Scaling transition for nonlinear random fields
with long-range dependence

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Abstract. We obtain a complete description of anisotropic scaling limits and the existence of scaling transition for nonlinear functions (Appell polynomials) of stationary linear random fields on $\mathbb{Z}^2$ with moving average coefficients decaying at possibly different rate in the horizontal and vertical direction. The paper extends recent results on scaling transition for linear random fields in [30], [31].

Keywords: scaling transition; anisotropic long-range dependence; fractionally integrated random field; Appell polynomials; multiple Itô-Wiener integral; fractional Brownian sheet

1 Introduction

\cite{30} introduced the notion of scaling transition for stationary random field (RF) $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ on $\mathbb{Z}^2$ in terms of partial sums limits

\begin{equation}
D_{\lambda, \gamma}^{-1} \sum_{(t, s) \in K_{[\lambda x, \lambda^\gamma y]}} X(t, s) \overset{\text{fdd}}{\to} V_\gamma(x, y), \quad (x, y) \in \mathbb{R}^2_+, \quad \lambda \to \infty, \quad \gamma > 0 \quad (1.1)
\end{equation}

where $D_{\lambda, \gamma} \to \infty$ is normalization and $K_{[\lambda x, \lambda^\gamma y]} := \{(t, s) \in \mathbb{Z}^2 : 1 \leq t \leq \lambda x, 1 \leq s \leq \lambda^\gamma y\}$ is a family of rectangles whose sides grow at possibly different rate $O(\lambda)$ and $O(\lambda^\gamma)$ and $\gamma > 0$ is arbitrary. See the end of this section for all unexplained notation. RF $X$ is said to exhibit \textit{scaling transition} at $\gamma_0 > 0$ if the limit RFs $V_\gamma \equiv V_\gamma^X$ in (1.1) do not depend on $\gamma$ for $\gamma > \gamma_0$ and $\gamma < \gamma_0$ and are different up to a multiplicative constant, viz.,

\begin{equation}
V_\gamma^X \overset{\text{fdd}}{=} V_+^X \quad (\forall \gamma > \gamma_0), \quad V_\gamma^X \overset{\text{fdd}}{=} V_-^X \quad (\forall \gamma < \gamma_0), \quad V_+^X \overset{\text{fdd}}{\neq} a V_-^X \quad (\forall a > 0). \quad (1.2)
\end{equation}

In such case, RF $V_-^{X_{\gamma_0}}$ is called the \textit{well-balanced} while RFs $V_+^{X_{\gamma_0}}$ and $V_-^{X_{\gamma_0}}$ the \textit{unbalanced} scaling limits of $X$.

It appears that scaling transition is a new and general feature of spatial dependence which occurs for many isotropic and anisotropic RF on $\mathbb{Z}^2$ with long-range dependence (LRD). It was established for a class of
aggregated α-stable autoregressive models \[30\], a class of Gaussian LRD RFs \[31\], and some RFs arising by aggregation of network traffic and random-coefficient time series models in telecommunications and economics; see \[11\], \[22\], \[26\], \[27\], also \(30\), Remark 2.3). The unbalanced limits \(V_\pm\) in these studies have a very special dependence structure (either independent or invariant rectangular increments along one of the coordinate axes) and coincide in the Gaussian case with a fractional Brownian sheet (FBS) \(B_{H_1,H_2}\) with one of the two parameters \(H_1,H_2 \in (0,1]\) equal to 1/2 or 1.

The above mentioned works deal with linear RF models written as sums (stochastic integrals) w.r.t. i.i.d. ‘noise’. It is well-known that nonlinear RFs can display quite complicated nongaussian scaling behavior. See Dobrushin and Major \[9\], also \[1\], \[2\], \[13\], \[14\], \[16\], \[19\], \[21\], \[33\], \[34\] and the references therein.

The present paper establishes the existence of scaling transition for a class of nonlinear subordinated RFs:

\[
X(t,s) = G(Y(t,s)),
\]

where \(Y = \{Y(t,s); (t,s) \in \mathbb{Z}^2\}\) is a stationary linear LRD RF in \(1.4\) and \(G(x) = A_k(x), x \in \mathbb{R}\) is the Appell polynomial of degree \(k \geq 1\) (see Sec. 2 for the definition) with \(E G^2(Y(0,0)) < \infty, EG(Y(t,s)) = 0\). The (underlying) RF \(Y\) is written as a moving-average

\[
Y(t,s) = \sum_{(u,v) \in \mathbb{Z}^2} a(t-u,s-v)\varepsilon(u,v), \quad (t,s) \in \mathbb{Z}^2,
\]

in a standardized i.i.d. sequence \(\{\varepsilon(u,v); (u,v) \in \mathbb{Z}^2\}\) with deterministic moving-average coefficients such that

\[
a(t,s) \sim \text{const} (|t|^2 + |s|^{2q_2/q_1})^{-q_1/2}, \quad |t| + |s| \to \infty,
\]

where parameters \(q_1, q_2 > 0\) satisfy

\[
1 < Q := \frac{1}{q_1} + \frac{1}{q_2} < 2.
\]

In Theorems \[3.1\] \[3.5\] below, the moving-average coefficients \(a(t,s)\) may take a more general form in \(2.1\) including an ‘angular function’. Condition \(Q < 2\) guarantees that \(\sum_{(t,s) \in \mathbb{Z}} a(t,s)^2 < \infty\) or \(Y\) in \(1.4\) is well-defined, while \(Q > 1\) implies that \(\sum_{(t,s) \in \mathbb{Z}} |a(t,s)| = \infty\) (in other words, that RF \(Y\) is LRD). Note \(a(t,0) = O(|t|^{-q_1}), a(0,s) = O(|s|^{-q_2})\) decay at a different rate when \(q_1 \neq q_2\) in which case \(Y\) exhibits strong anisotropy. The form of moving-average coefficients in \(1.5\) implies a similar behavior of the covariance function \(r_Y(t,s) := EY(0,0)Y(t,s) = \sum_{(u,v) \in \mathbb{Z}^2} a(u,v)a(t+u,s+v),\) namely

\[
r_Y(t,s) \sim \text{const} (|t|^2 + |s|^{2p_2/p_1})^{-p_1/2}, \quad |t| + |s| \to \infty,
\]

where

\[
p_i := q_i(2 - Q), \quad i = 1,2.
\]

Note \(p_1/p_2 = q_1/q_2\) and the 1-1 correspondence between \((q_1,q_2)\) and \((p_1,p_2)\):

\[
q_i := \frac{p_i}{2}(1 + P), \quad i = 1,2, \quad \text{where} \quad P := \frac{1}{p_1} + \frac{1}{p_2}.
\]
implies that for any integer \(k \geq 1\) and \(P \not\in \mathbb{N}\)
\[
\sum_{(t,s) \in \mathbb{Z}^2} |r_Y(t, s)|^k = \infty \iff 1 \leq k < P,
\]
see Proposition 5.1. In the case when \(Y\) in (1.4) is Gaussian RF, \(r_Y^k(t,s)\) coincides with the covariance of the \(k\)th Hermite polynomial \(H_k(Y(t, s))\) of \(Y\) and the (nonlinear) subordinated RF \(X = H_k(Y)\) is LRD if condition (1.10) holds. A similar result is true for non-Gaussian moving-average RF \(Y\) in (1.4) and Hermite polynomial \(H_k\) replaced by Appell polynomial \(A_k\).

The following summary describes the main results of this paper.

(R1) Subordinated RFs \(X = A_k(Y), 1 \leq k < P\) exhibit scaling transition at the same point \(\gamma_0 := p_1/p_2 = q_1/q_2\) independent of \(k\).

(R2) The well-balanced scaling limit \(V_{\gamma_0} X\) of \(X = A_k(Y)\) is non-Gaussian unless \(k = 1\) and is given by a \(k\)-tuple Itô-Wiener integral.

(R3) Unbalanced scaling limits \(V^+_X = V^X_{\gamma}, \gamma > \gamma_0\) of \(X = A_k(Y)\) agree with FBS \(B_{H^+_k, 1/2}\) with Hurst parameter \(H^+_k \in (1/2, 1)\) if \(kp_2 > 1\), and with a ‘generalized Hermite slide’ \(V^X_{\gamma}(x,y) = xZ^+_k(y)\) if \(kp_2 < 1\), where \(Z^+_k\) is a self-similar process written as a \(k\)-tuple Itô-Wiener integral. A similar fact holds for unbalanced limits \(V^-_X = V^X_{\gamma}, \gamma < \gamma_0\).

(R4) For \(k > P\), RF \(X = A_k(Y)\) does not exhibit scaling transition and all scaling limits \(V^X_{\gamma}, \gamma > 0\) agree with Brownian sheet \(B_{1/2, 1/2}\).

(R5) In the case of Gaussian underlying RF \(Y\) in (1.4), the above conclusions hold for general nonlinear function \(G\) in (1.3) and \(k\) equal to the Hermite rank of \(G\).

The above list contains several new noncentral and central limit results. (R2), (R4) and (R5) are new in the ‘anisotropic’ case \(p_1 \neq p_2\) while (R3) is new even for linear RF \(X = A_1(Y) = Y\) (see Remark 3.1 concerning the terminology in (R3)). Similarly as in the case of linear models (see [30], [31]), unbalanced limits in (R3) have either independent or completely dependent increments along one of the coordinate axes. According to (R3), the sample mean of nonlinear LRD RF \(X = A_k(Y), 1 < k < P\) on rectangles \(K_{[\lambda, \lambda\gamma]}, \gamma \neq \gamma_0\) may have Gaussian or non-Gaussian limit distribution depending on \(k, \gamma\) and parameters \(p_1, p_2\), moreover, in both cases the variance of the sum \(\sum_{(t,s) \in K_{[\lambda, \lambda\gamma]}} X(t, s)\) grows faster than \(\lambda^{1+\gamma}\), or the number of summands. The dichotomy of the limit distribution in (R3) is related to the presence or absence of the vertical/horizontal LRD property of \(X\), see Remark 6.1. We also note that our proofs of the central limit results in (R3) and (R4) use rather simple approximation by \(m\)-dependent r.v.’s and do not require a combinatorial argument or Malliavin’s calculus as in [6], [24] and other papers.

The paper is organized as follows. Sec. 2 provides the precise assumptions on RFs \(Y\) and \(X\) and some known properties of Appell polynomials. Sec. 3 contains formulations of the main results (Theorems 3.1, 3.5).
as described in (R1)-(R5) above. Sec. 4 provides two examples of linear fractionally integrated RFs satisfying the assumptions in Sec. 2. Sec. 5 discusses some properties of generalized homogeneous functions and their convolutions used to prove the results. Sec. 6 discusses the asymptotic form of the covariance function and the asymptotics of the variance of anisotropic partial sums of subordinated RF $X = A_k(Y)$. All proofs are collected in Sec. 7 and 8.

**Notation.** In this paper, $\overset{d}{\rightarrow}$, $\overset{fd}{\rightarrow}$, $\overset{d}{=}$, $\overset{fd}{=} \equiv$ denote the weak convergence and equality of (finite-dimensional) distributions. $C$ stands for a generic positive constant which may assume different values at various locations and whose precise value has no importance. $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_+^2 := (0, \infty)^2$, $\mathbb{R}_+^0 := \mathbb{R}^2 \setminus \{(0, 0)\}$, $\mathbb{N}_+ := \{0, 1, \cdots \}$, $\mathbb{Z}^{2k} := \{(u_1, v_1), \cdots , (u_k, v_k)\} \in \mathbb{Z}^{2k}$ : $(u_i, v_i) \neq (u_j, v_j), 1 \leq i < j \leq k$, $k \in \mathbb{N}_+$, $|t|_+ := |t| \vee 1$ ($t \in \mathbb{Z}$).

## 2 Assumptions and preliminaries

**Assumption (A1)** $\{\varepsilon, \varepsilon(t, s), (t, s) \in \mathbb{Z}^2\}$ is an i.i.d. sequence with $E \varepsilon = 0, E\varepsilon^2 = 1$

**Assumption (A2)** $Y = \{Y(t, s), (t, s) \in \mathbb{Z}^2\}$ is a moving-average RF in (1.4) with coefficients

$$a(t, s) = \frac{1}{(||t||^2 + |s|^{2q_1})q_1/2} \left( L_0 \left( \frac{t}{(||t||^2 + |s|^{2q_1})^{1/2}} \right) + o(1) \right), \quad ||t|| + |s| \rightarrow \infty,$$

where $q_i > 0, i = 1, 2$ satisfy $Q = \sum_{i=1}^2 q_i^{-1} \in (1, 2)$ (see (1.6)) and $L_0(u) \geq 0, u \in [-1, 1]$ is a bounded piece-wise continuous function on $[-1, 1]$.

We refer to $L_0$ in (2.1) as the angular function. Particularly, for $q_1 = q_2, \rho = (||t||^2 + |s|^{2})^{1/2}$ and $\arccos(t/\rho)$ are the polar coordinates of $(t, s) \in \mathbb{R}^2$. Assumptions (A1)-(A2) imply $EY(0, 0)^2 = \sum_{(t,s) \in \mathbb{Z}^2} a(t, s)^2 < \infty$ and hence RF $Y$ in (1.4) is well-defined and stationary, with zero mean $EY(t, s) = 0$. Moreover, if $E|\varepsilon|^\alpha < \infty$ for some $\alpha > 2$ then $E|Y(t, s)|^\alpha < \infty$ follows by Rosenthal’s inequality; see e.g. ([14], Corollary 2.5.1).

Given a r.v. $\xi$ with $E|\xi|^k < \infty, k \in \mathbb{N}_+$, the $k$th Appell polynomial $A_k(x)$ relative to the distribution of $\xi$ is defined by $A_k(x) := (-i)^k d^k(e^{iu\xi}/Ee^{iu\xi})/du^k |_{u=0}$, see [2], [14] for various properties of Appell polynomials. In as follows, $A_k(\xi)$ stands for the r.v. obtained by substituting $x = \xi$ in the Appell polynomial $A_k(x)$ relative to the distribution of $\xi$. Particularly, if $E\xi = 0$ then $A_1(\xi) = \xi, A_2(\xi) = \xi^2 - E\xi^2, A_3(\xi) = \xi^3 - 3\xi E\xi^2 - E\xi^3$ etc. For standard normal $\xi \sim N(0, 1)$ the Appell polynomials $A_k(\xi) = H_k(\xi) = (-i)^k d^k e^{iu\xi + u^2/2}/du^k |_{u=0}$ agree with the Hermite polynomials.

**Assumption (A3)** For $k \in \mathbb{N}_+, E|\varepsilon|^{2k} < \infty$ and

$$X = \{X(t, s) := A_k(Y(t, s)), (t, s) \in \mathbb{Z}^2\},$$

where $A_k$ is the $k$th Appell polynomial relative to the (marginal) distribution of $Y(t, s)$ in (1.4).

We also use the representation of (2.2) via Wick products of noise variables (see [14], Ch. 14):

$$A_k(Y(t, s)) = \sum_{(u,v)k \in \mathbb{Z}^{2k}} a(t - u_1, s - v_1) \cdots a(t - u_k, s - v_k) : \varepsilon(u_1, v_1) \cdots \varepsilon(u_k, v_k):$$

(2.3)
By definition, for mutually distinct points \((u_j, v_j) \neq (u_{j'}, v_{j'})\) \((j \neq j', 1 \leq j, j' \leq i)\) the Wick product
\[ 
\varepsilon(u_1, v_1) \varepsilon(u_2, v_2) \cdots \varepsilon(u_i, v_i) := \prod_{j=1}^{i} A_{k_j}(\varepsilon(u_j, v_j))
\]
equals the product of independent r.v.'s \(A_{k_j}(\varepsilon(u_j, v_j)), 1 \leq j \leq i\). (\ref{2.3}) leads to the decomposition of (\ref{2.2}) into the ‘off-diagonal’ and ‘diagonal’ parts:
\[ 
A_k(Y(t, s)) = Y^\bullet_k(t, s) + Z(t, s),
\]
where
\[ 
Y^\bullet_k(t, s) := \sum_{(u,v)_k} a(t - u_1, s - v_1) \cdots a(t - u_k, s - v_k) \varepsilon(u_1, v_1) \cdots \varepsilon(u_k, v_k)
\]
and the sum \(\sum_{(u,v)_k}\) is taken over all \((u,v)_k = (u_1, v_1), \ldots, (u_k, v_k)\) \(\in \mathbb{Z}^{2k}\) such that \((u_i, v_i) \neq (u_j, v_j)\) \((i \neq j)\) (the set of such \((u,v)_k\) \(\in \mathbb{Z}^{2k}\) will be denoted by \(\mathbb{Z}^{2k}\)). By definition, the ‘diagonal’ part \(Z(t, s)\) in (\ref{2.4}) is given by the r.h.s. of (\ref{2.3}) with \((u,v)_k \in \mathbb{Z}^{2k}\) replaced by \((u,v)_k \in \mathbb{Z}^{2k} \setminus \mathbb{Z}^{2k}\). In most of our limit results, \(Z(t, s)\) is negligible and \(Y^\bullet_k(t, s)\) is the main term which is easier to handle compared to \(A_k(Y(t, s))\) in (\ref{2.4}). We also note that limit distributions of partial sums of ‘off-diagonal’ polynomial forms in i.i.d. r.v.’s were studied in [33], [14], [3] and other works.

Assumption (A4) \(\varepsilon(0,0) \overset{d}{=} Z\) and \(Y(0,0) \overset{d}{=} Z\) have standard normal distribution \(Z \sim N(0,1)\) and \(X(t, s) = G(Y(t, s))\), where \(G = G(x), x \in \mathbb{R}\) is a measurable function with \(EG(Z)^2 < \infty, EG(Z) = 0\) and Hermite rank \(k \geq 1\).

Assumptions (A1), (A2) and (A4) imply that \(Y\) in (\ref{1.4}) is a Gaussian RF. As noted above, under Assumption (A4) Appell polynomials \(A_k(x)\) coincide with Hermite polynomials \(H_k(x)\). Recall that the Hermite rank of a measurable function \(G : \mathbb{R} \rightarrow \mathbb{R}\) with \(EG(Z)^2 < \infty\) is defined as the index \(k\) of the lowest nonzero coefficient \(c_k\) in the Hermite expansion of \(G\), viz., \(G(x) = \sum_{j=k}^{\infty} c_j H_j(x) / j!\) where \(c_k \neq 0\).

Let \(L^2(\mathbb{R}^{2k})\) denote the Hilbert space of real-valued functions \(h = h((u, v)_k), (u,v)_k = (u_1, v_1, \ldots, u_k, v_k) \in \mathbb{R}^{2k}\) with finite norm \(||h||_k := \{\int_{\mathbb{R}^{2k}} h^2((u, v)_k)d(u,v)_k\}^{1/2}\), \(d(u,v)_k = du_1dv_1 \cdots du_kdv_k\). Let \(W = \{W(du, dv), (u,v) \in \mathbb{R}^2\}\) denote a real-valued Gaussian white noise with zero mean and variance \(EW(du, dv)^2 = dudv\). For any \(h \in L^2(\mathbb{R}^{2k})\) the \(k\)-tuple Itô-Wiener integral \(\int_{\mathbb{R}^{2k}} h((u, v)_k)d^kW = \int_{\mathbb{R}^{2k}} h(u_1, v_1, \ldots, u_k, v_k)W(du_1dv_1 \cdots W(du_k, dv_k)\) is well-defined and satisfies \(E\int_{\mathbb{R}^{2k}} h((u, v)_k)d^kW = 0, E(\int_{\mathbb{R}^{2k}} h((u,v)_k)d^kW)^2 \leq k!||h||_k^2\), see e.g. [14].

3 Main results

Recall the definitions \(p_i, P\) in [18], [19]; \(\gamma_0 = q_1/q_2 = p_1/p_2\). Denote
\[ 
S^X_{\lambda, \gamma}(x,y) := \sum_{(t,s) \in K_{[\lambda x, \lambda y]}} X(t,s), \quad S^X_{\lambda, \gamma} := S^X_{\lambda, \gamma}(1,1)
\]
Consider a RF
\[ 
V_{k, \gamma_0}(x, y) := \int_{\mathbb{R}^{2k}} h(x,y; (u,v)_k)d^kW, \quad (x,y) \in \mathbb{R}^2_+,
\]
where (c.f. (2.1))

\[ h(x, y; (u, v)_k) := \int_{(0, x] \times (0, y]} \prod_{\ell=1}^k a_\infty(t - u_\ell, s - v_\ell) dt ds, \quad \text{where} \]

\[ a_\infty(t, s) := (|t|^2 + |s|^{2q/|q|} - q^2/2L_0(t/(|t|^2 + |s|^{2q/|q|})^{1/2}), \quad (t, s) \in \mathbb{R}^2. \quad (3.3) \]

Theorem 3.1 (i) The RF \( V_{k, \gamma_0} \) in (3.2) is well-defined for \( 1 \leq k < P \) as Itô-Wiener stochastic integral and has zero mean \( \text{EV}_{k, \gamma_0}(x, y) = 0 \) and finite variance \( \text{EV}_{k, \gamma_0}^2(x, y) = k!\|h(x, y; \cdot)\|_k^2 \). Moreover, RF \( V_{k, \gamma_0} \) has stationary rectangular increments and satisfies the OSRF property:

\[ \{V_{k, \gamma_0}(\lambda x, \lambda \gamma_0 y), (x, y) \in \mathbb{R}^2_+\} \overset{\text{fdd}}{=} \{\lambda^{H(\gamma_0)} V_{k, \gamma_0}(x, y), (x, y) \in \mathbb{R}^2_+\}, \quad \forall \lambda > 0, \quad (3.4) \]

where \( H(\gamma_0) := 1 + \gamma_0 - kp_1/2 \).

(ii) Let RFs \( Y = A_k(Y) \) satisfy Assumptions (A1), (A2) and (A3)_k, \( 1 \leq k < P \). Then

\[ \text{Var}(S^{X}_{\lambda, \gamma_0}) \sim c(\gamma_0)\lambda^{2H(\gamma_0)}, \quad c(\gamma_0) := \|h(1, 1; \cdot)\|_k^2 \]

and

\[ n^{-H(\gamma_0)} S^{X}_{\lambda, \gamma_0}(x, y) \overset{\text{fdd}}{\longrightarrow} V_{\gamma_0}(x, y). \quad (3.6) \]

Next, we discuss the case \( k < P, \gamma \neq \gamma_0 \). This case is split into four subcases: (c1): \( \gamma > \gamma_0, k > 1/p_2, \) (c2): \( \gamma > \gamma_0, k < 1/p_2, \) (c3): \( \gamma < \gamma_0, k > 1/p_1, \) and (c4): \( \gamma < \gamma_0, k < 1/p_1 \) (the ‘boundary’ cases \( k = 1/p_i, i = 1, 2 \) are more delicate and omitted, see Remark 3.2 below). Cases (c3) and (c4) are symmetric to (c1) and (c2) and essentially follow by exchanging the coordinates \( t \) and \( s \). Introduce random processes \( Z^+_k \) and \( Z^-_k \) with one-dimensional time:

\[ Z^+_k(y) := \int_{\mathbb{R}^{2k}} h^+_y(y; (u, v)_k) d^k W, \quad Z^-_k(x) := \int_{\mathbb{R}^{2k}} h^-_x(x; (u, v)_k) d^k W, \quad x, y \geq 0, \quad (3.7) \]

where

\[ h^+_y(y; (u, v)_k) := \int_0^y \prod_{i=1}^k a_\infty(u_i, s - v_i) ds, \quad h^-_y(y; (u, v)_k) := \int_0^x \prod_{i=1}^k a_\infty(t - u_i, v_i) dt, \quad (3.8) \]

and \( a_\infty(t, s) \) is defined in (3.3).

Theorem 3.2 (i) Processes \( Z^+_k \) and \( Z^-_k \) in (3.7) are well-defined for \( 1 \leq k < 1/p_2 \) and \( 1 \leq k < 1/p_1 \), respectively, as Itô-Wiener stochastic integrals. They have zero mean, finite variance, stationary increments and are self-similar with respective indices \( H^{+}_{2k} := 1 - kp_2/2 \in (1/2, 1) \) and \( H^-_{1k} := 1 - kp_1/2 \in (1/2, 1) \).

(ii) Let RFs \( Y = A_k(Y) \) satisfy Assumptions (A1), (A2) and (A3)_k, \( 1 \leq k < 1/p_2 \). Then for any \( \gamma > \gamma_0 \)

\[ \text{Var}(S^{X}_{\lambda, \gamma_0}) \sim c(\gamma)\lambda^{2H(\gamma)}, \quad (3.9) \]

where \( H(\gamma) := 1 + \gamma H^{+}_{2k} \) and \( c(\gamma) := \|h^+_y(1; \cdot)\|_k^2 \). Moreover,

\[ \lambda^{-H(\gamma)} S^{X}_{\lambda, \gamma_0}(x, y) \overset{\text{fdd}}{\longrightarrow} x Z^+_k(y). \quad (3.10) \]
(iii) Let RFs $Y$ and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1 \leq k < 1/p_1$. Then for any $\gamma < \gamma_0$
\[ \text{Var}(S^X_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \]  (3.11)
where $H(\gamma) := \gamma + H^{-1}_{1k}$ and $c(\gamma) := \|h_-(1; \cdot)\|_k^2 > 0$. Moreover,
\[ \lambda^{-H(\gamma)}S^X_{\lambda,\gamma}(x,y) \overset{fdd}{\longrightarrow} yZ_k^-(x). \]  (3.12)

**Remark 3.1** Processes $Z_k^\pm$ in (3.7) have a similar structure and properties to generalized Hermite processes discussed in [3] except that (3.7) are defined as $k$-tuple Itô-Wiener integrals with respect to white noise in $\mathbb{R}^2$ and not in $\mathbb{R}$ as in [3]. Following the terminology in [28], RFs $xZ_k^+(y)$ and $yZ_k^-(x)$ may be called a *generalized Hermite slide* since they represent a random surface ‘sliding linearly to 0’ along one of the coordinate on the plane from a generalized Hermite process indexed by the other coordinate. In the Gaussian case $k = 1$, a generalized Hermite slide agrees with a FBS $B_{H_1,H_2}$ where one of the two parameters $H_1, H_2$ equals 1. Recall that a fractional Brownian sheet (FBS) $B_{H_1,H_2} = \{B_{H_1,H_2}(x,y), (x,y) \in \mathbb{R}^2 \}$ with parameters $0 < H_1, H_2 \leq 1$ is a Gaussian process with zero mean and covariance function
\[ \text{cov}(B_{H_1,H_2}(x_1,y_1), B_{H_1,H_2}(x_2,y_2)) = (1/4)(x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1})(y_1^{2H_2} + y_2^{2H_2} - |y_1 - y_2|^{2H_2}). \]  (3.13)

**Theorem 3.3** (i) Let RFs $Y$ and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1/p_2 < k < P$. Then for any $\gamma > \gamma_0$
\[ \text{Var}(S^X_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \]  (3.14)
where $H(\gamma) := H^+_{1k} + \gamma/2$, $H^+_1 := 1 + \gamma_0/2 - kp_1/2 \in (1/2,1)$ and $c(\gamma) := \int_{(0,1]^2 \times \mathbb{R}} ((a_\infty \ast a_\infty)(t_1 - t_2, s))^k dt_1 dt_2 ds > 0$. Moreover,
\[ \lambda^{-H(\gamma)}S^X_{\lambda,\gamma}(x,y) \overset{fdd}{\longrightarrow} c(\gamma)^{1/2}B_{H^+_1,1/2}(x,y). \]  (3.15)

(ii) Let RFs $Y$ and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$, $1/p_1 < k < P$. Then for any $\gamma < \gamma_0$
\[ \text{Var}(S^X_{\lambda,\gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \]  (3.16)
where $H(\gamma) := \gamma H^-_{2k} + 1/2$, $H^-_{2k} := 1 + 1/(2\gamma_0) - kp_2/2 \in (1/2,1)$ and $c(\gamma) := \int_{\mathbb{R} \times (0,1]^2} ((a_\infty \ast a_\infty)(t, s_1 - s_2))^k dt ds_1 ds_2 > 0$. Moreover,
\[ \lambda^{-H(\gamma)}S^X_{\lambda,\gamma}(x,y) \overset{fdd}{\longrightarrow} c(\gamma)^{1/2}B_{1/2,H^-_{2k}}(x,y). \]  (3.17)

**Remark 3.2** Note $H^+_{1k} = 1$ ($kp_2 = 1$) and $H^-_{2k} = 1$ ($kp_1 = 1$). We expect that the convergences (3.15) and (3.17) remain true (modulus a logarithmic correction of normalization) in the ‘boundary’ cases $kp_2 = 1$ and $kp_1 = 1$ of Theorem 3.3 (i) and (iii) and the limit RFs in these cases agree with FBS $B_{1,1/2}$ or $B_{1/2,1}$, respectively, having both parameters equal to 1 or 1/2.

The next theorem discusses the case $k > P$.  

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Theorem 3.4 Let RFs $Y$ and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3) and $k > P$. Then for any $\gamma > 0$

$$\text{Var}(S^X_{\lambda,\gamma}) \sim \sigma^2_X \lambda^{1+\gamma},$$

(3.18)

where $\sigma^2_X := \sum_{(t,s) \in \mathbb{Z}^2} \text{Cov}(X(0,0), X(t,s)) \in (0, \infty)$. Moreover,

$$\lambda^{-(1+\gamma)/2} S^X_{\lambda,\gamma}(x,y) \xrightarrow{fdd} \sigma_X B_{1/2,1/2}(x,y).$$

(3.19)

Our last theorem extends the above results to general function $G$ having Hermite rank $k$ and Gaussian underlying RF $Y$.

Theorem 3.5 Let $X = G(Y)$ satisfy Assumption (A4). Assume w.l.g. that $G$ has Hermite expansion

$$G(x) = H_k(x) + \sum_{j=1}^{\infty} c_j H_j(x)/j!.$$

(i) Let $1 \leq k < P$. Then RF $X$ satisfies all statements of Theorems 3.1-3.3.

(ii) Let $k > P$. Then RF $X$ satisfies the statements of Theorem 3.4.

According to Theorems 3.2-3.3 the unbalanced scaling limits $V^X_\pm$ of RF $X = A_k(Y)$ satisfying Assumptions (A1)-(A3) are given by

$$V^X_+(x,y) = \begin{cases} xZ^+_k(y), & k p_2 < 1, \\ c^{1/2}_+ B_{H^+_k,1/2}(x,y), & k p_2 > 1, \end{cases}$$

$$V^X_-(x,y) = \begin{cases} yZ^-_k(x), & k p_1 < 1, \\ c^{1/2}_- B_{H^-_k,1/2}(x,y), & k p_1 > 1, \end{cases}$$

(3.20)

where $c_+ \equiv c(\gamma) > 0$ are given constants. The covariance functions of RFs $V^X_\pm$ in (3.20) agree (modulus a constant) with the covariance of FBS $B_{H_1,H_2}$ where at least one of the two parameters $H_1, H_2$ equals 1 or 1/2, namely $(H_1, H_2) = (1, H^+_k)$ if $k p_2 < 1$, = $(H^+_k, 1/2)$ if $k p_2 > 1$ in the case of $V^X_+$, and $(H_1, H_2) = (H^-_k, 1)$ if $k p_1 < 1$, = $(1/2, H^-_k)$ if $k p_1 > 1$ in the case of $V^X_-$. These facts and the explicit form of the covariance of FBS, see (3.13), imply that $V_+ \xrightarrow{fdd} aV_-$ ($\forall a > 0$), for any $k, p_1, p_2$ in Theorems 3.2-3.3 yielding the following corollary.

Corollary 3.1 Let RF $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3) and $1 \leq k < P, k p_i \neq 1, i = 1, 2$. Then $X$ exhibits scaling transition at $\gamma_0 = p_1/p_2$.

4 Examples: fractionally integrated RFs

In this section we present two examples of linear fractionally integrated random fields $Y$ in $\mathbb{Z}^2$ satisfying Assumptions (A1) and (A2).
Example 1. Isotropic fractionally integrated random field. Introduce the (discrete) Laplace operator
\[
\Delta Y(t,s) := (1/4) \sum_{|u|+|v|=1} (Y(t+u,s+v) - Y(t,s))
\]
and a lattice isotropic fractionally integrated random field satisfying the equation:
\[
(-\Delta)^d Y(t,s) = \varepsilon(t,s),
\]
where \{\varepsilon(t,s), (t,s) \in \mathbb{Z}^2\} are standard i.i.d. r.v.’s, \(0 < d < 1/2\) is the order of fractional integration,
\[
(1 - z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j,
\]
\[
\psi_j(d) := \Gamma(j - d)/\Gamma(j + 1)\Gamma(-d).
\]
More explicitly,
\[
(-\Delta)^d Y(t,s) = \sum_{j=0}^{\infty} \psi_j(d)(1 + \Delta)^j Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} b(u,v)Y(t-u,s-v),
\]
where \(b(u,v) := \sum_{j=0}^{\infty} \psi_j(d)p_j(u,v)\) and \(p_j(u,v)\) are \(j\)-step transition probabilities of a symmetric nearest-neighbor random walk \(\{W_k, k = 0,1,\cdots\}\) on \(\mathbb{Z}^2\) with equal 1-step probabilities \(P(W_1 = (u,v)|W_0 = (0,0)) = 1/4, |u| + |v| = 1\). Note \(\sum_{(u,v)\in\mathbb{Z}^2} |b(u,v)| = \sum_{j=0}^{\infty} \psi_j(d) < \infty, d > 0\) and therefore the l.h.s. of (4.2) is well-defined for any stationary random field \(\{Y(t,s)\}\) with \(E|Y(0,0)| < \infty\). As shown in [18], for \(0 < d < 1/2\) a stationary solution of (4.2) with zero-mean and finite variance can be defined as a moving-average random field:
\[
Y(t,s) = (-\Delta)^{-d}\varepsilon(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a(u,v)\varepsilon(t-u,s-v),
\]
with coefficients
\[
a(u,v) = \sum_{j=0}^{\infty} \psi_j(-d)p_j(u,v)
\]
satisfying \(\sum_{(u,v)\in\mathbb{Z}^2} a(u,v)^2 < \infty\). Moreover, RF \(Y\) in (4.3) has an explicit spectral density \(f(x,y) = (2\pi)^{-2}2^{-2d}|(1 - \cos x) + (1 - \cos y)|^{-2d}, (x,y) \in [-\pi,\pi]^2\) which behaves as \(\text{const} \ (x^2 + y^2)^{-2d}\) as \(x^2 + y^2 \to 0\).

According to ([18], Proposition 5.1), the moving-average coefficients in (4.3) satisfy the isotropic asymptotics:
\[
a(t,s) = (A + o(1))(t^2 + s^2)^{-(1-d)}, \quad t^2 + s^2 \to \infty,
\]
where \(A := \pi^{-1}\Gamma(1-d)/\Gamma(d)\) and hence Assumption (A2) with \(q_1 = q_2 = 2(1-d) \in (1,2), Q = 1/(1-d) \in (1,2)\) and a constant angular function \(L_0(z) = A, z \in [-1,1]\).

Example 2. Anisotropic fractionally integrated random field. Consider the ‘discrete heat operator’
\[
\Delta_{1,2} Y(t,s) = Y(t,s) - \theta Y(t-1,s) - \frac{1-\theta}{2}(Y(t-1,s+1) + Y(t-1,s-1)), \quad 0 < \theta < 1
\]
and a fractionally integrated random field satisfying
\[
\Delta_{1,2}^d Y(t,s) = \varepsilon(t,s),
\]
where \(\{\varepsilon(t,s)\}\) are as in (4.1). Similarly to (4.3), a stationary solution of (4.5) can be written as a moving-average random field:
\[
Y(t,s) = \Delta_{1,2}^{-d}\varepsilon(t,s) = \sum_{(u,v)\in\mathbb{Z} \times \mathbb{Z}} a(u,v)\varepsilon(t-u,s-v),
\]
with coefficients
\[
a(u,v) = \psi_u(-d)q_u(v)
\]
where \( q_u(v) \) are \( u \)-step transition probabilities of a random walk \( \{W_u, u = 0, 1, \cdots \} \) on \( \mathbb{Z} \) with 1-step probabilities \( P(W_1 = v|W_0 = 0) = \theta \) if \( v = 0, = (1-\theta)/2 \) if \( v = \pm 1 \). As shown in [20], \( \sum_{(u,v)\in \mathbb{Z}^2} a(t,s)^2 < \infty \) and the RF in (4.6) is well-defined for any \( 0 < d < 3/4, \theta \in [0,1) \); moreover, the spectral density \( f(x,y) \) of (4.6) is singular at the origin: \( f(x,y) \approx \text{const.} (x^2 + (1-\theta)^2y^4/4)^{-d}, (x,y) \to (0,0) \).

**Proposition 4.1** For any \( 0 < d < 3/4, 0 < \theta < 1 \) the coefficients in (4.7) satisfy Assumption (A2) with \( q_1 = 3/2-d, q_2 = 2q_1 \) and a continuous angular function \( L_0(z), z \in [-1,1] \) given by

\[
L_0(z) = \begin{cases} 
\frac{z^{d-3/2}}{\Gamma(d)\sqrt{2\pi(1-\theta)}}, & 0 < z \leq 1, \\
0, & -1 \leq z \leq 0.
\end{cases}
\]

**Remark 4.1** [7], [12] discussed fractionally integrated RFs satisfying the equation

\[
\Delta_1^{d_1} \Delta_2^{d_2} Y(t,s) = \varepsilon(t,s),
\]

where \( \Delta_1 Y(t,s) := Y(t,s) - Y(t-1,s), \Delta_2 Y(t,s) := Y(t,s) - Y(t,s-1) \) are difference operators and \( 0 < d_1, d_2 < 1/2 \) are parameters. Stationary solution of (1.9) is a moving-average RF in \( Y(t,s) = \sum_{(u,v)\in \mathbb{Z}^2} a(u,v)\varepsilon(t-u,v-s) \) with coefficients \( a(u,v) := \psi_u(-d_1)\psi_v(-d_2) \). Following the proof of Theorem 3.1 one can show that for any \( \gamma > 0 \) the (normalized) partial sums process of RF Y in (1.9) tends to a FBS depending on \( d_1, d_2 \) only, viz., \( \lambda^{-\gamma H_1-\gamma H_2} S_{\gamma}(x,y) \stackrel{\text{fdd}}{\longrightarrow} c(d_1)c(d_2)B_{H_1,H_2}(x,y) \), where \( H_i = d_i + 1/2 \) and \( c(d_i) > 0 \) are some constants. See [31, Proposition 3.2] for related result. We conclude that the fractionally integrated RF in (1.9) featuring a ‘separation of LRD along coordinate axes’ does not exhibit scaling transition in contrast to models in (1.1) and (4.5).

## 5 Properties of convolutions of generalized homogeneous functions

For a given \( \omega > 0 \) denote

\[
\rho(t,s) := (|t|^2 + |s|^2/\omega)^{1/2}, \quad \rho_+(t,s) := 1 \vee \rho(t,s), \quad (t,s) \in \mathbb{R}^2.
\]

Let \( f(t,s) = \rho(t,s)^{-h}L(t/\rho(t,s)) \) where \( L = L(z), z \in [-1,1] \) is an arbitrary measurable function, then \( f(t,s) \) satisfies the scaling property: \( f(\lambda t, \lambda^{-\omega} s) = \lambda^{-h}\rho(t,s), (t,s) \in \mathbb{R}_0^2 \) for each \( \lambda > 0 \). Such functions are called generalized homogeneous functions (see [15]).

We use the notation \([\phi_1 * \phi_2](t,s) = \sum_{(u,v)\in \mathbb{Z}^2} \phi_1(u,v)\phi_2(t+u,s+v)\) for ‘discrete’ convolution of sequences \( \phi_i = \{\phi_i(u,v), (u,v) \in \mathbb{Z}^2\} \in L^2(\mathbb{Z}^2), i = 1, 2 \) and \((\psi_1 \ast \psi_2)(t,s) = \int_{\mathbb{R}^2} \psi_1(u,v)\psi_2(t+u,s+v)dudv\) for ‘usual’ convolution of functions \( \psi_i = \{\psi_i(u,v), (u,v) \in \mathbb{R}^2\}, i = 1, 2 \). Note the symmetry \( \phi_i(t,s) = \phi_i(t,-s), i = 1, 2 \) implies the symmetry \([\phi_1 \ast \phi_2](t,s) = [\phi_1 \ast \phi_2](t,-s), (\phi_1 \ast \phi_2)(t,s) = (\phi_1 \ast \phi_2)(t,-s)\) of convolutions.

Let \( B_\delta(t,s) := \{(u,v) \in \mathbb{R}^2 : |t-u| + |s-v| \leq \delta\}, B_\delta^c(t,s) := \mathbb{R}^2 \setminus B_\delta(t,s) \).

**Proposition 5.1** (i) For any \( \delta > 0, h > 0 \),

\[
\int_{B_\delta(0,0)} \rho(t,s)^{-h}dt ds < \infty \quad \iff \quad h < 1 + \omega
\]

(5.2)
and
\[
\begin{align*}
\int_{B^c_{[0,0]}} \rho(t,s)^{-h} dt ds < \infty \\
\sum_{(t,s) \in \mathbb{Z}^2} \rho_+(t,s)^{-h} < \infty
\end{align*}
\] 
\[ \iff h > 1 + \varpi. \] (5.3)

(ii) Let \( h_i > 0, i = 1,2, h_1 + h_2 > 1 + \varpi. \) Then there exists \( C > 0 \) such that for any \((t,s) \in \mathbb{R}^2 \)
\[
\begin{align*}
(\rho^{-h_1} \ast \rho^{-h_2})(t,s) & \leq C \rho(t,s)^{1+\varpi-h_1-h_2}, & h_i < 1 + \varpi, i = 1,2, \\
(\rho_+^{-h_1} \ast \rho_+^{-h_2})(t,s) & \leq C \rho_+(t,s)^{-h_2}, & h_2 < 1 + \varpi, h_1 > 1 + \varpi, \\
(\rho_+^{-h_1} \ast \rho_+^{-h_2})(t,s) & \leq C \rho_+(t,s)^{-h_1 \wedge h_2}, & h_1 > 1 + \varpi, i = 1,2.
\end{align*}
\] (5.4) (5.5) (5.6)

Moreover, inequalities (5.4) - (5.6) are also valid for ‘discrete’ convolution \([\rho_+^{-h_1} \ast \rho_+^{-h_2}](t,s), (t,s) \in \mathbb{Z}^2 \) with \( \rho(t,s) \) on the r.h.s. of (5.4) replaced by \( \rho_+(t,s) \).

(iii) Let \( a_i = a_i(t,s), (t,s) \in \mathbb{Z}^2; i = 1,2 \) satisfy \( a_i(t,s) = \rho_+(t,s)^{-h_i}(L_i(t/\rho_+(t,s)) + o(1)), |t| + |s| \to \infty, \)
where \( 0 < h_i < 1 + \varpi, h_1 + h_2 > 1 + \varpi, \) and \( L_i(u) \neq 0, u \in [-1,1] \) are bounded piecewise continuous functions, \( i = 1,2. \) Let \( a_{i\infty}(u,v) := \rho(u,v)^{-h_i}L_i(u/\rho(u,v)), (u,v) \in \mathbb{R}^2, i = 1,2. \) Then
\[
[a_1 \ast a_2](t,s) = \rho_+(t,s)^{1+\varpi-h_1-h_2}(L_{12}(\frac{t}{\rho_+(t,s)}) + o(1)), \quad |t| + |s| \to \infty,
\] (5.7)

where
\[
L_{12}(z) := (a_{1\infty} \ast a_{2\infty})(z,(1 - z^2)^{\varpi/2}) = \int_{\mathbb{R}^2} a_{1\infty}(u,v)a_{2\infty}(u + z, v + (1 - z^2)^{\varpi/2}) dudv
\] (5.8)
is a bounded continuous function on the interval \( z \in [-1,1]. \) Moreover, if \( L_1(z) = L_2(z) \geq 0 \) then \( L_{12}(z) \) in (5.8) is strictly positive on \([-1,1]. \)

(iv) Let \( b(t,s) := \rho_+(t,s)^{-h}(L(t/\rho_+(t,s)) + o(1)), |t| + |s| \to \infty, (t,s) \in \mathbb{Z}^2, b_{\infty}(t,s) := \rho(t,s)^{-h}L(t/\rho(t,s)), (t,s) \in \mathbb{R}^2 \) where \( 0 < h < 1 + \varpi \) and \( L(u) \geq 0, u \in [-1,1] \) is a continuous function. Then for any \( \gamma > 0 \)
\[
B_\lambda(\gamma) := \sum_{(t_i,s_i) \in K_{[\lambda,\lambda \gamma],1,2}} b(t_i - t_2, s_i - s_2) \sim C(\gamma)\lambda^{2H(\gamma)}, \quad \lambda \to \infty,
\] (5.9)

where
\[
H(\gamma) := \begin{cases} 
1 + \varpi - \frac{h}{2}, \\
1 + \gamma - \frac{h}{2\varpi}, \\
1 + \gamma - \frac{h}{2}, \\
\frac{1}{2} + \gamma - \frac{h(h-1)}{4\varpi},
\end{cases} \quad C(\gamma) := \begin{cases} 
\int_{[0,1]^4} b_{\infty}(t_1 - t_2, s_1 - s_2) dt_1 dt_2 ds_1 ds_2, & (I) \\
\int_{[0,1]^2} b_{\infty}(0, s_1 - s_2) ds_1 ds_2, & (II) \\
\int_{[0,1]^2 \times \mathbb{R}} b_{\infty}(t_1 - t_2, s) dt_1 dt_2 ds, & (III) \\
\int_{[0,1]^2} b_{\infty}(t_1 - t_2, 0) dt_1 dt_2, & (IV) \\
\int_{[\mathbb{R} \times [0,1]^2]} b_{\infty}(t_1, s_1 - s_2) dt_1 ds_1 ds_2, & (V)
\end{cases}
\] (5.10)
in respective cases (I): \( \gamma = \varpi, \) (II): \( \gamma > \varpi, h < \varpi, \) (III): \( \gamma > \varpi, h > \varpi, \) (IV): \( \gamma < \varpi, h < 1 \) and (V): \( \gamma < \varpi, h > 1. \)
6 Covariance structure of subordinated anisotropic RFs

In this section from Proposition 5.1 with \( \varpi = \gamma_0 \) we obtain the asymptotic form of the covariance function of \( r_X(t, s) := \text{EX}(0, 0)X(t, s) \) and the asymptotics of the variance of anisotropic partial sums \( S^X_{\lambda, \gamma} \) of subordinated RF \( X = A_k(Y) \).

**Proposition 6.1** Let RF \( X = A_k(Y) \) satisfy assumptions (A1), (A2) and (A3)\(_k\).

(i) Let \( 1 \leq k < P \). Then

\[
   r_X(t, s) = \rho(t, s)^{- kp_1} (L_X(t/\rho(t, s)) + o(1)), \quad |t| + |s| \to \infty, \tag{6.1}
\]

where \( L_X(z) := (a_\infty \ast a_\infty)^k(z, (1 - z^2)^{\gamma_0/2}), \ z \in [-1, 1] \) is a strictly positive continuous function and \( a_\infty \) is defined in (3.3). Moreover, \( X(t, s) = Y^* k(t, s) + Z(t, s) \), where \( Z(t, s) \) is defined in (2.5) and

\[
   r_Z(t, s) = O(\rho(t, s)^{-2q_1}) = o(\rho(t, s)^{-kp_1}), \quad |t| + |s| \to \infty. \tag{6.2}
\]

(ii) Let \( k > P \). Then

\[
   r_X(t, s) = O(\rho(t, s)^{-(kp_1 \wedge (2q_1)}), \quad |t| + |s| \to \infty. \tag{6.3}
\]

Clearly, (6.1) implies \( C_1 \rho(t, s)^{-kp_1} \leq r_X(t, s) \leq C_2 \rho(t, s)^{-kp_1} \) for all \( |t| + |s| > C_3 \) and some \( 0 < C_i < \infty, \ i = 1, 2, 3 \). The last fact together with Proposition 5.1 (i) implies the following corollary.

**Corollary 6.1** Let \( X = A_k(Y), \ 1 \leq k < P \) be the subordinated RF defined in Proposition 6.1 and satisfying the conditions therein.

(i) Let \( 1 \leq k < P \). Then \( \sum_{(t, s) \in \mathbb{Z}^2} |r_X(t, s)| = \infty \). Moreover, \( \sum_{s \in \mathbb{Z}} |r_X(0, s)| = \infty \iff kp_2 \leq 1 \) and \( \sum_{t \in \mathbb{Z}} |r_X(t, 0)| = \infty \iff kp_1 \leq 1 \).

(ii) Let \( k > P \). Then \( \sum_{(t, s) \in \mathbb{Z}^2} |r_X(t, s)| < \infty \).

**Remark 6.1** Following the terminology in [28], we say that a covariance stationary RF \( X = \{X(t, s), (t, s) \in \mathbb{Z}^2\} \) has vertical LRD property (respectively, horizontal LRD property) if \( \sum_{s \in \mathbb{Z}} |r_X(0, s)| = \infty \) (respectively, \( \sum_{t \in \mathbb{Z}} |r_X(t, 0)| = \infty \)). From Corollary 6.1 we see the dichotomy of the limit distribution in Theorems 3.2 - 3.3 at points \( kp_2 = 1 \) at \( kp_1 = 1 \) is related to the change of vertical and horizontal LRD properties of the subordinated RF \( X = A_k(Y) \).

**Corollary 6.2** Let \( X(t, s) = A_k(Y(t, s)) = Y^* k(t, s) + Z(t, s), \ 1 \leq k < P \) be the subordinated RF defined in Proposition 6.1 and satisfying the conditions therein. Then for any \( \gamma > 0 \)

\[
   \text{Var}(S^X_{\lambda, \gamma}) \sim \text{Var}(S^{Y^* k}_{\lambda, \gamma}) \sim c(\gamma)\lambda^{2H(\gamma)}, \quad \lambda \to \infty \tag{6.4}
\]

and

\[
   \text{Var}(S^Z_{\lambda, \gamma}) = O(\lambda^{1+\gamma}), \tag{6.5}
\]

where \( H(\gamma) \in ((1 + \gamma)/2, 1 + \gamma) \) and \( c(\gamma) \) are defined in Theorems 5.1 - 5.3.
7 Proofs of Theorems 3.1–3.5

We use the criterion in Proposition 7.1 for the convergence in distribution of off-diagonal polygonal forms towards Itô-Wiener integral which is a straightforward extension of ([14], Proposition 14.3.2).

Let $L^2(\mathbb{Z}^2k)$ be the class of all real functions $g = g((u, v)_k), (u, v)_k \in \mathbb{Z}^2k$ with $\sum_{(u, v)_k \in \mathbb{Z}^2k} g((u, v)_k)^2 < \infty$ and $Q_k(g) := \sum_{(u, v)_k} g((u, v)_k)\varepsilon(u_1, v_1) \cdots \varepsilon(u_k, v_k), g \in L^2(\mathbb{Z}^2k)$ be a $k$-tuple off-diagonal form in i.i.d. r.v.’s $\{\varepsilon(u, v)\}$ satisfying Assumption (A1). For $g_{\lambda, \gamma} \in L^2(\mathbb{Z}^2k) (\lambda > 0, \gamma > 0)$ define a step function $\tilde{g}_{\lambda, \gamma} \in L^2(\mathbb{R}^2k)$ by

$$\tilde{g}_{\lambda, \gamma}((u, v)_k) := \lambda^{k\gamma(1+\gamma_0^{-1})/2}g_{\lambda, \gamma}([\lambda^{\gamma/\gamma_0}u_1], [\lambda^{\gamma/\gamma_0}u_2], \ldots, [\lambda^{\gamma/\gamma_0}u_k], [\lambda^{\gamma}v_k]), \quad (u, v)_k \in \mathbb{R}^{2k}. \quad (7.1)$$

**Proposition 7.1** Assume that there exists $h_\gamma \in L^2(\mathbb{R}^{2k})$ such that $\lim_{\lambda \to \infty} \|\tilde{g}_{\lambda, \gamma} - h_\gamma\|_k \to 0$. Then $Q_k(g_{\lambda, \gamma}) \xrightarrow{d} \int_{\mathbb{R}^{2k}} h_\gamma((u, v)_k)d^kW (\lambda \to \infty)$.

**Proof of Theorem 3.2** Let $p(t, s) := (|t|^2 + |s|^{2/\gamma_0})^{1/2}, (t, s) \in \mathbb{R}^2$.

(i) Let us show that the stochastic integral $V_{k, \gamma_0}(x, y)$ is well-defined or $h(x, y; \cdot)\|_k < \infty$, where $h(x, y; (u, v)_k)$ is defined in (3.3). It suffices to consider the case $x = y = 1$. By Proposition 3.1 $\|h(1, 1; \cdot)\|_k = \sum_{(t, s)} = \infty$ holds. The self-similarity property in (3.4) follows by scaling properties $a_\alpha(t, s, u_0, v_0) = \lambda^\alpha a_\alpha(t, s, u, v)$, $\{W(d\alpha u, d\lambda^{\gamma \alpha} v)\}$ of the integrand and the white noise, and the change of variables rules for multiple Itô-Wiener integral, see [8], also ([14], Proposition 14.3.5).

(ii) Relation (3.5) is proved in Proposition 6.2. Let us prove (3.6). Recall the decomposition $X(t, s) = Y^{*k}(t, s) + Z(t, s)$ in Corollary 6.2. Using $\text{Var}(S^2_{\lambda, \gamma_0}) = O(\lambda^{1+\gamma_0}) = o(\lambda^{2H(\gamma_0)})$, see (6.5), relation (3.6) follows from

$$Q_k(g_{\lambda, \gamma_0}(x, y; \cdot)) = \lambda^{-H(\gamma_0)} \sum_{(t, s) \in K_{[x, \lambda, \gamma_0]}} Y^{*k}(t, s) \quad \text{fdd} \quad V_{\gamma_0}(x, y), \quad (7.2)$$

where

$$g_{\lambda, \gamma_0}(x, y; (u, v)_k) := \lambda^{-H(\gamma_0)} \sum_{(t, s) \in K_{[x, \lambda, \gamma_0]}} a(t - u_1, s - v_1) \cdots a(t - u_k, s - v_k), \quad (u, v)_k \in \mathbb{Z}^{2k}. \quad (7.3)$$

Using Proposition 7.1 and Cramér-Wold device, relation (7.2) follows from

$$\lim_{\lambda \to \infty} \|\sum_{(t, s) \in K_{[x, \lambda, \gamma_0]}} \theta_i(g_{\lambda, \gamma_0}(x, y; \cdot) - h(x_i, y_1; \cdot))\|_k = 0, \quad (7.4)$$

for any $m \geq 1$ and any $\theta_i \in \mathbb{R}, (x_i, y_1) \in \mathbb{R}^2, 1 \leq i \leq m$, where the limit function $h(x, y; (u, v)_k)$ is given in (3.3). We restrict the subsequent proof of (7.4) to the case $m = \theta_1 = 1, (x_1, y_1) = (x, y)$ since the general case of (7.4) follows analogously. Using (2.1), (7.3), (7.1) and notation $a_\lambda(t, s) := (\lambda^{-1} \rho(t, s))^{-q_1}(L_0(t/\lambda^{-1} \vee \rho(t, s)) + o(1)), \lambda \to \infty$ and $\lambda' := \lambda^{\gamma_0}$ similarly to (8.24) we get

$$\tilde{g}_{\lambda, \gamma_0}(x, y; (u, v)_k) = \int_{\mathbb{R}^2} \prod_{i=1}^k a_\lambda\left(\frac{|\lambda'| - |\lambda u_i|}{\lambda}, \frac{|\lambda' v_i| - |\lambda' u_i|}{\lambda'}\right)1([\lambda'], [\lambda' v_i]) \in (0, \lambda x] \times (0, \lambda' y))
$$

$$\to h(x, y; (u, v)_k) \quad (7.5)$$
point-wise for any \((u, v)_k \in \mathbb{R}^{2k}, (u_i, v_i) \neq (u_j, v_j) (i \neq j)\) fixed. We use a similar bound to (8.20), viz.,
\[
\frac{1}{\lambda} \vee \rho\left(\frac{[\lambda]-[\lambda u]}{\lambda}, \frac{[\lambda s]-[\lambda v]}{\lambda}\right) \geq cp(t - u, s - v), \quad \forall \ t, u, s, v \in \mathbb{R}, \ \exists c > 0, \quad (7.6)
\]

implying the dominated bound
\[
|\tilde{g}_{\lambda, y}(x, y; (u, v)_k)| \leq C \int_{0,2A} \prod_{t=1}^{k} \rho(t - u_i, s - v_i)^{-q_1} dt ds =: \bar{g}(x, y; (u, v)_k)
\]
with \(\|\bar{g}(x, y; \cdot)\|_k < \infty\) so that (7.4) follows from (7.5) and Proposition 5.1 by the dominated convergence theorem. Theorem 3.1 is proved. \ \Box

**Proof of Theorem 3.2.** As noted in Sec. 3, part (iii) follows by the same argument as part (ii) by exchanging the coordinates \(t\) and \(s\) and we omit the details.

(i) Let us show that the stochastic integral in (3.7) is well-defined or \(h_+ (y; \cdot)\|_k < \infty\), where \(h_+ (y; (u, v)_k)\) is defined in (3.8). Indeed by Proposition 5.1 \(\|h_+ (y; \cdot)\|_k = \int_{0,1} ((a_\infty * a_\infty)(0, s_1 - s_2))^k ds_1 ds_2 \leq C \int_{0,1}^2 \rho(0, s_1 - s_2)^{-k p_1} ds_1 ds_2 \leq C \int_{-1,1} |s|^{-k p_2} ds < \infty\) since \(k p_2 < 1\). The remaining facts in (i) follow similarly as in the proof of Theorem 3.1(ii).

(ii) Relation (3.9) is proved in Corollary 6.2. Similarly to the proof of (3.6), the weak convergence in (3.10) follows from
\[
Q_k (g_{\lambda, \gamma}(x, y; \cdot)) = \lambda^{-H(\gamma)} \sum_{(t, s) \in K_{[\lambda x, \lambda \gamma y]}} Y^k (t, s) \xrightarrow{fdd} x Z_+(y), \quad (7.7)
\]
where
\[
g_{\lambda, \gamma}(x, y; (u, v)_k) := \lambda^{-H(\gamma)} \sum_{(t, s) \in K_{[\lambda x, \lambda \gamma y]}} a(t - u_1, s - v_1) \cdots a(t - u_k, s - v_k), \quad (u, v)_k \in \mathbb{Z}^{2k}. \quad (7.8)
\]

Again, we restrict the proof of (7.7) to one-dimensional convergence at \((x, y) \in \mathbb{R}^2_+\). By Proposition 7.1 this follows from
\[
\lim_{\lambda \to \infty} \|\tilde{g}_{\lambda, \gamma}(x, y; \cdot) - x h_+(y; \cdot)\|_k = 0, \quad (7.9)
\]
where, with \(\lambda := \lambda', \lambda'' := \lambda'^{\gamma/\gamma_0} \gg \lambda, a_{\lambda'}(t, s) := (|\lambda''|^{-1} \vee \rho(t, s))^{-q_1} (L_0(t/|\lambda''|^{-1} \vee \rho(t, s))) + o(1),\)
\[
\bar{g}_{\lambda, \gamma}(x, y; (u, v)_k) = \int_{\mathbb{R}^2} \prod_{t=1}^{k} a_{\lambda''} (\frac{[\lambda]-[\lambda'' u]}{\lambda''}, \frac{[\lambda s]-[\lambda'' v]}{\lambda''}) 1((|\lambda|, |\lambda'|) \in (0, \lambda x) \times (0, \lambda y)) dt ds \rightarrow x h_+(y; (u, v)_k) \quad (7.10)
\]
point-wise for any \((u, v)_k \in \mathbb{R}^{2k}, (u_i, v_i) \neq (u_j, v_j) (i \neq j)\) fixed.

The dominating convergence argument to prove (7.9) from (7.10) uses Pratt’s lemma 29, as follows. Similarly to (7.6) note that
\[
\frac{1}{\lambda''} \vee \rho\left(\frac{[\lambda]-[\lambda'' u]}{\lambda''}, \frac{[\lambda s]-[\lambda'' v]}{\lambda''}\right) \geq cp((\lambda t/\lambda'') - u, s - v), \quad \forall \ t, u, s, v \in \mathbb{R}, \ \exists c > 0
\]
and hence
\[ |g_{\lambda,\gamma}(x,y; (u,v)_k)| \leq C \int_{(0,2\pi)^2} 1_{\{0,2\pi\}}(t) \rho((\lambda t/\lambda')^2) \, dt \, ds =: CG_{\lambda}((u,v)_k) \]

with \( C > 0 \) independent of \( \lambda > 0 \), \((u,v)_k \in \mathbb{R}^2 \). Clearly, \( \lim_{\lambda \to \infty} G_{\lambda}((u,v)_k) = G((u,v)_k) := 2x \int_{(0,2\pi)} \rho(-u_i, s - v_i)^{-q_i} \, ds \) point-wise in \( \mathbb{R}^2 \) and
\[
\|G_{\lambda}\|^2_k = \int_{(0,2\pi)^2} (\rho^{-q_1}(\lambda t/\lambda'))^2 (t_1, s_1) \, dt_1 \, ds_1 \, ds_2 \ni \|G\|^2 < \infty
\]
by Proposition 5.1 and condition \( 1 \leq k < 1/p_2 \), or \( p_2 = q_2(2 - Q) < 1/k \). Thus, application of \( \ref{20} \) proves \( \ref{7.9} \). Theorem 3.2 is proved. \( \square \)

To prove Theorem 3.3 we use approximation by \( m \)-dependent variables and the following CLT for triangular array of \( m \)-dependent r.v.’s.

**Lemma 7.1** Let \( \{\xi_n, 1 \leq i \leq N_n\}, n \geq 1 \) be a triangular array of \( m \)-dependent r.v.’s with zero mean and finite variance. Assume that: (L1) \( \xi_n, 1 \leq i \leq N_n \) are identically distributed for any \( n \geq 1 \), (L2) \( \xi_n := \xi_n1_{\{|\xi_n| \leq \tau n^{1/2}\}}, \alpha_{ni} := Cov(\xi_{ni}, \xi_{nj}) \). It suffices to show that for any \( \tau > 0 \) the following conditions in \( \ref{25} \) are satisfied: (O1) \( n^{-1/2} \sum_{i=1}^n \alpha_{ni} \to 0 \), (O2) \( n^{-1} \sum_{i,j=1}^n \sigma_{nj} \to \sigma^2 \), (O3) \( n^{-1} \sum_{i,j=1}^n \sigma_{nij} = O(1) \), and (O4) \( \sum_{i=1}^n P(|\xi_n| > \tau n^{1/2}) \to 0. \)

Consider (O1), or \( n^{1/2} \alpha_{ni} \to 0 \), \( \alpha_{ni} := \alpha_{ni}. \) We have \( 0 = n^{1/2}E\xi_n = n^{1/2} \alpha_{ni} + \kappa_n \), where \( \kappa_n := n^{1/2}|E\xi_n1(|\xi_n| > \tau n^{1/2})| \leq \tau^{-1}E\xi_n1(|\xi_n| > \tau n^{1/2}) \). Therefore, (O1) follows from
\[
E\xi_n^21(|\xi_n| > \tau n^{1/2}) \to 0. \tag{7.11}
\]
Using the Skorohod representation theorem \( \ref{32} \) w.l.g. we can assume that r.v.s \( \xi, \xi_n, n \geq 1 \) are defined on the same probability space and \( \xi_n \to \xi \) almost surely. The latter fact together with (L2) and Pratt’s lemma \( \ref{29} \) implies that \( E|\xi_n - \xi|^2 \to 0 \) and hence \( \tag{7.11} \) follows due to \( P(|\xi_n| > \tau n^{1/2}) \to 0, \) see \( \ref{23}, \) Ch.2, Prop.5.3). The above argument also implies (O4) since \( P(|\xi_n| > \tau n^{1/2}) \leq \tau^{-1}n^{-1}E\xi_n^21(|\xi_n| > \tau n^{1/2}) \) by Markov’s inequality. (O3) is immediate from (L1) and (L2). Finally, (O2) follows from (L3), (O1) and
\[
\sum_{1 \leq i,j \leq n, |i-j| \leq m} E(\xi_{ni} \xi_{nj} - \xi_{ni}^* \xi_{nj}^*) \to 0. \tag{7.12}
\]
Let \( \xi_{ni}^* := \xi_{ni} - \xi_{ni}^* \). Since \( E(\xi_{ni} \xi_{nj} - \xi_{ni}^* \xi_{nj}^*) \leq |E(\xi_{ni} \xi_{nj} - \xi_{ni}^* \xi_{nj}^* + \xi_{ni} \xi_{nj}^*)| \leq CE\xi_n^21(|\xi_n| > \tau n^{1/2}) \), relation \( \tag{7.12} \) follows from \( \ref{7.11} \). Lemma \( \ref{4.1} \) is proved. \( \square \)
Proof of Theorem 3.3 Again, we prove part (i) only since part (ii) follows similarly by exchanging the coordinates \( t \) and \( s \).

Relation (3.11) is proved in Proposition 5.1. Let us prove (3.12). Similarly as in the case of the previous theorems, we shall restrict ourselves with the proof of one-dimensional convergence at \((x, y) \in \mathbb{R}_2^2\). For \( m \geq 1, \lambda > 0 \) define stationary RFs

\[
X_m(t, s) := A_k(Y_m(t, s)), \quad \text{where} \quad Y_m(t, s) := \sum_{(u,v) \in \mathbb{Z}^2: |s-v| \leq |\lambda| m} a(t-u, s-v)\epsilon(u,v), \quad (7.13)
\]

and where \( A_k \) stands for the Appell polynomial of degree \( k \) relative to the distribution of \( Y_m(t, s) \). Note \( X_m(t_1, s_1) \) and \( X_m(t_2, s_2) \) are independent if \(|s_1 - s_2| > 2|\lambda| m\). Then

\[
S_{\lambda, \gamma}^X(x, y) := \sum_{(t,s) \in K[\lambda x, \lambda \gamma y]} X_m(t, s) = \sum_{i=0}^N U_{\lambda, m}(i) \quad (7.14)
\]

where \( N := \lfloor |\lambda y|/|\lambda| m \rfloor \) is \( O(\lambda^{-\gamma m}) \) and

\[
U_{\lambda, m}(i) := \sum_{1 \leq t \leq \lambda x} \sum_{i | \lambda| m < s \leq (i+1) |\lambda| m} X_m(t, s) \quad (7.15)
\]

Note \( U_{\lambda, m}(i) \) and \( U_{\lambda, m}(j) \) are independent provided \(|i-j| > 2m\) hence (7.14) is a sum of \( 2m \)-dependent r.v.\'s. The one-dimensional convergence in (5.12) follows from standard Slutsky’s argument (see e.g. [14], Lemma 4.2.1) and the following lemma. Theorem 3.3 is proved. \( \square \)

Lemma 7.2 Under the conditions and notation of Theorem 3.3 (i), for any \( \gamma > \gamma_0 \) and any \( m = 1, 2, \cdots \)

\[
\text{Var}(S_{\lambda, \gamma}^X(x, y)) \sim \sigma_m^2(x, y) \lambda^{2H(\gamma)} \quad \text{and} \quad \lambda^{-H(\gamma)} S_{\lambda, \gamma}^X(x, y) \xrightarrow{d} N(0, \sigma_m^2(x, y)) \quad \text{as} \quad \lambda \to \infty, \tag{7.16, 7.17}
\]

where \( \sigma_m^2(x, y) \) is defined in (7.19) below. Moreover,

\[
\lim_{m \to \infty} \limsup_{\lambda \to \infty} \lambda^{-2H(\gamma)} \text{Var}(S_{\lambda, \gamma}^X(x, y) - S_{\lambda, \gamma}^X(x, y)) = 0. \tag{7.18}
\]

Proof. By adapting the argument in the proof of (3.11) and Proposition 5.1 (iv), Case (III), we can show the limits

\[
\lambda^{-2H(\gamma)} \text{Var}(S_{\lambda, \gamma}^X(x, y)) \to y \int_{(0,x)^2 \times \mathbb{R}} ((a_{\infty,m} * a_{\infty,m})(t_1 - t_2, s))^k dt_1 dt_2 ds =: \sigma_m^2(x, y) \quad \tag{7.19}
\]

and

\[
\lambda^{-2H(\gamma)} \text{Var}(S_{\lambda, \gamma}^X(x, y) - S_{\lambda, \gamma}^X(x, y))
\]

\[
= \lambda^{-2H(\gamma)} \sum_{(t_1, s_1) \in K[\lambda x, \lambda \gamma y], i=1,2} \left\{ \text{Cov}(X(t_1, s_1), X(t_2, s_2)) - \text{Cov}(X(t_1, s_1), X_m(t_2, s_2)) - \text{Cov}(X_m(t_1, s_1), X(t_2, s_2)) + \text{Cov}(X_m(t_1, s_1), X_m(t_2, s_2)) \right\}
\]

\[
\to y \int_{(0,x)^2 \times \mathbb{R}} G_m(t_1 - t_2, s) dt_1 dt_2 ds, \quad \lambda \to \infty, \tag{7.20}
\]
where \( G_m(t, s) := ((a_∞ * a_∞)(t, s))^k - ((a_∞ * a_∞, m)(t, s))^k - ((a_∞, m * a_∞, m)(t, s))^k + ((a_∞, m * a_∞, m)(t, s))^k \) and 
\[
a_{∞, m}(t, s) := L_0(t/ρ(t,s))ρ(t,s)^{-q_1}1(|s| ≤ m), \quad (t, s) ∈ ℜ^2.
\] (7.21)
is a ‘truncated’ version of \( a_∞(t, s) \) in (3.3). Since \( |G_m(t, s)| ≤ 4(a_∞ * a_∞)(t, s))^k \) and \( G_m(t, s) \) vanishes with \( m → ∞ \) for any fixed \((t, s) ≠ (0, 0)\), (7.18) follows from (7.20) by the dominated convergence theorem.

The proof of (7.17) uses Lemma 7.1. Accordingly, let \( N_λ := [Λγ]/[Λ^n] \) and \( ξ_λ := λ^{-H(λ^n)}U_{λ,m}(i) \), where 
\( H(λ^n) = 1 + λ^n - kρ/2 \) is the same as in Thm 3.1 and \( U_{λ,m}(i) \) are \( 2m \)-dependent r.v.’s defined in (7.15). Note \( U_{λ,m}(i), i = 1, \ldots, N_λ - 1 \) are identically distributed and \( λ^{-H(λ^n)}N_λ^{1/2} \sim λ^{-H(γ/2)}y^{1/2} \). Thus, condition (L1) of Lemma 7.1 for \( ξ_λ, 1 ≤ i ≤ N_λ - 1 \) is satisfied and (L3) follows from \( Var(∑i=1^{N_λ-1}ξ_λ) \sim Var(S^X_{λ,γ}(x, y)) \sim c_m(γ)x^{2Hk}y \), see (7.16). Finally, condition (L2), or
\[
ξ_{λ,1} = λ^{-H(λ^n)}U_{λ,m}(1) → ξ, \quad Eξ^2 \sim Eξ^2
\] (7.22)
follows similarly as in Theorem 3.1 with the limit r.v. \( ξ \) given by the \( k \)-tuple Itô-Wiener integral:
\[
ξ := ∫_R^k \{ ∫_0^x ∫_0^1 \prod_{ℓ=1}^k a_{∞, m}(t - u_ℓ, s - v_ℓ) dt ds \} d^kW
\]
and \( a_{∞, m}(t, s) \) defined in (7.21). This proves (7.17) and Lemma 7.2 too. □

**Proof of Theorem 3.4** The proof is an adaptation of the proof of CLT in (14), Theorem 4.8.1) for sums of ‘off-diagonal’ polynomial forms with one-dimensional ‘time’ parameter. Define
\[
X_m(t, s) := A_k(Y_m(t, s)), \quad Y_m(t, s) := ∑_{(u,v)∈Z^2:|t-u|+|s-v|≤m} a(t - u, s - v)ε(u, v),
\] (7.23)
where \( A_k \) stands for the Appell polynomial of degree \( k \) relative to the distribution of \( Y_m(t, s) \). Note the truncation level \( m \) in (7.23) does not depend on \( λ \) in contrast to the truncation level \( m[λ^n] \) in (7.13). Similarly to Lemma 7.2 it suffices to prove for any \( γ > 0, m = 1, 2, \ldots \)
\[
Var(S^X_{λ,γ}(x, y)) \sim x y σ^2_{X,λ,γ} \sim λ^{-(1+γ)/2}S^X_{λ,γ}(x, y) \sim N(0, x y σ^2_{X,λ,γ})
\] (7.24)
\[
lim_{m→∞} lim sup_{λ→∞} λ^{-(1+γ)}Var(S^X_{λ,γ}(x, y) - S^X_{λ,γ}(x, y)) = 0.
\] (7.25)
where \( σ^2_{X,λ,γ} := ∑_{(s,t)∈Z^2} r_{X_m}(t, s) \) and \( r_{X_m}(t, s) := Cov(X_m(0, 0), X_m(t, s)) \). Note \( X_m(t_1, s_1) \) and \( X_m(t_2, s_2) \) are independent if \( |t_1 - t_2| + |s_1 - s_2| > 2m \). Therefore \( ∑_{(t,s)∈Z^2} r_{X_m}(t, s) < ∞ \) and (7.24) follows the CLT for \( m \)-dependent RFs, see (5). Consider (7.25), where we can put \( x = y = 1 \) w.l.g. We have \( λ^{-(1+γ)}Var(S^X_{λ,γ} - S^X_{λ,γ}) \) ≤ \( ∑_{(t,s)∈Z^2} φ_m(t, s) \), where \( φ_m(t, s) := Cov(X(0, 0) - X_m(0, 0), X(t, s) - X_m(t, s)) \). From (5.26), (5.27) and (8.29) we conclude that
\[
|Cov(X(0, 0), X(t, s))| + |Cov(X(0, 0), X_m(t, s))| + |Cov(X_m(0, 0), X_m(t, s))| ≤ C ρ(t, s)^{-(kp_1)^2q_1} \] as in (6.3), with \( C > 0 \) independent of \( m \). Therefore, \( |φ_m(t, s)| ≤ C ρ(t, s)^{-(kp_1)^2q_1} =: φ(t, s) \), where \( ∑_{(t,s)∈Z^2} φ(t, s) < ∞ \), see Proposition 5.1(i), also Corollary 6.1(ii). Thus, (7.24) follows by the dominated convergence theorem and the fact that \( lim_{m→∞} φ_m(t, s) = 0 \) for any \( (t, s) ∈ Z^2 \). Theorem 3.4 is proved. □
**Proof of Theorem 3.2** (i) Split $X = X_k + X'_k$, where $X'_k := \sum_{j=k+1} c_j X_j/j!$, $X_j(t, s) := H_j(Y(t, s))$. Since all statements of Theorems 3.1, 3.3 hold for RF $X_k = H_k(Y)$ and $\text{Cov}(X_k(t_1, s_1), X'_k(t_2, s_2)) = 0, \forall (t_1, s_1) \in \mathbb{Z}^2, i = 1, 2$, it suffices to show that
\[
\text{Var}(S_{\lambda, \gamma}^{X'_k}) = o(\lambda^{2H(\gamma)}), \quad \lambda \to \infty
\] (7.26)
for $H(\gamma)$ defined in Theorems 3.1, 3.3. By well-known properties of Hermite polynomials, $\text{Var}(S_{\lambda, \gamma}^{X'_k}) = \sum_{j=k+1} c_j^2 \text{Var}(S_{\lambda, \gamma}^{X_j})/(j!)^2$ and $\text{Var}(S_{\lambda, \gamma}^{X'_k}) = \sum_{j=k} c_j^2 \text{Var}(S_{\lambda, \gamma}^{X_j})/(j!)^2 \leq \sum_{j=k} c_j^2 r_Y(t_1 - t_2, s_1 - s_2) \leq j! \Sigma_{k+1}(\lambda)$, where $\Sigma_{k+1}(\lambda) := \sum_{j=k+1} c_j^2 r_Y(t_1 - t_2, s_1 - s_2)$ for $j \geq k + 1$ since $|r_Y(t, s)| \leq 1$ according to Assumption (A4). Hence, $\text{Var}(S_{\lambda, \gamma}^{X'_k}) \leq \sum_{j=k+1} c_j^2/(j!)^2 \Sigma_{k+1}(\lambda) \leq E\sigma(0, 0)^2 \Sigma_{k+1}(\lambda)$, where $\Sigma_{k+1}(\lambda) = o(\lambda^{2H(\gamma)})$ follows by Proposition 5.1. This proves (7.26) and part (i).

(ii) For large $K \in \mathbb{N}, K > k$, split $X = X_K + X'_K$, where $X_K(t, s) = \sum_{j=0}^{K} c_j H_j(Y(t, s))/j!$, and the last result extends to finite sums of Hermite polynomials, viz., $\lambda^{-(1+\gamma)/2} S_{\lambda, \gamma}^{X'_K} / j!$ as in the proof of part (i), implying $\text{Var}(S_{\lambda, \gamma}^{X'_K}) \leq C \epsilon_K \lambda^{1+\gamma}$ where $\epsilon_K := \sum_{j=k+1} c_j^2/j!$ can be made arbitrarily small by choosing $K$ large enough. On the other hand, by Theorem 3.4 $\lambda^{-(1+\gamma)/2} S_{\lambda, \gamma}^{X'_K} / j!$ is distributed as $\sigma X_j B_{1/2, 1/2}(x, y)$ for any $j \geq k$ and the last result extends to finite sums of Hermite polynomials, viz., $\lambda^{-(1+\gamma)/2} S_{\lambda, \gamma}^{X'_K} / j!$ is distributed as $\sigma X_K B_{1/2, 1/2}(x, y)$, where $\sigma^2_{X_K} = \sum_{j=0}^{K} \text{Cov}(X_K(t_1, s_1), X_K(t_2, s_2)) \to \sigma^2_X, K \to \infty$. See e.g. ([14], proof of Thm.4.6.1). The remaining details are easy. Theorem 3.4 is proved.

8 Proofs of Propositions 4.1, 5.1, 6.1 and Corollary 6.2

**Proof of Proposition 4.1** The transition probabilities $q_u(v)$ in (4.17) can be explicitly written in terms of binomial probabilities $\text{bin}(j, k; p) := (k/j)^p(1 - p)^{k-j}, k = 0, 1, \ldots, j = 0, 1, \ldots, k, 0 \leq p \leq 1$:
\[
q_u(v) = \sum_{j=0}^{u} \text{bin}(u - j, u; \theta) \text{bin}((v + j)/2, j; 1/2), \quad u \in \mathbb{N}, \ |v| \leq u.
\] (8.1)

Similarly to ([18], proof of Prop.4.1) we shall use the following version of the Moivre-Laplace theorem (Feller [10], ch.7, §2, Thm.1): There exists a constant $C$ such that $j \to \infty$ and $k \to \infty$ vary in such a way that
\[
\frac{(j - kp)^3}{k^2} \to 0,
\] (8.2)
then
\[
\left| \frac{1}{\sqrt{2\pi kp(1-p)}} \exp \left\{ -\frac{(jkp)^2}{2kp(1-p)} \right\} - 1 \right| < \frac{C}{k} + \frac{C|j - kp|^3}{k^2}.
\] (8.3)

Let us first explain the idea of the proof. Using (8.1) and replacing the binomial probabilities by Gaussian
densities according to (8.3) leads to

\[
a(u, v) \sim \frac{1}{2} \sum_{j=0}^{\lfloor u \rfloor} \frac{1}{\Gamma(d)} \frac{1}{u^{1-d}} \frac{1}{2\pi \theta(1-\theta)} e^{-u(j-(1-\theta)u)/2(\theta(1-\theta)u)} \frac{1}{\sqrt{j/u}} e^{-v^2/2j}
\]

\[
= \frac{1}{\Gamma(d)} \frac{1}{\sqrt{2\pi u^{3/2}-d}} \sum_{j=0}^{\lfloor u \rfloor} \frac{1}{u^{1}} \frac{1}{2\pi \theta(1-\theta)/u} e^{-u(j/u-(1-\theta)u)/2\theta(1-\theta)} \frac{1}{\sqrt{j/u}} e^{-v^2/u(2/j)} dx
\]

\[
\sim \frac{1}{\Gamma(d)} \frac{1}{\sqrt{2\pi u^{3/2}-d}} \int_{0}^{\infty} \frac{1}{2\pi \theta(1-\theta)/u} e^{-u(x-(1-\theta)u)/2\theta(1-\theta)} \frac{1}{\sqrt{x}} e^{-v^2/2x} dx
\]

\[
= \rho(u, v)^{d-3/2} L_0(z)
\]

with \(L_0(z)\) defined in (8.8). Here, factor 1/2 in front of the sum in the first line appears since \(\text{bin}((v + j)/2, j; 1/2) = 0\) whenever \(v + j\) is odd, in other words, by using Gaussian approximation for all (even and odd) \(j\) we double the sum and therefore must divide it by 2. Note also that in the third line, the Gaussian kernel \(\frac{1}{\sqrt{2\pi \theta(1-\theta)/u}} e^{-u(x-(1-\theta)u)/2\theta(1-\theta)}\) acts as a \(\delta\)-function at \(x = 1 - \theta\) when \(u \rightarrow \infty\).

Let us turn to a rigorous proof of the above asymptotics. For \((u, v) \in \mathbb{Z}^2, (u, v) \neq (0, 0),\) denote \(\varrho := (u^2 + v^2)^{1/2}, z := u/\varrho \in [-1, 1],\) then \(u = z\varrho, v^2 = \varrho^2(1 - z^2).\) It suffices to prove

\[
\varrho^{3/2-d} a(u, v) - L_0(z) \rightarrow 0 \quad \text{as } |u| + |v| \rightarrow \infty.
\]

By definition (see (147), (148), (8.4)) holds for \(u \leq 0, z \geq 0\) hence we can assume \(u \geq 1, z > 0\) in as follows. Moreover, for any \(\epsilon > 0\) there exists \(K > 0\) such that

\[
\varrho^{3/2-d} a(u, v) < \epsilon \quad \text{and} \quad L_0(z) < \epsilon \quad (\forall 1 \leq u < v^{9/5}, \varrho > K).
\]

The second relation in (8.5) is immediate by \(\lim_{z \to 0} L_0(z) = L_0(0) = 0\) and \(z = u/\varrho \leq \varrho^{9/10}/\varrho \rightarrow 0 \quad (\varrho \rightarrow \infty).\)

To prove the first relation we use Hoeffding’s inequality [17]. Let \(\text{bin}(j, k; p)\) be the binomial distribution. Then for any \(\tau > 0\)

\[
\sum_{0 \leq j \leq k: |j - kp| > \tau} \text{bin}(j, k; p) \leq 2e^{-2\tau^2}.
\]

(8.6) implies \(\text{bin}((v + j)/2, j; 1/2) \leq 2e^{-2v^2/j} \leq 2e^{-2v^2/u}\) for any \(|v| \leq u, 0 \leq j \leq u.\) Also note that \(1 \leq u \leq v^{9/5}\) implies \(v^2 \geq 2^{1/20} u^2 \varrho^{1/10}.\) Using these facts and (8.1) with \(\sum_{j=0}^{u} \text{bin}(u - j, u; \theta) = 1\) for any \(1 \leq u < v^{8/5}\) we obtain

\[
\varrho^{3/2-d} a(u, v) \leq C \varrho^{3/2-d} q_u(v) \leq C \varrho^{3/2-d} e^{-2v^2/u} \leq C \varrho^{3/2-d} e^{-\varrho^{1/10}} \rightarrow 0, \quad \varrho \rightarrow \infty,
\]

proving (8.5). Hence, it suffices to prove (8.4) for \(u \rightarrow \infty, 0 \leq v \leq u^{9/10}.\) Below, we give the proof for \(v\) even, the proof for \(v\) odd being similar. Denote

\[
\mathcal{D}^+(u, v) := \{0 \leq j \leq u/2 : |2j - u(1-\theta)| < u^{3/5} \text{ and } |v| < j^{3/5}\},
\]

\[
\mathcal{D}^-(u, v) := \{0 \leq j \leq u/2 : |2j - u(1-\theta)| \geq u^{3/5} \text{ or } |v| \geq j^{3/5}\}.
\]
Split \(a(u, v) = \psi_u(-d) \sum_{0 \leq j \leq u/2} \text{bin}(u - 2j, u; \theta) \text{bin}(v/2 + j, 2j; 1/2) = a^+(u, v) + a^-(u, v)\), where \(a^\pm(u, v) := \psi_u(-d) \sum_{j \in D^\pm(u, v)} \cdots\). It suffices to prove that
\[
g^{3/2-d}a^-(u, v) - L_0(z) \rightarrow 0 \quad \text{and} \quad g^{3/2-d}a^+(u, v) \rightarrow 0
\] (8.7) as \(u \rightarrow \infty, 0 \leq v \leq u^{5/9}\). To show the first relation in (8.7), let \(j_u^{*} := [u(1 - \theta)/2]\) and
\[
a^+(u, v) := \text{bin}(v/2 + j_u^{*}, 2j_u^{*}; 1/2)\psi_u(-d) \sum_{j \in D^+(u, v)} \text{bin}(u - 2j, u; \theta),
\]
then
\[
a^+(u, v) - a^+(u, v) = \psi_u(-d) \sum_{j \in D^+(u, v)} \text{bin}(u - 2j, u; \theta)\left(\text{bin}(v/2 + j_u^{*}, 2j_u^{*}; 1/2) - \text{bin}(v/2 + j, 2j; 1/2)\right).
\]
According to (8.3), for \(j \in D^+(u, v), j_u^{*} \in D^+(u, v)\)
\[
\text{bin}(v/2 + j, 2j; 1/2) = \frac{1}{\sqrt{\pi j}} e^{-v^2/4j} (1 + O(j^{-1/5})) = \frac{1}{\sqrt{\pi j}} e^{-v^2/4j} (1 + O(u^{-1/5})),
\]
\[
\text{bin}(v/2 + j_u^{*}, 2j_u^{*}; 1/2) = \frac{1}{\sqrt{\pi j_u^{*}}} e^{-v^2/4j_u^{*}} (1 + O(u^{-1/5})).
\]
Using \(c_- u < j < c_+ u, j \in D^+(u, v)\) for some \(c_+, c_- > 0\), and elementary inequalities we obtain that
\[
\left|\frac{1}{\sqrt{\pi j}} e^{-v^2/4j} - \frac{1}{\sqrt{\pi j_u^{*}}} e^{-v^2/4j_u^{*}}\right| \leq C u^{-7/10} e^{-cv^2/u}
\]
for all \(j \in D^+(u, v)\) and all \(u > 0\) large enough. Therefore since \(\sum_{j \in D^+(u, v)} \text{bin}(u - 2j, u; \theta) \leq 1\) we obtain
\[
g^{3/2-d}|a^+(u, v) - a^+(u, v)| \leq C g^{3/2-d} u^{-7/10 + d - 1 - cv^2/u} = g^{-1/5} L^*(z) \leq C g^{-1/5},
\]
where \(L^*(z) := C z^{-17/10} e^{-c \sqrt{(1/z)^2 - 1}}, z \in (0, 1]\) is a bounded function. As a consequence, it suffices to prove the first relation in (8.7) with \(a^+(u, v)\) replaced by \(a^+(u, v)\). This in turn follows from relations \(\frac{1}{\sqrt{\pi j_u^{*}}} e^{-v^2/4j_u^{*}} \sim \frac{1}{\sqrt{\pi u(1 - \theta)/2}} e^{-v^2/2u(1 - \theta)}, \psi_u(-d) \sim \Gamma(d^{-1}) u^{-d^{-1}},\) and
\[
\sum_{j \in D^+(u, v)} \text{bin}(u - 2j, u; \theta) \rightarrow 1/2 \quad \text{as} \quad u \rightarrow \infty,
\] (8.8)
each of which hold uniformly in \(0 \leq v \leq u^{5/9}\). Let us check (8.8) for instance. Since \(c_- u < j < c_+ u, j \in D^+(u, v)\) for some \(c_+, c_- > 0\), see above, so \(u^{5/9} = o(j^{3/5})\) and (8.8) follows from
\[
B'(u) \rightarrow 1/2 \quad \text{and} \quad B''(u) \rightarrow 0,
\] (8.9)
where \(B'(u) := \sum_{j=0}^{u} \text{bin}(u - j, u; \theta) 1(j \text{ is even}), B''(u) := \sum_{j=0}^{u} \text{bin}(u - j, u; \theta) 1(|j - u(1 - \theta)| \geq u^{3/5})\). Here, the first relation in (8.9) is obvious by well-known properties of binomial coefficients (??) while the second one follows from (8.6) according to which \(B''(u) \leq C e^{-2u^{1/5}} \rightarrow 0\). This proves the first relation in (8.7).
The proof of the second relation in \([8.1]\) uses Hoeffding’s inequality in \([8.6]\) in a similar way. We have
\[
a^-(u, v) \leq a_1^-(u, v) + a_2^-(u, v),
\]
where
\[
a_1^-(u, v) := \psi_u(-d) \sum_{0 \leq j \leq u;|j-u-\theta| > u^{3/5}} \binom{u-j, u; \theta} \leq C_u^d e^{-2u^{1/5}}
\]
implying
\[
e^{3/2-d} J_1^-(u, v) \leq C_u^{10/9} e^{(3/2-d)(d-1)e^{-2u^{1/5}}} \to 0 (u \to \infty) \quad \text{uniformly in } |v| \leq u^{5/9}.
\]
Finally,
\[
a_2^-(u, v) := \psi_u(-d) \sum_{0 \leq j \leq u;|j-u-\theta| \leq u^{3/5}, v \geq j^{3/5}} \binom{u-j, u; \theta} \leq C_u^d e^{-c_2 u^{1/5}}
\]
for some positive constants \(c_1, c_2 > 0\), implying
\[
e^{3/2-d} a_2^-(u, v) \leq C_u^{10/9} e^{(3/2-d)+d-c_2 u^{1/5}} \to 0 (u \to \infty) \quad \text{uniformly in } |v| \leq u^{5/9}.
\]
This proves \([8.1]\) and Proposition \([5.1]\) too. \(\square\)

Proof of Proposition \([5.7]\): With the notation \(\rho := \rho(t, s)\) we have that \(\{(t, s) \in \mathbb{R}^2, s \geq 0\} \ni (t, s) \mapsto (\rho(t, s)) \in [0, \infty) \times [0, 1]\) is a 1-1 mapping. Particularly, if \(\varpi = 1\) then \((\rho, \arccos(t/\rho))\) are the polar coordinates of \((t, s) \in \mathbb{R}^2, s \geq 0\). We use the inequality:
\[
\rho(t_1 + t_2, s_1 + s_2) \leq C_{\varpi} \sum_{i=1}^{2} \rho(t_i, s_i), \quad (t_i, s_i) \in \mathbb{R}^2, \quad i = 1, 2,
\]
which follows from \(\rho(t_1 + t_2, s_1 + s_2) \leq \sum_{i=1}^{2} \rho(t_i, s_i)\) with \(C_{\varpi} := 1 + 2^{1/\varpi-1}\).

(i) W.l.g., let \(\delta = 1\). Then
\[
\int_{B_1(0,0)} \rho(t, s)^{-h} dt ds \leq 4 \int_0^1 \rho(t, s)^{-h} du \int_0^{1/\varpi} (1 + u^{2/\varpi})^{-h/2} dv,
\]
where the inner integral is \(O(1)\) if \(h > \varpi, = O(u^{h-\varpi})\) if \(h < \varpi, = O(|\log u|)\) if \(h = \varpi, \) as \(u \to 0\). This proves \([5.1]\) and \([5.3]\) follows analogously.

(ii) After the change of variables: \(u \to \rho u, v \to \rho^{\varpi} v, \rho := \rho(t, s)\), we get
\[
(\rho^{-h_1} \ast \rho^{-h_2})(t, s) = \rho^{1+\varpi-h_1-h_2} \int_{\mathbb{R}^2} \rho(u, v)^{-h_1}(\rho((t/\rho) + u, (s/\rho^{\varpi}) + v)^{-h_2} du dv
\]
where
\[
\begin{align*}
I_1 & := \int_{B_\delta(0,0)} \rho(u, v)^{-h_1}((t/\rho) + u, (s/\rho^{\varpi}) + v)^{-h_2} du dv, \\
I_2 & := \int_{B_\delta^{-1}(0,0) \cap B_\delta^{-1}(-t/\rho, -s/\rho^{\varpi})} \cdots du dv, \\
I_{12} & := \int_{B_\delta^{-1}(0,0) \cap B_\delta^{-1}(-t/\rho, -s/\rho^{\varpi})} \cdots du dv
\end{align*}
\]
with \(\delta > 0\) such that \(B_\delta(0,0) \cap B_\delta(-t/\rho, -s/\rho^{\varpi}) = \emptyset\) for any \((t, s) \neq (0, 0)\). The integrals \(I_i \leq C, i = 1, 2\) by \([5.2]\) and \(0 < h_i < 1 + \varpi, i = 1, 2\). Next, by Hölder’s inequality with \(h := h_1 + h_2\),
\[
I_{12} \leq \int_{B_{\delta/(0,0)}} \rho(u, v)^{-h} du dv \leq C,
\]
in view of \([5.3]\) and \(\int_{B_\delta^{-1}(0,0) \cap B_\delta^{-1}(-t/\rho, -s/\rho^{\varpi})} \rho((t/\rho) + u, (s/\rho^{\varpi}) + v)^{-h} du dv = \int_{B_\delta(0,0)} \rho(u, v)^{-h} du dv\). This proves \([5.4]\).

Next, consider \([5.3]\), or the case \(0 < h_2 < 1 + \varpi < h_1\). By changing the variables as in \([8.11]\), we get
\[
(\rho^{-h_1} \ast \rho^{-h_2})(t, s) \leq \rho^{1+h_1-2}(I_1 + I_2 + I_{12}),
\]
where \(I_2, I_{12} < C\) are the same as in \([8.11]\), whereas
\[
I_1' := \int_{B_\delta(0,0)} (\rho^{-1} \vee \rho(u, v))^{-h_1}(\rho((t/\rho) + u, (s/\rho^{\varpi}) + v)^{-h_2} du dv.
\]
Note that if given small enough \( \delta > 0 \), then (8.10) implies \( \rho((t/\varrho) + u, (s/\varrho_\omega) + v)^{1/\omega} \geq 1 - \rho(u, v)^{1/\omega} \geq 1/2 \) for all \((u, v) \in B_\delta(0, 0)\), and hence \( I'_1 \leq C g^{h_1-1/\omega} \int_{\mathbb{R}^2} \rho_+ (u, v)^{-h_1} dudv \leq C g^{h_1-1/\omega} \) according to (5.3). Since \( \rho(t, s)^{1 + \omega - h_1 - h_2} = o(\rho(t, s)^{-h_2}) \) as \(|t| + |s| \to \infty\), the proof of (5.5) is complete.

Finally, consider (5.6). We follow the proof of (5.5) and get \((\rho_+^{-h_1} \ast \rho_+^{-h_2})(t, s) \leq g^{1 + \omega - h_1 - h_2}(I'_1 + I'_2 + I_{12})\) with the same \( I'_1 < C, I_{12} < C \), whereas

\[
I'_2 := \int_{B_{\delta_0}(-t/\varrho, -s/\varrho_\omega)} \rho(u, v)^{-h_1} (q^{-1} \lor \rho((t/\varrho) + u, (s/\varrho_\omega) + v))^{-h_2} dudv.
\]

For small enough \( \delta > 0 \), we have \( \rho(u, v)^{1/\omega} \geq 1 - \rho((t/\varrho) + u, (s/\varrho_\omega) + v)^{1/\omega} \geq 1/2 \) for all \((u, v) \in B_\delta(-t/\varrho, -s/\varrho_\omega)\), and hence \( I'_2 \leq C g^{h_2-1/\omega} \int_{\mathbb{R}^2} \rho_+ ((t/\varrho) + u, (s/\varrho_\omega) + v)^{-h_2} dudv \leq C g^{h_2-1/\omega} \) by (5.3).

Using \( \rho(t, s)^{1 + \omega - h_1 - h_2} = o(\rho(t, s)^{-h_1 h_2}) \) as \(|t| + |s| \to \infty\), we conclude (5.6). Extension of (8.4)–(5.6) to ‘discrete’ convolution \( [\rho_+^{-h_1} \ast \rho_+^{-h_2}](t, s) \) requires minor changes and we omit the details. This proves part (ii).

(iii) It suffices to show (5.7) for \((t, s) \neq (0, 0), s \geq 0\), in which case \( \rho_+(t, s) = \rho(t, s) \). We have \([a_1 \ast a_2](t, s) = \sum_{j=0}^1 [a_i^0 \ast a_i^2](t, s)\), where \( a_i^0(t, s) := \rho_+(t, s)^{-h_1} L_i(t/\rho_+(t, s)) \), \( a_i^1(t, s) := a_i(t, s) - a_i^0(t, s) = o(\rho_+(t, s)^{-h_1}) \), \( i = 1, 2 \). Clearly, (5.7) follows from

\[
\lim_{|t| + |s| \to \infty} \big| \rho(t, s)^{h_1 + h_2 - 1/\omega} [a_1^0 \ast a_2^0](t, s) - L_{12}(t/\rho(t, s)) \big| = 0 \quad (8.13)
\]

and

\[
[a_1^0 \ast a_2^0](t, s) = \rho_+(t, s)^{-1 + \omega - h_1 - h_2}, \quad (i, j) \neq (0, 0), \quad i, j = 0, 1, \quad |t| + |s| \to \infty. \quad (8.14)
\]

The proof of (8.14) mimics the proof of (8.13) and is omitted. To prove (8.13), write \([a_1^0 \ast a_2^0](t, s)\) as the integral: \([a_1^0 \ast a_2^0](t, s) = \int_{\mathbb{R}^2} a_1^0([u], [v]) a_2^0([u + t, [v] + s]) dudv\). \([a_1^0 \ast a_2^0](t, s) = \int_{\mathbb{R}^2} a_1^0([u], [v]) a_2^0([u + t, [v] + s]) dudv\). After the same change of variables \( u \to \varrho u, v \to \varrho_\omega v, \varrho := \rho(t, s) \) as in the proof of (ii) we obtain \([a_1^0 \ast a_2^0](t, s) = \varrho^{1 + \omega - h_1 - h_2} L_\varrho(t/\varrho)\), where

\[
L_\varrho(z) := \int_{\mathbb{R}^2} g_\varrho(u, v, z) dudv, \quad z \in [-1, 1] \quad (8.15)
\]

and where

\[
g_\varrho(u, v, z) := a_{1\varrho}(\tilde{u}, \tilde{v}) a_{2\varrho}(\tilde{u} + z, \tilde{v} + (1 - z^2)^{\omega/2}), \quad (8.16)
\]

with \( \tilde{u} := [\varrho u]/\varrho, \tilde{v} := [\varrho_\omega v]/\varrho_\omega \) and

\[
a_{i\varrho}(u, v) := (q^{-1} \lor \rho(u, v))^{-h_i} L_i (u/(q^{-1} \lor \rho(u, v))), \quad i = 1, 2 \quad (8.17)
\]

since \( s/\varrho_\omega = (1 - z^2)^{\omega/2} \) for \( z = t/\varrho \in [-1, 1], s \geq 0 \). Then with \( a_{i\infty}(u, v), i = 1, 2 \) defined by the statement of Prop. 5.1(iii) we get that

\[
g_\varrho(u, v, z) \to g_\infty(u, v, z) := a_{1\infty}(u, v) a_{2\infty}(u + z, v + (1 - z^2)^{\omega/2}) \quad (8.18)
\]

as \( \varrho = \rho(t, s) \to \infty \) (\(|t| + |s| \to \infty\)) for any fixed \((u, v, z) \in \mathbb{R}^2 \times [-1, 1]\) such that \((u, v) \not\in \{(0, 0), (-z, -(1 - z^2)^{\omega/2})\}\) and \( u/\rho(u, v), (u + z)/\rho(u + z, v + (1 - z^2)^{\omega/2}) \) being continuity points of \( L_1 \) and \( L_2 \) respectively. Let us prove that

\[
L_\varrho(z) \to L_{12}(z) \quad \text{as} \quad \varrho \to \infty \quad (8.19)
\]
uniformly in $z \in [-1, 1]$, which implies $\|L(q)/q - L_1(t/q)\| \leq \sup_{z \in [-1, 1]} |L(q)(z) - L_1(z)| = o(1)$ as $q \to \infty$. The uniform convergence in $[\delta, 1]$ follows if $\lim_{q \to \infty} L(q)(z) = L_1(z)$ holds for any $z \in [-1, 1]$, and every sequence $\{z_q\} \subset [-1, 1]$ tending to $z$: $\lim_{q \to \infty} z_q = z$. Choose $\delta > 0$ and split the difference $L(q)(z_q) - L_1(z) = I_1 + I_2 + I_12$, where

$$I_1 := \int_{B_\delta(0,0)} (g_q(u, v; z_q) - g_\infty(u, v; z)) \, dudv,$$

$$I_2 := \int_{B_\delta(-z,-z')} \cdots \, dudv, \quad I_{12} := \int_{B_\delta(0,0) \setminus B_\delta^c(-z,-z')} \cdots \, dudv$$

with the notation $z' := (1 - z^2)^{\omega/2}$. Note that $\rho(z, z') = 1$ and $\delta > 0$ is chosen small enough so that $B_\delta(0, 0) \cap B_\delta(-z, -z') = \emptyset$. Let us first check that $|I_i|, i = 1, 2$ can be made arbitrary small by taking sufficiently small $\delta$. Towards this end, we need the bound

$$|a_{i\theta}(\bar{u}, \bar{v})| \leq C_\rho(u, v)^{-h_1}, \quad (u, v) \in \mathbb{R}^2, \quad i = 1, 2. \tag{8.20}$$

Indeed, by (8.10), $\rho(u, v) \leq C_\omega(\rho(\bar{u}, \bar{v}) + \rho(u - \bar{u}, v - \bar{v}))$, where $|u - \bar{u}| \leq 2^{-1}, |v - \bar{v}| \leq 2^{-\omega}$ and hence $\rho(u - \bar{u}, v - \bar{v}) \leq \sqrt{2} \rho^{-1}$, with $C_\omega > 0$ dependent only on $\omega > 0$. Therefore, $\rho(u, v) \leq \sqrt{2} C_\omega(\rho(\bar{u}, \bar{v}) + \rho^{-1}) \leq 2\sqrt{2} C_\omega(\rho(\bar{u}, \bar{v}) \vee \rho^{-1})$ implying $\rho(\bar{u}, \bar{v}) \vee \rho^{-1} \geq (2\sqrt{2} C_\omega)^{-1} \rho(u, v)$, or (8.20) in view of the definition of $a_{i\theta}$ in (8.17). Using (8.20), it follows that Using (8.20) it follows that

$$\left| g_q(u, v; z_q) - g_\infty(u, v; z) \right| \leq C_\rho(u, v)^{-h_1}(\rho(u + z_q, v + z'_q)^{-h_2} + \rho(u + z, v + z')^{-h_2}). \tag{8.21}$$

From (8.21) we obtain $|I_1| \leq C \int_{B_\delta(0,0)} \rho(u, v)^{-h_1} \, dudv \leq C\delta^{1+\omega-h_1} = o(1)$ and similarly, $|I_2| \leq C\delta^{1+\omega-h_2} = o(1)$. Hence it suffices to show that $I_{12} \to 0$ as $z_q \to z$, viz., that for each $\delta > 0$

$$\int_{B_\delta(0,0) \setminus B_\delta^c(-z,-z')} |g_q(u, v; z_q) - g_\infty(u, v; z)| \, dudv \to 0 \quad \text{as } q \to \infty. \tag{8.22}$$

From (8.10), $\rho(u + z_q, v + z'_q)^{1+\omega} \geq 2 \geq (1/2) \rho(u + z, v + z')^{1+\omega}$ for all $(u, v) \in B_\delta(z, -z')$ and $\rho$ large enough that $\rho(z - z_q, z' - z'_q)^{1+\omega} \leq \delta^{1+\omega}/2$ (in view of $z_q \to z$). Hence and from (8.21) we obtain that the integrand in (8.22) is dominated on $B_\delta^c(0,0) \setminus B_\delta^c(-z,-z')$ by an integrable function independent of $\rho$, viz., $|g_q(u, v; z_q) - g_\infty(u, v; z)| \leq C(\rho(u, v)^{-h_1}(\rho(u + z, v + z')^{-h_2}$ since this integrand vanishes a.e. on $B_\delta^c(0,0) \setminus B_\delta^c(-z,-z')$ as $\rho \to \infty$, see (8.18), relation (8.22) follows by the dominated convergence theorem, proving (8.19). The continuity of $L_{12}$ (8.8) follows similarly by the dominated convergence theorem.

It remains to prove the strict positivity of $L_{12}$ in the case where $L_{1}(z) \equiv L_{2}(z) =: L(z) \geq 0$. Under assumption of piecewise continuity of $L$ and $L \neq 0$ a.e., we can find $0 < \varepsilon < 1$ and $\delta > 0$ such that $L(z) > \delta$ for any $|z - z_0| < \delta$. We also have $|u/\rho(u, v) - (u + z)/\rho(u + z, v + z')| \leq \rho(u, v)^{-1} + |1 - \rho(u + z, v + z')/\rho(u, v)| = O(\rho(u, v)^{-1+\omega})$ uniformly in $z \in [-1, 1]$ for $\rho(u, v) \geq 1$. Indeed, this follows from $|1 - (\rho(u + z, v + z')/\rho(u, v))^{1+\omega}| < \rho(u, v)^{-1+\omega}$ by (8.10), when combined with $|1-x| \leq \omega^{-1}(1 + \omega^{1+\omega}|1-x^\omega|$, $x > 0$ due to the mean value theorem if $0 < \omega < 1$. Hence for $\rho(u, v) \geq K \geq 1$ large enough, we have $|u/\rho(u, v) - (u + z)/\rho(u + z, v + z')| < \delta/2$. Next, we can find the interior point $(u_0, v_0)$ of $B_K(0,0)$ with $u_0/\rho(u_0, v_0) = z_0$. In view of continuity of $u/\rho(u, v)$, there exists $\varepsilon > 0$ such that $|z_0 - u_0/\rho(u, v)| < \delta/2$ holds
for all \((u, v) \in B_{\varepsilon}(u_0, v_0) \subseteq B_{\hat{R}}(0, 0)\). Consequently, \(L(u/\rho(u, v))L((u + z)/\rho(u + z, v + z')) \geq \delta^2 > 0\) for any \(z \in [-1, 1]\) and all \((u, v) \in B_{\varepsilon}(u_0, v_0)\). By \((8.10)\), we have \(\rho(u + z, v + z') \leq 2C_{\infty}\rho(u, v)\) for \(\rho(u, v) \geq 1\) and hence \(L_{12}(z) > \delta^2(2C_{\infty})^{-h^2} \int_{B_{\varepsilon}(u_0, v_0)} (\rho(u, v))^{-h_1 - h_2} du dv > 0\), proving \(L_{12}(z) > 0, z \in [-1, 1]\) and part (iii).

(iv) Rewrite the l.h.s. of \((5.9)\) as

\[
B_\lambda(\gamma) = \int_{\tilde{K}_{b,\lambda,\gamma}^2} b([t_1] - [t_2], [s_1] - [s_2]) dt_1 dt_2 ds_1 ds_2,
\]

where \(\tilde{K}_{b,\lambda,\gamma} := \{(t, s) \in \mathbb{R}^2 : ([t], [s]) \in K_{b,\lambda,\gamma}\}\).

Case (I): \(\gamma = \varpi\). By changing the variables in \((8.23)\) as \(t_i \rightarrow \lambda t_i, s_i \rightarrow \lambda \varpi s_i, i = 1, 2\), we obtain \(\lambda^{-2H(\varpi)} B_\lambda(\varpi) = \int_{\mathbb{R}^4} \tilde{b}_\lambda(t_1, t_2, s_1, s_2) dt_1 dt_2 ds_1 ds_2\), where

\[
\tilde{b}_\lambda(t_1, t_2, s_1, s_2) := b_\lambda(([\lambda t_1] - [\lambda t_2]) / \lambda, ([\lambda \varpi s_1] - [\lambda \varpi s_2]) / \lambda \varpi) \times 1((\lambda t_i, \lambda \varpi s_i) \in (0, \lambda) \times (0, \lambda \varpi], i = 1, 2) \]

with \(b_\lambda(t, s) := (\lambda^{-1} \vee \rho(t, s))^{-h}(L(t/\lambda^{-1} \vee \rho(t, s) + o(1))\) as \(\lambda \rightarrow \infty\). Then

\[
\tilde{b}_\lambda(t_1, t_2, s_1, s_2) \rightarrow b_\infty(t_1 - t_2, s_1 - s_2) 1((t_i, s_i) \in (0, 1]^2, i = 1, 2), \quad \lambda \rightarrow \infty
\]

point-wise for any \((t_1, t_2, s_1, s_2) \in \mathbb{R}^4, (t_1, s_1) \neq (t_2, s_2)\) fixed. The dominating bound

\[
\lambda^{-1} \vee \rho(([\lambda t_1] - [\lambda t_2]) / \lambda, ([\lambda \varpi s_1] - [\lambda \varpi s_2]) / \lambda \varpi) \geq C \rho(t_1 - t_2, s_1 - s_2)
\]

follows by the same arguments as \((5.20)\). These facts and the dominated convergence theorem justify the limit

\[
\lim_{\lambda \rightarrow \infty} \lambda^{-2H(\varpi)} B_\lambda(\varpi) = C(\varpi)\]

since the integral \(C(\varpi) \leq C \int_{(1, 1]^2} \rho(t, s)^{-h} dt ds < \infty\) in \((5.10)\) converges by Prop. 5.1 (i).

Case (II): \(\gamma > \varpi, \h \varpi\). By changing the variables in \((8.23)\) as \(t_i \rightarrow \lambda t_i, s_i \rightarrow \lambda \gamma s_i, i = 1, 2\), we obtain \(\lambda^{-2H(\gamma)} B_\lambda(\gamma) = \int_{\mathbb{R}^4} \tilde{b}_\lambda(t_1, t_2, s_1, s_2) dt_1 dt_2 ds_1 ds_2\), where

\[
\tilde{b}_\lambda(t_1, t_2, s_1, s_2) := b_\lambda(([\lambda t_1] - [\lambda t_2]) / \lambda \gamma, ([\lambda \gamma s_1] - [\lambda \gamma s_2]) / \lambda \gamma) \times 1((\lambda t_i, \lambda \gamma s_i) \in (0, \lambda) \times (0, \lambda \gamma], i = 1, 2) \]

with \(b_\lambda(t, s) := (\lambda^{-\gamma/\varpi} \vee \rho(t, s))^{-h}(L(t/\lambda^{-\gamma/\varpi} \vee \rho(t, s) + o(1))\) as \(\lambda \rightarrow \infty\). Hence since \(\gamma/\varpi > 1\) it follows that

\[
\tilde{b}_\lambda(t_1, t_2, s_1, s_2) \rightarrow b_\infty(0, s_1 - s_2) 1((t_i, s_i) \in (0, 1]^2, i = 1, 2), \quad \lambda \rightarrow \infty
\]

point-wise for any \((t_1, t_2, s_1, s_2) \in \mathbb{R}^4, s_1 \neq s_2\) fixed. Note \(b_\infty(0, s) = L(0)|s|^{-h/\varpi}\) is integrable on \([-1, 1]\) due to \(h < \varpi\). The limit \(\lim_{\lambda \rightarrow \infty} \lambda^{-2H(\varpi)} B_\lambda(\varpi) = C(\varpi)\) can be justified by the dominated convergence theorem using the bound

\[
\lambda^{-\gamma/\varpi} \vee \rho(([\lambda t_1] - [\lambda t_2]) / \lambda \gamma, ([\lambda \gamma s_1] - [\lambda \gamma s_2]) / \lambda \gamma) \geq \lambda^{-\gamma/\varpi} \vee \rho(0, ([\lambda \gamma s_1] - [\lambda \gamma s_2]) / \lambda \gamma) \geq C \rho(0, s_1 - s_2),
\]
which follows by the same arguments as (8.20).

Case (III): \( \gamma > \varpi, h > \varpi \). By changing the variables in (8.23) as \( t_i \to \lambda t_i, i = 1, 2, s_1 - s_2 \to \lambda \varpi s_1, s_2 \to \lambda \gamma s_2 \), we obtain \( \lambda^{-2H(\gamma)}B_\lambda(\gamma) = \int_{\mathbb{R}^2} \tilde{b}_\lambda(t_1, t_2, s_1, s_2)dt_1dt_2ds_1ds_2 \), where

\[
\tilde{b}_\lambda(t_1, t_2, s_1, s_2) := b_\lambda((\lambda t_1 - |\lambda t_2|)/\lambda, (|\lambda \varpi s_1 + \lambda \gamma s_2| - |\lambda \gamma s_2|)/\lambda \varpi) \\
\times 1(|\lambda t_1| \in (0, \lambda], i = 1, 2, |\lambda \varpi s_1 + \lambda \gamma s_2| \in (0, \lambda \gamma], |\lambda \gamma s_2| \in (0, \lambda \gamma])
\]
with \( b_\lambda(t, s) := (\lambda^{-1} \vee \rho(t, s))^{-h}(L(t/(\lambda^{-1} \vee \rho(t, s)) + o(1)) \) as \( \lambda \to \infty \). Then

\[
\tilde{b}_\lambda(t_1, t_2, s, u) \to b_\infty(t_1 - t_2, s_1)1((t_1, t_2, s_2) \in (0, 1]^3), \quad \lambda \to \infty
\]
for any \( t_1 \neq t_2, s_1 \in \mathbb{R} \setminus \{0\}, s_2 \in \mathbb{R} \setminus \{0, 1\} \) fixed since \( \gamma > \varpi \) implies \( 1(0 < |\lambda \varpi s_1 + \lambda \gamma s_2| \leq \lambda \gamma) \to 1(0 < s_2 < 1) \). The dominating bound

\[
\lambda^{-1} \vee \rho((|\lambda t_1| - |\lambda t_2|)/\lambda, (|\lambda \varpi s_1 + \lambda \gamma s_2| - |\lambda \gamma s_2|)/\lambda \varpi) \geq C\rho(t_1 - t_2, s_1)
\]
follows by the same arguments as (8.20), because \( (|\lambda \varpi s_1 + \lambda \gamma s_2| - |\lambda \gamma s_2|)/\lambda \varpi - s_1 \leq 2\lambda^{-\varpi} \). Then the dominated convergence in (5.9) is proved in view of \( C(\gamma) \leq C \int_{-1}^1 \int_{\mathbb{R}} \rho(t, s)^{-h}dt ds < \infty \).

Cases (IV) and (V) can be treated similarly to Cases (II) and (III) and we omit the details. Proposition 5.1 is proved.

**Proof of Proposition 6.7.** (i) We first prove (8.2). According to (2.3)

\[
Z(t, s) = \sum_{i=1}^{k-1} \sum_{(D_i)_{(u,v)}} \sum_{i} a(t - u_1, s - v_1)|D_i| \cdots a(t - u_i, s - v_i)|D_i|A_{|D_i|}(\varepsilon(u_1, v_1)) \cdots A_{|D_i|}(\varepsilon(u_i, v_i))
\]
where the sum \( \sum_{(D_i)} \) is taken over all partitions of \( \{1, 2, \cdots, k\} \) into \( i \) nonempty sets \( D_1, \cdots, D_i \) having cardinality \( |D_1| \geq 1, \cdots, |D_i| \geq 1, |D_1| + \cdots + |D_i| = k \). Thus, (8.21) is a decomposition of \( Z(t, s) = A_k(Y(t, s)) - Y^{*k}(t, s) \) into a sum of stationary ‘off-diagonal’ polynomial forms of order \( i < k \) in i.i.d. r.v. \( A_{|D_i|}(\varepsilon(u_\ell, v_\ell)), 1 \leq \ell \leq i \) with max\( (|D_1|, \cdots, |D_i|) \geq 2 \). From (8.21) it follows that

\[
|EZ(0,0)Z(t, s)| \leq C \sum_{i=1}^{k-1} \sum_{(D_i)} \prod_{t=1}^{i} (|a|^{d_t} \ast |a|^{d'_t})(t, s)
\]
where the second sum is taken over all collections \( (d)_i = (d_1, \cdots, d_i), (d'_i) = (d'_1, \cdots, d'_i) \) of integers \( d_t \geq 1, d'_t \geq 1 \) with \( \sum_{t=1}^{d_t} d_t = \sum_{t=1}^{d'_t} d'_t = k \) and satisfying \( \max_{1 \leq \ell \leq i} d_t \geq 2, \max_{1 \leq \ell \leq i} d'_t \geq 2 \). See [14], proof of Thm 14.2.1. Then \( a(t, s)^{d_t} \leq C\rho(t, s)^{-\beta_t}, a(t, s)^{d'_t} \leq C\rho(t, s)^{-\beta'_t} \) where \( \beta_t := d_t q_1, \beta'_t := d'_t q_1 \). By Proposition 5.1 (ii)

\[
|EZ(0,0)Z(t, s)| \leq C \sum_{i=1}^{k-1} \sum_{(D_i), (d'_i)} \prod_{t=1}^{i} \rho(t, s)^{-w_t},
\]
where

\[
w_t := \begin{cases} 
2q_1 - 1 - \gamma_0 = p_1, & \text{if } d_t = d'_t = 1, \\
q_1, & \text{if } d_t \geq 2, d'_t = 1 \text{ or } d_t = 1, d'_t \geq 2, \\
2q_1, & \text{if } d_t \geq 2, d'_t \geq 2.
\end{cases}
\]
Since \( 2q_1 - 1 - \gamma_0 < q_1 < 2q_1 \), we have that, for \( \max_1 \leq t \leq d \geq 2, \max_1 \leq t \leq d \geq 2 \), the exponents \( w_1 \) in (8.28) satisfy \( \sum_{t=1}^d w_t \geq 2q_1 \), implying (6.2). Since RFs \{Y^{*k}(t, s)\} and \{Z(t, s)\} are uncorrelated: 
\[
\text{Cov}(Y^{*k}(t, s), Z(u, v)) = 0 \text{ for any } (t, s), (u, v) \in \mathbb{Z}^2, \text{ relation } (6.1) \text{ follows from } (6.2) \text{ and } 
\]
\[
\text{Cov}(Y^{*k}(t, s), Y^{*k}(0, 0)) = \rho_Y(t, s)^k(1 + o(1)), \quad |t| + |s| \to \infty. \tag{8.29}
\]
To show (8.29), note that the difference \( |\rho_Y(t, s)^k - \text{Cov}(Y^{*k}(t, s), Y^{*k}(0, 0))| = \left| (a \ast a)^k(t, s) - \sum_{(u,v)} a(t + u_1, s + v_1) \cdots a(t + u_k, s + v_k) a(u_k, v_k) \right| \) satisfies the same bound as in (8.27) and therefore this difference is \( O(\rho(t, s)^{-2q_1}) = o(\rho_Y(t, s)^k) \) according to (6.2). This proves (8.29) and part (i). Part (ii) follows similarly using (6.2) and \( |\text{Cov}(Y^{*k}(t, s), Y^{*k}(0, 0))| \leq (|a| \ast |a|(t, s))^k \leq C \rho_+(t, s)^{-k\gamma_1} \). Proposition 6.1 is proved. \( \Box \)

**Proof of Corollary 6.2**  
Relation (6.5) follows from (6.2) and Proposition 5.1 (i) since the l.h.s. of (6.5) does not exceed \( \sum_{(t_1, s_1, t_2, s_2) \in K_{[\lambda, \lambda]}^2} |r_Z(t_1 - t_2, s_1 - s_2)| \leq \lambda^{1+\gamma} \sum_{(t, s) \in \mathbb{Z}^2} |r_Z(t, s)| \leq C \lambda^{1+\gamma} \sum_{(t, s) \in \mathbb{Z}^2} \rho_+(t, s)^{-2q_1} \) and the last sum converges by Proposition 5.1 (i) due to \( 2q_1 > 1 + \gamma_0 \). 

Relations (6.4) follow from (6.5), the orthogonality of \( \{Y^{*k}(t, s)\} \) and \{Z(t, s)\} and 
\[
\text{Var} \left( \sum_{(t, s) \in K_{[\lambda, \lambda]}} Y^{*k}(t, s) \right) \sim c(\gamma) \lambda^{2H(\gamma)}. \tag{8.30}
\]

In turn, (8.30) follows from 
\[
V_{\lambda, \gamma} := \sum_{(t_1, s_1, t_2, s_2) \in K_{[\lambda, \lambda]}^2} r_Y^k(t_1 - t_2, s_1 - s_2) \sim c(\gamma)^2 \lambda^{2H(\gamma)}. \tag{8.31}
\]

and the fact that the difference \( \left| \text{Var} \left( \sum_{(t, s) \in K_{[\lambda, \lambda]}} Y^{*k}(t, s) \right) - \sum_{(t_1, s_1, t_2, s_2) \in K_{[\lambda, \lambda]}^2} r_Y^k(t_1 - t_2, s_1 - s_2) \right| \) can be estimated as in (8.26) - (8.27) and therefore this difference is \( O(\lambda^{1+\gamma}) = o(\lambda^{2H(\gamma)}) \) as shown in (6.5).

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