Isotropic nonarchimedean $S$-arithmetic groups are not left orderable

Groupes $S$-arithmétiques non-archimédiens isotropes ne sont pas ordonnés à gauche

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Abstract

If $O$ is either $\mathbb{Z}[\sqrt{r}]$ or $\mathbb{Z}[1/r]$, where $r > 1$ is any square-free natural number, we show that no finite-index subgroup of $\text{SL}(2, O)$ is left orderable. (Equivalently, these subgroups have no nontrivial orientation-preserving actions on the real line.) This implies that if $G$ is an isotropic $F$-simple algebraic group over an algebraic number field $F$, then no nonarchimedean $S$-arithmetic subgroup of $G$ is left orderable. Our proofs are based on the fact, proved by B. Liehl, that every element of $\text{SL}(2, O)$ is a product of a bounded number of elementary matrices.

Résumé

Si $O$ est soit $\mathbb{Z}[\sqrt{r}]$ ou soit $\mathbb{Z}[1/r]$, où $r > 1$ est un entier positif sans carré, nous prouvons qu’aucun sous-groupe d’indice fini de $\text{SL}(2, O)$ n’est ordonné à gauche. (En d’autres mots, les sous-groupes d’indice fini de $\text{SL}(2, O)$ ne possèdent pas d’action non triviale sur la droite respectant l’orientation.) Cela implique que si $G$ est un groupe algébrique $F$-simple isotrope, défini sur un corps de nombres $F$, alors aucun sous-groupe $S$-arithmétique non-archimédienn de $G$ n’est ordonné à gauche. La démonstration est fondée sur le fait, due à B. Liehl, que chaque élément de $\text{SL}(2, O)$ est le produit d’un nombre borné de matrices élémentaires.
1. Introduction

It is known [8] that finite-index subgroups of SL(3, Z) or Sp(4, Z) are not left orderable. (That is, there does not exist a total order $\prec$ on any finite-index subgroup, such that $ab \prec ac$ whenever $b \prec c$.) More generally, if $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}$-rank $G \geq 2$, then no finite-index subgroup of $G_{\mathbb{Z}}$ is left orderable. It has been conjectured that the restriction on $\mathbb{Q}$-rank can be replaced with the same restriction on $\mathbb{R}$-rank, which is a much weaker hypothesis:

**Conjecture 1** If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}$-rank $G \geq 2$, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}}$ is left orderable.

It is natural to propose an analogous conjecture that replaces $\mathbb{Z}$ with a ring of $S$-integers, and has no restriction on the $\mathbb{R}$-rank:

**Conjecture 2** If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}[1/p_1, \ldots, 1/p_n]}$ is left orderable.

We prove Conjecture 2 under the additional assumption that $\mathbb{Q}$-rank $G \geq 1$:

**Theorem 3** If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}$-rank $G \geq 1$, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}[1/p_1, \ldots, 1/p_n]}$ is left orderable.

More generally, if $G$ is an $F$-simple algebraic group over an algebraic number field $F$, with $F$-rank $G \geq 1$, then no nonarchimedean $S$-arithmetic subgroup $\Gamma$ of $G$ is left orderable.

We also prove some cases of Conjecture 1 (with $\mathbb{Q}$-rank $G = 1$). For example, we consider $\mathbb{Q}$-forms of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$.

**Theorem 4** If $r > 1$ is any square-free natural number, then no finite-index subgroup $\Gamma$ of $\text{SL}(2, \mathbb{Z}[\sqrt{r}])$ is left orderable.

In geometric terms, the theorems can be restated as the nonexistence of orientation-preserving actions on the line:

**Corollary 5** If $\Gamma$ is as described in Theorem 3 or Theorem 4, then there does not exist any nontrivial homomorphism $\varphi: \Gamma \to \text{Homeo}^+(\mathbb{R})$.

Combining this corollary with an important theorem of É. Ghys [3] yields the conclusion that every orientation-preserving action of $\Gamma$ on the circle $S^1$ is of an obvious type; any such action is either virtually trivial or semiconjugate to an action by linear-fractional transformations, obtained from a composition $\Gamma \to \text{PSL}(2, \mathbb{R}) \hookrightarrow \text{Homeo}^+(S^1)$. See [4] for a discussion of the general topic of group actions on the circle.

It has recently been proved that certain individual arithmetic groups are not left orderable (see, e.g., [2]), but our results apparently provide the first new examples in more than ten years of arithmetic groups that have no left-orderable subgroups of finite index. They are also the only known such examples that have $\mathbb{Q}$-rank 1.

The theorems are obtained by reducing to the fact, proved by B. Liehl [5], that if $\mathcal{O} = \mathbb{Z}[1/(p_1 \ldots p_n)]$ or $\mathcal{O} = \mathbb{Z}[\sqrt{r}]$, then $\text{SL}(2, \mathcal{O})$ has bounded generation by unipotent elements. (That is, the fact that $\text{SL}(2, \mathcal{O})$ is the product of finitely many of its unipotent subgroups. For the general case of Theorem 3, we also note that $\Gamma$ contains a finite-index subgroup of $\text{SL}(2, \mathbb{Z}[1/p])$, for some prime $p$.) We are able to prove the same reduction for certain other groups:

**Theorem 6** Suppose $\Gamma$ is a finite-index subgroup of either

(i) $\text{SL}(2, \mathbb{Z}[1/r])$, for some natural number $r > 1$,

(ii) $\text{SL}(2, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field $F$, and $F$ is neither $\mathbb{Q}$ nor an imaginary quadratic extension of $\mathbb{Q}$,

(iii) an arithmetic subgroup of a quasi-split $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group $\text{SL}(3, \mathbb{R})$.

If $\varphi: \Gamma \to \text{Homeo}^+(\mathbb{R})$ is any homomorphism, and $U$ is any unipotent subgroup of $\Gamma$, then every $\varphi(U)$-orbit on $\mathbb{R}$ is bounded.

**Corollary 7** Suppose
- \( \Gamma \) is as described in Thm. 6, and
- \( \Gamma \) is commensurable to a group that has bounded generation by unipotent elements.

Then every homomorphism \( \varphi: \Gamma \to \text{Homeo}^+(\mathbb{R}) \) is trivial. Therefore, \( \Gamma \) is not left orderable.

Assuming a certain generalized Riemann Hypothesis, G. Cooke and P. J. Weinberger [1] proved that the groups described in part (ii) of Thm. 6 do have bounded generation by unipotent elements. Thus, if this generalized Riemann Hypothesis holds, then finite-index subgroups of these groups are not left orderable. See [5] for relevant results on bounded generation that do not rely on any unproved hypotheses, and see [6] for a recent discussion of bounded generation.

2. Proof of Theorem 6(i)

Notation 8 For convenience, let
\[
\pi = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \quad s = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}
\]
for \( u, v \in \mathbb{Z}[1/r] \) and \( s \in \{ r^n \mid n \in \mathbb{Z} \} \).

Suppose some \( \varphi(U) \)-orbit on \( \mathbb{R} \) is not bounded above. (This will lead to a contradiction.) Let us assume \( U \) is a maximal unipotent subgroup of \( \Gamma \).

Let \( V \) be a subgroup of \( \Gamma \) that is conjugate to \( U \), but is not commensurable to \( U \). Then \( V_0 \neq U_0 \). Because \( \mathbb{Q} \)-rank \( \text{SL}(2, \mathbb{Q}) = 1 \), this implies that \( V_0 \) is opposite to \( U_0 \). Therefore, after replacing \( U \) and \( V \) by a conjugate under \( \text{SL}(2, \mathbb{Q}) \), we may assume
\[
U = \{ \pi \mid u \in \mathbb{Z}[1/r] \} \cap \Gamma \quad \text{and} \quad V = \{ v \mid v \in \mathbb{Z}[1/r] \} \cap \Gamma.
\]

Because \( V \) is conjugate to \( U \), we know that some \( \varphi(V) \)-orbit is not bounded above. Let
\[
x_U = \sup \{ x \in \mathbb{R} \mid \text{the } \varphi(U)\text{-orbit of } x \text{ is bounded above} \} < \infty
\]
and
\[
x_V = \sup \{ x \in \mathbb{R} \mid \text{the } \varphi(V)\text{-orbit of } x \text{ is bounded above} \} < \infty.
\]
Assume, without loss of generality, that \( x_U \geq x_V \).

Fix some \( s = r^n > 1 \), such that \( \hat{s} \in \Gamma \), and let \( B = \langle \hat{s} \rangle U \). Because \( \langle \hat{s} \rangle \) normalizes \( U \), this is a subgroup of \( \Gamma \). Note that \( \varphi(B) \) fixes \( x_U \), so it acts on the interval \( (x_U, \infty) \). Since \( \varphi(B) \) is nonabelian, it is well known (see, e.g., [4, Thm. 6.10]) that some nontrivial element of \( \varphi(B) \) must fix some point of \( (x_U, \infty) \). In fact, it is not difficult to see that each element of \( \varphi(B) \) \( \setminus \) \( \varphi(U) \) fixes some point of \( (x_U, \infty) \). In particular, \( \varphi(\hat{s}) \) fixes some point \( x \) of \( (x_U, \infty) \).

The left-ordering of any additive subgroup of \( \mathbb{Q} \) is unique (up to a sign), so we may assume that
\[
\varphi(\pi_1)x < \varphi(\pi_2)x \iff u_1 < u_2 \quad \text{and} \quad \varphi(v_1)x < \varphi(v_2)x \iff v_1 < v_2.
\]
The \( \varphi(U) \)-orbit of \( x \) is not bounded above (because \( x > x_U \) ), so we may fix some \( u_0, v_0 > 0 \), such that
\[
\varphi(\pi_0)x < \varphi(\pi_0)x.
\]
For any \( v \in V \), there is some \( k \in \mathbb{Z}^+ \), such that \( v < s^{2k}v_0 \). Then, because \( \varphi(\hat{s}) \) fixes \( x \) and \( s^{-2k} < 1 \), we have
\[
\varphi(\pi)x < \varphi(s^{2k}v_0)x = \varphi(s^{-k}v_0 s^k)x = \varphi(s^{-k}) \varphi(v_0)x < \varphi(s^{-k}) \varphi(\pi_0)x = \varphi(s^{-2k} u_0)x < \varphi(\pi_0)x.
\]
So the \( \varphi(V) \)-orbit of \( x \) is bounded above by \( \varphi(\pi_0)x \). This contradicts the fact that \( x > x_U \geq x_V \).
3. Other parts of Theorem 6

(ii) The above proof of Case (i) needs only minor modifications to be applied with a ring \( O \) of algebraic integers in the place of \( \mathbb{Z}[1/r] \). (We choose \( s = \omega^n \), where \( \omega \) is a unit of infinite order in \( O \).) The one substantial difference between the two cases is that the left-ordering of the additive group of \( O \) is far from unique — there are infinitely many different orderings. Fortunately, we are interested only in left-orderings of \( U = \{ \overline{u} \mid u \in O \} \cap \Gamma \) that arise from an unbounded \( \varphi(U) \)-orbit, and it turns out that any such left-ordering must be invariant under conjugation by \( \overline{s} \). The left-ordering must, therefore, arise from a field embedding \( \sigma \) of \( F \) in \( \mathbb{C} \) (such that \( \sigma(s) \) is real whenever \( \overline{s} \in \Gamma \)), and there are only finitely many such embeddings. Hence, we may replace \( U \) and \( V \) with two conjugates of \( U \) whose left-orderings come from the same field embedding (and the same choice of sign).

(iii) A serious difficulty prevents us from applying the above proof to quasi-split \( \mathbb{Q} \)-forms of \( \text{SL}(3,\mathbb{R}) \). Namely, the reason we were able to obtain a contradiction is that if \( \overline{u_0} \) is upper triangular, \( v \) is lower triangular, \( \overline{s} \) is diagonal, and \( \lim_{k \to \infty} \overline{s}^{-k} \overline{u_0} \overline{s}^k = \infty \) under an ordering of \( \Gamma \), then \( \lim_{k \to \infty} \overline{s}^{-k} \overline{v} \overline{s}^k = 1 \). Unfortunately, the “opposition involution” of \( \text{SL}(3,\mathbb{R}) \) causes the calculation to result in a different conclusion in case (iii): if \( \overline{s}^{-k} \overline{u_0} \overline{s}^k \) tends to \( \infty \), then \( \overline{s}^{-k} \overline{v} \overline{s}^k \) also tends to \( \infty \). Thus, the above simple argument does not immediately yield a contradiction.

Instead, we employ a lemma of M. S. Raghunathan [7, Lem. 1.7] that provides certain nontrivial relations in \( \Gamma \). These relations involve elements of both \( U \) and \( V \); they provide the crucial tension that leads to a contradiction.

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References

[1] G. Cooke and P. J. Weinberger: On the construction of division chains in algebraic number rings, with applications to \( \text{SL}_2 \). Comm. Algebra 3 (1975), 481–524.
[2] M. K. Dąbkowski, J. H. Przytycki and A. A. Togha: Non-left-orderable 3-manifold groups (preprint). arXiv:math.GT/0302098
[3] É. Ghys: Actions de réseaux sur le cercle. Invent. Math. 137 (1999), no. 1, 199–231.
[4] É. Ghys: Groups acting on the circle, Enseign. Math. (2) 47 (2001), no. 3–4, 329–407.
[5] B. Liehl: Beschränkte Wortlänge in \( \text{SL}_2 \). Math. Z. 186 (1984), no. 4, 509–524.
[6] V. K. Murty: Bounded and finite generation of arithmetic groups, in: K. Dilcher, ed., Number theory (Halifax, NS, 1994), pp. 249–261. CMS Conf. Proc., #15, Amer. Math. Soc., Providence, RI, 1995.
[7] M. S. Raghunathan: On the congruence subgroup problem, II, Invent. Math. 85 (1986) 73–117.
[8] D. Witte: Arithmetic groups of higher \( \mathbb{Q} \)-rank cannot act on 1-manifolds. Proc. Amer. Math. Soc. 122 (1994), no. 2, 333–340.