THE IGUSA TODOROV FUNCTION FOR COMODULES

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ABSTRACT. We define the Igusa-Todorov function in the context of finite dimensional comodules and prove that a coalgebra is left qcF if and only if it is left semiperfect and its Igusa-Todorov function on each right finite dimensional comodule is zero.

1. Introduction

The Igusa-Todorov function (IT-function) appeared first in [1] and has been considered again in [5] and [6]. It is a new homological tool that generalises the notion of injective dimension (see Lemma 2.1). In [5], the authors proved that, for artinian rings, the selfinjectivity can be characterised by the nullity of the IT-function on each finitely generated module. In other words, they proved that a ring is quasi-Frobenius if and only if it is right artinian and its IT-function is zero on each finitely generated right module.

A coalgebra $C$ is said to be left (right) quasi-co-Frobenius (qcF) if every indecomposable injective right (left) $C$-comodule is projective. Since indecomposable projective comodules are finite dimensional, a left (right) qcF coalgebra is left (right) semiperfect, that is, all indecomposable injective right (left) comodules are finite dimensional.

We define here an IT-function in the context of finite dimensional comodules and we prove that a coalgebra $C$ is left qcF if and only if it is left semiperfect and its IT-function is zero.

Even if this equivalence can be seen as a possible dual version of the problem treated in [5], it is not strictly the case (we deal with coalgebras while the authors in [5] deal with artinian rings) and the proof given here uses quite different ideas and tools.

It is worth to remark that while the notion of quasi-Frobenius ring is left-right symmetric, the notion of qcF coalgebra is not. This applied to a coalgebra that is left and also right semiperfect will give an example of a coalgebra for which its IT-function of right comodules is zero, while its IT-function for left comodules is not (see Example 2.4).

In what follows we present a brief description of the contents of this paper.
In Section 2 we recall a few basic notations and give the definitions, main properties and examples needed to understand and treat the question.
In Section 3 we prove the main result mentioned above and some needed previous lemmas.
In Section 4 we use some computations with the IT-function, to deduce some well known facts about qcF coalgebras.

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2. The Igusa-Todorov function

2.1. Some notations. In this work, $C$ will be a coalgebra over a field $k$ and we will denote by $\mathcal{M}^C$ and $\mathcal{M}$ the categories of right and left comodules over $C$ respectively and by $\mathcal{M}_f^C$ and $\mathcal{M}_f$ the respective complete subcategories of finite dimensional comodules. Since $\mathcal{M}^C$ and $\mathcal{M}$ are Grothendieck categories, every object in them has an injective envelope (see for example [3]).

2.2. The Igusa-Todorov function on comodules. Let $C$ be a coalgebra and $K(C)$ be the free abelian group generated by all symbols $[M]$ with $M \in \mathcal{M}^C$ under the relations

1. $[A] - [B] - [C]$ if $A \cong B \oplus C$,
2. $[I]$ if $I$ is injective.

Then $K(C)$ is the free abelian group generated by all isomorphism classes of indecomposable non injective objects in $\mathcal{M}^C$. As the syzygy $\Omega^{-1}$ respects direct sums and sends injective comodules to 0, it gives rise to a group morphism (that we also call $\Omega^{-1}$) $\Omega^{-1} : K(C) \to K(C)$.

For any $M \in \mathcal{M}_f^C$, let $\langle M \rangle$ denote the subgroup of $K(C)$ generated by all the symbols $[N]$, where $N$ is an indecomposable non injective direct summand of $M$. Since the rank of $\Omega^{-1}(\langle M \rangle)$ is less or equal to the rank of $\langle M \rangle$, which is finite, it exists a non-negative integer $n$ such that the rank of $\Omega^{-1-n}(\langle M \rangle)$ is equal to the rank of $\Omega^{-i}(\langle M \rangle)$ for all $i \geq n$. Let $\varphi(M)$ denote the least such $n \in \mathbb{N}$.

The main properties of $\varphi$ are summarised in the following lemma, whose version for Artin algebras has been proved, almost all in [1] and the last in [6] and can be easily adapted to obtain the version for coalgebras.

**Lemma 2.1.** ([1], [6]) Let $C$ be a coalgebra and $M, N \in \mathcal{M}_f^C$.

1. If the injective dimension of $M$, $\text{id}M$, is finite, then $\text{id}M = \varphi(M)$.
2. If $M$ is indecomposable of infinite injective dimension, then $\varphi(M) = 0$.
3. $\varphi(N \oplus M) \geq \varphi(M)$.
4. $\varphi(M^k) = \varphi(M)$ if $k \geq 1$.
5. $\varphi(M) \leq \varphi(\Omega^{-1}M) + 1$, whenever $\Omega^{-1}M$ is finite dimensional.

In a similar way it is possible to define $\varphi$ on the category $\mathcal{M}_f$. We will use the same notation for both functions, when no confusion arises.

**Remark 2.2.** Note that $\varphi(M) = 0$ mains that $\text{rk}\Omega^{-n}(\langle M \rangle)$ remains constant for all $n \in \mathbb{N}$.

**Definition 2.3.** For a coalgebra $C$ we define

$$\dim_{\varphi}(\mathcal{M}_f) = \sup \{ \varphi(M) \text{ with } M \in \mathcal{M}_f \},$$

$$\dim_{\varphi}(\mathcal{M}_f^C) = \sup \{ \varphi(M) \text{ with } M \in \mathcal{M}_f^C \}.$$

The following example shows that $\dim_{\varphi}(\mathcal{M}_f)$ and $\dim_{\varphi}(\mathcal{M}_f^C)$ can be different.

**Example 2.4.** Consider the quiver

$$Q = \cdots \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$$
and let $C$ be the coalgebra whose elements are all paths in $kQ$ of length less or equal to one. Each comodule $M$ in $^C\mathcal{M}_f$ can be seen as a $kQ$-representation $(M_i,T_i)_{i\in \mathbb{N}}$ where $T_i: M_{i+1} \rightarrow M_i$ is such that $T_i.T_{i+1} = 0$, for all $i \in \mathbb{N}$. It is easy to check that every such representation can be decomposed into a sum of indecomposable representations of the form (see [9]):

$$
\begin{array}{cccccccc}
\cdot & \cdots & 0 & 0 & 0 & 0 & k & 1_k & 0 \\
\cdot & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Note that representations of the first type are injective, while representations of the second type are simple. As $\varphi(M \oplus I) = \varphi(M)$, whenever $I$ is injective, in order to prove that $\varphi(M) = 0$, for every comodule $M$ in $^C\mathcal{M}_f$, it is enough to show it for cosemisimple comodules. If we consider:

$$M = \cdots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{V_k} 0 \xrightarrow{0} \cdots \xrightarrow{0} V_1 \xrightarrow{0} V_0,$$

after applying $\Omega^{-1}$ to $M$ we obtain the representation

$$\Omega^{-1}(M) = \cdots \xrightarrow{0} 0 \xrightarrow{V_k} 0 \xrightarrow{0} \cdots \xrightarrow{0} V_1 \xrightarrow{0} V_0 \xrightarrow{0} 0.$$

Hence the ranks of $\langle \Omega^{-1}(M) \rangle$ and $\langle M \rangle$ are equal, and then, by induction (because $\Omega^{-1}(M)$ is cosemisimple), the ranks of $\langle \Omega^{-n}(M) \rangle$ and $\langle M \rangle$ are equal. Then $\varphi(M) = 0$ for every cosemisimple object in $^C\mathcal{M}_f$, so $\dim_{\varphi}(^C\mathcal{M}_f) = 0$.

On the other hand, right $C$-comodules can be seen as $Q'$-representations $(M_i,T_i)_{i\in \mathbb{N}}$ where $T_i: M_i \rightarrow M_{i+1}$ is such that $T_i.T_{i+1} = 0$, for all $i \in \mathbb{N}$ and

$$Q' = \cdot \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{2} \cdot \xrightarrow{1} \cdot \xrightarrow{0}.$$

Now, the right $C$-comodule:

$$M_n = \cdots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{k} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} 0
$$

(where the non zero vector space $k$ is placed in the vertex $n$), has injective dimension $n$, so $\varphi(M_n) = n$ and therefore $\dim_{\varphi}(^C\mathcal{M}_f^n) = \infty$. In particular $\dim_{\varphi}(^C\mathcal{M}_f) \neq \dim_{\varphi}(^C\mathcal{M}_f^n)$.

To finish this section, we recall the notions of quasi-co-Frobenius and semiperfect.

**Definition 2.5.** A coalgebra $C$ is said to be

- **left quasi-co-Frobenius**, shortly left qcF, if every injective right $C$-comodule is projective.
- **right quasi-co-Frobenius**, shortly right qcF, if $C^{op}$ is left qcF.
- **left semiperfect** if all injective envelopes of simple right $C$-comodules are finite dimensional. Equivalently, if the category of left $C$-comodules has enough projectives (see [3]).
- **right semiperfect** if $C^{op}$ is left semiperfect.
3. The main result

In this section we will prove the main result of this work, stating that a coalgebra $C$ is left $qcF$ if and only if it is left semiperfect and verifies $\dim_{\varphi}(\mathcal{M}_f^C) = 0$ (Theorem 3.4).

In order to do this, we start by proving some auxiliary results. In concrete, and Lemmas 3.1, 3.2 and 3.3 will be needed in the proof of Theorem 3.4.

**Lemma 3.1.** Let $C$ be a left semiperfect coalgebra with $\dim_{\varphi}(\mathcal{M}_f^C) = 0$. For any right simple $C$-comodule, $\text{Top}(E(S))$ is (defined and) simple.

**Proof.** As $E(S)$ is finite dimensional, its top is (defined and) cosemisimple. We will prove that it is simple.

Suppose there are two simple right $C$-comodules $S_1, S_2$ such that $S_1 \oplus S_2$ is a direct summand of $\text{Top}(E(S))$, and consider the following short exact sequences:

$$\sigma_1 : 0 \rightarrow K_1 \rightarrow E(S) \rightarrow S_1 \rightarrow 0$$

$$\sigma_2 : 0 \rightarrow K_2 \rightarrow E(S) \rightarrow S_2 \rightarrow 0$$

$$\sigma_3 : 0 \rightarrow K_3 \rightarrow E(S) \rightarrow S_1 \oplus S_2 \rightarrow 0$$

Note that $E(S)$ has simple socle, and $K_i$ is a non injective indecomposable comodule (since its socle is simple), for $i \in \{1, 2, 3\}$. Moreover, $K_i$ is non zero, for $i \in \{1, 2, 3\}$, since $S_1$ and $S_2$ are non zero. Also, it is clear that $K_i \nmid K_3$, for $i \in \{1, 2\}$. So $rk\langle\{[K_1],[K_2],[K_3]\}\rangle \geq 2$. Note also that $E(S)$ is the injective envelope of $K_i$, for $i \in \{1, 2, 3\}$.

If $S_1 \equiv S_2$, then $rk\langle\{[S_1],[S_2],[S_1 \oplus S_2]\}\rangle = 1$, then $\phi(K_1 \oplus K_2 \oplus K_3) \geq 1$, contradicting $dim_{\varphi}(\mathcal{M}_f^C) = 0$. If not, then $rk\langle\{[S_1],[S_2],[S_1 \oplus S_2]\}\rangle = 2$ but also $K_1 \nmid K_2$, so $rk\langle\{[K_1],[K_2],[K_3]\}\rangle = 3$, arising again to a contradiction. \hfill \Box

The following is a technical result.

**Lemma 3.2.** Let $f : \bigoplus_{i \in I} U_i \rightarrow V$ be an epimorphism of right $C$-comodules, where $V$ has simple top. Then for some $i \in I$, $f_i : U_i \rightarrow V$ defined by restricting $f$ to $U_i$ is surjective.

**Proof.** Since $V$ has simple top it has a unique maximal subcomodule $M$. If, for all $i \in I$, we have that $f_i$ is not surjective, then $\text{Im}(f_i) \subseteq M$, so $\text{Im}(f) = \sum_{i \in I} \text{Im}(f_i) \subseteq M$, contradicting that $f$ is surjective. \hfill \Box

**Lemma 3.3.** If $P$ is a projective object in the category of finite dimensional right comodules, then it is projective as a right comodule.

**Proof.** As $P$ is finite dimensional, its dual $P^*$ is an injective object in the category of finite dimensional left comodules. From Theorem 2.4.17 in [3], we deduce that $P^*$ is an injective left comodule and therefore $P$ is projective. \hfill \Box

**Theorem 3.4.** A coalgebra $C$ is left $qcF$ if and only if it is left semiperfect and $\dim_{\varphi}(\mathcal{M}_f^C) = 0$. 
Proof. Assume first that $C$ is left $qcF$. It is well known that then $C$ is left semiperfect (see [3]). In order to prove that $\dim_{\varphi}(M^C_f) = 0$, note that as $C$ is left $qcF$ every injective right $C$-comodule is projective, so the group morphism $\Omega^{-1} : K(C) \to K(C)$ is injective. Indeed, the injective envelope of any finite dimensional non injective right comodule $M$ will be the projective cover of $\Omega^{-1}(M)$, so $\Omega \circ \Omega^{-1} = id_{K(C)}$. This implies that $\Omega^{-1}$ preserve ranks of subgroups, so $\dim_{\varphi}(M^C_f) = 0$.

Suppose now that $C$ is left semiperfect and that $\dim_{\varphi}(M^C_f) = 0$. We will prove that any injective right $C$-comodule is projective.

Let $E$ be the injective envelope of a simple right $C$-comodule $S$. Assume that $E$ is not projective and consider the following commutative diagram:

\[
\begin{array}{c}
\sigma_1 : \\
0 \rightarrow X \rightarrow U \rightarrow E \rightarrow 0 , \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\sigma_2 : \\
0 \rightarrow E(X) \rightarrow E' \rightarrow E \rightarrow 0
\end{array}
\]

where

- $\sigma_1$ is a non zero short exact sequence (it does exist since $E$ is not projective), with
  - $X$ finite dimensional (see Lemma 3.3),
  - $U$ indecomposable (note that $U$ is finite dimensional -since $X$ and $E$ are- and then we can apply Lemma 3.2 and assume is indecomposable).
- $(E(X), \iota)$ is the injective envelope of $X$ and $\sigma_2 = \iota.\sigma_1$ the pushout.

Note that $E'$ is injective (since $E(X)$ and $E$ are) and that $X$ is not injective (since $\sigma_1 \neq 0$). Moreover, if we consider the following commutative diagram:

\[
\begin{array}{c}
\sigma_1 : \\
0 \rightarrow X \rightarrow U \rightarrow E \rightarrow 0 , \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\sigma_2 : \\
0 \rightarrow E(X) \rightarrow E' \rightarrow E \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\Omega^{-1}(X) \rightarrow \Omega^{-1}(U) \oplus E''
\end{array}
\]

we get from the snake lemma that $\Omega^{-1}(X) \cong \Omega^{-1}(U) \oplus E''$. So $[\Omega^{-1}(X)] = [\Omega^{-1}(U) \oplus E''] = [\Omega^{-1}(U)] + [E''] = [\Omega^{-1}(U)]$. Since $E \neq 0$, then no summand of $X$ is isomorphic to $U$. Therefore $\text{rk}\Omega^{-1}(X \oplus U) \leq \text{rk}(X \oplus U) - 1$ and then $\dim_{\varphi}(M^C_f) \geq 1$, a contradiction. Then $E$ is a projective module.

$\square$
4. Some Consequences

The characterization of $qcF$ coalgebras given by Theorem 3.4 allows us to prove some known results about them, by using the tools given by the IT-function.

**Proposition 4.1.**
If $C$ is a left $qcF$ coalgebra, then:

(a) Every indecomposable injective right $C$-comodule has simple top.

(b) Every indecomposable projective left $C$-comodule has simple socle.

**Proof.**
(a) It is an immediate consequence of Lemma 3.1, after applying Theorem 3.4.

(b) It is known that every indecomposable projective left $C$-comodule $P$ is finite dimensional (see [4]) and then $P^*$ is an indecomposable injective right $C$-comodule (see for example [3], Chapter 2). But then note that $\text{Top}(P^*) = (\text{Soc}(P))^*$. As $\text{Top}(P^*)$ is simple (and then finite dimensional) $\text{Soc}(P)$ also is simple.

**Proposition 4.2.** Let $C$ be a left $qcF$ coalgebra. Let $S_r$ be a set of representatives of the isomorphism classes of simple right $C$-comodules. We can define a function

$$\nu_r : S_r \to S_r, \quad \nu_r(S) = \text{Top}(E(S))$$

that turns out to be injective.

**Proof.** By Theorem 3.4 we know that $C$ is left semiperfect and that $\dim(\phi(M^C_f)) = 0$. From Lemma 3.1 we can define $\nu$, that will be proved injective. Suppose it is not. Then, there are two simple non isomorphic right $C$-comodules $S_1$ and $S_2$ such that $T_1 = \text{Top}(E(S_1))$ and $T_2 = \text{Top}(E(S_2))$ are isomorphic. So we have the following short exact sequences:

$$\sigma_1 : \quad 0 \to J_1 \to E(S_1) \to T_1 \to 0,$$

$$\sigma_2 : \quad 0 \to J_2 \to E(S_2) \to T_2 \to 0$$

with $\text{soc}(J_1) = S_1$ and $\text{soc}(J_2) = S_2$. Now, $J_1$ and $J_2$ are non injective indecomposable comodules (since they have simple socle and $\sigma_1$, $\sigma_2$ are not zero).

As $T_1 \cong T_2$ and $\dim(\phi(M^C_f)) = 0$, we get $J_1 \cong J_2$ and therefore their socles are isomorphic: $S_1 \cong S_2$.

The function $\nu$ appears in contexts of categories of modules with finitely many simple modules (where it is bijective) and is usually called the Nakayama permutation (see for example [7]).

Note that every cosemisimple coalgebra $C$ verifies $\dim(\phi(Cf)) = \dim(\phi(M^C_f)) = 0$, since every (right or left) comodule is injective. Next proposition deals with the non cosemisimple case.

**Proposition 4.3.** Let $C$ be a left $qcF$ coalgebra. If $C$ is indecomposable not simple, there are no simple injective right $C$-comodules.
Proof. Let $S$ be a simple injective right $C$-comodule. As $C$ is not simple, there is some simple right comodule $T \neq S$ (otherwise $C = E(S) = S$) with $\text{Ext}_C^1(S, T) \neq 0$ (see [2]).

Let $\sigma$ be a non splitting exact sequence from $T$ to $S$ and consider the following commutative diagram:

\[
\begin{array}{c}
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & S & \longrightarrow & 0 \\
\sigma & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & T & \longrightarrow & E(T) & \longrightarrow & X & \longrightarrow & 0 \\
\end{array}
\]

As $\sigma$ is non zero, we have that $f$ is non zero and then injective (since $S$ is simple). Moreover, as $E(T)$ has simple top (by Proposition 4.1), we get that $X$ is indecomposable. Thus, $S \cong X$, since $S$ is injective. We conclude that the injective dimension of $T$ is 1, contradicting the assumption that $\text{dim}_\varphi(M_f^C) = 0$.

□

Corollary 4.4. If $C$ is a left and right $qcF$ coalgebra, then $\nu_r$ is bijective.

Proof. As $C$ is left and right $qcF$, we have that $\nu_r : S_r \rightarrow S_r$ and $\nu_l : S_l \rightarrow S_l$ (defined similarly) are injective. Now define:

$\mu_r : S_r \rightarrow S_r, \quad \mu_r(S) = \text{soc}(P(S)),$

where $P(S)$ is the projective cover of $S$ (note that as $C$ is right semiperfect $M_f^C$ has enough projectives).

From the diagram

\[
\begin{array}{c}
0 & \longrightarrow & \text{Soc}(P(S)) & \xrightarrow{\iota} & E(\text{Soc}(P(S))) & \longrightarrow & \text{coker}(\iota) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{ker}(\pi) & \longrightarrow & P(S) & \xrightarrow{\pi} & S & \longrightarrow & 0 \\
\end{array}
\]

it is clear that $\nu_r$ and $\mu_r$ are mutually inverses.

□

It is known that a coalgebra is left (right) $qcF$ if and only if it is left (right) semiperfect and it generates the category of its left comodules (see [8]). After Theorem 3.6 this result can be restated as follows:

Corollary 4.5. Let $C$ be a left semiperfect coalgebra. $C$ generates the category of its left comodules if and only if $\text{dim}_\varphi(M_f^C) = 0$.

The question of whether the equivalence of Corollary 4.5 holds for non semiperfect coalgebras seems interesting.

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