A DICHTOMY PROPERTY FOR THE GRAPHS OF MONOMIALS

J. M. ALMIRA

Abstract. The discontinuous monomial functions with connected graph are characterized, and this result is used to prove that the graph of a discontinuous $n$-monomial function $f : \mathbb{R} \to \mathbb{R}$ is either connected or totally disconnected.

1. Motivation

F. B. Jones [4] proved in 1942, in a famous paper, the existence of additive discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ whose graph $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ is connected, and characterized them. These functions are extraordinary since their graphs are dense connected subsets of the plane, containing exactly one point in each vertical line $\{x\} \times \mathbb{R}$ [7]. In his paper the author also stated, without proof, that the graph of a discontinuous additive function must be connected or totally disconnected. For this result he just referenced another famous paper, by Hamel [5], but the proof is not there. Indeed, up to our knowledge, a proof of this dichotomy result has never appeared in the literature. In this note we characterize the discontinuous monomial functions $f : \mathbb{R} \to \mathbb{R}$ with connected graph and we use this characterization to prove that the graph of a discontinuous monomial is either connected or totally disconnected. These results should be a good starting point to prove that, for larger classes of functions, such as the generalized polynomials or exponential polynomials over the real line, the graphs of the elements of these sets are either connected or totally disconnected. To find examples of both situations is an easy corollary of the structure of these functions and Jones’s existence result of additive discontinuous functions with connected graph.

2. Main results

Recall that $f : \mathbb{R} \to \mathbb{R}$ is an $n$-monomial function if it is a solution of the so called monomial functional equation

$$\frac{1}{n!}\Delta^nf(x) = f(h) \quad (x, h \in \mathbb{R}).$$

It is known that $f$ satisfies (1) if and only if $f(x) = F(x, \cdots , x)$ for a certain multi-additive and symmetric function $F : \mathbb{R}^n \to \mathbb{R}$, and that $f$ is a polynomial function of degree at most $n$ (i.e., $f$ solves Fréchet’s functional equation $\Delta^ {n+1}f(x) = 0$) if and only if $f(x) = \sum_{k=0}^{n} f_k(x)$, where $f_k(x)$ is a $k$-monomial function for $k = 0, 1, \cdots , n$. (See, for example, [3], [6], for the proofs of these claims).

Recently, Almira and Abu-Helaiel characterized the topological closures of the graphs of monomial functions as follows [1, Theorem 2.7]:

Theorem 2.1 (Almira, Abu-Helaiel). Assume that $f : \mathbb{R} \to \mathbb{R}$ is a discontinuous $n$-monomial function and let $\Gamma f = G(f) \cap \mathbb{R}^2$ and let us consider the function $A_n(h) = f(h)/h^n$, for $h \neq 0$. Let $\alpha = \sup_{h \in \mathbb{R}^*} A_n(h)$ and $\beta = \inf_{h \in \mathbb{R}^*} A_n(h)$. Then:

(a) If $\alpha = +\infty$ and $\beta = -\infty$, then $\Gamma f = \mathbb{R}^2$. 

(b) If $\alpha = +\infty$ and $\beta \in \mathbb{R}$, then $\Gamma_f = \{(x,y) : y \geq \beta x^n\}$ if $n = 2k$ is an even number, and
$\Gamma_f = \{(x,y) : x \leq 0$ and $y \leq \beta x^n\} \cup \{(x,y) : x \geq 0$ and $y \geq \beta x^n\}$ if $n = 2k + 1$ is an odd number. In particular, if $\beta = 0$, we get the half space $\Gamma_f = \{(x,y) : y \geq 0\}$ for $n = 2k$ and the union of the first and third quadrants $\Gamma_f = \{(x,y) : xy \geq 0\}$, for $n = 2k + 1$.

c) If $\alpha \in \mathbb{R}$ and $\beta = -\infty$, then $\Gamma_f = \{(x,y) : y \leq \beta x^n\}$ if $n = 2k$ is an even number, and
$\Gamma_f = \{(x,y) : x \leq 0$ and $y \geq \beta x^n\} \cup \{(x,y) : x \geq 0$ and $y \leq \beta x^n\}$ if $n = 2k + 1$ is an odd number. In particular, if $\alpha = 0$, we get the half space $\Gamma_f = \{(x,y) : y \leq 0\}$ for $n = 2k$ and the union of the second and fourth quadrants $\Gamma_f = \{(x,y) : xy \leq 0\}$, for $n = 2k + 1$.

Furthermore, for all $n \geq 2$ there are examples of discontinuous $n$-monomial functions $f$ verifying each one of the claims (a), (b), (c) above.

We use this result to prove a dichotomy property for monomial functions.

**Theorem 2.2 (Dichotomy, for monomial functions $f : \mathbb{R} \to \mathbb{R}$).** Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous $n$-monomial function and let $\Gamma_f = \overline{G(f)}^{\mathbb{R}^2}$, and $\Omega_f = \operatorname{Int}(\Gamma_f)$. Then

(i) $G(f)$ is connected if and only if $G(f)$ intersects all continuum $K \subseteq \Gamma_f$ which is not contained in any set of the form $\partial \Omega_f \cup \{(x) \times \mathbb{R} \cap \Gamma_f\}$.

(ii) $G(f)$ is either connected or totally disconnected. Furthermore, both cases are attained by concrete examples, for every $n$.

**Remark 2.3.** Recall that continuum means connected and compact.

**Proof.** (i). Assume that $G(f)$ is not connected. Then $G(f) \subseteq U \cup V$ with $U, V$ open subsets of the real plane, $U \cap V = \emptyset$, $G(f) \cap U \neq \emptyset$ and $G(f) \cap V \neq \emptyset$. We may assume that $U$ is a domain, since the connected components of the open sets $\Omega \subseteq \mathbb{R}^2$ are open sets.

Now, it follows from the density of $G(f)$ in $\Gamma_f = \overline{G(f)}^{\mathbb{R}^2}$ that $V \cap \Gamma_f = \operatorname{Ext}_{\Gamma_f}(U \cap \Gamma_f)$, since otherwise we could find $\varepsilon > 0$ and $(x_0, y_0) \in \Gamma_f$ such that $B((x_0, y_0), \varepsilon) \cap \Gamma_f \subseteq \operatorname{Ext}_{\Gamma_f}(U) \setminus V$ and this would contradict that $G(f) \subseteq U \cup V$ since $G(f)$ has at least a point in $B((x_0, y_0), \varepsilon) \cap \Gamma_f$.

It follows that $\partial U$ contains a continuum which intersects two distinct vertical lines and it is not contained into any set of the form $\partial \Omega_f \cup \{\{(x) \times \mathbb{R} \cap \Gamma_f\}$, since otherwise $\partial U$ should contain the intersection of a vertical line with $\Gamma_f$, a fact which leads to a contradiction, since $G(f)$ is a graph and hence intersects all vertical lines. This proves that, if $G(f)$ intersects all continuum $K \subseteq \Gamma_f$ which is not contained at any set of the form $\partial \Omega_f \cup \{\{(x) \times \mathbb{R} \cap \Gamma_f\}$, then $G(f)$ is connected.

Let us now assume that $G(f)$ is connected and let $K \subseteq \Gamma_f$ be a continuum which is not contained at any set of the form $\partial \Omega_f \cup \{\{(x) \times \mathbb{R} \cap \Gamma_f\}$. If $K$ has non-empty interior then $G(f) \cap K \neq \emptyset$, since $G(f)$ is dense in $\Gamma_f$. If $\operatorname{Int}(K) = \emptyset$ and $(x_0, y_0), (x_1, y_1) \in K$ with $x_0 < x_1$, then, there exist $(x_0^*, y_0^*), (x_1^*, y_1^*) \in K$ with $x_0 \leq x_0^* < x_1^* \leq x_1$, and $K^*$ a subcontinuum of $K$ which connects $(x_0^*, y_0^*), (x_1^*, y_1^*)$ such that $K^*$ is a subset of $[x_0^*, x_1^*] \times \mathbb{R} \cap \Omega_f$. Obviously, $K^* \cap [x_0^*, x_1^*] \times \mathbb{R}$ separates $[x_0^*, x_1^*] \times \mathbb{R}$ and $\Gamma_f$ in two (or more) components, since $K^*$ does not intersect the frontier of $\Gamma_f$. Now, $G(f)$ contains at least a point of each of these components, since $G(f)$ is dense in $\Gamma_f$. It follows that $K^* \cap G(f) \neq \emptyset$, since $G(f)$ is connected, by hypothesis. Hence, if $G(f)$ is connected, then $G(f)$ intersects every continuum $K \subseteq \Gamma_f$ which is not contained in any set of the form $\partial \Omega_f \cup \{\{(x) \times \mathbb{R} \cap \Gamma_f\}$. This proves (i).

(ii). If $f$ is continuous then $G(f)$ is connected. Hence we assume that $f$ is a discontinuous $n$-monomial function.

As a first step, we reduce our study to the case of monomial functions with even degree, by demonstrating that $G(f)$ is connected if and only if $G(f)$ is connected, where $g(x) = xf(x)$.

The implication $G(f)$ connected implies $G(g)$ connected is trivial. Let us prove the other implication. Indeed, assume that $G(g)$ is connected with $g(x) = xf(x), f(x)$ a $(2k+1)$-monomial
function. Let $K$ be a continuum included into $\Gamma_f$ which is not contained into any set of the form $\partial \Gamma_f \cup ((\{x\} \times \mathbb{R}) \cap \Gamma_f)$ and assume, with no loss of generality, that $(0,0) \notin K$ (if $(0,0) \in K$ then $G(f)$ contains a point of $K$, so that this case is trivial). Then $F = \{(x,y) : (x,y) \in K\}$ is a continuum, $F \subseteq \Gamma_g$, and $F$ is not contained into any set of the form $\partial \Gamma_g \cup ((\{x\} \times \mathbb{R}) \cap \Gamma_g)$. Hence $(i)$ and the connectedness of $G(g)$ imply that there exists $x_0 \neq 0$ such that $(x_0,g(x_0)) = (x_0,0) f(x_0)) \in F$. Thus $(x_0, f(x_0)) \in K$ and $G(f)$ contains a point of $K$. It follows, again from $(i)$, that $G(f)$ is connected.

Let us thus assume (with no loss of generality) that $n = 2k$ is even. Theorem 2.1 implies that $\Gamma_f = \mathbb{R}^2$ or $\Gamma_f = \{(x,y) : y \geq \beta x^{2k}\}$ for a certain $\beta \in \mathbb{R}$.

If $G(f)$ is not connected, there exist a continuum $K \subseteq \Gamma_f$ with empty interior, which is not contained in any set of the form $\partial \Gamma_f \cup ((\{x\} \times \mathbb{R}) \cap \Gamma_f)$ and $(x_0,y_0),(x_1,y_1) \in K$ with $x_0 < x_1$, such that $G(f) \cap K = \emptyset$. Obviously, the continuum $K$ separates $[x_0,x_1] \times \mathbb{R}$ in several disjoint open subsets of $\Gamma_f$ (with the relative topology). Hence we can assume that $$(\{x_0,x_1\} \times \mathbb{R}) \cap \Gamma_f \setminus K = U_K \cup V_K,$$ with $U_K,V_K$ disjoint open subsets of $\Gamma_f$, $U_K$ connected, and $\alpha \times [\beta, +\infty) \subseteq U_K$ for certain $\alpha \in ]x_0,x_1[$ and $\beta > 0$.

Let us set $U_K^* = U_K \cup \{(x,y) \in \partial U_K : (x,y) \notin \partial V_K\}$, $V_K^* = V_K \cup \{(x,y) \in \partial V : (x,y) \notin \partial U_K\}$. Then $U_K^*,V_K^*$ are open connected subsets of $\Gamma_f$, $U_K^* \cap V_K^* = \emptyset$, $G(f) \cap [x_0,x_1] \times \mathbb{R} \subseteq U_K^* \cup V_K^*$, $K^* = \partial U_K^* \cap \partial V_K^*$ is a continuum which separates $[x_0,x_1] \times \mathbb{R} \cap \Gamma_f$ in exactly two disjoint open connected subsets of $\Gamma_f$, $U_K^{*\circ} = U_K^*$ and $V_K^{*\circ} = V_K^*$. $G(f) \cap K^{*\circ} \neq \emptyset$ and $G(f) \cap V_K^{*\circ} = \emptyset$. Furthermore, the relation $f(\lambda x) = \lambda^nf(x)$ for all $x \in \mathbb{R}$ and all $\lambda \in \mathbb{Q}$ implies that $G(f) \cap \varphi_\lambda(K^*) = \emptyset$ for all rational number $\lambda$, where $\varphi_\lambda(x,y) = (\lambda x, \lambda^n y)$.

Let us prove that the connected component of $G(f)$ which contains the point $(x,f(x))$ with $x \in ]x_0,x_1[$ is the set $\{(x,f(x))\}$. To prove this, we note that the sets $U_K^*,V_K^*$ separate any of these points from the points $(y,f(y))$ of the graph satisfying $y \notin [x_0,x_1]$. Thus it is only necessary to prove our claim, the following two cases:

**Case 1:** $(x,f(x)) \in U_K^*$. The density of $G(f)$ in $\Gamma_f$ implies there exist an infinite sequence of open intervals $[a_n,b_n] \subset [x_0,x_1]$ such that $a_n < x < b_n$, $\lim_{n \to \infty} |a_n - b_n| = 0$, $(a_n,f(a_n)),(b_n,f(b_n)) \in V_K^*$. Hence

$$K^*(a_n,b_n) = ((a_n,b_n) \times \mathbb{R} \cap U_K^{*\circ}) \cup K^* \cap \overline{U_K^{*\circ}}$$

is a sequence of connected subsets of the plane which separate the point $(x,f(x))$ from any other point $(y,f(y))$ with $y \neq x$, $y \in ]x_0,x_1[$, and $G(f) \cap K^*(a_n,b_n) = \emptyset$ for all $n$. Hence $\{(x,f(x))\}$ is the connected component which contains the point $(x,f(x))$.

**Case 2:** $(x,f(x)) \in V_K^*$. This case has an analogous proof to Case 1.

The proof ends now easily. Indeed, if $(x,f(x))$ is any point of $G(f)$, there exists $\lambda \in \mathbb{Q}$ such that $x \in ]\lambda x_0, \lambda^n x_1[$ (since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$) and we can use the arguments above with $\varphi_\lambda(K^*)$ instead of $K^*$.

Last claim of the theorem follows from the existence of discontinuous additive functions $f : \mathbb{R} \to \mathbb{R}$ with connected graph $G(f)$, a fact that was demonstrated by Jones by using a nontrivial set theoretical argument on ordinals [4, Theorems 4 and 5]. Indeed, assume that $f : \mathbb{R} \to \mathbb{R}$ is additive discontinuous and $G(f)$ is connected. Then $F(x) = x^{n-1}f(x)$ is a discontinuous $n$-monomial function with connected graph, since the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(x,y) = (x,x^{n-1}y)$ is continuous and transforms the graph of $f$ onto the graph of $F$.}$\square$

**Corollary 2.4** (Dichotomy, for additive functions $f : \mathbb{R} \to \mathbb{R}$). Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Then $G(f)$ is connected or totally disconnected. Furthermore, there exists discontinuous additive functions $f : \mathbb{R} \to \mathbb{R}$ with connected graph $G(f)$. 

}\]
Proof. Any additive function is a 1-monomial function.

Remark 2.5. If $G(f)$ is connected, we have two cases: either $f$ is continuous and $G(f) = V$ is a one-dimensional vector space, or $G(f)$ is a connected dense additive subgroup of $\mathbb{R}^2$.

The following theorem may be also of interest:

**Theorem 2.6** ($(d + 2)$-chotomy property of additive functions). If $f : \mathbb{R}^d \to \mathbb{R}$ an additive function, then:

(a) There exists $s \in \{0, 1, \ldots, d+1\}$ such that every connected component of $G(f)$ is a dense subgroup of an $s$-dimensional affine subspace of $\mathbb{R}^{d+1}$.

(b) All cases described in (a) are attained by concrete examples.

Proof. (a) We consider $f : \mathbb{R}^d \to \mathbb{R}$ an additive function and we set $G(f) = \{(x, f(x)) : x \in \mathbb{R}^d\}$. Obviously, the additivity of $f$ implies that $G(f)$ is an additive subgroup of $\mathbb{R}^{d+1}$. Let $G$ be the connected component of $G(f)$ which contains the zero element. Every connected component of $G(f)$ results from $G$ by a translation.

$G$ is a connected additive subgroup of $\mathbb{R}^{d+1}$. Hence, its topological closure $\overline{G}$ is also a connected subgroup of $\mathbb{R}^{d+1}$. It is known that the topological closure of any additive subgroup $H$ of $\mathbb{R}^{d+1}$ satisfies $\overline{H} = V \oplus \Lambda$ for a certain vector subspace $V$ of $\mathbb{R}^{d+1}$ and a discrete additive subgroup $\Lambda$ of $\mathbb{R}^{d+1}$ (see [9, Theorem 3.1] for a proof of this fact). It follows that $\overline{G} = V$ for a certain vector subspace $V$ of $\mathbb{R}^{d+1}$. Hence every connected component of $G(f)$ is a dense connected additive subgroup of the affine space $V + \{\tau\}$ for some $\tau \in \mathbb{R}^{d+1}$. Note that, if $V = \{0\}$ then $G(f)$ is totally disconnected and, if $V = \mathbb{R}^{d+1}$, then $G(f)$ is a connected dense additive subgroup of $\mathbb{R}^{d+1}$. All the other cases represent an intermediate situation. For example, if $f$ is continuous, then $G(f) = V$ is a $d$-dimensional vector subspace of $\mathbb{R}^{d+1}$.

(b) All cases described by Theorem 2.6 can be constructed easily, since all functions $f : \mathbb{R}^d \to \mathbb{R}$ of the form $f(x_1, \ldots, x_d) = A_1(x_1) + A_2(x_2) + \cdots + A_d(x_d)$, with $A_k : \mathbb{R} \to \mathbb{R}$ additive for each $k$, are additive, and we can use the dichotomy result for each one of these functions $A_k$, $k = 1, \ldots, d$.

Remark 2.7. While searching in the literature for a demonstration of Corollary 2.4, the author commented this question to Professor Laszlo Székeleyhidi, who also was unable to find the proof nowhere. Then, he got a very nice independent proof of the result [8]. Indeed, for $d = 1$ we get the dichotomy result as follows (this is Székeleyhidi’s idea): Let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the canonical projection $\pi(x, y) = x$, and let $W = \pi(G)$ be the projection of the connected component of $G(f)$ which contains the zero element. Then $\pi(G) = \{0\}$ or $\pi(G) = \mathbb{R}$, since the only connected subgroups of the real line are $\{0\}$ and $\mathbb{R}$. Thus, if $G(f)$ is not totally disconnected, then $\pi(G) = \mathbb{R}$, which implies $G = G(f)$ and hence, $G(f)$ is connected. Unfortunately, this simple proof seems to be very difficult to generalize for the case of monomial functions $f : \mathbb{R} \to \mathbb{R}$, since the graph of an $n$-monomial function is in general not an additive subgroup of $\mathbb{R}^2$. I hope this justifies to introduce the proof of Theorem 2.2.

**References**

1. J. M. Almira, Kh. F. Abu-Helaiel, A note on monomials, Mediterr. J. Math. **10** (2013), 779-789.
2. A. Czarnecki, M. Kulczycki and W. Lubański, On the connectedness of boundary and complement for domains, Ann. Polon. Mat. **103** (2) (2012) 89-91.
3. S. Czerwik, Functional equations and inequalities in several variables, World Scientific, 2002.
4. F. B. Jones, Connected and disconnected plane sets and the functional equation $f(x) + f(y) = f(x + y)$, Bull. Amer. Math. Soc. **48** (1942) 115-120.
5. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x + y) = f(x) + f(y)$, Math. Ann. 60 (1905) 459-462.

6. M. Kuczma, An introduction to the theory of functional equations and inequalities, Second Edition, Birkhäuser Verlag, 2009.

7. R. San Juan, Una aplicación de las aproximaciones diofánticas a la ecuación funcional $f(x_1 + x_2) = f(x_1) + f(x_2)$, Publicaciones del Inst. Matemático de la Universidad Nacional del Litoral 6 (1946) 221-224.

8. L. Székelyhidi, Remark on the graph of additive functions, manuscript, submitted, 2014.

9. M. Waldschmidt, Topologie des Points Rationnels, Cours de Troisième Cycle 1994/95 Université P. et M. Curie (Paris VI), 1995.

Jose Maria Almira
Departamento de Matemáticas, Universidad de Jaén, Spain
E.P.S. Linares, C/Alfonso X el Sabio, 28
23700 Linares (Jaén) Spain
e-mail address: jmalmira@ujaen.es