From Free Motion on a 3-Sphere to the Zernike System of Wavefronts Inside a Circular Pupil

K B Wolf
Instituto de Ciencias Físicas
Universidad Nacional Autónoma de México
Av. Universidad s/n, Cuernavaca, Mor. 62210, México
E-mail: bwolf@icf.unam.mx

Abstract. Classical or quantum systems that stem from a basic symmetry are seen to be special in having several important properties. The harmonic oscillator and the Bohr system are such. Recent research into the Zernike system provides reasons to include it in this privileged class. Here we show that free motion on the 3-sphere can be projected down to produce classical orbits or complete and orthogonal bases for wavefronts in a circular pupil. This line of inquiry has been pursued in company with N.M. Atakishiyev, G.S. Pogosyan, C. Salto-Alegre, and A. Yakhno.

1. Introduction
The study of physical systems with higher —also called hidden— symmetries, dominated much of the research of group-theoretical methods in mechanics, classical and quantum, during the second half of the twentieth century. The harmonic oscillator, Kepler-Bohr, Calogero and related systems were the favourite subjects. In retrospect it seems that the Zernike system [1, 2] was mostly missed out; we think it may be equally important.

In Refs. [3–9] we carried out various analyses from which in the present paper I will choose to present some of the results, starting with N-spheres in Sect. 2. Sect. 3 will reduce the dimension of the free equation of motion on the N-sphere to that of a non-standard Hamiltonian on the (N−1)-sphere. This defines the classical mechanical Zernike system in Sect. 4, and the wave or quantum mechanical Zernike in Sect. 5. In Sect. 6 we reconvene the constants of the motion whose Poisson brackets and corresponding operator commutators do not close into a Lie algebra, but into a cubic superintegrable Higgs algebra [10]. The concluding Sect. 7 adds a perspective on the inquiries that may be made on this remarkable system.

2. The group SO(N+1), Laplace-Beltrami and Zernike operators
In an (N+1)-dimensional ambient space of Cartesian coordinates \( \{s_i\}_{i=1}^{N+1} \in \mathbb{R}^{N+1} \), an N-sphere manifold \( S^N \) is given by the points satisfying \( |s|^2 := \sum_{i=1}^{N+1} s_i^2 = 1 \). The symmetry algebra of this manifold is generated by the \( 1/2N(N+1) \) operators,

\[
\Lambda_{i,j} = s_i \partial_{s_j} - s_j \partial_{s_i} = -\Lambda_{j,i}, \quad i, j \in \{1, 2, \ldots, N+1\}
\]
where $\partial_{s_i} := \partial / \partial s_i$, which are well known to close into the Lie algebra $\mathfrak{so}(N+1)$, and exponentiate to rotations $\exp(i \sum_{i,j=1}^{N+1} \alpha_{i,j} \Lambda_{i,j})$ in the Lie group $\text{SO}(N+1)$. Our previous work was especially dedicated to the case when $N = 3$, which has some interesting properties that are not generic for any other $N$; yet we can draw some conclusions that hold for generic $N \geq 2$.

The operators in (1) are skew-adjoint in the usual Hilbert space of square-integrable functions on the $N$-sphere, $L^2(S^N)$. Since this group is compact, it only has finite-dimensional irreducible representations, and since the sphere is a minimal coset space of the group, i.e., as manifolds $S^N = \text{SO}(N+1)/\text{SO}(N)$, the matrix representations so obtained can be characterized by only one integer $J \in \{0, 1, 2, \ldots \}$ through the quadratic Casimir [11], or Laplace-Beltrami, operator,

$$\Delta^{(N)}_{\text{LB}} := \frac{1}{2} \sum_{i,j=1}^{N+1} \Lambda^2_{i,j} = \Delta^{(N-1)}_{\text{LB}} + \sum_{i=1}^{N} \Lambda^2_{i,N+1} = |s|^2 \nabla^2 - (s \cdot \nabla)^2 - 2s \cdot \nabla,$$  \tag{2}

where $|s|^2 = \sum_{i=1}^{N+1} s_i^2$, $s \cdot \nabla_s = \sum_{i=1}^{N+1} s_i \partial_{s_i}$ and $\nabla^2_s = \sum_{i=1}^{N+1} \partial^2_{s_i}$.

On the $S^N$ manifold, where $|s|^2 = 1$ and the functions $\Psi(s)$ on the manifold are not functions of the radial coordinate $|s|$, the radial derivative terms $s \cdot \nabla_s$ will vanish from (2). The representations of the $\text{SO}(N+1)$ group characterized by the original Laplace-Beltrami invariant (2) act on spaces of hyper-spherical harmonics $\Psi^N_{J_1 \ldots J_n}(s)$, where the dots in the lower index stand for further representation row labels, and satisfy the eigenvalue equation

$$\Delta^{(N)}_{\text{LB}} \Psi^N_{J_1 \ldots J_n}(s) = J(J + N - 1) \Psi^N_{J_1 \ldots J_n}(s), \quad J \in \{0, 1, 2, \ldots \}. \tag{3}$$

On the other hand, the Zernike system was defined in the original paper of Frits Zernike [1], by the Schrödinger-type equation clearly related to (2) for $N = 3$ on the 2-plane $\xi = (x, y)$ and inside the unit circle, $|\xi| \leq 1$,

$$\tilde{Z}(\xi) \Psi(\xi) : = \left( \nabla^2 - (\xi \cdot \nabla)^2 - 2\xi \cdot \nabla \right) \Psi(\xi) = -E_J \Psi(\xi), \quad E_J = J(J+2), \tag{4}$$

and the requirement that the solutions be finite inside the pupil, $|\Psi(\xi)|_{|\xi| \leq 1} < \infty$. Other authors have questioned the source of this equation noting that building an orthonormal set of polynomial solutions can also be made recursively with the basic Schmidt method [12, 13]. The paper of Zernike with Brinkman [2] indicates that the connection with the 3-sphere was intuited perhaps at the very writing of the first article, but hidden from the readers’ view.

The spectrum and various other features of the Zernike system, once it is solved, would lead us to believe that this is, or is related to, the two-dimensional harmonic oscillator. In fact, serious attempts were done to relate both systems through finding two-term recurrence relations between neighbouring states but without success [14]: the two systems, 2-dim oscillator and Zernike system are fundamentally distinct.

3. Reduction of dimension $N$ to $N - 1$

The $R^{N+1}$ coordinates $\{s_i\}_{i=1}^{N+1}$, reduced to the $S^N$ sphere manifold $|s| = 1$, can be written in terms of Cartesian coordinates $\{\xi_i\}_{i=1}^{N}$ for an $N$-sphere, $\sum_{i=1}^{N} \xi_i^2 = 1$, and an angle, $\varphi \in S^1$, as

$$s_1 = \xi_N \cos \varphi, \quad s_2 = \xi_N \sin \varphi, \quad s_3 = \xi_{N-1}, \quad \ldots, \quad s_k = \xi_{N-k-2}, \quad \ldots, \quad s_{N+1} = \xi_1. \tag{5}$$

In terms of these coordinates, the Laplace-Beltrami operator (2) reads

$$\Delta^{(N)}_{\text{LB}} = \Delta^{(N-1)}_{\text{LB}} - \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial \xi_i} + \frac{1}{\xi_N} \frac{\partial}{\partial \xi_N} + \frac{1}{\xi_N^2} \frac{\partial^2}{\partial \varphi^2}, \tag{6}$$
where the lower-dimensional Laplace-Beltrami operator is built with the $N$ coordinates $\xi_i$, as $\Delta_{\text{LB}}^{(N-1)} = \frac{1}{2} \sum_{i,j=1}^{N} (\xi_i \partial_{\xi_j} - \xi_j \partial_{\xi_i})^2$, and we note that the coordinate $\varphi \in S^1$ is separated as a second derivative, which for functions $\sim \exp(im\varphi)$ has the spectrum $-m^2$, with integer $m \in \{0, \pm 1, \pm 2, \ldots\} = \mathbb{Z}$.

The change of coordinates (5) thus provides a reduction of $m$-classified hyperspherical harmonics $\Psi_{J,...,m}^N(s)$ in (3), through integration over $\varphi$, leaving the $(N-1)$-sphere coordinates $\{\xi_i\}_{i=1}^{N-1}$ in (5) as arguments, whose value is nonzero only for $m = 0$, namely

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{d}\varphi \Psi_{J,...,m}^N (s(\xi_i, \varphi)) = \delta_{m,0} \psi_{J,...,0}^N(\xi_i),$$

and satisfying an equation of the form (4) in the $N-1$ coordinates $\{\xi_i\}_{i=1}^{N-1}$, since $\xi_N^2 = 1 - \sum_{i=1}^{N-1} \xi_i^2$. We note that the remaining independent $\xi$-manifold thus provides Cartesian coordinates for the disk. Below, we shall search for solutions of the Zernike differential equation (4) by products of \textit{separated} functions of a single coordinate; on the disk, of radius and angle for $N-1 = 2$. At this point we renounce to use generic dimension $N$, and henceforth fix $N = 3$, so that free motion on $S^3$ will become the Zernike system defined by (4), that can be incarnated as a classical or quantum/wave mechanical system.

4. The classical Zernike system

The \textit{classical counterpart} of the Zernike system that naturally corresponds to the Schrödinger form of (4) on the 2-dim disk, is obtained by straightforward \textit{de-quantization}, i.e., keeping position coordinates $\vec{\xi} = (\xi_1, \xi_2)^\top = (x, y)^\top =: \vec{q}$ as such, or as radius and angle $r$ and $\phi$, and replacing gradients by the momentum 2-vector $\vec{p} = (p_x, p_y)^\top$, also as such or with radial and angular components $p_r$ and $p_\phi$,

$$\nabla \mapsto i\vec{p} = i\begin{pmatrix} p_x \\ p_y \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad r := |\vec{q}| = \sqrt{x^2 + y^2},$$

$$\nabla^2 \mapsto -(p_x^2 + p_y^2) = -\left( p_r^2 + \frac{p_\phi^2}{r^2} \right), \quad \vec{q} \cdot \nabla \mapsto i\vec{q} \cdot \vec{p} = i(xp_x + yp_y) = i rp_r.$$  \hspace{1cm} (8)

The classical Hamiltonian function $H$ is obtained from $\hat{Z}$ in (4) under Cartesian and polar coordinates as $H(\vec{q}, \vec{p}) = -\hat{Z}(\vec{q}, -i\nabla)$, so that [3]

$$H(\vec{q}, \vec{p}) := (p_x^2 + p_y^2) - (xp_x + yp_y)^2 - 2i(xp_x + yp_y)$$

$$= (1 - r^2)p_r^2 + p_\phi^2/r^2 + 2i rp_r,$$ \hspace{1cm} (10)

while its energy $E$ will be as in (4), although not quantized by $J$. The appearance of $i = \sqrt{-1}$ in this Hamiltonian is anomalous for classical Hamiltonians; yet we shall find that trajectories and energies will be real, so in a sense, this $i$ ‘disappears.’

Originally, Zernike allowed two parameters into the differential equation (4), $\alpha$ in front of the $(\vec{q} \cdot \nabla)^2$ term, and $\beta$ in front of the linear $\vec{q} \cdot \nabla$ term. Since he required a self-adjoint $\hat{Z}$ operator on the disk, he showed that these parameters had to be $\alpha = -1$ and $\beta = -2$ [1,4]. The classical system can incorporate these parameters to a Hamiltonian $H^{(\alpha, \beta)}(\vec{q}, \vec{p})$ within certain regions of the $(\alpha, \beta)$ plane that have been explored more thoroughly in Ref. [3]. To shorten discussions of various cases that are interesting in their own right, here it will suffice to take these $(\alpha, \beta)$ values as prescribed by (4).

To find the classical trajectory and velocity of a unit-mass particle subject to the Hamilton-Jacobi equation $H + \partial S/\partial t = 0$ with the Hamiltonian (10) in polar coordinates $(r, \phi)$, we
proposed [3] the action to be separated as $S(r, \phi) = R(r) + p_\phi \phi - Et$. From here—after some ingenious computations [3]—one finds the geometric $r(\phi)$ and dynamical $x(t), y(t)$ trajectories to be, modulo $\phi$ and $t$ initial values, respectively

$$r^2(\phi) = \frac{B/A}{1 - \varepsilon \cos 2\phi} \leq 1, \quad \text{i.e. an ellipse inside the pupil } r = 1,$$

$$x(t) = \frac{1}{\sqrt{2\varepsilon}} \left( (\varepsilon + 1) \frac{A + C \cos(4t\sqrt{E+1})}{2(E+1)} \pm \frac{B}{A} \right),$$

where $A = E + p_\phi^2$, $B = 2p_\phi^2$, $C = \sqrt{(E - p_\phi^2)^2 - 4p_\phi^2}$. The trajectories are thus elliptic, with eccentricity

$$\varepsilon = \frac{C}{A} = \sqrt{(E - p_\phi^2)^2 - 4p_\phi^2} / (E + p_\phi^2).$$

In Fig. 1 we show trajectories obtained from (11), with equal-time marks derived from (12). The period between two consecutive passes of the mass point through the same position is easily determined from (12) to be $T = \pi / (2(E+1))$. This fact distinguishes the classical Zernike system from the harmonic oscillator, whose oscillation period is characteristically independent of the energy. We recall Bertrand’s theorem [15] which states that only the harmonic oscillator and Kepler systems result in closed, elliptic trajectories. His theorem however, hinges on Hamiltonians of the standard form $\sim \frac{1}{2} p^2 + V(q)$; in the Zernike Hamiltonian (10), the quartic term $(\vec{q} \cdot \vec{p})^2$ contravenes this assumption. Still, it is noteworthy that elliptic trajectories occur in this particular non-standard Hamiltonian.

![Figure 1](image-url)

Figure 1. Elliptic trajectories in the classical Zernike system for values of angular momentum and energy $(p_\phi, E) = (3.35), (3.20), \text{ and (3.15). These trajectories are real and remain inside the unit disk; they belong to the upper-left region of allowed trajectories in the next figure.}

The conditions on the eccentricity $0 \leq \varepsilon \leq 1$, and $r^2 \geq 0$ in (11), restrict the ranges of energy $E$ and angular momentum $p_\phi$ as shown in Fig. 2 to $0 \leq |p_\phi| \leq \sqrt{E+1} - 1$ in the upper-left ($E \geq 0, |p_\phi| \geq 0$) allowed region, and to $|p_\phi| \geq \sqrt{E+1} + 1$ for the right ($E \geq -1, |p_\phi| \geq 1$) region. Alternatively, the allowed ranges of the energy $E$ are $E \geq p_\phi^2 + 2|p_\phi|$ for $|p_\phi| \geq 0$ and $-1 \leq E \leq p_\phi^2 - 2|p_\phi|$ for $|p_\phi| \geq 1$. The ellipses degenerate into circles ($\varepsilon = 0$) when $E = p_\phi^2 \pm 2|p_\phi|$ on the boundary parabolas of Fig. 2, and into lines ($\varepsilon = 1$) when $p_\phi = 0$ or $E = -1$, again bounding the two regions of the figure. In the upper-left allowed region of the figure, the semi-major and semi-minor axes are $\mu_{maj}$ and $\mu_{min}$, and the area $= \pi \mu_{maj} \mu_{min}$ inside the elliptical...
orbits are [3]

\[
\mu_{\text{maj}} - \mu_{\text{min}} = \sqrt{\frac{2p_\phi^2}{(E+p_\phi^2)}} \leq 1, \quad \text{area} = \pi \frac{|p_\phi|}{\sqrt{E+1}} < \pi.
\] (14)

I interpret that the allowed right-side region in Fig. 2 \((E \geq -1, |p_\phi| \geq (E+1)^2)\), is un-physical because any large angular momentum \(p_\phi\) can ‘coexist’ with null energy \(E = 0\), and is ‘non-Zernike’ because the area in (14) is greater than \(\pi\) so the trajectory ellipse contains the unit disk. This was shown in [3] to stem from the projection of free motion from a hyperboloid on or outside the Zernike’s pupil.

Figure 2. Regions of the angular momentum—energy \(|p_\phi|, E\) plane where the trajectories in the classical Zernike system are allowed (white) or forbidden (grey) to form proper, real and closed ellipses. The allowed region on the right may be seen as unphysical, because it contains trajectories unbounded angular momentum, but small or negative energy.

5. The quantum/wave Zernike system

It would appear that we can follow the usual quantization recipe, where two-dimensional position and momentum \((q, p) \in \mathbb{R}^4\) are replaced by operators \((q, -i\nabla_q)\), that are essentially self-adjoint in the Hilbert space of functions \(f(q)\) that are square-integrable over the full plane \(\mathbb{R}^2\). Yet in the Zernike system the range of positions is the disk \(D = \{q||q|^2 \leq 1\}\). Marshalling the proper definitions, the case can be made to indeed define a Dirac basis of positions, with kets \(|q\rangle\) on the disk and thereby build a Hilbert space [16]. Alternatively, one may advance the argument that the classical Zernike Hamiltonian (10) is quantized on the basis of replacing \(\vec{p}^2\) and \(\vec{q} \cdot \vec{p}\) by their ‘quantum’ counterparts, \(-\nabla^2\) and \(-i(\vec{q} \cdot \nabla)\), in a particular ‘Zernike’ ordering scheme \(Z\). Thus the quantization from (10) to (4) need not make reference to any of the possible quantization rules, but only self-adjointness on the disk. And we end up using the position space restricted to the disk, and addressing the coordinates that permit separable solutions to the differential equation (4).

To address this question we return to Sect. 3, particularizing (5) to \(N = 3\), and propose two
out of the six orthogonal coordinate systems for the 3-sphere $S^3$ [17–19]. We call them

\[
\text{System I: cylindrical} \quad \mathfrak{so}(4) \supset \mathfrak{so}(2)^{(\gamma)} \oplus \mathfrak{so}(2)^{(\phi_2)}
\]

\[
\begin{align*}
& s_1 = \cos \gamma \cos \varphi, \\
& s_2 = \cos \gamma \sin \varphi, \\
& s_3 = \sin \gamma \cos \phi_2, \\
& s_4 = \sin \gamma \sin \phi_2, \\
& 0 < \gamma < \frac{1}{2} \pi, \\
& 0 \leq \varphi, \phi_2 < 2\pi,
\end{align*}
\]

\[
\text{System II: spherical} \quad \mathfrak{so}(4) \supset \mathfrak{so}(3)^{(\gamma, \theta)} \supset \mathfrak{so}(2)^{(\varphi)}
\]

\[
\begin{align*}
& s_1 = \sin \chi \sin \theta \cos \varphi, \\
& s_2 = \sin \chi \sin \theta \sin \varphi, \\
& s_3 = \sin \chi \cos \theta, \\
& s_4 = \cos \chi, \\
& 0 < \theta, \chi < \pi, \\
& 0 \leq \varphi < 2\pi,
\end{align*}
\]

where we are singling out the angle $\varphi \in S^1$ over which the integration took place in Sect. 3.

The $\mathfrak{so}(4)$ Laplace-Beltrami operator $\Delta_{\text{ILB}}^{(3)}$ can be written in each coordinate system as a second-order differential operator [20–23], which evinces that the two systems yield separable solutions, namely

\[
\Delta_{\text{ILB}}^{(3)} = \frac{\partial^2}{\partial \gamma^2} + (\cot \gamma - \tan \gamma) \frac{\partial}{\partial \gamma} + \frac{1}{\sin^2 \gamma} \frac{\partial^2}{\partial \phi_2^2} + \frac{1}{\cos^2 \gamma} \frac{\partial^2}{\partial \varphi^2},
\]

\[
\Delta_{\text{IIIB}}^{(3)} = \frac{\partial^2}{\partial \chi^2} + 2 \cot \chi \frac{\partial}{\partial \chi} + \frac{1}{\sin^2 \chi} \left( \frac{\partial^2}{\partial \theta^2} + \frac{\partial}{\partial \theta} \left( \frac{\partial^2}{\partial \theta^2} \right) \right).
\]

The integration over $\varphi$ of the states on the 3-sphere, brings them down to the 2-sphere. For System I, these yield the original Zernike solutions [1]; up to normalization $C_{n,m}$ they are

\[
\Psi_{n,m}^{\text{I}}(x,y) = C_{n,m} e^{-i \frac{\pi}{2} m (x^2+y^2)^{\frac{1}{2}}} P_{n_m}^{(m,0)}(1-2(x^2+y^2)) e^{i \theta \phi_2},
\]

projected from the 2-sphere in polar coordinates $x = \sin \gamma \sin \phi, y = \sin \gamma \cos \phi$ for the upper half-sphere $z = \cos \gamma \geq 0$, and with the special Jacobi polynomials $P_{n_m}^{(m,0)}(\chi)$. These original Zernike solutions are shown arranged in an $(n,m)$ pyramid in Fig. 3. Here $n \in Z_0^+$ is the principal quantum number, and $m \in Z$ the angular momentum number, restricted to $|m| \leq n$ —as in the 2-dimensional harmonic oscillator; finally, the radial quantum number is $n_r := \frac{1}{2} (n-|m|) \in Z_0^+$.

For System II, the solutions on the sphere after projection are, again up to normalization,

\[
\Psi_{n_1,n_2}^{\text{II}}(x,y) = C_{n_1,n_2} (1-x^2)^{\frac{1}{2} n_1} C_{n_2}^{n_1+1}(x) P_{n_1} \left( \frac{y}{\sqrt{1-x^2}} \right),
\]

where the coordinates are $x = \cos \chi$ and $y = \sin \chi \cos \theta$ on the disk, projected from the upper half-sphere $z = \sin \chi \sin \theta \geq 0$ on ranges $\chi \in [\gamma, \theta_0]$, with Gegenbauer $C_{n_2}^{n_1+1}(\chi)$ and Legendre $P_{n_1}(\eta)$ polynomials. These solutions are orthogonal and complete on $\mathcal{L}^2(D)$; they were found in [4], and are shown in Fig. 4, arranged as a pyramid with the edges given by the eigen-labels $n_1, n_2 \in Z_0^+$, that sum to the principal quantum number $n = n_1+n_2 = J \in Z_0^+$ —as in the 2-dimensional harmonic oscillator separated in Cartesian coordinates.

Here we have included only the two coordinate systems in (15), where the surface of the projecting 2-sphere is a parametrized polar grid, such that the equator or one meridian coincide with the circle boundary of the disk, respectively. There is a continuum of System II coordinates where the poles of that sphere are allowed around that boundary [9], and there is a 2-parameter continuum of elliptic coordinates on the sphere such that one of the coordinate lines coincide with the disk boundary; the solutions are Heun polynomials in the two elliptic Jacobi coordinates [7], with alternative expressions in terms of their trigonometric representation [8]. Moreover, one more significant result is that the overlap coefficients, which allow us to express System II solutions (19) in terms of System I ones (18) turn out to be special Clebsch-Gordan coefficients [5,6], expressible by Hahn and Racah polynomials of restricted type. In Ref. [9] we followed the two subalgebra reductions at the heads of (15) to show how one should expect these coefficients using purely Lie-algebraic arguments.
Figure 3. The original Zernike solutions (18) on the unit disk, arranged in a pyramid \((n, m)\) by the principal quantum number \(n\), of energy \(E_n = n(n + 2)\), and angular momentum \(m\) \([1, 4]\).

Figure 4. New Zernike solutions to the quantum/wave equation (19) on the unit disk arranged as a pyramid by the Cartesian quantum numbers \((n_1, n_2)\), energy \(E_n\) with \(n = n_1 + n_2\) \([4]\).

6. Algebraic structure: constants of the motion

A set of interesting properties of systems with a higher (but ‘hidden’) symmetry pertain its separability in more than a single set of coordinates. This is in turn related to the existence of constants of the motion beyond the obvious one —in this case— of angular momentum, in their classical and/or quantum models.

6.1. The classical system

The action function \(S(r, \phi)\) that was used to solve for the trajectories out of the Hamilton-Jacobi equation \(H + \partial S/\partial t = 0\), assumed that the Hamiltonian (10) is separable in polar coordinates \((r, \phi)\), proposing that it is separated as \(S(r, \phi) = R(r) + p_\phi \phi - Et\) \([3]\). This reveals that the angular momentum observable

\[
I_1 := p_\phi = xp_y - yp_x,
\]

is a rather evident constant of the motion. It will generate rotations through the exponential Poisson operator \(\exp(\alpha\{I_1, \phi\})\). When the action function is separated following other coordinate systems, as \(S(\theta, \phi) = S_1(\theta) + S_2(\phi)\) being the Hamilton-Jacobi action, with \(S_2(\phi) = p_\phi \phi\); this will render the equation into two equations bound by a separation constant.

Reverting to Cartesian coordinates, it was found \([3]\) that, together with (20),

\[
I_2 := (1 - x^2 - y^2)p_y^2 + 2iyyp_y,
\]

\[
I_3 := (1 - x^2 - y^2)p_x^2 + 2ixyp_x,
\]

are constants of the motion serving that role. The Zernike Hamiltonian (10) can be expressed in terms of them as \(H = I_1^2 + I_2 + I_3\), while we note that they have again the imaginary unit \(i\) hanging out. We re-defined three linearly independent constants out of these as follows,

\[
J_1 := \frac{1}{2}I_1 = \frac{1}{2}(xp_y - yp_x),
\]

\[
J_2 := \frac{1}{2}(I_3 - I_2) = \frac{1}{2} \left(1 - (x^2 + y^2)\right)(p_x^2 - p_y^2) + i(xp_y - yp_x),
\]

\[
J_3 := \{J_1, J_2\} = \left(1 - (x^2 + y^2)\right)p_xp_y + i(xp_y + yp_x),
\]
These functions Poisson-commute with the Zernike Hamiltonian (10), \( \{ \hat{J}_i, H \} = 0 \), but do not commute with each other; rather, they close into a ‘not-quite-su(2)’ structure:

\[
\{ \hat{J}_1, \hat{J}_2 \} = \hat{J}_3, \quad \{ \hat{J}_3, \hat{J}_1 \} = \hat{J}_2, \quad \{ \hat{J}_2, \hat{J}_3 \} = 2 \hat{J}_1 (H - 4 \hat{J}_1^2 + 2). \tag{26}
\]

While \( \hat{J}_1 \) generates rotations between \( \hat{J}_2 \) and \( \hat{J}_3 \), the last Poisson bracket is cubic in the generator \( \hat{J}_1 \), and the algebraic structure is called a cubic Higgs algebra [10].

### 6.2. The quantum/wave system

The Zernike Schrödinger equation (4) was written on the 2-sphere in Sect. 2. There the rotation \( \text{SO}(3) \) generators are \( \hat{L}_i := -(\xi_i \partial_k - \xi_k \partial_i) \), for \( i, j, k \) cyclic permutation of 1,2,3. Again, except for \( \hat{L}_3 \), the other two are not symmetries of (4). And there is the matter of a measure factor \( A(r) = (1 - r^2)^{1/4} \) to pass between self-adjoint operators on the sphere and on the disk, so that \( \hat{W} := A\hat{Z}A^{-1} \) is the quantum Zernike Hamiltonian operator,

\[
\hat{W} = (1 - x^2)\partial_{xx} - 2xy\partial_{xy} + (1 - y^2)\partial_{yy} - 2(x\partial_x + y\partial_y) + \frac{1}{4}(1 - x^2 - y^2)^{-1} + \frac{3}{4}. \tag{27}
\]

The separation constants of the differential equation in the two coordinate systems yield operators that commute with this \( \hat{W} \) [4]. They are the quantum/wave versions of (23)–(25), albeit with a definite quantization rule

\[
\hat{J}_1 = \frac{1}{2}(y\partial_x - x\partial_y),
\hat{J}_2 = -\frac{1}{2}\left(1 - (x^2 + y^2)\right)(\partial_{xx} - \partial_{yy}) + x\partial_x - y\partial_y, \tag{29}
\hat{J}_3 = -\left(1 - (x^2 + y^2)\right)\partial_{xy} + y\partial_x + x\partial_y. \tag{30}
\]

They also close into an algebra which is not a linear Lie algebra, but includes, as in the classical case (26), the cubic nonlinearity of a superintegrable Higgs algebra [10],

\[
[\hat{J}_1, \hat{J}_2] = \hat{J}_3, \quad [\hat{J}_3, \hat{J}_1] = \hat{J}_2, \quad [\hat{J}_2, \hat{J}_3] = 2(\hat{Z} - 4\hat{J}_1^2)\hat{J}_1, \tag{31}
\]

which can be compared with (26).

### 7. Concluding remarks

We have given a brief account of the Zernike system as a projection of free motion on the 3-sphere. This system is distinct from that of the 2-dimensional harmonic oscillator. The classical model of the oscillator, we should remind ourselves, is the shadow of a point on a rotating sphere, projected orthogonally on a plane. There, the angular velocity of the sphere is constant but the radius of the sphere is a (square-root) function of the energy; in the Zernike system, this radius is fixed while the angular velocity of the rotation of the sphere is proportional to the energy. The two systems are thus palpably different.

Much of the work in the references of our group has been left out of this recount. The pair of parameters \( (\alpha, \beta) \) whose \((-1, -2)\) case defines the original Zernike system, can be made to range freely, with classically allowed systems restricted to certain regions as in Fig. 2. In that case, free motion on hyperboloids will lead to dynamics where trajectories will be conic curves [3]. The sphere, we should also remind the reader, is the momentum space of 3-dimensional Helmholtz fields [24]; its transformations and aberrations are of interest in optics. Beside the model-specific analysis, the Zernike system also provides a fertile field for studies in orthogonal polynomial bases for functions on the unit disk. And higher polynomials of the Askey-Wilson scheme appear in
the interbasis expansions. The closest complex sibling to the group of motions on the 3-sphere, is on 3-hyperboloids, forming the Lorentz group $SO(3,1)$.

The Lie group $SO(4) = SU(2)^{(1)} \otimes SU(2)^{(2)}$ of rotations and motions of the 3-sphere is the only semisimple Lie group that splits with a correspondingly rich structure and physical realizations: it is germane to angular momentum coupling, to the two-dimensional harmonic oscillator, as well as to the discrete optics of Cartesian and polar pixellated screens [25]. And finally, this group describes a physical system that conforms to a Higgs, rather than a Lie, algebra [3,4].

**Acknowledgments**

I acknowledge the help of Ms. Juana Angelina Romero Vergara in forming the figures. Thanks are due to project AG-100119, Óptica Matemática awarded by the Dirección General de Asuntos del Personal Académico, Universidad Nacional Autónoma de México. Previous work has benefited from coauthor’s funds, particularly, Alexander Yakhno thanks the support of project PRO-SNI-2019 Universidad de Guadalajara.

**References**

[1] F Zernike 1934 Beugungstheorie des Schneidenverfahrens und Seiner Verbesserten Form der Phasenkontrastmethode *Physica* 1 689

[2] F Zernike and H C Brinkman 1935 Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome, *Verh. Akad. Wet. Amst. (Proc. Soc. Sci.)* 38 161

[3] G S Pogosyan, K B Wolf and A Yakhno 2017 Superintegrable classical Zernike system, *J. Math. Phys.* 58 072901

[4] G S Pogosyan, C Salto-Alegre, K B Wolf and A Yakhno 2017 Quantum superintegrable Zernike system, *J. Math. Phys.* 58 072101

[5] G S Pogosyan, K B Wolf and A Yakhno 2017 New separated polynomial solutions to the Zernike system on the unit disk and interbasis expansion *J. Opt. Soc. Am. A* 34 1844

[6] N M Atakishiyev, G S Pogosyan, K B Wolf and A Yakhno 2017 Interbasis expansions in the Zernike system, *J. Math. Phys.* 58 103505

[7] N M Atakishiyev, G S Pogosyan, K B Wolf and A Yakhno 2018 Elliptic basis for the Zernike system: Heun function solutions *J. Math. Phys.* 59 073503 doi: 10.1063/1.5030759

[8] N M Atakishiyev, G S Pogosyan, K B Wolf and A Yakhno 2019 On elliptic trigonometric form of the Zernike system and polar limits, *Physica Scripta* 59 https://doi.org/10.1088/1402-4896/aafebc

[9] N M Atakishiyev, G S Pogosyan, K B Wolf and A Yakhno 2019 Spherical geometry, Zernike’s separability, and interbasis expansion coefficients, *J. Math. Phys.* 60 101701 doi: 10.1063/1.5099974

[10] P W Higgs 1979 Dynamical symmetries in a spherical geometry, *J. Phys. A* 12 309

[11] R L Anderson and K B Wolf 1970 Complete sets of functions on homogeneous spaces with compact stabilizers, *J. Math. Phys.* 11 3176

[12] A B Bhatia and E Wolf 1954 On the circle polynomials of Zernike and related orthogonal sets, *Math. Proc. Cambridge Phil. Soc.* 50 40

[13] M Born and E Wolf 1999, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light* 7th ed. (Cambridge University Press) p. 986

[14] M E H Ismail and R Zhang 2016, Classes of bivariate orthogonal polynomials, *SIGMA* 12 021 arXiv:1502.07256.

[15] J Bertrand 1873 Théorème relatif au mouvement d’un point attiré vers un centre fixe, *Comptes Rendues Acad. Sci.* 77 849

[16] E Celeghini, M Gadella and M A del Olmo 2019 Zernike functions, rigged Hilbert spaces, and potential applications, *J. Math. Phys.* 60 083508

[17] C Grosche, G S Pogosyan and A N Sissakian 1995 Path integral discussion for Smorodinsky-Winternitz potentials II. Two- and three-dimensional sphere, *Fortschr. Phys.* 43 453

[18] C Grosche, Kh H Karayan, G S Pogosyan and A N Sissakian 1997 Quantum motion on the three-dimensional sphere: the ellipse-cylindrical basis, *J. Phys. A: Math. Gen.* 30 1629

[19] A A Izmest’ev, G S Pogosyan, A N Sissakian and P. Winternitz 1999 Contraction of Lie algebras and separation of variables. N-dimensional sphere, *J. Math. Phys.* 40 1549

[20] E G Kahnis and W. Miller Jr. 1986 Separation of variables on n-dimensional Riemannian manifolds. I. The n-sphere $S_n$ and Euclidean n-space $R_n$, *J. Math. Phys.* 27 1721 doi:10.1063/1.527088.
[21] E G Kalnins, W Miller Jr. and G. S. Pogosyan 1996 Superintegrability and associated polynomial solutions. Euclidean space and sphere in two-dimensions space and sphere, *J. Math. Phys.* 37 6439

[22] G S Pogosyan, A N Sissakian and P. Winternitz 2002 Separation of variables and Lie algebra contractions. Applications to special functions, *Phys. Part. Nuclei* 33 Suppl. 1 S123

[23] W Miller Jr., E G Kalnins, and G S Pogosyan 2006 Exact and quasi-exact solvability of second-order superintegrable systems. I. Euclidean space preliminaries, *J. Math. Phys.* 47 033502

[24] P González-Casanova and K B Wolf 1995 Interpolation of solutions to the Helmholtz equation, *Num. Meth. Part. Diff. Eqs.* 11 77

[25] K B Wolf 2010 Discrete systems and signals on phase space, *Appl. Math. & Inf. Science* 4 141