The Einstein Action for Algebras of Matrix Valued Functions — Toy Models

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3 April 1996

Abstract

Two toy models are considered within the framework of noncommutative differential geometry. In the first one, the Einstein action of the Levi–Civita connection is computed for the algebra of matrix valued functions on a torus. It is shown that, assuming some constraints on the metric, this action splits into a classical-like, a quantum-like and a mixed term. In the second model, an analogue of the Palatini method of variation is applied to obtain critical points of the Einstein action functional for $M_4(\mathbb{R})$. It is pointed out that a solution to the Palatini variational problem is not necessarily a Levi–Civita connection. In this model, no additional assumptions regarding metrics are made.

PACS: 02.10.Tq, 04.20.Fy

I. Introduction

The goal of this note is to analyse the behaviour of a noncommutative analogue of the Einstein–Hilbert action functional on two toy models. General definitions and constructions employed to study those models are provided in the next section.

In Section III, we present some results regarding the computation of the Einstein action of the Levi–Civita connection for $C^\infty(T^m) \otimes M_n(\mathbb{R})$, the algebra of matrix valued functions on an $m$-tori. The approach proposed there is analogous to the ‘derivation based’ approach to the calculation of the Yang–Mills (Maxwell) action for an algebra of matrix valued functions that was carried out in [1] (see Section V in [1], cf. [2] and the sections 4 and 5 of [3]). We choose our manifold $M$ to be an $m$-tori because it is a compact Abelian group, and we want integrals over $M$ to be finite, and Der($C^\infty(T^m)$) to be commutative as a Lie algebra and free as a $C^\infty(T^m)$-module. We also assume that the metric, understood as a pairing of derivations, has its values in the centre of an algebra. Consequently, the results presented there can be interpreted as concerning the ‘commutative part’ of noncommutative geometry. (Indeed, the Levi–Civita connections for $M_n(\mathbb{R})$ can be interpreted as the torsion part of the flat connection on $SL(n, \mathbb{R})$ given by the left translations; see Remark [14].)

In Section IV, which is the main part of this work, we study a toy model that is based on the algebra $M_4(\mathbb{R})$ of 4 by 4 real matrices, and the 2-dimensional Lie subalgebra $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ of Der($M_4(\mathbb{R})$). It is a simple but quite computable model that is presented with the aim of

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providing hints on how to approach more complicated situations. For this model, we derive an
analogue of the Einstein vacuum field equation, and apply the Palatini method of variation to
obtain critical points. We find a solution to the Palatini variational problem that is, in general,
neither metric nor torsion free. Yet, this solution (a connection $\nabla^g$ determined by a metric $g$)
turns out to be Ricci flat for all metrics. Consequently, the Einstein field equation, which
depends essentially on the Ricci curvature (see (7)), is automatically satisfied, and $(g, \nabla^g)$ is a
critical point of the Einstein action functional for any metric $g$. The value of this functional
at any such critical point is the same (zero). Thus we obtain a result that partially reflects
the classical geometric phenomenon that, for any Riemannian 2-manifold, the Einstein–Hilbert
action computed for a metric $g$ and its Levi–Civita connection does not depend on $g$ (see 9.1.10
in [4]). The key difference between Section IV and the preceding one is that in Section IV we
no longer assume that the metric is centre valued.

The proofs or calculations are rather straightforward and are often omitted here for the
sake of brevity. Except for Proposition 4, the letter $k$ will denote a field, $A$ a unital associative
$k$-algebra, $Z(A)$ its centre, and $\Omega(A)$ a differential graded algebra $A \oplus \bigoplus_{n \geq 1} \text{Hom}_{Z(A)}(\wedge^n \mathcal{L}, A)$
with the differential defined as in Proposition 4. The Einstein convention of summing over
repeating indices is assumed.

II. Preliminaries

Let $\mathcal{L}$ be a $Z(A)$-submodule and Lie subalgebra of $\text{Der}(A)$ such that $\mathcal{L} \otimes_{Z(A)} A$ is a finitely
generated projective right $A$-module. (Compare with the notion of a Lie–Cartan pair introduced in [3].)

**Definition 1** A linear map $\nabla : \mathcal{L} \otimes_{Z(A)} \Omega^r(A) \to \mathcal{L} \otimes_{Z(A)} \Omega^{r+1}(A)$ is called a connection
on $\mathcal{L}$ iff

$$\forall X \in \mathcal{L}, \alpha \in \Omega(A) : \nabla(X \otimes_{Z(A)} \alpha) = (\nabla X)\alpha + X \otimes_{Z(A)} d\alpha .$$

**Definition 2** The endomorphism $\nabla^2 \in \text{End}_{\Omega(A)}(\mathcal{L} \otimes_{Z(A)} \Omega(A))$ is called the curvature of a con-
nnection $\nabla$.

**Remark 3** The notions of connection and curvature defined above are equivalent to the usual
notions of noncommutative connection and curvature on a projective module (e.g., see Sec-
tion III.B of [1]). In this case, the projective right $A$-module is $\mathcal{L} \otimes_{Z(A)} A$.

**Proposition 4** (cf. p.369 in [5]) Let $A$ be an associative unital algebra over a commutative
ring $k$. Let $\mathcal{L}_k$ be a $k$-Lie subalgebra of the space of all $k$-derivations of $A$, and let $\mathcal{E}$ be any right
A-module admitting a connection (see Remark [3]). If $\Omega(A)$ is a differential graded subalgebra of $A \oplus \bigoplus_{n \geq 1} \text{Hom}_k(\Lambda^n L_k, A)$ with the differential given by (see the first section in [2]):

$$(d\alpha)(X_0, X_1, \cdots, X_n) = \sum_{0 \leq i \leq n} (-)^i X_i \alpha(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$$

$$+ \sum_{0 \leq r < s \leq n} (-)^{r+s} \alpha([X_r, X_s], X_0, \cdots, X_{r-1}, X_{r+1}, \cdots, X_{s-1}, X_{s+1}, \cdots, X_n),$$

then

$$\forall \xi \in \mathcal{E}, X, Y \in L_k : (\nabla^2 \xi)(X, Y) = ([\nabla X, \nabla Y] - \nabla_{[X, Y]})(\xi),$$

where, as in the classical differential geometry, $\nabla_{\xi}$ denotes $(\nabla \xi)(Z)$.

**Definition 5** A map $g : (\mathcal{L} \otimes_{Z(A)} A) \times (\mathcal{L} \otimes_{Z(A)} A) \to A$ is called a pseudo-Riemannian metric on $\mathcal{L}$ iff it satisfies the following conditions:

1) $\forall a, b \in A, \xi, \eta \in \mathcal{L} \otimes_{Z(A)} A : g(a\xi, \eta b) = ag(\xi, \eta)b$, where the left module structure on $\mathcal{L} \otimes_{Z(A)} A$ is given by $a(X_i \otimes_{Z(A)} c_i) = X_i \otimes_{Z(A)} ac_i$.

2) $\forall X, Y \in \mathcal{L} : g(X, Y) = g(Y, X)$ (symmetry condition).

3) The induced map $\tilde{g} : \mathcal{L} \otimes_{Z(A)} A \ni \xi \mapsto g(., \xi) \in \Omega^1(A)$ is an isomorphism of right $A$-modules.

**Definition 6** A connection on $\mathcal{L}$ is said to be compatible with $g$ iff

$$\forall X, Y, Z \in \mathcal{L} : Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

**Definition 7** The $Z(A)$-bilinear map

$$T_{\nabla} : \mathcal{L} \times \mathcal{L} \ni (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y] \otimes_{Z(A)} 1 \in \mathcal{L} \otimes_{Z(A)} A$$

is called the torsion of $\nabla$.

**Definition 8** Let $\mathcal{R}_{\nabla}(X, Y)$ be the $Z(A)$-homomorphism given by the formula:

$$\mathcal{L} \ni Z \mapsto \mathcal{R}_{\nabla}(X, Y)(Z) = (\nabla^2 Y)(Z, X) \in \mathcal{L} \otimes_{Z(A)} A,$$

and let $\mathcal{T}_E \in \text{Hom}_A(\text{Hom}_{Z(A)}(\mathcal{L}, \mathcal{L} \otimes_{Z(A)} A), A)$. We call the $Z(A)$-linear map

$$\text{Ric}_{\nabla} : \mathcal{L} \ni X \mapsto \mathcal{T}_E(\mathcal{R}_{\nabla}(X, .)) \in \Omega^1(A)$$

the Ricci curvature of $\nabla$. 
Remark 9 When $L$ or $A$ is a finitely generated projective $Z(A)$-module,

$$\mathrm{Hom}_{Z(A)}(L, L \otimes_{Z(A)} A) = \mathrm{End}_{Z(A)}(L) \otimes_{Z(A)} A$$

(see Proposition 2 in II.4[7]), and we can choose $T_E$ to be a trace on $\mathrm{End}_{Z(A)}(L)$ tensored over $Z(A)$ with $id_A$.

Definition 10 Let $\mathcal{M}(L)$ and $\mathcal{C}(L)$ denote the space of all pseudo-Riemannian metrics on $L$, and the space of all connections on $L$ respectively. The functional $E : \mathcal{M}(L) \times \mathcal{C}(L) \rightarrow k$ given by the formula:

$$E(g, \nabla) = -(\tau_g \circ T_E)(\tilde{g}^{-1} \circ \text{Ric}_{\nabla})$$

where $\tau_g : A \rightarrow k$ is a metric dependent trace, is called the Einstein action functional on $L$.

Remark 11 With an appropriate choice of $\tau_g$ and $T_E$ (cf. Proposition [15]), the functional $E$ coincides for $A = C^\infty(M)$ and $L = \text{Der}(C^\infty(M))$ with the standard Einstein–Hilbert action functional on $M$, for any (paracompact) manifold $M$ admitting a (pseudo-)Riemannian metric.

III. The case of $Z(A)$-valued Metrics

One can apply the same reasoning as in the case of classical differential geometry to obtain:

Proposition 12 (cf. Section 9 in [8]) Let $g(L, L) \subseteq Z(A)$ or $\nabla_L L \subseteq L$. Then there exists at most one metric compatible connection that is torsion free. If it exists, it is given by the formula:

$$\nabla_X Y = \frac{1}{2} \tilde{g}^{-1} (X g(Y, .) + Y g(X, .) - d g(X, Y) + g([X,Y], .) + g([., X], Y) + g([., Y], X)).$$

(1)

A connection given by (1), will be called the Levi–Civita connection of $g$, and denoted by $\nabla_g$.

For the rest of this section, we work under the assumption that $g(L, L) \subseteq Z(A)$. The mathematical model considered here is practically identical with a model constructed in [8, 10]. (Compare (2) with (3.24) in [9], and Proposition [15] with (3.23), (3.21) in [9] and (3.26) in [10].) Note that there is an extra term in (3.24)[9] that is absent in Proposition [15] due to our assumption that the classical and the algebraic derivations are orthogonal to each other. Let us also mention that the dual point of view regarding linear connections (i.e., where the space of 1-forms rather than $L \otimes_{Z(A)} A$ is taken as a starting point) was studied in [11] and [12] also in the context of ‘matrix geometry’ (see Section 4.3 in [11] and Section 3 in [12]).

To begin with, let us consider the algebra of matrices $M_n(\mathbb{R})$ and $L = \text{Der}(M_n(\mathbb{R})) = \mathfrak{sl}(n, \mathbb{R})$. Since $\mathfrak{sl}(n, \mathbb{R})$ is an $(n^2 - 1)$-dimensional vector space over $\mathbb{R}$ (the centre of the algebra of matrices), the endomorphisms of this space are simply $(n^2 - 1) \times (n^2 - 1)$ matrices, and we can choose $T_E$ to be the usual matrix trace tensored with $id_{M_n(\mathbb{R})}$ (see Remark [3]).

\footnote{For duality issues of this kind see Section 6 in [8].}
Proposition 13 Let $A = M_n(\mathbb{R})$, $\mathcal{L} = \text{Der}(M_n(\mathbb{R}))$, $\tau_g = \frac{1}{n} \det g^{ij} \frac{1}{2} \text{Tr}$, and $T_E = \text{Tr} \otimes \text{id}_{M_n(\mathbb{R})}$. Also, let $\{E_i\}$ be any basis of $\mathfrak{sl}(n, \mathbb{R})$. Then

$$E(g, \nabla_g) = g^{jp}(K_{jp} + \frac{1}{2}g^{il}g_{rk}c^k_{ip}c^k_{lj})\sqrt{|\det g|},$$

where $K$ is the Killing metric on $SL(n, \mathbb{R})$, and $\{K_{jp}, g_{ij}, g^{kl}, c^k_{ip}\}$ are defined by the formulas $K_{jp} = g(E_j, E_p)$, $g_{rk} = g(E_r, E_k)$, $g^{pr}g_{rk} = \delta^r_k$, $\{E_i, E_p\} = c^i_{jp}E_r$ respectively.

Proof. A direct computation using the symmetry of a metric and the anti-symmetry of the Lie algebra structure constants. \hfill \Box

Remark 14 Let $g$ be a left invariant metric on $SL(n, \mathbb{R})$. Then $g$ can be identified with a metric on $\text{Der}(M_n(\mathbb{R}))$. The connection on $SL(n, \mathbb{R})$ given by the left translations is compatible with all left invariant metrics. The torsion part of this connection is given by the formula (see 44 in [13]):

$$\mathfrak{E}^i_{jk} := \frac{1}{2}(Q^i_{jk} + g^{il}g_{kn}Q^l_{jk} + g^{il}g_{kn}Q^n_{jl}) = \frac{1}{2}(c^i_{kj} + g^{il}g_{kn}c^l_{kj} + g^{il}g_{kn}c^n_{lj})$$

If the coefficients of the $M_n(\mathbb{R})$–Levi–Civita connection of $g$ are defined by the equality $\nabla_{E_j} E_k = \Gamma^i_{jk} E_i$, then $\Gamma^i_{jk} = \mathfrak{E}^i_{jk} \in \mathbb{R}$. (Caution: One often defines the Christoffel symbols by the relation $\nabla_{E_j} E_k = \Gamma^i_{jk} E_i$. In this notation, which is compatible with the notation used in [13], the aforementioned relationship between the Christoffel symbols of the noncommutative connection and the torsion part of the classical connection can be equivalently written as $\tilde{\Gamma}^i_{jk} = \mathfrak{E}^i_{jk} + c^i_{jk}$. In general, if $\text{Der}(A)$ equals the Lie algebra of some Lie group $G$, then the noncommutative torsion free $g$-compatible connection on $\text{Der}(A)$ coincides in the above sense with the torsion part of the flat connection on $G$ given by the left translations. \hfill \Diamond

Our next step is to consider an algebra of matrix valued functions. The module of derivations of such an algebra splits into two direct sum components in the following way (cf. Lemma 2.1 in [4]):

$$\text{Der}(C^\infty(M) \otimes M_n(\mathbb{R})) = \text{Der}(C^\infty(M)) \otimes \mathbb{R} \oplus C^\infty(M) \otimes \text{Der}(M_n(\mathbb{R})).$$

For $M = T^m$, this module is a free $C^\infty(T^m)$-module of dimension $m + n^2 - 1$. Consequently, its algebra of endomorphisms is simply the algebra of matrices $M_{m+n^2-1}(C^\infty(T^m))$, and, again, we can choose $T_E$ to be the usual matrix trace (with values in $C^\infty(T^m)$) tensored with $\text{id}_{M_n(\mathbb{R})}$ (see Remark 9).

Proposition 15 Let $A = C^\infty(T^m) \otimes M_n(\mathbb{R})$, $\mathcal{L} = \text{Der}(A)$, $\tau_g = \frac{1}{n} \int_{T^m} |\det g|^{1/2} \text{Tr}$, and $T_E = \text{Tr} \otimes \text{id}_{M_n(\mathbb{R})}$. Assume also that there exists a basis $\{E_i\}_{i \in \{1, \ldots, m\}}$ of $\text{Der}(C^\infty(T^m))$, and a basis $\{E_j\}_{j \in \{m+1, \ldots, m+n^2-1\}}$ of $\text{Der}(M_n(\mathbb{R}))$ such that

$$g_{ij} = \begin{cases} 0 & \text{for } i \leq m \text{ and } j > m \\ g_c(E_i, E_j) & \text{for } i, j \leq m \\ g_q(E_i, E_j) & \text{for } i, j > m, \end{cases}$$
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where \( g_c \) is a classical (pseudo-)Riemannian metric on \( T^m \), and \( g_q \) is a function that to each point of \( T^m \) assigns a metric on \( \text{Der}(M_n(\mathbb{R})) \). Then

\[
E(g, \nabla_g) = \int_{T^m} R_c \sqrt{|\det g_c|} \sqrt{|\det g_q|} + \int_{T^m} R_q \sqrt{|\det g_q|} \sqrt{|\det g_c|} + \int_{T^m} R_{mixed} \sqrt{|\det g|},
\]

where \( R_c \) is the classical scalar curvature of \( g_c \), \( R_q \) is a (point dependent) scalar curvature of \( \nabla_{g_q} \) (i.e. \( R_q = \nabla_E(\tilde{\nabla}_q^{-1} \circ \text{Ric} \nabla_{g_q}) \)), and \( R_{mixed} \) is a function on \( T^m \) that is a sum of mixed terms of the kind

\[
g^{AB} \frac{\partial}{\partial x^\mu} \left( g^{\mu\nu} \frac{\partial g_{AB}}{\partial x^\nu} \right), \quad \mu, \nu \leq m, \quad A, B > m.
\]

**Proof.** Very much like the proof of Proposition 13.

As we see from Proposition 15, the assumption that the metric \( g \) is block-diagonal allows us to split the Einstein action of the Levi–Civita connection into the following three terms:

1. A classical-like term that differs from the usual Einstein–Hilbert action on \( T^m \) only by the ‘quantum volume element’ \( \sqrt{|\det g_q|} \).
2. A quantum-like term that is equal to the integral over \( T^m \) of the (point dependent) Einstein action of the Levi–Civita connection on \( M_n(\mathbb{R}) \).
3. A mixed term that involves the derivatives of \( g_q \).

If \( g_q \) is constant over \( T^m \), then \( R_{mixed} = 0 \), and the expression (3) simplifies to

\[
E(g, \nabla_g) = \sqrt{|\det g_q|} E(g_c) + E(g_q) \text{vol}(T^m, g_c),
\]

where \( E(g_c) \) is the usual Einstein–Hilbert action, \( E(g_q) \) is the action computed in Proposition 13, and \( \text{vol}(T^m, g_c) \) is the volume of \( T^m \) with respect to the metric \( g_c \).

Finally, let us remark that the Yang–Mills (Maxwell) action calculated in [1] for a similar algebra also splits into a classical-like, a quantum-like and a mixed term.

**IV. The Case of A-valued Metrics**

Let us now lift the assumption that \( g(\mathcal{L}, \mathcal{L}) \subseteq Z(A) \), and consider a toy model based on the following data:

\[
A = M_4(\mathbb{R}), \quad \mathcal{L} = \mathfrak{so}(2) \oplus \mathfrak{so}(2), \quad \tau_g = \frac{1}{4} \sqrt{|\det g|} \text{Tr}_{M_4(\mathbb{R})} \quad \text{and} \quad \nabla_E = \text{Tr} \otimes \text{id}_{M_4(\mathbb{R})}.
\]

We view \( \mathcal{L} \) as a Lie subalgebra of \( \text{Der}(M_4(\mathbb{R})) \) generated by \( \hat{F}_1 := [F_1, \cdot] \), \( \hat{F}_2 := [F_2, \cdot] \), where \( F_1 := \left( \begin{smallmatrix} F & 0 \\ 0 & 0 \end{smallmatrix} \right), \quad F_2 := \left( \begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix} \right), \quad F := \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \). A metric \( g \) is treated as an element of \( GL_2(M_4(\mathbb{R})) = GL_8(\mathbb{R}) \). An assumption that \( g \) is symmetric reads \( g(\hat{F}_i, \hat{F}_j) =: g_{ij} = g_{ji} := \ldots \)
instance, if

\[ g(\hat{F}_j, \hat{F}_i), \]

or equivalently \( g = g^T \), where \( T \) is the transpose in the algebra of \( 2 \times 2 \) matrices. As to the inverse of \( g \), denoted by \( g^{-1} = (\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}) \), in general we do not have \( g^{12} = g^{21} \). For instance, if

\[
g = \begin{pmatrix} 2I_2 & 0 & I_2 & I_2 \\ 0 & I_2 & I_2 & 0 \\ I_2 & I_2 & I_2 & 0 \\ I_2 & 0 & 0 & I_2 \end{pmatrix},
\]

where \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), then \( g^{12} \neq g^{21} \). Very much as we did before, we define the Christoffel symbols by \( \nabla_{\hat{F}_j} \hat{F}_i = \hat{F}_k \otimes \Gamma^k_{ji} \), the curvature coefficients by \( (\nabla^2 \hat{F}_i)(\hat{F}_k, \hat{F}_j) = \hat{F}_m \otimes R^m_{kij} \), and the Ricci curvature coefficients by \( (Ric \hat{F}_i)(\hat{F}_j) = R_{ji} \). It is straightforward to verify that \( R^m_{kij} = \Gamma^m_{kji} - \Gamma^m_{nj} \Gamma^i_{kji} + [F_i, \Gamma^m_{kj}] - [F_j, \Gamma^m_{ki}] \) (see Proposition 4), \( R_{kj} = R^i_{kij} \), and

\[
E(g, \nabla) = -\frac{1}{4} \sqrt{|\det g|} \operatorname{Tr}_{M_4(\mathbb{R})}(g^{ik}R_{kj}) = -\frac{1}{4} \sqrt{|\det g|} \operatorname{tr}(g^{-1}r),
\]

where \( \operatorname{tr} \) denotes the usual trace on \( M_8(\mathbb{R}) \), and \( r := (R^{11}_{21} R^{12}_{21} R^{12}_{22}) \). In what follows, rather than look for a Levi-Civita connection, we will use an analogue of the Palatini method of variation (see 21.2 in [14], cf. 5.4–5.5 in [15]) and find critical points of the Einstein action functional. First, let us determine the equation equivalent to the condition that the variation of \( E \) with respect to \( g \) vanish. At any critical point, for an arbitrary matrix \( h \in M_8(\mathbb{R}) \) with the coefficients satisfying \( h_{12} = h_{21} \in M_4(\mathbb{R}) \) we must have:

\[
\frac{d}{ds} E(g + sh, \nabla)|_{s=0} = 0.
\]

After substituting (4) into (3), and carrying out the differentiation, we obtain:

\[
\frac{1}{2} \operatorname{tr}(g^{-1}r)\operatorname{tr}(hg^{-1}) = \operatorname{tr}(hg^{-1}rg^{-1}).
\]

Now, since (3) must be true for any matrix \( h \) such that \( h_{12} = h_{21} \), we can conclude that

\[
\forall i, j \in \{1, 2\} : \frac{1}{2} \operatorname{tr}(g^{-1}r)(g^{ij} + g^{ji}) = g^{ik}R_{kl}g^{lj} + g^{jk}R_{kl}g^{li}.
\]

Multiplying both sides by \( g_{im} \), and then taking the trace, we find that

\[
8\operatorname{tr}(g^{-1}r) = 2\operatorname{tr}(g^{-1}r).
\]

Hence, very much as in the classical general relativity, we have \( \operatorname{tr}(g^{-1}r) = 0 \). Consequently,

\[
g^{-1}rg^{-1} + (g^{-1}rg^{-1})^T = 0.
\]

The formula (6) is an analogue of the Einstein vacuum field equation. (Observe that the implication (5) \( \Rightarrow \) (3) is also true.)
Our next step is to find an explicit form of the equations equivalent to the vanishing of the variation of $E$ with respect to $\nabla$. Let $\nabla + sA$ denote a connection on $\mathcal{L}$ whose Christoffel symbols are $\Gamma^k_{ji} + sA^k_{ji}$, where $s \in \mathbb{R}$, $A^k_{ji} \in M_4(\mathbb{R})$, $i, j, k \in \{1, 2\}$. Then the condition that

$$\frac{d}{ds}E(g, \nabla + sA)|_{s=0} = 0 \text{ for any } A$$

is equivalent to the following 8 equations:

\begin{align}
&g^{11}\Gamma^2_{12} + \Gamma^1_{22}g^{22} + [\Gamma^1_{12} + F_2, g^{21}] = 0 \\
&g^{11}\Gamma^2_{11} + \Gamma^1_{21}g^{22} + [\Gamma^1_{11} + F_1, g^{21}] = 0 \\
&g^{22}\Gamma^1_{22} + \Gamma^2_{12}g^{11} + [\Gamma^2_{22} + F_2, g^{12}] = 0 \\
&g^{22}\Gamma^1_{21} + \Gamma^1_{21}g^{11} + [\Gamma^2_{11} + F_1, g^{12}] = 0 \\
&g^{22}\Gamma^1_{12} - \Gamma^2_{22}g^{22} - g^{12}\Gamma^2_{12} - \Gamma^2_{12}g^{21} - [F_2, g^{22}] = 0 \\
&g^{11}\Gamma^2_{22} - \Gamma^1_{12}g^{11} - g^{21}\Gamma^1_{22} - \Gamma^1_{22}g^{12} - [F_2, g^{11}] = 0 \\
&g^{22}\Gamma^1_{11} - \Gamma^2_{21}g^{22} - g^{12}\Gamma^2_{11} - \Gamma^1_{11}g^{21} - [F_1, g^{22}] = 0 \\
&g^{11}\Gamma^2_{21} - \Gamma^1_{11}g^{11} - g^{21}\Gamma^1_{21} - \Gamma^1_{21}g^{12} - [F_1, g^{11}] = 0
\end{align}

It is straightforward to verify that a connection $\nabla^g$ (not to be confused with the Levi–Civita connection $\nabla_g$) whose Christoffel symbols are given by

$$\Gamma^1_{22} = \Gamma^1_{21} = -g^{11} \quad \Gamma^2_{22} = -F_2 - g^{12} \quad \Gamma^1_{12} = -F_2 + g^{21}$$

$$\Gamma^2_{22} = \Gamma^2_{21} = g^{22} \quad \Gamma^1_{21} = -F_1 - g^{12} \quad \Gamma^1_{11} = -F_1 + g^{21}$$

satisfies (8–13), and has vanishing Ricci curvature ($\text{Ric}_{\nabla^g} = 0$). We have thus arrived at the following:

**Proposition 16** Let $A$, $\mathcal{L}$, $\tau_g$, and $\mathcal{T}_E$ be as above. Then

$$\forall g \in \mathcal{M}(\mathcal{L}) \exists \nabla^g \in \mathcal{C}(\mathcal{L}) \text{ such that } (g, \nabla^g) \text{ is a critical point of } E.$$

Furthermore, the value of $E$ at any critical point is zero.

**Remark 17** The value of the functional $E$ calculated at non-critical points is not necessarily zero. For example, take a metric $g_0 = g_0^{-1}$ with the components

$$(g_0)_{11} = (g_0)_{22} = 0, \quad (g_0)_{12} = (g_0)_{21} = \begin{pmatrix} I_2 & 0 \\ 0 & K \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and take a connection $\nabla_0$ whose only non-vanishing Christoffel symbol is $(\Gamma_0)^1_{11} = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $E(g_0, \nabla_0) = -1$. \diamond
Remark 18 Note that for $\tau_g$ equal to $Tr_{M_4(\mathbb{R})}$ rather than to $\frac{1}{4} \log \det g^{\frac{1}{2}} Tr_{M_4(\mathbb{R})}$ the field equation (8) and the equations (8–15) are still satisfied. Also, $tr(g^{-1} r)$ still equals zero at any critical point. Consequently, Proposition 16 remains valid as well, if we replace $\frac{1}{4} \log \det g^{\frac{1}{2}} Tr_{M_4(\mathbb{R})}$ by the usual trace on $M_4(\mathbb{R})$. ◊

Remark 19 The fact that the functional $\tilde{E} : \mathcal{M}(\mathcal{L}) \ni g \mapsto E(g, \nabla^g) \in \mathbb{R}$ equals identically zero is a reflection of the same effect that we observe for the usual 2-torus. We might try to push this analogy even further and say that we think of a circle $S^1$ as a Lie group generated by $\mathfrak{so}(2)$, and replace $C^\infty(S^1)$ by $M_2(\mathbb{R})$ for which $\mathfrak{so}(2)$ is the space of all derivations satisfying $X(a^T) = (Xa)^T$. Then it is natural to replace $C^\infty(T^2)$ by $M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) = M_4(\mathbb{R})$. ◊

Observe that although $\nabla^g$ functions as if it were a Levi–Civita connection, it is in general neither metric nor torsion free (e.g., take $g$ to be the identity matrix of $GL_8(\mathbb{R})$). It is perhaps worth emphasizing, however, that the metric compatibility condition, which can be equivalently written as:

$$g^{pj} \Gamma^n_{ji} + \Gamma^n_{pj} g^{jn} + [F_i, g^{jn}] = 0, \quad i, p, n \in \{1, 2\},$$

is not very different from the formulas (8–15). It would be interesting to find a functional on $\mathcal{M}(\mathcal{L}) \times \mathcal{C}(\mathcal{L})$ that not only would coincide with the usual Einstein–Hilbert functional (or whose equations of motion would agree with the standard ones) in the case of classical geometry, but also would yield, through its variation with respect to connection, the metric compatibility condition.

Acknowledgments: This work was supported in part by a visiting fellowship at the International Centre for Theoretical Physics in Trieste, and the KBN grant 2 P301 020 07. It is my great pleasure to thank Andrzej Borowiec, Marc Rieffel and Tsou Sheung Tsun for their manifold assistance and encouragement, Michel Dubois-Violette for an electronic mail discussion, and Giovanni Landi and Andrzej Sitarz for their help in proofreading the manuscript.

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