The universe seen at different scales

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Abstract

A large–scale smoothed–out model of the universe ignores small–scale inhomogeneities, but the averaged effects of those inhomogeneities may alter both observational and dynamical relations at the larger scale. This article discusses these effects, and comments briefly on the relation to gravitational entropy.

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1 Different scale descriptions: coarse–graining the gravitational field

Any mathematical description of a physical system depends on an \textit{averaging scale} characterizing the nature of the envisaged model. This averaging scale is usually hidden from view: it is taken to be understood. Thus, when a fluid is described as a continuum, this assumes one is using an averaging scale large enough that the size of individual molecules is negligible. If the averaging scale is close to molecular scale, small changes in the position or size of the averaging volume lead to large changes in the measured density and velocity of the matter, as individual molecules are included or excluded from the reference volume. Then the fluid approximation is no longer applicable; rather one is using a detailed description of the fluid where individual molecules are represented. Usual work referring to the fluid density and velocity assumes a medium–size averaging scale: not so small that molecular effects matter, but not so large that spatial gradients in the properties of the fluid are significant.

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The actual averaging scale, or rather the acceptable range of averaging scales, is not explicitly stated but is in fact a key feature underlying the description used, and hence the effective macroscopic dynamical laws investigated. Indeed, different types of physics (particle physics, atomic physics, molecular physics, macroscopic physics, astrophysics) correspond to different assumed averaging scales. Thus, instead of referring to a density function $\rho$, one should really refer to a function $\rho_L$; the density averaged over volumes characterized by scale length $L$. The key point about the fluid approximation is that, provided this length scale is in the appropriate domain, then its actual value does not matter; i.e. when it is in this range, then changing $L$ by a factor of 10, 100, or even much more makes no difference: the measured density and average velocity will not change. But if you change $L$ by a very large amount until outside this range, this is no longer true. Hence, there is a range of validity $L_1 < L < L_2$ where the fluid approximation holds [3] and explicit mention of the associated averaging scale may be omitted.

In electromagnetic theory, polarization effects result from a large-scale field being applied to a medium with many microscopic charges. The macroscopic field $E$ differs from the point-to-point microscopic field, which acts on the individual charges because of a fluctuating internal field $E_i$, the total internal field at each point being $D = E + E_i$ ([43], p.116). Spatially averaging, one regains the average field because the internal field cancels out: $E = \langle D \rangle$, indeed this is how the macroscopic field is defined (implying invariance of the background field under averaging: $E = \langle E \rangle$). On a microscopic scale, however, the detailed field $D$ is the effective physical quantity, and so is the field "measured" by electrons and protons at that scale. Thus, the way different test objects respond to the field crucially depends on their scale (a macroscopic device will measure the averaged field).

Now, exactly the same issue arises with regard to the gravitational field. Applications such as the solar system tests of general relativity theory, and in particular Einstein’s triumphant prediction of light bending by the Sun, are at solar system scales. We apply gravitational theory, however, at many other scales: to star clusters, galaxies, clusters of galaxies, and large-scale structures (walls and voids), as well as to black holes (occurring at solar system and star cluster scales, and possibly at much smaller scales).

Cosmology utilizes the largest scale averaging envisaged in astrophysics: a representative scale is assumed that is a significant fraction of the Hubble scale, and the cosmological velocity and density functions are defined by averaging on such scales ([25], p.111). Einstein first introduced the fluid approximation in his 1917 static universe model, as well as a highly idealized macroscopic model of the large-scale (smoothed) geometry of the universe. This geometrical idealization was then canonized via Milne's *cosmological principle* ([68], p.408), or a somewhat more general *Copernican principle* ([39], pp.134, 350);
the resulting locally isotropic, constant–curvature Robertson–Walker geometries ([39], Sect. 5.3) are nowadays taken to be a good description of the known region of the universe. The best justification of this assumption is the measured high degree of isotropy of the cosmic blackbody background radiation, taken together with a Copernican assumption (see [39], pp.351–3, for the argument in the case of exact isotropy, and [65] or [33], Sect. 8.5 for the case of almost–isotropy).

However, a range of scales of description are relevant to cosmology. There are levels of approximation in modelling the universe, each with a hidden averaging scale. One can have a description in which every star is represented, or every galaxy (the stars averaged over), or only the largest scale cosmological structures (even galaxies averaged over, as in the fluid approximation). A typical cosmological simulation of “dark matter” gravitational clustering uses Newtonian theory and resolves fluid elements that still contain $10^{60}$ dark matter particles. This implicit coarse-graining can be made explicit within a Newtonian kinetic description: introducing filtering scales for a distribution of $N$ self–gravitating particles in phase space reveals that the washed out small–scale degrees of freedom are represented by additional force terms that account for the dynamical coupling to these degrees of freedom [14]; they can also be modelled by phenomenological noise and/or stochastic forces [15], [51], and can lead to drastic qualitative changes of the system. However, this kind of calculation would be much more difficult in a General Relativity context.

The General Relativistic cosmological perturbation solutions used to study structure formation embody two interacting levels: the background (zero–order) model, almost always a Robertson–Walker metric, and the perturbed (first–order) model representing the growth of inhomogeneities, represented by a perturbed Robertson–Walker metric. The question then is how do models on two or more different scales relate to each other in Einstein’s gravitational theory [26]. This is a difficult issue both because of the non–linearity of Einstein’s equations, and because of the lack of a fixed background spacetime – one of the core features of Einstein’s theory. This causes major problems in defining suitable averaging processes as needed in studying these processes. While there have been many analyses of this problem, there are still issues to be resolved in relation both to observations and dynamics, and in how this relates to gravitational entropy and the arrow of time.

2 Non–commutativity of averaging and observations

The usual analysis of cosmological observations is based on the Mattig equations relating apparent magnitude and redshift [63], [25], derived from analyzing the behaviour of null geodesics in Robertson–Walker spacetimes. In
terms of the Sachs optical scalar equations, for hypersurface-orthogonal null geodesics in a general spacetime, the basic equations are:

\[ \frac{d\theta}{dv} = -R_{ab}K^aK^b - 2\sigma^2 - \frac{1}{2}\theta^2 ; \quad \frac{d\sigma_{mn}}{dv} = -E_{mn} , \]  

where \( \theta \) is the rate of expansion of the null geodesics with tangent vector \( K^a = dx^a/dv \) and affine parameter \( v \), \( \sigma_{mn} \) is their shear, \( R_{ab} \) the Ricci tensor, and \( E_{mn} \) a matrix of Weyl tensor components ([39], p.88; [64], pp.108–9). In the idealized Robertson–Walker case, the Weyl tensor \( C_{abcd} \) vanishes, but the Ricci tensor is non–zero, being given via the Einstein field equations from the matter present. Thus, \( C_{abcd} = 0 \Rightarrow E_{mn} = 0 \), and the relevant solutions are shear–free:

\[ \sigma^2 = 0 \Rightarrow \frac{d\theta}{dv} = -R_{ab}K^aK^b - \frac{1}{2}\theta^2 . \]  

Integration gives the Mattig relations applicable to Friedmann universe models ([64], pp.134–7), also elegantly obtainable from the geodesic deviation equations with vanishing Weyl Tensor [34].

However, in the real universe, observations take place via null geodesics lying in the empty spacetime between galaxies (you can’t see the further galaxy, if there is one in the foreground). Thus, the real situation in a universe with no intergalactic medium (all the matter is concentrated in galaxies) is the opposite of that above: in the region of spacetime traversed by the geodesics, the Ricci tensor vanishes, so

\[ \frac{d\theta}{dv} = -2\sigma^2 - \frac{1}{2}\theta^2 , \]  

but the non–zero Weyl tensor (the tidal field caused by nearby matter) generates shear that then causes focussing. Thus, the microscopic description of the focussing \( \sigma \neq 0, R_{ab} = 0, E_{mn} \neq 0 \) is radically different from the macroscopic one \( \sigma = 0, R_{ab} \neq 0, E_{mn} = 0 \), and the area distance–redshift relation may be expected to be different on microscopic scales (i.e. the small solid angle bundles of null geodesics actually used in observations of individual objects), as compared with macroscopic scales (averaging over large solid angles).

Various proposals have been made to deal with this. The most popular is the Dyer–Roeder distance [23,24], obtained by assuming only a fraction \( f \) of the total mass density is encountered by the light–rays but ignoring the shear. Thus, in (2) one replaces \( R_{ab}K^aK^b \) by \( fR_{ab}K^aK^b \) and works out the corresponding area distance ([64], pp.138–143; [21]). This may be a good approximation if galaxies are embedded in a fairly uniform intergalactic medium.
of dark matter, but clearly does not take shear effects properly into account. How good it is will depend on the nature of clustering in the universe and how the averaged distribution impacts along the line of sight [49].

One can approach the topic in other ways: for example by using stochastic methods [4], or detailed examination of geodesics in Swiss–Cheese universe models [44,5]. It has been suggested that energy conservation will imply that the effect averages out over the entire sky [69], but this calculation assumed that areas of a bundle of null geodesics were the same in the perturbed and background models, which will not be the case when one takes the effect of caustics into account [29]. Indeed, areas increase slower than in a Robertson–Walker model in the empty spaces between matter, where the Ricci term is zero, and faster in the high–density regions where matter is concentrated, so one might think these effects cancel out. However, the strongly lensed rays soon go through a caustic and emerge highly divergent, so that areas are rapidly increasing again. It is plausible that on average the overall effect is always an increase in area, that is a lesser area distance than in the smooth background model.

The potential importance of this effect is in relation to the interpretation of the Supernova data [46,60,2], which is usually taken to imply the existence of a cosmological constant or quintessence causing acceleration of the universe at recent times [50]. Kantowski [45] has obtained analytic expressions for distance–redshift relations that have been corrected for the effects of inhomogeneities in the density. The values of the density parameter and cosmological constant inferred from a given set of observations depends on the fractional amount of matter in inhomogeneities and can significantly differ from those obtained by using the Mattig relations. As an example, a determination of $\Omega_0$ made by applying the homogeneous distance–redshift relation to SN 1997ap at $z = 0.83$ could be as much as 50% lower than its true value. It could be that the apparent acceleration term detected is at least partly due to this optical effect: focussing of null geodesics is different in a lumpy universe than in a smooth one. Clearly, this effect needs careful investigation.

3 Non–commutativity of averaging and dynamics

The key–point in considering dynamical effects is that the two processes involved in relating the field equations at different scales do not commute [26]. These processes are:

\[ \text{E: calculating the Einstein tensor } G_{1ab} := R_{1ab} - \frac{1}{2} R g_{1ab} \text{ from a metric tensor } g_{1ab}, \text{ and, hence, determining the quantity } E_{1ab} := G_{1ab} - \kappa T_{1ab} \text{ for } g_{1ab}, \text{ where } T_{1ab} \text{ is the matter tensor appropriate to the scale represented by } g_{1ab}; \]
averaging the metric tensor $g_{1ab}$ to produce a smoothed metric tensor $g_{2ab}$:

$$g_{2ab} = \langle g_{1ab} \rangle$$

and the matter tensor $T_{1ab}$ to produce a corresponding smoothed matter tensor $T_{2ab}$:

$$T_{2ab} = \langle T_{1ab} \rangle.$$  

Now in general the averaging process does not commute with taking derivatives: for a function $g$, usually $\partial_i \langle g \rangle \neq \langle \partial_i g \rangle$ (see equations (8), (9) below for specific examples). Furthermore the inverse metric $g^{ab}_2$ (non-linearly dependent on the metric tensor components $g_{1ab}$) is not the smoothed version of $g^{ab}_1$. The resulting Christoffel terms $\Gamma_{2bc}^a$ are therefore not the smoothed version of $\Gamma_{1bc}^a$; hence the Ricci tensor components $R_{2ab}$, non-linearly dependent on $\Gamma_{2bc}^a$, are not the smoothed versions of $R_{1ab}$. Extra non-linearities occur in calculating the Einstein tensor $G_{2ab} = R_{2ab} - \frac{1}{2} R g_{2ab}$ from the Ricci tensor $R_{2ab}$. Thus, if you smooth first and then calculate the field equations, you get a different answer than if you calculate the field equations first and then smooth; symbolically $A(E(g_{1ab})) \neq E(A(g_{1ab}))$.

Suppose the field equations are true at the first scale: $E_{1ab} = 0$, then they will not be true at the second scale: $E_{2ab} := G_{2ab} - \kappa T_{2ab} \neq 0$. Thus, there will be an extra term in the equations at the smoother scale. We can either regard it as an extra term on the left-hand-side,

$$G_{2ab} - E_{2ab} = \kappa T_{2ab}, \quad (4)$$

representing a modified curvature term, or as an extra term on the right-hand-side,

$$G_{2ab} = \kappa T_{2ab} + E_{2ab}, \quad (5)$$

where it is regarded as an extra contribution to the matter tensor. Which is the more appropriate interpretation depends on the context.

Szekeres [67] developed a polarization formulation for a gravitational field acting in a medium, in analogy to electromagnetic polarization. He showed that the linearized Bianchi identities for an almost flat spacetime may be expressed in a form that is suggestive of Maxwell’s equations with magnetic monopoles. Assuming the medium to be molecular in structure, it is shown how, on performing an averaging process on the field quantities, the Bianchi identities must be modified by the inclusion of polarization terms resulting from the induction of quadrupole moments on the individual “molecules”. A model of a medium whose molecules are harmonic oscillators is discussed and constitutive equations are derived. This results in the form:

$$E^{2ab} = Q^{abcd}_{\;\;\;\; cd}, \quad (6)$$
that is $E_{2ab}$ is expressed as the double divergence of an effective quadrupole gravitational polarization tensor $Q^{abcd}$ with suitable symmetries:

$$Q^{abcd} = Q^{[ab][cd]} = Q^{cdab}. \quad (7)$$

Gravitational waves are demonstrated to slow down in such a medium.

The problem with such averaging procedures is that they are not covariant. They can be defined in terms of the background unperturbed space, usually either flat spacetime or a Robertson–Walker geometry, and so will be adequate for linearized calculations where the perturbed quantities can be averaged in the background spacetime (although even here the gauge problem arises, see below). But the procedure is inadequate for non–linear cases, where the integral needs to be done over a generic lumpy (non–linearly perturbed) spacetime that are not “perturbations” of a high–symmetry background. However, it is precisely in these cases that the most interesting effects will occur.

The only tensor integrals that are well–defined over a generic spacelike surface or spacetime region (and one interesting issue is which of these one should use) are for scalars [9,10], unless one uses the bitensors associated with Synge’s world function [66], based on parallel propagation along geodesics, to compare tensors at different points in a normal neighbourhood. The problem is that they cannot be used for averaging the metric tensor, for it is the metric tensor itself that defines the parallel propagation used in this process, and so is left invariant by it (since $g_{abc} = 0$). So, one has to devise a procedure in which either the field equations are represented only in terms of scalars, possible for example if one takes components relative to a covariantly uniquely defined tetrad, or else bitensors are used to define averages of quantities other than the metric.

Zalaletdinov has taken this issue seriously, and provided the only sustained such attempt based on bitensors [71]. He proposes a macroscopic description of gravitation based on a covariant spacetime averaging procedure. The geometry of the macroscopic spacetime follows from averaging Cartan’s structure equations, leading to a definition of correlation tensors. Macroscopic field equations (averaged Einstein equations) can be derived in this framework. It is claimed that use of Einstein’s equations with a hydrodynamic stress–energy tensor means neglecting all gravitational field correlations, and a system of macroscopic gravity equations is given when the correlations are taken into consideration. This approach has not won many adherents, but is nevertheless a systematic and coherent attempt to set up the problem generically.
4 Gravitational radiation

So far, there are two main applications of dynamical averaging. The first is to the issue of gravitational radiation. When electromagnetic radiation is present, one can characterize it as the high–frequency part of the electromagnetic field [25], and assign to it an energy density and momentum. This leads to an energy–momentum tensor that then serves as a source of curvature in the Einstein field equations. An obvious question is if one can do the same for gravitational radiation: can one identify it locally, and then assign to it an energy density and momentum? If so, there should be a form of the gravitational equations where this high–frequency part of the gravitational field acts as an effective source of spacetime curvature. But this is a version of the problem described above: it is just the definition of a contribution $E_{2ab}$ to the macroscopic gravitational field due to the fine–scale structure of the high–frequency radiation.

The problem is that gravitational radiation is only easily determined in linearly perturbed spacetimes; in more general spacetimes it is not easy to define the gravitational radiation part of the curvature, except near infinity in asymptotically flat spacetimes. Isaacson [41,42] considered the case of linear perturbations about flat spacetime, determining the backreaction due to the gravitational radiation in this case. He obtained a close analogy with the electromagnetic situation: the ‘shortwave approximation’ shows how the stress–energy in the waves creates background curvature ([53], Sect. 35.13). A similar process can be applied to gravitational radiation in cosmological backgrounds, and backreaction by low–frequency gravitational radiation has been discussed by Dautcourt [20]. To understand non–linear phenomena in gravitational radiation, the possibility of solitonic solutions and caustics should also be of concern, since these phenomena are presumably easier to detect.

5 Cosmology: Backreaction

The second application is to understand the nature of the backreaction of perturbations in cosmology [11]. Unlike the gravitational radiation case, where one averages over tensor perturbations, here one first thinks of averaging over scalar quantities like the density or the rate of expansion, in order to get control on cosmological parameters in an inhomogeneous universe model. As long as one works with exact equations for the evolution of those fields in a given foliation of spacetime, such an averaging procedure is covariant, e.g. for idealized cases like dust or a perfect fluid we can work in the ‘covariant fluid gauge’ [7,8]. For these cases generalized forms of Friedmann’s equations can be employed to study backreaction [9,10]. As soon as we invoke explicit
model assumptions, e.g. perturbation theory, one runs directly into the gauge problem for cosmological perturbations, as well as the covariance question mentioned above.

The gauge problem is the following: when you perturb a smooth background cosmological metric $\bar{g}_{ab}$ to obtain a perturbed metric $g_{ab} = \bar{g}_{ab} + h_{1ab}$, the inverse relation is not unique: there is no agreed averaging or fitting process that will give back a unique background metric $\bar{g}_{ab}$ back from the “lumpy” metric $g_{ab}$ [32]. Some other smooth metric $\bar{g}_{2ab}$ could have been chosen as the background metric instead, leading to a different definition of the perturbations: $h_{2ab} := g_{ab} - \bar{g}_{2ab}$, instead of $h_{1ab} := g_{ab} - \bar{g}_{1ab}$. The choice of background metric $\bar{g}_{ab}$ for a specific “lumpy” metric $g_{ab}$ is called a ‘gauge choice’. The backreaction problem will look very different if described in terms of different gauges.

The best way to look at this is to think of a gauge choice as a mapping of a smooth background metric $\bar{g}_{1ab}$ into the lumpy universe with metric $g_{ab}$ [30]. At each point in the real spacetime the density perturbation $\delta \bar{\rho}$ is then defined by $\delta \bar{\rho} := \rho - \bar{\rho}$, where $\rho$ is the actual density at that point, and $\bar{\rho}$ the background density at the same point. The key–issue is the choice of surfaces of constant time in the perturbed spacetime, conventionally taken to represent the image of surfaces of constant density of the background spacetime. It then becomes clear that one can for example set the density perturbation to zero by choosing the mapping so that the surfaces of constant background density $\bar{\rho}$ are the same as the surfaces of constant real density: for then at each point $\rho = \bar{\rho} \Rightarrow \delta \rho = 0$. However, with this choice, the fluid flow lines will not be orthogonal to the surfaces of constant density, so there will still be a non–zero density variation measured by comoving observers. Gauge issues arising in treating multi–component fluids raise extra issues because of the multiple possible choices of reference velocity field [22].

The remedy to this disconcerting behaviour is to choose gauge invariant variables, for example a set of covariantly defined variables that vanish in the background spacetime [30]. While many studies have been carried out for quantifying backreaction effects in cosmology, where the smoothed–out effect of the small–scale perturbations causes extra terms in the Friedmann equations for the background metric, none have been done that both fully and clearly take the gauge issue into account and go beyond linear order. This is a key–issue waiting to be resolved. One certainly wants to go at least to second order in understanding the effects of non–linear perturbations, and while linear perturbations are well–understood, there are still many competing second order methods without a proper consensus on their implications emerging yet. Many of the crucial results at linear order no longer hold, for example scalar, vector and tensor perturbations are no longer independent of each other at second order [18,48], and then the backreaction in turn affects
the perturbations themselves [52].

Isaacson’s method mentioned above has been used in the cosmological context [35], as has Zalaletdinov’s [19]. In Zalaletdinov’s approach to the averaging problem in cosmology, the Einstein field equations on cosmological scales are modified by appropriate gravitational correlation terms. For a spatially homogeneous and isotropic macroscopic spacetime, the correlation tensor is of the form of a spatial curvature term. However, it is not clear how this approach relates to the gauge problem.

There is no doubt that interesting effects occur. How can we design a strategy that allows both making contact with the well-developed inventory of Friedmannian cosmology and quantifying backreaction effects? Cosmological parameters like the rate of expansion or the mass density are to be considered as volume-averaged quantities, and only these can be compared with cosmological observations. For this reason we expect that the relevant parameters are intrinsically scale-dependent unlike the situation in a Friedmannian cosmology. Averaging scalar characteristics on a Riemannian spatial domain delivers the effective dynamical sources that an observer would measure, but although he measures within the lumpy spacetime, he – due to a lack of better standards – is going to interpret his observations within a Friedmannian fitting model. This suggests a logical division of the averaging problem into 1) calculating averages in the real manifold, and 2) determining the mapping between averages in the real manifold and averages in the Friedmannian model. The first averaging is straightforward for scalars, as we mentioned above, and it encounters what we may call non-commutativity of averaging and time-evolution: this is a purely kinematical property that can be expressed, for a scalar field \( \psi \), through the rule

\[
\partial_t \langle \psi \rangle - \langle \partial_t \psi \rangle = \langle \theta \psi \rangle - \langle \theta \rangle \langle \psi \rangle .
\]

The fluctuation part on the right-hand-side of this rule produces the kinematical backreaction, which is now studied in the context of the dark energy problem (see below). The second “averaging” is more adequately thought of as a rescaling of the tensorial geometry. A (Lagrangian) smoothing as opposed to (Eulerian) rescaling of the metric on regional spatial domains has been proposed by Buchert and Carfora [12], using a global Ricci deformation flow for the metric. The smoothing of geometry implies a renormalization of averaged spatial variables, determining the effective cosmological parameters as they appear in the Friedmannian fitting model. Two effects that quantify the difference between background and real parameters were identified: curvature backreaction and volume effect [13]. Both are the result of an inherent non-commutativity of averaging and spatial rescaling. In this way we look at the averaging problem in two directions in function space: time-evolution (as a deformation in direction of the extrinsic curvature of the space sections en-
coding the kinematical variables), and scale—“evolution” (as a deformation in
direction of the intrinsic 3–Ricci curvature). With regard to a proper relation
of those averages to observations, however, the possibility of averaging on the
lightcone has to be seriously considered [32].

Employing such a logical split, it is reasonable to ask whether the universe
described by a kinematically averaged model accelerates, independently of the
question of whether we think that the universe accelerates because we may
be using the wrong fitting model. With regard to the dark energy problem,
the question of whether a cosmological constant is needed in the standard
fitting model is then related to all of the effects mentioned above, while the
question addressed to the realistic model only depends on the quantitative
importance of the kinematical backreaction compared with the other averaged
sources [36,17,61,47]. It should be emphasized that in Newtonian cosmology
global kinematical backreaction is absent, since (for Euclidean space sections)
the fluctuating source term is a total divergence and thus vanishes for the
periodic–boundary architecture of Newtonian models [16].

The key–issue is whether these backreaction effects are significant in cosmo-
logy. On the one hand, they might play a significant role in the inflationary
era [54,38]. On the other, it can possibly help explain the apparent dynamical
existence either of dark energy and/or of dark matter as effective terms in the
macroscopic dynamics at recent times. Various papers suggest the effect may
indeed be significant, for example the observed acceleration of the universe
could possibly be the result of the backreaction of cosmological perturbations
rather than the effect of a negative–pressure dark energy fluid [70], [47]. How-
ever, other studies obtain different results [62], [17], [55], [56], [61]. Gauge
effects are problematic [6,37], and many astrophysicists doubt the effect is sig-
nificant. The issue still has to be resolved. In any case, a detailed investigation
of backreaction effects helps to improve fitting models on regional scales for a
better interpretation of observational data.

6 Entropy and coarse–graining

Related to all this is the puzzling question of gravitational entropy. The spon-
taneous structure growth in the expanding universe due to gravitational at-
traction appears to be contrary to all the statements about entropy in standard
textbooks [27]. This must somehow be related to the nature of the entropy of
the gravitational field itself, not just the entropy of matter in a gravitational
field.

Now the key–feature regarding the entropy of matter, as clearly explained by
Penrose [58,59], is that it is associated with the loss of information that occurs
with any coarse-grained description of matter. The most likely macroscopic states will be those that correspond to the largest numbers of microscopic states; that is to the largest volumes of phase space. This is made clear in Boltzmann’s definition of entropy: \( S = k \ln V \Gamma \) where \( k \) is Boltzmann’s constant and \( V \Gamma \) the volume of phase space with points indistinguishable from each other by means of macroscopic observations of some macro (coarse–grained) variable to some accuracy \( \varepsilon \). The dynamics of the system is accompanied by an increase of this entropy as the representative point in phase space moves from less probable to more probable states.

One might therefore expect that a proper definition of gravitational entropy would similarly be related to some kind of coarse–graining of the gravitational field. However, most attempts at definitions of gravitational entropy in the cosmological context (e.g. [57], [1]) build on Penrose’s proposal [58,59] that it be related to the magnitude of the Weyl tensor, with no introduction of coarse–graining. This is quite puzzling, given the persuasiveness of Penrose’ arguments that in the case of matter descriptions, entropy is always related to such coarse–graining. In our view this is one of the most fundamental missing aspects of gravitational theory: a satisfactory relation of gravitational entropy for a general gravitational field in terms of a coarse–grained description of that field, therefore relating to all the issues mentioned in the preceding sections.

A promising start has been made by Hosoya et al. [40]: if we are only concerned with averaging the matter inhomogeneities on an inhomogeneous geometry, one can deduce an entropy measure for the distinguishability of the density distribution from its average value directly from the non–commutativity rule:

\[
\partial_t \langle \varrho \rangle - \langle \partial_t \varrho \rangle = -\frac{1}{V} \partial_t S\{\varrho||\langle \varrho \rangle\} \quad ; \quad S\{\varrho||\langle \varrho \rangle\} := \int \varrho \ln \frac{\varrho}{\langle \varrho \rangle},
\]

where the functional \( S\{\varrho||\langle \varrho \rangle\} \) is known in information theory as the Kullback–Leibler relative entropy, spatial averaging and integration is performed over a domain with volume \( V \).

The conjecture has been made [40] that this functional is, after a sufficient period of time, always globally increasing. This (in view of canonical considerations, e.g. in isolated Markovian systems) counter–intuitive statement is justified in a self–gravitating system because gravity is long–range, the averaging domain is not isolated, and gravity invokes a negative feedback: structural inhomogeneities are amplified due to gravitational instability. We may expect that the information content in the matter inhomogeneities is always increasing.

Given such a definition, the problem is to determine whether increasing total entropy (in the gravitational field and in the matter distribution) occurs al-
ways, or whether this is true only for special initial conditions. As discussed by Penrose [58,59], it seems plausible that the latter is the case, with the arrow of time in physics arising from boundary conditions at the start and end of the universe: specifically, the Weyl tensor taking a special form at the start of the expansion of the universe but a generic form at the end. The specific details of this proposal have never been clarified, and it is possible that the relation is not due to the Weyl tensor per se but rather due to a spatial integral of the divergence of the Electric part of the Weyl tensor (Ellis and Tavakol, unpublished). A further problem is then relating the arrow of time for structure growth in the universe to that for electromagnetic and gravitational radiation [31]. Here again coarse–graining is crucial, for this relates to the kind of multi–scale description of the gravitational field envisaged by Isaacson, as discussed above.

Entropy and the associated arrow of time are fundamental to macroscopic physics. Their foundations in relation to microphysics remain mysterious in the case of general gravitational fields. The entropy of black holes is of course well understood, but this is an extreme case that does not by itself help us understand the relation of entropy to spontaneous structure formation in the expanding universe. Until this is solved, we cannot claim to properly understand the nature of entropy in the cosmological context [28].

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