Twin-field quantum key distribution can tolerate arbitrary loss-independent misalignment error

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In this work, we apply two methods to improve performance of a practical twin-field quantum key distribution system. Firstly, to improve the secure key rate, we apply the error rate in X basis, Y basis and Z basis to precisely characterize the quantum channel. Secondly, we apply the advantage distillation method to further improve the secure key rate and the secure key transmission distance. Compared with the previous analysis result given by Maeda, Sasaki and Koashi [Nature Communication 10, 3140 (2019)], the secure key rate obtained by our analysis method will be increased at least 7%. By increasing the loss-independent misalignment error to 12%, the previous analysis method can not overcome the rate-distance bound. However, our analysis method can still overcome the rate-distance bound when the misalignment error is as large as 41%. More surprisingly, we prove that twin-field quantum key distribution can generate positive secure key even if the misalignment error arbitrary close to 50%, thus our analysis method can significantly improve the performance of a practical twin-field quantum key distribution system.

I. INTRODUCTION

Quantum key distribution (QKD) [1] is the art of sharing information-theoretical secure key between two different remote parties Alice and Bob. Under the perfect quantum devices preparation, the eavesdropper Eve can not get the secure key information even if she has unlimited computation and storage power [2–4]. Unfortunately, a practical QKD system is usually composed of imperfect devices, and the practical QKD system may be attacked [3] by utilizing imperfect quantum state preparation and measurement devices. To avoid the detector side channel attack [6, 7], measurement-device-independent QKD (MDI-QKD) protocol was proposed [8, 9]. In MDI-QKD protocol, the ideal quantum states are randomly prepared in Alice and Bob’s side, and the two quantum states will be transmitted to untrusted Charlie to apply Bell state measurement. To beat the Pirandola-Laurenza-Ottaviani-Banchi (PLOB) bound [11], twin-field QKD (TF-QKD) [12] was proposed. TF-QKD can be applied to overcome the rate scaling from $\eta$ to $\sqrt{\eta}$ with a relatively simple setup, which is a variant of the MDI-QKD protocol. In the TF-QKD protocol, Charlie simply conducts an interference measurement to learn the relative phase between Alice and Bob, and the secure key can be generated with in-phase and anti-phase measurement outcomes respectively. More recently, many intensive studies have been devoted to achieving information theoretic proofs of variants of TF-QKD protocols [13–18], and several practical TF-QKD systems have been widely implemented in labs and field tests [19–24].

To analyze security of MDI-QKD protocols [3, 11], Alice and Bob should randomly prepare phase-randomized pulses in Z basis, X basis and Y basis respectively, where Z basis consists of $|0\rangle$ and $|1\rangle$, X basis consists of $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$, and Y basis consists of $|r\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|t\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$. Note that the bit error rate in Z basis, X basis and Y basis can be applied to characterize the bit error rate and phase error rate about the quantum channel. Based on this quantum channel characterization, upper bound of the secure key information leaked to Eve can be strictly estimated. However, to estimate the bit error rate in X basis and Y basis, TF-QKD needs to generate a non-classical optical state [18, 23], which will be hard to be realized in current technology. Fortunately, Maeda, Sasaki and Koashi proposed the operator dominance method [18] to estimate the bit error rate in X basis by preparing phase-randomized weak coherent state. Combining the error rate in Z basis with the error rate in X basis, the practical quantum channel in TF-QKD can be characterized, and the secure key rate can be analyzed with the entanglement distillation and purification method [2, 3, 18].

In this work, we apply two methods to improve the performance of a practical TF-QKD system. In the first aspect, to improve the secure key rate, we apply the error rate in X basis, Y basis and Z basis to precisely characterize the practical quantum channel. By applying the information-theoretical security analysis method, we prove that the secure key rate can be improved by comparing with the previous analysis result [18]. In the second aspect, we apply advantage distillation (AD) method to further improve the secure key rate and the secure
key transmission distance. AD method was initially proposed in classical cryptography theory [26], then it has been widely used in different QKD protocols [27–30] to improve the error tolerance. More recently, we analyze security of the practical BB84-QKD [1], six-state-QKD [31] and MDI-QKD [3, 10] systems by combining the AD method [4] with the decoy-state method [32–34], and the analysis results demonstrate that AD method can significantly improve the performance of different practical QKD systems [35]. Inspired by our previous work, we propose in classical cryptography theory [26], then it has been restricted by the quantum bit error rate in three different bases.

To improve the maximal tolerable error rate, the repetition code protocol based AD method has been proposed [4]. In the repetition code protocol, Alice and Bob split their raw key into blocks of $b$ bits $x_0, x_1, \ldots, x_{b-1}$ and $y_0, y_1, \ldots, y_{b-1}$ respectively. Alice privately generates a random bit $c \in \{0, 1\}$, and sends the message $m = m_0, m_1, \ldots, m_{b-1} = x_0 \oplus c, x_1 \oplus c, \ldots, x_{b-1} \oplus c$ to Bob through an authenticated classical channel. Bob accepts the block if and only if $m_0 \oplus y_0, m_1 \oplus y_1, \ldots, m_{b-1} \oplus y_{b-1} \in \{0, 0, \ldots 0, 1, 1, \ldots 1\}$. If Alice and Bob accept the block, they keep the first bit $x_0$ and $y_0$ as the raw key. Finally, Alice and Bob will apply the error correction and privacy amplification algorithms to generate the final secure key.

Based on the repetition code protocol, the secure key rate $\bar{R}$ can be modified as the following inequality [4]

$$\bar{R} \geq \max_b \min_{\lambda_0, \lambda_1, \lambda_2, \lambda_3} \frac{1}{b} p_{\text{succ}}[1 - (\lambda_0 + \lambda_1)H(\frac{\lambda_0}{\lambda_0 + \lambda_1}) - (\lambda_2 + \lambda_3)H(\frac{\lambda_2}{\lambda_2 + \lambda_3}) - H(\lambda_0 + \lambda_1)],$$

where

$$\lambda_0 = \frac{(\lambda_0 + \lambda_1)^b + (\lambda_0 - \lambda_1)^b}{2p_{\text{succ}}},$$

$$\lambda_1 = \frac{(\lambda_0 + \lambda_1)^b - (\lambda_0 - \lambda_1)^b}{2p_{\text{succ}}},$$

$$\lambda_2 = \frac{(\lambda_2 + \lambda_3)^b + (\lambda_2 - \lambda_3)^b}{2p_{\text{succ}}},$$

$$\lambda_3 = \frac{(\lambda_2 + \lambda_3)^b - (\lambda_2 - \lambda_3)^b}{2p_{\text{succ}}},$$

$p_{\text{succ}} = (\lambda_0 + \lambda_1)^b + (\lambda_2 + \lambda_3)^b$ is the success probability of the AD protocol. Since the AD protocol parameter $b$ can be controlled by Alice and Bob, they can choose the optimal $b$ to improve the secure key rate. Note that this secure key rate is based on the single photon state preparation, which can not be directly applied in the practical QKD system with weak coherent pulse or phase-randomized weak coherent pulse preparation.

III. THREE VIRTUAL PROTOCOLS TO ANALYZE THE QUANTUM CHANNEL IN TF-QKD

Based on the TF-QKD protocol proposed by Maeda, Sasaki and Koashi [13], Alice and Bob generate four different pulses with the signal state modulation and the testing state modulation respectively. In the signal state modulation, Alice and Bob randomly prepare the weak
coherent pulse with amplitude $\sqrt{\mu}$ or $-\sqrt{\mu}$. In the testing state modulation, Alice and Bob randomly prepare the phase-randomized weak coherent pulse with intensities $\nu_1$, $\nu_2$ and 0 respectively. For every pair of pulses received from Alice and Bob, Charlie announces whether the phase difference was successfully detected. When the phase difference was detected, Charlie further announces whether it was in-phase when detector $D_1$ clicks or anti-phase when detector $D_2$ clicks. After receiving Charlie’s measurement outcomes, Alice and Bob will apply the signal state to generate the sifted key. In the in-phase measurement outcome case, Alice and Bob will respectively generate the random bit $a$ with the signal state preparation $\langle -1 \rangle_{\sqrt{\mu}}$. In the anti-phase measurement outcome case, Alice and Bob will respectively generate the random bit $a$ with the signal state preparation $|\langle -1 \rangle_{\sqrt{\mu}}\rangle$ and $\langle (\langle + \rangle_{\sqrt{\mu}}\rangle$ respectively.

By applying the entanglement based protocol, the signal state preparation in Alice’s side can be illustrated with the following quantum states:

$$
|\psi_1\rangle_{AC_A} = \frac{|0\rangle_A|\sqrt{\mu}\rangle_C + |1\rangle_A|\mu\rangle_C}{\sqrt{2}},
|\psi_2\rangle_{AC_A} = \frac{|0\rangle_A|\sqrt{\mu}\rangle_C - |1\rangle_A|\mu\rangle_C}{\sqrt{2}},
|\psi_3\rangle_{AC_A} = \frac{|0\rangle_A|\mu\rangle_C + |1\rangle_A|\sqrt{\mu}\rangle_C}{\sqrt{2}}.
$$

After preparing one of the quantum states $|\psi_i\rangle_{AC_A}$, $i = \{0, 1, 2, 3\}$, Alice will measure the first quantum state, and the second quantum state will be transmitted to the quantum channel. By considering all of the states preparation in Eq. (6), the second quantum state $\rho_{C_A}$ can be given by

$$
\rho_{C_A} = tr_A |\psi_1\rangle_{AC_A} \langle \psi_1|_{AC_A} = tr_A |\psi_2\rangle_{AC_A} \langle \psi_2|_{AC_A} = tr_A |\psi_3\rangle_{AC_A} \langle \psi_3|_{AC_A} = \frac{1}{2}(N|\sqrt{\mu}\rangle_{AC_A} + |\mu\rangle_{C_A} - |\mu\rangle_{C_A} - N|\sqrt{\mu}\rangle_{C_A}).
$$

Based on this analysis result, we find that Eve can not distinguish the four quantum states $|\psi_i\rangle_{AC_A}$, $i = \{0, 1, 2, 3\}$ by only attacking the second quantum state. Thus she can only apply the same operation with all of the four quantum states preparation. Similarly, the signal state preparation in Bob’s side can also be illustrated with the following quantum states:

$$
|\psi_1\rangle_{BC_B} = \frac{|0\rangle_B|\sqrt{\mu}\rangle_C + |1\rangle_B|\mu\rangle_C}{\sqrt{2}},
|\psi_2\rangle_{BC_B} = \frac{|0\rangle_B|\sqrt{\mu}\rangle_C - |1\rangle_B|\mu\rangle_C}{\sqrt{2}},
|\psi_3\rangle_{BC_B} = \frac{|0\rangle_B|\mu\rangle_C + |1\rangle_B|\sqrt{\mu}\rangle_C}{\sqrt{2}}.
$$

In the first virtual protocol, we consider the following quantum states preparation in Alice and Bob’s side

$$
\text{Alice’s side: } \rho_{A}\text{ is } |0\rangle_A|\sqrt{\mu}\rangle_C + |1\rangle_A|\mu\rangle_C,
\text{Bob’s side: } \rho_{B}\text{ is } |0\rangle_B|\sqrt{\mu}\rangle_C + |1\rangle_B|\mu\rangle_C.
$$

Note that this state preparation has also been analyzed in Ref [18]. After preparing these quantum states, Alice and Bob will measure the first quantum state, and the second quantum state will be transmitted to Charlie. Thus the quantum state shared between Alice, Bob and Charlie can be given by

$$
\langle 0\rangle_{A}|\sqrt{\mu}\rangle_{C_A} + |1\rangle_{A}|\mu\rangle_{C_A} \otimes \langle 0\rangle_{B}|\sqrt{\mu}\rangle_{C_B} + |1\rangle_{B}|\mu\rangle_{C_B}
\begin{aligned}
= & \frac{1}{2}(|0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\sqrt{\mu}\rangle + |\mu\rangle_1|\mu\rangle_{C_A,C_B}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\sqrt{\mu}\rangle - |\mu\rangle_1|\sqrt{\mu}\rangle_{C_A,C_B}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\mu\rangle + |\mu\rangle_1|\sqrt{\mu}\rangle_{C_A,C_B}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\mu\rangle - |\mu\rangle_1|\sqrt{\mu}\rangle_{C_A,C_B}.
\end{aligned}
$$

After measuring the received quantum states $C_A$ and $C_B$ with a 50:50 beam-splitter and a pair of photon detectors $D_1$ and $D_2$, Charlie announces whether the phase difference was successfully detected, and this quantum state will be transformed to

$$
\langle 0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\begin{aligned}
= & \frac{1}{2}(|0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\sqrt{\mu}\rangle + |\mu\rangle_1|\mu\rangle_{D_1,D_2}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\sqrt{\mu}\rangle + |\mu\rangle_1|\mu\rangle_{D_1,D_2}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\mu\rangle + |\mu\rangle_1|\mu\rangle_{D_1,D_2}
+ |0\rangle_0|0\rangle + |1\rangle_1|1\rangle_{AB}
\langle |\mu\rangle - |\mu\rangle_1|\mu\rangle_{D_1,D_2}.
\end{aligned}
$$

where the detection results that detector $D_1$ clicks and detector $D_2$ clicks respectively demonstrate the in-phase and anti-phase measurement outcomes in Charlie’s side. Here, for the simplicity of the discussion, we assume that they are no channel losses. Correspondingly, the quantum states ($|\sqrt{\mu}\rangle_0 + |\mu\rangle_1|\mu\rangle_0_{D_1,D_2}$ and ($|\sqrt{\mu}\rangle_0 + |\mu\rangle_1|\mu\rangle_0_{D_1,D_2}$) demonstrate the in-phase measurement outcome, while quantum states ($|\sqrt{\mu}\rangle_0 - |\mu\rangle_1|\mu\rangle_0_{D_1,D_2}$ and ($|\sqrt{\mu}\rangle_0 - |\mu\rangle_1|\mu\rangle_0_{D_1,D_2}$) demonstrate the anti-phase measurement outcome. Note that the detection result can be controlled by Eve, we can simply assume the click of detector $D_1$ demonstrates the Bell state $\frac{1}{\sqrt{2}}(|0\rangle_0 + |1\rangle_1)_{AB}$ preparation in Alice and Bob’s side, while the click of detector $D_2$ demonstrates the Bell state $\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_0)_{AB}$ preparation in Alice and Bob’s side. By applying the time reversed entanglement technique, the Bell states $\frac{1}{\sqrt{2}}(|0\rangle_0 + |1\rangle_1)_{AB}$ and $\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_0)_{AB}$ can be assumed to be prepared in Charlie’s side, then the two quantum states will be transmitted to Alice and Bob to perform Z basis or X basis measurement. Based on this analysis method, the in-phase measurement outcome demonstrate the Bell state preparation

$$
\frac{1}{\sqrt{2}}(|0\rangle_0 + |1\rangle_1)_{AB} = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle - \rangle)_{AB}.
$$
Suppose Alice and Bob apply Z basis measurement, Z basis error is defined to be an event where the pair was found in either state \( |0\rangle_A |1\rangle_B \) or \( |1\rangle_A |0\rangle_B \). Suppose Alice and Bob apply X basis measurement, X basis error is defined to be an event where the pair was found in either state \( |+\rangle_A |-\rangle_B \) or \( |-\rangle_A |+\rangle_B \).

Similarly, the anti-phase measurement outcome demonstrates the Bell state preparation
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) = \frac{1}{\sqrt{2}} ((+)|-\rangle_B + |-\rangle_A |+\rangle_B).
\]

(13)

Suppose Alice and Bob apply Y basis measurement, Y basis error is defined to be an event where the pair was found in either state \( |0\rangle_A |0\rangle_B \) or \( |1\rangle_A |1\rangle_B \). Suppose Alice and Bob apply X basis measurement, X basis error is defined to be an event where the pair was found in either state \( |+\rangle_A |-\rangle_B \) or \( |-\rangle_A |+\rangle_B \).

In a practical TF-QKD experiment, the bit error rate in Z basis can be directly tested, but the bit error rate in X basis and Y basis can not be directly observed. To analyze the bit error rate in X basis, the quantum state shared between Alice, Bob and Charlie can be rewritten as
\[
\frac{1}{\sqrt{2}} (|0\rangle_A \sqrt{|C\rangle + |1\rangle_A} |1\rangle_B - |\sqrt{|C\rangle - |1\rangle_A} |0\rangle_B) \otimes \frac{1}{\sqrt{2}} (|0\rangle_B \sqrt{|C\rangle + |1\rangle_B} |1\rangle_A - |\sqrt{|C\rangle - |1\rangle_B} |0\rangle_A).
\]

(14)

\(c_+ := e^{-\mu} \cosh \mu\), \(c_- := e^{-\mu} \sinh \mu\), \(|\sqrt{|C\rangle + |1\rangle}\rangle = ((/\sqrt{|C\rangle + |1\rangle})/2\sqrt{|C\rangle + |1\rangle} + (/\sqrt{|C\rangle + |1\rangle}))\langle/\sqrt{|C\rangle + |1\rangle} + (/\sqrt{|C\rangle + |1\rangle}))2\sqrt{|C\rangle + |1\rangle}). Based on this state preparation, the X basis error occurs with probability \(p_{\text{even}} = c_+^2 + c_-^2 = e^{-2\mu} \cosh 2\mu\), and the optical pulses are sent in the following quantum state
\[
p_{\text{even}} \rho_{\text{even}} = c_+^2 \sqrt{|C\rangle + |1\rangle} |\sqrt{|C\rangle + |1\rangle}\rangle |\sqrt{|C\rangle + |1\rangle}\rangle C_{AB} + c_-^2 \sqrt{|C\rangle + |1\rangle} |\sqrt{|C\rangle + |1\rangle}\rangle |\sqrt{|C\rangle + |1\rangle}\rangle C_{AB}.
\]

(15)

To analyze the error rate in X basis, the counting rate with this non-classical optical state preparation can be analyzed with the operator dominance method [18].

In the second virtual protocol, we consider the following quantum state preparation in Alice and Bob’s side

**Alice’s side:** \(\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle + |1\rangle} A |0\rangle_B)\)

**Bob’s side:** \(\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B)\)

(16)

Based on this state preparation, the quantum state shared between Alice, Bob and Charlie can be given by
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle + |1\rangle} A |0\rangle_B) \otimes \frac{1}{\sqrt{2}} (|0\rangle_B |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B).
\]

(17)

After measuring the received quantum states \(C_A\) and \(C_B\), Charlie announces whether the phase difference was successfully detected, and this quantum state will be transformed to
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle + |1\rangle} A |0\rangle_B) \otimes \frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B).
\]

(18)

Note that the quantum states \(\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle + |1\rangle} A |0\rangle_B\) and \(\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B\) demonstrate the in-phase measurement outcome. Since the detection result can be assumed to be controlled by Eve, we can simply assume the click of detector \(D_1\) demonstrates the Bell state \(\frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)\) preparation in Alice and Bob’s side. By applying the time reversed entanglement technique, the Bell state \(\frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)\) can be assumed to be prepared in Charlie’s side, then the two quantum states will be transmitted to Alice and Bob to perform Z basis or Y basis measurements. Based on this analysis method, the in-phase measurement outcome demonstrate the following Bell state preparation
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) = \frac{1}{\sqrt{2}} (|r\rangle |t\rangle + |t\rangle |r\rangle)_{AB}.
\]

(19)

Suppose Alice and Bob apply Y basis measurement, Y basis error is defined to be an event where the pair was found in either state \( |r\rangle |t\rangle \) or \( |t\rangle |r\rangle \). To analyze the bit error rate in Y basis with the Bell state \(\frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)\) preparation, the quantum state shared between Alice, Bob and Charlie can be rewritten as
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle + |1\rangle} A |0\rangle_B) \otimes \frac{1}{\sqrt{2}} (|0\rangle_B |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B).
\]

(20)

Based on this state preparation, we can interpret that the Y basis error occurs with probability \(p_{\text{even}}\) and the optical pulses are sent in state \(\rho_{\text{even}}\).

In the third virtual protocol, we consider the following quantum state preparation

**Alice’s side:** \(\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B)\)

**Bob’s side:** \(\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B)\)

(21)

The quantum state shared between Alice, Bob and Charlie can be given by
\[
\frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle + |1\rangle} B + |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B) \otimes \frac{1}{\sqrt{2}} (|0\rangle_A |\sqrt{|C\rangle - |1\rangle} B - |\sqrt{|C\rangle - |1\rangle} A |0\rangle_B).
\]

(22)
formed to
\begin{align}
\frac{1}{4}[(|0\rangle|0\rangle + |1\rangle|1\rangle)_{AB}((\sqrt{2}\mu)|0\rangle + |\sqrt{2}\mu\rangle)|D_1D_2 + (|0\rangle|0\rangle - |1\rangle|1\rangle)_{AB}((\sqrt{2}\mu)|0\rangle - |\sqrt{2}\mu\rangle)|D_1D_2 + (|0\rangle|1\rangle + |1\rangle|0\rangle)_{AB}(-i|0\rangle|\sqrt{2}\mu\rangle + i|0\rangle - |\sqrt{2}\mu\rangle)|D_1D_2 + (|0\rangle|1\rangle - |1\rangle|0\rangle)_{AB}(-i|0\rangle|\sqrt{2}\mu\rangle - i|0\rangle - |\sqrt{2}\mu\rangle)|D_1D_2]
\end{align}

Note that the quantum states \((-i|0\rangle|\sqrt{2}\mu\rangle + i|0\rangle - |\sqrt{2}\mu\rangle)\) demonstrate the anti-phase measurement outcome. Since the detection result can be assumed to be controlled by Eve, we can simply assume the click of detector \(D_2\) demonstrates the Bell state \(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)_{AB}\). By applying the time reversed entanglement technique, the Bell state \(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)_{AB}\) can be assumed to be prepared in Charlie’s side, then the two quantum states will be transmitted to Alice and Bob to perform Z basis or Y basis measurements. Based on this analysis method, the anti-phase measurement outcome demonstrate the Bell state preparation
\begin{align}
\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)_{AB} = -\sqrt{2}(|r\rangle|r\rangle - |t\rangle|t\rangle)_{AB}
\end{align}

Suppose Alice and Bob apply Y basis measurement, Y basis error is defined to be an event where the pair was found in either state \(|r\rangle|t\rangle\) or \(|t\rangle|r\rangle\). To analyze the bit error rate in Y basis with Bell state \(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)_{AB}\) preparation, the quantum state shared between Alice, Bob and Charlie can be rewritten as
\begin{align}
|0\rangle_A|\sqrt{\mu}|C_A + |1\rangle_A|\sqrt{-\mu}|C_A \otimes |0\rangle_B|\sqrt{\mu_c}|C_B - |1\rangle_B|\sqrt{-\mu}|C_B
\end{align}

\begin{align}
= (\sqrt{\mu_c}|r\rangle_A|\sqrt{\mu_{even}}|C_A + \sqrt{-\mu_c}|t\rangle_A|\sqrt{\mu_{odd}}|C_A) \\
\otimes (\sqrt{\mu_c}|r\rangle_B|\sqrt{\mu_{add}}|C_B + \sqrt{-\mu_c}|t\rangle_B|\sqrt{\mu_{even}}|C_B)
\end{align}

Based on this state preparation, we can interpret that Y basis error occurs with probability \(p_{even}\) and the optical pulses are sent in state \(\rho_{even}\).

Combining with this three virtual protocols, we can prove that the error rate in X basis is equal to the error rate in Y basis no matter Alice and Bob prepare the Bell state \(\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\) or \(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)\). Combining this analysis result with the analysis result given by [18], we can precisely estimate the quantum channel parameters \(\lambda_i\), \(i = 0, 1, 2, 3\) in the following section.

\section{Security of TF-QKD with AD}

By applying the previous analysis result with the signal state modulation in Alice and Bob’s side, we need to analyze the error rate \(E_{SU}^{ZZ}\) in Z basis, the error rate \(E_{SU}^{XX}\) in X basis and the error rate \(E_{SU}^{YY}\) in Y basis respectively. In a practical TF-QKD experiment, \(E_{SU}^{ZZ}\) can be directly calculated by testing part of the in-phase and anti-phase measurement outcomes. However, the nonclassical optical state \(\rho_{even}\) is hard to be realized in current technology, the bit error rate \(E_{SU}^{XX}\) and \(E_{SU}^{YY}\) can not be directly estimated in a practical TF-QKD system.

Fortunately, Maeda, Sasaki and Koashi proposed the operator dominance method to estimate \(E_{SU}^{XX}\) [18], where the number of testing states forming the linear combination to approximate the non-classical optical state \(\rho_{even}\).

In the previous section, we prove that \(E_{SU}^{XX} = E_{SU}^{YY}\), thus the practical quantum channel can be precisely characterized with \(E_{SU}^{XX}, E_{SU}^{YY}\) and \(E_{SU}^{ZZ}\). To analyze security of the entanglement based TF-QKD protocol with the information-theoretical analysis method, the relationship between \(E_{SU}^{XX}, E_{SU}^{YY}, E_{SU}^{ZZ}\) and \(\lambda_i\) can be given by the following equations.

\[
\begin{align}
\lambda_1 + \lambda_3 &= E_{SU}^{XX}, \\
\lambda_2 + \lambda_3 &= E_{SU}^{ZZ}, \\
\lambda_1 + \lambda_2 &= E_{SU}^{YY},
\end{align}
\]

where \(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1\). Based on these equations, all of the quantum channel parameters \(\lambda_i, i = 0, 1, 2, 3\) can be accurately solved.

In the TF-QKD protocol, only the signal state can be used to generate the final security key. Since the quantum channel parameters can be solved in Eq. (26), Eq. (4) can be modified with the following inequality by applying the AD method

\[
R_{TF} \geq \max_b \frac{1}{b|\text{succ}|_{E_{SU}^{YY}}[S(A|E) - H(A|B)]
\]

\[
= \max_b \frac{1}{b|\text{succ}|_{E_{SU}^{YY}}[(1 - (\lambda_0 + \lambda_1))H(\frac{\lambda_0}{\lambda_0 + \lambda_1}) - (\lambda_2 + \lambda_3)H(\frac{\lambda_0}{\lambda_2 + \lambda_3})] - fh(E_{SU}^{YY})],
\]

where \(Q_{SU}^{YY}\) is the counting rate by considering Alice and Bob prepare the signal states, \(E_{SU}^{YY} = E_{SU}^{ZZ\mu}b\) is the error rate after the AD protocol, \(f > 1\) is the error correction efficiency, \(\eta_{\text{succ}} = E_{SU}^{ZZ\mu}b + (1 - E_{SU}^{ZZ\mu}b)\) is the success probability of the AD method in the practical TF-QKD system.

By applying the operator dominance method with asymptotic key length, the error rate in X basis \(E_{SU}^{XX}\) and the error rate in Y basis \(E_{SU}^{YY}\) can be given by [18]

\[
E_{SU}^{XX} = E_{SU}^{YY} = C_1(1 + \sqrt{(C_2 + C_4)C_3})^2,
\]

where \(C_1 = \frac{1}{1 - e^{-2\eta_1d} + e^{-2\eta_2d}}, C_2 = \frac{e^{-2\nu_1(\nu_1 - \nu_2)}}{\nu_2}, C_3 = \frac{1}{\nu_1 e^{-2\eta_1d} \sum_{k=1}^{\infty} \frac{\mu^{k+1}}{k+1} e^{-2\eta_1d}}, C_4 = \frac{1}{\nu_1 e^{-2\eta_1d}} (1 - e^{-2\eta_1d} + e^{-2\eta_2d})\), max(\(\mu, \nu_2\)) < \nu_1.

Note that, we assume each detector has a dark count probability of \(p_d\), which amounts to the effective probability \(d = 2p_d - p_d^2\) from the two detectors. By considering the channel transmission efficiency, the overall transmissivity from Alice (Bob) to Charlie’s detection is \(\eta\).

Based on the simulation parameters given by [18], we calculate the secure key rate \(R_{TF}\) as a function of secure key rate transmission distance \(L\) between Alice and Bob.
We assume a fiber loss of 0.2 dB/km, a loss-independent misalignment error of \( e_d = 0.03 \), error correction efficiency \( f = 1.1 \), each detector has a detection efficiency \( \eta_d = 0.3 \) and dark count probability \( p_d = 10^{-8} \). The overall transmissivity from Alice (Bob) to Charlie’s detection is then \( \eta = \eta_d 10^{-0.01L} \). In the asymptotic limit, the counting rate of Alice and Bob’s signal states \( Q^Z_{uu} \) can be given by

\[
Q^Z_{uu} = 1 - e^{-2\eta}\eta + e^{-2\eta}d,
\]

Correspondingly, the error rate in Z basis \( E^{ZZ}_{\mu\mu} \) can be given by

\[
E^{ZZ}_{\mu\mu} = \frac{\eta(1-e^{-2\eta})+2e^{-2\eta}d}{Q^Z_{uu}}.
\]

Based on these simulation parameters and the optimal \( b \) values, we calculate the secure key rate \( R_{TF} \) as a function of transmission distance between Alice and Bob in Figure 1. By comparing with the previous analysis result give by [18], we find that both of the two analysis results can overcome the PLOB bound \(-log(1-\eta_d 10^{-0.02L})\), but our analysis method can generate the secure key rate at least 7% higher than the previous method even without utilizing the AD method. By applying the AD method, the simulation result demonstrate that the maximal secure key transmission distance can be significantly improved from 430 km to 480 km.

By increasing the loss-independent misalignment error to \( e_d = 0.12 \), we calculate the secure key rate \( R_{TF} \) as a function of transmission distance between Alice and Bob in Figure 2. By comparing with the previous analysis result give by [18], we find that our analysis result can overcome the PLOB bound, but the secure key rate with the previous analysis method can not overcome the PLOB bound at any transmission distance. The secure key rate obtained by our method is at least 1.5 times higher than that of the original method, and the maximal secure key transmission distance can be improved from 345 km to 466 km.

By increasing the loss-independent misalignment error to \( e_d = 0.3 \), we calculate the secure key rate \( R_{TF} \) as a function of transmission distance between Alice and Bob in Figure 3. By comparing with the previous analysis result give by [18], we find that the secure key obtained by our method is at least 85 times higher than that of the original method, and the maximal secure key transmission distance can be sharply improved from 82 km to 448 km.

By increasing the loss-independent misalignment error to \( e_d = 0.41 \), we calculate the secure key rate \( R_{TF} \) as a function of transmission distance between Alice and Bob in Figure 4. By comparing with the previous analysis result give by [18], we find that our analysis result can still overcome the PLOB bound, but the previous analysis method can not generate positive secure key rate even at 0 km transmission distance. The maximal secure key transmission distance can also be reached to 448 km.

By increasing the loss-independent misalignment error to \( e_d = 0.49 \), we calculate the secure key rate \( R_{TF} \) as a function of transmission distance between Alice and Bob in Figure 5. More surprisingly, our simulation result demonstrate that the maximal secure key transmission distance can also be reached to 447 km.

From the simulation result, we find that our analysis method can generate positive secure key even if the loss-independent misalignment error arbitrary close to 0.5, the reason for which is that \( E^{XX}_{\mu\mu} \) and \( E^{YY}_{\mu\mu} \) have no correlation with \( e_d \) in the security analysis model. This is quite different from the BB84-QKD system or the original MDI-QKD system, where all of the error rate in different bases will be increased close to 0.5 by increasing \( e_d \) close to 0.5. However, in a TF-QKD system, we can prove that the error rate \( E^{ZZ}_{\mu\mu} \) will be increased close to 0.5 by increasing \( e_d \) close to 0.5, but the error rate \( E^{XX}_{\mu\mu} \) and \( E^{YY}_{\mu\mu} \) will be remain unchanged. By applying the AD method with \( b > 1 \), the error rate \( E^{ZZ}_{\mu\mu} = \frac{E_{bb}}{E_{bb} + (1-E_{ZZ})^b} \) will be smaller than \( E^{ZZ}_{\mu\mu} \) at the cost of increasing \( \lambda_0 \) and \( \Lambda_1 \). Thus, if \( b \) is large enough, we can also generate positive secure key rate \( R_{TF} \) even if the loss-independent misalignment error \( e_d \) arbitrary close to 0.5.

V. DISCUSSION

In a practical TF-QKD system, by combining the AD method with the information-theoretical security analysis method, we prove that both of the maximal transmission distance and the secure key rate can be sharply improved. More surprisingly, the numerical simulation result demonstrate that TF-QKD can generate positive secure key even if the loss-independent misalignment error arbitrary close to 50%, thus our analysis method can significantly improve the performance of a practical TF-QKD system. In the future research, it will be interesting to experimentally realize the AD method in a practical TF-QKD system, especial with a high loss-independent misalignment error.

Since the AD method only modify the classical post-processing step, we can simply analyze the finite key length with the existed method. More precisely, based on the quantum asymptotic equipartition property [36], the leftover hash lemma [37] and the Chernoff bound [18, 38], the statistical fluctuation of the bit error rate and the secure key rate in finite-key length can be efficiently analyzed in the future research.

Source codes of the plots are available from the corresponding authors on request.

Data availability

The data that support the findings of this study are available from the corresponding authors on request.
(a) The relationship between the transmission distance and the secure key rate, the blue line is the secure key rate given by [18], the red line is the secure key rate given by this work, and the green line is the PLOB bound.

(b) The relationship between the transmission distance and the optimal $b$ values. From 0 km to 411 km, the optimal $b$ value is 1, thus we do not need to utilize the AD method in these transmission distances.

FIG. 1. Results of TF-QKD protocol with $e_d = 0.03$.

(a) The relationship between the transmission distance and the secure key rate, the blue line is the secure key rate given by [18], the red line is the secure key rate given by this work, and the green line is the PLOB bound.

(b) The relationship between the transmission distance and the optimal $b$ values. From 0 km to 417 km, the optimal $b$ value is 2.

FIG. 2. Results of TF-QKD protocol with $e_d = 0.12$.

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Author Contributions

Hong-Wei Li and Qing-Yu Cai conceived the project. Hong-Wei Li, Chun-Mei Zhang and Rui-Qiang Wang performed the calculation and analysis. Hong-Wei Li wrote the paper.

Competing Financial Interests

The authors declare no competing financial interests.

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(b) The relationship between the transmission distance and the optimal $b$ values. From 0 km to 414 km, the optimal $b$ value is 3.

FIG. 3. Results of TF-QKD protocol with $e_d = 0.3$.

(a) The relationship between the transmission distance and the secure key rate, the red line is the secure key rate given by this work, and the green line is the PLOB bound.

(b) The relationship between the transmission distance and the optimal $b$ values. From 0 km to 433 km, the optimal $b$ value is 5.

FIG. 4. Results of TF-QKD protocol with $e_d = 0.41$.

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(a) The relationship between the transmission distance and the secure key rate, the red line is the secure key rate given by this work, and the green line is the PLOB bound.

(b) The relationship between the transmission distance and the optimal $b$ values. From 0 km to 417 km, the optimal $b$ value is 36.

FIG. 5. Results of TF-QKD protocol with $e_d = 0.49$.

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