Abstract

In this paper, the author considers the numerical computation of CVA for large systems by Monte Carlo methods. He introduces two types of stochastic mesh methods for the computations of CVA. In the first method, stochastic mesh method is used to obtain the future value of the derivative contracts. In the second method, stochastic mesh method is used only to judge whether future value of the derivative contracts is positive or not. He discusses the rate of convergence to the real CVA value of these methods.

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1 Introduction

The credit valuation adjustment (CVA) is, by definition, the difference between the risk-free portfolio value and the true portfolio value that takes into account default risk of the counterparty. In other words, CVA is the market value of counterparty credit risk. After the financial crisis in 2007-2008, it has been widely recognized that even major financial institutions may default. Therefore, the market participants has become fully aware of counterparty credit risk. In order to reflect the counterparty credit risk in the price of over-the-counter (OTC) derivative transactions, CVA is widely used in the financial institutions today.

Although Duffie-Huang [3] has already introduced the basic idea of CVA in 1990’s, several people reconsidered the theory of CVA related to collateralized derivatives (cf. [4]) and also efficient numerical calculation methods appeared(cf. [10]).

There are two approaches to measuring CVA: unilateral and bilateral (cf. [6]). Under the unilateral approach, it is assumed that the bank that does the CVA analysis is default-free. CVA measured in this way is the current market value of future losses due to the counterparty’s potential default. The problem with unilateral CVA is that both the bank and the counterparty require a premium for the credit risk they are bearing and can
never agree on the fair value of the trades in the portfolio. Therefore, we have to consider not only the market value of the counterparty’s default risk, but also the bank’s own counterparty credit risk called debit value adjustment (DVA) in order to calculate the correct fair value. Bilateral CVA (it is calculated by netting unilateral CVA and DVA) takes into account the possibility of both the counterparty default and the own default. It is thus symmetric between the own company and the counterparty, and results in an objective fair value calculation.

Mathematically, unilateral CVA and DVA are calculated in the same way, and bilateral CVA is the difference of them. So we focus on the calculation of unilateral CVA in this paper.

CVA is measured at the counterparty level and there are many assets in the portfolio generally. Therefore, we have to be involved in the high dimensional numerical problem to obtain the value of CVA. This is one of the reasons why CVA calculation is difficult. On the other hand, each payoff usually depends only on a few assets. We will focus on this property and suggest an efficient calculation methods of CVA in the present paper.

Let us consider the portfolio consist of the contracts on one counterparty. Let \( X^{(m)}(t) \) be \( \mathbb{R}^{N_m} \)-valued stochastic processes, \( m = 0, 1, \ldots, M \). We think that \( X(t) = (X^{(0)}(t), \ldots, X^{(M)}(t)) \) is an underlying process. We consider the model that the macro factor is determined by \( X^{(0)}(t) \), and the payoff of each derivative at maturity \( T_k, k = 1, \ldots, K \), is the form of \( \sum_{m=1}^{M} \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) \).

Let \( T = T_K \) be the final maturity of all the contracts in the portfolio. Let \( \tau \) be the default time of the counterparty, \( \lambda(t) \) be its hazard rate process, \( L(t) \) be the process of loss when the default takes place at time \( t \), and \( D(t, T) \) be the discount factor process from \( t \) to \( T \). We assume that \( D(0, t) \) is the function of \( X^{(0)}(t) \) and that \( L(t), \lambda(t) \) and \( \exp(-\int_0^t \lambda(s)ds) \) are the function of \( X(t) \).

Let \( \tilde{V}_0(t) \) be total value of all contracts in the portfolio at time \( t \) under the assumption that counterparty is default free. Then \( \tilde{V}_0(t) \) is given by

\[
\tilde{V}_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geq t} D(t, T_k) \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t],
\]

where \( E \) denotes the expectation with respect to the risk neutral measure. Then unilateral CVA on this portfolio is the restructuring cost when the counterparty defaults. So unilateral CVA is given by

\[
\text{CVA} = E[L(\tau)D(0, \tau)1_{\{\tau < T\}}(\tilde{V}_0(\tau) \vee 0)]
= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s)ds)\lambda(t)D(0, t)(\tilde{V}_0(t) \vee 0)dt]
= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s)ds)\lambda(t)(V_0(t) \vee 0)dt],
\]

where

\[
V_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geq t} F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t],
\]

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and \( F_{m,k} \) is a function such as
\[
F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) = D(0, T_k) \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)).
\]

Since \( L(t) \exp(- \int_0^t \lambda(s) ds) \lambda(t) \) is a function of \( X(t) \), we denote it by \( g(t, X(t)) \). Then CVA is given by the following form.

\[
\text{CVA} = E[\int_0^T g(t, X(t)) (E[\sum_{m=1}^M \sum_{k \geq t} F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t] \vee 0) dt].
\]  

(2)

Now we prepare the mathematical setting. Let \( M \geq 1 \) be fixed, \( N_m \geq 1, m = 1, \ldots, M, N = N_0 + \cdots + N_M, N_m = N_0 + N_m, \) and \( \tilde{N} = \max_{m=1, \ldots, M} \tilde{N}_m. \)

Let \( W_0 = \{w \in C([0, \infty); \mathbb{R}^d); \ w(0) = 0\} \), \( \mathcal{F} \) be the Borel algebra over \( W_0 \) and \( \mu \) be the Wiener measure on \( (W_0, \mathcal{F}) \). Let \( B^i(t, w) = w^i(t), \ (t, w) \in [0, \infty) \times W_0. \) Then \( \{(B^1(t), \ldots, B^d(t); t \in [0, \infty)\} \) is a \( d \)-dimensional Brownian motion. Let \( B^0(t) = t, \ t \in [0, \infty). \)

Let \( V_i^{(0)} \in C^\infty_b(\mathbb{R}^{N_0}; \mathbb{R}^{N_0}), V_i^{(m)} \in C^\infty_b(\mathbb{R}^{N_0} \times \mathbb{R}^{N_m}; \mathbb{R}^{N_m}), i = 0, \ldots, d, m = 1, \ldots, M. \) Here \( C^\infty_b(\mathbb{R}^m; \mathbb{R}^n) \) denotes the space of \( \mathbb{R}^n \)-valued smooth functions defined in \( \mathbb{R}^m \) whose derivatives of any order are bounded. We regard elements in \( C^\infty_b(\mathbb{R}^n; \mathbb{R}^n) \) as vector fields on \( \mathbb{R}^n. \)

Now let us consider the following Stratonovich stochastic differential equations.

\[
X^{(0)}(t, x_0) = x_0 + \sum_{i=0}^d \int_0^t V_i^{(0)}(X^{(0)}(s, x_0)) \circ dB_i(s),
\]  

(3)

\[
X^{(m)}(t, \tilde{x}_m) = x_m + \sum_{i=1}^d \int_0^t V_i^{(m)}(X^{(0)}(s, x_0), X^{(m)}(s, \tilde{x}_m)) \circ dB_i(s),
\]  

(4)

where \( x_m \in \mathbb{R}^{N_m}, \tilde{x}_m = (x_0, x_m) \in \mathbb{R}^{N_0} \times \mathbb{R}^{N_m}, m = 1, \ldots, M. \)

Let \( \tilde{X}^{(m)}(t, \tilde{x}_m) = (X^{(0)}(t, x_0), X^{(m)}(t, \tilde{x}_m)) \) and \( \tilde{V}_i^{(m)} \in C^\infty_b(\mathbb{R}^{N_0} \times \mathbb{R}^{N_0}, \mathbb{R}^{N_0} \times \mathbb{R}^{N_m}), i = 0, \ldots, d, m = 1, \ldots, M \) be

\[
\tilde{V}_i^{(m)}(\tilde{x}_m) = \begin{pmatrix}
V_i^{(0)}(x_0) \\
V_i^{(m)}(\tilde{x}_m)
\end{pmatrix}.
\]

Then we have

\[
\tilde{X}^{(m)}(t, \tilde{x}_m) = \tilde{x}_m + \sum_{i=0}^d \int_0^t \tilde{V}_i^{(m)}(\tilde{X}^{(m)}(t, \tilde{x}_m)) \circ dB_i(s).
\]  

(5)

There is a unique solution \( \tilde{X}^{(m)}(t, \tilde{x}_m) \) to this equation. Then \( X(t, x) \), \( x \in \mathbb{R}^N \) also satisfies the solution to the following Stratonovich stochastic differential equation.

\[
X(t, x) = x + \sum_{i=0}^d \int_0^t V_i(X(s, x)) \circ dB_i(s),
\]  

(6)
where $V_i, i = 1, \ldots, d$ is

$$V_i(x) = \begin{pmatrix}
V_i^{(0)}(x_0) \\
V_i^{(1)}(\bar{x}_1) \\
\vdots \\
V_i^{(M)}(\bar{x}_M)
\end{pmatrix}. $$

We assume that vector fields $V_i, i = 1, \ldots, d$, satisfy condition (UFG) stated in the section 2. Let $E_m$ be defined by (11) in Section 2. By (9), if $\tilde{x}_m \in E_m$, the distribution law of $X^{(m)}(t, \tilde{x}_m)$ under $\mu$ has a smooth density function $p^{(m)}(t, \tilde{x}_m, \cdot) : \mathbb{R}^{\tilde{N}_m} \to [0, \infty)$ for $t > 0, m = 1, \ldots, M$.

Let $x^* = (x^*_0, \ldots, x^*_M) \in \mathbb{R}^N$. We assume that the underlying asset process is $X(t) = X(t, x^*)$. We also assume that

$$\tilde{x}_m^* = (x^*_0, x^*_m) \in E_m, m = 1, \ldots, M.$$ 

Let $\mathcal{D}(\mathbb{R}^n)$ denotes the space of functions on $\mathbb{R}^n$ given by

$$\mathcal{D}(\mathbb{R}^n) = \{f \in C^2(\mathbb{R}^n); \|\partial_\alpha f\|_\infty < \infty, \text{for } 1 \leq |\alpha| \leq 2\},$$

where $\|f\|_\infty = \sup\{|f(x)|; x \in \mathbb{R}^n\}$.

$Lip(\mathbb{R}^n)$ denotes the space of Lipschitz continuous functions on $\mathbb{R}^n$, and we define a semi-norm $\|\cdot\|_{Lip}$ on $Lip(\mathbb{R}^n)$ by

$$\|f\|_{Lip} = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad f \in Lip(\mathbb{R}^n).$$

Let $\mathcal{M}(\mathbb{R}^n)$ be the linear subspace of $Lip(\mathbb{R}^n)$ spanned by $\{f \vee g; f, g \in \mathcal{D}(\mathbb{R}^n)\}$.

We define linear operators $P_t : Lip(\mathbb{R}^N) \to Lip(\mathbb{R}^N), t \geq 0$, by

$$(P_t f)(x) = E^\mu[f(X(t, x))], \quad f \in Lip(\mathbb{R}^N),$$

and $P_t^{(m)} : Lip(\mathbb{R}^{\tilde{N}_m}) \to Lip(\mathbb{R}^{\tilde{N}_m}), t \geq 0, m = 1, \ldots, M$, by

$$(P_t^{(m)} f)(\tilde{x}_m) = E^\mu[f(\tilde{X}^{(m)}(t, \tilde{x}_m))], \quad f \in Lip(\mathbb{R}^{\tilde{N}_m}).$$

We remind that $L(t) \exp(-\int_0^t \lambda(s)ds)\lambda(t)$ is represented by

$$L(t) \exp(-\int_0^t \lambda(s)ds)\lambda(t) = g(t, X(t, x^*)).$$

We assume that $g : [0, T] \times \mathbb{R}^N \to [0, \infty)$ satisfies the following two conditions.

1. $g(t, x)$ is differentiable in $t$ and there is an integer $n_1$, and a constant $C_1 > 0$ such that

$$\sup_{t \in [0, T]} \left| \frac{\partial}{\partial t} g(t, x) \right| \leq C_1(1 + |x|^{n_1}), \quad x \in \mathbb{R}^N.$$
(2) \( g(t, x) \) is 2-times continuously differentiable in \( x \) and there is an integer \( n_2 \), and a constant \( c_2 > 0 \) such that

\[
\sup_{t \in [0,T]} | \frac{\partial^\alpha}{\partial x^\alpha} g(t, x) | \leq c_2 (1 + |x|^{n_2}), \quad x \in \mathbb{R}^N
\]

for any multi index \( |\alpha| \leq 2 \).

We assume that a discounted payoff functions \( F_{m,k}, m = 1, \ldots, M, k = 1, \ldots, K \) in equation (2) belong to \( \mathcal{M} (\mathbb{R}^{\bar{N}_m}) \). Under the assumptions above, CVA \( c_0 \) is given by

\[
c_0 = E^\mu \left[ \int_0^T \{ g(t, X(t, x^*)) E^{\mu} \left[ \sum_{m=1}^M \sum_{k:T_k \geq t} F_{m,k}(\bar{X}^{(m)}(T_k, \tilde{x}^*_m)) \} | \mathcal{F}_t \} \right] \right] dt. \tag{7}
\]

We will introduce numerical calculation methods by Monte Carlo simulation for \( c_0 \).

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and \( X_\ell : [0, \infty) \times \Omega \to \mathbb{R}^N, \ell = 1, 2, \ldots, \) be continuous stochastic processes such that each probability law on \( C([0, \infty); \mathbb{R}^N) \) of \( X_\ell(\cdot) \) under \( P \) is the same as that of \( X(\cdot, x^*) \) for all \( \ell = 1, 2, \ldots, \) and that \( \sigma \{ X_\ell(t); t \geq 0 \}, \ell = 1, 2, \ldots, \) are independent.

Let us define projections \( \pi_m : \mathbb{R}^N \to \mathbb{R}^{\bar{N}_m}, m = 1, \ldots, M, \) by \( \pi_m(x) = \tilde{x}_m = (x_0, \ldots, x_m) \), and define \( \varepsilon_0 > 0 \) by \( \varepsilon_0 = \min_{1 \leq k \leq K} (T_k - T_{k-1}) \). We define random linear operators (stochastic mesh operators) \( Q^{(m)}_{t,T_k,\varepsilon} = Q^{(m,L,\omega)}_{t,T_k,\varepsilon} \) by

\[
(Q^{(m,L,\omega)}_{t,T_k,\varepsilon} f)(\tilde{x}_m) = \begin{cases} 
\frac{1}{L} \sum_{\ell=1}^L \frac{f(X^{(m)}_{\ell}(T_k))p^{(m)}_{\ell}(T_k-t,\tilde{x}_m, \pi_m(X_\ell(T_k)))}{q^{(m,L,\omega)}_{\ell,T_k}(\pi_m(X_\ell(T_k)))}, & 0 \leq t < T_k - \varepsilon, \\
f(\tilde{x}_m), & T_k - \varepsilon \leq t \leq T_k, \\
0, & T_k < t \leq T.
\end{cases}
\]

where

\[
q^{(m,L,\omega)}_{\ell,T_k}(y_m) = \frac{1}{L} \sum_{\ell=1}^L p^{(m)}(T_k - t, \pi_m(X_\ell(t)), y_m)).
\]

Let \( \Pi \) denotes the set of partitions \( \Delta = \{ t_0, t_1, \ldots, t_n \} \) such that \( 0 = t_0 < t_1 < \ldots < t_n = T \) and that \( \{ T_k; k = 1, \ldots, K \} \subset \Delta \). Let \( |\Delta| = \max_{i=1,\ldots,n} (t_{i+1} - t_i) \). We define estimators \( \hat{c}_i = \hat{c}_i(\varepsilon, \Delta, L) : \Omega \to \mathbb{R}, i = 1, 2, \) in the following.

\[
\hat{c}_1(\varepsilon, \Delta, L)(\omega) = \frac{1}{L} \sum_{\ell=1}^L \sum_{i=0}^{n-1} (t_{i+1} - t_i) g(t_i, X_\ell(t_i)) \left( \sum_{m=1}^M \sum_{k:T_k \geq t_i+1} Q^{(m,L,\omega)}_{\ell,T_k,\varepsilon} F_{m,k}(\pi_k(X_\ell(t_i))) \cap 0 \right), \tag{8}
\]

and

\[
\hat{c}_2(\varepsilon, \Delta, L)(\omega) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ g(t_i, X(t_i, x^*)) \left( \sum_{m=1}^M \sum_{k:T_k \geq t_i+1} F_{m,k}(\pi_k(X(T_k, x^*))) \right) \right] \times \left[ \sum_{m=1}^M \sum_{k:T_k \geq t_i+1} (Q^{(m,L,\omega)}_{\ell,T_k,\varepsilon} F_{m,k}(\pi_k(X(t_i, x^*)))) \right] \geq 0]. \tag{9}
\]

Our main results are following.
Theorem 1 Let \( \alpha_0 = (1 + \delta)(N + 1)\ell_0/4 \vee 1 \). Let \( \{\varepsilon_L\}^\infty_{L=1} \subset (0, \varepsilon_0) \) be a sequence and suppose that there is a constant \( C_0 \in (0, \infty) \) such that \( \varepsilon_L \leq C_0 L^{-\frac{1+\delta}{2\alpha+1}}, L \geq 1 \). Then there exists a constant \( C_1 \in (0, \infty) \) such that

\[
E_P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_0|] \leq C_1 (L^{-\frac{1}{1+\alpha_0}} + \Delta)
\]

for any \( L \geq 1 \) and \( \Delta \in \Pi \).

Theorem 2 Let \( \alpha_1 = (1 + \delta)(N + 1)\ell_0/2 \vee 1 \), and let \( \{\varepsilon_L\}^\infty_{L=1} \subset (0, \varepsilon_0) \) be a sequence such that there is a constant \( C_0 \in (0, \infty) \), such that \( \varepsilon_L \leq C_0 L^{-\frac{1+\delta}{2\alpha+1}}, L \geq 1 \). Suppose that there are constants \( \gamma \in (0, 1] \) and \( C_\gamma \in (0, \infty) \) such that

\[
\sup_{\Delta} \sum_{i=0}^{n-1} (t_{i+1} - t_i) \mu(|N \sum_{m=1}^M \sum_{k:T_k \geq t_{i+1}} (P_{T_k-t_i}^{m}) F_{m,k}(\pi_m X(t_i, x^*))| \leq \theta) \leq C_\gamma \theta^\gamma
\]

for all \( \theta \in (0, 1] \).

Then then there exists a constant \( C_1 \in (0, \infty) \) such that

\[
P(\tilde{\Omega}(L)) \to 1, \quad L \to \infty,
\]

and

\[
1_{\tilde{\Omega}(L)}|\hat{c}_2(\varepsilon_L, \Delta, L) - c_0| \leq C_1 (L^{-1+\frac{(1-\delta)}{2\alpha+1}} + \Delta)
\]

for any \( L \geq 1 \), and \( \Delta \in \Pi \).

Remark 3 Let \( \tilde{\Omega}'(L) \) be

\[
\tilde{\Omega}'(L) = \left\{ \omega \in \Omega; |\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leq CL^{-\frac{1-\delta}{1+\alpha_0}} \right\}.
\]

Then by Theorem 2 we see that

\[
P(\tilde{\Omega}'(L)) \to 1, \quad L \to \infty,
\]

and

\[
1_{\tilde{\Omega}(L)}|\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leq CL^{-\frac{1-\delta}{1+\alpha_0}}.
\]

Theorem 2 shows that the estimation of \( \hat{c}_2 \) may be better than \( \hat{c}_1 \).

We can compute the estimators \( \hat{c}_i, i = 1, 2 \), practically in the following way. First, we generate a family of independent paths

\[
X_1 = \{X_\ell(t); 0 \leq t \leq T, \ell = 1, 2, \ldots, L \}.
\]

Next, by using \( X_1 \), we compute

\[
(Q_{t_i, T_k}^{m, L, \omega}) F_{m,k}(\pi_k (X_m(t_i))), \text{ for every } k \text{ such that } T_k > t_i.
\]

Then our estimator \( \tilde{c}_1 \) is

\[
\tilde{c}_1 = \frac{1}{L} \sum_{\ell=1}^L \sum_{i=0}^{n-1} g(t_i, X_\ell(t_i)) \left( \sum_{k:T_k \geq t_{i+1}} (Q_{t_i, T_k}^{m, L, \omega}) F_{m,k}(\pi_k (X_\ell(t_i))) \lor 0 \right) (t_{i+1} - t_i).
\]
We used the same paths for Monte Carlo simulation and construction of Stochastic mesh operator.

For \( \tilde{c}_2 \), we generate another independent family of independent paths

\[
X_2 = \{X'_m(t); 0 \leq t \leq T, m = 1, 2, \ldots, M \},
\]

and we compute

\[
\tilde{c}_2 = \frac{1}{M} \sum_{m=1}^{M} \sum_{i=0}^{n-1} g(t_i, X'_m(t_i)) \left\{ \sum_{k: T_k \geq t_{i+1}} F_{m,k}(X'_m(T_k)) \right\}
\times 1_{\left\{ \sum_{k: T_k \geq t_{i+1}} (Q_{t,T,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k(X'_m(t_i))) \geq 0 \right\}}(t_{i+1} - t_i).
\]

In the above computations of \( \tilde{c}_1 \) and \( \tilde{c}_2 \), we use the values of \( X_\ell(t_i), t_i \in \Delta \), only. As for the computation of \( \tilde{c}_2 \), we do not use \( Q_{t,T,\varepsilon}^{(m,L,\omega)} F_{m,k} \) explicitly. We use \( Q_{t,T,\varepsilon}^{(m,L,\omega)} F_{m,k} \) only to judge whether \( (P_{T_k-t}^{(k)} F_{m,k})(\pi_k(X'_m(t_i))) > 0 \) or not. So the approximation has no error when the signs of \( (Q_{t,T,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k X(t)) \) and \( (P_{T_k-t}^{(k)} F_{m,k})(\pi_k X'_m(t_i)) \) are the same, even if there are large differences between them.

## 2 Structure of Vector Fields

Let \( A = \bigcup_{k=1}^{\infty} \{0, 1, \ldots, d\}^k \) and \( A^* = A \setminus \{0\} \). For \( \alpha \in A \), let \( |\alpha| = k \) if \( \alpha = (\alpha^1, \ldots, \alpha^k) \in \{0, 1, \ldots, d\}^k \), and let \( \|\alpha\| = |\alpha| + \text{card}\{1 \leq i \leq |\alpha|; \alpha^i = 0\} \). Also, for each \( m \geq 1 \), \( A^*_{\leq m} = \{ \alpha \in A^*; \|\alpha\| \leq m \} \).

We define vector fields \( V_{[\alpha], \alpha \in A} \), inductively by

\[
V_{[\alpha]} = V_i, \quad i = 0, 1, \ldots, d,
\]

\[
V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \quad i = 0, 1, \ldots, d.
\]

Here \( \alpha * i = (\alpha^1, \ldots, \alpha^k, i) \) for \( \alpha = (\alpha^1, \ldots, \alpha^k) \) and \( i = 0, 1, \ldots, d \).

We assume that a system of vector fields \( \{V_i; i = 0, 1, \ldots, d\} \) satisfies the following condition (UFG).

(UFG) There is an integer \( \ell_0 \) and there are functions \( \varphi_{\alpha,\beta} \in C_0^\infty(\mathbb{R}^N) \), \( \alpha \in A^* \), \( \beta \in A^*_{\leq \ell_0} \), satisfying the following.

\[
V_{[\alpha]} = \sum_{\beta \in A^*_{\leq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}, \quad \alpha \in A^*.
\]

**Proposition 4** A system of vector fields \( \{\tilde{V}_i^{(m)}; i = 0, 1, \ldots, d\} \) also satisfies the (UFG) condition.

**Proof.** We prove following by induction on \( |\alpha| \).

\[
V_{[\alpha]}(f \circ \pi_m) = (\tilde{V}_i^{(m)}) f \circ \pi_m, \quad f \in C_0^\infty(\mathbb{R}^{\tilde{N}_m}), \quad (10)
\]
for any $\alpha \in \mathcal{A}$ and $m = 1, \ldots, M$.
It is trivial in the case of $|\alpha| = 1$. By the assumption for induction,
\begin{align*}
v_{[\alpha \pi_i]}(f \circ \pi_m) &= (V_{[\alpha]} V_i - V_i V_{[\alpha]})(f \circ \pi_m) \\
&= V_{[\alpha]} ((V_i^{(m)} f) \circ \pi_m) - V_i ((V_{[\alpha]}^{(m)} f) \circ \pi_m) \\
&= (V_i^{(m)} (V_{[\alpha]}^{(m)} f)) \circ \pi_m - (V_i^{(m)} (V_{[\alpha]}^{(m)} f)) \circ \pi_m.
\end{align*}
So we have \[10\]. From (UFG) condition, we have
\[
v_{[\alpha]}(f \circ \pi_m) = \sum_{\beta \in \mathcal{A}_{\leq t_0}} \varphi_{\alpha, \beta} (V_{[\beta]} f) \circ \pi_m.
\]
Let $j_m : \mathbb{R}^{\tilde{N}_m} \to \mathbb{R}^N$ be
\[
j_m(\tilde{x}_m) = (x_0, 0, \ldots, 0, x_m, 0, \ldots, 0).
\]
Then
\[
V_i^{(m)} f = (V_{[\alpha]}^{(m)} f) \circ \pi_m \circ j_m \\
= (\sum_{\beta \in \mathcal{A}_{\leq t_0}} \varphi_{\alpha, \beta} (V_{[\beta]}^{(m)} f) \circ \pi_m) \circ j_m = \sum_{\beta \in \mathcal{A}_{\leq t_0}} (\varphi_{\alpha, \beta} \circ j_m) V_{[\beta]}^{(m)} f.
\]
So we have our assertion.

Let $A_m(\tilde{x}_m) = (A^{i,j}_{m}(\tilde{x}_m))_{i,j=1,\ldots,\tilde{N}_m}$, $t > 0$, $\tilde{x}_m \in \mathbb{R}^{\tilde{N}_m}$, be a $\tilde{N}_m \times \tilde{N}_m$ symmetric matrix given by
\[
A^{i,j}_{m}(\tilde{x}_m) = \sum_{\alpha \in \mathcal{A}_{\leq t_0}} V_{m,[\alpha]}^{i}(\tilde{x}_m) V_{m,[\alpha]}^{j}(\tilde{x}_m), \quad i, j = 1, \ldots, \tilde{N}_m.
\]
Let $h_m(\tilde{x}_m) = \det A_m(\tilde{x}_m), \tilde{x}_m \in \mathbb{R}^{\tilde{N}_m}$ and
\[
E_m = \{ \tilde{x}_m \in \mathbb{R}^{\tilde{N}_m}; h_m(\tilde{x}_m) > 0 \}.
\]
By \[8\], we see that if $\tilde{x}_m \in E_m$, the distribution law of $\tilde{X}^{(m)}(t, \tilde{x}_m)$ under $\mu$ has a smooth density function $p^{(m)}(t, \tilde{x}_m, \cdot) : \mathbb{R}^{\tilde{N}_m} \to [0, \infty)$ for $t > 0$. Moreover, by \[9\] we see that $\int_{E_m} p^{(m)}(t, \tilde{x}_m, y) dy = 1, \tilde{x}_m \in E_m$. We have $p_m(t, \tilde{x}_m, y) = 0, y \in E_m^c$ by \[9\].
3 Preparations

In this section, we use the notation in [7]. We have the following Lemma similarly to the proof of [7] Lemma 8 (3).

Lemma 5 For any $\Phi \in D_{\infty}^1$, $\alpha \in A^*_\ell$, let

$$(D^{(\beta)}\Phi)(t, x) = (D\Phi(t, x), k^{\beta}(t, x))_H$$

and

$$\Phi_\alpha(t, x) = \sum_{\beta \in A^*_\ell} t^{-\|\alpha\|/2} \{ -D^{(\beta)}\Phi(t, x)M^{-1}_{\alpha\beta}(t, x)$$

$$- \sum_{\gamma_{\ell_1} \in A^*_\ell} \Phi(t, x)M^{-1}_{\alpha\gamma_{\ell_1}}(t, x)D^{(\beta)}M^{-1}_{\gamma_{\ell_2}}(t, x)M^{-1}_{\ell_{\ell_2}}(t, x)$$

$$+ \Phi(t, x)M^{-1}_{\alpha\beta}(t, x))D^*k^{\beta}(t, x)\}, \quad t > 0, x \in \mathbb{R}^N.$$

Then

$$E^\mu[\Phi(t, x)(V_\alpha f)(X(t, x))] = t^{-\|\alpha\|/2}E^\mu[\Phi_\alpha(t, x)f(X(t, x))],$$

and

$$\sup_{t \in [0, T], x \in \mathbb{R}^N, p \in (1, \infty)} E[|\Phi_\alpha(t, x)|^p] < \infty.$$

Let $\varphi$ be a smooth function such that

$$\varphi(z) = \begin{cases} 1, & z \geq 1 \\ 0, & z < 0, \end{cases} \quad (12)$$

$$\varphi'(z) \geq 0. \quad (13)$$

Let $\varphi_m(z) = \varphi(mz)$ and $\bar{\varphi}$ be

$$\bar{\varphi}_m(z) = \int_0^z \varphi_m(z')dz'. \quad (14)$$

Then for any $z \in \mathbb{R}$,

$$\bar{\varphi}_m(z) \rightarrow z \lor 0, \quad m \rightarrow \infty.$$

Lemma 6 If $\Phi \in D_{\infty}^1$, then $|\Phi| \in D_{\infty}^1$.

Proof. Let $\bar{\psi}_m(z) = \bar{\varphi}_m(z) + \varphi_m(-z)$. Then for any $z \in \mathbb{R}$,

$$\bar{\psi}_m(z) \rightarrow |z|, \quad m \rightarrow \infty,$$

and $|\bar{\psi}_m'(z)| \leq 1$. We have

$$D(\bar{\psi}_m(\Phi(t, x))) = \bar{\psi}_m(\Phi(t, x))D\Phi(t, x)$$

$$= (\varphi_m(\Phi(t, x)) - \varphi_m(-\Phi(t, x)))D\Phi(t, x), \quad m \geq 1.$$
Then \(\{D(\tilde{\psi}_{n}(\Phi(t, x)))\}_{n=1}^{\infty}\) is a Cauchy sequence in \(L^{p}(W_{0}, \mathcal{L}_{2}(H; \mathbb{R}))\), \(p > 1\), because
\[
\|D(\tilde{\psi}_{n}(\Phi(t, x))) - D(\tilde{\psi}_{m}(\Phi(t, x)))\|_{H} \leq 1_{\{\Phi(t, x) \in [0, 1/m]\}}\|D\Phi(t, x)\|_{H}, \quad n \geq m \geq 1.
\]
Because \(D : D_{p}^{1} \rightarrow L^{p}(W_{0}, \mathcal{L}_{2}(H; \mathbb{R}))\) is a closed operator, we have \(|\Phi(t, x)| \in D_{p}^{1}\), for any \(p > 1\). So we have the assertion.

Let us denote \(\|\nabla F\|_{\infty} = \sup_{x \in \mathbb{R}^{N}} |(\frac{\partial F}{\partial x_{1}}(x), \ldots, \frac{\partial F}{\partial x_{N}}(x))|\), \(F \in C^{\infty}(\mathbb{R}^{N})\).

**Lemma 7** Let \(T > 0\). Then there exists a \(C > 0\) such that
\[
E[|g(t, X(t, x^{*}))(P_{T-t}F)(X(t, x^{*})) \vee 0 - g(s, X(s, x^{*}))(P_{T-s}F)(X(s, x^{*})) \vee 0|] \leq C\|\nabla F\|_{\infty} \int_{s}^{t} (r^{-1/2} + (T - r)^{-1/2})dr,
\]
for any \(F \in C_{b}^{\infty}(\mathbb{R}^{N})\) and any \(0 < s < t < T\).

**Proof.** Let \(\{M(t)\}_{0 \leq t \leq T}\) be
\[
M(t) = E^{\mu}[F(X(T, x^{*}))|\mathcal{F}_{t}] = (P_{T-t}F)(X(t, x^{*})).
\]
\(\{M(t)\}_{0 \leq t \leq T}\) is a \(\{\mathcal{F}_{t}\}_{t \geq 0}\)-martingale. Let \(Y(t) = g(t, X(t, x^{*})), 0 \leq t \leq T\). Let
\[
L_{t} = \frac{\partial}{\partial t} + V_{0} + \frac{1}{2} \sum_{i=1}^{d} V_{i}^{2}.
\]
By Itô formula,
\[
Y(t)\tilde{\psi}_{m}(M(t)) = Y(s)\tilde{\psi}_{m}(M(s)) + \int_{s}^{t} Y(r)\tilde{\psi}_{m}(M(r))dM(r)
+ \frac{1}{2} \int_{s}^{t} Y(r)\tilde{\psi}_{m}'(M(r))d(M)(r)
+ \int_{s}^{t} \tilde{\psi}_{m}(M(r))dY(r) + \int_{s}^{t} d\langle Y, \tilde{\psi}_{m}(M) \rangle(r).
\]
Note that,
\[
M(t) = M(s) + \sum_{j=1}^{d} \int_{s}^{t} V_{j}(P_{T-r}F)(X(r, x^{*}))dB^{j}(r),
\]
\[
\langle M \rangle(t) = \langle M \rangle(s) + \sum_{j=1}^{d} \int_{s}^{t} (V_{j}(P_{T-r}F)(X(r, x^{*})))^{2}dr,
\]
\[
Y(t) = Y(s) + \sum_{j=1}^{d} \int_{s}^{t} (V_{j}g)(X(r, x^{*}))dB^{j}(r) + \int_{s}^{t} (L_{t}g)(X(r, x^{*}))dr,
\]
and
\[
\langle Y, \tilde{\psi}_{m}(M) \rangle(t) = \langle Y, \tilde{\psi}_{m}(M) \rangle(s) + \sum_{j=1}^{d} \int_{s}^{t} (V_{j}g)(X(r, x^{*}))(V_{j}(P_{T-r}F))(X(r, x^{*}))dr.
\]
So we have
\[ E^\mu[[Y(t)\varphi_m(M(t)) - Y(s)\varphi_m(M(s))]] \]
\[ = \frac{1}{2} \sum_{j=1}^d \int_s^t E^\mu[[Y(r)\varphi_m((P_{T-r}F)(X(r, x^*)))(V_j(P_{T-r}F)(X(r, x^*)))^2] \, dr \]
\[ + \int_s^t E^\mu[[\varphi_m(M(r))(Ltg)(X(r, x^*))] \, dr + \sum_{j=1}^d \int_s^t E^\mu[|(V_jg)(V_j(P_{T-r}F))(X(r, x^*))]| \, dr. \]

Now by the definition of \( \varphi_m \) and \( g \), we have,
\[ \int_s^t E^\mu[[\varphi_m(M(r))(Ltg)(X(r, x^*))] \, dr \leq \int_s^t E^\mu[[M(r)]^{1/2} E^\mu[[Ltg)(X(r, x^*))]^{1/2} \, dr \]
\[ \leq \sup_{t \in [0, T]} E^\mu[[Ltg)(X(r, x^*))]^{1/2} \int_s^t E^\mu[[M(r)]^{1/2} \, dr. \]

By Burkholder’s inequality,
\[ \int_s^t E^\mu[[M(r)]^{1/2} \, dr \leq E^\mu[(M)_r^{1/2}(t - s) \leq \sup_{r \in (s, t)} \|V_j(P_{T-r}F)\|_\infty(t - s). \]

On the other hand, we have,
\[ \sum_{j=1}^d \int_s^t E^\mu[|(V_jg)(X(r, x^*))|] \, dr \]
\[ \leq \|V_j(P_{T-r}F)\|_\infty \sum_{j=1}^d \int_s^t E^\mu[|(V_jg)(X(r, x^*))]| \, dr \]
\[ \leq \sum_{j=1}^d \sup_{r \in (s, t)} \|V_j(P_{T-r}F)\|_\infty(t - s), \]
for any \( F \in C^\infty_0(\mathbb{R}^N) \) and any \( 0 < s < t < T \).

On the other hand
\[ \varphi'_m((P_{T-r}F)(x^*)) (V_j(P_{T-r}F)(x^*))^2 \]
\[ = (V_j(\varphi_m \circ (P_{T-r}F)))(x^*) (V_j(P_{T-r}F))(x^*) \]
\[ = V_j(\varphi_m \circ (P_{T-r}F)(x^*))V_j(P_{T-r}F)(x^*) - \varphi_m \circ (P_{T-r}F)(x^*)V_j^2(P_{T-r}F)(x^*). \]

Notice that \( \varphi'_m \geq 0 \), we have
\[ E^\mu[[Y(r)\varphi_m((P_{T-r}F)(X(r, x^*)))\varphi_m((P_{T-r}F)(X(r, x^*))][ ] \]
\[ = E^\mu[[Y(r)\varphi'_m((P_{T-r}F)(X(r, x^*)))\varphi_m((P_{T-r}F)(X(r, x^*))][ ] \]
\[ = I_{1,j}(r, f) - I_{2,j}(r, f), \]
where
\[ I_{1,j}(r, F) = E^\mu[[g(r, X(r, x^*))]|V_j(\varphi_m \circ (P_{T-r}F)V_j(P_{T-r}F))(X(r, x^*))], \]
Let \( I_{2,j}(r, F) = E^\mu[[g(r, X(r, x^*))]|\varphi_m \circ (P_{T-r}F)(X(r, x^*))]V_j^2(P_{T-r}F)(X(r, x^*))]. \)

Let \( \Phi_g(r, x) = |g(r, X(r, x^*))| \). Then by Lemma \([6]\) \( \Phi_g \in D^1_p \). Let \( \Phi_{g,i}(r, x), i = 1, \ldots, N \) be defined by the formula of Lemma \([3]\). Then we have

\[
I_{1,j}(r, F) = r^{-1/2} E^\mu[\Phi_{g,j}(r, x)\varphi_m \circ (P_{T-r}F)(X(r, x^*))]V_j(P_{T-r}F)(X(r, x^*)),
\]

and

\[
\sup_{t \in [0, T], x \in \mathbb{R}^N} E^\mu[|\Phi_{g,i}(t, x)|^p] < \infty.
\]

Then there exists a constant \( C > 0 \) such that

\[
|I_{1,j}(r, F)| \leq Cr^{-1/2}\|V_j(P_{T-r}F)\|_\infty.
\]

Also we have

\[
|I_{2,j}(r, F)| \leq CE^\mu[|g(r, X(r, x^*))|]V_j^2(P_{T-r}F)\|_\infty,
\]

for any \( F \in C^\infty_b(\mathbb{R}^N) \) and any \( 0 < r < T \).

Let vector field \( V_j \) be represented by \( V_j = \sum_{i=1}^N v_j^i(x) \frac{\partial}{\partial x_i} \). Then we have

\[
V_j(P_{T-r}F)(x) = \sum_{i=1}^N \sum_{k=1}^N v_j^i(x)(T_{\Phi_{k,i}}(T-r) \frac{\partial F}{\partial x_i})(x),
\]

where \( \Phi_{k,i}(t, x) = \frac{\partial x^k(t, x)}{\partial x_i} \) and

\[
(T_{\Phi_{k,i}}(t)F)(x) = E^\mu[\Phi_{k,i}(t, x)F(X(t, x))].
\]

Moreover, we have

\[
V_j^2(P_{T-r}F)(x) = \sum_{i=1}^N \sum_{k=1}^N (V_jv_j^i(x)(T_{\Phi_{k,i}}(T-r) \frac{\partial F}{\partial x_i}))(x) + v_j^i(x)(V_jT_{\Phi_{k,i}}(T-r) \frac{\partial F}{\partial x_i})(x).
\]

Then by Corollary 9 of \([7]\), since \( \Phi_{k,i} \in K_0 \) and there is a constant \( C > 0 \) such that

\[
\|V_j(P_{T-r}F)\|_\infty \leq C\|\nabla F\|_\infty,
\]

and

\[
\|V_j^2(P_{T-r}F)\|_\infty \leq C(T-r)^{-1/2}\|\nabla F\|_\infty,
\]

for any \( F \in C^\infty_b(\mathbb{R}^N), j = 1, \ldots, d \), and any \( 0 < r < T \).

So we have

\[
E[[g(t, X(t, x^*))]|\varphi_m((P_{T-t}F)(X(t, x)) - g(s, X(s, x^*))|\varphi_m((P_{T-s}F)(X(s, x)))]
\leq C\|\nabla F\|_\infty \int_s^t (r^{-1/2} + (T-r)^{-1/2})dr.
\]

Letting \( m \to \infty \), we have our assertion.
Corollary 8 Let $T > 0$. There exists a constant $C > 0$ such that
\[
E[|g(t, X(t, x^*)) (P_{T-t} F)(X(t, x^*)) \vee 0 - g(s, X(s, x^*)) (P_{T-s} F)(X(s, x^*)) \vee 0|] 
\leq C\|F\|_{\text{Lip}} \int_s^t (r^{-1/2} + (T - r)^{-1/2}) dr,
\]
for any $F \in \text{Lip}(\mathbb{R}^N)$ and any $0 < s < t < T$.

Proof. For $F \in \text{Lip}(\mathbb{R}^N)$, there exists $F_m \in C_b^\infty(\mathbb{R}^N)$, $m = 1, 2, \ldots$, such that $\|\nabla F_m\| \leq \|F\|_{\text{Lip}}$ and $F_m(x) \to F(x)$, for any $x \in \mathbb{R}^N$. So we obtain the result from Lemma [7].

Lemma 9 Let $m = 1, \ldots, M$, and $T > 0$. There exists a constant $C > 0$ such that
\[
E^\mu[|g(t, X(t, x^*)) (P_{T-t}^{(m)} h)(\tilde{X}^{(m)}(t, \tilde{x}_m^*)) - h(\tilde{X}^{(m)}(t, \tilde{x}_m^*))|] 
\leq C(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T - t),
\]
and
\[
E^\mu[|g(t, X(t, x^*)) (P_{T-t}^{(m)} (h \vee 0))(\tilde{X}^{(m)}(t, \tilde{x}_m^*)) - (h \vee 0)(\tilde{X}^{(m)}(t, \tilde{x}_m^*))|] 
\leq C(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T - t).
\]
for any $h \in C_b^\infty(\mathbb{R}^{\tilde{N}_m})$, $t \in [0, T)$.

Proof. [15] follows from Itô’s formula. So we show [16].
Let $\varphi_k, k = 1, \ldots$, are as defined in [14]. Let
\[
\tilde{L}_m = \tilde{V}_0^{(m)} + \frac{1}{2} \sum_{i=1}^d (\tilde{V}_i^{(m)})^2.
\]
By Itô’s formula
\[
\varphi_k(h(\tilde{X}^{(m)}(T, \tilde{x}_m^*))) - \varphi_k(h(\tilde{X}^{(m)}(t, \tilde{x}_m^*)))
= \int_t^T \varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))) (\tilde{V}_i^{(m)} h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*)) dB^{m,i}(s)
+ \int_t^T \varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))) (\tilde{L}_m h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*)) ds
+ \frac{1}{2} \int_t^T \varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))) (\tilde{V}_i^{(m)} h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*))^2 ds
\]
So we have
\[
E^\mu[\varphi_k(h(\tilde{X}^{(m)}(T, \tilde{x}_m^*))) | \tilde{F}^{(m)}_t] - \varphi_k(h(\tilde{X}^{(m)}(t, \tilde{x}_m^*)))
= \int_t^T E^\mu[\varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))) (\tilde{L}_m h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*)) | \tilde{F}^{(m)}_t] ds
+ \frac{1}{2} \sum_{i=1}^{\tilde{d}_m} \int_t^T E[\varphi_k'(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))) ((\tilde{V}_i^{(m)} h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*)))^2] | \tilde{F}^{(m)}_t] ds
\]

Notice that $\varphi'_k \geq 0$, then we have
\begin{align*}
E[\|g(t, X(t, x^*))E^\mu[\varphi_k(h(\tilde{X}^{(m)}(T, \tilde{x}_m^*))) - \varphi_k(h(\tilde{X}^{(m)}(t, \tilde{x}_m^*)))]] & \leq E[\|g(t, X(t, x^*))\|]\tilde{L}_m h\|_\infty (T - t) \\
& + \frac{1}{2} \sum_{i=1}^{\tilde{d}_m} \int_t^T E[\|g(t, X(t, x^*))\|\varphi'_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*)))(\tilde{V}_i^{(m)} h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))^2)\|ds.
\end{align*}

On the other hand, we have
\begin{align*}
\varphi'_k(h(\tilde{x}_m^*))(\tilde{V}_i^{(m)} h(\tilde{x}_m^*))^2 &= \tilde{V}_i^{(m)}(\varphi_k \circ h)(\tilde{x}_m^*) (\tilde{V}_i^{(m)} h)(\tilde{x}_m^*) \\
& = \tilde{V}_i^{(m)} ((\varphi_k \circ h)(\tilde{x}_m^*)V_i^{(m)} h(\tilde{x}_m^*)) - (\varphi_k \circ h)(\tilde{x}_m^*)V_i^{(m)} h(\tilde{x}_m^*)^2 h(\tilde{x}_m^*).
\end{align*}

Let $\Phi_g(t, x) = \|g(t, X(t, x^*))\|$ and $\Phi_{g,i}(t, x^*), i = 1, \ldots, N$ be defined by the formula of Lemma 5. Then it follows that
\begin{align*}
& \quad \|E^\mu[\Phi_g(t, x)\tilde{V}_i^{(m)} ((\varphi_k \circ h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*))V_i^{(m)} h(\tilde{X}^{(m)}(s, \tilde{x}_m^*)))\|_\infty \\
& \leq C_s^{-1/2}E^\mu[\|\Phi_{g,i}(t, x^*)\|((\varphi_k \circ h)\tilde{V}_i^{(m)} h)\|_\infty \leq C_s^{-1/2}\|\tilde{V}_i^{(m)} h\|_\infty,
\end{align*}

and
\begin{align*}
\|E^\mu[\Phi_g(t, x)(\varphi_k \circ h)(\tilde{X}^{(m)}(s, \tilde{x}_m^*))(\tilde{V}_i^{(m)} h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))^2 h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))]\|_\infty \leq C\|\tilde{V}_i^{(m)} h\|_\infty.
\end{align*}

So we have
\begin{align*}
\frac{1}{2} \sum_{i=1}^{\tilde{d}_m} \int_t^T E[\|g(t, X(t, x^*))\|\varphi'_k(h(\tilde{X}^{(m)}(s, \tilde{x}_m^*)))(\tilde{V}_i^{(m)} h(\tilde{X}^{(m)}(s, \tilde{x}_m^*))^2)\|ds \\
& \leq \int_t^T C'(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty) (1 + s^{-1/2})ds \\
& \leq C'(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T - t)(1 + (T^{1/2} + t^{1/2})^{-1}).
\end{align*}

Letting $k \to \infty$, we have the assertion.

\textbf{Corollary 10} Let $m = 1, \ldots, M, T > 0$ and $F \in \mathcal{M}(\hat{R}^{\tilde{N}_m})$. There exists a constant $C > 0$ such that
\begin{align*}
E^\mu[\|g(t, X(t, x^*))((P_{T,t}^{(m)}) F)\|_\infty (\pi_m X(t, x^*)) - F(\pi_m X(t, x^*))]\| \leq C(T - t),
\end{align*}
for any $t \in [0, T]$.

\textbf{Proof.} Notice that $\pi_m X(t, x^*) = \tilde{X}^{(m)}(t, \tilde{x}_m^*)$ and Lemma 9 is valid for $h \in \hat{D}(\hat{R}^{\tilde{N}_m})$. On the other hand, for $F \in \mathcal{M}(\hat{R}^{\tilde{N}_m})$, we have the expression that
\begin{align*}
F = \sum_{k=1}^{K_F} a_k (f_k \lor g_k) = \sum_{k=1}^{K_F} a_k ((f_k - g_k) \lor 0 + g_k),
\end{align*}
\begin{align*}
a_k \in \mathbb{R}, f_k, g_k \in \hat{D}(\hat{R}^{\tilde{N}_m}), k = 1, \ldots, K_F. \text{ So our assertion follows from Lemma 9.}
4 Discretization

Let $c_\Delta, \Delta \in \Pi$, be given by
\[
    c_\Delta = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu[g(t_i, X(t_i, x^*))\{\sum_{m=1}^{M} \sum_{k=0}^{K} (P^{(m)}_{T_k - t_i} F_{m,k})(\pi_m X(t_i, x^*))\} \lor 0].
\]

Let $i(k), k = 1, \ldots, K$, be such that $T_k = t_{i(k)}$. Then we have $c_\Delta$ is as follows.
\[
    c_\Delta = \sum_{k=1}^{K} \sum_{i=i(k-1)}^{i(k)-1} (t_{i+1} - t_i) E^\mu[g(t_i, X(t_i, x^*))\{\sum_{m=1}^{M} \sum_{k'=k}^{K} (P^{(m)}_{T_k - t_i} F_{m,k'}) (\pi_m X(t_i, x^*))\} \lor 0]
\]

Let $F_t^{(\infty)}$, $t \geq 0$, be sub $\sigma$-algebra of $\mathcal{F}$ given by
\[
    F_t^{(\infty)} = \sigma\{X_\ell(s); s \in [0,t], \ell = 1,2,\ldots\}.
\]

**Proposition 11** There exists a constant $C > 0$ such that
\[
    |c_0 - c_\Delta| \leq C|\Delta|, \quad \Delta \in \Pi.
\]

**Proof.** Let
\[
    \tilde{F}_k(x) = \sum_{m=1}^{M} \sum_{k'=k}^{K} (P^{(m)}_{T_{k'} - T_k} F_{m,k'}) (\pi_m x), k = 1, \ldots, K.
\]

Then by Lemma 7, there is a constant $C > 0$ such that
\[
    c_0 - c_\Delta \leq \sum_{k=1}^{K} \sum_{i=i(k-1)}^{i(k)-1} \int_{t_i}^{t_{i+1}} \left| E^\mu[g(t, X(t, x^*))\left( (P_{T_k - t} \tilde{F}_k)(X(t, x^*)) \lor 0 \right)] - E^\mu[g(t_i, X(t_i, x^*)) (P_{T_k - t_i} \tilde{F}_k)(X(t_i, x^*)) \lor 0] \right| dt
\]
\[
    \leq C \int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t} (r^{-1/2} + (T_k - r)^{-1/2}) dr
\]
\[
    \leq C |\Delta| \sum_{k=1}^{K} \sum_{i=i(k-1)}^{i(k)-1} \int_{t_i}^{t_{i+1}} (r^{-1/2} + (T_k - r)^{-1/2}) dr.
\]

So we have
\[
    |c_0 - c_\Delta| \leq C |\Delta| \sum_{k=1}^{K} \int_{T_k-1}^{T_k} (r^{-1/2} + (T_k - r)^{-1/2}) dr.
\]

So the assertion follows.
5 Property of Stochastic Mesh Operator

To estimate the stochastic mesh operator, we use the following estimation of transition kernel $p^{(m)}(t, x_m, x)$ obtained by Proposition 8 of [9].

Proposition 12 Let $\delta_0^{(m)}$ be given by

$$\delta_0^{(m)} = \left(3 \Phi_m \left( \sup_{x \in \mathbb{R}^{\tilde{N}_m}} \sum_{i=1}^{d} |V_i^{(m)}(x)|^2 \right) \right)^{-1},$$

then for any $T > 0$, and $m = 1, \ldots, M$, there is a $C > 0$ such that

$$p^{(m)}(t, x, y) \leq Ct^{-(\tilde{N}_m + \ell_0)/2} h_m(x) - 2(\tilde{N}_m + \ell_0) \exp\left(-\frac{2\delta_0^{(m)}}{t}|y-x|^2\right), \quad t \in (0, T], \ x, y \in E_m,$$

and

$$p^{(m)}(t, x, y) \leq Ct^{-(\tilde{N}_m + \ell_0)/2} h_m(y) - 2(\tilde{N}_m + \ell_0) \exp\left(-\frac{2\delta_0^{(m)}}{t}|y-x|^2\right), \quad t \in (0, T], \ x, y \in E_m.$$

In particular, for any $T > 0, m = 1, \ldots, M$, and $q \geq 1$, there is a $C > 0$ such that

$$p^{(m)}(t, x, y) \leq Ct^{-(\tilde{N}_m + \ell_0)/2} h_m(x) - 2(\tilde{N}_m + \ell_0) (1 + |x|^2)^q (1 + |y|^2)^{-q}, \quad t \in (0, T], \ x, y \in E_m.$$

Let $\nu_t^{(m)}(dx) = p^{(m)}(t, x_m^*, x)dx$. From Proposition 13, 21 and Proposition 15 (1) of [9], we have the followings.

Proposition 13 Let $t > 0, f \in L^2(E_m; d\nu_t^{(m)})$ and $t > s \geq 0$. Then we have

$$E^P\left[\left|Q^{(m,L,\omega)}_{s,t} f(x)\right|^2 | \mathcal{F}_s^{(\infty)} \right] = \left(P^{(m)}_{s,t} f\right)(x), \quad \nu_t^{(m)} - \text{a.e.} \ x \in E_m,$$

and

$$E^P\left[\left|Q^{(m,L,\omega)}_{s,t} f(x) - \left(P^{(m)}_{s,t} f\right)(x)\right|^2 | \mathcal{F}_s^{(\infty)} \right] \leq \frac{1}{L} \int_{E_m} \frac{p^{(m)}(t-s, x, y)^2 |f(y)|^2}{q^{(m,L,\omega)}_{s,t}(y)} dy.$$

Proposition 14 Let $\delta \in (0, 1)$ then there exists a $C > 0$ such that

$$\left(1 \over L \sum_{\ell=1}^{L} E^P\left[\left|Q^{(m)}_{t,T_k,\ell} f(x_m)(X_{\ell}(t)) - \left(P^{(m)}_{t,T_k} f\right)\left(\pi_m(X_{\ell}(t))\right)\right|^2\right]\right)^{1/2} \leq CL^{-(1-\delta)/2} (T_k - t)^{-(1+\delta)(\tilde{N}_m + \ell_0)/4} \left(\int_{E_m} f(y)^2 (1 + |y|^2)^{-\tilde{N}_m} dy\right)^{1/2},$$

for any $\varepsilon > 0$ any $m = 1, \ldots, M$, and any $f \in Lip(R^{\tilde{N}_m})$.

Proposition 15 Let

$$Z_L^{(m,k)}(t; \delta) = \sup_{y \in R^{\tilde{N}_m}} \left|q^{(m,L,\omega)}_{t,T_k}(y) - p^{(m)}(T_k, \bar{x}, y)\right| \left(L^{-1/(1-\delta)} + p^{(m)}(T_k, \bar{x}, y)\right)^{(1-\delta)/2}.$$
Then we have the followings.
(1) For any $\delta \in (0, 1)$, and $p > 1$ , there is a $C_{p, \delta} > 0$ such that

$$E^P[(L^{(1-\delta^2)/2} \tilde Z^{(m,k)}_L(T_k - \varepsilon; \delta))^p]^{1/p} \leq C_{p,\delta} \varepsilon^{-5d_0 L^{-p\delta^2/2+1/p}}$$

for any $\varepsilon \in (0, T_k], k = 1, \ldots, K$, and $L \geq 1$.
(2) Let $\delta \in (0, 1), t \in (0, T_k)$ and $\varepsilon \in (0, T)$. If $L^{(1-\delta^2)/2} \tilde Z^{(m,k)}_L(t; \delta) \leq 1/4$, and $p^{(m)}(T_k, x_0, y) \geq L^{-(1-\delta)}$, then

$$\frac{1}{2} \leq \frac{q^{(m,k,L)}(y)}{p^{(m)}(T_k, \tilde x^*, y)} \leq 2,$$

for any $t \in (0, T_k - \varepsilon], \quad k = 1, \ldots, K$. and $L \geq 1$.

Now we introduce the following sets and functions. Let $B^{(m,k)}(t, \delta, L) \in \mathcal{F}$, $\varphi_{m,k,L}, m = 1, \ldots, M, k = 1, \ldots, K$, be given by

$$B^{(m,k)}(t, \delta, L) = \{ \omega \in \Omega; L^{(1-\delta^2)/2} \tilde Z^{(m,k)}_L(t; \delta) \leq 1/4 \},$$

and

$$\varphi_{m,k,L}(y) = 1_{\{ y \in E_m; p^{(m)}(T_k, x_0, y) > L^{-(1-\delta)} \}}.$$

Let $d^{(m,k)}_{i,\varepsilon,L} : [0, T] \times E \times \Omega \rightarrow [0, \infty), i = 1, 2, 3$, be the measurable functions given by

$$d^{(m,k)}_{1,\varepsilon,L}(t, x) = |\langle Q^{(m,L,\omega)}_{t,T_k,\varepsilon} (1 - \varphi_{m,k,L}) F_{m,k} \rangle (\pi_m(x)) - (p^{(m)}_{T_k-t} (1 - \varphi_{m,k,L}) F_{m,k} (\pi_m(x)))|1_{[0,T_k-\varepsilon)}(t),$$

$$d^{(m,k)}_{2,\varepsilon,L}(t, x) = 1_{B^{(m,k)}(T_k-\varepsilon,\delta,L)} |\langle Q^{(m,L,\omega)}_{t,T_k,\varepsilon} \varphi_k F_{m,k} \rangle (\pi_m(x)) - (p^{(m)}_{T_k-t} \varphi_{m,k,L} F_{m,k} (\pi_m(x)))|1_{[0,T_k-\varepsilon)}(t),$$

$$d^{(m,k)}_{3,\varepsilon,L}(t, x) = |\langle Q^{(m,L,\omega)}_{t,T_k,\varepsilon} F_{m,k} \rangle (\pi_m(x)) - (p^{(m)}_{T_k-t} F_{m,k} (\pi_m(x)))|1_{[T_k-\varepsilon,T_k)}(t)$$

$$= |F_{m,k}(\pi_m(X(T_k, x^*))) - (p^{(m)}_{T_k-t} F_{m,k} (\pi_m(x)))|1_{[T_k-\varepsilon,T_k)}(t), \quad k = 1, \ldots, K.$$

Let $p(t, x, dy)$ be the transition kernel of $X(t, x)$.

**Proposition 16** Let $\delta \in (0, 1)$. Then there exists a constant $C > 0$ such that

$$\int_E E^P \left[ d^{(m,k)}_{1,\varepsilon,L}(t, x) |g(t, x)|p(t, x^* , dx) \right] \leq C L^{-(1-\delta)} 1_{[0,T_k-\varepsilon)}(t), \quad (18)$$

$$\int_{E_m} E^P \left[ d^{(m,k)}_{2,\varepsilon,L}(t, x)^2 |p(t, x^*, dx) \right]^{1/2} \leq C L^{-(1-\delta)/2} (T_k - t)^{-(1+\delta)(\bar{N}+1)/6}/1_{[0,T_k-\varepsilon)}(t), \quad (19)$$

and

$$\int_{E_m} d^{(m,k)}_{3,\varepsilon,L}(t, x) |g(t, x)|p(t, x^*, dx) \leq C (T_k - t) 1_{[T_k-\varepsilon,T_k)}(t). \quad (20)$$

for any $\varepsilon \in (0, \varepsilon_0), t \in (0, T_k], L \geq 1, m = 1, \ldots, M$, and $k = 1, \ldots, K$. 


Proof. Equation (20) follows from Lemma 6. So we will show (18) and (19). Note that if \( t \geq T_k - \varepsilon \), both sides are 0 in (18) and (19). So we will consider the case \( t < T_k - \varepsilon \). By Proposition 13, we have

\[
\int_E E^P [g^{(m,k)}_{1,\varepsilon,L}(t, \pi_m(x))] |g(t, x)| p(t, x^*, dx)
\]

Using Hölder’s inequality for \( p = \frac{1}{\delta}, q = \frac{1}{1-\delta} \),

\[
\int_E (P_{T_k-t}^{(m)}(1 - \varphi_{m,k,L})|F_{m,k}|)(\pi_m(x))|g(t, x)| p(t, x^*, dx)
\]

\[
\leq \{ \int_E (P_{T_k-t}^{(m)}(1 - \varphi_{m,k,L})|F_{m,k}|)(\pi_m(x))1^{1-\delta}p(t, x^*, dx) \}^{1-\delta}
\]

\[
\times \{ \int_E |g(t, x)|^{1/\delta} p(t, x^*, dx) \}^\delta
\]

\[
\leq \{ \int_{E_m} (1 - \varphi_{m,k,L}(\bar{y}_m))1^{1-\delta}|F_{m,k}(\bar{y}_m)|1^{1-\delta}p(\pi_m(x^*, \bar{y}_m)dy_m \}^{1-\delta}
\]

\[
\times \{ \int_E |g(t, x)|^{1/\delta} p(t, x^*, dx) \}^\delta
\]

\[
\leq L^{-(1-\delta)^2} \{ \int_{E_m} |F_{m,k}(\pi_m(y))|1^{(1-\delta)}p^{(m)}(T_k, \pi_m(x^*, \bar{y}_m)dy_m \}^{1-\delta}
\]

\[
\times \{ \int_E |g(t, x)|^{1/\delta} p(t, x^*, dx) \}^\delta
\]

We used \((1 - \varphi_{m,k,L}(\bar{y}_m))1^{(1-\delta)}p^{(m)}(T_k, \pi_m(x^*, \bar{y}_m)1^{(1-\delta)} \leq L^{-(1-\delta)^2} \) in the last inequality. So we have Equation (18).

Next we will show Equation (19). Noting that from \( B^{(m,k)}(T_k - \varepsilon, \delta, L) \subset B^{(m,k)}(t, \delta, L), t \in [0, T_k - \varepsilon], k = 1, \ldots, K, \) and \( L \geq 1, \)

\[
d_{\varepsilon,L}^{(m,k)}(t, x) \leq 1_{B^{(m,k)}(t, \delta, L)}(Q_{t,T_k,\varepsilon}^{(m)}(1 - \varphi_{m,k,L})F_{m,k})(\pi_m(x)) - (P_{T_k-t}^{(m)}(1 - \varphi_{m,k,L})F_{m,k})(\pi_m(x))\|
\]

Since Proposition 15, \( 1_{B^{(m,k)}(t, \delta, L)}q_{T_k}^{(m,L,\omega)}(\bar{y}_m)^{-1} \leq 2p^{(m)}(T_k, \pi_m(x^*, \bar{y}_m)^{-1}. \) And by Proposition 13, we have

\[
1_{B^{(m,k)}(t, \delta, L)}E^P [((Q_{t,T_k,\varepsilon}^{(m)}(1 - \varphi_{m,k,L})F_{m,k})(\pi_m(x)) - (P_{T_k-t}^{(m)}(1 - \varphi_{m,k,L})F_{m,k})(\pi_m(x))^2 | \mathcal{F}_t]
\]

\[
\leq 1_{B^{(m,k)}(t, \delta, L)} \frac{1}{L} \int_{E_m} \left\{ \frac{|\varphi_{m,k,L}(\bar{y}_m)|^2}{q_{T_k}^{(m,L,\omega)}(\bar{y}_m)} \right\}^2 \frac{p^{(m)}(T_k - t, \bar{x}_m, \bar{y}_m)2}{dy_m}
\]

\[
\leq 2 \frac{1}{L} \int_{E_m} \frac{(|\varphi_{m,k,L}(\bar{y}_m)|^2 p^{(m)}(T_k - t, \bar{x}_m, \bar{y}_m)^2}{dy_m}
\]

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Then we have
\[
\left( \int_E E^P[d_{2,\varepsilon,L}(t,x)^2]p(t,x^*,dx) \right)^{1/2} \leq \left( \int_E E^P[1_{B^{(m,k)}(t,\delta,L)}E^P[(Q^{(m)}_{t,T_k} \tilde{\varphi}_{m,k,L} F_{m,k})(x)] - (P^{(m)}_{T_k-t} \tilde{\varphi}_{m,k,L} F_{m,k})(x)]^2|\mathcal{F}_t]|p(t,x^*,dx) \right)^{1/2}
\]
\[
\leq \left( \frac{2}{L} \int_{E_m} \frac{|\tilde{\varphi}_{m,k,L} F_{m,k}(y)|^2}{p^{(m)}(T_k, \pi_m(x^*), \tilde{y}_m)} \left( \int_E p^{(m)}(T_k - t, \tilde{x}_m, \tilde{y}_m)^{(1-\delta)+(1+\delta)}p(t,x^*,dx)d\tilde{y}_m \right)^{1/2}
\right)
\]
\[
\leq \left( \frac{2}{L} \int_{E_m} \frac{|\tilde{\varphi}_{m,k,L} F_{m,k}(y)|^2}{p^{(m)}(T_k, \pi_m(x^*), \tilde{y}_m)} \left( \int_{E_m} p^{(m)}(t, \pi_m(x^*) \tilde{x}_m) p^{(m)}(T_k - t, \tilde{x}_m, \tilde{y}_m)^{(1+\delta)/\delta} d\tilde{x}_m \right)^{1/2} d\tilde{y}_m \right)^{1/2}.
\]
Let \( q \geq \hat{N} \). From Lemma [12], there exists a constant \( C > 0 \) such that
\[
p^{(m)}(T_k - t, \tilde{x}_m, \tilde{y}_m) \leq C(T_k - t)^{-(\hat{N}_m + 1)\epsilon_0/2} h_m(\tilde{x}_m)^{-(\hat{N}_m + 1)\epsilon_0} (1 + |\tilde{x}_m|^2)^q (1 + |\tilde{y}_m|^2)^{-q}.
\]
We set \( C_1 \) as
\[
C_1 = \sup_{t \in [0,T]} \max_{m=1,\ldots,\hat{N}} \left( \int_{E_m} h_m(x)^{-(\hat{N}_m + 1)\epsilon_0(1+\delta)/\delta} (1 + |x|^2)^q (1+\delta)^q p^{(m)}(t, x^*, x) dx \right)^{\delta/2}
\]
\[
\times \left( \int_E |g(t,x)| dx \right)^{1/2}.
\]
\( C_1 \) is bounded by Proposition 3 of [9]. Then since \( \varphi_{m,k,L}(y)p^{(m)}(T_k, \tilde{x}_m, y)^{-\delta} \leq L^\delta \), we have
\[
\int_E E^P[d_{2,\varepsilon,L}(t,x)^2]^{1/2}|g(t,x)|p(t,x^*,dx)
\]
\[
\leq C_1 L \int_{E_m} p^{(m)}(T_k, \tilde{x}_m, \tilde{y}_m)^{-\delta} |\varphi_{m,k,L} F_{m,k}(\tilde{y}_m)|^2 (1 + |\tilde{y}_m|^2)^{-q(1+\delta)} d\tilde{y}_m (T_k - t)^{-(1+\delta)(\hat{N}_m + 1)\epsilon_0/2},
\]
\[
\leq C_1 L^{-(1-\delta)}(T_k - t)^{-(1+\delta)(\hat{N}_m + 1)\epsilon_0/2} \int_{E_m} |F_{m,k}(\tilde{y}_m)|^2 (1 + |\tilde{y}_m|^2)^{-q(1+\delta)} d\tilde{y}_m.
\]
Since \( q \geq \hat{N} \), and \( F_{m,k} \) is Lipschitz continuous,
\[
\int_{\mathbb{R}^{\delta_m}} |F_{m,k}(\tilde{y}_m)|^2 (1 + |\tilde{y}_m|^2)^{-q(1+\delta)} d\tilde{y}_m < \infty.
\]
So we have the assertion.

Let \( a, b, \alpha, \beta \geq 0 \), and \( a_i, b_i, \alpha_i, \beta_i \geq 0, i = 1, 2 \). Let \( \phi^{(k)}(t, \varepsilon; a, \alpha, b, \beta) \) and \( \hat{e}(\varepsilon, \gamma), t \in [0, T_k) \) be
\[
\phi^{(k)}(t, \varepsilon; a, \alpha, b, \beta) = a(T_k - t)^{-\alpha} 1_{[0,T_k-\varepsilon]}(t) + b(T_k - t)^{\beta} 1_{[T_k-\varepsilon,T_k]}(t),
\]
\[
\hat{e}(\varepsilon, \gamma) = \begin{cases}
\varepsilon^{-(\gamma-1)}, & \gamma > 1,
\log \varepsilon, & \gamma = 1,
1, & 0 \leq \gamma < 1.
\end{cases}
\]
Proposition 17 There exists a constant $C > 0$ such that

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k; T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; a, \alpha, b, \beta) \leq C(a \hat{\varepsilon}(\varepsilon, \alpha) + b\varepsilon^{\beta+1}),$$

(21)

and,

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k; T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; a_1, \alpha_1, b_1, \beta_1)\left(\sum_{k; T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; a_2, \alpha_2, b_2, \beta_2)\right) \leq C\left(a_1a_2\hat{\varepsilon}(\varepsilon, \alpha_1 + \alpha_2) + a_1b_2\varepsilon^{\beta_2+1} + a_2b_1\varepsilon^{\beta_1+1} + b_1b_2\varepsilon^{\beta_1+\beta_2+1}\right),$$

(22)

for any $\varepsilon > 0$.

Proof. Let us take $i_{(k)}$ as $t_{i_{(k)}} = T_k, k = 1, \ldots, K$. If $t_i \in [T_{k-1}, T_k]$ and $k' > k$ then $T_{k'} - t_i > \varepsilon$. So notice that

$$1_{[T_{k'} - \varepsilon, T_{k'})}(t_i) = 0, \quad \text{for } i_{(k-1)} \leq i \leq i_{(k)} - 1, k' > k.$$

(23)

So we have

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k; T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; a, \alpha, b, \beta) = \sum_{k=1}^{K} \sum_{t_{i_{(k-1)}}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta) = \sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta) \leq \sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta)$$

and

$$\sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta) = \sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta) \leq \sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta)$$

because $(T_{k'} - t_i)^{-\alpha} \leq (T_{k'} - t)^{-\alpha}$ for $t_i \leq t \leq t_{i+1}$.

On the other hand,

$$\sum_{k=1}^{K} \sum_{k' = k}^{K} \int_{T_{k'} - \varepsilon}^{T_{k'} - t} (T_{k'} - t)^{-\alpha} dt \leq K^2 \hat{\varepsilon}(\varepsilon, \alpha),$$

and

$$\sum_{k=1}^{K} \sum_{i_{(k-1)}}^{i_{(k)}-1} \sum_{k' = k}^{K} \phi^{(k')}(t_i, \varepsilon; a, \alpha, b, \beta),$$

because $(T_{k'} - t_i)^{-\alpha} \leq (T_{k'} - t)^{-\alpha}$ for $t_i \leq t \leq t_{i+1}$.

So we have Equation (21).

Next we show Equation (22).
where
\[ I_{i,1}^{(k)} = \sum_{k_1, k_2 = k}^{K} a_1 b_1 (T_{k_1} - t_i)^{-\alpha_1} (T_{k_2} - t_i)^{-\alpha_2} 1_{[0, T_{k_1} - \varepsilon]}(t_i) 1_{[0, T_{k_2} - \varepsilon]}(t_i), \]
\[ I_{i,2}^{(k)} = \sum_{k_1, k_2 = k}^{K} a_1 b_2 (T_{k_1} - t_i)^{-\alpha_1} (T_{k_2} - t_i)^{\beta_2} 1_{[0, T_{k_1} - \varepsilon]}(t_i) 1_{[0, T_{k_2} - \varepsilon]}(t_i), \]
\[ I_{i,3}^{(k)} = \sum_{k_1, k_2 = k}^{K} a_2 b_1 (T_{k_2} - t_i)^{-\alpha_2} (T_{k_1} - t_i)^{\beta_1} 1_{[0, T_{k_2} - \varepsilon]}(t_i) 1_{[0, T_{k_1} - \varepsilon]}(t_i), \]
\[ I_{i,4}^{(k)} = \sum_{k_1, k_2 = k}^{K} b_1 b_2 (T_{k_1} - t_i)^{\beta_1} (T_{k_2} - t_i)^{\beta_2} 1_{[0, T_{k_1} - \varepsilon]}(t_i) 1_{[0, T_{k_2} - \varepsilon]}(t_i). \]

Note that (23) and
\[ 1_{[0, T_{k_1} - \varepsilon]}(t_i) 1_{[T_{k_1} - \varepsilon, T_k]}(t_i) = \begin{cases} 0, & k_1 \leq k \\ 1, & k_1 > k, \end{cases} \]
we have for \( i \in \{i_{(k-1)}, \ldots, i_{(k)}\} \),
\[ I_{i,1}^{(k)} = \sum_{k_1 = k+1}^{K} a_1 b_2 (T_{k_1} - t_i)^{-\alpha_1} (T_k - t_i)^{\beta_2} 1_{[T_{k_1} - \varepsilon, T_k]}(t_i) \]
\[ \leq K(T_{k_1} - T_k)^{-\alpha_1} a_1 b_2 (T_k - t_i)^{\beta_2} 1_{[T_{k_1} - \varepsilon, T_k]}(t_i). \]

We have the followings similarly.
\[ \sum_{i=0}^{n-1} (t_{i+1} - t_i) I_{i,1}^{(k)} \leq a_1 a_2 \sum_{k_1, k_2 = k}^{K} \int_{t_{i_{(k-1)}}}^{t_{i_{(k)}}} (T_{k_1} \wedge T_{k_2} - t)^{-(\alpha_1 + \alpha_2)} dt, \]
\[ \sum_{i=0}^{n-1} (t_{i+1} - t_i) I_{i,2}^{(k)} \leq C a_1 b_2 \varepsilon^{\beta_2 + 1}, \]
\[ \sum_{i=0}^{n-1} (t_{i+1} - t_i) I_{i,3}^{(k)} \leq C a_2 b_1 \varepsilon^{\beta_1 + 1}, \]
\[ \sum_{i=0}^{n-1} (t_{i+1} - t_i) I_{i,4}^{(k)} \leq b_1 b_2 \varepsilon^{\beta_1 + \beta_2 + 1}. \]

So we obtain (22). \( \blacksquare \)

6 Proof of Theorem 18 and Theorem 2

Theorem 18 There exists a constant \( C > 0 \) such that
\[ E^P[\|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta\|] \leq C \left( L^{-1-\delta/2} \varepsilon \left( \varepsilon_1 (1 + \delta)(\tilde{N} + 1)\ell_0/4 \right) + \varepsilon^2 \right), \quad L \geq 1. \]
Proof.

\[ E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta|] \]

\[ \leq \frac{1}{L} \sum_{\ell=1}^{L} \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} \left| E^P[g(t_i, X(t_i, x)) \langle Q_{t_i, T_k, \varepsilon}^{(m,L, \omega)} F_{m,k} \rangle (\pi_k(X(t_i))) - (P_{T_k-t_i}^{(k)} F_{m,k}) (\pi_k(X(t_i)))\rangle]\right|.

Then by Schwartz’s inequality,

\[ \frac{1}{L} \sum_{\ell=1}^{L} \left| E^P[g(t_i, X(t_i, x)) \langle Q_{t_i, T_k, \varepsilon}^{(m,L, \omega)} F_{m,k} \rangle (\pi_k(X(t_i))) - (P_{T_k-t_i}^{(k)} F_{m,k}) (\pi_k(X(t_i)))\rangle]\right| \]

\[ \leq \frac{1}{L} \sum_{\ell=1}^{L} E^P[|g(t_i, X(t_i, x))| \langle Q_{t_i, T_k, \varepsilon}^{(m,L, \omega)} F_{m,k} \rangle (\pi_k(X(t))) - (P_{T_k-t_i}^{(k)} F_{m,k}) (\pi_k(X(t)))|1_{[0,T_k-\varepsilon]}] \]

\[ + C(T_k - t)1_{[T_k-\varepsilon, T_k)} \]

\[ \leq \left( \frac{1}{L} \sum_{\ell=1}^{L} E^P[|g(t_i, X(t_i, x))|^2 |1_{[0,T_k-\varepsilon]}] \right)^{1/2} + C(T_k - t)1_{[T_k-\varepsilon, T_k)} \]

By Proposition 14

\[ E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta|] \]

\[ \leq C \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k:T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; L^{-1/2}, (1 + \delta)(\bar{N} + 1)\ell_0/4, 1, 1). \]

By Proposition 17 we have the assertion.

Lemma 19 Let \( a, b \in \mathbb{R} \) and \( c, \theta > 0 \). Then we have

\[ c|a|1_{\{b \geq 0\}} - 1_{\{a \geq 0\}} | \leq c|b - a|1_{\{b - a \geq \theta\}} + c\theta 1_{\{|a| < \theta\}} \]

Proof. If \(|a| > |a - b|\), then

\[ 1_{\{b \geq 0\}} - 1_{\{a \geq 0\}} = 0. \]

So we see that

\[ |a|(|1_{\{b \geq 0\}} - 1_{\{a \geq 0\}}| \]

\[ \leq |a|1_{\{|a| \leq |a - b|\}} \]

\[ \leq |a - b|1_{\{|a - b| \geq \theta\}} + |a|1_{\{|a| < \theta\}}. \]
**Theorem 20** Let $\delta \in (0, 1), p > 1$. Suppose that there is $\gamma \in (0, 1]$ and $C_\gamma > 0$ such that

$$
\sup_\Delta \sum_{i=0}^{n-1} (t_{i+1} - t_i) \mu(\sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} (P_{T_k-t_i}^{(m)} F_{m,k})(\pi_mX(t_i, x^*)) \leq \theta) \\
\leq C_\gamma \theta^\gamma, \theta \in (0, 1).
$$

Then there exists a constant $C > 0$, $\Omega(L, \varepsilon) \in \mathcal{F}$, such that

$$
P(\Omega(L, \varepsilon)^c) \leq C \left( L^{-\frac{(1-\delta)^2}{2}} \left( L^{-\frac{(1-\delta)^2}{2}} \varepsilon \bigg( (1 - \delta^2)(\tilde{N} + 1)\ell_0/2 \bigg) + \varepsilon^{3(1-\delta)/2} \right)^\frac{\delta(1+\gamma)}{2+\gamma} \right),
$$

and

$$
1_{\Omega(L, \varepsilon)} |\hat{c}_2(\varepsilon, \Delta, L) - c_\Delta| \\
\leq C \left( L^{-\frac{(1-\delta)^2}{2}} \left( L^{-\frac{(1-\delta)^2}{2}} \varepsilon \bigg( (1 - \delta^2)(\tilde{N} + 1)\ell_0/2 \bigg) + \varepsilon^{3(1-\delta)/2} \right)^\frac{\delta(1+\gamma)}{2+\gamma} \right),
$$

for $L \geq 1$.

**Proof.** In this proof, we denote $X(t, x^*)$ by $X(t)$ for simplicity. Let

$$
\tilde{B}_\varepsilon = \bigcap_{m=1}^{M} \bigcap_{k=1}^{K} B^{(m,k)}(T_k - \varepsilon, \delta, L).
$$

Let $F_{P,i} : \mathbb{R}^N \rightarrow \mathbb{R}$ be given by

$$
F_{P,i}(x) = \sum_{m=1, \ldots, M}^{M} \sum_{k:T_k \geq t_{i+1}} (P_{T_k-t_i}^{(m)} F_{m,k})(\pi_m x),
$$

and let $F_{Q,i} : \mathbb{R}^N \rightarrow \mathbb{R}$ be given by

$$
F_{Q,i}(x) = \sum_{m=1, \ldots, M}^{M} \sum_{k:T_k \geq t_{i+1}} (Q_{t_i,T_k,\varepsilon}^{(m)} F_{m,k})(\pi_m x).
$$

Then

$$
1_{\tilde{B}_\varepsilon} |\hat{c}_2(\varepsilon, \Delta, L) - c_\Delta| \\
\leq 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1} - t_i) |E^\mu| |g(t_i, X(t_i))| \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} F_{m,k}(\pi_k X(T_k)) (1_{\{F_{Q,i}(X(t_i)) \geq 0\}} - 1_{\{F_{P,i}(X(t_i)) \geq 0\}}),
$$

$$
\leq 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1} - t_i) |E^\mu| |g(t_i, X(t_i))| |F_{P,i}(X(t_i))| 1_{\{F_{Q,i}(X(t_i)) \geq 0\}} - 1_{\{F_{P,i}(X(t_i)) \geq 0\}},
$$

since $|g(t_i, X(t_i))| (1_{\{F_{Q,i}(X(t_i)) \geq 0\}} - 1_{\{F_{P,i}(X(t_i)) \geq 0\}})$ is $\mathcal{F}_t$ measurable.

Applying Lemma 19 to $a = F_{P,i}(X(t_i))$, $b = F_{Q,i}(X(t_i))$, and $c = |g(t_i, X(t_i))|$, we have

$$
1_{\tilde{B}_\varepsilon} |\hat{c}_2(\varepsilon, \Delta, L) - c_\Delta| \leq I_1 + I_2,
$$

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where
\[
I_1 = 1 \mathbb{B}_\varepsilon \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ |g(t_i, X(t_i))| |F_{Q,i}(X(t_i)) - F_{P,i}(X(t_i))| 1_{\{|F_{Q,i}(X(t_i)) - F_{P,i}(X(t_i))| > \theta\}} \right],
\]
\[
I_2 = 1 \mathbb{B}_\varepsilon \sum_{i=0}^{n-1} (t_{i+1} - t_i) \theta E^\mu \left[ |g(t_i, X(t_i))| 1_{\{|F_{P,i}(X(t_i))| \leq \theta\}} \right].
\]

By Hölder's inequality,
\[
I_2 \leq \theta E^\mu \left[ |g(t_i, X(t_i))|^{1/\delta} \right] \mu \left( |F_{P,i}(X(t_i))| \leq \theta \right)^{1-\delta} \leq C \theta \mu \left( |F_{P,i}(X(t_i))| \leq \theta \right).
\]

From the assumption, we have
\[
I_2 \leq C \theta^{(\gamma+1)(1-\delta)}.
\]

Next we will estimate $I_1$.
\[
|F_{Q,i}(X(t_i)) - F_{P,i}(X(t_i))| \leq \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} \left( d^{(m,k)}_{1,\varepsilon,L} (t_i, X(t_i)) + d^{(m,k)}_{2,\varepsilon,L} (t_i, X(t_i)) + d^{(m,k)}_{3,\varepsilon,L} (t_i, X(t_i)) \right)
\]
\[
I_1 \leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ |g(t_i, X(t_i))| \right]
\]
\[
\sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} \left( d^{(m,k)}_{1,\varepsilon,L} (t_i, \pi_m(X(t_i))) + d^{(m,k)}_{2,\varepsilon,L} (t_i, \pi_m(X(t_i)) + d^{(m,k)}_{3,\varepsilon,L} (t_i, \pi_m(X(t_i))) \right)
\]
\[
\times 1 \left\{ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} \left( d^{(m,k)}_{1,\varepsilon,L} (t_i, \pi_m(X(t_i))) + d^{(m,k)}_{2,\varepsilon,L} (t_i, \pi_m(X(t_i)) + d^{(m,k)}_{3,\varepsilon,L} (t_i, \pi_m(X(t_i))) > \theta/2 \right) \right\}.
\]

$I_1$ is dominated by
\[
I_1 \leq I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4},
\]
where
\[
I_{1,1} = \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} E^\mu \left[ |g(t_i, X(t_i))| d^{(m,k)}_{1,\varepsilon,L} (t_i, \pi_m(X(t_i))) \right],
\]
\[
I_{1,2} = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} |g(t_i, X(t_i))| d^{(m,k)}_{2,\varepsilon,L} (t_i, \pi_m(X(t_i))) \right]
\]
\[
\times \left\{ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} \left( d^{(m,k)}_{1,\varepsilon,L} (t_i, \pi_m(X(t_i))) + d^{(m,k)}_{3,\varepsilon,L} (t_i, \pi_m(X(t_i))) > \theta/2 \right) \right\},
\]
\[
I_{1,3} = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} |g(t_i, X(t_i))| d^{(m,k)}_{3,\varepsilon,L} (t_i, \pi_m(X(t_i))) \right]
\]
\[
\times \left\{ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} d^{(m,k)}_{2,\varepsilon,L} (t_i, \pi_m(X(t_i))) > \theta/2 \right\},
\]
\[
I_{1,4} = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ \sum_{m=1}^M \sum_{k: T_k \geq t_{i+1}} |g(t_i, X(t_i))| d^{(m,k)}_{3,\varepsilon,L} (t_i, \pi_m(X(t_i))) \right]
\]
From Proposition 16,
\[
E^P[I_{1,1}] = \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} E^P[d_{1,\varepsilon, L}^{(m,k)}(t, \tilde{x}_m)|g(t_i, x)|p(t_i, x^*, x)dx
\]
\[
\leq C \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k:T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; L^{-(1-\delta)^3}, 0, 0, 0)
\]
\[
\leq C \hat{\epsilon} \left( \varepsilon, L^{-(1-\delta)^3} \right).
\]

Next, we will estimate \( I_{1,2} \). By Hölder’s inequality
\[
I_{1,2} \leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} d_{2,\varepsilon, L}^{(m,k)}(t_i, \pi_m(X(t_i)))^2 \right]^{1/2}
\]
\[
\times E^\mu \left[ |g(t_i, X(t_i))|^2 \right]^{\delta/2} E^\mu \left[ \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} (d_{1,\varepsilon, L}^{(m,k)}(t, \pi_m(X(t)) + d_{3,\varepsilon, L}^{(m,k)}(t, \pi_m(X(t)) > \theta/2)) \right]^{(1-\delta)/2},
\]
\[
\leq C \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu \left[ \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} d_{2,\varepsilon, L}^{(m,k)}(t_i, \pi_m(X(t_i)))^2 \right]^{1/2}
\]
\[
\times \left( \frac{2}{\theta} E^\mu \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} (d_{1,\varepsilon, L}^{(m,k)}(t_i, \pi_m(X(t_i)) + d_{3,\varepsilon, L}^{(m,k)}(t_i, \pi_m(X(t_i)))) \right)^{(1-\delta)/2}.
\]

So we have
\[
E^P[I_{1,2}] \leq C \sqrt{\frac{2}{\theta}} \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^P \left[ \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} d_{2,\varepsilon, L}^{(m,k)}(t_i, x)^2 p^{(m)}(t_i, \tilde{x}_m, x)dx \right]^{1/2}
\]
\[
\times E^P \left[ \sum_{m=1}^{M} \sum_{k:T_k \geq t_{i+1}} d_{1,\varepsilon, L}^{(m,k)}(t_i, x) + d_{3,\varepsilon, L}^{(m,k)}(t_i, x) \right]^{(1-\delta)/2}p^{(m)}(t_i, \tilde{x}_m, x)dx \right] \right]^{(1-\delta)/2}
\]
\[
\leq C \theta^{-(1-\delta)/2} \sum_{i=0}^{n-1} (t_{i+1} - t_i) \left( \sum_{k:T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; L^{-(1-\delta)^2/2}, 1 - \delta^2)(\tilde{N} + 1)\ell_0/4, 0, 0, 0) \right)
\]
\[
\times \left( \sum_{k:T_k \geq t_{i+1}} \phi^{(k)}(t_i, \varepsilon; L^{-(1-\delta)^4/2}, 0, 1, (1 - \delta)^2) \right).
\]

By Proposition 17,
\[
E^P[I_{1,2}] \leq C \theta^{-1/2} \left( L^{-(1-\delta)^3} \hat{\epsilon} \left( \varepsilon, (1 - \delta^2)(\tilde{N} + 1)\ell_0/4 \right) + L^{-(1-\delta)^2/2} \epsilon^3(1-\delta/2) \right).
\]
Similarly, we have

$$E^P[I_{1,3}] \leq C\theta^{-(1-\delta)/2} \sum_{i=0}^{n-1} (t_{i+1} - t_i) E \left[ \sum_{m=1}^{M, k: T_k \equiv t_{i+1}} \int_{E_m} d_2^{(k)}(t_i, x)^2 p^{(k)}(t_i, \tilde{x}_k, x) dx \right]^{(1-\delta)}$$

$$\leq C\theta^{-(1-\delta)/2} \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{k: T_k \equiv t_{i+1}} \phi^{(k)}(t_i, \varepsilon; L^{-(1-\delta)^2/2}, (1 - \delta^2)(\tilde{N} + 1)\ell_0/4, 0, 0)^2$$

$$\leq C\theta^{-1} L^{-(1-\delta)^2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right).$$

It follows easily that

$$E^P[I_{1,4}] \leq C\varepsilon^2.$$

Notice that

$$\theta^{-(1-\delta)/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/4 \right) \leq \theta^{-1} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right),$$

we have

$$E^P[I] \leq C \left( \theta^{\gamma+1} + \theta^{-1} L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right).$$

In particular if we take $\theta = \theta_L$ as

$$\theta_L = O \left( L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right)^{\frac{1}{2+\gamma}},$$

then we have

$$E^P[I] \leq C \left( L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right)^{(1+\gamma)/(2+\gamma)}.$$

Let $\tilde{\Omega}(L, \varepsilon)$ be

$$\tilde{\Omega}(L, \varepsilon) = \tilde{B}_\varepsilon \cap \{ \omega \in \Omega;$$

$$I \leq C \left( \left( L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right)^{(1-\delta)(1+\gamma)} \right)^{\frac{1}{2+\gamma}}.$$}

From Proposition 13, we have

$$P(\tilde{\Omega}(L, \varepsilon)^c) \leq C \left( \left( L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right)^{\frac{\delta(1+\gamma)}{2+\gamma}}$$

$$+ \left( \varepsilon^{-5\ell_0} L^{-12(1+\delta^2)/2+1/p} \right),$$

and

$$1_{\tilde{\Omega}(L, \varepsilon)} \hat{e}_2(\varepsilon, \Delta, L) - c_\Delta \leq C \left( L^{-(1-\delta)^2/2} \left( L^{-(1-\delta)^2/2} \hat{e} \left( \varepsilon, (1-\delta^2)(\tilde{N} + 1)\ell_0/2 \right) + \varepsilon^{3(1-\delta)/2} \right) \right)^{(1-\delta)(1+\gamma)}.$$

**Corollary 21** *Theorem 1 and Theorem 2 follow from the Theorem 18 and Theorem 19.*
7 Numerical Example

Let \( \{B(t); t \geq 0\} \) be 1 dimensional Brownian motion. Let \( t_i = i/n, i = 0, \ldots, n \). Let \( c \) be

\[
c = E\left[ \int_0^1 (E[B(1)|F_t] \lor 0) \, dt \right] = \frac{2}{3\sqrt{2\pi}}.
\]

Let \( c_\Delta \) be the discretization of \( c \), such that

\[
c_\Delta = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E[B(t_i) \lor 0].
\]

We approximate \( c \) as Remark 3, where \( F(x) = x \). Let \( X_1 = \{X_\ell(t_i); i = 0, 1, \ldots, n\}_{\ell=1}^L \) be i.i.d sample paths of \( \{B(t_i); i = 0, 1, \ldots, n\} \). We compute \( Q^{(L,\omega)}_t \) and \( \hat{c}_1 \) by using of paths \( X_1 \).

\[
\hat{c}_1 = \frac{1}{L} \sum_{\ell=1}^L \sum_{i=0}^{n-1} \left( Q^{(L,\omega)}_{t_i,t} F((X_\ell(t_i))) \lor 0 \right) (t_{i+1} - t_i).
\]

Let \( X_2 = \{X_\ell'(t_i); i = 0, 1, \ldots, n\}_{\ell=1}^L \) be another i.i.d sample paths of \( \{B(t_i); i = 0, 1, \ldots, n\} \). We compute \( \hat{c}_2 \) by

\[
\hat{c}_2 = \frac{1}{L_0} \sum_{\ell_0=1}^{L_0} \sum_{i=0}^{n-1} \left( F(X_\ell'(T)) \right) 1_{\{Q^{(L,\omega)}_{t_0,T}(X_\ell'(t_i)) \geq 0\}} (t_{i+1} - t_i).
\]

We have \( c \approx 0.2659615203 \). When we take \( n = 100 \), we have \( c_\Delta \approx 0.2638855365 \). We also take \( L_0 = 10000 \) and \( L = 100, 200, 400, 800, 1600, 3200, 6400 \). We replicate 100 estimators of \( \hat{c}_i, i = 1, 2 \) for each \( L \). Let "Average \( i \)" denote the average and "Standard Deviation \( i \)" denote the unbiased standard deviation of these 100 estimators of \( \hat{c}_i, i = 1, 2 \). We show the numerical result in Table 1, we show graph of "Average \( i, i = 1, 2 \)" and \( c_\Delta \) in Figure 1 and graph of "Standard Deviation \( i, i = 1, 2 \)" in Figure 2. We see in Figure 1 that both \( \hat{c}_1 \) and \( \hat{c}_2 \) are close to \( c_\Delta \), but we see in Figure 2 that \( \hat{c}_2 \) is more stable than \( \hat{c}_1 \).

| \( L \) | Average 1 | Average 2 | Standard Deviation 1 | Standard Deviation 2 |
|-------|-----------|-----------|----------------------|----------------------|
| 100   | 0.2654599783 | 0.2632060491 | 0.0334099140 | 0.0066656532 |
| 200   | 0.2655180439 | 0.2643180950 | 0.0244301557 | 0.0064812199 |
| 400   | 0.2641632386 | 0.2646417384 | 0.0168528548 | 0.0064796412 |
| 800   | 0.2661557058 | 0.2648673390 | 0.0114840865 | 0.0064426983 |
| 1600  | 0.2658622710 | 0.2649728715 | 0.0092734158 | 0.0064583976 |
| 3200  | 0.2659440890 | 0.2650330723 | 0.0071782226 | 0.0064616852 |
| 6400  | 0.2648318867 | 0.2650702303 | 0.0050607508 | 0.0064608708 |

Table 1: Average and Standard Deviation
Figure 1: Average

Figure 2: Standard Deviation
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