EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE OPERATOR RICCATI EQUATION. A GEOMETRIC APPROACH

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ABSTRACT. We introduce a new concept of unbounded solutions to the operator Riccati equation $A_1 X - X A_0 - X V X + V^* = 0$ and give a complete description of its solutions associated with the spectral graph subspaces of the block operator matrix $B = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix}$. We also provide a new characterization of the set of all contractive solutions under the assumption that the Riccati equation has a contractive solution associated with a spectral subspace of the operator $B$. In this case we establish a criterion for the uniqueness of contractive solutions.

1. INTRODUCTION

In the present article we address the problem of a perturbation of invariant subspaces of self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and related questions of the existence and uniqueness of solutions to the operator Riccati equation.

Given a self-adjoint operator $A$ and its closed invariant subspace $\mathcal{H}_0 \subset \mathcal{H}$ we set $A_i = A|_{\mathcal{H}_i}$, $i = 0, 1$ with $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Assuming that the perturbation $V$ is off-diagonal with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ consider the self-adjoint operator

$$B = A + V = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix}$$

where $V$ is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$. It is well known (see, e.g., [3], [5], [8]) that the Riccati equation

$$A_1 X - X A_0 - X V X + V^* = 0$$ (1.1)

has a bounded solution $X : \mathcal{H}_0 \to \mathcal{H}_1$ iff its graph

$$G(\mathcal{H}_0, X) := \{ x_0 \oplus X x_0 | x_0 \in \mathcal{H}_0 \}$$ (1.2)

is an invariant subspace for the operator $B$. It might happen, however, that the operator $B$ has invariant subspaces that are the graphs of closed densely defined unbounded operators $X : \mathcal{H}_0 \to \mathcal{H}_1$, and the problem of more general solutions to the Riccati equation naturally arises.

In the present article we introduce the new concept of unbounded (closed densely defined) operator solutions to the Riccati equation and we obtain a geometric criterion for their existence (Corollary 4.5) resulting in the complete description of the bijective correspondence between solutions of the Riccati equation and the $B$-invariant graph subspaces.

Among all solutions to the Riccati equation, those corresponding to the spectral subspaces of the operator $B$, i.e., the solutions $X$ such that $G(\mathcal{H}_0, X) = \text{Ran} E_B(\Delta)$, the range of the spectral projection corresponding to some Borel set $\Delta \subset \mathbb{R}$, are of particular

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interest. Using the Douglas-Pearcy theorem [12], we prove that a solution to the Riccati equation is associated with a spectral subspace iff it is an isolated point in the set of all its solutions (Theorem 5.3).

Revisiting the case of bounded solutions, we give a complete description of the set of all contractive solutions (\( \| X \| \leq 1 \)) to the Riccati equation, provided that the Riccati equation has a contractive solution which is associated with a spectral subspace (Lemma 6.1 and Theorem 6.2). This result substantially generalizes the recent uniqueness theorem due to Adamyan, Langer, and Tretter [2].

In the forthcoming publications [18, 19, 20] we prove a number of new existence and uniqueness results for solutions of the Riccati equation assuming some conditions on the spectra of the operators \( A_0 \) and \( A_1 \). Also we obtain sharp estimates for the norm of these solutions. These estimates are related to the study of the subspace perturbation problem [17].

To avoid getting into technical issues that may obscure the basic ideas of the work, we assume in this paper that the operator \( A \) and the perturbation \( V \) are bounded. In some cases this hypothesis can easily be relaxed to handle the case of unbounded \( A \)'s and even unbounded perturbations \( V \) as well. The extension to unbounded operators will be presented elsewhere.

The article is organized as follows. In Section 2 we collect some known facts about two closed subspaces of a separable Hilbert space \( \mathcal{H} \). In Section 3 we give a particularly simple proof of the Halmos theorem [14] providing a criterion for a closed subspace of the Hilbert space \( \mathcal{H} \) to be the graph \( \mathcal{G}(\mathcal{H}_0, X) \) of a closed densely defined operator \( X \) from a closed subspace \( \mathcal{H}_0 \subset \mathcal{H} \) to its orthogonal complement \( \mathcal{H}_1 = \mathcal{H}_0^\perp \). In Section 4 we formulate and prove a general criterion for the solvability of the operator Riccati equation in the class of closed densely defined (not necessarily bounded) operators. The structure of the set of all solutions to the Riccati equation is analyzed in Section 5 from the topological point of view. Finally, Section 6 is devoted to the thorough analysis of the set of all contractive solutions going beyond the one undertaken recently in [2]. In particular, we establish a general criterion for a contractive solution which is associated with a spectral subspace of the operator matrix \( B \) to be unique with no additional assumptions on the spectra of the operators \( A_0 \) and \( A_1 \).

A few words about the notations used throughout the paper. Given a linear operator \( A \) on a Hilbert space \( \mathcal{H} \), by \( \text{spec}(A) \) we denote the spectrum of \( A \). If not explicitly stated otherwise, \( \mathcal{H}^\perp \) denotes the orthogonal complement in \( \mathcal{H} \) of a subspace \( \mathcal{H} \subset \mathcal{H} \), i.e., \( \mathcal{H}^\perp = \mathcal{H} \cap \mathcal{H}^\perp \). The identity operator on \( \mathcal{H} \) is denoted by \( I_{\mathcal{H}} \). The notation \( \mathcal{B}(\mathcal{H}, \mathcal{L}) \) is used for the Banach algebra of bounded operators from the Hilbert space \( \mathcal{H} \) to the Hilbert space \( \mathcal{L} \). Finally, we write \( \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H}) \).

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2. Geometry of Two Subspaces of the Hilbert Space

In this section we collect some facts about pairs of closed subspaces of a separable Hilbert space. Although most of them are well known, they are scattered in the literature and frequently formulated in a different form which does not fit the context of the present paper. Without any attempt to give a complete overview of the whole work done in this
direction we mention the pioneering work of Friedrichs [13], M. Krein, Krasnoselsky, and Milman [21], [22], Dixmier [10], [11], Davis [9], and Halmos [14]. Some of the results described in this section admit an extension to the case of Banach spaces. We refer the interested reader to the papers [22] and [15].

**Definition 2.1.** Let \((P, Q)\) be an ordered pair of orthogonal projections in \(\mathcal{H}\). We use the standard notation as introduced by Halmos [14] (see also [26])

\[
\mathcal{M}_{pq} := \{ f \in \mathcal{H} \mid Pf = pf, Qf = qf \}, \quad p, q = 0, 1
\]

\[
\mathcal{M}'_0 := \text{Ran } P \ominus (\mathcal{M}_{10} \oplus \mathcal{M}_{11})
\]

\[
\mathcal{M}'_1 := \text{Ran } P^\perp \ominus (\mathcal{M}_{00} \oplus \mathcal{M}_{01})
\]

\[
\mathcal{M}' := \mathcal{M}'_0 \oplus \mathcal{M}'_1
\]

\[
P' := \left|_{\mathcal{M}'} P
\right.
\]

\[
Q' := \left|_{\mathcal{M}'} Q
\right.
\]

To avoid possible confusion we will often write \(\mathcal{M}_{pq}(P, Q)\) instead of the shorthand notation \(\mathcal{M}_{pq}\) to emphasize that the canonical decomposition of the Hilbert space \(\mathcal{H}\) is considered with respect to the ordered pair \((P, Q)\).

Following Halmos [14] we call the pair \((P', Q')\) the generic part of the pair \((P, Q)\). Roughly speaking, \((P', Q')\) is the non-commuting part of \((P, Q)\). Indeed, if \(P\) and \(Q\) commute, then \(P' = Q' = 0\).

**Theorem 2.2.** Let \((P, Q)\) be an ordered pair of orthogonal projections in the Hilbert space \(\mathcal{H}\). Then the space \(\mathcal{H}\) admits the (canonical) orthogonal decomposition

\[
\mathcal{H} = \mathcal{M}_{00} \oplus \mathcal{M}_{01} \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{11} \oplus \mathcal{M}'.
\]

(2.1)

With respect to this decomposition the projections \(P\) and \(Q\) read

\[
P = 0 \oplus 0 \oplus I_{\mathcal{M}_{10}} \oplus I_{\mathcal{M}_{11}} \oplus P',
\]

\[
Q = 0 \oplus I_{\mathcal{M}_{01}} \oplus 0 \oplus I_{\mathcal{M}_{11}} \oplus Q'.
\]

With respect to the decomposition \(\mathcal{M}' = \mathcal{M}'_0 \oplus \mathcal{M}'_1\) the projections \(P'\) and \(Q'\) read

\[
P' = \begin{pmatrix} I_{\mathcal{M}'_0} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q' = W^* \begin{pmatrix} \cos^2 \Theta & \sin \Theta \cos \Theta \\ \sin \Theta \cos \Theta & \sin^2 \Theta \end{pmatrix} W,
\]

where \(\Theta\) is a (unique) positive semidefinite angle operator in \(\mathcal{M}'_0\) such that

\[
\sin^2 \Theta = P'(I_{\mathcal{M}'} - Q')P'|_{\mathcal{M}'_0};
\]

spec(\(\Theta\)) \(\subset [0, \pi/2]\) but 0 and \(\pi/2\) are not eigenvalues of \(\Theta\). The unitary operator matrix

\[
W : \mathcal{M}'_0 \oplus \mathcal{M}'_1 \rightarrow \mathcal{M}'_0 \oplus \mathcal{M}'_1
\]

reads

\[
W = \begin{pmatrix} I_{\mathcal{M}'_0} & 0 \\ 0 & W \end{pmatrix}
\]

where \(W \in \mathcal{B}(\mathcal{M}'_1, \mathcal{M}'_0)\) is the unitary operator from the polar decomposition

\[
P'^\perp Q' P'|_{\mathcal{M}'_0} = W^* \left((P'^\perp Q' P'|_{\mathcal{M}'_0})^* P'^\perp Q' P'|_{\mathcal{M}'_0}\right)^{1/2}.
\]
In particular, the difference $Q' - P'$ of the generic parts of the projections $P$ and $Q$ can be represented with respect to the decomposition $\mathfrak{M}' = \mathfrak{M}_0' \oplus \mathfrak{M}_1'$ in the form
\[
Q' - P' = \mathcal{W}^* \begin{pmatrix}
\sin \Theta & 0 \\
0 & \sin \Theta
\end{pmatrix}
\begin{pmatrix}
-\sin \Theta & \cos \Theta \\
\cos \Theta & \sin \Theta
\end{pmatrix}
\mathcal{W}.
\] (2.3)
and hence
\[
\|Q' - P'\| = \|\sin \Theta (Q', P')\|.
\] (2.4)

In a slightly different form Theorem 2.2 was proven by Davis [9] and Halmos in [14]. An alternative, simple and direct proof of this theorem was given by Amrein and Sinha [4].

Theorem 2.2 has been proved to be of great importance in a number of problems related to pairs of orthogonal projections. In particular, it was successfully used for the study of the operator algebras generated by a pair of orthogonal projections (see [26], [27] and references therein).

In the next section we will study the graph subspaces associated with an orthogonal decomposition of the Hilbert space and we will revisit Theorem 2.2 which allows to perform the subsequent analysis in a particularly simple manner.

3. GRAPH SUBSPACES

**Definition 3.1.** Let $\mathfrak{S}_0$ be a closed subspace of a Hilbert space $\mathfrak{S}$ and $X$ a closed densely defined (possibly unbounded) operator from $\mathfrak{S}_0$ to $\mathfrak{S}_1 = \mathfrak{S}_0^\perp$ with domain $\text{Dom}(X)$. The closed linear subspace
\[
\mathfrak{S}(\mathfrak{S}_0, X) = \{x \in \mathfrak{S} | x = x_0 \oplus Xx_0, \ x_0 \in \text{Dom}(X) \subset \mathfrak{S}_0\}
\]
is called the graph subspace of $\mathfrak{S}$ associated with the pair $(\mathfrak{S}_0, X)$ or, in short, the graph of $X$.

One easily checks that
\[
\mathfrak{S}(\mathfrak{S}_0, X)^\perp = \mathfrak{S}(\mathfrak{S}_0^\perp, -X^*).
\] (3.1)

We start with presenting a fairly simple and partly known result (see [14]) that characterizes the graph subspaces in terms of the canonical decomposition (2.1).

**Theorem 3.2.** Let $P$ and $Q$ be orthogonal projections in a Hilbert space $\mathfrak{S}$. The subspace $\text{Ran} Q$ is a graph subspace $\mathfrak{S}(\text{Ran} P, X)$ associated with some closed densely defined (possibly unbounded) operator $X : \text{Ran} P \to \text{Ran} P^\perp$ with $\text{Dom}(X) \subset \text{Ran} P$ iff the subspaces $\mathfrak{M}_{01}(P, Q)$ and $\mathfrak{M}_{10}(P, Q)$ in the canonical decomposition of the Hilbert space $\mathfrak{S}$ (2.1) are trivial, i.e.,
\[
\mathfrak{M}_{01}(P, Q) = \mathfrak{M}_{10}(P, Q) = \{0\}.
\] (3.2)
For given orthogonal projection $P$ the correspondence between the closed subspaces $\text{Ran} Q$ satisfying (3.2) and closed densely defined operators $X : \text{Ran} \to \text{Ran} P^\perp$ is one-to-one.

**Proof.** “$\Rightarrow$” Part. Assume (3.2). Let $P'$ and $Q'$ be generic parts of the projections $P$ and $Q$, respectively. From (2.2) it follows that $\text{Ran} Q'$ given by
\[
\text{Ran} Q' = \{x \in \mathfrak{M}' | \cos \Theta x_0 + W^* \sin \Theta x_0, \ x_0 \in \mathfrak{M}_0'\}
\]
is a graph subspace of the generic subspace $\mathfrak{M}' = \text{Ran} P' \oplus (\text{Ran} P')^\perp$. More explicitly,
\[
\text{Ran} Q' = \mathfrak{S}(\text{Ran} P', \mathcal{W}^* \tan \Theta)
\]
with

\[ \text{Dom}(\tan \Theta) = \{ x_0 \in \text{Ran } P' | x_0 = P'x \text{ for some } x \in \text{Ran } Q' \}. \]

Introducing the operator \( X \) from \( \text{Ran } P \) to \( \text{Ran } P \perp \) with

\[ \text{Dom}(X) = \text{Dom}(\tan \Theta) \oplus (\text{Ran } P \ominus \mathfrak{M}'_0) \]

by

\[ Xx = \begin{cases} W^* \tan \Theta x, & x \in \mathfrak{M}'_0, \\ 0, & x \in \text{Ran } P \ominus \mathfrak{M}'_0 \end{cases} \]

yields \( \text{Ran } Q = \mathcal{G}(\text{Ran } P, X) \) since (3.2) holds.

"Only If" Part. Assume that \( \text{Ran } Q \) is a graph subspace associated with a closed densely defined operator \( X \), i.e., \( \text{Ran } Q = \mathcal{G}(\text{Ran } P, X) \). To prove (3.2) it suffices to establish that the points \( \pm 1 \) are not eigenvalues of \( Q - P \), i.e., \( \text{Ker}(Q - P \pm I_0) = 0 \).

Suppose to the contrary that, say, \( +1 \) is an eigenvalue of \( Q - P \), that is,

\[ (Q - P)f = f, \quad 0 \neq f \in \text{Ran } P, \]

and, hence, by (3.1), \( f \) admits the decomposition

\[ f = x + Xx - X^*y + y \]

for some \( x \in \text{Dom}(X) \subset \mathfrak{M} \) and \( y \in \text{Dom}(X^*) \subset \text{Ran } P \perp \). By inspection

\[ (Q - P)(x + Xx - X^*y + y) = Xx + X^*y. \]

Therefore, combining (3.5), (3.6), and (3.7) yields

\[ Xx + X^*y = x + Xx - X^*y + y \]

and

\[ 2X^*y = x + y, \]

which is only possible if \( x = y = 0 \) and, thus, \( f = 0 \). Hence, the point \( +1 \) is not an eigenvalue for \( Q - P \).

One proves that \( -1 \) is not an eigenvalue of \( Q - P \) in a similar way.

The last statement of the theorem follows from the fact that if two closed graph subspaces \( \mathcal{G}(\text{Ran } P, X_1) \) and \( \mathcal{G}(\text{Ran } P, X_2) \) coincide iff \( X_1 = X_2 \) (see, e.g., [14]).

**Remark 3.3.** Under the hypothesis of Theorem 3.2

\[ \mathfrak{M}_{11}(P, Q) = \text{Ker } X, \]

\[ \mathfrak{M}_{00}(P, Q) = \text{Ker } X^* = (\text{Ran } X)^\perp. \]

The first representation holds by the definition (3.4) of the operator \( X \) and (3.9) follows from (3.8) by duality argument (3.1).

The following reformulation of Theorem 3.2 distinguishes the cases of the graph subspaces associated with bounded and unbounded operators \( X \), respectively.

**Corollary 3.4.** Assume Hypothesis 2.1 Then:
(i) The inequality \( \|P - Q\| < 1 \) holds true iff \( \text{Ran} \, Q \) is a graph subspace associated with the subspace \( \text{Ran} \, P \) and some bounded operator \( X \in \mathcal{B}(\text{Ran} \, P, \text{Ran} \, P^\perp) \), that is, \( \text{Ran} \, Q = \mathcal{G}(\text{Ran} \, P, X) \). In this case

\[
(3.10) \quad \|X\| = \frac{\|P - Q\|}{\sqrt{1 - \|P - Q\|^2}}
\]

and

\[
(3.11) \quad \|P - Q\| = \frac{\|X\|}{\sqrt{1 + \|X\|^2}}
\]

(ii) \( \mathfrak{M}_{10}(P, Q) = \mathfrak{M}_{01}(P, Q) = \{0\} \) and \( \|P - Q\| = 1 \) iff \( \text{Ran} \, Q \) is a graph subspace associated with the subspace \( \text{Ran} \, P \) and an unbounded operator \( X \) from \( \text{Ran} \, P \) to \( \text{Ran} \, P^\perp \), i.e., \( \text{Ran} \, Q = \mathcal{G}(\text{Ran} \, P, X) \).

Proof. (i). By Theorem \ref{thm:graph_subspace}

\( \text{Ran} \, Q \) is a graph subspace with respect to the projection \( P \) if and only if \( \mathfrak{M}_{10}(P, Q) = \mathfrak{M}_{01}(P, Q) = \{0\} \), and hence

\[
\|P - Q\| = \|P' - Q'\|
\]

where \( (P', Q') \) is the generic part of the pair \( (P, Q) \). By Theorem \ref{thm:operator_angle}

\[
\|P' - Q'\| = \|\sin \Theta(P', Q')\|
\]

where \( \Theta(P', Q') \) is the operator angle between the subspaces \( \text{Ran} \, P' \) and \( \text{Ran} \, Q' \) and, moreover,

\[
\text{Ran} \, Q' = \mathcal{G}(\text{Ran} \, P', W^* \tan \Theta(P', Q')), \quad \text{Ran} \, Q = \mathcal{G}(\text{Ran} \, P, X),
\]

where \( X \) is the extension of \( W^* \tan \Theta(P', Q') \) given by \( (3.3), (3.4) \).

Clearly, \( X \) is bounded iff the operator \( \tan \Theta(P', Q') \) is bounded. The equality \( (3.4) \) implies

\[
(3.12) \quad \|X\| = \|\tan \Theta(P', Q')\|
\]

and then \( (3.11) \) is a consequence of the trigonometric identity

\[
\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}, \quad \theta \in [0, \pi/2)
\]

combining \( (6.22), (6.23) \), and \( (3.12) \) which proves (i).

(ii). The operator \( X \) is unbounded iff \( \pi/2 \in \text{spec}(\Theta(P', Q')) \). In this case \( \|P' - Q'\| = \|P - Q\| = 1 \) by \( (6.23) \) and \( (3.12) \) which proves (ii).

\[ \square \]

Remark 3.5. Part (i) with inequality sign instead of the equality \( (3.11) \) is well known. A proof can be found, e.g., in \cite[Theorem 1]{16} or \cite[Lemma 2.3]{5}.

Remark 3.6. The orthogonal projection \( Q \) onto the graph subspace \( \mathcal{G}(\text{Ran} \, P, X) \) corresponding to a closed densely defined operator \( X : \text{Ran} \, P \to \text{Ran} \, P^\perp \) can be written as \( 2 \times 2 \) operator matrix with respect to the orthogonal decomposition \( \mathcal{D} = \text{Ran} \, P \oplus \text{Ran} \, P^\perp \)

\[
(3.13) \quad Q = \begin{pmatrix}
(I_{P_0} + X^*X)^{-1} & (I_{P_0} + X^*X)^{-1}X^*
\end{pmatrix}
\]

where \( I_{P_0} = \text{Ran} \, P \) and the bar denotes the closure. The operator entries of \( (3.13) \) are bounded operators since \( \text{Dom}(X^*X) \subset \text{Dom}(X) \) is a core for \( X \) (see, e.g., \cite{16}).
4. Riccati Equation

The main purpose of this section is to introduce a concept of closed densely defined (possibly unbounded) operator solutions to the Riccati equation and to provide a geometric criterion of their existence.

Throughout this section we adopt the following hypothesis.

**Hypothesis 4.1.** Assume that the separable Hilbert space $\mathcal{H}$ is decomposed into the orthogonal sum of two subspaces

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1.$$  \hspace{1cm} (4.1)

Assume, in addition, that $B$ is a self-adjoint operator represented with respect to the decomposition (4.1) as a $2 \times 2$ operator block matrix

$$B = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix},$$

where $A_i \in \mathcal{B}(\mathcal{H}_i), \ i = 0, 1,$ are bounded self-adjoint operators in $\mathcal{H}_i$ while $V \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_0$. More explicitly, $B = A + V$, where $A$ is the bounded diagonal self-adjoint operator,

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

and the operator $V = V^*$ is an off-diagonal bounded operator

$$V = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}.$$

The notion of strong and weak bounded solutions to the Riccati equation with unbounded operator coefficients was introduced in [3] (cf. [24]). In our case where the operator coefficients are bounded but solutions are allowed to be unbounded we use the following definition.

**Definition 4.2.** Assume Hypothesis 4.1. A closed densely defined (possibly unbounded) operator $X$ from $\mathcal{H}_0$ to $\mathcal{H}_1$ with $\text{Dom}(X)$ is called a weak solution to the Riccati equation

$$A_1X - XA_0 - XVX + V^* = 0$$  \hspace{1cm} (4.2)

if for any $x \in \text{Dom}(X)$ and any $y \in \text{Dom}(X^*)$

$$\langle A_1y, Xx \rangle - \langle X^*y, A_0x \rangle - \langle X^*y, VXx \rangle + \langle Vy, x \rangle = 0.$$  \hspace{1cm} (4.3)

A closed densely defined (possibly unbounded) operator $X$ from $\mathcal{H}_0$ to $\mathcal{H}_1$ with $\text{Dom}(X)$ is called a strong solution to the Riccati equation (4.2) if

$$\text{Ran}(A_0 + VX)|_{\text{Dom}(X)} \subset \text{Dom}(X)$$

and

$$A_1X - X(A_0 + VX)x + V^*x = 0 \quad \text{for any} \quad x \in \text{Dom}(X).$$

Obviously, if $X$ is a bounded operator, then the Riccati equation (4.2) can be understood as an operator equality.

The notions of weak and strong solutions to the Riccati equation are in fact equivalent. The precise statement is as follows.
Lemma 4.3. Assume Hypothesis 4.1. A closed densely defined (possibly unbounded) operator $X$ from $\mathcal{S}_0$ to $\mathcal{S}_1$ with $\text{Dom}(X)$ is a weak solution to the Riccati equation (4.2) iff

$$A_1 X x - X (A_0 + V X) x + V^* x = 0 \quad \text{for any} \quad x \in \text{Dom}(X),$$

i.e., $X$ is a strong solution to (4.2).

Proof. Assume that $X$ is a weak solution to the Riccati equation (4.2), i.e., (4.3) holds for any $x \in \text{Dom}(X)$ and any $y \in \text{Dom}(X^*)$. Then

$$(y, A_1 X x + V^* x) = (X^* y, A_0 x + V X x),$$

which implies, in particular, that $A_0 x + V X x \in \text{Dom}(X^{**})$. Since $X$ is closed and densely defined, one infers $X^{**} = X$ and, therefore, $A_1 X x + V^* x = X (A_0 + V X) x$ for all $x \in \text{Dom}(X)$.

The converse statement is obvious. \qed

As a consequence of Lemma (4.3) we obtain the following theorem.

Theorem 4.4. Assume Hypothesis 4.1. A closed densely defined (possibly unbounded) operator $X$ from $\mathcal{S}_0$ to $\mathcal{S}_1$ with $\text{Dom}(X)$ is a weak solution to the Riccati equation (4.2) iff the graph subspace $\mathcal{G}(\mathcal{S}_0, X)$ is invariant for the operator $B$.

Proof. First, assume that $\mathcal{G}(\mathcal{S}_0, X)$ is invariant for $B$. Then

$$B (x \oplus X x) = (A_0 x + V X x) \oplus (A_1 X x + V^* x) \in \mathcal{G}(\mathcal{S}_0, X) \quad \text{for any} \quad x \in \text{Dom}(X).$$

In particular, $A_0 x + V X x \in \text{Dom}(X)$ and

$$A_1 X x + V^* x = X (A_0 x + V X x) \quad \text{for all} \quad x \in \text{Dom}(X).$$

Hence,

$$(y, V^* x + A_1 X x) = (y, X (A_0 x + V X x)) \quad \text{for all} \quad x \in \text{Dom}(X) \quad \text{and} \quad y \in \text{Dom}(X^*),$$

which proves that $X$ is a weak solution to the Riccati equation (4.2).

To prove the converse statement assume that $X$ is a weak solution to the Riccati equation (4.2), i.e., (4.3) holds for any $x \in \text{Dom}(X)$ and any $y \in \text{Dom}(X^*)$. From Lemma 4.3 it follows that

$$A_0 x + V X x \in \text{Dom}(X)$$

and

$$A_1 X x + V^* x = X (A_0 x + V X x), \quad x \in \text{Dom}(X),$$

which proves that the graph subspace $\mathcal{G}(\mathcal{S}_0, X)$ is $B$-invariant. \qed

The next statement is an immediate corollary of Theorems 3.4 and 4.4.

Corollary 4.5. Assume Hypothesis 4.1. Let $\mathcal{G}$ be a closed $B$-invariant subspace of the Hilbert space $\mathcal{S}$ and $P$ and $Q$ denote the orthogonal projections in $\mathcal{G}$ respectively onto $\mathcal{S}_0$ and $\mathcal{G}$. Then:

(i) The inequality

$$\| P - Q \| < 1$$

holds iff $\mathcal{G}$ is a graph subspace, $\mathcal{G} = \mathcal{G}(\mathcal{S}_0, X)$ where $X$ is a bounded solution to the Riccati equation (4.2). In this case equalities (5.10) and (5.11) hold true.
(ii) The equality
\[ \| P - Q \| = 1 \]
holds and
\[ \mathcal{M}_{01}(P, Q) = \mathcal{M}_{10}(P, Q) = \{0\}, \]
iff \( G \) is a graph subspace, \( G = \mathcal{G}(\mathcal{H}_0, X) \), where \( X \) is a closed densely defined unbounded weak solution to the Riccati equation (4.2).

We present an example where the Riccati equation has an unbounded solution.

Example 4.6. Let \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) where \( \mathcal{H}_0 = \mathcal{H}_1 = L^2(0, 1) \). Let \( \Lambda \) be the multiplication operator in \( L^2(0, 1) \),
\[ (\Lambda f)(\lambda) = \lambda f(\lambda), \quad f \in L^2(0, 1), \]
and \( A_0 = -\Lambda, A_1 = \Lambda, \) and \( V = \Lambda^2 \). In this case the Riccati equation (4.2) being of the form
\[ \Lambda X + X\Lambda - X\Lambda^2 X + \Lambda^2 = 0 \]
has a unbounded self-adjoint solution \( X = f(\Lambda) \) where
\[ f(\lambda) = -\frac{1 + \sqrt{1 + \lambda^2}}{\lambda}. \]

5. Solutions Associated With Spectral Subspaces

The structure of the set of solutions to the Riccati equation associated with spectral subspaces of the operator \( B \) can be studied based on the Douglas-Pearcy theorem [12, Theorem 3] on invariant subspaces of normal operators.

Theorem 5.1. Let \( T \) be a bounded self-adjoint operator in a Hilbert space \( \mathcal{H} \) and \( Q \) an orthogonal projection onto a closed \( T \)-invariant subspace of \( \mathcal{H} \). Then the following are equivalent:

(i) \( \operatorname{Ran} Q \) is a spectral subspace of the operator \( T \), i.e., there is a Borel set \( \Delta \subset \mathbb{R} \) such that \( Q = E_T(\Delta) \), where \( E_T(\Delta) \) denotes the spectral projection of \( T \) corresponding to the set \( \Delta \);

(ii) \( \| Q - P \| = 1 \) for any orthogonal projection \( P \) in \( \mathcal{H} \), \( P \neq Q \), such that \( \operatorname{Ran} P \) is \( T \)-invariant;

(iii) \( \dim \mathcal{M}_{10}(P, Q) + \dim \mathcal{M}_{01}(P, Q) > 0 \) for any orthogonal projection \( P \neq Q \) in \( \mathcal{H} \) such that \( \operatorname{Ran} P \) is \( T \)-invariant;

(iv) \( Q \) is an isolated point (in the operator norm topology) of the set of all orthogonal projections onto all \( T \)-invariant subspaces.

Proof. The equivalence of (i), (ii), and (iv) is proven in [12]. The implication (iii) \( \Rightarrow \) (ii) is implied by the decomposition (2.1). Thus, we will only prove the implication (i) \( \Rightarrow \) (iii).

Assume that (i) holds. Suppose to the contrary that (iii) does not hold. That is, \( Q \) is a spectral projection for \( T \) such that \( \mathcal{M}_{10}(P, Q) = \mathcal{M}_{01}(P, Q) = \{0\} \) for some orthogonal projection \( P \neq Q \) such that \( \operatorname{Ran} P \) is \( T \)-invariant. Since \( T \) is self-adjoint, the subspace \( \operatorname{Ran} P \) is reducing for \( T \) and thus \( TP = PT \). Therefore, (see, e.g., [6, Theorem 6.3.2]) \( P \) commutes with all spectral projections of \( T \). In particular, \( PQ = QP \). Since \( \mathcal{M}_{10}(P, Q) = \mathcal{M}_{01}(P, Q) = \{0\} \), we conclude that \( P = Q \), a contradiction. Thus, (i) implies (iii).
In the following we will use a concept of the generalized convergence of closed operators (see [16, Section IV.2]). The generalized convergence is a natural extension to the case of unbounded operators of a notion of uniform convergence. For convenience of the reader we recall its definition adapted to the present context.

**Definition 5.2.** A sequence \( \{X_n\}_{n \in \mathbb{N}} \) of closed densely defined operators \( X_n \) from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) converges in the generalized sense to a closed operator \( X \) from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) if

\[
\lim_{n \to \infty} \|Q_n - Q\| = 0,
\]

where \( Q_n \) and \( Q \) are orthogonal projections onto the graph subspaces \( \mathcal{G}(\mathcal{H}_0, X_n) \) and \( \mathcal{G}(\mathcal{H}_0, X) \), respectively.

The following statement characterizes the set of solutions to the Riccati equation (4.2) associated with spectral subspaces of the operator \( B \).

**Theorem 5.3.** Assume Hypothesis [4,\( ^t \)]. Denote by \( \mathcal{X} \) the set of all (weak) solutions to the Riccati equation (4.2). Then:

(i) If \( X \in \mathcal{X} \) and the invariant graph subspace \( \mathcal{G}(\mathcal{H}_0, X) \) is a spectral subspace of the operator \( B \), i.e., \( \mathcal{G}(\mathcal{H}_0, X) = \ker E_B(\Delta) \) for some Borel set \( \Delta \subset \mathbb{R} \), then \( X \) is an isolated point of the set \( \mathcal{X} \) in the topology of the generalized convergence of operators.

(ii) If \( X \in \mathcal{X} \) is a bounded operator, then the invariant graph subspace \( \mathcal{G}(\mathcal{H}_0, X) \) is a spectral subspace iff \( X \) is an isolated point of the set \( \mathcal{X} \cap \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \) in the operator norm topology, i.e., there is a neighborhood of \( X \) in \( \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \) where the Riccati equation (4.2) has no solutions except \( X \).

**Proof.** (i) Let \( \mathcal{G}(\mathcal{H}_0, X) \) be a spectral subspace for \( B \) and let \( Q \) denote the orthogonal projection in \( \mathcal{H}_0 \) onto \( \mathcal{G}(\mathcal{H}_0, X) \). Suppose to the contrary that \( X \) is not an isolated solution, i.e., there is a sequence \( \{X_n\}_{n \in \mathbb{N}} \) of solutions to (4.2) such that \( X_n \neq X \), \( n \in \mathbb{N} \) and

\[
\lim_{n \to \infty} \|Q_n - Q\| = 0,
\]

where \( Q_n \), \( n \in \mathbb{N} \) denote the orthogonal projections in \( \mathcal{H}_0 \) onto the \( B \)-invariant graph subspaces \( \mathcal{G}(\mathcal{H}_0, X_n) \). By Theorem 5.1 this contradicts the assumption that \( \mathcal{G}(\mathcal{H}_0, X) \) is a spectral subspace for \( B \) which completes the proof of (i).

(ii) Since by Theorem IV.2.23 in [16] the generalized convergence of bounded operators implies its uniform convergence, the “only if” part follows from (i). Therefore, we only prove the “if” part.

Let \( X \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \) be an isolated bounded solution to (4.2). Suppose that \( \mathcal{G}(\mathcal{H}_0, X) \) is not a spectral subspace for \( B \). By Theorem 5.1(iv) this implies that there is a sequence of orthogonal projections \( Q_n \), \( n \in \mathbb{N} \) such that \( \ker Q_n \) is \( B \)-invariant and

\[
\lim_{n \to \infty} \|Q_n - Q\| = 0,
\]

where \( Q \) is the orthogonal projection onto \( \mathcal{G}(\mathcal{H}_0, X) \). Equation (5.1) means that

\[
\|Q_n - Q\| < 1 - \frac{\|X\|}{\sqrt{1 + \|X\|^2}}
\]

for \( n \in \mathbb{N} \) large enough. Therefore,

\[
\|Q_n - P\| \leq \|Q_n - Q\| + \|Q - P\| = \|Q_n - Q\| + \frac{\|X\|}{\sqrt{1 + \|X\|^2}} < 1,
\]

\( n \in \mathbb{N} \) large enough, where \( P \) denotes the orthogonal projection in \( \mathcal{H}_0 \) onto \( \mathcal{H}_0 \). By Theorem 5.2 for those \( n \in \mathbb{N} \), \( \ker Q_n = \mathcal{G}(\mathcal{H}_0, X_n) \) for some \( X_n \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0) \) where \( X_n \) is
a solution to (4.2) by Theorem 4.4. Finally, by Theorem IV.2.23 in [16] equality (5.1) implies
\[ \lim_{n \to \infty} \| X_n - X \| = 0, \]
which contradicts the assumption that \(X\) is an isolated solution.

6. CONTRACTIVE SOLUTIONS ASSOCIATED WITH SPECTRAL SUBSPACES: UNIQUENESS CRITERIA

Corollaries 3.4 and 4.5 imply that under Hypothesis 4.1 the Riccati equation (4.2) has a contractive solution \(X\) if the subspaces \(\mathcal{H}_0\) and \(\mathcal{S}(\mathcal{H}_0, X)\) are invariant for \(A\) and \(B = A + V\), respectively, and the orthogonal projections \(P\) and \(Q\) onto these subspaces satisfy \(\| P - Q \| \leq \sqrt{2}/2\). It is also known [7] that under the same hypothesis the Riccati equation (4.2) has a contractive solution if there exists a self-adjoint involution \(J\) in \(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1\) such that
\[ BJ = JB \]
and the subspace \(\mathcal{H}_1\) is maximal \(J\)-nonnegative, that is, \(\mathcal{H}_1\) is not properly contained in another \(J\)-nonnegative subspace. In principle, these criteria provide complete although somewhat implicit characterization of the set \(\Delta\) of all possible contractive solutions for the Riccati equation (4.2). The main goal of this section is to obtain new characterization of the set \(\Delta\) under the assumption that the Riccati equation has at least one contractive solution associated with a spectral subspace of the operator matrix \(B\). As a by-product of this new description we get some uniqueness results generalizing those obtained in [7].

We start by stating an auxiliary result describing two contractions such that the orthogonal projections onto their graphs commute.

Lemma 6.1. Assume Hypothesis 4.1. Let \(X, Y \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)\). \(\| X \| \leq 1, \| Y \| \leq 1\) be two contractions such that the orthogonal projections in \(\mathcal{H}\) onto their graphs \(\mathcal{S}(\mathcal{H}_0, X)\) and \(\mathcal{S}(\mathcal{H}_0, Y)\) commute. Then
\[ Y|_{\mathcal{L}} = -X|_{\mathcal{L}}, \quad Y|_{\mathcal{L}^\perp} = X|_{\mathcal{L}^\perp}, \]
where
\[ \mathcal{L} = \text{Ker}(I_{\mathcal{H}_0} + Y^*X) = \text{Ker}(I_{\mathcal{H}_0} + X^*Y) \]
is a subspace of \(\text{Ker}(I_{\mathcal{H}_0} - X^*X) \cap \text{Ker}(I_{\mathcal{H}_0} - Y^*Y)\) and
\[ \mathcal{L}^\perp = \mathcal{H}_0 \oplus \mathcal{L}. \]
Moreover,
\[ \mathcal{L} = \text{Ker}(X + Y) \ominus (\text{Ker}(X) \cap \text{Ker}(Y)) \]
and
\[ \mathcal{L}^\perp = \text{Ker}(X - Y). \]

Proof. Note that \(x \in \text{Ker}(I_{\mathcal{H}_0} + Y^*X)\) means that
\[ \| x \|^2 = -(Y^*X, x) = -(X, Yx), \]
which holds if and only if
\[ Yx = -Xx \]
and
\[ x \in \text{Ker}(I_{H_0} - X^*X) \cap \text{Ker}(I_{H_0} - Y^*Y) \]
since both \( X \) and \( Y \) are contractions. Hence
\[ (6.5) \quad \text{Ker}(I_{H_0} + Y^*X) = \text{Ker}(X + Y) \cap \text{Ker}(I_{H_0} - X^*X) \cap \text{Ker}(I_{H_0} - Y^*Y). \]

By symmetry,
\[ \text{Ker}(I_{H_0} + X^*Y) = \text{Ker}(X + Y). \]

Therefore, we have proven equalities (6.2), and the first equality in (6.1).

Given an arbitrary \( x \in L^\perp \), one concludes that
\[ (x, y) = 0 \quad \text{for any} \; y \in L. \]

By (6.5)
\[ y \in \text{Ker}(I_{H_0} - X^*X) \text{ and, hence,} \]
\[ (x, y) + (Xx, Xy) = 0, \quad y \in L, \]
which means that
\[ (6.6) \quad (x \oplus Xx) \perp (y \oplus Xy), \quad y \in L. \]

By hypothesis the orthogonal projections onto \( G(H_0, X) \) and \( G(H_0, Y) \) commute. This implies in particular that
\[ (G(H_0, X) \ominus \Xi) \perp \big(G(H_0, Y) \ominus \Xi\big), \]
where \( \Xi = G(H_0, X) \cap G(H_0, Y) \). Introducing the subspace
\[ N = P_0\big(G(H_0, X) \ominus \Xi\big), \]
where \( P_0 \) denotes the canonical projection from \( H_0 \oplus H_1 \) onto \( H_0 \), one proves by inspection that
\[ (6.7) \quad N \subset L. \]

In particular, (6.6) and (6.7) imply that
\[ (6.8) \quad (x \oplus Xx) \perp (y \oplus Xy), \quad \text{for any} \; x \in L^\perp \text{ and any} \; y \in N. \]

Since
\[ G(H_0, X) \ominus \Xi = \{ y \oplus Xy \mid y \in N\} \]
and \( x \oplus Xx \in G(H_0, X) \), condition (6.8) means that
\[ x \oplus Xx \in \Xi, \quad \text{for any} \; x \in L^\perp. \]

Therefore, by the definition of the subspace \( \Xi \),
\[ Xx = Yx, \quad \text{for all} \; x \in L^\perp, \]
proving the second equality in (6.1) and the following inclusion
\[ L^\perp \subset \text{Ker}(X - Y). \]

It remains to check the opposite inclusion
\[ (6.9) \quad L^\perp \supset \text{Ker}(X - Y). \]

Let \( x \in \text{Ker}(X - Y) \) admit the representation
\[ x = v + w, \]
where \( v \in L \) and \( w \in L^\perp \). Then
\[ (6.10) \quad 0 = (X - Y)x = 2Xv, \]
using (6.1). Since \( \mathcal{L} \subset \text{Ker}(I_{\mathcal{H}_0} - X^*X) \), by (6.10)
\[
0 = v - X^*Xv = v,
\]
that is, \( x = w \in \mathcal{L}^\perp \), proving (6.9). Thus, (6.4) holds.

Finally, we prove (6.3). First, we notice that (6.2) and (6.5) imply that if \( x \in \mathcal{L} \) then \( x \in \text{Ker}(X + Y) \) and for any \( y \in \text{Ker}(X) \)
\[
(x, y) = (X^*X x, y) = (X x, X y) = 0
\]
Similarly, \( (x, y) = 0 \) for any \( y \in \text{Ker}(Y) \). Hence, \( x \) is orthogonal to \( \text{Ker}(X) \cap \text{Ker}(Y) \) and
\[
(6.11)
\]
\[
\mathcal{L} \subset \text{Ker}(X + Y) \oplus (\text{Ker}(X) \cap \text{Ker}(Y)).
\]
Suppose that the inverse inclusion does not hold. Then by (6.11) there is a nonzero \( y \in \text{Ker}(X + Y) \oplus (\text{Ker}(X) \cap \text{Ker}(Y)) \)
orthogonal to \( \mathcal{L} \), i.e. \( y \in \mathcal{L}^\perp \). By the second equality in (6.1) we will have \( Xy = Yy \) which contradicts (6.12). Hence equality (6.3) holds true.

The proof is complete.

Given any contractive solution \( X \) associated with a spectral subspace, Lemma 6.1 allows one to provide a complete characterization of the set of all contractive solutions to the Riccati equation in the sense that all contractive solutions \( Y \) to the Riccati equation (4.2) are in one-to-one correspondence with the closed subspaces
\[
(6.13)
\]
reducing both the operators \( A_0 \) and \( VX \). An explicit description of this correspondence is a content of Theorem 6.2 below. In particular, this theorem provides an efficient criterion for a contractive solution \( X \) associated with a spectral subspace of \( \mathcal{B} \) to be unique.

**Theorem 6.2.** Assume Hypothesis 4.1 and suppose that \( X \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \), \( \|X\| \leq 1 \) is a contractive solution to the Riccati equation (4.2) such that \( \mathcal{S}(\mathcal{H}_0, X) \) is a spectral subspace of the operator \( \mathcal{B} \). Denote by \( \mathcal{S} \) the set of all contractive solutions to the Riccati equation (4.2) and by \( \mathcal{M} \) the lattice of all closed subspaces of the Hilbert space \( \mathcal{H}_0 \). Then the mapping \( \mathcal{T}_X : \mathcal{S} \rightarrow \mathcal{M} \) introduced by
\[
(6.14)
\]
is one-to-one and the image of \( \mathcal{T}_X \) coincides with the set \( \mathcal{R} \) of all closed subspaces \( \mathcal{L} \subset \mathcal{H}_0 \) satisfying (6.13) and reducing both the operators \( A_0 \) and \( VX \).

In particular, if \( X \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \), is a contractive solution to the Riccati equation (4.2) associated with a spectral subspace of the operator \( \mathcal{B} \), then \( X \) is a unique contractive solution to (4.2) iff
\[
(6.15)
\]
\[
\ker(I_{\mathcal{H}_0} - X^*X) \cap \ker(VX - V^*) = \{0\}.
\]

**Proof.** Let \( Y \in \mathcal{S} \) be arbitrary. Since the graph of \( X \) is a spectral subspace of \( \mathcal{B} \), the orthogonal projections onto the graphs of \( X \) and \( Y \) commute. Then by Lemma 5.1
\[
(6.15)
\]
which proves, in particular, that the mapping \( \mathcal{T}_X \) is one-to-one.

It remains to prove that
\[
\text{Ran} \mathcal{T}_X = \mathcal{R}.
\]
We start with the proof of the inclusion

\[(6.16) \quad \text{Ran } \mathcal{T}_X \subset \mathcal{R}.\]

First, we prove that the subspace

\[(6.17) \quad \mathcal{L} = \mathcal{T}_X(Y), \quad Y \in \mathcal{S}\]

reduces \(A_0 + VX\). That is, we need to establish that \(\mathcal{L}\) and \(\mathcal{L}^\perp\) are \((A_0 + VX)\)-invariant subspaces.

The fact that \(\mathcal{L}^\perp\) is \((A_0 + VX)\)-invariant can be proven as follows. Taking into account that both \(X\) and \(Y\) satisfy the Riccati equation (4.2) and by Lemma 6.1 \((X - Y)x = 0\) for \(x \in \mathcal{L}^\perp\), a simple computation shows that

\[(X - Y)(A_0 + VX)x = 0 \quad \text{for any } x \in \mathcal{L}^\perp.\]

Applying Lemma 6.1 again yields \((A_0 + VX)x \in \mathcal{L}^\perp\) for any \(x \in \mathcal{L}^\perp\) which proves that \(\mathcal{L}^\perp\) is \((A_0 + VX)\)-invariant.

Next we establish that \(\mathcal{L}\) is \((A_0 + VX)\)-invariant. Since \(\mathcal{L}^\perp\) is \((A_0 + VX)\)-invariant, the subspace \(\mathcal{L}\) is invariant for the operator \(A_0 + X^*V^*\). Note that the operator \((A_0 + X^*V^*)(I_{\mathcal{H}_0} + X^*)\) is self-adjoint. This fact is proven in [23], [25] but alternatively can easily be seen from the identity

\[ (x + Xx, B(x + Xx)) = (x, (A_0 + X^*V^*)(I_{\mathcal{H}_0} + X^*)x) \quad \text{for any } x \in \mathcal{H}_0. \]

Taking into account that by Lemma 6.1 \(X^*X|\mathcal{L} = I_{\mathcal{L}}\), one concludes that

\[ (I_{\mathcal{H}_0} + X^*)((A_0 + VX)x = (A_0 + X^*V^*)(I_{\mathcal{H}_0} + X^*)x = 2(A_0 + X^*V^*), \quad x \in \mathcal{L}, \]

which implies \((A_0 + VX)x \in \mathcal{L}\), proving that \(\mathcal{L}\) is also \((A_0 + VX)\)-invariant. Thus we have proven that \(\mathcal{L}\) reduces the operator \(A_0 + VX\).

The same arguments hold for the operator \(A_0 + VY\). In particular, the subspace \(\mathcal{L}\) reduces the operator \(A_0 + VY\).

Now we are ready to prove inclusion (6.16). Combining the facts that \(\mathcal{L}\) reduces \(A_0 + VX\) as well as \(A_0 + VY\) and that \(X|_{\mathcal{L}} = -Y|_{\mathcal{L}}\) implies that \(\mathcal{L}\) reduces the operators \(A_0, VX,\) and \(VY\). In particular,

\[ 0 = (A_1Y - YA_0 - YY + V^*)x \]
\[ = (-A_1Xx + XA_0x - VX^2x + V^*)x \]
\[ = 2(XVX - V^*)x, \quad x \in \mathcal{L}, \]

proving that

\[(6.18) \quad \mathcal{L} \subset \text{Ker}(XVX - V^*),\]

and hence (6.17) holds, since \(\mathcal{L} \subset \text{Ker}(I_{\mathcal{H}_0} - X^*)\) by Lemma 6.1. Thus, \(\mathcal{L} = \mathcal{T}_X(Y) \subset \mathcal{R}\) which proves the inclusion (6.16).

In order to complete the proof of the theorem it remains to prove the opposite inclusion

\[(6.19) \quad \mathcal{R} \subset \text{Ran } \mathcal{T}_X.\]

Let \(\mathcal{L} \subset \mathcal{R}\) be arbitrary. Introduce the contraction \(Y\) by setting

\[(6.20) \quad Y|_{\mathcal{L}} = -X|_{\mathcal{L}}, \quad \text{and} \quad Y|_{\mathcal{L}^\perp} = X|_{\mathcal{L}^\perp}. \]

We need to show that \(Y \in \mathcal{S}\) and that \(\mathcal{T}_X(Y) = \mathcal{L}\).
For $x \in L^\perp$ one obtains
\[(A_1Y - YA_0 - YYV + V^*)x = (A_1X - XA_0 - XVX + V^*)x = 0\]
using the invariance of $L^\perp$ with respect to the operators $A_0$ and $VX$, the fact that $X$ solves the Riccati equation (4.2), and the second equality in (6.20).

Using the invariance of $L$ with respect to the operators $A_0$ and $VX$, and the first equality in (6.20), for $x \in L$ one obtains
\[(A_1Y - YA_0 - YYV + V^*)x = (A_1X - XA_0 - XVX + V^*)x = 0\]
for $x \in L$.

Since $L \subset \mathbb{R}$, and hence $L \subset \text{Ker}(XVX - V^*)$, for $x \in L$ one concludes that $(XVX - V^*)x = 0$. Therefore,
\[(A_1Y - YA_0 - YYV + V^*)x = (A_1X + XA_0 - XVX + V^*)x,
\]
which is zero, since $X$ solves the Riccati equation (4.2). Hence,
\[(A_1Y - YA_0 - YYV + V^*)x = 0, \quad x \in L.
\]
Thus, we constructed a contractive solution $Y$ to the Riccati equation (4.2), which yields $Y \in S$. Applying Lemma 6.1 implies that
\[Y|_{\mathcal{T}_X(Y)} = -X|_{\mathcal{T}_X(Y)}, \quad Y|_{\mathcal{T}_X(Y)^\perp} = X|_{\mathcal{T}_X(Y)^\perp}.
\]
and
\[\mathcal{T}_X(Y) \subset \text{Ker}(I_{\mathcal{H}_0} - X^*X).
\]
Since $L \subset \mathcal{R}$ one also concludes that
\[L \subset \text{Ker}(I_{\mathcal{H}_0} - X^*X).
\]
Combining (6.20), (6.21), (6.22), and (6.23) proves that
\[\mathcal{T}_X(Y) = L.
\]
Thus, inclusion (6.19) is proven. The proof is complete.

**Remark 6.3.** Notice that the subspace
\[L' = \{x \oplus Xx \mid x \in L\},
\]
where $L$ stands for any subspace referred to in Theorem 6.2, is simultaneously $A$- and $B$-invariant and, therefore, it can be split from the further considerations if necessary.

**Proof.** On the one hand,
\[A(x \oplus Xx) = A_0x \oplus A_1Xx \]
\[= A_0x \oplus XA_0x, \quad x \in L
\]
since $x \in L \subset \text{Ker}(XVX - V^*)$ taking into account that $X$ solves (4.2), proving that $L'$ is also $A$-invariant.

Since $X$ solves the Riccati equation (4.2), for any $x \in \mathcal{H}_0$, in particular, for $x \in L$ one has
\[B(x \oplus Xx) = (A_0 + VX)x \oplus (V^* + A_1X)x \]
\[= (A_0 + VX)x \oplus X(A_0 + VX)x,
\]
which proves that $L'$ is also $B$-invariant, since $L$ is $(A_0 + VX)$-invariant by hypothesis. As an immediate corollary of Theorem 6.2 we get the following uniqueness results.
Corollary 6.4. Assume Hypothesis 4.1. Let $X \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$, $\|X\| \leq 1$ be a contractive solution to the Riccati equation (4.2) such that $\mathcal{H}(\mathcal{H}_0, X)$ is a spectral subspace of the operator $\mathcal{B}$.

(i) If $X$ is a strictly contractive operator, i.e., $\|Xx\| < \|x\|$ for any $x \in \mathcal{H}_0$, $x \neq 0$, then $X$ is a unique contractive solution to (4.2).

(ii) If $\ker(I_{\mathcal{H}_0} - X^*X) \cap \ker(\text{Im}(VX)) = \{0\}$, then $X$ is a unique contractive solution to (4.2). In particular, if $VX$ is a dissipative operator with positive imaginary part, then $X$ is a unique contractive solution to (4.2).

Proof. (i) If $X$ is a strictly contractive operator, then $\ker(I_{\mathcal{H}_0} - X^*X) = \{0\}$. Hence (6.14) holds and by Theorem 6.2 the operator $X$ is the unique contractive solution to (4.2).

(ii) Suppose that $Y$ is a contractive solution of the Riccati equation (4.2). Introducing the subspace $\mathcal{L} = \ker(I_{\mathcal{H}_0} + Y^*X)$, by Theorem 6.2 one concludes that $\mathcal{L} \subset \ker(I_{\mathcal{H}_0} - X^*X) \cap \ker(XVX - V^*)$.

In particular, $X^*(XVX - V^*)x = 0$, $x \in \mathcal{L}$.

By Theorem 6.2 $\mathcal{L}$ reduces the operator $VX$. In particular, $X^*VX = VX$, $x \in \mathcal{L}$.

Combining (6.25) and (6.26) yields

$$(XVX - V^*)x = 0, \quad x \in \mathcal{L},$$

that is, $x = 0$ for any $x \in \mathcal{L}$, since $\ker(VX - X^*V^*) = \{0\}$ by hypothesis. Hence $\mathcal{L} = \ker(I_{\mathcal{H}_0} + Y^*X) = \{0\}$ which proves that $Y = X$ using Lemma 6.1, completing the proof.

Remark 6.5. Statement (i) of Corollary 6.4 concerning the spectral subspaces $\text{Ran} \mathcal{E}_B(\Delta)$ associated with closed Borel sets $\Delta$ of the real axis appeared first in [2] with a somewhat different strategy of the proof based on a description of maximal $J$-non-negative subspaces in a Krein space.

Remark 6.6. Some different uniqueness results for Riccati equations in finite-dimensional Hilbert spaces were obtained in [8]. Note that the property for a solution to the Riccati equation to be isolated is related to its stability [7]. Stability of invariant subspaces is studied in [1].

To illustrate the statement of Theorem 6.2 suppose that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are copies of the same Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}$. Assume that $A_0 = A_1 = 0$ and $V = I_\mathcal{H}$ is the identity operator in $\mathcal{H}$. Then the Riccati equation (4.2) (after the appropriate identification of the copies $\mathcal{H}_0$ and $\mathcal{H}_1$) reads as $X^2 = I_\mathcal{H}$ and it obviously has the solution $X = I_\mathcal{H}$ associated with the eigenspace of $\mathcal{B}$ corresponding to the eigenvalue one. Therefore, $X$ is an isolated point in the set of all solutions. Obviously,

$$\ker(I_\mathcal{H} - X^*X) \cap \ker(XVX - V^*) = \mathcal{H}.$$
\[ VX = I_{\mathcal{R}}, \text{ and } \mathfrak{R} \text{ reduces } A_{0} = 0. \] Therefore, by Theorem 5.3, all solutions to the Riccati equation can be uniquely parameterized by closed subspaces \( \mathfrak{L} \subset \mathfrak{R}. \) The case of \( \mathfrak{L} = \{0\} \) corresponds to the identity solution \( X = I_{\mathfrak{R}}, \) the case \( \mathfrak{L} = \mathfrak{R} \) corresponds to the solution \( \bar{X} = -X = -I_{\mathfrak{R}} \) which is also isolated being associated with the eigenspace of \( B \) corresponding to the eigenvalue negative one. If \( \dim \mathfrak{R} > 1, \) all the other solutions to the Riccati equation \( X^{2} = I_{\mathfrak{R}} \) can be uniquely parameterized by the nontrivial subspaces \( \mathfrak{L} \subset \mathfrak{R} \) of nonzero codimension and thus correspond to invariant subspaces of the operator \( B \) which are not spectral ones. All those solutions \( Y \) are unitary self-adjoint operators in \( \mathfrak{R} \) different from \( I_{\mathfrak{R}} \) and \( -I_{\mathfrak{R}}, \) with \( \mathfrak{L} = \operatorname{Ker}(Y + I_{\mathfrak{R}}), \) and hence

\[
\operatorname{spec}(B|_{\beta(0, Y)}) = \{1, -1\}.
\]

It is worth to note that given a solution \( X \) to the Riccati equation (4.2), in contrast to the hypothesis of Theorem 5.2 not necessarily contractive and not necessarily associated with a spectral subspace of the operator matrix \( B, \) the set \( \mathfrak{R} \) of all closed subspaces of \( \mathfrak{H} \) reducing both the operators \( A_{0} \) and \( VX \) and satisfying (5.13) admits a dual description in terms of the corresponding subspaces of \( \mathfrak{H} \).

In order to formulate the precise statement we introduce the set \( \mathfrak{R}_{s} \) of all closed subspaces \( \mathfrak{L}_{s} \subset \mathfrak{H}_{s} \) reducing both \( A_{1} \) and \( V^{*}X^{*} \) and satisfying

\[
\mathfrak{L}_{s} \subset \operatorname{Ker}(I_{\mathfrak{H}_{s}} - XX^{*}) \cap \operatorname{Ker}(X^{*}V^{*}X^{*} - V).
\]

**Proposition 6.7.** Let \( X \) be a solution to the Riccati equation (4.2). Then, under the above notations, the sets \( \mathfrak{R} \) and \( \mathfrak{R}_{s} \) are in one-to-one correspondence under the mapping

\[
\mathfrak{L} \mapsto X\mathfrak{L}, \quad \mathfrak{L} \in \mathfrak{R}.
\]

In particular, the inverse mapping is given by \( \mathfrak{L}_{s} \mapsto X^{*}\mathfrak{L}_{s} \), \( \mathfrak{L}_{s} \in \mathfrak{R}_{s}. \)

**Proof.** Let \( \mathfrak{L} \in \mathfrak{R} \) be arbitrary. Set \( \mathfrak{L}_{s} = X\mathfrak{L}. \) For any \( x^{*} \in \mathfrak{L}_{s} \) there is a unique \( x \in \mathfrak{L} \) such that \( x^{*} = Xx. \) To prove this, suppose to the contrary that there is another element \( y \neq x \) in \( \mathfrak{L} \) such that \( Xy = x^{*}. \) Then \( X(x - y) = 0 \) and thus \( X^{*}X(x - y) = 0. \) But \( x, y \in \operatorname{Ker}(I_{\mathfrak{H}_{s}} - XX^{*}) \) and, therefore, \( X^{*}X(x - y) = x - y \neq 0. \) A contradiction.

If \( x_{s} = Xx, \) then

\[
(A_{1} - V^{*}X^{*})x_{s} = (A_{1} - V^{*}X^{*})Xx = (A_{1}X - V^{*})x = X(A_{0} - VX)x \in \mathfrak{L}_{s}
\]

by successive use of (6.13), the hypothesis that \( X \) solves the Riccati equation (4.2), and that \( \mathfrak{L} \) is obviously \( (A_{0} - VX) \)-invariant. Thus \( \mathfrak{L}_{s} \) is \( (A_{1} - V^{*}X^{*}) \)-invariant. Moreover, \( A_{1}x_{s} = A_{1}XXx = XA_{0}x = \mathfrak{L}_{s} \), since \( x \in \mathfrak{L} \subset \operatorname{Ker}(XX^{*} - V^{*}) \) by (6.13) and \( X \) solves (4.2). Thus \( \mathfrak{L}_{s} \) in addition is \( A_{1} \)-invariant, proving that \( \mathfrak{L}_{s} \) is invariant for both \( A_{1} \) and \( V^{*}X^{*}. \) Further, we note that \( x_{s} = Xx \in \operatorname{Ker}(I_{\mathfrak{H}_{s}} - XX^{*}) \) and hence a simple computation

\[
(V - X^{*}V^{*}X^{*})x_{s} = (V - X^{*}V^{*}X^{*})Xx = (X^{*}XV^{*}X^{*})x = X^{*}(XX^{*} - V^{*})x = 0
\]

proves inclusion (6.27).

Now we claim that \( \mathfrak{L}_{s}^{\perp} = \mathfrak{H}_{s} \cap \mathfrak{L}_{s} \) is invariant for \( V^{*}X^{*}. \) To show this choose arbitrary \( y \in \mathfrak{L}_{s}^{\perp} \) and \( x \in \mathfrak{L}_{s}. \) Then

\[
(V^{*}X^{*}y, x) = (y, XVx) = (y, XX^{*}V^{*}X^{*}x)
\]

since \( VX = X^{*}V^{*}X^{*}x. \) The subspace \( \mathfrak{L}_{s} \) is invariant for \( V^{*}X^{*} \) by (6.27) and by (6.27) we get

\[
XX^{*}V^{*}X^{*}x \in \mathfrak{L}_{s}.
\]
Thus, $(V^*X*y, x) = 0$. Since $x$ and $y$ are arbitrary, this implies that $V^*X*y \in \Sigma_+^A$. Therefore, we proved that $\Sigma_+$ reduces both $A_1$ and $V^*X^*$. Thus, $X\Sigma \in \mathbb{R}^+$ and the mapping $\Sigma \rightarrow X\Sigma$ maps $\mathbb{R}$ onto $\mathbb{R}_{++}$. By symmetry we also conclude that this mapping is one-to-one.

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