Discord and non-classicality in probabilistic theories

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Quantum discord quantifies non-classical correlations in quantum states. We introduce discord for states in causal probabilistic theories, inspired by the original definition proposed in Ref. [17]. We show that the only probabilistic theory in which all states have null discord is classical probability theory. Non-null discord is then not just a quantum feature, but a generic signature of non-classicality.

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Non-locality plays a crucial role in quantum foundations. Entanglement is indeed the source of the most striking quantum paradoxes, since Schrödinger’s cat paper [1], or the incompleteness argument by Einstein, Podolsky and Rosen [2]. Any attempt to retain locality of physical properties is doomed to give up a realistic interpretation as proved by Bell’s inequality argument [3, 4]. Additional arguments against the existence of local realistic theories compatible with the quantum statistics were later proved [5–8]. The approach to the discussion on non-classical aspects of quantum theory, such as entanglement and non-locality, radically changed in recent years, due to the increasing interest in information processing within general probabilistic theories [9–14] as a new viewpoint for looking at quantum and classical theories from the outside. The notion of entanglement can be easily extended to general probabilistic theories, being just the negation of the separability property. From this point of view, classical theory is not the only one forbidding entangled states (see e.g. [15]). However, besides the widely studied feature of non-locality (with or without entanglement [16]), quantum theory exhibits also another form of non-classical correlations which is quantified by the quantum discord [17]. Discord is widely studied in the literature both as a resource for information processing [18] and as an ubiquitous feature in quantum statistical mechanics [19, 20]. However, discord has not been explored yet in the framework of general probabilistic theories.

The definition of quantum discord was originally motivated by the analysis of states describing a quantum system and a pointer in a measurement, at a time just after the unitary interaction of the system with the pointer and the environment [17]. In the particular situation where the environment broadcasts the measurement outcome, the information carried by the pointer has strongly classical features, which are not present in the general case of any bipartite quantum state. Loosely speaking, a quantum state has null discord when it resembles a bipartite pointer-system state in the peculiar situation described above. On the other hand, the operational interpretation of quantum discord is more obscure, and many subsequent papers tackled this point by providing physical and informational consequences of non-null discord [19, 21, 22], or by criticizing the definition [20]. Information theoretically, discord is interpreted as the amount of entanglement consumed in state merging [23].

In this paper we introduce a definition of discord in general causal probabilistic theories [14], namely theories where no signalling form the future holds. The definition reduces to a geometric measure of discord [24] in the quantum case, while null discord states coincide with the customary ones. We then prove that in theories where the separability condition coincides with the null discord condition, the set of states is simplicial, namely all pure states are jointly perfectly discriminated. In this case, only entangled non-null discord states may exist. If no entangled states are allowed, then the theory is classical. Consequently, there exist states with non-null discord in all causal probabilistic theories apart from the classical one. Non-null discord is not a quantum feature, but more generally a precise signature of non-classicality of the theory. Thus, our result strengthens the interpretation of non-null discord as non-classicality of correlations.

The operational interpretation of discord of Ref. [22] relies on a definition in terms of mutual information, thus depending on the notion of von Neumann entropy. However, there is no unique extension of the von Neumann entropy for general probabilistic theories [12, 13, 25]. Therefore we need an alternative definition, relying on purely operational concepts. For this purpose, we will first introduce a definition of null discord, extending a necessary and sufficient condition stated in Ref. [17]. We will then define discord of a state ρ in operational theories as the minimum operational distance between ρ and states with null discord.

We briefly remind here the definitions of quantum discord introduced in Refs. [17]. Given a composite quantum system AB in state ρAB, the quantum mutual information between A and B is defined as follows

\[ I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \]  

(1)

where \( S(\rho_X) := -\text{Tr}[\rho_X \log \rho_X] \) is the Von Neumann entropy of \( \rho_X \), and \( \rho_A \) and \( \rho_B \) are the marginal states \( \text{Tr}_B[\rho_{AB}] \) and \( \text{Tr}_A[\rho_{AB}] \), respectively. The quantum mutual information can be considered as an index of corre-
Quantum discord is then defined as

$$D_{\Pi^A}(\rho_{AB}) = I(A : B) - J(\Pi^A : B).$$ (3)

The dependence on the measurement $\Pi^A$ is usually removed by taking the minimum over possible von Neumann measurements. We remark that quantum discord is an asymmetric quantity, due to the definition of $J(A : B)$, and in general $D(\rho) \geq 0$ for any $\rho$. In Ref. \[17\] a necessary and sufficient condition for null discord was introduced. We now provide the condition as stated in Ref. \[24\]

**Theorem 1** A state $\rho$ of system AB has null discord if and only if there exists a von Neumann measurement $\Pi^A$ with $\Pi_k = |\psi_k\rangle\langle\psi_k|$ on system A such that

$$\sum_{k=1}^{d_A} (\Pi_k \otimes I_B) \rho (\Pi_k \otimes I_B) = \rho_{AB}.\quad (4)$$

The condition of theorem 1 will be taken as the definition of null discord state for the purpose of generalizing the notion of discord to the scenario of general probabilistic theories. We will now briefly review the framework of operational probabilistic theories introduced in Ref. \[11\], and recently adopted for an operational axiomatization of Quantum Theory \[14\]. Systems and tests are the primitive notions of an operational theory. A test represents one use of a physical device, like a Stern-Gerlach magnet, a beam splitter, or a photon counter. When the test is performed, it produces an outcome $i$ in some set $X$. A test can then be viewed as the collection of all events labeled by outcomes in the set $X$. Every test is represented by a sequence $\{a_j\}_{j \in Y}$ and an input-output system, respectively. These labels establish the rules for connecting two tests, namely tests $\{a'_j\}_{j \in X}$ and $\{b'_j\}_{j \in Y}$ can be connected in a sequence $\{b'_j \circ a'_j\}_{(i,j) \in X \times Y}$ only if the output of the first test $\{a'_j\}_{j \in X}$ is of the same type as the input system of the second one $\{b'_j\}_{j \in Y}$. Systems are denoted by capital letters, like A, B, C, ... We reserve the letter I for the trivial system. A test with input or output system I is called a preparation test or an observation test, respectively. Two systems A and B can be composed in parallel, obtaining a third system $C := AB$. Parallel composition is commutative ($AB = BA$) and associative ($A(BC) = (AB)C$), and the trivial system acts as a unit with respect to composition ($AI = IA = A$). An operational theory is specified by a collection of systems, closed under composition, and by a collection of tests, closed under parallel and sequential composition. An operational theory is probabilistic if every test $\{p_i\}_{i \in X}$ from the trivial system I to itself is a probability distribution over $X$, and both parallel and sequential composition of two events from the trivial system to itself are given by the product of probabilities: $p_i \otimes q_j = p_i \circ q_j = p_i q_j$.

In a probabilistic theory, a preparation-event $\rho_i$ for system A defines a function $\hat{\rho}_i$ sending observation-events of A to probabilities: $\hat{\rho}_i : \mathcal{E}(A) \to [0,1]$, $a_j \mapsto a_j \circ \rho_i$. Likewise, an observation-event $a_j$ defines a function $\hat{a}_j$ from preparation-events to probabilities $\hat{a}_j : \mathcal{S}(A) \to [0,1]$, $\rho_i \mapsto a_j \circ \rho_i$. Two observation-events (preparation-events) are equivalent if they define the same function. We will call the corresponding equivalence classes states (effects). Since states (effects) are functions from effects (states) to probabilities, one can take linear combinations of them. This defines two real vector spaces $\mathcal{S}_R(A)$ and $\mathcal{E}_R(A)$, and here we restrict our attention to the case where such spaces are finite dimensional. In this case, by construction one has $D_A := \dim(\mathcal{S}_R(A)) = \dim(\mathcal{E}_R(A))$.

The scenario depicted up to now entails a wide variety of possible theories, and we want now to restrict it as slightly as possible. In particular, throughout the paper we will consider causal theories \[10, 11\], which are defined as follows.

**Definition 1 (Causal theory)** A theory is causal if for every preparation-test $\{p_i\}_{i \in X}$ and every observation-test $\{a_j\}_{j \in Y}$ on system A the marginal probability

$$p_i := \sum_{j \in Y} a_j \circ \rho_i$$

is independent of the choice of the observation-test $\{a_j\}_{j \in Y}$. Precisely, if $\{a_j\}_{j \in Y}$ and $\{b_k\}_{k \in Z}$ are two different observation-tests, then one has

$$\sum_{j \in Y} a_j \circ \rho_i = \sum_{k \in Z} b_k \circ \rho_i.$$

Causal theories have a simple characterization given by the following equivalent condition

**Lemma 1 (Characterization of causal theories)** A theory is causal if and only if for every system $A$ there is a unique deterministic effect $e_A$.

In this framework a separable state for a bipartite system AB is given by the following:

**Definition 2** A state $\rho$ of system AB is separable if it is a convex combination factorized states, in formula

$$\rho = \sum_{i \in \mathbb{Z}} p_i \rho_i \otimes \sigma_i \quad (5)$$

where $\rho_i$ and $\sigma_i$ are states of systems A and B, respectively, and $\{p_i\}_{i \in \mathbb{Z}}$ is a probability distribution.

Notice that we use the symbol $\otimes$ to denote the composition of local states. However, in general causal theories without local discriminability the state space of a composite system is strictly larger than the tensor product.
of the state spaces of the component systems. However, separable states by definition lie in the subspace defined by the tensor product of states of A and states of B.

We now introduce the notion of pure state, which plays an important role in the derivation of our results.

**Definition 3 (Pure and mixed states)** A state is pure if it cannot be written as a convex combination of other states. A state that is not pure is mixed.

Finally, we need to introduce the notion of perfectly distinguishable states.

**Definition 4 (Perfectly distinguishable states)** The states \( \{\rho_i\}_{i \in X} \) are perfectly distinguishable if there is a test \( \{a_i\}_{i \in X} \) such that \( a_i \circ \rho_i = \delta_{ij} \). The test \( \{a_i\}_{i \in X} \) is called a discriminating test.

Now we have all the ingredients that are needed to export the notion of null-discord state as provided by the condition of Eq. (1) in the scenario of causal operational probabilistic theories. First we will introduce the notion of objective information, that can be viewed as an extension of the notion of element of reality provided by Einstein, Podolski and Rosen in their famous paper [2]. Notice however that we avoid here any reference to the notion of real information and reality.

**Definition 5 (Objective information)** We say that a test \( \{\mathcal{A}_i\}_{i \in X} \) provides objective information about state \( \rho \) if it fulfills the following requirements

1. The test \( \{\mathcal{A}_i\}_{i \in X} \in \Sigma(A) \) is repeatable, namely \( \mathcal{A}_i \circ \mathcal{A}_j = \delta_{ij} \).

2. The state \( \rho \) is not disturbed by the test \( \{\mathcal{A}_i\}_{i \in X} \), namely \( \mathcal{A}_i \circ \rho = \rho \) for \( \mathcal{A}_i = \sum_{i \in X} \mathcal{A}_i \).

Equivalently, we will say that \( \rho \) encodes objective information about the test \( \{\mathcal{A}_i\}_{i \in X} \).

This definition provides indeed the notion of a test that can extract information from a system without disturbing its state, thus leaving the same information accessible to further observers. As a consequence of the definition, we can prove the following results.

**Lemma 2** If \( \{\mathcal{A}_i\}_{i \in X} \) provides objective information about the state \( \rho \), then the states \( \rho_i := \mathcal{A}_i \circ \rho/ (e \circ \mathcal{A}_i \circ \rho) \) are perfectly distinguishable by the test \( a_i := e \circ \mathcal{A}_i \).

**Proof.** Trivially follows from property 1.

**Lemma 3** If \( \{\mathcal{A}_i\}_{i \in X} \) provides objective information about the state \( \rho \), then \( \rho = \sum_{i \in X} p_i \rho_i \), where \( p_i := (e \circ \mathcal{A}_i \circ \rho) \).

**Proof.** By the property 2 we have \( \rho = \mathcal{A} \circ \rho = \sum_{i \in X} \mathcal{A}_i \circ \rho = \sum_{i \in X} p_i \rho_i \). Along with lemma 2 this proves the thesis.

Finally, we introduce the following definition that accounts for those situations where the objective information encoded in a state cannot be further refined.

**Definition 6 A test \( \{\mathcal{A}_i\}_{i \in X} \) provides complete objective information about the state \( \rho \) if it provides objective information about \( \rho \) and the state \( \rho_i \) is pure for all \( i \).

We will now define discord for operational probabilistic theories in three steps.

1. We define null discord states.

2. We define the operational distance between two states of a generic probabilistic theory.

3. We define the discord \( D(\rho) \) of a bipartite state \( \rho \) as the minimum of the distance between \( \rho \) and the set of states with null discord.

Let us now define null discord states as follows.

**Definition 7 (Null discord states)** In a causal operational probabilistic theory, a bipartite state \( \rho \) has null discord if and only if it satisfies the following conditions

1. \( \rho \) is separable,

2. there exists a test \( \{\mathcal{A}_i\}_{k \in X} \) on system A that provides complete objective information about the state \( e_B \circ \rho \), and such that \( \{\mathcal{A}_i \otimes \mathcal{F}_B\}_{k \in X} \) provides objective information on \( \rho \).

Notice that the notion of null discord states is not symmetric with respect to the exchange of systems A and B. In the following we will follow the rule that null discord states encode objective information on system A. We now state an equivalent condition for null discord.

**Theorem 2** If the state \( \rho \) has null discord, then it can be expressed as follows

\[
\rho = \sum_{k \in X} q_k (\psi_k \circ \sigma_k),
\]

where \( \{\psi_k\}_{k \in X} \) is a set of jointly perfectly distinguishable pure states and \( \{q_k\}_{k \in X} \) is a probability distribution.

**Proof.** Since \( \rho \) is separable \( \rho = \sum_{j \in Y} p_j \rho_j \otimes \tau_j \). Moreover we have \( \nu := e_B \circ \rho = \sum_{j \in Y} p_j \rho_j \). Since there exists a test \( \{\mathcal{A}_k\}_{k \in X} \) that provides complete objective information on \( \nu \) we have \( \mathcal{A}_k \circ \nu = \sum_{j \in Y} p_j \mathcal{A}_k \circ \rho_j = q_k \psi_k \), with \( \psi_k \) pure states. This means that \( \mathcal{A}_k \circ \rho_j = p_k \psi_k \) with \( \sum_j p_j \psi_k =: q_k \), namely all vectors \( \mathcal{A}_k \circ \rho_j \) are parallel to each other and to the vector \( \psi_k \). Since the test \( \{\mathcal{A}_k \otimes \mathcal{F}\}_{k \in X} \) provides objective information for \( \rho \) we have \( \sum_{k \in X} (\mathcal{A}_k \otimes \mathcal{F}) \circ \rho = \sum_{j \in Y} \sum_{k \in X} p_j (\mathcal{A}_k \circ \rho_j \otimes \tau_j) = \)
Thus, exploiting the fact that \( \sigma_k \circ \rho_j = p_{jk} \psi_k \), the latter expression becomes

\[
\rho_{AB} = \sum_{k \in X} q_k (\psi_k \otimes \sigma_k),
\]

with \( \{ \psi_k \}_{k \in X} \) perfectly distinguishable states by hypothesis, and \( \sigma_k := 1/q_k \sum_{j \in Y} p_j p_{jk} \tau_j \).

Let us now proceed to the operational definition of a distance between states [11]. The operational distance between \( \rho_0 \) and \( \rho_1 \) is defined through the minimum error probability \( p_{err}^m \) in discrimination of \( \rho_0 \) and \( \rho_1 \) provided that their prior probability is 1/2, namely

\[
\| \rho_1 - \rho_0 \|_{\text{op}} := 1 - 2p_{err}^m.
\]

Let \( \Omega_{AB} \) be the set of states of \( AB \) with null discord. We can finally define the operational discord in a generic probabilistic theory as follows.

**Definition 8** Given a probabilistic theory and a bipartite system \( AB \), we define the discord \( D(\rho) \) of the state \( \rho \) as follows

\[
D(\rho) := \min_{\sigma \in \Omega_{AB}} \| \rho - \sigma \|_{\text{op}}.
\]

The present definition is hardly reducible to the standard notion of discord in the quantum case. However, it is strictly related to a geometric notion of discord proposed in [24]. Moreover, for the purpose of the main results of the present paper, what matters is the definition of null-discord states, which on the other hand coincides with the standard one in the quantum case. We now prove the main result of the present paper.

**Theorem 3** In a causal probabilistic theory where all separable states have null discord the set of normalized states for every system is a simplex.

**Proof.** The proof consists in showing that equivalence of null discord and separability in a causal probabilistic theory, implies that all states of any system in the theory are convex combinations of the same set of perfectly distinguishable states. Consider an arbitrary separable state \( \rho := \sum_{i \in Z} p_i (\rho_i \otimes \tau_i) \) with system B equivalent to system A. By the hypotheses of equivalence of separability and null discord, exploiting Eq. (8) we can write

\[
\rho = \sum_{i \in Z} p_i (\rho_i \otimes \tau_i) = \sum_{k \in X} q_k (\psi_k \otimes \sigma_k).
\]

In particular, we can consider states \( \rho \) such that \( \{ \rho_i \}_{i \in Z} \) and \( \{ \tau_i \}_{i \in Z} \) are complete sets of linearly independent states of systems A and B, respectively. Now, by Eq. (8), if the test providing complete objective information about the state \( \nu := \epsilon_B \circ \rho \) is \( \{ \sigma_k \}_{k \in X} \), then the observation-test \( \{ a_k \}_{k \in X} \) with \( a_k := \epsilon_A \circ \sigma_k \) is such that \( a_k \circ \psi_k = \delta_{kk'} \). Applying \( (a_k \otimes I_B) \) on state \( \rho \), by Eq. (9) we have that \( q_k \sigma_k = \sum_{i \in Z} p_i (a_k \circ \rho_i) \tau_i \), and substituting this expression into Eq. (10) we have

\[
\sum_{i \in Z} p_i (\rho_i \otimes \tau_i) = \sum_{i \in Z} q_i \sum_{k \in X} (a_k \circ \rho_i) (\psi_k \otimes \tau_i).
\]

Finally, by linear independence of \( \tau_i \), for any \( i \) we have \( \rho_i = \sum_{k \in X} (a_k \circ \rho_i) \psi_k \). Now, by hypothesis of completeness of \( \{ \rho_i \}_{i \in Z} \), we can write any state \( \lambda \) of system A as \( \lambda = \sum_{i \in Z} c_i \rho_i \), where \( c_i \) are real numbers. Substituting the expansion of \( \rho_i \) as a combination of states \( \psi_k \) into the latter formula, we obtain \( \lambda = \sum_{k \in X} d_k \psi_k \), where we defined \( d_k := \sum_{i \in Z} c_i (a_k \circ \rho_i) \). Since \( 0 \leq (a_k | \rho) = d_k \), and \( 1 = e_A \circ \lambda = \sum_{k \in X} d_k \), we conclude that \( \{ d_k \}_{k \in X} \) is a probability distribution. Hence we obtained that any state \( \lambda \) of system A can be written as a convex combination of the same set of perfectly distinguishable pure states \( \{ \psi_k \}_{k \in X} \).

As a consequence of theorem 3 either the theory enjoys local tomography [10] [11] and then it is classical probability theory, or it allows for entangled states—having non-null discord—despite being simplicial. We can then conclude that the only theory where no state has non-null discord is classical probability theory.

In conclusion we introduced the notion of objective information in causal theories, and used it to define null-discord states in the general operational probabilistic framework. The notion of discord is then introduced in terms of the minimum operational distance between a given state and the set of null-discord states. These definitions allowed us to prove that a theory where all separable states have null discord must have simplicial state sets. Now, either the theory enjoys local tomography and then it is classical, or it contains entangled states, having non-null discord. As a consequence, the only theory where no state has non-null discord is classical probability theory. In view of this result, we can justify the widespread identification of discord as a quantifier of non-classical correlations. Therefore, discord is not at all a signature of quantumness, but one should rather say that the absence of discord represents a singular feature of classical probability theory among all causal probabilistic theories. Finally, we want to point out that the notion of objective information introduced in this paper with the purpose of generalizing the notion of element of reality of Ref. 2 is a useful tool in the context of operational probabilistic theories, with a possible application to the extension of the notion of non-locality without entanglement in this framework.

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