EXTREMAL FUNCTIONS FOR MOSER-TRUDINGER TYPE INEQUALITY 
ON COMPACT CLOSED 4-MANIFOLDS

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Abstract. Given a compact closed four dimensional smooth Riemannian manifold, we prove existence of extremal functions for Moser-Trudinger type inequality. The method used is Blow-up analysis combined with capacity techniques.

Keywords: Moser-Trudinger inequality, Blow-up analysis, Capacity, Extremal function, Green function.

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CONTENTS
1. Introduction 1
2. Notations and Preliminaries 4
3. Proof of Theorem 1.1 6
3.1. Concentration behavior and profile of $u_k$ 6
3.2. Pohozaev type identity and application 10
3.3. Blow-up analysis 13
3.4. Capacity estimates 17
3.5. The test function 25
3.6. Proof of Theorem 1.1 27
4. Proof of Theorem 1.2 27
References 28

1. INTRODUCTION

It is well-known that Moser-Trudinger type inequalities are crucial analytic tools in the study of partial differential equations arising from geometry and physics.

In fact, much work has been done on such inequalities and their applications in the last decades, see for example, [1], [3], [4], [6], [8], [18], [22], and the references therein.

There are two important objects in the study of Moser-Trudinger type inequalities: one is to find the best constant and the other is to determine whether there exist extremal functions.

For the best constant there are the celebrated work of Moser[19] and the extension to higher order derivatives by Adams [1] on flat spaces. In the context of curved spaces Fontana has extended the results of Adams, see [9].

To mention results about extremal functions, we cite the results of Carleson and Chang [5], Flucher [10] and Lin [16] in the Euclidean case and the results of Li [14], [15] in the curved one. In [14] and [15] the author have proved the existence of an extremal function for the classic Moser-Trudinger inequality on a compact manifold under a constraint involving only the first derivatives.

In this paper, we will extend the results of Li to a compact closed four dimensional smooth Riemannian manifold under a constraint involving the Laplacian. More precisely we prove the following Theorems.
Theorem 1.1. Let \((M, g)\) be a compact closed smooth 4-dimensional Riemannian manifold. Then setting
\[
\mathcal{H}_1 = \{ u \in H^2(M) : \bar{\pi} = 0, \int_M |\Delta_g u|^2 dV_g = 1 \}
\]
we have that
\[
\sup_{u \in \mathcal{H}_1} \int_M e^{32\pi^2 u^2} dV_g
\]
is attained.

On the 4-dimensional manifold \((M, g)\), the so-called Paneitz operator, which is defined in terms of the scalar curvature \(R_g\) and the Ricci tensor \(\text{Ric}_g\) as
\[
P^4_g u = \Delta^2_g u + \text{div}_g \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) u \quad u \in C^\infty(M),
\]
plays an important role in conformal geometry see [4], [6], [7], [8], [11], [20], [21]. In particular, the relation between the Paneitz operator and the \(Q\)-curvature, which is defined as
\[
Q_g = -\frac{1}{12} (\Delta_g R_g - R^2_g + 3|\text{Ric}_g|^2), \tag{1.1}
\]
is of great interest. It is well-known that Moser-Trudinger inequalities involving \(P^4_g\) play an important role in the problem of prescribing constant \(Q\)-curvature see [8], [12], [20]. Therefore it is worth having an extension of Theorem 1.1 concerning the Paneitz operator as well. Our next result goes in this direction. More precisely we have the following.

Theorem 1.2. Let \((M, g)\) be a compact closed smooth 4-dimensional Riemannian manifold. Assuming that \(P^4_g\) is non-negative and \(\ker P^4_g \simeq \mathbb{R}\), then setting
\[
\mathcal{H}_2 = \{ u \in H^2(M) : \bar{\pi} = 0, < P^4_g u, u > = 1 \}
\]
we have
\[
\sup_{u \in \mathcal{H}_2} \int_M e^{32\pi^2 u^2} dV_g
\]
is attained.

Remark 1.3. Since the leading term of \(P^4\) (for the definition see the Section 2) is \(\Delta^2\) then the two Theorems are quite similar. We point out that the same proof is valid for both except some trivial adaptations, hence we will give a full proof of Theorem 1.1 only and sketch the proof of Theorem 1.2 in the last section.

Remark 1.4. We mention that due to a result by Gursky, see [11] if both the Yamabe class \(Y(g)\) and \(\int_M Q_g dV_g\) are non-negative, then we have that \(P^4_g\) is non-negative and \(\ker P^4_g \simeq \mathbb{R}\).

We are going to describe our approach to prove Theorem 1.1. We will use Blow-up analysis. First of all we take a sequence \((\alpha_k)_k\) such that \(\alpha_k \nearrow 32\pi^2\), and by using Direct Methods of the Calculus of variations we can find \(u_k \in \mathcal{H}_1\) such that
\[
\int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in \mathcal{H}_1} \int_M e^{\alpha u^2} dV_g.
\]
see Lemma 3.1. Moreover using the Lagrange multiplier rule we have that \((u_k)_k\) satisfies the equation:
\[
\Delta^2_g u_k = \frac{u_k}{\lambda_k} e^{\alpha_k u^2_k} - \gamma_k \tag{1.2}
\]
for some constants \(\lambda_k\) and \(\gamma_k\).

Now it is easy to see that if there exists \(\alpha > 32\pi^2\) such that \(\int_M e^{\alpha u^2} dV_g\) is bounded, then by using Lagrange formula, Young’s inequality and Rellich compactness Theorem, we obtain that the weak limit of \(u_k\) becomes an extremizer. On the other hand if
\[
c_k = \max_M |u_k| = |u_k|(x_k)
\]
is bounded, then from standard elliptic regularity theory $u_k$ is compact, thus converges uniformly to an extremizer. Hence assuming that Theorem 1.1 does not hold, we get

1) $$\forall \alpha > 32\pi^2 \lim_{k \to +\infty} \int_M e^{\alpha u_k^2} dV_g \to +\infty$$

2) $$c_k \to +\infty$$

We will follow the same method as in [14] up to some extents.

In [14], the function sequence we studied is the following:

$$-\Delta_g u_k = \frac{u_k}{\lambda_k} e^{\alpha'_k u_k^2} - \gamma_k,$$

where $\alpha'_k \nearrow 4\pi$, and $u_k$ attains $\sup_{M} \int_M |\nabla_g u_k|^2 dV_g = 1$, $\bar{u} = \int_M e^{\alpha'_k u_k^2} dV_g$. We also assumed $c_k \to +\infty$. Then we have

$$2\alpha_k c_k (u_k(x_k + r_k x) - c_k) \to -2 \log(1 + \pi |x|^2)$$

for suitable choices of $r_k$, $x_k$. Next we proved the following

$$\lim_{k \to +\infty} \int_{\{u_k \leq A\}} |\nabla_g u_k|^2 dV_g = \frac{1}{A} \forall A > 1,$$

which implies that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g = \mu(M) + \lim_{k \to +\infty} \frac{\lambda_k}{c_k^2},$$

and that $c_k u_k$ converges to some Green function weakly. In the end, we got an upper bound of $\frac{\Delta \mu}{c^2}$ via capacity.

Remark 1.5. (1.3) was first discovered by Struwe in [23].

Remark 1.6. (1.4) also appeared in [2].

However there are two main differences between the present case and the one in [14]. One is that there is no direct maximum principle for equation (1.2) and the other one is that truncations are not allowed in the space $H^2(M)$. Hence to get a counterpart of (1.3) and (1.4) is not easy.

To solve the first difficulty, we replace $c_k (u_k(x_k + r_k x) - c_k)$ with $\beta_k (u_k(exp_x(r_k x)) - c_k)$, where

$$1/\beta_k = \int_M \frac{|u_k|}{\lambda_k} e^{\alpha_k u_k^2} dV_g.$$

By using the strength of the Green representation formula, we get that the profile of $u_k$ is either a constant function or a standard bubble. The second difficulty will be solved by applying capacity and Pohozaev type identity. In more detail we will prove that $\beta_k u_k \to G$ (see Lemma 3.6) which satisfies

$$\left\{ \begin{array}{l} \Delta^2 G = \tau (\delta_{x_0} - Vol_g(M)) \\ \int_M G = 0. \end{array} \right.$$ 

for some $\tau \in (0, 1]$. Then we can derive from a Pohozaev type identity (see Lemma 3.7) that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g Vol_g(M) + \lim_{k \to +\infty} \tau^2 \frac{\lambda_k}{\beta_k^2}.$$

In order to apply the capacity, we will follow some ideas in [12]. Concretely, we will show that up to a small term the energy of $u_k$ on some annulus is bounded below by the Euclidean one (see Lemma 3.10). Moreover one can prove the existence of $U_k$ (see Lemma 3.11) such that the energy of $U_k$ is comparable to the Euclidean energy of $u_k$, and the Dirichlet datum and Neumann datum of $U_k$ at the boundary of the annulus are also comparable to those of $u_k$. In this sense,
we simplify the calculation of capacity in [15]. Now using capacity techniques we get \( \frac{\delta}{\rho_k} \to d \) and \( d \tau = 1 \), see Proposition 3.12. Furthermore we have that
\[
\lim_{k \to +\infty} r^2 \frac{\lambda_k}{\beta_k} \leq \frac{\pi^2}{6} e^{\frac{3}{2} + 32\pi^2 S_0}.
\]
Hence we arrive to
\[
\sup_{u \in \mathcal{H}_1} \int_W e^{32\pi^2 u^2} dV_g \leq Vol_g(M) + \frac{\pi^2}{6} e^{\frac{3}{2} + 32\pi^2 S_0}.
\] (1.5)
In the end, we will find test functions in order to contradict (1.5). We will simplify the arguments in [14]. Indeed we use carefully the regular part of \( G \) to avoid cut-off functions and hence making the calculations simpler.

The plan of the paper is the following: In Section 2 we collect some preliminary results regarding the existence of the Green functions for \( \Delta_p \) and \( P_g^4 \), and associated Moser-Trudinger type inequality. In Section 2 we prove Theorem 1.1. This Section is divided into six subsections. In the first one, we deal with concentration behavior and the profile of the blowing-up sequence. The second one is concerned about the derivation of a Pohozaev type identity and its application. In subsection 3 we perform the Blow-up analysis to get either the zero function or a standard bubble in the limit. In the subsection 4, we deal with the capacity estimates to get an upper bound. And in the subsection 5, we construct test functions. In the last subsection we show how to reach a contradiction. The last Section is concerned about the sketch of the proof of Theorem 1.2.

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2. Notations and Preliminaries

In this brief section we collect some useful notations, and state a lemma giving the existence of the well-known Moser-Trudinger inequality for the operator \( P_g^4 \) when it is non-negative.

The plan of the paper is the following: In Section 2 we collect some preliminary results regarding the existence of the Green functions for \( \Delta_p \) and \( P_g^4 \), and associated Moser-Trudinger type inequality. In Section 2 we prove Theorem 1.1. This Section is divided into six subsections. In the first one, we deal with concentration behavior and the profile of the blowing-up sequence. The second one is concerned about the derivation of a Pohozaev type identity and its application. In subsection 3 we perform the Blow-up analysis to get either the zero function or a standard bubble in the limit. In the subsection 4, we deal with the capacity estimates to get an upper bound. And in the subsection 5, we construct test functions. In the last subsection we show how to reach a contradiction. The last Section is concerned about the sketch of the proof of Theorem 1.2.

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2. Notations and Preliminaries

In this brief section we collect some useful notations, and state a lemma giving the existence of the Green functions of \( \Delta_p \) and \( P_g^4 \), with the asymptotics near the singularity. We also give a version of Adams inequality on the a manifold due to Fontana and an analogue of the well-known Moser-Trudinger inequality for the operator \( P_g^4 \) when it is non-negative.

In the following, \( B_r(x) \) stands for the metric ball of radius \( r \) and center \( x \) in \( M \), \( B'(p) \) and stands for the Euclidean ball of center \( p \) and radius \( r \). We also denote with \( d_g(x,y) \) the metric distance between two points \( x \) and \( y \) of \( M \). \( H^2(M) \) stands for the usual Sobolev space of functions on \( M \), i.e functions which are in \( L^2 \) together with their first and second derivatives. \( W^{2,q}(M) \) denotes the usual Sobolev spaces of functions which are in \( L^q(M) \) with their first and second derivatives. Large positive constants are always denoted by \( C \), and the value of \( C \) is allowed to vary from formula to formula and also within the same line. \( M^2 \) stands for the cartesian product \( M \times M \), while \( Diag(M) \) is the diagonal of \( M^2 \). Given a function \( u \in L^1(M) \) \( \bar{u} \) denotes its average on \( M \), that is \( \bar{u} = (Vol_g(M))^{-1} \int_M u(x) dV_g(x) \) where \( Vol_g(M) = \int_M dV_g \).

\( A_k = o_k(1) \) means that \( A_k \to 0 \) as the integer \( k \to +\infty \).
\( A_\delta = o_\delta(1) \) means that \( A_\delta \to 0 \) as the real number \( \delta \to 0 \).
\( A_{k,\delta} = o_{k,\delta}(1) \) means that \( A_{k,\delta} \to 0 \) as \( k \to +\infty \) first and after the real number \( \delta \to 0 \).
\( A_k = O(B_k) \) means that \( A_k \leq CB_k \) for some fixed constant \( C \).
\( inj_g(M) \) stands for the injectivity radius of \( M \).
\( dV_g \) denotes the Riemannian measure associated to the metric \( g \).
\( dS_g \) stands for the surface measure associated to \( g \).

Given a metric \( g \) on \( M \), and \( x \in M \), \( g(x) \), stands for determinant of the matrix with entries \( g_{ij}(x) \) where \( g_{ij}(x) \) are the components of \( g(x) \) in some system of coordinates.
\( \Delta_0 \) stands for the Euclidean Laplacian and \( \Delta_g \) the Laplace-Beltrami with respect to the background metric \( g \).

As mentioned before we begin by stating a lemma giving the existence of the Green function of \( \Delta^2 \) and \( P_g^4 \), and their asymptotics near the singularities.
Lemma 2.1. We have that the Green function \( F(x, y) \) of \( \Delta_g^2 \) exists in the following sense:

a) For all functions \( u \in C^2(M) \), we have

\[
    u(x) - \bar{u} = \int_M F(x, y) \Delta_g^2 u(y) dV_g(y) \quad x \neq y \in M
\]

b) \( F(x, y) = H(x, y) + K(x, y) \)

is smooth on \( M^2 \setminus \text{Diag}(M^2) \), \( K \) extends to a \( C^{1+\alpha} \) function on \( M^2 \) and

\[
    H(x, y) = \frac{1}{8\pi^2} f(r) \log \frac{1}{r}
\]

where, \( r = d_g(x, y) \) is the geodesic distance from \( x \) to \( y \); \( f(r) \) is a \( C^\infty \) positive decreasing function, \( f(r) = 1 \) in a neighborhood of \( r = 0 \) and \( f(r) = 0 \) for \( r \geq \text{inj}_g(M) \). Moreover we have that the following estimates holds

\[
    |\nabla_g F(x, y)| \leq C \frac{1}{d_g(x, y)} \quad |\nabla_g^2 F(x, y)| \leq C \frac{1}{d_g(x, y)^2}.
\]

Proof. For the proof see [6] and the proof of Lemma 2.3 in [17].

Lemma 2.2. Suppose \( \ker P_g^4 \simeq \mathbb{R} \). Then the Green function \( Q(x, y) \) of \( P_g^4 \) exists in the following sense:

a) For all functions \( u \in C^2(M) \), we have

\[
    u(x) - \bar{u} = \int_M Q(x, y) P_g^4 u(y) dV_g(y) \quad x \neq y \in M
\]

b) \( Q(x, y) = H_0(x, y) + K_0(x, y) \)

is smooth on \( M^2 \setminus \text{Diag}(M^2) \), \( K \) extends to a \( C^{2+\alpha} \) function on \( M^2 \) and

\[
    H(x, y) = \frac{1}{8\pi^2} f_0(r) \log \frac{1}{r}
\]

where, \( r = d_g(x, y) \) is the geodesic distance from \( x \) to \( y \); \( f_0(r) \) is a \( C^\infty \) positive decreasing function, \( f_0(r) = 1 \) in a neighborhood of \( r = 0 \) and \( f_0(r) = 0 \) for \( r \geq \text{inj}_g(M) \).

Proof. For the proof see Lemma 2.1 in [20].

Next we state a Theorem due to Fontana[9].

Theorem 2.3. ([9]) There exists a constant \( C = C(M) > 0 \) such that the following holds

\[
    \int_M e^{32\pi^2 u^2} dV_g \leq C \quad \text{for all } u \in H^2(M) \quad \text{such that} \quad \int_M |\Delta_g^2 u| dV_g = 1.
\]

Moreover this constant is optimal in the sense that if we replace it by any \( \alpha \) bigger then the integral can be maken as large as we want.

Next we state a Moser-Trudinger type inequality corresponding to \( P_g^4 \) when it is non-negative. The proof can be found in [20] where it is proven for every \( P_g^4 \) (where \( P_g^4 \) stands for higher order Paneitz operator).

Proposition 2.4. Suppose that \( P_g^4 \) is non-negative and that \( \ker P_g^4 = \mathbb{R} \), then there exists a constant \( C = C(M) > 0 \) such that

\[
    \int_M e^{32\pi^2 u^2} dV_g \leq C \quad \text{for all } u \in H^2(M) \quad \text{such that} \quad \langle P_g^4 u, u \rangle = 1.
\]
3. Proof of Theorem 1.1

Lemma 3.1. Let $\alpha_k$ be an increasing sequence converging to $32\pi^2$. Then for every $k$ there exists $u_k \in H_1$ such that
\[ \int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in H_1} \int_M e^{\alpha_k u^2} dV_g. \]
Moreover $u_k$ satisfies the following equation
\[ \Delta_g^2 u_k = \frac{1}{\lambda_k} u_k e^{\alpha_k u_k^2} - \gamma_k \] (3.1)
where
\[ \lambda_k = \int_M u_k^2 e^{\alpha_k u_k^2} dV_g \]
and
\[ \gamma_k = \frac{1}{\lambda_k \text{Vol}_g(M)} \int_M u_k e^{\alpha_k u_k^2} dV_g. \]
Moreover we have $u_k \in C^\infty(M)$. 

Proof. First of all using the inequality in Theorem 2.3, one can check easily that the functional
\[ I_k(u) = \int_M e^{\alpha_k u^2} dV_g; \]
is weakly continuous. Hence using Direct Methods of the Calculus of Variations we get the existence of maximizer say $u_k$. On the other hand using the Lagrange multiplier rule one get the equation (3.1). Moreover integrating the equation (3.1) and after multiplying it by $u_k$ and integrating again, we get the value of $\gamma_k$ and $\lambda_k$ respectively. Moreover using standard elliptic regularity we get that $u_k \in C^\infty(M)$. Hence the Lemma is proved.

Now we are ready to give the proof of Theorem 1.1. From now on we suppose by contradiction that Theorem 1.1 does not hold. Hence from the same considerations as in the Introduction we have that:

1) \[ \forall \alpha > 32\pi^2 \lim_{k \to +\infty} \int_M e^{\alpha u_k^2} dV_g \to +\infty \] (3.2)
2) \[ c_k = \max_M |u_k| = |u_k|(x_k) \to +\infty \]

We will divide the reminder of the proof into six subsections.

3.1. Concentration behavior and profile of $u_k$. This subsection is concerned about two main ingredients. The first one is the study of the concentration phenomenon of the energy corresponding to $u_k$. The second one is the description of the profile of $\beta_k u_k$ as $k \to +\infty$, where $\beta_k$ is given by the relation
\[ 1/\beta_k = \int_M \frac{|u_k|}{\lambda_k} e^{\alpha_k u_k^2} dV_g. \]
We start by giving an energy concentration lemma which is inspired from P.L.Lions’ work.

Lemma 3.2. $u_k$ verifies:
\[ u_k \to 0 \text{ in } H^2(M) \]
and
\[ |\Delta_g u_k| \to \delta_{x_0} \]
for some $x_0 \in M$. 

Proof. First of all from the fact that \( u_k \in H_1 \) we can assume without loss of generality that
\[
 u_k \rightharpoonup u_0 \quad \text{in} \quad H^2(M).
\] (3.3)

Now let us show that \( u_0 = 0 \).

We have the trivial identity
\[
 \int_M |\Delta_g (u_k - u_0)|^2 dv_g = \int_M |\Delta_g u_k|^2 dv_g + \int_M |\Delta_g u_0|^2 dv_g - 2 \int_M \Delta_g u_k \Delta_g u_0 dv_g.
\]

Hence using the fact that \( \int_M |\Delta_g u_k|^2 dv_g = 1 \) we derive
\[
 \int_M |\Delta_g (u_k - u_0)|^2 dv_g = 1 + \int_M |\Delta_g u_0|^2 dv_g - 2 \int_M \Delta_g u_k \Delta_g u_0 dv_g.
\]

So using (3.3) we get
\[
 \lim_{k \to 0} \int_M |\Delta_g (u_k - u_0)|^2 dv_g 1 - \int_M \Delta_g u_0 \Delta_g u_0 dv_g \leq 1 - \int_M \Delta_g u_0 \Delta_g u_0 dv_g.
\]

Now suppose that \( u_0 \neq 0 \) and let us argue for a contradiction. Then there exists some \( \beta \leq 1 \) such that for \( k \) large enough the following holds
\[
 \int_M |\Delta_g (u_k - u_0)|^2 dv_g < \beta.
\]

Hence using Fontana’s result see Theorem 2.3 we obtain that
\[
 \int_M e^{\alpha_1 (u_k - u_0)^2} dv_g \leq C \quad \text{for some} \quad \alpha_1 > 32\pi^2.
\]

Now using Cauchy inequality one can check easily that
\[
 \int_M e^{\alpha_2 u_k^2} dv_g \leq C \quad \text{for some} \quad \alpha_2 > 32\pi^2.
\]

Hence reaching a contradiction to (3.2).

On the other hand without lost of generality we can assume that
\[
 |\Delta_g u_k| dv_g \rightharpoonup \mu.
\]

Now suppose \( \mu \neq \delta_p \) for every \( p \in M \) and let us argue for a contradiction to (3.2) again. First of all let us take a cut-off function \( \eta \in C_0^\infty (B_\delta (x)) \), \( \eta = 1 \) on \( B_x (\frac{\delta}{2}) \) where \( x \) is a fixed point in \( M \) and \( \delta \) a fixed positive and small number.

We have that
\[
 \limsup_{k \to +\infty} \int_{B_\delta (x)} |\Delta_0 \eta \tilde{u}_k|^2 dv_g < 1.
\]

Now working in a normal coordinate system around \( x \) and using standard elliptic regularity theory we get
\[
 \int_{B^4 (\tilde{x})} |\Delta_0 \eta \tilde{u}_k|^2 dv_g \leq (1 + o_1 (1)) \int_{B_\delta (x)} |\Delta_g u_k|^2 dv_g;
\]

where \( \tilde{x} \) is the point corresponding to \( x \) in \( \mathbb{R}^4 \) and \( \eta \tilde{u}_k \) the expression of \( \eta u_k \) on the normal coordinate system. Hence for \( \delta \) small we get
\[
 \int_{B^4 (\tilde{x})} |\Delta_0 \eta \tilde{u}_k|^2 dv_g < 1
\]

Thus using the Adams result see [1] we have that
\[
 \int_{B^4 (\tilde{x})} e^{\tilde{\alpha} (\eta \tilde{u}_k)^2} dx \leq C \quad \text{for some} \quad \tilde{\alpha} > 32\pi^2.
\]

Hence using a covering argument we infer that
\[
 \int_M e^{\tilde{\alpha} u_k^2} dv_g \leq C \quad \text{for some} \quad \tilde{\alpha} > 32\pi^2,
\]

so reaching a contradiction. Hence the Lemma is proved. \( \square \)
Lemma 3.3. We have the following hold:
\[
\lim_{k \to +\infty} \lambda_k = +\infty, \quad \lim_{k \to +\infty} \gamma_k = 0.
\]

Proof. Let \( N > 0 \) be large enough. By using the definition of \( \lambda_k \) we have that
\[
\lambda_k = \int_M u_k^2 e^{\alpha_k u_k^2} dV_g \geq N^2 \int_{\{u_k \geq N\}} e^{\alpha_k u_k^2} dV_g = N^2 (\int_M e^{\alpha_k u_k^2} dV_g - \int_{\{u_k \leq N\}} e^{\alpha_k u_k^2} dV_g).
\]
On the other hand
\[
\lim_{k \to +\infty} \left( \int_M e^{\alpha_k u_k^2} dV_g - \int_{\{u_k \leq N\}} e^{\alpha_k u_k^2} dV_g \right) = \lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g - Vol_g(M).
\]
Hence using the fact that
\[
\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in \mathcal{H}_1} \int_M e^{2\pi^2 u^2} dV_g > Vol_g(M)
\]
we have that 1) holds. Now we prove 2). using the definition of \( \gamma_k \), we get
\[
|\gamma_k| \leq \frac{N}{\lambda_k} Ne^{2\pi^2 N^2} + \frac{1}{Vol_g(M) N}.
\]
Hence by using point 1 and letting \( k \to +\infty \) and after \( N \to +\infty \) we get point 2. So the Lemma is proved.

Next let us set
\[
\tau_k = \int_M \frac{\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2}.
\]
One can check easily the following

Lemma 3.4. With the definition above we have that \( 0 \leq \beta_k \leq c_k \), \( |\tau_k| \leq 1 \) and \( \beta_k \gamma_k \) is bounded. Moreover up to a subsequence and up to changing \( u_k \) to \( -u_k \)
\[
\tau_k \to \tau \geq 0.
\]

The next Lemma gives some Lebesgue estimates on Ball in terms of the radius with constant independent of the Ball. As a corollary we get the profile of \( \beta_k u_k \) as \( k \to +\infty \).

Lemma 3.5. There are constants \( C_1(p) \) and \( C_2(p) \) depending only on \( p \) and \( M \) such that, for \( r \) sufficiently small and for any \( x \in M \) there holds
\[
\int_{B_r(x)} |\nabla^2 g \beta_k u_k|^p dV_g \leq C_2(p) r^{4-2p};
\]
and
\[
\int_{B_r(x)} |\nabla g \beta_k u_k|^p dV_g \leq C_1(p) r^{4-p}
\]
where, respectively, \( p < 2 \), and \( p < 4 \).

Proof. First of all using the Green representation formula we have
\[
u_k(x) = \int_M F(x, y) \Delta^2 g u_k dV_g(y) \quad \forall x \in M.
\]
Hence using the equation we get
\[
u_k(x) = \int_M F(x, y) \left( \frac{1}{\lambda_k} u_k e^{\alpha_k u_k^2} \right) dV_g(y) = \int_M F(x, y) \gamma_k dV_g(y).
\]
Now by differentiating with respect to \( x \) for every \( m = 1, 2 \) we have that
\[
|\nabla^m u_k(x)| \leq \int_M |\nabla^m F(x, y)| \left( \frac{1}{\lambda_k} \right) |u_k| e^{\alpha_k u_k^2} dV_g(y) + \int_M |\nabla^m F(x, y)| |\gamma_k|.
\]
Hence we get
\[
|\nabla^m (\beta_k u_k(x))] \leq \int_M |\nabla^m F(x, y)| \beta_k \left( \frac{1}{\lambda_k} \right) |u_k| e^{\alpha_k u_k^2} dV_g(y) + \int_M |\nabla^m F(x, y)| \beta_k |\gamma_k|.
\]
Taking the \( p \)-th power in both side of the inequality and using the basic inequality
\[
(a + b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{for} \quad a \geq 0 \quad \text{and} \quad b \geq 0
\]
we obtain
\[
|\nabla^m g(\beta_k u_k(x))|^p \leq 2^{p-1} \left[ \int_M |\nabla^m_g F(x, y)|\beta_m \left( \frac{1}{\lambda_k} \right) |u_k|e^{\alpha_k u_k^2}dV_g(y) \right]^p + 2^{p-1} \left[ \int_M |\nabla^m_g F(x, y)|\beta_k |\gamma_k| \right]^p
\]
Now integrating both sides of the inequality we obtain
\[
\int_{B_r(x)} |\nabla^m g(\beta_k u_k(z))|dV_g(z) \leq 2^{p-1} \left[ \int_M |\nabla^m_g F(z, y)|\beta_k \left( \frac{1}{\lambda_k} \right) |u_k|e^{\alpha_k u_k^2}dV_g(y) \right]^p dV_g(z)
\]
\[
+ 2^{p-1} \int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|\beta_k |\gamma_k| \right]^p dV_g(z).
\]

First let us estimate the second term in the right hand side of the inequality
\[
\int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|\beta_k |\gamma_k| \right]^p dV_g(z) \leq C \int_{B_r(x)} \sup_{y \in M} \frac{1}{d_g(z, y)^p} dV_g(z) \leq C(M)r^{4-mp}
\]
Thanks to the fact that \( \beta_k \gamma_k \) is bounded, to the asymptotics of the Green function and to Jensen’s inequality. Now let us estimates the second term. First of all we define the following auxiliary measure
\[
m_k = \beta_k \left( \frac{1}{\lambda_k} \right) |u_k|e^{\alpha_k u_k^2}dV_g
\]
We have that \( m_k \) is a probability measure. On the other hand we can write
\[
\int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|\beta_k \left( \frac{1}{\lambda_k} \right) |u_k|e^{\alpha_k u_k^2}dV_g(y) \right]^p dV_g(z) = \int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|dm_k(y) \right]^p dV_g(z).
\]

Now by using Jensen’s inequality we have that
\[
\left[ \int_M |\nabla^m_g F(z, y)|dm_k(y) \right]^p \leq \left[ \int_M |\nabla^m_g F(z, y)|^p dm_k(y) \right]
\]
Thus with the (3.4) we have that
\[
\int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|\beta_k \left( \frac{1}{\lambda_k} \right) |u_k|e^{\alpha_k u_k^2}dV_g(y) \right]^p dV_g(z) \leq \int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|^p dm_k(y) \right] dV_g(z).
\]
Now by using again the same argument as in the first term we obtain
\[
\int_{B_r(x)} \left[ \int_M |\nabla^m_g F(z, y)|^p dm_k(y) \right] dV_g(z) \leq C(M)r^{4-mp}.
\]
Hence the Lemma is proved. \( \square \)

Next we give a corollary of this Lemma.

**Corollary 3.6.** We have \( \beta_k u_k \rightarrow G W^{2,p}(M) \) for \( p \in (1, 2) \), \( \beta_k u_k \rightarrow G \) smoothly in \( M \setminus B_\delta(x_0) \) where \( \delta \) is small and \( G \) satisfies
\[
\begin{align*}
\Delta^2_g G &= \tau(\delta x_0 - \frac{1}{\tau \omega_g(M)}) \quad \text{in} \ M; \\
\frac{\Delta^2 G}{G} &= 0
\end{align*}
\]
Moreover
\[
G(x) = \frac{\tau}{8\pi^2} \log \frac{1}{r} + \tau S(x)
\]
with $r = d_g(x, x_0)$. $S = S_0 + S_1(x)$, $S_0 = S(x_0)$ and $S \in W^{2,q}(M)$ for every $q \geq 1$.

Proof. By Lemma 3.5 we have that

$$\beta_k u_k \to G \quad W^{2,p}(M) \quad p \in (1, 2)$$

On the other hand using Lemma 3.2 we get $e^{\alpha_k u_k^2}$ is bounded in $L^p(M \setminus B_\delta(x_0))$. Hence the standard elliptic regularity implies that

$$\beta_k u_k \to G \quad \text{smoothly in } M \setminus B_\delta(x_0).$$

(3.5)

So to end the proof of the proposition we need only to show that

$$\frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} \to \tau \delta_{x_0}.$$  

(3.6)

To do this let us take $\varphi \in C^\infty(M)$ then we have

$$\int_M \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = \int_{M \setminus B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g + \int_{B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g$$

Using (3.5) we have that

$$\int_{M \setminus B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = O\left(\frac{1}{\lambda_k}\right).$$

On the other hand, we can write inside the ball $B_\delta(x_0)$

$$\int_{B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = (\varphi(x_0) + o_\delta(1)) \left( \int_{B_\delta(x_0)} \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g \right) = (\varphi(x_0) + o_\delta(1)) \left( \tau - \int_{M \setminus B_\delta(x_0)} \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g \right)$$

Now using again (3.5) we derive

$$\int_{M \setminus B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = O\left(\frac{1}{\lambda_k}\right).$$

Hence we arrive to

$$\int_{B_\delta(x_0)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = \tau \varphi(x_0) + o_{k, \delta}(1).$$

Thus we get

$$\int_M \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = O\left(\frac{1}{\lambda_k}\right) + \tau \varphi(x_0) + o_{k, \delta}(1).$$

Hence from Lemma 3.3 we conclude the proof of claim (3.6) and of the Corollary too. □

3.2. Pohozaev type identity and application. As it is already said in the introduction this subsection deals with the derivation of a Pohozaev type identity. And as corollary we give the limit of $\int_M e^{\alpha_k u_k^2} dV_g$ in terms of $Vol_g(M)$, $\lambda_k$, $\beta_k$ and $\tau$.

Lemma 3.7. Setting $U_k = \Delta_g u_k$ we have the following holds

$$-\frac{2}{\alpha_k \lambda_k} \int_{B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g = -\delta \int_{\partial B_\delta(x_k)} U_k^2 dS_g - \delta \int_{\partial B_\delta(x_k)} \nabla_g u_k \nabla_g U_k dV_g + 2 \int_{\partial B_\delta(x_k)} U_k \frac{\partial u_k}{\partial r} dS_g + 2 \delta \int_{\partial B_\delta(x_k)} \frac{\partial U_k}{\partial r} \frac{\partial u_k}{\partial r} dS_g + \int_{B_\delta(x_k)} O(r^2) \nabla_g u_k \nabla_g U_k dV_g + \int_{B_\delta(x_k)} O(r^2) U_k^2 dV_g + \int_{B_\delta(x_k)} e^{\alpha_k u_k^2} O(r^2) dV_g - \frac{\delta}{2 \lambda_k \alpha_k} \int_{\partial B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g + O\left(\frac{\delta}{\beta_k^2}\right).$$

where $\delta$ is small and fixed real number.
Proof. The proof relies on the divergence formula and the asymptotics of the metric \( g \) in normal coordinates around \( x_k \).

By the definition of \( U_k \) we have that
\[
\begin{align*}
\Delta_g u_k &= U_k \\
\Delta_g U_k &= \frac{\partial}{\partial r} e^{\alpha_k u_k} - \gamma_k.
\end{align*}
\]

The first issue is to compute \( \int_{B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \Delta_g u_k dV_g \) in two different ways, where \( r(x) = d_g(x, x_k) \).

On one side we obtain
\[
\int_{B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \Delta_g u_k dV_g = - \int_{B_\lambda(x_k)} (\nabla_g U_k \nabla_g u_k + \frac{\partial}{\partial r} \nabla_g U_k \nabla_g u_k) dV_g + \int_{\partial B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \frac{\partial u_k}{\partial r} dS_g.
\]

On the other side we get
\[
\int_{B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \Delta_g u_k dV_g = \int_{B_\lambda(x_k)} \frac{\partial U_k}{\partial r} U_k dV_g
\]
\[
= \int_0^{2\pi} \int_{\partial B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \sqrt{|g|} r^4 dS d\theta
\]
\[
= \frac{\delta}{\beta_k} \int_{\partial B_\lambda(x_k)} U_k^2 dS_g - 2 \int_{B_\lambda(x_k)} U_k^2 dV_g = - \int_{B_\lambda(x_k)} (\nabla_g U_k \nabla_g u_k + \frac{\partial}{\partial r} \nabla_g U_k \nabla_g u_k) dV_g + \int_{\partial B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \frac{\partial u_k}{\partial r} dS_g + \int_{B_\lambda(x_k)} O(r^2) U_k^2 dV_g.
\]

Thus we have
\[
\frac{\delta}{\beta_k} \int_{\partial B_\lambda(x_k)} e^{\alpha_k u_k^2} dS_g = \frac{2}{\lambda_k \alpha_k} \int_{B_\lambda(x_k)} e^{\alpha_k u_k^2} (1 + O(r^2)) dV_g
\]
\[
= - \int_{B_\lambda(x_k)} (\nabla_g U_k \nabla_g u_k + \frac{\partial}{\partial r} \nabla_g U_k \nabla_g u_k) dV_g + \int_{\partial B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \frac{\partial u_k}{\partial r} dS_g + O(\frac{\delta}{\beta_k^2}).
\]

Hence by summing this two last lines we arrive to
\[
\frac{\delta}{\beta_k} \int_{\partial B_\lambda(x_k)} e^{\alpha_k u_k^2} dS_g = \frac{2}{\lambda_k \alpha_k} \int_{B_\lambda(x_k)} e^{\alpha_k u_k^2} dV_g + \frac{\delta}{2} \int_{\partial B_\lambda(x_k)} U_k^2 dS_g - 2 \int_{B_\lambda(x_k)} U_k^2 dV_g
\]
\[
= - \int_{B_\lambda(x_k)} (2 \nabla_g U_k \nabla_g u_k + \frac{\partial}{\partial r} \nabla_g U_k \nabla_g u_k) dV_g + 2 \int_{\partial B_\lambda(x_k)} \frac{\partial U_k}{\partial r} \frac{\partial u_k}{\partial r} dS_g + \int_{B_\lambda(x_k)} O(r^2) U_k^2 dV_g + \int_{B_\lambda(x_k)} e^{\alpha_k u_k^2} O(r^2) dV_g + O(\frac{\delta}{\beta_k^2}).
\]

On the other hand using the same method one can check easily that
\[
\int_{B_\lambda(x_k)} \frac{\partial}{\partial r} \nabla_g u_k \nabla_g U_k dV_g = \delta \int_{B_\lambda(x_k)} \nabla_g u_k \nabla_g U_k dV_g - 4 \int_{B_\lambda(x_k)} \nabla_g u_k \nabla_g U_k dV_g + \int_{\partial B_\lambda(x_k)} O(r^2) \nabla_g u_k \nabla_g U_k dV_g
\]
\[
= - \int_{B_\lambda(x_k)} \nabla_g U_k \nabla_g u_k dV_g + \int_{\partial B_\lambda(x_k)} U_k \frac{\partial u_k}{\partial r} dS_g.
\]

and
\[
\int_{B_\lambda(x_k)} \nabla_g U_k \nabla_g u_k dV_g = - \int_{B_\lambda(x_k)} U_k \nabla_g u_k dV_g + \int_{\partial B_\lambda(x_k)} U_k \frac{\partial u_k}{\partial r} dS_g
\]
\[
= - \int_{B_\lambda(x_k)} U_k^2 dV_g + \int_{\partial B_\lambda(x_k)} U_k \frac{\partial u_k}{\partial r} dS_g.
\]
Thus the Lemma is proved □

Corollary 3.8. We have that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} = Vol_g(M) + \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2}.$$ 

Moreover we have that

$$\tau \in (0, 1).$$

Proof. First of all we have that the sequence $$(\frac{\lambda_k}{\beta_k^2})_k$$ is bounded. Indeed using the definition of $$\beta_k$$ we have that

$$\frac{\lambda_k}{\beta_k^2} = \frac{1}{\lambda_k} \left( \int_M |u_k| e^{\alpha_k u_k^2} dV_g \right)^2.$$ 

Hence using Jensen’s inequality we obtain

$$\frac{\lambda_k}{\beta_k^2} \leq \frac{1}{\lambda_k} \int_M e^{\alpha_k u_k^2} dV_g \int_M u_k^2 e^{\alpha_k u_k^2} dV_g.$$ 

Thus using the definition of $$\lambda_k$$ we have that

$$\frac{\lambda_k}{\beta_k^2} \leq \int_M e^{\alpha_k u_k^2} dV_g.$$ 

On the other hand one can check easily that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in H^1} \int_M e^{32\pi^2 u^2} dV_g < \infty.$$ 

Hence we derive that $$(\frac{\lambda_k}{\beta_k^2})_k$$ is bounded. So we can suppose without lost of generality that $$(\frac{\lambda_k}{\beta_k^2})_k$$ converges.

Now from Lemma 3.7 we have that

$$\lim_{k \to +\infty} \int_{B_k(x_k)} e^{\alpha_k u_k^2} dV_g = 16\pi^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} \left( \delta \beta_k U_k \right)^2 dS_g$$

$$= +\delta \int_{\partial B_k(x_k)} \nabla_g (\beta_k u_k) \nabla_g (\beta_k U_k) dS_g - 2 \int_{\partial B_k(x_k)} (\beta_k U_k) \frac{\partial (\beta_k u_k)}{\partial r} dS_g$$

$$- 2\delta \int_{\partial B_k(x_k)} \frac{\partial (\beta_k U_k)}{\partial r} \frac{\partial (\beta_k u_k)}{\partial r} dS_g + O(\delta).$$

So using Lemma 3.6 we obtain

$$\lim_{k \to +\infty} \int_{B_k(x_k)} e^{\alpha_k u_k^2} dV_g = 16\pi^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} \left( \delta \beta_k G \right)^2 dS_g$$

$$= +\delta \int_{\partial B_k(x_0)} \nabla_g G \nabla_g (\Delta G) dS_g - 2 \int_{\partial B_k(x_0)} \Delta G \frac{\partial G}{\partial r} dS_g$$

$$- 2\delta \int_{\partial B_k(x_0)} \frac{\partial \Delta G}{\partial r} \frac{\partial G}{\partial r} dS_g + O(\delta).$$

Moreover by trivial calculations we get

$$\int_{\partial B_k(x_0)} |\Delta G|^2 dS_g = \frac{\tau^2}{8\pi^2\delta} + O(1);$$
\[
\int_{\partial B_1(x_0)} \nabla g \nabla g (\Delta g) dS_g = -\frac{\tau^2}{8\pi^2 \delta} + O(1);
\]
\[
\int_{\partial B_1(x_0)} \Delta g \frac{\partial G}{\partial r} = \frac{\tau^2}{16\pi^2} + O(\delta);
\]
and
\[
\int_{\partial B_1(x_0)} \frac{\partial \Delta g}{\partial r} \frac{\partial G}{\partial r} dS_g = -\frac{\tau^2}{8\pi^2 \delta} + O(1)
\]
Hence with this we obtain
\[
\lim_{k \to +\infty} \int_{B_\delta(x_0)} e^{\alpha_k u_k^2} dV_g = \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} + O(\delta).
\]
On the other hand we have that
\[
\int_{M} e^{\alpha_k u_k^2} dV_g = \int_{B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g + \int_{M \setminus B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g.
\]
Moreover by Lemma 3.2 we have that
\[
\int_{M \setminus B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g = Vol_g(M) + o_k(\delta).
\]
Thus we derive that
\[
\lim_{k \to +\infty} \int_{M} e^{\alpha_k u_k^2} dV_g = Vol_g(M) + \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} + o_k(1).
\]
Hence letting \( \delta \to 0 \) we obtain
\[
\lim_{k \to +\infty} \int_{M} e^{\alpha_k u_k^2} dV_g = Vol_g(M) + \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2}.
\]
Now suppose \( \tau = 0 \) then we get
\[
\lim_{k \to +\infty} \int_{M} e^{\alpha_k u_k^2} dV_g = Vol_g(M).
\]
On the other hand we have that
\[
\lim_{k \to +\infty} \int_{M} e^{\alpha_k u_k^2} dV_g = \sup_{u \in H_1} \int_{M} e^{32\pi^2 u^2} dV_g > Vol_g(M);
\]
hence a contradiction. Thus \( \tau \neq 0 \) and the Corollary is proved. \( \square \)

3.3. Blow-up analysis. In this subsection we perform the Blow-up analysis and show that the asymptotic profile of \( u_k \) is either the zero function or a standard Bubble.

First of all let us introduce some notations.

We set
\[
r_k^4 = \frac{\lambda_k}{\beta_k^2} e^{-\alpha_k u_k^2}.
\]
Now for \( x \in B^\times_\delta(0) \) with \( \delta > 0 \) small we set
\[
w_k(x) = 2\alpha_k \beta_k (u_k(\exp_{x_k}(r_k x)) - c_k);
\]
\[
v_k(x) = \frac{1}{\beta_k^2} u_k(\exp_{x_k}(r_k x));
\]
\[
g_k(x) = (\exp_{x_k}^*) (r_k x).
\]
Next we define
\[
d_k = \frac{\beta_k}{\beta_k^2} d = \lim_{k \to +\infty} d_k.
\]
Proposition 3.9. The following hold:
We have

\[
\begin{align*}
&\text{if } d < +\infty \text{ then } w_k \to w(x) := \frac{4}{d} \log \left( \frac{1}{1 + \sqrt{\frac{d}{4}}|x|^2} \right) \text{ in } C^2_{\text{loc}}(\mathbb{R}^d); \\
&\text{and} \\
&\text{if } d = +\infty \text{ then } w_k \to 0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^d).
\end{align*}
\]

Proof. First of all we recall that

\[
g_k \to dx^2 \text{ in } C^2_{\text{loc}}(\mathbb{R}^d).
\]

Since \((\frac{1}{\sqrt{\beta x}}, \frac{1}{\sqrt{\gamma y}})\) are bounded and \(c_k \to +\infty\), then we infer that \(r_k \to 0 \text{ as } k \to 0\).

Now using the Green representation formula for \(\Delta_g^2\) (see Lemma 2.1) we have that

\[
u_k(x) = \int_M F(x, y) \Delta_g^2 u_k dV_g(y) \quad \forall x \in M.
\]

Now using equation and differentiating with respect to \(x\) we obtain that for \(m = 1, 2\)

\[
|\nabla_g^m u_k(x)| \leq \int_M |\nabla_g^m F(x, y)| \left| \frac{u_k}{\lambda_k} e^{\alpha x u_k^2} - \gamma_k \right| dV_g(y).
\]

Hence from the fact that \(\beta_k \gamma_k\) is bounded see Lemma 3.4 we get

\[
|\nabla_g^m u_k(x)| \leq \int_M |\nabla_g^m F(x, y)| \left| \frac{u_k}{\lambda_k} e^{\alpha x u_k^2} \right| dV_g(y) + O(\beta_k^{-1}).
\]

Now for \(y_k \in B_{Lr_k}(x_k), \ L > 0 \text{ fixed}\) we write that

\[
\int_M |\nabla_g^m F(y_k, y)| \left| \frac{u_k}{\lambda_k} e^{\alpha x u_k^2} \right| dV_g(y) = O\left( r_k^{-m} \int_{M \setminus B_{Lr_k}(y_k)} \left| \frac{u_k}{\lambda_k} e^{\alpha x u_k^2} \right| dV_g(y) \right)
\]

\[
+ O\left( \frac{c_k}{\lambda_k} e^{\alpha x \gamma_k^2} \int_{B_{Lr_k}(y_k)} dV_g(y) \right) = O(r_k^{-m} \beta_k^{-1}).
\]

thanks to the fact that \(|u_k| \leq c_k\) to the definition of \(r_k\).

Now it is not worth remarking that \(c_k = u_k(x_k)\) since we have taken \(\tau \geq 0\) (see Lemma 3.4).

Hence we have that

\[
w_k(x) \leq w_k(0) = 0 \quad \forall x \in \mathbb{R}^d.
\]

So we get from the estimate above that \(w_k\) is uniformly bounded in \(C^2(K)\) for every compact subset \(K\) of \(\mathbb{R}^d\). Thus by Arzelà-Ascoli Theorem we infer that

\[
w_k \to w \in C^1_{\text{loc}}(\mathbb{R}^d).
\]

Clearly \(w\) is a Lipschitz function since the constant which bounds the gradient of \(w_k\) is independent of the compact set \(K\).

On the other hand from the Green representation formula we have for \(x \in \mathbb{R}^4\) fixed and for \(L\) big enough such that \(x \in B^L(0)\)

\[
u_k(\exp_{x_k}(r_k x)) = \int_M F(\exp_{x_k}(r_k x), y) \Delta_g^2 u_k(y) dV_g(y).
\]

Now remarking that

\[
u_k(x_k) = u_k(\exp_{x_k}(r_k 0));
\]

we have that

\[
u_k(\exp_{x_k}(r_k x)) - \nu_k(x_k) = \int_M (F(\exp_{x_k}(r_k x), y) - F(\exp_{x_k}(0), y)) \Delta_g^2 u_k(y) dV_g(y).
\]
Hence using (3.1) we obtain

\[ u_k(\exp_{x_k}(r_kx)) - u_k(x_k) = \int_M \left( F(\exp_{x_k}(r_kx), y) - F(\exp_{x_k}(0), y) \right) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) \]

\[ - \int_M \left( F(\exp_{x_k}(r_kx), y) - F(\exp_{x_k}(0), y) \right) (\gamma_k) dV_g(y). \]

Now setting

\[ I_k(x) = \int_{B_{L_r}(x_k)} (F(\exp_{x_k}(r_kx), y) - F(\exp_{x_k}(0), y)) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y); \]

\[ \Pi_k(x) = \int_{M \setminus B_{L_r}(x_k)} (F(\exp_{x_k}(r_kx), y) - F(\exp_{x_k}(0), y)) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) \]

and

\[ \Pi_k(x) = \int_M (F(\exp_{x_k}(r_kx), y) - F(\exp_{x_k}(0), y)) (\gamma_k) dV_g(y); \]

we find

\[ u_k(\exp_{x_k}(r_kx)) - u_k(x_k) = I_k(x) + \Pi_k(x) + \Pi_k(x). \]

So using the definition of \( w_k \) we arrive to

\[ w_k = 2\alpha_k \beta_k (I_k(x) + \Pi_k(x) + \Pi_k(x)). \]

Now to continue the proof we consider two cases:

**Case 1: \( d < +\infty \)**

First of all let us study each of the terms \( 2\alpha_k \beta_k I_k(x), 2\alpha_k \beta_k \Pi_k(x), 2\alpha_k \beta_k \Pi_k(x) \) separately. Using the change of variables \( y = \exp_{x_k}(r_kz) \) we have

\[ 2\alpha_k \beta_k I_k(x) = \int_{B^L(0)} (F(\exp_{x_k}(r_kx), \exp_{x_k}(r_kz)) - F(\exp_{x_k}(0), \exp_{x_k}(r_kz))) \frac{2\alpha_k \beta_k u_k(\exp_{x_k}(r_kz))}{\lambda_k} e^{\alpha_k u_k^2(\exp_{x_k}(r_kz))} = \lambda_k dV_g(\beta_k(x_k)). \]

Hence using the definition of \( r_k \) and \( v_k \) one can check easily that the following holds

\[ 2\alpha_k \beta_k I_k(x) = 2\alpha_k \int_{B^L(0)} (G(\exp_{x_k}(r_kx), \exp_{x_k}(r_kz)) - G(\exp_{x_k}(0), \exp_{x_k}(r_kz))) v_k(z) e^{\frac{d}{2}(w_k(z)(1+v_k))} dV_g(\beta_k(x_k)). \]

Moreover from the asymptotics of the Green function see Lemma 2.1 we have that

\[ 2\alpha_k \beta_k I_k(x) = 2\alpha_k \int_{B^L(0)} \left( \frac{1}{8\pi^2} \log \frac{|z|}{|x-z|} + K_k(x, z) \right) v_k(z) e^{\frac{d}{2}(w_k(z)(1+v_k))} dV_g(\beta_k(x_k)). \]

where

\[ K_k(x, z) = [K(\exp_{x_k}(r_kx), \exp_{x_k}(r_kz)) - (K(\exp_{x_k}(0), \exp_{x_k}(r_kz))]. \]

Hence since \( K \) is of class \( C^1 \) on \( M^2 \) and \( g_k \to d^2 \) in \( C^2_{loc}(\mathbb{R}^4) \) and \( v_k \to 1 \) then letting \( k \to +\infty \) we derive

\[ \lim_{k \to +\infty} 2\alpha_k \beta_k I_k(x) = 8 \int_{B^L(0)} \log \frac{|z|}{|x-z|} e^{d_w(z)} dV_g(z). \]

Now to estimate \( \alpha_k \beta_k \Pi_k(x) \) we write for \( k \) large enough

\[ \alpha_k \beta_k \Pi_k(x) = \int_{M \setminus B_{L_r}(x_k)} \frac{1}{8\pi^2} \log \frac{d_g(\exp_{x_k}(0), y)}{d_g(\exp_{x_k}(r_kx), y)} \frac{2\alpha_k \beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) \]

\[ + \int_{M \setminus B_{L_r}(x_k)} \tilde{K}_k(x, y) \frac{2\alpha_k \beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y), \]

where

\[ \tilde{K}_k(x, y) = (K(\exp_{x_k}(r_kx), y) - K(\exp_{x_k}(0), y)). \]
Taking the absolute value in both sides of the equality and using the change of variable \( y = \exp_{x_k}(r_k z) \) and the fact that \( K \in C^1 \) we obtain,

\[
|2\alpha_k \beta_k \Pi_k(x)| \leq \int_{\mathbb{R}^4 \setminus B^L(0)} 8 \left| \log \left( \frac{|z|}{|x - z|} \right) \right| v_k(z) e^{\frac{R}{2}(u_k(z) + v_k(z))} dV_g(z) + Lr_k \int_{M \setminus B_{Lr_k}(x_k)} \frac{2\alpha_k \beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y).
\]

Hence letting \( k \to +\infty \) we deduce that

\[
\limsup_{k \to +\infty} |2\alpha_k \beta_k \Pi_k(x)| = o_L(1).
\]

Now using the same method one proves that

\[
2\alpha_k \beta_k \Pi_k(x) \to 0 \quad \text{as} \quad k \to +\infty.
\]

So we have that

\[
w(x) = \int_{B^L(R)} 8 \log \left( \frac{|z|}{|x - z|} \right) e^{dw(z)} dz + \lim_{k \to +\infty} 2\alpha_k \beta_k \Pi_k(x).
\]

Hence letting \( L \to +\infty \) we obtain that \( w \) is a solution of the following integral equation

\[
w(x) = \int_{\mathbb{R}^4} 8 \log \left( \frac{|z|}{|x - z|} \right) e^{dw(z)} dz.
\]

(3.10)

Now since \( w \) is Lipschitz then the theory of singular integral operator gives that \( w \in C^1(\mathbb{R}^4) \). Since

\[
\lim_{k \to +\infty} \int_{B_{Lr_k}(x_k)} \frac{2\alpha_k \beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g = 64\pi^2 \int_{B^L(0)} e^{dw(x)} dx.
\]

and

\[
\int_{B_{Lr_k}(x_k)} \frac{2\alpha_k \beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g \leq 64\pi^2,
\]

then we get

\[
\lim_{L \to +\infty} \int_{B^L(0)} e^{dw(x)} dx = \int_{\mathbb{R}^4} e^{dw(x)} dx \leq 1.
\]

Now setting

\[
\hat{w}(x) = \frac{d}{4} w(x) + \frac{1}{4} \log \left( \frac{8\pi^2 d}{3} \right);
\]

we have that \( \hat{w} \) satisfies the following conformally invariant integral equation

\[
\hat{w}(x) = \int_{\mathbb{R}^4} \frac{6}{8\pi^2} \log \left( \frac{|z|}{|x - z|} \right) e^{\hat{w}(z)} dz + \frac{1}{4} \log \left( \frac{8\pi^2 d}{3} \right),
\]

(3.11)

and

\[
\int_{\mathbb{R}^4} e^{4\hat{w}(x)} dx < +\infty.
\]

Hence from the classification result by X.Xu see Theorem 1.2 in [25] we derive that

\[
\hat{w}(x) = \log \left( \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \right)
\]

for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^4 \).

From the fact that

\[
w(x) \leq w(0) = 0 \quad \forall x \in \mathbb{R}^4;
\]

we obtain

\[
\hat{w}(x) \leq \hat{w}(0) = \frac{1}{4} \log \left( \frac{8\pi^2 d}{3} \right) \quad \forall x \in \mathbb{R}^4.
\]

Then we derive

\[
x_0 = 0, \quad \lambda = 2 \left( \frac{8\pi^2 d}{3} \right)^{-\frac{1}{2}}.
\]
Hence by trivial calculations we get
\[ w(x) = 4 \frac{d}{d} \log \left( \frac{1}{1 + \sqrt{\frac{4}{\pi}} |x|^2} \right). \]

**Case 2:** \( d = +\infty. \)

In this case using the same argument we get
\[
\limsup_{k \to +\infty} |\alpha_k \beta_k \Pi_k(x)| = o_k(1);
\]
and
\[ \alpha_k \beta_k \Pi_k(x) = o_k(1), \]
Now let us show that
\[ \alpha_k \beta_k I_k(x) = o_k(1). \]
By using the same arguments as in Case 1 we get
\[ \alpha_k \beta_k I_k(x) = \int_{B^L(0)} \left( \frac{1}{8\pi^2} \log \frac{|z|}{|x-z|} + K_k(x,z) \right) v_k(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z). \]

Now since \( K \) is \( C^1 \) we need only to show that
\[ \int_{B^L(0)} \frac{1}{8\pi^2} \log \frac{|z|}{|x-z|} v_k(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z) = o_k(1). \]

By using the trivial inequality
\[ \int_{B_{k^{-\ln(1)}}(x_k)} u_k^2 e^{\alpha_k u_k^2} dV_g \leq 1; \]
and the change of variables as above, we obtain
\[ \int_{B^L(0)} v_k^2(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z) = O(\frac{1}{d_k}) = o_k(1). \]

On the other hand using the property of \( v_k \) one can check easily that
\[ \int_{B^L(0)} v_k(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z) = \int_{B^L(0)} v_k^2(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z) + o_k(1). \]

Thus we arrive to
\[ \int_{B^L(0)} \frac{1}{8\pi^2} \log \frac{|z|}{|x-z|} v_k(z) e^{d_k(w_k(z)(1+v_k(z)))} dV_{g_k}(z) = o_k(1) \]
So we get
\[ \alpha_k \beta_k I_k(x) = o_k(1) \]
Thus letting \( k \to +\infty \), we obtain
\[ w(x) = 0 \quad \forall x \in \mathbb{R}^4. \]

Hence the Proposition is proved. \( \square \)

3.4. **Capacity estimates.** This subsection deals with some capacity-type estimates which allow us to get an upper bound of \( \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} \). We start by giving a first Lemma to show that we can basically work on Euclidean space in order to get the capacity estimates as already said in the Introduction.

**Lemma 3.10.** There is a constant \( B \) which is independent of \( k, L \) and \( \delta \) s.t.
\[
\int_{B^\delta(0) \setminus B^L(0)} |(1 - B|x|^2)\Delta_0 \tilde{u}_k|^2 dx \leq \int_{B^\delta(x_k) \setminus B^{L \cdot \tilde{c}_k}(x_k)} |\Delta_0 u_k|^2 dV_g + \frac{J_1(k, L, \delta)}{\beta_k^2},
\]
where
\[ \tilde{u}(x) = u_k(exp_{x_k}(x)). \]
Moreover we have that
\[ \lim_{\delta \to 0} \lim_{k \to +\infty} J_1(k, L, \delta) = 0. \]
Proof. First of all by using the definition of $\Delta_g$ we get

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_c (\sqrt{|g|} g^{rs} \partial_c)$$

we get

$$|\Delta_g \beta_k \tilde{u}_k|^2 = |g^{rs} \beta_k \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s} + O(|\nabla \beta_k \tilde{u}_k)|^2$$

$$= |g^{rs} \beta_k \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2 + O(|\nabla^2 \beta_k \tilde{u}_k||\nabla \beta_k \tilde{u}_k|) + O(|\nabla \beta_k \tilde{u}_k|)^2$$

On the other hand using the fact that (see Corollary 3.6)

$$\beta_k \tilde{u}_k \sim \tilde{G} \quad in \quad W^{2,p}(M)$$

where $p \in (1, 2)$; and $\tilde{G}(x) = G(exp_{x_0}(x))$; we obtain

$$\int_{B^+(0) \setminus B^{L^r}(0)} O(|\nabla^2 \beta_k \tilde{u}_k||\nabla \beta_k \tilde{u}_k|) + O(|\nabla \beta_k \tilde{u}_k|^2) \leq C ||\tilde{G}||_{W^{1,2}(B^+(0) \setminus B^{L^r}(0))}$$

$$= J_2(k, L, \delta),$$

and it is clear that

$$\lim_{\delta \to 0} \lim_{k \to +\infty} J_2(k, L, \delta) = 0$$

Now let us estimate $\int_{B^+(0) \setminus B^{L^r}(0)} |g^{rs} \beta_k \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2. \quad To \quad do \quad this, \quad we \quad first \quad write \quad the \quad inverse \quad of \quad the \quad metric \quad in \quad the \quad following \quad form

$$g^{rs} = \delta^{rs} + A^s$$

with

$$|A^s| \leq C |x|^2.$$

We can write

$$|g^{rs} \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2 |\Delta_0 \tilde{u}_k|^2 + 2 \sum_{p,q} A^{pq} \Delta_0 \tilde{u}_k \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q} + \sum_{r,s,p,q} A^r A^{pq} \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s} \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}$$

Furthermore we derive

$$\sum_{p,q} 2 \int_{B^+(0) \setminus B^{L^r}(0)} |A^{pq} \Delta_0 \tilde{u}_k \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}| dV_g \leq C \int_{B^+(0) \setminus B^{L^r}(0)} (|x|^2 |\Delta_0 \tilde{u}_k|^2 + \sum_{p,q} |x|^2 |\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}|^2) dx$$

On the other hand we have that

$$\sum_{p,q} \int_{B^+(0) \setminus B^{L^r}(0)} |x|^2 |\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}|^2 dx \int_{B^+(0) \setminus B^{L^r}(0)} |x|^2 |\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}| \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s} \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q} dx$$

$$+ \int_{B^+(0) \setminus B^{L^r}(0)} O(|\nabla \tilde{u}_k||\nabla^2 \tilde{u}_k|) dx + \int_{\partial(B^+(0) \setminus B^{L^r}(0))} |x|^2 \frac{\partial \tilde{u}_k}{\partial x^r} \frac{\partial^2 \tilde{u}_k}{\partial x^q \partial x^s} \frac{\partial}{\partial x^p} \frac{\partial}{\partial r} dS$$

$$+ \int_{\partial(B^+(0) \setminus B^{L^r}(0))} |x|^2 \frac{\partial \tilde{u}_k}{\partial x^r} \frac{\partial^2 \tilde{u}_k}{\partial x^q \partial x^s} \frac{\partial}{\partial x^p} \frac{\partial}{\partial r} dS.$$}

So setting

$$\frac{J_3(k, L, \delta)}{\beta_k^2} = \int_{B^+(0) \setminus B^{L^r}(0)} O(|\nabla \tilde{u}_k||\nabla^2 \tilde{u}_k|) dx + \int_{\partial(B^+(0) \setminus B^{L^r}(0))} |x|^2 \frac{\partial \tilde{u}_k}{\partial x^r} \frac{\partial^2 \tilde{u}_k}{\partial x^q \partial x^s} \frac{\partial}{\partial x^p} \frac{\partial}{\partial r} dS$$

$$+ \int_{\partial(B^+(0) \setminus B^{L^r}(0))} |x|^2 \frac{\partial \tilde{u}_k}{\partial x^r} \frac{\partial^2 \tilde{u}_k}{\partial x^q \partial x^s} \frac{\partial}{\partial x^p} \frac{\partial}{\partial r} dS$$

We obtain

$$\sum_{p,q} \int_{B^+(0) \setminus B^{L^r}(0)} |x|^2 |\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}|^2 = \int_{B^+(0) \setminus B^{L^r}(0)} |x|^2 |\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}|^2 dx + \frac{J_3(k, L, \delta)}{\beta_k^2}$$

Moreover we have that

$$\lim_{\delta \to 0} \lim_{k \to +\infty} J_3(k, L, \delta) = 0.$$
Hence we get
\[ 2 \sum_{p,q} \int_{B^\delta(0) \setminus B^{Lr_k}(0)} |A^p q \Delta_0 \bar{u}_k|^2 \leq C \int_{B^\delta(0) \setminus B^{Lr_k}(0)} \bar{u}_k^2 \leq C \int_{B^\delta(0) \setminus B^{Lr_k}(0)} \Delta_0 \bar{u}_k^2 dx + \frac{J_4(k,L,\delta)}{\beta_k^2}, \]
with
\[ \lim_{\delta \to 0} \lim_{k \to +\infty} J_4(k,L,\delta) = 0. \]

On the other hand using similar arguments we get
\[ \int_{B^\delta(0) \setminus B^{Lr_k}(0)} \sum_{r,s,p,q} A^p \Delta s \Delta r q \bar{u}_k \leq C \int_{B^\delta(0) \setminus B^{Lr_k}(0)} |\Delta_0 \bar{u}_k|^2 dx + \frac{J_5(k,L,\delta)}{\beta_k^2}, \]
with
\[ \lim_{\delta \to 0} \lim_{k \to +\infty} J_5(k,L,\delta) = 0. \]

So we arrive to
\[ \int_{B_1(x_k) \setminus B_{Lr_k}(x_k)} |\Delta_0 u_k|^2 dv_g \leq \int_{B^\delta(0) \setminus B^{Lr_k}(0)} (1 + C |x|^2 + C |x|^4) |\Delta_0 \bar{u}_k|^2 dx + \frac{J_6(k,L,\delta)}{\beta_k^2}; \]
with
\[ \lim_{\delta \to 0} \lim_{k \to +\infty} J_6(k,L,\delta) = 0. \]

Hence we can find a constant \( B_1 \) independent of \( k, L \) and \( \delta \) s.t.
\[ \int_{B_1(x_k) \setminus B_{Lr_k}(x_k)} |\Delta_0 u_k|^2 dv_g \geq \int_{B^\delta(0) \setminus B^{Lr_k}(0)} (1 - B_1 |x|^2) |\Delta_0 \bar{u}_k|^2 dx + \frac{J_7(k,L,\delta)}{\beta_k^2}. \]

So setting
\[ J_1(k,L,\delta) = -J_7(k,L,\delta) \quad \text{and} \quad B = B_1 \]
we have proved the Lemma.

Next we give a technical Lemma

**Lemma 3.11.** There exists a sequence of functions \( U_k \in W^{2,2}(B^\delta(0) \setminus B^{Lr_k}(0)) \) s.t
\[ U_k|_{\partial B^\delta(0)} = \frac{1}{\beta_k} \log \delta + S_0, \quad U_k|_{\partial B^{Lr_k}(0)} = \frac{w(L)}{2\alpha_k \beta_k} + c_k; \]
and
\[ \frac{\partial U_k}{\partial r}|_{\partial B^\delta(0)} = -\frac{\tau}{8\pi^2 \beta_k}, \quad \frac{\partial U_k}{\partial r}|_{\partial B^{Lr_k}(0)} = \frac{w'(L)}{2\alpha_k \beta_k r_k}. \]

Moreover there holds
\[ \lim_{\delta \to 0} \lim_{k \to +\infty} \beta_k^2 \left( \int_{B^\delta(0) \setminus B^{Lr_k}(0)} |\Delta_0 (1 - B |x|^2) U_k|^2 dx - \int_{B^\delta(0) \setminus B^{Lr_k}(0)} (1 - B |x|^2) \Delta_0 \bar{u}_k^2 dx \right) = 0. \]

**Proof.** First of all let us set
\[ h_k(x) = u_k(exp_{x_k}(r_k x)). \]
and \( u'_k \) to be the solution of
\[
\begin{dcases}
\frac{\partial^2 u'_k}{\partial x^2} = \Delta_0^2 h_k \\
\frac{\partial u'_k}{\partial n}|_{\partial B_2 L} = \frac{\partial h_k}{\partial n}|_{\partial B_2 L}, \quad u'_k|_{\partial B_2 L(0)} = h_k|_{\partial B_2 L(0)} \\
\frac{\partial u'_k}{\partial n}|_{\partial B_2 L(0)} = \frac{1}{2\alpha_k \beta_k} \frac{\partial h_k}{\partial n}|_{\partial B_2 L(0)}, \quad u'_k|_{\partial B_2 L(0)} = \frac{w}{2\alpha_k \beta_k}|_{\partial B_2 L(0)}.
\end{dcases}
\]
Next let us define
\[ U'_k = \begin{dcases} \frac{u'_k(x)}{r_k} \quad Lr_k \leq |x| \leq 2Lr_k \\
\tilde{u}_k(x) \quad 2Lr_k \leq |x|. \end{dcases} \]
Clearly we have that
\[ \lim_{k \to +\infty} \int_{B_2 Lr_k(0) \setminus B^{Lr_k}(0)} (1 - B |x|^2)(|\Delta_0 U'_k|^2 - |\Delta_0 \bar{u}_k|^2) dx = 0, \]
and
\[
\lim_{k \to +\infty} |U'_k - \bar{u}'_k|_{C^0(\mathbb{R}^2 \setminus \mathbb{B}_r(0) \cup \mathbb{B}_\sigma(0) = 0.}
\]

Now let \( \eta \) be a smooth function which satisfies
\[
\eta(t) = \begin{cases} 
1 & t \leq 1/2 \\
0 & t > 2/3 
\end{cases}
\]
and set
\[
G_k = \eta \left( \frac{|x|}{\delta} \right) (\bar{u}_k - \tau S_0 + \frac{\tau}{8\pi^2} \log |x|) - \frac{\tau}{8\pi^2} \log |x| + \tau S_0.
\]
Then we have that
\[
G_k \to -\frac{\tau}{8\pi^2} \log |x| + \tau S_0 + \tau \eta \left( \frac{|x|}{\delta} \right) \bar{S}_1(x);
\]
where \( \bar{S}_1(x) = S_1(\exp_{x_0}(x)) \).
Furthermore we obtain
\[
\beta_k \bar{u}_k - G_k \to \tau \left( 1 - \eta \left( \frac{|x|}{\delta} \right) \right) \bar{S}_1(x),
\]
then
\[
\lim_{\varepsilon \to 0} \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 \beta_k \bar{u}_k|^2 \, dx - \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 G_k|^2 \, dx \leq \Sigma.
\]
where
\[
\Sigma = \sqrt{\int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 (1 - \eta(\frac{|x|}{\delta})) \bar{S}_1(x)|^2 \, dx \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 (G - \frac{1}{8\pi^2} \log |x| + \eta(\frac{|x|}{\delta})) \bar{S}_1(x))|^2 \, dx}
\]
\[
\leq C \delta \sqrt{|\log \delta|}.
\]
So we get
\[
\lim_{\varepsilon \to 0} \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 \beta_k \bar{u}_k|^2 \, dx - \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 G_k|^2 \, dx \leq C \delta \sqrt{|\log \delta|}.
\]
Hence setting
\[
U_k = \begin{cases} 
U'_k(x) & |x| \leq \frac{\delta}{2} \\
G_k(x) & \delta/2 \leq |x| \leq \delta
\end{cases}
\]
we have proved the Lemma. \( \square \)

**Proposition 3.12.** We have the following holds
\[
\tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} \leq \frac{\pi^2}{6} e^{\frac{1}{2} + 2s^2} S_2,
\]
and
\[
d\tau = 1.
\]

**Proof.** First using Lemma 3.10 and Lemma 3.11 we get
\[
\int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 (1 - B|x|^2) U_k|^2 \, dx \leq 1 - \int_{M \setminus B_{\epsilon}(x_0)} |\Delta w|^2 + \int_{M \setminus B_{\epsilon}(x_0)} |\Delta G|^2 + J_0(k, L, \delta) \tag{3.12}
\]
with
\[
\lim_{\delta \to 0} \lim_{k \to +\infty} J_0(k, L, \delta) = 0.
\]
Next we will apply capacity to give a lower boundary of \( \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 (1 - B|x|^2) U_k|^2 \, dx \).
Hence we need to calculate
\[
\Phi|_{B^{\epsilon/2}(0)} = P_1, \Phi|_{B_{\infty}(0)} = P_2, \frac{\partial \Phi}{\partial B_{\infty}(0)} = Q_1, \frac{\partial \Phi}{\partial B_{\infty}(0)} = Q_2 \int_{B^{\epsilon}(0) \setminus B^{\epsilon/2}(0)} |\Delta_0 \Phi|^2 \, dx,
\]
where \( P_1, P_2, Q_1, Q_2 \) are constants.

It is obvious that the infimum is attained by the function \( \Phi \) which satisfies
\[
\begin{align*}
\Delta_0^2 \Phi &= 0 \\
\Phi|_{\partial B^r(0)} &= P_1, \quad \Phi|_{\partial B^n(0)} = P_2, \quad \frac{\partial \Phi}{\partial r}|_{\partial B^r(0)} = Q_1, \quad \frac{\partial \Phi}{\partial r}|_{\partial B^n(0)} = Q_2.
\end{align*}
\]

Moreover we can require the function \( \Phi \) to be of the form
\[
\Phi = A \log r + Br^2 + \frac{C}{r^2} + D,
\]
where \( A, B, C, D \) are all constants which satisfies the following linear system of equations
\[
\begin{align*}
A \log r + Br^2 + \frac{C}{r^2} + D &= P_1 \\
A \log R + Br^2 + \frac{C}{R^2} + D &= P_2 \\
\frac{A}{r} + 2Br - 2\frac{C}{r^2} &= Q_1 \\
\frac{A}{r} + 2BR - 2\frac{C}{r^2} &= Q_2
\end{align*}
\]

Now by straightforward calculations we obtain the explicit expression of \( A \) and \( B \)
\[
\begin{align*}
A &= \frac{P_1 - P_2 + \frac{\delta}{r}Q_1 + \frac{\delta}{r^2}RQ_2}{\log r + \frac{\delta}{r^2}} \\
B &= -2P_1 - P_2 - \frac{Q_1}{R^2} + \frac{Q_2}{(R^2 + r^2) \log r + \delta}
\end{align*}
\]

Where \( \delta = \frac{R^2 - r^2}{R^2 + r^2} \). Furthermore we have
\[
\int_{B^n(0) \setminus B^r(0)} |\Delta_0 \Phi|^2 \, dx = -8\pi^2 A^2 \log r/R + 32\pi^2 AB(R^2 - r^2) + 32\pi^2 B^2(R^4 - r^4) \quad (3.13)
\]

In our case in which we have that \( R = \delta \), \( r = Lr_k \),
\[
P_1 = c_k + \frac{w(L)}{2\alpha k \beta_k} + O(r_kc_k) \quad P_2 = \frac{\log \delta + \tau S_0 + O(\delta \log \delta)}{\beta_k} \\
Q_1 = \frac{w'(L) + O(r_kc_k)}{2\alpha k \beta_k r_k} \quad Q_2 = -\frac{\tau + O(\delta \log \delta)}{8\pi^2 \beta_k \delta}
\]

Then by the formula giving \( A \) we obtain by trivial calculations
\[
A = \frac{c_k + \frac{N_k + \frac{\log \delta}{\beta_k}}{\beta_k}}{-\log \delta + \log L + \frac{\log \frac{\alpha k \beta_k}{4}}{4} + 1 + O(r_k^2)}
\]

where
\[
N_k = \frac{w(L)}{2\alpha_k} - \tau S_0 + \frac{w'(L)L}{4\alpha_k} - \frac{\tau}{16\pi^2} + O(\delta \log \delta) + O(r_kc_k^2).
\]

Moreover using the the fact that the sequence \( \frac{\Delta_0}{\beta_k} \) is bounded it is easily seen that
\[
A = O\left(\frac{1}{\beta_k}\right).
\]

Furthermore using the formula of \( B \) we get still by trivial calculations
\[
B = \frac{-2c_k + \alpha \log \frac{\alpha k \beta_k}{4} + O\left(\frac{1}{\beta_k}\right)}{\frac{\delta^2(-\alpha k c_k^2 + \log \frac{\Delta_0}{\beta_k})}{\beta_k}}
\]

and then
\[
B = O\left(\frac{1}{\beta_k}\right) \frac{1}{\delta^2}.
\]
Now let compute $8\pi^2 A^2 \log r/R$. By using the expression of $A$, $r$ and $R$, we have that

$$-8\pi^2 A^2 \log \left( \frac{r}{R} \right) = -8\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\beta_k} \right)^2 \left( \log \frac{\lambda_k}{\beta_k c_k} - \frac{\alpha_k c_k^2}{4} \right) - \log \delta + \log L,$$

Now using the relation

$$\left( \frac{\alpha_k c_k^2}{4} \right)^2 \left( 1 - \frac{1}{\alpha_k c_k^2} \left( -4 \log \delta + 4 \log L + \log \frac{\lambda_k}{\beta_k c_k} + 4 + O(r_k^2) \right) \right)^2 =$$

$$\left( - \log \delta + \log L + \frac{\log \frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} + 1 + O(r_k^2) \right)^2$$

we derive

$$-8\pi^2 A^2 \log \left( \frac{r}{R} \right) = -8\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\alpha_k c_k^2} \right)^2 \left( 1 - \frac{1}{\alpha_k c_k^2} \left( -4 \log \delta + 4 \log L + \log \frac{\lambda_k}{\beta_k c_k} + 4 + O(r_k^2) \right) \right)^2$$

$$\times \left( \frac{\log \frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} - \log \delta + \log L \right).$$

On the other hand using Taylor expansion we have the following identity

$$\left( 1 - \frac{1}{\alpha_k c_k^2} \left( -4 \log \delta + 4 \log L + \log \frac{\lambda_k}{\beta_k c_k} + 4 + O(r_k^2) \right) \right)^{-2} = 1 + 2 \frac{\log \frac{\lambda_k}{\beta_k c_k} + 4 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2}$$

$$+ O\left( \frac{\log^2 c_k}{c_k} \right);$$

hence we get

$$-8\pi^2 A^2 \log \left( \frac{r}{R} \right) = -8\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\alpha_k c_k^2} \right)^2 \left( \log \frac{\lambda_k}{\beta_k c_k} - \frac{\alpha_k c_k^2}{4} - \log \delta + \log L \right)$$

$$\times \left( 1 + 2 \frac{\log \frac{\lambda_k}{\beta_k c_k} + 4 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} + O\left( \frac{\log^2 c_k}{c_k} \right) \right).$$

On the other hand using the relation

$$-8\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\alpha_k c_k^2} \right)^2 \left( \log \frac{\lambda_k}{\beta_k c_k} - \frac{\alpha_k c_k^2}{4} - \log \delta + \log L \right) =$$

$$32\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\beta_k} \right)^2 \left( 1 - \frac{\log \frac{\lambda_k}{\beta_k c_k} - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} \right)^2$$

we obtain

$$-8\pi^2 A^2 \log \left( \frac{r}{R} \right) = 32\pi^2 \left( c_k + \frac{N_k + \frac{N_k \pi^2}{2} \log \delta}{\beta_k} \right)^2 \left( 1 + 2 \frac{\log \frac{\lambda_k}{\beta_k c_k} + 4 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} + O\left( \frac{\log^2 c_k}{c_k} \right) \right)$$

$$\times \left( 1 - \frac{\log \frac{\lambda_k}{\beta_k c_k} - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} \right).$$

Moreover using again the trivial relation

$$(1 + 2 \frac{\log \frac{\lambda_k}{\beta_k c_k} + 4 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} + O\left( \frac{\log^2 c_k}{c_k} \right)) \left( 1 - \frac{\log \frac{\lambda_k}{\beta_k c_k} - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} \right) =$$

$$(1 + \frac{\log \frac{\lambda_k}{\beta_k c_k} + 8 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} + O\left( \frac{\log^2 c_k}{c_k} \right)) \left( 1 - \frac{\log \frac{\lambda_k}{\beta_k c_k} - 8 \log \delta + 4 \log L}{\alpha_k c_k^2} \right).$$
we arrive to 

$$-8\pi^2 A^2 \log \left( \frac{R}{r} \right) = \frac{32\pi^2}{\alpha_k} \left( c_k + \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} \right)^2 \left( 1 + \frac{\log \frac{\lambda_k}{\beta c_k}}{\alpha_k c_k^2} + \frac{8 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} \right) + O \left( \frac{\log^2 c_k}{c_k^4} \right)$$

On the other hand one can check easily that the following holds

$$\int \left( c_k^2 + \lambda_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} \right)^2 \left( 1 + \log \frac{\lambda_k}{\beta c_k} + \frac{8 - 4 \log \delta + 4 \log L}{\alpha_k c_k^2} \right) + O \left( \frac{\log^2 c_k}{c_k^4} \right) =$$

$$\left( c_k^2 + \frac{\log \frac{\lambda_k}{\beta c_k}}{\alpha_k} + \frac{8 - 4 \log \delta + 4 \log L}{\alpha_k} \right) + 2c_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} + O \left( \frac{\log c_k}{c_k^2} \right) + O \left( \frac{1}{\beta_k^2} \right);$$

thus we obtain

$$-8\pi^2 A^2 \log \left( \frac{R}{r} \right) = \frac{32\pi^2}{\alpha_k} \left( c_k^2 + \frac{\log \frac{\lambda_k}{\beta c_k}}{\alpha_k} + \frac{8 - 4 \log \delta + 4 \log L}{\alpha_k} \right)$$

$$+ \frac{32\pi^2}{\alpha_k} \left( O \left( \frac{\log c_k}{c_k^2} \right) + O \left( \frac{1}{\beta_k^2} \right) \right)$$

Furthermore using the relation

$$\left( c_k^2 + \frac{\log \frac{\lambda_k}{\beta c_k}}{\alpha_k} + \frac{8 - 4 \log \delta + 4 \log L}{\alpha_k} \right) + 2c_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} + O \left( \frac{\log c_k}{c_k^2} \right) + O \left( \frac{1}{\beta_k^2} \right) =$$

$$\left( c_k^2 + \frac{1}{\alpha_k} \log \lambda_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} - \frac{4}{\alpha_k} \log \delta + \frac{1}{8\pi^2} \frac{dk_\tau \log \delta + 2dk_\tau \log \delta + 2d_k N_k}{\alpha_k} + \frac{4 \log L}{\alpha_k} + \frac{8}{\alpha_k} + o_k(1) \right)$$

we get

$$-8\pi^2 A^2 \log \left( \frac{R}{r} \right) = \frac{32\pi^2}{\alpha_k} \left( c_k^2 + \frac{1}{\alpha_k} \log \lambda_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} - \frac{4}{\alpha_k} \log \delta + \frac{1}{4\pi^2} \frac{dk_\tau \log \delta + 2dk_\tau \log \delta + 2d_k N_k}{\alpha_k} + \frac{4 \log L}{\alpha_k} + \frac{8}{\alpha_k} \right)$$

$$+ \frac{32\pi^2}{\alpha_k} \left( O \left( \frac{\log c_k}{c_k^2} \right) + o_k(1) \right)$$

(3.14)

Next we will evaluate \( \int_{M \setminus B_\delta(x_0)} \Delta_\delta G \Delta_\delta G \, dV_g \). We have that by Green formula

$$\int_{M \setminus B_\delta(x_0)} \Delta_\delta G \Delta_\delta G \, dV_g = \int_{M \setminus B_\delta(x_0)} G \Delta_\delta^2 G \, dV_g - \int_{\partial B_\delta(x_0)} \frac{\partial G}{\partial r} \Delta_\delta G + \int_{\partial B_\delta(x_0)} c \frac{\partial \Delta_\delta G}{\partial r} \, dV.$$

Thus using the equation solved by \( G \) we get

$$\int_{M \setminus B_\delta(x_0)} \Delta_\delta G \Delta_\delta G \, dV_g = - \frac{\tau}{\mu(M)} \int_{M \setminus B_\delta(p)} G \, dV_g - \frac{\tau^2}{64\pi^2} \int_{\partial B_\delta(x_0)} \frac{\partial(- \log r)}{\partial r} \Delta_\delta(- \log r)$$

$$+ \int_{\partial B_\delta(x_0)} \left( - \frac{\tau}{8\pi^2} \log r + S_0 \right) \frac{\partial \Delta_\delta(- \log r)}{\partial r} + O(\delta \log \delta)$$

Hence we obtain

$$\int_{M \setminus B_\delta(x_0)} \Delta_\delta G \Delta_\delta G \, dV_g = - \frac{\tau^2}{16\pi^2} - \frac{\tau^2}{8\pi^2} \log \delta + \tau^2 S_0 + O(\delta \log \delta),$$

Now let us set

$$P(L) = \int_{B_{\epsilon(t)}} |\Delta_0 w|^2 \, dx/(2 \times 32\pi^2)^2.$$ 

Hence using (3.12), (3.13), (3.14), we derive that

$$\frac{32\pi^2}{\alpha_k} \left( c_k^2 + \frac{1}{\alpha_k} \log \lambda_k \frac{N_k + \frac{\pi}{\alpha_k} \log \delta}{\beta_k} - \frac{4}{\alpha_k} \log \delta + \frac{1}{4\pi^2} \frac{dk_\tau \log \delta + 2dk_\tau \log \delta + 2d_k N_k}{\alpha_k} + \frac{4 \log L}{\alpha_k} + \frac{8}{\alpha_k} \right)$$

$$\leq c_k^2 \left( - \frac{\tau^2}{16\pi^2} - \frac{\tau^2}{8\pi^2} \log \delta + \tau S_0 + O(\delta \log \delta) + o_k(1) \right) + \delta^2 O(c_k^2 AB) + \delta^4 O(c_k^2 B^2).$$
Moreover by isolating the term \( \frac{32\pi^2}{\alpha_k^2} \log \frac{\lambda_k}{\beta_k c_k} \) in the left and transposing all the other in the right we get

\[
\frac{32\pi^2}{\alpha_k^2} \log \frac{\lambda_k}{\beta_k c_k} \leq \frac{1}{8\pi^2} (d_k^2 \tau^2 - \frac{64}{\alpha_k} d_k \tau + (\frac{32\pi}{\alpha_k})^2 \log \delta - \frac{32\pi^2}{\alpha_k} (2d_k N_k + \frac{4 \log L}{\alpha_k} + \frac{8}{\alpha_k}) - d_k^2 (P(L) + \tau S_0) - \frac{\tau^2}{16\pi^2} + O(\delta \log \delta) + o_k(1)) + \delta^2 O(c_k^2 AB) + \delta^4 O(c_k^2 B^2).
\]

(3.15)

Hence using the trivial identity

\[
\log \frac{\lambda_k}{\beta_k^2} = \log \frac{\lambda_k}{\beta_k c_k} + \log d_k
\]

we get

\[
\frac{32\pi^2}{\alpha_k^2} \log \frac{\lambda_k}{\beta_k^2} \leq \frac{1}{8\pi^2} (d_k^2 \tau^2 - \frac{64}{\alpha_k} d_k \tau + (\frac{32\pi}{\alpha_k})^2 \log \delta - \frac{32\pi^2}{\alpha_k} (2d_k N_k + \frac{2 + 4 \log L}{\alpha_k} + \frac{2}{\alpha_k}) - d_k^2 (P(L) + \tau S_0) - \frac{\tau^2}{16\pi^2} + O(\delta \log \delta) + o_k(1)) + \frac{32\pi^2}{\alpha_k^2} \log d_k + O(d_k^2).
\]

Now suppose \( d = +\infty \), letting \( \delta \to 0 \), then we have that

\[
\lim_{k \to +\infty} \log \frac{\lambda_k}{\beta_k^2} = -\infty,
\]

thus we derive

\[
\lim_{k \to +\infty} \frac{\lambda_k}{\beta_k^2} = 0
\]

Hence using Corollary 3.8 we obtain a contradiction. So \( d \) must be finite.

On the other hand one can check easily that the following holds

\[
\frac{32\pi^2}{\alpha_k^2} \log \frac{\lambda_k}{\beta_k^2} \leq \frac{1}{8\pi^2} (d_k \tau - \frac{32\pi^2}{\alpha_k})^2 \log \delta + O(1)(d_k^2 + d_k + \log d_k) + O(1).
\]

Hence we derive

\[
d_k \tau \to 1;
\]

otherwise we reach the same contradiction. So we have that

\[
d \tau = 1.
\]

Hence by using this we can rewrite \( B \) as follows

\[
B = \frac{-2c_k + \delta(-\frac{1}{8\pi^2 c_k\delta^2} - \frac{\delta}{4}) + O(1/c_k)}{\delta^2(-\alpha_k c_k^2) + O(1)} = \frac{o_k(1)}{c_k}.
\]

Thus we obtain

\[
32\pi^2 AB(R^2 - r^2) + 32\pi^2 B^2(R^2 - r^4) = \frac{o_k(1)}{c_k^2}.
\]

On the other hand since \( d < +\infty \), we have that by Lemma 3.9

\[
w = -\frac{4 \log(1 + \sqrt{\frac{2}{6} |x|^2})}{d}.
\]

Moreover by trivial calculations we get

\[
P(L) = \frac{1}{96d^2\pi^2} + \frac{\log(1 + \sqrt{\frac{2}{6} \pi L^2})}{16d^2\pi^2}.
\]

Furthermore by taking the limit as \( k \to +\infty \) in (3.15) we obtain

\[
\lim_{k \to +\infty} \log \frac{\lambda_k}{\beta_k c_k} \leq -\frac{25}{3} + 4d\tau + 2d^2\tau^2 + 32\pi^2 S_0 + \frac{4\sqrt{\frac{2}{6} \pi L^2}}{1 + \sqrt{\frac{2}{6} \pi L^2}} + 2\log(1 + \sqrt{\frac{d}{6} \pi L^2}) - 4 \log L
\]
Now letting $L \to +\infty$, we get
\[
\lim_{k \to +\infty} \log \frac{\lambda_k}{\beta_k c_k} \leq \frac{5}{3} - \log 6 + \log \pi^2 + \log d.
\]
Hence by remarking the trivial identity
\[
\lim_{k \to +\infty} \frac{\lambda_k}{\beta_k c_k} \frac{1}{d} \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k}
\]
we get
\[
\tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k} \leq \frac{\pi^2}{6} e^{\frac{A}{2} + 32\pi^2 S_0}.
\]
So the proof of the proposition is done. □

3.5. **The test function.** This Subsection deals with the construction of some test functions in order to reach a contradiction.

Now let $\epsilon > 0$, $c > 0$, $L > 0$ and set
\[
f_\epsilon(x) = \begin{cases} 
\frac{c + \frac{L + B d_\infty(x, x_0)^2 - 4 \log(1 + \lambda d_\infty(x, x_0)^2)}{64\pi^2 c} + \frac{S(x)}{c}}{G(x)} & d_\infty(x, x_0) \leq L\epsilon \\
\frac{d_\infty(x, x_0)}{G(x)} & d_\infty(x, x_0) > L\epsilon
\end{cases}
\]
where
\[
\lambda = \frac{\pi}{\sqrt{6}}, \quad B = -\frac{4}{L^2 e^2 (1 + \lambda L^2)}
\]
and
\[
A = -64\pi^2 c^2 - BL^2 e^2 - 8 \log(L\epsilon) + 4 \log(1 + \lambda L^2).
\]

**Proposition 3.13.** We have that for $\epsilon$ small, there exist suitable $c$ and $L$ such that
\[
\int_M |\Delta_g f_\epsilon|^2 dV_g = 1;
\]
and
\[
\limsup_{\epsilon \to 0} \int_M e^{32\pi^2 (f_\epsilon - \tilde{f}_\epsilon)^2} dV_g > \text{Vol}(M) + \frac{\pi^2}{6} e^{\frac{A}{2} + 32\pi^2 S_0}.
\]

**Proof.** First of all using the expansion of $g$ in normal coordinates we get
\[
\int_{B_{L\epsilon}(0)} |\Delta_g f_\epsilon|^2 dV_g = \int_{B_{L\epsilon}(0)} |\Delta_0 \tilde{f}_\epsilon|^2 (1 + O(L\epsilon)^2) dx + \int_{B_{L\epsilon}(0)} O(\epsilon^2 |\nabla_0 \tilde{f}_\epsilon|^2) dx
\]
where
\[
\tilde{f}_\epsilon(x) = f_\epsilon(\exp_{x_0}(x)).
\]
On the other hand by direct calculations we obtain
\[
\int_{B_{L\epsilon}(0)} |\Delta_0 \tilde{f}_\epsilon|^2 dx = \frac{12 + \lambda L^2 (30 + \lambda L^2 (21 + \lambda L^2)) + 6(1 + \lambda L^3)^3 \log(1 + \lambda L^2)}{96c^2 (1 + \lambda L^2)^3 \pi^2}
\]
Hence we arrive to
\[
\int_{B_{L\epsilon}(0)} |\Delta_g f_\epsilon|^2 dV_g = (1 + O(L\epsilon)^2) \frac{12 + \lambda L^2 (30 + \lambda L^2 (21 + \lambda L^2)) + 6(1 + \lambda L^3)^3 \log(1 + \lambda L^2)}{96c^2 (1 + \lambda L^2)^3 \pi^2}
\]
\[
= \frac{\frac{A}{2} + 4 \log(1 + \lambda L^2) + O(\epsilon^2) + O((L\epsilon)^2 \log L\epsilon)}{32\pi^2 \epsilon^2}
\]
Furthermore, by direct computation, we have
\[
\int_{B_{L\epsilon}(0)} r^2 |\nabla_0 \tilde{f}_\epsilon|^2 dx = O\left(\frac{L^4 \epsilon^4}{c^2}\right).
\]
Moreover using Green formula we get
\[
\int_{M \setminus B_{L\epsilon}(x_0)} |\Delta_g G|^2 dV_g = \int_{M \setminus B_{L\epsilon}(x_0)} G dV_g - \int_{\partial B_{L\epsilon}(x_0)} \frac{\partial G}{\partial r} \Delta_g G dS_g + \int_{\partial B_{L\epsilon}} G \frac{\partial \Delta_g G}{\partial r} dS_g
\]
\[
= -\frac{1}{10\pi} + S_0 - \frac{\log L\epsilon}{8\pi} + O(L\epsilon \log L\epsilon)
\]
Now let us find a condition to have $\int_M |\Delta_g f_\epsilon|^2 dV_g = 1$. By trivial calculations we can see that it is equivalent to

$$\frac{1}{32\pi^2 c^2} \left( -\frac{5}{3} + 2 \log(1 + \lambda L^2) + 32\pi^2 S_0 - 4 \log L\epsilon + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon) \right) = 1.$$

i.e.

$$32\pi^2 c^2 = -\frac{5}{3} + 2 \log(1 + \lambda L^2) + 32\pi^2 S_0 - 4 \log L\epsilon + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon).$$

Hence by (3.16) $\Lambda$ take the following form

$$\Lambda = \frac{10}{3} - 64\pi^2 S_0 + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon).$$

On the other hand it is easily seen that

$$\int_{B_{L\epsilon}(x_0)} f_\epsilon dV_g = O(c(L\epsilon)^4);$$

and

$$\int_{M \setminus B_{L\epsilon}(x_0)} f_\epsilon dV_g = -\int_{B_{L\epsilon}} \frac{G}{c} = O\left(\frac{(L\epsilon)^4 \log L\epsilon}{c}\right).$$

hence

$$\tilde{f}_\epsilon = O(c(L\epsilon)^4).$$

Furthermore by trivial calculations one gets that in $B_{L\epsilon}(x_0)$

$$(f_\epsilon - \tilde{f}_\epsilon)^2 \geq c^2 + \frac{2}{64\pi^2} \left( \Lambda + B r^2 - 4 \log(1 + \lambda(\frac{L\epsilon}{r})^2) + 64\pi^2 S_0 + O(L\epsilon) + O(c^2(L\epsilon)^4) \right)$$

$$= c^2 + \frac{5}{48\pi^2} - \log(1 + \lambda \frac{r^2}{c^2}) + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon) + O(c^2(L\epsilon)^4);$$

hence

$$\int_{B_{L\epsilon}(x_0)} e^{32\pi^2 f_\epsilon} dV_g \geq (1 + O(L\epsilon)^2) \int_{B_{L\epsilon}(x_0)} e^{32\pi^2 \left( c^2 + \frac{5}{48\pi^2} - \log(1 + \lambda \frac{r^2}{c^2}) \right) + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon) + O(c^2(L\epsilon)^4)} dV_g$$

$$= e^{\frac{5}{48\pi^2} + 32\pi^2 c^2 + O\left(\frac{1}{L^2}\right) + O(c^2(L\epsilon)^4)} \left( \frac{\pi^2}{1 + \lambda \frac{r^2}{c^2}} + O(L\epsilon)^2 \right)$$

$$= \pi^2 e^{\frac{5}{48\pi^2} + 32\pi^2 c^2 \pi^2} \left( 1 + O\left(\frac{1}{L^2}\right) + O(L\epsilon \log L\epsilon) + O(L\epsilon)^2 \right)$$

on the other hand

$$\int_{M \setminus B_{L\epsilon}(x_0)} e^{32\pi^2 (f_\epsilon - \tilde{f}_\epsilon)^2} dV_g \geq \int_{M \setminus B_{L\epsilon}(x_0)} (1 + 32\pi^2 (f_\epsilon - \tilde{f}_\epsilon)^2) dV_g$$

$$\geq \text{Vol}(M \setminus B_{L\epsilon}(x_0)) + \int_{M \setminus B_{L\epsilon}(x_0)} \frac{32\pi^2 G^2 dV_g + O(c(L\epsilon)^4)}{c^2}$$

$$= \text{Vol}(M) + \frac{32\pi^2 G^2 dV_g}{c^2} + O((L\epsilon)^4 \log L\epsilon).$$

Thus we arrive to

$$\int_M e^{32\pi^2 f_\epsilon} dV_g \geq \text{Vol}(M) + \frac{\pi^2 e^{\frac{5}{48\pi^2} + 32\pi^2 c^2}}{6} + \frac{\int_{M \setminus B_{L\epsilon}(x_0)} 32\pi^2 G^2 dV_g}{c^2} + O(L\epsilon \log(L\epsilon)) + O\left(\frac{1}{L^2}\right) + O(c^2(L\epsilon)^4);$$

and factorizing by $\frac{1}{c^2}$ we get

$$\int_M e^{32\pi^2 (f_\epsilon - \tilde{f}_\epsilon)^2} dV_g \geq \text{Vol}(M) + \frac{\pi^2 e^{\frac{5}{48\pi^2} + 32\pi^2 c^2}}{6} S_0 + \frac{1}{c^2} \left( \int_M 32\pi^2 G^2 dV_g + O(c^2 L\epsilon \log(L\epsilon)) + O\left(\frac{c^2}{L^2}\right) + O(c^4(L\epsilon)^4) \right).$$
On the other hand setting
\[ L = \log \frac{1}{\epsilon} \]
we get
\[ O(c^2 L \log(L \epsilon)) + O(\frac{c^2}{L^2}) + O(c^4 (L \epsilon)^4) \to 0 \text{ as } \epsilon \to 0. \]
Hence the Proposition is proved.

3.6. **Proof of Theorem 1.1.** This small subsection is concerned about the proof of Theorem 1.1. First of all by corollary we have that
\[ \lim_{k \to +\infty} \int_M \epsilon^k u_k^2 = \text{Vol}_g(M) + \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k} \]
with \( \tau \neq 0. \)
On the other hand from Proposition 3.12 we get
\[ \tau^2 \lim_{k \to +\infty} \frac{\lambda_k}{\beta_k} \leq \frac{\pi^2}{6} e^{\frac{5}{4} + 32\pi^2 S_0}. \]
Hence we obtain
\[ \lim_{k \to +\infty} \int_M \epsilon^k u_k^2 \leq \text{Vol}_g(M) + \frac{\pi^2}{6} e^{\frac{5}{4} + 32\pi^2 S_0}. \]
Thus using the relation
\[ \lim_{k \to +\infty} \int_M \epsilon^k u_k^2 dV_g = \sup_{u \in H_1} \int_M \epsilon^{2\pi^2 u^2} dV_g, \]
we derive
\[ \sup_{u \in H_1} \int_M \epsilon^{2\pi^2 u^2} dV_g \leq \text{Vol}_g(M) + \frac{\pi^2}{6} e^{\frac{5}{4} + 32\pi^2 S_0}. \]
On the other hand from Proposition 3.13 we have the existence of a family of function \( f_\epsilon \) such that
\[ \int_M |\Delta_g f_\epsilon|^2 dV_g = 1; \]
and
\[ \lim_{\epsilon \to 0} \sup_{u \in H_1} \int_M \epsilon^{2\pi^2 (f_\epsilon - f_{\epsilon_0})^2} dV_g > \text{Vol}(M) + \frac{1}{6} e^{\frac{5}{4} + 32\pi^2 S_0 \pi^2}. \]
Hence we reach a contradiction. So the proof of Theorem 1.1 is completed. \( \square \)

4. **Proof of Theorem 1.2**

As already said in the Introduction, in this brief Section we will explain how the proof of Theorem 1.1 remains valid for Theorem 1.2. First of all we remark that all the analysis above have been possible due to the following facts
1) \( \int_M |\Delta_g u|^2 dV_g \) is an equivalent norm to the standard norm of \( H^2(M) \) on \( H_1. \)
2) The existence of the Green function for \( \Delta_g^2. \)
3) The result of Fontana.
On the other hand we have a counterpart of 2) and 3). Moreover it is easy to see that \( \langle P_g^4 u, u \rangle \) is also an equivalent norm to the standard norm of \( H^2(M) \) on \( H_2. \) Notice that for a blowing-up sequence \( u_k \) we have that
\[ \langle P_g^4 u_k, u_k \rangle = \int_M |\Delta_g u_k|^2 dV_g + o_k(1); \] (4.1)
then it is easy to see that the same proof is valid up to the subsection of test functions. Notice that (4.1) holds for the test functions \( f_\epsilon \), then it is easy to see that continuing the same proof we get Theorem 1.2.
References

[1] Adams D.R., *A sharp inequality of J.Moser for higher order derivatives*, Ann. math 128 (1988) 385-398.

[2] Adimurthi and O.Druet, *Blow up analysis in dimension 2 and a sharp form of Trudinger Moser inequality*, Comm. PDE 29 No 1-2, (2004), 295-322.

[3] Beckner W., *Sharp Sobolev inequalities on the sphere and the Moser-trudinger inequality*, Ann. Math 138(1) (1993) 213-242.

[4] Chang S.Y.A., *The Moser-Trudinger inequality and applications to some problems in conformal geometry*, in Nonlinear Partial differential equations in Differential Geometry (Park City, UT, 1992) IAS/Park City Mathematics series, Vol. 2(American Mathematical society, Providence, ri, 1996), pp. 65-125.

[5] Carleson L, Chang S.Y.A., *On the existence of an extremal function for an inequality of J.Moser*, Bull. Sci. Math 110 (1986) 113-127.

[6] Chang S.Y.A., Yang P.C., *Extremal metrics of zeta functional determinants on 4-manifolds*, ann. of Math. 142(1995), 171-212.

[7] Chang S.Y.A., Gursky M.J., Yang P.C., *A conformally invariant sphere theorem in four dimensions*, Publ. Math. Inst. Hautes etudes Sci. 9892003), 105-143.

[8] Djadli Z., Malchiodi A., *Existence of conformal metrics with constant Q-curvature*, to appear in ann. of Math.

[9] Fontana L., *Sharp bordeline sobolev inequalities on compact Riemannian manifolds*, comment. Math.Helv. 68 (1993) 415-454.

[10] Flucher M., *Extremal functions for Trudinger-Moser type inequality in 2 dimensions*, Comment. math.Helv. 67 (1992) 471-497.

[11] Gursky M., *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE*, Comm. Math. Phys. 207-1 (1999), 131-143.

[12] Li J., Li Y., Liu P., *The Q-curvature on a 4-dimensional Riemannian manifold (M, g) with \( \int_M Q dV_g = 8\pi^2 \)*, preprint.

[13] Li Y., Liu P., *Moser-Trudinger inequality on the boundary of compact of compact Riemannian surface*, Math. Z. 250 (2) (2005) 363-386.

[14] Li Y., *Moser-Trudinger inequality on compact Riemannian manifolds of dimension two*, J.Partial Differential equations 14(2) (2001) 1289-1319.

[15] Li Y., *The extremal functions for Moser-trudinger inequality on compact Riemannian manifolds*, Sci. China Series A. Math. 48 (2005) 18-648.

[16] Lin K.C., *Extremal functions for moser’s inequality*, trans. Amer. Math. Soc. 348 (1996) 2663-2671.

[17] Malchiodi A., *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math., to appear.

[18] Malchiodi A., Ndiaye C.B., *Some existence results for the Toda system on closed surfaces*, preprint, 2005

[19] Moser J., *A Sharp form of an inequality of Trudinger*, Ind. univ. math. j. 20 (1971) 1077-1091.

[20] Ndiaye C.B., *Constant Q-curvature metrics in arbitrary dimension* preprint 2006.

[21] Paneitz S., *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[22] Pohozaev S.I., *The Sobolev embedding in the case pl = n*. Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964-1965, Mathematics Section, 158-170, Moskov. Energet. Inst., Moscow, 1965

[23] Struwe M., *Critical points of embedding of \( H^1_0 \) into Orlicz space*, Ann. Inst. Henri., 5(5):425-464, 1988.
[24] Trudinger N.S., on embedding into orlicz space and some applications, J. Math. mech. 17 (1967) 473-484.

[25] Xu Xingwang., Uniqueness and non-existence theorems for conformally invariant equations, Journal of Functional Analysis. 222(2005) 1-28.

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