Revisiting the ACVI Method for Constrained Variational Inequalities

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Abstract

ACVI is a recently proposed first-order method for solving variational inequalities (VIs) with general constraints. Yang et al. (2022) showed that the gap function of the last iterate decreases at a rate of $O\left(\frac{1}{\sqrt{K}}\right)$ when the operator is $L$-Lipschitz, monotone, and at least one constraint is active. In this work, we show that the same guarantee holds when only assuming that the operator is monotone. To our knowledge, this is the first analytically derived last-iterate convergence rate for general monotone VIs, and overall the only one that does not rely on the assumption that the operator is $L$-Lipschitz. Furthermore, when the sub-problems of ACVI are solved approximately, we show that by using a standard warm-start technique the convergence rate stays the same, provided that the errors decrease at appropriate rates. We further provide empirical analyses and insights on its implementation for the latter case.

1 Introduction

Variational Inequalities (VIs) are a class of problems that generalize standard minimization problems, including constrained optimization problems, to problems such as finding equilibria in zero-sum games and general-sum games (Cottle and Dantzig, 1968; von Neumann and Morgenstern, 1947; Rockafellar, 1970). The (constrained) VI problem seeks a point $x^* \in \mathcal{X}$ such that:

$$\langle x - x^*, F(x^*) \rangle \geq 0, \quad \forall x \in \mathcal{X}, \quad (cVI)$$

where $\mathcal{X}$ is a subset of the Euclidean $n$-dimensional space $\mathbb{R}^n$, and where $F: \mathcal{X} \mapsto \mathbb{R}^n$ is a continuous map. Their generality is owing to the fact that $F$ is a vector field in the general case, in sharp contrast to standard minimization, where $F \equiv \nabla f$ is restricted to only represent gradient fields. The increased expressivity underlies their practical relevance to a wide range of emerging applications in machine learning, such as (i) multi-agent games (Goodfellow et al., 2014; Vinyals et al., 2017), (ii) robustification of single-objective problems, which yields min-max formulations (Szegedy et al., 2014; Mazuelas et al., 2020; Christiansen et al., 2020; Rothenhäusler et al., 2018), and (iii) statistical approaches to modeling complex multi-agent dynamics in stochastic and adversarial environments.

Such generality comes, however, at a price, in that solving for equilibria is significantly more challenging than solving for optima. In particular, the Jacobian of $F$ is no longer symmetric in the general case, which implies possible rotational trajectories in the parameter spaces (Korpelevich, 1976), which can lead to limit cycles (Hsieh et al., 2021). Moreover, in sharp contrast to standard minimization, the last iterate can be quite far from the solution even though the average iterate converges to the solution (Chavdarova et al., 2019). This has motivated recent efforts to focus specifically on convergence analysis of the last iterate of the VI optimization methods. Indeed, if the last iterate does not converge to the solution in some theoretical settings, then there is no hope that such a method will converge in highly nonconvex settings such as GANs or multi-agent reinforcement learning (Goodfellow et al., 2014; Vinyals et al., 2017). Despite substantial recent progress in this regard (see, e.g., Tseng, 1995; Daskalakis et al., 2018; Diakonikolas, 2020; Golowich et al., 2020b; Chavdarova et al., 2021a; Gorbunov et al., 2022a, see also § 2), currently, an analytical derivation of the rate of convergence of the last-iterate of the first-order VI methods for the large class of monotone VIs (defined in § 3) is still missing. This hinders the advancement of the VI methods, for example, hindering the development of accelerated methods. All of the known rates of convergence for monotone VIs assume that the operator satisfies the globally Lipschitz condition (see § 2). Since some applications do not satisfy this assumption (see for example the discussion in Patel and Berahas, 2022), it is an important challenge to derive last-iterate convergence rates while relying solely on the monotonicity of the operator $F$.  

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Motivated by the lack of a first-order method for the cVI problem with general constraints, Yang et al. (2022) recently proposed the ADMM-based Interior Point Method for Constrained VIs (ACVI), a first-order, primal-dual method for solving cVI. ACVI generalizes the alternating direction method of multipliers (ADMM) method (Glowinski and Marroco, 1975; Gabay and Mercier, 1976), an algorithmic paradigm that is central to large-scale optimization (Boyd et al., 2011; Tibshirani, 2017), but which has been little explored in the cVI context. Yang et al. (2022) showed that the gap function of the last iterate of ACVI decreases at a rate of $O\left(\frac{1}{\sqrt{K}}\right)$ when the operator is $L$-Lipschitz, monotone, and at least one constraint is active. It is, however, an open problem to determine if the method converges while assuming only that the operator is monotone. Moreover, in some cases, the sub-problems of ACVI may be cumbersome to solve analytically. Hence, a natural question that arises is if we can show convergence when sub-problems are solved approximately, and in particular understanding what assumptions are needed for the approximation errors.

In summary, our focus is the following two questions:

- **Does the last iterate of ACVI converge when the operator is monotone, and not necessarily $L$-Lipschitz?**
- **Does ACVI converge when the sub-problems are solved approximately?**

In this paper, we answer the former question affirmatively. More precisely, we prove that the last iterate of ACVI converges at a rate of $O\left(\frac{1}{\sqrt{K}}\right)$ under a sufficient condition that the errors decrease at appropriate rates. We evaluate inexact ACVI empirically on 2D and high-dimensional games and show how multiple inexact but computationally cheap iterations can yield faster wall-clock convergence than fewer exact ones.

Finally, we provide a detailed study of a special case of the problem class that ACVI can solve. In particular, we focus on the case when the inequality constraints are simple, in the sense that projection on those inequalities is fast to compute. Such problems arise relatively often in machine learning; e.g., whenever the constraint set is an $L_p$-ball, with $p \in \{1, 2, \infty\}$ as in adversarial training (Goodfellow et al., 2015). We show that the same convergence rate holds for this variant of ACVI. Moreover, we show empirically that when using this method on the problem of training a constrained GAN on the MNIST (Lecun and Cortes, 1998) dataset we find that it convergences faster than the projected variants of the standard VI methods.

In summary, our main contributions are as follows:

- We show that the gap function of the last iterate of ACVI (Yang et al., 2022, Algorithm 1 therein) decreases at a rate of $O\left(\frac{1}{\sqrt{K}}\right)$ for monotone variational inequalities, without relying on the assumption that the operator is $L$-Lipschitz.
- We combine a standard warm-start technique with ACVI, and propose a precise variant of ACVI with approximate solutions, named inexact ACVI—see Algorithm 1. We show that inexact ACVI recovers the same rate of convergence as ACVI, provided that the errors decrease at appropriate rates.
- We propose a variant of ACVI designed for the cases when the inequality constraints are relatively simple so that the projection is fast to compute—see Algorithm 2. We derive a corollary of the former result, which guarantees the convergence; moreover, the central path is not needed in this case, which simplifies the convergence analysis.
- Empirically, we: (i) verify the benefits of warm-start of the inexact ACVI; (ii) observe that I-ACVI can be faster than other methods by taking advantage of cheaper approximate steps; (iii) train a constrained GAN on MNIST and show the projected version of...
ACVI is faster to converge than other methods; and (iv) provide visualizations contrasting the different ACVI variants.

2 Related Works

Last-iterate convergence of first-order methods on VI-related problems. It is known that the last and average iterate can be quite far apart for algorithms that aim to solve VIs; see examples in (Chavdarova et al., 2019). Thus, an extensive line of work has focused on establishing last-iterate convergence for bilinear or strongly monotone games (see, e.g., Tseng, 1995; Malitsky, 2015; Facchinei and Pang, 2003; Daskalakis et al., 2018; Liang and Stokes, 2019; Gidel et al., 2019; Azizian et al., 2020; Thekumparampil et al., 2022; Alacaoglu et al., 2022). We note also the work of Diakonikolas (2020), who focuses on cocoercive operators. Continuous-time analyses have also been important in the study of last-iterate convergence, particularly through the use of Lyapunov functions (Ryu et al., 2019; Bot et al., 2020; Rosca et al., 2021; Chavdarova et al., 2021a; Bot et al., 2022). For monotone VIs, (i) Golowich et al. (2020b,a) established a lower bound of $O\left(\frac{1}{\sqrt{K}}\right)$ for $\varphi$-stationary canonical linear iterative ($\varphi$-SCLI) first-order methods (Arjevani et al., 2016); (ii) Golowich et al. (2020b) obtained a rate in terms of the gap function relying on first- and second-order smoothness of $F$, (iii) Gorbunov et al. (2022a) and Gorbunov et al. (2022b) obtained a rate of $O\left(\frac{1}{F}\right)$ for extragradient (Korpelevich, 1976) and optimistic GDA (Popov, 1980), respectively—in terms of reducing the squared norm of the operator, relying on first-order smoothness of $F$—while using computer-assisted proof; and (iv) Gidel et al. (2020b) and Chavdarova et al. (2021a) provided a best-iterate rate for OGDA while assuming first-order smoothness of $F$. Daskalakis and Panageas (2019) focused on zero-sum convex-concave constrained problems and provided an asymptotic convergence guarantee for the last iterate of the optimistic multiplicative weights update (OMWU) method. For constrained and monotone VIs with $L$-Lipschitz operator, Cai et al. (2022) recently showed that both the last iterate of extragradient and of optimistic GDA have a rate of convergence that matches the lower bound, using a computer-aided proof. Gidel et al. (2017) considered strongly convex-concave zero-sum games with strongly convex constraint set and analyzed the convergence of the Frank-Wolfe method (Frank and Wolfe, 1956; Jaggi, 2013; Lacoste-Julien and Jaggi, 2015).

Interior point methods for VIs. Interior point (IP) methods are a broad class of algorithms for solving problems that are constrained by general inequality and equality constraints. One of the most commonly used classes of IP methods is the primal-dual interior point method with a logarithmic barrier. IP methods are typically solved using Newton’s method, which iteratively approaches the solution from within the feasible region. Several works extend IP methods for constrained VI problems. Among these, Nesterov and Nemirovski (Chapter 7, 1994) studies their extension to VI problems, while relying on Newton’s method. Further, an extensive line of works discusses specific settings (e.g., Chen et al., 1998; Qi and Sun, 2002; Qi et al., 2000; Fan and Yan, 2010). On the other hand, Goffin et al. (1997) described a second-order cutting-plane method for solving pseudomonotone VIs with linear inequalities. Nonetheless, although these methods enjoy fast convergence in terms of the number of iterations, each iteration requires computing second-order derivatives, which becomes computationally prohibitive for large-scale problems. Recently, Yang et al. (2022) derived the aforementioned ACVI method which combines interior-point methods and the ADMM method, resulting in a first-order method that is able to handle general constraints.

3 Preliminaries

Notation. We use bold small and bold capital letters to denote vectors and matrices, respectively. Curly capital letters denote sets. We let $[n]$ denote $\{1, \ldots, n\}$ and let $e$ denote vector of all 1’s. The Euclidean norm of $v$ is denoted by $\|v\|$, the inner product in Euclidean space by $\langle \cdot, \cdot \rangle$, and $\odot$ denotes element-wise product.

Problem. The constraint set $C \subseteq \mathcal{X}$ is defined as an intersection of finitely many inequalities and linear equalities:

$$C = \{x \in \mathbb{R}^n | \varphi_i(x) \leq 0, \, i \in [m], \, Cx = d\}, \quad (CS)$$

where each $\varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $C \in \mathbb{R}^{p \times n}$, $d \in \mathbb{R}^p$, where $\text{rank}(C) = p$ is the rank of $C$. With abuse of notation, let $\varphi$ be the concatenated $\varphi_i(\cdot)$, $i \in [m]$. We assume that each of the inequality constraints is convex and assume $\varphi_i \in C^1(\mathbb{R}^n)$, $i \in [m]$. We define the following sets:

$$C_\varphi \triangleq \{x \in \mathbb{R}^n | \varphi(x) \leq 0\},$$
$$C_\varphi^c \triangleq \{x \in \mathbb{R}^n | \varphi(x) < 0\},$$
$$C_{=\varphi} \triangleq \{y \in \mathbb{R}^n | Cy = d\};$$

thus the relative interior of $C$ is $\text{int} \ C \triangleq C_\varphi \cap C_{=\varphi}$. We make the standard assumptions that $\text{int} \ C \neq \emptyset$ and that $C$ is compact.

In the following, we list the definitions and assumptions we refer to later on. We define these for a general domain $S$, and by setting $S \equiv \mathbb{R}^n$ and $\mathcal{X} \equiv \mathcal{X}$, these refer to the unconstrained and constrained settings, respectively.

Definition 1 (monotone operators). An operator $F : \mathcal{X} \supseteq S \rightarrow \mathbb{R}^n$ is monotone on $S$ if:

$$\langle x - x', F(x) - F(x') \rangle \geq 0, \quad \forall x, x' \in S.$$
We will use the standard gap function as a convergence measure, which requires $S$ to be compact for it to be defined.

**Definition 2** (gap function). Given a candidate point $x' \in \mathcal{X}$ and a map $F: \mathcal{X} \to \mathbb{R}^n$ where $\mathcal{S}$ is compact, the gap function $G: \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$G(x', S) \triangleq \max_{x \in S} \langle F(x'), x - x \rangle.$$

**Definition 3** ($\sigma$-approximate solution). Given a map $F: \mathcal{X} \to \mathbb{R}^n$ and a positive scalar $\sigma$, $x \in \mathcal{X}$ is said to be a $\sigma$-approximate solution of $F(x) = 0$ if $\|F(x)\| \leq \sigma$.

Since the $y$ sub-problem of ACVI is a standard minimization problem, we will also rely on the following definition.

**Definition 4** ($\varepsilon$-minimizer). Given a minimization problem $\min h(x), s.t. x \in S$, and a fixed positive scalar $\varepsilon$, a point $x \in S$ is said to be an $\varepsilon$-minimizer if:

$$h(\hat{x}) \leq h(x) + \varepsilon, \quad \forall x \in S.$$

Appendix A provides additional background, including a presentation of the KKT system for (cVI), guarantees under which the central path exists, and some additional VI definitions that are helpful for the discussion.

## 4 Convergence of Exact and Inexact ACVI for Monotone VIs

In this section we present our main theoretical results: (i) the rate of convergence of the last iterate of ACVI under the assumption that the operator $F$ is monotone; and (ii) the corresponding convergence rate when the sub-problems are solved approximately. In the latter case we analyze the inexact ACVI algorithm, which is stated in Algorithm 1; see Appendix A for a presentation of the ACVI algorithm. For simplicity, we will state both of these results for a fixed $\mu_{-1}$, but these theorems hold without knowing the exact value of $\mu_{-1}$, as discussed in Appendix B.3 in (Yang et al., 2022). Thus, both algorithms are parameter-free.

### 4.1 Last-iterate convergence of exact ACVI

**Theorem 1** (Last-iterate convergence rate of ACVI—Algorithm 1 in (Yang et al., 2022)). Given a continuous operator $F: \mathcal{X} \to \mathbb{R}^n$, assume: (i) $F$ is monotone on $\mathcal{C}_\varepsilon$, as per Def. 1; (ii) either $F$ is strictly monotone on $\mathcal{C}$ or one of $\varphi_i$ is strictly convex; and (iii) $F$ is $L$-Lipschitz on $\mathcal{X}$, that is, $\|F(x) - F(x')\| \leq L \|x - x'\|$, for all $x, x' \in \mathcal{X}$ and some $L > 0$. Let $(\{x_{K_t}^i\}, \{y_{K_t}^i\}, \lambda_{K_t}^i)$ denote the last iterate of Algorithm 1, run with sufficiently small $\mu_{-1}$; and let $\sigma_k$ and $\varepsilon_k$ denote the approximation errors at step $k$ of lines 8 and 9 (as per Def. 3 and 4), respectively. Further, suppose:

$$\lim_{K \to \infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} (k(\varepsilon_k + \sigma_k)) = < +\infty. $$

Then for all $t \in [T]$ and $K \in \mathbb{N}_+$, it holds that:

$$G(x_K, C) \leq O\left(\frac{1}{\sqrt{K}}\right), \quad \|x_K - y_K\| \leq O\left(\frac{1}{\sqrt{K}}\right).$$

Note that—as is also the case for Theorem 1—Theorem 2 gives a nonasymptotic convergence guarantee. More precisely, while the condition involving the sequences $\{\varepsilon_k\}_{k=1}^{K+1}$ and $\{\sigma_k\}_{k=1}^{K+1}$ requires the given expression to be summable, the convergence rate is nonasymptotic as it holds for any $K$. Appendix B gives specific details on the constants that appear in the convergence rates of Theorem 2 and provides its proof. Appendix C discusses further details of the implementation of Algorithm 1, and we will present an extensive analysis of the effect of warm-starting later in § 6.

### 4.2 Last-iterate convergence rate of inexact ACVI

For some problems, the equation in line 8, or the convex optimization problem in line 9 of ACVI—Algorithm 3—may not have an analytic solution (or the exact solution may be expensive to obtain). Here we consider solving these two problems approximately, using warm-starting. This technique requires that at each iteration, we set the initial variable $x$ and $y$ to be the solution at the previous step when solving the $x$ and $y$ sub-problems, respectively; see the detailed description in Algorithm 1. The following Theorem—inspired by (Schmidt et al., 2011)—establishes that when the errors in the calculation of the sub-problems satisfy certain conditions, the last-iterate convergence rate of inexact ACVI recovers that of (exact) ACVI.

**Theorem 2** (Last-iterate convergence rate of Inexact ACVI (I-ACVI)). Given a continuous operator $F: \mathcal{X} \to \mathbb{R}^n$, assume: (i) $F$ is monotone on $\mathcal{C}_\varepsilon$, as per Def. 1; (ii) either $F$ is strictly monotone on $\mathcal{C}$ or one of $\varphi_i$ is strictly convex; and (iii) $F$ is $L$-Lipschitz on $\mathcal{X}$, that is, $\|F(x) - F(x')\| \leq L \|x - x'\|$, for all $x, x' \in \mathcal{X}$ and some $L > 0$. Let $(\{x_{K_t}^i\}, \{y_{K_t}^i\}, \lambda_{K_t}^i)$ denote the last iterate of Algorithm 1, run with sufficiently small $\mu_{-1}$; and let $\sigma_k$ and $\varepsilon_k$ denote the approximation errors at step $k$ of lines 8 and 9 (as per Def. 3 and 4), respectively. Further, suppose:

$$\lim_{K \to \infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} (k(\varepsilon_k + \sigma_k)) = < +\infty. $$

Then for all $t \in [T]$ and $K \in \mathbb{N}_+$, it holds that:

$$G(x_K, C) \leq O\left(\frac{1}{\sqrt{K}}\right), \quad \|x_K - y_K\| \leq O\left(\frac{1}{\sqrt{K}}\right).$$

Specialization of ACVI for Simple Inequality Constraints

While ACVI is able to handle general constraints, it is a relatively common scenario in machine learning applications that the inequality constraints are fairly simple, in the sense that the projection can be computed cheaply. This is the case when we have $L_{\infty}$-ball constraints for
Figure 1: Convergence of ACVI, and I-ACVI with different parameters on the (2D-BG) problem: illustrations with the central path (depicted in yellow). For all the methods, we show the $y$-iterates, initialized at the same point. Each subsequent bullet on the trajectory depicts the (exact or approximate) solution at the end of the inner loop (when $k = K - 1$). The Nash equilibrium (NE) of the game is represented by a yellow star, and the constraint set is the interior of the red square. Fig. (a): As we decay $\mu$, the solutions of the inner loop of ACVI follow the central path. As $\mu \to 0$ the solution of the inner loop of ACVI converges to the NE. Fig. (b, c, d): When $x$ and $y$ sub-problems are both solved approximately (see the I-ACVI algorithm) with a finite $K$ and $\ell$, the iterates might not converge as the approximation error increases (and $K$ decreases). See § 6 for a discussion.

Algorithm 1 Inexact ACVI (I-ACVI) pseudocode.

1: **Input:** operator $F$: $X \to \mathbb{R}^n$, constraints $Cx = d$ and $\varphi_i(x) \leq 0$, $i = [m]$, hyperparameters $\mu_0$, $\beta > 0$, $\delta \in (0, 1)$, inner optimizers $A_x$ (e.g. EG, GDA, OGDA) and $A_y$ (GD), for the $x$ and $y$ sub-problems, resp.; number of outer and inner loop iterations $T$ and $K$, resp.
2: **Initialize:** $y_0^{(0)} \in \mathbb{R}^n$, $\lambda_0^{(0)} \in \mathbb{R}^n$  
3: $P_c \triangleq I - C^{\top}(CC^{\top})^{-1}C$  
4: $d_c \triangleq C^{\top}(CC^{\top})^{-1}d$  
5: for $t = 0, \ldots, T - 1$
6: $\mu_t = \delta \mu_{t-1}$
7: for $k = 0, \ldots, K - 1$
8: Set $x_{k+1}^{(t)}$ to be a $\sigma_{k+1}$-approximate solution of: $x + \frac{1}{\beta}P_cF(x) - P_c y_{k}^{(t)} + \frac{1}{\beta}P_c \lambda_{k}^{(t)} - d_c = 0$ (w.r.t. $x$), through running $\ell_x^{(t)}$ steps of $A_x$, where each time $x$ is initialized to be the solution of the previous step $x_{k}^{(t)}$
9: Set $y_{k}^{(t)}$ to be an $\varepsilon_{k+1}$-minimizer of $\min_y - \mu \sum_{i=1}^{m} \log (-\varphi_i(y)) + \frac{\beta}{2} \left\| y - x_{k+1} - \frac{1}{\beta} \lambda_{k}^{(t)} \right\|^2$, obtained through running $\ell_y^{(t)}$ steps of $A_y$, where each time $y$ is initialized to be the solution of the previous step $y_{k}^{(t)}$
10: $\lambda_{k+1}^{(t)} = \lambda_{k}^{(t)} + \beta (x_{k+1}^{(t)} - y_{k}^{(t)})$
11: **end for**
12: $(y_{k}^{(t+1)}, \lambda_{k}^{(t+1)}) \triangleq (y_{K}^{(t)}, \lambda_{K}^{(t)})$
13: **end for**

which the projection can be obtained through simple clipping. Projections onto the $L_2$ and $L_1$-balls can also be obtained efficiently, through simple normalization for $L_2$ and a $\mathcal{O}(n \log(n))$ algorithm for $L_1$ (Duchi et al., 2008).

In those cases, the $y$ sub-problem of ACVI can be replaced by projection onto the set defined by the inequalities. On the other hand, the $x$ sub-problem can still take into account the equality constraints in the general case; or if there are no equality constraints, it is easy to see that such problem setting will only simplify the $x$ sub-problem since $P_c \equiv I$, and $d_c \equiv 0$.

In this section, we specifically focus on this special case in which there are simple inequalities.

The P-ACVI Algorithm: omitting the log barrier. Suppose that the given inequality constraints can be satisfied with a projection $\Pi_C(\cdot): \mathbb{R}^n \to \mathcal{C}_\leq$ that is fast to compute. In that case, we no longer need the log barrier, hence we omit the use of $\mu$ and remove the outer loop of ACVI over $t \in [T]$. By differentiating the remaining expression of the $y$ sub-problem with respect to $y$ and setting it to zero, we obtain:

$$\argmin_{y} \frac{\beta}{2} \left\| y - x_{k+1} - \frac{1}{\beta} \lambda_{k} \right\|^2 = x_{k+1} + \frac{1}{\beta} \lambda_{k}.$$ 

This implies that line 9 of the exact ACVI algorithm (given
in Appendix A) can be replaced with the solution of the $y$ sub-problem without the inequality constraints, and then projecting so as to satisfy the inequality constraints. We have:

$$y_{k+1} = \Pi_{\leq}(x_{k+1} + \frac{1}{\beta} \lambda_k),$$

where the inequality constraints $\varphi_i(\cdot)$ are included in the projection. We present the resulting procedure in detail in Algorithm 2, which we refer to as P-ACVI as it uses projections. In this scenario with simple $\varphi$, the $y$ sub-problem is always solved exactly; nonetheless, when in addition the $x$ sub-problem is solved approximately, we refer to this combination of P-ACVI and I-ACVI as PI-ACVI.

Figure 2: Intermediate iterates of PI-ACVI (Algorithm 2) on the 2D minmax game (2D-BG). The boundary of the constraint set is shown in red. Fig. (b) depicts the $y_k$ (from line 7 in Algorithm 2) which we obtain through projections. In Fig. (a), each spiral corresponds to iteratively solving the $x_k$ sub-problem for $\ell = 20$ steps (line 6 in Algorithm 2). Jointly, these two trajectories of $x$ and $y$ iterates illustrate the ACVI dynamics: the two unconstrained $x$ and a constrained $y$ “collaborate” and eventually converge to the same point. Additional similar experiments that include other methods as well are available in Appendix D.

Algorithm 2 P-ACVI: ACVI with simple inequalities.

1: Input: operator $F$: $\mathcal{X} \rightarrow \mathbb{R}^n$, constraints $Cx = d$ and projection operator $\Pi_{\leq}$ for the inequality constraints, hyperparameter $\beta > 0$, and number of iterations $K$.
2: Initialize: $y_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}^n$
3: $P_e \triangleq I - C^T(CC^T)^{-1}C$ where $P_e \in \mathbb{R}^{n \times n}$
4: $d_e \triangleq C^T(CC^T)^{-1}d$ where $d_e \in \mathbb{R}^n$
5: for $k = 0, \ldots, K - 1$ do
6: Set $x_{k+1}$ to be the solution of: $x + \frac{1}{\beta} P_e F(x) - P_e y_k + \frac{1}{\beta} P_e \lambda_k - d_e = 0$ (w.r.t. $x$)
7: $y_{k+1} = \Pi_{\leq}(x_{k+1} + \frac{1}{\beta} \lambda_k)$
8: $\lambda_{k+1} = \lambda_k + \beta(x_{k+1} - y_{k+1})$
9: end for

Last-iterate convergence of P-ACVI. The following theorem shows that the P-ACVI—Algorithm 2—has the same last-iterate rate as ACVI. Given the proof of Theorem 1—which focuses on a more general setting than P-ACVI—its proof follows readily; see Appendix B for details. We state it as a separate theorem, as it cannot be deduced directly from the statement of the former.

Theorem 3 (Last-iterate convergence rate of P-ACVI—Algorithm 2). Given a continuous operator $F$: $\mathcal{X} \rightarrow \mathbb{R}^n$, assume $F$ is monotone on $C_m$, as per Def. 1. Let $(x_K, y_K, \lambda_K)$ denote the last iterate of Algorithm 2. Then for all $K \in \mathbb{N}_+$, it holds that:

$$G(x_K, C) \leq O(\frac{1}{\sqrt{K}}), \text{ and } ||x^K - y^K|| \leq O(\frac{1}{\sqrt{K}}).$$

Remark 1. Notice that Theorem 3 requires fewer assumptions than Theorem 1. This follows from the fact that the central path is removed in the P-ACVI Algorithm. Hence, assumption (ii) in Theorem 1—which is used to guarantee the existence of the central path (see Appendix A)—is no longer needed.

6 Experiments

Methods. We compare ACVI, Inexact-ACVI (I-ACVI), and Projected-Inexact-ACVI (PI-ACVI) with projected variants of Gradient Descent Ascent (P-GDA), Extragradient (Korpelevich, 1976) (P-EG), Optimistic-GDA (Popov, 1980) (P-OGDA), and Lookahead-Minmax (Zhang et al., 2019; Chavdarova et al., 2021b) (P-ILA). For simplicity, we always use GDA as an inner optimizer for I-ACVI, PI-ACVI, and P-ACVI.

Problems. We study the empirical performance of the above methods on three different problems:

• **2D bilinear game**: a version of the bilinear game with $L_\infty$ constraints:

$$\min_{x_1 \in \Delta} \max_{x_2 \in \Delta} x_1^T x_2, \quad (2D-BG)$$

where $\Delta = \{x \in \mathbb{R} | -0.4 \leq x \leq 2.4\}$.

• **High-dimensional bilinear game**: a bilinear game where each player is a 500-dimensional vector. The iterates are constrained to the probability simplex. A parameter $\eta \in (0, 1)$ controls the rotational component of the game (when $\eta = 1$ the game is a potential game, when $\eta = 0$ the game is a Hamiltonian game):

$$\min_{x_1 \in \Delta} \max_{x_2 \in \Delta} \eta x_1^T x_1 + (1 - \eta) x_1^T x_2 - \eta x_2^T x_2, \quad (HBG)$$

$\Delta = \{x \in \mathbb{R}^{500} | x \geq 0, \text{ and } e^T x = 1\}$.

• **MNIST.** We train GANs on the MNIST (Lecun and Cortes, 1998) dataset. Similarly to Yang et al. (2022), we use linear inequality constraints and no equality con-
6.2 Projected-Inexact-ACVI

2D bilinear game. In Fig. 2 we show the dynamics of Projected Inexact ACVI (PI-ACVI) on the 2D game defined by (2D-BG). Compared to ACVI in Fig. 1, the iter-
ates converge to the solution without following the central path. A comparison with other optimizers is available in Appendix D.

**MNIST.** In Fig. 4 we compare PI-ACVI and baselines on the (C-GAN) game trained on the MNIST dataset. The projections are obtained using the greedy projection algorithm (Beck, 2017). Since ACVI was derived primarily for handling general constraints, a natural question that arises is how it (and its variants) perform when the projection is fast to compute. We observe that although the projection is fast to compute for these experiments, PI-ACVI still converges faster relative to the projection-based methods. Moreover, it gives more consistent improvements over the standard GDA baseline, compared to the projected extra-gradient method, which improves upon GDA in cases when the rotational component of $F$ is large.

### 6.3 Effect of Warm-up on I-ACVI and PI-ACVI

**I-ACVI.** The illustrative experiments in Fig. 1 motivate increasing the selected number of iterations $K$ only at the first iteration $t = 0$—which we denote with $K_0$, so as to ensure that the early iterates are close to the central path. As a reminder, the $K$ steps (corresponding to the loop line 7 in Algorithm 1) are bringing the iterates closer to the central path as $K \rightarrow \infty$ (see Appendix B for details). After those $K_0$ steps, $\mu$ is decayed, which moves the solution of the problem along the central path.

For I-ACVI, from Fig. 3(c)—where $\ell$ is fixed to 10—we observed that regardless of the selected value of $K_0$, (that is selected $K$ at any $t > 0$), it can be compensated by a large enough $K_0$.

**PI-ACVI.** We similarly study the impact of the warmup technique for the PI-ACVI method (Algorithm 2). Compared to I-ACVI, this method omits the outer loop over $t \in \mathbb{N}$. Hence, instead of varying $K_0$, we experiment with increasing the first $\ell$ at iteration $k = 0$, denoted by $\ell_0$. In Fig. 4 we solve the constrained MNIST problem with PI-ACVI using either $\ell_0 = 500$ or $\ell_0 = 100$, $\ell_+$ is set to 20 in both cases. We observe that using a larger $\ell_0$ leads to notably faster convergence.

**Overall conclusion.** We consistently observed that using a large $K_0$ or I-ACVI, or large $\ell_0$ for PI-ACVI, aids the convergence. On the other hand, $\ell$ and $K_+$ for I-ACVI, or $l_+$ for PI-ACVI have a lesser influence. See Appendix D for additional experiments and discussions.

### 7 Conclusion

In this paper, we revisit the state-of-the-art ACVI method derived to handle general constraints in the setting of constrained variational inequality problems. We showed that the last iterate of ACVI converges at a rate of order $O(1/\sqrt{K})$ for monotone VIs. Our theoretical result is novel in that it does not rely on a first-order smoothness assumption for the operator. As the sub-problems of ACVI cannot always be solved in closed form and may need to be solved approximately, we also studied an inexact ACVI (I-ACVI) algorithm that uses warm-starting for its sub-problems. We show that the last iterate of I-ACVI recovers the same convergence rate as ACVI for monotone VIs, provided that the errors stay bounded. Motivated by machine learning applications, we focused on the special case in which the inequality constraints are simple in the sense that, their projection is fast to compute. We derived a variant of ACVI, referred to as P-ACVI, and showed that P-ACVI converges with a $O(1/\sqrt{K})$ rate. Finally, our experiments (i) provided insights into the dynamics of I-ACVI when the sub-problems are solved approximately: (ii) we showed the impact of the warm-start technique, and (iii) we demonstrated the advantages of these ACVI variants relative to the standard projection-based algorithms.

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A Additional Background

In this section, we give additional background, such as: (i) some relevant definitions—see Appendix A.1, (ii) details of the ACVI method including its derivation, required for the proofs—see Appendix A.2, as well as (iii) descriptions of the baseline methods used in § 6 of the main part—see Appendix A.4.

A.1 Additional VI definitions and equivalent formulations

Here we give the complete statement of the definition of an $L$-Lipschitz operator for completeness.

**Definition 5 (L-Lipschitz operator).** Let $F : \mathcal{X} \supseteq S \to \mathbb{R}^n$ be an operator, we say that $F$ satisfies $L$-first-order smoothness on $S$ if $F$ is an $L$-Lipschitz map; that is, there exists $L > 0$ such that:

$$
\|F(x) - F(x')\| \leq L \|x - x'\|, \quad \forall x, x' \in S.
$$

To define cocoercive operators—mentioned in the discussions of the related work, we will first introduce the inverse of an operator.

Viewing an operator $F : \mathcal{X} \to \mathbb{R}^n$ as the graph $GrF = \{(x, y) | x \in \mathcal{X}, y = F(x)\}$, its inverse $F^{-1}$ is defined as $GrF^{-1} = \{(y, x) | (y, x) \in GrF\}$ (see, e.g., Ryu and Yin, 2022).

**Definition 6 ($\frac{1}{\mu}$-cocoercive operator).** An operator $F : \mathcal{X} \supseteq S \to \mathbb{R}^n$ is $\frac{1}{\mu}$-cocoercive (or $\frac{1}{\mu}$-inverse strongly monotone) on $S$ if its inverse graph $F^{-1}$ is $\mu$-strongly monotone on $S$, that is,

$$
\exists \mu > 0, \quad \text{s.t.} \quad \langle x - x', F(x) - F(x') \rangle \geq \mu \|F(x) - F(x')\|^2, \forall x, x' \in S.
$$

It is star $\frac{1}{\mu}$-cocoercive if the above holds when setting $x' \equiv x^*$ where $x^*$ denotes a solution, that is:

$$
\exists \mu > 0, \quad \text{s.t.} \quad \langle x - x^*, F(x) - F(x^*) \rangle \geq \mu \|F(x) - F(x^*)\|^2, \forall x \in S, x^* \in S^*_x, F.
$$

Notice that cocoercivity implies monotonicity, and is thus a stronger assumption.

In the following, we will make use of the natural and normal mappings of an operator $F : \mathcal{X} \to \mathbb{R}^n$, where $\mathcal{X} \subseteq \mathbb{R}^n$. We denote the projection to the set $\mathcal{X}$ with $\Pi_\mathcal{X}$. Following similar notation as in (Facchinei and Pang, 2003), the natural map $F^{\text{NAT}}_\mathcal{X} : \mathcal{X} \to \mathbb{R}^n$ is defined as:

$$
F^{\text{NAT}}_\mathcal{X} \triangleq x - \Pi_\mathcal{X}(x - F(x)), \quad \forall x \in \mathcal{X}, \quad \text{(F-NAT)}
$$

whereas the normal map $F^{\text{NOR}}_\mathcal{X} : \mathbb{R}^n \to \mathbb{R}^n$ is:

$$
F^{\text{NOR}}_\mathcal{X} \triangleq F(\Pi_\mathcal{X}(x)) + x - \Pi_\mathcal{X}(x), \quad \forall x \in \mathbb{R}^n. \quad \text{(F-NOR)}
$$

Moreover, we have the following solution characterizations:

(i) $x^* \in S^*_x, F$ iff $F^{\text{NAT}}_\mathcal{X}(x^*) = 0$.

(ii) $x^* \in S^*_x, F$ iff $\exists x' \in \mathbb{R}^n$ s.t. $x^* = \Pi_\mathcal{X}(x')$ and $F^{\text{NOR}}_\mathcal{X}(x') = 0$.

A.2 Details on ACVI

For completeness, herein we recall the ACVI algorithm and show its derivation; see (Yang et al., 2022) for details. We will use these equations also for the proofs of our main results.

**Derivation of ACVI.** We first restate the cVI problem in a form that will allow us to derive an interior-point procedure. By the definition of cVI it follows (see §1.3 in Facchinei and Pang, 2003) that:

$$
x \in S^*_x, F \iff \begin{cases} w = x \\
x = \arg \min_z F(w)^T z \\
s.t. \quad \varphi(z) \leq 0 \\
Cz = d \end{cases} \iff \begin{cases} F(x) + \nabla \varphi^\top(x) \lambda + C^\top \nu = 0 \\
x = d \\
0 \leq \lambda \perp \varphi(x) \leq 0. \quad \text{(KKT)}
\end{cases}
$$
where $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ are dual variables. Recall that we assume that $\text{int } C \neq \emptyset$, thus, by the Slater condition (using the fact that $\varphi_i(x), i \in [m]$ are convex) and the KKT conditions, the second equivalence holds, yielding the KKT system of cVI. Note that the above equivalence also guarantees the two solutions coincide; see Facchinei and Pang (2003, Prop. 1.3.4 (b)).

Analogous to the method described in § 3, we add a log-barrier term to the objective to remove the inequality constraints and obtain the following modified version of (KKT):

\[
\begin{cases}
  \mathbf{w} = \mathbf{x} \\
  \mathbf{x} = \text{argmin } \mathbf{F}(\mathbf{w})^\top \mathbf{z} - \mu \sum_{i=1}^{m} \log (-\varphi_i(\mathbf{z})) \\
  \text{s.t. } \mathbf{Cz} = \mathbf{d}
\end{cases}
\]

\[
\Rightarrow \begin{cases}
  F(\mathbf{x}) + \nabla \varphi^\top \mathbf{x} \lambda + C^\top \nu = 0 \\
  \lambda \odot \varphi(\mathbf{x}) + \mu \mathbf{e} = 0 \\
  \mathbf{Cx} - \mathbf{d} = 0 \\
  \varphi(\mathbf{x}) < 0, \lambda > 0,
\end{cases}
\]

(KKT-2)

with $\mu > 0$, $\mathbf{e} \triangleq [1, \ldots, 1]^\top \in \mathbb{R}^m$. The equivalence holds by the KKT and the Slater condition.

The update rule at step $k$ is derived by the following sub-problem:

\[
\begin{align*}
\min_{\mathbf{x}} & F(\mathbf{w}_k)^\top \mathbf{x} - \mu \sum_{i=1}^{m} \log (-\varphi_i(\mathbf{x})) \\
\text{s.t. } & \mathbf{Cx} = \mathbf{d},
\end{align*}
\]

where we fix $\mathbf{w} = \mathbf{w}_k$. Notice that (i) $\mathbf{w}_k$ is a constant vector in this sub-problem, and (ii) the objective is split, making ADMM a natural choice to solve the sub-problem. To apply an ADMM-type method, we introduce a new variable $\mathbf{y} \in \mathbb{R}^n$ yielding:

\[
\begin{align*}
\min_{\mathbf{x}, \mathbf{y}} & F(\mathbf{w}_k)^\top \mathbf{x} + I[\mathbf{Cx} = \mathbf{d}] - \mu \sum_{i=1}^{m} \log (-\varphi_i(\mathbf{y})) \\
\text{s.t. } & \mathbf{x} = \mathbf{y}
\end{align*}
\]

(ACVI:sub-problem)

where:

\[
I[\mathbf{Cx} = \mathbf{d}] \triangleq \begin{cases}
0, & \text{if } \mathbf{Cx} = \mathbf{d} \\
+\infty, & \text{if } \mathbf{Cx} \neq \mathbf{d}
\end{cases}
\]

is a generalized real-valued convex function of $\mathbf{x}$.

As in Algorithm 1, for ACVI we also have the same projection matrix:

\[
P_c \triangleq \mathbf{I} - \mathbf{C}^\top (\mathbf{C} \mathbf{C}^\top)^{-1} \mathbf{C},
\]

(\(P_c\))

and:

\[
d_c \triangleq \mathbf{C}^\top (\mathbf{C} \mathbf{C}^\top)^{-1} \mathbf{d},
\]

(\(d_c\)-EQ)

The augmented Lagrangian of (ACVI:sub-problem) is thus:

\[
\mathcal{L}_\beta(\mathbf{x}, \mathbf{y}, \lambda) = F(\mathbf{w}_k)^\top \mathbf{x} + I[\mathbf{Cx} = \mathbf{d}] - \mu \sum_{i=1}^{m} \log(-\varphi_i(\mathbf{y})) + \langle \lambda, \mathbf{x} - \mathbf{y} \rangle + \frac{\beta}{2} \| \mathbf{x} - \mathbf{y} \|^2,
\]

(AL)

where $\beta > 0$ is the penalty parameter. Finally, using ADMM, we have the following update rule for $\mathbf{x}$ at step $k$:

\[
\mathbf{x}_{k+1} = \text{arg min } \mathcal{L}_\beta(\mathbf{x}, \mathbf{y}_k, \lambda_k)
\]

(Def-X)

\[
= \text{arg min } \mathbf{x} \in \mathbb{C} \frac{\beta}{2} \| \mathbf{x} - \mathbf{y}_k + \frac{1}{\beta} (F(\mathbf{w}_k) + \lambda_k) \|^2.
\]

This yields the following update for $\mathbf{x}$:

\[
\mathbf{x}_{k+1} = P_c \left( \mathbf{y}_k - \frac{1}{\beta} (F(\mathbf{w}_k) + \lambda_k) \right) + d_c.
\]

(X-EQ)
For \( y \) and the dual variable \( \lambda \), we have:
\[
y_{k+1} = \arg\min_y \mathcal{L}_\beta(x_{k+1}, y, \lambda_k) \tag{Def-Y}
\]
\[
= \arg\min_y \left( -\mu \sum_{i=1}^m \log \left( -\varphi_i(y) \right) + \frac{\beta}{2} \left\| y - x_{k+1} - \frac{1}{\beta} \lambda_k \right\|^2 \right) \tag{Y-EQ}
\]

To derive the update rule for \( w, w_k \) is set to be the solution of the following equation:
\[
w + \frac{1}{\beta} P_c F(w) - P_c y_k - \frac{1}{\beta} P_c \lambda_k - d_c = 0. \tag{W-EQ}
\]

The following theorem ensures the solution of (W-EQ) exists and is unique, see Appendix B in (Yang et al., 2022) for proof.

**Theorem 4 (W-EQ: solution uniqueness).** If \( F \) is monotone on \( C = \), the following statements hold true for the solution of (W-EQ):

1. it always exists,
2. it is unique, and
3. it is contained in \( C = \).

Finally, notice that \( w \) is redundant, since \( w_k = x_{k+1} \), and it is thus removed.

**The ACVI algorithm.** Algorithm 3 describes the (exact) ACVI algorithm.

### Algorithm 3 ACVI pseudocode.

1: **Input:** operator \( F : \mathcal{X} \to \mathbb{R}^n \), constraints \( Cx = d \) and \( \varphi_i(x) \leq 0, i = [m] \), hyperparameters \( \mu, \beta > 0, \delta \in (0, 1) \), number of outer and inner loop iterations \( T \) and \( K \), resp.
2: **Initialize:** \( y_0^{(0)} \in \mathbb{R}^n, \lambda_0^{(0)} \in \mathbb{R}^n \)
3: \( P_c \triangleq \mathcal{I} - C^T (CC^T)^{-1} C \)
4: \( d_c \triangleq C^T (CC^T)^{-1} d \)
5: for \( t = 0, \ldots, T - 1 \) do
6: \( \mu_t = \delta \mu_{t-1} \)
7: for \( k = 0, \ldots, K - 1 \) do
8: Set \( x_{k+1}^{(t)} \) to be the solution of: \( x + \frac{1}{\beta} P_c F(x) - P_c y_k^{(t)} + \frac{1}{\beta} P_c \lambda_k^{(t)} - d_c = 0 \) (w.r.t. \( x \))
9: \( y_{k+1}^{(t)} = \arg\min_y \mu \sum_{i=1}^m \log \left( -\varphi_i(y) \right) + \frac{\beta}{2} \left\| y - x_{k+1}^{(t)} - \frac{1}{\beta} \lambda_k^{(t)} \right\|^2 \)
10: \( \lambda_{k+1}^{(t)} = \lambda_k^{(t)} + \beta (x_{k+1}^{(t)} - y_{k+1}^{(t)}) \)
11: end for
12: \( (y_0^{(t+1)}, \lambda_0^{(t+1)}) \triangleq (y_K^{(t)}, \lambda_K^{(t)}) \)
13: end for

## A.3 Existence of the central path

In this section, we discuss the results that establish guarantees of the existence of the central path. Let:
\[
L(x, \lambda, \nu) \triangleq F(x) + \nabla \varphi^T(x) \lambda + C^T \nu, \quad \text{and} \quad h(x) = C^T x - d.
\]

For \( (\lambda, w, x, \nu) \in \mathbb{R}^{2m+n+p} \), let
\[
G(\lambda, w, x, \nu) \triangleq \begin{pmatrix} w \circ \lambda \\ w + \varphi(x) \\ L(x, \lambda, \nu) \\ \sum_{i=1}^m \log \left( -\varphi_i(x) \right) \\ 1^T(x) \end{pmatrix} \in \mathbb{R}^{2m+n+p},
\]
and

\[ H(\lambda, w, x, \nu) \triangleq \begin{pmatrix} w + \varphi(x) \\ L(x, \lambda, \nu) \\ h(x) \end{pmatrix} \in \mathbb{R}^{m+n+p}. \]

Let \( H_{++} \triangleq H(\mathbb{R}^{2m}_+ \times \mathbb{R}^{n+p} \times \mathbb{R}^p) \).

By (Corollary 11.4.24, Facchinei and Pang, 2003) we have the following proposition.

**Proposition 1** (sufficient condition for the existence of the central path). If \( F \) is monotone, either \( F \) is strictly monotone or one of \( \varphi_1 \) is strictly convex, and \( C \) is bounded. The following four statements hold for the functions \( G \) and \( H \):

1. \( G \) maps \( \mathbb{R}^{2m}_+ \times \mathbb{R}^{n+p} \) homeomorphically onto \( \mathbb{R}^{m}_+ \times H_{++} \);
2. \( \mathbb{R}^{m}_+ \times H_{++} \subseteq G(\mathbb{R}^{2m}_+ \times \mathbb{R}^{n+p}) \);
3. for every vector \( \alpha \in \mathbb{R}^{m}_+ \), the system

\[
H(\lambda, w, x, \nu) = 0, \quad w \circ \lambda = \alpha
\]

has a solution \((\lambda, w, x, \nu) \in \mathbb{R}^{2m}_+ \times \mathbb{R}^{n+p} \); and
4. the set \( H_{++} \) is convex.

### A.4 Saddle-point optimization methods

In this section, we describe in detail the saddle point methods that we compare with in the main paper in § 6. We denote the projection to the set \( X \) with \( \Pi_X \), and when the method is applied in the unconstrained setting \( \Pi_X \equiv I \).

For an example of the associated vector field and its Jacobian, consider the following constrained zero-sum game:

\[
\min_{x_1 \in X_1} \max_{x_2 \in X_2} f(x_1, x_2), \quad \text{(ZS-G)}
\]

where \( f : X_1 \times X_2 \to \mathbb{R} \) is smooth and convex in \( x_1 \) and concave in \( x_2 \). As in the main paper, we write \( x \triangleq (x_1, x_2) \in \mathbb{R}^n \). The vector field \( F : X \to \mathbb{R}^n \) and its Jacobian \( J \) are defined as:

\[
F(x) = \begin{bmatrix} \nabla_{x_1} f(x) \\ -\nabla_{x_2} f(x) \end{bmatrix}, \quad J(x) = \begin{bmatrix} \nabla^2_{x_1} f(x) & \nabla_{x_1} \nabla_{x_1} f(x) \\ -\nabla^2_{x_1} f(x) & -\nabla^2_{x_2} f(x) \end{bmatrix}.
\]

In the remaining of this section, we will only refer to the joint variable \( x \), and (with abuse of notation) the subscript will denote the step. Let \( \gamma \in [0, 1] \) denote the step size.

**(Projected) Gradient Descent Ascent (GDA).** The extension of gradient descent for the cVI problem is **gradient descent ascent** (GDA). The GDA update at step \( k \) is then:

\[
x_{k+1} = \Pi_X(x_k - \gamma F(x_k)). \quad \text{(GDA)}
\]

**(Projected) Extragradient (EG).** EG (Korpelevich, 1976) uses a “prediction” step to obtain an extrapolated point \( x_{k+1/2} \) using GDA: \( x_{k+1/2} = \Pi_X(x_k - \gamma F(x_k)) \), and the gradients at the extrapolated point are then applied to the current iterate \( x_k \):

\[
x_{k+1} = \Pi_X \left( x_k - \gamma F \left( \Pi_X(x_k - \gamma F(x_k)) \right) \right). \quad \text{(EG)}
\]

In the original EG paper, (Korpelevich, 1976) proved that the EG method (with a fixed step size) converges for monotone VIs, as follows.

**Theorem 5** (Korpelevich (1976)). Given a map \( F : X \to \mathbb{R}^n \), if the following is satisfied:

1. the set \( X \) is closed and convex,
2. \( F \) is single-valued, definite, and monotone on \( X \)–as per **Def. 1**,
3. **F** is L-Lipschitz—as per Asm. 5.

then there exists a solution \( x^* \in X \), such that the iterates \( x_k \) produced by the EG update rule with a fixed step size \( \gamma \in (0, \frac{1}{L}) \) converge to it, that is \( x_k \to x^* \), as \( k \to \infty \).

Facchinei and Pang (2003) also show that for any convex-concave function \( f \) and any closed convex sets \( x_1 \in X_1 \) and \( x_2 \in X_2 \), the EG method converges (Facchinei and Pang, 2003, Theorem 12.1.11).

**(Projected) Optimistic Gradient Descent Ascent (OGDA).** The update rule of Optimistic Gradient Descent Ascent (OGDA) (Popov, 1980) is:

\[
x_{n+1} = \Pi_X(x_n - 2\gamma F(x_n) + \gamma F(x_{n-1})).
\]  

**OGDA**

**(Projected) Lookahead–Minmax (LA).** The LA algorithm for min-max optimization (Chavdarova et al., 2021b), originally proposed for minimization by Zhang et al. (2019), is a general wrapper of a “base” optimizer where, at every step \( t \):

(i) a copy of the current iterate \( \tilde{x}_n \) is made: \( \tilde{x}_n \leftarrow x_n \),

(ii) \( \tilde{x}_n \) is updated \( k \geq 1 \) times, yielding \( \tilde{\omega}_{n+k} \), and finally

(iii) the actual update \( x_{n+1} \) is obtained as a point that lies on a line between the current \( x_n \) iterate and the predicted one \( \tilde{x}_{n+k} \):

\[
x_{n+1} \leftarrow x_n + \alpha (\tilde{x}_{n+k} - x_n), \quad \alpha \in [0, 1].
\]  

**LA**

In this work, we use solely GDA as a base optimizer for LA, and denote it with LAk-GDA.
B Missing Proofs

In this section, we provide the proofs of Theorems 1, 2 and 3, stated in the main part.

B.1 Proof of Theorem 1: Last-iterate convergence of ACVI for Monotone Variational Inequalities

Recall from Theorem 1 that we have the following assumptions:

- $F$ is monotone on $C_\infty$, as per Def. 1; and
- either $F$ is strictly monotone on $C$ or one of $\varphi_i$ is strictly convex.

B.1.1 Setting and notations

Before we proceed with the lemmas needed for the proof of Theorem 1, herein we introduce some definitions and notations.

Sub-problems and definitions. We remark that the ACVI derivation—given in Appendix A.2—is helpful for following the proof herein. Recall from it, that in order to derive the update rule for $x$ we introduced a new variable $w$ and the relevant subproblem that yields the update rule for $x$ includes a term $(F(w), x)$, where $F$ is evaluated at some fixed point. As the proof relates the $x^{(t)}$ iterate of ACVI with the solution $x^{\star}$ of (KKT-2), in the following we will define two different maps each with fixed $w \equiv x^{\mu_t}$ and $w \equiv x^{(t)}_{k+1}$. That is, for convenience, we define the following maps from $\mathbb{R}^n$ to $\mathbb{R}^n$:

\[
\begin{align*}
    f^{(t)}(x) & \equiv F(x^{\mu_t})^\top x + 1(Cx = d), \quad (f^{(t)})_k \\
    f^{(t)}_k(x) & \equiv F(x^{(t)}_{k+1})^\top x + 1(Cx = d), \quad \text{and} \quad (f^{(t)})_k \\
    g^{(t)}(y) & \equiv -\mu_t \sum_{i=1}^m \log (-\varphi_i(y)), \quad (g^{(t)})
\end{align*}
\]

where $x^{\mu_t}$ is a solution of (KKT-2) when $\mu = \mu_t$, and $x^{(t)}_{k+1}$ is the solution of the $x$ sub-problem in ACVI at step $(t, k)$—see line 8 in Algorithm 3. Note that the existence of $x^{\mu_t}$ is guaranteed by the existence of the central path; see Appendix A.3. Also, notice that $f^{(t)}$, $f^{(t)}_k$ and $g^{(t)}$ are all convex functions. In the following, unless otherwise specified, we drop the superscript $(t)$ of $x^{(t)}_{k+1}$, $f^{(t)}$, $f^{(t)}_k$ and subscript $t$ of $\mu_t$ to simplify the notation.

In the remaining of this section, we introduce the notation of the solution points of the above KKT systems, as well as that of the ACVI iterates.

Let $y^{\mu} = x^{\mu}$. In this case, from (KKT-2) we can see that $(x^{\mu}, y^{\mu})$ is an optimal solution of:

\[
\begin{align*}
    \min_{x,y} f(x) + g(y) \\
    \text{s.t.} \quad x = y
\end{align*}
\]

There exists $\lambda^{\mu} \in \mathbb{R}^n$ such that $(x^{\mu}, y^{\mu}, \lambda^{\mu})$ is a KKT point of $(f\text{-Pr})$. By Prop. 1, $x^{\mu} = y^{\mu}$ converges to a solution of (KKT). We denote this solution by $x^{\star}$. Then $(x^{\mu}, y^{\mu}, \lambda^{\mu})$ converges to the KKT point of (ACVI:sub-problem) with $w_k = x^{\star}$. Let $(x^{\star}, y^{\star}, \lambda^{\star})$ denote this KKT point, where $x^{\star} = y^{\star}$.

On the other hand, let us denote with $(x^{\mu}_k, y^{\mu}_k, \lambda^{\mu}_k)$ the KKT point of the analogous problem of $f_k(\cdot)$ of:

\[
\begin{align*}
    \min_{x,y} f_k(x) + g(y) \\
    \text{s.t.} \quad x = y
\end{align*}
\]

Note that the KKT point $(x^{\mu}_k, y^{\mu}_k, \lambda^{\mu}_k)$ is guaranteed to exist by Slater’s condition. Also, recall from the derivation of ACVI, that $(f_k\text{-Pr})$ is “non-symmetric” for $x, y$ when using ADMM-like approach, in the sense that: when we derive the update rule for $x$ we fix $y$ to $y_k$ (see Def-X), but when we derive the update rule for $y$ we fix $x$ to $x_{k+1}$ (see Def-Y). This fact is used later in (L2-1) and L2-2 in Lemma 2 for example.
Then, for the solution point which we denoted with \((x_k^*, y_k^*, \lambda_k^*)\) we also have that \(x_k^* = y_k^*\). By noticing that the objective above is equivalent to \(F(x_{k+1})^T x + \mathbb{1}(C x = d) - \mu t \sum_{i=1}^m \log (-\varphi_i(y))\), it follows that the above problem \((f_k\text{-Pr})\) is an approximation of:

\[
\begin{align*}
\min_x & \langle F(x_{k+1}), x \rangle + \mathbb{1}(C x = d) + \mathbb{1}(\varphi(y) \leq 0) \\
\text{s.t.} & \quad x = y
\end{align*}
\]

(f_k\text{-Pr-2})

where, as a reminder, the constraint set \(C \subseteq X\) is defined as an intersection of finitely many inequalities and linear equalities:

\[
C = \{ x \in \mathbb{R}^n | \varphi_i(x) \leq 0, \, i \in [m], \, C x = d \},
\]

(CS)

where each \(\varphi_i : \mathbb{R}^n \to \mathbb{R}, \, C \in \mathbb{R}^{p \times n}, \, d \in \mathbb{R}^p\), and we assumed \(\text{rank}(C) = p\).

In fact, when \(\mu \to 0^+\), corollary 2.11 in CHU (1998) guarantees that \((x_k^*, y_k^*, \lambda_k^*)\) converges to a KKT point of problem \((f_k\text{-Pr-2})\)—which immediately follows here since \((f_k\text{-Pr-2})\) is a convex problem. Let \((x_k^*, y_k^*, \lambda_k^*)\) denote this KKT point, where \(x_k^* = y_k^*\).

**Summary.** To conclude, \((x^*, y^*, \lambda^*)\)—the solution of \((f\text{-Pr})\), converges to \((x^*, y^*, \lambda^*)\), a KKT point of \((ACVI\text{-sub-problem})\) with \(w_k = x^*\), where \(x^* = y^* \in S^*_C, F; \, (x_k^*, y_k^*, \lambda_k^*)\) converges to \((x^*_k, y^*_k, \lambda^*_k)\)—a KKT point of problem \((f_k\text{-Pr-2})\), where \((x_k^*, y_k^*, \lambda_k^*)\) (in which \(x_k^* = y_k^*\)) is a KKT point of problem \((f_k\text{-Pr})\). Table 1 summarizes the notation for convenience.

| Solution point | Description | Problem |
|----------------|-------------|---------|
| \((x^*, y^*, \lambda^*)\) | cVI solution, more precisely \(x^* = y^* \in S^*_C, F\) | \((cVI)\) |
| \((x^{\mu i}, y^{\mu i}, \lambda^{\mu i})\) | central path point, also solution point of the sub-problem with fixed \(F(x^{\mu i})\) | \((f\text{-Pr})\) |
| \((x^{\mu i}_k, y^{\mu i}_k, \lambda^{\mu i}_k)\) | solution point of the sub-problem with fixed \(F(x^{(t)}_{k+1})\) where the indicator function is replaced with \(\log\)-barrier | \((f_k\text{-Pr})\) |
| \((x^*_k, y^*_k, \lambda^*_k)\) | solution point of the sub-problem with fixed \(F(x^{(t)}_{k+1})\) | \((f_k\text{-Pr-2})\) |

Table 1: Summary of the notation used for the solution points of the different problems. \((f_k\text{-Pr})\) is an approximation of \((f_k\text{-Pr-2})\) which replaces the indicator function with \(\log\)-barrier. The \(t\) emphasizes that these solution points change for different \(\mu(t)\). Where clear from the context that we focus on a particular step \(t\), we drop the super/sub-script \(t\) to simplify the notation. See Appendix B.1.1.

**B.1.2 Intermediate results**

We will repeatedly use the following proposition that relates the output differences of \(f_k(\cdot)\) and \(f(\cdot)\), defined above.

**Proposition 2** (Relation between \(f_k\) and \(f\)). If \(F\) is monotone, then \(\forall k \in \mathbb{N}\), we have that:

\[
f_k(x_{k+1}) - f_k(x^\mu) \geq f(x_{k+1}) - f(x^\mu).
\]

**Proof of Proposition 2.** It suffices to notice that:

\[
f_k(x_{k+1}) - f_k(x^\mu) - (f(x_{k+1}) - f(x^\mu)) = \langle F(x_{k+1}) - F(x^\mu), x_{k+1} - x^\mu \rangle.
\]

The proof follows by applying the definition for monotonicity to the right-hand side.  

We will use the following lemmas.
Lemma 1. For all \( x \) and \( y \), we have:

\[
f(x) + g(y) - f(x^\mu) - g(y^\mu) + (\lambda^\mu, x - y) \geq 0, \tag{L1-f}
\]

and:

\[
f_k(x) + g(y) - f_k(x_k^\mu) - g(y_k^\mu) + (\lambda_k^\mu, x - y) \geq 0. \tag{L1-f_k}
\]

Proof. The Lagrange function of \((f_{-Pr})\) is:

\[
L(x, y, \lambda) = f(x) + g(y) + \langle \lambda, x - y \rangle.
\]

And by the property of KKT point, we have:

\[
L(x^\mu, y^\mu, \lambda) \leq L(x^\mu, y^\mu, \lambda^\mu) \leq L(x, y, \lambda), \quad \forall (x, y, \lambda),
\]

from which \((L1-f)\) follows.

\((L1-f_k)\) can be shown in an analogous way. \(\square\)

The following lemma lists some simple but useful facts that we will use in the following proofs.

Lemma 2. For the problems \((f_{-Pr})\), \((f_k_{-Pr})\) and the \( x_k, y_k, \lambda_k \) of Algorithm 3, we have:

\[
0 \in \partial f_k(x_{k+1}) + \lambda_k + \beta(x_{k+1} - y_k), \tag{L2-1}
\]

\[
0 \in \partial g(y_{k+1}) - \lambda_k - \beta(x_{k+1} - y_{k+1}), \tag{L2-2}
\]

\[
\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_{k+1}), \tag{L2-3}
\]

\[
-\lambda^\mu \in \partial f(x^\mu), \tag{L2-4}
\]

\[
-\lambda_k^\mu \in \partial f_k(x_k^\mu), \tag{L2-5}
\]

\[
\lambda^\mu \in \partial g(y^\mu), \tag{L2-6}
\]

\[
\lambda_k^\mu \in \partial g(y_k^\mu), \tag{L2-7}
\]

\[
x^\mu = y^\mu, \tag{L2-8}
\]

\[
x_k^\mu = y_k^\mu. \tag{L2-9}
\]

Remark 2. Since \( g \) is differentiable, \( \partial g \) could be replaced by \( \nabla g \) in Lemma 2. Here we use \( \partial g \) so that the results could be easily extended to Lemma 19 for the proofs of Theorem 3, where we replace the current \( g(y) \) by the indicator function \( 1(\varphi(y) \leq 0) \), which is non-differentiable.

Proof of Lemma 2. We can rewrite \((AL)\) as:

\[
L_\beta(x, y, \lambda) = f^k(x) + g(y) + (\lambda, x - y) + \frac{\beta}{2} \|x - y\|^2. \tag{re-AL}
\]

\((L2-1)\) and \((L2-2)\) follow directly from \((\text{Def-X})\) and \((\text{Def-Y})\), resp. \((L2-3)\) follows from line 10 in Algorithm 3, and \((L2-4)-(L2-9)\) follows by the property of the KKT point. \(\square\)

We also define the following two maps (whose naming will be evident from the inclusions shown after):

\[
\hat{\nabla} f_k(x_{k+1}) \triangleq -\lambda_k - \beta(x_{k+1} - y_k), \quad \text{and} \quad \hat{\nabla} g(y_{k+1}) \triangleq \lambda_k + \beta(x_{k+1} - y_{k+1}). \tag{\hat{\nabla} f_k, \hat{\nabla} g}
\]

Then, from \((L2-1)\) and \((L2-2)\) it follows that:

\[
\hat{\nabla} f_k(x_{k+1}) \in \partial f_k(x_{k+1}) \quad \text{and} \quad \hat{\nabla} g(y_{k+1}) \in \partial g(y_{k+1}). \tag{1}
\]

We continue with two equalities for the dot products involving \( \hat{\nabla} f_k \) and \( \hat{\nabla} g \).
Lemma 3. For the iterates $x_{k+1}$, $y_{k+1}$, and $\lambda_{k+1}$ of the ACVI—Algorithm 3—we have:
\[
\langle \nabla g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, y - y_{k+1} \rangle, \tag{L3-1}
\]
and
\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x + y \rangle + \beta (y_{k+1} + y - y_{k+1} + x_{k+1} - x). \tag{L3-2}
\]

Proof of Lemma 3. The first part of the lemma (L3-1), follows trivially by noticing that $\nabla g(y_{k+1}) = \lambda_{k+1}$.

For the second part, from (L2-3), $\langle \nabla f_k \rangle$ and $\nabla g$ we have:
\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x \rangle = -\langle \lambda_k + \beta (x_{k+1} - y_k), x_{k+1} - x \rangle
\]
and
\[
\langle \nabla g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, y - y_{k+1} \rangle.
\]

Adding these together yields (L3-2). \qed

The following Lemma further builds on the previous Lemma 3, and upper-bounds some dot products involving $\nabla f_k$ and $\nabla g$ with a sum of only squared norms.

Lemma 4. For the $x_{k+1}$, $y_{k+1}$, and $\lambda_{k+1}$ iterates of the ACVI—Algorithm 3—we have:
\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x^\mu \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^\mu \rangle + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle
\leq \frac{1}{2\beta} \| \lambda_k - \lambda^\mu \|^2 - \frac{1}{2\beta} \| \lambda_{k+1} - \lambda^\mu \|^2 + \frac{\beta}{2} \| y^\mu - y_k \|^2 - \frac{\beta}{2} \| y^\mu - y_{k+1} \|^2
\]
and
\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x^\mu \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^\mu \rangle + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle
\leq \frac{1}{2\beta} \| \lambda_k - \lambda^\mu \|^2 - \frac{1}{2\beta} \| \lambda_{k+1} - \lambda^\mu \|^2 + \frac{\beta}{2} \| y^\mu - y_k \|^2 - \frac{\beta}{2} \| y^\mu - y_{k+1} \|^2.
\]

Proof of Lemma 4. For the left-hand side of the first part of Lemma 4:
\[
LHS = \langle \nabla f_k(x_{k+1}), x_{k+1} - x^\mu \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^\mu \rangle + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle,
\]
we let $(x, y, \lambda) = (x^\mu, y^\mu, \lambda^\mu)$ in (L3-2), and using the result of that lemma we get that:
\[
LHS = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x^\mu + y^\mu \rangle + \beta (y_{k+1} + y_k, x_{k+1} - x^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle,
\]
and since $x^\mu = y^\mu$ (L2-8):
\[
LHS = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} \rangle + \beta (y_{k+1} + y_k, x_{k+1} - x^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle
\]
\[
= -\langle \lambda_{k+1} - \lambda^\mu, x_{k+1} - y_{k+1} \rangle + \beta (y_{k+1} + y_k, x_{k+1} - x^\mu),
\]
where in the last equality we combined the first and last term together. Using (L2-3) that $\frac{1}{\beta} (\lambda_{k+1} - \lambda_k) = (x_{k+1} - y_{k+1})$ yields (for the second term above, we add and subtract $y_{k+1}$ in its second argument, and use $x^\mu = y^\mu$):
\[
LHS = -\frac{1}{\beta} (\lambda_{k+1} - \lambda^\mu, \lambda_{k+1} - \lambda_k) + (y_{k+1} + y_k, \lambda_{k+1} - \lambda_k) - \beta (y_{k+1} + y_k, y_{k+1} + y_k + y^\mu) \tag{2}
\]
Using the 3-point identity—that for any vectors \( a, b, c \) it holds \( \langle b - a, b - c \rangle = \frac{1}{2}(\|a - b\|^2 + \|b - c\|^2 - \|a - c\|^2) \)—for the first term above we get that:

\[
\langle \lambda_{k+1} - \lambda^\mu, \lambda_{k+1} - \lambda_k \rangle = \frac{1}{2} \left( \|\lambda_{k+1} - \lambda^\mu\|^2 + \|\lambda_{k+1} - \lambda_k\|^2 - \|\lambda_{k+1} - \lambda^\mu\|^2 \right),
\]

and similarly,

\[
\langle -y_{k+1} + y_k, -y_{k+1} + y^\mu \rangle = \frac{1}{2} \left( \|y_{k+1} - y^\mu\|^2 - \|y_{k+1} - y^\mu\|^2 - \|y_{k+1} + y_k\|^2 \right),
\]

and by adding these together we get:

\[
LHS = \frac{1}{2\beta} \|\lambda_{k+1} - \lambda^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} - y^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2 + \langle -y_{k+1} + y_k, \lambda_{k+1} - \lambda_k \rangle.
\]

On the other hand, (L3-1) which states that \( \langle \nabla g(y_{k+1}), y_{k+1} - y \rangle + \langle \lambda_{k+1}, -y_{k+1} + y \rangle = 0 \), also asserts:

\[
\langle \nabla g(y_k), y_k - y \rangle + \langle \lambda_k, -y_k + y \rangle = 0.
\]

Letting \( y = y_k \) in (L3-1), and \( y = y_{k+1} \) in (4), and adding them together yields:

\[
\langle \nabla g(y_{k+1}) - \nabla g(y_k), y_{k+1} - y_k \rangle + \langle \lambda_{k+1} - \lambda_k, -y_{k+1} + y_k \rangle = 0.
\]

By the monotonicity of \( \partial g \) we know that the first term of the above equality is non-negative. Thus, we have:

\[
\langle \lambda_{k+1} - \lambda_k, -y_{k+1} + y_k \rangle \leq 0.
\]

Lastly, plugging it into (3) gives the first inequality of Lemma 4.

The second inequality of Lemma 4 follows similarly.

The following Lemma upper-bounds the sum of (i) the difference of \( f(\cdot) \) evaluated at \( x_{k+1} \) and at \( x^\mu \) and (ii) the difference of \( g(\cdot) \) evaluated at \( y_{k+1} \) and at \( y^\mu \); up to a term that depends on \( x_{k+1} - y_{k+1} \) as well. Recall that \( (x^\mu, y^\mu, \lambda^\mu) \) is a point on the central path.

**Lemma 5.** For the \( x_{k+1}, y_{k+1}, \) and \( \lambda_{k+1} \) \( \Lambda \)CVI—Algorithm 3—we have:

\[
f(x_{k+1}) + g(y_{k+1}) - f(x^\mu) - g(y^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle \\
\leq \frac{1}{2\beta} \|\lambda_k - \lambda^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} - y^\mu\|^2 \\
+ \frac{\beta}{2} \|y_{k+1} - y^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \\
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2
\]

(L5)

**Proof of Lemma 5.** From the convexity of \( f_k(x) \) and \( g(y) \); from proposition 2 on the relation between \( f_k(\cdot) \) and \( f(\cdot) \) which asserts that \( f(x_{k+1}) - f(x^\mu) \leq f_k(x_{k+1}) - f_k(x^\mu) \); as well as from Eq. (1) which asserts that \( \nabla f_k(x_{k+1}) \in \partial f_k(x_{k+1}) \) and \( \nabla g(y_{k+1}) \in \partial g(y_{k+1}) \); it follows for the LHS of Lemma 5 that:

\[
f(x_{k+1}) + g(y_{k+1}) - f(x^\mu) - g(y^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle \\
\leq f_k(x_{k+1}) + g(y_{k+1}) - f_k(x^\mu) - g(y^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle \\
\leq \langle \nabla f_k(x_{k+1}), x_{k+1} - x^\mu \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^\mu \rangle + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle
\]

(6)
Finally, by plugging in the first part of Lemma 4, Lemma 5 follows, that is:

\[
\begin{align*}
    f(x_{k+1}) + g(y_{k+1}) - f(x^*) - g(y^*) + \langle \lambda^*, x_{k+1} - y_{k+1} \rangle \\
    \leq \frac{1}{2\beta} \|x_{k+1} - x^*\|^2 - \frac{1}{2\beta} \|x_{k+1} - x_k\|^2 \\
    + \beta \|y_k + y^*\|^2 - \beta \|y_{k+1} + y^*\|^2 \\
    - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2.
\end{align*}
\] (7)

The following theorem upper bounds the analogous quantity but for \(f_k(\cdot)\) (instead of \(f\) as does Lemma 5), and further asserts that the difference between the \(x_{k+1}\) and \(y_{k+1}\) iterates of exact ACVI (Algorithm 3) tends to 0 asymptotically. The inequality in Theorem 6 plays an important role later when deriving the nonasymptotic convergence rate of ACVI.

**Theorem 6** (Asymptotic convergence of \((x_{k+1} - y_{k+1})\) of ACVI). For the \(x_{k+1}, y_{k+1},\) and \(\lambda_{k+1}\) iterates of the ACVI—Algorithm 3—we have:

\[
f_k(x_{k+1}) - f_k(x^*_k) + g(y_{k+1}) - g(y^*_k) \leq \|\lambda_{k+1}\| \|x_{k+1} - y_{k+1}\| + \beta \|y_{k+1} - y_k\| \|x_{k+1} - x^*_k\| \to 0, \quad \text{(T6-fk-UB)}
\]

and

\[
x_{k+1} - y_{k+1} \to 0, \quad \text{as } k \to \infty.
\]

**Proof of Lemma 6.** Recall from (L1-f) of Lemma 1 that by setting \(x \equiv x_{k+1}, y \equiv y_{k+1}\) it asserts that:

\[
f(x_{k+1}) - f(x^*) + g(y_{k+1}) - g(y^*) + \langle \lambda^*, x_{k+1} - y_{k+1} \rangle \geq 0.
\]

Further, notice that the LHS of the above inequality overlaps with that of (L5). This implies that the RHS of (L5) has to be non-negative. Hence, we have that:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \leq \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 \\
+ \frac{\beta}{2} \|y_{k+1} + y_k\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2.
\] (8)

Summing over \(k = 0, \ldots, \infty\) gives:

\[
\sum_{k=0}^{\infty} \left( \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \right) \leq \frac{1}{2\beta} \|\lambda_0 - \lambda^*\|^2 + \frac{\beta}{2} \|y_0 + y^*\|^2,
\]

from which we deduce that \(\lambda_{k+1} - \lambda_k \to 0\) and \(y_{k+1} - y_k \to 0\).

Also notice that by simply reorganizing (8) we have:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \\
\leq \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \\
= \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2.
\] (9)

where the second inequality follows because the norm is non-negative.

From (9) we can see that \(\|\lambda_k\|\) and \(\|y_k - y^*\|^2\) are bounded for all \(k\), as well as \(\|\lambda_k\|\). Recall that:

\[
\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_{k+1}) - \frac{1}{2\beta} (x_{k+1} - x^*) + \beta(-y_{k+1} + y^*),
\]
where in the last equality we add and subtract $x^\mu = y^\mu$. Combining this with the fact that $\lambda_{k+1} - \lambda_k \to 0$ (see above), we deduce that $x_{k+1} - y_{k+1} \to 0$ and that $x_{k+1} - x^\mu$ is also bounded.

Using the convexity of $f_k(\cdot)$ and $g(\cdot)$ for the LHS of Theorem 6 we have:

$$\text{LHS} = f_k(x_{k+1}) - f_k(x_k^\mu) + g(y_{k+1}) - g(y_k^\mu) \leq \langle \nabla f_k(x_{k+1}), x_{k+1} - x_k^\mu \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y_k^\mu \rangle.$$

Using (L3-2) with $x \equiv x_k^\mu$, $y \equiv y_k^\mu$ we have:

$$\text{LHS} \leq -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x_k^\mu + y_k^\mu \rangle + \beta \langle -y_{k+1} + y_k, x_{k+1} - x_k^\mu \rangle.$$

Hence, it follows that:

$$f_k(x_{k+1}) - f_k(x_k^\mu) + g(y_{k+1}) - g(y_k^\mu) \leq -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} \rangle + \beta \langle -y_{k+1} + y_k, x_{k+1} - x_k^\mu \rangle$$

$$\leq \|\lambda_{k+1}\| \|x_{k+1} - y_{k+1}\| + \beta \|y_{k+1} - y_k\| \|x_{k+1} - x_k^\mu\|,$$

where the last inequality follows from Cauchy-Schwarz.

Recall that $C$ is compact and $D$ is the diameter of $C$:

$$D \triangleq \sup_{x,y \in C} \|x - y\|.$$

Thus, we have:

$$\|y_{k+1} - y_k^\mu\| = \|y_{k+1} - y^\mu\| + \|y^\mu - y_k^\mu\| \leq \|y_{k+1} - y^\mu\| + D,$$

which implies that $\|y_k - y_k^\mu\|$ are bounded for all $k$. Since:

$$\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_{k+1}) = \beta(x_{k+1} - x_k^\mu) + \beta(-y_{k+1} + y_k^\mu),$$

we deduce that $x_{k+1} - x_k^\mu$ is also bounded. Thus, we have (T6-$f_k$-UB).

The following lemma states an important intermediate result that ensures that $\frac{1}{\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} - y_k\|^2$ does not increase.

**Lemma 6** (non-increment of $\frac{1}{\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2$). For the $x_{k+1}$, $y_{k+1}$, and $\lambda_{k+1}$ iterates of the ACVI—Algorithm 3—we have:

$$\frac{1}{\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \leq \frac{1}{\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_{k-1}\|^2. \tag{L6}$$

**Proof of Lemma 6.** (L3-2) gives:

$$\langle \nabla f_{k-1}(x_k), x_k - x \rangle + \langle \nabla g(y_k), y_k - y \rangle$$

$$= -\langle \lambda_k, x_k - y_k - x + y \rangle + \beta \langle -y_k + y_{k-1}, x_k - x \rangle. \tag{10}$$

Letting $(x, y, \lambda) = (x_k, y_k, \lambda_k)$ in (L3-2) and $(x, y, \lambda) \equiv \frac{1}{\beta} (x_{k+1}, y_{k+1}, \lambda_{k+1})$ in (10), and adding them together, and
using (L2-3) yields:
\[
\langle \nabla f_k (x_{k+1}) - \nabla f_{k-1} (x_k), x_{k+1} - x_k \rangle + \langle \nabla g(y_{k+1}) - \nabla g(y_k), y_{k+1} - y_k \rangle \\
= - \langle \lambda_{k+1} - \lambda_k, x_{k+1} - y_{k+1} - x_k + y_k \rangle + \beta (-y_{k+1} + y_k - (-y_k + y_{k-1}), x_{k+1} - x_k) \\
= - \frac{1}{\beta} \langle \lambda_{k+1} - \lambda_k, \lambda_{k+1} - \lambda_k - (\lambda_k - \lambda_{k-1}) \rangle \\
+ \langle -y_{k+1} + y_k + (y_k - y_{k-1}), \lambda_{k+1} - \lambda_k + \beta y_{k+1} - (\lambda_k - \lambda_{k-1} + \beta y_k) \rangle \\
= \frac{1}{2\beta} \left[ ||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2 \right] \\
+ \frac{\beta}{2} \left[ ||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2 - ||-y_{k+1} + y_k - (-y_k + y_{k-1})||^2 \right] \\
+ \langle -y_{k+1} + y_k - (-y_k + y_{k-1}), \lambda_{k+1} - \lambda_k - (\lambda_k - \lambda_{k-1}) \rangle \\
= \frac{1}{2\beta} \left[ ||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2 \right] \\
+ \frac{\beta}{2} \left[ ||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2 \right] \\
+ \langle -y_{k+1} + y_k - (-y_k + y_{k-1}), \lambda_{k+1} - \lambda_k - (\lambda_k - \lambda_{k-1}) \rangle \\
\leq \frac{1}{2\beta} \left( ||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2 \right) + \frac{\beta}{2} \left( ||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2 \right). \\
\]

By the convexity of \( f_k \) and \( f_{k-1} \), we have
\[
\langle \nabla f_k (x_{k+1}), x_{k+1} - x_k \rangle \geq f_k(x_{k+1}) - f_k(x_k), \\
\langle \nabla f_{k-1} (x_k), x_{k+1} - x_k \rangle \geq f_{k-1}(x_k) - f_{k-1}(x_{k+1}).
\]

Adding them together gives that:
\[
\langle \nabla f_k (x_{k+1}) - \nabla f_{k-1} (x_k), x_{k+1} - x_k \rangle \geq f_k(x_{k+1}) - f_{k-1}(x_{k+1}) - f_k(x_k) + f_{k-1}(x_k) \\
= \langle F(x_{k+1}) - F(x_k), x_{k+1} - x_k \rangle \geq 0.
\]

Thus by the monotonicity of \( F \) and \( \nabla g \), (L6) follows.

\[\square\]

**Lemma 7.** If \( F \) is monotone on \( C_\infty \), then for Algorithm 3, we have:
\[
f_K (x_{K+1}) + g(y_{K+1}) \leq f_K (x^\mu_K) - f_K (x^\mu_K) - f_K (x^\mu_K) \leq \frac{1}{\sqrt{\beta}} \frac{\Delta^\mu}{K + 1} + \left( 2\sqrt{\Delta^\mu} + \frac{1}{\sqrt{\beta}} \right) \sqrt{\frac{\Delta^\mu}{K + 1}} + \frac{\sqrt{\Delta^\mu}}{\beta (K + 1)}, \tag{L7-1}
\]

and
\[
||x_{K+1} - y_{K+1}|| \leq \sqrt{\frac{\Delta^\mu}{\beta (K + 1)}}, \tag{L7-2}
\]

where \( \Delta^\mu \triangleq \frac{1}{2} ||\lambda_0 - \lambda^\mu||^2 + \beta ||y_0 - y^\mu||^2 \).

**Proof of Lemma 7.** Summing (8) over \( k = 0, 1, \ldots, K \) and using the monotonicity of \( \frac{1}{2\beta} ||\lambda_{k+1} - \lambda_k||^2 + \frac{\beta}{2} ||y_{k+1} + y_k||^2 \) from Lemma 6, we have:
\[
\frac{1}{\beta} ||\lambda_{K+1} - \lambda_K||^2 + \beta ||y_{K+1} + y_K||^2 \\
\leq \frac{1}{K + 1} \sum_{k=0}^{K} \left( \frac{1}{2\beta} ||\lambda_{k+1} - \lambda_k||^2 + \frac{\beta}{2} ||y_{k+1} + y_k||^2 \right) \\
\leq \frac{1}{K + 1} \left( \frac{1}{\beta} ||\lambda_0 - \lambda^\mu||^2 + \beta ||y_0 + y^\mu||^2 \right). \tag{11}
\]
From which we deduce that:
\[
\beta \|x_{K+1} - y_{K+1}\| = \|\lambda_{K+1} - \lambda_K\| \leq \sqrt{\frac{\beta \Delta^\mu}{(K + 1)}},
\]
\[
\| - y_{K+1} + y_K\| \leq \sqrt{\frac{\Delta^\mu}{\beta (K + 1)}}.
\]

On the other hand, (9) gives:
\[
\frac{1}{2\beta} \|\lambda_{K+1} - \lambda\|^2 + \frac{\beta}{2} \| - y_{K+1} + y\|^2 \leq \frac{1}{2} \Delta^\mu.
\]

Hence, we have:
\[
\|\lambda_{K+1} - \lambda\| \leq \sqrt{\beta \Delta^\mu},
\]
\[
\| - y_{K+1} + y\| \leq \sqrt{\frac{\Delta^\mu}{\beta}}.
\]

Furthermore, we have:
\[
\|y_{K+1} - y^\mu\| = \|y^{K+1} - y^*\| + \|y^* - y^\mu\| \leq \sqrt{\frac{\Delta^\mu}{\beta}} + D,
\]
\[
\|x_{K+1} - x^\mu\| = \|\frac{1}{\beta}(\lambda_{K+1} - \lambda_K) - (-y_{K+1} + y^\mu)\|
\leq \frac{1}{\beta} \|\lambda_{K+1} - \lambda_K\| + \|y_{K+1} - y^\mu\|
\leq \frac{1}{\beta} \sqrt{\frac{\Delta^\mu}{K + 1}} + \sqrt{\frac{\Delta^\mu}{\beta}} + D,
\]
and
\[
\|\lambda_{K+1}\| \leq \|\lambda\| + \|\lambda_{K+1} - \lambda\| \leq \|\lambda\| + \sqrt{\beta \Delta^\mu}.
\]

Then using (T6-fk-UB) in Lemma 6, we have (L24-1).

B.1.3 Proving Theorem 1

We are now ready to prove Theorem 1. Here we give a nonasymptotic convergence rate of Algorithm 3.

**Theorem 7** (Restatement of Theorem 1). *Given a continuous operator \( F: \mathcal{X} \to \mathbb{R}^n \), assume that:

(i) \( F \) is monotone on \( C \), as per Def. 1;

(ii) \( F \) is either strictly monotone on \( C \) or one of \( \phi_i \) is strictly convex.*

Let \((x^{(t)}_K, y^{(t)}_K, \lambda^{(t)}_K)\) denote the last iterate of Algorithm 3, run with sufficiently small \( \mu - 1 \). Then \( \forall t \in [T] \) and \( \forall K \in \mathbb{N}_+ \), we have
\[
\mathcal{G}(x_K, C) \leq \frac{2}{\sqrt{\beta} K} \Delta + 2 \left( \frac{1}{\sqrt{\beta}} \|\lambda\| + \sqrt{\beta} D + 1 + M \right) \sqrt{\frac{\Delta}{K}}.
\]

and
\[
\|x^K - y^K\| \leq 2 \sqrt{\frac{\Delta}{\beta K}},
\]

where \( \Delta \triangleq \frac{1}{\beta} \|\lambda_0 - \lambda^*\|^2 + \beta \|y_0 - y^*\|^2 \) and \( D \triangleq \sup_{x, y \in C} \|x - y\| \), and \( M \triangleq \sup_{x \in C} \|F(x)\| \).
Proof of Theorem 7. Note that
\[ (f_k \text{-Pr-2}) \iff \min_{x \in C} \langle F(x_{k+1}), x \rangle \]
\[ \iff \max_{x \in C} \langle F(x_{k+1}), x_{k+1} - x \rangle \]
\[ \iff G(x_{k+1}, C), \]
from which we deduce
\[ G(x_{k+1}, C) = \langle F(x_{k+1}), x_{k+1} - x^+_K \rangle, \forall k. \quad (14) \]

For any \( K \in \mathbb{N} \), by (CHU, 1998) we know that
\[ x^\mu_K \to x^*_K, \]
\[ g(y_{K+1}) - g(y^\mu_K) \to 0, \]
\[ \Delta^\mu \to \frac{1}{\beta} \| \lambda_0 - \lambda^* \|^2 + \beta \| y_0 - y^* \|^2 = \Delta. \quad (15) \]

Thus, there exists \( \mu_1 > 0 \), s.t. \( \forall 0 < \mu < \mu_1 \),
\[ \| x^\mu_K - x^*_K \| \leq \sqrt{\frac{\Delta^\mu}{K+1}}, \]
\[ |g(y_{K+1}) - g(y^\mu_K)| \leq \sqrt{\frac{\Delta^\mu}{K+1}}. \]

Combining with Lemma 7, we have that
\[ \langle F(x_{K+1}), x_{K+1} - x^\mu_K \rangle = f_K(x_{K+1}) - f_K(x^\mu_K) \]
\[ \leq \frac{1}{\sqrt{\beta}} \frac{\Delta^\mu}{K+1} + \left( 2\sqrt{\Delta^\mu} + \frac{1}{\sqrt{\beta}} \| \lambda^\mu \| + \sqrt{\beta D} \right) \sqrt{\frac{\Delta^\mu}{K+1}} + g(y^\mu_K) - g(y_{K+1}) \]
\[ \leq \frac{1}{\sqrt{\beta}} \frac{\Delta^\mu}{K+1} + \left( 2\sqrt{\Delta^\mu} + \frac{1}{\sqrt{\beta}} \| \lambda^\mu \| + \sqrt{\beta D + 1} \right) \sqrt{\frac{\Delta^\mu}{K+1}}. \]

Using the above inequality, we have
\[ G(x_{K+1}, C) = \langle F(x_{K+1}), x_{K+1} - x^*_K \rangle \]
\[ = \langle F(x_{K+1}), x_{K+1} - x^\mu_K \rangle + \langle F(x_{K+1}), x^\mu_K - x^*_K \rangle \]
\[ \leq \langle F(x_{K+1}), x_{K+1} - x^\mu_K \rangle + \| F(x_{K+1}) \| \| x^\mu_K - x^*_K \| \]
\[ \leq \frac{1}{\sqrt{\beta}} \frac{\Delta^\mu}{K+1} + \left( 2\sqrt{\Delta^\mu} + \frac{1}{\sqrt{\beta}} \| \lambda^\mu \| + \sqrt{\beta D + 1 + M} \right) \sqrt{\frac{\Delta^\mu}{K+1}}. \]

Moreover, by (15), we can choose small enough \( \mu_1 \) so that
\[ G(x_{K+1}, C) \leq \frac{2}{\sqrt{\beta}} \frac{\Delta}{K+1} + 2 \left( 2\sqrt{\Delta} + \frac{1}{\sqrt{\beta}} \| \lambda \| + \sqrt{\beta D + 1 + M} \right) \sqrt{\frac{\Delta}{K+1}}, \]
and
\[ \| x_{K+1} - y_{K+1} \| \leq 2 \sqrt{\frac{\Delta}{\beta(K+1)}}, \quad (16) \]
where (16) uses (L7-2) in Lemma 7.
B.2 Proof of Theorem 2: Last-iterate convergence of inexact ACVI for Monotone Variational Inequalities

B.2.1 Useful lemmas from previous works

The following lemma is Lemma 1 from Schmidt et al. (2011).

**Lemma 8 (Lemma 1 in Schmidt et al. (2011)).** Assume that the nonnegative sequence \( \{u_k\} \) satisfies the following recursion for all \( k \geq 1 \):

\[
 u_k^2 \leq S_k + \sum_{i=1}^{k} \lambda_i u_i,
\]

with \( \{S_k\} \) an increasing sequence, \( S_0 \geq u_0^2 \) and \( \lambda_i \geq 0 \) for all \( i \). Then, for all \( k \geq 1 \), it follows:

\[
 u_k \leq \frac{1}{2} \sum_{i=1}^{k} \lambda_i + \left( S_k + \left( \frac{1}{2} \sum_{i=1}^{k} \lambda_i \right)^2 \right)^{1/2}.
\]

**Proof.** We prove the result by induction. It is true for \( k = 0 \) (by assumption). We assume it is true for \( k - 1 \), and we denote by \( v_{k-1} = \max\{u_1, \ldots, u_{k-1}\} \). From the recursion we deduce:

\[
 (u_k - \lambda_k/2)^2 \leq S_k + \frac{\lambda_k^2}{4} + v_{k-1} \sum_{i=1}^{k-1} \lambda_i,
\]

leading to

\[
 u_k \leq \frac{\lambda_k}{2} + \left( S_k + \frac{\lambda_k^2}{4} + v_{k-1} - 1 \sum_{i=1}^{k-1} \lambda_i \right)^{1/2},
\]

and thus

\[
 u_k \leq \max \left\{ v_{k-1} - \frac{\lambda_k}{2} + \left( S_k + \frac{\lambda_k^2}{4} + v_{k-1} - 1 \sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \right\}.
\]

Let \( v_{k-1}^{*} \triangleq \frac{1}{2} \sum_{i=1}^{k} \lambda_i + \left( S_k + \left( \frac{1}{2} \sum_{i=1}^{k} \lambda_i \right)^2 \right)^{1/2} \). Note that

\[
 v_{k-1} = \frac{\lambda_k}{2} + \left( S_k + \frac{\lambda_k^2}{4} + v_{k-1} - 1 \sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \iff v_{k-1} = v_{k-1}^{*}.
\]

Since the two terms in the max are increasing functions of \( v_{k-1} \), it follows that if \( v_{k-1} \leq v_{k-1}^{*} \), then \( v_k \leq v_{k-1}^{*} \). Also note that

\[
 v_{k-1} \geq \frac{\lambda_k}{2} + \left( S_k + \frac{\lambda_k^2}{4} + v_{k-1} - 1 \sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \iff v_{k-1} \geq v_{k-1}^{*}.
\]

From which we deduce that if \( v_{k-1} \geq v_{k-1}^{*} \), then \( v_k \leq v_{k-1} \), and the induction hypotheses ensure that the property is satisfied for \( k \).

In the convergence rate analysis of inexact ACVI-Algorithm 1, we need the following definition (Bertsekas et al., 2003):

**Definition 7 (\( \varepsilon \)-subdifferential).** Given a convex function \( \psi : \mathbb{R}^n \to \mathbb{R} \) and a positive scalar \( \varepsilon \), a vector \( a \in \mathbb{R}^n \) is called an \( \varepsilon \)-subgradient of \( \psi \) at a point \( x \in \text{dom}\psi \) if

\[
 \psi(z) \geq \psi(x) + (z - x)^\top a - \varepsilon, \quad \forall z \in \mathbb{R}^n.
\]

*(\( \varepsilon \)-G)*

The set of all \( \varepsilon \)-subgradients of a convex function \( \psi \) at \( x \in \text{dom}\psi \) is called the \( \varepsilon \)-subdifferential of \( \psi \) at \( x \), and is denoted by \( \partial_{\varepsilon}\psi(x) \).

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B.2.2 Intermediate results

We first give some lemmas which will be used in the proof of Theorem 2.

In the following proofs, we assume $\varepsilon_0 = \sigma_0 = 0$. To state a lemma analogous to Lemma 2 but for the inexact ACVI, we need the following two lemmas.

**Lemma 9.** In inexact ACVI-Algorithm 1, for each $k$, $\exists r_{k+1} \in \mathbb{R}^n$, $\|r_{k+1}\| \leq \frac{2\varepsilon_{k+1}}{\beta}$, s.t.

$\beta(x_{k+1} + \frac{1}{\beta}\lambda_k - y_{k+1} - r_{k+1}) \in \partial_{\varepsilon_{k+1}} g(y_{k+1}).$

**Proof of Lemma 9.** We first recall some properties of $\varepsilon$-subdifferentials (see, eg. (Bertsekas et al., 2003), Section 4.3 for more details). $x$ is an $\varepsilon$-minimizer (see Def. 4) of a convex function $\psi$ if and only if $0 \in \partial_{\varepsilon}\psi(x)$. Let $\psi = \psi_1 + \psi_2$, where both $\psi_1$ and $\psi_2$ are convex, we have $\partial_{\varepsilon}\psi(x) \subseteq \partial_{\varepsilon}\psi_1(x) + \partial_{\varepsilon}\psi_2(x)$. If $\psi_1(x) = \frac{\beta}{2} \|x - z\|^2$, then

$$\partial_{\varepsilon}\psi_1(x) = \left\{ y \in \mathbb{R}^n \mid \frac{\beta}{2} \|x - z - \frac{y}{\beta}\|^2 \leq \varepsilon \right\}$$

$$= \left\{ y \in \mathbb{R}^n, y = \beta x - \beta z + \beta r \mid \frac{\beta}{2} \|r\|^2 \leq \varepsilon \right\}.$$

Let $\psi_2 = g$ and $z = x_{k+1} + \frac{1}{\beta}\lambda_k$, then $y_{k+1}$ is an $\varepsilon_{k+1}$-minimizer of $\psi_1 + \psi_2$. Thus we have $0 \in \partial_{\varepsilon_{k+1}} \psi(y_{k+1}) \subseteq \partial_{\varepsilon_{k+1}} \psi_1(y_{k+1}) + \partial_{\varepsilon_{k+1}} \psi_2(y_{k+1})$. Hence, there is an $r_{k+1}$ such that

$$\beta(x_{k+1} + \frac{1}{\beta}\lambda_k - y_{k+1} - r_{k+1}) \in \partial_{\varepsilon_{k+1}} g(y_{k+1}) \quad \text{with} \quad \|r_{k+1}\| \leq \frac{2\varepsilon_{k+1}}{\beta}.$$

**Lemma 10.** In inexact ACVI-Algorithm 1, for each $k$, $\exists q_{k+1} \in \mathbb{R}^n$, $\|q_{k+1}\| \leq \sigma_{k+1}$, s.t.

$$x_{k+1} + q_{k+1} = \arg \min_x \left\{ f_k(x) + \frac{\beta}{2} \|x - y_k + \frac{1}{\beta}\lambda_k\|^2 \right\},$$

where $L_\beta$ is the augmented Lagrangian of problem $(f_k$-$Pr)$.

**Proof of Lemma 10.** By the definition of $x_{k+1}$ (see line 8 of inexact ACVI-Algorithm 1 and Def. 3) we have

$$x_{k+1} + q_{k+1} = -\frac{1}{\beta} Pf_c F(x) + P_c y_k - \frac{1}{\beta} P_c \lambda_k + d_c$$

$$= \arg \min_\lambda L_\beta(x, y_k, \lambda_k),$$

where $L_\beta$ is the augmented Lagrangian of problem, which is given in AL (note that $w_k = x_{k+1}$). ($f_k$-$Pr$). And from the above equation (17) follows.

Similar to Lemma 2, and using Lemma 9 and Lemma 10, we give the following lemma for inexact ACVI-Algorithm 1.

**Lemma 11.** For the problems $(f$-$Pr)$, $(f_k$-$Pr)$ and inexact ACVI-Algorithm 1, we have

$$0 \in \partial f_k(x_{k+1} + q_{k+1}) + \lambda_{k+1} + \beta(x_{k+1} - y_k) + \beta q_{k+1} \quad \text{(L11-1)}$$

$$0 \in \partial_{\varepsilon_{k+1}} g(y_{k+1}) - \lambda_k - \beta(x_{k+1} - y_{k+1}) + \beta r_{k+1} \quad \text{(L11-2)}$$

$$\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_{k+1}) \quad \text{(L11-3)}$$

$$-\lambda^\nu \in \partial f(x^\nu), \quad \text{(L11-4)}$$

$$-\lambda^\mu_k \in \partial f_k(x^\mu_k), \quad \text{(L11-5)}$$

$$\lambda^\mu = \nabla g(y^\mu), \quad \text{(L11-6)}$$

$$\lambda^\mu_k = \nabla g(y^\mu_k), \quad \text{(L11-7)}$$

$$x^\mu = y^\mu, \quad \text{(L11-8)}$$

$$x^\mu_k = y^\mu_k \quad \text{(L11-9)}$$
We define the following two maps (whose naming will be evident from the inclusions shown after):
\[
\hat{\nabla} f_k(x_{k+1} + q_{k+1}) \equiv -\lambda_k - \beta(x_{k+1} - y_k) - \beta q_{k+1}, \quad \text{and} \quad \text{noisy-} \hat{\nabla} f_k
\]
\[
\hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) \equiv \lambda_k + \beta(x_{k+1} - y_k) - \beta r_{k+1}. \quad \text{noisy-} \hat{\nabla} g
\]

Then, from (L2-1) and (L2-2) it follows that:
\[
\hat{\nabla} f_k(x_{k+1} + q_{k+1}) \in \partial f_k(x_{k+1} + q_{k+1}) \quad \text{and} \quad \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) \in \partial \varepsilon_{k+1} g(y_{k+1}). \quad (18)
\]

The following lemma is analogous to Lemma 3 but refers to the noisy case.

**Lemma 12.** For the iterates \(x_{k+1}, y_{k+1}, \text{and} \lambda_{k+1}\) of the inexact ACVI—Algorithm 1—we have:
\[
\langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, y - y_{k+1} \rangle - \beta \langle r_{k+1}, y_{k+1} - y \rangle, \quad \text{(L12-1)}
\]

and
\[
\langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}), x_{k+1} - x \rangle + \langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}), y_{k+1} - y \rangle
\]
\[
= -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x + y \rangle + \beta \langle -y_{k+1} + y_k, x_{k+1} - x \rangle - \beta \langle q_{k+1}, x_{k+1} - x \rangle - \beta \langle r_{k+1}, y_{k+1} - y \rangle. \quad \text{(L12-2)}
\]

**Proof of Lemma 12.** From (L11-3), (noisy-\(\hat{\nabla} f_k\)) and (noisy-\(\hat{\nabla} g\)) we have:
\[
\langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}), x_{k+1} - x \rangle = -\langle \lambda_{k+1}, x_{k+1} - y_k, x_{k+1} - x \rangle
\]
\[
= -\langle \lambda_{k+1}, x_{k+1} - x \rangle + \beta \langle -y_{k+1} + y_k, x_{k+1} - x \rangle,
\]

and
\[
\langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) + \beta r_{k+1}, y_{k+1} - y \rangle = -\langle \lambda_{k+1}, y - y_{k+1} \rangle.
\]

Adding these together yields:
\[
\langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}) + \beta q_{k+1}, x_{k+1} - x \rangle + \langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) + \beta r_{k+1}, y_{k+1} - y \rangle = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x + y \rangle
\]
\[
+ \beta \langle -y_{k+1} + y_k, x_{k+1} - x \rangle.
\]

Rearranging the above two equations, we obtain (L12-1) and (L12-2). \(\Box\)

The following lemma is analogous to Lemma 4 but refers to the noisy case.

**Lemma 13.** For the \(x_{k+1}, y_{k+1}, \text{and} \lambda_{k+1}\) iterates of the inexact ACVI—Algorithm 1—we have:
\[
\langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}), x_{k+1} - x' \rangle + \langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}), y_{k+1} - y' \rangle + \langle \lambda', x_{k+1} - y_{k+1} \rangle
\]
\[
\leq \frac{1}{2\beta} \|\lambda_k - \lambda_k'\|^2 - \frac{1}{2\beta} \|\varepsilon_{k+1}, y_{k+1} - y'\|^2 + \frac{\beta}{2} \|y' - y_k\|^2 - \frac{\beta}{2} \|y' - y_{k+1}\|^2
\]
\[
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_k - y_{k+1}\|^2
\]
\[
- \beta \langle r_{k+1} - r_k, y_{k+1} - y_k \rangle - \beta \langle \varepsilon_{k+1}, y_{k+1} - y' \rangle + \varepsilon_k + \varepsilon_{k+1} - \beta \langle q_{k+1}, x_{k+1} - x' \rangle,
\]

and
\[
\langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}), x_{k+1} - x' \rangle + \langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}), y_{k+1} - y' \rangle + \langle \lambda', x_{k+1} - y_{k+1} \rangle
\]
\[
\leq \frac{1}{2\beta} \|\lambda_k - \lambda_k'\|^2 - \frac{1}{2\beta} \|\varepsilon_{k+1}, y_{k+1} - y'\|^2 + \frac{\beta}{2} \|y' - y_k\|^2 - \frac{\beta}{2} \|y' - y_{k+1}\|^2
\]
\[
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_k - y_{k+1}\|^2
\]
\[
- \beta \langle r_{k+1} - r_k, y_{k+1} - y_k \rangle - \beta \langle \varepsilon_{k+1}, y_{k+1} - y' \rangle + \varepsilon_k + \varepsilon_{k+1} - \beta \langle q_{k+1}, x_{k+1} - x' \rangle.
Proof of Lemma 13. For the left-hand side of the first part of Lemma 13:

\[ \text{LHS} = \langle \nabla f_k(x_{k+1} + q_{k+1}), x_{k+1} - x_k^\mu \rangle + \langle \nabla \varepsilon_k, g(y_{k+1}), y_{k+1} - y_k^\mu \rangle + \langle \lambda_k^\mu, x_{k+1} - y_{k+1} \rangle, \]

we let \((x, y, \lambda) = (x^\mu, y^\mu, \lambda^\mu)\) in (L12-2), and using the result of that lemma we get that:

\[ \text{LHS} = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x^\mu + y^\mu \rangle + \beta(-y_{k+1} + y_k, x_{k+1} - x^\mu) - \beta(q_{k+1}, x_{k+1} - x^\mu) \]
\[ -\beta(r_{k+1}, y_{k+1} - y^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle, \]

and since \(x^\mu = y^\mu\) (L2-8):

\[ \text{LHS} = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} \rangle + \beta(-y_{k+1} + y_k, x_{k+1} - x^\mu) + \langle \lambda^\mu, x_{k+1} - y_{k+1} \rangle \]
\[ -\beta(q_{k+1}, x_{k+1} - x^\mu) - \beta(r_{k+1}, y_{k+1} - y^\mu) \]
\[ = -\langle \lambda_{k+1} - \lambda^\mu, x_{k+1} - y_{k+1} \rangle + \beta(-y_{k+1} + y_k, x_{k+1} - x^\mu) \]
\[ -\beta(q_{k+1}, x_{k+1} - x^\mu) - \beta(r_{k+1}, y_{k+1} - y^\mu) \]

where in the last equality we combined the first and third term together. Using (L11-3) that \(\frac{1}{2}(\lambda_{k+1} - \lambda_k) = x_{k+1} - y_{k+1}\) yields (for the second term above, we add and subtract \(y_{k+1}\) in its second argument, and use \(x^\mu = y^\mu\)):

\[ \text{LHS} = -\frac{1}{\beta} \langle \lambda_{k+1} - \lambda^\mu, \lambda_{k+1} - \lambda_k \rangle + \beta(-y_{k+1} + y_k, \lambda_{k+1} - \lambda_k) - \beta(-y_{k+1} + y_k, -y_{k+1} + y^\mu) \]
\[ -\beta(q_{k+1}, x_{k+1} - x^\mu) - \beta(r_{k+1}, y_{k+1} - y^\mu) \]  \hspace{1cm} (19)

Using the 3-point identity, that for any vectors \(a, b, c\) it holds \(\langle b - a, b - c \rangle = \frac{1}{2}(\|a - b\|^2 + \|b - c\|^2 - \|a - c\|^2)\), for the first term above we get that:

\[ \langle \lambda_{k+1} - \lambda^\mu, \lambda_{k+1} - \lambda_k \rangle = \frac{1}{2}(\|\lambda_k - \lambda^\mu\| + \|\lambda_{k+1} - \lambda_k\| - \|\lambda_{k+1} - \lambda^\mu\|) \]

and similarly,

\[ \langle -y_{k+1} + y_k, -y_{k+1} + y^\mu \rangle = \frac{1}{2}(\|-y_k + y^\mu\| - \|-y_{k+1} + y^\mu\| - \|-y_{k+1} + y_k\|) \]

and by plugging these into (19) we get:

\[ \text{LHS} = \frac{1}{2\beta} \|\lambda_k - \lambda^\mu\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda^\mu\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 \]
\[ + \frac{\beta}{2} \|-y_k + y^\mu\|^2 - \frac{\beta}{2} \|-y_{k+1} + y^\mu\|^2 - \frac{\beta}{2} \|-y_{k+1} + y_k\|^2 \]
\[ + \langle -y_{k+1} + y_k, \lambda_{k+1} - \lambda_k \rangle - \beta(q_{k+1}, x_{k+1} - x^\mu) - \beta(r_{k+1}, y_{k+1} - y^\mu) \]  \hspace{1cm} (20)

On the other hand, (L12-1) which states that \(\langle \nabla \varepsilon_k, g(y_{k+1}), y_{k+1} - y \rangle + \langle \lambda_{k+1}, -y_{k+1} + y \rangle = -\beta(r_{k+1}, y_{k+1} - y)\), also asserts:

\[ \langle \nabla \varepsilon_k, g(y_k), y_k - y \rangle + \langle \lambda_k, -y_k + y \rangle = -\beta(r_k, y_k - y) \]  \hspace{1cm} (21)

Letting \(y = y_k\) in (L12-1), and \(y = y_{k+1}\) in (21), and adding them together yields:

\[ \langle \nabla \varepsilon_{k+1}, g(y_{k+1}) - \nabla \varepsilon_k, g(y_k), y_{k+1} - y_k \rangle + \langle \lambda_{k+1} - \lambda_k, -y_{k+1} + y_k \rangle = -\beta(r_{k+1} - r_k, y_{k+1} - y_k) \]  \hspace{1cm} (22)

By the definition of \(\varepsilon\)-subdifferential as per Def.7 we have:

\[ \varepsilon_k + g(y_{k+1}) \geq g(y_k) + \langle \nabla \varepsilon_k, g(y_k), y_{k+1} - y_k \rangle, \text{ and} \]
\[ \varepsilon_{k+1} + g(y_k) \geq g(y_{k+1}) + \langle \nabla \varepsilon_{k+1}, g(y_{k+1}), y_k - y_{k+1} \rangle. \]
Adding together the above two inequalities, we obtain:

\[
\langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) - \hat{\nabla} \varepsilon_k g(y_k), y_{k+1} - y_k \rangle \geq -\varepsilon_{k+1} - \varepsilon_k. 
\]

(23)

Combining (22) and (23), we deduce:

\[
\langle \lambda_{k+1} - \lambda_k - y_{k+1} + y_k \rangle \leq -\beta \langle r_{k+1} - r_k, y_{k+1} - y_k \rangle + \varepsilon_{k+1} + \varepsilon_k. 
\]

(24)

Lastly, plugging it into (20) gives the first inequality of Lemma 13.

The second inequality of Lemma 13 follows similarly.

\[\square\]

**Lemma 14.** For the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the inexact ACVI—Algorithm 1—we have:

\[
f_k(x_{k+1} + q_{k+1}) + g(y_{k+1}) - f_k(x^*) - g(y^*) + \langle \lambda^*, x_{k+1} + q_{k+1} - y_{k+1} \rangle 
\leq \frac{1}{2\beta} \| \lambda_k - \lambda^* \|^2 - \frac{1}{2\beta} \| \lambda_{k+1} - \lambda^* \|^2 + \frac{\beta}{2} \| y_k + y^* \|^2 - \frac{\beta}{2} \| y_{k+1} + y^* \|^2 
- \frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 - \frac{\beta}{2} \| y_{k+1} + y_k \|^2 - \beta \langle r_{k+1} - r_k, y_{k+1} - y_k \rangle - \beta \langle r_{k+1}, y_{k+1} - y^* \rangle + \varepsilon_{k+1} + 2\varepsilon_k. 
\]

(25)

Proof of Lemma 14. From the convexity of \(f_k(x)\) and \(g(y)\) and Eq. (18) which asserts that \(\hat{\nabla} f_k(x_{k+1} + q_{k+1}) \in \partial f_k(x_{k+1} + q_{k+1})\) and \(\hat{\nabla} \varepsilon_{k+1} g(y_{k+1}) \in \partial \varepsilon_{k+1} g(y_{k+1})\), it follows for the LHS of Lemma 14 that:

\[
f_k(x_{k+1} + q_{k+1}) + g(y_{k+1}) - f_k(x^*) - g(y^*) + \langle \lambda^*, x_{k+1} + q_{k+1} - y_{k+1} \rangle 
\leq \langle \hat{\nabla} f_k(x_{k+1} + q_{k+1}), x_{k+1} + q_{k+1} - x^* \rangle + \langle \hat{\nabla} \varepsilon_{k+1} g(y_{k+1}), y_{k+1} - y^* \rangle + \varepsilon_{k+1} + \langle \lambda^*, x_{k+1} + q_{k+1} - y_{k+1} \rangle. 
\]

Finally, by plugging in the first part of Lemma 13 and using (noisy-\(\hat{\nabla} f_k\)), Lemma 14 follows, that is:

\[
f_k(x_{k+1} + q_{k+1}) + g(y_{k+1}) - f(x^*) - g(y^*) + \langle \lambda^*, x_{k+1} + q_{k+1} - y_{k+1} \rangle 
\leq \frac{1}{2\beta} \| \lambda_k - \lambda^* \|^2 - \frac{1}{2\beta} \| \lambda_{k+1} - \lambda^* \|^2 + \frac{\beta}{2} \| y_k + y^* \|^2 - \frac{\beta}{2} \| y_{k+1} + y^* \|^2 
- \frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 - \frac{\beta}{2} \| y_{k+1} + y_k \|^2 - \beta \langle r_{k+1} - r_k, y_{k+1} - y_k \rangle - \beta \langle r_{k+1}, y_{k+1} - y^* \rangle + \varepsilon_{k+1} + 2\varepsilon_k. 
\]

(25)

The following theorem upper bounds the analogous quantity but for \(f_k(\cdot)\) (instead of \(f\)), and further asserts that the difference between the \(x_{k+1}\) and \(y_{k+1}\) iterates of inexact ACVI (Algorithm 1) tends to 0 asymptotically.
Theorem 8 (Asymptotic convergence of \((x_{k+1} - y_{k+1})\) of I-ACVI). Assume that \(\sum_{i=1}^{\infty} (\sigma_i + \sqrt{\varepsilon_i}) < +\infty\), then for the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the inexact ACVI—Algorithm 1—we have:

\[
f_k(x_{k+1} + q_{k+1}) - f_k(x^\mu_k) + g(y_{k+1}) - g(y^\mu_k) \leq \|\lambda_{k+1} - \lambda_k\|^2 + \beta \|y_{k+1} - y_k\| \cdot \|x_{k+1} - x^\mu_k\| + \beta \|y_{k+1} - y^\mu_k\| + \epsilon_{k+1} \rightarrow 0,
\]

and

\[
x_{k+1} - y_{k+1} \rightarrow 0, \quad \text{as } k \rightarrow \infty.
\]

Proof of Lemma 8. Recall from (L1-f) of Lemma 1 that by setting \(x \equiv x_{k+1} + q_{k+1}, y \equiv y_{k+1}\) it asserts that:

\[
f(x_{k+1} + q_{k+1}) - f(x^\mu) + g(y_{k+1}) - g(y^\mu) + (\lambda^\mu, x_{k+1} + q_{k+1} - y_{k+1}) \geq 0.
\]

Further, notice that the LHS of the above inequality overlaps with that of (L14). This implies that the RHS of (L14) has to be non-negative. Hence, we have that:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \leq \frac{1}{2\beta} \|\lambda_k - \lambda^\mu\|^2 \leq \frac{\beta}{2} \|y_k + y^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} + y^\mu\|^2
\]

\[
- \beta(r_{k+1} - r_k, y_{k+1} - y_k) - \beta(r_{k+1}, y_{k+1} - y^\mu) + \epsilon_k + 2\varepsilon_{k+1} - \|q_{k+1}, \lambda_k - \lambda^\mu + \beta(x_{k+1} - y_k) + \beta(x_{k+1} - x^\mu)\|.
\]

Recall that \(\|r_{k+1}\| \leq \sqrt{\frac{2\varepsilon_{k+1}}{\beta}}\) and \(\|q_{k+1}\| \leq \sigma_{k+1}\) (see Lemma 9 and Lemma 10), by Cauchy-Schwarz inequality we have:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \leq \frac{1}{2\beta} \|\lambda_k - \lambda^\mu\|^2 \leq \frac{\beta}{2} \|y_k + y^\mu\|^2 - \frac{\beta}{2} \|y_{k+1} + y^\mu\|^2
\]

\[
+ \sqrt{2\beta(\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k})} \|y_{k+1} - y_k\| + \sqrt{2\beta(\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k})} \|y_{k+1} - y^\mu\| + \epsilon_k + 2\varepsilon_{k+1} + \sigma_{k+1} \|\lambda_k - \lambda^\mu + \beta\| x_{k+1} - y_k\| + \beta\| x_{k+1} - x^\mu\|.
\]

Summing over \(k = 0, \ldots, \infty\), we have:

\[
\sum_{k=0}^{\infty} \left(\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2\right) \leq \frac{1}{2\beta} \|\lambda_0 - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_0 + y^\mu\|^2
\]

\[
+ \sqrt{2\beta} \sum_{k=0}^{\infty} \left(\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k}\right) \|y_{k+1} - y_k\| + \sqrt{2\beta} \|y_{k+1} - y^\mu\| + 3 \sum_{k=0}^{\infty} \epsilon_k
\]

\[
+ \sum_{k=0}^{\infty} \sigma_{k+1} \|\lambda_k - \lambda^\mu + \beta\| x_{k+1} - y_k\| + \beta\| x_{k+1} - x^\mu\|.
\]
Also notice that by simply reorganizing (25) we have:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda^\mu\|^2 + \frac{\beta}{2} \|\lambda_k - \lambda^\mu\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|\lambda_k - \lambda^\mu\|^2
\]

\[
\leq \frac{1}{2\beta} \|\lambda_k - \lambda^\mu\|^2 + \frac{\beta}{2} \|\lambda_{k+1} - \lambda^\mu\|^2 - \frac{1}{2\beta} \|\lambda_k - \lambda_{k+1}\|^2 - \frac{\beta}{2} \|\lambda_k - \lambda^\mu\|^2
\]

\[
+ \sqrt{2\beta(\sqrt{\epsilon_{k+1}} + \sqrt{\epsilon_k})} \|y_{k+1} - y_k\| + \sqrt{2\beta\epsilon_{k+1}} \|y_{k+1} - y^\mu\|
\]

\[
+ \epsilon_k + 2\epsilon_{k+1} + \sigma_{k+1}(\|\lambda_k - \lambda^\mu\|^2 + \beta \|x_{k+1} - y_k\| + \beta \|x_{k+1} - x^\mu\|)
\]

\[
\leq \frac{1}{2\beta} \|\lambda_0 - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_0 - y^\mu\|^2
\]

\[
+ \sqrt{2\beta(\sqrt{\epsilon_{i+1}} + \sqrt{\epsilon_i})} \|y_{i+1} - y_i\| + \sqrt{2\beta\epsilon_{i+1}} \|y_{i+1} - y^\mu\|
\]

\[
+ \sum_{i=0}^{k} \epsilon_i + 2 \sum_{i=0}^{k} \epsilon_{i+1} + \sum_{i=0}^{k} \sigma_{i+1}(\|\lambda_i - \lambda^\mu\|^2 + \beta \|x_{i+1} - y_i\| + \beta \|x_{i+1} - x^\mu\|),
\]

where the second inequality follows because the norm is non-negative.

From the above inequality we deduce:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_{k+1} - y^\mu\|^2
\]

\[
\leq \frac{1}{2\beta} \|\lambda_{k+1} - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_{k+1} - y^\mu\|^2
\]

\[
+ \sqrt{2\beta(\sqrt{\epsilon_{i+1}} + \sqrt{\epsilon_i})} \|y_{i+1} - y_i\| + \sqrt{2\beta\epsilon_{i+1}} \|y_{i+1} - y^\mu\|
\]

\[
+ \sum_{i=0}^{k} \epsilon_i + 2 \sum_{i=0}^{k} \epsilon_{i+1} + \sum_{i=0}^{k} \sigma_{i+1}(\|\lambda_i - \lambda^\mu\|^2 + \beta \|x_{i+1} - y_i\| + \beta \|x_{i+1} - x^\mu\|),
\]

where the first inequality is the Cauchy-Schwarz inequality.

Using (L11-3) and (L11-8), we have:

\[
\|x_{i+1} - x^\mu\| = \|y_{i+1} - y^\mu + x_{i+1} - y_{i+1}\| \leq \|y_{i+1} - y^\mu\| + \frac{1}{\beta} \|\lambda_{i+1} - \lambda_i\|
\]

\[
\leq \|y_{i+1} - y^\mu\| + \frac{1}{\beta} \|\lambda_{i+1} - \lambda^\mu\| + \frac{1}{\beta} \|\lambda_i - \lambda^\mu\|,
\]

\[
\|x_{i+1} - y_i\| \leq \|x_{i+1} - x^\mu\| + \|y_i - y^\mu\| \leq \|y_i - y^\mu\| + \|y_{i+1} - y^\mu\| + \frac{1}{\beta} \|\lambda_{i+1} - \lambda^\mu\| + \frac{1}{\beta} \|\lambda_i - \lambda^\mu\|,
\]

\[
\|y_{i+1} - y_i\| \leq \|y_{i+1} - y^\mu\| + \|y_i - y^\mu\|.
\]
Now we set 
\[ u_i \triangleq \beta \| y_i - y^* \| + \lambda_i - \lambda^* \]
and Lemma 834
\[ S_{k+1} \triangleq 2 \| \lambda_0 - \lambda^* \|^2 + 2 \beta^2 \| y_0 - y^* \|^2 + 4 \beta (2 \sqrt{\beta} \sqrt{\varepsilon_i} + \beta \sigma_i) \| y_0 - y^* \| + 12 \sigma_i \| \lambda_0 - \lambda^* \| + 12 \beta \sum_{i=1}^{k+1} \varepsilon_i \]
From which we deduce:
\[ (\| \lambda_{k+1} - \lambda^* \| + \beta \| y_{k+1} - y^* \|)^2 \]
\[ \leq 2 \| \lambda_0 - \lambda^* \|^2 + 2 \beta^2 \| y_0 - y^* \|^2 + 4 \beta (2 \sqrt{\beta} \sqrt{\varepsilon_i} + \beta \sigma_i) \| y_0 - y^* \| + 12 \sigma_i \| \lambda_0 - \lambda^* \| + 12 \beta \sum_{i=1}^{k+1} \varepsilon_i \]
\[ + 4 \beta \sum_{i=1}^{k+1} \left( \sqrt{\varepsilon_i + 3 \lambda^* \varepsilon_i} + (2 \sigma_i + 3 \sigma_{i+1}) \right) \left( \beta \| y_i - y^* \| + \| \lambda_i - \lambda^* \| \right) . \]
Now we set 
\[ u_i \triangleq \beta \| y_i - y^* \| + \lambda_i - \lambda^* , \]
and Lemma 834
\[ S_{k+1} \triangleq 2 \| \lambda_0 - \lambda^* \|^2 + 2 \beta^2 \| y_0 - y^* \|^2 + 4 \beta (2 \sqrt{\beta} \sqrt{\varepsilon_i} + \beta \sigma_i) \| y_0 - y^* \| + 12 \sigma_i \| \lambda_0 - \lambda^* \| + 12 \beta \sum_{i=1}^{k+1} \varepsilon_i \]
to get:

$$u_{k+1} \leq \frac{1}{2} \sum_{i=1}^{k+1} \lambda_i + \left( S_{k+1} + \left( \frac{1}{2} \sum_{i=1}^{k+1} \lambda_i \right)^2 \right)^{1/2},$$

(32)

where we set the RHS of (32) to be $A_{k+1}$.

Note that when $\sum_{i=1}^{\infty} (\sigma_i + \sqrt{\varepsilon_i}) < +\infty$, we have $A^\mu \triangleq \lim_{k \to +\infty} A_k < +\infty$, and

$$\|y_k - y^\mu\| \leq \frac{1}{\beta} A^\mu,$$  \hspace{1cm} (33)

$$\|\lambda_k - \lambda^\mu\| \leq A^\mu.$$  \hspace{1cm} (34)

Using Eq. (26) we could further get:

$$\|x_k - x^\mu\| \leq \frac{3}{\beta} A^\mu.$$  \hspace{1cm} (35)

Combining (29),(30) and (31) with (26) and using the above inequalities, we have:

$$\sum_{k=0}^{\infty} \left( \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \right)$$

$$\leq \frac{1}{2\beta} \|\lambda_0 - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_0 + y^\mu\|^2$$

$$+ \sqrt{2\beta} \sum_{k=0}^{\infty} \left( (\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k}) \cdot \frac{2}{\beta} A^\mu + \sqrt{\varepsilon_{k+1}} \cdot \frac{1}{\beta} A^\mu \right) + 3 \sum_{k=0}^{\infty} \varepsilon_k$$

$$+ \sum_{k=0}^{\infty} \sigma_{k+1} \left( A^\mu + \beta \cdot \frac{4}{\beta^2} A^\mu + \beta \cdot \frac{3}{\beta} A^\mu \right)$$

$$\leq \frac{1}{2\beta} \|\lambda_0 - \lambda^\mu\|^2 + \frac{\beta}{2} \|y_0 + y^\mu\|^2 + 5 \sqrt{\frac{2}{\beta}} A^\mu \sum_{k=0}^{\infty} \varepsilon_k + 3 \sum_{k=0}^{\infty} \varepsilon_k + 8 A^\mu \sum_{k=1}^{\infty} \sigma_k,$$  \hspace{1cm} (36)

from which we can see that $\lambda_{k+1} - \lambda_k \to 0$ and $y_{k+1} - y_k \to 0$.

Recall that:

$$\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_{k+1}),$$

from which we deduce $x_{k+1} - y_{k+1} \to 0$.

Using the convexity of $f_k(\cdot)$ and $g(\cdot)$ for the LHS of Theorem 8 we have:

$$\text{LHS} = f_k(x_{k+1} + q_{k+1}) - f_k(x^\mu_k) + g(y_{k+1}) - g(y^\mu_k)$$

$$\leq \langle \nabla f_k(x_{k+1} + q_{k+1}), x_{k+1} - x^\mu_k \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^\mu_k \rangle + \varepsilon_{k+1}$$

Using (L12-2) with $x \equiv x^\mu_k, y \equiv y^\mu_k$ we have:

$$\text{LHS} \leq -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x^\mu_k \rangle + \beta(-y_{k+1} + y_k, x_{k+1} - x^\mu_k)$$

$$- \beta(q_{k+1}, x_{k+1} - x^\mu_k) - \beta(r_{k+1}, y_{k+1} - y^\mu_k) + \varepsilon_{k+1},$$

Hence, it follows that:

$$f_k(x_{k+1} + q_{k+1}) - f_k(x^\mu_k) + g(y_{k+1}) - g(y^\mu_k) \leq -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} \rangle + \beta(-y_{k+1} + y_k, x_{k+1} - x^\mu_k)$$

$$- \beta(q_{k+1}, x_{k+1} - x^\mu_k) - \beta(r_{k+1}, y_{k+1} - y^\mu_k) + \varepsilon_{k+1}$$

$$\leq ||\lambda_{k+1}|| \cdot ||x_{k+1} - y_{k+1}|| + \beta ||y_{k+1} - y_k|| \cdot ||x_{k+1} - x^\mu_k||$$

$$+ \beta \sigma_{k+1} ||x_{k+1} - x^\mu_k|| + \sqrt{2\varepsilon_{k+1} \beta} ||y_{k+1} - y^\mu_k|| + \varepsilon_{k+1},$$
where the last inequality follows from Cauchy-Schwarz.

Recall that $C$ is compact and $D$ is the diameter of $C$:

\[ D \triangleq \sup_{x, y \in C} ||x - y||. \]

Combining with (31), we have:

\[ ||y_{k+1} - y_k^*|| = ||y_{k+1} - y^\mu|| + ||y^\mu - y_k^*|| \leq \frac{1}{\beta} A^\mu + D, \tag{37} \]

which implies that $||y_k - y_k^*||$ are bounded for all $k$. Similarly, using (29), we deduce:

\[ ||x_{k+1} - x_k^\mu|| = ||x_{k+1} - x^\mu|| + ||x^\mu - x_k^\mu|| \leq \frac{3}{\beta} A^\mu + D, \tag{38} \]

which implies that $x_{k+1} - x_k^\mu$ is also bounded. Note that when $\sum_{i=1}^{\infty} (\sigma_i + \sqrt{\varepsilon_i}) < +\infty$, we have $\lim_{k \to \infty} \sigma_k = \lim_{k \to \infty} \varepsilon_k = 0$. Thus, we have (T8-$f_k$-UB).

**Lemma 15.** Assume that $F$ is $L$-Lipschitz. For the $x_{k+1}$, $y_{k+1}$, and $\lambda_{k+1}$ iterates of the ACVI—Algorithm 3—we have:

\[
\frac{1}{2\beta} ||\lambda_{k+1} - \lambda_k||^2 + \frac{\beta}{2} ||y_{k+1} + y_k||^2 \\
\leq \frac{1}{2\beta} ||\lambda_k - \lambda_{k-1}||^2 + \frac{\beta}{2} ||y_k + y_{k-1}||^2 \\
+ (\sigma_{k+1} + \sigma_k) \left( \beta ||y_k - y_{k-1}|| + (2\beta + L) ||y_{k+1} - y_k|| + \left( \frac{2 + L}{\beta} \right) ||\lambda_{k+1} - \lambda_k|| + \left( \frac{L + 1}{\beta} \right) ||\lambda_k - \lambda_{k-1}|| \right) \\
+ \sqrt{2\beta(\sqrt{\varepsilon_k} + \sqrt{\varepsilon_{k+1}})} ||y_{k+1} - y_k|| + \varepsilon_k + \varepsilon_{k+1}. \tag{L15} \]

**Proof of Lemma 15.** (L12-2) gives:

\[
\langle \hat{\nabla} f_{k-1}(x_k + q_k), x_k - x \rangle + \langle \nabla \epsilon_k g(y_k), y_k - y \rangle \\
= -\langle \lambda_k, x_k - y_k - x + y \rangle + \beta(-y_k + y_{k-1}, x_k - x) - \beta(q_k, x_k - x) - \beta(r_k, y_k - y). \tag{39} \]

Letting $(x, y, \lambda) = (x_k, y_k, \lambda_k)$ in (L12-2) and $(x, y, \lambda) \equiv (x_{k+1}, y_{k+1}, \lambda_{k+1})$ in (39), and adding them together, and
using (L11-3), we have

\[
\langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_{k-1} (x_k + q_k), x_{k+1} - x_k \rangle + \langle \nabla e_{k+1}, g (y_{k+1}) - \nabla e_k, g (y_k), y_{k+1} - y_k \rangle \\
= - (\lambda_{k+1} - \lambda_k, x_{k+1} - y_{k+1} - x_k + y_k) + \beta (-y_{k+1} + y_k - (-y_k + y_{k-1}), x_{k+1} - x_k) \\
- \beta (q_{k+1} - q_k, x_{k+1} - x_k) - \beta (r_{k+1} - r_k, y_{k+1} - y_k) \\
= - \frac{1}{\beta} (\lambda_{k+1} - \lambda_k, x_{k+1} - \lambda_k - (\lambda_k - \lambda_{k-1})) \\
+ (-y_{k+1} + y_k + (y_k - y_{k-1}), \lambda_{k+1} - \lambda_k + \beta y_{k+1} - (\lambda_k - \lambda_{k-1} + \beta y_k)) \\
- \beta (q_{k+1} - q_k, x_{k+1} - x_k) - \beta (r_{k+1} - r_k, y_{k+1} - y_k) \\
= \frac{1}{2 \beta} [||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2 - ||\lambda_{k+1} - \lambda_k - (\lambda_k - \lambda_{k-1})||^2] \\
+ \beta \frac{1}{2} (||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2 - ||-y_{k+1} + y_k - (-y_k + y_{k-1})||^2) \\
+ \beta (q_{k+1} - q_k, x_{k+1} - x_k) - \beta (r_{k+1} - r_k, y_{k+1} - y_k) \\
= \frac{1}{2 \beta} (||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2) + \beta \frac{1}{2} (||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2) \\
- \frac{1}{2 \beta} (||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2) - \beta (q_{k+1} - q_k, x_{k+1} - x_k) - \beta (r_{k+1} - r_k, y_{k+1} - y_k) \\
\leq \frac{1}{2 \beta} (||\lambda_k - \lambda_{k-1}||^2 - ||\lambda_{k+1} - \lambda_k||^2) + \beta \frac{1}{2} (||-y_k + y_{k-1}||^2 - ||-y_{k+1} + y_k||^2) \\
+ \beta (\sigma_{k+1} + \sigma_k) ||x_{k+1} - x_k|| + \sqrt{2\beta} (\sqrt{e_k} + \sqrt{e_{k+1}}) ||y_{k+1} - y_k||.
\]

Using the monotonicity of \( f_k \) and \( f_{k-1} \), we deduce:

\[
\langle \nabla f_k (x_{k+1} + q_{k+1}), x_k + q_k - (x_{k+1} + q_{k+1}) \rangle + f_k (x_{k+1} + q_{k+1}) \leq f_k (x_k + q_k), \\
\langle \nabla f_{k-1} (x_k + q_k), x_{k+1} + q_{k+1} - (x_k + q_k) \rangle + f_{k-1} (x_k + q_k) \leq f_{k-1} (x_{k+1} + q_{k+1}).
\]

Adding together the above two inequalities and rearranging the terms, we have:

\[
\langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_{k-1} (x_k + q_k), x_{k+1} + q_{k+1} - (x_k + q_k) \rangle + f_k (x_{k+1} + q_{k+1}) - f_{k-1} (x_k + q_k) \leq 0,
\]

which gives:

\[
\langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_{k-1} (x_k + q_k), x_{k+1} + q_{k+1} - x_k \rangle \\
\geq \langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_{k-1} (x_k + q_k), q_{k+1} - q_k \rangle \\
+ f_k (x_{k+1} + q_{k+1}) - f_{k-1} (x_k + q_k) \\
= \langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_{k-1} (x_k + q_k), q_{k+1} - q_k \rangle \\
+ (F(x_{k+1}) - F(x_k), x_{k+1} + q_{k+1} - x_k - q_k) \\
\geq (-\lambda_k - \beta (x_{k+1} - y_k) - \beta q_{k+1} - (\lambda_{k-1} - \beta (x_k - y_{k-1}) - \beta q_k), q_k - q_{k+1}) \\
+ (F(x_{k+1}) - F(x_k), x_{k+1} + q_{k+1} - x_k - q_k) \\
\geq \langle \lambda_{k-1} - \lambda_k - \beta (x_{k+1} - y_k) + \beta (x_k - y_{k-1}), q_k - q_{k+1} \rangle \\
+ (F(x_{k+1}) - F(x_k), x_{k+1} + q_{k+1} - x_k - q_k) \\
\geq - (\sigma_{k+1} + \sigma_k) (||\lambda_{k-1} - \lambda_k - \beta (x_{k+1} - y_k) + \beta (x_k - y_{k-1})|| + \|F(x_{k+1}) - F(x_k)\|),
\]

where the second inequality uses (noisy-\( \nabla f_k \)), the penultimate inequality uses the nonnegativity of \((q_k - q_{k+1}, q_k - q_{k+1})\), and the last inequality follows from the monotonicity of \(F\), the Cauchy-Schwarz inequality and the fact that \(\|q_k\| \leq \sigma_k\).
Note that by (L11-3) we have:

\[
\begin{align*}
\lambda_{k-1} - \lambda_k - \beta(x_{k+1} - y_k) + \beta(x_k - y_{k-1}) &= \beta(y_k - x_k - (x_{k+1} - y_k) + (x_k - y_{k-1})) \\
&= \beta(2y_k - x_{k+1} - y_{k-1}) \\
&= \beta((y_k - y_{k-1}) - (y_{k+1} - y_k) - (x_{k+1} - y_{k+1})) \\
&= \beta((y_k - y_{k-1}) - (y_{k+1} - y_k)) - (\lambda_{k+1} - \lambda_k), \tag{43}
\end{align*}
\]

\[
x_{k+1} - x_k = x_{k+1} - y_{k+1} + y_{k+1} - y_k + y_k - x_k = \frac{1}{\beta}(\lambda_{k+1} - \lambda_k) + y_{k+1} - y_k + \frac{1}{\beta}(\lambda_k - \lambda_{k-1}). \tag{44}
\]

Using (42), (43), (44) and the L-smoothness property of \( F \), we get:

\[
\begin{align*}
\langle \nabla f_k (x_{k+1} + q_{k+1}) - \nabla f_k (x_k + q_k), x_{k+1} - x_k \rangle \\
&\geq - (\sigma_{k+1} + \sigma_k) (\beta \langle y_k - y_{k-1} \rangle + \beta \|y_k - y_k\| + \|\lambda_{k+1} - \lambda_k\| + L \|x_{k+1} - x_k\|) \\
&\geq - (\sigma_{k+1} + \sigma_k) \left( \beta \|y_k - y_{k-1}\| + (\beta + L) \|y_{k+1} - y_k\| + \left( 1 + \frac{L}{\beta} \right) \|\lambda_{k+1} - \lambda_k\| + \frac{L}{\beta} \|\lambda_k - \lambda_{k-1}\| \right).
\end{align*}
\]

Combining the above inequality with (23) and (41), and using (33) and (35), it follows that:

\[
\begin{align*}
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \\
&\leq \frac{1}{2\beta} \|\lambda_k - \lambda_{k-1}\|^2 + \frac{\beta}{2} \|y_k + y_{k-1}\|^2 \\
&+ (\sigma_{k+1} + \sigma_k) \left( \beta \|y_k - y_{k-1}\| + (2\beta + L) \|y_{k+1} - y_k\| + \left( 2 + \frac{L}{\beta} \right) \|\lambda_{k+1} - \lambda_k\| + \frac{L}{\beta} \|\lambda_k - \lambda_{k-1}\| \right) \\
&+ \sqrt{2\beta} (\sqrt{\epsilon_k} + \sqrt{\epsilon_{k+1}}) \|y_{k+1} - y_k\| + \epsilon_k + \epsilon_{k+1}.
\end{align*}
\]

\[\square\]

**Lemma 16.** If \( \lim_{K \to +\infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\epsilon_k}) < +\infty \), then we have:

\[
\begin{align*}
\sum_{k=1}^{\infty} \sigma_k + \sqrt{\epsilon_k} < +\infty, \tag{45}
\sum_{k=1}^{\infty} k\epsilon_k < +\infty, \tag{46}
\sigma_K + \sqrt{\epsilon_k} &\leq O \left( \frac{1}{\sqrt{K}} \right). \tag{47}
\end{align*}
\]

**Proof.** Let \( T_K \triangleq \lim_{K \to +\infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\epsilon_k}) \). If \( \lim_{K \to +\infty} T_K < +\infty \), then by Cauchy’s convergence test, \( \forall p \in \mathbb{N}_+, T_{K+p} - T_K \to 0, K \to +\infty \).
where the second term
\[
\frac{p}{\sqrt{K + p} \sqrt{K + p + p}} \sum_{k=K+2}^{K+p+1} k(\sigma_k + \sqrt{\varepsilon_k}) \leq \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\varepsilon_k}) \rightarrow 0, \quad K \rightarrow +\infty, \quad \forall p \in \mathbb{N}_+.
\]

Thus for any \( p \in \mathbb{N}_+ \), we have
\[
\frac{1}{\sqrt{K} + p} \sum_{k=K+2}^{K+p+1} k(\sigma_k + \sqrt{\varepsilon_k}) \rightarrow 0, \quad K \rightarrow +\infty.
\]

From which we deduce that for any \( p \in \mathbb{N}_+ \),
\[
\sum_{k=K+2}^{K+p+1} (\sigma_k + \sqrt{\varepsilon_k}) \leq \frac{\sqrt{K + p}}{K + 2} \sum_{k=K+2}^{K+p+1} (\sigma_k + \sqrt{\varepsilon_k}) \leq \frac{\sqrt{K + p}}{K + 2} \frac{1}{\sqrt{K + p}} \sum_{K+2}^{K+p+1} k(\sigma_k + \sqrt{\varepsilon_k}) \rightarrow 0, \quad \forall K \rightarrow +\infty.
\]

Again by Cauchy’s convergence test, we have
\[
\sum_{k=1}^{\infty} \sigma_k + \sqrt{\varepsilon_k} < +\infty,
\]
which is (45).

Note that \( \lim_{K \to \infty} T_K = T_0 + \sum_{k=0}^{\infty} T_{k+1} - T_k \). And
\[
T_{k+1} - T_k = O\left( \sqrt{K} (\sigma_k + \sqrt{\varepsilon_k}) \right) \geq O(k\varepsilon_k).
\]

Thus by the comparison test, we have
\[
\sum_{k=1}^{\infty} k\varepsilon_k < +\infty,
\]
\[
\sigma_K + \sqrt{\varepsilon_k} \leq O\left( \frac{1}{\sqrt{K}} \right),
\]
which gives (46), (47).

**Lemma 17.** Assume that \( F \) is monotone on \( C_\varepsilon \), and \( \lim_{K \to \infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\varepsilon_k}) < +\infty \), then for the inexact ACVI—Alg. 1, we have:

\[
f_K(x_{K+1} + q_{K+1}) + g(y_{K+1}) - f_K(x^*_K) - g(y^*_K)
\leq (\| \lambda^\mu \| + 4A + \beta D) \frac{E^\mu}{\beta \sqrt{K}} + (3A^\mu + \beta D) \sigma_{k+1} + \sqrt{2\beta} \left( \frac{A^\mu}{\beta} + D \right) \sqrt{\varepsilon_{k+1} + \varepsilon_{k+1}}, \quad \text{(L17-1)}
\]

and
\[
\| x_{K+1} - y_{K+1} \| \leq \frac{E^\mu}{\beta \sqrt{K}}, \quad \text{(L17-2)}
\]

where \( A^\mu \) is defined in Theorem 8.
Proof of Lemma 17. First, we define: \( \Delta^\mu \triangleq \frac{1}{\beta} \| \lambda_0 - \lambda^\mu \|^2 + \beta \| y_0 - y^\mu \|^2 \).

Summing (25) over \( k = 0, 1, \ldots, K \), we have:

\[
\sum_{i=0}^{K} \left( \frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 + \frac{\beta}{2} \| y_{k+1} + y_k \|^2 \right) \\
\leq \frac{1}{2\beta} \| \lambda_0 - \lambda^\mu \|^2 + \frac{\beta}{2} \| y_0 + y^\mu \|^2 \\
+ \sum_{i=0}^{K} \sqrt{2\beta} (\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k}) \| y_{k+1} - y_k \| + \sum_{i=0}^{K} \sqrt{2\beta} \varepsilon_{k+1} \| y_{k+1} - y^\mu \| \\
+ \sum_{k=0}^{K} \varepsilon_k + 2 \sum_{k=0}^{K} \varepsilon_{k+1} + \sum_{k=0}^{K} \sigma_{k+1} (\| \lambda_k - \lambda^\mu \| + \beta \| x_{k+1} - y_k \| + \beta \| x_{k+1} - x^\mu \|) \\
\leq \frac{1}{2\beta} \| \lambda_0 - \lambda^\mu \|^2 + \frac{\beta}{2} \| y_0 + y^\mu \|^2 \\
+ 2 \sum_{k=0}^{K} \sqrt{\frac{2}{\beta}} A^\mu (\sqrt{\varepsilon_{k+1}} + \sqrt{\varepsilon_k}) + \sum_{k=0}^{K} \sqrt{\frac{2}{\beta}} A^\mu \sqrt{\varepsilon_{k+1}} \\
+ \sum_{k=0}^{K} \varepsilon_k + 2 \sum_{k=0}^{K} \varepsilon_{k+1} + \sum_{k=0}^{K} \sigma_{k+1} \left( A + \beta \cdot \frac{4}{\beta} A^\mu + \beta \cdot \frac{3}{\beta} A^\mu \right) \\
\leq \Delta^\mu + 5 \sqrt{\frac{2}{\beta}} A^\mu \sum_{k=1}^{K} \sqrt{\varepsilon_k} + 8 A^\mu \sum_{k=1}^{K+1} \sigma_{k+1} + 3 \sum_{k=1}^{K+1} \varepsilon_k, \tag{48}
\]

where the penultimate inequality follows from (30), (33), (34) and (35), and \( A^\mu \) is defined in Theorem 8.

Recall that Lemma 15 gives:

\[
\frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 + \frac{\beta}{2} \| y_{k+1} + y_k \|^2 \\
\leq \frac{1}{2\beta} \| \lambda_k - \lambda_{k-1} \|^2 + \frac{\beta}{2} \| y_k + y_{k-1} \|^2 \\
+ (\sigma_{k+1} + \sigma_k) \left( \beta \| y_{k} - y_{k-1} \| + (2\beta + L) \| y_{k+1} - y_k \| + \left( 2 + \frac{L}{\beta} \right) \| \lambda_{k+1} - \lambda_{k} \| + \left( \frac{L}{\beta} + 1 \right) \| \lambda_k - \lambda_{k-1} \| \right) \\
+ \sqrt{2\beta} (\sqrt{\varepsilon_k} + \sqrt{\varepsilon_{k+1}}) \| y_{k+1} - y_k \| + \varepsilon_k + \varepsilon_{k+1}.
\]

Let:

\[
\delta_{k+1} \triangleq (\sigma_{k+1} + \sigma_k) \left( \beta \| y_{k} - y_{k-1} \| + (2\beta + L) \| y_{k+1} - y_k \| + \left( 2 + \frac{L}{\beta} \right) \| \lambda_{k+1} - \lambda_{k} \| + \left( \frac{L}{\beta} + 1 \right) \| \lambda_k - \lambda_{k-1} \| \right) \\
+ \sqrt{2\beta} (\sqrt{\varepsilon_k} + \sqrt{\varepsilon_{k+1}}) \| y_{k+1} - y_k \| + \varepsilon_k + \varepsilon_{k+1}. \tag{\delta}
\]

Then the above inequality could be rewritten as:

\[
\frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 + \frac{\beta}{2} \| y_{k+1} + y_k \|^2 \\
\leq \frac{1}{2\beta} \| \lambda_k - \lambda_{k-1} \|^2 + \frac{\beta}{2} \| y_k + y_{k-1} \|^2 + \delta_{k+1},
\]

which gives:

\[
\frac{1}{2\beta} \| \lambda_{K+1} - \lambda_K \|^2 + \frac{\beta}{2} \| y_{K+1} + y_K \|^2 \\
\leq \frac{1}{2\beta} \| \lambda_K - \lambda_{K-1} \|^2 + \frac{\beta}{2} \| y_K + y_{K-1} \|^2 + \sum_{i=k}^{K} \delta_{i+1}. \tag{49}
\]
Combining (49) with (48), we obtain:

\[
K \left( \frac{1}{2\beta} \| \lambda_{K+1} - \lambda_K \|^2 + \frac{\beta}{2} \| -y_{K+1} + y_K \|^2 \right) \\
\leq \sum_{i=0}^{K} \left( \frac{1}{2\beta} \| \lambda_{k+1} - \lambda_k \|^2 + \frac{\beta}{2} \| -y_{k+1} + y_k \|^2 \right) + \sum_{i=0}^{K-1} \sum_{j=k+1}^{K} \delta_{j+1} \\
\leq \Delta^\mu + 5\sqrt{\frac{2}{\beta}A \sum_{k=1}^{K+1} \sqrt{\varepsilon_k} + 8A \sum_{k=1}^{K+1} \sigma_i + 3 \sum_{k=1}^{K+1} \varepsilon_k + \sum_{k=1}^{K+1} k\delta_{k+1}.} 
\] (50)

We define:

\[
a_{k+1} \triangleq (\sigma_{k+1} + \sigma_k) \left( 1 + \frac{L}{\beta} \right), 
\]

\[
b_{k+1} \triangleq (\sigma_{k+1} + \sigma_k) \left( 2 + \frac{L}{\beta} \right) + \sqrt{2} \left( \sqrt{\varepsilon_{k+1} + \sqrt{\varepsilon_k}} \right), 
\]

\[
u_{k+1}' \triangleq \| \lambda_{k+1} - \lambda_k \| + \beta \| y_{k+1} - y_k \|. 
\]

Note that:

\[
\delta_{k+1} \leq \varepsilon_k + \varepsilon_{k+1} + (\sigma_{k+1} + \sigma_k) \left( 1 + \frac{L}{\beta} \right) \frac{\| \lambda_k - \lambda_{k+1} \| + \beta \| y_k - y_{k-1} \|}{u_k'} \\
+ \left( (\sigma_{k+1} + \sigma_k) \left( 2 + \frac{L}{\beta} \right) + \sqrt{2} \left( \sqrt{\varepsilon_{k+1} + \sqrt{\varepsilon_k}} \right) \right) \frac{\| \lambda_{k+1} - \lambda_k \| + \beta \| y_{k+1} - y_k \|}{u_{k+1}'}.
\]

from which we deduce that:

\[
\sum_{k=1}^{K} k\delta_{k+1} \leq \sum_{k=1}^{K} (ka_{k+1}u_k' + (k+1)b_{k+1}u_{k+1}') + \sum_{k=1}^{K} k(\varepsilon_k + \varepsilon_{k+1}) \\
\leq \sum_{k=1}^{K+1} (a_{k+1} + b_k)k' u_k' + 2 \sum_{k=1}^{K+1} k\varepsilon_k \\
= \sum_{k=1}^{K+1} \left( (\sigma_{k+1} + \sigma_k) \left( 1 + \frac{L}{\beta} \right) + (\sigma_{k-1} + \sigma_k) \left( 2 + \frac{L}{\beta} \right) + \sqrt{2} \left( \sqrt{\varepsilon_{k-1} + \sqrt{\varepsilon_k}} \right) \right) k' u_k' + 2 \sum_{k=1}^{K+1} k\varepsilon_k, 
\] (51)

where we define

\[
c_k \triangleq (\sigma_{k+1} + \sigma_k) \left( 1 + \frac{L}{\beta} \right) + (\sigma_{k-1} + \sigma_k) \left( 2 + \frac{L}{\beta} \right) + \sqrt{2} \left( \sqrt{\varepsilon_{k-1} + \sqrt{\varepsilon_k}} \right). 
\]

(c)

Note that by Cauchy-Schwarz inequality, we have:

\[
\frac{K}{4\beta} u_{k+1}'^2 = \frac{K}{4\beta} \left( \| \lambda_{k+1} - \lambda_k \| + \beta \| y_{k+1} - y_k \| \right)^2 \leq \frac{K}{2\beta} \left( \| \lambda_{K+1} - \lambda_K \|^2 + \beta^2 \| y_{K+1} + y_K \|^2 \right). 
\]

Combining this inequality with (50), (53), and letting:

\[
B_{k+1} \triangleq \Delta^\mu + 5\sqrt{\frac{2}{\beta}A \sum_{k=1}^{K+1} \sqrt{\varepsilon_k} + 8A^\mu \sum_{k=1}^{K+1} \sigma_i + 3 \sum_{k=1}^{K+1} \varepsilon_k}, 
\]

(B)
gives:
\[ u_{k+1}^2 \leq \frac{4\beta}{K} \left( B_{k+1} + 2 \sum_{k=1}^{K+1} k\varepsilon_k \right) + \frac{4\beta}{K} \sum_{k=1}^{K+1} k\epsilon_k u_k'. \]

Using Lemma 8, we obtain:
\[ u_{k+1}' \leq \frac{1}{\sqrt{K}} \left( \frac{2\beta}{K} \sum_{k=1}^{K+1} kc_k + \left( \frac{4\beta}{\sqrt{K} \sum_{k=1}^{K+1} k\epsilon_k} \right)^2 \right)^{\frac{1}{2}}. \]  \hspace{1cm} (54)

Using the assumption that \( \lim_{K \to +\infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\varepsilon_k}) < +\infty \) and (45) in Lemma 16, we have \( B_{k+1} \) is bounded; using (46), we know that \( E_{k+1} \) in the RHS of (54) is bounded.

Let \( E^\mu = \lim_{k \to \infty} E_k \), then by (54) we have
\[ \beta \| x_{K+1} - y_{K+1} \| = \| \lambda_{K+1} - \lambda_K \| \leq \frac{E^\mu}{\sqrt{K}}, \]
\[ \| y_{K+1} - y_K \| \leq \frac{E^\mu}{\beta \sqrt{K}}. \]  \hspace{1cm} (55, 56)

On the other hand, (33), (34) and (35) gives:
\[ \| x_k - x^\mu_k \| \leq \| x_k - x^\mu \| + \| x^\mu - x^\mu_k \| \leq \frac{3}{\beta} A^\mu + D, \]
\[ \| y_k - y^\mu_k \| \leq \| y_k - y^\mu \| + \| y^\mu - y^\mu_k \| \leq \frac{1}{\beta} A^\mu + D, \]
\[ \| \lambda_{k+1} \| \leq \| \lambda_{k+1} - \lambda^\mu \| + \| \lambda^\mu \| \leq A^\mu + \| \lambda^\mu \|. \]

Plugging these into (T8- \( f_k \)-UB) yields (L17-1).

\[ \square \]

**B.2.3 Proving Theorem 2**

We are now ready to prove Theorem 2. Here we give a nonasymptotic convergence rate of Algorithm 1.

**Theorem 9** (Restatement of Theorem 2). Given an continuous operator \( F : \mathcal{X} \to \mathbb{R}^n \), assume that:

(i) \( F \) is monotone on \( C_\infty \), as per Def. 1;

(ii) \( F \) is \( L \)-Lipschitz on \( \mathcal{X} \);

(iii) \( F \) is either strictly monotone on \( C \) or one of \( \varphi_i \) is strictly convex.

Let \( (x_{iK}^{(t)}, y_{iK}^{(t)}, \lambda_{iK}^{(t)}) \) denote the last iterate of Algorithm 1, run with sufficiently small \( \mu_{-1} \). Then \( \forall t \in [T] \) and \( \forall K \in \mathbb{N}_+ \), Further, suppose:
\[ \lim_{K \to +\infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K+1} k(\sigma_k + \sqrt{\varepsilon_k}) < +\infty. \]

We define
\[ \lambda_i \triangleq 4\beta \left( \frac{2}{\beta} \left( \sqrt{\varepsilon_{i+1}} + 3\sqrt{\varepsilon_i + \sqrt{\varepsilon_{i-1}}} \right) + (2\sigma_i + 3\sigma_{i+1}) \right), \]
\[ S_{k+1}^* \triangleq 2 \| \lambda_0 - \lambda^* \|^2 + 2 \beta^2 \| y_0 - y^* \|^2 + 4 \beta (\sqrt{2 \beta} \sqrt{\epsilon_1} + \beta \sigma_1) \| y_0 - y^* \| + 12 \beta \sigma_1 \| \lambda_0 - \lambda^* \| + 12 \beta \sum_{i=1}^{k+1} \epsilon_i, \]

\[ A_{k+1}^* \triangleq \frac{1}{2} \sum_{i=1}^{k+1} \lambda_i + \left( S_{k+1}^* + \left( \frac{1}{2} \sum_{i=1}^{k+1} \lambda_i \right)^2 \right)^{1/2}, \]

and

\[ A \triangleq \lim_{k \to +\infty} A_k^* < +\infty. \]

We define

\[ c_k \triangleq ((\sigma_{k+1} + \sigma_k) \left( 1 + \frac{L}{\beta} \right) + (\sigma_{k-1} + \sigma_k) \left( 2 + \frac{L}{\beta} \right) + \sqrt{\frac{\sigma_k}{\beta}} (\sqrt{\epsilon_k} + \sqrt{\epsilon_{k-1}}), \]

\[ \Delta \triangleq \frac{1}{\beta} \| \lambda_0 - \lambda^* \|^2 + \beta \| y_0 - y^* \|^2, \]

\[ B_{k+1}^* \triangleq \Delta + 5 \sqrt{\frac{2}{\beta}} A \sum_{k=1}^{K+1} \sqrt{\epsilon_k} + 8 A \sum_{k=1}^{K+1} \sigma_k + 3 \sum_{k=1}^{K+1} \epsilon_k, \]

\[ E_{k+1}^* \triangleq \frac{2 \beta}{\sqrt{K}} \sum_{k=1}^{K+1} k c_k + \left( 4 \beta \left( B_{k+1}^* + 2 \sum_{k=1}^{K+1} k \epsilon_k \right) + \left( \frac{2 \beta}{\sqrt{K}} \sum_{k=1}^{K+1} k c_k \right)^2 \right)^{1/2}, \]

and

\[ E = \lim_{k \to \infty} E_k^*. \]

Then we have

\[ G(x_{K+1}, C) \leq (2 \| \lambda^* \| + \beta D + 1 + M) \frac{E}{\beta \sqrt{K}} + (4A + \beta D + M) \sigma_{k+1} + \sqrt{2 \beta} \left( \frac{2A}{\beta} + D \right) \sqrt{\epsilon_{k+1} + \epsilon_k + 1} \]

\[ = O \left( \frac{1}{\sqrt{K}} \right). \]

and

\[ \| x_{K+1} - y_{K+1} \| \leq \frac{2E}{\beta \sqrt{K}}, \]

where \( \Delta \triangleq \frac{1}{\beta} \| \lambda_0 - \lambda^* \|^2 + \beta \| y_0 - y^* \|^2 \) and \( D \triangleq \sup_{x, y \in C} \| x - y \|, \) and \( M \triangleq \sup_{x \in C} \| F(x) \|. \)

**Proof of Theorem 9.** Note that

\[ (f_k-\text{Pr-2}) \iff \min_{x \in C} \langle F(x_{k+1}), x \rangle \]

\[ \iff \max_{x \in C} \langle F(x_{k+1}), x_{k+1} - x \rangle \]

\[ \iff G(x_{k+1}, C), \]

from which we deduce

\[ G(x_{k+1}, C) = \langle F(x_{k+1}), x_{k+1} - x^*_k \rangle, \forall k. \] (57)
For any $K \in \mathbb{N}$, by (CHU, 1998) we know that

$$x^K_{\mu} \to x_K^\ast,$$

$$g(y_{K+1}) - g(y^K_{\mu}) \to 0,$$

$$\Delta^\mu \to \frac{1}{\beta} ||x_0 - x^\ast||^2 + \beta ||y_0 - y^\ast||^2 = \Delta,$$

$$A^\mu \to A,$$

$$E^\mu \to E.$$ (58)

Therefore, there exists $\mu_{-1} > 0$, s.t.$\forall 0 < \mu < \mu_{-1}$,

$$\|x^K_{\mu} - x_K^\ast\| \leq \frac{E^\mu}{\beta \sqrt{K}},$$

$$|g(y_{K+1}) - g(y^K_{\mu})| \leq \frac{E^\mu}{\beta \sqrt{K}}.$$ (59)

Combining with Lemma 17, and using the we have that

$$\langle F(x_{K+1}), x_{K+1} - x^K_{\mu}\rangle$$

$$= \langle F(x_{K+1}), x_{K+1} + q_{K+1} - x^K_{\mu}\rangle - \langle F(x_{K+1}), q_{K+1}\rangle$$

$$= f_K (x_{K+1} + q_{K+1}) - f_K (x^K_{\mu}) - (F(x_{K+1}), q_{K+1})$$

$$\leq (\|x^\mu\| + 4A^\mu + \beta D) \frac{E^\mu}{\beta \sqrt{K}} + (3A^\mu + \beta D) \sigma_{k+1} + \sqrt{2\beta} \left(\frac{A^\mu}{\beta} + D\right) \sqrt{\varepsilon_{k+1} + \varepsilon_{k+1}}$$

$$\quad + \|q_{K+1}\| \|F(x_{K+1})\| + g(y_{K+1}) - g(y^K_{\mu})$$

$$\leq (\|x^\mu\| + 4A^\mu + \beta D + 1) \frac{E^\mu}{\beta \sqrt{K}} + (3A^\mu + \beta D + M) \sigma_{k+1} + \sqrt{2\beta} \left(\frac{A^\mu}{\beta} + D\right) \sqrt{\varepsilon_{k+1} + \varepsilon_{k+1}}.$$ (60)

Using the above inequality, we have

$$G(x_{K+1}, C) = \langle F(x_{K+1}), x_{K+1} - x^K_{\mu}\rangle$$

$$= \langle F(x_{K+1}), x_{K+1} - x^K_{\mu}\rangle + \langle F(x_{K+1}), x^K_{\mu} - x_K^\ast\rangle$$

$$\leq (\|x^\mu\| + 4A^\mu + \beta D + 1) \frac{E^\mu}{\beta \sqrt{K}} + (3A^\mu + \beta D + M) \sigma_{k+1} + \sqrt{2\beta} \left(\frac{A^\mu}{\beta} + D\right) \sqrt{\varepsilon_{k+1} + \varepsilon_{k+1}}.$$ (61)

Moreover, by (58), we can choose small enough $\mu_{-1}$ so that

$$G(x_{K+1}, C) \leq (2 \|x^\ast\| + 5A + \beta D + 1 + M) \frac{E}{\beta \sqrt{K}} + (4A + \beta D + M) \sigma_{k+1} + \sqrt{2\beta} \left(\frac{2A}{\beta} + D\right) \sqrt{\varepsilon_{k+1} + \varepsilon_{k+1}}.$$ (61)

$$\|x_{K+1} - y_{K+1}\| \leq \frac{2E}{\beta \sqrt{K}},$$ (62)

where (62) uses (L17-2) in Lemma 17. By (50), (61) and (62), we draw the conclusion.
B.3 Proof of Theorem 3

B.3.1 Setting and notations

We define the following maps from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$f(x) \triangleq F(x^*)^T x + \mathbb{1}(Cx = d), \quad (f_k)$$
$$f_k(x) \triangleq F(x_{k+1})^T x + \mathbb{1}(Cx = d), \quad (f)$$
$$g(y) \triangleq \mathbb{1}(\varphi(y) \leq 0), \quad (g)$$

where $x^*$ is a solution of (KKT). Let $y^* = x^*$. Then $(x^*, y^*)$ is an optimal solution of $(f, Pr)$. Let us denote with $(x^*_k, y^*_k, \lambda^*_k)$ the KKT point of $(f_k, Pr)$. Note that in this case, the problem $(f_k, Pr)$ is equivalent to $(f_k, Pr-2)$.

B.3.2 Intermediate results

In P-ACVI-Algorithm 2, by the definition of $y_{k+1}$ (line 7 of Algorithm 2), $y^*_k$ and $y^*$ we immediately know that

$$g(y_{k+1}) = g(y^*_k) = g(y^*) = 0. \quad (63)$$

The intermediate results for the proofs of Theorem 1 still hold in this case only with a little modification, and the proofs of them are very close to the previous ones. To avoid redundancy, we omit these proofs.

**Proposition 3** (Relation between $f_k$ and $f$). If $F$ is monotone, then $\forall k \in \mathbb{N}$, we have that:

$$f_k(x_{k+1}) - f_k(x^*) \geq f(x_{k+1}) - f(x^*).$$

**Lemma 18.** For all $x$ and $y$, we have

$$f(x) + g(y) - f(x^*) - g(y^*) + \langle \lambda^*, x - y \rangle \geq 0, \quad (L18-f)$$

and:

$$f_k(x) + g(y) - f_k(x^*_k) - g(y^*_k) + \langle \lambda^*_k, x - y \rangle \geq 0. \quad (L18-f_k)$$

The following lemma lists some simple but useful facts that we will use in the following proofs.

**Lemma 19.** For the problems $(f, Pr)$, $(f_k, Pr)$ and Algorithm 2, we have

$$0 \in \partial f_k(x_{k+1}) + \lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_k), \quad (L19-1)$$
$$0 \in \partial g(y_{k+1}) - \lambda_k - \beta(x_{k+1} - y_k), \quad (L19-2)$$
$$\lambda_{k+1} - \lambda_k = \beta(x_{k+1} - y_k), \quad (L19-3)$$
$$-\lambda^* \in \partial f(x^*), \quad (L19-4)$$
$$-\lambda^*_k \in \partial f_k(x^*_k), \quad (L19-5)$$
$$\lambda^* \in \partial g(y^*), \quad (L19-6)$$
$$\lambda^*_k \in \partial g(y^*_k), \quad (L19-7)$$
$$x^* = y^*, \quad (L19-8)$$
$$x^*_k = y^*_k, \quad (L19-9)$$

As in Appendix B.1.2, we also define $\nabla f_k(x_{k+1})$ and $\nabla g(y_{k+1})$ by $(\nabla f_k)$ and $(\nabla g)$, resp.

Then, from (L19-1) and (L19-2) it follows that:

$$\nabla f_k(x_{k+1}) \in \partial f_k(x_{k+1}) \quad \text{and} \quad \nabla g(y_{k+1}) \in \partial g(y_{k+1}). \quad (64)$$
Lemma 20. For the iterates \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) of the P-ACVI—Algorithm 2—we have:

\[
\langle \nabla g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, y - y_{k+1} \rangle,
\]

(65)

and

\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y \rangle = -\langle \lambda_{k+1}, x_{k+1} - y_{k+1} - x + y \rangle
+ \beta \langle -y_{k+1} + y_k, x_{k+1} - x \rangle.
\]

(66)

Lemma 21. For the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the P-ACVI—Algorithm 2—we have:

\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x^* \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^* \rangle + \langle \lambda^*, x_{k+1} - y_{k+1} \rangle
\leq \frac{1}{2\beta} \|\lambda_k - \lambda^*\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda^*\|^2 + \frac{\beta}{2} \|y^* - y_k\|^2 - \frac{\beta}{2} \|y^* - y_{k+1}\|^2
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_k - y_{k+1}\|^2,
\]

and

\[
\langle \nabla f_k(x_{k+1}), x_{k+1} - x^* \rangle + \langle \nabla g(y_{k+1}), y_{k+1} - y^* \rangle + \langle \lambda_k^*, x_{k+1} - y_{k+1} \rangle
\leq \frac{1}{2\beta} \|\lambda_k - \lambda_k^*\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k^*\|^2 + \frac{\beta}{2} \|y_k - y^*\|^2 - \frac{\beta}{2} \|y_k - y_{k+1}\|^2
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_k - y_{k+1}\|^2.
\]

Lemma 22. For the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the P-ACVI—Algorithm 2—we have:

\[
f(x_{k+1}) + g(y_{k+1}) - f(x^*) - g(y^*) + \langle \lambda^*, x_{k+1} - y_{k+1} \rangle
\leq \frac{1}{2\beta} \|\lambda_k - \lambda^*\|^2 - \frac{1}{2\beta} \|\lambda_{k+1} - \lambda^*\|^2
+ \frac{\beta}{2} \|y_k + y^*\|^2 - \frac{\beta}{2} \|y_{k+1} + y^*\|^2
- \frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 - \frac{\beta}{2} \|y_{k+1} + y_k\|^2
\]

(L22)

The following theorem upper bounds the analogous quantity but for \(f_k(\cdot)\) (instead of \(f\)), and further asserts that the difference between the \(x_{k+1}\) and \(y_{k+1}\) iterates of P-ACVI (Algorithm 2) tends to 0 asymptotically.

Theorem 10 (Asymptotic convergence of \(x_{k+1} - y_{k+1}\) of P-ACVI). For the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the P-ACVI—Algorithm 2—we have:

\[
f_k(x_{k+1}) - f_k(x_k^*) \leq \|\lambda_{k+1}\| \|x_{k+1} - y_{k+1}\| + \beta \|y_{k+1} - y_k\| \|x_{k+1} - x_k^*\| \to 0,
\]

(T10-UB)

and

\[
x_{k+1} - y_{k+1} \to 0, \quad \text{as } k \to \infty.
\]

Lemma 23. For the \(x_{k+1}, y_{k+1}\), and \(\lambda_{k+1}\) iterates of the P-ACVI—Algorithm 2—we have:

\[
\frac{1}{2\beta} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta}{2} \|y_{k+1} + y_k\|^2 \leq \frac{1}{46\beta} \|\lambda_k - \lambda_{k-1}\|^2 + \frac{\beta}{2} \|y_k + y_{k-1}\|^2.
\]

(L23)
Lemma 24. If $F$ is monotone on $C = \mathbb{R}^n$, then for Algorithm 2, we have:

$$f_K(x_{K+1}) - f_K(x_K^*) \leq \frac{\Delta}{\sqrt{\beta} (K+1)} + \left(2\sqrt{\Delta} + \frac{1}{\sqrt{\beta}} \|\lambda^*\| + \sqrt{\beta} D\right) \sqrt{\frac{\Delta}{K+1}}, \tag{L24-1}$$

and

$$\|x_{K+1} - y_{K+1}\| \leq \sqrt{\frac{\Delta}{\beta (K+1)}}, \tag{L24-2}$$

where $\Delta \triangleq \frac{1}{\beta} \|\lambda_0 - \lambda^*\|^2 + \beta \|y_0 - y^*\|^2$.

### B.3.3 Proving Theorem 3

We are now ready to prove Theorem 3. Here we give a nonasymptotic convergence rate of P-ACVI-Algorithm 2.

**Theorem 11 (Restatement of Theorem 3).** Given a continuous operator $F : \mathcal{X} \rightarrow \mathbb{R}^n$, assume $F$ is monotone on $C = \mathbb{R}^n$, as per Def. 1. Let $(x_K, y_K, \lambda_K)$ denote the last iterate of Algorithm 3. Then $\forall K \in \mathbb{N}_+$, we have

$$G(x_K, C) \leq \frac{\Delta}{\sqrt{\beta} K} + \left(2\sqrt{\Delta} + \frac{1}{\sqrt{\beta}} \|\lambda^*\| + \sqrt{\beta} D\right) \sqrt{\frac{\Delta}{K}}, \tag{na-If-Rate}$$

and

$$\|x^K - y^K\| \leq \sqrt{\frac{\Delta}{\beta K}}, \tag{67}$$

where $\Delta \triangleq \frac{1}{\beta} \|\lambda_0 - \lambda^*\|^2 + \beta \|y_0 - y^*\|^2$ and $D \triangleq \sup_{x, y \in C} \|x - y\|$, and $M \triangleq \sup_{x \in C} \|F(x)\|$.

**Proof of Theorem 3.** Note that

$$(f_k\text{-Pr-2}) \iff \min_{x \in C} \langle F(x_{k+1}), x \rangle$$

$$\iff \max_{x \in C} \langle F(x_{k+1}), x_{k+1} - x \rangle$$

$$\iff G(x_{k+1}, C),$$

from which we deduce

$$G(x_{k+1}, C) = \langle F(x_{k+1}), x_{k+1} - x_k^* \rangle = f_K(x_{K+1}) - f_K(x_K^*), \forall k. \tag{68}$$

Combining with Lemma 24, we obtain (na-If-Rate) and (67). \qed
C Details On The Implementation

In this section, we provide the details on the implementation of the results presented in § 6 in the main part, as well as those of the additional results presented in Appendix D. In addition, we provide the source code through the following anonymous link: https://github.com/mpagli/Revisiting-ACVI.

C.1 Implementation details for the 2D-BG game

Recall that we defined the 2D bilinear game as:

$$\min_{x_1 \in \Delta} \max_{x_2 \in \Delta} x_1 x_2 \quad \text{where} \quad \Delta = \{ x \in \mathbb{R} | -0.4 \leq x \leq 2.4 \}. \quad (2D-BG)$$

To avoid confusion in the notation, in the reminder of this section we rename the players in (2D-BG) as $p_1$ and $p_2$:

$$\min_{p_1 \in \Delta} \max_{p_2 \in \Delta} p_1 p_2 \quad \text{where} \quad \Delta = \{ p \in \mathbb{R} | -0.4 \leq p \leq 2.4 \}$$

In the following, we list the I-ACVI and P-ACVI implementations.

I-ACVI. For I-ACVI (Algorithm 1) we set $\beta = 0.5, \mu = 3, K = 20, \ell = 20, \delta = 0.5$ and use a learning rate of 0.1.

PI-ACVI. For PI-ACVI we set $\beta = 0.5, K = 20, \ell = 20$, and use a learning rate of 0.1.

C.2 Implementation details for the HBG game

Solution and relative error. The solution of (HBG) is $x^* = \frac{1}{500} e$, with $e \in \mathbb{R}^{1000}$. As a metric of the experiments on this problem we use the relative error: $\varepsilon_k(x_k) = \frac{\|x_k - x^*\|}{\|x^*\|}$.

Experiments of Fig.3.a showing CPU time to reach a fixed relative error. The target relative error is 0.02. We set the step size of GDA, EG, and OGDA to 0.3, and use $k = 5$ and $\alpha = 0.5$ for LA-GDA. For I-ACVI, we set $\beta = 0.5, \mu_1 = 10^{-6}, \delta = 0.8, \lambda_0 = 0, K = 10, \ell = 10$ and the step size is 0.05.

Experiments of Fig.3.b showing number of iterations to reach a fixed relative error. Hyperparameters are the same as for Fig.3.a. We vary the rotation “strength” $(1 - \eta)$, with $\eta \in (0, 1)$.

Experiments of Fig.3.c showing the impact of $K_0$. For this experiment we see, for various pairs $(K_0, K_+)$ how many iterations are required to reach a relative error smaller than $10^{-4}$. We set $\beta = 0.5, \mu = 1e - 6, \delta = 0.8, T = 5000$ and 0.05 as learning rate. We experiment with $K_0 \in \{5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130\}$ and $K_+ \in \{1, 5, 10, 20, 30, 40, 50, 60, 70\}$.

C.3 Implementation details for the C-GAN game

For the experiments on the MNIST dataset, we use the source code of Chavdarova et al. (2021b) for the baselines and we build on it to implement PI-ACVI (Algorithm 2). For completeness, we provide an overview of the implementation.

Models. We used the DCGAN architectures (Radford et al., 2016), listed in Table 2, and the parameters of the models are initialized using PyTorch default initialization. For experiments on this dataset, we used the non-saturating GAN loss as proposed in (Goodfellow et al., 2014):

$$L_D = \mathbb{E}_{\tilde{x}_d \sim p_d} \log \left( D(\tilde{x}_d) \right) + \mathbb{E}_{\tilde{z} \sim p_z} \log \left( 1 - D(G(\tilde{z})) \right) \quad (L-D)$$

$$L_G = \mathbb{E}_{\tilde{z} \sim p_z} \log \left( D(G(\tilde{z})) \right), \quad (L-G)$$

where $G(\cdot), D(\cdot)$ denote the generator and discriminator, resp., and $p_d$ and $p_z$ denote the data and the latent distributions (the latter predefined as normal distribution).
Table 2: DCGAN architectures (Radford et al., 2016) used for experiments on MNIST. With “conv.” we denote a convolutional layer and “transposed conv.” a transposed convolution layer (Radford et al., 2016). We use $ker$ and $pad$ to denote kernel and padding for the (transposed) convolution layers, respectively. With $h \times w$ we denote the kernel size. With $c_{in} \rightarrow y_{out}$ we denote the number of channels of the input and output, for (transposed) convolution layers. The models use Batch Normalization (Ioffe and Szegedy, 2015) layers.

Details on the PI-ACVI implementation. When implementing PI-ACVI on MNIST, we set $\beta = 0.5$, and $K = 5000$, we use $\ell_+ = 20$ and $\ell_0 \in \{100, 500\}$. We consider only inequality constraints (and there are no equality constraints), therefore the matrices $P_i$ and $d_c$ are identity and zero, respectively. As inequality constraints, we use 100 randomly generated linear inequalities for the Generator and 100 for the Discriminator.

Projection details. Suppose the linear inequality constraints for the Generator are $A\theta \leq b$, where $\theta \in \mathbb{R}^n$ is the vector of all parameters of the Generator, $A = (a_1^T, \ldots, a_{100}^T)^T \in \mathbb{R}^{100 \times n}$, $b = (b_1, \ldots, b_{100}) \in \mathbb{R}^{100}$. We use the greedy projection algorithm described in (Beck, 2017). A greedy projection algorithm is essentially a projected gradient method, it is easy to implement in high-dimension problems, and it has a convergence rate of $O(1/\sqrt{K})$. See Chapter 8.2.3 in (Beck, 2017) for more details. Since the dimension $n$ is very large, at each step of the projection, one could only project $\theta$ to one hyperplane $a_i^T x = b_i$ for some $i \in \mathcal{I}(\theta)$, where

$$\mathcal{I}(\theta) \triangleq \{j | a_j^T \theta > b_j\}.$$ 

For every $j \in \{1, 2, \ldots, 100\}$, let

$$S_j \triangleq \{x | a_j^T x \leq b_j\}.$$ 

The greedy projection method chooses $i$ so that $i \in \arg\max\{dist(\theta, S_i)\}$. Note that as long as $\theta$ is not in the constraint set $C_\leq = \{x | Ax \leq b\}$, $i$ would be in $\mathcal{I}(\theta)$. Algorithm 4 gives the details of the greedy projection method we use for the baseline, written for the Generator only for simplicity; the same projection method is used for the Discriminator as well.

Metrics. We describe the metrics for the MNIST experiments. We use the two standard GAN metrics, Inception Score (IS, Salimans et al., 2016) and Fréchet Inception Distance (FID, Heusel et al., 2017). Both FID and IS rely on a pre-trained classifier and take a finite set of $\hat{m}$ samples from the generator to compute these. Since MNIST has greyscale images, we used a classifier trained on this dataset and used $\hat{m} = 5000$.

Metrics: IS. Given a sample from the generator $\tilde{x}_g \sim p_g$—where $p_g$ denotes the data distribution of the generator—IS uses the softmax output of the pre-trained network $p(y|\tilde{x}_g)$ which represents the probability that $\tilde{x}_g$ is of class $c_i$, $i \in 1 \ldots C$, i.e., $p(y|\tilde{x}_g) \in [0, 1]^C$. It then computes the marginal class distribution $p(y) = \int p(y|\tilde{x}_g)p_g(\tilde{x}_g)$. IS measures the Kullback–Leibler divergence $\mathbb{D}_{KL}$ between the predicted conditional label distribution $p(y|\tilde{x}_g)$ and the marginal class distribution $p(y)$. More precisely, it is computed as follows:

$$IS(G) = \exp \left( \mathbb{E}_{\tilde{x}_g \sim p_g} \left[ \mathbb{D}_{KL}(p(y|\tilde{x}_g)||p(y)) \right] \right) = \exp \left( \frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}} \sum_{c=1}^{C} p(y_c|x_i) \log \frac{p(y_c|x_i)}{p(y_c)} \right).$$ 

(IS)
Algorithm 4 Greedy projection method for the baseline.

1: Input: \( \theta \in \mathbb{R}^n, A = (a_1^T, \ldots, a_{100}^T)^T \in \mathbb{R}^{100 \times n}, b = (b_1, \ldots, b_{100}) \in \mathbb{R}^{100}, \varepsilon > 0 \)
2: while True do
3: \( I(\theta) \triangleq \{ j | a_j^T \theta > b_j \} \)
4: if \( I(\theta) = \emptyset \) or \( \max_{j \in I(\theta)} \frac{|a_j^T \theta - b_j|}{\|a_j\|} < \varepsilon \) then
5: \( \text{break} \)
6: end if
7: choose \( i = \arg \max_{j \in I(\theta)} \frac{|a_j^T \theta - b_j|}{\|a_j\|} \)
8: \( \theta \leftarrow \theta - \frac{|a_i^T \theta - b_i|}{\|a_i\|^2} a_i \)
9: end while
10: Return: \( \theta \)

It aims at estimating (i) if the samples look realistic i.e., \( p(\tilde{y} | \tilde{x}_g) \) should have low entropy, and (ii) if the samples are diverse (from different ImageNet classes) i.e., \( p(\tilde{y}) \) should have high entropy. As these are combined using the Kullback–Leibler divergence, the higher the score is, the better the performance.

**Metrics:** FID. Contrary to IS, FID compares the synthetic samples \( \tilde{x}_g \sim p_g \) with those of the training dataset \( \tilde{x}_d \sim p_d \) in a feature space. The samples are embedded using the first several layers of a pretrained classifier. It assumes \( p_g \) and \( p_d \) are multivariate normal distributions, and estimates the means \( m_g \) and \( m_d \) and covariances \( C_g \) and \( C_d \), respectively for \( p_g \) and \( p_d \) in that feature space. Finally, FID is computed as:

\[
D_{\text{FID}}(p_d, p_g) \approx \mathcal{D}_2\left((m_d, C_d), (m_g, C_g)\right) = \|m_d - m_g\|^2_2 + Tr\left(C_d + C_g - 2(C_d C_g)^{\frac{1}{2}}\right),
\]

where \( \mathcal{D}_2 \) denotes the Fréchet Distance. Note that as this metric is a distance, the lower it is, the better the performance.

**Hardware.** We used the Colab platform (https://colab.research.google.com/) and Nvidia T4 GPUs.
D Additional Experiments and Analyses

In this section, we provide complementary experiments associated with the three games introduced in the main paper: (2D-BG), (HBG), and (C-GAN). We also provide an additional study of the robustness of I-ACVI to bad conditioning by introducing a version of the (HBG) game, see § D.3 for more details.

D.1 Additional results for I-ACVI on the 2D-BG game

For completeness, in Fig. 5 we show the trajectories for the \( x \) iterates—complementary to the \( y \)-iterates’ trajectories depicted in Fig. 1 of the main part.

![Complementary illustrations to those in Fig. 1 of the main part: depicting here the trajectories of the \( x \) iterates. We compare the convergence of ACVI, and I-ACVI with different parameters on the (2D-BG) problem, while depicting the central path as well (shown in yellow). Each subsequent bullet on the trajectory depicts the (exact or approximate) solution at the end of the inner loop (when \( k = K - 1 \)). The Nash equilibrium (NE) of the game is represented by a yellow star, and the constraint set is the interior of the red square.](image)

D.2 Additional results for PI-ACVI on the 2D-BG game

In this section, we provide complementary visualization to Fig. 2 in the main paper. We (i) compare with other methods in Fig. 6 and (ii) show PI-ACVI trajectories for various hyperparamters in Fig. 7.

**PI-ACVI vs. baselines.** In Fig. 6 we can observe the behavior of projected gradient descent ascent, projected extragradient, projected lookahead, and projected proximal point on the simple 2D constrained bilinear game (2D-BG), we use the same learning rate of 0.2 for all methods. In Fig. 7 we show trajectories for PI-ACVI for \( \ell \in \{1, 4, 10, 100\} \), \( \beta = 0.5 \), \( K = 150 \) and a learning rate 0.2.

![Comparison of Projected Gradient Descent Ascent (P-GDA), extragradient (P-EG) (Korpelevich, 1976), Lookahead (P-LA) (Chavdarova et al., 2021b) and Proximal-Point (P-PP) on the (2D-BG) game. For P-PP we solve the inner proximal problem through multiple steps of GDA and use warm-start (the last PP solution is used as a starting point of the next proximal problem). All those methods progress slowly when hitting the constraint. Those trajectories can be contrasted with PI-ACVI in Fig. 7.](image)
Figure 7: PI-ACVI (Algorithm 2) for different choices of $\ell$. Top row: Trajectories for the $x$ iterates. Bottom row: Trajectories for the $y$ iterates. For $\ell = 1$ the trajectory for the $y$ iterates is similar to the one of P-GDA (see Fig. 6), as we increase $\ell$ we observe how relatively few iterations are required for convergence compared to baselines.

D.3 Additional results on the HBG game

In this section, we (i) provide complementary experiments to Fig. 3 from the main paper, as well as (ii) analyze the robustness of I-ACVI against bad conditioning.

CPU time to reach a given relative error. In Fig. 8 we extend the x-axis of Fig. 3.a from the main paper for I-ACVI. We see that unlike baselines, I-ACVI remains fast even when the target relative error is very small. This is due to the fact that I-ACVI uses cheaper approximate steps for lines 8 and 9 of Algorithm 1.

Impact of $K_0$. In Fig. 9 we show, for each $(K_0, K_\infty)$ the CPU time required to reach a relative error of $10^{-4}$. Those times are highly correlated with the number of iterations shown in Fig. 3.c of the main paper.

Figure 8: Comparison between I-ACVI and other baselines used in § 6 of the main part. CPU time (in seconds; y-axis) to reach a given relative error (x-axis); while the rotational intensity is fixed to $\eta = 0.05$ in (HBG) for all methods. I-ACVI is much faster to converge than other methods, including ACVI.
Figure 9: **Impact of $K_0$:** joint impact of the number of inner-loop iterations $K_0$ at $t = 0$, and different choices of inner-loop iterations for $K_+$ at any $t > 0$, on the CPU-time needed to reach a fixed relative error of $10^{-4}$. A large enough $K_0$ can compensate for a small $K_+$.

**Impact of conditioning.** We modify the (HBG) game to study the impact of conditioning, hence, we propose the following version:

$$\min_{x_1 \in \triangle} \max_{x_2 \in \triangle} x_1^T Dx_2,$$

(HBG-v2)

where \( \triangle = \{ x_i \in \mathbb{R}^{500} | x_i \geq 0, \text{ and } e_i^T x_i = 1 \} \), and \( D = \text{diag}(\alpha_1, \ldots, \alpha_{500}) \).

The solution of this game depends on the \( \{ \alpha_i \}_{i=1}^{500} \):

$$x_1^* = x_2^* = \frac{1}{\sum_{i=1}^{500} 1/\alpha_i} \begin{pmatrix} 1/\alpha_1 \\ 1/\alpha_2 \\ \vdots \\ 1/\alpha_{500} \end{pmatrix}$$

We define the conditioning $\kappa$ as the ratio between the largest and smallest $\alpha_i$: $\kappa \triangleq \frac{\alpha_{\text{max}}}{\alpha_{\text{min}}}$. In our experiments we select $\alpha_i$ linearly interpolated between 1 and $\alpha_{\text{max}}$ (e.g. using the `np.linspace` function). We set $\alpha_{\text{min}} = 1$ and vary $\alpha_{\text{max}} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We compare projected extragradient (P-EG) with I-ACVI. For P-EG, we obtained better results when using smaller learning rates $\gamma$ for larger $\alpha_{\text{max}}$: $\gamma = 0.3 \times 0.9^{\alpha_{\text{max}}}$. For I-ACVI we set $\beta = 0.5$, $\mu = 10^{-5}$, $\delta = 0.5$, $\gamma = 0.003$, $K = 100$ and $T = 200$. We vary $\ell$ depending on $\alpha_{\text{max}}$: $\ell = 20$ for $\alpha_{\text{max}} \in \{1, 2, 3\}$, $\ell = 50$ for $\alpha_{\text{max}} \in \{4, 5, 6\}$, and $\ell = 100$ for $\alpha_{\text{max}} \in \{7, 8, 9, 10\}$. We compare the CPU times required to reach a relative error of 0.02 in Fig. 10. We observe that I-ACVI is more robust to bad conditioning than P-EG. As $\kappa \to 0$, P-EG is failing to converge in an appropriate time despite reducing the learning rate. For I-ACVI, keeping the same learning rate and only increasing $\ell$ is enough to compensate for smaller $\kappa$ values. One can speculate that I-ACVI is more robust thanks to (i) the $y$-problem (line 9 in Algorithm 1) not depending on $F(x)$, hence being relatively immune to the problem itself, and (ii) the $x$-problem (line 8 in Algorithm 1) being “regularized” by $y_k$ and $\lambda_k$.

**D.4 Additional results on the C-GAN game**

In this section, we show complementary results to our constrained GAN MNIST experiments. In Fig. 11 we further show the impact of $\ell_0$ on the convergence speed by training different PI-ACVI models with $\ell_0 \in \{20, 50, 100, 200, 400, 600, 800, 1000\}$, all other hyperparameters being equal — setting $\ell_+ = 10$. We compare in Fig. 12 the obtained curves for $\ell_0 = 400$ with projected-GDA (P-GDA), and verify that — similarly to Fig. 4 of the main paper for which $\ell_+ = 20$ — PI-ACVI is here as well outperforming significantly P-GDA. This shows that PI-ACVI is relatively unaffected by $\ell_+$ as opposed to $\ell_0$. 
Figure 10: **Experiment on conditioning:** CPU time to reach a relative error of 0.02 on the (HBG-v2) game, for different conditioning values $\kappa$. While P-EG is struggling to converge when the conditioning is bad (small $\kappa$), I-ACVI on the other hand can cope relatively well.

Figure 11: **Effect of $\ell_0$ on FID and IS:** On the MNIST datasets, comparison of various runs of PI-ACVI for different $\ell_0$. All other hyperparameters are equal: $\ell_+ = 10$, $\beta = 0.5$, see §C for more details. (a) and (b): we observe the importance of $\ell_0$, despite $\ell_+ = 10$ being relatively small we still converge fast to a solution — in terms of both FID ($\downarrow$) and IS ($\uparrow$) — given $\ell_0$ large enough. All curves are obtained by averaging over two seeds.

Figure 12: **PI-ACVI vs P-GDA on (C-GAN) MNIST:** On the MNIST datasets, comparison of P-GDA and PI-ACVI. For PI-ACVI, we set $\ell_0 = 400$ and $\ell_+ = 10$. (a) and (b): in both FID ($\downarrow$) and IS ($\uparrow$), PI-ACVI converges faster than P-GDA. The difference with Fig. 4 from the main paper is that we use $\ell_+ = 10$ instead of $\ell_+ = 20$. This shows that PI-ACVI is relatively robust to different values of $\ell_+$. 