ON A FAMILY OF SINGULAR CONTINUOUS MEASURES
RELATED TO THE DOUBLING MAP

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Abstract. Here, we study some measures that can be represented by infinite Riesz products of 1-periodic functions and are related to the doubling map. We show that these measures are purely singular continuous with respect to Lebesgue measure and that their distribution functions satisfy super-polynomial asymptotics near the origin, thus providing a family of extremal examples of singular measures, including the Thue–Morse measure.

1. Introduction

The Lebesgue decomposition theorem states that any positive regular Borel measure $\mu$ on $\mathbb{R}^n$ has a unique decomposition $\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$ relative to Lebesgue measure, where $\mu_{pp}$, $\mu_{ac}$ and $\mu_{sc}$ are mutually singular as measures. Here, $\mu_{pp}$ is pure point (the Bragg part), while $\mu_{ac}$ is an absolutely continuous and $\mu_{sc}$ is a singular continuous measure. We call a measure pure if it has only one of these parts. For example, with respect to Lebesgue measure, a Dirac measure is purely pure point, and Lebesgue measure is purely absolutely continuous. Purely singular continuous measures often arise in the study of dynamical systems, such as those associated to constant-length substitutions; see [35, 1, 6] for general background.

Since the work by Wiener [41], the spectrum of a sequence has been identified as an important quantity, and Mahler [31] studied the first example of singular continuity, the Thue–Morse (TM) measure, by which we mean, in modern terminology, the diffraction measure $\hat{\gamma}_t$ of the classic TM sequence over the alphabet $\{-1, 1\}$; see [15] for some historical background. This measure is 1-periodic, and its restriction to the fundamental domain $[0, 1)$ can be identified with the (dynamical) spectral measure of maximal type in the orthocomplement of the pure point sector of the TM shift dynamical system; see [6, 35] for details. Recall that the TM sequence is the bi-infinite sequence given by $t(0) = 1$, for non-negative $n$ by the recursions $t(2n) = t(n)$ and $t(2n + 1) = -t(n)$, and for negative integers by $t(n) = t(-n - 1)$. The corresponding diffraction measure, $\hat{\gamma}_t$, is the Fourier transform of the autocorrelation measure.

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$^1$This should not be confused with the unique, ergodic probability measure on the TM shift space (the orbit closure of the TM sequence), which is sometimes also called TM measure in the literature.
\[ \gamma_t = \sum_{m \in \mathbb{Z}} \eta_t(m) \delta_m, \]

where \( \eta_t(m) \) is the volume-averaged autocorrelation coefficient

\[ \eta_t(m) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} t(n) t(n+m). \]

These coefficients satisfy \( \eta_t(0) = 1 \) and \( \eta_t(-m) = \eta_t(m) \) for \( m \in \mathbb{N} \), as well as the recursions

\[ \eta_t(2m) = \eta_t(m) \quad \text{and} \quad \eta_t(2m+1) = -\frac{1}{2}(\eta_t(m) + \eta_t(m+1)) \]

for \( m \in \mathbb{N}_0 \); see [6, Sec. 10.1] and references therein. Perhaps the most interesting property of \( \hat{\gamma}_t \) in our context is the fact that it can be represented as an infinite Riesz product; see [42, Ch. V.7] for general background on such measures. Indeed, as a measure on \( \mathbb{R} \), one has

\[ \hat{\gamma}_t = \prod_{\ell \geq 0} \left( 1 - \cos(2^{\ell+1} \pi \cdot) \right), \]

which is to be understood as the limit of a vaguely converging sequence of absolutely continuous measures on \( \mathbb{R} \); see [6, Sec. 10.1] for a detailed discussion, which is based on the original work by Mahler and Kakutani, as well as [25, 5, 7] for results on its scaling properties. Since this measure is 1-periodic, hence of the form \( \hat{\gamma}_t = \mu_{TM} \ast \delta_\mathbb{Z} \) with \( \mu_{TM} = \hat{\gamma}_t|_{(0,1)} \), it is natural and more convenient to view it as a finite measure on the 1-torus, where it is a probability measure, and work with weak convergence. This is our point of view from now on, where the autocorrelation coefficients \( \eta_t(m) \) agree with the Fourier–Stieltjes coefficients of the measure \( \mu_{TM} \) on \( \mathbb{T} \). This simplifies various steps from a technical perspective, and is perfectly adequate for a complete study, including hyperuniformity aspects [25, 39], which have recently gained importance in the physical sciences [2, 32, 7, 33] and beyond [12, 13, 4, 36].

Taking the above infinite Riesz product as a starting point, we investigate the spectral properties of measures that can be represented as infinite Riesz products

\[ \prod_{\ell \geq 0} h(2^\ell x), \]

where \( h \) is a non-negative continuous function that is 1-periodic. Such products are connected with the doubling map \( x \mapsto T x := 2x \) on the 1-torus, \( \mathbb{T} \), the latter represented as [0,1) with addition modulo 1. We denote the corresponding topological dynamical system by \( (\mathbb{T}, T) \). This doubling system, from a spectral perspective, is more complicated than successive multiplication by an integer \( \geq 3 \); see [3] for a related example with Stern’s diatomic sequence. Another difficulty emerges from the observation that the Fourier–Stieltjes coefficients of a general Riesz product do not have such a simple recursive structure as those of \( \mu_{TM} \).

To deal with this class systematically, we will thus assume some additional symmetry conditions on \( h \). In particular, we require \( g := h/2 \) to be a \( g \)-function in the sense of Keane [29], where we have \( g(x) + g(x + \frac{1}{2}) = 1 \) for all \( x \in \mathbb{T} \). Under this condition, the corresponding Riesz product (if it exists as a vague limit) is in fact a \( g \)-measure. Since the seminal work of Keane [29], \( g \)-measures have played an important role in the development
of the thermodynamic formalism, pioneered by work of Ruelle [37], Ledrappier [30], Walters [40] and Bowen [10], to name just a few. They are also intimately connected to a class of stochastic processes, known as chains with infinite connections [19] or chains of infinite order [27]. We refer to [8, 24, 28, 38] and references therein for more on probabilistic aspects of $g$-measures. The methods developed in the context of $g$-measures have found applications in fields such as diffraction [29], wavelets [17, 18, 23], multifractal analysis [5, 21, 34] and learning models [16]. More general Riesz products than those presented above may fall into the class of $G$-measures, a generalisation of $g$-measures that was introduced by Brown and Dooley in [14]; see also [22].

Given a $g$-function, the question under which condition there is a unique associated $g$-measure that is also the vague limit of the corresponding Riesz product has attracted considerable attention; compare [24, 28, 40] among many others. In most cases, it is assumed that $g$ is strictly positive, notable exceptions are [18, 29, 30]. Since $g(0) = 0$ for the TM measure and since the hyperuniformity of the TM sequence depends on this property, we do not want to make such a restriction. For a few examples of $g$-functions with zeros, illustrating that there may or may not be a unique $g$-measure, we refer to [29, Sec. 4] and [18, Sec. VII].

In this note, we will not assume prior knowledge on $g$-measures in order to make it more accessible for readers coming from a number-theoretic angle. In particular, all the relevant notions will be introduced in Section 2 and the exposition is mostly self-contained. Some basic results on $g$-measures that by now can be considered folklore will be proved in an elementary fashion for the reader’s convenience. We point to the relevant literature as we go along. In the following, we collect some of this ‘folklore’ and state it in two theorems for easier reference.

Here, we denote the space of continuous functions on $T$ by $C(T)$. The first theorem builds on a powerful result from [29]; compare [18] for an alternative approach.

**Theorem 1.1.** Let $g \in C(T)$ be a $g$-function of summable variation and assume that one of the following properties holds.

1. The function $g$ has at most one zero in $T$;
2. The function $g$ has only finitely many zeros in $T$, none of which wanders into a periodic orbit under $T$; or:
3. All zeros of $g$ lie in $\left[\frac{1}{4}, \frac{3}{4}\right]$, sparing at least one of the boundary points.

Then, the probability measures on $T$ defined by the densities $g_n(x) := 2^n \prod_{k=0}^{n-1} g(2^k x)$, as $n \to \infty$, converge weakly to a probability measure on $T$, denoted by $\mu_g$, and $\mu_g$ is strongly mixing on $(T, T)$.

Below, we call a $g$-function **good** when one of the assumptions (and thus the conclusion) of Theorem 1.1 holds. Since both $\mu_g$ and Lebesgue measure $\lambda$ are ergodic, invariant probability measures for $(T, T)$, they are either equal or mutually singular. In fact, by standard properties of $g$-measures, we can fully characterise the spectral type of $\mu_g$ for good $g$-functions as follows.

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2See Definition 2.3 below for this notion.
Theorem 1.2. Let $g \in \mathcal{C}(\mathbb{T})$ be a good $g$-function, and $\mu_g$ the associated measure. Then, we are in one of the following three cases.

(ac): $\mu_g = \lambda_L$ if and only if $g$ is constant on $\mathbb{T}$, which means $g \equiv \frac{1}{2}$.

(pp): $\mu_g = \delta_0$ if and only if $g\left(\frac{1}{2}\right) = 0$.

(sc): $\mu_g$ is singular continuous with respect to $\lambda_L$ otherwise.

As we are aiming at generalisations of the TM measure, we will impose another property. We say that a good $g$-function $g$ exhibits power-law scaling if there exist positive constants $c_1, c_2, \theta_1$ and $\theta_2$ such that

$$c_1 x^{\theta_1} \leq g(x) \leq c_2 x^{\theta_2}$$

holds for all $x \in \left[0, \frac{1}{2}\right]$. This in particular implies that $g(0) = 0$ is the unique zero of $g$ in the interval $\left[0, \frac{1}{2}\right]$.

Example 1.3. Guiding examples of such $g$-functions are given by $g_t(x) = \frac{1}{2}(1 - \cos(2\pi x))$, which induces the TM measure, by the tent map

$$g_\wedge(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1 - x), & \frac{1}{2} < x < 1, \end{cases}$$

and by the square root inspired function

$$g_{\sqrt{\cdot}}(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq \frac{1}{4}, \\ 1 - \sqrt{|x - \frac{1}{4}|}, & \frac{1}{4} < x \leq \frac{3}{4}, \\ \sqrt{1 - x}, & \frac{3}{4} < x < 1. \end{cases}$$

These three functions exemplify the three behaviours of $g$-functions with power-law scaling at 0. Indeed, at $x = 0$, the derivative of $g_t$ vanishes, $g_\wedge$ has finite derivative, and $g_{\sqrt{\cdot}}$ has undefined (infinite) derivative; see the top row of Figure 1 for graphs of these functions.

Next, we examine the distribution function $F_g(x) := \mu_g([0, x])$. After proving that $F_g(x)$ is strictly increasing with $x$, which is a generalisation of this known property in the TM case, we show that the scaling of $F_g(x)$ near 0 is super-polynomial. This is our main result.

Theorem 1.4. Let $g \in \mathcal{C}(\mathbb{T})$ be a good $g$-function with power-law scaling. Then, one has

$$\log(F_g(x)) \asymp -\log_2(x)^2 \quad \text{as} \quad x \to 0^+,$$

where $\log_2$ denotes the logarithm to base 2. In particular, $F_g(x)$ decays faster than any power of $x$ as $x \to 0^+$.

This result may be viewed as a first step of a more general scaling analysis at arbitrary $x$, as the one in \[\text{[3]}\] for the TM measure, which will be a substantial task for the future. Here, we notice that the result of Theorem 1.4 is of particular relevance in number theory, for asymptotical results related to automatic sequences, and in the quantitative theory of order and local fluctuations, as recently studied in the context of hyperuniformity; see \[39, 12, 13\].
The remainder of this paper is organised as follows. In Section 2, using specific properties of $g$-functions, we derive Theorem 1.1 as a consequence of a more general result, which is interesting on its own as well. In Section 3, we characterise the pure point part of general $g$-measures and prove Theorem 1.2. Finally, Section 4 contains properties of the distribution function of $\mu_g$ as well as a proof of Theorem 1.4, together with more precise upper bounds on the scaling near 0, thus extending a result that is well known \[25, 7\] for $\mu_{\text{TM}}$. 

Figure 1. The probability densities $g_1, g_2, g_3, g_6$ and $g_{11},$ from the sequence $(g_n)_{n \in \mathbb{N}}$ that converges to $\mu_g$, for the three $g$-functions of Example 1.3.
2. Some properties of $g$-measures

As above, let $\mathbb{T}$ denote the 1-torus, written as $[0, 1)$ with addition modulo 1, and let $g \geq 0$ be a $g$-function on $(\mathbb{T}, T)$, where $T$ is the doubling map $Tx = 2x$. This means that $g$ is Borel measurable and that we have $g(x) + g(x + \frac{1}{2}) = 1$ for all $x \in \mathbb{T}$. Given a $g$-function on $(\mathbb{T}, T)$, we define the transfer operator $\varphi_g$ on a real-valued function $f$ on $\mathbb{T}$ by

$$(\varphi_g f)(x) = \sum_{y \in \mathbb{T}^{-1}(x)} g(y) f(y) = g\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + g\left(\frac{x + 1}{2}\right) f\left(\frac{x + 1}{2}\right).$$

**Definition 2.1.** A shift-invariant Borel probability measure $\mu$ on $\mathbb{T}$ is called a $g$-measure if $(\varphi_g)_* \mu = \mu$ holds, where $((\varphi_g)_* \mu)(f) := \mu(\varphi_g f)$ for $f \in C(\mathbb{T})$.

The operator $\varphi_g$ preserves the normalisation of a measure, hence maps probability measures to probability measures. A short calculation yields $(\varphi_g)_* \lambda_L = 2g\lambda_L$, where $\lambda_L$ is Lebesgue measure on $\mathbb{T}$ and $2g\lambda_L(f) = 2 \int_{\mathbb{T}} g(x) f(x) \, dx$. In particular, $\lambda_L$ itself is a $g$-measure precisely if $g \equiv \frac{1}{2}$ almost-surely. More generally, we obtain the following property; compare [23, Prop. 1].

**Lemma 2.2.** For $g_k(x) = 2^k \prod_{j=0}^{k-1} g(2^j x)$ and any $k \in \mathbb{N}$, one has $(\varphi_g^k)_* \lambda_L = g_k \lambda_L$. In particular, $g_k$ is a probability density on $\mathbb{T}$.

**Proof.** First, we observe inductively that

$$(\varphi_g^k f)(x) = 2^{-k} \sum_{y \in \mathbb{T}^{-k}(x)} g_k(y) f(y).$$

For every $f \in C(\mathbb{T})$, we now obtain

$$\int_0^1 f(x) g_k(x) \, dx = \int_0^1 (\varphi_g^k f)(x) \, dx = ((\varphi_g)_* \lambda_L)(f)$$

by a straightforward calculation. Since $(\varphi_g^k)_* \lambda_L = g_k \lambda_L$ is a probability measure on $\mathbb{T}$, it follows that $g_k$ is a probability density. $\square$

Interpreting $(g_n)_{n \in \mathbb{N}}$ as a sequence of probability measures on $\mathbb{T}$, by the Banach–Alaoglu theorem, it must have an accumulation point $\mu_g$ in the weak topology. Assume for a moment that $g_n$ converges to $\mu_g$. Then, it is easily checked that $\mu_g$ is in fact a $g$-measure. Below, we will give a sufficient condition for weak convergence of $(g_n)_{n \in \mathbb{N}}$, though this first needs some preparation via a suitable concept. To formulate the latter, let $f \in C(\mathbb{T})$ and set

$$f[\delta] = \max_{|x-y| \leq \delta} |f(x) - f(y)|$$

for $\delta > 0$. Note that the maximum is indeed attained, as $f$ is uniformly continuous, and that $\lim_{\delta \to 0^+} f[\delta] = 0$.

**Definition 2.3.** We say that $f \in C(\mathbb{T})$ is of summable variation if

$$f[\delta] := \sum_{j=0}^{\infty} f[2^{-j} \delta] < \infty,$$
for some (equivalently all) \( \delta > 0 \).

**Remark 2.4.** Note that \( f[\delta] \) is increasing monotonically in \( \delta \). It is straightforward to verify that \( f_\delta \to 0 \) as \( \delta \to 0^+ \) whenever \( f \) is of summable variation, and also that every Hölder-continuous function is of summable variation. The condition that \( f \) is of summable variation appears also under the term *Dini continuous* in the literature. For the comparison of our results with the literature on \( g \)-measures, it is noteworthy that, if \( f > 0 \), it is of summable variation precisely if \( \log(f) \) has this property.

Even in the case that \( g > 0 \), additional regularity assumptions are needed to conclude that there is a unique \( g \)-measure associated to it; see [11] for an example that shows that continuity alone does not suffice. Summable variation has become a standard assumption, and indeed guarantees both uniqueness and uniform convergence of \( \varphi^n_g f \) to a constant when \( g > 0 \); compare [10]. It does not immediately yield such a strong conclusion if \( g \) is allowed to have zeros, but has proved useful also in this more general setting [15]. The reason is that summable variation guarantees that, for each \( f \in C(T) \), the family \( \{\varphi^n_g f\}_{n \in \mathbb{N}} \) is uniformly equicontinuous. Indeed, a straightforward calculation shows that

\[
(\varphi_g f)[\delta] \leq 2 |f| \left[ g\left[ \frac{\delta}{2} \right] + f\left[ \frac{\delta}{2} \right] \right],
\]

from which equicontinuity is quickly deduced if \( g \) has summable variation. Hence, \( \{\varphi^n_g f\}_{n \in \mathbb{N}} \) has a uniformly converging subsequence by the Arzela–Ascoli theorem.

**Remark 2.5.** The term *summable variation* is somewhat more natural in the context of shift spaces. This is the classic setup in the context of the thermodynamic formalism [37, 40]. In fact, there is a natural identification of \((T, T)\) with the one-sided shift \((X, S)\), with the space \( X = \{0, 1\}^\mathbb{N} \) and with \( S \) denoting the left shift action, \((Sx)_n = x_{n+1}\) for all \( n \in \mathbb{N} \). The identification is given by the semi-conjugation \((x_n)_{n \in \mathbb{N}} \mapsto \sum_{j=1}^{\infty} x_j 2^{-j}\). On the complement of dyadic points, this map has a well-defined inverse, given by the 2-adic expansion of elements in \([0, 1)\). With a few modifications, much of the analysis in this paper can also be performed on \((X, S)\); compare [5]. We have chosen to work with \((T, T)\) because this seems more natural for the applications we have in mind, that is, diffraction and hyperuniformity results.

The following is a mild adaptation of a result in [29]. The only difference is that, instead of differentiability, we use the weaker summable variation property to obtain equicontinuity of the sequence \( \{\varphi^n_g f\}_{n \in \mathbb{N}} \). The rest of the proof remains unchanged. Parts of the following proposition have also been covered by [8, Thm. 2.1].

**Proposition 2.6 ([29]).** Let \( g \in C(T) \) be a \( g \)-function of summable variation, and assume that one of the following properties is satisfied, namely

1. that \( g \) has at most one zero in \( T \),
2. that \( g \) has only finitely many zeros in \( T \), none of which wanders into a periodic orbit under the map \( T \), or
3. that all zeros of \( g \) lie in \([\frac{1}{4}, \frac{3}{4})\) or in \([\frac{1}{4}, \frac{3}{4}]\).
Then, for all $f \in C(T)$, the sequence of functions $(\phi_k^n f)_{k \in \mathbb{N}}$ converges, uniformly on $T$, to a constant, denoted by $\mu_g(f)$. Further, the mapping $f \mapsto \mu_g(f)$ defines a strongly mixing probability measure on $T$, where $\mu_g$ is the unique $g$-measure induced by $g$. \hfill \Box

**Remark 2.7.** Proposition 2.6 also follows from results by Conze and Raugi [18]. In fact, under the assumption of summable variation, they give a very nice characterisation for the uniqueness of $g$-measures and the uniform convergence of $\phi_k^n g f$ in terms of proximality. Given $d \in \mathbb{N}$, a closed subset $F \subseteq T$ is said to be $d$-invariant if, for each $x \in F$ and $y \in T^{-d}(x)$ such that $g_n(y) > 0$, it follows that $y \in F$. The function $g$ is said to be $d$-proximal if any two closed $d$-invariant subsets of $T$ intersect. It was shown in [18, Secs. V and VI] (for the interval $[0, 1]$ instead of $T$) that a $g$-measure is unique precisely if $g$ is 1-proximal, and that $(\phi_k^n f)_{k \in \mathbb{N}}$ converges uniformly to a constant if and only if $g$ is $d$-proximal for all $d \in \mathbb{N}$. Indeed, it is straightforward to verify that every $g$-function that satisfies any of the properties listed in Proposition 2.6 is in fact $d$-proximal for all $d \in \mathbb{N}$. We can also use this characterisation to give weaker conditions under which the conclusion of Proposition 2.6 holds. For example, we can replace (2) and (3) by

(2') the set $g^{-1}(1)$ is finite and does not contain a complete $T$-orbit,

(3') all zeros of $g$ lie in $[\frac{1}{6}, \frac{2}{3})$ or in $(\frac{1}{6}, \frac{5}{6}]$,

respectively. Since Proposition 2.6 in its present form is sufficient for our purposes, we leave the proof of this stronger result to the interested reader — the major ingredients can be found in [18], some details will be given in [26]. \hfill \Box

With Proposition 2.6 at hand, Theorem 1.1 becomes a direct consequence as follows.

**Proof of Theorem 1.1.** Under the assumptions of Proposition 2.6, we obtain

$$(g_n \lambda_L)(f) = \lambda_L(\phi_n^n f) \xrightarrow{n \to \infty} \mu_g(f),$$

for all $f \in C(T)$, where we have used Lemma 2.2 and dominated convergence. \hfill \Box

3. The pure point part of $\mu_g$

Next, we want to understand the Lebesgue decomposition $\mu_g = \mu_{g,pp} + \mu_{g,sc} + \mu_{g,ac}$. To this end, let us first summarise some useful properties of $g$-measures.

**Proposition 3.1.** The $g$-measure $\mu_g$ of a good $g$-function satisfies the following properties.

1. Either one has $\mu_g = \lambda_L$, or $\mu_g$ is singular with respect to $\lambda_L$. In particular, when $\mu_g \neq \lambda_L$, the absolutely continuous part of $\mu_g$ vanishes.

2. One has $\mu_g = \lambda_L$ if and only if $g(x) = \frac{1}{2}$ for a.e. $x \in T$. Within $C(T)$, this means that $g$ is identical to the constant function $\frac{1}{2}$.

3. Either one has $\mu_{g,pp} = \mu_g$, or $\mu_g$ has no pure point part.

4. The measure $\mu_g$ is always of pure type.
Proof. Properties (1) and (2) immediately follow from Theorem\textsuperscript{1.1} and the observation made before Theorem\textsuperscript{1.2}. To establish (3), we simply verify \((\varphi_g)_\ast (\mu_{g,pp}) = ((\varphi_g)_\ast \mu_g)_{pp} = \mu_{g,pp}\). Then, the uniqueness stated in Proposition\textsuperscript{2.6} gives the claim, and (4) becomes a straightforward consequence. □

More generally, \(\mu_g\) is always of pure type whenever it is unique for a given \(g\)-function. This follows from the fact that \(\mu_{g,\alpha}\) is separately invariant under \((\varphi_g)_\ast\) for \(\alpha \in \{ac, sc, pp\}\) if \(\mu_g\) is invariant [8, Thm. 1.2]; compare [20, Lem. 2.2].

In what follows, we explore conditions for the existence of a pure point part. Since some of the observations hold in full generality, without extra complications with the proofs, we drop the assumption that \(g\) is a good \(g\)-function for a while. Recall from [29] that the set of \(g\)-measures coincides precisely with the set of \(T\)-invariant measures on \(T\). From this, we conclude that the pure point part of an arbitrary \(g\)-measure is supported on complete (forward) \(T\)-orbits, the latter denoted by \(O_T(x) := \{T^kx : k \in \mathbb{N}_0\}\) in what follows. In fact, it is also known that \(\mu_{g,pp}\) is supported on \(g^{-1}(1)\) [18]. We give the precise statement with a short proof for convenience.

Proposition 3.2. Suppose \(\mu_g(\{x\}) > 0\). Then, \(x\) is a periodic point of \(T\), and we have \(g(y) = 1\) together with \(\mu_g(\{y\}) = \mu_g(\{x\})\) for every \(y \in O_T(x)\).

Proof. Since \(\mu_g\) is \(T\)-invariant, we have \(\mu_g(\{Tx\}) = \mu_g(\{x\}) + \mu_g(\{x + \frac{1}{2}\})\), which implies

\[0 < \mu_g(\{x\}) \leq \mu_g(\{Tx\}) \leq \cdots \leq \mu_g(\{T^n x\})\]

for all \(n \in \mathbb{N}\). This shows that \(O_T(x)\) must be finite, and that \(\mu_g\) is ultimately constant on this orbit. Hence, there exist \(j, k \in \mathbb{N}\) with \(j < k\) such that \(T^jx = T^kx\). But then, we either have \(T^{j-1}x = T^{k-1}x\) or \(T^{j-1}x = T^{k-1}x + \frac{1}{2}\). The latter case is impossible because

\[\mu_g(\{T^{k-1}x\}) = \mu_g(\{T^kx\}) = \mu_g(\{T^{k-1}x\}) + \mu_g(\{T^{k-1}x + \frac{1}{2}\})\]

implies that \(\mu_g(\{T^{k-1}x + \frac{1}{2}\}) = 0\), which contradicts \(\mu_g(\{T^{j-1}x\}) \geq \mu_g(\{x\}) > 0\). We conclude that \((T^kx)_{k \in \mathbb{N}_0}\) is periodic and \(\mu_g\) is constant on \(O_T(x)\).

Since \(\mu_g\) is a regular Borel measure, every integrable function \(h\) satisfies

\[\mu_g(h) = (\varphi_g)_\ast \mu_g(h) = \mu_g(\varphi_g h)\]

In particular, for \(y \in O_T(x)\) and \(h = 1_{\{y\}}\), we obtain \(\varphi_g 1_{\{y\}} = g(y) 1_{\{Ty\}}\) and thus also \(\mu_g(1_{\{y\}}) = g(y) \mu_g(1_{\{Ty\}})\). This implies \(g(y) = 1\) for all \(y \in O_T(x)\). □

Remark 3.3. It is not difficult to verify that some kind of converse of the statement in Proposition\textsuperscript{3.2} also holds. Indeed, whenever \(x = T^p x\) and \(g(T^j x) = 1\) for all \(0 \leq j \leq p - 1\), the measure \(\mu = \frac{1}{p} \sum_{j=0}^{p-1} \delta_{T^j x}\) is a \(g\)-measure. This observation can be used to construct examples of \(g\)-functions that give rise to more than one \(g\)-measure; compare [18, 29]. ◇

Returning to the case of good \(g\)-functions, we are now ready to prove Theorem\textsuperscript{1.2}.
Proof of Theorem 1.2. The ac case is clear from properties (1) and (2) of Proposition 3.1. Next, assume that \( \mu_g \) has a non-trivial pure point part and let \( x \in \mathbb{T} \) be such that \( \mu_g(\{x\}) > 0 \). By Proposition 3.2 there is some \( p \in \mathbb{N} \) with \( T^p x = x \) and \( g(T^j x) = 1 \) for all \( 0 \leq j \leq p - 1 \). We now proceed with a case distinction as in Theorem 1.1.

First, assume that \( g \) has only one zero in \( \mathbb{T} \), say \( z \). Then, there is precisely one \( x \in \mathbb{T} \) with \( g(x) = 1 \), namely \( x = z + \frac{1}{2} \). But this implies \( p = 1 \), hence \( Tx = x \), and thus \( x = 0 \). Since this point is unique and \( \mu_g \) is a probability measure on \( \mathbb{T} \), we get \( \mu_g = \delta_0 \) in this case.

Second, assume that \( g \) has only finitely many zeros, none of which wanders into a periodic orbit. However, if \( \mu_g(\{x\}) > 0 \), we know that \( z = x + \frac{1}{2} \) must be a zero of \( g \) with \( Tx = Tz \), which is impossible because \( x \) is a periodic point of \( T \), and \( \mu_{g,pp} = 0 \) in this case.

Third, assume that all zeros of \( g \) lie in \( \left[\frac{1}{4}, \frac{3}{4}\right] \) or in \( \left(\frac{1}{4}, \frac{3}{4}\right) \), and let \( I \) be either of these intervals. If \( x \neq 0 \), there is some \( k \in \mathbb{N} \) such that \( T^k x \in I \). But then, \( z = T^k x + \frac{1}{2} \notin I \) with \( g(z) = 0 \), a contradiction. So, we do not get any further location beyond the one from our first case.

Together, this shows that, for a good \( g \)-function, \( \mu_g(\{x\}) > 0 \) is only possible for \( x = 0 \), in which case \( g(\frac{1}{2}) = 0 \). Conversely, assume \( g(\frac{1}{2}) = 0 \), which is equivalent to \( g(0) = 1 \). Then, \( (\varphi_g f)(0) = f(0) \) for every \( f \in C(\mathbb{T}) \). Since \( \varphi_g f \) converges pointwise to \( \mu_g(f) \), this yields

\[
\mu_g(f) = \lim_{k \to \infty} (\varphi_g f)(0) = f(0),
\]

and thus \( \mu_g = \delta_0 \), which settles the pp case.

Finally, if \( \mu_g \neq \mu_{g,pp} \), Proposition 3.1 implies that the measure \( \mu_g \) cannot have any pure point part, and is singular relative to \( \lambda_\mathbb{T} \). The statement for the ac case is then clear.

Remark 3.4. If \( \mu \) is a general, \( T \)-invariant probability measure on \( \mathbb{T} \), we can conclude that \( \mu_{ac} = c \lambda_\mathbb{T} \) for some \( c \in [0, 1] \). First, it is clear that each of \( \mu_{pp} \), \( \mu_{ac} \) and \( \mu_{sc} \) is separately \( T \)-invariant, so consider \( \nu = \mu_{ac} \). With \( f_n(x) := e^{2\pi i n x} \) and a simple calculation, we obtain the Fourier–Stieltjes coefficients of \( \nu \) as

\[
\hat{\nu}(n) = \nu(f_n) = \nu(f_n \circ T) = \nu(f_{2n}) = \hat{\nu}(2n)
\]

for \( n \in \mathbb{Z} \), where \( \hat{\nu}(0) = \nu(\mathbb{T}) \). Any \( n \neq 0 \) has a unique representation as \( n = 2^k(2m + 1) \). Then, by a standard application of the Riemann–Lebesgue lemma, the above doubling relation forces \( \hat{\nu}(2m + 1) = 0 \) for all \( m \in \mathbb{Z} \) and thus \( \hat{\nu}(n) = 0 \) for all \( n \neq 0 \). This implies our claim with \( c = \nu(\mathbb{T}) \in [0, 1] \), by the uniqueness of the Fourier–Stieltjes coefficients.

4. The Distribution Function \( F_g(x) \)

In this section, we examine the distribution function \( F_g(x) := \mu_g([0, x]) \). We first prove that \( F_g(x) \) is strictly increasing with \( x \) if \( g \) has at most countably many zeros. This generalises the classic result [10, Lem. 2.1] that \( \mu_g \) has full support if \( g > 0 \). Then, we restrict to \( g \)-functions with power-law scaling and prove an effective result in showing that the scaling of \( F_g(x) \) near zero is super-polynomial. In particular, we prove Theorem 1.4. This is also related to the analysis of hyperuniform structures, and shows that further connections to number-theoretic
questions exist. In particular, the deviation from a power-law scaling, as known from the TM sequence (see [25, 7] and references therein), is not at all unusual.

**Theorem 4.1.** Let $g \in C(\mathbb{T})$ be a good $g$-function with at most countably many zeros, and suppose $g(\frac{1}{2}) \neq 0$. Then, the distribution function $F_g(x)$ is strictly increasing in $x$. In particular, every open interval has positive $\mu_g$-measure.

**Proof.** It suffices to show that $\mu_g(I) > 0$ for every interval of the form $I = [2^{-k}j, 2^{-k}(j + 1)]$, with $k \in \mathbb{N}$ and $0 \leq j \leq 2^k - 1$. Since $\mu_g$ is continuous as a measure, weak convergence $\int_{n \to \infty} \mu_g \Rightarrow \mu_g$ yields

$$
\mu_g(I) = \lim_{n \to \infty} \int_I g_n(x) \, dx = \lim_{n \to \infty} \int_I g_k(x) g_{n-k}(2^k x) \, dx
$$

$$
= \lim_{n \to \infty} \int_0^1 2^{-k} g_k(2^{-k}(j + y)) g_{n-k}(y) \, dy = \mu_g(f),
$$

where $x \mapsto f(x) = 2^{-k} g_k(2^{-k}(j + x))$ defines a non-negative function that is bounded by 1 and has at most countably many zeros. Set $A_n = \{ x \in \mathbb{T} : f(x) \geq \frac{1}{2} \}$ for $n \in \mathbb{N}$ and $B = \{ x \in \mathbb{T} : f(x) = 0 \}$. Clearly, $\mathbb{T} = \bigcup_{n \in \mathbb{N}} A_n \cup B$. Since $B$ is countable and $\mu_g$ continuous, we obtain

$$
1 = \mu_g(\mathbb{T}) \leq \sum_{n \in \mathbb{N}} \mu_g(A_n),
$$

and thus $\mu_g(A_n) > 0$ for some $n \in \mathbb{N}$. For this choice of $n$, we find

$$
\mu_g(I) = \mu_g(f) \geq \frac{1}{n} \mu_g(A_n) > 0.
$$

Since $k$ and $j$ were arbitrary, this implies that $F_g(x)$ is strictly increasing in $x$. \qed

The following result provides inequalities as well as asymptotics that immediately imply Theorem 4.1.

**Theorem 4.2.** Let $g \in C(\mathbb{T})$ be a good $g$-function with power-law scaling. Let $c_1, c_2, \theta_1$ and $\theta_2$ be positive constants with $\theta_2 \leq \theta_1$ such that, for all $x \in [0, \frac{1}{2}]$,

$$
c_1 x^{\theta_1} \leq g(x) \leq c_2 x^{\theta_2}.
$$

Then, with $s = \min\{1, c_1\}$, $S = \max\{1, c_2\}$ and $\kappa = \mu_g(\left[\frac{1}{2}, 1\right]) > 0$, one has

$$
\kappa s 2^{-2\theta_1} x^{-\frac{\theta_1}{2}} \log_2(x) \frac{\log_2(s)}{2} < F_g(x) < x^{-\frac{\theta_1}{2}} \log_2(x) x^{-\frac{\theta_1}{2}} \log_2(s).
$$

In particular, $F_g(x)$ decays faster than any power of $x$ as $x \to 0^+$.

**Proof.** Consider an interval $I_m := [0, 2^{-m}]$ for $m \in \mathbb{N}$. Let $\mu_n = g_n \lambda_\cdot$, and recall that the sequence $(g_n \lambda_\cdot)_{n \in \mathbb{N}}$ converges weakly to $\mu_g$ by Theorem 1.1. The inequalities (1) give upper and lower bounds on $g_m(x)$ for $x$ close to zero. For $y \in [0, 1]$, we have

$$
g_m(2^{-m} y) = 2^m \prod_{k=0}^{m-1} g(2^{-m} y) \leq 2^m \prod_{k=0}^{m-1} c_2 2^{(k-m)\theta_2} = (2c_2)^m 2^{-\frac{m(m+1)}{2}} \theta_2.
$$
Similarly, for the lower bound, we find

\[ g_m(2^{-m}y) = 2^m \prod_{k=0}^{m-1} g(2^{k-m}y) \geq 2^m \prod_{k=0}^{m-1} c_1 2^{(k-m)\theta_1} y^{\theta_1} = (2c_1)^m 2^{-m(m+1)/2} \theta_1 y^{\theta_1}. \]

We use these bounds on \( g_m(2^{-m}y) \) to establish upper and lower bounds on \( \mu_n(I_m) \), and then apply the portmanteau theorem \cite[Thm. 2.1]{9} for \( x \in [2^{-(m+1)}, 2^{-m}] \); that is, that

\[ \limsup_{n \to \infty} \mu_n(I_{m+1}) \leq F_g(x) \leq \liminf_{n \to \infty} \mu_n(I_m). \]

For \( x \in [2^{-(m+1)}, 2^{-m}] \), we have both

\[ 2^{-(m+1)} \leq x < 2^{-m} \quad \text{and} \quad -\log_2(x) - 1 \leq m < -\log_2(x). \]

We use these four inequalities freely in what follows. Further, for \( x \in I_{m+1} \), we have \( x < 2^{-m} \), which is the key inequality in the proof of the lower bound.

Given \( n > m \), we use \( g_n(x) = g_m(x) g_{n-m}(2^m x) \) to obtain

\[
\mu_n(I_m) = \int_0^{2^{-m}} g_m(x) g_{n-m}(2^m x) \, dx = 2^{-m} \int_0^{1} g_m(2^{-m} y) g_{n-m}(y) \, dy
\]

\[
\leq c_2^m 2^{-m(m+1)/2} \theta_2 \int_0^{1} g_{n-m}(y) \, dy = c_2^m 2^{-m(m+1)/2} \theta_2,
\]

where the last step follows because \( g_{n-m} \) is a probability density on \([0,1]\). Using the last bound and the inequalities from above, we thus get

\[ \liminf_{n \to \infty} \mu_n(I_m) \leq c_2^m 2^{-m^2/2 - m \theta_2} < x^{-\theta_2} \log_2(x) x^{-\theta_2} - \log_2(S), \]

where \( S = \max\{1, c_2\} \) and thus \( \log_2(S) \geq 0 \).

For the lower bound, we obtain

\[
\mu_n(I_m) = 2^{-m} \int_0^{1} g_m(2^{-m} y) g_{n-m}(y) \, dy \geq c_1^m 2^{-m(m+1)/2} \theta_1 \int_0^{1} y^{\theta_1} g_{n-m}(y) \, dy
\]

\[
\geq c_1^m 2^{-m(m+1)/2} \theta_1 \int_0^{1/2} 2^{-m \theta_1} g_{n-m}(y) \, dy \xrightarrow{n \to \infty} \kappa c_1^m 2^{-m^2/2 - m \theta_2/2}.
\]

Then, incrementing \( m \), we get

\[ \limsup_{n \to \infty} \mu_n(I_{m+1}) \geq \kappa c_1^{m+1} 2^{-(m+1)^2/2 -(m+1)\theta_2} > \kappa s 2^{-2\theta_1} x^{-\theta_2} \log_2(x) x^{\theta_2} - \log_2(s), \]

where \( s = \min\{1, c_1\} \) and thus \( \log_2(s) \leq 0 \). Combining (2) with (3) and applying the portmanteau theorem as described above provides the desired result. \( \square \)

**Corollary 4.3.** Under the assumptions of Theorem 4.2, we have

\[-\frac{\theta_1}{2} \log_2(x)^2 \left(1 + O\left(\frac{1}{\log_2(x)}\right)\right) \leq \log(F_g(x)) \leq -\frac{\theta_2}{2} \log_2(x)^2 \left(1 + O\left(\frac{1}{\log_2(x)}\right)\right) \]

as \( x \to 0^+ \). \( \square \)
Let us see what this gives for our three guiding examples.

**Example 4.4.** Consider our three examples from Example 1.3. They all share the additional symmetry relation $g(x) = g(1 - x)$, for all $x \in [0, \frac{1}{2}]$. This implies the same symmetry for $g_n$, for all $n \in \mathbb{N}$, and hence $\kappa = \mu_g([\frac{1}{2}, 1]) = \frac{1}{2}$. For the TM measure $\mu_{TM}$, we have $g_t(x) = \frac{1}{2}(1 - \cos(2\pi x))$. For $x \in \left[0, \frac{1}{2}\right]$, one has $4x^2 \leq g_t(x) \leq \pi^2 x^2$, so Theorem 4.2 gives

$$2^{-5} x^{-\log_2(x)} x^5 < F_{g_t}(x) < x^{-1-2\log_2(\pi)}$$

which should be compared with [7, Thm. 5.2], where a slightly stronger result was derived by using more specific properties of the TM measure.

For the tent map, we have $g_\wedge(x) = 2x$, so

$$2^{-3} x^{-\frac{1}{2}\log_2(x)} x^\frac{5}{2} < F_{g_\wedge}(x) < x^{-\frac{1}{2}\log_2(x)} x^{-\frac{3}{2}}$$

Finally, for the square root map $g_{\sqrt{}}(x)$, we get $\sqrt{x} \leq g_{\sqrt{}}(z) \leq \sqrt{2}x$ for $x \in \left[0, \frac{1}{2}\right]$, hence

$$2^{-2} x^{-\frac{1}{2}\log_2(x)} x^\frac{5}{4} < F_{g_{\sqrt{}}}(x) < x^{-\frac{1}{2}\log_2(x)} x^{-\frac{3}{4}}$$

which shows the common structure of all three cases. ♦

It will now be interesting to extend the full scaling analysis of [5] to these guiding examples, and the general family treated above.

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