Gröbner-Shirshov bases for extensions of algebras*

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Abstract. An algebra $R$ is called an extension of the algebra $M$ by $B$ if $M^2 = 0$, $M$ is an ideal of $R$ and $R/M \cong B$ as algebras. In this paper, by using the Gröbner-Shirshov bases, we characterize completely the extensions of $M$ by $B$. An algorithm to find the conditions of an algebra $A$ to be an extension of $M$ by $B$ is obtained.

Keywords: algebra, module, Gröbner-Shirshov bases, extension

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1 Preliminaries

Let $k$ be a field, $X$ a set, $X^*$ the monoid of all words on $X$ and $X^+$ the free semigroup of nonempty words on $X$. We denote $k\langle X_+ \rangle$ the $k$-span of all nonempty words in $X$. As we know, $k\langle X_+ \rangle$ is a free associative algebra without identity and $k\langle X \rangle$ a free associative algebra with identity. It is clear that, for every algebra $A$ (not necessarily with 1), we have $A \cong k\langle X_+ \rangle/I$ for some $X$ and ideal $I$ of $k\langle X_+ \rangle$. For a word $w \in X^*$, we denote the length of $w$ by $\text{deg}(w)$. Let $X^*$ be a well ordered set. Let $f \in k\langle X \rangle$ with the leading word $\bar{f}$. We say that $f$ is monic if $\bar{f}$ has coefficient 1.

Definition 1.1 ([11], see also [2], [3]) Let $f$ and $g$ be two monic polynomials in $k\langle X \rangle$ and $<$ a well order on $X^*$. Then, there are two kinds of compositions:

1. If $w$ is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $\text{deg}(\bar{f}) + \text{deg}(\bar{g}) > \text{deg}(w)$, then the polynomial $(f, g)_w = fb - ag$ is called the intersection composition of $f$ and $g$ with respect to $w$.

2. If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the inclusion composition of $f$ and $g$ with respect to $w$.

Definition 1.2 ([2], [3], cf. [11]) Let $S \subseteq k\langle X \rangle$ with each $s \in S$ monic. Then the composition $(f, g)_w$ is called trivial modulo $S$ if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$.

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\(a_i, b_i \in X^* \text{ and } \overline{a_is_ib_i} < w. \) If this is the case, then we write
\[(f, g) \equiv 0 \pmod{(S, w)}\]
In general, for \(p, q \in k\langle X \rangle\), we write
\[p \equiv q \pmod{(S, w)}\]
which means that \(p - q = \sum \alpha_i a_is_ib_i\), where each \(\alpha_i \in k, a_i, b_i \in X^* \text{ and } \overline{a_is_ib_i} < w. \)

**Definition 1.3** ([2], [3], cf. [11]) We call the set \(S\) with respect to the well order a Gröbner-Shirshov set (basis) in \(k\langle X \rangle\) if any composition of polynomials in \(S\) is trivial relative to \(S\).

A well order \(>\) on \(X^*\) is monomial if it is compatible with the multiplication of words, that is, for \(u, v \in X^*\), we have
\[u > v \Rightarrow w_1uw_2 > w_1vw_2, \text{ for all } w_1, w_2 \in X^*.\]

A standard example of monomial order on \(X^*\) is the deg-lex order to compare two words first by degree and then lexicographically, where \(X\) is a linearly ordered set.

The following lemma was proved by Shirshov [11] for the free Lie algebras (with deg-lex ordering) in 1962 (see also Bokut [2]). In 1976, Bokut [3] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For commutative polynomials, this lemma is known as the Buchberger’s Theorem in [5] and [6].

**Lemma 1.4** (Composition-Diamond Lemma) Let \(k\) be a field, \(A = k\langle X\rangle = K\langle X \rangle/\text{Id}(S)\) and \(<\) a monomial order on \(X^*\), where \(\text{Id}(S)\) is the ideal of \(k\langle X \rangle\) generated by \(S\). Then the following statements are equivalent:

(i) \(S\) is a Gröbner-Shirshov basis.

(ii) \(f \in \text{Id}(S) \Rightarrow \bar{f} = a\bar{s}b\) for some \(s \in S\) and \(a, b \in X^*\).

(iii) \(\text{Irr}(S) = \{u \in X^*|u \neq a\bar{s}b, s \in S, a, b \in X^*\}\) is a basis of the algebra \(A = k\langle X\rangle\).

The following lemma comes from [8] ([8], Lemma 4.2) which is essentially the same as Lemma 1.4

**Lemma 1.5** (Composition-Diamond Lemma) Let \(k\) be a field, \(S \subseteq K\langle X_+\rangle\) be monic, \(A = k\langle X_+\rangle = K\langle X_+\rangle/\text{Id}(S)\) and \(<\) a monomial order on \(X^+\), where \(\text{Id}(S)\) is the ideal of \(k\langle X_+\rangle\) generated by \(S\). Then the following statements are equivalent:

(i) \(S\) is a Gröbner-Shirshov basis.

(ii) \(f \in \text{Id}(S) \Rightarrow \bar{f} = a\bar{s}b\) for some \(s \in S\) and \(a, b \in X^*\).

(iii) \(\text{Irr}(S) = \{u \in X^+|u \neq a\bar{s}b, s \in S, a, b \in X^*\}\) is a basis of the algebra \(A = k\langle X_+\rangle\).
Remark: Suppose that $S \subseteq K\langle X_+ \rangle$ is a Gröbner-Shirshov basis. In the (iii) of Lemma 1.3 $1 \notin \text{Irr}(S)$ but not the case in the Lemma 1.4.

For convenience, we identify a relation $u = f_u$ of an algebra presented by generators and relations with the polynomial $u - f_u$ in the corresponding free algebra.

The concepts of the extension of algebras was invented by Hochschild [9] (also see [7] and [10]).

Definition 1.6 Let $k$ be a field, $M, B, \mathcal{R}$ $k$-algebras (not necessarily with 1). Then $\mathcal{R}$ is called an extension of $M$ by $B$ if $M^2 = 0$, $M$ is an ideal of $\mathcal{R}$ and $\mathcal{R}/M \cong B$ as algebras.

In [9], such an extension is called a singular extension.

2 Characterizations of extensions of algebras

Let $M, B$ be $k$-algebras, $M^2 = 0$ and $M$ a $B$-bimodule.

By a factor set $\{(b, b')|b, b' \in B\}$ of $B$ in $M$ we mean that $\{(b, b')|b, b' \in B\}$ is a subset of $M$ such that the function $(b, b')$ is $k$-bilinear.

Let $I, J$ be linearly ordered sets, $\{b_i|i \in I\}$, $\{m_j|j \in J\}$ $k$-bases of $B$ and $M$, respectively and $\{(b, b')|b, b' \in B\}$ a factor set of $B$ in $M$. Denote

$$A = E_k(M, B, (b_p, b_q)) = k\langle\{m_j\}, \{b_i\}\rangle + |S|$$

where $S = \{b_p b_q = [b_p b_q] + (b_p, b_q), b_p m_j = [b_p m_j], m_j b_p = [m_j b_p], m_j m_l = 0, p, q \in I, j, l \in J\}$ and for example, $[b_p b_q] = \sum_{i \in I} \alpha_{pq}^i b_i$, $\alpha_{pq}^i \in k$, is the product in $B$.

We order the set $\{(m_j) \cup \{b_i\}\}^+$ by the deg-lex order.

Equipping the above concepts, we have the following theorems which give characterizations of extensions of algebras.

Theorem 2.1 Let $M, B$ be $k$-algebras, $M^2 = 0$, $M$ a $B$-bimodule, $\{b_i|i \in I\}$, $\{m_j|j \in J\}$ $k$-bases of $B$ and $M$, respectively and $\{(b, b')|b, b' \in B\}$ a factor set of $B$ in $M$. Let $A = E_k(M, B, (b_p, b_q)) = k\langle\{m_j\}, \{b_i\}\rangle + |S|$, where $S = \{b_p b_q = [b_p b_q] + (b_p, b_q), b_p m_j = [b_p m_j], m_j b_p = [m_j b_p], m_j m_l = 0, p, q \in I, j, l \in J\}$. Then, $S$ is a Gröbner-Shirshov bases for $A$ if and only if the factor set satisfies the following condition in $M$: for any $p, q, r \in I$

$$b_p(b_q, b_r) - (b_p b_q, b_r) + (b_p, b_q b_r) - (b_p, b_q)b_r = 0$$

i.e., the function $(b, b')$ is a cocycle.

Moreover, if this is the case, $A = E_k(M, B, (b_p, b_q)) = k\langle\{m_j\}, \{b_i\}\rangle + |S|$ is an extension of $M$ by $B$ in a natural way.

Proof: Suppose that (1) holds. Then the possible compositions in $S$ are related to the following ambiguities:

$$w = b_p b_q b_r, \ m_j b_p b_q, \ b_p b_q m_j, \ m_j m_l b_p, \ b_p m_j m_l, \ b_p m_j b_q, \ m_j b_p m_l.$$
For \( w = m_1n_1m_n, \ b_1m_1b_2, \ b_2m_2b_3, \ b_3m_3b_4, \ m_jb_pm_l, \) by noting that \( [[m_jm_i]m_n] = 0, [[m_jm_i]b_p] = [m_j[b_pm_n]], \) thus, for any \( j, l, n \in J, \) \( q_j \) is trivial modulo \( S. \)

For \( w = m_jb_pb_q, \) since \( [[m_jb_pb_q]] = [m_jb_pb_q] \) and \( m_j(b_p, b_q) = 0, \) we have
\[
(m_jb_p - [m_jb_p], b_q - [b_pb_q] - (b_p, b_q)) = -[m_jb_pb_q] + m_j(b_pb_q) \equiv 0 \mod(S, w)
\]

Similarly, for \( w = b_pb_qm_j, \) the correspondent composition is trivial modulo \( S. \)

Thus, all compositions in \( S \) are trivial modulo \( S. \)

Conversely, if \( S \) is a Gröbner-Shirshov bases for \( A, \) then, by noting that for \( w = b_pb_qb_r, \) the compositions are trivial modulo \( S, \) we can easily check that the condition (1) holds in \( M = (M + \text{Id}(S))/\text{Id}(S). \) Then, by using Lemma 1.5, the algebra \( \overline{M} \) is the same as \( M. \)

Moreover, assume that \( S \) is a Gröbner-Shirshov bases for \( A. \) Then, by Lemma 1.5, each element \( r \in A \) can be uniquely written as \( r = m + b, \) where \( m \in M, \) \( b \in B. \) Hence, \( A = M \oplus B \) as \( k \)-modules with the following multiplication: for any \( m, m' \in M, \) \( b, b' \in B, \)
\[
(m + b) \cdot (m' + b') = mb' + bm' + (b, b') + bb'.
\]

From this it follows that \( M \) is an ideal of \( A \) and \( A/M \cong B \) as algebras.

**Theorem 2.2** Let \( M, B, \mathcal{R} \) be \( k \)-algebras with \( M^2 = 0. \) If \( \mathcal{R} \) is an extension of \( M \) by \( B \) and \( \sigma : \mathcal{R}/M \to B, \) \( r_b + M \to b \) an algebra isomorphism, then \( M \) is a \( B \)-bimodule in a natural way: for any \( b \in B, \)
\[
b \cdot m = r_bm, \quad m \cdot b = mr_b \tag{2}
\]

and there exists a factor set \( \{(b, b')|b, b' \in B\} \) of \( B \) in \( M \) such that for any \( b, b', b'' \in B, \)
\[
b(b', b'') - (bb', b'') + (b, b'b'') - (b, b)b'' = 0.
\]

Moreover, \( \mathcal{R} \cong A \) as algebras, where \( A = E_k(M, B, (b_p, b_q)) = k\langle\{m_j\}, \{b_i\}\rangle + S, \)
\[
\{m_j\}, \{b_i\} \text{ lin bases of } M \text{ and } B \text{ respectively and } S = \{b_pb_q = [b_pb_q], (b_p, b_q), b_pm_m = b_pm_m, m_jb_p = [m_jb_p], m_jm_l = 0, p, q \in I, j, l \in J\}.
\]

**Proof.** Clearly, as \( k \)-modules, \( \mathcal{R} = M \oplus C, \) where \( C = \sum_{i \in I} kr_{bi} \) with a basis \( \{r_{bi}|i \in I\}. \) Thus, for any \( b, b' \in B, \) we have \( r_b + r_{b'} = r_{b+b'} \) because \( \sigma \) is an algebra isomorphism. Since \( (r_b + M)(r_{b'} + M) = r_br_{b'} + M = r_{b+b'} + M, \) there exists a unique \( (b, b') \in M \)
such that \( r_b r_{b'} = r_{b'} + (b, b') \). Then, it is easy to see that \( M \) is a \( B \)-bimodule with the module operations \( (2) \) and the function \((b, b')\) is \( k \)-bilinear. For example, for any \( b, b', b'' \in B, m \in M \), \((bb') \cdot m = r_{b'} m = (r_{b'} - (b, b')) m = (r_{b'} m) = r_b(r_{b'} m) = b \cdot (b' \cdot m)\).

Also, since \( r_b (r_{b'} + r_{b''}) = r_b r_{b'} + r_b r_{b''} = r_b r_{b' + b''} \), we have \((b, b' + b'') = (b, b') + (b, b'')\).

Moreover, by noting that \((r_b r_{b'}) r_{b''} = r_b (r_{b'} r_{b''})\), we know that the factor set \( \{(b, b')|b, b' \in B\} \) satisfies
\[
b(b', b'') - (bb', b'') + (b, b'b'') - (b, b')b'' = 0.
\]

Now, we prove that \( \mathcal{R} \cong A \) as algebras. For any \( m, m' \in M, c, c' \in C \),
\[
(m + c) \cdot (m' + c') = mc' + cm' + (c, c') + cc'
\]
where \((c, c') = \sum_{p, q} \alpha_p \alpha_q (b_p, b_q) \) if \( c = \sum_p \alpha_p r_{b_p}, c' = \sum_q \alpha_q' r_{b_q} \). Then, by the proof of Theorem 2.1, it is easy to see that \( \tau : \mathcal{R} \rightarrow A \) by \( \tau(m_j) = m_j, \tau(r_b) = b_i \) is an algebra isomorphism.

By Theorem 2.1 and Theorem 2.2, we have the following theorem.

**Theorem 2.3** Let \( M, B, \mathcal{R} \) be \( k \)-algebras with \( M^2 = 0 \). Then \( \mathcal{R} \) is an extension of \( M \) by \( B \) if and only if \( M \) is a \( B \)-bimodule and there exists a factor set \( \{(b, b')|b, b' \in B\} \) of \( B \) in \( M \) such that for any \( b, b', b'' \in B \),
\[
b(b', b'') - (bb', b'') + (b, b'b'') - (b, b')b'' = 0
\]
and \( \mathcal{R} \cong A = E_k(M, B, (b_p, b_q)) = k\langle \{m_j\}, \{b_i\} \mid S \rangle \), where \( \{b_i\}_I \) and \( \{m_j\}_J \) are any linear bases of \( B \) and \( M \) respectively, \( S = \{b_p b_q = [b_p, b_q] + (b_p, b_q), b_p m_j = [b_p, m_j], m_j b_p = [m_j b_p], m_j m_l = 0, p, q \in I, j, l \in J\} \).

Now, we consider the general case when the algebra \( B \) is presented by generators and relations.

Let \( M, B \) be \( k \)-algebras, \( M^2 = 0 \), \( M \) a \( B \)-bimodule, \( B = k\langle X_+ | R \rangle \), where \( R \) is a Gröbner-Shirshov bases for \( B \) with the deg-lex order \( <_B \) on \( X^+ \).

For convenience, we can assume that \( R \) is a minimal Gröbner-Shirshov bases in a sense that the leading monomials are not contained each other as subwords, in particular, they are pairwise different. Let \( R = \{u - f_u | u \in \Lambda \} \), where \( u \) is the leading term of the polynomial \( h_u = u - f_u \) in \( k\langle X_+ \rangle \). Let
\[
A = E_k(M, X, (u)) = k\langle \{m_j\} \cup X \mid S_1 \rangle
\]
where \( \{m_j\}_J \) is a basis of \( M \), \( S_1 = \{u = f_u + (u), u \in \Lambda, x m_j = [x, m_j], m_j x = [m_j x], m_j m_l = 0, x \in X, j, l \in J\} \) and \( \{(u)|u \in \Lambda\} \subseteq M \). For convenience, we also call \( \{(u)|u \in \Lambda\} \) a factor set of \( B \) in \( M \).

We order the set \( \{m_j\}_J \cup X \) also by the deg-lex order which extends \( <_B \) and satisfies \( x > m_j \) for any \( x \in X \) and \( j \in J \).

In \( S_1 \), the possible compositions are related to the following ambiguities:
\[
w_1 = w, m_j u, u m_j, m_j m_l m_n, m_j m_l x, x m_j m_l, x m_j x', m_j x m_l
\]
where \( w \) is an ambiguity appeared in \( R \).

For \( w_1 = m_jm_jm_{m_j} \), \( m_jm_jx, xm_jm_j, xm_jx' \), \( m_jxm_j \), by noting that \([m_jm_jm_{m_j}] = [m_jm_{m_j}m_j] = 0, \; [m_jm_jx] = [m_jxm_j] = 0, \; [xm_jm_j] = [x[m_jm_j]] = 0, \; [m_jx]m_j = [m_jxm_j] = 0 \) and \([xm_jx'] = [x[m_jx']]\) for any \( j, l, n \in J, x, x' \in X \), the corresponding compositions are trivial modulo \( S_1 \).

For \( w_1 = m_ju, \; u = x_1 \cdots x_t \), we have

\[
(m_jx_1 - [m_jx_1], u - (f_u + (u)))_{w_1} = -[m_jx_1]x_2 \cdots x_t + m_j(f_u + (u)) \\
\equiv -[\cdots[[m_jx_1]x_2 \cdots x_t] + m_jf_u + m_j(u) \equiv -[m_ju] + [m_jf_u] \equiv 0 \text{ mod}(S_1, w_1)
\]

Similarly, for \( w_1 = um_j \), we have \((u - (f_u + (u)), xm_j - [xm_j])_{w_1} \equiv 0 \text{ mod}(S_1, w_1)\).

Since \( R \) is a minimal Gröbner-Shirshov bases, all compositions in \( R \) are only intersection ones.

Now, for \( w_1 = w = u_1c = du_2, \; u_1, u_2 \in \Lambda, \; c, d \in X^+ \), we have

\[
(u_1 - (f_u + (u_1)), u_2 - (f_u + (u_2)))_{w_1} = -(f_u + (u_1))c + d(f_u + (u_2)) \\
= (df_u - f_u)c + (d(u_2) - (u_1)c)
\]

Since the composition \((h_{u_1}, h_{u_2})_{w_1} \) in \( R \) is trivial modulo \( R \), it follows that in \( k\langle X_+ \rangle \),

\[
(h_{u_1}, h_{u_2})_{w_1} = df_u - f_u c = \sum \alpha_i a_i (u_i' - f_{u_i'}) b_i \tag{3}
\]

where each \( \alpha_i \in k, \; a_i, b_i \in X^+, \; u_i' \in \Lambda \) and \( a_i(u_i' - f_{u_i'}) b_i = a_i u_i' b_i < w_1 \). Therefore, in \( k\langle \{m_j \} \cup X_+ \rangle \), we have

\[
(u_1 - (f_u + (u_1)), u_2 - (f_u + (u_2)))_{w_1} = (df_{u_2} - f_{u_2} c) + (d(u_2) - (u_1)c) \\
= \sum \alpha_i a_i (u_i' - f_{u_i'}) b_i + \sum \alpha_i a_i (u_i') b_i - ((u_1)c - d(u_2))
\]

where \( \alpha_i(u_i' - f_{u_i'}) b_i = a_i u_i' b_i < w_1 \). From this it follows that

\[
(u_1 - (f_u + (u_1)), u_2 - (f_u + (u_2)))_{w_1} \equiv g(u_1, u_2)_{w_i} \text{ mod}(S_1, w_1) \tag{4}
\]

where \( g(u_1, u_2)_{w_i} (u) = \sum \alpha_i a_i (u_i') b_i - ((u_1)c - d(u_2)) \in M \) is a function of \( \{u \mid u \in \Lambda \} \).

In fact, by (3), we have an algorithm to find the function \( g(u_1, u_2)_{w_i} \).

Therefore, we have the following theorem.

**Theorem 2.4** Let \( B = k\langle X_+ | R \rangle \), where \( R = \{u - f_u | u \in \Lambda \} \) is a minimal Gröbner-Shirshov bases for \( B \) and \( u \) the leading term of the polynomial \( h_u = u - f_u \) in \( k\langle X_+ \rangle \). Let \( M \) be a \( k \)-module, \( M^2 = 0, \; M \) a \( B \)-bimodule, \{\( m_j \}\}, \( J \) a \( k \)-basis of \( M \) and \( \{u \mid u \in \Lambda \} \) a factor set of \( B \) in \( M \). Let \( A = E_k(M, X, (u)) = k\langle \{m_j \} \cup X_+ | S_1 \rangle \) where \( S_1 = \{u = f_u + (u), \; xm_j = [xm_j], \; m_jx = [m_jx], \; m_jm_i = 0, \; u \in \Lambda, \; x \in X, j, l \in J \} \). Then, \( S_1 \) is a Gröbner-Shirshov bases for \( A \) if and only if

\[
\{g(u_1, u_2)_{w_i} (u) | (h_{u_1}, h_{u_2})_{w} \text{ is a composition in } R \} = 0 \tag{5}
\]

where \( g(u_1, u_2)_{w_i} (u) \) is defined by (4). Moreover, if this is the case, \( A \) is an extension of \( M \) by \( B \) in a natural way.
Proof. Assume that $S_1$ is a Gröbner-Shirshov bases for $A$. Then, by the previous statements, for any composition $(U_1, U_2) \in R$, $g_{(u_1, u_2)} = 0$ in $\overline{M} = (M + \text{Id}(S))/\text{Id}(S)$. Then, by using Lemma 1.5, the algebra $\overline{M}$ is the same as $M$. So, the (5) holds. Conversely, if (5) holds, then it is clear that $S_1$ is a Gröbner-Shirshov bases for $A$.

We need only to prove that $A$ is an extension of $M$ by $B$ in a natural way if (5) holds. By Lemma 1.5 each element $r \in A$ can be uniquely written as $r = m + b$, where $m \in M$, $b \in B$ is $R$-irreducible. Hence, $A = M \oplus B$ as $k$-modules with the following multiplication: for any $m, m' \in M$, $b, b' \in B$, 

$$(m + b) \cdot (m' + b') = m'b + bm' + (u_{b,b'}) + bb'$$

where $bb' \equiv [bb'] \mod(R)$, $[bb']$ is $R$-irreducible and $(u_{b,b'}) \in M$ is a function of $\{(u)|u \in \Lambda\}$. From this it follows that $M$ is an ideal of $A$ and $A/M \cong B$ as algebras. 

**Theorem 2.5** Let $B = k(X_+|R)$, where $R = \{u - f_u|u \in \Lambda\}$ is a minimal Gröbner-Shirshov bases for $B$ and $u$ the leading term of the polynomial $h_u = u - f_u$ in $k(X_+)$. Let $M, R$ be $k$-algebras with $M^2 = 0$. If $R$ is an extension of $M$ by $B$ and $\sigma : R/M \rightarrow B$, $r_x + M \mapsto x$ an algebra isomorphism, then $M$ is a $B$-bimodule with a natural way: for any $x \in X$, $m \in M$,

$$x \cdot m = r_x m, \quad m \cdot x = mr_x$$

and there exists a factor set $\{(u)|u \in \Lambda\}$ of $B$ in $M$ such that

$$\{g_{(u_1, u_2)}(u)|(h_{u_1}, h_{u_2})_w \text{ is a composition in } R) = 0$$

where $g_{(u_1, u_2)}(u)$ is defined by (4). Moreover, $R \cong A$ as algebras, where $A = E_k(M, X, (u)) = k\langle\{m_j\}_J \cup \{X\}\rangle + |S_1\rangle$, $\{m_j\}_J$ a $k$-basis of $M$ and $S_1 = \{u = f_u + (u), x m_j = [x m_j], m_j x = [m_j x], m_j m_l = 0, u \in \Lambda, x \in X, j, l \in J\}$.

**Proof.** It is clear that $M$ is a $B$-bimodule under the given operations.

For any $u \in \Lambda$, since $r_u + M = r_{f_u} + M$, there exists a unique $(u) \in M$ such that $r_u = r_{f_u} + (u)$. By noting that each composition $(h_{u_1}, h_{u_2})_w$ in $R$, $(h_{u_1}, h_{u_2})_w \equiv 0 \mod(R, w)$, we have that $g_{(u_1, u_2)}(u) = 0$ in $M$ by (3).

Now, by using the proof of Theorem 2.4, $S_1$ is a Gröbner-Shirshov bases for the algebra $A$ and then the result follows. 

The following theorem, which gives a complete characterization of an algebra to be an extension of $M$ by $B$, follows from Theorem 2.4 and Theorem 2.5

**Theorem 2.6** Let $M, B, R$ be $k$-algebras with $M^2 = 0$, $B = k(X_+|R)$, where $R = \{u - f_u|u \in \Lambda\}$ is a minimal Gröbner-Shirshov bases for $B$ and $u$ the leading term of the polynomial $h_u = u - f_u$ in $k(X_+)$. Then $R$ is an extension of $M$ by $B$ if and only if $M$ is a $B$-bimodule and there exists a factor set $\{(u)|u \in \Lambda\}$ of $B$ in $M$ such that

$$\{g_{(u_1, u_2)}(u)|(h_{u_1}, h_{u_2})_w \text{ is a composition in } R) = 0$$

and $R \cong E_k(M, X, (u)) = k\langle\{m_j\}_J \cup \{X\}\rangle + |S_1\rangle$, where $g_{(u_1, u_2)}(u)$ is defined by (4), $\{m_j\}_J$ a $k$-basis of $M$ and $S_1 = \{u = f_u + (u), x m_j = [x m_j], m_j x = [m_j x], m_j m_l = 0, u \in \Lambda, x \in X, j, l \in J\}$.
3 Applications

Let $M, B, \mathcal{R}$ be $k$-algebras with $M^2 = 0$. The previous theorems give an answer to how to find the conditions which makes $\mathcal{R}$ to be an extension of $M$ by $B$. We call the condition (4) the extension condition of Theorem 3.1

Let $\mathfrak{a}$-algebra, universal envelope of the free Lie algebra and Grassman algebra respectively.

As results, by using the extension conditions, let us give some examples. We give the characterization of the extension of $M$ by $B$ when the $B$ is cyclic algebra, free commutative algebra, universal envelope of the free Lie algebra and Grassman algebra respectively.

Theorem 3.1 Let $M, B, \mathcal{R}$ be $k$-algebras with $M^2 = 0$ and $B = k\langle\{x\}_+ | x^n = f(x)\rangle$ a cyclic algebra, where $n$ is a natural number and $f(x)$ is a polynomial of degree less than $n$ such that $f(0) = 0$. Then, $\mathcal{R}$ is isomorphic to an extension of $M$ by $B$ if and only if $M$ is a $B$-bimodule, there exists an $m \in M$ such that $mx = xw$ and $\mathcal{R} \cong E_k(M, x, m) = k\langle\{m_j\}_j \cup \{x\}_+ | S\rangle$, where $\{m_j\}_j$ is any $k$-basis of $M$ and $S = \{x^n = f(x) + m, \, mx = [xm_j], m_jx = [m_jx], m_jm_l = 0, \, j, l \in J\}$.

Proof. Clearly, $R = \{x^n = f(x)\}$ is a Gröbner-Shirshov bases for $B$. We need only to consider the composition in $S$: $(x^n - f(x) - m, x^n - f(x) - m)_w$, $w = x^n + 1$. Thus, we obtain the extension condition: $mx = xw$. Now, by Theorem 2.6, the result follows. □

Theorem 3.2 Let $M, B, \mathcal{R}$ be $k$-algebras with $M^2 = 0$. Let $X = \{x_i | i \in I\}$, $I$ a well ordered set and $B = k\langle X | x_i x_q = x_q x_p, \, p > q, \, p, q \in I \rangle$ the free commutative algebra generated by $X$. Then, $\mathcal{R}$ is isomorphic to an extension of $M$ by $B$ if and only if $M$ is a $B$-bimodule, there exists a factor set $\{(x_p, x_q) | p > q, \, p, q \in I\}$ of $B$ in $M$ such that for any $p, q, r \in I, \, p > q > r$,

$$(x_q, x_r)p - x_p(x_q, x_r) + x_q(x_p, x_r) - (x_p, x_r)q + (x_p, x_q)r - x_q(x_p, x_q) = 0 \quad (6)$$

and $\mathcal{R} \cong E_k(M, X, (x_p, x_q)) = k\langle\{m_j\}_j \cup X \rangle, \, m_j \in M \rangle S\rangle$, where $\{m_j\}_j$ is any $k$-basis of $M$ and $S = \{x_p x_q = x_q x_p + (x_p, x_q), \, x_p m_j = [x_p, m_j], m_j x_p = [m_j x_p], m_j m_l = 0, \, p > q, \, p, q \in I, \, j, l \in J\}$.

Proof. Let $R = \{x_p x_q = x_q x_p | p > q, \, p, q \in I\}$. Then, for the deg-lex order on $X^+$, $R$ is clearly a Gröbner-Shirshov bases for $B$ and only one kind of compositions are in $R$, i.e.,

$$(x_p x_q - x_q x_p, x_q x_r - x_r x_q)_w, \, w = x_p x_q x_r, \, p, q, r \in I, \, p > q > r.$$ Then, in $S$, by calculating the composition $(x_p x_q - x_q x_p - (x_p, x_q), \, x_q x_r - x_r x_q - (x_q, x_r))_w, \, w = x_p x_q x_r, \, p, q, r \in I, \, p > q > r$, we obtain the extension condition (6). Now, by Theorem 2.6, the result follows. □

The following theorem is a generation of Theorem 3.2.

Theorem 3.3 Let $M, B, \mathcal{R}$ be $k$-algebras with $M^2 = 0$. Let $X = \{x_i | i \in I\}$, $I$ be a well ordered set, $L = \text{Lie}_k(X | [x_p x_q] = \sum_{i \in I} \alpha_{pq}^{i} x_i, \, p, q \in I)$ a Lie algebra and $B = U(L) = k\langle X, R \rangle$ the universal envelop of the Lie algebra $L$, where each $\alpha_{pq}^{i} \in k$ and $R = \{x_p x_q = x_q x_p + [x_p x_q] | p > q, \, p, q \in I\}$. Then, $\mathcal{R}$ is isomorphic to an extension of $M$
by $B$ if and only if $M$ is a $B$-bimodule, there exists a factor set $\{(x_p, x_q)|p, q \in I\}$ of $B$ in $M$ such that for any $p, q, r \in I$, $p > q > r$,

\[(x_q, x_r)x_p + x_q(x_p, x_r) + (x_p, x_q)x_r + \left([x_qx_r], x_p\right) + (x_q, [x_p]x_r) + ([x_p]x_q, x_r)\]

\[-(x_p, x_r)x_q - x_p(x_q, x_r) - x_r(x_p, x_q) = 0\]

and $R \cong E_k(M, X, (x_p, x_q)) = k\langle\{m_j\}angle \cup X|_+|S\rangle$, where $(x_p, x_q) = -(x_q, x_p)$, $(x_p, x_p) = 0$ for any $p, q \in I$, $\{m_j\}$ is any $k$-basis of $M$ and $S = \{x_p x_q = x_q x_p + [x_p]x_q + (x_p, x_q), x_p m_j = [x_p m_j], m_j x_p = [m_j x_p], m_j m_i = 0, p > q, p, q \in I, j, l \in J\}$.

**Proof.** By [4], for the deg-lex order on $X^+$, $R$ is a Gröbner-Shirshov bases for $B$ and only one kind of compositions are in $R$, i.e., $(x_p x_q - x_q x_p - [x_p]x_q, x_q x_r - x_r x_q - [x_q]x_r)_w, w = x_p x_q x_r, p, q, r \in I$, $p > q > r$. Since, in $S$,

\[(x_p x_q - x_q x_p - [x_p]x_q, x_q x_r - x_r x_q - [x_q]x_r)_w\]

\[= -(x_q x_p x_r + [x_p]x_q x_r + (x_p, x_q)x_r) + (x_p x_r x_q + x_p [x_q]x_r + x_p (x_q, x_r))\]

\[\equiv -(x_q x_p x_r + [x_p]x_q x_r + (x_p, x_q)x_r) + (x_p x_q x_r + [x_q]x_r) + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

\[\equiv -x_q x_p x_r + [x_p]x_q x_r + [x_q]x_r + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

\[\equiv -x_q x_p x_r + [x_p]x_q x_r + [x_q]x_r + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

\[\equiv -x_q x_p x_r + [x_p]x_q x_r + [x_q]x_r + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

\[\equiv -x_q x_p x_r + [x_p]x_q x_r + [x_q]x_r + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

\[\equiv -x_q x_p x_r + [x_p]x_q x_r + [x_q]x_r + \left([x_p]x_q x_r + (x_p, x_q)x_r\right)\]

and by using Jacobi identity $\left([x_p]x_r x_q + [x_p x_q]x_r\right) + [x_r]x_p x_q = 0$,

\[\equiv (x_q x_p x_r + [x_p]x_q x_r + (x_p, x_q)x_r) + (x_p x_q x_r + [x_q]x_r) + (x_p x_q x_r + [x_q]x_r)\]

\[\equiv ([x_p]x_q x_r + [x_p x_q]x_r) + [x_r]x_p x_q + (x_p x_q x_r) + (x_p x_q x_r)\]

\[\equiv ([x_p]x_q x_r + [x_p x_q]x_r) + [x_r]x_p x_q + (x_p x_q x_r) + ([x_r]x_p x_q)\]

\[\equiv [x_p x_r]x_q + (x_q x_p x_r) + (x_p x_q x_r) + ([x_r]x_p x_q)\]

we have the extension condition mentioned in the theorem. Now, by Theorem 2.6, the result follows. □

**Theorem 3.4** Let $M, B, R$ be $k$-algebras with $M^2 = 0$. Let $X = \{x_i\}_i \in I$, $I$ be a well ordered set, $B = k\langle X^+|R\rangle$ the Grassman algebra, where $R = \{x^2 = 0, x_p x_q = -x_q x_p|p > q, p, q \in I\}$. Then, $R$ is isomorphic to an extension of $M$ by $B$ if and only if $M$ is a $B$-bimodule, there exists a factor set $\{(x_p, x_q)|p \geq q, p, q \in I\}$ of $B$ in $M$ such that for any $p, q, r \in I$, $p > q > r$,

\[(x_q, x_r)x_p - x_r(x_q, x_r) + (x_q, x_r)x_p - x_q(x_q, x_r) = 0\]

\[(x_q, x_r)x_p - x_q(x_q, x_r) + (x_q, x_r)x_p - x_q(x_q, x_r) = 0\]

\[(x_q, x_r)x_p + (x_p, x_r)x_q - x_q(x_p, x_r) - x_p(x_q, x_r) - x_r(x_p, x_q) + (x_q, x_r)x_p - x_q(x_q, x_r) = 0\]
and $\mathcal{R} \cong E_k(M, X, (x_p, x_q)) = k(\{m_j\}_{j \in J} \cup X) + |S|$, where $\{m_j\}_{j}$ is any $k$-basis of $M$ and

\[ S = \{ x_q^2 = (x_q, x_q), \quad x_p x_q = -x_q x_p + (x_p, x_q), \quad x_q m_j = [x_q m_j], m_j x_q = [m_j x_q], m_j m_l = 0, p > q, \quad p, q \in I, j, l \in J \} \]

**Proof.** By [4], for the deg-lex order on $X^+$, $R$ is a Gröbner-Shirshov bases for $B$ and the possible compositions in $R$ are related to the following ambiguities:

\[ w_1 = x_q^2 x_r, \quad w_2 = x_q x_r^2, \quad w_3 = x_p x_q x_r, \quad p, q, r \in I, \quad p > q > r. \]

Then, in $S$, by calculating the corresponding compositions:

\[ (x_q^2 - (x_q, x_q), x_q x_r + x_r x_q - (x_q, x_r), x_r^2 - (x_q, x_q)) w_1, \quad (x_q x_r + x_r x_q - (x_q, x_r), x_r^2 - (x_q, x_q)) w_2, \quad ((x_p x_q + x_q x_p - (x_p, x_q), x_q x_r + x_r x_q - (x_q, x_r)) w_3, \]

respectively, we obtain the extension conditions mentioned in the theorem. Now, by Theorem 2.6, the result follows.

\[ \square \]

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