Low-Energy QCD and Ultraviolet Renormalons

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Abstract

We discuss the contribution of ultraviolet (UV) renormalons in QCD to two–point functions of quark current operators. This explicitly includes effects due to the exchange of one renormalon chain as well as two chains. It is shown that, when the external euclidean momentum of the two–point functions becomes smaller than the scale $\Lambda_L$ associated with the Landau singularity of the QCD one–loop running coupling constant, the positions of the UV renormalons in the Borel plane become true singularities in the integration range of the Borel transform. This introduces ambiguities in the evaluation of the corresponding two–point functions. The ambiguities associated with the leading UV renormalon singularity are of the same type as the contribution due to the inclusion of dimension $d = 6$ local operators in a low–energy effective Lagrangian valid at scales smaller than $\Lambda_L$. We then discuss the inclusion of an infinite number of renormalon chains and argue that the previous ambiguity hints at a plausible approximation scheme for low–energy QCD, resulting in an effective Lagrangian similar to the one of the extended Nambu-Jona-Lasinio (ENJL) model of QCD at large $N_c$.

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1 Introduction.

The origin of renormalons in quantum field theory goes back to early work by 't Hooft [1], Lautrup [2] and Parisi [3]. They made the observation that in renormalizable theories like for example quantum electrodynamics (QED), there exists a class of Feynman diagrams which give rise to a characteristic pattern in the coefficients of the large-order terms in the perturbation series; these coefficients have an $n!$-growth with the same sign indicating that the corresponding series are not Borel summable. In QED this growth originates in the large momentum integration region of virtual photons dressed with vacuum polarization insertions which leads to singularities in the associated Borel plane; the so-called UV renormalons [1, 2, 5–7]. In quantum chromodynamics (QCD), it is the low momentum integration region of virtual gluons dressed with running couplings what is the source of the non integrable singularities in the Borel plane; the so-called IR renormalons [1,8,9]. The appearance of these singularities is perhaps not surprising since in the extreme kinematic regimes in question it is not expected that these theories are well described by a simple perturbative expansion in the coupling constant.

There are two good reasons to concentrate on the class of Feynman graphs which are at the origin of the renormalon singularities. One is the fact that in massless theories, like QCD with massless quarks, the breaking of conformal invariance is encoded in the $\beta$–function. If one wants to find hints from perturbation theory on the possible origin of scales it seems appropriate to focus one's attention on the summation of the infinite subset of graphs associated with the renormalization of the coupling constant. The other reason is the opportunity that is offered to explore issues which have to do with the analyticity (or rather, lack of it) in the coupling constant.

The study of renormalon properties in gauge theories is at present an active field of research. The rôle of IR renormalons in the operator product expansion of two–point functions and their relationship with non–perturbative inverse power corrections has been extensively discussed in the literature [10,11]. This and further discussions which originate in the work of ref. [17], has led to a new point of view concerning renormalons in QCD. The focus is now on the possibility that systematic study in a given hadronic process might suggest generic non–perturbative effects of a universal nature. The basic idea is that genuinely non–perturbative effects ought to cure [Aany disease which appears when perturbation theory results are analytically continued [19]. This applies to the case of IR renormalons and, as recently suggested by Vainshtein and Zakharov(V-Z) [20,21], perhaps to the much less explored rôle of UV renormalons in QCD as well.

In QED (or for that matter in any renormalizable theory which is not asymptotically free) the UV renormalons cause a real obstruction to defining the theory in the ultraviolet from a resummation of the perturbative series. In earlier work by Parisi on $\phi^4$ theories [7] it was argued that it ought to be possible to mimic the contribution from large momenta (namely UV renormalons) by means of insertions of local dimension six and higher operators [7]. Specifically, local here means that all the physics at momentum larger than a certain scale $k$ is encoded in higher dimensional composite operators that are local on a scale of $1/k$. In other words, these composite operators are products of fields evaluated at the same point but suppressed by inverse powers of the cut–off momentum $k$. On these grounds, one

\[\text{For a comprehensive review of the subject and a collection of articles previous to 1990 see ref. [4].}\]
would expect that by adding carefully–adjusted higher–dimensional operators to the initial
Lagrangian one ought to be able to remove all the UV renormalons from QED to all orders
in the coupling constant \[ \mu \]. The intuitive reason is that QED can always be thought of
as the low–energy limit of a larger asymptotically free theory (like e.g. \( SU(5) \)) for which,
in principle, UV renormalons are not a problem. Of course this means adding an infinite
tower of higher dimensional operators which will make QED an effective nonrenormalizable
theory, in agreement with its alleged triviality \[ \mu \]. Notice that the previous argument is
entirely perturbative. In other words, the higher dimensional operators in question are still
considered as irrelevant. If, on the other hand, a few of them turned out to be relevant, it
might be possible to truncate the list of operators to just the relevant ones. This is actually
what happens in the quenched ladder approximation of QED \[ \mu \] where four–fermion opera-
tors of the Nambu–Jona-Lasinio type \[ \mu \] (NJL) turn out to become relevant operators and
consequently are kept in the Lagrangian on an equal footing as the four dimensional ones.

In this connection we wish to point out that there has also been an ongoing struggle in the
lattice community in trying to clarify if an abelian theory in the strong coupling regime has
anything to do with the dynamics of models à la Nambu–Jona-Lasinio. Although this point
is still unclear \[ \mu \], there seems to be some evidence that four–fermion operators may play an
indispensable role for understanding the QED non–perturbative (i.e. ultraviolet) dynamics.

In QCD, and provided that the external momenta of a given Green’s function are not
in regions of exceptional momenta \[ \mu \], the UV renormalons are not an obstruction to inte-
grate virtual Euclidean momenta. Therefore one does not expect them to be at the origin
of fundamental ambiguities. In their recent work \[ \mu \], V-Z have found however that the
contribution to the Adler function from the leading UV renormalon coming from the ex-
change of two chains of vacuum polarization self–energy loops in an Abelian–like model was,
contrary to naive expectations, dominant over the contribution coming from a single chain.
Furthermore, they also argued that the contribution from more and more chains should be
equally important and that, consequently, the actual value of the residue of the leading UV
renormalon was ill defined. They then concluded that, at the phenomenological level, this
could be taken as an indication that a new type of ambiguity may appear , which in the
particular case of a two–point function with Euclidean momentum \( Q^2 \), shows up as possible
\( 1/Q^2 \)-like contributions.

Interestingly enough, the leading UV behaviour found by V-Z was shown to originate in
the contribution from the insertion of \( d = 6 \) four–fermion operators. The appearance of these
four–fermion operators, much like those of the Nambu–Jona-Lasinio model \[ \mu \], is rather
intriguing. It is known that there are extensions of this model which, when taken as models
of QCD in the large–\( N_c \) limit at intermediate scales \( \lesssim O(1\text{GeV}) \), are rather successful \[ \mu \] in predicting low–energy physics (like for instance the \( L_i \) coupling constants of the Lagrangian
of Gasser and Leutwyler \[ \mu \].) However, the possible connection between these models and
QCD has remained so far a mystery.

There are some alternative routes to NJL–type models which have also been suggested to
describe low–energy QCD \[ \mu \]. Although not yet comparable at the level of phenomeno-
logical success, they are nevertheless interesting in the sense that they represent relatively
small departures from perturbative QCD and, hence, they ensure that at least in some limit
they are likely to be related to QCD. Needless to say, the tough problem they face is to
show that the departures from perturbative QCD are big enough and in the right direction to explain the observed phenomenology at low energies.

A very popular model for instance is that of a “freezing” coupling constant [31]. Although certainly economic and rather successful, this approach (at least in its most naïve version) runs into conflict with spontaneous chiral symmetry breaking as we shall show with a particular example in the Appendix B.

Another alternative approach to low–energy QCD is the one which has recently appeared in refs. [32]. These authors discuss an interesting attempt at describing spontaneous chiral symmetry breaking in QCD using a variational approach on resummed perturbation theory. We are curious to know how this variational approach could be related to the properties of large orders in the coupling constant, i.e. to renormalons, and hence to some of the results that will be obtained in this paper.

The aim of this article is to study more closely and within a specific class of QCD diagrams the interplay between the insertion of four–fermion operators and the leading UV renormalon contributions; as well as the possible impact of this relationship on bridging the gap between QCD and the low–energy chiral effective Lagrangian. This we do by explicitly studying UV renormalon effects in two–point functions of colour–singlet vector currents and of colour–singlet pseudoscalar currents. The currents which we consider are light quark currents of the flavour $SU(3)_L \times SU(3)_R$ group. Their associated two–point functions correspond to physical observables in hadron physics. We are not so much interested in UV renormalons as a source of possible $1/Q^2$–like ambiguities in two–point function QCD sum rules, but rather in their possible relevance to genuinely non–perturbative effects which they may signal when the external Euclidean momentum $Q^2$ in a two–point function is taken much smaller than the characteristic QCD scale. With the restriction to a one–loop $\beta$ function this means $Q^2 < \Lambda^2_L$, where $\Lambda^2_L$ is the Landau pole. Although this is not the conventional situation (wherein $Q^2$ is always assumed to be very large), we think that taking $Q^2$ small is unavoidable if one ever wants to make any contact with a low–energy effective Lagrangian for QCD, as this is the range of momentum which the effective Lagrangian is supposed to describe. We find that the ambiguities generated by the leading UV renormalon in this regime of low $Q^2$ hint at the existence of non–perturbative effective four–fermion local operators to describe low–energy physics, much the same as the study of the ambiguities generated by the IR renormalons hint at the existence of vacuum condensates of composite operators in the OPE. Moreover, these non–perturbative contributions turn out to be exactly like those of the extended NJL model mentioned earlier, with the four–fermion operators normalized by an energy scale which is momentum–independent, i.e. a constant. This constant which plays the rôle of a cutoff in the Euclidean momentum integrals appears to be related to the characteristic QCD scale and it seems natural to identify it with $\Lambda_\chi$, the chiral symmetry breaking scale [33]. At the level of a one–loop $\beta$ function this identification implies that $\Lambda_\chi \simeq \Lambda_L$.

Most of the technical part of the paper has to do with the calculation of the contribution to the residue of the leading UV renormalon coming from the effective charge exchange of one and two–chains of gluon self–energy–like graphs in these two–point functions. The concept of a QCD effective charge at the one loop level is reviewed in Section 3, where we also define the QCD “amputated” action which we use as a calculational framework. For the sake of simplicity we work in the chiral limit where the light $u, d, s$ quark masses are neglected. In this limit the two–point functions of flavour non–singlet axial–vector currents and scalar currents
are trivially related to those of vector currents and pseudoscalar currents respectively; and it is therefore sufficient to study the two types of two–point functions which we do. Since, eventually, we are interested in a comparison with an “all–orders” analysis of QCD in the large–$N_c$ limit, we have kept track explicitly of the $N_c$ factors which appear at the various stages of the calculations. The calculations reported in Section 3 are made in two ways. One method uses the Gegenbauer expansion technique of conventional Feynman diagrams; the other makes use of the operator product expansion technique following refs. [20, 21]. We find that the dominant rôle played by the dimension–six four–fermion operators is indeed universal and comes about in the same way in the two channels which we have studied. The specific discussion of the calculations at low $Q^2$ values is done in Section 4, and it is followed by Section 5 which is dedicated to the conclusions and outlook.

2 The QCD Effective Charge and Renormalon Calculus.

In QED, the infinite subset of radiative corrections summed in the Dyson series generated by the one–particle–irreducible vacuum polarization self–energy function $\Pi_R(k^2)$ defines an effective charge which is universal, gauge–, and scheme–independent to all orders in perturbation theory:

$$\alpha_{\text{eff}}(k^2) = \frac{e^2}{4\pi} \frac{1}{1 + \Pi_R(k^2)} = \frac{e^2}{4\pi} \frac{1}{1 + \Pi(k^2)}, \quad (2.1)$$

where $e$ and $\Pi(k^2)$ denote bare quantities. The extension of a similar effective charge concept to QCD is of fundamental importance for renormalon calculus, if one wants to identify unambiguously the infinite subset of gluon self–energy–like radiative corrections that one is summing in the replacement of a gluon propagator by a so-called “renormalon chain”. Recently, there has been substantial progress in this direction. The theoretical framework which has enabled this progress is the so–called pinch technique [34–37]. The pinch technique is a well–defined algorithm for the rearrangement of conventional gauge–dependent one–loop $n$–point functions to construct individually gauge–independent “one–loop” $n$–point–like functions. This rearrangement of perturbation theory is based on a systematic use of the tree level Ward identities of the theory to cancel in Feynman amplitudes all factors of longitudinal four–momentum associated with gauge fields propagating in loops. In the case of QCD, the resulting effective charge has the form [36]

$$\alpha_{\text{eff}}(k_E^2) = \frac{g_R^2}{4\pi} \frac{1}{1 - \Pi_R(k^2)}, \quad (2.2)$$

where $k_E^2 \equiv -k^2 \geq 0$ for $k^2$–spacelike, and $\Pi_R(k^2)$ is a gauge invariant gluon self–energy–like two–point function, which in the $\overline{MS}$–scheme in particular, and at the one loop level is given by the expression

$$\Pi_{\overline{MS}}(k^2) = \frac{g_{\overline{MS}}^2}{4\pi^2} \left\{ \left( -\frac{11}{6} + \frac{1}{3} n_f \right) \frac{1}{2} \log \frac{-k^2}{\mu^2} + \frac{1}{36} N_c - \frac{5}{18} n_f \right\}. \quad (2.3)$$

The coefficient $-\frac{11}{6} + \frac{1}{3} n_f$ of the logarithmic term in this equation is precisely the first coefficient $\beta_1$ of the QCD $\beta$–function. The effective charge encodes therefore the physics of the $\beta$–function; in particular the scale breaking property. Since, eventually, we are interested in the appearance of physical scales in QCD, it seems natural to focus our attention on the
properties of those diagrams of perturbation theory generated by the insertion of effective charge–exchanges. Of course, we do that at the simplest level of keeping only the one loop dependence of the $\beta$–function. As far as one is only interested in general qualitative features this should not be a serious limitation.

With the concept of an effective charge in hand it is possible to define a framework to do QCD renormalon calculations in the sector of light quark flavours. It is this framework which we next describe.

Let $\Gamma(v,a,s,p)$ be the full QCD generating functional of the Green’s functions of quark currents in the presence of external vector $v$, axial–vector $a$, scalar $s$, and pseudoscalar $p$ matrix field sources:

$$e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}G_\mu \exp \left( -i \int dx \frac{1}{4} G^{(a)}_{\mu\nu}(x) G^{(a)\mu\nu}(x) \right) \times$$

$$\int \mathcal{D}\bar{q}Dq \exp \left( i \int d^4 x \bar{q}(x)i \not{D}q(x) \right),$$

(2.4)

where $\bar{q} = (\bar{u}, \bar{d}, \bar{s})$, and $\not{D}$ denotes the QCD Dirac operator in the presence of the external sources

$$\not{D} \equiv \gamma^\mu [\partial_\mu + ig_s G_\mu(x)] - i \gamma^\mu [v_\mu(x) + \gamma_5 a_\mu(x)] + i [s(x) - i \gamma_5 p(x)];$$

(2.5)

$G_\mu$ is the gluon gauge field colour matrix ($a = 1, 2, 3, \ldots 8$)

$$G_\mu(x) \equiv \sum_a \frac{\lambda^{(a)}}{2} G^{(a)}_\mu(x);$$

(2.6)

and

$$G^{(a)}_{\mu\nu}(x) = \partial_\mu G^{(a)}_\nu - \partial_\nu G^{(a)}_\mu - g_s f_{abc} G^{(b)}_\mu G^{(c)}_\nu;$$

(2.7)

the eight gluon field strength tensor components. The factor $Z$ in eq. (2.4) is such that $\Gamma(0,0,0,0) = 1$.

The QCD contributions which we shall consider in the renormalon calculations presented below are the ones generated by an “amputated” generating functional $e^{i\Gamma(v,a,s,p)}$ defined as follows:

$$e^{i\tilde{\Gamma}(v,a,s,p)} =$$

$$\frac{1}{Z} \int \mathcal{D}\bar{q}Dq \exp \left\{ \int d^4 x \bar{q}(x) \left[ i \gamma^\mu \left( \partial_\mu - i(v_\mu + \gamma_5 a_\mu) \right) + i (s - i \gamma_5 p) \right] q(x) \right\}$$

$$\times \exp \left\{ -g_R^2 \sum_{a,b} \int d^4 x d^4 y \bar{q}(x) \gamma^\mu \lambda^{(a)} 2 q(x) i \Delta^{ab}_{\mu\nu}(x - y) \bar{q}(y) \gamma^\nu \lambda^{(b)} 2 q(y) \right\},$$

(2.8)

with $i \Delta^{ab}_{\mu\nu}(x - y)$ the Fourier transform of the renormalized gluon propagator–like function:

$$i \Delta^{ab}_{\mu\nu}(x - y) = \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot (x - y)} i \delta^{ab} \left\{ \left( -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) d_R(k^2) - \xi k_\mu k_\nu \right\},$$

(2.9)

and $d_R(k^2)$ the effective charge defined in eq. (2.2) i.e.

$$g^2_R d_R(k^2) = \frac{g^2_R}{1 - \Pi_R(k^2)} \equiv 4 \pi \alpha_{\text{eff}}(k^2).$$

(2.10)
The function $i \Delta^{ab}_{R \mu \nu}(x - y)$ results from having integrated out those gluonic interactions which, order by order in perturbation theory, contribute to the QCD effective charge as successive powers of gluon self–energy–like two–point functions evaluated at the one loop level. In terms of a characteristic scale, like e.g. the $\Lambda_{\overline{MS}}$ scale of the $\overline{MS}$–scheme, we have

$$\alpha_{\text{eff}}(k^2_E) = \frac{1}{2 \pi} \log \frac{k^2_E}{\Lambda^2_{\overline{MS}}},$$

where

$$\beta_1 = -\frac{11}{6} N_c + \frac{1}{3} n_f \quad \text{and} \quad [34–37] \quad c^2 = \exp \left\{ \frac{67 N_c - 10 n_f}{33 N_c - 6 n_f} \right\}.$$  \hspace{1cm} (2.11)

The numerical value of the constant $c$ in different relevant limits for $N_c$ and $n_f$ does not change much: $c(N_c = 3, n_f = 3) = 2.87, c(N_c \to \infty) = 2.76, c(n_f \to \infty) = 2.30$. We shall often refer to the scale

$$\Lambda_L = c \times \Lambda_{\overline{MS}},$$

as the Landau pole. Notice that, with $\Lambda_{\overline{MS}} \sim 300 - 400$ MeV, as determined phenomenologically, $\Lambda_L$ turns out to be numerically of the same size as the chiral symmetry breaking scale: $\Lambda_L \simeq \Lambda_{\chi} \simeq 1$GeV. Although $c$ and $\Lambda_{\overline{MS}}$ are scheme dependent, the combination $\Lambda_L$ is scheme independent. For simplicity we shall work in a subtraction scheme in which the corresponding $c$ is unity, or equivalently, wherein $g_R(\mu) = g_{\overline{MS}}(\mu/c)$; and from now on we shall drop the subscript “R” from $g_R(\mu)$.

We consider that the framework described above is a net improvement with respect to previous approaches to renormalon calculus in QCD. The existence of an effective charge in QCD, with properties analogous to those of the QED effective charge, as recently emphasized in refs. [36] and [37], provides the basic feature that allows us to select a minimal class of well defined contributions to QCD renormalons. In the so called naive non–abelianization procedure (NNA) [38], it is suggested to perform first QED–like calculations —ignoring the non–abelian gluonic interactions— and replace in the results thus obtained the number of light quark flavours, $n_f$, coming from QED–like vacuum polarization insertions, by

$$n_f \to n_f - \frac{11}{2} N_c.$$  \hspace{1cm} (2.14)

Another procedure often adopted is to consider abelian–like gluonic interactions dressed with the asymptotic running coupling at the scale of the virtual gluonic Euclidean momentum, and then advocate a “large $\beta_1$ expansion”. None of these prescriptions, included the “amputated” generating functional in (2.8) which we are proposing, has been justified as yet in terms of a well defined approximation within QCD itself. The merit however of the “amputated” effective action in (2.8) is that it selects a minimum set of a well defined class of QCD contributions; and in principle it could be improved by considering higher loop contributions to the gluon self–energy–like function $\hat{\Pi}_R$ which governs the gluon propagator–like function $i \Delta^{ab}_{R \mu \nu}(x - y)$; as well as by taking into consideration more and more non–local interaction (three–point–like, four–point–like, ...) terms.

As an illustrative example, we dedicate the rest of this Section to the calculation of the contribution to the Adler function induced by the renormalon effects which result from the

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\footnote{See e.g. ref. [39] and references therein.}
exchange of an effective charge chain in the “amputated” generating functional \( \tilde{\Gamma}(v, a, s, p) \) in (2.8). With

\[
V^\mu = \frac{1}{2} (\bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d)
\]

(2.15)

the vector–isovector quark current, and

\[
\Pi^{\mu\nu}(q) = i \int d^4 x e^{i q \cdot x} \langle 0 | T \{ V^\mu(x) V^\nu(0) \} | 0 \rangle = - \left( g^{\mu\nu} q^2 - q^\mu q^\nu \right) \Pi(q^2)
\]

(2.16)

the associated correlation function, the Adler function is defined as the logarithmic derivative of \( \Pi(q^2) \): (\( Q^2 \equiv -q^2 \), with \( Q^2 > 0 \) for \( q^2 \)–spacelike)

\[
A(Q^2) \equiv -Q^2 \frac{\partial \Pi}{\partial Q^2}.
\]

(2.17)

In perturbative QCD calculations, \( Q^2 \) is supposed to be larger than the characteristic QCD scale, which in our case is \( \Lambda^2 \) in eq. (2.13). In terms of Feynman–like diagrams, the calculation in question can be represented by diagrams like the one in Fig. 1:

![Fig. 1](image)

**Fig. 1** One of the three Feynman–like diagrams with an effective-charge chain which induce renormalon effects in the Adler function.

The chain of bubbles in Fig. 1 corresponds to the replacement of the ordinary free gluon propagator by the full gluon propagator–like function

\[
\frac{-i \left( g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right)}{k^2 + i\epsilon} (-ig_s)^2 \Rightarrow -i \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{4\pi \alpha_{\text{eff}}(k_E^2)}{k_E^2 - i\epsilon} - i\xi \frac{k_\mu k_\nu (-ig_s)^2}{k^2 + i\epsilon},
\]

(2.18)

where \( k_E \) denotes the Euclidean virtual momentum carried by the chain; and \( \xi \) is the same covariant gauge parameter as in eq. (2.3). This replacement is in fact the net effect, in momentum space, of the interaction term in eq. (2.8) when evaluated to its lowest non trivial order.

An interesting function to consider then is the functional derivative

\[
\frac{\delta \Pi(Q^2)}{\delta \alpha_{\text{eff}}(k_E^2)}.
\]

(2.19)
Up to overall normalization factors, it corresponds to the forward elastic scattering amplitude of an off–shell vector–isovector quark current off an off–shell gluon evaluated at the one loop level as illustrated in Fig. 2.

Once this function is known, the full calculation of renormalon effects induced by one effective charge chain exchange is simply given by the integral

$$\Pi(Q^2) = \int_0^\infty dk_E^2 \frac{\delta \Pi(Q^2)}{\delta \alpha_{\text{eff}}(k_E^2)} \alpha_{\text{eff}}(k_E^2). \quad (2.20)$$

It turns out that the kernel given by the functional derivative above can be extracted from early papers on QED by Baker and Johnson [40]. It can be easily adapted to QCD by including the appropriate colour factors and it has the following form ($C_F = \frac{N_c^2 - 1}{2N_c}$):

$$\frac{\delta \Pi(Q^2)}{\delta \alpha_{\text{eff}}(k_E^2)} = N_c \frac{C_F}{32\pi^3} k_E^2 \times \begin{cases} \frac{1}{Q^2} \Xi \left( \frac{k_E^2}{Q^2} \right) & \text{for } k_E^2 \leq Q^2, \\ \frac{1}{k_E^2} \Xi \left( \frac{Q^2}{k_E^2} \right) & \text{for } k_E^2 \geq Q^2, \end{cases} \quad (2.21)$$

clearly illustrating the UV \(\leftrightarrow\) IR symmetry of the kernel. Notice that renormalization does not affect the calculation of this kernel. At this level QCD with massless quarks still has its full conformal invariance properties, which is reflected in the symmetries of the kernel above.

The function $\Xi(z)$ has the simple integral representation [40]

$$\Xi(z) = 1 + 4 \int_0^z dy \left( 1 - \frac{1}{z}y \right)^2 \frac{1}{1 + y} \log y. \quad (2.22)$$

In the interval $0 \leq z \leq 1$ it is a monotonically decreasing function from $\Xi(0) = 1$ to $\Xi(1) = \frac{11}{3} - \frac{1}{9}\pi^2 = 0.2802...$, and it has the Taylor expansion:

$$\Xi(z) = 1 + \frac{4}{3}z \left( \frac{1}{3} \log z - \frac{11}{18} \right) - \frac{4}{3}z^2 \left( \frac{1}{12} \log z - \frac{13}{144} \right) + \cdots. \quad (2.23)$$
For the purpose of renormalon calculations, it is natural to separate the integration region in eq. (2.20) into an infrared–dominated region where \( 0 \leq k_E^2 \leq Q^2 \), and an ultraviolet–dominated region where \( Q^2 \leq k_E^2 \leq \infty \). The integral in the infrared region is in fact ill–defined because the function \( \alpha_{\text{eff}}(k_E^2) \) blows up at the Landau pole where \( k_E^2 = \Lambda_L^2 \). As a consequence of this, there is an obstruction as a matter of principle to reconstructing the full non–perturbative answer from resummed perturbation theory. This intuitive argument can be refined and made more systematic with the use of Borel resummation techniques \[4\]. Let us discuss the most salient features of the integral in eq. (2.20):

- The limit \( \alpha_{\text{eff}}(k_E^2) = \alpha(\mu^2) \).

In this limit we find the well known perturbative result corresponding to the one gluon exchange contribution to the Adler function, namely

\[
A(Q^2) = \frac{N_c}{16\pi^2} \frac{2}{3} \left( 1 + \frac{3}{4} C_F \frac{\alpha(\mu^2)}{\pi} \right).
\]

Notice that in this limit the integral in the region from \( 0 \leq k_E^2 \leq Q^2 \) contributes a constant to \( \Pi(Q^2) \) and hence nothing to the Adler function; i.e. it is renormalized away. One–gluon exchange with no gluon self–energy–like correction (no bubble) in it is not sensitive to the infrared region. Only the first term \( \Xi(k_E^2 \to 0) = 1 \) in the expansion of the \( \Xi \) function in the UV region, where \( k_E^2 > Q^2 \), contributes to the Adler function in this limit.

- Infrared renormalons.

Once we start summing bubbles corresponding to the expansion of \( \alpha_{\text{eff}}(k_E^2) \) in powers of gluon self–energy–like insertions, we find that the leading infrared renormalon is induced by the first term, \( \Xi(k_E^2 \to 0) = 1 \), in the expansion of the \( \Xi \) function in the IR region. Making the change of variables

\[
w/2 = -b_0 \alpha(Q^2) \log k_E^2/Q^2, \quad \text{where} \quad b_0 = -\beta_1/2\pi
\]

in eq. (2.20), one obtains

\[
\Pi(Q^2)|_{\text{IR}} = \frac{N_c}{16\pi^2} C_F \frac{1}{2\pi b_0} \int_0^\infty dw \ e^{-\frac{w}{b_0} \alpha(\mu^2)} \frac{1}{2 - w} \left( \frac{Q^2}{\mu^2} \right)^{-w},
\]

which leads to the following contribution to the Adler function

\[
A(Q^2)|_{\text{IR}} = \frac{N_c}{16\pi^2} C_F \frac{1}{2\pi b_0} \int_0^\infty dw \ e^{-\frac{w}{b_0} \alpha(\mu^2)} \frac{w}{2 - w} \left( \frac{Q^2}{\mu^2} \right)^{-w}.
\]

This expression is already in the form of a Borel transform. An expansion around \( w = 0 \) would generate the characteristic \( n! \) behaviour of the perturbative expansion in \( \alpha(\mu^2) \) which we mentioned in the Introduction. As already discussed by other authors [4, 18], this is in fact only true for a one–loop \( \beta \) function. In the case of a two–loop \( \beta \) function, the relevant pole is not at \( \Lambda_L^2 \) [4], but the physics is pretty much the same.
we find that the leading IR renormalon contribution to the Adler function appears as a pole in the Borel plane at $w = 2$. There is no term in the IR expansion of the $\Xi \left( \frac{k^2}{Q^2} \right)$ function which leads to a pole at $w = 1$. From the previous change of variables one also sees that low values of $w$ are associated with momenta around the large scale $Q^2$, where perturbation theory is expected to give a good description of the dynamics. However, as $w$ goes up (and it has to go all the way up to infinity) one enters deeper and deeper into the IR region, where perturbation theory must fail or else QCD would not describe the spectrum of hadronic bound states that are observed. The singularity at $w = 2$ exhibits this in its crudest form. More precisely, this singularity implies an ambiguity in the perturbative evaluation of the Adler function. Therefore, the analytic continuation in $w$ in eq. (2.27), or equivalently the resummation of perturbation theory into the effective charge $\alpha_{\text{eff}}(k^2/Q)$ of eq. (2.20), that one has tried in order to obtain the full solution has failed. The ambiguity is encoded in the unavoidable prescription to skip the pole; and is of the form $\sim e^{-1/\Lambda_{\text{QCD}}(Q)}$

\[ \delta A(Q^2)|_{\text{IR}} \sim \frac{N_c}{16\pi^2} \frac{C_F}{\beta_1} \left( \frac{\Lambda_{\text{QCD}}}{Q} \right)^4, \]  

(2.28)

i.e., an ambiguity which has the same $1/Q^4$ pattern as the gluon condensate contribution that appears in the OPE evaluation of the Adler function [12, 13]. Since the Adler function must be an unambiguous physical observable, there must exist another contribution that cancels (2.28). In other words, (Borel) resummed perturbation theory requires the presence of the gluon condensate (which is also ambiguous for the same reason) to combine with the result of eq. (2.27) and yield a final well-defined answer [9]. This is an example of how all-orders perturbation theory is capable of “hinting” at non-perturbative dynamics.

Following analogous steps, the insertion of higher powers in $k^2/Q^2$ from the IR expansion of the $\Xi$–function in the $0 \leq k^2/Q^2 \leq Q^2$ integrand produces contributions to higher singularities in the Borel plane corresponding to higher order IR renormalons located at $w = 3$, $w = 4$, and so on. However, the higher power terms of the $\Xi$–function only generate partial contributions to the higher order IR renormalons. For example, for the IR renormalon located at $w = 3$, there will also be further contributions originating in the three–point–like non–local term which will appear as a correction to the “amputated” generating functional in (2.8). This new term will generate diagrams like e.g. the one shown in Fig. 3 which we are not taking into account here.
The total contribution to higher order IR renormalons is associated with higher dimensional condensates with a larger power in $1/Q^2$ than the one in (2.28), and in this sense they are non-leading.

- Ultraviolet renormalons.

The integral in eq. (2.20), in contrast to the behaviour in the infrared region discussed above, is well behaved in the ultraviolet region where $Q^2 \leq k_E^2 \leq \infty$, provided that $Q^2 \gg \Lambda_L^2$. Nonetheless it also has associated with it a Borel representation, the only difference being that now the poles will occur at negative values of the Borel variable $w$. As in the case of infrared renormalons, the further away the pole is from the origin $w = 0$ the smaller will be the $n$-th coefficient accompanying $\alpha(Q^2)^n$ in the perturbative series. This gives rise to a hierarchy of ultraviolet renormalons. The leading one, within the exchange of one power of the effective charge, is the one induced by the $\mathcal{O}(z)$–term in the UV expansion ($k_E^2 > Q^2$) of the $\Xi$–function in eq. (2.23). With the change of variables $w = b_0 \alpha(Q^2) \log k_E^2/Q^2$ one obtains

$$
\Pi(Q^2)|_{UV} = -\frac{N_c}{16\pi^2} C_F \left\{ \frac{1}{9} b_0 \alpha(Q^2) \right\} \int_0^\infty dw e^{-\frac{w}{b_0 \alpha(Q^2)}} \left( \frac{\mu^2}{Q^2} \right)^w \left\{ \frac{1}{1 + w^2} + \frac{11}{6} \frac{1}{1 + w} \right\},
$$

(2.29)

where the leading $\frac{1}{(1+w)^2}$ term is the one induced by the term $\frac{4}{3} \left( \frac{Q^2}{k_E^2} \frac{1}{3} \log \frac{Q^2}{k_E^2} \right)$ in eq. (2.23). The corresponding contribution to the Adler function is then

$$
\mathcal{A}(Q^2)|_{UV} = \frac{N_c}{16\pi^2} C_F \left\{ \frac{1}{9} b_0 \alpha(Q^2) \right\} \int_0^\infty dw e^{-\frac{w}{b_0 \alpha(Q^2)}} \left\{ \frac{1}{1 + w^2} + \frac{5}{6} \frac{1}{1 + w} - \frac{11}{6} \right\},
$$

(2.30)

where we have also scaled $\mu^2$ at $Q^2$. The result of the leading term $\frac{1}{(1+w)^3}$ agrees with the one obtained in refs. [20, 21, 42] using other methods. One sees that the singularity is at $w = -1$, i.e. outside the integration range. Therefore, unlike the case of IR renormalons, one cannot argue now that the singularity is an obstruction to do the integral. This is also clear from the above change of variables since for $0 \leq w \leq \infty$ one never leaves the region of very large momentum where perturbation theory is supposed to be a faithful description of the dynamics.

Contributions to higher–order UV renormalons can be obtained from the insertion of successive higher power terms in the UV expansion of the $\Xi$–function in the $k_E^2$ integrand in eq. (2.20) for $Q^2 \leq k_E^2 \leq \infty$. The next Section is fully dedicated to the study of UV renormalons in two–point functions.

\footnote{In fact, in the particular case of the IR renormalon located at $w = 3$, and with no extra hard $\alpha(Q^2)$ correction, we expect a cancellation of the various partial contributions to the residue of the pole at $w = 3$, in accordance with the fact that there is no three gluon condensate contribution in the OPE to the Adler function as proved in refs. [43, 44].}
3 UV Renormalon Contributions to Two–Point Functions.

We shall discuss in this Section the calculations of the contributions to two–point functions of quark current operators induced by the leading UV renormalon generated by the exchange of one and two QCD effective charge chains. The two–point functions which we shall consider are the Adler function already discussed in the previous Section, and the two–point function associated with the divergence of the axial–current:

\[ \partial^\mu A_\mu(x) \equiv (m_d + m_u) \langle \bar{d}(x)i\gamma_5 u(x) \rangle ; \]  

(3.1)

where the overall quark mass factor will only be kept so as to make the pseudoscalar current density renormalization invariant. All the calculations presented here have been made in the chiral limit where the masses of the light \( u, d \), and \( s \) quarks are set to zero. The two–point function associated with the divergence of the axial current is

\[ \Psi_5(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 \mid T \{ \partial^\mu A_\mu(x) \partial^\nu A_\nu^\dagger(0) \} \mid 0 \rangle . \]  

(3.2)

The equivalent of the Adler function here is the second derivative of \( \Psi_5(q^2) \),

\[ \mathcal{P}(Q^2) = \frac{Q^2}{(m_u + m_d)^2} \frac{\partial^2 \Psi_5(q^2)}{(\partial Q^2)^2} . \]  

(3.3)

This second derivative is, like the Adler function in eq. (2.17), independent of external subtractions.

3.1 The Vector Two–Point Function.

As mentioned in Section 3 we have carried out the UV renormalon calculations using two different methods. One is the operator product expansion (OPE) technique, which we discuss below, following rather closely the QED work of Vainshtein and Zakharov [20,21]. The other method uses ordinary Feynman diagrams directly and the Gegenbauer polynomial expansion of Euclidean propagators. For the vector two–point function, we shall discuss this second method only in connection with the exchange of two effective charge chains.

3.1.1 OPE Calculation: One Chain.

Let us start with the contribution from the UV renormalon generated by the exchange of one effective charge chain, to which we shall refer for short as the contribution of a “one–renormalon chain” or even “one–chain”. The contribution of the “one–renormalon chain” to the external vector field vacuum polarization tensor of eq. (2.16) which we have discussed in Section 2 can also be written as follows:

\[ \Pi^{\mu\nu}(q) v^\mu v^\nu = -\frac{i}{2} \int_{k^2 \geq Q^2} \frac{d^4k}{(2\pi)^4} \frac{4\pi \alpha_{\text{eff}}(k_E^2)}{k^2} \langle v \mid T \{ \lambda^a \bar{q}(0) \gamma_\mu \lambda^a q(x) \} \mid v \rangle , \]  

(3.4)

where the external vector field \( v \) is the same as the one appearing in the Dirac operator in eq. (2.17) and \( T \) is the time–ordered operator

\[ T \equiv i k^2 \int d^4xe^{ik\cdot x} T \{ : \bar{q}(x)\gamma_\mu \lambda^a \frac{1}{2} q(0) : \bar{q}(0) \gamma_\mu \lambda^a \frac{1}{2} q(x) : \} \] .  

(3.5)
Following [20,21] we now perform the operator product expansion on the operator $\mathcal{T}$ by using the Schwinger background field technique, explained e.g. in ref. [45]. The first nonvanishing contributions comes from the dimension–six operators:

$$
\mathcal{T} = \frac{4}{3k^2} \bar{q}(0) \gamma^\mu \left[ -\frac{g_{\text{eff}}(k^2_E)}{2N_c} D^\alpha G_{\alpha\mu}(0) + C_F D^\alpha F_{\alpha\mu}(0) \right] q(0),
$$

where $F_{\mu\nu}(x)$ is the field strength tensor associated with the external vector source. With this expression for the operator $\mathcal{T}$ one can easily compute the integral in (3.4). [Recall eq. (2.16), and notice that at this order only the $F_{\alpha\mu}$ term in (3.6) contributes.] This results in the expression

$$
\Pi(q^2) = -\frac{N_c}{16\pi^2} \frac{2}{9} Q^2 \int_0^\infty \frac{dk_E^2}{k_E^2} \frac{\alpha_{\text{eff}}(k_E^2)}{\pi} \log(k_E^2/Q^2),
$$

where the momentum in the argument of the logarithm has been cut off at the scale $k_E^2$ that appears in the OPE in eq. (3.6). The steps of the calculation are illustrated in Fig. 4 below:

**Fig. 4** The leading UV renormalon contribution from a one effective charge exchange, evaluated with the operator product expansion technique, is the one generated by the dimension six vertex like operators simulated by the form factor in the lower figure.

The UV renormalon contribution to the Adler function in (2.17) can now be readily obtained. Using the change of variables $w = b_0 \alpha(Q^2) \log(k_E^2/Q^2)$ results in

$$
\mathcal{A}(Q^2) |_{\text{one chain}} = \frac{N_c}{16\pi^2} C_F \frac{4}{9} \frac{1}{2\pi b_0} \int_0^\infty \frac{dw}{(1+w)^2} e^{-\frac{w}{b_0 \alpha(Q^2)}}
$$

which, if expanded in perturbation theory, yields

$$
\mathcal{A}(Q^2) |_{\text{one chain}} \sim - \sum_{\text{Large } n} \frac{N_c}{16\pi^2} C_F \frac{4}{9} \frac{1}{2\pi b_0} \left(-b_0 \alpha(Q^2)\right)^n n!.
$$

Notice that the result in (3.8) is in agreement with the one we found in eq. (2.30) for the leading behaviour of the Borel transform around the singularity $w = -1$. Our result coincides also with the one obtained in refs. [20,21] in the abelian limit where $\frac{k^\mu}{Q^2} \to 1$. 

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3.1.2 OPE Calculation: Two Chains.

In an analogous way one can deal with the more complicated contributions coming from “two renormalon chains”, as illustrated in Fig. 5. Two different new structures emerge from the box–like diagrams:

\[
\mathcal{T}^{\text{Box}} = \mathcal{T}_1^{\text{Box}} + \mathcal{T}_2^{\text{Box}}
\]

\[
= - \frac{g_{\text{eff}}^2(k_E^2)}{2k^2} \left[ \bar{q}(0)\gamma^\mu\gamma^\lambda\gamma^\alpha \frac{\lambda^a}{2} q(0)\bar{q}(0)\gamma_\mu\gamma_\lambda\gamma_\alpha \frac{\lambda^b}{2} q(0) 
- \bar{q}(0)\gamma^\mu\gamma^\lambda\gamma^\alpha \frac{\lambda^a}{2} q(0)\bar{q}(0)\gamma_\alpha\gamma_\lambda\gamma_\mu \frac{\lambda^b}{2} q(0) \right].
\]

\[\text{Fig. 5} \text{ The leading UV renormalon contribution from the exchange of two effective charges. The evaluation with the operator product expansion technique, generates a dimension six four–fermion operator simulated by the box form factor in the lower figure.}\]

Using the operator identity

\[
(\gamma^\mu\gamma^\lambda\gamma^\alpha) \otimes (\gamma_\alpha\gamma_\lambda\gamma_\mu) = 10 \gamma^\mu \otimes \gamma_\mu - 6 \gamma^\mu\gamma_5 \otimes \gamma_\mu\gamma_5,
\]

one can immediately see that the abelian limit yields the result

\[
\mathcal{T}^{\text{Box}}_{\text{abelian}} = - \frac{6 g_{\text{eff}}^2(k_E^2)}{k^2} \bar{q}(0)\gamma^\mu\gamma_5 q(0)\bar{q}(0)\gamma_\mu\gamma_5 q(0);
\]

i.e. there is a cancellation of the \(\gamma^\mu \otimes \gamma_\mu\) structure. Equation (3.12) was also obtained by the authors of refs. [20, 21]. However, in the large–\(N_c\) limit, the previous cancellation does not take place. Instead one sees that \(\mathcal{T}_2^{\text{Box}}\) is proportional to \(N_c\) whereas \(\mathcal{T}_1^{\text{Box}}\) goes like \(1/N_c\), i.e. it is suppressed by two powers of \(N_c\) relative to \(\mathcal{T}_2^{\text{Box}}\). Consequently, at large–\(N_c\),

\[
\mathcal{T}^{\text{Box}} \simeq \mathcal{T}_2^{\text{Box}} \simeq \frac{N_c \ g_{\text{eff}}^2(k_E^2)}{4k^2} \left\{ 10 \ \bar{q}(0)\gamma^\mu \frac{\lambda^a}{2} q(0)\bar{q}(0)\gamma_\mu \frac{\lambda^a}{2} q(0) 
- 6 \ \bar{q}(0)\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q(0)\bar{q}(0)\gamma_\mu\gamma_5 \frac{\lambda^a}{2} q(0) \right\}.
\]

\[\text{(3.13)}\]
Upon comparing with the one–renormalon chain contribution of eq. (3.6) we see that the use of the equations of motion for the gluon field in that expression also produces a four–fermion operator of the form (3.13) which is however suppressed relative to the one in eq. (3.13) by a factor $1/N_c^2$ and hence we neglect it. Therefore, we find that large–$N_c$ selects as the only operator contributing at the “two–renormalon–chain” level the one in eq. (3.13). If we now Fierz the operator $T^\Box_2$, and use the identity

$$a \left( \bar{q} \gamma^\mu \frac{\lambda^a}{2} q \right)^2 \quad \text{and} \quad b \left( \bar{q} \gamma^\mu \gamma^5 \frac{\lambda^a}{2} q \right)^2 \equiv$$

$$a - b \left[ \sum_{a,b} \left( \bar{q}_L^a \gamma^\mu q_R^b \right)^2 \right] - 2 \left( a + b \right) \sum_{a,b} \left( \bar{q}_L^a q_R^b \right) \left( \bar{q}_R^b q_L^a \right), \quad (3.14)$$

we find that the result is almost like the four–fermion operators which appear in the ENJL model of large–$N_c$ QCD \[28, 29]. There is however a very important difference, namely that the dimension–six $T^\Box_2$ operator which appears here is suppressed by the momentum scale $k^2_E$ and not by a constant scale like $\Lambda^2$. We shall come back to this important issue in the next Section. Here we shall limit ourselves to a numerical observation concerning the relative size of the scalar–like coupling $G_S \sim 2(a + b)$ and vector–like coupling $G_V \sim (a - b)/2$ which appear in eq. (3.14). Plugging in the values $a = 10$, and $b = 6$ obtained in eq. (3.13) results in a ratio $G_S/G_V = 16$. For comparison we recall that a cut–off one–gluon exchange yields $b = 0, G_S/G_V = 4$, and the abelian version of refs. \[21\], which has $a = 4/3, b = -3$, would yield $G_S/G_V = -20/13$, a negative value !. The phenomenologically favoured value of ref. \[28\] is $G_S/G_V \approx 1$.

Let us come back now to the contribution of “two renormalon chains” to the Adler function. The relevant diagram is the one depicted in Fig. 5, where the black square stands for the insertion of the operator (3.13). As already noticed in \[20, 21\] a subtlety arises at this point, for if one wishes to keep the relations (3.4) and (3.5), in order to avoid double counting one has to multiply the right hand side of eq. (3.4) by a factor $1/2$. After Fierz–ing the result in (3.13), only the vector part contributes of course, and one obtains (recall eq. (2.16) and the fact that we are working in the large–$N_c$ limit)

$$\Pi(q^2) = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi} \right)^2 \left( -\frac{1}{9} \right) Q^2 \int_{Q^2}^{\infty} \frac{dk^2_E}{k^4_E} \left[ \alpha_{\text{eff}}(k^2_E) \right]^2 \log^2(k^2_E/Q^2) \quad (3.15)$$

for the external vector vacuum polarization function, and

$$\mathcal{A}(Q^2) |_{\text{two chains}} = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \frac{2}{9} \int_{0}^{\infty} dw \frac{w^{n}}{(1 + w)^3} \quad (3.16)$$

for the Adler function, where $w = b_0 \alpha(Q^2) \log(k^2_E/Q^2)$.

At large orders one finds

$$\mathcal{A}(Q^2) |_{\text{two chains}} \sim \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \frac{2}{9} \sum_{\text{Large } n} \left( -b_0 \alpha(Q^2) \right)^n (n + 1)! . \quad (3.17)$$

It turns out that, both expressions (3.9) and (3.17) are comparable in the large–$N_c$ limit and of the same order as the one–loop “parton” graph. However, (3.17) is leading at large orders

\[i.e. \text{ a penguin–like diagram.}\]
of perturbation theory, i.e. for large \( n \). In this sense one can view the “two renormalon chain” contribution as a selection of a subset of the whole set of diagrams that are leading at large–\( N_c \). This simplification may be welcome in the sense that, so far, even the leading contribution at large–\( N_c \) has proven itself to be already too difficult to deal with in QCD. We shall comment on this again in the next Section.

### 3.1.3 Two Chains with the Gegenbauer Expansion Technique.

We want to present another technique for calculating the “two renormalon chain” contribution to the Adler function which uses directly the Feynman diagrams generated by the “amputated” effective action in (2.8). Further limitation to leading contributions in the large–\( N_c \) limit, and the fact that we are only interested in the leading UV renormalon contribution restricts the class of possible diagrams with two chain exchanges to the one in Fig. 6, with a Feynman gauge–like coupling i.e., in this diagram only the \( g^{\mu\nu} \) term in (2.9) is operational.

![Diagram of two effective charges exchange](image)

**Fig. 6** Diagram representing the exchange of two effective charges. A convenient routing of momenta to evaluate the contribution to the leading UV renormalon using the Gegenbauer expansion technique is indicated in the figure.

With the routing of momenta indicated in Fig. 6, the contribution to the vector two–point function reads as follows:

\[
\begin{align*}
\Pi^{\mu\nu}(q) &= N_c \frac{1}{2} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \int \frac{d^4 k_3}{(2\pi)^4} \\
&\quad \times \left(4\pi\right)^2 \int_0^\infty dz_1 e^{-z_1/\alpha(\mu^2)} \int_0^\infty dz_2 e^{-z_2/\alpha(\mu^2)} \frac{g_{\alpha\alpha'}(\mu^2)^b_0 z_1}{(k_2 - k_1)^2(1+b_0 z_1)} \frac{g_{\beta\beta'}(\mu^2)^b_0 z_2}{(k_3 - k_2)^2(1+b_0 z_2)} \\
&\quad \times \text{tr} \left[ \gamma^\mu \gamma^\alpha k_1 \gamma^\beta k_2 \gamma^\alpha' (k_3 - q) \gamma^\beta' (k_2 - q) \gamma^\alpha' (k_1 - q) \right] \\
&\quad \times \frac{k_1^2 k_2^2 k_3^2 (k_1 - q)^2 (k_2 - q)^2 (k_3 - q)^2}{k_1^2 k_2^2 k_3^2 (k_1 - q)^2 (k_2 - q)^2 (k_3 - q)^2},
\end{align*}
\]

(3.18)

where we have used the identity

\[
\alpha_{\text{eff}}(k_E^2) = \int_0^\infty dz e^{-z/\alpha(\mu^2)} \left( \frac{\mu^2}{k_E^2} \right)^b_0 z.
\]

(3.19)
In order to calculate this integral, it is convenient to first introduce Dirichlet variables \[^7\]:

\[
    w_1 = b_0 z_1, \quad w_2 = b_0 z_2; \quad w = w_1 + w_2, \quad v = \frac{w_2}{w}; \quad (3.20)
\]

and expand the denominators which depend on differences of momenta in Gegenbauer polynomials \[^7\]

\[
    \frac{1}{(k - p)^{2\lambda}} \to \frac{-1}{(k_E^2 - 2k_E p_E \cos \theta + p_E)^{2\lambda}} =
\]

\[
    -\frac{1}{k_E^2} \Theta \left(1 - \frac{p_E}{k_E} \right) \sum_{n=0}^{\infty} C_n^\lambda(\cos \theta) \left(\frac{p_E}{k_E} \right)^n + \frac{1}{k_E^2} \Theta \left(1 - \frac{k_E}{p_E} \right) \sum_{n=0}^{\infty} C_n^\lambda(\cos \theta) \left(\frac{k_E}{p_E} \right)^n ,
\]

where the \( C_n^\lambda(z) \), \([C_n^1(z) \equiv C_n(z)]\) are Gegenbauer polynomials with the orthogonality property:

\[
    \int d\Omega_k C_n^\lambda(\hat{a} \cdot \hat{k}) C_m^\lambda(\hat{b} \cdot \hat{k}) = \delta_{n,m} \frac{\lambda}{\lambda + n} C_n^\lambda(\hat{a} \cdot \hat{b}) ,
\]

with \( C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{\pi^{\lambda+1}} \), and \( d\Omega_k \) the solid angular element resulting from

\[
    \int d^{(4-\epsilon)}k \to i \frac{2\pi^{(2-\epsilon)/2}}{\Gamma(2-\epsilon/2)} (k_E)^{3-\epsilon} dk_E d\Omega_k .
\]

The trace in the numerator of eq. (3.18) also has to be expressed in terms of powers of the Euclidean momenta \( k_1E, k_2E, k_3E, Q \equiv q_E \); and powers of \( \cos \theta_{i0} \equiv \cos(\hat{k}_i \cdot \hat{q}) \), \( i = 1, 2, 3 \); and \( \cos \theta_{ij} \equiv \cos(\hat{k}_i \cdot \hat{k}_j) \). The Dirac trace results in at most three powers of \( \cos \theta \)'s; and therefore, from the orthogonality properties of the Gegenbauer polynomial, only a few terms give non-zero contributions to the solid angle integrals \( \int d\Omega_{k_1} d\Omega_{k_2} d\Omega_{k_3} \). With further restriction to those terms which can contribute to the leading UV renormalon, one finds that only terms with two powers of \( \hat{q} \) and terms with no powers of \( \hat{q} \) in the Dirac trace are relevant, with contributions proportional to \( Q^4 k_{2E}^2 \) and \( -\frac{2}{3} Q^4 k_{2E}^2 \) respectively. This results in a contribution to \( \Pi(q^2) \) from the leading UV renormalon generated by the exchange of two chains:

\[
    \Pi(q^2) = \frac{1}{3} \left(1 - \frac{2}{3}\right) \frac{N_c}{10\pi^2} \left(\frac{N_c}{2} \cdot \frac{1}{2\pi b_0}\right)^2 \int_0^{\infty} dw \ w e^{-w} \left(\frac{1}{q_0^2(a^2)}\right) \times
\]

\[
    \int_{Q^2}^{\infty} \frac{dk_{2E}^2}{k_{2E}^2} \left(\frac{Q^2}{k_{2E}^2}\right)^{1+w} \int_{k_{1E}^2}^{k_{2E}^2} \frac{dk_{1E}^2}{k_{1E}^2} \int_{Q^2}^{k_{1E}^2} \frac{dk_{2E}^2}{k_{2E}^2} \left(\frac{Q^2}{\mu^2}\right)^{-w} \w
\]

\[
    = \frac{N_c}{16\pi^2} \left(\frac{N_c}{2} \cdot \frac{1}{2\pi b_0}\right)^2 \frac{1}{9} \int_0^{\infty} dw \ w e^{-w} \left(\frac{1}{q_0^2(a^2)}\right) \left(\frac{Q^2}{\mu^2}\right)^{-w} \int_0^1 dx \ x^w \log^2 \frac{1}{x} . \quad (3.25)
\]

The corresponding contribution to the Adler function, scaled to \( \mu^2 = Q^2 \), is then

\[
    \mathcal{A}(Q^2)_{\text{two chains}} = \frac{N_c}{16\pi^2} \left(\frac{N_c}{2} \cdot \frac{1}{2\pi b_0}\right)^2 \left(-\frac{2}{9}\right) \int_0^{\infty} dw \ e^{-w} \left(\frac{1}{q_0^2(a^2)}\right) \frac{1}{(1 + w)^3} . \quad (3.26)
\]

\(^7\)The change to the Dirichlet variables \( w \) and \( v \) automatically implements the convolution property of Laplace transforms.

\(^8\)Useful properties of Gegenbauer polynomials in connection with Feynman diagram calculations can be found, e.g., in ref. [4].
in agreement with the OPE result in eq. (3.16).

The final result of the leading UV renormalon contribution from the exchange of one and two powers of the QCD effective charge to the Adler function $A(Q^2)$; i.e., the sum of one and two renormalon chains, is then given by the expression:

$$A(Q^2) = \frac{N_c}{16\pi^2} \frac{N_c}{2} \frac{1}{2\pi b_0} \frac{4}{9} \int_0^\infty dw e^{-\frac{w}{\alpha_0(Q^2)}} \left\{ \frac{1}{(1+w)^2} - \frac{N_c}{2} \frac{1}{2\pi b_0} \frac{1}{2} \frac{1}{(1+w)^3} \right\},$$

where the first term in the second line corresponds to the contribution from the one renormalon chain, and the second term to the contribution from the two renormalon chain.

### 3.2 The Pseudoscalar Two–Point Function

We shall next discuss the contribution of the leading UV renormalon to the two–point function $\Psi_5(q^2)$ defined by eqs. (3.2) and (3.1), calculated also at the level of one and two chains. Here we shall first describe the calculations we have done using the Gegenbauer expansion technique.

#### 3.2.1 One Chain with the Gegenbauer Expansion Technique.

There are three possible diagrams with a one renormalon chain contributing to the two–point function $\Psi_5(q^2)$. Much the same as in the case of the vector two–point function, if we restrict ourselves to the contribution from the leading UV renormalon, it is then sufficient to consider the contribution from the one renormalon chain exchange in Fig. 7 with a Feynman gauge–like coupling of the renormalon chain.

**Fig. 7** Diagram representing the exchange of one effective charge. A convenient routing of momenta to evaluate the contribution to the leading UV renormalon using the Gegenbauer expansion technique is the one indicated in the figure.

With the routing of momenta indicated in Fig 7, the contribution to $\Psi_5(q^2)$ in large–$N_c$ is then as follows:

$$\Psi_5(q^2) = -i(m_u + m_d)^2 N_c \frac{N_c}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} -i g_{\text{eff}} \frac{[-(k_1 - k_2)^2]^2}{(k_1 - k_2)^2 + i\epsilon} \times$$
Using the identity

\[-i (-ig_{0}^{\text{eff}} [-((k_{1} - k_{2})^{2})])^{2} = \frac{i}{4\pi} \int e^{-z \bar{\partial}_{\alpha}(w)} \left( \frac{\mu^{2}}{k_{1E} - k_{2E}} \right)^{1+b_{0}e} dz ; \tag{3.29}\]

and expanding the denominators which depend on differences of momenta in Gegenbauer polynomials as in subsection [3.2.1] one obtains the expression

\[
\Psi_{5}(q^{2}) = \frac{(m_{u} + m_{d})^{2}}{\mu^{2}} \frac{N_{c}}{16\pi^{2}} \frac{N_{c}}{2} \frac{1}{2\pi b_{0}} \int_{0}^{\infty} dw e^{-w} \frac{1}{b_{0}\alpha(w)} \left( \frac{\mu^{2}}{Q^{2}} \right)^{1+w} \times \\
\int_{Q^{2}}^{\infty} \frac{dk_{1E}^{2}}{k_{1E}^{2}} \left( \frac{Q^{2}}{k_{1E}^{2}} \right)^{1+w} \int_{Q^{2}}^{\infty} \frac{dk_{2E}^{2}}{k_{2E}^{2}} \int d\Omega_{k_{1}} \int d\Omega_{k_{2}} \sum C_{n_{1}}^{(1+w)}(\cos \theta_{12}) \left( \frac{k_{2E}}{k_{1E}} \right)^{1} \times \\
\sum C_{n_{2}}^{(1+w)}(\cos \theta_{10}) \left( \frac{Q}{k_{1E}} \right)^{n_{1}} \sum C_{n_{2}}^{(1+w)}(\cos \theta_{20}) \left( \frac{Q}{k_{2E}} \right)^{n_{2}} \times \\
16 \left[ k_{1E}^{2}k_{2E}^{2} - k_{1E}^{2}k_{2E}^{2}Q \cos \theta_{20} - k_{2E}^{2}k_{1E}^{2}Q \cos \theta_{10} + k_{1E}^{2}k_{2E}^{2}Q \cos \theta_{10} \cos \theta_{20} \right] , \tag{3.30}\]

where \( w = b_{0}z \) and \( \cos \theta_{i0} = \cos(k_{i} \cdot \hat{q}) \) for \( i = 1,2 \) and \( \cos \theta_{12} = \cos(k_{1} \cdot k_{2}) \).

The leading UV renormalon contribution in eq. (3.30) comes solely from the \( l = 0 \) term and the \( k_{1E}k_{2E}Q^{2} \cos \theta_{10} \cos \theta_{20} \) term resulting from the Dirac trace. The angular integrals can then be trivially done, with the result

\[
\Psi_{5}(q^{2})|_{\text{UV}}^{\text{one chain}} = \frac{(m_{u} + m_{d})^{2}}{\mu^{2}} \frac{N_{c}}{16\pi^{2}} \frac{N_{c}}{2} \frac{1}{2\pi b_{0}} \int_{0}^{\infty} dw e^{-w} \frac{1}{b_{0}\alpha(w)} \left( \frac{\mu^{2}}{Q^{2}} \right)^{1+w} \times \\
\int_{Q^{2}}^{\infty} \frac{dk_{1E}^{2}}{k_{1E}^{2}} \left( \frac{Q^{2}}{k_{1E}^{2}} \right)^{1+w} \times 4Q^{4} \log \frac{k_{1E}^{2}}{Q^{2}} ; \tag{3.31}\]

and therefore

\[
\Psi_{5}(q^{2})|_{\text{UV}}^{\text{one chain}} = (m_{u} + m_{d})^{2} \frac{\mu^{2}}{\mu^{2}} \frac{N_{c}}{16\pi^{2}} \frac{N_{c}}{2} \frac{1}{2\pi b_{0}} \int_{0}^{\infty} dw e^{-w} \frac{1}{b_{0}\alpha(w)} \left( \frac{Q^{2}}{\mu^{2}} \right)^{1-w} \times 4 \left( \frac{1}{1+w} \right)^{2} . \tag{3.32}\]

Taking two derivatives with respect to \( Q^{2} \) and scaling \( \mu^{2} = Q^{2} \) we get the corresponding contribution to the pseudoscalar–pseudoscalar \( \mathcal{P}(q^{2}) \) correlation function:

\[
\mathcal{P}(q^{2})|_{\text{UV}}^{\text{one chain}} = \frac{N_{c}}{16\pi^{2}} \frac{N_{c}}{2} \frac{1}{2\pi b_{0}} \int_{0}^{\infty} dw e^{-w} \frac{1}{b_{0}\alpha(Q^{2})} \left( \frac{8}{1+w} \right)^{2} . \tag{3.33}\]

### 3.2.2 Two Chains with the Gegenbauer Expansion Technique.

As in the case of the two–chain evaluation of the Adler function in subsection [3.1.3] the relevant diagram is the one in Fig. 6 with now external pseudoscalar sources, and with a Feynman
gauge–like coupling in the renormalon chains. With the routing of momenta indicated in Fig. 6 the contribution to $\Psi_5(q^2)$ can be readily written as follows

$$
\Psi_5(q^2) = i(m_u + m_d)^2 N_c \left( \frac{N_c}{2} \right)^2 \pi^2 \left( \frac{\pi}{2} \right)^4 \int_{Q^2} \frac{d^2 k_1^2}{k_1^2} \int_{Q^2} \frac{d^2 k_2^2}{k_2^2} \int_{Q^2} \frac{d^2 k_3^2}{k_3^2} \int_{Q^2} \frac{d^2 k_4^2}{k_4^2} \int_{Q^2} \frac{d^2 k_5^2}{k_5^2} \times d\Omega_{k_1} \int d\Omega_{k_2} \int d\Omega_{k_3} \left[ i \gamma_5 k_1 \gamma^\mu k_2 \gamma^\nu k_3 \gamma_5 (k_3 - q) \gamma_\nu (k_2 - q) \gamma_\mu (k_1 - q) \right] \times \left( \frac{4\pi}{b_0} \right)^2 \int_0^\infty dw_1 e^{-w_1 \frac{1}{b_0(\alpha^2)}} \int_0^\infty dw_2 e^{-w_2 \frac{1}{b_0(\alpha^2)}} \times \sum_n C_{n_1} \cos(\theta_{10}) \left( \frac{Q}{k_1 E} \right)^{n_1} \sum_n C_{n_2} \cos(\theta_{20}) \left( \frac{Q}{k_2 E} \right)^{n_2} \sum_n C_{n_3} \cos(\theta_{30}) \left( \frac{Q}{k_3 E} \right)^{n_3} \times \frac{(\mu^2)^{w_1 + w_2}}{(k_2 E)^{1+w_1+w_2}} \frac{1}{(k_1 E)^{1+w_1(\cos 12)}} \left( \frac{k_1 E}{k_2 E} \right)^{l_1} \frac{1}{(k_2 E)^{1+w_2(\cos 12)}} \left( \frac{k_2 E}{k_3 E} \right)^{l_2}. \quad (3.34)
$$

The configurations with $k_{2E} \leq k_{1E}$ in the Gegenbauer expansion have been dropped because they do not contribute to the leading UV renormalon. The evaluation of the trace gives the rather simple result

$$
\text{tr} [\ldots] = -64 \left( k_{1E}^2 - k_{1E}Q \cos \theta_{10} \right) \left( k_{2E}^2 - k_{2E}Q \cos \theta_{20} \right) \left( k_{3E}^2 - k_{3E}Q \cos \theta_{30} \right). \quad (3.35)
$$

To proceed further we introduce Dirichlet variables, as in eqs. (3.20). The integral over the variable $v$ can be trivially done; and the contribution to the leading UV renormalon comes only from the terms with $l_1 = l_2 = n_2 = 0$, and $n_1 = n_3 = 1$. The integral then simplifies a lot, with the result

$$
\frac{1}{(m_u + m_d)^2} \Psi_5(q^2) = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \right)^2 \frac{4Q^2}{(2\pi b_0)^2} \int_0^\infty dw \left. \frac{1}{b_0(\alpha^2)} \times \int_{Q^2} \frac{d^2 k_2^2}{k_2^2} \left( \frac{Q}{k_2 E} \right)^{1+w} \log \frac{Q}{k_2 E} \log \frac{Q}{k_2 E} \right)^w \times \int_{Q^2} \frac{d^2 k_3^2}{k_3^2} \left( \frac{k_3 E}{k_2 E} \right)^2 \left( \frac{k_3 E}{k_2 E} \right)^{l_2}. \quad (3.36)
$$

Taking two derivatives with respect to $Q^2$ and scaling $\mu^2$ at $\mu^2 = Q^2$ we get as a final result for the two–chain contribution to the pseudoscalar–pseudoscalar $\mathcal{P}(q^2)$ correlation function the expression:

$$
\mathcal{P}(q^2)_{\text{two chains}} = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \right)^2 \frac{1}{2(\pi b_0)^2} \left( \frac{1}{\alpha_{\text{eff}}(k_2^2)} \right) \frac{1}{(1 + w)^3} \quad (3.37)
$$

### 3.2.3 The One and Two Chain Calculations with the OPE Technique.

The calculation of the pseudoscalar two–point function with the OPE technique is done along the same lines as the one described earlier for the vector two–point function. Recalling the definitions (2.5), (3.1), (3.2), (3.3) one finds that

$$
\Psi_5(q^2) = -\frac{1}{2} \int_{k_2^2 \geq Q^2} \frac{d^4 k}{(2\pi)^4} \frac{4\pi \alpha_{\text{eff}}(k_2^2)}{k_2^2} \langle p \mid T \mid p \rangle \quad (3.38)
$$
where $\mathcal{T}$ is the same as in eq. (3.5), and $|p\rangle$ is the pseudoscalar “particle” annihilated by the external field $p(x)$ of eq. (2.5). Contrary to what happened for the vector two-point function, now at the level of the “one renormalon chain” one does find a priori a dimension–four operator in the expansion of the operator $\mathcal{T}$ which, in the large–$N_c$ limit, reads

$$\mathcal{T}_{\text{dim.}4} = 3N_c \bar{q}(0)i\gamma_5 q(0) \装(0); \quad (3.39)$$

as well as some new dimension–six operators

$$\mathcal{T}_{\text{dim.}6} = \frac{N_c}{k^2} \bar{q}(0)i\gamma_5 q(0) \left[ 2 \partial^2 \装(0) + 4 \装(0)^3 \right] \quad (3.40)$$

besides, of course, the term $D^\alpha G_{\mu\nu}$ of eq. (3.6) which, being subleading in $N_c$, we shall not consider any further. Dimension four operators do not contribute to $\mathcal{P}(Q^2)$ [20, 21]. As for the contribution of the dimension six operators of eq. (3.40), the relevant diagram is the one analogous to the vertex form factor depicted in Fig. 4 and yields

$$\mathcal{P}(Q^2)|_{\text{one chain}} = \frac{4N_c^2}{(4\pi)^3} Q^2 \int_0^\infty \frac{dk^2_E}{k^4_E} \alpha_{\text{eff}}(k^2_E) \log(k^2_E/Q^2) \quad (3.41)$$

which, with the change of variable $w = b_0 \alpha(Q^2) \log(k^2_E/Q^2)$, reads

$$\mathcal{P}(Q^2)|_{\text{one chain}} = \frac{N_c}{16\pi^2} \frac{N_c}{2} \frac{1}{2\pi b_0} \int_0^\infty dw e^{-w / b_0 \alpha(Q^2)} \frac{8}{(1 + w)^2}, \quad (3.42)$$

in agreement with the result (3.33) obtained with the Gegenbauer expansion technique.

When considering two renormalon chains, the four–fermion operators of eq. (3.13) also contribute to $\mathcal{P}(Q^2)$ through diagrams analogous to the ones in Fig. 5, with the result

$$\mathcal{P}(Q^2) = \frac{8N_c^3}{(4\pi)^3} Q^2 \int_0^\infty \frac{dk^2_E}{k^4_E} \left[ \alpha_{\text{eff}}(k^2_E) \right]^2 \log^2(k^2_E/Q^2), \quad (3.43)$$

or with the above change of variables

$$\mathcal{P}(Q^2) = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \int_0^\infty dw e^{-w / b_0 \alpha(Q^2)} \frac{w}{(1 + w)^3}, \quad (3.44)$$

also in agreement with the result obtained in eq. (3.37)

The final result of the leading UV renormalon contribution from the exchange of one and two powers of the QCD effective charge to the function $\mathcal{P}(Q^2)$; i.e., the sum of one and two renormalon chains, is then given by the expression:

$$\mathcal{P}(Q^2) = \frac{N_c}{16\pi^2} \frac{N_c}{2} \frac{1}{2\pi b_0} \int_0^\infty dw e^{-w / b_0 \alpha(Q^2)} \left\{ \frac{1}{(1 + w)^2} - \frac{N_c}{2} \frac{1}{2\pi b_0} \frac{1}{(1 + w)^3} \right\}, \quad (3.45)$$

where the first term in the second line corresponds to the contribution from the one renormalon chain, and the second term to the contribution from the two renormalon chains.
3.2.4 Comments on the two–point function calculations.

We shall conclude this Section with various comments based on the previous calculations.

- **Large–$N_c$ equivalences.**

  Recall that $b_0 \equiv -\beta_1 / 2\pi$, and $\beta_1 = -11 \pi^2 / 6 N_c + \frac{1}{3} n_f$. Since we are working to leading order in the $1/N_c$–expansion, we have the following equivalences:

  \[
  \frac{1}{2\pi b_0} C_F \rightarrow \frac{1}{2\pi b_0} \frac{N_c}{2} \rightarrow \frac{3}{11}.
  \]  

  (3.46)

  We keep the second algebraic form explicitly in most of the results because it shows better the origin of the various factors and the fact that we are neglecting non–leading contributions.

- **Large order behaviour.**

  The large order behaviour in perturbation theory of the Adler function and the pseudoscalar correlation function $P(Q^2)$ can be read off from the results in eqs. (3.27) and (3.45) above. They are:

  \[
  A(Q^2) \big|_{\text{UV}} \sim -\frac{N_c}{16\pi^2} \frac{4}{33} (-1)^{n-1} n! \left(1 - \frac{3}{44} n\right) (b_0 \alpha_s(Q^2))^n, 
  \]  

  (3.47)

  and

  \[
  P(Q^2) \big|_{\text{UV}} \sim -\frac{N_c}{16\pi^2} \frac{24}{11} (-1)^{n-1} n! \left(1 - \frac{6}{22} n\right) (b_0 \alpha_s(Q^2))^n. 
  \]  

  (3.48)

  As already discussed above, these results show that, asymptotically, the effect of the two UV renormalon chains is leading with respect to the one UV renormalon chain. Notice also that the overall coefficient of the pseudoscalar $P(Q^2) \big|_{\text{UV}}$–function is an order of magnitude larger than the corresponding one in the Adler $A(Q^2) \big|_{\text{UV}}$–function. Taking the results (3.47,3.48) at face value, one would infer that the dominance of the two renormalon chain in the pseudoscalar correlation function happens at earlier $n$ values than in the vector correlation function.

- **The expansion in terms of local operators.**

  As we have seen from the previous calculations, one and two chains of bubbles generate at a large Euclidean scale $k_E$ a set of local operators (on the scale $1/k_E$) of increasing dimensionality, i.e.,

  \[
  \mathcal{T} = \sum_{i \geq 6} \frac{c_i(k_E)}{(k_E)^{n_i-4}} \mathcal{O}_i(0),
  \]  

  (3.49)

  where $n_i = \text{dimension of the operator } \mathcal{O}_i(0)$. In this equation $c_i(k_E)$ are the corresponding Wilson coefficients to be obtained as a power series in $\alpha_{\text{eff}}(k_E^2)$. The crucial observation, already made in refs. [20, 21], is that the physics starts with operators $\mathcal{O}_i$ of dimension not smaller than six. As explained in [21] operators of dimension smaller than six have their contribution to the two–point functions buried in the infinite renormalization constant already present at the parton level and that drops out once enough derivatives with respect to the external momenta are taken in the corresponding two–point function.
Furthermore the expansion of eq. (3.49) is meaningful, in the sense that contributions from operators $O_i$ of higher dimension give rise to singularities in the Borel plane further away from the origin, i.e. their $n$–th coefficient in the asymptotic perturbative expansion is more and more suppressed.

• **Diagrams with increasing complexity.**

As one considers contributions coming from diagrams with an increasing complexity, i.e. more and more chains, the different Wilson coefficients $c_i(k_E)$ in (3.49) have more and more powers of $\alpha_{\text{eff}}(k_E)$. One can easily see that integrals like

$$I_p = Q^2 \int_Q^\infty \frac{dK^2}{K^4} \left[ \alpha_{\text{eff}}(k_E) \right]^p \log^2(k^2/Q^2),$$

(3.50)

which will appear in the two–point functions (3.15) and (3.43), give a contribution to the leading term in the perturbative asymptotic expansion that is independent of $p$ and, consequently, powers of $\alpha_{\text{eff}}(k_E)$ are not suppressed. To see this just use

$$\left[ \alpha_{\text{eff}}(k_E) \right]^p = \left[ \alpha(Q^2) \right]^p \frac{\Gamma(n+p)}{\Gamma(p)}$$

(3.51)

and insert it in eq. (3.50). After the $k$ integration one finds

$$I = \frac{1}{\Gamma(p)} \sum_{n=0}^\infty \frac{(-b_0)^n}{n!} \alpha(Q^2)^{n+p} \Gamma(n+p) \Gamma(n+3).$$

(3.52)

At large $n$ one can shift $n \to n - p$ to obtain

$$I_p = \frac{1}{\Gamma(p)(-b_0)^p} \sum_{n, \text{large}} \frac{(-b_0)^n}{n!(n+1)!} \left[ \frac{\Gamma(n+3-p)}{n(n+1)\Gamma(n+1-p)} \right].$$

(3.53)

One can see that the expression between square brackets goes to unity at large $n$, independent of $p$. The factor $1/\Gamma(p)$ out front affects the residue but not the position of the singularity on the Borel plane at $-1/b_0$. In fact one does not expect any suppression from large values of $p$ in the final contribution, as the coefficient accompanying the $\alpha_{\text{eff}}(k_E)^p$ term a priori may behave as $p! [20,21]$. This effectively renders the residue of the UV renormalon incalculable.

Our discussion has been simplified by not including the effect of anomalous dimensions of the operators $O_i$ of eq. (3.43). They introduce contributions such as $[\alpha(Q^2) \log(k^2/Q^2)]^m$ for an arbitrary power $m$ in the integral (3.50). Taking into account anomalous dimensions makes the analysis more cumbersome but does not change the main conclusion about the dominance of four–fermion operators (see refs. [4,21] for details).

• **What message does one learn from these calculations?**

For the class of QCD contributions which are obtained with the “amputated” generating functional of eq. (2.8) we argue that, when the momentum transfer in the effective
charges exchanged are very large, only two types of effective local operators can appear, depending on whether the large momentum flows to the external vertex or not. The local operators are respectively vertex–like operators which connect two fermions to the relevant external source (like the case illustrated in Fig. 4,) and four–fermion box–like operators (like the case illustrated in Fig. 5.) As we have seen the contribution from the dimension $d = 6$ four–fermion operators dominates because of the two powers of $\log \frac{k^2}{Q^2}$ factors from each fermion loop. In the large $N_c$ limit, the only $d = 6$ four–fermion operators which are allowed by the chiral symmetry properties of perturbation theory are the two operators of the ENJL model. Four–fermion tensor operators like

$$\sum_{i,j=\text{flavour}} \left( \bar{\psi}^j_L \sigma^{\mu\nu} \psi^j_R \right) \left( \bar{\psi}^i_R \sigma_{\mu\nu} \psi^i_L \right),$$

which is chirally invariant and where colour is summed over within the parentheses, can be seen to vanish identically for instance by employing the identity $\sigma^{\mu\nu}(1 \pm \gamma_5) = \sigma^{\mu\nu} \pm i2\epsilon^{\mu\nu\rho\lambda} \sigma_{\rho\lambda}$. 

In theories like QED, where the UV renormalon can be interpreted as the signal that non–perturbative contributions must exist, there are indications from lattice simulations that four–fermion operators may indeed play an important rôle in non–perturbative dynamics [26]. In QCD it is difficult to imagine how any non–perturbative dynamics may emerge from the study of UV renormalons since they involve only very high momenta. Our point of view is that the richness revealed by the UV renormalon structure may perhaps be used as well in QCD to learn about generic features of non–perturbative physics provided one studies a regime where there is some infrared momenta involved. The only possible choice in the two–point functions which we are studying seems to be the regime where the external momenta $Q^2$ is taken to be very small instead of very large as is usually assumed. One can think of many physical processes which involve integrals over the external field Euclidean momenta $Q^2$ of these two–point functions all the way down to zero. The hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon is a well known example. (See e.g. ref. [17] and references therein.) It is this “atypical” situation of small external momenta which we next want to explore.

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\footnote{We thank J. Bijnens for pointing this out to us.}
4 The fate of UV–Renormalons in Two–Point Functions at low $Q^2$ values.

We wish now to discuss what happens to the UV renormalon contributions to two–point functions when the external Euclidean momenta $Q^2$ becomes smaller and smaller. We shall take as a starting point the contribution of two chains to the vector two–point function in eq. (3.15). One can readily obtain from it the following leading contribution to the Adler function

$$\Pi(q^2) = \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi} \right)^2 (-\frac{1}{9}) Q^2 \int_{Q^2}^{\infty} \frac{dk_E^2}{k_E^2} \left[ \alpha_{\text{eff}}(k_E^2) \right]^2 \log^2(k_E^2/Q^2) .$$

(4.1)

This integral is well defined as long as $Q^2 > \Lambda_L^2$; however, for $Q^2 < \Lambda_L^2$ there is a pole in the integration range. Let us examine this more closely. With the change of variables $w = \alpha(\mu^2)b_0 \log(k^2_E/Q^2)$, we can rewrite eq. (3.16) as follows

$$A(Q^2) = -\frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \frac{1}{b_0 \alpha(\mu^2)} \int_0^\infty dw e^{-\frac{w}{b_0 \alpha(\mu^2)}} \frac{w^2}{(1 + wQ + w)^2} .$$

(4.2)

where $\alpha(\mu^2)$ is the running coupling constant defined at a scale $\mu^2 > \Lambda_L^2$, where a perturbative expansion in powers of $\alpha(\mu^2)$ makes sense a priori. In this equation

$$w_Q \equiv \alpha(\mu)b_0 \log(Q^2/\mu^2) ,$$

(4.3)

and in the range $Q^2 \leq \Lambda_L^2$ one finds that $w_Q \leq -1$ so that, indeed, the above integrand has a double pole at $w = -1 - w_Q$, i.e. within the integration range. Consequently this integral will be ambiguous depending on how one goes around the double pole. This ambiguity turns out to be $\mu$–independent and reads

$$\delta A(Q^2)_{Q^2 < \Lambda_L^2} = -K \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \frac{Q^2}{\Lambda_L^2} \left( \frac{1}{9} \log^2 \frac{\Lambda_L^2}{Q^2} - 2 \log \frac{\Lambda_L^2}{Q^2} \right) ,$$

(4.4)

where $K$ is an unknown constant parameterizing the ambiguity and of $O(N_c^0)$.

We think this is quite a remarkable result! The ambiguity turns out to be of the same type as the insertion in the vector current two–point function $\Pi(Q^2)$ of the local $d = 6$ composite operator

$$\mathcal{O} = -\frac{8\pi^2 G_V}{N_c \Lambda^2} \sum_{a,b=\text{flavour}} \left[ (q^a_L \gamma^\mu q^b_L) (q^b_L \gamma_\mu q^a_L) + (L \rightarrow R) \right]$$

(4.5)

provided one interprets $\Lambda_L^2$ as the momentum cutoff in the loops and $\Lambda^2 \approx \Lambda_L^2$. This is one of the four–fermion operators appearing in the ENJL model \cite{28,29} with the identification

$$4G_V \equiv -K \left( \frac{1}{2\pi b_0} \right)^2 .$$

(4.6)

The floating scale $k_E^2$ in eq. (3.13) has now turned into a hard scale $\Lambda_L^2$.

\footnote{One can easily check that if $Q^2 \geq \Lambda_L^2$ one may choose $\mu = Q^2$ and integrate by parts to recover eq. (3.16).}
We can repeat the same calculation starting from the integral in eq. (3.7) for the contribution of the one–chain renormalon. The ambiguity is again \( \mu \)-independent but now it has one power less of the log \( \Lambda_L \):

\[
\delta \Pi(Q^2) \big|_{Q^2<\Lambda_L^2} \sim \frac{N_c^2}{b_0} \frac{Q^2}{\Lambda_L^2} \log \frac{\Lambda_L^2}{Q^2}
\]  

(4.7)

and consequently this ambiguity is subleading with respect to that of eq. (4.4) in the low–\( Q^2 \) limit. In Section 3 we have seen that, for \( Q^2 \) large, the two–chain renormalon dominates over the one–chain one at large orders of perturbation theory. What we see now is that, for \( Q^2 \) small, a similar dominance effect appears as well, and it manifests itself in the form of an extra log \( \frac{\Lambda_L^2}{Q^2} \) in the order of the ambiguity. The ambiguity in (4.7) can also be reproduced by the insertion of a local \( d=6 \) operator, which in this case is

\[
\sim \frac{N_c}{b_0 \Lambda_L^2} \bar{q}(x) \gamma^\mu \partial^\nu F_{\alpha \mu}(x) q(x)
\]  

(4.8)

with \( F_{\alpha \mu}(x) \) being the field strength tensor of the external vector field \( \nu_\mu(x) \). The normalization out front is of \( \mathcal{O}(N_0^2) \) but is ambiguous. The operator in eq. (4.8) again coincides with the effective operator at the scale \( k_E \) of eq. (3.6) but with the important difference that now there is a constant \( \Lambda_L^2 \) scale in the denominator.

If one defines

\[
\alpha(Q^2) \equiv \frac{\alpha(\mu^2)}{(1 + b_0 \alpha(\mu^2) \log(Q^2/\mu^2))}
\]  

(4.9)

for \( Q^2 < \Lambda_L^2 \), i.e. if one uses analytic continuation, expressions such as eq. (4.2) acquire a more familiar form. Notice that the \( \alpha(Q^2) \) defined in this way becomes negative for \( Q^2 < \Lambda_L^2 \). By making the change of variables

\[
w' = - w \frac{\alpha(Q^2)}{\alpha(\mu^2)}
\]  

(4.10)

one can check that eq. (4.2) becomes

\[
\mathcal{A}(Q^2) = - \frac{N_c^3}{9(4\pi)^4 b_0^2} \frac{1}{\alpha(\mu^2)} \int_0^\infty dw' e^{-\frac{w'}{b_0 \alpha(\mu^2)}} \frac{w'^2}{(1 - w')^2},
\]  

(4.11)

and the relevant expansion parameter is now \( | \alpha(Q^2) | \), the absolute value of \( \alpha(Q^2) \). In the ordinary situation where \( Q^2 > \Lambda_L^2 \) one can follow the same steps and show that a plus sign appears instead in the denominator, i.e. it is of the form \( 1 + w'^2 \), as it should be. Of course in this case eq. (4.9) is not merely a definition but the true solution of the renormalization group equation.

We shall next argue that it is not so surprising to find that UV renormalons become real singularities when \( Q^2 \leq \Lambda_L^2 \). It is helpful for this purpose to look at the \( Q^2-k_E^2 \) plot in Fig. 8.
The integration regions $0 \leq k^2_E \leq Q^2$ and $Q^2 \leq k^2_E \leq \infty$ shown in Fig. 8 are at the origin of the appearance of IR renormalons and UV renormalons in two–point functions, respectively. (Fig. 8 corresponds to the precise case of the one–chain renormalon, but it can be easily generalized to an arbitrary number of chains by introducing one more dimension for each renormalon chain.) For reference, we also show in this plot the lines $Q^2 = \Lambda^2_L$ and $k^2_E = \Lambda^2_L$. In the conventional study of IR renormalons, the external Euclidean momenta is always chosen to be $Q^2 \gg \Lambda^2_L$; however, regardless of how large $Q^2$ is taken, there will always be a region in the virtual $k^2_E$ integration which is below $\Lambda^2_L$ and which leads to IR renormalon poles in the Borel plane. As discussed in Section 2 their appearance is welcome because they reflect the limitations of perturbation theory and indicate the presence of non–perturbative $\frac{1}{Q^2}$ power corrections, as indeed the OPE in the physical vacuum suggests. What we are proposing here is another way to approach the non–perturbative regime of QCD. We want to explore the kind of ambiguities which appear when the external $Q^2$ is chosen to be below $\Lambda^2_L$. Then, with $Q^2 < \Lambda^2_L$, there is always a region in the virtual $Q^2 \leq k^2_E \leq \infty$ integration region which is also below $\Lambda^2_L$, (the darker triangle in Fig. 8.) It is precisely this integration region which is responsible for the promotion of the UV renormalons to poles on the right hand side of the Borel plane. From considering IR renormalons we are learning what type of corrections to the perturbative evaluation of two–point functions to expect, coming from the large $Q^2$. 

**Fig. 8** Integration regions of the virtual Euclidean momentum $K^2_E$ for a fixed Euclidean external momenta $Q^2$. 

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asymptotic regime. In the case of UV renormalons we are learning what type of effective local operators govern the low–energy physics of the same two–point function, when the virtual Euclidean momenta $k_E^2$ is integrated all the way from $\infty$ down to the low–$Q^2$ regime. The perturbative approach to these two non–perturbative limits is very different, and it is therefore not surprising that the physics in the two cases is also different. We insist on the fact that both limits involve an integration region of virtual momenta below the Landau scale; and in that respect there is no reason to expect one limit to be more physical than the other.

From another point of view let us notice that eq. (4.9) implies that $\alpha(Q^2) \to 0^-$ as $Q^2 \to 0$. This means that the behaviour of $\alpha(Q^2)$ extracted by analytic continuation in the low–energy region is governed by the trivial IR fixed point at $\alpha(Q^2) = 0^-$. In this sense it is pretty much like an abelian theory, only that the expansion parameter is $|\alpha(Q^2)| = -\alpha(Q^2) > 0$; (see eq. (4.11)). We know that an abelian theory is not asymptotically free and consequently that its UV renormalons sit on the right hand side of the Borel plane. This is what happens here as well. It is then reasonable that, as seen from the extreme low–energy end, there may be local higher dimensional operators suppressed by the large scale $\Lambda_L$ since it is the only scale available. As the energy goes up and eventually goes over this scale $\Lambda_L$, the dynamics is ultimately governed by the true UV fixed point of QCD at $\alpha(Q^2) = 0^+$, as follows from its nonabelian character. As far as renormalons are concerned, if we had to exemplify this, a similar situation could be like having QED embedded in a nonabelian GUT model like, e.g., $SU(5)$.

We turn next to the discussion of the pseudoscalar two–point function in the same low–$Q^2$ regime. The analysis runs parallel to the vector one. As far as the two–chain renormalon is concerned one sees that the leading result eq. (3.43) is proportional to the one in the vector channel, eq. (3.15). One therefore concludes that the ambiguity is

$$\delta P(Q^2)_{\text{two chains}} = -\mathcal{K} \frac{N_c}{16\pi^2} \left( \frac{N_c}{2} \frac{1}{2\pi b_0} \right)^2 \frac{Q^2}{\Lambda^2_L} \left( 8 \log^2 \frac{\Lambda^2_L}{Q^2} + \mathcal{O}\left( \log \frac{\Lambda^2_L}{Q^2} \right) \right)$$

(4.12)

with $\mathcal{K}$ the same constant appearing in eq. (4.4). This ambiguity can also be reproduced by a local four–fermion operator; an operator exactly of the form of the scalar–pseudoscalar four–fermion operator which appears in the ENJL model:

$$\mathcal{O} = \frac{8\pi^2 G_S}{N_c \Lambda^2_L} \sum_{a,b=\text{flavour}} \left( \bar{q}_R^a(x) q_L^b(x) \right) \left( \bar{q}_L^a(x) q_R^b(x) \right).$$

(4.13)

The identification with the resulting perturbative ambiguity requires now

$$G_S \equiv -4\mathcal{K} \left( \frac{1}{2\pi b_0} \frac{N_c}{2} \right)^2.$$

(4.14)

Assuming that the ambiguity $\mathcal{K}$ is a measure of the final non–perturbative value for the coefficients of the ENJL–like four–fermion operators would lead to the conclusion that $G_S/G_V = 16$. However, one should not forget that this result has been obtained at the two–chain renormalon level. As we shall next show, higher number of chains also contribute

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11 The one–loop coupling $b_0 \alpha(Q^2) = (\log Q^2/\Lambda^2_L)^{-1}$ is invariant under the change $\alpha(Q^2) \to -\alpha(Q^2)$, $b_0 \to -b_0$. 

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with the same $Q^2/\Lambda^2_L$ dependence as in eqs. (4.1,13), with the result that not even the ratio $G_S/G_V$ can be fixed from the lowest non–trivial two–chain calculation.

When we examined UV renormalons in the high–$Q^2$ regime in 3.2.4 we saw that diagrams with an increasing number of chains are not suppressed with respect to the contribution coming from two chains. This feature persists when $Q^2 < \Lambda^2_L$. Let us consider the integral in (3.50) again, but now for $Q^2 < \Lambda^2_L$:

$$I_p = N_c Q^2 \int_{Q^2}^{\Lambda^2_L} \frac{dk^2_E}{k^2_E} \left[ N_c \alpha_{\text{eff}}(k^2_E) \right]^p \log^2 \frac{k^2_E}{Q^2},$$ (4.15)

where we have introduced convenient factors of $N_c$ to match the dependence of a leading contribution at large $N_c$. Defining $w = -\alpha(Q^2) b_0 \log(k^2_E/Q^2)$, where $\alpha(Q^2) < 0$ is given by equation (4.9), one finds

$$I_p = N_c \left[ \frac{N_c \alpha(Q^2)}{-b_0 \alpha(Q^2)} \right]^p \int_0^\infty dw \ e^{-\frac{w}{b_0 \alpha(Q^2)}} \frac{w^2}{(1-w)^p}. \quad (4.16)$$

This integral has a pole at $w = 1$ of multiplicity $p$. Therefore the ambiguity is given by

$$\delta I_p = \tilde{K} N_c \left[ \frac{N_c \alpha(Q^2)}{-b_0 \alpha(Q^2)} \right]^p \left( \frac{d}{dw} \right)^{p-1} \left[ e^{-\frac{w}{b_0 \alpha(Q^2)}} w^2 \right]_{w=1}. \quad (4.17)$$

Each time the derivative $d/dw$ acts on the exponential it brings down a factor $(b_0 \alpha(Q^2))^{-1}$, hence a power of $\log(Q^2/\Lambda^2_L)$. When $Q^2 \to 0$, the leading contribution is given by

$$\delta I_p = -\frac{1}{(p-1)!} \tilde{K} N_c \left( \frac{N_c}{b_0} \right)^p \left( \log \frac{\Lambda^2_L}{Q^2} + \cdots \right), \quad (4.18)$$

where $\cdots$ stands for subleading terms when $Q^2 \ll \Lambda^2_L$. Consequently one obtains once more the $Q^2$ dependence of eqs. (4.4) and (4.13), and therefore the ambiguity can be reproduced by the insertion of the ENJL operators (4.5) and (4.13), although with different coefficients $G_S, G_V$. The calculation of these coefficients for an infinite number of chains looks like a much harder problem since there is no obvious parameter that dictates that a higher number of chains should give a smaller contribution. Therefore there is no reason to believe the relation $G_S = 16G_V$ will be preserved beyond two chains and, for instance, the $G_S$ and $G_V$ corresponding to the three–chain double–box diagram come in the ratio $G_S/G_V = 64$. From a practical point of view, one may just as well leave the coefficients $G_S$ and $G_V$ as free phenomenological parameters to be fitted to some experimental input [28, 29]. This is just the usual situation. Renormalons help one to make good guesses for parameterizing non–perturbative physics, but they are not capable of producing quantitative results: in our case, $G_S$ and $G_V$ appear to be incalculable quantities.

There is another point of view, concerning the ratio $G_S/G_V$, which has been advocated by the authors of ref. [18]. Their claim is that the consistency of the successes of the QCD sum rules with the ENJL model requires $G_S \gg G_V$. If that was the case, one may then perhaps be tempted to take more seriously the quantitative perturbation theory results above. The analysis in ref. [18], however, shows that the success of the ENJL model in predicting the $O(p^4)$ $L_i$ coupling constants of the chiral Lagrangian hinges on the fact that $G_V$ has to be as large as $G_S$ [28]. A detailed phenomenological analysis of this issue is under investigation [19].

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12This persists even after use of a two–loop $\beta$ function. See Appendix A.
13It is in particular the constants $L_5$ and $L_8$ what requires the axial coupling constant of the constituent quarks $g_A$ to be $g_A \geq 0.6$ and hence $G_V \sim G_S$. 

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5 Conclusions and Outlook.

The analysis of ultraviolet renormalons, in the way we just described, seems helpful in bridging the gap between high energy and low energy in QCD. It offers an arena where it begins to be possible to discuss generic dynamical patterns of the effective low energy chiral Lagrangian of QCD. At high energies UV renormalons hint at the emergence of dimension $d = 6$ four–fermion operators from the large orders of perturbation theory \[20, 21\]. What we have seen here is that, at low energies, they may even “feel” the existence of the scale $\Lambda_X$ in the normalization of these four–fermion operators. There is a correspondence between the leading $n!$ behaviour at large orders of perturbation theory and the leading $Q^2 \to 0$ behaviour as both are due to the insertion of the same effective operator.

Perturbation theory (to all orders) and analyticity in the coupling constant has led us this far. Let us now turn to some physical comments. The picture which emerges is obviously incomplete as, so far, there is no signal of spontaneous chiral symmetry breaking or confinement. Neither of these features should be expected however from perturbation theory arguments alone, which after all is the framework of renormalon calculus. As we have seen, the analysis of UV renormalons in the regime of low external $Q^2$ momenta provides a plausible link with the ENJL model. This model however is only supposed to be meaningful in the event of a clear separation between the confinement scale $\Lambda_{\text{conf}}$ and the spontaneous chiral symmetry scale $\Lambda_X$, as it is an effective Lagrangian description in between the two scales with $\Lambda_{\text{conf}} < \Lambda_X$. That these two scales may be separated widely enough seems to be backed up by the phenomenological success of the constituent chiral quark model \[33\], which follows naturally from the ENJL model, and wherein there is a successful description in terms of “constituent massive quarks”. For this to be possible within the UV renormalon approach which has been advocated above, the coupling constant $G_S$ in the $d = 6$ four–fermion operator \[1.13\] has to be larger than unity. It is very difficult to justify why $G_S$ should have this size within perturbation theory alone. However we know that there is a (rather similar!) case — the gluon condensate — where the ambiguities foreseen in perturbation theory via the IR renormalon analysis are finally realized by a rather large vacuum expectation value of non–perturbative origin. There also, renormalons only signal the appearance of non–perturbative contributions in the form of $Q^{-4}$ terms, and it is then a matter of non–perturbative dynamics what finally makes the numerator large. It is not unreasonable to assume that the coupling constants $G_S$ and $G_V$ of the four–fermion operators which govern the leading UV renormalon effects may behave similarly to the gluon condensate and turn out to have large values as well in the real world.

We have seen that, at least within the framework of the “amputated” effective action in \[2.8\], four–fermion operators govern the properties of the leading UV renormalon. Assuming that, eventually beyond perturbation theory, chiral symmetry is spontaneously broken by the dimension–six scalar four–fermion operator with $G_S \geq 1$, the question one may ask is: can one safely neglect operators with dimensions higher than six? Although they are associated with subleading renormalons in perturbation theory; beyond perturbation theory the counting of dimensions changes for the dimension–six scalar four–fermion operator with $G_S \geq 1$. Its effects become suppressed only by a $1/\log(\Lambda_X^2/Q^2)^2$ instead of $Q^2/\Lambda_X^2$. This enhancement partially takes place also in scalar four–fermion operators with higher dimensions. The net effect there, however, is that these only modify somewhat the relationship between the mass of the scalar particle and the mass of the constituent quark ($M_S \neq 2M_Q$) but they do not change the spectrum or the interactions \[30\]. In the vector channel, higher–than–six
dimensional operators only modify the results of the ENJL model at values of $Q^2 \sim M^2$ but not at lower energy.

More work than what we have presented here is clearly needed and seems worthwhile. What is at stake is that it may be finally possible to understand why the ENJL model, and hence the constituent chiral quark model, does so well in describing low-energy QCD phenomenology \[28, 29\]. We think that the patterns unraveled so far must not be just a coincidence.

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Appendix A

In this appendix we shall outline how to generalize our previous results to the case of a two–coefficient $\beta$ function such as

$$\alpha(\mu^2)\beta(\alpha(\mu^2)) = -b_0\alpha(\mu^2)^2 - b_1\alpha(\mu^2)^3.$$  \hfill (A.1)

The use of this $\beta$ function is interesting since i) it has a finite radius of convergence in $\alpha$ (it is a polynomial), and ii) any $\beta$ function can be brought to this form by making a perturbative redefinition of the coupling constant \[1\].

The equation

$$\frac{d\alpha(\mu^2)}{d\log(\mu^2/Q^2)} = -b_0\alpha(\mu^2)^2 - b_1\alpha(\mu^2)^3$$ \hfill (A.2)

can be integrated to yield

$$\frac{1}{b_0} \left( \frac{1}{\alpha(\mu^2)} - \frac{1}{\alpha(Q^2)} \right) = \log(\mu^2/Q^2) + \frac{b_1}{b_0} \log \frac{\alpha(Q^2)}{\alpha(\mu^2)} + \frac{b_1}{b_0}. \hfill (A.3)$$

This equation has a Landau pole at

$$\Lambda_L^2 = \mu^2 e^{-\frac{1}{b_0\alpha(\mu^2)}} \left( 1 + \frac{b_0}{b_1\alpha(\mu^2)} \right)^{\frac{b_1}{b_0}}.$$ \hfill (A.4)

Notice that equation (A.3) implies that

$$\alpha(\mu^2) \simeq \frac{1}{b_0 \log(\mu^2/\Lambda_L^2)}, \hfill (A.5)$$

both when $\mu^2 \to +\infty$ and $\mu^2 \to 0$.

The integral we want to study is

$$I = \int_Q^\infty \frac{dk_E^2}{k_E^2} \left( \frac{k_E^2}{Q^2} \right)^n \left[ \alpha(k_E^2) \right]^p \log^2 \frac{k_E^2}{Q^2}, \hfill (A.6)$$

for $n$ a negative integer with $n \leq -1$ and $p$ a natural number, $p \geq 2$.

Making the change of variables \[2\]

$$z \equiv \frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(Q^2)} \left( \frac{b_1}{b_0} \right)^n,$$ \hfill (A.7)

where $z_n \equiv n/b_0 < 0$ and $Q^2 \ll \Lambda_L^2$, so $\alpha(Q^2) \simeq (b_0 \log(Q^2/\Lambda_L^2))^{-1}$, one obtains

$$I = \left[ \frac{1}{\alpha(Q^2)} - \frac{z}{z_n \left( \frac{1}{\alpha(Q^2)} + \frac{b_1}{b_0} \right)^{1-p}} \right]^{1-p} \left[ -\frac{z}{z_n b_0 \alpha(Q^2)} - \frac{z b_1}{z_n b_0^2} - \frac{b_1}{b_0} \log \left( 1 - \frac{z}{z_n} \right) \right]^2. \hfill (A.8)$$
where \( \delta_n \equiv nb_1/b_0^2 < 0 \).

Since \( z_n < 0 \), the integration region contains two singularities that, in the limit \( \alpha(Q^2) \to 0^- \), collapse to a branch point at \( z = z_n \). Therefore

\[
\delta I = \alpha(Q^2)^{p-1} \int_0^{-\infty} \frac{dz}{-n} \text{Disc} \left\{ \frac{e^{-z} \left( \frac{1}{\alpha(Q^2)} + \frac{b_1}{b_0} \right)}{(1 - \frac{z}{z_n})^{p+\delta_n}} \left[ - \frac{z}{z_n b_0 \alpha(Q^2)} + \frac{b_1}{b_0} \log \left( 1 - \frac{z}{z_n} \right) \right]^{2} \right\}
\]

hence

\[
\delta I \simeq \alpha(Q^2)^{p-1} \int_0^{-\infty} \frac{dz}{-n} \text{Disc} \left\{ \frac{e^{-z} \left( \frac{1}{\alpha(Q^2)} + \frac{b_1}{b_0} \right)}{(1 - \frac{z}{z_n})^{p+\delta_n}} \right\}
\]

which can still be approximated as

\[
\delta I \simeq \frac{\alpha(Q^2)^{p-3}}{b_0^2} \int_0^{-\infty} \frac{dz}{-n} e^{-z} \left( \frac{1}{\alpha(Q^2)} + \frac{b_1}{b_0} \right) \text{Disc} \left\{ 1 - \frac{z}{z_n} \right\}'.
\]

Given that \( \text{Disc} \left( 1 - \frac{z}{z_n} \right)^{p-\delta_n} = \hat{K}(1 - e^{-i2\pi\delta_n}) \Gamma(1 - p - \delta_n) \frac{Q^2}{\Lambda^2} \log^2 \frac{\Lambda^2}{Q^2} + \cdots \), where \( \hat{K} \) is an arbitrary constant, one obtains

\[
\delta I \simeq \frac{\hat{K}(1 - e^{-i2\pi\delta_n})}{-n(b_1/b_0)^{\delta_n}} \left[ z_n^{p+\delta_n} \Gamma(1 - p - \delta_n) \frac{Q^2}{\Lambda^2} \log^2 \frac{\Lambda^2}{Q^2} \right] + \cdots.
\]

where the ellipses stand for terms subleading in the limit \( Q^2/\Lambda^2 \to 0 \). One can check that the \( b_1 \to 0 \) limit of the previous expression is equivalent to eq. (1.18) in the text, which was obtained in the case \( b_1 = 0 \).

One finally sees from eq. (1.12) that the leading contribution \( n = -1 \) produces again an ambiguity \( \sim (Q^2/\Lambda^2)^2 \log^2(\Lambda^2/Q^2) \), exactly as in the \( b_1 = 0 \) case discussed in the text. Therefore our conclusions are also valid in the general case of the \( \beta \) function (A.1).

**Appendix B**

We would like to comment here on the issue of the “freezing” of the strong coupling constant at low energy.

Sometimes one finds in the literature proposals for a non-perturbative behaviour of the coupling constant in the low-energy region of the form

\[
\tilde{\alpha}(Q^2) = \frac{1}{b_0 \log Q^2 + \Lambda^2}.
\]

This form clearly guarantees the right high-\( Q^2 \) behaviour dictated by perturbation theory. It can be obtained from the following RG equation:

\[
Q^2 \frac{d\tilde{\alpha}(Q^2)}{dQ^2} = \alpha(\tilde{Q}^2) \beta(\tilde{\alpha}(Q^2)).
\]
with
\[
\beta(\tilde{\alpha}(Q^2)) = - b_0 \tilde{\alpha}(Q^2) \left[ 1 - \frac{C^2}{\Lambda^2} e^{-\frac{1}{b_0 \tilde{\alpha}(Q^2)}} \right].
\]

(B.3)

The constant $C$ is supposed to encode the non–perturbative dynamics. As $C^2 \to 0$ one recovers the usual situation in perturbation theory at one loop. In this equation $C^2 > \Lambda^2 > 0$ so that $\tilde{\alpha}(Q^2)$ does not have a Landau pole.

What we would like to point out here is that chiral symmetry breaking is a strong constraint for this proposal, at least in the simplest form of eq. (B.1), as we now explain.

If we take the Adler function $A(Q^2)$ (defined in the main text) one knows from general arguments of spontaneous chiral symmetry breaking in the large–$N_c$ limit that $A(Q^2) \sim Q^2$ as $Q^2 \to 0$ (see 14), and in particular $A(0) = 0$.

In perturbation theory (i.e. at large $Q^2$) the Adler function is given by
\[
A(Q^2) = \frac{1}{8\pi^2} \left( 1 + \frac{\alpha(Q^2)}{\pi} \right),
\]

(B.4)

where $\alpha(Q^2)$ is the usual perturbative coupling constant at one loop, say. If $\tilde{\alpha}(Q^2)$ is supposed to incorporate all the non–perturbative dynamics by deviating from $\alpha(Q^2)$ as $Q^2 \to 0$ one would expect that
\[
A(Q^2) = \frac{1}{8\pi^2} \left( 1 + \frac{\tilde{\alpha}(Q^2)}{\pi} \right),
\]

(B.5)

would be the natural answer. However we saw that freezing leads to $\tilde{\alpha}(Q^2) \to$ constant ($> 0$) as $Q^2 \to 0$ and therefore $A(0) \neq 0$, in conflict with chiral symmetry.

The problem seems rather generic for coupling constants obeying an equation such as (B.2) for if $A(0)$ has to vanish, $\tilde{\alpha}(Q^2)$ has to become negative at low $Q^2$. But since $\tilde{\alpha}(Q^2)$ is positive at high $Q^2$ it follows, if $\tilde{\alpha}(Q^2)$ is a continuous function of its variable, that there must be an intermediate point at which $\tilde{\alpha}(Q^2)$ is small and its slope is positive, which is impossible since its slope is the $\beta$ function which, for small $\tilde{\alpha}(Q^2)$, has to be negative according to asymptotic freedom.

We would like to emphasize that the argument we presented above is not intended as a “NO-GO” theorem but as a sort of illustrative warning signal. One way out is to invalidate eq. (B.5), because more powers of $\tilde{\alpha}(Q^2)$ cannot be neglected or for any other reason. For instance, there are models that are successful in describing diffractive phenomena in which it is argued that the freezing coupling constant (B.1) also comes with a $Q^2$-dependent dynamical mass for the gluon. In this case eq. (B.5) does not hold and one may then be safe. As it turns out, however, in this case it takes a more detailed analysis of the Schwinger–Dyson equations to reveal that, actually, these models do not seem capable of generating enough chiral symmetry breaking in the end (54).

Another possibility is that the freezing of the coupling constant may have a more sophisticated dynamical origin. For instance, if one allows that the RG equation be governed by a $\beta$ function that depends on both $\tilde{\alpha}(Q^2)$ and $Q^2$, $\beta(\tilde{\alpha}(Q^2), Q^2)$, then eq. (B.5) may as a matter

\[14\text{In the large-$N_c$ limit only one–meson states contribute to } A(Q^2). \text{ Since these states are not massless in the chiral limit, dimensional analysis imposes that } A(Q^2) \sim Q^2/M^2 \text{ with } M \text{ the corresponding meson mass.}\]

\[15\text{As a matter of fact } \tilde{\alpha}(0)/\pi \lesssim 0.3 \text{ (3)}, \text{ i.e. a relatively small correction to unity.}\]
of fact reproduce $A(0) = 0$. Just as an existing proof, let us quote one such $\tilde{\alpha}(Q^2)$:

$$
\tilde{\alpha}(Q^2) = -\pi e^{-\frac{Q^2}{\Lambda^2}} + \frac{e^{-\frac{\Lambda^2}{Q^2}}}{b_0 \log \left( a + \frac{Q^2}{\Lambda^2} \right)}
$$

(B.6)

for arbitrary parameters $\Lambda^2$ and $a > 1$.

Therefore our conclusion is that spontaneous chiral symmetry breaking imposes rather severe restrictions on the idea of freezing. One should check that enough symmetry breaking can be produced before any argument relying on a particular freezing coupling constant is put forward.
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