A framework for the non-equilibrium thermodynamics of glasses is discussed. It also explains the non-equilibrium thermodynamics of a black hole isolated from matter. The first and second laws of black dynamics and black hole thermodynamics are shown to coincide, while the third laws deal with different physical issues.

1 Introduction

Thermodynamics is the old science that describes the flow of energy in systems with many atoms. Works started by Carnot, Kelvin and Clausius showed that these laws are very general. This universality led to the formulation of the first and second law of thermodynamics, that apply to a vast amount of systems, such as gases and crystals.

During half a century there was still a problem with the application to glasses. In this field there were classical paradoxes related to the so-called Ehrenfest relations and Prigogine-Defay ratio. The solution of this problem is discussed below. Due to its inherent non-equilibrium nature a glass is far from equilibrium. To describe it in a thermodynamic treatment one has to take into account at least one additional system parameter, the self-generated effective temperature, and its conjugate variable, the configurational entropy.

After having realized how thermodynamics should be formulated for glasses, we have investigated the situation for black holes. For this problem various aspects of the dynamics are known, and it was generally expected that the laws for black hole dynamics would coincide with the laws of black hole thermodynamics. We nevertheless felt that the proper connection between black hole dynamics and standard thermodynamics had not been made, and this will be clarified below.

Two years have passed since our letter on the black hole thermodynamics appeared. Till now the reaction of workers in gravitation was reserved, though researchers in other areas were attracted by its unifying concept. We feel that the blame should be put on the lack of basic thermodynamic training in physics teaching programs. Indeed, for many scholars thermodynamics is not much more than Gibbsian equilibrium thermodynamics. If asked “what is the second law?” the answer is often $dU = TdS - pdV$, which, however, is the combination of the first law ($dU = dQ + dW$ with $dW = -pdV$ for a fluid) and the second law in case of equilibrium, where the heat added to the system satisfies $dQ = TdS$. When mentioning the issue of thermodynamics of a black hole isolated from matter to people working in gravitation, a standard reaction is: “But I can put it in an insulating box, and then I can apply equilibrium thermodynamics”. To us it is worrisome that such an unrealistic setup is considered to be a satisfactory explanation of the...
issue. Of course, what is at stake is the non-equilibrium thermodynamics of the black hole in the presence of cosmic background radiation, and it is found that it is not much different from the one for a black body. Even the question “Would you also claim that for a Newtonian star the background radiation should be taken into account?” has a simple answer: “Yes, in principle it should, but since the surface temperature is then much larger, one could say that the outside has zero temperature”. And this statement is just equivalent to saying that, by assumption, the Newtonian star is isolated, which makes it different from a star immersed in a gas cloud. Conceptually it is also important that requiring the outside to have (practically) zero temperature also excludes the situation where only some interior part of a star is discussed. So, for obvious reasons, the picture is self-consistent.

In short, we wish to stress that thermodynamics is not some nasty issue that can be best avoided, rather it embodies the heart of the physics under consideration, also, within appropriate time-windows, for a wide range of problems in the field of gravitation, such as black holes, Newtonian stars, globular star clusters, etc.

To explain the issue we shall first consider the non-equilibrium thermodynamics of glasses, and then apply those insights to show that known results for isolated black holes perfectly fit within this framework.

2 Glasses

Thermodynamics for systems far from equilibrium has long been a field of confusion. A typical application is window glass. Such a system is far from equilibrium: a cubic micron of glass is neither a crystal nor an ordinary under-cooled liquid. It is an under-cooled liquid that, in the glass formation process, has fallen out of its meta-stable liquid equilibrium. There is thus a separation of timescales between the fast processes, often called $\beta$-processes, and the basically quenched $\alpha$- or configurational processes.

Until our recent works on this field, the general consensus reached after more than half a century of research was: **Thermodynamics does not work for glasses, because there is no equilibrium.** This conclusion was mainly based on the failure to understand the Ehrenfest relations and the related Prigogine-Defay ratio. It should be kept in mind that, so far, the approaches leaned very much on equilibrium ideas. We shall stress that such approaches are not applicable, due to the inherent non-equilibrium character of the glassy state. Even the quoted conclusion itself is actually confusing, since thermodynamics should also hold outside equilibrium.

Thermodynamics is the most robust field of physics. Its failure to describe the glassy state is quite unsatisfactory, since up to 25 decades in time can be involved. Naively we expect that each decade has its own dynamics, basically independent of the other ones. We have found support for this point in models that can be solved exactly. Thermodynamics then means a description of system properties under smooth enough non-equilibrium conditions, where the dynamics is slow enough to build a quasi-equilibrium state.
2.1 Two-temperature thermodynamics

A state that slowly relaxes to equilibrium is characterized by the time elapsed so far, sometimes called “age” or “waiting time”. For glassy systems this is of special relevance. We shall restrict to systems with one diverging time scale. They are systems with first-order-type transitions, with (smoothed out) discontinuous order parameter, though usually there is no latent heat.

The picture to be investigated in this work starts by describing a non-equilibrium state characterized by three parameters, namely \( T, p \) and the age \( t \). By exactly solving the dynamics of certain model systems \( \ldots \), it was found that in the thermodynamics the non-equilibrium nature shows up only through an effective temperature \( T_e(t) \); also numerical data on the slow evolution of a binary Lennard-Jones glass were recently interpreted in terms of an effective temperature \( \ldots \). For a set of smoothly related cooling experiments \( T_i(t) \) at pressures \( p_i \), one may express the effective temperature as a continuous function: \( T_e(t) \rightarrow T_e(T, p) \). This sets a surface in \((T, T_e, p)\) space, that becomes multi-valued if one first cools, and then heats. For covering the whole space one needs to do many experiments, e.g., at different pressures and different cooling rates. The results should agree with findings from heating experiments and aging experiments. Thermodynamics amounts to giving differential relations between observables at nearby points in this space. In principle also an effective pressure could be needed. If there are many long time scales, also several effective temperatures would be needed. We shall restrict ourselves to the simplest case of one effective temperature.

Of special importance is the thermodynamics of a thermal body at temperature \( T_2 \) in a heat bath at temperature \( T_1 \). This could apply to mundane situations such as a cup of coffee, or an ice-cream, in a room. There are also two entropies, \( S_1 \) and \( S_2 \). Notice that there are also two time-scales: the time-scale for heat to leave the cup is much larger than the time-scale for equilibrating that heat in the room. It is this separation of time-scales that makes spontaneous difference in temperatures possible. The change in heat of such a system obeys

\[
\begin{align*}
\mathrm{d}Q &= \mathrm{d}Q_1 + \mathrm{d}Q_2 \\
&\leq T_1 \mathrm{d}S_1 + T_2 \mathrm{d}S_2.
\end{align*}
\]

A similar two-temperature approach proves to be relevant for glassy systems. The known exact results on the thermodynamics of systems without currents can be summarized by the very same change in heat

\[
\begin{align*}
\mathrm{d}Q &= T \mathrm{d}S_{\text{ep}} + T_e \mathrm{d}I
\end{align*}
\]

where \( S_{\text{ep}} \) is the entropy of the fast or equilibrium processes (\( \beta \)-processes) and \( I \) the configurational entropy of the slow or configurational processes (\( \alpha \)-processes). This object is also known as information entropy or complexity. Notice that it deviates from the standard definition of the configurational entropy \( S_c \) as the entropy of the glass minus the one of the vibrational modes of the crystal. The latter definition also incorporates fast non-vibrational processes.

It is both surprising and satisfactory that a glass can be described by the same general law. If, in certain systems, also an effective pressure or field would be needed, then \( \mathrm{d}Q \) is expected to keep the same form, but \( \mathrm{d}W \) would change from its standard value \(-p \mathrm{d}V\) for liquids. Such an extension could be needed to describe a larger class of systems.
For a glass forming liquid the first law \(dU = dQ + dW\) becomes

\[
dU = TdS_{ep} + T_e I - pdV
\]  

(2)

It is appropriate to define the generalized free enthalpy

\[
G = U - TS_{ep} - T_e I + pV
\]  

(3)

This is not the standard form, since \(T_e \neq T\). It satisfies

\[
dG = -S_{ep}dT - IdT_e + Vdp
\]  

(4)

The total entropy is

\[
S = S_{ep} + I
\]  

(5)

The second law requires \(dQ \leq TdS\), leading to

\[
(T_e - T)dI \leq 0
\]  

(6)

which merely says that heat goes from high to low temperatures.

Since \(T_e = T_e(T, p)\), and both entropies are functions of \(T\), \(T_e\) and \(p\), the expression (6) yields a specific heat

\[
C_p = \left. \frac{\partial Q}{\partial T} \right|_p = \left. \frac{\partial S_{ep}}{\partial T} \right|_{T_e, p} + \left. \frac{\partial S_{ep}}{\partial T_e} \right|_p \left. \frac{\partial T_e}{\partial T} \right|_p + \left. \frac{\partial I}{\partial T} \right|_{T_e, p} + \left. \frac{\partial I}{\partial p} \right|_{T_e, T, p} \frac{\partial T_e}{\partial T} \right|_p
\]  

(7)

In the glass transition region all factors, except \(\partial T_e/\partial T\), are basically constant. This leads to

\[
C_p = C_1 + C_2 \frac{\partial T_e}{\partial T} \bigg|_p
\]  

(8)

Precisely this form has been assumed half a century ago by Tool [1] as starting point for the study of caloric behavior in the glass formation region, and has often been used for the explanation of experiments [412]. It is a direct consequence of eq. (6).

2.2 Modified Maxwell relation, Ehrenfest relation and Prigogine-Defay ratio

For a smooth sequence of cooling procedures of a glassy liquid, eq. (2) implies a modified Maxwell relation between macroscopic observables such as \(U(T, p) \rightarrow U(T, T_e(T, p), p)\) and \(V\). This solely occurs since \(T_e\) is a non-trivial function of \(T\) and \(p\) for the smooth set of experiments under consideration. Most Maxwell relations involve the entropy, which is difficult to observe. The modified Maxwell relation between the observables \(U\) and \(V\) reads

\[
\left. \frac{\partial U}{\partial p} \right|_T + \left. p \frac{\partial V}{\partial p} \right|_T + \left. T \frac{\partial V}{\partial T} \right|_p = \left. T \frac{\partial I}{\partial T} \right|_{T_e, p} + \left. \frac{\partial I}{\partial p} \right|_{T_e, T} - T \left. \frac{\partial I}{\partial p} \right|_{T_e, T} \left. \frac{\partial T_e}{\partial T} \right|_p + T_e \left. \frac{\partial I}{\partial p} \right|_{T_e, T}
\]  

(9)

In equilibrium \(T_e = T\), so the right hand side vanishes, and the standard form is recovered.

In the glass transition region a glass forming liquid exhibits smeared jumps in the specific heat \(C_p = \partial(U + pV)/\partial T|_p\), the expansivity \(\alpha = \partial \ln V/ \partial p|_T\), and the compressibility \(\kappa = -\partial \ln V/\partial p|_T\). If one forgets about the smearing, one may consider them as true discontinuities, yielding an analogy with continuous phase transitions of the classical type.
Following Ehrenfest one may take the derivative of $\Delta V(T, p_g(T)) = 0$. The result is

$$
\Delta \alpha = \Delta \kappa \frac{dp_g}{dT}
$$

(10)

The conclusion drawn from half a century of research on glass forming liquids is that this relation is never satisfied. This has very much hindered progress on a thermodynamical approach. However, from a theoretical viewpoint it is hard to imagine that something could go wrong when just taking a derivative. We have pointed out that this relation is indeed satisfied automatically, but it is important to say what is meant by $\kappa$ in the glassy state.

Previous claims about the violation of the first Ehrenfest relation can be traced back to the equilibrium thermodynamics idea that there is one, ideal $\kappa$, to be inserted in (10). Indeed, investigators always considered cooling curves $V(T, p_i)$ at a set of pressures $p_i$ to determine $\Delta \alpha$ and $dp_g/dT$. However, $\Delta \kappa$ was always determined in another way, often from measurements of the speed of sound. In equilibrium such alternative determinations would yield the same outcome. In glasses this is not the case: the speed of sound is a short-time process, which only measures short-time relaxation. Therefore alternative procedures should be avoided, and only the cooling curves $V(T, p_i)$ should be used. They constitute a liquid surface $V_{\text{liquid}}(T, p)$ and a glass surface $V_{\text{glass}}(T, p)$ in $(T, p, V)$ space. These surfaces intersect, and the first Ehrenfest relation is no more than a mathematical identity about the intersection line of these surfaces. It is therefore a tautology, as was also stressed by McKenna.

The second Ehrenfest relation follows from differentiating $\Delta U(T, p_g(T)) = 0$. The obtained relation will also be satisfied automatically. However, one then eliminates $\partial U/\partial p$ by means of the Maxwell relation (9). We obtain

$$
\frac{\Delta C_p}{T_g V} = \Delta \alpha \frac{dp_g}{dT} + \frac{1}{V} \left(1 - \frac{\partial T_e}{\partial T} \right) \frac{dI}{dT}
$$

(11)

The last term, involving the “total” derivative $dI/dT = \partial I/\partial T + (\partial I/\partial p)(dp_g/dT)$ of the configurational entropy along the glass transition line, is new. Its prefactor only vanishes at equilibrium, in which case the standard Ehrenfest relation is recovered. The equality $T_e(T, p_g(T)) = T$ implies the useful identity

$$
\frac{dT_e}{dT} = \frac{\partial T_e}{\partial T} \bigg|_p + \frac{\partial T_e}{\partial p} \bigg|_T \frac{dp_g}{dT} = 1
$$

(12)

Historically one has defined so-called the Prigogine-Defay ratio

$$
\Pi = \frac{\Delta C_p \Delta \kappa}{TV(\Delta \alpha)^2}
$$

(13)

This looks like an equilibrium quantity, and for equilibrium transitions it should be equal to unity. Assuming that at the glass transition a number of unspecified parameters undergo a phase transition, Davies and Jones derived that $\Pi \geq 1$, while DiMarzio showed that in that case the correct value is $\Pi = 1$. In glasses typical experimental values are reported in the range $2 < \Pi < 5$. It was therefore...
generally expected that $\Pi \geq 1$ is a strict inequality arising from the requirement of mechanical stability.

However, since the first Ehrenfest relation is satisfied, it holds that

$$
\Pi = \frac{\Delta C_p}{TV\Delta \alpha (dp_g/dT)} = 1 + \frac{1}{V\Delta \alpha} \left( 1 - \frac{\partial T_c}{\partial T} \right) \frac{dT}{dp}
$$

Depending on the smooth set of experiments to be performed, $dp_g/dT$ can be small or large: $\Pi$ depends on the set of experiments. As a result, it can also be below unity. Rehage-Oels found $\Pi = 1.09 \approx 1$ at $p = 1 \text{ kbar}$, using a short-time value for $\kappa$. Reanalyzing their data we find from (14), where the correct $\kappa$ has been inserted, a value $\Pi = 0.77$, which indeed is below unity. The commonly accepted inequality $\Pi \geq 1$ is based on the equilibrium assumption of a unique $\kappa$. Our theoretical arguments and the Rehage-Oels data show that such assumptions are incorrect.

Further steps involve fluctuation formula and the fluctuation-dissipation theorem. They are modified outside equilibrium, and effective temperatures also occur there. In simple model systems all these effective temperatures coincide, and this is expected to hold for a subclass of systems.

3 Black Holes

Black holes were anticipated by Laplace in 1798. He considered the gravitational escape problem from a mass $M$. Equating kinetic $mv^2/2$ to the potential energy $GMm/r$ he observed, for light, the critical escape radius $R_S = 2GM/c^2$. Exactly this radius shows up in the Schwarzschild solution of the Einstein equations. The metric for a neutral, non-rotating black hole reads

$$
ds^2 = \left(1 - \frac{R_S}{r}\right)c^2 dt^2 - \frac{1}{1 - R_S/r} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{15}
$$

For spherical light waves one has $ds = d\theta = d\phi = 0$, implying a radial speed $dr/dt = (1 - R_S/r)c$, which vanishes at the ‘horizon’ $r = R_S$.

The connection between black holes and thermodynamics goes back to Bekenstein. He introduced in 1973 the notion of black hole entropy, proportional to its area, in dimensionless units. Since Hawking later fixed the numerical prefactor, the result is now called the Bekenstein-Hawking entropy

$$
S_{BH} = \frac{k_B A}{4L_p^2} = \frac{\pi R_S^2 k_B c^3}{\hbar G} \tag{16}
$$

The presence of $\hbar$ calls for a quantum mechanical interpretation, and the quantum evaporation of black holes was demonstrated by Hawking. This underlined the physical relevance of Bekenstein’s approach. The black hole radiates as a black body at Hawking temperature

$$
T_H = \frac{\hbar G k_B}{2\pi c^3 k_B} = \frac{\hbar c^3}{8\pi GMk_B} \tag{17}
$$

where the second equality holds for a non-rotating, neutral black hole. All possible particles are emitted at this temperature; for large black holes, however, $T_H$ is so
small, that in practice only massless particles (photons, gravitons, and possibly neutrino’s) are emitted.

A black hole has no hair, i.e. it can be characterized by a few parameters, namely its mass $M$, charge $q$ and angular momentum $J$. This is reminiscent of fluids, that can be characterized by temperature and pressure. From the mass formula for charged, rotating black holes it is known for long that the energy $U = Mc^2$ satisfies

$$dU = \frac{\kappa}{8\pi} dA + \Omega \cdot dJ + \phi dq$$

where $\kappa$ is the surface tension, $A = 4\pi R^2$ the area, $\Omega$ the horizon’s angular velocity, and $\phi$ the electrostatic potential at the horizon. This law holds when adding matter to one given black hole, but also when comparing two different black holes. These two very different applications suggest a universal validity, and a thermodynamic description.

These two fundamental results led Bardeen, Carter and Hawking to formulate “the four laws of black hole dynamics”. The zeroth law states that the surface tension $\kappa$ is constant at the black hole surface, just like the temperature is the same everywhere in an equilibrium system. The first law is given in eq. (19). Since the last two terms corresponds to work terms, one may write this relation in the suggestive form

$$dU = T_H dS_{BH} + dW$$

This formulation is sometimes referred to as the first law of black holes thermodynamics, but so far it is only a rewriting of (18). Bekenstein had already discussed the generalized second law of black hole dynamics: the total entropy $S = S_{BH} + S_m$, where $S_m$ is the entropy of the matter outside the black hole, cannot decrease:

$$dS \geq 0$$

The third law concerns “extremal” black holes, the ones that are maximally rotating and/or are maximally charged, and have $\kappa = 0$. The third law of black hole dynamics says that black holes with $T_H = 0$ cannot be reached by a finite number of steps.

As indicated by their name, the “four laws of black hole dynamics” look similar to the laws of thermodynamics, though a precise connection was not established. Only for a black hole put in a not-too-large box there is equilibrium, and the standard laws of equilibrium thermodynamics apply. During last 25 years the question has been to relate these laws to the standard laws of thermodynamics.

From the viewpoint of a condensed matter physicist, the literature on black hole thermodynamics is somewhat confusing. First of all, one should define the system for which a thermodynamic description is to be given. A natural choice is to consider as system the black hole and a “Gedanken” sphere around it of, say, a hundred times the Schwarzschild radius. One could also consider the whole universe as an isolated container. Our next objection concerns the non-general formulation of the first and second law. This has already be discussed in a more general context above.
Having defined the system, one should determine its entropy. For the Gedanken sphere with the black hole in it, eq. (19) applies. The entropy occurring in eq. (20) is \( S = S_{BH} + S_{m} \). The latter is the entropy of the cosmic background matter outside the black hole but inside the Gedanken sphere, and expected to scale with the sphere’s volume. The radiation generated by the hole will quickly leave the system and go to the heat bath around it; this is described by a \( dQ < 0 \).

If, on the other hand, the whole universe is considered as system, then \( dQ = 0 \). If no work is done, this implies that \( dU = 0 \), saying that energy radiated from the hole is still inside the system. In that case eq. (19) does not describe the change of the system’s energy, it only says something about the black hole. The total entropy is now \( S = S_{BH} + S_{m} \), and the second law of thermodynamics indeed says that \( dS \geq 0 \).

We conclude that eq. (19) and (20) are both valid, though they should not be applied simultaneously. They refer to different situations, that is to say, to different time scales. When only the black hole and its Gedanken sphere are considered, this describes the radiation emitted in per unit time. When considering the change in entropy of the whole universe, one tacitly assumes time scales so large that all information on the emitted radiation has been lost. Notice that the negative curvature of the universe, leading to exponential divergence of nearby trajectories, can establish this loss of information even though the emitted photons are hardly scattered.

A final, severe, objection against the current formulation of thermodynamics for black holes is: what is the heat bath? When working with one temperature, \( T_{H} \), this is by definition the bath temperature, and normally also the temperature of the object. This can only apply to a black hole in equilibrium with its own Hawking radiation, which is an unstable and thus unphysical situation; such an approach can also not deal with black holes of different size. Physically there is one, and only one choice for the bath: for a black hole far away from matter, the heat bath is the cosmic microwave background radiation, that presently has temperature \( T_{CMB} = 2.73 \, K \) (we neglect here the small CMB-fluctuations). So the actual problem deals with a system of which the dynamics “lives” at a second temperature, namely \( T_{H} \). As for glasses, this is a two-temperature situation, in agreement with the time scale argument: Black holes heavier than \( 10^{-18} M_{\odot} = 10^{15} g \) need more time to evaporate than the present age of the universe. For them the evaporation process, as seen by far-away observers, is so slow, that equilibration of the cosmic background radiation is a fast process.

The slow evaporation processes occur at the Hawking temperature and have entropy \( S_{BH} \). The entropy of the background radiation outside the back hole but inside the Gedanken sphere is very small. Moreover, the emitted radiation will immediately leave this sphere and there is no ‘back reaction’, so \( dS_{m} = 0 \). This simplifies eq. (19) to

\[
dQ = T_{H} dS_{BH}
\]

The standard first law \( dU = dQ + dW \) thus indeed reproduces (19), notwithstanding its non-equilibrium thermodynamic nature with \( T_{BH} \neq T_{CMB} \). According to the
second law \( (20) \), \( S_{BH} \) has to satisfy

\[
(T_{CMB} - T_H) \, dS_{BH} \geq 0
\]  

(22)

Hawking radiation leads to \( dS_{BH} < 0 \). Eq. (22) is thus satisfied as long as \( T_H > T_{CMB} \), but not below that. One might think that \( T_{CMB} \) plays no physical role whatsoever, and only shows up as determinator in the second law. However, the real point is that we not yet considered absorption of background radiation by the black hole. The absorption rate will be proportional to the area times the energy density, i.e., \( \sim M^2 T_{CMB}^4 \). The quantum absorption process is simply the time-reversed evaporation process. For non-rotating, neutral holes Hawking radiation leads to a mass loss

\[
\dot{M} = -\alpha_{em} \frac{\hbar c^4}{G^2 M^2} \rightarrow \dot{T}_H = \frac{(8\pi)^3 \alpha_{em} Gk_B^3}{\hbar^2 c^5} T_H^4
\]  

(23)

The dimensionless constant \( \alpha_{em} \) depends on the type of particles present, and their absorption probabilities, called “oscillator strengths” in solid state systems. \( T_H \) enters through the Bose-Einstein occupation numbers (for bosons, in particular photons) or Fermi-Dirac occupation numbers (for fermions). For an uncharged, non-rotating black hole Page finds \( \alpha = 5.246 \times 10^{-4} \) in the high-frequency limit, and \( 0.181 \times 10^{-4} \) in the low frequency limit. For absorption by the black hole of a photon (or a particle) from the cosmic background, the time-reversed problem shows up. It thus holds that \( \alpha_{abs}(T) = \alpha_{em}(T) \), no matter the character of the particle content; for simplicity we shall now replace both by a constant. The only difference between the two situations is the temperature occurring in the occupation numbers: for Hawking emission it is \( T_H \), while for cosmic background absorption it is \( T_{CMB} \). The combined processes of Hawking emission and background photon absorption thus yields for a neutral, non-rotation black hole

\[
\dot{T}_H = \frac{(8\pi)^3 \alpha Gk_B^3}{\hbar^2 c^5} (T_H^4 - T_{CMB}^4)
\]  

(24)

It exhibits an instability at \( T_H = T_{CMB} \), related to the fact that the “Bekenstein-Hawking” specific heat \( C_{BH} = dU/dT_H \) is negative. If there is equilibrium, and \( T_{CMB} \) is changed a little, then \( T_H \) branches away from it.

There are two regimes. In the “classical” regime \( T_H < T_{CMB} \) the black hole absorbs more radiation than it emits. Its entropy will increase, and \( dQ = T_H dS_{BH} > 0 \), but this is still in accord with the second law \( (22) \). In the “quantum” regime \( T_H > T_{CMB} \) the black hole emits more than it absorbs. Now it holds that \( dS_{BH} < 0 \), confirming again that heat flows from high to low temperature.

In analogy with glasses, one can define the apparent specific heat \( C = \partial U/\partial T_{CMB} = \dot{U}/\dot{T}_{CMB} \). For black holes this object is less natural because the background temperature cannot be changed by hand. However, \( C \) does have a meaning in our expanding universe. Due to the decrease of \( T_{CMB} \), there will be less and less background energy to be absorbed. Eq. (24) tells us that, provided \( T_{CMB} \) decays quicker than \( t^{-1/3} \), a black hole will reach its maximal size at some moment \( t = t_0 \) where the temperatures match, \( T_H = T_{CMB} = T_0 \); from then on it will shrink, and its Hawking temperature rises. In our matter dominated Universe
one has $T_{CMB} \sim t^{-2/3}$, so finally the black hole will evaporate. Around $t_0$ the apparent specific heat takes a form independent of $T_{CMB}$, viz.

$$C = k_B(t - t_0)/\tau,$$

with characteristic time scale the quantum time $\tau = \hbar/[(16\pi)^2\alpha k_B T_0]$. In the classical regime ($t < t_0$) $C$ is negative, while in the quantum regime it is positive.

The third law of black hole dynamics is related to extremal black holes, that have $T_H = 0$. The third law of thermodynamics, however, concerns the vanishing of the entropy for $T_{CMB} \to 0$. We have seen already that finally all black holes evaporate, thereby lowering their configurational entropy very much, in accord with Planck’s third law. Notice that this has nothing to do with the third law of black hole dynamics. What happens ultimately with the black hole has been the focus of studies by ’t Hooft.

The entropy change of the universe, now assumed to be a thermal bath, is found as for black body radiation. Changes of the energy and entropy of the black hole surroundings due to absorption and emission by the black hole are $\dot{U}_m \sim \alpha (T_H^4 - T_{CMB}^4)$, $\dot{S}_m \sim (4/3)(T_H^3 - T_{CMB}^3)$, which leads to

$$\frac{dS_m}{dS_{BH}} = \frac{T_H dS_m}{dU} = -\frac{T_H dS_m}{dU_m} = -\frac{4T_H(T_H^3 - T_{CMB}^3)}{3(T_H^4 - T_{CMB}^4)}$$

(25)

This yields the entropy production $\dot{S} = \dot{S}_m + \dot{S}_{BH}$

$$\dot{S} = \frac{(8\pi)^2\alpha k_B^2}{3\pi \hbar} \frac{(T_H^4 + 2T_H T_{CMB} + 3T_{CMB}^2)(T_H - T_{CMB})^2}{T_H^3 T_{CMB}}$$

(26)

Next we consider the whole universe as our system, so the entropy of the universe $S_m$ has to be taken into account. While $S = S_m + S_{BH}$ is the total entropy, eq. (21) becomes $dQ = T_{CMB} dS_m + T_H dS_{BH}$. As $dQ = 0$, the second law again leads to (22), but with different entropy production. For the present case we find the new result

$$\dot{S} = \frac{(8\pi)^2\alpha k_B^2}{\hbar} \frac{T_H^4 - T_{CMB}^4)(T_H - T_{CMB})}{T_H^3 T_{CMB}}$$

(27)

This expression exceeds eq. (23), which referred to radiation that was still located near the black hole. The difference is due to loss of information on the emitted radiation.

Our approach thus incorporates the known properties of dynamics, and shows how the generalized second law comes into the play. Both the formation and evaporation of black holes leads to an increase of the entropy of the whole universe. Our picture involves the standard zero-entropy formulation of the third law of thermodynamics, thus putting aside the third law of black hole mechanics. To the best of our knowledge, there is no contradiction with the occurrence of negative specific heats.

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A framework for the non-equilibrium thermodynamics of glasses is discussed. It also explains the non-equilibrium thermodynamics of a black hole isolated from matter. The first and second laws of black dynamics and black hole thermodynamics are shown to coincide, while the third laws deal with different issues.

Introduction. After having realized how thermodynamics should be formulated for glasses, we have investigated the situation for black holes. For this problem various aspects of the dynamics are known, and it was generally expected that the laws for black hole dynamics and black hole thermodynamics would coincide.

Two years have passed since our letter on the black hole thermodynamics appeared. Till now the reaction of workers in gravitation was reserved, though researchers in other areas were attracted by its unifying concept. We feel that the blame should be put on the lack of basic thermodynamic training in physics teaching programs. Indeed, for many scholars thermodynamics is not much more than Gibbsian equilibrium thermodynamics.

Glasses. Thermodynamics for systems far from equilibrium has long been a field of confusion. A typical application is window glass. Such a system is far from equilibrium: a cubic micron of glass is neither a crystal nor an ordinary under-cooled liquid. It is an under-cooled liquid that, in the glass formation process, has fallen out of its meta-stable liquid equilibrium. There is thus a separation of timescales between the fast processes, often called \( \beta \)-processes, and the basically quenched \( \alpha \)- or configurational processes.

Two-temperature thermodynamics. A state that slowly relaxes to equilibrium is characterized by the time elapsed so far, sometimes called “age” or “waiting time”. For glassy systems this is of special relevance. We shall restrict to systems with one diverging time scale. They are systems with first-order-type transitions, with nearly discontinuous order parameter, though usually there is no latent heat.

The picture to be investigated in this work starts by describing a non-equilibrium state characterized by three parameters, namely \( T, p \) and the age \( t \). By exactly solving the dynamics of certain model systems, it was found that in the thermodynamics the non-equilibrium nature shows up only through an effective temperature \( T_e(t) \); also numerical data on the slow evolution of a binary Lennard-Jones glass were recently interpreted in terms of an effective temperature.

Of special importance is the thermodynamics of a thermal body at temperature \( T_2 \) in a heat bath at temperature \( T_1 \). This could apply to mundane situations such as a cup of coffee, or an ice-cream, in a room. There are also two entropies, \( S_1 \) and \( S_2 \). Notice that there are also two time-scales: the time-scale for heat to leave the cup is much larger than the time-scale for equilibrating that heat in the room. It
is this separation of time-scales that makes spontaneous difference in temperatures possible. The change in heat is \( dQ = dQ_1 + dQ_2 \leq T_1 dS_1 + T_2 dS_2 \).

A similar two-temperature approach proves to be relevant for glassy systems. The known exact results on the thermodynamics of systems without currents can be summarized by the very same change in heat
\[
\bar{d}Q = \bar{d}Q_1 + \bar{d}Q_2 \leq T_1 dS_1 + T_2 dS_2.
\]

Black Holes. The Bekenstein-Hawking entropy reads, with \( R_S = 2GM/c^2 \),
\[
S_{BH} = \frac{k_B A}{4 L_p^2} = \frac{\pi R_S^2 k_B c^3}{\hbar G} \tag{1}
\]
The presence of \( \hbar \) calls for a quantum mechanical interpretation, and the quantum evaporation of black holes occurs at the Hawking temperature
\[
T_H = \frac{\hbar G \kappa}{2 \pi c^3 k_B} = \frac{\hbar c^3}{8 \pi GM k_B} \tag{2}
\]
where the second equality holds for a non-rotating, neutral black hole. All possible particles are emitted at this temperature; for large black holes, however, \( T_H \) is so small, that in practice only massless particles (photons, gravitons, and possibly neutrino’s) are emitted.

A black hole has no hair, i.e. it can be characterized by a few parameters, namely its mass \( M \), charge \( q \) and angular momentum \( J \). For charged, rotating black holes, the energy \( U = Mc^2 \) satisfies
\[
dU = \frac{\kappa}{8 \pi} \, dA + \Omega \cdot dJ + \phi \, dq \tag{3}
\]
where \( \kappa \) is the surface tension, \( A = 4 \pi R_S^2 \) the area, \( \Omega \) the horizon’s angular velocity, and \( \phi \) the electrostatic potential at the horizon. This law holds when adding matter to one given black hole, but also when comparing two different black holes. These two very different applications suggest a universal, thermodynamic description.

These two fundamental results led Bardeen, Carter and Hawking to formulate “the four laws of black hole dynamics”. The zeroth law states that the surface tension \( \kappa \) is constant at the black hole surface, just like the temperature is the same everywhere in an equilibrium system. The first law is given in eq. (3). Since the last two terms are work terms, one may write it in the suggestive form
\[
dU = T_H dS_{BH} + dW \tag{4}
\]
Bekenstein had already discussed the \textit{generalized second law of black hole dynamics}: the total entropy \( S = S_{BH} + S_m \), where \( S_m \) is the entropy of the matter outside the black hole, cannot decrease:
\[
dS \geq 0 \tag{5}
\]
The third law concerns “extremal” black holes, that are maximally rotating and/or are maximally charged, and have \( \kappa = 0 \). The third law of black hole dynamics says that black holes with \( T_H = 0 \) cannot be reached by a finite number of steps.
During last 25 years the question has been to relate these laws to the standard laws of thermodynamics.

From the viewpoint of a condensed matter physicist, the literature on black hole thermodynamics is somewhat confusing. First of all, one should define the system for which a thermodynamic description is to be given. A natural choice is to consider as system the black hole and a “Gedanken” sphere around it of, say, a hundred times the Schwarzschild radius. One could also consider the whole universe as an isolated container.

Having defined the system, one should determine its entropy. For the Gedanken sphere with the black hole in it, eq. (4) applies. The entropy occurring in eq. (5) is $S = S_{BH} + S_{Gs}$. The latter is the entropy of the cosmic background matter outside the black hole but inside the Gedanken sphere, and expected to scale with the sphere’s volume. The radiation generated by the hole will quickly leave the system and go to the heat bath around it; this is described by $\bar{d}Q < 0$.

If, on the other hand, the whole universe is considered as system, then $\bar{d}Q = 0$. The total entropy is now $S = S_{BH} + S_{m}$, and the second law of thermodynamics indeed says that $dS \geq 0$.

We conclude that eq. (4) and (5) are both valid, though they should not be applied simultaneously. They refer to different situations, that is to say, to different time scales. When only the black hole and its Gedanken sphere are considered, this describes the radiation emitted in per unit time. When considering the change in entropy of the whole universe, one tacitly assumes time scales so large that all information on the emitted radiation has been lost.

A final, severe, objection against the current formulation of thermodynamics for black holes is: what is the heat bath? Physically there is only one choice: for a black hole far away from matter, the heat bath is the cosmic microwave background radiation, that presently has temperature $T_{CMB} = 2.73 K$ (we neglect here the small CMB-fluctuations). So the actual problem deals with a system of which the dynamics “lives” at a second temperature, namely $T_H$.

The resolution. The slow evaporation processes occur at the Hawking temperature and have entropy $S_{BH}$. The entropy of the background radiation outside the back hole but inside the Gedanken sphere is very small. Moreover, the emitted radiation will immediately leave this sphere and there is no “back reaction”, so $dS_{m} = 0$, implying

$$dQ = T_H dS_{BH} \quad (6)$$

The standard first law $dU = dQ + dW$ thus indeed reproduces (5), notwithstanding its non-equilibrium thermodynamic nature with $T_{BH} \neq T_{CMB}$. According to the second law (5), $S_{BH}$ has to satisfy

$$(T_{CMB} - T_H) dS_{BH} \geq 0 \quad (7)$$

There also occurs absorption of background radiation by the black hole. The absorption rate will be proportional to the area times the energy density, i.e., $\sim M^2 T_{CMB}^4$. The quantum absorption process is simply the time-reversed evaporation process. The combined processes of Hawking emission and background photon
absorption thus yields for a neutral, non-rotation black hole

\[ \dot{T}_H = \frac{(8\pi)^3 \alpha Gk_B^3}{h^2 c^5} (T_H^4 - T_{CMB}^4) \]  

(8)

It exhibits an instability at \( T_H = T_{CMB} \), related to the fact that the “Bekenstein-Hawking” specific heat \( C_{BH} = dU/dT_H \) is negative. There are two regimes. In the “classical” regime \( T_H < T_{CMB} \) the black hole absorbs more radiation than is emitted. Its entropy will increase, and \( dQ = T_H dS_{BH} > 0 \), but this is still in accord with the second law (11). In the “quantum” regime \( T_H > T_{CMB} \) the black hole emits more than it absorbs. Now it holds that \( dS_{BH} < 0 \), confirming again that heat flows from high to low temperature.

The third law of black hole dynamics is related to extremal black holes, that have \( T_H = 0 \). The third law of thermodynamics, however, concerns the vanishing of the entropy for \( T_{CMB} \to 0 \). We have seen already that finally all black holes evaporate, thereby lowering their configurational entropy very much, in accord with Planck’s third law. This has nothing to do with the third law of black hole dynamics.

The entropy change of the universe, now assumed to be a thermal bath, is found as for black body radiation [1].

\[ \dot{S} = \frac{(8\pi)^2 a k_B^2}{3\pi h} \left( \frac{T_H^4 + 2T_H T_{CMB} + 3T_{CMB}^2}{T_H^2} \right) (T_H - T_{CMB})^2 \]  

(9)

Next we consider the whole universe as our system, so the entropy of the universe \( S_m \) has to be taken into account. While \( S = S_m + S_{BH} \) is the total entropy, one has \( dQ = T_{CMB} dS_m + T_H dS_{BH} \). As \( dQ = 0 \), the second law again leads to (9), but with different entropy production. For the present case we find

\[ \dot{S} = \frac{(8\pi)^2 a k_B^2}{h} \left( \frac{T_H^4 - T_{CMB}^4}{T_H^2 T_{CMB}^2} \right) (T_H - T_{CMB}) \]  

(10)

This exceeds eq. (9), which referred to radiation that was still located near the black hole. The difference is due to loss of information on the emitted radiation.

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