Integrable open-boundary conditions for the $q$-deformed supersymmetric $U$ model of strongly correlated electrons

Anthony J. Bracken, Xiang-Yu Ge *, Yao-Zhong Zhang † and Huan-Qiang Zhou ‡

Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia

A general graded reflection equation algebra is proposed and the corresponding boundary quantum inverse scattering method is formulated. The formalism is applicable to all boundary lattice systems where an invertible R-matrix exists. As an application, the integrable open-boundary conditions for the $q$-deformed supersymmetric $U$ model of strongly correlated electrons are investigated. The diagonal boundary K-matrices are found and a class of integrable boundary terms are determined. The boundary system is solved by means of the coordinate space Bethe ansatz technique and the Bethe ansatz equations are derived. As a sideline, it is shown that all R-matrices associated with a quantum affine superalgebra enjoy the crossing-unitarity property.

PACS numbers: 71.20.Fd, 75.10.Jm, 75.10.Lp

I. INTRODUCTION

In the last decade, much attention has been paid to the study of strongly correlated electron systems. In particular, the discovery of high-$T_c$ superconductivity has greatly stimulated investigations on various electron lattice models in one dimension (1D), which are exactly solvable by means of the coordinate Bethe ansatz method or the quantum inverse scattering method (QISM). The most known and studied integrable electron systems are perhaps the Hubbard and the supersymmetric $t$-$J$ models. Other integrable correlated electron theories of interest include, for instance, the so-called extended Hubbard model [1], the Hubbard-like model [2], the supersymmetric $U$ model [3] and its eight-state version [4], and the $q$-deformed supersymmetric $U$ model [5,6]. These models have been extensively investigated in the literature (c.f.: [7–9]).

On the other hand, one of the recent developments in the theory of completely integrable lattice models is Sklyanin’s work [10] on the boundary QISM, which may be used to describe integrable systems on a finite interval with independent boundary conditions on each end. The important ingredient in this boundary QISM is the new algebraic structure, the reflection equation (RE) algebra. The solutions to the REs are called boundary K-matrices which in turn give rise to boundary conditions compatible with the bulk model integrability.

In Sklyanin’s formulation on the boundary QISM, the R-matrices are assumed to enjoy $P$-, $T$- and crossing-symmetry. Sklyanin’s results are generalized by Mezincescu et al [11] to treat cases where, instead of having the separate $P$- and $T$-symmetry, the R-matrices satisfy, in addition to the crossing-symmetry, the less restrictive condition of the combined $PT$ invariance. It is noted by de Vega et al [12] that the formalism of Mezincescu et al is actually also applicable even if the R-matrices only have the so-called crossing-unitarity, a weaker version of the crossing-symmetry.

Generalizations to the following two cases seem very interesting: (i) the graded or supersymmetric case, and (ii) the case where the R-matrices do not obey any constraint conditions (except the unitarity condition). Many attempts have been made in the literature concerning the extension (i). However, very few authors treat the grading properly from the beginning to the end. Therefore, in the eyes of the present authors most known results in the literature are not very satisfactory and a fully graded or supersymmetric formalism is desirable. Concerning (ii), two important examples are the R-matrices introduced in [9] corresponding to the models of two coupled and three coupled XY spin chains introduced in [4], to which the formalism developed in [10,11] does not apply.

In section II of this paper, we shall fulfill the two aims of extensions. More specifically, we shall formulate a fully supersymmetric boundary QISM which is applicable to any cases where an invertible R-matrix exists. We introduce a very general graded RE algebra and show that this algebra indeed leads to a commuting family of the boundary transfer matrices. Throughout the procedure, no spectral parameter multiplicativity/additivity of the graded quantum Yang-Baxter equation (QYBE) has been assumed and no constraint conditions on the R-matrices (except the always

*E-mail: xg@maths.uq.edu.au
†Queen Elizabeth II Fellow. E-mail: yzz@maths.uq.edu.au
‡On leave of absence from Dept of Physics, Chongqing University, Chongqing 630044, China. E-mail: hqzhou@cqu.edu.cn
satisfied unitarity property) been imposed. Our formalism is a supersymmetric generalization of that developed in [13,14] for the bosonic (or non-supersymmetric) case, where open-boundary integrable conditions for the models of two coupled and three coupled one-dimensional $XY$ spin chains have been constructed, respectively. We then use, in section III, our results to study integrable open-boundary conditions for the $q$-deformed supersymmetric $U$ model [8] of strongly correlated electrons. The quantum integrability of the model in the bulk has been established in [8] by embedding the bulk model Hamiltonian into a one-parameter family of commuting transfer matrices formed from a $U_q[gl(2|1)]$ invariant R-matrix. We solve the graded REs for the diagonal boundary K-matrices and determine the open-boundary integrable model. The boundary model Hamiltonian is shown to be related to the second derivative of the boundary transfer matrix. In section IV, we solve this boundary model by the coordinate space Bethe ansatz method and derive the Bethe ansatz equations. In the Appendix, we show that all R-matrices associated with a quantum affine superalgebra possess the crossing-unitarity.

II. GRADED REFLECTION EQUATIONS AND TRANSFER MATRIX: GENERAL FORMULATION

In this section, we establish a very general RE algebra. We shall not assume spectral-parameter multiplicativity/additivity of the graded QYBE, although for the special example we shall discuss in section III the spectral parameters are multiplicative/additive. Throughout the procedure, no constraint conditions have been assumed on the R-matrices.

To begin with, let $V$ be a finite-dimensional $\mathbb{Z}_2$ graded linear superspace. Let $R \in End(V \otimes V)$ be a solution to the graded QYBE

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \quad (II.1)$$

Here $R_{jk}(u)$ denotes the matrix on $V \otimes V \otimes V$ acting on the $j$-th and $k$-th superspaces and as an identity on the remaining superspace. The variables $u_1$, $u_2$ and $u_3$ are spectral parameters. The tensor product should be understood in the graded sense, that is the multiplication rule for any homogeneous elements $x, y, x', y' \in V$ is given by

$$(x \otimes y)(x' \otimes y') = (-1)^{|x||x'|} (xx' \otimes yy') \quad (II.2)$$

where $|x|$ stands for the grading of element $x$: i.e. $|x| = 0$ if $x$ is even (or bosonic) and $|x| = 1$ if $x$ is odd (or fermionic). Let $P$ be the graded permutation operator in $V \otimes V$. Then $P(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for all homogeneous $x, y \in V$ and $R_{21}(u) = P_{12}R_{12}(u)P_{12}$. We form the monodromy matrix $T(u)$ for an $L$-site lattice chain

$$T(u) = L_{0L}(u) \cdots L_{01}(u), \quad (II.3)$$

where $L_{0j}(u) \equiv R_{0j}(u, 0)$, where subscript 0 labels the auxiliary superspace $\mathbb{V}$. Indeed, one may show that $T(u)$ generates a representation of the graded quantum Yang-Baxter algebra,

$$R_{12}(u_1, u_2) \frac{1}{T}(u_1) \frac{2}{T}(u_2) = \frac{2}{T}(u_2) \frac{1}{T}(u_1)R_{12}(u_1, u_2), \quad (II.4)$$

where $\frac{1}{X} = X \otimes 1$ and $\frac{2}{X} = 1 \otimes X$, for any matrix $X \in End(V)$.

In order to construct integrable electronic models with open boundary conditions, we need to introduce an appropriate graded RE algebra. We introduce two associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the R-matrices and the relations

$$R_{12}(u_1, u_2) \frac{1}{T_-}(u_1)R_{21}(u_2, -u_1) \frac{2}{T_-}(u_2) = \frac{2}{T_-}(u_2)R_{12}(u_1, -u_2) \frac{1}{T_-}(u_1)R_{21}(-u_2, -u_1), \quad R_{21}^{st_i}(u_2, u_1) \frac{1}{T_+}(u_1)\tilde{R}_{12}(-u_1, u_2) \frac{2}{T_+}(u_2) = \frac{2}{T_+}(u_2)\tilde{R}_{21}(-u_2, u_1) \frac{1}{T_+}(u_1)R_{12}^{ist_i}(-u_1, -u_2), \quad (II.5)$$

where we have defined new objects $\tilde{R}$ and $\tilde{R}$ through the relations

$$\tilde{R}_{12}^{st_i}(u_1, -u_2)R_{21}^{st_i}(u_2, -u_1) = 1, \quad \tilde{R}_{21}^{ist_i}(u_2, -u_1)R_{12}^{ist_i}(u_1, -u_2) = 1 \quad (II.6)$$

and, as defined in the Appendix, $st_i$ stands for the supertransposition taken in the $i$-th space, whereas $ist_i$ is the operation inverse to $st_i$. As will become clear below, one of the important steps towards formulating a correct
formalism for the graded or supersymmetric case is to introduce, in the second RE in (II.5) below, the inverse operation of the supertransposition. The introduction of this inverse operation is essential because, applying the same supertransposition operation twice does not in general give an identity operation. For instance, applying $st_i$, $i = 1, 2$, twice to the R-matrix $R(u)$ does not yield $R(u)$:

$$
\{ R(u)^{st_i} \}^{st_i} \neq R(u), \quad i = 1, 2.
$$

(II.7)

In all cases, the quantum R-matrices possess the unitarity:

$$
R_{12}(u_1, u_2)R_{21}(u_2, u_1) = 1.
$$

(II.8)

We now show that the second RE in (II.5) is indeed the correct “conjugation” to the first one, so that the boundary transfer matrices defined as usual constitute a commuting family. Following Sklyanin’s arguments [10], one defines the boundary transfer matrix $t(u)$ as

$$
t(u) = \text{str}(T_+(u)T_-(u)),
$$

(II.9)

where $\text{str}$ denotes the supertrace taken over the auxiliary superspace $V$. Then it can be shown that

$$
[t(u_1), t(u_2)] = 0.
$$

(II.10)

The proof is elementary. We nevertheless present the details. By means of the commutativity of operators in $T_+$ and $T_-$,

$$
t_1(u_1)t_2(u_2) = \text{str}_1\{ T_+ (u_1) \ T_-(u_1) \} \text{str}_2\{ T_+ (u_2) \ T_-(u_2) \}
$$

$$
= \text{str}_1\{ T_+^{st_1} (u_1) \ T_-^{st_1} (u_1) \} \text{str}_2\{ T_+ (u_2) \ T_- (u_2) \}
$$

$$
= \text{str}_2\{ T_+^{st_2} (u_1) \ T_-^{st_2} (u_1) \} \text{str}_1\{ T_+ (u_2) \ T_- (u_2) \} = \cdots.
$$

Inserting a variant of the first expression in (II.4) for “1” into the supertrace,

$$
\cdots = \text{str}_2\{ T_+^{st_1} (u_1) \ T_-^{st_1} (u_1) \} \text{str}_1\{ R_{12}(-u_1, u_2) R_{21}^{st_1} (u_2, -u_1) \ T_-^{st_1} (u_1) \ T_- (u_2) \} = \cdots,
$$

then applying the supertransposition,

$$
\cdots = \text{str}_2\{ T_+^{st_1} (u_1) \hat{R}_{12} (-u_1, u_2) \ T_-^{st_2} (u_2) \} \text{str}_1\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1}
$$

$$
= \text{str}_2\{ T_+^{st_1} (u_1) \hat{R}_{12} (-u_1, u_2) \ T_-^{st_2} (u_2) \}^{st_1} \text{str}_1\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1}
$$

$$
= \cdots,
$$

and, finally, inserting the unitarity property into the supertrace one obtains

$$
\cdots = \text{str}_2\{ T_+^{st_1} (u_1) \hat{R}_{12} (-u_1, u_2) \ T_-^{st_2} (u_2) \}^{st_1} \text{str}_1\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1}
$$

$$
\times R_{12}(u_1, u_2)\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1}
$$

$$
= \text{str}_2\{ R_{21}^{st_1} (u_1) \ T_-^{st_1} (u_1) \hat{R}_{12} (-u_1, u_2) \ T_-^{st_2} (u_2) \}^{st_1} \text{str}_1\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1}
$$

$$
\times R_{12}(u_1, u_2)\{ T_- (u_1) R_{21}(u_2, -u_1) \ T_- (u_2) \}^{st_1} = \cdots.
$$

Applying the RE algebra (II.5),

1 One can always normalize the R-matrices so that the right hand side of (II.8) equals to 1
\[
\cdots = s t r_{12} \left\{ T_{+}^{1/2} (u_2) T_{21}^{1/2} (-u_2, u_1) T_{-}^{1/2} (u_1) \right\} (u_1) R_{12}^{ist_1} (u_1) R_{12}^{ist_2} (-u_1, -u_2) \right] \]

\[
= s t r_{12} \left\{ T_{+}^{1/2} (u_2) T_{21}^{1/2} (-u_2, u_1) T_{-}^{1/2} (u_1) \right\} (u_1) R_{12}^{ist_1} (u_1) R_{12}^{ist_2} (-u_1, -u_2) \right]
\]

Repeating the whole chain of transformation in the reverse order and keeping in mind the second expression in (II.6) for \(1\), one ends up with \(t(21) / t(u_1)\), as required.

One can obtain a class of realizations of the superalgebras \(T_+\) and \(T_-\) by choosing \(T_{\pm}(u)\) to be the form

\[
T_{-}(u) = T_{-}(u) K_{-}(u) T_{-}^{-1} (-u), \quad T_{+}(u) = K_{+}(u)
\]

with

\[
T_{-}(u) = R_{0N}(u) \cdots R_{01}(u),
\]

where \(N\) is any node between 1 and \(L\) (including 1 and \(L\)), and \(K_{\pm}(u)\), called boundary K-matrices, are representations of \(T_k\). In this realization, the elements of the matrices \(K_{\pm}(u)\) are all Grassmann numbers. Note that the K-matrices \(K_{\pm}(u)\) satisfy the same relations as \(T_{\pm}(u)\), respectively. That is, the K-matrices fulfill the following graded REs deduced from (II.5)

\[
R_{12}(u_1, u_2) \frac{1}{1} K_{-1}(u_1) R_{21}(u_1, u_2) \frac{2}{2} K_{-1}(u_2) R_{12}(u_1, u_2) \frac{2}{2} K_{-1}(u_1) R_{21}(-u_2, u_1),
\]

\[
R_{21}^{ist_1} (u_1) R_{12}^{ist_2} (u_2) = K_{+1}^{ist_1} (u_1) R_{12}^{ist_2} (u_1, u_2) \frac{1}{1} K_{+1}^{ist_1} (u_1) R_{12}^{ist_2} (-u_1, -u_2).
\]

The REs (II.13) and their realizations (II.13) in the Grassmann algebra are generalizations of those introduced in [13]. Note that no constraint conditions have been imposed on the R-matrices. Therefore our graded REs apply to any case where an invertible R-matrix exists.

Let us now consider the important “special” case in which the R-matrices are related to finite dimensional representations \(\pi_v\) of the quantum affine superalgebra \(U_q (\mathfrak{g}(k))\) \((k = 1, 2)\) for generic \(q\), where \(\mathfrak{g}\) is any simple Lie superalgebra. In the Appendix, we show that all such R-matrices enjoy the crossing-unitarity properties [see (A.11) or (A.14)]. One can show that there exists a more general realization of the superalgebra \(T_{+}\)

\[
T_{+}^{ist_1}(u) = T_{+}^{ist_1}(u) K_{+}^{ist_1}(u), \quad T_{+}(u) = R_{0L}(u) \cdots R_{0N+1}(u).
\]

In the sequel, without loss of generality, we shall choose \(N = L\) so that \(T_{+}(u) \equiv K_{+}(u)\). As the spectral parameters in the QYBE are now multiplicative/additivity, the K-matrices fulfill the REs of the following form,

\[
R_{12}(u_1 - u_2) \frac{1}{1} K_{-1}(u_1) R_{21}(u_1 + u_2) \frac{2}{2} K_{-1}(u_2) R_{12}(u_1 + u_2) \frac{1}{1} K_{-1}(u_1) R_{21}(u_1 - u_2),
\]

\[
R_{12}(u_1 - u_2) \frac{1}{1} K_{+1}^{ist_1}(u_1) R_{21}(u_1 - u_2) \frac{2}{2} K_{+1}^{ist_2}(u_2) R_{12}(u_1 - u_2) \frac{1}{1} K_{+1}(u_1) R_{21}(-u_1 + u_2),
\]

where in deriving the second relation we have applied the operation \(ist_1 st_2\) on both side of the second equation of (II.13).

By (II.6),

\[
\tilde{R}_{21}^{ist_1 st_2} (-u_1 - u_2) = \left((R_{21} (-u_1 - u_2)^{-1} \right)^{-1}) st_2,
\]

\[
\tilde{R}_{12}^{ist_1 st_2} (-u_1 - u_2) = \left((R_{12} (-u_1 - u_2)^{-1} \right)^{-1}) st_1.
\]

With the help of (A.11), (A.3) and (A.12) (one should identify \(z = q^{2g}\) to convert the multiplicative spectral parameter \(z\) into the additive one \(u\)), one can show that the second RE in (II.13) becomes

\[
R_{12}(u_1 + u_2) \frac{1}{1} K_{+}(u_1) R_{21}(-u_1 - u_2) \frac{2}{2} q \hat{M} \frac{1}{1} \frac{2}{2} K_{+}(u_2)
\]

\[
= \hat{M} K_{+}(u_2) R_{12}(-u_1 - u_2) \frac{2}{2} q \hat{M} \frac{1}{1} K_{+}(u_1) R_{21}(-u_1 + u_2),
\]

where \(M = \pi_v(q^{-2g})\) is the so-called crossing matrix and \(g\) is defined as in the Appendix. The RE (II.17) coincides, in the case of \(k = 1\), with the one used in [13] and is applicable to all cases whose R-matrices are related to the quantum affine superalgebra \(U_q (\mathfrak{g}(k))\) for generic \(q\).
Let $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ denote fermionic creation and annihilation operators with spin $\sigma$ at site $j$, which satisfy the anti-commutation relations given by $\{c_{j,\sigma},c_{j,\tau}\} = \delta_{\sigma\tau}\delta_{ij}$, where $i, j = 1, 2, \cdots, L$ and $\sigma, \tau = \uparrow, \downarrow$. We consider the $q$-deformed supersymmetric $U$ model with boundary terms described by the following Hamiltonian:

$$H = \sum_{j=1}^{L-1} H^Q_{j,j+1} + B_L + B_R,$$

(III.1)

where $H^Q_{j,j+1}$ is the bulk Hamiltonian density of the $q$-deformed supersymmetric $U$ model. Then

$$H^Q_{j,j+1} = -\sum_\sigma \left( c_{j,\sigma}^\dagger c_{j+1,\sigma} + h.c. \right) \exp\left( -\frac{1}{2} (\bar{\eta} - \sigma \gamma) n_{j,-\sigma} - \frac{1}{2} (\eta + \sigma \gamma) n_{j+1,-\sigma} \right)$$

$$+ \frac{U}{2} (n_{j,\uparrow} n_{j,\downarrow} + n_{j+1,\uparrow} n_{j+1,\downarrow})$$

$$+ t_p (c_{j,\uparrow}^\dagger c_{j+1,\downarrow} c_{j+1,\uparrow} + h.c.) + \mu \ n_j + \mu^{-1} n_{j+1},$$

(III.2)

and $B_L$ and $B_R$ are left and right boundary terms, respectively, given by

$$B_L = -\frac{\mu - \mu^{-1}}{2 \sinh \gamma (2-\xi_\pm)} \left( \begin{array}{c} \sinh \gamma n_{L\uparrow} n_{L\downarrow} - e^{-\gamma (1-\frac{\xi_\pm}{2})} n_{L\uparrow} \\ \sinh \frac{\gamma}{2} n_{L\downarrow} \end{array} \right),$$

$$B_R = -\frac{\mu - \mu^{-1}}{2 \sinh \gamma (2-\xi_\pm)} \left( \begin{array}{c} \sinh \gamma n_{R\uparrow} n_{R\downarrow} - e^{\gamma (1-\frac{\xi_\pm}{2})} n_{R\downarrow} \\ \sinh \frac{\gamma}{2} n_{R\uparrow} \end{array} \right),$$

(III.3)

where $n_{j,\sigma}$ is the density operator $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$, $n_j = n_{j,\uparrow} + n_{j,\downarrow}$ and

$$t_p = \frac{U}{2} = \epsilon (2e^{-\eta} (\cosh \eta - \cosh \gamma))^{\frac{1}{2}}, \quad \epsilon = \pm,$$

$$\mu = \sqrt{\frac{\sinh (\eta - \gamma)/2}{\sinh (\eta + \gamma)/2}};$$

(III.4)

$\xi_\pm$ are some parameters describing the boundary effects. It is interesting to note that the Hamiltonian (III.1) becomes $U_q[gl(2|1)]$-invariant in the limits $\xi_- \to -\infty$, $\xi_+ \to \infty$.

We shall establish quantum integrability for the system defined by the Hamiltonian (III.1), by applying the general formalism developed in the previous section. To this end, let us first of all recall some basic results of the $q$-deformed supersymmetric $U$ model with the periodic boundary conditions. In [5], it was shown that the bulk Hamiltonian (III.2) of the model commutes with the bulk transfer matrix $\tau(u)$, which is the supertrace of the monodromy matrix $T(u)$ with the local monodromy matrix $L_{0j}(u) = R_{0j}(u)$. That is,

$$\tau(u) = \text{str}(T(u)), \quad T(u) = R_{0L}(u) \cdots R_{01}(u),$$

(III.5)

where the quantum R-matrix $R(u) \equiv P\tilde{R}(u)$, with

$$\tilde{R}(u) = \frac{q^u - q^{2\alpha}}{1 - q^{u+2\alpha}} \tilde{P}_1 + \tilde{P}_2 + \frac{1 - q^{u+2\alpha+2}}{q^u - q^{2\alpha+2}} \tilde{P}_3,$$

(III.6)

where $\tilde{P}_i$, $i = 1, 2, 3$ are the projection operators whose explicit formulae may be found in [5].

In order to describe integrable systems with the boundary conditions different from the periodic ones, we first solve the REs for the two boundary K-matrices $K_\pm(u)$. For our purpose, we only look for solutions where $K_\pm(u)$ are diagonal. After complicated algebraic manipulations, we find

$$K_-(u) = \frac{1}{\sinh \frac{\gamma}{2} \sinh \frac{\gamma (1-\xi_-)}{2}} \left( \begin{array}{cccc} A_-(u) & 0 & 0 & 0 \\ 0 & B_-(-u) & 0 & 0 \\ 0 & 0 & B_-(u) & 0 \\ 0 & 0 & 0 & C_-(u) \end{array} \right),$$

(III.7)
where

\[ A_-(u) = e^{\gamma u} \sinh \frac{\gamma (\xi_- + u)}{2} \sinh \frac{\gamma (u - 2 + \xi_-)}{2}, \]
\[ B_-(u) = \sinh \frac{\gamma (\xi_- - u)}{2} \sinh \frac{\gamma (u - 2 + \xi_-)}{2}, \]
\[ C_-(u) = e^{-\gamma u} \sinh \frac{\gamma (\xi_- - u)}{2} \sinh \frac{\gamma (-u - 2 + \xi_-)}{2}. \]

(III.8)

As is shown in the Appendix, the R-matrix (III.6) satisfies the crossing-unatritiy condition. This implies that there is an isomorphism between the graded REs for \( K_+ \) and \( K_- \):

\[ K_+(u) = MK_-(u - 1), \]

(III.9)

where \( M \) is given by (up to an overall factor)

\[ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\gamma} & 0 \\ 0 & 0 & 0 & e^{2\gamma} \end{pmatrix}. \]

(III.10)

Therefore we may choose the boundary K-matrix \( K_+(u) \) as

\[ K_+(u) = \begin{pmatrix} A_+(u) & 0 & 0 & 0 \\ 0 & B_+(u) & 0 & 0 \\ 0 & 0 & C_+(u) & 0 \\ 0 & 0 & 0 & D_+(u) \end{pmatrix}, \]

(III.11)

with

\[ A_+(u) = e^{-\gamma (u-1)} \sinh \frac{\gamma (2 - 2\alpha - \xi_+ - u)}{2} \sinh \frac{\gamma (2\alpha + \xi_+ + u)}{2}, \]
\[ B_+(u) = e^{-2\gamma} \sinh \frac{\gamma (-2\alpha - \xi_+ + u)}{2} \sinh \frac{\gamma (2\alpha + \xi_+ + u)}{2}, \]
\[ C_+(u) = \sinh \frac{\gamma (-2\alpha - \xi_+ + u)}{2} \sinh \frac{\gamma (2\alpha + \xi_+ + u)}{2}, \]
\[ D_+(u) = e^{\gamma (u-1)} \sinh \frac{\gamma (-2\alpha - \xi_+ + u)}{2} \sinh \frac{\gamma (2\alpha + \xi_+ - u)}{2}. \]

(III.12)

Indeed, we have checked that this \( K_+ \) matrix constitutes a solution to the graded REs (III.17).

To show that the Hamiltonian (III.11) can be embedded into the boundary transfer matrix \( t(u) \) constructed in section II is an involved algebraic manipulation. This is because the superter of \( K_+(0) \) is equal to zero. So at best we can only expect that the Hamiltonian (III.11) appears as the second derivative of the boundary transfer matrix with respect to the spectral parameter \( u \), at \( u = 0 \).

Let us expand the local monodromy matrix \( L_{0j}(u) \) up to the second order in the spectral parameter \( u \),

\[ L_{0j}(u) = (1 + H_{j0}u + \frac{1}{2!} B_{j0}u^2 + \cdots) L_{0j}(0). \]

(III.13)

Substituting this expression into the boundary transfer matrix \( t(u) \), and after a lengthy but straightforward algebraic calculation, one finds

\[ t(u) = C_1 u + C_2 (H + \text{const.}) u^2 + \cdots, \]

(III.14)

where \( C_i (i = 1, 2, \cdots) \) are some scalar functions of the boundary constant \( \xi_+ \). Then it can be shown that up to some additive constants the Hamiltonian (III.11) is related to the second derivative of the boundary transfer matrix,

\[ H = -\frac{q^{\alpha + 1} - q^{-\alpha - 1}}{\ln q} H^R, \]

\[ H^R = \frac{\partial^2 (0)}{4(V + 2W)} + \sum_{j=1}^{L-1} H^R_{0j} \gamma_{0j} + \frac{1}{2} K^R_{0} (0) + \frac{1}{2 (V + 2W)} \left[ \text{str}_0 \left( \frac{0}{0} K_+ (0) G_{L0} \right) \right] + 2 \text{str}_0 \left( \frac{0}{0} K_+ (0) H_{L0}^R \right) + \text{str}_0 \left( \frac{0}{0} K_+ (0) (H_{L0}^R)^2 \right). \]

(III.15)
where

\[ V = \text{str}_0 K'_+(0), \quad W = \text{str}_0 \left( \frac{\delta}{K_+ (0) H^R_{L0}} \right), \]

\[ H^R_{i,j} = P_{i,j} R^R_{i,j}(0), \quad G_{i,j} = P_{i,j} R^R_{i,j}(0), \tag{III.16} \]

if we make the following identifications:

\[ q = e^\gamma, \quad \frac{q^{\alpha+1} - q^{-\alpha-1}}{q^\gamma - q^{-\gamma}} = e^{-\eta}. \tag{III.17} \]

Thus, we have shown that the Hamiltonian (III.1) of the \( q \)-deformed supersymmetric \( U \) model with the boundary terms \( B_L \) and \( B_R \) is related to a class of commuting transfer matrices. As a result, the system has an infinite number of higher conserved currents which are involutive with each other, and therefore the system under study is completely integrable.

### IV. THE BETHE ANSATZ EQUATIONS

Having established the quantum integrability of the model, let us in this section solve it by using the coordinate space Bethe ansatz method. Following [17,16,14], we assume that the eigenfunction of Hamiltonian (III.1) takes the form

\[ |\Psi\rangle = \sum_{\{p(x_1, x_2)\}} \Psi_{\sigma_1, \ldots, \sigma_N}(x_1, \ldots, x_N) e^{i x_1 \sigma_1} \cdots e^{i x_N \sigma_N} |0\rangle, \tag{IV.1} \]

where the summation is taken over all permutations and negations of \( k_1, \ldots, k_N \), and \( Q \) is the permutation of the \( N \) particles such that \( 1 \leq x_{Q1} \leq \cdots \leq x_{QN} \leq L \). The symbol \( \epsilon_p \) is a sign factor \( \pm 1 \) and changes its sign under each 'mutation'. Substituting the wavefunction into the eigenvalue equation \( H |\Psi\rangle = E |\Psi\rangle \), one gets

\[ A_{\ldots, \sigma_i, \ldots}(\ldots, k_j, k_i, \ldots) = S_{ij}(k_i, k_j) A_{\ldots, \sigma_j, \ldots}(\ldots, k_i, k_j, \ldots), \]

\[ A_{\ldots, \sigma_i, \ldots}(\ldots, -k_j, \ldots) = s^L(k_j; p_{1\sigma_i}) A_{\ldots, \sigma_i, \ldots}(\ldots, k_j, \ldots), \]

\[ A_{\ldots, \sigma_i, \ldots}(\ldots, -k_j, \ldots) = s^R(k_j; p_{L\sigma_i}) A_{\ldots, \sigma_i, \ldots}(\ldots, k_j, \ldots), \tag{IV.2} \]

where \( S_{ij}(k_i, k_j) \) are the two-particle scattering matrices,

\[ S_{ij}(k_i, k_j)^{11}_{12} = S_{ij}(k_i, k_j)^{22}_{21} = 1, \]

\[ S_{ij}(k_i, k_j)^{12}_{12} = S_{ij}(k_i, k_j)^{21}_{21} = \sin(\lambda_i - \lambda_j) \sin(i \gamma), \]

\[ S_{ij}(k_i, k_j)^{12}_{21} = e^{-i (\lambda_i - \lambda_j)} \sin(i \gamma), \]

\[ S_{ij}(k_i, k_j)^{21}_{12} = e^{i (\lambda_i - \lambda_j)} \sin(i \gamma) \tag{IV.3} \]

with \( \lambda_j \) being suitable particle rapidities related to the quasi-momenta \( k_j \) of the electrons by [3]

\[ k(\lambda) = \begin{cases} \Theta(\lambda, a), & \epsilon = +, \\ \pi - \Theta(\lambda, a), & \epsilon = - \end{cases} \]

\[ \Theta(\lambda, a) = 2 \arctan(\coth a \tan \lambda), \]

\[ a = \frac{1}{4} \left\{ \ln \left[ \frac{\sinh \frac{1}{2}(\eta + \gamma)}{\sinh \frac{1}{2}(\eta - \gamma)} \right] - \gamma \right\}, \tag{IV.4} \]

and \( s^L(k_j; p_{1\sigma_i}), s^R(k_j; p_{L\sigma_i}) \) are the boundary scattering matrices,
the R-matrices are associated with finite-dimensional representations of the quantum affine superalgebra. Moreover, the Bethe ansatz equations are derived by means of the coordinate space Bethe ansatz approach. This problem can be solved using the algebraic Bethe ansatz method. The Bethe ansatz equations are

\[
\begin{align*}
\mathcal{S}^L(k_j; p_{1\sigma}) & = \frac{1 - p_{1\sigma} e^{ik_j}}{1 - p_{1\sigma} e^{-ik_j}} \\
\mathcal{S}^R(k_j; p_{L\sigma}) & = \frac{1 - p_{L\sigma} e^{-ik_j}}{1 - p_{L\sigma} e^{ik_j}} e^{2ik_j(L+1)}
\end{align*}
\]

(IV.5)

with

\[
\begin{align*}
p_{1\alpha} & \equiv p_1 = -\mu^{-1} + \frac{\mu - \mu^{-1}}{2 \sinh \frac{\gamma(1 - \xi_{1\alpha})}{2}} e^{-\gamma(1 - \xi_{1\alpha})}, \\
p_{L\alpha} & \equiv p_L = -\mu + \frac{\mu - \mu^{-1}}{2 \sinh \frac{\gamma(1 - \xi_{L\alpha})}{2}} e^{\gamma(1 - \xi_{L\alpha})}.
\end{align*}
\]

(IV.6)

Then, the diagonalization of Hamiltonian (III.1) reduces to solving the following matrix eigenvalue equation

\[
T_j t = t, \quad j = 1, \ldots, N,
\]

(IV.7)

where \( t \) denotes an eigenvector on the space of the spin variables and \( T_j \) takes the form

\[
T_j = S^{-}_j(k_j) s^L(-k_j; p_{1\sigma}) R^{-}_j(k_j) R^+_j(k_j) s^R(k_j; p_{L\sigma}) S^+_j(k_j)
\]

(IV.8)

with

\[
\begin{align*}
S^+_j(k_j) & = S_{j,N}(k_j, k_N) \cdots S_{j,j+1}(k_j, k_{j+1}), \\
S^-_j(k_j) & = S_{j,j-1}(k_j, k_{j-1}) \cdots S_{j,1}(k_j, k_1), \\
R^-_j(k_j) & = S_{1,j}(k_1, -k_j) \cdots S_{j-1,j}(k_{j-1}, -k_j), \\
R^+_j(k_j) & = S_{j+1,j}(k_{j+1}, -k_j) \cdots S_{N,j}(k_N, -k_j).
\end{align*}
\]

(IV.9)

This problem can be solved using the algebraic Bethe ansatz method. The Bethe ansatz equations are

\[
\begin{align*}
e^{ik_j 2(L+1)} \zeta(k_j; p_1) \zeta(k_j; p_L) & = \frac{1}{\prod_{j=1}^N \sin(\lambda_j - \lambda_j + \frac{i\gamma}{2}) \sin(\lambda_j + \lambda_j + \frac{i\gamma}{2})} \\
& \quad \cdot \frac{\prod_{\alpha=1}^M \sin(\Lambda_\alpha - \lambda_j + \text{sign}(\Delta_{\alpha})) \sin(\Lambda_\alpha + \lambda_j + \text{sign}(\Delta_{\alpha}))}{\sin(\Lambda_\alpha - \lambda_j - \frac{i\gamma}{2}) \sin(\Lambda_\alpha + \lambda_j + \frac{i\gamma}{2})}
\end{align*}
\]

(IV.10)

where \( \zeta(k; p) = (1 - pe^{-ik})/(1 - pe^{ik}) \). The energy eigenvalue \( E \) of the model is given by \( E = -2 \sum_{j=1}^N \cos k_j \) (up to an unimportant additive constant, which we have dropped).

V. CONCLUSION

We have proposed a very general graded RE algebra and developed the corresponding boundary QISM. In formulating our general formalism in section II we do not impose any constraint conditions on the quantum R-matrices, and therefore our formalism applies to all lattice boundary systems where an invertible R-matrix exists. Two nontrivial examples are the supersymmetric versions of the models of two coupled and three coupled one-dimensional XY spin chains [3], where the fermionic R-matrices do not possess the so-called crossing-unitarity and so no isomorphism between the boundary K-matrices \( K_- \) and \( K_+ \) exists. We have also considered the important “special” case in which the R-matrices are associated with finite-dimensional representations of the quantum affine superalgebra \( U_q[\mathcal{G}^{(k)}] \) for generic \( q \). For such a case the two REs are isomorphic to each other since, as is shown in the Appendix, all such R-matrices enjoy the crossing-unitarity property.

We have applied our general formalism to study integrable open-boundary conditions for the q-deformed supersymmetric \( U \) model of strongly correlated electrons. The quantum integrability of the boundary system is established by the fact that the corresponding Hamiltonian may be embedded into a one-parameter family of commuting transfer matrices. Moreover, the Bethe ansatz equations are derived by means of the coordinate space Bethe ansatz approach. This provides us with a basis for computing the finite-size corrections to the low-lying energies in the system, which
in turn allow us to use the boundary conformal field theory technique to study the critical properties of the boundary model.

As mentioned in section III, the Hamiltonian \([18]\) becomes \(U_q[gl(2|1)]\)-invariant in the limits \(\xi_- \to -\infty\) and \(\xi_+ \to \infty\). It is very interesting to see whether or not the Bethe states constructed above constitute the highest weight states for \(U_q[gl(2|1)]\). This property is crucial in understanding the completeness of the Bethe states. A similar problem has been studied \([18]\) for the \(U_q[gl(2|1)]\)-invariant open \(t-J\) chain. It seems interesting to note that although the completeness problem for the Bethe states are well studied in the periodic case for the Hubbard model and the supersymmetric \(t-J\) model \([13]\), it remains largely unexplored in the non-periodic (or open) boundary case. Another interesting question is to test the Bethe ansatz equations using the algebraic Bethe ansatz approach.

Y.-Z.Z and H.-Q.Z are supported by Australian Research Council, University of Queensland New Staff Research Grant and External Support Enabling Grant. H.-Q.Z would like to thank the Department of Mathematics, University of China and Sichuan Young Investigators Science and Technology Fund.

**APPENDIX A: ON THE CROSSING-UNITARITY**

So far in literature, the discussions on the crossing-unitarity of R-matrices have been on a case by case basis and the crossing-unitarity of a given R-matrix has been checked by brute force.

In this appendix, we show that all R-matrices associated with finite-dimensional representations of the quantum affine superalgebra \(U_q[\hat{G}^{(k)}]\) \((k = 1, 2)\) for generic \(q\), where \(\hat{G}\) is any simple Lie superalgebra, enjoy the crossing unitarity property.

Let us first of all recall some facts about the affine superalgebra \(\hat{G}\). Then

\[
\hat{G} = \frac{\mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{r}}{\mathfrak{r}},
\]

where \(\mathfrak{g}\) is any simple Lie superalgebra, enjoy the crossing unitarity property.

We shall not give the defining relations for \(U_q[\hat{G}^{(k)}]\), but mention that the actions of coproduct and antipode on its generators \(\{h_i, e_i, f_i, d, 0 \leq i \leq r\}\) are given by

\[
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta(d) = d \otimes 1 + 1 \otimes d,
\]

\[
\Delta(e_i) = e_i \otimes q^{-\frac{\delta}{2}} + q^{\frac{\delta}{2}} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-\frac{\delta}{2}} + q^{\frac{\delta}{2}} \otimes f_i,
\]

\[
S(a) = -q^{-\hat{\rho}} a q^\hat{\rho}, \quad \forall a = d, h_i, e_i, f_i.
\]

Define an automorphism \(D_z\) of \(U_q[\hat{G}^{(k)}]\) by

\[
D_z(e_i) = z^{\delta_0} e_i, \quad D_z(f_i) = z^{-\delta_0} f_i, \quad D_z(h_i) = h_i, \quad D_z(d) = d.
\]

Then

\[
S^2(a) = q^{-2\rho} D_{\frac{\delta}{2}+} (a) q^{2\rho}, \quad S^{-2}(a) = q^{2\rho} D_{\frac{\delta}{2}+} (a) q^{-2\rho}, \quad \forall a \in U_q[\hat{G}^{(k)}],
\]

which can be checked on the generators [remembering that the simple roots associated with \(e_0, f_0\) are \(\alpha_0 = \pm(\frac{1}{k} \delta - \psi)\), respectively, where \(\delta = (0, 0, 1)]\). We define the right dual module \(V^\ast\) and left dual module \(*V\) of \(V\) by
\[ \pi_V(a) = \pi_V(S(a))^st, \quad \pi_V(a) = \pi_V(S^{-1}(a))^st, \quad (A.7) \]

respectively. Here \( st \) is the supertransposition operation defined by
\[ (A_{ab})^{st} = (-1)^{|a|+|b|}A_{ba}. \quad (A.8) \]

Note that in general \((A_{ab})^{st} = (-1)^{|a|+|b|}A_{ab} \neq A_{ab}\). Let \( ist \) be the inverse operation of \( st \) such that \((A_{ab})^{ist} = ((A_{ab})^{st})^{ist} = A_{ab}\). Then
\[ (A_{ab})^{ist} = (-1)^{|b|(|a|+|b|)}A_{ba} = (-1)^{|a|+|b|}(A_{ab})^{st}, \quad (A.9) \]

or \( A^{ist} = \theta A^{st} \theta \), where \( \theta \) is a diagonal matrix with elements \( \theta_{ab} = (-1)^{|a|}\delta_{ab} \).

By arguments similar to those in \([20]\), one can show that
\[ R^{V,W}(z) = (R^{V,W}(z)^{-1})^{st_1}, \quad R^{V,W}(z) = (R^{V,W}(z)^{-1})^{st_2}. \quad (A.10) \]

It follows from the representations for \( R^{V,W}(z) \) and \( R^{V,W}(z) \) that for any pair of finite dimensional \( U_q[G^{(k)}] \)-modules \( V \) and \( W \), the R-matrix satisfies the following crossing-unitarity relations
\[ (((R^{V,W}(z)^{-1})^{st_1})^{-1})^{st_1} = (\pi_V(q^{-2p}) \otimes 1_W)((R^{V,W}(zq^{-\hat{\gamma}}))^{st_1}(\pi_V(q^{2p}) \otimes 1_W), \]
\[ (((R^{V,W}(z)^{-1})^{st_2})^{-1})^{st_2} = (1_V \otimes \pi_W(q^{2p}))((R^{V,W}(zq^{\hat{\gamma}}))^{st_2}(1_V \otimes \pi_W(q^{-2p})). \quad (A.11) \]

Note also that
\[ (\pi_V(q^{\pm 2p}) \otimes \pi_W(q^{\pm 2p}))R^{V,W}(z) = R^{V,W}(z)(\pi_V(q^{\pm 2p}) \otimes \pi_W(q^{\pm 2p})). \quad (A.12) \]

Let us remark that if one uses the opposite coproduct and antipode of \( U_q[G^{(k)}] \),
\[ \hat{\Delta}(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \hat{\Delta}(d) = d \otimes 1 + 1 \otimes d, \]
\[ \hat{\Delta}(e_i) = e_i \otimes q^{b_i} + q^{-b_i} \otimes e_i, \quad \hat{\Delta}(f_i) = f_i \otimes q^{b_i} + q^{-b_i} \otimes f_i, \]
\[ \hat{S}(a) = -q^{\hat{a}} a q^{-\hat{a}}, \quad \forall a = d, h_i, e_i, f_i, \quad (A.13) \]

and denote the corresponding R-matrix by \( \hat{R}(z) \), then the similar arguments as above give rise to the following crossing-unitarity relations:
\[ (((\hat{R}^{V,W}(z)^{-1})^{st_1})^{-1})^{st_1} = (\pi_V(q^{2p}) \otimes 1_W)((\hat{R}^{V,W}(zq^{\hat{\gamma}}))^{st_1}(\pi_V(q^{-2p}) \otimes 1_W), \]
\[ (((\hat{R}^{V,W}(z)^{-1})^{st_2})^{-1})^{st_2} = (1_V \otimes \pi_W(q^{-2p}))((\hat{R}^{V,W}(zq^{\hat{\gamma}}))^{st_2}(1_V \otimes \pi_W(q^{2p})). \quad (A.14) \]

[1] F.H.L. Essler, V.E. Korepin, K. Schoutens, Phys. Rev. Lett. 68 (1992) 2960; 70 (1993) 73.
[2] R.Z. Bariev, J. Phys. A: Math. Gen. 24 (1991) L919.
[3] A.J. Bracken, M.D. Gould, J.R. Links, Y.-Z. Zhang, Phys. Rev. Lett. 74 (1995) 2768.
[4] M.D. Gould, Y.-Z. Zhang, H.-Q. Zhou, preprint [cond-mat/9709129].
[5] R.Z. Bariev, A. Klümper, J. Zittartz, Europhys. Lett. 32 (1995) 85.
[6] M.D. Gould, K.E. Hibberd, J.R. Links, Y.-Z. Zhang, Phys. Lett. A212 (1996) 156.
[7] F.H.L. Essler, V.E. Korepin, Exactly solvable models of strongly correlated electrons, World Scientific, 1994.
[8] H.-Q. Zhou, J. Phys. A: Math. Gen. 29 (1996) 5509; Phys. Lett. A221 (1996) 104; H.-Q. Zhou, D.-M. Tong, Phys. Lett. A232 (1997) 377.
[9] G. Bedürftig, H. Frahm, J. Phys. A: Math. Gen. 28 (1995) 4453; P.B. Ramos, M.J. Martins, Nucl. Phys. B474 (1996) 678; M.P. Pfannmüller, H. Frahm, Nucl. Phys. B479 (1996) 575; K.E. Hibberd, M.D. Gould, J.R. Links, Phys. Rev. B54 (1996) 8430.
[10] E.K. Sklyanin, J. Phys. A: Math.Gen. 21 (1988) 2375.
[11] L. Mezincescu, R. Nepomechie, J. Phys. A: Math. Gen. 24 (1991) L17; Int. J. Mod. Phys. A6 (1991) 5231.
[12] H.J. de Vega, A. González-Ruiz, J. Phys. A: Math. Gen. 26 (1993) L519; Mod. Phys. Lett. A9 (1994) 2207.
[13] H.-Q. Zhou, Phys. Rev. B54 (1996) 41; ibid B53 (1996) 5089.
[14] A.J. Bracken, X.-Y. Ge, Y.-Z. Zhang, H.-Q. Zhou, preprint cond-mat/9710171.
[15] J.R. Links, M.D. Gould, Int. J. Mod. Phys. B10 (1996) 3461.
[16] Y.-Z. Zhang, H.-Q. Zhou, preprint cond-mat/9707263.
[17] H. Asakawa, M. Suzuki, J. Phys. A: Math. Gen. 29 (1996) 225;
    M. Shiroishi, M. Wadati, J. Phys. Soc. Jpn. 66 (1997) 1.
[18] A. González-Ruiz, Nucl. Phys. B424 (1994) 553.
[19] F.H.L. Essler, V.E. Korepin, K. Schoutens, Nucl. Phys. B384 (1992) 431;
    A. Foerster, M. Karowski, Nucl. Phys. B408 (1993) 512.
[20] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, Lett. Math. Phys. 19 (1990) 133;
    I.B. Frenkel, N.Yu. Reshetikhin, Commun. Math. Phys. 146 (1992) 1.