LOCALIZING VIRTUAL STRUCTURE SHEAVES BY COSECTIONS

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Abstract. We construct a cosection localized virtual structure sheaf when a Deligne-Mumford stack is equipped with a perfect obstruction theory and a cosection of the obstruction sheaf.

1. Introduction

According to Schubert [25], enumerative geometry is about finding the number of geometric figures of fixed type satisfying certain given conditions. A typical way of solving an enumerative problem consists of constructing a moduli space parameterizing all geometric figures of fixed type and then finding the number of intersection points of the subsets defined by the given conditions. The latter part is called an intersection theory and a solid mathematical theory was developed by Fulton [13] in 1970s through a systematic use of normal cones and Gysin maps. In particular, the intersection rings for smooth schemes were rigorously defined and Riemann-Roch theorems were established for schemes. Fulton’s intersection theory was updated for Deligne-Mumford stacks by Vistoli [37] and for Artin stacks by Kresch [27].

However moduli spaces are often very singular and do not behave well under deformation. To deal with this issue, the theory of virtual fundamental classes was developed in 1990s by Li-Tian [30] and Behrend-Fantechi [2]. A Deligne-Mumford stack \( X \) has its intrinsic normal cone \( \mathcal{C}_X \) which is locally defined as the quotient stack \( C_{U/M}/T_M|_U \) for an étale \( U \to X \) and a closed embedding \( U \hookrightarrow M \) into a smooth \( M \). When \( \mathcal{C}_X \) is embedded into a vector bundle stack \( \mathcal{E}_X \) which is locally the quotient \( E_1/E_0 \) for vector bundles \( E_1, E_0 \), the virtual fundamental class is defined as the intersection

\[
[X]^{\text{vir}} = 0_{\mathcal{E}_X}[\mathcal{C}_X] \in A_*(X)
\]

of \( \mathcal{C}_X \) with the zero section of \( \mathcal{E}_X \). Many nice properties such as deformation invariance can be deduced under reasonable assumptions [24] and since 1995, important enumerative invariants in algebraic geometry have been constructed as integrals on the virtual cycles \( [X]^{\text{vir}} \) on suitable moduli spaces \( X \), including the Gromov-Witten, Donaldson-Thomas, Pandharipande-Thomas invariants and more.

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The computation of these virtual invariants is known to be very hard because there are not many tools to handle the virtual cycle \([X]^{\text{vir}}\). When there is an action of \(\mathbb{C}^*\) on \(X\), under suitable assumptions, the virtual cycle is localized to the fixed locus \(X^{\mathbb{C}^*}\) by the torus localization formula (cf. \[16\])

\[
[X]^{\text{vir}} = \frac{[X^{\mathbb{C}^*}]^{\text{vir}}}{e(N^{\text{vir}})}
\]

which has been most effective for actual computations so far. Recently, another localization of the virtual cycle \([X]^{\text{vir}}\) was discovered \[19\] which also turned out to be quite useful \[4, 6, 7, 8, 10, 11, 14, 15, 17, 18, 21, 22, 26, 32, 33\]. It says that when there is a morphism \(\sigma: \mathcal{E}_X \to \mathcal{O}_X\), called a cosection, the virtual cycle is localized to the zero locus \(X(\sigma)\) of \(\sigma\), i.e. we have a class \([X]^{\text{vir}}_\text{loc} \in A_*(X(\sigma))\) which equals \([X]^{\text{vir}}\) when pushed to \(X\). The two localizations were combined in \[5\].

Recently there arose a demand to lift the theory of virtual cycles in Chow groups to algebraic K-groups with applications towards physical and geometric representation theory. In \[2, 29\], the K-theoretic virtual fundamental class was defined as

\[
[0]^{\text{vir}}_X = 0^1_{E_1}[\mathcal{O}_{C_1}] = [\mathcal{O}_X \otimes_{\mathcal{O}_{E_1}} \mathcal{O}_{C_1}] \in K_0(X)
\]

and called the virtual structure sheaf of \(X\). Here \(K_0(X)\) is the Grothendieck group of coherent sheaves on \(X\) with relations \([F] = [F'] + [F'']\) whenever we have an exact sequence \(0 \to F' \to F \to F'' \to 0\); \(C_1 = \mathcal{E}_X \times_{\mathcal{E}_X} E_1\) when \(\mathcal{E}_X = E_1/E_0\) is a global resolution by vector bundles. In order to lift some of the results on the virtual cycles and invariants to the setting of algebraic K-groups, it is necessary to develop K-theoretic techniques to handle the virtual structure sheaf \([0]^{\text{vir}}_X\) such as the cosection localization.

In this paper, we prove the following.

**Theorem 1.1.** *(Theorem 5.2, Proposition 5.2, Proposition 5.3)* Let \(X\) be a Deligne-Mumford stack equipped with a perfect obstruction theory \(\phi: E \to \mathbb{L}_X\) and a cosection \(\sigma: h^1(E^\vee) \to \mathcal{O}_X\) whose zero locus is denoted by \(X(\sigma)\). Assume \(E\) admits a global resolution \([E^{-1} \to E^0]\) by locally free sheaves. Then there is a cosection localized virtual structure sheaf

\[
[0]^{\text{vir}}_{X, \text{loc}} \in K_0(X(\sigma))
\]

such that \(\imath_*[0]^{\text{vir}}_{X, \text{loc}} = [0]^{\text{vir}}_X \in K_0(X)\) where \(\imath: X(\sigma) \to X\) denotes the inclusion. Moreover, \([0]^{\text{vir}}_{X, \text{loc}}\) is independent of the choice of the resolution \([E^{-1} \to E^0]\) and is deformation invariant.

By \[19, \S 4\], the cone \(C_1 \subset E_1\) has (reduced) support in

\[
E_1(\sigma) = E_1|_{X(\sigma)} \cup \ker(\sigma: E_1|_U \to h^1(E^\vee)|_U \to \mathcal{O}_U)
\]

where \(U = X - X(\sigma)\). We define the cosection localized Gysin map (cf. Theorem 4.1)

\[
0^1_{E_1, \sigma}: K_0(E_1(\sigma)) \to K_0(X(\sigma))
\]
and the cosection localized virtual structure sheaf \([O^{\text{vir}}_{X,\text{loc}}]\) is defined as
\[
O^{\text{vir}}_{X,\text{loc}} = 0^1_{E_1,\sigma}[O_{C_1}] \in K_0(X(\sigma)).
\]

Using the cosection localized virtual structure sheaf \([O^{\text{vir}}_{X,\text{loc}}]\) \(\in K_0(X(\sigma))\), we can define the cosection localized virtual Euler characteristic, even when \(X\) is not proper, as long as \(X(\sigma)\) is proper.

**Definition 1.2.** The cosection localized virtual Euler characteristic of a class \(\beta \in K^0(X)\) is defined as
\[
\chi^{\text{vir}}_{\text{loc}}(X, \beta) = \chi(X(\sigma), \beta \cdot O^{\text{vir}}_{X,\text{loc}}) = \sum_i (-1)^i \dim H^i(X(\sigma), \beta \cdot O^{\text{vir}}_{X,\text{loc}}).
\]

As an application, we lift the results of [6, 7] to the K-theoretic setting. In particular, we define the K-theoretic Fan-Jarvis-Ruan-Witten invariant as the Euler characteristic of the virtual structure sheaf on the moduli space of spin curves with sections.

The layout of this paper is as follows. In §2, we collect useful facts. In §3, we prove that the virtual structure sheaf vanishes if there is a surjective cosection \(\sigma : \text{Ob}_X \to O_X\). In §4, we construct the cosection localized Gysin map \(0^1_{E_1,\sigma}\). In §5, we define the cosection localized virtual structure sheaf \([O^{\text{vir}}_{X,\text{loc}}]\) and prove key properties. In §6, we apply the cosection localized virtual structure sheaf to construct a K-theoretic FJRW invariant and discuss K-theoretic Landau-Ginzburg/Calabi-Yau correspondence.

In this paper, all schemes or Deligne-Mumford stacks are separated and Noetherian of finite type over the complex number field \(\mathbb{C}\). When \(X \hookrightarrow Y\) is a closed embedding, its normal cone is denoted by \(C_{X/Y}\). When \(f : X \to Y\) is a morphism and \(F\) is a coherent sheaf on \(Y\), \(F|_Y\) denotes the pullback \(f^*F\).

### 2. Preliminaries

In this section, we collect useful facts about algebraic K-groups and cosection localization.

#### 2.1. Algebraic K-theory

For a Deligne-Mumford stack \(X\), \(K_0(X)\) (resp. \(K^0(X)\)) denotes the Grothendieck group generated by coherent sheaves (resp. locally free sheaves) on \(X\) with relations \([F] = [F'] + [F'']\) whenever we have an exact sequence \(0 \to F' \to F \to F'' \to 0\).

For a projective morphism \(f : X \to Y\), the pushforward
\[
f_* : K_0(X) \longrightarrow K_0(Y)
\]
is defined by the right derived direct image \(Rf_*\) as
\[
f_*[F] = [Rf_*F] = \sum_i (-1)^i[R^i f_*F].
\]
The pullback

\[ f^* : K_0(Y) \to K_0(X) \]

for a morphism \( f : X \to Y \) is defined by the left derived inverse image \( Lf^* \) as

\[ f^*[G] = [Lf^*G] = [\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}G] = \sum_i (-1)^i [\text{Tor}^{f^{-1}\mathcal{O}_Y}_i (\mathcal{O}_X, f^{-1}G)] \]

when the sum is finite, i.e. \( \text{Tor}^{f^{-1}\mathcal{O}_Y}_i (\mathcal{O}_X, f^{-1}G) \neq 0 \) for only finitely many \( i \). Tensor product of locally free sheaves makes \( K_0(X) \) a commutative ring and \( K_0(X) \) a module over \( K_0(X) \).

For a Deligne-Mumford stack \( X \), let \( X_{\text{red}} \) be the reduced stack of \( X \) and \( \iota : X_{\text{red}} \to X \) denote the inclusion.

**Proposition 2.1.** For a Deligne-Mumford stack \( X \), \( \iota_* : K_0(X_{\text{red}}) \to K_0(X) \) is an isomorphism.

**Proof.** Let \( I \) be the ideal sheaf defining \( X_{\text{red}} \) in \( X \), i.e. it is the ideal of nilpotents. Since \( X \) is Noetherian by assumption, \( I^n = 0 \) for some \( n > 0 \). Hence, if \( F \) is a coherent sheaf on \( X \),

\[ [F] = \sum_{r=0}^{n-1} [I^r F/I^{r+1}F] \in K_0(X) \]

and \( I^r F/I^{r+1}F \) are coherent sheaves on \( X_{\text{red}} \). Hence \( \iota_* \) is surjective. In fact, the assignment

\[ (2.1) \quad [F] \mapsto \sum_{r=0}^{n-1} [I^r F/I^{r+1}F] \]

is obviously a right inverse of \( \iota_* \) and hence \( \iota_* \) is injective. Therefore the proof will be complete once we show that \( (2.1) \) defines a homomorphism of \( K_0(X) \) into \( K_0(X_{\text{red}}) \). This follows from the following lemma. \( \square \)

**Lemma 2.2.** Under the notation of Proposition 2.1, if \( 0 \to F' \to F \to F'' \to 0 \) is an exact sequence of coherent sheaves on \( X \), we have an equality

\[ \sum_r [I^r F/I^{r+1}F] = \sum_r [I^r F'/I^{r+1}F'] + \sum_r [I^r F''/I^{r+1}F''] \in K_0(X_{\text{red}}). \]

**Proof.** Let \( F_r = I^r F \) and \( F''_r = I^r F'' \). Let

\[ F'_r = \ker(I^r F \to I^r F'') \]

so that we have a commutative diagram of exact sequences

\[
\begin{array}{cccccccc}
0 & \to & F'_{r+1} & \to & F_{r+1} & \to & F''_{r+1} & \to & 0 \\
| & & | & & | & & | & & |
0 & \to & F'_r & \to & F_r & \to & F''_r & \to & 0
\end{array}
\]
which gives us an equality
\[
\sum_r [I^r F/I^{r+1}F] = \sum_r [I^r F'/I^{r+1}F'] = \sum_r [F'_r/F'_{r+1}].
\]
Hence it suffices to prove that
\[
\sum_r [F'_r/F'_{r+1}] = \sum_r [I^r F'/I^{r+1}F'].
\]
Since \(F'_r \supset F'_1 \supset \cdots \supset F'_n \supset 0\) and \(F' = I^0 F' \supset IF' \supset \cdots \supset I^n F' = 0\) are two filtrations of \(F'\), they have a common refinement by Schreier’s theorem (cf. [28]) and hence we have the equality (2.2). □

**Proposition 2.3.** [3, Lemma 17] For a Deligne-Mumford stack \(X\), \(K_0(X)\) is generated by the classes \([\mathcal{O}_Z]\) for integral closed substacks \(Z\) in \(X\).

**Proof.** By Proposition 2.1, we may assume \(X\) is reduced and consider only sheaves supported on reduced stacks. Now the proof of [3, Lemma 17], by induction on the dimension of the support, also proves the proposition. □

**Proposition 2.4.** [3, Proposition 7] Let \(X\) be a Deligne-Mumford stack and \(Z\) be a closed substack. Let \(U = X - Z\). Then the inclusions
\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
& \searrow & \downarrow j \\
& & U
\end{array}
\]
induce a complex
\[
(2.3) \quad K_0(Z) \xrightarrow{i_\ast} K_0(X) \xrightarrow{j_\ast} K_0(U) \longrightarrow 0
\]
which is exact at \(K_0(U)\). If every coherent sheaf on \(U\) extends to a coherent sheaf on \(X\) (e.g. \(X\) is a scheme), then (2.3) is exact.

**Proof.** \(j_\ast\) is surjective by Proposition 2.3 since \(j_\ast[\mathcal{O}_Z] = [\mathcal{O}_Z]\) for an integral \(Z \subset U\) where \(\bar{Z}\) denotes the closure of \(Z\) in \(X\). When every coherent sheaf on \(U\) extends to a coherent sheaf on \(X\), the exactness of (2.3) is proved by the same proof of [3, Proposition 7]. □

For instance, when \(X\) has resolution property, every sheaf of \(\mathcal{O}_U\)-modules, where \(U \subset X\) open, can be extended to a sheaf of \(\mathcal{O}_X\)-modules. When the Keel-Mori coarse moduli of \(X\) is a separated scheme, \(X\) has resolution property [36].

### 2.2. Virtual structure sheaf
Let \(X\) be a Deligne-Mumford stack equipped with a perfect obstruction theory \(\phi : E \to \mathbb{L}_X\), i.e. \(\phi\) is a morphism in the derived category of quasi-coherent sheaves on \(X\) such that \(h^0(\phi) : h^0(E) \to h^0(\mathbb{L}_X)\) is an isomorphism, \(h^{-1}(\phi) : h^{-1}(E) \to h^{-1}(\mathbb{L}_X)\) is surjective, and \(E\) is locally isomorphic to a 2-term complex of locally free sheaves, concentrated in degrees \(-1\) and \(0\). Here \(\mathbb{L}_X = \tau_{\geq -1}^\perp \mathbb{L}_X\) is the truncated cotangent complex of \(X\).

To avoid discussion on algebraic K-theory of Artin stacks, let us suppose in this paper that \(E\) admits a global resolution by a 2-term complex \([E^{-1} \to E^0]\) of locally free sheaves on \(X\). Let \([E_0 \to E_1]\) denote the dual with
localized virtual fundamental class $X$. If $\pi^* E_1$ over $E_1$ gives us the Koszul resolution $\wedge \pi^* E_1^\vee$ of the sheaf $\mathcal{O}_X$ on $E_1$. Hence \([2.4]\) can be rephrased as

$$[\mathcal{O}_X] = [\wedge \pi^* E_1^\vee \otimes \mathcal{O}_{E_1}]$$

2.3. **Localizing virtual fundamental classes by cosections.** Suppose the obstruction sheaf $Ob_X := h^1(E_1^\vee)$ admits a cosection $\sigma : Ob_X \to \mathcal{O}_X$. Let $X(\sigma)$ denote the locus where $\sigma$ is not surjective, i.e. the closed stubstack defined by the ideal sheaf $\sigma(Ob_X) \subset \mathcal{O}_X$. By \([19, \S 4]\), the (reduced) support of $C_1$ is contained in

$$E_1(\sigma) = E_1|_{X(\sigma)} \cup \ker(\sigma : E_1 \to Ob_X \to \mathcal{O}_X).$$

The ordinary Gysin map $0^!_{E_1} : A_*(E_1) \to A_*(X)$ can be localized to a homomorphism (cf. \([19, \S 2]\))

$$0^!_{E_1,\sigma} : A_*(E_1(\sigma)) \to A_*(X(\sigma))$$

such that $\iota_* 0^!_{E_1,\sigma} = 0^!_{E_1}$ where $\iota : X(\sigma) \to X$ is the inclusion map.

The cosection localized virtual cycle is then obtained by

$$[X]_{\text{loc}}^\vee = 0^!_{E_1,\sigma}[C_1] \in A_*(X(\sigma))$$

which satisfies many nice properties such as $\iota_* [X]_{\text{loc}}^\vee = [X]^\vee$ and deformation invariance. When $X$ is not proper, $[X]^\vee$ is not properly supported in general. If $X(\sigma)$ is proper, then we can still define integrals on the cosection localized virtual fundamental class $[X]_{\text{loc}}^\vee$.

In the subsequent sections, we will show that the virtual structure sheaf $[\mathcal{O}_X] \in K_0(X)$ is also localized to $X(\sigma)$ by cosection.
3. A vanishing result for virtual structure sheaves

In this section, we show that if there is a surjective cosection \( \sigma : \text{Ob}_U \rightarrow \mathcal{O}_U \) on a Deligne-Mumford stack \( U \), then the virtual structure sheaf \( \mathcal{O}_U^{\text{vir}} \) is zero.

Let \( U \) be a Deligne-Mumford stack and let \( \phi : E_U \rightarrow \mathbb{L}_U \) be a perfect obstruction theory on \( U \) where \( \mathbb{L}_U = \tau_{\geq -1}L_U \) is the truncated cotangent complex on \( U \), i.e. \( h^0(\phi) \) is an isomorphism and \( h^{-1}(\phi) \) is surjective. By [2], we have a closed embedding

\[
\mathfrak{N}_U = h^1/h^0(\mathbb{L}_U') \hookrightarrow h^1/h^0(\mathcal{E}_U') =: \mathcal{E}_U
\]

of the virtual normal sheaf \( \mathfrak{N}_U \) into the vector bundle stack \( \mathcal{E}_U \). In this section, we make following assumptions.

**Assumption 3.1.** (1) There is a surjective homomorphism \( \sigma_U : \text{Ob}_U \rightarrow \mathcal{O}_U \) where \( \text{Ob}_U = h^1(E_U') \) is the obstruction sheaf.

(2) There is a 2-term complex \([E^{-1} \rightarrow E^0]\) of locally free sheaves on \( U \) which is isomorphic to \( E_U \) in the derived category \( D(\mathcal{O}_U) \).

Let \( E_i = (E^{-i})' \) be the dual bundle of \( E^{-i} \) for \( i = 0, 1 \) so that \( E_U' = [E_0 \rightarrow E_1] \). Then \( \mathcal{E}_U = h^1/h^0(E_U') \) is the quotient stack \([E_1/E_0]\). Since the abelian hull of the intrinsic normal cone \( \mathcal{C}_U \) is the intrinsic normal sheaf \( \mathfrak{N}_U \), we have an embedding

\[
\mathcal{C}_U \hookrightarrow \mathfrak{N}_U \hookrightarrow \mathcal{E}_U
\]

uniquely determined by the perfect obstruction theory \( \phi \). The natural homomorphism \( E_1 \rightarrow \mathcal{E}_U = [E_1/E_0] \) induces the Cartesian square

\[
\begin{array}{ccc}
C_1 & \longrightarrow & E_1 \\
\downarrow & & \downarrow \\
\mathcal{E}_U' & \longrightarrow & \mathcal{E}_U.
\end{array}
\]

Since \( \text{Ob}_U \) is the cokernel of \( E_U' = [E_0 \rightarrow E_1] \), we have a surjective homomorphism

\[
(3.1) \quad E_1 \longrightarrow \text{Ob}_U \xrightarrow{\sigma_U} \mathcal{O}_U.
\]

By [19, §4], \( C_1 \) has support in the subbundle

\[
E_1' \equiv \ker(E_1 \rightarrow \mathcal{O}_U).
\]

By Proposition 2.1, we may think of \( \mathcal{O}_{C_1} \) as a sheaf on \( E_1' \) because

\[
(3.2) \quad [\mathcal{O}_{C_1}] \in K_0(E_1').
\]

For the computation of \( \mathcal{O}_U^{\text{vir}} \), we may use the Koszul resolution of \( \mathcal{O}_U \) as an \( \mathcal{O}_{E_1} \)-module. If we denote the vector bundle projection of \( E_1 \) by \( \pi : E_1 \rightarrow U \), the tautological section \( \mathcal{O}_{E_1} \rightarrow \pi^*E_1 \) induces a complex

\[
0 \longrightarrow \wedge^r \pi^*E_1' \longrightarrow \cdots \longrightarrow \wedge^2 \pi^*E_1' \longrightarrow \pi^*E_1' \longrightarrow \mathcal{O}_{E_1} \longrightarrow 0
\]
where \( r = \text{rank}(E_1) \). As is well known, this complex is a locally free resolution of \( \mathcal{O}_U \) on \( E_1 \), and \( \text{Tor}^E_1(\mathcal{O}_{C_1}, \mathcal{O}_U) \) is the \((-i)\)-th cohomology of the complex

\[
0 \to \wedge^r \pi^* E_1^\vee \otimes \mathcal{O}_{E_1} \mathcal{O}_{C_1} \to \cdots \to \wedge^1 \pi^* E_1^\vee \otimes \mathcal{O}_{E_1} \mathcal{O}_{C_1} \to \mathcal{O}_{C_1} \to 0.
\]

In this section, we prove the following.

**Proposition 3.2.** Let \( \sigma_U : \text{Ob}_U \to \mathcal{O}_U \) be a surjective homomorphism. Then the virtual structure sheaf \([\mathcal{O}^\text{vir}_U] \in K_0(U)\) is zero.

**Proof.** Let \( E'_1 \) be the kernel of the surjective homomorphism \((3.1)\) so that we have an exact sequence of vector bundles

\[
0 \to E'_1 \to E_1 \to \mathcal{O}_U \to 0
\]

which induces an exact sequence of vector bundles

\[
(3.3) 0 \to \wedge^i E'_1^\vee \to \wedge^i E_1^\vee \to \wedge^i E'_1^\vee \to 0
\]

for each \( i \), where the superscript \( \vee \) denotes the dual vector bundle. Recall that the normal cone \( C_1 \) has support in \( E'_1 \) by [19]. Pulling back \((3.3)\) by \( \pi \) and tensoring with \( \mathcal{O}_{C_1} \) give us a short exact sequence

\[
(3.4) 0 \to \wedge^i \pi^* E_1^\vee \otimes \mathcal{O}_{E_1} \mathcal{O}_{C_1} \to \wedge^i E_1^\vee \otimes \mathcal{O}_{E_1} \mathcal{O}_{C_1} \to 0
\]

of complexes of coherent sheaves on \( C_1 \subset E'_1 \). Then we have

\[
\wedge^i \pi^* E_1^\vee \otimes \mathcal{O}_{C_1} \cong (\wedge^i \pi^* E_1^\vee \otimes \mathcal{O}_{E_1}) \otimes \mathcal{O}_{C_1} = \wedge^i \pi^* E'_1^\vee \otimes \mathcal{O}_{C_1}
\]

where \( \pi' : E'_1 \to U \) is the bundle projection. Let

\[
A_j = \text{Tor}^E_1(\mathcal{O}_U, \mathcal{O}_{C_1}) \quad \text{and} \quad A'_j = \text{Tor}^{E'_1}(\mathcal{O}_U, \mathcal{O}_{C_1})
\]

so that

\[
(3.5) [\mathcal{O}^\text{vir}_U] = \sum_j (-1)^j A_j.
\]

The long exact sequence associated to \((3.4)\) gives us the exact sequence

\[
(3.6) 0 \to A_r \to A'_r \to A'_{r-1} \to \cdots \to A'_0 \to A_1 \to A'_1 \to 0
\]

and an isomorphism \( A_0 = A'_0 \) which imply

\[
\sum_j (-1)^j A_j = \sum_j (-1)^j A'_{j-1} + \sum_j (-1)^j A'_j = 0.
\]

Therefore we have the vanishing

\[
(3.7) [\mathcal{O}^\text{vir}_U] = \sum_j (-1)^j A_j = 0 \in K_0(U).
\]

This proves the proposition. \(\square\)
In many natural situations, the cosection \( \sigma \) is not surjective. In the subsequent sections, we will show that if we let \( X(\sigma) \) denote the zero locus of a cosection \( \sigma : \text{Ob}_X \rightarrow \mathcal{O}_X \) of the obstruction sheaf \( \text{Ob}_X \) on a Deligne-Mumford stack \( X \), the virtual structure sheaf \( \mathcal{O}_X^{\text{vir}} \) localizes to the substack \( X(\sigma) \).

4. Cosection localized Gysin map

Let \( X \) be a Deligne-Mumford stack and \( E_1 \) be a vector bundle on \( X \). Let \( \sigma : E_1 \rightarrow \mathcal{O}_X \) be a homomorphism of \( \mathcal{O}_X \)-modules and \( X(\sigma) \) be the zero locus of \( \sigma \), i.e. the closed substack defined by the ideal sheaf \( \sigma(E_1) \subset \mathcal{O}_X \).

Let \( U = X - X(\sigma) \) and let \( E_1' \) denote the closure in \( E_1 \) of \( \ker(\sigma : E_1|_U \rightarrow \mathcal{O}_U) \), where the latter is a subbundle of \( E_1|_U \) of corank 1. Let

\[
E_1(\sigma) := E_1|_{X(\sigma)} \cup E_1'.
\]

The purpose of this section is to construct the following cosection localized Gysin map.

**Theorem 4.1.** Under the notation as above, we have a homomorphism

\[
0^!_{E_1, \sigma} : K_0(E_1(\sigma)) \rightarrow K_0(X(\sigma))
\]

which satisfies

\[
i_* \circ 0^!_{E_1, \sigma} = 0^!_{E_1} \circ j_* : K_0(E_1(\sigma)) \rightarrow K_0(X)
\]

where \( i : X(\sigma) \rightarrow X \) and \( j : E_1(\sigma) \rightarrow E_1 \) denote the inclusion maps while

\[
0^!_{E_1}[F] = [\mathcal{O}_X \otimes_{\mathcal{O}_{E_1}}^L F] = \sum_i (-1)^i \text{Tor}^E_1_i(\mathcal{O}_X, F).
\]

**Proof.** Let \( \rho : \tilde{X} \rightarrow X \) be the blowup of \( X \) along \( X(\sigma) \) so that we have a surjective homomorphism

\[
\tilde{\sigma} : \tilde{E}_1 \rightarrow \mathcal{O}_{\tilde{X}}(D)
\]

where \( \tilde{E}_1 = \rho^* E_1 \) and \( -D \) is the exceptional divisor. Let \( \tilde{E}_1' \) be the kernel of \( \tilde{\sigma} \). Let \( \rho' : D \rightarrow X(\sigma) \) denote the restriction of \( \rho \) to \( D \) and \( \tilde{\rho} : \tilde{E}_1' \rightarrow E_1(\sigma) \) be the map induced by the projection

\[
\tilde{E}_1 = E_1 \times_X \tilde{X} \rightarrow E_1.
\]

Let \( \tilde{\rho}' : \tilde{E}_1'|_D \rightarrow E_1|_{X(\sigma)} \) denote the restriction of \( \tilde{\rho} \) to \( \tilde{E}_1'|_D \) and let

\[
i : E_1|_{X(\sigma)} \rightarrow E_1(\sigma) \quad \text{and} \quad i' : \tilde{E}_1'|_D \rightarrow \tilde{E}_1'
\]

denote the inclusion maps.
For any coherent sheaf $F$ on $E_1(\sigma)$, let $\tilde{F}$ be any coherent sheaf on $\tilde{E}_1$ such that there is an epimorphism
\begin{equation}
(4.6) \quad \tilde{\rho}^* F \longrightarrow \tilde{F}
\end{equation}
and that
\[ \tilde{F}|_{\tilde{E}_1|_{\tilde{X}-D}} = F|_{E_1|_{U}} \]
where we identified $\tilde{E}_1|_{\tilde{X}-D}$ with $E_1|_{U}$. For instance, we may choose $\tilde{F} = \tilde{\rho}^* F$.

By adjunction together with (4.6), we have homomorphisms
\[ \eta_{F, \tilde{F}} : F \longrightarrow \tilde{\rho}_* \tilde{\rho}^* F \longrightarrow \tilde{\rho}_* \tilde{F}. \]
Since $\tilde{F}$, $\tilde{\rho}_* \tilde{\rho}^* F$ and $\tilde{\rho}_* \tilde{F}$ all coincide over $U$, we have
\[ [\ker \eta_{F, \tilde{F}}] - [\operatorname{coker} \eta_{F, \tilde{F}}] \in K_0(E_1|_{X(\sigma)}). \]
Since $\tilde{\rho}$ is an isomorphism over $U$, we have
\[ [R^i \tilde{\rho}_* \tilde{F}] \in K_0(E_1|_{X(\sigma)}) \]
for each $i \geq 1$. We let
\begin{equation}
(4.7) \quad R_{F, \tilde{F}} := [\ker \eta_{F, \tilde{F}}] - [\operatorname{coker} \eta_{F, \tilde{F}}] - \sum_{i \geq 1} (-1)^i [R^i \tilde{\rho}_* \tilde{F}] \in K_0(E_1|_{X(\sigma)}).
\end{equation}

We now define the cosection localized Gysin map by
\begin{equation}
(4.8) \quad 0^!_{E_1, \sigma}[F] := \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{F}) + 0^!_{E_1|_{X(\sigma)}, \tilde{X}} R_{F, \tilde{F}} \in K_0(X(\sigma))
\end{equation}
where $D : K_0(\tilde{X}) \rightarrow K_0(D)$ denotes the Gysin map
\[ D \cdot [F] = [\mathcal{O}_D \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{F}] = \sum_i (-1)^i [\operatorname{Tor}_i^X (\mathcal{O}_D, \mathcal{F})] \]
for the Cartier divisor $D$ of $\tilde{X}$. To complete the proof, we will show the following:

(i) (4.3) is independent of the choice of $\tilde{F}$.

(ii) If $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is an exact sequence of coherent sheaves on $E_1(\sigma)$, then
\begin{equation}
(4.9) \quad 0^!_{E_1, \sigma}[F] = 0^!_{E_1, \sigma}[F'] + 0^!_{E_1, \sigma}[F''].
\end{equation}

(iii) (4.3) holds.

To prove (i), since all $\tilde{F}$ are quotients of $\tilde{\rho}^* F$, it suffices to show
\begin{equation}
(4.10) \quad \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{\rho}^* F) + 0^!_{E_1|_{X(\sigma)}, \tilde{X}} R_{F, \tilde{F}} = \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{F}) + 0^!_{E_1|_{X(\sigma)}, \tilde{X}} R_{F, \tilde{F}}.
\end{equation}

Let $G$ be the kernel of (4.6) so that we have an exact sequence
\[ 0 \longrightarrow \tilde{\imath}_L^* G \longrightarrow \tilde{\rho}^* F \longrightarrow \tilde{F} \longrightarrow 0 \]
with $[G] \in K_0(\tilde{E}_1|_{\tilde{D}})$. Then we have
\[ \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{\rho}^* F) - \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{F}) = \rho'_*(D \cdot 0^!_{\tilde{E}_1, \tilde{X}} \tilde{\imath}_L^* G) \]
and break it into exact sequences as follows:

\[
\text{From (4.16), we have a commutative diagram of exact sequence s}
\]

\[
\text{(4.19)}
\]

\[
\text{(4.20)}
\]

\[
\text{and} \, (4.13)
\]

\[
R_F, \bar{\rho}^* F = R_{F, \bar{F}} - \bar{\rho}'_s[|G] \in K_0(E_1|X(\sigma)).
\]

To prove (4.13), we consider the long exact sequence

\[
0 \to \bar{\rho}'_s G \to \bar{\rho}'_s \bar{\rho}^* F \to \bar{\rho}'_s \bar{\rho}^* F \to R^1 \bar{\rho}'_s G \to R^1 \bar{\rho}'_s \bar{\rho}^* F \to R^1 \bar{\rho}'_s \bar{\rho}^* F \to \cdots
\]

and break it into exact sequences as follows:

\[
0 \to \bar{\rho}'_s G \to \bar{\rho}'_s \bar{\rho}^* F \to H \to 0
\]

\[
0 \to H \to \bar{\rho}'_s \bar{\rho}^* F \to L \to 0
\]

\[
0 \to L \to R^1 \bar{\rho}'_s G \to R^1 \bar{\rho}'_s \bar{\rho}^* F \to R^1 \bar{\rho}'_s \bar{\rho}^* F \to \cdots
\]

Note that (4.17) consists only of sheaves on $E_1|X(\sigma)$ and we have an equality

\[
\sum_{i \geq 1} (-1)^i[R^i \bar{\rho}' F] = \sum_{i \geq 1} (-1)^i[R^i \bar{\rho}' F] + \sum_{i \geq 1} (-1)^i[R^i \bar{\rho}'_s G] + [L] \in K_0(E_1|X(\sigma)).
\]

From (4.15), we have a commutative diagram of exact sequences

\[
\begin{array}{c}
0 \to 0 \to F \xrightarrow{id} F \to 0 \\
0 \to \bar{\rho}'_s G \to \bar{\rho}'_s \bar{\rho}^* F \to H \to 0
\end{array}
\]

which gives an exact sequence

\[
0 \to \ker(\eta_{F, \bar{\rho}^* F}) \to \ker(\nu) \to \bar{\rho}'_s G \to \coker(\eta_{F, \bar{\rho}^* F}) \to \coker(\nu) \to 0
\]

of coherent sheaves on $E_1|X(\sigma)$. Hence we have

\[
\text{(4.19)}
\]

From (4.16), we have a commutative diagram of exact sequences

\[
\begin{array}{c}
0 \to F \xrightarrow{id} F \to 0 \\
0 \to H \to \bar{\rho}'_s \bar{\rho}^* F \to L \to 0
\end{array}
\]

which gives an exact sequence

\[
0 \to \ker \nu \to \ker(\eta_{F, \bar{\rho}^* F}) \to 0 \to \coker \nu \to \coker(\eta_{F, \bar{\rho}^* F}) \to L \to 0
\]

of coherent sheaves on $E_1|X(\sigma)$. Hence we have

\[
\text{(4.20)}
\]

\[
\text{[ker}(\eta_{F, \bar{\rho}^* F})] - [\coker(\eta_{F, \bar{\rho}^* F})] = [\ker(\nu)] - [\coker(\nu)] - [\bar{\rho}'_s G] \in K_0(E_1|X(\sigma)).
\]
Upon adding \((4.19)\) with \((4.20)\) and subtracting \((4.18)\), we obtain
\begin{equation}
R_{F,\hat{\rho}^*F} = R_{F,\tilde{\rho}^*F} - \hat{\rho}_*\{G\} \in K_0(E_1|_{X(\sigma)})
\end{equation}
as desired. This proves (i).

Next we prove (ii). Let \(0 \to F' \to F \to F'' \to 0\) be an exact sequence of coherent sheaves on \(E_1(\sigma)\). Since \(\hat{\rho}^*\) is right exact, we have an exact sequence
\begin{equation}
\hat{\rho}^*F' \to \hat{\rho}^*F \to \hat{\rho}^*F'' \to 0.
\end{equation}
Let \(\tilde{F}'\) be the image of the first arrow and let
\begin{equation}
\tilde{F} = \hat{\rho}^*F, \quad \tilde{F}'' = \hat{\rho}^*F''
\end{equation}
so that we have an exact sequence
\begin{equation}
0 \to \tilde{F}' \to \tilde{F} \to \tilde{F}'' \to 0
\end{equation}
of coherent sheaves on \(\tilde{E}_1\). When applied to the long exact sequence
\begin{equation}
0 \to \hat{\rho}_*\tilde{F}' \to \hat{\rho}_*\tilde{F} \to \hat{\rho}_*\tilde{F}'' \to R^1\hat{\rho}_*\tilde{F}' \to R^1\hat{\rho}_*\tilde{F} \to R^1\hat{\rho}_*\tilde{F}'' \to \cdots
\end{equation}
associated to \((4.22)\), the same argument that we used above to prove \((4.21)\) gives us the equality
\begin{equation}
R_{F',\tilde{F}'} + R_{F'',\tilde{F}''} = R_{F,\tilde{F}} \in K_0(E_1|_{X(\sigma)}).
\end{equation}
On the other hand, \((4.22)\) gives us the equality \([\tilde{F}] = [\tilde{F}'] + [\tilde{F}''] \in K_0(\tilde{E}_1)\) and hence
\begin{equation}
\rho'_*(D \cdot 0^1_{E_1}) = \rho'_*(D \cdot 0^1_{E_1}) + \rho'_*(D \cdot 0^1_{E_1}) \in K_0(X(\sigma)).
\end{equation}
Combining \((4.23)\) and \((4.24)\), we obtain \((4.9)\).

Finally we prove \((4.3)\). By its definition \((4.7)\),
\begin{equation}
\hat{j}_*R_{F,\tilde{F}} = [F] - \rho_*[\tilde{F}] \in K_0(E_1)
\end{equation}
where \(j = j \circ \hat{i} : E_1|_{X(\sigma)} \to E_1\) is the inclusion map. Hence
\begin{equation}
\iota_*0_{E_1|_{X(\sigma)}}^1 R_{F,\tilde{F}} = 0_{E_1}^1 j_*R_{F,\tilde{F}} = 0_{E_1}^1 [F] - 0_{E_1}^1 \rho_*[\tilde{F}] \in K_0(X).
\end{equation}
On the other hand, we have
\begin{equation}
\iota_*\rho'_*(D \cdot 0^1_{E_1}) = \rho_*\iota_*\rho_*[\tilde{F}] = \rho_*\iota_*\rho'_*(D \cdot 0^1_{E_1}) + \rho_*\iota_*\rho'_*(D \cdot 0^1_{E_1}) = \rho_*\iota_*\rho_*[\tilde{F}]
\end{equation}
where \(\iota : D \to \tilde{X}\) is the inclusion. Now \((4.3)\) follows from \((4.25)\), \((4.26)\) and \((4.5)\). This completes the proof. \(\square\)

The following is a basic example.
Example 4.2. Let $X$ be a smooth variety of dimension $n$ and $E$ be a vector bundle on $X$ of rank $n$. Let $\sigma : E \to \mathcal{O}_X$ be a cosection whose zero locus $X(\sigma)$ consists of one simple point $p$. Then

\[(4.27) \quad 0^!_{E,\sigma}[\mathcal{O}_X] = (-1)^n[\mathcal{O}_p] \in K_0(\{p\}).\]

To see it, let $\rho : \tilde{X} \to X$ be the blowup of $X$ at $p$. Let $\rho^*E \cong \mathbb{P}^{n-1}$ be the exceptional divisor so that we have an exact sequence

\[0 \to \tilde{E}^i \to \rho^*E \to \mathcal{O}_{\tilde{X}}(D) \to 0\]

whose restriction to the exceptional divisor is

\[0 \to \tilde{E}^i|_{\mathbb{P}^{n-1}} \to \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus n} \to \mathcal{O}_{\mathbb{P}^{n-1}}(1) \to 0\]

so that $\tilde{E}^i|_{\mathbb{P}^{n-1}} = T^\vee_{\mathbb{P}^{n-1}}(1)$. By the Whitney sum formula, $c_{n-1}(\tilde{E}^i|_{\mathbb{P}^{n-1}}) = (-1)^{n-1}c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}$ and hence

\[D \cdot 0^!_{E,\sigma}[\mathcal{O}_{\tilde{X}}] = -\mathbb{P}^{n-1} \cdot 0^!_{E^i}[\mathcal{O}_{\tilde{X}}] = (-1)^n[\mathcal{O}_p].\]

Since $\rho^*\mathcal{O}_X = \mathcal{O}_{\tilde{X}}$, $\rho_*\rho^*\mathcal{O}_X = \rho_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i\rho_*\mathcal{O}_{\tilde{X}} = 0$ for $i \geq 1$, we have $R\rho_*\mathcal{O}_{\tilde{X}} = 0$ and

\[0^!_{E,\sigma}[\mathcal{O}_X] = \rho'_*(D \cdot 0^!_{E^i}[\mathcal{O}_{\tilde{X}}]) = (-1)^n[\mathcal{O}_p] \in K_0(\{p\})\]

by (4.8).

5. Cosection localized virtual structure sheaf

Let $X$ be a Deligne-Mumford stack. Let $\phi : E \to \mathbb{L}_X$ be a perfect obstruction theory and $\sigma : \text{Ob}_X = h^1(E^\vee) \to \mathcal{O}_X$ be a homomorphism, called a cosection of the obstruction sheaf $\text{Ob}_X$. Let $X(\sigma)$ be the zero locus of $\sigma$, i.e. the closed substack of $X$ defined by the ideal sheaf $\mathcal{O}_{\text{Ob}_X} \subset \mathcal{O}_X$.

We assume that $E$ admits a global resolution

\[E^{-1} \to E^0\]

by a 2-term complex of locally free sheaves on $X$. The intrinsic normal cone $\mathcal{E}_X$ is canonically embedded into the intrinsic normal sheaf $\mathcal{R}_X$ which in turn embeds into

\[\mathcal{E}_X = h^1/h^0(E^\vee) = [E_1/E_0]\]

by $h^1/h^0(\phi^\vee)$ where $E_i$ is the dual of $E^{-i}$. The fiber product

\[\begin{array}{ccc}
C_1 & \longrightarrow & E_1 \\
\downarrow & & \downarrow \\
\mathcal{E}_X & \longrightarrow & \mathcal{E}_X
\end{array}\]

defines a cone $C_1$ in $E_1$ and we have

\[[X]^{\text{vir}} = 0^!_{\mathcal{E}_X}[\mathcal{E}_X] = 0^!_{E_1}[C_1] \in A_*(X),\]
\[ [\mathcal{O}^{\text{vir}}_{X}] = 0^!_{E_1}[\mathcal{O}_{C_1}] = [\mathcal{O}_X \otimes_{\mathcal{O}_{E_1}}^{L} \mathcal{O}_{C_1}] \in K_0(X). \]

In this section, we prove the following.

**Theorem 5.1.** Suppose a Deligne-Mumford stack \( X \) is equipped with a perfect obstruction theory \( \phi : E \to \mathbb{L}_X \) and a cosection \( \sigma : \text{Ob}_X \to \mathcal{O}_X \) whose zero locus is denoted by \( X(\sigma) \). Assume \( E \) admits a global resolution \([E^{-1} \to E^0]\) by locally free sheaves. We define the cosection localized virtual structure sheaf to be

\[ [\mathcal{O}^{\text{vir}}_{X, \text{loc}}] = 0^!_{E_1, \varphi}[\mathcal{O}_{C_1}] \in K_0(X(\sigma)) \]

where \( \varphi : E_1 \to h^1(E^\vee) = \text{Ob}_X \to \mathcal{O}_X \) is the composition of the quotient map \( E_1 \to h^1(E^\vee) \) with \( \sigma \). It satisfies

\[ \iota_*[\mathcal{O}^{\text{vir}}_{X, \text{loc}}] = [\mathcal{O}^{\text{vir}}_{X}] \in K_0(X) \]

where \( \iota : X(\sigma) \to X \) denotes the inclusion.

We will further show that the cosection localized virtual structure sheaf is deformation invariant and independent of all the choices such as a resolution \([E^{-1} \to E^0]\) of the perfect obstruction theory on \( X \).

**Proof.** We use the notation of Theorem 4.1. By [19, §4], the cone \( C_1 \subset E_1 \) has (reduced) support in \( E_1(\sigma) \) defined in (4.4). By Proposition 2.1, \( \mathcal{O}_{C_1} \) gives us a class \( [\mathcal{O}_{C_1}] \in K_0(E_1(\sigma)) \).

Applying the cosection localized Gysin map constructed in Theorem 4.1 we define the cosection localized virtual structure sheaf by

\[ [\mathcal{O}^{\text{vir}}_{X, \text{loc}}] = 0^!_{E_1, \varphi}[\mathcal{O}_{C_1}] \in K_0(X(\sigma)) \]

where \( \varphi \) is the composition \( E_1 \to h^1(E^\vee) = \text{Ob}_X \xrightarrow{\sigma} \mathcal{O}_X \).

We prove (5.2). Since \( [\mathcal{O}^{\text{vir}}_{X}] = 0^!_{E_1}[\mathcal{O}_{C_1}] \), by (4.3), we have

\[ \iota_*[\mathcal{O}^{\text{vir}}_{X, \text{loc}}] = \iota_* 0^!_{E_1, \varphi}[\mathcal{O}_{C_1}] = 0^!_{E_1}[\mathcal{O}_{C_1}] = [\mathcal{O}^{\text{vir}}_{X}] \in K_0(X) \]

as desired. \( \square \)

**Proposition 5.2.** The cosection localized structure sheaf \([\mathcal{O}^{\text{vir}}_{X, \text{loc}}] \in K_0(X(\sigma))\) is independent of the choice of a global resolution \([E^{-1} \to E^0]\) of the perfect obstruction theory \( E \) of \( X \).

**Proof.** By the proof of [2] Proposition 5.3], it suffices to consider the case where the two resolutions are related by surjective homomorphisms

\[
\begin{array}{ccc}
\tilde{E}^\vee & \xrightarrow{\tilde{E}_0} & \tilde{E}_1 \\
\downarrow & & \downarrow \\
E^\vee & \xrightarrow{E_0} & E_1.
\end{array}
\]

If we let \( C_1 \subset E_1 \) be the pullback of the intrinsic normal cone \( \mathcal{C}_X \subset E_1/E_0 \) of \( X \) to \( E_1 \), then \( \tilde{C}_1 = C_1 \times_{E_1} \tilde{E}_1 \) is the pullback of \( \mathcal{C}_X \) to \( \tilde{E}_1 \). Then
\[ [\mathcal{O}_{C_1}] = q^* [\mathcal{O}_{C_1}] \]
where \( q : \tilde{E}_1 \to E_1 \) denotes the projection. By the definition of \( \theta_{E_1, \sigma} \) above, it is straightforward to see that
\[ \theta_{E_1, \tilde{\sigma}}[\mathcal{O}_{C_1}] = \theta_{E_1, \tilde{\sigma}}[\mathcal{O}_{C_1}] \]
where \( \tilde{\sigma} \) is the composition \( \tilde{E}_1 \to E_1 \xrightarrow{\sigma} \mathcal{O}_X \). \( \square \)

**Remark 5.3.** Exactly the same holds in the algebraic cobordism group of \( X \) instead of the algebraic K-group \( K_0(X) \). See \cite{35} for a virtual fundamental class in the algebraic cobordism group of \( X(\sigma) \). We have the cosection localized virtual class \( [X]^{\text{vir}}_{X, \text{loc}} \) in the algebraic cobordism group of \( X(\sigma) \). Details will appear in a subsequent paper.

**Definition 5.4.** Let \( X \) be a Deligne-Mumford stack equipped with a perfect obstruction theory \( \phi : E \to \mathbb{L}_X \) and a cosection \( \sigma : \text{Ob}_X = h^1(E^\vee) \to \mathcal{O}_X \). Let \( [\mathcal{O}_{X, \text{loc}}^{\text{vir}}] \in K_0(X(\sigma)) \) be the cosection localized virtual structure sheaf of \( X \). Suppose the zero locus \( X(\sigma) \) of the cosection \( \sigma \) is proper. The cosection localized virtual Euler characteristic of a class \( \beta \in K_0(X) \) is defined as
\[ (5.4) \chi_{\text{loc}}^{\text{vir}}(X, \beta) = \chi(X(\sigma), \beta \cdot \mathcal{O}_{X, \text{loc}}^{\text{vir}}) = \sum_i (-1)^i \dim H^i(X(\sigma), \beta \cdot \mathcal{O}_{X, \text{loc}}^{\text{vir}}). \]

We next prove that the cosection localized structure sheaf is deformation invariant. Let \( X \to S \) be a morphism of stacks, where \( X \) is a Deligne-Mumford stack and \( S \) is a smooth Artin stack. Let \( v : Z \to W \) be a regular embedding of schemes that fits into a Cartesian square
\[
\begin{array}{ccc}
Y & \to & X \\
\downarrow & & \downarrow \\
Z & \to & W
\end{array}
\]
Suppose we have relative perfect obstruction theories \( \phi : E \to \mathbb{L}_{X/S} \) and \( \phi' : E' \to \mathbb{L}_{Y/S} \) that fit into a morphism of distinguished triangles
\[
(5.5)
\begin{array}{ccc}
E|_Y & \to & E' & \to & N_{Z/W|Y}[1] \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \xrightarrow{\cong} \\
\mathbb{L}_{X/S}|_Y & \to & \mathbb{L}_{Y/S} & \to & \mathbb{L}_{Y/X} \xrightarrow{+1},
\end{array}
\]
where \( N_{Z/W} \) is the normal bundle of \( Z \) in \( W \).

As above, we assume the (relative) perfect obstruction theory \( E \) admits a global resolution
\[ (5.6) [E^{-1} \to E^0] \]
by locally free sheaves \( E^{-1} \) and \( E^0 \) on \( X \). Since \( E' \) is the cone of the morphism \( N_{Z/W|Y}^{\vee} \to E|_Y \), \( E' \) has the global resolution
\[ (5.7) [E^{-1}|_Y \oplus N_{Z/W|Y}^{\vee} \to E^0|_Y]. \]
From the distinguished triangle $\mathbb{L}_S|_X \rightarrow \mathbb{L}_X \rightarrow \mathbb{L}_{X/S} \rightarrow \cdots$, we find that

\[
E^{-1} \rightarrow E^0 \oplus T^y_S|_X
\]

is an (absolute) perfect obstruction theory of $X$. Similarly,

\[
E^{-1}|_Y \oplus N^y_{Z/W}|_Y \rightarrow E^0|_Y \oplus T^y_S|_Y
\]

is an (absolute) perfect obstruction theory of $Y$. Recall that the (absolute) obstruction sheaf $\text{Ob}_X$ of $X$ is the cokernel of the dual of (5.8) and the obstruction sheaf $\text{Ob}_Y$ of $Y$ is that of the dual of (5.9). Hence we have an exact sequence

\[
N_{Z/W}|_Y \rightarrow \text{Ob}_Y \rightarrow \text{Ob}_X|_Y \rightarrow 0.
\]

Let $\sigma : \text{Ob}_X \rightarrow \mathcal{O}_X$ be a cosection for $X$ and let

$\sigma' : \text{Ob}_Y \rightarrow \text{Ob}_X|_Y \xrightarrow{\sigma|_Y} \mathcal{O}_Y$

be the induced cosection for $Y$. Let $\mathcal{O}^\text{vir}_{X,\text{loc}}$ (resp. $\mathcal{O}^\text{vir}_{Y,\text{loc}}$) be the cosection localized virtual structure sheaf of $X$ (resp. $Y$) with respect to the perfect obstruction theory (5.8) of $X$ (resp. (5.9) of $Y$). Let $X(\sigma)$ (resp. $Y(\sigma)$) denote the zero locus of the cosection $\sigma$ and (resp. $\sigma'$). Then by construction, $Y(\sigma) = X(\sigma) \times_X Y$ which fits into the Cartesian square

\[
\begin{array}{ccc}
Y(\sigma) & \xrightarrow{t} & X(\sigma) \\
\downarrow & & \downarrow \\
Z & \xrightarrow{u} & W,
\end{array}
\]

since $Y = Z \times_W X$. We then have the Gysin map

$\nu^! : K_0(X(\sigma)) \rightarrow K_0(Y(\sigma))$

defined by

\[
\nu^![\mathcal{O}_A] = 0^!_{N_{Z/W}([\mathcal{O}_{C_Y(\sigma) \cap A}/A]} \in K_0(Y(\sigma)), \quad A \subset X(\sigma).
\]

In fact, if we choose a finite locally free resolution $\mathcal{O}^W_Z$ of $v_*\mathcal{O}_Z$ on $W$, then the Gysin map $\nu^!$ is given by the tensor product

\[
\nu^![F] = [\mathcal{O}^W_Z|_{X(\sigma)} \otimes_{\mathcal{O}_{X(\sigma)}} F], \quad F \in K_0(X(\sigma))
\]

by [29, Remark 1].

To avoid discussion about K-theory of Artin stacks, let us assume that there is a smooth morphism $M \rightarrow S$ and a closed embedding $X \subset M$ over $S$. This is always possible if, for instance, $X \rightarrow S$ is quasi-projective.

**Proposition 5.5.** *Under the above assumptions, we have*

$\nu^![\mathcal{O}^\text{vir}_{X,\text{loc}}] = [\mathcal{O}^\text{vir}_{Y,\text{loc}}] \in K_0(Y(\sigma)).$
Proof. The proof essentially follows from that of \cite[Theorem 5.2]{19}, using the work of \cite{24}. We outline the arguments. By \cite{13}, there is a deformation $\mathcal{M} \to \mathbb{P}^1$ whose fibers are $\mathcal{M}$ except the central fiber $C_{X/M}$ which is the normal cone of $X$ in $\mathcal{M}$. In \cite[below (7)]{24}, the authors constructed a double deformation space over $\mathbb{P}^1 \times \mathbb{P}^1$, that is the deformation of $\mathcal{M}$ to the normal cone $C_{Y \times \mathbb{P}^1 / \mathcal{M}}$ of $Y \times \mathbb{P}^1$. We denote this double deformation space by $\mathcal{W}$.

By its construction, $\mathcal{W}$ is flat over $\mathbb{P}^1 \times \mathbb{P}^1 - \{(0,0)\}$, where $(0,0) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the special point having the following properties: The fiber of $\mathcal{W}$ over $(1,0) \in \mathbb{P}^1 \times \mathbb{P}^1$ is $C_{Y/M}$; the flat specialization along $\mathbb{P}^1 \times \{0\}$ (over $(0,0)$) is $C_{Y/C_{X/M}}$.

Because $X$ is of finite type, an easy argument shows that we can find a smooth birational model $U \to \mathbb{P}^1 \times \mathbb{P}^1$, isomorphic over $\mathbb{P}^1 \times \mathbb{P}^1 - \{(0,0)\}$, so that $\mathcal{W} \times \mathbb{P}^1 \times \mathbb{P}^1 (\mathbb{P}^1 \times \mathbb{P}^1 - \{(0,0)\}) \times \mathbb{P}^1 \times \mathbb{P}^1 U$ extends to a family $\tilde{\mathcal{W}} \subset \mathcal{W} \times \mathbb{P}^1 \times \mathbb{P}^1 U$, flat over $U$. Consequently, we can find a chain $\Sigma = \cup_{i=1}^n \Sigma_i$ of $\mathbb{P}^1$’s in $U$, lying over $\mathbb{P}^1 \times \{0\}$, and two points $a \in \Sigma_1$ and $b \in \Sigma_n$ so that $\tilde{\mathcal{W}}|_a = C_{Y/M}$, $\tilde{\mathcal{W}}|_b = C_{Y/C_{X/M}}$.

We thus obtain a rational equivalence

\[(5.13) \quad [C_{Y/M}] = [C_{Y/C_{X/M}}].\]

Recall that \eqref{5.8} and \eqref{5.9} are perfect obstruction theories for $X$ and $Y$ respectively. Let $E_i$ denote the dual of $E^{-i}$ for $i = 0, 1$. Since the intrinsic normal cone is $\mathcal{C}_X = C_{X/M} / T_M|_X$, the cone $C_X := C_1 = \mathcal{C}_X \times \mathcal{E}_X E_1$ equals

\[C_{X/M} \times_X (E_0 \oplus T_S|_X) / T_M|_X \subset E_1(\sigma)\]

where $\sigma : E_1 \to Ob_X \xrightarrow{\sigma} \mathcal{O}_X$ is the cosection of $E_1$ induced by $\sigma$. Hence we have

\[(5.14) \quad [\mathcal{O}^{vir}_{X, \text{loc}}] = 0^1_{E_1, \mathcal{Z}} [\mathcal{O}_{C_{X/M} \times_X (E_0 \oplus T_S|_X) / T_M|_X}] \in K_0(X(\sigma)).\]

By \eqref{5.10}, the restriction $\sigma' : E_1|_Y \to \mathcal{O}_Y$ of $\sigma$ factors through $\sigma' : Ob_Y \to \mathcal{O}_Y$ defined above. By definition, we have the equality $E_1(\sigma)|_Y = E_1|_Y(\sigma')$ and a Cartesian diagram

\[
\begin{array}{ccc}
E_1(\sigma)|_Y & \longrightarrow & E_1(\sigma) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Z & \xrightarrow{v} & W
\end{array}
\]
Similarly to the case of $X$ above, the cone $C_Y = \mathcal{C}_Y \times_{\mathcal{E}_Y} (E_1|_Y \oplus N_{Z/W}|_Y)$ equals

$$C_{Y/M} \times_Y (E_0|_Y \oplus T_S|_Y)/T_M|_Y \subset E_1(\mathcal{L})|_Y \oplus N_{Z/W}|_Y$$

so that

$$\begin{equation}
0^!_{E_1|_Y \mathcal{L}} = 0^!_{E_1|_Y \oplus N_{Z/W}|_Y} [O_{C_{Y/M} \times_Y (E_0|_Y \oplus T_S|_Y)/T_M|_Y}]
= 0^!_{E_1|_Y \oplus N_{Z/W}|_Y} [O_{C_{Y/M} \times_Y (E_0|_Y \oplus T_S|_Y)/T_M|_Y}]
\end{equation}$$

(5.15)

since $N_{Z/W}|_Y$ lies in the kernel of $\sigma'$ by (5.10).

It was proved in [19, §5] that the rational equivalence (5.13) takes place in the kernel of a cosection extended over the double deformation space. Therefore (5.15) equals

$$0^!_{E_1|_Y \mathcal{L}} = 0^!_{E_1|_Y \oplus N_{Z/W}|_Y} [O_{C_{X/M} \times_X (E_0|_Y \oplus T_S|_X)/T_M|_X}].$$

By Lemma 5.6 below, we have $0^!_{E_1|_Y \mathcal{L}} \circ v^! = v^! \circ 0^!_{E_1 \mathcal{L}}$. Hence, $[O^!_{Y|\text{loc}}]$ equals

$$v^! 0^!_{E_1|_Y \mathcal{L}} [O_{C_{X/M} \times X (E_0|_Y \oplus T_S|_X)/T_M|_X}] = v^! [O^!_{X|\text{loc}}]$$

by (5.14). This proves the proposition.

It remains to prove the following.

**Lemma 5.6.** With the notation as above, $0^!_{E_1|_Y \mathcal{L}} \circ v^! = v^! \circ 0^!_{E_1 \mathcal{L}}$.

**Proof.** Let $\rho : \tilde{X} \to X$ denote the blowup of $X$ along $X(\sigma)$ with exceptional divisor $-D$ so that we have an exact sequence

$$0 \to \tilde{E}_1 \to E_1 \to O_{\tilde{X}}(D) \to 0$$

where $\tilde{E}_1 = \rho^* E_1$. Let $A \subset E_1(\sigma)$ be an integral substack. If $A \subset E_1|_X(\sigma)$, the lemma simply says $v^! \circ 0^!_{E_1|_Y} = 0^!_{E_1|_Y} \circ v^!$ which is straightforward since both $v^!$ and $0^!_{E_1}$ are tensor products (cf. (5.12)). If $A \not\subset E_1|_X(\sigma)$, let $\tilde{A} \subset \tilde{E}_1$ be the proper transform of $A$ so that

$$0^!_{E_1|_Y} [O_A] = \rho^*(D \cdot 0^!_{\tilde{E}_1|_X} [O_{\tilde{A}}] + 0^!_{E_1|_X(\sigma)} [O_A] \to \rho_* O_{\tilde{A}}].$$

Since $v^! [O_A] = [O^W_{Z|E_1(\mathcal{L})} \otimes O_A]$ where $O^W_Z$ is a finite locally free resolution of $v_* O_Z$ on $W$, we have

$$0^!_{E_1|_Y \mathcal{L}} v^! [O_A] = \rho^*(D \cdot 0^!_{\tilde{E}_1|_X} [O^W_{Z|E_1} \otimes O_{\tilde{A}}] + 0^!_{E_1|_X(\sigma)} [O^W_{Z|E_1} |_{E_1|_X(\sigma)} \otimes [O_A] \to \rho_* O_{\tilde{A}}].$$

Since $O^W_Z$ is a complex of locally free sheaves on $W$ and the Gysin maps are tensor products, we can pull out $O^W_Z$ to obtain

$$0^!_{E_1|_Y \mathcal{L}} v^! [O_A] = O^W_{Z|X(\sigma)} \otimes \rho^*(D \cdot 0^!_{\tilde{E}_1|_X} [O_{\tilde{A}}] + 0^!_{E_1|_X(\sigma)} [O_A] \to \rho_* O_{\tilde{A}}] = 0^!_{E_1|_Y \mathcal{L}} v^! [O_A] = v^! 0^!_{E_1 \mathcal{L}} [O_A]$$

as desired. 

\qed
As an application of Proposition 5.5, we will prove a principle of conservation of numbers for the cosection localized virtual Euler characteristics. Let \( t \in \mathbb{P}^1 \) be a closed point and \( \pi : X \to \mathbb{P}^1 \) be a morphism of Deligne-Mumford stacks, equipped with a relative perfect obstruction theory

\[
\phi : E \to \mathbb{L}_{X/\mathbb{P}^1}.
\]

Let \( \sigma : Ob_{X/\mathbb{P}^1} = h^1(E^\vee) \to \mathcal{O}_X \) be a cosection of the relative obstruction sheaf whose zero locus is denoted by \( X(\sigma) \). We assume that \( X(\sigma) \) is proper over \( \mathbb{P}^1 \) although \( X \) may not be proper over \( \mathbb{P}^1 \). The restriction

\[
\phi_t : E_t \to \mathbb{L}_{X_t}
\]

of \( \phi \) to \( X_t = t \times_{\mathbb{P}^1} X \) with \( E_t = E|_{X_t} \) is a perfect obstruction theory of \( X_t \) and we have an absolute perfect obstruction theory

\[
\bar{\phi} : \bar{E} \to \mathbb{L}_X
\]

of \( X \) defined by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{\phi} & E \\
\| & \downarrow & \downarrow \\
\pi^*\mathbb{L}_{\mathbb{P}^1} & \xrightarrow{\bar{\phi}} & \mathbb{L}_X \\
\end{array}
\]

of distinguished triangles. The first row of (5.16) gives us an exact sequence

\[
\mathcal{O}_X \to Ob_{X/\mathbb{P}^1} = h^1(E^\vee) \to Ob_X = h^1(\bar{E}^\vee) \to 0.
\]

We further assume that the cosection \( \sigma : Ob_{X/\mathbb{P}^1} \to \mathcal{O}_X \) descends to a cosection

\[
\bar{\sigma} : Ob_X \to \mathcal{O}_X
\]

so that we have the cosection localized virtual structure sheaf \([\mathcal{O}^\text{vir}_{X_t,\text{loc}}] \in K_0(X(\bar{\sigma})).\) Likewise, the homomorphism \( \sigma_t : Ob_{X_t} \to \mathcal{O}_{X_t} \) induced by \( \sigma \) gives us the cosection localized virtual structure sheaf \([\mathcal{O}^\text{vir}_{X_t,\text{loc}}] \in K_0(X_t(\sigma_t))\).

**Corollary 5.7.** Under the above assumptions, for any \( \beta \in K^0(X(\bar{\sigma})) \)

\[
\chi^\text{vir}_{\text{loc}}(X_t, t^*_t \beta) = \chi(X_t(\sigma_t), \mathcal{O}^\text{vir}_{X_t,\text{loc}} \cdot t^*_t \beta) = \chi(X(\bar{\sigma}), t_* \mathcal{O}^\text{vir}_{X_t,\text{loc}} \cdot \beta)
\]

is independent of \( t \in \mathbb{P}^1 \), where \( t_* : X_t(\sigma_t) \hookrightarrow X(\bar{\sigma}) \).

**Proof.** By Proposition 5.5, we have

\[
t^!\mathcal{O}^\text{vir}_{X_t,\text{loc}} = [\mathcal{O}^\text{vir}_{X_t,\text{loc}}] \in K_0(X_t(\sigma_t))
\]

where \( t : \{ t \} \to \mathbb{P}^1 \) is the inclusion. Then we have

\[
t_* t^!\mathcal{O}^\text{vir}_{X_t,\text{loc}} = t_* t^!\mathcal{O}^\text{vir}_{X_t,\text{loc}} \in K_0(X(\bar{\sigma})).
\]

Since \([\mathcal{O}_t] \in K_0(\mathbb{P}^1)\) is independent of \( t \in \mathbb{P}^1 \), \( t_* t^!\mathcal{O}^\text{vir}_{X_t,\text{loc}} \) is independent of \( t \). Hence

\[
\chi(X(\bar{\sigma}), t_* \mathcal{O}^\text{vir}_{X_t,\text{loc}} \cdot \beta), \quad \beta \in K^0(X(\bar{\sigma}))
\]

is independent of \( t \) as desired. \( \square \)
Another way to prove Corollary 5.7 is to use a cosection localized Riemann-Roch as follows. For schemes, we have the following cosection localized version of virtual Grothendieck-Riemann-Roch (cf. [12, Theorem 3.3]).

**Theorem 5.8.** Let \( f : X \to Y \) be a morphism of schemes with \( Y \) smooth. Let \( \phi : E \to L \) be a perfect obstruction theory of \( X \) and \( \sigma : \text{Ob}_X \to \mathcal{O}_X \) be a cosection. Suppose the restriction \( f' : X(\sigma) \to Y \) of \( f \) to the zero locus \( X(\sigma) \) of the cosection \( \sigma \) is proper. Let \( T_X^{\text{vir}} = [E_0] - [E_1] \in K^0(X) \) be the virtual tangent bundle where \( [E_0 \to E_1] \) is the dual of \( E \). Then for \( \beta \in K^0(X) \), we have

\[
\text{ch}(f'_* (\beta \cdot O_{X,\text{vir}}^{\text{loc}})) \cdot \text{td}(T_Y) \cap [Y] = f'_* (\text{ch}(\beta) \cdot \text{td}(T_X^{\text{vir}}) \cap [X]^{\text{vir, loc}}).
\]

In particular, if \( Y \) is a point and \( X(\sigma) \) is proper, the cosection localized virtual Euler characteristic of \( \beta \) is

\[
\chi_{\text{loc}}^{\text{vir}}(X, \beta) = \int_{[X]^{\text{vir, loc}}} \text{ch}(\beta) \cdot \text{td}(T_X^{\text{vir}}).
\]

We omit the proof since we are not going to use it.

For a quasi-projective Deligne-Mumford stack \( X \), we choose a smooth projective Deligne-Mumford stack \( M \) and an immersion \( i : X \to M \). Then we have

\[
\chi_{\text{loc}}^{\text{vir}}(X_t, i^* \beta) = \chi(M, i_* (O_{X_t,\text{loc}}^{\text{vir}} \cdot \beta), \beta \in K^0(M).
\]

Applying the Kawasaki-Riemann-Roch for the smooth Deligne-Mumford stack \( M \) together with (5.18), we can express the right side of (5.20) in terms of cosection localized virtual integrals on inertia substacks \( X_\mu \). Since the cosection localized virtual fundamental classes are deformation invariant, we find that the left side of (5.20) is independent of \( t \).

6. **Application to K-theoretic Landau-Ginzburg/Calabi-Yau correspondence**

In this section, we apply the cosection localized virtual structure sheaf to a K-theoretic Landau-Ginzburg/Calabi-Yau correspondence.

6.1. **K-theoretic FJRW invariant.** The cosection localized virtual structure sheaf enables us to define the K-theoretic FJRW invariant in algebraic geometry. To simplify the discussion, we focus on the Fermat quintic case

\[
\sum_{i=1}^5 z_i^5 : \mathbb{C}^5 \to \mathbb{C}.
\]

Let \( S \) be the moduli space of 5-spin curves, i.e. triples \((C, L, p)\) of

1. a 1-dimensional projective Deligne-Mumford stack \( C \) with at worst nodal singularities whose stabilizer groups at nodes or marked points are \( \mathbb{Z}_5 \) which acts on a node \((zw = 0)\) as \( \zeta \cdot (z, w) = (\zeta z, \zeta^{-1} w) \) for \( \zeta^5 = 1 \),
2. a line bundle \( L \) on \( C \), and
(3) an isomorphism \( p : L^5 \to \omega_C^{\log} \).

Assume all the marked points are narrow, i.e. the action of the stabilizer group \( \mathbb{Z}_5 \) on the fiber of \( L \) at a marked point is not trivial.

Let \( X \) be the Deligne-Mumford stack of quadruples \((C, L, p, x)\) where \((C, L, p) \in S \) and \( x \in H^0(L) \otimes \mathbb{Z}_5 \). By [6, 7], there is a perfect obstruction theory on \( X \) and a cosection \( \sigma : \text{Ob}_X \to \mathcal{O}_X \) whose zero locus \( X(\sigma) \) is the proper Deligne-Mumford stack \( S \). As the relative obstruction theory of \( X/S \) can be presented by a two-term complex of vector bundles on \( X \), we can apply Theorem 5.1 to obtain the cosection localized virtual structure sheaf

\[
\left[ \mathcal{O}_{X, \text{loc}}^{\text{vir}} \right] \in K_0(S).
\]

To obtain numerical invariants, we take the Euler characteristic.

**Definition 6.1.** The K-theoretic Fan-Jarvis-Ruan-Witten invariant is defined by

\[
\chi_{\text{loc}}^{\text{vir}}(X, \beta) = \chi(S, \mathcal{O}_{X, \text{loc}}^{\text{vir}} \cdot \beta), \quad \beta \in K^0(S).
\]

Since \( S \) is a smooth Deligne-Mumford stack, the Kawasaki-Riemann-Roch theorem enables us to express the numerical K-theoretic FJRW invariant as (cosection localized) virtual integrals.

In [9], based on the Polishchuk-Vaintrob construction of Witten’s top Chern class [34], Chiodo constructed a K-theory class \( Ke(E^\vee, \tau^\vee) \in K_0(S) \) whose top Chern class

\[
c_{\text{top}}(E, \tau) = \text{ch}(Ke(E^\vee, \tau^\vee)) \text{td}(E)^{-1} \in A_*(S)
\]

coincides with the cosection localized virtual fundamental class \([X]^{\text{vir}}_{\text{loc}} \) in \( A_*(S) \) by [7, §5.2] and [9, Theorem 5.4.1]. We have the following comparison result.

**Proposition 6.2.** Chiodo’s K-theory class \( Ke(E^\vee, \tau^\vee) \) coincides with the cosection localized virtual structure sheaf \([\mathcal{O}_{X, \text{loc}}^{\text{vir}}] \) in \( K_0(S) \).

**Proof.** The same argument as in the proof of [7, Proposition 5.10] also proves this proposition. Indeed, since the pushforward and the localized Gysin map \( 0_{E^\vee_{1, \sigma}} \) commute, it suffices to prove the proposition on the blowup \( \tilde{X} \) of \( X \) along \( X(\sigma) \) where we have an exact sequence

\[
0 \longrightarrow \tilde{E}_1^t \longrightarrow \tilde{E}_1 \longrightarrow \mathcal{O}_{\tilde{X}}(D) \longrightarrow 0.
\]

Here \(-D\) denotes the exceptional divisor. By deforming this exact sequence to the split case \( \tilde{E}_1 = \tilde{E}_1^t \oplus \mathcal{O}_{\tilde{X}}(D) \), the proposition is reduced to the case of vanishing cosection \( \tilde{E}_1^t \to 0 \) and the case where cosection is the natural inclusion \( \mathcal{O}_{\tilde{X}}(D) \to \mathcal{O}_{\tilde{X}} \). Each of these cases is easy to check. \( \square \)
6.2. GSW model for K-theoretic Gromov-Witten invariant. Consider the Fermat quintic Calabi-Yau 3-fold
\[ Y = (\sum_{i=1}^{5} z_i^5 = 0) \subset \mathbb{P}^4. \]

Let \( Z \) be the deformation of \( \mathbb{P}^4 \) to the normal cone \( \mathcal{O}_Y(5) \) of \( Y \) in \( \mathbb{P}^4 \), i.e. \( Z \) is the complement of the proper transform of \( \{0\} \times \mathbb{P}^4 \) in the blowup of \( \mathbb{P}^1 \times \mathbb{P}^4 \) along \( \{0\} \times Y \). Let
\[ p : Z \longrightarrow \mathbb{P}^1 \times \mathbb{P}^4 \longrightarrow \mathbb{P}^1 \]
denote the composition. Then \( Z_t = t \times_{\mathbb{P}^1} Z \) is \( \mathbb{P}^4 \) for \( t \neq 0 \) and \( \mathcal{O}_Y(5) \) for \( t = 0 \).

Let \( N = \overline{M}_g(Y, d) \) (resp. \( M = \overline{M}_g(\mathbb{P}^4, d) \)) be the moduli space of stable maps to \( Y \) (resp. \( \mathbb{P}^4 \)). Let \( M^p = \overline{M}_g(\mathbb{P}^4, d)^p \) denote the moduli space of pairs \((f, p)\) where \((f : C \to \mathbb{P}^4) \in M \) and \( p \in H^0(f^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C) \).

Let \( X = \overline{M}_g(Z/\mathbb{P}^1, d) \) be the moduli space of stable maps of genus \( g \) and degree \( d \) to the fibers of \( p \). Let
\[ \pi : X = \overline{M}_g(Z/\mathbb{P}^1, d)^p \longrightarrow \mathbb{P}^1 \]
denote the moduli space of pairs \((f, p)\) where \((f : C \to Z) \in \tilde{X} \) and \( p \in H^0(f^*\mathcal{O}_{\mathbb{P}^1}(-5) \otimes \omega_C) \) (cf. \[6, \S 4.1\]). It is straightforward that \( X_t \) is \( M^p \) for \( t \neq 0 \) and the central fiber \( X_0 \) is the moduli space of triples \((f, s, p)\) where \((f : C \to Y) \in N = \overline{M}_g(Y, d) \), \( s \in H^0(f^*\mathcal{O}_Y(5)) \) and \( p \in H^0(f^*\mathcal{O}_Y(-5) \otimes \omega_C) \).

By \[6, \S 4\], all the assumptions of Proposition \[5.5\] are satisfied and hence we find that \( t!\mathcal{O}_{X_t, \text{loc}} = t!\mathcal{O}_{X_t, \text{loc}}^\text{vir} \in K_0(N) \) for all \( t \in \mathbb{P}^1 \) and
\[ \chi(N, \mathcal{O}_{M^p, \text{loc}}^{\text{vir}}) = \chi(N, \mathcal{O}_{X_0, \text{loc}}^{\text{vir}}). \]

Here \( X(\sigma) = N \times \mathbb{P}^1 \). On the other hand, the proof of \[6, \text{Theorem 5.7}\] together with Example \[5.2\] above proves
\[ \chi(N, \mathcal{O}_{X_0, \text{loc}}^{\text{vir}}) = (-1)^{5d-g+1} \chi(N, \mathcal{O}_{N}^{\text{vir}}) = (-1)^{5d-g+1} \chi^{\text{vir}}(N). \]

We therefore proved the following.

**Proposition 6.3.** The cosection localized virtual Euler characteristic
\[ \chi_{\text{loc}}^{\text{vir}}(\overline{M}_g(\mathbb{P}^4, d)^p) := \chi(N, \mathcal{O}_{M^p, \text{loc}}^{\text{vir}}) \]
of \( M^p \) is equal to the K-theoretic Gromov-Witten invariant \( \chi(N, \mathcal{O}_N^{\text{vir}}) \) of \( N \) up to sign by
\[ \chi_{\text{loc}}^{\text{vir}}(\overline{M}_g(\mathbb{P}^4, d)^p) = (-1)^{5d-g+1} \chi^{\text{vir}}(N). \]

The Landau-Ginzburg/Calabi-Yau correspondence predicts that the FJRW invariant of the Landau-Ginzburg model \[6.1\] is equivalent to the GW invariant of the Calabi-Yau 3-fold \( Y \) via variable changes, analytic continuations, and symplectic transformations. Proposition \[6.3\] may be useful for a K-theoretic LG/CY correspondence.
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