Lacunary ideal convergence in probabilistic normed spaces

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Abstract. An ideal $I$ is a family of subsets of positive integers $\mathbb{N}$ which is closed under
taking finite unions and subsets of its elements. A sequence $(x_k)$ of real numbers is said
to be lacunary $I$-convergent to a real number $\ell$, if for each $\varepsilon > 0$ the set
\[
\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - \ell| \geq \varepsilon \}
\]
belongs to $I$. The aim of this paper is to study the notion of lacunary $I$-convergence in
probabilistic normed spaces as a variant of the notion of ideal convergence. Also lacunary
$I$-limit points and lacunary $I$-cluster points have been defined and the relation between
them has been established. Furthermore, lacunary-Cauchy and lacunary $I$-Cauchy se-
quences are introduced and studied. Finally, we provided example which shows that our
method of convergence in probabilistic normed spaces is more general.

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convergence.

1. Introduction

Steinhaus [54] and Fast [17] independently introduced the notion of statistical conver-
genence for sequences of real numbers. Over the years and under different names statistical
convergence has been discussed in the theory of Fourier analysis, ergodic theory and num-
ber theory. Later on it was further investigated from various points of view. For example,
statistical convergence has been investigated in summability theory by (Connor [10], Fridy
[19], Šalát [49]), number theory and mathematical analysis by (Buck [3], Mitrinović et al.,
[46]), topological groups (Çakalli [4, 5]), topological spaces (Di Maio and Kočinac [33]),
function spaces (Caserta and Kočinac [7]), locally convex spaces (Maddox [42]), measure
theory (Cheng et al., [8], Connor and Swardson [11], Miller [45]). Fridy and Orhan [20] in-
troduced the concept of lacunary statistical convergence. Some work on lacunary statistical
convergence can be found in ([3, 24, 27, 41]).

Kostyrko, et. al [35] introduced the notion of $I$-convergence as a generalization of sta-
tistical convergence which is based on the structure of an admissible ideal $I$ of subset of
natural numbers $\mathbb{N}$. Kostyrko, et. al [36] gave some of basic properties of $I$-convergence
and dealt with extremal $I$-limit points. Further details on ideal convergence can be found
in ([6, 11, 15, 16, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 57, 58]), and many others. The no-
tion of lacunary ideal convergence of real sequences was introduced in ([3, 55]) and Hazarika
([24, 25]), was introduced the lacunary ideal convergent sequences of fuzzy real numbers
and studied some properties. Debnath [13] introduced the notion lacunary ideal convergence in

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convergent sequences $N^J$ be denoted by of non-negative integers with $k$ led to the development of the area now called probabilistic metric spaces. This is Sherstnev subsets of $N^I$. Kumar [37] studied Rahmat [48] studied the ideal convergence in probabilistic normed spaces and V. Kumar and refer ([2, 12, 22, 23, 33, 38, 40, 51, 52]). Subsequently, Mursaleen and Mohiuddine [47] and the definition of Sherstnev as a special case. For an extensive view on this subject, we Alsina et al. [1] presented a new definition of probabilistic normed space which includes [53] who first used this idea of Menger to introduce the concept of a PN space. In 1993, $\varepsilon > 0$ A sequence $x = (x_k)$ is said to be statistically convergent to $\ell$ if for every $\varepsilon > 0$:

\[ \delta \left( \{ k \in N : |x_k - \ell| \geq \varepsilon \} \right) = 0. \]

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and $S$ denotes the set of all statistically convergent sequences.

\[ \delta (E) = \lim_{n \to \infty} \frac{1}{n} |\{ k \leq n : k \in E \}| \text{ exists.} \]

**Definition 1.1.** A sequence $x = (x_k)$ is said to be statistically convergent to $\ell$ if for every $\varepsilon > 0$.

\[ \delta \left( \{ k \in N : |x_k - \ell| \geq \varepsilon \} \right) = 0. \]
Definition 1.2. Let $I \subset 2^\mathbb{N}$ be a non-trivial ideal. A real sequence $x = (x_k)$ is said to be lacunary $I$-convergent or $I_\theta$-convergent to $L \in \mathbb{R}$ if, for every $\varepsilon > 0$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{hr} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$ 

$L$ is called the $I_\theta$-limit of the sequence $x = (x_k)$, and we write $I_\theta \lim x = L$.

In this paper we study the concept of lacunary $I$-convergence in probabilistic normed spaces. We also define lacunary $I$-limit points and lacunary $I$-cluster points in probabilistic normed space and prove some interesting results.

2. Basic definitions and notations

Now we recall some notations and basic definitions that we are going to use in this paper.

Definition 2.1. A distribution function (briefly a d.f.) $F$ is a function from the extended reals $(-\infty, +\infty)$ into $[0, 1]$ such that

(a) it is non-decreasing;
(b) it is left-continuous on $(-\infty, +\infty)$;
(c) $F(-\infty) = 0$ and $F(+\infty) = 1$.

The set of all d.f.'s will be denoted by $\Delta$. The subset of $\Delta$ consisting of proper d.f.'s, namely those elements $F$ such that $\ell^+ F(-\infty) = F(-\infty) = 0$ and $\ell^- F(+\infty) = F(+\infty) = 1$ will be denoted by $D$. A distance distribution function (briefly, d.d.f.) is a d.f. $F$ such that $F(0) = 0$. The set of all d.d.f.'s will be denoted by $\Delta^+$, while $D^+ := D \cap \Delta^+$ will denote the set of proper d.d.f.'s.

Definition 2.2. A triangular norm or, briefly, a $t$-norm is a binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the following conditions (see [34]):

(T1) $T$ is commutative, i.e., $T(s, t) = T(t, s)$ for all $s$ and $t$ in $[0, 1]$;
(T2) $T$ is associative, i.e., $T(T(s, t), u) = T(s, T(t, u))$ for all $s, t$ and $u$ in $[0, 1]$;
(T3) $T$ is nondecreasing, i.e., $T(s, t) \leq T(s', t)$ for all $t \in [0, 1]$ whenever $s \leq s'$;
(T4) $T$ satisfies the boundary condition $T(1, t) = t$ for every $t \in [0, 1]$.

$T^*$ is a continuous $t$-conorm, namely, a continuous binary operation on $[0, 1]$ that is related to a continuous $t$-norm through $T^*(s, t) = 1 - T(1 - s, 1 - t)$. Notice that by virtue of its commutativity, any $t$-norm $T$ is nondecreasing in each place. Some examples of $t$-norms $T$ and its $t$-conorms $T^*$ are: $M(x, y) = \min\{x, y\}, \Pi(x, y) = x \cdot y$ and $M^*(x, y) = \max\{x, y\}, \Pi^*(x, y) = x + y - x \cdot y$.

Using the definitions just given above Sherstnev [53] defined a PN space as follows:

Definition 2.3. A triplet $(X, \nu, T)$ is called a probabilistic normed space (in short PNS) if $X$ is a real vector space, $\nu$ is a mapping from $X$ into $D$ and for $x \in X$, the d.f. $\nu(x)$ is denoted by $\nu_x$, $\nu_x(t)$ is the value of $\nu_x$ at $t \in \mathbb{R}$ and $T$ is a $t$-norm. $\nu$ satisfies the following conditions:

(i) $\nu_x(0) = 0$;
(ii) $\nu_x(t) = 1$ for all $t > 0$ if and only if $x = 0$;
(iii) $\nu_{ax}(t) = \nu_x \left( \frac{t}{|a|} \right)$ for all $a \in \mathbb{R} \setminus \{0\}$;
(iv) $\nu_{x+y}(s + t) \geq T(\nu_x(s), \nu_y(t))$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$. 
Let \((X, ||.||)\) be a normed space and \(\mu \in D\) with \(\mu(0) = 0\) and \(\mu \neq \epsilon_0\), where
\[
\epsilon_0(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
1, & \text{if } t > 0
\end{cases}
\]
For \(x \in X, t \in \mathbb{R}\), if we define
\[
\nu_x(t) = \mu\left(\frac{t}{||x||}\right), x \neq 0,
\]
then in [10], it is proved that \((X, \nu, T)\) is a PN space in the sense of Definition 2.3. Alsina et al. [1] gave new definition of a PN-Space. Before giving this, we recall for the reader’s convenience the concept of a triangle function, that of a PN space from the point of view of the new definition.

**Definition 2.4.** A triangle function is a mapping \(\tau\) from \(\Delta^+ \times \Delta^+\) into \(\Delta^+\) such that, for all \(F, G, H, K\) in \(\Delta^+\),

1. \(\tau(F, \epsilon_0) = F\);
2. \(\tau(F, G) = \tau(G, F)\);
3. \(\tau(F, G) \leq \tau(H, K)\) whenever \(F \leq H, G \leq K\);
4. \(\tau(\tau(F, G), H) = \tau(F, \tau(G, H))\).

Particular and relevant triangle functions are the functions \(\tau_T, \tau_T^*\) and those of the form \(\Pi_T\) which, for any continuous \(t\)-norm \(T\), and any \(x > 0\), are given by
\[
\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) : u + v = x\},
\]
\[
\tau_T^*(F, G)(x) = \inf\{T^*(F(u), G(v)) : u + v = x\}
\]
and
\[
\Pi_T(F, G)(x) = T(F(x), G(x))
\]

**Definition 2.5.** ([1]) A probabilistic normed space is a quad-rupe \((X, \nu, \tau, \tau^*)\), where \(X\) is a real linear space, \(\tau\) and \(\tau^*\) are continuous triangle functions such that \(\tau \leq \tau^*\) and the mapping \(\nu : X \to \Delta^+\) called the probabilistic norm, satisfies for all \(p \leq q\) in \(X\), the conditions

1. \((PN1)\) \(\nu_p = \epsilon_0\) if and only if \(p = \theta\) (\(\theta\) is the null vector in \(X\));
2. \((PN2)\) \(\forall p \in X, \nu_{-p} = \nu_p\);
3. \((PN3)\) \(\nu_{p+q} \geq \nu_p + \nu_q\);
4. \((PN4)\) \(\forall a \in [0, 1], \nu_p \leq \tau^*(\nu_{ap}, \nu_{(1-a)p})\).

If a PN space \((X, \nu, \tau, \tau^*)\), satisfies the following condition
\[
(\tilde{S}) \forall p \in X, \forall \lambda \in \mathbb{R} \setminus \{0\}, \forall t > 0, \nu_{ap}(t) = \nu_p\left(\frac{t}{\lambda}\right),
\]
then it is called a \(\tilde{S}\)erstnev PN space; the condition \((\tilde{S})\) implies that the best-possible selection for \(\tau^*\) is \(\tau^* = \tau_M\), which satisfies a stricter version of (PN4), namely,
\[
\forall a \in [0, 1], \nu_p = \tau_M(\nu_{ap}, \nu_{(1-a)p})
\]

**Definition 2.6.** A Menger PN space under \(T\) is a PN space \((X, \nu, \tau, \tau^*)\) denoted by \((X, \nu, T)\), in which \(\tau = \tau_T\) and \(\tau^* = \tau_T^*\), for some continuous \(t\)-norm \(T\) and its \(t\)-conorm \(T^*\).

**Lemma 2.1** ([10]). The simple space generated by \((X, ||.||)\) and by \(\mu\) is a Menger PN space under \(M\) and also a \(\tilde{S}\)erstnev PN space. Here \(M(x, y) := \min\{x, y\}\).

For further study, by a PN space we mean a PN space in the sense of Definition 2.3. We now give a quick look on the characterization of convergence and Cauchy sequences on these spaces.
Let \((X, \nu, T)\) be a PN space and \(x = (x_k)\) be a sequence in \(X\). We say that \((x_k)\) is convergent to \(\ell \in X\) with respect to the probabilistic norm \(\nu\) if for each \(\varepsilon > 0\) and \(\alpha \in (0, 1)\) there exists a positive integer \(m\) such that \(\nu_{x_k - \ell}(\varepsilon) > 1 - \alpha\) whenever \(k \geq m\). The element \(\ell\) is called the ordinary double limit of the sequence \((x_k)\) and we shall write \(\nu - \lim x_k = \ell\) or \(x_k \overset{\nu}{\to} \ell\) as \(k \to \infty\).

A sequence \((x_k)\) in \(X\) is said to be Cauchy with respect to the probabilistic norm \(\nu\) if for each \(\varepsilon > 0\) and \(\alpha \in (0, 1)\) there exists a positive integer \(M = M(\varepsilon, \alpha)\) such that \(\nu_{x_k-x_p}(\varepsilon) > 1 - \alpha\) whenever \(k, p \geq M\).

**Definition 2.7.** Let \((X, \nu, T)\) be an probabilistic normed space, and let \(r \in (0, 1)\) and \(x \in X\). The set
\[
B(x, r; t) = \{y \in X : \nu_{y-x}(t) > 1 - r\}
\]
is called open ball with center \(x\) and radius \(r\) with respect to \(t\).

Throughout the paper, we denote \(I\) is an admissible ideal of subsets of \(\mathbb{N}\) and \(\theta = (k_r)\), respectively, unless otherwise stated.

3. Main results

We now obtain our main results.

**Definition 3.1.** Let \(I \subset 2^\mathbb{N}\) and \((X, \nu, T)\) be an PNS. A sequence \(x = (x_k)\) in \(X\) is said to be \(I_\theta\)-convergent to \(L \in X\) with respect to the probabilistic norm \(\nu\) if, for every \(\varepsilon > 0\) and \(\alpha \in (0, 1)\) the set
\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-L}(\varepsilon) \leq 1 - \alpha\right\} \in I.
\]

\(L\) is called the \(I_\theta\)-limit of the sequence \(x = (x_k)\) in \(X\), and we write \(I_\theta^\nu - \lim x = L\).

**Example 3.1.** Let \((\mathbb{R}, |\cdot|)\) denote the space of all real numbers with the usual norm, and let \(T(a, b) = ab\) for all \(a, b \in [0, 1]\). For all \(x \in \mathbb{R}\) and every \(t > 0\), consider \(\nu_{x}(t) = \frac{1}{t + |x|}\). Then \((\mathbb{R}, \nu, T)\) is an PNS. If we take \(I = \{A \subset \mathbb{N} : \delta(A) = 0\}\), where \(\delta(A)\) denotes the natural density of the set \(A\), then \(I\) is a non-trivial admissible ideal. Define a sequence \(x = (x_k)\) as follows:
\[
x_k = \begin{cases} 1, & \text{if } k = i^2, i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}
\]

Then for every \(\alpha \in (0, 1)\) and for any \(\varepsilon > 0\), the set
\[
K = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k}(\varepsilon) \leq 1 - \alpha\right\}
\]
will be a finite set. Hence, \(\delta(K) = 0\) and consequently \(K \in I\), i.e., \(I_\theta^\nu - \lim x = 0\).

**Lemma 3.1.** Let \((X, \nu, T)\) be an PNS and \(x = (x_k)\) be a sequence in \(X\). Then, for every \(\varepsilon > 0\) and \(\alpha \in (0, 1)\) the following statements are equivalent:

(i) \(I_\theta^\nu - \lim x = L\),

(ii) \(\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-L}(\varepsilon) \leq 1 - \alpha\right\} \in I\),

(iii) \(\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-L}(\varepsilon) > 1 - \alpha\right\} \in F(I),\)

(iv) \(I_\theta - \lim \nu_{x_k-L}(\varepsilon) = 1\).
**Theorem 3.1.** Let \((X, \nu, T)\) be an PNS and if a sequence \(x = (x_k)\) in \(X\) is \(I_\theta\)-convergent to \(L \in X\) with respect to the probabilistic norm \(\nu\), then \(I_\theta - \lim x = L\) is unique.

**Proof.** Suppose that \(I_\theta - \lim x = L_1\) and \(I_\theta - \lim x = L_2\) \((L_1 \neq L_2)\). Given \(\alpha > 0\) and choose \(\beta \in (0, 1)\) such that

\[
T(1 - \beta, 1 - \beta) > 1 - \alpha.
\]

Then for \(\varepsilon > 0\), define the following sets:

\[
K_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left( \frac{\varepsilon}{2} \right) \leq 1 - \beta \right\},
\]

\[
K_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left( \frac{\varepsilon}{2} \right) \leq 1 - \beta \right\},
\]

Since \(I_\theta - \lim x = L_1\), using Lemma 2.1., we have \(K_1 \in I\). Also, using \(I_\theta - \lim x = L_2\), we get \(K_2 \in I\). Now let

\[
K = K_1 \cup K_2.
\]

Then \(K \in I\). This implies that its complement \(K^c\) is a non-empty set in \(F(I)\). Now if \(r \in K^c\), let us consider \(r \in K_1^c \cap K_2^c\). Then we have

\[
\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left( \frac{\varepsilon}{2} \right) > 1 - \beta \quad \text{and} \quad \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left( \frac{\varepsilon}{2} \right) > 1 - \beta.
\]

Now, we choose a \(s \in \mathbb{N}\) such that

\[
\nu_{x_s - L_1} \left( \frac{\varepsilon}{2} \right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left( \frac{\varepsilon}{2} \right) > 1 - \beta
\]

and

\[
\nu_{x_s - L_2} \left( \frac{\varepsilon}{2} \right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left( \frac{\varepsilon}{2} \right) > 1 - \beta
\]

e.g., consider \(\max \{\nu_{x_k - L_1} \left( \frac{\varepsilon}{2} \right), \nu_{x_k - L_2} \left( \frac{\varepsilon}{2} \right) : k \in J_r\}\) and choose that \(k\) as \(s\) for which the maximum occurs. Then from (2.1), we have

\[
\nu_{L_1 - L_2}(\varepsilon) \geq T \left( \nu_{x_s - L_1} \left( \frac{\varepsilon}{2} \right), \nu_{x_s - L_2} \left( \frac{\varepsilon}{2} \right) \right) > T(1 - \beta, 1 - \beta) > 1 - \alpha.
\]

Since \(\alpha > 0\) is arbitrary, we have \(\nu_{L_1 - L_2}(\varepsilon) = 1\) for all \(\varepsilon > 0\), which implies that \(L_1 = L_2\). Therefore, we conclude that \(I_\theta - \lim x\) is unique.

Here, we introduce the notion of \(\theta\)-convergence in an PNS and discuss some properties.

**Definition 3.2.** Let \((X, \nu, T)\) be an PNS. A sequence \(x = (x_k)\) in \(X\) is \(\theta\)-convergent to \(L \in X\) with respect to the probabilistic norm \(\nu\) if, for \(\alpha \in (0, 1)\) and every \(\varepsilon > 0\), there exists \(r_0 \in \mathbb{N}\) such that

\[
\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha
\]

for all \(r \geq r_0\). In this case, we write \(\nu^\theta - \lim x = L\).

**Theorem 3.2.** Let \((X, \nu, T)\) be an PNS and let \(x = (x_k)\) in \(X\). If \(x = (x_k)\) is \(\theta\)-convergent with respect to the probabilistic norm \(\nu\), then \(\nu^\theta - \lim x = L\) is unique.
Proof. Suppose that $\nu^0 - \lim x = L_1$ and $\nu^\beta - \lim x = L_2$ $(L_1 \neq L_2)$. Given $\alpha \in (0, 1)$ and choose $\beta \in (0, 1)$ such that $T(1 - \beta, 1 - \beta) > 1 - \alpha$. Then for any $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_1$. Also, there exists $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_2$. Now, consider $r_o = \max \{r_1, r_2\}$. Then for $r \geq r_o$, we will get a $s \in \mathbb{N}$ such that

$$\nu_{x_s - L_1}(\frac{\varepsilon}{2}) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}(\frac{\varepsilon}{2}) > 1 - \beta$$

and

$$\nu_{x_s - L_2}(\frac{\varepsilon}{2}) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2}(\frac{\varepsilon}{2}) > 1 - \beta.$$

Then, we have

$$\nu_{L_1 - L_2}(\varepsilon) \geq T\left(\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right)\right) > T(1 - \beta, 1 - \beta) > 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$.

**Theorem 3.3.** Let $(X, \nu, T)$ be an PNS and let $x = (x_k)$ in $X$. If $\nu^\gamma - \lim x = L$, then $I^\nu_o - \lim x = L$.

Proof. Let $\nu^\gamma - \lim x = L$, then for every $\varepsilon > 0$ and given $\alpha \in (0, 1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Therefore the set

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \subseteq \{1, 2, \ldots, n_0 - 1\}.$$

But, with $I$ being admissible, we have $B \in I$. Hence $I^\nu_o - \lim x = L$.

**Theorem 3.4.** Let $(X, \nu, T)$ be an PNS and $x = (x_k), y = (y_k)$ be two sequence in $X$.

(i) If $I^\nu_o - \lim x_k = L_1$ and $I^\nu_o - \lim y_k = L_2$, then $I^\nu_o - \lim(x_k \pm y_k) = L_1 \pm L_2$;

(ii) If $I^\nu_o - \lim x_k = L$ and $a$ be a non-zero real number, then $I^\nu_o - \lim ax_k = aL$. If $a = 0$, then result is true only if $I$ is an admissible of $\mathbb{N}$.

Proof. (i) We have proved that, if $I^\nu_o - \lim x_k = L_1$ and $I^\nu_o - \lim y_k = L_2$, then $I^\nu_o - \lim(x_k + y_k) = L_1 + L_2$, only. The proof of the other part follows similarly.

Take $\varepsilon > 0, \alpha \in (0, 1)$ and choose $\beta \in (0, 1)$ such that the condition (3.1) holds. If we define

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}(\frac{\varepsilon}{2}) \leq 1 - \beta \right\},$$
and
\[ A_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L_2} \left( \frac{\varepsilon}{2} \right) \leq 1 - \beta \right\}. \]

Then \( A_1^c \cap A_2^c \in F(I) \). We claim that
\[ A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}. \]

Let \( n \in A_1^c \cap A_2^c \). Now, using (3.1), we have
\[ \frac{1}{h_r} \sum_{n \in J_r} \nu_{(x_n - L_1) + (y_n - L_2)}(\varepsilon) \geq T \left( \frac{1}{h_r} \sum_{n \in J_r} \nu_{x_n - L_1} \left( \frac{\varepsilon}{2} \right) + \frac{1}{h_r} \sum_{n \in J_r} \nu_{y_n - L_2} \left( \frac{\varepsilon}{2} \right) \right), \]
\[ > T(1 - \beta, 1 - \beta) > 1 - \alpha. \]

Hence
\[ A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}. \]

As \( A_1^c \cap A_2^c \in F(I) \), so
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) \leq 1 - \alpha \right\} \in I. \]

Therefore \( I_0^r - \lim(x_k + y_k) = L_1 + L_2. \)

(ii) Suppose \( a \neq 0 \). Since \( I_0^r - \lim x_k = L \), for each \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \), the set
\[ A(\varepsilon, \alpha) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) < 1 - \alpha \right\} \in F(I). \]

If \( n \in A(\varepsilon, \alpha) \), the we have
\[ \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L} \left( \frac{\varepsilon}{|a|} \right), \]
\[ \geq T \left( \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), \nu_0 \left( \frac{\varepsilon}{|a|} - \varepsilon \right) \right), \]
\[ \geq T \left( \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), 1 \right) \geq \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha. \]

Hence
\[ A(\varepsilon, \alpha) \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\} \]
and
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\} \in F(I). \]

It follows that
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) \leq 1 - \alpha \right\} \in I. \]

Hence \( I_0^r - \lim ax_k = aL \).
Next suppose that \( a = 0 \). Then for each \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \), we have
\[
\frac{1}{h_r} \sum_{k \in J_r} \nu_{\alpha x_k - \alpha L}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_0(\varepsilon) = 1 - 1 - \alpha,
\]
it follows that \( \nu^\theta - \lim x = \ell \). Hence from Theorem 3.3, \( I_\nu^\theta - \lim x = \ell \).

**Theorem 3.5.** Let \((X, \nu, T)\) be an PNS and let \( x = (x_k) \) in \( X \). If \( \nu^\theta - \lim x = L \), then there exists a subsequence \((x_{m_k})\) of \( x = (x_k) \) such that \( \nu - \lim x_{m_k} = L \).

**Proof.** Let \( \nu^\theta - \lim x = L \). Then, for every \( \varepsilon > 0 \) and given \( \alpha \in (0, 1) \), there exists \( r_0 \in \mathbb{N} \) such that
\[
\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha
\]
for all \( r \geq r_0 \). Clearly, for each \( r \geq r_0 \), we can select an \( m_k \in J_r \) such that
\[
\nu_{x_{m_k} - L}(\varepsilon) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.
\]
It follows that \( \nu - \lim x_{m_k} = L \).

**Definition 3.3.** Let \((X, \nu, T)\) be an PNS and let \( x = (x_k) \) be a sequence in \( X \). Then,
1. An element \( L \in X \) is said to be \( I_\theta \)-limit point of \( x = (x_k) \) if there is a set \( M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N} \) such that the set \( M' = \{r \in \mathbb{N} : m_k \in J_r \} \notin I \) and \( \nu^\theta - \lim x_{m_\nu} = L \).
2. An element \( L \in X \) is said to be \( I_\theta \)-cluster point of \( x = (x_k) \) if for every \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \), we have
\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \notin I.
\]

Let \( \Lambda^\theta_I(x) \) denote the set of all \( I_\theta \)-limit points and \( \Gamma^\theta_I(x) \) denote the set of all \( I_\theta \)-cluster points in \( X \), respectively.

**Theorem 3.6.** Let \((X, \nu, T)\) be an PNS. For each sequence \( x = (x_k) \) in \( X \), we have
\[
\Lambda^\theta_I(x) \subset \Gamma^\theta_I(x).
\]

**Proof.** Let \( L \in \Lambda^\theta_I(x) \), then there exists a set \( M \subset \mathbb{N} \) such that \( M' \notin I \), where \( M \) and \( M' \) are as in the Definition 2.3., satisfies \( \nu^\Lambda - \lim x_{m_\nu} = L \). Thus, for every \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \), there exists \( r_0 \in \mathbb{N} \) such that
\[
\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_{m_k} - L}(\varepsilon) > 1 - \alpha
\]
for all \( r \geq r_0 \). Therefore,
\[
B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \supseteq M' \\setminus \{m_1, m_2, \ldots, m_\nu\}.
\]
Now, with \( I \) being admissible, we must have \( M' \\setminus \{m_1, m_2, \ldots, m_\nu\} \notin I \) and as such \( B \notin I \). Hence \( L \in \Gamma^\theta_I(x) \).

**Theorem 3.7.** Let \((X, \nu, T)\) be an PNS. For each sequence \( x = (x_k) \) in \( X \), the set \( \Gamma^\theta_I(x) \) is closed set in \( X \) with respect to the usual topology induced by the probabilistic norm \( \nu^\theta \).
Proof. Let $y \in I_0^\nu(x)$. Take $\varepsilon > 0$ and $\alpha \in (0,1)$. Then there exists $L_0 \in I_0^\nu(x) \cap B(y, \alpha, \varepsilon)$. Choose $\delta > 0$ such that $B(L_0, \delta, \varepsilon) \subset B(y, \alpha, \varepsilon)$. We have

$$G = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-y}(\varepsilon) > 1 - \alpha \right\}$$

$$\supset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-L_0}(\varepsilon) > 1 - \delta \right\} = H.$$

Thus $H \notin I$ and so $G \notin I$. Hence $y \in I_0^\nu(x)$.

**Theorem 3.8.** Let $(X, \nu, T)$ be an PNS and let $x = (x_k)$ in $X$. Then the following statements are equivalent:

1. $L$ is a $I_0$-limit point of $x$,
2. There exist two sequences $y$ and $z$ in $X$ such that $x = y + z$ and $\nu^\theta - \lim y = L$ and \( \{ r \in \mathbb{N} : k \in J_r, z_k \neq \emptyset \} \in I \), where $\emptyset$ is the zero element of $X$.

**Proof.** Suppose that (1) holds. Then there exist sets $M$ and $M'$ as in Definition 2.3, such that $M' \notin I$ and $\nu^\theta - \lim x_{m_k} = L$. Define the sequences $y$ and $z$ as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in J_r; r \in M', \\ L, & \text{otherwise.} \end{cases}$$

and

$$z_k = \begin{cases} \emptyset, & \text{if } k \in J_r; r \in M', \\ x_k-L, & \text{otherwise.} \end{cases}$$

It suffices to consider the case $k \in J_r$ such that $r \in \mathbb{N} \setminus M'$. Then for each $\alpha \in (0,1)$ and $\varepsilon > 0$, we have $\nu_{y_k-L}(\varepsilon) = 1 > 1 - \alpha$. Thus, in this case,

$$1 = \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k-L}(\varepsilon) = 1 > 1 - \alpha.$$

Hence $\nu^\theta - \lim y = L$. Now $\{ r \in \mathbb{N} : k \in J_r, z_k \neq \emptyset \} \subset \mathbb{N} \setminus M'$ and so $\{ r \in \mathbb{N} : k \in J_r, z_k \neq \emptyset \} \notin I$.

Now, suppose that (2) holds. Let $M' = \{ r \in \mathbb{N} : k \in J_r, z_k = \emptyset \}$. Then, clearly $M' \in F(I)$ and so it is an infinite set. Construct the set $M = \{ m_1 < m_2 < ... < m_k < ... \} \subset \mathbb{N}$ such that $m_k \in J_r$ and $z_{m_k} = \emptyset$. Since $x_{m_k} = y_{m_k} = \lim y = L$ we obtain $\nu^\theta - \lim x_{m_k} = L$. This completes the proof.

**Theorem 3.9.** Let $(X, \nu, T)$ be an PNS and $x = (x_k)$ be a sequence in $X$. Let $I$ be an admissible ideal in $\mathbb{N}$. If there is a $I_0^\nu$-convergent sequence $y = (y_k)$ in $X$ such that $\{ k \in \mathbb{N} : y_k \neq x_k \} \in I$ then $x$ is also $I_0^\nu$-convergent.

**Proof.** Suppose that $\{ k \in \mathbb{N} : y_k \neq x_k \} \in I$ and $I_0^\nu - \lim y = \ell$. Then for every $\alpha \in (0,1)$ and $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k-L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

For every $0 < \alpha < 1$ and $\varepsilon > 0$, we have

$$r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k-L}(\varepsilon) \leq 1 - \alpha$$

(3.2)
\[ \{ k \in \mathbb{N} : y_k \neq x_k \} \cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L} (\varepsilon) \leq 1 - \alpha \right\}. \]

As both the sets of right-hand side of (2.2) is in \( I \), therefore we have that
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu (x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu (x_k - L, t) \geq \varepsilon \right\} \in I. \]

This completes the proof of the theorem.

**Definition 3.4.** Let \((X, \nu, T)\) be an PNS. A sequence \(x = (x_k)\) in \(X\) is said to be \(\theta\)-Cauchy sequence with respect to the probabilistic norm \(\nu\) if, for every \(\varepsilon > 0\) and \(\alpha \in (0, 1)\), there exist \(r_0, m \in \mathbb{N}\) satisfying
\[ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m} (\varepsilon) > 1 - \varepsilon \]
for all \(r \geq r_0\).

**Definition 3.5.** Let \(I\) be an admissible ideal of \(\mathbb{N}\). Let \((X, \nu, T)\) be an PNS. A sequence \(x = (x_k)\) in \(X\) is said to be \(I_\theta\)-Cauchy sequence with respect to the probabilistic norm \(\nu\) if, for every \(\varepsilon > 0\) and \(\alpha \in (0, 1)\), there exists \(m \in \mathbb{N}\) satisfying
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m} (\varepsilon) > 1 - \varepsilon \right\} \in F (I) \]

**Definition 3.6.** Let \(I\) be an admissible ideal of \(\mathbb{N}\). Let \((X, \nu, T)\) be an PNS. A sequence \(x = (x_k)\) in \(X\) is said to be \(I_\theta^*\)-Cauchy sequence with respect to the probabilistic norm \(\nu\) if, there exists a set \(M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}\) such that the set \(M' = \{r \in \mathbb{N} : m_k \in J_r \} \subset F (I)\) and the subsequence \((x_{m_k})\) of \(x = (x_k)\) is a \(\theta\)-Cauchy sequence with respect to the probabilistic norm \(\nu\).

The following theorem is an analogue of Theorem 3.3, so the proof omitted.

**Theorem 3.10.** Let \(I\) be an admissible ideal of \(\mathbb{N}\). Let \((X, \nu, T)\) be an PNS. If a sequence \(x = (x_k)\) in \(X\) is \(\theta\)-Cauchy sequence with respect to the probabilistic norm \(\nu\), then it is \(I_\theta\)-Cauchy sequence with respect to the same norm.

The proof of the following theorem’s proof is similar to that of Theorem 3.5.

**Theorem 3.11.** Let \((X, \nu, T)\) be an PNS. If a sequence \(x = (x_k)\) in \(X\) is \(\theta\)-Cauchy sequence with respect to the probabilistic norm \(\nu\), then there is a subsequence of \(x = (x_k)\) which is ordinary Cauchy sequence with respect to the same norm.

The following theorem can be proved easily using similar techniques as in the proof of Theorem 3.6.

**Theorem 3.12.** Let \(I\) be an admissible ideal of \(\mathbb{N}\). Let \((X, \nu, T)\) be an PNS. If a sequence \(x = (x_k)\) in \(X\) is \(I_\theta^*\)-Cauchy sequence with respect to the probabilistic norm \(\nu\), then it is \(I_\theta\)-Cauchy sequence as well.

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