Positivity in $T$-equivariant $K$-theory of flag varieties associated to Kac-Moody groups II

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We prove sign-alternation of the structure constants in the basis of structure sheaves of opposite Schubert varieties in the torus-equivariant Grothendieck group of coherent sheaves on the flag varieties $G/P$ associated to an arbitrary symmetrizable Kac-Moody group $G$, where $P$ is any parabolic subgroup. This generalizes the work of Anderson-Griffeth-Miller from the finite case to the general Kac-Moody case, and affirmatively answers a conjecture of Lam-Schilling-Shimozono regarding the signs of the structure constants in the case of the affine Grassmannian.

1 Introduction

Let $G$ be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots and $G^{\text{min}} \subset G$ be the minimal Kac-Moody group as in [Ku2 §7.4]. Let $B$ be the standard (positive) Borel subgroup, $B^-$ the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus, and $W$ the Weyl group [Ku2 Chapter 6]. Let $\bar{X} = G/B$ be the ‘thick’ flag variety (introduced by Kashiwara) which contains the standard flag variety $X = G^{\text{min}}/B$. Let $T$ be the adjoint torus, i.e., $T := H/Z(G^{\text{min}})$, where $Z(G^{\text{min}})$ denotes the center of $G^{\text{min}}$ and let $R(T)$ denote the representation ring of $T$. For any $w \in W$, we have the Schubert cell $C_w := BwB/B \subset \bar{X}$, the opposite Schubert cell $C^-w := B^-wB/B \subset \bar{X}$, the opposite Schubert variety $X^-w := \overline{C^-w} \subset \bar{X}$. When $G$ is a (finite dimensional) semisimple group, it is referred to as the finite case.

Let $K^0_{T}(\bar{X})$ denote the Grothendieck group of $T$-equivariant coherent sheaves on $\bar{X}$. Then, $\{[O_{X^w}]\}_{w \in W}$ forms an $R(T)$-‘basis’ of $K^0_{T}(\bar{X})$ (where infinite sums are allowed), i.e., $K^0_{T}(\bar{X}) = \prod_{w \in W} R(T)[O_{X^w}]$. We express the product in $K^0_{T}(\bar{X})$ by:

$$[O_{X^u}] \cdot [O_{X^v}] = \sum_{w \in W} d^w_{u,v} [O_{X^w}], \text{ for unique } d^w_{u,v} \in R(T).$$

The following result is our main theorem (cf. Theorem 5.3). This was conjectured first by Griffeth-Ram [GR] in the finite case (2004), proven in the finite case by Anderson-Griffeth-Miller [AGM] (2011), and then conjectured in the general Kac-Moody case by Kumar [Ku1] (2012).

**Theorem 1.1.** For any $u, v, w \in W$, 

$$(-1)^{\ell(w)+\ell(u)+\ell(v)}d^w_{u,v} \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \ldots, (e^{-\alpha_r} - 1)]$$

where $\{\alpha_1, \ldots, \alpha_r\}$ are the simple roots.
Let $P$ be any standard parabolic subgroup of $G$ of finite type (cf. [Ku2, Definition 6.1.18]). (Recall that a parabolic subgroup is said to be of finite type if its Levi subgroup is finite dimensional.) We may express, in $K^0_\mathcal{T}(G/P)$,

$$[\mathcal{O}_{X_P^w}] \cdot [\mathcal{O}_{X_P^w}] = \sum_{w \in W^P} d^w_{u,v}(P)[\mathcal{O}_{X_P^w}],$$

for unique $d^w_{u,v}(P) \in \mathbb{R}(T)$,

where $W^P$ is the set of minimal length representatives of $W/W^P$, $W^P$ is the Weyl group of $P$, and $X^w_{P} := \overline{B^{-w}P/P} \subset G/P$ is the opposite Schubert variety.

Let $\pi : G/B \to G/P$ be the standard ($T$-equivariant) projection. Then, $\pi$ is a locally trivial fibration (with fiber the smooth projective variety $P/B$) and hence flat (cf. [Ku2, Chapter 7]). Thus, we have

$$\pi^*[\mathcal{O}_{X_P^w}] = [\mathcal{O}_{\pi^{-1}(X^w_P)}] = [\mathcal{O}_{X^w}] .$$

Since $\pi^* : K^0_\mathcal{T}(G/P) \to K^0_\mathcal{T}(G/B)$ is a ring homomorphism, we have $d^w_{u,v} = d^u_{v,v}(P)$ for any $u, v, w \in W^P$ and hence Theorem 1.3 immediately generalizes from the case of $G/B$ to the case of $G/P$, and we obtain:

**Theorem 1.2.** For any standard parabolic subgroup $P$ of $G$ of finite type, and any $u, v, w \in W^P$,

$$(-1)^{\ell(u) + \ell(v) + \ell(w)} d^w_{u,v}(P) \in \mathbb{Z}_{\geq 0}[(e^{\alpha_1} - 1), \ldots, (e^{\alpha_r} - 1)].$$

Theorems 1.1 and 1.2 also apply to ordinary (non-equivariant) $K$-theory. Let $K^0(\tilde{X})$ denote the Grothendieck group of coherent sheaves on $\tilde{X}$. Then, we have $K^0(\tilde{X}) = \prod_{w \in W} \mathbb{Z}[\mathcal{O}_{\tilde{X}^w}]$. Further, the map

$$Z \otimes_{R(T)} K^0_\mathcal{T}(\tilde{X}) \to K^0(\tilde{X}), \quad 1 \otimes [\mathcal{O}_{\tilde{X}^w}] \mapsto [\mathcal{O}_{\tilde{X}^w}]$$

is an isomorphism, where we view $Z$ as an $R(T)$-module via evaluation at 1. Similar results apply to $G/P$.

Write, in $K^0_\mathcal{T}(G/P)$, for $u, v \in W^P$,

$$[\mathcal{O}_{X_P^w}] \cdot [\mathcal{O}_{X_P^w}] = \sum_{w \in W^P} a^w_{u,v}(P)[\mathcal{O}_{X_P^w}],$$

for unique $a^w_{u,v}(P) \in \mathbb{Z}$.

Then, by the above, along with Theorem 1.2 we have:

**Theorem 1.3.** For any standard parabolic subgroup $P$ of $G$ of finite type, and any $u, v, w \in W^P$,

$$(-1)^{\ell(w) + \ell(u) + \ell(v)} a^w_{u,v}(P) \in \mathbb{Z}_{\geq 0}.$$

The following conjecture of Lam-Schilling-Shimozono [LSS, Conjectures 7.20 (ii) and 7.21 (iii)] is a special case of Theorem 1.3.

Let $G = \hat{SL}_N$ be the affine Kac-Moody group associated to $SL_N$, and let $P$ be its standard maximal parabolic subgroup. Let $\tilde{X} = G/P$ be the corresponding infinite Grassmannian. Then, $K^0(\tilde{X})$ has the structure sheaf ‘basis’ $\{ [\mathcal{O}_{X^w}] \}_{u \in W^P}$ over $\mathbb{Z}$, where $W$ is the (affine) Weyl group of $G$ and $W^P$ is the set of minimal coset representatives in $W/W_o$ ($W_o$ being the finite Weyl group of $SL_N$). Write, for any $u, v \in W^P$,

$$[\mathcal{O}_{X^w}] \cdot [\mathcal{O}_{X^v}] = \sum_{w \in W^P} b^w_{u,v}[\mathcal{O}_{X^w}],$$

for unique integers $b^w_{u,v}$.

Then, the following is a special case of Theorem 1.3.
Corollary 1.4 (Conjectured by Lam-Schilling-Shimozono).
\[ (-1)^{\ell(u)+\ell(v)+\ell(w)}b_{u,v}^w \in \mathbb{Z}_{\geq 0}. \]

The proof of Theorem 1.1 follows closely the work of Anderson-Griffeth-Miller [AGM] and Kumar [Ku1]. Though several technical details had to be carefully addressed, e.g., Proposition 7.7 proving the rational singularities of \( \mathcal{Z} \). Letting \( K^0_T(X) \) denote the Grothendieck group of finitely supported \( T \)-equivariant coherent sheaves on \( X \), there is a natural pairing
\[ \langle \cdot, \cdot \rangle : K^0_T(\bar{X}) \otimes K^0_T(X) \to R(T), \]
coming from the \( T \)-equivariant Euler-Poincaré characteristic (cf. (1)). Under this pairing, the bases \( \{ [\mathcal{O}_X] \} \) of \( K^0_T(\bar{X}) \) and \( \{ [\xi_w] \} \) of \( K^0_T(X) \) are dual (cf. Proposition 3.10), where \( \xi_w := \mathcal{O}_{X_w}(-\partial X_w) \) and \( \partial X_w := X_w \setminus C_w \). (By Lemma 3.5, \( \xi_w \) is indeed a basis of \( K^0_T(X) \).)

Then, realizing the product structure constants in \( K^0_T(\bar{X}) \) as the coproduct structure constants of the dual basis in \( K^0_T(X) \) (cf. Lemma 3.2) allows the use of the above pairing and duality to express the structure constants in terms of certain cohomology groups. Following [AGM], we introduce the ‘mixing space’ \( \bar{X}_\pi \), which is a bundle over a product of projective spaces \( \bar{F} \) with fiber \( X \). This allows for the reduction from \( T \)-equivariant \( K \)-theory to non-equivariant \( K \)-theory. Then, we introduce the ‘mixing group’ \( \Gamma \) (cf. Definition 4.11) whose action is sufficient to ensure a certain transversality needed to prove our main result.

Theorem 5.1 is our main technical result. Part a) allows the structure constants to be realized as the Euler characteristic of a certain sheaf, while part b) shows that this sheaf has cohomology which is zero in all but a single term of known degree, immediately resulting in a determination of the sign of the Euler characteristic and hence the structure constants.

Part a) of Theorem 5.1 is proved in Section 6. The proof relies on some local \( \mathcal{F}or \) vanishing results which were proven in [Ku1], as well as a reduction to finite dimensional schemes, where a transversality result due to Sierra is crucially used (cf. Theorem 6.1).

The proof of the more difficult part b) of Theorem 5.1 is the content of Sections 7 through 9. In Section 7 we introduce the crucial scheme \( Z \) which comes with a projection to the mixing group \( \Gamma \). We also introduce a desingularization \( \bar{Z} \) of \( Z \) and prove that \( Z \) is irreducible, normal and has rational singularities (cf. Propositions 7.5 and 7.7). We further introduce the divisor \( \partial Z \subset Z \) which is shown to be Cohen-Macaulay (cf. Proposition 7.8). It is on the fibers of the projection \( \pi : Z \to \Gamma \) that the computation of the cohomology of the previously mentioned sheaf occurs.

In Section 8 the rational singularities of \( Z \) are used to apply a relative version of the Kawamata-Viehweg vanishing theorem (cf. Theorem 5.1) to show that \( R^i\pi_*\omega_Z(\partial Z) = 0 \) for all \( i > 0 \) (cf. Corollary 8.8). Finally, in Section 9, this vanishing of the higher direct images along with the semicontinuity theorem is used to prove vanishing of the relevant cohomology along the fibers of \( \pi : Z \to \Gamma \) (cf. Theorem 9.2) and thus conclude the proof of part b) of Theorem 5.1.

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2 Notation

We work over the field \( \mathbb{C} \) of complex numbers. By a variety we mean an algebraic variety over \( \mathbb{C} \) which is reduced, but not necessarily irreducible.
Let $G$ be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots (as opposed to completed along the positive roots as in [Ku2, Chapter 6]) and $G^{\min} \subset G$ be the minimal Kac-Moody group as in [Ku2, §7.4]. Let $B$ be the standard (positive) Borel subgroup, $B^-$ the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus, and $W$ the Weyl group [Ku2, Chapter 6]. Let
\[ \bar{X} = G/B \]
be the ‘thick’ flag variety which contains the standard flag variety $X = G^{\min}/B$.

If $G$ is not of finite type, $\bar{X}$ is an infinite dimensional non quasi-compact scheme [K, §4] and $X$ is an ind-projective variety [Ku2, §7.1]. The group $G^{\min}$ and, in particular, the maximal torus $H$ acts on $\bar{X}$ and $X$.

Let $T$ be the adjoint torus, i.e., $T := H/Z(G^{\min})$, where $Z(G^{\min})$ denotes the center of $G^{\min}$. (Recall that by [Ku2 Lemma 6.2.9(c)], $Z(G^{\min}) = \{ h \in H : e^{\alpha_i}(h) = 1 \text{ for all the simple roots } \alpha_i \}$.)

Then, the action of $H$ on $\bar{X}$ (and on $X$) descends to an action of $T$.

For any $w \in W$ we have the Schubert cell
\[ C_w := BwB/B \subset X, \]
the Schubert variety
\[ X_w := \overline{C_w} = \bigsqcup_{w' \leq w} C_{w'} \subset X, \]
the opposite Schubert cell
\[ C^w := B^-wB/B \subset \bar{X}, \]
and the opposite Schubert variety
\[ X^w := \overline{C^w} = \bigsqcup_{w \geq w'} C_{w'} \subset \bar{X}, \]
all endowed with the reduced subscheme structures. Then, $X_w$ is a (finite dimensional) irreducible projective subvariety of $X$ and $X^w$ is a finite codimensional irreducible subscheme of $\bar{X}$ [Ku2, §7.1] and [K, §4]. We denote by $Z_w$ the Bott-Samelson-Demazure-Hansen (BSDH) variety as in [Ku2, §7.1.3], which is a $B$-equivariant desingularization of $X_w$ [Ku2 Proposition 7.1.15]. Further, $X_w$ is normal, Cohen-Macaulay (CM for short) and has rational singularities [Ku2, Theorem 8.2.2].

We also define the boundary of the Schubert variety by
\[ \partial X_w := X_w \setminus C_w \]
with the reduced subscheme structure. Then, $\partial X_w$ is pure of codimension 1 in $X_w$ and is CM (See the proof of Proposition 7.8).

For any $u \leq w$, we have the Richardson variety
\[ X_u^w := X^u \cap X_w \subset X \]
endowed with the reduced subvariety structure. By [Ku1, Proposition 6.6], $X_u^w$ is irreducible, normal and CM. We denote by $Z_u^w$ the $T$-equivariant desingularization of $X_u^w$ as in [Ku1, Theorem 6.8]. By [KuS Theorem 3.1], $X_u^w$ has rational singularities (in fact it has Kawamata log terminal singularities).
We denote the representation ring of $T$ by $R(T)$. Let $\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha'_1, \ldots, \alpha'_r\} \subset \mathfrak{h}$ the set of simple coroots, and $\{s_1, \ldots, s_r\}$ the corresponding set of simple reflections, where $\mathfrak{h} := \text{Lie}(H)$. Let $\rho \in \mathfrak{h}^*$ be any integral weight satisfying 
\[
\rho(\alpha'_i) = 1, \quad \text{for all } 1 \leq i \leq r.
\]

When $G$ is a finite dimensional semisimple group, $\rho$ is unique, but for a general Kac-Moody group $G$, it may not be unique.

For any integral weight $\lambda$ let $C_\lambda$ denote the one-dimensional representation of $H$ on $\mathbb{C}$ given by $h \cdot v = \lambda(h)v$ for $h \in H, v \in \mathbb{C}$. By extending this action to $B$ we may define, for any integral weight $\lambda$, the $G$-equivariant line bundle $\mathcal{L}(\lambda)$ on $\bar{X}$ by
\[
\mathcal{L}(\lambda) := G \times^B \mathbb{C}_{-\lambda},
\]
where for any representation $V$ of $B$, $G \times^B V := (G \times V)/B$ where $B$ acts on $G \times V$ via $(g, v) \cdot b = (gb, b^{-1}v)$ for $g \in G, v \in V, b \in B$. Then, $G \times^B V$ is the total space of a $G$-equivariant vector bundle over $X$, with projection given by $(g, v)B \mapsto gB$. We also define the bundle 
\[
e^\lambda := \bar{X} \times C_\lambda,
\]
which while trivial when viewed as a non-equivariant line bundle, is equivariantly non-trivial with the diagonal action of $H$.

## 3 The Grothendieck group

For a quasi-compact scheme $Y$, an $\mathcal{O}_Y$-module is called coherent if it is finitely presented as an $\mathcal{O}_Y$-module and any $\mathcal{O}_Y$-submodule of finite type admits a finite presentation.

A subset $S \subset W$ is called an ideal if for all $x \in S$ and $y \leq x$ we have $y \in S$. We say that an $\mathcal{O}_S$-module $S$ is coherent if $S|_{V^S}$ is a coherent $\mathcal{O}_{V^S}$-module for every finite ideal $S \subset W$, where $V^S$ is the quasi-compact open subset of $X$ defined by
\[
V^S := \bigcup_{w \in S} wU^-B/B, \quad \text{where } U^- := [B^{-1}, B^{-1}].
\]

Let $K_0^T(\bar{X})$ denote the Grothendieck group of $T$-equivariant coherent $\mathcal{O}_{\bar{X}}$-modules. Since the coherence condition on $S$ is imposed only for $S|_{V^S}$ for finite ideals $S \subset W$, $K_0^T(\bar{X})$ can be thought of as the inverse limit of $K_0^T(V^S)$ as $S$ varies over all finite ideals of $W$ (cf. \cite{KS} §2).

We define
\[
K_0^T(X) := \text{Limit}_{n \to \infty} K_0^T(X_n),
\]
where $\{X_n\}_{n \geq 1}$ is the filtration of $X$ giving the ind-projective variety structure and $K_0^T(X_n)$ is the Grothendieck group of $T$-equivariant coherent sheaves on $X_n$.

For any $u \in W$, $\mathcal{O}_{X^u}$ is a coherent $\mathcal{O}_{\bar{X}}$-module by \cite{KS} §2. From \cite{KS} comment after Remark 2.4 we have:

**Lemma 3.1.** $\{[\mathcal{O}_{X^u}]\}$ forms a basis of $K_0^T(\bar{X})$ as an $R(T)$-module (where we allow arbitrary infinite sums).

**Lemma 3.2.** $\{[\mathcal{O}_{X^u}]\}$ forms a basis of $K_0^T(X)$ as an $R(T)$-module.
Proof. This follows from [CG, §5.2.14 and Theorem 5.4.17].

The following lemma is due to Kashiwara-Shimozono [KS] Lemma 8.1.

Lemma 3.3. Any $T$-equivariant coherent sheaf $S$ on $V^u$ admits a free resolution in $\text{Coh}_T(O_{V^u})$:

$$0 \to S_n \otimes O_{V^u} \to \cdots \to S_1 \otimes O_{V^u} \to S_0 \otimes O_{V^u} \to S \to 0,$$

where $S_k$ are finite dimensional $T$-modules, $V^u := uU - B/B \subset \bar{X}$, and $\text{Coh}_T(O_{V^u})$ denotes the abelian category of $T$-equivariant coherent $O_{V^u}$-modules.

We define a pairing

$$\langle \ , \ \rangle : K_T^0(\bar{X}) \otimes K_T^0(X) \to R(T),$$

$$\langle [S], [F] \rangle = \sum_i (-1)^i \chi_T(X_n, \mathcal{T}or^O_{X}i(S, F)),$$

where $S$ is a $T$-equivariant coherent sheaf on $\bar{X}$ and $F$ is a $T$-equivariant coherent sheaf on $X$ supported on $X_n$ for some $n$, where $\chi_T$ represents the $T$-equivariant Euler-Poincaré characteristic. By [Ku1, Lemma 3.5] this is well defined.

Definition 3.4. We define $\xi_w$ to be the ideal sheaf of $\partial X_w$ in $X_w$, where $\partial X_w$ is given the reduced subscheme structure:

$$\xi_w := O_{X_w}(-\partial X_w).$$

Lemma 3.5. $\{[\xi_w]\}$ forms a basis of $K^0_T(X)$ as an $R(T)$-module.

Proof. This follows since $[\xi_w] = [O_{X_w}] + \sum_{w' < w} r_{w'}[O_{X_w'}]$ for some $r_{w'} \in R(T)$ and the fact that $[O_{X_w}]$ is a basis of $K^0_T(X)$ (cf. Lemma 3.2).

Proposition 3.6. $\omega_{X_w} = e^{-\rho} \mathcal{L}(-\rho)\xi_w$, where $\omega_{X_w}$ is the dualizing sheaf of $X_w$.

Proof. This follows from [GK] Proposition 2.2 since the same proof works for general Kac-Moody groups.

From [Ku1, Lemma 5.5] we have the following result:

Lemma 3.7. For any $u, w \in W$, we have

$$\mathcal{T}or^O_i(O_{X^u}, O_{X_w}) = 0, \forall i > 0.$$

We now prove:

Lemma 3.8. For any $u \in W$ and any finite union of Schubert varieties $Y = \bigcup_{i=1}^j X_w$, we have

$$\mathcal{T}or^O_i(O_{X^u}, O_Y) = 0, \forall i > 0.$$

In particular, for any $u, w \in W$ we have

$$\mathcal{T}or^O_i(O_{X^u}, O_{\partial X_w}) = 0, \forall i > 0.$$
Proof. We proceed by double induction on the dimension of $Y$ (i.e. the largest dimension of the irreducible components of $Y$) and the number of irreducible components of $Y$. If $\dim Y = 0$, then $Y = X_e$ and so the result follows from Lemma 3.7. Now, suppose that $\dim Y = d$ and $Y$ has $k$ irreducible components. If $k = 1$ then the result follows from Lemma 3.7, so we may assume that $k \geq 2$. Let $Y_1 = X_{w_1}$ be an irreducible component of $Y$ and let $Y_2$ be the union of all the other irreducible components.

By [KuS, Proposition 5.3 and its proof] $X$ is Frobenius split compatibly splitting its Schubert varieties $X_u$ and also Richardson varieties $X^w_u$ in $X$.

Thus, any finite intersection $X_{u_1} \cap \ldots \cap X_{u_n} \cap X^{v_1} \cap \ldots \cap X^{v_m}$ is reduced for any $n \geq 1$. (\*)

(In the proof here we have only used $X_{u_1} \cap \ldots \cap X_{u_n}$ to be reduced, but the more general assertion here will be used later in the paper.)

The short exact sequence of sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0$$

yields the long exact sequence

$$\ldots \to \mathcal{F} or^{O_X}_{i+1}(O_{X^w}, \mathcal{O}_{Y_1 \cap Y_2}) \to \mathcal{F} or^O_{i}(O_{X^w}, \mathcal{O}_Y) \to \mathcal{F} or^{O_X}_{i}(O_{X^w}, \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2}) \to \ldots$$

Now, since $Y_2$ has less than $k$ irreducible components, induction on the number of irreducible components gives

$$\mathcal{F} or^O_{i}(O_{X^w}, \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2}) = 0, \ \forall i > 0.$$ 

Since $Y_1 \cap Y_2$ is reduced and of dimension less than $d$, induction on the dimension gives

$$\mathcal{F} or^O_{i}(O_{X^w}, \mathcal{O}_{Y_1 \cap Y_2}) = 0, \ \forall i > 0.$$ 

Together, these imply the lemma. 

Lemma 3.9. For any $u, w \in W$, we have

$$\mathcal{F} or^O_{i}(O_{X^w}, \xi_w) = 0, \ \forall i > 0.$$ 

Proof. Applying Lemmas 3.7 and 3.8 the desired result follows from the long exact sequence for $\mathcal{F} or$. 

Proposition 3.10. For any $u, w \in W$ we have

$$\langle [O_{X^w}], [\xi_w] \rangle = \delta_{u,w}.$$ 

Proof. By definition

$$\langle [O_{X^w}], [\xi_w] \rangle = \sum_i (-1)^i \chi_T(X_n, \mathcal{F} or^O_{i}(O_{X^w}, \xi_w))$$

where $n$ is taken such that $n \geq \ell(w)$. Thus, by Lemma 3.9 we have

$$\langle [O_{X^w}], [\xi_w] \rangle = \chi_T(X_n, O_{X^w} \otimes_{O_X} \xi_w).$$
By Lemma 3.8 we have the sheaf exact sequence
\[ 0 \to \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \xi_w \to \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_w} \to \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \mathcal{O}_{\partial X_w} \to 0. \]
Observe that by (\(*\))
\[ \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_w} = \mathcal{O}_{X^n \cap X_w}, \]
and similarly for \( \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \mathcal{O}_{\partial X_w}. \) Thus,
\[ \chi_T(X_n, \mathcal{O}_{X^n} \otimes_{\mathcal{O}_X} \xi_w) = \chi_T(X_n, \mathcal{O}_{X^n \cap X_w}) - \chi_T(X_n, \mathcal{O}_{X^n \cap \partial X_w}). \]

When non-empty, \( X^u \cap X_w \) is irreducible by [KuI Proposition 6.6], and thus \( X^u \cap \partial X_w = \bigcup_{u \leq w' < w} X^u \cap X_{w'} \) is a connected projective variety when non-empty, since \( u \in X^u \cap X_{w'} \) for all \( u \leq w' < w. \)

If \( u \not< w \) we have \( X^u \cap X_w = \emptyset, \) so assume \( u \leq w. \) In this case \( X^u \cap X_w \neq \emptyset. \) Now, if \( u = w \) then \( X^u \cap \partial X_w = \emptyset, \) while if \( u < w \) then \( X^u \cap \partial X_w \neq \emptyset. \) By [KuS Corollary 3.2],
\[ H^i(X_n, \mathcal{O}_{X^n \cap X_w}) = 0, \ \forall i > 0. \]
Using an inductive argument similar to Lemma 3.8 we obtain
\[ H^i(X_n, \mathcal{O}_{X^n \cap Y}) = 0, \ \forall i > 0, \]
where \( Y \) is any finite union of Schubert varieties. Taking \( Y = X^u \cap \partial X_w \) and combining the above implies the proposition (here we use that, when non-empty, \( X^u \cap Y \) is connected). \( \square \)

4 The mixing space and mixing group

In this section we realize the product structure constants of \( \{[\mathcal{O}_{X^n}]\} \) in \( K_0^T(\bar{X}) \) as the coproduct structure constants of the dual basis \( \{[\xi_u]\} \) in \( K_0^T(X) \) (Lemma 4.2). We then introduce the mixing space \( X_\mu \), which is a bundle over a product of projective spaces \( \mathbb{P}, \) with fiber \( X. \) This allows for the reduction from \( T\)-equivariant \( K\)-theory to non-equivariant \( K\)-theory. Using the pairing and duality introduced in the previous section, we realize the structure constants in terms of certain cohomology groups (cf. Lemma 4.10). Finally, we introduce the mixing group \( \Gamma \) whose action is sufficient to allow for a transversality result necessary to prove part a) of our main technical result (Theorem 5.1).

**Definition 4.1.** (Structure constants \( d_{u,v}^w \)) By Lemma 3.1 in \( K_0^T(\bar{X}) \) we have:
\[ [\mathcal{O}_{X^n}] \cdot [\mathcal{O}_{X^n}] = \sum_{w \in W^H} d_{u,v}^w [\mathcal{O}_{X^n}], \ \text{for unique } d_{u,v}^w \in R(T), \] (2)
where infinitely many of \( d_{u,v}^w \) may be nonzero.

**Lemma 4.2.** Write, in \( K_0^T(X \times X) \) under the diagonal action of \( T \) on \( X \times X, \)
\[ \Delta^* [\xi_u] = \sum_{u,v} e_{u,v}^w [\xi_u \boxtimes \xi_v], \ \text{for } e_{u,v}^w \in R(T), \] (3)
where \( \Delta : X \to X \times X \) is the diagonal map. Then, \( e_{u,v}^w = d_{u,v}^w. \)
Proof. Let \( \bar{\Delta} : \bar{X} \to \bar{X} \times \bar{X} \) be the diagonal map, and note that \( \bar{\Delta}^*[\mathcal{O}_{X^*} \otimes \mathcal{O}_{X^*}] = [\mathcal{O}_{X^*}] \cdot [\mathcal{O}_{X^*}] \).

Hence, we have:

\[
\begin{align*}
    d^w_{u,v} &= \langle \sum_{w' \in W} d_{u,v}^{w'}[\mathcal{O}_{X^*}, [\xi_{w'}]], \text{ by Proposition 3.10} \\
    &= \langle [\mathcal{O}_{X^*}] \cdot [\mathcal{O}_{X^*}], [\xi_{w}] \rangle, \text{ by 2} \\
    &= \langle \bar{\Delta}^*[\mathcal{O}_{X^*} \otimes \mathcal{O}_{X^*}], [\xi_{w}] \rangle \\
    &= \langle [\mathcal{O}_{X^*} \otimes \mathcal{O}_{X^*}], \Delta_*[\xi_{w}] \rangle \\
    &= \langle [\mathcal{O}_{X^*} \otimes \mathcal{O}_{X^*}], \sum_{u',v'} e^{w}_{u',v'}[\xi_{u'} \otimes \xi_{v'}] \rangle, \text{ by 3} \\
    &= e^w_{u,v}, \text{ by Proposition 3.10}.
\end{align*}
\]

We now prepare to define the mixing space. Let \( \mathbb{P} := (\mathbb{P}^N)^r \) where \( r = \dim T \) and \( N \) is some large fixed integer. Let \( [N] = \{0, 1, \ldots, N\} \) and let \( j = (j_1, j_2, \ldots, j_r) \in [N]^r \). We define

\[
    \mathbb{P}^j := \mathbb{P}^{N-j_1} \times \cdots \times \mathbb{P}^{N-j_r}
\]

and similarly define

\[
    \mathbb{P}_j := \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_r}.
\]

We also define the boundary of \( \mathbb{P}_j \) by

\[
    \partial \mathbb{P}_j := (\mathbb{P}^{j_1-1} \times \mathbb{P}^{j_2} \times \cdots \times \mathbb{P}^{j_r}) \cup \cdots \cup (\mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_r-1} \times \mathbb{P}^{j_r-1}),
\]

where we interpret \( \mathbb{P}^{-1} := \emptyset \) to be the empty set. Throughout this paper we will identify \( T \simeq (\mathbb{C}^*)^r \) via \( t \mapsto (e^{-\alpha_1}(t), \ldots, e^{-\alpha_r}(t)) \).

**Definition 4.3.** (Mixing space \( X_{\mathbb{P}} \)) Let \( E(T)_{\mathbb{P}} := (\mathbb{C}^{N+1} \setminus \{0\})^r \) be the total space of the standard principal \( T \)-bundle \( E(T)_{\mathbb{P}} \to \mathbb{P} \). We can view \( E(T)_{\mathbb{P}} \to \mathbb{P} \) as a finite dimensional approximation of the classifying bundle for \( T \). Define \( X_{\mathbb{P}} := E(T)_{\mathbb{P}} \times_T X \) and let \( p : X_{\mathbb{P}} \to \mathbb{P} \) be the Zariski-locally trivial fibration with fiber \( X = G^{\text{min}}/B \).

For any \( T \)-scheme \( V \) we define \( V_{\mathbb{P}} := E(T)_{\mathbb{P}} \times_T V \).

For the rest of this paper we use the notation \( Y := X \times X \) and \( \bar{Y} := \bar{X} \times \bar{X} \). Further, we let \( Y_{\mathbb{P}} := E(T)_{\mathbb{P}} \times_T Y \simeq X_{\mathbb{P}} \times X_{\mathbb{P}} \), where \( T \) acts diagonally on \( Y \).

Note that for any \( u, v \in W \), \( (X_u)_{\mathbb{P}} \) and \( (X_u \times X_v)_{\mathbb{P}} \) are CM, as they are fiber bundles over \( \mathbb{P} \), and hence locally a product of CM schemes. Thus, they have dualizing sheaves.
Definition 4.4. We define the sheaf \((\xi_u)_\mathbb{P}\) on \(X_\mathbb{P}\) by
\[
(\xi_u)_\mathbb{P} := (e^\rho L(\rho))_\mathbb{P} \otimes \omega_{(X_u)_\mathbb{P}},
\]
where \((e^\rho L(\rho))_\mathbb{P}\) is defined by
\[
E(T)_\mathbb{P} \times T e^\rho L(\rho) \to X_\mathbb{P}.
\]

Proposition 4.5. \(K_0(X_\mathbb{P}) := \text{Limit}_{n \to \infty} K_0((X_n)_\mathbb{P})\) is a free module over the ring \(K_0(\mathbb{P}) = K_0(\mathbb{P})\) with basis \(\{[(\xi_u)_\mathbb{P}]\}\).

Thus, \(K_0(X_\mathbb{P})\) has \(\mathbb{Z}\)-basis
\[
\{p^*([O_{\mathbb{P}}]) \cdot [((\xi_u)_\mathbb{P})]\}_{j \in [N]^r, w \in W},
\]
where, as above, \(p : X_\mathbb{P} \to \mathbb{P}\) is the projection.

Proof. This follows from \([CG, \text{§5.2.14 and Theorem 5.4.17}]\) as well as the fact that the transition matrix between the basis \(\{[O_{X_w}]\}\) and \(\{[\xi_w]\}\) is upper triangular with invertible diagonal entries. \(\square\)

Definition 4.6. We define the sheaf \(\tilde{\xi}_u \boxtimes \tilde{\xi}_v\) on \(Y_\mathbb{P}\) by
\[
\tilde{\xi}_u \boxtimes \tilde{\xi}_v := (e^{2\rho} L(\rho) \boxtimes L(\rho))_\mathbb{P} \otimes \omega_{(X_u \times X_v)_\mathbb{P}},
\]
where \((e^{2\rho} L(\rho) \boxtimes L(\rho))_\mathbb{P}\) is defined by
\[
E(T)_\mathbb{P} \times T (e^{2\rho} L(\rho) \boxtimes L(\rho)) \to Y_\mathbb{P}.
\]

The diagonal map \(\Delta : X \to Y\) gives rise to the embedding \(\tilde{\Delta} : X_\mathbb{P} \to Y_\mathbb{P}\). Thus, by the previous proposition, we may write
\[
\tilde{\Delta}^*[(\xi_u)_\mathbb{P}] = \sum_{u,v \in W, j \in [N]^r} c_{u,v}^w(j) \hat{p}^*[O_{\mathbb{P}}] : [\tilde{\xi}_u \boxtimes \tilde{\xi}_v] \in K_0(Y_\mathbb{P}), \text{ for unique } c_{u,v}^w(j) \in \mathbb{Z}, \tag{4}
\]
where \(\hat{p} : Y_\mathbb{P} \to \mathbb{P}\) is the projection.

The following lemma makes precise the reduction from \(T\)-equivariant \(K\)-theory of \(\tilde{X}\) to the ordinary \(K\)-theory of the mixing space.

Lemma 4.7. For any \(u, v, w \in W\) we can choose \(N\) large enough and express
\[
d_{u,v}^w = \sum_{j \in [N]^r} d_{u,v}^w(j)(e^{-\alpha_1} - 1)^{j_1} \cdots (e^{-\alpha_r} - 1)^{j_r},
\]
where \(d_{u,v}^w(j) \in \mathbb{Z}\).

Then,
\[
d_{u,v}^w(j) = (-1)^{|j|} c_{u,v}^w(j), \text{ where } |j| := \sum_i j_i.
\]

Proof. This follows from Lemma 4.2 and \([CG, \text{Lemma 6.2}]\) (see also \([AGM, \text{§3}]\)). \(\square\)

Lemma 4.8. For any coherent sheaf \(S\) on \(\mathbb{P}\) and any \(u, v \in W\) we have:
a) \( \mathcal{T} or^O_{i} (\bar{p}^*(S), \xi_u \boxtimes \xi_v) = 0, \forall i > 0, \)

b) \( \mathcal{T} or^O_{i} (\bar{p}^*(S), \mathcal{O}(X^* \times X^*)_{P}) = 0, \forall i > 0, \)

where \( \bar{p} : \bar{Y}_P \to \mathbb{P} \) is the projection.

**Proof.** As the statements are local in \( P \), we may replace \( \bar{Y}_P \) by \( U \times Y \), for some open set \( U \subset \mathbb{P} \). Then,

\[
\bar{p}^* S \simeq S \boxtimes \mathcal{O}_Y,
\]

\[
\xi_u \boxtimes \xi_v \simeq \mathcal{O}_U \boxtimes (\xi_u \boxtimes \xi_v)\]

\[
\mathcal{O}_P(\bar{X} \times \bar{X})_{P} \simeq \mathcal{O}_U \boxtimes (\mathcal{O}_X \boxtimes \mathcal{O}_X).
\]

Applying the above, followed by the Kunneth formula, gives, for part a),

\[
\mathcal{T} or^O_{i} (\bar{p}^*(S), \xi_u \boxtimes \xi_v) = 0, \forall i > 0.
\]

A similar computation gives part b). \( \square \)

**Lemma 4.9.** For any coherent sheaf \( S \) on \( \mathbb{P} \) and any \( u \in W \) we have:

\[
\mathcal{T} or^O_{i} (\bar{p}^*(S), \hat{\Delta}_u((\xi_u)_P)) = 0, \forall i > 0,
\]

where, as earlier, \( \bar{p} : Y_P \to \mathbb{P} \) is the projection.

**Proof.** As before, since the statement is local in \( P \), we may replace \( Y_P \) by \( U \times Y \), for some open set \( U \subset \mathbb{P} \). Then,

\[
\bar{p}^* S \simeq S \boxtimes \mathcal{O}_Y,
\]

\[
\hat{\Delta}_u((\xi_u)_P) = \mathcal{O}_U \boxtimes \Delta_\ast (\xi_u).
\]

Now, proceed as in the proof of Lemma 4.8. \( \square \)

**Lemma 4.10.** With notation as in (4) we have

\[
c^w_{u,v}(j) = \langle [\mathcal{O}_P(\bar{X} \times \bar{X})_{P}], \hat{\Delta}_u((\xi_u)_P) \rangle, \]

where the pairing

\[
\langle , \rangle : K_0(Y_P) \otimes K_0(Y_P) \to \mathbb{Z}
\]

is defined (similar to (4)) by

\[
\langle [S], [F] \rangle = \sum_i (-1)^i \chi(Y_P, \mathcal{T} or^O_{i} (S, F)),
\]

where \( \chi \) denotes the Euler-Poincaré characteristic, and (as earlier) the map \( \hat{p} : Y_P \to \mathbb{P} \) denotes the projection.
Proof. First, we have
\[
\langle [\mathcal{O}_{(X^u \times X^v)_p}], \hat{\rho}^\ast [\mathcal{O}_{p_j}(-\partial P_j)] \cdot \hat{\Delta}_\ast ((\xi_w)_p) \rangle = \langle \hat{\rho}^\ast [\mathcal{O}_{p_j}(-\partial P_j)] \cdot [\mathcal{O}_{(X^u \times X^v)_p}], \hat{\Delta}_\ast ((\xi_w)_p) \rangle,
\]
where \( \hat{\rho} : \bar{Y}_p \to \mathbb{P} \) denotes the projection. To see this, first take a locally free finite resolution of \( \mathcal{O}_{(X^u \times X^v)_p} \) on a quasi-compact open subset of \( \bar{Y}_p \) and a locally free finite resolution of \( \mathcal{O}_{p_j}(-\partial P_j) \) on \( \mathbb{P} \). Then, use the fact that for a locally free sheaf \( F \) on a quasi-compact open subset of \( \bar{Y}_p \) and a locally free sheaf \( G \) on \( \mathbb{P} \), we have
\[
\langle F, \hat{\rho}^\ast (G) \cdot \hat{\Delta}_\ast ((\xi_w)_p) \rangle = \langle \bar{\rho}^\ast (G) \cdot F, \hat{\Delta}_\ast ((\xi_w)_p) \rangle.
\]
Now, the lemma follows from (4), Lemma 4.8, Proposition 3.10 and [Ku1, Identity 20].

We now introduce the mixing group \( \Gamma \) which acts on \( \bar{Y}_p \).

**Definition 4.11.** (Mixing group \( \Gamma \)) Let \( T \) act on \( B \) via
\[
t \cdot b = tbt^{-1}
\]
for \( t \in T, b \in B \). This action induces a natural action of \( \Delta T \) on \( B \times B \). Consider the ind-group scheme over \( \mathbb{P} \):
\[
(B^2)_p = E(T)_p \times T B^2 \to \mathbb{P}.
\]
Let \( \Gamma_0 \) denote the group of global sections of \( (B^2)_p \) under pointwise multiplication. Since \( GL(N+1)^r \) acts canonically on \( (B^2)_p \) in a way compatible with its action on \( \mathbb{P} \), it also acts on \( \Gamma_0 \) via inverse pull-back. We define the mixing group \( \Gamma \) to be the semi direct product \( \Gamma := \Gamma_0 \rtimes GL(N + 1)^r \):
\[
1 \to \Gamma_0 \to \Gamma \to GL(N + 1)^r \to 1.
\]
By [Ku1] Lemmas 4.7 and 4.8 (more precisely, the paragraph following these lemmas), we have the following two lemmas:

**Lemma 4.12.** \( \Gamma \) is connected.

**Lemma 4.13.** For any \( \bar{e} \in \mathbb{P} \) and any \((b,b') \) in the fiber of \( (B^2)_p \) over \( \bar{e} \) there exists a section \( \gamma \in \Gamma_0 \) such that \( \gamma(\bar{e}) = (b,b') \).

We define the action of \( \Gamma \) on \( \bar{Y}_p \) via
\[
(\gamma, g) \cdot [e, (x, y)] = [ge, \gamma(g) \cdot (x, y)]
\]
for \( \gamma \in \Gamma_0, g \in GL(N + 1)^r, e \in E(T)_p, \) and \((x, y) \in \bar{Y}, \) where the action of \( \Gamma_0 \) is via the standard action of \( B \times B \) on \( \bar{Y} = \bar{X}^2 \). It follows from Lemma 4.13 that the orbits of the \( \Gamma \)-action on \( \bar{Y}_p \) are precisely equal to \( \{(C_w \times C'_w)_p\} \).

5 Statement of main results

We now come to our main technical result.

**Theorem 5.1.** For general \( \gamma \in \Gamma \) and any \( u, v, w \in W, j \in [N]^r \) we have:
a) For all $i > 0$,
\[
\mathcal{O}_y \circ_i^y \left( \gamma_* \mathcal{O}_{(X^x \times X^y)^y}, \hat{p}^* (\mathcal{O}_{\mathcal{P}_j} (-\partial \mathcal{P}_j)) \otimes \mathcal{O}_{\mathcal{Y}_y} \tilde{\Delta}_s((\xi_w)\mathcal{P}) \right) = 0.
\]

b) Assume $c_{u,v}(j) \neq 0$. For all $p \neq |j| + \ell(u) - \ell(v)$,
\[
H^p \left( \hat{Y}_\mathcal{P}, \gamma_* \mathcal{O}_{(X^x \times X^y)^y} \otimes \mathcal{O}_{\mathcal{Y}_y} \left( \hat{p}^* (\mathcal{O}_{\mathcal{P}_j} (-\partial \mathcal{P}_j)) \otimes \mathcal{O}_{\mathcal{Y}_y} \tilde{\Delta}_s((\xi_w)\mathcal{P}) \right) \right) = 0,
\]

where $|j| := \sum_{i=1}^j j_i$.

Proof. Deferred to the later sections. Part a) is proved in Section 6, while part b) is proved in Section 9.

Since $\Gamma$ is connected, Lemmas 4.10 and Theorem 5.1 together give:

**Corollary 5.2.** $(-1)^{\ell(u) - \ell(v) + |j|} c_{u,v}(j) \in \mathbb{Z}_{\geq 0}$.

As an immediate consequence of Corollary 5.2 and Lemma 4.7 we get:

**Theorem 5.3.** For any symmetrizable Kac-Moody group $G$ and any $u, v, w \in W$, the structure constants of the product in the basis $\{[\mathcal{O}_{X^x}]\}$ in $K^0(\mathcal{X})$ satisfy
\[
(-1)^{\ell(u) + \ell(v) + |j|} c_{u,v} = \sum_{|j| = 0}^\infty \sum_{\ell(u) + \ell(v) + |j|} c_{u,v} \in \mathbb{Z}_{\geq 0}[[e^{-\alpha_1} - 1], \ldots, (e^{-\alpha_r} - 1]].
\]

### 6 Proof of part a) of Theorem 5.1

The key tool used to prove part a) of Theorem 5.1 is the following transversality result, taken from [AGM, Theorem 2.3] (originally due to Sierra)

**Theorem 6.1.** Let $X$ be a variety with a left action of an algebraic group $G$ and let $\mathcal{F}$ be a coherent sheaf on $X$. Suppose that $\mathcal{F}$ is homologically transverse to the closures of the $G$-orbits on $X$. Then, for each coherent sheaf $\mathcal{E}$ on $X$, there is a Zariski-dense open set $U \subseteq G$ such that $\mathcal{F} \circ_i \mathcal{F}(\mathcal{F}, \mathcal{E}) = 0$ for all $i \geq 1$ and all $g \in U$.

Theorem 6.1 part a) is a particular case of the following slightly more general result by taking $\mathcal{E} = \hat{p}^* (\mathcal{O}_{\mathcal{P}_j} (-\partial \mathcal{P}_j)) \otimes \mathcal{O}_{\mathcal{Y}_y} \tilde{\Delta}_s((\xi_w)\mathcal{P})$.

**Theorem 6.2.** Let $w \in W$ and let $\mathcal{E}$ be a coherent sheaf on $(Y_w) := (X_w \times X_w)$. Then, for general $\gamma \in \Gamma$ and any $u, v \in W$ we have:
\[
\mathcal{F} \circ_i \left( \gamma_* \mathcal{O}_{(X^x \times X^y)^y}, \mathcal{E} \right) = 0, \quad \forall i > 0.
\]

Proof. We first show that for any $\gamma \in \Gamma$,
\[
\mathcal{F} \circ_i \left( \mathcal{O}_{(X^x \times X^y)^y}, \gamma_* \mathcal{E} \right) = \mathcal{F} \circ_i \left( \mathcal{O}_{(Y_w)^y} \otimes \mathcal{O}_{\mathcal{Y}_y} \mathcal{O}_{(X^x \times X^y)^y}, \gamma_* \mathcal{E} \right).
\]

Since $\gamma_* \mathcal{E}$ is a coherent sheaf on $(Y_w)$, we can replace $\gamma_* \mathcal{E}$ by $\mathcal{E}$ itself.
As the assertion is local on $\mathbb{P}$ we may assume $\bar{Y}_p \simeq \mathbb{P} \times \bar{Y}$ and that

$$\mathcal{O}_{(X \times X')_p} \simeq \mathcal{O}_p \boxtimes (\mathcal{O}_{X} \boxtimes \mathcal{O}_{X'})$$

$$\mathcal{O}_{(Y_w)_p} \simeq \mathcal{O}_p \boxtimes (\mathcal{O}_{X_p} \boxtimes \mathcal{O}_{X_w}).$$

To simplify notation let $A = \mathcal{O}_{Y_p}$, $B = \mathcal{O}_{(Y_w)_p}$, $M = \mathcal{O}_{(X \times X')_p}$ and $N = \mathcal{E}$. Take an $A$-free resolution $F_* \to M$ on an open subset of $\bar{Y}_p$ of the form $(V^{u'} \times V^{v'})_p$, where $V^u$ is defined as in Lemma 3.3. Then, the homology of the chain complex $B \otimes_A F_*$ is by definition $\mathcal{J}or_4^4(M)$. Moreover,

$$\mathcal{J}or_4^4(M, B) = 0, \forall i > 0.$$

Indeed, locally on $\mathbb{P}$,

$$\mathcal{J}or_4^4(M, B) = \mathcal{J}or_4^4_{\mathcal{O}_p} \big( \mathcal{O}_p \boxtimes \mathcal{O}_{(X \times X')_p}, \mathcal{O}_p \boxtimes \mathcal{O}_{(X_w \times X_w)} \big)$$

$$= \bigoplus_{j+k=i} \mathcal{J}or_4^4_j(\mathcal{O}_p, \mathcal{O}_p) \otimes \mathcal{J}or_4^4_k(\mathcal{O}_{(X \times X')_p}, \mathcal{O}_{(X_w \times X_w)})$$

$$= 0, \forall i > 0$$

by Lemma 3.7 and the Kunneth formula. Hence, $B \otimes_A F_*$ is a $B$-free resolution of $B \otimes_A M$.

Thus, the homology of the chain complex $N \otimes_B (B \otimes_A F_*)$ is equal to $\mathcal{J}or_4^4(B \otimes_A M, N)$; but,

$$N \otimes_B (B \otimes_A F_*) = (N \otimes_B B) \otimes_A F_* = N \otimes_A F_*,$$

so the homology is also equal to $\mathcal{J}or_4^4(M, N)$. Hence,

$$\mathcal{J}or_4^4(B \otimes_A M, N) = \mathcal{J}or_4^4(M, N)$$

as desired. This proves (5).

Now, by Lemma 4.16 the closures of the $\Gamma$-orbits of $(Y_w)_p$ are precisely $(X_x \times X_y)_p$, where $x, y \leq w$. Equation (5) implies that the sheaf $\mathcal{F}$ defined by

$$\mathcal{F} := \mathcal{O}_{(Y_w)_p} \otimes \mathcal{O}_{Y_p} \mathcal{O}_{(X \times X')_p}$$

is homologically transverse to the $\Gamma$-orbit closures in $(Y_w)_p$. Indeed, since $\mathcal{O}_{(X_x \times X_y)_p}$ is a coherent $\mathcal{O}_{(Y_w)_p}$-module when $x, y \leq w$, equation (5) gives

$$\mathcal{J}or_4^4_{(Y_w)_p}(\mathcal{F}, \mathcal{O}_{(X_x \times X_y)_p}) = \mathcal{J}or_4^4_{(Y_w)_p}(\mathcal{O}_{(X \times X')_p}, \mathcal{O}_{(X_x \times X_y)_p})$$

$$= 0, \forall i > 0$$

by Lemma 3.7 and the Kunneth formula.

Thus, by Theorem 6.1 (with $G = \Gamma, X = (Y_w)_p, \mathcal{E}$ and $\mathcal{F}$ as above), we conclude that for general $\gamma \in \Gamma$,

$$\mathcal{J}or_4^4_{(Y_w)_p}(\mathcal{O}_{(Y_w)_p} \otimes \mathcal{O}_{Y_p} \mathcal{O}_{(X \times X')_p}, \mathcal{O}_{(X_x \times X_y)_p}, \mathcal{O}_{\gamma \ast \mathcal{E}}) = 0, \forall i > 0. \quad (6)$$
Here we note that although $\Gamma$ is infinite dimensional, the action of $\Gamma$ on $(Y_w)_p$ factors through the action of a finite dimensional quotient group of $\Gamma$.

Now, \[\eqref{eq:action}\] gives

$$\mathcal{F}or_{i}^{\mathcal{O}_{Y_w}}(\mathcal{O}_{(X_{w_{1}} \times X_{w_{2}})_p}, \gamma, \mathcal{E}) = 0, \quad \forall i > 0,$$

which is equivalent to the desired vanishing.

7 The schemes $Z$ and $\partial Z$

For $u, v \leq w$ we use the notation $X_{u,v}^w := X_{u,w} \times X_{v,w}$. We also write $X_{w}^2 := X_{w} \times X_{w}$. For any $j \in [N]^r$, we let $X_{w_j}^w$ denote the inverse image of $P_j$ through the map $E(T)_p \times T X_w \to \mathbb{P}$.

Similarly, for $u, v \leq w$ we write $Z_{u,v}^w := Z_{u,w} \times Z_{v,w}$, where $Z_{w}^w$ is the $T$-equivariant desingularization of $X_{w}$ as in \[\text{[Ku1, Theorem 6.8]}\]. We also write $Z_{w}^2 := Z_{w} \times Z_{w}$, where $Z_{w}$ is a BSDH variety as in \[\text{[Ku2, \S 7.1.3]}\]. For any $j \in [N]^r$, we let $Z_{w_j}^w$ denote the inverse image of $P_j$ through the map $E(T)_p \times T Z_w \to \mathbb{P}$.

The action of $B$ on $Z_w$ factors through the action of a finite dimensional quotient group $\bar{B}$ containing the maximal torus $H$. Further, the action of $\Gamma$ on $(X_{w}^2)_p$ descends to an action of the finite dimensional quotient group

$$\bar{\Gamma} := \bar{\Gamma}_0 \times GL(N + 1)^r,$$

where $\bar{\Gamma}_0$ is the group of global sections of the bundle $E(T)_p \times T (\bar{B}^2) \to \mathbb{P}$.

From \[\text{[Ku1, Lemmas 6.11 and 6.12]}\] we have

Lemma 7.1. Let $u, v \leq w$. The map

$$m : \bar{\Gamma} \times (X_{u,v}^w)_p \to (X_{w}^2)_p, \quad m(\gamma, x) = \gamma \cdot \pi_2(x)$$

is flat, where $\pi_2 : (X_{u,w} \times X_{v,w})_p \to (X_{w}^2)_p$ is induced from the canonical map $X_{u,w} \times X_{v,w} \to X_{w}^2$.

Lemma 7.2. Let $u, v \leq w$. The map

$$\tilde{m} : \bar{\Gamma} \times (Z_{u,v}^w)_p \to (Z_{w}^2)_p, \quad \tilde{m}(\gamma, x) = \gamma \cdot \tilde{\pi}_2(x)$$

is smooth, where $\tilde{\pi}_2 : (Z_{u,w} \times Z_{v,w})_p \to (Z_{w}^2)_p$ is induced from the canonical map $Z_{u,w} \times Z_{v,w} \to Z_{w}^2$.

Define $\tilde{Z}$ to be the fiber product

$$\tilde{Z} := (\bar{\Gamma} \times (Z_{u,v}^w)_p) \times (X_{w_j}^w) \tilde{\Delta}((X_{w_j})_p)$$

and $\tilde{Z}$ to be the fiber product

$$\tilde{Z} := (\bar{\Gamma} \times (Z_{u,v}^w)_p) \times (Z_{w_j}^w) \tilde{\Delta}((Z_{w_j})_p)$$

as in the commutative diagram:
In the above diagram, \( \Box \) denotes a fiber square. Note that the maps \( \theta \) and \( \beta \) above are desingularizations. The maps \( \pi : Z \to \tilde{\Gamma} \) and \( \tilde{\pi} : \tilde{Z} \to \tilde{\Gamma} \) are induced by the projections onto the first factor. The map \( f : \tilde{Z} \to Z \) is defined by \( f := \theta \circ \tilde{\iota} \). It is clear that the image of \( f \) is indeed contained inside of \( Z \) using commutativity of the diagram, along with the fact that \( \beta(\tilde{\Delta}((Z_w)_j)) = \tilde{\Delta}((X_w)_j) \).

We define the boundary of \((X_w)_j\) by
\[
\partial((X_w)_j) := (\partial X_w)_j \cup (X_w)_{\partial \wp_j}
\]
and similarly define the boundary of \((Z_w)_j\) by
\[
\partial((Z_w)_j) := (\partial Z_w)_j \cup (Z_w)_{\partial \wp_j},
\]
where \( \partial Z_w := \varphi^{-1}(\partial X_w) \) and \( \varphi : Z_w \to X_w \) denotes the desingularization.

We define the boundary of \( Z \) by
\[
\partial Z := (\tilde{\varphi} \times (X_w^\prime)_{\wp}) \times (X_w)_j \tilde{\Delta}(\partial((X_w)_j))
\]
and similarly define the boundary of \( \tilde{Z} \) by
\[
\partial \tilde{Z} := (\tilde{\varphi} \times (Z_w^\prime)_{\wp}) \times (Z_w)_j \tilde{\Delta}(\partial((Z_w)_j)).
\]
Observe that \( f^{-1}(\partial Z) = \partial \tilde{Z} \) is the scheme-theoretic inverse image.

We will need the following lemmas, which are restatements of [Ku1, Lemmas 7.2 and 7.3] respectively ([Ku1, Lemma 7.3] is originally from [FP, Lemma on page 108]).

**Lemma 7.3.** Let \( f : W \to X \) be a flat morphism from a pure-dimensional CM scheme \( W \) of finite type over \( \mathbb{C} \) to a CM irreducible variety \( X \) and let \( Y \) be a closed CM subscheme of \( X \) of pure codimension \( d \). Set \( Z := f^{-1}(Y) \). If \( \text{codim}_W(Z) \geq d \) then equality holds and \( Z \) is CM.
Lemma 7.4. Let \( f : W \to X \) be a morphism from a pure-dimensional CM scheme \( W \) of finite type over \( \mathbb{C} \) to a smooth irreducible variety \( X \) and let \( Y \) be a closed CM subscheme of \( X \) of pure codimension \( d \). Set \( Z := f^{-1}(Y) \). If \( \text{codim}_W(Z) \geq d \) then equality holds and \( Z \) is CM.

Proposition 7.7. The scheme \( Z \) is normal, irreducible, and CM, of dimension
\[
\dim Z = |j| + \ell(w) - \ell(u) - \ell(v) + \dim \tilde{\Gamma}.
\]

The scheme \( \tilde{Z} \) is irreducible, and the map \( f: \tilde{Z} \to Z \) is a proper birational map. Hence, the scheme \( \tilde{Z} \) is a desingularization of \( Z \).

Proof. By [11] Chapter III, Corollary 9.6], the fibers of \( m \) are pure of dimension
\[
\dim \tilde{\Gamma} + \dim(X^u,v)_\mathbb{P} - \dim(X^2)_\mathbb{P}.
\]
Since the fibers of \( \mu \) are the same as those of \( m \), applying loc. cit. to \( \mu \) gives that \( Z \) is pure of dimension
\[
\dim Z = \dim \tilde{\Gamma} + \dim(X^u,v)_\mathbb{P} - \dim(X^2)_\mathbb{P} + \dim \tilde{\Delta}((X^u)_j)
= |j| + \ell(w) - \ell(u) - \ell(v) + \dim \tilde{\Gamma}.
\]

The remainder of the proof of the proposition follows from the proof of [11] Proposition 7.4 and Lemma 7.5 (Note that there is a slight difference in the definition of \( Z \) and \( \tilde{Z} \) between ours and that in [11]). Further, the scheme \( \tilde{ZZ} \) is non-singular, since \( \tilde{\mu} \) is a smooth morphism with non-singular base.

Lemma 7.6. Let \( G \) be a group acting on a set \( X \) and let \( Y \subset X \). Consider the action map \( m: G \times Y \to X \). For \( x \in X \) denote the orbit of \( x \) by \( O(x) \) and the stabilizer by \( \text{Stab}(x) \). Then, \( \text{Stab}(x) \) acts on the fiber \( m^{-1}(x) \) and \( \text{Stab}(x) \setminus m^{-1}(x) \simeq O(x) \cap Y \).

Proof. It is easy to check that
\[
m^{-1}(x) = \{(g, h^{-1}x) : h \in G, \ h^{-1}x \in Y, \ g \in \text{Stab}(x) \cdot h\}.
\]
Thus, \( \text{Stab}(x) \) acts on \( m^{-1}(x) \) by left multiplication on the left component. Since every element of \( O(x) \cap Y \) is of the form \( h^{-1}x \) for some \( h \in G \), the second projection \( m^{-1}(x) \to O(x) \cap Y \) is surjective. This map clearly factors through the quotient to give a map \( \text{Stab}(x) \setminus m^{-1}(x) \to O(x) \cap Y \). To show this induced map is injective, note first that each class has a representative of the form \( (h, h^{-1}x) \).

Proposition 7.7. The scheme \( Z \) has rational singularities.

Proof. Since \( \mu \) is flat and \( \tilde{\Delta}((X^u)_j) \) has rational singularities, by [11] Théorème 5] it is sufficient to show that the fibers of \( \mu \) are disjoint unions of irreducible varieties with rational singularities (in fact, the fibers of \( \mu \) are irreducible, but we do not provide a proof here as we do not need this fact).

Let \( x \in \tilde{\Delta}((C^u)_j) \), where \( w' \leq w \). Then, by Lemmas 7.6 and 1.13 we have \( \text{Stab}(x) \setminus \mu^{-1}(x) \simeq (X^u \cap C^u) \times X^v \setminus (C^u) \mathbb{P} \), where \( \text{Stab}(x) \) is taken with respect to the action of \( \tilde{\Gamma} \) on \( (X^2)_\mathbb{P} \). By [11] Proposition 3, §2.5], the quotient map \( \tilde{\Gamma} \to \text{Stab}(x) \setminus \tilde{\Gamma} \) is locally trivial in the étale topology.

Consider the pullback diagram:

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\[ \mu^{-1}(x) \subseteq \bar{\Gamma} \times (X_wu,v)_P \]
\[ \text{Stab}(x) \setminus \mu^{-1}(x) \subseteq (\text{Stab}(x) \setminus \bar{\Gamma}) \times (X_wu,v)_P \]

Since the right vertical map is a locally trivial fibration in the étale topology, the left vertical map is too. Now, \( \text{Stab}(x) \setminus \mu^{-1}(x) \simeq (X_u \cap C_w \times X_v \cap C_w)_P \) has rational singularities by [KM, Theorem 3.1]. Further, \( \text{Stab}(x) \) being smooth and \( \mu^{-1}(x) \rightarrow \text{Stab}(x) \setminus \mu^{-1}(x) \) being locally trivial in the étale topology, we get that \( \mu^{-1}(x) \) is a disjoint union of irreducible varieties with rational singularities by [KM, Corollary 5.11].

**Proposition 7.8.** The scheme \( \partial \mathcal{Z} \) is pure of codimension 1 in \( \mathcal{Z} \) and is CM.

**Proof.** Using the fact that \( \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) and \( \hat{\Delta}((\partial X_w)_j) \) are equidimensional, applying [H, Chapter III, Corollary 9.6] to their irreducible components gives that \( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) and \( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_j) \) are both pure of dimension

\[ \dim \bar{\Gamma} + \dim (X_wu,v)_P - \dim (X_wu,v)_P + \dim \hat{\Delta}((X_w)_j) - 1. \]

Hence, \( \partial \mathcal{Z} \) is pure of codimension 1 in \( \mathcal{Z} \).

A similar argument also gives that \( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_{\partial \bar{P}_j}) \) is pure of dimension

\[ \dim \bar{\Gamma} + \dim (X_wu,v)_P - \dim (X_wu,v)_P + \dim \hat{\Delta}((X_w)_j) - 2. \]

Next we show that \( \partial \mathcal{Z} \) is CM. Since \( (X_w)_{\partial \bar{P}_j} \) is a locally trivial fibration over \( \mathbb{P}_j \), it is locally a product of CM schemes and hence is CM. Also, since \( \partial \mathcal{Z} = \mathbb{P}_j \) is CM, we similarly have that \( (X_w)_{\partial \bar{P}_j} \) and hence \( \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) is CM. Now, applying Lemma 7.3 to \( \mu : \mathcal{Z} \rightarrow \hat{\Delta}((X_w)_j) \) gives that \( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) is CM, since \( \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) and \( \mu^{-1}(\hat{\Delta}((X_w)_{\partial \bar{P}_j})) = (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \) are of pure codimension 1 in \( \hat{\Delta}((X_w)_j) \) and \( \mathcal{Z} \) respectively.

Observe that \( \partial X_w \) is CM. To prove this, use Proposition 3.6 and the argument as in the proof of [Ku, Corollary 10.5] by taking an embedding of \( X_w \) into a smooth projective variety. Thus, \( (\partial X_w)_j \) is locally a product of CM schemes and hence is CM, and hence so is \( \hat{\Delta}((\partial X_w)_j) \). Now, Lemma 7.3 applied to \( \mu : \mathcal{Z} \rightarrow \hat{\Delta}((X_w)_j) \) gives that

\[ (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_j) \]

is CM since \( \hat{\Delta}((\partial X_w)_j) \) and \( \mu^{-1}(\hat{\Delta}((\partial X_w)_j)) = (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_j) \) are of pure codimension 1 in \( \hat{\Delta}((X_w)_j) \) and \( \mathcal{Z} \) respectively.

The intersection

\[ (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_j) \]

\[ = \left( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((X_w)_{\partial \bar{P}_j}) \right) \cap \left( (\bar{\Gamma} \times (X_wu,v)_P) \times (X_wu,v)_P \hat{\Delta}((\partial X_w)_j) \right) \]

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is of pure codimension 1 in both \((\tilde{\Gamma} \times (X^u_w)_p) \times (X^2_w)_p \setminus \Delta((\partial X_w)_j)\) and \((\tilde{\Gamma} \times (X^u_v)_p) \times (X^2_v)_p \setminus \Delta((\partial X_w)_j)\). Now, Exercise 18.13 of \[\text{[2]}\] Exercise 18.13 \gives that the union \(\partial Z\) is CM if the intersection \((\tilde{\Gamma} \times (X^u_w)_p) \times (X^2_w)_p \setminus \Delta((\partial X_w)_j)\) is. But Lemma \[\text{[2]}\] Lemma 7.3 \applied to \(\mu : Z \to \tilde{\Delta}((X_w)_j)\) \gives that \((\tilde{\Gamma} \times (X^u_w)_p) \times (X^2_w)_p \setminus \Delta((\partial X_w)_j)\) is CM, since \(\tilde{\Delta}((\partial X_w)_j)\) and \(\mu^{-1}(\tilde{\Delta}((\partial X_w)_j))\) are of pure codimension 2 in \(\tilde{\Delta}((X_w)_j)\) and \(Z\) respectively. □

Lemma 7.9. The morphism \(\mu : Z \to \tilde{\Delta}((X_w)_j)\) is affine.

Proof. Since \(\tilde{\Delta}((X_w)_j)\) is a closed subscheme of \((X^2_w)_p\), it suffices to show that the map \(m : \tilde{\Gamma} \times (X^u_w)_p \to (X^2_w)_p\) is an affine morphism. Now, if \(f : X \to Y\) is an affine morphism and \(Z \subset X\) is a closed subscheme then \(f|_Z : Z \to Y\) is clearly an affine morphism. Thus, it suffices to show that \(\tilde{\Gamma} \times (X^2_w)_p \to (X^2_w)_p\) is an affine morphism. Further, if \(X\) and \(Y\) are total spaces of principal \(T\)-bundles and \(f : X \to Y\) is a \(T\)-equivariant map, then \(f\) is affine if \(f : X/T \to Y/T\) is affine. Thus, it suffices to show that

\[
\tilde{\mu} : \tilde{\Gamma} \times (E(T)_p \times X^2_w) \to E(T)_p \times X^2_w
\]

is affine. Recall that \(\tilde{\Gamma} = \tilde{\Gamma}_0 \times GL(N+1)^r\) and \(\tilde{\mu}\) is given by \(\tilde{\mu}((\sigma, g, (e, x)) = (g, \sigma(g) \cdot x)\), where \(\sigma \in \tilde{\Gamma}_0, g \in GL(N+1)^r, e \in E(T)_p, x \in X^2_w\). Write \(\tilde{\mu}\) as a composite \(\tilde{\mu} = \mu_3 \circ \mu_2 \circ \mu_1\) where:

\[
\begin{align*}
\mu_1 : \tilde{\Gamma}_0 \times GL(N+1)^r \times (E(T)_p) \times X^2_w &\to \tilde{\Gamma}_0 \times E(T)_p \times X^2_w, \quad (\sigma, g, e, x) \mapsto (\sigma, g \cdot e, x) \\
\mu_2 : \tilde{\Gamma}_0 \times E(T)_p \times X^2_w &\to \bar{B}^2 \times E(T)_p \times X^2_w, \quad (\sigma, e, x) \mapsto (\sigma(e), e, x) \\
\mu_3 : \bar{B}^2 \times E(T)_p \times X^2_w &\to E(T)_p \times X^2_w, \quad ((b_1, b_2), (e, y)) \mapsto (e, b_1 x, b_2 y).
\end{align*}
\]

As the composition of two affine morphisms is affine, it suffices to prove that \(\mu_1, \mu_2, \mu_3\) are affine. Moreover, if \(f : X \to Y\) is affine then \(f \circ \text{Id}_Z : X \times Z \to Y \times Z\) is affine for any scheme \(Z\). Hence, it suffices to prove that the following maps \(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\) are affine:

\[
\begin{align*}
\tilde{\mu}_1 : GL(N+1)^r \times (E(T)_p) &\to (E(T)_p, (g, e) \mapsto g \cdot e) \\
\tilde{\mu}_2 : \tilde{\Gamma}_0 \times E(T)_p &\to \bar{B}^2 \times E(T)_p, \quad (\sigma, e) \mapsto (\sigma(e), e) \\
\tilde{\mu}_3 : \bar{B}^2 \times X^2_w &\to X^2_w, \quad ((b_1, b_2), (x, y)) \mapsto (b_1 x, b_2 y).
\end{align*}
\]

1. \(\tilde{\mu}_1\) is affine: Since \(E(T)_p = (C^{N+1} \setminus \{0\})^r\), it suffices to prove that \(\tilde{\theta} : GL(N+1) \times (C^{N+1} \setminus \{0\}) \to C^{N+1} \setminus \{0\}, (g, v) \mapsto g \cdot v\) is affine. Now, consider the map \(\tilde{\theta} : GL(N+1) \times C^{N+1} \to C^{N+1}, (g, v) \mapsto g \cdot v\). Since both the domain and codomain are affine, \(\tilde{\theta}\) is an affine morphism. Moreover, \(\theta = \tilde{\theta}|_{\tilde{\theta}^{-1}(C^{N+1} \setminus \{0\})}\). Thus, \(\theta\) is affine.

2. \(\tilde{\mu}_2\) is affine: Take an affine open subset \(U \subset E(T)_p\). Then, \(\bar{B}^2 \times U\) is an affine open subset in \(\bar{B}^2 \times E(T)_p\). Now, \(\mu_2^{-1}(\bar{B}^2 \times U) = \tilde{\Gamma}_0 \times U\). Since \(\tilde{\Gamma}_0\) is affine, so is \(\tilde{\Gamma}_0 \times U\). Thus, \(\tilde{\mu}_2\) is affine.

3. \(\tilde{\mu}_3\) is affine: It suffices to prove that \(\delta : \bar{B} \times X^2_w \to (b, x) \mapsto b x\) is affine. Take an affine open subset \(V \subset X^2_w\). Then, \(\delta^{-1}(V) = \bigcup_{b \in \bar{B}} (b, b^{-1} V)\). Consider the scheme isomorphism \(i : \bar{B} \times X^2_w \to \bar{B} \times (b, x) \mapsto (b, b \cdot x)\). Then, \(i(\delta^{-1}(V)) = \bar{B} \times V\). But, since \(\bar{B} \times V\) is affine, so is \(\delta^{-1}(V)\). Thus, \(\delta\) is an affine morphism and hence so is \(\tilde{\mu}_3\).
Let $\pi: Z \to \bar{\Gamma}$ denote the projection onto the first factor and $\pi_1: \partial Z \to \bar{\Gamma}$ denote the restriction of $\pi$ to $\partial Z$. We define the fibers

$$N_\gamma := \pi^{-1}(\gamma) \simeq \gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}((X_w)_j)$$

and

$$M_\gamma := \pi_1^{-1}(\gamma) \simeq \gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}(\partial((X_w)_j)).$$

**Corollary 7.10.** Assume that $c_{w,v}^u(j) \neq 0$, where $c_{w,v}^u(j)$ are defined by (4). Then, for general $\gamma \in \bar{\Gamma}$, we have that $N_\gamma$ (defined by (7)) is CM of pure dimension. In fact, for $\gamma \in \bar{\Gamma}$ such that

$$\dim N_\gamma = \dim Z - \dim \bar{\Gamma} = |j| + \ell(w) - \ell(u) - \ell(v),$$

$N_\gamma$ is CM, and this condition is satisfied for general $\gamma$.

Similarly, if $|j| + \ell(w) - \ell(u) - \ell(v) > 0$, for general $\gamma \in \bar{\Gamma}$, we have that $M_\gamma$ (defined by (8)) is CM of pure codimension 1 in $N_\gamma$. If $|j| + \ell(w) - \ell(u) - \ell(v) = 0$, for general $\gamma \in \bar{\Gamma}$, we have that $M_\gamma$ is empty.

**Proof.** First we show that $\pi$ is surjective when $c_{w,v}^u(j) \neq 0$. From the definition of $\pi$ we have that

$$\text{Im } \pi = \left\{ \gamma \in \bar{\Gamma} : \gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}((X_w)_j) \neq \emptyset \right\}.$$

Since $\bar{\Gamma}$ is connected, Lemma [4,10] along with the assumption $c_{w,v}^u(j) \neq 0$ gives that $\gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}((X_w)_j) \neq \emptyset$ for any $\gamma \in \bar{\Gamma}$. Since $\gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}((X_w)_j) = \gamma((X_{w,v}^u)^p) \cap \tilde{\Delta}((X_w)_j)$, we get that $\pi$ is surjective.

Now, since $Z$ is CM, applying Lemma [7,4] to $\pi$ gives that if

$$\text{codim}_Z(N_\gamma) = \dim \bar{\Gamma},$$

then $N_\gamma$ is CM. By [S, Chapter I, §6.3, Theorem 1.25], this condition holds for $\gamma$ in a dense open subset of $\bar{\Gamma}$. Thus, $N_\gamma$ is CM for $\gamma$ satisfying $\dim N_\gamma = \dim Z - \dim \bar{\Gamma}$, and this condition is satisfied for general $\gamma$.

Next, we show that if $|j| + \ell(w) - \ell(u) - \ell(v) > 0$ then $\pi_1$ is surjective. First note that since $\pi_1$ is projective, if it is not surjective, its image is a proper closed subset of $\bar{\Gamma}$. Thus, for general $\gamma \in \bar{\Gamma}$, $M_\gamma = \emptyset$, i.e., $N_\gamma \subset Z \setminus \partial Z$. As $\mu$ is an affine morphism by Lemma [7,9] $Z \setminus \partial Z$ is affine. But, $N_\gamma$ is projective of positive dimension (since $|j| + \ell(w) - \ell(u) - \ell(v) > 0$) which gives a contradiction.

In particular, there is at least one irreducible component of $\partial Z$ on which $\pi_1$ is surjective. The other irreducible components are mapped to closed subsets of $\bar{\Gamma}$. Let $U$ be the complement of the union of the images of the irreducible components on which $\pi_1$ is not surjective. Then, $U$ is open, since there are only finitely many irreducible components. Applying [S, Chapter I, §6.3, Theorem 1.25] to $\pi_1$ on each irreducible component of $\partial Z$ which surjects onto $\bar{\Gamma}$ and then intersecting with $U$ gives that for general $\gamma \in \bar{\Gamma}$

$$\text{codim}_{\partial Z}(M_\gamma) = \dim \bar{\Gamma}. \quad (10)$$

Thus, by Lemma [7,4] $M_\gamma$ is CM. Moreover, (9) and (10) together imply that $M_\gamma$ is pure of codimension 1 in $N_\gamma$.

In the case where $|j| + \ell(w) - \ell(u) - \ell(v) = 0$, we have $\dim Z = \dim \bar{\Gamma}$, so that $\dim \partial Z < \dim \bar{\Gamma}$. Thus, $\text{Im } \pi_1$ is a proper closed subset of $\bar{\Gamma}$, and hence $M_\gamma$ is empty for general $\gamma$. \qed
Corollary 7.11. Assume that $c^w_{u,i}(j) \neq 0$. Then, for general $\gamma \in \Gamma$, we have

$$\mathcal{E}xt^i_{\mathcal{O}_{\gamma}}(\mathcal{O}_{\gamma}(-M_\gamma), \omega_{\gamma}) = 0, \ \forall i > 0.$$  

Proof. By [I] Proposition 11.33 and Corollary 11.43] we have

$$\mathcal{E}xt^i_{\mathcal{O}_{\gamma}}(\mathcal{O}_{\gamma}, \omega_{\gamma}) = 0, \text{ unless } i = 1,$$

and, of course,

$$\mathcal{E}xt^i_{\mathcal{O}_{\gamma}}(\mathcal{O}_{\gamma}, \omega_{\gamma}) = 0, \ \forall i > 0.$$

Hence, the desired result follows from the long exact $\mathcal{E}xt$ sequence associated to the short exact sequence

$$0 \to \mathcal{O}_{\gamma}(-M_\gamma) \to \mathcal{O}_{\gamma} \to \mathcal{O}_{M_\gamma} \to 0.$$  

\hfill \Box

8 Application of Kawamata-Viehweg Vanishing

In this section we assume that $c^w_{u,i}(j) \neq 0$.

Let $f : X \to Y$ be a proper morphism between schemes, with $X$ irreducible, and let $\mathcal{M}$ be a line bundle on $X$. Then, $\mathcal{M}$ is said to be $f$-nef if it has nonnegative intersection with every curve contained in a fiber of $f$. The line bundle $\mathcal{M}$ is said to be $f$-big if rank $f_*(\mathcal{M}^k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where $n$ is the dimension of a general fiber of $f$. A Weil divisor $D \subset X$ has normal crossings if all of its irreducible components intersect transversely. It is easy to see that $\partial Z_w$ and hence $\partial((Z_w)_j)$ are normal crossings divisors in $Z_w$ and $(Z_w)_j$ respectively.

We will need the following relative Kawamata-Viehweg vanishing theorem from [AGM, Theorem 2.4], which was originally extracted from [EV, Corollary 6.11].

Theorem 8.1. Let $f : \mathcal{Z} \to Z$ be a proper surjective morphism of varieties, with $\mathcal{Z}$ nonsingular. Let $\mathcal{M}$ be a line bundle on $\mathcal{Z}$ such that $\mathcal{M}^N(-D)$ is $f$-nef and $f$-big for a normal crossing divisor $D = \sum_{j=1}^r a_j D_j$, where $0 < a_j < N$ for all $j$. Then,

$$R^i f_*(\mathcal{M} \otimes \omega_{\mathcal{Z}}) = 0, \ \forall i > 0.$$  

Definition 8.2. We define the sheaf

$$\omega_{\mathcal{Z}}(\partial \mathcal{Z}) := \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \omega_{\mathcal{Z}}).$$

Theorem 8.3. We have $f_*(\omega_{\mathcal{Z}}(\partial \mathcal{Z})) = \omega_{\mathcal{Z}}(\partial \mathcal{Z})$, where $\mathcal{Z}, \mathcal{Z}, \partial \mathcal{Z}, \partial \mathcal{Z}$ and the morphism $f : \mathcal{Z} \to Z$ are defined after Lemma 7.2. (Observe that since $Z$ is CM, the dualizing sheaf $\omega_{\mathcal{Z}}$ makes sense.)

Proof. First, we claim

$$\mathcal{O}_{\mathcal{Z}}(\partial \mathcal{Z}) \simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(f^* \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \mathcal{O}_{\mathcal{Z}}), \quad (11)$$

where $\mathcal{O}_{\mathcal{Z}}(\partial \mathcal{Z}) := \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \mathcal{O}_{\mathcal{Z}})$. To see this, first note that since $f^{-1}(\partial \mathcal{Z}) = \partial \mathcal{Z}$ is the scheme-theoretic inverse image (cf. §6), the natural morphism $f^* \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}) \to \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z})$ is
surjective (cf., e.g. [Stacks Tag 01HJ Lemma 25.4.7]). As \( f \) is a desingularization (cf. Proposition 7.3), the kernel of this morphism is supported on a proper closed subset of \( Z \) and hence is a torsion sheaf. This implies that the dual map \( \mathcal{O}_Z(\partial Z) \to \mathcal{H}om_{\mathcal{O}_Z}(f^*\mathcal{O}_Z(-\partial Z), \mathcal{O}_Z) \) is an isomorphism, proving (11).

To complete the proof of the theorem, we compute:

\[
f_*(\omega_Z \otimes \mathcal{O}_Z(\partial Z)) = f_*(\mathcal{O}_Z) = \mathcal{O}_Z(\partial Z)
\]

\[
f_*(\omega_Z \otimes \mathcal{O}_Z(\partial Z)) = f_*(\omega_Z \otimes \mathcal{H}om_{\mathcal{O}_Z}(f^*\mathcal{O}_Z(-\partial Z), \mathcal{O}_Z)), \text{ by (11)}
\]

\[
= f_*\mathcal{H}om_{\mathcal{O}_Z}(f^*\mathcal{O}_Z(-\partial Z), \omega_Z)
\]

\[
= \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z(-\partial Z), f_*\omega_Z), \text{ by adjunction (cf. [H Chapter II, §5])}
\]

\[
= \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z(-\partial Z), \omega_Z), \text{ by Proposition 7.7 and [KM] Theorem 5.10}
\]

\[
= \omega_Z(\partial Z)
\]

**Lemma 8.4.** The homogeneous line bundle \( L(\rho)|_{X_w} \) (cf. §2) has a section with zero set exactly equal to \( \partial X_w \).

**Proof.** Consider the Borel-Weil homomorphism (cf. [Ku2 §8.1.21]) \( \chi_w : L(\rho)^* \to H^0(X_w, L(\rho)) \) which is given by \( \chi_w(f)(gB) = [g, f(ge_\rho)] \), where \( e_\rho \) is the highest weight vector of the irreducible highest weight \( G^{\text{min}} \)-module \( L(\rho) \) with highest weight \( \rho \).

Consider the section \( \chi_w(e_{w\rho}^*) \), where \( e_{w\rho} \) is the weight vector of \( L(\rho) \) with weight \( w\rho \) and \( e_{w\rho}^* \in L(\rho)^* \) is the linear form which takes the value 1 on \( e_{w\rho} \) and 0 on any weight vector of \( L(\rho) \) of weight different from \( w\rho \). Let \( w' \leq w \) and \( b \in B \). We have

\[
\chi_w(e_{w\rho}^*)(bw'B) = [bw', e_{w\rho}(bw'e_\rho)].
\]

Now, for \( w' < w \), we have \( w'\rho > w\rho \) by [Ku2 Lemma 8.3.3]. Thus, \( e_{w\rho}^*(bw'e_\rho) = 0 \) for any \( b \in B \) and \( w' < w \). For \( w' = w \), we have \( e_{w\rho}^*(bw'e_\rho) \neq 0 \) for any \( b \in B \). Hence, \( \chi_w(e_{w\rho}^*) \) has zero set precisely equal to \( \partial X_w \). \( \square \)

**Lemma 8.5.** There is an ample line bundle \( L \) on \( (X_w)_j \) with a section with zero set exactly equal to \( \partial((X_w)_j) \).

**Proof.** By the previous lemma, the ample line bundle \( L(\rho)|_{X_w} \) has a section with zero set exactly equal to \( \partial X_w \). The \( T \)-equivariant line bundle \( (e_{w\rho}L(\rho))|_{X_w} \) gives rise to the line bundle

\[
(e_{w\rho}L(\rho)|_{X_w})_j := E(T)_{\mathbb{P}_j} \times^T ((e_{w\rho}L(\rho))|_{X_w})
\]
on \( (X_w)_j \). Then, the section \( \theta \) defined by \( [e, x] \mapsto [e, 1_{w\rho} \otimes \chi_w(e_{w\rho}^*)x] \) for \( e \in E(T)_{\mathbb{P}_j} \) and \( x \in X_w \) has zero set exactly equal to \( \partial((X_w)_j) \), where \( 1_{w\rho} \) is a nonzero element of the line bundle \( e_{w\rho} \) (cf. §2).
Now, let $\mathcal{H}$ be an ample line bundle on $\mathbb{P}_j$ with a section $\sigma$ with zero set exactly $\partial \mathbb{P}_j$ and consider the bundle
\[ \mathcal{L} := (e^{\varphi p} \mathcal{L}(\rho)|_{X_w})_j \otimes p^*(\mathcal{H}^N), \]
where $p : (X_w)_j \to \mathbb{P}_j$ is the projection. Let $\hat{\sigma}$ represent the pullback of $\sigma$. Then, the section $\theta \otimes (\hat{\sigma})^N$ has zero set exactly equal to $\partial((X_w)_j)$. Furthermore, by [KM Proposition 1.45], if $N$ is large enough, $\mathcal{L}$ is ample.

**Lemma 8.6.** Let $f : \tilde{Z} \to Z$ be a proper birational map between normal, irreducible varieties and let $\pi : Z \to \Gamma$ be a surjective proper morphism. Let $\tilde{\mathcal{L}}$ be a $\pi$-big line bundle on $Z$. Then, the pullback line bundle $f^*\tilde{\mathcal{L}}$ is $\tilde{\pi}$-big, where $\tilde{\pi} = \pi \circ f : \tilde{Z} \to \Gamma$.

**Proof.** It suffices to show that rank $\tilde{\pi}^*(f^*\tilde{\mathcal{L}}_k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where $n$ is the dimension of a general fiber of $\tilde{\pi}$.

By [S] Chapter 1, §6.3, Theorem 7, the dimension of a general fiber of $\pi$ and the dimension of a general fiber of $\tilde{\pi}$ are both equal to $\dim Z - \dim \Gamma$.

Now, $\tilde{\pi}^*(f^*\tilde{\mathcal{L}}_k) = \pi^*_j(f^*\tilde{\mathcal{L}}_k) \simeq \pi^*_j(\tilde{\mathcal{L}}_k)$ by the projection formula, since $Z$ is normal. Hence, rank $\tilde{\pi}^*(f^*\tilde{\mathcal{L}}_k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where $n$ is the dimension of a general fiber of $\pi$, since $\tilde{\mathcal{L}}$ is $\pi$-big. This proves the lemma.

**Theorem 8.7.** Assume that $c^{w, u}(j) \neq 0$. Then, for all $i > 0$, we have $R^i\tilde{\pi}_*\omega_{\tilde{Z}}(\partial\tilde{Z}) = 0$ and $R^if_*\omega_{\tilde{Z}}(\partial\tilde{Z}) = 0$, where $\tilde{\pi} : \tilde{Z} \to \tilde{\Gamma}$ is the projection onto the first factor and $f : \tilde{Z} \to Z$ is defined after Lemma 7.2.

**Proof.** Since $f$ is proper and birational by Proposition 7.3, it is surjective. By the proof of Corollary 7.10, the projection $\pi : Z \to \Gamma$ is also surjective. Thus, $\tilde{\pi} = \pi \circ f$ is also surjective. Moreover, $\tilde{\pi}$ is proper since it is the restriction of the projection $\tilde{\Gamma} \times (Z_{w,v})_p \to \tilde{\Gamma}$ to the closed subset $\tilde{\mathcal{Z}}$, where $(Z_{w,v})_p$ is projective. Let $\mathcal{M}$ denote the line bundle $\mathcal{M} := \mathcal{O}_{\tilde{Z}}(\partial\tilde{Z})$ (observe that $\partial\tilde{Z}$ is a divisor in the non-singular scheme $\tilde{Z}$). By Theorem 8.1, it suffices to find a normal crossings divisor $D \subset \tilde{Z}$ such that the line bundle $\mathcal{M}^N(-D)$ is $\tilde{\pi}$-nef, $f$-nef, $\tilde{\pi}$-big, and $f$-big and $D = \sum_{j=1}^\ell a_jD_j$, with $0 < a_j < N$.

By Lemma 8.5, we may choose an ample line bundle $\mathcal{L}$ on $\tilde{\Delta}((X_w)_j)$ with a section with zero set exactly equal to $\tilde{\Delta}(\partial((X_w)_j))$. Let $\varphi : \tilde{\Delta}((Z_w)_j) \to \tilde{\Delta}((X_w)_j)$ denote the desingularization. Since $\tilde{\Delta}((Z_w)_j)$ is a normal crossings divisor in $(Z_{w,v})_p$, it follows that $\partial\tilde{Z} = (\Gamma \times (Z_{w,v})_p) \times (Z_{w,v})_p$ and $\tilde{\Delta}(\partial((Z_w)_j))$ is a normal crossings divisor in $\tilde{Z}$. Write
\[ \partial\tilde{Z} = D_1 + \cdots + D_\ell \]
where $D_i$ are the irreducible components. Since $\tilde{\mu}^*\varphi^*\mathcal{L}$ has a section with zero set precisely equal to $\partial\tilde{Z}$, it follows that
\[ \tilde{\mu}^*\varphi^*\mathcal{L} = \mathcal{O}_{\tilde{Z}}(b_1D_1 + \cdots + b_\ell D_\ell) \]
for some positive integers $b_1, \ldots, b_\ell$.

Let $D$ be the divisor $D := a_1D_1 + \cdots + a_\ell D_\ell$ where $a_i := N - b_i$ for some integer $N$ greater than all the $b_i$'s. Since $\partial\tilde{Z}$ has normal crossings, so does $D$. Then,
\[ \mathcal{M}^\frac{N}{a_i}(-D) = \mathcal{O}_{\tilde{Z}}(b_1D_1 + \cdots + b_\ell D_\ell) = \tilde{\mu}^*\varphi^*\mathcal{L}. \]

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Hence, the ample line bundle $L_{\pi}$ preserves exactness, so we have, for general $\gamma$:

\[ \text{Clearly, } \overline{M}_{\bar{\gamma}} \text{ have an exact sequence } \]

By Theorem 6.2, for general $\gamma$ tensoring with $\tilde{\Delta}$:\

**Proof.** Since $\hat{\pi}$, the fibers of $\tilde{\Delta}((X_w)_j)$ pull back to a $\pi$-big line bundle $\mu^*L$ on $Z$. Now, $\hat{\pi}$-bigness of $M^\hat{\pi}(-D) = \hat{\mu}^*\varphi^*L = f^*\mu^*L$ follows from Lemma 8.6 and Proposition 7.5. $\square$

Theorems 8.3 and 8.4 together with the Grothendieck spectral sequence give

**Corollary 8.8.** For all $i > 0$ we have $R^i\pi_*\omega_Z(\partial Z) = 0$.

## 9 Proof of part b) of Theorem 5.1

Recall the definitions of $N_{\gamma}$ and $M_{\gamma}$ from equations (7) and (8).

We also define

\[ M^{1}_{\gamma} := \gamma((X_{w,v}^u)_{\bar{p}}) \cap \Delta((X_w)_{\partial Y_j}) \]

and

\[ M^{2}_{\gamma} := \gamma((X_{w,v}^u)_{\bar{p}}) \cap \Delta((\partial X_w)_j), \]

so that

\[ M_{\gamma} = M^{1}_{\gamma} \cup M^{2}_{\gamma}. \]

**Lemma 9.1.** For general $\gamma \in \Gamma$, the sheaf $\gamma_*\mathcal{O}_{(X^u \times X^v)_{\bar{p}}} \otimes_{\mathcal{O}_{Y_p}} \left( \hat{\mu}^*(\mathcal{O}_{Y_p}(-\partial Y_j)) \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \right)$ is supported on $N_{\gamma}$ and is equal to the sheaf $\mathcal{O}_{\overline{N}_{\gamma}}(-M_{\gamma})$, where $\hat{p} : Y_p \to \mathbb{P}$ is the projection.

**Proof.** Since $\hat{p} : Y_p \to \mathbb{P}$ is flat, we have a short exact sequence

\[ 0 \to \hat{\mu}^*(\mathcal{O}_{Y_p}(-\partial Y_j)) \to \hat{\mu}^*\mathcal{O}_{Y_p} \to \hat{\mu}^*\mathcal{O}_{\partial Y_j} \to 0. \]

By Lemma 6.5 tensoring with $\hat{\Delta}_s((\xi_w)_p)$ over $\mathcal{O}_{Y_p}$ preserves exactness of the above sequence, so we have an exact sequence

\[ 0 \to \hat{\mu}^*(\mathcal{O}_{Y_p}(-\partial Y_j)) \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \to \hat{\mu}^*\mathcal{O}_{Y_p} \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \to \hat{\mu}^*\mathcal{O}_{\partial Y_j} \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \to 0. \]

By Theorem 6.2 for general $\gamma \in \Gamma$, tensoring the above sequence with $\gamma_*\mathcal{O}_{(X^u \times X^v)_{\bar{p}}}$ over $\mathcal{O}_{Y_p}$ preserves exactness, so we have, for general $\gamma \in \Gamma$, an exact sequence

\[ \gamma_*\mathcal{O}_{(X^u \times X^v)_{\bar{p}}} \otimes_{\mathcal{O}_{Y_p}} \left( \hat{\mu}^*(\mathcal{O}_{Y_p}(-\partial Y_j)) \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \right) \to \\
\gamma_*\mathcal{O}_{(X^u \times X^v)_{\bar{p}}} \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \to \gamma_*\mathcal{O}_{(X^u \times X^v)_{\bar{p}}} \otimes_{\mathcal{O}_{Y_p}} \left( \hat{\mu}^*\mathcal{O}_{\partial Y_j} \otimes_{\mathcal{O}_{Y_p}} \hat{\Delta}_s((\xi_w)_p) \right) \to 0. \]

Next, consider the exact sequence

\[ 0 \to \mathcal{O}_{\overline{N}_{\gamma}}(-M_{\gamma}) \to \mathcal{O}_{\overline{N}_{\gamma}}(-M^2_{\gamma}) \to \mathcal{O}_{M^{1}_{\gamma}(-M^1_{\gamma} \cap M^2_{\gamma})} \to 0. \]
Comparing (12) with (13) we see that it is sufficient to show that
\[ \gamma_* O_{(X^u \times X^v)^p} \otimes O_{Y_p} \left( \hat{p}^* O_{P_j} \otimes O_{Y_p} \Delta_* ((\xi_w)_p) \right) = O_{N, (-M^2_j)} \] (14)
and
\[ \gamma_* O_{(X^u \times X^v)^p} \otimes O_{Y_p} \left( \hat{p}^* O_{\partial P_j} \otimes O_{Y_p} \Delta_* ((\xi_w)_p) \right) = O_{M^1_j \cap M^2_j}. \] (15)

To prove (14) and (15) consider the short exact sequence
\[ 0 \to \Delta_* ((\xi_w)_p) \to \Delta_* (O_{(X_w)_p}) \to \Delta_* (O_{(\partial X_w)_p}) \to 0. \] (16)
Tensor the sequence (16) with \( \hat{p}^* (O_{P_j}) \) over \( O_{Y_p} \), which preserves exactness by the proof of Lemma 4.4 and then tensor with \( \gamma_* O_{(X^u \times X^v)^p} \) over \( O_{Y_p} \), which preserves exactness by Theorem 6.2. For closed subschemes \( X \) and \( Y \) of a scheme \( Z \),
\[ O_X \otimes O_Z O_Y = O_{X \cap Y}. \]
Thus,
\[ \gamma_* O_{(X^u \times X^v)^p} \otimes O_{Y_p} \left( \hat{p}^* (O_{P_j}) \otimes O_{Y_p} \Delta_* (O_{(X_w)_p}) \right) = O_{N, (-M^2_j)} \]
and
\[ \gamma_* O_{(X^u \times X^v)^p} \otimes O_{Y_p} \left( \hat{p}^* (O_{\partial P_j}) \otimes O_{Y_p} \Delta_* (O_{(\partial X_w)_p}) \right) = O_{M^1_j \cap M^2_j}. \]
Thus, (14) follows.

Similarly, (15) follows by tensoring the sequence (16) with \( \hat{p}^* (O_{\partial P_j}) \) over \( O_{Y_p} \), which preserves exactness by the proof of Lemma 4.4 and then tensoring with \( \gamma_* O_{(X^u \times X^v)^p} \) over \( O_{Y_p} \), which preserves exactness by Theorem 6.2. This completes the proof.

By the previous lemma, Theorem 5.1 part b) is equivalent to the following theorem:

**Theorem 9.2.** For general \( \gamma \in \bar{\Gamma} \) and any \( u, v, w \in W, j \in [N]^r \) such that \( c^w_u(j) \neq 0 \) we have
\[ H^p(N, \omega_{N, \gamma}(-M^1_j)) = 0. \]
for all \( p \neq |j| + \ell(w) - \ell(u) - \ell(v) \).

**Proof.** First, the theorem is equivalent to the statement that for general \( \gamma \in \bar{\Gamma} \),
\[ H^p(N, \omega_{N, \gamma}(-M^1_j)) = 0, \forall p > 0. \] (17)
To see this, observe that:
\[ H^p(N, \omega_{N, \gamma}(-M^1_j)) = H^p(N, \mathcal{H} om_{O_{N, \gamma}} (O_{N, \gamma}(-M^1_j), \omega_{N, \gamma})) \]
\[ \simeq Ext^p_{O_{N, \gamma}} (O_{N, \gamma}(-M^1_j), \omega_{N, \gamma}) \]
\[ \simeq H^{n-p}(N, \omega_{N, \gamma}(-M^1_j))^*. \]
where \( n := \dim N = |j| + \ell(w) - \ell(u) - \ell(v) \), the isomorphism \( \varphi_1 \) follows by Corollary \( 7.11 \) and the local to global Ext spectral sequence [Go, Théorème 7.3.3, Chap. II], and the isomorphism \( \varphi_2 \) follows by Corollary \( 7.10 \) and Serre duality [H, Chap. III, Theorem 7.6].

We now prove (17), which implies the theorem. By [S, Chapter I, §6.3, Theorem 1.25] and [H, Chap. III, Exercise 10.9], there is a nonempty open subset \( U \subset \bar{\Gamma} \) such that \( \pi : \pi^{-1}(U) \rightarrow U \) is flat. (Observe that, by the proof of Corollary 7.10, \( \pi \) is surjective.) We prove that \( \omega_Z(\partial Z) \) is flat over \( U \):

To show this, let \( A = \mathcal{O}_U, B = \mathcal{O}_{\pi^{-1}(U)}, \) and \( M = \omega_Z(\partial Z)|_{\pi^{-1}(U)} \). By taking stalks, we immediately reduce to showing that for an embedding of local rings \( A \subset B \) such that \( A \) is regular and \( B \) is flat over \( A \), we have that \( M \) is flat over \( A \). Now, to prove this, let \( \{x_1, \ldots, x_d\} \) be a minimal set of generators of the maximal ideal of \( A \). Let \( K\bullet = K\bullet(x_1, \ldots, x_d) \) be the Koszul complex of the \( x_i \)'s over \( A \). Then, recall that a finitely generated \( B \)-module \( N \) is flat over \( A \) iff \( K\bullet \otimes_A N \) is exact except at the extreme right, i.e., \( H^i(K\bullet \otimes_A N) = 0 \) for \( i < d \) [E, Corollary 17.5 and Theorem 6.8]. Thus, by hypothesis, \( K\bullet \otimes_A B \) is exact except at the extreme right and hence the \( x_i \)'s form a \( B \)-regular sequence [E, Theorem 17.6]. Now, since \( \mathcal{O}_Z \) and \( \mathcal{O}_{\partial Z} \) are CM, we have that \( \mathcal{O}_Z(-\partial Z) \) is a CM \( \mathcal{O}_Z \)-module.

This proves that \( M \) is flat over \( A \), as desired.

Thus, by Corollary 8.8 and the semicontinuity theorem [Ko, Theorem 13.1] to prove (17), it is sufficient to show that for general \( \gamma \in \bar{\Gamma} \), that

\[
\omega_Z(\partial Z)|_{\pi^{-1}(\gamma)} \cong \omega_{N\gamma}(M\gamma). \tag{18}
\]

To prove this, observe that since \( U \) is smooth and \( Z \) and \( \partial Z \) are CM, and the assertion is local in \( U \), it suffices to observe (cf. [H, Corollary 11.35]) that for a nonzero function \( \theta \) on \( U \), the sheaf

\[
S/\theta \cdot S = \mathcal{H}om_{\mathcal{O}_{\mathbb{Z}\theta}}(\mathcal{O}_Z(-\partial Z)/\theta \cdot \mathcal{O}_Z(-\partial Z), \omega_{\mathbb{Z}\theta}),
\]

where \( \mathbb{Z}_{\theta} \) denotes the zero scheme of \( \theta \) in \( Z \) and the sheaf \( S := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z(-\partial Z), \omega_Z) \). Choosing \( \theta \) to be in a local coordinate system and using induction and the above result, the desired conclusion (18) is obtained.
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