Theory and Realization of GPS Orbit Integration

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Abstract  The algorithm of using one-day-arc to integrate \(n\)-days-arc by orbit overlaying is discussed in detail. An example is given, which proves that the orbit integration method can improve the precision of the orbit efficiently, especially in the determination of a local area’s orbit.

Keywords  orbit integration; one-day-arc; \(n\)-days-arc

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Introduction

The orbit published by the IGS is the integrated result of seven IGS analysis center (CODE, EMR, ESA, GFZ, JPL, NGS, and SIO) independent results. Because of the terminational effect of the one-day-arc, the error in both terminal of the orbit is often larger by comparison, and the orbit of each day is discontinuous. To improve the quality of the orbit, each analysis center needs to integrate one-day-arc into \(n\)-days-arc and provide the middle section of the \(n\)-days-arc for the integration of \(n\)-analysis-center to gain the precise IGS orbit. Generally, the middle section of the orbit by the orbit integration method has higher accuracy than by the one-day-arc method. This research field is still in the elementary stage in China.

1  Orbit parameter and orbit integration method

Suppose that the normal equation system including the orbit parameters is:

\[
N_i \cdot \Delta \hat{p}_i = b_i \quad (i = 1, 2, \cdots, n)
\]

where \(i\) is the number of days; \(N_i = B_i^T \cdot P_i \cdot B_i\) is the normal equation matrix; \(b_i = B_i^T \cdot \mathbf{l}_i\); \(B_i\) is the designed matrix; \(P_i\) is the weight matrix; and \(\mathbf{l}_i\) is the column matrix including observational value. The final estimation value of the parameter is gained by the formula below:

\[
\hat{\mathbf{p}}_i = \mathbf{p}_i^0 + \Delta \hat{\mathbf{p}}_i
\]

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m orbit parameters, it can be classified into three parts.

1) The continuous orbit parameters \( E_{ik}, i = 1, 2, \ldots, n; k = 1, 2, \ldots, 6 \), where it has the Kepler orbital elements \( E_i = (E_{1i}, E_{2i}, \ldots, E_{6i}) = (a, e, i, \Omega, \omega, u) \) for the \( i \)th day on epoch \( t_{0i} \).

2) The dynamics parameter \( q_{ik}, i = 1, 2, \ldots, n; k = 1, 2, \ldots, m_1 \), which is the orbit parameter used to model the disturbing force by the sunlight pressure radiation.

3) The pseudo-random parameters \( S_{ik}, i = 1, 2, \ldots, n; k = 1, 2, \ldots, m_2 \), which describe the change of the speed in a certain direction and on a certain epoch.

For a certain satellite on the \( i \)th day, the adopted parameters \( m = m_1 + m_2 + 6 \) can be described as follows:

\[
Q_i = (O_{1i}, O_{2i}, \ldots, O_{6i}) = (E_{1i}, E_{2i}, \ldots, E_{6i}, q_{1i}, q_{12i}, \ldots, q_{1m_1}, s_{1i}, s_{21i}, s_{22i}, \ldots, s_{2m_2}, s_{12i})
\]

The integrated orbit \( r_c(t) \) can be defined as:

\[
r_c(t) = r(t; O_{c1}, O_{c2}, \ldots, O_{cm}) = r(t; E_{c1}, E_{c2}, \ldots, E_{c6}, q_{c1}, q_{c2}, \ldots, q_{c_{m_1}}, s_{c1}, s_{c2}, \ldots, s_{c_{m_2}})
\]

The vector \( r_c(t) \) can describe the whole arcs for \( n \) days as a function with a suite of Kepler orbital elements, a suite of dynamics parameters by the number of \( m_1 \), and \( n \) suites of pseudo-random parameters by the number of \( m_2 \).

The usual method used in the orbit computation process is to overlay the computed orbits of several days to fit a long orbit; the number of days is usually three. At present, the method is based on the normal equation overlaying. Fig.1 shows the numerical procedure. The advantage of this method is that it can focus on the complicated data processing for the single period of time each day, and the result can be kept in the normal equation. The three-days-arc is based on the normal equations of one-day-arc. The final orbit computation is to overlay the normal equations using least squares method to get the final orbit.

2 Integrate the continuous one-day-arc into \( n \)-days-arc

Integrating the continuous one-day-arcs into \( n \)-days-arc can be considered as the integration of the consanguineous elements and the dynamics parameters, and the integration of the additional pseudo-random parameters. For convenience of discussion, the integration of the additional pseudo-random parameters is neglected.

The six orbital elements \( E_i = (a, e, i, \Omega, \omega, u) \) on the epoch \( r \) of the \( i \)th day are identical with the satellite position \( r_i(t) \) and its velocity \( \dot{r}_i(t) \) on the same epoch. So if we want to describe the orbit of the \((i+1)\)th day through the orbit parameters of the \( i \)th day, we need to use the continuity of the satellite position and its velocity on the boundary everyday. Therefore, the \((6+m_1)\) associated condition equations are:

\[
r_i(t_{i+1}) = r_{i+1}(t_{i+1})
\]

\[
\dot{r}_i(t_{i+1}) = \dot{r}_{i+1}(t_{i+1})
\]

\[
q_i = q_{i+1} = q_c
\]

Through the linearization of the equations above, we can get the linear transform equations from the \((i+1)\)th day to the \( i \)th day as follows:

\[
r_i(t_0) + \sum_{k=1}^{m_1} \frac{\partial r_i(t_0)}{\partial E_{ik}} \Delta E_{ik} + \sum_{k=1}^{m_1} \frac{\partial r_i(t_0)}{\partial q_{ik}} \Delta q_{ik} = r_{i+1}(t_0)
\]

\[
\dot{r}_i(t_0) + \sum_{k=1}^{m_1} \frac{\partial \dot{r}_i(t_0)}{\partial E_{ik}} \Delta E_{ik} + \sum_{k=1}^{m_1} \frac{\partial \dot{r}_i(t_0)}{\partial q_{ik}} \Delta q_{ik} = \dot{r}_{i+1}(t_0)
\]

\[
q_i(t_0) + \Delta q_i = q_{i+1}(t_0)
\]

Put the position vector and the velocity vector into a column matrix:

\[
r_i = \begin{bmatrix} r_i(t_0) \\ \dot{r}_i(t_0) \end{bmatrix}
\]

![Fig.1 Procedure of 3-d orbit integration](image-url)
The matrix forms of condition Eqs.(8) and (9) can be described as follows:  
\[
\begin{bmatrix}
H_{i+1} & Q_{i+1} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta E_{i+1} \\
\Delta q_{i+1}
\end{bmatrix}
= 
\begin{bmatrix}
H_i & Q_i \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta E_i \\
\Delta q_i
\end{bmatrix}
+ 
\begin{bmatrix}
r v_{i 0} - r v_{i+1 0} \\
q_{i 0} - q_{i+1 0}
\end{bmatrix}
\]
(12)
where \( i \) is the number of days; \( H_i \) is the transform matrix from the consanguineous elements to the initial coordinates and velocity on the epoch \( t_{i+1}; Q_i \) is the partial derivative of the dynamic parameter; \( \Delta E_i \) is the estimation of the consanguineous orbit parameter on the \( i \)th day; \( \Delta q_i \) is the estimation of the dynamic parameter on the \( i \)th day; \( E_{i 0} \) is the corresponding a priori consanguineous parameters of the one-day-arc \(( t = t_{i+1}); \) \( r v_{i 0} \) is the corresponding a priori position and velocity of the one-day-arc \(( t = t_{i+1}); \) and \( q_{i 0} \) is the a priori dynamic parameter of the one-day-arc, where \( q_{i 0} = (q_{i 1}, q_{i 2}, \ldots, q_{i m}) \).

To get the transform equation, we need to dispose Eq.(12), considering \( \Delta \hat{\beta} = [\Delta E_{i+1} q_{i+1}]^T \), and
\[
\begin{bmatrix}
H_{i+1} & Q_{i+1} \\
0 & I
\end{bmatrix}
^{-1}
= 
\begin{bmatrix}
H_{i+1}^{-1} - H_{i+1}^{-1}Q_{i+1} \\
0 & I
\end{bmatrix}
\]
(13)
So we can get:
\[
\begin{bmatrix}
\Delta E_{i+1} \\
\Delta q_{i+1}
\end{bmatrix}
= 
\begin{bmatrix}
K_{i+1,i} & L_{i+1,i} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta E_i \\
\Delta q_i
\end{bmatrix}
+ 
\begin{bmatrix}
M_{i+1,i} \\
N_{i+1,i}
\end{bmatrix}
\]
(14)
where \( K_{i+1,i} = H_{i+1}^{-1} \cdot H_i; \) \( L_{i+1,i} = H_{i+1}^{-1} \cdot (Q_i - Q_{i+1}); \) \( M_{i+1,i} = H_{i+1}^{-1} \cdot \{( r v_{i 0} - r v_{i+1 0}) - Q_{i+1} \cdot (q_{i 0} - q_{i+1 0})\}; \) \( N_{i+1,i} = q_{i 0} - q_{i+1 0}. \)

The parameters of the \((i+1)\)th day can be described through the ones of the first day by the transform sequence like Eq.(14). Through the recursive algorithm as follows, we can get:
\[
\begin{bmatrix}
\Delta E_{i+1} \\
\Delta q_{i+1}
\end{bmatrix}
= 
\begin{bmatrix}
K_{i+1,i} & L_{i+1,i} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta E_i \\
\Delta q_i
\end{bmatrix}
+ 
\begin{bmatrix}
M_{i+1,i} \\
N_{i+1,i}
\end{bmatrix}
\]
(15)
where \( K_{i+1,i} = K_{i+1,i}, \) \( K_{i,i} \); \( L_{i+1,i} = L_{i+1,i} + K_{i+1,i} \cdot L_{i,1}; \) \( M_{i+1,i} = M_{i+1,i} + L_{i+1,i} \cdot (q_{i 0} - q_{i+1 0}) + K_{i+1,i} \cdot M_{i,1}; \) \( N_{i+1,i} = q_{i 0} - q_{i+1 0}. \)

Before the overlaying and integrating of the normal equation, the corresponding normal equation system on the \((i+1)\)th day should be transformed according to the transform principle as follows:

\[ \Delta y + e = X \Delta \hat{\beta}, \quad D(\Delta y) = \sigma^2 P^{-1} \]
(16)
\[ X^T PX \Delta \hat{\beta} = X^T P \Delta y \rightarrow N \Delta \hat{\beta} = b \]
(17)

Taking the parameter transform equation into account, \( \Delta \hat{\beta} \) in Eq.(17) can be described as:
\[ \Delta \hat{\beta} = B \Delta \hat{\beta} + d \beta \]
(18)
where \( B \) is \( u \times u \) transform matrix; and \( d \beta \) is \( u \times 1 \) column vector.

Taking the Eq.(18) into Eq.(16), we have:
\[ \Delta y - X \Delta \hat{\beta} + e = XB \Delta \hat{\beta} \]
(19)
So the normal equation is:
\[ B^T X^T PB \Delta \hat{\beta} = B^T X^T P \Delta y, \quad \Delta \hat{\beta} = \Delta y - X \Delta \hat{\beta} \]
(20)
Combining Eq.(17), we have:
\[ \tilde{N} \Delta \hat{\beta} = \tilde{b} \]
(21)
where \( \tilde{N} = B^T N B, \quad \tilde{b} = B(b - N \Delta \hat{\beta}). \)

Another optional method is to keep the whole orbit parameters \( E \) and \( q_{i,i,i} = 2,3, \ldots, n \) in the integration normal equation, and connect these parameters with the ones on the first day by introducing the limited condition equation.

3 Data analysis of the orbit integration

To test the feasibility of the orbit integration method based on overlaying the normal equations as mentioned above, we carry on partial extending on the program of parameter estimation which is based on the coordinate pattern, and add the function of integrating the continuous one-day-arc into \( n \)-days-arc.

The orbit determination results from October 15th to October 17th in 2003 are used to integrate the orbit. These orbit determination results are obtained by using five continuous tracking stations to estimate the orbit, and it uses the broadcast ephemeris as the a priori orbit while estimating the one-day-arc. By comparing the one-day-arc with the final orbit of IGS, we know that because the data of ground tracking stations used to determine orbit is limited and these tracking stations locate in the local areas, the visual arc of the GPS satellite by the ground tracking stations each day is limited. There are not enough observation
data for orbit constraint. The quality of one-day orbit determination for some satellites is poor, with the worst being tens of meters. For those satellites where the tracking time is longer, the precision of orbit determination is approximately within five meters. Then carrying on the orbit integration based on the orbit overlaying method, the comparison of the integrated 3-day-arc and the precise orbit of IGS is as follows.

| PRN | Total  | ΔR    | ΔA    | ΔC    |
|-----|--------|-------|-------|-------|
| 1   | 1.338  | 0.567 | 1.611 | 0.567 |
| 2   | 0.923  | 0.579 | 1.422 | 0.462 |
| 3   | 1.282  | 0.672 | 2.014 | 0.646 |
| 4   | 0.889  | 0.613 | 1.170 | 0.790 |
| 5   | 0.971  | 0.677 | 1.438 | 0.551 |
| 6   | 1.177  | 0.803 | 1.767 | 0.625 |
| 7   | 0.856  | 0.615 | 1.112 | 0.765 |
| 8   | 0.916  | 0.600 | 1.317 | 0.645 |
| 9   | 1.667  | 0.751 | 2.611 | 0.951 |
| 10  | 0.947  | 0.636 | 1.228 | 0.884 |
| 11  | 1.163  | 0.516 | 1.790 | 0.767 |
| 12  | 1.224  | 0.718 | 1.751 | 0.955 |
| 13  | 0.886  | 0.648 | 1.231 | 0.649 |
| 14  | 1.220  | 0.599 | 1.779 | 0.966 |
| 15  | 0.735  | 0.443 | 1.085 | 0.499 |
| 16  | 1.182  | 0.625 | 1.813 | 0.705 |
| 17  | 0.701  | 0.476 | 1.023 | 0.444 |
| 18  | 0.935  | 0.712 | 1.278 | 0.690 |
| 19  | 0.789  | 0.582 | 1.034 | 0.677 |
| 20  | 1.623  | 1.245 | 2.340 | 0.934 |
| 21  | 0.881  | 0.589 | 1.291 | 0.561 |
| 22  | 0.798  | 0.541 | 1.200 | 0.424 |
| 23  | 0.883  | 0.461 | 1.347 | 0.551 |
| 24  | 1.123  | 0.588 | 1.664 | 0.815 |
| 25  | 0.946  | 0.622 | 1.432 | 0.495 |
| 26  | 0.974  | 0.525 | 1.538 | 0.457 |
| 27  | 1.016  | 0.671 | 1.387 | 0.847 |
| 28  | 1.142  | 0.535 | 1.745 | 0.754 |

In this form, PRN is the satellite number. Total means the total accuracy of a satellite; ΔR is the radial accuracy of a satellite; ΔA is the tangential accuracy; and ΔC is the normal accuracy of a satellite. As we can see in the form, the accuracy of the 3-day-arc after orbit integration has improved obviously than that of one-day-arc. The accuracy of the 3-day-arc is about 1 m. Its normal error is the smallest. The radial error is bigger than normal error, and the tangential error is the biggest. The reason why the accuracy of orbit has improved obviously after orbit integration is that the quality of the 3-day-arc using three one-day-arcs to integrate the orbit based on the orbit overlaying method is the same as directly using observation data of three days to determine the orbit. For one-day-arc in the local orbit determination, the visual arcs of some GPS satellites by the ground tracking station are very limited. In some cases, there are even no observation data for some satellites in some stations. Without enough observed data to restrict, the result of the one-day-arc may diverge. Therefore, the quality of the one-day-arc for some satellites is poor. After orbit integration, the visual arcs of GPS satellites for three days by the ground tracking station have increased a lot. In the process of orbit determination using one-day observation data, the tracking of the satellite by the ground station is centered in a certain arc, whereas for the 3-day-arc, from the view of the long arc, it is considered that there is discontinuous tracking of the satellite by all the ground tracking stations in the whole 3-day-arc, which is effective to avoid the divergence of the orbit determination result. This can be explained in Fig.2.

To compare the orbit accuracy of each day in the 3-day-arc after orbit integration, we use the figure below:

![Fig.2 Influence analysis to local orbit determination by orbit integration](image)

From Figs.3-5, we can see that for the 3-day-arc, which is from the orbit integration using three one-day-arcs, the orbit quality of the second day is the best on the whole. This is why at present many IGS analysis centers (AC) adopt three one-day-arcs to integrate the orbit and then use the middle-day-arc as the final provided precise orbit. However, we should take note that the differences between the second-day-
 arc and the first-day-arc or the third-day-arc of the 3-day-arc in the figure above are not obvious. Therefore, for local orbit determination, the effect of orbit integration is not only to keep the continuity of the orbit arc, but more importantly to obviously improve the accuracy of orbit determination. This is very important to establish our independent navigation and positioning system and develop space technology. For some political reasons, it is impossible for our ground tracking station to cover the whole world, but only to adopt the local area orbit determination. When the accuracy of a one-day-arc is poor, the orbit integration method can improve the accuracy of orbit determination or orbit control obviously.

4 Conclusions

In this study, the orbit integration method is used to carry on the detailed algorithm derivation for integrating the one-day-arc into n-days-arc. An example is used to prove that for the local area orbit determination, the effect of orbit integration is not only to keep the continuity of the orbit, but more importantly to improve the orbit accuracy obviously. The orbit integration process is the prerequisite premise to establish the data analysis center for satellite navigation. Considering the actual demand for the development of our second navigation system, we should strengthen the research of the orbit integration.

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