Numerical solution of the Vlasov-Maxwell system of equations for cylindrical plasmas

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Abstract. We solved numerically the Vlasov-Maxwell system of equations for a bounded cylindrical and radial inhomogeneous plasma which is confined by a strong magnetic field directed along the axis cylinder. Through this solution we found numerically the radial structure of the axial electric field corresponding to the high frequency fundamental transverse magnetic mode propagating in the cylindrical wave guide. Our result shows that the intensity of the electric field tends to be higher in those regions where the plasma is denser and also the field presents oscillations with intensities that decrease and vanish at the radial plasma boundary. This behavior could be relevant in the design of efficient modern plasma based particle accelerators that use the axial electric field to achieve this task.

1. Introduction

The matter in the plasma state contains a large number of charged particles whose state of thermodynamic equilibrium is characterized by the absence of macroscopic electric fields due to the Debye shielding effect. When plasmas interact with external electromagnetic fields, induced currents are generated in the system which are generally related to the electric field by a linear response function such as the conductivity. However, if displacement currents are also considered, the corresponding response function is given by the dielectric tensor. These response functions defines the temporal evolution of the electromagnetic fields in the plasma system and must be calculated according to the internal dynamics of the particles. Thus, the study of electromagnetic fields propagation in plasmas is an important issue in plasma theory since it provides information about the plasma state [1].

Theoretical calculations of the electrical response functions of the plasma requires a model of the plasma particles dynamics which coupled to a model of the electromagnetic fields evolution leads to the response functions mentioned above. In this sense, a kinetic model providing a set of equations that describe the statistical distribution function in the phase space of the particles is desirable in contrast with fluid macroscopic models. These kinetic models provide microscopic information of the particles dynamics which is not contained in a macroscopic model [1, 2]. In this context the kinetic Vlasov equations [3] describe the dynamics of the particles distribution function and together with the Maxwell equations for the electromagnetic fields offers a description of the plasma system [4]. In general terms, laboratory plasmas are inhomogeneous and bounded systems since they are contained inside of metallic conductor devices and confined by magnetic fields. The study of this kind of systems leads to a boundary
IOP Conf. Series: Journal of Physics: Conf. Series 1247 (2019) 012005 doi:10.1088/1742-6596/1247/1/012005

value problem for the electromagnetic fields propagating in the system. Several works have dealt with detailed studies for these kind of plasma bounded and inhomogeneous systems [5–15]. In this paper we are interested in find the radial structure of the fundamental transverse magnetic mode of high frequency propagating in a cylindrical inhomogeneous plasma with the simplifying assumption that the plasma is confined with a strong magnetic field in the sense that the transverse particles dynamics is neglected in the system. The work of [10] extends this study and consider the case of radially inhomogeneous cylindrical waveguide magnetized plasma without the strong magnetic field simplifying assumption. In our work we follow the approach developed by [5,12] and the numerical scheme used in [11,15]. The paper is organized as follows. In section 2 we illustrate the Vlasov-maxwell equations for the system. In section 3 we show the main conclusions of our work.

2. Vlasov-Maxwell equations for the system

2.1. Basic Assumptions

In this work we study the electromagnetic wave propagation in a completely ionized plasma which is confined in a long cylindrical and metallic waveguide through a strong and uniform external magnetic field applied in the axial direction. This strong axial field induces a small which is confined in a long cylindrical and metallic waveguide through a strong and uniform electric fields

In this paper we are interested in find the radial structure of the fundamental transverse magnetic modes solution for the system and the corresponding numerical result. Finally we state the section 2 we illustrate the Vlasov-maxwell equations for the system. In section 3 we show the numerical solution of these equations. In particular in sect 3.1 we show the transverse magnetic modes solution for the system and the corresponding numerical result. Finally we state the main conclusions of our work.

2. Vlasov-Maxwell equations for the system

2.1. Basic Assumptions

In this work we study the electromagnetic wave propagation in a completely ionized plasma which is confined in a long cylindrical and metallic waveguide through a strong and uniform external magnetic field applied in the axial direction. This strong axial field induces a small cyclotron radius for electrons such as the Larmor radius of the particles \( r_L \) is much smaller than the waveguide radius \( a \), i.e: \( r_L = (m_e v_L)(qB)^{-1} \ll a \) where \( v_{\text{perp}} \) is the perpendicular particles velocity respect to the magnetic field \( B \) and \( m \) is the particles mass. Thus, we neglect in our model the transverse dynamics of electrons and consider only the dependence of the particles distribution function respect to the axial cylinder direction [5,6,12]. Considering the large masses difference between electrons and ions, we neglect the movement of ions respect to electrons; thus, the only role of ions is to neutralize the total charge of the system. Therefore, our analysis is valid for propagating waves in the plasma with frequencies range given by \( \omega^2_{pe} < \omega^2_e < \omega^2_0 \sim \omega_{ce} \), where \( \omega_{pe} \) is the plasma frequency for particles of \( \alpha \) species (\( \alpha = e \rightarrow \text{electrons}, \alpha = i \rightarrow \text{ions} \)): \( \omega^2_{pe} = (4\pi n_0 q^2_e)(m_e)^{-1} \) and \( \omega_{ce} = (q_e B)(m_e c)^{-1} \).

The symmetry of the system implies the use of clyndrical coordinates for the space coordinates, thus, \((x) \rightarrow (r, \theta, z) \), where \( z \) labels the axial direction which corresponds also to the direction of the external confining magnetic field. When the plasma species \( \alpha \) are in a state of thermodynamic equilibrium (labeled by the sub-index 0) their statistical distribution function \( f_{\alpha 0} \) are independent of time, the Debye shielding implies the absence of macroscopic electric fields \( E_0 \) and the magnetic field \( B_0 \) is given by the constant external confining magnetic field \( :f_{\alpha 0}(x,v) = f_{\alpha 0}(r,v) = g_{\alpha 0}(r) F_{\alpha 0}(v), E_0(x) = 0, B_0(x) = B_0 \hat{z} \), where the function \( g_{\alpha 0}(r) = \int_{-\infty}^{\infty} f_{\alpha 0}(r,v) dv = (n_0(r))(n_0(0))^{-1} \), is a dimensionless function determined from the experiment and gives the radial distribution of the particles \( \alpha \) satisfying the condition \( g_{\alpha 0}(0) = 1 \) [5,6,11]. Since \( g_{\alpha 0} \) depends only of \( r \) implies that the plasma is uniform respect to the axial direction and respect to the angular direction, i.e., the equilibrium distribution function does not depend on the axial and angular direction. On the other hand, the function \( F_{\alpha 0}(v) \) is the axial velocities equilibrium distribution function corresponding to the Maxwell-Boltzman distribution. Since there are not currents in the equilibrium state we set: \( J_0(x) = 0 \).

2.2. Vlasov-Maxwell linearized equations

Taking into account that the Vlasov-Maxwell system of equations constitute a non-linear system, we linearize this set of equations by studying the behavior of small perturbations of the distribution function and the fields respect to the plasma equilibrium state. The equilibrium variables are indexed by a subindex 0 and the perturbation by a subindex 1: \( f_\alpha(x,v,t) = f_{\alpha 0}(x,v) + f_{\alpha 1}(x,v,t), E(x,t) = E_1(x,t), B(x,t) = B_0(x) + B_1(x,t) \). Using
these considerations and neglecting small quadratic powers of \( f_{\alpha 1}(x, v, t), E_1(x, t), B_1(x, t) \) we obtain the Vlasov-Maxwell linearized set of equations \([5,6,11,12,15]\),

\[
\frac{\partial f_{\alpha 1}}{\partial t} + v \frac{\partial f_{\alpha 1}}{\partial z} = -\frac{q_\alpha}{m_\alpha} g_{\alpha 0}(r) E_{1z} \frac{\partial F_{\alpha 0}}{\partial v},
\]

(1)

\[
\nabla \cdot E_1 = 4\pi \sum_\alpha n_\alpha (0) q_\alpha \int f_{\alpha 1} dv,
\]

(2)

\[
\nabla \times B_1 - \frac{1}{c} \frac{\partial E_1}{\partial t} = 4\pi \sum_\alpha n_\alpha (0) q_\alpha \int v f_{\alpha 1} dv,
\]

(3)

\[
\nabla \times E_1 + \frac{1}{c} \frac{\partial B_1}{\partial t} = 0,
\]

(4)

\[
\nabla \cdot B_1 = 0.
\]

(5)

3. Numerical solution of the Vlasov-Maxwell equations

In order to solve the Vlasov-Maxwell system of equations (1), (2), (3), (4), (5) we use the method of integral transforms of Fourier-Bessel-Laplace developed for this kind of systems by \([5,12]\). This method reduces the differential equations to a set of algebraic equations to be solved respect to the corresponding transforms. In particular the method of this work makes use of an expansion of the radial cylindrical coordinates as a Bessel series, Fourier series for the angle \( \theta \) and Fourier integral for the axial coordinate \( z \). Thus, we use the following orthonormal set of functions in the intervals \( 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad -\infty \leq z \leq \infty \):

\[
\begin{align*}
Y_{ml}^k (r, \theta, z) &= \frac{J_m(P_{ml}r)}{\sqrt{2\pi a} J_{m+1}(X_{ml})} \exp (im\theta) \exp (ikz); \\
m &= 0, \pm 1, \pm 2, \ldots; \quad l = 1, 2, 3, \ldots; \quad -\infty \leq k \leq \infty
\end{align*}
\]

(6)

\[
J_m(X) \quad \text{are the Bessel functions of first kind and order } m, \quad P_{ml} \quad \text{is the radial wavenumber which is determined by the perfect conductor boundary condition at the cylinder walls } J_m(P_{ml}a) = 0. \quad \text{Therefore } P_{ml}a = X_{ml}, \quad \text{where } X_{ml} \quad \text{are the zeros of the equation } J_m(X) = 0 \quad \text{and } k \quad \text{is the axial wavenumber. Applying this expansion to Eq (1), we obtain a temporal differential equation for the Fourier-Bessel coefficients of the distribution function } [5]. \quad \text{Multiplying the corresponding differential equation by } \exp (i\omega t) \quad \text{and integrating respect } t \text{ at the interval } [0, \infty], \quad \text{after collecting terms we obtain the Fourier-Bessel-Laplace coefficients for the distribution function } [5,12,15]
\]

\[
f_{\alpha 1ml}(k, v, \omega) = f_{\alpha 1ml}(k, v, t = 0) - \frac{q_\alpha}{m_\alpha} \frac{dF_{\alpha 0}(v)}{dv} \sum_{l=1}^{\infty} \frac{E_{1z ml'}(k, \omega) C_{\alpha ml'}^l}{i(kv - \omega)},
\]

(8)

where \( f_{\alpha 1ml}(k, v, t = 0) \) \quad \text{are the Fourier-Bessel-Laplace for the initial perturbation. We also defined the convolution coefficients as:}

\[
C_{\alpha ml'}^l = \frac{2}{a^2 J_{m+1}(x_{ml}) J_{m+1}(x_{ml'})} \int_0^a dr r J_m(p_{ml}r) J_m(p_{ml'}r) g_{\alpha 0}(r).
\]

(9)

In order to calculate these convolution coefficients by a numerical integration algorithm we used an experimental density profile obtained for these kind of systems. This function is given by \( g_{\alpha 0}(r) = (1 + (\beta r)^2 a^{-2})^{-1} \) \([5]\) where \( \beta \) is a parameter chosen to fix the experimental data that we select as \( \beta = 3 \) \([5]\). This density profile describes a plasma with more particles near to the cylinder center.
3.1. Transverse magnetic modes propagating in the cylindrical inhomogeneous plasmas

In the last section the Fourier-Bessel-Laplace coefficients for the distribution function perturbation. Taking into account that the perturbed current density is a statistical moment of the perturbed distribution function, we follow a similar expansion procedure to find the Fourier-Bessel-Laplace of the current density and the electric field. This procedure allows us to find the linear relation between the current density and the electric field which originates the conductivity tensor and then the corresponding axial component of the dielectric tensor $\varepsilon_z (r, k, \omega)$ (transverse dynamics is neglected). Assuming harmonic variation for the fields: $E(x, t) = E_\omega (x) \exp (-i \omega t)$, we obtain the axial component of the wave equation $z$ in cylindrical coordinates:

$$\frac{d^2}{dr^2} E_{zm} (r) + \frac{1}{r} \frac{d}{dr} E_{zm} (r) - \left\{ [1 - g_{e0} (r) I (k, \omega)]^2 (\omega) + \frac{m^2}{r^2} \right\} E_{zm} (r) = 0. \tag{10}$$

where the axial component of the dielectric tensor is given by $\varepsilon_z (r, k, \omega) = [1 - g_{e0} (r) I (k, \omega)]$ [5, 12], and we defined the function,

$$I (k, \omega) = \frac{\omega_p^2 (0)}{k^2} \int_{-\infty}^{\infty} \frac{dF_{e0}(v)}{dv} \frac{dF_{e0}(v)}{dv} dv. \tag{11}$$

Since the cylindrical walls are perfect conductor, the tangential electric field component must vanish at these boundaries. Thus, the solution of the spatial part for the axial electric field must satisfy $E_{zm} (a) = 0$. Thus, and according to this boundary value problem the wave angular frequency $\omega$ must be chosen in such a way $E_{zm} (r)$ accomplish this boundary condition $E_{zm} (a) = 0$. This constitutes an eigenvalue problem where for each value of the wavenumber $k$ we obtain through the dispersion relation the eigenfrequencies $\omega_{mn} (k)$, with $n = 1, 2, 3, ...$ labeling the different radial modes for each $k$: $E_{zm}^n (r) = E_{zmk}^n (r) [5, 6, 12, 15]$.

![Figure 1. Radial structure of the axial electric field for the fundamental transverse magnetic mode at the high frequencies range for several values of $ka = 4, 30, 50$](image)
This eigenvalue problem was solved through the implementation of the numerical scheme developed by [16] which allows to find the coefficients defining the field structure together with the corresponding wave frequencies.

Figure 1 shows the radial structure of the axial electric field corresponding to the fundamental mode $m = 0$ and $n = 1$ at the high frequency range for different values of $ka$ ($ka = 4, 30, 50$) obtained through our numerical solution. This figure illustrates that the electric field is high near to the cylinder center where the plasma is more concentrated. The field also shows the presence of oscillations with decaying amplitude until the cylinder boundary where the electric field is null as expected from the boundary condition. We also note from the figure that the field intensity is higher for longer wavenumbers $ka$. This implies that for particles accelerators based in plasmas this axial electric field can transfer energy to charged particles producing an axial acceleration which could be more effective near the waveguide center.

Conclusions
In summary, using a Fourier-Bessel-Laplace expansion method we solve the linearized Vlasov-Maxwell equations for a radially inhomogeneous plasma confined in a cylindrical waveguide through a strong axial magnetic field. After the calculation of the dielectric tensor we obtained the wave equation for the axial electric and solved numerically the radial structure of the axial fundamental and high frequency transverse magnetic mode. Our result shows that the axial electric field is more intense in those regions where the plasma density is higher. This spatial behavior of the field can be an important issue to be considered in design of particles accelerators using plasma based technologies.

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