A PRIORI ESTIMATES FOR ELLIPTIC PROBLEMS VIA LIOUVILLE TYPE THEOREMS

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Dedicated to Professor Patrizia Pucci on the occasion of her 65th birthday, with deep gratitude, esteem and affection.

Abstract. In this paper we prove a priori estimates for positive solutions of elliptic equations of the $p$-Laplacian type on arbitrary domains of $\mathbb{R}^N$, when a nonlinearity depending on the gradient is considered. Also the case of systems with very general nonlinearities is considered. Our main theorems extend previous results by Polacik, Quitter and Souplet in [26] in which either the case $p = 2$ with a nonlinearity depending on the gradient or the $p$-Laplacian case with a nonlinearity not depending on the gradient is treated. The technique is based on the use of a method developed in [26] whose main tools are rescaling arguments combined with a key “doubling” property, which is different from the celebrated blow up technique due to Gidas and Spruck in [16]. A discussion on the sharpness of the main result in the scalar case is presented.

1. Introduction. Motivated by [26], in this paper we prove pointwise a priori estimates for local positive solutions of elliptic nonlinear problems involving the $p$-Laplacian operator and such that the nonlinearity $f$ depends also on the gradient, precisely we deal with the following problem

$$-\Delta_p u = f(x,u,Du) \quad \text{in } \Omega,$$

where $\Omega \subseteq \mathbb{R}^N$ is an arbitrary domain, $1 < p < N$ and $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function satisfying some natural growth conditions. In particular, we are mainly interested in the following prototype for problem (1)

$$-\Delta_p u = a(x) u^q - b(x) u^s |Du|^\theta \quad \text{in } \Omega,$$

where $q, \theta > 0$, $s \geq 0$ and throughout the paper $a(x)$, $b(x)$ will be continuous functions with $a(x)$ positive.

2010 Mathematics Subject Classification. Primary: 35J92, 35J70; Secondary: 35J47.

Key words and phrases. A priori estimates, elliptic problems, $p$-$q$-Laplacian systems.

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Equation (2), in the Laplacian case, namely when $p = 2$, and in the subcase when $a(x) = 1, b(x) = b$ and $s = 0$, is the stationary case of the following parabolic model in an arbitrary domain of $\mathbb{R}^N$, possibly $\Omega = \mathbb{R}^N$,

$$
\begin{cases}
  u_t - \Delta u = |u|^{q-1}u - b|Du|^\theta, & t > 0, \ x \in \Omega, \\
  u(t, x) = 0, & t > 0, \ x \in \partial \Omega, \\
  u(0, x) = \phi(x) \geq 0, & x \in \Omega,
\end{cases}
$$

with $q > 1, \theta \geq 1, b > 0$, which was introduced in [4] by Chipot, Weissler in order to investigate the possible effect of a damping gradient term on global existence or nonexistence. Indeed, the dynamics of this equation reflects the competition between the reaction term $u^q$, which may cause blow-up, and the gradient term $|Du|^\theta$ which will fight against blow-up.

Moreover, the equation in (3) is a model in population dynamics proposed in [30] by Souplet to describe the evolution of the population density of a biological species, under the effect of certain natural mechanism. In particular, the dissipative gradient term $-b|Du|^\theta$, $b > 0$, represents the action of a predator which destroys the individuals during their displacements (it is assumed that the preys are not vulnerable at rest). For further details, we refer to the survey [31]. Later, Bartier in [2] studied the generalization of the parabolic problem (3) given by

$$
\begin{cases}
  u_t - \Delta u = |u|^{q-1}u - bu^s|Du|^\theta, & t > 0, \ x \in \Omega, \\
  u(t, x) = 0, & t > 0, \ x \in \partial \Omega, \\
  u(0, x) = \phi(x) \geq 0, & x \in \Omega,
\end{cases}
$$

where $q > 1, \theta > 0, s \geq 0, \theta + s \geq 1, b \geq 0$ and $\Omega$ a regular domain of $\mathbb{R}^N$, possibly unbounded. Bartier proposed the following variant of model (3), to interpret the destruction term $u^s|Du|^\theta$. The main idea of Bartier is based on what follows: the number of preys contained in a small volume $dx$ is $udx$, the number of deaths during $dt$ is given by $Pdtudx$ where $P$ is the probability of a predator of meeting a prey per time unit. Since the flux of preys is proportional to $Du$ (Fick’s law), if one assumes that the greed of the predator is stimulated by the displacements of the preys, as well as their concentration, then one is reasonably led to a probability function $P = P(u, |Du|)$, increasing in $u$ and $|Du|$, for instance $P = bu^s|Du|^\theta$.

A detailed discussion both in the case $\Omega = \mathbb{R}^N$ and in the case $\Omega = B_R$ of the elliptic equation associated to (3), when $b$ is a function, can be found in [31].

The local estimates for positive solution of (1) we are interested in have the following form

$$
u(x) + |Du(x)|^{\mu_1} \leq C(1 + dist^{-\mu_2}(x, \partial \Omega)), \quad x \in \Omega, \quad (5)$$

with $\mu_1, \mu_2 > 0$ depending on the parameters of the equation. Estimates of the type above are those that Serrin and Zou in [29] call universal a priori estimates, because they are independent of the solutions and do not need any boundary conditions. Polacik, Quittet and Souplet study in [26] new connections between Liouville type theorems, namely nonexistence of non trivial nonnegative solutions in the whole $\mathbb{R}^N$ or in the half-space, and local properties of nonnegative solutions of (1). In particular in [26], they develop a general method for derivation of pointwise a priori estimates of local solutions of the type (5), from Liouville type theorems, where local stands for an arbitrary domain and without any boundary conditions.

As concerns Liouville theorems, we recall that the critical exponent of Sobolev embeddings $p^* = Np/(N-p)$ is optimal for Liouville theorems for elliptic equations
of the type $-\Delta_p u = u^q$, $u \geq 0$, while Serrin exponent $p_* = p(N-1)/(N-p)$ is optimal for Liouville theorems for elliptic inequalities of the type $-\Delta_p u \geq u^q$, $u \geq 0$. Of course $p_* < p^*$. For further results in this direction we refer to [29] and [22]-[24]. By using the technique in [26], but adapted to the $p$-Laplacian case with a nonlinearity of type (2) as corollary our main theorem we obtain the following.

Corollary 1. Let $\Omega$ be an arbitrary domain of $\mathbb{R}^N$, let $0 \leq s < q$. Suppose

$$0 < p - 1 < q < p^* - 1, \quad 0 < \theta < \frac{(q-s)p}{q+1}. \quad (6)$$

Then there exist $C = C(p, N, q, s, \theta) > 0$ (independent of $\Omega$ and $u$) such that for any nonnegative solution $u$ of the equation (2) in $\Omega$ with $a, b \in L^\infty(\Omega)$, it holds

$$u + |Du|^{p/((q+1)}} \leq C(1 + \text{dist}^{-p/(q-p+1)}(x, \partial\Omega)), \quad x \in \Omega. \quad (7)$$

In particular, if $\Omega = B_R \setminus \{0\}$ for some $R > 0$, then

$$u + |Du|^{p/((q+1)}} \leq C(1 + |x|^{-p/(q-p+1)}), \quad 0 < x \leq \frac{R}{2}. \quad (8)$$

These estimates extend previous results of Polacik, Quitter and Souplet in [26], in particular they consider either problem (1) with $p = 2$ and $f$ of the type (2) but with $s = 0$ or problem (1) but with a nonlinearity not depending on the gradient. In particular, in [26] the authors introduced a method based on arguments of rescaling combined with a key doubling property, that is different from the classical rescaling method of Gidas and Spruck in 1981 developed in [16] (see also [17]).

As discussed in [26], an important consequence of results of this kind is that theorems that provide uniform estimates (in norm) of the solutions are substantially equivalent to Liouville theorems. Indeed, since the celebrated works by Gidas and Spruck [16, 17], it is known that the non-existence of a uniform estimate for solutions of certain Dirichlet problems for nonlinear elliptic equations in regular bounded domains produces a non-trivial solution of a limit problem defined in $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$, but this contradicts the assertion of an appropriate Liouville type theorem. The difficulty of this technique, known as blow up technique, is that, even at present, Liouville type theorems in the half-space under rather general hypotheses are not known in the literature. Therefore, after the work of Gidas and Spruck, in literature a series of works have been developed with hypotheses, mainly of a geometrical type on the domain cfr. [1, 6], to avoid the case of the half-space.

In this direction, we mention a paper due to Ruiz [28] in which instead of assuming geometrical conditions on the domain, he produces a slight modification of the blowup technique of Gidas and Spruck in [16], precisely Ruiz applies the rescaling argument around a fixed point in $\Omega$ rather than in a sequence of suitable points of $\Omega$, furthermore he uses a consequence of Harnack’s inequality. For this reason the range in which the exponent $q$ can vary is smaller, precisely $q \in (p-1, p_* - 1)$ instead of $q \in (p-1, p^* - 1)$, but nonlinearities depending also on the gradient can be handled. A recent existence result for Dirichlet problems, which extends that of Ruiz, is contained in [10]. For other existence results for boundary Dirichlet problems with gradient terms, we refer to the pioneering paper by Ghergu and Radulescu [13] with a linear growth in the gradient and to [14], [15].

In [26], Polacik, Quitter and Souplet prove a doubling lemma in order roughly to avoid again the half-space case. Based on this doubling property, the authors start the rescaling procedure to prove local estimates of solutions of superlinear
problems. The idea, by contradiction, is that if the estimate (5) fails, then the violating sequence of solutions \( u_k \) will be increasingly large along a sequence of points \( x_k \) such that each \( x_k \) has a suitable neighborhood, where the relative growth of \( u_k \) remains controlled. After appropriate rescaling, one can blow up the sequence of neighborhoods and pass to the limit to obtain a bounded solution of a limit problem in the whole of \( \mathbb{R}^N \). A Liouville type theorem give the required contradiction.

We point out that, in (2) the presence in the same term both of the gradient and of a power of \( u \) creates several difficulties, even the management of the high number of parameters involved is quite demanding.

In Section 5 we consider general \( p, q \)-Laplacian elliptic systems with nonlinearities depending on the gradient, precisely the vectorial version of (2), which extend the well known Lane-Emden systems. For a classification of radial solutions of elliptic systems with gradient terms we refer to [12].

Here, we give some a priori estimates for nonnegative solutions \((u, v)\) of elliptic systems, where \((u, v)\) is considered a nonnegative solution if \( u \geq 0 \) and \( v \geq 0 \), which extends previous results in [26]. For simplicity, we state only a corollary relative to the Laplacian operator.

**Corollary 2.** Let \( \Omega \) be an arbitrary domain of \( \mathbb{R}^N, N \geq 2 \). Assume

\[
p_1, q_1 > 1, \quad 0 \leq s_1 < q_1, \quad 0 \leq r_1 < p_1
\]

and

\[
q_1 - s_1 > \frac{\theta_1(p_1 q_1 + 1 + 2 p_1) + \theta_2(p_1 q_1 + 1 + 2 q_1) + 2 s_2(q_1 + 1)}{2(p_1 + 1)},
\]

\[
p_1 - r_1 > \frac{\gamma_1(p_1 q_1 + 1 + 2 q_1) + \gamma_2(p_1 q_1 + 1 + 2 p_1) + 2 r_2(p_1 + 1)}{2(q_1 + 1)}.
\]

If either

\[
q_1 \geq p_1, \quad q_1 \left( \frac{p_1 N - 2}{2} - 1 \right) \leq \frac{N}{2}, \quad (11)
\]

or

\[
p_1 \geq q_1, \quad p_1 \left( q_1 \frac{N - 2}{2} - 1 \right) \leq \frac{N}{2}, \quad (12)
\]

then there exist \( C = C(N, q_1, p_1, s_1, s_2, r_1, r_2, \theta_1, \theta_2, \gamma_1, \gamma_2) > 0 \) (independent of \( \Omega \) and \( u \)) such that for any nonnegative solution \((u, v)\) of

\[
\begin{aligned}
\begin{cases}
-\Delta u = v^{q_1} - v^{s_1} u^{s_2} |Dv|^{\theta_1}|Du|^{\theta_2} & \text{in } \Omega, \\
-\Delta v = u^{p_1} - u^{r_1} v^{r_2} |Du|^{\gamma_1}|Dv|^{\gamma_2} & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

the following estimates holds

\[
\begin{aligned}
&u + |Du|^\alpha/(\alpha + 1) \leq C(1 + \text{dist}^{-\alpha}(x, \partial \Omega)), \quad x \in \Omega, \\
v + |Dv|^\beta/(\beta + 1) \leq C(1 + \text{dist}^{-\beta}(x, \partial \Omega)), \quad x \in \Omega,
\end{aligned}
\]

where

\[
\alpha = \frac{2(q_1 + 1)}{p_1 q_1 - 1}, \quad \beta = \frac{2(p_1 + 1)}{p_1 q_1 - 1}.
\]

We point out that conditions (11) and (12) are not optimal, cfr. Remark 2 but guarantee the validity of Liouville theorem for system

\[
\begin{cases}
-\Delta u = v^{q_1} \\
-\Delta v = u^{q_1}.
\end{cases}
\]
It is well known that a crucial role for existence and non-existence of positive solutions of (17) is played by the Sobolev hyperbola and in the case of inequalities by the Serrin curve, which roughly plays the role of the Serrin exponent. For a detailed discussion in this direction we refer to Section 4 in [26].

The paper is organized as follows. In Section 2, some preliminary results are stated such as doubling lemma and Liouville type results for elliptic inequalities, equation and for systems of inequalities. Then, Section 3 is devoted to prove the main a priori estimate result for local solutions of (1) given in Theorem 3.1 which will be applied to (2). Section 3 contains also the proof of Corollary 1. In Section 4, a discussion on the sharpness of condition (6) is presented. Finally, in Section 5, a priori estimates for local solutions of $p, q$-Laplacian elliptic systems with very general nonlinearities are investigated. In particular, the main result of Section 5 is represented by Theorem 5.1.

2. Preliminaries. In this section we state some useful results, which are crucial in the proofs of the main theorems of the paper, given in Section 3 for equation (1) and in Section 5 for systems. The first result we state is the doubling lemma due to Polacik, Quitter and Souplet in [26]. As discuss in [26], it is an extension of an idea of [19], where a similar doubling property in time was used to estimate blowup rates of nonglobal solutions of certain nonlinear parabolic problems. But in [26], it is the first time it is applied to elliptic equation. In particular, Polacik, Quitter and Souplet describes the doubling lemma as follow: in the Euclidean case, it states roughly the following. Consider a real function $M$, defined on a domain $\Omega$ of $\mathbb{R}^N$ and locally bounded that takes at some point $y$ a value larger than the inverse of the distance from $y$ to $\partial \Omega$. Then, for at least one point $x$ where $M$ is similarly large, $M$ cannot double its size in the ball centered at $x$ and of radius the inverse of $M(x)$.

Lemma 2.1. (Theorem 5.1,[26]) Let $(X,d)$ be a complete metric space, and let $\emptyset \neq D \subset \Sigma \subset X$ with $\Sigma$ closed. Set $\Gamma = \Sigma \setminus D$. Finally, let $M : D \to (0, \infty)$ be bounded on compact subsets of $D$, and fix a real $k > 0$. If $y \in D$ is such that

$$M(y) > 2k \operatorname{dist}^{-1}(y, \Gamma), \quad (18)$$

then there exist $x \in D$ such that

$$M(x) > 2k \operatorname{dist}^{-1}(x, \Gamma), \quad M(x) \geq M(y), \quad (19)$$

and

$$M(z) \leq 2M(x) \text{ for all } z \in D \cup \overline{B}_X(x, kM^{-1}(x)).$$

Remark 1. In the subcase $X = \mathbb{R}^N$ and $\Omega$ open subset of $\mathbb{R}^N$, put $D = \Omega$, $\Sigma = \overline{D}$, $\Gamma = \partial \Omega$. Then we have $\overline{B}(x, k, M^{-1}(x)) \subset D$. Indeed, since $D$ is open, (19) implies that

$$\operatorname{dist}(x, D^c) = \operatorname{dist}(x, \Gamma) > 2kM^{-1}(x).$$

Now, we state some Liouville type results both for elliptic equations and for inequalities relative to the model problem in which the nonlinearity is exactly a power of the solution. As discussed before, Liouville theorems represent one of the main tools in the proof of Theorem 3.1 in the next section.

Theorem 2.2. (Corollary II,[29]) (i) The inequality $-\Delta_p u \geq 0$ has a non-constant positive solution in $\mathbb{R}^N$ if and only if $N > p$.

(ii) Assume $N > p$. Then the generalized Lane-Emden equation $-\Delta_p u = u^q$ has a positive solution in $\mathbb{R}^N$ if and only if $q \geq p^* - 1$. 

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(iii) Assume $N > p$. Then the differential inequality $-\Delta_p u \geq u^q$ has a positive solution in $\mathbb{R}^N$ if and only if $q > p^*_s - 1$.

Case (iii) is an immediate consequence of Theorem I proved by Serrin and Zou in [29] and it was first proved by Mitidieri and Pohozaev in [22], while cases (i) and (ii) are consequences of Theorem II in [29]. Some of the above results have previously been established in [25] for radially symmetric solutions.

For Liouville type results devoted to equations and inequalities depending on the gradient of the type $-\Delta_p u \geq |Du|^q$, $u \geq 0$, $q, \theta > 0$ we refer to [21], [24], [8], [9], [11], [7], [18] and the references therein.

Finally, we state a Liouville result for a system of $p$-Laplacian elliptic inequalities in the case when the nonlinearity is a pure power.

**Theorem 2.3.** ([Theorem 22.1, [24]]) Let

$$1 < p, q < N, \quad p_1 > p - 1, \quad q_1 > q - 1. \quad (20)$$

If

$$\max \left\{ \frac{qq_1 + p(q - 1)}{p_1 q_1 - (p - 1)(q - 1)}, \frac{N - p}{p - 1}, \frac{pp_1 + q(p - 1)}{p_1 q_1 - (p - 1)(q - 1)} \right\} \geq 0 \quad (21)$$

then the problem

$$\begin{cases}
-\Delta_p u \geq u^{q_1}, & x \in \mathbb{R}^N, \\
-\Delta_q v \geq u^{p_1}, & x \in \mathbb{R}^N, \\
u \geq 0, & \quad u \not\equiv 0, \quad x \in \mathbb{R}^N, \\
v \geq 0, & \quad v \not\equiv 0, \quad x \in \mathbb{R}^N,
\end{cases} \quad (22)$$

has no solutions $(u, v) \in (W^{1,p}_{loc}(\mathbb{R}^N) \cap L^{p_1}_{loc}(\mathbb{R}^N)) \times (W^{1,q}_{loc}(\mathbb{R}^N) \cap L^{q_1}_{loc}(\mathbb{R}^N))$.

3. **Main result.** In this section we prove the main result of the paper which is pointwise a priori estimates of local solutions of (1) under a general growth condition on the nonlinearity $f$.

The next result extends Theorem 6.1 proved by Polacik, Quittner and Souplet in [26] relative to equation (1) in the subcase $p = 2$ and $s = 0$.

**Theorem 3.1.** Let $\Omega$ be an arbitrary domain of $\mathbb{R}^N$, $N \geq 2$. Let

$$1 < p < N, \quad p - 1 < q < p^* - 1, \quad 0 \leq s < q.$$ 

Assume that $f : \Omega \times \mathbb{R}^N_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Caratheodory function and that there exist $q_1 \in (0, q)$, $s$ and $S$ with

$$0 \leq s \leq S < \min \left\{ q, \frac{(q + 1)(s + 1)}{p} - 1 \right\}, \quad (23)$$

$$0 < \theta < \Theta := \frac{(q - S)p}{q + 1}(< p), \quad (24)$$

such that

$$-C_1 \left( 1 + t^{q_1} + t^s |\eta|^q \right) \leq f(x, t, \eta) \leq C_1 \left( 1 + t^q + t^s |\eta|^\theta \right) \quad (25)$$

for all $x \in \Omega$ and for $t \geq 0$ and $\eta \in \mathbb{R}^N$, with $C_1 > 0$. Suppose that for all $x \in \overline{\Omega}$, if $\Omega$ is bounded,

$$\lim_{t \to \infty, \Omega \ni z \to x} t^{-q} f(z, t, t^{(q+1)/p} \eta) = l(x) \in (0, \infty) \quad (26)$$
uniformly for $\eta$ bounded. Moreover, if $\Omega$ is unbounded, then assume that (26) holds also for $x = \infty$.

Then there exist $C = C(p,N,f) > 0$ (independent of $\Omega$ and $u$) such that for any nonnegative solution $u$ of (1) the estimate (7) holds.

In particular, if $\Omega = B_R \setminus \{0\}$ for some $R > 0$, then (8) is in force.

Proof. We follow the argument in the proof of Theorem 6.1 in [26].

Put $\alpha = p/(q - p + 1)$. Assume that the estimate (7) fails. Then there exist sequences $\Omega_k \subseteq \Omega$, $y_k \in \Omega_k$ and $u_k$ solution of (1) on $\Omega_k$, such that the ratio
\[
u_k(y_k) = \frac{u_k(y_k) + |Du_k(y_k)|^{\alpha/(\alpha + 1)}}{1 + \text{dist}^{-\alpha}(y_k, \partial\Omega_k)}
\]
is unbounded, so that
\[
\frac{(u_k(y_k) + |Du_k(y_k)|^{\alpha/(\alpha + 1)})^{1/\alpha}}{(1 + \text{dist}^{-\alpha}(y_k, \partial\Omega_k))^{1/\alpha}} \to \infty,
\]
as $k \to \infty$. Now, by setting
\[
M_k := u_k^{1/\alpha} + |Du_k|^{1/(\alpha + 1)}, \quad k \geq 1,
\]
we can suppose that
\[
M_k(y_k) \geq c(u_k(y_k) + |Du_k(y_k)|^{\alpha/(\alpha + 1)})^{1/\alpha} \geq 2k(1 + \text{dist}^{-\alpha}(y_k, \partial\Omega_k))^{1/\alpha} \geq 2k \text{dist}^{-1}(y_k, \partial\Omega_k),
\]
where $c$ is a positive constant. By Lemma 2.1 and Remark 1, it follows that there exists $x_k \in \Omega_k$ such that
\[
M_k(x_k) > 2k \text{dist}^{-1}(x_k, \partial\Omega_k), \quad M_k(x_k) \geq M_k(y_k),
\]
and for all $z$ such that $|z - x_k| \leq kM_k^{-1}(x_k)$, then
\[
M_k(z) \leq 2M_k(x_k).
\]
Now we rescale $u_k$ by setting
\[
v_k(y) = \lambda_k^2 u_k(z), \quad z = x_k + \lambda_k y, \quad |y| \leq k, \quad \text{with } \lambda_k = M_k^{-1}(x_k).
\]
Since $M_k(x_k) \geq M_k(y_k) > 2k$ we also have by (31)
\[
\lambda_k \to 0, \quad k \to \infty.
\]
In particular, since $(\partial/\partial y_i)v_k(y) = \lambda_k^{\alpha + 1} (\partial/\partial z_i)u_k(z)$, we immediately have
\[
\Delta_p v_k = \text{div}_y([Dv_k]^{p-2} Dv_k) = \lambda_k^{\alpha(p-1)+p} \Delta_p u_k,
\]
so that, since $u_k$ is a solution of (1), then $v_k$ is a solution of
\[
-\Delta_p v_k = f_k(v_k(y)), \quad |y| \leq k,
\]
with
\[
f_k(v_k(y)) = \lambda_k^{\alpha(p-1)+p} f(x_k + \lambda_k y, \lambda_k^{-\alpha} v_k, \lambda_k^{-\alpha-1} Dv_k)
\]
Moreover, from (27), (30) and the definition of $\lambda_k$,
\[
[v_k^{1/\alpha} + |Du_k|^{1/(\alpha + 1)}](0) = \lambda_k[u_k^{1/\alpha} + |Du_k|^{1/(\alpha + 1)}](0) = \lambda_k M_k(x_k) = 1,
\]
and, when $|y| \leq k$,
\[
[v_k^{1/\alpha} + |Du_k|^{1/(\alpha + 1)}](y) = \lambda_k[u_k^{1/\alpha} + |Du_k|^{1/(\alpha + 1)}](z) = \lambda_k M_k(z) \leq 2\lambda_k M_k(x_k) = 2.
\]
In particular, by virtue of assumption (25) on \( f \), we have for all \(|y| \leq k\)
\[
f_k(v_k(y)) \geq -C_1 \lambda_k^{(p-1)+p} \left( 1 + \lambda_k^{-\alpha q_1} v_k^q + \lambda_k^{-\alpha \rho (\alpha+1)} v_k^\rho |Dv_k|^{\theta} \right)
= -C_1 \lambda_k^\epsilon \left( \lambda_k^{(p-1)+p-\epsilon} + \lambda_k^{\alpha (q-q_1) - \epsilon} v_k^{q_1} 
+ \lambda_k^{p-\theta - \alpha (\theta - p + 1 + s) - \epsilon} v_k^\rho |Dv_k|^{\theta} \right)
= -C_1 \lambda_k^\epsilon \left( \lambda_k^{(p-1)+p-\epsilon} + \lambda_k^{\alpha (q-q_1) - \epsilon} v_k^{q_1} 
+ \lambda_k^{\alpha - \epsilon} v_k^\rho |Dv_k|^{\theta} \right)
\geq -C_1 \lambda_k^\epsilon,
\]
where
\[
\epsilon = \min \{ \alpha (p-1) + p, \alpha (q-q_1) + p, \alpha \} > 0, \quad \alpha_1 = p - \theta - \alpha (\theta - p + 1 + s).
\]

We observe that \( \alpha_1 > 0 \) thanks to (24) and (23). Indeed, condition (23) forces that
\[
p - 1 - s < (q-S) p / (q+1),
\]
so that we have two possibilities, either \( 0 < \theta \leq p - 1 - s \) or \( p - 1 - s < \theta < (q-S) p / (q+1) \).
In the first case, we have \( \theta - p + 1 + s \leq 0 \) thus \( \alpha_1 > 0 \) for all \( \alpha > 0 \), in the second case \( \theta - p + 1 + s > 0 \) then \( \alpha_1 > 0 \) if and only if \( \alpha < (p-\theta) / (\theta - p + 1 + s) \), namely, by the choice of \( \alpha \),
\[
\frac{p}{q-p+1} < \frac{p-\theta}{\theta - p + 1 + s}
\]
that is
\[
\theta < \frac{(q-S) p}{q+1}
\]
which follows immediately from \( \theta < \Theta \) and \( S \geq s \). On the other hand, being \( \alpha (p-1) + p - \alpha S - (\alpha+1) \Theta = 0 \), from (25) we have
\[
f_k(v_k(y)) \leq C_1 \lambda_k^{(p-1)+p} \left( 1 + \lambda_k^{-\alpha q_1} v_k^q + \lambda_k^{-\alpha S - (\alpha+1) \Theta} v_k^{\Theta} |Dv_k|^{\Theta} \right)
= C_1 \lambda_k^{p q / (q-p+1) + v_k^{\rho} + v_k^{2 \rho} |Dv_k|^{\Theta}},
\]
(35)

hence, thanks to (34) and since \( q > p - 1 \), we have for all \( k \) large
\[
-C_1 \lambda_k^\epsilon \leq f_k(v_k(y)) \leq C', \quad |y| \leq k, \quad C' > 0.
\]
(36)

By using standard regularity results (cf. [20], [32]) since \( v_k \) and \( Dv_k \) are bounded on compact subsets of \( \Omega \) by (34), we deduce that there exist a positive constant \( \beta \) such that \( v_k \) is bounded in \( C^{1,\beta}(\Omega) \), namely \( v_k \) is bounded in \( C^{1,\beta}_{loc}(\mathbb{R}^N) \). Therefore \( v_k \) converges in \( C^{1,\beta}_{loc}(\mathbb{R}^N) \) to a certain function \( v \geq 0 \). Moreover, by letting \( k \to \infty \) in (33) we have that \( v(0)^{1/(\alpha+1)} + |Dv(0)|^{1/\alpha} = 1 \) thus \( v \) is a nontrivial furthermore, by letting \( k \to \infty \) in (32) and (36), we have that \( v \) satisfies \( -\Delta_p v \geq 0 \) in \( \mathbb{R}^N \), so by the strong maximum principle for the \( p \)-Laplacian (see [27, 33]) it results \( v(y) > 0 \) for all \( y \in \mathbb{R}^N \).

Fix \( y \in \mathbb{R}^N \) and denote
\[
\mu_k = \lambda_k^{-\alpha} v_k(y), \quad \eta_k = v_k^{-(\alpha+1)/\alpha} Dv_k(y).
\]
Consequently, since $\lambda_k^{-\alpha-1}v_k = \mu_k^{(\alpha+1)/\alpha} \eta_k$, we obtain
\[
f_k(v_k(y)) = v_k^{p-1+\frac{\alpha}{p}} \mu_k^{-p+1-\frac{\alpha}{p}} f(x_k + \lambda_k y, \mu_k, \mu_k^{(\alpha+1)/\alpha} \eta_k) = v_k^q \mu_k^{-q} f(x_k + \lambda_k y, \mu_k, \mu_k^{(q+1)/p} \eta_k),
\]
by the choice of $\alpha$. Note that as $k \to \infty$, we have $\mu_k \to \infty$ being $v(y) > 0$ in $\mathbb{R}^N$ and $\eta_k$ bounded from (34).

If $(x_k)_k$ is bounded, then we may assume that $x_k \to \bar{x} \in \overline{\Omega}$ by extracting a further subsequence, and assumption (26) implies that
\[
f_k(v_k(y)) \to l(\bar{x})v(y)^q \quad \text{as} \quad k \to \infty. \tag{37}
\]
Otherwise, if $\Omega$ is unbounded and $x_k \to \infty$ as $k \to \infty$ (along some subsequence), then the additional assumption on $f$ implies that (37) still holds with $\bar{x} = \infty$. Consequently by letting $k \to \infty$ in (32), thank to (37), we obtain that $v$ verifies
\[-\Delta_p v = l(\bar{x})v(y)^q \quad \text{in} \quad \mathbb{R}^N.
\]
This contradicts the Liouville type Theorem 2.2 (ii) being $q < p^* - 1$ and concludes the proof of the theorem. \hfill \square

We point out that when $f$ is exactly a power the estimates (7) can be given more precise, see Theorem 3.3 in [26].

Theorem 3.1 can be applied to prototype (2), this application is exactly Corollary 1 in the Introduction. We give now its proof.

Proof of Corollary 1. It is enough to apply Theorem 3.1 with
\[f(x,u,\eta) = a(x)u^q - b(x)u^s|\eta|^\theta,\]
where $(x,u,\eta) \in \Omega \times \mathbb{R}^+_x \times \mathbb{R}^N$ with $q$, $s$, $\theta$ satisfying (6), $b$ is a continuous function with $b \in L^\infty(\Omega)$, whenever $\Omega$ is unbounded, and $a$ is a positive continuous function in $\Omega$ such that $a \in C(\overline{\Omega})$ if $\Omega$ is bounded, otherwise
\[
\lim_{|x| \to \infty} a(x) =: l \in (0, \infty)
\]
if $\Omega$ is unbounded. For such a function $f$, the left side of assumption (25) is trivially verified for all $q_1 < q$ with $C_1 = \|b(x)\|_{\infty}$, indeed $a(x) > 0$ so that
\[a(x)u^q - b(x)u^s|\eta|^\theta \geq -b(x)u^s|\eta|^\theta \geq -\|b\|_{\infty}u^s|\eta|^\theta.\]
About the right side of assumption (25) it holds with $S = s$ since we have
\[a(x)u^q - b(x)u^s|\eta|^\theta \leq \|a\|_{\infty}u^q + \|b\|_{\infty}u^s|\eta|^\theta.\]
Concerning (26), we have
\[
\lim_{t \to \infty, \Omega \ni z \to x} t^{-q} f(z, t, t^{(q+1)/p} \eta) = \lim_{t \to \infty, \Omega \ni z \to x} t^{-q} a(z)t^q - t^{-q} b(z)t^{s+\theta(q+1)/p} |\eta|^\theta = \lim_{t \to \infty, \Omega \ni z \to x} a(z) - b(z)t^{s+\theta(q+1)/p} |\eta|^\theta = a(x),
\]
since we have used that $s - q + \theta(q+1)/p < 0$, $a \in L^\infty(\Omega)$ and $\eta$ bounded. Thus (26) holds choosing $l(x) = a(x)$ thanks to the properties of $a(x)$.
4. On the sharpness of Theorem 3.1. The condition on $\theta$ in Theorem 3.1 is optimal (up to the equality case), as shown by the following counterexample, already discussed in [26] when $s = 0$. Indeed, if hold the following condition

$$\theta > \frac{(q-s)p}{q+1}, \quad (38)$$

then there are solutions of equation (2), with $a(x) \equiv 1$, which does not satisfy the a priori estimate (8).

Precisely, let $u(r) = u(|x|) = |x|^{-\theta} = r^{-\theta}$, $r > 0$ and $\beta > 0$ defined in $B_R \setminus \{0\}$, with $R > 0$ to be chosen. Obviously

$$-\Delta_{\nu}u = \beta^{p-1}[N - 1 - (\beta + 1)(p - 1)]r^{-\beta(p-1)-p}. \quad (39)$$

On the other hand

$$u^q - b(x)u^\theta |Du|^\theta = r^{-\beta q} - b(x)\beta^{q}r^{-\beta s - \theta(q + 1)},$$

Consequently $u$ is a radial solution of (2) if

$$\beta^{p-1}[N - p - \beta(p - 1)]r^{-\beta(p-1)-p} = r^{-\beta q} - b(x)\beta^{q}r^{-\beta s - \theta(q + 1)},$$

which holds if and only if

$$b(x) = \beta^{-1}r^{(\beta + s - q) + \theta} - \beta^{p-1-\theta}[N - p - \beta(p - 1)]r^{\beta(q + s - p - 1) - p + \theta}.$$ 

Consequently $b(x) \in C^1(B_R)$, $R > 0$, if

$$\begin{cases}
\beta(\theta + s - p + 1) - p + \theta > 0, \\
\beta(\theta + s - q) + \theta > 0.
\end{cases} \quad (40)$$

By (38) we have $\theta > p - s - 1$ so that system (40) admits a solution if

$$\frac{p - \theta}{\theta + s - p + 1} < \beta < \frac{\theta}{q - \theta - s}, \quad \text{if } q - \theta > s, \quad (41)$$

$$\frac{p - \theta}{\theta + s - p + 1} < \beta, \quad \text{if } q - \theta \leq s. \quad (42)$$

Condition (38) when $q - \theta > s$ forces that

$$\frac{p - \theta}{\theta + s - p + 1} < \frac{\theta}{q - \theta - s}$$

so that the interval given for $\beta$ in (41) is not empty.

Therefore by choosing $\beta$ such that either (41) or (42) holds and with $p$, $q$, $s$, $\theta$ satisfy all the assumption of Theorem 3.1 with

$$f(x, u, \eta) = u^q - b(x)u^\theta |\eta|^\theta$$

hold except assumption $\theta < (q-s)p/(q+1)$.

We claim that condition (8) fails. Indeed,

$$u + |Du|^{p/(q+1)} = r^{-\beta} + \beta^{p/(q+1)}r^{-(\beta + 1)p/(q+1)}.$$ 

Consequently condition (8) reduces to prove the existence of some $C > 0$ such that

$$r^{-\beta}(1 + \beta^{p/(q+1)}r^{[\beta(q+p-1)-p]/(q+1)+\beta} - Cr^{\beta-p/(q-p+1)}) \leq C \quad (43)$$

Thanks to condition (38) we have

$$\frac{p - \theta}{\theta + s - p + 1} < \frac{p}{q + 1 - p} < \frac{\theta}{q - \theta - s}.$$
when \( q - \theta - s > 0 \), thus in this case choosing
\[
\beta \in \left( \frac{p}{q + 1 - p}, \frac{\theta}{q - \theta - s} \right)
\]
or simply
\[
\beta > \frac{p}{q + 1 - p}
\]
when \( q - \theta - s \leq 0 \), we get
\[
1 + \beta^{p/(q+1)} r^{\beta(q-p+1)/(q+1)} \beta - Cr^{\beta-p/(q-p+1)} \to 1 \quad \text{as} \quad r \to 0^+
\]
so that inequality (43) cannot hold since the left hand side goes to infinity as \( r \to 0^+ \).

Concerning the equality case the situation is not clear, as emphasized by Polacik, Quittner and Souplet before the proof of Theorem 6.1 in [26]. For instance the above example does not give any contradiction if \( \theta = (q-s)p/(q+1) \), in this case
\[
p - \theta
\]
and
\[
q - \theta - s = \frac{q - s}{q + 1}(q - p + 1) > 0,
\]
thus only (41) can occur and necessarily
\[
\beta = \frac{p}{q - p + 1}.
\]
Consequently
\[
b(x) = \left( \frac{p}{q - p + 1} \right)^{-\theta} \left[ 1 + \frac{p^{p-1}[N(p-1) - q(N-p)]}{(q - p + 1)p} \right] := b_0
\]
with \( b_0 > 0 \) since \( q < p^* - 1 \). In turn equation (2), in this particular case, becomes
\[
-\Delta_p u = u^q - b_0 u^s |Du|^{\frac{(q-s)p}{q-p+1}}.
\]
This equation when \( p = 2, s = 0 \) is exactly that discussed in details in [31]. In particular, we point out that the existence or nonexistence of solutions of the associated Dirichlet problem when \( \Omega \) is a ball of \( \mathbb{R}^N \) or in all of \( \mathbb{R}^N \), depends on the “size” of \( b_0 \), cfr. Section 4 in [31] (iii) when \( \Omega = \mathbb{R}^N \) and (ii) when \( \Omega = B_R, R > 0 \), respectively.

5. Elliptic systems. As observed by Polacik, Quittner and Souplet in [26], the universal estimates (7) and (8) can be obtained also for systems, by using the same technique, precisely we consider the following system
\[
\begin{align*}
-\Delta_p u &= f_1(x, u, v, Du, Dv) \quad \text{in } \Omega, \\
-\Delta_q v &= f_2(x, u, v, Du, Dv) \quad \text{in } \Omega,
\end{align*}
\]
where \( 1 < p, q < N \), \( f_1 \) and \( f_2 \) are Caratheodory functions such that there exist \( p_1, q_1 \) satisfying (20) and \( \bar{q}_1 \), \( \bar{p}_1 \) with \( \bar{q}_1 \in (0, q_1), \bar{p}_1 \in (0, p_1) \), furthermore there exist exponents \( s_i, r_i, \theta_i, \gamma_i, S_i, R_i, \Theta_i, \Gamma_i \geq 0, \) \( i = 1, 2 \), with
\[
\begin{align*}
\theta_1 + \theta_2 &> 0, \quad \gamma_1 + \gamma_2 > 0, \quad S_1 < q_1, \quad R_1 < p_1, \\
s_i &\leq S_i, \quad r_i \leq R_i, \quad \theta_i < \Theta_i, \quad \gamma_i < \Gamma_i,
\end{align*}
\]
satisfying
\[
q_1 - s_1 = \frac{\Theta_1 \alpha_1 + \Theta_2 \alpha_2 + S_2 [qq_1 + p(q - 1)]}{pp_1 + q(p - 1)},
\]
\[
p_1 - r_1 = \frac{\Gamma_1 \alpha_2 + \Gamma_2 \alpha_1 + R_2 [pp_1 + q(p - 1)]}{qq_1 + p(q - 1)},
\]
where
\[\alpha_1 = p_1 q_1 + p - 1 + pp_1, \quad \alpha_2 = p_1 q_1 + q - 1 + qq_1.\]

Assume also that there exist \(C_1, C_2 > 0\) such that
\[-C_1 (1 + v^{\eta_1} + v^{s_1} u^{s_2} |\eta|^{\theta_1} |\xi|^{\sigma_1}) \leq f_1 (x, u, v, \eta, \xi) \leq C_1 (1 + v^{\eta_1} + v^{s_1} u^{s_2} |\eta|^{\theta_2} |\xi|^{\sigma_1}),\]
and
\[-C_2 (1 + u^{\Gamma_1} + u^{r_1} v^{r_2} |\eta|^{\gamma_1} |\xi|^{\tau_1}) \leq f_2 (x, u, v, \eta, \xi) \leq C_2 (1 + u^{\Gamma_1} + u^{r_1} v^{r_2} |\eta|^{\Gamma_1} |\xi|^{\tau_1}),\]
for all \(x \in \Omega\) and for \(u, v \geq 0\) and \(\eta, \xi \in \mathbb{R}^N\).

Finally, there exist a positive constant \(C\) such that
\[
f_1 (x, u, v, \eta, \xi) \geq C v^{\eta_1},
\]
\[
f_2 (x, u, v, \eta, \xi) \geq C u^{\Gamma_1}
\]
for all \(u, v\) sufficiently large and \(\eta, \xi \in \mathbb{R}^N\).

The following prototype
\[
\begin{align*}
-\Delta_p u &= v^{\eta_1} - v^{s_1} u^{s_2} |Dv|^{\theta_1} |Du|^{\sigma_1} \quad \text{in } \Omega, \\
-\Delta_q v &= u^{\Gamma_1} - u^{r_1} v^{r_2} |Du|^{\gamma_1} |Dv|^{\tau_1} \quad \text{in } \Omega.
\end{align*}
\]
satisfies all the assumptions above with \(s_i = S_i, r_i = R_i, \theta_i < \Theta_i\) and \(\gamma_i < \Gamma_i\) \(i = 1, 2\) and \(\Theta_i, \Gamma_i\) obtained from (46) and (47). In particular,
\[0 \leq s_1 < q_1, \quad 0 \leq r_1 < p_1,
\]
while from (45), (46) and (47) it immediately follows that
\[
q_1 - s_1 > \frac{\theta_1 \alpha_1 + \theta_2 \alpha_2 + s_2 [qq_1 + p(q - 1)]}{pp_1 + q(p - 1)},
\]
\[
p_1 - r_1 > \frac{\gamma_1 \alpha_2 + \gamma_2 \alpha_1 + r_2 [pp_1 + q(p - 1)]}{qq_1 + p(q - 1)}.
\]
System (49) is a generalization of the celebrated Lane-Emden system (17). Because of the technique used, we need to use Liouville theorems for systems. Since we assume (48), we need a Liouville type Theorem for a system of inequalities, for this reason we make use of Theorem 2.3.

Finally, we point out, that conditions (46) and (47), in the scalar case of (49), that is equation (2), produces exactly the definition of \(\Theta\) in (24), indeed \(u = v\) and it is enough to take \(q_1 = p_1, \theta_1 = \gamma_1 = 0\) and \(s_1 = r_1 = 0\).

The next theorem extends Theorem 4.1 and 7.3 in [26].

**Theorem 5.1.** Let \(\Omega\) be an arbitrary domain of \(\mathbb{R}^N, N \geq 2\). Assume (20), (21) and let all the hypothesis above hold for \(f_1\) and \(f_2\).

Then there exist \(C = C(p, q, N, f_1, f_2) > 0\) (independent of \(\Omega\) and \(u\)) such that for any nonnegative solution \((u, v)\) of (44) the estimates (14) and (15) hold with
\[
\alpha = \frac{q_1 q + p(q - 1)}{p_1 q_1 - (q - 1)(p - 1)}, \quad \beta = \frac{p_1 p + q(p - 1)}{p_1 q_1 - (q - 1)(p - 1)}.
\]
Proof. We follow the argument in the proof of Theorem 7.3 in [26]. Assume that the estimates (14) and/or (15) fail. Then there exist sequences \( \Omega_k \subseteq \Omega \), \( y_k \in \Omega_k \) and \((u_k, v_k)\) solution of (1) on \( \Omega_k \), such that, as in (28)

\[
M_k(y_k) \geq 2k \text{dist}^{-1}(y_k, \partial \Omega_k)
\]

where now

\[
M_k := u_k^{1/\alpha} + v_k^{1/\beta} + |Du_k|^{1/(\alpha+1)} + |Dv_k|^{1/(\beta+1)}, \quad k \geq 1.
\]

By Lemma 2.1 and Remark 1, it follows that there exists \( x_k \in \Omega_k \) such that

\[
M_k(x_k) > 2k \text{dist}^{-1}(x_k, \partial \Omega_k), \quad M_k(x_k) \geq M_k(y_k),
\]

and for all \( z \) such that \( |z - x_k| \leq kM_k^{-1}(x_k) \), then

\[
M_k(z) \leq 2M_k(x_k).
\]

Now we rescale \((u_k, v_k)\) by setting \( \lambda_k = M_k^{-1}(x_k) \),

\[
\tilde{u}_k(y) = \lambda_k^\alpha u_k(y), \quad \tilde{v}_k(y) = \lambda_k^\beta v_k(y), \quad z = x_k + \lambda_k y, \quad |y| \leq k.
\]

Since \( M_k(x_k) \geq M_k(y_k) > 2k \) we also have by (31)

\[
\lambda_k \to 0, \quad k \to \infty.
\]

In particular, as in Theorem 3.1, we immediately have

\[
\Delta_p \tilde{u}_k = \text{div}_y(|D\tilde{u}_k|^{p-2} D\tilde{u}_k) = \lambda_k^\alpha M(p-1)+p \Delta_P u_k,
\]

\[
\Delta_q \tilde{v}_k = \text{div}_y(|D\tilde{v}_k|^{q-2} D\tilde{v}_k) = \lambda_k^\beta M(q-1)+q \Delta_q v_k,
\]

so that, since \((u_k, v_k)\) is a solution of (44), then \((\tilde{u}_k, \tilde{v}_k)\) is a solution, in \(|y| \leq k\), of

\[
\begin{cases}
-\Delta_p \tilde{u}_k = f_{1,k}(\tilde{u}_k, \tilde{v}_k), \\
-\Delta_q \tilde{v}_k = f_{2,k}(\tilde{u}_k, \tilde{v}_k),
\end{cases}
\]

with

\[
f_{1,k}(\tilde{u}_k, \tilde{v}_k) = \lambda_k^\alpha (p-1)+p f_1(x_k + \lambda_k y, \lambda_k^{-\alpha} \tilde{u}_k, \lambda_k^{-\beta} \tilde{v}_k, \lambda_k^{-\alpha-1} D\tilde{u}_k, \lambda_k^{-\beta-1} D\tilde{v}_k)
\]

and

\[
f_{2,k}(\tilde{u}_k, \tilde{v}_k) = \lambda_k^\beta (q-1)+q f_2(x_k + \lambda_k y, \lambda_k^{-\alpha} \tilde{u}_k, \lambda_k^{-\beta} \tilde{v}_k, \lambda_k^{-\alpha-1} D\tilde{u}_k, \lambda_k^{-\beta-1} D\tilde{v}_k).
\]

As in the proof of Theorem 3.1, from (53), (55) and the definition of \( \lambda_k \),

\[
[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta} + |D\tilde{u}_k|^{1/(\alpha+1)} + |D\tilde{v}_k|^{1/(\beta+1)}](0) = \lambda_k M_k(x_k) = 1,
\]

and, when \(|y| \leq k\),

\[
[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta} + |D\tilde{u}_k|^{1/(\alpha+1)} + |D\tilde{v}_k|^{1/(\beta+1)}](y) = \lambda_k M_k(z) \leq 2\lambda_k M_k(x_k) \leq 2.
\]

By using the estimates from below and from above for \( f_1 \) and \( f_2 \) we obtain

\[
f_{1,k}(\tilde{u}_k, \tilde{v}_k) \geq -C_1 \lambda_k^{\tau_1} \left( \lambda_k^{\alpha(p-1)+\gamma} + \lambda_k^{\gamma} \tilde{u}_k^{\tau_1} \tilde{v}_k^{\tau_2} + \lambda_k^{\beta(p-1)+\gamma} \tilde{v}_k^{\tau_1} \tilde{u}_k^{\tau_2} |D\tilde{u}_k|^\gamma |D\tilde{v}_k|^\gamma \right),
\]

\[
f_{2,k}(\tilde{u}_k, \tilde{v}_k) \geq -C_2 \lambda_k^{\tau_2} \left( \lambda_k^{\beta(q-1)+\gamma} + \lambda_k^{\gamma} \tilde{v}_k^{\tau_1} \tilde{u}_k^{\tau_2} + \lambda_k^{\beta(q-1)+\gamma} \tilde{u}_k^{\tau_1} \tilde{v}_k^{\tau_2} |D\tilde{u}_k|^\gamma |D\tilde{v}_k|^\gamma \right)
\]

where

\[
\tau_1 = \alpha(p-1) - \beta \tau_1 + \rho \quad \text{and} \quad \tau_2 = \beta(q-1) - \alpha \tau_1 + q,
\]

\[
\sigma_1 = \alpha(p-1 - s_2 - \theta_2) - \beta(s_1 + \theta_1) + p - \theta_1 - \theta_2,
\]

\[
\sigma_2 = \beta(q-1 - r_2 - \gamma_2) - \alpha(r_1 + \gamma_1) + q - \gamma_1 - \gamma_2.
\]
and \( \epsilon_1 = \min\{\alpha(p - 1) + p, \tau_1, \sigma_1\} \), \( \epsilon_2 = \min\{\beta(q - 1) + q, \tau_2, \sigma_2\} \). In particular, \( \tau_1, \tau_2 > 0 \) from (52) and \( \beta_1 < q_1, \beta_2 < p_1 \), while \( \sigma_1 > 0, \sigma_2 > 0 \) by (50) and (51) respectively, so that \( \epsilon_1, \epsilon_2 > 0 \), consequently for \( k \) large, being \( \lambda_k \to 0 \) as \( k \to \infty \), then \( f_{1,k} \geq -C_1\lambda_k^{\epsilon_1} \) and \( f_{2,k} \geq -C_2\lambda_k^{\epsilon_2} \) by virtue of (59).

On the other hand, being \( \alpha(p - 1) - \beta q_1 + p = 0 \) and \( \beta(q - 1) - \alpha p_1 + q = 0 \) by (52), furthermore since

\[
\alpha(p - 1 - S_2 - \Theta_2) - \beta(S_1 + \Theta_1) + p - \Theta_1 - \Theta_2 = 0,
\]

and

\[
\beta(q - 1 - R_2 - \Gamma_2) - \alpha(R_1 + \Gamma_1) + q - \Gamma_1 - \Gamma_2 = 0
\]

thanks to (46) and (47) respectively, we have

\[
f_{1,k}(\tilde{u}_k, \tilde{v}_k) \leq C_1 \left( \lambda_k^{\alpha(p-1)+p} + \tilde{v}_k^{q_1} + \tilde{v}_k^{S_1} \tilde{u}_k^{S_2} |D\tilde{v}_k|^\Theta_1 |D\tilde{u}_k|^{\Theta_2} \right)
\]

\[
f_{2,k}(\tilde{u}_k, \tilde{v}_k) \leq C_2 \left( \lambda_k^{\beta(q-1)+q} + \tilde{v}_k^{p_1} + \tilde{u}_k^{R_1} \tilde{v}_k^{R_2} |D\tilde{u}_k|^{\Gamma_1} |D\tilde{v}_k|^{\Gamma_2} \right)
\]

Hence, thanks to (59) and since \( p_1 > p - 1, q_1 > q - 1 \), we have for all \( k \) large, with \( C' > 0 \)

\[
-C_1\lambda_k^{\epsilon_1} \leq f_{1,k}(\tilde{u}_k, \tilde{v}_k) \leq C', \quad -C_2\lambda_k^{\epsilon_2} \leq f_{2,k}(\tilde{u}_k, \tilde{v}_k) \leq C', \quad |y| \leq k. \quad (60)
\]

By using elliptic \( L^q \)-estimates and standard embeddings, we deduce that some subsequence of \((\tilde{u}_k, \tilde{v}_k)\) converges in \( C_{1,\infty}^1(\mathbb{R}^N) \) to a certain function \((u, v)\) with \( u \geq 0 \) and \( v \geq 0 \). Moreover, by letting \( k \to \infty \) in (58) we have that

\[
u(0)^{1/(\alpha+1)} + v(0)^{1/(\beta+1)} + |Du(0)|^{1/\alpha} + |Dv(0)|^{1/\beta} = 1
\]

thus \((u, v)\) is nontrivial. By letting \( k \to \infty \) in (57), thanks to condition (48), we obtain that \((u, v)\) verifies

\[
\begin{aligned}
-\Delta_p u &\geq C_1 v^{q_1}, & \text{in } \mathbb{R}^N, \\
-\Delta_q v &\geq C_2 u^{p_1}, & \text{in } \mathbb{R}^N,
\end{aligned}
\]

with \( C_1, C_2 > 0 \). This contradicts Theorem 2.3, whose validity is guaranteed by (20) and (21), and concludes the proof of the theorem.

**Corollary 3.** Let \( \Omega \) be an arbitrary domain of \( \mathbb{R}^N, N \geq 2 \). Assume (20) and (21). Assume that all the hypothesis above hold for the parameters \( q_1, p_1, s_1, s_2, \theta_1, \theta_2, \gamma_1 \) and \( \gamma_2 \).

Then there exist \( C = C(p, q, N, p_1, q_1, s_1, s_2, \theta_1, \theta_2, \gamma_1, \gamma_2) > 0 \) (independent of \( \Omega \) and \( u ) such that for any nonnegative solution \((u, v)\) of (49) the estimates (14) and (15) hold where \( \alpha \) and \( \beta \) are defined in (52).

**Proof.** It is enough to apply Theorem 5.1 with \( s_i = S_i, r_i = R_i, \theta_i < \Theta_i \) and \( \gamma_i < \Gamma_i \) \( i = 1, 2 \) and \( \Theta_i, \Gamma_i \) obtained from (46) and (47), as discussed before.

**Remark 2.** Actually for Corollary 3, condition (20) and (21) are not optimal since the limit system for (49) is no more (61), but

\[
\begin{aligned}
-\Delta_p u &= u^{q_1}, & \text{in } \mathbb{R}^N, \\
-\Delta_q v &= u^{p_1}, & \text{in } \mathbb{R}^N.
\end{aligned}
\]

Consequently, the estimates (14) and (15) continue to be valid provided that a Liouville type theorem for (bounded) solutions of (62) is available. We refer a result in this direction to [5]. As in [26], conditions (20) and (21) can be replaced
by the sentence “Assume that (62) does not admit bounded nonnegative nontrivial solutions”.

Proof of Corollary 2. It is enough to apply Corollary 3 with \( p = q = 2 \).

Acknowledgments. R. Filippucci is member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). R. Filippucci was partly supported by the Italian MIUR project Variational methods, with applications to problems in mathematical physics and geometry (2015KB9WPT_009). The manuscript was realized within the auspices of the INdAM-GNAMPA Project 2018 titled Problemi nonlineari alle derivate parziali (Prot_U-UFMBAZ-2018-000384), and of the Fondo Ricerca di Base di Ateneo-Esercizio 2015 of the University of Perugia, titled Non esistenza di soluzioni intere.

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Received November 2018; revised November 2018.

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