HOW MUCH OF THE HILBERT FUNCTION DO WE REALLY NEED TO KNOW?

JÁNOS KOLLÁR

Abstract. The aim of this lecture is to describe several examples where the leading coefficient of a Hilbert function tells us everything we need.

The starting point is the following theorem, whose proof—though not its precise statement—is in [Har77 III.9.9].

Old Theorem 1. Let $f : X \to S$ be a projective morphism and $F$ a coherent sheaf on $X$. Then

1. $s \mapsto \chi(X_s, F_s(m))$ is a lower semicontinuous function on $S$ for $m \gg 1$.
2. If $S$ is connected and reduced, then $F$ is flat over $S$ $\iff$ the above function $s \mapsto \chi(X_s, F_s(m))$ is constant on $S$ for every $m$.

Thus one can establish flatness by computing the Hilbert function of the individual fibers $F_s$. Note that the fibers over points carry no information about the nilpotent directions in the base, so the restriction to reduced $S$ is necessary in (2).

In practice it is frequently quite hard to determine the whole Hilbert function $\chi(X, F(m))$ for a coherent sheaf $F$ on a proper scheme $X$, but it turns out that there are many interesting situations where it is enough to know the leading coefficient of $\chi(X, F(m))$ to guarantee flatness. The first such general result I know of is due to Hironaka [Hir58]; see also [Har77 III.9.11]. The projective case of the theorem can be formulated as follows.

Old Theorem 2. Let $T$ be a connected, regular, 1-dimensional scheme and $X \subset \mathbb{P}_T^N$ a closed subscheme, flat over $T$. Then

1. $t \mapsto \deg(\text{red } X_t)$ is a lower semicontinuous function on $T$.
2. If the reduced fibers $\text{red } X_t$ are normal then the following are equivalent.
   (a) $t \mapsto \deg(\text{red } X_t)$ is constant on $T$,
   (b) $t \mapsto \chi(\text{red } X_t, \mathcal{O}_{\text{red } X_t}(m))$ is constant for every $m$ and
   (c) the fibers $X_t$ are reduced.

The leading coefficient of $\chi(\text{red } X_t, \mathcal{O}_{\text{red } X_t}(m))$ equals $\deg(\text{red } X_t)/(\dim X_t)!$, thus we can informally summarize the above theorem by saying that “the leading coefficient determines flatness.”

We are looking for theorems of this type. The first part should be a general assertion that some invariants related to Hilbert functions are lower or upper semicontinuous on the base. Then, under some geometric assumptions, we aim to show that constancy of the leading coefficient—usually given as the volume of a divisor as in [LS1]—implies constancy of the whole Hilbert function, hence flatness.

Each of the next 5 sections outlines such results. A detailed treatment of the claims in Section 1–2 will appear in [Kol15]. Sections 3–4 summarize some of the theorems of [Kol13a, BdJ14, Kol14] while Section 5 is taken from [FKL15].
There are, unfortunately, two distinct definitions of canonical models in use.

**Definition 3** (Canonical models). Let \((X, \Delta)\) be a proper log canonical pair such that \(K_X + \Delta\) is big. As in [KM98, 3.50], its canonical model is the unique log canonical pair \((X^c, \Delta^c)\) such that \(K_{X^c} + \Delta^c\) is ample and
\[
\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + |m\Delta|)) \cong \sum_{m \geq 0} H^0(X^c, \mathcal{O}_{X^c}(mK_{X^c} + |m\Delta^c|)).
\]
There is a natural birational map
\[
\phi: (X, \Delta) \rightarrow (X^c, \Delta^c).
\]

On the other hand, if \(X\) is a proper variety with arbitrary singularities, then one can take a resolution \(X^r \rightarrow X\) and its canonical model \((X^r)^c\). Since this is independent of the choice of \(X^r\), it is frequently called the canonical model of \(X\). I suggest to call it the canonical model of resolutions of \(X\) and denote by \(X^{ct}\). More generally, let \(X\) be a proper, pure dimensional scheme over a field. Start with any resolution \(X^r \rightarrow \text{red}X\) and let \(X^{ct}\) denote the disjoint union of the canonical models of those components that are of general type. With a slight abuse of terminology, there is a natural map
\[
\phi: X \rightarrow X^{ct},
\]
which is birational on the general type components and not defined on the others.

If \(X\) has log canonical singularities then both variants are defined. Note that \(X^c \cong X^{ct}\) if \(X\) has only canonical singularities but not in general.

**Definition 4** (Simultaneous canonical model). Let \(f: X \rightarrow S\) be a proper morphism of pure relative dimension \(n\). One can define a simultaneous canonical model of resolutions \(f^{scr}: X^{scr} \rightarrow S\). If we also have a divisor \(\Delta\) on \(X\) such that the fibers \((X_s, \Delta_s)\) are log canonical, then one can also define a simultaneous canonical model \(f^{sc}: (X^{sc}, \Delta^{sc}) \rightarrow S\). These are given by diagrams
\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^{scr} \\
\downarrow f & \nearrow & \nearrow f^{scr} \\
S & \downarrow & \downarrow \phi \\
& \nearrow & \nearrow f^{sc} \\
& & (X, \Delta) \\
& & (X^{sc}, \Delta^{sc})
\end{array}
\]
where \(f^{scr}\) and \(f^{sc}\) are flat, proper and each \(\phi_s: X_s \rightarrow X^{scr}_s\) is the canonical model of the resolutions of \(X_s\) (resp. each \(\phi_s: (X_s, \Delta_s) \rightarrow (X^{sc}_s, \Delta^{sc}_s)\) is the canonical model of \((X_s, \Delta_s)\)).

**Note:** We need the additional assumption that \(f^{sc}: (X^{sc}, \Delta^{sc}) \rightarrow S\) be locally stable, equivalently, that \(K_{X^{sc}/S} + \Delta\) be \(\mathbb{Q}\)-Cartier. See [Kol13b, Kol15] for discussions about this condition. If the fibers \(X^{sc}_s\) have canonical singularities then \(K_{X^{sc}/S}\) is automatically \(\mathbb{Q}\)-Cartier, thus we did not need to assume this for simultaneous canonical models of resolutions.

**Theorem 5** (Numerical criterion for simultaneous canonical models 1). Let \(S\) be a connected, seminormal scheme of char 0 and \(f: X \rightarrow S\) a morphism of pure relative dimension \(n\). Then

1. \(s \mapsto \text{vol}(K_{X^r})\) is a lower semicontinuous function on \(S\) and
2. \(f: X \rightarrow S\) has a simultaneous canonical model of resolutions iff this function is constant (and positive).
Part (1) was first observed and proved in [Nak86, Nak87].

The following is a similar result for normal lc pairs, but the lower semicontinuity of Theorem 5 changes to upper semicontinuity.

**Theorem 6** (Numerical criterion for simultaneous canonical models II). Let $S$ be a connected, seminormal scheme of char 0 and $f : (X, \Delta) \to S$ a flat morphism whose fibers $(X_s, \Delta_s)$ are log canonical. Then

1. $s \mapsto \text{vol}(K_{X_s} + \Delta_s)$ is an upper semicontinuous function on $S$ and
2. $f : (X, \Delta) \to S$ has a simultaneous canonical model iff this function is constant.

Notes. Strictly speaking, part (2) needs the assumption that the fibers $(X_s, \Delta_s)$ have a canonical model. This is conjectured to be true and it is known in many cases, for instance when $(X_s, \Delta_s)$ is klt.

A stronger version of the theorem assumes only that each fiber is normal in codimension 1 and has log canonical normalization.

A key ingredient of the proof of Theorems 5, 6 is the following characterization of simultaneous canonical models.

**Proposition 7.** Let $X$ be a smooth proper variety of dimension $n$. Let $Y$ be a normal, proper variety birational to $X$ and $D$ an effective $\mathbb{Q}$-divisor on $Y$ such that $K_Y + D$ is $\mathbb{Q}$-Cartier, nef and big. Then

1. $\text{vol}(K_X) \leq \text{vol}(K_Y + D) = (K_Y + D)^n$ and
2. equality holds iff $D = 0$ and $Y$ has canonical singularities.

## 2. SIMULTANEOUS CANONICAL MODIFICATIONS

For surfaces, the existence criterion of simultaneous canonical modifications is proved in [KSB88, Sec.2]. In higher dimensions we need to work with a sequence of intersection numbers and with their lexicographic ordering.

**Definition 8.** Let $X$ be a proper scheme of dimension $n$ and $A, B \in \mathbb{R}$-Cartier divisors on $X$. Their sequence of intersection numbers is

$$I(A, B) := \left( (A^n), \ldots, (A^{n-i} \cdot B^i), \ldots, (B^n) \right) \in \mathbb{R}^{n+1}.$$

For two divisors, the relevant Hilbert function is the 2-variable polynomial $h(u, v) := \chi(X, \mathcal{O}_X(uA + vB))$ and the above intersection numbers are the coefficients of the leading homogeneous term, which has degree $= \dim X$.

The lexicographic ordering is denoted by $(a_0, \ldots, a_n) \preceq (b_0, \ldots, b_n)$. (This holds if either $a_i = b_i$ for every $i$ or there is an $r \leq n$ such that $a_i = b_i$ for $i < r$ but $a_r < b_r$.) For polynomials we define an ordering

$$f(t) \preceq g(t) \iff f(t) \leq g(t) \forall t \gg 0.$$

Note that $\sum a_i t^{n-i} \preceq \sum b_i t^{n-i}$ iff $(a_0, \ldots, a_n) \preceq (b_0, \ldots, b_n)$. Thus

$$I(A, B) \preceq I(A', B') \iff (mA + B)^n \leq (mA' + B')^n \forall m \gg 0.$$

**Definition 9** (Simultaneous canonical modification). Let $Y$ be a scheme over a field $k$. (We allow $Y$ to be reducible and nonreduced.) Its canonical modification is a morphism $p : Y_{\text{can}} \to Y$ such that $Y_{\text{can}} \to Y$ is proper, birational, $Y_{\text{can}}$ has canonical singularities and $K_{Y_{\text{can}}}$ is ample over $Y$.

Let $\Delta$ be an effective divisor on $Y$. A canonical modification is a morphism $p : (Y_{\text{can}}, \Delta_{\text{can}}) \to (Y, \Delta)$ where $p$ is proper, birational, $\Delta_{\text{can}} = p_*^{-1} \Delta$, $(Y_{\text{can}}, \Delta_{\text{can}})$
is canonical and $K_{Y,can} + \Delta_{can}$ is ample over $Y$. A canonical modification is unique and it exists iff the following conditions hold:

1. The reduced scheme $\text{red} Y$ is smooth at the generic points of $\text{Supp} \Delta$ and all coefficients in $\Delta$ are in the interval $[0,1]$.

Let $f: X \to S$ be a morphism of pure relative dimension $n$ and $\Delta$ an effective divisor on $Y$. A simultaneous canonical modification is a proper morphism $p: (X^\text{scan}, \Delta^\text{scan}) \to (X, \Delta)$ such that $f \circ p: (X^\text{scan}, \Delta^\text{scan}) \to S$ is locally stable and $p_s: (X^\text{scan}_s, \Delta^\text{scan}_s) \to (X_s, \Delta_s)$ is the canonical modification for every $s \in S$.

Let $S$ be a connected, seminormal scheme of char $0$, $f: X \to S$ a morphism of pure relative dimension $n$, $H$ an $f$-ample divisor class and $\Delta$ an effective divisor on $X$ such that $(X_s, \Delta_s)$ satisfies the assumptions (1) for every $s \in S$. Thus the canonical modifications $p_s: (X^\text{can}_s, \Delta^\text{can}_s) \to (X_s, \Delta_s)$ exist.

**Theorem 10** (Numerical criterion for simultaneous canonical modification). With the above notation,

1. $s \mapsto I(p^*_sH_s, K_{X,can} + \Delta_{can})$ is a lexicographically lower semicontinuous function on $S$ and
2. $f: (X, \Delta) \to S$ has a simultaneous canonical modification iff this function is constant.

There is also a similar condition for simultaneous log canonical and semi-log-canonical modifications but these only apply when $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier. The following example illustrates the problems that occur in general.

**Example 11.** In $\mathbb{P}^2$ consider a line $L \subset \mathbb{P}^2$ and a family of degree 8 curves $C_t$ such that $C_0$ has 4 nodes on $L$ plus an ordinary 6-fold point outside $L$ and $C_t$ is smooth and tangent to $L$ at 4 points for $t \neq 0$.

Let $\pi_t: S_t \to \mathbb{P}^2$ denote the double cover of $\mathbb{P}^2$ ramified along $C_t$. Note that $K_{S_t} = \pi_t^*O(1)$, thus $(K^2_{S_t}) = 2$. For each $t$, the preimage $\pi_t^{-1}(L)$ is a union of 2 curves $D_t + D'_t$. Our example is the family of pairs $(S_t, D_t)$. We claim that

1. there is a log canonical modification $(S^lc_t, D^lc_t) \to (S_t, D_t)$ for every $t$ and
2. $(K^lc_t + D^lc_t)^2 = 1$ for every $t$ yet
3. there is no simultaneous log canonical modification.

If $t \neq 0$ then $S_t$ is smooth and $D_t$ is smooth. Furthermore $D_t, D'_t$ meet transversally at 4 points, thus $(D_t \cdot D'_t) = 4$. Using $((D_t + D'_t)^2 = 2$, we obtain that $(D^2_t) = -3$. Thus $(K^2_{S_t} + D^2_t) = 1$.

If $t = 0$ then $S_0$ is singular at 5 points. $D_0, D'_0$ meet transversally at 4 singular points of type $A_1$, thus $(D_0 \cdot D'_0) = 2$. This gives that $(D^2_0) = -1$. Thus $(K^2_{S_0} + D^2_0) = 3$. The pair $(S_0, D_0)$ is lc away from the preimage of the 6-fold point. Let $q: T_0 \to S_0$ denote the minimal resolution of this point. The exceptional curve $E$ is smooth, has genus 2 and $(E^2) = -2$. Thus $K_{T_0} = q^*K_{S_0} - 2E$ hence $(T_0, E + D_0)$ is the log canonical modification of $(S_0, D_0)$ and

$$(K_{T_0} + E + D_0)^2 = (q^*K_{S_0} - E + D_0)^2 = (K_{S_0} + D_0)^2 + (E^2) = 1.$$ 

Thus $(K^lc_t + D^lc_t)^2 = 1$ for every $t$.

Nonetheless, the log canonical modifications do not form a flat family. Indeed, such a family would be a family of surfaces with ordinary nodes, so the relative
canonical class would be a Cartier divisor. However, \((K^2_{S_t}) = 2\) for \(t \neq 0\) but 
\((K^2_{S_0}) = (q^*K_{S_0} - 2E)^2 = -6\).

3. Families of Cartier divisors

Example 12. Consider the family of quadric surfaces 
\[ X := (x_1^2 - x_2^2 + x_3^2 - t^2x_0^2 = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1. \]
The fiber \(X_0\) is a cone, the other fibers are smooth. Consider the Weil divisors 
\[ D := (x_1 - x_2 = x_3 - tx_0 = 0) \quad \text{and} \quad E := (x_1 + x_2 = x_3 - tx_0 = 0). \]
The fibers \(D_t, E_t\) form a pair of intersecting lines on \(X_t\) for every \(t\). It is easy to compute that 
\[
(1) \quad (aD_0 + bE_0)^2 = \frac{1}{2}(a + b)^2 \geq 2ab = (aD_t + bE_t)^2 \quad \text{and} \\
(2) \quad \text{equality holds iff } a = b \text{ iff } aD + bE \text{ is Cartier.}
\]

We aim to prove that this example is quite typical, as far as intersection numbers are concerned. (It is, however, special in that the equations define the restrictions \(D_t\) unambiguously. In general, if \(D\) is effective, the sheaf theoretic restriction \(\mathcal{O}_X(-D)|_{X_s}\) may have embedded points. As long as the fibers are smooth in codimension 1, such embedded points appear only in codimension \(\geq 2\), so there is a well-defined Weil divisor that can be thought of as the restriction \(D_t\).

The following result was conjectured in [Koll13a] and proved there for log canonical fibers. The extension to normal fibers is done in [BdJ14].

Theorem 13 (Numerical criterion of Cartier divisors, weak form). Let \(C\) be a smooth, irreducible curve and \(f : X \to C\) a proper, flat family of normal varieties of dimension \(n\). Let \(D\) be a Weil divisor on \(X\) such that its restriction \(D_c\) is an ample Cartier divisor for every \(c\). Then

\[
(1) \ c \mapsto (D^c) \text{ is an upper semicontinuous function on } C \quad \text{and} \\
(2) \ D \text{ is a Cartier divisor on } X \text{ iff the above function is constant.}
\]

Ampleness is needed for \(n \geq 3\), the main reason is that \((-D)^n = (-1)^n(D^n)\). Thus, on a 3-fold, ample divisors behave anti-symmetrically while divisors pulled-back form a surface behave symmetrically.

The following general form is proved in [Kol13a], building on the earlier results of [Kol13a, BdJ14].

Theorem 14 (Numerical criterion of Cartier divisors). Let \(S\) be a connected, reduced scheme over a field, \(f : X \to S\) a flat, proper morphism of pure relative dimension \(n\) with \(S_2\) fibers and \(Z \subset X\) a closed subset such that \(\text{codim}_{X_s}(Z \cap X_s) \geq 2\) for every \(s \in S\). Let \(L_U\) be an invertible sheaf on \(U := X \setminus Z\) and assume that the restriction \(L_U|_U\) extends to an invertible sheaf \(L_s\) on \(X_s\) for every \(s \in S\). Then

\[
(1) \ s \mapsto (H^{n-2}_L \cdot L^s) \text{ is an upper semicontinuous function on } S \quad \text{and} \\
(2) \ L_U \text{ extends to an invertible sheaf } L \text{ on } X \text{ iff the above function is constant.}
\]
Furthermore, if \(L_s\) is ample for every \(s\) then

\[
(3) \ s \mapsto (L^s) \text{ is an upper semicontinuous function on } S \quad \text{and} \\
(4) \ L_U \text{ extends to an invertible sheaf } L \text{ on } X \text{ iff the above function is constant.}
\]
Note that taking \((H^2)^{n-2}\) in (1) is equivalent to restricting to the intersection of \(n-2\) very ample divisors. In particular, the assumptions in (1) do not depend on singularities of the fibers that appear in codimension \(\geq 3\). This is a key point in the proof of Theorems 13–14 to be discussed next.

4. Grothendieck–Lefschetz theorems for the local Picard group

Let us recall the form given in [Gro68].

**Old Theorem 15** (Grothendieck–Lefschetz). [Gro68, XIII.2.1] Let \((x \in X)\) be an excellent local scheme, \(x \in D \subset X\) a Cartier divisor. Set \(U := X \setminus \{x\}\), \(U_D := D \setminus \{x\}\) and let \(L_U\) be a line bundle on \(U\) such that \(L_U|_{U_D} \cong \mathcal{O}_{U_D}\).

\((\ast)\) Assume that \(\text{depth}_x \mathcal{O}_D \geq 3\).

Then \(L_U \cong \mathcal{O}_U\).

For our purposes, three aspects of this theorem are worth thinking about.

- It does not imply the usual Lefschetz theorem for hyperplane sections since a cone over a smooth projective variety is usually only \(S_2^1\) at its vertex.
- We would like to apply it to families of varieties over a smooth curve \(f : X \rightarrow C\) with \(D\) being a fiber. In this context assuming that the fibers are \(S_2^1\) is natural but \(S_3\) is not. For instance, log canonical (and semi-log-canonical) varieties are \(S_2^1\) but frequently not \(S_3\).
- The original form of the theorem assumes only that \(L\) is a rank 1 reflexive sheaf and in that setting the assumption \((\ast)\) is optimal. However, in many potential applications we know by induction that \(L\) is locally free on \(U\). The following strengthening was conjectured in [Kol13a] and proved there for log canonical fibers. The extension to normal fibers is done in [BdJ14], aside from some \(p\)-torsion questions in characteristic \(p\). The general form below is established in [Kol14]. Conjecturally, the result should hold for any excellent local scheme, but the current proofs do not work in mixed characteristic.

**New Theorem 16.** Let \((x \in X)\) be a local scheme that is essentially of finite type over a field and \(x \in D \subset X\) a Cartier divisor. Set \(U := X \setminus \{x\}\), \(U_D := D \setminus \{x\}\) and let \(L_U\) be a line bundle on \(U\) such that \(L_U|_{U_D} \cong \mathcal{O}_{U_D}\).

\((\ast\ast)\) Assume that \(\text{depth}_x \mathcal{O}_D \geq 2\) and \(\text{dim}_x D \geq 3\).

Then \(L_U \cong \mathcal{O}_U\).

17 (Proof of the old form). Let \(t\) be a defining equation of \(D\) and write \(L_D := L_U|_{U_D}\). The sequence \(0 \rightarrow L_U \rightarrow L_U \rightarrow L_D \cong \mathcal{O}_{U_D} \rightarrow 0\) gives

\[
\begin{array}{cccc}
H^0(U, L_U) & \rightarrow & H^0(U, L_U) & \rightarrow & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) \\
H^1(U, L_U) & \rightarrow & H^1(U, L_U) & \rightarrow & H^1(U_D, L_D \cong \mathcal{O}_{U_D}).
\end{array}
\]

The assumption \(\text{depth}_x \mathcal{O}_D \geq 3\) implies that \(H^1(U_D, \mathcal{O}_{U_D}) = 0\) (see [Gro67, Sec.3]) and so the map \(t : H^1(U, L_U) \rightarrow H^1(U, U_L)\) is surjective. Next, \(\dim U \geq 4\) implies that \(H^1(U, L_U)\) has finite length (see [Gro68, VIII.2.3]), which implies that the map \(t : H^1(U, L_U) \rightarrow H^1(U, L_U)\) is an isomorphism.

Therefore \(r : H^0(U, L_U) \rightarrow H^0(U_D, L_D)\) is surjective and the constant 1 section of \(L_D \cong \mathcal{O}_{U_D}\) lifts back to a nowhere-zero section of \(L_U\). \(\square\)
The vanishing $H^1(U_D, \mathcal{O}_{U_D}) = 0$ is pretty much equivalent to depth$_x \mathcal{O}_D \geq 3$, so the argument does not work if depth$_x \mathcal{O}_D = 2$.

Bhatt and de Jong observed that one can go around this problem in positive characteristic as follows. Assume that $X$ is normal and let $X^+ \to X$ denote the normalization of $X$ in an algebraic closure of its field of functions $k(X)$. Then $X^+$ is non-Noetherian but it is CM by [HH92]. We can lift everything back to $X^+$, apply the above proof and then descend to $X$ at the end. There are several foundational issues to deal with while working on $X^+$ (see [BdJ14]) and the descent proves only that $L^m_U \cong \mathcal{O}_U$ for some $m > 0$.

It is technically simpler to view $\mathcal{O}_{X^+}$ as a quasi-coherent sheaf on $X$ and work with it; see [Kol15].

Lifting back to characteristic 0 is easier. The extension to the non-normal case relies on the structure theory of the local Picard group developed in [Kol14].

5. Variation of $\mathbb{R}$-divisors

This topic has the same spirit as the previous ones and it is also used in the proofs of the theorems in Section 1–2.

**Definition 18.** Let $X$ be a proper, normal algebraic variety of dimension $n$ over a field $K$ and $D$ an $\mathbb{R}$-divisor on $X$. The Hilbert function of $D$ is the function

$$H(X, D) : m \mapsto h^0(mD) := \dim_K H^0(X, \mathcal{O}_X(\lfloor mD \rfloor));$$

defined for all $m \in \mathbb{R}$. If $D$ is an ample Cartier divisor then $H(X, D)$ agrees with the usual Hilbert polynomial whenever $m \gg 1$ is an integer, but in general $H(X, D)$ is not a polynomial, not even if $D$ is a $\mathbb{Z}$-divisor and $m \in \mathbb{Z}$. The simplest numerical invariant associated to the Hilbert function is the volume of $D$, defined as

$$\text{vol}(D) := \limsup_{m \to \infty} \frac{h^0(mD)}{m^n/n!}. \quad (18.1)$$

The volume is preserved by $\mathbb{R}$-linear equivalence but the Hilbert function is not; see Example 20. If $E$ is an effective $\mathbb{R}$-divisor, then clearly

$$h^0(mD - mE) \leq h^0(mD) \leq h^0(mD + mE)$$

holds for every $m > 0$, hence $\text{vol}(D - E) \leq \text{vol}(D) \leq \text{vol}(D + E)$.

We claim that, although the volume does not determine the Hilbert function, the only way to change the Hilbert function by subtracting or adding an effective divisor is to change the volume.

**Theorem 19.** [FKL15] Let $X$ be a proper, normal algebraic variety over a perfect field, $D$ a big $\mathbb{R}$-divisor on $X$ and $E$ an effective $\mathbb{R}$-divisor on $X$. Then

(Subtraction version.) The following are equivalent.

(1$^-$) $\text{vol}(D - E) = \text{vol}(D)$.

(2$^-$) $h^0(mD - mE) = h^0(mD)$ for all $m > 0$.

(3$^-$) $E \leq N_\sigma(D)$, the negative part of the Zariski–Nakayama-decomposition.

(Addition version.) The following are equivalent.

(1$^+$) $\text{vol}(D + E) = \text{vol}(D)$.

(2$^+$) $h^0(mD + mE) = h^0(mD)$ for all $m > 0$.

(3$^+$) $\text{Supp}(E) \subseteq \mathcal{B}_{\text{div}}^+(D)$, the divisorial part of the augmented base locus of $D$.  

\[ \]
Example 20. Let $S \to \mathbb{P}^1$ be a minimal ruled surface with a negative section $E \subset S$ and a positive section $C \subset S$ that is disjoint from $E$. Let $F_1, \ldots, F_4$ be distinct fibers. Then $C \sim_\mathbb{Q} C + (F_1 - F_2) + \sqrt{2}(F_3 - F_4)$.

Note that $[mc + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4)]$ has negative intersection with $E$ for all real $m > 0$. This implies that, for every $m > 0$ we have

$$h^0(S, mc + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4)) < h^0(S, mc).$$

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Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu