CLASSIFICATION OF EQUIVARIANT VECTOR BUNDLES
OVER TWO-SPHERE

MIN KYU KIM

Abstract. We classify equivariant topological complex vector bundles over
two-sphere under a compact Lie group (not necessarily effective) action. It is
shown that nonequivariant Chern classes and isotropy representations at (at
most) three points are sufficient to classify equivariant vector bundles except
a few cases. To do it, we calculate homotopy of the set of equivariant clutch-
ing maps. In other papers, we also give classifications over two-torus, real
projective plane, Klein bottle.

1. Introduction

In topology, nonequivariant complex vector bundles can be classified just by
calculating their Chern classes under a dimension condition \[P\]. However, there
is no such general result on equivariant vector bundles. Instead, a few results on
extreme cases are known. Let us mention four of them. Let a compact Lie group
\(G\) act on a topological space \(X\).

- If \(G\) is trivial, then \(G\)-vector bundles over \(X\) are just nonequivariant vector
  bundles, and are classified by their Chern classes in \(H^*(X)\).
- In [At, Proposition 1.6.1], [S, p. 132], \(G\)-vector bundles over a free \(G\)-space
  \(X\) are in one-to-one correspondence with nonequivariant vector bundles
  over \(X/G\), and are classified by Chern classes in \(H^*(X/G)\).
- For a closed subgroup \(H\) of \(G\) and any point \(x\) in \(X = G/H\), \(G\)-vector
  bundles over \(X\) are classified by their isotropy representations at \(x\) which
  are contained in \(\text{Rep}(G_x)\). For this, see [S, p. 130], [B, Proposition II.3.2].
In [CKMS], \(G\)-vector bundles over \(X = S^1\) are classified by their isotropy
representations at (at most) two points.

Since invariants live in \(H^*(X)\), \(H^*(X/G)\), \(\text{Rep}(G_x)\), these results might be consid-
ered to have three different types. In this paper, we classify equivariant topological
complex vector bundles over two-sphere under a compact Lie group (not necessarily
effective) action. Readers will see those different types at once in it. By develop-
ing ideas and machineries of it, classification over two-torus is given in [Ki1] which
shows that our method is not restrictive. Classifications over real projective plane
and Klein bottle are also obtained as corollaries in [Ki2], [Ki3]. After these, clas-
sification of equivariant holomorphic vector bundles over Riemann sphere under a
holomorphic complex reductive group (not necessarily effective) action will follow
which is an equivariant version of Grothendieck’s Theorem for holomorphic vector
bundles on Riemann sphere \(\mathbb{C}\), and is the motivation for the paper.

To state main results, we need introduce several notations. It is well-known that
a topological action on \(S^2\) by a compact Lie group is conjugate to a linear action
[Io, Theorem 1.2], [CK]. Let a compact Lie group \(G\) act linearly (not necessarily
effectively) on the unit sphere \(S^2\) in \(\mathbb{R}^3\) through a representation \(\rho: G \to O(3)\).
Let \(\text{Vect}_G(S^2)\) be the set of isomorphism classes of topological complex \(G\)-vector

2000 Mathematics Subject Classification. Primary 57S25, 55P91; Secondary 20C99.
Key words and phrases. equivariant vector bundle, equivariant homotopy, representation.
bundles over $S^2$ with the given $G$-action. For a bundle $E$ in $\text{Vect}_G(S^2)$ and a point $x$ in $S^2$, denote by $E_x$ the isotropy $G_x$-representation on the fiber at $x$. It is needed to decompose $\text{Vect}_G(S^2)$ as the sum of more simple subsemigroups. For this, some terminologies are introduced. Put $H = \ker \rho$, i.e. the kernel of the $G$-action on $S^2$. Let $\text{Irr}(H)$ be the set of characters of irreducible complex $H$-representations which has a $G$-action defined as

$$(g \cdot \chi)(h) = \chi(g^{-1}hg)$$

for $\chi \in \text{Irr}(H)$, $g \in G$, $h \in H$. Sometimes, we also use the notation $\text{Irr}(H)$ to denote the set of isomorphism classes of irreducible complex $H$-representations themselves. For $\chi \in \text{Irr}(H)$, an $H$-representation is called $\chi$-isotypical if its character is a multiple of $\chi$. We slightly generalize this concept. For $\chi \in \text{Irr}(H)$ and a compact Lie group $K$ satisfying $H \triangleleft K < G$ and $K \cdot \chi = \chi$, a $K$-representation $W$ is called $\chi$-isotypical if $\text{res}^K_H W$ is $\chi$-isotypical, and denote by $\text{Vect}_K(S^2, \chi)$ the set

$$\left\{ [E] \in \text{Vect}_K(S^2) \mid E_x \text{ is } \chi\text{-isotypical for each } x \in S^2 \right\}$$

where $S^2$ delivers the restricted $K$-action. In [CKMS], the (isotypical) decomposition of a $G$-bundle is defined, and from this a semigroup isomorphism is constructed to satisfy

$$\text{Vect}_G(S^2) \cong \bigoplus_{\chi \in \text{Irr}(H)/G} \text{Vect}_{G_\chi}(S^2, \chi)$$

where $G_\chi$ is the isotropy subgroup of $G$ at $\chi$. As a result, our classification is reduced to $\text{Vect}_{G_\chi}(S^2, \chi)$ for each $\chi \in \text{Irr}(H)$. Details are found in [CKMS] Section 2.

The classification of $\text{Vect}_{G_\chi}(S^2, \chi)$ is highly dependent on the $G_\chi$-action on the base space $S^2$, i.e. on the image $\rho(G_\chi)$, and classification is actually given by case by case according to $\rho(G_\chi)$. So, we need to describe $\rho(G_\chi)$ in a moderate way. For this, we would list all possible $\rho(G_\chi)$’s up to conjugacy, and then assign an equivariant simplicial complex structure of $S^2$ to each finite $\rho(G_\chi)$. Cases of nonzero-dimensional $\rho(G_\chi)$ are relatively simple and separately dealt with as special cases. First, let us define some polyhedra. Let $P_m$ for $m \geq 3$ be the regular $m$-gon on $xy$-plane in $\mathbb{R}^3$ whose center is the origin and one of whose vertices is $(1,0,0)$. Then,

1. $|\mathcal{K}_m|$ is defined as the boundary of the convex hull of $P_m$, $S = (0,0,-1)$, $N = (0,0,1)$,
2. $|\mathcal{K}_T|$ is defined as the tetrahedron which is the boundary of the convex hull of four points $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, $(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, and which is inscribed to $|\mathcal{K}_3|$,
3. $|\mathcal{K}_I|$ is defined as an icosahedron which has the origin as the center.

With these, denote natural simplicial complex structures on $|\mathcal{K}_m|$, $|\mathcal{K}_T|$, $|\mathcal{K}_I|$ by $\mathcal{K}_m$, $\mathcal{K}_T$, $\mathcal{K}_I$, respectively. Then, it is well-known that each closed subgroup of $\text{SO}(3)$ is conjugate to one of the following subgroups $\mathcal{K}$ Theorem 11:

1. $\mathbb{Z}_n$ generated by the rotation $a_n$ through the angle $2\pi/n$ around $z$-axis,
2. $D_n$ generated by $a_n$ and the rotation $b$ through the angle $\pi$ around $x$-axis,
3. the tetrahedral group $T$ which is the rotation group of $|\mathcal{K}_T|$,
4. the octahedral group $O$ which is the rotation group of $|\mathcal{K}_4|$,
5. the icosahedral group $I$ which is the rotation group of $|\mathcal{K}_I|$,
6. $\text{SO}(2)$ which is the set of rotations around $z$-axis,
7. $\text{O}(2)$ which is defined as $\langle \text{SO}(2), b \rangle$,
8. $\text{SO}(3)$ itself.

Note that $T \subset O$, and pick an element $o_0$ of $O \setminus T$ so that $O = \langle T, o_0 \rangle$. And, denote by $Z$ the centralizer $\{\text{id}, -\text{id}\}$ of $O(3)$. In Section 2 it is shown that each closed
subgroup of $\text{O}(3)$ is conjugate to an $R$-entry of Table 11 (there is no literature on this as far as the author knows). In the table, the notation $\times$ means internal direct product of two subgroups in $\text{O}(3)$. Henceforward, it is assumed that $\rho(G_\chi) = R$ for some $R$. Let $\text{O}(3)$ and their subgroups act naturally on $\mathbb{R}^3$. To each finite $R$, we assign a simplicial complex $\mathcal{K}_R$ of Table 11 where $\mathcal{K}_O$ is defined in the below. Each $|K_R|$ is invariant under the natural $R$-action on $\mathbb{R}^3$ as shown in Section 3 and is also invariant under the $G_\chi$-action defined through $\rho$. So, we assume that $|K_R|$ delivers the $G_\chi$-action, and that $|K_R|$ delivers the $G_\chi$-action inherited from it. Henceforward, we consider $|K_R|$ as the base space instead of the usual two-sphere $S^2$ when $R$ is finite.

| $R$ | $K_R$ | $D_R$ | $d^{-1}$ |
|-----|-------|-------|---------|
| $D_n$, $n > 1$ | $K_n$ | $[e^0, b(e^0)]$ | $S$ |
| $Z_m$ | $K_n$ | $[e^0]$ | $S$ |
| $D_n \times Z$, odd $n$ | $K_{2n}$ | $[e^0, b(e^0)]$ | $S$ |
| $\langle a_n, -b \rangle$, odd $n$ | $K_{2n}$ | $[e^0]$, $[v^1] \cup [v^1, b(e^1)]$ | $S$ |
| $Z_m \times Z$, odd $n$ | $K_{2n}$ | $[e^0]$ | $S$ |
| $D_n \times Z$, even $n$ | $K_n$ | $[e^0, b(e^0)]$ | $S$ |
| $\langle a_n, -b \rangle$, even $n$ | $K_n$ | $[e^0]$ | $S$ |
| $\langle -a_n, b \rangle$, odd $n/2$, $n > 2$ | $K_{n/2}$ | $b(e^0)$ | $S$ |
| $\langle -a_n, b \rangle$, even $n/2$, $n > 2$ | $K_n$ | $[e^0, b(e^0)]$ | $S$ |
| $Z_m \times Z$, even $n$, $n > 2$ | $K_n$ | $[e^0]$ | $S$ |
| $\langle -a_n, -b \rangle$, odd $n/2$, $n > 2$ | $K_{n/2}$ | $[e^0]$ | $S$ |
| $\langle -a_n, -b \rangle$, even $n/2$, $n > 2$ | $K_n$ | $[e^0]$ | $S$ |
| $O(3)$ | $K_T$ | $[e^0, b(e^0)]$ | $b(f^{-1})$ |
| $I$ | $K_O$ | $b(e^0)$ | $b(f^{-1})$ |
| $(T, -a_0)$ | $K_T$ | $[e^0, b(e^0)]$ | $b(f^{-1})$ |
| $T \times Z$ | $K_O$ | $b(e^0)$ | $b(f^{-1})$ |
| $O \times Z$ | $K_O$ | $[e^0, b(e^0)]$ | $b(f^{-1})$ |
| $I \times Z$ | $K_I$ | $[e^0, b(e^0)]$ | $b(f^{-1})$ |
| $O(3)$ | $O(2) \times Z$ | $\{v^0\}$ | $v^0$ |
| $SO(2), -b$ | $\{v^0\}$ | $v^0$ |
| $SO(2), -a_2$ | $\{v^0\}$ | $v^0$ |
| $SO(3)$ | $\{v^0\}$ | $v^0$ |

Table 1.1. $\mathcal{K}_R$, $D_R$, $d^{-1}$ for closed subgroup $R$

In dealing with equivariant vector bundles over two-sphere, we need to consider isotropy representations at a few points (at most three points) of $|K_R|$. To specify those points, we introduce some more notations. When $m \geq 3$, denote by $v^i$ the vertex exp ($\frac{2\pi i}{m}$) $\in \mathbb{R}^2$ of $K_m$, and by $e^i$ the edge of $K_m$ connecting $v^i$ and $v^{i+1}$ for $i \in \mathbb{Z}_m$. These notations are illustrated in Figure 10.1. When we use the notation $\mathbb{Z}_m$ to denote an index set, it is just the group $\mathbb{Z}/m\mathbb{Z}$ of integers modulo $m$. In Section 3, $K_m$, $v^i$, $e^i$ for $m = 1, 2$ are also defined. We would define similar notations for $K_T$, $K_I$. For $K_T$, $K_I$, pick two adjacent faces in each case, and call them $f^{-1}$ and $f^0$. And, label vertices of $f^{-1}$ as $e^i$ for $i \in \mathbb{Z}_3$ to satisfy

1. $v^0$, $v^1$, $v^2$ are arranged in the clockwise way around $f^{-1}$,
2. $v^0$, $v^1$ are contained in $f^{-1} \cap f^0$,

For $i \in \mathbb{Z}_3$, denote by $e^i$ be the edge connecting $v^i$ and $v^{i+1}$, and by $f^i$ the face which is adjacent to $f^{-1}$ and contains the edge $e^i$. We distinguish the superscripts $-1$ and
2 only for \( f^1 \), i.e., \( f^{-1} \neq f^2 \) in contrast to \( v^{-1} = v^2, e^{-1} = e^2 \). These notations are illustrated in Figure 4.1(a). Here, we define one more simplicial complex denoted by \( K_\circ \) which is the same simplicial complex with \( K_i \) but has the same convention of notations \( v^i, e^i, f^{-i} \) with \( K_T, K_I \). Also, put \( |K_\circ| = |K_4| \). These notations are illustrated in Figure 4.2(a). With these notations, we explain for \( D \) fundamental domain \( D \) of three points \( x, y \) in \( K \). And, let \( d^0 \) and \( d^1 \) be boundary points of \( D \) such that \( d^0 \) is nearer to \( v^0 \) than \( d^1 \). Here, we define one more point \( d^{-1} \) for each finite \( D \) which is listed in the fourth column in Table 1.1. If \( R \) is one-dimensional, then denote by \( D \) the one point set \( \{v^0 = (1, 0, 0)\} \), and let \( d^{-1}, d^0, d^1 \) be equal to \( S, v^0, v^0 \), respectively. Similarly, if \( R \) is three-dimensional, then denote by \( D \) the one point set \( \{v^0 = (1, 0, 0)\} \), and let \( d^{-1}, d^0, d^1 \) be all equal to \( v^0 \). So far, we have defined \( d^{-1}, d^0, d^1 \) for each \( R \). Then, \( \{S, N\} \) or \( \{d^{-1}, d^0, d^1\} \) are wanted points according to \( R \), and we will consider the restriction \( E|_{\{S, N\}} \) or \( E|_{\{d^{-1}, d^0, d^1\}} \) for each \( E \) in \( \text{Vect}_{G_\circ}(S^2, \chi) \). Define the following semigroup which will be shown to be equal to the set of all the restrictions:

**Definition 1.1.** For \( \chi \in \text{Irr}(H) \), assume that \( \rho(G_\chi) = R \) for some \( R \) of Table 1.1

1. If \( R = \mathbb{Z}_n, (a_n, -b), \text{SO}(2), (\text{SO}(2), -b) \), then let \( A_{G_\chi}(S^2, \chi) \) be the semigroup of pairs \((W_S, W_N)\) in \( \text{Rep}(G_\chi)^2 \) satisfying
   - \( W_S \) is \( \chi \)-isotypical,
   - \( \text{res}^{G_\chi}_{G_\circ}(W_S) \cong \text{res}^{G_\circ}_{G_\chi}(W_N) \) for \( x = d^0, d^1 \).
   And, let \( p_{\text{vect}} : \text{Vect}_{G_\chi}(S^2, \chi) \to A_{G_\chi}(S^2, \chi) \) be the semigroup homomorphism defined as \( [E] \to (E_S, E_N) \).

2. Otherwise, let \( A_{G_\chi}(S^2, \chi) \) be the semigroup of triples \((W_{d^{-1}}, W_{d^0}, W_{d^1})\) in \( \text{Rep}((G_\chi)_{d^{-1}}) \times \text{Rep}((G_\chi)_{d^0}) \times \text{Rep}((G_\chi)_{d^1}) \) satisfying
   - \( W_{d^{-1}} \) is \( \chi \)-isotypical,
   - \( W_{d^0} \cong gW_{d^1} \) if there exists \( g \in G_\chi \) such that \( gd^0 = d^1 \),
   - \( \text{res}^{G_\chi}_{G_\circ}((G_\chi)_{d^{-1}})(x) \cong \text{res}^{G_\chi}_{G_\circ}((G_\chi)_{d^0})((G_\chi)_{d^1})(x) \) for any two points \( x, x' \) of three points \( d^{-1}, d^0, d^1 \).
   And, let \( p_{\text{vect}} : \text{Vect}_{G_\chi}(S^2, \chi) \to A_{G_\chi}(S^2, \chi) \) be the semigroup homomorphism defined as \( [E] \to (E_{d^{-1}}, E_{d^0}, E_{d^1}) \).

See Definition S.1 for the superscript \( g \). Well-definedness of \( p_{\text{vect}} \) is proved in Lemma S.2 and Lemma 11.1. Put \( I = \{0, 1\} \) and \( I^+ = \{-1, 0, 1\} \). And, denote a triple \((W_{d^{-1}}, W_{d^0}, W_{d^1})\) by \((W_g)_{g \in I^+}\). Especially, if \( \rho(G_\chi) = \text{SO}(3), O(3) \), then \( A_{G_\chi}(S^2, \chi) \) is equal to the set

\[
\left\{ (W_g)_{g \in I^+} \in \text{Rep}((G_\chi)_{\chi})^3 \mid \text{\( W_g \)'s are the same \( \chi \)-isotypical representation} \right\},
\]

and \( p_{\text{vect}} \) becomes an isomorphism by classification over homogeneous space.

Now, we can state main results. Let \( c_1 : \text{Vect}_{G_\chi}(S^2, \chi) \to H^2(S^2) \) be the map defined as \( [E] \to c_1([E]) \). Denote by \( \text{lr} \) the number \([G_\chi]/[G_\chi(D_R)]\) where \( G_\chi(D_R) \) is the subgroup of \( G_\chi \) preserving \( D_R \).

**Theorem A.** Assume that \( \rho(G_\chi) = R \) is equal to one of \( \mathbb{Z}_n, D_n, T, O, I \). Then, \( p_{\text{vect}} \) is surjective, and

\[
p_{\text{vect}} \times c_1 : \text{Vect}_{G_\chi}(S^2, \chi) \to A_{G_\chi}(S^2, \chi) \times H^2(S^2)
\]
is injective. More precisely, bundles in $p_{\text{vect}}^{-1}(W)$ for each $W$ in $A_{G_\chi}(S^2, \chi)$ have all different Chern classes, and $c_1(p_{\text{vect}}^{-1}(W))$ is equal to $\{ \chi(id)(1_Rk_0 + k) \mid k \in \mathbb{Z} \}$ where $k_0$ is dependent on $W$.

Theorem B. Assume that $p(G, \chi) = R$ is equal to one of $\mathbb{Z}_n \times \mathbb{Z}$ with odd $n$, $\{-a_n\}$ with even $n/2$. For each $W$ in $A_{G_\chi}(S^2, \chi)$, its preimage $p_{\text{vect}}^{-1}(W)$ has exactly two elements which have the same Chern class. Also, $[E \oplus E_1] \neq [E \oplus E_2]$ for any bundles $E, E_1, E_2$ in $\text{Vect}_{G_\chi}(S^2, \chi)$ such that $p_{\text{vect}}([E_1]) = p_{\text{vect}}([E_2])$ and $[E_1] \neq [E_2]$.

Theorem C. Assume that $p(G, \chi) = R_{1, 1}$, $R_{1, 2}$ for some $R$ appearing in Theorem A, B. Assume that $\chi|_R = R_{1, 2}$ is equal to one of $\mathbb{Z}_2 \times \mathbb{Z}$ with odd $n$, $\{R_{1, 2}\}$ with even $n/2$. For each $W$ in $A_{G_\chi}(S^2, \chi)$, its preimage $p_{\text{vect}}^{-1}(W)$ has exactly two elements which have the same Chern class. Also, $[E \oplus E_1] \neq [E \oplus E_2]$ for any bundles $E, E_1, E_2$ in $\text{Vect}_{G_\chi}(S^2, \chi)$ such that $p_{\text{vect}}([E_1]) = p_{\text{vect}}([E_2])$ and $[E_1] \neq [E_2]$.

Theorem D. Assume that $p(G, \chi) = R$ for some $R$ appearing in Theorem A, B. Then, $\text{Vect}_{G_\chi}(S^2, \chi)$ is isomorphic to $\text{Vect}_R(S^2)$ as semigroups, and $\text{Vect}_R(S^2)$ is generated by line bundles. Also, $A_R(S^2, \text{id})$ is generated by all the elements with one-dimensional entries. The number of such elements is equal to

$$\begin{cases} |R_\chi| \times |R_N| & \text{if } R = \mathbb{Z}_n, \\ |R_{d-1}| \times |R_{d}| \times |R_{d}| & \text{if } R \neq \mathbb{Z}_n \end{cases}$$

where $R_\chi$ is the isotropy subgroup at $x \in S^2$ and $R_N$ denote simply by $\text{id}$ the trivial character of the trivial group.

In Section 13, we calculate Chern classes of line bundles in $\text{Vect}_R(S^2)$, and from this we obtain $k_0$. Also in Section 13 we explain for the reason why we prove the isomorphism of Theorem D only for $R$’s appearing in Theorem A, B.

This paper is organized as follows. In Section 2, we list all closed subgroups of $O(3)$ up to conjugacy. In Section 3, we give an equivariant simplicial complex structure $K_R$ on $S^2$ according to finite $R$, and investigate equivariance of $S^2$ by calculating isotropy subgroups at vertices and barycenters of $[K_R]$. Section 4-12 are divided into three parts. In Section 4-9 we first deal with cases when $K_R = K_T$, $K_O$, $K_1$. In Section 10 we introduce a new simplicial complex denoted by $K_\mathcal{R}$ which is just a disjoint union of faces of $K_R$, and consider $[K_R]$ as a quotient space of $[K_\mathcal{R}]$. Also, we consider an equivariant vector bundle over $S^2$ as an equivariant clutching construction of an equivariant vector bundle over $[K_\mathcal{R}]$. For this, we define equivariant clutching map. In Section 5, we investigate relations among $\text{Vect}_{G_\chi}(S^2, \chi)$, $A_{G_\chi}(S^2, \chi)$, and homotopy of the set of equivariant clutching maps. From these relations, it is shown that our classification in most cases is obtained by calculation of the homotopy. In Section 6 we develop our machinery called equivariant pointwise gluing which glues an equivariant vector bundle over a finite set through a map called equivariant pointwise clutching map. In calculation of the homotopy, equivariant pointwise clutching map plays a key role because an equivariant clutching map can be considered as a continuous collection of equivariant pointwise clutching maps. Here, the concept of representation extension enters with which equivariant pointwise gluing is described in the language of representation theory. In this way, the homotopy is technically related to $A_{G_\chi}(S^2, \chi)$, and calculation of it becomes reduced to calculation of a relative (nonequivariant) homotopy. So, we prove a lemma on relative homotopy in Section 7. In Section 5 we prove technical lemmas needed in dealing with equivariant clutching maps through equivariant pointwise clutching maps. In Section 9 we prove main theorems for cases when $K_R = K_T$, $K_O$, $K_1$. In Section 10-11 we second deal with cases when
\( \mathcal{K}_R = \mathcal{K}_m \) for some \( m \in \mathbb{N} \). In Section 10 we rewrite what we have done in Section 4 [5, 8] to be in accordance with \( \mathcal{K}_m \). In Section 11 we prove main theorems for the cases when \( \mathcal{K}_R = \mathcal{K}_m \). In Section 12 we third prove Theorem [4] for cases when \( R \) is one-dimensional. In Section 13 we prove Theorem [1]. Section 14 is the appendix on representation extension.

2. Closed subgroups of \( O(3) \)

In this section, we list all closed subgroups of \( O(3) \) up to conjugacy. In this section, the notation \( \times \) is internal direct product of two subgroups in \( O(3) \). Since \( Z \) is the centralizer of \( O(3) \), \( O(3) = SO(3) \times Z \) and this gives the exact sequence

\[
0 \to Z \to O(3) \xrightarrow{\text{pr}} SO(3) \to 0
\]

where \( \text{pr}_{SO(3)} \) is the identity map. First, we deal with finite subgroups.

**Proposition 2.1.** Let \( R \) be a finite subgroup of \( O(3) \) such that \( R \not\subseteq SO(3) \).

1. If \( \text{pr}(R) \) is conjugate to \( \mathbb{Z}_n \), then
   \[
   R \text{ is conjugate to } \begin{cases} \mathbb{Z}_n \times Z & \text{if } n \text{ is odd,} \\ \mathbb{Z}_n \times Z \text{ or } \langle -a_n \rangle & \text{if } n \text{ is even.} \end{cases}
   \]

2. If \( \text{pr}(R) \) is conjugate to \( D_n \), then
   \[
   R \text{ is conjugate to } \begin{cases} D_n \times Z \text{ or } \langle a_n, -b \rangle & \text{if } n \text{ is odd,} \\ D_n \times Z, \langle a_n, -b \rangle, \langle -a_n, b \rangle, \langle -a_n, -b \rangle & \text{if } n \text{ is even.} \end{cases}
   \]

3. If \( \text{pr}(R) \) is conjugate to \( T \), then \( R \) is conjugate to \( T \times Z \).

4. If \( \text{pr}(R) \) is conjugate to \( O \), then \( R \) is conjugate to \( O \times Z \) or \( \langle T, -o_0 \rangle \).

5. If \( \text{pr}(R) \) is conjugate to \( I \), then \( R \) is conjugate to \( I \times Z \).

**Proof.** We may assume that \( \text{pr}(R) \) is equal to one of \( \mathbb{Z}_n, D_n, T, O, I \). Denote by \( R_{\text{rot}} \) the subgroup \( R \cap SO(3) \). Then, \( R_{\text{rot}} \) is an index two subgroup of \( R \). Denoting \( \text{pr}(R) \) by \( K \), \( R_{\text{rot}} \subset K \) because \( \text{pr}_{SO(3)} \) is the identity map. Also, since the preimage \( \text{pr}^{-1}(g) \) of \( g \in SO(3) \) is equal to \( \{g, -g\} \), it is obtained that \( R \subset K \times Z \). Here are two possibilities. First, if \( Z = \ker \text{pr} \subset R \), then \( |R| = 2|K| \) so that

\[
(2.1) \quad R = K \times Z
\]

from \( R \subset K \times Z \). Second, if \( Z \not\subseteq R \), then \( \text{pr}|_R \) is injective and \( |R| = |K| \) so that

\[
(2.2) \quad |K| = 2|R_{\text{rot}}|
\]

because \( R_{\text{rot}} \) is an index two subgroup of \( R \). For an element \( g_0 \in K - R_{\text{rot}} \) and its preimage \( \text{pr}^{-1}(g_0) = \{g_0, -g_0\} \), \( R \) is equal to \( \langle R_{\text{rot}}, g_0 \rangle \) or \( \langle R_{\text{rot}}, -g_0 \rangle \) because \( K = \langle R_{\text{rot}}, g_0 \rangle \) is the injective image of \( R \). However, if \( R = \langle R_{\text{rot}}, g_0 \rangle \), then \( R \subset SO(3) \) and this contradicts the assumption. So, we obtain

\[
(2.3) \quad R = \langle R_{\text{rot}}, -g_0 \rangle.
\]

In the remaining proof, we apply \((2.1), (2.2), (2.3)\) to each possible \( K \).

If \( K = Z_n \) with odd \( n \) and \( Z \subset R \), then \( R = Z_n \times Z \). If \( K = Z_n \) with odd \( n \) and \( Z \not\subseteq R \), then \( K \) has no index two subgroup so that there exists no possible \( R_{\text{rot}} \) in \( K \). That is, there exists no such \( R \). Therefore, a proof of (1) for odd \( n \) is obtained.

If \( K = Z_n \) with even \( n \) and \( Z \subset R \), then \( R = Z_n \times Z \). If \( K = Z_n \) with even \( n \) and \( Z \not\subseteq R \), then there is the unique index two subgroup \( \langle a_n^2 \rangle \) of \( K \) which should be equal to \( R_{\text{rot}} \). Since \( a_n \in K - R_{\text{rot}} \), \( R = \langle R_{\text{rot}}, -a_n \rangle = \langle -a_n \rangle \) and this is the proof of (1) for even \( n \).

If \( K = D_n \) with odd \( n \) and \( Z \subset R \), then \( R = D_n \times Z \). If \( K = D_n \) with odd \( n \) and \( Z \not\subseteq R \), then there is the unique index two subgroup \( Z_n \) of \( D_n \) which should
be equal to \( R_{rot} \). Since \( b \in K - R_{rot}, R = \langle R_{rot}, -b \rangle = \langle a_n, -b \rangle \) and this is a proof of (2) for odd \( n \).

If \( K = D_n \) with even \( n \) and \( Z \subset R \), then \( R = D_n \times Z \).

If \( K = D_n \) with even \( n \) and \( Z \not\subset R \), then index two subgroups of \( D_n \) are \( \mathbb{Z}_n, \langle a^2_n, b \rangle, \langle a^2, a_n b \rangle \) which are candidates for \( R_{rot} \) for some \( R \). If \( R_{rot} = \mathbb{Z}_n \), then \( R = \langle R_{rot}, -b \rangle = \langle a_n, -b \rangle \).

If \( R_{rot} = \langle a^2_n, b \rangle \), then \( R = \langle R_{rot}, -a_n \rangle = \langle -a_n, b \rangle \). If \( R_{rot} = \langle a^2, a_n b \rangle \), then \( R = \langle R_{rot}, -a_n \rangle = \langle -a_n, a_n b \rangle = \langle -a_n, -b \rangle \). Therefore, a proof of (2) for even \( n \) is obtained.

If \( K = T \) or \( I \), then it is well-known that \( K \) has no index two subgroup because \( T \cong A_4 \) and \( I \cong A_5 \). So, \( Z \not\subset R \) can not happen. Therefore, we obtain a proof for (3) and (5).

If \( K = O \) and \( Z \subset R \), then \( R = O \times Z \). If \( K = O \) and \( Z \not\subset R \), then \( T \) is the only index two subgroup of \( O \), so \( R_{rot} = T \).

Second, we deal with closed nonzero-dimensional subgroups of \( O(3) \).

Proposition 2.2. Let \( R \) be a nonzero-dimensional closed subgroup of \( O(3) \) such that \( R \not\subset SO(3) \).

1. If \( \text{pr}(R) \) is conjugate to \( SO(2) \), then \( R \) is conjugate to the group \( SO(2) \times Z = \langle SO(2), -a_2 \rangle \).

2. If \( \text{pr}(R) \) is conjugate to \( O(2) \), then \( R \) is conjugate to \( \langle SO(2), -b \rangle \) or \( O(2) \times Z \).

3. If \( \text{pr}(R) \) is conjugate to \( SO(3) \), then \( R \) is conjugate to \( O(3) \).

Proof of this is done in a similar way with Proposition 2.1. We have explained for \( R \)-entry of Table 1.1. Here, we calculate some isotropy subgroups for later use.

Lemma 2.3. For each one-dimensional \( R \) of Table 1.1 and its natural action on \( S^2 \), isotropy subgroups \( R_{v^0} \) and \( R_{v^0} \cap R_S \) are calculated as in Table 2.1.

| \( R \) | \( R_{v^0} \) | \( R_{v^0} \cap R_S \) |
|---|---|---|
| \( O(2) \times Z \) | \( \langle b, -a_2 \rangle \) | \( \langle -a_2 \rangle \) |
| \( (SO(2), -b) \) | \( \langle -a_2 \rangle \) | \( \langle -a_2 \rangle \) |
| \( (SO(2), -a_2) \) | \( \langle -a_2 \rangle \) | \( \langle -a_2 \rangle \) |
| \( O(2) \) | \( \langle b \rangle \) | \( \langle \text{id} \rangle \) |
| \( SO(2) \) | \( \langle \text{id} \rangle \) | \( \langle \text{id} \rangle \) |

Table 2.1. Isotropy subgroups at \( v^0 = (1, 0, 0) \) for closed one-dimensional subgroups of \( O(3) \)

3. Equivariant simplicial complex structures for finite subgroups of \( O(3) \)

Let a finite \( R \) of Table 1.1 act naturally on \( S^2 \). In this section, we explain for \( K_R \) and \( D_R \)-entries of Table 1.1 and investigate equivariance of \( |K_R| \) by calculating
isotropy subgroups at some points. We do these for \( \text{pr}(R) = Z_n, D_n \), and then for \( \text{pr}(R) = T, O, I \).

First, we define \( K_m \) and its \( v^i, e^i \) for \( m = 1, 2 \) as promised in Introduction. Denote by \( K_1 \) the simplicial complex which is the same with \( K_4 \) but has the following notations:

\[
v_0 = (1, 0, 0), \\
b(e_0) = (-1, 0, 0), \\
[v_0, b(e_0)] = [(1, 0, 0), (0, 1, 0)] \cup [(0, 1, 0), (-1, 0, 0)], \\
|e_0| = P_4.
\]

And, denote by \( K_2 \) the simplicial complex which is the same with \( K_4 \) but has the following notations:

\[
v^i = \exp \left( \frac{\pi i \sqrt{-1}}{4} \right), \\
b(e^i) = \exp \left( \frac{2\pi (2i + 1) \sqrt{-1}}{4} \right), \\
|e^i| = [v^i, b(e^i)] \cup [b(e^i), v^{i+1}]
\]

for \( i \in \mathbb{Z}_2 \). Also, let \( |K_1| \) and \( |K_2| \) be equal to \(|K_4|\). Here, we remark that if \( K_R = K_1 \) and \( D_R = |e_0| \) in Table 3.1 then \( d^0 \) and \( d^1 \) are defined as \( v^0 \). So, we have finished defining \( K_m \) for all natural number \( m \). Similarly, denote by \( P_m \) for \( m = 1, 2 \) the regular polygon \( P_4 \). For reader’s convenience, we list all possible \( R \)'s with \( K_R = K_1 \) or \( K_2 \).

| \( R \) | \( n \) | \( K_R \) |
|---|---|---|
| \( D_n \) | 2 | \( K_2 \) |
| \( Z_m \) | 1 | \( K_1 \) |
| \( Z_m \) | 2 | \( K_2 \) |
| \( D_n \times Z \), odd \( n \) | | \( K_2 \) |
| \( \langle a_n, -b \rangle \), odd \( n \) | 1 | \( K_2 \) |
| \( Z_n \times Z \), odd \( n \) | 1 | \( K_2 \) |
| \( D_n \times Z \), even \( n \) | 2 | \( K_2 \) |
| \( \langle a_n, -b \rangle \), even \( n \) | 2 | \( K_2 \) |

Table 3.1. Groups with \( K_R = K_1 \) or \( K_2 \)

**Lemma 3.1.** There are eight \( R \)'s in Table 3.1 which satisfy \( \text{pr}(R) = Z_n, D_n \) and \( K_R = K_1, K_2 \). They are listed in Table 3.1.

Now, we explain for \( K_R \)-entry of Table 3.1 when \( \text{pr}(R) = Z_n, D_n \). Denoting by \( m_R \) the number \(|R|/|R_{v^0}|\), we can check that \( K_R \) is equal to \( K_{m_R} \). Since \( R \cdot v^0 \) is equal to the set of vertices of \( K_{m_R} \) in \( xy \)-plane, \( |K_R| \) and \( K_R \) are \( R \)-invariant for each \( R \). Here, we calculate \( R_{v^0} \).

**Lemma 3.2.** For each finite \( R \) of Table 3.1 such that \( \text{pr}(R) = Z_n, D_n \), isotropy subgroups \( R_{v^0} \), \( R_{e^0} \cap R_S \) are calculated as in Table 3.2 when \( R \) acts naturally on \( S^2 \).

**Proof.** By calculation, \( R_{v^0} = \langle b \rangle \) for \( R = D_n \times Z \) with odd \( n \), and \( R_{v^0} = \langle -a_n^{1/2}, b \rangle \) for \( R = D_n \times Z \) with even \( n \). Other \( R \)'s are subgroups of one of these two. So, \( R_{v^0} = R \cap (D_n)_{v^0} \) is easily calculated, and from this the group \( R_{v^0} \cap R_S \) is also obtained.

We calculate other isotropy subgroups.
Since the observed that

Here, we explain for

Isotropy subgroup at

Lemma 3.3. Let $R$ be a finite group in Table 1.1 such that $\text{pr}(R) = Z_n$, $D_n$. Isotropy subgroups $R_{b(e^0)}$, $R_{b(e^0)} \cap R_S$, $R_x$ are calculated as in Table 3.3 where $x$ is a point in $P_{m,n} - \{v^i, b(e^i) | i \in Z_{m,n} \}$.

Proof. Since the $R$-action on $|K_R|$ is simplicial, any element $g$ in $R_x$ fixes the whole $P_{m,n}$. Therefore, $R_x = \langle \text{id} \rangle$ or $\langle -a_n/2 \rangle$ for even $n$, and it is easy to calculate $R_x$.

Observe that $R_{b(e^0)}$ is isomorphic to a subgroup of $Z_2 \times Z_2$ and that $R_x \subset R_{b(e^0)}$ and $R_{b(e^0)}/R_x \cong \langle \text{id} \rangle$ or $Z_2$, we can calculate $R_{b(e^0)}$ case by case.

Remark 3.4. In the cases of $\langle a_n, -b \rangle$ with odd $n$ and $\langle -a_n, -b \rangle$ with odd $n/2$, it is observed that $|R_{a_n}| > |R_{b(e^0)}|$, so $R$ does not act transitively on $b(e^i)$'s. And, in the case of $\langle -a_n, -b \rangle$ with odd $n/2$, we additionally calculate $R_{b(e^i)} = \langle -a_n^{-1}, a_n^3b \rangle$.

Here, we explain for $D_R$.

Lemma 3.5. For each finite $R$ of Table 1.1 such that $\text{pr}(R) = Z_n$, $D_n$, the $R$-orbit of the $D_R$-entry in Table 1.2 covers $P_{m,n}$, and $D_R$ is a minimal path satisfying such a property. So, any interior point $x$ of $D_R$ is not moved to other point in $D_R$ by $R$. 

| $R$          | $K_R$  | $R_{b(e^0)}$ | $R_{b(e^0)} \cap R_S$ | $R_x$ |
|--------------|--------|--------------|------------------------|-------|
| $D_n$, $n > 1$ | $K_n$  | $\langle a_n \rangle$ | $\langle a_n \rangle$ | $\langle a_n \rangle$ |
| $Z_n$        | $K_n$  | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |
| $D_n \times Z$, odd $n$ | $K_{2n}$ | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle a_n \rangle$ |
| $\langle a_n, -b \rangle$, odd $n$ | $K_{2n}$ | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle a_n \rangle$ |
| $Z_n \times Z$, odd $n$ | $K_{2n}$ | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |
| $D_n \times Z$, even $n$ | $K_n$  | $\langle -a_n^{-2}, a_n \rangle$ | $\langle -a_n^{-2}, a_n \rangle$ | $\langle -a_n^{-2}, a_n \rangle$ |
| $\langle a_n, -b \rangle$, even $n$ | $K_n$  | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle -a_n^{n+1}, a_n \rangle$ | $\langle a_n \rangle$ |
| $\langle -a_n, -b \rangle$, odd $n/2$, $n > 2$ | $K_{n/2}$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ |
| $\langle a_n, -b \rangle$, even $n/2$ | $K_n$  | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ |
| $\langle a_n, -b \rangle$, odd $n/2$, $n > 2$ | $K_{n/2}$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ | $\langle -a_n^{n/2+1}, a_n^2 \rangle$ |
| $\langle -a_n, -b \rangle$, even $n/2$ | $K_n$  | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |
| $Z_n \times Z$, even $n$, $n > 2$ | $K_n$  | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |
| $\langle -a_n \rangle$, odd $n/2$, $n > 2$ | $K_{n/2}$ | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |
| $\langle -a_n \rangle$, even $n/2$ | $K_n$  | $\langle id \rangle$   | $\langle id \rangle$   | $\langle id \rangle$ |

Table 3.2. Isotropy subgroup at $v^0$ for $\text{pr}(R) = Z_n$ or $D_n$. 

Table 3.3. Isotropy subgroups for $\text{pr}(R) = Z_n$, $D_n$. 

Lemma 3.9. For cases when $Z$ refers. In Table 3.4, $R$ such a property. Proof. First, observe that $|R_{v}| = |R_{b(e')}|$ if and only if $R$ acts transitively on $b(e')$'s because $R$ acts transitively on $v$'s and $|K_{R}|$ has the same number of $v$'s and $b(e')$'s. And, $R$ acts transitively on $b(e')$'s if and only if the $R$-orbit of $[e^0]$ cover $P_{m,v}$. By Table 3.2 and 3.3 $|R_{v}| = |R_{b(e')}|$ except two cases $(a_{n},-b)$ with odd $n$ and $(-a_{n},-b)$ with odd $n/2$. If $|R_{v}| = |R_{b(e')}|$ and $R_{b(e')}/R_{x} \cong Z_{2}$, then $[v^{0},b(e^{0})]$ can be moved to $[b(e^{0}),v^{1}]$ by $R$ so that the $R$-orbit of $[v^{0},b(e^{0})]$ covers $P_{m,v}$. In the other side, if $|R_{v}| = |R_{b(e')}|$ and $R_{b(e')}$ is a reflection through $R$, then the $R$-orbit of $[v^{0},b(e^{0})]$ does not cover $P_{m,v}$. If $|R_{v}| < |R_{b(e')}|$, then the $R$-orbit of $[e^{0}]$ does not cover $P_{m,v}$ because $R$ does not act transitively on $b(e')$'s, but the $R$-orbit of $[b(e^{0}),v^{1}] \cup [v^{1},b(e^{1})]$ covers $P_{m,v}$, because $R$ acts transitively on $v$'s. So, the remaining of proof is done by comparing Table 3.2 with Table 3.3. \[\square \square \]

Now, we repeat these arguments for $R$'s satisfying $\text{pr}(R) = T, O, I$.

**Lemma 3.6.** For each finite $R$ of Table 3.1 such that $\text{pr}(R) = T, O, I$, $|K_{R}|$ is $R$-invariant. And, $R$ act transitively on vertices, edges, faces of $K_{R}$, respectively.

**Remark 3.7.** For later use, we need understand the case of $R = T \times Z$ because it is not equal to the full symmetry of $|K_{R}| = |K_{O}|$. In this case, $R_{d(f-1)}$ is equal to $T_{d(f-1)} \cong Z_{2}$. Also, $-a_{2}^{2}, -a_{2}^{2}b$ are in $T \times Z$ because $b, a_{2}^{2}$ are reflections through the $xy$-plane, $xz$-plane, respectively. \[\square \]

**Lemma 3.8.** For each finite $R$ of Table 3.1 such that $\text{pr}(R) = T, O, I$, isotropy subgroups $R_{v}, R_{b(e')}$, $R_{x}$ are in Table 3.4 where $x$ is an interior point of $[v^{0},b(e^{0})]$. In the table, the notation $\equiv$ is used when an isotropy subgroup is not equal to $Z_{n} = \langle a_{n} \rangle$ or $D_{n} = \langle a_{n}, b \rangle$ for some $n$ but isomorphic to one of them.

**Lemma 3.9.** For each finite $R$ of Table 3.1 such that $\text{pr}(R) = T, O, I$, the $R$-orbit of the $D_{R}$-entry in Table 3.1 covers $|K_{R}^{(1)}|$, and $D_{R}$ is a minimal path satisfying such a property.

We summarize all results of Section 2 in Table 3.4 so that it will be repeatedly referred. In Table 3.4 $R_{d^{-1}}$ is also calculated. In Section 3.11 we need the following lemma on isotropy subgroups:

**Lemma 3.10.** For each finite $R$ of Table 3.1 such that $R \neq Z_{n}, \langle a_{n}, b \rangle$ for any $n$, isotropy subgroups $R_{d^{-1}}$'s satisfy

1. $R_{d^{-1}} \cap R_{b} = R_{d^{-1}d'}$ and $[d^{-1}, d'] \subset |K_{R}|^{R_{d^{-1}d'} R_{d'}}$ for $i \in I$,
2. $R_{d'} \cap R_{d} = R_{D_{R}}$ and $D_{R} \subset |K_{R}|^{R_{d} R_{d'} R_{d'}}$ where $R_{X}$ for a subset $X$ of $|K_{R}|$ is the subgroup of $R$ fixing $X$.

Proof. For the natural $R$-action on $S^{2}$ and two points $x \neq \pm x'$ of $S^{2}$, we have

(*)

$R_{x} \cap R_{x'} = R_{C}$ and $C = (S^{2})^{R_{x} R_{x'}}$

where $C$ is the great circle containing $x$ and $x'$. From this, we easily obtain proof for cases when $K_{R} = K_{T}, K_{O}, K_{I}$, or $K_{m}$ with $m \geq 3$. Then, there are remaining 8 cases by Lemma 3.1. Four cases of these are $Z_{n}$ or $\langle a_{n}, b \rangle$ for some $n$ so that four cases are remaining. We can apply (*) to three of remaining four. The remaining case is $Z_{n} \times Z$ with $n = 1$. Proof for this is easy. \[\square \square \]

4. Equivariant clutching construction

Let a compact Lie group $G_{X}$ act linearly (not necessarily effectively) on $S^{2}$ through a representation $\rho : G_{X} \to O(3)$. Assume that $\rho(G_{X}) = R$ for some finite $R$ in Table 3.1. In the below, our treatment is different according to $R$. First, we deal with cases when $\text{pr}(R) = T, O, I$ in Section 4.9. In a similar way, we deal
with cases when \( \text{pr}(R) = \mathbb{Z}_n, \mathbb{D}_n \) in Section 10–11. Then, cases of one-dimensional \( R \)'s are dealt with.

Assume that \( \text{pr}(R) = T, O, I \). Let \( \mathcal{K}_R \) be the simplicial complex \( \Pi_{f \in \mathcal{K}_R} f \), i.e. the disjoint union of faces of \( \mathcal{K}_R \). In this section, we would consider \( |\mathcal{K}_R| \) as the quotient of the underlying space \( |\mathcal{K}_R| = \Pi_{f \in \mathcal{K}_R} |f| \). And, we would consider an equivariant vector bundle over \( |\mathcal{K}_R| \) as an equivariant clutching construction of an equivariant vector bundle over \( |\mathcal{K}_R| \). For this, we would define equivariant clutching map and its generalization preclutching map. And, we state equivalent conditions under which a preclutching map is an equivariant clutching map. Before these, we need introduce notations on some relevant simplicial complexes. Since we should deal with various cases at the same time, these notations are necessary. Examples of the notations are illustrated in Figure 1.1 and Figure 1.2.

First, we define some notations on \( \mathcal{K}_R \) and \( \mathcal{K}_R \). We denote simply by \( \pi, |\pi| \) natural quotient maps from \( \mathcal{K}_R \), \( |\mathcal{K}_R| \) to \( \mathcal{K}_R, |\mathcal{K}_R| \), respectively. By definition, \( |\pi| \mid_{|f|} \) is bijective for each face \( f \in \mathcal{K}_R \). From this, the \( G \times \chi \)-actions on \( \mathcal{K}_R \), \( |\mathcal{K}_R| \) induce \( G \times \chi \)-actions on \( \mathcal{K}_R \), \( |\mathcal{K}_R| \) so that \( \pi, |\pi| \) are equivariant, respectively. We use notations \( v, \bar{v}, f, \bar{f}, x, \bar{x} \) to denote a vertex, an edge, a face of \( \mathcal{K}_R \), respectively. We use the notation \( \bar{v} \) to denote an arbitrary point of \( |\mathcal{K}_R| \). When \( v, \bar{v}, f, \bar{f}, x, \bar{x} \) are understood, we use notations \( v, e, f, x \) to denote images \( \pi(v), \pi(e), \pi(f), \pi([x]) \), respectively. Denote by \( f^{-1} \), \( f \) faces of \( \mathcal{K}_R \) such that \( \pi(f^{-1}) = f^{-1} \) and \( \pi(f) = f \), and denote by \( v, e \) simplices of \( f^{-1} \) such that \( \pi(v) = v \) and \( \pi(e) = e \). And, denote by \( d^0, d^1 \) points of \( f^{-1} \) such that \( |\pi|(d^0) = d^0 \) and \( |\pi|(d^1) = d^1 \), and denote by \( D_R \) the

| \( R \) | \( \mathcal{K}_R \) | \( D_R \) | \( R_{\emptyset} \) | \( R_{(e, v)} \) | \( R_e \) | \( R_{d^0} \|
|---|---|---|---|---|---|---|
| \( \mathbb{D}_n, n > 1 \) | \( \mathcal{K}_n \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{odd } n \) | \( \mathcal{K}_{2n} \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{odd } n \) | \( \mathcal{K}_{2n} \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{even } n \) | \( \mathcal{K}_n \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{even } n \) | \( \mathcal{K}_n \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{even } n \) | \( \mathcal{K}_n \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( \mathbb{D}_n \times \mathbb{Z}, \text{even } n \) | \( \mathcal{K}_n \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( T \times Z \) | \( \mathcal{K}_T \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( O \times Z \) | \( \mathcal{K}_O \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( O(3) \times Z \) | \( \mathcal{K}_O \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( SO(2), a \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |
| \( SO(2), a \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) | \( |v^0, b(e^0)| \) | \( |b \) |

**Table 3.4.** \( \mathcal{K}_R, D_R \), and isotropy subgroups
path $|\pi|^{-1}(D_R) \cap \{f^{-1}\}$, i.e. $D_R = [d^0, d^1]$. Define the integer $j_R$ as the cardinality of $\pi^{-1}(v^i)$ for $i \in Z_3$, i.e. $j_R = 3, 4, 5$ according to $K_R = K_T, K_O, K_I$, respectively. Let $B$ be the subset $\{b(f) \mid f \in K_R\}$ of $|K_R|$ on which $R$ (and $G_v$) acts transitively by Lemma 3.6, and $B$ is often confused with $|\pi|(B) = \{b(f) \mid f \in K_R\}$. So far, we have defined superscript $i$ for simplices in $K_R$. Next, we define $\bar{x}_j$ with subscript $j$ for any point $\bar{x}$ of $|K_R^{(1)}|$.

**Notation 4.1.**

1. For a vertex $\bar{v}$ in $K_R^{(1)}$ and $v = \pi(\bar{v})$, we label vertices in $\pi^{-1}(v)$ with $\bar{v}_j$ to satisfy
   i) $\pi^{-1}(v) = \{\bar{v}_j \mid j \in Z_{j_R}\}$,
   ii) $\bar{v}_0 = \bar{v}$,
   iii) in each face $f_j$ containing $\bar{v}_j$ for $j \in Z_{j_R}$, we can take a small neighborhood $U_j$ of $\bar{v}_j$ so that $|\pi|(U_j)$’s are arranged in the counterclockwise way around $v$.

2. For a non-vertex $\bar{x}$ in $K_R^{(1)}$ and $x = |\pi|(\bar{x})$, we label two points in $|\pi|^{-1}(x)$ with $\{\bar{x}_j \mid j \in Z_2\}$ to satisfy $\bar{x}_0 = \bar{x}$.

---

**Figure 4.1.** Relation between $K_R$, $\hat{K}_R$, $\hat{L}_R$ in the case when $K_R = K_T$ and $j_R = 3$
For simplicity, we denote \((\bar{v}^i)_j, (\bar{d}^i)_j\) by \(\bar{v}^i_j, \bar{d}^i_j\), respectively.

We need introduce two more simplicial complexes. Denote by \(\bar{L}_R\) and \(\hat{L}_R\) the 1-skeleton \(\bar{K}^{(1)}_R\) of \(\bar{K}_R\) and the disjoint union \(\bigcup_{\bar{e} \in \bar{L}_R} \bar{e}\), respectively. Then, \(\bar{L}_R\) is a subcomplex of \(\bar{K}_R\), and can be regarded as a quotient of \(\hat{L}_R\). These relations are expressed by two natural simplicial maps

\[
\iota_{\bar{L}}: \bar{L}_R \rightarrow \bar{K}_R, \quad p_{\bar{L}}: \hat{L}_R \rightarrow \bar{L}_R
\]

where \(\iota_{\bar{L}}\) is the inclusion and \(p_{\bar{L}}\) is the quotient map whose preimage of each vertex and edge of \(\bar{L}_R\) consists of two vertices and one edge of \(\hat{L}_R\), respectively. Two maps on underlying spaces are denoted by

\[
\iota_{|\bar{L}|}: |\bar{L}_R| \rightarrow |\bar{K}_R|, \quad p_{|\bar{L}|}: |\hat{L}_R| \rightarrow |\bar{L}_R|.
\]

The \(G_\chi\)-actions on \(\bar{K}_R, |\bar{K}_R|\) naturally induce \(G_\chi\)-actions on these relevant simplicial complexes \(\bar{L}_R, \hat{L}_R\) and their underlying spaces. We need introduce notations on simplices of \(\bar{L}_R\) and points of \(\bar{L}_R\). We use notations \(\bar{v}\) and \(\hat{e}\) to denote a vertex and an edge of \(\bar{L}_R\), respectively. And, we use the notation \(\hat{x}\) to denote an arbitrary point in \(|\bar{L}_R|\). When \(\bar{v}, \hat{e}, \hat{x}\) are understood, we use notations \(\bar{v}, \hat{e}, \hat{x}\) to denote \(p_{\bar{L}}(\bar{v}), p_{\bar{L}}(\hat{e}), p_{|\bar{L}|}(\hat{x})\), respectively. Two edges \(\hat{e}, \hat{e}'\) of \(\bar{L}_R\) (and their images \(\hat{e}, \hat{e}'\) in \(\hat{L}_R\)) are called adjacent if \(\hat{e} \neq \hat{e}'\) and \(\pi(p_{\bar{L}}(\hat{e})) = \pi(p_{\bar{L}}(\hat{e}'))\). And, two faces \(\hat{f}, \hat{f}'\) of \(\hat{K}_R\) are called adjacent if their images \(f, f'\) are adjacent.

Next, we introduce superscript \(i\) and subscripts \(+, -\) for vertices and edges of \(\hat{L}_R\). Before it, we introduce a simplicial map. Let \(c: \hat{L}_R \rightarrow \bar{L}_R\) be the simplicial map whose underlying space map \(|c|: |\hat{L}_R| \rightarrow |\bar{L}_R|\) is defined as

\[
\text{for any adjacent } \hat{c}, \hat{c}' \in \hat{L}_R, \text{ each point } \hat{x} \text{ in } |\hat{c}| \text{ is sent to the point } |c|(\hat{x}) \text{ in } |\hat{c}'| \text{ to satisfy } |\pi|(|c|(\hat{x})) = |\pi|(|c|(c(\hat{x}))).
\]

For example, \(\hat{e}\) and \(c(\hat{e})\) are adjacent for any edge \(\hat{e}\) in \(\hat{L}_R\). Easily, \(c\) and \(|c|\) are \(G_\chi\)-equivariant. For notational simplicity, we define \(c\) also on edges of \(\bar{L}_R\) to satisfy \(c(p_{\bar{L}}(\hat{e})) = p_{\bar{L}}(c(\hat{e}))\) for each edge \(\hat{e}\).

**Notation 4.2.**

1. For a vertex \(\bar{v}\) in \(\bar{L}_R\), we label two vertices in \(p_{\bar{L}}^{-1}(\bar{v})\) with \(\bar{v}_\pm\) to satisfy
   \[
   p_{\bar{L}}(c(\bar{v}_+)) = \bar{v}_1, \quad \text{and} \quad p_{\bar{L}}(c(\bar{v}_-)) = \bar{v}_{-1} = \bar{v}_{ijR-1}.
   \]

2. For a non-vertex \(\hat{x}\) in \(\bar{L}_R\), we label the point in \(p_{\bar{L}}^{-1}(\hat{x})\) with \(\hat{x}_+\) or \(\hat{x}_-\), i.e.
   \[
   \hat{x}_+ = \hat{x}_-.
   \]

For simplicity, denote \(\hat{x}_\pm\) for \(\hat{x} = \hat{v}^i, \hat{v}^i_j, \hat{d}^i\) by \(\hat{v}^i_\pm, \hat{v}^i_\pm, \hat{d}^i_\pm, \hat{d}^i_\pm\), respectively. So, if \(\hat{d}^i\) is a barycenter of an edge, then \(\hat{d}^i_+ = \hat{d}^i_-\). And, denote by \(\hat{e}^i\) the edge in \(\hat{L}_R\) such that \(p_{\bar{L}}(\hat{e}^i) = \hat{e}^i\) for \(i \in \mathbb{Z}_3\).

Until now, we have finished introducing notations in Figure 4.1. By using these notations, we introduce one-dimensional fundamental domain in \(|\hat{L}_R|\). For \(D_R = [\hat{d}^0_+, \hat{d}^1_-] \subseteq |\bar{K}_R|\), we define \(\hat{D}_R\) in \(|\hat{L}_R|\) as \([\hat{d}^0_+, \hat{d}^1_-]\) so that \(p_{|\bar{L}|}(\hat{D}_R) = D_R\). And, denote by \(\hat{D}_R\) the set \((|\pi| \circ p_{\bar{L}})^{-1}(D_R)\) in \(|\hat{L}_R|\) which is equal to

\[
[d^0_+, d^1_-] \bigcup |c|(d^0_+, d^1_-) \bigcup \bigcup_{v \in D_R} (\pi \circ p_{\bar{L}})^{-1}(v).
\]

The union of thick points and edges of Figure 4.2 (b) is \(\hat{D}_R\) in the case of \(R = T \times Z\).

For convenience in calculation, we parameterize each edge of \(|\hat{L}_R|\) linearly by \(s \in [0, 1]\) to satisfy

1. \(\hat{v}_+ = 0, \hat{v}_- = 1, b(\hat{e}) = 1/2\) for each vertex \(\bar{v}\) of \(\bar{K}_R\) and each edge \(\hat{e}\) of \(\hat{L}_R\).
2. \(|c|(s) = 1 - s\) for each edge \(\hat{e}\) of \(\hat{L}_R\) and \(s \in [\hat{e}]\).
We repeatedly use this parametrization.

Now, we describe an equivariant vector bundle over $|\mathcal{K}_R|$ as an equivariant clutching construction of an equivariant vector bundle over $|\mathcal{K}_R|$. Let $V_B$ be a $G_x$-vector bundle over $B$ such that $(\text{res}^{G_x}_{H} V_B)_{|b(f)}$ is $\chi$-isotypical at each $b(f)$ in $B$. If we denote by $V_f$ the isotropy representation of $V_B$ at each $b(f)$, then $V_B \cong G_x \times (G_x)_{|b(f)} V_f$ because $G_x$ acts transitively on $B$. And, $\text{res}^{(G_x)_{|b(f)}} V_f$'s are all isomorphic because they are all $\chi$-isotypical. We define $\text{Vect}_{G_x} (|\mathcal{K}_R|, \chi)_{V_B}$ as the set

$$\{ [E] \in \text{Vect}_{G_x} (|\mathcal{K}_R|, \chi) \mid E|_B \cong V_B \}.$$  

Similarly, $\text{Vect}_{G_x} (|\mathcal{K}_R|, \chi)_{V_B}$ is defined. Observe that $\text{Vect}_{G_x} (|\mathcal{K}_R|, \chi)_{V_B}$ has the unique element $[F_{V_B}]$ for the bundle $F_{V_B} = G_x \times (G_x)_{|f^{-1}} (|f^{-1}| \times V_{f^{-1}})$ because
regard $\Phi$ as the map $\tilde{\cdot}$ more precisely. Let $F$ be a vector bundle constructed by gluing the pull-back bundle $F_{V_e}$ with respect to $V_e$. Then, the quotient $\bar{F}$ of $F$ inherited from $F_{V_e}$ along edges through $\Phi$ can be also constructed by gluing $\bar{F}_{V_e}$ along edges through $\Phi$, i.e.,

$$
\Phi(u, u') = \Phi(\tilde{u}) \sim \Phi(\tilde{u}')$

for any two $u, u'$ in $\tilde{F}_{V_e}$, $u \sim u'$ if and only if $\tilde{\Phi}(u) \sim \tilde{\Phi}(u')$.

Then, the quotient $\bar{F}_{V_e}$ delivers the equivariant vector bundle structure inherited from $F_{V_e}$, and trivially $(\bar{F}_{V_e} / \sim') \cong \hat{E}$. Let us describe the relation on vectors in $\bar{F}_{V_e}$ more precisely. By using trivialization (4.1) of $\bar{F}_{V_e}$, the quotient $\bar{F}_{V_e} / \sim'$ can be also constructed by gluing $\bar{F}_{V_e}$ along edges through

$$
\varphi_{\tilde{e}} : [\tilde{e}] \rightarrow \text{Iso}(V_{\tilde{e}})\text{Iso}(\hat{F})
$$

for each edge $\tilde{e}$, $\hat{e} \in [\tilde{e}]$, $u \in V_{\tilde{e}}$ where $\tilde{e} = p_E(\tilde{e})$ and $\hat{e} \in \hat{F}$. Here, the notation Iso with no subscript means the set of nonequivariant isomorphisms. The union $\Phi = \bigcup_{\tilde{e} \in \hat{F}} \varphi_{\tilde{e}}$ is called an equivariant clutching map of $\hat{E}$ with respect to $V_{\hat{E}}$. The relation $\sim'$ on vectors in $\bar{F}_{V_e}$ is defined by $\Phi$, and the quotient $\bar{F}_{V_e} / \sim'$ is denoted by $\bar{F}_{V_e} / \Phi$. And, the equivariant vector bundle $\bar{F}_{V_e} / \Phi$ is called determined by $\Phi$ with respect to $V_{\hat{E}}$. When we use the phrase ‘with respect to $V_{\hat{E}}$’, it is assumed that we use the bundle $\bar{F}_{V_e}$ and its trivialization (4.1) in gluing. Equivariance of $[\pi]$ and $\bar{A}$ guarantees equivariance of $\Phi$, i.e.

$$(g \cdot \Phi)(\hat{x}) = g\Phi(g^{-1}\hat{x})g^{-1} = \Phi(\hat{x})$$

for all $g \in G, \hat{x} \in [\hat{L}_R]$. We denote by $p_{\Phi}$ the quotient map from $\bar{F}_{V_e}$ to $\bar{F}_{V_e} / \Phi$. Here, note that $\Phi$ is defined on $[\hat{L}_R]$. That is why we define $\hat{L}_R$. Sometimes, we regard $\Phi$ as the map

$$
p_{\Phi}[\bar{F}_{V_e}] \rightarrow p_{\Phi}[\bar{F}_{V_e}], \quad (\hat{x}, u) \mapsto ([c](\hat{x}), \Phi(\hat{x})u)
$$
by using trivialization \( \mathbf{1} \) for each \( \langle \hat{x}, u \rangle \in |\hat{c}| \times V_f \) where \( \hat{c} \in \hat{f} \). An equivariant clutching map of some bundle in \( \text{Vect}_{\mathcal{G}_\chi}(|K_R|, \chi)_{V_B} \) with respect to \( V_B \) is called simply an equivariant clutching map with respect to \( V_B \), and let \( \hat{\Omega}_V \) be the set of all equivariant clutching maps with respect to \( V_B \). In the next section, we will see that we need calculate the (nonequivariant) homotopy \( \pi_0(\hat{\Omega}_V) \) to classify equivariant vector bundles. To do it, we need to restrict an equivariant clutching map in \( \Omega_{V_B} \) to \( D_R \). We explain for this. Let \( \hat{\Omega}_{D_R, V_B} \) be the set

\[ \{ \Phi|_{D_R} \mid \Phi \in \Omega_{V_B} \} . \]

If two equivariant clutching maps coincide on \( D_R \), then they are identical by equivariance and definition of one-dimensional fundamental domain. So, the restriction map \( \Omega_{V_B} \to \Omega_{D_R, V_B} \), \( \Phi \mapsto \Phi|_{D_R} \) is bijective, and we obtain a bijection \( \pi_0(\hat{\Omega}_V) \cong \pi_0(\hat{\Omega}_{D_R, V_B}) \) between two homotopies. It is conceivable that it is easier to deal with \( \Omega_{D_R, V_B} \) than \( \Omega_{V_B} \) because of smaller domain of definition. This is why we restrict an equivariant clutching map to \( D_R \). We call a map \( \Phi \) in \( \Omega_{V_B} \) the extension of \( \Phi|_{D_R} \) in \( \Omega_{D_R, V_B} \). And, denote the bundle \( F_{V_B}/\Phi \) also by \( F_{V_B}/\Phi|_{D_R} \).

Next, we define preclutching map, a generalization of equivariant clutching map. Let \( C^0(|\hat{L}_R|, V_B) \) be the set of continuous functions \( \Phi \) on \( |\hat{L}_R| \) satisfying \( \Phi|_{\hat{c}}(\hat{x}) \in \text{Iso}(V_f, V_{f'}) \) for each \( \hat{c} \) and \( \hat{x} \in |\hat{c}| \) where \( \hat{c} \in \hat{f} \), \( c(\hat{c}) \in f' \). Note that we can define the quotient \( F_{V_B}/\Phi \) also for any \( \Phi \in C^0(|\hat{L}_R|, V_B) \) as we have done in (4.2), though \( F_{V_B}/\Phi \) need not deliver a suitable equivariant vector bundle structure or even nonequivariant vector bundle structure. Let \( C^0(D_R, V_B) \) be the set

\[ \left\{ \Phi|_{D_R} \mid \Phi \in C^0(|\hat{L}_R|, V_B) \right\} . \]

A function \( \Phi \) in \( C^0(|\hat{L}_R|, V_B) \) or a function \( \Phi|_{D_R} \) in \( C^0(D_R, V_B) \) is called a preclutching map with respect to \( V_B \). Then, it is a natural question under which conditions a preclutching map becomes an equivariant clutching map. We can answer this question for a preclutching map in \( C^0(|\hat{L}_R|, V_B) \). A preclutching map \( \Phi \) in \( C^0(|\hat{L}_R|, V_B) \) is an equivariant clutching map with respect to \( V_B \) if and only if it satisfies the following conditions:

1. \( \Phi|_{|\hat{c}|}(\hat{x}) = \Phi(\hat{x})^{-1} \) for each \( \hat{x} \in |\hat{L}_R| \),
2. For each vertex \( \hat{v} \in \hat{K}_R \),
   \[ \Phi(\hat{v}_{j\chi -1,+}) \cdots \Phi(\hat{v}_{j,+}) = \text{id} \]
   for \( j \in \mathbb{Z}_{\chi} \).
3. \( \Phi(g \hat{x}) = g \Phi(\hat{x}) g^{-1} \) for each \( \hat{x} \in |\hat{L}_R| \), \( g \in G_\chi \).

We explain for this more precisely. As a slight generalization of the classical result [AI, p. 20, 21], if \( \Phi \) satisfies Condition N1, N2., then the quotient \( F_{V_B}/\Phi \) becomes a nonequivariant vector bundle though it need not be an equivariant vector bundle. Moreover, if \( \Phi \) also satisfies Condition E1., then \( F_{V_B}/\Phi \) becomes an equivariant vector bundle so that \( \Phi \) is an equivariant clutching map with respect to \( V_B \). We will answer the same question for a preclutching map in \( C^0(D_R, V_B) \) in Section 5.

5. Relations among \( \text{Vect}_{\mathcal{G}_\chi}(S^2, \chi)_{V_B} \), \( A_{\mathcal{G}_\chi}(S^2, \chi) \), \( \pi_0(\hat{\Omega}_{V_B}) \)

In this section, we investigate relations among

\[ \text{Vect}_{\mathcal{G}_\chi}(S^2, \chi)_{V_B} \], \( A_{\mathcal{G}_\chi}(S^2, \chi) \), \( \pi_0(\hat{\Omega}_{V_B}) \).

Our classification of the paper is based on these relations. Before it, we state two basic facts on equivariant vector bundles. First, two equivariantly homotopic equivariant clutching maps give isomorphic equivariant vector bundles.
Lemma 5.1. For two maps $\Phi$ and $\Phi'$ in $\Omega_{V_B}$, if $\Phi$ and $\Phi'$ are homotopic in $\Omega_{V_B}$, i.e. $[\Phi] = [\Phi']$ in $\pi_0(\Omega_{V_B})$, then $[F_{V_B}/\Phi] = [F_{V_B}/\Phi']$ in $\text{Vect}_{G_\chi}(S^2, \chi)_{V_B}$.

Proof. This is a slight generalization of the classical result [At; Lemma 1.4.6. and Section 1.6]. So, we omit the proof. □ □

Lemma 5.1 gives a sufficient condition for isomorphism. Sometimes, we need an equivalent condition. When we consider a map in $\Omega_{V_B}$ as defined on $p_{\mid L}^*F_{V_B}$, we have the following equivalent condition:

Lemma 5.2. For any $\Phi$ and $\Phi'$ in $\Omega_{V_B}$, $[F_{V_B}/\Phi] = [F_{V_B}/\Phi']$ in $\text{Vect}_{G_\chi}(S^2, \chi)_{V_B}$ if and only if there is a $G_\chi$-isomorphism $\Theta : F_{V_B} \rightarrow F_{V_B}$ such that $(p_{\mid L}^*\Theta)\Phi = \Phi'(p_{\mid L}^*\Theta)$ where $p_{\mid L}^*\Theta : p_{\mid L}^*F_{V_B} \rightarrow p_{\mid L}^*F_{V_B}$ is the pull-back of $\Theta$.

\[
\begin{array}{ccc}
p_{\mid L}^*F_{V_B} & \xrightarrow{p_{\mid L}^*\Theta} & p_{\mid L}^*F_{V_B} \\
\Phi & \downarrow & \Phi' \\
p_{\mid L}^*F_{V_B} & \xrightarrow{p_{\mid L}^*\Theta} & p_{\mid L}^*F_{V_B}
\end{array}
\]

Proof. This is also a slight generalization of the classical result [At; p. 22, (ii) and Section 1.6]. First, we prove sufficiency. Let $A : F_{V_B}/\Phi \rightarrow F_{V_B}/\Phi'$ be a $G_\chi$-isomorphism. Then, we can show that there exists the $G_\chi$-isomorphism $\overline{A} : F_{V_B} \rightarrow F_{V_B}$ satisfying the following commutative diagram

\[
\begin{array}{ccc}
F_{V_B} & \xrightarrow{A} & F_{V_B} \\
p_{\theta} & \downarrow & \downarrow p_{\theta'} \\
F_{V_B}/\Phi & \xrightarrow{A} & F_{V_B}/\Phi'.
\end{array}
\]

From this, $(p_{\mid L}^*\overline{A})\Phi = \Phi'(p_{\mid L}^*\overline{A})$ is obtained.

Next, we prove necessity. If we put $\overline{A} = \Theta$, there exists the unique $G_\chi$-isomorphism $A$ satisfying (5.1) by the assumption. So, we obtain a proof. □ □

Consider the map $\pi_0 : \pi_0(\Omega_{V_B}) \rightarrow \text{Vect}_{G_\chi}(S^2, \chi)_{V_B}$ mapping $[\Phi]$ to $[F_{V_B}/\Phi]$. This is well-defined by Lemma 5.1 and also surjective because each bundle in $\text{Vect}_{G_\chi}(S^2, \chi)_{V_B}$ can be considered as an equivariant clutching construction. Then, the map $\pi_0 : \pi_0(\Omega_{V_B}) \rightarrow A_{G_\chi}(S^2, \chi)$ defined as $\pi_0 = p_{\text{vect}} \circ \pi_0$ satisfies the following diagram:

\[
\begin{array}{ccc}
\pi_0(\Omega_{V_B}) & \xrightarrow{\pi_0} & \text{Vect}_{G_\chi}(S^2, \chi)_{V_B} \\
p_{\text{vect}} & \downarrow & \downarrow \pi_{\text{vect}} \\
A_{G_\chi}(S^2, \chi)
\end{array}
\]

Let $p_{\text{vect}} : \Omega_{V_B} \rightarrow \pi_0(\Omega_{V_B})$ be the natural quotient map. For different elements in $A_{G_\chi}(S^2, \chi)$, their preimages through $(p_{\text{vect}} \circ p_{\text{vect}})^{-1}$ do not intersect each other so that we obtain a decomposition of $\Omega_{V_B}$. We describe this decomposition more precisely. For each $(W_{d'}^\prime)_{d' \in I^+} \in A_{G_\chi}(S^2, \chi)$, put

\[
V_B = G_\chi \times (G_\chi)_{d-1} \times W_{d-1}, \quad F_{V_B} = G_\chi \times (G_\chi)_{d-1} \times |f^{-1}| \times W_{d-1},
\]

and by using these define $\Omega_{(W_{d'}^\prime)_{d' \in I^+}}$ as the subset $(p_{\text{vect}} \circ p_{\text{vect}})^{-1}((W_{d'}^\prime)_{d' \in I^+})$ of $\Omega_{V_B}$. Henceforward, we will use these $V_B$ and $F_{V_B}$ whenever we deal with $\Omega_{(W_{d'}^\prime)_{d' \in I^+}}$.\]
Then, given a bundle $V_B$, the set $\Omega_{V_B}$ is equal to the disjoint union
\[
(W_d')_{i \in \mathbb{Z}^+} \cup \Omega(W_{d'}')_{i \in \mathbb{Z}^+}.
\]
Since $\Omega_{V_B}$ and $\Omega_{D_{R,V_B}}$ are in one-to-one correspondence, we may consider $\pi_0$, $p_\Omega$, $p_\pi$, as defined also on $(\pi_0(\Omega_{D_{R,V_B}}), \pi_0(\Omega_{D_{R,V_B}}), \Omega_{D_{R,V_B}})$, respectively. So, if we define $\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+}$ as the subset $(p_\Omega \circ p_\pi)^{-1}((W_d')_{i \in \mathbb{Z}^+})$ of $\Omega_{D_{R,V_B}}$, then we obtain the decomposition of $\Omega_{D_{R,V_B}}$
\[
(W_d')_{i \in \mathbb{Z}^+} \cup \Omega_{D_{R,V_B}}(W_{d'}')_{i \in \mathbb{Z}^+}.
\]
From this decomposition, it suffices to focus on $\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+}$ to understand $\Omega_{D_{R,V_B}}$.

Now, we state a classification result.

**Proposition 5.3.**

1. Assume that $\pi_0(\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+})$ is nonempty and $c_1 : p_\Omega(\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+}) \to H^2(S^2, \mathbb{Z})$, $[\Phi]_{D_R} \mapsto c_1(F_{V_B}/\Phi_{D_R})$ is injective for each $(W_{d'})_{i \in \mathbb{Z}^+}$ in $A_{G_{x}}(S^2)$. Then, $p_{\text{vect}}$ is surjective, and $p_{\text{vect}} \times c_1 : \text{Vect}_{G_{x}}(S^2, \chi) \to A_{G_{x}}(S^2, \chi) \times H^2(S^2)$ is injective.

2. Assume that $\pi_0(\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+})$ consists of exactly one element for each $(W_{d'})_{i \in \mathbb{Z}^+}$ in $A_{G_{x}}(S^2, \chi)$. Then, $p_{\text{vect}}$ is an isomorphism.

**Proof.** We prove only (1), and (2) is easier. Surjectivity of $p_{\text{vect}}$ is trivial by assumption. For arbitrary $[E'] \neq [E]$ in $\text{Vect}_{G_{x}}(S^2, \chi)$, if $p_{\text{vect}}([E]) \neq p_{\text{vect}}([E'])$, then there is nothing to prove. Assume that there are $[E] \neq [E']$ in the set $\text{Vect}_{G_{x}}(S^2, \chi)$ satisfy $p_{\text{vect}}([E]) = p_{\text{vect}}([E']) = (W_{d'})_{i \in \mathbb{Z}^+}$ for some $(W_{d'})_{i \in \mathbb{Z}^+}$. Then, it suffices to show $c_1(E) \neq c_1(E')$ to prove injectivity. Put $V_B = G_{x} \times (G_{x})_{m-1}W_{d-1}$. Then, $E|_{B} \cong E'|_{B} \cong V_B$ because their isotropy representations at $d^{-1}$ are all $W_{d-1}$. That is, $[E]$ and $[E']$ are in $\text{Vect}_{G_{x}}(S^2, \chi)V_B$. Since $\pi_0$ is surjective, $E \cong \Gamma_{V_B}/\Phi$ and $E' \cong \Gamma_{V_B}/\Phi'$ for some $\Phi$ and $\Phi'$ in $\Omega_{V_B}$, especially in $\Omega(\Gamma_{V_B})_{i \in \mathbb{Z}^+}$. By Lemma 5.1, $[\Phi] \neq [\Phi']$ in $\pi_0(\Omega_{\Gamma_{V_B}}(W_{d'}'))$ because $[E] \neq [E']$. So, $c_1(\Gamma_{V_B}/\Phi) \neq c_1(\Gamma_{V_B}/\Phi')$ by assumption. Therefore, we obtain a proof for injectivity. □ □

By this lemma, we only have to calculate $\pi_0(\Omega_{D_{R,V_B}}(W_{d'})_{i \in \mathbb{Z}^+})$ to classify equivariant vector bundles in many cases. In fact, we can apply this lemma except the case when $R = \rho(G_{x})$ is equal to $\mathbb{Z}_{m} \times \mathbb{Z}$ with odd $n$ or $\langle -an \rangle$ with even $n/2$ as we shall see in Section 11. When we cannot apply this lemma, we should apply Lemma 5.2 directly.

6. Equivariant pointwise clutching map

Let $\Phi$ be an equivariant clutching map determining $E$ in $\text{Vect}_{G_{x}}(S^2, \chi)_{V_B}$, i.e. the map $\Phi$ glues $\Gamma_{V_B}$ along $L_{R}$ to give $E$. Let us investigate this gluing process pointwise. For each $\bar{x} \in L_{R}$ and $x = \pi(|\bar{x}|)$, let $\bar{x} = \pi^{-1}(x) = \{\bar{x}_j | \bar{x}_j \in \mathbb{Z}_m\}$ for some $m$. Then, the map $\Phi$ glues the $(G_{x})_{\bar{x}}$-representation $(\text{res}_{(G_{x})_{\bar{x}}}^{G_{x}} \Gamma_{V_B})|_{\bar{x}}$ along $\bar{x}$ to give the $(G_{x})_{\bar{x}}$-representation $E_{\bar{x}}$, and we call this process equivariant pointwise gluing. Here, note that $(G_{x})_{\bar{x}} < (G_{x})_{x}$ for each $j \in \mathbb{Z}_m$ and

\[
(6.1) \quad \text{res}_{(G_{x})_{\bar{x}}}^{G_{x}} E_{\bar{x}} \cong (\Gamma_{V_B})_{\bar{x}}
\]

by equivariance of $\Phi$. In dealing with equivariant clutching maps, technical difficulties occur in equivariant pointwise gluings because gluing by $\Phi$ can be considered
as just a continuous collection of equivariant pointwise gluings at points in $|L|$. In this section, we prove results on equivariant pointwise gluing. To deal with equivariant pointwise gluing, we need the concept of representation extension. For compact Lie groups $N_1 < N_2$, let $W_2$ be an $N_2$-representation and $W_1$ be an $N_1$-representation. Then, $W_2$ is called a representation extension or an $N_2$-extension of $W_1$ if $\text{res}^N_{N_1}W_2 \cong W_1$. For example, $E_x$ is an $(G_x)_x$-extension of $(F_{V_{2j}})_x$ for each $j \in \mathbb{Z}_m$ by (6.1). And, let $\text{ext}^N_{N_1}W_1$ be the set

$$\{W_2 \in \text{Rep}(N_2) \mid \text{res}^N_{N_1}W_2 \cong W_1\}.$$

Let us investigate equivariant pointwise gluings more precisely under a little bit general setting. Let a compact Lie group $N_2$ act on a finite set $\mathfrak{x} = \{x_j \mid j \in \mathbb{Z}_m\}$ for $m \geq 2$, and let $N_0$ and $N_1$ be the kernel of the action and the isotropy subgroup $(N_2)_{x_0}$, respectively. Let $F$ be an $N_2$-vector bundle over $\mathfrak{x}$. Consider an arbitrary map

$$\psi : \mathfrak{x} \to \Pi_{j \in \mathbb{Z}_m} \text{Iso}(F_{x_j}, F_{x_{j+1}})$$

such that $\psi(x_j) \in \text{Iso}(F_{x_j}, F_{x_{j+1}})$. Call such a map pointwise preclutching map with respect to $F$. By using $\psi$, we glue $F_{x_j}$’s, i.e. a vector $u$ in $F_{x_j}$ is identified with $\psi(x_j)u$ in $F_{x_{j+1}}$ for each $j$. Let $F/\psi$ be the quotient of $F$ through this identification, and let $p_{\psi} : F \to F/\psi$ be the quotient map. Let $i_{\psi} : F_{x_0} \to F/\psi$ be the composition of the natural injection $i_{x_0} : F_{x_0} \to F$ and the quotient map $p_{\psi}$.

$$\begin{equation}
\begin{array}{ccc}
F_{x_0} & \xrightarrow{i_{\psi}} & F \\
\downarrow & & \downarrow \psi \\
F/\psi & \xrightarrow{p_{\psi}} & F
\end{array}
\end{equation}$$

We would find conditions on $\psi$ under which the quotient $F/\psi$ inherits an $N_2$-representation structure from $F$ and the map $i_{\psi}$ becomes an $N_1$-isomorphism from $F_{x_0}$ to $\text{res}^N_{N_1}(F/\psi)$. For notational simplicity, denote

$$\psi(x_{j'}) \cdots \psi(x_{j+1})\psi(x_j)u$$

by $\psi^{j'-j+1}u$ for $u \in F_{x_j}$ and $j \leq j'$ in $\mathbb{Z}$.

**Lemma 6.1.** For a pointwise preclutching map $\psi$ with respect to $F$, the quotient $F/\psi$ carries an $N_2$-representation structure so that $p_{\psi}$ is $N_2$-equivariant and $i_{\psi}$ is an $N_1$-isomorphism if and only if the following conditions hold:

1. $\psi^m = \text{id}$ in $\text{Iso}(F_{x_j})$ for each $j \in \mathbb{Z}_m$. So, $\psi^k$ is well-defined for all $k \in \mathbb{Z}_m$.
2. $\psi^{j_3-j_1} = g\psi(x_{j_1})g^{-1}$ in $F_{x_{j_1}}$ for each $j_1 \in \mathbb{Z}_m$, $g \in N_2$ when $g^{-1}x_{j_1} = x_{j_2}$ and $g^{j_2+j_3} = x_{j_3}$ for some $j_2, j_3 \in \mathbb{Z}_m$.

**Proof.** To begin with, it is obvious that $i_{\psi}$ is surjective. And, note that $i_{\psi}$ is injective if and only if Condition (1) holds.

Next, we show that $F/\psi$ carries an $N_2$-representation structure such that $p_{\psi}$ is equivariant if and only if Condition (2) holds. The possible group action on $F/\psi$ to guarantee equivariance of $p_{\psi}$ is as follows:

$$g \cdot u = p_{\psi}(gu)$$

for each $g \in N_2, u \in F/\psi, u \in p_{\psi}^{-1}(u)$. This is well-defined if and only if $p_{\psi}(gu) = p_{\psi}(gv\psi(u))$ for each $l \in \mathbb{Z}$, i.e. $\psi^k gu = gv^j u$ for some $k$. Putting $\bar{u} = gu$, this can be written as $\psi^k \bar{u} = \psi^j \bar{u}$. Since $\psi^j g^{-1} = (gv^j g^{-1})^l$, this holds for each $l$ if and only if it holds for $l = 1$, i.e. Condition (2). This gives a proof. $\square$
A pointwise preclutching map satisfying (1), (2) of Lemma 6.1 is called an *equivariant pointwise clutching map* with respect to $F$. Let $\mathcal{A}$ be the set of all equivariant pointwise clutching maps with respect to $F$, and we topologize $\mathcal{A}$ with the subspace topology of $\prod_{j \in \mathbb{Z}_m} \text{Iso}(F_{x_j}, F_{x_{j+1}})$. An $N_2$-representation $W$ is called determined by $\psi \in \mathcal{A}$ with respect to $F$ if $W \cong F/\psi$. In the next corollary, we can see that representation extension is related to equivariant pointwise clutching map and especially guarantees nonemptiness of $\mathcal{A}$.

**Corollary 6.2.** For an $N_2$-extension $W$ of $F_{x_0}$, assume that there exists an $N_2$-morphism $p : F \to W$ such that $p|_{F_{x_j}}$ is a nonequivariant isomorphism for each $j \in \mathbb{Z}_m$. Let $\psi$ be the pointwise preclutching map defined by $\psi(x_j) = (p|_{F_{x_{j+1}}})^{-1} \circ (p|_{F_{x_j}})$ for $j \in \mathbb{Z}_m$. Then, $\psi$ is in $\mathcal{A}$, and there is an $N_2$-isomorphism $i : W \to F/\psi$ such that $p_0 = i \circ p$, i.e. $\psi$ determines $W$ with respect to $F$. Especially, $\mathcal{A}$ is nonempty.

**Proof.** Easy check. \qed

To calculate $\pi_0(\Omega_{Vx})$ later, we need to understand topology of $\mathcal{A}$. For this, we would consider $F/\psi$ as an additional structure over the fixed $F_{x_0}$ for each $\psi \in \mathcal{A}$ as follows:

**Lemma 6.3.** Define a operation $\star_\psi$ of $N_2$ on $F_{x_0}$ as $g \star_\psi u = \iota_\psi^{-1} p_\psi(gu)$ for $\psi \in \mathcal{A}$, $g \in N_2$, $u \in F_{x_0}$. Then,

1. $\star_\psi$ is an $N_2$-action on $F_{x_0}$ such that $\iota_\psi : (F_{x_0}, \star_\psi) \to F/\psi$ is an $N_2$-isomorphism.
2. $g \star_\psi u = \psi^{m-1} gu = gu^k u$ when $g \bar{x}_0 = \bar{x}_j$ and $g \bar{x}_k = \bar{x}_0$. Especially, $g \star_\psi u = gu$ for $g \in N_1$.

Henceforth, we consider $F/\psi$ as $(F_{x_0}, \star_\psi)$.

**Proof.** To prove (1), we show that $\iota_\psi(g \star_\psi u) = g \cdot \iota_\psi(u)$ for each $g \in N_2$, $u \in F_{x_0}$ as follows:

\[ \iota_\psi(g \star_\psi u) = \iota_\psi(\iota_\psi^{-1} p_\psi(gu)) = p_\psi(gu), \]
\[ g \cdot \iota_\psi(u) = g \cdot p_\psi(u) = p_\psi(gu). \]

Since $F/\psi$ already delivers an $N_2$-action and $\iota_\psi$ is bijective, this shows that $\star_\psi$ is an action. Therefore, we prove (1).

Next, we prove (2). Note that $\iota_\psi^{-1} p_\psi|_{F_{x_0}}$ is an identity. Using this,

\[ g \star_\psi u = \iota_\psi^{-1} p_\psi(gu) = \iota_\psi^{-1} p_\psi(\psi^{m-1} gu) = \psi^{m-1} gu. \]

Also, $\psi^{m-1} gu = g\psi^k u$ by Lemma 6.1(2). Therefore, we prove (2). \qed

To obtain more precise results on $\mathcal{A}$, we assume that the $N_2$-bundle $F$ over $x$ satisfies one of the following two conditions:

1. $N_2$ fixes $x = \{\bar{x}_j \mid j \in \mathbb{Z}_m\}$ with $m = 2$, and $F_{x_0} \cong F_{x_1}$.
2. $N_2$ acts transitively on $x = \{\bar{x}_j \mid j \in \mathbb{Z}_m\}$ with $m \geq 2$.

These conditions are not too restrictive as the following example shows:

**Example 6.1.** Given the bundle $F_{Vn}$ over $|\mathcal{K}_R|$ of Section 4 let $x = \pi^{-1}(x) = \{\bar{x}_j \mid j \in \mathbb{Z}_m\}$ for each $x \in |\mathcal{L}_R|$, $x = \pi(\dot{x})$, some $m \geq 2$. Put $N_2 = (G_x)_{\bar{x}}$ and $F = (\text{res}_{G_{\bar{x}}})_{\bar{x}, F_{Vn}}|_{\bar{x}}$. By Table 3.3 and definition of $F_{Vn}$, we can check that $F$ satisfies Condition F1. or F2. according to $\dot{x}$. Given a map $\Phi$ in $\Omega_{Vn}$, we can define a map $\psi$ in $\mathcal{A}$ as follows:

1. if $\bar{x}$ is a vertex, then $\psi_\dot{x}(\bar{x}_j) = \Phi(\bar{x}_{j+})$ for each $j \in \mathbb{Z}_{jn}$,
2. if $\bar{x}$ is not a vertex and $\bar{x} = p_1(\hat{x})$, then
\[ \psi(\bar{x}_0) = \Phi(\bar{x}) \quad \text{and} \quad \psi(\bar{x}_1) = \Phi(\hat{c}(\bar{x})). \]
Then, we have \( F/\psi \cong (F_{\mathcal{V}}/\Phi)_x \) for each \( x \).

Let \( \mathcal{A}' \) be the set
\[
\{ \psi(\bar{x}_j) \mid \psi \in \mathcal{A} \},
\]
and let \( \mathcal{A}^{j,j'} \) be the set
\[
\{ (\psi(\bar{x}_j), \psi(\bar{x}_{j'})) \mid \psi \in \mathcal{A} \}.
\]
In the below, it will be witnessed that \( \mathcal{A} \) is homeomorphic to \( \mathcal{A}' \) or \( \mathcal{A}^{j,j'} \) in many cases.

**Lemma 6.4.** Assume that \( F \) satisfies Condition F1. Then, \( \mathcal{A} \) is homeomorphic to nonempty \( \mathcal{A}^0 = \text{Iso}_{N^2}(F_{x_0}, F_{x_1}) \) through the the evaluation map
\[
\mathcal{A} \rightarrow \mathcal{A}^0, \; \psi \mapsto \psi(\bar{x}_0).
\]

**Proof.** By Lemma [6.1](1), each \( \psi \in \mathcal{A} \) satisfies \( \psi^2 = \text{id} \) so that \( \psi \) is determined by \( \psi(\bar{x}_0) \). From this, \( \mathcal{A} \) is homeomorphic to \( \mathcal{A}^0 \) through the evaluation. And, \( \psi \) satisfies Lemma [6.1](2) if and only if \( \psi(\bar{x}_j) \) is \( \text{Iso}_{N^2}(F_{x_j}, F_{x_{j+1}}) \)-valued because \( N^2 \) fixes \( \bar{x}_0, \bar{x}_1 \). Nonemptiness is guaranteed by Condition F1.

In the remaining of this section, we assume that \( F \) satisfies Condition F2. so that \( F \cong N^2 \times N^1 \bar{F}_{x_0} \). Under this assumption, the zeroth homotopy of \( \mathcal{A} \) will be related to \( \text{ext}^N_{N^1} F_{x_0} \). For this, we need a technical lemma. We would express a map \( \psi \) in \( \mathcal{A} \) as \( m \) endomorphisms of \( F_{x_0} \). This will be useful in dealing with \( \mathcal{A} \).

For each \( j \in \mathbb{Z}_m \), pick an element \( g_j \in N^2 \) such that \( g_j \bar{x}_j = \bar{x}_0 \), and denote by \( \mathbf{g} \) the \( m \)-tuple \((g_j)_{j \in \mathbb{Z}_m} \). Also, express a pointwise preclutching map \( \psi \) by the \( m \)-tuple \((\psi(\bar{x}_j))_{j \in \mathbb{Z}_m} \). For each \( \psi = (\psi(\bar{x}_j))_{j \in \mathbb{Z}_m} \), define the map
\[
\psi_\mathbf{g} : \bar{x} \rightarrow \text{Iso}(F_{x_0}), \; \bar{x}_j \mapsto g_{j+1}^{-1} \psi(\bar{x}_j) g_j^{-1},
\]
and express it by the \( m \)-tuple \((g_{j+1} \psi(\bar{x}_j) g_j^{-1})_{j \in \mathbb{Z}_m} \). And, denote by \( \mathcal{A}_\mathbf{g} \) the set
\[
\{ \psi_\mathbf{g} \mid \psi \in \mathcal{A} \}.
\]
Here, we consider an action of \( \text{Iso}_{N^1}(F_{x_0}) \) on these \( \psi_\mathbf{g} \)’s. For \( B \in \text{Iso}_{N^1}(F_{x_0}) \), denote by \( B \psi_\mathbf{g} \) the \( m \)-tuple
\[
(Bg_{j+1} \psi(\bar{x}_j) g_j^{-1} B^{-1})_{j \in \mathbb{Z}_m}.
\]
Then, this action preserves \( \mathcal{A}_\mathbf{g} \) as follows:

**Lemma 6.5.**

1. For \( \psi \in \mathcal{A} \) and \( B \in \text{Iso}_{N^1}(F_{x_0}) \), put
\[
\psi'(\bar{x}_j) = g_{j+1}^{-1}Bg_{j+1} \psi(\bar{x}_j) g_j^{-1} B^{-1} g_j
\]

for each \( j \in \mathbb{Z}_m \) so that \( \psi' \) satisfies \( B \psi_\mathbf{g} B^{-1} = \psi_\mathbf{g}' \). Then, \( \psi' \in \mathcal{A} \).

2. For two elements \( \psi, \psi' \) in \( \mathcal{A} \), \( F/\psi \cong F/\psi' \) as \( N^2 \)-representation if and only if \( B \psi_\mathbf{g} B^{-1} = \psi_\mathbf{g}' \) for some \( B \in \text{Iso}_{N^1}(F_{x_0}) \).

**Proof.** To prove (1), we would show that \( \psi' \) satisfies two conditions of Lemma [6.1](1) of Lemma [6.1]

Condition (2) of Lemma [6.1] is written as \((\psi')^{j_1-j_2}u = g g'(\bar{x}_{j_2}) g^{-1} u \) in \( F_{x_{j_1}} \) for \( j_1 \in \mathbb{Z}_m \), \( g \in N^2 \) when \( g \bar{x}_{j_1} = \bar{x}_{j_1} \) and \( g \bar{x}_{j_2+1} = \bar{x}_{j_2} \). Here,
\[
g g'(\bar{x}_{j_2}) g^{-1} = g(g_{j_2+1}^{-1}Bg_{j_2+1} \psi(\bar{x}_{j_2}) g_{j_2}^{-1} B^{-1} g_j g^{-1})
\]
\[
= (gg_{j_2+1}^{-1}Bg_{j_2+1}g^{-1})(g \psi(\bar{x}_{j_2}) g^{-1})(gg_{j_2+1}^{-1}Bg_{j_2+1}g^{-1}).
\]
Note that $g_{j_2+1}^{-1} = g_{j_2}^{-1} g_{j_2+1}^{-1}$ and $g_{j_2}^{-1} = g_{j_2}^{-1} g_{j_2+1}^{-1}$ where $g_{j_2} g_{j_2+1}$ and $g_{j_2} g_{j_2+1}$ are in $N_1$. With these,
\[ g_{j_2+1}^{-1} B g_{j_2+1} g^{-1} = g_{j_2}^{-1} (g_{j_2} g_{j_2+1}) B (g_{j_2} g_{j_2+1})^{-1} g_{j_2} = g_{j_2}^{-1} B g_{j_2}, \]
and
\[ g_{j_2}^{-1} B^{-1} g_{j_2} g^{-1} = g_{j_2}^{-1} (g_{j_2} g_{j_2+1}) B^{-1} (g_{j_2} g_{j_2+1})^{-1} g_{j_2} = g_{j_2}^{-1} B^{-1} g_{j_2}, \]
because $B \in \text{Iso}_{N_1}(F_{x_0})$. So,
\[
g_{j_2} \psi(\bar{x}_{j_2}) g^{-1} = (g_{j_2}^{-1} B g_{j_2})(g_{j_2}^{-1} B^{-1} g_{j_2}) = (g_{j_2}^{-1} B g_{j_2}) \psi_{j_2-j_1}(g_{j_2}^{-1} B^{-1} g_{j_2}) = (g_{j_2}^{-1} B g_{j_2}) \psi_{j_2-j_1}(g_{j_2-j_1}^{-1} B g_{j_2-j_1}) \cdots \]
\[\cdots (g_{j_2+1}^{-1} B^{-1} g_{j_2+1} \psi(\bar{x}_{j_2}) g_{j_2}^{-1} B g_{j_2}) = \psi_{j_2-j_1} \cdots \psi_{j_2-j_2}(g_{j_2-j_1}^{-1} B g_{j_2-j_1}) \]
in $F_{x_{j_1}}$, and this proves (1).

Next, we prove (2). For sufficiency, assume that $F/\psi \cong F/\psi'$. Considering $F/\psi$ as $(F_{x_0}, \ast_\psi)$ by Lemma 6.3 there exists an $N_2$-isomorphism $B : (F_{x_0}, \ast_\psi) \to (F_{x_0}, \ast_{\psi'})$, and especially $B \in \text{Iso}_{N_1}(F_{x_0})$ by Lemma 6.3 (2). So, $B g_j \ast_\psi g_0^{-1} u = g_j \ast_{\psi'} B g_0^{-1} u$ for each $j \in Z_m$, $u \in F_{x_0}$, and if we substitute $B^{-1} u$ into $u$, this is written as
\[
(B g_j \psi_{j_0}^{-1} B^{-1}) u = (g_j \psi_{j_0}^{-1} B^{-1}) u
\]
by Lemma 6.3 (2). Substituting $j = 1$ in this, we obtain $B g_1 \psi_{j_0}^{-1} B^{-1} = g_1 \psi_{j_0}^{-1} B^{-1}$ on $F_{x_0}$. By using this and mathematical induction, we would show $B \psi_{j} B^{-1} = \psi_{j}$. Assume that $B g_j \psi_{j_0}^{-1} B^{-1} = g_j \psi_{j_0}^{-1} B^{-1}$ on $F_{x_0}$ from $j = 1$ to $j = k - 1$. Then,
\[
(g_k \psi_{g_{k-1}^{-1}} (g_{k-1} \psi_{g_{k-2}^{-1}}) \cdots (g_1 \psi_{g_0}^{-1})
\]
\[= g_k \psi_{g_{k-1}^{-1}} (g_{k-1} \psi_{g_{k-2}^{-1}}) \cdots (g_1 \psi_{g_0}^{-1})
\]
\[= B g_k \psi_{g_{k-1}^{-1}} B^{-1}
\]
\[= (B g_k \psi_{g_{k-1}^{-1}} B^{-1})(B g_{k-1} \psi_{g_{k-2}^{-1}} B^{-1}) \cdots (B g_1 \psi_{g_0}^{-1} B^{-1})
\]
\[= (B g_k \psi_{g_{k-1}^{-1}} B^{-1})(B g_{k-1} \psi_{g_{k-2}^{-1}} B^{-1}) \cdots (g_1 \psi_{g_0}^{-1}).
\]
Comparing the first line with the last, we have $B g_k \psi_{g_{k-1}^{-1}} B^{-1} = g_k \psi_{g_{k-1}^{-1}}$. By mathematical induction, we obtain $B g_j \psi_{j_0}^{-1} B^{-1} = g_j \psi_{j_0}^{-1} B^{-1}$ on $F_{x_0}$ for each $j \in Z_m$, and this shows $B \psi_{j} B^{-1} = \psi_{j}$.

Last, we prove necessity of (2). By assumption, we have $B g_{j+1} \psi_{j}^{-1} B^{-1} = g_{j+1} \psi_{j}^{-1} B^{-1}$ for each $j \in Z_m$ on $F_{x_0}$. Then,
\[
B g_{j} \psi_{j}^{-1} B^{-1}
\]
\[= (B g_{j} \psi_{j_0}^{-1} B^{-1})(B g_{j-1} \psi_{j_0}^{-1} B^{-1}) \cdots (B g_1 \psi_{g_0}^{-1} B^{-1})
\]
\[= (g_j \psi_{g_{j_0}^{-1}} (g_{j_1} \psi_{g_{j_2}^{-1}}) \cdots (g_1 \psi_{g_0}^{-1})
\]
\[= g_j \psi_{g_{j_0}^{-1}} B^{-1}
\]
for each $j \in Z_m$ on $F_{x_0}$. Acting these on $B g_0 u$ for $u \in F_{x_0}$, we obtain
\[B g_j \psi_{j}^{-1} B u = g_j \psi_{j}^{-1} B u\]
because \( B \in \text{Iso}_{N_1}(F_{z_0}) \). And, this is written as \( Bg_j \ast \psi \cdot u = g_j \ast \psi \cdot Bu \) by Lemma 6.3.(2). This means that \( B \) is equivariant for \( \langle N_1, g_j \rangle_{j \in \mathbb{Z}_m} \) so that \( B \) is an \( N_2 \)-isomorphism between \( (F_{z_0}, \ast_\psi) \) and \( (F_{z_0}, \ast_\psi) \). Therefore, we obtain proof. ∎ ∎

This lemma means that each orbit of \( \mathcal{A}_g \) under the \( \text{Iso}_{N_1}(F_{z_0}) \)-action corresponds to a representation. More precisely, we have the following:

**Theorem 6.6.** Assume that \( F \) satisfies Condition F2. Then, the map

\[
\pi_0(\mathcal{A}) \rightarrow \text{ext}_{N_1}^{N_2} F_{z_0}, \quad [\psi] \mapsto F/\psi
\]

is bijective. So, \( \mathcal{A} \) is nonempty if and only if \( F_{z_0} \) has an \( N_2 \)-extension.

**Proof.** To begin with, we show that each \( N_2 \)-extension of \( F_{z_0} \) is obtained as \( F/\psi \) for some \( \psi \in \mathcal{A} \). Let \( W \) be an \( N_2 \)-extension of \( F_{z_0} \) such that \( \text{res}_{N_1} N_2 W \) is equal to \( F_{z_0} \). Consider a map \( p : N_2 \times N_1 F_{z_0} \rightarrow W \), \( [g, u] \mapsto gu \). Since \( F \cong N_2 \times N_1 F_{z_0} \), the map \( p \) satisfies conditions of Corollary 6.2. Therefore, there exists \( \psi \) such that \( F/\psi \cong W \) by Corollary 6.2. That is, surjectivity is proved.

Next, we show that two \( \psi \) and \( \psi' \) in \( \mathcal{A} \) are connected by a path in \( \mathcal{A} \) if and only if \( F/\psi \cong F/\psi' \). First, we show sufficiency. Let \( \psi_t \) for \( t \in [0,1] \) be a continuous path in \( \mathcal{A} \) such that \( \psi_0 = \psi \) and \( \psi_1 = \psi' \). Then, all \( (F_{z_0}, \ast_{\psi_t}) \)'s are isomorphic because the path of their characters are in the discrete space \( \text{Rep}(N_2) \). Second, we show necessity. If \( F/\psi \cong F/\psi' \), then Lemma 6.5 says that \( B\psi B^{-1} = \psi' \) so that \( \psi' \) is connected in \( \mathcal{A}_g \) by a path because \( \text{Iso}_{N_1}(F_{z_0}) \) is a product of general linear groups by Schur’s Lemma. Since the map from \( \mathcal{A} \) to \( \mathcal{A}_g \) sending \( \psi \) to \( \psi_g \) is a homeomorphism, \( \psi \) and \( \psi' \) are connected by a path in \( \mathcal{A} \), and we obtain necessity. Therefore, we obtain injectivity.

Nonemptiness is clear from the one-to-one correspondence. ∎ ∎

Denote by \( \mathcal{A}_{\psi} \) the path-component of \( \mathcal{A} \)

\[
\{ \psi' \in \mathcal{A} \mid F/\psi \cong F/\psi' \}
\]

and by \( \mathcal{A}_{\psi,g} \) the path-component of \( \mathcal{A}_g \)

\[
\{ \psi'_g \in \mathcal{A}_g \mid F/\psi \cong F/\psi' \}
\]

Note that \( \mathcal{A}_{\psi} \) and \( \mathcal{A}_{\psi,g} \) are homeomorphic. To calculate homotopy of equivariant clutching maps in next sections, we need to calculate \( \pi_1(\mathcal{A}_{\psi}) \) for each \( \psi \in \mathcal{A} \). To do it, we would investigate the shape of \( \mathcal{A}_{\psi} \).

**Lemma 6.7.** For each \( \psi \in \mathcal{A} \), \( \mathcal{A}_{\psi} \) is homeomorphic to

\[
\text{Iso}_{N_1}(F/\psi)/\text{Iso}_{N_2}(F/\psi).
\]

**Proof.** In Lemma 6.5.(2), we have shown that \( \mathcal{A}_{\psi,g} \) is equal to the orbit of \( \psi'_g \) by \( \text{Iso}_{N_1}(F_{z_0}) \). To understand this orbit, we need to know the isotropy subgroup \( \text{Iso}_{N_1}(F_{z_0})_{\psi_g} \), i.e.

\[
\{ B \in \text{Iso}_{N_1}(F_{z_0}) \mid B\psi_g B^{-1} = \psi_g \}.
\]

By definition, \( B\psi_g B^{-1} = \psi_g \) is expressed as \( Bg_{j+1}\psi(x_j)g_{j-1}B^{-1} = g_{j+1}\psi(x_j)g_{j-1} \) for each \( j \in \mathbb{Z}_m \). Since \( g_{j+1}\psi g_{j-1} = g_{j+1}g_{j-1}g_{j+1}\psi g_{j-1} = (g_{j+1}g_{j-1})\psi k \) on \( F_{z_0} \) for some \( k \) by Lemma 6.3.(2), this is written as

\[
B(g_{j+1}g_{j-1} \ast \psi B^{-1} u) = g_{j+1}g_{j-1} \ast \psi u.
\]

Since \( g_0 \in N_1 \), we have \( N_2 = \langle N_1, g_jg^{-1}_{j-1} \rangle_{j \in \mathbb{Z}_m} \). Therefore, \( B \in \text{Iso}_{N_1}(F_{z_0})_{\psi_g} \) if and only if \( B \in \text{Iso}_{N_1}(F_{z_0}, \ast_{\psi}) \). Therefore, \( \mathcal{A}_{\psi,g} \) is homeomorphic to the quotient \( \text{Iso}_{N_1}(F/\psi)/\text{Iso}_{N_2}(F/\psi) \) because \( (F_{z_0}, \ast_\psi) \cong F/\psi \). ∎ ∎
In some special cases, we can understand $\mathcal{A}_\psi$ more precisely. Denote by $\mathcal{A}_\psi^0$, the set
\[ \{ \psi'(\bar{x}_j) \mid \psi' \in \mathcal{A}_\psi \}, \]
and by $\mathcal{A}_\psi^{j, 0}$ the set
\[ \{ (\psi'(\bar{x}_j), \psi'(\bar{x}_{j+1})) \mid \psi' \in \mathcal{A}_\psi \} \].

**Proposition 6.8.** Assume that $N_2 = \langle N_0, a_0 \rangle$ with some $a_0 \in N_2$ such that $a_0 \bar{x}_j = \bar{x}_{j+1}$ for each $j \in \mathbb{Z}_m$ so that $N_2/N_0 \cong \mathbb{Z}_m$ and $N_1 = N_0$. Then,

1. A pointwise preclutching map $\psi$ with respect to $F$ is in $\mathcal{A}$ if and only if $\psi' = \psi(0) \in \text{Irr}(2)$ for each $j \in \mathbb{Z}_m$.

2. $\mathcal{A}_\psi$ is homeomorphic to $\mathcal{A}_\psi^0$, $\mathcal{A}_\psi^{j, 0}$ for any $\psi \in \mathcal{A}$, respectively.

3. If $F_{x_0}$ is $N_0$-isotypical, then $\mathcal{A}_\psi$ is simply connected for each $\psi \in \mathcal{A}$.

**Proof.** For (1), we only have to show that $\psi$ satisfies two conditions of Lemma 6.1 if and only if $\psi' = \psi(0) \in \text{Irr}(2)$ for each $j \in \mathbb{Z}_m$. First, we prove sufficiency. If we substitute each $a_0^j$ into $g$ in Condition (2) of Lemma 6.1, $\psi(0)$ is $N_0$-equivariant for each $j \in \mathbb{Z}_m$. Also, if we substitute $a_0^j$ into $g$ in the same formula, then we obtain $\psi(\bar{x}_j) = a_0^j \psi(\bar{x}_0) a_0^{-j}$. Next, we prove necessity. Note that $\psi(\bar{x}_j) = a_0^j \psi(\bar{x}_0) a_0^{-j}$ is $N_0$-equivariant for each $j \in \mathbb{Z}_m$ because $N_0$ is normal in $N_2$ and $\psi(\bar{x}_0)$ is $N_0$-equivariant. An arbitrary $g$ in $N_2$ is expressed as $a_0^j h$ for some $h \in N_0, j \in \mathbb{Z}_m$. Then, since $g^{-1} \bar{x}_j = \bar{x}_{j+1}$ and $g \bar{x}_j = \bar{x}_j$, with $\bar{j} = j - j, \bar{j}_2 = j + 1$, for each $j \in \mathbb{Z}_m$, we have
\[
g \psi(\bar{x}_j) g^{-1} u = (a_0^j h) \psi(\bar{x}_j) (a_0^j h)^{-1} u = a_0^j \psi(\bar{x}_j) a_0^{-j} u = a_0^{j + j_2} \psi(\bar{x}_0) a_0^{-j} \bar{j}_2 u = a_0^j \psi(\bar{x}_0) a_0^{-j} \bar{j}_2 u = \psi(\bar{x}_j) u
\]
for each $u \in F_{x_0}$, so that $\psi$ satisfies Condition (2) of Lemma 6.1. Therefore, we obtain a proof of (1).

(2) is easy because a map $\psi$ in $\mathcal{A}$ is determined by $\psi(\bar{x}_0)$.

Before we prove (3), we prove that the $N_0$-character of $F_{x_0}$ is fixed by $N_2$. Assume that $F_{x_0}$ is $\chi$-isotypical for some $\chi \in \text{Irr}(N_0)$. By Theorem 6.6, existence of $\psi$ guarantees existence of an $N_0$-extension, say $V$. For each $g \in N_2$, we have
\[
g F_{x_0} \cong \text{res}^N_{g N_0 g^{-1}} g V \cong \text{res}^N_{g N_0} g V \cong \text{res}^N_{N_0} V \cong F_{x_0}
\]
by using normality of $N_0$. By Definition 8.1, $g F_{x_0}$ is $(g \cdot \chi)$-isotypical so that $g \cdot \chi = \chi$ for each $g$.

Now, we prove (3). Put $F_{x_0} \cong U \bar{l}$ for some $l \in N$ and $U \in \text{Irr}(N_0)$. We already know that the character of $U$ is fixed by $N_2$. Let $\bar{U}$ be an $N_2$-extension of $U$ whose existence is guaranteed by Theorem 6.1. By Corollary 14.2, $F/\psi$ is isomorphic to $(l_0 \bar{U} \otimes \Omega(0)) \otimes \cdots \otimes (l_{m-1} \bar{U} \otimes \Omega(m - 1))$ for some $l_k$s in $\mathbb{Z}$ so that $l = \sum l_k$. Schur’s Lemma says that $\text{Iso}_{N_0}(F/\psi) \cong \text{Iso}_{N_0}(U) \cong GL(l, \mathbb{C})$ and $\text{Iso}_{N_2}(F/\psi) \cong GL(l_0, \mathbb{C}) \times \cdots \times GL(l_{m-1}, \mathbb{C})$. So, $\text{Iso}_{N_0}(F/\psi)/\text{Iso}_{N_2}(F/\psi)$ is homeomorphic to $GL(l, \mathbb{C}) / GL(l_0, \mathbb{C}) \times \cdots \times GL(l_{m-1}, \mathbb{C})$. By long exact sequence of homotopy of a fibration,
\[
\begin{align*}
\rightarrow \pi_1 \left( GL(l_0, \mathbb{C}) \times \cdots \times GL(l_{m-1}, \mathbb{C}) \right) \xrightarrow{i} \pi_1 \left( GL(l, \mathbb{C}) \right) \\
\rightarrow \pi_1 \left( GL(l, \mathbb{C}) / GL(l_0, \mathbb{C}) \times \cdots \times GL(l_{m-1}, \mathbb{C}) \right) \rightarrow 0.
\end{align*}
\]
Lemma 6.11. useful lemma on $p$

And, we have the isomorphism
\[
\psi = \langle \bar{\psi}, \psi \rangle = \langle \bar{\psi}, \psi \rangle \quad \text{so that the restricted bundle trivially satisfies Condition F1.}
\]
And, it is trivial that
\[
\psi = \langle \bar{\psi}, \psi \rangle \quad \text{be considered to glue each pair of fibers of } F.
\]

Proof. If we restrict the action to $(N_0, a_0)$, then the proof is the same with Proposition [6.3] (2).

Proposition 6.10. Assume that $N_2 = (N_0, \alpha_1, \alpha_2, \alpha_3)$ with some $\alpha_i$’s in $N_2$ such that
\[
\alpha_1 \bar{x}_j = \bar{x}_{j+1}, \quad \alpha_2 \bar{x}_j = \bar{x}_{j+2}, \quad \alpha_3 \bar{x}_j = \bar{x}_{j+3}
\]
for each $j \in \mathbb{Z}_m$. So, $N_2/N_0 \cong \mathbb{Z}_m$ and $N_1 = (N_0, b_0 a_0)$. Then, $A, A_\psi$ are homeomorphic to $A^0, A^0_\psi$ for any $\psi \in A$, respectively.

Proof. It suffices to show that each $\psi$ in $A$ is determined by $\psi(x_0), \psi(x_1)$. Substituting $g = \alpha_i$ in Lemma [6.1] (2), we obtain
\[
\psi(x_2) = \alpha_3 \psi(x_0) \alpha_3^{-1}, \quad \psi(x_1) = \alpha_2 \psi(x_3) \alpha_2^{-1},
\]
so we obtain a proof.

We often need to restrict our arguments on $A$ to $\{x_j, \bar{x}_j\}$ as follows: let $A_{j,j'}$ with $j \neq j'$ be the set of equivariant pointwise clutching maps with respect to the $N_2(x_j, x_{j'})$-bundle $\left(\text{res}_{j,j'}^{N_2} F\right)|_{(x_j, x_{j'})}$ where $N_2(x_j, x_{j'})$ is the subgroup preserving $\{x_j, \bar{x}_j\}$. Here, each $\psi$ in $A_{j,j'}$ is defined on $\{x_j, \bar{x}_j\}$, and satisfies
\[
\psi(x_j) \in \text{Iso}(F_{x_j}, F_{\bar{x}_j}) \quad \text{and} \quad \psi(\bar{x}_j) \in \text{Iso}(F_{\bar{x}_j}, F_{x_j}).
\]
We need to know if the restricted bundle $\left(\text{res}_{j,j'}^{N_2} F\right)|_{(x_j, x_{j'})}$ satisfies Condition F1. or F2. If $F$ satisfies Condition F1., then
\[
\text{res}_{j,j'}^{N_2} F \quad \text{so that the restricted bundle trivially satisfies Condition F1.}
\]
And, it is trivial that $A = A_{j,j'}$. If $F$ satisfies Condition F2., then $\left(\text{res}_{j,j'}^{N_2} F\right)|_{(x_j, x_{j'})}$ satisfies Condition F2. because $N_2(x_j, x_{j'})$ acts transitively on $\{x_j, \bar{x}_j\}$. We obtain a useful lemma on $A_{j,j'}$.

Lemma 6.11. The map $\text{res}_{j,j'} : A \to A_{j,j'}$, $\psi \mapsto \text{res}_{j,j'}(\psi)$ is well-defined where
\[
\text{res}_{j,j'}(\psi)(x_j) = \psi^{-j}(x_j), \quad \text{res}_{j,j'}(\psi)(\bar{x}_j) = x_{j+1} \quad (\mod m).
\]
And, we have the isomorphism
\[
\text{res}_{j,j'}^{N_2} F \quad \cong \quad \text{res}_{j,j'}^{N_2} F \quad \text{for each } \psi \in A.
\]

Proof. By Lemma [6.1], we can check that $\text{res}_{j,j'}(\psi)$ is in $A_{j,j'}$. And, the injection from $\left(\text{res}_{j,j'}^{N_2} F\right)|_{(x_j, x_{j'})}$ to $F$ induces the isomorphism through $p_{\text{res}_{j,j'}(\psi)}$ and $\psi^0$.

Since any $\psi$ in $A$ glues all fibers of $F$ to obtain a single vector space $F/\psi$, $\psi$ might be considered to glue each pair of fibers of $F$. That is, $\psi$ determines the function $\psi$ defined on $\bar{x} \times \bar{x} - \Delta$ sending a pair $(\bar{x}, \bar{x}')$ to the element $\psi(\bar{x}, \bar{x}')$ in $\text{Iso}(F_{\bar{x}}, F_{\bar{x}'})$ such that each $u$ in $F_{\bar{x}}$ is identified with $\psi(\bar{x}, \bar{x}') u$ in $F_{\bar{x}'}, \psi(\bar{x}, \bar{x}')$ satisfies $p_0(u) = p_0(\psi(\bar{x}, \bar{x}') u)$ where $\Delta$ is the diagonal. Call $\psi$ the saturation of $\psi$. Since the index $j$ is not used in defining $\psi$, it is often convenient to use $\psi$ instead of $\psi$. Denote by $A$ the set $\{\psi \mid \psi \in A\}$, and call it the saturation of $A$. And, denote $F/\psi, p_0$ also by $F/\psi, p_\psi$, respectively.
7. SOME LEMMAS ON FUNDAMENTAL GROUPS

In this section we prove three lemmas needed to calculate homotopy of equivariant clutching maps. Two of them are just rewriting of Schur’s Lemma. The other is on relative homotopy.

**Lemma 7.1.** For \( \chi \in \text{Irr}(H) \), let \( W \) be a \( \chi \)-isotypical \( H \)-representation. For the natural inclusion \( i : \text{Iso}_H(W) \rightarrow \text{Iso}(W) \), the map

\[
i_* : \pi_1(\text{Iso}_H(W)) \rightarrow \pi_1(\text{Iso}(W))
\]

is injective and equal to the multiplication by \( \chi(id) \) up to sign.

*Proof.* An example is given instead of a detailed proof. Let \( V \), a \( \chi \)-representation whose character is equal to \( \chi \), be an \( H \)-representation whose existence is guaranteed by Theorem 14.1. Then, \( W = V \). Assume that \( \chi(id) = 3 \). By Schur’s Lemma, \( \text{Iso}_H(W) \cong \text{GL}(2, \mathbb{C}) \) and \( i \) is a map from \( \text{GL}(2, \mathbb{C}) \) to \( \text{GL}(6, \mathbb{C}) \) such that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & b & 0 \\ c & 0 & 0 & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & c & 0 & 0 & d \end{pmatrix}.
\]

Then, proof for this case is easily followed. \( \square \)

**Lemma 7.2.** For \( \chi \in \text{Irr}(H) \), let \( N_2 \) be a compact Lie group such that \( H \triangleleft N_2 \) and \( N_2/H \cong \mathbb{Z}_m \), and let \( W \) be an \( N_2 \)-representation such that \( \text{res}_{H}W \) is \( \chi \)-isotypical. For the natural inclusion \( i : \text{Iso}_{N_2}(W) \rightarrow \text{Iso}_H(W) \), the map \( i_* : \pi_1(\text{Iso}_{N_2}(W)) \rightarrow \pi_1(\text{Iso}_H(W)) \) is surjective.

*Proof.* As in [6.3], the character \( \chi \) is fixed by \( N_2 \). Let \( U \) be an irreducible \( H \)-representation whose character is equal to \( \chi \), and let \( \tilde{U} \) be an \( N_2 \)-extension of \( U \) whose existence is guaranteed by Theorem 14.1. Then, \( W \) can be decomposed as \( \bigoplus_{k \in \mathbb{Z}_m} l_k \tilde{U}_k \) for some nonnegative integers \( l_k \)'s by Corollary 14.2 where we denote by \( U_k \) the representation \( (\tilde{U} \otimes \Omega(k)) \). Then, Schur’s Lemma says that

\[
\text{Iso}_{N_2}(W) \cong \text{Iso}_{N_2}(l_0 \tilde{U}_0) \times \cdots \times \text{Iso}_{N_2}(l_{m-1} \tilde{U}_{m-1}),
\]

\[
\cong \text{GL}(l_0, \mathbb{C}) \times \cdots \times \text{GL}(l_{m-1}, \mathbb{C}),
\]

\[
\text{Iso}_{H}(W) \cong \text{GL}(l, \mathbb{C})
\]

where \( l = \sum l_k \). Since \( i \) is the natural inclusion from to \( \text{GL}(l_0, \mathbb{C}) \times \cdots \times \text{GL}(l_{m-1}, \mathbb{C}) \) to \( \text{GL}(l, \mathbb{C}) \), we obtain a proof. \( \square \)

Here, recall a notation.

**Notation 7.3.** Let \( X \) be a topological space. For two points \( y_0 \) and \( y_1 \) in \( X \) and a path \( \gamma : [0, 1] \rightarrow X \) such that \( \gamma(0) = y_0 \) and \( \gamma(1) = y_1 \), denote by \( \overline{\gamma} \) the function defined as

\[
\overline{\gamma} : \pi_1(X, y_0) \rightarrow \pi_1(X, y_1), \quad [\sigma] \mapsto [\gamma^{-1} \cdot \sigma \cdot \gamma].
\]

**Lemma 7.4.** Let \( X \) be a path connected topological space with an abelian \( \pi_1(X) \). Let \( A \) and \( B \) be path connected subspaces of \( X \). Also, let \( \iota_1 \) and \( \iota_2 \) denote inclusions from \( A \) and \( B \) to \( X \), respectively. Pick two points \( y_0 \in A, y_1 \in B \), and a path \( \gamma : [0, 1] \rightarrow X \) such that \( \gamma(0) = y_0 \) and \( \gamma(1) = y_1 \). Then, we have a one-to-one correspondence

\[
\Pi : \pi_1(X, y_1) / \left\{ \overline{\gamma_1, \pi_1(A, y_0)} + \iota_2 \pi_1(B, y_1) \right\} \rightarrow [0, 1], [0, 1]; X, A, B,
\]

\[
[\sigma] \mapsto [\gamma, \sigma].
\]
Proof. The only issue is well-definedness of $II$ and $II^{-1}$ which is just a tedious check. 

8. Equivariant clutching maps on one-dimensional fundamental domain

Assume that $\rho(G_\chi) = R$ for some finite $R$ in Table 1.1 such that $pr(R) = T, O, I$. In this section, we find conditions on a preclutching map $\Phi_{D_R}$ in $C^0(D_R, V_R)$ to guarantee that $\Phi_{D_R}$ be the restriction of an equivariant clutching map as promised in Section 4. By using this, we show that $\Omega \bar{\chi} or 2$ according to whether $\bar{\chi}$ is a vertex or not. Then, let $A_x$ be the set of equivariant pointwise clutching maps with respect to the $(G_\chi)_x$-bundle $(res^{G_\chi}_{G_\chi|x} F_{\bar{\chi}x})|_{E}^{-1}(x)$ for each $\bar{x} \in \bar{L}(x)$ and $x = |\bar{x}(\bar{x})$, and prove some lemmas on them. Then, we apply results of Section 6 in dealing with $\Omega \bar{\chi}$. Concrete calculation of homotopy of equivariant clutching maps for each case is done in the next section.

First, we define a set $A_x$ of equivariant pointwise clutching maps for each $\bar{x}$ in $|\bar{L}(x)|$. For each $\bar{x}$ in $|\bar{L}(x)|$ and $x = |\bar{x}(\bar{x})$, put $x = |\bar{x}(\bar{x})|$, and denote the identified points $\bar{x}$ in $|\bar{L}(x)|$ by $\bar{x}$. Then, let $A_x$ be the set of equivariant pointwise clutching maps with respect to the $(G_\chi)_x$-bundle $(res^{G_\chi}_{G_\chi|x} F_{\bar{\chi}x})|_{E}$, i.e. $N_2 = (G_\chi)_x$ and $F = (res^{G_\chi}_{G_\chi|x} F_{\bar{\chi}x})|_{E}$ in notations of Section 6. Here, we need to explain for codomain of maps in $A_x$. For each $\bar{x} \in |\bar{L}(x)|$ and $\psi_x \in A_x$, $\psi_x(\bar{x})$ is in

$$Iso\left(\left(F_{\bar{\chi}x}\right)_{x_1},\left(F_{\bar{\chi}x}\right)_{x_1+1}\right)$$

for $j \in Z_{j_R}$ or $Z_2$. If $\bar{x}_j \in |\bar{f}|$ and $\bar{x}_{j+1} \in |\bar{f}'|$ for some $\bar{f}$, $\bar{f}'$, then $\psi_x(\bar{x}_j)$ is henceforth regarded as in $Iso(V_f, V_f')$. This is justified because we have fixed the trivialization $(res^{G_\chi}_{G_\chi|x} F_{\bar{\chi}x})|_{E} = |\bar{f}| \times V_f$ for each face $\bar{f} \in K_R$. We define one more set $A_x$ of equivariant pointwise clutching maps for each edge $|e|$ and its images $\bar{e} = p_{\bar{L}(\bar{e})}$, $|e| = |\bar{e}(\bar{e})|$. For each $\bar{x}$ in $|\bar{e}|$ and $\bar{x} = p_{\bar{L}(\bar{e})}(\bar{x})$, put

$$x'_0 = \bar{x}, \quad x'_1 = p_{\bar{L}(\bar{e})}(|e|), \quad x' = \{x'_j|j \in Z_2\}.$$ 

Consider the set $A_x$ of equivariant pointwise clutching maps with respect to the $(G_x)|_{|e|}$-bundle $(res^{G_x}_{G_x|x} F_{\bar{\chi}x})|_{|e|}$ where $(G_x)|_{|e|}$ is the subgroup of $G_x$ fixing $|e|$, i.e. $N_2 = (G_x)|_{|e|}$ and $F = (res^{G_x}_{G_x|x} F_{\bar{\chi}x})|_{|e|}$ in notations of Section 6. As for $A_x$, each map $\psi_x$ in $A_x$ is considered to satisfy

$$\psi_x(x'_0) \in Iso(V_f, V_f') \quad and \quad \psi_x(x'_1) \in Iso(V_f', V_f)$$

when $x'_0 \in |\bar{f}|$ and $x'_1 \in |\bar{f}'|$. Here, observe that $A_x$ is in one-to-one correspondence with $A_y$ for any two $\bar{x}, \bar{y}$ in $|e|$, i.e. an element $\psi_y$ in $A_y$ and an element $\psi_y$ in $A_y$ are corresponded to each other when $\psi_x(x'_j) = \psi_y(y'_j)$ for $j \in Z_2$. This is because the $(G_x)|_{|e|}$-bundle $(res^{G_x}_{G_x|x} F_{\bar{\chi}x})|_{|e|}$ is all isomorphic regardless of $\bar{x} \in |e|$. It is very useful to identify all $A_x$’s for $\bar{x} \in |e|$ in this way, so we denote the identified set by $A_e$. That is, each element $\psi_x$ in $A_e$ is considered as contained in $A_x$ for any $\bar{x} \in |e|$ according to the context.

Next, we would define a $G_x$-action on saturations. First, we define notations on saturations. For each $x = |\bar{x}(\bar{x})$ and $\psi_x \in A_x$, denote saturations of $A_x$ and $\psi_x$ by $A_x$ and $\psi_x$, respectively. Since index set is irrelevant in defining $A_x$, the saturation depends not on $\bar{x}$ but on $x$. This is why we use the subscript $x$ instead of $\bar{x}$. For any $g \in G_x$, the function $g \cdot \psi_x$ is contained in $A_{gx}$ where $g \cdot \psi_x$ is defined as

$$(g \cdot \psi_x)(\bar{y}, \bar{y}') = g \psi_x(g^{-1} \bar{y}, g^{-1} \bar{y}')g^{-1}$$
for any $\bar{y} \neq \bar{y}'$ in $\pi^{-1}(gx)$. That is, we obtain $gA = \bar{A}_g$. Especially, it is easily shown that $g \cdot \bar{\psi}_x = \bar{\psi}_x$ for each $g \in (G_X)_x$ by equivariance of $\psi_x$. From this, it is noted that if $g'x = gx$ for some $g', g \in G_X$, then $g' \cdot \bar{\psi}_x = \bar{\psi}_x$ because $g' = g(g^{-1}g')$ with $g^{-1}g' \in (G_X)_x$, i.e. $g \cdot \bar{\psi}_x$ is dependent on $gx$. We have defined a $G_X$-action on saturations. Since $A_x$ and $\bar{A}_x$ are in one-to-one correspondence, $\mathbb{A}_x$'s also deliver the $G_X$-action induced from the $G_{\bar{X}}$-action on $A_{\bar{X}}$'s, i.e. $g \cdot \bar{\psi}_x \in \bar{A}_{\bar{X}}$ for each $\bar{\psi}_x \in \bar{A}_x$ is defined and $A_{\bar{X}} = gA_x$. Here, we prove a useful lemma on this action. Before it, we explain for the superscript $g$, and state an elementary fact.

**Definition 8.1.** Let $K$ be a closed subgroup of a compact Lie group $G$. For a given element $g \in G$ and $W \in \text{Rep}(K)$, the $gKg^{-1}$-representation $\vartheta W$ is defined to be the vector space $W$ with the new $gKg^{-1}$-action

$$gKg^{-1} \times W \rightarrow W; \quad (k, u) \mapsto g^{-1}kgu$$

for $k \in gKg^{-1}$, $u \in W$.

**Lemma 8.2.** Let $G$ be a compact Lie group acting on a topological space $X$. And, let $E$ be an equivariant vector bundle over $X$. Then, $\vartheta E_x \cong E_{gx}$ for each $g \in G$ and $x \in X$. Also,

$$\text{res}^G_{G'_x \cap G_x} E_x \cong \text{res}^G_{G'_x \cap G_x} E_{x'}$$

for any two points $x, x' \in X$.

By Lemma 8.2 and Lemma 8.4, $p_{\text{vect}}$ is well-defined in cases when $\text{pr}(R) = T, O, I$.

**Lemma 8.3.** For each $\bar{x} \in |\mathcal{L}_R|$, $x = |\pi(\bar{x})|$, $g \in G_X$, $\bar{\psi}_x \in A_x$, we have an $(G_X)_{gx}$-isomorphism

$$(\text{res}^G_{(G_X)_x} F_{V_{gx}})|_{gx}/g \cdot \bar{\psi}_x \cong \vartheta (\text{res}^G_{(G_X)_x} F_{V_{gx}})|_{gx}/\bar{\psi}_x).$$

**Proof.** Put $L = (g, (G_X)_x)$, and let $k_0 \in \mathbb{N}$ be the smallest natural number satisfying $g^{k_0} \in (G_X)_x$ where such a number exists because $R = p(G_X)$ is finite. To prove this lemma by using Lemma 8.2, we would construct an $L$-bundle $\bar{F}$ over the orbit $Lx = \{x, gx, \ldots, g^{k_0-1}x\}$. Put $\bar{F} = (\text{res}^G_L F_{V_x})|_{Lx}$ over $Lx$, and consider the nonequivariant bundle $F$ over the orbit $Lx$ whose fiber $F_{gx}$ at $g^kx$ is equal to $\bar{F}|_{g^kx}/(g^k \cdot \bar{\psi}_x)$ for $k = 0, \ldots, k_0 - 1$. And, let $P : \bar{F} \rightarrow F$ be the map such that the restriction of $P$ to $\bar{F}|_{g^kx}$ is equal to $p_{g^k \cdot \bar{\psi}_x}$ where $p_{g^k \cdot \bar{\psi}_x} : \bar{F}|_{g^kx} \rightarrow \bar{F}|_{g^kx}/(g^k \cdot \bar{\psi}_x)$ is the quotient map (6.2). Then, we would define an $L$-action on $\bar{F}$ as $lu = P(lu)$ for each $l \in L, u \in \bar{F}$, and any $\bar{u} \in P^{-1}(u)$ so that $P$ becomes $L$-equivariant. As long as this is well-defined, it is easily shown that it becomes an action because it is defined by the $L$-action on $\bar{F}$ through $P$. So, we would prove well-definedness of the action. For this, it suffices to show $P(l\bar{u}) = P(lu')$ for each $l \in L$ and each $\bar{u}, u'$ in $\bar{F}$ satisfying $P(\bar{u}) = P(u')$. If $\bar{u} \in (F_{V_x})_{g^kx}$ and $u' \in (F_{V_x})_{g^kx}$ for some $j, j'$, then $P(\bar{u}) = P(u')$ is written as

$$(g^k \cdot \bar{\psi}_x)(g^k \bar{x}_j, g^k \bar{x}_{j'}) \bar{u} = u'.$$

Note that $l\bar{u} \in (F_{V_x})_{g^kx}$ and $l\bar{u}' \in (F_{V_x})_{g^kx}$. And, put $l = g^k l'$ with $l' \in L_{g^kx}$ and some integer $k'$ so that $g^{k+k'}x = g^{k+k'}x$ because $l'$ fixes $g^kx$. Remembering that the restriction of $P$ to $\bar{F}|_{g^kx} = \bar{F}|_{g^k+k'x}$ is equal to $p_{g^k+k' \cdot \bar{\psi}_x}$, $P(l\bar{u}) = P(lu')$ is
shown as
\[(g^{k+k'} \cdot \tilde{\psi}_x)(l g^k \tilde{x}_j, l g^k \tilde{x}_j') l \tilde{u} = g^k (g^k \cdot \tilde{\psi}_x)(g^{-k'} l g^k \tilde{x}_j, g^{-k'} l g^k \tilde{x}_j') g^{-k'} l \tilde{u} = g^k l' l^{-1} (g^k \cdot \tilde{\psi}_x)(l' g^{k'} \tilde{x}_j, l' g^{k'} \tilde{x}_j') l \tilde{u} = g^{k'} l' (l'^{-1} g^{k'} \cdot \tilde{\psi}_x)(g^{k'} \tilde{x}_j, g^{k'} \tilde{x}_j') \tilde{u} = l (g^k \cdot \tilde{\psi}_x)(g^{k'} \tilde{x}_j, g^{k'} \tilde{x}_j') \tilde{u} = l \tilde{u}
\]

where we use (*) in the last line. So, we obtain well-definedness of \(L\)-action on \(\mathcal{F}\).
By definition, isotropy representations \(\mathcal{F}_x\) and \(\mathcal{F}_{\tilde{x}}\) are equal to representations
\[(\text{res}_{(G^x)_{\tilde{x}}}^G F_{\tilde{V} x})|_{/\tilde{\psi}_x} \text{ and } (\text{res}_{(G^x)_{\tilde{x}}}^G F_{\tilde{V} x})|_{/\tilde{\psi}_x}\]
respectively. Then, the lemma follows from Lemma 8.2.

To investigate \(\Omega_{\mu_0}^{(w_{\mu_0})\varepsilon_{i+1}}\), we need prove a basic lemma on relations between \(\mathcal{A}_{\varepsilon_1}\) and \(\mathcal{A}_{\varepsilon_1}^0\) for \(\tilde{x} = \overline{\varepsilon_1}\) or \(\tilde{\varepsilon}_{i+1}\). Also, we prove lemmas on evaluation of equivariant pointwise clutching maps.

**Lemma 8.4.** Put \(\psi^k = |x|^{-1}(v^k) = \{x_j | x_j = \overline{v}_j^k\text{ for }j \in \mathbb{Z}_{2j}\} \text{ for }k = i, i + 1\).

And, put \(v^i = \{v_0^i, v_1^i\}\) and \(v^{i+1} = \{v_0^{i+1}, v_1^{i+1}\}\) so that \(v_i, v_i^{i+1} \in c(\overline{\varepsilon}_i)\).

1. \(\mathcal{A}_x \subset \mathcal{A}_{\varepsilon_i}\) for each interior \(x\) in \(|\varepsilon_i|\).
2. \(\mathcal{A}_x^0 = \mathcal{A}_x^i\) for each interior \(\tilde{x} \neq b(\varepsilon_i)\) in \(|\varepsilon_i|\). Moreover, \(\mathcal{A}_{\varepsilon_i} = \mathcal{A}_{b(\varepsilon_i)}\) if \((G^x)_{b(\varepsilon_i)} = (G^x)_x\) for \(x = |\varepsilon|(\tilde{x})\).
3. \(\mathcal{A}_0 \subset \mathcal{A}_{\varepsilon_i}^0\) and \(\mathcal{A}_{\varepsilon_i}^{n-1} \subset \mathcal{A}_{\varepsilon_i}^0\).
4. For each \(\psi_v\) in \(\mathcal{A}_{\varepsilon_i}^0\), we have \(\psi_v(v_0^i) = \psi_v(v_1^i)\) for the unique \(\psi_v \in \mathcal{A}_{\varepsilon_i}^0\), and
\[
\text{res}_{(G^x)_{\varepsilon_i}}^G F_{\tilde{V} x}|_{/\psi_v} \cong (\text{res}_{(G^x)_{\varepsilon_i}}^G F_{\tilde{V} x})|_{/\psi_v^i}.
\]
5. For each \(\psi_{\varepsilon_{i+1}}\) in \(\mathcal{A}_{\varepsilon_{i+1}}\), we have \(\psi_{\varepsilon_{i+1}}(v_i^{i+1}) = \psi_{\varepsilon_{i+1}}(v_1^{i+1})\) for the unique \(\psi_{\varepsilon_{i+1}} \in \mathcal{A}_{\varepsilon_{i+1}}\), and
\[
\text{res}_{(G^x)_{\varepsilon_{i+1}}}^G F_{\tilde{V} x}|_{/\psi_{\varepsilon_{i+1}}} \cong (\text{res}_{(G^x)_{\varepsilon_{i+1}}}^G F_{\tilde{V} x})|_{/\psi_{\varepsilon_{i+1}}}.
\]

**Proof.** (1) follows from \((G^x)_{\varepsilon_i} \subset (G^x)_x\) when \(x = |\varepsilon|(\tilde{x})\). (2) follows from \((G^x)_{\varepsilon_i} = (G^x)_x\). Similarly, since \((G^x)_{\varepsilon_i} \subset G_x(v^n)\) and \((G^x)_{\varepsilon_i} \subset G_x(v^{n+1})\), (3) holds by Lemma 6.11 where \(G_x(v^n)\) and \(G_x(v^{n+1})\) are subgroups of \(G_x\) preserving \(v^n\) and \(v^{n+1}\), respectively. The first statement of (4) follows by (3) and Lemma 6.13. By Lemma 6.14, we have the second statement of (4). Similarly, (5) is also obtained.

**Lemma 8.5.** Assume that \(\text{pr}(R) = T, O, I\) and \(R \neq T \times Z\). For each vertex \(\tilde{v}\) in \(\tilde{K}_R\), the evaluation map \(\mathcal{A}_v \to \mathcal{A}_v^0\), \(\psi_v \mapsto \psi_v(\tilde{v})\) is homeomorphic.

**Proof.** Put \(v = |\varepsilon|(\tilde{v})\). In these cases, \(R_v \cong \mathbb{Z}_m\) or \(D_m\), with \(m = j_R\) by Table 3.4 and proof is done by Proposition 6.8 (2) and 6.9.

**Lemma 8.6.** Assume that \(R = T \times Z\). For each vertex \(\tilde{v}\) in \(\tilde{K}_R\), the evaluation map
\[
\mathcal{A}_v \to \mathcal{A}_v^0 \times \mathcal{A}_v^{-1}, \quad \psi_v \mapsto (\psi_v(v_0^i), \psi(v_1^{i+1}))
\]
is homeomorphic.

**Proof.** Put \(v = |\varepsilon|(\tilde{v})\). In these cases, \(R_v \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) by Table 3.4 and proof is done by Proposition 6.10.
Now, we state conditions on a preclutching map \( \Phi_{D_R} \) in \( C^0(\hat{D}_R, V_B) \) to guarantee that \( \Phi_{D_R} \) be the restriction of an equivariant clutching map. When \( R = T \times Z \), pick an element \( t_0 \in (G_x)_{(j-1)} \) such that \( t_0 v^0 = v^1 \). So, \( t_0 \) satisfies \( t_0 v^0_j = v^1_j \) and \( t_0 v^0_{j, +} = v^1_{j, +} \) for \( j \in \mathbb{Z}_4 \) as illustrated in Figure 12. Also, we define a terminology.

**Definition 8.7.** For \( \hat{x} \in |\hat{L}_R|, x = |\pi|(\hat{x}), \psi_x \in A_x \), \( \Phi \in C^0(|\hat{L}_R|, V_B) \), the map \( \psi_x \) or its saturation \( \bar{\psi}_x \) is called determined by \( \Phi \) if \( \Phi \) satisfies the following condition:

\[
\bar{\psi}_x \left( p_{|\hat{L}_R|}(\hat{x}), p_{|\hat{L}_R|}(|c|(|\hat{x}|)) \right) = \Phi(\hat{x})
\]

for each \( \hat{x} \in (|\pi| \circ p_{|\hat{L}_R|})^{-1}(x) \). The condition is concretely written as

1. if \( \hat{x} \) is a vertex, then
   \[
   \psi_x(\hat{x}_j) = \Phi(\hat{x}_j, +) \quad \text{and} \quad \psi_x^{-1}(\hat{x}_j) = \Phi(\hat{x}_j, -)
   \]
   for each \( j \in \mathbb{Z}_{|R|} \),
2. if \( \hat{x} \) is not a vertex, then
   \[
   \psi_x(\hat{x}_0) = \Phi(\hat{x}_0, +) \quad \text{and} \quad \psi_x(\hat{x}_1) = \Phi(|c|(\hat{x}_0, +)).
   \]

**Theorem 8.8.** Assume that \( \text{pr}(R) = T, O, I \). Then, a preclutching map \( \Phi_{D_R} \) in \( C^0(\hat{D}_R, V_B) \) is in \( \Omega_{\hat{D}_R, V_B} \) if and only if there exists the unique \( \psi_x \in A_x \) for each \( \hat{x} \in \hat{D}_R \) and \( x = |\pi|(\hat{x}) \) satisfying

E2. \( \psi_x \left( p_{|\hat{L}_R|}(\hat{x}), p_{|\hat{L}_R|}(|c|(|\hat{x}|)) \right) = \Phi_{D_R}(\hat{x}) \) for each \( \hat{x} \in \hat{D}_R \),

E3. for each \( \hat{x}, \hat{x}' \in \hat{D}_R \) and their images \( x = |\pi|(\hat{x}), x' = |\pi|(\hat{x}') \), if \( x' = gx \) for some \( g \in G_x \), then \( \psi_{x'} = g \cdot \psi_x \).

The set \( \left\{ \psi_x \right\}_{x \in D_R} \) is called determined by \( \Phi_{D_R} \).

**Proof.** For necessity, it suffices to construct a map \( \Phi \) in \( \Omega_{V_B} \) such that \( \Phi|_{D_R} = \Phi_{D_R} \).

For this, we would define \( \Phi \) on \( \hat{D}_R \), and then extend the domain of definition of \( \Phi \) to the whole \( |\hat{L}_R| \). First, we define \( \Phi \) on \( \hat{D}_R \) so that each \( \psi_x \) is determined by \( \Phi \), i.e. each \( \psi_x \) and \( \Phi \) satisfy Definition 8.7. Then, Condition E2. says that \( \Phi|_{\hat{D}_R} = \Phi_{D_R} \).

Next, we define \( \Phi(\hat{x}) = g^{-1} \Phi(g\hat{x})g \) for each \( \hat{x} \in |\hat{L}_R| \) and some \( g \in G_x \) such that \( g\hat{x} \) is in \( D_R \). We need prove well-definedness of this. Assume that \( \hat{y} = g\hat{x} \) and \( \hat{y}' = g'\hat{x} \) are in \( \hat{D}_R \) for two elements \( g, g' \) in \( G_x \) such that \( \hat{y} = g\hat{y}' \hat{y} \hat{g}' \hat{y} \). And, let \( y \) and \( y' \) be images of \( \hat{y} \) and \( \hat{y}' \) through \( |\pi| \circ p_{|\hat{L}_R|} \), respectively. Then, \( y' = g'g^{-1}y \). These give us \( \hat{y}' = (g'g^{-1}) \cdot \hat{y} \) by Condition E3. From this, we obtain

\[
g^{-1} \Phi(\hat{y})g' = g^{-1} \hat{y}' \left( p_{|\hat{L}_R|}(\hat{y}'), p_{|\hat{L}_R|}(|c|(|\hat{y}'|)) \right)g'
\]

\[
= g^{-1} \bar{\psi}_y \left( p_{|\hat{L}_R|}(\hat{y}), p_{|\hat{L}_R|}(|c|(|\hat{y}|)) \right)g
\]

\[
= g^{-1} \Phi(\hat{y})g
\]

where we use equivariance of \( |c| \). So, well-definedness is proved. It is easily checked that \( \Phi \) satisfies Condition N1., N2., E1. Therefore, \( \Phi \) is the wanted equivariant clutching map.

For sufficiency, assume that \( \Phi|_{\hat{D}_R} = \Phi|_{\hat{D}_R} \) for some \( \Phi \in \Omega_{V_B} \). Then, we should choose the unique \( \psi_x \in A_x \) for each \( \hat{x} \in \hat{D}_R \) satisfying Condition E2. and E3. When we show that \( \psi_x \)'s satisfy Condition E3., the condition \( x' = gx \) holds only if \( x', x \) are equal points or \( x' \) and \( x \) are different vertices by definition of one-dimensional fundamental domain. The second situation happens only in the case of \( R = T \times Z \) by Table 1.1, and in other cases \( D_R \) contains only one vertex. So, proof for sufficiency is different according to \( R \).

First, assume that \( R \neq T \times Z \). At each \( \hat{x} \in \hat{D}_R \), the unique \( \psi_x \) in \( A_x \) is determined by \( \Phi \) because \( \Phi \) satisfies Condition N1., N2., E1. Moreover, it can be checked that
ψ_x is the unique element satisfying Condition E2. for each \( \bar{x} \) by Lemma 8.5. And, as we have seen in the above, we do not need to consider Condition E3. in these cases.

Second, assume that \( R = T \times Z \). At each \( \bar{x} \in \bar{D}_R \), the unique \( \psi_x \) in \( A_x \) is determined by \( \Phi \) because \( \Phi \) satisfies Condition N1., N2., E1. Easily, these \( \psi_x \)'s satisfy Condition E2., E3. so that it remains to show their uniqueness. For \( \bar{x} \in \bar{D}_R - \{ \bar{0}, \bar{v}_1 \} \), \( \psi_x \) is the unique element satisfying Condition E2. by Lemma 6.4. Condition E2. says that \( \psi_x \) satisfies Condition E2. so that it remains to show their uniqueness. For \( \bar{x} \in \bar{D}_R - \{ \bar{0}, \bar{v}_1 \} \), \( \psi_x \) is the unique element satisfying Condition E2. by Lemma 6.4.

Formula (*) say that two values \( \psi_{\bar{0}}(\bar{v}_1) \) and \( \psi_{\bar{0}}^{-1}(\bar{v}_1) \) are determined by \( \Phi_{\bar{D}_R} \). And, Condition E3. says that \( \phi_{\bar{0}} \) is the unique element satisfying Condition E2. by Lemma 6.4. And, this means that \( \psi_{\bar{0}} \) is unique by Lemma 8.6. By Condition E3., uniqueness of \( \psi_{\bar{0}} \) is also obtained.

Remark 8.9. This theorem holds even though we might omit the word ‘unique’ in the statement of the theorem because uniqueness is not used in the proof of necessity.

By using Theorem 8.8. we would describe \( \Omega_{\bar{D}_R,V_B} \) through \( A_x \)'s. Define the following set of equivariant pointwise clutching maps on \( d^0 \)'s.

**Definition 8.10.** Denote by \( \bar{A}_{G_\chi}(S^2,V_B) \) the set

\[
\{ (\bar{\psi}_x)_{x \in I} \mid \bar{\psi}_x \in A_{\bar{x}} \text{ and } \bar{\psi}_{\bar{0}} = g \cdot \bar{\psi}_{\bar{0}} \text{ if } d^1 = gd^0 \text{ for some } g \in G_\chi \}.
\]

An element \( (\bar{\psi}_x)_{x \in I} \) in \( \bar{A}_{G_\chi}(S^2,V_B) \) is determined by \( \Phi_{\bar{D}_R} \in \Omega_{\bar{D}_R,V_B} \) if \( \psi_{\bar{0}} \)'s and \( \Phi_{\bar{D}_R} \) satisfy Condition E2. and E3. of Theorem 8.8. Also, a triple \( (W_{\bar{d}})_{i \in I} \) in \( A_{G_\chi}(S^2,\chi) \) is determined by \( (\bar{\psi}_x)_{x \in I} \) in \( \bar{A}_{G_\chi}(S^2,V_B) \) if \( W_{\bar{d}} \) is determined by \( \bar{\psi}_x \) with respect to \( (\res_{G_\chi}^{G_\chi}(\bar{V}_{B})) \) for each \( i \in I \).

By Theorem 8.8. we can see that for each \( \Phi_{\bar{D}_R} \in \Omega_{\bar{D}_R,V_B} \) there exists an element \( (\bar{\psi}_x)_{x \in I} \) in \( \bar{A}_{G_\chi}(S^2,V_B) \) which is determined by \( \Phi_{\bar{D}_R} \). In fact, it can be checked that this element is unique by the proof of Theorem 8.8.

**Corollary 8.11.** The set \( \Omega_{\bar{D}_R,V_B} \) is equal to the set

\[
\left\{ \Phi_{\bar{D}_R} \in C^0(\bar{D}_R,V_B) \mid \Phi_{\bar{D}_R}(\bar{x}) \in A_{\bar{x}}^0 \text{ for each } \bar{x} \in [\bar{0},\bar{v}_1], \text{ and } \Phi_{\bar{D}_R}(d^0_+) = \psi_{\bar{0}}(d^0), \Phi_{\bar{D}_R}(d^1_-) = \psi_{\bar{0}}^{-1}(d^1) \text{ for some } (\bar{\psi}_x)_{x \in I} \in \bar{A}_{G_\chi}(S^2,V_B) \right\}.
\]

**Proof.** To prove this corollary, we would rewrite Theorem 8.8. by using \( A_{\bar{x}} \) and \( \bar{A}_{G_\chi}(S^2,\chi) \). By Theorem 8.8. a preclutching map \( \Phi_{\bar{D}_R} \) is in \( \Omega_{\bar{D}_R,V_B} \) if and only if a set \( \Psi = (\bar{\psi}_x)_{x \in D_R} \) is determined by \( \Phi_{\bar{D}_R} \). As we have seen in the proof of the theorem, \( gx = x' \) with \( x \neq x' \) in Condition E3. is possible only when \( x \) and \( x' \) are \( d^0 \)'s (of course, more precisely when they are vertices). So, \( \Psi \) is determined by \( \Phi_{\bar{D}_R} \) if and only if \( (\bar{\psi}_x)_{x \in I} \) satisfies Condition E2. and E3. of Theorem 8.8. Here, \( (\bar{\psi}_x)_{x \in I} \) satisfies Condition E2. if and only if

\[
(\ast) \quad \Phi_{\bar{D}_R}(d^0_+) = \psi_{\bar{0}}(d^0) \quad \text{and} \quad \Phi_{\bar{D}_R}(d^1_-) = \psi_{\bar{0}}^{-1}(d^1).
\]

Lemma 6.3. says that \( (\ast) \) implies \( \Phi_{\bar{D}_R}(\bar{x}) \in A^0_{\bar{x}} \) for \( \bar{x} = d^0_+, d^1_- \). So, \( (\ast) \) could be redundantly rewritten as

\[
(\ast\ast) \quad \Phi_{\bar{D}_R}(d^0_+) = \psi_{\bar{0}}(d^0), \quad \Phi_{\bar{D}_R}(d^1_-) = \psi_{\bar{0}}^{-1}(d^1), \quad \text{and} \quad \Phi_{\bar{D}_R}(\bar{x}) \in A^0_{\bar{x}}
\]
for \( \hat{x} = \tilde{d}_i^+ \), \( \tilde{d}_i^- \). And, \((\psi_{\tilde{d}_i})_{i \in I}\) satisfies Condition E3. if and only if
\[
(***) \quad (\psi_{\tilde{d}_i})_{i \in I} \in \tilde{A}_G(S^2, V_B).
\]
Next, we deal with \( \psi_{\bar{\delta}} \)'s in \( (\psi_{\bar{\delta}})_{\bar{\delta} \in D_R} \). They satisfy Condition E2. if and only if \( \psi_{\bar{\delta}}(\bar{x}) = \Phi_{D_R}(\bar{x}^+)_i \) for each \( \bar{x} \in \bar{D}_R = \{ \bar{d}^i | i \in I \} \). And, this is satisfied if and only if \( \psi_{\bar{\delta}}(\bar{x}^+)_i \in \tilde{A}_0 \) = \( \tilde{A}_0^{(\bar{\delta})} \). and we have chosen \( \psi_{\bar{\delta}} \)'s such that \( \psi_{\bar{\delta}}(\bar{x}) = \Phi_{D_R}(\bar{x}^+)_i \) for each \( \bar{x} \). In summary, three conditions of this, (**), (***), (****) are equivalent conditions for \( \Psi \) to be determined by \( \Phi_{D_R} \). Therefore, we obtain a proof. \( \square \)

By using this corollary, we would show nonemptiness of \( \Omega_{D_R}(W_{\bar{d}})_{i \in I^+} \). For this, we need a lemma.

**Lemma 8.12.** For each \( (W_{\bar{d}})_{i \in I^+} \in A_G(S^2, \chi) \), if we put
\[
V_B = G_\chi \times_{(G_{\chi})_{d-i}} W_{d-i}, \quad F_B = G_\chi \times_{(G_{\chi})_{d-i}} ((\bar{f}^{-1}) \times W_{d-i}),
\]
then each \( A_{\bar{\delta}} \) for \( \bar{x} \in [\bar{d}^i, \bar{d}^i] \) is nonempty. And, we can pick an element \((\psi_{\bar{\delta}})_i \in I \) in \( A_G(S^2, V_B) \) which determines \( (W_{\bar{d}})_{i \in I^+} \).

**Proof.** For each \( i \in I \), put \( \kappa = |\pi|^{-1}(d^i) = (\bar{x}_j | \bar{x}_j = \tilde{d}_j^i \) for \( j \in Z_m \)) for \( m = j_R \) or 2. If we put \( F_i = (\text{res}_{(G_{\chi})_{d-i}}F_B)|_\chi \) and \( N_2 = (G_{\chi})_{\bar{\delta}} \), then \( N_2 = (G_{\chi})_{\bar{\delta}} \). Example 6.4 says that \( F_i \) satisfies Condition F1. or F2. Definitions of \( F_i \) and \( F_B \) say that
\[
(F_i)_{x_0} \cong \text{res}_{(G_{\chi})_{d-i}}|_{\bar{\delta}=1}(G_{\chi})_{d-i} W_{d-i}^{-1}
\]
because \( (G_{\chi})_{x_0} = (G_{\chi})_{[d-i, d]} = (G_{\chi})_{[d-i, d]} \) is equal to \( (G_{\chi})_{[d-i, d]} \) by Lemma 3.10. This implies
\[
(F_i)_{x_0} \cong \text{res}_{(G_{\chi})_{d-i}}|_{\bar{\delta}=1}(G_{\chi})_{d-i} W_{d-i}^{-1}
\]
by Definition 3.11 (4), i.e. \( W_{d-i} \) is a \( (G_{\chi})_{d-i} \)-extension of \( (F_i)_{x_0} \). So, Theorem 6.6 says that \( A_{\bar{\delta}} \) is nonempty. Moreover, we obtain nonemptiness of \( A_{\bar{\delta}} \) for \( \bar{x} \in [\bar{d}^i, \bar{d}^i] \) by Lemma 3.14. To prove the second statement, pick an element \( \tilde{\psi}_{\bar{\delta}} \) in \( A_{\bar{\delta}} \) which determines \( W_{d-i} \). If \( d^i = g \cdot d^i \) for some \( g \in G_{\chi}, \) the element \( g \cdot \tilde{\psi}_{\bar{\delta}} \) in \( A_{\bar{\delta}} \) satisfies
\[
F_i/g \cdot \tilde{\psi}_{\bar{\delta}} \cong g W_{\bar{\delta}} \cong W_{d-i}
\]
by Lemma 3.8 and Definition 3.11 (2). So, we may assume that \( \tilde{\psi}_{\bar{\delta}} = g \cdot \tilde{\psi}_{\bar{\delta}} \). And, \( W_{d-i} \cong V_{d-i} \) by definition of \( V_B \). Then, \((\psi_{\bar{\delta}})_i \in I \) is in \( A_G(S^2, V_B) \) which determines \( (W_{\bar{d}})_{i \in I^+} \). Therefore, we obtain a proof. \( \square \)

**Proposition 8.13.** For each \( (W_{\bar{d}})_{i \in I^+} \in A_G(S^2, \chi) \), the set \( \Omega_{D_R}(W_{\bar{d}})_{i \in I^+} \) is nonempty.

**Proof.** Put \( V_B = G_\chi \times_{(G_{\chi})_{d-i}} W_{d-i} \) so that \( \Omega_{D_R}(W_{\bar{d}})_{i \in I^+} \) is contained in \( \Omega_{D_R}(V_B) \). First, we would describe \( \Omega_{D_R}(W_{\bar{d}})_{i \in I^+} \) by using Corollary 8.11. By Lemma 8.12 we can pick an element \((\psi_{\bar{\delta}})_i \in I \) in \( A_G(S^2, V_B) \) which determines \( (W_{\bar{d}})_{i \in I^+} \). By Theorem 8.8 and Definition 8.10 each element \( \Phi_{D_R} \) in \( \Omega_{D_R}(W_{\bar{d}})_{i \in I^+} \) satisfies
\[
\Phi_{D_R}(d_i^+) = \psi_{\bar{\delta}}(d_i^+), \quad \Phi_{D_R}(d_i^-) = \psi_{\bar{\delta}}^{-1}(d_i^-)
\]
for some \((\psi_{\bar{\delta}})_i \in I \) in \( A_G(S^2, V_B) \), i.e. \((\psi_{\bar{\delta}})_i \in I \) is determined by \( \Phi_{D_R} \). Easily, \( \psi_{\bar{\delta}}^i \in (A_{\bar{\delta}})_{\bar{\delta}} \) for \( i \in I \) because \((\psi_{\bar{\delta}})_i \) should determine \( (W_{\bar{d}})_{i \in I^+} \). Moreover, Lemma 8.4 says that both \( \psi_{\bar{\delta}}(d_i^+) \) and \( \psi_{\bar{\delta}}^{-1}(d_i^-) \) are in \( (A_{\bar{\delta}})^0 \) which determine \( \text{res}_{(G_{\chi})_{d-i}} W_{\bar{\delta}} \) and \( \text{res}_{(G_{\chi})_{d-i}} W_{d-i} \), respectively. Since \( (G_{\chi})_{d-i} = (G_{\chi})_{d-i} \cap (G_{\chi})_{d} \) by Lemma 3.10 and these two representations are isomorphic by definition of \( A_G(S^2, \chi) \), Theorem 6.6 says that \( \psi_{\bar{\delta}}(d_i^+) \) and \( \psi_{\bar{\delta}}^{-1}(d_i^-) \) (of course, also \( \psi_{\bar{\delta}}(d_i^+) \) and \( \psi_{\bar{\delta}}^{-1}(d_i^-) \)) are in the same component \( (A_{\bar{\delta}})^0 \) of \( (A_{\bar{\delta}})^0 \) for some \( \psi_{\bar{\delta}} \in A_{\bar{\delta}} \).
Since \( \psi_{R}(d^{0}) \) and \( \psi_{R}(d^{3}) \) do exist, such a \( \psi_{\varnothing} \) exists and \( (A_{\varnothing})_{\psi_{\varnothing}} \) is nonempty. So, \( \Omega_{\hat{D}_{R},(W_{a}),i+1}^{0} \) is expressed as

\[
\Phi_{\hat{D}_{R}} \in C^{0}(\hat{D}_{R},V_{B}) \quad \text{for each } \hat{x} \in [\hat{d}^{0},\hat{d}^{3}], \text{and}
\]

\[
\Phi_{\hat{D}_{R}}(\hat{d}^{0}) = \psi_{\varnothing}^{\varnothing}(d^{0}), \quad \Phi_{\hat{D}_{R}}(\hat{d}^{3}) = \psi_{\varnothing}^{-1}(d^{3}) \quad \text{for some } (\psi_{\varnothing}^{\varnothing})_{i} \in \bar{A}_{G}(S^{2},V_{B})
\]

such that \( \psi_{\varnothing}^{\varnothing} \in (A_{\varnothing})_{\psi_{\varnothing}} \) for \( i \in I \).

However, we can construct a continuous function \( \Phi_{\hat{D}_{R}}' : [\hat{d}^{0},\hat{d}^{3}] \rightarrow (A_{\varnothing})_{\psi_{\varnothing}} \subset \text{Iso}(V_{f^{-1}},V_{f}) \) such that \( \Phi_{\hat{D}_{R}}'(\hat{d}^{0}) = \psi_{\varnothing}(d^{0}) \) and \( \Phi_{\hat{D}_{R}}'(\hat{d}^{3}) = \psi_{\varnothing}^{-1}(d^{3}) \) because \( \psi_{\varnothing}(d^{0}) \) and \( \psi_{\varnothing}^{-1}(d^{3}) \) are in the nonempty path connected \( (A_{\varnothing})_{\psi_{\varnothing}} \). Since \( \Phi_{\hat{D}_{R}}' \) is contained in the set \( \text{S.1} \), we obtain a proof.

\[ \square \]

9. Proof for cases when \( \text{pr}(\rho(G_{\chi})) = T, O, I \)

Now, we are ready to calculate homotopy of equivariant clutching maps.

**Proposition 9.1.** Assume that \( \text{pr}(R) = T, O, I \) and \( R \neq T, O, I \). Then, Theorem \( \text{C} \) holds for these \( R \)'s.

**Proof.** We prove the proposition only for the case of \( \rho(G_{\chi}) = T \times Z \). Proof for other cases are similar. We would show that \( \tau_{0}(\Omega_{\hat{D}_{R},(W_{a}),i+1})^{0} \) is one point set for each \( (W_{a})_{i} \in A_{G}(S^{2},\chi) \) by Proposition \( \text{S.3} \). Put

\[
V_{B} = G_{\chi} \times_{(G_{\chi})_{i+1}} W_{d^{-1}}, \quad FV_{B} = G_{\chi} \times_{(G_{\chi})_{i+1}} ([f^{-1}] \times W_{d^{-1}})
\]

for each \( (W_{a})_{i} \in A_{G}(S^{2},\chi) \).

By Proposition \( \text{S.13} \) we know that \( \tau_{0}(\Omega_{\hat{D}_{R},(W_{a}),i+1})^{0} \) is nontrivial. For two arbitrary \( \Phi_{\hat{D}_{R}} \) and \( \Phi_{\hat{D}_{R}}' \) in \( \Omega_{\hat{D}_{R},(W_{a}),i+1}^{0} \), let \( (\psi_{R})_{i} \in I \) and \( (\psi_{R}')_{i} \in I \) in \( \bar{A}_{G}(S^{2},V_{B}) \) be two elements determined by \( \Phi_{\hat{D}_{R}} \) and \( \Phi_{\hat{D}_{R}}' \) respectively. We would construct a homotopy connecting \( \Phi_{\hat{D}_{R}} \) and \( \Phi_{\hat{D}_{R}}' \) in \( \Omega_{\hat{D}_{R},(W_{a}),i+1}^{0} \). First, we show that we may assume that \( (\psi_{R})_{i} = (\psi_{R}')_{i} \) for \( i \in I \). Since \( \psi_{\varnothing} \) and \( \psi_{\varnothing}' \) for \( i \in I \) determine the same element \( \bar{W}_{B} \), these two are in the same path component of \( \tilde{A}_{B} \) by Theorem \( \text{S.9} \). Take paths \( \gamma_{i} : [0,1] \rightarrow \tilde{A}_{B} \) for \( i \in I \) such that \( \gamma_{i}(0) = \psi_{\varnothing}^{\varnothing} \) and \( \gamma_{i}(1) = \psi_{\varnothing} \).

In the case of \( R = T \times Z \), we have \( d^{0} = t_{0} \cdot d^{0} \) and \( t_{0} \cdot \gamma_{0}(0) = \psi_{\varnothing}^{\varnothing} \), and \( (t_{0} \cdot \gamma_{0})(1) = \psi_{\varnothing} \) by Definition \( \text{S.10} \). So, we may also assume that \( \gamma_{i} = t_{0} \cdot \gamma_{0}^{i} \). Recall that the parametrization on \( [\bar{d}^{0}] = [\bar{d}^{0}_{\varnothing},\bar{d}^{1}_{\varnothing}] \) by \( s \in [0,1] \) satisfies \( \bar{v}_{+}^{0} = 0, \bar{b}(\bar{d}^{0}) = 1/2, \bar{v}_{-}^{1} = 1 \). We construct a homotopy \( L(s,t) : [\bar{d}^{0},\bar{d}^{1}] \times [0,1] \rightarrow A_{G}(S^{2},V_{B}) \) as

\[
L(s,t) = \gamma_{i}^{i}(1 - 3s)t(\bar{d}^{0}) \quad \text{for } s \in [0,\frac{1}{3}],
\]

\[
L(s,t) = \varphi^{t}(3s - 1) \quad \text{for } s \in [\frac{1}{3},\frac{2}{3}],
\]

\[
L(s,t) = \gamma_{i+1}^{i+1}(3s - 2)t^{-1}(\bar{d}^{1+1}) \quad \text{for } s \in [\frac{2}{3},1].
\]

Then, \( L \) for each \( t \in [0,1] \) is in \( \Omega_{\hat{D}_{R},(W_{a}),i+1}^{0} \) corollary \( \text{S.11} \) and \( L \) connects \( \Phi_{\hat{D}_{R}} \) with \( L_{1} \) in \( \Omega_{\hat{D}_{R},(W_{a}),i+1}^{0} \) which determines \( (\psi_{\varnothing})_{i} \). So, if we put \( \Phi_{\hat{D}_{R}}' = L_{1} \), then we may assume that \( \Phi_{\hat{D}_{R}} \) and \( \Phi_{\hat{D}_{R}}' \) determine the same element \( (\psi_{\varnothing})_{i} \) in \( \bar{A}_{G}(S^{2},V_{B}) \).

Now, we construct a homotopy between \( \Phi_{\hat{D}_{R}} \) and \( \Phi_{\hat{D}_{R}}' \). As in the proof of Proposition \( \text{S.10} \), \( \psi_{\varnothing}(d^{0}) \) and \( \psi_{\varnothing}^{-1}(d^{3}) \) are in the same component \( (A_{\varnothing})_{\psi_{\varnothing}}^{0} \) for some element \( \psi_{\varnothing} \) in \( A_{\varnothing} \), and \( \Phi_{\hat{D}_{R}}, \Phi_{\hat{D}_{R}}' \) have values in \( (A_{\varnothing})_{\psi_{\varnothing}}^{0} \). Here, note that \( (A_{\varnothing})_{\psi_{\varnothing}}^{0} \) is
simply connected by Lemma 3.8 and Proposition 8.8 (3). By simply connectedness, we can obtain a homotopy $L'(s, t) : [d^1, d^-1] \times [0, 1] \to A^0_{δ}$ as

$$L'(s, t) = \Phi_{δ_R}(s), \quad L'(s, 0) = \Phi_{δ_R}(s), \quad L'(s, 1) = \Phi_{δ_R}(s)$$

for $s, t \in [0, 1]$. Then, $L'$ connects $Φ_{δ_R}$ and $Φ_{δ_R}$ in $Ω_{δ_R(W d_i)_{λ}}$ by Corollary 8.11. Therefore, we obtain a proof. Here, we remark that simply connectedness is critical in obtaining $L'$.

Proposition 9.2. Assume that $R$ is equal to one of $T$, $O$, $I$. Then, Theorem 4 holds for these cases.

Proof. When $ρ(G, X)$ is given, put

$$V_R = G \times (G_{\gamma})_{λ-1}, \quad V_{FR} = G \times (G_{\gamma})_{λ-1} \times (\bar{f}^{-1} \times W_d)$$

for each $(W_d)_{λ} \in A_{G_{\gamma}}(S^2, X)$. We can pick an element $(ψ_i)_{λ} \in A_{G_{\gamma}}(S^2, V_R)$ which determines $(W_d)_{λ}$ by Lemma 8.12. In these cases, we have

$$D_R = [ψ^0, b(ψ^0)], \quad (G)(ψ_{λ})_{σ}/H \cong Z_{ijn}, \quad (G)(b(ψ^0))/H \cong Z_{ijn}$$

by Table 3.4. By using these, path components of $A_{ρ, ψ_{λ}}$, $A_{b(ψ^0)}$ are simply connected by Proposition 6.3 (3) and we have $A^0_{b(ψ^0)} = Iso_H(V_{f^1}, V_{f^0})$ by Lemma 6.4. Here, $A_{ρ, ψ_{λ}}$, $A_{b(ψ^0)}, A_{ρ}$ are all nonempty by Lemma 8.12. Since $d^1 \neq d^0$ for any $g \in G, ψ^0$ and $ψ'_{λ}$ in the set 8.11 have no relation. So, the set (8.11) is rewritten as

$$\left\{ Φ_{δ_R} \in C^0(€, V_{R}), \quad Φ_{δ_R}(ϕ) \in Iso_H(V_{f^1}, V_{f^0}) \right. \quad \text{for each } ϕ \in [ψ^0, b(ψ^0)],$$

and $Φ_{δ_R}(ψ^0) \in (A_{ρ, ψ_{λ}})_{ψ_{λ}}^0, Φ_{δ_R}(b(ψ^0)) \in (A_{b(ψ^0)})_{ψ_{λ}}^0 \left\} \right.$

Since path components $(A_{ρ, ψ_{λ}})_{ψ_{λ}}^0$ and $(A_{b(ψ^0)})_{ψ_{λ}}^0$ are simply connected, the homotopy $π_0(Ω_{δ_R(W d_i)_{λ}})$ is in one-to-one correspondence with the homotopy $π_1(Iso_H(V_{f^1}, V_{f^0}))$ by Lemma 4.1 and the injection $i_0$ from $Ω_0$ to $Ω_{δ_R(W d_i)_{λ}}$ induces the bijection

$$π_0(ω) : π_0(Ω_0) \to π_0(Ω_{δ_R(W d_i)_{λ}})$$

where $Ω_0$ is defined as

$$\left\{ Φ_{δ_R} \in C^0(€, V_{R}), \quad Φ_{δ_R}(ϕ) \in Iso_H(V_{f^1}, V_{f^0}) \right. \quad \text{for each } ϕ \in [ψ^0, b(ψ^0)],$$

and $Φ_{δ_R}(ψ^0) = ψ_{λ}(ψ^0)$, $Φ_{δ_R}(b(ψ^0)) = ψ_{λ}(b(ψ^0)) \right\}.$

Now, we would show that Chern classes of equivariant vector bundles determined by different classes in $π_0(Ω_{δ_R(W d_i)_{λ}})$ with respect to $V_R$ are all different by calculating Chern class. For this, we use the parametrization on $L_R$ introduced in Section 4. For notational simplicity, put $* = ψ_{λ}(ψ^0)$ and $* = ψ_{λ}(b(ψ^0))$. Let $γ_0 : [0, 1/2] \to Iso_H(V_{f^1}, V_{f^0})$ be an element of the set $Ω_0$ such that $γ_0(1/2) = *$ and $γ_0(0) = *$. Also, let $σ_0 : [0, 1/2] \to Iso_H(V_{f^1}, V_{f^0})$ be a loop such that $σ_0(1/2) = *$ and $[σ_0]$ is a generator of $π_1(Iso_H(V_{f^1}, V_{f^0}), *).$ Here, we introduce some notations.

Notation 9.3. For some $H$-representations $W_1$ and $W_2$, let $X_1$ and $X_2$ be two spaces $Iso_H(W_1, W_2)$ and $Iso_H(W_2, W_1)$, respectively. For paths $δ_1, δ_2 : [a, b] \to X_1$ satisfying $δ_1(b) = δ_2(a)$, denote by $δ_1, δ_2 : [a, b] \to X_1$ the path defined by

$$\left\{ (δ_1, δ_2)(t) = \begin{cases} δ_1(a + 2(t - a)), & t \in [a, \frac{a + b}{2}], \\ δ_2(a + 2(t - a + b)), & t \in [\frac{a + b}{2}, b]. \end{cases} \right\}.$$
For paths $\delta_3 : [a, b] \to X_1$ and $\delta_4 : [b, c] \to X_1$ satisfying $\delta_3(b) = \delta_4(b)$, denote by $\delta_1 \vee \delta_2 : [a, c] \to X_1$ the path defined by

$$(\delta_3 \vee \delta_4)(t) = \begin{cases} \delta_3(t), & t \in [a, b], \\ \delta_4(t), & t \in [b, c]. \end{cases}$$

Also, for a path $\delta : [a, b] \to X_1$ denote by $\delta^* : [1 - b, 1 - a] \to X_2$ the path $\delta^*(t) = \delta(1 - t)^{-1}$.

Note that $\sigma_0, \gamma_0 \in \Omega_0 \subset \Omega_{D_R, (W_{\phi})_1^{+}}$. We would show that the difference between $c_1(F_{V_{\hat{h}}}/\sigma_0, \gamma_0)$ and $c_1(F_{V_{\hat{h}}}/\gamma_0)$ is $12\chi(\text{id})$, $24\chi(\text{id})$, $60\chi(\text{id})$ according to $R = T$, $O$, $1$, respectively. Here, $l_{R} = 12, 24, 60$ because the number of edges of $K_T, K_O, K_1$ is $6, 12, 30$ and $D_R$ is a half of an edge at each case, respectively. For the calculation, we need to describe the equivariant clutching maps precisely. Let $\Phi = \cup \varphi_\hat{e}$ in $\Omega_{D_R, (W_{\phi})_1^{+}}$ be the extension of $\gamma_0$, i.e. $\varphi_\hat{e}(t) = \gamma_0(t)$ for $t \in [0, 1/2]$. By Condition N1, $\varphi_{e(\hat{e}^0)}(t) = \gamma_0(1 - t)^{-1}$ for $t \in [1/2, 1]$. Pick an element $c_0$ of $(G_X)_{b(\hat{e}^0)}$ such that $c_0b(\hat{e}^0) = b(e(\hat{e}^0))$. By equivariance of $\Phi$,

$$
\varphi_{e(\hat{e}^0)}(t) = \Phi(t) = c_0\Phi(e_0^{-1}t)e_0^{-1} = c_0\varphi_\hat{e}(e_0^{-1}t)e_0^{-1} = c_0\gamma_0(t)e_0^{-1}
$$

for $t \in [0, 1/2]$. Again by Condition N1,

$$
\varphi_\hat{e}(t) = \varphi_{e(\hat{e})}(1 - t)^{-1} = c_0\gamma_0(1 - t)^{-1}e_0^{-1}
$$

for $t \in [1/2, 1]$. This gives $\varphi_\hat{e} = \gamma_0 \vee c_0\gamma_0^*c_0^{-1}$ and $\varphi_{e(\hat{e})} = c_0\gamma_0^*c_0^{-1} \vee \gamma_0^*$. Let $\Phi' = \cup \varphi'_\hat{e}$ be the extension of $\sigma_0, \gamma_0$. Then,

$$
\varphi'_\hat{e} = \sigma_0, \gamma_0 \vee c_0\gamma_0^*c_0^{-1} \text{ and }
\varphi'_{e(\hat{e})} = c_0\sigma_0, \gamma_0^*c_0^{-1} \vee \gamma_0^*c_0^{-1}
$$

by using $(\sigma_0, \gamma_0)^* = \gamma_0^*c_0^{-1}$. Replacing only $\varphi_\hat{e}$ and $\varphi_{e(\hat{e})}$ of $\Phi$ with $\varphi'_\hat{e}$ and $\varphi'_{e(\hat{e})}$, we obtain a new nonequivariant clutching map, say $\Phi_1$. Here, $\Phi_1$ becomes an nonequivariant clutching map because $\Phi_1 = \Phi$ on vertices of $\hat{L}_R$. The difference $c_1(F_{V_{\hat{h}}}/\Phi_1) - c_1(F_{V_{\hat{h}}}/\Phi)$ is equal to $\pm 2\chi(\text{id})$ (say $+ 2\chi(\text{id})$) by Lemma 7.1 because $\varphi'_{e(\hat{e})}$ contains two generators $\sigma_0$ and $c_0\sigma_0^*c_0^{-1}$. We would repeat this argument for other edges. Pick edges $\hat{e}$ and $e(\hat{e})$ such that $\{\hat{e}, e(\hat{e})\} \neq \{\hat{e}^0, e(\hat{e}^0)\}$. Then, there exists the unique $g_0 \in G_X$ up to $H$ such that $g_0e(\hat{e}^0) = \hat{e}$. By equivariance,

$$
\varphi_\hat{e} = g_0 \cdot \varphi_\hat{e} = g_0 \cdot (\gamma_0 \vee c_0\gamma_0^*c_0^{-1}) = g_0\gamma_0g_0^{-1} \vee g_0c_0\gamma_0^*c_0^{-1}g_0^{-1},
$$

and

$$
\varphi'_{e(\hat{e})} = g_0\sigma_0, \gamma_0^*c_0^{-1} \vee g_0c_0\sigma_0^*c_0^{-1}g_0^{-1}.
$$

Replacing only $\varphi_\hat{e}$ and $\varphi_{e(\hat{e})}$ of $\Phi_1$ with $\varphi'_\hat{e}$ and $\varphi'_{e(\hat{e})}$, we obtain a new nonequivariant clutching map, say $\Phi_2$, is obtained, and the difference $c_1(F_{V_{\hat{h}}}/\Phi_2) - c_1(F_{V_{\hat{h}}}/\Phi_1)$ is also equal to $2\chi(\text{id})$ because $g_0$ preserves the orientation. In this way, if we replace all $\varphi_\hat{e}$ of $\Phi$ with $\varphi'_\hat{e}$ to obtain $\Phi'$, then the difference $c_1(F_{V_{\hat{h}}}/\Phi') - c_1(F_{V_{\hat{h}}}/\Phi)$ is equal to $l_R\chi(\text{id})$. Proposition 5.3 and these calculations prove the proposition. □ □
10. Equivariant clutching maps when \( \text{pr}(\rho(G_X)) = \mathbb{Z}_n, D_n \)

Assume that \( \rho(G_X) = R \) for some finite \( R \) in Table 1.1 such that \( \text{pr}(R) = \mathbb{Z}_n, D_n \). We redefine notations introduced for cases of \( \text{pr}(R) = T, O, I \) to be in accordance with these cases, and mimic what we have done in Section 4, 5, 8 with those notations. In our treatment of cases when \( R = \mathbb{Z}_n, (a_n, -b) \), there are some differences with other cases which are caused by existence of fixed points.

First, we rewrite Section 4. In these cases, \( K_R = K_{m_R} \) with \( m_R = n/2, n, 2n \) by Table 1.1. Denote by \( K_{R,S} \) the lower simplicial subcomplex of \( K_R \) whose underlying space \( |K_{R,S}| \) is equal to \( |K_R| \cap \{(x, y, z) \in \mathbb{R}^3 | z \leq 0\} \), and by \( K_{R,N} \) the upper part. Let \( L_R \) be the subcomplex \( K_{R,S} \cap K_{R,N} \) of \( K_R \) lying on the equator \( z = 0 \). Put \( B = \{S, N\} \subset |K_R| \) on which \( R \) (and \( G_X \)) acts. Since each \( |K_{R,q}| \) for \( q \in B \) has a simple equivariant structure for \( R_q \), we consider \( K_R \) as the union of two pieces \( K_{R,q} \) for \( q \in B \), not of all faces of \( K_R \). Denote by \( \bar{K}_R \) the disjoint union \( \coprod_{q \in B} K_{R,q} \), and denote by \( \bar{K}_{R,q} \) the subcomplex \( K_{R,q} \) of \( K_R \) so that \( |\bar{K}_R| = \coprod_{q \in B} |K_{R,q}| \). We denote simply by \( \pi \) and \( |\pi| \) natural quotient maps from \( \bar{K}_R \) and \( |\bar{K}_R| \) to \( K_R \) and \( |K_R| \), respectively. Let \( \bar{L}_R = \pi^{-1}(L_R) \), and let \( \bar{L}_{R,q} \) be the subcomplex \( L_R \cap \bar{K}_{R,q} \) of \( \bar{K}_R \) for \( q \in B \). The subset \( |\pi|^{-1}(B) \) in \( |\bar{K}_R| \) is often confused with \( B \) in \( |K_R| \). The injection from \( \bar{L}_R \) to \( \bar{K}_R \) and its underlying space map are denoted by \( \iota_{\bar{L}} \) and \( \iota_{\bar{K}} \), respectively. The \( G_X \)-actions on \( K_R, |\bar{K}_R| \) induce \( G_X \)-actions on \( L_R, K_R, \bar{L}_R \), and their underlying spaces. We do not need define \( \bar{L}_R \). Here, we introduce the following notations:

\[
\begin{align*}
\bar{e}_q^i &= \bar{L}_{R,q} \cap \pi^{-1}(e_q^i), \\
\bar{e}_q^j &= \bar{L}_{R,q} \cap \pi^{-1}(e_q^j), \\
\bar{D}_R &= |\bar{L}_{R,S}| \cap |\pi|^{-1}(D_R), \\
\bar{D}_R &= |\pi|^{-1}(D_R), \\
\bar{d}_R^j &= |\bar{L}_{R,S}| \cap |\pi|^{-1}(d_R^j)
\end{align*}
\]
for \( q \in B, \ i \in \mathbb{Z}_{m,n}, \ i' \in I \). Some notations are illustrated in Figure 10.1. Also, let \( c : \mathcal{L}_R \to \mathcal{L}_R \) be the simplicial map whose underlying space map \( |c| : |\mathcal{L}_R| \to |\mathcal{L}_R| \) satisfies
\[
\bar{x} \neq |c|(\bar{x}) \quad \text{and} \quad |\pi|(\bar{x}) = |\pi|(|c|(\bar{x}))
\]
for each \( \bar{x} \in |\mathcal{L}_R| \). And, put \( \bar{x}_0 = \bar{x} \) and \( \bar{x}_1 = |c|(\bar{x}) \) for each \( \bar{x} \in |\mathcal{L}_R| \). Now, we describe an equivariant vector bundle over \( |\mathcal{K}_R| \) as equivariant clutching construction by using an equivariant vector bundle over \( |\mathcal{K}_R| \). Pick a \( G_\chi \)-vector bundle \( V_B \) over \( B \) such that \( (\text{res}^G_{H}|B)|_q \) is \( \chi \)-isotypical at each \( q \) in \( B \). We denote by \( V_q \) the \( \chi \)-isotypical at each \( q \) in \( B \). Then, we have
\[
V_B \cong G_\chi \times_{(G_\chi)_q} V_S \quad \text{and} \quad \text{res}^G_{H}|V_B) \cong \text{res}^G_{H}|V_S)
\]
when \( R \neq \mathbb{Z}_n, \langle a_n, -b \rangle \). Define \( \text{Vect}_{G_\chi}(|\mathcal{K}_R|, \chi)V_B \) and \( \text{Vect}_{G_\chi}(|\mathcal{K}_R|, \chi)V_B \) as before. We observe that \( \text{Vect}_{G_\chi}(|\mathcal{K}_R|, \chi)V_B \) has the unique element \( |F_{V_B}| \) for the bundle
\[
F_{V_B} = \left\{ \bigsqcup_{q \in B} |\mathcal{K}_R,q| \times V_q \right\} \quad \text{if} \ R = \mathbb{Z}_n, \langle a_n, -b \rangle, \quad \text{and} \quad \left\{ \bigsqcup_{q \in B} |\mathcal{K}_R,q| \times V_q \right\} \quad \text{if} \ R \neq \mathbb{Z}_n, \langle a_n, -b \rangle.
\]
Henceforward, we use trivializations
\[
|\mathcal{K}_R,q| \times V_q \quad \text{for} \ \left( \text{res}^G_{H}|(G_\chi)_q) F_{V_B}\right)|\mathcal{K}_R,q)
\]
\[
|\mathcal{K}_R,q| \times V_q \quad \text{and} \quad \left( \text{res}^G_{H}|(G_\chi)_q) F_{V_B}\right)|\mathcal{K}_R,q)
\]
for each \( q \) and \( i \). Then, each \( E \) in \( \text{Vect}_{G_\chi}(|\mathcal{K}_R|, \chi)V_B \) can be constructed by gluing \( F_{V_B} \cong |\pi|^*E \) along \( |\mathcal{L}_R,q| \)'s through
\[
|\mathcal{L}_R,q| \times V_q \to |\mathcal{L}_R,q'| \times V_{q'}, \quad (\bar{x}, u) \mapsto ((|c|(\bar{x}), \varphi_q(\bar{x}))u)
\]
via continuous maps
\[
\varphi_q : |\mathcal{L}_R,q| \to \text{Iso}(V_q, V_{q'})
\]
for \( \bar{x} \in |\mathcal{L}_R,q|, \ u \in V_q, \ B = \{q, q'\} \) as we have done in Section 4. The union \( \Phi = \bigcup_{q \in B} \varphi_q \) is called an equivariant clutching map of \( E \) with respect to \( V_B \). This construction of \( E \) is denoted by \( F_{V_B}/\Phi \). The map \( \Phi \) is defined on \( |\mathcal{L}_R| \), and we also regard \( \Phi \) as a map
\[
i^*_{|\mathcal{L}_B}\Phi_{V_B} \to \iota^*_{|\mathcal{L}_B}\Phi_{V_B}, \quad (\bar{x}, u) \mapsto (|c|(\bar{x}), \Phi(\bar{x}))u
\]
by using trivialization \( 10.2 \) for each \( q \in B, \ (\bar{x}, u) \in |\mathcal{L}_R,q| \times V_q \). Also, \( \Phi \) should be equivariant. An equivariant clutching map of some bundle in \( \text{Vect}_{G_\chi}(|\mathcal{K}_R|, \chi)V_B \) with respect to \( V_B \) is called simply an equivariant clutching map with respect to \( V_B \), and let \( \Omega_{V_B} \) be the set of all equivariant clutching maps with respect to \( V_B \).
And, if we define \( \Omega_{D_{R,V_B}} \) as the set
\[
\{ \Phi|_{D_{R,V_B}} \mid \Phi \in \Omega_{V_B} \}
\]
then the restriction map \( \Omega_{V_B} \to \Omega_{D_{R,V_B}} \) is bijective, and \( \pi_0(\Omega_{V_B}) \cong \pi_0(\Omega_{D_{R,V_B}}) \).
We call an equivariant clutching map \( \Phi \) the extension of \( \Phi|_{D_{R}} \). And, denote also by \( F_{V_B}/\Phi|_{D_{R}} \) the bundle \( F_{V_B}/\Phi \). Let \( C^0(|\mathcal{L}_R|, V_B) \) be the set of functions \( \Phi \) defined on \( |\mathcal{L}_R| \) satisfying \( \Phi|\mathcal{L}_R,q\) for each \( \bar{x} \in |\mathcal{L}_R,q| \) and \( B = \{q, q'\} \). We can define the quotient \( F_{V_B}/\Phi \) for any \( \Phi \) in \( C^0(|\mathcal{L}_R|, V_B) \). And, let \( C^0(|\mathcal{L}_R|, V_B) \) be the set
\[
\{ \Phi|_{D_{R}} \mid \Phi \in C^0(|\mathcal{L}_R|, V_B) \}
\]
A function \( \Phi \) in \( C^0(|\mathcal{L}_R|, V_B) \) or a function \( \Phi|_{D_{R}} \) in \( C^0(|\mathcal{L}_R|, V_B) \) is called a pre-clutching map with respect to \( V_B \). A pre-clutching map \( \Phi \) in \( C^0(|\mathcal{L}_R|, V_B) \) is an equivariant clutching map with respect to \( V_B \) if and only if it satisfies the following conditions:
\[
N' \ : \ \Phi(|c|(\bar{x})) = \Phi(\bar{x})^{-1} \quad \text{for each} \ \bar{x} \in \mathcal{L}_R,
\]
E1'. $\Phi(\bar{x}) = g\Phi(x)g^{-1}$ for each $\bar{x} \in |\mathcal{L}_R|, g \in G_\chi$.

More precisely, if $\Phi$ satisfies Condition N1', then the quotient $F_{V_B}/\Phi$ becomes a nonequivariant vector bundle. And, if $\Phi$ also satisfies Condition E1', then $F_{V_B}/\Phi$ becomes an equivariant vector bundle, i.e. $\Phi$ is an equivariant clutching map with respect to $V_B$.

Second, we rewrite Section 5. We obtain Lemma 5.1 and 5.2 by replacing $p_{|\zeta|$ with $j_{|\zeta|}$. Define $v_0, p_{\alpha}, p_{\beta}$ as before by replacing $\Omega_{D_R, V_B}$ with $\Omega_{D_R, V_B}$. We decompose $\Omega_{D_R, V_B}$. When $R = Z_n$, $(a_n, -b)$, denote a pair $(W_S, W_X)$ in $A_G(S^2, \chi)$ by $(W_q)e \in B$. And, pick the $G_\chi$-vector bundle $V_B$ over $B$ such that $V_B|_{q} = W_q$ for each $(W_q)e \in B \in A_G(S^2, \chi)$ and $q \in B$. Define $\Omega_{D_R, (W_q)e} = (p_{\alpha} \circ p_{\beta})^{-1}(\langle W_q \rangle)$. Then, we obtain $\Omega_{D_R, V_B} = \Omega_{D_R, (W_q)e}$ because $V_B$ and $(W_q)e \in B$ can be regarded as the same $G_\chi$-bundle over $B$. When $R \neq Z_n$, $(a_n, -b)$, put

$$V_B = G_\chi \times_{(G_\chi)_d} W_d^{-1}, \quad F_{V_B} = G_\chi \times_{(G_\chi)_d} ([f^{-1}] \times W_d^{-1})$$

for each $(W_q)e \in A_G(S^2, \chi)$. Then, we obtain the decomposition of $\Omega_{D_R, V_B}$ by replacing $\Omega_{D_R, (W_q)e}^{-1}$ of 5.3 with $\Omega_{D_R, (W_q)e}^{-1}$. Proposition 5.3 holds if

1. we replace $(W_q)e \in \Omega_{D_R, V_B}$ with $(W_q)e \in B$, $\Omega_{D_R, (W_q)e}$ when $R = Z_n$, $(a_n, -b)$, respectively.

2. we replace $\Omega_{D_R, (W_q)e}^{-1}$ with $\Omega_{D_R, (W_q)e}^{-1}$, when $R \neq Z_n$, $(a_n, -b)$.

Third, we rewrite Section 8. To begin with, we define some sets of equivariant pointwise clutching maps. For each $\bar{x} \in |\mathcal{L}_R|$ and $x = |\pi|(|\bar{x}|)$, let $\mathfrak{x} = |\pi|^{-1}(x) = \{\bar{x}_j|j \in Z_2\}$, and let $A_x$ be the set of equivariant pointwise clutching maps with respect to $(G_\chi)_e$-bundle $F_{V_B}|_{\mathfrak{x}}$. Also for each edge $\bar{e}$ in $|\mathcal{L}_R|, \pi(\bar{e}) = e, \bar{x} \in |\mathcal{L}_R|$, put $\mathfrak{x} = |\pi|^{-1}(x) = \{\bar{x}_j|j \in Z_2\}$, and let $A_e$ be the set of equivariant pointwise clutching maps with respect to $(G_\chi)_e$-bundle $F_{V_B}|_{\mathfrak{x}}$, where $A_e$'s for different $\bar{x}$'s are identified as in Section 8. We can define saturations on $A_x$ and elements of it, and also group actions on saturations. Evaluation maps $A_x \to A^0_x$ and $A_e \to A^0_e$ are bijective for any $\bar{x} \in |\mathcal{L}_R|, \bar{e} \in \mathcal{L}_R$. With these, we can rewrite Theorem 8.11

**Theorem 10.1.** Assume that $pr(R) = Z_n, D_n$. Then, a preclutching map $\Phi_{D_R}$ in $C^0(D_R, V_B)$ is in $\Omega_{D_R, V_B}$ if and only if there exists the unique $\psi_x \in A_x$ for each $\bar{x} \in \mathcal{D}_R$ satisfying the following conditions:

E2'. $\psi_x(\bar{x}) = \Phi_{D_R}(\bar{x})$ for each $\bar{x} \in D_R$,

E3'. for each $\bar{x}, \bar{x} \in D_R$ and $x = |\pi|(|\bar{x}|), x' = |\pi|(|\bar{x}'|)$, if $x' = gx$ for some $g \in G_\chi$, then $\psi_x' = g \cdot \psi_x$.

The set $(\psi_x)_{\bar{x} \in D_R}$ is called determined by $\Phi_{D_R}$.

In cases when $R = Z_n, (a_n, -b)$, Theorem 10.1 is sufficient to calculate the homotopy $\pi_0(\Omega_{D_R, V_B})$, but in other cases we need more. In the remaining of this section, assume that $R \neq Z_n, (a_n, -b)$. With exactly the same definition of $A_{G_\chi}(S^2, V_B)$, we can rewrite Corollary 8.11

**Corollary 10.2.** The set $\Omega_{D_R, V_B}$ is equal to the set

$$\left\{ \Phi_{D_R} \in C^0(D_R, V_B) \mid \Phi_{D_R}(\bar{x}) \in A^0_x \text{ for each } \bar{x} \text{ and any } \bar{e} \text{ such that } \bar{x} \in |\mathcal{L}_R|, \text{ and } \Phi_{D_R}(\bar{d}) = \psi_{\bar{d}}(\bar{d}), \Phi_{D_R}(\bar{d}) = \psi_{\bar{d}}(\bar{d}) \text{ for some } \langle \psi_{\bar{d}} \rangle_{\bar{e} \in \mathcal{L}_R} \in A_{G_\chi}(S^2, V_B) \right\}$$

By using this, we would show nonemptiness of $\Omega_{D_R, (W_q)e}$. For this, we need a lemma.
Lemma 10.3. For each \((W_{d_i})_{i \in I^+} \in \mathcal{A}_{G_x}(S^2, \chi)\), if we put
\[ V_B = G_x \times (G_x)_{d_i} \cdot W_{d_i}^{-1}, \quad F_{V_B} = G_x \times (G_x)_{d_i} \cdot \langle \bar{K}_{R,S} \rangle \times W_{d_i}^{-1}, \]
then each \(A_x\) for \(\bar{x} \in [d^0, d^1]\) is nonempty. And, we can pick an element \((\psi_{d_i})_{i \in I^+} \in \mathcal{A}_{G_x}(S^2, V_B)\) which determines \((W_{d_i})_{i \in I^+}\).

**Proof.** For each \(i \in I, \) put \(\bar{x} = [\pi^{-1}(d^j)] = [\bar{x}_j|\bar{x}_j = d^i_j\) for \(j \in \mathbb{Z}_2\)]. If we put \(F_1 = (\text{res}_{(G_x)_z}^{G_x} F_{V_B})_{|x}\) and \(N_2 = (G_x)_{d^i}\), then \(N_1 = (G_x)_{d^i}\). By Table \[3.4\] and \[10.1\], at least one of \(F_i\)’s satisfies Condition \(F2\). except two cases \(R = Z_n \times Z\) with odd \(n\), \((-a_n)\) with even \(n/2\). Then, we obtain nonemptiness as in Lemma \[8.12\] in these cases. In the remaining two cases, \((G_x)_{x} = H\) for each \(\bar{x} \in [d^0, d^1]\) by Table \[3.4\] and \(F_i\)’s satisfy Condition \(F1\) because \(\text{res}_{(G_x)_z}^{G_x} V_S \cong \text{res}_{(G_x)_z}^{G_x} V_N\) by \[10.1\]. So, we obtain nonemptiness by Lemma \[6.4\]. The second statement is proved as in Lemma \[8.12\]. □ □

By using these, we obtain nonemptiness of \(\Omega_{D_R,(W_{d_i})_{i \in I^+}}\).

**Proposition 10.4.** For each \((W_{d_i})_{i \in I^+} \in \mathcal{A}_{G_x}(S^2, \chi)\), the set \(\Omega_{D_R,(W_{d_i})_{i \in I^+}}\) is nonempty.

11. **Proof for cases when \(\text{pr}(\rho(G_x)) = Z_n, D_n\)**

In this section, we calculate homotopy of equivariant clutching maps in cases when \(\text{pr}(R) = Z_n, D_n\). We parameterize each edge \([c_b^s] \in [\bar{K}_{R,S}]\) for \(0 \leq i \leq m_R - 1\) by the interval \([i, i + 1] \) linearly to satisfy \(v_j^s \mapsto i\) when \(m_R \geq 3\). The vertex \(v_j^s\) is parameterized by 0 or \(m_R\) according to context. First, we deal with cases when \(R = Z_n, (a_n, -b)\). Similarly to Lemma \[8.2\] \(\), we easily obtain the following lemma:

**Lemma 11.1.** Assume that \(R = Z_n, (a_n, -b)\). Then,
\[ \text{res}_{(G_x)_z}^{G_x} E_S \cong \text{res}_{(G_x)_z}^{G_x} E_N \]
for each \(E \in \text{Vect}_{G_x}([K_R], \chi)\) for \(x \in [L_R]\).

**Proof.** In these cases, the great circle passing through \(N, S, x\) is fixed by \((G_x)_z\) because \(G_x\) fixes \(B\). From this, we obtain a proof by Lemma \[8.2\] □ □

This lemma says that \(p_{\text{vect}}\) is well-defined in cases when \(R = Z_n, (a_n, -b)\). In other cases, \(p_{\text{vect}}\) is well-defined by Lemma \[8.2\] and Lemma \[8.10\]. For a path \(\gamma\) defined on \([0, 1]\), define a path \(\gamma_{-s}\) on \([i, i + 1]\) as \(\gamma_{-s}(t) = \gamma(t - s)\) for \(i \in Z\).

**Proposition 11.2.** Assume that \(R = Z_n\). Then, Theorem \[A\] holds for the case.

**Proof.** For simplicity, we prove the proposition only when \(m_R = n \geq 3\). Other cases are similarly proved. For each \((W_q)_{q \in B} \in \mathcal{A}_{G_x}(S^2, \chi)\), pick the \(G_x\)-bundle \(V_B\) over \(B\) such that \(V_B|_{q} = W_q\) for \(q \in B\). Proof is similar to Proposition \[9.2\].

In this case, \((G_x)_x = H\) for each point \(x \in [L_R]\) by Table \[3.4\] so that \(A_x = A_x\) and \(A_x^0 = \text{Iso}_H(V_S, V_N)\) by Lemma \[6.4\] for each \(x \in \bar{L}_{R,S}\) and \(\bar{e} \in \bar{L}_{R,S}\) which are nonempty by Lemma \[10.3\]. Let \(g_0\) be an element in \(G_x\) such that \(\rho(g_0) = a_n\), i.e. \(g_0d^0 = d^1\). Similarly to Corollary \[10.2\] \(\), \(\Omega_{D_R,(W_q)_{q \in B} = \Omega_{D_R,V_B}\) is equal to
\[ \{ \Phi_{D_R} \in C^0(D_R, V_B) | \Phi_{D_R}(\bar{x}) \in \text{Iso}_H(V_S, V_N) \text{ for each } \bar{x} \in D_R, \]
\[ \text{and } \Phi_{D_R}(d^1) = g_0\Phi_{D_R}(d^0)g_0^{-1} \} \]
by Theorem \[10.1\] Pick a point \(* \in \text{Iso}_H(V_S, V_N)\), and put \(*^t = g_0 \ast g_0^{-1}\). Since \(\text{Iso}_H(V_S, V_N)\) is path-connected, we may assume that each element \(\Phi_{D_R}\) in \(\Omega_{D_R,V_B}\)
has * at $d^0, d^1$, respectively. Then, it is easy that $\pi_0(\Omega_{D_R,V_R}) \cong \mathbb{Z}$ by Lemma 7.4.

Let $\gamma_0 : \bar{D}_R = [0, 1] \to \Iso_H(V_S, V_N)$ be an arbitrary element of $\Omega_{D_R,V_R}$ such that $\gamma_0(0) = *$, $\gamma_0(1) = *$; and let $\sigma_0 : [0, 1] \to \Iso_H(V_S, V_N)$ be a loop such that $\sigma_0(0) = \sigma_0(1) = *$ and $[\sigma_0]$ is a generator of $\pi_1(\Iso_H(V_S, V_N), *)$. So, $\sigma_0.\gamma_0$ is contained in $\Omega_{D_R,V_R}$. We would show that the difference $c_1(F_{V_R}/\sigma_0.\gamma_0) - c_1(F_{V_R}/\gamma_0)$ is equal to $n\chi(id)$ up to sign where $l_R = n$. For this, we need to describe the equivariant clutching maps precisely. Let $\Phi = \cup_{q \in B} \varphi_q$ be the extension of $\gamma_0$, i.e. $\varphi_S(t) = \gamma_0(t)$ for $t \in [0, 1]$. By equivariance of $\Phi$,

$$\varphi_S(t) = g_0^i \varphi_S(t - i) g_0^{-i}$$

for $0 \leq i \leq n - 1$ and $t \in [i, i + 1)$. That is,

$$\varphi_S = \gamma_0 \lor (g_0(\gamma_0)^{-1} \lor \cdots \lor (g_0^{-1}(\gamma_0 - n + 1)g_0^{-n + 1})$$

And, let $\Phi' = \cup_{q \in B} \varphi_q'$ be the extension of $\sigma_0.\gamma_0$. Then,

$$\varphi_S' = (\sigma_0.\gamma_0) \lor (g_0(\sigma_0.\gamma_0)^{-1} \lor \cdots \lor (g_0^{-1}(\sigma_0.\gamma_0 - n + 1)g_0^{-n + 1})$$

Here, $g_0^i(\sigma_0.\gamma_0)^{-i} g_0^{-i}$ for $0 \leq i \leq n - 1$ is equal to

$$g_0^i(\sigma_0)^{-i} g_0^{-i} \cdot g_0(\gamma_0)^{-i} g_0^{-i}$$

and we can move $g_0^i(\sigma_0)^{-i} g_0^{-i}$ in (11.1) (nonequivariantly) homotopically to $\sigma_0.\gamma_0$. For example,

$$[(\sigma_0.\gamma_0) \lor (g_0(\sigma_0)^{-1} g_0^{-1} : g_0(\gamma_0)^{-1} g_0^{-1})]$$

$$= [(\sigma_0.\gamma_0 - g_0(\gamma_0)^{-1} g_0^{-1})]$$

$$= [(\sigma_0.\gamma_0^{-1}(g_0(\gamma_0)^{-1} g_0^{-1}) \lor (g_0(\gamma_0)^{-1} g_0^{-1})]$$

$$= [(\sigma_0.\gamma_0) \lor (g_0(\gamma_0)^{-1} g_0^{-1})].$$

Since $\varphi_S'$ contains $n \sigma_0$'s, difference between Chern classes is $n\chi(id)$ up to sign by Lemma 7.4. And, we can conclude that $c_1$ on $\Omega_{D_R,V_R}$ is injective and its image is equal to the set $\{\chi(id)(nk + k_0) \mid k \in \mathbb{Z}\}$ where $k_0$ is dependent on the pair. Therefore, we obtain a proof by Proposition 5.3(1). \hfill \Box \Box 

**Proposition 11.3.** Assume that $R = \langle a_n, -b \rangle$. Then, Theorem 4 holds for the case.

**Proof.** For simplicity, we prove the proposition only when $m_R \geq 3$. Other cases are similarly proved. For each $(W_q)_{q \in B}$ in $AG_\chi(S^2, \chi)$, pick the $G_\chi$-bundle $V_B$ over $B$ such that $V_B|q = W_q$ for $q \in B$.

In this case, $(G_\chi)_x = H$ and $A_{\phi}^0 = \Iso_H(V_S, V_N)$ for each interior $x \in \bar{D}_R$ and its image $x = |\pi(\bar{x})|$ by Table 4.4 and Lemma 6.4. Here, $\Iso_H(V_S, V_N)$ is nonempty by Lemma 11.3. Similarly, $A_{\phi}^0 = \Iso_{G_\chi}(V_S, V_N)$ is nonempty for $\bar{x} = d^0, d^1$ and its image $x = |\pi(\bar{x})|$. Similarly to Corollary 10.2, $\Omega_{D_R, (W_q)_{q \in B}} = \Omega_{D_R,V_R}$ is equal to

$$\left\{ \Phi_{D_R} \in C^0(\bar{D}_R, V_B) \mid \Phi_{D_R}(\bar{x}) \in \Iso_H(V_S, V_N) \text{ for each } \bar{x} \in \bar{D}_R, \right.$$

$$\left. \text{and } \Phi_{D_R}(\bar{x}) \in A_{\phi}^0 \text{ for } \bar{x} = d^0, d^1 \right\}$$

by Theorem 10.1. Then, we obtain a proof by Lemma 7.4 because the inclusion from $\Iso_{G_\chi}(V_S, V_N)$ to $\Iso_H(V_S, V_N)$ for $x = d^0, d^1$ induces surjection in the level of fundamental groups by Lemma 7.2. \hfill \Box \Box
Proposition 11.4. Assume that $R$ is equal to one of the following:

- $D_n \times Z$, even $n$,
- $\langle -a_n, -b \rangle$, odd $n/2$,
- $\langle -a_n, b \rangle$, odd $n/2$,
- $Z_n \times Z$, even $n$,
- $\langle -a_n \rangle$, odd $n/2$.

Then, Theorem C holds for these cases.

Proof. Proof is similar to Proposition 9.2. □ □

Proposition 11.5. Assume that $R = D_n$. Then, Theorem C holds for the case where $l_R = 2n$.

Proof. Proof is similar to Proposition 10.2. □ □

Proposition 11.6. Assume that $R$ is equal to one of the following:

- $D_n \times Z$, odd $n$,
- $\langle -a_n, b \rangle$, even $n/2$,
- $\langle -a_n, -b \rangle$, even $n/2$.

Then, Theorem C holds for these cases.

Proof. For each $(W_d)_{i \in I^+}$ in $A_{G_\chi}(S^2, \chi)$, put $V_B = G_\chi \times (G_\chi)_{a-1} W_{d-1}$. In these cases, $A_B$ is nonempty for each $\tilde{x} \in \tilde{D}_R$ by Lemma 10.3. Let us investigate $A_B$ more precisely. First, $(G_\chi)_{a-1} = H$ and $A^0_B = \text{Iso}_H(V_S, V_\chi)$ for each interior $\tilde{x} \in \tilde{D}_R$ and its image $x = [\pi(x)]$ by Table 11.4 and Lemma 6.4. Also, $(G_\chi, x) \cong \mathbb{Z}_2$ for $x = d^0$ or $d^1$ (say it $d^0$), and $\rho((G_\chi)_{a-1})$ has an element of the form $-a_n b$ fixing $S$ for some $i$ by Table 11.4. From this, $(G_\chi)_{a-1}$ fixes the great circle containing $S, N, d^0$ so that $V_S$ and $V_N$ are $(G_\chi)_{a-1}$-isomorphic as in Lemma 11.1. And, $A^0_B$ is homeomorphic to $\text{Iso}_{(G_\chi)_{a-1}}(V_S, V_\chi)$ by Lemma 6.4. Pick an element $\psi_{2i}$ in $A_B$ which determines $W_\psi$. Note that $d^0$ and $d^1$ are not in a $G_\chi$-orbit because one is a vertex and the other is not a vertex in $D_R$ by Table 11.4. Similarly to 8.13 of Proposition 8.13 $\Omega_{D_R(w_{d+1})_{i \in I^+}}$ is equal to

\[ \left\{ \Phi_{D_R} \in C^0(\tilde{D}_R, V_B) \mid \Phi_{D_R}(\tilde{x}) \in \text{Iso}_H(V_S, V_N) \text{ for each } \tilde{x} \in \tilde{D}_R, \right\} \]

by Theorem 10.1 which is nonempty by Proposition 10.4. Here, $(A_B)_{\psi_{2i}}$ is simply connected by Proposition 6.8 because $(G_\chi)_{a-1}/H \cong \mathbb{Z}_2$ and $d^1$, $\rho(d^0)$ are in a $G_\chi$-orbit by Table 11.4. And, the inclusion from $\text{Iso}_{(G_\chi)_{a-1}}(V_S, V_\chi)$ to $\text{Iso}_H(V_S, V_N)$ induces surjection in the level of fundamental groups by Lemma 11.2. Therefore, we obtain a proof by Lemma 7.3. □ □

Now, we deal with remaining two cases $R = Z_n \times Z$ with odd $n$ or $\langle -a_n \rangle$ with even $n/2$. For a path $\gamma$ defined on $[0, 1]$, define a path $\gamma''$ on $[0, 1]$ as $\gamma''(t) = \gamma(1-t)$ for $t \in [0, 1]$.

Proposition 11.7. Assume that $R = Z_n \times Z$ with odd $n$ or $\langle -a_n \rangle$ with even $n/2$.

For each triple $(W_d)_{i \in I^+} \in A_{G_\chi}(S^2, \chi)$, we have

\[ \pi_0(\Omega_{D_R(w_{d+1})_{i \in I^+}}) \cong \mathbb{Z}. \]

And, $c_1(F_{V_B}/\Phi)$ is constant for $\Phi \in \Omega_{D_R(w_{d+1})_{i \in I^+}}$ when $V_B = G_\chi \times (G_\chi)_{a-1} W_{d-1}$.

Proof. In these cases, $A_B$ is nonempty for each $\tilde{x} \in \tilde{D}_R$ by Lemma 11.5. Since $(G_\chi)_{a-1} = H$ for each $\tilde{x} \in \tilde{D}_R$ and its image $x = [\pi(x)]$ by Table 11.4, the triple satisfies

\[ W_{d-1} = V_S \text{ and } W_{d+1} \cong \text{res}_{H}^{(G_\chi)_{a-1}} V_S \]

for $i \in I$ by Definition 11.4. i.e. each triple is determined by the third entry $W_{d-1}$.

So, $\Omega_{D_R(w_{d+1})_{i \in I^+}} = \Omega_{D_R(w_{d+1})_{i \in I^+}}$ because $V_B$ is also determined by $W_{d-1}$. And, $A^0_B = A^0_{\psi_{2i}} \cong \text{Iso}_H(V_S, V_N)$ for $x \in [\mathcal{L}_R, S]$ for $\psi_{2i} \in \mathcal{L}_R, S$ by Lemma 8.4 so that $\pi_0(A^0_B) \cong \mathbb{Z}$ by Schur’s Lemma because $V_S$ is $H$-isotypical.
Let $g_0 \in G_{\chi}$ be an element such that

1. $\rho(g_0) = -a_n^{n+1/2}$ if $\rho(G_{\chi}) = Z \times Z$ with odd $n$,
2. $\rho(g_0) = -a_n^{n/2+1}$ if $\rho(G_{\chi}) = (-a_n)$ with even $n/2$.

In both cases, $g_0^{\sigma_i} = \tilde{\sigma}_N^{i+1}$ for each $i \in \mathbb{Z}_{m_R}$ in $\mathcal{L}_R$. Similarly to Corollary 10.2, $\Omega_{D_R,(V_{\varphi})_{i+1}} = \Omega_{D_R,V_{\varphi}}$ is equal to

$$\{ \Phi_{D_R} \in C^0(D_R,V_B) \mid \Phi_{D_R}(\bar{x}) \in \text{Iso}_H(V_S,V_N) \text{ for each } \bar{x} \in D_R,$$

and $\Phi_{D_R}(\bar{d}) = g_0 \Phi_{D_R}^{-1}(d^0) g_0^{-1}$

by Theorem 10.1. Pick an element $* \in D^0_{V_B} = \text{Iso}_H(V_S,V_N)$, and put $* = g_0 *^{-1} g_0^{-1}$ in $D^0_{V_B} = \text{Iso}_H(V_S,V_N)$. Since Iso$_H(V_S,V_N)$ is path-connected, we may assume that each element $\Phi_{D_R}$ in $\Omega_{D_R,V_{\varphi}}$ satisfies

$$\Phi_{D_R}(\bar{d}) = * \text{ and } \Phi_{D_R}(\bar{d}) = *'.$$

So, $\pi_0(\Omega_{D_R,V_{\varphi}}) \cong \mathbb{Z}$ by Lemma 7.3 because $\pi_1(\text{Iso}_H(V_S,V_N)) \cong \mathbb{Z}$.

Next, we calculate the first Chern class. Our calculation is done in a similar way with Proposition 11.2. Pick an arbitrary element $\Phi_{D_R}$ in $\Omega_{D_R,V_{\varphi}}$ such that $\Phi_{D_R}(\bar{d}) = *$ and $\Phi_{D_R}(\bar{d}) = *'$. Put $\gamma(t) = \Phi_{D_R}(t)$ on $\mathbb{Z}$, $t \in [0,1]$. Take a loop $\sigma_0 : [0,1] \to \text{Iso}_H(V_S,V_N)$ such that $\sigma_0(0) = \sigma_0(1) = *$ and $[\sigma_0]$ is a generator of $\pi_1(\text{Iso}_H(V_S,V_N),*)$. To prove our result, we only have to show that $c_1(F_{V_B}/\sigma_0\gamma) = c_1(F_{V_B}/\gamma)$ where $\sigma_0\gamma$ is in $\Omega_{D_R,V_{\varphi}}$ by Corollary 10.2. Let $\Phi = \cup_{q \in B} \varphi_q$ be the extension of $\gamma$, i.e. $\varphi_S(t) = \gamma(t)$ on $t \in [0,1]$. Then, equivariance of $\Phi$ shows that

$$\varphi_S(t) = g_0 \varphi_S(g_0^{-1}) g_0^{-1} \quad * = g_0 \varphi_S(g_0^{-1})^{-1} g_0^{-1}$$

for $t \in [1,2]$. That is, $\varphi_S = \gamma \wedge g_0^{-1} g_0^{-1}$ on $[0,2]$. By using this, if $\Phi' = \cup_{q \in B} \varphi_q'$ is the extension of $\sigma_0\gamma$, then we have

$$\varphi_S' = \sigma_0\gamma \wedge g_0((\sigma_0^{-1}) \cdot \gamma^{-1})^{-1} g_0^{-1}$$

on $[0,2]$. Here, two generators $[\sigma_0]$ in $\pi_1(\text{Iso}_H(V_S,V_N),*)$ and $[g_0(\sigma_0)^{-1} g_0^{-1}]$ in $\pi_1(\text{Iso}_H(V_S,V_N),*)$ cancel each other in $\pi_1(\text{Iso}_H(V_S,V_N))$. This is in contrast with the calculation of Proposition 11.2. Since the whole $\varphi_S$ and $\varphi_S'$ are equivariantly determined by $\gamma$ and $\sigma_0\gamma$ as in the proof of Proposition 11.2, we obtain

$$c_1(F_{V_B}/\sigma_0\gamma) = c_1(F_{V_B}/\gamma).$$

The proposition says that Proposition 5.3 does not applies to these two cases. So, we need to apply Lemma 5.2 to these cases. For this, we prove two technical lemmas.

**Lemma 11.8.** Assume that $R = \mathbb{Z}_n \times Z$ with odd $n$ or $(-a_n)$ with even $n/2$. For each $(W_{\varphi})_{i+1} \in \mathcal{A}_{G_{\chi}}(S^2,\chi)$, put $V_B = G_{\chi} \times (G_{\chi})_{i+1} W_{d-1}$. Let $\eta_S : [K_{R,S}] \times V_S \to [K_{R,S}] \times V_S$ be a $(G_{\chi})_{i}$-isomorphism. For an element $g$ in $G_{\chi}$ such that $gS = N$, let $\eta_N : [K_{R,N}] \times V_N \to [K_{R,N}] \times V_N$ be defined by $\eta_N(x) = g\eta_S(g^{-1} x) g^{-1}$. Then,

1. $\eta = \cup_{q \in B} \eta_q$ is the unique $G_{\chi}$-isomorphism of $F_{V_B}$ extending $\eta_S$.
2. For a map $\Phi = \cup_{q \in B} \varphi_q$ in $\Omega_{V_B}$, the map $\Phi' = \cup_{q \in B} \varphi_q'$ with $\varphi_q' = \eta_q \varphi_q \eta_q^{-1}$ is also an equivariant clutching map satisfying the following commutative diagram.
two elements for each
variantly extended to all
η
We can extend
η
for each
η
t, i
for each
η
to satisfy
G
Assume that
Lemma 11.10.

Proof. Well-definedness is an only issue here, and normality of \((G_\chi)_S = (G_\chi)_N\) in \(G_\chi\) is used for this.

Remark 11.9.

(1) In this lemma, \(\eta\) gives a \(G_\chi\)-isomorphism between \(F_{V_B}/\Phi\) and \(F_{V_B}/\Phi'\).

(2) Let \(\eta_{S,t}\) for \(t \in [0,1]\) be a homotopy of \((G_\chi)_S\)-isomorphisms of \(|\mathcal{K}_{R,S}| \times V_S\).

Let \(\Phi'_t = \bigcup_{q \in B} \phi'_q,t\) for each \(t \in [0,1]\) be the equivariant clutching map determined by \(\eta_{S,t}\) and \(\Phi\) in this lemma. Then, \(F_{V_B}/\Phi'_0\) and \(F_{V_B}/\Phi'_1\) are \(G_\chi\)-isomorphic by Lemma 11.1.

Lemma 11.10. Assume that \(R = \mathbb{Z}_n \times \mathbb{Z}\) with odd \(n\) or \((-a_n)\) with even \(n/2\).

For each \((W_d')_{i \in t}\) in \(A_{G_\chi}(S^2,\chi)\), put \(V_B = G_\chi \times (G_\chi)_N^{-1} W_{d-1}\). Then, each \(G_\chi\)-isomorphism \(\eta = \bigcup_{q \in B} \eta_q\) of \(F_{V_B}\) is equivariantly homotopic to a \(G_\chi\)-isomorphism \(\eta' = \bigcup_{q \in B} \eta'_q\) of \(F_{V_B}\) such that

\[\eta'_q|_{e_0^q \cup |e_1^q|} = \gamma \land (\gamma^t)_{-1}\]

for some loop \(\gamma : D_R = [0,1] \to \text{Iso}_H(V_S)\) satisfying \(\gamma(\bar{e}_1^q) = \gamma(\bar{e}_1^q) = \text{id}\) where \((\gamma^t)_{-1}\) is defined on \([1,2]\).

Proof. Note that \(R_S = \mathbb{Z}_{mn/2}\) for both cases, and pick an element \(g \in (G_\chi)_S\) such that \(g_{\bar{e}_1^q} = \bar{e}_1^q\). Consider \(|\mathcal{K}_{R,S}|\) as the quotient \(|\mathcal{L}_{R,S}| \times [0,1]/|\mathcal{L}_{R,S}| \times 0\), and parameterize points of it by \((\bar{x},t)\) with \(\bar{x} \in |\mathcal{L}_{R,S}|\) and \(t \in [0,1]\). Let \(\gamma_i : [1/2,1] \to \text{Iso}_H(V_S)\) be paths such that

\[\gamma_i(1/2) = \eta_S(\bar{e}_1^q), \quad \gamma_i(1) = \text{id}, \quad \gamma(t) = g_1\gamma_0(t)g_1^{-1}\]

for each \(t, i \in \{0,2\}\). Then, define

\[\eta'_S : |\mathcal{L}_{R,S}| \times [0,1/2] \cup \left( \bigcup_{i=0,2} (|\bar{e}_1^q\times [1/2,1]) \right) \longrightarrow \text{Iso}_H(V_S)\]

to satisfy

\[\eta'_S(\bar{x},t) = \eta_S(\bar{x},2t) \quad \text{for} \ t \in [0,1/2],\]

\[\eta'_S(\bar{e}_1^q, t) = \gamma_i(t) \quad \text{for} \ t \in [1/2,1], i \in \{0,2\}.\]

We can extend \(\eta'_S\) to \(|\mathcal{L}_{R,S}||\mathcal{E}_{R,S}|\) as a map from \(S^1\) to \(\text{Iso}_H(V_S)\). Let \(\alpha : |\bar{e}_0^q| \cup |\bar{e}_1^q| \to \text{Iso}_H(V_S)\) be the restriction \(\eta'_S|_{\bar{e}_0^q \cup |\bar{e}_1^q|}\). Note that \(\alpha(\bar{e}_0^q) = \alpha(\bar{e}_1^q) = \text{id}\).

Since \(\eta'_S|_{|\mathcal{E}_{R,S}|}\) is the \(G_\chi\)-orbit of \(\alpha\), \(\alpha\) should be also (nonequivariantly) homotopically trivial. Therefore, we may assume that \(\alpha = \gamma \land (\gamma^t)_{-1}\) for some loop \(\gamma : D_R \to \text{Iso}_H(V_S)\) satisfying \(\gamma(\bar{e}_1^q) = \gamma(\bar{e}_1^q) = \text{id}\).

Now, we can prove Theorem 11.3.

Proof of Theorem 11.3. For each \((W_d')_{i \in t}\), put \(V_B = G_\chi \times (G_\chi)_N^{-1} W_{d-1}\). Since \(\Omega_D = \partial_{\Omega_D}(W_d')_{i \in t}\) as we have seen in the proof of Proposition 11.7, we have \(p_{\text{vect}}((W_d')_{i \in t}) = \text{Vect}_{G_\chi}(S^2,\chi)_{V_B}\). So, it suffices to show that the set \(\text{Vect}_{G_\chi}(S^2,\chi)_{V_B}\) has exactly two elements for each \(V_B\) to prove the first statement.
Let $\Phi = \cup_{q \in B} \varphi_\eta$ be a map in $\Omega_{B, \varphi_\eta}$. Put $\Phi(d^0) = \ast$ and $\Phi(d^1) = \ast'$. Let $g_0 \in G_\varphi$ be the element in the class of the proposition such that $g_0 e_i = e_i$ for each $i$. We would determine which classes of $\bar{\Omega}_{D_r, B, \varphi}$ give isomorphic equivariant vector bundles. Take a loop $\eta : D_R = [0, 1] \to \text{Iso}_H(V_S)$ such that $\eta_0(0) = \eta_0(1) = \ast$ and $[\eta_0]$ is a generator of $\pi_1(\text{Iso}_H(V_S), \ast)$. For each $G_\varphi$-isomorphism $\eta = \cup_{q \in B} \eta_{i_q}$ of $F_{V, B}$, we may assume that $\eta_S(\varphi_{\bar{\varphi}}|_{\varphi_{\bar{\varphi}}}) = \sigma_0^{i_k} \otimes (\sigma_0^{i_k})_{s_i}$ for some $j \in \mathbb{Z}$ by Lemma 4.4.11. Put

$$\varphi_S(x) = \eta_N([c(x)]) \varphi_S(x) \eta_S(x)^{-1}$$

for all $x \in D_R$. By equivariance,

$$\eta_N([c(x)]) = g_0^{-1} \eta_S(g_0 \varphi(x)) g_0 = g_0^{-1} \eta_S(x + 1) g_0.$$

Since $\eta_N(\varphi_{\bar{\varphi}}|_{\varphi_{\bar{\varphi}}}) = \sigma_0^{i_k} \otimes (\sigma_0^{i_k})_{s_i}$,

$$\varphi_S(x) = g_0^{-1} \sigma_0^{i_k} g_0 \varphi_S(x) \sigma_0^{-i_k}$$

for $x \in D_R$. And, this path is nonequivariantly homotopic to $\varphi_S(x) \sigma_0(x)^{-2j}$ in $[0, 1]$. Since each class in $\Omega_{D_r, V, B}$ is nonequivariantly homotopic to one of $\varphi_S(x) \sigma_0(x)^{i_k}$ for $i \in \mathbb{Z}$ by Proposition 4.4.14, this says that the equivariant vector bundle determined by an equivariant clutching map on $D_R$ with respect to $V_B$ depends only on the parity of $i$ by Lemma 5.3. Therefore, $\text{Vect}_{G_\varphi}(S^2, \chi)_{V, B}$ have two different $G_\varphi$-bundles. By the arguments on parity, we similarly obtain the second statement. 

12. Proof for cases when $\text{pr}(\rho(G_\varphi)) = \text{SO}(2), \text{O}(2)$

Assume that $\rho(G_\varphi) = R$ for some one-dimensional $R$ in Table 4.4.1. In these cases, $S^2$ can not have equivariant simplicial structure. So, we introduce new notations. Consider the disjoint union $S^2 = S_0^2 \sqcup S_1^2$ of the lower and upper hemispheres $S_0^2$ and $S_1^2$, and denote by $S_0^2$ and $S_1^2$ hemispheres $S_0^2$ and $S_1^2$ in $S^2$, respectively. Denote by $\pi$ the natural quotient maps from $S^2$ to $S^2$. Denote by $S_0$ and $S_1$ boundaries of $S_0^2$ and $S_1^2$ in $S^2$, respectively. And, denote by $S^2$ the disjoint union $S_0^2 \sqcup S_1^2$, which is the preimage of the equator through $\pi$. Put $B = \{S, N\} \subset S^2$ on which $R$ acts. The subset $\pi^{-1}(B)$ in $S^2$ is often confused with $B$ in $S^2$. Denote by $\pi^0$ be the point $\pi^{-1}(\pi^0)$ in $S^2$ for $q \in B$ where $\pi^0 = (1, 0, 0)$. Note that if $R = \text{SO}(2), \langle \text{SO}(2), \langle -b \rangle \rangle$, then $R$ fixes $B$, and otherwise $R$ acts transitively on $B$. We can redefine notations $V_B, F_{V_B}$, $\Phi, F_{V_B}/\Phi, \varphi, p_3, p\varphi, \Omega_{V, B}, \cdots$ of Section 4.4 by replacing $[K_R], [K_R], [K_{R, q}]$, $[L_R], [L_{R, q}]$ with $S^2, S_0^2, S_1^2, S_0$, respectively. Here, notations for cases when $R = \text{SO}(2)$, $\langle \text{SO}(2), \langle -b \rangle \rangle$ are redefined in the same way with cases when $R = Z_n$, $\langle a_n, -b \rangle$ and $R \neq \text{SO}(2), \langle \text{SO}(2), \langle -b \rangle \rangle$ are redefined in the same way with cases when $R = Z_n$, $\langle a_n, -b \rangle$ and $R \neq \text{SO}(2), \langle \text{SO}(2), \langle -b \rangle \rangle$.

Put $x = \{x_j \mid j \in \mathbb{Z}^2\}$ with $x_0 = \bar{v}_0$ and $x_1 = v_0$. Let $A_{\varphi}$ be the set of equivariant pointwise clutching maps with respect to the $(G_\varphi)_{\varphi}$-bundle $(\text{res}^{\varphi}_{\varphi}, F_{V_B})$.

**Proposition 12.1.** Assume that $R$ is equal to one of the following:

$$\text{O}(2) \times Z, \quad \text{O}(2), \quad \langle \text{SO}(2), \langle -a_2 \rangle \rangle.$$

Then, Theorem 4.4 holds for these cases.

Proof. Put $V_B = G_\varphi \times (\varphi, \chi)^{-1} W_{d^{-1}}$ for each $(W_d)_{d^{-1}}$ in $A_{\varphi}$. By Proposition 5.3.2, we only have to show that $\sigma_0(\varphi_{\varphi})_{d^{-1}}$ consists of exactly one element for each $(W_d)_{d^{-1}}$. First, we show nonemptiness of $A_{\varphi}$. For those $R$'s, the $(G_\varphi)_{\varphi}$-bundle $F = (\text{res}^{\varphi}_{\varphi}, F_{V_B})$ satisfies Condition F2. By Table 5.3.4 since $d^0 = d^1 = v_0$, we have $W_{d^0} \cong W_{d^1}$ which is an $(G_{\varphi})_{\varphi}$ extension of $F_{\varphi}$.
By definition of $W_d$ as in the proof of Lemma 8.12. Pick an element $\psi_0$ in $A_{\psi_0}$ which determines $W_d$.

Consider the evaluation map at $\omega^0$:

$$\Omega_{(W_d)} \rightarrow (A_{\psi_0})^0 \psi_0, \quad \Phi \mapsto \Phi(\omega^0).$$

By definition of $\Omega_{(W_d)}$, the isotropy representation $(F_{V_B}/\Phi)_{\omega}$ for each $\Phi$ is isomorphic to $W_d$ so that $\Phi(\omega^0)$ is contained in $(A_{\psi_0})^0 \psi_0$, i.e., the evaluation map is well-defined. We show that it is a one-to-one correspondence. For those $R$'s, $G$ acts transitively on $S^1$. So, each map $\Phi$ in $\Omega_{V_B}$ is determined by $\Phi(\omega^0)$ through equivariance, and the evaluation is injective. For each $\psi$ in $(A_{\psi_0})^0 \psi_0$, we construct a map $\Phi$ to satisfy

$$\Phi(g\omega^0) = g\psi(\omega^0)g^{-1},$$

for each $g \in G$, especially $\Phi|_B = \psi$. We can show that this is a well-defined equivariant clutching map with respect to $V_B$ so that the evaluation is surjective. And, it induces the bijection from $\pi_0(\Omega_{(W_d)})$ to a one point set $\pi_0((A_{\psi_0}) \psi_0)$. Therefore, we obtain a proof.

Proposition 12.2. Assume that $R$ is equal to one of $\text{SO}(2), (\text{SO}(2), -b)$. Then, Theorem [C] holds for these cases.

Proof. Pick a $G$-vector bundle $V_B$ over $B$ such that $V_B|_q = W_q$ for each pair $(W_q)_{q \in B} \in A_G(S^2, \chi)$ and $q \in B$. By Proposition 5.3(2), we only have to show that $\pi_0(\Omega_{(W_q)})$ consists of exactly one element for each $(W_q)_{q \in B}$. For those $R$'s, the $(G, \chi)$-bundle $F = (\text{res}^G_{\text{res}}(F_{V_B}))\chi$ satisfies Condition F1. By Table 5.1 and Proposition 6.4. Consider the evaluation map at $\omega^0$:

$$\Omega_{(W_q)} \rightarrow (A_{\psi_0})^0 \psi_0, \quad \Phi \mapsto \Phi(\omega^0).$$

In these cases, $G$ acts transitively only on $S^1$, but each map $\Phi$ in $\Omega_{V_B}$ is determined by $\Phi(\omega^0)$ through equivariance and inverse. It can be shown that the evaluation is bijective as in Proposition 12.3. And, it induces the bijective map from $\pi_0(\Omega_{(W_q)})$ to $\pi_0((A_{\psi_0})^0 \psi_0)$ which is one-point set. Therefore, we obtain a proof.

13. Equivariant line bundles over effective $G \times H$-actions

In this section, we prove Theorem 11 and calculate Chern classes. Since $H$ is the kernel of the $G$-action on $S^2$, $S^2$ delivers the $G$-action. Since $\rho(G) = R$ for some $R$ of Table 1.1, by assumption, we may assume that $G \times H$ is equal to $R$ and the $G \times H$-action is equal to the $R$-action on $S^2$.

Proof of Theorem 11 We prove this only for the case when $R = Z_n$ because other cases are proved similarly. Let $\bar{U}$ be the $H$-representation with the character $\chi$. Let $U$ be a $(G, \chi)$-extension of $U$ for $q \in B$ whose existence is guaranteed by Theorem 14.1. Pick a bundle $E$ in $\text{Vect}_G(S^2, \chi)$, and put $(W_q)_{q \in B} = \rho_{\text{sect}}(E)$. Then, $W_q$'s are direct sums of $U \otimes \Omega(l_0)$ and $U \otimes \Omega(l_1)$ of $W_S$ and $W_N$, respectively. Define $(W'_q)_{q \in B}$ by

$$W'_S = \bar{U} \otimes \Omega(l_0) \quad \text{and} \quad W'_N = \bar{U} \otimes \Omega(l_1).$$
By definition of $A_{G_n}(S^2, \chi)$, the pair $(W_q')_{q \in B}$ is contained in $A_{G_n}(S^2, \chi)$. Since $p^{-1}_{\text{vect}}((W_q')_{q \in B})$ is nonempty by Theorem $\text{[A]}$, there exists a bundle $L$ with rank $\chi(id)$ in $\text{Vect}_{G_n}(S^2, \chi)$. Existence of $L$ proves the isomorphism by $\text{[CKMS Lemma 2.2]}$. By similar arguments, we can show that $A_R(S^2, \text{id})$ is generated by all the elements with one-dimensional entries. By using this and Theorem $\text{[A]}$ $\text{[B]}$, we can show that $\text{Vect}_R(S^2)$ is generated by line bundles.

Now, we calculate the number of elements in $A_R(S^2, \text{id})$ with one-dimensional entries. By Definition $\text{[13]}$, each pair $(W_q)_{q \in B}$ in $\text{Rep}(R)^2$ is in $A_R(S^2, \text{id})$, i.e. there is no relation between $W_S$ and $W_N$. Since the number of one-dimensional representations in $\text{Rep}(R)$ is equal to $n = |R_S| = |R_N|$, we obtain a proof. □

**Remark 13.1.** We explain for the reason why we prove the isomorphism of Theorem $\text{[D]}$ only for $R$’s appearing in Theorem $\text{[A]}$ $\text{[B]}$. In the proof of Theorem $\text{[D]}$ existence of $(G_n)_{q}$-extensions of $U$ for all $q$ or $(G_n)_{d}$-extensions of $U$ for all $i$ is critical according to $R$. But, such existence is not guaranteed if $R_q$ or $R_d$ is isomorphic to $D_{2m}$ for some $m$ by $\text{[CMS Corollary 3.5.(2)]}$, and almost all $R$’s appearing in Theorem $\text{[C]}$ satisfy that $R_q$ or $R_d$ is isomorphic to $D_{2m}$ for some $q$ or $i$ according to $R$. So, we can not obtain the isomorphism for such $R$’s. The inextensibility does not happen in dealing with equivariant vector bundles over circle. □

In the below, we use the notation $W$ to denote an element in $A_R(S^2, \text{id})$ with one-dimensional entries. We would calculate Chern classes of line bundles in $\text{Vect}_R(S^2)$. Especially, we would calculate $k_0(W)$ to stress its dependency. By Theorem $\text{[A]}$, $k_0(W)$ is determined up to $l_{R} \cdot \mathbb{Z}$, i.e. $k_0(W)$ lives in $\mathbb{Z}_n$. More precisely, Theorem $\text{[A]}$ says that $k_0(W)$ is congruent modulo $l_{R}$ to $c_1(L)$ for any line bundle $L \in p^{-1}_{\text{vect}}(W)$. So, we will calculate $c_1(L) \mod l_{R}$ for one bundle $L$ in $p^{-1}_{\text{vect}}(W)$. In doing so, $c_1(L)$ is expressed by using $W = (L_q)_{q \in B}$ or $W = (L_d^i)_{i \in I^+}$ according to $R$. When $n \in \mathbb{N}$ is understood, put $\xi_0 = \exp(2\pi \sqrt{-1}i/n)$, and let $\Omega(l)$ for $l \in \mathbb{Z}_n$ be the one-dimensional $\mathbb{Z}_n$-representation satisfying $a_n \cdot v = \xi_0^l v$ for each $v \in \mathbb{C}$. Then, we have the following well-known result:

**Lemma 13.2.** Assume that $R = \mathbb{Z}_n$. For any line bundle $L$ in $\text{Vect}_R(S^2)$, if $L_q \cong \Omega(l_q)$ for $q \in B$, $l_q \in \mathbb{Z}_n$, then $c_1(L) \equiv l_N - l_S \mod n$.

We obtain similar results for cases when $R = D_n$, $\mathbb{Z}_n \times Z$ with odd $n$, or $\{-a_n\}$ with even $n/2$.

**Lemma 13.3.** Assume that $R = D_n$. For any line bundle $L$ in $\text{Vect}_R(S^2)$, if $L_q \cong \Omega(l_q)$ for $q \in B$, $l_q \in \mathbb{Z}_n$, then $l_N \equiv -l_S \mod n$, and $c_1(L)$ is congruent modulo $2n$ to

$$\begin{cases} -2l_S & \text{if } L_d^i \cong L_d^j, \\ -2l_S + n & \text{if } L_d^i \not\cong L_d^j. \end{cases}$$

**Proof.** The first statement easily follows from the relation $ba_n b^{-1} = a_n^{-1}$. To prove the second statement, we would construct line bundles $L'$ in $\text{Vect}_R(S^2)$ such that $L_q' \cong \Omega(l_q)$. Pick the $R$-bundle $V_B$ over $B$ such that $a_n \cdot v = \xi_0^l v$ and $b \cdot v = v$ for $q \in B$ and $v \in V_q = \mathbb{C}$. And, define $F_{V_B}$ as $H_{q \in B} [\tilde{K}_{R,q}] \times V_q$ such that $g \cdot (\tilde{x}, v) = (g \cdot \tilde{x}, g \cdot v)$ for $g \in R$, $q \in B$, $\tilde{x} \in [\tilde{K}_{R,q}]$, $v \in V_q$. We calculate $A_x$. Let $\varphi_0$ be the element id in $\text{Iso}(V_S, V_N) = \text{Iso}(\mathbb{C})$. Then, we can show the following:
Remark 13.4. We explain for how to calculate \( k_0(\mathbf{W}) \) for cases when \( R = T, O, I \). Let \( \mathbf{W} = p_{\text{vect}}(L) \) for some line bundle \( L \in \text{Vect}_R(S^2) \). Then, it suffices to calculate \( c_1(L) \pmod{l_R} \) for \( R, T, O, I \). Note \( l_R = |R| \) in these cases. For a 2-Sylow subgroup \( P \) of \( R \), observe that the restricted \( P \)-action on \( S^2 \) is conjugate to \( D_m \) for some \( m \). Then, we can calculate \( c_1(L) \pmod{2m} \) by applying Lemma 13.3 to \( \text{res}^R_P L \) where \( |D_m| = 2m \). For other prime number \( p \) dividing \( |R| \) and a \( p \)-Sylow subgroup \( P \) of \( R \), observe that the restricted \( P \)-action on \( S^2 \) is conjugate to \( Z_p \), and we can calculate \( c_1(L) \pmod{p} \) by applying Lemma 13.2 to \( \text{res}^R_P L \) where \( |Z_p| = p \). So, we can calculate \( c_1(L) \pmod{l_R} \) by Chinese Remainder Theorem because \( l_R = |R| \). 

Lemma 13.5. Assume that \( R = \mathbb{Z}_n \times \mathbb{Z} \) with odd \( n \). For any line bundle \( L \) in \( \text{Vect}_R(S^2) \), if \( L_q \cong \Omega(l_q) \) for \( q \in B, l_q \in \mathbb{Z}_n \), then \( L_N \cong l_S \pmod{n} \), and \( c_1(L) \) is trivial.

Proof. The first statement easily follows because \( a_n \) and \(-\text{id}\) commute in \( R \). To prove the second statement, we would construct a line bundle \( L' \) in \( \text{Vect}_R(S^2) \) such that \( L'_q \cong \Omega(l_q) \). Pick the \( R \)-bundle \( V_B \) over \( B \) such that

\[
a_n \cdot v = \xi^0_n v \quad \text{and} \quad -\text{id} \cdot v = v
\]

for \( q \in B \) and \( v \in V_q = \mathbb{C} \). And, define \( F_{V_B} \) as \( \Pi_{q \in B} [\mathcal{K}_{R,q}] \times V_q \) such that

\[
g \cdot (\bar{x}, v) = (g \cdot \bar{x}, g \cdot v)
\]

for \( g \in R, q \in B, \bar{x} \in [\mathcal{K}_{R,q}], v \in V_q \). Then, we can define the equivariant clutching map \( \Phi \) with respect to \( V_B \) which satisfies \( \Phi(\bar{x}) = \text{id} \in \text{Iso}(V_S, V_N) = \text{Iso}(\mathbb{C}) \) for each \( \bar{x} \in \mathcal{L}_{R,S} \). For \( L' = F_{V_B}/\Phi \), the Chern class \( c_1(L') \) is trivial. Since two equivariant vector bundles in \( \text{Vect}_R(S^2) \) with the same isotropy representation at each \( d^i \) have the same Chern class by Theorem 13, we obtain a proof.

Lemma 13.6. Assume that \( R = \langle -a_n \rangle \) with even \( n/2 \). For any line bundle \( L \) in \( \text{Vect}_R(S^2) \), if \( L_q \cong \Omega(l_q) \) for \( q \in B, l_q \in \mathbb{Z}_{n/2} \), then \( L_N \equiv l_S \pmod{n/2} \), and \( c_1(L) \) is trivial.
**Proof.** Similarly to Lemma 13.3 and Lemma 13.5, the first statement can be proved by using the fact that $R$ is a cyclic group. To prove the second statement, we would construct a line bundle $L'$ in $\text{Vect}_R(S^2)$ such that $L'_q \cong \Omega(l_q)$. Put $c_0 = \exp \left( \frac{2\pi i}{m} a_0 \right)$ so that $c_0^2 = \xi_0^2$ where $\xi_0 = \exp \left( \frac{2\pi i}{m} r \right)$. Pick the $R$-bundle $V_B$ over $B$ such that

$$-a_n \cdot v = c_0 v$$

for $q \in B$ and $v \in V_q = \mathbb{C}$. And, define $F_{V_B}$ as $\Pi_{q \in B} [K_{R,q}] \times V_q$ such that

$$g \cdot (\tilde{x}, v) = (g \cdot \tilde{x}, g \cdot v)$$

for $g \in R$, $q \in B$, $\tilde{x} \in [K_{R,q}]$, $v \in V_q$. Then, the remaining is the same with Lemma 13.5.

**14. APPENDIX: REPRESENTATION EXTENSION**

Let $N_0$ and $N_2$ be compact Lie groups such that $N_0 \triangleleft N_2$ and $N_2/N_0 \cong \mathbb{Z}_m$. Let $a_0$ be a fixed generator of $N_2/N_0$, and let $\Omega(l)$ be the representation defined by

$$N_2/N_0 \times C \to C, \quad (a_0, z) \mapsto \exp(2\pi i \sqrt{-1}/m)z$$

for $l \in \mathbb{Z}_m$. We also consider $\Omega(l)$ to be an $N_2$-representation via the projection $N_2 \to N_2/N_0$. Then, we obtain the following result from [CMS].

**Theorem 14.1.** For $U \in \text{Irr}(N_0)$, if the character of $U$ is fixed by $N_2$, then there exists an $N_2$-extension of $U$. If $\hat{U}$ is an $N_2$-extension of $U$, then the number of mutually nonisomorphic $N_2$-extensions of $U$ is $m$ and they are $\hat{U} \oplus \Omega(l)$ for $l \in \mathbb{Z}_m$.

**Proof.** By [CMS] Theorem 3.2., $U$ has $m$ mutually nonisomorphic $N_2$-extensions. Call one of them $\hat{U}$. By [CMS] Proposition 3.1. and its proof, each extension of $U$ is expressed as $\hat{U} \oplus \Omega(l)$ for some $l \in \mathbb{Z}_m$.

**Corollary 14.2.** Let $U$ be an irreducible $N_0$-representation whose character is fixed by $N_2$, and $\hat{U}$ be an $N_2$-extension of $U$. If $W$ be an $N_2$-representation such that $\text{res}_{N_0}^N \hat{U}$ is $U$-isotypical, $W$ is a direct sum of $\hat{U} \oplus \Omega(l)$’s.

**Proof.** First, we prove that the induced representation $\text{ind}_{N_0}^N \hat{U}$ is isomorphic to the direct sum $\oplus_{l \in \mathbb{Z}_m} (\hat{U} \oplus \Omega(l))$. By Frobenius reciprocity, $\text{Hom}_{N_2}(\hat{U} \oplus \Omega(l), \text{ind}_{N_0}^N \hat{U}) \cong \text{Hom}_{N_0}(\text{res}_{N_0}^N \hat{U} \oplus \Omega(l), U)$ is one-dimensional, and this means that each $\hat{U} \oplus \Omega(l)$ for $l \in \mathbb{Z}_m$ is a subrepresentation of $\text{ind}_{N_0}^N \hat{U}$ by Schur’s Lemma. So, $\text{ind}_{N_0}^N \hat{U}$ is isomorphic to the direct sum $\oplus_{l \in \mathbb{Z}_m} (\hat{U} \oplus \Omega(l))$.

We may assume that $W$ is irreducible. We only have to show that $W$ is one of $\hat{U} \oplus \Omega(l)$’s. Since $\text{res}_{N_0}^N \hat{U}$ is $U$-isotypical, $\text{res}_{N_0}^N W \cong IU$ for some integer $l$. By Frobenius reciprocity, $\text{Hom}_{N_2}(W, \text{ind}_{N_0}^N \hat{U}) \cong \text{Hom}_{N_0}(\text{res}_{N_0}^N W, U)$. Since $\text{res}_{N_0}^N W$ is isomorphic to $lU$, we obtain that $\text{Hom}_{N_0}(\text{res}_{N_0}^N W, U)$ is $l$-dimensional by Schur’s Lemma. But, since $\hat{U} \oplus \Omega(l)$’s are all different and $W$ is irreducible, Schur’s Lemma says that $\text{Hom}_{N_0}(W, \text{ind}_{N_0}^N \hat{U})$ is at most one-dimensional, i.e. $l \leq 1$. Therefore, $l$ is equal to 1, and this gives a proof.

**References**

[At] M. F. Atiyah, *K-Theory*, Addison-Wesley, 1989.

[B] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[BD] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, 98, Springer-Verlag, New York, 1985.

[CKMS] J.-H. Cho, S. S. Kim, M. Masuda, D. Y. Suh, *Classification of Equivariant Complex Vector Bundles over a Circle*, J. Math. Kyoto Univ. 41 (2001), 517-534.
[CK] A. Constantin, B. Kolev, The theorem of Kerekjarto on periodic homeomorphisms of the disk and the sphere, Enseign. Math. (2) 40 (1994), 193–204.

[CMS] J.-H. Cho, M. Masuda, D. Y. Suh, Extending representations of $H$ to $G$ with discrete $G/H$, J. Korean Math. Soc. 43 (2006), 29–43.

[G] A. Grothendieck, Sur la classification des fibres holomorphes sur la sphere de Riemann, Am. J. Math. (79) (1957), 121–138.

[Ki1] M. K. Kim, Classification of equivariant vector bundles over two-torus, arXiv:1005.0682v2.

[Ki2] M. K. Kim, Classification of equivariant vector bundles over real projective plane, preprint.

[Ki3] M. K. Kim, Classification of equivariant vector bundles over Klein bottle, in preparation.

[Ko] B. Kolev, Sous-groupes compacts d’homeomorphismes de la sphere, Enseign. Math. (2) 52 (2006), no. 3-4, 193–214.

[P] F. P. Peterson, Some remarks on Chern classes, Ann. Math. (2) 69 (1959), 414–420.

[R] E. G. Rees, Notes on geometry, Universitext, Springer-Verlag, Berlin-New York, 1983.

[S] G. Segal, Equivariant K-theory, Inst. Hautes Études Sci. Publ. Math. 34 (1968), 129–151.

Department of Mathematics Education, Gyeongin National University of Education, San 59-12, Gyesan-dong, Gyeong-gu, Incheon, 407-753, Korea

E-mail address: mkkim@kias.re.kr