A SHORT PROOF OF AN IDENTITY FOR CUBIC PARTITIONS

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Abstract. In this note, we will give a short proof of an identity for cubic partitions.

1. Introduction

Let \( p(n) \) denote the number of the unrestricted partitions of \( n \), defined by
\[
\sum_{n=0}^{\infty} q^n = \prod_{n=0}^{\infty} \frac{1}{1-q^n}.
\]
One of the celebrated results about \( p(n) \) is the theorem which was proved by Watson [11]: if \( k \geq 1 \), then for every nonnegative integer \( n \)
\[
p(5^kn + r_k) \equiv 0 \pmod{5^k},
\]
(1.1)
where \( r_k \) is the reciprocal modulo \( 5^k \) of 24. Recently, the notion of cubic partitions of a natural number \( n \), named by Kim [6], was introduced by Chan [1] in connection with Ramanujan’s cubic continued fraction. By definition, the generating function of the number of cubic partitions of \( n \) is
\[
\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})}.
\]
(1.2)
Chan [1] from the Ramanujan’s cubic continued fraction
\[
v(q) := \frac{q^{\frac{1}{3}}}{1 + \frac{q + q^3}{1 + \frac{q^2 + q^4}{1 + \ldots}}} \quad |q| \leq 1
\]
derived an elegant identity: let \( x(q) = q^{-\frac{1}{3}}v(q) \), then
\[
\frac{1}{x(q)} - q^{\frac{1}{3}} - 2q^{\frac{2}{3}}x(q) = \frac{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_\infty (q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty}{(q^{\frac{2}{3}}; q^{\frac{2}{3}})_\infty (q^6; q^6)_\infty},
\]
where we set for \( |q| \leq 1 \), \( (c; q)_\infty := \prod_{n=0}^{\infty} (1 - cq^k) \). From this he obtained the generating function for \( a(3n + 2) \) [1, Theorem 1]:

**Theorem 1.1.**
\[
\sum_{n=0}^{\infty} a(3n + 2)q^n = \frac{3(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^5 (q^2; q^2)_\infty^4}.
\]
(1.3)
Moreover he use this and the method of Hirschhorn and Hunt derived the following congruences [2, Theorem 1] analogous to (1.1):

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\]
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**Theorem 1.2.** For every $k \geq 1$ and nonnegative integer $n$,
\[
a(3^\alpha n + c_\alpha) \equiv 0 \pmod{3^\beta + \delta(\alpha)},
\]  
where $c_\alpha$ is the reciprocal modulo $3^\alpha$ of 8, and $\delta(\alpha) = 1$ if $\alpha$ is even and $\delta(\alpha) = 0$ otherwise.

In this note, we will give a short proof of Theorem 1.2 and outline the proof of Theorem 1.2. The proof of Theorem 1.1 by Chan used identities involved Ramanujan’s cubic continued fraction. The proof of Theorem 1.2 used the $H$ operators on formal power series used by Hirschhorn and Hunt [5]. Our proofs will use meromorphic modular functions on $\Gamma_0(6)$ and $\Gamma_0(18)$.

2. Preliminaries

Let $\mathbb{H} := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ denote the complex upper half plane, for a positive integer $N$, define the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ by $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \right\}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ act on the complex upper half plane by the linear fractional transformation $\gamma z := \frac{az + b}{cz + d}$. Let $f(z)$ be a function on $\mathbb{H}$ which satisfies $f(\gamma z) = f(z)$, if $f(z)$ is meromorphic on $\mathbb{H}$ and at all the cusps of $\Gamma_0(N)$, then we call $f(z)$ a meromorphic modular function with respect to $\Gamma_0(N)$. The set of all such functions is denoted by $\mathcal{M}_0(\Gamma_0(N))$.

Dedekind’s eta function is defined by $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, where $q = e^{2\pi i z}$ and $\text{Im}(z) > 0$. A function $f(z)$ is called an eta-product if it can be written in the form of $f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z)$, where $N$ is a natural number and $r_\delta$ is an integer. The following fact which is due to Gordon, Hughes [4] and Newman [10] is useful to verify whether an eta-product is a modular function.

**Proposition 2.1.** If $f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z)$ is an eta-product with $\frac{1}{2} \sum_{\delta \mid N} r_\delta = 0$ satisfies the conditions:
\[
\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}, \quad \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \quad \prod_{\delta \mid N} \delta^{r_\delta} \in \mathbb{Q}^2,
\]
then $f(z)$ is in $\mathcal{M}_0(\Gamma_0(N))$.

The following formula which is due to Ligozat [7] gives the analytic orders of an eta-product at the cusps of $\Gamma_0(N)$.

**Proposition 2.2.** Let $c, d$ and $N$ be positive integers with $d \mid N$ and $(c, d) = 1$. If $f(z)$ is an eta-product satisfying the conditions in Proposition 2.1 for $N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is
\[
\frac{N}{24} \sum_{\delta \mid N} \left( \frac{d}{\delta} \right)^2 r_\delta.
\]

Let $p$ be a prime, and $f(q) = \sum_{n \geq 0} a(n)q^n$ be a formal power series, we define $U_p f(q) = \sum_{pn \geq 0} a(pm)q^p$. If $f(z) \in \mathcal{M}_0(\Gamma_0(N))$, then $f(z)$ has an expansion at the point $i\infty$ of the form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ where $q = e^{2\pi iz}$ and $\text{Im}(z) > 0$. We call this expansion the Fourier series of the $f(z)$. Moreover we define $U_p f(z)$ to be the result of applying $U_p$ to the Fourier series $f(z)$.

We use the results on the $U_3$-operator (we write $U$ for $U_3$ in the following) acting on modular functions on $\mathcal{M}_0(\Gamma_0(6))$ and $\mathcal{M}_0(\Gamma_0(18))$ stated by Gordon and Hughes [4]. We know that $\Gamma_0(6)$ has 4 cusps, represented by $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6} (= i\infty)$, $\Gamma_0(18)$ has 8 cusps, represented by $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18} (= i\infty)$. 


By Ligozat’s formula on the analytic orders of an eta-product, if \( f(z) \) is in \( \mathcal{M}_0(\Gamma_0(N)) \), then \( f(z) \) has the same order at the cusps which have the same denominators. The order of \( U_3(f(z)) \) at a cusp \( r \) of \( \Gamma_0(6) \) is denoted by \( \text{ord}_r U(f) \), and the order of \( f(z) \) at a cusp of \( s \) of \( \Gamma_0(18) \) is denoted by \( \text{ord}_s f \).

**Proposition 2.3.** Let \( f(z) \) be an eta-product in \( \mathcal{M}_0(\Gamma_0(18)) \), then \( U_3(f(z)) \) is in \( \mathcal{M}_0(\Gamma_0(6)) \), and

\[
\text{ord}_0 U(f) \geq \min(\text{ord}_0 f, \text{ord}_{\frac{1}{3}} f), \quad \text{ord}_{\frac{1}{3}} U(f) \geq \min(\text{ord}_{\frac{1}{3}} f, \text{ord}_{\frac{1}{6}} f),
\]

\[
\text{ord}_{\frac{1}{6}} U(f) \geq \frac{1}{3} \text{ord}_{\frac{1}{6}} f, \quad \text{ord}_{\frac{1}{18}} U(f) \geq \frac{1}{3} \text{ord}_{\frac{1}{18}} f.
\]

Moreover, \( U(f) \) has no poles on \( \mathbb{H} \) except the cusps.

### 3. Proof of Theorem 1.1

Let the eta-product

\[
F := F(z) = \frac{\eta(9z) \eta(18z)}{\eta(z) \eta(2z)},
\]

put \( N = 18 \), we find \( F(z) \) satisfies the conditions of Newman-Gordon-Hughes’s theorem i.e. Proposition 2.1 so \( F(z) \) is in \( \mathcal{M}_0(\Gamma_0(18)) \). We use Ligozat’s formula to calculate the orders of \( F(z) \) at the cusps \( \frac{d}{6} \), for \( d = 1, 2, 3, 6, 9, 18 \). We give the calculation of the case of \( d = 1 \) as an example:

\[
\text{ord}_0 F = \frac{18}{24} \times \frac{(1, 9)^2}{\delta} \times \frac{1}{\delta} \sum_{\delta|18} \left( (1, \frac{18}{1}) \times (1 + \frac{(1, 18)^2}{2} \times (-1) + \frac{(1, 1)^2}{2}) \times (-1) \right)
\]

\[
= -1.
\]

Similar calculations give

\[
\text{ord}_{\frac{1}{3}} F = -1, \quad \text{ord}_{\frac{2}{3}} F = 0, \quad \text{ord}_{\frac{1}{6}} F = 0, \quad \text{ord}_{\frac{1}{9}} F = 1, \quad \text{ord}_{\frac{1}{18}} F = 1.
\]

By Proposition 2.3, the orders of \( U(F) \) at the cusps of \( \Gamma_0(6) \) satisfy

\[
\text{ord}_0 U(F) \geq -1, \quad \text{ord}_{\frac{1}{3}} U(F) \geq -1, \quad \text{ord}_{\frac{1}{6}} U(F) \geq 1, \quad \text{ord}_{\frac{1}{18}} U(F) \geq 1
\]

and \( U(F) \) is holomorphic on \( \mathbb{H} \). We note that the poles of \( U(F) \) only appear at the cusps 0 and \( \frac{1}{2} \).

We define another eta-product

\[
A := A(z) = \frac{\eta^4(3z) \eta^4(6z)}{\eta^4(z) \eta^4(2z)}.
\]

By Proposition 2.1, we find that \( A \) is in \( \mathcal{M}_0(\Gamma_0(6)) \). Ligozat’s formula on the orders of an eta-product gives

\[
\text{ord}_0 A = -1, \quad \text{ord}_{\frac{1}{3}} A = -1, \text{ord}_{\frac{1}{6}} A = 1, \quad \text{ord}_{\frac{1}{18}} A = 1
\]
and $A$ is holomorphic and non-zero elsewhere. Since the Riemann surface $(\mathbb{H} \cup \mathbb{Q} \cup i\infty)/\Gamma_0(6)$ has genus 0, $\mathcal{M}_0(\Gamma_0(6))$ has one generator as a field. The orders of $A$ show that $U(F) = cA$. Since

$$
F = q^3 \prod_{n=1}^{\infty} \frac{(1-q^{3n})(1-q^{18n})}{(1-q^n)(1-q^{2n})} = q + q^2 + 3q^3 + 4q^4 + 9q^5 + 12q^6 + 23q^7 + 31q^8 + 54q^9 + \ldots,
$$

$$
A = q \prod_{n=1}^{\infty} \frac{(1-q^{3n})^4(1-q^{6n})^4}{(1-q^n)^4(1-q^{2n})^4} = q + 4q^2 + 18q^3 + 52q^4 + \ldots.
$$

So $U(F) = 3q + 12q^2 + 54q^3 + \ldots$. The comparison of the coefficients of $U(f)$ and $A$ shows that $c = 3$, so $U(F) = 3A$. On the other hand,

$$
F = q^3 \prod_{n=1}^{\infty} \frac{(1-q^{3n})(1-q^{18n})}{(1-q^n)(1-q^{2n})} = \left( \sum_{n=1}^{\infty} a(n-1)q^n \right) \prod_{n=1}^{\infty} (1-q^{3n})(1-q^{6n}).
$$

Apply $U$-operator again on both sides of the above, we have

$$
U(F) = 3A = \left( \sum_{n=0}^{\infty} a(3n-1)q^n \right) \prod_{n=1}^{\infty} (1-q^{3n})(1-q^{6n}). \quad (3.1)
$$

Put

$$
A = q \prod_{n=1}^{\infty} \frac{(1-q^{3n})^4(1-q^{6n})^4}{(1-q^n)^4(1-q^{2n})^4}
$$

into the above, we obtain the identity:

$$
\sum_{n=0}^{\infty} a(3n+2)q^n = 3 \frac{(q^3; q^3)^3_\infty (q^6; q^6)^3_\infty}{(q; q)_\infty^3 (q^2; q^2)_\infty^3}. \quad (3.2)
$$

Which is Theorem 1.1.

We outline the proof of Theorem 1.2. The ideal is similar to the paper [12]. Firstly we apply the proposition 2.3 to $A^i$ and $FA^i$ for $i \geq 1$, we can express $A^i$ (resp. $FA^i$) as a polynomial in $A$ of degree at most $3i$ (resp. $3i+1$). This is the part corresponding to the Proposition 1 and Proposition 2 in [2]. Next we use the initial values of $A^i$ to calculate the elementary symmetric functions $\sigma_i$ ($i = 1, 2, 3$) of $U(A^{i}(t = 0, 1, 2))$ which are polynomials in $A$ with integers as coefficients. Then by the Newton recurrence for power sums, we get for all $i \geq 3$

$$
U(A^i) = \sigma_1 U(A^{i-1}) - \sigma_2 U(A^{i-2}) + \sigma_3 U(A^{i-3})
$$

Hence for $i \geq 1$, $U(A^i) \in \mathbb{Z}[A]$. Moreover $U(FA^i)$ satisfies the same recurrence as $U(A^i)$ also $U(FA^i)$ is in $\mathbb{Z}[A]$ and By induction we obtain the lower bounds of 3-adic orders of these coefficients. The last step is almost the same as the Proposition 3 and the Theorem 4.

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