On extensions of Myers’ theorem

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Abstract

Let $M$ be a compact Riemannian manifold and $h$ a smooth function on $M$. Let $\rho^h(x) = \inf_{|v|=1} (Ric_x(v, v) - 2Hess(h)_x(v, v))$. Here $Ric_x$ denotes the Ricci curvature at $x$ and $Hess(h)$ is the Hessian of $h$. Then $M$ has finite fundamental group if $\Delta^h - \rho^h < 0$. Here $\Delta^h =: \Delta + 2\nabla h$ is the Bismut-Witten Laplacian. This leads to a quick proof of recent results on extension of Myers’ theorem to manifolds with mostly positive curvature. There is also a similar result for noncompact manifolds.

An early result of Myers says a complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group. See e.g. [9]. Since then efforts have been made to get the same type of result but to allow a little bit of negativity of the curvature (see Bérard and Besson[2]). Wu [12] showed that Myers’ theorem holds if the manifold is allowed to have negative curvature on a set of small diameter, while Elworthy and Rosenberg [8] considered manifolds with some negative curvature on a set of small volume, followed by recent work of Rosenberg and Yang [10]. We use a method of Bakry [1] to obtain a result given in terms of the potential kernel related to $\rho(x) = \inf_{|v|=1} Ric_x(v, v)$, which gives a quick probabilistic proof of recent results on extensions of Myers’ theorem. Here $Ric_x$ denotes the Ricci curvature at $x$.

Let $M$ be a complete Riemannian manifold, and $h$ a smooth real-valued function on it. Assume $\text{Ric} - 2\text{Hess}(h)$ is bounded from below, where $\text{Hess}(h)$

*Research supported in part by NATO Collaborative Research Grants Programme 0232/87 and by SERC grant GR/H67263. 1991 Mathematical subject classification 60H30,53C21
is the hessian of $h$. Denote by $\Delta^h$ the Bismut-Witten Laplacian (with probabilistic sign convention) defined by: $\Delta^h = \Delta + 2L_{\nabla h}$ on $C^\infty_K$ the space of smooth differential forms with compact support. Here $L_{\nabla h}$ is the Lie derivative in direction of $\nabla h$. Then the closure of $\Delta^h$ is a negative-definite self-adjoint differential operator on $L^2$ functions (or $L^2$ differential forms) with respect to $e^{2h}dx$ for $dx$ the standard Lebesgue measure on $M$. We shall use the same notation for $\Delta^h$ and its closure. By the spectral theorem there is a heat semigroup $P^h_t$ satisfying the following heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2} \Delta^h u(x,t).$$

We shall denote by $P^h_t \phi$ the solution with initial value $\phi$. For clarity, we also use $P^{h,1}_t$ for the corresponding heat semigroup for one forms. Then for a function $f$ in $C^\infty_K$,

$$dP^h_t f = P^{h,1}_t(df). \quad (1)$$

On $M$ there is a h-Brownian motion $\{F_t(x) : t \geq 0\}$, i.e. a path continuous strong Markov process with generator $\frac{1}{2} \Delta^h$ for each starting point $x$. For a fixed point $x_0 \in M$, we shall write $x_t = F_t(x_0)$. Then $P^h_t f(x) = Ef(F_t(x))$ for all bounded $L^2$ functions.

Let $\{W^h_t(-), t \geq 0\}$ be the solution flow to the following covariant equation along h-Brownian paths $\{x_t\}$:

$$\begin{align*}
\frac{\partial}{\partial t} W^h_t(v_0) &= -\frac{1}{2} \text{Ric}_{x_t} (W^h_t(v_0), -)^\# + \text{Hess}(h)_{x_t} (W^h_t(v_0), -)^\#, \\
W^h_0(v_0) &= v_0, \quad v_0 \in T_{x_0}M.
\end{align*} \quad (2)$$

Here $^\#$ stands for the adjoint. The solution flow $W^h_t$ is called the Hessian flow.

Let $\phi$ be a bounded 1-form, then for $x_0 \in M$, and $v_0 \in T_{x_0}M$

$$E\phi(W^h_t(v_0)) = P^{h,1}_t \phi(v_0). \quad (3)$$

if $\text{Ric} - 2 \text{Hess}(h)$ is bounded from below. See e.g. [4] and [5].

Formula (3) gives the following estimates on the heat semigroup:

$$|P^{h,1}_t \phi| \leq |\phi|_{\infty} E|W^h_t|. \quad (4)$$
Let $\rho^h(x) = \inf_{|v|=1} (\text{Ric}_x(v,v) - 2\text{Hess}(h)(x)(v,v))$ and write $\rho$ for $\rho^h$ if $h = 0$. Then covariant equation (2) gives:

$$E|W^h_t| \leq E e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \quad (5)$$

as in [4].

Let $P_t^{\rho^h}$ be the $L^2$ semigroup generated by the Schrödinger operator $\frac{1}{2}\Delta_h - \frac{1}{2}\rho^h$. Then

$$P_t^{\rho^h} f(x_0) = E \left[ f(x_t) e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right]$$

by the Feynman-Kac formula. So equation (5) is equivalent to

$$E|W^h_t| \leq P_t^{\rho^h} 1.$$

Let $Uf$ be the corresponding potential kernel defined by:

$$Uf(x_0) = \int_0^\infty E \left[ f(x_t) e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right] dt.$$

Following Bakry’s paper [1], we have the following theorem:

**Theorem 1** Let $M$ be a complete Riemannian manifold with $\text{Ric} - 2\text{Hess}(h)$ bounded from below. Suppose

$$\sup_{x \in K} U1(x) < \infty \quad (6)$$

for each compact set $K$. Then $M$ has finite $h$-volume (i.e. $\int_M e^{2h(x)} dx < \infty$), and finite fundamental group.

**Proof:** We follow [1]. Let $f \in C^\infty_K$, then $Hf = \lim_{t \to \infty} P_t f$ is an $L^2$ harmonic function. Assume $h\text{-vol}(M) = \infty$, then $Hf = 0$. We shall prove this is impossible. Let $f, g \in C^\infty_K$, then:

$$\int_M (P_t^h f - f)ge^{2h} dx$$

$$= \int_M \int_0^t \left( \frac{\partial}{\partial s} P_s^h f \right) ge^{2h} dx ds$$

$$= \int_0^t \int_M \langle \nabla P_s^h f, \nabla g \rangle e^{2h} dx ds$$
\[
\leq |\nabla f|_\infty \int_M |\nabla g| \left( \int_0^t E|W^h_s| ds \right) e^{2h} dx, \\
\leq |\nabla f|_\infty \left( \sup_{x \in \text{sup}(g)} \int_0^\infty E|W^h_s| ds \right) |\nabla g|_{L^1}, \\
\leq c |\nabla f|_\infty |\nabla g|_{L^1}.
\]

Here \( c = \sup_{x \in \text{sup}(g)} [U1(x)] = \sup_{x \in \text{sup}(g)} \left( \int_0^\infty E \left( e^{-\frac{1}{2} \int_0^t \rho^h(F_s(x)) ds} \right) dt \right) \), and \( \text{sup}(g) \) denotes the support of \( g \).

Next take \( f = h_n \), for \( h_n \) an increasing sequence of smooth functions approximating 1 with \( 0 \leq h_n \leq 1 \) and \( |\nabla h_n| \leq \frac{1}{n} \), see e.g. [1].

Then
\[
\int_M (P^h_t h_n - h_n) e^{2h} dx \leq c \frac{1}{n} |\nabla g|_{L^1}.
\]

First let \( t \) go to infinity, then let \( n \to \infty \) to obtain:
\[
- \int_M e^{2h} dx \leq 0.
\]

This gives a contradiction with a suitable choice of \( g \). So we conclude \( h-Vol(M) < \infty \).

Let \( p: \tilde{M} \to M \) be the universal covering space for \( M \) with induced Riemannian metric on \( \tilde{M} \). For \( p(\tilde{x}) = x \), let \( \{ \tilde{F}_t(\tilde{x}), t \geq 0 \} \) be the horizontal lift of \( \{ F_t(x) \} \) to \( \tilde{M} \). Denote by \( \tilde{\text{Ric}} \) the Ricci curvature on \( \tilde{M} \), \( \tilde{h} \) the lift of \( h \) to \( \tilde{M} \), and \( \tilde{\rho}^h \) the corresponding lower bound for \( \tilde{\text{Ric}} - 2 \text{Hess}(\tilde{h}) \). Then the induced \( \{ \tilde{F}_t(\tilde{x}), t \geq 0 \} \) is a \( h \)-Brownian motion on \( \tilde{M} \). See e.g. [4]. Note also \( \tilde{\rho}^h \) satisfies
\[
\sup_{\tilde{x} \in \tilde{K}} \int_0^\infty E \left( e^{-\frac{1}{2} \int_0^t \tilde{\rho}^h(\tilde{F}_s(\tilde{x})) ds} \right) dt = \sup_{x \in p(\tilde{K})} \int_0^\infty E \left( e^{-\frac{1}{2} \int_0^t \rho^h(F_s(x)) ds} \right) dt
\]
for any \( \tilde{K} \subset \tilde{M} \) compact. The same calculation as above will show that \( \tilde{M} \) has finite \( h \)-volume, therefore \( p \) is a finite covering and so \( M \) has finite fundamental group.

Let \( r(x) \) be the Riemannian distance between \( x \) and a fixed point of \( M \) and take \( h \) to be identically zero:

**Corollary 2** Let \( M \) be a complete Riemannian manifold with
\[
\text{Ric}_x > - \frac{n}{n-1} \frac{1}{r^2(x)}, \quad \text{when } r > r_0
\]

\( \text{Corollary 2} \)
for some \( r_0 > 0 \). Then the manifold is compact if \( \sup_{x \in K} U1(x) < \infty \) for each compact set \( K \).

**Proof:** This is a consequence of the result of [3]: A complete Riemannian manifold with (7) has infinite volume. See also [11]. ■

For another extension of Myers' compactness theorem, see [3] where a diameter estimate is also obtained.

In the following we shall assume \( M \) is compact and get the following corollary:

**Corollary 3** Let \( M \) be a compact Riemannian manifold and \( h \) a smooth function on it. Then \( M \) has finite fundamental group if \( \Delta^h - \rho^h < 0 \).

**Proof:** Let \( \lambda_0 \) be the minimal eigenvalue of \( \Delta^h - \rho^h \). Then

\[
\lim_{t \to \infty} \frac{1}{t} \sup_M \log E \left( e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) \leq \lambda_0 < 0.
\]

See e.g. [7]. Thus there is a number \( T_0 \) such that if \( t \geq T_0 \),

\[
\sup_M E \left( e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) \leq e^{\lambda_0 t}.
\]

Therefore

\[
\sup_M \int_0^\infty E \left( e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) dt \\
\leq \int_0^{T_0} E \left( e^{-\frac{1}{2} \inf_{x \in M} [\rho^h(x)] t} \right) dt + \int_{T_0}^\infty E \left( e^{-\frac{1}{2} \int_0^t \rho^h(x_s) ds} \right) dt \\
< \infty.
\]

The result follows from theorem 1. ■

Let \( d_h = e^h de^{-h} \). It has adjoint \( \delta_h = e^{-h} \delta e^h \) on \( L^2(M, dx) \). Let \( \Box_h \) be the Witten Laplacian defined by:

\[
\Box_h = -(d_h + \delta_h)^2.
\]

By conjugacy of \( \Box_h \) on \( L^2(M, dx) \) with \( \Delta^h \) on \( L^2(M, e^{2h} dx) \), the condition "\( \Delta^h - \rho^h < 0 \)" becomes:

\[
\Box_h - \rho^h < 0
\]
on $L^2(M, dx)$. On the other hand,

\[ \Box_h = \Delta - ||dh||^2 - \Delta h. \]

See e.g. [6]. This gives: a compact manifold has finite fundamental group if

\[ \Delta - ||dh||^2 - \Delta h - \rho^h < 0 \]

on $L^2(M, dx)$.

Corollary 3 leads to the following theorem from [10]: Let $\mathcal{N} = \mathcal{N}(K, D, V, n)$ be the collection of $n$-dimensional Riemannian manifolds with Ricci curvature bounded below by $K$, diameter bounded above by $D$, and volume bounded below by $V$.

**Corollary 4 (Rosenberg& Yang)** Choose $R_0 > 0$. There exists $a = a(\mathcal{N}, R_0)$ such that a manifold $M \in \mathcal{N}$ with $\text{vol}\{x : \rho(x) < R_0\} < a$ has finite fundamental group. Here ”vol” denotes the volume of the relevant set.

**Proof:** Let $h = 0$ in corollary 3. Then under the assumptions in the corollary, $\Delta - \rho < 0$ according to [8]. The conclusion follows from corollary 3. \hfill \blacksquare

**Acknowledgement:** The author is grateful to Professor D. Elworthy and Professor S. Rosenberg for helpful comments and encouragement.

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