EIGENVALUE ANALYSIS OF THE LAX OPERATOR FOR THE
ONE-DIMENSIONAL CUBIC NONLINEAR DEFOCUSING
SCHRÖDINGER EQUATION

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Abstract. We characterize the location and number of eigenvalues for the Lax operator associated to the one-dimensional cubic nonlinear defocusing Schrödinger equation. With the help of a newly discovered unitary matrix, the analysis reduces to the study of the spectral problem for a unitarily equivalent operator, which involves only the amplitude and the phase velocity of the potential. Examples of potentials with special amplitude and phase velocity are investigated.

Keywords: Cubic nonlinear defocusing Schrödinger equation, nonzero boundary condition, Lax operator, one-dimensional Dirac operator, Sturm-Liouville eigenvalue problem, spectral analysis

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1. Introduction

We consider the following one-dimensional defocusing cubic nonlinear Schrödinger (NLS) equation

\[ i\partial_t q + \partial_{xx} q = 2|q|^2 q, \]

where \( q = q(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) denotes the unknown wave function. By the seminal paper by Zakharov-Shabat [17], the (NLS) equation can be (formally) formulated in the Lax pair form

\[ \partial_t L = PL - LP, \]

where \( L \) denotes the self-adjoint Lax operator with the potential \( q \)

\[ L = L_q = \begin{pmatrix} i\partial_x & -iq \\ iq & -i\partial_x \end{pmatrix}, \]

and \( P \) is the following skewadjoint differential operator

\[ P = i \begin{pmatrix} 2\partial_x^2 - |q|^2 & -q\partial_x - \partial_x q \\ \overline{q}\partial_x + \partial_x \overline{q} & -2\partial_x^2 + |q|^2 \end{pmatrix}. \]

Here the application of the operator \( \partial_x \overline{q} \) on a function \( f \) is understood as \( \partial_x (\overline{q} f) \). Let \( U(t', t) \) be the unitary family generated by the skewadjoint operator \( P \), then by virtue of (1.2), one can relate the operators \( L(t) := L_{q(t,x)} \) and \( L(t') = L_{q(t',x)} \) at different times by

\[ L(t) = U^*(t', t) L(t') U(t', t), \]

such that the spectrum of the Lax operator \( L(t) \) is (formally) invariant under the evolutionary NLS-flow (1.1). In particular, the eigenvalues of \( L(0) \) at the initial
time $t = 0$ are the eigenvalues of $L(t)$ for all the time $t$ (as long as the solution exists). In the present paper we will analyze the eigenvalues of the Lax operator $L_q$ for a class of nowhere vanishing bounded potentials $q$, and in the following we will largely ignore the time dependence.

In the (classical) setting of decaying potentials:

$$q(x) \to 0 \text{ as } |x| \to \infty,$$

the spectral problem of the Lax operator $L_q$ and the associated direct/inverse scattering transform have been extensively studied in the literature, cf. the book [2]. If one assumes the nonzero boundary condition for $q$ at infinity:

(1.4) $$|q(x)| \to 1 \text{ as } |x| \to \infty,$$

the equation (1.1) possesses a family of soliton solutions $e^{-2it} q_c(x - 2ct)$ where

$$q_c(x) = \sqrt{1 - c^2} \tanh \left( \sqrt{1 - c^2} x \right) + ic, \quad -1 < c < 1,$$

and the corresponding Lax operator $L_q$ has a unique simple eigenvalue $-c$. These soliton solutions are called dark/black solitons in nonlinear optics, which travels at the speed $2|c| < 2$, and there are no soliton solutions with traveling speed bigger than 2. Due to the experimental relevance of the problem (1.1)-(1.4), the study of the spectral problem of the Lax operator $L_q$ under the assumption (1.4) (i.e. the one-dimensional Dirac operator with nonzero rest mass) has attracted much attention, cf. [1, 4, 5, 6, 7, 8, 9, 18]. In particular in the classical framework where

$$q(x) - 1 \in \mathcal{S}(\mathbb{R})$$

is a Schwartz function, Faddeev-Takhtajan [8] studied the self-adjoint operator $L_q$, and showed that its essential spectrum is $(-\infty, -1] \cup [1, \infty)$ and there are at most countably many simple real eigenvalues $\{\lambda_m\}$ inside $(-1, 1)$. More recently, Demontis et al. [7] studied rigorously the inverse scattering transform if

$$q(x) \to e^{i\theta \pm} \in \mathbb{S}^1 \text{ as } x \to \pm \infty$$

sufficiently fast in the sense that $(1 + x^2)(q(x) - e^{i\theta \pm}) \in L^1(\mathbb{R}^\pm)$. Under the stronger decay assumption

(1.5) $$\left(1 + x^4\right)(q(x) - e^{i\theta \pm}) \in L^1(\mathbb{R}^\pm),$$

they showed that there are only finitely many discrete eigenvalues which belong to the spectral gap $(-1, 1)$. It was shown recently in [12] that in the low-regularity finite-energy setting

$$q(x) \in L^2_{\text{loc}}(\mathbb{R}) \text{ with } |q|^2 - 1, \partial_x q \in H^{-1}(\mathbb{R}),$$

the essential spectrum of the Lax operator $L_q$ is $(-\infty, -1] \cup [1, \infty)$, and the spectrum outside the essential spectrum consists of isolated simple eigenvalues in $(-1, 1)$. However, under this weak assumption, there might be eigenvalues embedded in the essential spectrum. A specific kind of piecewise constant potentials

$$q(x) = \begin{cases} 
e^{-i\theta} & x < -R, \\ A e^{i\varphi} & -R < x < R, \\ e^{i\theta} & x > R, \end{cases}$$

has been considered in [4], and the authors there estimated the location of the discrete eigenvalues inside the spectral gap $(-1, 1)$ by considering the relation between $A$ and $\cos(\varphi)$. In particular, if $A < 1$, then there is at least one discrete eigenvalue.
Finally we mention here a recent work [3] assuming the nonzero asymmetric boundary condition on the potential

\[ q(x) \rightarrow q_\pm, \text{ as } x \rightarrow \pm \infty, \text{ with } |q_+| \neq |q_-|. \]

In spite of its physical and mathematical relevance, the spectral theory for the Lax operator \( L = \begin{pmatrix} i\partial_x & -iq \\ iq & -i\partial_x \end{pmatrix} \) with general potentials \( q \) is still far from satisfactory. Here we propose a new idea to study the operator \( L \), when the potentials are assumed to be nowhere vanishing, bounded and with finite phase velocity:

\[ q = |q|e^{i\varphi} \in L^\infty(\mathbb{R}; \mathbb{C}), \quad |q| > 0, \quad \partial_x \varphi \in L^\infty(\mathbb{R}; \mathbb{R}). \]

By straightforward calculations, the following unitary matrix which we believe to be new

\[ M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{i}{2}(\varphi - \frac{\pi}{2})} & e^{\frac{i}{2}(\varphi - \frac{\pi}{2})} \\ e^{-\frac{i}{2}(\varphi - \frac{\pi}{2})} & -e^{\frac{i}{2}(\varphi - \frac{\pi}{2})} \end{pmatrix} : H^s(\mathbb{R}; \mathbb{C}^2) \rightarrow H^s(\mathbb{R}; \mathbb{C}^2), \quad s = 0, 1, \]

transforms the Lax operator \( L : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) to the following unitarily equivalent operator

\[ \mathcal{L} = MLM^* = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix} : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2), \]

where the two real-valued functions \( u_\pm \) read

\[ u_\pm := \frac{1}{2} \partial_x \varphi \pm |q| \in L^\infty(\mathbb{R}; \mathbb{R}). \]

This proves

**Lemma 1.1** (Unitary equivalence between \( L \) and \( \mathcal{L} \)). For nowhere vanishing bounded potentials \( q \) with finite phase velocity such that \( u_\pm \in L^\infty(\mathbb{R}; \mathbb{R}) \), the operator \( L : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) and the operator \( \mathcal{L} : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) are unitarily equivalent:

\[ L = M^* \mathcal{L} M. \]

Hence it suffices to study the spectral problem for \( \mathcal{L} \). Simply by integration by parts, one can show that for any \( c \in \mathbb{R} \), \( u_- + c \) (resp. \( u_+ - c \)) controls the size \( c - \lambda \) (resp. \( c + \lambda \)), where \( \lambda \) is any eigenvalue of \( L \), in the following sense:

**Theorem 1.1** (Location of eigenvalues of \( \mathcal{L} \)). Let \( u_\pm \in L^\infty(\mathbb{R}; \mathbb{R}) \). If \( \lambda \in \mathbb{R} \) is an eigenvalue of the operator \( \mathcal{L} : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) given in (1.7), then \( \lambda \) satisfies, for all \( c \in \mathbb{R} \),

\[ c - \lambda \leq \|(u_- + c)^+\|_{L^\infty} \quad \text{or} \quad c + \lambda \leq \|(u_+ - c)^-\|_{L^\infty}. \]

In particular if \( u_+ - u_- \geq c > 0 \), there are no eigenvalues of \( \mathcal{L} \) in \((-c, c)\).

In the above, \( f^+, f^- \) denote the positive and negative parts of a real-valued function \( f \) respectively. Due to the unitary equivalence between the Lax operator \( L \) and the operator \( \mathcal{L} \), we have immediately

**Corollary 1.1** (Location of eigenvalues of \( L \)). Let \( q = q(x) \) satisfy (1.6). The eigenvalues \( \lambda \in \mathbb{R} \) of the operator \( L : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) given in (1.3) satisfy, for all \( c \in \mathbb{R} \),

\[ c - \lambda \leq \left\| \left( \frac{1}{2} \partial_x \varphi - |q| + c \right)^+ \right\|_{L^\infty} \quad \text{or} \quad c + \lambda \leq \left\| \left( \frac{1}{2} \partial_x \varphi + |q| - c \right)^- \right\|_{L^\infty}. \]
In particular, if \(|q| \geq c + \frac{1}{2}|\partial_x \varphi|\) pointwise for some \(c > 0\), then there are no eigenvalues of \(L\) in \((-c, c)\).

Theorem 1.1 will be proved in Section 2.

If one assumes the following boundary condition for the nowhere vanishing bounded potentials \(q\) given (1.6):
\[
|q(x)| \to 1 \quad \text{and} \quad \partial_x \varphi(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\]
which is stronger than (1.4), then \(u_\pm\) given in (1.8) satisfy
\[
u_{\pm}(x) \to \pm 1 \quad \text{as} \quad |x| \to \infty.
\]

Under the smallness condition
\[
\| (u_+ - 1)^- \|_{L^\infty} + \| (u_- + 1)^+ \|_{L^\infty} < 2,
\]
by Theorem 1.1, the eigenvalues \(\lambda \in (-1, 1)\) of \(L, \mathcal{L}\) are located
either in \(I_- := (-1, -1 + \| (u_+ - 1)^- \|_{L^\infty})\) or in \(I_+ := [1 - \| (u_- + 1)^+ \|_{L^\infty}, 1)\),
where \(I_-\) and \(I_+\) are disjoint. We have the following characterization of the numbers of eigenvalues inside \(I_-\) or \(I_+\) respectively:

**Theorem 1.2** (Number of eigenvalues). Let \(q = q(x)\) satisfy (1.6)-(1.9)-(1.10). The following holds true:

1. Let \(m \in \mathbb{N}_0 \cup \{\infty\}\) denote the number of negative eigenvalues \(\mu\) of the Sturm-Liouville eigenvalue problem

   \[
   -\partial_x \left( \frac{1}{1 - u_-} \partial_x \psi \right) - (1 - u_+) \psi = \mu \psi, \quad x \in \mathbb{R}.
   \]

   Then there are precisely \(m\) eigenvalues \(\lambda\) inside \(I_-\) of the operators \(L, \mathcal{L}\), counting by multiplicity.

2. Let \(l \in \mathbb{N}_0 \cup \{\infty\}\) denote the number of negative eigenvalues \(\nu\) of the Sturm-Liouville eigenvalue problem

   \[
   -\partial_x \left( \frac{1}{1 + u_+} \partial_x \psi \right) - (1 + u_-) \psi = \nu \psi, \quad x \in \mathbb{R}.
   \]

   Then there are precisely \(l\) eigenvalues \(\lambda\) inside \(I_+\) of the operators \(L, \mathcal{L}\), counting by multiplicity.

We will prove Theorem 1.2 in Section 3 by use of the min-max principle. In Section 3 we will also show the (non-)existence of eigenvalues for the Lax operator with some special potentials, as applications of Theorem 1.1 and Theorem 1.2.

At the end of the introduction part we give some remarks below.

**Remark 1.1** (Eigenvalue problems (1.11) and (1.12)). The eigenvalue problems (1.11) and (1.12) are formulated strongly here. It is however sufficient to consider their weak formulations as follows
\[
\int_{\mathbb{R}} \left( \frac{1}{1 - u_-} \partial_x \varphi \right) \partial_x \varphi - (1 - u_+) \varphi \varphi \right) \, dx = \mu \int_{\mathbb{R}} \varphi \varphi \, dx, \quad \forall \varphi \in H^1(\mathbb{R}),
\]
\[
\int_{\mathbb{R}} \left( \frac{1}{1 + u_+} \partial_x \varphi \right) \partial_x \varphi - (1 + u_-) \varphi \varphi \right) \, dx = \nu \int_{\mathbb{R}} \varphi \varphi \, dx, \quad \forall \varphi \in H^1(\mathbb{R}).
\]

If \(u_+ \geq 1\) (resp. \(u_- \leq -1\)), then (1.11) (resp. (1.12)) has no negative eigenvalues, and hence Theorem 1.2 implies that all the eigenvalues \(\lambda \in (-1, 1)\) of the
operators $L, \mathcal{L}$ lie in $I_+$ (resp. $I_-$). This partially recovers results in Theorem 1.1 with $c = 1$.

One can use the Rayleigh-Ritz method to show the existence of negative eigenvalues for (1.11) or (1.12). See [13, Theorem 10.23] for more explanations for the method, and see Example 3.2 in Section 3 below for the existence result of negative eigenvalues of (1.11) or (1.12) as an application of the Rayleigh-Ritz method.

Remark 1.2 (Functions $u_{\pm}$). The compressible Euler equations

\begin{equation}
\begin{cases}
\partial_t \rho + 2 \partial_x (\rho v) = 0, \\
\partial_t (\rho v) + 2 \partial_x (\rho v^2) + \partial_x p = 0,
\end{cases}
\end{equation}

where $(\rho, v) : \mathbb{R} \times \mathbb{R} \to [0, \infty) \times \mathbb{R}$ denote the unknown density and velocity functions respectively, are used to describe the motion of compressible fluids. For example, (1.13) together with the pressure law (up to a constant) $p = P_\lambda(\rho) := \rho^2$ governs the flow of (a class of) polytropic gases, and in the study of its dynamics, the two Riemannian invariants $u_{\pm} := \frac{1}{2} v \pm \sqrt{p}$ played an important role, cf. the book [15].

If $q = |q|e^{i\varphi} \neq 0$ everywhere, then one can write the NLS equation (1.1) in its hydrodynamic form (1.13) for the corresponding density and velocity functions given by $(\rho, v) := (|q|^2, \partial_x \varphi)$, and the pressure law reads $p = P_\lambda(\rho) + P_q(\rho)$, where $P_q(\rho) := -\frac{1}{2} \rho \partial_x (\frac{\partial_x \varphi}{\rho})$ denotes the so-called quantum pressure. The corresponding functions $u_{\pm} = \frac{1}{2} \partial_x \varphi \pm |q|$ given in (1.8) have also been used to study NLS-related problems in the literature, e.g. in the study of hydrodynamic optical soliton tunneling in [16], and in the study of the semiclassical limit in [10, 11].

Observe that if $u_{\pm}$ satisfy

$$-1 \leq u_- < u_+ \leq 1, \quad u_{\pm}(x) \to \pm 1 \text{ as } |x| \to \infty, \quad \|1 - u_+\|_{L^\infty} + \|1 + u_-\|_{L^\infty} < 2,$$

then the corresponding potential $q = |q|e^{i\varphi}$ satisfies the assumptions (1.6)-(1.9)-(1.10). In [11], under some further strong assumptions such as boundedness, single critical points and decay assumptions on $(u_+ + u_-)$ and $(u_+ - u_- + 2)$, Jin used the WKB method to calculate the asymptotic number of eigenvalues of $L^h := (\begin{pmatrix} i h \partial_x & -i h \varphi \\ i h \varphi & -i h \partial_x \end{pmatrix})$, $q_h(x) = |A(x)| e^{i \frac{\varphi}{\sqrt{h}}}$ in the semiclassical limit $h \to 0$:

$$\frac{1}{\pi h} \int_{\mathbb{R}} \left( \sqrt{(1-u_+(x))(1-u_-(x))} + \sqrt{(1+u_+(x))(1+u_-(x))} \right) \mathrm{d}x.$$  

We believe that the unitarily equivalent formulation $\mathcal{L}$ of $L$ will give new observations to interesting NLS-related problems.

Remark 1.3 (Example of potentials with piecewise-constant amplitude and phase velocity). We will show that (see Example 3.3 in Section 3 below), for potentials with piecewise-constant amplitude and phase velocity

\begin{equation}
q(x) = \begin{cases}
e^{i\theta} & x < a, \\
A e^{i(\varphi + (x-a))} & a \leq x \leq b, \\
e^{i(\varphi +(b-a))} & x > b,
\end{cases}
\end{equation}

where $(a, b) \subset \mathbb{R}, \theta \in [0, 2\pi)$, $A > 0$ and $v \in \mathbb{R}$, the Lax operator $L$ has

- No eigenvalues if $A \geq 1$ and $\frac{1}{2}|v| \leq A - 1$. In particular, the case with large amplitude $A \geq 1$ and constant phase $v = 0$ is included: $q = e^{i\theta} \begin{cases}
A & \text{on } [a, b], \\
1 & \text{otherwise}.
\end{cases}$
where \( \delta \) for all test functions \( \phi \in X \). LIAO AND M. PLUM

global-in-time solution of the NLS equation (1.1) with the initial data given in (1.14):

\[ q(x) = (1 + (A - 1)1_{[a,b]}(x))e^{i(q + v_0 \chi(x))}, \]

where \( 1_{[a,b]} \) denotes the characteristic function on \([a,b]\) and \( \chi \) denotes the Lipschitz continuous function \((x - a)1_{[a,b]} + (b - a)1_{(b,\infty)}\), satisfy

\[ q \in L^2_{\text{loc}}(\mathbb{R}), \quad |q|^2 - 1 = (A^2 - 1)1_{[a,b]} \in L^2(\mathbb{R}), \]

\[ \partial_x q = e^{i(q + v_0 \chi)}((A - 1)(\delta_a - \delta_b) + iAv1_{[a,b]}) \in H^{-1}(\mathbb{R}), \]

where \( \delta_a, c \in \mathbb{R} \) denotes the Dirac function such that \( \langle \delta_a, \varphi \rangle_{\mathcal{D}'(\mathbb{R})}, \mathcal{D}(\mathbb{R}) = \varphi(c) \) holds for all test functions \( \varphi \in \mathcal{D}(\mathbb{R}) \). Hence by [12, Theorem 1.5] there exists a unique global-in-time solution of the NLS equation (1.1) with the initial data given in (1.14) (in the sense given there), and by [12, Theorem 1.6] the corresponding (simple) eigenvalues inside \((-1,1)\) of the Lax operator given above preserve for all the time.

Organization of the paper. We will prove Theorem 1.1 in Section 2, for general nowhere vanishing bounded potentials with finite phase velocity.

In Subsection 3.1 we will prove Theorem 1.2 for nowhere vanishing bounded potentials with unit-size amplitude and vanishing phase velocity at infinity. In Subsection 3.2 we will give some interesting examples as illustration of Theorem 1.1 and Theorem 1.2.

In the appendix we will consider a specific kind of potentials where \( u_+ = 1 \) or \( u_- = -1 \), and the eigenvalues will be characterized via a family of compact operators. It is also of independent interest.

2. General case of nowhere vanishing bounded potentials with finite phase velocity

In this section we will study the eigenvalue problem for the Lax operator

\[ L = \begin{pmatrix} i\partial_x & -iq \\ iq & -i\partial_x \end{pmatrix}, \]

in the (general) case of nowhere vanishing bounded potentials with finite phase velocity given in (1.6):

\[ q = |q|e^{i\varphi} \in L^\infty(\mathbb{R}; \mathbb{C}), \quad |q| > 0, \quad \partial_x \varphi \in L^\infty(\mathbb{R}; \mathbb{R}). \]

By the unitary equivalence between \( L \) and \( \mathcal{L} \) given in Lemma 1.1, it suffices to study the eigenvalue problem for the operator \( \mathcal{L} \), which will be reformulated into \( \lambda \)-nonlinear eigenvalue problems in Subsection 2.1. We will analyze these \( \lambda \)-nonlinear eigenvalue problems to derive estimates for the eigenvalues of \( L, \mathcal{L} \) given in Theorem 1.1 in Subsection 2.2.
2.1. **Eigenvalue problem of** $\mathcal{L}$. We consider the eigenvalue problem for $\mathcal{L}$:

$$\mathcal{L}\Psi = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix} \Psi = \lambda \Psi,$$

with $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$.

We are going to reformulate it in different interesting cases. We notice the following symmetry

$$\begin{pmatrix} u_+, u_-, \lambda, \Psi_1, \Psi_2 \end{pmatrix} \mapsto \begin{pmatrix} -u_-, -u_+, -\lambda, -\Psi_2, \Psi_1 \end{pmatrix},$$

in this spectral problem $\mathcal{L}\Psi = \lambda \Psi$, which corresponds to the symmetry

$$\begin{pmatrix} q, \lambda, \psi_1, \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} -\bar{q}, -\lambda, \psi_2, \psi_1 \end{pmatrix}$$

in the spectral problem for the Lax operator $\mathcal{L}\psi = \lambda\psi$, with $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

2.1.1. **Case** $u_\pm = \pm 1$. If $\lambda \neq 1$, then the above spectral problem (2.1) reads simply

$$\begin{cases} (\lambda - 1)\Psi_1 = i\partial_x \Psi_2, \\ -\partial_{xx}\Psi_2 - (\lambda^2 - 1)\Psi_2 = 0, \end{cases}$$

That is, the second component $\Psi_2$ solves the spectral problem for the free Schrödinger operator $-\partial_{xx}$ with the spectral parameter $(\lambda^2 - 1)$, and the first component $\Psi_1$ is given by

$$\frac{1}{\lambda - 1}i\partial_x \Psi_2.$$

By a similar argument or by the symmetry property (2.2), if $\lambda \neq -1$, then the spectral problem (2.1) reads

$$\begin{cases} -\partial_{xx}\Psi_1 - (\lambda^2 - 1)\Psi_1 = 0, \\ (\lambda + 1)\Psi_2 = i\partial_x \Psi_1, \end{cases}$$

i.e. the first component $\Psi_1$ solves the spectral problem for the free Schrödinger operator $-\partial_{xx}$ with the spectral parameter $(\lambda^2 - 1)$, and the second component $\Psi_2$ is given by

$$\frac{1}{\lambda + 1}i\partial_x \Psi_1.$$

2.1.2. **Case** $u_- = -1$. If $\lambda \neq 1$, then the spectral problem (2.1) reads

$$\begin{cases} (\lambda - 1)\Psi_1 = i\partial_x \Psi_2, \\ -\partial_{xx}\Psi_2 - (\lambda - 1)(\lambda + u_+)\Psi_2 = 0, \end{cases}$$

and it suffices to consider the $\lambda$-nonlinear eigenvalue problem for $\Psi_2$:

$$-\partial_{xx}\phi - (\lambda - 1)(\lambda + u_+)\phi = 0,$$

with the first component $\Psi_1$ given by

$$\frac{1}{\lambda - 1}i\partial_x \phi.$$

Obviously $\lambda = 1$ is not an eigenvalue of $\mathcal{L}$ in this case $u_- = -1$.

2.1.3. **Case** $u_+ = 1$. Similarly as above or by the symmetry property (2.2), if $\lambda \neq -1$, then the spectral problem (2.1) reads

$$\begin{cases} -\partial_{xx}\Psi_1 - (\lambda + 1)(\lambda + u_-)\Psi_1 = 0, \\ (\lambda + 1)\Psi_2 = i\partial_x \Psi_1, \end{cases}$$

and $\lambda = -1$ is not an eigenvalue of $\mathcal{L}$ in this case $u_+ = 1$.

2.1.4. **General case of** $u_\pm \in L^\infty(\mathbb{R}; \mathbb{R})$ and $\lambda$ such that $\lambda + u_- \neq 0$ on $\mathbb{R}$. By straightforward calculations, the spectral problem (2.1) reads as

$$\begin{cases} (\lambda + u_-)\Psi_1 = i\partial_x \Psi_2, \\ -\partial_x \left( \frac{1}{\lambda + u_-} \partial_x \Psi_2 \right) - (\lambda + u_+)\Psi_2 = 0. \end{cases}$$
2.1.5. General case of \( u_\pm \in L^\infty(\mathbb{R};\mathbb{R}) \) and \( \lambda \) such that \( \lambda + u_+ \neq 0 \) on \( \mathbb{R} \). As above, \( (2.1) \) becomes

\[
\begin{cases}
-\partial_x \left( \frac{1}{\lambda + u_-} \partial_x \Psi_1 \right) - (\lambda + u_-) \Psi_1 = 0, \\
(\lambda + u_+) \Psi_2 = i \partial_x \Psi_1.
\end{cases}
\]

To conclude, we have

**Lemma 2.1** (Reformulation of the eigenvalue problem of \( L \)). The eigenvalue problem \( L \Psi = \lambda \Psi \) reads,

1. if \( \lambda + u_- \neq 0 \) on \( \mathbb{R} \), as

\[
\begin{aligned}
-\partial_x \left( \frac{1}{\lambda + u_-} \partial_x \phi \right) - (\lambda + u_-) \phi = 0,
\end{aligned}
\]

\[\text{together with } \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda + u_-} i \partial_x \phi \\ \phi \end{pmatrix}.\]

2. if \( \lambda + u_+ \neq 0 \) on \( \mathbb{R} \), as

\[
\begin{aligned}
-\partial_x \left( \frac{1}{\lambda + u_+} \partial_x \phi \right) - (\lambda + u_+) \phi = 0,
\end{aligned}
\]

\[\text{together with } \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \frac{1}{\lambda + u_+} i \partial_x \phi \end{pmatrix}.\]

2.2. **Proof of Theorem 1.1.** In this subsection we prove Theorem 1.1. We consider first \( c = 1 \), and for notational simplicity we introduce two real-valued functions

\[
V_\pm = u_\pm + 1 = \frac{1}{2} v \pm (|q| - 1).
\]

From now on we assume that \( u_\pm \in L^\infty(\mathbb{R};\mathbb{R}) \), and hence \( V_\pm \in L^\infty(\mathbb{R};\mathbb{R}) \), and we consider the eigenvalue problem of the operator

\[
L = \begin{pmatrix} 1 - V_- & i \partial_x \\ i \partial_x & -1 - V_+ \end{pmatrix} : H^1(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2).
\]

We decompose \( V_\pm \) into their positive and negative parts respectively

\[V_\pm = (V_\pm)^+ - (V_\pm)^- , \text{ with } (V_\pm)^+ = \max\{V_\pm, 0\}, \ (V_\pm)^- = \max\{-V_\pm, 0\} .\]

If \( \lambda \) is an eigenvalue of \( L \) such that

\[1 - \lambda > \|(V_-)^+\|_{L^\infty} , \]

then

\[-(\lambda + u_-) = 1 - \lambda - V_- = 1 - \lambda - (V_-)^+ + (V_-)^- > 0 \text{ on } \mathbb{R} .\]

By Lemma 2.1, the eigenvalue problem \( L \Psi = \lambda \Psi \) reads as the \( \lambda \)-nonlinear eigenvalue problem (2.7):

\[
-\partial_x \left( \frac{1}{1 - \lambda - V_-} \partial_x \phi \right) + (1 + \lambda + V_+) \phi = 0.
\]

We test it by \( \bar{\phi} \) to derive

\[
\int_\mathbb{R} \left( \frac{1}{1 - \lambda - V_-} |\partial_x \phi|^2 + (1 + \lambda + V_+) |\phi|^2 \right) \, dx = 0.
\]

This yields

\[0 \geq \int_\mathbb{R} (1 + \lambda + V_+) |\phi|^2 \, dx \geq (1 + \lambda - \|(V_+)^-\|_{L^\infty}) \|\phi\|_{L^2}^2 .\]
We then have shown for $\lambda$ an eigenvalue of $L$,
\[
\frac{1}{\lambda} > \|(V_-)^+\|_{L^\infty} \implies 1 + \lambda \leq \|(V_+)^-\|_{L^\infty}.
\]
Hence equivalently (or following the same argument)
\[
1 + \lambda > \|(V_+)^-\|_{L^\infty} \implies 1 - \lambda \leq \|(V_-)^+\|_{L^\infty}.
\]
We conclude the following unconditional statement for the eigenvalues $\lambda$ of $L$:
\[
1 - \lambda \leq \|(u_- + 1)^+\|_{L^\infty} \quad \text{or} \quad 1 + \lambda \leq \|(u_+ - 1)^-\|_{L^\infty}.
\]
More generally, $\forall c \in \mathbb{R}$, we can replace $\pm 1$ and $u_\pm \mp 1$ by $\pm c$ and $u_\pm \mp c$ respectively in the above arguments. This completes the proof of Theorem 1.1.

3. Special case of potentials under nonzero boundary conditions

In this section we will consider the Lax operator $L = \left( i\frac{\partial}{\partial x} - iq \right) : H^1(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2)$ with bounded potentials $q = q(x) : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ satisfying
\[
q = |q|e^{i\varphi} \in L^\infty(\mathbb{R}; \mathbb{C}), \quad |q| > 0, \quad \partial_x \varphi \in L^\infty(\mathbb{R}; \mathbb{R}),
\]
\[
|q(x)| \to 1 \quad \text{and} \quad \partial_x \varphi(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]
Thus the potentials $u_\pm = \frac{1}{2i}\partial_x \varphi \mp |q|$ of its unitarily equivalent operator $L = \left( -u_- i\frac{\partial}{\partial x}, u_+ i\frac{\partial}{\partial x} \right) : H^1(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2)$ satisfy
\[
u \in L^\infty(\mathbb{R}; \mathbb{R}), \quad u_\pm(x) \to \pm 1 \quad \text{as} \quad |x| \to \infty.
\]
As in Subsection 2.2 we introduce
\[
V_\pm = u_\pm \mp 1 \in L^\infty(\mathbb{R}; \mathbb{R}), \quad \text{which satisfy} \quad V_\pm(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\]
and we decompose $V_\pm$ into their positive and negative parts respectively
\[
V_\pm = (V_\pm)^+ - (V_\pm)^-, \quad \text{with} \quad (V_\pm)^+ = \max\{V_\pm, 0\}, \quad (V_\pm)^- = \max\{-V_\pm, 0\}.
\]
Under the smallness assumption (1.10):
\[
\|(V_+)^-\|_{L^\infty} + \|(V_-)^+\|_{L^\infty} < 2,
\]
by Theorem 1.1, all the eigenvalues $\lambda \in (-1, 1)$ of $L, \mathcal{L}$ satisfy
\[
either \lambda \in I_- = (-1, -1 + \|(V_+)^-\|_{L^\infty}) \quad \text{or} \quad \lambda \in I_+ = [1 - \|(V_-)^+\|_{L^\infty}, 1),
\]
where $I_+$ and $I_-$ are disjoint. For notational convenience we introduce further two intervals
\[
J_- := [-1, 1 - \|(V_-)^+\|_{L^\infty}), \quad J_+ := (-1 + \|(V_+)^-\|_{L^\infty}, 1],
\]
so that $I_- \subset J_-$ and $I_+ \subset J_+$. The eigenvalues $\lambda \in (-1, 1)$ have been discussed in Subsection 2.2, where we have shown
\[
\lambda \in J_- \Rightarrow \lambda + u_- = \lambda - 1 + V_- < 0 \quad \text{on} \quad \mathbb{R},
\]
\[
\lambda \in J_- = J_- \setminus \{-1\} \text{is an eigenvalue of} \quad \mathcal{L} \Rightarrow \lambda \in I_-,
\]
and similarly (or equivalently)
\[
\lambda \in J_+ \Rightarrow \lambda + u_+ = \lambda + 1 + V_+ > 0 \quad \text{on} \quad \mathbb{R},
\]
\[
\lambda \in J_+ = (-1, 1) \setminus I_- \text{is an eigenvalue of} \quad \mathcal{L} \Rightarrow \lambda \in I_+ = (-1, 1) \setminus J_-.
\]
This section is organized as follows. In Subsection 3.1 we are going to prove Theorem 1.2, which characterizes the number of the eigenvalues of \(L, \mathcal{L}\) inside \((-1, 1)\). Some examples showing the (non-)existence of eigenvalues will be given in Subsection 3.2.

In the appendix some interesting characterization of the eigenvalues inside \((-1, 1)\) in the special case \(V_-=0\) or \(V_+=0\) will be discussed.

3.1. Proof of Theorem 1.2. In this subsection we are going to consider the eigenvalues of \(L\) inside \((-1, 1)\), under the assumptions (3.2)-(3.3). By Lemma 2.1 and (3.4)-(3.5), it suffices to consider the eigenvalues \(\lambda \in \mathcal{J}_-\) of the eigenvalue problem (2.7):

\[
-\partial_x \left( \frac{1}{1 - V_- - \lambda} \partial_x \phi \right) + (1 + V_+ + \lambda) \phi = 0, \quad x \in \mathbb{R},
\]

and the eigenvalues \(\lambda \in \mathcal{J}_+\) of the eigenvalue problem (2.8):

\[
-\partial_x \left( \frac{1}{1 + V_+ + \lambda} \partial_x \phi \right) + (1 - V_- - \lambda) \phi = 0, \quad x \in \mathbb{R}.
\]

Our goal is to prove Theorem 1.2, that is, the number of eigenvalues inside \(\mathcal{J}_-\) of (3.6) (resp. \(\mathcal{J}_+\) of (3.7)) is the same as the number of negative eigenvalues of the Sturm-Liouville eigenvalue problem (1.11) (resp. (1.12)) reading as follows:

\[
-\partial_x \left( \frac{1}{2 - V_-} \partial_x \psi \right) + V_+ \psi = \mu \psi, \quad x \in \mathbb{R},
\]

resp.

\[
-\partial_x \left( \frac{1}{2 + V_+} \partial_x \psi \right) - V_- \psi = \nu \psi, \quad x \in \mathbb{R}.
\]

By compact perturbation arguments, we deduce from the decay property of \(V_{\pm}\) in (3.2) that the essential spectra of both (3.8) and (3.9) are contained in \(\{0, \infty\}\).

We first do some preparations for the proof of Theorem 1.2. We consider, for \(\lambda \in \mathcal{J}_- = [-1, 1 - \|(V_+)\|_{L^\infty})\), the eigenvalue problem (noticing \(1 - V_- - \lambda > 0\) on \(\mathbb{R}\) in this case)

\[
-\partial_x \left( \frac{1}{1 - V_- - \lambda} \partial_x \psi \right) + (1 + V_+ + \lambda) \psi = \mu \psi, \quad x \in \mathbb{R},
\]

and for \(\lambda \in \mathcal{J}_+ = (-1 + \|(V_+)\|_{L^\infty}, 1]\), the eigenvalue problem (noticing \(1 + V_+ + \lambda > 0\) on \(\mathbb{R}\) in this case)

\[
-\partial_x \left( \frac{1}{1 + V_+ + \lambda} \partial_x \psi \right) + (1 - V_- - \lambda) \psi = \nu \psi, \quad x \in \mathbb{R}.
\]

Here, \(\lambda\) is regarded as a given parameter, and \(\mu\) (resp. \(\nu\)) is the eigenvalue parameter. In particular,

\[
\lambda = -1 \Rightarrow \text{the two eigenvalue problems (3.8), (3.10) are the same},
\]

\[
\lambda = 1 \Rightarrow \text{the two eigenvalue problems (3.9), (3.11) are the same}.
\]

We define the Rayleigh quotients for \(\psi \in H^1(\mathbb{R})\setminus\{0\}\)

\[
R(\lambda, \psi) := \frac{\int_{\mathbb{R}} \left[ \frac{1}{1 - V_- - \lambda} |\partial_x \psi|^2 + (1 + V_+ + \lambda) |\psi|^2 \right] dx}{\int_{\mathbb{R}} |\psi|^2 dx}
\]
for the eigenvalue problem (3.10), and
\[ (3.13) \quad S(\lambda, \psi) := \frac{\int_{\mathbb{R}} \left[ \frac{1}{1 + V_+ + \lambda} |\partial_x \psi|^2 + (1 - V_- - \lambda)|\psi|^2 \right] dx}{\int_{\mathbb{R}} |\psi|^2 dx} \]
for the eigenvalue problem (3.11), and respectively the Rayleigh extremal values
\[ (3.14) \quad \mu_j(\lambda) := \inf_{U \subset H^1(\mathbb{R}) \text{ subspace}, \psi \in U \setminus \{0\}} \max_{\dim(U) = j} R(\lambda, \psi), \quad j \in \mathbb{N}, \]
\[ (3.15) \quad \nu_j(\lambda) := \inf_{U \subset H^1(\mathbb{R}) \text{ subspace}, \psi \in U \setminus \{0\}} \max_{\dim(U) = j} S(\lambda, \psi), \quad j \in \mathbb{N}. \]

We first observe that for all \( \lambda_0 \in (-1 + \|(V_+)^-\|_{L^\infty}, 1 - \|(V_-)^+\|_{L^\infty}) = J_- \cap J_+ \), we have
\[ (3.16) \quad \mu_j(\lambda_0) \geq 1 - \|(V_+)^-\|_{L^\infty} + \lambda_0 > 0, \quad \forall j \in \mathbb{N}, \quad \text{(by (3.12) and (3.14))}, \]
\[ (3.17) \quad \nu_j(\lambda_0) \geq 1 - \|(V_-)^+\|_{L^\infty} - \lambda_0 > 0, \quad \forall j \in \mathbb{N}, \quad \text{(by (3.13) and (3.15))}. \]
By the decay property of \( V_\pm(x) \to 0 \) as \( |x| \to \infty \), we observe that the essential spectrum of (3.10) (resp. (3.11)) is contained in \([1 + \lambda, \infty)\) (resp. \([1 - \lambda, \infty)\)). Thus, cf. [13, Theorem 10.33], one has
\[ (3.18) \]
If \( \mu_j(\lambda) < 1 + \lambda \), then \( \mu_j(\lambda) \) is the \( j \)-th eigenvalue of (3.10),
If \( \nu_j(\lambda) < 1 - \lambda \), then \( \nu_j(\lambda) \) is the \( j \)-th eigenvalue of (3.11).

In particular, we observe that
\[ (3.19) \]
If \( \mu_j(\lambda_j) = 0 \) for some \( \lambda_j \in J_- \), \( j \in \mathbb{N} \), then \( \lambda_j \) is an eigenvalue of (3.6);
If \( \nu_j(\lambda_j) = 0 \) for some \( \lambda_j \in J_+ \), \( j \in \mathbb{N} \), then \( \lambda_j \) is an eigenvalue of (3.7).

**Lemma 3.1** (Properties of \( \mu_j, \nu_j \)). For \( j \in \mathbb{N} \), the mappings
\[ \mu_j = \mu_j(\lambda) : J_- = [-1, 1 - \|(V_-)^+\|_{L^\infty}) \to \mathbb{R} \]
and \( \nu_j = \nu_j(\lambda) : J_+ = (-1 + \|(V_+)^-\|_{L^\infty}, 1) \to \mathbb{R} \)
are continuous, and \( \mu_j \) (resp. \( \nu_j \)) is strictly increasing (resp. decreasing) in \( \lambda \).

**Proof.** Let \( \lambda \in J_- \), and we are going to show the continuity of the mapping \( \mu_j \) at \( \lambda \). For any \( \lambda \in J_- \), and any fixed \( \psi \in H^1(\mathbb{R}) \setminus \{0\} \), we estimate
\[ |R(\lambda, \psi) - R(\lambda, \psi)| \leq \frac{|\lambda - \lambda_0|}{1 - \|(V_-)^+\|_{L^\infty}} \frac{1}{\int_{\mathbb{R}} \left[ (1 - V_- - \lambda)(1 - V_+ - \lambda) \right] dx} \int_{\mathbb{R}} \left( \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda} \right) |\partial_x \psi|^2 + |\psi|^2 dx \]
\[ \leq \frac{|\lambda - \lambda_0|}{\int_{\mathbb{R}} |\psi|^2 dx} \left( \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda} \right) \int_{\mathbb{R}} \left[ (1 - V_- - \lambda) |\partial_x \psi|^2 + (1 + V_+ + \lambda)|\psi|^2 \right] dx \]
\[ + \left[ \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda} \right] \int_{\mathbb{R}} |\psi|^2 dx \]
\[ = |\lambda - \lambda_0| \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda} R(\lambda, \psi) + \left[ \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda} \right] \int_{\mathbb{R}} |\psi|^2 dx. \]
If \( \lambda \in J_- \) satisfies \( |\lambda - \lambda| < \frac{1}{2} (1 - \|(V_-)^+\|_{L^\infty} - \lambda) \) and hence \( \frac{1}{1 - \|(V_-)^+\|_{L^\infty} - \lambda| < \frac{1}{2} (1 - \|(V_-)^+\|_{L^\infty} - \lambda), \) then there exist two positive constants \( C_1, C_2 \)
which are independent of $\tilde{\lambda}$ and $\psi$ (depending only on $\lambda$, $\| (V_+)^- \|_{L^\infty}$, $\| (V_-)^+ \|_{L^\infty}$), such that

$$| R(\lambda, \psi) - R(\tilde{\lambda}, \psi) | \leq | \lambda - \tilde{\lambda} | \left( C_1 R(\lambda, \psi) + C_2 \right).$$

Thus for $\tilde{\lambda} \in J_-$ satisfying $| \lambda - \tilde{\lambda} | < \min \left\{ \frac{1}{2} (1 - \| (V_-)^+ \|_{L^\infty} - \lambda), \frac{1}{4} \right\}$, we have

$$(1 - | \lambda - \tilde{\lambda}| C_1) R(\lambda, \psi) - | \lambda - \tilde{\lambda}| C_2 \leq R(\tilde{\lambda}, \psi) \leq (1 + | \lambda - \tilde{\lambda}| C_1) R(\lambda, \psi) + | \lambda - \tilde{\lambda}| C_2,$$

which, together with (3.14), implies for each $j \in \mathbb{N}$,

$$(1 - | \lambda - \tilde{\lambda}| C_1) \mu_j(\lambda) - | \lambda - \tilde{\lambda}| C_2 \leq \mu_j(\tilde{\lambda}) \leq (1 + | \lambda - \tilde{\lambda}| C_1) \mu_j(\lambda) + | \lambda - \tilde{\lambda}| C_2.$$  

This implies the continuity of the mapping $\mu_j$ at $\lambda$.

By (3.12) and (3.14),

$$\mu_j(\lambda) = (1 + \lambda) + \inf_{U \subset \mathcal{H}^1(\mathbb{R}) \text{ subspace, } \dim(U) = j} \max_{\psi \in U \setminus \{0\}} \frac{\int_{\mathbb{R}} \left[ \frac{1}{1 - V_+ - \lambda} | \partial_x \psi|^2 + V_+ | \psi|^2 \right] \, dx}{\int_{\mathbb{R}} | \psi|^2 \, dx}.$$  

Hence $\mu_j(\lambda)$ is strictly increasing in $\lambda \in J_-$.

Analogously, we can argue exactly as above, with $\lambda, \tilde{\lambda}, V_+, V_-, R, \mu$ replaced by $-\lambda, -\tilde{\lambda}, -V_-, -V_+, S, \nu$ respectively, to derive the continuity and the monotonicity of the mapping $\nu_j$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $m \in \mathbb{N}_0 \cup \{\infty\}$ denote the number of negative eigenvalues $\mu$ of the eigenvalue problem (3.8), that is, the eigenvalue problem (3.10) with $\lambda = -1$. We take these negative eigenvalues

$$(3.20) \quad \mu_j(-1) < 0, \quad j = 1, \cdots, m \text{ if } m \in \mathbb{N} \text{ or } j \in \mathbb{N} \text{ if } m = \infty.$$  

By Lemma 3.1 and (3.16), for each such $j$, there exists a unique $\lambda_j \in (-1, 0) \subset \tilde{J}_-$ such that

$$(3.21) \quad \mu_j(\lambda_j) = 0.$$  

By (3.19), $\lambda_j \in \tilde{J}_-$ is an eigenvalue of (3.6), and hence of the operator $\mathcal{L}$ (by Lemma 2.1), which implies $\lambda_j \in I_-$ (by (3.4)). Furthermore, if for some $j$ and $k$, the functions $\mu_j, \cdots, \mu_{j+k}$ have the same zero with $\lambda_j = \cdots = \lambda_{j+k}$, that is, 0 is a $(k+1)$-fold eigenvalue of (3.10) for this value of $\lambda = \lambda_j$, then the associated eigenfunctions of the eigenvalue problem (3.8) are linearly independent. Thus, counting by multiplicity, we obtain at least $m$ eigenvalues inside $I_-$ of the operator $\mathcal{L}$.

Vice versa, when $\lambda \in I_- \subset \tilde{J}_-$ is an eigenvalue of $\mathcal{L}$, then $\lambda$ is an eigenvalue of (3.6), and hence $\mu = 0$ is an eigenvalue of (3.10). Furthermore, the eigenvalue $\mu = 0$ is below the essential spectrum $[1 + \lambda, \infty)$. Thus, by (3.18), there exists some $j \in \mathbb{N}$ such that $\mu_j(\lambda) = 0$. By the strict monotonicity in Lemma 3.1, we have $\mu_j(-1) < 0$. By (3.18), $\mu_j(-1)$ is a negative eigenvalue of (3.8), which implies $j \in \{1, \cdots, m\}$ if $m \in \mathbb{N}$. Since the zero $\lambda_j$ of $\mu_j$ is unique, we obtain $\lambda = \lambda_j$. If $\lambda$ has multiplicity $(k+1)$ as an eigenvalue of $\mathcal{L}$, then $\mu = 0$ has multiplicity $(k+1)$ as an eigenvalue of (3.10), and hence $\mu_{j+k}(\lambda) = 0$ for some $j \in \mathbb{N}$. As before we find $\lambda = \lambda_j = \cdots = \lambda_{j+k}$. This implies, counting by multiplicity, that
the operator \( L \) has at most \( m \) eigenvalues in \( I_- \). This completes the proof of (1) in Theorem 1.2 if \( m \in \mathbb{N} \). The above arguments show that it is also true if \( m = \infty \).

Part (2) in Theorem 1.2 follows exactly as above, if we replace \( \lambda, m, \mu, I_-, J_- \) by \( -\lambda, l, \nu, I_+, J_+ \) respectively.

\[ \square \]

3.2. Examples. In this subsection we will give some examples, where the Lax operator \( L \) in (1.3), or equivalently its unitary equivalence \( \tilde{L} \) in (1.7), may or may not possess eigenvalues inside the spectral gap \(( -1, 1)\).

In the following examples we will always assume the nonzero boundary condition at infinity (1.9) for the nowhere vanishing bounded potential \( q \), so that

\[ V_\pm \in L^\infty(\mathbb{R}; \mathbb{R}), \quad V_\pm(x) \to 0 \text{ as } |x| \to \infty, \]

where \( V_\pm \) are related to \( q \) as follows:

\[ q = |q|e^{i\varphi}, \quad u_\pm = \frac{1}{2}(\partial_x \varphi) \pm |q|, \quad V_\pm = u_\pm \mp 1. \]

In order to show the existence of eigenvalues in some examples we will assume further the smallness assumption (1.10):

\[ \| (V_+)^+ \|_{L^\infty} + \| (V_-)^+ \|_{L^\infty} < 2. \]

**Example 3.1** (No eigenvalues). For the potentials of the following type

\[ q = |q|e^{i\varphi} \in L^\infty(\mathbb{R}), \quad |q| - 1 \geq 0, \quad \varphi(x) = \int_{x_0}^x 2(|q| - 1) \, dy, \]

for some real number \( x_0 \in \mathbb{R} \), we have \( \frac{1}{2} \partial_x \varphi = |q| - 1 \) and

\[ u_+ = 2|q| - 1 \geq 1, \quad u_- = -1. \]

Therefore by Theorem 1.1 (with \( c = 1 \)) there are no eigenvalues of the Lax operator \( L \) inside \((-1, 1)\). If we assume further the decay condition (1.5) at infinity: \((1 + x^4)(q - e^{i\theta \pm}) \in L^1(\mathbb{R}^+)\) for some \( \theta_\pm \in \mathbb{R} \), then by [7] there are only finitely many discrete eigenvalues which are located in \((-1, 1)\), and hence the spectrum of \( L \) consists only of essential spectrum \((-\infty, -1] \cup [1, \infty)\).

**Example 3.2** (Existence of eigenvalues). (i) Let the potentials \( V_\pm \) satisfy (3.22)-(3.23) and the following assumption:

\[ \int_{-R}^R V_+(x) \, dx < 0 \text{ and } V_+ \leq 0 \text{ outside } [-R, R], \]

for some \( R > 0 \).

We are going to show the existence of negative eigenvalues for (1.11) (i.e. (3.8)):

\[ -\partial_x \left( \frac{1}{2 - V_-^+} \partial_x \psi \right) + V_+ \psi = \mu \psi \]

by the Rayleigh-Ritz method, cf. [13, Theorem 10.23]. More precisely, we can try to prove that the eigenvalue problem (3.25) has at least \( n \) negative eigenvalues (with \( n \in \mathbb{N} \) chosen as we like): We have to select linearly independent \( \psi_1, \ldots, \psi_n \in H^1(\mathbb{R}) \), compute the numbers

\[ A_{jk} := \int_{\mathbb{R}} \left( \frac{1}{2 - V_-^+} \partial_x \psi_j \partial_x \psi_k + V_+ \psi_j \psi_k \right) \, dx, \quad j, k = 1, \ldots, n, \]
and show that the matrix $A = (A_{jk})$ has $n$ negative eigenvalues (by explicit calculation or by computer assistance). If $A$ turns out to have less than $n$ negative eigenvalues, $n$ was chosen too large, and we can retry with smaller $n$.

In the following we will just choose $n = 1$ to show the existence of eigenvalues.

We take $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\psi} = 1$ on $(-1, 1)$ and $\tilde{\psi} = 0$ outside $(-2, 2)$. Let $\psi(x) = \tilde{\psi}(\frac{x}{k})$, $x \in \mathbb{R}$, where $k \geq R$ is chosen large enough such that (by virtue of the assumptions (3.23) and (3.24))

$$A_{11} = \int_R \left( \frac{1}{2 - V_-} |\partial_x \psi|^2 + V_+ |\psi|^2 \right) dx$$

$$\leq \frac{1}{2 - \|V_+\|_{L^\infty}} \left( \frac{1}{k} \|\partial_x \tilde{\psi}\|_{L^2}^2 \right) + \left( \int_R V_+ dx \right) < 0.$$  

Hence there exists at least one negative eigenvalue $\mu$ of (3.25) (i.e. (1.11)), and thus there exists at least one eigenvalue $\lambda \in I_- = (-1, -1 + \|V_+\|_{L^\infty})$ of the operator $L, \mathcal{L}$ (by Theorem 1.2).

(ii) Analogously, if the potentials $V_\pm$ satisfy (3.22)-(3.23) and the assumption:

$$\int_{-R}^R V_-(x) dx > 0 \text{ and } V_- \geq 0 \text{ outside } [-R, R],$$

for some $R > 0$, then there exists at least one negative eigenvalue $\nu$ of the eigenvalue problem (1.12) (i.e. (3.9)):

$$-\partial_x \left( \frac{1}{2 + V_+} \partial_x \psi \right) - V_- \psi = \nu \psi,$$

and hence the operator $L$ has at least one eigenvalue $\lambda \in I_+ = (1 - \|V_-\|^+_{L^\infty}, 1]$.

**Example 3.3** (Potentials with piecewise-constant amplitude and phase velocity).

We take the potentials of the following form

$$q(x) = \begin{cases} 
    e^{i\theta} & x < a, \\
    Ae^{i(\theta + \psi(x-a))} & a \leq x \leq b, \\
    e^{i(\theta + \psi(b-a))} & x > b,
\end{cases}$$

where $(a, b)$ is an arbitrary nonempty finite interval on $\mathbb{R}$, and $\theta \in [0, 2\pi)$, $A > 0$ and $\psi \in \mathbb{R}$ are real numbers.

If $A \geq 1$ and $\psi \in \mathbb{R}$ such that

$$\frac{1}{2} |\psi| \leq A - 1,$$

then $u_+ \geq 1$ and $u_- \leq -1$, and hence by Theorem 1.1 (with $c = 1$) the operator $L$ has no eigenvalues inside $(-1, 1)$, and thus no eigenvalues by [7] since obviously $q$ in (3.27) satisfies (1.5) with $\theta_- = \theta$ and $\theta_+ = \theta + \psi(b-a)$.

In the following we will take

$$A \in (0, 1) \text{ and } \frac{1}{2} |\psi| < A + 1,$$

and show the existence of exactly one or two eigenvalue(s) of $L$ in different cases:

(I) Exactly two simple eigenvalues, located in $(-1, -\frac{1}{2}v - A]$ and $[-\frac{1}{2}v + A, 1)$ respectively, if

$$A \in (0, 1) \text{ and } \frac{1}{2} |\psi| < 1 - A.$$
We are going to show that there are indeed exactly two eigenvalues.\(\mu\)egration by parts shows that any negative eigenvalue
\[1\]
where the two positive constants \(k, \kappa\) lies in \((0, 2 - 2A)\), such that the smallness condition (3.23) holds:
\[\| (V_+)^- \|_{L^\infty} + \| (V_-)^+ \|_{L^\infty} = \varepsilon + \delta = 2 - 2A < 2.\]
We conclude from Example 3.2 that in this case there are at least two eigenvalues of the Lax operator \(L\), located in \((-1, -1 + \varepsilon)\) and \([1 - \delta, 1)\) respectively.
We are going to show that there are indeed exactly two eigenvalues.

We consider the eigenvalue problem (1.11) (i.e. (3.25) above). An integration by parts shows that any negative eigenvalue \(\mu\) lies in \((-\varepsilon, 0)\). More precisely, (3.25) reads as
\[
\partial_{xx} \psi = k^2 \psi \quad \text{outside} \quad [a, b],
\]
\[- \partial_{xx} \psi = \kappa^2 \psi \quad \text{on} \quad [a, b],
\]
where the two positive constants \(k, \kappa\) read as
\[k = \sqrt{-2\mu} < \sqrt{2\varepsilon} < 2 \quad \text{and} \quad \kappa = \sqrt{(2 - \delta)(\mu + \varepsilon)} < \sqrt{\varepsilon(2 - \delta)}.\]
We then search for the non-trivial solution of the following form which are continuous at \(a\) and \(b\):
\[
\psi(x) = \begin{cases}
\frac{e^{iak} + de^{-iak}}{e^{iak} - de^{-iak}} e^{kx} & \text{for} \quad x \leq a, \\
\frac{e^{i\kappa x} + de^{-i\kappa x}}{e^{i\kappa x} - de^{-i\kappa x}} e^{\kappa x} & \text{for} \quad x \in [a, b], \\
\frac{e^{i\kappa x} + de^{-i\kappa x}}{e^{i\kappa x} - de^{-i\kappa x}} e^{-kx} & \text{for} \quad x \geq b.
\end{cases}
\]
The continuity conditions of \(\frac{1}{2\varepsilon} \partial_x \psi\) at \(a\) and \(b\) read then as
\[
\begin{pmatrix}
e^{iak} \left(\frac{1}{2\varepsilon} + \frac{i\kappa - \frac{1}{\kappa}}{2}\right) & e^{-iak} \left(\frac{1}{2\varepsilon} - \frac{i\kappa - \frac{1}{\kappa}}{2}\right) \\
(\frac{1}{2\varepsilon} + \frac{i\kappa + \frac{1}{\kappa}}{2}) e^{i\kappa x} & (\frac{1}{2\varepsilon} - \frac{i\kappa + \frac{1}{\kappa}}{2}) e^{-i\kappa x}
\end{pmatrix}
\begin{pmatrix}
e \\
d
\end{pmatrix} = \begin{pmatrix}0 \\
0
\end{pmatrix}.
\]
In order to have a non-trivial solution, the determinant of the matrix on the lefthand side should vanish:
\[
0 = \cot((b - a)\kappa) - \frac{1}{2\varepsilon(2 - \delta)} f(\kappa),
\]
where
\[f(\kappa) := \frac{(4 - \delta)\kappa^2 - \varepsilon(2 - \delta)^2}{\kappa \sqrt{\varepsilon(2 - \delta) - \kappa^2}},\]
for \(\kappa \in (0, \sqrt{\varepsilon(2 - \delta)})\). It is straightforward to check that the function \(f(\kappa)\) has a strictly positive derivative
\[
\frac{(2 - \delta)(\delta \kappa^2 + \varepsilon(2 - \delta)^2)}{\kappa^2 \sqrt{\varepsilon(2 - \delta) - \kappa^2}} > 0 \quad \text{for} \quad \kappa \in (0, \sqrt{\varepsilon(2 - \delta)}),
\]
and \(f(\kappa)\) tends to \(-\infty\) as \(\kappa \to 0_+\) while to \(+\infty\) as \(\kappa \to (\sqrt{\varepsilon(2 - \delta)})_-.\)
Hence there exists a unique \(\kappa \in (0, \sqrt{\varepsilon(2 - \delta)})\) such that the above equality holds, and hence the eigenvalue problem (1.11) has a single negative eigenvalue \(\mu = -(\varepsilon - \frac{k^2}{2})\). Thus the operator \(L\) has a single eigenvalue inside \((-1, -1 + \varepsilon)\).
Replacing \(V_+\) by \(-V_+\), that is, exchanging \(\varepsilon\) and \(\delta\), the above argument shows that the eigenvalue problem (1.12) (i.e. (3.26) above) has a unique negative eigenvalue, and thus the operator \(L\) has a single eigenvalue inside \([1 - \delta, 1)\).
To conclude, if \( A \in (0, 1) \) and \( \frac{1}{2}|v| < 1 - A \), the spectra of the operator \( L \) associated to the potential (3.27) consists of essential spectrum \((-\infty, -1) \cup [1, \infty)\) and two simple eigenvalues, located in \((-1, -\frac{1}{2}v - A)\) and \([-\frac{1}{2}v + A, 1)\), respectively.

(II) Exactly one simple eigenvalue, located in \((-1, -\frac{1}{2}v - A)\), if

\[-1 - A < \frac{1}{2}v \leq A - 1 < 0.\]

In this case, \( V_\pm \) are given in (3.30), with the parameter \( \epsilon \in [2 - 2A, 2) \subset (0, 2) \) and \( \delta \in (-2A, 0) \). We analyze the eigenvalue problem (1.11) as in the above case (I), and derive a single negative eigenvalue \( \mu = -\langle \epsilon - \frac{\delta^2}{4} \rangle \). Since the eigenvalue problem (1.12) has no negative eigenvalues for \( V_- < 0 \), the operator \( L \) has exactly one eigenvalue located in \((-1, 1 + \epsilon) = (-1, -\frac{1}{2}v - A)\).

(III) Exactly one simple eigenvalue, located in \([-\frac{1}{2}v + A, 1)\) if

\[0 < 1 - A \leq \frac{1}{2}v < 1 + A.\]

It follows by exchanging the values of \( \epsilon \) and \( \delta \) in the above case (II), that the eigenvalue problem (1.12) has a single negative eigenvalues and (1.11) has no negative eigenvalues, and hence \( L \) has exactly one eigenvalue located in \([1 - \delta, 1) = [-\frac{1}{2}v + A, 1)\).

**Example 3.4** (Piecewise-constant case, continued). We take the potentials of the following form

\[(3.31) \quad q(x) = \begin{cases} 
(e^{i\theta} & \text{if } x < -2, \\
(1 - \frac{1}{2}\delta)e^{i(\theta + \delta(x + 2))} & \text{if } x \in [-2, -1], \\
(1 - \frac{1}{2}\epsilon)e^{i(\theta + \delta - \epsilon(x - 1))} & \text{if } x \in [1, 2], \\
e^{i(\theta + \delta - \epsilon)} & \text{if } x > 2,
\end{cases}\]

where \( \theta \in [0, 2\pi) \) and \( 0 < \epsilon, \delta \) such that \( \epsilon + \delta < 2 \). Then the corresponding \( V_\pm \) are supported on disjoint intervals as follows

\( V_+(x) = \begin{cases} -\epsilon & \text{on } [1, 2], \\
\delta & \text{on } [-2, -1], \\
0 & \text{otherwise}, \end{cases} \)

\( V_-(x) = \begin{cases} \delta & \text{on } [-2, -1], \\
0 & \text{otherwise}. \end{cases} \)

We remark that in view of Example 3.3, it is easy to see that the operator \( L \) has no eigenvalues if \( \epsilon = \delta = 0 \), and has exactly one eigenvalue if \( \epsilon = 0, \delta \in (0, 2) \) or \( \epsilon \in (0, 2), \delta = 0 \). In the following we follow exactly the arguments in Example 3.3 to show briefly that \( L \) has exactly two eigenvalues if \( \epsilon, \delta > 0 \) and \( \epsilon + \delta < 2 \).

We can rewrite the eigenvalue problem (3.8) with \( \mu \in (-\epsilon, 0) \) as

\[\begin{align*}
\partial_{xx}\psi &= k^2\psi \text{ on } (-\infty, -2) \cup (-1, 1) \cup (2, \infty), \\
-\partial_{xx}\psi &= \kappa_1^2\psi \text{ on } [-2, -1], \\
-\partial_{xx}\psi &= \kappa_2^2\psi \text{ on } [1, 2],
\end{align*}\]

where \( k = \sqrt{-2\mu} \in (0, \sqrt{2\epsilon}) \), \( \kappa_1 = i\sqrt{-\mu(2 - \delta)} = i\sqrt{1 - \frac{\delta}{2}}k \in \mathbb{R}^+ \), and \( \kappa_2 = \sqrt{2(\mu + \epsilon)} = \sqrt{2\epsilon - k^2} \in (0, \sqrt{2\epsilon}) \). And we search for a non-trivial solution \( \psi \in H^1(\mathbb{R}) \), such that \( \psi \) itself as well as \( \frac{1}{2\sqrt{\epsilon}}\partial_x\psi \) are continuous at \( \pm 1 \) and \( \pm 2 \). By a long but straightforward calculation, the fact that the determinant of the
4 × 4 matrix generated by these continuity conditions vanishes is equivalent to the following equality

\[
0 = g(k; \varepsilon, \delta), \quad \text{where}\ g(k; \varepsilon, \delta) = k^2 - \varepsilon + k\sqrt{2\varepsilon - k^2} \cot(\sqrt{2\varepsilon - k^2}) + \frac{\varepsilon \delta e^{-4k}}{4 - \delta + 2\sqrt{2}(2 - \delta) \coth(\sqrt{1 - \frac{2}{4}k})}.
\]

We claim that there exists a unique zero \( k \in (0, \sqrt{2\varepsilon}) \) of \( g(k; \varepsilon, \delta) \). Indeed, the existence of a zero \( k \in (0, \sqrt{2\varepsilon}) \) follows immediately from the fact that \( g(0_+) < 0 \), and \( g > 0 \) on \( [\sqrt{2\varepsilon}, \sqrt{2\varepsilon}] \) (since \( \sqrt{2\varepsilon - k^2} \leq \sqrt{2} < \frac{\varepsilon}{2} \) for \( k \in [\sqrt{2\varepsilon}, \sqrt{2\varepsilon}] \)). The uniqueness is however non-trivial, and we consider the cases \( \varepsilon \leq 0.04 \) and \( \varepsilon \geq 0.04 \) separately:

- **Case \( \varepsilon \leq 0.04 \).** In this case one may check that \( g'(k) > 0 \) on \( (0, \sqrt{2\varepsilon}) \) and hence there exists a unique zero \( k \in (0, \sqrt{2\varepsilon}) \).

- **Case \( \varepsilon \geq 0.04 \).** In this case we use a computer-assisted proof to show the uniqueness. More precisely, we cover the parameter range \( \varepsilon \geq 0.04, \delta \geq 0, \varepsilon + \delta \leq 2 \) by (finitely many) squares with side-length 0.03, and on each square \( S = [\varepsilon, \varepsilon] \times [\delta, \delta] \) we evaluate \( g(I_j; S), j = 1, \ldots, N \) using interval arithmetic [14], where \( \{I_j\}_{j=1}^N \) is a subdivision of \( [0, \sqrt{2\varepsilon}] \). For all squares \( S \), the interval evaluation showed that there exists a unique index \( j_0 \in \{1, \ldots, N\} \) such that \( g(\cup_{j \leq j_0-1} I_j; S) \subset (-\infty, 0), g(\cup_{j \geq j_0+1} I_j; S) \subset (0, \infty) \) and \( g(\cup_{j_0} I_j; S) \subset (0, \infty) \). This proves (rigorously) the uniqueness of the zero \( k \in (0, \sqrt{2\varepsilon}) \) of \( g(k; \varepsilon, \delta) \), for all \( (\varepsilon, \delta) \in S \). The covering property of the squares \( S \) gives the desired result. Note that we have to exclude a “small” parameter range near \( \varepsilon = 0 \) since the singularity of \( g \) at \( \varepsilon = 0 \) is difficult to capture by interval evaluations.

Analogously we consider the eigenvalue problem (3.9) with \( \nu \in (-\delta, 0) \). The existence of non-trivial solution \( \psi \in \mathcal{H}^1(\mathbb{R}) \) is again equivalent to the equation \( 0 = g(k; \delta, \varepsilon) \) for \( k \in (0, \sqrt{2\delta}) \). Hence there exists a unique zero \( k \in (0, \sqrt{2\delta}) \).

To conclude, if \( \varepsilon, \delta > 0 \) satisfy \( \varepsilon + \delta < 2 \), then the operator \( L \) has exactly two simple eigenvalues, located in \( (-1, -1 + \varepsilon] \) and \( [1 - \delta, 1) \) respectively.

**Appendix A. Characterization of eigenvalues when \( u_+ = 1 \) or \( u_- = -1 \) by a family of compact operators**

In the appendix we consider the eigenvalue problem

\[
L\Psi = \lambda\Psi, \quad L = \begin{pmatrix} -u_- & i\partial_x \\ i\partial_x & -u_+ \end{pmatrix}: \mathcal{H}^1(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2), \quad \lambda \in (-1, 1),
\]

when the potentials \( u_\pm \) satisfy

\[
\begin{align*}
u_- &= -1, \quad V_+ = u_+ - 1 \in L^\infty(\mathbb{R}; \mathbb{R}), \\
&\text{such that } V_+(x) \to 0 \text{ as } |x| \to \infty \text{ and } \|(V_+)^-\|_{L^\infty} < 2,
\end{align*}
\]

or

\[
\begin{align*}
u_+ &= 1, \quad V_- = u_- + 1 \in L^\infty(\mathbb{R}; \mathbb{R}), \\
&\text{such that } V_-(x) \to 0 \text{ as } |x| \to \infty \text{ and } \|(V_-)^+\|_{L^\infty} < 2.
\end{align*}
\]
If $u_--1$, then $\lambda + u_--0$ for all $\lambda \in (-1,1)$, and by Lemma 2.1 it is equivalent to consider the eigenvalue problem (2.7) with $\lambda \in (-1,1)$:

\[
\frac{1}{1-\lambda}(-\partial_{xx} + 1 - \lambda^2)\phi + V_+ \phi = 0.
\]

Similarly if $u_+1$, then $\lambda + u_+\neq0$ for all $\lambda \in (-1,1)$, and it is equivalent to consider the eigenvalue problem (2.8) with $\lambda \in (-1,1)$:

\[
\frac{1}{1+\lambda}(-\partial_{xx} + 1 - \lambda^2)\phi - V_- \phi = 0.
\]

Theorem 1.1 and Theorem 1.2 imply that in the case (A.1), all the eigenvalues of (A.3) (and hence of $L$) inside $(-1,1)$ lie in $I_- = (-1,-1 + \|V_+\|_{L^\infty})$, and the number of eigenvalues in $I_-$ is the same as the number of all negative eigenvalues of (1.11):

\[
-\frac{1}{2}\partial_{xx} \psi + V_+ \psi = \mu \psi,
\]

counted by multiplicity, while for the case (A.2), all the eigenvalues of (A.4) (and hence of $L$) inside $(-1,1)$ lie in $I_+ = [1 - \|V_-\|_{L^\infty},1)$, and the number of eigenvalues in $I_+$ is the same as the number of all negative eigenvalues of (1.12):

\[
-\frac{1}{2}\partial_{xx} \psi - V_- \psi = \nu \psi,
\]

counted by multiplicity.

In this appendix we are going to study the eigenvalue problems (A.3) and (A.4) for $\lambda \in (-1,1)$ directly, and we reformulate them as

\[
(-\partial_{xx} + 1 - \lambda^2)^{-1}V_+ \phi = -\frac{1}{1-\lambda} \phi,
\]

and

\[
(-\partial_{xx} + 1 - \lambda^2)^{-1}(-V_-) \phi = -\frac{1}{1+\lambda} \phi.
\]

In Subsection A.1 we will introduce a symmetric and compact operator $K_\beta$, $\beta > 0$ on $H^1(\mathbb{R})$. The above formulation (A.7)-(A.8) of the eigenvalue problems motivates us to define the operator

\[
K_\beta = (-\partial_{xx} + \beta)^{-1}V : H^1(\mathbb{R}) \to H^1(\mathbb{R})
\]

where $\beta > 0$ is a fixed positive constant, and $V \in L^\infty(\mathbb{R})$ vanishes as infinity.

Then the operator $K_\beta$ is a symmetric and compact operator, when we endow $H^1(\mathbb{R})$ with the inner product

\[
\langle u, v \rangle_\beta := \langle \partial_x u, \partial_x v \rangle_{L^2(\mathbb{R})} + \beta \langle u, v \rangle_{L^2(\mathbb{R})}.
\]

It has an ONB of eigenfunctions and an associated eigenvalue sequence converging to 0. We take all the negative eigenvalues

\[
\{\gamma_j(\beta)\}_{j \in M} \subset (-\infty, 0),
\]
and order them non-decreasingly
\[ |\gamma_1(\beta)| \geq |\gamma_2(\beta)| \geq \cdots > 0. \]
Here the set \( M \) can be \( \mathbb{N} \) or a finite set \( \{1, \ldots, m\} \) or the empty set \( \emptyset \).
As before, we denote by \( V^+ \) resp. \( V^- \) the positive resp. negative part of \( V \), and we have the following properties of the set \( M \) and the function \( \gamma_j, j \in M \).

**Lemma A.1** (Properties of the set \( M \) and the function \( \gamma_j \)). The following holds true:

(i) The set \( M = M^{(\beta)} \) is independent of \( \beta \in \mathbb{R}^+ \).

(ii) For each \( j \in M \), the function \( \gamma_j : (0, \infty) \to (-\infty, 0) \) is strictly increasing and continuous, and for any \( \beta \in (0, \infty) \),

\[
(\text{A.12}) |\gamma_j(\beta)| \leq \frac{\| V^- \|_{L^\infty}}{\beta},
\]

\[
(\text{A.13}) \limsup_{\beta \to \beta^+} \frac{|\gamma_j(\beta) - \gamma_j(\beta)|}{|\beta - \beta|} \leq \frac{\| V^- \|_{L^\infty}}{\beta^2}.
\]

Moreover, \( \gamma_j \) is Lipschitz continuous on \([\delta, \infty)\), with Lipschitz constant \( \delta^{-2}\| V^- \|_{L^\infty} \), for every \( \delta > 0 \).

**Proof.** This lemma follows by a similar argument as in the proof of Lemma 3.1.

For \( \phi \in H^1(\mathbb{R}) \setminus \{0\} \) and \( \beta > 0 \), we derive from (A.9) and (A.10) that

\[
\frac{\langle K_{\beta} \phi, \phi \rangle_{\beta}}{\langle \phi, \phi \rangle_{\beta}} = \frac{\langle V \phi, \phi \rangle_{L^2}}{\| \partial_x \phi \|_{L^2}^2 + \beta \| \phi \|_{L^2}^2}.
\]

Poincaré’s min-max principle (recalling (3.14)-(3.15)) implies for any \( j \in M^{(\beta)} \),

\[
(\text{A.14}) \gamma_j(\beta) = \min_{U \subset H^1(\mathbb{R}) \setminus \{0\}} \max_{\dim(U) = j} \max_{\phi \in U \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\| \partial_x \phi \|_{L^2}^2 + \beta \| \phi \|_{L^2}^2}.
\]

The minimum is attained at

\[
U = U^{(\beta)}_j := \text{Span}\{\phi_1(\beta), \ldots, \phi_j(\beta)\},
\]

where \( \phi_i(\beta) \) denotes an eigenfunction of \( K_\beta \) associated with \( \gamma_i(\beta) \), and \( \phi_1(\beta), \ldots, \phi_j(\beta) \) are chosen \( \langle \cdot, \cdot \rangle_\beta \)-orthonormal.

Let \( \beta_1 \in (0, \infty) \) and \( j \in M^{(\beta_1)} \), so that \( \gamma_j(\beta_1) < 0 \). Then (A.14) implies

\[
\frac{\langle V \phi, \phi \rangle_{L^2}}{\| \partial_x \phi \|_{L^2}^2 + \beta_1 \| \phi \|_{L^2}^2} < 0, \quad \forall \phi \in U^{(\beta_1)}_j \setminus \{0\},
\]

and thus \( \langle V \phi, \phi \rangle_{L^2} < 0, \forall \beta \in (0, \infty) \), \( \forall \phi \in U^{(\beta_1)}_j \setminus \{0\} \).

By the \( L^2 \)-compactness of the unit sphere in the finite-dimensional space \( U_j^{(\beta_1)} \), we conclude that

\[
\max_{\phi \in U^{(\beta_1)}_j \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\| \partial_x \phi \|_{L^2}^2 + \beta \| \phi \|_{L^2}^2} = \max_{\phi \in U^{(\beta_1)}_j, \| \phi \|_{L^2} = 1} \frac{\langle V \phi, \phi \rangle_{L^2}}{\| \partial_x \phi \|_{L^2}^2 + \beta} < 0,
\]
and hence
\[
\min_{U \subset H^1(\mathbb{R}) \text{ subspace, } \dim(U) = j} \max_{\phi \in U \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} < 0, \quad \forall \beta \in (0, \infty).
\]
This implies \( j \in M^{(\beta)} \) for all \( \beta \in (0, \infty) \), and we have proved \( M^{(\beta_1)} \subset M^{(\beta)} \) for all \( \beta_1, \beta \in (0, \infty) \). The assertion (i) follows.

Let \( j \in M \) and \( 0 < \beta < \beta_1 < \infty \), so that
\[
\frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} < \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta_1 \|\phi\|_{L^2}^2} < 0, \quad \forall \phi \in U^{(\beta_1)}_j \setminus \{0\}.
\]
By the above compactness argument again we deduce
\[
\max_{\phi \in U^{(\beta_1)}_j \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} < \max_{\phi \in U^{(\beta_1)}_j \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta_1 \|\phi\|_{L^2}^2} = \gamma_j(\beta_1).
\]
This implies \( \gamma_j(\beta) < \gamma_j(\beta_1) \) by the min-max principle, and hence \( \gamma_j : (0, \infty) \to (-\infty, 0) \) is strictly increasing.

Let \( \beta \in (0, \infty) \) and we are going to show the continuity of \( \gamma_j \) at \( \beta \). From the above argument we know that, for all \( \phi \in U^{(\beta)}_j \setminus \{0\} \), \( \langle V \phi, \phi \rangle_{L^2} < 0 \) and hence
\[
\langle V^+ \phi, \phi \rangle_{L^2} < \langle V^- \phi, \phi \rangle_{L^2},
\]
and thus
\[
\|\langle V \phi, \phi \rangle_{L^2} \| = \langle V^- \phi, \phi \rangle_{L^2} - \langle V^+ \phi, \phi \rangle_{L^2} \leq \|V^-\|_{L^\infty} \|\phi\|_{L^2}^2.
\]
This implies
\[
\frac{\|\langle V \phi, \phi \rangle_{L^2} \|}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} \leq \frac{\|V^-\|_{L^\infty}}{\beta}, \quad \forall j \in M, \quad \forall \phi \in U^{(\beta)}_j \setminus \{0\},
\]
and hence the estimate (A.12) by the min-max principle (A.14). Furthermore, we calculate for all \( \beta \in (0, \infty) \) and all \( \phi \in H^1(\mathbb{R}) \setminus \{0\} \) that
\[
\frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} \leq \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta_1 \|\phi\|_{L^2}^2} \leq \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} \leq \frac{\|V^-\|_{L^\infty} \|\beta - \tilde{\beta}\| \beta \beta}{\beta \beta}.
\]
We immediately have
\[
\gamma_j(\beta) = \max_{\phi \in U^{(\beta)}_j \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} \geq \max_{\phi \in U^{(\beta)}_j \setminus \{0\}} \frac{\langle V \phi, \phi \rangle_{L^2}}{\|\partial_x \phi\|_{L^2}^2 + \beta \|\phi\|_{L^2}^2} \frac{\|V^-\|_{L^\infty} \|\beta - \tilde{\beta}\|}{\beta \beta}.
\]
We apply the min-max principle in (A.14) to derive
\[
\gamma_j(\beta) \geq \gamma_j(\tilde{\beta}) - \frac{\|V^-\|_{L^\infty} \|\beta - \tilde{\beta}\|}{\beta \beta}.
\]
As \( \beta, \tilde{\beta} \in (0, \infty) \) are chosen arbitrarily, we have similarly
\[
\gamma_j(\tilde{\beta}) \geq \gamma_j(\beta) - \frac{\|V^-\|_{L^\infty} \|\beta - \tilde{\beta}\|}{\beta \beta}.
\]
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Hence
\[ |\gamma_j(\beta) - \gamma_j(\tilde{\beta})| \leq \frac{\|V^-\|_{L^\infty}^2}{\beta \tilde{\beta}} |\beta - \tilde{\beta}|, \]
and the estimate (A.13) follows. The Lipschitz continuity of \( \gamma_j \) on \( [\delta, \infty) \) with \( \delta > 0 \) follows correspondingly. \( \square \)

A.2. Eigenvalues \( \lambda \in (-1, 1) \) of \( \mathcal{L} \) when \( u_- = -1 \) or \( u_+ = 1 \). Assume (A.1), and we now come back to the eigenvalue problem (A.3), i.e. (A.7) for \( \lambda \in (-1, 1) \). For any \( \beta > 0 \), let \( \{\gamma_j^+ (\beta)\}_{j \in M_+} \) denote all the negative eigenvalues of the symmetric and compact operator (recalling the definitions (A.9)-(A.11))
\[ K_{\tilde{\beta}}^+ = (-\partial_{xx} + \beta)^{-1} V_+ : H^1(\mathbb{R}) \to H^1(\mathbb{R}). \]

We define the even function
\[ \Gamma_j^+ (\lambda) := \gamma_j^+(1 - \lambda^2), \quad \lambda \in (-1, 1), \]
and we are indeed searching for \( \lambda_j \in (-1, 1) \) such that
\begin{equation}
(A.16) \quad \Gamma_j^+ (\lambda_j) = -\frac{1}{1 - \lambda_j}.
\end{equation}

By Lemma A.1, the function \( \Gamma_j^+ : (-1, 1) \to (-\infty, 0) \) is strictly increasing in \( \lambda \in (-1, 0) \) while strictly decreasing in \( \lambda \in (0, 1) \), and for any \( \lambda \in (-1, 1) \), using (A.12) and (A.13),
\begin{equation}
(A.17) \quad |\Gamma_j^+ (\lambda)| \leq \frac{\|(V_+)^-\|_{L^\infty}}{1 - \lambda^2},
\end{equation}
\begin{align*}
\limsup_{\lambda \to \lambda} \frac{|\Gamma_j^+ (\lambda) - \Gamma_j^+ (\tilde{\lambda})|}{|\lambda - \tilde{\lambda}|} &= \limsup_{\lambda \to \lambda} \frac{|\gamma_j^+(1 - \lambda^2) - \gamma_j^+(1 - \tilde{\lambda}^2)|}{|\lambda^2 - \tilde{\lambda}^2|} |\lambda + \tilde{\lambda}| \\
&\leq 2|\lambda| \cdot \|(V_+)^-\|_{L^\infty} \left(1 - \lambda^2\right)^2.
\end{align*}
\begin{equation}
(A.18)
\end{equation}

If \( \|(V_+)^-\|_{L^\infty} < 2 \), then for all \( \lambda \in (0, 1) \), we derive from (A.18) that
\[ \limsup_{\lambda \to \lambda} \frac{|\Gamma_j^+ (\lambda) - \Gamma_j^+ (\tilde{\lambda})|}{|\lambda - \tilde{\lambda}|} < \frac{4\lambda}{(1 - \lambda^2)^2} < \frac{1}{(1 - \lambda)^2}, \]
that is,
\begin{equation}
(A.19) \quad \limsup_{\lambda \to \lambda} \frac{|\Gamma_j^+ (\lambda) - \Gamma_j^+ (\tilde{\lambda})|}{|\lambda - \tilde{\lambda}|} < \left| \frac{d}{d\lambda} \left( \frac{1}{1 - \lambda} \right) \right|, \quad \forall \lambda \in (0, 1).
\end{equation}

Notice that the function \( -\frac{1}{1 - \lambda} : [-1, 1) \to (-\infty, -\frac{1}{2}] \) is a strictly decreasing function of \( \lambda \in [-1, 1) \), with its maximum \( -\frac{1}{2} \) evaluated at \( \lambda = -1 \). Let \( M_+^* \) denote the subset of \( M_+ \) defined by
\begin{equation}
(A.20) \quad M_+^* = \{ j \in M_+ | \gamma_j^+(0_+) < -\frac{1}{2} \}.
\end{equation}

We claim that for any \( j \in M_+^* \), there exists a unique solution \( \lambda_j \in (-1, 1) \) of (A.16). Indeed, on one side, for any \( j \in M_+^* \),
\[ \Gamma_j^+ ((-1)_+) = \gamma_j^+(0_+) < -\frac{1}{2} = \left( -\frac{1}{1 - \lambda} \right)_{|\lambda=\lambda_1}. \]
On the other side, there exists \( \lambda_0 \in [0, 1) \) so that
\[
\Gamma_j^+(\lambda_0) \geq -\frac{1}{1 - \lambda_0},
\]
since, using (A.17), if \( \|(V_+)^-\|_{L^\infty} \leq 1 \), we can take \( \lambda_0 = 0 \) so that
\[
\Gamma_j^+(\lambda_0) = -|\Gamma_j^+(\lambda_0)| \geq -\|(V_+)^-\|_{L^\infty} \geq -1 = -\frac{1}{1 - \lambda_0},
\]
while if \( \|(V_+)^-\|_{L^\infty} \in (1, 2) \), we can take \( \lambda_0 = \|(V_+)^-\|_{L^\infty} - 1 \in (0, 1) \) so that
\[
\Gamma_j^+(\lambda_0) = -|\Gamma_j^+(\lambda_0)| \geq -\frac{1 + \lambda_0}{1 - (\lambda_0)^2} = -\frac{1}{1 - \lambda_0}.
\]
By the continuity of the functions \( \Gamma_j^+ \) and \(-\frac{1}{1 - \lambda}\), there exists \( \lambda_j \in (-1, \lambda_0) \) so that (A.16) holds. The uniqueness of \( \lambda_j \in (-1, 1) \) follows from the strict monotonicity properties of the functions \( \Gamma_j^+|_{(-1,0)} \), \( \Gamma_j^+|_{(0,1)} \) and \(-\frac{\lambda}{1 - \lambda}|_{(-1,1)} \) and (A.19).

This unique solution \( \lambda_j \in (-1, 1) \) of (A.16) is an eigenvalue of (A.3), i.e. (A.7), with the eigenspace the same as the one associated with the negative eigenvalue \( \gamma_j^+(1 - \lambda_j^2) \) of the operator \( K_{1-\lambda_j^2}^+ \).

Conversely, let \((\lambda, \phi) \in (-1, 1) \times (H^1(\mathbb{R})\setminus\{0\})\) denote any eigenpair of the eigenvalue problem (A.3) and hence of (A.7), then \( \phi \) is an eigenfunction associated with the (negative) eigenvalue \(-\frac{1}{1 - \lambda_j} \in (-\infty, -\frac{1}{2})\) of the operator \( K_{1-\lambda_j}^+ \). Thus there exists \( j \in M_+ \) so that (A.16) holds:
\[
\gamma_j^+(1 - \lambda^2) = -\frac{1}{1 - \lambda}.
\]
Since \( \gamma_j^+: (0, \infty) \rightarrow (-\infty, 0) \) is strictly increasing, we have indeed \( j \in M_+^l \):
\[
\gamma_j^+(0^+) < \gamma_j^+(1 - \lambda^2) = -\frac{1}{1 - \lambda} < -\frac{1}{2}.
\]

The case (A.2) can be considered analogously: We replace \( V_+ \) by \( V_- \), and for any \( \beta > 0 \) we denote by \( \{\gamma_j^- (\beta)\}_{j \in M_-} \) all the negative eigenvalues of the operator
\[
K_\beta = (-\partial_{xx} + \beta)^{-1}(-V_-): H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}).
\]
Let \( \Gamma_j^-(\lambda) = \gamma_j^- (1 - \lambda^2) \) for \( \lambda \in (-1, 1) \), and we search for \( \lambda_j \in (-1, 1) \) such that
\[
(A.21) \quad \Gamma_j^- (\lambda_j) = -\frac{1}{1 + \lambda_j}.
\]
For any \( j \in M_+^l = \{j \in M_- \mid \gamma_j^-(0^+) < -\frac{1}{2}\} \), the existence of a unique solution \( \lambda_j \in (-1, 1) \) follows similarly as for the case (A.1): The function \( \Gamma_j^-: (-1, 1) \mapsto (-\infty, 0) \) is continuous, strictly increasing in \( \lambda \in (-1, 0) \), strictly decreasing in \( \lambda \in (0, 1) \) and bounded as \( |\Gamma_j^- (\lambda)| \leq \frac{\|\nabla V_+\|_{L^\infty}}{1 - \lambda} \), while the function \(-\frac{1}{1 + \lambda}\): \((-1, 1) \rightarrow (-\infty, -\frac{1}{2})\) is continuous and strictly increasing, with its maximum \(-\frac{1}{2}\) attained at \( \lambda = 1 \), and they satisfy
\[
\Gamma_j^- (1^-) < -\frac{1}{2}, \quad \Gamma_j^- (\lambda_0) \geq -\frac{1}{1 + \lambda_0} \text{ for some } \lambda_0 \in (-1, 0],
\]
\[
\limsup_{\lambda \rightarrow \lambda_j^-} \frac{|\Gamma_j^- (\lambda) - \Gamma_j^- (\lambda_j)|}{|\lambda - \lambda_j|} < \left| \frac{d}{d\lambda} \left( \frac{1}{1 + \lambda} \right) \right|, \quad \forall \lambda \in (-1, 0).
\]
This unique solution \( \lambda_j \in (-1, 1) \) of (A.21) is an eigenvalue of (A.4), i.e. (A.8), with the eigenspace the same as the one associated with the negative eigenvalue.
\[ \gamma_j \left( 1 - \lambda_j^2 \right) \] of the operator \( K_{1-\lambda_j^2} \). Conversely, for any eigenpair \((\lambda, \phi) \in (-1, 1) \times H^1(\mathbb{R}) \setminus \{0\}\) of (A.4), i.e. (A.8), \( \phi \) is an eigenfunction associated with the eigenvalue \(-\frac{1}{\gamma_j} \) of the operator \( K_{1-\lambda_j^2} \), and hence there exists \( j \in M_- \) so that

\[ \gamma_j \left( 1 - \lambda_j^2 \right) = -\frac{1}{\gamma_j}. \]

To conclude, we have proved

**Theorem A.1** (Eigenvalues inside \((-1, 1)\) of \( L, \mathcal{L} \) when \( u_- = -1 \) or \( u_+ = 1 \)). Let \( q(x) \) satisfies (1.6)-(1.9)-(1.10). The following holds true:

- If \( u_- = -1 \) and \( \{\gamma_j^+ (\beta)\}_{j \in M_-^+}, \beta > 0 \) denote all the negative eigenvalues of the operator \( K_{1-\lambda_j^2}^+ = (-\partial_{xx} + \beta)^{-1}(u_- - 1) \) such that \( \gamma_j^+(0) < -\frac{1}{2} \), then for all \( j \in M_-^+ \), there exists a unique fixed point \( \gamma_j \in (-1, 1) \) of the mapping

\[ 1 + \frac{1}{\gamma_j^+ (1-\lambda_j^2)}, \] \[ \{\lambda_j\}_{j \in M_-^+} \] is the set of eigenvalues of \( L, \mathcal{L} \) in \((-1, 1)\), with the eigenspace of \( \lambda_j \) coinciding with the eigenspace associated with the negative eigenvalue \( \gamma_j^+ (1 - \lambda_j^2) \) of the operator \( K_{1-\lambda_j^2}^+ \).

- If \( u_+ = 1 \) and \( \{\gamma_j^- (\beta)\}_{j \in M_-^+}, \beta > 0 \) denote all the negative eigenvalues of the operator \( K_{1-\lambda_j^2}^- = (-\partial_{xx} + \beta)^{-1}(-u_+ - 1) \) such that \( \gamma_j^- (0) < -\frac{1}{2} \), then for all \( j \in M_-^+ \), there exists a unique fixed point \( \lambda_j \in (-1, 1) \) of the mapping

\[ -1 + \frac{1}{\gamma_j^- (1-\lambda_j^2)}, \] \[ \{\lambda_j\}_{j \in M_-^+} \] is the set of eigenvalues of \( L, \mathcal{L} \) in \((-1, 1)\), with the eigenspace of \( \lambda_j \) coinciding with the eigenspace associated with the negative eigenvalue \( \gamma_j^- (1 - \lambda_j^2) \) of the operator \( K_{1-\lambda_j^2}^- \).

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