COUNTING CONICS IN COMPLETE INTERSECTIONS

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Abstract. We count the number of conics through two general points in complete intersections when this number is finite and give an application in terms of quasi-lines.

1. Introduction

Let $X$ be a complex projective manifold of dimension $n$. A quasi-line $l$ in $X$ is a smooth rational curve $f : \mathbb{P}^1 \hookrightarrow X$ such that $f^*T_X$ is the same as for a line in $\mathbb{P}^n$, i.e., is isomorphic to
\[ O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus n-1}. \]

Let $X$ be a smooth projective variety containing a quasi-line $l$. Following Ionescu and Voica [IV03], we denote by $e(X, l)$ the number of quasi-lines which are deformations of $l$ and pass through two given general points of $X$. We denote by $e_0(X, l)$ the number of quasi-lines which are deformations of $l$ and pass through a general point $x$ of $X$ with a given general tangent direction at $x$. Note that one always has $e_0(X, l) \leq e(X, l)$, but in general the inequality may be strict [IN03, p.1066].

1.1. Theorem. Let $X \subset \mathbb{P}^{n+r}$ be a general smooth $n$-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$. Assume moreover that
\[ d_1 + \cdots + d_r = \frac{n + 1}{2} + r. \]

Then
(1) the family of conics contained in $X$ is a nonempty, smooth and irreducible component of the Chow scheme $C(X)$,
(2) a general conic $C$ contained in $X$ is a quasi-line of $X$ and
\[ e_0(X, C) = e(X, C) = \frac{1}{2} \prod_{i=1}^r (d_i - 1)! d_i!. \]

The numerical assumption $d_1 + \cdots + d_r = (n + 1)/2 + r$ assures that if $C$ is a conic in $X$, then $-K_X \cdot C = n + 1$. This numerical condition is of course necessary for a curve to be a quasi-line. Note that varieties appearing in our theorem are Fano varieties of dimension $n$ and index $(n+1)/2$; they are well known to be the boundary Fano varieties with Picard number one being conic-connected (see [IR07], Theorem 2.2).

Using a degeneration argument, one can strengthen parts of the statement.

1.2. Corollary. Let $X \subset \mathbb{P}^{n+r}$ be a smooth $n$-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$. If $d_1 + \cdots + d_r = (n + 1)/2 + r$, the variety $X$ contains a conic that is a quasi-line.

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By a theorem of Ionescu [Ion05], we obtain an immediate application of the theorem to formal geometry. Before stating it, we recall that a subvariety $Y$ of a variety $X$ is G3 in $X$ if the ring $K(X|_Y)$ of formal-rational functions of $X$ along $Y$ is equal to $K(X)$.

### 1.3. Corollary

Let $X \subset \mathbb{P}^{n+r}$ be a general smooth $n$-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$ such that $d_1 + \cdots + d_r = (n+1)/2 + r$. Then any general conic $C$ contained in $X$ is G3 in $X$. In particular, if $(X, C)$ and $(X', C')$ are two such pairs such that the formal completions $X_C$ and $X_{C'}$ are isomorphic as formal schemes, there exists an isomorphism from $X$ to $X'$ sending $C$ to $C'$.

When this note was almost finished, we learned from L. Manivel that A. Beauville had obtained the formula $e(X, l) = \frac{1}{2} \prod_{i=1}^{r} (d_i - 1)!d_i!$ as a consequence of his computation of the quantum cohomology algebra $H^*(X, \mathbb{Q})$ of a complete intersection [Bea95]. We provide here a completely elementary proof. We end this note by mentioning a similar question where no elementary proof seems to be known.

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## 2. Proofs

We start by explaining the enumerative argument in the simplest case.

### 2.1. A well known example

Suppose that $X = \{s = 0\}$ is a smooth cubic threefold in $\mathbb{P}^4$. A general conic $C$ in $X$ is a quasi-line [BB00 Thm.3.2]. The basic idea of our proof is that counting conics in $X$ through $p$ and $q$ can be reduced to counting 2-planes $\pi$ through $p$ and $q$ such that the restriction $s_{\pi}$ is a product of a polynomial of degree two and some residual polynomial. We will explain how to do this in general below, in the case of the cubic threefold we can use a geometric construction.

It is a classical fact that the lines in $X$ form an irreducible smooth family of dimension two and that there are exactly six lines passing through a general point of $X$ [AK77 Prop.1.7]. Fix now two general points $p$ and $q$ in $X$, then the line $[pq]$ intersects $X$ in a third point $u$. For every line $l \subset X$ through $u$ there exists a unique plane $\pi_l$ containing $l$ and $[pq]$. The intersection $X \cap \pi_l$ is the union of $l$ and a residual conic $C$. Since $l$ does not pass through $p$ and $q$, the conic $C$ passes through $p$ and $q$. Vice versa the linear span of a conic $C \subset X$ passing through $p$ and $q$ is a 2-plane $\pi_C$ containing the line $[pq]$. Since $C$ does not pass through $u$, the residual line passes through $u$. Thus the conics through $p$ and $q$ are in bijection with the lines through $u$, so $e(X, C) = 6$.

Suppose now that we are in the general situation of Theorem [11]. We always assume that $X \subset \mathbb{P}^{n+r}$ is a general smooth $n$-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$ with $d_i \geq 2$ for all $i$ and

$$d_1 + \cdots + d_r = \frac{n+1}{2} + r.$$ 

Let $l \subset X$ be a smooth rational curve contained in $X$. Then

$$-K_X \cdot l = (n + r + 1 - (d_1 + \cdots + d_r)) \deg(l) = \frac{n+1}{2} \deg(l)$$
therefore $-K_X \cdot l = n + 1$ if and only if $l$ is a conic.

2.2. The main step. For any general points $p$ and $q$ of $X$, there exists a conic contained in $X$ passing through $p$ and $q$.

Fix two distinct points in $\mathbb{P}^{n+r}$, say $p = [1 : 0 : \cdots : 0]$ and $q = [0 : 0 : \cdots : 1]$. Suppose that $X$ is a general complete intersection with equations

$$(s_1 = 0) \cap (s_2 = 0) \cap \cdots \cap (s_r = 0)$$

passing through $p$ and $q$, where each $s_i \in H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d_i))$ is general among sections vanishing at $p$ and $q$.

Suppose there is a conic $C$ contained in $X$, passing through $p$ and $q$ and let $\pi_C$ the projective 2-plane generated by $C$. If $s_C$ denotes the equation defining $C$ in $\pi_C$, there exists for each $i = 1, \ldots, r$ a $\tilde{s}_i \in H^0(\pi_C, \mathcal{O}_{\pi_C}(d_i - 2))$ (defining the residual curve) such that

$$(s_i)|_{\pi_C} = s_C \cdot \tilde{s}_i.$$ 

Since $X$ is general, it does not contain the 2-plane $\pi_C$ [DM98, Thm. 2.1]. Therefore $(s_i)|_{\pi_C}$ and $\tilde{s}_i$ are not zero for at least one $i$.

Conversely, let $\pi$ be a projective 2-plane containing $p$ and $q$ and assume there exists a non-zero $s_C \in H^0(\pi, \mathcal{O}_{\pi}(2))$ vanishing at $p$ and $q$ and, for each $i = 1, \ldots, r$, there exists a $\tilde{s}_i \in H^0(\pi, \mathcal{O}_{\pi}(d_i - 2))$ such that

$$(s_i)|_{\pi} = s_C \cdot \tilde{s}_i,$$

then the conic $(s_C = 0)$ is obviously contained in $X$.

Consider now the projective space of dimension $n + r - 2$ parametrizing the projective 2-planes in $\mathbb{P}^{n+r}$ containing $p$ and $q$. Fixing homogeneous coordinates $[a_1 : \cdots : a_{n+r-1}]$ on this space, let

$$\pi_{[a_1: \cdots: a_{n+r-1}]} = \{[x : za_1 : \cdots : za_{n+r-1} : y] | [x : z : y] \in \mathbb{P}^2\}$$

be such a 2-plane. Then

$$(s_i)|_{\pi_{[a_1: \cdots: a_{n+r-1}]}(x, z, y)} = \sum_{k=0}^{d_i} \sum_{a=0}^{k} s_{a,k}^i a^k y^{k-a} z^{d_i-k}$$

where $s_{a,k}^i$ is a homogeneous polynomial of degree $d_i - k$ in the variables $a_1, \ldots, a_{n+r-1}$.

The equation of an irreducible conic in this plane that passes through $p$ and $q$ is

$$s_C = s_2 z^2 + s_1 xz + s'_1 yz + xy.$$ 

So for each $i = 1, \ldots, r$, the equation $(s_i)|_{\pi} = s_C \cdot \tilde{s}_i$ can be written explicitly

$$\sum_{k=0}^{d_i} \sum_{a=0}^{k} s_{a,k}^i a^k y^{k-a} z^{d_i-k} = (s_2 z^2 + s_1 xz + s'_1 yz + xy) \times \sum_{k=0}^{d_i-2} \sum_{a=0}^{k} s_{a,k}^i a^k y^{k-a} z^{d_i-2-k}.$$ 

Thus we have to solve the equations

$$\tilde{s}_{a,k}^i = s_2 \tilde{s}_{a,k}^i + s_1 \tilde{s}_{a-1,k-1}^i + s'_1 \tilde{s}_{a,k-1}^i + \tilde{s}_{a-1,k-2}^i$$

for any $0 \leq k \leq d_i$ and $0 \leq a \leq k$.

Let us first solve this system (whose unknown variables are $s_2$, $s_1$, $s'_1$ defining the conic and the $\tilde{s}_{a,k}^i$’s defining the residual curve) for each $i$ separately. Note that $X$ passes
through \( p \) and \( q \) if and only if \( s^i_{0,d_i} = s^i_{d_i,d_i} = 0 \). Therefore writing the \( d_i - 1 \) equations \( s^i_{a,d_i} = s^i_{a-1,d_i-2} \) for \( 1 \leq a \leq d_i - 1 \) provides the \( \tilde{s}^i_{a-1,d_i-2} \)'s.

Considering the equations corresponding to \((a,k) = (0,d_i - 1)\) and \((d_i - 1, d_i - 1)\) allows to find \( s_2 \) and \( s_1 \). Considering then the equations corresponding to \((a,k) = (1,d_i - 1)\) and \((0,d_i - 2)\) gives \( \tilde{s}^i_{0,d_i-3} \) and \( s^i_1 \) (in particular this determines the conic, if it exists!). Write down successively the equations for \((a,k), a = 1, \ldots , k - 1, k = d_i - 1, \ldots , 2\) to find all the \( \tilde{s}^i_{a,k} \)'s (this determines the residual curve \((\tilde{s}^i = 0)\)).

Therefore, the \( r \) systems have a common solution if and only if the remaining equations for each system are satisfied and the corresponding conic is the same for each \( i \). For each \( i \), the remaining equations are “universal formulas” (meaning the coefficients just depend on the equations defining \( X \)) corresponding to \((a,k) = (0,d_i - 3), \ldots (0,0)\) and \((a,k) = (d_i - 2, d_i - 2), \ldots (1,1)\). This gives \( 2d_i - 4 \) equations of respective degrees 3, \ldots , \( d_i \) and \( 2, 3, \ldots , d_i - 1 \) in the variables \( a_1, \ldots , a_{n+r-1} \). The \( 3r - 3 \) equations saying that the conic is the same for each \( i = 1, \ldots , r \) are \( 2r - 2 \) equations of degree 1 and \( r - 1 \) equations of degree 2 in the variables \( a_1, \ldots , a_{n+r-1} \).

Altogether, using the relation \( d_1 + \cdots + d_r = (n + 1)/2 + r \), this gives exactly \( n + r - 2 \) homogeneous equations in the variables \( a_1, \ldots , a_{n+r-1} \). We therefore get at least one solution. Moreover since \( X \) is general, the coefficients \( s^i_{a,k} \) appearing in the initial equations are general. Since they completely determine the remaining \( n + r - 2 \) homogeneous equations, these equations are general. Thus the space of solutions is smooth and of the expected dimension, so there are exactly \( \frac{1}{2} \prod_{i=1}^r (d_i - 1)!d_i! \) solutions by Bezout’s theorem.

Let us briefly indicate how the same method gives the number of conics contained in \( X \), passing through \( p \) and tangent to the line \((pq)\). With the above notations, we have \( s^i_{d_i-1,d_i} = s^i_{d_i,d_i} = 0 \) and we have to solve the \( r \) systems

\[
\sum_{k=0}^{d_i} \sum_{a=0}^{k} s^i_{a,k} x^a y^{k-a} z^{d_i-k} = (s_2 z^2 + s_1 x z + s_1^2 y z + y^2) \times \sum_{k=0}^{d_i-2} \sum_{a=0}^{k} \tilde{s}^i_{a,k} x^a y^{k-a} z^{d_i-2-k}
\]

which means

\[
s^i_{a,k} = s_2 \tilde{s}^i_{a,k} + s_1 \tilde{s}^i_{a-1,k-1} + s_1 \tilde{s}^i_{a,k-1} + \tilde{s}^i_{a,k-2}
\]

for any \( 0 \leq k \leq d_i \) and \( 0 \leq a \leq k \). The remaining details are left to the reader.

2.3. The space of conics is irreducible. Let \( \mathbb{G}(2, n + r) \) be the Grassmannian of projective 2-planes contained in \( \mathbb{P}^{n+r} \) and \( E \) be the tautological rank 3-bundle on \( \mathbb{G}(2, n + r) \). The Hilbert scheme of conics in \( \mathbb{P}^{n+r} \) is the projectivisation \( \mathbb{P} \) of \( S^2 E^* \). Denote by \( \varphi : \mathbb{P}(S^2 E^*) \to \mathbb{G}(2, n + r) \) the natural map. We have an exact sequence on \( \mathbb{P}(S^2 E^*) \):

\[
(*) \quad 0 \to \bigoplus_{i=1}^r \varphi^* S^{d_i-3} E^* \otimes \mathcal{O}_{\mathbb{P}(S^2 E^*)}(-1) \to \bigoplus_{i=1}^r \varphi^* S^{d_i} E^* \to \mathcal{Q} \to 0
\]

defining a vector bundle \( \mathcal{Q} \) of rank \( n + 1 + 3r \). Since \( X \) is a complete intersection \((s_1 = 0) \cap (s_2 = 0) \cap \cdots \cap (s_r = 0)\), the \( s_i \)'s induce by restriction to 2-planes, pull-back and projection onto \( \mathcal{Q} \) a section of \( \mathcal{Q} \) whose zero locus \( Z \) is precisely the set of conics

\footnote{In this article we follow the convention that the projectivisation of a vector bundle \( E \) is the variety of lines of \( E \).}
contained in $X$. Since $E^*$ is globally generated, the images of sections $(s_1, \ldots, s_r)$ give a vector space $V \subseteq H^0(\mathbb{P}(S^2 E^*), \mathcal{Q})$ that globally generates $\mathcal{Q}$. Applying Bertini’s theorem to this subspace we see that the zero locus of a general section in $V$ is smooth. Since $X$ is supposed to be a general complete intersection, $Z$ is smooth and proving its irreducibility reduces to showing that $h^0(Z, \mathcal{O}_Z) = 1$. By the Koszul resolution of $\mathcal{O}_Z$, it is enough to show that for any $1 \leq j \leq \text{rk} \mathcal{Q}$

$$h^j(\mathbb{P}(S^2 E^*), \wedge^j \mathcal{Q}^*) = 0.$$  

Using the exact sequence $(\ast)$, this easily reduces to showing that for any $1 \leq j \leq \text{rk} \mathcal{Q}$ and any $0 \leq k \leq j$,

$$H^k(\mathbb{P}(S^2 E^*), \wedge^k (\bigoplus_{i=1}^r \varphi^* S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r \varphi^* S^{d_i-2} E) \otimes \mathcal{O}_{\mathbb{P}(S^2 E^*)}(1))) = 0.$$  

Since the higher direct images with respect to $\varphi$ vanish, it is sufficient to show that for any $1 \leq j \leq \text{rk} \mathcal{Q}$ and for any $0 \leq k \leq j$, we have

$$H^k(\mathbb{G}(2, n + r), \wedge^k (\bigoplus_{i=1}^r \varphi^* S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r \varphi^* S^{d_i-2} E) \otimes S^2 E))) = 0.$$  

This will follow from Bott’s theorem applied on $\mathbb{G}(2, n + r)$. Indeed, using Schur functor notation, let $S_b E$ be an irreducible factor appearing in the decomposition of

$$\wedge^k (\bigoplus_{i=1}^r \varphi^* S^{d_i} E) \otimes S^{j-k}((\bigoplus_{i=1}^r \varphi^* S^{d_i-2} E) \otimes S^2 E)$$  

where $b = (b_1, b_2, b_3)$ is a triple of integers $b_1 \geq b_2 \geq b_3 \geq 0$. By the Littlewood-Richardson rule, we get

$$(\ast\ast) \quad b_2 + b_3 \geq k - r \text{ and } b_3 \geq k - (d_1 + \ldots + d_r) - r = k - (n + 1)/2 - 2r.$$  

On the other hand by Bott’s theorem, the whole cohomology of $S_b E$ vanishes except maybe in the following cases:

1. $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 0, 0)$ with $b_1 \geq n + r - 1$,
2. $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 1, 0)$ with $b_1 \geq n + r - 1$,
3. $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 1, 1)$ with $b_1 \geq n + r - 1$,
4. $k = 2(n + r - 2)$ with $b_2 \geq n + r$ and $b_3 = 0, 1, 2$,
5. $k = 3(n + r - 2)$ with $b_3 \geq n + r + 1$.

The case $n = 3$ has been dealt with by Bădescu, Beltrametti and Ionescu [BB100], so we may assume $n \geq 5$ since $n$ is odd.

In the first three cases, we get $k - r = n - 2 \geq 3 > b_2 + b_3 = 0, 1, 2$, which is excluded by $(\ast\ast)$. In case (4), since $n \geq 5$, we get $k - (n + 1)/2 - 2r = 3(n - 3)/2 > b_1 = 0, 1, 2$, which is again excluded by $(\ast\ast)$. Case (5) is also excluded since we are only interested in the situation where $k \leq \text{rk} \mathcal{Q} = n + 1 + 3r$, but $3(n + r - 2) > n + 1 + 3r$ when $n \geq 5$.

We obtain the following corollary of the proof.

**2.1. Corollary.** Let $X \subset \mathbb{P}^{n+r}$ be a general smooth $n$-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$. Assume moreover that

$$d_1 + \ldots + d_r \leq \frac{n + 1}{2} + r$$  

and $n \geq 5$. Then the family of conics contained in $X$ is a nonempty, smooth and irreducible component of the Chow scheme $\mathcal{C}(X)$.

Let us also mention that Harris, Roth and Starr have shown the irreducibility of the space of smooth rational curves of arbitrary degree $e$ for general hypersurfaces of low degree $d$ [HRS04].
2.4. Conics are quasi-lines. By the first step, there exists a conic $C$ passing through two general points. Such a conic is necessarily smooth: a line $d$ contained in $X$ and passing through a general point satisfies

$$T_X|_d \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus \frac{n-3}{2}} \oplus O_{\mathbb{P}^1}(n+1),$$

so an easy dimension count shows that two general points are not connected by a chain of two lines. Thus $C$ smooth and its deformations with a fixed point cover a dense open subset in $X$. This implies that the normal bundle $N_{C/X}$ is ample [Deb01, Prop.4.10] and since $-K_X \cdot C = n + 1$, the curve $C$ is a quasi-line.

2.5. Proof of the Corollary 1.3. The irreducibility of the variety of conics gives $e_0(X,l) = e(X,l) = \frac{1}{2} \prod_{i=1}^r (d_i - 1)!d_i!$. The equality $e_0(X,l) = e(X,l)$ implies that general conics are G3 in $X$ [Ion05, Cor. 4.6], in particular [Ion05, Cor. 4.7, Cor.1.9] apply.

3. A similar question

Using exactly the same method as developed in §2.3, one can prove the following result, left to the reader.

3.1. Proposition. Let $X_d \subset \mathbb{P}^{n+1}$ be a general smooth $n$-dimensional hypersurface of degree $d$. Then, for $n \geq 7$ and $d \leq n + 1$, the family of conics contained in $X_d$ is a nonempty, smooth and irreducible component of dimension $3n - 2d + 1$ of the Chow scheme $C(X_d)$.

In the case of $d = n + 1$, there is a finite number of conics passing through a general point of $X_{n+1}$. Let us denote by $N_{n+1}$ this number. It seems that there are no known elementary method to compute this number. A general formula comes from the calculation of some Gromov-Witten invariants using mirror symmetry and an ordinary differential equation introduced by Givental. The following lines were written while reading [JNS04] and [Jin05].

3.2. Proposition. (Coates, Givental - Jinzenji, Nakamura, Suzuki) Let $X_n \subset \mathbb{P}^n$ be a general smooth hypersurface of degree $n$ in $\mathbb{P}^n$. Let $N_n$ be the number of conics passing through a general point of $X_n$. Then

$$N_n = \frac{(2n)!}{2^n + 1} - \frac{(n!)^2}{2}.$$

Let us briefly explain where this result comes from. If $a, b, c$ et $d$ are four integers, let $\langle O_aO_bO_cO_d \rangle_d$ be the Gromov-Witten invariant counting the number (possibly infinite) of rational curves of degree $d$ contained in $X_n$ and meeting 3 general subspaces of $\mathbb{P}^n$, of respective codimension $a, b$ and $c$. When $a, b$ or $c$ are equal to 1, each such rational curve has to be counted $d$ times since the intersection of a degree $d$ curve intersects a general hyperplane in $d$ points. Since a general line meets $X_n$ in $n$ points, we get that $N_n = \langle O_1O_1O_{n-1} \rangle_2/4n$. In [Jin05] are introduced some constants $\tilde{L}_{n+1,n,d}$, called “structure constants of the quantum cohomology ring of $X_n$”. They satisfy the following formula:

$$\sum_{m=0}^{n-1} \tilde{L}_{n+1,n,d}^m = n \prod_{j=1}^{n-1} (jw + (n - j))$$
and
\[ \sum_{m=0}^{n-2} \tilde{L}^{n+1,n,2}_m w^m = \sum_{j_2=0}^{n} \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \tilde{L}^{n+1,n,1}_{j_1} \tilde{L}^{n+1,n,1}_{j_2+1} w^{j_1-j_0} \left( \frac{1 + w}{2} \right)^{j_2-j_1}. \]

It is also shown in [Jin05] that for every integer \( m, 0 \leq m \leq n - 2 \), we have
\[ \tilde{L}^{n+1,n,2}_m = \langle \mathcal{O}_1 \mathcal{O}_{n-1-m} \mathcal{O}_{m+1} \rangle^2 / n. \]

Then the proposition follows by evaluating the \( w^{n-2} \) coefficient in the second formula above, the \( w^{n-1} \) coefficient in the first and putting \( w = 2 \).

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