A new geometrical approach to metric-affine gravity

F. F. Faria *
Centro de Ciências da Natureza,
Universidade Estadual do Piauí,
64.002-150 Teresina, PI, Brazil

Abstract

Here we consider a metric-affine theory of gravity in which the gravitational Lagrangian is the scalar curvature. The matter action is allowed to depend also on the torsion and the nonmetricity, which are considered as the field variables together with the metric. We find the field equations of the theory and show that they reduce to Einstein equations in vacuum.

* fff@uespi.br
1 Introduction

It is well known that general relativity (GR) is consistent with several solar system experimental tests. However, the standard GR seems unable to explain some gravity phenomena on both atomic and cosmological scales. A number of alternative theories of gravity have been proposed to explain such phenomena but none have been completely successful so far. One of these theories is the metric-affine gravity (MAG), which is a generalization of GR based on a spacetime with an asymmetric and not metric compatible connection.

In MAG, the antisymmetric part of the connection is related to torsion [1], whereas the covariant derivative of the metric construct out of the connection is related to nonmetricity [2]. The sources of torsion and nonmetricity are the matter spin and strain (dilation plus shear) currents [3], respectively. The geometrical concepts of torsion and nonmetricity have concrete physical interpretations as lattice defects in the three-dimensional theory of elastic continua [4].

The field equations of MAG can be derived by means of a metric-affine variational principle in which the metric and the connection are considered as independent variables. Comparison with the metric variational principle of GR suggests the choice of the scalar curvature as the MAG gravitational field Lagrangian. The adoption of this Lagrangian together with a matter Lagrangian independent of the connection leads to the Einstein field equations of GR [6]. However, assuming dependence of the matter Lagrangian on the connection, which is the most natural choice, yields an unphysical MAG model [7]. The usual procedures to solve such a problem are either to generalize the gravitational part of the Lagrangian by adding some extra terms [7], or to interpret MAG as a gauge theory by introducing the coframe \( e^\mu = e_{a\mu} dx^a \), besides the metric and the connection, as a field variable [8].

In this article, we address a new geometrical approach to MAG in which we consider the scalar curvature as the gravitational Lagrangian and the metric, the torsion and the nonmetricity as the field variables. In section 2 we describe the MAG spacetime and the formalism of MAG which we shall use throughout this article. In section 3 we derive the MAG field equations for the independent metric, torsion and nonmetricity. In section 4 we analyze the vacuum solution of the MAG field equations. Finally, in section 5 we present our conclusions.
2 MAG spacetime

In GR, gravity is represented by the Riemann spacetime, in which a free particle follows the geodesic trajectory

\[
d\frac{d^2x^\lambda}{d\tau^2} +\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,
\]  

(1)

where

\[
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}),
\]

(2)

is the Levi-Civita connection, \(g_{\mu\nu}\) is the spacetime metric and \(\tau\) is a suitable affine parameter along the geodesic, which can be taken to represent the particle proper time. Both \(g_{\mu\nu}\) and \(\Gamma^{\lambda}_{\mu\nu}\) are symmetric in the \(\mu\nu\) indices.

The covariant derivative of the metric, the torsion and the curvature construct out of the Levi-Civita connection are given by

\[
\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu\rho}g_{\rho\nu} - \Gamma^{\sigma}_{\lambda\nu\rho}g_{\rho\mu} = 0,
\]

(3)

\[
\check{T}^{\lambda}_{\mu\nu} = \check{\Gamma}^{\lambda}_{\mu\nu} - \check{\Gamma}^{\lambda}_{\nu\mu} = 0,
\]

(4)

\[
\check{R}^{\alpha}_{\mu\beta\nu} = \partial_\beta \check{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu \check{\Gamma}^{\alpha}_{\mu\beta} + \check{\Gamma}^{\sigma}_{\alpha\beta} \check{\Gamma}^{\sigma}_{\mu\nu} - \check{\Gamma}^{\sigma}_{\alpha\sigma} \check{\Gamma}^{\sigma}_{\mu\beta},
\]

(5)

respectively. This means that in Riemann spacetime lengths and angles are preserved under parallel displacement, infinitesimal parallelograms are closed and geodesics deviate from each other. This spacetime geometry seem to be fully confirmed experimentally. However, indications from the inflationary model [9] and the string theory [10] points to a non-Riemannian geometry in the early universe and on the Planck scale, respectively. The simplest spacetime having such geometry is MAG spacetime.

The MAG spacetime is endowed with an asymmetric connection \(\Gamma^{\lambda}_{\mu\nu}\) presenting nonmetricity

\[
Q_{\lambda\mu\nu} = -\nabla_\lambda g_{\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^{\rho}_{\mu\lambda}g_{\rho\nu} + \Gamma^{\rho}_{\nu\lambda}g_{\rho\mu},
\]

(6)

torsion

\[
T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}.
\]

(7)

and curvature

\[
\check{R}^{\alpha}_{\mu\beta\nu} = \partial_\beta \check{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu \check{\Gamma}^{\alpha}_{\mu\beta} + \check{\Gamma}^{\sigma}_{\alpha\beta} \check{\Gamma}^{\sigma}_{\mu\nu} - \check{\Gamma}^{\sigma}_{\alpha\sigma} \check{\Gamma}^{\sigma}_{\mu\beta}.
\]

(8)
The nonmetricity, which is symmetric in its last two indices, is responsible for the change in lengths and angles under parallel displacement, whereas the torsion, which is antisymmetric in its last two indices, is responsible for the breaking of infinitesimal parallelograms. The only symmetry of the curvature tensor is $R^\alpha_{\mu\beta\nu} = -R^\alpha_{\mu\nu\beta}$.

In order to get a general expression for the metric-affine connection $\Gamma^\lambda_{\mu\nu}$, we add to equation (6) the same equation with the indices $\mu$ and $\lambda$ changed, subtract the same equation with the indices $\nu$ and $\lambda$ changed, and find

$$Q_{\lambda\mu\nu} + Q_{\mu\lambda\nu} - Q_{\nu\mu\lambda} = -\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda}$$
$$+ \Gamma_\rho_{\mu\lambda} g_{\rho\nu} + \Gamma_\rho_{\nu\lambda} g_{\rho\mu} + \Gamma_\rho_{\lambda\mu} g_{\rho\nu}$$
$$+ \Gamma_\rho_{\nu\mu} g_{\rho\lambda} - \Gamma_\rho_{\mu\nu} g_{\rho\lambda} - \Gamma_\rho_{\lambda\nu} g_{\rho\mu}.$$  (9)

With some manipulation, we can write this equation as

$$\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + K^\lambda_{\mu\nu} + N^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + W^\lambda_{\mu\nu},$$  (10)

where $\hat{\Gamma}^\lambda_{\mu\nu}$ is the Levi-Civita connection,

$$K^\lambda_{\mu\nu} = \frac{1}{2} \left( T^\lambda_{\mu\nu} - T^\lambda_{\mu\nu} - T^\lambda_{\nu\mu} \right)$$  (11)

is the contortion tensor,

$$N^\lambda_{\mu\nu} = \frac{1}{2} \left( Q^\lambda_{\mu\nu} + Q^\lambda_{\nu\mu} - Q^\lambda_{\mu\nu} \right)$$  (12)

is the nonmetric part of the connection, and

$$W^\lambda_{\mu\nu} = K^\lambda_{\mu\nu} + N^\lambda_{\mu\nu}$$  (13)

is the distortion tensor. The contortion tensor is antisymmetric in its first two indices, the nonmetric part of the connection is symmetric in its last two indices and the distortion tensor is asymmetric.

By substituting equation (10) into equation (8), we can write the curvature of the metric-affine connection as

$$R^\alpha_{\mu\beta\nu} = \hat{R}^\alpha_{\mu\beta\nu} + \hat{\nabla}_\beta W^\alpha_{\mu\nu} - \hat{\nabla}_\nu W^\alpha_{\mu\beta} + W^\alpha_{\rho\beta} W^\rho_{\mu\nu} - W^\alpha_{\rho\nu} W^\rho_{\mu\beta},$$  (14)

where $\hat{R}^\alpha_{\mu\beta\nu}$ is the curvature of GR (Riemann tensor) and $\hat{\nabla}_\nu$ is the covariant derivative constructed out of the Levi-Civita connection. Contracting the curvature tensor (14) in the $\alpha\beta$ indices gives the Ricci tensor

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + \hat{\nabla}_\alpha W^\alpha_{\mu\nu} - \hat{\nabla}_\nu W^\alpha_{\mu\alpha} + W^\alpha_{\rho\alpha} W^\rho_{\mu\nu} - W^\alpha_{\rho\nu} W^\rho_{\mu\alpha},$$  (15)
which is asymmetric. The scalar curvature \( R = g^{\mu\nu} R_{\mu\nu} \) reads

\[
R = g^{\mu\nu} \left( \overset{\circ}{\nabla}_\alpha W^\alpha_{\mu\nu} - \overset{\circ}{\nabla}_\nu W^\alpha_{\mu\alpha} + W^\alpha_{\rho\alpha} W^\rho_{\mu\nu} - W^\alpha_{\rho\nu} W^\rho_{\mu\alpha} \right).
\] (16)

Note that it is possible to construct another Ricci tensor \( R'_{\mu\nu} \) by contracting the curvature tensor (14) in its first two indices. However, the scalar curvature is uniquely defined by equation (16), since \( g_{\mu\nu} \) is symmetric and \( R'_{\mu\nu} \) is antisymmetric.

### 3 MAG field equations

The MAG field equations can be obtained by the variational principle

\[
\delta S_g + \delta S_m = 0,
\] (17)

where \( S_g \) is the action of the gravitational field and \( S_m \) is the action of the matter field, which includes matter and all fields that interact with the gravitational field.

The MAG gravitational action adopted here is given by

\[
S_g = -\frac{1}{2kc} \int d^4x \sqrt{-g} R,
\] (18)

where \( k = 8\pi G/c^4 \) (\( G \) is the gravitational constant and \( c \) is the speed of light in vacuum) and \( R = R(g, T, Q) \). Using equation (16) and neglecting surface terms (which do not contribute to the field equations), we find that equation (18) reduces to

\[
S_g = -\frac{1}{2kc} \int d^4x \sqrt{-g} g^{\mu\nu} \left[ \overset{\circ}{\nabla}_\mu W^\alpha_{\rho\alpha} W^\rho_{\nu\alpha} - \overset{\circ}{\nabla}_\nu W^\alpha_{\rho\alpha} W^\rho_{\mu\alpha} \right].
\] (19)

Varying the gravitational action (19) with respect to the metric tensor, contortion tensor (which is equivalent to varying with respect to the torsion tensor) and the nonmetric part of the connection (which is equivalent to
varying with respect to the nonmetricity tensor) gives

\[ \delta S_g = -\frac{1}{2kc} \int d^4x \sqrt{-g} \left[ \left( \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} - kU_{\mu\nu} \right) \delta g^{\mu\nu} \right. \]

\[ + \left( \delta^\lambda_{[\mu} W_{\nu\rho]}^\rho + \delta^\lambda_{[\mu} W_{\rho]\nu]}^\rho - W_{[\mu
u]}^\lambda - W_{[\nu\mu]}^\lambda \right) \delta K_{\mu\nu}^\lambda \]

\[ + \left( \delta^\lambda_{(\mu} W_{\nu)}^\rho \rho + g_{\mu\nu} W^\rho_{\rho} - W_{(\mu\nu)}^\lambda - W_{(\nu\mu)}^\lambda \right) \delta N_{\lambda\mu\nu}^\rho \], \quad (20) \]

where

\[ U_{\mu\nu} = \frac{1}{k} \left[ W^\alpha_{\rho(\nu} W^\rho_{\mu)\rho} \alpha - W^\alpha_{\alpha\rho} W^\rho_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} \left( W^\alpha_{\rho\sigma} W^\rho_{\sigma\rho} \alpha - W^\alpha_{\alpha\rho} W^\rho_{\rho\sigma} \right) \right]. \quad (21) \]

The symbols ( ) and [ ] around the indices denote symmetrization and antisymmetrization, respectively. For example, \( A_{\mu\nu} = \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}) \) and \( A_{[\mu\nu]} = \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu}) \).

The MAG matter action that we consider is given by

\[ S_m = \frac{1}{c} \int d^4x \mathcal{L}_m, \quad (22) \]

where \( \mathcal{L}_m = \mathcal{L}_m (\Psi, g, T, Q) \) is the Lagrangian density of the matter field \( \Psi \). The variation of the matter action (22) with respect to the metric tensor, contortion tensor and the nonmetric part of the connection reads

\[ \delta S_m = \frac{1}{2c} \int d^4x \sqrt{-g} \left[ T_{\mu\nu} \delta g^{\mu\nu} + S^\lambda_{\mu\nu} \delta K_{\mu\nu}^\lambda + \Sigma^\lambda_{\mu\nu} \delta N_{\lambda\mu\nu}^\rho \right], \quad (23) \]

where

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \quad (24) \]

is the matter energy-momentum tensor,

\[ S^\lambda_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta K_{\mu\nu}^\lambda} \quad (25) \]

is the matter spin tensor, and

\[ \Sigma^\lambda_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta N_{\lambda\mu\nu}^\rho} \quad (26) \]
is the matter strain tensor, which can be split into a trace part (dilation)

\[ \Sigma^\lambda_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \Sigma^\lambda_{\rho}, \]  

and a traceless part (shear)

\[ \tilde{\Sigma}^\lambda_{\mu\nu} = \Sigma^\lambda_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \Sigma^\lambda_{\rho}. \]  

Note that the matter energy-momentum tensor is symmetric, the matter spin tensor is antisymmetric in its first two indices and the matter strain tensor is symmetric in its last two indices. An example of a physical matter source of MAG is the hyperfluid \cite{12}, which is a classical model of a continuous medium with energy-momentum, spin and strain. A different model of a perfect fluid with the same properties was proposed in \cite{13}.

From equations (17), (20) and (23), we find the MAG field equations

\[ \circ R^\mu_{\nu\rho\sigma} - \frac{1}{2} g_{\mu\nu} \circ R = k (T^\mu_{\nu\rho\sigma} + U^\mu_{\nu\rho\sigma}), \]  

\[ \delta^\lambda_{[\mu} W^\rho_{\nu]} = \delta^\lambda_{[\nu} W^\rho_{\mu]} - W^\lambda_{[\mu} - W^\lambda_{[\nu} = k S^\mu_{\nu\lambda}, \]  

\[ \delta^\lambda_{(\mu} W^\rho_{\nu)\lambda} + g_{\mu\nu} W^\rho_{\lambda} - W^\lambda_{(\mu} - W^\lambda_{(\nu} = k \Sigma^\lambda_{\mu\nu}. \]  

The first MAG field equations (29) are a generalization of the Einstein equations with a correction to the energy-momentum tensor from the spin and strain contributions to the spacetime geometry. The second MAG field equations (30) and the third MAG field equations (31) are algebraic relations linking spin and strain with torsion and nonmetricity. Note that even though the torsion and the nonmetricity do not propagate outside the matter, the spin and the strain of the matter influence the geometry also outside the matter through the metric tensor.

### 4 Vacuum solution

In order to obtain the vacuum solution to MAG field equations, we have to resolve the combined system of algebraic equations (30) and (31). Taking into account equations (11-13), we can rewrite equations (30-31) as

\[ T^\lambda_{\nu\mu} + 2 \delta^\lambda_{[\nu} T_{\mu]} + Q^\lambda_{[\nu\mu]} + \delta^\lambda_{[\nu} Q^\rho_{\mu]} - \delta^\lambda_{[\nu} Q_{\mu]} = k S^\lambda_{\nu\mu}. \]  

6
\[
T_{(\mu\nu)}^\lambda + \delta^\lambda_{(\mu} T_{\nu)}^\lambda - g_{\mu\nu} T^\lambda - Q^\lambda_{\mu\nu} + \frac{1}{2} \delta^\lambda_{(\mu} Q_{\nu)}^\rho + \frac{1}{2} g_{\mu\nu} Q^\lambda = k \Sigma^\lambda_{\mu\nu},
\] (33)

where \( T^\lambda = T^\rho^\lambda \) is the trace of the torsion tensor and \( Q^\lambda = Q^\lambda^\rho^\rho \) is the Weyl vector.

With some manipulation, we find that equation (32) is equivalent to
\[
T^\lambda_{\nu\mu} + Q^\lambda_{[\nu\mu]} = k \left( S^\lambda_{\mu\nu} + \delta^\lambda_{(\mu} S_{\nu)}^\rho \right), \tag{34}
\]
\[
T^\lambda_{\nu\mu} + 2T^\lambda_{[\nu\mu]} = k \left( S^\lambda_{\mu\nu} + 2S^\lambda_{[\mu\nu]} \right). \tag{35}
\]

Similarly, we can see that equation (33) is equivalent to
\[
Q^\lambda_{\mu\nu} - T_{(\mu\nu)}^\lambda - \frac{1}{3} g_{\mu\nu} T^\lambda - \frac{1}{3} \delta^\lambda_{(\mu} T_{\nu)} = k \left[ -\Sigma^\lambda_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left( \Sigma^\lambda_{\rho} - \frac{2}{3} \Sigma^\rho^\lambda_{\rho} \right) \right. \\
\left. + \frac{2}{3} \delta^\lambda_{(\mu} \Sigma_{\nu)}^\rho \right], \tag{36}
\]
\[
Q^\lambda_{\mu\nu} + 2Q_{(\mu\nu)}^\lambda - \frac{2}{3} g_{\mu\nu} T^\lambda - \frac{4}{3} \delta^\lambda_{(\mu} T_{\nu)} = k \left[ -\Sigma^\lambda_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left( \Sigma^\lambda_{\rho} - \frac{2}{3} \Sigma^\rho^\lambda_{\rho} \right) \right. \\
\left. + \frac{2}{3} \delta^\lambda_{(\mu} \Sigma_{\nu)}^\rho + \delta^\lambda_{(\mu} \Sigma_{\nu)}^\rho - 2 \Sigma_{(\mu\nu)}^\lambda \right]. \tag{37}
\]

In vacuum, where \( T_{\mu\nu} = S^\lambda_{\mu\nu} = \Sigma^\lambda_{\mu\nu} = 0 \), equations (34-37) reduces to
\[
T^\lambda_{\nu\mu} + Q^\lambda_{[\nu\mu]} = 0, \tag{38}
\]
\[
T^\lambda_{\nu\mu} + 2T^\lambda_{[\nu\mu]} = 0, \tag{39}
\]
\[
Q^\lambda_{\mu\nu} - T_{(\mu\nu)}^\lambda - \frac{1}{3} g_{\mu\nu} T^\lambda - \frac{1}{3} \delta^\lambda_{(\mu} T_{\nu)} = 0, \tag{40}
\]
\[
Q^\lambda_{\mu\nu} + 2Q_{(\mu\nu)}^\lambda - \frac{2}{3} g_{\mu\nu} T^\lambda - \frac{4}{3} \delta^\lambda_{(\mu} T_{\nu)} = 0. \tag{41}
\]

By contracting the \( \lambda\nu \) indices in equations (38-41), we obtain
\[
T_\mu + \frac{1}{2} Q^\rho_{\rho\mu} - \frac{1}{2} Q_\mu = 0, \tag{42}
\]
\[
Q^\rho_{\rho\mu} - \frac{2}{3} T_\mu = 0, \tag{43}
\]
\[
2Q^\rho_{\rho\mu} + Q_\mu - 4T_\mu = 0. \tag{44}
\]
whose only solution is

\[ T_\mu = Q_\mu = Q^\rho_{\rho\mu} = 0. \] (45)

Thus equations (40-41) reduces to

\[ Q^\lambda_{\mu\nu} - T_{(\mu\nu)}^\lambda = 0, \] (46)

\[ Q^\lambda_{\mu\nu} + 2Q_{(\mu\nu)}^\lambda = 0. \] (47)

Combining equations (38-39) and equations (46-47) gives

\[ T^\lambda_{\mu\nu} = Q^\lambda_{\mu\nu} = 0, \] (48)

and, consequently, \( U_{\mu\nu} = 0 \). Therefore, in vacuum, the first MAG field equations (29) reduces to the Einstein equations

\[ \overset{\circ}{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \overset{\circ}{R} = 0. \] (49)

Thus we conclude that MAG is equivalent to GR in vacuum.

5 Final remarks

In this article, we proved that is possible obtain a physical MAG model by the variational principle with the scalar curvature as the gravitational Lagrangian. In most MAG approaches, the gravitational action and the matter action are varied with respect to the metric and the connection. However, the variation of the total action (with the scalar curvature as the gravitational Lagrangian) with respect to the connection constrains possible forms of matter that the theory can describe leading to inconsistent field equations. We solved this problem by varying the gravitational action construct out of the scalar curvature and the matter action with respect to the torsion and the nonmetricity instead of the connection. As a result we have obtained consistent field equations which reduces to Einstein equations in vacuum. It should be important to investigate the solutions of these field equations in matter with spin and strain. This question is under consideration now.

Acknowledgments

The author would like to thank J. B. Fromiga for useful discussions.
References

[1] E. Cartan, C. R. Acad. Sci. (Paris) 174, 593 (1922).

[2] H. Weyl, Sitzungsber. Königl. Preuss. Akad. Wiss., 465 (1918).

[3] F. W. Hehl, G. D. Kerlick and P. Von Der Heyde, Z. Naturf. 31A, 524 (1976).

[4] E. Kröner, “Continuum theory of defects”, in Physics of Defects, edited by R. Balian et al. (North-Holland, Amsterdam, 1981).

[5] A.D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967).

[6] R. M. Wald, General Relativity (The University of Chicago Press, Chicago, 1984).

[7] F. W. Hehl, E. A. Lord and L. L. Smalley, Gen. Rel. Grav. 13, 1037 (1981).

[8] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne’eman, Found. Phys. 19, 1075 (1989).

[9] A. H. Guth, Phys. Rev. D 23, 347 (1981); A. Linde, Phys. Lett. B 108, 389 (1982).

[10] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B 158, 316 (1985); C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. 262, 593 (1985).

[11] J.A. Schouten, Ricci-Calculus (North-Holland, Amsterdam, 1954).

[12] Y.N. Obukhov and R. Tresguerres, Phys. Lett. A 184, 17 (1993).

[13] L.L. Smalley and J.P. Krisch, J. Math. Phys. 36, 778 (1995).