Phases of a two dimensional large-$N$ gauge theory on a torus

Gautam Mandal$^1$ and Takeshi Morita$^2$

$^1$Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai 400 005, INDIA

$^2$Crete Center for Theoretical Physics
Department of Physics
University of Crete, 71003 Heraklion, Greece

email: mandal@theory.tifr.res.in, takeshi@physics.uoc.gr

Abstract

We consider two-dimensional large $N$ gauge theory with $D$ adjoint scalars on a torus, which is obtained from a $D+2$ dimensional pure Yang-Mills theory on $T^{D+2}$ with $D$ small radii. The two dimensional model has various phases characterized by the holonomy of the gauge field around non-contractible cycles of the 2-torus. We determine the phase boundaries and derive the order of the phase transitions using a method, developed in an earlier work (hep-th/0910.4526), which is nonperturbative in the 'tHooft coupling and uses a $1/D$ expansion. We embed our phase diagram in the more extensive phase structure of the $D+2$ dimensional Yang-Mills theory and match with the picture of a cascade of phase transitions found earlier in lattice calculations (hep-lat/0710.0098). We also propose a dual gravity system based on a Scherk-Schwarz compactification of a D2 brane wrapped on a 3-torus and find a phase structure which is similar to the phase diagram found in the gauge theory calculation.


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1 Introduction and Summary

Gauge theories on spaces with compact directions have been studied for a long time. As a prototypical example, $d + 1$ dimensional Yang-Mills theory at a finite temperature $T$ corresponds to a compactification of the (Euclidean) time direction on a circle of length $\beta = 1/T$. It is obviously important to study such a compactification to understand the physics of confinement/deconfinement transitions [1]. More generally, one can consider Yang-Mills theory on a compact space $\Sigma$. If the volume of $\Sigma$ is finite, there is no sharp phase transition, but for an $SU(N)$ gauge theory in the large $N$ limit there are sharply demarcated phases depending on the shape and size parameters of $\Sigma$. In case the compact space is a torus, the phase diagram as a function of various radii (and coupling) reveals a rich phase structure [2, 3], including a cascade of phase transitions in which the “Polyakov” loops along various non-contractible cycles become non-zero in succession as the radii are reduced [4, 5]. Most of these studies are numerical (in the lattice or in the continuum) or, in some cases, based on holography (see Section 6 for references and more details). One of the motivations of the present paper is to investigate these questions analytically in a simple situation, as explained below, by using and extending the “large $D$” technique developed in [6].

To elaborate further, let us consider a Euclidean $d + D$-dimensional gauge theory on a $d + D$-dimensional torus with radii $L_\mu$.

$$S = \int_0^\beta dt \left( \prod_{M=1}^{D+d-1} \int_0^{L_M} dx^M \right) \frac{1}{4g_{d+D}^2} \text{Tr} F_{\mu\nu}^2. \quad (1)$$

Here the length of the temporal circle is denoted as $L_0 = \beta$ and the rest are denoted as $L_M, M = 1, \ldots, d + D - 1$. The phases of (1) are characterized by Wilson lines around the $d + D$ noncontractible cycles of the torus:

$$W_\mu = \text{Tr} U_\mu \equiv \frac{1}{N} \text{Tr} P \left( \exp \left[ i \int_0^{L_\mu} A_\mu dx^\mu \right] \right), \quad (2)$$

1In this paper, we will not consider the contribution of the $\theta$ term.

2Our notation for spacetime coordinates is: $\{x^0 \equiv t, x^M\}, M = 1, \ldots, d + D - 1$. We will further split the $d + D - 1$ coordinates into $d$ ‘large’ dimensions $\{x_0, x^i\}, i = 1, \ldots, d - 1$ and $D$ ‘small’ dimensions $x^I, I = 1, 2, \ldots D$ (the meaning of ‘large’ and ‘small’ is explained below).
where no sum over $\mu$ is intended. These Wilson loops transform nontrivially under the centre symmetry\(^3\). For sufficiently large radii $L_\mu$, all $W_\mu$ vanish, signifying unbroken centre symmetry. In this phase, local gauge-invariant observables are independent of $L_\mu$ in the strict large $N$ limit \cite{3,10,11}. Since $W_0$ can be interpreted as $\exp[-S_q]$, where $S_q$ is the action for a static quark, the phase with $\langle W_0 \rangle = 0$ exhibits confinement. As is well-known, as $\beta$ is reduced (i.e. the temperature is increased), below a certain critical value $\beta_c$, $\langle W_0 \rangle$ becomes non-zero, signaling a deconfinement transition together with a breaking of the centre symmetry $Z_N^{d+D} \rightarrow Z_N^{d+D-1}$. In this phase, the observables can depend on $\beta$ but are still independent of $L_i$ \cite{10}. It has been argued from lattice studies (see \cite{3} and Section 6 for a review) that as the other radii are successively reduced, one has a cascade of analogous symmetry breaking transitions $Z_N^{d+D-1} \rightarrow Z_N^{d+D-2} \rightarrow \ldots \rightarrow 1$.

While it would be fascinating to study all the above phases analytically, in this paper we will be able to study the phases of a $D + 2$ dimensional pure Yang Mills theory on $T^{D+2}$ (i.e. \cite{11} with $d = 2$) in which a $D$-dimensional torus (with radii $L^i/(2\pi)$, $i = 1, 2, \ldots, D$) is taken as small (ensuring broken $Z_N$ symmetries in those directions), leaving the remaining $d = 2$ directions (including time) of variable size. Such a theory is given by a Kaluza-Klein reduction\(^4\) of \cite{11} on the small $T^D$, and is described by the following action:

$$S = \int_0^\beta dt \int_0^L dx \langle Tr \left( \frac{1}{2g^2} F_{01}^2 + \sum_{I=1}^{D} \frac{1}{2} (D_\mu Y^I)^2 + \frac{m^2}{2} (Y^I)^2 \right. \left. - \sum_{I,J} \frac{g^2}{4} [Y^I, Y^J] [Y^I, Y^J] - \sum_{I,J} \frac{g}{2} \frac{g'}{2} [Y^I, Y^J] \right) \rangle. \quad (3)$$

Here $Y^I$ comes from the gauge field components $A_{I+1}$ and the covariant derivative is defined as $D_\mu = \partial_\mu - i[A_\mu, \cdot]$. A naive KK reduction leads to

\(^3\)Centre symmetry \cite{7,8} is generated by quasiperiodic ‘gauge transformations’ $\alpha(x^\mu) = \exp[2\pi i(n_\mu x^\mu/L_\mu)A]$, where $A = $ diag$[1/N, 1/N, \ldots, 1/N, (1-N)/N]$. The quasiperiodicity is up to phases $h_\mu = \exp[2\pi i n_\mu/N]$, $\mu = 0, 1, \ldots, d+D-1$ which parametrize $d+D$ copies of the centre of $SU(N)$, $Z_N^{d+D}$. The $\alpha(x^\mu)$ are valid gauge transformations locally, and leave local colour-singlets e.g. $tr F_{\mu\nu}^2$ invariant; in particular they commute with the hamiltonian. However, under the $\alpha$-transformations $W_\mu \rightarrow h_\mu W_\mu$. A non-zero value of $\langle W_\mu \rangle$ implies spontaneous symmetry breaking of the centre symmetry in the $\mu$-direction.

\(^4\)Kaluza Klein reduction is tricky for gauge theories \cite[10,11], since in the confined phase the KK modes can have energies $\sim 1/(NL)$, which become arbitrarily low at large $N$. The fractional modes, equivalent to the ‘long string’ modes of \cite{12}, can be understood as arising from mode shifts of charged fields in the presence of Wilson lines whose eigenvalues are uniformly distributed along a circle (see Section 2 for an explicit verification for this statement). In the deconfined phase, however, the KK modes have energies $\sim 1/L$, like in ordinary field theories, and KK reduction proceeds as usual.
massless $Y^I$’s and $g = g'$; however, a mass $m$ for the adjoint scalars as well as radiative splitting between $g$ and $g'$ is induced from loops of KK modes (see Appendix A and Section 2, respectively, for more details).

We should remark that (3) can either be regarded as a step towards understanding the full phase diagram of the $d + D$ dimensional theory (1), or be understood as a two-dimensional gauge theory in its own right. The spirit of the latter approach is to provide an example of an analytically solvable low-dimensional gauge theory in the limit of a large number of adjoint scalars (the $d = 1$ theory was discussed in [3]). Equation (3), from this viewpoint, provides a bosonic counterpart of the Gross-Neveu model [13] where $D$ plays the role of $N_f$, and the $SO(D)$-invariant bilinear $\sum_{I=1}^{D} Y^I a I$ of Section 2.1 plays the role of $SU(N_f)$-invariant fermion bilinears such as $\sum_{i=1}^{N_f} \bar{\psi}_i \psi_i$ of the Gross-Neveu model.

Note that in this second point of view, where we regard the action (3) as an independent theory in its own right, the mass $m$ can be taken to be arbitrary. In particular, if the mass is sufficiently large ($m^2 \gg g^2 N D, g'^2 N D$), the phase structure can be determined perturbatively [14, 3]. However, as we will see in appendix A, if we regard (3) as a KK reduction of (1) on $T^D$, then the mass $m$ of the adjoint scalars is much lower than the scales mentioned above and the theory is not amenable to such perturbative methods. One of the goals of the present work is to provide a nonperturbative\(^5\) analysis of (3) valid for any value of mass (including $m = 0$), based on the ‘large $D$’ method developed in the previous work [6].

A few additional comments are in order:

(i) KK gauge theories have important applications to phenomenology \([15, 16, 17, 18]\). Theories such as (3) provide important toy models in this context. In particular, issues such as different running of gauge couplings in the compactified and decompactified theories can be examined in such models. We will encounter some of these issues in Section 2.

(ii) Gauge theories on compact spaces can sometimes have gravity duals. The deconfinement transition in $\mathcal{N} = 4$ super Yang Mills theory on $S^3 \times S^1$ \([19, 20]\) is a weak coupling continuation of the gravitational Hawking-Page transition \([21, 22]\). Similar correspondences for two-dimensional supersymmetric gauge theories on tori were analyzed in \([23, 34]\). In this paper, we will obtain (3) with $D = 8$ from a Scherk-Schwarz compactification of a three dimensional super Yang-Mills theory with sixteen supercharges, which corresponds to the world volume theory of D2 branes. The latter theory has

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\(^{5}\)in the ’tHooft coupling.

\(^{6}\)For an extensive list of correspondences between low dimensional gauge theories and gravitational systems, see \([24, 25, 26]\).
an AdS/CFT dual [24], which leads to the construction of a gravity dual for (3) in a sense defined in Section 5. As we will see, the gravity analysis will complement our knowledge of the phase structure from the gauge theory analysis.

(iii) Large $N$ two dimensional gauge theories themselves are interesting objects in the context of string theory and QCD. For instance, confinement/deconfinement type transitions [28] have been analytically found in 2D QCD with heavy adjoint scalars [14]. In addition, stringy excitations and glueball spectra have been obtained in 2D models in [29, 30, 51, 32, 33]. Thus, these models are good laboratories for real QCD. Our study, in fact, has a direct relevance for [32]; we hope to return to the issue of glueball spectrum discussed in this reference.

The principal result in this paper is the determination of some parts of the phase diagram of the two-dimensional theory (3) at weak coupling. The result is summarized in figure 4. The second result is the gravity analysis which complements the phase diagram at strong coupling, which is presented in figure 5.

The plan of the paper is as follows.

In Sections 2, 3 and 4 we analyse the model (3) at weak coupling by using the $1/D$ expansion [34] developed in [6]. We find that the nature and the order of the confinement/deconfinement type transition depends on whether the size $L$ of the spatial circle is large or small. For large $L$ (which corresponds to the $\text{Tr}V = 0$ phase), we find (see Section 2) a single first order transition, thus providing analytical evidence for earlier lattice studies (see Section 5.1 for further details). On the other hand for small $L$ (corresponding to the $\text{Tr}V \neq 0$ phase), the analysis in [6] is valid and, as detailed in Section 3, the transition consists of two higher order phase transitions. The phase diagram is summarised in Section 4 in figure 4 and is in agreement with those from the lattice studies of [2, 5, 4]. We compare our results with these lattice studies and with [3] in Section 5.

To supplement our $1/D$ analysis of the gauge theory, we consider in Section 5 a dual gravity theory, obtained from D2 branes wrapped on a 3-torus with a Scherk Schwarz circle. Although the gravity results pertain to thermodynamics in the strongly coupled regime, we observe that the phase structure is qualitatively similar to that of the weak coupling gauge theory. This allows us to arrive at a conjectured phase diagram in figure 5, which suggests a particular way of connecting the phase boundaries of figure 4.

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7One of the motivations for this work was to construct a gauge theory dual to a dynamical Gregory-Laflamme transition. We will discuss this issue in a forthcoming publication [27].
In Section 6.3 we comment on the dependence of the order of phase transition on the topology of the compact space. In Appendix A we discuss the masses of the adjoint scalars which appear from integration of the KK modes at the one loop level. In Appendix B we fill in some details needed in Section 2 for integrating out the adjoint scalars. In Appendix C we discuss the influence of the mass term in (3) on the phase diagram. In Appendix D we provide important details of our gravity analysis.

2 Confinement/deconfinement type transition in large radius torus

In this section, we will analyse the confinement/deconfinement type transition in (3) for large \(L\). First, (in Section 2.1) we will integrate out the adjoint scalars \(Y^I\) in a \(1/D\) expansion, in a manner similar to \([6]\), leading to the effective hamiltonian (23) in the large \(L\) limit. Next (in Section 2.2) we use, at \(L \to \infty\), the results of \([14, 35, 36]\), who studied this effective action in slightly different contexts, to determine the phase structure of our theory (3) at large \(L\). The justification for extrapolating their result to finite \(L\), as detailed below, comes from the phenomenon of large \(N\) volume independence \([9, 10, 11]\) which is valid as long as \(L\) is large enough to ensure \(\text{Tr} V = 0\).

To keep the analysis simple, in this section we consider (3) with \(m = 0\), and defer the case of nonzero mass to appendix C. As we will find there, the inclusion of the mass term does not change the qualitative structure of the different phases.

2.1 Large \(D\) saddle point

In this section, we will generalize the analysis of the \(0 + 1\)-dimensional gauge theory \([6]\) to the \(d = 2\) model (3) and show that if we consider the number \(D\) of adjoint scalars to be large, the theory can be considered to be in the vicinity of a large \(D\) saddle point \([5]\) and various quantities such as the free energy and the mass gap etc. can be computed around the saddle point in a \(1/D\) expansion.

As mentioned above, in this section we will consider (3) with \(m = 0\). Introducing an auxiliary field \(B_{ab}\), the path-integral of the gauge theory can

\[\text{Tr} V = 0\]
be rewritten as
\[
Z = N \int DBDA_\mu D Y^I e^{-S(B, A, Y)},
\]
\[
S(B, A, Y) = \int_0^\beta \int_0^L dt \int_0^L dx \left[ \frac{1}{2 g^2} F_{01}^2 + \frac{1}{2} \left( D_\mu Y^I_a \right)^2 - i \frac{1}{2} B_{ab} Y^I_a Y^I_b + \frac{1}{4 g^2} B_{ab} M^{-1}_{ab,cd} B_{cd} \right],
\]
where \( N \) is a constant factor and we have used the following matrix,
\[
M_{ab,cd} = -\frac{1}{4} \left\{ \text{Tr}[\lambda_a, \lambda_c][\lambda_b, \lambda_d] + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d) \right\},
\]
\( \lambda^a (a = 1, \cdots, N^2 - 1) \) being the generators of \( SU(N) \). Our approach, similar to the one-dimensional case, will be as follows. We will integrate out the \( Y^I \)’s to obtain an effective action for \( A_\mu \) and \( B_{ab} \), and find a saddle point solution for \( B_{ab} \) (for given \( A_\mu \)) in a large \( D \) limit. The effective action for \( A_\mu \) will be essentially obtained by substituting the saddle point value of \( B_{ab} \) in this effective action.

As in [6], it is convenient to decompose \( B_{ab} \) as the sum of a trace piece (independent of \( x, t \)) and an orthogonal part:
\[
B_{ab}(t) = i \Delta^2 \delta_{ab} + g' b_{ab}(t, x),
\]
where \( b_{ab} \) satisfies \( \int dt \int dx b_{aa} = 0 \). Such a decomposition, into a large diagonal piece and a small off-diagonal fluctuation, will be \textit{a posteriori} justified by finding a saddle point for \( B_{ab} \) of the form \( \langle B_{ab} \rangle = i \Delta_0^2 \delta_{ab} \), where \( \Delta_0 \) is a real constant (depending only on the ’tHooft coupling).

With this decomposition, the action reduces to
\[
S_0 = -\frac{\beta LN \Delta^4}{8 g^2},
\]
\[
S_1 = \int_0^\beta \int_0^L dt \int_0^L dx \left( \frac{1}{2 g^2} F_{01}^2 + \frac{1}{4} b_{ab} M^{-1}_{ab,cd} b_{cd} + \frac{1}{2} \left( D_\mu Y^I_a \right)^2 + \frac{1}{2} \Delta^2 Y^I_a \right),
\]
\[
S_{int} = -\int_0^\beta \int_0^L dt \int_0^L dx \left( \frac{i g'}{2} b_{ab} Y^I_a Y^I_b \right),
\]
where we have used \( M^{-1}_{ab,cd} \delta_{cd} = \delta_{ab}/2N \). Let us now take a large \( D \) limit,
\[
g, g' \to 0, \quad N, D \to \infty \quad s.t. \quad \tilde{\lambda} \equiv g^2 DN, \quad \tilde{\lambda}' \equiv g'^2 DN \quad \text{fixed.}
\]
To the leading order of this expansion, we can ignore the interaction term $S_{\text{int}}$. In that case, the integration of $b_{ab}$ will contribute just a numerical factor and we will ignore it. Integrating over the $Y^I$'s in $S_1$, we then get the following leading result for the partition function

$$Z = \int \mathcal{D}A \mathcal{D}\Delta e^{-S_{\text{eff}}[A,\Delta]},$$

$$S_{\text{eff}}[A,\Delta] = -\frac{\beta LN\Delta^4}{8g^2} + \int_0^\beta dt \int_0^L dx \frac{1}{2g^2} F_{01}^a F_{01}^a + \delta S_{\text{eff}},$$

where $\delta S_{\text{eff}}$ is the 1-loop contribution from the $Y^I$-integration:

$$\delta S_{\text{eff}}[A,\Delta] = \frac{D}{2} \log \det (-D_\mu^2 + \Delta^2).$$

It is difficult to evaluate this last quantity in general. However, using arguments similar to [3], we will find that:

(a) If $\Delta^2 \gg \tilde{\lambda} \equiv g^2 DN$ (this assumption will be justified in (19)), terms in $\delta S_{\text{eff}}$ involving derivatives of $A_\mu$, e.g. terms involving the gauge field strength and its covariant derivatives are suppressed.

(b) If $L\Delta \gg 1^{10}$ we can approximately treat $A_0$ and $A_1$ as commuting matrices.

The argument for assertion (a) can be sketched briefly as follows. Consider the simplest of the 1-loop diagrams (figure 1) which contribute to (10) and has only two external gauge field insertions:

![Figure 1: A simple Feynman diagram contributing to (10).](image)

9In the one dimensional model ($d = 1$), the next order of the $1/D$ expansion has been evaluated in [6]. There, such $1/D$ corrections do not change the nature of the phase structure. We can expect that the same thing will happen in our two dimensional gauge theory also. Hence we do not evaluate the $1/D$ corrections in this article.

10This assumption will be justified later in what we will define as the ‘large $L$ regime’ $L \gtrsim L_c$ (see [33] and [32]), since $L_c\Delta$ will turn out be large, using (19).
For large $\Delta$, the $Y^I$-propagator in the loop carrying momentum $p$ can be expanded in powers of $p^2/\Delta^2$. The first term in that expansion goes as $ND(k_\mu k_\nu - k^2 g_{\mu\nu})/\Delta^2$ (in the Feynman gauge), where the factors of $N$ and $D$ come from the $Y^I$-loop, and $k$ is the external momentum going into the loop. This term amounts to a correction, to the $F^{20}_{01}/g^2$ term, of the form $(1 + O(\tilde{\lambda}/\Delta^2))$, as claimed above. We will ensure below that $\tilde{\lambda}/\Delta^2 \ll 1$.

The argument for assertion (b) will follow a posteriori after we proceed with the assumption that $A_\mu$ are constant commuting matrices. Under this assumption, as detailed in appendix B, the one-loop term becomes

$$\delta S_{\text{eff}} = \frac{D\beta L}{8\pi^2} \left[ N^2 (-\pi \Lambda^2 + \pi (\Lambda^2 + \Delta^2) \log (\Lambda^2 + \Delta^2) - \pi \Delta^2 \log \Delta^2) ight. \right.$$

$$- \left. \sum_{(k,l) \neq (0,0)} |\text{Tr}(U^k V^l)|^2 \frac{4\pi\Delta}{\sqrt{16k^2 + (Ll)^2}} K_1 \left( \Delta \sqrt{16k^2 + (Ll)^2} \right) \right],$$

(11)

Here we have used a momentum cut-off $\Lambda$ to regulate the determinant and used the notation $U = e^{i\beta A_0}$ and $V = e^{iLA_1}$. $K_1$ is the modified Bessel function of the second kind. Using the asymptotic expansion $K_1(z) = \sqrt{\pi/2} e^{-z} + \cdots$ for large $z$ (justified below) and omitting some irrelevant divergent terms, we get

$$\delta S_{\text{eff}} = \frac{DN^2\beta L}{8\pi} \left[ (\Lambda^2 + \Delta^2) \log \left( 1 + \frac{\Lambda^2}{\Delta^2} \right) + \Delta^2 \log \left( \frac{\Lambda^2}{\Delta^2} \right) \right]$$

$$- \frac{D}{\sqrt{2\pi}} \left[ L \sqrt{\frac{\Delta}{\beta}} e^{-\Delta \beta} |\text{Tr}U|^2 + \beta \sqrt{\frac{\Delta}{L}} e^{-\Delta L} |\text{Tr}V|^2 \right] + \cdots,$$

(12)

where the $\cdots$ terms represent higher order terms in $e^{-\Delta \beta}, e^{-\Delta L}$. Since we are interested in large $L$, it is obvious why higher order terms in $e^{-\Delta L}$ should be ignored. We now have an a posteriori justification for having ignored terms involving commutators $[U, V]$; the smallest gauge invariant such term would have at least 2 $U$s and 2 $V$s and hence are expected to be of the same order as the $U^2 V^2$ term in (11), and hence can be ignored. Higher order terms in $e^{-\Delta \beta}$ are ignored because they will be small in the region of parameter space in which interesting phase structures will appear. We will ensure this at the end of Section 2.2 (see comment (a) below).

\footnote{The cut off $\Lambda$ should be smaller than $M_{KK}$, which is the inverse length scale of the $D$-dimensional compactification torus.}
In the $L \to \infty$ limit we can ignore the term in (12) involving $V$. Let us integrate $\Delta$ from (9) under this assumption and derive the effective action for the $U$-variable. The saddle point equation with respect to $\Delta^2$ reads as

$$\frac{-\beta L \Delta^2}{2\lambda'} + \frac{\beta L}{4\pi} \log \left(1 + \frac{\Lambda^2}{\Delta^2}\right) + \frac{L}{\sqrt{2\pi}} \sqrt{e^{-\Delta\beta} \left(1 - \frac{1}{2\Delta\beta}\right)} \left|\frac{1}{N} \text{Tr} U\right|^2 + \cdots = 0.$$  \hspace{1cm} (13)

In the $\text{Tr} U = 0$ phase (which is realized for sufficiently large $\beta$), $\Delta = \Delta_0$ is determined implicitly from the equation

$$\tilde{\lambda}' = \frac{2\pi \Delta_0^2}{\log \left(1 + \frac{\Lambda^2}{\Delta_0^2}\right)}.$$  \hspace{1cm} (14)

This equation can be viewed as a renormalization condition which assigns a $\Lambda$-dependence to $\tilde{\lambda}'$ such that the physical mass scale $\Delta_0$ is held fixed. Here $\Delta_0$ plays a role analogous to $\Lambda_{QCD}$ and its choice specifies the theory. We will, in fact, choose it in such a way as to ensure the condition

$$\tilde{\lambda} \sim \tilde{\lambda}' \text{ at } \Lambda = M_{KK}. \hspace{1cm} (15)$$

As $\beta$ decreases, $\text{Tr} U$ eventually becomes non-zero. However, near criticality $\beta \Delta \gg 1$ (this is justified because of (19) and (32)). Therefore, we can solve (13) for $\Delta$ in the form $\Delta_0 + O(\exp[-\beta \Delta_0])$. We will write the explicit solution only for $\Lambda \gg \Delta$:

$$\Delta = \Delta_0 + \frac{1}{2\pi \Delta_0^2/\lambda' + 1} \sqrt{\frac{2\pi \Delta_0}{\beta} e^{-\Delta_0 \beta} \left|\frac{1}{N} \text{Tr} U\right|^2 + \cdots}.$$ \hspace{1cm} (16)

where

$$\Delta_0 = \sqrt{\frac{\tilde{\lambda}'}{2\pi W} \left(\frac{2\pi \Lambda^2}{\lambda'}\right)}, \hspace{1cm} (17)$$

12 This follows from the fact that the distinction between $g$ and $g'$ in (3) vanishes in the original pure YM theory (1). In order to fix the precise coefficient of the renormalization condition (15), we need to evaluate the contribution of the mass (52) and the running of $g$ and $g'$ for scales larger than $M_{KK}$. The value of mass and the running of the couplings depend on the details of the higher dimensional theory. If we are interested in the two dimensional gauge theory (3) itself as in the comment (iii) in the introduction, the renormalization condition that replaces (15) is arbitrary. However, even in that case, the qualitative nature of the phase structures in this paper does not change as long as $(\Delta_0^2 + m^2)/\tilde{\lambda} \gg 1$ is satisfied.
and $W(z)$ is the Lambert’s W function defined implicitly by the equation $z = W(z)e^{W(z)}$. Note that, by using an expansion $W(z) = \log z - \log(\log z) + \cdots$ for large $z$, we obtain

$$
\Delta_0 = \sqrt{\frac{\lambda'}{2\pi} \log \left( \frac{2\pi \Lambda^2}{\lambda'} \right)} + \cdots. \quad (18)
$$

Imposing the renormalization condition \[(15),\] we then obtain a relation\[13]

$$
\Delta_0^2 / \lambda' \sim 1 \frac{1}{2\pi} \log \left( \frac{2\pi M^2_{KK}}{\lambda} \right) \gg 1.
$$

(19)

Thus we confirmed that the assertion (a) above is satisfied. Substituting \[(16)\] in $S_{eff}$ in \[(9)\] and ignoring the term involving $\text{Tr} V$, we get the following effective action for $A_\mu$:

$$
\frac{S(A)}{DN^2} = C(\lambda', \Delta_0) + \int_0^\beta dt \int_0^L dx \left( \frac{1}{2\lambda N} F_{01}^a \right)^2 - \frac{1}{\beta} \sqrt{2\pi} \frac{\Delta_0}{\lambda'} e^{-\Delta_0^2} \left| \frac{1}{N} \text{Tr} U \right|^2 + \cdots,
$$

(20)

$$
C(\lambda', \Delta_0) = \frac{\beta L \Delta_0^2}{8\pi} \left( 1 + \frac{\pi \Delta_0^2}{\lambda'} \right), \quad (21)
$$

where the terms $\cdots$ are higher order in the same sense as in \[(12)\]. We have used $\int dx dt |\text{Tr} U|^2 = L\beta |\text{Tr} U|^2$ which is correct up to derivatives of $U$ which occur at $O(\lambda' / \Delta_0^2)$ and are small, as argued above.

In the limit of $L \to \infty$\[14\] we can choose the gauge $A_1 = 0$. Solving the Gauss’s law condition in this gauge, we get $A_0 = (g^2 / \partial_x) \phi$ where $\phi \equiv -1/2 [Y^i, \partial Y^I]$ is the charge density; in the large $D$ saddle point, especially for large enough $\Delta_0$, the condensate is static and temporal fluctuations of $\phi$, and consequently, of $A_0$, are suppressed. As a result, we can write

$$
\int dx \, dt \, \text{Tr} F_{01}^2 = \beta \int dx \, \text{Tr} (\partial_x A_0)^2 = \beta^{-1} \int dx \, \text{Tr} |\partial_x U|^2,
$$

\[13\] The inequality follows from the condition $\tilde{\lambda} / M^2_{KK} \ll 1$, which is necessary for a KK reduction. This condition implies that the dimensionless 'tHooft coupling is small: $\lambda_D + 2 M^2_{KK} \ll 1$.

\[14\] Although we are apparently taking $L \to \infty$ here, as we discuss at the end of the next subsection, we can extend these results to finite $L$ which is large enough to ensure vanishing of $\text{Tr} V$. Note that if $\text{Tr} V \neq 0$, the $1/D$ expansion does not work at $L \to \infty$ as argued in \[37\], where such a situation is discussed in the presence of $R$-symmetry chemical potential.
where

\[ U(x) = P \exp[i \int_0^\beta dt A_0(x,t)] = \exp[i \beta A_0(x)]. \]  

(22)

By rescaling \( x \to x' = \tilde{\lambda} \beta x \) we eventually get an effective action in terms of \( U(x) \)

\[ S/DN^2 = C(\tilde{\lambda}', \Delta_0) + \int_{-\infty}^\infty dx \left[ \frac{1}{2N} \Tr \left( |\partial_x U|^2 \right) - \frac{\xi}{N^2} |\Tr U|^2 \right], \]  

(23)

where

\[ \xi = \sqrt{\frac{\Delta_0}{2\pi \lambda^2 \beta^3}} e^{-\Delta_0 \beta}. \]  

(24)

Note that \( \xi \) is a monotonically decreasing function of \( \beta \).

Equations similar to (23), (24) have earlier been derived in [14] who consider a two-dimensional gauge theory with heavy adjoint scalars of mass \( m \). In their equations \( m \) appears in place of \( \Delta_0 \). We could have, in fact, derived the above effective action (23) as follows: the large \( D \) saddle point generates a dynamical mass \( \Delta_0 \) for the adjoint scalars which turns the theory into a massive adjoint scalar QCD; once this is established, we can use the method of [14] with \( m = \Delta_0 \), to arrive at (23). The agreement with the results of [14] provides an additional check on our derivation. Adjoint scalar QCD with large scalar mass has also been considered by [3] who have independently derived equations analogous to (23), (24); our method of derivation follows their derivation closely, except that our mass is dynamically generated, as mentioned above. The discussion in appendix C involving arbitrary \( m \) relates the two extreme cases of large mass and zero mass.

### 2.2 The phase transition at Large \( L \)

Phase transitions in the system (23) have been discussed in [14, 35, 36]. We will adopt their result to infer about phase transitions in our two-dimensional gauge theory (3) at large \( L \) (the range of \( L \) is defined in (33)). For completeness, we will briefly review some of the results in these papers.

If we regard the coordinate \( x \) in (23) as time, it becomes the quantum mechanics of a single unitary matrix, a subject that has been extensively studied [38, 39, 41]. Phases of such a model can be described by the behaviour of the eigenvalue density

\[ \rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(\theta - \theta_i(x)), \]  

(25)
where \( \exp (i \theta_i(x)) \) are the eigenvalues of (22).

The hamiltonian of this system (regarding \( x \) as time) can be written as

\[
H = \int d\theta \left( \frac{1}{2} \rho v^2 + \frac{\pi^2}{6} \rho^3 - \xi |u_1|^2 - \frac{1}{24}, \right) \tag{26}
\]

where we have ignored the constant term \( C \) in (23). Here \( v = \partial_\theta \Pi \), and \( \Pi(\theta, x) \) is the canonical conjugate of \( \rho(\theta, x) \). We have also used the notation \( u_n = \frac{1}{N} \text{Tr} U^n, u_{-n} = (u_n)^* \). Note that \( u_n(x) \) are moments of the eigenvalue density (25):

\[
\rho(\theta, x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} u_n(x) e^{-in\theta}. \tag{27}
\]

To study the various equilibrium phases of the system, we study static solutions of (26), which [14, 35, 36] are given by \( v(\theta) = 0 \) and

\[
\rho(\theta) = \frac{\sqrt{2}}{\pi} \left( \sqrt{E + 2\xi \rho_1 \cos \theta} \right), \tag{28}
\]

where \( \rho_1 = u_1 = u_1^* \) is the first moment (real in this case), which must self-consistently satisfy (see (27))

\[
\int_0^{2\pi} d\theta \rho(\theta) \cos \theta = \rho_1. \tag{29}
\]

The constant \( E \) is fixed by solving the normalization condition

\[
\int_0^{2\pi} d\theta \rho(\theta) = 1. \tag{30}
\]

Depending on the value of the constant \( \xi \) in (26), Eqn. (30) may not determine \( E \) uniquely. In general, we obtain three branches \( E(\xi) \), depending on whether (28) describes a uniform, non-uniform or gapped eigenvalue distribution (see figure 2).
Figure 2: Configurations of eigenvalue density $\rho(\theta)$ in the unitary matrix model. The left plot is the uniform distribution (corresponding to $\rho_1 = 0$ in (28)), the middle one is the non-uniform distribution ($|E/(2\xi\rho_1)| \geq 1$) and the right one is the gapped distribution ($|E/(2\xi\rho_1)| \leq 1$).

The value of $E(\xi)$ in these three cases are different. The thermodynamic stability for the various branches of the function $E(\xi)$ is analyzed [14, 35, 36] by comparing the values of the Euclidean Hamiltonian (26) which can also be regarded as the free energy. They can be summarised as follows:

- Independent of the value of $\xi$, the uniform solution always exists. We call this phase as I.
- For $\xi < \xi_0 = 0.227$, only one solution (phase I) exists and is stable.
- At $\xi = \xi_0$, there is nucleation of two gapped solutions. One is unstable (phase II) and another is meta-stable (phase III).
- At $\xi = \xi_1 = 0.23125$, a GWW type phase transition [45, 38] occurs in phase II and the gapped solution becomes a solution with non-uniform distribution (Phase IV).
- At $\xi = \xi_2 = 0.237$, there is a first order phase transition between the phases I and III. Above $\xi_2$, the phase III is stable and the phase I is meta-stable.
- At $\xi = \xi_3 = 1/4$, phase IV merges into phase I, and the uniform solution becomes unstable beyond $\xi_3$.

These are summarised in figure 3.
Figure 3: Free energy \( F \) vs \( \xi \) in the four phases. The gapped and non-uniform solutions here are numerically evaluated. Since \( \xi \) is a monotonically increasing function of temperature (see (24)), the uniform distribution (Phase I) is stable at low temperatures and the gapped distribution (Phase III) is stable at higher temperature. A first order phase transition between these two phases happens at \( \xi_2 \).

Using (24), we can read off the critical temperatures corresponding to these transition points:

\[
\beta_m \equiv \beta(\xi_m) = \frac{3}{2\Delta_0} W \left[ \frac{2}{3(2\pi \xi_m^2)^{1/3}} \left( \frac{\Delta_0^2}{\lambda} \right)^{2/3} \right] \\
\approx \frac{1}{\Delta_0} \left( \log \left( \frac{\Delta_0^2}{\lambda} \right) - 1.53 - \log \xi_m \right), \quad m = 0, 1, 2, 3. \tag{31}
\]

Here Lambert’s W function is employed again. In the second step we have assumed \( \Delta_0^2/\lambda \gg 1 \) and \( \xi_m = O(1) \).

As we come down from \( \beta = \infty \) (go up in temperature), there is a first order phase transition at \( \beta_2 \) from the centre symmetric phase (\( \text{Tr} U = 0 \)) to the broken symmetry phase (\( \text{Tr} U \neq 0 \)) at an inverse temperature

\[
\beta_{cr} \equiv \beta_2 \approx \frac{1}{\Delta_0} \log \left( \frac{\Delta_0^2}{\lambda} \right). \tag{32}
\]

Several comments are in order here:

(a) The assumption, used in Section 2.1, that \( e^{-\Delta \beta} \) is small, is correct in the parameter region we are interested in. The interesting phase structures appear in the regime \( \xi \sim O(1) \). Thus since \( \lambda/\Delta_0^2 \ll 1 \) (see (19)), \( e^{-\Delta \beta} \sim e^{-\Delta_0 \beta} \ll 1 \) from (31). Therefore the terms in (11) involving \( U^n, n = 2, 3, ... \) are suppressed.
(b) The Euclidean model (3) is symmetric under the interchange of \((t, \beta) \leftrightarrow (x, L)\). Hence, similarly to (32), we can deduce a phase transition in \(L\) from the \(\text{Tr}V = 0\) phase to \(\text{Tr}V \neq 0\) (at large enough \(\beta\)) at a critical length \(L_{cr} = \beta_{cr}\).

(c) The existence of a finite \(L_{cr}\) above confirms that the transition which we found at \(L \to \infty\) and \(\beta = \beta_{cr}\) between \(\text{Tr}U = 0\) and \(\text{Tr}U \neq 0\) indeed happens in the \(\text{Tr}V = 0\) phase. Therefore the expression for \(\beta_{cr}\) is valid even at finite \(L\) as long as \(\text{Tr}V = 0\), since large \(N\) volume independence \([9, 10]\) ensures that gauge invariant quantities like the free energy and vev of Wilson loop operators do not depend on \(L\) in the \(\text{Tr}V = 0\) phase. Thus, the correct definition of ‘large \(L\)’ in this section is

\[
L \gg L_{cr} = \beta_{cr},
\]

which ensures that we are in the \(\text{Tr}V = 0\) phase. \(\beta_{cr}\) is defined in (32).

(d) By considering the interchange \(\beta \leftrightarrow L\), we can claim that, if there is no direct transition from \(\text{Tr}U = \text{Tr}V = 0\) phase to \(\text{Tr}U \neq 0\ \text{Tr}V \neq 0\) phase, the two transition line \(\beta = \beta_{cr}\) and \(L = L_{cr}\) meet at \(\beta = L = \beta_{cr}\). See figure 4.\(^{15}\)

(e) As we discuss in appendix C, the non-zero mass in (3) does not change the qualitative nature of the phase structure.

3 Phase transitions at small \(L\)

In this section, we discuss the phase structure for small \(L\) \((L \ll L_{cr})\). In this case we can dimensionally reduce the theory (see footnote 4) to obtain the action (42) with \(d = 1\). Hence we can use the analysis in \([6]\)\(^{16}\) where the phase structure has been studied by using the \(1/D\) expansion. The phases are characterized by the eigenvalue density (27). Here we summarise the results of \([6]\).

- \(\beta > \beta_{c1}\): The stable solution is given by \(u_n = 0\) \((n \geq 1)\). The eigenvalues of \(A_0\) are distributed uniformly.

\(^{15}\)Note that a transition line between \(\text{Tr}U = \text{Tr}V = 0\) phase and \(\text{Tr}U \neq 0\ \text{Tr}V \neq 0\) phase, if it exists (e.g. as in the third circled option in figure 4), could depend on \(L\) and \(\beta\). However the gravity analysis in Section 5 suggests that there is no such transition in our model, consistent with the ‘cascade picture’ reviewed in Section 6. In this case, as represented by the second joining option in figure 4, \(L > L_{cr}\) can be relaxed to \(L > L_{cr}\).

\(^{16}\)In \([6]\) we had considered massless adjoint scalars. We generalize the results to non-zero mass in Appendix C.
• $\beta_{c1} > \beta > \beta_{c2}$: The stable solution is given by $u_1 \neq 0$, $u_n = 0 \ (n \geq 2)$. The eigenvalue distribution is non-uniform and gapless.

• $\beta_{c2} > \beta$: The stable solution is given by $u_n \neq 0 \ (n \geq 1)$. The eigenvalue distribution is gapped.

• The phase transition at $\beta = \beta_{c1}$ is of second order and the transition at $\beta = \beta_{c2}$ is a third order (GWW type) transition.

The critical temperatures are calculated up to $O(1/D)$ in \cite{6} as

$$\beta_{c1} \tilde{\lambda}_{1}^{1/3} = \log \tilde{D} \left( 1 + \frac{1}{D} \left( \frac{203}{160} - \frac{\sqrt{5}}{3} \right) \right),$$

$$\beta_{c2} \tilde{\lambda}_{1}^{1/3} - \beta_{c1} \tilde{\lambda}_{1}^{1/3} = \log \tilde{D} \left[ -\frac{1}{6} + \frac{1}{\tilde{D}} \left( -\frac{499073}{460800} + \frac{203\sqrt{5}}{480} \right) \log \tilde{D} - \frac{1127\sqrt{5}}{1800} + \frac{85051}{76800} \right],$$

(34) (35)

where $\tilde{D} = D + 1$ and $\tilde{\lambda}_1 = (g')^2N(D+1)/L$. In the $\beta$-$L$ plane the transition lines appear as curves $\beta \propto L^{1/3}$ passing through the origin. Since our analysis is valid only for $L \ll L_{cr}$, we should trust these transition lines only in that region, as we have depicted in figure 4. By using the $\beta \leftrightarrow L$ reflection symmetry, we can also infer phase transition lines for $\beta \ll \beta_{cr}$ described by $L \propto \beta^{1/3}$, as shown in figure 4.

Contrary to the case of large $L$, an intermediate non-uniform phase exists at small $L$. A similar feature in the context of higher order confinement/deconfinement type phase transitions has been seen in \cite{20}.

We should mention that considerations in this section are valid up to $\tilde{X} \lesssim \lambda_{max}$ where $\lambda_{max} = L/\beta^3$ for $\beta \ll \beta_{cr}$, and $\lambda_{max} = \beta/L^4$ for $L \ll L_{cr}$ \cite{6}, which can be large close to the origin (See footnote \cite{64} also.). We will come back to this point in Section 5.

4 Phases of 2d gauge theory on $T^2$

In the last two sections, we have studied confinement/deconfinement type transitions in the model (4) for large and small values of the spatial size $L$.
We have found that the nature of the transition depends on $L$. We can summarise these results in figure 4 where we supplemented our calculations with the reflection symmetry $\beta \leftrightarrow L$ of the model.

![Diagram](image)

Figure 4: Phase structure of the 2d gauge theory at weak coupling (defined below). There are essentially 4 phases characterized by non-zero values of various Wilson lines. The inner region, with both Wilson lines non-zero, includes 2 additional phases in which the eigenvalue distribution is gapless but non-uniform. The orders of the phase transitions (1st, 2nd, 3rd) are indicated. Our analysis does not apply to the region enclosed by the dotted lines. Possible connections between the phase boundaries across this region are suggested in the inset (where boundaries of the intermediate phases are omitted for simplicity). A similar diagram is proposed in [3] for the model (3) with large mass for the adjoint scalars (see Section 6.2 for details). As we will see in figure 5, the gravity analysis conforms to the second pattern. We will see in section 6.1 that the second pattern is also supported by lattice studies.

Weak coupling in the above diagram (figure 4), for large $L$ (or large $\beta$), is defined by $\tilde{\lambda}/\Delta^2 \ll 1$ (see assumption (a) below (10)). In case the 2d gauge theory is obtained from a KK reduction, the above notion of weak coupling translates to $\lambda \ll M_{KK}^2$ (see (19) and footnote 12). For small $L$ (or $\beta$), the coupling should satisfy $\lambda' \lesssim \beta/L^3$ (or $L/\beta^3$) to validate the additional KK reduction to one dimension; as remarked at the end of Section 3, this limit on $\lambda'$ can be quite large close to the origin of figure 4.

As mentioned in the Introduction, our model (3) can be regarded as a dimensional reduction of a $D+2$ dimensional pure Yang-Mills theory compactified on a small $T^D$. For the dimensional reduction to work (see footnote...
$W_I = \text{Tr} U_I$ ($I = d, \cdots, d + D - 1$) must be non-zero (which is ensured by a sufficiently small size of the $D$-dimensional torus). Therefore we can regard the phase structure in figure 4 as a part of the $D+2$ dimensional pure Yang-Mills theory in the $W_I \neq 0$ phase. Such a Yang-Mills theory on $T^3$ and $T^4$ have been studied in lattice gauge theory and we will compare our results with these studies in Section 6.

Since our phase structure is derived through the $1/D$ expansion, it is not a priori obvious whether the result should be valid for small $D$. However, at least for small $L$, the comparison with numerical studies [46, 47, 48], as explained in [6], turns out to be remarkably good even for small $D$. For example, for $D = 2$ the $1/D$ expansion, performed up to an accuracy of $O(1/D)^2$ reproduces numerical results within the expected 25%. Thus we believe that the phase structure in the large $L$ region also should be qualitatively correct for small $D$ ($D \geq 2$).

5 The phase structure from gravity

In the previous sections, we evaluated the phase structure of the bosonic gauge theory (3) at weak coupling. In this analysis, it was difficult to figure out the phase structure of the middle region, namely where both $\beta$ and $L$ have intermediate values. In particular, it was not clear how the various phase boundaries in figure 4 are connected.

In this section, we attempt to construct a gravitational dual of our system along the lines of Witten’s realization of the 4d pure Yang-Mills theory [22]. We consider IIA supergravity on $R^7 \times T^3$ ($T^3 = S^1_\beta \times S^1_\lambda \times S^1_\lambda^2$) and put $N$ D2 branes on the $T^3$. The AdS/CFT duality in this context is discussed in [24] and more details are provided in Appendix D. The dual gauge theory is 3 dimensional $\mathcal{N} = 8$ $SU(N)$ super Yang-Mills on $T^3$, with the identifications $(t, x_1, x_2) = (t + \beta, x_1 + L, x_2 + L_2)$. In order to complete the definition of the theory, we need to choose a boundary condition of the fermions along each compact direction. Let us impose an anti-periodic boundary condition on the fermions on the $x_2$ cycle. If $L_2$ is sufficiently small ($1/L_2 \gg \lambda_3 = g_3^2 N [18]$), we can use a dimensional reduction to the two-dimensional torus $S^1_{\beta} \times S^1_{\lambda^2}$: owing to the anti-periodic boundary condition along $L_2$, all fermions would acquire a mass proportional to $1/L_2$ and we can ignore them [19].

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[18] This implies $1/L_2 \gg \sqrt{\lambda_2}$.

[19] In fact, even the scalars would acquire a mass at one-loop, as in [22]. However, unlike in [22], the scalar mass does not become infinite as $L_2 \to 0$. From the 2d gauge theory perspective, the scalar mass renormalization due to fermion loops schematically goes as $m_Y^2 = g_3^2 N \int d^2 p \delta(p + m) = \lambda_2 \frac{1}{L_2} A = \lambda_2$, where we have used a fermion mass $m = 1/L_2$. 

19
One would, thus, expect the gravitational system, for small $L_2$, to describe a dual of (3) with $D = 8$. Unfortunately, however, the gravity solutions are not valid in the small $L_2$ region ($L_2 \ll 1/\lambda_3$) since stringy corrections become important (see appendix D.1.2). In case of $L_2 \gg 1/\lambda_3$, since the fermions are not decoupled, the D2 brane theory will depend on the boundary conditions of fermions along the $t$ and $x_1$ directions. There are 4 choices of boundary conditions: (AP,AP), (AP,P), (P,AP), (P,P), where P denotes the periodic boundary condition and AP denotes the anti-periodic one. Phase diagrams of the gravity theories for different spin structures are worked out in Appendix D and presented in figures 5, 7 and 8. The salient features are:

(i) The phase structures in the gravity analysis depend on the boundary conditions.

(ii) Only the gravity analysis with (P,P) boundary condition is reliable as a prediction for gauge theory through the arguments in appendix D.4. The phase structure in this case is shown in figure 5. It predicts the second joining pattern in figure 4.

(iii) The phase transitions in the gravity description in figure 5 are Gregory-Laflamme (type) transitions [49, 50, 51] and are expected to be of the first order, at least for large $L_2$ [23, 5].

(iv) The gravity analysis for small $\beta$ and $L$ is not reliable.

and an uv cut-off for the 2d theory $\Lambda = 1/L_2$. For a more precise calculation see appendix A. Since the scalars remain light compared to the KK scale, we must keep them in the Lagrangian as in (3).

20 This is a common problem in the construction of holographic duals of non-supersymmetric gauge theories. Since the gauge theory coupling constant $\lambda_3$ is greater than the KK scale ($1/L_2$) in the region of validity of gravity, the gravity description has been likened (cf. [26], p. 196-197) to strong coupling lattice gauge theory, the small $L_2$ limit being regarded as analogous to the continuum limit. Interesting results, including the qualitative predictions in [22], and results in AdS/QCD, have been obtained using this philosophy. We will use the small $L_2$ extrapolation of our gravity results in this spirit.

21 The gravity calculation with $\beta = L$ in the (P,P) case has been studied in [25, 5].
Figure 5: Conjectured phase structure of the gauge theory from the gravity analysis. We used a large $L_2$ and the (P,P,AP) spin structure of the fermions on the three-dimensional torus (with AP boundary condition on the Scherk-Schwarz (SS) circle). TrW is the Polyakov loop operator along the SS circle. The gravity analysis is reliable only in the region above the dotted line. The transitions in this diagram are predicted to be first order phase transitions. (See appendix D.4)

Comparing figures 4 and 5, we can see that both diagrams share some common features. Both have four phases in similar parameter regions. In particular, the behaviour of the transition lines for large $\beta$ and large $L$ is the same. The line $BC$ in figure 5 is independent of $L$. This is consistent with large $N$ volume independence [9, 10, 52], since TrV = 0 on both sides of $BC$. Similar remarks apply to the line $AB$ as well.

In the small $\beta, L$ region too the two phase diagrams share similarities. In figure 5 two phase transition lines emanate from the point $D$ towards low values of $\beta, L$. However it is not clear from the gravity analysis how to continue towards the origin. On the other hand, in figure 4 the region near the origin $O$ can be computed reliably and the two (double) lines $OD_1$ and $OD_2$ can be identified as a continuation of the two phase transition lines mentioned above. We should note that the phase structure in the small $\beta, L$ region of figure 4 can be calculated from gauge theory even at strong coupling, up to $\tilde{\lambda} \lesssim \lambda_{\text{max}}$ where $\lambda_{\text{max}} = L/\beta^3$ for $\beta \ll \beta_{\text{crit}}$, and $\lambda_{\text{max}} = \beta/L^3$ for $L \ll L_{\text{cr}}$; as $\lambda$ grows stronger, the calculable region becomes narrower.

In addition to the above similarities, various details of the phases in figure
are also similar. Recall that the phases in the gauge theory are characterized by three solutions: uniform, non-uniform and gapped. Correspondingly, three solutions (uniform black string, non-uniform black string and localized black hole) play a key role in the discussion of Gregory-Laflamme transitions in gravity. For large $L$, the free energy of these solutions in the gauge theory are related as shown in figure 3. A similar relation has been found in gravity \[53\], in the case where the GL transition is of first order. On the other hand, if the GL transitions are of higher order, it consists of two transitions: a transition between a uniform black string and a non-uniform one, and another transition between the non-uniform black string and a localized black hole \[50\]. This is precisely similar to the higher order phase transitions in the gauge theory, which we have observed in the small $L$ case.

An important consequence of the gravity analysis is that we can guess how the phase transitions in figure 4 are connected. In particular, it indicates that there is no direct transition between $\text{Tr} U = \text{Tr} V = 0$ and $\text{Tr} U \neq 0, \text{Tr} V \neq 0$ phase. In \[5\], it was pointed out that this property has also been observed in large $N$ pure Yang-Mills theories on 4 and 3 dimensional tori in the lattice calculations of \[2, 54, 55\]. Thus the gravity analysis is in agreement with the lattice calculation. More details of the lattice calculation are presented in Section 6.

In summary, the full phase diagram of the two dimensional gauge theory (3) may be obtained by combining the results from gauge theory and gravity. The result would be given by figure 4 with the second joining possibility. 22

6 Relation to other works

In this section, we detail some of the remarks made in the Introduction regarding previous works.

6.1 Comparison with lattice studies

Large $N$ Yang-Mills theories on tori have been studied using lattice methods in \[2, 54, 55, 4\], in $d = 3$ and 4 dimensions. Reference \[4\] contains a nice summary of these works. We describe some of the salient features below (see also figure 5).

(a) If we start from $d = 4$ pure Yang Mills theory on a Euclidean torus

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22The extrapolation involved in this conclusion has additional support from the lattice calculations mentioned above.
$T^4$ with $L_3 < L_2 < L_1 < L_0$ then for all radii large enough the centre symmetry $Z_{N}^3$ (see footnote \[3\]) is unbroken and all the Wilson loops $W_\mu$ vanish. This phase is called the 0\(_c\) phase. In this completely unbroken phase, the thermodynamics in the large $N$ limit does not depend on any one of the lengths $L_\mu$ \[3\] \[11\].

(b) As $L_3$ is decreased, below a certain value $L_3^c$ there is a phase transition to a new phase where the centre symmetry is broken to $Z_N^2$ and $W_3$ develops a non-zero expectation value. The other Wilson loops $W_0, W_1$ and $W_2$ still vanish. This phase is called the 1\(_c\) phase in which there is no dependence on the lengths $L_\mu, \mu = 0, 1, 2$ which characterize the directions of unbroken centre symmetry.

(c) If $L_2$ is now decreased, maintaining $L_2 > L_3$, a new phase 2\(_c\) appears below $L_2^c$ (which is a function of $L_3$) where $W_2$ becomes non-zero. The centre symmetry is broken to $Z_N^2$, with non-zero values of $W_2, W_3$ while $W_0, W_1$ still vanish.

(d) Proceeding similarly, a phase 3\(_c\) is reached when $L_1$ is reduced below a critical value $L_1^c(L_2, L_3)$, and the phase 4\(_c\) are reached when, finally, $L_0$ is reduced below $L_0^c(L_1, L_2, L_3)$.

(e) Thus, $d = 4$ pure YM theory (with $L_3 < L_2 < L_1 < L_0$) exhibits a cascade of transitions

$$0_c \rightarrow 1_c \rightarrow 2_c \rightarrow 3_c \rightarrow 4_c.$$  

(f) In case of $d = 3$ pure Yang Mills theory (with $L_2 < L_1 < L_0$) the sequence of transitions works similarly, leading to a cascade

$$0_c \rightarrow 1_c \rightarrow 2_c \rightarrow 3_c.$$  

(g) It was found in \[2, 54\] that the cascade of transitions persists even when all radii are the same. E.g., in case of the system in (f) with $L_1 = L_2 = L_3 = L$, for high enough $L$ all $W_\mu = 0$; as $L$ is reduced below a certain critical value $L_c$, only one of the $W_\mu$’s picks up a non-zero value \[2\], and the 3D cubic symmetry group spontaneously breaks down to the symmetry group appropriate to a square lattice.

(h) Generally speaking, it was found in these works that the Wilson lines $W_\mu$ change from zero to non-zero one by one; two or more Wilson lines never simultaneously change from zero to non-zero values.

(i) **Order of phase transitions** \[24\] There is ample evidence that the first of the cascade of transitions $Z_N^{d+D} \rightarrow Z_N^{d+D-1}$ is first order. In the case

\[23\text{Since all directions are equivalent in the Euclidean space, the ordering chosen here is arbitrary and all arguments below can be repeated with any other ordering.}\]

\[24\text{We thank Rajamani Narayanan for pointing out some references in this paragraph.}\]
of $0_c \to 1_c$ transitions in four-dimensional Yang Mills theories on $T^4$ such evidences are presented directly in [56] and indirectly, assuming large $N$ volume independence, in [57, 58, 59]. Evidences for the first order nature of the $0_c \to 1_c$ and $1_c \to 2_c$ phase transitions have been presented for Yang Mills theory on $T^3$ in [60], which indicates that the first two transitions $Z_N^{d+D} \to Z_N^{d+D-1} \to Z_N^{d+D-2}$ are also first order. Our gauge theory analysis presents analytic evidence that the first order nature continues till the transition $Z_N^2 \to Z_N^1$ (which is $2_c \to 3_c$ in the notation of pure Yang Mills theory on $T^3$), whereas the last transition $Z_N \to 1$ in which the centre symmetry is completely broken, occurs (in the parameter region of Section 3) in two steps through a second and a third order phase transitions. The last statement, first derived in [6], is corroborated by the numerical work in [46].

Let us compare the above with the phase diagram in figure 4, which describes phases of the theory (3). Note that the theory (3) with $D = 2$ is precisely the one obtained after the steps (a)-(b) described above, in which we reduce a $d = 4, D = 0$ theory (pure YM theory in four dimensions) on two small circles of length $L_2, L_3$. To make the correspondence more explicit, we identify $L_1 = L, L_0 = \beta, W_1 = \text{Tr}V, W_0 = \text{Tr}U$. It is easy to see that the phase transitions in the $L < \beta$ region of figure 3 precisely correspond to the phase transitions described in (c)-(d) above. The top right part of figure 4 (above $AB_1B_2C$) represents the $2_c$ phase. The region above the line $OD_1B_2A$ (or its mirror image: the region to the right of the line $OD_2B_2C$) corresponds to the $3_c$ phase. The enclosed region $OD_1D_2O$ corresponds to the $4_c$ phase. The phase transition across $AB_1$ from right to left corresponds to the transition $2_c \to 3_c$; the phase transition across $OD_1$ from above corresponds to $3_c \to 4_c$ (see figure 3).

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25In other parameter regions $3_c \to 4_c$ can be a single first order phase transition [6, 60].
26We will assume that the essential features of the large $D$ calculation which led to this phase diagram remain valid for $D = 2$. This was certainly true in the $d = 1$ case discussed in [6].
27The $\beta < L$ region is the mirror image; it corresponds to the sequence similar to (c)-(d) resulting from an ordering $L_0 < L_1$.
28We are treating the non-uniform phase as part of the $4_c$ phase here.
Figure 6: Cascade of phase transitions for pure Yang-Mills theory on a $T^4$ (adapted from [4]; see the points (a)-(e) above for more details). Our results in this paper, for the theory (3) with $D = 2$, describe the indicated phases $2c, 3c,$ and $4c$. The $0c \rightarrow 1c$ transition is found to be first order from lattice studies in [56]; the $2c \rightarrow 3c$ transition is found to be first order from our analysis; the $3c \rightarrow 4c$ transition for an asymmetric torus is found in our analysis to be a double (2nd order + 3rd order) transition for an appropriate parameter regime (see figure 4), although, it can be a single first order transition at other regimes [6, 60].

The comments (g)-(h) above have a direct bearing on the possible joining pattern in figure 4. Out of the three possible joining patterns shown in the insets, the first and the third patterns allow a direct transition $2c \rightarrow 4c$ and are, hence, inconsistent with the comment (h). Thus, consistency with the lattice results described in this subsection uniquely pick up the second joining pattern. As we showed in Section 5, the same joining pattern is also picked up uniquely in figure 5 through the analysis of the gravity dual.

A more quantitative comparison of our work with the above lattice studies is left for the future.

We should mention another important lattice study [61] which deals with super Yang Mills theories in $d = 2$ and is closely related to the work presented here and in [6]. In parameter ranges where the two theories coincide, our phase diagrams agree (see section 5 of [61]). See also related numerical works about the center symmetry breaking in super Yang-Mills [62, 63].

### 6.2 Comparison with earlier analytical studies with massive adjoint scalars

Reference [8] considers the theory (3) in the limit $m \gg \lambda^2$. Our figure is similar to figure 13 of [3], except that our figure is obtained for any mass (including $m = 0$) where their figure is for the large mass limit. The reason for the agreement is the appearance of a dynamical mass $\Delta$ for the adjoint scalars in our model, as we have explained above.
In figure 14 of [3] an interpolation between small and large radii is proposed on the basis of some analytical estimates for the intermediate radii. Our figure although similar to this figure, differs in one crucial respect. The phase transition line BC in our figure is horizontal throughout, as it must be according to large $N$ volume independence arguments [9, 10]. Since the line BC is entirely in the $\text{Tr} V = 0$ phase, the transition temperature $\beta_c$ cannot depend on $L$; hence BC must be horizontal. This property is violated by the corresponding line (the intermediate radius segment) of figure 14 in [3], which should have been horizontal according to the above argument.

6.3 Comparison with Yang-Mills theories on different topologies

We have found that the nature (in particular, order) of the confinement/deconfinement type transition at $\beta_c$ in the two dimensional gauge theory [3] at a fixed radius $L$ depends on the value of $L$. Since this theory can be obtained from a pure Yang-Mills theory on a small $T^D \times S^1_\beta \times S^1_\beta$, it is interesting to compare this result with Yang-Mills theories on compact spaces with other topologies. We present such a comparison in Table 6.3.

| $\mathcal{M}$           | type of phase transition |
|-------------------------|--------------------------|
| small $T^D \times$ small $S^1$ | 2nd+3rd                  |
| small $T^D \times$ large $S^1$ | 1st                      |
| small $S^2$             | 2nd+3rd                  |
| small $S^3$             | 1st                      |

Table 1: Confinement/deconfinement type transitions in pure Yang-Mills theories on $S^1_\beta \times \mathcal{M}$. Here “small $S^1$” and “small $T^D$” refer to sizes small enough to ensure (a) that the $Z_N$ symmetries in the $S^1$ and $T^D$ directions, respectively, are broken, and (b) that all the KK modes can be integrated out. “Large $S^1$” ensures that the $Z_N$ symmetry along the $S^1$ is not broken.

Because of the difficulty of the analysis of Yang-Mills theory, only weakly coupled Yang-Mills theories on $S^2$ and $S^3$ [64, 65] have been studied. In these cases, all the spatial components of the gauge field have a mass proportional to $1/R$, where $R$ is the radius of the sphere. These massive gauge fields can be integrated out perturbatively if the radius is sufficiently small ($R \Lambda_{QCD} \ll 1$). Reference [64, 65] derived effective potentials for $A_0$ up to three loop

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29In order to apply $1/D$ expansion, we need $D \geq 1$ in the small $T^D \times$ small $S^1$ case and $D \geq 2$ in the small $T^D \times$ large $S^1$ case.
order and found the transition in the $S^3$ case to be first order \[64\]. On the other hand, the transition in the $S^2$ case consists of second and third order transitions as we found in the small $T^D$ case \[65\]. Note that the higher order transitions for small $S^2$ is expected to change to a first order transition in a strong coupling regime according to lattice studies \[66\].

Thus the nature of the transition depends not only on the size but also on the topology of the compact space. It would be interesting to understand the origin of these differences.

### 7 Conclusions

In this work, we have computed the phase diagram of two dimensional Yang-Mills theory with adjoint scalars \[3\], which can be obtained from a KK reduction of a higher dimensional pure Yang-Mills theory. We treated the case of massless adjoint scalars in detail, outlining the generalization to arbitrary non-zero mass in Appendix \[C\] and found the phase diagram in figure \[4\]. At large spatial radius, there is a first order confinement/deconfinement phase transition, whereas at small spatial radius, there are two closely spaced phase transitions: (a) a second order phase transition from the ‘confined’ phase to a ‘non-uniform’ phase (non-uniform eigenvalues of the Polyakov line), followed by (b) a third order phase transition from the ‘non-uniform’ phase to a ‘gapped’ phase. Our calculations, based on the large $D$ method \[6\], provide an analytical derivation of the dependence of the thermodynamic behaviour on the size of the spatial box, which is anticipated on the basis of lattice studies and gauge/gravity duality.

We have also considered the phase transitions in the gauge theory from the viewpoint of a gravity dual, based on a scaling limit of Scherk-Schwarz compactification of a D2 brane on a 3-torus. Although there is no overlapping region of validity of the gauge theory and gravity descriptions, the analysis of the gravity dual leads us to conjecture a certain specific completion of the phase diagram in the gauge theory, as in figure \[5\]. In performing this analysis, we encountered an inherent problem with the holographic analysis, namely a dependence of the physics on the fermion boundary conditions, which was absent in the gauge theory description (see Appendix \[D\]). Indeed this problem is related to a more general problem in the holographic description of QCD \[67\]. We discuss this problem further in \[68\].

We matched our findings from gauge theory regarding the order of various phase transitions with those from a gravity analysis in Section \[5\] and with those from lattice studies in Section \[6\].

Note that the method of integrating out the adjoint scalars using a $1/D$
expansion works equally well in higher dimensional \((d \geq 3)\) gauge theories \([42]\), leading to an effective action for the gauge field as shown in \([43]\). However it is difficult to evaluate the dynamics of this model because of the existence of dynamical gluons. This is a crucial difference from the lower dimensional cases \((d = 0, 1, 2)\). In addition, the \(d\) dimensional model \([42]\) typically appears through a KK reduction of a \(d + D\) dimensional (super) Yang Mills theory, but for \(d \geq 3\) the mass of the adjoint scalars induced from loops of KK modes is large (see appendix A); hence the contribution of the adjoint scalars may be not relevant for \(d \geq 3\).

The investigations in the present paper were partly motivated by a desire to understand a gauge theory dual to a dynamical Gregory-Laflamme transition. The considerations in this paper provide a step towards understanding this issue; details of this will appear elsewhere \([27]\).

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A Mass for \(Y^I\) from one-loop of the KK modes

If we consider a \(d + D\) dimensional Yang-Mills on \(T^{d+D}\) \([1]\) and consider a dimensional reduction by taking the radii of \(T^D\) to be small, we will classically obtain a \(d\) dimensional gauge theory coupled to \(D\) massless adjoint scalars. However, if we consider quantum effects, the action would be modified. One of the relevant corrections is that the adjoint scalars would acquire mass as...
in \([12]\). In this appendix, we evaluate the mass at a one-loop level \([30]\).

Starting from the \(d + D\) dimensional action \([11]\), we can derive a one-loop effective action for the constant diagonal components of \(A^\mu = (a_1^\mu, \ldots, a_n^\mu)\) by integrating out all the other modes \([1, 69, 3]\).

\[
S_{\text{eff}} = \left( \sum_{\mu=0}^{d+D-1} L_\mu \right) \frac{d + D - 2}{\pi^{(d+D)/2}} \Gamma((d + D)/2) \times \sum_{i,j} \sum_{\{k_\nu\} \neq \{0\}} \exp \left( i \sum_\nu k_\nu L_\nu (a_i^\nu - a_j^\nu) \right) \left( \sum_\nu k_\nu^2 L_\nu^2 \right)^{(d+D)/2}.
\]

Here the sum \(\sum_{\{k_\nu\} \neq \{0\}}\) includes all integers \(k_\nu\) except \(k_0 = \cdots = k_{d+D-1} = 0\). Let us now take \(L_\mu = (\mu = 0, \ldots, d - 1)\) large and \(L_I = (I = d, \ldots, d + D - 1)\) small and derive a low energy effective theory by using this expression. Gauge invariance implies that the effective action in this situation will be given by

\[
S_d = \int_0^\beta dt \prod_{i=1}^{d-1} \int_0^{L_i} dx^i \text{Tr} \left( \frac{1}{4g_d^2} F_{\mu\nu}^2 + \sum_{I=1}^D \frac{1}{2} (D_{\mu} Y^I)^2 - \frac{g_d^2}{4} \int \frac{d^D Y^I}{\pi^{(d+D)/2}} \right) \sum_{\{k_\nu\} \neq \{0\}} \left| \text{Tr} e^{i g_d \sum_\nu J_\nu J_I Y^I} \right|^2.
\]

Here we have rewritten \(A_{d+I-1} = g_d Y_I\) and \(L_{d+I-1} = I_I\). \(g_d\) and \(g'_d\) are the same as \(g\) and \(g'\) of \([3]\); they satisfy \(g_d^2 / \prod_{I=1}^D l_I = g'_d = g_d^2\) at a physical scale \(\mu \gg 1/l_I\) (cf. \([12]\)).

If \(l_I\) are small and the long string modes are suppressed (see footnote \([3]\)), we can treat \(Y^I\) perturbatively. Then we can expand the exponentials in \([31]\) and obtain a quadratic term in \(Y^I\) as

\[
g_d^2 N \left( \prod_{I=1}^D l_I \right) (d + D - 2) \frac{\Gamma((d + D)/2)}{\pi^{(d+D)/2}} \sum_{\{k_\nu\} \neq \{0\}} \frac{k_\nu^2 l_I^2}{\left( \sum_\nu k_\nu^2 l_I^2 \right)^{(d+D)/2}} \frac{\text{Tr}(Y^I)^2}{2}.
\]

Other interaction terms from the exponential would be suppressed by small \(l_I\). If we take all the \(l_I\) to have a common value \(l_{KK} = 1/M_{KK}\), we obtain a mass for \(Y^I\) as

\[
m^2 = g_d^2 N M_{KK}^2 (d + D - 2) \frac{\Gamma((d + D)/2)}{\pi^{(d+D)/2}} \sum_{\{k_\nu\} \neq \{0\}} \frac{1}{D} \frac{1}{\left( k_1^2 + \cdots + k_D^2 \right)^{(d+D)/2 - 1}}.
\]

\(^{30}\) We do not use the large \(D\) limit in this appendix.
This sum would diverge for $d \leq 2$ and hence needs to be regulated e.g. by using a prescription $(\vec{k})^2 \equiv k_1^2 + \cdots + k_D^2 \leq (\Lambda_{UV}/M_{KK})^2$, where $\Lambda_{UV}$ is a cut-off scale. This would imply a non-trivial RG flow of the mass. Let us first consider the case of $d = 2$. For $\Lambda_{UV} \gg M_{KK}$, the regulated sum approximately gives $m^2(\Lambda_{uv}) = Ag^2N \log(\Lambda_{uv}/M_{KK})$ where $A$ is a numerical constant. Let us choose a renormalization scale $\mu \gtrsim O(M_{KK})$. We can then define a renormalized mass at the scale $\mu$ as

$$\tilde{m}^2 \equiv m^2(\Lambda_{uv}) - A g^2 N \log(\Lambda_{uv}/\mu) = A' \lambda_2,$$

where $A' = A \log(\mu/M_{KK}), \lambda_2 = g^2 N$. Note that the running of the mass will stop below $\mu < M_{KK}$.

For $d = 1$, a similar analysis gives

$$\tilde{m}^2 \sim \lambda_3/M_{KK}.$$

For $d \geq 3$, the sum in (39) is convergent, leading to $m \sim \lambda_3^{1/2} M_{KK}^{(d-2)/2}$ which is much larger than the typical QCD scale, e.g. for $d = 3$ the QCD scale is $O(\lambda_3)$ whereas the adjoint mass is $O(\lambda_3^{1/2} M_{KK}^{1/2})$. Thus if $M_{KK}$ is large, the adjoint scalars will not contribute to the QCD dynamics. It means that only the $d$ dimensional gauge field dominates the dynamics.

So far we have considered the mass correction from the KK modes in pure Yang-Mills theory. We can extend this calculation to the KK reduction of supersymmetric theories with a Scherk-Schwarz compactification e.g. to the derivation of (3) with $D = 8$ from the D2 theory. In this case, we need to evaluate the contribution of loops of adjoint scalars as well as fermions. However, it can be shown that we obtain a similar mass $\sim O(\lambda_3^{1/2} M_{KK}^{(d-2)/2})$ even in this case.

B Derivation of effective potential (11)

In this appendix, we show the derivation of the effective action (11) from (10) with the assumptions (a) and (b) below (10). We can evaluate (10) in

31 An ultraviolet cutoff typically breaks gauge symmetry. The calculation discussed here can be repeated avoiding such problems, by using dimensional regularization.

32 For $d = 2$, the mass of the adjoint scalar from the KK modes is finite ($O(\lambda_2)$) but the dynamical mass $\Delta$ is (logarithmically) larger than $\lambda_2$ (see (19)). Hence one may naively think that the adjoint scalar would be irrelevant as for $d \geq 3$. However this is not correct, since the phase structure of the 2d pure Yang-Mills on $T^2$ is trivial and is always confined. Therefore the contribution of the (logarithmically) heavy adjoint scalars is important in the 2 dimensional gauge theory.
a general $d$ dimensional gauge theory, described by the action
\[
S = \int_0^\beta dt \prod_{i=1}^{d-1} \int_0^{L_i} dx^i \text{Tr} \left( \frac{1}{4g^2} F_{\mu\nu}^2 + \sum_{l=1}^{D} \frac{1}{2} (D_\mu Y^l)^2 + \frac{m^2}{2} Y_{IJ} \right) - \sum_{I,J} \frac{g_f^2}{4} [Y^I, Y^J] [Y^I, Y^J].
\] (42)

This model can be identified with (37) if we take $L_I$ small and choose the mass and couplings appropriately. We will first discuss the general $d$-dimensional case and apply the results to $d = 2$ later.

Let us first set $m = 0$. Then, through a similar calculation as in Section 2.1, we obtain a generalization of (10) (the calculation closely follows [3]). In this section, we will use the more general notation $(L_0, L_1)$ for $(\beta, L)$.

\[
\delta S[A, \Delta] = \frac{D}{2} \log \det (-D^2 + \Delta^2) = \frac{D}{2} \text{Tr} \sum_{\{n_\mu\}} \log \left( \sum_{\mu=0}^{d-1} \left( \frac{2\pi n_\mu}{L_\mu} + A_\mu \right)^2 + \Delta^2 \right)
\]
\[
= \frac{D}{2} \frac{L_0 \cdots L_{d-1}}{(2\pi)^d} \text{Tr}_{adj} \sum_{\{k_\mu\}} P_{\{k_\mu\}}(\Delta, \{L_\mu\}) e^{i \sum_{\mu} k_\mu L_\mu A_\mu},
\] (43)

where $P_{\{k_\mu\}}$ is
\[
P_{\{k_\mu\}}(\Delta, \{L_\mu\}) = \int_0^{2\pi/L_0} da_0 \cdots \int_0^{2\pi/L_{d-1}} da_{d-1} \sum_{\{n_\mu\}} \log \left( \sum_{\mu=0}^{d-1} \left( \frac{2\pi n_\mu}{L_\mu} + a_\mu \right)^2 + \Delta^2 \right) e^{-i \sum_{\mu} k_\mu L_\mu a_\mu}
\]
\[
= \sum_{\{n_\mu\}} \int_{n_0}^{n_0+2\pi/L_0} da_0 \cdots \int_{n_{d-1}}^{n_{d-1}+2\pi/L_{d-1}} da_{d-1} \log \left( \sum_{\mu=0}^{d-1} a_\mu^2 + \Delta^2 \right) e^{-i \sum_{\mu} k_\mu L_\mu a_\mu}
\]
\[
= \int_{-\infty}^{\infty} da_0 \cdots \int_{-\infty}^{\infty} da_{d-1} \log \left( \sum_{\mu=0}^{d-1} a_\mu^2 + \Delta^2 \right) e^{-i \sum_{\mu} k_\mu L_\mu a_\mu}.\] (44)

Let us now evaluate $P_{\{k_\mu\}}$ in the $\{k_\mu\} = \{0\}$ and $\{k_\mu\} \neq \{0\}$ cases separately.
In the $\{k_\mu\} \neq \{0\}$ case, $P_{\{k_\mu\}}(\Delta, \{L_\mu\})$ becomes,

$$P_{\{k_\mu\}}(\Delta, \{L_\mu\})$$

$$= \int_{-\infty}^{\infty} da_0 \cdots \int_{-\infty}^{\infty} da_{d-1} \log \left( \sum_{\mu=0}^{d-1} a_\mu^2 + \Delta^2 \right) e^{-i \sum_{\mu} k_\mu L_\mu a_\mu}$$

$$= \int_{-\infty}^{\infty} da_0 \cdots \int_{-\infty}^{\infty} da_{d-1} \lim_{\epsilon \to 0} \left[ -\log \epsilon - \gamma - \int_{\epsilon}^{\infty} \frac{d\alpha}{\alpha} e^{-\alpha (\sum_{\mu} a_\mu^2 + \Delta^2)} \right] e^{-i \sum_{\mu} k_\mu L_\mu a_\mu}$$

$$= -\pi^{d/2} \int_{0}^{\infty} \frac{d\alpha}{\alpha^{d/2+1}} e^{-\Delta^2 - \frac{1}{4\pi} \sum_{\mu} (k_\mu L_\mu)^2}$$

$$= -\frac{2 (2\pi \Delta)^{d/2}}{\left( \sqrt{\sum_{\mu} (L_\mu k_\mu)^2} \right)^{d/2}} K_{d/2} \left( \Delta \sqrt{\sum_{\mu} (L_\mu k_\mu)^2} \right)$$

$$= -\frac{2 (2\pi \Delta)^{d/2}}{\left( \sqrt{\sum_{\mu} (L_\mu k_\mu)^2} \right)^{d/2}} \sqrt{\frac{\pi}{2\Delta}} \exp \left( -\Delta \sqrt{\sum_{\mu} (L_\mu k_\mu)^2} \right) + \cdots, \quad (45)$$

where $K_{d/2}$ is the modified Bessel function of the second kind and we have used $K_{\alpha}(z) = \sqrt{\pi/2e^{-z}} + \cdots$ for large $z$ in the last equation.

For $\{k_\mu\} = \{0\}$, we find

$$P_{\{0\}}(\Delta, \{L_\mu\}) = \int_{-\infty}^{\infty} da_0 \cdots \int_{-\infty}^{\infty} da_{d-1} \log \left( \sum_{\mu=0}^{d-1} a_\mu^2 + \Delta^2 \right)$$

$$= \int_{0}^{\Lambda} da_\Omega a^{d-1} \log \left( a^2 + \Delta^2 \right), \quad (46)$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the $d$ dimensional unit sphere and $\Lambda$ is a cut off.

Finally we turn on the mass term. In this case, we can obtain the results by replacing $\Delta \to \sqrt{\Delta^2 + m^2}$ in (45) and (46). Note that the assumption (a) and (b) below (10) should be modified accordingly.

### B.1 Effective action for $d = 2$

We now consider the special case of $d = 2$. In this case, we can evaluate $P_{\{0\}}$ as

$$P_{\{0\}}(\Delta, \beta, L) = 2\pi \int_{0}^{\Lambda} da \ a \log \left( a^2 + \Delta^2 \right)$$

$$= -\pi \Lambda^2 + \pi \left( \Lambda^2 + \Delta^2 \right) \log \left( \Lambda^2 + \Delta^2 \right) - \pi \Delta^2 \log \Delta^2. \quad (47)$$
By using this result and (45) to (43), we obtain the effective action (11). Note that Tr $\text{ad}_j e^{i(k\beta A_0 + IL_1)}$ in (43) becomes $|\text{Tr} (U^k V^l)|^2$ by using the assumption (b).

### B.2 Effective potential for higher dimensional models

It is easy to generalize the derivation of the effective potential (20) for the two dimensional gauge theory in Section 2.1 to the $d$ dimensional gauge theory (42) for large $L_\mu$ \[3\]. By using the results in the previous section, we obtain

$$ S = \prod_{\mu=0}^{d-1} \int_0^{L_\mu} dx_\mu \left( \frac{1}{4g^2} \text{Tr} F^2_{\mu\nu} - \frac{DN^2}{(2\pi)^{\frac{d-1}{2}}} \sum_{\mu=0}^{d-1} \frac{(\Delta^2_0 + m^2)^{\frac{d-1}{2}}}{L^2_{\mu}} e^{-\sqrt{\Delta^2_0 + m^2} L_{\mu}} \left| \frac{1}{N} \text{Tr} e^{iL_\mu A_\mu} \right|^2 \right) $$

$$ + DN^2 L_0 \cdots L_{d-1} \left( -\frac{\Delta^4_0}{8\lambda^2} + \frac{1}{2(2\pi)^d} P_0 \left( \sqrt{\Delta^2_0 + m^2}, \{L_\mu\} \right) \right). \quad (48) $$

Here $\Delta_0$ is defined as a solution of the saddle point equation

$$ \frac{\Delta^2}{2\lambda'} = \frac{\Omega_d}{(2\pi)^d} \int_0^{\Lambda} da a^{d-1} \frac{a^{d-1}}{a^2 + \Delta^2 + m^2}. \quad (49) $$

Note that we can always find a unique positive solution $\Delta_0$ from this equation.

### C The phase structure of massive model.

In this appendix, we study the two dimensional gauge theory (3) with a mass term for the adjoint scalars. As we mentioned in the introduction and elaborated in appendix A, such a mass term generically arises from KK loops. We discuss here how the results of the massless case in Section 2 and Section 3 are modified. We will show that the mass does not change the qualitative nature of the phase structure.

#### C.1 Large L case

By using the results in B.2, we generalize the effective action for Tr$U$ (23) to the massive case. The resulting effective action is again given by (23) with...
different values of $\xi$ and $C$:

$$\xi = \sqrt{\frac{\Delta_0^2 + m^2}{2\pi \tilde{\lambda}^2 \beta^3}} e^{-\beta \sqrt{\Delta_0^2 + m^2}},$$

(50)

$$C(\tilde{\lambda}', \Delta_0) = \frac{\beta L \Delta_0^2}{8\pi} \left(1 + \frac{\pi \Delta_0^2}{\tilde{\lambda}'}\right) + \frac{\beta L m^2}{8\pi} \left(1 + \frac{2\pi \Delta_0^2}{\tilde{\lambda}'}\right).$$

(51)

The dynamical mass $\Delta_0$ is a solution of

$$\tilde{\lambda}' = \frac{2\pi \Delta_0^2}{\log \left(1 + \frac{\Lambda^2}{\Delta_0^2 + m^2}\right)}.$$  

(52)

For large $\Lambda$, $\Delta_0$ becomes

$$\Delta_0 = \sqrt{\frac{\tilde{\lambda}'}{2\pi} W \left(\frac{2\pi \Lambda^2}{\tilde{\lambda}'} e^{\frac{2\pi m^2}{\tilde{\lambda}'}}\right) - m^2}.$$  

(53)

Therefore we can use the same analysis as in Section 2.2 and obtain the same phase structure with the following modified transition temperatures

$$\beta_m = \frac{3}{2\sqrt{\Delta_0^2 + m^2}} W \left[\frac{2}{3(2\pi \xi_m^2)^{1/3}} \left(\frac{\Delta_0^2 + m^2}{\lambda}\right)^{2/3}\right].$$

(54)

Although the mass changes the explicit values of the transition temperatures and some other physical quantities, the qualitative nature of the phase structure is not modified, as we have mentioned.

Note that (52) implies that $\Delta_0$ becomes smaller as $m$ increases for fixed $\tilde{\lambda}'$ and $\Lambda$. Therefore, for heavy mass, we can ignore the dynamical mass $\Delta_0$ compared to $m$ and our calculations reproduce the heavy mass QCD results in [14, 3].

### C.2 Small $L$ case

If $L$ is small enough in (3), we can integrate out all the non-zero momentum modes in the $L$ direction (see footnote 4) and obtain a one dimensional model (42) with $d = 1$. In this case, the mass of the adjoint scalars induced at one loop from the KK modes is proportional to $(\tilde{\lambda} L)^{1/2}$ (see (41)). Hence we can ignore it for small enough $L$ and the results in [6] shown in Section 3 would still be valid. Although the contribution from the mass would be small, it may be valuable to confirm that the results in [6] are not modified qualitatively.
Starting from (42) with \( d = 1 \), we obtain an effective potential through a similar calculation as in B.34:

\[
S_{\text{eff}}(\Delta, \{u_n\}) / DN^2 = -\frac{\beta \Delta^4}{8 \lambda_1} + \frac{\beta \sqrt{\Delta^2 + m^2}}{2} + \sum_{n=1}^{\infty} \left( \frac{1}{D} - e^{-n \beta \sqrt{\Delta^2 + m^2}} \right) |u_n|^2.
\]

(55)

Here the third term is a contribution of the Vandermonde determinant \[6\]. Then the saddle point equation for \( \Delta^2 \) becomes

\[
\frac{\Delta^2}{\lambda_1} = \frac{1}{\sqrt{\Delta^2 + m^2}} + 2 \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{\Delta^2 + m^2}} e^{-n \beta \sqrt{\Delta^2 + m^2}} \right) |u_n|^2.
\]

(56)

The solution of this equation at low temperatures \((e^{-\beta \sqrt{\Delta^2 + m^2}} \ll 1)\) is

\[
\Delta^2 = \Delta_0^2 + \frac{4 \lambda_1}{3 \Delta_0^2 + 2m^2} e^{-\beta \sqrt{\Delta_0^2 + m^2}} |u_1|^2 + \cdots,
\]

(57)

where

\[
\Delta_0^2 = \frac{m^2}{3} \left( f(m) + f(m)^{-1} - 1 \right),
\]

\[
f(m) = \frac{1}{2^{1/3} m^2} \left( 27 \lambda_1^2 - 4 m^6 + \sqrt{27 \lambda_1^2 (27 \lambda_1^2 - 4m^6)} \right)^{1/3}.
\]

(58)

Note that, although \(f(m)\) is a complex for \( m^6 \geq 27 \lambda_1^2 / 4 \), \( \Delta_0^2 \) is always real and positive. By substituting this solution to (55), we obtain an effective action for \( u_n \),

\[
S_{\text{eff}}(\{u_n\}) / DN^2 = -\frac{\beta \Delta_0^4}{8 \lambda_1} + \frac{\beta \sqrt{\Delta_0^2 + m^2}}{2} + a |u_1|^2 + b |u_1|^4 + \cdots
\]

(59)

where

\[
a = \left( \frac{1}{D} - e^{-\beta \sqrt{\Delta_0^2 + m^2}} \right), \quad b = \frac{\beta \lambda_1}{3 \Delta_0^2 + 2m^2} e^{-2\beta \sqrt{\Delta_0^2 + m^2}}.
\]

(60)

One important fact is that \( b \) is always positive. It has been shown that, in this case, the confinement/deconfinement type transition always consists of

\[34\]Contrary to the \( d \geq 2 \) case, the condition (a) and (b) below (10) are not required for a derivation of the effective action in the \( d = 1 \) case. Thus the \( 1/D \) analysis would be valid as long as the effective dimensionless coupling \( \lambda_1 \beta^3 \) does not scale with \( D \).
the two transitions (2nd+3rd) as in the massless case \[6, 20\]. These critical temperatures are given by the solutions of \(a = 0\) and \(a/(2b) = -1/4\) respectively, and evaluated as

\[
\beta_{c_1} = \frac{1}{\sqrt{\Delta^2_0 + m^2}} \log D, \\
\beta_{c_2} = \beta_{c_1} - \frac{\tilde{\lambda}_1}{2D} \frac{\log D}{3\Delta^2_0 + 2m^2}.
\]

Therefore as in the large \(L\) case, the qualitative nature of the phase transition is not modified by the mass term\[35\].

Although we have evaluated only the leading order of the \(1/D\) expansion in this section, the result does not change even if we include the next order.

## D Phase structure of D2 branes on a 3-torus

In this appendix, we consider the gravitational system of Section 5 in detail. We first review some generalities for Dp branes with arbitrary \(p\).

\[35\] Note that our results based on the \(1/D\) expansion disagree with the results in \[3\]. Reference \[3\] studied the same model \[55\] by using a large mass approximation, which is supposed to be valid if \(\tilde{\lambda}_1/m^3 \ll 1\), up to three-loop order and concluded that the confinement/deconfinement transition in this model would be a single first order transition. The difference would presumably arise from the fact that the \(1/D\) expansion employed here evaluates the model in a non-trivial vacuum characterized by the non-zero \(\Delta\), whereas the large mass analysis of \[3\] is performed as a perturbation around the trivial vacuum. Numerical studies analogous to \[46\] but performed for massive adjoint scalars \[47\] should be able to provide further insight into this issue.
D.1 $D_p$ branes wrapped on $p + 1$-Torus

D.1.1 The solutions

The geometry of a black $D_p$ brane on a $p$-torus $x_i \in (0, L_i)$, $(i = 1, \cdots, p)$ in the Maldacena limit (assuming Euclidean time $t \in (0, \beta)$) \cite{24} is given by

$$ds^2 = \alpha' \left[ F(u) \left( f(u)dt^2 + \sum_{i=1}^{p} dx_i dx_i \right) + \frac{du^2}{F(u) f(u)} + G(u)d\Omega^2_{8-p} \right],$$

$$e^{\phi} = \frac{(2\pi)^{2-p} \lambda_{p+1}}{N} [F(u)]^{(p-3)/2},$$

$$F(u) = \frac{u^{(7-p)/2}}{d_p \lambda_{p+1}}, \quad G(u) = \sqrt{d_p \lambda_{p+1}} u^{(p-3)/2}, \quad f(u) = 1 - \left( \frac{u_0}{u} \right)^{(7-p)},$$

$$d_p = 2^{(7-2p)\pi(9-3p)/2} \Gamma \left( \frac{7-p}{2} \right), \quad \lambda_{p+1} = g_{p+1}^2 N,$$ (62)

where $g_{p+1}^2$ is the $p + 1$ dimensional YM coupling, which, in the bulk theory, can be regarded as specifying a boundary condition for the dilaton field. The dimensionless YM coupling, defined by

$$g_{\text{eff}}^2 = (g_{p+1}^2 N) u^{p-3} = (e^{\phi} N)^{2/(7-p)},$$ (63)

is given directly in terms of the dilaton; its dependence on $u$ for $p \neq 3$ reflects the running of the gauge coupling. The scalar curvature is given by

$$\alpha' R = 1/g_{\text{eff}}.$$

Since time is Euclidean, $u \in (u_0, \infty)$. The smoothness condition at $u = u_0$ relates the inverse temperature $\beta$ to $u_0$ as follows:

$$\frac{\beta}{2\pi} = \frac{\sqrt{d_p \lambda_{p+1}}}{7-p} u_0^{(p-3)/2}.$$ (65)

The classical action of the black $D_p$ brane is \cite{24, 23, 5}

$$S/N^2 = C_p \lambda_{p+1}^{\frac{2-p}{2}} L_1 \cdots L_p \beta \left( -\beta^{-2(7-p)/(5-p)} + H_{\text{reg}}(U) \right),$$

$$C_p = \frac{5-p}{2^{11-2p}\pi(13-3p)/2 \Gamma((9-p)/2)d_p^{2(7-p)/(5-p)}},$$

$$a_p = \frac{7-p}{4\pi d_p^{1/2}}, \quad d_p = 2^{(7-2p)\pi(9-3p)/2} \Gamma \left( \frac{7-p}{2} \right),$$

$$H_{\text{reg}}(U) = \left( \frac{2a_p}{\sqrt{\lambda_{p+1}}} \right)^{2(7-p)/(5-p)} U^{7-p},$$ (66)

37
which is evaluated by expressing the on-shell action as a regularized integral of $\sqrt{g}e^{-2\phi}R$ over the range $u_0 \leq u \leq U$.

In the following we will also be interested in AdS solitons [22, 70] which can be obtained from (62) by moving the coefficient $f(u)$ from $dt^2$ to one of the $x_i$’s, say to $dx^2_p$:

$$ds^2 = \alpha' \left[ F(u) \left( f(u)dx^2_p + dt^2 + \sum_{i=1}^{p-1} dx_i dx_i \right) + \frac{du^2}{F(u)f(u)} + G(u)d\Omega^2_{8-p} \right].$$  \hspace{1cm} (67)

The smoothness condition at $u = u_0$ now gives a condition analogous to (65) where $\beta$ is replaced by $L_p$. Thus this solution has a contractible $x_p$-cycle (which wraps around a so-called ‘cigar’ geometry on the $(x_p, u)$ plane) along which the fermions must obey the anti-periodic (AP) boundary condition.

The regularized classical action evaluated on such a classical configuration is given by

$$S/N^2 = C_p \lambda^{p+1}_p L_1 \cdots L_p \beta \left( -L_p^{-\frac{2(7-p)}{6-p}} + H_{\text{reg}}(U) \right),$$  \hspace{1cm} (68)

where the notations are the same as before.

In a toroidal, Euclidean, spacetime, the time direction is on a similar footing as any other direction. Thus, the difference between the black brane and the solitonic solution is only in the labelling of the contractible cycle (location of the ‘cigar’ geometry). Hence we will sometimes refer to both black branes as well as AdS solitons as just $D_p$ solutions. In the following sections, we will consider different $D_p$ solutions (with $p = 0, 1, 2$) which wrap on (are localized along) various cycles and, in order to distinguish them, we will use the following notation:

$D_{pL_0,L_1,\ldots,L_p}$ denotes a $D_p$ solution which (i) has a contractible $L_0$ cycle (that winds around the ‘cigar’) and (ii) wraps on the $L_1, \ldots, L_p$ cycles; this is a black brane. Similarly $D_{pL_p,L_0,L_1,\ldots,L_{p-1}}$ denotes an AdS soliton in which the roles of $t$ and $x^p$ are flipped, as in (67).

**D.1.2 Validity of supergravity**

The solutions described in the previous section are leading order supergravity solutions. When we consider a black $D_p$ brane solution (62), the gravity analysis is reliable, if the parameters satisfy the following conditions [51]:

1. The typical length scale of the black $D_p$ brane near the horizon (see, e.g. (64)) is given by $l = \left( \alpha' \sqrt{d_p \lambda^{p+1}_p u_0^{(p-3)/2}} \right)^{1/2}$. In order to sup-
press stringy excitations, we should satisfy \( l \gg \sqrt{\alpha'} \). From (65), this condition is equivalent to
\[
\lambda_{p+1} \beta^{3-p} \gg 1 \quad (p \leq 5).
\] (69)

2. The mass of the winding mode along an \( L_i \) cycle is given by
\[
M_{wi} = \left( \alpha' u_0^{(7-p)/2} / \sqrt{d_p \lambda_{p+1}} \right)^{1/2} L_i / \alpha'.
\]
In order to suppress the winding mode, we must have \( M_{wi} l \gg 1 \). This condition gives
\[
\lambda_{p+1}^{1/2} L_i^{3-p} \gg \beta \quad (p \leq 5).
\] (70)

If this condition is violated and if the fermion on the brane satisfies the periodic (P) boundary condition along the \( L_i \) cycle, we can perform a \( T \)-duality along this direction and reassess the validity of supergravity in the dual frame\(^{36}\). After the \( T \)-duality, the black \( D_p \) solution becomes a smeared black \( D(p-1) \) brane solution, which is composed of uniformly distributed \( D(p-1) \) branes on the dual \( L_i \) cycle\(^{23, 71, 72}\). Then the condition (70) is replaced by \( \tilde{M}_{wi} l \gg 1 \), where
\[
\tilde{M}_{wi} \equiv \left( \alpha' u_0^{(7-p)/2} / \sqrt{d_p \lambda_{p+1}} \right)^{-1/2} (2\pi)^2 / L_i
\]
is the mass of the winding mode on the dual cycle. This condition gives
\[
L_i \ll \beta.
\] (71)

Note that the first condition (69) does not change under the \( T \)-duality (the value of the classical action is also invariant).

If, instead of a black \( D_p \) brane, we consider a solitonic solution which is obtained from the black brane by flipping \( t \leftrightarrow x_i \), then the conditions for the validity of supergravity are simply obtained by replacing \( \beta \) and \( L_i \) in the above conditions.

We will discuss these criteria below in some detail in the parameter regime of interest.

D.2 Phase transitions of the \( D_p \) solutions

Using the on-shell classical actions (66) and (68), we can determine various phase transitions \(^{25, 26, 51}\), as we will show now.

\(^{36}\)If the fermion satisfies an anti-periodic (AP) boundary condition along the \( L_i \) cycle, the theory is mapped to a type 0 theory through the \( T \)-duality. Then the bulk theory involves a tachyon and how the holographic description of gauge theory works in this frame is unclear. We thank Shiraz Minwalla for pointing this out.
D.2.1 GL transition

The Gregory-Laflamme (GL) transition \[49, 50, 51\] was originally found in the context of black rings in \(D \geq 5\) which were found to be unstable unless wrapped on a sufficiently small circle. In the context of black \(p\)-branes, the GL instability shows up as follows \[23, 71, 72\]. Suppose a black \(p\)-brane is wrapped on a circle of length \(L_1\), along which the fermion satisfies the periodic boundary condition. If \(L_1\) is so small that it violates \(70\), we need the \(T\)-dualized description in terms of a uniformly smeared black \((p - 1)\)-brane on the dual circle of length \(L'_1 = (2\pi)^2 / L_1\). If \(L'_1\) is large enough, the smeared black \((p - 1)\)-brane undergoes a GL transition leading to a black \(p - 1\) brane localized on the dual cycle. The transition can be studied dynamically, as well as thermodynamically. To study the latter, let us interpret \((66)\) without the regulator term (see Eq. (9) of \[24\]) as the Euclidean action (above extremality) of the smeared black \((p - 1)\)-brane:

\[
S_{p}/N^2 = -C_p \lambda_{p+1}^{\frac{5-p}{2}} L_1 \cdots L_p \beta^\frac{2(7-p)}{5-p}.
\]  

(72)

Here the action is evaluated in the \(Dp\)-frame (recall that the action is invariant under \(T\)-duality).

The value of the Euclidean action (above extremality) for the localized black \(p - 1\) brane solution is approximately (see below) given by \[5\]

\[
S_{p-1}/N^2 = -C_{p-1} \lambda_{p}^{\frac{4}{2}} L_2 \cdots L_p \beta^\frac{2(8-p)}{6-p}.
\]  

(73)

For small enough \(L_1\) (large enough \(L'_1\)), this is smaller than \(72\). The transition between the uniform and localized \(p - 1\) branes happens when \(72\) equals \(73\):

\[
\frac{\beta}{L_1} = \left( \frac{C_p}{C_{p-1}} \right)^{(5-p)(6-p)/4} \sqrt{\lambda_{p+1} L_1^{3-p}}.
\]  

(74)

Here we have used \(\lambda_p = \lambda_{p+1}/L_1\). In \(73\) one has used the approximation that the horizon size is much smaller than \(1/L_p\). Since this is strictly not true near the phase transition point (where the horizon size is of the order of \(1/L_1\)), the estimate \(74\) is approximate. See \[5\] for some details of this approximation.

Note that there are several arguments that this transition would be first order \[23, 71, 5\].

\[37\] Improvements to this approximation are discussed in \[71\].
D.2.2 GL type transition in the soliton sector

For bosonic theories in Euclidean spacetimes, the $\beta \leftrightarrow L_p$ interchange is a symmetry, provided all other radii are left unchanged. This is a symmetry of fermionic theories as well, provided the spin structures along $t$ and $x^p$ respect this symmetry. Thus, if there is a GL transition given by (74), there must be a GL type transition between a uniformly distributed solitonic $(p-1)$ brane to a localized solitonic $(p-1)$ brane when $1/L_1$ becomes too large. The transition is given by (74) with a $\beta \leftrightarrow L_p$ interchange:

\[
\frac{L_p}{L_1} = \left( \frac{C_p}{C_{p-1}} \right)^{(5-p)(6-p)/4} \sqrt{\lambda_{p+1} L_1^{4-p}}. \tag{75}
\]

D.2.3 The Scherk-Schwarz (SS) transition

This is a transition between the black $p$ brane configuration (62) and its solitonic counterpart (67). If we use the same large $U$ regulator for both the solutions, then (66) and (68) can easily be compared. The regulator term is the same in both the actions and can be ignored while comparing the two. Equating the two Euclidean actions, we get the transition temperature

\[
\beta = L_p. \tag{76}
\]

Similarly, we can consider another solitonic solution by replacing $L_p \leftrightarrow L_k$, if $L_k$ is also an AP circle. Then further transitions will happen at $\beta = L_k$ and $L_p = L_k$.

Note that this transition is also expected to be first order [26].

D.3 D2 branes for various spin structures: generalities

In this section, we apply the general properties of the $Dp$ solutions studied in the previous sections to the D2 brane on the 3-torus: $(t, x_1, x_2) = (t + \beta, x_1 + L_1, x_2 + L_2)$ and make a prediction for the phase structure of the 2 dimensional gauge theory (3). As mentioned in Section 3, we fix AP boundary condition for fermions along the $x_2$ circle. Then, there are 4 choices of boundary conditions for the fermions along the $t$ and $x_1$ directions: (AP,AP), (AP,P), (P,AP), (P,P), where P denotes the periodic boundary condition and AP denotes the anti-periodic one. We will evaluate the phase structure of the gravitational system with these 4 boundary conditions. (Since the result in the (P,AP) case can be obtained from (AP,P) by exchanging $\beta \leftrightarrow L_1$, we will show the results in the (AP,P) case only.)

\[\text{The notation } L_1 \text{ in this section represents what is called } L \text{ in the main text.}\]
The order parameters of this theory are the Wilson loop operators winding around each cycle:

\[
\text{Tr} U = \frac{1}{N} \text{Tr} P \exp \left( i \int_0^\beta A_0 dt \right), \quad \text{Tr} V = \frac{1}{N} \text{Tr} P \exp \left( i \int_0^{L_1} A_1 dx_1 \right), \quad \text{Tr} W = \frac{1}{N} \text{Tr} P \exp \left( i \int_0^{L_2} A_2 dx_2 \right).
\]

(77)

If the gravity solution has a contractible cycle (i.e. it wraps around a ‘cigar’), the expectation value of the corresponding Wilson loop operator is non-zero. If the solution is localized on a cycle, then also the expectation value of the corresponding Wilson loop operator is non-zero. However, if the solution wraps around a non-contractible cycle, the expectation value of the corresponding Wilson loop operator vanishes \[23\].

In order to derive the phase structure corresponding to figure 4, we will evaluate the phase structure of supergravity for each boundary condition by changing $\beta$ and $L_1$ with a fixed $L_2$, which is related to a cut off scale of the 2 dimensional gauge theory \[3\] (see footnote \[20\]).

From now on, we use units such that $\lambda_3 = 1$.

### D.3.1 D2 on (AP,AP,AP) torus

We consider the phase structure of D2 branes on a 3-torus with (AP,AP,AP) boundary conditions. In this case, three solutions appear: $D2_{\beta(L_1,L_2)}$, $D2_{L_1(\beta,L_2)}$ and $D2_{L_2(\beta,L_1)}$. (Recall the notation at the end of section D.1.1) In this case, the theory does not have the P circle and only the SS transition \[76\] happens. The phase structure is shown in figure 7.

As we have argued in appendix D.1.2 we need to check the validity of the gravity solutions. Let us consider the solitonic solution $D2_{L_2(\beta,L_1)}$. From \[69\], $L_2 \gg 1$ is required (in units where $\lambda_3 = 1$) to suppress the stringy excitations around the tip of the cigar. We need to check the condition related to the winding modes also. In order to suppress the winding modes, $L_1^{\beta/2}$, $\beta L_1 \gg L_2$ are required from \[70\]. The phase boundary $AB$ and $BC$ are given by $L_1 = L_2$ and $\beta = L_2$, and these conditions are satisfied on these boundaries in the large $L_2$ case. Thus the $D2_{L_2(\beta,L_1)}$ phase is always reliable, if $L_2$ is large.

Next we consider the black brane solution $D2_{\beta(L_1,L_2)}$. From \[69\], $\beta \gg 1$ is required. Thus this solution is not reliable when $\beta \sim O(1)$. We can see that the condition \[70\] for the winding modes is satisfied in the region surrounded by the phase boundaries and $\beta \gg 1$. Therefore the $D2_{\beta(L_1,L_2)}$ solution is reliable in the region indicated in figure 7. Similarly the solitonic solution $D2_{L_1(\beta,L_2)}$ is reliable in the region indicated in figure 4.
Figure 7: Phase structure of the D2 brane on \( T^3 \) with (AP,AP,AP) boundary condition for large \( L_2 \). The gravity analysis is valid above the dotted line.

Summing up these tests for the validity of the gravity analysis, the phase structure is reliable if \( L_2 \) is large and \( \beta \) and \( L_1 \) are above the dotted line in figure 7. This is, of course, a problem, since we are interested in the results in the \( L_2 \to 0 \) limit as we mentioned in Section 5. We will discuss this problem in appendix D.3.

### D.3.2 D2 on (AP,P,AP) torus

In this case, 4 solutions appear: \( D_{2\beta(L_1,L_2)} \), \( D_{2\beta(L_1,L_2)} \), \( D_{1\beta(L_2)} \) and \( D_{1\beta(L_2)} \). The phase structure for large \( L_2 \) is shown in figure 8. The phase boundaries are given as

\[
AB : L_1 = \left( \frac{C_1}{C_2} \right)^2 L_2^{2/3}, \quad EC : \beta = L_2, \quad BO : \beta = \left( \frac{C_2}{C_1} \right)^3 L_1^{3/2}.
\]

Note that \( AB \) is the GL type transition (75), \( BO \) is the GL transition (74) and \( EC \) is the SS transition (76).

Through similar tests for the validity of gravity as before, we find that the gravity analysis is valid only in the region above the dotted line in figure 8 (Again the gravity analysis in a small \( L_2 \) is invalid.)
D.3.3 D2 on (P,P,AP) torus

In this case, 4 solutions appear: $D_{2L_2(L_1)}$, $D_{1L_2(L_1)}$, $D_{1L_2(L_1)}$ and $D_{0L_2}$. The phase structure for a large $L_2$ is shown in figure 5. The phase boundaries are given by

$$AB : L_1 = \left( \frac{C_1}{C_2} \right)^2 L_2^{2/3}, \quad BD : \beta = L_1, \quad DO : \beta = \left( \frac{C_0}{C_1} \right)^{5/2} L_2^{1/2} L_1^{1/4}.$$  \hspace{1cm} (79)

These are GL type transitions (75). Other lines can be obtained by $\beta \leftrightarrow L_1$. The gravity analysis is valid only in the region above the dotted line in figure 5 for large $L_2$.

We will adopt the phase structure in this boundary conclusion as a prediction for the gauge theory. The reason will be explained in the next section.

D.4 Conjectured phase diagram of 2D bosonic gauge theory

In the preceding subsections, we have obtained various phase structures from gravity for different spin structures (i.e., for different fermion boundary conditions). Since these results are valid only for large $L_2$, we need to extrapolate
them to small $L_2$, where the bosonic gauge theory should appear (see footnotes 20 and 22).

From the viewpoint of the two-dimensional bosonic gauge theory, a holographic correspondence with the various gravity phase diagrams described in this section is problematic since the latter have a dependence on fermionic boundary conditions, while no such dependence obviously exists for the two-dimensional gauge theory (such dependences, however, exist for the three-dimensional SYM theory, which is more directly related to the gravity description). Indeed one such problem was pointed out in [67]. This argument is further developed in [68]. The arguments presented in these papers indicate that the phase transition in the bosonic gauge theory cannot be regarded as a continuation of the SS transition of gravity. Thus, in order to have a smooth continuation of phase boundaries between the holographic description and the two-dimensional gauge theory, we should avoid choosing the $(AP, AP, AP)$ and $(AP, P, AP)$ boundary conditions, in which the SS transition appear. For this reason, we choose the $(P, P, AP)$ case to read off the predicted phase structure from gravity.

Note that all the transitions in the $(P, P, AP)$ case are of the GL-type, and are supposed to be first order phase transitions [23, 51, 5]. In this case, in addition to the uniformly distributed solitonic $(p-1)$ brane and localized solitonic $(p-1)$ brane discussed in [D.2.2], non-uniformly distributed solitonic $(p-1)$ brane, which is always unstable, appears. These three solutions would correspond to the uniform, non-uniform and gapped distribution in the gauge theory (see figure [2]). Indeed, the free energies of these gravity solutions are expected to satisfy a similar relation to those in the gauge theory shown in figure [3] through numerical study in general relativity [53]. This fact also supports our prediction from the gravity analysis.

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