Sinc Numerical Methods for Time Nonlocal Parabolic Equation

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Abstract. In recent years, more and more researchers have paid attention to the study of non-local problems. The numerical method for initial-boundary value problems of time nonlocal parabolic equations is established in this paper. The time nonlocal operator is discretized by finite difference method, and spatial differential operators is discretized by Sinc-Galerkin method. Then fully discrete scheme (D-SD scheme) for solving one-dimensional time nonlocal parabolic equation is obtained. Numerical example shows the effectiveness and superiority of the scheme for solving non-local problems.

1. Introduction

In recent years, nonlocal models have often been used to describe long memory processes or some nonlocal natural phenomena in materials science, ecology, neurology, and so on[1-4]. Non-local problems can be divided into two forms: the first is that the reaction term of the model is a integration of space or time variables, and the second is that the boundary conditions of the model have non-local operators [6,7]. This paper will focus on the numerical methods for the first form of non-local problems. Consider the following initial-boundary value problems for time-nonlocal Parabolic Equations:

\[
\begin{align*}
Z_\delta u(x,t) - \Delta u(x,t) &= f(x,t) &\Omega \times [0,T], \\
u(x,t) &= 0 &\partial \Omega \times [0,T], \\
u(x,t) &= g(x,t) &\Omega \times [-\delta,0].
\end{align*}
\]

Where \(\Omega\) is a bounded polygon domain in \(R^d\), \(\partial \Omega\) is its boundary; \(g(x,t)\) and \(f(x,t)\) are given functions. When a nonnegative symmetric kernel function \(\rho_\delta(s) = \rho_\delta(\|s\|)\) is given, the related nonlocal operator \(Z_\delta\) is defined as follows:

\[
Z_\delta v(t) = \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^\delta (v(t) - v(t-s)) s \rho_\delta(s) ds
\]

Generally, the exact analytical solution of non-local diffusion equation can not be obtained, so finding numerical methods becomes an urgent need to solve such equation. In the numerical calculation of non-local problems, the finite element method and the finite difference method are commonly used to discretize the non-local operator [8, 9]. In 2017, Chen et al. [10] established the full discrete scheme of the space-time nonlocal problem by using the finite difference method to discretize the time nonlocal operator, and using the Fourier spectrum method to discretize the space nonlocal operator, and carried on the numerical simulation. In 2017, Du et al. [11] used the classical Galerkin finite element method
to discretize the spatial variables of time non-local parabolic equation, obtained the semi-discrete scheme, and obtained the error estimates according to the numerical smoothness. Du et al. [12] used the implicit Runge-Kutta (IRK) method to discretize time nonlocal operators in 2019, and carried out numerical simulation. In the same year, Du et al. [13] discretized the time integral by orthogonal finite difference method, applied stable ETD approximation to space, obtained the full discrete numerical scheme of the nonlocal Allen-Cahn (NAC) equation, and pointed out that the approximate solution always converged to the classical Allen-Cahn solution.

Sinc numerical method has the characteristics of high accuracy and exponential convergence speed, and has been widely used in the numerical solution of differential equations. For example, Zarebnia and Sajjadian [14] proposed the Sinc-Galerkin method for Troesch problem. The convergence of the method is analyzed and it is proved that the method is exponentially convergent. Some numerical examples are implemented, and the numerical results are compared with the homotopy perturbation method, Laplace method, perturbation method and spline method. In 2014, El-Gamel and Mohsen et al. [15] proposed a Sinc method for solving biharmonic problems. The literature shows that the Sinc method is a very effective tool for numerical solution of this equation. Even if the singular points appear on the boundary, the Sinc method can get good results. By comparison, the accuracy of Sinc method is higher than that of LM, LGSM, DM and BSM. However, there are few papers on Sinc method to solve non-local problems. In this paper, the Sinc method is applied to the numerical solution of the initial-boundary value problem for one-dimensional time-nonlocal parabolic equation. Combining the difference approximation of time nonlocal operator, the numerical scheme of Sinc-Galerkin method is constructed. Experiment shows that this method is of high accuracy and exponential convergence.

2. Time Semi-Discrete Scheme and Its Local Truncation Error

We use a uniform time step \( \tau = T / S \) to divide the interval \([0, T]\), \( t_m = m\tau, m = 0, 1, 2, \cdots, S \). For discrete time nonlocal operators, interval \((0, \delta)\) is also divided by step size \( \tau \). Getting grid nodes \( t_k = k\tau, k = 0, 1, \cdots, r, \delta = r\tau + \delta_0 \), where \( \delta_0 \in (0, \tau] \). Noting intervals \( I_k = ((k - 1)\tau, k\tau) \), \( 1 \leq k \leq r \), \( I_{r+1} = (r\tau, \delta) \). Then there is

\[
Z_\delta u(x, t_m) = Z_\delta u^m(x) = \sum_{k=0}^{r+1} \int_{I_k} (u(x, t_m) - u(x, t_m - s)s \rho_\delta(s))ds.
\]

where \( u^m(x) = u(x, t_m) \). If the finite difference method is used to discretize \( Z_\delta u^m(x) \) [10], then we have the approximate formula

\[
Z_\delta u^m(x) \approx Z_\delta^1 u^m(x) = \sum_{k=1}^{r+1} w_k \frac{u^m(x) - u^{m-k}(x)}{k\tau}.
\]

where \( m = 1, 2, \cdots, S \), and \( w_k = \int_{I_k} s^2 \rho_\delta(s)ds \), \( k = 1, 2, \cdots, r + 1 \).

It is known from reference [12] that the local truncation error of formula (3) is

\[
\left| Z_\delta u(x, t_m) - Z_\delta^1 u^m(x) \right| \leq C \tau \left\| u_m(x, t) \right\|_{\mathcal{C}^1([-\delta, T])}.
\]

By substituting formula (3) into equation (1), a time semi-discrete scheme for the initial-boundary value problem (1) of one-dimensional time nonlocal parabolic equation is obtained

\[
\begin{align*}
\sum_{k=1}^{r+1} w_k \frac{u^m(x) - u^{m-k}(x)}{k\tau} & - u^m_{xx}(x) = f^m(x), \quad x \in (a, b), \\
u^m(a) &= u^m(b) = 0, \quad m = 0, 1, 2, \cdots, S, \\
u^m(x) &= g^m(x), \quad m = -\delta, -\delta_r, -\delta_{r-1}, \cdots, -1.
\end{align*}
\]

and the local truncation error is \( O(\tau) \).
3. Full Discrete Schemes for One-Dimensional Time Nonlocal Parabolic Equation

According to the theory of Sinc function approximation [16] and Ref. [17], assume that the approximate solution of problem (5) be

\[ u^m(x) = \sum_{j=-M}^{N} u^m_j S_j(x), \]  

where

\[ S_j(x) = S(jh) \circ \phi(x), \quad S(x, h)(x) = \text{Sinc}\left(\frac{x - kh}{h}\right), \quad \phi(x) = \ln\left(\frac{x - a}{b - x}\right), \]

\[ x_j = \phi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}, \quad N = \left\lfloor \frac{\alpha_0 M + 1}{\beta_0 M} \right\rfloor, \quad h = \frac{\pi d}{\alpha_0 M}, \quad u^m_j \approx u(x_j, t_m). \]

\( M \) is the given positive integer, \( \alpha_0, \beta, d \) are given constants and their values are given in Ref.[16]. Here the inner product is defined as

\[ (f, g) = \int_a^b f(x)g(x)w(x)dx, \]

and select \( w = \frac{1}{\phi} \). By the Galerkin method and the properties of Sinc function[16,17], then we can obtain the Sinc-Galerkin fully discrete scheme for one-dimensional time nonlocal problem (1):

\[ \sum_{k=1}^{r+1} W_k u_{m-k}^m - \sum_{j=-M}^{N} \left[ \frac{\delta^{(2)}_{jk}}{h^2} + \frac{\delta^{(1)}_{jk}}{h} \left( \frac{\phi''(x_j)}{(\phi'(x_j))^2} + 2 \frac{1}{\phi'(x_j)} \right) \right] u^m_j = \sum_{k=-M}^{N} W_k u^m_{m-k} + f^m(x_j). \]  

(7)

Where

\[ \delta^{(0)}_{jk} = [S(jh) \circ \phi(x)]_{x=x_j}, \quad \delta^{(1)}_{jk} = \frac{d}{d\phi} [S(jh) \circ \phi(x)]_{x=x_j}, \quad \delta^{(2)}_{jk} = \frac{d^2}{d\phi^2} [S(jh) \circ \phi(x)]_{x=x_j}, \]

\[ \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \]

\[ \begin{cases} \frac{\pi^2}{3}, & j = k, \\ -2(-1)^{k-j} \frac{1}{(k-j)^2}, & j \neq k, \end{cases} \]

\( j, k = -M, \ldots, N \).

Let’s simply note this scheme as D-SG scheme, its matrix form is

\[ \left[ \sum_{k=1}^{r+1} W_k I^{(0)} - \frac{1}{h^2} I^{(2)} D((\phi')^2) - \frac{1}{h} I^{(1)} D(\phi')^2 + \frac{2}{h} D(\phi')^2 \right] \tilde{u}^m = \sum_{k=1}^{r+1} W_k \tilde{u}^m_{m-k} + \tilde{f}^m. \]  

(8)

where

\[ \tilde{u}^m = (u^m_{-M}, u^m_{-M+1}, \ldots, u^m_0, \ldots, u^m_N)^T, \]

\[ \tilde{f}^m = (f^m(x_{-M}), f^m(x_{-M+1}), \ldots, f^m(0), \ldots, f^m(x_N))^T, \]

\[ I^{(l)} = [\delta^{(l)}_{jk}], l = 0, 1, 2, \quad D(g) = \text{diag}(g(x_j), j = -M, \cdots, N). \]
4. Numerical Experiment

For the one-dimensional nonlocal parabolic equation initial-boundary value problem (1), we select \( \Omega = (0,1), f(x,t) = 0, g(x,t) = (x-x^2)e^t \), \( \rho_s(s) = (\alpha+1)s^{\alpha-2}\delta^{-\alpha-1}, \alpha > -1 \), then

\[
w_k = \left(\frac{\tau}{\delta}\right)^{\alpha+1} \left[k^{\alpha+1} - (k-1)^{\alpha+1}\right], \quad k = 1, 2, ..., r, \quad w_{r+1} = 1 - \left(\frac{r+1}{\delta}\right)^{\alpha+1}.
\]

When \( \delta \to 0 \), the problem degenerates into the classical diffusion problem, and the exact solution is

\[
u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n\pi} e^{-n^2\pi^2t} \sin(n\pi x).
\] (9)

Table 1 gives the numerical solution of the D-SG scheme at \( x = 0.5 \) when \( \delta = 0.0001, \alpha = 0.8, T = 0.5, M \) takes different values, and the exact solution of the degeneration problem (14) at this point. We can see that, when \( M = 32 \), the accuracy of numerical solution is very good, so we take \( M = 32, N = 3M / 2, h = \pi / \sqrt{3M} \) in the following calculation.

Tab.1 The Comparison of numerical solution of D-SG scheme with exact solution of degradation problem for different \( M \)

| \( \alpha \) | \( M \) | \( N = \frac{3M}{2} \) | \( h = \frac{\pi}{\sqrt{3M}} \) | \( x = 0.5 \) | numerical solution | exact solution of (14) |
|---|---|---|---|---|---|---|
| 0.8 | 2 | 3 | 1.283 | 1.3996e-03 |
| | 4 | 6 | 0.907 | 1.6475e-03 |
| | 8 | 12 | 0.641 | 1.8195e-03 |
| | 16 | 24 | 0.453 | 1.8737e-03 |
| | 32 | 48 | 0.3206 | 1.8918e-03 |

Fig. 1 shows the comparison between the numerical solution obtained by formula (35) and the exact solution of degenerate problem (14) when \( M = 32, N = 3M / 2, h = \pi / \sqrt{3M}, \alpha = 0.8, T = 0.5, \delta \) takes different values. It can be seen that with the decrease of \( \delta \), the numerical solution of problem (12) is more and more close to the exact solution of degenerate problem (14). When \( \delta = 0.0001 \), the numerical solution of the problem (12) almost coincides with the exact solution of the degenerate problem (14) (Table 1 also shows this), which shows that D-SG Scheme is reliable.

Fig.1 The comparison of the numerical solutions of nonlocal problem with the exact solution of the classical diffusion problem when \( \delta \to 0 \)
Fig. 2 (a) is the numerical solution $u(x,t)$ of the problem (12) when $\delta = 0.8, \alpha = 0.5, T = 1.0$, Fig. 2 (b) is the initial value $g(x,t)$, and Fig. 2 (c) is the graph that draws $u(x,t)$ and $g(x,t)$ together. We can see the particularity of the non-local model.

In this example, the value of $\delta$ has an influence on the definition domain of the initial value function $g(x,t)$ and the kernel function $\rho_\delta(s)$. Fig. 3 shows the numerical solution at $T = 0.5$ when $\alpha = 0.5$, $\delta = 0.1, 0.5, 1.0$, respectively. It can be seen that with the increase of $\delta$, the solution $u(x,T)$ also increases. The value of $\alpha$ only influence the kernel function $\rho_\delta(s)$. Fig. 4 shows the curve of $u(0.5,0.5)$ versus $\alpha$ when $\delta = 0.5$. We can see that, the function value first increases with the increase of $\alpha$, and then decreases with the increase of $\alpha$ after about $\alpha = 0.4$. 

Fig. 2 Numerical solutions and initial values when $\delta = 0.8, \alpha = 0.5, T = 1.0$

Fig. 3 The influence of the values of $\delta$ on the numerical solutions when $T = 0.5$

Fig. 4 The curve of $u(0.5,0.5)$ versus $\alpha$
5. Conclusion

In this paper, the Sinc numerical method for one-dimensional time nonlocal parabolic equation is studied. The time nonlocal operator is discretized by finite difference method. Considering the good numerical approximation property of Sinc function, the spatial differential operator is discretized by the Sinc-Galerkin method. Then Sinc-Galerkin fully discrete scheme (DG scheme) is established. The numerical example shows that, when $\delta \to 0$, the numerical solution of the scheme tend to the exact solution of the degeneration problem, which accords with the theoretical analysis; and shows the effectiveness of the scheme proposed in this paper. The influence of parameters on the numerical solution is analysed, and the mechanism of nonlocal operators is explored. The results of this paper show that the Sinc numerical method is effective for solving non-local parabolic equations, and can be used for reference in the establishment of numerical methods for other non-local problems.

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