HOMOTOPY INVARIANTS AND TIME EVOLUTION
IN (2+1)-DIMENSIONAL GRAVITY *

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Abstract: We establish the relation between the $ISO(2,1)$ homotopy invariants, and the polygon representation of $(2+1)$-dimensional gravity. The polygon closure conditions, together with the $SO(2,1)$ cycle conditions, are equivalent to the $ISO(2,1)$ cycle conditions for the representations $\rho : \pi_1(\Sigma_{g,N}) \to ISO(2,1)$. Also, the symplectic structure on the space of invariants is closely related to that of the polygon representation. We choose one of the polygon variables as internal time and compute the Hamiltonian, then perform the Hamilton-Jacobi transformation explicitly. We make contact with other authors’ results for $g = 1$ and $g = 2$ ($N = 0$).

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1. Introduction

This article is the first of a set of two, which pursues the ultimate goal of finding non-perturbative solutions of quantum gravity in $2 + 1$ spacetime dimensions. The classical theory will be reviewed here, with an emphasis on the issue of time and dynamical evolution.

In one approach to quantum gravity, one focuses on finding a complete set of gauge invariants, or “observables”, which span the so-called “reduced phase space”. One then postulates that the wave function is a square integrable function on the reduced configuration space and promotes the canonical brackets to commutators. Since the “gaUGE symmetries” of gravity include translations in time, all observables must be constants of the motion. This leads to one aspect of the problem of time in quantum gravity: How does one formulate dynamical questions in such a reduced phase space and, do the answers to such questions depend on the choice of time which one makes in order to formulate them?

Following Witten’s work [1], many authors have recently turned to the toy model of (2+1)-dimensional gravity to investigate this and other longstanding problems of quantum gravity in a simpler setting [2]. Witten found that the spacetimes with topology $\Sigma_g \times [0, 1]$ can be labeled by representations of the fundamental group $\pi_1(\Sigma_g) \to ISO(2, 1)$. The $ISO(2, 1)$ scalars derived from elements of such a representation, or homotopy invariants, form an over-complete set of phase space observables, which are related by non-linear constraints. To complete Witten’s programme at the classical level, one must solve these constraints to identify the reduced phase space explicitly, and solve the dynamical problem.

The task of solving the constraints and deriving the reduced phase space was accomplished by Urrutia and one of us (F.Z.) for genus one [3], based on the work of Nelson, Regge and F.Z. [4]. Nelson and Regge have since found the reduced phase space in the case of $SO(2, 2)$ for genus 2 [5], and implicitly for $g > 2$ [6].

Although the homotopy invariants provide a complete phase space with only constants of the motion, that does not mean that there is no dynamics in (2 + 1)-dimensional gravity: as Moncrief has emphasized [7], the observables should be interpreted as Hamilton-Jacobi variables [8], which are related to the initial data for the dynamical system. Moncrief showed that the homotopy invariants can be written in relation to the ADM variables and the extrinsic curvature time $K = t$ [9]. By inverting this relation, one obtains the ADM variables as functions of the homotopy invariants and time. He carried out this procedure in the case of genus one, thereby solving the time evolution problem in that case. Moncrief’s work for $g = 1$ was carried over to the quantum theory by Carlip [10] and Anderson [11], with different orderings leading to different quantum theories.

To pursue this program beyond genus one, we turn to a different formulation of (2 + 1)-dimensional gravity. The explicit solution of the time-evolution problem for genus g and N particles was given by one of us (HW) in terms of time-dependent global variables, which define a polygon representation of spacetime. Since these variables evolve in time, they are not observable in the same sense as the homotopy invariants, yet they are what one would intuitively call “observable”, in the sense that they can be measured by an observer [12]. Carlip [13] has suggested that the variables of the polygon representation are related to the $ISO(2, 1)$ homotopy invariants, yet it has not been clear, until now, exactly how.

In this article, we will construct a representation of $\pi_1(\Sigma_g) \to ISO(2, 1)$ explicitly as a function of the polygon variables. We will then choose an internal time, compute the Hamil-
tonian and perform the Hamilton-Jacobi transformation. This work is prerequisite to the discussion of quantum dynamics in (2+1)-dimensional gravity, which we leave to the following article.

The next two sections are devoted to a review of the Witten formalism, and of the polygon representation. The two formalisms are compared in Sec. 4. We choose an internal time in Sec. 5, compute the corresponding Hamiltonian, then perform the Hamilton-Jacobi transformation in Sec. 6. Two examples (genus one and genus two) are worked out in Sec. 7, where we make contact with the work of other authors.
2. ISO(2,1) Homotopy Invariants

Three-dimensional spacetimes are flat everywhere the source term vanishes, as a consequence of Einstein’s equations [14]: any open ball with trivial topology and no sources is isomorphic to an open ball in Minkowski space. The spacetime itself is isomorphic to a subset of (2+1)-dimensional Minkowski space, with boundary points identified by elements of a subgroup $\Gamma \subset ISO(2,1)$. This subset is a cylinder with polygonal base, which is the basis of the “polygon representation”. Since (2 + 1)-dimensional spacetimes are flat, the holonomy maps $\pi_1(\Sigma_{g,N})$ into a subgroup of $ISO(2,1)$: this subgroup is precisely $\Gamma$, as we will see below. For their relation to the first homotopy group, we will refer to elements of $\Gamma$ as “$ISO(2,1)$ homotopies” in what follows. Superspace is the space of these subgroups $\Gamma$ modulo $ISO(2,1)$ conjugacy; for a genus $g$ surface with $N=0$ punctures it is locally isomorphic to the $(12g-12)$-dimensional cotangent bundle of Teichmüller space [15].

We restrict our discussion in this section to $N = 0$, for simplicity; the extension to $N \neq 0$ will be considered later on. The fundamental group $\pi_1(\Sigma_g)$ is finite-generated: One can choose a basis of $2g$ loops $\{u_i, v_i, i = 1, ..., g\}$ of $\pi_1(\Sigma_g)$, where each $u_i$ intersects $v_i$ once (Fig. 2.1). We will use the characters $M_i$ and $E_i$, respectively, for the $SO(2,1)$ and $T(2,1)$ projections of the homotopies $\rho(u_i)$:

$$\rho(u_i) = \begin{pmatrix} M_i & E_i \\ 0 & 1 \end{pmatrix}$$

(2.1)

There is a symplectic structure on the space of $ISO(2,1)$ homotopies, inherited from the brackets of the $ISO(2,1)$ connection field, which derive from the Chern-Simons action [1,16]. The equivalence of sets of homotopies by $ISO(2,1)$ conjugacy represents a global Poincaré symmetry which is generated, in the Hamiltonian formulation, by six first-class constraints: the so-called “cycle conditions” (Eqn. (4.13) below). Specifically, the global Poincaré transformations are the usual Lorentz frame transformations, together with the translations of the base-point $O$ of the homotopy group. If one thinks of this point as the “observer”, then timelike translations of $O$ should be interpreted as “time evolution”. Observables are Poincaré invariants and can be constructed by taking any products of homotopies $\rho(u_i) \subset \Gamma$ and constructing the two invariants $L_1$ and $L_2$ as follows [17], [3]:

$$L_1 = tr(M)$$

(2.2)

$$L_2 = P \cdot E$$

(2.3)

where

$$P^a = \frac{1}{2} \epsilon^{abc} M_{cb}$$

(2.4)

These invariants, for all loops in $\pi_1(\Sigma_g)$, are Witten’s observables. They are clearly not all independent, since the fundamental group of $\Sigma_g$ is finite generated. As Nelson and Regge showed, one can construct a finite but still redundant set of invariants which are related by given non-linear constraints. One then solves these constraints and finds an ideal of the constraint...
algebra; the symplectic structure on this ideal is inherited from the symplectic structure on the space of ISO(2, 1) matrices, leading to a true reduced phase space. This program is difficult to carry out explicitly, but has been completed for genus two by Nelson and Regge [5]. The group ISO(2, 1) can be replaced in all the above by SO(3, 1), SO(2, 2) and their supersymmetric generalizations [4], [18]; the reduced phase space differs in each case.

In this approach, one is solving all of the constraints explicitly, and seeking a reduced phase space spanned by variables that are invariant under the ISO(2, 1) transformations, which include time translations. How can one formulate dynamical questions in this picture? To see what has happened, we will consider the elementary example of a free non-relativistic particle, in parametrized form:

\[ \{P_i, X_j\} = \delta_{ij} \]  
\[ \{P_T, T\} = 1 \]  
\[ P_T + \frac{p^2}{2m} \approx 0 \]

The Hamiltonian is a linear combination of the constraint with an arbitrary parameter, which determines the “time” parametrization, and can be any function on phase space, \( \lambda(x_i, p_i, T, P_T; t) \). The “weak equality” \( \approx \) used for the constraint reminds one that the equality cannot be used inside a Poisson bracket, since its bracket with other variables is not zero (it is the generator of time translations) [19].

\[ H \equiv \lambda(P_T + \frac{p^2}{2m}) \approx 0 \]  
\[ \frac{dT}{dt} = \{H, T\} \approx \lambda \]  
\[ \frac{dx_i}{dt} = \{H, x_i\} \approx \lambda \frac{p_i}{m} \]  
\[ \frac{dp_i}{dt} \approx 0 \]

Combining (2.9) and (2.10) gives the more familiar result

\[ \frac{dx_i}{dT} \approx \frac{p_i}{m} \]

Following the Witten-Nelson-Regge approach described above, one would construct a maximum set of independent “gauge” invariants. The dimension of the reduced phase space is equal to the number of variables minus the number of constraints, minus the number of symmetries, in this case six for \( x_i \) and \( p_i \), plus two for \( T \) and \( P_T \), minus two for the Hamiltonian

5
constraint and the corresponding symmetry. Any set of six independent invariants can be chosen; a simple choice could be the following:

\[ P_i = p_i \]  
\[ X_i = x_i - \frac{p_i}{m} T \]  

The reduced system inherits the following brackets and Hamiltonian

\[ \{X_i, P_j\} = \delta_{ij} \]  
\[ H = 0 \]

All observables are constants of the motion.

Of course (2.13)-(2.16) is just the Hamilton-Jacobi formalism for a free particle. \( X_i \) is the initial value of the dynamical variable \( x_i \); the time evolution is given by the way in which \( T \) appears explicitly in the relation (1.14), namely by inverting it:

\[ x_i(T) = X_i + \frac{p_i}{m} T \]

Thus, although the variables \( X_i, P_i \) are a complete set of labels for the trajectories, knowing these variables is not sufficient to answer dynamical questions.

We pause here to comment on the difference between “initial conditions” and “dynamical invariants”. While initial conditions can only be measured on the initial slice, dynamical invariants can be measured anywhere along the history of the system. In this sense, the knowledge of dynamical invariants (energy, angular momentum, etc.) provides important information about the dynamics. Since the homotopy invariants are independent of the base-point \( O \) of \( \pi_1(\Sigma_g) \), they are not attached to any “initial surface” and one would naturally think that they are dynamical invariants, rather than initial conditions. Having found as many invariants as there are phase space dimensions, one would conclude that (2 + 1)-dimensional gravity is static. The reason why this is not the case, is that in the parametrized formalism “time” is one of the dynamical variables which can be measured anywhere along the trajectory of the system, therefore initial conditions, such as \( x_i(T) - (p_i/m)T \), can be measured by observers away from the initial slice. So the difference between “initial conditions” and “dynamical invariants” does not exist in the parametrized formalism, until one has specified a choice of internal time. We will see later that when “time” is taken to be the size of a loop of \( \pi_1(\Sigma_g) \), some of the invariants appear as initial conditions, much as \( X_i(T) \) in Eqn. (2.14), while others are independent of \( T \) and therefore are dynamical invariants. Which observables are initial conditions, and which are dynamical invariants, depends on the choice of internal time. Since (2 + 1)-dimensional gravity is dynamical, there cannot be an internal time where all of the homotopy invariants are dynamical invariants. Nevertheless, the knowledge of a complete set of invariants for (2 + 1)-dimensional gravity has proven to be extremely valuable in understanding the physical content of the theory.
3. The Polygon Representation

The solution of the time-evolution problem of $(2 + 1)$-dimensional gravity, for spacetimes with the topology $\Sigma g,N \times (0,1)$ was originally derived [12] from an exact lattice theory [20]. This solution gives a representation of spacetime by means of a polygon embedded in Minkowski space, with identifications of boundary points by elements of $\Gamma \subset ISO(2,1)$. A one-to-one map between this set of polygons and the flat spacetimes was given in [21]. We will first review the constrained Hamiltonian system, then explain how it is related to $(2 + 1)$-dimensional gravity. Later in this article, we will show that the polygon representation is closely related to the $ISO(2,1)$ Chern–Simons theory, in particular, the Poisson brackets of the homotopy invariants will be recovered from the brackets (3.1) and (3.2).

The phase space variables are three-vectors $E(\mu), \mu = 1, \ldots, 2g + N$, and Lorentz matrices in $2 + 1$ dimensions, $M(\mu)$, with the following Poisson brackets.

$$\{E^a(\mu), E^b(\mu)\} = \epsilon^{abc} E_c(\mu) \quad (3.1)$$

$$\{E^a(\mu), M^b_{\,c}(\mu)\} = \epsilon^{abd} M_{dc}(\mu) \quad (3.2)$$

All other brackets are zero, in particular any brackets of variables with different loop indices ($\nu \neq \mu$) vanish, and so do all brackets of $M$'s among themselves. The dynamics and gauge symmetries are generated by six first-class constraints $J^a \approx 0$ and $P^a \approx 0$, which have a Poincaré algebra with the brackets given above.

$$J \equiv \sum_{\mu} (I - M^{-1}(\mu)) E(\mu) \approx 0 \quad (3.3)$$

$$P^a \equiv \frac{1}{2} \epsilon^{abc} W_{cb} \approx 0 \quad (3.4)$$

where

$$W = \left( M(1)M^{-1}(2)M^{-1}(1)M(2) \right) \cdots \left( M(2g - 1)M^{-1}(2g) \right) M^{-1}(2g - 1)M(2g) \cdots M(2g + N) \quad (3.5)$$

and the constraints which define the rest-mass of each point source,

$$H(\mu) \equiv P^2(\mu) + \sin^2 (\Omega(\mu)) \approx 0 \quad (3.6)$$

($\Omega(\mu) = 4\pi Gm(\mu); \mu = 2g + 1, \ldots, 2g + N$). The condition $W \approx I$, related to the “cycle condition” of Riemann surfaces, must be satisfied by the generators of any representation of the fundamental group, $\pi_1(\Sigma_g)$. We will consider only faithful representations in $SO(2,1)$. In this way, we select among the various sheets of solutions to (3.5) the one with maximal
Euler class, which gives physically acceptable spacetimes: solutions on other sheets represent spacetimes with large “negative mass” singularities, \( \Omega_n = \Omega - 4\pi n/G \) [1], [12].

The first constraint has an important geometrical interpretation: The vectors \( E(\mu) \), and their images \( M^{-1}(\mu)E(\mu) \) under the corresponding \( SO(2,1) \) matrices, form a closed polygonal contour in Minkowski space. Because of this geometrical interpretation, we refer to the constrained system (3.1)- (3.6) as the “polygon representation”. We will require that the three-vectors \( E(\mu) \) and all diagonals of the polygon are spacelike; this is necessary and sufficient for the spacetime to admit a spacelike foliation locally in time [21] (it cannot admit a spacelike foliation for all \( t \) [22]).

The Hamiltonian for this parametrized system is a linear combination of the constraints. The constraints \( J^a \approx 0 \) generate Lorentz transformations; the constraints \( P^a \approx 0 \) are the energy-momentum constraints and the corresponding parameters in the Hamiltonian are the “lapse-shift functions”; their geometrical interpretation will be made clear shortly. If one chooses a fixed frame, the Hamiltonian constraint becomes

\[
H \equiv \sum_a N_a P^a + \sum_\mu N(\mu)H(\mu) \approx 0 \quad (3.7)
\]

The matrices \( M(\mu) \) are constants of the motion, since \( H \) depends only on \( M \)'s and brackets among \( M \)'s are zero; on the other hand, the time derivative of a three-vector \( E(\mu) \) is a function of \( M \)'s, and therefore a constant of the motion. Thus,

\[
\frac{dM(\mu)}{dt} = 0 \quad (3.8)
\]

\[
\frac{d^2E(\mu)}{dt^2} = 0 \quad (3.9)
\]

These variables represent a spacetime which we will construct in a moment. First note that the edges of the polygon (3.3) come in pairs, with the elements of a pair identified by the corresponding matrix (for example, \( E(1) \) and \( -M^{-1}(1)E(1) \)). Let \( N \) be a timelike three-vector at one corner of the polygon. This corner belongs to two edges, each of which is identified to another one. Following these identifications, the corner is mapped to two other corners and \( N \) to two timelike vectors (such as \( M^{-1}(1)N \)). These corners and timelike vectors each belong to one other edge, which is identified to another edge of the polygon, etc. In this way each corner of the polygon is endowed with a timelike three-vector, which is the image of \( N \) under the corresponding identification. This procedure terminates when an identification takes one back to a corner of the polygon which had already been endowed with a timelike vector. This last identification returns the same timelike three-vector as was already there, if and only if the \( SO(2,1) \) cycle condition (3.4) holds. A corner which represents the location of a point source is not identified to any other, but lies at the intersection of two identified edges. The Lorentz matrix which identifies these two edges is elliptic (a rotation); the axis of this rotation is proportional to the energy-momentum three-vector of the particle and the angle is related to its mass. We place a timelike line parallel to this axis at each such corner. We will show next how to construct the spacetime from the initial polygon and these timelike lines.

For concreteness, consider the example of genus one. The polygon is a quadrilateral figure in Minkowski space, \( ABCD \) [Figure 3.1]. The edge \( AB \), represented by the three-vector \( E(1) \),
is mapped to $DC$ by the matrix $M^{-1}(1)$, and $BC$ to $AD$ by $M^{-1}(2)$. A vector $N$ at $A$ becomes $M^{-1}(1)N$ at $D$, following the identification $AB \rightarrow DC$, and $M(2)N$ at $B$ following $BC \rightarrow AD$. From $D$, the identification $BC$, $AD$ leads to $M(2)M^{-1}(1)N$ at $C$, and from $B$ the identification $AB \rightarrow DC$ leads to $M^{-1}(1)M(2)N$ at $C$. The consistency condition is that the two ways of carrying $N$ to $C$ give the same result, in this case that $M(2)M^{-1}(1)N = M^{-1}(1)M(2)N$. This condition is equivalent to $M(1)M^{-1}(2)M^{-1}(1)M(2) = I$, which is precisely the cycle condition for genus one.

Given the polygon, $p(0)$, and a timelike line at each corner, one constructs the spacetime as follows. One first constructs a family of polygons $p(t)$ by sliding each corner of the polygon along the corresponding timelike line, for a proper time $t$. For small enough $t_0$, the region of Minkowski space which lies inside these polygons for $0 \leq t \leq t_0$, is a truncated cylinder with polygonal base; let its timelike walls be identified in pairs, for instance the wall $E(1; t)$ is identified with the wall $M^{-1}(1)E(1; t)$. In this way one constructs a three-manifold with topology $\Sigma_{g, N} \times (0, 1)$ and no curvature, i.e. a solution of $(2 + 1)$-dimensional gravity from $t = 0$ to $t = t_0$. It can be shown that all solutions of $(2 + 1)$-dimensional gravity, with the topology and sources considered here, can be obtained in this way, and that the observable properties of the spacetime constructed are independent of the choice of $N$ [21].

In the example of genus one, the spacetime is defined by the worldlines of the four corners, $A(t)$, $B(t)$, $C(t)$, $D(t)$, $0 \leq t \leq t_0$. Since the edges and diagonals are spacelike, by hypothesis, there exists a spacelike surface which includes all four points at a given $t$; it can further be shown that one can choose a surface $\sigma_t(t)$ that is differentiable at the identified edges [21]. Note that the choice of surface $\sigma_t(t)$ is not an observable property of this spacetime. On the other hand, the three-vector $E(1; t)$ is observable: If an observer at $A$ were to extend a straight stick of length $\|E(1; t)\|$, in the direction of $E(1; t)$, the two ends of the stick would touch. To measure $M(1)$, the observer can send a friend with a $(2 + 1)$-dimensional “gyroscope” around the same loop and measure its “rotation” when he returns from travelling around the loop. We emphasize that the polygon variables are not associated to any particular choice of slicing of the spacetime.

If one considers one handle of a genus $g$ solution, the velocities can be written in closed form as follows. Let $\mu = 1, 2$ be the two loops corresponding to this handle. From the brackets (3.1-3.2) and the constraints (3.4) one finds, using the $SO(2, 1)$-invariance of the structure constants $\epsilon^{abc}$,

$$\frac{dE(1)}{dt} \approx \frac{1}{2} \left( M(1)M(2)M^{-1}(1) - I \right)N \quad (3.10)$$

$$\frac{dE(2)}{dt} \approx \frac{1}{2} \left( M(2)M^{-1}(1) - M(1)M(2)M^{-1}(1) \right)N \quad (3.11)$$

This result can also be derived directly from the geometrical picture which we have introduced above. The right hand side of equation (3.10) is the difference between the timelike image of $N$ at the tip of the segment $E(1)$ and the vector $N$, which we have chosen to be based at the origin of $E(1)$. Thus, the constrained Hamiltonian system generates precisely the dynamical evolution which we have given by constructing the spacetime from a polygon and identifications.

The picture which we have just presented, with the phase space variables $E(\mu; t)$ and $M(\mu)$, looks like an ordinary classical system in parametrized form; in fact if it were not for
the constraints (3.4) and the mapping class group symmetry (related to the arbitrary choice
of $2g + N$ generators of the homotopy group, [23]), it would be closely related to a system
$2g + N$ free particles. We will discuss next the relation between this formulations and the
$ISO(2,1)$ homotopy invariants. We will first give the algebraic relation between the homotopies
and the $E - M$ variables, then in sections 5 and 6 we will find the time-dependent canonical
transformation which relates Witten’s invariant formalism to the polygon representation.
4. ISO(2,1) Homotopies from the Polygon Variables

A spacetime with topology $\Sigma_g \times (0, 1)$ can be represented by the ISO(2,1) homotopies for $2g$ basis loops $u_i, v_i$, as explained in Sec. 2. The same spacetime can be represented by a polygon embedded in (2+1)-dimensional Minkowski space, as we saw in Sec. 3. One handle of a slice of the spacetime is represented by four edges of the polygon. For the first handle, the edges are [Figure 4.1]: $E(1), E(2), M^{-1}(1)E(1)$, and $M^{-1}(2)E(2)$. In this representation, the loops $u_1$ and $v_1$ are represented as follows. The loop $u_1$ is the path from an arbitrary base point $O$ to a point $X$ on the edge $E(1)$, continuing through the identified point $X'$ on the edge $M^{-1}(1)E(1)$ to close the loop at $O$. Similarly, $v_1$ is a path from $O$ to $Y$ on the edge $-M^{-1}(2)E(2)$, continuing from the identified point $Y'$ on the edge $E(2)$ to $O$ (the reversed orientation for $v_1$ is conventional).

Having defined the loops $u_1, v_1$, it is now a straightforward task to construct a representation of $\pi_1(\Sigma_g) \to ISO(2,1)$. We begin with the homotopy $\rho(u_1)$. This is a Poincaré transformation, which can be represented as a four-by-four matrix as (Sec. 2)

$$\rho(u_1) = \begin{pmatrix} M_1 & E_1 \\ 0 & 1 \end{pmatrix} \quad (4.1)$$

The Lorentz projection of this homotopy, $M_1$, is the matrix for parallel-transport along the loop back to the base-point $O$; for the loop $u_1$ this is the matrix $M(1)$. The translation component $E_1$ is the path integral of the dreibein, parallel-transported back along the path to $O$. This is also the displacement of an observer which travels once around the loop, expressed in the frame at the base-point. In the case of the loop $u_1$, this is the displacement $OX + M(1)X'O$, where $X$ is the point at which the loop crosses the edge $E(1)$ and $X'$ is the identified point on the edge $-M^{-1}(1)E(1)$. If we denote by $A$ the origin of the vector $E(1)$, this displacement can be written as

$$E = OA + AX + M(1)(X'D + DO)$$
$$= \left( I - M(1) \right) OA + M(1) \left( M^{-1}(1)E(1) - E(1) - E(2) \right) \quad (4.2)$$

Thus, the ISO(2,1) element for the loop $u_1$ based at $O$, is

$$\rho(u_1) = \begin{pmatrix} M(1) & (I - M(1))E(1) - M(1)E(2) + (I - M(1))OA \\ 0 & 1 \end{pmatrix} \quad (4.3)$$

Similarly, one finds

$$\rho(v_1) = \begin{pmatrix} M^{-1}(2) & (I - M^{-1}(1) - M^{-1}(2))E(1) + (I - M^{-1}(2))E(2) + (I - M^{-1}(2))OA \\ 0 & 1 \end{pmatrix} \quad (4.4)$$

Each such ISO(2,1) matrix leads to two Poincaré invariants: the trace, $M^a_a$, and the scalar product of the translation component with the "rotation vector", $P^a = (1/2)\epsilon^{abc}M_{cb}$. For a genus $g$ universe, there are $2g$ loops, $u_i$ and $v_i$. Taking all possible products of the
corresponding ISO(2, 1) matrices and constructing the corresponding invariants, one obtains an infinite set of invariants. Clearly they are not all independent: for instance, \(tr(M)\) and \(tr(M^2)\) are related by a Cayley-Hamilton-like identity [4]. In general, one can find the complete set of relations among the different invariants, and attempt to solve these constraints explicitly in order to identify the reduced phase space. In practice, this task is very difficult [4], [5], [6], and has yet to be completed explicitly in the general case (the number of independent invariants is \(12g - 12\), the dimension of the reduced phase space). We will return to this problem in the case of genus two, where the reduced phase space is known explicitly [5].

For the loop \(u_1v_1\), the two Poincaré invariants are

\[
L_1(u_1v_1) = tr(M(2)M^{-1}(1)) - 1 \tag{4.5}
\]

\[
L_2(u_1v_1) = E(1) \cdot P(2\mathbf{I}) + E(2) \cdot P(2\mathbf{I}) \tag{4.6}
\]

where

\[
P^a(2\mathbf{I}) = \frac{1}{2} \epsilon^{abc}(M(2)M^{-1}(1))_{cb} \tag{4.7}
\]

One may check that this is a constant of the motion by using the expressions (3.10), (3.11) for the velocities:

\[
\frac{dL_2(u_1v_1)}{dt} = P(2\mathbf{I}) \cdot \frac{d(E(1) + E(2))}{dt} = P(2\mathbf{I}) \cdot \frac{(M(2)M^{-1}(1) - I)N}{2} = 0 \tag{4.8}
\]

since \(P(2\mathbf{I})\) is invariant under \(M(2)M^{-1}(1)\).

The ISO(2, 1) homotopy for a loop which surrounds a particle is computed as follows. Let \(w_1\) be the loop which goes from \(O\), to a point \(X\) on the edge \(E(2g + 1)\), then back to \(O\) through the corresponding point \(X'\) on the identified edge \(M^{-1}(2g + 1)E(2g + 1)\). Proceeding as above, one obtains the homotopy

\[
\rho(w_1) = \begin{pmatrix} M(2g + 1) & (I - M(2g + 1))E(2g + 1) \\ 0 & (I - M(2g + 1))OF \end{pmatrix} \tag{4.9}
\]

where \(F\) is the base point of the three-vector \(E(2g + 1)\). Note that since \(A\) was defined as the base point of \(E(1)\), and the vectors \(E(1), E(2), -M^{-1}(1)E(1), \ldots, -M^{-1}(2)E(2)\) represent the section of polygon which lies between \(A\) and \(F\), one has

\[
OF = OA + AF \tag{4.10}
\]

\[
AF = (I - M^{-1}(1)E(1) + \cdots + (I - M^{-1}(2g))E(2g) \tag{4.11}
\]

12
\[ J(1) + J(2) + \cdots + J(2g) \]  
where \( J(\mu) = E(\mu) - M^{-1}(\mu)E(\mu) \).

From the geometric picture in which we have represented both the ISO(2, 1) homotopies and the \((E - M)\) variables, one expects that the cycle condition for the ISO(2, 1) homotopies,

\[
(\rho(u_1)\rho(v_1)\rho(u_1^{-1})\rho(v_1^{-1})) \cdots (\rho(u_g)\rho(v_g)) \\
\rho(u_g^{-1})\rho(v_g^{-1})\rho(w_1)\rho(w_2) \cdots \rho(w_N) \approx I
\]  
would be equivalent to the six Poincaré constraints,

\[
J \equiv \sum_{\mu} (I - M^{-1}(\mu))E(\mu) \approx 0
\]  

To check this explicitly, one simply computes the product of ISO(2, 1) matrices in the order (4.13), using the relations (4.3), (4.4), etc. This takes a bit of work, but the result is simple: one finds

\[
\rho(u_1)\rho(v_1)\rho(u_1^{-1})\rho(v_1^{-1}) \cdots \rho(w_N) = \begin{pmatrix} W & -WJ + (I - W)OA \\ 0 & 1 \end{pmatrix}
\]  

where \( W \) is the SO(2, 1) “cycle”:

\[
W = \left( M(1)M^{-1}(2)M^{-1}(1)M(2) \right) \cdots \left( M(2g - 1)M^{-1}(2g) \right) \\
M^{-1}(2g - 1)M(2g) M(2g + 1)M(2g + 2) \cdots M(2g + N)
\]  

In the next sections, we will choose a definition of internal time, compute the corresponding Hamiltonian, then carry out the Hamilton-Jacobi transformation explicitly.
5. Time

Since the variables \(E(\mu; \tau)\) depend linearly on the evolution parameter \(\tau\), a natural choice for an internal time variable, or “clock”, is any one component of such a vector, such as \(E^x(1)\). We will consider separately the closed universes with \(N = 0, g \geq 2\), then the closed and open universes with \(N \geq 2, g \geq 0\). The other cases can be treated in a similar fashion, except for the torus \((N = 0, g = 1)\) which will be discussed in Sec. 7.1.

5.1 Time and the Hamiltonian for Empty Genus \(g\) Universes

For \(N = 0\), the \(SO(2, 1)\) projections of the \(ISO(2, 1)\) homotopies generate a Fuchsian group [24]; this implies that the \(SO(2, 1)\) matrices \(M(1), \cdots\) are boosts. Therefore, one can choose a frame where \(P(1)\) is parallel to the \(x\)-axis.

\[
P^t(1) = 0 \quad (5.1)
\]

\[
P^y(1) = 0 \quad (5.2)
\]

and solve these simultaneously with the corresponding constraints,

\[
J^t = 0 \quad (5.3)
\]

\[
J^y = 0 \quad (5.4)
\]

In this gauge, \(M(1)\) is a pure boost in the \(y - t\)-plane,

\[
M(1) = \begin{pmatrix}
\cosh(b) & 0 & \sinh(b) \\
0 & 1 & 0 \\
\sinh(b) & 0 & \cosh(b)
\end{pmatrix} \quad (5.5)
\]

and the constraints \(J^t = J^y = 0\),

\[
J \equiv \left( I - M^{-1}(1) \right) E(1) + \sum_{\mu > 1} \left( I - M^{-1}(\mu) \right) E(\mu) \approx 0 \quad (5.6)
\]

split into the \(x\)-component, for which the first term vanishes thanks to (5.5),

\[
J^x \equiv \sum_{\mu > 1} \left( I - M^{-1}(\mu) \right)^{x} E^a(\mu) \approx 0 \quad (5.7)
\]

and two equations which we solve for \(E^t(1)\) and \(E^y(1)\):

\[
\begin{pmatrix}
E^t(1) \\
E^y(1)
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2} & \frac{\sinh(b)}{2(1 - \cosh(b))} \\
\frac{\sinh(b)}{2(1 - \cosh(b))} & -\frac{1}{2}
\end{pmatrix} \begin{pmatrix}
J^t_1 \\
J^y_1
\end{pmatrix} \quad (5.8)
\]
where

\[ J_1 \equiv \sum_{\mu>1} (I - M^{-1}(\mu))E(\mu) \] (5.9)

With this choice of frame, the component \( E^x(1) \), which we would like to use as a clock, is canonically conjugate to the boost parameter \( b \). Indeed,

\[ \{E^a(\mu), P^b(\mu)\} = \frac{1}{2} \left( \eta^{ab}trM(\mu) - M^{ba}(\mu) \right) \] (5.10)

so that, for \( a = b = x \) and \( \mu = 1 \), one finds

\[ \{E^x(1), P^x(1)\} = \cosh(b) \] (5.11)

or, using \( P^x(1) = \sinh(b) \),

\[ \{E^x(1), b\} = 1 \] (5.12)

\( P^x(1) \) is not bounded from below, but if we choose as internal time

\[ T = -\frac{E^x(1)}{P^x(1)} \] (5.13)

then the conjugate variable is \( P_T = -\cosh(b) \) and the Hamiltonian is

\[ H = -P_T = \frac{1}{2} \left( trM(1) - 1 \right) \] (5.14)

which is bounded from below. This is of course one of many possible choices for the time/hamiltonian pair. In \((2 + 1)\)-dimensional gravity there is no outstanding choice of “clock”. In the classical theory all choices of time lead to the same equations of motion, but this is not generally true in the quantum theory [25], [26]; we will discuss this point further in the second article.

The Hamiltonian (5.14) is given in terms of the variables \( M(\mu), \mu > 1 \), by solving the constraint

\[ W = \left( M(1)M^{-1}(2)M^{-1}(1)M(2) \right) \cdots \left( M(2g-1)M^{-1}(2g)M^{-1}(2g-1)M(2g) \right) \] (5.15)

for \( M(1) \) (a boost of magnitude \( b \), with axis \( x \)). One finds

\[ H = \frac{P_t(2)Q_t - P_y(2)Q_y}{P_y^2(2) - P_t^2(2)} \] (5.16)

where

15
Q^a = \frac{1}{2} \epsilon^{abc} \left( M(2)(M(3)M^{-1}(4)M^{-1}(3)M(4)) \ldots \right.
\left. (M(2g - 1)M^{-1}(2g)M^{-1}(2g - 1)M(2g)) \right)_{cb}

(5.17)

Altogether, we have imposed three gauge conditions out of six Poincaré symmetries. The remaining phase space variables are \(E(\mu), M(\mu); \mu = 2, \ldots, 2g\). Their time-evolution is generated by the Hamiltonian (5.16), and they are subject to the three remaining first class constraints, which commute with the Hamiltonian. These are

\[ J^x \equiv \sum_{\mu > 1} \left( I - M^{-1}(\mu) \right)_a^x E^a(\mu) \approx 0 \quad (5.18) \]

and, from \(M(1)M^{-1}(2)M^{-1}(1) \approx \left( M(2)M(3)M^{-1}(4)M^{-1}(3)M(4) \ldots M(2g) \right)^{-1} \),

\[ P^x(2) - Q^x \approx 0 \quad (5.19) \]

\[ P^2(2) - Q^2 \approx 0 \quad (5.20) \]

One could fix the remaining three gauge symmetries, which are generated by (5.19)-(5.20), and eliminate \(E(2)\) and \(M(2)\), thereby reducing the theory to an ordinary Hamiltonian system with \(6 \times (2g - 2)\) independent phase space variables. However, this would be of little practical interest in what follows, and would make the formalism significantly more cumbersome.

Using the Hamiltonian (5.16), one can compute the velocities of any function on phase space. For example, consider the edge vector \(E(3)\) of the polygon: By computing the bracket with the Hamiltonian and then using the constraints (5.15), one finds

\[ \frac{dE(3)}{dt} = \frac{(M^{-1}(2) - M(3)M(4)M^{-1}(3)M^{-1}(2))(TrM(2) - M(1)M^{-1}(2)M^{-1}(1))}{2(P^2_y(2) - P^2_t(2))} \mathcal{P}_x P(2) \quad (5.21) \]

where

\[ \mathcal{P}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.22) \]

This velocity can also be computed in the parametrized formalism defined in Sec. 3; this allows us to compute the lapse-shift vector which corresponds to our choice of gauge: one finds that \(dE(3)/dt = N^a\{P_a, E(3)\}\) when \(N\) is given by

\[ N = \left( \frac{M(1)M^{-1}(2)M^{-1}(1) - TrM(2)}{P^2_y(2) - P^2_t(2)} \right) M(1)M^{-1}(2)M^{-1}(1)\mathcal{P}_x P(2) \quad (5.23) \]
\[ P_a = \frac{1}{2} e^{abc} W_{cb} = 0 \quad (5.24) \]

\[ W = \left( M(1)M^{-1}(2)M^{-1}(1)M(2) \right) \cdots \left( M(2g-1)M^{-1}(2g)M^{-1}(2g-1)M(2g) \right) \quad (5.25) \]

Of course if one had chosen any other \( E(\mu) \) to carry out this calculation, one would have found the same lapse-shift vector \( N \). For instance, one can show (with some work) that the velocity \( dE(2)/dt = N^a \{ P_a, E(2) \} \) is equal to \( \{ H, E(2) \} \) when \( N \) is given by \( 5.23 \).

5.2 Closed Universes with \( N \geq 2 \) Particles

Since the methods and derivations for open and closed universes with particles are conceptually identical to those of the first sub-section, we will run through the next cases rapidly and refer the reader to the literature for details. The motion of any particle in an otherwise unspecified geometry can be obtained from the Hamiltonian \( (2.7) \). One finds, for \( \mu = 2g + 1 \) and \( N(2g + 1) = N^t/(4 \cos \Omega(2g + 1))P^t(2g + 1) \) (see ref. [12]),

\[ \frac{dE(2g + 1)}{dt} = \left( 0, - \frac{P^x(2g + 1)}{P^t(2g + 1)}, \frac{P^y(2g + 1)}{P^t(2g + 1)} \right) \quad (5.26) \]

Similarly for particle \( \mu = 2g + 2 \), one may choose a parametrization of its worldline, \( N(2g + 2) \), that is such that its velocity lie in the \((x - y)\)-plane. One chooses a frame where \( M(2g + 2) \) is a pure rotation \( (P^x(2g + 2) = P^y(2g + 2) = 0) \), and uses the residual freedom to rotate around the \( t \)-axis to set \( P^y(2g + 1) = 0 \). Combining these gauge choices, the observer is at rest with respect to particle \((2g + 2)\), and sees particle \((2g + 1)\) moving along the \( x \)-axis at a constant velocity. The \( x \)-component of the second particle could then be chosen as “clock”, however this would complicate the task of obtaining the Hamiltonian function, because these gauge choices require computing Dirac brackets between the matrix \( M(2g + 1) \) and the vector \( E(2g + 1) \). If we choose, instead, to use the \( x \)-component of the canonical variable \( X(2g + 1) \) (Sec. 6.1) and \( T = X^x(2g + 1)/P^x(2g + 1) \), one finds, as before, \( H = tr(M(2g + 1)) \). \( H \) must be calculated as a function of the other matrices, by solving the cycle conditions for \( M(2g + 1) \), given that \( M(2g + 2) \) is a pure rotation with angle \( \Omega(2g + 2) \).

\[ H = tr \left( M(2g + 2) \cdots M(2g + N)M(1)M^{-1}(2)M^{-1}(2g)M^{-1}(2g-1)M(2g) \right) \quad (5.27) \]

where

\[ M(2g + 2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Omega(2g + 2)) & \sin(\Omega(2g + 2)) \\ 0 & -\sin(\Omega(2g + 2)) & \cos(\Omega(2g + 2)) \end{pmatrix} \quad (5.28) \]
5.3 Open Universes with \( N \geq 2 \) Particles

If the universe has the topology \( \mathbb{R}^2 \), then the geometry at infinity approximates that of a cone with a helical shift \([27]\). In this case, there is a well-defined direction for the flow of time, which is given by the axis of the cone. The deficit angle of this cone is the total energy of the universe, and the helical shift its total angular momentum. The \( SO(2,1) \) “cycle”, which was required to be equal to the identity matrix in order that the universe may close, is now a matrix which is generally \emph{not} equal to the identity, but corresponds to parallel transport around a loop at infinity. It is usually required that its axis be timelike, in which case there are no closed timelike curves \([28], [29]\); it is then a rotation and the angle of this rotation is the deficit angle which corresponds to the conical geometry at infinity, i.e., the energy. So the Hamiltonian is the following function of the matrices \( M(\mu) \):

\[
H = \arccos \left( \frac{\text{tr}W - 1}{2} \right)
\]  

(5.29)

where \( W \) is the cycle, equal to the left-hand side of Equation (3.7) (but it is no longer required to be trivial: \( W \neq I \)).
6. The Hamilton-Jacobi Transformation

We will only consider the case \( N = 0, g \geq 2 \); the generalization to \( N \neq 0 \) and \( g \geq 0 \) is straightforward. We will perform three successive changes of variables, to relate the time-dependent variables \( E(\mu; T), M(\mu), \mu = 2, ..., 2g \) to the homotopy invariants. The first transformation is a change to canonical variables \( X(\mu; T), P(\mu) \). We then perform the Hamilton-Jacobi transformation, with the internal time which we introduced in the previous section. Finally, the homotopy invariants will be computed from the Hamilton-Jacobi variables.

6.1 Canonical Variables

The canonical phase space variables are defined as follows.

\[
X(\mu) = \frac{1}{P^2(\mu)} \left( P(\mu) \wedge J(\mu) - \frac{2(E(\mu) \cdot P(\mu))P(\mu)}{\text{tr} M(\mu) - 1} \right) \quad (6.1)
\]

where, by definition,

\[
J(\mu) = (I - M^{-1}(\mu))E(\mu) \quad (6.2)
\]

\[
(A \wedge B)^a = \epsilon^{abc} A_b B_c \quad (6.3)
\]

and the “momentum” variables are

\[
P_a(\mu) = \frac{1}{2} \epsilon_a{}^bc M_{cb}(\mu) \quad (6.4)
\]

One may check that the brackets of the variables \( X(\mu), P(\mu) \) are canonical, by using (3.1)-(3.2):

\[
\{P^a(\mu), X^b(\mu)\} = \delta^b_a \quad (6.5)
\]

The change of variables defined by (6.1)-(6.4) is regular. The inverse is given by either one of the following two sets of relations, depending on \( M(\mu) \). If \( M(\mu) \) is a boost, then

\[
E(\mu) = \frac{M^{1/2}P \wedge (P \wedge X)}{P \sqrt{2P^2 + 1 - 2}} - \frac{\sqrt{P^2 + 1}(P \cdot X)P}{P^2} \quad (6.6)
\]

where the index \( \mu = 1, 2, ..., 2g \) was omitted in the right hand side, for clarity, and

\[
(M^{1/2})_a^b = \delta_a^b + \sqrt{\frac{\sqrt{P^2 + 1} - 1}{2P^2}} P_a \epsilon^{ca} \quad (6.7)
\]

\[
+ \left( \sqrt{\frac{\sqrt{P^2 + 1} + 1}{2}} - 1 \right) \left( \delta_a^b - \frac{P_a P_b}{P^2} \right)
\]
If $M(\mu)$ is a rotation, as for particles,

$$E(\mu) = \frac{M^{1/2}P \wedge (P \wedge X)}{P \sqrt{2 - 2\sqrt{1 - P^2}}} - \frac{\sqrt{1 - P^2} (P \cdot X) P}{P^2} \quad (6.8)$$

$$(M^{1/2})^a_b = \delta^a_b + \sqrt{\frac{1 - \sqrt{1 - P^2}}{2P^2}} P_c \epsilon^{ca} b$$

$$+ \left( \sqrt{\frac{1 - P^2}{2}} + 1 \right) \left( \delta^a_b - \frac{P^a P_b}{P^2} \right) \quad (6.9)$$

The matrices $M(\mu)$ are given as a function of $P(\mu)$ as follows. For a boost,

$$M^a_b = \delta^a_b + P_c \epsilon^{ca} b + \left( \sqrt{1 + P^2} - 1 \right) \left( \delta^a_b - \frac{P^a P_b}{P^2} \right) \quad (6.10)$$

and for a rotation,

$$M^a_b = \delta^a_b + P_c \epsilon^{ca} b + \left( \sqrt{1 - P^2} - 1 \right) \left( \delta^a_b - \frac{P^a P_b}{P^2} \right) \quad (6.11)$$

The constraints $J \approx 0, P \approx 0$ can be written in terms of the canonical variables, using (6.6)-(6.11) and

$$(I - M^{-1}(\mu)) E(\mu) = J(\mu) = X(\mu) \wedge P(\mu) \quad (6.12)$$

The $SO(2,1)$ constraints become simply

$$\sum_\mu J(\mu) \approx 0 \quad (6.13)$$

while the translation constraints $P \approx 0$ are defined implicitly in terms of $P(\mu)$ by (6.10) and (6.11). The explicit form of the function $P(P(\mu)) \approx 0$ can be computed explicitly.

6.2 The Hamilton-Jacobi Transformation

Having found the Hamiltonian in the previous section, it is a straightforward task to carry out the Hamilton-Jacobi transformation from the $X(\mu), P(\mu)$ variables. The generating function is given in terms of the new momenta $P'(\mu)$ and the old variables $X(\mu)$ as the “identity operator” minus the “time evolution operator”:

$$G(X, P'; T) = P' \cdot X - \int_0^T H(P) dt \quad (6.14)$$

where the old momenta in the Hamiltonian function must be replaced in terms of the new ones by inverting the relations which define the canonical transformation, namely
\[
P(\mu) = \frac{\partial G}{\partial X(\mu)} \quad (6.15)
\]
\[
X'(\mu) = \frac{\partial G}{\partial P'(\mu)} \quad (6.16)
\]

Since the Hamiltonian is a function of \(P\)'s only, the task of replacing \(P(\mu)\) in terms of \(P'(\mu)\), usually the main difficulty in deriving the generating function, is trivial: \(P(\mu) = P'(\mu)\). The integral over time is also easy, since the Hamiltonian does not depend explicitly on time. Thus,

\[
G(X, P'; T) = P' \cdot X - \left( \frac{P'_i(2)Q'_i - P'_y(2)Q'_y}{P'_y(2) - P'_i(2)} \right) T \quad (6.17)
\]

where

\[
Q'^a = \frac{1}{2} \epsilon^{abc} \left( M'(2)M'(3)M'^{-1}(4)M'^{-1}(3)M'(4) \cdots M'(2g) \right)_{cb} \quad (6.18)
\]

The new Hamiltonian is \(K = H + \partial G/\partial T = 0\), and the new canonical variables \(X'(\mu), P'(\mu)\) are given in terms of the old ones by the equations (6.15)-(6.16), i.e. \(P'(\mu) = P(\mu)\) and

\[
X'(\mu) = X(\mu) - V(\mu)T \quad (6.19)
\]

where

\[
V(\mu) = \{ \frac{P_i(2)Q_i - P_y(2)Q_y}{P'_y(2) - P'_i(2)} X(\mu) \} \quad (6.20)
\]

In these expressions, one should remember that \(Q\) and \(P'(2)\) are related by the first-class constraints (5.25).

6.3 Time Evolution in the Hamilton-Jacobi Variables

The Hamilton-Jacobi variables \(X'(\mu), \mu > 1\), are constants of the motion. In fact, one goes from the old variables to the new ones precisely by subtracting the time evolution out of \(X(\mu; T)\), so that \(X'(\mu) = X(\mu; 0)\). In order to derive the time-evolution of the dynamical variables, one inverts the relations which define the Hamilton-Jacobi variables in terms of the time-dependent ones:

\[
X(\mu; T) = X'(\mu) + \{ H, X(\mu) \} T \quad (6.21)
\]

Of course this is the usual way that the time evolution is extracted from the Hamilton-Jacobi formalism, only that it appears particularly simple in this case because the Hamiltonian depends only on the momenta. We will show next that the homotopy invariants are functions of the Hamilton-Jacobi variables which we have just derived.
6.4 Homotopy Invariants as Functions of the Hamilton-Jacobi Variables

As we have seen, the homotopy invariants can be separated in two types (Equations 4.5-4.6). The first are traces of the Lorentz projections of the \( ISO(2,1) \) homotopies, such as:

\[
L_1(u_1v_1) = tr \left( M(2)M^{-1}(1) \right) - 1
\]  

(6.22)

The other are the scalar products of the vector components of the \( ISO(2,1) \) homotopies, with the duals of their Lorentz projections, such as:

\[
L_2(u_1v_1) = E(1) \cdot P(2\bar{I}) + E(2) \cdot P(2\bar{I})
\]  

(6.23)

The Poisson brackets of these observables can be computed directly using the brackets of the polygon variables \( E(\mu), M(\mu) \). For instance, one finds with some algebraic work

\[
\{L_1(\rho), L_1(\rho')\} = 0 \forall \rho, \rho' \in \pi_1(\Sigma_g)
\]  

(6.24)

\[
\{L_1(u_1), L_2(v_1)\} = L_1(u_1v_1) - L_1(u_1v_1^{-1})
\]  

(6.25)

\[
\{L_2(u_1), L_2(v_1)\} = L_2(u_1v_1) - L_2(u_1v_1^{-1})
\]  

(6.26)

These are the usual expressions for the brackets of the homotopy invariants, when they are calculated directly from the field theory (see, e. g., Nelson and Regge [17]).

In the expressions (6.22)-(6.23), one identifies the explicit dependence on time by replacing \( E(1) \) by its expression (5.13) from gauge-fixing. For example,

\[
L_2(u_1v_1) = E(2) \cdot P(2\bar{I}) + A'P^t(2\bar{I}) - BP^y(2\bar{I}) - P^x(1)P^x(2\bar{I})T
\]  

(6.27)

\[
= E'(2) \cdot P'(2\bar{I}) + A'P'^t(2\bar{I}) - B'P'^y(2\bar{I})
\]  

(6.28)

where

\[
A = \frac{(1 - \cosh(b))J_1^t + \sinh(b)J_1^y}{2(1 - \cosh(b))},
\]  

(6.29)

\[
B = \frac{\sinh(b)J_1^t + (1 - \cosh(b))J_1^y}{2(1 - \cosh(b))},
\]  

(6.30)

\[
J_1 = \sum_{\mu > 1} \left( I - M^{-1}(\mu) \right) E(\mu)
\]  

(6.31)
Equation (6.28) gives $L_2(u_1v_1)$ in terms of the Hamilton-Jacobi variables $X'(\mu), P'(\mu), \mu = 2, ..., 2g + N$, while (6.27) gives the same invariant in terms of the dynamical variables and time. All of the homotopy invariants can be similarly expressed in terms of the Hamilton-Jacobi variables. Since these form a complete description of the phase space, the expressions can be inverted to give the Hamilton-Jacobi variables as functions of the homotopy invariants. The time-evolution problem is solved by expressing the invariants as explicit functions of time.

Altogether, we have gone from the $E(\mu), M(\mu)$ variables to $X(\mu), P(\mu)$ with a change of variables, then applied the Hamilton-Jacobi transformation, and finally with another regular change of variables we recovered the homotopy invariants. The $E(\mu), M(\mu)$ variables, modulo the constraints and gauge symmetry, span the same reduced phase space as the homotopy invariants, which shows that the two formulations of (2+1)-dimensional gravity are closely related, at least classically. We now turn to two examples. The example of the torus is reasonably well understood already [7]; we use it mainly to throw a bridge with existing literature. For the genus two example, on the other hand, this is the first explicit solution of the Hamilton-Jacobi problem.
7. Examples

Most recent work on (2 + 1)-dimensional gravity (classical and quantum) has focused on spacetimes with the topology \( T^2 \times (0,1) \), largely because this is the case where the time evolution problem has been solved explicitly in York’s extrinsic time [7].

The reader interested in a clear illustration of the arguments of the previous sections should turn to the second example (genus 2), which is more representative of the general case. This subsection is designed specifically for those who have worked, or have the intention of working, on the problem of \( T^2 \times (0,1) \) gravity.

We will review the example of the torus in our formalism, emphasizing its particularities, then compare with the literature.

The reduced variables for the torus are two three-vectors \( E^{(1)}, E^{(2)} \), and two Lorentz matrices, with the constraints

\[
J \equiv E^{(1)} + E^{(2)} - M^{-1}(1)E^{(1)} - M^{-1}(2)E^{(2)} \approx 0
\]

\[
P^a \equiv \frac{1}{2} \epsilon^{abc} \left( M(1)M^{-1}(2)M^{-1}(1)M(2) \right)_{cb} \approx 0
\]

The first constraint states that the four vectors in the sum form a closed quadrilateral [Figure 7.1]. We choose two loops which form a basis of the homotopy group, as follows. One, \( u \), begins from a point \( O \) inside the quadrilateral, crosses through the edge \( E^{(1)} \) at \( X \) and returns to \( O \) from the identified point \( X' \). The other, \( v \), crosses through the edge \( M^{-1}(2)E^{(2)} \) at \( Y \) and continues through the identified point \( Y' \). The corresponding ISO(2,1) homotopies are

\[
\rho(u) = OX + M(1)X'0 = \begin{pmatrix} M(1) & -M(1)M^{-1}(2)E^{(2)} + (I - M(1))OA \\ 0 & 1 \end{pmatrix}
\]

\[
\rho(v) = \begin{pmatrix} M^{-1}(2) & -M^{-1}(2)E^{(1)} + (I - M^{-1}(2))OA \\ 0 & 1 \end{pmatrix}
\]

These matrices generate a representation of \( \pi_1(\Sigma_g) \to ISO(2,1) \) and, consequently, they satisfy the cycle condition.

\[
\rho(u)\rho(v)\rho(u^{-1})\rho(v^{-1}) = I
\]

Indeed, computing the product (7.5) explicitly, one finds

\[
\begin{pmatrix} M(1)M^{-1}(2)M^{-1}(1)M(2) & -(M(1)M^{-1}(2)M^{-1}(1)M(2))J \\ +I - M(1)M^{-1}(2)M^{-1}(1)M(2)OA \end{pmatrix}
\]

24
This identity states that the closed path \( OX + X'O + OY + Y'O - OX - YO - Y'O = 0 \) is contractible, i.e. belongs to the trivial homotopy class [Figure 7.1]. Note that the cycle condition for the \( ISO(2, 1) \) homotopies summarizes both the cycle condition for \( SO(2, 1) \) homotopies and the closure condition for the polygon.

The number of independent observables should be equal to the dimension of the cotangent bundle of Teichmüller space, namely 4. For the torus, the \( SO(2, 1) \) cycle condition states that the matrices \( M(1) \) and \( M(2) \) commute, only two independent conditions. Also, given that these matrices commute (and therefore have parallel axes), the dot product of the closure condition with either one of the rotation vectors, say \( P(1) \cdot J \), is identically zero, since \( P(1) \) is parallel to the common axis of \( M(1) \) and \( M(2) \), and

\[
P(1) \cdot J = P(1) \cdot \left( I - M^{-1}(1) \right) E(1) + P(1) \cdot \left( I - M^{-1}(2) \right) E(2) \equiv 0 \quad (7.7)
\]

So the number of independent constraints is really four, not six, and the number of degrees of freedom is \( 12 - 2 \times 4 = 4 \).

The four homotopy invariants are

\[
L_1(u) = Tr M(1) \quad (7.8)
\]
\[
L_2(u) = P(1) \cdot E(2) \quad (7.9)
\]
\[
L_1(v) = Tr M(2) \quad (7.10)
\]
\[
L_2(v) = P(2) \cdot E(1) \quad (7.11)
\]

We will fix the frame by setting

\[
P^u(1) = P^t(1) = 0 \quad (7.12)
\]

and solving the two independent constraints \( J \approx 0 \) for

\[
\begin{pmatrix}
E^t(1) \\
E^y(1)
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{2} & \frac{\sinh(b)}{2(1 - \cosh(b))} \\
\frac{\sinh(b)}{2(1 - \cosh(b))} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
J^t(2) \\
J^y(2)
\end{pmatrix} \quad (7.13)
\]

This leaves the variables \( E^x(1) \) and \( E^a(2) \), \( a=0,1,2 \), and the constraints \( P^a \approx 0 \) which are now equivalent to

\[
P^y(2) \approx 0 \quad (7.15)
\]
\[
P^t(2) \approx 0 \quad (7.16)
\]
The internal time must be a variable which does not commute with these constraints - the choice \( T = -E^x(1)/P^x(1) \) analogous to that of Sec. 4 does not satisfy this criterion. One may choose instead, for example, \( T = -(\cos(\alpha)X^x(2) + \sin(\alpha)X^y(2)) \), where \( \alpha \) is a parameter not equal to zero. This fixes the gauge freedom for the constraint \( P^y = 0 \). There is one remaining gauge freedom, which one can fix by choosing the gauge condition \( X^t(2) = 0 \). The Hamiltonian corresponding to \( T \) is

\[
H = \cos \alpha P^x(2) + \sin \alpha P^y(2) \approx \cos \alpha P^x(2)
\]  

(7.17)

For \( \alpha = \pi/2 \), the Hamiltonian vanishes and the reduced phase space variables \( X^x(1), X^x(2), P^x(1) \) and \( P^x(2) \) are Hamilton-Jacobi invariants.

We close this section with a short dictionary to translate between our results and other authors’ results for the torus (in the extrinsic curvature time).

The extrinsic time is alternatively written \( \tau = tr(K) \), or \( q_0 \).

The moduli are written as a complex number \( m \), or specifically as \( m = m_1 + im_2 = q_1 + iq_2 \). They are conjugate to \( p = p_1 + ip_2 \). The moduli and their momenta are given in terms of the \( ISO(2,1) \) homotopies \( \rho(u) = \Lambda_1 \) and \( \rho(v) = \Lambda_2 \) by the relations [10]

\[
m = \left( a + \frac{i\lambda}{\tau} \right)^{-1} \left( b + \frac{i\mu}{\tau} \right)
\]  

(7.18)

\[
p = -i\tau \left( a - \frac{i\lambda}{\tau} \right)^2
\]  

(7.19)

where \( a, b, \lambda, \mu \) are defined by

\[
\rho(u) = \begin{pmatrix}
\cosh \lambda & 0 & \sinh \lambda & 0 \\
0 & 1 & 0 & a \\
\sinh \lambda & 0 & \cosh \lambda & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

(7.20)

\[
\rho(v) = \begin{pmatrix}
\cosh \mu & 0 & -\sinh \mu & 0 \\
0 & 1 & 0 & b \\
-\sinh \mu & 0 & \cosh \mu & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

(7.21)

This gives us the correspondence with our variables at \( t = 0 \). Using (4.3)-(4.4), one finds

\[
E(1; t = 0) = (0, -b, 0)
\]  

(7.22)

\[
M(1) = \begin{pmatrix}
\cosh \lambda & 0 & \sinh \lambda \\
0 & 1 & 0 \\
\sinh \lambda & 0 & \cosh \lambda
\end{pmatrix}
\]  

(7.23)

\[
E(2; t = 0) = (0, -a, 0)
\]  

(7.24)
Note that the solution is singular at $t = 0$, in the sense that the torus has vanishing area there. One easily checks that the area is positive for all $t \neq 0$, by constructing the spacetime as explained in the beginning of this section.

The extrinsic time $\tau$, itself, is a function of the polygon variables: It is the trace of the extrinsic curvature of a constant curvature slice which must respect the identification conditions. This is a straightforward but lengthy exercise in geometry. One finds

$$\tau = \sqrt{\frac{(E(1) \cdot E(2))A + E^2(1)B + E^2(2)C + \sqrt{\Delta}}{4A^2 - 16BC}}$$  \hfill (7.26)$$

where

$$A = \cosh(\mu)\cosh(\mu + \lambda) - \cosh(\mu) - \cosh(\mu + \lambda) - 1$$  \hfill (7.27)$$

$$B = \cosh(\mu) \cosh(\mu + \lambda) - 1 - \sinh(\mu) \sinh(\mu + \lambda)$$  \hfill (7.28)$$

$$C = \cosh(\mu) - 1$$  \hfill (7.29)$$

$$\Delta = (A^2 - BC)E^2(1)E^2(2) + 4BC(E(1) \cdot E(2))^2$$  \hfill (7.30)$$

$$+ B^2E^4(1) + C^2E^4(2) + 2(E(1) \cdot E(2))(ABE^2(1) + ACE^2(2))$$

The Hamiltonian which corresponds to this choice of time is given, in various notations, as

$$H = \frac{\sqrt{q_0^2(p_1^2 + p_2^2)}}{q_0}$$  \hfill (7.31)$$

$$H = \frac{|a\mu - b\lambda|}{\tau}$$  \hfill (7.32)$$

$$H = \frac{(E(1) \cdot P(1))\sinh^{-1}(P(1))}{\tau P(1)} + \frac{(E(2) \cdot P(2))\sinh^{-1}(P(2))}{\tau P(2)}$$  \hfill (7.33)$$

Finally, the invariants are

$$E(1) \cdot P(2) = -a \sinh \lambda$$  \hfill (7.34)$$
\[ E(2) \cdot P(1) = b \sinh \mu \]  \hspace{1cm} (7.35)

\[ trM(1) = 1 + 2 \cosh \lambda \]  \hspace{1cm} (7.36)

\[ trM(2) = 1 + 2 \cosh \mu \]  \hspace{1cm} (7.37)

As we have seen, the torus is a rather singular case in many respects. The case of genus two is more representative of the general case \( g > 1, N \neq 0 \).

7.2 The Genus Two Universe

The reduced variables are the three-vectors \( E(1), E(2), E(3), E(4), M(1), M(2), M(3), M(4) \), and the constraints are

\[ J \equiv \sum_{\mu=1}^{4} \left( I - M^{-1}(\mu) \right) E(\mu) \approx 0 \]  \hspace{1cm} (7.38)

\[ P^a \equiv \frac{1}{2} \varepsilon^{abc} \left( M(1)M^{-1}(2)(M(2)M(3)M^{-1}(4)M^{-1}(3)M(4) \right)_{cb} \approx 0 \]  \hspace{1cm} (7.39)

Given a choice of basis loops \( u_1, v_1, u_2, v_2 \) [Figure 7.2], one computes the corresponding \( ISO(2,1) \) homotopies as explained in Sec. 3.

\[ \rho(u_1) = \begin{pmatrix} M(1) & (I - M(1))E(1) - M(1)E(2) + (I - M(1))OA \\ 0 & 1 \end{pmatrix} \]  \hspace{1cm} (7.40)

\[ \rho(v_1) = \begin{pmatrix} M(1) & (I - M(1))E(1) - M(1)E(2) + (I - M(1))OA \\ 0 & 1 \end{pmatrix} \]  \hspace{1cm} (7.41)

\[ \rho(u_2) = \begin{pmatrix} M(3) & (M^{-1}(4) - I - M(3)M^{-1}(4))E(4) + (M^{-1}(3) - I)E(3) + (I - M(3))OA \\ 0 & 1 \end{pmatrix} \]  \hspace{1cm} (7.42)

\[ \rho(v_2) = \begin{pmatrix} M^{-1}(4) & (M^{-1}(4) - M^{-2}(4))E(4) \\ 0 & 1 \end{pmatrix} \]  \hspace{1cm} (7.43)

The observables are \( ISO(2,1) \) invariants taken from these matrices and any products of them. A convenient choice of 30 invariants was proposed by Nelson and Regge [5], who
also gave their brackets and the remaining non-linear relations (the number of phase space variables for $g = 2$ is 12). We are now in a position where we can compute the 30 invariants in terms of the reduced variables $E(\mu)$ and $M(\mu)$. In [Table 7.1] we give the 15 loops chosen by Nelson and Regge and the two invariants for each loop. We use the condensed notation $P^a(2\bar{1}) = (1/2)\epsilon^{abc}(M(1)M^{-1}(2))_{cb}$. In [5], Nelson and Regge worked with the group $SO(2, 2)$ rather than $ISO(2, 1)$, which amounts to having a negative cosmological constant. However the Poincaré limit $\Lambda \to 0$ was carried out explicitly in ref [3]. The invariants $L_1$ and $L_2$ in this article are equivalent to the traces $a_I \to a_{XV}$ used in [5] in the limit $\Lambda \to 0$.

In the expressions from Table 7.1, one identifies the explicit dependence on time by replacing $E(1)$ by its expression from gauge-fixing, Eqn. (5.13), as in the example from Sec. 6:

$$L_2(u_1v_1) = E(2) \cdot P(2\bar{1}) + AP^x(2\bar{1}) - BP^y(2\bar{1}) - P^x(1)P^x(2\bar{1})T$$  \hspace{1cm} (6.27)

Given the algebraic relations between the variables, one must still check that the Poisson brackets and constraints are consistent. The Poisson brackets were shown to be consistent in the general case in Sec. 6.4. As for the constraints, the $ISO(2, 1)$ matrices $u_i$ and $v_i$ are required to satisfy the cycle condition, specifically:

$$\rho(u_1v_1u_1^{-1}v_1^{-1}u_2v_2u_2^{-1}v_2^{-1}) = I$$  \hspace{1cm} (7.44)

By computing this product explicitly with the relations (7.40)-(7.43), one finds

$$\begin{pmatrix} W & -WJ + (I - W)OA \\ 0 & 1 \end{pmatrix} = I$$  \hspace{1cm} (7.45)

Since the dimension of the reduced phase space for genus two is $12g - 12 = 12$, the 30 invariants given in the table cannot all be independent. Indeed, the complete set of relations among them was proposed by Nelson and Regge. To complete the proof of equivalence of the formalisms, one must show that these relations are satisfied by the expressions given in the table above, when the variables satisfy the constraints (7.38)-(7.39). This is a trivial consequence of the fact that the matrices $\rho(u_i), \rho(v_i)$ generate a representation of $\pi_1(\Sigma_g) \to ISO(2, 1)$.
Conclusion

In (2+1)-dimensional gravity, the physical degrees of freedom can be parametrized by a representation $\rho : \pi_1(\Sigma_{g,N}) \to ISO(2,1)$. We have shown how elements $\rho(u_i)$ can be written in terms of the finite set of variables $\{E(\mu), M(\mu); \mu = 1, \cdots, 2g + N\}$ of the polygon representation. The ISO(2,1) cycle conditions for the representations $\rho$ are equivalent to the SO(2,1) cycle conditions $P^a \approx 0$ together with the polygon closure relations $J^a \approx 0$, and the symplectic structure on the reduced space of ISO(2,1) invariants can be recovered from the Poisson brackets of the polygon variables. Thus, the homotopy invariants and the polygon variables represent the same set of solutions of (2+1)-dimensional gravity, and the phase space of ISO(2,1) invariants is closely related to the Hamilton-Jacobi phase space of the polygon representation.

The choice of an internal time in the polygon representation permits an explicit calculation of the Hamiltonian for any genus. On the other hand, the virtue of working with the extrinsic slicing is that one finds expressions which are similar to those encountered in the ADM formalism for (3+1)-dimensional gravity. We will pay the price of losing this similarity, and in exchange hope to push further the quantization programme for (2+1)-dimensional gravity.

The method which we used in this article is limited to flat spacetimes, since the reduced variables $E(\mu), M(\mu)$ have yet to be generalized to de Sitter gravity (one possible way to do this was sketched in [30]). Nevertheless, the problem of time is conceptually similar in (2+1)-dimensional de Sitter gravity and we expect the conclusion to generalize: The homotopy invariants are most likely related to the Hamilton-Jacobi phase space for any of the exactly soluble theories discussed by Horowitz [31], as well as their supersymmetric generalizations. A generalization to flat (3+1)-dimensional gravity (with topological degrees of freedom) could be carried out following Horowitz and the reduction to global $E - M$ variables, to define a three-dimensional polygonal cell embedded in $\mathbb{R}^{3+1}$ [32]. One might also consider the loop space variables of Ashtekar et al. [33]. The loop variables are related to the homotopy invariants discussed in this article, but are path-dependent and uncountable, among other important differences.

The quantum analogue of the Hamilton-Jacobi formalism is the Heisenberg picture quantization; observables are constants but depend explicitly on time. Three highly non-trivial problems come up. One is the invariance of the wave function with respect to the mapping class symmetry. So far the only examples of symmetric wave functions for (2 + 1)-dimensional gravity are for the simplest cases of two particles scattering on $\mathbb{R}^2$ [34] and the quantum torus [10]. The second problem is that of ordering the operators in the solution of the Heisenberg equations of motion in a way which is consistent with the mapping class group; this difficulty was emphasized by Carlip [10], who found a consistent ordering of the relations giving the homotopy invariants for $g = 1$ as functions of the ADM variables, where the wave functions transform non-trivially under the modular group. The third is the so-called “problem of time”, which can be summarized as follows. In order to formulate dynamical questions, a choice of “internal time” must be made. The question then is whether the answer to a given question depends on the choice of time. This problem is linked to the ordering ambiguities and the problem of consistency with the mapping class symmetry. Thus, although the classical theory of (2+1)-dimensional gravity is essentially solved, the status of the quantum theory is far less clear; we examine it in greater detail in the following article.
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Figure 2.1 A basis for the homotopy group of $\Sigma_g$ is represented. Note that each $u_i$ ($i = 1, \ldots, g$) intersects only the corresponding $v_i$, at only one point. These loops are not independent: the last one ($v_g$) can be deformed into a combination of the other $2g-1$.

Figure 3.1 The torus (initial surface) is represented by a parallelogram with opposite sides identified. At each corner, one has an image of the timelike line $N$. The evolution of this torus is obtained by sliding each corner along the corresponding line.

Figure 4.1 One handle of a genus $g$ surface is represented by four edges of a polygon, with identifications $E(1) \sim -M^{-1}(1)E(1), E(2) \sim -M^{-1}(2)E(2)$. The loops $u_1$ and $v_1$ cross only at $O$, which can be chosen arbitrarily on $\Sigma_g$.

Figure 7.1 The path $uvu^{-1}v^{-1}$ on the torus is contractible. To see this, inflate the area which is squeezed between the double lines until it becomes a single line surrounding the four corners. Since these four corners are identified, this line is a small circle around the “vertex”: the point of $T^2$ which maps onto the four corners. This small circle is contractible, since there is no topological obstruction at the vertex.

Figure 7.2 The genus two spacetime is represented by an octagon with the identifications $E(\mu) \sim -M^{-1}(\mu)E(\mu)$, $\mu = 1, 2, 3, 4$. The four loops $u_1, v_1, u_2, v_2$, which intersect only at $O$, form a basis of the homotopy group based at $O$.

Table 7.1 The 30 homotopy invariants proposed by Nelson and Regge are given in terms of the polygon variables.