RATIONALITY OF EULER-CHOW SERIES AND FINITE GENERATION OF COX RINGS

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To Blaine Lawson on the occasion of his 70th birthday.

1. INTRODUCTION

1.1. Euler-Chow series. Zariski was the first to propose to compute the dimension of linear systems in \( \mathbb{P}^2 \) passing for a fixed number of points in general position with a given multiplicity. There has been a lot of work in that direction since then. We can say that in general it is an important and interesting problem to compute the dimensions of different linear systems. It turns out to be a very hard problem as we will see that this is “uncomputable” in some sense if we have 9 or more general points in \( \mathbb{P}^2 \).

It has also been of great interest to study the use of topological invariants on moduli spaces and mathematical physics in the last two or three decades. In particular we are interested in the class of invariants for projective varieties arising from the Euler characteristic of their Chow varieties.

In the case of the blow up of \( \mathbb{P}^2 \) at a finite number of points the problem posed by Zariski merges with the topological invariant, more precisely, with computing the Euler characteristic of Chow varieties of this variety.

We start by introducing the Euler-Chow series in general and then we see what form it takes in the particular case that we are interested here. We can take any of the equivalence relations we have for cycles; here we take homological equivalence. The reader can look at other cases in [EK2].

In particular it is worth mentioning that the series can be defined in different categories with resulting series generalizing well known series. For instance, in the category of pure motives, modulo some relations, our series is called the Motivic Chow series and generalizes the motivic zeta series of Yves André. It is also interesting to see that in the category of algebraic varieties over a field \( k \), our series takes a form that generalizes the Motivic Zeta series of Kapranov [EK1], and this also generalizes the Motivic Zeta series of André Weil. We shall see that the Euler-Chow series (in codimension 1) is the Hilbert series for \( X \) if \( \text{Pic}(X) = \mathbb{Z} \).

A series that can be defined in different categories and generalizes important series, like the ones mentioned, is a series that can be very interesting...
to define and study in the most general setting where it can be defined. One can guess that much information can be obtained when this is done.

In this paper we consider the Euler-Chow series, whose coefficients are the Euler characteristic of Chow varieties instead of motives or classes in the category of varieties over $k$. Among other things, we are interested in the case where $X$ is the blow up of $\mathbb{P}^2$ at a finite number of points in general position.

For these cases it is also worth saying that there is a relation between the series and the Cox ring, as it will be shown.

The Euler-Chow series has been computed for some examples. In the cases where it is possible to compute, it has turned out to be rational. Once we have a rational function as the generating function, we can compute, with a little algebra, the coefficients that are the Euler characteristic of Chow varieties of the variety where the series is taken place. For the examples we have worked out here, the Euler characteristics of Chow varieties are the dimensions of complete linear systems. In particular we can solve the problem posed by Zariski for all multiplicities when the number of points is less than 9.

**Definition 1.1.** Given a projective variety $X$, let $\lambda$ be a nontorsion element in its 2p-homology group $H_{2p}(X, \mathbb{Z})$. Consider $M$ the monoid in $H_{2p}(X, \mathbb{Z})$ given by algebraic classes of effective cycles. We consider $M$ a multiplicative monoid by $t^a t^b = t^{a+b}$, with $a, b \in M$.

The $p$-dimensional Euler-Chow Series of $X$ is defined as (cf. [E])

$$E_p(X) = \sum_{\lambda \in M} \chi(C_{p,\lambda}(X)) \cdot t^\lambda \in \mathbb{Z}[[M]]$$

where $C_{p,\lambda}(X)$ is the Chow variety parametrizing effective algebraic $p$-cycles homologous to $\lambda$, $\chi(C_{p,\lambda}(X))$ denotes its Euler characteristic, and $\mathbb{Z}[[M]]$ is the ring of functions from $M$ to $\mathbb{Z}$ with the convolution product. In other words, if $f \in \mathbb{Z}[[M]]$ with $f(\lambda) = a_\lambda$ then we can write

$$\mathbb{Z}[[M]] = \left\{ f = \sum_{\lambda \in M} a_\lambda \cdot t^\lambda \mid a_\lambda \in \mathbb{Z} \right\}$$

and the product on $\mathbb{Z}[[M]]$ is the convolution: if $f = \sum_{\lambda \in M} a_\lambda \cdot t^\lambda$ and $g = \sum_{\beta \in M} a_\beta \cdot t^\beta$, then $f \cdot g = \sum_{\delta} \left( \sum_{\lambda+\beta=\delta} a_\lambda \cdot a_\beta \right) \cdot t^\delta$ which is well defined as long as the product operation $\times : M \times M \to M$ has finite fibres. We denote by $\mathbb{Z}[M]$ the ring contained in $\mathbb{Z}[[M]]$ given by the elements with only a finite number of $a_\lambda$ not zero. Equivalently, $\mathbb{Z}[M]$ is the monoid ring associated to $M$.

We say that $f \in \mathbb{Z}[[M]]$ is rational if there are two elements $g, h$ in $\mathbb{Z}[M]$, not all zero, such that $g \cdot f = h$. Similarly, we say that $f \in \mathbb{Z}[[M]]$ is algebraic if there exist $a_0, a_1, \ldots, a_d \in \mathbb{Z}[M]$, not all zero, such that

$$a_0 + a_1 f + \ldots + a_d f^d = 0.$$
If \( f \) is not rational or algebraic, we call \( f \) irrational or transcendental.

Note that the monoid \( M \) has no additive inverse, i.e., \( D_1 + D_2 = 0 \) in \( M \) if and only if \( D_1 = D_2 = 0 \), which is a consequence of \( X \) being projective. Indeed, the projectivity of \( X \) implies that

\[
\bigcap_{l=1}^{\infty} \sum_{k=1}^{l} M^* = \emptyset,
\]

where \( M^* = M \setminus \{0\} \). Therefore, the monoid ring \( \mathbb{Z}[M] \) contains an ideal

\[
I = \left\{ \sum a_D \cdot t^D : D \in M^*, a_D \in \mathbb{Z} \right\}
\]

satisfying \( \cap I^l = \{0\} \) and

\[
\mathbb{Z}[[M]] = \lim_{\leftarrow} \mathbb{Z}[M]/I^l = \widehat{\mathbb{Z}[M]}
\]

is the completion of \( \mathbb{Z}[M] \) along \( I \).

Let us start with a simple example, the case of dimension zero. As before, let \( X \) be a projective variety. Since we are considering elements \( \lambda \) in the zero group of homology, we have that \( \lambda \) must be equal to a nonnegative integer, and it is well known that \( C_{0,d}(X) \) is isomorphic to the \( d \)-fold symmetric product \( SP^d(X) \). In this case, the 0-dimensional Euler-Chow Series is

\[
E_0(X) = \sum_{i=0}^{\infty} \chi(SP^d(X)) \cdot t^d
\]

and a result of Macdonald [M] shows that \( E_0(X) \) is given by rational function \( E_0(X) = (1/(1-t))^{\chi(X)} \).

Another familiar instance arises in the case of divisors. Let \( X \) be a smooth projective variety of dimension \( n \) satisfying \( H^1(O_X) = 0 \). Then \( \text{Pic}_0(X) = \{0\} \) and \( \text{Pic}(X) \) is a subgroup of \( H^2(X, \mathbb{Z}) \) and hence finitely generated. Let \( \text{Div}_+(X) \) be the space of effective divisors on \( X \) and let

\[
(1.2) \quad M = M_X = \text{Div}_+(X)/\sim
\]

be the monoid of effective divisors modulo linear equivalence. Observe that

A.- Given \( L \in \text{Pic}(X) \), then \( \dim H^0(X, L) \neq 0 \) if and only if \( L = O(D) \) for some effective divisor \( D \).

B.- Under the given hypothesis, homological and linear equivalence coincide, and two effective divisors \( D \) and \( D' \) are homologically equivalent if and only if they are in the same linear system. Therefore, \( C_{n-1,\lambda}(X) = \mathbb{P}H^0(X, O(D)) \) and hence \( \chi(C_{n-1,\lambda}(X)) = h^0(X, O(D)) \) with \( [D] = \lambda \).

Thus, the \( (n-1) \)-dimensional Euler-Chow series is

\[
(1.3) \quad E_{n-1}(X) = \sum_{D \in M} h^0(X, O(D)) \cdot t^D := E^1(X)
\]
From now on, we denote by $E^1(X)$ the $(n-1)$-dimensional Euler-Chow series of a projective variety $X$ of dimension $n$ and focus on $E^1(X)$ exclusively. $M = M_X$ always refers to the monoid of effective divisors on $X$.

1.2. **Rationality of Euler-Chow series.** In general, $E^1(X)$ is very hard to compute. It is only computed for some very special varieties $X$ (cf. [ELF]). It seems that in all the cases where $E^1(X)$ is “computable”, it turns out to be a rational function. Rationality of Euler-Chow series has been studied in [EK1], [ES] and [EK2]. The case of abelian varieties was worked out in [EH], and for toric varieties in [E]. It is suspected that it has much to do with the Cox ring of the variety. Indeed, in the known cases including toric varieties and the blow-ups of $\mathbb{P}^2$ at points lying on a line, the Cox ring or the total coordinate ring (cf. [C] and [EKW])

$$\text{Cox}(X) = \bigoplus_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D))$$

is noetherian. We have the following simple fact:

**Fact 1.2.** Let $X$ be a smooth projective variety of dimension $n$ whose Pic($X$) is a finitely generated free abelian group. If Cox($X$) is finitely generated, then $E^1(X)$ is rational.

The first purpose of our paper is to investigate the relation between the rationality of $E^1(X)$ and the finite generation of Cox($X$). Given the above fact, it is natural to ask whether the converse of the above statement holds:

**Question 1.3.** Under the same hypothesis as above, does rationality of $E^1(X)$ imply finite generation of Cox($X$)?

The answer to this question is negative. To construct a counterexample, we need the result of Y. Hu and S. Keel that characterizes a variety with finitely generated Cox ring geometrically [H-K]:

**Theorem (Hu-Keel).** Let $X$ be a smooth projective variety of dimension $n$ whose Pic($X$) is a finitely generated free abelian group. Then Cox($X$) is finitely generated if and only if $X$ is a Mori dream space (MDS).

In order to explain what a MDS is, we need to introduce a few basic concepts in birational geometry:

**Definition 1.4.** Let $\text{NE}_k(X) \subset H^{2n-2k}(X, \mathbb{R})$ be the cone of effective algebraic cycles of dimension $k$ on $X$. That is, it is the smallest closed real cone in $H^{2n-2k}(X, \mathbb{R})$ containing all the effective algebraic cycles of dimension $k$. For convenience, we write $\text{NE}^k(X) = \text{NE}_{n-k}(X)$. So $\text{NE}^1(X)$ is the smallest closed real cone containing all the effective divisors in $H^2(X, \mathbb{R})$. Namely, it is the closure $\text{Conv}(M_X)$ of the convex hull of $M_X$ in $H^2(X, \mathbb{R})$. It is usually called the cone of (pseudo-)effective divisors or effective cone of divisors on $X$. 
The nef cone \( NM^k(X) = NE^k(X)^\vee \) is the dual of \( NE^k(X) \) in \( H^{2k}(X, \mathbb{R}) \). In particular, \( NM^1(X) \subset H^2(X, \mathbb{R}) \) is the smallest closed real cone containing all the numerically effective (nef) divisors and it is a subcone of \( NE^1(X) \) by Kleiman’s criterion.

A divisor \( D \) is semi-ample if the complete linear series \( |mD| \) is base point free for some \( m \in \mathbb{Z}^+ \).

Let \( L \) be a linear system on a smooth projective variety. For a general member \( D \in L \), we can write
\[
D = D_f + D_\mu
\]
as a sum of two effective divisors, where \( D_f \) is the fixed part of \( L \) satisfying \( D_f \subset D' \) for every \( D' \in L \) and \( D_\mu \) is the moving part of \( L \) satisfying \( \dim(D_\mu \cap D') < \dim(D') \) for \( D' \in L \) general. We call a divisor \( D \) movable if \( D_f = \emptyset \) for a general member of \( |D| \). The smallest closed real cone in \( H^2(X, \mathbb{R}) \) containing all the movable divisors is called the movable cone of \( X \) and denoted by \( Mov(X) \).

**Definition 1.5.** A \( \mathbb{Q} \)-factorial projective variety \( X \) is a MDS if

1. The nef cone \( NM^1(X) \) is generated by finitely many semi-ample divisors.
2. There exists a finite collection of birational maps \( f_i : X_i \rightarrow X \) such that \( f_i \) is an isomorphism in codimension one, \( X_i \) is \( \mathbb{Q} \)-factorial, \( NM^1(X_i) \) is generated by finitely many semi-ample divisors and the movable cone \( Mov(X) = \bigcup (f_i)_* NM^1(X_i) \).

All toric and Fano varieties are MDS. So their Euler-Chow series are rational. In the case of toric varieties, explicit computation where made in [1].

For \( X \) to be a MDS, we see that its nef cone \( NM^1(X) \), a priori, has to be rational polyhedral. Another necessary condition for \( X \) to be a MDS is that \( NE^1(X) \) is also rational polyhedral. This is clear if we apply Hu-Keel’s theorem since \( NE^1(X) \) is obviously rational polyhedral if \( Cox(X) \) is finitely generated. We can also see this directly from MD1 and MD2: for an effective divisor \( D \) and a movable divisor \( D' \) on \( X \), \( D + ND' \) is movable for \( N \gg 1 \) if \( D' \) is big when restricted to \( D \); thus, if \( D \notin Mov(X) \), it must be contracted by some \( |D'| \) for \( D' \in Mov(X) \) and then it is easy to see that such \( D' \)'s span a rational polyhedral cone.

For a smooth projective surface \( X \), \( NM^1(X) \) and \( NE^1(X) \) are dual to each other and every movable divisor on \( X \) is nef. Therefore, MD1 is sufficient for surfaces to have finitely generated Cox rings. That is, when \( \dim X = 2 \), \( Cox(X) \) is finitely generated if and only if its nef cone is rational polyhedral and every nef divisor on \( X \) is semi-ample. Our counterexample to Question 1.3 is exactly a smooth projective surface \( X \) with rational polyhedral cones \( NE^1(X) \) and \( NM^1(X) \) and a nef divisor that is not semi-ample.
Theorem 1.6. Let $S$ be a smooth quartic surface in $\mathbb{P}^3$ and $X = \text{Bl}_p S$ be the blow-up of $S$ at a point $p \in S$. Then $\text{Cox}(X)$ is not finitely generated and $E_1(X)$ is rational for $(S, p)$ general.

It has been brought to our attention that this example has already appeared in the work of Artebani and Laface [A-L]. Our argument for the infinite generation of $\text{Cox}(X)$ is identical to theirs. So we do not pretend any originality in these parts. We have kept our argument for the readers’ convenience. However, our proof for the rationality of $E_1(X)$ is new, to the best of our knowledge.

Despite the above counterexample, we still expect that Question 1.3 holds true for a certain class of varieties. We tentatively make the following conjecture:

Conjecture 1.7. Let $X$ be a smooth rationally connected projective variety of dimension $n$. Then $E_1(X)$ is rational if and only if $\text{Cox}(X)$ is finitely generated.

Note that $\text{Pic}(X)$ is automatically finitely generated and free if $X$ is a smooth rationally connected projective variety.

So far we do not have much evidence supporting the conjecture. But in the examples we have where $X$ is rational and $\text{Cox}(X)$ is known to be infinitely generated, we can always prove that $E_1(X)$ is irrational. And these examples are interesting in their own rights.

Our first example is the blow-up $X$ of $\mathbb{P}^2$ at 9 or more points in general position, corresponding to Zariski’s problem mentioned at the very beginning. This is probably the “simplest” surface whose Cox rings are not finitely generated. It has been suspected that its Euler-Chow series is not rational for some time. But the irrationality of $E_1(X)$ has not been established until very recently. Shun-ichi Kimura notified us that there is a paper in preparation where it is proved that $E_1(X)$ is irrational [KKT].

They based their proof on the well-known fact that $\text{NE}_1(X)$ is not a rational polyhedral cone (actually not even a polyhedral cone) for such $X$. To show that $E_1(X)$ is irrational, it suffices to prove the following algebraic result:

Theorem (Kimura-Kuroda-Takahashi). Let $M$ be a submonoid of $\mathbb{Z}^m$ satisfying [11]. Then the cone associated to a series $\sum a_D t^D \in \mathbb{Z}[M]$, i.e., the smallest closed real cone in $\text{Span}_\mathbb{R} M$ containing $\{D : a_D \neq 0\}$, is a rational polyhedron if $\sum a_D t^D$ is rational. Consequently, $\text{NE}_1(X)$ is a rational polyhedron if $E_1(X)$ is rational for a smooth projective variety $X$ of dimension $n$ with $\text{Pic}(X) \cong \mathbb{Z}^m$. In particular, $E_1(X)$ is irrational for the blow-up $X$ of $\mathbb{P}^2$ at 9 or more general points.

We generalize this result in two theorems [1.8] and [1.9]. The proofs in these two theorems are completely different to the one in their theorem, in fact in one of our example their result does not apply to the example, as we will see. Before stating these two theorems we should mention that there
are two corollaries that give general geometric criteria for the series to be transcendental, Corollary 3.3 in page 16 and Corollary 3.2 in page 14, they are stated and proved in section 3.

Now, we are ready to state the two theorems mentioned above

**Theorem 1.8.** For every pair of integers \( p > 1 \) and \( r \geq 3 \), let \( q_0(r, p) \) be the minimal positive integer greater than \( r \) and satisfying Inequality (3.17). Then \( E_1(X) \) is transcendental in the following cases:

1. \( X \) is the blow-up of \(( \mathbb{P}^{r-1} )^{p-1}\) at \( \Lambda \), where \( r \geq 3 \), \( p \geq 2 \), \( \Lambda \) is a finite set of points in \(( \mathbb{P}^{r-1} )^{p-1}\) and contains \( q_0(r, p) \) points in very general position.
2. \( X \) is the blow-up of the product \( \mathbb{P}^{r_1-1} \times \cdots \times \mathbb{P}^{r_p-1} \) at a finite set \( \Lambda \), where \( p \geq 2 \), \( \Lambda \) lies on a linear subspace \(( \mathbb{P}^{r_0-1} )^{p-1}\) with \( 3 \leq r_0 \leq \min_{i=1}^{p}(r_i) \) and contains \( q_0(r_0, p) \) points in very general position as points of \(( \mathbb{P}^{r_0-1} )^{p-1}\).

For the case when \( X \) is the blow up of \( \mathbb{P}^2 \) at \( \Lambda \), we can say that in a way \( E_1(X) \) is uncomputable if \( \Lambda \) consists of \( r \geq 9 \) points in general position. On the other hand, it should be pointed out that \( \text{Bl}_\Lambda \mathbb{P}^2 \) can still be a MDS if the points in \( \Lambda \) are not in general position. For example, if \( \Lambda \) consists of points lying on a line, \( X = \text{Bl}_\Lambda \mathbb{P}^2 \) is a MDS and \( E_1(X) \) has been computed by E. Javier Elizondo and Shun-ichi Kimura in \( [EK2] \) using its motivic version, the Chow motivic series.

Our second theorem is

**Theorem 1.9.** \( E_1(X) \) is transcendental in the following cases:

1. \( X \) is the blow-up of \( \mathbb{P}^2 \) at a finite set \( \Lambda \), where \( \Lambda \) contains the intersection of two general cubic curves.
2. \( X \) is the blow-up of \( \mathbb{P}^3 \) at a finite set \( \Lambda \), where \( \Lambda \) contains the intersection of three general quadrics.
3. \( X \) is the blow-up of \( \mathbb{P}^r \) at a finite set \( \Lambda \), where \( \Lambda \) lies on a linear subspace \( \mathbb{P}^2 \subset \mathbb{P}^r \) containing the intersection of two general cubics.
4. \( X \) is the blow-up of \( \mathbb{P}^r \) at a finite set \( \Lambda \), where \( \Lambda \) lies on a linear subspace \( \mathbb{P}^3 \subset \mathbb{P}^r \) containing the intersection of three general quadrics.

All these cases are known to have infinitely generated Cox rings: (1) is due to S. Mukai \( [Mu1] \), (2) is a variation of (1) due to A. Prendergast-Smith \( [P] \) and (3) and (4) are basically due to B. Hassett and Y. Tschinkel \( [H-T, \text{Example 1.8}] \). Hassett-Tschinkel’s example is of special interest to us. The blow-up \( X \) of \( \mathbb{P}^3 \) at finitely many points \( \Lambda \) lying on a plane \( P \) has rational polyhedral \( \text{NE}_1(X) \), which is obviously generated by the proper transform of \( P \) and the exceptional divisors. On the other hand, \( \text{NM}_1(X) \) is not polyhedral under the hypothesis of Theorem 1.9 \( [H-T] \). Thus, this gives us a smooth projective variety \( X \) with rational polyhedral \( \text{NE}_1(X) \), non polyhedral \( \text{NM}_1(X) \) and hence infinitely generated Cox ring \( \text{Cox}(X) \). The theorem of Kimura-Kuroda-Takahashi cannot be directly applied in this case.
1.3. Euler-Chow series of Del Pezzo surfaces. The second purpose of this paper is to compute $E_1(X)$ for Del Pezzo surfaces. Although it is known that $E_1(X)$ is rational for Del Pezzo surfaces, it is only computed for $X$ the blow-up of $\mathbb{P}^2$ up to 3 points, as these are toric varieties and they were computed in [E]. Here we will try to develop a recursive formula for $E_1(X)$ when $X$ is the blow-up of $\mathbb{P}^2$ at $r \leq 8$ general points and carry out the computation for $r \leq 4$. This computation also involves quadratic transforms, which feature prominently in our proof of Theorem 1.8. An important observation should be said. It is not understood the behavior of the series with respect to blow-ups, we hope to be able to understand this behavior with the computations carried out here.

The following sections are organized as follows: In §2 we prove Theorem 1.6 and state some open questions.

In §3 we state some criteria for the series being transcendental. In particular we prove Corollary 3.5 and Corollary 3.2 which give a geometric criteria for this. We also prove a proposition that shows a nice formula that relate the Euler-Chow series for a birational map between two varieties under certain conditions. It is also proved Theorem 1.8 and Theorem 1.9 which were stated in this introduction. We will go one step further to revisit the Mukai’s counterexamples to Hilbert’s 14th problem and show the irrationality of the corresponding Euler-Chow series in Mukai’s examples.

Finally in §4 we compute the Euler-Chow series in codimension one of Del Pezzo surfaces. The Euler-Chow series for $r \geq 4$ are first computed in this paper, while the cases $1 \leq r \leq 3$ were computed in a different way using toric varieties, see [E].

Here we should mention that it is not understood how the series behave under the blow up. We hope the computation presented here will allow to see this behavior since the computations are based in the geometry of the surface.

It is also important to keep in mind that at the end we are able to compute the Euler characteristic of Chow varieties.

Conventions. We work exclusively over $\mathbb{C}$. In the rest of the paper if $X$ is a variety of dimension $n$, then $E_{n-1}(X)$ sometimes will be denoted by $E^1(X)$.

2. Blow-ups of Quartic $K3$

2.1. Proof of Theorem 1.6. Let $L$ be the hyperplane divisor on $S$ and $C \in |L|$ be the curve cut out by the tangent plane of $S$ at $p$. Then $C$ is a quartic plane curve with exactly one node for $p \in S$ general. Let $\tilde{C} \subset X$ be the proper transform of $C$ under the blow-up $\pi : X \to S$. Obviously, $\tilde{C} = \tilde{L} - 2E$, where $\tilde{L} = \pi^*L$ and $E \subset X$ is the exceptional divisor of $\pi$.

Since $\tilde{C}$ is irreducible and $\tilde{C}^2 = 0 \geq 0$, $\tilde{C}$ is nef. Indeed, it is easy to see that $NM^1(X)$ is generated by $\tilde{C}$ and $\tilde{L}$ and $NE^1(X)$ is generated by $\tilde{C}$ and
We claim that $\hat{C}$ is not semi-ample. That is, $h^0(\mathcal{O}_X(n\hat{C})) = 1$ for all $n \in \mathbb{Z}^+$. From the exact sequence
\begin{equation}
0 \to H^0(\mathcal{O}_X((n - 1)\hat{C})) \to H^0(\mathcal{O}_X(n\hat{C})) \to H^0(\mathcal{O}_{\hat{C}}(n\hat{C})),
\end{equation}
we see that $h^0(\mathcal{O}_X(n\hat{C})) = 1$ as long as
\begin{equation}
H^0(\mathcal{O}_{\hat{C}}(n\hat{C})) = 0
\end{equation}
for all $n \in \mathbb{Z}^+$. Note that
\begin{equation}
E + \hat{C} = K_X + \hat{C} = K_{\hat{C}}
\end{equation}
in $\text{Pic}(\hat{C})$ by adjunction, where $K_X$ and $K_{\hat{C}}$ are the canonical divisors of $X$ and $\hat{C}$, respectively. Therefore,
\begin{equation}
\hat{C} = K_{\hat{C}} - E = K_{\hat{C}} - q_1 - q_2
\end{equation}
in $\text{Pic}(\hat{C})$, where $q_1$ and $q_2$ are the two points on $\hat{C}$ over $p$. Therefore, (2.2) holds as long as $K_{\hat{C}} - q_1 - q_2$ is non torsion in $\text{Pic}(\hat{C})$.

**Lemma 2.1.** For a general quartic $K3$ surface $S$ and a general point $p \in S$, $K_{\hat{C}} - q_1 - q_2$ is non torsion.

**Proof.** We fix a plane $\Lambda \in \mathbb{P}^3$ and consider $W \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$ consisting of all quartic surfaces $S$ tangent to $\Lambda$. Obviously, we have a dominant rational map $W \dashrightarrow V_{4,2}$ sending $S$ to $S \cap \Lambda$, where $V_{d,g}$ is the Severi variety parametrizing nodal plane curves of degree $d$ and genus $g$. And $V_{4,2}$ in turn dominates the moduli space of genus 2 curves with two unmarked points via the map sending $C$ to $(\hat{C}, q_1, q_2)$, where $\hat{C}$ is the normalization of $C$ and $q_1$ and $q_2$ are the two points on $\hat{C}$ over the node $p \in C$. In summary, we have dominant maps
\begin{equation}
W \dashrightarrow V_{4,2} \to \mathcal{M}_{2,2}//\Sigma_2
\end{equation}
where $\mathcal{M}_{g,n}$ is the moduli space of genus $g$ curves with $n$ marked points and its quotient by the symmetric group $\Sigma_n$ on the $n$ marked points is the moduli space of genus $g$ curves with $n$ unmarked points. Obviously, $K_{\hat{C}} - q_1 - q_2$ is non torsion for a general point $(\hat{C}, q_1, q_2)$ of $\mathcal{M}_{2,2}$. \qed

Therefore, $\hat{C}$ is nef and not semi-ample and $X$ is not a MDS. It follows that $\text{Cox}(X)$ is not finitely generated by the theorem of Hu-Keel. However, its Euler-Chow series $E_1(X)$ can be explicitly computed as follows and it turns out to be rational.

We write
\begin{equation}
E_1(X) = \sum_{a,b \geq 0} h^0(a\hat{C} + bE)t_1^at_2^b
\end{equation}
\begin{equation}
= \left( \sum_{a \geq 0} + \sum_{b > a} + \sum_{2a > b} + \sum_{2a < b} \right) h^0(a\hat{C} + bE)t_1^at_2^b
\end{equation}
where $t_1 = t^\hat{C}$ and $t_2 = t^E$.

We have proved that $h^0(a\hat{C}) = 1$ for $a \geq 0$. Hence

\[ h^0(a\hat{C} + bE) = 1 \quad \text{when } a = 0 \text{ or } b = 0. \]

And it is trivial that

\[ h^0(a\hat{C} + bE) = h^0(a\hat{L}) = 2a^2 + 2 \quad \text{when } b \geq 2a > 0. \]

When $2a > b > 0$, we have

\[ h^0(a\hat{C} + bE) - h^1(a\hat{C} + bE) + h^2(a\hat{C} + bE) = 2ab - a - \frac{b(b-1)}{2} + 2 \]

by Riemann-Roch. We have the vanishing

\[ h^2(a\hat{C} + bE) = h^0(-a\hat{C} - (b-1)E) = 0 \]

since $(-a\hat{C} - (b-1)E)\hat{L} < 0$ as long as $a > 0$. Also $h^1(a\hat{C} + bE) = 0$ for $2a > b > 1$ since $a\hat{C} + (b-1)E$ is ample in this case. When $2a > b = 1$, we have

\[ H^1(O_X((a-1)\hat{C} + E)) \to H^1(O_X(a\hat{C} + E)) \to H^1(O_{\hat{C}}(a\hat{C} + E)) \to H^2(O_X((a-1)\hat{C} + E)) \to 0, \]

where $h^1(O_{\hat{C}}(a\hat{C} + E)) = h^0(O_{\hat{C}}((1-a)\hat{C}))$. Note that $h^0(O_{\hat{C}}((1-a)\hat{C})) = 0$ for $a \neq 1$ by Lemma 2.1. When $a = 1$, (2.11) becomes

\[ 0 \to H^1(O_X(\hat{C} + E)) \to H^1(K_{\hat{C}}) \to H^2(O_X(E)) \to 0 \]

and hence $H^1(\hat{C} + E) = 0$. Then $H^1(a\hat{C} + E) = 0$ for all $a > 0$ by induction using (2.11). In conclusion, $h^1(a\hat{C} + bE) = h^2(a\hat{C} + bE) = 0$ and hence

\[ h^0(a\hat{C} + bE) = 2ab - a - \frac{b(b-1)}{2} + 2 \quad \text{when } 2a > b > 0. \]

**Remark 2.2.** Even without Hu-Keel’s theorem, we can directly see that $\text{Cox}(X)$ is not finitely generated by this computation. Setting $b = 1$ in (2.13), we obtain

\[ h^0(a\hat{C} + E) = a + 2 \]

for all $a \geq 1$. It follows that the map

\[ H^0(\hat{C}) \otimes H^0((a-1)\hat{C} + E) \to H^0(a\hat{C} + E) \]
RATIONALITY OF EULER-CHOW SERIES

is not surjective and hence there exists an irreducible curve \( D_a \in (a\hat{C} + E) \) for each \( a \geq 1 \). The ideal generated by \( \{ D_a : a \in \mathbb{Z}^+ \} \subset \text{Cox}(X) \) is obviously not finitely generated.

Combining (2.6), (2.7), (2.8) and (2.13), we can compute \( E_1(X) \). Although the computation is not hard, we are not going to carry it out as it is not very inspiring. All we need for Theorem 1.6 is to show that \( E_1(X) \) is a rational function. For this purpose, we simply write

\[
E_1(X) = \sum_i \sum_{(a, b) \in \mathbb{N}^2 \cap \mathbb{Z}^2} P_i(a, b) t^a t^b
\]

where \( N_i \) are a finite collection of closed rational polyhedral cones in \( \mathbb{R}^2 \), \( N_i \cap \mathbb{Z}^2 \) are the lattice points contained in \( N_i \) and \( P_i(a, b) \) are polynomials in \( a \) and \( b \). Here we allow \( N_i \) to be degenerated, i.e., to be contained in a linear subspace. For example, the last term of (2.6) can be written as

\[
\sum_{2a \geq b > 0} - \sum_{2a = b \geq 0} - \sum_{2a > b \geq 0} + \sum_{a = b = 0}
\]

Therefore, the rationality of \( E_1(X) \) follows if we can show

**Proposition 2.3.** For a closed rational polyhedral cone \( N \) in \( \mathbb{R}^n \) and a polynomial \( P(t) \in \mathbb{Z}[t_1, t_2, ..., t_n] \), the series

\[
\sum_{D \in N \cap \mathbb{Z}^n} P(D) t^D \in \mathbb{Z}[[M]]
\]

is rational, where \( t^D = t_1^{d_1} t_2^{d_2} ... t_n^{d_n} \) for \( D = (d_1, d_2, ..., d_n) \) and \( M \) is a submonoid of \( \mathbb{Z}^n \) containing \( N \cap \mathbb{Z}^n \) and satisfying (1.1).

The way we prove Proposition 2.3 also gives an algorithm to compute the series (2.18), which we will need later for the computation of Euler-Chow series of Del Pezzo surfaces.

First, for each \( P(t) \in \mathbb{Z}[t_1, t_2, ..., t_n] \), there exists a differential operator

\[
Q = \sum_{i=1}^{m} f_i(t) \frac{\partial^{D_i}}{\partial t^{D_i}}
\]

such that

\[
\sum_{D \in N \cap \mathbb{Z}^n} P(D) t^D = Q \left( \sum_{D \in N \cap \mathbb{Z}^n} t^D \right)
\]

where \( f_i(t) \in \mathbb{Z}[t_1, t_2, ..., t_n], D_i \in \mathbb{N}^n \) and

\[
\frac{\partial^D}{\partial t^D} = \frac{\partial^{d_1 + d_2 + ... + d_n}}{\partial t_1^{d_1} \partial t_2^{d_2} ... \partial t_n^{d_n}}
\]

for \( D = (d_1, d_2, ..., d_n) \). Therefore, to show the rationality of (2.18), it suffices to show that of

\[
\sum_{D \in N \cap \mathbb{Z}^n} t^D.
\]
Lemma 2.4. Suppose that \( N \subset \mathbb{R}^n \) is a closed rational simplicial cone of dimension \( m \leq n \), i.e., it is generated by \( m \) linearly independent rational vectors \( \{v_1, v_2, \ldots, v_m \in \mathbb{Q}^n\} \). Then (2.22) is a rational function in \( \mathbb{Z}[[M]] \) with \( M \) a submonoid of \( \mathbb{Z}^n \) containing \( N \cap \mathbb{Z}^n \) and satisfying (1.1).

Proof. After replacing \( v_i \) by \( \lambda v_i \in \mathbb{Z}^n \) for some \( \lambda \in \mathbb{Z}^+ \), we may assume \( v_i \in \mathbb{Z}^n \). Let

\[
\Sigma_N = \mathbb{Z}^n \cap \left\{ \sum_{i=1}^{m} a_i v_i : 0 \leq a_i < 1 \right\}.
\]

Clearly, \( \Sigma_N \) is a finite set and every \( v \in N \cap \mathbb{Z}^n \) can be uniquely written as

\[
v = w + \sum_{i=1}^{m} \lambda_i v_i
\]

for some \( w \in \Sigma_N \) and \( \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{N} \). Thus

\[
\sum_{D \in N \cap \mathbb{Z}^n} t^D = \left( \sum_{w \in \Sigma_N} t^w \right) \prod_{i=1}^{m} \frac{1}{1 - t v_i}
\]

is a rational function. \( \square \)

To show that (2.22) is rational for an arbitrary rational polyhedral cone \( N \), it suffices to subdivide \( N \) into a finite union of simplicial cones which meet along faces [S]. This proves Proposition 2.3 and hence \( E_1(X) \) is rational for a general pair \((S, p)\).

2.2. Some further comments. If the pair \((S, p)\) fails to be general, the corresponding \( \text{Cox}(X) \) might still be finitely generated. It is interesting to study how \( \text{Cox}(X) \) and \( E_1(X) \) vary as \((S, p)\) does. To set this up, let us consider

\[
\begin{aligned}
B &= \{(S, p) : S \text{ is a smooth quartic surface with } \text{Pic}(S) = \mathbb{Z}, \\
p &\in S \text{ is a point such that the tangent plane of } S \text{ at } p \text{ cuts out}
\end{aligned}
\]

on \( S \) a curve \( C \in |L| \) with a single node\} \subset |\mathcal{O}_{p3}(4)| \times \mathbb{P}^3

and the universal family \( \mathcal{S} = \{(S, p, q) : q \in S\} \subset B \times \mathbb{P}^3 \) over \( B \).

Clearly, \( S/B \) has a section \( P \) given by the map \( B \to S \) sending \((S, p)\) to \((S, p, p)\). Let \( \mathcal{X} \) be the blow-up of \( S \) along \( P \). Obviously, at each point \( b = (S, p) \in B \), the fiber \( \mathcal{X}_b \) of \( \mathcal{X}/B \) at \( b \) is exactly the blow-up \( \text{Bl}_p S \).

Question 2.5. What is the set \( \Delta_M = \{b \in B : X_b \text{ is a MDS}\} \) in \( B \)? Is it Zariski closed in \( B \)?

It is tempting to think that \( \Delta_M \) consists of \((S, p)\) with the corresponding \( K_C - q_1 - q_2 \in \text{Pic}(C)_{\text{tors}} \). This, however, is unlikely to be true by a naive dimension count: the subvariety

\[
\{(S, p) \in B : K_C - q_1 - q_2 \text{ is an n-torsion}\}
\]
has codimension 2 in $B$ while the subvariety
\[(2.28) \quad \Delta_{M,n} = \{(S, p) \in B : h^0(a\tilde{C}) = 1 \text{ for } 0 \leq a < n, h^0(n\tilde{C}) > 1\}\]
has negative expected dimension for $n$ sufficiently large.

It is clear that $\Delta_M$ is the union of $\Delta_{M,n}$ and hence, a priori, is a countable union of subvarieties of $B$.

Likewise, we want to know how $E_1(X_b)$ varies:

**Question 2.6.** Is $E_1(X_b)$ a rational function for all $b \in B$?

3. Transcendental Euler-Chow Series

3.1. Transcendence criteria. We will obtain our first transcendence criterion based upon the following algebraic result.

To make the statement as general as possible, we work with $R[M]$ and $R[[M]]$ instead of $\mathbb{Z}[M]$ and $\mathbb{Z}[[M]]$ for an arbitrary integral domain $R$. The rationality and algebraicity of $f(t) \in R[[M]]$ are defined in an obvious way.

**Proposition 3.1.** Let $M$ be a submonoid of $\mathbb{Z}^m$ satisfying \((1.1)\), $J$ be a subset of $M$ and $R$ be an integral domain. Suppose that there is a collection \(\{\delta_{\alpha} \in \text{Hom}_\mathbb{Z}(\mathbb{Z}^m, R) : \alpha \in A\}\) satisfying
\begin{itemize}
  \item the minimum
  \begin{equation}
  \varepsilon_{\alpha} = \min_{D \in J} \delta_{\alpha}(D)
  \end{equation}
  exists for every $\alpha \in A$;
  \item \(\{J_{\alpha} : \alpha \in A\}\) is an infinite set, where
  \begin{equation}
  J_{\alpha} = \{D \in J : \delta_{\alpha}(D) = \varepsilon_{\alpha}\}
  \end{equation}
  for $\alpha \in A$.
\end{itemize}

Then
\begin{equation}
  f(t) = \sum_{D \in J} a_D t^D \in R[[M]]
\end{equation}
is transcendental as long as $a_D \neq 0$ for every $D \in J$.

**Proof.** Basically, each $\delta \in \text{Hom}_\mathbb{Z}(\mathbb{Z}^m, R)$ makes $R[M]$ into a graded ring by
\begin{equation}
  R[M] = \bigoplus_{d \in \mathbb{R}} \left( \bigoplus_{\delta(D) = d} R t^D \right).
\end{equation}
We can call $\delta(D)$ the weight of $t^D$ under this grading.

Let
\begin{equation}
  f_{\alpha}(t) = \sum_{D \in J_{\alpha}} a_D t^D \in R[[M]].
\end{equation}
Since \(\{J_{\alpha} : \alpha \in A\}\) is an infinite set, the set \(\{f_{\alpha}(t) : \alpha \in A\} \subset R[[M]]\) is also infinite since $a_D \neq 0$ for all $D \in J$. 

Suppose that \( f(t) \) is algebraic. Then there exists a nonzero polynomial \( F(t, x) \in R[M, x] = R[M][x] \) such that \( F(t, f(t)) = 0 \) in \( R[[M]] \). We write

\[
F(t, x) = \sum_{D \in M} \sum_{k=0}^{\infty} b_{D,k} t^D x^k
\]

where \( b_{D,k} \in R \) vanishes outside of a finitely many pairs \((D, k)\).

Let \( \Pi \) be a subset of \( R[M, x] \) given by

\[
\Pi = \left\{ G(t, x) = \sum_{D \in M} \sum_{k=0}^{\infty} c_{D,k} t^D x^k : \quad c_{D,k} = b_{D,k} \text{ or } c_{D,k} = 0 \text{ for each pair } (D, k) \right\}.
\]

(3.7)

Obviously, \( \Pi \) is a finite set.

For each \( \alpha \in A \), we let

\[
\mu_{\alpha} = \min_{b_{D,k} \neq 0} \left( \delta_{\alpha}(D) + k \varepsilon_{\alpha} \right)
\]

and

\[
G_{\alpha}(t, x) = \sum_{\delta_{\alpha}(D) + k \varepsilon_{\alpha} = \mu_{\alpha}} b_{D,k} t^D x^k.
\]

(3.9)

Obviously, \( G_{\alpha}(t, x) \neq 0 \), \( G_{\alpha}(t, x) \in \Pi \) and we see that \( G_{\alpha}(t, f_{\alpha}(t)) = 0 \) by collecting the terms of \( F(t, f(t)) \) of the lowest weight \( \mu_{\alpha} \) under the grading given by \( \delta_{\alpha} \).

Since \( \Pi \) is finite and \( \{ f_{\alpha}(t) \} \) is infinite, there exists \( G(t, x) \neq 0 \in \Pi \) such that \( G(t, g(t)) = 0 \) for infinitely many different \( g(t) \in R[[M]] \). In other words, the polynomial \( G(t, x) \) has infinitely many roots in \( R[[M]] \). Obviously, this is impossible for an integral domain \( R[[M]] \). This proves that \( f(t) \) is transcendental. \( \square \)

**Corollary 3.2.** Let \( X \) be a smooth projective variety of dimension \( n \) with \( \text{Pic}(X) \cong \mathbb{Z}^m \). If there are infinitely effective divisors \( D \in \text{Pic}(X) \) each generating an extreme ray of the effective cone \( \text{NE}^1(X) \) of \( \text{Pic}(X) \), then \( E^1(X) \) is transcendental. Moreover, for every smooth projective variety \( Y \) that dominates \( X \) via a birational regular map \( \pi : Y \to X \), \( E^1(Y) \) is transcendental.

**Proof.** Assume that \( A \) is the set of all effective classes in \( M \) that generate extreme rays of \( \text{NE}^1(X) \). Since \( \text{NE}^1(X) \) is strongly convex, there exists \( \delta_{\alpha} \in \text{NM}^{n-1}(X) \) for each \( \alpha \in A \), such that \( \delta_{\alpha}(D) \geq 0 \) for all \( D \in \text{NE}^1(X) \) and \( \delta_{\alpha}(D) = 0 \) if and only if \( D \) lies on the ray \( [\alpha] \) generated by \( \alpha \). The corresponding \( \varepsilon_{\alpha} \) and \( J_{\alpha} \) defined by (3.1) and (3.2) are exactly \( \varepsilon_{\alpha} = 0 \) and \( J_{\alpha} = [\alpha] \cap M \). Here we are trying to apply Proposition 3.1 with \( J = M \).

Obviously, \( \{ J_{\alpha} : \alpha \in A \} \) is an infinite set. Consequently,

\[
\sum_{D \in M} a_D t^D
\]

(3.10)
is transcendental provided that \( a_D \neq 0 \) for all \( D \in M \). It follows that \( E^1(X) \) is transcendental.

For \( Y \) dominating \( X \) via a birational regular map \( \pi : Y \to X \), it is enough to apply the same argument as above with \( \pi^*\delta_\alpha \).

In the case that \( \pi : Y \to X \) is a composite of blow-ups at finitely many points, with the following lemma, we can also deduce the transcendence of \( E^1(Y) \) from that of \( E^1(X) \).

**Lemma 3.3.** Let \( \pi : Y \to X \) be the blow-up of a projective variety \( X \) of dimension \( \geq 2 \) at a point \( P \) and \( \text{Pic}(X) \simeq \mathbb{Z}^m \). Let \( D \subset X \) be an effective divisor and \( \tilde{D} \) be its proper transform under \( \pi \). Assume that \( D \) lies on an extreme ray of the effective cone \( \text{NE}^1(X) \) of \( \text{Pic}(X) \) and \( h^0(D) = 1 \). Then \( \tilde{D} \) also lies on an extreme ray of \( \text{NE}^1(Y) \) and \( h^0(\tilde{D}) = 1 \).

**Proof.** Let \( U = X \setminus \{P\} \) and \( j : U \to Y \) be the inverse of \( \pi \). Assume that \( \tilde{D} \) is linearly equivalent to a sum of two effective divisors \( \tilde{D}_1 + \tilde{D}_2 \). It suffices to show that \( \tilde{D}_1 \) and \( \tilde{D}_2 \) are linearly dependent in \( \text{Pic}(Y) \) and \( \tilde{D} = \tilde{D}_1 + \tilde{D}_2 \) as divisors.

First, \( j^{-1}(\tilde{D}) \sim j^{-1}(\tilde{D}_1) + j^{-1}(\tilde{D}_2) \). Let \( D_i \) be the Zariski closure in \( X \) of \( j^{-1}(\tilde{D}_i) \) for \( i = 1, 2 \). Thus \( D \sim D_1 + D_2 \). As \( h^0(D) = 1 \), thus as divisors \( D = D_1 + D_2 \). Moreover, as \( D \) lies on an extreme ray of \( \text{NE}^1(X) \), take \( D_0 \) to be the generator of the extreme ray, thus \( h^0(D_0) = 1 \) and as divisors \( D_i = \lambda_i D_0 \) for \( i = 1, 2 \), where \( \lambda_i \in \mathbb{Z}^{\geq 0} \).

Let \( \tilde{D}_0 \) be the proper transform of \( D_0 \) under \( \pi \). Clearly \( \tilde{D} = (\lambda_1 + \lambda_2)\tilde{D}_0 \) and \( \tilde{D}_i - \lambda_i \tilde{D}_0 \geq 0 \). As \( \tilde{D} \sim \tilde{D}_1 + \tilde{D}_2 \), thus \( \tilde{D}_i = \lambda_i \tilde{D}_0 \) for \( i = 1, 2 \), and \( \tilde{D} = \tilde{D}_1 + \tilde{D}_2 \).

**Remark 3.4.** The condition \( h^0(D) = 1 \) is dispensable.

Based upon Proposition 3.1 we can deduce another criterion with the following observation: for an arbitrary nonzero effective divisor \( F \) of \( X \),

\[
E^1(X) \text{ is transcendental} \iff (1 - t^F)E^1(X) \text{ is transcendental}.
\]

Note that

\[
(3.11) \quad (1 - t^F)E^1(X) = (1 - t^F) \sum h^0(D)t^D = \sum (h^0(D) - h^0(D - F))t^D,
\]

thus the nonzero terms \( t^D \) of \( (1 - t^F)E^1(X) \) satisfy that \( h^0(D) > h^0(D - F) \).

For each effective divisor \( F \neq 0 \) on \( X \), let \( L_F \subset M \) be the submonoid

\[
(3.12) \quad L_F = \{ D \in \text{Pic}(X) : h^0(X, D) > h^0(X, D - F) \}.
\]

Or equivalently, \( L_F \) consists of effective divisors \( D \) such that a general member of \( |D| \) meets \( F \) properly and hence \( |D| \) does not have \( F \) in its fixed part.

If the cone \( \text{Conv}(L_F) \subset H^2(X, \mathbb{R}) \) has infinitely many extreme rays generated by classes in \( L_F \), then we can apply Proposition 3.1 similarly to the
proof of Corollary [3.2] to conclude that
\[(3.13) \sum_{D \in L_F} a_D t^D \]
is transcendental provided that \(a_D \neq 0\) for all \(D \in L_F\). It follows that \((1 - t^F)E^1(X)\) and hence \(E^1(X)\) are transcendental.

**Corollary 3.5.** Let \(X\) be a smooth projective variety of dimension \(n\) with \(\text{Pic}(X) \cong \mathbb{Z}^m\). If there is an effective divisor \(F\) on \(X\) such that there are infinitely many \(E \in L_F\) each generating an extreme ray of the cone \(\text{Conv}(L_F)\), then \(E^1(X)\) is transcendental. Moreover, for every smooth projective variety \(Y\) that dominates \(X\) via a birational regular map \(Y \to X\), \(E^1(Y)\) is transcendental.

In some special cases to be shown as follows, it is even true that
\[(1 - t^F)E^1(X) = E^1(F).\]

**Proposition 3.6.** Let \(X\) be the blow-up of \(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_p}\) at a finite set \(\Lambda\) of points, where \(r_i \geq 2, \forall i\). Assume that \(\Lambda\) lies on a linear subspace \(P\) of codimension 1, i.e. the pull-back of some hyperplane \(H_{i_0} \subset \mathbb{P}^{r_{i_0}}\). Let the closed immersion \(i : \widehat{P} \subseteq X\) be the proper transform of \(P\). Then
\[(3.14) (1 - t^{\widehat{P}})E^1(X) = E^1(\widehat{P}).\]

In particularly, under the natural isomorphism \(\text{Pic}(X) \cong \text{Pic}(\widehat{P})\), the monoid \(L_{\widehat{P}} = \{D \in \text{Pic}(X) : h^0(X, D) > h^0(X, D - \widehat{P})\}\), is isomorphic to the effective monoid \(\text{NE}^1(\widehat{P})\). Moreover, for every \(D \in L_{\widehat{P}}\),
h^0(\widehat{P}, i^*O(D)) = h^0(X, D) - h^0(X, D - \widehat{P}).

The proof of Proposition [3.6] is based upon the following facts.

**Fact 3.7.** Let \(r_i \geq 2\) for \(1 \leq i \leq p\) and \(X\) be the blow-up of \(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_p}\) at a finite set \(\Lambda = \{P_1, \ldots, P_n\}\) giving exceptional divisors \(E_1, \ldots, E_n\). Assume that \(H_i\) is the pull-back of the hyperplane class of \(\mathbb{P}^{r_i}\). Then
\[(3.15) h^0(X, \sum_{i=1}^{p} a_i H_i + \sum_{j=1}^{n} b_j E_j) = h^0(X, \sum_{i=1}^{p} a_i H_i + \sum_{j=1}^{n} b_j \frac{|b_j|}{2} E_j);\]
Let \(b_j \geq 0\), the space \(H^0(X, \sum_{i=1}^{p} a_i H_i - \sum_{j=1}^{n} b_j E_j)\) can be identified with
\[(3.16) \{s \in \otimes_{i=1}^{p} H^0(\mathbb{P}^{r_i}, a_i H_i) \mid \text{mult}(s, P_j) \geq b_j, \forall j\}.\]

**Proof of Proposition [3.6]** We may assume that \(P\) is the pull-back of a hyperplane defined by \(x_{r_1} = 0\) of \(\mathbb{P}^{r_1}\).

Assume that \(D \in L_{\widehat{P}}\). Choose \(s \in H^0(X, D) \setminus H^0(X, D - \widehat{P})\), clearly \(i^*s\) is a nonzero section of \(H^0(\widehat{P}, i^*O(D))\), thus \(i^*O(D) \in \text{Eff}(\widehat{P})\). Conversely, assume that \(i^*O(D) \in \text{Eff}(\widehat{P})\). As sections of \(D\) and \(i^*O(D)\) can be identified
with polynomials of multiple degrees in $\delta_i$, thus for $\forall s \in H^0(\hat{\mathbb{P}}, i^*\mathcal{O}(D))$, there exists a canonical section $s_1 \in H^0(X, D)\setminus H^0(X, D - \hat{\mathbb{P}})$ such that $i^*s_1 = s$. Thus $D \in L_{\hat{\mathbb{P}}}$.

To prove the equation, we assume that $D = \sum_{i=1}^{p} a_i H_i + \sum_{j=1}^{n} b_j E_j \in L_{\hat{\mathbb{P}}}$.

There are three cases: (1) $a_i = 0, \forall i$; (2) $\exists a_i > 0$ for some $i$ and some of the $b_j$’s are positive; (3) $\exists a_i > 0$ for some $i$ and $b_j \leq 0, \forall j$. In Case (1), $h^0(D) > 0 \implies b_j \geq 0, \forall j$ and the proposition holds trivially. Case (2) can be reduced to Case (3) by Equation (3.15).

Thus it suffices to prove the case that $D = \sum_{i=1}^{p} a_i H_i - \sum_{j=1}^{n} b_j E_j$, where $\forall a_i, b_j \geq 0$. Identifying $H^0(X, D)$ with the space of multi-graded homogeneous polynomials in $p = 1$, then

$$H^0(X, D - \hat{\mathbb{P}}) \simeq \{s \in H^0(X, D) \mid x_1 \text{ is a factor of } s\}$$

$$\frac{H^0(X, D)}{H^0(X, D - \hat{\mathbb{P}})} \simeq \{s \in \bigotimes_{i=1}^{p} H^0(\mathbb{P}^{p-1}; a_i H_i) \mid \mult(s, P_j) \geq b_j, \forall j\},$$

where $\delta_{i1} = 1$ and $\delta i_1 = 0$ if $i \neq 1$. Obviously the latter is isomorphic to $H^0(\hat{\mathbb{P}}, i^*\mathcal{O}(D))$ by applying (3.16) again. \qed

3.2. Mukai’s construction and the generalization. In [Mu2], Mukai has constructed a family of smooth projective varieties with infinitely generated Cox ring; by establishing an isomorphism between the Cox ring of these varieties and the invariant ring of an action of Nagata type, he thus obtained a family of counterexamples to Hilbert’s 14th problem.

[Mu2] Theorem 3] can be reformulated as follows:

**Theorem** (Mukai). Let $r > 2$ and $X$ be the blow-up of $(\mathbb{P}^{p-1})^{p-1}$ at $q > r$ points in very general position. Assume that

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{q - r} \leq 1.$$  \hspace{1cm} (3.17)

Then $\text{Cox}(X)$ is infinitely generated. When $p = 2$, it is the result in [Mu1].

By “$n$ points $\{P_1, \cdots, P_n\} \in \mathbb{P}^{r-1}$ in very general position”, we mean that any $r$ points of $\{P_1, \cdots, P_n\}$ after any finite sequence of Cremona transformations span $\mathbb{P}^{r-1}$. Here a Cremona transformation is a birational map of the form $\sigma_1 \circ \Psi \circ \sigma_2$, where $\sigma_1, \sigma_2 \in \text{Aut}(\mathbb{P}^{r-1})$, and

$$\Psi : \mathbb{P}^{r-1} \dasharrow \mathbb{P}^{r-1}, \hspace{1cm} (x_1, \cdots, x_r) \mapsto \left(\frac{1}{x_1}, \cdots, \frac{1}{x_r}\right).$$

By “$n$ points $\{P_1, \cdots, P_n\} \in (\mathbb{P}^{p-1})^{p-1}$ in very general position”, we mean that for $1 \leq i \leq p - 1$, the $i^{th}$ components $\{P_1^{(i)}, \cdots, P_n^{(i)}\} \in \mathbb{P}^{r-1}$ are in very general position.

Note that a key fact in the proof of [Mu2] Theorem 3] is [Mu1] Lemma 3], which says that exceptional divisors are dispensable as generators of the Cox ring. We observe that the proof of [Mu1] Lemma 3] implies that exceptional divisors generate extreme rays of the effective cone of $X$. Thus the proof of
Theorem 3.2 implies that the effective cone of $X$ has infinitely many extreme rays. Therefore, $E^1(X)$ is transcendental by Corollary 3.2.

**Corollary 3.8.** Let $r > 2$ and $X$ be the blow-up of $(\mathbb{P}^{r-1})^{p-1}$ at $q$ points in very general position, where $p, q, r$ satisfy Inequality (3.17). Then the effective cone of $X$ has infinitely many extreme rays and the Euler-Chow series $E^1(X)$ is transcendental.

With Corollary 3.2 and Proposition 3.6, Corollary 3.8 can be generalized in two directions as follows.

**Theorem (Theorem 1.8).** For every pair of integers $p > 1$ and $r \geq 3$, let $q_0(r, p)$ be the minimal positive integer greater than $r$ and satisfying Inequality (3.17). Then $E^1(X)$ is transcendental in the following cases:

1. $X$ is the blow-up of $(\mathbb{P}^{r-1})^{p-1}$ at $\Lambda$, where $r \geq 3$, $p \geq 2$, $\Lambda$ is a finite set of points in $(\mathbb{P}^{r-1})^{p-1}$ and contains $q_0(r, p)$ points in very general position.

2. $X$ is the blow-up of the product $\mathbb{P}^{r_1-1} \times \cdots \times \mathbb{P}^{r_p-1}$ at a finite set $\Lambda$, where $p \geq 2$, $\Lambda$ lies on a linear subspace $(\mathbb{P}^{p_0-1})^{p-1}$ with $3 \leq r_0 \leq \min_{i=1}^{p-1} (r_i)$ and contains $q_0(r_0, p)$ points in very general position as points of $(\mathbb{P}^{p_0-1})^{p-1}$.

**Proof.** Case (1) is a direct consequence of Corollary 3.8 and Corollary 3.2. Case (2) follows from Case (1) and Proposition 3.6. Note that Proposition 3.6 can be applied inductively on dimension to the proper transform of every linear subspace containing $(\mathbb{P}^{p_0-1})^{p-1}$.

**3.3. Elliptic fibration.** The purpose of this subsection is to prove that $E^1(X)$ is transcendental for some elliptic fibration.

**Theorem (Theorem 1.9).** $E^1(X)$ is transcendental in the following cases:

1. $X$ is the blow-up of $\mathbb{P}^2$ at a finite set $\Lambda$, where $\Lambda$ contains the intersection of two general cubic curves.

2. $X$ is the blow-up of $\mathbb{P}^3$ at a finite set $\Lambda$, where $\Lambda$ contains the intersection of three general quadrics.

3. $X$ is the blow-up of $\mathbb{P}^r$ at a finite set $\Lambda$, where $\Lambda$ lies on a linear subspace $\mathbb{P}^2 \subset \mathbb{P}^r$ containing the intersection of two general cubics.

4. $X$ is the blow-up of $\mathbb{P}^r$ at a finite set $\Lambda$, where $\Lambda$ lies on a linear subspace $\mathbb{P}^3 \subset \mathbb{P}^r$ containing the intersection of three general quadrics.

**Proof.** With Proposition 3.6, Case (3) and (4) follows from Case (1) and (2) respectively. The proof of Case (1) and (2) makes use of the facts that $X$ is an elliptic fibration over $\mathbb{P}^1$ or $\mathbb{P}^2$.

Case (2) is a consequence of Corollary 3.5 by setting $F = \hat{Q}$ and Proposition 3.9, the latter showing that there are infinitely many $(-1)$-curves on the proper transform $\hat{Q}$ of a general member of the net of quadrics. Case (1) follows from a similar proof as Proposition 3.9.
Proposition 3.9. Let $X$ be the blow up of $\mathbb{P}^3$ at the base locus $\Lambda$ of a general net of quadrics in $\mathbb{P}^3$ and let $\hat{Q}$ be the proper transform of a general member of the net. Then

$$L_{\hat{Q}} = \{ D \in \text{Pic}(X) : h^0(X, D) > h^0(X, D - \hat{Q}) \},$$

as a submonoid of $L_{\hat{Q}}$ under the injection $\text{Pic}(X) \hookrightarrow \text{Pic}(\hat{Q})$, contains infinitely many $(-1)$-curves on $\hat{Q}$.

Proof. Let $H, E_1, E_2, ..., E_8$ be the generators of $\text{Pic}(X)$, where $H$ is the pullback of the hyperplane divisor and $E_1, E_2, ..., E_8$ are the exceptional divisors of the blow-up $X \to \mathbb{P}^3$.

The net of quadrics gives a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with $\Lambda$ the indeterminacy locus. Blowing up $\Lambda$ gives a regular map $f : X \to \mathbb{P}^2$, which is a fibration of elliptic curves with sections $E_1, E_2, ..., E_8$. Each fiber of $f$ is the proper transform of the intersection of two quadrics of the net and the pull back $f^{-1}(\Gamma)$ of a line $\Gamma \subset \mathbb{P}^2$ is the proper transform of a quadric of the net. So $\hat{Q} = f^{-1}(\Gamma)$ for a general line $\Gamma \subset \mathbb{P}^2$.

Let $X_\eta$ be the generic fiber of $f : X \to \mathbb{P}^2$, $J(X_\eta) = \text{Pic}_0(X_\eta)$ be the Jacobian of $X_\eta$ and $A \subset J(X_\eta)$ be the subgroup of $J(X_\eta)$ generated by $H, E_1, E_2, ..., E_8$. For each $a \in A$, we have an automorphism $\phi_a : X_\eta \to X_\eta$ by taking $p$ to $p + a$; $\phi_a$ corresponds to a birational self map $\phi_a : X \dashrightarrow X$. More explicitly, for each $a = dH + m_1E_1 + m_2E_2 + ... + m_8E_8$ satisfying $4d + m_1 + m_2 + ... + m_8 = 0$, $\phi_a : X \dashrightarrow X$ is a birational map sending $p \in X_b$ to $p + a \in X_b$ on a general fiber $X_b$ of $f$. Obviously, $\phi_a$ preserves the fibration $X/\mathbb{P}^2$, i.e., $f \circ \phi_a = f$.

This map can be extends to all irreducible fibers of $f$. And since $f$ has only finitely many reducible fibers, $\phi_a$ is an isomorphism

$$\phi_a : X \setminus f^{-1}(\Delta) \xrightarrow{\sim} X \setminus f^{-1}(\Delta)$$

in codimension one, where $\Delta \subset \mathbb{P}^2$ is the finite set of points $b$ with reducible fiber $X_b$. In particular, $\phi_a$ induces an isomorphism $\phi_a : \hat{Q} \xrightarrow{\sim} \hat{Q}$.

For each $a \in A$, $G_a = \phi_a(E_1)$ is a rational section of $f$. And since $\phi_a$ is an isomorphism on $\hat{Q}$, $G_a \cdot \hat{Q}$ is a $(-1)$-curve on $\hat{Q}$. Clearly, $G_a$ meets $\hat{Q}$ properly and hence $G_a \in L_{\hat{Q}}$. And the orbit $\{ G_a : a \in A \}$ of $E_1$ under the action of $A$ is obviously infinite. We are done. \qed

4. Euler-Chow Series of Del Pezzo surfaces

4.1. Some basic facts on $\text{Bl}_A \mathbb{P}^2$. In the computation of Euler-Chow series of Del Pezzo surfaces, we will need some statements on divisors of the surfaces to be discussed as follows. In this subsection, we assume that $X$ is the blow-up of $\mathbb{P}^2$ at $\Lambda$ with $\Lambda$ being either $r \leq 9$ points in general position or the intersection of two general cubics. Note that $-K_X$ is nef and effective. If $r \leq 8$, then $X$ is a Del Pezzo surface and $-K_X$ is ample.
Lemma 4.1. For an effective divisor $D$, $-K_X D > 0$ unless $r = 9$ and $D = -dK_X$ for some $d \geq 0$. For an integral curve $D \subset X$, $D^2 \geq -1$ and

- $D$ is nef if $D^2 \geq 0$;
- $D$ is a $(-1)$-curve if $D^2 = -1$.

Proof. When $r \leq 8$, $-K_X$ is ample, thus $-K_X D > 0$.

When $r = 9$, assume that $D = dH - \sum_{i=1}^{9} m_i E_i$ and $C$ is a generic member of $|-K_X|$. Note that $C$ is a smooth elliptic curve under our assumptions on $\Lambda$. Suppose that $K_X D = 0$ and $D$ is not a multiple of $C$. Replacing $D$ by $D - \lambda C$, we may assume that $D$ meets $C$ properly. Restricting $D$ to $C$, let $P_i = C \cap E_i$, then

$$O_X(D)|_C = 3dH - \sum_{i=1}^{9} m_i P_i = 0 \text{ in Pic}(C).$$

When $\Lambda$ is a set of 9 points in general position, $P_1, P_2, ..., P_9$ are 9 general points on $C$; therefore, there are no relations between them and $H$ in Pic$(C)$ and (4.1) cannot happen. When $\Lambda$ is the intersection of two general cubics, the only relation between $P_1, P_2, ..., P_9$ and $H$ is

$$3H - \sum_{i=1}^{9} P_i = 0$$

in Pic$(C)$. That is, $D = -dK_X$. Contradiction.

Let $D$ be an integral curve. Since $(K_X + D)D = 2p_a(D) - 2 \geq -2$ and $K_X D \leq 0$, we conclude that $D^2 \geq -2$, where $p_a(D)$ is the arithmetic genus of $D$. And $D^2 = -2$ only if $K_X D = 0$, which can only happen when $r = 9$ and $D = -dK_X$; but then we have $D^2 = 0$. If $D^2 \geq 0$, obviously $D$ is nef; if $D^2 = -1$, then $p_a(D) = 0$ and $D$ is a $(-1)$-curve. $\square$

Lemma 4.2. Let $D$ be a nonzero nef divisor on $X$. Then

- $D$ is effective.
- $H^1(D) = 0$ and

$$h^0(D) = \frac{(D - K_X)D}{2} + 1,$$

unless $\Lambda$ is the intersection of two general cubic curves and $D = -dK_X$ for some $d > 0$.

- $h^0(D) > 1$ unless $\Lambda$ is a set of 9 points in general position and $D = -dK_X$ for some $d > 0$.

Proof. By Riemann-Roch,

$$h^0(D) - h^1(D) + h^2(D) = \frac{(D - K_X)D}{2} + 1.$$

By Serre duality, $h^2(D) = h^0(K_X - D)$. Since $D$ is nef and $-K_X$ is nef, $\frac{(D - K_X)D}{2} \geq 0$ and $H(K_X - D) < 0$ for every ample divisor $H$, which implies
that $K_X - D$ is not effective and thus $h^2(D) = h^0(K_X - D) = 0$. Therefore,

\begin{equation}
\label{eq:5}
h^0(D) = \frac{(D - K_X)D}{2} + 1 + h^1(D) \geq 1,
\end{equation}

and $D$ is effective.

By Lemma \ref{lem:4.1} $-K_X D > 0$ unless $r = 9$ and $D = -dK_X$. If $-K_X D > 0$, clearly $h^0(D) > 1$. Then $(D - K_X)^2 > 0$, thus $D - K_X$ is big and nef. We have $H^1(D) = 0$ by Kawamata-Viehweg vanishing theorem and \eqref{eq:3} follows.

If $-K_X D = 0$, $r = 9$ and $D = -dK_X$. If $\Lambda$ is a set of 9 general points, $h^0(D) = 1$ and we still have $h^1(D) = 0$ and \eqref{eq:3}. If $\Lambda$ is the intersection of two general cubics and $D = -dK_X$, $h^0(D) > 1$.

\begin{lemma}
\label{lem:4.3}
Every effective divisor $D$ on $X$ can be uniquely written as
\begin{equation}
\label{eq:6}
D = A + m_1 I_1 + m_2 I_2 + \ldots + m_a I_a
\end{equation}

for some nef divisor $A$ and some set of disjoint $(-1)$-curves $I_1, I_2, \ldots, I_a$ such that $AI_k = 0$ for all $k$ and $m_1, m_2, \ldots, m_a \in \mathbb{Z}^+$.\end{lemma}

\begin{proof}
Let $D_f$ be the fixed part of $|D|$ and write $D = D_\mu + D_f$ as in \eqref{eq:5}. Note that $D_\mu$ is nef as it is easy to verify that $C \cdot D_\mu \geq 0$ for every integral curve $C$. We let

\begin{equation}
\label{eq:7}
D = D_\mu + D_f = A + F
\end{equation}

where $A \supset D_\mu$, $F \subset D_f$ and $A$ is nef and maximal in the sense that $A + F'$ is not nef for every nonzero effective divisor $F' \subset F$.

First, every irreducible component $I$ of $F$ is a $(-1)$-curve. By our choice of $A$ and $F$, $I$ cannot be a multiple of $-K_X$. By Lemma \ref{lem:4.1} $I$ is either a $(-1)$-curve or a nef divisor. In the latter case, $h^0(I) \geq 2$ by Lemma \ref{lem:4.2} and $I$ cannot lie in the fixed part of $|D|$. Contradiction.

Second, $I_1 \cdot I_2 = 0$ for two distinct irreducible components $I_1$ and $I_2$ of $F$. Otherwise, if $I_1 \cdot I_2 > 0$, then $I_1 + I_2$ is nef and $A + I_1 + I_2$ is nef. This contradicts with our choice of $A$.

Last, $A \cdot I = 0$ for every irreducible component $I$ of $F$. Otherwise, if $A \cdot I > 0$, then $A + I$ is nef, this contradicts with our choice of $A$.

In conclusion,

\begin{equation}
\label{eq:8}
D = A + F = A + m_1 I_1 + m_2 I_2 + \ldots + m_a I_a
\end{equation}

with required properties. Clearly, this representation of $D$ is unique since $m_k = -DI_k$ for all $k$. \hfill \Box

\begin{remark}
Actually we have proved that for every effective divisor $D$, let $S$ be the set of $(-1)$-curves $I$ satisfying $D \cdot I < 0$. Then $A = D - \sum_{I \in S} (I \cdot D) I$ is nef; $A \cdot I = 0$ for every $I \in S$; $I_1 \cdot I_2 = 0$ for $I_1, I_2 \in S$.
\end{remark}

\begin{lemma}
\label{lem:4.5}
Let $D$ be an effective divisor on $X$. Then $H^1(D) = 0$ and \eqref{eq:3} holds if and only if $DI \geq -1$ for all $(-1)$-curves $I \subset X$.
\end{lemma}
4.2. Euler-Chow series.

in general position, and let

where

\[ I \]

\[ B \]

\[ O \]

\[ D \]

\[ H \]

\[ E \]

\[ F \]

\[ H^1(O_X(D)) \rightarrow H^1(O_{I_j}(-m_j)) \rightarrow H^2(O_X(D - I_j)), \]

where the last term vanishes because \( D - I_j \) is effective. If \( H^1(D) = 0 \), then \( H^1(O_{I_j}(-m_j)) = 0 \); as \( m_j \in \mathbb{Z}^+ \), thus \( m_j = 1 \).

Conversely, suppose that \( m_i = 1 \) for all \( i \). We observe that if \( B \cdot I = 0 \) and \( H^1(B) = 0 \), then \( H^1(B + I) = 0 \) by the exact sequence

\[ H^1(O_X(B)) \rightarrow H^1(O_X(B + I)) \rightarrow H^1(O_I(-1)). \]

So we can inductively show that \( H^1(D) = H^1(A + I_1 + I_2 + ... + I_a) = 0 \). □

Lemma 4.6. Let \( D \) be an effective divisor on \( X \). Suppose that \( r \geq 2 \). Then

- \( D \) is nef if and only if \( DI \geq 0 \) for all \((-1\)-curves \( I \subset X \).
- \( D \) is ample if and only if \( DI > 0 \) for all \((-1\)-curves \( I \subset X \) and \( D \) is not a multiple of \(-K_X\) when \( r = 9 \).
- \( D \) is ample if and only if

\[ D = m(-K_X) + F \]

for some \( m \in \mathbb{Z}^+ \) and some nef divisor \( F \) that is not ample and not a multiple of \(-K_X\) when \( r = 9 \).

4.2. Euler-Chow series. Let \( X = P_r \) be the blow-up of \( \mathbb{P}^2 \) at \( r \leq 8 \) points in general position. and let \( E_1(X) = f_r(t_0, t_1, ..., t_r) \) where \( t_0 = t^H \) and \( t_i = t^{E_i} \) for \( i = 1, 2, ..., r \). Our aim is to develop a recursive formula for \( E_1(X) \).

For each set \( S = \{I_1, I_2, ..., I_a\} \) of disjoint \((-1\)-curves), let \( M_S \) be the monoid \( \{ \sum_{i=1}^a m_i I_i | m_i \in \mathbb{Z}^+, \forall i \} \). Since every effective divisor \( D \) on \( X \) is of the form \( (4.6) \) and \( h^0(D) = h^0(A) \) by Lemma 4.3, thus

\[ E_1(X) = \sum_{S} \sum_{I \in M_S} \sum_{A \in S^\perp} h^0(A) t^{A+I} = \sum_{S} \left( \sum_{I \in M_S} t^I \cdot \sum_{A \in S^\perp} h^0(A) t^A \right) \]

where \( S \) runs over all sets of disjoint \((-1\)-curves), \( S^\perp \) is the group of divisors \( B \) satisfying \( BI = 0 \) for all \( I \in S \), and \( \sum_{I \in M_S} t^I = \frac{t_1^{i_1} t_2^{i_2} ... t_r^{i_r}}{(1-t_1)(1-t_2) ... (1-t_r)} \).

So naturally we turn to consider the series

\[ N_X(t) = \sum_{A \in S} h^0(A) t^A = \sum_{A \in S} \left( \frac{(A - K_X)A}{2} + 1 \right) t^A. \]

Let \( g_r(t_0, t_1, ..., t_r) = N_X(t) \) for \( X = P_r \). We first express \( f_r \) in terms of \( g_r \).

Note that for each \( S \) in \((4.12)\), there is a map \( \pi_S : X \rightarrow X_S \) of Del Pezzo surfaces given by contracting the \((-1\)-curves in \( S \). Obviously, \( S^\perp = \)
defined by sending $r$ where the action of the symmetric group $\Sigma_X$ of lines on the group of which we will omit due to its tedious nature. In [H, V, 4.10.1], $\Phi$ is called

(4.15)

A $T$ $X$ $E$ $q$ $^2 = (S^* 1 \text{ or } 2 \times S)$ or $S$ $\ast 1$ or $2$. By Lemma 4.7, every set of disjoint $(-1)$-curves on $P_r$, we consider the group $\Phi \subset \text{Aut}(\text{Pic}(P_r))$ generated by $\Sigma_r$ and $\varphi_{abc}$ for all $1 \leq a \neq b \neq c \leq r$, where the action of the symmetric group $\Sigma_r$ of $\{1, 2, ..., r\}$ on $\text{Pic}(X)$ is defined by sending $H \mapsto H$ and $E_i \mapsto E_{\sigma(i)}$ for $\sigma \in \Sigma_r$, and $\varphi_{abc}$ is given by

$$
\varphi_{abc}(H) = 2H - E_a - E_b - E_c
$$

$\varphi_{abc}(E_a) = H - E_b - E_c$

$\varphi_{abc}(E_b) = H - E_c - E_a$

$\varphi_{abc}(E_c) = H - E_a - E_b$ and

$\varphi_{abc}(E_i) = E_i$ for $i \neq a, b, c$.

Indeed, for Del Pezzo surfaces $P_r$ with $3 \leq r \leq 8$, $\Phi$ are Weyl groups $A_2 \times A_1, A_4, D_5, E_6, E_7, E_8$ respectively.

**Lemma 4.7.** Let $\Pi$ be the set of $(-1)$-curves on $X$. Then

- $\Phi(\Pi) = \Pi$, i.e., $\Phi$ induces a permutation of $\Pi$. Indeed, $\Phi \cong \text{Aut}(\Pi)$ can be identified with the group of bijections $\varphi : \Pi \to \Pi$ preserving the intersection pairing, i.e., $\varphi(I_1)\varphi(I_2) = I_1I_2$ for all $I_1, I_2 \in \Pi$.

- For every subset $S$ of disjoint $(-1)$-curves, there exists $\varphi \in \Phi$ such that $\varphi(S) = S_k$ or $T$, where $k = r - \#S$.

The proof of the above lemma follows the argument for [H, V, Ex 4.15], which we will omit due to its tedious nature. In [H, V, 4.10.1], $\Phi$ is called the group of automorphisms of the configuration of lines on $X$, as $\Pi$ consists of lines on $X$ under the map $X \to \mathbb{P}^N$ by $| - K_X|$.

By Lemma 4.7, every set of disjoint $(-1)$-curves lies in the orbit of $S_k$ or $T$ under $\Phi$. With this in mind, we can rewrite (4.12) as

$$
f_r(t_0, t_1, ..., t_r)
$$

(4.16)

$\sum_{k=0}^{r} \frac{1}{|\Phi_{S_k}|} \sum_{\varphi \in \Phi} \varphi \left( g_k(t_0, t_1, ..., t_k) \prod_{j=k+1}^{r} \frac{t_j}{1 - t_j} \right)
$

$+ \sum_{\varphi \in \Phi} \frac{1}{|\Phi_T|} \varphi \left( q(t_0t_1^{-1}, t_0t_2^{-1}) \prod_{j=3}^{r} \frac{t_j}{1 - t_j} \right)
$
where $\Phi_{S_k}$ and $\Phi_T$ are subgroups of $\Phi$ consisting of $\varphi$ with $\varphi(S_k) = S_k$ and $\varphi(T) = T$ respectively.

4.3. Computation of $N_X(t)$. To compute $N_X(t)$, we can apply the algorithm in the proof of Proposition 2.3.

Let

$$
\rho_r(t_0, t_1, \ldots, t_r) = L_X(t) = \sum_{A \in \text{nef}} t^A.
$$

As in (2.20), because of (4.13), there exists a second order differential operator $Q$ such that $N_X(t) = Q(L_X(t))$, where $Q$ is defined as follows:

$$
Q = \left( \frac{t_0^2}{2} \frac{\partial^2}{\partial t_0} + 2t_0 \frac{\partial}{\partial t_0} \right) - \frac{1}{2} \sum_{k=1}^{r} t_k \frac{\partial^2}{\partial t_k} + 1.
$$

As a result, the computation of $N_X(t)$ comes down to that of $L_X(t)$, the formal sum of $t^A$ over all the lattice points $A$ in the nef cone $\text{NM}^1(X) \subset H^2(X, \mathbb{R})$ of $X$. As $\text{NM}^1(X)$ is a rational polyhedral cone given by Lemma 4.6, thus we can follow the algorithm in the proof of Proposition 2.3 and compute $L_X(t)$ by subdividing $\text{NM}^1(X)$ into simplicial cones.

Now we compute $L_X(t)$ for $X = P_r$ when $r \leq 4$.

Case $r = 1$. $\text{NM}^1(P_1) = \{a_0H + a_1(H - E_1) | a_0, a_1 \in \mathbb{Z}_{\geq 0}\}$. Therefore, (4.19)

$$
\rho_1(t_0, t_1) = \frac{1}{(1-t_0)(1-t_0/t_1)}.
$$

Case $r = 2$. $\text{NM}^1(P_2) = \{a_0H + a_1(H - E_1) + a_2(H - E_2) | a_0, a_1, a_2 \in \mathbb{Z}_{\geq 0}\}$. Therefore,

$$
\rho_2(t_0, t_1, t_2) = \frac{1}{(1-t_0)(1-t_0/t_1)(1-t_0/t_2)}.
$$

Case $r = 3$. $\text{NM}^1(P_3) = \{a_0H - \sum_{i=1}^{3} a_iE_i | a_i \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq 3; a_0 \geq a_i + a_j, 1 \leq i \neq j \leq 3\}$, thus $\text{NM}^1(P_3) = \{a_0H + \sum_{i=1}^{3} a_i(H - E_i) | a_i \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq 3\} \cup \{a_2(H - E_2) | a_2 \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq 3\}$.

Therefore,

$$
\rho_3(t_0, t_1, t_2, t_3) = \frac{1}{(1-t_0)(1-t_0/t_1)(1-t_0/t_2)(1-t_0/t_3)}
$$

$$
+ \left( \frac{1}{(1-t_0)(1-t_0/t_1)(1-t_0/t_2)(1-t_0/t_3)} - \frac{1}{(1-t_0/t_1)(1-t_0/t_2)(1-t_0/t_3)} \right).
$$

Case $r = 4$. $\text{NM}^1(P_4) = \{a_0H - \sum_{i=1}^{4} a_iE_i | a_i \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq 4; a_0 \geq a_i + a_j, 1 \leq i \neq j \leq 4\}$. Note that a decomposition of $\text{NM}^1(P_4)$ as a union of simplicial rational polyhedral can be given as follows: (4.22)

$$
\text{NM}^1(P_4) = C_1 \cup C_2 \cup \bigcup_{i=1}^{4} (C_{3,i} \cup C_{4,i}),
$$

where (all the $b_i$’s are assumed to go through $\mathbb{Z}_{\geq 0}$ unless otherwise pointed)

$$
C_1 = \text{NM}^1(P_4) \cap \{ \sum_{j=1}^{4} a_j \leq a_0 \} = \{b_0H + \sum_{j=1}^{4} b_j(H - E_j) | \cdots \},
$$

$$
C_2 = \text{NM}^1(P_4) \cap \{ \sum_{j=1}^{4} a_j = a_0 \} = \{b_0H + \sum_{j=1}^{4} b_j(H - E_j) | \cdots \},
$$

$$
C_{3,i} = \text{NM}^1(P_4) \cap \{ a_i = b_i \},
$$

$$
C_{4,i} = \text{NM}^1(P_4) \cap \{ a_i = b_i + 1 \}.
$$
\[ C_2 = N M^1(P_4) \cap \left\{ \sum_{j=1}^{4} a_j > a_0; \sum_{j=1}^{4} a_j \leq a_i + a_0, \forall i > 0 \right\} \]

\[ = \{ b_0(-K) + \sum_{j=1}^{4} b_j(H - E_j) \mid b_0 > 0 \}, \]

\[ C_{3-i} = N M^1(P_4) \cap \left\{ a_i = \min_{j=1}^{4}(a_j); \sum_{j=1}^{4} a_j > a_i + a_0; \sum_{j=1}^{4} a_j \leq a_0 + 2a_i \right\} \]

\[ = \{ b_0(-K) + b_i(-K - H) + \sum_{j=1}^{4} b_j(H - E_j) - b_i(H - E_i) \mid b_i > 0 \}, \]

\[ C_{4-i} = N M^1(P_4) \cap \left\{ a_i = \min_{j=1}^{4}(a_j); \sum_{j=1}^{4} a_j > a_0 + 2a_i \right\} = \{ b_0(-K - H) + b_i(-K - H + E_i) + \sum_{j=1}^{4} b_j(H - E_j) - b_i(H - E_i) \mid b_i > 0 \}. \]

Clearly any two sub-cones as above do not intersect unless both are of the form \( C_{3-i} \). Assume that the set of indices \( \{i, j, k, l\} \) is exactly \( \{1, 2, 3, 4\} \). We list all possible intersection sub-cones as follows:

\[ C_{3-ij} = C_{3-i} \cap C_{3-j} \]

\[ = \{ b_0(-K) + b_i(-K - H) + b_k(H - E_k) + b_l(H - E_l) \mid b_i > 0 \}; \]

\[ C_{3-ijk} = C_{3-i} \cap C_{3-j} \cap C_{3-k} \]

\[ = \{ b_0(-K) + b_i(-K - H) + b_l(H - E_l) \mid b_i > 0 \}; \]

\[ C_{3-ijkl} = \cap_{i=1}^{4} C_{3-i} = \{ b_0(-K) + b_1(-K - H) \mid b_1 > 0 \}. \]

Denote by \( F_C(t) \) the sum \( \sum_{\varpi \in C} t^{\varpi} \) over a lattice \( C \). Therefore,

\[ (4.23) \quad \rho_4(t_0, \cdots, t_4) = F_{C_1}(t) + F_{C_2}(t) + \sum_{i=1}^{4} (F_{C_{3-i}}(t) + F_{C_{4-i}}(t)) \]

\[ - \sum_{1 \leq i \neq j \leq 4} F_{C_{3-ij}}(t) + \sum_{1 \leq i \neq j \neq k \leq 4} F_{C_{3-ijk}}(t) - F_{C_{3-ijkl}}(t) \]

\[ = \left( \frac{1}{1 - t_0} + \frac{t-K}{1 - t-K} \right) \cdot e_4 + \frac{t-K-H \cdot (e_3 - e_2 + e_1 - e_0)}{(1 - t-K)(1 - t-K-H)} \]

\[ + \frac{1}{1 - t-K-H} \sum_{i=1}^{4} \left( \frac{t-K-H+E_i}{1 - t-K-H+E_i} \cdot \frac{1 - t_0/t_i}{\prod_{j=1}^{4}(1 - t_0/t_j)} \right), \]

where \( e_0 = 1, e_k \) is the elementary symmetric polynomial of degree \( k \) in \( (x_1, \cdots, x_4), \) and \( x_k = \frac{1}{1-t_0/t_k} \) for \( 1 \leq k \leq 4. \)
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