The critical points of lattice QCD with a non–zero quark density
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Abstract

We study the interplay of quark number density and chiral symmetry in lattice QCD. We suggest that both are controlled by the eigenvalue spectrum of the fermionic propagator matrix, which shapes the pattern of zeros of the partition functions. The onset in the quark current would be triggered by the lowest lying eigenvalue, the chiral transition by the density of zeros, the two critical points being distinct in full QCD, and coincident in the quenched approximation. Our preliminary estimate for the critical point in full QCD in the infinite coupling limit compares favourably with the predictions of the strong coupling expansions and of numerical simulations based on exact, alternative representations of the partition function. Several reasons of perplexity however remains, which are briefly discussed.

Lattice QCD at non–zero quark number[1] is a poorly understood subject, despite the success of numerous calculations exploiting approximation schemes or simpler models. Many of the difficulties come from the particular structure of the Lagrangean: as the building blocks of the lattice QCD Lagrangean are quarks fields, a finite density on the lattice is realized by a coupling to a quark density, as opposed to a nucleon density of nuclear physics models. In practical numerical work, this induces several technical problems, which are described in detail in past publications. From a phenomenological point of view, it is not obvious that an excess of 3N quarks would produce the same physical effects as an excess of N baryons and it might be useful to keep this in mind, especially when facing unexpected results[2].

This warning issued, the rest of this note will only deal with the current formulation of lattice QCD at non–zero quark number. The numerical analysis poses specific problems, since the action is complex. The only method which at the moment has the potential to deal with it, proposed by Barbour some years ago, is based on the representation of the partition function Z as

\[ Z = \langle \text{Det}(P - \exp(\mu)) \rangle \]

where \(P\) is the fermionic propagator matrix[3].

The quark number density is an interesting observable which can be easily computed within this approach. The common feature of the results for the number density [4,5], as reviewed by Ukawa at this meeting, is a rather early onset \(\mu_o\), definitively smaller than the critical point for chiral symmetry restoration expected around \(m_N/3\) and a saturation threshold, beyond which the particle density is one. \(\mu_o\) can be rather accurately measured in the strong coupling limit, where we find \(\mu_o \simeq m_\pi/2\). High statistics simulations confirm
that also at intermediate coupling the number density is sensitive to the pion mass [5].

In the following we only discuss the subcritical region at zero temperature in the infinite coupling limit, where the results they can be contrasted with the predictions of the strong coupling expansions [6] and of an alternative, exact representation of the partition function [7].

We do not discuss here the problems connected with the saturation threshold, which hampers the observation of the Stefan-Boltzmann behaviour deep in the hot and dense phase, we just mention that possible solutions might be found in the framework of the lattice improved/perfect actions discussed here by Karsch, Wiese and T. D. Lee.

The early onset: the density of states of the fermionic operator and the pion mass. Gibbs proved that the onset of the number density on isolated configurations is controlled by the lowest eigenvalue of the fermion propagator matrix, and argued that the lowest eigenvalues is half the pion mass [8]. The results reviewed above suggest that this holds true also in the ensemble average. Alternative scenarios can be proposed and we postpone their discussion to a lengthier presentation. Here we merely sketch an argument which suggests that the persistency of this result in the statistical ensemble at zero temperature is compatible with the symmetries of the system, so ergodicity problems, if any, are not obviously manifest.

Consider the determinant on an isolated configuration, and the partition function after averaging over the statistical ensemble: $Det = \prod_{i=1}^{6V}(z - \lambda_i)$, $Z = \prod_{i=1}^{6V}(z - \alpha_i)$. The $\lambda_i$’s are the eigenvalues of the fermionic propagator matrix, $z$ is the fugacity $e^\mu$ and the $\alpha_i$’s the zeros of the partition function in the complex fugacity plane: the zeros of the partition function can be seen as the “proper” ensemble average of the eigenmodes of the fermionic propagator matrix. From the determinant we obtain the number density on isolated configurations at zero temperature [8]: $J_0 = 1/V \sum_{1<|\lambda_i|<e^\mu} 1.$, and we note that an analogous expression holds true in the ensemble average as well, provided that we trade the eigenvalues with the zeros of the partition function. To know the fate of the onset of the current in the statistical ensemble we only need to monitor $\min \{|\lambda_i|’s \}$, the contribution of each pole to the current being, configuration by configuration, constant. Consider now that the Z3 symmetry, well verified in high statistics simulations [5], imposes $Z = \prod_{i=1}^{24V}(z^3 - \beta_i)$, i.e. the Z3 symmetry constraints the arguments of the zeros, but not their modulus. We have indeed checked that, configuration by configuration, $\min \{\ln |\lambda| \} \simeq m_\pi/2$ : it may well be that the onset in the current in the full ensemble is the same as the onset on isolated configurations, whose origin is clear.

In a sense, this is a straightforward result: it says that the signal in the
fermion number density is triggered by the lowest eigenmode in the spectrum of the fermionic propagator matrix. This result is the one expected for non confining theories, like Gross Neveu, where the lowest state defines indeed the dynamical fermion mass [9]. In this formulation and within this approach to lattice QCD the lowest eigenvalue gives half the pion mass.

The chiral transition, and random matrix models

The possibility of observing another phase transition at \( \mu_c > \mu_o \), and a non trivial distribution of zeros of the partition function are closely related, as we can read off the expression for the number density. How would these transitions at \( \mu_o \) and \( \mu_c \) relate with chiral symmetry restoration?

A very useful laboratory for the study of chiral symmetry is offered by random matrix models. In the case of QCD at finite density, we learnt from the work of Stephanov that the transition would show at a physical \( \mu_c \) in the full model, but at half the pion mass in the quenched approximation [10]. The applications of these results to QCD would suggest that the onset for the quark number at half the pion mass would also restore the chiral symmetry in the quenched model, because of the simultaneous occurrence of quarks and conjugate quarks in the system, while in full QCD the chiral symmetry, would be restored at the “correct” \( \mu_c \) [6,7] thanks to the rearrangements of the eigenvalues produced by the richer dynamics—this relates also to the different nature of critical phenomena in quenched and full models.

This discussion suggests a natural numerical strategy, whose preliminary outcome is shown in Figure 1. These results, obtained on a \( 6^4 \) and \( 8^4 \) lattice with a bare quark mass = .1, are preliminary, and are just meant as an illustration of the simple idea presented above. We see that the two statistical ensembles (quenched and full) show the same extrema, which defines a common critical region for the quenched and the full model. This agrees with the results of [13], where the “forbidden” region of quenched QCD was found to be coincident with the metastable region of full QCD, as computed in strong coupling expansion. However, contrary to the expectations of [13], it seems impossible to measure the real chiral transition point in the quenched approximation, while in the full model a peak in the eigenvalues distribution shows up in correspondence with the expected critical point. Needless to say, it would be very interesting to study the relation of the spectrum of

1 Given the prominent role of the density of states evaluated at zero \( \mu \), the suggestion would naturally arise to give a second try to various partial quenched schemes [11]. These approximations were dismissed in the past because of the threshold at half the pion mass, but, as we have shown, this threshold is not necessarily related with the chiral transition. Particularly interesting could be their application to the recent proposal by Kogut and collaborators [12].

2The is reminiscent of other observations: quenched and full model can be completely different, but even dramatic qualitative differences are realized by subtle numerical effects.
Summary The picture suggested by the above discussions is as follows: on isolated configurations the analysis of the fermionic propagator matrix gives a signal for the current in correspondence to its lowest state. This seems to survive the statistical ensemble average: there are two main reasons for this, first, that the real part of the pole is stable, second, that for each pole the amplitude of the contribution to the current is constant, and equal 1, so no cancellation occurs. Thus the onset for the quark number is the lowest real part of the zero of the partition function, or, equivalently, the real part of the logarithm of the lowest eigenmode of the fermionic propagator matrix. This a straightforward result in non–confining models, while in QCD the result $\mu_o \simeq m_\pi/2$, and the nature of the region $\mu_o < \mu < \mu_c$, deserve further investigation.

We haven’t found any indication of systematic errors associated with the method or lack of ergodicity in the algorithm which can offer an alternative explanations of these observations. Still, they cannot be excluded. The best way to address this issue, in our opinion, is to cross check with the results of alternative formulations\textsuperscript{12}.

Within this formulation quenched and full QCD share the same critical region. However, the simple dynamics of the quenched model cannot build up any structure in the density of modes, so no new transition appears after
statistical averaging, and the results from random matrix models suggest that the onset of the current restores chiral symmetry. The structure in the spectrum is instead apparent in full QCD, where we attempted, not unsuccessfully, an estimate of the chiral transition point.

These observations would predict a non–zero critical density at the zero temperature chiral transition, and might be related with the presence of diquarks in the region $\mu_o < \mu < \mu_c$. But, again, cross checks with other formulations are mandatory in order to disentangle possible numerical artifacts from predictions amenable to an experimental verification.

This note reports on work in progress with I. Barbour, S. E. Morrison, E. G. Klepfish and J. B. Kogut. I wish to thank my collaborators and acknowledge valuable discussions with F. Karsch. I would like to thank the Physics Department of the University of Bielefeld for its hospitality, and the High Energy Group of HLRZ/Jülich, particularly K. Schilling, for support during the initial stages of this project. The calculations were done at HLRZ/KFA Jülich. This work was partially supported by Nato grant no. CRG 950896.

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