Second-Order Optimality Conditions for Vector Problems with Continuously Fréchet Differentiable Data and Second-Order Constraint Qualifications

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Abstract In the present paper, we consider the inequality constrained vector problem with continuously Fréchet differentiable objective functions and constraints. We obtain second-order necessary optimality conditions of Karush–Kuhn–Tucker type for weak efficiency. A new second-order constraint qualification of Zangwill type is introduced. It is applied in the optimality conditions. Some connections with other constraint qualifications are established.

Keywords Vector optimization · Second-order KKT optimality conditions · Second-order constraint qualifications · Nonsmooth optimization · $C^1$ problems with inequality constraints

Mathematics Subject Classification 90C46 · 90C29

1 Introduction

What is the life of the man without his desire to make the things in the best way? One of the tools for doing this is Mathematics and Optimization, in particular. There is a famous quote by the brilliant Swiss mathematician Leonhard Euler: “Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear”. Karush–Kuhn–Tucker optimality conditions are one of the greatest results in Optimization.

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Constraint qualifications (CQ) play an important role in the necessary conditions. In this paper, we investigate second-order conditions and second-order constraint qualifications (SOCQ). They are connected to second-order local approximations of the feasible set. Historically, the first SOCQ is due to McCormick [1]. SOCQ of different types appeared in the papers by Ben-Tal [2], Ben-Tal and Zowe [3], Kawasaki [4] (Guignard type), Aghezzaf and Hachimi [5,6] (Abadie and Guignard ones). Generalizations of Ban-Tal’s CQ were applied in Karush–Kuhn–Tucker (KKT) conditions by Penot [7], Jimenez and Novo [8]. Optimality conditions were also obtained by Maciel et al. [9] using two types SOCQ. All mentioned authors investigated twice Fréchet differentiable problems or twice continuously Fréchet differentiable ones. SOCQ were applied in KKT conditions for problems, whose data have locally Lipschitz gradient, in several works: Yang [10] (extension of McCormick’s constraint qualification), Maeda [11] (CQ of Abadie type), Ginchev et al. [12] (CQ of Kuhn–Tucker type). Second-order KKT conditions with the help of a SOCQ of Mangasarian–Fromovitz type for problems with continuously Fréchet differentiable data were derived by Ivanov [13]. The authors of some works obtained KKT conditions under the assumption that a first-order CQ holds (see, for example, the paper by Andreani et al. [14], where the constant rank condition appeared).

In the papers by Ginchev and Ivanov [15] and by Ivanov [16], the authors obtained various second-order optimality conditions for the scalar nonlinear programming problem with inequality constraints. All results are derived for functions, which do not satisfy the standard assumptions of second-order Fréchet differentiability. In the most results, the objective function and the constraints are continuously Fréchet differentiable, and the standard second-order directional derivative is applied. Some more optimality conditions for the vector problem with continuously Fréchet differentiable data were obtained also by Ivanov [13]. In the present work, we continue the investigations given there. We introduce a new SOCQ, which is analogous to the Zangwill CQ [17,18]. It is more general than the constraint qualification, introduced in [16]. We obtain second-order KKT necessary optimality conditions for weak local minimum of the vector problem in terms of this CQ. Our results fit to problems with continuously Fréchet differentiable data. In our knowledge, it is an open question to apply second-order constraint qualifications for such problems. Corollary 4.1 generalizes the first-order KKT necessary conditions for scalar problems in terms of Guignard CQ [19]. In the cited works, the authors did not obtain such results, because they do not consider problems with arbitrary Fréchet differentiable data like a lot of the first-order known results. They consider problems with twice Fréchet differentiable or at least data, whose gradient is locally Lipschitz. The second-order linearizing cone that we define is different from the second-order linearizing cone from the paper by Kawasaki [4]. It is also different from the linearizing cone, which is defined by Aghezzaf and Hachimi [5], but in principal both cones are similar. Our cone is a projection of this one.

2 Preliminaries

In this section, we recall the definitions of some preliminary notions and notations. Denote by $\mathbb{R}$ the set of reals and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be the extended real line. Denote by $\text{cl} \ S$ the closed hull of the set $S$ and by $\text{conv} \ S$ the convex hull of $S$. 
Consider the vector problem

\[
\text{Minimize } f(x) \ \text{subject to } g(x) \leq 0, \quad (VP)
\]

where \( f : X \to \mathbb{R}^n \) and \( g : X \to \mathbb{R}^m \) are given vector functions, defined on some open set \( X \), and \( X \subseteq \mathbb{R}^s \). Denote by \( S \) the set of feasible points, that is,

\[
S := \{ x \in X : g_i(x) \leq 0, \quad i = 1, 2, \ldots, m \}.
\]

For every feasible point \( x \in S \), let \( I(x) \) be the set of active constraints

\[
I(x) := \{ i \in \{ 1, 2, \ldots, m \} : g_i(x) = 0 \}.
\]

Throughout this paper, we use the following notations comparing the vectors \( x \) and \( y \) with components \( x_i \) and \( y_i \) in finite-dimensional spaces:

- \( x < y \), if \( x_i < y_i \) for all indices \( i \);
- \( x \leq y \), if \( x_i \leq y_i \) for all indices \( i \);
- \( x \leq y \), if \( x_i \leq y_i \) for all indices \( i \) with at least one being strict.

**Definition 2.1** A feasible point \( \bar{x} \in S \) is called a weak local Pareto minimizer, or weakly efficient solution, iff there exists a neighbourhood \( U \ni \bar{x} \) such that there is no \( x \in U \cap S \) with \( f(x) < f(\bar{x}) \).

**Definition 2.2** A direction \( d \) is called critical at the point \( x \in S \), iff

\[
\nabla f_j(x)d \leq 0 \quad \text{for all } j \in \{ 1, 2, \ldots, n \} \quad \text{and} \quad \nabla g_i(x)d \leq 0 \quad \text{for all } i \in I(x).
\]

For a feasible point \( \bar{x} \) and a direction \( d \), denote by \( J(\bar{x}, d) \) and \( K(\bar{x}, d) \) the following sets:

\[
J(\bar{x}, d) := \{ j \in \{ 1, 2, \ldots, n \} : \nabla f_j(\bar{x})d = 0 \},
K(\bar{x}, d) := \{ i \in I(\bar{x}) : \nabla g_i(\bar{x})d = 0 \}.
\]

These notations are sensible only if \( \bar{x} \) is a local minimizer and \( d \) is a critical direction.

**Definition 2.3** Let the function \( h : X \to \mathbb{R} \) with an open domain \( X \subseteq \mathbb{R}^s \) be Fréchet differentiable at the point \( x \in X \). Then, the second-order directional derivative \( h''(x, u) \) of \( h \) at the point \( x \in X \) in direction \( u \in \mathbb{R}^n \) is defined as an element of \( \mathbb{R} \) by the equality

\[
h''(x, u) := \lim_{t \to +0} 2t^{-2} [h(x + tu) - h(x) - t \nabla h(x)u].
\]

The function \( h \) is called second-order directionally differentiable on \( X \), iff the derivative \( h''(x, u) \) exists for each \( x \in X \) and any direction \( u \in \mathbb{R}^n \), and it is finite.
**Definition 2.4** ([18]) Let the function \( h : X \to \mathbb{R} \) with an open domain \( X \subseteq \mathbb{R}^s \) be Fréchet differentiable at the point \( x \in X \). Then, \( h \) is said to be pseudoconvex at \( x \in X \), iff

\[
y \in X, \ h(y) < h(x) \quad \text{imply} \quad \nabla h(x)(y - x) < 0.
\]

If it is Fréchet differentiable on \( X \), then \( h \) is called pseudoconvex on \( X \), when \( h \) is pseudoconvex at each \( x \in X \). If the function \(-h\) is pseudoconvex, then \( h \) is called pseudoconcave.

The following definition is due to Ginchev and Ivanov [20]:

**Definition 2.5** Consider a function \( h : X \to \mathbb{R} \) with an open domain \( X \), which is Fréchet differentiable at \( x \in X \) and second-order directionally differentiable at \( x \in X \) in every direction \( y - x \) such that \( y \in X, h(y) - h(x) \leq 0 \). The function \( h \) is called second-order pseudoconvex at \( x \in X \), iff for all \( y \in X \), the following implications hold:

\[
\begin{align*}
h(y) < h(x) & \implies \nabla h(x)(y - x) \leq 0; \\
h(y) < h(x), \ \nabla h(x)(y - x) = 0 & \implies h''(x, y - x) < 0.
\end{align*}
\]

Suppose that \( h \) is Fréchet differentiable on \( X \) and second-order directionally differentiable at every \( x \in X \) in each direction \( y - x \) such that \( y \in X, h(y) - h(x) = 0 \). The function \( h \) is called second-order pseudoconvex at \( x \in X \), iff \(-h\) is second-order pseudoconvex, then \( h \) is called second-order pseudoconcave.

Roughly speaking, if the function is pseudoconvex, then \( h(y) < h(x) \) implies that there exists \( \delta > 0 \) with \( h[y + t(y - x)] < h(x) \) for every \( t \in ]0, \delta[ \). If the function is second-order pseudoconvex, then this claim also holds. In particular, the relations \( h(y) < h(x), \nabla h(x)(y - x) = 0 \) imply that there exists \( \delta > 0 \) with \( h[y + t(y - x)] < h(x) \) for every \( t \in ]0, \delta[ \). The graph of a second-order pseudoconvex function is similar to the graph of a pseudoconvex one (see [20, Theorem3]).

It follows from this definition that every Fréchet differentiable pseudoconvex function is second-order pseudoconvex. The converse does not hold.

The following simple example from the paper [20] shows the essence of second-order pseudoconvexity:

**Example 2.1** Consider the function of one variable \( h : \mathbb{R} \to \mathbb{R} \), defined as follows:

\[
h(x) := \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}
\]

It is not pseudoconvex at \( x = 0 \), because \( h'(0) = 0 \), and for \( h(y) < h(x) \) we have \( h'(x)(y - x) = 0 \). This function is pseudoconvex at every other point. On the other hand, the function is second-order pseudoconvex, because

\[
h''(x, y - x) = h''(0, y) = -2y^2 < 0.
\]
We can construct from Example 2.1 more second-order pseudoconvex functions.

**Example 2.2** Consider the function of two variables \( f : \mathbb{R}^2 \to \mathbb{R} \), defined by the equality

\[
f(x_1, x_2) := h(x_1^2 - x_2),
\]

where \( h \) is the function from Example 2.1. We prove that it is second-order pseudoconvex without being pseudoconvex.

Let \( f(y) < f(x) \), where \( x = (x_1, x_2) \), \( y = (y_1, y_2) \). Since \( h \) is strictly monotone increasing,

\[
x_2 - y_2 < x_1^2 - y_1^2. \tag{1}
\]

Therefore

\[
\nabla f(x)(y - x) = h'(x_1^2 - x_2)(2x_1y_1 - 2x_1^2 + x_2 - y_2) \\
\leq -h'(x_1^2 - x_2)(y_1 - x_1)^2 \leq 0.
\]

Consequently, if \( x_2 \neq x_1^2 \), then \( h'(x_1^2 - x_2) > 0 \) and, by (1), \( \nabla f(x)(y - x) < 0 \).

Let \( x_2 = x_1^2 \). Then, \( h'(x_1^2 - x_2) = 0 \) and \( \nabla f(x)(y - x) = 0 \). Therefore, \( f \) is not pseudoconvex. The function of one variable \( \varphi(t) = t^2 \) is convex. Hence, its epigraph is also convex. Then, the claim \( y \) is a point from the epigraph implies that, for all \( t \in [0, 1] \), \( y + t(y - x) \) is also a point from the epigraph. Therefore,

\[
f''(x, y - x) = \lim_{t \to +0} 2 \frac{f[x + t(y - x)]}{t^2} = -2 [2x_1(y_1 - x_1) + (x_2 - y_2)]^2 \leq 0.
\]

We prove that \( f''(x, y - x) < 0 \). Suppose the contrary: \( f''(x, y - x) = 0 \). Then, we have

\[
2x_1(y_1 - x_1) + (x_2 - y_2) = 0.
\]

By (1) we obtain \( 0 < - (x_1 - y_1)^2 \leq 0 \), which is impossible. Hence, the function \( f \) is second-order pseudoconvex.

**3 A New Second-Order Constraint Qualification of Zangwill Type**

Consider the problem (VP) and the following conditions:

\[
\begin{array}{l}
\text{The functions } g_i, \quad i \notin I(\bar{x}) \text{ are continuous at } \bar{x}. \\
\text{The functions } f_j, \quad j = 1, 2, \ldots, n, \quad g_i, \quad i \in I(\bar{x}) \text{ belong to the class } C^1(X). \\
\text{If } \nabla f_j(\bar{x})d = 0, \text{ then there exists } f''_j(\bar{x}, d). \\
\text{If } \nabla g_i(\bar{x})d = 0, \quad i \in I(\bar{x}), \text{ then there exists } g''_i(\bar{x}, d).
\end{array}
\]

(C)
Recall that a function \( h : X \to \mathbb{R} \) belongs to the class \( C^1(X) \), iff its gradient map exists, and it is continuous. Such function is usually called continuously Fréchet differentiable or simply continuously differentiable.

For every feasible point \( x \) and direction \( d \), consider the sets:

\[
A(x, d) := \{ z \in \mathbb{R}^n : \forall i \in K(x, d) \exists \delta_i > 0 \text{ with } g_i(x + td + 0.5t^2z) \leq 0, \forall t \in [0, \delta_i] \},
\]

\[
B(x, d) := \{ z \in \mathbb{R}^n : \nabla g_i(x)z + g_i''(x, d) \leq 0 \text{ for each } i \in K(x, d) \}.
\]

By definition, \( A(x, d) = B(x, d) = \mathbb{R}^n \), if \( K(x, d) = \emptyset \).

**Proposition 3.1** Let \( \bar{x} \) be a feasible point for the Problem (VP) and \( d \) be a direction. Suppose that all functions \( g_i, i \in I(\bar{x}) \) are continuously Fréchet differentiable, and there exist \( g_i''(\bar{x}, d), i \in I(\bar{x}) \), provided that \( \nabla g_i(\bar{x})d = 0 \). Then, \( A(\bar{x}, d) \subseteq B(\bar{x}, d) \).

**Proof** Let \( i \in I(\bar{x}) \) with \( \nabla g_i(\bar{x})d = 0 \) and \( z \in A(\bar{x}, d) \). We prove that \( z \in B(\bar{x}, d) \).

There exists \( \delta_i > 0 \) such that

\[
g_i(\bar{x} + td + 0.5t^2z) - g_i(\bar{x}) \leq 0, \quad \forall t \in [0, \delta_i]. \tag{2}
\]

Consider the function of one variable \( \varphi_i(t) := g_i(\bar{x} + td + 0.5t^2z) \). Since \( X \) is open and \( \bar{x} \) is feasible, there exists a number \( \delta_i > 0 \) such that, \( \varphi_i \) is defined for all numbers \( t \) with \( -\delta_i < t < \delta_i \). The following equality holds:

\[
\varphi_i'(t) = \nabla g_i(\bar{x} + td + 0.5t^2z)(d + tz).
\]

Therefore \( \varphi_i'(0) = \nabla g_i(\bar{x})d \). Consider the differential quotient

\[
2t^{-2}[\varphi_i(t) - \varphi_i(0) - t\varphi_i'(0)] = 2t^{-2}[g_i(\bar{x} + td + 0.5t^2z) - g_i(\bar{x}) - t\nabla g_i(\bar{x})d].
\]

Let us choose an arbitrary sequence \( \{t_k\}_{k=1}^{\infty} \) of positive numbers, which converges to \( 0 \). According to the mean-value theorem, for every positive integer \( k \) there exists \( \theta_k^i \in [0, 1] \) with

\[
g_i(\bar{x} + t_k d + 0.5t_k^2z) = g_i(\bar{x} + t_k d) + \nabla g_i(\bar{x} + t_k d + 0.5t_k^2\theta_k^i z)(0.5t_k^2z). \tag{3}
\]

It follows from \( g_i \in C^1 \) and (3) that

\[
\varphi_i''(0, 1) = \lim_{k \to +\infty}[
\nabla g_i(\bar{x} + t_k d + 0.5t_k^2\theta_k^i z)z
+2t_k^{-2}(g_i(\bar{x} + t_k d) - g_i(\bar{x}) - t_k \nabla g_i(\bar{x})d)] = \nabla g_i(\bar{x})z + g_i''(\bar{x}, d).
\]

Therefore

\[
\nabla g_i(\bar{x})z + g_i''(\bar{x}, d) = \varphi_i''(0, 1). \tag{4}
\]
It follows from (2) and (4) that

$$
\nabla g_i(\bar{x})z + g_i''(\bar{x}, d) = \lim_{t \to +0} 2t^{-2}[g_i(\bar{x} + td + 0.5t^2z) - g_i(\bar{x})] \leq 0,
$$

which proves that $A(\bar{x}, d) \subseteq B(\bar{x}, d)$. \qed

The following example shows that the converse claim of Proposition 3.1 does not hold.

**Example 3.1** Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$, defined by $g(x_1, x_2) := x_1^3$. Choose $\bar{x} = (0, 0), d = (1, 0)$. We have

$$
A(\bar{x}, d) = \{z \in \mathbb{R}^2 : \nabla g(\bar{x})d = 0 \implies \exists \delta > 0 \text{ with } g(\bar{x} + td + 0.5t^2z) \leq 0, \ \forall t \in ]0, \delta[\}
$$

$$
B(\bar{x}, d) = \{z \in \mathbb{R}^2 : \nabla g(\bar{x})d = 0 \implies \nabla g(\bar{x})z + g''(\bar{x}, d) \leq 0\},
$$

$\nabla g(\bar{x}) = (0, 0), g''(\bar{x}, d) = 0$. If $z = (1, 0)$, then $g(\bar{x} + td + 0.5t^2z) > 0$ for all $t > 0$, Therefore $z \in B(\bar{x}, d), \text{ but } z \notin A(\bar{x}, d)$.

**Definition 3.1** Consider a function of one variable $\varphi : ]-a, a[ \to \mathbb{R}$, which is Fréchet differentiable at the point $t = 0$, and there exists its second-order right derivative

$$
\varphi''(0, 1) := \lim_{t \to +0} 2t^{-2}\left[\varphi(t) - \varphi(0) - t\varphi'(0)\right].
$$

Then, we call $\varphi$ second-order locally pseudoconcave at $t = 0$ on the right, iff there exists $\delta > 0$ such that

$$
\varphi(t) > \varphi(0), \ 0 < t < \delta \ \text{ implies } \varphi'(0) \geq 0,
$$

$$
\varphi(t) > \varphi(0), \ 0 < t < \delta, \ \varphi'(0) = 0 \ \text{ implies } \varphi''(0, 1) > 0.
$$

The condition the constraint functions $g_i, i \in I(\bar{x})$ to be pseudoconcave at $\bar{x}$, where $\bar{x}$ is a local minimizer, is called the weak reverse constraint qualification [18, p. 253]. The respective second-order condition is the assumption that $g_i, i \in I(\bar{x})$ are second-order pseudoconcave at $\bar{x}$. This condition is weaker than the respective first-order one, because every pseudoconvex function is second-order pseudoconvex, but the inverse claim is not true. The CQ that the functions of one variable $\varphi_i(t)$, which are defined by the equality

$$
\varphi_i(t) := g_i(\bar{x} + td + 0.5t^2z), \ t \in \mathbb{R}
$$

(5)

are second-order locally pseudoconcave at $t = 0$ on the right, is a weaker second-order CQ.

**Proposition 3.2** Let the constraint functions satisfy Conditions (C). Suppose that $\bar{x}$ is a feasible point for (VP) and $d$ is an arbitrary direction. Let for every $z \in \mathbb{R}^n$ the functions of one variable $\varphi_i, i \in K(\bar{x}, d)$, defined by (5), be second-order locally pseudoconcave at the point $t = 0$ on the right. Then,

$$
A(\bar{x}, d) = B(\bar{x}, d).
$$
Proof According to Proposition 3.1, it is enough to prove that $B(\bar{x}, d) \subseteq A(\bar{x}, d)$. If $K(\bar{x}, d) = \emptyset$, then the claim is obvious. We prove the proposition by contradiction. Suppose that there is $z \in B(\bar{x}, d)$ with $z \notin A(\bar{x}, d)$. It follows from $z \notin A(\bar{x}, d)$ that there exists $j \in K(\bar{x}, d)$ and a sequence $\{t_k\}_{k=1}^\infty$, $t_k \to +0$, which consists of positive numbers, with the property $\varphi_j(t_k) > \varphi_j(0)$ for each positive integer $k$. By second-order local pseudoconcavity, we obtain that $\varphi'_j(0, 1) > 0$, which implies that $z \notin B(\bar{x}, d)$, a contradiction.

If $\bar{x}$ is a feasible point, then the set $B(\bar{x}, d)$ is closed, but $A(\bar{x}, d)$ is not.

Example 3.2 Let $S := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2 \leq 0\}$, $\bar{x} = (0, 0)$, $d = (0, 0)$. Then $(1, z_2) \in A(\bar{x}, d)$ with $z_2$ arbitrary positive number, but $(1, 0) \notin A(\bar{x}, d)$. Therefore, $A(\bar{x}, d)$ is not closed.

Definition 3.2 We introduce the condition $cl A(\bar{x}, d) = B(\bar{x}, d)$ under the assumption that the inactive constraints are continuous.

In next section, we show that the condition $cl A(\bar{x}, d) = B(\bar{x}, d)$ is a SOCQ. We could name it the second-order Zangwill CQ, because it is a second-order analog of the Zangwill CQ $cl Z(\bar{x}) = L(\bar{x})$ (see [17]). Here

$$L(\bar{x}) := \{d \in \mathbb{R}^n : \nabla g_i(\bar{x})d \leq 0, \ i \in I(\bar{x})\}$$

is the linearizing cone of the problem (VP) at the feasible point $\bar{x}$, and

$$Z(\bar{x}) := \{d \in \mathbb{R}^n : \exists \delta > 0 \text{ such that } \bar{x} + td \in S, \ \forall t \in [0, \delta]\}$$

is the cone of the feasible directions to $S$ at $\bar{x}$ under the assumption that the inactive constraints are continuous at the feasible point $\bar{x}$. In the case when $d = 0$, the set $A(\bar{x}, d)$ reduces to the cone of feasible directions $Z(\bar{x})$ and $B(\bar{x}, d)$ reduces to the linearizing cone $L(\bar{x})$. We will see in next section why this case is special.

4 Second-Order Karush–Kuhn–Tucker Necessary Conditions for Weak Local Minimum

Theorem 4.1 (Primal conditions) Let $\bar{x}$ be a weak local minimizer of the problem (VP) and $d$ be a critical direction. Suppose that Conditions (C) are satisfied. Assume that the constraint qualification $cl A(\bar{x}, d) = B(\bar{x}, d)$ holds. Then, there does not exist a vector $z$ such that

$$\nabla f_j(\bar{x})z + f''_j(\bar{x}, d) < 0, \ j \in J(\bar{x}, d), \quad (6)$$

$$\nabla g_i(\bar{x})z + g''_i(\bar{x}, d) \leq 0, \ i \in K(\bar{x}, d). \quad (7)$$

Proof Conditions (6) and (7) can be considered as a system of inequalities. This system contains at least one inequality, because $\bar{x}$ is a weak local solution. Assume the contrary that there exists a vector $z$, which satisfies (6) (7). Therefore $z \in B(\bar{x}, d)$. Consider the following cases:
1) Let \( i \in K(\bar{x},d) \). It follows from the condition \( \text{cl } A(\bar{x},d) = B(\bar{x},d) \) that there exists a sequence \( \{z_l\}_{l=1}^\infty \), converging to \( z \), such that \( z_l \in A(\bar{x},d) \). Take an arbitrary positive integer \( l \). Suppose that it is fixed. Therefore, there exists a number \( \delta_l > 0 \) with \( g_i(\bar{x} + td + 0.5t^2z_l) \leq 0 \) for every \( t \in ]0, \delta_l[ \).

2) Suppose that \( i \in I(\bar{x}) \setminus K(\bar{x},d) \). We have \( \nabla g_i(\bar{x})d < 0 \). Therefore, \( \varphi'_i(0) < 0 \), where \( \varphi_i(t) := g_i(\bar{x} + td + 0.5t^2z_l) \). It follows from here that there exists \( \delta_i > 0 \) with \( \varphi_i(t) < \varphi_i(0) \), that is, \( g_i(\bar{x} + td + 0.5t^2z_l) < g_i(\bar{x}) = 0 \) for all \( t \in ]0, \delta_i[ \).

3) For every \( i \in \{1, 2, \ldots, m\} \setminus I(\bar{x}) \) is satisfied the inequality \( g_i(\bar{x}) < 0 \). According to the assumption that \( g_i \) is continuous, there exists \( \delta_i > 0 \) such that

\[
\text{for } \delta_i \in ]0, \delta_i[.
\]

Thus, we obtain from all these cases that the point \( \bar{x} + td + 0.5t^2z_l \) is feasible for all sufficiently small positive numbers \( t \).

We consider two cases concerning the objective function:

1) Let \( \nabla f_j(\bar{x})d < 0 \). Define the function of one variable

\[
\psi_j(t) := f_j(\bar{x} + td + 0.5t^2z_l).
\]

Then \( \psi'_j(0) < 0 \) and hence, there exists \( \varepsilon_j > 0 \) with \( f_j(\bar{x} + td + 0.5t^2z_l) < f_j(\bar{x}) \) for arbitrary \( t \in ]0, \varepsilon_j[ \).

2) Let \( j \in J(\bar{x},d) \), that is, \( \nabla f_j(\bar{x})d = 0 \). Since \( X \) is open and \( \bar{x} \) is feasible, there exists a number \( \varepsilon_j > 0 \) such that \( \psi_j \) is defined for all numbers \( t \) with \( -\varepsilon_j < t < \varepsilon_j \). The following equality holds:

\[
\psi'_j(t) = \nabla f_j(\bar{x} + td + 0.5t^2z_l)(d + tz_l).
\]

Therefore \( \psi'_j(0) = \nabla f_j(\bar{x})d \). Consider the differential quotient

\[
2t^{-2}[\psi_j(t) - \psi_j(0) - t\psi'_j(0)] = 2t^{-2}[f_j(\bar{x} + td + 0.5t^2z_l) - f_j(\bar{x}) - t\nabla f_j(\bar{x})d].
\]

Let us choose an arbitrary sequence \( \{t_k\}_{k=1}^\infty \) of positive numbers, which converges to 0. According to the mean–value theorem, for every positive integer \( k \), there exists \( \theta_j^k \in ]0, 1[ \) with

\[
f_j(\bar{x} + t_kd + 0.5t_k^2z_l) = f_j(\bar{x} + t_kd) + \nabla f_j(\bar{x} + t_kd + 0.5t_k^2\theta_j^kz_l)(0.5t_k^2z_l). \tag{8}
\]

It follows from \( f \in C^1 \) and (8) that

\[
\psi''_j(0, 1) = \lim_{k \to +\infty} [\nabla f_j(\bar{x} + t_kd + 0.5t_k^2\theta_j^kz_l)z_l + 2t_k^{-2}(f_j(\bar{x} + t_kd) - f_j(\bar{x}) - t_k\nabla f_j(\bar{x})d)] = \nabla f_j(\bar{x})z_l + f_j''(\bar{x}, d).
\]

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Lemma 4.1 Let the point \( \bar{x} \) be a weak local solution of the problem (VP) and \( d \) be a nonzero critical direction. Suppose that the functions \( f, g_i, i \in I(\bar{x}) \) are Fréchet differentiable, the functions \( g_i, i \notin I(\bar{x}) \) are continuous, and in the case when \( \nabla f_j(\bar{x})d = 0, i \in I(\bar{x}) \), there exist the second-order directional derivatives \( f''_j(\bar{x}, d) \) or \( g''_i(\bar{x}, d) \), respectively. Then, a necessary and sufficient condition for the existence of Lagrange multipliers

\[
\lambda = (\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad \mu = (\mu_1, \ldots, \mu_m), \quad \lambda \geq 0, \quad \mu \geq 0,
\]

which satisfy KKT conditions

\[
\mu_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m, \quad \nabla L(\bar{x}) = 0
\]

\[
L''(\bar{x}, d) = \sum_{j=1}^{n} \lambda_j f''_j(\bar{x}, d) + \sum_{i \in I(\bar{x})} \mu_i g''_i(\bar{x}, d) \geq 0, \quad (11)
\]

where \( L \) is the Lagrange function \( L := \sum_{j=1}^{n} \lambda_j f_j + \sum_{i=1}^{m} \mu_i g_i \), is the condition that both systems (9) and (10) are not solvable.

Therefore \( \nabla f_j(\bar{x}) z_l + f''_j(\bar{x}, d) = \psi''_j(0, 1) \). It follows from (6) that

\[
\nabla f_j(\bar{x}) z_l + f''_j(\bar{x}, d) < 0
\]

for all sufficiently large numbers \( l \). Therefore \( \psi''_j(0, 1) < 0 \). By \( j \in J(\bar{x}, d) \), we have

\[
\lim_{t \to +0} 2[\psi_j(t) - \psi_j(0)]/t^2 < 0,
\]

which implies that there exists \( \epsilon_j > 0 \) such that \( f_j(\bar{x} + td + 0.5t^2 z_l) < f_j(\bar{x}) \) for arbitrary \( t \in [0, \epsilon_j] \).

Taking into account both cases, we get a contradiction to the hypothesis that \( \bar{x} \) is a weak local minimizer, since the inequality \( f_j(\bar{x} + td + 0.5t^2 z_l) < f_j(\bar{x}) \) is satisfied for all \( t \in [0, \epsilon] \), where \( \epsilon \) is the minimal among the positive numbers \( \epsilon_j \) and \( \delta_j \). \( \Box \)

Let us consider the system with unknowns \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R} \), where \( d \) is an arbitrary critical direction:

\[
\begin{align*}
\nabla f_j(\bar{x}) u + v f''_j(\bar{x}, d) &< 0, & j = 1, 2, \ldots, n \\
\nabla g_i(\bar{x}) u + v g''_i(\bar{x}, d) &\leq 0, & i \in I(\bar{x}), \\
v &> 0,
\end{align*}
\]

and the system with an unknown \( u \in \mathbb{R}^n \):

\[
\nabla f_j(\bar{x}) u < 0, \quad j = 1, 2, \ldots, n, \quad \nabla g_i(\bar{x}) u \leq 0, \quad i \in I(\bar{x}), \quad (10)
\]

where \( \bar{x} \) and \( d \) are a given point and a direction, respectively.
Proof Let $d$ be an arbitrary critical direction. Consider the linear programming problem

$$\begin{align*}
\text{Maximize } & 0 \\
\text{subject to } & \nabla f_j(\bar{x})u + v f''_j(\bar{x}, d) \leq -1, \ j = 1, 2, \ldots, n \\
& \nabla g_i(\bar{x})u + v g''_i(\bar{x}, d) \leq 0, \ i \in I(\bar{x}) \\
& v \geq 0.
\end{align*}$$

Its dual is the following one:

$$\begin{align*}
\text{Minimize } & -\sum_{j=1}^n \lambda_j \\
\text{subject to } & \sum_{j=1}^n \lambda_j \nabla f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \nabla g_i(\bar{x}) = 0, \\
& \sum_{j=1}^n \lambda_j f''_j(\bar{x}, d) + \sum_{i \in I(\bar{x})} \mu_i g''_i(\bar{x}, d) \geq 0, \\
& \lambda \geq 0, \ \mu_i \geq 0, \ \forall i \in I(\bar{x}).
\end{align*}$$

Suppose that the systems (9) and (10) have no solutions. Therefore, the primal problem is not solvable, because it is infeasible. According to duality theorem in linear programming, the dual problem also is not solvable. Since $\lambda = 0$, $\mu_i = 0$, $i \in I(x)$ is a feasible point, the dual problem is unbounded from below. Therefore, there exist Lagrange multipliers, which satisfy the second-order Karush–Kuhn–Tucker conditions.

Conversely, let there exist Lagrange multipliers, which satisfy KKT conditions. Therefore, the dual problem has no solutions, because its objective function is unbounded from below over the feasible set. It follows from duality theorem that the primal problem is unsolvable, because it is infeasible. Therefore, there are no vector $u \in \mathbb{R}^n$ and a number $v > 0$, which form a solution of the system (9), there is no vector $u \in \mathbb{R}^n$, which forms a feasible point for the primal problem together with the number $v = 0$. We obtain from here that the system (10) is inconsistent.

Let $S$ be a given set. The Bouligand tangent cone (or the contingent cone) [18] of the set $S$ at the point $x \in \text{cl } S$ is defined as follows:

$$\begin{align*}
T(S, x) := \{ u \in \mathbb{R}^n : \exists \{t_k\}, t_k > 0, t_k \to +0, \exists \{u_k\} \subset \mathbb{R}^n, \\
& u_k \to u \text{ such that } x + t_k u_k \in S \text{ for all positive integers } k \}.
\end{align*}$$

If $S$ is the feasible set of the problem (VP), then the condition $L(\bar{x}) = T(S, \bar{x})$ is called the Abadie CQ [21].

**Theorem 4.2 (Dual conditions)** Let $\bar{x}$ be a weak local minimizer for the problem (VP), and let $d$ be a nonzero critical direction. Suppose that Conditions (C) are satisfied. Suppose that the constraint qualification $\text{cl } A(\bar{x}, d) = B(\bar{x}, d)$ and the Abadie CQ hold. Then there exist non-negative Lagrange multipliers $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda \neq 0$ and $\mu = (\mu_1, \ldots, \mu_m)$, which satisfy the second-order KKT conditions (11).

Proof Let $d$ be an arbitrary critical direction. We prove that the system (9) has no solutions. Let us suppose the contrary: The system (9) is solvable, and let $(u, v)$ be
an arbitrary solution. It follows from here that there exists a point $z$, which satisfies conditions (6) and (7). This is a contradiction to Theorem 4.1.

We prove that the system (10) has no solutions. Assume the contrary and let $u \in \mathbb{R}^n$ be a solution. Therefore, by the definition of the linearizing cone, $u \in L(\bar{x})$. It follows from Abadie CQ that $u \in T(S, \bar{x})$. Let $F$ be the cone

$$F := \{ d : \nabla f_j(\bar{x})d < 0, \ j = 1, 2, \ldots, n \}.$$ 

It is known [18, Theorem 6.6.1] that $F \cap T(S, \bar{x}) = \emptyset$. On the other hand, by the Abadie CQ we have $u \in F \cap T(S, \bar{x})$, which is a contradiction.

It follows from our arguments up to here that both systems (9) and (10) are not consistent. Then, according to Lemma 4.1, there exist Lagrange multipliers, which satisfy the second-order KKT conditions.

Consider the scalar problem

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, \ i = 1, 2, \ldots, m,$$

(P)

where the real functions $f, g_i, i = 1, 2, \ldots, m$ are defined on some open set $X$ and $X \subseteq \mathbb{R}^s$.

The closed and convex hull of the Bouligand tangent cone is called the pseudotangent cone [19], that is,

$$PT(S, x) := \text{cl} (\text{conv} T(S, x)).$$

If $S$ is the feasible set of the problem (P), then the condition $L(\bar{x}) = PT(S, \bar{x})$ is called the Guignard CQ [19].

**Corollary 4.1 (Dual conditions for the scalar problem (P))**

Let $\tilde{x}$ be a local minimizer for the scalar problem (P) and $d$ be a nonzero critical direction. Suppose that Conditions (C) are satisfied, the constraint qualification $\text{cl} \ A(\tilde{x}, d) = B(\tilde{x}, d)$ and the Guignard CQ hold. Then, there exist non-negative Lagrange multipliers $\mu_1, \ldots, \mu_m$, which satisfy the second-order KKT conditions (11) with $n = 1$.

**Proof** We should prove only the part that the system (10) has no solutions. Assume the contrary, and let $u \in \mathbb{R}^n$ be a solution. Therefore, by the definition of the linearizing cone, $u \in L(\tilde{x})$. It follows from Guignard CQ that $u \in PT(S, \tilde{x})$. On the other hand, it is well-known (see Lemma 5.1.2 from the book [22]) that

$$\nabla f(\tilde{x})d \geq 0, \ \forall d \in T(S, \tilde{x}),$$

where $T(S, \tilde{x})$ is the Bouligand tangent cone to the feasible set $S$ at $\tilde{x}$. It follows from here that $\nabla f(\tilde{x})d \geq 0$ for all $d \in PT(S, \tilde{x})$. In particular, $\nabla f(\tilde{x})u \geq 0$, which contradicts the assumption that $u$ is a solution of the system (10). \qed
Remark 4.1 In the case when $d = 0$, Corollary 4.1 reduces to the first-order KKT optimality conditions with Guignard CQ. Indeed, this direction is critical, the second-order derivatives exist, and they are equal to zero. The set $A(\bar{x}, d)$ coincides with the cone of the feasible directions, the set $B(\bar{x}, d)$ coincides with the linearizing cone, the second-order Zangwill CQ reduce to Zangwill CQ, the conditions (11) reduce to KKT necessary optimality conditions. Therefore, these necessary conditions are a particular case of Corollary 4.1.

5 Conclusions

We obtain primal and dual second-order KKT necessary optimality conditions for scalar and vector problems. A new second-order constraint qualification is introduced, and it is applied in our results. The SOCQ is of Zangwill type. We consider problems with $C^1$ data. Some sufficient and other necessary conditions for such problems were derived in the papers by Ginchev and Ivanov [15], by Ivanov [13,16].

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