MODULI SPACES OF PAIRS OVER PROJECTIVE STACKS

ELENA ANDREINI

Abstract. Let $X$ a projective stack over an algebraically closed field $k$ of characteristic 0. Let $E$ be a generating sheaf over $X$ and $O_X(1)$ a polarization of its coarse moduli space $X$. We define a notion of pair which is the datum of a non vanishing morphism $\Gamma \otimes E \to F$ where $\Gamma$ is a finite dimensional $k$ vector space and $F$ is a coherent sheaf over $X$. We construct the stack and the moduli space of semistable pairs. The notion of semistability depends on a polynomial parameter and it is dictated by the GIT construction of the moduli space.

1. Introduction

Recently a lot of attention has been drawn by sheaf theoretic curve counting theories of projective threefolds. Among them Pandharipande-Thomas invariants \cite{14} are computed via integration over the virtual fundamental class of the moduli space of the so called stable pairs. The moduli spaces used to compute PT invariants are a special case of
moduli spaces of coherent systems introduced by Le Potier in [11]. A coherent system is the datum of a pair \((F, \Gamma)\), where \(F\) is a pure \(d\)-dimensional coherent sheaf and \(\Gamma \subset H^0(X, F)\) is a subspace of its global sections. The moduli spaces are constructed as projective varieties via GIT techniques. The GIT stability condition is equivalent to a modified Gieseker stability, where the Hilbert polynomial is corrected by a contribution proportional to \(\dim \Gamma\) and to a polynomial stability parameter. A coherent system can be reconstructed from the associated evaluation morphism \(\text{ev} : \Gamma \otimes \mathcal{O}_X \to F\). In this note we study a similar moduli problem over projective stacks. If we work over an algebraically closed field \(k\) of characteristic zero, a projective stack is a stack with projective coarse moduli space that can be embedded into a smooth proper Deligne-Mumford stack. Any projective stack admits a generating sheaf \(\mathcal{E}\), namely a locally free sheaf whose fibers carry every representation of the automorphism group of the underlying point. We propose a notion of pair on projective stacks which is a natural generalization of the evaluation morphism in the setting of [12]. The pair is defined as a non vanishing morphism

\[ \phi : \Gamma \otimes \mathcal{E} \to \mathcal{F} \]

where \(\Gamma\) is a finite dimensional \(k\)-vector space and \(\mathcal{F}\) is a coherent sheaf on \(X\). Note that \(\phi\) determines a morphism

\[ \text{ev}(\phi) : \Gamma \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{E}^\vee \]

which we don’t require to be injective on global sections. Such a definition is reasonable if we think of projective stacks which are banded gerbes. In that case, there are no morphisms \(\Gamma \otimes \mathcal{O}_X \to \mathcal{F}\) if \(\mathcal{F}\) is not a pullback from the coarse moduli space. Then in this case by twisting coherent sheaves by the generating sheaf we get a richer theory than the theory of the coarse moduli space. We give a notion of semistability which depends on a Hilbert polynomial of pairs, defined as the sum of the usual Hilbert polynomial of \(\mathcal{F} \otimes \mathcal{E}^\vee\) plus a term depending on a polynomial \(\delta\). We follow closely two papers dealing with very similar moduli problems: [7] and [16]. In this note we make the exercise of checking that the proofs extend to our setting, by using results on sheaves over projective stacks proven in [12]. We construct the stack of semistable pairs as a global quotient stack and we obtain its coarse moduli space with GIT techniques.

Conventions. In this paper we work over an algebraically closed field \(k\) of characteristic zero. By an algebraic stack we mean an algebraic stack over \(k\) in the sense of [3]. By a Deligne-Mumford stack we mean an algebraic stack over \(k\) in the sense of [4]. We assume moreover all stacks and schemes unless otherwise stated are are noetherian of finite type over \(k\). For sheaves on stacks we refer to [10].

Notations. When dealing with sheaves we often adopt the notation in [6]. We denote by \(X \xrightarrow{\pi} X \to k\) a projective stack. We choose a polarization \(\mathcal{O}_X(1)\) on the coarse moduli space. Given a sheaf \(\mathcal{F}\) over \(X\) we often denote by \(\mathcal{F}(m)\) the sheaf \(\mathcal{F} \otimes \pi^* \mathcal{O}_X(m)\).

2. Introductory Material

2.1. Recall on projective stacks and on generating sheaves.

Definition 2.1. A projective stack is Deligne-Mumford stack with projective coarse moduli scheme and a locally free sheaf which is a generating sheaf in the sense of [13].

For the reader’s convenience we recall the notion of generating sheaf following [12].
Definition 2.2 (Generating sheaf). A locally free sheaf $\mathcal{E}$ is said to be a generator for a quasi coherent sheaf $\mathcal{F}$ is the adjunction morphism (left adjoint to the identity $\pi_*\mathcal{F} \otimes \mathcal{E}^\vee \xrightarrow{i_!} \pi_*\mathcal{F} \otimes \mathcal{E}^\vee$)

(1) \[ \theta_\mathcal{E}(\mathcal{F}) : \pi^*\pi_*\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F} \]

is surjective. It is a generating sheaf of $X$ if it is a generator for every quasi coherent sheaf on $X$.

A characterization of generating sheaves can be given by making use of a relative (to the base) ampleness notion for locally free sheaves on stacks introduced in [13].

Definition 2.3. A locally free sheaf on $X$ is $\pi$-ample if and only if for every geometric point of $X$ the representation on the fiber of the stabilizer group at that point is faithful.

Definition 2.4. A locally free sheaf $\mathcal{E}$ on $X$ is $\pi$-very ample if for any geometric point of $X$ at that point the representation of the stabilizer group on the fiber at that point contains every irreducible representation.

Proposition 2.5 ([8], 5.2). Let $\mathcal{E}$ be a $\pi$-ample sheaf on $X$, then there is a positive integer $r$ such that the locally free sheaf $\bigoplus_{i=0}^r \mathcal{E}^\otimes i$ is $\pi$-very ample.

Proposition 2.6 ([13], 5.2). A locally free sheaf on a Deligne-Mumfords stack $X$ is a generating sheaf if and only if it is $\pi$-very ample.

We now come to the definition of projective stack. In [8] it shown that for a proper Deligne-Mumford stack over a field the following characterizations are equivalent.

Theorem 2.7 ([8] Corollary 5.4). Let $X \rightarrow k$ be a proper Deligne-Mumford stack. Then the following are equivalent:

1) the stack $X$ has projective coarse moduli space is a quotient stack;
2) the stack $X$ has a projective coarse moduli scheme and there exists a generating sheaf;
3) the stack $X$ has a closed embedding in a smooth proper Deligne-Mumford stack over $k$ and has a projective coarse moduli scheme.

The third statement is used in [8] as a definition.

Definition 2.8 ([8] Definition 5.5). A stack $X \rightarrow k$ is projective if it admits a closed embedding into a smooth Deligne-Mumford stacks proper over $k$ and has projective coarse moduli space.

Definition 2.9 (Functors $F_\mathcal{E}$ and $G_\mathcal{E}$). Let $\mathcal{E}$ be a locally free sheaf on $X$. Let $F_\mathcal{E} : \mathcal{QCoh}_{X/S} \rightarrow \mathcal{QCoh}_{X/S}$ be the functor mapping $\mathcal{F} \mapsto \pi_*\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ and let $G_\mathcal{E} : \mathcal{QCoh}_{X/S} \rightarrow \mathcal{QCoh}_{X/S}$ be a second functor mapping $\mathcal{F} \mapsto \pi^*\mathcal{F} \otimes \mathcal{E}$.

Remark 2.10. The functor $F_\mathcal{E}$ is exact because both $\otimes \mathcal{E}^\vee$ and $\pi_*$ are exact functors. On the other hand $G_\mathcal{E}$ is not exact unless $\pi^*$ is exact, i.e. $\pi$ is flat. Examples of stacks with flat map to the coarse moduli scheme are flat gerbes (e.g. [10] Definition 3.5) over schemes or stacks root of line bundles (see [1] or [2]).

Remark 2.11. The notation $F_\mathcal{E}$ is the same as in [13] but $G_\mathcal{E}$ there corresponds to $G_\mathcal{E} \circ F_\mathcal{E}$ here.
Notation 2.12. We denote by

\[ \iota_{\mathcal{E}}(\mathcal{F}) : \mathcal{F} \otimes \mathcal{O}_X \to \mathcal{F} \otimes \text{End}(\mathcal{E}) \]

the injective morphism mapping a section to its tensor product with the identity endomorphism of \( \mathcal{E} \).

In this paper we use the following notion of slope. Given a coherent sheaf \( \mathcal{F} \) of dimension \( d \)

\[ \hat{\mu}_{\mathcal{E}}(\mathcal{F}) = \frac{\alpha_{d-1}(\mathcal{F} \otimes \mathcal{E})}{\alpha_d(\mathcal{F} \otimes \mathcal{E}^\vee)} \]

We often denote the multiplicity \( \alpha_d(\mathcal{F} \otimes \mathcal{E}^\vee) \) by \( r_{\mathcal{E},\mathcal{F}} \).

3. Setting up the moduli problem

In the following \( \pi : X \to X \to k \) will be a smooth projective stack with coarse moduli scheme \( X \) over an algebraically closed field \( k \). We will fix a polarization \((\mathcal{O}_X(1), \mathcal{E})\) and a rational polynomial \( \delta \) such that \( \delta(m) \geq 0 \) for \( m >> 0 \).

Definition 3.1. A pair \((\mathcal{F}, \phi)\) is a non-trivial morphism

\[ \phi : \Gamma \otimes \mathcal{E} \to \mathcal{F}, \]

where \( \Gamma \) is a finite dimensional \( k \)-vector space, \( \mathcal{F} \) is a coherent sheaf of dimension \( d \), \( d \in \mathbb{N} \), \( d \leq \text{dim } X \), and \( \mathcal{E} \) is the fixed generating sheaf. A morphism between two pairs \((\mathcal{F}, \phi), (\mathcal{F}', \phi')\) is a commutative diagram

\[ \begin{array}{ccc} \Gamma \otimes \mathcal{E} & \xrightarrow{\phi} & \mathcal{F} \\ \downarrow{\lambda} & & \downarrow{\alpha} \\ \Gamma \otimes \mathcal{E} & \xrightarrow{\phi'} & \mathcal{F}' \end{array} \]

where \( \lambda \in \mathbb{C}^* \) and \( \alpha \) is a morphism of coherent sheaves.

Note that we can relate the notion of pairs to a stacky version of coherent systems of [11]. For the reader's convenience we recall the definition of coherents systems on schemes given by LePotier.

Definition 3.2. [11] Def. 4.1 Let \( X \) be a smooth projective variety of dimension \( n \). A coherent system of dimension \( d \) is a pair \((\Gamma, F)\), where \( F \) is a coherent sheaf of dimension \( d \) over \( X \) and \( \Gamma \subseteq H^0(F) \) is a vector subspace.

We extend this notion to projective Deligne-Mumford stacks.

Definition 3.3. Let \( \mathcal{X} \) be a projective Deligne-Mumford stack over \( k \). A twisted coherent system on \( \mathcal{X} \) is a pair \((\Gamma, \mathcal{F})\) where \( \mathcal{F} \) is a coherent sheaf over \( \mathcal{X} \) and \( \Gamma \subseteq H^0(\mathcal{F} \otimes \mathcal{E}^\vee) \) is a vector subspace.

A pair \((\mathcal{F}, \phi)\) determines a subspace of \( H^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee) \) given by the image of \( \Gamma \) along \( H^0(\text{ev}(\phi)) \) where

\[ \text{ev}(\phi) : \Gamma \otimes \mathcal{O}_X \xrightarrow{\iota_{\mathcal{E}}(\Gamma)} (\Gamma \otimes \mathcal{E}) \otimes \mathcal{E}^\vee \xrightarrow{\phi \otimes \mathcal{E}^\vee} \mathcal{F} \otimes \mathcal{E}^\vee \]
is obtained from $\phi$ by applying the functor $- \otimes \mathcal{E}^\vee$ and by composing by the inclusion morphism $\iota_\mathcal{E}(\Gamma)$. Note that $H^0(ev(\phi))$ is not necessarily injective on global sections. Hence the twisted coherent system determined as above is $(W, \mathcal{F})$ with $\dim W < \dim \Gamma$. Conversely, let us consider a twisted pair $(\Gamma, \mathcal{F})$. Let

$$ev : \Gamma \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{E}^\vee.$$  

be the corresponding evaluation morphism. It is possible to associate to (7) the pair $(\mathcal{F}, \phi)$ obtained by applying the functor $- \otimes \mathcal{E}^\vee$ and by composing with $\text{Tr} : \text{End}(\mathcal{E}) \to \mathcal{O}_X$. It is not hard to see that $ev(\phi(ev)) = ev$ and that $\phi(ev(\phi)) = \phi$.

We also give the definition of family of pairs. Let $S$ be a scheme of finite type over $k$. Let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ be the natural projections.

**Definition 3.4.** A pair parametrized by $S$ is a $S$-flat coherent sheaf $\mathcal{F}$ over $X \times S$ and a homomorphism

$$\phi_S : \pi_X^* \Gamma \otimes \mathcal{E} \to \mathcal{F}$$

such that for any closed point $s$ of $S$

$$\phi_S(s) : \pi_X^* \Gamma \otimes \mathcal{E}(s) \to \mathcal{F}(s)$$

is a pair.

3.1. (Semi)stability. We define a parameter-dependent Hilbert polynomial for a stable pair in the following way.

**Definition 3.5.** The Hilbert polynomial of a pair $(\mathcal{F}, \phi)$ is

$$P_\mathcal{E}(\mathcal{F}, \phi) := P(F_\mathcal{E}(\mathcal{F})) + \epsilon(\phi)\delta$$

where $\epsilon(\phi) = 1$ if $\phi \neq 0$ and $0$ otherwise. The reduced Hilbert polynomial is

$$p_\mathcal{E}(\mathcal{F}, \phi) := \frac{P_\mathcal{E}(\mathcal{F}, \phi)}{r_{F_\mathcal{E}(\mathcal{F})}}.$$  

**Remark 3.6.** Note that in the above definition following [12] we do not use the Hilbert polynomial $P(\mathcal{F})$ of the sheaf on the stack, rather the Hilbert polynomial $P(F_\mathcal{E}(\mathcal{F})) = P(\mathcal{F} \otimes \mathcal{E}^\vee)$. The reason is particularly evident if we consider sheaves on gerbes. In this case the non twisted Hilbert polynomial of any sheaf which is not a pull-back from the coarse moduli space vanishes. For more details see [12].

We will define (semi)stability by using the Hilbert polynomial introduced above. We need some more preliminary remarks and notations.

**Definition 3.7.** Let $(\mathcal{F}, \phi)$ be a stacky pair. Any subsheaf $\mathcal{F}' \subset \mathcal{F}$ defines a induced homomorphism $\phi' : \Gamma \otimes \mathcal{E} \to \mathcal{F}'$ which is equal to $\phi$ if $\text{Im} \phi \subseteq \mathcal{F}'$ and zero otherwise. The corresponding quotient $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ also inherits an induced homomorphism $\phi'' : \Gamma \otimes \mathcal{E} \to \mathcal{F}''$ which is defined as the composition of $\phi$ with the quotient map. Note that it is the zero morphism if and only if $\text{Im} \phi \subseteq \mathcal{F}'$.

**Remark 3.8.** The Hilbert polynomial of stacky pairs is additive on short exact sequences.
Definition 3.9. A stacky pair is δ (semi)stable if for any saturated submodule $\mathcal{F}' \subset \mathcal{F}$

\[(11) \quad P_\varepsilon(\mathcal{F}', \phi')(\leq) r_{F_\varepsilon(\mathcal{F})} p(F_\varepsilon(\mathcal{F}), \phi)\]

Definition 3.10. A δ semi stable pair parametrized by $S$ is a stacky pair over $S$ such that for every closed point of $S$ the pair $(\mathcal{F}(s), \phi|_{\pi_X^*\mathcal{E}(s)})$ is a δ semistable pair.

3.2. Properties of δ semi stable pairs. We note that δ semi stability implies purity of the underlying sheaf of the pair.

Proposition 3.11. Let $(\mathcal{F}, \phi)$ be a δ semi stable pair. Then $\mathcal{F}$ is pure.

Proof. Let us assume that $\mathcal{F}$ is not pure. Let $\mathcal{F} = T_{d-1}(\mathcal{F})$ be the element of the torsion filtration of maximal dimension. Then

\[(12) \quad P_\varepsilon(\mathcal{F}) + \epsilon(\mathcal{F}) \leq \frac{r_{F_\varepsilon(\mathcal{F})}}{r_{F_\varepsilon(\mathcal{F})}} (p + \delta) = 0\]

where the equality on the r.h.s. holds because $\mathcal{F}$ is a sheaf at most of dimension $d - 1$. It follows that $\mathcal{F} = 0$. □

For δ semistability we have characterizations and properties analogous to usual Gieseker semi stability for sheaves on schemes. We list some of them

Proposition 3.12. Let $(\mathcal{F}, \phi)$ be a stacky pair. Then the following conditions are equivalent:

1. for all proper subsheaves $\mathcal{F}' \subset \mathcal{F}$

\[P_\varepsilon(\mathcal{F}', \phi')(\leq) r_{F_\varepsilon(\mathcal{F})} p_\varepsilon(\mathcal{F}, \phi);\]

2. $(\mathcal{F}, \phi)$ is (semi)stable;

3. for all proper quotient sheaves $\mathcal{F} \rightarrow \mathcal{F}''$ with $\alpha_d(F_\varepsilon(\mathcal{F})) > 0$

\[P_\varepsilon(\mathcal{F}'', \phi'')^{(\geq)} r_{F_\varepsilon(\mathcal{F}'')} p_\varepsilon(\mathcal{F}, \phi);\]

4. for all proper purely $d$-dimensional quotient sheaves $\mathcal{F} \rightarrow \mathcal{F}''$ with $\alpha_d(F_\varepsilon(\mathcal{F})) > 0$

\[P_\varepsilon(\mathcal{F}'', \phi'')^{(\geq)} r_{F_\varepsilon(\mathcal{F}'')} p_\varepsilon(\mathcal{F}, \phi);\]

Proof. The proof is very similar to [6] Prop. 1.2.6. We use additivity of the ranks and of the modified Hilbert polynomials on short exact sequences. Note that if inequality \[(11)\] holds for saturated subsheaves, it also holds for arbitrary subsheaves. Indeed, let $\mathcal{F}' \subset \mathcal{F}$ be a not necessarily saturated subsheaf. Then if $\text{Im} \phi$ is contained in $\mathcal{F}'$, then $\text{Im} \phi \subset \mathcal{F}'^s$, where $\mathcal{F}'^s$ is the saturation of $\mathcal{F}'$ in $\mathcal{F}$. Moreover $P_\varepsilon(\mathcal{F}') \leq P_\varepsilon(\mathcal{F}'^s)$. □

Lemma 3.13. Let $(\mathcal{F}, \phi), (\mathcal{G}, \psi)$ be two δ semistable pairs such that $p_\varepsilon((\mathcal{F}, \phi)) > p_\varepsilon((\mathcal{G}, \psi))$. Then $\text{Hom}(\mathcal{F}, \phi), (\mathcal{G}, \psi)) = 0$.

Proof. Let us assume there is a non zero morphism $(\alpha, \lambda)$. Let $\mathcal{H} = \text{Im} \alpha$. By semi stability we get

\[(13) \quad p_\varepsilon(\mathcal{F}, \phi) \leq p_\varepsilon(\mathcal{H}, \phi_H) = p_\varepsilon(\mathcal{H}, \psi_H) \leq p_\varepsilon(\mathcal{G}, \psi),\]

where $\phi_H$ and $\psi_H$ are the induced homomorphisms. Inequality \[(13)\] contradicts the assumption. □

Lemma 3.14. Let $\alpha : (\mathcal{F}, \phi) \rightarrow (\mathcal{G}, \psi)$ be a homomorphism between δ stable pairs of the same reduced Hilbert polynomial. Then $\alpha$ is 0 or an isomorphism.

Proof. Analogous to Lemma 1.6 in [16]. Cfr. also [6] Proposition 1.2.7. □
Corollary 3.15. Let \((\mathcal{F}, \delta)\) be a semi stable pair. Then \(\text{End}((\mathcal{F}, \delta))\) is a finite dimensional division algebra. Since we work over an algebraically closed field \(k\), \(\text{End}((\mathcal{F}, \delta)) \cong k\).

Proof. Same as [6] Cor. 1.2.8. □

3.3. Harder-Narasimhan and Jordan-Hölder filtration.

Proposition 3.16. Let \((\mathcal{F}, \phi)\) be a pair such that \(\mathcal{F}\) is pure. Then it admits a unique Harder-Narasimhan filtration
\begin{equation}
0 \subset HN_0(\mathcal{F}, \phi) \subset \ldots \subset HN_{l-1}(\mathcal{F}, \phi) \subset HN_l(\mathcal{F}, \phi) = (\mathcal{F}, \phi)
\end{equation}
such that each \(\text{gr}^{HN}_i(\mathcal{F}, \phi) = HN_i(\mathcal{F}, \phi)/HN_{i-1}(\mathcal{F}, \phi)\) is \(\delta\) semistable and if \(p_i := p_\mathcal{E}(\text{gr}^{HN}_i(\mathcal{F}, \phi))\) then
\begin{equation}
p_{\max}(\mathcal{F}, \phi) = p_1 > p_2 > \ldots > p_l = p_{\min}(\mathcal{F}, \phi)
\end{equation}

Proof. The proof proceeds as in [6] Theorem 1.3.4. Indeed, it is possible to find a subsheaf \(\mathcal{F}_0 \subseteq \mathcal{F}\) such that it is not contained in any subsheaf \(\mathcal{F}'\) of \(\mathcal{F}\) with \(p_\mathcal{E}(\mathcal{F}_0, \phi_0) < p_\mathcal{E}(\mathcal{F}', \phi')\), where \(\phi_0\) and \(\phi'\) are the induced homomorphisms. This implies the existence part. Uniqueness is proven by using Lemma 3.13. □

Proposition 3.17. Let \((\mathcal{F}, \phi)\) be a \(\delta\) semistable pair with reduced Hilbert polynomial \(p\). Then there is a Jordan-Hölder filtration
\begin{equation}
0 = JH_0(\mathcal{F}, \phi) \subset JH_1(\mathcal{F}, \phi) \subset \ldots \subset JH_l(\mathcal{F}, \phi) = (\mathcal{F}, \phi)
\end{equation}
such that each \(\text{gr}^{JH}_i(\mathcal{F}, \phi) = JH_i(\mathcal{F}, \phi)/JH_{i-1}(\mathcal{F}, \phi)\) is \(\delta\) stable with reduced Hilbert polynomial \(p\). The graded object \(\text{gr}^{JH}(\mathcal{F}, \phi) = \oplus_i \text{gr}^{JH}_i(\mathcal{F}, \phi)\) is independent of the choice of the filtration. Note that it inherits an induced homomorphism \(\text{gr}^{JH}(\phi) : \Gamma \otimes \mathcal{E} \rightarrow \text{gr}^{JH}(\mathcal{F}, \phi)\).

Proof. The proof is the same as [6] 1.5. The same arguments holds because of additivity the modified Hilbert polynomial on short exact sequences. □

Remark 3.18. It is not hard to see that \(\text{gr}^{JH}(\phi)\) is non trivial if \(\phi\) is not, and its image is contained in only one summand of \(\text{gr}^{JH}(\mathcal{F}, \phi)\).

Definition 3.19. Two \(\delta\) semistable pairs are said to be \(S\)-equivalent if their Jordan-Hölder graded objects are isomorphic.

Remark 3.20. From now on we will always assume that \(\delta\) is strictly positive and \(\phi\) is non vanishing otherwise \(\delta\) (semi)stability reduces to usual Gieseker (semi)stability for sheaves on stacks.

We introduce a symbol which is convenient to restate the (semi)stability condition when assuming that the homomorphism of a pair is non vanishing.

Definition 3.21. Let \((\mathcal{F}, \phi)\) be a stacky pair. For any exact sequence
\begin{equation}
0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0
\end{equation}
let
\[\epsilon(\mathcal{F}') := \begin{cases} 1 & \text{if } \text{Im } \phi \subseteq \mathcal{F}' \\ 0 & \text{otherwise} \end{cases}\]
and
\[\epsilon(\mathcal{F}'') := 1 - \epsilon(\mathcal{F}')\]
With the above definition we can restate the \( \delta \) (semi)stability condition for stacky pairs \((\mathcal{F}, \phi)\). Indeed \((\mathcal{F}, \phi)\) is semi stable if and only if for every saturated subsheaf \( \mathcal{F}' \)

\[
P_{\mathcal{E}}(\mathcal{F}') + \epsilon(\mathcal{F}')\delta \leq \frac{r_{F_{\mathcal{E}}}(\mathcal{F})}{r_{F_{\mathcal{E}}}(\mathcal{F})}(P_{\mathcal{E}}(\mathcal{F}) + \delta)
\]

4. Boundedness

In this section we prove boundedness of the family of \( \delta \) semistable pairs. Since we want to use a GIT construction similar to \cite{17} and \cite{16} we take \( \deg \delta < \dim X \).

**Proposition 4.1.** Let \( P \) be a fixed polynomial of degree \( d < \dim X \). Then the family of \( \delta \) semistable pairs with Hilbert polynomial \( P \) is bounded.

**Proof.** Let \( \mathcal{F} \) be a family of coherent sheaves over \( X \). According to \cite{12} Corollary 4.17 \( \mathcal{F} \) is bounded if and only if \( F_{\mathcal{E}}(\mathcal{F}) \) is bounded over \( X \). We use \cite{15} Theorem 1.1, according to which a family \( \mathcal{F} \) of sheaves over a projective scheme \( X \) with fixed Hilbert polynomial is bounded if and only there exists a constant \( C \) such that for any \( F \in \mathcal{F} \)

\[
\mu_{\text{max}}(F) \leq C.
\]

Let \( \phi : \mathcal{E} \otimes \Gamma \to \mathcal{F} \) be a \( \delta \) semistable pair. Let \( \text{Supp}(\mathcal{F}) = Y \). Let us assume first that \( \text{Im} \phi \not\subseteq HN_{l-1}(\mathcal{F}) \), where \( HN_{l-1}(\mathcal{F}) \) is the maximal proper subsheaf in the Harder-Narasimhan filtration. Then the composition

\[
\Gamma \otimes \mathcal{E} \otimes \mathcal{O}_Y \to \mathcal{F} \to gr_1^H N(\mathcal{F})
\]

is a non zero morphism between sheaves of pure dimension \( d \). This implies that

\[
\mu_{\text{min}}(\mathcal{E} \otimes \mathcal{O}_Y) \leq \mu_{\text{min}}(\mathcal{F}).
\]

We note first that \( \mu_{\text{min}}(\mathcal{E} \otimes \mathcal{O}_Y) \geq \mu_{\text{min}}(\pi_*\text{End}(\mathcal{E}) \otimes \mathcal{O}_Y) \) where \( Y = \pi(Y) \). The reason is that not all the quotient sheaves of \( F_{\mathcal{E}}(\mathcal{F}) \) are obtained as images by the functor \( F_{\mathcal{E}} \) of quotient sheaves of \( \mathcal{F} \) (cfr.\cite{12} Remark 3.15). We want to find a lower bound for \( \mu_{\text{min}}(\mathcal{O}_Y) \). This is provided by a result proven in \cite{11}.

**Corollary 4.2** (\cite{11} Corollary 2.13). Let \( X \) be a projective scheme over \( k \). Let \( S \) be a subscheme of pure dimension \( d \) and of degree \( k \). Then \( \mu_{\text{min}}(\mathcal{O}_S) \) is bounded from below by a constant which only depends on \( d, k \) and \( X \).

We observe that \( Y = \pi \text{Supp} \mathcal{F} = \text{Supp} F_{\mathcal{E}}(\mathcal{F}) \) is a purely \( d \)-dimensional subscheme of degree \( \leq r_{\mathcal{E}}^2 \mathcal{F} \). Then

\[
\mu_{\text{min}}(\mathcal{F}) \geq \mu_{\text{min}}(\mathcal{O}_Y) + \mu_{\text{min}}(\pi_*\text{End}(\mathcal{E})) \geq A + \mu_{\text{min}}(\pi_*\text{End}(\mathcal{E})) := B
\]

for some constant \( A \) which only depends on \( X \) and on the fixed polynomial \( P \). By the barycenter formula for the slope this implies that

\[
\mu_{\text{max}}(\mathcal{F}) \leq \max\{r_{\mathcal{E},\mathcal{F}} \mu_{\mathcal{E}}(\mathcal{F}) - (r_{\mathcal{E},\mathcal{F}} - 1) B, \mu_{\mathcal{E}}(\mathcal{F})\}.
\]

Boundedness of \( \mu_{\text{max}}(F_{\mathcal{E}}(\mathcal{F})) \) follows from Lemma 4.3. Let us consider now the case where \( \text{Im} \phi \subseteq HN_{l-1}(\mathcal{F}) \). Then by \( \delta \) semistability

\[
p_{\mathcal{E}}(HN_{l-1}(\mathcal{F})) \leq p_{\mathcal{E}}(\mathcal{F}),
\]

which in turn implies that

\[
p_{\mathcal{E}}(\mathcal{F}) \leq p_{\mathcal{E}}(gr_1(\mathcal{F}))
\]

and by the barycenter formula that

\[
\mu_{\text{max}}(\mathcal{F}) \leq \mu_{\mathcal{E}}(\mathcal{F}).
\]
Summing up we get
\[ \mu_{\text{max}}(F_\ell(\mathcal{F})) \leq \max\{\hat{\mu}_\ell(\mathcal{F}), r_{\ell,\mathcal{F}}\mu_\ell(\mathcal{F}) - (r_{\ell,\mathcal{F}} - 1)B\} + \tilde{m} \deg \mathcal{O}_X(1), \]
where the inequality is a consequence of the above estimates and of Lemma 4.3. \qed

**Lemma 4.3.** Let \( \mathcal{F} \) be a coherent sheaf on \( X \) of pure dimension \( d \). Then if \( \mu_{\text{max},\ell}(\mathcal{F}) \) (\( \mu_{\text{min},\ell}(\mathcal{F}) \)) is bounded from above (below), then also \( \mu_{\text{max}}(F_\ell(\mathcal{F})) \) (\( \mu_{\text{max}}(F_\ell(\mathcal{F})) \)) is bounded from above (below).

**Proof.** The proof is similar to [12] Proposition 4.24. Let \( \mathcal{F} \subset F_\ell(\mathcal{F}) \) be the maximal destabilizing subsheaf. Let us consider the morphism
\[ \pi^*F \otimes \mathcal{E} \longrightarrow \pi^*(\pi_*\mathcal{F} \otimes \mathcal{E}) \otimes \mathcal{E} \longrightarrow \mathcal{F}. \]
The right arrow is surjective by definition of generating sheaf. Let \( \mathcal{F} \) be the subsheaf corresponding to the image of the composition. By applying the functor \( F_\ell \) we get the surjective morphism
\[ \mathcal{F} \otimes \pi_*\text{End}(\mathcal{E}) \longrightarrow F_\ell(\mathcal{F}). \]

Let \( \tilde{m} > 1 \) be an integer number such that \( \pi_*\text{End}(\mathcal{E}) \) is generated by global sections. Let \( N = h^0(\pi_*\text{End}(\mathcal{E})/\tilde{m}) \). Then there is a surjective morphism \( \mathcal{F} \otimes \mathcal{O}_X^N(-\tilde{m}) \longrightarrow F_\ell(\mathcal{F}) \). Note that since \( \mathcal{F} \) is semistable, so is \( \mathcal{F}(-\tilde{m}) \). Moreover, for any \( k \in \mathbb{N} \), \( \mathcal{F}(-\tilde{m})^\oplus_k \) is also semistable. By composition we get the surjective morphism
\[ \mathcal{F}(-\tilde{m})^\oplus_N \longrightarrow \mathcal{F} \otimes \pi_*\text{End}(\mathcal{E}) \longrightarrow F_\ell(\mathcal{F}). \]

Then by semistability of \( \mathcal{F}(-\tilde{m})^\oplus_N \)
\[ \hat{\mu}_{\text{max}}(F_\ell(\mathcal{F})) \leq \hat{\mu}_\ell(\mathcal{F}) + \tilde{m} \deg \mathcal{O}_X(1) \leq \hat{\mu}_{\text{max},\ell}(\mathcal{F}) + \tilde{m} \deg \mathcal{O}_X(1). \]
Hence if \( \hat{\mu}_{\text{max},\ell}(\mathcal{F}) \) is bounded from above, \( \hat{\mu}_{\text{max}}(F_\ell(\mathcal{F})) \) is also bounded from above. Boundeness from below of \( \hat{\mu}_{\text{max}}(F_\ell(\mathcal{F})) \) also follows. \qed

### 4.1. Rephrasing semistability in terms of number of global sections.
We apply here a result due Le Potier and Simpson (cfr. e.g. [6] Corollary 3.3.1 and 3.3.8) in order to get a bound on the number of global sections of \( F_\ell(\mathcal{F}) \), where \( \mathcal{F} \) is a coherent sheaf on \( X \) of pure dimension \( d \). We state it for sheaves on \( X \) obtained by applying the functor \( F_\ell \) to some sheaf on \( X \).

**Corollary 4.4.** Let \( \mathcal{F} \) be a \( d \)-dimensional coherent sheaf over \( X \). Let \( r = r_{\ell,\mathcal{F}} \) be the multiplicity of \( F_\ell(\mathcal{F}) \). Let \( C := r(r + d)/2 \). Then
\[ h^0(\mathcal{F} \otimes \mathcal{E}) \leq \frac{r - 1}{r} \cdot 1 \cdot \frac{1}{d!} [\hat{\mu}_{\text{max}}(F_\ell(\mathcal{F})) + C - 1 + m]^d_+ + \frac{1}{r} \cdot \frac{1}{d!} [\hat{\mu}(F_\ell(\mathcal{F})) + C - 1 + m]^d_+, \]
where \( [x]_+ = \max\{0, x\} \).

We will need the above estimate in order to give the following characterization of semistability.

**Proposition 4.5.** For \( m > 0 \) for any pure pair \((\mathcal{F}, \phi)\) the following properties are equivalent

1. \((\mathcal{F}, \phi)\) is \( \delta \) (semi)-stable,
2. \( P(m) \leq h^0(\mathcal{F} \otimes \mathcal{E}(m)) \) and for any subsheaf \( \mathcal{F}' \subset \mathcal{F} \) with \( 0 < r_{F_\ell(\mathcal{F})} < r_{F_\ell(\mathcal{F})} \)
\[ h^0(\mathcal{F}' \otimes \mathcal{E}(m)) + \epsilon(\mathcal{F}')\delta(m)(\leq) < \frac{r_{F_\ell(\mathcal{F})}}{r_{F_\ell(\mathcal{F})}}(P(m) + \delta(m)) \]
(3) for any quotient $\mathcal{F} \to \mathcal{F}'$ with $0 < r_{F_{\bar{\delta}}(\mathcal{F}')} < r_{F_{\bar{\delta}}(\mathcal{F})}$

$$\frac{r_{F_{\bar{\delta}}(\mathcal{F})}}{r_{F_{\bar{\delta}}(\mathcal{F})}}(P(m) + \delta(m))(\leq) < h^0(\mathcal{F}' \otimes \mathcal{E}'(m)) + \epsilon(\mathcal{F}')(\delta(m))$$

PROOF. We prove $(1) \implies (2)$. The family of sheaves underlying the family of $\delta$ semistable pairs on $\mathcal{X}$ is bounded. This is equivalent to the family of sheaves on $X$ obtained by applying the functor $F_{\bar{\delta}}$ being bounded (cfr. [12 Corollary 4.17]). Hence there exists $m$ such that for any $\mathcal{F}$ in a $\delta$ semistable pair $P_{\bar{\delta}}(m) = h^0(F_{\bar{\delta}}(\mathcal{F})(m))$. Let $\mathcal{F}' \subseteq \mathcal{F}$ be an arbitrary subsheaf of $\mathcal{F}$. Let us assume that inequality (24) gives $\hat{\mu}_{max}(F_{\bar{\delta}}(\mathcal{F})) \leq \hat{\mu}_{max}(F_{\bar{\delta}}(\mathcal{F})) + \tilde{\delta} \deg \Theta_X(1)$. We distinguish two cases:

A) $\hat{\mu}(F_{\bar{\delta}}(\mathcal{F}')) \geq \hat{\mu}_\delta(\mathcal{F}) - (r - 1)\tilde{\delta} \deg \Theta_X(1) - C \cdot r - \delta_1 r$;

B) $\hat{\mu}(F_{\bar{\delta}}(\mathcal{F}')) \leq \hat{\mu}_\delta(\mathcal{F}) - (r - 1)\tilde{\delta} \deg \Theta_X(1) - C \cdot r - \delta_1 r$;

where $C = r(r + d)/2$ and $r = r_{\bar{\delta}, X}$. If $\mathcal{F}'$ is of type $A$, then $\hat{\mu}(F_{\bar{\delta}}(\mathcal{F}'))$ is bounded from below. We observe that we can assume that $\mathcal{F}'$ is saturated (which implies that also $F_{\bar{\delta}}(\mathcal{F}')$ is saturated, because the functor $F_{\bar{\delta}}$ maps torsion filtrations to torsion filtrations, cfr. [12 Corollary 3.17]). Indeed for any sheaf $\mathcal{H}$ on $X P_{\bar{\delta}}(\mathcal{H}) \leq P(\mathcal{H}^\vee)$, where $\mathcal{H}^\vee$ is the saturation of $\mathcal{H}$. Then the family of sheaves of type $A$ is bounded by Grothendieck Lemma for stacks (see [12 Lemma 4.13]). As a consequence the number of Hilbert polynomials of the family is finite and there exists an integer number $m_0$ such that for any $m \geq m_0$ and for any subsheaf $\mathcal{F}'$ of type $A$ $P(\mathcal{F}' \otimes \mathcal{E}'(m)) = h^0(\mathcal{F}' \otimes \mathcal{E}'(m))$ and

$$P(\mathcal{F}'(m) \otimes \mathcal{E}(m)) + \epsilon(\mathcal{F}) \delta(m) \quad (\leq) \quad \frac{r_{F_{\bar{\delta}}(\mathcal{F}')}}{r_{F_{\bar{\delta}}(\mathcal{F})}}[p(m) + \delta(m)] \iff$$

$$P(\mathcal{F}' \otimes \mathcal{E}'(m)) + \epsilon(\mathcal{F}) \delta(m) \quad (\leq) \quad \frac{r_{F_{\bar{\delta}}(\mathcal{F}')}}{r_{F_{\bar{\delta}}(\mathcal{F})}}[p + \delta].$$

Let us consider now sheaves of type $B$. We get $h^0(\mathcal{F}' \otimes \mathcal{E}'(m)) \leq \frac{r' - 1}{r'} \cdot \frac{1}{d!} \left[ \hat{\mu}_{max}(F_{\bar{\delta}}(\mathcal{F}')) + C' - 1 + m \right]^d + \frac{1}{r'} \cdot \frac{1}{d!} \left[ \hat{\mu}(F_{\bar{\delta}}(\mathcal{F}')) \right] + C' + 1 + m]_+^d$,

where $C' = r'(r + d)/2$ and $r' = r_{\bar{\delta}, X}$. This in turn implies

$$\frac{h^0(\mathcal{F}' \otimes \mathcal{E}'(m))}{r_{\bar{\delta}, \mathcal{F}'}{\bar{\delta}}} \leq \frac{r - 1}{r} \cdot \frac{1}{d!} \left[ \hat{\mu}(\mathcal{F}) + \tilde{\delta} \deg \Theta_X(1) + C - 1 + m]_+^d \right.$$ 

$$+ \frac{1}{r} \cdot \frac{1}{d!} \left[ \hat{\mu}(F_{\bar{\delta}}(\mathcal{F}')) \right] + (1 - r)C - (r - 1)\tilde{\delta} \deg \Theta_X(1) - 1 - \delta_1 r + m]_+^d$$

$$\leq \frac{m^d}{d!} + \frac{m^{d-1}}{(d - 1)!} \left( \hat{\mu}(\mathcal{F}) - 1 - \delta_1 \right) + \ldots$$

where $\delta_1$ is the degree $d - 1$ coefficient of $\delta$ and .... stay for lower degree polynomials. We can conclude that

$$\frac{1}{r_{\bar{\delta}, \mathcal{F}'}}(h^0(\mathcal{F}' \otimes \mathcal{E}'(m)) + \epsilon(\mathcal{F}') \delta(m)) \leq \frac{P(m)}{r} \leq \frac{P(m) + \delta(m)}{r}.$$

If inequality (24) gives a different upper bound it is possible to apply the same arguments used above, except that suitably modified bounds have to be chosen to define sheaves of type $A$ and $B$ as on page 10.
(2) \(\Rightarrow\) (3) Let \(F''\) be any quotient of \(F\) with \(0 < r_{F_E(F'')} < r_{F_E(F)}\). Let \(F' \subseteq F\) denote the corresponding kernel. Then

\[ h^0(F'' \otimes \mathcal{E}^\vee(m)) + \epsilon(F'')\delta(m) \geq h^0(F \otimes \mathcal{E}^\vee(m)) - h^0(F'' \otimes \mathcal{E}^\vee(m)) + \delta(m) - \epsilon(F'')\delta(m) \]

\[ \geq \frac{1}{r}(rP_{E}(m) - r'P_{E}(m) + r\delta(m) - r'\delta(m)) \]

where \(r = r_{F_E(F)}, r' = r_{F_E(F')}, r'' = r_{F_E(F'')}\).

(3) \(\Rightarrow\) (1). Let us show first that the underlying sheaves of \(\delta\) semistable pairs satisfying (3) form a bounded family. Let \((\mathcal{F}, \phi)\) be such a pair. Let \(F_{\min}\) the minimal destabilizing quotient of \(F\). Then by hypothesis

\[ \frac{P_{E}(m) + \delta(m)}{r_{E,F}} - \frac{\epsilon(F_{\min})\delta(m)}{r_{E,F_{\min}}} \leq \frac{h^0(F_{\min} \otimes \mathcal{E}^\vee)}{r_{E,F_{\min}}} \]

\[ \leq \frac{1}{d}[\mu_{\delta}(F_{\min}) + \bar{m}_{\text{deg}} \mathcal{O}(1) + C - 1 + m]_{+}^{d/3} \]

where the second inequality can be deduced from Corollary 4.4. It follows that \(\hat{\mu}_{\delta,\min}(F)\) is bounded from below. By Lemma 4.3, \(\hat{\mu}_{\min}(F_{E}(F))\) is also bounded from below or equivalently \(\hat{\mu}_{\max}(F_{E}(F))\) is bounded from above, which implies boundedness for sheaves satisfying (3). Let us consider now an arbitrary quotient \(F''\) of \(F\). Then either \(\hat{\mu}_{\delta}(F'') > \hat{\mu}_{\delta}(F) + \delta_{1}/r\) or \(\hat{\mu}_{\delta}(F'') \leq \hat{\mu}_{\delta}(F) + \delta_{1}/r\). In the first case a strict inequality for Hilbert polynomials is satisfied, implying stability. In the second case \(\hat{\mu}_{\delta}(F'')\) is bounded from above. By Grothendieck Lemma for stacks we get that the family of pure dimensional quotients satisfying the second inequality is bounded. Hence there exists \(m\) large enough such that for any sheaf \(F\) satisfying (3) \(h^0(F'' \otimes \mathcal{E}^\vee(m)) = P_{E}(F''(m))\). Therefore

\[ P(F'' \otimes \mathcal{E}^\vee(m)) + \epsilon(F'')\delta(m) \geq \frac{P_{E}(m) + \delta(m)}{r''} \]

(32)

where \(r = r_{E,F}\) and \(r'' = r_{E,F''}\). Eventually we remark that from the proofs of (2) and (3) equality holds if and only if the subsheaf or the quotient sheaf is destabilizing. \(\square\)

We recall a technical result originally due to Le Potier which will be used in Theorem 6.16. The proof can be found in [12] Lemma 6.10 and generalizes [6] Proposition 4.4.2.

**Lemma 4.6.** Let \(F\) be a coherent sheaf over \(X\) that can be deformed to a sheaf of the same dimension \(d\). Then there is a pure \(d\)-dimensional sheaf \(\mathcal{S}\) on \(X\) with a map \(F \rightarrow \mathcal{S}\) such that the kernel is \(T_{d-1}(F)\) and \(P_{E}(F) = P_{E}(\mathcal{S})\).

5. The parameter space

By Proposition 4 we know that the family of sheaves obtained by applying the functor \(F_{E}\) to the underlying family of sheaves of \(\delta\) semistable pairs with fixed Hilbert polynomial is bounded. Therefore there exists an integer number \(m_{0}\) such that for any \(m \geq m_{0}\) and for any \(\delta\) semistable pair \((\mathcal{F}, \phi)\), \(F_{E}(\mathcal{F})(m)\) is generated by global sections. Let \(V\) be a \(k\) vector space of dimension equal to \(P_{E}(\mathcal{F})(m)\). There is a surjective morphism

\[ q: V \otimes \mathcal{E}(-m) \longrightarrow \pi^{*}\pi_{*}(\mathcal{F} \otimes \mathcal{E}^\vee) \otimes \mathcal{E} \longrightarrow \mathcal{F} \]
obtained by applying the functor $G_{\xi}$ to

$$V \xrightarrow{\sim} H^0(F_{\xi}(\mathcal{F}(m))) \to F_{\xi}(\mathcal{F}(m))$$

and composing with $\theta_{\xi}(\mathcal{F}) : G_{\xi}(F_{\xi}(\mathcal{F})) \to \mathcal{F}$. The morphism $q$ corresponds to a closed point of $\tilde{Q} := Quot(V \otimes \mathcal{E}(-m), P_{\xi}(\mathcal{F}))$ (existence of Quot schemes on Deligne-Mumford stacks follows from [13]). Up to changing $m$ we can assume that also $F_{\xi}(\mathcal{E})(m) = \pi_{\ast} \mathcal{E}(\mathcal{E})(m)$ is generated by global sections. Under this assumption we get a surjection

$$(34) \quad H := H^0(\pi_{\ast} \mathcal{E}(\mathcal{E})(m)) \otimes \mathcal{E}(-m) \longrightarrow \mathcal{E}.$$ 

To any pair $(\mathcal{F}, \phi)$ we can associate a commutative diagram

$$(35) \quad H \times \Gamma \otimes \mathcal{E}(-m) \xrightarrow{\tilde{a}} V \otimes \mathcal{E}(-m) \xrightarrow{q} \Gamma \otimes \mathcal{E} \xrightarrow{\phi} \mathcal{F}$$

where $\tilde{a} := a \otimes \mathcal{E}(-m)$ and $a \in \text{Hom}(H \times \Gamma, V)$. In this section we will show that we can use as parameter space a suitable locus of $N \times \tilde{Q}$, where $N$ is the projectivization of the vector space $\text{Hom}(H \times \Gamma, V)$. We start by recalling some useful facts.

5.1. Quot schemes on projective stacks. Let $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{f} S$ be a projective stack. Let $\tilde{Q}$ denote $\text{Quot}_{\mathcal{X}/S}(V \otimes \mathcal{E}(-m), P)$ and let $Q$ denote $\text{Quot}_{\mathcal{X}/S}(F_{\xi}(V \otimes \mathcal{E}(-m)), P)$. By [13] Proposition 6.2 and [12] 4.20 there is a closed embedding $\iota : \tilde{Q} \to Q$. We know that for any $l \in \mathbb{N}$ big enough there is a closed embedding into the Grassmannian

$$(36) \quad \iota : \text{Quot}_{\mathcal{X}/S}(F_{\xi}(V \otimes \mathcal{E}(-m)), P) \hookrightarrow \text{Grass}(f_{\ast}F_{\xi}(V \otimes \mathcal{E}(l - m)), P(l))$$

given by the very ample line bundles $\text{det} f_{Q_{U}U}(l)$, where $U$ is the universal quotient sheaf over $Q$ and $f_{Q}$ is defined as in the following cartesian diagram

$$(37) \quad \begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & X \\
\pi_{\tilde{Q}} & & \\
\mathcal{X} & \xrightarrow{\tilde{\iota}} & X_{\tilde{Q}} \xrightarrow{\tilde{\iota}} X_{Q} \xrightarrow{\iota} G \xrightarrow{k_{l}} P \xrightarrow{k} S \\
\pi & & \\
\mathcal{X} & \xrightarrow{\pi} & X \xrightarrow{f} \mathcal{X} \\
\pi_{Q} & & \\
\tilde{Q} & \xrightarrow{\iota} & Q \xrightarrow{k_{l}} G \xrightarrow{k} P \xrightarrow{k} S \\
\end{array}$$

where $G = \text{Grass}(f_{\ast}F_{\xi}(V \otimes \mathcal{E}(l - m)), P(l))$, $\mathbb{P} = \mathbb{P}(\wedge^{P(l)}(f_{\ast}F_{\xi}(V \otimes \mathcal{E}(l - m))))$, $k_{l}$ is the Plücker embedding and $k_{l}$ is the Grothendieck embedding. A result in [12] identifies a (relatively) very ample line bundle giving an equivariant embedding into the projective space $\mathbb{P}(\wedge^{P(l)}(f_{\ast}F_{\xi}(V \otimes \mathcal{E}(l - m))))$.

**Proposition 5.1** ([12] Prop 6.2). *The class of invertible sheaves*

$$(38) \quad L_{l} := \text{det}(f_{Q_{U}}(F_{\xi}(U)(l)))$$

is very ample for $l$ big enough, where we refer for notation to diagram (37) and $U$ is the universal quotient bundle over $\tilde{Q}$. 
As in [3] pag. 101, \( \tilde{U} \) has a natural \( GL(V) \)-linearization induced by the universal automorphism of \( GL(V) \). As observed in [12] Lemma 6.3 there is an induced linearization of \( L_t \) as its formation commutes with arbitrary base change.

5.2. Identification of the parameter space. Let \( N := \mathbb{P}(\text{Hom}(H \times \Gamma, V)^\vee) \) be the projective space of morphisms \( H \times \Gamma \to V \), which is polarized by \( \mathcal{O}_N(1) \).

**Lemma 5.2.** Let \( (\mathcal{F}, \phi) \) be a \( \delta \) semistable pair over \( X \) with \( P_\epsilon(\mathcal{F}) = P \). Then it determines a pair \( (a, q) \) in \( N \times \tilde{Q} \) such that

\[ q \circ a = \phi \circ \tilde{ev} \]

and such that \( H^0(F_\epsilon(q(m)) \circ \varphi_\epsilon(V \otimes \mathcal{O}_X)) \) is an isomorphism, where \( \varphi_\epsilon(V \otimes \mathcal{O}_X) : V \otimes \mathcal{O}_X \hookrightarrow V \otimes \text{End}(\mathcal{E}) \) is the multiplication by the identity endomorphism.

**Proof.** Since \( \mathcal{F} \) is bounded, then for \( m \) large enough there is a surjection

\[
\pi^* H^0(F_\epsilon(\mathcal{F})(m)) \otimes \mathcal{E} \xrightarrow{G_\epsilon(\tilde{ev})} \pi^* F_\epsilon(\mathcal{F})(m) \otimes \mathcal{E} \xrightarrow{\theta_\epsilon(\mathcal{F})} \mathcal{F}(m)
\]

Moreover, \( P = \text{h}^0(F_\epsilon(\mathcal{F})(m)) \). Hence, by choosing an isomorphism \( V \xrightarrow{\sim} H^0(F_\epsilon(\mathcal{F})(m)) \) and by tensoring by \( \mathcal{O}_X(-m) \) we get a quotient

\[
V \otimes \mathcal{E}(-m) \to \mathcal{F}
\]

namely an element of \( \tilde{Q} \). Let us consider \( \phi : \mathcal{E} \otimes \Gamma \to \mathcal{F} \). By twisting by \( \pi^* \mathcal{O}_X(-m) \), by applying the functor \( F_\epsilon \) and by taking global sections we get

\[
H^0(F_\epsilon(\mathcal{E})(m)) \times \Gamma \to H^0(F_\epsilon(\mathcal{F})(m))
\]

By choosing again an isomorphism \( V \xrightarrow{\sim} H^0(F_\epsilon(\mathcal{F})(m)) \) we get a morphism \( \tilde{a} : H \times \Gamma \to V \). Note that \( a \) and \( \lambda \tilde{a} \), \( \lambda \in \mathbb{C}^* \), come from isomorphic pairs. Indeed for any \( \lambda \in \mathbb{C}^* \) \( (\mathcal{F}, \phi) \simeq (\mathcal{F}, \lambda \mathcal{F}) \) by definition. We observe moreover \( H^0(F_\epsilon(q(m)) \circ \varphi_\epsilon(V \otimes \mathcal{O}_X)) \) is an isomorphism by construction. The same is true for the relation \( q \circ a = \phi \circ \tilde{ev} \). \( \square \)

We characterize now the locus in \( N \times \tilde{Q} \) containing pairs \( (a, q) \) yielding a pair \( (\mathcal{F}, \phi) \).

**Proposition 5.3.** There is a closed subscheme \( \mathcal{W} \subseteq N \times \tilde{Q} \) with the following property: given a pair \( (a, q) \in N \times \tilde{Q} \) the map \( q \circ \tilde{a} \) factors through \( \tilde{ev} \) iff \( (a, q) \in \mathcal{W} \).

**Proof.** Same as [16] Prop. 3.4. \( \square \)

**Definition 5.4.** We define \( \mathcal{Z} \) to be the closure in \( \mathcal{W} \) of the open locus of points \( (a, q) \) such that \( q(V \otimes \mathcal{E}(-m)) \) is pure.

6. GIT Construction

We come now to the GIT construction of the moduli space of \( \delta \) semistable pairs. We observe that \( \mathbb{C}^* \subset GL(V, k) \) acts trivially on both \( N \) and \( \tilde{Q} \). As far as the GIT problem is concerned we can consider the action of the group \( PGL(V, k) \) or \( SL(V, k) \) (\( SL(V) \) from now on). Indeed \( PGL(V) \) is a quotient of \( SL(V) \) by a finite subgroup. As a consequence, up to taking finite tensor powers, the line bundles linearized for the actions of the two groups are the same. There is a natural action of the group \( SL(V) \) on \( \mathcal{Z} \) defined as follows

\[
g \cdot (a, q) \mapsto (g^{-1} \cdot a, q \cdot g).
\]
The line bundles $L_{i}$ of Lemma 5.1 and $\mathcal{O}_{\mathcal{X}}(1)$ have a natural $SL(V)$ linearization. We choose as linearized line bundle for the GIT construction

\[(43) \quad \mathcal{O}_{\mathcal{X}}(n_1, n_2) := \pi_{Q_{i}}^* L_{i}^{n_1} \otimes \pi_{\mathcal{X}}^* \mathcal{O}_{\mathcal{X}}(1)^{n_2}|_{\mathcal{Z}}.\]

We choose $n_1$ and $n_2$ as

\[(44) \quad \frac{n_1}{n_2} = \frac{P_{\mathcal{X}}(l)\delta(m) - P_{\mathcal{X}}(m)\delta(l)}{P_{\mathcal{X}}(m) + \delta(m)}.\]

**Definition 6.1.** Let $\mathcal{R} \subseteq \mathcal{Z}$ be the subset of points corresponding to $\delta$ semistable pairs and such that $H^0(F_{\mathcal{X}}(q(m)) \circ t_{\mathcal{X}}(V \otimes \mathcal{O}_{\mathcal{X}}))$ is an isomorphism.

**Lemma 6.2.** The subset $\mathcal{R}$ of Definition 6.1 is open and $SL(V)$ invariant. Moreover there is an open subset $\mathcal{R}^{ss} \subseteq \mathcal{R}$ corresponding to $\delta$ stable pairs.

**Proof.** The proof is almost the same as [16] Definition/Lemma 3.5. The proof relies on the fact that the set of Hilbert polynomials of purely dimensional quotients destabilizing the pair corresponding to some $(a, q) \in \mathcal{Z}$ is finite. To prove this for projective stacks one need two ingredients. The first is the Grothendieck Lemma for stacks. The second is the fact that given a family of projective stacks and a coherent sheaf over it there exists a finite stratification such that the restriction of the given sheaf to each stratum is flat (cfr. [12] Proposition 1.13). Recall that the Hilbert polynomial over a projective stack is constant in families. \(\Box\)

**Definition 6.3.** We define $\overline{\mathcal{R}}$ as the closure of $\mathcal{R}$.

We define the functor of semistable pairs.

**Definition 6.4.** Let the functor

\[(45) \quad \mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P) : (\text{Sch}/k)^{op} \to (\text{Sets})\]

be defined as follows. For any $S$ in $(\text{Sch}/k) \mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)(S)$ is the set of isomorphism classes of families $((\mathcal{F}, \phi)$ of $\delta$ semistable pairs parametrized by $S$ as defined in Definition 3.10, such that for any closed point $s$ of $S$ $((\mathcal{F}(s), \phi|_{\pi_{\mathcal{X}}^* \mathcal{E}(s)})$ has Hilbert polynomial $P$. For any $f : S' \to S$

\[\mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)(f) : \mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)(S) \to \mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)(S')\]

takes a family $((\mathcal{F}, \phi)$ over $S$ to $((f \times 1_{\mathcal{X}})^* \mathcal{F}, (f \times 1_{\mathcal{X}})^* \phi)$ over $S'$. We define $\mathbb{M}_{\mathcal{X}, \delta}^{ss}(\mathcal{E}, P)$ as the subfunctor parametrizing $\delta$ stable pairs.

**Theorem 6.5.** Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1), \mathcal{E})$ be a polarized smooth projective stack. Then the functor $\mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)$ is isomorphic to $[\mathbb{R}^{ss}/GL(V)]$, where $\mathbb{R}^{ss} \subseteq \overline{\mathcal{R}}$ is the subset of GIT semistable points.

**Proof.** The proof uses a quite standard machinery (cfr. [6] Lemma 4.3.1, [5] Proposition 3.9, [17] Theorem 4.0.7). We sketch it. We will show that there is an invertible functor of categories fibered in groupoids between $[\mathbb{R}^{ss}/GL(V)]$ and $\mathbb{M}_{\mathcal{X}, \delta}(\mathcal{E}, P)$. Let us show that $\xi$ exists. An object of $[\mathbb{R}^{ss}/GL(V)]$ over $S$ is a diagram

\[(46) \quad \begin{array}{ccc}
P & \longrightarrow & \mathbb{R}^{ss} \\
\downarrow & & \downarrow \\
S & & \end{array}\]

where the horizontal arrow is $GL(V)$ equivariant. By pulling back the universal family over $\overline{\mathcal{R}} \times \mathcal{X}$ we get a $\delta$ semistable pair $\phi : \mathcal{E} \otimes \Gamma \to \mathcal{F}$ over $P \times \mathcal{X}$ with an isomorphism...
$V \xrightarrow{\sim} H^0(F_{\mathcal{E}}(\mathcal{F}(m))(s))$ for any closed point $s$ of $S$. The pair comes with a $GL(V)$ linearization. Therefore both sheaves and the morphism between them descend to $S$. The same is true for a morphism between $\delta$ semistable pairs over $P \times X$. We show that there exists a functor $\eta$ in the opposite direction. Let us draw a diagram to fix the notations:

\[(47)\]
\[
P \times X \xrightarrow{p \times 1_X} S \times X.
\]

Let us consider a $\delta$ semistable pair over $X \times S \phi : \mathcal{E} \otimes \Gamma \to \mathcal{F}$. Since $\mathcal{E}$ and $\mathcal{F}$ are flat over $S$ by definition of family, $\pi_S \mathcal{E}_{\text{nd}}(m)$ and $\pi_S \mathcal{F} \otimes \mathcal{E}^\vee(m)$ are locally free $\mathcal{O}_S$-modules and the morphisms

\[(48)\]
\[
\pi_S^* \mathcal{A} \otimes \mathcal{E} := \pi_S^* \pi_S^* (\Gamma \otimes \mathcal{E}_{\text{nd}}(\mathcal{E})(m)) \otimes \mathcal{E} \to \mathcal{E}(m)
\]

and

\[(49)\]
\[
\pi_S^* \mathcal{V} \mathcal{B} \otimes \mathcal{E} := \pi_S^* \pi_S^* (\mathcal{F} \otimes \mathcal{E}^\vee(m)) \otimes \mathcal{E} \to \mathcal{F}(m)
\]

are surjective. Note that $\mathcal{A} \simeq H \times \Gamma \otimes \mathcal{O}_S$, $H := H^0(\mathcal{X}, \mathcal{E}_{\text{nd}}(\mathcal{E})(m))$ Let $P := \text{Isom}(V, \mathcal{B})$ be the frame bundle of $\mathcal{B}$ (see [6] Example 4.2.3 for the definition). Then $p : P \to S$ is a $GL(V)$ principal bundle.

There is a commutative diagram over $P \times X$

\[(50)\]
\[
\begin{array}{cccc}
H \times \Gamma \otimes \mathcal{E} \otimes \mathcal{O}_{P \times X} & \longrightarrow & V \otimes \mathcal{E} \otimes \mathcal{O}_{P \times X} \\
\downarrow & & \downarrow \\
(p \times 1_X)^* \mathcal{E}(m) & \longrightarrow & (p \times 1_X)^* \mathcal{F}(m)
\end{array}
\]

where

i) the right vertical arrow corresponds to a morphism $P \to \tilde{Q}$ and it is obtained by composing with $(p \times 1_X)^* \mathcal{F}(m)$ with $\pi_P^* \mathcal{h}^{-1}$, where $\mathcal{h}$ is the universal isomorphism over $P$;

ii) the upper horizontal arrow corresponds to a morphism $P \to N$ and it is obtained by post-composing the natural morphism $H \times \Gamma \otimes \mathcal{O}_{P \times X} \to (p \times 1_X)^* \pi_S^* \mathcal{B}$ with $\pi_P^* \mathcal{h}^{-1}$.

By Cohomology and base change for Deligne-Mumford stacks (see [12] Theorem 1.7) there are isomorphisms $H \times \Gamma \otimes \mathcal{O}_P \xrightarrow{\sim} p^* \mathcal{A}$ and $V \otimes \mathcal{O}_P \xrightarrow{\sim} p^* \mathcal{B}$. Let $p$ be a closed point of $P$. Then

\[(51)\]
\[
H \times \Gamma \simeq p^* \mathcal{A}|_p \simeq H^0(F_{\mathcal{E}}(\mathcal{E} \otimes \Gamma(m))(p))
\]

\[
V \simeq p^* \mathcal{B}|_p \simeq H^0(F_{\mathcal{F}}(\mathcal{F}(m))(p))
\]

Then the family of diagram [47] with the property [51] and its natural $GL(V)$ linearization provides a $GL(V)$-equivariant morphism to $\mathbb{R}_{ss}^n$. Such a morphism descends to a morphism $S \to [\mathbb{R}_{ss}^n/GL(V)]$. In a similar way we can reconstruct morphisms. Let $f : S \to T$ be a morphism, let $(\mathcal{F}, \phi)$ and $(\mathcal{F}', \phi')$ be families of pairs over $S \times X$ and
Let \( T \times X, \tau : (\mathcal{F}, \phi) \to (p \times 1_X)^*(\mathcal{F}', \phi') \) be a morphism over \( S \times X \). By cohomology and base change along the diagram
\[
\begin{array}{ccc}
S \times X & \xrightarrow{f} & T \times X \\
\downarrow & & \downarrow \\
S & \xrightarrow{u} & T
\end{array}
\]
we see that the families produce \( GL(V) \) principal bundles \( P \) and \( P' \) such that \( P \sim f^*P' \). By cohomology and base change along the diagram
\[
\begin{array}{ccc}
f^*P' \times X & \xrightarrow{u} & P' \times X \\
\downarrow & & \downarrow \\
f^*P' & \xrightarrow{f} & P'
\end{array}
\]
we see that there is a canonical isomorphism between the pair over \( P \) and the pullback of the pair over \( P' \). Then we get two isomorphic \( GL(V) \) equivariant maps \( g : P \to R_{ss} \) and \( g' : f^*P' \to R_{ss} \) such that \( g' \circ u = g \). We conclude that the functor \( \eta \) is defined. It is straightforward to check that \( \xi \) and \( \eta \) are the inverse of each other. \( \square \)

**Lemma 6.6.** Let \( M \) be a scheme which is a categorical quotient for the \( SL(V) \) action on \( R \). Then it corepresent the functor \( M_{\chi, \delta}(E, P) \).

**Proof.** The functors \( M_{\chi, \delta}(E, P) \) and \( [R/GL(V)] \) are isomorphic. Hence to corepresent either functor is the same. \( \square \)

We state the theorem relating GIT semistability of points of the parameter space to \( \delta \) semistability of the corresponding pairs. The proof will be given by Theorems 6.16 and 6.17.

**Theorem 6.7.** For \( l \) large enough the subset of points in the closure \( \overline{R} \) of \( R \) which are semistable with respect to \( O_{\mathbb{Z}}(n_1, n_2) \) and its \( SL(V) \) linearization coincides with the subset of points corresponding to \( \delta \) semistable pairs.

**Theorem 6.8.** Let \( (X, O_X(1), E) \) be a polarized smooth Deligne-Mumford stack. Then there exists a moduli space \( M_{\chi, \delta}(E, P) \) of \( \delta \) semistable pairs. Two pairs correspond to the same point in \( M_{\chi, \delta}(E, P) \) if and only if they are \( S \)-equivalent. Moreover there is an open subset \( M_{\chi, \delta}^s(E, P) \) corresponding to stable pairs. It is a fine moduli space for \( \delta \) semistable pairs.

**Proof.** The proof is an in [16] Theorem 3.8. We proved in Lemma 6.6 that a categorical quotient of \( cl \) for the \( SL(V) \) action corepresent the functor \( M_{\chi, \delta}(E, P) \). General GIT theory and Theorem 6.7 provide a categorical quotient of \( \mathcal{R} \) which in particular is a good quotient. By using arguments analogous to [7] Proposition 3.3 or [6] Theorem 4.3.3 one proves that closure of the orbit of a \( \delta \) semistable pair \( (\mathcal{F}, \phi) \) contains the pair \( (gr^{JH}(\mathcal{F}), gr^{JH}(\phi)) \). Moreover the orbit of \( (\mathcal{F}, \phi) \) is closed if and only if \( (\mathcal{F}, \phi) \) is polystable. In proving the last fact a semicontinuity result in used, which also holds for Deligne-Mumford stacks (cfr. [12] Theorem 1.8). The second statement follows then from properties of good quotients. Moreover there is a universal family over \( X \times M_{\chi, \delta}^s(E, P) \), which is implied by the existence of a line bundle over \( R^s \) of weight 1 for the action of \( \mathbb{G}_m \subseteq GL(V) \) (cfr. [8] Section 4.6). Such a line bundle is provided by \( O_N(1) \). \( \square \)
6.1. **GIT computations.** We give in the following proposition numerical conditions implied by GIT semistability. We recall some useful standard results. The action of a 1-parameter subgroup of $SL(V)$ λ can be diagonalized. Hence $V$ splits in eigenspaces for the eigenvectors of λ. The subspaces $V_{≤ n} = \oplus_{i=1}^{n} V_i \subseteq V$ give an ascending filtration.

Let $q : V \otimes \mathcal{E}(-m) \to \mathcal{F}$ a closed point of $\tilde{Q}$. We get a corresponding filtration $\mathcal{F}_{≤ n}$ of $\mathcal{F}$, where $\mathcal{F}_{≤ n} = q(V_{≤ n} \otimes \mathcal{E}(-m))$. Let $\mathcal{F}_n = \mathcal{F}_{≤ n}/\mathcal{F}_{≤ n-1}$ the graduate pieces. Let

$$\mathcal{F} : V \otimes \mathcal{E}(-m) \to \oplus_{i=1}^{n} \mathcal{F}_{≤ n}.$$  

be a closed point of $\tilde{Q}$. Then we have the following result generalizing [6] Lemma 4.4.3 and proven in [12] Lemma 6.11.

**Lemma 6.9.** The quotient $[\mathcal{F}]$ is $\lim_{l \to 0} \lambda(l)q$ in the sense of the Hilbert-Mumford criterion.

As in the case of scheme the action of λ on the fiber of $L_l$ is characterized as follows.

**Lemma 6.10 ([12] Lem. 6.12).** The action of $G_{m,k}$ via the representation λ on the fiber of $L_l$ at the point $[\mathcal{F}]$ is given by the weight

$$\sum_n P_{\mathcal{F}}(\mathcal{F}_n(l)).$$

We fix some notation. Let $W := H^0(\mathcal{O}_X(l - m))$. We denote by $q'$ the following morphism induced by $q$:

$$q' : V \otimes W \xrightarrow{H^0(\varphi_{\mathcal{F}}(l))} H^0(\mathcal{F} \otimes \mathcal{E}'(l))$$

and by $q''$ its $r$-th antisymmetric product

$$q'' : \wedge^r (V \otimes W) \to \det H^0(\mathcal{F} \otimes \mathcal{E}'(l)).$$

**Proposition 6.11.** Let $(a, q)$ be a point in $\mathcal{F}$. For $l$ large enough $(a, q)$ is GIT semistable with respect to $\mathcal{O}_{\mathcal{F}}(n_1, n_2)$ if and only if the following holds. For any non trivial subspace $U \subseteq V$ we have

$$\dim(U[n_1 P(l) - n_2](≤)P(m)[n_1 \dim q'(U \otimes W) - \epsilon(U)n_2]$$

where $\epsilon(U) = 1$ if $\im a \subseteq U$ and 0 otherwise.

**Proof.** We use Hilbert-Mumford criterion. Let $\lambda : \mathbb{C}^* \to SL(V)$ be a one parameter subgroup. Let us choose a basis of $V v_1, \ldots, v_p$ such that $v_i \cdot \lambda(t) = t^{\gamma_i} v_i$, $\gamma_{i+1} \geq \gamma_i$, $i = 1 \ldots, p - 1$. Recall that λ is completely specified by a weight vector $(\gamma_1, \ldots, \gamma_p) \in \mathbb{Z}^p$. Moreover $\sum_{i=1}^{p} \gamma_i = 0$. The number $\mu(q, \lambda)$ is computed by Lemma 6.10. Let us compute $\mu(a, \lambda)$, $a \in N = \text{Proj}(\text{Sym} \times \text{Hom}(H \times \Gamma, V)^{\vee})$. Sections of $\mathcal{O}_N(1)$ are $\text{Hom}(H \times \Gamma, V)^{\vee}$. Let $\xi : H \times \Gamma \to V$ be a homomorphism. We can write it in matricial form as $\xi = \sum_{i,j} \alpha_{ij} w_j^T \otimes v_i$, $v_j \in H \times \Gamma$. Then

$$\mu(a, \lambda) = \max \{\gamma_i | a_{ij} \neq 0\}.$$  

Equivalently $\mu(a, \lambda) = \gamma_i$ where $i = \min\{i | \im a \subseteq \langle v_1, \ldots, v_i \rangle\}$. By the Hilbert-Mumford criterion (semi) stability of $(a, q)$ requires that

$$\mu(q'', \lambda)n_1 + \mu(a, \lambda)n_2(≥)0.$$  

Given a base $v_1, \ldots, v_p$ of $V$, let consider weight vectors of the form

$$\gamma^{(i)} = (i - p, \ldots, i - p, i, \ldots, i)$$
Any other weight vector can be expressed as a finite non negative linear combination of weight vectors in this class. In fact
\[ \gamma_k = \sum_{i=1}^{p-1} \left( \frac{\gamma_{i+1} - \gamma_i}{p} \right)^{\gamma_{(i)}}. \]

Let us start by evaluating \( \mu(q, \lambda) \). By Lemma 6.10 we get
\[ \mu(q, \lambda) = \sum_{n} nP(F_n(l)) = (i - p) \sum_{n \leq i} P(F_n(l)) + i \sum_{i < n \leq p} P(F_n(l)) = \]
\[ (62) \quad i \rho - p \psi(i), \]
where \( \rho = h^0(\mathcal{F} \otimes \mathcal{E}^\vee(l)) \) and \( \psi(i) = \sum_{n \leq i} P(F_n(l)) = \dim(q'((v_1, \ldots, v_i) \otimes \mathcal{W}). \) Here the second equality holds because the graded pieces are also bounded. Let us come to \( \mu(a, \lambda) \). Note that \( \mu(a, \lambda) = i - p \) if im \( a \subseteq \langle v_1, \ldots, v_i \rangle \), and that \( \mu(a, \lambda) = i \) otherwise. More compactly, \( \mu(a, \lambda) = i - \epsilon(i)p_i \), where \( \epsilon(i) = 1 \) if im \( a \subseteq \langle v_1, \ldots, v_i \rangle \), otherwise.

Summing up the Hilbert-Mumford criterion implies
\[ (63) \quad i(p n_1 - n_2)(\leq)p(\psi(i) n_1 - \epsilon(i) n_2) \]
We observe that the above equation does neither depend on the base of \( V \) we chose nor on the particular weight vector we evaluated the Hilbert-Mumford criterion on. Let \( U \subseteq V \) the vector subspace generated by \( v_1, \ldots, v_i \). Since \( \rho = P_{\mathcal{E}}(l) \) and \( \dim V = P_{\mathcal{E}}(m) \) the inequality (63) can be rewritten as
\[ (64) \quad \dim U(P_{\mathcal{E}}(l)n_1 - n_2)(\leq) \dim V(\dim(q'(U \otimes \mathcal{W}))n_1 - \epsilon(U)n_2) \]
where \( \epsilon(U) = 0 \) if im \( a \subseteq U \) and 0 otherwise. \( \square \)

**Notation 6.12.** Let \( U \subseteq V \) be a sub vector space. We denote by \( \mathcal{F}_U \) the subsheaf of \( \mathcal{F} \) generated by \( q(\mathcal{G}_\mathcal{E}(U)) \).

**Corollary 6.13.** For any GIT-semistable point \( (a, q) \) the induced morphism
\[ q(m) \circ \iota_{\mathcal{E}}(V \otimes \mathcal{O}_X) : V \otimes \mathcal{O}_X \to V \otimes \text{End} \mathcal{E} \to \mathcal{F} \otimes \mathcal{E}^\vee(m) \]
is injective. In particular \( \dim(V \cap h^0(F_{\mathcal{E}}(\mathcal{F})(m))) \leq h^0(F_{\mathcal{E}}(\mathcal{F})(m)) \). Moreover \( q' \) is injective and for any subspace \( U \subseteq V \) \( \dim q'(U \otimes \mathcal{W}) \leq h^0(\mathcal{F}_U(l)) \) where \( \mathcal{F}_U = q(U(-m)) \).

**Proof.** Let \( U \subseteq V \) be the kernel of \( q(m) \circ \iota_{\mathcal{E}}(V \otimes \mathcal{O}_X) \). Then \( \epsilon(U) = 0 \), otherwise \( \phi \) would be zero. Moreover \( \dim q'(U \otimes \mathcal{W}) = 0 \). Substituting in equation (64) we get \( \dim U \leq 0 \) because \( n_1 \rho - n_2 \geq 0 \) by the choice (44). \( \square \)

We give another numerical characterization for GIT semi-stability.

**Proposition 6.14.** For sufficiently large \( l \) a point \( (a, q) \) is GIT-semistable if for any \( U \subseteq V \) the following polynomial equation holds
\[ (65) \quad \dim U(P_{\mathcal{E}}(l))n_1 - n_2)(\leq) \dim V(\dim(P_{\mathcal{E}}(\mathcal{F}_U(l)))n_1 - \epsilon(U)n_2) \]

**Proof.** It is enough to prove that equation (65) implies (64). Note that subsheaves of the form \( \mathcal{F}_U \) are bounded. Hence for \( l \) large enough \( P_{\mathcal{E}}(\mathcal{F}_U(l)) = h^0(\mathcal{F}_U \otimes \mathcal{E}^\vee(l)) = \dim q'(U \otimes \mathcal{W}) \). We are left to show that \( \epsilon(U) = 1 \iff \epsilon(\mathcal{F}_U) = 1 \). For any \( U \subseteq V \) \( \epsilon(U) = 1 \Rightarrow \epsilon(\mathcal{F}_U) = 1 \). Let us prove the opposite implication. It can happen that \( \text{Im } a \subseteq U \) but \( \text{Im } \phi \subseteq \mathcal{F}_U \). Let \( U' \) be the subspace of \( V \) generated by \( U \) and by \( \text{Im } a \). Then the inequality holds with \( \dim U' \) replacing \( \dim U \), hence it holds a fortiori for \( \dim U \). \( \square \)
Proposition 6.15. For sufficiently large $l$ a point $(a,q)$ is GIT-semistable if and only if for any $U \subseteq V$ the following polynomial equation holds
\[(66) \ P(dim \ U + \epsilon(F_U) \delta(m)) + \delta(dim \ U - \epsilon(F_U)P(m))(\leq)P_{\mathcal{F}_U}(P(m) + \delta(m))\]

PROOF. The proof is as in [16]. Take $l$ large enough such that inequality (65) holds as an inequality of polynomials. Then put
\[(67) \ \frac{n_1}{n_2} = \frac{P_{\mathcal{F}}(l) \delta(m) - P_{\mathcal{F}}(m) \delta(l)}{P(m) + \delta(m)}\]

\[\square\]

Theorem 6.16. For sufficiently large $l$ if a point $(a,q)$ in $\mathcal{M}$ is GIT-semistable then the corresponding pair $(\mathcal{F}, \phi)$ is $\delta$ semi-stable and $H^0(q(m) \circ \iota_{\mathcal{E}}(V \otimes \mathcal{O}_X))$ is an isomorphism. In particular any GIT-semistable point corresponds to a pair with torsion-free sheaf.

PROOF. Note that by Corollary [6.1.2] $V \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{E}^\vee(m)$ is injective. Hence for dimensional reasons $V \to H^0(\mathcal{F} \otimes \mathcal{E}^\vee(m))$ is an isomorphism. Let $\mathcal{F}' \subseteq \mathcal{F}$ be a subsheaf. Let $U = V \cap H^0(\mathcal{F}'(q))$. Then put $U = h^0(\mathcal{F}'(q))$. Let $\mathcal{F}_U \subseteq \mathcal{F}'$ as in Notation [6.1.2]. We observe that $\epsilon(\mathcal{F}_U) = 1$ iff $\epsilon(\mathcal{F}') = 1$. The “if” direction holds because $\epsilon(\mathcal{F}') = 1$ implies $\epsilon(U) = 1$ since $U = V \cap H^0(\mathcal{F}_U(\mathcal{F}'))$. This in turn implies $\epsilon(\mathcal{F}_U) = 1$. By taking the leading coefficients in the polynomial equation (66) we get
\[(68) \ \dim U + \epsilon(\mathcal{F}') \delta(m) \leq \frac{r'}{r}(P_\mathcal{E}(m) + \delta(m))\]

By Lemma [4.6] there exists a morphism $\psi : \mathcal{F} \to \mathcal{H}$, where $\mathcal{H}$ is pure, the kernel is a torsion subsheaf $\mathcal{F}$ and $P_{\mathcal{F}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{H})$. There is an induced homomorphism $\phi_H : \mathcal{E} \otimes \Gamma \to \mathcal{H}$ which is non vanishing. If it was, then (68) would be violated for $\mathcal{F}' = \mathcal{F}$. Let $\mathcal{H}'$ be a quotient of $\mathcal{H}$, and $\mathcal{H}'$ the corresponding kernel. Let $\mathcal{H}'$ the image of $\mathcal{F}$ in $\mathcal{H}'$ and let $\mathcal{F}'$ be the corresponding kernel. Then
\[h^0(\mathcal{H}' \otimes \mathcal{E}^\vee(m)) + \epsilon(\mathcal{H}') \delta(m) \geq h^0(\mathcal{F}' \otimes \mathcal{E}^\vee) + \epsilon(\mathcal{F}') \delta(m) \geq \dim V + \delta(\mathcal{F}') - (\epsilon(\mathcal{F}') + \delta(\mathcal{F}')) \geq P_{\mathcal{E}}(m) + \delta(m) - \frac{r_{\mathcal{E},\mathcal{F}'}(P_\mathcal{E}(m) + \delta(m))}{r_{\mathcal{E},\mathcal{F}}(P_\mathcal{E}(m) + \delta(m))} = \frac{r_{\mathcal{E},\mathcal{F}}(P_\mathcal{E}(m) + \delta(m))}{r_{\mathcal{E},\mathcal{F}}(P_\mathcal{E}(m) + \delta(m))}\]

It follows that $(\mathcal{H}, \phi_H)$ is $\delta$ semistable by Proposition [4.5] therefore belongs to a bounded family. Consequently it is $m$-regular. Let us consider the sheaf $\psi(\mathcal{F}) \subseteq \mathcal{H}$. It is a quotient of $\mathcal{F}$. By (69) $h^0(\psi(\mathcal{F}) \otimes \mathcal{E}^\vee(m)) \geq h^0(\mathcal{H} \otimes \mathcal{E}^\vee(m)) = P_{\mathcal{E}}(m)$, while the opposite inequality is obvious. Hence $V \otimes \mathcal{E}^\vee(-m) \to \mathcal{H}$ is surjective, which implies $\mathcal{F} \to \mathcal{H}$ is surjective. By semistability of $\mathcal{H}$ and $P_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{H})$ it follows that $\psi$ is an isomorphism. Eventually, it is an obvious consequence that $V \to H^0(\mathcal{H} \otimes \mathcal{E}^\vee(m))$ is an isomorphism, because $V \to H^0(\mathcal{F} \otimes \mathcal{E}^\vee(m))$ is. \[\square\]

We prove now the inverse implication.

Theorem 6.17. Let $(\mathcal{F}, \phi)$ be a $\delta$-semistable pair such that $H^0(\mathcal{F}_\mathcal{E}(q(m)) \circ \iota_{\mathcal{E}}(V \otimes \mathcal{O}_X)) : V \to H^0(\mathcal{F}_\mathcal{E}(\mathcal{F}))$ is an isomorphism. Then the corresponding point is GIT-semistable.
PROOF. If \( \delta \) stability holds we can prove (66) by only considering the leading coefficients. In this case, given \( \mathcal{F}' \subseteq \mathcal{F} \) we define \( U = V \cap H^0(\mathcal{F}' \otimes \mathcal{E}^\vee(m)) \). Then \( \dim U \leq h^0(\mathcal{F}' \otimes \mathcal{E}^\vee(m)) \). Recall \( \epsilon(\mathcal{F}') = 1 \Leftrightarrow \epsilon(\mathcal{F}_U) \) where \( \mathcal{F}_U \) is the sheaf generated by \( U \) via \( q(m) \otimes \mathcal{E}^\vee \circ \varphi_E \). This inequality and the \( \delta \) stability condition yield
\[
\dim U + \epsilon(\mathcal{F}') \delta(m) < \frac{r_{\delta,E}}{r_{\delta,F}} (P_{\delta}(m) + \delta(m)),
\]
which implies (66) as a polynomial inequality. The rest of the proof taking into account \( \delta \) semistability proceeds exactly as in [16] Theorem 4.7. In particular a pair \((\mathcal{F}, \phi)\) is GIT stable only if it is \( \delta \) stable. \( \square \)

REFERENCES

[1] Gromov-Witten theory of Deligne-Mumford stacks, Amer. J. Math., 2008, vol. 130, n.5, pgg. 1337–1398.
[2] Using stacks to impose tangency conditions on curves, Cadman, C., Amer. J. Math., vol. 129, pgg. 405–427, 2007.
[3] M. Artin, Versal deformations and algebraic stacks, \textit{Invent. Math.} Volume 27 (1974), 165–189.
[4] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, \textit{Inst. Hautes Études Sci. Publ. Math.}, Number 36 (1969), 75–109.
[5] Moduli of ADHM Sheaves and Local Donaldson-Thomas Theory, Diaconescu, D.-E., \texttt{arXiv:0801.0820} [math.AG].
[6] The geometry of moduli spaces of sheaves, Huybrechts, D. and Lehn, M., Cambridge Mathematical Library, Cambridge University Press, 2-nd edition, 2010.
[7] Framed modules and their moduli, Huybrechts, D., Lehn, M., Internat. J. Math., von. 6, n. 2, pgg. 297–324, 1995.
[8] On the geometry of Deligne-Mumford stacks, Kresch, A., in Algebraic geometry-Seattle 2005, Part 1, Proc. Sympos. Pure Math., vol. 80, pgg. 259–271, 2009, Amer. Math. Soc., Providence, RI.
[9] Moduli spaces of sheaves in mixed characteristic, Langer, A., Duke Math. J., vol. 124, n. 3, pgg. 571–586, 2004.
[10] Champs algébriques, Laumon, G. and Moret-Bailly, L., vol. 39, Springer-Verlag, Berlin, 2000.
[11] Systèmes cohérents et structures de niveau, Le Potier, J., Astérisque, n. 214, 1993.
[12] Moduli Spaces of Semistable Sheaves on Projective Deligne-Mumford Stacks, Nironi, F., \texttt{arXiv:0811.1949} [math.AG].
[13] Quot functors for Deligne-Mumford stacks, Olsson, M. and Starr, J., Special issue in honor of Steven L. Kleiman, Comm. Algebra, vol. 31, n. 8, pags. 1231–1264, 2003.
[14] Curve counting via stable pairs in the derived category, Pandharipande, R., Thomas, R., Invent. Math, 2009, vol. 178, pgg. 407-447.
[15] Moduli of representations of the fundamental group of a smooth projective variety. I, Simpson, C. T., Inst. Hautes Études Sci. Publ. Math., n. 79, 1994, pgg. 47–129.
[16] Moduli Spaces of Stable Pairs in Donaldson-Thomas Theory, Wandel, M., \texttt{arXiv:1011.3328} [math.AG].
[17] Higher rank stable pairs and virtual localization, Sheshmani, A., \texttt{arXiv:1011.6342} [math.AG].