Vector quantile regression beyond correct specification

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Abstract

This paper studies vector quantile regression (VQR), which is a way to model the dependence of a random vector of interest with respect to a vector of explanatory variables so to capture the whole conditional distribution, and not only the conditional mean. The problem of vector quantile regression is formulated as an optimal transport problem subject to an additional mean-independence condition. This paper provides a new set of results on VQR beyond the case with correct specification which had been the focus of previous work. First, we show that even under misspecification, the VQR problem still has a solution which provides a general representation of the conditional dependence between random vectors. Second, we provide a detailed comparison with the classical approach of Koenker and Bassett in the case when the dependent variable is univariate and we show that in that case, VQR is equivalent to classical quantile regression with an additional monotonicity constraint.

Keywords: vector quantile regression, optimal transport, duality.

1 Introduction

Vector quantile regression was recently introduced in [4] in order to generalize the technique of quantile regression when the dependent random variable

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is multivariate. Quantile regression, pioneered by Koenker and Bassett [10], provides a powerful way to study dependence between random variables assuming a linear form for the quantile of the endogenous variable $Y$ given the explanatory variables $X$. It has therefore become a very popular tool in many areas of economics, program evaluation, biometrics, etc. However, a well-known limitation of the approach is that $Y$ should be scalar so that its quantile map is defined. When $Y$ is multivariate, there is no canonical notion of quantile, and the picture is less clear than in the univariate case[1].

The approach proposed in [4] is based on optimal transport ideas and can be described as follows. For a random vector vector $Y$ taking values in $\mathbb{R}^d$, we look for a random vector $U$ uniformly distributed on the unit cube $[0, 1]^d$ and which is maximally correlated to $Y$, finding such a $U$ is an optimal transport problem. A celebrated result of Brenier [2] implies that such an optimal $U$ is characterized by the existence of a convex function $\varphi$ such that $Y = \nabla \varphi(U)$. When $d = 1$, of course, the optimal transport map of Brenier $\nabla \varphi = Q$ is the quantile of $Y$ and in higher dimensions, it still has one of the main properties of univariate quantiles, namely monotonicity. Thus Brenier’s map $\nabla \varphi$ is a natural candidate to be considered as the vector quantile of $Y$, and one advantage of such an approach is the pointwise relation $Y = \nabla \varphi(U)$ where $U$ is a uniformly distributed random vector which best approximates $Y$ in $L^2$.

If, in addition, we are given another random vector $X$ capturing a set of observable explanatory variables, we wish to have a tractable method to estimate the conditional quantile of $Y$ given $X = x$, that is the map $u \in [0, 1]^d \mapsto Q(x, u) \in \mathbb{R}^d$. In the univariate case $d = 1$, and if the conditional quantile is affine in $x$ i.e. $Q(x, u) = \alpha(u) + \beta(u)x$, the quantile regression method of Koenker and Bassett gives a constructive and powerful linear programming approach to compute the coefficients $\alpha(t)$ and $\beta(t)$ for any fixed $t \in [0, 1]$, dual to the linear programming problem:

$$\sup_{(U_t)} \{\mathbb{E}(U_t Y) : U_t \in [0, 1], \mathbb{E}(U_t) = (1 - t), \mathbb{E}(XU_t) = \mathbb{E}(X)\}. \quad (1.1)$$

Under correct specification, i.e. when the true conditional quantile is affine in $x$, this variational approach estimates the true coefficients $\alpha(t)$ and $\beta(t)$. In [4], we have shown that in the multivariate case as well, when the true vector quantile is affine in $x$, one may estimate it by a variational problem which consists in finding the uniformly distributed random variable $U$ such

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1There is actually an important literature that aims at generalizing the notion of quantile to a multidimensional setting and various different approaches have been proposed; see in particular [1], [9], [12] and the references therein.
that $\mathbb{E}(X|U) = \mathbb{E}(X)$ (mean independence) and maximally correlated with $Y$.

The purpose of the present paper is to understand what these variational approaches tell about the dependence between $Y$ and $X$ in the general case i.e. without assuming any particular form for the conditional quantile. Our main results are the following:

- **A general representation of dependence**: we will characterize the solution of the optimal transport problem with a mean-independence constraint from [4] and relate it to a relaxed form of vector quantile regression. To be more precise, our theorem 3.3 below will provide the following general representation of the distribution of $(X,Y)$:

$$Y \in \partial \Phi_X^*(U) \quad \text{with} \quad X \mapsto \Phi_X(U) \text{ affine},$$

$$\Phi_X(U) = \Phi_X^{**}(U) \quad \text{almost surely},$$

$$U \text{ uniformly distributed on } [0,1]^d, \quad \mathbb{E}(X|U) = \mathbb{E}(X),$$

where $\Phi_x^{**}$ denotes the convex envelope of $u \mapsto \Phi_x(u)$ for a fixed $x$, and $\partial$ denotes the subdifferential. The main ingredients are convex duality and an existence theorem for optimal dual variables. The latter is a non-trivial extension of Kantorovich duality: indeed, the existence of a Lagrange multiplier associated to the mean-independence constraint is not straightforward and we shall prove it thanks to Komlos’ theorem (theorem 3.2). Vector quantile regression is under correct specification if $\Phi_x(u)$ is convex for all $x$ in the support, in which case one can write

$$Y = \nabla \Phi_X(U) \quad \text{with} \quad \Phi_X(.) \text{ convex,} \quad X \mapsto \Phi_X(U) \text{ affine},$$

$$U \text{ uniformly distributed on } [0,1]^d, \quad \mathbb{E}(X|U) = \mathbb{E}(X).$$

While our previous paper [4] focused on the case with correct specification, the results we obtain in the present paper are general.

- **A precise link with classical quantile regression in the univariate case**: it was shown in [4] that in the particular case when $d = 1$ and under correct specification, classical quantile regression and vector quantile regression are equivalent. Going beyond correct specification here, we shall see that the optimal transport approach is equivalent (theorem 4.9) to a variant of (1.1) where one further imposes the monotonicity constraint that $t \mapsto U_t$ is nonincreasing (which is consistent with the fact that the true quantile $Q(x,t)$ is nondecreasing with respect to $t$).
The paper is organized as follows. In section 2 introduces vector quantiles through optimal transport. Section 3 is devoted to a precise, duality based, analysis of the vector quantile regression beyond correct specification. Finally, we shall revisit in section 4 the univariate case and then carefully relate the Koenker and Bassett approach to that of [4].

2 Vector quantiles and optimal transport

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be some nonatomic probability space, and let \((X,Y)\) be a random vector, where the vector of explanatory variables \(X\) is valued in \(\mathbb{R}^N\) and the vector of dependent variables \(Y\) is valued in \(\mathbb{R}^d\).

2.1 Vector quantiles by correlation maximization

The notion of vector quantile was recently introduced by Ekeland, Galichon and Henry [5], Galichon and Henry [8] and was used in the framework of quantile regression in our companion paper [4]. The starting point for this approach is the correlation maximization problem

\[
\max \{ \mathbb{E}(V \cdot Y), \ \text{Law}(V) = \mu \} \tag{2.1}
\]

where \(\mu := \text{uniform}(0,1)^d\) is the uniform measure on the unit \(d\)-dimensional cube \([0,1]^d\). This problem is equivalent to the optimal transport problem which consists in minimizing \(\mathbb{E}(|Y - V|^2)\) among uniformly distributed random vectors \(V\). As shown in the seminal paper of Brenier [2], this problem has a solution \(U\) which is characterized by the condition

\[
Y = \nabla \varphi(U)
\]

for some (essentially uniquely defined) convex function \(\varphi\) which is again obtained by solving a dual formulation of (2.1). Arguing that gradients of convex functions are the natural multivariate extension of monotone nondecreasing functions, the authors of [5] and [8] considered the function \(Q := \nabla \varphi\) as the vector quantile of \(Y\). This function \(Q = \nabla \varphi\) is by definition the Brenier’s map, i.e. the optimal transport map (for the quadratic cost) between the uniform measure on \([0,1]^d\) and \(\text{Law}(Y)\):

**Theorem 2.1. (Brenier’s theorem)** If \(Y\) is a squared-integrable random vector valued in \(\mathbb{R}^d\), there is a unique map of the form \(T = \nabla \varphi\) with \(\varphi\) convex on \([0,1]^d\) such that \(\nabla \varphi \# \mu = \text{Law}(Y)\), this map is by definition the vector quantile function of \(Y\).

We refer to the textbooks [15], [16] and [14] for a presentation of optimal transport theory, and to [7] for a survey of applications to economics.
2.2 Conditional vector quantiles

Take a \(N\)-dimensional random vector \(X\) of regressors, \(\nu := \text{Law}(X,Y)\), \(m := \text{Law}(X)\), \(\nu^x \otimes m\) where \(m\) is the law of \(X\) and \(\nu^x\) is the law of \(Y\) given \(X = x\). One can consider \(Q(x,u) = \nabla \varphi(x,u)\) as the optimal transport between \(\mu\) and \(\nu^x\), under some regularity assumptions on \(\nu^x\), one can invert \(Q(x,.)\): \(Q(x,.)^{-1} = \nabla_y \varphi(x,.)^*\) (where the Legendre transform is taken for fixed \(x\)) and one can define \(U\) through

\[ U = \nabla_y \varphi^*(X,Y),\ Y = Q(X,U) = \nabla_u \varphi(X,U) \]

\(Q(X,.)\) is then the conditional vector quantile of \(Y\) given \(X\). There is, as we will see in dimension one, a variational principle behind this definition:

- \(U\) is uniformly distributed, independent from \(X\) and solves:

\[
\max \{ \mathbb{E}(V \cdot Y), \ \text{Law}(V) = \mu, V \perp \perp X \} \tag{2.2}
\]

- the conditional quantile \(Q(x,.)\) and its inverse are given by \(Q(x,u) = \nabla_u \varphi(x,u), F(x,y) = \nabla_y \psi(x,y)\) (the link between \(F\) and \(Q\) being \(F(x,Q(x,u)) = u\)), the potentials \(\psi\) and \(\varphi\) are convex conjugates (\(x\) being fixed) and solve

\[
\min \int \varphi(x,u)m(dx)\mu(du) + \int \psi(x,y)\nu(dx,dy) : \psi(x,y) + \varphi(x,u) \geq y \cdot u.
\]

Note that if the conditional quantile function is affine in \(X\) and \(Y = Q(X,U) = \alpha(U) + \beta(U)X\) where \(U\) is uniform and independent from \(X\), the function \(u \mapsto \alpha(u) + \beta(u)x\) should be the gradient of some function of \(u\) which requires

\[
\alpha = \nabla \varphi, \ \beta = Db^T
\]

for some potential \(\varphi\) and some vector-valued function \(b\) in which case, \(Q(x,.)\) is the gradient of \(u \mapsto \varphi(u) + b(u) \cdot x\). Moreover, since quantiles are gradients of convex potentials one should also have

\[
u \in [0,1]^d \mapsto \varphi(u) + b(u) \cdot x\] is convex.
3 Vector quantile regression

In the next paragraphs, we will impose a parametric form of the dependence of the vector quantile $Q(x, u)$ with respect to $x$. More specifically, we shall assume that $Q(x, u)$ is affine in $x$. In the scalar case ($d = 1$), this problem is called quantile regression; we shall investigate that case in section 4 below.

3.1 Correlation maximization

Without loss of generality we normalize $X$ so that it is centered

$$\mathbb{E}(X) = 0.$$  

Our approach to vector quantile regression is based on a variation of the correlation maximization problem [22] where the independence constraint is replaced by a mean-independence constraint, that is

$$\max \{ \mathbb{E}(V \cdot Y), \ \text{Law}(V) = \mu, \ \mathbb{E}(X|V) = 0 \}.$$  

(3.1)

where $\mu = \text{uniform}([0, 1]^d)$ is the uniform measure on the unit $d$-dimensional cube.

An obvious connection with the specification of vector quantile regression (i.e. the validity of an affine in $x$ form for the conditional quantile) is given by:

**Proposition 3.1.** If $Y = \nabla \varphi(U) + Db(U)^T X$ with

- $u \mapsto \varphi(u) + b(u) \cdot x$ convex and smooth for m-a.e $x$,
- $\text{Law}(U) = \mu$, $\mathbb{E}(X|U) = 0$,

then $U$ solves (3.1).

**Proof.** This result follows from [4], but for the sake of completeness, we give a proof:

$$Y = \nabla \Phi_X(U), \ \text{with} \ \Phi_X(t) = \varphi(t) + b(X) \cdot t.$$  

Let $V$ be such that $\text{Law}(V) = \mu$, $\mathbb{E}(X|V) = 0$, then by Young’s inequality

$$V \cdot Y \leq \Phi_X(V) + \Phi_X^*(Y)$$

but $Y = \nabla \Phi_X(U)$ implies that

$$U \cdot Y = \Phi_X(U) + \Phi_X^*(Y)$$

so taking expectations gives the desired result.  

$\square$
3.2 Duality

From now on, we do not assume a particular form for the conditional quantile and wish to study which information (3.1) can give regarding the dependence of $X$ and $Y$. Once again, a good starting point is convex duality. As explained in details in [4], the dual of (3.1) takes the form

$$\inf_{(\psi, \varphi, b)} \mathbb{E}(\psi(X, Y) + \varphi(U)) : \psi(x, y) + \varphi(t) + b(t) \cdot x \geq t \cdot y. \quad (3.2)$$

where $U$ is any uniformly distributed random vector on $[0, 1]^d$ i.e. $\text{Law}(U) = \mu = \text{uniform}([0, 1]^d)$ and the infimum is taken over continuous functions $\psi \in C(\text{spt}(\nu), \mathbb{R})$, $\varphi \in C([0, 1]^d, \mathbb{R})$ and $b \in C([0, 1]^d, \mathbb{R}^N)$ satisfying the pointwise constraint

$$\psi(x, y) + \varphi(t) + b(t) \cdot x \geq t \cdot y, \quad \forall (x, y, t) \in \text{spt}(\nu) \times [0, 1]^d. \quad (3.3)$$

Since for fixed $(\varphi, b)$, the largest $\psi$ which satisfies the pointwise constraint in (3.2) is given by the convex function

$$\psi(x, y) := \max_{t \in [0, 1]^d} \{t \cdot y - \varphi(t) - b(t) \cdot x\}$$

one may equivalently rewrite (3.2) as the minimization over continuous functions $\varphi$ and $b$ of

$$\int \max_{t \in [0, 1]} \{ty - \varphi(t) - b(t) \cdot x\} \nu(dx, dy) + \int_{[0, 1]^d} \varphi(t) \mu(dt).$$

We claim now that the infimum over continuous functions $(\varphi, b)$ coincides with the one over smooth or simply integrable functions. Indeed, let $b \in L^1((0, 1)^d)^N$, $\varphi \in L^1((0, 1)^d)$ and $\psi$ such that (3.3) holds. Let $\varepsilon > 0$ and, extend $\varphi$ and $b$ to $Q_\varepsilon := [0, 1]^d + B_\varepsilon$ ($B_\varepsilon$ being the closed Euclidean ball of center 0 and radius $\varepsilon$):

$$\varphi_\varepsilon(t) := \begin{cases} \varphi(t), & \text{if } t \in (0, 1)^d \\ \frac{1}{\varepsilon}, & \text{if } t \in Q_\varepsilon \setminus (0, 1)^d \end{cases}, \quad b_\varepsilon(t) := \begin{cases} b(t), & \text{if } t \in (0, 1)^d \\ 0, & \text{if } t \in Q_\varepsilon \setminus (0, 1)^d \end{cases}$$

and for $(x, y) \in \text{spt}(\nu)$:

$$\psi_\varepsilon(x, y) := \max \left(\psi(x, y), \max_{t \in Q_\varepsilon \setminus (0, 1)^d} (t \cdot y - \frac{1}{\varepsilon})\right)$$

then by construction $(\psi_\varepsilon, \varphi_\varepsilon, b_\varepsilon)$ satisfies (3.3) on $\text{spt}(\nu) \times Q_\varepsilon$. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a centered, smooth probability density supported on $B_1$, and
define the mollifiers \( \rho_\delta := \delta^{-d} \rho(\frac{x}{\delta}) \), then for \( \delta \in (0, \varepsilon) \), defining the smooth functions \( b_{\varepsilon, \delta} := \rho_\delta \ast b_\varepsilon \) and \( \varphi_{\varepsilon, \delta} := \rho_\delta \ast \varphi_\varepsilon \), we have that \((\psi_\varepsilon, \varphi_{\varepsilon, \delta}, b_{\varepsilon, \delta})\) satisfies (3.3). By monotone convergence, \( \int \psi_\varepsilon \, d\nu \) converges to \( \int \psi \, d\nu \), moreover

\[
\lim_{\delta \to 0} \int_{(0,1)^d} \varphi_{\varepsilon, \delta} = \int_{(0,1)^d} \varphi_\varepsilon = \int_{(0,1)^d} \varphi.
\]

we deduce that the value of the minimization problem (3.2), can indifferently be obtained by minimizing over continuous, smooth or \( L^1 \) \( \varphi \) and \( b \)'s. The existence of optimal \( (L^1) \) functions \( \psi, \varphi \) and \( b \) is not totally obvious and is proven in the appendix under the following assumptions:

- the support of \( \nu \), is of the form \( \text{spt}(\nu) := \overline{\Omega} \) where \( \Omega \) is an open bounded convex subset of \( \mathbb{R}^N \times \mathbb{R}^d \),
- \( \nu \in L^\infty(\Omega) \),
- \( \nu \) is bounded away from zero on compact subsets of \( \Omega \) that is for every \( K \) compact, included in \( \Omega \) there exists \( \alpha_K > 0 \) such that \( \nu \geq \alpha_K \text{ a.e.} \) on \( K \).

**Theorem 3.2.** Under the assumptions above, the dual problem (3.2) admits at least a solution.

### 3.3 Vector quantile regression as optimality conditions

Let \( U \) solve (3.1) and \((\psi, \varphi, b)\) solve its dual (3.2). Recall that, without loss of generality, we can take \( \tilde{\psi} \) convex given

\[
\psi(x, y) = \sup_{t \in [0,1]^d} \{ t \cdot y - \varphi(t) - b(t) \cdot x \}. \tag{3.4}
\]

The constraint of the dual is

\[
\psi(x, y) + \varphi(t) + b(t) \cdot x \geq t \cdot y, \quad \forall (x, y, t) \in \Omega \times [0,1]^d, \tag{3.5}
\]

and the primal-dual relations give that, almost-surely

\[
\psi(X, Y) + \varphi(U) + b(U) \cdot X = U \cdot Y. \tag{3.6}
\]

Which, since \( \psi \), given by (3.4), is convex, yields

\[
(-b(U), U) \in \partial \psi(X, Y), \quad \text{or, equivalently} \quad (X, Y) \in \partial \psi^*(-b(U), U).
\]

Problems (3.1) and (3.2) have thus enabled us to find:
\begin{itemize}
  \item $U$ uniformly distributed with $X$ mean-independent from $U$,
  \item $\phi : [0, 1]^d \to \mathbb{R}$, $b : [0, 1]^d \to \mathbb{R}^N$ and $\psi : \Omega \to \mathbb{R}$ convex,
\end{itemize}

such that $(X,Y) \in \partial \psi^*(-b(U), U)$. Specification of vector quantile regression rather asks whether one can write $Y = \nabla \phi(U) + Db(U)^T X := \nabla \Phi_X(U)$ with $u \mapsto \Phi_x(u) := \phi(u) + b(u)x$ convex in $u$ for fixed $x$. The smoothness of $\phi$ and $b$ is actually related to this specification issue. Indeed, if $\phi$ and $b$ were smooth then (by the envelope theorem) we would have

$$
Y = \nabla \phi(U) + Db(U)^T X = \nabla \Phi_X(U).
$$

But even smoothness of $\phi$ and $b$ are not enough to guarantee that the conditional quantile is affine in $x$, which would also require $u \mapsto \Phi_x(u)$ to be convex. Note also that if $\psi$ was smooth, we would then have

$$
U = \nabla_Y \psi(X,Y), \quad -b(U) = \nabla_X \psi(X,Y)
$$

so that $b$ and $\psi$ should be related by the vectorial Hamilton-Jacobi equation

$$
\nabla_x \psi(x, y) + b(\nabla_y \psi(x, y)) = 0.
\tag{3.7}
$$

In general (without assuming any smoothness), define

$$
\psi_x(y) = \psi(x, y).
$$

We then have, thanks to (3.5)-(3.6)

$$
U \in \partial \psi_Y(Y) \text{ i.e. } Y \in \partial \psi_X^*(U).
$$

The constraint of (3.2) also gives

$$
\psi_x(y) + \Phi_x(t) \geq t \cdot y
$$

since Legendre Transform is order-reversing, this implies

$$
\psi_x \geq \Phi_x^*
\tag{3.8}
$$

hence

$$
\psi_x^* \leq (\Phi_x)^{**} \leq \Phi_x
$$

(where $\Phi_x^{**}$ denotes the convex envelope of $\Phi_x$). Duality between (3.1) and (3.2) thus gives:
Theorem 3.3. Let $U$ solve (3.1), $(\psi, \varphi, b)$ solve its dual (3.2) and set $\Phi_x(t) := \varphi(t) + b(t) \cdot x$ for every $(t, x) \in [0, 1]^d \times \text{spt}(m)$ then

$$\Phi_X(U) = \Phi^*_X(U) \text{ and } U \in \partial \Phi^*_X(Y) \text{ i.e. } Y \in \partial \Phi^*_X(U). \quad (3.9)$$

almost surely.

Proof. From the duality relation (3.6) and (3.8), we have

$$U \cdot Y = \psi_X(Y) + \Phi_X(U) \geq \Phi^*_X(Y) + \Phi_X(U)$$

so that $U \cdot Y = \Phi^*_X(Y) + \Phi_X(U)$ and then

$$\Phi^*_X(U) \geq U \cdot Y - \Phi^*_X(Y) = \Phi_X(U).$$

Hence, $\Phi_X(U) = \Phi^*_X(U)$ and $U \cdot Y = \Phi^*_X(Y) + \Phi^*_X(U)$ i.e. $U \in \partial \Phi^*_X(Y)$ almost surely, and the latter is equivalent to the requirement that $Y \in \partial \Phi^*_X(U)$. $\square$

The previous theorem thus gives the following interpretation of the correlation maximization with a mean independence constraint (3.1) and its dual (3.2). These two variational problems in duality lead to the pointwise relations (3.9) which can be seen as best approximations of a specification assumption:

$$Y = \nabla \Phi_X(U), \ (X, U) \mapsto \Phi_X(U) \text{ affine in } X, \text{ convex in } U$$

with $U$ uniformly distributed and $\mathbb{E}(X|U) = 0$. Indeed in (3.9), $\Phi_X$ is replaced by its convex envelope, the uniform random variable $U$ solving (3.1) is shown to lie a.s. in the contact set $\Phi_X = \Phi^*_X$ and the gradient of $\Phi_X$ (which may not be well-defined) is replaced by a subgradient of $\Phi^*_X$.

4 The univariate case

We now study in detail the case when the dependent variable $X$ is scalar, i.e. $d = 1$. As before, let $(\Omega, \mathcal{F}, \mathbb{P})$ be some nonatomic probability space and $Y$ be some univariate random variable defined on this space. Denoting by $F_Y$ the distribution function of $Y$:

$$F_Y(\alpha) := \mathbb{P}(Y \leq \alpha), \ \forall \alpha \in \mathbb{R}$$

the quantile function of $Y$, $Q_Y = F_Y^{-1}$ is the generalized inverse of $F_Y$ given by the formula:

$$Q_Y(t) := \inf\{\alpha \in \mathbb{R} : F_Y(\alpha) > t\} \text{ for all } t \in (0, 1). \quad (4.1)$$

Let us now recall two well-known facts about quantiles:
• $\alpha = Q_Y(t)$ is a solution of the convex minimization problem

$$\min_{\alpha} \{\mathbb{E}((Y - \alpha)_+) + \alpha(1 - t)\}$$

(4.2)

• there exists a uniformly distributed random variable $U$ such that $Y = Q_Y(U)$. Moreover, among uniformly distributed random variables, $U$ is maximally correlated to $Y$ in the sense that it solves

$$\max\{\mathbb{E}(VY), \text{Law}(V) = \mu\}$$

(4.3)

where $\mu := \text{uniform}([0, 1])$ is the uniform measure on $[0, 1]$.

Of course, when $\text{Law}(Y)$ has no atom, i.e. when $F_Y$ is continuous, $U$ is unique and given by $U = F_Y(Y)$. Problem (4.3) is the easiest example of optimal transport problem one can think of. The decomposition of a random variable $Y$ as the composed of a monotone nondecreasing function and a uniformly distributed random variable is called a polar factorization of $Y$, the existence of such decompositions goes back to Ryff [13] and the extension to the multivariate case (by optimal transport) is due to Brenier [2].

We therefore see that there are basically two different approaches to study or estimate quantiles:

• the local or "$t$ by $t$" approach which consists, for a fixed probability level $t$, in using directly formula (4.1) or the minimization problem (4.2) (or some approximation of it), this can be done very efficiently in practice but has the disadvantage of forgetting the fundamental global property of the quantile function: it should be monotone in $t$,

• the global approach (or polar factorization approach), where quantiles of $Y$ are defined as all nondecreasing functions $Q$ for which one can write $Y = Q(U)$ with $U$ uniformly distributed; in this approach, one rather tries to recover directly the whole monotone function $Q$ (or the uniform variable $U$ that is maximally correlated to $Y$), in this global approach, one should rather use the optimization problem (4.3).

Let us assume now that, in addition to the random variable $Y$, we are also given a random vector $X \in \mathbb{R}^N$ which we may think of as being a list of explanatory variables for $Y$. We are therefore interested in the dependence between $Y$ and $X$ and in particular the conditional quantiles of $Y$ given $X = x$. In the sequel we shall denote by $\nu$ the joint law of $(X, Y)$, $\nu := \text{Law}(X, Y)$ and assume that $\nu$ is compactly supported on $\mathbb{R}^{N+1}$ (i.e. $X$ and
Y are bounded). We shall also denote by $m$ the first marginal of $\nu$ i.e. $m := \Pi_X \# \nu = \text{Law}(X)$. We shall denote by $F(x, y)$ the conditional cdf:

$$F(x, y) := \mathbb{P}(Y \leq y | X = x)$$

and $Q(x, t)$ the conditional quantile

$$Q(x, t) := \inf\{\alpha \in \mathbb{R} : F(x, \alpha) > t\}.$$ 

For the sake of simplicity we shall also assume that:

- for $m$-a.e. $x$, $t \mapsto Q(x, t)$ is continuous and increasing (so that for $m$-a.e. $x$, identities $Q(x, F(x, y)) = y$ and $F(x, Q(x, t)) = t$ hold for every $y$ and every $t$),

- the law of $(X, Y)$ does not charge nonvertical hyperplanes i.e. for every $(\alpha, \beta) \in \mathbb{R}^{1+N}$, $\mathbb{P}(Y = \alpha + \beta \cdot X) = 0$.

Finally we denote by $\nu_x$ the conditional probability of $Y$ given $X = x$ so that $\nu = m \otimes \nu_x$.

### 4.1 A variational characterization of conditional quantiles

Let us define the random variable $U := F(X, Y)$, then by construction:

$$\mathbb{P}(U < t | X = x) = \mathbb{P}(F(x, Y) < t | X = x) = \mathbb{P}(Y < Q(x, t) | X = x) = F(x, Q(x, t)) = t.$$ 

From this elementary observation we deduce that

- $U$ is independent from $X$ (since its conditional cdf does not depend on $x$),

- $U$ is uniformly distributed,

- $Y = Q(X, U)$ where $Q(x, .)$ is increasing.

This easy remark leads to a sort of conditional polar factorization of $Y$ with an independence condition between $U$ and $X$. We would like to emphasize now that there is a variational principle behind this conditional decomposition. Recall that we have denoted by $\mu$ the uniform measure on $[0, 1]$. Let us consider the variant of the optimal transport problem (4.3)
where one further requires $U$ to be independent from the vector of regressors $X$:
\[
\max \{ \mathbb{E}(VY), \text{Law}(V) = \mu, V \perp \perp X \}.
\] (4.4)

which in terms of joint law $\theta = \text{Law}(X, Y, U)$ can be written as
\[
\max_{\theta \in I(\nu, \mu)} \int u \cdot y \theta(dx, dy, du) \tag{4.5}
\]

where $I(\mu, \nu)$ consists of probability measures $\theta$ on $\mathbb{R}^{N+1} \times [0,1]$ such that
the $(X,Y)$ marginal of $\theta$ is $\nu$ and the $(X,U)$ marginal of $\theta$ is $m \otimes \mu$. Problem
(4.5) is a linear programming problem and our assumptions easily imply that it possesses solutions, moreover its dual formulation (see [4] for details) reads
as the minimization of
\[
\inf J(\varphi, \psi) = \int \varphi(x,u)m(dx)\mu(du) + \int \psi(x,y)\nu(dx,dy) \tag{4.6}
\]
among pairs of potentials $\varphi, \psi$ that pointwise satisfy the constraint
\[
\varphi(x,u) + \psi(x,y) \geq uy. \tag{4.7}
\]

Rewriting $J(\varphi, \psi)$ as
\[
J(\varphi, \psi) = \int \left( \int \varphi(x,u)\mu(du) + \int \psi(x,y)\nu^\varphi(dy) \right)m(dx)
\]
and using the fact that the right hand side of the constraint (4.7) has no
dependence in $x$, we observe that (4.6) can actually be solved "by $x$". More
precisely, for fixed $x$ in the support of $m$, $\varphi(x,.)$ and $\psi(x,.)$ are obtained by solving
\[
\inf \int f(u)\mu(du) + \int g(y)\nu^\varphi(dy) : f(u) + g(y) \geq uy
\]
which appears naturally in optimal transport and is well-known to admit a
solution which is given by a pair of convex conjugate functions (see [15] [16]).
In other words, the infimum in (4.6) is attained by a pair $\varphi$ and $\psi$ such that
for $m$-a.e. $x$, $\varphi(x,.)$ and $\psi(x,.)$ are conjugate convex functions:
\[
\varphi(x,u) = \sup_y \{ uy - \psi(x,y) \}, \quad \psi(x,y) := \sup_u \{ uy - \varphi(x,u) \}.
\]
Since $\varphi(x,.)$ is convex it is differentiable a.e. and then $\partial_u \varphi(x,u)$ is defined
for a.e. $u$, moreover $\partial_u \varphi(x,.) \# \mu = \nu^\varphi$; hence $\partial_u \varphi(x,.)$ is a nondecreasing
map which pushes $\mu$ forward to $\nu^\varphi$: it thus coincides with the conditional quantile
\[
\partial_u \varphi(x,t) = Q(x,t) \text{ for } m\text{-a.e. } x \text{ and every } t. \tag{4.8}
\]

We then have the following variational characterization of conditional quantiles

\[
\partial_u \varphi(x,t) = Q(x,t) \text{ for } m\text{-a.e. } x \text{ and every } t. \tag{4.8}
\]
Theorem 4.1. Let \( \varphi \) and \( \psi \) solve (4.6). Then for m.a.e. \( x \), the conditional quantile \( Q(x,.) \) is given by:

\[
Q(x,.) = \partial_u \varphi(x,.)
\]

and the conditional cdf \( F(x,.) \) is given by:

\[
F(x,.) = \partial_y \psi(x,.)
\]

Let now \( \theta \) solve (4.5), there is a unique \( U \) such that Law\((X,Y,U) = \theta \) (so that \( U \) is uniformly distributed and independent from \( X \)) and it is given by \( Y = \partial_u \varphi(X,U) \) almost surely.

Proof. The fact that identity (4.8) holds for every \( t \) and m.a.e. \( x \) comes from the continuity of the conditional quantile. The second identity comes from the continuity of the conditional cdf. Now, duality tells us that the maximum in (4.5) coincides with the infimum in (4.6), so that if \( \theta \in I(\mu,\nu) \) is optimal for (4.6) and \((\tilde{X},\tilde{Y},\tilde{U})\) has law \( \theta^2 \), we have

\[
E(\tilde{U}\tilde{Y}) = E(\varphi(\tilde{X},\tilde{U}) + \psi(\tilde{X},\tilde{Y})).
\]

Hence, almost surely

\[
\tilde{U}\tilde{Y} = \varphi(\tilde{X},\tilde{U}) + \psi(\tilde{X},\tilde{Y}).
\]

which, since \( \varphi(x,.) \) and \( \psi(x,.) \) are conjugate and \( \varphi(x,.) \) is differentiable, gives

\[
\tilde{Y} = \partial_u \varphi(\tilde{X},\tilde{U}) = Q(\tilde{X},\tilde{U}). \tag{4.9}
\]

Since \( F(x,.) \) is the inverse of the conditional quantile, we can invert the previous relation as

\[
\tilde{U} = \partial_y \psi(\tilde{X},\tilde{Y}) = F(\tilde{X},\tilde{U}). \tag{4.10}
\]

We then define

\[
U := \partial_y \psi(X,Y) = F(X,Y),
\]

then obviously, by construction Law\((X,Y,U) = \theta \) and \( Y = \partial_u \varphi(X,U) = Q(X,U) \) almost surely. If Law\((X,Y,U) = \theta \), then as observed above, necessarily \( U = F(X,Y) \) which proves the uniqueness claim.

\footnote{The fact that there exists such a triple follows from the nonatomicity of the underlying space.}
To sum up, thanks to the two problems \((4.5)\) and \((4.6)\), we have been able to find a *conditional polar factorization* of \(Y\) as

\[ Y = Q(X, U), \quad Q \text{ nondecreasing in } U, \text{ } U \text{ uniform, } U \perp X. \]  

(4.11)

One obtains \(U\) thanks to the correlation maximization with an independence constraint problem \((4.4)\) and one obtains the primitive of \(Q(X,\cdot)\) by the dual problem \((4.6)\).

In this decomposition, it is very demanding to ask that \(U\) is independent from the regressors \(X\), in turn, the function \(Q(X,\cdot)\) is just monotone non-decreasing. In practice, the econometrician rather looks for a specific form of \(Q\) (linear in \(X\) for instance), which by duality will amount to relaxing the independence constraint. We shall develop this idea in details in the next paragraphs and relate it to classical quantile regression.

### 4.2 Quantile regression: from specification to quasi-specification

From now on, we normalize \(X\) to be centered i.e. assume (and this is without loss of generality) that

\[ \mathbb{E}(X) = 0. \]

We also assume that \(m := \text{Law}(X)\) is nondegenerate in the sense that its support contains some ball centered at \(\mathbb{E}(X) = 0\).

Since the seminal work of Koenker and Bassett [10], it has been widely accepted that a convenient way to estimate conditional quantiles is to stipulate an affine form with respect to \(x\) for the conditional quantile. Since a quantile function should be monotone in its second argument, this leads to the following definition

**Definition 4.2.** *Quantile regression is under correct specification if there exist \((\alpha, \beta) \in \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R}^N)\) such that for \(m\)-a.e. \(x\)

\[ t \mapsto \alpha(t) + \beta(t) \cdot x \text{ is increasing on } [0,1] \]  

(4.12)

and

\[ Q(x, t) = \alpha(t) + x \cdot \beta(t), \]  

(4.13)

for \(m\)-a.e. \(x\) and every \(t \in [0,1]\). If \((4.12)\) and \((4.13)\) hold, quantile regression is under correct specification with regression coefficients \((\alpha, \beta)\).

Specification of quantile regression can be characterized by
Proposition 4.3. Let $(\alpha, \beta)$ be continuous and satisfy (4.12). Quantile regression is under correct specification with regression coefficients $(\alpha, \beta)$ if and only if there exists $U$ such that

$$Y = \alpha(U) + X \cdot \beta(U) \text{ a.s., Law}(U) = \mu, \ U \perp X.$$  

(4.14)

Proof. The fact that specification of quantile regression implies decomposition (4.14) has already been explained in paragraph 4.1. Let us assume (4.14), and compute

$$F(x, \alpha(t) + \beta(t) \cdot x) = \mathbb{P}(Y \leq \alpha(t) + \beta(t) x | X = x)$$

$$= \mathbb{P}(\alpha(U) + x \cdot \beta(U) \leq \alpha(t) + \beta(t) x | X = x)$$

$$= \mathbb{P}(U \leq t | X = x) = \mathbb{P}(U \leq t) = t$$

so that $Q(x, t) = \alpha(t) + \beta(t) \cdot x$. \hfill \qed

Koenker and Bassett showed that, for a fixed probability level $t$, the regression coefficients $(\alpha, \beta)$ can be estimated by quantile regression i.e. the minimization problem

$$\inf_{(\alpha, \beta) \in \mathbb{R}_{1+}^N} \mathbb{E}(\rho_t(Y - \alpha - \beta \cdot X))$$  

(4.15)

where the penalty $\rho_t$ is given by $\rho_t(z) := tz^- + (1 - t)z^+$ with $z^-$ and $z^+$ denoting the negative and positive parts of $z$. For further use, note that (4.15) can be conveniently be rewritten as

$$\inf_{(\alpha, \beta) \in \mathbb{R}_{1+}^N} \{\mathbb{E}((Y - \alpha - \beta \cdot X)_+) + (1 - t)\alpha\}. \tag{4.16}$$

As already noticed by Koenker and Bassett, this convex program admits as dual formulation

$$\sup \{\mathbb{E}(U_t Y) : U_t \in [0, 1], \ \mathbb{E}(U_t) = (1 - t), \ \mathbb{E}(U_t X) = 0\}. \tag{4.17}$$

An optimal $(\alpha, \beta)$ for (4.16) and an optimal $U_t$ in (4.17) are related by the complementary slackness condition:

$$Y > \alpha + \beta \cdot X \Rightarrow U_t = 1, \text{ and } Y < \alpha + \beta \cdot X \Rightarrow U_t = 0. \tag{4.18}$$

Note that $\alpha$ appears naturally as a Lagrange multiplier associated to the constraint $\mathbb{E}(U_t) = (1 - t)$ and $\beta$ as a Lagrange multiplier associated to $\mathbb{E}(U_t X) = 0$. Since $\nu = \text{Law}(X, Y)$ gives zero mass to nonvertical hyperplanes, we may simply write

$$U_t = 1_{\{Y > \alpha + \beta \cdot X\}} \tag{4.19}$$
and thus the constraints $\mathbb{E}(U_t) = (1 - t)$, $\mathbb{E}(XU_t) = 0$ read

$$\mathbb{E}(1_{\{Y > \alpha + \beta \cdot X\}}) = \mathbb{P}(Y > \alpha + \beta \cdot X) = (1 - t), \quad \mathbb{E}(X1_{\{Y > \alpha + \beta \cdot X\}}) = 0$$ (4.20)

which simply are the first-order conditions for (4.16).

Any pair $(\alpha, \beta)$ which solves the optimality conditions (4.20) for the Koenker and Bassett approach will be denoted

$$\alpha = \alpha^{QR}(t), \beta = \beta^{QR}(t)$$

and the variable $U_t$ solving (4.17) given by (4.19) will similarly be denoted $U_t^{QR}$

$$U_t^{QR} := 1_{\{Y > \alpha^{QR}(t) + \beta^{QR}(t) \cdot X\}}.$$ (4.21)

Note that in the previous considerations the probability level $t$ is fixed, this is what we called the "$t$ by $t$" approach. For this approach to be consistent with conditional quantile estimation, if we allow $t$ to vary we should add an additional monotonicity requirement:

**Definition 4.4.** Quantile regression is under quasi-specification if there exists for each $t$, a solution $(\alpha^{QR}(t), \beta^{QR}(t))$ of (4.20) (equivalently the minimization problem (4.15)) such that $t \in [0, 1] \mapsto (\alpha^{QR}(t), \beta^{QR}(t))$ is continuous and, for $m$-a.e. $x$

$$t \mapsto \alpha^{QR}(t) + \beta^{QR}(t) \cdot x$$ is increasing on $[0, 1]$. (4.22)

A first consequence of quasi-specification is given by

**Proposition 4.5.** If quantile regression is under quasi-specification and if we define $U^{QR} := \int_0^1 U_t^{QR} \, dt$ (recall that $U_t^{QR}$ is given by (4.21)) then:

- $U^{QR}$ is uniformly distributed,
- $X$ is mean-independent from $U^{QR}$ i.e. $\mathbb{E}(X|U^{QR}) = \mathbb{E}(X) = 0$,
- $Y = \alpha^{QR}(U^{QR}) + \beta^{QR}(U^{QR}) \cdot X$ a.s.

Moreover $U^{QR}$ solves the correlation maximization problem with a mean-independence constraint:

$$\max \{ \mathbb{E}(VY), \ \text{Law}(V) = \mu, \ \mathbb{E}(X|V) = 0 \}. \quad (4.23)$$

\[^3\text{Uniqueness will be discussed later on.}\]
Proof. Obviously

\[ U_t^{QR} = 1 \Rightarrow U^{QR} \geq t, \text{ and } U^{QR} > t \Rightarrow U_t^{QR} = 1 \]

hence \( \mathbb{P}(U^{QR} \geq t) \geq \mathbb{P}(U_t^{QR} = 1) = \mathbb{P}(Y > \alpha^{QR}(t) + \beta^{QR}(t) \cdot X) = (1-t) \)

and \( \mathbb{P}(U^{QR} > t) \leq \mathbb{P}(U_t^{QR} = 1) = (1-t) \) which proves that \( U^{QR} \) is uniformly distributed and \( \{U^{QR} > t\} \) coincides with \( \{U_t^{QR} = 1\} \) up to a set of null probability. We thus have \( \mathbb{E}(X\mathbf{1}_{U^{QR} > t}) = \mathbb{E}(XU_t^{QR}) = 0 \), by a standard approximation argument we deduce that \( \mathbb{E}(Xf(U^{QR})) = 0 \) for every \( f \in C([0,1],\mathbb{R}) \) which means that \( X \) is mean-independent from \( U^{QR} \).

As already observed \( U^{QR} > t \) implies that \( Y > \alpha^{QR}(t) + \beta^{QR}(t) \cdot X \) in particular \( Y \geq \alpha^{QR}(U^{QR} - \delta) + \beta^{QR}(U^{QR} - \delta) \cdot X \) for \( \delta > 0 \), letting \( \delta \to 0^+ \) and using the continuity of \( (\alpha^{QR}, \beta^{QR}) \) we get \( Y \geq \alpha^{QR}(U^{QR}) + \beta^{QR}(U^{QR}) \cdot X \). The converse inequality is obtained similarly by remaking that \( U^{QR} < t \) implies that \( Y \leq \alpha^{QR}(t) + \beta^{QR}(t) \cdot X \).

Let us now prove that \( U^{QR} \) solves (4.23). Take \( V \) uniformly distributed, such that \( X \) is mean-independent from \( V \) and set \( V_t := \mathbf{1}_{\{V > t\}} \), we then have \( \mathbb{E}(XV_t) = 0 \), \( \mathbb{E}(V_t) = (1-t) \) but since \( U_t^{QR} \) solves (4.17) we have \( \mathbb{E}(V_t Y) \leq \mathbb{E}(U_t^{QR} Y) \). Observing that \( V = \int_0^1 V_t dt \) and integrating the previous inequality with respect to \( t \) gives \( \mathbb{E}(V Y) \leq \mathbb{E}(U^{QR} Y) \) so that \( U^{QR} \) solves (4.23).

\[ \square \]

Let us continue with a uniqueness argument for the mean-independent decomposition given in proposition 4.5:

**Proposition 4.6.** Let us assume that

\[ Y = \alpha(U) + \beta(U) \cdot X = \overline{\alpha(U)} + \overline{\beta(U)} \cdot X \]

with:

- both \( U \) and \( \overline{U} \) uniformly distributed,
- \( X \) is mean-independent from \( U \) and \( \overline{U} \): \( \mathbb{E}(X|U) = \mathbb{E}(X|\overline{U}) = 0 \),
- \( \alpha, \beta, \overline{\alpha}, \overline{\beta} \) are continuous on \([0,1] \),
- \((\alpha, \beta)\) and \((\overline{\alpha}, \overline{\beta})\) satisfy the monotonicity condition (4.12),

then

\[ \alpha = \overline{\alpha}, \ \beta = \overline{\beta}, \ \ U = \overline{U}. \]
Proof. Let us define for every $t \in [0, 1]$

$$
\varphi(t) := \int_0^t \alpha(s)ds, \quad b(t) := \int_0^t \beta(s)ds.
$$

Let us also define for $(x,y)$ in $\mathbb{R}^{N+1}$:

$$
\psi(x,y) := \max_{t \in [0,1]} \{ ty - \varphi(t) - b(t) \cdot x \}
$$

thanks to monotonicity condition (4.12), the maximization program above is strictly concave in $t$ for every $y$ and $m$-a.e. $x$. We then remark that $Y = \alpha(U) + \beta(U) \cdot X = \varphi'(U) + b'(U) \cdot X$ exactly is the first-order condition for the above maximization problem when $(x,y) = (X,Y)$. In other words, we have

$$
\psi(x,y) + b(t) \cdot x + \varphi(t) \geq ty, \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}^N \times \mathbb{R} \quad (4.24)
$$

with and equality for $(x,y,t) = (X,Y,U)$ i.e.

$$
\psi(X,Y) + b(U) \cdot X + \varphi(U) = UY, \quad \text{a.s.} \quad (4.25)
$$

Using the fact that $\text{Law}(U) = \text{Law}(\overline{U})$ and the fact that mean-independence gives $\mathbb{E}(b(U) \cdot X) = \mathbb{E}(b(\overline{U}) \cdot X) = 0$, we have

$$
\mathbb{E}(UY) = \mathbb{E}(\psi(X,Y)+b(U) \cdot X + \varphi(U)) = \mathbb{E}(\psi(X,Y)+b(\overline{U}) \cdot X + \varphi(\overline{U})) \geq \mathbb{E}(\overline{U}Y)
$$

but reversing the role of $U$ and $\overline{U}$, we also have $\mathbb{E}(UY) \leq \mathbb{E}(\overline{U}Y)$ and then

$$
\mathbb{E}(\overline{U}Y) = \mathbb{E}(\psi(X,Y) + b(\overline{U}) \cdot X + \varphi(\overline{U}))
$$

so that, thanks to inequality (4.24)

$$
\psi(X,Y) + b(U) \cdot X + \varphi(U) = \overline{U}Y, \quad \text{a.s.}
$$

which means that $\overline{U}$ solves $\max_{t \in [0,1]} \{ tY - \varphi(t) - b(t) \cdot X \}$ which, by strict concavity admits $U$ as unique solution. This proves that $U = \overline{U}$ and thus

$$
\alpha(U) - \overline{U}(U) = (\overline{\beta}(U) - \beta(U)) \cdot X
$$

taking the conditional expectation of both sides with respect to $U$, we then obtain $\alpha = \overline{\alpha}$ and thus $\beta(U) \cdot X = \overline{\beta}(U) \cdot X$ a.s.. We then compute

$$
F(x, \alpha(t) + \beta(t) \cdot x) = \mathbb{P}(\alpha(U) + \beta(U) \cdot X \leq \alpha(t) + \beta(t) \cdot x | X = x) \\
= \mathbb{P}(\alpha(U) + \beta(U) \cdot x \leq \alpha(t) + \beta(t) \cdot x | X = x) \\
= \mathbb{P}(U \leq t | X = x)
$$
and similarly $F(x, \alpha(t) + \beta(t) \cdot x) = \mathbb{P}(U \leq t | X = x) = F(x, \alpha(t) + \beta(t) \cdot x).$ Since $F(x, .)$ is increasing for $m$-a.e. $x$, we deduce that $\beta(t) \cdot x = \overline{\beta}(t) \cdot x$ for $m$-a.e. $x$ and every $t \in [0,1]$. Finally, the previous considerations and the nondegeneracy of $m$ enable us to conclude that $\beta = \overline{\beta}$.

**Corollary 4.7.** If quantile regression is under quasi-specification, the regression coefficients $(\alpha^{QR}, \beta^{QR})$ are uniquely defined and if $Y$ can be written as

$$Y = \alpha(U) + \beta(U) \cdot X$$

for $U$ uniformly distributed, $X$ being mean independent from $U$, $(\alpha, \beta)$ continuous such that the monotonicity condition (4.12) holds then necessarily

$$\alpha = \alpha^{QR}, \beta = \beta^{QR}.$$

To sum up, we have shown that quasi-specification is equivalent to the validity of the factor linear model:

$$Y = \alpha(U) + \beta(U) \cdot X$$

for $(\alpha, \beta)$ continuous and satisfying the monotonicity condition (4.12) and $U$, uniformly distributed and such that $X$ is mean-independent from $U$. This has to be compared with the decomposition of paragraph 4.1 where $U$ is required to be independent from $X$ but the dependence of $Y$ with respect to $U$, given $X$, is given by any nondecreasing function of $U$.

### 4.3 Global approaches and duality

Now we wish to address quantile regression in the case where neither specification nor quasi-specification can be taken for granted. In such a general situation, keeping in mind the remarks from the previous paragraphs, we can think of two natural approaches.

The first one consists in studying directly the correlation maximization with a mean-independence constraint (4.23). The second one consists in getting back to the Koenker and Bassett $t$ by $t$ problem (4.17) but adding as an additional global consistency constraint that $U_t$ should be nonincreasing with respect to $t$:

$$\sup \{ \mathbb{E} \left( \int_0^1 U_t Y dt \right) : U_t \text{ nonincr.}, U_t \in [0,1], \mathbb{E}(U_t) = (1 - t), \mathbb{E}(U_t X) = 0 \}$$

(4.26)
Our aim is to compare these two approaches (and in particular to show that the maximization problems (4.23) and (4.26) have the same value) as well as their dual formulations. Before going further, let us remark that (4.23) can directly be considered in the multivariate case whereas the monotonicity constrained problem (4.26) makes sense only in the univariate case.

As proven in [4], (4.23) is dual to
\[
\inf_{(\psi,\varphi,b)} \{ \mathbb{E}(\psi(X,Y)) + \mathbb{E}(\varphi(U)) : \psi(x,y) + \varphi(u) \geq uy - b(u) \cdot x \} \tag{4.27}
\]
which can be reformulated as:
\[
\inf_{(\varphi,b)} \int t \in [0,1] \max(ty - \varphi(t) - b(t) \cdot x) \nu(dx,dy) + \int_0^1 \varphi(t)dt \tag{4.28}
\]
in the sense that
\[
\sup(4.23) = \inf(4.27) = \inf(4.28). \tag{4.29}
\]

The existence of a solution to (4.27) is not straightforward and is established under appropriate assumptions in the appendix directly in the multivariate case. The following result shows that there is a \(t\)-dependent reformulation of (4.23):

**Lemma 4.8.** The value of (4.23) coincides with
\[
\sup \{ \mathbb{E} \left( \int_0^t U_t Y dt \right) : U_t \text{ nonincr.}, U_t \in \{0,1\}, \ 1 - t, \ 1 \} \tag{4.30}
\]

**Proof.** Let \(U\) be admissible for (4.23) and define \(U_t := 1_{\{U>t\}}\) then \(U = \int_0^1 U_t dt\) and obviously \((U_t)\) is admissible for (4.30), we thus have \(\sup(4.23) \leq \sup(4.30)\). Take now \((V_t)\) admissible for (4.30) and let \(V := \int_0^1 V_t dt\), we then have
\[
V > t \Rightarrow V_t = 1 \Rightarrow V \geq t
\]
since \(\mathbb{E}(V_t) = (1 - t)\) this implies that \(V\) is uniformly distributed and \(V_t = 1_{\{V>t\}}\) a.s. so that \(\mathbb{E}(X1_{\{V>t\}}) = 0\) which implies that \(X\) is mean-independent from \(V\) and thus \(\mathbb{E}(\int_0^1 V_t Y dt) \leq \sup(4.23)\). We conclude that \(\sup(4.23) = \sup(4.30)\).

Let us now define
\[
\mathcal{C} := \{ u : [0,1] \mapsto [0,1], \ \text{nonincreasing} \}
\]

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Let \((U_t)_t\) be admissible for (4.26) and set
\[
v_t(x,y) := \mathbb{E}(U_t|X=x, Y=y), \quad V_t := v_t(X,Y)
\]
it is obvious that \((V_t)_t\) is admissible for (4.26) and by construction \(\mathbb{E}(V_tY) = \mathbb{E}(U_tY)\). Moreover the deterministic function \((t,x,y) \mapsto v_t(x,y)\) satisfies the following conditions:

for fixed \((x,y)\), \(t \mapsto v_t(x,y)\) belongs to \(C\), \(4.31\)

and for a.e. \(t \in [0,1]\),
\[
\int v_t(x,y)\nu(dx,dy) = (1-t), \quad \int v_t(x,y)x\nu(dx,dy) = 0. \quad (4.32)
\]
Conversely, if \((t,x,y) \mapsto v_t(x,y)\) satisfies (4.31)-(4.32), \(V_t := v_t(X,Y)\) is admissible for (4.26) and \(\mathbb{E}(V_tY) = \int v_t(x,y)y\nu(dx,dy)\). All this proves that
\[
\sup(4.26) \text{ coincides with } \sup_{(t,x,y) \mapsto v_t(x,y)} \int v_t(x,y)y\nu(dx,dy)dt \text{ subject to: } (4.31)-(4.32) \quad (4.33)
\]

**Theorem 4.9.**
\[
\sup(4.23) = \sup(4.26).
\]

**Proof.** We know from lemma 4.8 and the remarks above that
\[
\sup(4.23) = \sup(4.30) \leq \sup(4.26) = \sup(4.33).
\]
But now we may get rid of constraints (4.32) by rewriting (4.33) in sup-inf form as
\[
\sup_{v_t \text{ satisfies } 4.31} \inf_{(\alpha,\beta)} \int v_t(x,y)(y - \alpha(t) - \beta(t)x)\nu(dx,dy)dt + \int_0^1 (1-t)\alpha(t)dt.
\]
Recall that one always have \(\sup\inf \leq \inf\sup\) so that \(\sup(4.33)\) is less than
\[
\inf_{(\alpha,\beta)} \sup_{v_t \text{ satisf. } 4.31} \int v_t(x,y)(y - \alpha(t) - \beta(t)x)\nu(dx,dy)dt + \int_0^1 (1-t)\alpha(t)dt \leq \inf_{(\alpha,\beta)} \int \left( \sup_{v_t \in \mathcal{C}} \int_0^1 v(t)(y - \alpha(t) - \beta(t)x)dt \right) \nu(dx,dy) + \int_0^1 (1-t)\alpha(t)dt.
\]
It follows from Lemma 4.10 below that, for \(q \in L^1(0,1)\) defining \(Q(t) := \int_0^t q(s)ds\), one has
\[
\sup_{v_t \in \mathcal{C}} \int_0^1 v(t)q(t)dt = \max_{t \in [0,1]} Q(t).
\]

22
So setting $\varphi(t) := \int_0^t \alpha(s)\,ds$, $b(t) := \int_0^t \beta(s)\,ds$ and remarking that integrating by parts immediately gives

$$
\int_0^1 (1-t)\alpha(t)\,dt = \int_0^1 \varphi(t)\,dt
$$

we thus have

$$
\sup_{v \in C} \int_0^1 v(t)(y - \alpha(t) - \beta(t)x)\,dt + \int_0^1 (1-t)\alpha(t)\,dt = \max_{t \in [0,1]} \{ty - \varphi(t) - b(t)x\} + \int_0^1 \varphi(t)\,dt.
$$

This yields\footnote{The functions $\varphi$ and $b$ constructed above vanish at 0 and are absolutely continuous but this is by no means a restriction in the minimization problem \textbf{(4.28)} as explained in paragraph \textbf{3.2}.}

$$
\sup_{(\varphi,b)} (4.33) \leq \inf_{(\varphi,b)} \int_{t \in [0,1]} \max (ty - \varphi(t) - b(t) \cdot x)\nu(dx,dy) + \int_0^1 \varphi(t)\,dt = \inf (4.28)
$$

but we know from \textbf{(4.29)} that $\inf (4.28) = \sup (4.28)$ which ends the proof.

In the previous proof, we have used the elementary result (proven in the appendix).

**Lemma 4.10.** Let $q \in L^1(0,1)$ and define $Q(t) := \int_0^t q(s)\,ds$ for every $t \in [0,1]$, one has

$$
\sup_{v \in C} \int_0^1 v(t)q(t)\,dt = \max_{t \in [0,1]} Q(t).
$$

**Appendix**

**Proof of Lemma 4.10**

Since $1_{[0,t]} \in C$, one obviously first has

$$
\sup_{v \in C} \int_0^t v(s)q(s)\,ds \geq \max_{t \in [0,1]} \int_0^t q(s)\,ds = \max_{t \in [0,1]} Q(t).
$$

Let us now prove the converse inequality, taking an arbitrary $v \in C$. We first observe that $Q$ is absolutely continuous and that $v$ is of bounded variation (its
derivative in the sense of distributions being a bounded nonpositive measure which we denote by \( \eta \), integrating by parts and using the definition of \( C \) then give:

\[
\int_0^1 v(s)q(s)ds = -\int_0^1 Q\eta + v(1^-)Q(1)
\leq (\max_{[0,1]} Q) \times (-\eta([0, 1]) + v(1^-)Q(1))
= (\max_{[0,1]} Q)(v(0^+) - v(1^-)) + v(1^-)Q(1)
= (\max_{[0,1]} Q)v(0^+) + (Q(1) - \max_{[0,1]} Q)v(1^-)
\leq \max_{[0,1]} Q.
\]

**Proof of theorem 3.2**

Let us denote by \((0, \overline{y})\) the barycenter of \( \nu \):

\[
\int_\Omega x \nu(dx, dy) = 0, \quad \int_\Omega y \nu(dx, dy) =: \overline{y}
\]

and observe that \((0, \overline{y}) \in \Omega \) (otherwise, by convexity, \( \nu \) would be supported on \( \partial \Omega \) which would contradict our assumption that \( \nu \in L^\infty(\Omega) \)).

We wish to prove the existence of optimal potentials for the problem

\[
\inf_{\psi,\varphi,b} \int_\Omega \psi(x,y) d\nu(x,y) + \int_{[0,1]^d} \varphi(u) d\mu(u)
\]

subject to the pointwise constraint that

\[
\psi(x,y) + \varphi(u) \geq u \cdot y - b(u) \cdot x, \quad (x,y) \in \overline{\Omega}, \quad u \in [0,1]^d.
\]

Of course, we can take \( \psi \) that satisfies

\[
\psi(x,y) := \sup_{u \in [0,1]^d} \{u \cdot y - b(u) \cdot x - \varphi(u)\}
\]

so that \( \psi \) can be chosen convex and 1 Lipschitz with respect to \( y \). In particular, we have

\[
\psi(x,\overline{y}) - |y - \overline{y}| \leq \psi(x,y) \leq \psi(x,\overline{y}) + |y - \overline{y}|.
\]

The problem being invariant by the transform \((\psi, \varphi) \rightarrow (\psi + C, \psi - C) \) (\( C \) being an arbitrary constant), we can add as a normalization the condition that

\[
\psi(0, \overline{y}) = 0.
\]
This normalization and the constraint (4.35) imply that
\[ \varphi(t) \geq t \cdot \overline{y} - \psi(0, \overline{y}) \geq -|\overline{y}|. \] (4.38)

We note that there is one extra invariance of the problem: if one adds an affine term \( q \cdot x \) to \( \psi \) this does not change the cost and neither does it affect the constraint, provided one modifies \( b \) accordingly by substracting to it the constant vector \( q \). Take then \( q \) in the subdifferential of \( x \mapsto \psi(x, \overline{y}) \) at \( 0 \) and change \( \psi \) into \( \psi - q \cdot x \), we obtain a new potential with the same properties as above and with the additional property that \( \psi(.,\overline{y}) \) is minimal at \( x = 0 \), and thus \( \psi(x,\overline{y}) \geq 0 \), together with (4.36) this gives the lower bound
\[ \psi(x,y) \geq -|y - \overline{y}| \geq -C \] (4.39)
where the bound comes from the boundedness of \( \Omega \) (from now one, \( C \) will denote a generic constant maybe changing from one line to another).

Now take a minimizing sequence \( (\psi_n, \varphi_n, b_n) \in C(\overline{\Omega}, \mathbb{R}) \times C([0, 1]^d, \mathbb{R}) \times C([0, 1]^d, \mathbb{R}^N) \) where for each \( n \), \( \psi_n \) has been chosen with the same properties as above, since \( \varphi_n \) and \( \psi_n \) are bounded from below (\( \varphi_n \geq -|\overline{y}| \) and \( \psi_n \geq C \)) and since the sequence is minimizing, we deduce immediately that \( \psi_n \) and \( \varphi_n \) are bounded sequences in \( L^1_\text{loc} \).

Let us now prove that \( b_n \) is bounded in \( L^1 \), for this take \( r > 0 \) such that \( B_{2r}(0, \overline{y}) \) is included in \( \Omega \). For every \( x \in B_r(0) \), any \( t \in [0, 1]^d \) and any \( n \) we then have
\[ -b_n(t) \cdot x \leq \varphi_n(t) - t \cdot \overline{y} + \|\psi_n\|_{L^\infty(B_r(0,\overline{y}))} \leq C + \varphi_n(t) \]
maximizing in $x \in B_r(0)$ immediately gives

$$|b_n(t)|r \leq C + \varphi_n(t).$$

From which we deduce that $b_n$ is bounded in $L^1$ since $\varphi_n$ is.

From Komlos’ theorem (see [11]), we may find a subsequence such that the Cesaro means

$$\frac{1}{n} \sum_{k=1}^{n} \varphi_k, \quad \frac{1}{n} \sum_{k=1}^{n} b_k$$

converge a.e. respectively to some $\varphi$ and $b$. Clearly $\psi$, $\varphi$ and $b$ satisfy the linear constraint (4.35), and since the sequence of Cesaro means $(\psi_n', \varphi_n', b_n') := n^{-1} \sum_{k=1}^{n} (\psi_k, \varphi_k, b_k)$ is also minimizing, we deduce from Fatous’ Lemma

$$\int_{\Omega} \psi(x,y) d\nu(x,y) + \int_{[0,1]^{d'}} \varphi(u) d\mu(u) \leq \liminf_n \int_{\Omega} \psi_n'(x,y) d\nu(x,y) + \int_{[0,1]^{d'}} \varphi_n'(u) d\mu(u) = \inf(4.34)$$

which ends the existence proof.

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