UNIFORMIZATION THEORY AND 2D GRAVITY
I. LIOUVILLE ACTION AND INTERSECTION NUMBERS

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ABSTRACT

This is the first part of an investigation concerning the formulation of 2D gravity in the
framework of the uniformization theory of Riemann surfaces. As a first step in this direction
we show that the classical Liouville action appears in the expression of the correlators of
topological gravity. Next we derive an inequality involving the cutoff of 2D gravity and
the background geometry. Another result, still related to uniformization theory, concerns a
relation between the higher genus normal ordering and the Liouville action. We introduce
operators covariantized by means of the inverse map of uniformization. These operators
have interesting properties including holomorphicity. In particular they are crucial to show
that the chirally split anomaly of CFT is equivalent to the Krichever-Novikov cocycle and
vanishes for deformation of the complex structure induced by the harmonic Beltrami differ-
entials. By means of the inverse map we propose a realization of the Virasoro algebra on
arbitrary Riemann surfaces and find the eigenfunctions for the holomorphic covariant oper-
ators defining higher order cocycles and anomalies which are related to \( W \)-algebras. Finally
we face the problem of considering the positivity of \( e^\sigma \), with \( \sigma \) the Liouville field, by proposing an explicit construction for the Fourier modes on compact Riemann surfaces. These
functions, whose underlying number theoretic structure seems related to Fuchsian groups
and to the eigenvalues of the Laplacian, are quite basic and may provide the building blocks
to properly investigate the long-standing uniformization problem posed by Klein, Koebe and
Poincaré.
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1 Introduction

The last few years have witnessed remarkable progress in 2D quantum gravity [1–6]. Important connections between non-perturbative 2D gravity, generalized KdV hierarchies, topological and Liouville gravity have been discovered [7–9] (see [10] for reviews). This interplay has provided a good opportunity for important progress both in physics and mathematics.

Despite these results there are still important unsolved problems. For example a direct and satisfactory proof of the equivalence of the different models of 2D quantum gravity is still lacking. In particular in the continuum formulation of 2D gravity the higher genus correlators have not yet been worked out. One of the problems concerns the integration on the moduli space. The Schottky problem hinders integration on the Siegel upper plane. Analogously to the relations between intersection theory on the moduli space and KdV [7–9], the solution of the Schottky problem is based on deep relationships between algebraic geometry and integrable systems. In particular Shiota and Mulase [11] proved that, according to the Novikov conjecture, a matrix in the Siegel upper plane is a Riemann period matrix if and only if the corresponding $\tau$-function satisfies Hirota’s bilinear relations. Apparently it is technically impossible to satisfy these constraints in performing the integration on the Siegel upper plane. Similar aspects are intimately linked with 2D gravity. In particular, results from matrix models, where the integration on the moduli space is implicitly performed, suggest that at least in some cases the integrand is a total derivative.

Many of the results concerning the continuum formulation of 2D gravity have been derived by considering its formulation in Minkowskian space whereas Liouville theory is intrinsically Euclidean. Furthermore, most of the results are known on the sphere and the torus. On the other hand the Liouville equation is the condition of constant negative curvature. Thus to get some insight into Liouville theory it is necessary to concentrate the analysis of the continuum formulation of 2D gravity on surfaces with negative Euler characteristic. A crucial step in setting the mathematical formalism for this purpose has been made by Zograf
and Takhtajan [12, 13]. Starting with the intention of proving a conjecture by Polyakov, they showed that Liouville theory is strictly related to uniformization theory of Riemann surfaces. In particular it turns out that the Liouville action evaluated on the classical solution is the Kähler potential for the Weil-Petersson metric on the Schottky space. It seems that the results in [12, 13] have relevant implications for 2D gravity which are still largely unexplored. In our opinion they will serve as a catalyst to formulate a “quantum geometrical” approach to 2D gravity in the framework of uniformization theory. Some aspects concerning uniformization theory and 2D gravity have been considered in [14].

The present paper is the first part of a work whose basic aim is to attempt to bring together the different branches of mathematical technology in order to investigate 2D gravity in a purely geometrical context.

The organization of the paper is as follows. In section 2 we summarize basic facts about the uniformization theory of Riemann surfaces and the Liouville equation. The points we will consider include the explicit construction of differentials in terms of theta functions. This is particularly useful because by means of essentially two theorems it is possible to recover and understand, by some pedagogical “theta gymnastics”, a lot of basic facts concerning Riemann surfaces. Furthermore, we will discuss in detail the properties of the Poincaré metric in relation to Liouville equation and uniformization theory. One of the aims of this section is to clarify some points concerning the properties of the Liouville field. For example note that in current literature the field $e^{\gamma\sigma}$ is sometimes considered as a $(1, 1)$-differential, which is in contradiction with the fact that, since $g = e^{\gamma\sigma} \hat{g}$, with $\hat{g}$ a background metric, $e^{\gamma\sigma}$ must be a $(0, 0)$-differential. This point is related to the difficulties arising in the definition of conformal weights in Liouville theory.

In section 3 we introduce a set of operators which are covariantized by means of the inverse of the uniformization map and give their chiral (polymorphic) and non chiral eigenfunctions. We will see that these operators have important properties including holomorphicity. Next we consider the cocycles associated to the above operators. In particular, the cocycle associated to the covariantized third derivative is the Fuchsian form of the Krichever-Novikov (KN) cocycle. In this framework we show that the normal ordering for operators defined on $\Sigma$ is related to classical Liouville theory. Another result concerns the equivalence between the chirally split anomaly of CFT and the KN cocycle. In particular it turns out that this anomaly vanishes under deformation of the complex structure induced by the harmonic Beltrami differentials. These results suggest to consider higher order anomalies in the framework
of uniformization theory of vector bundles on Riemann surfaces ($W$-algebras).

In section 4 we consider a sort of higher genus generalization of the Killing vectors which is based on the properties of the Poincaré metric and of the inverse map of uniformization. This analysis will suggest a realization of Virasoro algebra on arbitrary Riemann surfaces.

In section 5 we introduce an infinite set of regular functions which can be considered as “building-blocks” to develop the higher genus Fourier analysis. We argue that the number theoretic structure underlying the building-blocks is related to the uniformization problem. In particular we investigate the structure of the eigenvalues of the Laplacian. Besides its mathematical interest one of the aims of this investigation is to provide a suitable tool to recognize the modes of the Liouville field $\sigma$. This is an attempt to face the problem, usually untouched in the literature, of considering metric positivity in performing the quantization of Liouville theory.

In section 6 we find a direct link between Liouville and topological gravity. In particular we will show that the first tautological class, which enters in the correlators of topological gravity [6–8], has the classical Liouville action as potential, in particular

$$\kappa_1 = \frac{i}{2\pi^2} \partial \sigma^{(h)}_{cl}.$$  \hspace{1cm} (1.1)

In the last section we discuss important aspects concerning the role of the Poincaré metric chosen as background. Furthermore, by means of classical results on univalent functions, we derive an interesting inequality involving the cutoff of 2D gravity and the background geometry whose consequences should be further investigated.

## 2 Uniformization Theory And Liouville Equation

In this section we introduce background material for later use. In particular we begin by giving a procedure to explicitly construct any meromorphic differential defined on a Riemann surface. After that we introduce basic facts concerning the uniformization theory [15,16] and the Liouville equation. Then we investigate the properties of the inverse map of uniformization, and consider the linearized version of the Schwarzian equation\{\bar{J}_{h}^{1}, z\} = T^{F}$, with $T^{F}$ the Liouville stress tensor (or Fuchsian projective connection). We conclude the section with some remarks on the standard approach to Liouville gravity.
2.1 Differentials On $\Sigma$: Explicit Construction

Let us start by recalling some basic facts about the space of the $(p,q)$-differentials $T^{p,q}$. Let $\{(U_\alpha,z_\alpha)|\alpha \in I\}$ be an atlas with harmonic coordinates on a Riemann surface $\Sigma$. A differential in $T^{p,q}$ is a set of functions $f \equiv \{f_\alpha(z_\alpha,\bar{z}_\alpha)|\alpha \in I\}$ where each $f_\alpha$ is defined on $U_\alpha$. These functions are related by the following transformation in $U_\alpha \cap U_\beta$

$$f_\alpha(z_\alpha,\bar{z}_\alpha)(dz_\alpha)^p(d\bar{z}_\alpha)^q = f_\beta(z_\beta,\bar{z}_\beta)(dz_\beta)^p(d\bar{z}_\beta)^q, \quad f \in T^{p,q},$$

(2.1)

that is $f_\alpha$ transforms as $\partial_{z_\alpha}^{p} \partial_{\bar{z}_\alpha}^{q}$.

In the case of the Riemann sphere $\hat{C} \equiv C \cup \{\infty\}$, all the possible transition functions $z_- = g_{-+}(z_+)$ between the two patches $(U_+,z_\pm)$ of the standard atlas are holomorphically equivalent to $g_{-+}(z_+) = z_+^{-1}$, that is $\hat{C}$ has one complex structure only (no moduli). Therefore giving $f_+(z_+,\bar{z}_+)$ fixes $f_-(z_-,\bar{z}_-)$ and vice versa.

In the higher genus case fixing a component of $f$ in a patch is not sufficient to uniquely fix the other functions in $f \equiv \{f_\alpha(z_\alpha,\bar{z}_\alpha)|\alpha \in I\}$. As an example we consider the case of meromorphic $n$-differentials $f^{(n)}$ on a compact Riemann surface of genus $h$. The Riemann-Roch theorem guarantees that it is possible to fix the points in $^1$ $\operatorname{Div} f^{(n)}$ up to (in general) $h$ zeroes, say $P_1, \ldots, P_h$, whose position is fixed by $P_{h+1}, \ldots, P_p, Q_1, \ldots, Q_q, \quad q = p - 2n(h - 1)$, the conformal structure of $\Sigma$ and (in general) on the choice of the local coordinates. An instructive way to see this is to explicitly construct $f^{(n)}$. In order to do this we first recall some facts about theta functions. Let us denote by $\Omega$ the $\beta$-period matrix

$$\Omega_{jk} \equiv \oint_{\beta_j} \omega_k,$$

(2.2)

where $\omega_1, \ldots, \omega_h$ are the holomorphic differentials with the standard normalization

$$\oint_{\alpha_j} \omega_k = \delta_{jk},$$

(2.3)

$\alpha_k, \beta_k$ being the homology cycles basis. The theta function with characteristic reads

$$\Theta^{[a\,\,b]}(z|\Omega) = \sum_{k \in \mathbb{Z}^h} e^{\pi i (k+a) \cdot \Omega \cdot (k+a) + 2\pi i (k+a) \cdot (z+b)} , \quad \Theta(z|\Omega) \equiv \Theta^{[0\,\,0]}(z|\Omega),$$

(2.4)

where $z \in C^h, \quad a, b \in R^h$. When $a_k, b_k \in \{0, 1/2\}$, $\Theta^{[a\,\,b]}(-z|\Omega) = (-1)^{4a \cdot b} \Theta^{[a\,\,b]}(z|\Omega)$. The $\Theta$-function is multivalued under a lattice shift in the $z$-variable

$$\Theta^{[a\,\,b]}(z + n + \Omega \cdot m|\Omega) = e^{-\pi i m \cdot \Omega \cdot m - 2\pi i m \cdot z + 2\pi i (a \cdot n - b \cdot m)} \Theta^{[a\,\,b]}(z|\Omega).$$

(2.5)

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1Let $\{P_k\} (\{Q_k\})$ be the set of zeroes (poles) of a $n$-differential $f^{(n)}$. The formal sum $\operatorname{Div} f^{(n)} \equiv \sum_{k=1}^{p} P_k - \sum_{k=1}^{q} Q_k$ and $\deg f^{(n)} \equiv p - q$, define the divisor and the degree of $f^{(n)}$ respectively. It turns out that $\deg f^{(n)} = 2n(h - 1)$. 

5
Let us now introduce the prime form $E(z, w)$. It is a holomorphic $-1/2$-differential both in $z$ and $w$, vanishing for $z = w$ only

$$E(z, w) = \frac{\Theta [g] (I(z) - I(w)|\Omega)}{h(z)h(w)}.$$  \hspace{1cm} (2.6)

Here $h(z)$ denotes the square root of $\sum_{k=1}^{h} \omega_k(z) \theta_{uk} [g] (u|\Omega) |_{u_k=0}$; it is the holomorphic $1/2$-differential with non singular (i.e. $\partial_{uk} \Theta [g] (u|\Omega) |_{u_k=0} \neq 0$) odd spin structure $[g]$. The function $I(z)$ in (2.6) denotes the Jacobi map

$$I_k(z) = \int_{P_0}^{z} \omega_k, \quad z \in \Sigma,$$  \hspace{1cm} (2.7)

with $P_0 \in \Sigma$ an arbitrary base point. This map is an embedding of $\Sigma$ into the Jacobian

$$J(\Sigma) = \mathbb{C}^h / L_\Omega, \quad L_\Omega = \mathbb{Z}^h + \Omega \mathbb{Z}^h.$$  \hspace{1cm} (2.8)

By (2.5) it follows that the multivaluedness of $E(z, w)$ is

$$E(z + n \cdot \alpha + m \cdot \beta, z) = e^{-\pi im \cdot \Omega \cdot m - 2\pi im \cdot (I(z) - I(w))} E(z, w).$$  \hspace{1cm} (2.9)

In terms of $E(z, w)$ one can construct the following $\frac{h}{2}$-differential with empty divisor

$$\sigma(z) = \exp \left( - \sum_{k=1}^{h} \oint_{\alpha_k} \omega_k(w) \log E(z, w) \right),$$  \hspace{1cm} (2.10)

whose multivaluedness is

$$\sigma(z + n \cdot \alpha + m \cdot \beta) = e^{\pi i (h-1) m \cdot \Omega \cdot m - 2\pi im \cdot (\Delta - (h-1)I(z))} \sigma(z),$$  \hspace{1cm} (2.11)

where $\Delta$ is (essentially) the vector of Riemann constants [17]. Finally we quote two theorems:

a. **Abel Theorem** [15]. A necessary and sufficient condition for $\mathcal{D}$ to be the divisor of a meromorphic function is that

$$I(\mathcal{D}) = 0 \mod (L_\Omega) \text{ and } \deg \mathcal{D} = 0.$$  \hspace{1cm} (2.12)

b. **Riemann vanishing theorem** [17]. The function

$$\Theta \left( I(z) - \sum_{k=1}^{h} I(P_k) + \Delta |\Omega \right), \quad z, P_k \in \Sigma,$$  \hspace{1cm} (2.13)

either vanishes identically or else it has $h$ zeroes at $z = P_1, \ldots, P_h$. 

We are now ready to explicitly construct the differential $f^{(n)}$ defined above. First of all note that
\[ \tilde{f}^{(n)} = \sigma(z)^{2n-1} \frac{\prod_{k=h+1}^{p} E(z, P_k)}{\prod_{j=1}^{p-2n(h-1)} E(z, Q_j)}, \tag{2.14} \]
is a multivalued $n$-differential with $\text{Div} \tilde{f}^{(n)} = \sum_{k=h+1}^{p} P_k - \sum_{k=1}^{p-2n(h-1)} Q_k$. Therefore we set
\[ f^{(n)}(z) = g(z)\tilde{f}^{(n)}, \tag{2.15} \]
where, up to a multiplicative constant, $g$ is fixed by the requirement that $f^{(n)}$ be singlevalued. From the multivaluedness of the $E(z, w)$ and $\sigma(z)$ it follows that, up to a multiplicative constant
\[ g(z) = \Theta (I(z) + \mathcal{D}|\Omega), \tag{2.16} \]
with
\[ \mathcal{D} = \sum_{k=h+1}^{p} I(P_k) - \sum_{k=1}^{p-2n(h-1)} I(Q_k) + (1-2n)\Delta. \tag{2.17} \]
By Riemann vanishing theorem $g(z)$ has just $h$-zeroes $P_1, \ldots, P_h$ fixed by $\mathcal{D}$. Thus the requirement of singlevaluedness also fixes the position of the remainder $h$ zeroes. To make manifest the divisor in the RHS of (2.15) we first recall that the image of the canonical line bundle $K$ on the Jacobian of $\Sigma$ coincides with $2\Delta$ [17]. On the other hand, since
\[ [K^n] = \left[ \sum_{k=1}^{p} P_k - \sum_{k=1}^{p-2n(h-1)} Q_k \right], \tag{2.18} \]
by Abel theorem we have\(^2\)
\[ \text{Div} \Theta (I(z) + \mathcal{D}|\Omega) = \text{Div} \Theta \left( I(z) - \sum_{k=1}^{h} I(P_k) + \Delta|\Omega \right), \tag{2.19} \]
and by Riemann vanishing theorem
\[ \text{Div} \Theta (I(z) + \mathcal{D}|\Omega) = \sum_{k=1}^{h} I(P_k). \tag{2.20} \]

Above we have considered harmonic coordinates. If one starts with arbitrary coordinates
\[ ds^2 = \bar{g}_{ab} dx^a dx^b, \tag{2.21} \]
\(^2\)The square brackets in (2.18) denote the divisor class associated to the line bundle $K^n$. Two divisors belong to the same class if they differ by a divisor of a meromorphic function.
the harmonic ones, defining the conformal form \(2g_{zz}|dz|^2\), are determined by the Beltrami equation\(^3\) \(\tilde{g}^{\frac{1}{2}}\epsilon_{ac}\tilde{g}^{\frac{1}{2}}\partial_b z = i\partial_a z\). Therefore we can globally choose \(ds^2 = e^\phi|dz|^2\), \(e^\phi = 2g_{zz}\). This means that with respect to the new set of coordinates \(\{(U_\alpha, z_\alpha)|\alpha \in I\}\) the metric is in the conformal gauge \(ds^2_\alpha = e^{\phi_\alpha}|dz_\alpha|^2\) in each patch. That is the functions in 
\(\phi \equiv \{\phi_\alpha(z_\alpha, \bar{z}_\alpha)|\alpha \in I\}\) are related by the following transformation in \(U_\alpha \cap U_\beta\)
\[\phi_\alpha(z_\alpha, \bar{z}_\alpha) = \phi_\beta(z_\beta, \bar{z}_\beta) + \log |dz_\beta/dz_\alpha|^2 .\] (2.22)

By a rescaling \(ds^2 \rightarrow d\tilde{s}^2 = \rho ds^2\) it is possible to set, at least in one patch, \(d\tilde{s}^2_\alpha = |dz_\alpha|^2\).
Since \(\rho \in T_{0,0}\), there is at least one patch \((U_\gamma, z_\gamma)\) where \(d\tilde{s}^2_\gamma \neq \text{cst}|dz_\gamma|^2\). Finally we recall that a property of the metric is positivity. Thus if \(g = e^\sigma \hat{g}\) with \(\hat{g}\) a well-defined metric, then \(e^\sigma \in C^\infty_+\) where \(C^\infty_+\) denotes the subspace of positive smooth functions in \(T_{0,0}\). Later, in the framework of the “Liouville condition”, we will discuss some aspects concerning the structure of the boundary of \(C^\infty_+\).

### 2.2 Uniformization And Poincaré Metric

Let us denote by \(D\) either the Riemann sphere \(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\), the complex plane \(\mathbb{C}\), or the upper half plane \(H = \{w \in \mathbb{C}|\text{Im} w > 0\}\). The uniformization theorem states that every Riemann surface \(\Sigma\) is conformally equivalent to the quotient \(D/\Gamma\) with \(\Gamma\) a freely acting discontinuous group of fractional transformations preserving \(D\).

Let us consider the case of Riemann surfaces with universal covering \(H\) and denote by \(J_H\) the complex analytic covering \(J_H : H \rightarrow \Sigma\). In this case \(\Gamma\) (the automorphism group of \(J_H\)) is a finitely generated Fuchsian group \(\Gamma \subset PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{I, -I\}\) acting on \(H\) by linear fractional transformations
\[w \in H, \quad \gamma \cdot w = \frac{aw + b}{cw + d} \in H, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset PSL(2, \mathbb{R}).\] (2.23)

By the fixed point equation
\[w_\pm = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c},\] (2.24)
it follows that \(\gamma \neq I\) can be classified according to the value of \(|\text{tr} \gamma|:\)

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\(^3\)In isothermal coordinates the metric reads \(ds^2 = e^\phi ((dx)^2 + (dy)^2)\), where \(z = x + iy\). Note that considering \(x = \text{cst}\) as an isothermal curve, \(y = \text{cst}\) corresponds to the curve of heat flow.
1. **Elliptic.** $|\text{tr } \gamma| < 2$, $\gamma$ has one fixed point on $H$ ($w_\pm = \pi_\pm \notin \mathbb{R}$) and $\Sigma$ has a branched point $z$ with index $q^{-1} \in \mathbb{N}\{0, 1\}$ where $q^{-1}$ is the finite order of the stabilizer of $z$.

2. **Parabolic.** $|\text{tr } \gamma| = 2$, then $w_\pm = w_+ \in \mathbb{R}$ and the Riemann surface has a puncture. The order of the stabilizer is now infinite, that is $q^{-1} = \infty$.

3. **Hyperbolic.** $|\text{tr } \gamma| > 2$, the fixed points are distinct and lie on the real axis, thus $w_\pm \notin H$. These group elements represent handles of the Riemann surface and can be represented in the form $(\gamma w - w_+)/(\gamma w - w_-) = e^\lambda (w - w_+)/(w - w_-)$, $e^\lambda \in \mathbb{R}\{0, 1\}$.

Note that if $\Gamma$ contains elliptic elements then $H/\Gamma$ is an orbifold. Furthermore, since the parabolic points do not belong to $H$, point $J_H(w_+)$ corresponds to a deleted point of $\Sigma$. By abuse of language we shall call both the elliptic and the parabolic points ramified punctures.

A Riemann surface isomorphic to the quotient $H/\Gamma$ has the Poincaré metric $\hat{g}$ as the unique metric with scalar curvature $R_{\hat{g}} = -1$ compatible with its complex structure. This implies the uniqueness of the solution of the Liouville equation on $\Sigma$. The Poincaré metric on $H$ is

$$ds^2 = \frac{|dw|^2}{(\text{Im } w)^2}. \quad (2.25)$$

Note that $PSL(2, \mathbb{R})$ transformations are isometries of $H$ endowed with the Poincaré metric.

An important property of $\Gamma$ is that it is isomorphic to the fundamental group $\pi_1(\Sigma)$. Uniformizing groups admit the following structure. Suppose $\Gamma$ uniformizes a surface of genus $h$ with $n$ punctures and $m$ elliptic points with indices $2 \leq q_i^{-1} \leq q_2^{-1} \leq \ldots \leq q_m^{-1} < \infty$. In this case the Fuchsian group is generated by $2h$ hyperbolic elements $H_1, \ldots, H_{2h}$, $m$ elliptic elements $E_1, \ldots, E_m$ and $n$ parabolic elements $P_1, \ldots, P_n$, satisfying the relations

$$E_i^{q_i^{-1}} = I, \quad \prod_{i=1}^m E_i \prod_{k=1}^n P_k \prod_{j=1}^h \left( H_{2j-1} H_{2j} H_{2j-1}^{-1} H_{2j}^{-1} \right) = I, \quad (2.26)$$

where the infinite cyclicity of parabolic fixed point stabilizers is understood.

Setting $w = J_H^{-1}(z)$ in (2.25), where $J_H^{-1}: \Sigma \rightarrow H$ is the inverse of the uniformization map, we get the Poincaré metric on $\Sigma$

$$ds^2 = 2\tilde{g}_{zz}|dz|^2 = e^{\varphi(z, \bar{z})}|dz|^2, \quad (2.27)$$

where

$$e^{\varphi(z, \bar{z})} = \frac{|J_H^{-1}(z)|^2}{(\text{Im } J_H^{-1}(z))^2}, \quad (2.28)$$
which is invariant under \( SL(2, \mathbb{R}) \) fractional transformations of \( J_H^{-1}(z) \). Since

\[
R_g = -\hat{g}^{z\bar{z}} \partial_z \partial_{\bar{z}} \log \hat{g}_{z\bar{z}}, \quad \hat{g}^{z\bar{z}} = 2e^{-\varphi}, \tag{2.29}
\]

the condition \( R_g = -1 \) is equivalent to the Liouville equation

\[
\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{1}{2} e^{\varphi(z, \bar{z})}, \tag{2.30}
\]

whereas the field \( \tilde{\varphi} = \varphi + \log \mu, \mu > 0, \) defines a metric of constant curvature \(-\mu\). Notice that the metric \( g_{z\bar{z}} = e^{\sigma + \varphi}/2 \), with \( \partial_z \partial_{\bar{z}} \sigma = 0 \), has scalar curvature \( R_g = -e^{-\sigma} \). However recall that non constant harmonic functions do not exist on compact Riemann surfaces.

### 2.3 The Liouville Condition

If \( g \) is a (in general non singular) metric on a Riemann surface of genus \( h \) with \( n \) parabolic points we have\(^4\)

\[
\int_\Sigma \sqrt{g} R_g = 2\pi \chi(\Sigma), \tag{2.31}
\]

where

\[
\chi(\Sigma) = 2 - 2h - n. \tag{2.32}
\]

A peculiarity of parabolic points is that they do not belong to \( \Sigma \), so that the singularities in the metric and in the Gaussian curvature at the punctures do not appear in \( g \) and \( R \) as functions on \( \Sigma \).

Let \( \Sigma \) be a \( n \)-punctured Riemann surface of genus \( h \) and with elliptic points \( (z_1, \ldots, z_m) \). Its Poincaré area is \(^{[16]}\)

\[
\int_\Sigma \sqrt{g} = 2\pi \left( 2h - 2 + n + \sum_{k=1}^m (1 - q_k) \right), \tag{2.33}
\]

where \( q_k^{-1} \in \mathbb{N}\backslash\{0,1\} \) denotes the ramification index of \( z_k \).

Let us choose the coordinates in such a way that the metric be in the conformal form \( ds^2 = 2g_{z\bar{z}} |dz|^2 \). In this case \( g_{z\bar{z}} = e^{\sigma} \hat{g}_{z\bar{z}} \) (here we set \( \gamma = 1 \)) where \( e^{\sigma} \in \mathcal{C}_+^{\infty} \) and \( \hat{g}_{z\bar{z}} = e^{\varphi/2} \) is the Poincaré metric. Since

\[
R_g = -2e^{-\sigma-\varphi} \partial_z \partial_{\bar{z}} (\varphi + \sigma) = -e^{-\sigma} \left( 1 + 2e^{-\varphi} \partial_z \partial_{\bar{z}} \sigma \right), \tag{2.34}
\]

\(^4\)In the following the \( \frac{|dz \wedge d\bar{z}|}{2} \) term in the surface integrals is understood.
by (2.31) it follows that
\[ \int_{\Sigma} \sqrt{g} R_g = - \int_{\Sigma} \partial_{\bar{z}} \partial_z (\varphi + \sigma) = - \int_{\Sigma} \partial_{\bar{z}} \partial_z \varphi = 2\pi \chi(\Sigma). \] (2.35)

Eq. (2.35) follows from the fact that \( \log f \in T^{0,0} \), for \( f \in C^\infty_+ \). This shows that for admissible metric the contribution to \( \chi(\Sigma) \) comes from the transformation property of \( \varphi \), whereas terms such as \( \int_{\Sigma} \sigma \partial_{\bar{z}} z \), \( \sigma \in C^\infty_+ \), are vanishing. A necessary condition in order that \( e^{\sigma} \hat{g} \) be an admissible metric on \( \Sigma \) is that \( \sigma \) satisfies the Liouville condition
\[ \int_{\Sigma} \sigma \partial_{\bar{z}} z = 0, \] (2.36)
which is a weaker condition than \( e^{\sigma} \in C^\infty_+ \). This trivial remark is useful to understand the structure of the boundary of space of admissible metrics \( e^{\sigma} \hat{g} \). Actually it is possible to add delta-like singularities at the scalar curvature leaving the Euler characteristic unchanged. That is these additional singularities do not imply additional punctures on the surface. In particular there are positive semidefinite \((1,1)\)-differentials \( g_{zz} = e^{\sigma} \hat{g}_{zz} \) that, since \( \int_{\Sigma} \sqrt{g} R_g = \int_{\Sigma} \sqrt{\hat{g}} R_{\hat{g}} \), can be considered as degenerate metrics. An interesting case is when \( \sigma(z) = -4\pi G(z, w) \) where \( G \) is Green’s function for the scalar Laplacian with respect to the Poincaré metric \( \hat{g}_{zz} = e^\varphi/2 \). The Green function takes real values and has the behaviour \( -\frac{1}{2\pi} \log |z-w| \) as \( z \to w \), moreover
\[ -\partial_{\bar{z}} \partial_z G(z, w) = \delta^{(2)}(z - w) - \frac{\sqrt{g}}{2 \int_{\Sigma} \sqrt{g}}, \] (2.37)
where the \( -\sqrt{g}/2 \int_{\Sigma} \sqrt{g} = e^\varphi/8\pi \chi \) term is due to the constant zero-mode of the Laplacian. Thus, in spite of the logarithm singularity of \( G \), the contribution to \( \int_{\Sigma} G_{zz} \) coming from the delta function is cancelled by the contribution due to the \( e^\varphi/8\pi \chi \) term. Therefore \( \int_{\Sigma} \partial_{\bar{z}} \partial_z G(z, w) = 0 \) and \( \chi(\Sigma) \) is unchanged whereas the scalar curvature becomes
\[ R_{\hat{g}}(z, \bar{z}) = -1 \to R_{\hat{g}}(z, \bar{z}) = -e^{4\pi G(z, w)} \left( 1 + 8\pi e^{-\varphi(z, \bar{z})} \delta^{(2)}(z - w) + \frac{1}{4\chi(\Sigma)} \right). \] (2.38)

Note that similar remarks extend to the positive definite metric \( e^{2\pi G(z, w) + \varphi} \). We conclude this digression by stressing that one can modify the metric by adding singularities in such a way that the Euler characteristic changes. In this case one should try to define a new surface with additional punctures where the \((1,1)\)-differential is an admissible metric.

### 2.4 Chiral Factorization And Polymorphicity

By means of chiral (in general polymorphic) functions it is possible to construct regular and non vanishing differentials \( f^{(n,n)} \in T^{n,n} \). An important example is given by the expression
of the Poincaré metric in terms of the inverse map $J_{\mu}^{-1}$ (which is a chiral polymorphic scalar function) or in terms of solutions of the uniformization equation (cfr. (2.70)). Conversely, a factorized form $f^{(n,n)} = g_1(z)g_2(\bar{z})$ enforces us to consider chiral differentials whose degree is fixed by $n$ and the topology of $\Sigma$. It is easy to see that the only change that $g_1$ and $g_2$ can undergo after winding around the homology cycles of $\Sigma$ is to get a constant multiplicative factor. However these differentials have the same degree as singlevalued differentials, that is $\deg g_1 = \deg g_2 = 2n(h - 1)$.

Later we will see that similar aspects force us to consider non Abelian monodromy for the chiral function arising in the construction of the Poincaré metric.

### 2.5 $\mu, \chi(\Sigma)$ And The Liouville Equation

Here we consider some aspects of the Liouville equation. We start by noticing that by Gauss-Bonnet it follows that if $\int_{\Sigma} e^\varphi > 0$, then the equation

$$\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{\mu}{2} e^{\varphi(z, \bar{z})},$$

has no solutions on surfaces with $\text{sgn} \chi(\Sigma) = \text{sgn} \mu$. In particular, on the Riemann sphere with $n \leq 2$ punctures\(^5\) there are no solutions of the equation

$$\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{1}{2} e^{\varphi(z, \bar{z})}, \quad \int_{\Sigma} e^\varphi > 0,$$

(2.40)

The metric of constant curvature on $\hat{\mathbb{C}}$

$$ds^2 = e^{\varphi_0} |dz|^2, \quad e^{\varphi_0} = \frac{4}{(1 + |z|^2)^2},$$

(2.41)

satisfies the Liouville equation with the “wrong sign”, that is

$$R_{\varphi_0} = 1 \quad \rightarrow \quad \partial_z \partial_{\bar{z}} \varphi_0(z, \bar{z}) = -\frac{1}{2} e^{\varphi_0(z, \bar{z})}.$$  

(2.42)

If one insists on finding a solution of eq.(2.40) on $\hat{\mathbb{C}}$, then inevitably one obtains at least three delta-singularities

$$\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{1}{2} e^{\varphi(z, \bar{z})} - 2\pi \sum_{k=1}^{n} \delta^{(2)}(z - z_k), \quad n \geq 3.$$  

(2.43)

\(^5\)The 1-punctured Riemann sphere, i.e. $\mathbb{C}$, has itself as universal covering. For $n = 2$ we have $J_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}\backslash\{0\}$, $z \mapsto e^{2\pi iz}$. Furthermore, $\mathbb{C}\backslash\{0\} \cong \mathbb{C} / < T_1 >$, where $< T_1 >$ is the group generated by $T_1 : z \mapsto z + 1$.  

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Since $\sigma = \log(\varphi - \varphi_0)$ does not satisfy the Liouville condition, the $(1, 1)$-differential $e^\varphi$ is not an admissible metric on $\hat{C}$. Furthermore, since the unique solution of the equation $\varphi_{zz} = e^\varphi / 2$ on the Riemann sphere is $\varphi = \varphi_0 + i\pi$, to consider the Liouville equation on $\hat{C}$ gives the unphysical metric $-e^{\varphi_0}$.

This discussion shows that in order to find a solution of eq.(2.40) one needs at least three punctures, that is one must consider eq.(2.40) on the surface $\Sigma = \hat{C}\setminus\{z_1, z_2, z_3\}$ where the term $2\pi \sum_{k=1}^{3} \delta^{(2)}(z - z_k)$ does not appear simply because $z_k \notin \Sigma$, $k = 1, 2, 3$. In this case $\chi(\Sigma) = -1$, so that sgn $\chi(\Sigma) = -$sgn $\mu$ in agreement with Gauss-Bonnet.

### 2.6 The Inverse Map And The Uniformization Equation

Let us now consider some aspects of the Liouville equation (2.30). As we have seen the Poincaré metric on $\Sigma$ is

$$e^{\varphi(z, \bar{z})} = \frac{|J_H^{-1}(z)'|^2}{(\text{Im} J_H^{-1}(z))^2},$$

(2.44)

which is invariant under $SL(2, \mathbb{R})$ fractional transformations of $J_H^{-1}$. This metric is the unique solution of the Liouville equation.

An alternative expression for $e^\varphi$ follows by considering as universal covering of $\Sigma$ the Poincaré disc $\Delta = \{z||z| < 1\}$. Let us denote by $J_\Delta : \Delta \to \Sigma$ the map of uniformization. Since the map from $\Delta$ to $H$ is

$$w = i \frac{1 - z}{z + 1}, \quad z \in \Delta, \quad w \in H,$$

(2.45)

we have

$$e^\varphi = 4 \frac{|J_\Delta^{-1}'|^2}{(1 - |J_\Delta^{-1}|^2)^2} = 4|J_\Delta^{-1}'|^2 \sum_{k=0}^{\infty} (k + 1)|J_\Delta^{-1}|^{2k}. \quad (2.46)$$

Both (2.44) and (2.46) make it evident that from the explicit expression of the inverse map we can find the dependence of $e^\varphi$ on the moduli of $\Sigma$. Conversely we can express the inverse map (to within a $SL(2, \mathbb{C})$ fractional transformation) in terms of $\varphi$. This follows from the Schwarzian equation

$$\{J_H^{-1}, z\} = T^F(z),$$

(2.47)

where

$$T^F(z) = \varphi_{zz} - \frac{1}{2} \varphi_z^2,$$

(2.48)

is the classical Liouville energy-momentum tensor (or Fuchsian projective connection) and

$$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2 = -2(f')^{\frac{1}{2}}((f')^{-\frac{1}{2}})', \quad (2.49)$$
is the Schwarzian derivative. Note that eq.(2.30) implies that
\[ \partial \bar{\zeta} T^F = 0. \] (2.50)

In the conformal gauge the metric can be written as \( g_{\bar{z}z} = e^{\varphi + \gamma \sigma} / 2, \ e^{\gamma \sigma} \in C_+^\infty. \) In this case the stress tensor
\[ T^\gamma = (\varphi + \gamma \sigma)_{\bar{z}z} - \frac{1}{2} (\varphi_z + \gamma \sigma_z)^2, \] (2.51)
satisfies the equation
\[ \partial \bar{\zeta} T^\gamma = - e^{\varphi + \gamma \sigma} \partial \bar{\zeta} R_{\varphi + \gamma \sigma}. \] (2.52)

Therefore \( T^\gamma \) is not chiral unless \( R_{\varphi + \gamma \sigma} \) is an antiholomorphic function. Of course the only possibility compatible with the fact that \( e^{\gamma \sigma} \in C_+^\infty \) is \( R_{\varphi + \gamma \sigma} = \text{cst}. \) Another aspect of the stress tensor is that \( SL(2, \mathbb{C}) \) transformations of \( J_{H^1} \), while changing the Poincaré metric, leave \( T^F \) invariant.

On punctured surfaces there are non trivial global solutions of the equation \( \sigma_{\bar{z}z} = 0, \) so that in this case \( \partial \bar{\zeta} T^\gamma = -\gamma \partial \bar{\zeta} (\varphi_z \sigma_z) = -\frac{\gamma}{2} e^{\varphi} \sigma_z. \) Furthermore, since \( \partial \bar{\zeta} R_{\varphi + \gamma \sigma} = \frac{\gamma}{\beta} \partial \bar{\zeta} R_{\varphi + \beta \sigma}, \) with \( \beta \) an arbitrary constant, we have
\[ \partial \bar{\zeta} T^\gamma = -\frac{\gamma}{\beta} e^{\varphi + \gamma \sigma} \partial \bar{\zeta} R_{\varphi + \beta \sigma}. \] (2.53)

Let us define the **covariant Schwarzian operator**

\[ S_f^{(2)} = 2(f')^{1/2} \partial_z (f')^{-1} \partial_z (f')^{1/2}, \] (2.54)

mapping \(-1/2\)- to \(3/2\)-differentials. Since
\[ S_f^{(2)} \cdot \psi = \left(2 \partial_z^2 + \{f, z\}\right) \psi, \] (2.55)
the Schwarzian derivative can be written as
\[ \{f, z\} = S_f^{(2)} \cdot 1. \] (2.56)

The operator \( S_f^{(2)} \) is invariant under \( SL(2, \mathbb{C}) \) fractional transformations of \( f, \) that is
\[ S_{\gamma \cdot f}^{(2)} = S_f^{(2)}, \quad \gamma \in SL(2, \mathbb{C}). \] (2.57)

Therefore, if the transition functions of \( \Sigma \) are linear fractional transformations, then \( \{f, z\} \) transforms as a quadratic differential. However, except in the case of projective coordinates,
the Schwarzian derivative does not transform covariantly on $\Sigma$. This is evident by (2.56) since in flat spaces only (e.g. the torus) a constant can be considered as a $-\frac{1}{2}$-differential.

Let us consider the equation
\[ S^{(2)}_f \cdot \psi = 0. \] (2.58)

To find two independent solutions we set
\[ (f')^{\frac{1}{2}} \partial_z (f')^{-1} \partial_z (f')^{\frac{1}{2}} \psi_1 = (f')^{\frac{1}{2}} \partial_z (f')^{-1} \partial_z cst = 0, \] (2.59)

and
\[ (f')^{\frac{1}{2}} \partial_z (f')^{-1} \partial_z (f')^{\frac{1}{2}} \psi_2 = (f')^{\frac{1}{2}} \partial_z cst = 0, \] (2.60)

so that the solutions of (2.58) are
\[ \psi_1 = cst (f')^{-\frac{1}{2}}, \quad \psi_2 = cst f(f')^{-\frac{1}{2}}. \] (2.61)

Since $\psi_2/\psi_1 = cst f$, to find the solution of the Schwarzian equation $\{f, z\} = g$ is equivalent to solve the linear equation
\[ \left( 2 \partial_z^2 + g(z) \right) \psi = 0. \] (2.62)

We stress that the “constants” in the linear combination $\phi = a \psi_1 + b \psi_2$ admit a $\bar{z}$-dependence provided that $\partial_z a = \partial_z b = 0$.

The inverse map is locally univalent, that if $z_1 \neq z_2$ then $J^{-1}_H(z_1) \neq J^{-1}_H(z_2)$. A related characteristic of $J^{-1}_H$ is that under a winding of $z$ around non trivial cycles of $\Sigma$ the point $J^{-1}_H(z) \in H$ moves from a representative $D$ of the fundamental domain to an equivalent point of another representative of $D$. On the other hand, since $\Gamma$ is the automorphism group of $J_H$, it follows that after winding around non trivial cycles of $\Sigma$ the inverse map transforms in the linear fractional way
\[ J^{-1}_H \rightarrow \gamma \cdot J^{-1}_H = \frac{a J^{-1}_H + b}{c J^{-1}_H + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma. \] (2.63)

However note that (2.57) guarantees that, in spite of the polymorphicity (2.63), the classical Liouville stress tensor $T^F = S^{(2)}_{J^{-1}_H} \cdot 1$ is singlevalued.

As we have seen one of the important properties of the Schwarzian derivative is that the Schwarzian equation (2.47) can be linearized. Thus if $\psi_1$ and $\psi_2$ are linearly independent solutions of the uniformization equation
\[ \left( \frac{\partial^2}{\partial z^2} + \frac{1}{2} T^F(z) \right) \psi(z) = 0, \] (2.64)

This property of $J^{-1}_H$ makes evident its univalence as function on $\Sigma$. 

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6This property of $J^{-1}_H$ makes evident its univalence as function on $\Sigma$. 

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then $\psi_2/\psi_1$ is a solution of eq.(2.47). That is, up to a $SL(2, \mathbb{C})$ linear fractional transformation, we have

$$J_H^{-1} = \psi_2/\psi_1.$$  \hfill(2.65)

Indeed by (2.59,2.60) it follows that

$$\psi_1 = (J_H^{-1})^{-\frac{1}{2}}, \quad \psi_2 = (J_H^{-1})^{-\frac{1}{2}} J_H^{-1}.$$  \hfill(2.66)

are independent solutions of (2.64). Another way to prove (2.65) is to write eq.(2.64) in the equivalent form

$$(f')^{\frac{1}{2}} \partial_z (f')^{-1} \partial_z (f')^{\frac{1}{2}} \psi = 0, \quad f \equiv J_H^{-1},$$  \hfill(2.67)

and then to set $z = J_H(w)$. In this case (2.64) becomes the trivial equation on $H$

$$w'^{3/2} \partial_w^2 \phi = 0.$$  \hfill(2.68)

For any choice of the two linearly independent solutions we have $\phi_2/\phi_1 = w$ up to an $SL(2, \mathbb{C})$ transformation. Going back to $\Sigma$ we get $J_H^{-1} = \psi_2/\psi_1$.

Note that any $SL(2, \mathbb{R})$ transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$  \hfill(2.69)

induces a linear fractional transformation of $J_H^{-1}$. Therefore the invariance of $e^\varphi$ under $SL(2, \mathbb{R})$ linear fractional transformations of $J_H^{-1}$ corresponds to its invariance for $SL(2, \mathbb{R})$ linear transformations of $^T \psi_1, \psi_2$. This leads us to express $e^{-k\varphi}$ as

$$e^{-k\varphi} = (-4)^{-k} \left( \bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1 \right)^{2k},$$  \hfill(2.70)

in particular, when $2k$ is a non negative integer, we get

$$e^{-k\varphi} = 4^{-k} \sum_{j=-k}^{k} (-1)^j C_{2k}^{j+k} \bar{\psi}_1^{k+j} \psi_1^{k-j} \psi_2^{j} \bar{\psi}_2^{2-j}, \quad 2k \in \mathbb{Z}^+, \quad C_{2k}^{j} = \frac{k!}{j!(k-j)!}.$$  \hfill(2.71)

On the other hand, since we can choose $\psi_2 = \psi_1 f \psi_1^{-2}$, we have

$$e^{-k\varphi} = (-4)^{-k} |\psi|^4 k \left( \int \psi^{-2} - \int \bar{\psi}^{-2} \right)^{2k}, \quad \forall k,$$  \hfill(2.72)

Note that the Poincaré metric is invariant under $SL(2, \mathbb{R})$ fractional transformations of $J_H^{-1}$ whereas the Schwarzian derivative $T^F(z) = \{J_H^{-1}, z\}$ is invariant for $SL(2, \mathbb{C})$ transformations of $J_H^{-1}$. Thus the identification $J_H^{-1} = \psi_2/\psi_1$ is up to a $SL(2, \mathbb{C})$ transformation.
\[ \psi = a \psi_1 (1 + b \int \psi_1^{-2}), \quad a \in \mathbb{R} \setminus \{0\}, \quad b \in \mathbb{R}. \] (2.73)

We note that the ambiguity in the definition of \( f^z \psi^{-2} \) reflects the polymorphicity of \( J_H^{-1} \). This property of \( J_H^{-1} \) implies that, under a winding around non trivial loops, a solution of (2.64) transforms in a linear combination involving itself and another (independent) solution.

It is easy to check that
\[ \left( \partial_z^2 + \frac{1}{2} T^F(z) \right) e^{-\varphi/2} = 0, \]
which shows that the uniformization equation has the interesting property of admitting singlevalued solutions. The reason is that the \( \bar{z} \)-dependence of \( e^{-\varphi/2} \) arises through the coefficients \( \psi_1 \) and \( \psi_2 \) in the linear combination of \( \psi_1 \) and \( \psi_2 \).

Since \( [\partial_{\bar{z}}, S^{(2)}_{J_H^{-1}}] = 0 \), the singlevalued solutions of the uniformization equation are
\[ \left( \partial_z^2 + \frac{1}{2} T^F(z) \right) \partial_z e^{-\varphi/2} = 0, \quad l = 0, 1, \ldots. \] (2.75)
Thus, since \( e^{-\varphi} \) and \( e^{-\varphi} \psi_2 \) are linearly independent solutions of eq.(2.64), their ratio solve the Schwarzian equation
\[ \{ \varphi_2, z \} = T^F(z). \] (2.76)

Higher order derivatives \( \partial_z^l e^{-\varphi/2}, \ l \geq 2 \), are linear combinations of \( e^{-\varphi/2} \) and \( e^{-\varphi/2} \psi_2 \) with coefficients depending on \( T^F \) and its derivatives; for example
\[ \partial_z^2 e^{-\varphi/2} = - \frac{T^F}{2} e^{-\varphi/2}. \] (2.77)

In particular if \( \psi_2(z) = T^F \psi_1(z) \) then, in spite of the fact that \( T^F \) is not a constant on \( \Sigma \), \( \psi_1 \) and \( \psi_2 \) are linearly dependent solutions of eq.(2.64). A check of the linear dependence of \( \psi_1 \) from \( \psi_2 \) follows from the fact that
\[ \{ \psi_2/\psi_1, z \} = \{ T^F, z \} = 0 \neq T^F(z). \] (2.78)

Let us show what happens if one sets \( J_H^{-1} = \psi_1/\psi_2 \) without considering the remark made in the previous footnote. As solutions of the uniformization equation, we can consider \( \psi_1 = e^{-\varphi/2} \) and an arbitrary solution \( \psi_2 \) such that \( \partial_z \left( \psi_2/\psi_1 \right) = 0 \). Since \( \partial_z \left( e^{-\varphi/2}/\psi_2 \right) \neq 0 \), in spite of the fact that \( \{ e^{-\varphi/2}/\psi_2, z \} = T^F \), we have \( J_H^{-1} \neq \psi_1/\psi_2 \).

We conclude the analysis of the uniformization equation by summarizing some useful expressions for the Liouville stress tensor
\[ T^F = \{ J_H^{-1}, z \} = \{ \varphi_2, z \} = 2 \left( J_H^{-1} \right)^{\frac{1}{2}} \partial_z \frac{1}{J_H^{-1}} \partial_z \left( J_H^{-1} \right)^{\frac{1}{2}} \cdot 1 = 2 e^{\varphi/2} \partial_z e^{-\varphi} \partial_{\bar{z}} e^{\varphi/2} \cdot 1. \]
\[
\frac{1}{2} \left( e^{-\phi/2} \right)^{\prime} \left( e^{-\phi/2} \right)^{-1} \frac{\partial z}{\psi_2} \left( e^{-\phi/2} \right)^{\prime} - \frac{1}{2} \frac{\partial z}{\psi_2} \left( e^{-\phi/2} \right)^{\prime} = -2e^{\phi/2} \left( e^{-\phi/2} \right)^{\prime} = -2\psi^{-1}\psi'',
\]
(2.79)

with \( \psi \) given in (2.73) and \( \psi_2 \) an arbitrary solution of eq.(2.64) such that \( \partial_z \left( e^{-\phi/2}/\psi_2 \right) \neq 0 \).

### 2.7 Remarks On \( e^{\varphi_A} = \frac{|A'|^2}{(\text{Im } A)^2} \)

Sometimes in current literature it is stated that the solution of the Liouville equation is

\[
e^{\varphi_A} = \frac{|A'|^2}{(\text{Im } A)^2},
\]
(2.80)

with \( A \) a generic holomorphic function. However the uniqueness of the solution of the Liouville equation implies that \( A(z) \) is the inverse map of the uniformization which is unique up to \( SL(2, \mathbb{R}) \) fractional transformations. Let us show what happens if \( A \) is considered to be an arbitrary well-defined chiral function on a compact Riemann surface. First of all according to the Weierstrass gap theorem a meromorphic function \( f^{(0)} \), with divisor in general position, has at least \( h + 1 \) zeroes\(^8\). Thus, since \( \text{deg } f^{(0)} = 0 \), \( f^{(0)} \) has at least \( h + 1 \) poles. Since \( \partial_z z^{-1} \sim \pi \delta^{(2)}(z) \), it follows that if \( A(z) \) were a well-defined 0-differential then it would induce singularities in the scalar curvature at the divisor of \( A \), so that \( R_{\varphi_A} \neq -1 \). Furthermore \( e^{\varphi_A} \) itself is singular when \( \text{Im } A(z) = 0 \) and will degenerate for the zeroes of \( A'(z) \). Therefore, in order that \( \varphi_A \) be the solution of the Liouville equation

\[
R_{\varphi_A} = -1,
\]
(2.81)

the field \( A(z) \) must be a chiral and linearly polymorphic function. In particular, under the action of the fundamental group \( \pi_1(\Sigma) \), \( A(z) \) must transform in a linear fractional way with the coefficients of the transformation in the Fuchsian group \( \Gamma \) whose elements are fixed by the moduli of \( \Sigma \).

Notice that \( A(z) \) cannot simply be a holomorphic nowhere vanishing function with constant multivaluedness. In this case the monodromy is Abelian. From the analytic point of view the commutativity of group monodromy has the effect of giving a metric with singularities. This, for example, follows from the fact that if \( A \rightarrow \text{cst } A \), then \( A'/A \) would be a well-defined one-differential so that

\[
\#\text{zeroses } (A'/A) = \#\text{poles } (A'/A) + 2(h - 1) > 0.
\]
(2.82)

---

\( ^8 \)The restriction to “points in general position” means that we are not considering Weierstrass points in \( \text{Div } f^{(0)} \).
On the other hand since $A$ must be a holomorphic nonvanishing function it follows that $\text{Div} A' = \text{Div}(A'/A)$. Thus $e^{\varphi A}$ would be a degenerate metric since, if $\text{Im} A(P) \neq 0$, $\forall P \in \text{Div} A'$, then

$$\# \text{ zeroes } (e^{\varphi A}) = 4(h-1) > 0. \quad (2.83)$$

Unfortunately no one has succeeded in writing down the explicit form of the inverse map in terms of the moduli of $\Sigma$. This is the uniformization problem. In section 5 we will introduce a new set of Fourier modes in higher genus whose properties suggest that they are strictly related to the underlying Fuchsian group.

### 2.8 On The Standard Approach To Liouville Gravity

We conclude this section by considering some aspects concerning the standard approach to Liouville gravity. Let us begin by noticing that if one parametrizes the metric in the form $g = e^{\gamma \sigma} \hat{g}$, where $\hat{g}$ is a background metric (in particular $\hat{g}$ is a positive definite $(1,1)$-differential), then $e^{\gamma \sigma} \in C^\infty$. The Liouville action in harmonic coordinates reads

$$S = \int_{\Sigma} \left( |\partial_z \sigma|^2 + \frac{1}{\gamma} \sqrt{\hat{g}} R_{\hat{g}} \sigma + \frac{\mu}{2\gamma^2} \sqrt{\hat{g}} e^{\gamma \sigma} \right). \quad (2.84)$$

Sometimes it is stated that at the classical level $\gamma \sigma$ transforms as in (2.22) whereas after quantization the logarithm term is multiplied by a constant related to the central charge. Actually, to perform the surface integration $\hat{g}$ must be a $(1,1)$-differential and $e^{\gamma \sigma} \in T^{0,0}$. Therefore it is unclear what the meaning of $S$ is if one considers $e^{\gamma \sigma} \in T^{1,1}$. Another standard choice is to set $d\hat{s}^2 = \text{cst}|dz|^2$ on a patch. Once again, since it is not possible to make this choice on the whole manifold, with this prescription the surface integral is undefined. A way to (partly) solve these problems is to set (formally) $\hat{g}_{zz} = 1$ and then consider $ds^2 = e^{\gamma \sigma}|dz|^2$, so that $e^{\gamma \sigma} \in T^{1,1}$. In this case $R_{\hat{g}}$ is formally zero and the integrand in (2.84) reduces to

$$F = |\partial_z \sigma|^2 + \frac{\mu}{2\gamma^2} e^{\gamma \sigma}. \quad (2.85)$$

The transition from the integrand in (2.84) to $F$ is implicitly assumed by some authors (see for example equations (1.2) and (2.1) in the interesting paper [18]). However, since $|\partial_z \sigma|^2$ does not transform covariantly, one must add “boundary terms” to $\int_{\Sigma} F$ in order to get a well-defined action. This has been done in [12]. In the case of surfaces with punctures this boundary term corresponds to a regularization term which is crucial in fixing the scaling properties of $S$. This regularization procedure is related to the fact that negatively curved
surfaces (the realm of Liouville theory) have both ultraviolet and infrared cutoffing properties. The coupling between cutoff, regularization terms in the Liouville action, modular anomaly is a highly non trivial (and interesting) aspect which is related to the properties of univalent functions (e.g. Koebe 1/4-theorem) that we will discuss in the last section in the context of 2D quantum gravity. We notice that this subject is related to classical and quantum chaos on Riemann surfaces.

In the operator formulation of CFT one considers the solutions of the classical equation of motion with allowed singularities at the points where the in and out vacua are placed. This allows one to consider non trivial solutions of the equation $\phi_{z\bar{z}} = 0$. In Liouville theory it is not possible to compute the asymptotics of the stress tensor by standard CFT techniques. The reason is that the OPE in CFT is based on free fields techniques where $\langle X(z)X(w) \rangle \sim -\log(z - w)$. This explains why it is difficult to recognize what the vacuum of Liouville theory is.

The known results in quantum Liouville theory essentially concern the formulation on the sphere and the torus. To get insight about the continuum formulation of 2D gravity in higher genus is an outstanding problem. To understand the difficulties that one meets with respect to the $\h = 0, 1$ cases we summarize few basic facts.

$h \leq 1$ A feature of $\hat{\mathcal{C}}$ with respect to higher genus surfaces is that its universal covering is $\hat{\mathcal{C}}$ itself. Therefore metrics on $\hat{\mathcal{C}}$ and on its universal covering coincide. This partly explains why in this case computations are easier to be done. In the torus case “Liouville theory” is free, the reason is that the metric of constant curvature $e^\varphi$ satisfies the equation $\varphi_{z\bar{z}} = 0$, that is $\varphi = \text{cst}$. Therefore to quantize 2D gravity on the torus it is sufficient to use standard CFT techniques and to impose positivity on the Liouville field. This can easily be done because the Fourier modes on the torus are explicitly known$^9$.

$h \geq 2$ In the higher genus case the metric of constant curvature on $\Sigma$ has a richer geometrical structure with respect to the Poincaré metric on its universal covering $H$. Thus in quantizing the theory we must consider the geometry of the moduli space or, which is the same, the geometry of Fuchsian groups (which is encoded in $J^{-1}_H$). To get results in a way similar to those derived on $\hat{\mathcal{C}}$ we have to shift our attention from $\Sigma$ to the upper half plane whose Poincaré metric is explicitly known (see (2.25)). In this case

$^9$In section 5 we will consider the problem of formulating higher genus Fourier analysis.
we are not considering the underlying topology and geometry of the terms in the genus expansion. Nevertheless non perturbative results enjoy similar properties. This similarity suggests to formulate a non perturbative approach to 2D gravity based on a sort of path-integral formulation of 2D gravity on $H$. The reason for this is that both $H$ and the Poincaré metric on it are universal (non perturbative) objects underlying the full genus expansion.

3 Covariant Holomorphic Operators, Classical Liouville Action And Normal Ordering

Here we introduce a set of operators $S_{J^{-1}}^{(2k+1)}$ corresponding to $\partial_2^{(2k+1)}$ covariantized by means of $J_{H}^{-1}$. We stress that univalence of $J_{H}^{-1}$ implies that these operators are holomorphic. Next, we derive the chiral (polymorphic) and non chiral eigenfunctions for a set of operators related to $S_{J^{-1}}^{(2k+1)}$. An interesting property of these operators is that the (generalized) harmonic Beltrami differentials $\mu_{harm}^{(2k+1)}$ (see eq.(3.26)) are in their kernel

$$S_{J^{-1}}^{(2k+1)} \mu_{harm}^{(2k+1)} = 0. \quad (3.1)$$

We consider the cocycles associated to $S_{J^{-1}}^{(2k+1)}$. In this framework we show that the normal ordering for operators defined on $\Sigma$ is related to classical Liouville theory. The univalence of $J_{H}^{-1}$ allows us to get time-independence and locality for the cocycles.

The holomorphicity of the covariantization above allows us to show that the chirally split anomaly of CFT reduces to the Krichever-Novikov ($KN$) cocycle. This suggests to introduce higher order anomalies given as surfaces integrals of $(1, 1)$-forms defined in terms of $S_{J_{H}^{-1}}^{(2k+1)}$. These anomalies are related to the uniformization theory of vector bundles on Riemann surfaces ($W$-algebras). Remarkably, eq.(3.1) implies that these anomalies (including the standard chirally split anomaly) vanish in the case one considers deformation of the complex structure induced by (generalized) harmonic Beltrami differentials.

3.1 Higher Order Schwarzian Operators

Let us start by noticing that since

$$e^{-k\varphi} = |J_{H}^{-1}'|^{-2k} \left( \frac{J_{H}^{-1} - J_{H}^{-1}}{2i} \right)^{2k}, \quad (3.2)$$
it follows that the negative powers of the Poincaré metric satisfy the higher order generalization of eq.(2.74)
\[ S^{(2k+1)}\cdot e^{-k\varphi} = 0, \quad k = 0, \frac{1}{2}, 1, \ldots, \] (3.3)
with \( S^{(2k+1)}_J \) the higher order covariant Schwarzian operator
\[ S^{(2k+1)}_J = (2k + 1)(f')^k \partial_z (f')^{-1} \partial_z (f')^{-1} \ldots \partial_z (f')^{-1} \partial_z (f')^k, \] (3.4)
where the number of derivatives is \( 2k + 1 \). We stress that univalence of \( J^{-1}_H \) implies holomorphicity of the \( S^{(2k+1)}_J \) operators. Eq.(3.3) is manifestly covariant and singlevalued on \( \Sigma \). Furthermore it can be proved that the dependence of \( S^{(2k+1)}_f \) on \( f \) appears only through \( S^{(2)}_f \cdot 1 = \{ f, z \} \) and its derivatives; for example
\[ S^{(3)}_{J^{-1}_H} = 3 \left( \partial^3 z + 2T^F \partial_z + T^F' \right), \] (3.5)
which is the second symplectic structure of the KdV equation. A nice property of the equation \( S^{(2k+1)}_{J^{-1}_H} \cdot \tilde{\psi} = 0 \) is that its projection on \( H \) is the trivial equation
\[ (2k + 1)w'^{2k+1} \partial_w^{2k+1} \psi = 0, \quad w \in H, \] (3.6)
where \( w = J^{-1}_H(z) \). This makes evident why only for \( k > 0 \) it is possible to have finite expansions of \( e^{-k\varphi} \) such as in eq.(2.71). The reason is that the solutions of eq.(3.6) are \( \{ w^j | j = 0, \ldots, 2k \} \) so that the best thing we can do is to consider linear combinations of positive powers of the non chiral solution \( \text{Im} w \) which is just the square root of inverse of the Poincaré metric on \( H \).

The \( SL(2, \mathbb{C}) \) invariance of the Schwarzian derivative implies that
\[ S^{(2k+1)}_{J^{-1}_H} = S^{(2k+1)}_{\varphi z}, \quad k = 0, \frac{1}{2}, 1, \ldots. \] (3.7)
On the other hand by Liouville equation
\[ S^{(2k+1)}_{J^{-1}_H} = S^{(2k+1)}_{\varphi z} = (2k + 1)e^{k\varphi} \partial_{\varphi z} e^{-\varphi} \partial_z e^{-\varphi} \ldots \partial_z e^{-\varphi} \partial_{\varphi z} e^{k\varphi}, \quad k = 0, \frac{1}{2}, 1, \ldots. \] (3.8)
In the following we will use this property of \( S^{(2k+1)}_{J^{-1}_H} \) to construct the eigenfunctions for the operator
\[ Q^{(2k+1)}_{\varphi z} = S^{(2k+1)}_{\varphi z} \left( 2\varphi z e^{-\varphi} \right)^{2k+1}, \] (3.9)
and for its chiral analogous
\[ Q^{ch(2k+1)}_{J^{-1}_H} = S^{(2k+1)}_{J^{-1}_H} \left( \partial_z \log J^{-1}_H \right)^{-2k-1}. \] (3.10)
3.2 Eigenfunctions Of $Q_{\varphi_{\bar{z}}}^{(2k+1)}$ And $Q_{J_{H}^{-1}}^{ch(2k+1)}$

Since

$$[\partial_{\bar{z}}, S_{J_{H}^{-1}}^{(2k+1)}] = 0,$$  \hspace{1cm} (3.11)

it follows that besides $e^{-k\varphi}$ other singlevalued solutions of $S_{J_{H}^{-1}}^{(2k+1)} \cdot \psi = 0$ have the form $\partial_{\bar{z}} e^{-k\varphi}$. However notice that by (3.7) it follows that the set of singlevalued differentials

$$\psi_{l} = (2\varphi_{\bar{z}})^{l} e^{-k\varphi}, \hspace{1cm} l = 0, \ldots, 2k;$$  \hspace{1cm} (3.12)

is a basis of solutions of $S_{J_{H}^{-1}}^{(2k+1)} \cdot \psi = 0$. To see this it is sufficient to substitute $\psi_{l}$ in the RHS of (3.8) and systematically use the Liouville equation

$$e^{-\varphi} \partial_{\bar{z}} (2\varphi_{\bar{z}})^{l} = l(2\varphi_{\bar{z}})^{l-1}.$$  \hspace{1cm} (3.13)

In the intersection of two patches $(U, z)$ and $(V, w)$ the field $\psi_{l}$ transforms as

$$(2\varphi_{\bar{z}}(z, \bar{z}))^{l} e^{-k\varphi(f(z, \bar{z}))} = \left(2\varphi_{\bar{w}}(w, \bar{w}) + 2 \bar{w}_{z}/(\bar{w}_{\bar{z}})^{2}\right)^{l} e^{-k\varphi_{f}(w, \bar{w})} w_{\bar{z}}^{-k} \bar{w}_{\bar{z}}^{l},$$  \hspace{1cm} (3.14)

that is $\psi_{l}$ decomposes into a sum of solutions for the covariant operator $S_{J_{H}^{-1}}^{(2k+1)}$ written in the patch $V$.

Let us consider the chiral (i.e. such that $\partial_{\bar{z}} \phi_{l}(-k) = 0$) solutions of

$$S_{J_{H}^{-1}}^{(2k+1)} \cdot \phi_{l}(-k) = 0.$$  \hspace{1cm} (3.15)

We have $\phi_{l}(-k) = (J_{H}^{-1})^{l}(J_{H}^{-1})^{-k} l = 0, \ldots, 2k$. Note that $\phi_{l}(-k)$ and $\psi_{l}$ have the common property of generating other solutions either by changing patch (eq.(3.14) for $\psi_{l}$) or, in the case of $\phi_{l}(-k)$, by running around cycles. For example, in the case $k = 1/2$, starting from $\phi_{l}^{-1/2)}$, to get the other linearly independent solution $\phi_{2}^{-1/2)}$ it is sufficient to perform a nontrivial winding around $\Sigma$.

Since

$$S_{J_{H}^{-1}}^{(2k+1)} \cdot (2\varphi_{\bar{z}})^{2k+l} e^{-k\varphi} = \lambda_{l} (2\varphi_{\bar{z}})^{l-1} e^{(k+1)\varphi}, \hspace{1cm} l \in \mathbb{Z},$$  \hspace{1cm} (3.16)

where

$$\lambda_{l} = (2k + 1)(2k + l)(2k + l - 1) \ldots (l + 1), \hspace{1cm} l \in \mathbb{Z},$$  \hspace{1cm} (3.17)

it follows that the singlevalued differentials

$$\psi_{l} = (2\varphi_{\bar{z}})^{l-1} e^{(k+1)\varphi}, \hspace{1cm} l \in \mathbb{Z},$$  \hspace{1cm} (3.18)
are eigenfunctions of \( Q_{\varphi^2}^{(2k+1)} \)
\[
Q_{\varphi^2}^{(2k+1)} \cdot \psi_l = \lambda_l \psi_l, \quad l \in \mathbb{Z}. \tag{3.19}
\]

Note that \( \psi_{-2k}, \ldots, \psi_0 \) are the zero modes of \( Q_{\varphi^2}^{(2k+1)} \). Furthermore, eq. (3.19) is invariant under the substitution, \( \psi_l \rightarrow F \psi_l \), where \( F \) is an arbitrary solution of
\[
\partial_z F = 0. \tag{3.20}
\]

Since the Liouville stress tensor satisfies the equations
\[
\partial_z \partial_n \bar{\psi} = 0, \quad n = 0, 1, 2, \ldots,
\]
the general solution of (3.20) depends on \( \bar{T}^F \) and its derivatives. However, taking into account polymorphic differentials, the general solution of eq. (3.20) has the form
\[
F \equiv F \left( \overline{\psi}_k, \overline{\psi}'_k, \overline{\psi}''_k, \ldots \right), \quad k = 1, 2, \ldots
\tag{3.21}
\]

where \( \overline{\psi}_1 \) and \( \overline{\psi}_2 \) are two solutions of
\[
Q_{\varphi^2}^{(2k+1)} \cdot \psi_l = \lambda_l \psi_l, \quad l \in \mathbb{Z}, \tag{3.22}
\]

such that
\[
\partial_z \overline{\psi}_1 = \partial_z \overline{\psi}_2 = 0 \quad \text{and} \quad \partial_z \left( \overline{\psi}_1 / \overline{\psi}_2 \right) \neq 0.
\]

The differentials
\[
\psi^{ch}_l = \left( J^{-1}_H \right)^{l-1} \left( J^{-1}_H \right)^{k+1}, \quad l \in \mathbb{Z}, \tag{3.23}
\]
are the chiral analogous of \( \psi_l \). Indeed they satisfy the equation
\[
Q_{\varphi^2}^{(2k+1)} \cdot \psi^{ch}_l = \lambda_l \psi^{ch}_l, \quad l \in \mathbb{Z}, \tag{3.24}
\]

where \( \lambda_l \) is given in (3.17). A property of \( S_{J^{-1}_H}^{(2k+1)} \) is
\[
\left[ S_{J^{-1}_H}^{(2k+1)}, \psi^{(n)} \right] = 0, \tag{3.25}
\]

with \( \psi^{(n)} \) a holomorphic \( n \)-differential. This implies that the (generalized) harmonic Beltrami differentials satisfy the equation
\[
\mathcal{S}_{J^{-1}_H}^{(2k+1)} \mu_{\text{harm}}^{(2k+1)} = 0, \quad \mu_{\text{harm}}^{(2k+1)} = \overline{\psi}^{(2k+1)} e^{-k \varphi}. \tag{3.26}
\]

This equation will be useful in recovering the kernel of the chirally split anomaly arising in CFT. Note that \( \mu_{\text{harm}}^{(3)} \) is a standard harmonic Beltrami differential.

The operator \( \mathcal{S}_{f_{BA}}^{(2k+1)} \), where
\[
\partial_z f_{BA} = e_{BA}^{-1}, \tag{3.27}
\]
with \( e_{BA} \) a \textit{Baker-Akhiezer vector field}, appears in the formulation of the covariant KdV in higher genus [19, 20]. The non holomorphic operators \( \mathcal{S}_{f_{BA}}^{(2k+1)} \), \( k \in \mathbb{Z}_+ \) define cocycles on Riemann surfaces [19]. Similar operators, not related to uniformization, have been considered also in [21] (see also [22]).
3.3 Normal Ordering On $\Sigma$ And Classical Liouville Action

Let us now consider the meromorphic $n$-differentials $\psi^{(n)}_j$ proposed in [23]. They are the higher genus analogous of the Laurent monomials $z_j^{1-n}$. By means of $\psi^{(n)}_j$ we can expand holomorphic differentials on $\Sigma \backslash \{P_+, P_-\}$. Their relevance for an operator approach which mimics the radial quantization on the Riemann sphere has been shown in [24].

In terms of local coordinates $z_\pm$ vanishing at $P_\pm \in \Sigma$ the basis reads

$$\psi^{(n)}_j(z) = a^{(n)}_j z_\pm^{j-s(n)} (1+O(z_\pm)) (dz_\pm)^n, \quad s(n) = \frac{h}{2} - n(h-1),$$

(3.28)

where $j \in \mathbb{Z} + h/2$ and $n \in \mathbb{Z}$. The $(dz_\pm)^n$ term has been included to emphasize that $\psi^{(n)}_j(z)$ transforms as $\partial^n_z$. By the Riemann-Roch theorem $\psi^{(n)}_j$ is uniquely determined by fixing one of the constants $a^{(n)}_j$ (to choose the value of $a^{(n)}_j+$ fixes $a^{(n)}_j-$ and vice versa). In the following we set $a^{(n)}_j+ = 1$. There are few exceptions to (3.28) concerning essentially the $h=1$ and $n=0,1$ cases [23]. The expression of this basis in terms of theta functions reads [24]

$$\psi^{(n)}_j(z) = C^{(n)}_j \Theta \left( I(z) + D^{i;n} \Omega \right) \frac{\sigma(z)^{2n-1} E(z, P_+)^j-s(n)}{E(z, P_-)^j+s(n)},$$

(3.29)

where

$$D^{i;n} = (j-s(n)) I(P_+) - (j+s(n)) I(P_-) + (1-2n) \Delta,$$

(3.30)

and the constant $C^{(n)}_j$ is fixed by the condition $a^{(n)}_j+ = 1$.

Let us introduce the following notation for vector fields and quadratic differentials

$$e_k \equiv \psi^{(-1)}_k, \quad \Omega^k \equiv \psi^{(2)}_{-k}.$$  

(3.31)

Note that (3.28) furnishes a basis for the $1-2s(n) = (2n-1)(h-1)$ holomorphic $n$-differentials on $\Sigma$

$$\mathcal{H}^{(n)} = \left\{ \psi^{(n)}_k | s(n) \leq k \leq -s(n) \right\}.$$  

(3.32)

In particular the quadratic holomorphic differentials are

$$\mathcal{H}^{(2)} = \left\{ \Omega^{k+1-h_0} | k = 1, \ldots, (3h-3) \right\}, \quad h_0 \equiv \frac{3}{2} h.$$  

(3.33)

Let $C$ be a homologically trivial contour separating $P_+$ and $P_-$. The dual of $\psi^{(n)}_j$ is defined by

$$\frac{1}{2\pi i} \oint_C \psi^{(n)}_k \psi^{(n)}_j = \delta^k_j,$$

(3.34)

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which gives
\[\psi_j^{(n)} = \psi_{-j}^{(1-n)}.\] (3.35)

By means of the operator \(S_{j,\mu}^{(2k+1)}\) we define the quantity
\[\chi_F^{(2k+1)} \left( \psi_i^{(-k)}, \psi_j^{(-k)} \right) = \frac{1}{24(2k+1)\pi i} \oint_C \psi_i^{(-k)} S_{j,\mu}^{(2k+1)} \psi_j^{(-k)}, \quad k = 0, 1, 2, \ldots\] (3.36)

that for \(k = 1\) is the Fuchsian KN cocycle
\[\chi_F^{(1)} (e_i, e_j) = \frac{1}{24} \left[ \frac{1}{2} \left( e_i e_j''' - e_i''' e_j \right) + T_F (e_i e_j' - e_j e_i') \right], \quad e_j \equiv \psi_j^{(-1)}.\] (3.37)

Notice that
\[\chi_F^{(2k+1)} (e_i, e_j) = \frac{1}{24(2k+1)\pi i} \oint_C \psi_i^{(-k)} S_{j,\mu}^{(2k+1)} \psi_j^{(-k)} = 0, \quad \forall i, j,\] (3.38)

where
\[f_j(z) = \int_z^1 \left[ \psi_j^{(-k)} \right]^{-\frac{1}{k}}, \quad \text{for} \ k = 1, 2, 3, \ldots, \quad f_j(z) = \psi_j^{(0)}, \quad \text{for} \ k = 0.\] (3.39)

An arbitrary KN cocycle has the form
\[\chi^{(3)} (e_i, e_j) = \chi_F^{(3)} (e_i, e_j) + \sum_{k=1}^{3h-3} a_k \oint C \Omega^{k+1-h_0} [e_i, e_j], \quad h_0 = \frac{3}{2} h.\] (3.40)

The cocycle \(\chi^{(3)} (e_i, e_j)\) defines the central extension \(\hat{V}_\Sigma\) of the \(h_0\) graded-algebra \(V_\Sigma\) of the meromorphic vector fields \(\{e_j | j \in \mathbb{Z} + h/2\}\). In particular the commutator in \(\hat{V}_\Sigma\) is
\[\left[ e_i, e_j \right] = \sum_{s=-h_0}^{h_0} C_{ij}^s e_{i+j-s} + t \chi^{(3)} (e_i, e_j), \quad \left[ e_i, t \right] = 0,\] (3.41)

where
\[C_{ij}^s = \frac{1}{2\pi i} \oint C \Omega^{i+j-s} [e_i, e_j].\] (3.42)

Two important properties of \(\chi^{(3)} (e_i, e_j)\) are locality
\[\chi^{(3)} (e_i, e_j) = 0, \quad \text{for} \ |i + j| > 3h,\] (3.43)

and “time-independence”. Time-independence of \(\chi^{(3)}\) means that the contribution to the cocycle is due only to the residue of the integral at the point \(P_+ (\tau = 0)\) or, equivalently at \(P_- (\tau = \infty)\) (here we are considering \(\tau = e^t\) where \(t\) is the time parameter introduced by Krichever and Novikov which parametrizes the position of the contour \(C\) on \(\Sigma\)).
Let us expand $T^F$ in terms of the $3h - 3$ holomorphic differentials

$$T^F = \{ J^{-1}_H, z \} = T_\Sigma + \sum_{k=1}^{3h-3} \lambda^{(F)}_k \Omega_{k+1-h_0},$$

where $T_\Sigma$ denotes the holomorphic projective connection on $\Sigma$ obtained from the symmetric differential of the second-kind with bi-residue 1 and zero $\alpha$-periods (see [17] for the explicit expression of $T_\Sigma$).

In the case of Schottky uniformization we have

$$T^S = \{ J^{-1}_\Omega, z \} = T_\Sigma + \sum_{k=1}^{3h-3} \lambda^{(S)}_k \Omega_{k+1-h_0},$$

where $J_\Omega : \Omega \to \Sigma$, with $\Omega \subset \hat{C}$ the region of discontinuity of the Schottky group. The constants $\lambda^{(F)}_k$ and $\lambda^{(S)}_k$ are the (higher genus) Fuchsian and Schottkian accessory parameters. The Schottkian cocycle is

$$\chi^{(2k+1)}_S (\psi_i^{(-k)}, \psi_j^{(-k)}) = \frac{1}{24(2k+1)\pi i} \oint_C \psi_i^{(-k)} S^{(2k+1)}_{\lambda_i^{(-1)}} \psi_j^{(-k)}, \quad k = 0, 1, 2, \ldots,$$

that for $k = 1$ reduces to the Schottkian $KN$ cocycle

$$\chi^{(3)}_S (e_i, e_j) = \frac{1}{24\pi i} \oint_C \left[ \frac{1}{2} (e_i e''_j - e'_i e'_j) + T^S (e'_i - e'_j) \right].$$

The choice of the $KN$ cocycle fixes the normal ordering of operators in higher genus [23]. In particular the normal ordering associated to $\chi^{(3)}_F (e_i, e_j)$ and $\chi^{(3)}_S (e_i, e_j)$ depends on the accessory parameters $\lambda^{(F)}_k$ and $\lambda^{(S)}_k$ respectively. On the other hand these parameters are related to $S^{(h)}_{cl}$ which denotes the Liouville action evaluated on the classical solution [12].

To write down $S^{(h)}$ we must consider the Schottky covering of $\Sigma$. In this approach the relevant group is the Schottky group $G \subset PSL(2, \mathbb{C})$. Let $L_1, \ldots, L_h$ be a system of generators for $G$ of rank $h > 1$ and $D$ a fundamental region in the region of discontinuity $\Omega \subset \hat{C}$ of $G$ bounded by $2h$ disjoint Jordan curves $C_1, C'_1, \ldots, C_h, C'_h$ such that $C'_i = -L_i(C_i)$. These curves correspond to a cutting of $\Sigma \cong \Omega/\mathcal{G}$ along the $\alpha$-cycles. The Liouville action has the form [12]

$$S^{(h)} = \int_D d^2 z (\partial_z \varphi \partial_{\bar{z}} \varphi + \exp \varphi) - \frac{i}{2} \sum_{i=1}^h \int_{C_i} \varphi \left( \frac{T''_i}{L''_i} d\bar{z} - \frac{L''_i}{L'_i} dz \right) +$$

$$+ \frac{i}{2} \sum_{i=2}^h \int_{C_i} \log |L''_i|^2 \frac{T''_i}{L'_i} d\bar{z} + 4\pi \sum_{i=2}^h \log \left| \frac{(1 - \lambda_i)^2}{\lambda_i (a_i - b_i)^2} \right|, \quad (3.48)$$

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where \(a_i, b_i \in \hat{\mathbb{C}}\) are the attracting and repelling fixed points of \(L_i\) (it is possible to assume that \(a_1 = 0, a_2 = 1\) and \(b_1 = \infty\)) while \(\lambda_i\) is defined by the normal form
\[
\frac{L_i z - a_i}{L_i z - b_i} = \lambda_i \frac{z - a_i}{z - b_i}, \quad 0 < |\lambda_i| < 1. \tag{3.49}
\]

We now quote the main results in [12]. The first one concerns the quadratic holomorphic differential \(\Omega = T^F - T^S\) considered as a 1-form on Schottky space \(S\). It turns out that
\[
\Omega = \frac{1}{2} \partial S^{(h)}_{cl}, \tag{3.50}
\]
where \(\partial\) is the holomorphic component of the exterior differentiation operator on \(S\). Furthermore
\[
\lambda^{(F)}_k - \lambda^{(S)}_k = \frac{1}{2} \frac{\partial S^{(h)}_{cl}}{\partial z_k}, \quad k = 1, \ldots, 3h - 3, \tag{3.51}
\]
where \(\{z_k\}\) are the coordinates on \(S\). Another result in [12] is
\[
\frac{1}{2} \overline{\partial \partial S^{(h)}_{cl}} = -i \omega_{WP}. \tag{3.52}
\]
where \(\omega_{WP}\) is the Weil-Petersson 2-form on \(S\). Furthermore, since \(\overline{\partial T^S} = 0\), it follows that
\[
\overline{\partial T^F} = -i \omega_{WP}. \tag{3.53}
\]

From the above results it follows that the difference between Fuchsian and Schottkian higher order cocycles depends on the classical Liouville action. In particular for the \(KN\) cocycle we have
\[
\chi^{(3)}_F (e_i, e_j) - \chi^{(3)}_S (e_i, e_j) = \frac{1}{48\pi i} \sum_{k=1}^{3h-3} \frac{\partial S^{(h)}_{cl}}{\partial z_k} \oint_C \Omega^{k+1-h_0} [e_j, e_i]. \tag{3.54}
\]
Similar relations hold for the Virasoro algebra on punctured Riemann spheres. Eq.(3.54) clarifies how classical Liouville theory is connected with quantum aspects of operators defined on Riemann surfaces. Let us notice that (3.54) can be generalized to the case of higher order cocycles. Also in this case the difference between Fuchsian and Schottkian cocycles depends on the Liouville action.

The investigation above solves the problem posed in [19] about time-independence and locality of the cocycles defined by covariantization. This follows from the fact that, since the divisor of the vector field \(\frac{1}{J_H^{(i)}}\) is empty, the integrand in (3.36) has no poles outside \(P_\pm\).
3.4 Diffeomorphism Anomaly And The KN Cocycle

Let us now consider the chirally split form of the diffeomorphism anomaly [25]
\[ A(\mu; e) + \overline{A(\mu; e)}, \] (3.55)
where
\[ A(\mu; e) = \frac{1}{24\pi} \int_{\Sigma} \left[ \frac{1}{2} (e\partial_z^3 \mu - \mu \partial_z^3 e) + T (e\partial_z \mu - \mu \partial_z e) \right], \] (3.56)
with \( e \) a vector field. Here \( T \) denotes an arbitrary projective connection. We now show that \( A(\mu; e) \) reduces to the KN cocycle. To do this we first introduce some results on the deformation of the complex structure of Riemann surfaces [24]. For a short introduction to this subject and related topics see for example [26]; for more details see [27, 28].

To parametrize different metrics we consider Beltrami differentials with discontinuities along a closed curve. Let \( P_+ \) be a distinguished point of \( \Sigma \) and \( z_+ \) a local coordinate such that \( z_+(P_+) = 0 \). Let us denote by \( \Sigma^+ \) the disc defined by \( z_+ \leq 1 \), and by \( A \subset \Sigma^+ \) an annulus whose centre is \( P_+ \). Let \( \Sigma^- \) be the surface defined by
\[ \Sigma^+ \cup \Sigma^- = \Sigma, \quad \Sigma^+ \cap \Sigma^- = A. \] (3.57)
We now perform a change of coordinate
\[ z_+ \rightarrow Z = z_+ + \epsilon e_k(z_+), \quad z_+ \in A, \quad \epsilon \in \mathbb{C}, \] (3.58)
with \( e_k \equiv \psi_k^{(-1)} \) a KN vector field. Identifying the new annulus with the previous collar on \( \Sigma^+ \) we get a new surface \( \tilde{\Sigma} \) whose metric reads
\[ g(\mu_k) = \rho(z, \overline{z})|dz + \mu_k d\overline{z}|^2, \] (3.59)
where the Beltrami differential is
\[ \mu_k(P) = \begin{cases} \epsilon \partial_z e_k, & \text{if } P \in \Sigma^+; \\ 0, & \text{otherwise.} \end{cases} \] (3.60)
The KN holomorphic differentials \( \Omega^j \) form a dual basis with respect to \( \mu_k \). Indeed integrating by parts we have
\[ \frac{1}{\pi} \int_{\Sigma} \Omega^j \mu_k = \epsilon \delta_k^j. \] (3.61)
Since \( e_k \sim z_+^{h_0+1} \ldots \), it follows that for \( k \geq h_0 \) we only change the coordinate \( z_+ \), whereas \( e_{h_0-1} (h_0 \equiv 3h/2) \) changes \( z_+ \) and moves \( P_+ \). For \( k \leq -h_0 + 1 \), \( e_k \) is holomorphic
on $\Sigma \setminus \{P_+\}$, so $\tilde{\Sigma}$ is isomorphic to $\Sigma$ because the variation induced in the annulus can be reabsorbed in a holomorphic coordinate transformation on $\Sigma \setminus \Sigma^+$. For $|k| \leq h_0 - 2$ the vector field $e_k$ has poles both in $P_+$ and $P_-$. This change in $\Sigma$ corresponds to an infinitesimal moduli deformation. Notice that the dimension of the space of these vector fields is just $3h - 3$.

We are now ready to show that the anomaly $\mathcal{A}(\mu; e)$ reduces to the KN cocycle. First of all notice that by choosing (3.60) for the Beltrami differential in (3.56), the domain of the surface integral (3.56) reduces to $\Sigma^+$. Then we write $\mathcal{A}(\mu_k; e)$ in the useful form

$$\mathcal{A}(\mu_k; e) = \frac{\epsilon}{24\pi} \int_{\Sigma^+} \frac{e}{v} \partial_z \left( v \partial_z \left( v \partial_z \left( \frac{e_k}{v} \right) \right) \right),$$

(3.62)

where $v$ satisfies the equation

$$\frac{1}{2} \left( \frac{v'}{v} \right)^2 - \frac{v''}{v} = T,$$

(3.63)

that for $T = T^F$ has solution

$$v = \frac{1}{J_{H^{-1}}^T}.$$  

(3.64)

We now use the univalence of $J_{H^{-1}}^1$. Indeed this guarantees that the obstruction for the reduction of (3.62) to a contour integral around $\partial \Sigma^+$ (which is homologically equivalent to the $C$-contour in (3.34)) comes only from possible poles of $e$ in $\Sigma^+$. As we have seen the univalence of $J_{H^{-1}}^1$ implies the holomorphicity of $S_{J_{H^{-1}}^1}^{(2k+1)}$. It has been just this property of $S_{J_{H^{-1}}^1}^{(2k+1)}$ which has suggested to write the integrand in $\mathcal{A}(\mu_k; e)$ in the form (3.62).

Since any diffeomorphism can be expressed in terms of the KN vectors $e_j$, it is sufficient to consider $\mathcal{A}(\mu_k; e_j)$ instead of $\mathcal{A}(\mu_k; e)$. By the remarks above it follows that$^{10}$

$$\mathcal{A}(\mu_k; e_j) = \frac{\epsilon}{2} \chi^{(3)}_F(e_j, e_k), \quad j \geq h_0 - 1.$$  

(3.65)

For $j \leq h_0 - 2$, the vector field $e_j$ has poles at $z = P_+$ and $\mathcal{A}(\mu_k; e_j)$ can be expressed as a linear combination of KN cocycles.

Note that the Wess-Zumino condition for $\mathcal{A}(\mu_k; e_j)$ corresponds to the cocycle identity for $\chi_F^{(3)}(e_j, e_k)$. On the other hand writing $\chi^{(3)}_F(e_j, e_k)$ in terms of theta functions (the explicit form of $e_k \equiv \psi_{-1}^k$ is given in (3.29)) one should get some constraints on the period matrix from the cocycle identity that presumably are connected with the Hirota bilinear relation. Thus the Wess-Zumino condition for $\mathcal{A}(\mu_k; e_j)$ seems to be related to the Schottky problem. We do not perform such analysis here, however we stress that the cocycle condition for $\chi^{(3)}_F(e_j, e_k)$ involves, besides the period matrix, the Fuchsian accessory parameters.

$^{10}$Note that for $T \neq T^F$ we have a similar relation.
Eq. (3.65) suggests to define the higher order anomalies

\[
A \left( \mu_j^{(2k+1)}, \psi_{i}^{(-k)} \right) = \frac{1}{24(2k+1)\pi} \int_{\Sigma} \psi_{i}^{(-k)} R_{j_{H}^{(2k+1)}}^{(2k+1)} \psi_{j}^{(-k)}, \quad k = 0, 1, 2, \ldots, \tag{3.66}
\]

with

\[
R_{j}^{(2k+1)} = (2k + 1)(f')^{k} \partial_{z}(f')^{-1}\partial_{\bar{z}}(f')^{-1} \ldots \partial_{z}(f')^{-1}\partial_{\bar{z}}(f')^{k}, \tag{3.67}
\]

where the number of derivatives is \(2k+1\). Notice that the generalized Beltrami differentials are

\[
\mu_{k}^{(2k+1)}(P) = \begin{cases} 
\epsilon \partial_{z} \psi_{j}^{(-k)}, & \text{if} \ P \in \Sigma^{+}; \\
0, & \text{otherwise.} 
\end{cases} \tag{3.68}
\]

Here the deformation of the complex structure of vector bundles on Riemann surfaces is provided by the space of differentials

\[
\tilde{H}^{(k+1)} = \left\{ \psi_{j}^{(-k)} | 1 - s(-k) \leq j \leq s(-k) - 1 \right\}, \tag{3.69}
\]

which is the dual space to \(H^{(k+1)}\) defined in (3.32).

By construction higher order anomalies are related to higher order cocycles in a way similar to eq. (3.65). In this case we must consider \(W\)-algebras and the moduli space of vector bundles on Riemann surfaces. We notice that the explicit expression of the \(KN\)-differentials in terms of theta functions given in (3.29) provides a useful tool to investigate this subject.

We now show that in the case one uses the generalized harmonic Beltrami differentials

\[
\mu_{\text{harm}, j}^{(2k+1)} = \overline{\psi}_{j}^{(k+1)} e^{-k\varphi}, \quad j \in [s(k + 1), -s(k + 1)], \quad s(k) = h/2 - (k)(h - 1), \tag{3.70}
\]

the anomalies (including the standard chirally split anomaly) vanish

\[
A \left( \mu_{\text{harm}, j}^{(2k+1)}, \psi_{i}^{(-k)} \right) = \frac{1}{24(2k+1)\pi} \int_{\Sigma} \psi_{i}^{(-k)} S_{J_{H}^{(2k+1)}}^{(2k+1)} \mu_{\text{harm}, j}^{(2k+1)} = 0, \quad k = 0, 1, 2, \ldots. \tag{3.71}
\]

This follows simply because

\[
S_{J_{H}^{(2k+1)}}^{(2k+1)} \mu_{\text{harm}, j}^{(2k+1)} = 0. \tag{3.72}
\]

## 4 Virasoro Algebra On \(\Sigma\)

Here we consider a sort of higher genus generalization of the Killing vectors. This generalization follows from an investigation of the kernel of the \(KN\) cocycle. This analysis will suggest two possible realizations of the Virasoro algebra on \(\Sigma\) without central extension based on the Poincaré metric and \(J_{H}^{1}\). Finally a higher genus realization of the Virasoro cocycle \(\chi_{kj} = \delta_{k-j}(j^3 - j)/12\) is given.
4.1 Higher Genus Analogous Of Killing Vectors

In genus zero the KN cocycle reduces to $\chi_{kj}$ which vanishes for $j = -1, 0, 1, \forall k$. This reflects the $SL(2, \mathbb{C})$ symmetry of the Riemann sphere due to the three Killing vectors. For $h \geq 2$ do not exist chiral holomorphic $-k$-differentials, with $k = 1, 2, \ldots$. The reason is that in this case

$$\deg \psi_j^{(-k)} = 2k(1 - h) < 0,$$

so that $\psi_j^{(-k)}$ has at least $2k(h - 1)$ poles. Nevertheless by eq.(3.16) the cocycles have the following property

$$\chi_F^{(2k+1)} \left( \psi^{(-k)}_i, (2\varphi z)^l e^{-k\varphi} \right) = 0, \quad \forall i, \quad k = 0, 1, 2, \ldots, \quad l = 0, \ldots, 2k.$$  

In particular

$$\chi_F^{(3)} \left( e_i, -\varphi \right) = \chi_F^{(3)} \left( e_i, 2\varphi z e^{-\varphi} \right) = \chi_F^{(3)} \left( e_i, (2\varphi z)^2 e^{-\varphi} \right) = 0, \quad \forall i.$$  

Thus, in spite of the fact that for $h \geq 2$ Killing vectors do not exist, the non-chiral vectors $(2\varphi z)^l e^{-\varphi}$, $l = 0, 1, 2$, can be seen as their higher genus generalization. Let us make some remarks on this point. The Killing vectors are the solutions of the equation

$$\partial_z v = 0.$$  

In the case of the Riemann sphere we can choose the standard atlas $(U_{\pm}, z_{\pm})$ with $z_- = z_{\pm}^{-1}$ in the intersection $U_+ \cap U_-$. For the component of a vector field $v \equiv \{v^+(z_+), v^-(z_-)\}$, we have $v^+(z_+)(dz_+)^{-1} = v^-(z_-)(dz_-)^{-1}$, that is $v^-(z_-) = -z^2_v(z_-)^{-1}$). Therefore if

$$v^+_l(z_+) = z^l_+,$$

then $v^-_l(z_-) = -z^{-l+2}$ and the solutions of eq.(4.4) are $v_0, v_1, v_2$. To understand what happen in higher genus, we first note that besides eq.(4.4) these vector fields are solutions of the covariant equation

$$S_f^{(3)} \cdot v = 0, \quad f(z) \equiv \int^z v_0^{-1}.$$  

Indeed in $U_+ \cap U_-$ eq.(4.6) reads

$$\partial^3_{z_+} v^+(z_+) = z^2_+ \partial_{z_-} \left( z^2_- \partial_{z_-} \left( z_-^{-2} v^-(z_-) \right) \right) = 0,$$

whose solutions coincide with the solutions of eq.(4.4). This relationship between the zero modes of $\partial_z$ and $S_f^{(3)}$ extends to the case of $-k$-differentials, $k = 0, 1/2, 1, \ldots$. In particular
on the Riemann sphere the $2k + 1$ chiral solutions of the equation $S_f^{(2k+1)} \cdot \phi^{(-k)} = 0$, where $\phi^{(-k)}$ are $-k$-differentials, coincide with the zero modes of the $\partial_\bar{z}$ operator

$$S_f^{(2k+1)} \cdot \phi^{(-k)} = 0 \quad \rightarrow \quad \partial_\bar{z} \phi^{(-k)} = 0, \quad k = 0, \frac{1}{2}, 1, \ldots, \quad (4.8)$$

whose solutions are $\phi_l^{(-k)} = \{\phi_l^{(-k)+}, \phi_l^{(-k)-}\}$, $l = 0, 1, 2$, where

$$\phi_l^{(-k)+}(z_+) = z_+^l, \quad \phi_l^{(-k)-}(z_-) = (-1)^k z_-^{2k-l}. \quad (4.9)$$

The higher genus generalization of (4.8) reads

$$S_{i_H^{(2k+1)}} \cdot \phi^{(-k)} = 0 \quad \rightarrow \quad \partial_\bar{z} \phi^{(-k)} = 0, \quad k = 0, \frac{1}{2}, 1, \ldots, \quad (4.10)$$

whose solutions are\(^{11}\)

$$\phi_l^{(-k)} = \left(\frac{J_{i_H}^{-1}}{J_{-1}^{-1}}\right)^l, \quad l = 0, 1, \ldots, 2k. \quad (4.11)$$

Thus a possible choice for the higher genus analogous of the Killing vectors are the polymorphic vector fields

$$\phi_0^{(-1)} = \frac{1}{J_{-1}^{1}}, \quad \phi_1^{(-1)} = \frac{J_{i_H}^{-1}}{J_{-1}^{1}}, \quad \phi_2^{(-1)} = \left(\frac{J_{i_H}^{-1}}{J_{-1}^{1}}\right)^2. \quad (4.12)$$

Similarly to the case of the Killing vectors, $\phi_0^{(-1)}$, $\phi_1^{(-1)}$ and $\phi_2^{(-1)}$ are zero modes for $\chi^{(3)}_F$. More generally

$$\chi_{F}^{(2k+1)} \left(\psi_i^{(-k)}, \phi_l^{(-k)}\right) = 0, \quad \forall i, \quad k = 0, 1, 2, \ldots, \quad l = 0, \ldots, 2k. \quad (4.13)$$

However if singlevaluedness is required we must relax the chirality condition and instead of $\phi_l^{(-k)}$ we must consider the non chiral differentials $(2\varphi_\bar{z})^l e^{-k \varphi}$, $l = 0, 1, \ldots, 2k$.

### 4.2 Realization Of The Virasoro Algebra On $\Sigma$

The previous discussion suggests a higher genus realization of the Virasoro algebra. We first consider two realizations of this algebra without central extension. In the first case we have

$$[L_j, L_k] = (k - j)L_{j+k}, \quad L_k = (2\varphi_\bar{z})^{k+1} e^{-\varphi} \partial_\bar{z}. \quad (4.14)$$

\(^{11}\)However note that $\partial_\bar{z} \phi_l^{(-k)} = 0, \forall l.$
Similarly we can realize the centreless Virasoro algebra on $\Sigma$ considering as generators the polymorphic chiral vector fields

$$L^{ch}_k = \frac{(J^{-1}_H)^{k+1}}{J^{-1}_{H'}} \partial_z, \quad \overline{L}^{ch}_k = \frac{(J^{-1}_H)^{k+1}}{J^{-1}_{H'}} \partial_{\bar{z}},$$

so that

$$[L^{ch}_j, L^{ch}_k] = (k - j) L^{ch}_{j+k}, \quad [\overline{L}^{ch}_j, \overline{L}^{ch}_k] = (k - j) \overline{L}^{ch}_{j+k}, \quad [L^{ch}_j, \overline{L}^{ch}_k] = 0. \quad (4.15)$$

Observe that the holomorphic operators $S^{(2k+1)}_{J^{-1}_H}$ can be expressed in terms of the above generators

$$S^{(2k+1)}_{J^{-1}_H} = (2k+1) \left( J^{-1}_{H'} \right)^{k+1} L^{ch}_{-1} \left( J^{-1}_H \right)^k = (2k+1) e^{(k+1)\varphi} L^{ch}_{-1} e^{k\varphi}. \quad (4.16)$$

The structure of the generators $L^{ch}_k$ suggests the generalization

$$L_k = v_k \partial_z, \quad v_k(z) = \frac{f^{k+1}(z)}{f'(z)}, \quad (4.17)$$

with $f(z)$ an arbitrary meromorphic function. In this case we can define the cocycle

$$\chi(v_k, v_j) = \frac{1}{24\pi i} \oint_{C_0} v_k S^{(3)}_f v_j = \frac{j^3 - j}{12} \delta_{k,-j}, \quad (4.18)$$

where $C_0$ encircles a simple zero of $f$. Thus we have

$$[L_j, L_k] = (k - j) L_{j+k} + \frac{j^3 - j}{12} \delta_{k,-j}. \quad (4.19)$$

To define a cocycle depending only on the homological class of the contour we consider

$$f = e^{\int \omega}, \quad (4.20)$$

with $\omega$ a 1-differential. A possible choice is to consider the third-kind differential with poles at $P_\pm$ and with periods over all cycles imaginary

$$\omega = \partial_z \log \frac{E(z, P_+)}{E(z, P_-)} - 2\pi i \sum_{j,k=1}^h \left( \text{Im} \int_{P_+}^{P_-} \omega_j \right) \Omega^{(2)}_{jk}^{-1} \omega_k(z), \quad (4.21)$$

where $\Omega^{(2)}$ denotes the imaginary part of the Riemann period matrix. In this case one can substitute the contour $C_0$ in (4.19) with the contour $C$ in (3.34). However we notice that the integrand of (4.19) with $f$ given in (4.21) is not singlevalued for $j \neq -k$. An interesting possibility to investigate this aspect is to set $f = f_{n,m}$ where the $f_{n,m}$'s are defined in the next section.
5 Liouville Field And Higher Genus Fourier Analysis

In the standard approach to 2D gravity the Liouville field is considered as a free field. However there is a substantial hindrance in the CFT approach to Liouville gravity. Namely, since the metric \( g = e^\sigma \hat{g} \) must be well-defined, \( e^\sigma \) must be an element of \( \mathcal{C}_+^\infty \). If \( \sigma \) were considered as a free scalar field then the metric would take non positive values as well. One of the aims of this section is to investigate uniformization theory and then provide the mathematical tools to face the problem of metric positivity in considering Liouville gravity on higher genus Riemann surfaces.

A long-standing problem in uniformization theory is to express the uniformizing Fuchsian group in terms of the Riemann period matrix \( \Omega \). A possibility is to write \( J_{H}^{-1} \) in terms of \( \Omega \). In this case, after going around non trivial cycles, the coefficients in the Möbius transformation of \( J_{H}^{-1} \) are given in terms of \( \Omega \). Unfortunately to write explicitly \( J_{H}^{-1} \) seems to be an outstanding problem.

These aspects are related to the problem of finding the eigenfunctions of the Laplacian on \( \Sigma \). Actually, there is a strict relationship between the eigenvalues of the Laplacian and geodesic lengths. These lengths are related to the trace of hyperbolic Fuchsian elements. A way to investigate this argument is by Selberg trace formula. However one should try to investigate the problem of uniformization in a more direct (analytic) way. Here we introduce a new set of functions on \( \Sigma \) whose structure seems related to the uniformizing Fuchsian group.

5.1 Positivity And Fourier Analysis On Riemann Surfaces

A possible way to construct functions in \( \mathcal{C}_+^\infty \) is to consider the ratio of two suitable \((p, q)\)-differentials. For \( p = q = 1 \), besides the Poincaré metric, we can use any positive quadratic form like \( \sum_{j,k=1}^{h} \omega_j A_{jk} \omega_k \). An alternative is to attempt to define the higher genus analogous of the Fourier modes. In the following we adopt this approach.

Let
\[
G = \frac{e^{g-\bar{g}} + e^{\bar{g}-g}}{2} = \cos \left(2 \text{Im} \ g \right),
\]
be a function on a compact Riemann surface \( \Sigma \). We will see that in order that \( G \) be a non trivial regular function in \( \mathcal{T}^{0,0} \), it is necessary that after winding around the homology cycles of \( \Sigma \), the function \( g \) transforms with an additive term whose imaginary part be a non-vanishing element in \( \pi \mathbb{Z} \). Such a multivaluedness is crucial for the construction of well-
defined regular functions on $\Sigma$. We will see that there are infinitely many functions, labelled by $2h$ integers $(n, m) \in \mathbb{Z}^{2h}$, with the properties of $g$ whose existence is strictly related to the positive definiteness of the imaginary part of the Riemann period matrix.

5.2 Real Multivaluedness And $\text{Im} \Omega > 0$

We begin by considering the holomorphic differential

$$\omega(z) = \sum_{k=1}^{h} A_k \omega_k(z), \quad A = a + ib, \quad (a, b) \in \mathbb{R}^{2h},$$

(5.2)

where $\omega_1, \ldots, \omega_h$, are the holomorphic differentials with the standard normalization (2.3).

After winding around the cycle

$$c_{q,p} = p \cdot \alpha + q \cdot \beta, \quad (q, p) \in \mathbb{Z}^{2h},$$

(5.3)

the function $f(z) = e^{\int_{z_0}^{z} \omega}$ transforms into

$$f(z + c_{q,p}) = \exp \left( \sum_{k=1}^{h} (p_k + \sum_{l=1}^{h} q_l \Omega_{kl}) A_k \right) f(z).$$

(5.4)

We constrain the multivaluedness factor in (5.4) to be real for arbitrary $(q, p) \in \mathbb{Z}^{2h}$; that is we require that the imaginary part of the exponent in (5.4) be an integer multiple of $\pi$

$$\sum_{k=1}^{h} b_k \int_{\alpha_j} \omega_k = \pi n_j, \quad j = 1, \ldots, h, \quad n_j \in \mathbb{Z},$$

(5.5)

$$\sum_{k=1}^{h} a_k \Omega_{kj}^{(2)} + \sum_{k=1}^{h} b_k \Omega_{kj}^{(1)} = \pi m_j, \quad j = 1, \ldots, h, \quad m_j \in \mathbb{Z},$$

(5.6)

where

$$\Omega_{kj}^{(1)} \equiv \text{Re} \Omega_{kj}, \quad \Omega_{kj}^{(2)} \equiv \text{Im} \Omega_{kj}.$$  

(5.7)

Thus after winding around $\alpha_j$ we have

$$f(z + \alpha_j) = \exp(a_j + i\pi n_j) f(z),$$

(5.8)

whereas around $\beta_j$

$$f(z + \beta_j) = \exp \left[ \sum_{k=1}^{h} \left( a_k \Omega_{kj}^{(1)} - \pi n_k \Omega_{kj}^{(2)} \right) + i\pi m_j \right] f(z).$$

(5.9)
Eqs. (5.5, 5.6) show an interesting connection between a fundamental property of Riemann surfaces and the existence of regular functions with real multivaluedness. Namely, for each fixed set of integers \((n, m) \neq (0, 0)\), positivity of \(\Omega^{(2)}\) guarantees the existence of a non trivial solution of eqs. (5.5, 5.6). We have

\[
a_k = \frac{\det \Omega^{(2;k)}}{\det \Omega^{(2)}}, \quad b_k = \pi n_k, \quad n_k \in \mathbb{Z},
\]

where \(\Omega^{(2;k)}\) is obtained by the matrix \(\Omega^{(2)}\) after the substitutions

\[
\Omega^{(2)}_{kj} \rightarrow \pi \left( m_j - \sum_{i=1}^{h} n_i \Omega^{(1)}_{ij} \right), \quad j = 1, \ldots, h.
\]

For practical reasons we change the notation of \(f\) and \(\omega\)

\[
f_{n,m}(z) = \exp \int_{P_0}^{z} \omega_{n,m}, \quad \omega_{n,m}(z) = \sum_{k=1}^{h} \left( \frac{\det \Omega^{(2;k)}}{\det \Omega^{(2)}} + i\pi n_k \right) \omega_k(z).
\]

We now illustrate some interesting properties of the functions \(f_{n,m}\).

### 5.3 Eigenfunctions

Let \((n, m) \in \mathbb{Z}^{2h}\) be fixed and consider the scalar Laplacian \(\Delta_{g,0} = -g^z\bar{z} \partial_z \partial_{\bar{z}}\) with respect to the degenerate metric

\[
ds^2 = 2g_{z\bar{z}}|dz|^2, \quad g_{z\bar{z}} = \frac{|\omega_{n,m}|^2}{2A},
\]

where \(A\) normalizes the area of \(\Sigma\) to 1 (see eq. (5.26)). The functions

\[
\psi_k(z, \bar{z}) = \frac{1}{\sqrt{2}} \left[ \left( \frac{f_{n,m}(z)}{\bar{f}_{n,m}(z)} \right)^k + \left( \frac{f_{n,m}(z)}{\bar{f}_{n,m}(z)} \right)^k \right], \quad k = 0, 1, 2, \ldots,
\]

with \((n, m)\) fixed, are eigenfunctions of \(\Delta_{g,0}\) with eigenvalues

\[
\Delta_{g,0} \psi_k(z, \bar{z}) = \lambda_k \psi_k(z, \bar{z}), \quad \lambda_k = 2Ak^2, \quad k = 0, 1, 2, \ldots.
\]

Note that

\[
2A \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \zeta(1) = \frac{\pi^2}{6}, \quad 4A^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \zeta(2) = \frac{\pi^4}{90}.
\]

The orthonormality of the eigenfunctions

\[
\int_{\Sigma} \sqrt{g} \psi_j \psi_k = \delta_{jk},
\]
follows from the fact that \(|\omega_{n,m}(z)|^2 \exp k \left( \int^z \omega_{n,m} - \int^\overline{z} \omega_{n,m} \right)\) is a total derivative.

Let us notice that for \(k \notin \mathbb{Z}\) the functions \(\left( \frac{f_{n,m}}{f_{n,m}} \right)^k\) are in general (see below) not well-defined. This shows that arbitrary powers of well-defined scalar functions can be multivalued around the homology cycles. In this sense the possible values of \(k\) in (5.14) are fixed by “boundary conditions”. This aspect should be taken into account in considering operators such as \(e^{a\phi}\) in Liouville and conformal field theories.

We stress that in general there are other eigenfunctions besides \(\psi_k, k \in \mathbb{N}\). For example when all the \(2m_j's\) and \(2n_j's\) are integer multiple of an integer \(N\) then the eigenfunctions include \(\psi_{k/N}, k \in \mathbb{N}\) whose eigenvalue is \(2Ak^2/N^2\). More generally one should investigate whether the period matrix has some non trivial number theoretic structure. For example the problem of finding the possible solutions of the equation

\[
\omega_{n',m'}(z) = c \cdot \omega_{n,m}(z),
\]

with both \((n, m)\) and \((n', m')\) in \(\mathbb{Z}^{2h}\) and \(c\) a (in general complex) constant, is strictly related to the numerical properties of \(\Omega\).

On the other hand if eq.(5.18) has non trivial solutions (by trivial we mean \(c \in \mathbb{Q}\)) then

\[
\frac{f_{n',m'}(z)}{f_{n',m'}(z)} = e^{c \int^z \omega_{n,m} - \int^\overline{z} \omega_{n,m} - \frac{\left( f_{n,m}(z) \right)^k}{f_{n,m}(z)}}, \quad c \in \mathbb{C} \setminus \mathbb{Q}, \quad k \in \mathbb{Q}.
\]

Therefore

\[
\Delta_{g,0} \phi_c(z, \overline{z}) = 2A|c|^2 \phi_c(z, \overline{z}), \quad \Delta_{g,0} = -2A|\omega_{n,m}|^{-2} \partial_z \partial_{\overline{z}},
\]

where

\[
\phi_c(z, \overline{z}) = \frac{1}{\sqrt{2}} \left( \frac{f_{n',m'}(z)}{f_{n',m'}(z)} + \frac{\overline{f_{n',m'}(z)}}{f_{n',m'}(z)} \right).
\]

The relevance of these functions resides in the non trivial number theoretic (chaotic) structure of the eigenvalues

\[
\lambda_c = 2A|c|^2.
\]

To understand this it is sufficient to write eq.(5.18) in the more transparent form

\[
m_j' = \sum_{k=1}^{h} \Omega_{jk} n_k' = \tau \left( m_j - \sum_{k=1}^{h} \Omega_{jk} n_k \right), \quad j = 1, \ldots, h.
\]

To each \((n, m)\) and \((n', m')\) satisfying (5.23) corresponds a possible value of \(c\).
Since (5.18) (or equivalently (5.23)) are equations on $\Omega$ one should think that they are related to the Hirota bilinear relations or equivalently to the Fay trisecant identity\(^{12}\). As well-known these must be satisfied by $\Omega$ (Schottky’s problem). These remarks indicate that the aspects of number theory underlying the structure of Fuchsian groups should be related to integrable systems such as the KP hierarchy.

The construction of the functions $h_{n,m} = f_{n,m}/f_{n,m}$ indicates that the solutions of the equation

\[
\partial_z \psi(z, \bar{z}) = \mu(z) \bar{\psi}(z, \bar{z}),
\]

with $\mu$ a holomorphic differential and $\psi$ a smooth function are $\mu = \omega_{n,m}$ and $\psi = h_{n,m}$. It seems that equations such as (5.18) and (5.24) provide analytic tools to investigate aspects of number theory, structure of moduli space, chaotic spectrum etc.

### 5.4 Multivaluedness, Area And Eigenvalues

To evaluate $A$ we can use the Riemann bilinear relations

\[
2A = \frac{i}{2} \int_R \omega_{n,m} \wedge \bar{\omega}_{n,m} = -\text{Im} \sum_{j=1}^{h} \oint_{\alpha_j} \omega_{n,m} \oint_{\beta_j} \bar{\omega}_{n,m} = \sum_{l,k=1}^{h} (a_k a_l + b_k b_l) \Omega_{lk}^{(2)},
\]

and by eqs.(5.5,5.6)

\[
A = \frac{\pi}{2} \sum_{l=1}^{h} \left\{ a_l \left( m_l - \sum_{k=1}^{h} \Omega_{l}^{(1)} n_k \right) + \pi \sum_{k=1}^{h} n_k \Omega_{lk}^{(2)} n_k \right\}.
\]

The multivaluedness of $f_{n,m}$ is related to $A$ (the area of the metric $|\omega_{n,m}(z)|^2$). In particular, after winding around the cycle $c_{n,-m} = -m \cdot \alpha + n \cdot \beta$, we have

\[
\mathcal{P}_{n,-m} f_{n,m}(z) = e^{-2A} f_{n,m}(z),
\]

where $\mathcal{P}_{q,p}$ is the winding operator

\[
\mathcal{P}_{q,p} g(z) = g(z + c_{q,p}).
\]

\(^{12}\)Recall that the Fay trisecant identity is the higher genus generalization of the Cauchy (= bosonization) formula

\[
\prod_{i<j} (z_i - z_j) (w_i - w_j) \prod_{ij} (z_i - w_j) = (-1)^n (n-1)! \det \left( \frac{1}{z_i - w_j} \right).
\]
Comparing (5.27) with (5.15) we get the following relationship connecting multivaluedness, area and eigenvalues
\[
\lambda_k = \frac{1}{\pi} \log \frac{f_{n,m}(z)}{f_{n,m}(z + k^2 c_{n,m})}.
\] (5.29)
Thus we can express the action of the Laplacian in terms of the winding operator. This relationship between eigenvalues and multivaluedness is reminiscent of a similar relation arising between geodesic lengths (Fuchsian dilatation) and eigenvalues of the Poincaré Laplacian (Selberg trace formula).

5.5 Genus One

One of the properties of the \(f_{n,m}\)'s is that in the case of the torus the functions
\[
\phi_{n,m}(z, \bar{z}) = \frac{1}{\sqrt{2}} \left( \frac{f_{n,m}(z)}{f_{n,m}(\bar{z})} + \frac{\overline{f}_{n,m}(\bar{z})}{f_{n,m}(\bar{z})} \right), \quad (n, m) \in \mathbb{Z}^2,
\] (5.30)
coincide with the well-known eigenfunctions for the Laplacian \(-2\partial_z \overline{\partial}_z\). To prove this we choose the coordinate \(z = x + \tau y\) with \(\tau = \tau^{(1)} + i\tau^{(2)}\) the torus period matrix. Eq.(5.10) gives
\[
a = \frac{\pi(m - n\tau^{(1)})}{\tau^{(2)}}, \quad b = \pi n,
\] (5.31)
thus, choosing \(P_0 = 0\), we get
\[
\phi_{n,m}(z, \bar{z}) = \sqrt{2}\cos 2\pi(nx + my), \quad (n, m) \in \mathbb{Z}^2,
\] (5.32)
and
\[
\lambda_{n,m} = 2\pi^2(m - \tau n)(m - \overline{\tau} n)/\tau^{(2)}^2, \quad (n, m) \in \mathbb{Z}^2.
\] (5.33)

5.6 Remarks

Let us make further remarks about \(h_{n,m} = f_{n,m}/\overline{f}_{n,m}\) in (5.14). First of all note that in considering \(h_{n,m}^k (= h_{kn,km})\) as eigenfunctions of the scalar Laplacian, the indices \((n, m)\) are fixed whereas in the case of the torus the eigenfunctions are \(h_{n,m}\) with \((n, m)\) running in \(\mathbb{Z}^2\). Thus if we insist on using \(h_{n,m}^k\) with fixed \((n, m)\) also on the torus, then we will lose infinitely many eigenfunctions of the Laplacian \(-2\partial_z \overline{\partial}_z\). Therefore, for analogy with the torus case, a complete set of eigenfunctions in higher genus should be labelled by \((n, m) \in \mathbb{Z}^{2h}\). However the structure of eq.(5.23) suggests that there is a constraint on the values of \((n, m)\) which should reduce the number of indexcens to \(h\).
Unfortunately it is very difficult to recognize a complete set of eigenfunctions in higher genus. This question is related to the problem of finding the explicit dependence of the Poincaré metric \( e^\phi \) on the moduli\(^{13} \). The reason is that if in the torus case the complete set of eigenfunctions should reduce to \( \{ \phi_{n,m} \} \) then for analogy the Laplacian on higher genus surfaces must be definite with respect to the constant curvature metric, that is the Poincaré metric. Really, each metric in the form \( g_{zz}^{(p)} = \sum_{j,k=1}^h \omega_j A_{jk}^{(p)} \bar{\omega}_k \), with \( A^{(p)} \) a positive definite matrix, reduces to the constant curvature metric on the torus. However \( \det' \Delta_{g^{(p)},0} \) and \( \det' \Delta_{g^{(q)},0} \) are related by the Liouville action for the Liouville field \( \sigma = \log g_{zz}^{(p)}/g_{zz}^{(q)} \) in the background metric \( g^{(q)} \).

Notice that starting from the requirement of real multivaluedness, that is the “Dirac condition” \( (5.5,5.6) \), we end in a natural way with a set of functions with important properties. In particular, since this condition is the basic feature underlying the construction of eigenfunctions in the case of the torus, it seems that the Dirac condition is a guidance to formulate Fourier analysis on higher genus Riemann surfaces as well. Thus the properties of the \( f_{n,m} \)'s suggest that they are a sort of “building-blocks” to construct a complete set of eigenfunctions for the scalar Laplacian of a well-defined metric. In particular the set of real functions

\[
\mathcal{F} = \left\{ \cos (2 \text{Im} g_{n,m}(z)), \sin (2 \text{Im} g_{n,m}(z)) \bigg| (n, m) \in \mathbb{Z}^{2h} \right\}, \quad g_{n,m}(z) \equiv \int_{P_0}^z \omega_{n,m}, \quad (5.34)
\]

resemble higher genus Fourier modes. Furthermore, by the analogy with the torus case, one should investigate whether

\[
\lambda_{n,m} = 2 \sum_{l,k=1}^h (a_k a_l + b_k b_l), \quad (n, m) \in \mathbb{Z}^{2h}, \quad (5.35)
\]

are eigenvalues of the Laplacian with respect to some metrics. Note that the term \( a_k a_l + b_k b_l \) in \( (5.35) \) appears in the expression for the area of the metric \( |\omega_{n,m}(z)|^2 \) (see \( (5.25) \)). On the other hand \( \Omega^{(2)}/\det \Omega^{(2)} \) reduces to 1 in genus 1, therefore still by analogy with the torus case one should consider possible candidates for eigenvalues also the following quantities

\[
\mu_{n,m} = \frac{2}{\det \Omega^{(2)}} \sum_{l,k=1}^h (a_k a_l + b_k b_l) \Omega^{(2)}_{lk} = \frac{4 A_{n,m}}{\det \Omega^{(2)}}, \quad (n, m) \in \mathbb{Z}^{2h}, \quad (5.36)
\]

\(^{13}\)As we have seen this would be equivalent to finding the explicit dependence of \( J_{H^{-1}} \) on the moduli of \( \Sigma \) and then to solving long-standing problems in the theory of uniformization, Fuchsian groups etc..
with \( A_{n,m} \equiv A \) where the dependence of the area \( A \) on \((n,m)\) is given in (5.26). Other quantities that should be evaluated are

\[
Z_h(\Omega) = \prod_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \lambda_{n,m},
\]

(5.37)

and

\[
\tilde{Z}_h(\Omega) = \prod_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \mu_{n,m},
\]

(5.38)

that on the torus reduce to the determinant of the Laplacian

\[
Z_1(\tau) = \tilde{Z}_1(\tau) = \tau^{(2)}|\eta(\tau)|^4.
\]

(5.39)

It is possible to get some insight on \( Z_h(\Omega) \) and \( \tilde{Z}_h(\Omega) \) by investigating the behaviour of the \( \lambda_{n,m} \)'s and \( \mu_{n,m} \)'s under pinching of the separating and non-separating cycles of \( \Sigma \). This can be done because the behaviour of the period matrix near the boundary of the moduli space is well-known. In particular the structure of the \( \lambda_{n,m} \)'s and \( \mu_{n,m} \)'s seems to be suitable to recover the eigenvalues of the Laplacian on the torus in the “first” component of the boundary of the compactified moduli space. However we do not perform such analysis here.

Another possible investigation concerning the results in this section is the analysis of the subspace of the differentials in \( T^p,q \) made up of the scalar functions \( h_{n,m}, \tilde{h}_{n,m} \) suitably combined with products of the \( KN \) differentials \( \psi_k^{(p)} \) and \( \psi_l^{(q)} \).

Going back to the construction of functions in \( C^\infty_+ \) we notice that, considering the functions in the set \( F \) as Fourier modes on higher genus Riemann surfaces, the Liouville field can be expanded as

\[
\sigma(z, \bar{z}) = \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} f_{n,m}, \quad \bar{a}_{n,m} = a_{-n,-m}.
\]

(5.40)

6 Liouville Action And Topological Gravity

In this section we show that the classical Liouville action appears in the intersection numbers on moduli space. These numbers are the correlators of topological gravity as formulated by Witten [6, 7]. This result provides an explicit relation between topological and Liouville gravity.
6.1 Compactified Moduli Space

We now introduce the moduli space of stable curves $\overline{M}_h$, that is the Deligne-Mumford compactification of moduli space. $\overline{M}_h$ is a projective variety and its boundary $D = \overline{M}_h \setminus M_h$, called the compactification divisor, decomposes into a union of divisors $D_0, \ldots, D_{[h/2]}$ which are subvarieties of complex codimension one.

A Riemann surface $\Sigma$ belongs to $D_{k>0} \cong \overline{M}_h - k \times \overline{M}_k$ if it has one node separating it into two components of genus $k$ and $h-k$. The locus in $D_0 \cong \overline{M}_{h-1,2}$ consists of surfaces that become, on removal of the node, genus $h-1$ double punctured surfaces. Surfaces with multiple nodes lie in the intersections of the $D_k$.

The compactified moduli space $\overline{M}_{h,n}$ of Riemann surfaces with $n$-punctures $z_1, \ldots, z_n$ is defined in an analogous way to $\overline{M}_h$. The important point now is that the punctures never collide with the node. Actually the configurations with $(z_i - z_j) \to 0$ are stabilized by considering them as the limit in which the $n$-punctured surface degenerates into a $(n-1)$-punctured surface and the three punctured sphere.

Let us go back to the space $\overline{M}_h$. The divisors $D_k$ define cycles and thus classes in $H^2(\overline{M}_h, \mathbb{Q})$. It turns out that the components of $D$ together with the divisor associated to $\frac{[\omega_{WP}]}{2\pi^2}$ provide a basis for $H^2(\overline{M}_h, \mathbb{Q})$. The main steps to prove this are the following. First of all recall that the Weil-Petersson Kähler form $\omega_{WP}$ extends as a closed form to $\overline{M}_h$ [29], in particular [30]

$$\frac{[\omega_{WP}]}{2\pi^2} \in H^2(\overline{M}_h, \mathbb{Q}),$$

which by Poincaré duality defines a cycle $D_{WP}/2\pi^2$ in $H_{6h-8}(\overline{M}_h, \mathbb{Q})$. The next step is due to Harer [31] who proved that $H_2(\overline{M}_h, \mathbb{Q}) = \mathbb{Q}$ so that by Mayer-Vietoris

$$H_2(\overline{M}_h, \mathbb{Q}) = \mathbb{Q}^{[h/2]+2}.$$  \hfill (6.2)

In [30] Wolpert constructed a basis of 2-cycles $C_k, k = 0, \ldots, [h/2] + 1$ for $H_2(\overline{M}_h, \mathbb{Q})$ and computed the intersection matrix

$$A_{jk} = C_j \cdot D_k, \quad j, k = 0, \ldots, [h/2] + 1,$$  \hfill (6.3)

where $D_{[h/2]+1} \equiv D_{WP}/2\pi^2$. The crucial result in [30] is that $A_{jk}$ is not singular so that the classes associated to $D_k, k = 0, \ldots, [h/2] + 1$ are a basis for $H_{6h-8}(\overline{M}_h, \mathbb{Q})$.

Let us now define the universal curve $\mathcal{C}\overline{M}_{h,n}$ over $\overline{M}_{h,n}$. It is built by placing over each point of $\overline{M}_{h,n}$ the Riemann surface which that point denotes. Of course $\overline{M}_{h,1}$ can be...
identified with $C\mathcal{M}_h$. More generally $\mathcal{M}_{h,n}$ can be identified with $C_n(\mathcal{M}_h) \setminus \{\text{sing}\}$ where $C_n(\mathcal{M}_h)$ denotes the $n$-fold fiber product of the $n$-copies $C(1)\mathcal{M}_h, \ldots, C(n)\mathcal{M}_h$ of the universal curve over $\mathcal{M}_h$ and $\{\text{sing}\}$ is the locus of $C_n(\mathcal{M}_h)$ where the punctures come together.

Finally we define $K_{C/M}$ as the cotangent bundle to the fibers of $C\mathcal{M}_{h,n} \to \mathcal{M}_{h,n}$, it is built by taking all the spaces of $(1,0)$-forms on the various $\Sigma$ and pasting them together into a bundle over $C\mathcal{M}_{h,n}$.

6.2 $\langle \kappa_{d_1-1} \cdots \kappa_{d_n-1} \rangle$

Let $\Sigma$ be a Riemann surface in $\mathcal{M}_{h,n}$. The cotangent space $T^*\Sigma|_{z_i}$ varies holomorphically with $z_i$ giving a holomorphic line bundle $L_{(i)}$ on $\mathcal{M}_{h,n}$. Considering the $z_i$ as sections of the universal curve $C\mathcal{M}_{h,n}$ we have $L_{(i)} = z_i^* (K_{C/M})$.

Let us consider the intersection numbers $\langle 6,7 \rangle$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\mathcal{M}_{h,n}} c_1(L_{(1)})^{d_1} \wedge \cdots \wedge c_1(L_{(n)})^{d_n}, (6.4)$$

where the power $d_i$ denotes the $d_i$-fold wedge product. Notice that, since $c_1(L_{(i)})$ is a two-form, $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ does not depend on the ordering and, by dimensional arguments, it may be nonvanishing only if the charge conservation condition $\sum d_i = 3h - 3 + n$ is satisfied. Moreover, due to the orbifold nature of $\mathcal{M}_{h,n}$, the intersection numbers will generally be rational.

Related to the $\tau$’s there are the so-called Mumford tautological classes [32]. Let $\pi: \mathcal{M}_{h,1} \to \mathcal{M}_h$ be the projection forgetting the puncture. The tautological classes are

$$\kappa_l = \pi_* (c_1(L)^{l+1}) = \int_{\mathcal{M}_{h,1}} c_1(L)^{l+1}, \quad l \in \mathbb{Z}^+, p \in \mathcal{M}_h, (6.5)$$

where $L$ is the line bundle whose fiber is the cotangent space to the one marked point of $\mathcal{M}_{h,1}$. The $\kappa$’s correlation functions are $\langle \kappa_{s_1} \cdots \kappa_{s_n} \rangle = \langle \wedge_{i=1}^n \kappa_{s_i}, \mathcal{M}_h \rangle$. To get the charge conservation condition we must take into account the fact that integration on the fibre in (6.5) decreases one (complex) dimension so that $\kappa_l$ is a $(l,l)$-form on the moduli space. It follows that the nonvanishing condition for the intersection numbers $\langle \kappa_{s_1} \cdots \kappa_{s_n} \rangle$ is $\sum s_i = 3h - 3$.

There are relationships between the $\kappa$’s and $\tau$’s correlators. For example performing the integral over the fiber of $\pi: \mathcal{M}_{h,1} \to \mathcal{M}_h$, we have

$$\langle \tau_{3h-2} \rangle = \int_{\mathcal{M}_{h,1}} c_1(L)^{3h-2} = \int_{\mathcal{M}_h} \kappa_{3h-3} = \langle \kappa_{3h-3} \rangle. (6.6)$$
To find the general relationships between the $\kappa$'s and $\tau$'s correlators it is useful to write 

$$\langle \kappa_{d_1-1} \cdots \kappa_{d_n-1} \rangle = \int_{c_n(M_h)} c_1 \left( \hat{L}^{(1)} \right)^{d_1} \wedge \cdots \wedge c_1 \left( \hat{L}^{(n)} \right)^{d_n}, \quad (6.7)$$

where $\hat{L}^{(i)} = \pi^* \left( K_{c(i)/M} \right)$ and $\pi : c_n(M_h) \to c_n(M_h)$ is the natural projection. Then notice that $c_n(M_h)$ and $\overline{M}_{h,n}$ differ for a divisor at infinity only. This is the unique difference between $\langle \kappa_{d_1-1} \cdots \kappa_{d_n-1} \rangle$ and $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ as defined in (6.4) and (6.7). Thus it is possible to get relations for arbitrary correlators.

### 6.3 $\kappa_1 = \frac{i}{2\pi^2} \partial \bar{\partial} S^{(h)}_{cl}$

We now show how the scalar Laplacian defined with respect to the Poincaré metric enters in the expression of the first tautological class on the moduli space. In order to do this we first introduce the determinant line bundles on the moduli space $M_h$

$$\lambda_n = \det \mathrm{ind} \mathcal{G}_n. \quad (6.8)$$

They are the maximum wedge powers of the space of holomorphic $n$-differentials. The line bundles $\lambda_1 (\equiv \lambda_H)$ and $\lambda_2$ are the Hodge and the canonical line bundles respectively.

In [33] it has been shown that $\kappa_1 = \omega_{\mathcal{W}P}/\pi^2$ thus, by standard results on $\omega_{\mathcal{W}P}$, we have

$$\kappa_1 = \frac{6i}{\pi} \partial \bar{\partial} \log \left( \frac{\det \Omega^{(2)}}{\det \Delta_{\bar{g},0}} \right), \quad (6.9)$$

where $\partial, \bar{\partial}$ denote the holomorphic and antiholomorphic components of the external derivative $d = \partial + \bar{\partial}$ on the moduli space. Therefore the first tautological class can be seen as the curvature form (that is $\kappa_1 = 12 c_1(\lambda_H)$ in $M_h$) of the Hodge line bundle $(\lambda_H; \langle, \rangle_Q)$ endowed with the Quillen norm

$$\langle \omega, \omega \rangle_Q = \frac{\det \Omega^{(2)}}{\det \Delta_{\bar{g},0}}, \quad \omega = \omega_1 \wedge \cdots \wedge \omega_h. \quad (6.10)$$

As we have seen the Liouville action (3.48) evaluated on the classical solution is a potential of $\omega_{\mathcal{W}P}$ projected onto the Schottky space. Thus in this space

$$\kappa_1 = \frac{i}{2\pi^2} \partial \bar{\partial} S^{(h)}_{cl}, \quad (6.11)$$

which provides a direct link between Liouville and topological gravity.
7 Cutoff In 2D Gravity And The Background Metric

Here we apply classical results on univalent (schlicht) functions in order to derive an inequality involving the cutoff of 2D gravity and the background geometry.

7.1 Background Dependence In The Definition Of The Quantum Field

An important aspect arising in quantum gravity is the problem of the choice of the cutoff. In 2D quantum gravity a related problem appears when we consider the norm of the Liouville field $\sigma$ defined by

$$g = e^\sigma \hat{g},$$

(7.1)

with $\hat{g}$ a background metric that we suppose to be in the conformal form $ds^2 = 2\hat{g}_{zz}|dz|^2$. The choice of the background is an important step as it defines the classical solution. To explain this point more thoroughly, we first consider the relationship between the scalar curvatures

$$\sqrt{\hat{g}} \Delta_{\hat{g},0} = \sqrt{g} R_g - \sqrt{\hat{g}} R_{\hat{g}},$$

(7.2)

where

$$\Delta_{\hat{g},0} = -\hat{g}^{zz} \partial_z \partial_{\bar{z}},$$

(7.3)

is the scalar Laplacian for the conformal metric. When $R_g = cst < 0$, $\sigma = \sigma_{cl}$ of eq.(7.2) is the solution of the classical equation of motion defined by the Liouville action in the background metric $\hat{g}$. Thus both the solution of the equation of motion and the splitting

$$\sigma = \sigma_{cl}[\hat{g}] + \sigma_{qu}[\hat{g}],$$

(7.4)

are background dependent. The background dependence appears in the path-integral formulation of Liouville gravity where the measure $Dg\sigma$, defined by the scalar product

$$||\delta \sigma||^2_g = \int_{\Sigma} \sqrt{\hat{g}} e^\sigma |\delta \sigma|^2,$$

(7.5)

is not translationally invariant.

Let us now choose the Poincaré metric as background

$$\hat{g}_{zz} = \frac{e^\varphi}{2},$$

(7.6)
where \( \varphi \) is given in (2.44). Before investigating its role in defining the quantum cutoff let us notice that since

\[
\sigma_{cl}[\hat{g} = e^\varphi] = 0,
\]

the \( \sigma \) field in (7.4) reduces to a full quantum field and the Liouville action for \( \sigma \), written with respect to the background metric \( \hat{g} \) and evaluated on the classical solution, reduces to the area of \( \hat{g} \) which is just the topological number \(-2\pi \chi(\Sigma)\).

### 7.2 The Cutoff In \( z \)-Space

In [3] it was conjectured that the Jacobian that arises in using the translation invariant measure

\[
||\delta \sigma||^2 = \int_\Sigma \sqrt{\hat{g}}|\delta \sigma|^2,
\]

is given by the exponential of the Liouville action with modified coefficients. Arguments in support of this conjecture may be found in [34]. Some aspects of this conjecture are related to the choice of the regulator. We now show how the choice of the Poincaré metric \( ds^2 = e^\varphi|dz|^2 \) as background makes it possible to find an inequality involving the quantum cutoff and classical geometry. Including the Liouville field \( \sigma \) we have

\[
ds^2 = e^\phi|dz|^2, \quad \phi = \sigma + \varphi, \quad \sigma = \sigma_{cl}[e^\varphi] + \sigma_{qu}[e^\varphi] = \sigma_{qu}[e^\varphi].
\]

It is well-known that the cutoff in \( z \) space \((\Delta z)^2_{\text{min}}\) is \( z \)-dependent

\[
(\Delta s)^2_{\text{min}} = e^\phi(\Delta z)^2_{\text{min}} = \epsilon,
\]

that is

\[
(\Delta z)^2 \geq \epsilon e^{-\sigma - \varphi}.
\]

We stress that the cutoff arises already at the classical level. As an example we consider a Riemann sphere with \( n \geq 3 \) punctures (we choose the standard normalization \( z_{n-2} = 0, z_{n-1} = 1 \) and \( z_n = \infty \))

\[
\Sigma = \mathbb{C}\setminus\{z_1, \ldots, z_{n-3}, 0, 1\}.
\]

Near a puncture the Poincaré metric has the following behaviour

\[
\varphi(z) \sim -2\log|z - z'| - 2\log|\log|z - z'||.
\]

Note that \( e^\varphi \) is well-defined on the punctured surface: to deleting the point provides a sort of “topological” cutoff for \( \varphi \) which is related to the univalence of the inverse map of uniformization.
The topological cutoff is related to the covariance of the Poincaré metric. To understand this it is instructive to write down the Liouville action on the Riemann spheres with \( n \)-punctures [12]

\[
S^{(0,n)}(0,r) = \lim_{r \to 0} S^{(0,n)}(0) = \lim_{r \to 0} \left[ \int_{\Sigma_r} \left( \partial_z \phi \partial_\bar{z} \phi + e^\phi \right) + 2\pi(n\log r + 2(n-2)\log|\log r|) \right],
\]

(7.14)

where the field \( \phi \) is in the class of smooth functions on \( \Sigma \) with the boundary condition given by the asymptotic behaviour (7.13). Eq.(7.14) shows that already at the classical level the Liouville action needs a regularization whose effect is to cancel the contributions coming from the non covariance of \( |\phi_z|^2 \notin T^{1,1} \) and provides a modular anomaly for the Liouville action which is strictly related to the geometry of the moduli space [13]. This classical geometric context is the natural framework to understand the relationships between covariance, regularization and modular anomaly. In particular the relation between regularization and conformal weight in this framework is similar to the analogous relation which arises in CFT where the scaling behaviour is fixed by normal ordering and regularization. The fact that classical Liouville theory encodes a quantum feature such as regularization may be connected to the fact that for the canonical transformation relating a particle moving in a Liouville potential to a free particle, the effective quantum generating function is identical to its classical counterpart [35] (no normal ordering problems). Furthermore, as we have seen, the link between classical Liouville theory and normal ordering appears also in the analysis of cocycles.

### 7.3 Univalent Functions And \((\Delta z)^2_{min}\)

The correlators of the one dimensional string have the structure (see [36] for notation)

\[
G(p_1, \ldots, p_N) = F(p_1, \ldots, p_N) + \frac{A(p_1, \ldots, p_N)}{\sum \epsilon(p_i) + 2b},
\]

(7.15)

where the reason for the denominator, instead of the usual delta-function, is that the Liouville mode represents a positively defined metric. It seems that the boundaries of the space of the “half-infinite” configuration space are related to the inequalities that naturally appear in the theory of univalent functions and in particular to their role in the uniformization theory which, as it has been shown in [12], is strictly related to classical Liouville theory.
Let us consider a simply connected domain \( D \) of \( \hat{\mathbb{C}} \) with more than one boundary point. The Poincaré metric on \( D \) reads

\[
e^{\varphi_D(z, \bar{z})} = 4 \frac{|f'_D(z)|^2}{(1 - |f_D(z)|^2)^2}, \tag{7.16}
\]

where \( f_D : D \to \Delta \) is a conformal mapping. We are interested in the bounds of \( e^{\varphi_D} \). By an application of the Schwarz lemma it can be proved that

\[
e^{\varphi_D(z, \bar{z})} (\Delta_D z)^2 \leq 4, \tag{7.17}
\]

where \( \Delta_D z \) denotes the Euclidean distance between \( z \) and the boundary \( \partial D \). The lower bound is

\[
e^{\varphi_D(z, \bar{z})} (\Delta_D z)^2 \geq 1, \tag{7.18}
\]

where now it is assumed that \( \infty \notin D \). By eq.(2.72) we can express the metric in terms of the Schrödinger wave functions satisfying (2.64) so that (7.18) reads

\[
\Delta_D z \geq \frac{1}{2i} \left[ \int \psi^{-2} - \int \bar{\psi}^{-2} \right]. \tag{7.19}
\]

Eq.(7.18) follows from the Koebe one-quarter theorem [37] stating that the boundary of the map of \( |z| < 1 \) by any univalent and holomorphic function \( f \) is always at an Euclidean distance not less than \( 1/4 \) from \( f = 0 \). Thus if \( D \) is the unit disc and \( f(0) = 0 \) we have \( |f(z)| \geq 1/4 \). We stress that (7.18) resembles a sort of geometric uncertainty relation. In particular (7.17) and (7.18) can be considered as infrared and ultraviolet cutoff respectively.

Let us now consider the cutoff on \( D \). By (7.11) and (7.17) it follows that

\[
(\Delta z)^2 \geq \frac{\epsilon}{4} e^{-\sigma_D} (\Delta_D z)^2, \tag{7.20}
\]

which relates the quantum cutoff to the background geometry.

A related result concerns the Nehari theorem [38]. It states that a sufficient condition for the univalence of a function \( g \) is

\[
e^{-\psi} \{|g, z| \} \leq 2, \quad |z| < 1, \tag{7.21}
\]

whereas the necessary condition is

\[
e^{-\psi} \{|g, z| \} \leq 6, \quad |z| < 1, \tag{7.22}
\]
where $e^\psi = (1 - |z|^2)^{-2}$. It can be shown that the constant 2 in (7.21) cannot be replaced by any larger one. Eqs.(7.21,7.22) are inequalities between the Poincaré metric and the modulus of the Schwarzian derivative of a univalent function which is related to the stress tensor.

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