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that, if $\theta_0$ is $R$-monomial, then so is $\theta_r$.

In fact Theorems 1.2.2 and 1.2.3 are corollaries of the more general Theorems 4.1.1 and 5.1.1 where we also prove the functorality of the resolution for a certain kind of morphism called *chain-preserving smooth morphisms* (see section 2.5 for the definition).

**Remark 1.2.4.** Although all proofs and results of this paper are set in the analytic category, we see no reason why it could not be done in the algebraic category. The proofs should follow the same arguments, but with different local technicalities. We stress that we have not verified in detail this adaptation.

For proving these Theorems, we introduce a new kind of blowing-up center and a new invariant:

- **The $\theta$-admissible center**: Intuitively, a center $C$ is $\theta$-admissible at a point $p$ in $C$ (see subsection 3.1 for the precise definition) if there exists a local decomposition $\theta_p = \theta_{tr} + \theta_{inv}$ (as $\mathcal{O}_p$-modules) of the singular distribution $\theta_p$ into two singular distributions $\{\theta_{tr}, \theta_{inv}\}$ such that:
  
  - The singular distribution $\theta_{tr}$ is totally transversal to $C$, i.e. no vector of $T_pC$ is contained in the subspace of $T_pM$ generated by $\theta_{tr}$;
  
  - The singular distribution $\theta_{inv}$ is everywhere tangent to $C$, i.e. the center $C$ is invariant by $\theta_{inv}$.
Later, we formalize this intuitive interpretation (see Proposition 3.4.1). An admissible blowing-up with a $\theta$-admissible center is called a $\theta$-admissible blowing-up. This notion is defined for arbitrary singular distributions, but is particularly important for $R$-monomial singular distributions because of the following result:

**Theorem 1.2.5.** Let $(M, \theta, E)$ be a $R$-monomial $d$-foliated manifold and:

$$\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$$

a $\theta$-admissible blowing-up. Then $\theta'$ is $R$-monomial.

Which is proved in section 3 Theorem 3.1.1

**Remark 1.2.6.** The definition of $\theta$-admissible center seems to have a much wider range of application. For example, if $\theta$ has leaf dimension one and has only canonical singularities (see 1.1.2 of [Mc] for the definition), then $\theta'$ has only canonical singularities if, and only if, the blowing-up is $\theta$-admissible (this follows from fact 1.2.8 of [Mc] and Proposition 3.4.1 below).

- **The $tg$-order:** In addition to the classical invariants (see, for example, the works of [Hi, BM1, V1, W1, W2, Ko]), we introduce a new invariant called the $tg$-order (abbreviation for tangency order) attached to each point $p$ in $M$ and denoted by $\nu_p(\theta, \mathcal{I})$ (see section 2.3 for the precise definition). This invariant gives a measure of the order of tangency between an ideal sheaf $\mathcal{I}$ and a singular distribution $\theta$, even if the objects are singular. When $\theta = \text{Der}_M$, the order of tangency coincides with the usual multiplicity of the ideal sheaf.

### 1.3 Example

We give a simple example in order to illustrate the difficulty of the problem. We work over the $\mathbb{Z}$-monomial foliated ideal sheaf $(M, \theta, \mathcal{I}, E) = (\mathbb{C}^3, \theta, \mathcal{I}, \emptyset)$, where $\theta$ is a $\mathbb{Z}$-monomial singular distribution generated by the regular vector-field $X = \frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$ and $\mathcal{I}$ is an ideal generated by $(x, y)$.
If we consider the admissible blowing-up of order one \( \sigma : (M', \theta', \mathcal{I}', E') \rightarrow (M, \theta, \mathcal{I}, E) \) with center \( \mathcal{C} = V(x, y) \) we obtain a resolution of \( \mathcal{I} \). By another hand, the transform of the singular distribution \( \theta \) (in this case, the adapted analytic strict transform and the strict transform coincides) restricted to the \( x \)-chart is generated by the vector-field:

\[
X' = x \frac{\partial}{\partial z} + z(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y})
\]

which is not \( \mathbb{Z} \)-monomial (indeed the linear part is nilpotent). So, this naive attempt breaks \( \mathbb{Z} \)-monomiality. Intuitive, this happens because the center \( \mathcal{C} \) is tangent to the orbit of the vector field \( X \) at the origin and, thus, \( \mathcal{C} \) is not \( \theta \)-admissible.

So, let \( \sigma : (M', \theta', \mathcal{I}', E') \rightarrow (M, \theta, \mathcal{I}, E) \) be the admissible blowing-up of order one with center \( \mathcal{C} = V(x, y, z) \). The only interesting chart is the \( z \)-chart, where we obtain:

\[
I^* = (x' z', y' z') \\
X^* = \frac{1}{z} (z' \frac{\partial}{\partial z} - x' \frac{\partial}{\partial x} - y' \frac{\partial}{\partial y}) + \frac{\partial}{\partial x} \\
I' = (x', y') \\
X' = z' \frac{\partial}{\partial z} + (z' - x') \frac{\partial}{\partial x} - y' \frac{\partial}{\partial y}
\]

where \( I^* \) and \( X^* \) stands for the pull-back of the ideal sheaf and the vector-field respectively. We claim that \( \theta' \) is \( \mathbb{Z} \)-monomial. Indeed, if we consider the change of coordinates:

\[
(\tilde{x}, \tilde{y}, \tilde{z}) = (2x' - z', y', z')
\]

we obtain that the vector-field \( X^* \) on this new coordinates is given by:

\[
X' = \tilde{z} \frac{\partial}{\partial \tilde{z}} - \tilde{x} \frac{\partial}{\partial \tilde{x}} - \tilde{y} \frac{\partial}{\partial \tilde{y}}
\]

Now, let \( \sigma : (M'', \theta'', \mathcal{I}'', E'') \rightarrow (M', \theta', \mathcal{I}, E') \) be the admissible blowing-up of order one with center \( \mathcal{C}' = V(x', y') \). Once again, we obtain a resolution of \( \mathcal{I}' \) that breaks \( \mathbb{Z} \)-monomiality because the singular distribution \( \theta' \) restricted to the \( x' \)-chart is generated by the vector-field:

\[
X'' = x'' z'' \frac{\partial}{\partial z''} + (z'' - x'') x'' \frac{\partial}{\partial x''} - z'' y'' \frac{\partial}{\partial y''}
\]

which is not \( \mathbb{Z} \)-monomial (indeed the linear part is nilpotent). Intuitive, this happens because the vector-field \( X' \) is singular in the origin and transverse to the center \( \mathcal{C}' \) everywhere.
else and, thus, $C'$ is not $\theta'$-admissible.

So, let $\sigma : (\mathcal{M}', \theta', \mathcal{I}', E') \rightarrow (\mathcal{M}', \theta', \mathcal{I}', E')$ be the admissible blowing-up of order one with center $C' = V(x', y', z')$. The only interesting chart is the $z'$-chart, where we obtain:

$$I'' = (x'', y'') \quad X'' = z'' \frac{\partial}{\partial z''} + (1 - 2x'') \frac{\partial}{\partial x''} - 2y'' \frac{\partial}{\partial y''}$$

We leave to the reader the verification that $X''$ is $Z$-monomial. We finally claim that a third blowing-up with center $C'' = V(x'', y'')$ gives a resolution of $I''$ such that $\theta''$ is $Z$-monomial. The crucial intuitive reason is that the vector-field $X''$ is everywhere transverse to the center $C''$ which implies that $C$ is $\theta$-admissible. We leave the details to the interested reader.

### 1.4 Applications and Related Problems

In this subsection we present one application and two related themes of research. We remark that a further application concerning a dynamical system problem may be found in the thesis [Bel], section 8.

**Application: Resolution in families.**

A smooth family of ideal sheafs is given by a quadruple $(B, \Lambda, \pi, \mathcal{I})$ where:

- The ambient space $B$ and the parameter space $\Lambda$ are two smooth analytic manifolds;
- The morphism $\pi : B \rightarrow \Lambda$ is smooth;
- The ideal sheaf $\mathcal{I}$ is coherent and everywhere non-zero over $B$.

Given $\lambda \in \Lambda$, the set $\pi^{-1}(\lambda)$ is a regular sub-manifold of $B$ called fiber. A point $\lambda_0 \in \Lambda$ is called an exceptional value of a smooth family of ideal sheaf $(B, \Lambda, \pi, \mathcal{I})$ if the fiber $\pi^{-1}(\lambda_0)$ is contained in $V(\mathcal{I})$.

Many works have addressed resolution process for families of ideal sheafs. By this, we intuitively mean a resolution of $(B, \mathcal{I}, \emptyset)$ that preserves, in some way, the structure of family. The precise meaning of resolution in families is not unique in the literature (see e.g
In the context of this work, a smooth family of ideal sheaves \((B, \Lambda, \pi, \mathcal{I})\) gives rise to a foliated ideal sheaf \((B, \theta, \mathcal{I}, \emptyset)\), where \(\theta\) is the maximal regular distribution such that \((D\pi)\theta = 0\). This motivates another possible definition of resolution in families:

**Uniform Resolution in Families of Ideal Sheaves:** An *uniform resolution* of a smooth family of ideal sheaves \((B, \Lambda, \pi, \mathcal{I})\) is a sequence \(\tilde{\sigma} = (\sigma_1, \ldots, \sigma_r)\) of admissible blowings-up of order one:

\[
(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B, \theta, \mathcal{I}, \emptyset)
\]

such that \(\mathcal{I}_r = \mathcal{O}_{B_r}\) and \(\theta_r\) is \(\mathbb{Z}\)-monomial.

This kind of resolution in families has originally been introduced at [DR] in the context of smooth families of planar foliations by curves, where it is an essential step in Roussarie’s program for the existential part of Hilbert 16th Problem. This approach is also similar to the one adopted at [V2], where it is proved the existence of an uniform resolution in families for the case \(\dim \Lambda = 1\) (under the hypotheses that the morphism \(\pi\) is flat over \(V(\mathcal{I})\)).

The existence of an uniform resolution in families would give rise to a resolution (in some sense) “uniform” in the parameter space. In particular, the study of the fibers of the resolution (i.e. of the morphism \(\sigma_r \circ \ldots \circ \sigma_1 \circ \pi\)) may be useful for equiresolution and bifurcation theory. In particular, it might give rise to a stratification of the parameter space in the same sense given in [ENV].

With the results of this work, we can prove the existence of an uniform resolution for a smooth family of ideal sheaves when \(\dim \Lambda = \dim B - 1\) (it is a trivial consequence of Theorem 5.1.1). Furthermore, we can eliminate exceptional values of a smooth family of ideal sheaves preserving the family structure (see the Theorem 1.4.1 below). This can be seen as a first step in the solution of the problem of uniform resolution in families.
**Theorem 1.4.1.** Let \((B, \Lambda, \pi, \mathcal{I})\) be a smooth family of ideal sheafs such that all fibers are connected. Then, there exists a smooth family of ideal sheafs \((B', \Lambda', \pi', \mathcal{I}')\) and two proper analytic maps \(\sigma : B' \to B\) and \(\tau : \Lambda' \to \Lambda\) such that:

i ) The smooth family of ideal sheafs \((B', \Lambda', \pi', \mathcal{I}')\) has no exceptional value;

ii ) The following diagram:

\[
\begin{array}{ccc}
B' & \xrightarrow{\pi'} & \Lambda' \\
\downarrow{\sigma} & & \downarrow{\tau} \\
B & \xrightarrow{\pi} & \Lambda
\end{array}
\]

commutes;

iii ) For any relatively compact open subset \(B_0\) of \(B\), there exists a sequence of invariant admissible blowings-up of order one for \((B, B_0, \theta, \mathcal{I}, \emptyset)\):

\[
(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that \(\sigma|_{\pi^{-1}B_0} = \sigma_1 \circ \cdots \circ \sigma_r\) and \(\mathcal{I}', \mathcal{O}_{B_r} = \mathcal{I}_r\);

iv ) For any relatively compact open subset \(\Lambda_0\) of \(\Lambda\), there exists a sequence of admissible blowings-up:

\[
(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)
\]

such that \(\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ \cdots \circ \tau_r\).

The proof of this Theorem can be find in the Appendix section 6.

**Related Problem:** Equiresolution.
Classically, a equiresolution is a concept that is defined independently of a family struc-
ture (we refer to [V3] for the classical definition). Nevertheless, in [ENV] the authors shift
the focus from classical equiresolutions to resolution of families.

Given a smooth family of ideal sheafs \((B, \Lambda, \pi, \mathcal{I})\), an *equiresolution* of \((B, \Lambda, \pi, \mathcal{I})\) is a
sequence of admissible blowing-ups \(\bar{\sigma} = (\sigma_1, ..., \sigma_r)\) of order one:

\[
(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B, \text{Der}_{B}, \mathcal{I}, \emptyset)
\]

such that:

- The morphism \(\pi' = \pi \circ \sigma_1 \circ ... \circ \sigma_r : B_r \rightarrow \Lambda\) is smooth;
- The varieties \(V(\mathcal{I}_r) \cap (\pi')^{-1}(\lambda)\) are smooth varieties for all \(\lambda \in \Lambda\);
- Let \(E_r = (E^{(1)}_r, ..., E^{(s)}_r)\) and consider \(\vec{i}_t = (i_1, ..., i_t)\), where \(t\) is a number smaller or
equal to \(s\) and each \(i_j\) is also a number smaller or equal to \(s\). We denote by \(F_{\vec{i}_t}\) the
intersection of divisors:

\[
F_{\vec{i}_t} = E^{(i_1)} \cap ... \cap E^{(i_t)}
\]

With this notation, the restricted morphism \(\pi \circ \sigma_1 \circ ... \circ \sigma_r : F_{\vec{i}_t} \rightarrow \Lambda\) is smooth for
all indexes \(\vec{i}_t\).

In [ENV], the authors describe the necessary conditions that a fixed resolution algorithm
needs to verify, called condition (AE), so that a equiresolution may be obtained. The in-
tuitive interpretation of the condition (AE) is the following: the centers of the resolution
algorithm should “spread evenly” over the parameter space \(\Lambda\).

In the context of this work, consider a foliated ideal sheaf \((B, \theta, \mathcal{I}, E)\) such that:

- The projection of \(\theta\) generates all the derivations of \(\Lambda\), i.e. \(d\pi(\theta) = \text{Der}_{\Lambda}\);
- The singular distribution \(\theta\) is everywhere tangent to \(E\).
We remark that the choice of such a singular distribution $\theta$ is not unique. We claim that if there exists a resolution of $(B, \theta, I, E)$ by $\theta$-invariant centers, then there exists a equiresolution of the family. This interpretation may help the identification of algorithms respecting condition $(AE)$. In other words, we believe that the techniques of this work may help to find a resolution of singularities satisfying condition $(AE)$, provided that such a resolution exists. We hope to discuss more about this idea in a forthcoming paper.

**Related Problem: Monomialization of maps.**

An analytic map $\Phi : M \rightarrow N$ is *monomial* if at every point $p$ in $M$, there exists a system of coordinates $(x) = (x_1, \ldots, x_m)$ over $O_p$ and $(y) = (y_1, \ldots, y_n)$ of $O_{\Phi(p)}$ such that:

$$
\Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x)) = (\prod_{j=1}^{n} x_j^{q_{1,j}}, \ldots, \prod_{j=1}^{n} x_j^{q_{n,j}})
$$

where the exponents $q_{i,j}$ are natural numbers such that the matrix:

$$
\begin{bmatrix}
q_{1,1} & \cdots & q_{1,n} \\
\vdots & \ddots & \vdots \\
q_{n-d,1} & \cdots & q_{n-d,n}
\end{bmatrix}
$$

is of maximal rank.

The problem is the following (see a more precise formulation in e.g. [Ki, Bel, Cu1]): given an analytic map $\Phi : M \rightarrow N$ such that $d\Phi$ is generically of maximal rank then, up to a sequence of blowings-up in $M$ and $N$, can we assume that the map $\Phi : M \rightarrow N$ is monomial?

The best results, up to our knowledge, are given in a series of articles of Cutkosky [Cu1, Cu2, Cu3] (where local uniformization for any dimension and global monomialization for maps from three folds to surfaces is proved) and an article of Dan Abramovich, Jan Denef and Kalle Karu [ADK] (where monomialization by modifications is proved). Nevertheless, the problem in all its generality is still not solved.
We believe that the present work may be an useful tool for tackling this problem. This believe comes from the fact that the problem of monomialization of maps is closely related to the concept of “monomialization” of Fitting-ideals, which is connected with reduction of singularities of foliations. In other words, we believe that the present result would allow to resolve ideals without losing good properties of the Fitting Ideals associated to it.

We remark that this idea seems to induce new results for low dimension (because reduction of singularities of vector-fields is only known for dimension of the ambient space smaller or equal to 3 - see [P]). We also speculate that the results of this work could lead to new ways of tackling the problem in higher dimension. We hope to discuss more about these ideas in a forthcoming paper.

2 Main Objects

In this section we present the most important objects of the work. We remark that the reader may advance to section 3 after subsections 2.1 – 2.3. Subsections 2.4 – 2.7 are important for sections 4 and 5.

2.1 The $R$-monomial singular distribution

Given a ring $R$ such that $\mathbb{Z} \subset R \subset \mathbb{K}$ and a point $p$ in $M$, we recall that a $d$-singular distribution $\theta$ is $R$-monomial at $p$ if there exists a local coordinate system $x = (x_1, ..., x_n)$ and a coherent set of generators $\{X_1, ..., X_d\}$ of $\theta_p$ such that:

- Either $X_i = \frac{\partial}{\partial x_i}$, or;
- $X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,j} \in R$.

In this case, we say that $x = (x_1, ..., x_n)$ is a $R$-monomial coordinate system and $\{X_1, ..., X_d\}$ is a $R$-monomial basis of $\theta_p$. A singular distribution is $R$-monomial if it is $R$-monomial in all points. A foliated manifold $(M, \theta, E)$ (respectively a foliated ideal sheaf $(M, \theta, I, E)$) is $R$-monomial if $\theta$ is $R$-monomial.
Examples:

- Any regular distribution is a $\mathbb{Z}$-monomial singular distribution;

- We say that a $d$-singular distribution $\theta$ is $R$-monomially integrable at $p$ if there exists a local coordinate system $x = (x_1, ..., x_n)$ and $n - d$ monomial functions $\lambda_i = \prod_{j=1}^{n} x_j^{q_{i,j}}$ for $1 \leq i \leq n - d$ with exponents $q_{i,j} \in R$ such that:
  
  - Each $\lambda_i$ is a first-integral for all vector-fields contained in $\theta_p$, and;
  
  - The matrix:

$$
(q_{i,j}) := \begin{bmatrix}
q_{1,1} & \cdots & q_{1,n} \\
\vdots & \ddots & \vdots \\
q_{n-d,1} & \cdots & q_{n-d,n}
\end{bmatrix}
$$

is of maximal rank.

Furthermore, we say that a singular distribution $\theta$ is full $R$-monomially integrable if for each point $p$ in $M$, if $X$ is a vector-field over $p$ such that $X(\lambda_i) \equiv 0$ for all $i$, then $X$ is an element of $\theta_p$.

Claim: Given a singular distribution $\theta$:

I ) If it is full $R$-monomially integrable, then it is $R$-monomial;

II ) If it is $R$-monomial, then it is $R$-monomially integrable.

The proof of the above Claim can be find in [Bel], Lemma 2.2.2.

An important property of $R$-monomiality is that it is an open condition:

Lemma 2.1.1. The $R$-monomiality is an open condition i.e. if $\theta$ is $R$-monomial at $p$ in $M$, then there exists an open neighborhood $U$ of $p$ such that $\theta$ is $R$-monomial at every point $q$ in $U$.

Proof. Let $\theta$ be a $R$-monomial $d$-singular distribution over $p \in M$. There exists an open set $U \subset M$ containing $p$, a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ defined over $U$ and a $R$-monomial basis $\{X_1, ..., X_d\}$ such that $X_i$ is defined over $U$ for all $i \leq d$. We claim that $\theta$
is $R$-monomial at every point $q \in U$.

Fix $q \in U$. There exists $\xi = (\xi_1, ..., \xi_n) \in \mathbb{K}^n$ such that $q = \xi$ in the coordinate system $x = (x_1, ..., x_n)$.

First, suppose that all vector-fields:

$$X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$$

are singular at $p$. Without loss of generality, suppose that $\xi = (\xi_1, ..., \xi_t, 0, ..., 0)$, where $\xi_i \neq 0$ for all $i \leq t$. Consider the matrix:

$$A = \begin{bmatrix}
\alpha_{1,1} & \cdots & \alpha_{1,t} \\
\vdots & \ddots & \vdots \\
\alpha_{d,1} & \cdots & \alpha_{d,t}
\end{bmatrix}$$

And let $s$ be its rank. Without loss of generality, we assume that:

$$A = \begin{bmatrix}
D & B \\
0 & 0
\end{bmatrix}$$

where $D$ is a $s \times s$-diagonal matrix, $B$ is a $s \times d - s$-matrix and both matrices have only elements in $R$. This implies that:

- $X_i = \alpha_{i,i} x_i \frac{\partial}{\partial x_i} + \sum_{j=s}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,i} \neq 0$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ for all $i > s$.

And all $\alpha_{i,j} \in R$. Now, taking the change of coordinates $(y_1, ..., y_n) = (x_1 - \xi_1, ..., x_n - \xi_n)$ we obtain:

- $X_i = \alpha_{i,i} (y_i + \xi_i) \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j} (y_j + \xi_j) \frac{\partial}{\partial y_j} + \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i > s$.

And $q = (0, ..., 0)$ at this coordinate system. We proceed with three coordinate changes:
• First change: let \( y_i = \xi_i(-1 + \exp(\alpha_{i,i} \bar{y}_i)) \) for all \( i \leq s \) and \( y_i = \bar{y}_i \) otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

\[
\frac{\partial}{\partial \bar{y}_i} = \alpha_{i,i}(y_i + \xi_i) \frac{\partial}{\partial y_i}
\]

for all \( i < s \). This implies that:

- For \( i \leq s \), we have that
  \[ X_i = \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(\bar{y}_j + \xi_j) \frac{\partial}{\partial \bar{y}_j} + \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}; \]
- For \( i > s \), we have that
  \[ X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}. \]

In what follows, we drop the bars;

• Second change: let \( y_i = -\xi_i + (\bar{y}_i + \xi_i) \exp(\sum_{j=1}^{s} \alpha_{j,i} \bar{y}_j) \) if \( s < i \leq t \) and \( \bar{y}_i = y_i \) otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

\[
\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(y_j + \xi_j) \frac{\partial}{\partial y_j}
\]

for all \( i < s \). This implies that:

- For \( i \leq s \), we have that
  \[ X_i = \frac{\partial}{\partial y_i} + \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}; \]
- For \( i > s \), we have that
  \[ X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}. \]

In what follows, we drop the bars;

• Third change: let \( y_i = \bar{y}_i \exp(\sum_{j=1}^{s} \alpha_{j,i} \bar{y}_j) \) if \( i > t \) and \( \bar{y}_i = y_i \) otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

\[
\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}
\]

for all \( i < s \) and

\[
\bar{y}_i \frac{\partial}{\partial \bar{y}_i} = y_i \frac{\partial}{\partial y_i}
\]

for \( i > t \). This implies that:

- For \( i \leq s \), we have that
  \[ X_i = \frac{\partial}{\partial y_i}; \]
- For \( i > s \), we have that
  \[ X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}. \]
which forms a $R$-monomial basis.

Now, suppose that for $i \leq r$, the vector-field $X_i$ is non-singular at $p$. Without loss of generality, $X_i = \frac{\partial}{\partial x_i}$ and $X_j(x_i) \equiv 0$ whenever $i \leq r$ and $j > r$. In particular, when we make the translation $(y_1, ..., y_n) = (x_1 - \xi_1, ..., x_n - \xi_n)$, we have that $X_i = \frac{\partial}{\partial y_i}$ for $i \leq r$.

Consider the quotient $O_U/(x_1, ..., x_r)$. It is another regular ring with a $R$-monomial singular distribution $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ that is all singular over the origin. Using the first part of the proof, there exists a change of coordinates in $O_q/(x_1, ..., x_r)$ that turns $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ into a $R$-monomial basis. Moreover, this coordinate change is invariant by the first $r$-coordinates. Taking the equivalent change in $O_q$, we conclude the Lemma.

\[ \Box \]

### 2.2 The adapted analytic strict transform of $\theta$

Let $(M, \theta, E)$ be a $d$-foliated manifold and $\sigma : (M', E') \longrightarrow (M, E)$ an admissible blowing-up with exceptional divisor $F$. Consider the sheaf of $O_{M'}$-modules $BlDer_{M'} := O(-F) \otimes_{O_{M'}} Der_{M'}$. There exists a mapping from $Der_{M'}$ to $BlDer_{M'}$:

\[ \zeta : Der_{M'} \longrightarrow BlDer_{M'} \]

which, given an open subset $U$ of $M'$, associates to a vector-field $X \in Der_{M'}(U)$ the element $\zeta(X) = 1 \otimes X \in BlDer_{M'}(U)$. Notice that this mapping is injective.

Given a sub-sheaf $\omega$ of $Der_{M'}$, we abuse notation and denote by $\zeta(\omega)$ the sub-sheaf of $BlDer_{M'}$, with the structure of a $O_{M'}$-module, generated by the image of $\omega$. Reciprocally, given a sub-sheaf $\omega$ of $BlDer_{M'}$, we denote by $\zeta^{-1}(\omega)$ the sub-sheaf of $Der_{M'}$ defined in each open set $U$ of $M'$ by the following elements:

\[ \zeta^{-1}(\omega)_U = \{ X \in Der_U; \zeta(X) \in \omega_U \} \]

Since the blowing-up $\sigma : M' \longrightarrow M$ is a morphism, it gives rise to a mapping on the structural sheafs $\sigma^* : O_M \longrightarrow O_{M'}$. Abusing notation, this morphism also gives rise to an application:

\[ \sigma^* : Der_M \longrightarrow BlDer_{M'} \]
which, given an open subset $U$ of $M$, associates to a vector-field $X$ of $\text{Der}_U$ the element $\sigma^*(X) = (\frac{1}{f} \otimes fX^*)$, where the principal ideal $(f)$ is equal to $\mathcal{O}(F)\mathcal{O}_{\sigma^{-1}(U)}$ and $X^*$ is the pull-back of the derivation (i.e. $X^*(\sigma^* f) = \sigma^* X(f)$).

Remark 2.2.1. Since the pull-back of an analytic vector-field under blowings-up have at most poles of order one, the map $\sigma^* : \text{Der}_M \rightarrow \text{BlDer}_M$ is well-defined.

The image $\sigma^*(\theta)$ is a coherent sub-sheaf of the sheaf of $\mathcal{O}_M$-modules $\text{BlDer}_M$. We remark that $\theta^*$ is also a morphism of Lie-algebras. We now define two possible transforms of $\theta$:

- The total transform of $\theta$ is given by $\theta^* := \sigma^*(\theta)$;
- The analytic strict transform of $\theta$ is given by $\theta^a := \zeta^{-1}(\theta^*)$.

Whenever $\zeta^{-1}(\theta^*)$ is isomorphic to $(\theta^*)$, we abuse notation and write $\theta^* = \zeta^{-1}(\theta^*)$.

We claim that the analytic strict transform is an involutive $d$-singular distributions (not necessarily tangent to $E'$). Indeed:

Lemma 2.2.2. The sub-sheaf $\theta^a$ is an involutive $d$-singular distribution. Moreover, consider a point $q$ of $M'$ and let $p = \sigma(q)$ and $\{X_1, ..., X_d\}$ be a coherent set of generators of $\theta_p$. Then $\theta^a_q$ has a coherent set of generators $\{Y_i, Z_j, W_k\}$ with $i = 1, ..., r$, $j = 1, ..., s$ ($r + s = d_p$) and $k = 1, ..., t$, where:

- $Y_i = (\mathcal{O}(F)X_i^*).\mathcal{O}_q$ whenever $X_i^*.\mathcal{O}_q$ is not analytic;
- $Z_j = X_j^*.\mathcal{O}_q$ whenever $X_j^*.\mathcal{O}_q$ is analytic;
- $W_k = \mathcal{O}(-F)\sum \gamma_{i,k} Y_i$ for some $\Gamma_{\theta,k} \in \mathcal{O}_U^r$ such that $W_k \notin \langle Y_i, Z_j \rangle$.

We prove this Lemma in the end of this section.

Now, consider the involutive $n$-singular distribution $\text{Der}_{M'}(-\log F)$ of $\text{Der}_M$ composed by all the derivations leaving the exceptional divisor $F$ invariant. The adapted analytic strict transform of $\theta$ is defined as $\theta^{a,ad} = \theta^a \cap \text{Der}_{M'}(-\log F)$. It follows from Oka’s Theorem that $\theta^{a,ad}$ is an involutive $d$-singular distribution.
Proof. (Lemma 2.2.2)

- Coherence: If \( q \) is a point outside the exceptional divisor \( F \), the result is clear because \( \sigma \) is a local isomorphism and, thus, \( \zeta : \theta^a_q \rightarrow \theta^s_q \) is a local isomorphism. So, consider the point \( q \) contained in \( F \) and let \( p = \sigma(q) \). If \( \{ X_1, ..., X_{d_p} \} \) is a coherent set of generators of \( \theta_p \), then it is clear that:

\[
\theta^s_q = < \sigma^*(\zeta(X_1)), ..., \sigma^*(\zeta(X_{d_p})) >. \mathcal{O}_q = < \left( \frac{1}{f} \otimes fX_1^* \right), ..., \left( \frac{1}{f} \otimes fX_{d_p}^* \right) >. \mathcal{O}_q
\]

Take \( U \) a sufficiently small neighborhood of \( q \) and \((x, y) = (x, y_1, ..., y_{n-1})\) a coordinate system such that \( f = x \) and \( \theta^a_U = < \left( \frac{1}{x} \otimes xX_1^* \right), ..., \left( \frac{1}{x} \otimes xX_{d_p}^* \right) >. \mathcal{O}_U \). Notice that whenever \( X_i^*. \mathcal{O}_U \) is an analytic vector-field: \( \left( \frac{1}{x} \otimes xX_i^*. \mathcal{O}_U \right) = (1 \otimes X_i^*. \mathcal{O}_U) \). Reorganizing the set of generators, we can suppose that \( \sigma^*_U = < \left( \frac{1}{x} \otimes Y_1 \right), ..., \left( \frac{1}{x} \otimes Y_r \right), (1 \otimes Z_1), ..., (1 \otimes Z_s) > \) where \( r + s = d_p \), \( Y_i = xX_i^*. \mathcal{O}_U \) (such that \( Y_\zeta(0, y) \not= 0 \)) and \( Z_i = X_i^*. \mathcal{O}_U \).

Let \( \mathcal{R} \) be the sub-module of relations of \( \{ Y_i|_{x=0} \} \), i.e. the \( r \)-tuples \((f_1, ..., f_r) \in \mathcal{O}^*_U \) such that \((\sum_{i=1}^r f_i Y_i)|_{x=0} \equiv 0 \). It is easy to see that this is the same sub-module of relations of \( \{ Y_\zeta(x)|_{x=0}, Y_\zeta(y)|_{x=0} \}_{i \leq r, j \leq n-1} \). Thus, by the Oka’s Theorem (see Theorem 6.4.1 of [Ho]), \( \mathcal{R} \) is finitely generated: \( \mathcal{R} = (F_1, ..., F_t) \) where \( F_i = (f_{1,i}, ..., f_{r,i}) \).

In particular, for every \( j \leq t \), \( \sum f_{i,j} Y_i \) is divisible by \( x \). So, for each \( F_j \), we have that:

\[
\sum_{i=1}^r (f_{i,j} \otimes Y_i) = \left( \frac{1}{x} \otimes \sum_{i=1}^r f_{i,j} Y_i \right) =: (1 \otimes W_j)
\]

We claim that \( \{ Y_i, Z_j, W_k \}_{i \leq r, j \leq s, k \leq t} \) generates \( \theta^a_U \), which implies the coherence. Indeed, consider \( X \in \theta^a_U \): we only need to check that \( \zeta(X) \in \{ \zeta(Y_i), \zeta(Z_j), \zeta(W_k) \}_{i \leq r, j \leq s, k \leq t} \).

We know there exists \( \alpha \in \mathcal{O}^*_U \) and \( \beta \in \mathcal{O}^*_U \) such that:

\[
\zeta(X) = (1 \otimes X) = \sum \alpha_i \left( \frac{1}{x} \otimes Y_i \right) + \sum \beta_j (1 \otimes Z_j)
\]

Now, \( \alpha_i = x\tilde{\alpha}_i(x, y) + \tilde{\alpha}_i(y) \) and thus:

\[
\zeta(X) = \sum \tilde{\alpha}_i(x, y)(1 \otimes Y_i) + \sum \beta_j (1 \otimes Z_j) + \sum \tilde{\alpha}_i(y) \left( \frac{1}{x} \otimes Y_i \right)
\]

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It is clear that \( \sum \alpha_i(y)Y_i \) is divisible by \( x \). This implies that \( (\bar{\alpha}) \subset R \). So, there exists \( \gamma \in O_U^t \) such that \( (\bar{\alpha}) = \sum \gamma_k F_k \). This finally implies that:

\[
\zeta(X) = \sum \tilde{\alpha}(x,y)(1 \otimes Y_i) + \sum \beta_j(1 \otimes Z_j) + \sum \gamma_k(1 \otimes W_k)
\]

- **Involutive:** For any point \( q \) of \( M' \), consider vector-fields \( X \) and \( Y \) contained in \( \theta^a_q \). Then the elements \( \zeta(X) \) and \( \zeta(Y) \) are contained in \( \theta^a_q \). Since \( \theta^a_q \) is closed under Lie brackets, necessarily \( [\zeta(X), \zeta(Y)] \in \theta^a_q \) and since the Lie bracket of two analytic derivations is still an analytic derivation, we deduce that \( [X, Y] \in \theta^a_q \).

- **Leaf dimension:** Since the blowing-up \( \sigma : M' \to M \) and the morphism \( \zeta : \theta^a \to \theta^* \) are local isomorphisms in an open and dense set, \( \theta^a \) has also leaf dimension \( d \).

\[ \square \]

### 2.3 Generalized \( k \)-Fitting Operation

Let \( (M, \theta, E) \) be a foliated manifold. The **generalized \( k \)-Fitting operation** (for \( k \leq d \)) is a mapping \( \Gamma_{\theta,k} \) that associates to each coherent ideal sheaf \( \mathcal{I} \) over \( M \) the ideal sheaf \( \Gamma_{\theta,k}(\mathcal{I}) \) whose stalk at each point \( p \) in \( M \) is given by:

\[
\Gamma_{\theta,k}(\mathcal{I}).O_p = < \{ \det[X_i(f_j)]_{i,j \leq k}; X_i \in \theta_p, f_j \in \mathcal{I}.O_p \} >
\]

where \( < S > \) stands for the ideal generated by the subset \( S \subset O_p \). The operation \( \Gamma_{\theta,1} \) will play an important role in this work and, for simplifying the notation, we denote it by \( \theta[\mathcal{I}] \).

**Remark 2.3.1.** If \( \mathcal{I} \) is a coherent ideal sheaf, then \( \Gamma_{\theta,k}(\mathcal{I}) \) is also coherent for every \( k \leq d \). This follows from the coherence of the singular distribution \( \theta \).

**Remark 2.3.2.** In this work, we mainly use the ideals sheaf of the form \( < \Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I} > \). In particular, we notice that if \( \theta = \text{Der}_M \) then the ideal sheaf \( < \Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I} > \) coincides with the usual \( k \)-Fitting ideal sheaf (see [Ko] for details on derivative ideal sheafs).

**Remark 2.3.3.** If \( \theta = \text{Der}_M \), the generalized 1-Fitting ideal sheaf coincides with the derivative ideal (see chapter 3.7 of [Ko] for details on derivative ideal sheafs).

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Lemma 2.3.4. A $d$-singular distribution $\theta$ is regular at a point $p$ in $M$ if, and only if, 

$$< \Gamma_{\theta,d}(m_p) + m_p > = \mathcal{O}_M,$$

where $m_p$ stands for the maximal ideal of the structural ideal $\mathcal{O}_p$.

Proof. First suppose that $\theta$ is a regular distribution in a point $p$ of $M$. In this case, there exists a coordinate system $x = (x_1, \ldots, x_n)$ of $\mathcal{O}_p$ and a coherent set of generators $\{X_1, \ldots, X_d\}$ of $\theta_p$ which, by the flow-box Theorem, can be assumed to be equal to $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$. Now, it is clear that the determinant of the matrix:

$$\begin{vmatrix}
X_1(x_1) & \ldots & X_1(x_d) \\
\vdots & \ddots & \vdots \\
X_d(x_1) & \ldots & X_d(x_d)
\end{vmatrix}$$

is one. Thus, $\Gamma_{\theta,d}(m_p)\mathcal{O}_p$ is equal to $\mathcal{O}_p$, which implies that $< \Gamma_{\theta,d}(m_p) + m_p >$ is equal to $\mathcal{O}_M$.

Now, suppose that $< \Gamma_{\theta,d}(m_p) + m_p >$ is equal to $\mathcal{O}_M$. This implies that $\Gamma_{\theta,d}(m_p)\mathcal{O}_p$ is equal to $\mathcal{O}_p$. So, there exists a coherent set of generators $\{X_1, \ldots, X_{d_p}\}$ of $\theta_p$ and a collection of functions $\{f_1, \ldots, f_d\} \subset m_p$ such that the determinant of the matrix:

$$\begin{vmatrix}
X_1(f_1) & \ldots & X_1(f_d) \\
\vdots & \ddots & \vdots \\
X_d(f_1) & \ldots & X_d(f_d)
\end{vmatrix}$$

is an unity of $\mathcal{O}_p$. In particular, this implies that the vector-fields $\{X_1, \ldots, X_d\}$ are regular and generates linearly independent vectors of $T_pM$. Since the leaf-dimension of $\theta$ is $d$, we conclude that $d_p$ may be taken equal to $d$ and the singular distribution $\theta$ is regular. \qed

Given a coherent ideal sheaf $\mathcal{I}$, we say that:

- $\mathcal{I}$ is invariant by $\theta$ or $\theta$-invariant if $\theta[\mathcal{I}] \subset \mathcal{I}$;

- $\mathcal{I}$ is totally transverse to $\theta$ or $\theta$-totally transverse if $\Gamma_{\theta,d}(\mathcal{I}) = \mathcal{O}_M$.

The following Lemma studies the behavior of the $k$-generalized Fitting Operations under blowings-up and is a crucial result for sections 4 and 5:
Lemma 2.3.5. Let \( \sigma : (M', \theta', E') \to (M, \theta, E) \) be an admissible blowing-up over a foliated ideal sheaf \((M, \theta, I, E)\). Then:

- \([\Gamma_{s, \theta}(I)]^* \subset \Gamma_{s, \theta^*}(I^*)\);
- \([\Gamma_{s, \theta}(I + I)]^* = \Gamma_{s, \theta^*}(I^*) + I^*\).

for all \(s \leq d\).

Remark 2.3.6. In the above Lemma, if \(\theta^*\) is a meromorphic singular distribution, there is a natural way to extend the definition of the operation \(\Gamma_{s, \theta^*}\) to the sheaf of meromorphic functions over \(M\).

Proof. Notice that, since \(\sigma^*: \text{Der}_M \to \text{BlDer}_M\) is a morphism, it is clear that:

\([\Gamma_{s, \theta}(I)]^* \subset \Gamma_{s, \theta^*}(I^*)\)

And, in particular \([\Gamma_{s, \theta}(I + I)]^* \subset \Gamma_{s, \theta^*}(I^*) + I^*\). To prove the other inclusion, fix a point \(q\) of \(M'\), let \(p = \sigma(q)\) and consider a coherent set of generators \(\{g_1, ..., g_t\}\) of \(\Gamma_{s, \theta^*}(I^*).O_q\). For simplicity, we assume that \(s = 1\) (the other cases follows from analogous reasons). We can chose the generators \(g_i\)'s of the following form:

\[g_i = \sum_j X_{i,j}^* \left( \sum_k a_{i,j,k} f_{i,j,k}^* \right)\]

where \(X_{i,j}\) are vector-fields of \(\theta_p\), \(a_{i,j,k}\) and \(f_{i,j,k}\) are functions in \(O_q\) and \(I.O_p\). This clearly implies that \(g_i\) is contained in the ideal \(([\Gamma_{1, \theta}(I)]^* + I^*).O_q\), which proves the other inclusion. This finally gives the desired result. \(\square\)

2.4 Chain of Ideal sheafs

A chain of ideal sheafs consists of a sequence \((I_i)_{i \in \mathbb{N}}\) such that:

- \(I_i\) is an ideal sheaf over \(O_M\);
- \(I_i \subset I_j\) if \(i \leq j\).

The length of a chain of ideal sheafs at a point \(p\) of \(M\) is the minimal number \(\nu_p \in \mathbb{N}\) such that \(I_i.O_p = I_{\nu_p}.O_p\) for all \(i \geq \nu_p\). We distinguish two cases:
• if \( I_p \circ O_p = O_p \), then the chain is said to be of type 1 at \( p \);
• if \( I_p \circ O_p \neq O_p \), then the chain is said to be of type 2 at \( p \).

Given a chain of ideal sheaf \((I_n)\), it is not difficult to see that the functions:

\[
\nu : M \rightarrow \mathbb{N}, \quad \text{type} : M \rightarrow \{1, 2\}
\]

\[
p \mapsto \nu_p, \quad p \mapsto \text{type}_p = \text{type of } (I_n) \text{ at } p
\]

are upper semi-continuous. So, given a subset \( U \) of \( M \), the definition of length and type naturally extends to \( U \) as follows:

• The length of \((I_n)\) at \( U \) is \( \nu_U := \sup\{\nu_p; p \in U\} \);
• The type of \((I_n)\) at \( U \) is \( \text{type}_U := \sup\{\text{type}_p; p \in U\} \).

Notice that \( \nu_U \) may be infinity. Nevertheless, if \( U \) is a relatively compact open subset of \( M \), \( \nu_U \) is necessarily finite.

Given a foliated ideal sheaf \((M, \theta, I, E)\), the tangency chain of the pair \((\theta, I)\) is defined as the following chain of ideal sheafs:

\[
\mathcal{T}_g(\theta, I) = \{H(\theta, I, i); i \in \mathbb{N}\}
\]

where the ideal sheafs \( H(\theta, I, i) \) are given by:

\[
\begin{cases}
H(\theta, I, 0) := I \\
H(\theta, I, i + 1) := H(\theta, I, i) + \theta[H(\theta, I, i)]
\end{cases}
\]

At each point \( p \) in \( M \), the length of this chain is called the tangent order (or shortly, the \( tg\)-order) at \( p \), and is denoted by \( \nu_p(\theta, I) \). The type of the chain is denoted by \( \text{type}_p(\theta, I) \).

**Remark 2.4.1.** Suppose that \( \theta \) is generated by a regular vector-field \( X \) and let \( \gamma_p \) be the orbit of \( X \) passing through a point \( p \) of \( V(I) \). In this simple case, we can interpret these invariants as follow:

• If the orbit \( \gamma_p \) is contained in the variety \( V(I) \), then the type of \((\theta, I)\) at \( p \) is two;
• If the orbit $\gamma_p$ is not contained in $V(\mathcal{I})$, then the type of $(\theta, \mathcal{I})$ at $p$ is one. Furthermore, the $tg$-order of $(\theta, \mathcal{I})$ is equal to the order of tangency between the orbit $\gamma_p$ and the variety $V(\mathcal{I})$ at $p$.

In other words, the type identifies the presence of invariant leaves and the $tg$-order measures the order of tangency between the leaves and the variety $V(\mathcal{I})$.

2.5 Chain-preserving smooth morphism

Given two foliated ideal sheafs $(M, \theta, \mathcal{I}, E_M)$ and $(N, \omega, \mathcal{J}, E_N)$, a morphism $\phi : M \to N$ is smooth with respect to $(M, \theta, \mathcal{I}, E_M)$ and $(N, \omega, \mathcal{J}, E_N)$ if:

• The morphism $\phi : M \to N$ is smooth;

• The set $\phi^{-1}(E_N)$ is equal to $E_M$;

• The ideal sheaf $\mathcal{J}.\mathcal{O}_M$ is equal to $\mathcal{I}$.

In this case, we abuse notation and denote the morphism as:

$$\phi : (M, \theta, \mathcal{I}, E_M) \to (N, \omega, \mathcal{J}, E_N)$$

Notice that this definition is independent of the singular distributions $\theta$ and $\omega$. We say that a smooth morphism $\phi : (M, \theta, \mathcal{I}, E_M) \to (N, \omega, \mathcal{J}, E_N)$ is chain-preserving if:

$$\mathcal{T}g(\omega, \mathcal{J}).\mathcal{O}_M = \mathcal{T}g(\theta, \mathcal{I})$$

i.e $H(\omega, \mathcal{J}, i)\mathcal{O}_M = H(\theta, \mathcal{I}, i)$ for all $i \in \mathbb{N}$.

Remark 2.5.1. A morphism may be chain preserving even if $\theta$ and $\omega$ are very “different”. This notion depends on the interaction between the singular distributions and the ideal sheafs.

We will further say that a smooth morphism $\phi : (M, \theta, \mathcal{I}, E_M) \to (N, \omega, \mathcal{J}, E_N)$ is $k$-chain-preserving if the morphism is chain preserving and $\theta$ and $\omega$ have leaf dimension equal to $k$. 

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Whenever we work with local foliated ideal sheaf, a morphism \( \phi : (M, M_0, \theta, \mathcal{I}, E_M) \rightarrow (N, N_0, \omega, \mathcal{J}, E_N) \) is \( k \)-chain-preserving smooth if:

\[
\phi|_{M_0} : (M_0, \theta, \mathcal{O}_{M_0}, \mathcal{I}, \mathcal{O}_{M_0}, E_M \cap M_0) \rightarrow (N_0, \omega, \mathcal{O}_{N_0}, \mathcal{J}, \mathcal{O}_{N_0}, E_N \cap N_0)
\]

is \( k \)-chain-preserving smooth.

### 2.6 Weak Resolutions and Resolution Functors

A weak-resolution of a foliated ideal sheaf \((M, \theta, \mathcal{I}, E)\) is a proper and analytic morphism:

\[
\sigma : M' \rightarrow M
\]

such that, for every relatively compact open subset \(M_0\) of \(M\), there exist a resolution of \((M, M_0, \theta, \mathcal{I}, E)\):

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that \(\sigma|_{\sigma^{-1}M_0} = \sigma_1 \circ \cdots \circ \sigma_r\).

A “good” resolution also respects a functorial property. More precisely, following [Ko] (see definition 3.31), we look for a functor \( \mathcal{R} \) that has:

- **input**: The category whose objects are foliated ideal sheafs \((M, \theta, \mathcal{I}, E_M)\) and whose morphisms are smooth morphisms;
- **output**: The category whose objects are admissible blowing-up sequences:

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)
\]

with specified admissible centers \(C_i\) and whose morphisms are given by the Cartesian product.

The functor \( \mathcal{R} \) is said to be a resolution functor if for all \((M, \theta, \mathcal{I}, E_M)\), it associates a resolution of \((M, \theta, \mathcal{I}, E_M)\) that commutes with smooth morphisms. One can define in the same manner the notion of resolution functor for local foliated ideal sheafs and of weak-resolution functors.
Remark 2.6.1. For such a functor to be well defined, we accept blowings-up with empty centers (isomorphisms).

2.7 The Hironaka’s Theorem

Let us state the version of Hironaka’s Theorem that we are going to use:

Theorem 2.7.1. (Hironaka): Let \((M, M_0, \theta, \mathcal{I}, E)\) be a local foliated ideal sheaf. Then there exists a resolution of \((M, M_0, \theta, \mathcal{I}, E)\):

\[
\mathcal{R}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that \(\mathcal{R}\) is a resolution functor that commutes with smooth morphisms.

Remark 2.7.2. The above Theorem is an interpretation of Theorem 1.3 of [BM2] or Theorems 2.0.3 and 6.0.6 of [W1] in the following sense:

- Neither of the Theorems need the notion of singular distribution;
- Theorem 1.3 of [BM2] is enunciated in algebraic category. But the paragraph before Theorem 1.1 of [BM2] justifies the analytic statement;
- In [BM2] and [W1], the authors work with marked ideal sheafs. We specialize their result to marked ideal sheafs with weight one. The reader may verify that the definition of Support and (weak) transform give rise to the interpretations formulated in this work;
- In order to stress the functorial property of the resolution, we have followed Kollor’s presentation (see [Ko]).

Remark 2.7.3. The functorial property implies an intuitive sense of “unicity”. For example, let \(C_i\) be the centers of \(\mathcal{R}(M, M_0, \theta, \mathcal{I}, E)\) and \(N\) a compact analytic manifold. Then \(C_i \times N\) are the centers of \(\mathcal{R}(M \times N, M_0 \times N, \omega, \mathcal{I}_O_{M \times N}, E \times N)\) for any singular distribution \(\omega\).

An important consequence of the functoriality is the following global version of Theorem 2.7.1.
Theorem 2.7.4. Let \((M, \theta, \mathcal{I}, E)\) be a foliated ideal sheaf. Then there exists a weak-resolution of \((M, \theta, \mathcal{I}, E)\):

\[
\mathcal{R}G(M, \theta, \mathcal{I}, E) = \sigma : \widetilde{M} \to M
\]

such that \(\mathcal{R}G(M, \theta, \mathcal{I}, E)\) is a weak-resolution functor that commutes with smooth morphisms.

The proof of Theorem 2.7.4 follows the same steps of Theorem 13.3 of \([BM1]\). We present the proof because the idea will be useful for us.

Proof. (Theorem 2.7.4): Let \((U_i)_{i \in \mathbb{N}}\) be an open cover of \(M\) by relatively compact subsets \(U_i\) of \(M\) such that \(U_i \subset U_{i+1}\). Theorem 2.7.1 guarantees the existence of a resolution \(\tilde{\sigma}_i = (\sigma_{i,1}, \ldots, \sigma_{i,r_i})\) for each \((M, U_i, \theta, \mathcal{I}, E)\). Consider the morphism, \(\sigma_i := \sigma_{i,1} \circ \ldots \circ \sigma_{i,r_i}\).

The inclusion \(\epsilon_i : U_i \to U_{i+1}\) is a smooth morphism and, by the functoriality of Theorem 2.7.1 there exists a smooth morphism \(\delta_i : U'_i \to U'_{i+1}\) such that the following diagram:

\[
\begin{array}{ccc}
U'_i & \overset{\delta_i}{\to} & U'_{i+1} \\
\sigma_i \downarrow & & \sigma_{i+1} \downarrow \\
U_i & \overset{\epsilon_i}{\to} & U_{i+1}
\end{array}
\]

commutes. It is clear that \(M\) is isomorphic to the direct limit of the \(U_i\), i.e. the disjoint union \(\bigsqcup U_i\) identified by the morphisms \(\epsilon_i\). Let \(M'\) be the direct limit of \(U'_i\) (identified by the morphisms \(\delta_i\)) and \(\sigma : M' \to M\) be the direct limit of \(\sigma\). By construction, \(\sigma|_{U'_i}\) coincides with \(\sigma_i\).

The functorial statement follows from the functoriality of each \(\sigma_i\). \(\square\)
3 The $\theta$-admissible blowing-up

3.1 Definition and Main result

Let $(M, \theta, E)$ be a $d$-foliated manifold and let $C$ be an analytic sub-manifold of $M$. Consider the reduced ideal sheaf $I_C$ that generates $C$, i.e. $V(I_C) = C$. We say that $C$ is a $\theta$-admissible center if:

- $C$ is a regular closed sub-variety;
- $C$ has SNC with $E$;
- There exists $0 \leq d_0 \leq d$ such that the $k$-generalized Fitting-ideal $\Gamma_{\theta,k}(I_C)$ is equal to the structural ideal $\mathcal{O}_M$ for all $k \leq d_0$ and is contained in the ideal sheaf $I_C$ otherwise.

We give a geometrical interpretation of $\theta$-admissible centers in Remark 3.3.2.

Examples:

- If $C$ is an admissible and $\theta$-invariant center, it is $\theta$-admissible;
- If $C$ is an admissible and $\theta$-totally transverse center, it is $\theta$-admissible;
- Let $(M, \theta, E) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, \emptyset)$ and $C = \{x = 0\}$. Then $C$ is a $\theta$-admissible center, but it is neither invariant nor totally transverse. Indeed, $\Gamma_{\theta,1}(I_C) = \mathcal{O}_M$ and $\Gamma_{\theta,2}(I_C) \subset I_C$.
- Let $(M, \theta, E) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, \emptyset)$ and $C = \{x^2 - z = 0\}$. Then $C$ is not a $\theta$-admissible center. Indeed, $\Gamma_{\theta,1}(I_C) = (x, z)$.

An admissible blowing-up $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ is $\theta$-admissible if the center $C$ is $\theta$-admissible. We emphasize two particular cases of $\theta$-admissible blowings-up:

- An admissible blowing-up $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ is $\theta$-invariant if the center $C$ is $\theta$-invariant (i.e $\theta[I_C] \subset I_C$);
- An admissible blowing-up $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ is $\theta$-totally transverse if the center $C$ is totally transverse to $\theta$ (i.e $\Gamma_{\theta,d}(I_C) = \mathcal{O}_M$).
A sequence \( \sigma = (\sigma_1, ..., \sigma_r) \) of \( \theta \)-admissible blowings-up is a sequence of admissible blowings-up:

\[
(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)
\]

such that \( \sigma_i : (M_{i+1}, \theta_{i+1}, E_{i+1}) \rightarrow (M_i, \theta_i, E_i) \) is a \( \theta_i \)-admissible blowing-up. A sequence \( \bar{\sigma} = (\sigma_1, ..., \sigma_r) \) of \( \theta \)-invariant blowings-up and of \( \theta \)-totally transverse blowings-up are defined analogously. A resolution of \((M, \theta, \mathcal{I}, E)\):

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)
\]

is said to be \( \theta \)-admissible (resp. \( \theta \)-invariant) if \( \sigma_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \rightarrow (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1}) \) is \( \theta_{i-1} \)-admissible (resp. \( \theta_{i-1} \)-invariant).

The following Theorem enlightens the interest of \( \theta \)-admissible blowings-up:

**Theorem 3.1.1.** Let \((M, \theta, E)\) be a \( R \)-monomial \( d \)-foliated manifold and:

\[
\sigma : (M', \theta', E') \rightarrow (M, \theta, E)
\]

an \( \theta \)-admissible blowing-up. Then \( \theta' \) is \( R \)-monomial.

The proof is divided in three parts. The two first subsections prove the existence of a “good” coordinate systems. The proof of the Theorem is given in subsection 3.4. An important Corollary of the proof of this Theorem is the following:

**Corollary 3.1.2.** Let \((M, \theta, E)\) be a \( d \)-foliated manifold such that \( \theta \) is regular and:

\[
\sigma : (M', \theta', E') \rightarrow (M, \theta, E)
\]

a \( \theta \)-invariant blowing-up. Then, \( \theta' \) is regular.

Which is proved in the end of this section.
3.2 Local coordinates for a $\theta$-invariant center

The main result of this subsection is the following:

**Proposition 3.2.1.** Let $(M, \theta, E)$ be a $R$-monomial $d$-foliated manifold and $C$ an invariant $\theta$-admissible center. Then, at each point $p \in C$, there exists a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ such that $I_{C, \mathcal{O}_p} = (x_1, ..., x_t)$.

In what follows, $C$ is always a $\theta$-invariant admissible center and, given a point $p$ of $M$, we denote by $I_C$ the ideal $I_{C, \mathcal{O}_p}$ when there is no risk of confusion on the point $p$.

The fundamental step for proving proposition 3.2.1 is the following result:

**Lemma 3.2.2.** Let $(M, \theta, E)$ be a $R$-monomial $d$-foliated manifold and $I$ a $\theta$-invariant regular coherent ideal sheaf. Given a point $p$ of $M$ and a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ with a $R$-monomial basis $\{X_1, ..., X_d\}$, there exists a set of generators $\{f_1, ..., f_t\}$ of $I := I_{C, \mathcal{O}_p}$ such that:

- $X_i(f_j) \equiv 0$ if $X_i$ is regular;
- $X_i(f_j) = K_{i,j}f_j$ for some $K_{i,j} \in R$, if $X_i$ is singular.

Let us see how this result proves proposition 3.2.1.

**Proof.** (Proposition 3.2.1) Take $p \in C$. Our proof is by induction on the pair $(d,n)$, where $d$ is the leaf dimension of $\theta_p$ and $n$ is the dimension of the ring $\mathcal{O}_p$.

Notice that for $d = 0$ or $n = 1$ the result is trivial (if $n = 1$, the support of the ideal is a point). By induction, suppose that for all $(d', n') < (d, n)$, where $<$ is the lexicographical order, there is always a $R$-monomial coordinate system $x = (x_1, ..., x_{n'})$ such that $I_C = (x_1, ..., x_t)$.

We prove it to $(d, n)$.

Fix a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ and $\{X_1, ..., X_d\}$ a $R$-monomial basis. By lemma 3.2.2, there exists a set of generators $\{f_1, ..., f_t\}$ of the ideal $I_C$ such that:
• $X_i(f_j) \equiv 0$ if $X_i$ is regular;

• $X_i(f_j) = K_{i,j}f_j$ for some $K_{i,j} \in R$, if $X_i$ is singular.

We have two cases to consider:

• Case I: Without loss of generality, suppose $X_1 = \frac{\partial}{\partial x_1}$ and that $X_j(x_1) = 0$ for all $j \neq 1$. Since $X_1(f_i) \equiv 0$ for all $i$, the set of generators is independent of $x_1$.

Let $U_p$ be an open neighborhood of $p$ such that the coordinate system $x = (x_1, ..., x_n)$ is well defined over $U_p$ and the vector-fields $X_i$ have representatives over $U_p$. Consider the quotient:

$$\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1)$$

The image of the distribution $\theta$ by $\Pi$ is a $R$-monomial involutive $(d - 1)$-singular distribution $\bar{\theta}$ given by the image of $X_i$, for $i > 1$. We denote the image of the coordinate system $x = (x_1, ..., x_n)$ by $\bar{x}$ as $\bar{x} = (\bar{x}_2, ..., \bar{x}_n)$. By induction, there exists a change of coordinates over $\mathcal{O}_{U_p}/(x_1)$ such that $I_c = (\bar{x}_2, ..., \bar{x}_t)$. Doing the equivalent change of coordinates in $\mathcal{O}_{U_p}$, since the change is invariant by $x_1$, we get $I_c = (x_2, ..., x_t)$.

• Case II: All vector-fields of the $R$-monomial basis $\{X_1, ..., X_d\}$ are singular:

$$X_i = \sum_{j=1}^{n} \alpha_{i,j}x_j \frac{\partial}{\partial x_j}$$

Since $I_c$ is regular, we can suppose that $f_1$ is regular and, without loss of generality, that $\frac{\partial}{\partial x_1}f_1(p) \neq 0$. Take the change of coordinates $\bar{x}_1 = f_1$ and $\bar{x}_i = x_i$ otherwise. In the new coordinates, we get:

$$X_i = \sum_{j=2}^{n} \alpha_{i,j}\bar{x}_j \frac{\partial}{\partial \bar{x}_j} + K_{1,i}\bar{x}_1 \frac{\partial}{\partial \bar{x}_1}$$

because $X_i(f_1) = K_{1,i}f_1$ for $K_{1,j} \in R$. Notice that $\{X_1, ..., X_d\}$ is also a $R$-monomial basis at this coordinate system. We drop the bars of this coordinate system in order to have simpler notation.
Let $U_p$ be an open neighborhood of $p$ such that the coordinate system $x = (x_1, ..., x_n)$ is well defined over $U_p$ and the vector-fields $X_i$ have representatives over $U_p$. Consider the quotient:

$$
\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1)
$$

Notice that $\mathcal{O}_{U_p}/(x_1)$ is an analytic manifold of dimension $n - 1$. The image of the distribution $\theta$ by $\Pi$ is a $R$-monomial involutive singular distribution $\tilde{\theta}$ given by the image of all $X_i$. Furthermore, $\tilde{\theta}$ satisfies one of the following conditions:

- Either $\tilde{\theta}$ is a $R$-monomial singular distribution of dimension $d$, or;
- $\tilde{\theta}$ is a $R$-monomial singular distribution of dimension $d - 1$ and we can assume $X_1 = x_1 \frac{\partial}{\partial x_1}$.

Either way, by induction, there exists a $R$-monomial coordinate system $\bar{x} = (\bar{x}_2, ..., \bar{x}_n)$ at $\mathcal{O}_{U_p}/(x_1)$ such that $\bar{I}_C = (\bar{x}_2, ..., \bar{x}_{t+1})$. Doing the equivalent change of coordinates in $\mathcal{O}_p$, since the change is invariant by $x_1$, we deduce the result.

\[ \square \]

In order to prove Lemma 3.2.2 we will need some preliminary definitions:

- Let $\hat{\mathcal{O}}_p$ denote the completion of $\mathcal{O}_p$ and fix a coordinate system $x = (x_1, ..., x_n)$. We introduce the topology of simple convergence in $\hat{\mathcal{O}}_p$, defined by a countably many semi norms:

$$
    f = \sum a_\alpha x^\alpha \longrightarrow |a_\alpha|
$$

Thus $f_i \longrightarrow f$ means that the coefficients of $x^\alpha$ in $f_i$ converges to the coefficient of $x^\alpha$ in $f$;

- Fixed a coordinate system $x = (x_1, ..., x_n)$, and given $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, let $\delta^\alpha$ be the derivation $\frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$. Given two functions $f, g \in \mathcal{O}_p$ we say that $g$ is contained in the Taylor expansion of $f$ at $p$ if, for all $\alpha$, either $\delta^\alpha g(p) = \delta^\alpha f(p)$ or $\delta^\alpha g(p) = 0$.

We also recall the following result (see section 6.3 and Theorems 6.3.4 and 6.3.5 of [Ho]):

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Proposition 3.2.3. Let $I$ be an ideal of $\mathcal{O}_p$ and $(f_n)_{n\in\mathbb{N}} \subset I$ be a sequence of analytic function germs which converges simply to an analytic function germ $f$. Then, $f \in I$.

We start the proof of Lemma 3.2.2 supposing that the distribution $\theta$ has leaf dimension 1. In the next Lemma, the coordinate system $(x, y) = (x, y_1, \ldots, y_n)$ is fixed:

Lemma 3.2.4. In the notation of Lemma 3.2.2, if $\theta_p$ has leaf dimension 1 and $\theta_p = \langle \frac{\partial}{\partial x} \rangle$, then there exists a set of generators $(h_1, \ldots, h_t)$ of $I$ such that $X(h_i) \equiv 0$. Moreover, if $(f_1, \ldots, f_r)$ is any set of generators of $I$, we can choose $(h_1, \ldots, h_t)$ such that each $h_j$ is contained in the Taylor expansion of a $f_i$ at $p$.

Proof. Take $(f_1, \ldots, f_r)$ any set of generators of $I$ and let $f := f_1$. Consider its Taylor expansion in $x$:

$$f = \sum_{i=0}^{\infty} h_i(y)x^i$$

Since $I$ is invariant by $X$, we have that $(f)_\# \subset I$ (we recall that $(f)_\#$ is the $\theta$-differential closure of the ideal $(f)$). We claim that $(h_i(y))_{i\in\mathbb{N}} = (f)_\#$.

Indeed, let us prove that $h_0(y) \in (f)_\#$ (the proof for the other coefficients is analogous). We set $g_0 = f$ and define recursively the expressions:

$$g_{i+1} := g_i - xX(g_i)\frac{1}{i}$$

It is easy to see that:

$$g_i = h_0(y) + \sum_{j=i}^{\infty} \beta_{i,j}h_j(y)x^j$$

for some $\beta_{i,j} \in \mathbb{K}$. It is clear that the sequence $(g_n)_n$ is contained in $I$ and converges simply to $h_0(y)$. By Proposition 3.2.3, this implies that $h_0(y) \in (f)_\# \subset I$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $h_i(y) \in I$ for all $i$. Thus $(h_i(y))_{i\in\mathbb{N}} \subset (f)_\#$.

Using again Proposition 3.2.3, it is clear that the ideal generated by $(h_i(y))_{i\in\mathbb{N}}$ contains $(f)_\#$. Moreover, since the structural ring is noetherian, we have that $(h_i(y))_{i\leq N} = (f)_\#$ for some $N \in \mathbb{N}$. Doing this for all the generators of $I$, we get the desired result. \qed
In the next Lemma, the $R$-monomial coordinate system $x = (x_1, ..., x_n)$ is fixed:

**Lemma 3.2.5.** In the notation of Proposition 3.2.2, if $\theta_p$ has leaf dimension 1 and $\theta_p = < X >$ where $X$ is a singular $R$-monomial vector-field, then there exists a set of generators $(h_1, ..., h_t)$ of $I$ such that $X(h_i) = K_i h_i$, for $K_i \in R$. Moreover, if $(f_1, ..., f_r)$ is any set of generators of $I$, we can choose $(h_1, ..., h_t)$ such that each $h_j$ is contained in the Taylor expansion of a $f_i$ at $p$.

**Proof.** Let $(f_1, ..., f_r)$ be a set of generators of $I$ and set $f = f_1$. Since the coordinate system is $R$-monomial we have that $X = \sum_{i=1}^n K_i x_i \partial_{x_i}^\alpha$ for $K_i \in R$. Taking any monomial $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ we get:

$$X(x^\alpha) = \sum_{i=1}^n K_i \alpha_i x^\alpha = K_\alpha x^\alpha$$

For some $K_\alpha \in R$ (because $\alpha_i \in \mathbb{Z}$ and $K_i \in R$). Since the number of different monomials is countable, there exists a countable set $R' \subset R$ such that $K_\alpha \in R'$, for all $\alpha \in \mathbb{Z}^n$. This allow us to rewrite the Taylor expansion of $f = f_1$ in the following form:

$$f(x) = \sum_{i \in \mathbb{N}} h_i(x)$$

with $h_i(x)$ such that $Xh_i(x) = K_i h_i(x)$, $K_i \in R'$ and $K_i \neq K_j$ whenever $i \neq j$. Moreover, since there exists a representative of $f$ convergent in a open neighborhood of $p$ (thus absolutely convergent), $h_i(x) \in \mathcal{O}_p$. We claim that $(h_i(x))_{i \in \mathbb{N}} = (f)$. Indeed, we show that $h_0 \in (f)$ (the others are analogous). Define $g_0 = f$ and:

$$g_1 := \frac{1}{K_0 - K_1} (K_1 f - X(f)) = \frac{1}{K_0 - K_1} [\sum_{i \in \mathbb{N}} K_i h_i(x) - K_1 \sum_{i \in \mathbb{N}} K_i h_i(x)] = h_0 + \sum_{i \geq 2} \beta_{i,1} h_i \in (f)$$

where $\beta_{i,1} = \frac{K_i - K_1}{K_0 - K_1}$. We define recursively:

$$g_n = \frac{1}{K_0 - K_n} (K_n g_{n-1} - X(g_{n-1})) = h_0 + \sum_{i \geq n+1} \beta_{i,n} h_i \in (f)$$

for non-zero constants $\beta_{i,n}$. It is clear that $(g_n) \subset I$ converges simply to $h_0(x)$. By the proposition 3.2.3 this implies that $h_0(x) \in (f)$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $(h_i(y)) \subset (f)$ for all $i$. 

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Using again proposition 3.2.3, it is clear that \((h_i(x))_{i \in \mathbb{N}}\) contains \((f)_\#\). Moreover, since the structural ring is noetherian, we have that \((h_i(x))_{i \leq N}\) is equal to \((f)_\#\) for some \(N \in \mathbb{N}\). Doing this for all \(f_i\) in the set of generators of \(I\), we get the desired result.

We are ready to prove Lemma 3.2.2.

**Proof.** (Lemma 3.2.2): We prove the result by induction on the leaf dimension of \(\theta\). Fix a \(R\)-monomial coordinate system \(x = (x_1, ..., x_n)\) and a \(R\)-monomial base \(\{X_1, ..., X_d\}\). Let \((f_1, ..., f_t)\) be a set of generators of \(I\) and assume by induction that the lemma is true for \(d' < d\).

By the induction hypotheses, we can assume without loss of generality that:

- \(X_i(f_j) \equiv 0\) if \(X_i\) is regular;
- \(X_i(f_j) = K_{i,j}f_j\) for some \(K_{i,j} \in R\), if \(X_i\) is singular.

for all \(i < d\). Now, by lemma 3.2.4 or 3.2.5, there exists another set of generators \((h_1, ..., h_l)\) such that:

- Either \(X_d(h_j) \equiv 0\) if \(X_d\) is regular, or;
- \(X_d(h_j) = K_{d,j}h_j\) for some \(K_{i,j} \in R\), if \(X_d\) is singular.

Furthermore, as each \(h_i\) is a part of the Taylor expansion of some \(f_j\), we have that:

- \(X_i(h_j) \equiv 0\) if \(X_i\) is regular;
- \(X_i(h_j) = K_{i,j}h_j\) for some \(K_{i,j} \in R\), if \(X_i\) is singular.

for all \(i \leq d\).



### 3.3 Local coordinates for a \(\theta\)-admissible center

The main result of this section is the following:

**Proposition 3.3.1.** Let \((M, \theta, E)\) be a \(d\)-foliated manifold and \(C\) a \(\theta\)-admissible center. Then, at each point \(p \in C\), there exists a coherent set of generators \(\{Y_i, Z_j\}\) of \(\theta_p\) with \(i = 1, ..., r\) and \(j = 1, ..., s\) such that:
• $I_C \cdot O_p$ is totally transverse to \{Y_i\};

• $I_C \cdot O_p$ is invariant by \{Z_j\};

• There exists a coordinate system $x = (x_1, ..., x_n)$ of $O_p$ such that: $I_C \cdot O_p = (x_1, ..., x_t)$, $Y_i = \frac{\partial}{\partial x_i}$ and $Z_j(x_i) = 0$ for $i = 1, ..., r$;

• If $\theta$ is $R$-monomial, then there exists a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ such that \{Y_i, Z_j\} is a $R$-monomial basis. Moreover, this coordinate system can be chosen so that $I_C \cdot O_p = (x_1, ..., x_t)$, $Y_i = \frac{\partial}{\partial x_i}$ and $Z_j(x_i) = 0$ for $i = 1, ..., r$.

In what follows, $C$ is always a $\theta$-admissible center and, given a point $p$ of $M$, we denote by $I_C$ the ideal $I_C \cdot O_p$ when there is no risk of confusion on the point $p$.

**Proof.** (Proposition 3.3.1): We prove this Proposition for $\theta$ a $R$-monomial singular distribution. In the general case, we only have to prove the first three statements, and it is not necessary to be careful with coordinate changes.

Fix a point $p \in C$ and take a $R$-monomial coordinate system $x = (x_1, ..., x_n)$ of $O_p$ and a $R$-monomial basis \{X_1, ..., X_d\} of $\theta_p$. If the center $C$ is invariant by $\theta$, the Proposition trivially follows from Proposition 3.2.2. So, suppose that the center $C$ is not invariant by $\theta$.

There exists a maximal integer $d_0 > 0$ such that $\Gamma_{\theta, d_0}(I_C) = O_M$. This implies that there exists $(f_1, ..., f_{d_0}) \subset I_C$ such that the determinant of the matrix:

$$A = \begin{vmatrix} X_1(f_1) & \cdots & X_1(f_{d_0}) \\ \vdots & \ddots & \vdots \\ X_{d_0}(f_1) & \cdots & X_{d_0}(f_{d_0}) \end{vmatrix}$$

is an unity of $O_p$. Without loss of generality, we assume that $X_i = \frac{\partial}{\partial x_i}$ for $i \leq d_0$ and $X_j(x_i) = 0$ for $i \leq d_0$ and $j > d_0$.

The next step is a change of coordinate system and $R$-monomial basis that diagonalizes the matrix $A$ in $O_p$. But we need to be careful with this process, so to not destroy the $R$-monomial structure.
Without loss of generality, we assume that \( X_i(f_i) \) is an unity for \( i \leq d_0 \). Consider the change of coordinates \( \bar{x}_1 = f_1 \) and \( \bar{x}_i = x_i \) otherwise. After the change we get:

\[
X_1 = U \frac{\partial}{\partial \bar{x}_1} \\
X_i(\bar{x}) = g_i(\bar{x})
\]

for some unit \( U \) of \( \mathcal{O}_p \). Notice that \( X_1 \) is equivalent to \( \frac{\partial}{\partial \bar{x}_1} \) and that \( \{ X_1, X_i - \frac{\partial}{\partial \bar{x}_1} X_1 \} \) is a \( R \)-monomial basis of this new coordinate system.

Repeating this process for all the others \( f_i \), with \( i \leq d_0 \) we can assume that \( x = (x_1, \ldots, x_n) \) is a \( R \)-monomial coordinate system of \( \mathcal{O}_p \) such that \( f_i = x_i \) and \( X_i = \frac{\partial}{\partial x_i} \) for \( i \leq d_0 \).

Let \( Y_i := X_i \) for \( i \leq d_0 \) and \( Z_j := X_{j+d_0} \) for \( j \leq d_p - d_0 \). It is clear that \( \{ Y_i \} \) is totally transverse to \( I_C \) and that \( \{ Y_i, Z_j \} \) is a \( R \)-monomial basis. Let us prove that \( I_C \) is invariant by \( \{ Z_j \} \): Since \( \mathfrak{I}_C \) is \( \theta \)-admissible, we conclude that \( \Gamma_{d_0+1}(I_C) \subset I_C \). In particular, taking \( Z = \sum h_j Z_j \) a \( \mathcal{O}_p \)-linear combination of the \( \{ Z_j \} \), we get:

\[
\det \begin{pmatrix}
Y_1(f_1) & \ldots & Y_1(f_{d_0}) & Y_1(g) \\
\vdots & \ddots & \vdots & \vdots \\
Y_{d_0}(f_1) & \ldots & Y_{d_0}(f_{d_0}) & Y_{d_0}(g) \\
Z(f_1) & \ldots & Z(f_{d_0}) & Z(g)
\end{pmatrix} \in I_C \quad \rightarrow \quad \det \begin{pmatrix}
\text{Id} & Y_i(g) \\
0 & Z(g)
\end{pmatrix} \in I_C
\]

So \( Z(g) \in I_C \) for every \( g \in I_C \) and we conclude that \( I_C \) is invariant by \( \{ Z_j \} \).

In this coordinate system, we have that \( I_C = (x_1, \ldots, x_{d_0}, h_1, \ldots, h_s) \) where \( h_i \) does not depend on \( (x_1, \ldots, x_{d_0}) \).

Let \( U_p \) be an open neighborhood of \( p \) such that the coordinate system \( x = (x_1, \ldots, x_n) \) is well defined over \( U_p \) and the vector-fields \( X_i \) have representatives over \( U_p \). Consider the map:

\[
\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1, \ldots, x_{d_0})
\]

We denote the image of the coordinate system \( x = (x_1, \ldots, x_n) \) under \( \Pi \) by \( \bar{x} = (\bar{x}_{d_0+1}, \ldots, \bar{x}_n) \). At this coordinate system the image \( \bar{I}_C \) of \( I_C \) is generated by \( (\bar{h}_1, \ldots, \bar{h}_s) \), and the image \( \bar{\theta} \) of
the singular distribution $\theta$ is generated by $\{\bar{Z}_j\}$. This implies that $\bar{I}_C$ is invariant by $\bar{\theta}$ and, by Proposition 3.2.1, there exists a change of coordinates such that $\bar{I}_C = (\bar{x}_{d_0+1}, \ldots, \bar{x}_t)$ and $\{\bar{Z}_j\}$ is a $R$-monomial basis of $\bar{\theta}$. Since neither $Z_j$ nor $h_i$ depends on $(x_1, \ldots, x_{d_0})$, using the equivalent change of coordinates in $O_p$ we get $I_C = (x_1, \ldots, x_t)$ and $\{Y_i, Z_j\}$ a $R$-monomial basis such that $Z_j(x_i) = 0$ for $i < d_0$.

\[\square\]

Remark 3.3.2. If a center $C$ is $\theta$-admissible, for each point $p$ in $C$, there exists two singular distributions germs $\theta_{inv}$ and $\theta_{tr}$ such that:

- The singular distribution $\theta_p$ is generated by $\{\theta_{inv}, \theta_{tr}\}$;
- The ideal $I_C$ is invariant by $\theta_{inv}$;
- The ideal $I_C$ is totally transverse by $\theta_{tr}$.

3.4 Proof of Theorem 3.1.1

We present a Proposition that trivially implies Theorem 3.1.1.

Proposition 3.4.1. Let $(M, \theta, E)$ be a $d$-foliated manifold, $C$ a $\theta$-admissible center and $\sigma : (M', \theta', E') \rightarrow (M, \theta, E)$ the blowing-up with center $C$. For a point $q$ in the exceptional divisor $F$, let $p = \sigma(q)$. Then there exists a coherent set of generators $\{Y_i, Z_j\}$ of $\theta_p$ with $i = 1, \ldots, r$ and $j = 1, \ldots, s$ (the same of Proposition 3.3.1) such that:

- The singular distribution $\theta'.O_q$ is generated by $\{O(F)Y_i^*, Z_j^*\}.O_q$.
- If the singular distribution $\theta$ is $R$-monomial, so is $\theta'$.

Proof. In the notation of the enunciate, consider the coordinate system $x = (x_1, \ldots, x_n)$ of $O_p$ and the coherent set of generators $\{Y_i, Z_j\}$ of $\theta_p$ given by Proposition 3.3.1. In this case, we have that $I_C := I_C.O_p = (x_1, \ldots, x_t)$ is totally transverse to $\{Y_i\}$ and invariant by $\{Z_j\}$.

Consider a vector-field $X$ contained in $\theta_p$:

$$X = \sum A_i \frac{\partial}{\partial x_i}$$
such that $I_C$ is invariant by $X$. This implies that $(A_i)_{i \leq t} \subset I_C$. After the blowing-up, without loss of generality, we can assume that $q$ is the origin of the $x_1$ chart:

$$(x_1, y_2, \ldots, y_t, x_{t+1}, \ldots, x_n) = (x_1, x_1 x_2, \ldots, x_1 x_t, x_{t+1}, \ldots, x_n)$$

In this chart, we get:

$$X^* = A_1^* \frac{\partial}{\partial x_1} + \sum_{i=2}^t \frac{1}{x_1} (A_i^* - A_1^* y_i) \frac{\partial}{\partial y_i} + \sum_{i=t+1}^n A_i^* \frac{\partial}{\partial x_i}$$

Since $(A_i)_{i \leq t} \subset I_C$, the function $\frac{1}{x_1} A_i^*$ is analytic for $i \leq t$. Thus, $X^*$ is analytic. In particular, this implies that $Z_j^*$ are all analytic.

In the other hand, the expressions of the blowing-up of the $Y_i$ are given by the following expressions:

- If $t = r$, we can always assume that $q$ is the origin of the $x_1$ chart:

$$Y_1^* = \frac{\partial}{\partial x_1} = \frac{1}{x_1} (x_1 \frac{\partial}{\partial x_1} - \sum_{i=t+1}^n y_i \frac{\partial}{\partial y_i})$$

$$Y_i^* = \frac{\partial}{\partial x_i} = \frac{1}{x_1} \frac{\partial}{\partial y_i} \quad (1)$$

- If $t > r$, then:

  - The point $q$ can be assumed to be the origin of the $x_1$ chart and the transform expressions are the same as in (1);

  - The point $q$ can be assumed to be the origin of the $x_t$ chart:

$$Y_i^* = \frac{\partial}{\partial x_i} = \frac{1}{x_t} \frac{\partial}{\partial y_i} \quad (2)$$

for all $i \leq r$.

Thus, they are all meromorphic and we must multiply by $O(F)$ exactly one time to get analytic vector-fields. Furthermore, we claim that $\{O(F).Y_i^*, Z_j^*\}.O_q$ is contained in $Der_{O_q}(-logF)$. Indeed:

- It is clear by the expressions (1) and (2) that $O(F)Y_i^*, O_q$ leaves $F = \{x_1 = 0\}$ invariant.
• Consider a vector-field $X$ contained in $\theta_p$ such that $I_c$ is invariant by $X$. Then:

$$[X^*(\mathcal{O}(F)) + \mathcal{O}(F)].\mathcal{O}_q = [X^*(I_c^*) + I_c^*].\mathcal{O}_q = (X(I_c) + I_c^*).\mathcal{O}_q = I_c^*.\mathcal{O}_q = \mathcal{O}(F).\mathcal{O}_q$$

Thus $Z_j^*.\mathcal{O}_q$ is contained in $\text{Der}_{\mathcal{O}_q}(-\log F)$.

By Lemma [2.2.2] the singular distribution $\theta^\alpha.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y_i^*, Z_j^*, W_k\}.\mathcal{O}_q$ where $W_k$ is a combination of $Y_i^*.\mathcal{O}_q$ that is analytic and not generated by $\{\mathcal{O}(F).Y_i^*, Z_j^*\}.\mathcal{O}_q$. We have two cases to consider:

i ) If $t = r$, then there exists a linear combination that generates $W_1 = \frac{\partial}{\partial x_1}$. But remark that $W_1$ is not contained in $\text{Der}_{\mathcal{O}_q}(-\log F)$ and $\mathcal{O}(F)Y_i^*.\mathcal{O}_q = x_1\frac{\partial}{\partial x_1}$ is the minimal multiple of $W_1$ contained in $\text{Der}_{\mathcal{O}_q}(-\log F)$. Thus: $\theta^\alpha.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y_i^*, Z_j^*\}.\mathcal{O}_q$.

ii ) If $t > r$, then it is clear by the expressions (1) and (2) that there is no possible $W_k$.

This implies that $\theta^\alpha.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y_i^*, Z_j^*\}.\mathcal{O}_q$.

Furthermore, if the $\theta$ is $R$-monomial, we can write $Z_j$ in one of the following forms:

$$Z_j = \sum_{i=1}^{n} \alpha_{i,j} x_i \frac{\partial}{\partial x_i}$$

$$Z_j = \frac{\partial}{\partial x_k}$$

with $\alpha_{i,j} \in R$ and $k_j > t$. Without loss of generality, we assume that $q$ is in the $x_1$-chart so to get:

$$Z_j^* = \sum_{i=j}^{t} (\alpha_{i,j} - \alpha_{1,j}) y_i \frac{\partial}{\partial y_i} + \sum_{i=t+1}^{n} \alpha_{i,j} x_i \frac{\partial}{\partial y_i}$$

$$Z_j^* = \frac{\partial}{\partial x_k}$$

which are $R$-monomial at the origin. Moreover, using the expressions (1) and (2), it is clear that $\{\mathcal{O}(F)Y_i^*, Z_j^*\}$ is a $R$-monomial basis at the origin. Now, the the same proof of Lemma [2.1.1] is enough to show that $\theta^\alpha$ is also $R$-monomial at $q$. $\square$

And we are finally ready to prove corollary 3.1.2:

**Proof.** (Corollary 3.1.2) By Lemma [2.3.4] a $d$-singular distribution $\theta$ is regular at a point $p$ of $M$ if, and only if, $\Gamma_{d,\theta}(m_p) + m_p = \mathcal{O}_M$, where $m_p$ is the maximal ideal of the structural ideal $\mathcal{O}_p$. 

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Now, consider a point $q$ of $M'$ and let $p = \sigma(q)$. Since $C$ is $\theta$-invariant, by Proposition 3.4.1 the singular distribution $\theta'$ is equal to the total transform $\theta^*$. Since $m_p^* \subset m_q$, by Lemma 2.3.5,

$$\Gamma_{d,\theta'}(m_q) = \Gamma_{d,\theta^*}(m_q) \supset \Gamma_{d,\theta^*}(m_p^*) \supset [\Gamma_{d,\theta}(m_p)]^* = \mathcal{O}_{M'}$$

which proves the result.

\[ \square \]

4 Resolution Theorem for an invariant ideal sheaf

4.1 Statement of the result

Our intention, is to prove the following result:

**Theorem 4.1.1.** Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local $d$-foliated ideal sheaf. Suppose that $\mathcal{I}_0$ is invariant by $\theta_0$, i.e. $\theta[\mathcal{I}]_0 \subset \mathcal{I}_0 \mathcal{O}_{M_0}$. Then, there exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:

$$\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i ) $\bar{\sigma} = (\sigma_r, ..., \sigma_1)$ is a sequence of $\theta$-invariant blowings-up (in particular, a sequence of $\theta$-admissible blowings-up);

ii ) The composition $\sigma = \sigma_1 \circ ... \circ \sigma_r$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;

iii ) If $\theta_0$ is $R$-monomial, then so is $\theta_r$;

iv ) If $\theta_0$ is regular, then so is $\theta_r$;

v ) $\mathcal{R}_{inv}$ is a resolution functor that commutes with chain-preserving smooth morphisms.

The proof of this Theorem is divided in two subsections. In the first, we prove a geometrical property of $\theta$-invariant centers. In the second, we use the functoriality of Hironaka’s Theorem to finish the proof of Theorem 4.1.1.
Furhtermore, the functoriality property $\mathcal{F}$ of Theorem 4.1.1 allows us to prove a global result just as in the Hironaka’s Theorem:

**Theorem 4.1.2.** Let $(M, \theta, \mathcal{I}, E)$ be a $d$-foliated ideal sheaf. Suppose that $\mathcal{I}$ is invariant by $\theta$. Then there exists a weak-resolution of $(M, \theta, \mathcal{I}, E)$:

$$\mathcal{R}\mathcal{G}_{\text{inv}}(M, \theta, \mathcal{I}, E) = \sigma : (\widetilde{M}, \widetilde{\theta}) \rightarrow (M, \theta)$$

such that:

i ) If $\theta$ is $R$-monomial, so is $\widetilde{\theta}$;

ii ) If $\theta$ is regular, so is $\widetilde{\theta}$;

iii ) $\sigma$ is an isomorphism over $M \setminus V(\mathcal{I})$;

iv ) $\mathcal{R}\mathcal{G}_{\text{inv}}(M, \theta, \mathcal{I}, E)$ is a weak-resolution functor that commutes with chain-preserving smooth morphisms.

The proof follows, mutatis mutandis, the same proof of Theorem 2.7.4

### 4.2 Geometric invariance

We briefly recall the notion of singular foliation. According to the Stefan-Sussmann Theorem (see [St, Su]) an involutive singular distribution $\theta$ is *integrable*, i.e. for all point $p$ in $M$, there exists an immersed locally closed sub manifold $(N, \phi)$ passing through $p$ such that:

- $D_q \phi(T_q N) = L_{\phi(q)}$ for all $q \in N$

where $L_q \subset TM_q$ is the linear subspace generated by $\theta_q$. The maximal connected submanifolds with respect to this property are called *leaves* and denoted by $\mathcal{L}$. The partition of $M$ into leaves is called the *singular foliation* generated by $\theta$ (not necessarily saturated).

We say that an ideal sheaf $\mathcal{I}$ is *geometrically invariant* by $\theta$ if every leaf of the foliation generated by $\theta$ that intersects $V(\mathcal{I})$ is totally contained in $V(\mathcal{I})$. The following example
shows that geometric invariance does not imply invariance:

Example: Consider \((M, \theta, E) = (\mathbb{R}^2, \frac{\partial}{\partial x}, \emptyset)\) and \(\mathcal{I} = (y x, y^2)\). Notice that \(\mathcal{I}\) is not invariant by \(\theta\), since \(\theta[\mathcal{I}] = (y)\). But \(\mathcal{I}\) is geometrically invariant by \(\theta\) because \(V(\mathcal{I}) = \{y = 0\}\) is a leaf of \(\theta\).

The following result gives the relation between these two notions of invariance:

**Lemma 4.2.1.** Let \(\theta\) be an involutive \(d\)-singular distribution and \(\mathcal{I}\) a coherent ideal sheaf.

- I) If \(\mathcal{I}\) is an ideal sheaf \(\theta\)-invariant, then \(\mathcal{I}\) is geometrically invariant by \(\theta\);
- II) If \(\mathcal{I}\) is a reduced ideal sheaf geometrically invariant by \(\theta\), then \(\mathcal{I}\) is \(\theta\)-invariant.

**Proof.** We start supposing that \(\theta\) is a 1-singular distribution. Take a point \(p\) in \(V(\mathcal{I})\) and let \(\mathcal{L}\) be the leaf of \(\theta\) through \(p\) (recall that \(\mathcal{L}\) is a sub-manifold of \(M\)).

- I): If \(\mathcal{L}\) is zero dimensional then it is clear that \(\mathcal{L} \subset V(\mathcal{I})\), so we assume that \(\mathcal{L}\) is one dimensional. In this case, for each point \(q\) in \(\mathcal{L} \cap V(\mathcal{I})\), the singular distribution \(\theta_q\) is generated by a regular vector-field \(X_q\) and, by Lemma 3.2.1, there exists a system of generators \(\{f_1, ..., f_s\}\) of \(\mathcal{I},\mathcal{O}_q\) such that \(X_q(f_i) \equiv 0\). This implies the existence of an open neighborhood \(U_q\) of \(q\) and a local coordinate system \((x, y) = (x, y_1, ..., y_{n-1})\) over \(U_q\), such that \(X_q = \frac{\partial}{\partial x}\) and \(\mathcal{I} = (f_1(y), ..., f_s(y))\). Thus \((\mathcal{L} \cap U_q) \cap V(\mathcal{I}) = \mathcal{L} \cap U_q\), and, since the choice of \(q\) in \(\mathcal{L}\) was arbitrary, \(\mathcal{L} \cap V(\mathcal{I})\) is an open subset of \(\mathcal{L}\). Furthermore, since \(\mathcal{L}\) is locally closed and \(V(\mathcal{I})\) is closed, \(\mathcal{L} \cap V(\mathcal{I})\) is a closed subset of \(\mathcal{L}\). Thus \(\mathcal{L} \subset V(\mathcal{I})\).

- II): We claim that \(V(\mathcal{I}) \subset V(\theta[\mathcal{I}])\). The claim implies the result because:

\[
\theta[\mathcal{I}] \subset \sqrt{\theta[\mathcal{I}]} \subset \sqrt{\mathcal{I}} = \mathcal{I}
\]

So, take \(p \in V(\mathcal{I})\) and let \(\mathcal{L}\) be the leaf of \(\theta\) passing through \(p\). If \(\mathcal{L}\) is zero dimensional, then all vector-fields germs of \(\theta_p\) are singular and it is clear that \(p \in V(\theta[\mathcal{I}])\), so we assume that \(\mathcal{L}\) is one dimensional. In this case \(\theta_p\) is generated by a regular vector-field \(X_p\). Consider \(f \in \mathcal{I},\mathcal{O}_p\): by hypotheses \(f|_{\mathcal{L}} \equiv 0\), which implies that \(X_p(f)|_{\mathcal{L}} = X_p(f|_{\mathcal{L}}) \equiv 0\). Since the choice of \(f \in \mathcal{I},\mathcal{O}_p\) is arbitrarily, \(p \in V(\theta[\mathcal{I}])\).
Now, we prove the result for \( \theta \) an involutive \( d \)-singular distribution. Take a point \( p \) in \( V(\mathcal{I}) \) and let \( \mathcal{L} \) be the leaf of \( \theta \) through \( p \) and \( \{X_1, ..., X_{d_p}\} \) be a set of coherent generators of \( \theta \) in a small neighborhood \( U_p \) of \( p \).

- **I)**: For a sufficiently small neighborhood \( U_p \) of \( p \), every point \( q \) in \( U_p \cap \mathcal{L} \) is the image of the flow \( (F_{t_1}^{X_1} \circ ... \circ F_{t_{d_p}}^{X_{d_p}})(p) = q \) for some \( (t_1, ..., t_{d_p}) \in \mathbb{R}^{d_p} \), where \( F_{t}^{X}(p) \) is the flow of the vector field \( X \) at time \( t \) and with initial point \( p \) (see Lemma 3.24 of [Mi]). Since \( X_i(\mathcal{I}, \mathcal{O}_{U_p}) \subset \mathcal{I}, \mathcal{O}_{U_p} \) by hypotheses, by the first part of the proof \( F_{t_i}^{X_i}(p) \in V(\mathcal{I}) \) for any \( t \). A recursive use of this argument implies that \( q \in V(\mathcal{I}) \). Thus, \( V(\mathcal{I}) \cap \mathcal{L} \) is open in \( \mathcal{L} \). Furthermore, since \( \mathcal{L} \) is locally closed and \( V(\mathcal{I}) \) is closed, \( \mathcal{L} \cap V(\mathcal{I}) \) is a closed subset of \( \mathcal{L} \). Thus \( \mathcal{L} \subset V(\mathcal{I}) \);

- **II)**: Take any vector-field \( X \) in \( \theta_p \) and let \( \gamma \) be the orbit of \( X \) at \( p \). Since \( \mathcal{L} \subset V(\mathcal{I}) \), it is clear that \( \gamma \subset V(\mathcal{I}) \) and, by the first part of the proof, \( X(\mathcal{I}, \mathcal{O}_p) \subset \mathcal{I}, \mathcal{O}_p \). Since the choice of the point and vector field is arbitrarily, we conclude that \( \theta[\mathcal{I}] \subset \mathcal{I} \).

4.3 Proof of Theorem [4.1.1]

By the Hironaka’s Theorem [2.7.1], there exists a resolution \( \bar{\sigma} = (\sigma_1, ..., \sigma_r) \) of \((M, M_0, \theta, \mathcal{I}, E)\):

\[
\mathcal{R}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

where \( \sigma_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \rightarrow (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1}) \) has center \( C_i \).

**Claim:** The admissible sequence of blowings-up \( \bar{\sigma} = (\sigma_1, ..., \sigma_r) \) is \( \theta \)-invariant.

**Proof.** Suppose by induction that the centers \( C_i \) are \( \theta_{i-1} \)-invariant for \( i < k \). We need to verify that \( C_k \) is also \( \theta_{k-1} \)-invariant (including for \( k = 1 \)).

First, notice that \( \mathcal{I}_{k-1} \) is invariant by \( \theta_{k-1} \). This follows from the induction hypotheses and a recursive use of the following lemma:
Lemma 4.3.1. Consider an admissible blowing-up of order one \( \sigma : (M', \theta', \mathcal{I}', E') \rightarrow (M, \theta, \mathcal{I}, E) \) with a center \( \mathcal{C} \) invariant by \( \theta \). Then \( \mathcal{I}' \) is invariant by \( \theta' \).

This Lemma is proved in the end of this section. We continue with the proof of the Claim: Since \( C_k \) is regular, by Lemma 4.2.1 we only need to verify that \( C_k \) is geometrically invariant by \( \theta_{k-1} \). We divide in two cases:

- First case: \( \theta_{k-1} \) has leaf dimension one. Let \( \mathcal{L} \) be a connected leaf of \( \theta_{k-1} \) with non-empty intersection with \( C_k \). We need to verify that \( \mathcal{L} \subset C_k \), which is clear if \( \mathcal{L} \) is zero-dimensional. So, assume that the leaf \( \mathcal{L} \) is one-dimensional and take a point \( p \) in \( C_k \cap \mathcal{L} \).

Locally, the singular distribution \( \theta_{k-1} \mathcal{O}_p \) is generated by an unique non-singular vector-field germ \( X_p \) with a representative in an open neighborhood \( U_p \) of \( p \). By the flow-box Theorem there exists a coordinate system \( (x, y) = (x, y_1, \ldots, y_{n-1}) \) in \( U_p \) such that \( X_p = \frac{\partial}{\partial x} \).

Furthermore, without loss of generality, \( U_p = V \times W \) where \( V \) is a domain of \( \mathbb{K}^{n-1} \) and \( W \) a domain of \( \mathbb{K} \) such that:

- The leaves of \( \theta \mathcal{O}_{U_p} \) are given by \( \{q\} \times W \), for every \( q \in V \);
- The divisor \( E_{k-1} \cap U_p \) is equal to \( E_V \times W \), where \( E_V \) is a SNC divisor over \( V \);
- There exist a natural smooth morphism \( \pi : V \times W \rightarrow V \).

By the coherence of \( \mathcal{I}_{k-1} \) and Proposition 3.2.2, without loss of generality, the ideal sheaf \( \mathcal{I}_{U_p} := \mathcal{I}_{k-1} \mathcal{O}_{U_p} \) has a finite set of generators \( \{f_1(y), \ldots, f_k(y)\} \) independent of \( x \).

Let \( g_i \in \mathcal{O}_V \) be functions such that \( g_i(y) = f_i(0, y) \) and \( \mathcal{J} \) be the ideal sheaf over \( \mathcal{O}_V \) generated by the \( (g_1(y), \ldots, g_t(y)) \): this clearly implies that \( \mathcal{J} \mathcal{O}_{V \times W} = \mathcal{I}_{U_p} \). Furthermore, the functorial statement of Hironaka’s Theorem 2.7.1 guarantees that the resolution of \((U_p, \mathcal{I} \mathcal{O}_{U_p}, E_{k-1} \cap U_p)\) and \((V, \mathcal{J}, E_V)\) commutes. This finally implies that \( \mathcal{C}_k = \pi(\mathcal{C}_k) \times W \) (see remark 2.7.3) and the intersection \( \mathcal{L} \cap \mathcal{C}_k \) must be open over \( \mathcal{L} \). By analyticity it is also closed and \( \mathcal{L} \subset \mathcal{C}_k \).
• Second case: $\theta_{k-1}$ has leaf dimensional $d$. Let $\mathcal{L}$ be a connected leaf of $\theta_{k-1}$ with non-empty intersection with $\mathcal{C}_k$. Take a point $p \in \mathcal{C}_k \cap \mathcal{L}$ and a coherent set of generators $\{X_1, ..., X_{d_p}\}$ of $\theta_\mathcal{O}_p$ with representatives defined in an open neighborhood $U_p$ of $p$. Without loss of generality, every point $q$ of $U_p \cap \mathcal{L}$ is contained in the image of the flux $(Fl_{t_1}^{X_1} \circ ... \circ Fl_{t_{d_p}}^{X_{d_p}})(p)$ for some $(t_1, ..., t_{d_p})$. 

If $\mathcal{L}_i$ is the leaf of $X_i$ passing through $p$, by the first part of the proof $\mathcal{L}_i \subset \mathcal{C}_k$. A recursive use of this argument implies that $(Fl_{t_1}^{X_1} \circ ... \circ Fl_{t_{d_p}}^{X_{d_p}})(p) \in \mathcal{C}_k$ for small enough $(t_1, ..., t_{d_p})$ which implies that $\mathcal{L} \cap \mathcal{C}_k$ is open over $\mathcal{L}$. By analyticity it is also closed, which implies that $\mathcal{L} \subset \mathcal{C}_k$.

Thus, by induction, $\tilde{\sigma} = (\sigma_1, ..., \sigma_r)$ is a sequence of $\theta$-invariant admissible blowings-up of order one. 

The functoriality statement of Theorem 4.1.1 is a direct consequence of the functoriality of Theorem 2.7.1. The $R$-monomiality statement is a direct consequence of Theorem 3.1.1 and the regularity statement is a direct consequence of Corollary 3.1.2.

To finish, we only need to prove Lemma 4.3.1.

Proof. (Proof of Lemma 4.3.1): Since $\mathcal{C}$ is a regular sub-manifold geometrically invariant by $\theta$, by Lemma 4.2.1 it is also invariant by $\theta$. Furthermore, by Lemma 2.3.5 we have that:

$$\theta[\mathcal{I}_\mathcal{C}] \subset \mathcal{I}_\mathcal{C} \implies \theta^*[\mathcal{O}(F)] \subset \mathcal{O}(F)$$

Moreover, $\sigma$ is a $\theta$-admissible blowing-up and, by Proposition 3.4.1 $\theta' = \theta^*$. Thus, again by Lemma 2.3.5

$$\theta'[\mathcal{I}'] + \mathcal{I}' = \theta^*[\mathcal{I}', \mathcal{O}(-F)] + \mathcal{I}' \subset \theta^*[\mathcal{I}^*\mathcal{O}(-F)] + \mathcal{I}' = \mathcal{I}'$$

$\square$
5 An ideal resolution subordinated to 1-foliations

5.1 Statement of the result

In this section we consider foliated ideal sheafs \((M, \theta, \mathcal{I}, E)\) such that \(\theta\) has leaf dimension one. In this case, our main result is the following:

**Theorem 5.1.1.** Let \((M, M_0, \theta, \mathcal{I}, E)\) be a local 1-foliated ideal sheaf. Then, there exists a resolution of \((M, M_0, \theta, \mathcal{I}, E)\):

\[
\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that:

i ) \(\bar{\sigma} = (\sigma_r, ..., \sigma_1)\) is a sequence of \(\theta\)-admissible blowings-up;

ii ) The composition \(\sigma = \sigma_1 \circ ... \circ \sigma_1\) is an isomorphism over \(M_0 \setminus V(\mathcal{I}_0)\);

iii ) If \(\theta_0\) is \(R\)-monomial, then so is \(\theta_r\);

iv ) \(\mathcal{R}_1\) is a resolution functor that commutes with 1-chain-preserving smooth morphisms.

v ) If \(\omega\) is a \(d\)-involutive distribution such that \(\mathcal{I}\) is \(\omega\)-invariant and \(\{\omega, \theta\}\) is an involutive \(d + 1\)-singular distribution, the sequence of blowings-up \(\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E)\) is \(\omega\)-invariant;

The functorial property [iv] of Theorem 5.1.1 allows us to prove a global result just as in the Hironaka’s Theorem:

**Theorem 5.1.2.** Let \((M, \theta, \mathcal{I}, E)\) be a 1-foliated ideal sheaf. Then there exists a weak-resolution of \((M, \theta, \mathcal{I}, E)\):

\[
\mathcal{R}_G(M, \theta, \mathcal{I}, E) = \sigma : (\tilde{M}, \tilde{\theta}) \rightarrow (M, \theta)
\]

such that:

i ) If \(\theta\) is \(R\)-monomial, so is \(\tilde{\theta}\);
\( ii \) \( \sigma \) is an isomorphism over \( M \setminus V(\mathcal{I}) \);

\( iii \) \( \mathcal{RG}_1(M, \theta, \mathcal{I}, E) \) is a weak-resolution functor that commutes with 1-chain-preserving smooth morphisms.

The proof follows, mutatis mutandis, the same proof of Theorem 2.7.4.

### 5.2 Proof of Theorem 5.1.1

Let us start giving the intuitive idea of the proof. Given a local 1-foliated ideal sheaf \((M, M_0, \theta, \mathcal{I}, E)\) the main invariant we consider is the pair:

\[
(\nu, t) := (\nu_{M_0}(\theta, \mathcal{I}), \text{type}_{M_0}(\theta, \mathcal{I}))
\]

where we recall that the \( \text{tg-order} \) \( \nu_{M_0}(\theta, \mathcal{I}) \) stands for the length of the tangency chain \( Tg(\theta, \mathcal{I}) \) over \( M_0 \) and the \( \text{type}_{M_0}(\theta, \mathcal{I}) \) stands for the type of this chain at \( M_0 \) (see section 2.4).

The proof of the Theorem relies on two steps:

- **First step**: \((\nu, 2) \rightarrow (\nu, 1)\);
- **Second step**: \((\nu, 1) \rightarrow (\nu - 1, 2)\).

which shows that this invariant drops. The following Propositions formalize the above steps:

**Proposition 5.2.1.** Let \((M, M_0, \theta, \mathcal{I}, E)\) be a local \( d \)-foliated ideal sheaf and suppose that \( \text{type}_{M_0}(\theta, \mathcal{I}) = 2 \). Then, there exists a sequence of \( \theta \)-invariant admissible blowings-up of order one:

\[
S_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that:

\( i \) \( \nu_{M_r}(\theta_r, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I}) \) and \( \text{type}_{M_r}(\theta_r, \mathcal{I}_r) = 1 \);
ii ) If $\omega$ is a $d'$-involutive distribution such that $\mathcal{I}$ is $\omega$-invariant and $\{\omega, \theta\}$ generates an involutive $d + d'$-singular distribution, the sequence of blowings-up is $\omega$-invariant;

iii ) If $\phi: (M, M_0, \theta, \mathcal{I}, E_M) \to (N, N_0, \omega, \mathcal{J}, E_N)$ is a chain-preserving smooth morphism, then there exists a chain-preserving smooth morphism $\psi: (M_r, \theta_r, \mathcal{I}_r, E_{M,r}) \to (N_r, \omega_r, \mathcal{J}_r, E_{N,r})$.

**Proposition 5.2.2.** Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local 1-foliated ideal sheaf and suppose that $\text{type}_{M_0}(\theta, \mathcal{I}) = 1$. Then, there exists a sequence of $\theta$-admissible blowings-up of order one:

$$S_2(M, M_0, \theta, \mathcal{I}, E): (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i ) $\nu_{M_r}(\theta_r, \mathcal{I}_r) < \nu_{M_0}(\theta, \mathcal{I})$;

ii ) If $\omega$ is a $d$-involutive distribution such that $\mathcal{I}$ is $\omega$-invariant and $\{\omega, \theta\}$ generates an involutive $d + 1$-singular distribution, the sequence of blowings-up is $\omega$-invariant;

iii ) If $\phi: (M, M_0, \theta, \mathcal{I}, E_M) \to (N, N_0, \omega, \mathcal{J}, E_N)$ is a 1-chain-preserving smooth morphism, then there exists a 1-chain-preserving smooth morphism $\psi: (M_r, \theta_r, \mathcal{I}_r, E_{M,r}) \to (N_r, \omega_r, \mathcal{J}_r, E_{N,r})$.

These two Propositions will be proved in the next two sections. For now, we assume them so to prove Theorem 5.1.1

**Proof.** (Theorem 5.1.1): Let $N$ be a relatively compact open subset of $M$. The tg-order and type $(\nu(N), t(N)) := (\nu_N(\theta, \mathcal{I}), \text{type}_N(\theta, \mathcal{I}))$ are well-defined.

In particular, if $N_1$ and $N_2$ are two relatively open subsets of $M$ such that $N_1 \subset N_2$, then $(\nu(N_1), t(N_1)) \leq (\nu_{N_2}(\theta, \mathcal{I}), \text{type}_{N_2}(\theta, \mathcal{I}))$ (where the order is lexicographically).

Fix $N$ a relatively compact open subset of $M$ such that $\overline{M_0} \subset N$. We claim that there
exists $\overline{M}_0 \subset N_0 \subset N$ a relatively compact open subset $N_0$ that satisfies $\overline{M}_0 \subset N_0 \subset \overline{N}_0 \subset N$ and a sequence of $\theta$-admissible blowings-up:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (N_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N), t(N))$.

We prove the claim: Take any relatively compact open subset $N_0$ satisfying $\overline{M}_0 \subset N_0 \subset \overline{N}_0 \subset N$. If $(\nu(N_0), t(N_0)) < (\nu(N), t(N))$, the claim is obvious, so assume that $(\nu(N_0), t(N_0)) = (\nu(N), t(N))$. By Propositions 5.2.1 or 5.2.2 applied to $(N, N_0, \theta, I, E)$, there exists a sequence of $\theta$-admissible blowings-up:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (N_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N_0), t(N_0)) = (\nu(N), t(N))$, which proves the claim.

As a matter of fact, the recursive use of this claim will prove the theorem: since the pair $(\nu, t)$ is bounded below by $(0, 1)$ one cannot recursively apply the claim an infinite number of times. Once the process stops, we restrict all blowings-up to $M_0$ and its transforms, which is well-defined because of the functoriality statements of Propositions 5.2.1 and 5.2.2.

The functoriality statements $[iv]$ and $[v]$ of the theorem follows directly from the functoriality statements $[ii]$ and $[iii]$ of Propositions 5.2.1 and 5.2.2. Furthermore, as all blowings-up are $\theta$-admissible, by Theorem 3.1.1 if $\theta, O_{M_0}$ is $R$-monomial, so will be its transforms.

5.3 Proof of Proposition 5.2.1

Consider a $d$-foliated ideal sheaf $(M, M_0, \theta, I, E)$ such that $\text{type}_{M_0}(\theta, I) = 2$. Let $\nu = \nu_{M_0}(\theta, I)$ and $\mathcal{C}l(I) := H(\theta, I, \nu)$ (see section 2.4). By Theorem 4.1.1 there exists a $\theta$-invariant resolution $\overline{\sigma} = (\sigma_1, \ldots, \sigma_r)$ of $(M, M_0, \theta, \mathcal{C}l(I), E)$:

$$(M_r, \theta_r, (\mathcal{C}l(I)_r), E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, (\mathcal{C}l(I)_1), E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, (\mathcal{C}l(I)_0), E_0)$$

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Claim 1: The sequence of blowings-up $\vec{\sigma}$ is $\theta$-admissible of order one for $(M, M_0, \theta, I, E)$. Furthermore:

$$(\text{Cl}(I))_j = \text{Cl}(I_j)$$

for all $j \leq r$.

The main step for proving the claim is the following Lemma:

Lemma 5.3.1. Let $\sigma : (M', \theta', I', E') \to (M, \theta, I, E)$ be an invariant $\theta$-admissible blowing-up of order one for $(M, \theta, \text{Cl}(I), E)$. Then:

$$H(\theta', I', i) = H(\theta, I, i)' = H(\theta, I, i)^*\mathcal{O}(-F)$$

for every $i \leq \nu$. In particular: $(\text{Cl}(I))' = \text{Cl}(I')$.

Which we prove in the end of this section. We now proceed with the proof of the Claim 1:

Proof. (Claim 1) Suppose by induction that, for $i < k$, the sequence $(\sigma_1, ..., \sigma_i)$ is admissible of order one for $(M, M_0, \theta, I, E)$ and:

$$(\text{Cl}(I))_j = \text{Cl}(I_j)$$

for $j \leq i$. We prove the result for $i = k$ (including $k = 1$). Since $\sigma_k$ is a blowing-up of order one for $(M_{k-1}, \theta_{k-1}, (\text{Cl}(I))_{k-1}, E_{k-1})$, by the induction hypotheses, it is also of order one for $(M_{k-1}, \theta_{k-1}, \text{Cl}(I_{k-1}), E_{k-1})$. Finally, since $I_{k-1} \subset \text{Cl}(I_{k-1})$, the blowing-up $\sigma_k$ is of order one for $(M_{k-1}, \theta_{k-1}, I_{k-1}, E_{k-1})$, which implies that $(\sigma_1, ..., \sigma_i)$ is of order one for $(M, M_0, \theta, I, E)$.

Now, by Lemma 5.3.1 and the induction hypotheses:

$$(\text{Cl}(I))_k = (\text{Cl}(I))'_k = (\text{Cl}(I))'_{k-1} = \text{Cl}(I_k)$$

This implies that $\vec{\sigma}$ gives rise to an invariant $\theta$-admissible sequence of blowings-up of order one for $(M, M_0, \theta, I, E)$:

$$(M_r, \theta_r, I_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, I_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, I_0, E_0)$$

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such that:

\[ H(\theta, \mathcal{I}_r, \nu) = \text{Cl}(\mathcal{I}_r) = (\text{Cl}(\mathcal{I}))_r = \mathcal{O}_{M_r} \]

which implies that \( \nu_{M_r}(\theta, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I}) \) and \( \text{type}_{M_r}(\theta, \mathcal{I}_r) = 1 \).

We now prove the functorial statement [ii] of the Proposition:

**Claim 2**: The ideal sheafs \( H(\mathcal{I}, \theta, i) \) are \( \omega \)-invariant for all \( i \in \mathbb{N} \).

**Proof.** We prove the result by induction on \( i \). For \( i = 0 \), the result follows by hypotheses, so assume the result proved for \( i = k \). Since \( \{\theta, \omega\} \) is an involutive singular distribution, the following calculation shows that the Claim 2 is valid for \( k + 1 \):

\[
\begin{align*}
\omega[H(\mathcal{I}, \theta, k + 1)] &= \omega[\theta[H(\mathcal{I}, \theta, k)] + H(\mathcal{I}, \theta, k)] \\ &= \theta[\omega[H(\mathcal{I}, \theta, k)] + \omega[H(\mathcal{I}, \theta, k)] + H(\mathcal{I}, \theta, k)] \\
&\subset H(\mathcal{I}, \theta, k + 1)
\end{align*}
\]

\[ \square \]

So, by part [iv] of Theorem 4.1.1, the resolution \( \bar{\sigma} = (\sigma_1, ..., \sigma_r) \) is also \( \omega \)-invariant, because the identity is a chain-preserving smooth morphism between \((M, M_0, \theta, \text{Cl}(\mathcal{I}), E)\) and \((M, M_0, \{\theta, \omega\}, \text{Cl}(\mathcal{I}), E)\). This proves the functorial statement [ii] of the Proposition.

We now prove the functorial statement [iii] of the Proposition:

Let \( \phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N) \) be a chain-preserving smooth morphism. Let \( \bar{\sigma} = (\sigma_1, ..., \sigma_r) \) and \( \bar{\tau} = (\tau_1, ..., \tau_r) \) be the sequences of blowings-up given in the Proposition (the length of the sequence may be chosen to be the same because of the functoriality of Theorem 4.1.1). Furthermore, for any ideal sheaf \( \mathcal{K} \) over \( N_{i-1} \), because of the functoriality of Theorem 4.1.1, we deduce that:

\[
(\sigma_i)^{\ast}(\mathcal{K} \cdot \mathcal{O}_{M_{i-1}}) = (\tau_i^{\ast}\mathcal{K}) \cdot \mathcal{O}_{M_i}
\]

In particular, if \( F_{M,i} \) is the exceptional divisor of the blowing-up \( \sigma_i : M_i \longrightarrow M_{i-1} \) and \( F_{N,i} \) is the exceptional divisor of the blowing-up \( \tau_i : N_i \longrightarrow N_{i-1} \), we have that:

\[
\mathcal{O}(-F_{N,i}) \cdot \mathcal{O}_{M_i} = \mathcal{O}(-F_{M,i})
\]

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Claim 3: The following equality holds:

\[ H(\mathcal{J}_i, \omega_i, j).\mathcal{O}_{M_i} = H(\mathcal{I}_i, \theta_i, j) \]

for \( i \leq r \) and \( j \in \mathbb{N} \).

Proof. Suppose by induction that \( H(\omega_i, \mathcal{J}_i, j).\mathcal{O}_{M_i} = H(\theta_i, \mathcal{I}_i, j) \) for \( i < k \) and any \( j \in \mathbb{N} \). Then:

\[
H(\omega_k, \mathcal{J}_k, j).\mathcal{O}_{M_k} = (\mathcal{O}(-F_{N,k})^*H(\omega_{k-1}, \mathcal{J}_{k-1}, j)).\mathcal{O}_{M_k} = \\
= \mathcal{O}(-F_{M,k})\sigma_k^*H(\theta_{k-1}, \mathcal{I}_{k-1}, j) = H(\theta_k, \mathcal{I}_k, j)
\]

for any \( j \in \mathbb{N} \), which proves Claim 3.

It is clear that Claim 3 implies the functoriality statement \([iii]\) of the Proposition.

To finish, we only need to prove Lemma 5.3.1.

Proof. (Lemma 5.3.1) First, notice that, since \( H(\theta, \mathcal{I}, i) \subset \mathcal{C}(\mathcal{I}) \) for \( i \leq \nu \), the blowing-up is also of order one for \((M, \theta, H(\theta, \mathcal{I}, i), E)\).

By hypotheses, the center \( \mathcal{C} \) is invariant by \( \theta \) and, by Proposition 3.3.1, the adapted analytic strict transform \( \theta' \) coincides with the total transform \( \theta^* \). Thus, if \( F \) is the exceptional divisor and \( \mathcal{J} \) is a coherent ideal sheaf, by Lemma 2.3.5:

\[
\theta'[\mathcal{O}(F)] \subset \mathcal{O}(F) \Rightarrow \mathcal{J}\theta'[(\mathcal{O}(-F))] \subset \mathcal{J}\mathcal{O}(-F)
\]

In particular, this implies that:

\[
\theta'[\mathcal{J}\mathcal{O}(-F)] + \mathcal{J}\mathcal{O}(-F) = \mathcal{O}(-F)(\theta'[\mathcal{J}] + \mathcal{J})
\]

Now, it rests to prove that the following equality:

\[
H(\theta', \mathcal{I}', i) = H(\theta, \mathcal{I}, i)^*\mathcal{O}(-F)
\]

is valid for all \( i \leq \nu \). Indeed, suppose by induction that the equality is valid for \( i < k \) (notice that for \( k = 0 \), the equality is trivial). Since the blowing-up is of order one for
\[ (M, \theta, H(\theta, \mathcal{I}, k), E), \text{ we have that:} \]
\[ H(\theta', \mathcal{I}', k) = H(\theta', \mathcal{I}', k - 1) + \theta' \left[ H(\theta', \mathcal{I}', k - 1) \right] = \]
\[ H(\theta', \mathcal{I}', k - 1) + \theta' \left[ H(\theta, \mathcal{I}, k - 1) \right] \cdot \mathcal{O}(-F) \]
\[ \mathcal{O}(-F) \text{ is an unity of } \mathcal{O}_p. \]

5.4 Proof of Proposition 5.2.2

Consider a 1-foliated ideal sheaf \((M, M_0, \theta, \mathcal{I}, E)\) such that \(\text{type}_{M_0}(\theta, \mathcal{I}) = 1\). Let \(\nu = \nu_{M_0}(\theta, \mathcal{I})\) and \(M_{tg}(\mathcal{I}) := H(\theta, \mathcal{I}, \nu - 1)\). By Theorem 2.7.1, there exists a resolution \(\tilde{\sigma} = (\sigma_1, ..., \sigma_r)\) of \((M, M_0, \theta, M_{tg}(\mathcal{I}), E)\):

\[
(M_r, \theta_r, (M_{tg}(\mathcal{I}))_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, (M_{tg}(\mathcal{I}))_0, E_0)
\]

Claim 1: the sequence of blowing-up \(\tilde{\sigma}\) is \(\theta\)-admissible. Furthermore, the center of blowing-up \(C_k\) are totally transverse to \(\theta_{k-1}\).

\textit{Proof.} Suppose by induction that, for \(i < k\):

a) The sequence \((\sigma_1, ..., \sigma_i)\) of blowing-up is \(\theta\)-admissible;

b) For \(p \in V((M_{tg}(\mathcal{I}))_i)\), there exists a coherent coordinate system \((x, y) = (x, y_1, ..., y_{n-1})\) of \(\mathcal{O}_p\) such that \(x \in M_{tg}(\mathcal{I})_i, \mathcal{O}_p\) and \(\frac{\partial}{\partial x}\) generates \(\theta_{i,p} := \theta_i \cdot \mathcal{O}_p\).

We prove the result for \(k\):

\begin{itemize}
  \item Step \(k = 1\). In this case, since the type of the tangency chain \(T_g(\theta, \mathcal{I})\) is one, for every \(p \in V((M_{tg}(\mathcal{I}))_0)\) the distribution \(\theta_p\) is generated by a non-singular vector-field. By the flow-box Theorem, there exists a coherent coordinate system \((x, y) = (x, y_1, ..., y_{n-1})\) of \(\mathcal{O}_p\) such that \(\theta_p\) is generated by \(X := \frac{\partial}{\partial x}\).
\end{itemize}

Furthermore, there exists \(g \in T_g(\theta, \mathcal{I}) \cdot \mathcal{O}_p\) such that \(X(g)\) is an unity of \(\mathcal{O}_p\). This
implies that \( g = xU(x, y) + h(y) \) where \( U(x, y) \) is an unity. Making the change of coordinates \( \bar{x} = g(x, y) \) and \( \bar{y} = y \) we get a coordinate system such that \( \bar{x} \in Tg(\theta, \mathcal{I}).\mathcal{O}_p \) and \( \theta_p \) is generated by \( X = V \frac{\partial}{\partial \bar{x}} \), where \( V = U(x, y) + xU_x(x, y) \) is an unity.

- Step \( k > 1 \). Take any \( p \in V((\mathcal{M}tg(\mathcal{I}))_{k-1}) \). Since the center \( \mathcal{C}_k \) of the blowing-up \( \sigma_k : M_k \rightarrow M_{k-1} \) is contained in \( V((\mathcal{M}tg(\mathcal{I}))_{k-1}) \), by the induction hypotheses \([b]\) it is also totally transverse to \( \theta \) at \( p \). This implies that the sequence \((\sigma_1, ..., \sigma_k)\) of blowing-up is \( \theta \)-admissible.

Consider \( q \in V((\mathcal{M}tg(\mathcal{I}))_k) \) and \( p = \sigma_k(q) \). If \( \sigma_k \) is a local isomorphism over \( q \), the result is trivial, so we assume that \( q \in F_k \). By the induction hypotheses \([ii]\), there exists a coherent coordinate system \((x, y) = (x, y_1, ..., y_{n-1})\) of \( \mathcal{O}_p \) such that \( x \in \mathcal{M}tg(\mathcal{I})_{k-1} \).\mathcal{O}_p \) and \( \frac{\partial}{\partial x} \) generates \( \theta_{k-1,p} \). Since \( \mathcal{C} \subset V(\mathcal{M}tg(\mathcal{I})_{k-1}) \), without loss of generality \( \mathcal{I}.\mathcal{O}_p = (x, y_1, ..., y_t) \) and \( q \) is the origin of the \( y_1 \)-chart. It is now easy to compute the transforms of the blowing-up at \( q \) and see that the induction hypotheses \([b]\) is valid for \( i = k \).

Using claim 1 together and Proposition \( [3.3.1] \) we deduce that:

\[
\theta_{k+1} = \mathcal{O}(F_k)\sigma_k^*(\theta_k) \tag{3}
\]

and, since the center is totally transverse:

\[
\theta_{k+1}[\mathcal{O}(F_k)] \subset \mathcal{O}(F_k) \tag{4}
\]

to simplify notation, define \((i\sigma_k) = \sigma_{i+1} \circ ... \circ \sigma_k \) for \( i < k \), \((k\sigma_k) = \text{id} \) and \( \bar{\sigma}_k = \sigma_1 \circ ... \circ \sigma_k \).

We also introduce:

\[
\mathcal{K}_k(\alpha) = \prod_{i=1}^{k-1}[(i\sigma_{k-1})^*\mathcal{O}(\alpha F_i)]
\]

Using the equation \( [3] \) recursively, we get that:

\[
\theta_k = \mathcal{K}_k(1)\bar{\sigma}_k^*\theta \tag{5}
\]
Using the equation (4) recursively, we get that:

\[ \theta_k(\mathcal{K}_k(\alpha)) \subset \mathcal{K}_k(\alpha) \]  

(6)

Furthermore, given an ideal sheaf \( \mathcal{J} \), equation (6) implies that:

\[ \theta_k[\mathcal{K}_k(\alpha)\mathcal{J}] + \mathcal{K}_k(\alpha)\mathcal{J} = \mathcal{K}_k(\alpha)(\mathcal{J} + \theta_k[\mathcal{J}]) \]  

(7)

Claim 2: the sequence of blowing-up \( \vec{\sigma} \) is of order one for \( (M, M_0, \theta, \mathcal{I}, E) \) and:

\[ H(\theta_k, \mathcal{I}_k, j) = \mathcal{K}_k(-1). \sum_{i=0}^{j} \mathcal{K}_k(i)\bar{\sigma}_k^*H(\theta_0, \mathcal{I}_0, i) \]  

(8)

for all \( j \leq \nu \).

Proof. Suppose by induction that, for \( k < k_0 \):

a ) The sequence \( (\sigma_1, ..., \sigma_i) \) of blowing-up is of order one for \( (M, M_0, \theta, \mathcal{I}, E) \);

b ) Equation (8) is valid for \( k < k_0 \).

We prove the result for \( k_0 \). Notice that the step \( k_0 = 0 \) is trivial, so we can treat only the case \( k_0 > 0 \):

- Step \( k_0 > 0 \). Using the induction hypotheses [ii] we deduce that:

\[ H(\theta_{k_0-1}, \mathcal{I}_{k_0-1}, j) = \mathcal{K}_{k_0-1}(-1). \sum_{i=0}^{j} \mathcal{K}_{k_0-1}(i)\bar{\sigma}_{k_0-1}^*H(\theta_0, \mathcal{I}_0, i) \subset \mathcal{K}_{k_0-1}(-1). \sum_{i=0}^{j} \bar{\sigma}_{k_0-1}^*H(\theta_0, \mathcal{I}_0, i) = \mathcal{K}_{k_0-1}(-1)\bar{\sigma}_{k_0-1}^*H(\theta_0, \mathcal{I}_0, j) \]

In particular:

\[ \mathcal{M}tg(\mathcal{I}_{k_0-1}) \subset (\mathcal{M}tg(\mathcal{I}_0))_{k_0-1} \]  

(9)

Which implies that \( \mathcal{C}_{k_0} \subset V(\mathcal{M}tg(\mathcal{I}_{k_0-1})) \). So the sequence of blowings-up \( (\sigma_1, ..., \sigma_{k_0}) \) is of order one for \( (M, M_0, \theta, \mathcal{I}, E) \).

We now verify the induction hypotheses [b] for \( k = k_0 \) by induction on \( j \). Indeed,
the formula is clearly true for \( j = 0 \), so consider it proved for \( j < j_0 \). We prove it for \( j = j_0 \). Indeed, by equation (7):

\[
H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0) = H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \theta_{k_0}[H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1)] = \\
= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \theta_{k_0}[\mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} H(\theta_{0}, \mathcal{I}_{0}, i)] = \\
= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \theta_{k_0}[\bar{\sigma}^*_{k_0} H(\theta_{0}, \mathcal{I}_{0}, i)]
\]

Now, using equation (3) and Lemma \ref{lem} we can continue the deduction:

\[
= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} (\theta[H(\theta_{0}, \mathcal{I}_{0}, i)]) = \\
= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} (H(\theta_{0}, \mathcal{I}_{0}, i + 1)) = \\
= \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} H(\theta_{0}, \mathcal{I}_{0}, i)
\]

So the formula is proved.

\[\square\]

Claim 2 implies that the sequence of \( \theta \)-admissible blowings-up \( \bar{\sigma} = (\sigma_1, \ldots, \sigma_r) \) of order one for \((M, M_0, \theta, \mathcal{M}t_g(\mathcal{I}), E)\) is also of order one for \((M, M_0, \theta, \mathcal{I}, E)\):

\[
(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)
\]

such that: **Claim 3**: The \( \text{tg-order} \) \( \nu(\theta_r, \mathcal{I}_r) \) is strictly smaller than \( \nu \).

**Proof.** Let \( \bar{\sigma} = \sigma_1 \circ \ldots \circ \sigma_r \) and we recall that \((\mathcal{M}t_g(\mathcal{I}))_r = \mathcal{O}_{M_r}\), which implies that \( \bar{\sigma}^* H(\theta_0, \mathcal{I}_0, \nu - 1) = \mathcal{K}_r(1) \). By claim 2, we deduce that:

\[
H(\theta_r, \mathcal{I}_r, \nu - 1) = \mathcal{K}_r(-1). \sum_{i=0}^{\nu-1} \mathcal{K}_r(i) \bar{\sigma}^* H(\theta_0, \mathcal{I}_0, i) = \\
= \mathcal{K}_r(-1). \sum_{i=0}^{\nu-2} \mathcal{K}_r(i) \bar{\sigma}^* H(\theta_0, \mathcal{I}_0, i) + \mathcal{K}_r(\nu - 2) = \\
= H(\theta_r, \mathcal{I}_r, \nu - 2) + \mathcal{K}_r(\nu - 2)
\]

which implies that:

\[
\theta_r[H(\theta_r, \mathcal{I}_r, \nu - 1)] + H(\theta_r, \mathcal{I}_r, \nu - 1) = H(\theta_r, \mathcal{I}_r, \nu - 1) + \theta_r[\mathcal{K}_r(\nu - 2)] \subset \\
\subset H(\theta_r, \mathcal{I}_r, \nu - 1) + \mathcal{K}_r(\nu - 2) = H(\theta_r, \mathcal{I}_r, \nu - 1)
\]

Which proves that the chain is stabilizing in at most \( \nu - 1 \) steps. \[\square\]
We now prove the functorial statement \([ii]\) of the proposition:

**Claim 4:** The ideal sheafs \(H(\mathcal{I}, \theta, i)\) are \(\omega\)-invariant for all \(i \in \mathbb{N}\).

**Proof.** The proof follows, mutatis mutatis, the same proof of Claim 2 contained in the proof of Proposition 5.1.

So, by part \([iv]\) of Theorem 4.1, the resolution \(\bar{\sigma} = (\sigma_1, ..., \sigma_r)\) is also \(\omega\)-invariant, because the identity is a chain-preserving smooth morphism between \((M, M_0, 0, M_{tg}(\mathcal{I}), E)\) and \((M, M_0, \omega, M_{tg}(\mathcal{I}), E)\).

We now prove the functorial statement \([iii]\) of the Proposition.

Let \(\phi : (M, M_0, \theta, \mathcal{I}, E_M) \rightarrow (N, N_0, \omega, \mathcal{J}, E_N)\) be a 1-chain-preserving smooth morphism. Consider \(\bar{\sigma} = (\sigma_1, ..., \sigma_r)\) and \(\bar{\tau} = (\tau_1, ..., \tau_r)\) the sequences of blowings-up given in the Proposition (the length of the sequence may be chosen to be the same because of the functoriality of Theorem 2.7.1). Furthermore, for any ideal sheaf \(\mathcal{K}\) over \(N_{i-1}\), because of the functoriality of Theorem 2.7.1, we deduce that:

\[(\sigma_i)^* (\mathcal{K}.\mathcal{O}_{M_{i-1}}) = (\tau_i^* \mathcal{K}).\mathcal{O}_{M_i}\]

In particular, if \(F_{M,i}\) is the exceptional divisor of the blowing-up \(\sigma_i : M_i \rightarrow M_{i-1}\) and \(F_{N,i}\) is the exceptional divisor of the blowing-up \(\tau_i : N_i \rightarrow N_{i-1}\), we have that:

\[\mathcal{O}(-F_{N,i}).\mathcal{O}_{M_i} = \mathcal{O}(-F_{M,i})\]

Furthermore, define \(\mathcal{K}_{M,k}(\alpha)\) and \(\mathcal{K}_{N,k}(\alpha)\) in the obvious way. We have that:

\[\mathcal{K}_{N,i}(\alpha).\mathcal{O}_{M_i} = \mathcal{K}_{M,i}(\alpha)\]

**Claim 5:** The following equality holds:

\[H(\mathcal{J}_i, \omega_i, j).\mathcal{O}_{M_i} = H(\mathcal{I}_i, \theta_i, j)\]

for \(i \leq r\) and \(j \in \mathbb{N}\).
Proof. Suppose by induction that \( H(\omega_i, J_i, j).O_{M_i} = H(\theta_i, I_i, j) \) for \( i < k_0 \) and any \( j \in \mathbb{N} \). Then:

\[
H(\omega_{k_0}, J_{k_0}, j).O_{M_{k_0}} = (K_{N,k_0}(-1) \sum_{i=0}^j K_{N,k_0}(i) \bar{\pi}_{k_0}^* H(\omega_0, J_0, i)).O_{M_{k_0}} = K_{M,k_0}(-1) \sum_{i=0}^j K_{M,k_0}(i) \bar{\sigma}_{k_0}^* H(\theta_0, I_0, i) = H(\theta_{k_0}, I_{k_0}, j)
\]

for any \( j \in \mathbb{N} \), which proves the claim. \[\square\]

It is clear that Claim 5 implies the functoriality statement \([iii]\) of the Proposition.

6 Appendix

6.1 Proof of Theorem 1.4.1

Consider the two foliated manifolds \((B, \theta, \emptyset)\) and \((\Lambda, \omega, \emptyset)\), where \( \omega = 0 \), and let \( I_\# \) be the smaller \( \theta \)-invariant ideal sheaf containing \( I \).

Claim: There exists an ideal sheaf \( J \) over \( \Lambda \) such that \( J.O_B = I_\# \).

Proof. Consider a point \( \lambda \) in \( \Lambda \) and let \( p \) be a point contained in the fiber \( \pi^{-1}(\lambda) \). Since \( \theta \) is regular, there exists a coordinate system \((x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_{n-d})\) of \( O_p \) such that \( \pi(x, y) = y \) and \( \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right\} \) is a coherent set of generators of \( \theta_p \).

Since \( I_\# \) is \( \theta \)-invariant, by Proposition 3.2.2 there exists a set of generators \( \{f_1(y), \ldots, f_s(y)\} \) of \( I_\# . O_p \). Let \( J_p \) be the ideal of \( O_{\lambda} \) generated by \( \{f_1(y), \ldots, f_s(y)\} \). Notice that this construction can be done for any point \( q \) contained in the fiber \( \pi^{-1}(\lambda) \), and generates an ideal \( J_q \).

By the construction of \( J_p \), there exists an open neighborhood \( U \) of \( p \) such that, for every \( q \) in \( U \cap \pi^{-1}(\lambda) \), \( J_q = J_p \). Furthermore, by analyticity, if \( (q_i) \) is a sequence of points in the fiber \( \sigma^{-1}(\lambda) \) that are converging to a point \( q \) such that \( J_{q_i} = J_{q_i} \), for all \( i \in \mathbb{N} \), then \( \{J_q \} \). Because of this two properties and the fact that the fiber \( \sigma^{-1}(\lambda) \) is connected, we conclude that the ideal \( J_p \) is independent of the point \( p \) in the fiber \( \pi^{-1}(\lambda) \).
Now, we only need to define $\mathcal{J}$ as the ideal sheaf locally given by $\mathcal{J}\mathcal{O}_\lambda = J_p$ for some $p$ in the fiber $\pi^{-1}(\lambda)$.

Notice that $\pi : B \longrightarrow \Lambda$ is a chain-preserving smooth morphism between $(B, \theta, \mathcal{I}_\#, \emptyset)$ and $(\Lambda, \omega, \mathcal{J}, \emptyset)$. By Theorem 4.1.2 there exists two proper analytic maps $\sigma : B' \longrightarrow B$ and $\tau : \Lambda' \longrightarrow \Lambda$ and a smooth map $\pi' : B' \longrightarrow \Lambda'$ such that:

- The morphism $\sigma : B' \longrightarrow B$ is a weak-resolution of $(B, \theta, \mathcal{I}_\#, \emptyset)$;
- The morphism $\tau : \Lambda' \longrightarrow \Lambda$ is a weak-resolution of $(\Lambda, \omega, \mathcal{J}, \emptyset)$;
- The following diagram:

\[
\begin{array}{c}
\xymatrix{B' \ar[r]^-{\pi'} \ar[d]_{\sigma} & \Lambda' \ar[d]^{\tau} \\
B \ar[u]_{\sigma} \ar[r]^-{\pi} & \Lambda}
\end{array}
\]

commutes.

Furthermore, given a relatively compact open subset $B_0$ of $B$, the proof of Proposition 5.2.1 guarantees that the sequence of invariant blowings-up $\tilde{\sigma} = (\sigma_1, \ldots, \sigma_r)$, where $\sigma_{|_{\sigma^{-1}B_0}} = \sigma_1 \circ \ldots \circ \sigma_r$:

\[
(B_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)
\]

is of order one for $(B, B_0, \theta, \mathcal{I}, \emptyset)$ and $\mathcal{I}_r$ is of type 1.

Define the ideal sheaf $\mathcal{I}'$ of $B'$ given as the direct limit of the controlled transforms $\mathcal{I}_r$ over all relatively compact open subsets $B_0$ of $B$. By construction $(B', \Lambda', \pi', \mathcal{I}')$ has no exceptional value and satisfies all hypotheses of the Theorem.

**Acknowledgments**

I would like to express my gratitude to my advisor, Professor Daniel Panazzolo, for the useful discussions, suggestions and revision of the manuscript. This work owns a great deal
to his influence. I would also like to express my gratitude to Professor Vincent Grandjean for the suggestions and revision of the manuscript. Finally, I would like to thank Orlando Villamayor and Santiago Encinas for the useful discussions on the subject.

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