Three dimensional strongly symmetric circulant tensors

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\textbf{Article history:}
Received 15 April 2015
Accepted 19 May 2015
Available online 18 June 2015
Submitted by R. Brualdi

\textbf{MSC:}
15A18
15A69

\textbf{Keywords:}
H-eigenvalue
Strongly symmetric tensor
Circulant tensor
Sum of squares
Positive semi-definiteness

\textbf{Abstract}
In this paper, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. We show that this condition can be a sufficient condition for such a tensor to be sum-of-squares in some cases. There are no PNS strongly symmetric circulant tensors to be found in numerical tests.

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1 The author’s work was partially supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 502111, 501212, 501913 and 15302114).
2 The author’s work was supported by the National Natural Science Foundation of China (Grant No. 11401539) and the Development Foundation for Excellent Youth Scholars of Zhengzhou University (Grant No. 1421315070).
1. Introduction

Let \( \mathcal{A} = (a_{i_1 \cdots i_m}) \) be an \( m \)th order \( n \) dimensional real symmetric tensor, where \( m \) is even, \( i_1, \cdots, i_m = 1, \cdots, n \). Entries \( a_{i_1 \cdots i_m} \) are real and invariant under any index permutation. Then \( \mathcal{A} \) corresponds to a homogeneous polynomial \( f(x) \) for \( x \in \mathbb{R}^n \), defined by

\[
f(x) = \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.
\]

If \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \), \( \mathcal{A} \) is called a positive semi-definite (PSD) tensor, and \( f \) is accordingly called a PSD polynomial. If \( f(x) \) can be decomposed to a sum of squares of polynomials of degree \( \frac{m}{2} \), \( \mathcal{A} \) is called a sum-of-squares (SOS) tensor, and \( f \) is accordingly called an SOS polynomial. It is clear that an SOS polynomial is positive semi-definite.

The identification of a PSD tensor is NP-hard and is also important both theoretically and practically [1,9,18]. Recently it was discovered that several easily checkable classes of special even order symmetric tensors are PSD, including even order symmetric diagonally dominated tensors [18], even order symmetric \( B_0 \) tensors [20], even order Hilbert tensors [23], even order symmetric M tensors [25], even order symmetric double \( B_0 \) tensors [11], even order symmetric strong \( H \) tensors [12,10], even order strong Hankel tensors [19], even order positive Cauchy tensors [3], etc. However, some kinds of tensors, such as Hankel tensors, are not easy to identify their positive semi-definiteness. So, we turn to find a sufficient and necessary condition of PSD for a new class of structured tensors.

In 1888, Hilbert [9] proved that only in the following three cases, a PSD homogeneous polynomial of degree \( m \) in \( n \) variables is an SOS polynomial: 1) \( m = 2 \); 2) \( n = 2 \); 3) \( m = 4 \) and \( n = 3 \). Hilbert proved that in all the other possible combinations of \( n \) and even \( m \), there are PSD non-SOS (PNS) homogeneous polynomials.\(^3\) However, Hilbert did not give an explicit example for PNS homogeneous polynomials. The first explicit example for PNS homogeneous polynomials was given by Motzkin [17] in 1967. More examples of PNS homogeneous polynomials can be found in [7,22]. In [13] and [4], sixth order three dimensional Hankel tensors and fourth order four dimensional Hankel tensors were studied respectively. No PNS Hankel tensors were found there.

Are there other special classes of even order symmetric PNS-free tensors, whose positive semi-definiteness is not easily checkable? A good candidate for such PNS-free tensors is the class of even order strongly symmetric circulant tensors.

Strongly symmetric tensors were introduced in [21]. An \( m \)th order \( n \) dimensional tensor \( \mathcal{A} = (a_{i_1 \cdots i_m}) \) is called a strongly symmetric tensor if

\[
a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}
\]

\(^3\) Chesi [6] used the abbreviation PNS for PSD non-SOS in 2007.
as long as \( \{i_1, \cdots, i_m\} = \{j_1, \cdots, j_m\} \). Note that a symmetric matrix is a strongly symmetric tensor of order 2. Hence, strongly symmetric tensors are also extensions of symmetric matrices.

Circulant tensors have applications in stochastic process and spectral hypergraph theory [5]. An \( m \)th order \( n \) dimensional tensor \( \mathcal{A} = (a_{i_1 \cdots i_m}) \) is called a circulant tensor if

\[
a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}
\]
as long as \( j_l \equiv i_l + 1 \mod n \) for \( l = 1, \cdots, m \).

In this paper, we consider even order three dimensional strongly symmetric circulant tensors. A general three dimensional strongly symmetric circulant tensor has only three independent entries: the diagonal entry \( d \), the off-diagonal entries of value \( u \) with two different indices in \( a_{i_1 \cdots i_m} \), and the off-diagonal entries of value \( c \) with three different indices in \( a_{i_1 \cdots i_m} \). Let \( \mathcal{A} = (a_{i_1 \cdots i_m}) \) be an \( m \)th order three dimensional strongly symmetric circulant tensor. Here \( m \) can be even or odd. We denote

\[
a_S \equiv a_{i_1 \cdots i_m}
\]
if \( S = \{i_1, \cdots, i_m\} \). Then there are seven cases for \( S: \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \) and \( \{1, 2, 3\} \). Since \( \mathcal{A} \) is also circulant, we have

\[
a_{\{1\}} = a_{\{2\}} = a_{\{3\}}, \quad \text{and} \quad a_{\{1, 2\}} = a_{\{2, 3\}} = a_{\{1, 3\}}.
\]

Let \( d = a_{\{1\}} = a_{\{2\}} = a_{\{3\}} \), \( u = a_{\{1, 2\}} = a_{\{2, 3\}} = a_{\{1, 3\}} \) and \( c = a_{\{1, 2, 3\}} \). Then we see that \( d \) is the diagonal entry of \( \mathcal{A} \): \( d = a_{1 \cdots 1} = a_{2 \cdots 2} = a_{3 \cdots 3} \), and an \( m \)th order three dimensional strongly symmetric circulant tensor has only three independent entries \( d, u \) and \( c \). Thus, we may denote a general three dimensional strongly symmetric circulant tensor \( \mathcal{A} = \mathcal{A}(m, d, u, c) \), where \( m \) is its order. When the context is clear, we only use \( \mathcal{A} \) to denote it.

In our discussion, we need the concept of H-eigenvalues of symmetric tensors, which was introduced in [18] and is closely related to positive semi-definiteness of even order symmetric tensors. Let \( \mathcal{T} = (t_{i_1 \cdots i_m}) \) be an \( m \)th order \( n \) dimensional real symmetric tensor and \( x \in \mathbb{R}^n \). Then \( \mathcal{T}x^{m-1} \) is a vector in \( \mathbb{R}^n \), with its \( i \)th component defined as

\[
(\mathcal{T}x^{m-1})_i = \sum_{i_2, \cdots, i_m=1}^n t_{i_2 \cdots i_m}x_{i_2} \cdots x_{i_m}.
\]
If there is \( x \in \mathbb{R}^n, x \neq 0 \) and \( \lambda \in \mathbb{R} \) such that for \( i = 1, \cdots, n \),

\[
(\mathcal{T}x^{m-1})_i = \lambda x_i^{m-1},
\]
then \( \lambda \) is called an H-eigenvalue of \( \mathcal{T} \) and \( x \) is called its associated H-eigenvector. When \( m \) is even, H-eigenvalues always exist. \( \mathcal{T} \) is PSD if and only if its smallest H-eigenvale
is nonnegative [18]. From now on, we denote the smallest H-eigenvalue of $A(m, d, u, c)$ as $\lambda_{\text{min}}(m, d, u, c)$.

Now, let $m = 2k$ be even. In Section 2, we show that there are two one-variable functions $M_c(u)$ and $N_c(u)$, such that $M_c(u) \geq N_c(u) \geq 0$, $A$ is SOS if and only if $d \geq M_c(u)$, and $A$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then three dimensional strongly symmetric PNS circulant tensors do not exist for such $u$ and $c$. We show that if $u, c \leq 0$ or $u = c > 0$, then $M_c(u) = N_c(u)$. Explicit formulae for $M_c(u) = N_c(u)$ are given there in these cases. Thus, it is PNS-free for such $u$ and $c$.

Note that $A$ is PSD or SOS if and only if $\alpha A$ is PSD or SOS, respectively. Thus, we only need to consider three cases that $c = 0$, $c = 1$ and $c = -1$.

In Section 3, we discuss the case that $c = 0$. In this case, for $u > 0$, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We show that $-N_0(1)$ is the smallest H-eigenvalue of $A(m, 0, 1, 0)$. Numerical tests show that $M_0(1) = N_0(1)$ for $m = 6, 8, 10, 12$ and 14.

In Section 4, we study the case that $c = -1$. We show that there is a $u_0 > 0$ such that if $u \leq u_0$, $N_{-1}(u)$ is linear and the explicit formula of $N_{-1}(u)$ can be given, and if $u > u_0$, $N_{-1}(u)$ is the smallest H-eigenvalue of a tensor with $u$ as a parameter. Numerical tests show that for $u > 0$, we still have $M_{-1}(u) = N_{-1}(u)$ for $m = 6, 8, 10$ and 12.

In Section 5, we study the case that $c = 1$. We show that there is a $v_0 < 0$ such that if $u \leq v_0$, $N_1(u)$ is linear and the explicit formula of $N_1(u)$ can be given, and if $u > v_0$, $N_{1}(u)$ is the smallest H-eigenvalue of a tensor with $u$ as a parameter. Numerical tests show that for $u \neq 1$, we still have $M_1(u) = N_1(u)$ for $m = 6, 8, 10$ and 12.

Some final remarks are made in Section 6.

2. Functions $M_c(u)$ and $N_c(u)$

In this section and the next three sections, we assume that $n = 3$ and $m = 2k$ is even. Let $A$ be an $m$th order three dimensional strongly symmetric circulant tensor. Then we may write $f(x) = Ax^m$ as

$$f(x) = d(x_1^m + x_2^m + x_3^m) + u \sum_{p=1}^{m-1} \binom{m}{p} \left( x_1^{m-p}x_2^p + x_1^{m-p}x_3^p + x_2^{m-p}x_3^p \right)$$

$$+ c \sum_{p=1}^{m-2} \sum_{q=1}^{m-p-1} \binom{m}{p} \binom{m-p}{q} x_1^{m-p-q}x_2^px_3^q. \tag{1}$$

We now establish two functions $M_c(u)$ and $N_c(u)$, in the following theorem. Recall that for an $m$th order $n$ dimensional tensor $A = (a_{i_1\ldots i_m})$, the sum of the absolute values of its $i$th off-diagonal entries, i.e.,

$$r_i = \sum_{i_2\ldots i_m=1}^{n} |a_{i_2i_3\ldots i_m}| - |a_{i_1i_2\ldots i_m}|.$$
If $a_{i...i} \geq r_i$ for $i = 1, \ldots, n$, then $\mathcal{A}$ is called diagonally dominated. It is shown in [18] that all the H-eigenvalues of a diagonally dominated tensor, if there is, are nonnegative. Furthermore, an even order symmetric diagonally dominated tensor is PSD [18] and SOS [2].

**Theorem 1.** Let $\mathcal{A}$ be an $m$th order three dimensional strongly symmetric circulant tensor. For given off-diagonal entries $u$ and $c$, we define

$$M_c(u) \equiv \inf\{d \in \mathbb{R} : \mathcal{A}(m, d, u, c) \text{ is SOS}\},$$

$$N_c(u) \equiv \inf\{d \in \mathbb{R} : \mathcal{A}(m, d, u, c) \text{ is PSD}\}.$$

Then, functions $M_c(u)$ and $N_c(u)$ are well-defined and convex. Furthermore, we have

$$0 \leq N_c(u) \leq M_c(u) \leq |u|(2^m - 2) + |c|(3^{m-1} - 2^m + 1). \quad (2)$$

**Proof.** Since $\mathcal{A}$ is a circulant tensor, then it has the same off-diagonal entry absolute value sum for different rows, i.e., $r_1 = r_2 = r_3$. By (1), this row sum is equal to the right hand side of (2). Thus, if $d$ is greater than or equal to this value, $\mathcal{A}$ is diagonally dominated and thus PSD and SOS. This shows that functions $M_c(u)$ and $N_c(u)$ are well-defined and the inequalities in (2) hold. As the set of PSD tensors and the set of SOS tensors are convex [14], $M_c(u)$ and $N_c(u)$ are convex. Since a necessary condition for an even order circulant tensor to be PSD is that its diagonal entry to be nonnegative [5], we have $N_c(u) \geq 0$ for all $u$ and $c$.

By definition, we have $N_c(u) \leq M_c(u)$. Clearly, if $M_c(u) = N_c(u)$, then $m$th order three dimensional PNS strongly symmetric circulant tensors do not exist for such $u$ and $c$. The theorem is proved. \qed

Theorem 1 means that $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then $m$th order three dimensional PNS strongly symmetric circulant tensors do not exist for such $u$ and $c$. If $M_c(u) = N_c(u)$, $u$ is called a PNS-free point for $c$.

For the convenience, we present formally three ingredients used in theoretical proofs of PNS-free points. If a point $u$ enjoys these ingredients, it is PNS-free.

**Definition 1.** Suppose that $n = 3$ and $m$ is even. Suppose that there is a number $M$ such that $f^*(\mathbf{x}) \equiv \mathcal{A}(m, M, u, c)\mathbf{x}^m$ is SOS for given $u$ and $c$, and a nonzero vector $\mathbf{x} \in \mathbb{R}^3$ such that $f^*(\mathbf{x}) = 0$. Then we call $M$ the critical value of $\mathcal{A}$ at $u$ and $c$, the SOS decomposition $f^*(\mathbf{x})$ the critical SOS decomposition of $\mathcal{A}$ at $u$ and $c$, and $\mathbf{x}$ the critical minimizer of $\mathcal{A}$ at $u$ and $c$.

**Theorem 2.** Let $u \in \mathbb{R}$. Then $u$ is PNS-free point for $c$ if $\mathcal{A}$ has a critical value $M$, a critical SOS decomposition $f^*(\mathbf{x})$ and a critical minimizer $\mathbf{x}$ at $u$ and $c$. 
Proof. Suppose that $A$ has a critical value $M$, a critical SOS decomposition $f^*(x)$ and a critical minimizer $\bar{x}$ at $u$. Then we have $M \geq M_c(u)$ by the definition of $M_c(u)$. If $d < M$, then
\[
  f(\bar{x}) = (d - M)(\bar{x}_1^m + \bar{x}_2^m + \bar{x}_3^m) + f^*(\bar{x}) < 0.
\]
This implies that $N_c(u) \geq M$ by the definition of $N_c(u)$. But $N_c(u) \leq M_c(u)$. Thus, $M_c(u) = N_c(u) = M$, i.e., $u$ is PNS-free point for $c$.  

Corollary 1. If $u, c \leq 0$, then
\[
  M_c(u) = N_c(u) = -u(2^m - 2) - c(3^{m-1} - 2^m + 1).
\]
Thus, it is PNS-free for such $u$ and $c$.

Proof. Suppose that $u, c \leq 0$. Let $M$ be the value of the right hand side of (2), and $\bar{x} = (1, 1, 1)^T$. If $d = M$, then $f(\bar{x}) = f^*(\bar{x})$ has an SOS decomposition as $A$ is an even order diagonally dominated symmetric tensor [2]. We also see that $f^*(\bar{x}) = 0$. The result follows.  

Corollary 2. If $u = c > 0$, then
\[
  M_c(u) = N_c(u) = u = c.
\]
Thus, it is PNS-free for such $u$ and $c$.

Proof. Suppose that $u = c > 0$. Let $M = u = c$, and $\bar{x} = (2, -1, -1)^T$. If $d = M$, then $f(x) = f^*(x) = (x_1 + x_2 + x_2)^m$ has an SOS decomposition. We also see that $f^*(\bar{x}) = 0$. The result follows.  

Corollary 3. If $u > 0$, then
\[
  M_0(u) = uM_0(1)
\]
and
\[
  N_0(u) = uN_0(1).
\]
Hence, for $c = 0$, it is PNS-free if and only if $M_0(1) = N_0(1)$.

Proof. Suppose that $u > 0$ and $d \geq uM_0(1)$. By (1), we have
\[
  f(x) = (d - uM_0(1))(x_1^m + x_2^m + x_3^m)
  + u \left(M_0(1)(x_1^m + x_2^m + x_3^m) + \sum_{p=1}^{m-1} \binom{m}{p} x_1^{m-p}x_2^p + x_1^m x_3^p + x_2^m x_3^{m-p} \right).
\]
We see that $f(x)$ is SOS. Hence, $M_0(u) = uM_0(1)$. Similarly, we may prove that $N_0(u) = uN_0(1)$. By these and Corollary 1, we have the last conclusion. □

As discussed in the Introduction, for the PNS-free problem, we only need to consider three cases: $c = 0, 1$ and $-1$.

3. $c = 0$

If $u \leq 0$, by Corollary 1, we have $M_0(u) = N_0(u) = -u(2^m - 2)$. If $u > 0$, by Corollary 3, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We only need to consider the case that $u = 1$.

**Proposition 1.** We have that $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$.

**Proof.** By [18], $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $\lambda_{\min}(m, d, 1, 0) \geq 0$. By the structure of circulant tensors, $\lambda_{\min}(m, d, 1, 0) = d + \lambda_{\min}(m, 0, 1, 0)$. Thus, $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $d \geq -\lambda_{\min}(m, 0, 1, 0)$. By the definition of $N_c(u)$, we have $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$. □

For $m = 6, 8, 10, 12$ and 14, we compute $M_0(1)$ and $N_0(1)$ by using softwares Matlab (YALMIP, GloptiPloy and SeDuMi) and Maple [8,15,16,24], respectively. We find for such $m$, $M_0(1) = N_0(1)$. The results are displayed in Table 1.

4. $c = -1$

If $u \leq 0$, then Corollary 1 indicates that $M_{-1}(u) = N_{-1}(u) = -u(2^m - 2) + (3^{m-1} - 2^m + 1)$. We now discuss the case that $u > 0$.

In this section and the next section, we denote that $\mathcal{B} = \mathcal{A}(m, 3^{m-1} - 2^m + 1, 0, -1)$ and $\mathcal{T} = \mathcal{A}(m, 2^m - 2, -1, 0)$. Then, $\mathcal{B}$ and $\mathcal{T}$ are obviously diagonally dominated. Hence, they are PSD and SOS [2]. And all of their H-eigenvalues are nonnegative.

**Theorem 3.** Let

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}),$$

Table 1

| $m$ | $M_0(1)$ | $N_0(1)$ |
|-----|----------|----------|
| 6   | 1.73734847173345 | 1.737348471777547 |
| 8   | 1.882980354978972 | 1.882980356780414 |
| 10  | 1.947977161918168 | 1.947977172341075 |
| 12  | 1.976878006619490 | 1.976878047128592 |
| 14  | 1.989722829997529 | 1.989723542124766 |
where \( \lambda_{\min}(\cdot) \) denotes the smallest H-eigenvalue. Then, \( \varphi(u) \leq 0 \). If \( \varphi(u) = 0 \), then we have

\[
N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2). \tag{4}
\]

If \( \varphi(u) < 0 \), then we have

\[
N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \tag{5}
\]

Furthermore, the set \( C = \{u : \varphi(u) = 0\} \) is a nonempty closed convex ray \((-\infty, u_0]\) for some \( u_0 \geq 0 \).

**Proof.** Let \( \bar{x} = (1, 1, 1)^T \). Then \( B \bar{x}^m = 0 \) and \( T \bar{x}^m = 0 \). Thus, \( (B - uT) \bar{x}^m = 0 \) for any \( u \). By [18], we see that

\[
\varphi(u) \equiv \lambda_{\min}(B - uT) \leq 0.
\]

If \( \varphi(u) = 0 \), we let \( d = 3^{m-1} - 2^m + 1 - u(2^m - 2) \). Then \( A(m, d, u, -1) = B - uT \). We have \( \lambda_{\min}(m, d, u, -1) = 0 \). This implies (4).

If \( \varphi(u) < 0 \), then, because

\[
f(x) = (d - (3^{m-1} - 2^m + 1) + u(2^m - 2))I \bar{x}^m + B \bar{x}^m - uT \bar{x}^m \geq 0,
\]

we have

\[
N_{-1}(u) = \inf \{d : \lambda_{\min}((d - (3^{m-1} - 2^m + 1) + u(2^m - 2))I + B - uT \geq 0\}
= (3^{m-1} - 2^m + 1) - u(2^m - 2) - \lambda_{\min}(B - uT)
= 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1).
\]

We have (5).

By Corollary 1, \( C \) is nonempty and \( u \in C \) as long as \( u \leq 0 \). By Theorem 1, \( N_{-1}(u) \) is a convex function. It follows, together with (4) and (5), that \( C \) is convex. Since \( \lambda_{\min} \) is a continuous function [18], \( C \) is closed. Since \( u \in C \) for any \( u \leq 0 \), \( C \) is a ray, with the form \((-\infty, u_0]\) for some \( u_0 \geq 0 \). \( \square \)

**Corollary 4.** Let \( u_0 \equiv \max \{\hat{u} : \varphi(\hat{u}) = 0\} \). Then \( u_0 \) is well-defined and \( u_0 \geq 0 \). Furthermore, for \( u \leq u_0 \), we have (4), and for \( u > u_0 \), we have (5).

**Proposition 2.** If \( M_{-1}(u_0) = N_{-1}(u_0) = 3^{m-1} - 2^m + 1 - u_0(2^m - 2) \), then for \( u \leq u_0 \), we have \( M_{-1}(u) = N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2) \).
Table 2

The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6$.

| $u$ | $M_{-1}(u)$ | $N_{-1}(u)$ |
|-----|-------------|-------------|
| 0.1 | 173.7999999999899 | 173.8 |
| 2   | 55.9999999995172  | 56         |
| $\frac{45}{16}$ | 5.62499991033116 | 5.625 |
| 5   | 9.42544641511067  | 9.4254465011842588 |
| 10  | 18.1121860822789  | 18.112186280892696 |
| 40  | 70.2326321651344   | 70.232638183914150 |
| 300 | 521.943237017699   | 521.94324013633004 |

Proof. By Theorem 1, $M_{-1}(u)$ is convex. By Corollary 1, $M_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2)$ for $u \leq 0$. Since $u_0 \geq 0$, the conclusion follows. \(\square\)

Proposition 3. Suppose $u_0 = \max\{\hat{u} : \varphi(\hat{u}) = 0\}$. Then, we have

$$0 \leq u_0 \leq \bar{u}_0(m) = \frac{3^{m-1} + 1}{2^m} - 1.$$  \(\text{(6)}\)

Proof. Since $B$ is PSD and has an H-eigenvalue 0, we have $\varphi(0) = 0$ and $u_0 \geq 0$.

On the other hand, we consider the case $u > \bar{u}_0$. Let $x_0 = (1, 1, -3)^\top$. We have

$$(B - \bar{u}_0 T)x_0^m = 0 \quad \text{and} \quad Tx_0^m = 2^m(3^m - 1).$$

Then,

$$(B - u T)x_0^m = (B - \bar{u}_0 T)x_0^m - (u - \bar{u}_0)Tx_0^m = -(u - \bar{u}_0)2^m(3^m - 1) < 0.$$ 

Hence, we have $\varphi(u) = \lambda_{\min}(B - u T) < 0$ when $u > \bar{u}_0$. Therefore, $u_0 \leq \bar{u}_0$. \(\square\)

For $m = 6, 8, 10, 12$ and $14$, we find that $B - \bar{u}_0 T$ is PSD. This shows that for such $m$, $\varphi(\bar{u}_0) = 0$, i.e.,

$$u_0 = \bar{u}_0(m) = \frac{3^{m-1} + 1}{2^m} - 1.$$  \(\text{(7)}\)

It remains a further research topic to show that $B - \bar{u}_0 T$ is PSD for all even $m$ with $m \geq 16$. If this is true, then (7) is true for all even $m$ with $m \geq 6$.

In Tables 2-5, the values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6, 8, 10, 12$ and $u = 0, 1, 2, \frac{45}{16}, 5, 10, 40, 300$, $u = 1, 3, \frac{483}{64}$, $10, 40, 140, 300$, $u = 1, 10, \frac{4665}{256}$, $20, 40, 140, 300$ and $u = 1, 20, \frac{43263}{1024}$, $60, 100, 140, 300$ are reported, respectively. We find for such $m$ and $u$, $M_{-1}(u) = N_{-1}(u)$.
Table 3
The values of $M^{-1}(u)$ and $N^{-1}(u)$ for $m = 8$.

| $u$ | $M^{-1}(u)$     | $N^{-1}(u)$     |
|-----|----------------|----------------|
| 1   | 1677.99999992219 | 1678           |
| 3   | 1169.9999999356  | 1170           |
| $483$ | 15.0937478786308  | 15.09375       |
| 10  | 19.7129359300341  | 19.7129361640501|
| 40  | 76.2023466001335  | 76.2023468071730|
| 140 | 264.500365037152  | 264.500382469583|
| 300 | 565.777184078832  | 565.777239551091|

Table 4
The values of $M^{-1}(u)$ and $N^{-1}(u)$ for $m = 10$.

| $u$ | $M^{-1}(u)$     | $N^{-1}(u)$     |
|-----|----------------|----------------|
| 1   | 17637.9999999549 | 17638          |
| 10  | 8439.99999783081  | 8440           |
| $4665$ | 36.445260348520   | 36.445125      |
| 20  | 39.907358817909   | 39.9075375522954|
| 40  | 78.8670625326286  | 78.8670809985488|
| 140 | 273.664775923815  | 273.664798232238|
| 300 | 585.341085323688  | 585.341145806726|

Table 5
The values of $M^{-1}(u)$ and $N^{-1}(u)$ for $m = 12$.

| $u$ | $M^{-1}(u)$     | $N^{-1}(u)$     |
|-----|----------------|----------------|
| 1   | 168957.9999979042 | 168958         |
| 20  | 91.171.9999999683  | 91.172         |
| $43261$ | 84.4971787852022  | 84.498046875   |
| 60  | 119.589505579120  | 119.589562756497|
| 100 | 198.664532858285  | 198.664684641639|
| 140 | 277.739708996851  | 277.739806526784|
| 300 | 594.040191670531  | 594.040294067366|

5. $c = 1$

Corollary 2 indicates that $M_1(1) = N_1(1) = 1$. Hence, we only need to consider the case that $u \neq 1$. Let $B$ and $T$ be the same as in the last section. We have the following theorem.

Theorem 4. Let

$$\psi(u) \equiv \lambda_{\min}(-uT - B).$$

Then, $\psi(u) \leq 0$. If $\psi(u) = 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2). \quad (8)$$

If $\psi(u) < 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$$

$$- \lambda_{\min}(m, -(3^{m-1} - 2^m + 1) - u(2^m - 2), u, 1). \quad (9)$$
Furthermore, if the set $C = \{ u : \psi(u) = 0 \}$ is nonempty, then it is a closed convex ray $(-\infty, v_0]$ for some $v_0 < 0$.

**Proof.** The proof of this theorem is similar to the proof of Theorem 3. However, we cannot apply Corollary 1 here. If $C$ is nonempty, we may show that $C$ is closed and convex as in the proof of Theorem 3.

If there is a $\hat{u} \leq 0$ such that $\lambda_{\min}(-\hat{u}T - B) = 0$, for $u \leq \hat{u} \leq 0$, we have

$$\psi(u) = \lambda_{\min}(-uT - B) = \lambda_{\min}(-\hat{u}T - B + (-u + \hat{u})T) \geq \lambda_{\min}(-\hat{u}T - B) = 0.$$ 

Hence, $\psi(u) = 0$ for all $u \leq \hat{u} \leq 0$. So if $C$ is not empty, it is a ray with the form $(-\infty, v_0]$ for some $v_0$. Clearly, $\psi(0) \equiv \lambda_{\min}(-B) < 0$ as $B$ is PSD and not a zero tensor. Hence, $v_0 < 0$.

The other parts of the proof are similar to the proof of Theorem 3. \qed

**Corollary 5.** If there is one point $\hat{u}$ such that $\psi(\hat{u}) = 0$, let $v_0 \equiv \max\{\hat{u} : \psi(\hat{u}) = 0\}$. Then for $u \leq v_0$, we have (8), and for $u \geq v_0$, we have (9).

We also have the following proposition.

**Proposition 4.** If $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$, then for $u \leq v_0$, we have $M_1(u) = N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$.

**Proof.** Suppose that $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$. By (3), if $u \leq v_0$ and $d = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$, we have

$$f^*(x) = -\bar{u}g_1(x) + g_2(x),$$

where

$$g_1(x) = A(m, 2^m - 2, -1, 0),$$

$$g_2(x) = A(m, -(3^{m-1} - 2^m + 1) - v_0(2^m - 2), v_0, 1)$$

and

$$\bar{u} \equiv u - v_0 \leq 0.$$ 

We see that $g_2(x)$ is equal to the critical SOS decomposition of $A$ at $c = 1$ and $u = v_0$, and $g_1(x)$ is equal to the critical SOS decomposition of $A$ at $c = 0$ and $u = -1$. Hence both
$g_1(x)$ and $g_2(x)$ are SOS polynomials. This implies that $f^*(x)$ is an SOS polynomial. Let $\bar{x} = (1, 1, 1)^T$, we see that $f^*(\bar{x}) = 0$. Then the conclusion follows from Theorem 2. \qed

**Proposition 5.** Suppose that $C$ is not empty. Let $v_0 = \max\{\hat{u} : \psi(\hat{u}) = 0\}$. Then, we have

$$v_0 \leq \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \quad (10)$$

**Proof.** Let $x_0 = (1, 1, -\frac{1}{2})^T$. We have

$$(-B - \bar{v}_0 T)x_0^m = 0 \quad \text{and} \quad Tx_0^m = 2^m + 1 - 2^{1-m}.$$  

Then,

$$(-B - u T)x_0^m = (-B - \bar{v}_0 T)x_0^m - (u - \bar{v}_0)T x_0^m = -(u - \bar{v}_0)(2^m + 1 - 2^{1-m}) < 0.$$  

Hence, we have $\psi(u) = \lambda_{\min}(-B - u T) < 0$ when $u > \bar{v}_0$. Therefore, $v_0 \leq \bar{v}_0$. \qed

By the similar discussion on $u_0$, we find that $-\bar{v}_0 T - B$ is PSD for $m = 6, 8, 10, 12$ and $14$. This shows that for such $m$, $\psi(\bar{v}_0) = 0$, i.e.,

$$v_0 = \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \quad (11)$$

This also shows that $C$ is not empty for such $m$. It remains a further research topic to show that $-\bar{v}_0 T - B$ is PSD for all even $m$ with $m \geq 16$. If this is true, then (11) is true for all even $m$ with $m \geq 6$.

In Tables 6–9, the values of $M_1(u)$ and $N_1(u)$ for $m = 6, 8, 10, 12$ and $u = -40, -10, -\frac{70}{11}, -5, -1, 10, 40, u = -40, -20, -\frac{686}{43}, -10, -1, 10, 40, u = -100, -40, -\frac{710}{19}, -20, -1, 10, 40$ and $u = -140, -100, -\frac{58266}{683}, -60, -1, 10, 40$ are reported, respectively. We find for such $m$ and $u$, $M_1(u) = N_1(u)$.
6. Final remarks

In Proposition 1, Theorems 3 and 4, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. For \( u, c \leq 0 \) and \( u = c > 0 \), we show that this condition is also sufficient for this tensor to be sum-of-squares. Numerical tests indicate that this is also true in the other cases.

How can \( B - \bar{u}_0 \mathcal{T} \) and \( -\bar{v}_0 \mathcal{T} - B \) be shown to be PSD for all even \( m \geq 6 \)? If these are true, then (7) and (11) are true for all even \( m \geq 6 \).

Finally, more efforts are needed to prove that this problem is PNS-free eventually.

Acknowledgement

We are thankful to the referee for his or her comments, which helped us to improve our paper.
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