Abstract. Geometry of the solution space of the self-dual Yang-Mills (SDYM) equations in Euclidean four-dimensional space is studied. Combining the twistor and group-theoretic approaches, we describe the full infinite-dimensional symmetry group of the SDYM equations and its action on the space of local solutions to the field equations. It is argued that owing to the relation to a holomorphic analogue of the Chern-Simons theory, the SDYM theory may be as solvable as 2D rational conformal field theories, and successful nonperturbative quantization may be developed. An algebra acting on the space of self-dual conformal structures on a 4-space (an analogue of the Virasoro algebra) and an algebra acting on the space of self-dual connections (an analogue of affine Lie algebras) are described. Relations to problems of topological and $N=2$ strings are briefly discussed.

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1 Introduction

In the past two decades significant progress in understanding integrable, conformal and topological quantum field theories in two dimensions has been achieved. In many respects this progress was related to the existence of an infinite number of symmetries making it possible not only to describe the space of classical solutions to 2D field equations, but also to advance essentially the nonperturbative quantization of 2D theories. Among symmetry algebras of 2D models, the most important role is played by the Virasoro and affine Lie algebras (see e.g. [1]-[4]). The use of transformation groups, their orbits and representations forms the basis of the dressing transformation method [3]-[10] and of the Kyoto’s school approach [1, 1, 12] to solvable equations of 2D and 3D field theories.

In four dimensions there also exist integrable, conformal and topological field theories, and naturally the following question arises: Can the methods and results of 2D theories be transferred to 4D theories? On the whole, the answer is positive for 4D integrable and topological field theories. At the same time, the knowledge of 4D conformal field theories (CFT) beyond the trivial case of free field theories is much less explicit, and not so many exact results are obtained (see e.g. [13]-[16] and references therein). Usually, one connects this with the fact that, unlike the 2D case, the conformal group in four dimensions is finite-dimensional, and constraints arising from conformal invariance are not sufficient for a detailed description of 4D CFT’s. One of our aims is to demonstrate that for a special subclass of CFT’s – integrable 4D CFT’s – this is a wrong impression based on the consideration of only local (manifest) symmetries.

There actually exists only one nonlinear integrable model in 4D described by the self-dual Yang-Mills (SDYM) equations defined on a 4-manifold with the self-dual Weyl tensor [17]-[19]. This unique theory is conformally invariant, and it is usually considered as a 4D analogue of the 2D WZNW theory. It is expected that many results of 2D rational CFT’s can be extended to the SDYM theory. This was discussed, for instance, in [20, 21], where the quantization of the SDYM theory on Kähler manifolds was considered. We shall give additional arguments in favour of the conjecture that the SDYM model is a good starting point for the development of 4D quantum CFT’s.

The main purpose of our paper is to describe all symmetries of the SDYM equations and, in particular, algebras generalizing the Virasoro and affine Lie algebras to the 4D case. In contrast with the WZNW model, most symmetries of the SDYM model are nonlocal. These symmetries are local symmetries of a holomorphic analogue of the Chern-Simons theory on a 6D twistor space, and the SDYM theory is connected with this model via the nonlocal Penrose-Ward transform. The use of this correspondence makes it possible to simplify considerably the investigation of symmetries of the SDYM equations. The lift from a 4D self-dual space to its 6D twistor space is useful for understanding correct degrees of freedom and correct symmetries of the SDYM theory. We show that just as in the case of the 2D WZNW theory these symmetries completely define the space of local solutions to the field equations and therefore the quantization of the SDYM theory is connected with the construction of representations of a symmetry algebra.

Roughly speaking, the symmetry group of a system of differential equations is the group that maps solutions of this system into one another. From this point of view the transformation groups of type $\text{Map}(X^3; G)$ (maps: space $X^3 \rightarrow$ group $G$, $\dim_{\mathbb{R}} X^3 = 3$) considered in [24, 21] are not symmetry groups, since in general their action does not preserve the solution space. The above-mentioned groups $\text{Map}(X^3; G)$ can be considered as “off-shell” symmetry groups reflecting only the field content of the theory and acting on free fields. These groups are not connected with the integrability and can be introduced in a space-time of an arbitrary dimension (see e.g. [22]).

Study of “on-shell” infinitesimal symmetries of the SDYM equations (in 4D Euclidean space) began from the papers [24] and was continued in [24]-[33]. In [24] it has been shown that the obtained infinitesimal symmetries form the affine Lie algebra $\mathfrak{g} \otimes \mathbb{C} \{\lambda, \lambda^{-1}\}$ when the gauge potential $A = A_\mu dx^\mu$ takes values in the Lie algebra $\mathfrak{g}$ of a group $G$. Ueno and Nakamura [26] have shown that on the solution space of the SDYM equations it is possible to define an infinitesimal action of a larger Lie algebra of holomorphic maps from a domain on the twistor space $Z$ of $\mathbb{R}^4$ into the algebra $\mathfrak{g}$. Takasaki [27] described this algebra in terms of Sato’s approach to soliton equations. But Crane [28] showed that in general the group corresponding to the Ueno-Nakamura algebra does not preserve the solution space of the SDYM equations and indicated a vagueness of geometrical meaning of these transformations. The above-mentioned infinitesimal symmetries do not exhaust all symmetries of the SDYM equations, which has been shown in the papers [30]-[22] where Virasoro-type symmetries were described. In this paper, we describe the full symmetry group of the SDYM equations.
2 Self-duality and manifest symmetries

2.1 Definitions and notation

We consider the Euclidean space $\mathbb{R}^4$ with the metric $\delta_{\mu\nu}$, a gauge potential $A = A_\mu dx^\mu$ and the Yang-Mills field $F = dA + A \wedge A$ with components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\mu, \nu, ... = 1, ..., 4$, $\partial_\mu := \partial/\partial x^\mu$. The fields $A_\mu$ and $F_{\mu\nu}$ take values in the Lie algebra $\mathfrak{g}$ of an arbitrary semisimple compact Lie group $G$. We suppose that $G$ is a matrix group $G \subset GL(n, \mathbb{C})$.

The SDYM equations have the form

$$\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = F_{\mu\nu}, \quad (2.1)$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in $\mathbb{R}^4$ and $\varepsilon_{1234} = 1$. Here, and throughout the paper, we use the Einstein summation convention, unless otherwise stated.

In this paper, we study the space of smooth local solutions to the SDYM equations (2.1). More precisely, we suppose $A_\mu$ to be smooth on an arbitrary open ball $U$ in $\mathbb{R}^4$, and we do not fix boundary conditions for $A_\mu$. As $U$ we shall also consider open subsets in $\mathbb{R}^4$ which can be dense subsets in $\mathbb{R}^4$ and can even coincide with $\mathbb{R}^4$. We shall consider smooth solutions of the SDYM equations on $U$ lying in an open neighbourhood of some fixed solution $A^0_\mu$, for instance, in a neighbourhood of the vacuum $A^0_\mu = 0$ (local solutions). The set of local solutions is an infinite-dimensional space and contains finite-dimensional moduli spaces of global solutions (instantons, monopoles etc.) as subspaces.

2.2 Gauge symmetries

Equations (2.1) are manifestly invariant under the group of gauge transformations

$$A_\mu \mapsto A^g_\mu = g^{-1} A_\mu g + g^{-1} \delta_\mu g, \quad F_{\mu\nu} \mapsto F^g_{\mu\nu} = g^{-1} F_{\mu\nu} g, \quad (2.2)$$

where $g = g(x) \in G, x \in U \subset \mathbb{R}^4$. For infinitesimal gauge transformations we have

$$\delta_\varphi A_\mu = D_\mu \varphi \equiv \partial_\mu \varphi + [A_\mu, \varphi], \quad (2.3)$$

where $\varphi(x) \in \mathfrak{g}, x \in U$.

The fields $A_\mu$ and $A^g_\mu$ differing by the gauge transformation (2.2) are considered to be equivalent. That is why gauge transformations are “trivial” symmetries.

2.3 Conformal symmetries

It is well-known that the SDYM equations (2.1) are invariant with respect to (w.r.t.) the group of conformal transformations of the space $\mathbb{R}^4$. This group is locally isomorphic to the group $SO(5, 1)$. On the coordinates $x^\mu$ and components $A_\mu$ of the gauge potential $A$ the group of conformal transformations acts in the following way:

- translations: $x^\mu \mapsto \tilde{x}^\mu = x^\mu + a^\mu$, $A_\mu(x^\nu) \mapsto \tilde{A}_\mu = A_\mu(x^\nu + a^\nu)$, \quad (2.4a)
- rotations: $x^\mu \mapsto \tilde{x}^\mu = a^\mu x^\nu$, $A_\mu(x^\nu) \mapsto \tilde{A}_\mu = (a^{-1})^\mu_\sigma A_\sigma(a^\nu x^\rho)$, \quad (2.4b)
- dilatations: $x^\mu \mapsto \tilde{x}^\mu = \epsilon^\alpha x^\mu$, $A_\mu(x^\nu) \mapsto \tilde{A}_\mu = \epsilon^{-\alpha} A_\mu(\epsilon^\alpha x^\nu)$, \quad (2.4c)
- special conformal transformations: $x^\mu \mapsto \tilde{x}^\mu = \frac{x^\mu + \alpha^\mu x^2}{1 + 2\alpha_\nu x^\nu + \alpha^2 x^2}$, $A_\mu(x^\nu) \mapsto \tilde{A}_\mu = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} A_\sigma(\tilde{x}^\nu)$, \quad (2.4d)

where $a = (a^\mu) \in SO(4), a^\mu, \alpha, \alpha^\mu \in \mathbb{R}$, $\alpha^2 := \alpha_\nu \alpha^\nu$, $x^2 := x_\mu x^\nu$.

For infinitesimal conformal transformations we have

$$\delta_N A_\mu = \mathcal{L}_N A_\mu \equiv N^\nu \partial_\nu A_\mu + A_\nu \partial_\mu N^\nu, \quad (2.5)$$

where $\mathcal{L}_N$ is the Lie derivative along a vector field $N$ and $N = N^\nu \partial_\nu$ is any generator of the 15-parameter conformal group,

$$X_a = \delta_{ab} \eta^b_{\mu\nu} x\nu, \quad Y_a = \delta_{ab} \tilde{\eta}^b_{\mu\nu} x\nu, \quad F_\mu = \partial_\mu, \quad P_\mu = \partial_\mu.$$
\[ K_\mu = \frac{1}{2} x^2 \partial_\mu - x_\mu x^\nu \partial_\nu, \quad D = x^\nu \partial_\nu. \] (2.6)

Here \{X_\mu\} and \{Y_\mu\}, \(a, b, \ldots = 1, 2, 3\), generate two commuting \(SO(3)\) subgroups in \(SO(4)\), \(P_\mu\) are the translation generators, \(K_\mu\) are the generators of special conformal transformations, \(D\) is the dilatation generator, \(\eta^a_{\mu\nu} = \{\epsilon^a_{bc}, \mu = b, \nu = c; \delta^a_{\mu\nu} = 4; \delta^a_{\nu\mu} = 4\}\) are the self-dual 't Hooft tensors and \(\bar{\eta}^a_{\mu\nu} = \{\epsilon^a_{bc}, \mu = b, \nu = c; -\delta^a_{\mu\nu} = 4; \delta^a_{\nu\mu} = 4\}\) are the anti-self-dual 't Hooft tensors.

**Remark.** It is well-known that for semisimple structure group \(G\) there are no local symmetries of the SDYM equations differing from the gauge and conformal symmetries described above.

### 3 Complex geometry of twistor spaces

#### 3.1 Complex structure on \(\mathbb{R}^4\)

To write down a linear system for eqs. (2.1) and to clarify its geometrical meaning, it is necessary to introduce a complex structure \(J\) on \(\mathbb{R}^4\) (and thus on any open subset \(U \subset \mathbb{R}^4\)). This means that we must introduce on \(\mathbb{R}^4\) a tensor \(J^\nu_\mu\) such that \(J^\nu_\mu J^\sigma_\nu = -\delta^\sigma_\mu\). It is well-known that all constant complex structures on \(\mathbb{R}^4\) are parametrized by the two-sphere \(S^2 \simeq SO(4)/U(2)\), and the most general form of \(J^\nu_\mu\) is

\[ J^\nu_\mu = s_a \eta^a_{\mu\sigma} \delta^\nu_\sigma, \] (3.1)

where real numbers \(s^a\) parametrize \(S^2\): \(s_a s^a = 1\). Using the identities for the 't Hooft tensors

\[ \bar{\eta}^a_{\mu\sigma} \bar{\eta}^b_{\sigma\nu} = \delta^{ab} \delta^\mu_\nu + \epsilon^{abc} \eta^c_{\mu\nu}, \] (3.2)

it can be shown that \(J^2 = -1\). The other admissible choice of the complex structure \(\bar{J}^\mu_\nu = s_a \eta^a_{\mu\sigma} \delta^\nu_\sigma\) corresponds to choosing the opposite orientation on \(\mathbb{R}^4\) and transition from self-duality to anti-self-duality equations.

Eigenvalues of the operator \(J = (J^\mu_\nu)\) (applied to vectors) are \(\pm i\), and we can introduce two subspaces in \(\mathbb{C}^4 = \mathbb{R}^4 \otimes \mathbb{C}\),

\[ V_{1,0} = \{V \in \mathbb{C}^4 : J^\mu_\nu V^\nu = i V^\mu\} , \quad V_{0,1} = \{V \in \mathbb{C}^4 : J^\mu_\nu V^\nu = -i V^\mu\}. \] (3.3a)

As a basis in \(V_{1,0}\) and \(V_{0,1}\) one may take vectors with the components

\[ \{V^{(1)}_{1,0}\} = \{\frac{1}{2}, -\frac{i}{2}, -\frac{1}{2}, \frac{i}{2}\}, \quad \{V^{(1)}_{0,1}\} = \{\frac{1}{2}, \frac{i}{2}, \frac{1}{2}, -\frac{i}{2}\}, \] (3.3b)

\[ \{\bar{V}^{(1)}_{1,0}\} = \{\frac{1}{2}, \frac{i}{2}, -\frac{1}{2}, -\frac{i}{2}\}, \quad \{\bar{V}^{(1)}_{0,1}\} = \{\frac{1}{2}, -\frac{i}{2}, -\frac{1}{2}, \frac{i}{2}\}, \] (3.3c)

where \(\lambda\) and \(\bar{\lambda}\) are local holomorphic and antiholomorphic coordinates on the sphere \(S^2\), \(\lambda = (s^1 + is^2)/(1 + s^3)\).

Using \(J\), one can introduce vector fields \(V^{(1)}_{A}\) and \(V^{(1)}_{B}\) of the type \((1, 0)\) and vector fields \(\bar{V}^{(1)}_{A}\) and \(\bar{V}^{(1)}_{B}\) of the type \((0, 1)\) w.r.t. \(J\), where \(A, B, \ldots = 1, 2\). We have

\[ \bar{V}^{(1)}_{1} = \bar{V}^{(1)}_{1,0} \partial_\mu = \frac{1}{2} (\partial_1 + i \partial_2) - \lambda \frac{1}{2} (\partial_3 + i \partial_4) = \partial_{\bar{y}^1} - \lambda \partial_{\bar{y}^2}, \] (3.4a)

\[ \bar{V}^{(1)}_{2} = \bar{V}^{(1)}_{0,1} \partial_\mu = \frac{1}{2} (\partial_1 - i \partial_2) + \lambda \frac{1}{2} (\partial_3 - i \partial_4) = \partial_{\bar{y}^2} + \lambda \partial_{\bar{y}^1}, \] (3.4b)

where

\[ y^1 = x^1 + ix^2, \quad y^2 = x^3 - ix^4, \quad \bar{y}^1 = x^1 - ix^2, \quad \bar{y}^2 = x^3 + ix^4 \] (3.5)

are the canonical complex coordinates on \(\mathbb{R}^4 \simeq \mathbb{C}^2\).

#### 3.2 Riemann sphere \(\mathbb{CP}^1\)

In (3.3) we have introduced the complex coordinate \(\lambda\) on \(S^2 \simeq \mathbb{CP}^1\), parametrizing complex structures on \(\mathbb{R}^4\). Using the stereographic projection \(S^2 \rightarrow \mathbb{R}^2\), one can introduce two coordinate patches \(\Omega_1 \simeq \mathbb{R}^2\) and \(\Omega_2 \simeq \mathbb{R}^2\) of the sphere with the coordinates

\[ \omega^1_1 = \frac{s^1}{1 + s^3} \quad \text{and} \quad \omega^2_1 = \frac{s^2}{1 + s^3} \quad \text{on} \quad \Omega_1, \quad \omega^1_2 = \frac{s^1}{1 - s^3} \quad \text{and} \quad \omega^2_2 = \frac{s^2}{1 - s^3} \quad \text{on} \quad \Omega_2, \] (3.6)
in which the metric on $S^2$ is conformally flat.

We introduce the standard complex structure $j$ on $S^2$ with the components

\[ j = (j_B^A), \quad j_B^A j^C_B = -\delta^A_C, \quad j_B^2 = -i \delta^1_B = -1 \]  

in the coordinates \{\omega_1^A\}. Now we can introduce vector fields, holomorphic and antiholomorphic w.r.t. $j$, on $\Omega_1$ as

\[ V^{(1)}_3 = \frac{1}{2} (\partial_{\omega^1_1} - i \partial_{\omega^1_2}) = \partial_{\lambda}, \quad j_B^A V^{(1)B}_3 = i V^{(1)A}_3, \tag{3.7} \]

\[ \bar{V}^{(1)}_3 = \frac{1}{2} (\partial_{\omega^1_1} + i \partial_{\omega^1_2}) = \partial_{\bar{\lambda}}, \quad j_B^A \bar{V}^{(1)B}_3 = -i \bar{V}^{(1)A}_3, \tag{3.7c} \]

where $\lambda = \omega^1_1 + i \omega^1_2$ is the complex coordinate on $\Omega_1 \simeq \mathbb{C}$. Analogously, we introduce the complex coordinate $\zeta = \omega^1_2 - i \omega^1_1$ on $\Omega_2 \simeq \mathbb{C}$ and vector fields $V^{(2)}_3 = \partial_{\zeta}$, $\bar{V}^{(2)}_3 = \partial_{\bar{\zeta}}$ on $\Omega_2$.

So the sphere $S^2$ can be covered by two coordinate patches $\Omega_1$, $\Omega_2$, with $\Omega_1$, the neighbourhood of $\lambda = 0$, and $\Omega_2$, the neighbourhood of $\lambda = \infty$. Let us fix $\alpha_1, \alpha_2$: $0 \leq \alpha_1 < 1 < \alpha_2 \leq \infty$ and put

\[ \Omega_1 = \{ \lambda \in \mathbb{C} : |\lambda| < \alpha_2 \}, \quad \Omega_2 = \{ \lambda \in \mathbb{C} \cup \infty : |\lambda| > \alpha_1 \}. \tag{3.9} \]

The sphere $S^2$, considered as a complex projective line $\mathbb{CP}^1 = \Omega_1 \cup \Omega_2$, is the complex manifold obtained by patching together $\Omega_1$ and $\Omega_2$ with the coordinates $\lambda$ and $\zeta$ related by $\zeta = \lambda^{-1}$ on $\Omega_1 \cap \Omega_2$. For example, if $\Omega_1 = \{ \lambda \in \mathbb{C} : |\lambda| < \infty \}$ and $\Omega_2 = \{ \lambda \in \mathbb{C} \cup \infty : |\lambda| > 0 \}$, $\Omega_1 \cap \Omega_2$ is the multiplicative group $\mathbb{C}^*$ of complex numbers $\lambda \neq \{0, \infty\}$.

### 3.3 Twistor space

We consider an open subset $U$ in $\mathbb{R}^4$. As a smooth manifold the twistor space $\mathcal{P} \equiv \mathcal{P}(U)$ of $U$ is a direct product of the spaces $U$ and $\mathbb{CP}^1$: $\mathcal{P} = U \times \mathbb{CP}^1$ and is the bundle of complex structures on $U$ \[\text{[19]}\]. This space can be covered by two coordinate patches:

\[ \mathcal{P} = U_1 \cup U_2, \quad U_1 = U \times \Omega_1, \quad U_2 = U \times \Omega_2, \tag{3.10a} \]

with the coordinates \{\(x^\mu, \lambda, \bar{\lambda}\) on $U_1$ and \{\(x^\mu, \zeta, \bar{\zeta}\) on $U_2$. The two-set open cover $\mathcal{D} = \{\Omega_1, \Omega_2\}$ of the Riemann sphere $\mathbb{CP}^1$ was described in §3.2. We shall consider the intersection $U_{12}$ of $U_1$ and $U_2$

\[ U_{12} := U_1 \cap U_2 = U \times (\Omega_1 \cap \Omega_2) \tag{3.10b} \]

with the coordinates $x^\mu \in U$, $\lambda, \bar{\lambda} \in \Omega_{12} := \Omega_1 \cap \Omega_2$. Thus, the twistor space $\mathcal{P}$ is a trivial bundle $\pi : \mathcal{P} \to U$ over $U$ with the fibre $\mathbb{CP}^1$, where $\pi : \{x^\mu, \lambda, \bar{\lambda}\} \to \{x^\mu\}$ is the canonical projection.

We shall also consider the twistor space $\mathcal{Z} \equiv \mathcal{Z}(\mathbb{R}^4)$ of $\mathbb{R}^4$ which as a smooth manifold is a direct product $\mathcal{Z} = \mathbb{R}^4 \times \mathbb{CP}^1$. The twistor space $\mathcal{P}$ is an open subset of $\mathcal{Z}$. In its turn, $\mathcal{Z} \simeq \mathbb{CP}^3 - \mathbb{CP}^1$ is an open subset in the space $\mathbb{CP}^3$ which is the twistor space of the sphere $S^4$. Formally, $\mathcal{P}$ coincides with $\mathcal{Z}$ if we take $U = \mathbb{R}^4$; that is why we denote the cover of $\mathcal{Z}$ by the same letters $U_1 = \mathbb{R}^4 \times \Omega_1, U_2 = \mathbb{R}^4 \times \Omega_2$. Since $\mathcal{P}$ is an open subset of $\mathcal{Z}$, a complex structure will be discussed for $\mathcal{Z}$.

Having the complex structure $J$ on $\mathbb{R}^4$ and the complex structure $j$ on $S^2$, we can introduce a complex structure $\mathcal{J} = (J, j)$ on $\mathcal{Z}$. The vector fields $\{V_a^{(1)}\}$ on $U_1$, introduced in (3.4), are vector fields of the type $(0,1)$ w.r.t. the complex structure $\mathcal{J}$. Vector fields $\{\bar{V}_a^{(2)}\}$ of the type $(0,1)$ on $U_2$ have the form

\[ \bar{V}_1^{(2)} = \zeta \partial_{g^1} - \partial_{g^2}, \quad \bar{V}_2^{(2)} = \zeta \partial_{g^2} + \partial_{g^1}, \quad \bar{V}_3^{(2)} = \partial_{\zeta}, \tag{3.11a, b, c} \]

and we have

\[ \bar{V}_1^{(1)} = \lambda \bar{V}_1^{(2)}, \quad \bar{V}_2^{(1)} = \lambda \bar{V}_2^{(2)}, \quad \bar{V}_3^{(1)} = -\bar{\lambda}^2 \bar{V}_3^{(2)} \tag{3.12a, b, c} \]

on $U_{12} = U_1 \cap U_2$.

Now we can introduce complex coordinates $\{z_1^A\}$ on $U_1$ and $\{z_2^A\}$ on $U_2$ as solutions of the equations $\bar{V}_a^{(1)}(z_1^A) = 0$ and $\bar{V}_a^{(2)}(z_2^A) = 0$. We have

\[ z_1^1 = y^1 - \lambda g^2, \quad z_1^2 = y^2 + \lambda g^1, \quad z_1^3 = \lambda, \tag{3.13a} \]

\[ z_2^1 = \zeta y^1 - \bar{\zeta}, \quad z_2^2 = \zeta y^2 + \bar{\zeta}, \quad z_2^3 = \zeta \tag{3.13b} \]
and on the intersection $U_{12}$ these coordinates are connected by the holomorphic transition function $f_{12}$

$$z_1^i = f_{12}^i(z_2^b) \Leftrightarrow z_1^1 = f_{12}^1(z_2^b) = \frac{z_2^1}{z_2^2}, \quad z_1^2 = f_{12}^2(z_2^b) = \frac{z_2^2}{z_2^3}, \quad z_1^3 = f_{12}^3(z_2^b) = \frac{1}{z_2^3}. \quad (3.13c)$$

From (3.13) it is not difficult to derive the formulae

$$\frac{\partial}{\partial \bar{z}_1^1} = \gamma_1 \bar{V}_1^{(1)}, \quad \frac{\partial}{\partial \bar{z}_1^2} = \gamma_1 \bar{V}_2^{(1)}, \quad \frac{\partial}{\partial \bar{z}_1^3} = \bar{V}_3^{(1)} + \bar{y}^2 \gamma_1 \bar{V}_1^{(1)} - \bar{y}^1 \gamma_1 \bar{V}_2^{(1)}, \quad (3.14a)$$

where $\gamma_1 = 1/(1 + \bar{\lambda} \lambda)$. Analogously, on $U_2$

$$\frac{\partial}{\partial \bar{z}_2^1} = \gamma_2 \bar{V}_1^{(2)}, \quad \frac{\partial}{\partial \bar{z}_2^2} = \gamma_2 \bar{V}_2^{(2)}, \quad \frac{\partial}{\partial \bar{z}_2^3} = \bar{V}_3^{(2)} - \bar{y}^1 \gamma_2 \bar{V}_1^{(2)} - \bar{y}^2 \gamma_2 \bar{V}_2^{(2)}, \quad (3.14b)$$

where $\gamma_2 = 1/(1 + \zeta \bar{\zeta})$.

It is easy to check that the local basis $(0,1)$-forms w.r.t. $J$ are

$$\bar{\theta}_{(1)}^1 = \gamma_1 (dy^1 - \bar{\lambda} dy^2), \quad \bar{\theta}_{(2)}^1 = \gamma_1 (dy^2 + \bar{\lambda} dy^1), \quad \bar{\bar{\theta}}_{(1)}^3 = d\bar{\lambda} \quad \text{on} \quad U_1, \quad (3.15a)$$

$$\bar{\theta}_{(2)}^3 = \gamma_2 (d\bar{y}^1 - \bar{\lambda} d\bar{y}^2), \quad \bar{\bar{\theta}}_{(2)}^3 = \gamma_2 (d\bar{y}^2 + \bar{\lambda} d\bar{y}^1), \quad \bar{\bar{\theta}}_{(1)}^3 = d\bar{\zeta} \quad \text{on} \quad U_2. \quad (3.15b)$$

The exterior derivative $d$ on $Z$ splits into $\bar{\partial}$ and $\bar{\bar{\partial}}$: $d = \partial + \bar{\partial}$, where

$$\bar{\partial} = d\bar{z}_1^a \frac{\partial}{\partial \bar{z}_1^a} = \bar{\theta}_{(1)}^a \bar{V}_a^{(1)} \quad \text{on} \quad U_1, \quad (3.16a)$$

$$\bar{\bar{\partial}} = d\bar{z}_2^a \frac{\partial}{\partial \bar{z}_2^a} = \bar{\theta}_{(2)}^a \bar{V}_a^{(2)} \quad \text{on} \quad U_2, \quad (3.16b)$$

and the operator $\bar{\partial}$ is connected with $\bar{\bar{\partial}}$ by means of complex conjugation. As usual $d^2 = \partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0$.

It follows from (3.12), (3.13) and (3.15) that as a complex manifold $Z$ is not a direct product $\mathbb{C}^2 \times \mathbb{C}P^1$, but is a nontrivial holomorphic vector bundle $p: Z \to \mathbb{C}P^1$. Moreover, from (3.12), (3.13) and (3.15) it follows that $Z$ coincides with a total space of the rank 2 holomorphic vector bundle $L^{-1} \oplus L^{-1}$ over $\mathbb{C}P^1$,

$$p: Z = L^{-1} \oplus L^{-1} \to \mathbb{C}P^1, \quad (3.17)$$

where $L$ is the tautological complex line bundle over $\mathbb{C}P^1$ with the transition function $\lambda^{-1}$, and the first Chern class $c_1(L)$ equals $-1$: $c_1(L) = -1$. Its dual $L^{-1}$ is isomorphic to the hyperplane bundle (Chern class $c_1(L^{-1}) = 1$) over $\mathbb{C}P^1$. The twistor space $\mathcal{P}$ of $U \subset \mathbb{R}^4$ is an open subset of $Z$ and $Z = L^{-1} \oplus L^{-1} \simeq \mathbb{C}P^3 - \mathbb{C}P^1$ is an open subset of $\mathbb{C}P^3$. Holomorphic sections of the bundle (3.17) are projective lines

$$\mathbb{C}P^1_y = \left\{ \lambda \in \Omega_1 : z_1^1 = y^1 + \lambda y^2, \quad z_1^2 = y^2 + \lambda y^1 \right\}$$

or

$$\zeta \in \Omega_2 : z_1^1 = \zeta y^1 + \bar{y}^2, \quad z_1^2 = \zeta y^2 + \bar{y}^1 \right\} \quad (3.18)$$

parametrized by the points $y = \{y^1, y^2, \bar{y}^1, \bar{y}^2\} \in \mathbb{C}^4$.

### 3.4 Real structure on twistor space

A real structure on the complex twistor space $Z$ is defined as an antiholomorphic involution $\tau: Z \to Z$, defined by the antipodal map $\lambda \mapsto -1/\lambda$ on the $\mathbb{C}P^1$ factor,

$$\tau(x^\mu, \lambda) = (x^\mu, -1/\bar{\lambda}), \quad \tau^2 = 1. \quad (3.19)$$

This involution takes the complex structure $J$ on $Z$ to its conjugate $-J$, i.e., it is antiholomorphic. It is obvious from the definition (3.19) that $\tau$ has no fixed points on $\mathcal{P} \subset Z$ but does leave the fibres $\mathbb{C}P^1_x, \quad x \in U$, of the bundle $\mathcal{P} \to U$ invariant. The same is true for the fibres $\mathbb{C}P^1_x$ of the bundle $Z \to \mathbb{R}^4$. Fibres $\mathbb{C}P^1_x$ of the bundle $\mathcal{P} \to U$ are also real holomorphic sections of the bundle (3.17) for which we have $\bar{y}^1 = y^1$, $\bar{y}^2 = -\bar{y}^2$ in (3.18), i.e., they are parametrized by $\{x^\mu\} = \{y^4, \bar{y}^A\} \in U$.

An extension of the involution $\tau$ to complex functions $f(x^\mu, \lambda)$ has the form

$$\tau: f(x, \lambda) \mapsto \tau(f(x, \lambda)) \equiv f_r(x, \lambda) := \overline{f(\tau(x, \lambda))} = f(x, -\lambda^{-1}). \quad (3.20)$$
In particular, for the complex coordinates \{z^a_1\} \text{ and } \{z^a_2\} \text{ on } Z \text{ we have }

\begin{align*}
\tau(z^1_1) &= z^2_1, \quad \tau(z^2_1) = -z^1_2, \quad \tau(z^3_1) = -z^3_2, \Leftrightarrow \\
\tau(z^a_1) &= B^a_0 z^b_1, \quad B^1_2 = 1, \quad B^2_3 = -1, \quad B^3_1 = -1.
\end{align*}

All the rest components of the constant matrix \(B = (B^a_0)\) are equal to zero.

Using (3.21), it is not difficult to verify that for the transition function (3.13c) compatible with the real structure \(\tau\), we have

\begin{equation}
\tau(f^a_{12}) = B^a_0 \tilde{f}_1^b,
\end{equation}

where \(\tilde{f}_{12}\) is the transition function inverse to \(f_{12}\)

\begin{equation}
\frac{z^1_2}{z^1_1} = \tilde{f}_{12}^1(z^b_1) = \frac{z^1_1}{z^1_2}, \quad \frac{z^2_2}{z^1_1} = \tilde{f}_{12}^2(z^b_1) = \frac{z^2_1}{z^1_1}, \quad \frac{z^3_2}{z^1_1} = \tilde{f}_{12}^3(z^b_1) = \frac{1}{z^1_1}.
\end{equation}

So all the holomorphic data are compatible with \(\tau\).

## 4 The Penrose-Ward correspondence

### 4.1 Complex vector bundles over \(U\) and \(\mathcal{P}\)

Let us consider a principal \(G\)-bundle \(P = P(U, G) = U \times G\) over \(U \subset \mathbb{R}^4\). Then, a gauge potential \(A = A_\mu dx^\mu\) (a connection 1-form) defines a connection \(D := d + A = dx^\mu (\partial_\mu + A_\mu)\) on the bundle \(P\), and the 2-form \(F = dA + A \wedge A = F_{\mu\nu} dx^\mu \wedge dx^\nu\) is the curvature of the connection form \(A\). We shall consider irreducible connections. Suppose a representation of \(G\) in the complex vector space \(\mathbb{C}^n\) is given. In the standard manner we associate with \(P\) the complex vector bundle

\[ E = P \times_G \mathbb{C}^n \cong U \times \mathbb{C}^n, \]

which is topologically trivial.

Using the projection \(\pi : \mathcal{P} \to U\) of the twistor space \(\mathcal{P}\) on \(U\), we can pull back \(E\) to a bundle \(E' := \pi^* E\) over \(\mathcal{P}\), and the pulled back bundle \(E'\) is trivial on the fibres \(\mathbb{C}P_1^1\) of the bundle \(\mathcal{P} \to U\). We can set components of \(\pi^* A\) along the fibres equal to zero and then the pulled back connection \(D'\) will have the form

\[ D' = D + d\lambda \partial_\lambda + d\bar{\lambda} \partial_{\bar{\lambda}} \text{ (on } U_1) = D + d\zeta \partial_\zeta + d\bar{\zeta} \partial_{\bar{\zeta}} \text{ (on } U_2). \]

### 4.2 Self-duality \(\Rightarrow\) holomorphy

The twistor space \(\mathcal{P}\) of the space \(U \subset \mathbb{R}^4\) is a complex three-dimensional manifold with the coordinates \(\{z^a_1\}\) on \(U_1 \subset \mathcal{P}\) and \(\{z^a_2\}\) on \(U_2 \subset \mathcal{P}\), \(\mathcal{P} = U_1 \cup U_2\). Using the (0,1)-forms (3.15), we introduce the (0,1) components \(B_a\) of the connection 1-form \(\pi^* A = A_\mu dx^\mu = B^{1,0} + B^{0,1} = \bar{B} + B\) by the formulae

\begin{align*}
\{B^{(1)}_1 := A_{\bar{y}} - \lambda A_y, \quad B^{(1)}_3 := A_{\bar{y}} + \lambda A_y, \quad B^{(1)}_3 := 0\} &\text{ on } U_1, \quad (4.1a) \\
\{B^{(2)}_1 := \zeta A_{\bar{y}}, \quad B^{(2)}_2 := \zeta A_y, \quad B^{(2)}_3 := 0\} &\text{ on } U_2. \quad (4.1b)
\end{align*}

Notice that \(B^{(1)}_1 = \lambda B^{(2)}_1\) on \(U_{12}\). One can also introduce the components \(B_{z^a_{1,2}}\) of \(B\) along the antiholomorphic vector fields \(\partial_{z^a_{1,2}}\) from (3.14),

\begin{align*}
\{B_{z^1_1} := \gamma_1 B^{(1)}_1, \quad B_{z^2_1} := \gamma_1 B^{(1)}_3, \quad B_{\bar{z}^1_1} := y^2 \gamma_1 B^{(1)}_1 - y^1 \gamma_1 B^{(1)}_3\} &\text{ on } U_1, \quad (4.2a) \\
\{B_{z^2_2} := \gamma_2 B^{(2)}_1, \quad B_{z^1_2} := \gamma_2 B^{(2)}_3, \quad B_{\bar{z}^2_2} := -\bar{y}^1 \gamma_2 B^{(2)}_1 - \bar{y}^2 \gamma_2 B^{(2)}_3\} &\text{ on } U_2. \quad (4.2b)
\end{align*}

Then we have \(\pi^* A = \bar{B} + B\) and

\begin{align*}
B &\equiv B^{0,1} = B_{z^a_1} dz^a_1 = B^{(1)}_a \bar{\theta}^{(1)}_a \text{ on } U_1, \quad (4.3a) \\
B &\equiv B^{0,1} = B_{z^a_2} dz^a_2 = B^{(2)}_a \bar{\theta}^{(2)}_a \text{ on } U_2. \quad (4.3b)
\end{align*}

Now we can introduce components of the connection \(D'\) on the complex vector bundle \(E'\) which are (0,1) components w.r.t. the complex structure \(\mathcal{J}\) on \(\mathcal{P}\),

\[ D' := \partial_B + \bar{\partial}_B, \quad \bar{\partial}_B = \bar{\partial} + B, \quad (4.4) \]
where the operator $\tilde{\partial}$ was introduced in (3.16), the $(0,1)$-form $B$ was introduced in (4.3) and the operator $\partial_B = \partial + B$ is the $(1,0)$ component of the operator $D'$.

**Remark.** In most cases we shall further write down formulae and equations in the trivialization over $\mathcal{U}_1 \subset \mathcal{P}$.

Let us consider the equations
\begin{equation}
\tilde{\partial}_B \xi = 0 \tag{4.5}
\end{equation}
on a smooth local section $\xi$ of the bundle $E'$. The local solutions of these equations are by definition the local holomorphic sections of the complex vector bundle $E'$. The bundle $E' \to \mathcal{P}$ is said to be holomorphic if eqs.(4.5) are compatible, i.e., $\tilde{\partial}_B^2 = 0 \Rightarrow$ the $(0,2)$ components of the curvature of $D'$ are equal to zero.

In the trivialization over $\mathcal{U}_1$, eqs. (4.5) are equivalent to the equations
\begin{align}
[(D_1 + iD_2) - \lambda (D_1 + iD_2)]\xi_1(x, \lambda, \bar{\lambda}) &= 0, \tag{4.6a} \\
[(D_3 - iD_4) + \lambda (D_1 - iD_2)]\xi_1(x, \lambda, \bar{\lambda}) &= 0, \tag{4.6b}
\end{align}
and analogously in the trivialization over $\mathcal{U}_2$. Equation (4.6c) simply means that $\xi_1$ is a function of $x^a$ and $\lambda$ (does not depend on $\bar{\lambda}$). If eq.(4.6c) is solved, the remaining two equations (4.6a,b) for $\xi_1(x, \lambda)$ are usually called the linear system for the SDYM equations. It is readily seen that the compatibility conditions $\tilde{\partial}_B^2 = 0$ of eqs.(4.6) are identical to the SDYM equations (2.1), which in the coordinates $\{y^1, y^2, \tilde{y}^1, \tilde{y}^2\}$ have the form
\begin{equation}
F_{y^1y^2} = 0, \quad F_{\tilde{y}^1\tilde{y}^2} = 0, \quad F_{y^1\tilde{y}^1} + F_{\tilde{y}^2y^2} = 0, \tag{4.7}
\end{equation}
i.e., eqs. (4.7) follow from the equations $\tilde{\partial}_B^2 = 0$. Therefore, if a gauge potential $A = A_\mu dx^\mu$ is a smooth solution of eqs.(4.7) on a domain $U$ in $\mathbb{R}^4$, there exist solutions of eqs.(4.5), and the bundle $E' \to \mathcal{P}$ is holomorphic.

For the cover $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$, eqs.(4.5) have a local solution $\xi_1$ over $\mathcal{U}_1$, a local solution $\xi_2$ over $\mathcal{U}_2$ and $\xi_1 = \xi_2$ on the overlap $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$ (i.e., it is a section over $\mathcal{P}$). We can always represent $\xi_1, \xi_2$ in the form $\xi_1 = \psi_1 \chi_1, \xi_2 = \psi_2 \chi_2$, where $G^c$-valued functions $\psi_1$ and $\psi_2$ nonsingular on $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy the equations
\begin{equation}
\tilde{\partial}_B \psi_1 = 0, \quad \tilde{\partial}_B \psi_2 = 0 \tag{4.8}
\end{equation}
on $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. The vector-functions $\chi_{1,2} \in \mathbb{C}^n$ are holomorphic on $\mathcal{U}_{1,2}$.

\begin{equation}
\tilde{V}^{(1)}_a \chi_1 = 0, \quad \tilde{V}^{(2)}_a \chi_2 = 0. \tag{4.9}
\end{equation}
It follows from (4.8) that
\begin{align}
(\partial_{y^1} \psi_1 - \lambda \partial_{\tilde{y}^2} \psi_1)\psi_1^{-1} &= (\partial_{\tilde{y}^1} \psi_2 - \lambda \partial_{y^2} \psi_2)\psi_2^{-1} = -(A_{y^1} - \lambda A_{\tilde{y}^2}), \tag{4.10a} \\
(\partial_{\tilde{y}^2} \psi_1 + \lambda \partial_{y^1} \psi_1)\psi_1^{-1} &= (\partial_{y^2} \psi_2 + \lambda \partial_{\tilde{y}^1} \psi_2)\psi_2^{-1} = -(A_{\tilde{y}^2} + \lambda A_{y^1}), \tag{4.10b} \\
\partial_x \psi_1 = \partial_x \psi_2 = 0. \tag{4.10c}
\end{align}
Moreover, the vector-functions $\chi_1$ and $\chi_2$ are related by
\begin{equation}
\chi_1 = F_{12} \chi_2 \tag{4.11}
\end{equation}
on $\mathcal{U}_{12}$, i.e.,
\begin{equation}
F_{12} := \psi_1^{-1} \psi_2. \tag{4.12}
\end{equation}
is the transition matrix in the bundle $E'$ and $F_{21} := \psi_2^{-1} \psi_1 = F_{12}^{-1}$. From eqs.(4.8), (4.10) it follows that $F_{12}$ is the holomorphic $G^c$-valued function on $\mathcal{U}_{12}$ with nonvanishing determinant.

**Remarks**

1. The matrices $\psi_1$ and $\psi_2$ are matrix fundamental solutions, i.e., the columns of $\psi_1, \psi_2$ form frame fields for $E'$ over $\mathcal{U}_1, \mathcal{U}_2$. In other words, matrix-valued functions $\psi_1, \psi_2$ define a trivialization of the bundle $E'$ over $\mathcal{U}_1, \mathcal{U}_2$. At the same time, $\chi_1 = \chi_1(z^1)$ and $\chi_2 = \chi_2(z^2)$ are Čech fibre coordinates of the bundle $E'$ over $\mathcal{U}_1$ and $\mathcal{U}_2$. The representation of $\xi_{1,2}$ in the form $\xi_1 = \psi_1 \chi_1, \xi_2 = \psi_2 \chi_2$ is simply an expansion of the sections $\xi_{1,2}$ in the basis sections $\psi_{1,2}$ with the components $\chi_{1,2}$ (see e.g. [9]).

2. The matrix-valued functions $\psi_{1,2}$ are $C^\infty$-functions on $\mathcal{U}_{1,2}$, and any transition matrix of the form (4.12) defines a bundle $E'$, which is topologically trivial, but holomorphically nontrivial, since $\psi_{1,2}$ are not holomorphic functions on $\mathcal{U}_{1,2}$. On the other hand, eqs.(4.10c) mean that the restriction of $E'$ to any real projective line $\mathbb{C}P^1_x (x \in U)$ is holomorphically trivial: $E'|_{\mathbb{C}P^1_x} \cong \mathbb{C}P^1_x \times \mathbb{C}^n$. 


4.3 Gauge transformations and holomorphic equivalence

It is easy to see that the local gauge transformations (2.2) of the gauge potential $A$ are induced by the transformations

$$\psi_1 \mapsto \psi_1^\alpha := g^{-1}(x)\psi_1, \quad \psi_2 \mapsto \psi_2^\alpha := g^{-1}(x)\psi_2,$$

and the transition matrix $F_{12} = \psi_1^{-1}\psi_2$ is invariant under these transformations because $(\psi_1^\alpha)^{-1}\psi_2^\alpha = \psi_1^{-1}\psi_2$.

On the other hand, the components $\{A_\mu\}$ of the gauge potential $A$ in (4.10) will not change after transformations

$$\psi_1 \mapsto \psi_1^{h_1} := \psi_1h_1^{-1}, \quad \psi_2 \mapsto \psi_2^{h_2} := \psi_2h_2^{-1},$$

where $h_1$ is any regular holomorphic $G^\mathbb{C}$-valued function on $U_1$ and $h_2$ is any regular holomorphic $G^\mathbb{C}$-valued function on $U_2$. This means that a class of holomorphically equivalent bundles over the twistor space $\mathcal{P}$ corresponds to a self-dual connection on $U$. Recall that holomorphic bundles with the transition matrices $F_{12}$ and $F_{12}$ are called holomorphically equivalent if

$$\tilde{F}_{12} = h_1F_{12}h_2^{-1}$$

for some regular matrices $h_1, h_2$ such that $h_1$ is holomorphic on $U_1$ and $h_2$ is holomorphic on $U_2$.

4.4 Unitarity conditions

It follows from eqs.(4.10) that in the general case the components $\{A_\mu\}$ of the gauge potential $A$ will take values in the Lie algebra $\mathfrak{g}^\mathbb{C}$, because $\psi_1, \psi_2$ are $G^\mathbb{C}$-valued. This is equivalent to the consideration of $A_\mu$ with values in the Lie algebra $\mathfrak{g}$, but with complex components $A_\mu^i$ in the expansion $A_\mu = A_\mu^i T_k$ in the generators $\{T_k\}$ of the Lie group $G$. If we want to consider real gauge fields, we have to impose additional reality conditions on the bundle $E'$ induced by the real structure $\tau$ on $\mathcal{P}$ (see §3.4) and by an automorphism $\tilde{\sigma}$ of the Lie algebra $\mathfrak{g}^\mathbb{C}$ such that $\mathfrak{g} = \{a \in \mathfrak{g}^\mathbb{C} : \tilde{\sigma}(a) = a, \tilde{\sigma}^2 = id\}$. Such a reality structure in the bundle $E'$ exists for any compact Lie group $G$ [19], and we shall describe it for the case $G = SU(n)$, $\mathfrak{g} = su(n)$.

Namely, in the case $\mathfrak{g} = su(n)$ we have $A_\mu^\dagger = -A_\mu$ and therefore

$$A_\mu^\dagger = -A_\mu^i, \quad A_\mu^{i*} = -A_\mu^i,$$

where $\dagger$ denotes Hermitian conjugation. Then the matrices $F_{12} \in SL(n, \mathbb{C})$ and $\psi_1, \psi_2 \in SL(n, \mathbb{C})$ have to satisfy on $U_{12}$ the following unitarity conditions (see e.g. [28]):

$$F_{12}(\tau(z^a_1)) = F_{12}(z^a_1),$$

$$\psi_1^\dagger(\tau(x, \lambda)) = \psi_2^{-1}(x, \lambda),$$

where the action of $\tau$ on the coordinates of the space $\mathcal{P}$ was described in §3.4.

**Remark.** For simplicity, we shall always consider the case $G = SU(n)$ when discussing real gauge fields.

Thus, starting from a bundle $E$ over $U \subset \mathbb{R}^4$ with a self-dual connection, we have constructed a topologically trivial holomorphic vector bundle $E'$ over $\mathcal{P}$ satisfying the conditions: (1) $E'$ is holomorphically trivial on each real projective line $\mathbb{CP}^1_x$, $x \in U$, in $\mathcal{P}$; (2) $E'$ has a real structure.

4.5 Riemann-Hilbert problems

Suppose we have a nonsingular matrix-valued function $F(x, \lambda) \in SL(n, \mathbb{C})$ on $\Omega_1 \cap \Omega_2 \subset \mathbb{CP}^1$ (see §3.2) depending holomorphically on $\lambda$ and smoothly on some parameters $\{x^\mu\}$. Then a parametric Riemann-Hilbert problem is to find matrix-valued functions $\psi_1, \psi_2 \in SL(n, \mathbb{C})$ on $\Omega_1 \cap \Omega_2$ such that $\psi_1$ can be extended continuously to a regular (i.e., holomorphic with a non-vanishing determinant) matrix-valued function on $\Omega_1$, $\psi_2$ can be extended to a regular matrix-valued function on $\Omega_2$ and

$$F(x, \lambda) = \psi_1^{-1}(x, \lambda)\psi_2(x, \lambda)$$

on $\Omega_1 \cap \Omega_2$.

It follows from the Birkhoff decomposition theorem (see e.g. [27]) that for a fixed $x$ any holomorphic on $\Omega_1 \cap \Omega_2$ nonsingular matrix-valued function $F$ admits a decomposition

$$F = \psi_1 \Lambda \psi_2,$$
where \(\psi_1, \psi_2\) are defined above and \(\Lambda\) is a diagonal matrix whose entries are integral powers \(k_i \in \mathbb{Z}\) of \(\lambda\), \(k_1 + \ldots + k_n = 0\). The \(k_i\)'s are unique up to permutation and are Chern classes of the holomorphic line bundles over \(\mathbb{CP}^1\) which occur in the decomposition of the holomorphic vector bundle over \(\mathbb{CP}^1\) with \(\mathcal{F}\) as a transition matrix (Grothendieck's theorem).

If \(\Lambda\) is the identity matrix, the decomposition (4.18) is called a solution to the Riemann-Hilbert problem. For these matrices \(\mathcal{F}\), the factorization is unique up to a transformation

\[
\psi_1(x, \lambda) \mapsto \psi_1^g = g^{-1}(x)\psi_1(x, \lambda), \quad \psi_2(x, \lambda) \mapsto \psi_2^g = g^{-1}(x)\psi_2(x, \lambda),
\]

for some matrix \(g(x) \in SL(n, \mathbb{C})\). So the Riemann-Hilbert problem can only be solved 'generically' and (4.17) may not have a solution for all values of the parameters \(x^\mu\). But if a factorization (4.17) exists at some \(x^\mu_0\), then it exists in an open neighbourhood \(U\) of \(x^\mu_0\). Usually, \(\Lambda \neq 1\) on a submanifold of codimension 1 (or more) of the parameter space. The points \(x^\mu\) for which \(\Lambda \neq 1\) are called jumping points, and projective lines \(\mathbb{CP}^1\) corresponding to these points are called jumping lines. In the twistor construction the jumping points \(x \in \mathbb{R}^4\) give rise to singularities in the SDYM potential \(A\). For details see e.g. [37, 40].

### 4.6 Holomorphy \(\Rightarrow\) self-duality

Suppose we have a topologically trivial holomorphic vector bundle \(E'\) over \(\mathcal{P}\) with the cover \(\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}\) and a transition matrix \(\mathcal{F}_{12}\) satisfying the unitarity condition (4.16b). Considering \(\mathcal{F}_{12}\) for fixed \(x^\mu \in U\), we obtain a parametric Riemann-Hilbert problem on \(\mathbb{CP}^1\). Then in a set of all possible transition matrices we choose those for which a solution of the Riemann-Hilbert problem exists.

After finding a Birkhoff decomposition (4.17) for \(\mathcal{F}_{12}\) we consider \((\mathcal{V}_a^{(1)}(x))\psi_1^{-1}\) and \((\mathcal{V}_a^{(1)}(x))\psi_2^{-1}\) as functions on \(\mathcal{U}_1\) and \(\mathcal{U}_2\) with values in the Lie algebra \(\mathfrak{su}(n)\). For definitions of the \((0,1)\) vector fields \(\mathcal{V}_a^{(1)}\), \(\mathcal{V}_a^{(2)}\) see §3. From the holomorphy of \(\mathcal{F}_{12}\) it follows that

\[
(\mathcal{V}_a^{(1)}(x))\psi_1^{-1} = (\mathcal{V}_a^{(1)}(x))\psi_2^{-1}
\]

on \(\mathcal{U}_{12}\). Notice that as functions on \(\mathbb{CP}^1\) the matrices \(\psi_1\) and \(\psi_2\) are regular on \(\Omega_1\) and \(\Omega_2\), respectively. Hence, \(\psi_{1,2}\) can be expanded on \(\Omega_1 \cap \Omega_2\) in powers of \(\lambda\):

\[
\psi_1(x, \lambda) = \sum_{n=0}^{\infty} \lambda^n \psi_1^n(x), \quad \psi_2(x, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} \psi_2^n(x).
\]

If we substitute the expansion of \(\psi_{1,2}\) in powers of \(\lambda\) into (4.20), both the sides of (4.20) must be linear in \(\lambda\), and we have

\[
(\partial_{\bar{\lambda}} - \partial_{\lambda} \bar{\lambda})(\psi_1^{-1}) = (\partial_{\bar{\lambda}} - \partial_{\lambda} \bar{\lambda})(\psi_2^{-1}) = -\lambda A_{\psi_1}^1(x) - \lambda A_{\psi_2}^2(x),
\]

where

\[
A_{\psi_i}^1 := -\text{Res}_{\lambda=0} \lambda^{-2}(\mathcal{V}_1^{(1)}(x))\psi_1^{-1} = -\int_{S^1} \frac{d\lambda}{2\pi i \lambda^2}(\mathcal{V}_1^{(1)}(x))\psi_2^{-1} = -\partial_{\bar{\lambda}} \psi_0^0(\psi_2^{-1})^{-1},
\]

\[
A_{\psi_i}^2 := \text{Res}_{\lambda=0} \lambda^{-2}(\mathcal{V}_2^{(1)}(x))\psi_2^{-1} = \int_{S^1} \frac{d\lambda}{2\pi i \lambda^2}(\mathcal{V}_2^{(1)}(x))\psi_2^{-1} = -\partial_{\bar{\lambda}} \psi_0^0(\psi_2^{-1})^{-1},
\]

\[
A_{\psi_1}^1 := \text{Res}_{\lambda=0} \lambda^{-1}(\mathcal{V}_1^{(1)}(x))\psi_1^{-1} = -\int_{S^1} \frac{d\lambda}{2\pi i \lambda}(\mathcal{V}_1^{(1)}(x))\psi_1^{-1} = -\partial_{\bar{\lambda}} \psi_0^0(\psi_1^{-1})^{-1},
\]

\[
A_{\psi_2}^2 := \text{Res}_{\lambda=0} \lambda^{-1}(\mathcal{V}_2^{(1)}(x))\psi_2^{-1} = -\int_{S^1} \frac{d\lambda}{2\pi i \lambda}(\mathcal{V}_2^{(1)}(x))\psi_1^{-1} = -\partial_{\bar{\lambda}} \psi_0^0(\psi_1^{-1})^{-1}.
\]

Here, the contour \(S^1 = \{\lambda \in \mathbb{CP}^1 : |\lambda| = 1\}\) circles once around \(\lambda = 0\) and the contour integral determines residue \(\text{Res}\) at the point \(\lambda = 0\).

The components \(\{A_{\mu}\}\) of the gauge potential defined by (4.23) satisfy the SDYM equations on \(U\) which are the compatibility conditions of eqs.(4.22). Thus, starting from a holomorphic matrix-valued function \(\mathcal{F}_{12}\) which is a transition matrix of a holomorphic vector bundle \(E'\) over the twistor space \(\mathcal{P}\), we have completed the procedure of reconstructing a gauge potential \(A\) which defines a self-dual connection on a complex vector bundle \(E\) over \(U \subset \mathbb{R}^4\). As it was explained in §3.3, the transformations (4.14), (4.15) of \(\mathcal{F}_{12}\) into a holomorphically equivalent transition matrix \(h_1\mathcal{F}_{12}h_2^{-1}\) do not change \(A_{\mu}\), and gauge transformations
A_\mu \mapsto A_\mu^g$ inducing the transformations (4.13) do not change $\mathcal{F}_{12}$. It follows from the twistor construction that a self-dual gauge potential $A$ is real-analytic.

To sum up, we have described a one-to-one correspondence between gauge equivalence classes of solutions to the SDYM equations on an open subset $U$ of the Euclidean 4-space and equivalence classes of holomorphic vector bundles $E'$ over the twistor space $\mathcal{P}$ satisfying the conditions: (i) bundles $E'$ are holomorphically trivial on each real projective line $\mathbb{CP}^1_x$, $x \in U$, in $\mathcal{P}$, (ii) each $E'$ has a real structure. This is the Euclidean version of Ward’s theorem [41, 39].

**Remark.** A twistor correspondence between self-dual gauge fields and holomorphic bundles also exists in a more general situation [44]. Let us consider a real oriented four-manifold $M$ with a metric $g$ of signature $(+++++)$. The 4-manifold $M$ is called self-dual if its Weyl tensor is self-dual. In [43] it was proved that the twistor space $\mathcal{Z} \equiv \mathcal{Z}(M)$ for a self-dual manifold $M$ is a complex analytic 3-fold. There is a natural one-to-one correspondence between self-dual bundles $E$ over $M$ (in particular, over $\mathbb{R}^4, S^4, T^4, \ldots$) and holomorphic vector bundles $E'$ over the twistor space $\mathcal{Z}$. In the general case, bundles $E$ and $E'$ are not topologically trivial, as it takes place in the case of Euclidean space $\mathbb{R}^4$, when $\mathcal{P} \subset \mathcal{Z}(\mathbb{R}^4) = \mathbb{R}^4 \times \mathbb{CP}^1$.

## 5 Holomorphic bundles in the Čech approach

We are going to analyse the twistor correspondence between self-dual complex vector bundles $E$ over $U \subset \mathbb{R}^4$ and holomorphic bundles $E'$ over $\mathcal{P}$ from the group-theoretic point of view, i.e., we want to describe groups acting on the space of transition matrices $\mathcal{F}_{12}$ of the bundles $E'$, on the space of self-dual gauge potentials $A$ and on the moduli space of self-dual gauge fields. In our discussion, we shall use the notion of **local groups**, (**local**) actions of (**local**) groups on sets, **germs**, **sheaves** and **Čech cohomology**, definitions of which are recalled in Appendices A, B and C.

In this section, we shall describe symmetries and the moduli space of all holomorphic vector bundles over $\mathcal{P}$. This means that we shall consider holomorphic bundles over $\mathcal{P}$ which are not necessarily holomorphically trivial over $\mathbb{CP}^1_x \rightarrow \mathcal{P}$, $x \in U$, and do not satisfy the unitarity condition (4.16b). As recalled in Appendices B and C, there is a one-to-one correspondence between the set of isomorphism classes of holomorphic bundles over a complex space $X$ and the Čech 1-cohomology set $H^1(X, \mathcal{H})$ of the space $X$ with values in the sheaf $\mathcal{H} = \mathcal{O}^G$ of germs of holomorphic maps from $X$ into the complex Lie group $G^\mathbb{C}$. We shall consider this correspondence for our case of the complex twistor space $\mathcal{P}$ and the group $G^\mathbb{C} = SL(n, \mathbb{C})$ and describe it from the group-theoretic point of view.

### 5.1 Moduli space of holomorphic bundles over the twistor space $\mathcal{P}$

We consider the two-set open cover $\mathcal{U} = \{U_1, U_2\}$ of $\mathcal{P}$ (see §3.3), where $U_1, U_2$ are Stein manifolds. For this cover we have the following q-simplexes $\langle \{U_{\alpha_0}, \ldots, U_{\alpha_q}\} : \langle U_1 \rangle, \langle U_2 \rangle, \langle U_1 \cup U_2 \rangle, \langle U_1 \cup U_2 \rangle \rangle$, supports $U_1, U_2, U_1 \cap U_2 := U_1 \cap U_2$ of which are nonempty sets. Further, a 0-cocycle of the cover $\mathcal{U}$ with the coefficients in the sheaf $\mathcal{H} = \mathcal{O}^{SL(n, \mathbb{C})}$ is a map $f$, which associates with any q-simplex $\langle U_{\alpha_0}, \ldots, U_{\alpha_q}\rangle$ a section of the sheaf $\mathcal{H}$ over $U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$: $f_{\alpha_0 \ldots \alpha_q} \equiv f(U_{\alpha_0}) \cap \ldots \cap U_{\alpha_q}) \in \mathcal{H}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_q})$. In other words, a 0-cocycle of the cover $\mathcal{U}$ with values in $\mathcal{H}$ is a collection $\{f_{\alpha_0 \ldots \alpha_q}\}$ of sections of the sheaf $\mathcal{H}$ over nonempty intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$. The set of 0-cochains is denoted by $C^0(\mathcal{U}, \mathcal{H})$ (see Appendix C). In the considered case we have the sets of 0-cochains $C^0(\mathcal{U}, \mathcal{H})$ and 1-cochains $C^1(\mathcal{U}, \mathcal{H})$.

The set $C^0(\mathcal{U}, \mathcal{H})$ is a group under a pointwise multiplication. For $h = \{h_1, h_2\} = \{f_1, f_2\} \in C^0(\mathcal{U}, \mathcal{H})$ we have

$$h_1 f = \{(h_1 f)_1, (h_2 f)_2\} := \{h_1 f_1, h_2 f_2\},$$

(5.1)

where $h_{\alpha}, f_{\alpha} \in \mathcal{H}(U_{\alpha}), \alpha = 1, 2$. The set $C^1(\mathcal{U}, \mathcal{H})$ of all 1-cochains forms a group under the following operation: if $h = \{h_{12}, h_{21}\} = \{f_{12}, f_{21}\} \in C^1(\mathcal{U}, \mathcal{H})$, then

$$h f = \{(h f)_{12}, (h f)_{21}\} := \{h_{12} f_{12}, h_{21} f_{21}\},$$

(5.2)

where $h_{12}, h_{21}, f_{12}, f_{21} \in \mathcal{H}(U_{12}) \equiv \Gamma(U_{12}, \mathcal{H})$. Notice that $h_{12}$ and $h_{21}$ ($f_{12}$ and $f_{21}$) are elements of two different groups $\mathcal{H}(U_{12}) : \{h_{12}, h_{21}\} \in \mathcal{H}(U_{12}) \times \mathcal{H}(U_{12})$.

For the two-set open cover $\mathcal{U}$, sets of 0- and 1-cochains are defined by the formulæ

$$Z^0(\mathcal{U}, \mathcal{H}) = \{h_1, h_2 \in C^0(\mathcal{U}, \mathcal{H}) : h_1 = h_2 \text{ on } U_{12}\},$$

(5.3)

$$Z^1(\mathcal{U}, \mathcal{H}) = \{h_{12}, h_{21} \in C^1(\mathcal{U}, \mathcal{H}) : h_{12} = h_{21}^{-1}\},$$

(5.4)

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and the space $Z^n(\mathcal{U}, \mathcal{H})$ coincides with the group $H^n(\mathcal{P}, \mathcal{H}) \equiv \Gamma(\mathcal{P}, \mathcal{H})$ of global sections of the sheaf $\mathcal{H}$. The set $Z^1(\mathcal{U}, \mathcal{H})$ is not a group for the non-Abelian sheaf $\mathcal{H}$.

Finally, two cocycles $F, \tilde{F} \in Z^1(\mathcal{U}, \mathcal{H})$ are said to be equivalent, $\tilde{F} \sim F$, if

$$\tilde{F}_{12} = h_1 F_{12} h_2^{-1},$$

(5.5)

for some element $h = \{h_1, h_2\} \in C^0(\mathcal{U}, \mathcal{H})$ restricted to $\mathcal{U}_{12}$. A set of equivalence classes of 1-cocycles $F$ with respect to the equivalence relation (5.5) is called a Čech 1-cohomology set and denoted by $H^1(\mathcal{U}, \mathcal{H})$. In the general case we should take the direct limit of these sets $H^1(\mathcal{U}, \mathcal{H})$ over successive refinement of cover $\mathcal{U}$ of $\mathcal{P}$ to obtain $H^1(\mathcal{P}, \mathcal{H})$, the Čech 1-cohomology set of $\mathcal{P}$ with coefficients in $\mathcal{H}$. But in our case $\mathcal{U}_1, \mathcal{U}_2$ are Stein manifolds and therefore $H^1(\mathcal{U}, \mathcal{H}) = H^1(\mathcal{P}, \mathcal{H})$. The cohomology set $H^1(\mathcal{P}, \mathcal{H})$ is identified with the set of all holomorphic vector bundles over $\mathcal{P}$ with the group $SL(n, \mathbb{C})$ which are considered up to equivalence (5.5), i.e., with the moduli space of holomorphic vector bundles $E'$.

### 5.2 Action of the group $C^0(\mathcal{U}, \mathcal{H})$ on the space $Z^1(\mathcal{U}, \mathcal{H})$

Suppose that we are given a cover $\{\mathcal{U}_\gamma\}$ of the space $\mathcal{P}$, $\gamma = 1, 2, ..., $ and the groups $C^0(\{\mathcal{U}_\gamma\}, \mathcal{H})$ and $C^1(\{\mathcal{U}_\gamma\}, \mathcal{H})$ of 0-cochains and 1-cochains. Let us define the following action of the group $C^0$ on the group $C^1$ (automorphism $\sigma_0(h, .)$):

$$\sigma_0(h, f)_{\alpha\beta} = h_\beta f_{\alpha\beta} h_\beta^{-1} \quad \text{(no summation),}$$

(5.6)

where $h = \{h_\alpha\} \in C^0(\{\mathcal{U}_\gamma\}, \mathcal{H})$, $f = \{f_{\alpha\beta}\} \in C^1(\{\mathcal{U}_\gamma\}, \mathcal{H})$. Now we can define a twisted homomorphism $\delta^0 : C^0 \to C^1$ of the group $C^0$ into the group $C^1$ by the formula

$$\delta^0(h)_{\alpha\beta} = h_\alpha h_\beta^{-1},$$

(5.7a)

where $\delta^0(h) = \{\delta^0(h)_{\alpha\beta}\} \in C^1(\{\mathcal{U}_\gamma\}, \mathcal{H})$. It is not difficult to see that

$$\delta^0(hg) = \delta^0(h)\sigma_0(h, \delta^0(g)),$$

(5.7b)

i.e., the homomorphism $\delta^0$ is “twisted” by $\sigma_0$. The twisted homomorphism $\delta^0$ permits one to define an action $\rho_0$ of the group $C^0$ on $C^1$ as on a set. The corresponding transformations act on $C^1$ by the formula

$$\rho_0(h, f) = \delta^0(h)\sigma_0(h, f) \iff \rho_0(h, f)_{\alpha\beta} = h_\alpha f_{\alpha\beta} h_\beta^{-1} \quad \text{(no summation),}$$

(5.8a)

$$\rho_0(gh, f) = \rho_0(g, \rho_0(h, f)),$$

(5.8b)

where $h, g \in C^0(\{\mathcal{U}_\gamma\}, \mathcal{H})$, $f \in C^1(\{\mathcal{U}_\gamma\}, \mathcal{H})$. Of course, in (5.6)-(5.8) it is implied that the components $h_\alpha$ of the element $h \in C^0$ are restricted to $\mathcal{U}_{\alpha\beta}$. It is not difficult to verify that the action (5.8) preserves the space of 1-cocycles $Z^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \subset C^1(\{\mathcal{U}_\gamma\}, \mathcal{H})$.

For a two-set open cover $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathcal{P}$, the action $\rho_0$ of the group $C^0$ on the space $Z^1(\mathcal{U}, \mathcal{H})$ of 1-cocycles has the form

$$\rho_0(h, F)_{12} = h_{12} F_{12} h_{21}^{-1},$$

(5.9)

where $h \in C^0(\mathcal{U}, \mathcal{H})$, $F \in Z^1(\mathcal{U}, \mathcal{H})$. As already said, the action (5.9) (the special case of (5.8)) preserves the space $Z^1$, and the quotient space $\rho_0(C^0) \backslash Z^1$ ($\rho_0(C^0)$ acts on $Z^1$ on the left), i.e., the space of orbits of the group $C^0$ in $Z^1$,

$$H^1(\mathcal{P}, \mathcal{H}) = H^1(\mathcal{U}, \mathcal{H}) := \rho_0(C^0(\mathcal{U}, \mathcal{H})) \backslash Z^1(\mathcal{U}, \mathcal{H}),$$

(5.10)

is the Čech 1-cohomology set.

### 5.3 Action of the group $C^1(\mathcal{U}, \mathcal{H})$ on the space $Z^1(\mathcal{U}, \mathcal{H})$

For a two-set open cover $\mathcal{U}$ of $\mathcal{P}$ one may define an automorphism $\sigma(h, .) : C^1 \to C^1$, $h \in C^1$, of the group of 1-cochains by the formula

$$\sigma(h, f)_{12} = h_{21} f_{12} h_{21}^{-1},$$

(5.11)

where $h, f \in C^1(\mathcal{U}, \mathcal{H})$, and a twisted homomorphism $\delta : C^1 \to C^1$ by the formula

$$\delta(h) = \{\delta(h)_{12}, \delta(h)_{21}\} = \{h_{12} h_{21}^{-1}, h_{21} h_{12}^{-1}\},$$

(5.12)

where $h, \delta(h) \in C^1(\mathcal{U}, \mathcal{H})$. With the help of the homomorphisms $\sigma$ and $\delta$ one can define the action of the group $C^1$ on itself as follows:

$$\rho(h, f) = \delta(h)\sigma(h, f) \iff \rho(h, f)_{12} = h_{12} f_{12} h_{21}^{-1},$$

(5.13a)
\[ \rho(gh, f) = \rho(g, \rho(h, f)), \]  
\[ (5.13b) \]

where \( g, h, f \in C^1(\mathcal{U}, \mathcal{H}) \). This action preserves the set \( Z^1(\mathcal{U}, \mathcal{H}) \) of 1-cocycles, and for a cocycle \( \mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H}) \) we have

\[ \tilde{\mathcal{F}}_{12} := \rho(h, \mathcal{F})_{12} = h_{12} \mathcal{F}_{12} h_{12}^{-1}. \]
\[ (5.14) \]

It is easy to see that \( \tilde{\mathcal{F}}_{21} := h_{21} \mathcal{F}_{21} h_{21}^{-1} = (h_{12} \mathcal{F}_{12} h_{12}^{-1})^{-1} = \tilde{\mathcal{F}}_{12}^{-1} \), i.e., \( \tilde{\mathcal{F}} \) is a 1-cocycle.

For \( h = \{h_{12}, h_{21}\} \in C^1(\mathcal{U}, \mathcal{H}), \) the matrices \( h_{12}, h_{21} \in SL(n, \mathbb{C}) \) are arbitrary holomorphic matrix-valued functions on \( \mathcal{U}_{12} \) and therefore with the help of the action (5.14) one can obtain any cocycle from \( Z^1(\mathcal{U}, \mathcal{H}) \). In other words, the action of \( C^1(\mathcal{U}, \mathcal{H}) \) on \( Z^1(\mathcal{U}, \mathcal{H}) \) is transitive, and the set \( Z^1 \) can be identified with a homogeneous space \( C^1 / C^1_{\Delta} \),

\[ Z^1(\mathcal{U}, \mathcal{H}) = C^1(\mathcal{U}, \mathcal{H}) / C^1_{\Delta}(\mathcal{U}, \mathcal{H}), \]
\[ (5.15a) \]

where

\[ C^1_{\Delta}(\mathcal{U}, \mathcal{H}) = \{ \{h_{12}, h_{21}\} \in C^1(\mathcal{U}, \mathcal{H}) : h_{21} = h_{12} \} \]
\[ (5.15b) \]

is the stability subgroup of the trivial cocycle \( \mathcal{F}_{12}^0 = 1 \). The group \( C^1_{\Delta}(\mathcal{U}, \mathcal{H}) \) is the kernel of the homomorphism (5.12). Thus, the group \( C^1(\mathcal{U}, \mathcal{H}) \) acts transitively on the space \( Z^1(\mathcal{U}, \mathcal{H}) \) of holomorphic bundles \( E' \) over \( \mathcal{P} \).

**Remark.** The description of the group \( C^1 \) and of its action on the space \( Z^1 \) of cocycles in terms of matrix-valued functions depends on a cover of the space \( \mathcal{P} \). For a general system of local trivializations with an open cover \( \{\mathcal{U}_\gamma\}, \gamma \in I \), the elements \( \mathcal{F} \) of \( Z^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \) must satisfy the conditions

\[ \mathcal{F}_{\alpha\gamma} = 1 \quad (\text{no summation}) \quad \text{on} \quad \mathcal{U}_\alpha, \quad \mathcal{F}_{\beta\gamma} = \mathcal{F}_{\alpha\beta}^{-1} \quad \text{on} \quad \mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta, \]
\[ (5.16a) \]

\[ \mathcal{F}_{\alpha\beta} \mathcal{F}_{\gamma\beta} \mathcal{F}_{\gamma\alpha} = 1 \quad (\text{no summation}) \quad \text{on} \quad \mathcal{U}_{\alpha\beta\gamma} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset. \]
\[ (5.16b) \]

Then \( C^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \) acts on \( \mathcal{F} \in Z^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \) as follows:

\[ \mathcal{F}_{\alpha\beta} \mapsto \tilde{\mathcal{F}}_{\alpha\beta} := \rho(h, \mathcal{F})_{\alpha\beta} = h_{\alpha\beta} \mathcal{F}_{\alpha\beta} h_{\beta\alpha}^{-1} \quad (\text{no summation}). \]
\[ (5.17) \]

It is easily checked that the conditions (5.16a) for \( \tilde{\mathcal{F}} \) are satisfied, and from the conditions (5.16b) imposed on \( \tilde{\mathcal{F}}_{\alpha\beta} \), it follows that

\[ h_{\alpha\beta}|_{\mathcal{U}_{\alpha\beta\gamma}} = h_{\alpha\gamma}|_{\mathcal{U}_{\alpha\beta\gamma}}. \]
\[ (5.18) \]

It simply means that \( h_{\alpha\beta} \) are defined on

\[ \bigcup_{\alpha, \beta \in I} \mathcal{U}_{\alpha\beta}, \]
\[ (5.19) \]

and we denote by \( \tilde{C}^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \) the subgroup of all elements \( h = \{h_{\alpha\beta}\} \in C^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) \) satisfying (5.18). Thus, we obtain

\[ Z^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) = \tilde{C}^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) / C^1_{\Delta}(\{\mathcal{U}_\gamma\}, \mathcal{H}), \]
\[ (5.20a) \]

where

\[ \tilde{C}^1_{\Delta}(\{\mathcal{U}_\gamma\}, \mathcal{H}) = \{ \{h_{\alpha\beta}\} \in \tilde{C}^1(\{\mathcal{U}_\gamma\}, \mathcal{H}) : h_{\beta\alpha} = h_{\alpha\beta} \} \]
\[ (5.20b) \]

is the stability subgroup of the trivial cocycle \( \mathcal{F}_{\beta\alpha}^0 = 1 \). For a two-set open cover \( \mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\} \) we have \( \tilde{C}^1(\mathcal{U}, \mathcal{H}) = C^1(\mathcal{U}, \mathcal{H}) \).

It follows from the definitions that the groups \( C^0(\mathcal{U}, \mathcal{H}) \) and \( C^1(\mathcal{U}, \mathcal{H}) \) are direct products

\[ C^0(\mathcal{U}, \mathcal{H}) = \mathcal{H}(\mathcal{U}_1) \times \mathcal{H}(\mathcal{U}_2) \equiv \Gamma(\mathcal{U}_1, \mathcal{H}) \times \Gamma(\mathcal{U}_2, \mathcal{H}) \ni \{h_1, h_2\}, \]
\[ (5.21a) \]

\[ C^1(\mathcal{U}, \mathcal{H}) = \mathcal{H}(\mathcal{U}_{12}) \times \mathcal{H}(\mathcal{U}_{12}) \equiv \Gamma(\mathcal{U}_{12}, \mathcal{H}) \times \Gamma(\mathcal{U}_{12}, \mathcal{H}) \ni \{h_{12}, h_{21}\}, \]
\[ (5.21b) \]

of the groups \( \mathcal{H}(\mathcal{U}_1), \mathcal{H}(\mathcal{U}_2) \) and \( \mathcal{H}(\mathcal{U}_{12}) \) of sections over \( \mathcal{U}_1, \mathcal{U}_2 \) and \( \mathcal{U}_{12} \) of the sheaf \( \mathcal{H} \). Respectively, \( C^1_{\Delta}(\mathcal{U}, \mathcal{H}) \) coincides with the diagonal subgroup in the group \( \mathcal{H}(\mathcal{U}_{12}) \times \mathcal{H}(\mathcal{U}_{12}) \), and \( Z^1(\mathcal{U}, \mathcal{H}) \) coincides with the subset of elements \( h = \{h_{12}, h_{21}^{-1}\} \) from the group \( C^1(\mathcal{U}, \mathcal{H}) \).

Collating formulae (5.10) and (5.15), we obtain that

\[ H^1(\mathcal{P}, \mathcal{H}) = \rho_0(C^0) \setminus C^1 / C^1_{\Delta}, \]
\[ (5.22) \]

i.e., the moduli space of holomorphic bundles \( E' \) over \( \mathcal{P} \) is parametrized by the double coset space (5.22). It is not difficult to see that the 1-cohomology set (5.22) is isomorphic to

(i) the set of \( C^1_{\Delta} \)-orbits in \( Y^1 := \rho_0(C^0) \setminus C^1, \)
(ii) the set of $C^0$-orbits in $Z^1 = C^1/C^1_\Delta$, 
(iii) the set of $C^1$-orbits in $Y^1 \times Z^1$,
where an action of $h \in C^1$ on $(y, z) \in Y^1 \times Z^1$ is defined by the formula
\[ C^1 \times (Y^1 \times Z^1) \ni (h, (y, z)) : (y, z) \mapsto (yh, \rho(h^{-1}, z)) \in Y^1 \times Z^1. \]

To sum up, for the space $Z^1(\mathcal{U}, \mathcal{H})$ of holomorphic bundles $E'$ over $\mathcal{P}$, the group $C^1(\mathcal{U}, \mathcal{H})$ of 1-cochains for the cover $\mathcal{U}$ with values in the sheaf $\mathcal{H}$ of non-Abelian groups acts on the transition matrices $\mathcal{F}_{12}$ of bundles $E'$ by the left multiplication on matrices $h_{12} \in H(\mathcal{U}_{12})$ and by the right multiplication on matrices $\tilde{h}_{12}^0 \in H(\hat{\mathcal{U}}_{12})$. This group acts on $Z^1$ transitively, and the space $Z^1$ is the coset space (5.15a) (or (5.20a) for an arbitrary cover of $\mathcal{P}$). So $C^1(\mathcal{U}, \mathcal{H})$ is the symmetry group of the space of holomorphic bundles $E'$ in the Čech approach. The moduli space $H^1(\mathcal{P}, \mathcal{H})$ of bundles $E'$ is the double coset space (5.22).

5.4 The group $\mathcal{H}(\mathcal{P})$ of automorphisms of the complex manifold $\mathcal{P}$

Let $X$ be a compact smooth manifold, $G$ a compact simple connected Lie group and $\text{Aut} G$ a group of automorphisms of the group $G$. Consider the group $\text{Map}(X; G)$ of smooth maps from $X$ into $G$ and the connected component of the unity $\text{Map}_0(X; G)$ of the group $\text{Map}(X; G)$. It is well-known that the group of automorphisms of the group $\text{Map}_0(X; G)$ is a semidirect product
\[ \text{Diff}(X) \ltimes \text{Map}(X; \text{Aut} G) \] (5.23)
of the diffeomorphism group $\text{Diff}(X)$ of the manifold $X$ and the group of automorphisms $\text{Map}(X; \text{Aut} G)$ (for proof see §3.4 in [37]).

As a set the space $Z^1(\mathcal{U}, \mathcal{H})$ considered above coincides with the group $\text{Map}(\mathcal{U}_{12}; \text{SL}(n, \mathbb{C}))$ of holomorphic maps from $\mathcal{U}_{12}$ into $\text{SL}(n, \mathbb{C})$ and it is an analogue of the group $\text{Map}_0(X; G)$. The group $C^1(\mathcal{U}, \mathcal{H})$ acting on the space $Z^1(\mathcal{U}, \mathcal{H})$ is respectively an analogue of the group of automorphisms $\text{Map}(X; \text{Aut} G)$. It is clear that there should be an analogue of the diffeomorphism group from (5.23), i.e., some group of transformations of the coordinates of the space $\mathcal{P}$ acting on the set $Z^1(\mathcal{U}, \mathcal{H})$.

Remember that as a smooth manifold the twistor space is $\mathcal{P} = U \times S^2$. At the same time, $\mathcal{P}$ is a complex 3-manifold, and in §3.3 we have introduced the complex coordinates $z_1 : \mathcal{U}_1 \to C^3$, $z_2 : \mathcal{U}_2 \to C^3$ on $\mathcal{P}$ and the holomorphic transition function $f_{12}$ connecting $z_1$ and $z_2$ on $\mathcal{U}_{12}$. Let $\eta : \mathcal{P} \to \mathcal{P}$ be an arbitrary transformation from the group $\mathcal{H}(\mathcal{P})$ of diffeomorphisms of the twistor space $\mathcal{P}$. Let us denote by $\mathcal{U}_1 := \eta(\mathcal{U}_1)$, $\mathcal{U}_2 := \eta(\mathcal{U}_2)$ the images of the open sets $\mathcal{U}_1, \mathcal{U}_2$ in $\mathcal{P}$. We have
\[ \eta(\mathcal{P}) = \eta(\mathcal{U}_1 \cup \mathcal{U}_2) = \eta(\mathcal{U}_1) \cup \eta(\mathcal{U}_2) = \tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2, \] (5.24a)
\[ \eta(\mathcal{U}_{12}) = \eta(\mathcal{U}_1 \cap \mathcal{U}_2) = \eta(\mathcal{U}_1) \cap \eta(\mathcal{U}_2) = \tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2, \] (5.24b)
since the map $\eta$ is a bijection.

Let us consider the restriction of the map $\eta$ to $\mathcal{U}_{12}$, i.e., the local diffeomorphism $\eta|_{\mathcal{U}_{12}} : \mathcal{U}_{12} \to \mathcal{P}$. On $\mathcal{U}_{12} = \eta(\mathcal{U}_{12})$ one can always introduce complex coordinates $\tilde{z}_1 : \mathcal{U}_{12} \to C^3$, $\tilde{z}_2 : \mathcal{U}_{12} \to C^3$ related by a holomorphic transition function $\tilde{f}_{12}$ such that the map $\eta|_{\mathcal{U}_{12}} : \mathcal{U}_{12} \to \mathcal{U}_{12}$ will be holomorphic in the chosen coordinates. In other words, domains $\mathcal{U}_{12}$ and $\mathcal{U}_{12}$ are biholomorphic and there exist holomorphic functions $\eta_1, \eta_2$ such that
\[ \tilde{z}_1^a \circ \eta = \eta_1^a(z_1^b), \quad \tilde{z}_2^a \circ \eta = \eta_2^a(z_2^b), \quad \tilde{z}_1^a = \tilde{f}_{12}^a(z_2^b). \] (5.25)
These maps form the (local) group $\mathcal{H}(\mathcal{U}_{12})$.

Having the group $\mathcal{H}(\mathcal{U}_{12})$ of local holomorphic maps $\eta|_{\mathcal{U}_{12}} : \mathcal{U}_{12} \to \mathcal{P}$, one can define its action on transition matrices $\mathcal{F}$ of holomorphic bundles over $\mathcal{P}$. But in this connection the following questions arise:

1. Is it possible to introduce on $\tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2$ complex coordinates $\tilde{z}_1 : \tilde{\mathcal{U}}_1 \to C^3$, $\tilde{z}_2 : \tilde{\mathcal{U}}_2 \to C^3$ related by a holomorphic transition function $\tilde{f}_{12}$?

2. Can the coordinates $\tilde{z}_1, \tilde{z}_2$ on $\tilde{\mathcal{U}}_{12}$ be extended to $\tilde{\mathcal{U}}_1, \tilde{\mathcal{U}}_2$ and will they be equivalent to the coordinates $\tilde{z}_1, \tilde{z}_2$?

The diffeomorphism group $\text{Diff}(\mathcal{P})$ acts not only on transition matrices of bundles $E'$ over $\mathcal{P}$, but also on the complex structure of the space $\mathcal{P}$. But a change of the complex structure of the space $\mathcal{P}$ leads to a change of the conformal structure and a metric on $U \subset \mathbb{R}^4$ by virtue of the twistor correspondence [17] [13]. If we are interested in symmetries of the SDYM equations on the space $U$ with a conformally flat metric,
then we have to consider only those diffeomorphisms \( \eta \in \text{Diff}(\mathcal{P}) \) which preserve the complex structure of \( \mathcal{P} \). These maps \( \eta : \mathcal{P} \to \mathcal{P} \) form the group of biholomorphic transformations of the space \( \mathcal{P} \) which we shall denote by \( \mathcal{H}(\mathcal{P}) \). It is a subgroup of the diffeomorphism group: \( \mathcal{H}(\mathcal{P}) \subseteq \text{Diff}(\mathcal{P}) \).

In the coordinates \( z_1, z_2, \bar{z}_1, \bar{z}_2 \) transformations \( \eta \in \mathcal{H}(\mathcal{P}) \) are defined by the holomorphic functions

\[
\bar{z}_1^{\eta}, \bar{z}_2^{\eta}, z_1^{\eta}, z_2^{\eta} = f_{12}^{\eta} (\bar{z}_2^{\eta}).
\]

(5.26)

Formulae (5.26) are not always convenient because there the coordinates \( z_\alpha \) are calculated at points \( p \in \mathcal{P} \), and the coordinates \( \bar{z}_\alpha \) are calculated at points \( q = \eta(p) \in \mathcal{P} \). It is often more convenient to define \( \eta \) by transition functions \( \eta_{\alpha\beta} \) from \( z_\alpha \) to \( \bar{z}_\beta \) in the domains \( U_\alpha \cap U_\beta \) (if \( U_\alpha \cap U_\beta \neq \emptyset \)), then \( z_\alpha \) and \( \bar{z}_\beta \) are calculated at the same points \( p \in U_\alpha \cap U_\beta \). For example, the conformal transformations (2.4) of the space \( \mathbb{R}^4 \) induce such holomorphic transformations of coordinates \( \{z_1^{\eta}\} \rightarrow \{\bar{z}_1^{\eta}\} \) of the twistor space \( \mathcal{Z} = \mathcal{Z}(\mathbb{R}^4) \) that on \( U_1 \cap U_2 \) we have

translating: \( \bar{z}_1 = z_1 + a_1 - \bar{a}_2 z_1^3, \bar{z}_2 = z_2 + a_2 + \bar{a}_1 z_1^3, \bar{z}_3 = z_3^3, \bar{z}_4 = z_4^3, \)

rotations induced by \( \{X_\alpha\} \): \( \bar{z}_1 = \frac{z_1}{a - bz_1^3}, \bar{z}_2 = \frac{z_2}{a - bz_1^3}, \bar{z}_3 = \frac{b + \bar{a}z_1^3}{a - \bar{b}z_1^3}, \bar{z}_4 = \frac{a}{a - \bar{b}z_1^3} \in SU_L(2), \)

rotations induced by \( \{Y_\alpha\} \): \( \bar{z}_1 = \frac{z_1}{1 + \alpha_1 z_1^3 + \alpha_2 z_1^3}, \bar{z}_2 = \frac{z_2}{1 + \alpha_1 z_1^3 + \alpha_2 z_1^3}, \bar{z}_3 = \frac{1 + \bar{a}_1 z_1^3 + \bar{a}_2 z_1^3}{1 + \alpha_1 z_1^3 + \alpha_2 z_1^3}, \bar{z}_4 = \frac{1 + \bar{a}_1 z_1^3 + \bar{a}_2 z_1^3}{1 + \alpha_1 z_1^3 + \alpha_2 z_1^3}, \)

dilatations: \( \bar{z}_1 = e^{z_1}, \bar{z}_2 = e^{z_2}, \bar{z}_3 = z_3^3, \bar{z}_4 = z_4^3, \)

Here \( a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{C}, \alpha \in \mathbb{R}. \)

5.5 Action of the group \( \mathcal{H}(\mathcal{P}) \) on the space \( \mathcal{Z}^1(\mathcal{U}, \mathcal{H}) \)

Action of the group \( \mathcal{H}(\mathcal{P}) \) of complex-analytic diffeomorphisms of the space \( \mathcal{P} \) on transition matrices of holomorphic bundles \( E' \) over \( \mathcal{P} \) is defined in the following way. We consider a two-set open cover \( \mathcal{U} = \{U_1, U_2\} \) of \( \mathcal{P} \) and a transition matrix \( \mathcal{F} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{H}) \) of a bundle \( E' \). After a transformation \( \mathcal{H}(\mathcal{P}) \ni \eta : \mathcal{P} \to \mathcal{P} \) we have a new cover \( \mathcal{U} = \{U_1, U_2\}, \mathcal{U}_1 = \eta(U_1), \mathcal{U}_2 = \eta(U_2) \). Let us consider the common refinement both of the covers. Denote

\[
\hat{U}_1 := U_1 \cap \mathcal{U}_1, \hat{U}_2 := U_2 \cap \mathcal{U}_2, \hat{U}_3 := U_3 \cap \hat{U}_1, \hat{U}_4 := U_4 \cap \hat{U}_2,
\]

(5.27a)

to give the refined cover \( \hat{\mathcal{U}} = \{\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4\} \).

The cocycle \( \mathcal{F} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{H}) \) induces the following 1-cocycle \( \hat{\mathcal{F}} \in \mathcal{Z}^1(\hat{\mathcal{U}}, \hat{\mathcal{H}}) \):

\[
\hat{\mathcal{F}} = \{\hat{F}_{12}, \hat{F}_{13}, \hat{F}_{14}, \hat{F}_{23}, \hat{F}_{24}, \hat{F}_{34}\} := \{1, \hat{F}_{12}(z_1^\eta), \hat{F}_{12}(z_1^\eta), \hat{F}_{12}(z_1^\eta), \hat{F}_{12}(z_1^\eta), 1\},
\]

(5.27b)

where \( \hat{F}_{\alpha\beta} \) is defined in \( \hat{U}_{\alpha\beta} := \hat{U}_\alpha \cap \hat{U}_\beta \) and

\[
\hat{U}_{12} := \hat{U}_1 \cap \hat{U}_2 = U_1 \cap U_2, \hat{U}_{13} := \hat{U}_1 \cap \hat{U}_3 = U_1 \cap U_3, \hat{U}_{14} := \hat{U}_1 \cap \hat{U}_4 = U_1 \cap U_4, \\
\hat{U}_{23} := \hat{U}_2 \cap \hat{U}_3 = U_2 \cap U_3, \hat{U}_{24} := \hat{U}_2 \cap \hat{U}_4 = U_2 \cap U_4, \hat{U}_{34} := \hat{U}_3 \cap \hat{U}_4 = U_3 \cap U_4.
\]

(5.28)

The cocycle \( \hat{\mathcal{F}} \) is equivalent to the cocycle \( \mathcal{F} \), and the group \( \mathcal{H}(\mathcal{P}) \) acts on \( \mathcal{F} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{H}) \) as follows:

\[
\mathcal{H}(\mathcal{P}) \ni \eta : \mathcal{F} \to \hat{\mathcal{F}} \to \rho(\eta, \mathcal{F}) \equiv \mathcal{F}^\eta = \{\hat{F}_{12}^\eta, \hat{F}_{13}^\eta, \hat{F}_{14}^\eta, \hat{F}_{23}^\eta, \hat{F}_{24}^\eta, \hat{F}_{34}^\eta\},
\]

\[
\hat{F}_{12}^\eta := 1, \hat{F}_{13}^\eta := \hat{F}_{12}(\eta_1^\eta(z_1)), \hat{F}_{14}^\eta := \hat{F}_{12}(\eta_2^\eta(z_1)), \\
\hat{F}_{23}^\eta := \hat{F}_{12}(\eta_1^\eta(z_1)), \hat{F}_{24}^\eta := \hat{F}_{12}(\eta_2^\eta(z_1)), \hat{F}_{34}^\eta := 1.
\]

(5.29)

In the general case cocycles \( \mathcal{F} \) and \( \mathcal{F}^\eta \) are not equivalent and therefore the group \( \mathcal{H}(\mathcal{P}) \) of biholomorphic transformations of the twistor space \( \mathcal{P} \) acts nontrivially on the space \( \mathcal{Z}^1(\mathcal{U}, \mathcal{H}) \). This action includes refining of the cover and a transition to an equivalent cocycle.
It is usually considered that elements $\eta \in \mathcal{H}(\mathcal{P})$ which are close to the identity do not move the covering sets. That is, if $\eta$ is close to the identity, it is possible to define the action of such $\eta \in \mathcal{H}(\mathcal{P})$ as follows:

$$\rho(\eta, \cdot) : \mathcal{F}_{12} \mapsto \rho(\eta, \mathcal{F}_{12}) = \mathcal{F}_{12}^\eta = \mathcal{F}_{12}(\eta(z_1)),$$

(5.30)
i.e., without using the refined cover $\hat{\mathcal{U}}$. In other words, the action of a neighbourhood of unity of the group $\mathcal{H}(\mathcal{P})$ maps $Z^1(\mathcal{U}, \mathcal{H})$ into itself. In what follows we shall study just this case.

Returning to §5.4 and to the beginning of §5.4, we come to the conclusion that the full group of continuous symmetries acting on the space $Z^1(\mathcal{U}, \mathcal{H})$ of holomorphic bundles $E'$ over $\mathcal{P}$ is a semidirect product

$$\mathcal{H}(\mathcal{P}) \ltimes C^1(\mathcal{U}, \mathcal{H})$$

(5.31)
of the group $\mathcal{H}(\mathcal{P})$ of holomorphic automorphisms of the space $\mathcal{P}$ and of the group $C^1(\mathcal{U}, \mathcal{H})$ of 1-cochains for the cover $\mathcal{U}$ with values in the sheaf $\mathcal{H}$ of holomorphic maps of the space $\mathcal{P}$ into the Lie group $SL(n, \mathbb{C})$.

6 Symmetries in holomorphic setting

6.1 Germs of sets and groups

Let $\mathcal{B}$ be a set with a marked point $e \in \mathcal{B}$. The element $e$ is called the unity. If $\mathcal{B}$ and $\mathcal{C}$ are sets with the marked points which we denote by the same letter $e$, then a homomorphism of the set $\mathcal{B}$ into the set $\mathcal{C}$ is such a map $\varphi : \mathcal{B} \to \mathcal{C}$ that $\varphi(e) = e$. The homomorphism $\mathcal{B} \to \mathcal{C}$ is said to be the isomorphism if it maps $\mathcal{B}$ onto $\mathcal{C}$ bijectively. The set $\text{Ker} \varphi = \varphi^{-1}(e)$ with the marked point $e$ is called the kernel of the homomorphism $\varphi$.

Let $X$ be a set with a marked point $e$, and let $Y_1, Y_2$ be two subsets of the set $X$ also containing the point $e$. The sets $Y_1, Y_2$ are called equivalent at the point $e$ if there exists such a neighbourhood $Y_3$ of this point that $Y_1 \cap Y_3 = Y_2 \cap Y_3$. The class of all sets equivalent to the set $Y_1$ is called the germ of this set at the point $e$ and denoted by $Y$. The sets $Y_1, Y_2, Y_3$ are representatives of the germ $Y$ of sets.

In Appendix A, a notion of group germs $\mathcal{G}$ based on the definition of germs of sets is introduced. Representatives of the group germ $\mathcal{G}$ are local groups, i.e., open neighbourhoods $\mathcal{G}$ of the identity $e \equiv 1$, which are closed under all group operations (multiplication, operation of inverse etc). In particular, we shall consider the germs $\mathcal{C}$ and $\mathcal{H}$ of the groups $C^1(\mathcal{U}, \mathcal{H})$ and $\mathcal{H}(\mathcal{P})$ described in §5.4.

6.2 Holomorphic triviality of bundles $E'$ on $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}$

Let us consider the twistor space $Z \equiv Z(\mathbb{R}^4)$ of $\mathbb{R}^4$ and the moduli space $H^1(Z, \mathcal{H})$ of holomorphic bundles $E'$ over the space $Z$. With the sheaf $\mathcal{H} = \mathcal{O}^{SL(n, \mathbb{C})}$ of holomorphic maps from $Z$ into the group $SL(n, \mathbb{C})$ one associates the sheaf $\mathcal{O}^{sl(n, \mathbb{C})}$ of holomorphic maps from $Z$ into the Lie algebra $sl(n, \mathbb{C})$. The Abelian group (by addition) of cohomologies $H^1(Z, \mathcal{O}^{sl(n, \mathbb{C})})$ of the space $Z$ with values in the sheaf $\mathcal{O}^{sl(n, \mathbb{C})}$ parametrizes infinitesimal deformations of the trivial bundle $E'_0 = Z \times \mathbb{C}^n$ and $\dim H^1(Z, \mathcal{O}^{sl(n, \mathbb{C})}) = \infty$, i.e., in an arbitrarily small neighbourhood of the trivial bundle $E'_0$ there exists an infinite number of holomorphically nontrivial bundles $E'$.

Let us fix an arbitrary point $x_0 \in \mathbb{R}^4$ and consider a real projective line $\mathbb{C}P^1_{x_0}$ embedded into $Z$. Now we consider the restriction $\mathcal{O}^{sl(n, \mathbb{C})}_{x_0} := \mathcal{O}^{sl(n, \mathbb{C})}|_{\mathbb{C}P^1_{x_0}}$ of the sheaf $\mathcal{O}^{sl(n, \mathbb{C})}$ to $\mathbb{C}P^1_{x_0}$ and the cohomology group $H^1(\mathbb{C}P^1_{x_0}, \mathcal{O}^{sl(n, \mathbb{C})}_{x_0})$ parametrizing infinitesimal deformations of the trivial holomorphic bundle $E'_{0|x_0} := \mathbb{C}P^1_{x_0} \times \mathbb{C}^n$. It is easily seen that

$$H^1(\mathbb{C}P^1_{x_0}, \mathcal{O}^{sl(n, \mathbb{C})}_{x_0}) = 0,$$

(6.1)
because $H^1(\mathbb{C}P^1, \mathcal{O}) = 0$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}P^1$. The equality (6.1) means that there exists a sufficiently small open neighbourhood $U \subset \mathbb{R}^4$ of the point $x_0$ and an open subset $M\ni e$ of the space $H^1(P, \mathcal{H})$, where $P \subset Z$ is the twistor space of $U$, such that for the bundles $E'$, representing points $E'_{x}$ of the space $M \subset H^1(P, \mathcal{H})$, their restriction $E'_{x}$ to $\mathbb{C}P^1_{x} \to P$ will be holomorphically trivial for any $x \in U$ (version of the Kodaira theorem). In other words, small enough deformations do not change the trivializability of the bundle $E'$ over real projective lines in a neighbourhood of a given projective line $\mathbb{C}P^1_{x_0}$ (for discussion see e.g. [3], [10]).

Projective lines $\{\mathbb{C}P^1_x\}_{x \in U}$ form a family of complex 1-manifolds parametrized by $x \in U$, and $\mathbb{C}P^1_x$ coincides with $\mathbb{C}P^1 \times \{x\}$ in the direct product $\mathbb{C}P^1 \times U \approx P$. We consider holomorphic bundles $E'$ over $P$ with transition matrices $F$ from $Z'(\mathcal{U}, \mathcal{H})$ and their restriction to $\mathbb{C}P^1_x \to P$, $x \in U$. Then a family of holomorphic maps $F_{12}(x, \lambda)$ from $U \times \Omega_{12}$ into $SL(n, \mathbb{C})$ determines a family of vector bundles $E'_x := E'|_{\mathbb{C}P^1_x}$ over $\mathbb{C}P^1_{x_0}$.
labelled by the parameters \( x \in U \subset \mathbb{R}^4 \). In this family there exists a marked family of homolorphically trivial bundles \( E'_0 = \mathbb{C}P^1_x \times \mathbb{C}^n \). Finally, we have introduced an open subset \( \mathcal{M} \) of the set \( H^1(\mathcal{P}, \mathcal{H}) \) (being an open neighbourhood of the marked point \( e \in H^1(\mathcal{P}, \mathcal{H}) \) of moduli of those bundles \( E' \) from \( H^1(\mathcal{P}, \mathcal{H}) \), which are holomorphically trivial on \( \mathbb{C}P^1_x \rightarrow \mathcal{P} \) for all \( x \in U \). With each point \( m \equiv [E'] \in \mathcal{M} \) one can associate a bundle \( \mathcal{E}_m := E'(m) \) over \( \mathcal{P} \). Then we have a family \( \{\mathcal{E}_m\}_{m \in \mathcal{M}} \) of holomorphic bundles over \( \mathcal{P} \), parametrized by \( m \in \mathcal{M} \). The marked point in this family is the trivial bundle \( \mathcal{E}_0 := E'(e) \) (the isomorphism class of the bundle \( E'_0 \)).

Let \( X \) be a complex space. Consider a family of holomorphic vector bundles of rank \( n \) with the base \( X \) and a family of complex parameters \( T_i \), i.e., a holomorphic vector bundle \( \mathcal{E} \) of rank \( n \) over \( X \times T_i \). The space \( T \) is called the base of deformation. For \( t \in T_i \), we denote by \( \mathcal{E}_t \) a bundle over \( X \) which is induced by restriction of \( \mathcal{E} \) to \( X \times \{t\} \) with a natural identification \( X \leftrightarrow X \times \{t\} \). In our case, we have a holomorphic vector bundle \( \mathcal{E} \) of rank \( n \) over \( \mathcal{P} \times \mathcal{M} \), \( \mathcal{M} \subset H^1(\mathcal{P}, \mathcal{H}) \).

Using the theorems \( \S \) one can consider sets equivalent to the set \( \mathcal{M} \), and a class of all open subsets in \( H^1(\mathcal{P}, \mathcal{H}) \), equivalent to the set \( \mathcal{M} \), defines the germ \( \mathcal{M} \) of this set at the point \( e \). Of course, the notion of equivalence is supplemented here by the demand that all representations \( \mathcal{M}, \mathcal{M}' \), ..., of the germ \( \mathcal{M} \) should be moduli spaces of those bundles from \( Z^1(\mathcal{U}, \mathcal{H}) \) which are holomorphically trivial on \( \mathbb{C}P^1_x, x \in U \). Let us stress that a choice of a concrete representation \( \mathcal{M}, \mathcal{M}', \ldots \) of the germ \( \mathcal{M} \) is not essential since a different choice gives equivalent deformations of the bundle \( E'_0 \). That is why in the modern deformation theory of complex spaces and holomorphic bundles as a base of deformation one takes not a set with a marked point \( e \) but the germ of this set at the marked point (see e.g. \( [10] \)).

Now we take a point \( m = [E'] \in \mathcal{M} \) and the transition matrix \( \mathcal{F}(m) \in Z^1(\mathcal{U}, \mathcal{H}) \) in the bundle \( E' \) representing this point. Acting on \( \mathcal{F}(m) \) by all possible elements of \( C^0(\mathcal{U}, \mathcal{H}) \) by formulae (5.8), (5.9), we obtain an orbit \( \rho_0(C^0)(\mathcal{F}(m)) \) of the point \( \mathcal{F}(m) \in Z^1(\mathcal{U}, \mathcal{H}) \) under the action \( \rho_0 \) of the group \( C^0(\mathcal{U}, \mathcal{H}) \). This orbit coincides with the space \( \mathcal{C}(\mathcal{U}, \mathcal{H}) := C^0(\mathcal{U}, \mathcal{H})/H^0(\mathcal{P}, \mathcal{H}) \), and we denote it by \( \mathcal{C}_m(\mathcal{U}, \mathcal{H}) \). Consider the union of orbits

\[
\mathcal{N} = \bigcup_{m \in \mathcal{M}} \mathcal{C}_m(\mathcal{U}, \mathcal{H}).
\]

The space \( \mathcal{N} \subset Z^1(\mathcal{U}, \mathcal{H}) \) is a bundle over \( \mathcal{M} \) associated with the principal fibre bundle \( P(\mathcal{M}, C^0) \),

\[
\mathcal{N} = P(\mathcal{M}, C^0(\mathcal{U}, \mathcal{H})) \times_{C^0(\mathcal{U}, \mathcal{H})} \mathcal{C}(\mathcal{U}, \mathcal{H}),
\]

and the group \( C^0 \) acts on \( \mathcal{N} \) on the left. The space \( \mathcal{N} \) is a neighbourhood of the unity \( \mathcal{F}^0 = 1 \) in the space \( Z^1(\mathcal{U}, \mathcal{H}) \). We consider an open subset \( \mathcal{N}' \subset Z^1(\mathcal{U}, \mathcal{H}) \) equivalent to \( \mathcal{N} \) and such that for all transition matrices \( \mathcal{F} \) from \( \mathcal{N}' \) there exists a solution of the Riemann-Hilbert problem (4.17) on \( \mathbb{C}P^1_x \) and \( \mathcal{F}^0 \in \mathcal{N}' \). Then we can introduce the germ \( \mathcal{N} \) of the set \( \mathcal{N} \) at the point \( \mathcal{F}^0 \) as a class of sets equivalent to \( \mathcal{N}' \).

The group \( C^0(\mathcal{U}, \mathcal{H}) \) acts on any representative \( \mathcal{N} \) of the germ \( \mathcal{N} \), and we have

\[
\mathcal{M} = \rho_0(C^0)\backslash \mathcal{N}.
\]

i.e., \( \mathcal{M} \) is a set of orbits of the group \( C^0 \) in the space \( \mathcal{N} \) (cf. (5.10)). By virtue of the Penrose-Ward correspondence described in \( \S \), there is a bijection between the space \( \mathcal{M} \) and the moduli space of real-analytic solutions to the SDYM equations on an open set \( U \subset \mathbb{R}^4 \) which are sufficiently close to the trivial solution \( A_0 = 0 \). A set of all such solutions is called the space of local solutions (a small open neighbourhood of the point \( A_0 = 0 \)). So, \( \mathcal{M} \) is bijective to the moduli space of local solutions to the SDYM equations with the marked point \( A_0 = 0 \). However, as a marked point in \( Z^1(\mathcal{U}, \mathcal{H}) \) one can choose a transition matrix \( \mathcal{F} \) of a bundle \( E' \) over \( \mathcal{P} \), holomorphically trivial on \( \mathbb{C}P^1_x \), which corresponds to a solution \( \hat{A} \) of the SDYM equations. Then one can consider bundles (trivial on \( \mathbb{C}P^1_x, x \in U \)) with transition matrices from an open neighbourhood \( \mathcal{N} \subset Z^1(\mathcal{U}, \mathcal{H}) \) of the point \( \hat{A} \) and the moduli space \( \mathcal{M} = \rho_0(C^0)\backslash \mathcal{N} \) of these bundles. This space \( \mathcal{M} \) will be bijective to the space of local solutions to the SDYM equations that are near the solution \( \hat{A} \).

### 6.3 Jumping points and jumping lines

Let us consider a holomorphic bundle \( E' \) over the twistor space \( Z = L^{-1} \oplus L^{-1} \simeq \mathbb{R}^4 \times \mathbb{C}P^1 \) such that its restriction to \( \mathcal{P} \subset Z \) belongs to the space \( \mathcal{N} \subset Z^1(\mathcal{U}, \mathcal{H}) \) introduced in \( \S \). In general the bundle \( E' \) will be holomorphically trivial on real projective lines \( \mathbb{C}P^1_x \) parametrized not by \( x \) from \( U \) but by \( x \) from a “wider” open set \( U' \supset U \). Those points \( x \) from \( \mathbb{R}^4 \), for which \( E'_{|\mathbb{C}P^1_x} \) are not holomorphically trivial, are called the jumping points, and projective lines \( \mathbb{C}P^1_x \) corresponding to them are called the jumping lines. In the Ward construction the jumping points give rise to singularities in the gauge potential \( A \). The set \( \mathbb{R}^4 - U' \)
of jumping points has codimension 1 (hypersurface) or more, i.e., the set $U'$ is an open dense subset in $\mathbb{R}^4$. Lines $\mathbb{CP}^1_x$ with $x \in U'$ are called generic lines, and semi-stability of the bundle $E'$ is equivalent to being trivial on the generic line. For more details see e.g. [33, 44].

Now we consider a holomorphic bundle $E''$ over $\mathcal{Z}$ such that its restriction $E''|_P$ to $\mathcal{P}$ belongs to $\mathcal{N}$, and $E''$ is nonequivalent to the bundle $E'$ considered above. So, $E'|_P$ and $E''|_P$ correspond to different points from the moduli space $\mathcal{M}$. The bundle $E''$ will be holomorphically trivial on $\mathbb{CP}^1_x$ with $x$ from an open set $U'' \supset U$ and in the general case $U' \neq U''$. In other words, subsets of jumping points for the bundles $E'$ and $E''$ do not coincide. At last, one can consider bundles $E''_{\text{inst}}$ over $\mathcal{Z}$ which have no jumping points in $\mathbb{R}^4 \subset S^4$. The restriction of $E''_{\text{inst}}$ to $\mathcal{P}$ belongs to $\mathcal{N}$, and instantons are parametrized by a subset $\mathcal{N}_{\text{inst}}$ in the set $\mathcal{N}$. It is clear that $\mathcal{N}_{\text{inst}} \subset \mathcal{N}$, and for a fixed topological charge the dimension of the moduli space $\mathcal{M}_{\text{inst}}$ is finite.

### 6.4 Representatives $\mathcal{M}_0$ and $\mathcal{N}_0$ of the germs $\mathcal{M}$ and $\mathcal{N}$

In §6.2 the germ $\mathcal{M}$ at the point $e$ of the set $\mathcal{M}$ and the germ $\mathcal{N}$ at the point $\mathcal{P}^0$ of the set $\mathcal{N}$ have been introduced. As an example, we shall describe some representatives $\mathcal{M}_0$ and $\mathcal{N}_0$ of these germs using the standard $\varepsilon$-$\delta$ language.

Consider the twistor space $\mathcal{P}$ for an open ball $U = \{x \in \mathbb{R}^4 : (x-x_0)^2 < r_0^2\}$ of the radius $r_0$ with a center at the point $x_0 \in \mathbb{R}^4$, the cover $\mathcal{U} = \{U_1, U_2\}$ of $\mathcal{P}$ and the space $Z^1(\mathcal{U}, \mathcal{H})$ of holomorphic vector bundles over $\mathcal{P}$. For the cover $\mathcal{O} = \{\Omega_1, \Omega_2\}$ of $\mathbb{CP}^1$ from §3.2, we consider the closure $\overline{\Omega}_{12} := \{\lambda \in \mathbb{C} : \alpha_1 \leq |\lambda| \leq \alpha_2\}$ of the open set $\Omega_{12} = \Omega_1 \cap \Omega_2$. Let $\overline{\mathcal{U}}$ be the closure of the open set $\overline{U} = \{x \in \mathbb{R}^4 : (x-x_0)^2 \leq r_0^2\}$. Then the closure of the open set $U_{12} = U \times \Omega_{12}$ is

$$
\overline{U}_{12} = \overline{U} \times \overline{\Omega}_{12},
$$

(6.5)

and $\overline{U}_{12}$ is a compact subset of the set $\mathcal{P}$.

We assume that matrix-valued transition functions $\mathcal{F}_{12}$ of bundles $E'$ are not only holomorphic on $U_{12}$, but also smooth on $\overline{U}_{12}$. This mild assumption can be replaced by the condition of holomorphy of $\mathcal{F}_{12}$ in an open $\delta$-neighbourhood of the set $\overline{U}_{12}$ with sufficiently small $\delta > 0$ [44]. Length $|\xi|$ of a vector $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$ is given by the formula $|\xi|^2 = \sum_{i} |\xi_i|^2 = \sum_{i} \xi_i \bar{\xi}_i$. We consider complex $n \times n$ matrices $A = (a_{ij})$ defining a linear transformation $A : \xi \mapsto A\xi$. For the matrices $A$ we define a norm $|A|$ by setting (see e.g. [44]):

$$
|A| := \max_{\xi \neq 0} \frac{|A\xi|}{|\xi|} = \max_{|\xi|=1} |A\xi|
$$

(6.6a)

Now let us introduce a norm $\| \cdot \|$ on the space $Z^1(\mathcal{U}, \mathcal{H})$ setting

$$
\| \mathcal{F} \| = \max_{z_1 \in \overline{U}_{12}} |\mathcal{F}_{12}(z_1)|
$$

(6.6b)

for $\mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H})$. Then $Z^1(\mathcal{U}, \mathcal{H})$ turns into a topological space.

It follows from the equality (6.1) discussed in §6.2 that there exists such a positive number $r_1(x)$ depending on $x \in \overline{U}$ that the bundle $E'_{x} = E'|_{\mathbb{CP}^1_x}$ will be holomorphically trivial if its transition matrix satisfies the condition

$$
\max_{\lambda \in \overline{\Omega}_{12}} |\mathcal{F}_{12}(x, \lambda)| - 1 < r_1(x).
$$

(6.7a)

The function $r_1(x) : \overline{U} \to \mathbb{R}$ can always be chosen smooth. It maps the compact space $\overline{U}$ into $\mathbb{R}$ and therefore

$$
r_1(x) \geq r_1 := \min_{x \in \overline{U}} r_1(x),
$$

(6.7b)

i.e., it is bounded from below. Moreover, one can always choose such a radius $r_0$ of an open ball $U$ that $r_1$ will be positive: $r_1 > 0$.

We fix the radius $r_0$ of an open ball $U$ and consider all $\mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H})$ such that

$$
\| \mathcal{F} - 1 \| \equiv \max_{z_1 \in \overline{U}_{12}} |\mathcal{F}_{12}(z_1)| - 1 < r_1,
$$

(6.8)

i.e., we consider the transition matrices $\mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H})$ close to the identity in the norm (6.6b). By virtue of (6.7b), all such transition matrices will satisfy the condition (6.7a) for any $x \in \overline{U}$ and therefore holomorphically nontrivial bundles $E'$ over $\mathcal{P}$, associated with them, will be holomorphically trivial on $\mathbb{CP}^1_x \hookrightarrow \mathcal{P}$ for all $x \in U$. 

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Notice that in the general case the action (5.9) of the group \( C^0(\mathcal{U}, \mathcal{H}) \) does not preserve the condition (6.8) on \( \mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H}) \), but it preserves the condition of holomorphic triviality of bundles \( E' \) on \( \mathbb{CP}^1_x \). As such, we can act by the group \( C^0(\mathcal{U}, \mathcal{H}) \) on the space of all \( \mathcal{F}'_s \) satisfying inequality (6.8) and “spread” this space over the space \( Z^1(\mathcal{U}, \mathcal{H}) \). As usual, two matrices \( \mathcal{F} \) and \( \mathcal{F}' \) satisfying the condition (6.8) are considered to be equivalent if they are connected by formula (5.9). Factorizing the space of all transition matrices satisfying (6.8) by this equivalence relation, we get a moduli space \( \mathcal{M}_0 \). The space \( \mathcal{M}_0 \) is one of representatives of the germ \( \mathcal{M} \) at the point \( e = [E'_0] \) of the moduli space of holomorphic bundles introduced in § 6.2.

Now, following § 6.2 we introduce the space

\[
\mathcal{N}_0 = \bigcup_{m \in \mathcal{M}_0} \mathcal{C}_m(\mathcal{U}, \mathcal{H}),
\]

(6.9)

obtained by the “spread” of \( \mathcal{F}(m) \) over the space \( Z^1(\mathcal{U}, \mathcal{H}) \) with the help of the action of the group \( C^0(\mathcal{U}, \mathcal{H}) \). We have (cf. (6.4))

\[
\mathcal{M}_0 = \rho_0(C^0) \backslash \mathcal{N}_0,
\]

(6.10)
i.e., \( \mathcal{M}_0 \) is the space of orbits of the group \( C^0(\mathcal{U}, \mathcal{H}) \) in the space \( \mathcal{N}_0 \). The space \( \mathcal{N}_0 \) is an open neighbourhood of \( \mathcal{F}' = 1 \) in the set \( Z^1(\mathcal{U}, \mathcal{H}) \) and is one of representatives of the germ \( \mathcal{N} \) at the point \( \mathcal{F}' = 1 \) of the space of holomorphic bundles described in § 6.2. So, for transition matrices \( \mathcal{F}_{12} \) from \( \mathcal{M}_0 \) the Birkhoff decomposition (4.17) exists for all \( x \in U \).

### 6.5 Symmetries of local solutions in the Čech approach

We consider the space \( Z^1(\mathcal{U}, \mathcal{H}) \) of holomorphic bundles \( \mathcal{F}' \) over \( \mathcal{P} \) and the open subset \( \mathcal{N} \) in \( Z^1(\mathcal{U}, \mathcal{H}) \) introduced in § 6.2. In §§ 5.1—5.5 we have defined the group \( \mathfrak{H}(\mathcal{P}, \mathcal{H}) := \mathfrak{H}(\mathcal{P}) \ltimes C^1(\mathcal{U}, \mathcal{H}) \) and described its action \( \rho \) on the space \( Z^1(\mathcal{U}, \mathcal{H}) \). This action, of course, does not map \( \mathcal{N} \) into itself (or into another representative of the germ \( \mathcal{N} \)), and one should consider a local action of the group \( \mathfrak{H}(\mathcal{P}, \mathcal{H}) \).

Let us consider an open neighbourhood \( \mathfrak{J} \) of the identity of the group \( \mathfrak{J}(\mathcal{P}) \), an open neighbourhood \( \mathfrak{C} \) of the identity of the group \( C^1(\mathcal{U}, \mathcal{H}) \) and an open neighbourhood \( \mathfrak{G} := \mathfrak{J} \ltimes \mathfrak{C} \) of the identity of the group \( \mathfrak{G}(\mathcal{P}, \mathcal{H}) \). As explained in the Appendix A and § 6.1, the local groups \( \mathfrak{J}, \mathfrak{C} \) and \( \mathfrak{G} \) are representatives of the germs \( \mathfrak{J}, \mathfrak{C}, \mathfrak{G} \) at the identity of the groups \( \mathfrak{J}(\mathcal{P}), C^1(\mathcal{U}, \mathcal{H}) \) and \( \mathfrak{G}(\mathcal{P}, \mathcal{H}) \), respectively. As local groups, \( \mathfrak{J}, \mathfrak{C} \) and \( \mathfrak{G} \) are isomorphic to the groups \( \mathfrak{J}(\mathcal{P}), C^1(\mathcal{U}, \mathcal{H}) \) and \( \mathfrak{G}(\mathcal{P}, \mathcal{H}) \).

The above-mentioned representatives of the germs \( \mathfrak{J}, \mathfrak{C} \) and \( \mathfrak{G} \) can always be chosen so that the local group \( \mathfrak{G} \) will map the set \( \mathcal{N} \) into itself. In more detail, there exists a subset \( \mathcal{N}' \) of \( \mathcal{N} \) (\( \mathcal{N}' \) is another representative of the germ \( \mathcal{N} \)) such that we have a map \( \rho : \mathfrak{G} \times \mathcal{N}' \to \mathcal{N} \). The map \( \rho : \mathcal{N}' \to \mathcal{N} \), where \( \mathcal{N}' = \{(a, \mathcal{F}) \in \mathfrak{G} \times \mathcal{N} : \rho(a, \mathcal{F}) \in \mathcal{N}\} \) is an open subset in \( \mathfrak{G} \times \mathcal{N} \) containing \( \{e\} \times \mathcal{N} \), is also defined. In this case, the properties \( \rho(e, \mathcal{F}) = \mathcal{F} \), \( \rho(a, \rho(b, \mathcal{F})) = \rho(ab, \mathcal{F}) \) etc. are fulfilled for all \( (a, \mathcal{F}) \in \mathcal{N}' \). In particular, the local group \( \mathfrak{J} \) of biholomorphisms acts on the space \( \mathcal{N} \) by formula (5.30) from § 5.5.

For the matrix local group \( \mathfrak{C} \) we introduce the diagonal subgroup

\[
\mathfrak{C}_\Delta := \mathfrak{C} \cap C^1(\mathcal{U}, \mathcal{H}),
\]

(6.11)

which is the local stability subgroup of the marked cocycle \( \mathcal{F}' \in \mathcal{N} \). For the definition of the group \( C^1(\mathcal{U}, \mathcal{H}) \) see (5.15b). Then, by repeating all the arguments of § 5.1 in terms of the local groups, we have

\[
\mathcal{N} \simeq \mathfrak{C}/\mathfrak{C}_\Delta,
\]

(6.12a)
i.e., \( \mathcal{N} \) is a coset space. In other words, for each representative \( \mathcal{N}' \) of the germ \( \mathcal{N} \) of the space of bundles, holomorphically trivial on \( \mathbb{CP}^1_x \to \mathcal{P} \), one can always choose a representative \( \mathfrak{C} \) of the germ \( \mathfrak{C} \) of the group of 1-cochains such that (6.12a) will take place. In fact, (6.12a) is a consequence of an isomorphism of germs

\[
\mathcal{N} \simeq \mathfrak{C}/\mathfrak{C}_\Delta.
\]

(6.12b)

Combining (6.12a) and (6.4), we obtain

\[
\mathcal{M} \simeq \rho_0(C^0) / \mathfrak{C}/\mathfrak{C}_\Delta,
\]

(6.13a)
i.e., the moduli space of local solutions to the SDYM equations is a double coset space. Again, (6.13a) is a consequence of the isomorphism of germs

\[
\mathcal{M} \simeq \rho_0(C^0) / \mathfrak{C}/\mathfrak{C}_\Delta.
\]

(6.13b)
Thus, the full group of continuous symmetries acting on the space $\mathcal{N}$ is a semidirect product

$$\mathcal{G} = \mathcal{H} \rtimes \mathcal{C}$$  \hspace{1cm} (6.14)

of the local group $\mathcal{H}$ of holomorphic automorphisms of the space $\mathcal{P}$ and of the local group $\mathcal{C}$ of 1-cochains of the cover $\mathcal{U}$ with values in the sheaf $\mathcal{H} = \mathcal{O}^{SL(n, \mathbb{C})}$ of holomorphic maps of the space $\mathcal{P}$ into the group $SL(n, \mathbb{C})$.

### 6.6 Unitarity conditions

As it was discussed in §4.4, the transition matrices $\mathcal{F}_{12}$ in holomorphic bundles $E' \to \mathcal{P}$ which are compatible with the real structure $\tau$ on $\mathcal{P}$ have to satisfy the additional condition (4.16b). Denote by

$$Z^1_1(\mathcal{U}, \mathcal{H}) := \left\{ \mathcal{F} \in Z^1(\mathcal{U}, \mathcal{H}) : \mathcal{F}_{12}^1(\tau(z_1)) = \mathcal{F}_{12}(z_1) \right\}$$  \hspace{1cm} (6.15)

a subset of transition matrices satisfying this unitarity conditions.

We should next define subgroups $C^0_\tau$ in $C^0$ and $C^1_\tau$ in $C^1$ such that their action, described by formulae (5.9) and (5.14), will preserve $Z^1_1(\mathcal{U}, \mathcal{H})$. It is not hard to see that

$$C^0_\tau(\mathcal{U}, \mathcal{H}) = \left\{ \{h_1, h_2\} \in C^0(\mathcal{U}, \mathcal{H}) : h^1_1(\tau(z_1)) = h^{-1}_2(1) \right\},$$  \hspace{1cm} (6.16)

$$C^1_\tau(\mathcal{U}, \mathcal{H}) = \left\{ \{h_{12}, h_{21}\} \in C^1(\mathcal{U}, \mathcal{H}) : h^1_{12}(\tau(z_1)) = h^{-1}_{21}(1) \right\}.$$  \hspace{1cm} (6.17)

Actions of these groups on $Z^1_1$ have the form

$$\mathcal{F}_{12} \mapsto \tilde{\mathcal{F}}_{12}(z_1) := \rho_0(h, \mathcal{F})_{12} = h_1(z_1)\mathcal{F}_{12}(z_1)h^1_1(\tau(z_1)), \quad h \in C^0_\tau,$$

$$\mathcal{F}_{12} \mapsto \tilde{\mathcal{F}}_{12}(z_1) := \rho(h, \mathcal{F})_{12} = h_{12}(z_1)\mathcal{F}_{12}(z_1)h^1_{12}(\tau(z_1)), \quad h \in C^1_\tau.$$  \hspace{1cm} (6.18)

By the definitions (6.16) and (6.17), $C^0_\tau(\mathcal{U}, \mathcal{H})$ and $C^1_\tau(\mathcal{U}, \mathcal{H})$ are real subgroups in $C^0(\mathcal{U}, \mathcal{H})$ and $C^1(\mathcal{U}, \mathcal{H})$, respectively.

The cocycles $\mathcal{F}_{12}$ and $\tilde{\mathcal{F}}_{12}$ from (6.18) define equivalent bundles $E' \sim \tilde{E}'$, $\mathcal{F}_{12} \sim \tilde{\mathcal{F}}_{12}$, and one can introduce a 1-cohomology set $H^1_1(\mathcal{U}, \mathcal{H})$ as a set of orbits of the group $\rho_0(C^0_\tau)$ in the space $Z^1_1(\mathcal{U}, \mathcal{H})$ of transition matrices compatible with the real structure $\tau$ on $\mathcal{P}$,

$$H^1_1(\mathcal{U}, \mathcal{H}) := \rho_0(C^0_\tau(\mathcal{U}, \mathcal{H})) \backslash Z^1_1(\mathcal{U}, \mathcal{H}) \subset H^1(\mathcal{U}, \mathcal{H}).$$  \hspace{1cm} (6.20)

For the cover $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ we have $H^1_1(\mathcal{P}, \mathcal{H}) = H^1_1(\mathcal{U}, \mathcal{H})$. So, the real structure $\tau$ on $\mathcal{P}$ induces a real structure on $H^1(\mathcal{P}, \mathcal{H})$, and $H^1_1(\mathcal{P}, \mathcal{H})$ is a set of real “points” of the space $H^1(\mathcal{P}, \mathcal{H})$ corresponding to the bundles $E'$ with the unitary structure (6.15).

Consider the action of the group $C^1_\tau$ on $Z^1_1$. As a stability subgroup of the element $\mathcal{F}^0 = 1$ compatible with the real structure we have the group

$$C^1_{\tau, \Delta} := C^1_\tau \cap C^1_\Delta = \left\{ \{h_{12}, h_{21}\} \in C^1_\tau(\mathcal{U}, \mathcal{H}) : h_{12} = h_{21} \right\},$$  \hspace{1cm} (6.21)

and the space $Z^1_1(\mathcal{U}, \mathcal{H})$ can be identified with the quotient space

$$Z^1_1(\mathcal{U}, \mathcal{H}) = C^1_\tau(\mathcal{U}, \mathcal{H}) / C^1_{\tau, \Delta}.$$  \hspace{1cm} (6.22)

The moduli space $H^1_1(\mathcal{P}, \mathcal{H})$ of holomorphic bundles $E'$ with the unitary structure coincides with the double coset space

$$H^1_1 := \rho_0(C^1_\tau) \backslash C^1_\tau / C^1_{\tau, \Delta},$$  \hspace{1cm} (6.23)

and this set is isomorphic to (i) the set of $C^1_{\tau, \Delta}$-orbits in $Y^1_1 := \rho_0(C^0_\tau) \backslash C^1_\tau$, (ii) the set of $C^0_\tau$-orbits in $Z^1_1$, (iii) the set of $C^1_\tau$-orbits in $Y^1_1 \times Z^1_1$.

As to the group $\mathcal{H}(\mathcal{P})$, the action of which on $Z^1_1(\mathcal{U}, \mathcal{H})$ was described in §5.5, one should choose in it a subgroup $\mathcal{H}_1(\mathcal{P})$ of those transformations $\eta \in \mathcal{H}(\mathcal{P})$ which are compatible with the real structure $\tau$ on $\mathcal{P}$. In terms of the functions $\eta_1$ and $\eta_2$ from (5.26) representing $\eta$ in the chosen coordinates it means that

$$\overline{\eta}^\tau_1(\tau(z_1)) = B^0_\eta \eta^0_2(z_2),$$  \hspace{1cm} (6.24)
where the coefficients $B_0^q$ are written down in (3.21). Thus, the symmetry group acting on the space $Z^1_\tau(\mathcal{U}, \mathcal{H})$ of holomorphic bundles $E'$ satisfying the unitarity conditions is the group

$$\mathfrak{G}_\tau(\mathcal{P}, \mathcal{H}) := \mathfrak{H}_\tau(\mathcal{P}) \ltimes C^1_\tau(\mathcal{U}, \mathcal{H}).$$

(6.25)

This group is a real subgroup in the group (5.31).

Further, going over to local solutions, we introduce a subset $\mathcal{N}_\tau$ of those transition matrices from $\mathcal{N}$ which satisfy the condition (4.16b), i.e., $\mathcal{N}_\tau := \mathcal{N} \cap Z^1_\tau(\mathcal{U}, \mathcal{H})$. One analogously introduces the moduli space $\mathcal{M}_\tau := \mathcal{M} \cap H^1_\tau(\mathcal{P}, \mathcal{H})$, the real local groups $\mathfrak{H}_\tau := \mathfrak{H} \cap \mathfrak{H}_\tau(\mathcal{P})$, $\mathfrak{C}_\tau := \mathfrak{C} \cap C^1_\tau(\mathcal{U}, \mathcal{H})$ and the germs $\mathfrak{H}_\tau, \mathfrak{C}_\tau$ corresponding to them. Then one obtains isomorphisms

$$\mathcal{N}_\tau \simeq \mathfrak{C}_\tau/\mathfrak{C}_\tau\Delta, \quad \mathcal{M}_\tau \simeq \rho_0(C^0_\tau)\backslash\mathfrak{C}_\tau/\mathfrak{C}_\tau\Delta,$$

(6.26)

corresponding to the isomorphisms (6.12), (6.13). At last, as the symmetry group of the space of real local solutions in the Čech approach one gets the local group

$$\mathfrak{G}_\tau = \mathfrak{H}_\tau \ltimes \mathfrak{C}_\tau,$$

(6.27)

which is a semidirect product of the local groups $\mathfrak{H}_\tau$ and $\mathfrak{C}_\tau$.

## 7 Holomorphic bundles: the Dolbeault description

### 7.1 Some definitions

The well-known Dolbeault theorem reduces a computation of cohomology spaces of a manifold $X$ with the coefficients in a sheaf of germs of holomorphic maps from $X$ into a complex Abelian group $\mathbb{T}$ to problems of calculus of $\mathbb{T}$-valued differential forms of the type $(0,q)$ on the manifold $X$ (isomorphism between Čech and Dolbeault cohomology groups) \(\text{[20]}\). We want to describe an analogue of the Dolbeault theorem for the sheaf $\mathcal{H}$ of germs of holomorphic maps of the space $\mathcal{P}$ into the non-Abelian group $SL(n, \mathbb{C})$, following mainly the papers \(\text{[12]}\). This will permit us to describe symmetries of the space of local solutions to the SDYM equations on $U \subset \mathbb{R}^4$. But first, let us recall some definitions for objects which will be considered below.

Let $K$ be a sheaf of groups and $\mathcal{A}$ a sheaf of sets on $X$. We shall say that $K$ acts on $\mathcal{A}$ if for any $x \in X$ the group $K_x$ acts on $\mathcal{A}_x$, and also this action is continuous in the topology of the sheaves $K$ and $\mathcal{A}$. It is said that $K$ transitively acts on $\mathcal{A}$, if $K_x$ transitively acts on $\mathcal{A}_x$ for each $x \in X$. In this case $\mathcal{A}$ can be identified with a quotient sheaf $K/K'$, where $K'$ is a sheaf of stability subgroups $K'_x$ and stalks of the sheaf $K/K'$ are quotient spaces $K_x/K'_x$. Conversely, if $K'$ is a subsheaf of subgroups in $K$, the sheaf $K/K'$ can be considered as a sheaf of sets with marked section $x \mapsto K'_x, x \in X$, on which $K$ transitively acts on the left.

### 7.2 The sheaves $\hat{S}, \hat{B}^{0,q}$ and $\hat{B}$

Consider the sheaf $\hat{S}$ of germs of smooth maps from $\mathcal{P}$ into the group $SL(n, \mathbb{C})$. The sheaf $\mathcal{H}$ of germs of holomorphic maps $\mathcal{P} \rightarrow SL(n, \mathbb{C})$ is a subsheaf of the sheaf $\hat{S}$, and there exists a canonical embedding $i : \mathcal{H} \rightarrow \hat{S}$. Consider also the sheaf $\hat{B}^{0,q}$ ($q = 1, 2, ...$) of germs of smooth $(0,q)$-forms on $\mathcal{P}$ with values in the Lie algebra $sl(n, \mathbb{C})$. Let us define a map $\hat{\delta}^0 : \hat{S} \rightarrow \hat{B}^{0,1}$ given for any open set $\mathcal{U}$ of the space $\mathcal{P}$ by the formula

$$\hat{\delta}^0 \hat{\psi} = -\langle \hat{\delta} \hat{\psi}, \hat{\psi} \rangle^{-1},$$

(7.1)

where $\hat{\psi} \in \hat{S}(\mathcal{U})$, $\hat{\delta}^0 \hat{\psi} \in \hat{B}^{0,1}(\mathcal{U})$, $\hat{\delta} = \partial + \bar{\partial}$. Let us also introduce an operator $\hat{\delta}^1 : \hat{B}^{0,1} \rightarrow \hat{B}^{0,2}$, defined for any open set $\mathcal{U} \subset \mathcal{P}$ by the formula

$$\hat{\delta}^1 \hat{B} = \bar{\partial} \hat{B} + \hat{B} \wedge \bar{\partial},$$

(7.2)

where $\hat{B} \in \hat{B}^{0,1}(\mathcal{U})$, $\hat{\delta}^1 \hat{B} \in \hat{B}^{0,2}(\mathcal{U})$. In other words, the maps of sheaves $\hat{\delta}^0 : \hat{S} \rightarrow \hat{B}^{0,1}$ and $\hat{\delta}^1 : \hat{B}^{0,1} \rightarrow \hat{B}^{0,2}$ are defined by means of localizations. In particular, on $\mathcal{U}_1 \subset \mathcal{P}$ we have

$$\hat{\delta}^0 \hat{\psi}_1 = -\langle \hat{V}_1^{(1)} \hat{\psi}_1, \hat{\psi}_1 \rangle^{-1} \hat{\psi}_1,$$

(7.1')

$$\hat{\delta}^1 \hat{B}_a^{(1)} = \hat{V}_a^{(1)} \hat{B}_b^{(1)} - \hat{V}_b^{(1)} \hat{B}_a^{(1)} + \hat{[B}_a^{(1)}, \hat{B}_b^{(1)}].$$

(7.2')

The sheaf $\hat{S}$ acts on the sheaves $\hat{B}^{0,q}$ ($q = 1, 2, ...$) with the help of the adjoint representation. In particular, for any open set $\mathcal{U} \subset \mathcal{P}$ we have

$$\hat{B} \mapsto \text{Ad}(\hat{\psi}, \hat{B}) = \hat{\psi}^{-1} \hat{B} \hat{\psi} + \hat{\psi}^{-1} \bar{\partial} \hat{\psi},$$

(7.3a)
where \( \hat{\psi} \in \hat{\mathcal{S}}(U) \), \( \hat{B} \in \hat{\mathcal{B}}^{0,1}(U) \), \( \hat{F} \in \hat{\mathcal{B}}^{0,2}(U) \).

Denote by \( \tilde{\mathcal{B}} \) the subsheaf in \( \hat{\mathcal{B}}^{0,1} \) consisting of germs of \( (0,1) \)-forms \( \hat{B} \) with values in \( sl(n, \mathbb{C}) \) such that \( \bar{\partial} \hat{B} = 0 \), i.e., sections \( \hat{B} \) over any open set \( U \) of the sheaf \( \hat{\mathcal{B}} = \text{Ker} \bar{\partial} \) satisfy the equations

\[ \bar{\partial} \hat{B} + \hat{B} \wedge \hat{B} = 0, \tag{7.4} \]

where \( \hat{B} \in \hat{\mathcal{B}}^{0,1}(U) \). So the sheaf \( \hat{B} \) can be identified with the sheaf of \( (0,1) \)-connections \( \bar{\partial}_B = \bar{\partial} + \hat{B} \) in the holomorphic bundle \( E' \) over \( \mathcal{P} \).

### 7.3 The sheaves \( \mathcal{S}, \mathcal{B}^{0,q} \) and \( \mathcal{B} \)

Recall that \( \mathcal{P} \) is the fibre bundle with fibres \( \mathbb{CP}^1 \) over the points \( x \) from \( U \subset \mathbb{R}^4 \), and the canonical projection \( \pi : \mathcal{P} \rightarrow U \) is defined. The typical fibre \( \mathbb{CP}^1 \) has the \( SU(2) \)-invariant complex structure \( j \) (see §3.2), and the vertical distribution \( V = \text{Ker} \pi_* \) inherits this complex structure. A restriction of \( V \) to each fibre \( \mathbb{CP}^1_x, \ x \in U \), is the tangent bundle to that fibre. The (flat) Levi-Civita connection on \( U \) generates the splitting of the tangent bundle \( T(\mathcal{P}) \) into a direct sum

\[ T(\mathcal{P}) = V \oplus H \tag{7.5} \]

of the vertical distribution \( V \) and the horizontal distribution \( H \).

Using the complex structures \( j, J \) and \( \mathcal{J} \) on \( \mathbb{CP}^1, U \) and \( \mathcal{P} \) respectively, one can split the complexified tangent bundle of \( \mathcal{P} \) into a direct sum

\[ T^c(\mathcal{P}) = (V^{1,0} \oplus H^{1,0}) \oplus (V^{0,1} \oplus H^{0,1}) \tag{7.6} \]

of vectors of type \((1,0)\) and \((0,1)\).

So we have the integrable distribution \( V^{0,1} \) of antiholomorphic vector fields with the basis \( V_{3}^{(1)} = \partial_3 \) on \( U_1 \subset \mathcal{P} \) and \( \hat{V}_{3}^{(2)} = \partial_3^\mathcal{J} \) on \( U_2 \subset \mathcal{P} \). The vector fields (3.4a), (3.4b) and (3.11a), (3.11b) form a basis in the normal bundle \( H^{0,1} \) of a line \( \mathbb{CP}^1_x \rightarrow \mathcal{P} \).

Having the canonical distribution \( V^{0,1} \) on the space \( \mathcal{P} \), we introduce the sheaf \( \mathcal{S} \) of germs of partially holomorphic maps \( \hat{\psi} : \mathcal{P} \rightarrow SL(n, \mathbb{C}) \), which are annihilated by vector fields from \( V^{0,1} \). In other words, sections of the sheaf \( \mathcal{S} \) over open subsets \( U \subset \mathcal{P} \) are \( SL(n, \mathbb{C}) \)-valued functions \( \hat{\psi} \) on \( U \), which satisfy the equations

\[ \partial_3 \hat{\psi} = 0 \text{ on } U \cap U_1, \quad \partial_3^\mathcal{J} \hat{\psi} = 0 \text{ on } U \cap U_2, \tag{7.7} \]

i.e., they are holomorphic along \( \mathbb{CP}^1_x \rightarrow \mathcal{P} \), \( x \in U \). It is obvious that the sheaf \( \mathcal{H} \) of holomorphic maps from \( \mathcal{P} \) into \( SL(n, \mathbb{C}) \), i.e., smooth maps which are annihilated by vector fields from \( V^{0,1} \oplus H^{0,1} \), is a subsheaf of \( \mathcal{S} \) and \( \mathcal{S} \) is a subsheaf of \( \mathcal{H} \).

Consider now the sheaves \( \hat{\mathcal{B}}^{0,q} \), introduced in §7.2. Let \( \mathcal{B}^{0,1} \) be the subsheaf of \( (0,1) \)-forms from \( \hat{\mathcal{B}}^{0,1} \) vanishing on the distribution \( V^{0,1} \). In components this means that for any open set \( U \subset \mathcal{P} \)

\[ B_3^{(1)} = 0 \text{ on } U \cap U_1, \quad B_3^{(2)} = 0 \text{ on } U \cap U_2, \tag{7.8} \]

where \( B_3^{(1)} \) belongs to the section of the sheaf \( \mathcal{B}^{0,1} \) over \( U_1 \), and \( B_3^{(2)} \) belongs to the section of the sheaf \( \mathcal{B}^{0,1} \) over \( U_2 \). So \( \mathcal{B}^{0,1} \) is the subsheaf of \( \hat{\mathcal{B}}^{0,1} \).

The map \( \bar{\partial} \), introduced in §7.3, induces a map \( \bar{\partial} : \mathcal{S} \rightarrow \mathcal{B}^{0,1} \), defined for any open set \( U \) of the space \( \mathcal{P} \) by the formula

\[ \bar{\partial} \hat{\psi} = - (\bar{\partial} \hat{\psi}) \psi^{-1}, \tag{7.9a} \]

where \( \hat{\psi} \in \hat{\mathcal{S}}(U) \), \( \bar{\partial} \hat{\psi} \in \hat{\mathcal{B}}^{0,1}(U) \). Analogously, the operator \( \bar{\partial} \) induces a map \( \bar{\partial} : \mathcal{B}^{0,1} \rightarrow \hat{\mathcal{B}}^{0,2} \), given for any open set \( U \subset \mathcal{P} \) by the formula

\[ \bar{\partial} B = \bar{\partial} B + B \wedge B, \tag{7.10a} \]

where \( B \in \mathcal{B}^{0,1}(U) \), \( \bar{\partial} B \in \hat{\mathcal{B}}^{0,2}(U) \). In particular, on \( U_1 \subset \mathcal{P} \) we have

\[ \bar{\partial} \hat{\psi}_1 = - \left\{ (V_a^{(1)} \hat{\psi}_1) \hat{\psi}_1^{-1} \right\} \hat{\theta}_a^{(1)}, \tag{7.9b} \]

\[ \bar{\partial} B^{(1)} = \frac{1}{2} \left\{ \bar{V}_a^{(1)} B_1^{(1)} - \bar{V}_a^{(1)} B_1^{(1)} + [B_a^{(1)}, B_1^{(1)}] \right\} \hat{\theta}_a^{(1)} \wedge \hat{\theta}_b^{(1)}, \tag{7.10b} \]

where the \((0,1)\)-forms \( \{ \hat{\theta}_a^{(1)} \} \) were introduced in §3.3. \( \hat{\psi}_1 \in \hat{\mathcal{S}}(U_1), B^{(1)} \in \mathcal{B}^{0,1}(U_1) \).
The sheaf $S$ acts on the sheaves $B^{0,1}$ and $\tilde{B}^{0,1}$ by means of the adjoint representation. In particular, for $B^{0,1}$ and $\tilde{B}^{0,2}$ we have the same formulae (7.3) with replacement $\hat{\psi}$ by $\psi \in S(U)$,

$$B \mapsto \text{Ad}(\psi, B) = \psi^{-1} B \psi + \psi^{-1} \bar{\partial} \psi,$$

$$\hat{F} \mapsto \text{Ad}(\hat{\psi}, \hat{F}) = \hat{\psi}^{-1} \hat{F} \hat{\psi},$$

where $B \in B^{0,1}(U)$, $\hat{F} \in \tilde{B}^{0,2}(U)$.

At last, let us denote by $B$ the subsheaf of $B^{0,1}$ consisting of germs of $sl(n, \mathbb{C})$-valued $(0,1)$-forms $B$ such that $\bar{\partial}^1 B = 0$, i.e., sections $B$ of the sheaf $B = \text{Ker} \bar{\delta}^1$ satisfy the equations

$$\bar{\partial} B + B \wedge B = 0.$$ 

In components for $B \in B^{0,1}(U_1)$ on the open set $U_1$ eqs. (7.12) have the form

$$\bar{V}_1^{(1)} B_2^{(1)} - \bar{V}_2^{(1)} B_1^{(1)} + [B_1^{(1)}, B_2^{(1)}] = 0, \quad \bar{V}_3^{(1)} B_1^{(1)} = 0, \quad \bar{V}_3^{(1)} B_2^{(1)} = 0,$$

since $B_3^{(1)} = 0$. We have analogous equations on $U_2 \subset \mathcal{P}$.

### 7.4 Exact sequences of sheaves

Let us consider the sheaves $\hat{S}$, $\tilde{B}^{0,1}$ and $\tilde{B}^{0,2}$. The triple $\{\hat{S}, \tilde{B}^{0,1}, \tilde{B}^{0,2}\}$ with the maps $\bar{\delta}^0$ and $\bar{\delta}^1$ is a resolution of the sheaf $\mathcal{H}$, i.e., the sequence of sheaves

$$1 \rightarrow \mathcal{H} \xrightarrow{i} \hat{S} \xrightarrow{\bar{\delta}^0} \tilde{B}^{0,1} \xrightarrow{\bar{\delta}^1} \tilde{B}^{0,2},$$

where $i$ is an embedding, is exact. For proof see [12]. Restricting $\bar{\delta}^0$ to $S \subset \hat{S}$ and $\bar{\delta}^1$ to $B^{0,1} \subset \tilde{B}^{0,1}$, we obtain the exact sequence of sheaves

$$1 \rightarrow \mathcal{H} \xrightarrow{i} S \xrightarrow{\bar{\delta}^0} B^{0,1} \xrightarrow{\bar{\delta}^1} \tilde{B}^{0,2},$$

where $1$ is the identity of the sheaf $\mathcal{H}$.

By virtue of the exactness of the sequence (7.13), we have

$$\bar{\delta}^0 \hat{S} = \text{Ker} \bar{\delta}^1 = \hat{B}.$$

Since $\bar{\delta}^0$ is the projection, connected with the action (7.3a) of the sheaf $\hat{S}$ on $B^{0,1}$, the sheaf $\hat{S}$ acts transitively with the help of $\text{Ad}$ on $\hat{B}$ and $\tilde{B} \simeq \hat{S}/\mathcal{H}$. Thus, we obtain the exact sequence of sheaves

$$1 \rightarrow \mathcal{H} \xrightarrow{i} \hat{S} \xrightarrow{\bar{\delta}^0} \hat{B} \xrightarrow{\bar{\delta}^1} 0.$$ 

For more details see [12]. Restricting the map $\bar{\delta}^0$ to $S$ and $\bar{\delta}^1$ to $B$, we obtain the exact sequence of sheaves

$$1 \rightarrow \mathcal{H} \xrightarrow{i} S \xrightarrow{\bar{\delta}^0} B \xrightarrow{\bar{\delta}^1} 0,$$

since $\bar{\delta}^0 S = \text{Ker} \bar{\delta}^1$ (the exactness of the sequence (7.14)), and $S$ acts on $B$ transitively ($B \simeq S/\mathcal{H}$). For sections of the sheaf $B$ over $U_1$ and $U_2$ we have

$$B_1^{(1)} = -(\bar{V}_1^{(1)} \psi_1) \psi_1^{-1}, \quad B_2^{(1)} = -(\bar{V}_2^{(1)} \psi_1) \psi_1^{-1}, \quad B_3^{(1)} = -(\bar{V}_3^{(1)} \psi_1) \psi_1^{-1} \equiv 0,$$

$$B_1^{(2)} = -(\bar{V}_1^{(2)} \psi_2) \psi_2^{-1}, \quad B_2^{(2)} = -(\bar{V}_2^{(2)} \psi_2) \psi_2^{-1}, \quad B_3^{(2)} = -(\bar{V}_3^{(2)} \psi_2) \psi_2^{-1} \equiv 0,$$

where $\psi_{1,2} \in S(U_{1,2})$, $B_{1,2}^{(1,2)} \in B(U_{1,2})$.

### 7.5 The group $H^0(\mathcal{P}, S)$ and the cohomology set $H^1(\mathcal{P}, S)$

Having the sheaf $S$ of partially holomorphic smooth maps from $\mathcal{P}$ into $SL(n, \mathbb{C})$ and the two-set open cover $\mathcal{U} = \{U_1, U_2\}$, we consider the groups of cochains

$$C^0(\mathcal{U}, S) = \{\text{maps } \psi_1 : U_1 \rightarrow S(U_1), \psi_2 : U_2 \rightarrow S(U_2)\} = S(U_1) \times S(U_2),$$

$$C^1(\mathcal{U}, S) = \{\text{maps } f_{12} : U_{12} \rightarrow S(U_{12}), f_{21} : U_{12} \rightarrow S(U_{12})\} = S(U_{12}) \times S(U_{12}),$$

where $U_{12} = U_1 \cap U_2$.
where $S(U)$ is a space of sections of the sheaf $S$ over an open set $U \subset \mathcal{P}$.

For 0- and 1-cocycles we have

$$Z^0(\mathcal{P}, S) = \{ \psi = \{ \psi_1, \psi_2 \} \in C^0(\mathcal{U}, S) : \psi_1 = \psi_2 \text{ on } \mathcal{U}_2 \},$$

$$Z^1(\mathcal{U}, S) = \{ f = \{ f_{12}, f_{21} \} \in C^1(\mathcal{U}, S) : f_{21} = f_{12}^{-1} \}. $$

(7.19a) \hspace{1cm} (7.19b)

By definition, $H^0(\mathcal{P}, S) := Z^0(\mathcal{P}, S) = \Gamma(\mathcal{P}, S)$. As usual, two cocycles $F, \tilde{F} \in Z^1(\mathcal{U}, S)$ are called equivalent if $\tilde{F}_{12} = \psi_1 F_{12} \psi_2^{-1}$ for some $\psi = \{ \psi_1, \psi_2 \} \in C^0(\mathcal{U}, S)$. A set of equivalence classes of 1-cocycles $F$ is the Čech 1-cohomology set $H^1(\mathcal{U}, S)$. For the considered cover $\mathcal{U}$ we have $H^1(\mathcal{P}, S) = H^1(\mathcal{U}, S)$.

By replacing the sheaf $H$ by the sheaf $S$ in the formulae of §5.2, one can define the action of the group $C^0(\mathcal{U}, S)$ on $C^1(\mathcal{U}, S)$ by automorphisms $\sigma_0$,

$$\sigma_0(\psi, f)_{12} = \psi_2 f_{12} \psi_2^{-1}, \quad \sigma_0(\psi, f)_{21} = \psi_1 f_{21} \psi_1^{-1},$$

$$\psi = \{ \psi_1, \psi_2 \} \in C^0(\mathcal{U}, S), \quad f = \{ f_{12}, f_{21} \} \in C^1(\mathcal{U}, S),$$

and define a twisted homomorphism $\delta^0 : C^0(\mathcal{U}, S) \rightarrow C^1(\mathcal{U}, S)$ by the formulae

$$\delta^0(\phi)_{12} = \phi_1 \phi_2^{-1}, \quad \delta^0(\phi)_{21} = \phi_2 \phi_1^{-1},$$

$$\delta^0(h \phi) = \delta^0(h) \sigma_0(h, \delta^0(\phi)),$$

(7.20) \hspace{1cm} (7.21)

where $\phi = \{ \phi_1, \phi_2 \} \in C^0(\mathcal{U}, S)$, $\delta^0(\phi) \in Z^1(\mathcal{U}, S) \subset C^1(\mathcal{U}, S)$. Then we have

$$H^0(\mathcal{P}, S) = \text{Ker} \delta^0,$$

(7.22)

and the image

$$\text{Im} \delta^0 = \delta^0(C^0(\mathcal{U}, S)) \subset Z^1(\mathcal{U}, S)$$

(7.23)

of the map $\delta^0$ corresponds to the marked element $e \in H^1(\mathcal{P}, S)$, i.e., to the class of smoothly trivial bundles over $\mathcal{P}$ which are holomorphically trivial over $\mathbb{C}P^1 \hookrightarrow \mathcal{P}$, $x \in U$. Transition matrices $F \in \text{Im} \delta^0$ have the form (4.17): $F_{12} = \psi_{1}^{-1}(x, \lambda) \psi_{2}(x, \lambda)$.

Finally, for $\psi \in C^0(\mathcal{U}, S)$, $F \in Z^1(\mathcal{U}, S)$, the formula

$$\rho_0(\psi, F) := \delta^0(\psi) \sigma_0(\psi, F) \iff \rho_0(\psi, F)_{12} = \psi_1 F_{12} \psi_2^{-1}$$

(7.24)

defines the action of the group $C^0(\mathcal{U}, S)$ on the set $Z^1(\mathcal{U}, S)$, and we obtain

$$H^1(\mathcal{U}, S) = \rho_0(C^0(\mathcal{U}, S)) \backslash Z^1(\mathcal{U}, S).$$

(7.25)

For the chosen cover $\mathcal{U}$ we have $H^1(\mathcal{P}, S) = H^1(\mathcal{U}, S)$.

### 7.6 Exact sequences of cohomology sets

From (7.15b) we obtain the exact sequence of cohomology sets $[12]$

$$e \rightarrow H^0(\mathcal{P}, \mathcal{H}) \overset{i_*}{\rightarrow} H^0(\mathcal{P}, \hat{S}) \overset{\delta^0}{\rightarrow} H^0(\mathcal{P}, \hat{B}) \overset{\delta^1}{\rightarrow} H^1(\mathcal{P}, \mathcal{H}) \overset{\varphi}{\rightarrow} H^1(\mathcal{P}, \hat{S}),$$

(7.26)

where $e$ is a marked element (identity) of the considered sets, and a homomorphism $\varphi$ coincides with the canonical embedding, induced by the embedding of sheaves $i : \mathcal{H} \rightarrow \hat{S}$. The kernel $\text{Ker} \varphi = \varphi^{-1}(e)$ of the map $\varphi$ coincides with a subset of those elements from $H^1(\mathcal{P}, \mathcal{H})$, which are mapped into the class $e \in H^1(\mathcal{P}, \hat{S})$ of topologically (and smoothly) trivial bundles. This means that representatives of the subset $\text{Ker} \varphi$ are those transition matrices $F \in Z^1(\mathcal{U}, \mathcal{H})$ for which there exists a splitting

$$F_{12} = \psi_1^{-1}(x, \lambda, \bar{\lambda}) \psi_2(x, \lambda, \bar{\lambda})$$

(7.27)

with smooth matrix-valued functions $\psi_1, \psi_2 \in \text{SL}(n, \mathbb{C})$.

Similarly, from (7.16) we obtain the exact cohomology sequence

$$e \rightarrow H^0(\mathcal{P}, \mathcal{H}) \overset{i_*}{\rightarrow} H^0(\mathcal{P}, \hat{S}) \overset{\delta^0}{\rightarrow} H^0(\mathcal{P}, \hat{B}) \overset{\delta^1}{\rightarrow} H^1(\mathcal{P}, \mathcal{H}) \overset{\varphi}{\rightarrow} H^1(\mathcal{P}, \hat{S}),$$

(7.28)

where a homomorphism $\varphi$ is an embedding, induced by the embedding of sheaves $i : \mathcal{H} \rightarrow \hat{S}$. The kernel $\text{Ker} \varphi = \varphi^{-1}(e)$ of the map $\varphi$ coincides with a subset of those elements from $H^1(\mathcal{P}, \mathcal{H})$, which are mapped into the class $e \in H^1(\mathcal{P}, S)$ of smoothly trivial bundles over $\mathcal{P}$, which are holomorphically trivial on any
various subsets of transition matrices and their moduli satisfying the unitarity condition.

In 7.7 Unitarity conditions

It follows from the exactness of the sequence (7.26) that $H$ with the base space $H$ of bundles. Transition matrices of such bundles have the form (7.27).

The set $su$ of bundles $E$ over $P$ to $\{\psi_1, \psi_2\} \in C^0(\mathcal{U}, S)$. The equality $B^{(1)} = B^{(2)}$ on $U \subset \mathbb{P}^1$ means that the (0,1)-form $B \in H^0(\mathcal{P}, B)$ is defined globally, follows from the identity $\bar{\partial} = \bar{\partial} = (\bar{\partial}^{-1}) \psi_2 + \psi_1^{-1} \bar{\partial} \psi_2 = \psi_1^{-1} \{(\bar{\partial} \psi_1) \psi_1^{-1} + (\bar{\partial} \psi_2) \psi_2^{-1}\} \psi_2 = 0.$ (7.31)

The group $S(\mathcal{P}) := H^0(\mathcal{P}, S) = Z^0(\mathcal{P}, S) = \Gamma(\mathcal{P}, S)$ of global sections of the sheaf $S$ acts on the set $H^0(\mathcal{P}, B)$ with the help of $Ad(g, \cdot)$ transformations

$$Ad(g, B) = g^{-1} B g + g^{-1} \bar{\partial} g,$$

where $g \in H^0(\mathcal{P}, S), B \in H^0(\mathcal{P}, B)$. Notice that from the definition (7.19a) of the group $H^0(\mathcal{P}, S)$ and from the Liouville theorem for $\mathbb{CP}_x \to \mathcal{P}$ it follows that the elements $g \in H^0(\mathcal{P}, S)$ do not depend on $\lambda$. Comparing (7.12) and (7.30) with (4.20)–(4.23), we conclude that the 0-cohomology set $H^0(\mathcal{P}, B)$ coincides with the space of (complex) local solutions to the SDYM equations on $U \subset \mathbb{R}^4$, the group $H^0(\mathcal{P}, S)$ coincides with the group of (complex) gauge transformations, and the quotient space $H^0(\mathcal{P}, B)/H^0(\mathcal{P}, S)$ coincides with the moduli space of (complex) local solutions to the SDYM equations on $U$.

The space $\ker \phi$ is a representative of the germ $\mathcal{M}$ at the point $e \in H^1(\mathcal{P}, H)$ of the moduli space of bundles $E'$ over $\mathcal{P}$, holomorphically trivial on $\mathbb{CP}_x \to \mathcal{P}, x \in U$. We will denote it by $\mathcal{M} := \ker \phi$; this set was described in detail in §4.3. From the exactness of the sequence (7.28) it follows that the set $\mathcal{M} = \ker \phi \subset H^1(\mathcal{P}, H)$ is bijective to the moduli space $H^0(\mathcal{P}, B)/H^0(\mathcal{P}, S)$ of (complex) solutions to the SDYM equations,

$$\mathcal{M} \simeq H^0(\mathcal{P}, B)/H^0(\mathcal{P}, S).$$

This correspondence is a non-Abelian analogue of the Dolbeault theorem about the isomorphism of (Abelian) Čech and Dolbeault 1-cohomology groups.

Remark. Using the sheaves $\hat{S}$ and $\hat{B}$, considered in §§7.2,7.4 and §7.6, one can introduce a Dolbeault 1-cohomology set $H^0_{\hat{S}}(\mathcal{P})$ as a set of orbits of the group $H^0(\mathcal{P}, \hat{S})$ in the set $H^0(\mathcal{P}, \hat{B})$, i.e.,

$$H^0_{\hat{S}}(\mathcal{P}) := H^0(\mathcal{P}, \hat{B})/H^0(\mathcal{P}, \hat{S}).$$

The set $H^0(\mathcal{P}, B)/H^0(\mathcal{P}, S)$ considered above is an open subset in the Dolbeault 1-cohomology set $H^0_{\hat{B}}(\mathcal{P})$. It follows from the exactness of the sequence (7.26) that $H^0_{\hat{B}}(\mathcal{P}) \simeq \ker \phi$, i.e., the moduli space $H^0_{\hat{B}}(\mathcal{P})$ of global solutions of eqs.(7.4) on $\mathcal{P}$ is bijective to the moduli space of holomorphic bundles over $\mathcal{P}$ which are trivial as smooth bundles. Transition matrices of such bundles have the form (7.27).

Using the bijection (7.33), we will identify the spaces $\mathcal{M}$ and $H^0(\mathcal{P}, B)/H^0(\mathcal{P}, S)$ and denote them by the same letter $\mathcal{M}$. It also follows from (7.33) that $H^0(\mathcal{P}, B)$ is a principal fibre bundle

$$H^0(\mathcal{P}, B) = P(\mathcal{M}, H^0(\mathcal{P}, S))$$

with the base space $\mathcal{M}$ and the structure group $H^0(\mathcal{P}, S)$.

7.7 Unitarity conditions

In §6.6 we discussed the imposition of a unitarity condition on transition matrices $F \in Z^1(\mathcal{U}, H)$ and defined various subsets of transition matrices and their moduli satisfying the unitarity condition.

As it has been discussed in §4.4, the matrices $\psi_1, \psi_2 \in SL(n, \mathbb{C})$ corresponding to gauge fields with values in the algebra $su(n)$ have to satisfy the condition (4.16c). The conditions (4.16a) for components of the gauge potential follow from (4.16c), (4.22) and (4.23). To satisfy these conditions, consider the following real subgroup $C^0(\mathcal{U}, S)$ (a real form) of the group $C^0(\mathcal{U}, S)$:

$$C^0(\mathcal{U}, S) := \left\{ \psi = (\psi_1, \psi_2) \in C^0(\mathcal{U}, S) : \psi_1(\tau(x, \lambda)) = \psi_2^{-1}(x, \lambda) \right\},$$

(7.36)
compatible with the real form $\tau$ on $P$. Of course, one can also define other real forms of the complex group $C^0(\mathcal{U}, S)$ assuming

$$\psi_1^\dagger(\tau(\lambda)) = \Pi \psi_2^{-1}(\tau(\lambda)),$$  \hfill (7.37)

where $\Pi$ is a diagonal matrix with $m$ copies of $+1$ and $n-m$ copies of $-1$. For all these subgroups the matrices $\delta^0(\psi^{-1}) = \psi_1^{-1}\psi_2 \in Z^1(\mathcal{U}, H)$ will satisfy the unitarity condition (4.16a) and therefore $\delta^0(\psi^{-1}) \in Z^1(\mathcal{U}, H)$.

The map $\delta^0 : C^0(\mathcal{U}, S) \to Z^1(\mathcal{U}, S)$ defines in $Z^1(\mathcal{U}, S)$ a subset of matrices $\psi_1^{-1}\psi_2$ with $\{\psi_1, \psi_2\} \in C^0(\mathcal{U}, S)$ which corresponds to the element $e \in H^1(\mathcal{P}, S)$. The set $H^1(\mathcal{P}, S)$ is defined analogously with the set $H^1(\mathcal{P}, S)$ (see §7.3). The kernel $\text{Ker} \varphi_r = \varphi_r^{-1}(e)$ of the map

$$\varphi_r := \varphi |_{H^1(\mathcal{P}, H)} : H^1(\mathcal{P}, H) \to H^1(\mathcal{P}, S)$$  \hfill (7.38)

coincides with the moduli space $\mathcal{M}_r$ of transition matrices $F \in Z^1(\mathcal{U}, H)$, for which there exists a Birkhoff decomposition (7.29) with $\psi_1, \psi_2$ satisfying the unitarity conditions (4.16c). The map $\delta^0$ associates with $\psi_1, \psi_2$ the global section (7.17), (7.30) of the sheaf $\mathcal{B}$ satisfying the unitarity condition (4.16a). We denote the space of all these solutions by $H^0_\varphi(\mathcal{P}, B)$. The matrices $g \in SL(n, \mathbb{C})$ from the group $H^0(\mathcal{P}, S)$ do not depend on $\lambda$, and the subgroup

$$H^0_\varphi(\mathcal{P}, S) = \{ g \in H^0(\mathcal{P}, S) : g^b = g^{-1} \}$$  \hfill (7.39)

of unitary matrices $g(x) \in SU(n)$ preserves the space $H^0_\varphi(\mathcal{P}, B)$. So we have a one-to-one correspondence between $\mathcal{M}_r$ and the moduli space $H^0_\varphi(\mathcal{P}, B)/H^0_\varphi(\mathcal{P}, S)$ of real local solutions to the SDYM equations,

$$\mathcal{M}_r \simeq H^0_\varphi(\mathcal{P}, B)/H^0_\varphi(\mathcal{P}, S).$$  \hfill (7.40)

\section{Symmetries in terms of smooth sheaves}

\subsection{Riemann-Hilbert problems from the cohomological point of view}

In §7.3 we described the twisted homomorphism $\delta^0 : C^0(\mathcal{U}, S) \to C^1(\mathcal{U}, S)$, the image of which $\text{Im} \delta^0 = \delta^0(C^0(\mathcal{U}, S))$ belongs to the set $Z^1(\mathcal{U}, S) \subset C^1(\mathcal{U}, S)$. More precisely, we have $\text{Im} \delta^0 \simeq C^0(\mathcal{U}, S)/H^0(\mathcal{P}, S)$, where the group $H^0(\mathcal{P}, S) = \text{Ker} \delta^0$ is a kernel of the map $\delta^0$. Hence $\delta^0(C^0(\mathcal{U}, S))$ can be identified with $C^0(\mathcal{U}, S)/H^0(\mathcal{P}, S)$, and

$$\delta^0 : C^0(\mathcal{U}, S) \to C^0(\mathcal{U}, S)/H^0(\mathcal{P}, S)$$  \hfill (8.1)

is a projection of the group $C^0(\mathcal{U}, S)$ onto the homogeneous space $Q := C^0(\mathcal{U}, S)/H^0(\mathcal{P}, S)$. So, the group $C^0(\mathcal{U}, S)$ can be considered as a principal fibre bundle

$$C^0(\mathcal{U}, S) = P(Q, H^0(\mathcal{P}, S))$$  \hfill (8.2)

with the structure group $H^0(\mathcal{P}, S)$ and the base space $Q \subset Z^1(\mathcal{U}, S)$, points of which correspond to smoothly trivial bundles.

As described in detail in §8.3, the space $Q$ contains as a subset the set $\mathcal{N}$ of those holomorphic bundles which are not only trivial as smooth bundles, but also holomorphically trivial on $\mathbb{CP}^1 \hookrightarrow P$, $x \in U$. The group $C^0(\mathcal{U}, S)$ acts on $Q$ transitively by formula (7.24) and therefore for any cocycle $F \in \mathcal{N} \subset Q$ there exists an element $\psi = \{\psi_1, \psi_2\} \in C^0(\mathcal{U}, S)$ such that the action $\rho_0(\psi, \cdot)$ transforms $F$ into $F^0 = 1$,

$$\rho_0(\psi, F)^{12} = \psi_1 F_{12} \psi_2^{-1} = 1 \Rightarrow F_{12} = \psi_1^{-1} \psi_2,$$  \hfill (8.3)

and to solve the Riemann-Hilbert problem means to find such an element $\psi$ from the group $C^0(\mathcal{U}, S)$. Of course, this element $\psi \in C^0(\mathcal{U}, S)$ is not unique; it is defined up to an element $g$ from the stability subgroup $H^0(\mathcal{P}, S)$ of the point $F^0 = 1$.

Indeed, if $\psi_1 F_{12} \psi_2^{-1} = 1$, then $(g^{-1} \psi_1) F_{12} (g^{-1} \psi_2)^{-1} = 1$ for any $g \in H^0(\mathcal{P}, S)$. In other words, to solve the Riemann-Hilbert problem means to define a section

$$s : \mathcal{N} \to C^0(\mathcal{U}, S)$$  \hfill (8.4)

over $\mathcal{N} \subset Q$ of the bundle (8.2). The section $s$ is not uniquely defined, and the group $H^0(\mathcal{P}, S)$ defines a transformation $g$ of the section $s$ into an equivalent section $s_g$.

**Remark.** It should be stressed that the cohomological description of the construction of solutions is applicable not only to the SDYM equations, but also to all equations integrable with the help of a Birkhoff decomposition of matrices on $\mathbb{CP}^1$ (the dressing method \cite{[10]}). For such equations, one can write an exact
Consider the restriction
\[ P(\mathcal{N}, H^0(\mathcal{P}, S)) := P(Q, H^0(\mathcal{P}, S)) \mid \mathcal{N} = (C^0)^{-1}(N) \] (8.5)
of the principal fibre bundle \( P(Q, H^0(\mathcal{P}, S)) \) to the subset \( \mathcal{N} \subset Q \). As described in \( \S \), the group \( C^0(\mathcal{U}, \mathcal{H}) \) acts on the space \( \mathcal{N} \) on the left, and this action can be lifted up to the action on \( P(\mathcal{N}, H^0(\mathcal{P}, S)) \), since this (left) action commutes with the (right) action of the group \( H^0(\mathcal{P}, S) \) on the space \( P(\mathcal{N}, H^0(\mathcal{P}, S)) \). Thus, we have the space \( P(\mathcal{M}, H^0(\mathcal{P}, S)) \) as a space of orbits of the group \( C^0(\mathcal{U}, \mathcal{H}) \) in the space \( P(\mathcal{N}, H^0(\mathcal{P}, S)) \).

At the same time, it follows from (7.35) that this space coincides with the space
\[ H^0(\mathcal{P}, B) = P(\mathcal{M}, H^0(\mathcal{P}, S)) \] (8.6b)
of (complex) local solutions to the SDYM equations.

Finally, it follows from (8.6) that the moduli space of (complex) local solutions to the SDYM equations is
\[ \mathcal{M} \simeq \rho_0(C^0(\mathcal{U}, \mathcal{H})) \backslash P(\mathcal{N}, H^0(\mathcal{P}, S)) / H^0(\mathcal{P}, S), \] (8.7)
i.e., \( \mathcal{M} \) is the biquotient space of the space \( P(\mathcal{N}, H^0(\mathcal{P}, S)) \) under the action of the groups \( C^0(\mathcal{U}, \mathcal{H}) \) and \( H^0(\mathcal{P}, S) \).

Using \( \S \), where we discussed the unitarity conditions in terms of \( \mathcal{F}_{12}, \psi \in C^0(\mathcal{U}, \mathcal{S}) \) etc., one can rewrite all formulae of \( \S \) in a way compatible with the real structure \( \tau \) on \( \mathcal{P} \). In particular, for the moduli space \( \mathcal{M}_\tau \) of (local) solutions to the SDYM equations we have
\[ \mathcal{M}_\tau \simeq \rho_0(C^0(\mathcal{U}, \mathcal{H}))/P(\mathcal{N}_\tau, H^0(\mathcal{P}, S))/H^0(\mathcal{P}, S). \] (8.8)

Then gauge fields take values in the Lie algebra \( su(n) \).

### 8.2 Action of the symmetry group \( \mathcal{G}_\tau \) on real solutions of the SDYM equations

We consider the cover \( \mathcal{U} = \{ \mathcal{U}_1, \mathcal{U}_2 \} \) of the twistor space \( \mathcal{P} \) and holomorphic bundles \( E' \in \mathcal{N}_\tau \subset Z_1(\mathcal{U}, \mathcal{H}) \). In \( \S \), the (local) action of the local group \( \mathcal{G}_\tau = \mathcal{G}_\tau \times \mathcal{E}_\tau \) on the space \( \mathcal{N}_\tau \simeq \mathcal{E}_\tau / \mathcal{E}_{\tau \triangle} \) was described. Let us choose an arbitrary transition matrix \( \mathcal{F}_{12} = \psi_1^{-1} \psi_2 \in \mathcal{N}_\tau \) and an element \( h = \{ \eta, a \} \in \mathcal{H}_\tau \times \mathcal{E}_\tau \). Consider the action \( \rho(h, \cdot) : \mathcal{F}_{12} \mapsto \mathcal{F}_{12}^h = \rho(h, \mathcal{F}_{12}) \). Since the local action preserves \( \mathcal{N}_{\tau} \), then \( \mathcal{F}^h \in \mathcal{N}_\tau \) and therefore there exists an element \( \psi^h = \{ \psi_1^h, \psi_2^h \} \in C^0(\mathcal{U}, \mathcal{S}) \) such that
\[ \mathcal{F}_{12}^h = (\psi_1^h)^{-1} \psi_2^h. \] (8.9)
Let us introduce \( \phi(h) = \{ \phi_1(h), \phi_2(h) \} \in C^0(\mathcal{U}, \mathcal{S}) \) by the formulae
\[ \phi_1(h) := \psi_1^h \psi_1^{-1}, \quad \phi_2(h) := \psi_2^h \psi_2^{-1}. \] (8.10)
Then we have a map
\[ \phi : \mathcal{G}_\tau \rightarrow C^0(\mathcal{U}, \mathcal{S}) \] (8.11)
of the group \( \mathcal{G}_\tau \) into the group \( C^0(\mathcal{U}, \mathcal{S}) \).

The elements \( \phi(h) = \{ \phi_1(h), \phi_2(h) \} \) of the group \( C^0(\mathcal{U}, \mathcal{S}) \) act by definition on \( \psi = \{ \psi_1, \psi_2 \} \in P(\mathcal{N}_\tau, H^0(\mathcal{P}, S)) \) as follows:
\[ h : \psi = \{ \psi_1, \psi_2 \} \mapsto \rho(h, \psi) = \psi^h = \{ \psi_1^h, \psi_2^h \} = \{ \phi_1(h) \psi_1, \phi_2(h) \psi_2 \}. \] (8.12)
From (7.11) it follows that \( B = \{ B^{(1)}, B^{(2)} \} \) is transformed by the formulae
\[ h : B^{(1)} \mapsto \phi_1(h) B^{(1)} \phi_1^{-1}(h) + \phi_1(h) \tilde{\partial} \phi_1^{-1}(h), \quad B^{(2)} \mapsto \phi_2(h) B^{(2)} \phi_2^{-1}(h) + \phi_2(h) \tilde{\partial} \phi_2^{-1}(h). \] (8.13b)
With the help of formulae (8.13), (4.22) and (4.23) it is not difficult to write down explicit formulae for transformations of components \( A_\mu \) of the gauge potential \( A \). We shall not do this.

Consider now a transformation
\[
\mathcal{F}_{12} \mapsto \mathcal{F}_{12}^h \mapsto \mathcal{F}_{12}^{f_h} = f_{12}h_{12}f_{12}^h. 
\] (8.14)

It is easy to see that
\[
B^{f_h} = \phi(fh)B\phi^{-1}(fh) + \phi(fh)\partial\phi^{-1}(fh) = \phi(f)B^h\phi^{-1}(f) + \phi(f)\partial\phi^{-1}(f) = 
\]
\[
= \phi(f)\phi(h)B(\phi(f)\phi(h))^{-1} + \phi(f)\phi(h)\partial(\phi(f)\phi(h))^{-1}. 
\] (8.15)

It follows from (8.14), (8.15) that
\[
\phi(fh) = \phi(f)\phi(h), 
\] (8.16)
i.e., the map (8.11) is a homomorphism of the local Lie group \( \mathfrak{g}_r \) into the group \( C^0_r(\mathcal{U}, S) \).

### 8.3 Gauge fixing and some formulae

The SDYM equations (4.7) for \( A_\mu \in sl(n, \mathbb{C}) \) imply that the components of the gauge potential can be written in the form
\[
A_{y^1} = \Theta^{-1}\partial_{y^1}\Theta, \quad A_{y^2} = \Theta^{-1}\partial_{y^2}\Theta, \quad A_{y^3} = \tilde{\Theta}^{-1}\partial_{y^3}\tilde{\Theta}, \quad A_{y^4} = \tilde{\Theta}^{-1}\partial_{y^4}\tilde{\Theta}, 
\] (8.17)
where \( \Theta \) and \( \tilde{\Theta} \) are some \( SL(n, \mathbb{C}) \)-valued functions on \( U \subset \mathbb{R}^4 \). One may perform the following gauge transformation:
\[
A_{y^1} \mapsto A_{y^1}^\Theta = \tilde{\Theta}A_{y^1}\Theta^{-1} + \tilde{\Theta}\partial_{y^1}\tilde{\Theta}^{-1} = 0, \quad A_{y^2} \mapsto A_{y^2}^\Theta = \tilde{\Theta}A_{y^2}\Theta^{-1} + \tilde{\Theta}\partial_{y^2}\tilde{\Theta}^{-1} = 0, 
\] (8.18a)
\[
A_{y^3} \mapsto A_{y^3}^\Phi = \Theta A_{y^3}\Phi^{-1} + \Theta\partial_{y^3}\Phi^{-1} = 0, \quad A_{y^4} \mapsto A_{y^4}^\Phi = \tilde{\Theta}A_{y^4}\Phi^{-1} + \tilde{\Theta}\partial_{y^4}\Phi^{-1} = 0, 
\] (8.18b)
where \( \Phi := \Theta\tilde{\Theta}^{-1} \in SL(n, \mathbb{C}) \), and thus fix the gauge \( A_{y^0}^\Theta = A_{y^0}^\Phi = 0 \) [23]-[27]. Then eqs.(4.7) are replaced by the matrix equations
\[
\partial_{y^1}(\Phi^{-1}\partial_{y^1}\Phi) + \partial_{y^2}(\Phi^{-1}\partial_{y^2}\Phi) = 0, 
\] (8.19)
which are the SDYM equations in the Yang gauge. Equations (8.19) are a 4D analogue of the 2D WZNW equations.

It is also possible to perform the gauge transformation
\[
A_{y^1} \mapsto \Theta A_{y^1}\Theta^{-1} + \Theta\partial_{y^1}\Theta^{-1} = \Phi\partial_{y^1}\Phi^{-1}, \quad A_{y^2} \mapsto \Theta A_{y^2}\Theta^{-1} + \Theta\partial_{y^2}\Theta^{-1} = \Phi\partial_{y^2}\Phi^{-1}, 
\] (8.20)
then eqs.(4.7) get converted into the equations
\[
\partial_{y^1}(\Phi\partial_{y^1}\Phi^{-1}) + \partial_{y^2}(\Phi\partial_{y^2}\Phi^{-1}) = 0. 
\] (8.21)

From the linear system (4.10) it is easy to see that
\[
\Theta = \psi_2^{-1}(\zeta = 0), \quad \tilde{\Theta} = \psi_1^{-1}(\lambda = 0), 
\] (8.22)
where the \( SL(n, \mathbb{C}) \)-valued function \( \psi_1 \) is defined on \( \mathcal{U}_1 \), and the \( SL(n, \mathbb{C}) \)-valued function \( \psi_2 \) is defined on \( \mathcal{U}_2 \). Equations (8.19) are the compatibility conditions of the linear system
\[
\partial_{y^1}\psi_1 - \lambda(\partial_{y^2} + \Phi^{-1}\partial_{y^2}\Phi)\psi_1 = 0, \quad \partial_{y^2}\psi_1 + \lambda(\partial_{y^1} + \Phi^{-1}\partial_{y^1}\Phi)\psi_1 = 0, 
\] (8.23)
obtained from (4.10) for \( \psi_1 \) by performing the gauge transformation \( \psi_1(x, \lambda) \mapsto \tilde{\psi}_1(x, \lambda) = \psi_1^{-1}(x, 0)\psi_1(x, \lambda) = \Theta(x)\psi_1(x, \lambda), \quad \lambda \in \Omega_1 \). Analogously, eqs.(8.21) are the compatibility conditions for the linear system
\[
\zeta(\partial_{y^1} + \Phi\partial_{y^1}\Phi^{-1})\tilde{\psi}_2 - \partial_{y^2}\tilde{\psi}_2 = 0, \quad \zeta(\partial_{y^2} + \Phi\partial_{y^2}\Phi^{-1})\tilde{\psi}_2 + \partial_{y^1}\tilde{\psi}_2 = 0, 
\] (8.24)
where \( \tilde{\psi}_2(x, \zeta) = \psi_2^{-1}(x, 0)\psi_2(x, \zeta) \) is well defined for \( \zeta \in \Omega_2 \).

We have \( \psi_1(x, \lambda = 0) = 1 \) and therefore
\[
\tilde{\psi}_1 = 1 + \lambda\Psi + O(\lambda^2) 
\] (8.25)
for some Lie algebra valued function $\Psi \in sl(n, \mathbb{C})$. By substituting (8.25) into (8.23), we find that

$$\Phi^{-1} \partial_{y^2} \Phi = \partial_{\bar{y}^1} \Psi, \quad \Phi^{-1} \partial_{\bar{y}^1} \Phi = -\partial_{y^2} \Psi. \tag{8.26}$$

Then after substitution (8.26) into (8.23), the compatibility conditions of the linear system (8.23) will be

$$\partial_{y^1} \partial_{y^2} \Psi + \partial_{y^2} \partial_{y^2} \Psi + [\partial_{y^1} \Psi, \partial_{y^2} \Psi] = 0. \tag{8.27}$$

Equations (8.27) are the SDYM equations in the so-called Leznov-Parkes form.

Notice that the condition

$$\psi_1(x, \lambda = 0) = 1, \tag{8.28}$$

leading to the gauge fixing $A_{\bar{y}^1} = A_{y^2} = 0$, can be imposed from the very beginning. Then the Birkhoff factorization (8.3) is unique, which corresponds to the choice of the fixed section (8.4) of the bundle (8.5). Nevertheless, the gauge (8.28) does not remove all degrees of freedom related to holomorphic transformations of the group $C^0(\mathcal{U}, \mathcal{H})$, and if we want to obtain the moduli space $\mathcal{M}$, we have to factorize $s(\mathcal{N}) \simeq \mathcal{N}$ w.r.t. the action of the subgroup in $C^0(\mathcal{U}, \mathcal{H})$ preserving the gauge (8.28). The same gauge may be used in the description of the moduli space $\mathcal{M}_\tau$ discussed in §8.3.

### 8.4 Generalization to self-dual manifolds

As has been mentioned in §6.6, the twistor correspondence between self-dual gauge fields and holomorphic bundles exists not only for the Euclidean space $\mathbb{R}^4$, but also for 4-manifolds $M$, the Weyl tensor of which is self-dual. Twistor spaces $\mathbb{Z} \equiv \mathbb{Z}(M)$ for such manifolds $M$ are three-dimensional complex spaces. The description of symmetries of local solutions to the SDYM equations can be easily generalized to this general case.

It can be done as follows. Fix an open set $U \subset M$ such that $\mathbb{Z}|_U \simeq U \times \mathbb{CP}^1$ and choose coordinates $x^\mu$ on $U$. Consider the restriction of the twistor bundle $\pi : \mathbb{Z} \rightarrow M$ to $U$ and put $\mathcal{P} := \mathbb{Z}|_U$. The space $\mathcal{P}$ is an open subset of $\mathbb{Z}$, and, as a real manifold, $\mathcal{P}$ is diffeomorphic to the direct product $U \times \mathbb{CP}^1$. Now a metric on $U$ is not flat, and a conformal structure on $U$ is coded into a complex structure $\mathcal{J}$ on $\mathcal{P}$. In this “curved” case we again have a natural one-to-one correspondence between solutions of the SDYM equations on $U$ and holomorphic bundles $\mathcal{E}'$ over $\mathcal{P}$, holomorphically trivial on (real) projective lines $\mathbb{CP}^1_x \rightarrow \mathcal{P}$, $\forall x \in U$.

In our group-theoretic analysis of the twistor correspondence we did not use the explicit form of the complex structure $\mathcal{J}$ on $\mathcal{P}$ and therefore did not use the explicit form of the metric on $U$. This explicit form was used only in some illustrating formulae, which can easily be generalized. That is why, all statements about local solutions and symmetry groups are also true for the SDYM equations on self-dual manifolds $M$.

Thus, as the local symmetry group we again obtain the group $\Theta_\tau = \mathcal{S}_\tau \ltimes \mathbb{C}_\tau$ from §§6–8 acting on the space of local solutions to the SDYM equations defined on a self-dual 4-manifold $M$.

### 9 Discussion

#### 9.1 What is integrability?

In books and papers on soliton equations one often poses the question: What is integrability? There is no general answer to this question, and usually one connects the integrability with the existence of Lax or zero curvature representations. Then non-Abelian cohomology, local groups and deformation theory of bundles with holomorphic or flat connections form the basis of integrability. In other words, there are always exact sequences of sheaves and cohomology sets of type (7.16), (7.28) hiding behind the integrability. This explains, in particular, why almost all integrable equations in two dimensions can be obtained by reductions of the SDYM equations (see e.g. [17], [48], [49] and references therein).

In [50], [51] generalized SDYM equations in dimension $D>4$ and their solutions have been considered. Some of these equations in dimension $D=4n$ [51], [52] are integrable, since with the help of the twistor approach these quaternionic-type SDYM equations can be rewritten as holomorphy conditions of the Yang-Mills bundle over an auxiliary (twistor) $(4n+2)$-space. The situation with the integrability of other generalized SDYM equations in $D>4$ is much less clear. Solutions of these equations, e.g. octonionic-type SDYM equations in $D=8$ [54], [55], were used in constructing solitonic solutions of string theories [54]. The modification of these generalized SDYM equations arising after replacement of commutators by Poisson brackets are considered in supermembrane theory (see e.g. [57]). At the moment it is not clear whether all these equations can be interpreted as an existence condition of flat or holomorphic connections in bundles over some auxiliary spaces. This interesting problem deserves further study.
9.2 Holomorphic Chern-Simons-Witten theory

Let us consider a smooth six-dimensional manifold $Z$ with an integrable almost complex structure $\mathcal{J}$. Then $Z$ is a complex 3-manifold, and one can introduce a cover $\{U_\alpha\}$ of $Z$ and coordinates $z_\alpha : U_\alpha \to \mathbb{C}^3$. Let $E'$ be a smooth complex vector bundle of rank $n$ over $Z$ and let $\tilde{B}$ be the $(0,1)$-component of a connection 1-form on the bundle $E'$. Suppose that $\tilde{B}$ satisfies the equations

$$\bar{\partial}\tilde{B} + \tilde{B} \wedge \tilde{B} = 0,$$

where $\bar{\partial}$ is the $(0,1)$ part of the exterior derivative $d = \partial + \bar{\partial}$. The special case of eqs.(9.1) on the twistor space $\mathcal{P}$ of $U \subset \mathbb{R}^4$ was considered in §7. Equations (9.1) mean that the $(0,2)$ part of the curvature of the bundle $E'$ is equal to zero: $F^{0,2} := \bar{\partial}^2 = (\partial + \bar{\partial})^2 = 0$ and, therefore, the bundle $E'$ is holomorphic. We shall call eqs.(9.1) defined on a complex 3-manifold $Z$ the field equations of holomorphic Chern-Simons-Witten (CSW) theory.

Equations (9.1) were suggested by Witten [58] for a special case of bundles over Calabi-Yau (CY) 3-folds $Z$ as equations of a holomorphic analogue of the ordinary Chern-Simons theory. Witten obtained eqs.(9.1) from open $N = 2$ topological string theories with a central charge $\hat{c} = 3$ (6D target space) and the CY restriction $c_1(Z) = 0$ arose from $N = 2$ superconformal invariance of a sigma model used in constructing the topological string theory. The connection of eqs.(9.1) with topological strings was also considered in [60]. Equations (9.1) on CY 3-folds were considered by Donaldson and Thomas [54] in the frames of program on extending the results of Casson, Floer, Jones and Donaldson to manifolds of dimension $D > 4$. Donaldson and Thomas [54] pointed out that one may try to consider a more general situation with eqs.(9.1) on complex manifolds $Z$ which are not Calabi-Yau ($c_1(Z) \neq 0$). This is important since the CY restriction cannot be imposed if one uses the twistor correspondence between 4D and 6D theories.

In §5 we considered the special case of the holomorphic CSW theory when field equations are defined not on an arbitrary complex 3-manifold, but on the twistor space $\mathcal{P}$ of $U \subset \mathbb{R}^4$. The manifold $\mathcal{P}$ can be covered by two charts, and in §5 we described the moduli space and symmetries of the holomorphic CSW theory in the Čech approach. In §7 (see formulae (7.13), (7.26) and (7.34)) we gave the Dolbeault description of this moduli space. This analysis of the moduli space and symmetries of the holomorphic CSW theory can be generalized without difficulties to an arbitrary complex 3-manifold $Z$.

9.3 $N = 2$ and $N = 4$ topological strings

The coupling of topological sigma models and topological gravity gives the above-mentioned $N = 2$ topological strings [60] which were further studied in [58, 59, 61]. They have critical dimension $D = 6$ and are related to topological sigma models with the 6D target space. There are two classes of such models, called A- and B-models. In the open string sector of the critical topological string theories there are A and B versions of these theories. The A-model is related to the ordinary Chern-Simons theory in 3 real dimensions and the B-model is related to the holomorphic Chern-Simons-Witten theory in 3 complex dimensions. We discuss only the B-model, the field equations for which coincide with eqs.(9.1) on a CY 3-fold.

Besides $N = 2$ topological strings with $\hat{c} = 3$ (6D target space) there are $N = 4$ topological strings with a central charge $\hat{c} = 2$ (4D target space) [61] and nontopological $N = 2$ strings (see e.g. [62-65] and references therein). In [62] it was shown that $N = 2$ strings are a special case of $N = 4$ topological strings. The $N = 2$ string theories describe quantum SDYM fields on a self-dual gravitational background [62, 63, 64, 65]. For heterotic $N = 2$ strings [60] besides SDYM fields there are also matter fields depending on the details of the construction.

Comparing the above-mentioned string theories and field theories corresponding to them, one obtains the following “commutative” diagram

\[
\begin{array}{ccc}
\text{Holomorphic CSW theory on complex 3-manifolds} & \rightarrow & \text{N = 4 topological strings} \\
\downarrow & & \downarrow \\
\text{Holomorphic CSW theory on twistor spaces of self-dual 4-manifolds} & \rightarrow & \text{SDYM theory on self-dual 4-manifolds}
\end{array}
\]

The arrows mean that one theory can be derived from another one. The difference between the holomorphic CSW theory on a general complex 3-manifold and the one defined on a twistor space $Z$ is stipulated by the

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existence in $Z$ of a bundle structure $\pi : Z \to M$ with a self-dual 4D manifold $M$ as a base space and $\mathbb{CP}^1$ as a typical fibre. In the general case, complex 3-spaces are arbitrary.

Into the box with the question-mark from (9.2) one cannot substitute ‘$N = 2$ topological strings’, since they are obtained from sigma models on CY 3-folds. One should substitute there some generalized \( \Omega \) meromorphic functions on $\mathbb{CP}^N$ = 4 topological strings and generalized $N = 2$ topological strings was pointed out in the papers [59, 64]. Ooguri and Vafa [59] gave reasons for possible equivalence of $N = 4$ topological strings and generalized $N = 2$ topological strings on the twistor space with a holomorphic (2,0)-form turned on. It would be very interesting to study this possibility.

### 9.4 Integrable 4D conformal field theories

It is well-known that the ordinary 3D Chern-Simons theory is connected with 2D conformal field theories if one supposes that a 3-manifold has the form $\Sigma \times \mathbb{R}$, where $\Sigma$ is a 2-manifold with or without a boundary [67, 68]. In particular, if $\Sigma$ has a boundary, the quantum Hilbert space $H_{\Sigma}$ is infinite-dimensional and is a representation space of the chiral algebra of CFT on $\Sigma$. Analogously, the holomorphic Chern-Simons-Witten theory on a complex 3-manifold $Z$ is connected with integrable 4D CFT’s on a self-dual 4-manifold $M$ if one supposes that $Z$ is the twistor space of $M$. This means that $Z$ is the bundle $\pi : Z \to M$ over $M$ with $\mathbb{CP}^1$ as a typical fibre. On $M$ it is possible to consider a CFT of fields of an arbitrary spin. Most of these CFT’s will describe free fields in a fixed background. By considering local solutions of field equations on $M$ we take an open set $U \subset M$ and consider the twistor space $\mathcal{P} = Z|_U$ of which is an open subset in $Z$.

In this paper, we actually discuss how the concrete nonlinear 4D CFT – the SDYM theory – is connected with the holomorphic CSW theory on the twistor space $\mathcal{P}$ of $U$. The SDYM model on an open ball $U \subset \mathbb{R}^4$ is a generalization of the WZNW model on the complex plane $\mathbb{C}$, and we mainly consider sets $U$ with the flat metric. We described symmetries of the SDYM model and the moduli space of self-dual gauge fields on $U$. Naturally, the following questions arise:

1. What is an analogue of affine Lie algebras of 2D CFT’s?

2. What is an analogue of the Virasoro algebra?

In this paper we have not discussed symmetry algebras yet. But knowing the symmetry groups of the SDYM equations, described in §§ 7.8, it is not difficult to write down the algebras corresponding to them.

A symmetry algebra of integrable 4D CFT’s is connected with the algebra $\mathcal{G}_h$ of functions that are holomorphic on $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathcal{P}$ and take values in the Lie algebra $\mathfrak{g}$ of a complex Lie group $\mathbb{G}$. The algebra $\mathcal{G}_h$ with pointwise commutators generalizes affine Lie algebras. The symmetry algebra is the algebra

$$C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}^g) \simeq \mathcal{G}_h \oplus \mathcal{g}_h \quad (9.3)$$

of 1-cochains of the cover $\mathcal{U} = \{ \mathcal{U}_1, \mathcal{U}_2 \}$ of the space $\mathcal{P}$ with values in the sheaf $\mathcal{O}_{\mathbb{P}^1}^g$ of holomorphic maps from $\mathcal{P}$ into the Lie algebra $\mathfrak{g}$. We mainly considered the case $\mathfrak{g} = sl(n, \mathbb{C})$. The algebra (9.3) was also considered by Ivanova [63].

Notice that the affine Lie algebra $\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ (without a central term) is the algebra of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{CP}^1 \simeq \mathbb{C}^* \cup \{0\} \cup \{\infty\}$ with the poles at $\lambda = 0, \lambda = \infty$ and holomorphic on $\Omega_{12} = \Omega_1 \cap \Omega_2 \simeq \mathbb{C}^*$. Hence, it is a subalgebra in the algebra

$$C^1(\Omega, \mathcal{O}_{\mathbb{P}^1}^{\mathfrak{g}}) \simeq \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \quad (9.4)$$

of 1-cochains of the cover $\Omega = \{ \Omega_1, \Omega_2 \}$ of $\mathbb{CP}^1$ with values in the sheaf of holomorphic maps from $\mathbb{CP}^1$ into the Lie algebra $\mathfrak{g}$. Thus, the algebra (9.3) is an analogue of the 2D affine Lie algebra (9.4). Notice that (central) extensions of the algebras (9.3) and (9.4) will appear after the transition to quantum theory.

### 9.5 The Čech description of the Virasoro algebra

Elements of the Virasoro algebra $Vir^0$ (with zero central charge) are meromorphic vector fields on $\mathbb{CP}^1$ having poles at the points $\lambda = 0, \lambda = \infty$ and holomorphic on the overlap $\Omega_{12} = \Omega_1 \cap \Omega_2 \simeq \mathbb{C}^* = \mathbb{CP}^1 - \{0\} - \{\infty\}$. This algebra has the following Čech description. Let us consider the sheaf $\mathcal{V}_{\mathbb{CP}^1}$ of holomorphic vector fields on $\mathbb{CP}^1$. Then for the space of Čech 1-cochains with values in $\mathcal{V}_{\mathbb{CP}^1}$ we have

$$C^1(\Omega, \mathcal{V}_{\mathbb{CP}^1}) \simeq Vir^0 \oplus Vir^0. \quad (9.5)$$
Notice that for \( \{v_{12}, v_{21}\} \in C^1(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1}) \) the antisymmetry condition cannot be imposed on cohomology indices of the holomorphic vector fields \( v_{12}, v_{21} \), since it is not preserved under commutation. So we have \( v_{21} \neq -v_{12} \) in the general case.

The space \( Z^1(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1}) \) of 1-cocycles of the cover \( \mathcal{O} = \{\Omega_1, \Omega_2\} \) of \( \mathbb{P}^1 \) with values in the sheaf \( \mathcal{V}_{\mathbb{P}^1} \), coincides with the algebra \( Vir^0 \) as a vector space, since

\[
Z^1(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1}) \simeq (Vir^0 \oplus Vir^0)/\text{diag}(Vir^0 \oplus Vir^0).
\] (9.6)

Further, by virtue of the equality

\[
H^1(\mathbb{C}P^1, \mathcal{V}_{\mathbb{P}^1}) = 0,
\] (9.7)

which means the rigidity of the complex structure of \( \mathbb{C}P^1 \), any element \( v \) from \( Vir^0 \simeq Z^1 \) can be represented in the form

\[
v = v_1 - v_2.
\] (9.8)

Here, \( v_1 \) can be extended to a holomorphic vector field on \( \Omega_1 \), and \( v_2 \) can be extended to a holomorphic vector field on \( \Omega_2 \).

It follows from (9.6)--(9.8) that the algebra \( Vir^0 \) is connected with the algebra

\[
C^0(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1})
\] (9.9)
of 0-cochains of the cover \( \mathcal{O} \) with values in the sheaf \( \mathcal{V}_{\mathbb{P}^1} \) by the (twisted) homomorphism

\[
\bar{\delta}^0 : C^0(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1}) \longrightarrow C^1(\mathcal{O}, \mathcal{V}_{\mathbb{P}^1}) \Leftrightarrow \bar{\delta}^0 \colon \{v_1, v_2\} \mapsto \{v_1 - v_2, v_2 - v_1\}.
\] (9.10a)

Just the cohomological nature of the algebra \( Vir^0 \) permits one to define its local action on Riemann surfaces of arbitrary genus and on the space of conformal structures of Riemann surfaces \([70]\). A central extension arises under an action of the Virasoro algebra on holomorphic sections of linear bundles over moduli spaces (quantization).

### 9.6 Infinitesimal deformations of self-dual conformal structures

Here we briefly answer the question of §9.4 about an analogue of the Virasoro algebra (without a central term).

In §§5.4, 5.5, 8.2 we described the local group \( \mathcal{H} \) of biholomorphisms of the twistor space \( \mathcal{P} \) and its action on the space of local solutions to the SDYM equations. To this group there corresponds the algebra (cf.(9.9), (9.10))

\[
C^0(\mathcal{U}, \mathcal{V}_P)
\] (9.11)
of 0-cochains of the cover \( \mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\} \) of \( \mathcal{P} \) with values in the sheaf \( \mathcal{V}_P \) (of germs) of holomorphic vector fields on \( \mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2 \). However, this algebra is not a correct generalization of the Virasoro algebra.

An analogue of the Virasoro algebra is the algebra \( \mathcal{V}_P(\mathcal{U}_{12}) \) of holomorphic vector fields on \( \mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathcal{P} \). It is a subalgebra of the algebra

\[
C^1(\mathcal{U}, \mathcal{V}_P) \simeq \mathcal{V}_P(\mathcal{U}_{12}) \oplus \mathcal{V}_P(\mathcal{U}_{12})
\] (9.12)
of 1-cochains of the cover \( \mathcal{U} \) with values in the sheaf \( \mathcal{V}_P \). Elements of the algebra \( C^1(\mathcal{U}, \mathcal{V}_P) \) are the collections of vector fields

\[
\chi = \{\chi_{12}, \chi_{21}\} = \{\chi_{12}^a \frac{\partial}{\partial z_1^a}, \chi_{21}^a \frac{\partial}{\partial z_2^a}\}
\] (9.13)

with ordered “cohomology indices”.

From the Kodaira-Spencer deformation theory \([74]\) it follows that the algebra (9.12) acts on the transition function \( f_{12} \) of the space \( \mathcal{P} \) (see §3.3) by the formula

\[
\delta f_{12}^a = \chi_{12}^a - \frac{\partial f_{12}^a}{\partial z_2^b} \chi_{21}^b \Leftrightarrow \delta f_{12} := \delta f_{12}^a \frac{\partial}{\partial z_1^a} = \chi_{12} - \chi_{21}.
\] (9.14)

Accordingly, one may define the following action of the algebra \( C^1(\mathcal{U}, \mathcal{V}_P) \) on the transition matrices \( \mathcal{F}_{12} \) of holomorphic bundles \( E' \) over the twistor space \( \mathcal{P} \):

\[
\delta \chi \mathcal{F}_{12} = \chi_{12}(\mathcal{F}_{12}) - \chi_{21}(\mathcal{F}_{12}).
\] (9.15)
The algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ acts on the transition function $f_{12}$ of the space $\mathcal{P}$ and on the transition matrices $\mathcal{F}_{12}$ of bundles $E'$ over $\mathcal{P}$ by formulæ (9.14),(9.15) via the twisted homomorphism

$$\delta^0 : C^0(\mathcal{U}, \mathcal{V}_P) \ni \{\chi_1, \chi_2\} \mapsto \{\chi_1 - \chi_2, \chi_2 - \chi_1\} \in C^1(\mathcal{U}, \mathcal{V}_P)$$

(9.16)

of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ into the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$.

Notice that $\delta f := \{\delta f_{12}, \delta f_21\} \in Z^1(\mathcal{U}, \mathcal{V}_P)$, and the quotient space

$$H^1(\mathcal{U}, \mathcal{V}_P) := Z^1(\mathcal{U}, \mathcal{V}_P)/\delta^0(C^0(\mathcal{U}, \mathcal{V}_P))$$

(9.17)

describes nontrivial infinitesimal deformations of the complex structure of $\mathcal{P}$. For a cover $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$, where $\mathcal{U}_1, \mathcal{U}_2$ are Stein manifolds, we have $H^1(\mathcal{P}, \mathcal{V}_P) = H^1(\mathcal{U}, \mathcal{V}_P)$. In contrast with the 2D case (9.7) now we have $H^1(\mathcal{P}, \mathcal{V}_P) \neq 0$. Hence, the transformations (9.14) of the transition function in general change the complex structure of $\mathcal{P}$ and therefore change the conformal structure on $U$. Recall that a conformal structure $[g]$ is called self-dual if the Weyl tensor for any metric $g$ in the conformal equivalence class $[g]$ is self-dual [13, 14]. In virtue of the twistor correspondence [13, 14] the moduli space of self-dual conformal structures on a 4-manifold $M$ is bijection to the moduli space of complex structures on the twistor space of $M$.

All algebras of infinitesimal symmetries of the self-dual gravity equations known by now (see e.g. [74] and references therein) are subalgebras in the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$. The action of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ (and the group $\delta(\mathcal{P})$ corresponding to it) transforms $f_{12}$ into an equivalent transition function and therefore preserves the conformal structure on $U$. At the same time, the action of the algebra $C^0(\mathcal{U}, \mathcal{V}_P)$ on transition matrices of holomorphic bundles $E' \to \mathcal{P}$ is not trivial.

If we want to define an action of the algebra $C^1(\mathcal{U}, \mathcal{V}_P)$ on the coordinates $\{z_1^a, z_2^a\}$, q-forms etc. we should define: 1) a sheaf $\mathcal{T}^{1,0}$ of $(1,0)$ vector fields on $\mathcal{P}$, holomorphic along fibres $\mathbb{C}P^1$ of the bundle $\mathcal{P} \to U$; 2) a sheaf $\mathcal{V}$ of $(0,1)$-forms $W$ on $\mathcal{P}$ with values in $\mathcal{T}^{1,0}$, vanishing on the distribution $V^{0,1}$ (see §7.3) and satisfying the equations

$$\delta \mathcal{V} = 0$$

(9.18)

on any open set $\mathcal{U} \subset \mathcal{P}$, where $\mathcal{W} \in \mathcal{W}(\mathcal{U})$. Then we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{V}_P \longrightarrow \mathcal{T}^{1,0} \longrightarrow \mathcal{W} \longrightarrow 0$$

(9.19)

and the corresponding exact sequence of cohomology spaces

$$0 \longrightarrow H^0(\mathcal{P}, \mathcal{V}_P) \longrightarrow H^0(\mathcal{P}, \mathcal{T}^{1,0}) \longrightarrow H^0(\mathcal{P}, \mathcal{W}) \longrightarrow H^1(\mathcal{P}, \mathcal{V}_P) \longrightarrow 0,$$

(9.20)

describing infinitesimal deformations of the complex structure of the twistor space $\mathcal{P}$.

From (9.20) it follows that for any element $\delta f \in Z^1(\mathcal{U}, \mathcal{V}_P) \subset Z^1(\mathcal{U}, \mathcal{T}^{1,0})$ there exists an element $\{\varphi_1, \varphi_2\} \in C^0(\mathcal{U}, \mathcal{T}^{1,0})$ such that

$$\delta f = \{\chi_{12} - \chi_{21}, \chi_{21} - \chi_{12}\} = \{\varphi_1 - \varphi_2, \varphi_2 - \varphi_1\} \in \delta^0(C^0(\mathcal{U}, \mathcal{T}^{1,0})).$$

(9.21)

Then for infinitesimal transformations of coordinates on $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ we have

$$\delta z_1^a := \varphi_1^a(z_1, \bar{z}_1), \quad \delta z_2^a := \varphi_2^a(z_2, \bar{z}_2).$$

(9.22)

To preserve the reality of the conformal structure on $U$, one should define real subalgebras of the algebras $C^1(\mathcal{U}, \mathcal{V}_P)$ and $C^0(\mathcal{U}, \mathcal{T}^{1,0})$ by analogy with §§6.6, 7.7. We shall not write down transformations of the metric and conformal structure on $U$, since this will require a lot of additional explanations. Details will be published elsewhere.

9.7 Quantization

Some problems related to the quantization of the SDYM model were discussed in [20, 74, 21]. The quantization was carried out in four dimensions in terms of q-valued fields $A_\mu$ or in terms of a $G$-valued scalar field by using the Yang gauge. But the obtained results are fragmentary; the picture is not complete and far from what we have in 2D CFT’s. Remembering the connection between 2D CFT’s and the ordinary 3D CS theory, one may come to the reasonable conclusion that the quantization of integrable 4D CFT’s may be much more successful if we use the 6D holomorphic CSW theory.

When quantizing the holomorphic CSW theory on the twistor space $\mathcal{P}$ one may use the results on the quantization of the ordinary CS theory (see e.g. [67, 68] and references therein) after a proper generalization.
We are mainly interested in quantizing the SDYM model. As such, we have to put \( \tilde{B}_3 = 0 \) in eqs.(9.1), which leads to the equations (cf.(7.12))
\[
\partial B + B \wedge B = 0
\]  
(9.23)
equivalent to the SDYM equations, as has been discussed in this paper. The comparison with the ordinary CS theory in the Hamiltonian approach shows that \( \lambda \) may be considered as (complex) time of the holomorphic CSW theory.

Further, one can use two standard approaches to the quantization of constrained systems: 1) one first solves the constraints and then performs the quantization of the moduli space; 2) one first quantizes the free theory and then imposes (quantum) constraints. The first approach will mainly be discussed. We shall write down the list of questions and open problems whose solutions are necessary to give the holomorphic CSW and the SDYM theories a status of quantum field theories.

1. One should rewrite a symplectic structure \( \tilde{\omega} \) on the space of gauge potentials or their relatives \( \tilde{\omega}(n) \) \cite{20, 24, 10} in terms of fields on the twistor space \( \mathcal{P} \). This 2-form \( \tilde{\omega} \) induces a symplectic structure \( \omega \) on the moduli space \( \mathcal{M} \) of solutions to eqs.(9.23), and the cohomology class \( [\omega] \in H^2(\mathcal{M}, \mathbb{R}) \) has to be integral.

2. Over the moduli space \( \mathcal{M} \) one should define a complex line bundle \( \mathcal{L} \) with the Chern class \( c_1(\mathcal{L}) = [\omega] \). Then \( \mathcal{L} \) admits a connection with the curvature 2-form equal to \( \omega \).

3. A choice of a complex structure \( J \) on the twistor space \( \mathcal{P} \) endows the moduli space \( \mathcal{M} \) with a complex structure which we shall denote by the same letter \( J \). Then the bundle \( \mathcal{L} \) over \( (\mathcal{M}, J) \) has a holomorphic structure, and a quantum Hilbert space of the SDYM theory can be introduced as the space \( H_{\mathcal{J}} \) of (global) holomorphic sections of \( \mathcal{L} \).

4. Is it possible to introduce the bundle \( \mathcal{L} \to \mathcal{M} \) as the holomorphic determinant line bundle \( \text{Det} \tilde{\partial}_B \) of the operator \( \tilde{\partial}_B = \tilde{\partial} + B \) on \( \mathcal{P} \)?

5. The action functional of the holomorphic CSW theory on a Calabi-Yau 3-fold has a simple form \( \text{CSW} \) analogous to the action of the standard CS theory. How should one modify this action if we go over to the case of an arbitrary complex 3-manifold?

6. One should lift the action of the symmetry groups and algebras described in this paper up to an action on the space \( H_{\mathcal{J}} \) of holomorphic sections of the bundle \( \mathcal{L} \) over \( \mathcal{M} \). What is an extension (central or not) of these groups and algebras? Finding of an extension of the algebra \( C^1(\mathcal{U}, \mathcal{O}_\mathcal{P}) \) is equivalent to finding a curvature of the bundle \( \mathcal{L} \) since this curvature represents a local anomaly.

7. What can be said about representations of the algebras \( C^1(\mathcal{U}, \mathcal{V}_\mathcal{P}) \) and \( C^1(\mathcal{U}, \mathcal{O}_\mathcal{P}) \)? Which of these representations are connected with the Hilbert space \( H_{\mathcal{J}} \)?

8. In the quantum holomorphic CSW and SDYM theories there exist Sugawara-type formulae, i.e., generators of the algebra \( C^1(\mathcal{U}, \mathcal{V}_\mathcal{P}) \) can be quadratically expressed in terms of generators of the algebra \( C^1(\mathcal{U}, \mathcal{O}_\mathcal{P}) \). This follows from the fact that any transformation of transition matrices of a holomorphic bundle \( \mathcal{E} \to \mathcal{P} \) under the action of the algebra \( C^1(\mathcal{U}, \mathcal{V}_\mathcal{P}) \) can be compensated by an action of the algebra \( C^1(\mathcal{U}, \mathcal{O}_\mathcal{P}) \). What are the explicit formulae connecting the generators of these algebras?

9. One should write down Ward identities resulting from the symmetry algebra \( C^1(\mathcal{U}, \mathcal{V}_\mathcal{P}) + C^1(\mathcal{U}, \mathcal{O}_\mathcal{P}) \). To what extent do these identities define correlation functions?

Clearly, to carry out this quantization program, it will be necessary to overcome a number of technical difficulties.

The general picture arising as a result of quantization of the SDYM model on a self-dual 4-manifold \( M \) and the holomorphic CSW theory on the twistor space \( \mathcal{Z} \) of \( M \) resembles the one that arises in the quantization of the ordinary CS theory and is as follows: Let \( \{ g \} \) be a self-dual conformal structure on a 4-manifold \( M \) and let \( \mathcal{J} \) be a complex structure on the twistor space \( \mathcal{Z} \) of \( M \). As has already been noted, there exists a bijection \( \{ 17, 19 \} \) between the moduli space of self-dual conformal structures on \( M \) and the moduli space \( \mathcal{X} \) of complex structures on \( \mathcal{Z} \). Let \( \mathcal{M} \) be a moduli space of solutions to the SDYM equations on \( M \) and let \( H_{\mathcal{J}} \) be the quantum Hilbert space of holomorphic sections of the line bundle \( \mathcal{L} \) over \( (\mathcal{M}, \mathcal{J}) \). The space \( H_{\mathcal{J}} \) depends on \( \mathcal{J} \in \mathcal{X} \) and one can introduce a holomorphic vector bundle
\[
p : \tilde{H} \to \mathcal{X}
\]
(9.24)
with fibres \( H_{\mathcal{J}} \) at the points \( \mathcal{J} \in \mathcal{X} \). Then one may put a question about the existence of a (projectively) flat connection in the bundle (9.24). If such a connection exists, then as a quantum Hilbert space one may take a space of covariantly constant sections of the vector bundle \( \tilde{H} \).
10 Conclusion

In this paper, the group-theoretic analysis of the Penrose-Ward correspondence was undertaken. Having used sheaves of non-Abelian groups and cohomology sets we have described the symmetry group acting on the space of local solutions to the SDYM equations and the moduli space $\mathcal{M}$ of local solutions. It has been shown that $\mathcal{M}$ is a double coset space. The full algebra of infinitesimal deformations of self-dual conformal structures on a 4-space $M$ has also been described. We have discussed the program of quantization of the SDYM model on $M$ based on the equivalence of this model to a subsector of the holomorphic CSW model on the twistor space $Z$ of $M$. There are a lot of open problems, which deserve further study.

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Appendix A. Actions of groups on sets

The left action of a group $\mathcal{G}$ on a set $\Upsilon$ is a map $\rho : \mathcal{G} \times \Upsilon \to \Upsilon$ with the following properties:

$$\rho(e, x) = x,$$

(A.1a)

$$\rho(a, \rho(b, x)) = \rho(ab, x),$$

(A.1b)

for any $x \in \Upsilon, a, b, e \in \mathcal{G}$. Here $e$ is the identity in the group $\mathcal{G}$. If we are given an action $\rho$ on a set $\Upsilon$, to any $a \in \mathcal{G}$ we can correspond a bijective transformation $\rho_a : \Upsilon \to \rho(a, \Upsilon)$ of the set $\Upsilon$ such that a map $\gamma : a \to \rho_a$ is a homomorphism of the group $\mathcal{G}$ into the group $S_\Upsilon$ of all permutations (bijective transformations) of the set $\Upsilon$. Conversely, any homomorphism $\gamma : \mathcal{G} \to S_\Upsilon$ defines the action of the group $\mathcal{G}$ on $\Upsilon$ by the formula

$$\rho(a, x) := \gamma(a)(x)$$

(A.2)

for any $a \in \mathcal{G}, x \in \Upsilon$. If $\Upsilon$ is a smooth manifold, then to define an action of $\mathcal{G}$ on $\Upsilon$ is equivalent to assigning a homomorphism $\gamma : \mathcal{G} \to \text{Diff}(\Upsilon)$ of the group $\mathcal{G}$ into the group of diffeomorphisms of the manifold $\Upsilon$.

Usually the left action of the group $\mathcal{G}$ is represented as a multiplication of elements from $\Upsilon$ by elements of the group $\mathcal{G}$ and written as $\rho(a, x) = ax, a \in \mathcal{G}, x \in \Upsilon$. One also considers the right action of the group $\mathcal{G}$ on $\Upsilon$ in the definition of which the condition (A.1b) is replaced by the condition

$$\rho(a, \rho(b, x)) = \rho(ba, x).$$

(A.1c)

Then the notation $\rho(a, x) = xa$ is used.

Recall that a space $\mathcal{G}$ is called a local group, if for elements $a, b$ sufficiently close to the identity $e$ (marked element) the multiplication $ab$ is defined, the inverse elements $a^{-1}, b^{-1}$ exist and all group axioms are fulfilled every time the objects participating in these axioms are defined. More precisely, a space $\mathcal{G}$ is called a local group if: 1) some element $e$ (identity) of $\mathcal{G}$ is chosen; 2) a neighbourhood $\mathcal{V} \subset \mathcal{G}$ of the element $e$ is chosen; 3) there is a map $\mathcal{V} \times \mathcal{V} \to \mathcal{G}$, $(a, b) \to ab$ (multiplication) satisfying the conditions $ea = ae = a$ and $(ab)c = a(bc)$ for $a, b, c, ab, bc \in \mathcal{V}$. From these conditions it follows that there exists a neighbourhood $\mathcal{W} \subset \mathcal{G}$ of the identity and a map $1 : \mathcal{W} \to \mathcal{W}, a \to a^{-1}$ (inversion) such that $aa^{-1} = a^{-1}a = e$. Choosing $\mathcal{V} = \mathcal{W} = \mathcal{G}$, one can consider any group $\mathcal{G}$ as a local group; this is why we use the same letter $\mathcal{G}$ for groups and for local groups.

If one replaces $\mathcal{G}$ and $\Upsilon$ by open subsets $\mathcal{G}' \subset \mathcal{G}, \Upsilon' \subset \mathcal{V} \cap \mathcal{G}'$ satisfying the condition $\mathcal{V}' \Upsilon' \subset \mathcal{G}'$, one obtains a local group $\mathcal{G}'$, called a restriction or a part of the initial one. Two local groups are called equivalent, if some of their parts coincide. The equivalence class of the local group $\mathcal{G}$ is called the germ of the group $\mathcal{G}$ at the point $e \in \mathcal{G}$ and denoted by $\mathcal{G}$.

An action of a group $\mathcal{G}$ on a set $\Upsilon$ can be localized if one considers $\mathcal{G}$ as a local group. Namely, let $\rho$ be an action of the group $\mathcal{G}$ on the set $\Upsilon$ and let $\mathcal{N}$ be an open subset in $\Upsilon$. The action $\rho$, generally speaking, does not map $\mathcal{N}$ into itself and therefore does not define an action of the whole group $\mathcal{G}$ on $\mathcal{N}$. However, an action of $\mathcal{G}$ as a local group is defined, i.e., a map $\rho : \mathcal{W} \to \mathcal{N}$ is defined, where $\mathcal{W} = \{ (a, x) \in \mathcal{G} \times \mathcal{N} : \rho(a, x) \in \mathcal{N} \}$ is an open subset in $\mathcal{G} \times \mathcal{N}$ containing $\{e\} \times \mathcal{N}$. Moreover, for any fixed point $x \in \mathcal{N}$ there exists a neighbourhood $\mathcal{V}$ of the identity in $\mathcal{G}$ and a neighbourhood $\mathcal{N}'$ of the point $x$ in $\mathcal{N}$ such that $\rho(\mathcal{V} \times \mathcal{N}') \subset \mathcal{N}'$.  


In a more general situation, a local action of a local group $\mathcal{G}$ on a set $\mathcal{N}$ is a map $\rho : \mathcal{W} \to \mathcal{N}$, where $\mathcal{W}$ is an open set in $\mathcal{G} \times \mathcal{N}$ containing $\{e\} \times \mathcal{N}$, and the properties (A.1) are satisfied for all $a, b \in \mathcal{G}, x \in \mathcal{N}$ for which both parts of the equality (A.1b) are defined. A local action $\rho$ of the local group $\mathcal{G}$ on the set $\mathcal{N}$ generates a local action of $\mathcal{G}$ on any open subset $\mathcal{N}' \subset \mathcal{N}$. This action is called a restriction of the action $\rho$ to the subset $\mathcal{N}'$. A local action of the group $\mathcal{G}$ is called globalizable if it is a localization of some global action of the group.

**Appendix B. Sheaves of (non-Abelian) groups**

Let us consider a topological space $X$ and recall the definitions of a presheaf and a sheaf of groups over $X$ (see e.g. \cite{44, 45}).

One has a presheaf $\{\mathcal{S}(U), r^U_V\}$ of groups over a topological space $X$ if with any nonempty open set $U$ of the space $X$ one associates a group $\mathcal{S}(U)$ and with any two open sets $U$ and $V$ with $V \subset U$ one associates a homomorphism $r^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$ satisfying the following conditions: (i) the homomorphism $r^U_U : \mathcal{S}(U) \to \mathcal{S}(U)$ is the identity map $\text{id}_U$; (ii) if $W \subset V \subset U$, then $r^W_V = r^U_W \circ r^U_V$.

A sheaf of groups over a topological space $X$ is a topological space $\mathcal{S}$ with a local homeomorphism $\pi : \mathcal{S} \to X$. This means that any point $s \in \mathcal{S}$ has an open neighbourhood $V$ in $\mathcal{S}$ such that $\pi(V)$ is open in $X$ and $\pi : V \to \pi(V)$ is a homeomorphism. A set $\mathcal{S}_x = \pi^{-1}(x)$ is called a stalk of the sheaf $\mathcal{S}$ over $x \in X$, and the map $\pi$ is called the projection. For any point $x \in X$ the stalk $\mathcal{S}_x$ is a group, and the group operations are continuous.

A section of a sheaf $\mathcal{S}$ over an open set $U$ of the space $X$ is a continuous map $s : U \to \mathcal{S}$ such that $\pi \circ s = \text{id}_U$. A set $\mathcal{S}(U) := \Gamma(U, \mathcal{S})$ of all sections of the sheaf $\mathcal{S}$ of groups over $U$ is a group. Corresponding to any open set $U$ of the space $X$ the group $\mathcal{S}(U)$ of sections of the sheaf $\mathcal{S}$ over $U$ and to any two open sets $U, V$ with $V \subset U$ the restriction homomorphism $r^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$, we obtain the presheaf $\{\mathcal{S}(U), r^U_V\}$ over $X$. This presheaf is called the canonical presheaf.

On the other hand, one can associate a sheaf with any presheaf $\{\mathcal{S}(U), r^U_V\}$. Let

$$\mathcal{S}_x = \lim_{x \in U} \mathcal{S}(U)$$

be a direct limit of sets $\mathcal{S}(U)$. There exists a natural map $r^U_x : \mathcal{S}(U) \to \mathcal{S}_x, x \in U$, sending elements from $\mathcal{S}(U)$ into their equivalence classes in the direct limit. If $s \in \mathcal{S}(U)$, then $s_x := r^U_x(s)$ is called a germ of the section $s$ at the point $x$, and $s$ is called a representative of the germ $s_x$. In other terms, two sections $s, s' \in \mathcal{S}(U)$ are called equivalent at the point $x \in U$ if there exists an open neighbourhood $V \subset U$ such that $s|_V = s'|_V$; the equivalence class of such sections is called the germ $s_x$ of section $s$ at the point $x$. Put

$$\mathcal{S} = \bigcup_{x \in X} \mathcal{S}_x$$

and let $\pi : \mathcal{S} \to X$ be a projection mapping points from $\mathcal{S}_x$ into $x$. The set $\mathcal{S}$ is equipped with a topology, the basis of open sets of which consists of sets $\{s_x, x \in U\}$ for all possible $s \in \mathcal{S}(U), U \subset X$. In this topology $\pi$ is a local homeomorphism, and we obtain the sheaf $\mathcal{S}$.

Let $X$ be a smooth manifold. Consider a complex (non-Abelian) Lie group $G = G^C$ and define a presheaf $\{\hat{\mathcal{S}}(U), r^U_V\}$ of groups by putting

$$\hat{\mathcal{S}}(U) := \{C^\infty\text{-maps } f : U \to G\}, \quad (B.1)$$

and using the canonical restriction homomorphisms $r^U_V$ when for $f \in \hat{\mathcal{S}}(U)$ its image $r^U_V(f)$ equals $f|_V \in \hat{\mathcal{S}}(V)$, $V \subset U$. To each elements $\alpha_x$ and $\beta_x$ from $\hat{\mathcal{S}}_x := r^U_x(\hat{\mathcal{S}}(U))$ one can correspond their pointwise multiplication $\alpha_x \beta_x$. To this presheaf $\{\hat{\mathcal{S}}(U), r^U_V\}$ there corresponds the sheaf $\hat{\mathcal{S}}$ of germs of smooth maps of the space $X$ into the group $G$.

Suppose now that $X$ is a complex manifold. Then one can define a presheaf $\{\mathcal{H}(U), r^U_V\}$ of groups assuming that

$$\mathcal{H}(U) := \{\text{holomorphic maps } h : U \to G\}, \quad (B.2)$$

and associate with it the sheaf $\mathcal{H} := \mathcal{O}^G$ of germs of holomorphic maps of the space $X$ into the complex Lie group $G$.  

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Appendix C. Cohomology sets and vector bundles

We shall consider a complex manifold $X$ and a sheaf $\mathcal{G}$ coinciding with either the sheaf $\mathcal{S}$ or the sheaf $\mathcal{H}$ introduced in Appendix B. So $\mathcal{G}$ is the sheaf of germs of smooth or holomorphic maps of the space $X$ into the complex Lie group $G$.

Čech cohomology sets $H^0(X, \mathcal{G})$ and $H^1(X, \mathcal{G})$ of the space $X$ with values in the sheaf $\mathcal{G}$ of groups are defined as follows [44, 45, 34].

Let there be given an open cover $\mathcal{U} = \{U_\alpha\}$, $\alpha \in I$, of the manifold $X$. The family $\langle U_0, ..., U_q \rangle$ of elements of the cover such that $U_0 \cap ... \cap U_q \neq \emptyset$ is called a q-simplex. The support of this simplex is $U_0 \cap ... \cap U_q$. Define a 0-cochain with coefficients in $\mathcal{G}$ as a map $f$ associating with $\alpha \in I$ a section $f_\alpha$ of the sheaf $\mathcal{G}$ over $U_\alpha$:

$$f_\alpha \in \mathcal{G}(U_\alpha) := \Gamma(U_\alpha, \mathcal{G}).$$

(A.1)

A set of 0-cochains is denoted by $C^0(\mathcal{U}, \mathcal{G})$ and is a group under the pointwise multiplication.

Consider now the ordered set of two indices $\langle \alpha, \beta \rangle$ such that $\alpha, \beta \in I$ and $U_\alpha \cap U_\beta \neq \emptyset$. Define a 1-cochain with coefficients in $\mathcal{G}$ as a map $f$ associating with $\langle \alpha, \beta \rangle$ a section of the sheaf $\mathcal{G}$ over $U_\alpha \cap U_\beta$:

$$f_{\alpha\beta} \in \mathcal{G}(U_\alpha \cap U_\beta) := \Gamma(U_\alpha \cap U_\beta, \mathcal{G}).$$

(A.2)

A set of 1-cochains is denoted by $C^1(\mathcal{U}, \mathcal{G})$ and is a group under the pointwise multiplication.

Subsets of cocycles $Z^q(\mathcal{U}, \mathcal{G}) \subset C^q(\mathcal{U}, \mathcal{G})$ for $q = 0, 1$ are defined by the formulae

$$Z^0(\mathcal{U}, \mathcal{G}) = \{ f \in C^0(\mathcal{U}, \mathcal{G}) : f_\alpha f_\beta^{-1} = 1 \text{ on } U_\alpha \cap U_\beta \neq \emptyset \},$$

(A.3)

$$Z^1(\mathcal{U}, \mathcal{G}) = \{ f \in C^1(\mathcal{U}, \mathcal{G}) : f_{\alpha\beta} = f_{\beta\alpha}^{-1} \text{ on } U_\alpha \cap U_\beta \neq \emptyset, f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \}. \tag{A.4}$$

It follows from $(A.3)$ that $Z^0(\mathcal{U}, \mathcal{G})$ coincides with the group $H^0(X, \mathcal{G}) := \mathcal{G}(X) = \Gamma(X, \mathcal{G})$ of global sections of the sheaf $\mathcal{G}$. The set $Z^1(\mathcal{U}, \mathcal{G})$ is not in general a subgroup of the group $C^1(\mathcal{U}, \mathcal{G})$. It contains the marked element 1, represented by the 1-cocycle $f_{\alpha\beta} = 1$ for any $\alpha, \beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$.

For $h \in C^0(\mathcal{U}, \mathcal{G})$, $f \in Z^1(\mathcal{U}, \mathcal{G})$ let us define an action $\rho_0$ of the group $C^0(\mathcal{U}, \mathcal{G})$ on the set $Z^1(\mathcal{U}, \mathcal{G})$ by the formula

$$\rho_0(h, f)_{\alpha\beta} = h_\alpha f_{\alpha\beta} h_\beta^{-1} \tag{A.5}$$

So we have a map $\rho_0 : C^0 \times Z^1 \ni (h, f) \mapsto \rho_0(h, f) \in Z^1$. A set of orbits of the group $C^0$ in $Z^1$ is called a 1-cohomology set and denoted by $H^1(\mathcal{U}, \mathcal{G})$. In other words, two cocycles $f, \tilde{f} \in Z^1$ are called equivalent, $f \sim \tilde{f}$, if

$$\tilde{f} = \rho_0(h, f) \tag{A.6}$$

for some $h \in C^0$, and by the 1-cohomology set $H^1 = \rho_0(C^0) \setminus Z^1$ one calls a set of equivalence classes of 1-cocycles. Finally, we should take the direct limit of these sets $H^1(\mathcal{U}, \mathcal{G})$ over successive refinement of the cover $\mathcal{U}$ of $X$ to obtain $H^1(X, \mathcal{G})$, the 1-cohomology set of $X$ with coefficients in $\mathcal{G}$. In fact, one can always choose a cover $\mathcal{U} = \{U_\alpha\}$ such that it will be $H^1(\mathcal{U}, \mathcal{G}) = H^1(X, \mathcal{G})$ and therefore it will not be necessary to take the direct limit of sets. This is realized, for instance, when the coordinate charts $U_\alpha$ are Stein manifolds (see e.g. [44, 45]).

Recall that $\mathcal{G}$ is the sheaf of germs of (smooth or holomorphic) functions with values in the complex Lie group $G$. Suppose we are given a representation of $G$ in $C^n$. It is well-known that any 1-cocycle $\{f_{\alpha\beta}\}$ from $Z^1(\mathcal{U}, \mathcal{G})$ defines a unique complex vector bundle $E'$ over $X$, obtained from the direct products $U_\alpha \times C^n$ by gluing with the help of $f_{\alpha\beta} \in G$. Moreover, two 1-cocycles define isomorphic complex vector bundles over $X$ if and only if the same element from $H^1(X, \mathcal{G})$ corresponds to them. Thus, we have a one-to-one correspondence between the set $H^1(X, \mathcal{G})$ and the set of equivalence classes of complex vector bundles of the rank $n$ over $X$. Smooth bundles are parametrized by the set $H^1(X, \mathcal{S})$ and holomorphic bundles are parametrized by the set $H^1(X, \mathcal{H})$, where the sheaves $\mathcal{S}$ and $\mathcal{H}$ were described in Appendix B. For more details see e.g. [44, 45].
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