Directed percolation in two dimensions: 
An exact solution

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Abstract

We consider a directed percolation process on an $M \times N$ rectangular lattice whose vertical edges are directed upward with an occupation probability $y$ and horizontal edges directed toward the right with occupation probabilities $x$ and 1 in alternate rows. We deduce a closed-form expression for the percolation probability $P(x, y)$, the probability that one or more directed paths connect the lower-left and upper-right corner sites of the lattice. It is shown that $P(x, y)$ is critical in the aspect ratio \( \alpha = M/N \) at a value $\alpha_c(x, y) = [1 - y^2 - x(1 - y)^2]/2y^2$ where $P(x, y)$ is discontinuous, and the critical exponent of the correlation length for $\alpha < \alpha_c(x, y)$ is $\nu = 2$.

Key words: Directed Percolation, Critical behavior.

An outstanding unsolved problem in stochastic processes is the consideration of directed percolation [1, 2]. Directed percolation is a Markovian bond percolation process in which bonds are directed such that only clusters with a “flow” are relevant. Very few exact results of directed percolation are known. In 1981 Domany and Kinzel [3] solved one version of a directed percolation where the occupation probability is fixed at unity in one spatial direction of a rectangular lattice. The problem was subsequently reformulated and solved as a random walk by one of us and Stanley [4]. However, the Domany-Kinzel model is essentially of a one-dimensional nature due to the restricted freedom
in one spatial direction. To uncover the genuine nature of a two-dimensional directed percolation it is necessary to relax this uni-directional restriction.

As a first step toward this goal we consider in this paper a directed percolation in which the unity percolation probability occurs in every other row of a rectangular lattice. We deduce a closed-form expression for the percolation probability and analyze its critical properties for large lattices.

We first describe our model. Consider a 2-dimensional rectangular net of \((M + 1) \times (2N + 1)\) sites with an aspect ratio

\[
\alpha = M/2N. \tag{1}
\]

Number the sites by \((m, n)\) with \(m = 0, 1, \ldots M, \ n = 0, 1, \ldots 2N\) as shown in Fig. 1. Consider a bond percolation process on the lattice with vertical edges occupied with a probability \(p_y = y\) and horizontal edges in the \(n\)-th row occupied with a probability

\[
p_x = 1, \quad n = \text{odd} \\
= x, \quad n = \text{even}. \tag{2}
\]

Direct edges in the upward direction and toward the right. Occupied edges form directed paths if traced along the arrows. In ensuing discussions we shall refer to percolation configurations as bond configurations. A bond configuration is percolating if it contains one or more directed paths connecting the two opposite corner sites \((0, 0)\) and \((M, 2N)\). A typical percolating configuration is shown in Fig. 1.

In a bond configuration there are \(n_x\) (resp. \(MN - n_x\)) occupied (resp. empty) horizontal edges, and \(n_y\) (resp. \(2(M + 1)N - n_y\)) occupied (resp. empty) vertical edges. Then the percolation probability, the probability that a bond configuration is percolating, is

\[
P_{M,2N}(x, y) = \sum_{\text{perc conf}} x^{n_x} (1 - x)^{MN - n_x} y^{n_y} (1 - y)^{2(M + 1)N - n_y} \tag{3}
\]

where the summation is restricted to percolating bond configurations. It is clear that \(0 \leq P_{M,2N}(x, y) \leq 1\) since the summation (3) is identically 1 if unrestricted. It is also clear that \(P_{\infty,2N}(x, y) = 1\) and \(P_{0,2N}(x, y) = 0\). Our interest is to investigate how does \(P\) change from 1 to 0 as \(\alpha\) varies, and whether the change is a sharp transition.
We state the main result as a Proposition:

**Proposition:**

For any \( x \in [0, 1] \) and \( y \in (0, 1) \), there exists a critical aspect ratio

\[
\alpha_c(x, y) = \frac{[1 - y^2 - x(1 - y)^2]}{2 y^2}
\]

such that

\[
\lim_{N \to \infty} P_{2\alpha N, 2N}(x, y) = \begin{cases} 
1 & \text{if } \alpha > \alpha_c(x, y) \\
0 & \text{if } \alpha < \alpha_c(x, y) \\
\frac{1}{2} & \text{if } \alpha = \alpha_c(x, y).
\end{cases}
\]

Moreover, for \( \alpha < \alpha_c(x, y) \), we have the asymptotic behavior

\[
P(2\alpha N, 2N) \sim e^{-2N/\xi}
\]

where

\[
\xi \sim (\alpha_c - \alpha)^{-2}.
\]

**Remarks:**

1. Equation (6) defines \( \xi \) as the correlation length and Eq. (7) gives the correlation length critical exponent \( \nu = 2 \).

2. For \( x = 1 \) our model reduces to the Domany-Kinzel model on an \((M + 1) \times (2N + 1)\) lattice and (4) leads to \( \alpha_c = (1 - y)/y \) in agreement with previous result. For \( x = 0 \) our model is again a Domany-Kinzel model but on an \((M + 1) \times (N + 1)\) lattice with a vertical edge occupation probability \( y^2 \). Our result gives the critical aspect ratio \( 2\alpha_c = (1 - y^2)/y^2 \) again in agreement with [3, 4].

**Proof of the Proposition:**

The main body of this paper is the proof of the Proposition.

There are \( 2N \) rows of vertical edges in the lattice. Number these rows from 1 to \( 2N \) starting from the bottom. An occupied vertical edge in a bond configuration is wet if it lies on a percolating path connecting \((0, 0)\) and \((M, 2N)\), and is primary wet if it is the first wet edge (in a row of vertical edges) counting from the left. In the bottom row of vertical edges in Fig. 1, for example, there are two wet edges and the primary wet edge is the one
connecting sites (1, 0) and (1, 1). In a percolating configuration there is one primary wet edge in every row and these edges carry an overall occupation probability \( y^{2N} \). Since a bond configuration is percolating whenever a vertical edge in the \( 2N \)-th row is primary wet, which can occur at any of the \( m \)-th horizontal positions \( m = 0, 1, \ldots, M \), we have

\[
P_{M,2N}(x, y) = y^{2N} \sum_{m=0}^{M} w_{m,2N}. \tag{8}
\]

Here \( y^{2n} w_{m,2n} \) is the probability that the primary wet edge in the \( (2n) \)-th row occurs at the horizontal position \( m \).

We first establish a Lemma:

**Lemma:**

\[
w_{m,2n} = \frac{1}{2\pi i} \oint \frac{dt}{t^{m+1}(1 - at + bt^2)^n} \tag{9}
\]

where the contour of integration is around the unit circle and

\[
a = 1 - y^2 + x(1 - y)^2, \quad b = x(1 - y)^2. \tag{10}
\]

**Proof of the Lemma:**

It is not difficult to see that the function \( w_{m,2n}(x, y) \) satisfies the recursion relation

\[
w_{m,2n} = \sum_{k=0}^{m} w_{k,2} w_{m-k,2n-2} \tag{11}
\]

and the initial condition

\[
w_{m,0} = \delta_{K_1}(m, 0). \tag{12}
\]

Define generating functions

\[
W_1(t) = \sum_{m=0}^{\infty} w_{m,2} t^m \tag{13}
\]

\[
W_2(t, s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w_{m,2n} t^m s^n. \tag{14}
\]

Substituting (11) into (14) and changing the order of summation by using \( \sum_{m=0}^{\infty} \sum_{n=0}^{m} = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \), we obtain after some rearrangement and the use of (12),

\[
W_2(t, s) = 1 + s W_1(t) W_2(t, s)
\]
which yields

\[ W_2(t, s) = \frac{1}{1 - sW_1(t)}. \]  

(15)

We can now invert (14) to obtain

\[ w_{m,2n} = \frac{1}{(2\pi i)^2} \oint dt \frac{ds}{t^{n+1} s^{n+1}} \left( \frac{1}{1 - sW_1(t)} \right)^n, \]

(16)

where the contour of integration is around the unit circle.

To compute \( W_1(t) \) we need to evaluate \( w_{m,2}(x, y) \) for an \((m + 1) \times 3\) lattice. There are now 2 rows of vertical edges. As aforementioned \( y^2w_{m,2} \) is the probability that \((0, 0)\) is connected to \((m, 2)\) with the primary wet vertical edge in the top row occurring at \(m\). However the primary wet vertical edge in the bottom row can be at any \(j\) in \(0 \leq j \leq m\). Denote the probability for this to occur by \( y^2l_j(1 - y)^{m-j}x^{m-j} \). Then we have

\[ w_{m,2} = \sum_{j=0}^{m} l_j(1 - y)^{m-j}x^{m-j}, \]  

(17)

where the factor \((1 - y)^{m-j}x^{m-j}\) ensures that the primary wet edge in the top row is at \(m\) as shown in Fig 2(a). Particularly, we have \( w_{0,2} = \lambda_0 = 1 \).

The factor \( l_j \) in (17) satisfies a recursive relation which can be written as

\[ l_j = (1 - y)l_{j-1} + y(1 - x)(1 - y)w_{j-1,2}, \quad j = 1, 2, \ldots, m. \]  

(18)

The two terms on the right-hand side of (18) arise from the two possibilities that the vertical edge connecting \((j - 1, 0)\) and \((j - 1, 1)\) is either empty (with probability \(1 - y\)) or occupied (with probability \(y\)) as shown in the two panels in Fig. 2(b). In the latter case the factor \((1 - x)(1 - y)\) ensures that the site \((j - 1, 1)\) is not on a percolating path.

To solve the coupled recursion relations (17) and (18), define the generating function

\[ \Lambda(t) = \sum_{j=0}^{\infty} \lambda_j t^j. \]
Multiplying (17) and (18) by $t^m$ and $t^{j-1}$, respectively, and summing over $m$ and $j - 1$ from 0 to $\infty$, we obtain after some manipulation

$$W_1(t) = \frac{y^2 \Lambda(t)}{1 - x(1 - y)t},$$

$$\frac{1}{t} \left( \Lambda(t) - 1 \right) = (1 - y) \Lambda(t) + y(1 - x)(1 - y)W_1(t). \quad (19)$$

This gives

$$W_1(t) = \frac{1}{1 - at + bt^2} \quad (20)$$

after eliminating $\Lambda(t)$ where $a, b$ are given in (10). The substitution of (20) into (16) establishes the Lemma.

We now continue the proof of the Proposition.

Substitute (9) into (8) and carry out the summation in $m$. This leads to

$$P_{M,2N}(x, y) = \frac{y^{2N}}{2\pi i} \oint_{C^+} \frac{dt}{(t - 1)(1 - at + bt^2)^N} \left( 1 - \frac{1}{t^{M+1}} \right) \quad (21)$$

where the contour $C^+$ is a circle enclosing the unit circle. Let $t_1$ and $t_2$ be the two roots of $1 - at + bt^2 = 0$, both of which are real. We have $t_1 t_2 = 1/b, t_1 + t_2 = a/b$, and hence

$$(t_1 - 1)(t_2 - 1) = t_1 t_2 - (t_1 + t_2) + 1 = \frac{1}{b} - \frac{a}{b} + 1 = \frac{y}{(1 - x)^2} > 0,$$

so both $t_1$ and $t_2$ lie outside the unit circle. We can therefore choose the radius of $C^+$ to be greater than 1 but smaller than both $t_1$ and $t_2$ so that $C^+$ encloses only the simple pole $t = 1$ in (21). It follows that the first term on the right-hand side of (21) picks up only the residue at $t = 1$ which is

$$\frac{y^{2N}}{(1 - a + b)^N} = 1,$$

and we obtain

$$P_{M,2N}(x, y) = 1 - I_{M,N}$$
where

\[ I_{M,N} = \frac{y^{2N}}{2\pi i} \oint_{C^+} \frac{dt}{(t-1)t^{M+1}(1-\alpha t + \beta t^2)^N}. \]  

(22)

Note that since \(|t| > 1\) along \(C^+\) \((22)\) leads to the expected result \(P_{\infty,2N} = 1\).

To further evaluate \(I_{M,N}\) we introduce \(z = 1/t\) to write

\[ I_{M,N} = \frac{y^{2N}}{2\pi i} \oint_{C^-} \frac{z^{M+2N}dz}{(z-1)(z^2 - \alpha z + \beta)^N}. \]  

(23)

where the contour \(C^-\) is now within the unit circle.

For \(M, N\) large and fixed aspect ratio \(\alpha = M/N\), we can rewrite \((23)\) as

\[ I_{M,N} = \frac{1}{2\pi i} \oint_{C^-} \frac{dz}{z - 1} \left[ f_{\alpha}(z) \right]^N \]  

(24)

where

\[ f_{\alpha}(z) = \frac{y^2z^{2+\alpha}}{z^2 - \alpha z + \beta}. \]

The integral \(I_{M,N}\) can be evaluated using the method of steepest descent [5, 6] by deforming the contour to pass a point \(z = z_0\) where \(f_{\alpha}(z)\) is stationary. To the leading order this gives \(I_{M,N} \sim [f_{\alpha}(z_0)]^N\). Moreover, since \(I_{M,N} \leq 1\), we must have \(f_{\alpha}(z_0) \leq 1\) with the equal sign holding at \(f_{\alpha}(z_0) = 1\). Thus a transition occurs at \(z_0 = 1\).

Now

\[ f'_{\alpha}(z) = \frac{y^2z^{1+\alpha}}{(z^2 - \alpha z + \beta)^2} \left[ \alpha z^2 - (1 + \alpha)\alpha z + (2 + \alpha)\beta \right] \]

and the stationary point \(z_0\) is determined by

\[ \alpha z_0^2 - (1 + \alpha)\alpha z_0 + (2 + \alpha)\beta = 0. \]

The critical condition \(z_0 = 1\) now gives

\[ \alpha = \frac{a - 2b}{1 - a + b} = \alpha_c(x, y) \]  

(25)

where \(\alpha_c(x, y)\) is given in [4]. It is readily verified that we have \((d\alpha/dz)_{z=1} < 0\) along \((25)\). Thus, for \(\alpha > \alpha_c(x, y)\), the stationary point \(z_0\) lies within
the unit circle so we can deform $C$ continuously to pass $z_0$, and obtain $I_{2\alpha N,N} = [f_\alpha(z_0)]^N \sim 0$. This gives $P_{2\alpha N,N}(x,y) \sim 1$ which establishes the first line of (5).

On the other hand, for $\alpha < \alpha_c(x,y)$, $z_0$ occurs outside the unit circle and when the contour $C$ is deformed to pass $z_0$ it must cross the simple pole at $z = 1$ and picks up the residue at the pole, which is equal to 1. This gives $I_{2\alpha N,N} \sim 1 - [f_\alpha(z_0)]^N$ and $P_{2\alpha N,N}(x,y) \sim [f_\alpha(z_0)]^N \sim 0$ for large $N$. This establishes the second line of (5).

For $\alpha = \alpha_c(x,y)$, $z_0$ is on the unit circle so the crossing of the contour at $z = 1$ picks up only half of the residue, namely, $1/2$. This establishes the third line of (5).

Finally, for $\alpha < \alpha_c(x,y)$, the method of steepest decent [5, 6] dictates that we have

$$[f_\alpha(z_0)]^N = e^{N\ln[f_\alpha(z_0)]} \sim e^{-N C_1(x,y)(z_0-1)^2} \sim e^{-N C_2(x,y)(\alpha - \alpha_c)^2}$$

where expressions of $C_1(x,y)$ and $C_2(x,y)$, which do not affect our conclusions, can be explicitly evaluated. This establishes the asymptotic behavior with $\xi = 2/[C_2(x,y)(\alpha - \alpha_c)^2]$.

We have completed the proof of the Proposition.

In summary, we have obtained a closed-form expression for the percolation probability $P_{M,2N}(x,y)$ for the directed percolation process in which the occupation probability is $y$ in the vertical direction and alternately $x$ and 1 in the horizontal direction. For $M, N$ large, the percolation probability exhibits a critical behavior at $\alpha = \alpha_c$. The correlation length $\xi$ for $\alpha < \alpha_c$ is found to diverge with the critical exponent $\nu = 2$. While these properties are similar to those found in the Domany-Kinzel model [3, 4], our analysis permits the relaxation of the restriction of unit occupation probability in one spatial direction. It is hoped that the analysis serves as the first step of further relaxation in percolation probabilities, eventually leading to an understanding of genuine 2-dimensional directed percolation processes.

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Figure Captions

Fig. 1. A typical percolating configuration on a $6 \times 5$ lattice ($M = 5, N = 2$). Open circles denote lattice sites. Oriented edges are occupied with weights shown. Empty edges carry weights $1 - x$ and $1 - y$ in horizontal and vertical directions respectively.

Fig. 2. Construction of recursion relations. (a) Construction of (17). (b) Construction of (18). Occupied edges are shown as oriented edges; dotted edges can be either occupied or empty. To each row of vertical edges there is an additional factor $y$ not shown.
Figure 1
Figure 2