I. INTRODUCTION

As one of the cornerstones of quantum mechanics, uncertainty relation describes the measurement limitation on two incompatible observables \( \text{[1]} \). It should be emphasized that the uncertainty relation actually states an intrinsic property of a quantum system, rather than a statement about the observational success of current technology. Uncertainty relation has deep connection with many special characters in quantum mechanics, such as Bell non-locality and entanglement \( \text{[2, 3]} \), which cannot occur in classical world. With rapid progress in quantum technology, such as quantum communication and quantum computation \( \text{[4, 5]} \), in recent years, it is important for us to know the fundamental limitations in the achievable accuracy of quantum measurement.

Note that there are two different types of uncertainty relations, one is the preparation uncertainty relation, which studies the minimal dispersion of two quantum observables before measurement \( \text{[6, 7]} \). The Robertson uncertainty relation \( \text{[7]} \), reads as \( \sigma(x)\sigma(p) \geq \hbar/2 \), is a typical example in this sense, where \( \sigma(x) \) and \( \sigma(p) \) are the standard deviations of position and momentum of a particle. For such uncertainty relation, the measurements of \( x \) and \( p \) are performed on an ensemble of identically prepared quantum systems. While in the original spirit of Heisenberg’s idea \( \text{[1]} \), the Heisenberg’s uncertainty principle should be based on the observer’s effect, which means that measurement of a certain system cannot be made without affecting the system. So this leads to the second type of uncertainty relation: measurement uncertainty relation, which studies to what extent the accuracy of position measurement of a particle is related to the disturbance of the particle’s momentum, so called the error-disturbance uncertainty relation \( \text{[8]} \). It is also called the error-tradeoff relation in the approximate joint measurements of two incompatible observables \( \text{[9, 10]} \).

Heisenberg’s error-tradeoff uncertainty relation for joint measurement is generally expressed as

\[
\varepsilon(A)\varepsilon(B) \geq C_{AB}
\]

(1)

where \( C_{AB} = |\langle [A,B] \rangle|/2, [A,B] = AB - BA \). However, it has been shown that this relation is not valid in some cases \( \text{[11]} \). For this reason, Ozawa and Hall proposed new measurement uncertainty relations which have been theoretically proven to be universally valid for any incompatible observables, respectively \( \text{[8, 12]} \). After that, Branciard proposed a new uncertainty relation, which is universally valid and tighter than the Ozawa’s relation \( \text{[10]} \). There are also other types of measurement uncertainty relations generalizing Heisenberg’s original idea, which can be found in Refs. \( \text{[13–18]} \). Experimental tests of the measurement uncertainty relations have been demonstrated in photonic \( \text{[19–24]} \), spin \( \text{[25–28]} \), and ion trap systems \( \text{[29]} \). All of these experiments are limited in discrete-variable systems. Up to now, experimental test of the measurement uncertainty relation based on continuous-variable system has not been reported.

In this paper, we present the first experimental test of the error-tradeoff relation for two incompatible vari-
One mode of EPR entangled state is used as signal state $\rho$ and two incompatible observables are taken as $A = \hat{x}_1$ and $B = \hat{p}_1$, respectively [Fig. 1(a)], where $\hat{x}_1 = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{p}_1 = (\hat{a} - \hat{a}^\dagger)/2i$ denote the amplitude and phase quadratures of $\rho$, respectively. Another mode of EPR entangled state is used as the meter state $\rho_M$. Two compatible observables $C$ and $D$ are measured simultaneously to approximate $A$ and $B$. The quality of the approximations are characterized by defining the root-mean-square errors $\varepsilon(A) = \langle (C - A)^2 \rangle^{1/2}$ and $\varepsilon(B) = \langle (D - B)^2 \rangle^{1/2}$. Ozawa’s error-tradeoff relation is expressed by \[ \varepsilon(A)\varepsilon(B) + \varepsilon(A)\sigma(B) + \sigma(A)\varepsilon(B) \geq C_{AB} \] (2)

where $\sigma(A)$ is the standard deviation of observable $A$. The Branciard’s error-tradeoff relation is given by \[ [\varepsilon^2(A)\sigma^2(B) + \sigma^2(A)\varepsilon^2(B)]^{1/2} + 2\varepsilon(A)\varepsilon(B) \sqrt{\sigma^2(A)\sigma^2(B) - C_{AB}^2} \geq C_{AB} \] (3)

III. EXPERIMENTAL IMPLEMENTATION AND RESULTS

In the experiment, an EPR entangled state with $-2.9$ dB squeezing and $3.9$ dB antisqueezing is prepared by a nondegenerate optical parametric amplifier (NOPA), as shown in Fig. 1(b), which consists of an a-cut type-II KTP crystal and a concave mirror \[ \rho \]. The front face of the KTP crystal is used as the input coupler, and the concave mirror with $50$ mm curvature serves as the output coupler. The front face of the KTP crystal is coated with the transmission of $42\%$ at $540$ nm and high reflectivity at $1080$ nm. The end face of the KTP crystal is antireflection coated for both $540$ nm and high reflectivity at $1080$ nm. In the measurement, a sample size of $5 \times 10^5$ data points is used for all quadrature measurements with sampling rate of $500$ K/s. The interference efficiency between signal and local oscillation is $99\%$ and the quantum efficiency of photodiodes is $99.6\%$.

At first, we consider a situation that the observable $A$ is measured accurately (error-free measurement of observable $A$), i.e., the optimal estimation $C = A$. The measured phase quadrature $D = \hat{p}_2$ is used to approximate the observable $B$. Because the amplitude quadrature $\hat{x}_1$ of $\rho$ and the phase quadrature $\hat{p}_2$ of $\rho_M$ are compatible, they can be measured simultaneously. The errors for approximating $A$ and $B$ are expressed as $\varepsilon(A) = \sqrt{\langle (C - A)^2 \rangle} = 0$, and $\varepsilon(B) = \sqrt{\langle (D - B)^2 \rangle} = \sqrt{\sigma^2(\hat{p}_2 - \hat{p}_1)} = e^{-r}/\sqrt{2}$, respectively. Since $\varepsilon(A) = 0$ and $\varepsilon(B) < \infty$, we have $\varepsilon(A)\varepsilon(B) = 0$.

\[ \varepsilon(A)\varepsilon(B) \geq \sqrt{\langle (C - A)^2 \rangle} \sqrt{\langle (D - B)^2 \rangle} \]

\[ \varepsilon(A)\varepsilon(B) + \varepsilon(A)\sigma(B) + \sigma(A)\varepsilon(B) \geq C_{AB} \]

\[ [\varepsilon^2(A)\sigma^2(B) + \sigma^2(A)\varepsilon^2(B)]^{1/2} + 2\varepsilon(A)\varepsilon(B) \sqrt{\sigma^2(A)\sigma^2(B) - C_{AB}^2} \geq C_{AB} \]
It is obvious that Heisenberg’s error-tradeoff uncertainty relation is violated. The Ozawa’s and Branciard’s relations are the same for $\varepsilon(A) = 0$, which are
\[
\sigma(A)\varepsilon(B) = \sqrt{1 + e^{-4\theta}}/4 \geq 1/4. \quad (5)
\]

The amplitude quadrature $\hat{x}_1$ of the signal state is measured by a homodyne detector HD1 in the time domain, as shown in Fig. 1(b). To evaluate the error $\varepsilon(B)$, we experimentally measure the observables $B$ and $D$, i.e. the phase quadratures $\hat{p}_1$ and $\hat{p}_2$, by two homodyne detectors (HD1 and HD2) simultaneously.

In our experiment, the achievable lower bound is limited by the quantum correlation of the EPR entangled state [Eq. (5)]. In order to demonstrate this property, we change the quantum correlation of signal state and meter state by changing the relative phase $\theta$ between the two mode of EPR entangled state. Thus, the error $\varepsilon(B) = \sqrt{\sigma^2(\hat{p}_2 - \hat{p}_1)}$ is measured in experiment.

When the relative phase $\theta = 0^\circ$ and $\theta = 360^\circ$, the minimum error is obtained [Fig. 2(a)] and the left-hand-side (LHS) of the relation reaches its minimum value [Fig. 2(b)], which is determined by the present squeezing level. When $\theta = 180^\circ$, the maximum error is obtained, which corresponds to the measurement of anti-correlated noise $\sqrt{\sigma^2(\hat{p}_2 + \hat{p}_1)}$. The results confirm that the Ozawa’s and Branciard’s relations are the same and valid for the error-free measurement of observable $A$.

Then, we test the error-tradeoff relation with nonzero errors. When both errors are not equal to zero, Ozawa’s and Branciard’s relations are different. In the experiment, we apply a linear operation on the signal mode, which is done by transmitting the signal mode through a lossy channel, as shown in the inset of Fig. 1(b). In this case, the amplitude and phase quadratures of the signal mode are changed to $\hat{x}'_1 = \sqrt{T}\hat{x}_1 + \sqrt{1-T}\hat{x}_v$ and $\hat{p}'_1 = \sqrt{T}\hat{p}_1 + \sqrt{1-T}\hat{p}_v$, respectively, after transmitted over the lossy channel, where $\hat{x}_v$ and $\hat{p}_v$ represent the amplitude and phase quadratures of vacuum. By choosing $C = \hat{x}_1$ and $D = \hat{p}_2$, which are compatible, the errors
FIG. 4: Uncertainty relation for mixed state. (a) The error \( \varepsilon(B) \) as a function of the transmission efficiency. (b) The LHS of the Ozawa’s and Branciard’s relation as a function of the transmission efficiency. The right hand side of the relations \( C_{AB} \) is indicated by the red line.

for the two incompatible observables \( A = \hat{x}_1 \) and \( B = \hat{p}_1 \) are \( \varepsilon(A) = \sqrt{\sigma^2(\hat{x}_1' - \hat{x}_1)} \) and \( \varepsilon(B) = \sqrt{\sigma^2(\hat{p}_2 - \hat{p}_1)} \), respectively.

In this case, the error \( \varepsilon(A) \) increases with the decreasing of channel efficiency, while the error \( \varepsilon(B) \) is not affected by the channel efficiency [Fig. 3(a)]. Heisenberg’s error-tradeoff uncertainty relation is violated when the transmission efficiency is higher than 0.3. While the Ozawa’s and Branciard’s relations are always valid [Fig. 3(b)]. By comparing the LHS of Ozawa’s and Branciard’s relation, we confirm that Branciard’s relation is tighter than Ozawa’s relation.

Finally, we demonstrate the error-tradeoff relation for mixed state, i.e., the state \( \rho \) transmitted over a lossy channel. Here, observables \( C = A = \hat{x}_1' \), \( B = \hat{p}_1' \), and \( D = \hat{p}_2 \) are chosen, and thus errors for the mixed state are \( \varepsilon(A) = 0 \) and \( \varepsilon(B) = \sqrt{\sigma^2(\hat{p}_2 - \hat{p}_1)} \), respectively. In this case, Ozawa’s and Branciard’s relations are the same. The error \( \varepsilon(B) \) and the LHS of the relation increase along with the decreasing of transmission efficiency as shown in Fig. 4(a) and 4(b), respectively. The error and LHS of the relation get the minimum value when the transmission efficiency is unit.

The predicted lower bounds for Heisenberg’s [Eq. (1)], Ozawa’s [Eq. (2)] and Branciard’s [Eq. (3)] error-tradeoff relations are compared in the plane \( (\varepsilon(A), \varepsilon(B)) \), as shown in Fig. 5. For the Heisenberg’s error-tradeoff uncertainty relation (bounded by the blue dashed curve), one of the error must be infinite when the other goes to zero. While in our experiment, for the case of error \( \varepsilon(A) = 0 \), the finite error \( \varepsilon(B) \) is observed (red circles), which violates the Heisenberg’s error-tradeoff uncertainty relation, yet satisfies the Ozawa’s and Branciard’s relation. For the case of nonzero errors, only one of the observed values satisfies the Heisenberg’s error-tradeoff uncertainty relation (the data with 0.2 transmission efficiency). Our experimental data do not reach the lower bound of the relations for the limitation of the experiment condition, for example the limited squeezing parameter.

FIG. 5: Lower bounds of the error-tradeoff relations. Blue dashed curve: the Heisenberg’s bound. Yellow dotted curve: the Ozawa’s bound. Gray solid curve: the Branciard’s bound. Red circles: experimental data for error free measurement of observable \( A \) as shown in Fig. 2. Black diamonds: experimental data for nonzero errors condition as shown in Fig. 3.

IV. CONCLUSION

We experimentally test the Heisenberg’s, Ozawa’s and Branciard’s error-tradeoff relations for continuous-variable observables, i.e., amplitude and phase quadratures of an optical mode. Especially, we investigate the error-tradeoff relation in case of zero error by using Gaussian EPR entangled state. Three different measurement apparatus are applied in our experiment, which are used to test the error-tradeoff relation for three different cases. The results demonstrate that the Heisenberg’s error-tradeoff uncertainty relation is violated in
some cases while the Ozawa’s and the Brinciard’s relations are valid. Our work is useful not only in understanding fundamentals of physical measurement but also in developing of continuous variable quantum information technology.

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