A positive solution to Hilbert’s 10th problem

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Abstract

Polynome codes and code evaluation; arithmetical theory frames; $\mu$-recursive race for decision; decision correctness; decision termination in Zermelo-Fraenkel set theory ZFC+ with axiom of choice and consistency provability; decision correctness in theory $T = \text{PR}$ of Primitive Recursion; comparison with the negative result of Matiyasevich; positive solution for each single diophantine polynomial in p. r. non-infinite-descent theory $\pi\text{R} = \text{PR} + (\pi)$.

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Introduction

Within theory $\text{ZFC}^+ = \text{ZFC} + \text{Con}_{\text{ZFC}}$ of Zermelo-Fraenkel set theory with axiom of choice $\text{AC}$, strengthened by formula $\text{Con}_{\text{ZFC}}$ which is to express $\text{ZFC}$’s internal, gödelised consistency, we solve Hilbert’s 10th problem positively: we organise decision of diophantine polynome codes—decision on overall non-nullity—as an enumerative $\mu$-recursive race for a (first) zero (counterexample), against race for a first internal $\text{ZFC}$-non-nullity proof for a given such polynomial code, given as the (nested) list of coefficients. Comparison with Matiyasevich’s negative solution of Hilbert’s 10th problem gives inconsistency of theory $\text{ZFC} + \text{Con}_{\text{ZFC}}$ whence self-inconsistency $\text{ZFC} \vdash \neg \text{Con}_{\text{ZFC}}$.

In a final section we plug our positive solution of the problem into the constructive framework of p. r. non-infinite descent theory $\pi \textbf{R} = \text{PR} + (\pi)$ out of Arithmetical Foundations in the References.
This is to give a decision algorithm for each single diophantine equation (in a uniform way), as asked in the original Hilbert’s 10th problem.

1 Hilbert’s 10th Problem

We attempt a positive solution to Hilbert’s 10th problem. In its original form it reads:

10. DETERMINATION OF THE SOLVABILITY OF A DIOPHANTINE EQUATION Given a diophantine equation with any number of unknown quantities and with rational integer numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

[translation quoted from Matiyasevich 1993.]

Formally, this text allows for a separate decision algorithm (“process”) for each diophantine polynomial. But it is clear that a decision-family must be uniform in a suitable sense.

Correctness of our alleged \( \mu \)-recursive decision algorithm \( \nabla_{\text{ZFC}} : \text{DIO} \rightarrow 2 = \{0, 1\} \) builds, within \( \text{ZFC}^+ \), on diophantine soundness inferred by Con\(_{\text{ZFC}}\) over \( \text{ZFC} \). Termination follows from (countable) Choice. This already within \( \text{ZFC} \). Together this gives the wanted decision \( \nabla = \nabla_{\text{ZFC}} \) within \( \text{ZFC}^+ \), of all polynome codes in \( \text{DIO} \subset \mathbb{N} \).

Comparison with Matiyasevich’s negative Theorem, unsolving Hilbert’s 10th Problem, theorem in particular of (classically quantified Arithmetical Theory) \( \text{ZFC}^+ \), gives a contradiction within \( \text{ZFC}^+ \), hence self-inconsistency of \( \text{ZFC} \), and from that in particular \( \omega \)-inconsistency.

In a final section we show correctness and irrefutable termination of localised decision \( \nabla[D] \)—for each single diophantine
polynomial $D = D(\vec{x})$—within the constructive framework of p.r. finite-descent-theory $\pi R = \pi R + \text{Con}_\pi R$ out of op. cit.

2 Polynome coding and code evaluation

Diophantine polynomials $D = D(\vec{x})$ are $\text{TeX}/\text{ASCII}$ coded into

$$D = D(\vec{x}) = \bigoplus_{m \geq 1} \mathbb{Z}[\xi_1, \ldots, \xi_m] = \bigoplus_{m \geq 1} \mathbb{Z}[\xi_1][\xi_2] \cdots [\xi_m]$$

as nested coefficient lists $\mathbb{Z}^{(*)} \subset \mathbb{N}$.

[The symbols $\xi_i$ are the indeterminates.]

Example:

$$D = D(\xi_1, \xi_2) = (2 \cdot \xi_1^0 + 3 \cdot \xi_1^1 - 4 \cdot \xi_1^3) \cdot \xi_2^0$$
$$+ (0 \cdot \xi_1^0 + 3 \cdot \xi_1^1 - 7 \cdot \xi_1^2) \cdot \xi_2^1 + (1 - 4 \cdot \xi_1) \cdot \xi_2^2$$

is coded 1-1 as (nested) coefficient list

$$\cup D = \langle \langle 2; 3; 0; 4 \rangle; \langle 0; 3; -7 \rangle; \langle 0 \rangle; \langle 1; -4 \rangle \rangle :$$

1 $\rightarrow$ DIO $= \text{by def} \mathbb{Z}^{(*)} \subset \mathbb{N}$:

defined element, point of DIO

PR evaluation of DIO codes:

Evaluation $ev = ev(d, \vec{x}) : DIO \times \mathbb{Z}^*$ is PR defined

$$ev(d, \langle \vec{x}; x_{m+1} \rangle) = ev(d, \langle x_1; \ldots; x_m; x_{m+1} \rangle)$$
$$= \text{def} ev(\text{horner}(d, x_{m+1}), \langle \vec{x} \rangle) :$$

$$DIO \times \mathbb{Z}^* \supset \mathbb{Z}[\vec{\xi}, \xi_{m+1}] \times \mathbb{Z}^{m+1} \xrightarrow{\cong} (\mathbb{Z}[\vec{\xi}][\xi_{m+1}] \times (\mathbb{Z}^m \times \mathbb{Z})$$
$$\cong (\mathbb{Z}[\vec{\xi}][\xi_{m+1}] \times \mathbb{Z}) \times \mathbb{Z}^m \xrightarrow{\text{horner} \times \text{id}} \mathbb{Z}[\vec{\xi}] \times \mathbb{Z}^m \xrightarrow{ev} \mathbb{Z},$$
recursively by iterative application of Horner’s schema to the hitherto trailing argument, until all of the arguments (constants or variables) are substituted into their corresponding indeterminates $\xi_j$.

Result then is the integer $\text{ev}(d, \vec{x})$, constant or integer variable.

For the example above, $D = D(\xi_1, \xi_2)$, with argument string $\langle x_1; x_2 \rangle := \langle 23; 64 \rangle \in \mathbb{Z}^*$, we get

$\text{ev}(d, \langle x_1; x_2 \rangle) = \text{ev}(\langle (2; 3; 0; 4); (0; 3; -7); (0); (1; -4) \rangle, \langle 23; 64 \rangle)$

$= \text{horner}( (((((-4 \cdot 64 + 1) \cdot \xi_1 + 0)) \cdot \xi_1 + (-7 \cdot 64 + 3) \cdot 64) \cdot \xi_1$

$\quad + ((4 \cdot 64 + 0) \cdot 64 + 3) \cdot 64 + 2; 23))$

$= (((((-4 \cdot 64 + 1) \cdot 23 + 0)) \cdot 23 + (-7 \cdot 64 + 3) \cdot 64) \cdot 23$

$\quad + ((4 \cdot 64 + 0) \cdot 64 + 3) \cdot 64 + 2$

First step: apply Horner’s schema to coefficient list $d \in \text{DIO}$ and (trailing) Argument $x_2$ : indeterminate $\xi_1$ is coded by list nesting and is seen as a constant, as an element of intermediate ring $\mathbb{Z}[\xi_1]$ :

$\mathbb{Z}[\xi_1, \xi_2] = \text{by def } \mathbb{Z}[\xi_1][\xi_2] = \text{by def } (\mathbb{Z}[\xi_1])[\xi_2]$.  

Last—here second—step: evaluation of $\mathbb{Z}[\xi_1]$ polynomial in remaining indeterminate $\xi_1$ on remaining argument $x_1$, by a last application of Horner’s schema.

3 Arithmetical frame theories

We consider here as frame theories—for our decision algorithm—on one hand classically quantified arithmetical theories $T = \text{Q} + \text{AC}$ with (countable) axiom of choice, as in particular Zermelo-Fraenkel set theory $T = \text{ZFC} = \text{ZF} + \text{AC}$. Frame then is the strengthening

$T^+ = T + \text{Con}_T = \text{ZFC} + \text{Con}_{\text{ZFC}}$.
of $\mathcal{T}$ by its own consistency-formula

$$\text{Con}_\mathcal{T} = \neg (\exists k \in \mathbb{N}) \text{Prov}_\mathcal{T}(k, \lnot \text{false})$$

$$= (\forall k) \neg \text{Prov}_\mathcal{T}(k, \lnot \text{false}) \quad \text{(Gödel)}$$

see Smorynski 1977 and op. cit.

Strengthening by this consistency formula will provide for correctness of our decision process (Hilbert).

On the other hand we take as frame the Free-Variables (categorical) theory $\mathcal{T} = \mathcal{PR} = \mathcal{PRA}$ of Primitive Recursion with predicate abstraction into subsets

$$(\chi = \chi(a) : A \to 2) \mapsto \{A : \chi\} = \{a \in A : \chi(a)\}$$

out of op. cit., $\mathcal{T} = \mathcal{S}$ in Smorynski’s notation, as well as descent theory $\pi \mathcal{R} = \pi \mathcal{R}^+ = \pi \mathcal{R} + \text{Con}_{\pi \mathcal{R}} :$ that theory is self-consistent, $\pi \mathcal{R} \vdash \text{Con}_{\pi \mathcal{R}},$ main result of op. cit.

4 A $\mu$-recursive race for decision

We define an enumerative race—for $d \in \text{DIO}$ thought passive, fixed, and $k \in \mathbb{N}$ running—for satisfaction of

$$\varphi_0(d, k) = \lceil \text{ev}(d, ct_* k) = 0 \rceil$$

against

$$\varphi_1(d, k) = \text{Prov}_\mathcal{T}(k, \lnot (\vec{x})\text{ev}(d, \vec{x}) \neq 0 \ldots) : \text{DIO} \times \mathbb{N} \to 2 = \{0, 1\},$$

$ct_* = ct_* k : \mathbb{N} \overset{\sim}{\longrightarrow} \mathbb{Z}^*,$ Cantor-type count, $\vec{x} \in \mathbb{Z}^*$ free under code.

This race towards termination is defined as a—formally partial—$\mu$-recursive mapping as follows within the theory $\mathcal{T}$ of partial PR maps, i.e. of (partially defined) $\mu$-recursive maps, cf. again op. cit.:

$$t = t(d) = \mu\{ k \mid \varphi_0(d, k) \lor \varphi_1(d, k) \} : \text{DIO} \to \mathbb{N}. \quad (*)$$
Decision candidate then is

\[
\nabla d = \begin{cases} 
0 & \text{if } \varphi_0(d, t(d)) \\
1 & \text{if } \varphi_1(d, t(d))
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } \text{ev}(d, \text{ct}_*(t(d))) = 0 \\
1 & \text{if } \text{Prov}_T(t(d), \lnot \text{ev}(d, \vec{x}) \neq 0) \\
\end{cases}
\]

: \text{DIO} \overset{(\text{id}, t)}{\to} \text{DIO} \times \mathbb{N} \to 2.

**Question:** Is \( \nabla \) well-defined as a partial map? In which frame?

Well-definedness of the decision within \( T^+ = \text{ZFC}^+ = \text{ZFC} + \text{Con}_{\text{ZFC}} = T + \text{Con}_T \):

\( T^+ \vdash \varphi_0(d, k) \land \varphi_1(d, k') \)

(cases-overlap Assumption)

\( \implies \text{ev}(d, \text{ct}_*k) = 0 \)

\( \land \text{Prov}_T(k', \lnot (\vec{x}) \text{ev}(d, \vec{x}) \neq 0) \)

\( \implies \text{Prov}_T(j(k, k'), \lnot \text{false}) \)

\( \implies \neg \text{Con}_T \implies \text{false}, \)

\( j = j(k, k') : \mathbb{N}^2 \to \mathbb{N} \text{ suitable.} \)

**Consequence:**

\( T^+ \vdash \neg [\varphi_0(d, k) \land \varphi_1(d, k')] : \text{DIO} \times \mathbb{N}^2 \to 2, \)

\( \nabla = \nabla_T(d) : \text{DIO} \to \mathbb{N} \text{ is well-defined as a (formally partial) } \mu\text{-recursive map, within } T^+ = T + \text{Con}_T. \)
Well-definedness of decision within descent theory \( \pi R \):

We consider now descent theory \( \pi R \) out of op. cit. strengthening \( PR \) by axiom (\( \pi \)) of non-infinite endo driven descending complexity with complexity values in polynomial semiring \( \mathbb{N}[\omega] \), and its logical properties, in particular soundness giving \( \pi R \vdash \text{Con}_{\pi R} \).

Decision \( \nabla = \nabla_{\pi R}(d) : \text{DIO} \to 2 \) is in fact well-defined as a partial PR map, within theory \( \pi R \), since—in parallel to the above case \( T = \text{ZFC} \):

\[
\pi R \vdash \varphi_0(d, k) \land \varphi_1(d, k')
\]

(cases-overlap Assumption)

\[
\implies \text{ev}(d, \text{ct}_* k) = 0
\land
\text{Prov}_{\pi R}(k', \lnot(\overline{x}) \text{ev}(d, \overline{x}) \neq 0^\top )
\implies \text{Prov}_{\pi R}(j(k, k'), \lnot\text{false}^\top )
\implies \text{"\neg Con}_{\pi R} \text{"} \implies \text{false},
\]

\( j = j(k, k') : \mathbb{N}^2 \to \mathbb{N} \) suitable.

The latter since \( \pi R \vdash \text{Con}_{\pi R} \).
Well-definedness of DIO-decision within PR itself

Decision $\nabla = \nabla_{\text{PR}}(d) : \text{DIO} \rightarrow 2$ is well-defined as a partial PR map, within theory $\text{P Ra}$ of partial PR maps since

$$\text{P Ra} \vdash \varphi_0(d, k) \land \varphi_1^{\text{DIO}}(d, k') \quad (\text{cases-overlap Assumption})$$

$$\iff \text{ev}(d, c t \ast k) = 0$$

$$\land \text{Prov}^{\text{DIO}}(k', \lceil (\vec{x}) \text{ ev}(d, \vec{x}) \neq 0 \rfloor)$$

$$\implies \text{Prov}^{\text{DIO}}(j(k, k'), \lceil \text{false} \rfloor)$$

$$\implies \text{false},$$

$$j = j(k, k') : \mathbb{N}^2 \rightarrow \mathbb{N} \text{ suitable.}$$

The latter by diophantine soundness of $\mathbf{T} = \text{PR}$, see Smoryński 1977, Theorem 4.1.4.

5 Decision Correctness

Decision Correctness, result-0-case:

$$\mathbf{T} \vdash [\varphi_0(d, t(d)) \implies \text{ev}(d, c t \ast t(d)) = 0]$$

$$\subseteq \text{true}_{\text{DIO}}^{(\text{id}, t)} : \text{DIO} \times \mathbb{N} \rightarrow 2 :$$

If race-for-decision $\nabla$ terminates on DIO-code $d$, with result 0, then (evaluation of) $d$ has (at least) one zero, namely

$$c t \ast t(d) \in \mathbb{N}.$$
Correctness, result-1-case:

\[ T \vdash \varphi_1(d, k) \implies \text{Prov}_{\text{DIO}}(k, \lnot \text{ev}(d, \vec{x}) \neq 0) \]
\[ \implies \text{ev}(d, \vec{x}) \neq 0 : (\text{DIO} \times \mathbb{N}) \times \mathbb{Z}^* \to 2, \]
\[ (d \in \text{DIO}, \ k \in \mathbb{N}, \ \vec{x} \in \mathbb{Z}^* \text{ all free}), \]

or, with quantifier decoration:

\[ T \vdash (\forall d \in \text{DIO})(\forall k \in \mathbb{N})(\forall \vec{x} \in \mathbb{Z}^*) \]
\[ [\varphi_1^T(d, k) \implies \text{Prov}_{\text{DIO}}(k, \lnot \text{ev}(d, \vec{x}) \neq 0) \]
\[ \implies \text{ev}(d, \vec{x}) \neq 0]. \]

If race-for-decision \( \nabla \) terminates on DIO-code \( d \), with result 1, then (evaluation of) \( d \) has no zeroes.

This because of Diophantine Soundness of \( T \), see Smoryński 1977, Theorem 4.1.4 again.

Correctness in result-1-case, under termination condition:

Substitution of \( t(d) \) for \( k \) in the above gives

\[ T^+, \pi_R, PR \vdash [\varphi_1^{\text{DIO}}(d, t) \implies \text{ev}(d, \vec{x}) \neq 0] \subseteq \text{true}_{\text{DIO} \times \mathbb{Z}^*}, \]
\[ d \in \text{DIO}, \ \vec{x} \in \mathbb{Z}^* \text{ both free}. \]

Correctness of \( \nabla(d) \) where defined, in both defined cases: in case of reaching result 0, as well as in case of reaching result 1.

[For partial maps \( f, g : A \to B, f \subseteq g \) designates inclusion of the graphs of \( f \) and \( g \).]

6 Termination

We show first
Pointwise non-derivability of non-termination:

For no diophantine point $d_0 : 1 \to \text{DIO}$ $T$ derives non-termination of $t$ at $d_0$.

Proof:

Assumption

\[
T \vdash (\vec{x}) \text{ev}(d_0, \vec{x}) \neq 0 \quad (\bullet)
\]

\[
\land (k) \neg \text{Prov}_T(k, \vec{r}(\vec{x}) \text{ev}(d_0, \vec{x}) \neq 0) ^\top
\]

\[
T \vdash \text{Prov}_T(\text{num} j, \vec{r}(\vec{x}) \text{ev}(d_0, \vec{x}) \neq 0 ^\top)
\]

\[
\land (k) \neg \text{Prov}_T(k, \vec{r}(\vec{x}) D(\vec{x}) \neq \mathbb{Z} 0 ^\top)
\]

a contradiction: appropriate $j$ is available from ($\bullet$) via derivation-to-Proof-internalisation (gödelisation).

[For the time being we consider $T$ as frame, not (yet) $T^+ = T + \text{Con}_T$.]

For $T = Q$ quantified, with (countable) axiom of choice $\text{ACC}$, in particular $Q = \text{PA} + \text{ACC}$ Peano Arithmetic with choice, we define the undecided part of DIO as

\[
\Psi = \Psi^Q = \{ d \in \text{DIO} : \forall k \text{ ev}(d, \text{ct}_* k) \neq 0 \\
\land \forall k \neg \text{Prov}_Q(k, \vec{r}(\vec{x}) \text{ev}(d, \vec{x}) \neq 0) ^\top \}
\]

\[
\subset \text{DIO} = \mathbb{Z}^{(\ast)} \subset \mathbb{N}.
\]

With this definition we get

\[
Q \vdash \Psi \neq \emptyset \implies \text{choice}_\Psi : 1 \to \Psi \subset \mathbb{N} \text{ total}
\]

\[
(\text{choice available by } \text{ACC} : \text{non-empty sets have defined points})
\]

\[
\implies \mu \{ d : \text{t}(d) \text{ non-terminating} \} : 1 \to \Psi \text{ total}.
\]

This means: the assumption of (formal) existence of a $d \in \text{DIO}$ for which decision race $t : \text{DIO} \to \mathbb{N}$ does not terminate, leads to a (defined) point

\[
d_0 : 1 \to \text{DIO}
\]
for which $t$ derivably does not terminate.

But this is **excluded** by pointwise non-derivability above of non-termination, within frame $Q$ assumed consistent.

So we have shown

$$Q, PA + ACC \vdash \Psi = \emptyset,$$

i.e.

$$Q \vdash (\forall d \in \text{DIO})[\exists k \text{ev}(d, ct_k) = 0 \lor \exists k \text{Prov}_{\text{DIO}}(k, \Gamma(\vec{x}) \text{ev}(d, \vec{x}^*) \neq 0)],$$

whence

**Termination Theorem:** $Q, \text{ZFC}, PA + ACC$ derive race $t$ to terminate on all diophantine codes $d$, on all $d \in \text{DIO} = \mathbb{Z}^{(\ast)}$.

### 7 Correct termination of decision $\triangledown$

In particular ($Q^+ = Q + ACC$ stronger than $Q$):

$Q^+$ derives

overall termination of $\mu$-recursive

termination race $t = t^Q(d) : \text{DIO} \rightarrow \mathbb{N}$:

$$Q^+ \vdash [(\forall d \in \text{DIO}) \ t(d) \in \mathbb{N} \text{ defined}]$$

**Hence,** by Decision Correctness within $Q^+$:

$Q^+$ derives

overall correct termination of $\mu$-recursive decision

$\triangledown : \text{DIO} \rightarrow 2$, **main result** here:

$$\triangledown(d) = \begin{cases} 
0 & \text{if } \text{ev}(d, t(d)) = 0 \\
[ \implies d \text{ has a zero } \vec{z} \in \mathbb{Z}^* ] & \\
1 & \text{if } \text{Prov}_{\text{DIO}}(t, \Gamma(\forall \vec{x}) \text{ ev}(d, \vec{x}) \neq 0^\triangledown) \\
[ \implies d \text{ has no zero } ] & 
\end{cases} : \text{DIO} \rightarrow 2.
8 Comparison with Matiyasevich’s negative result

Main result above says in terms of the theory $\text{TM}$ of TURING machines, by the established part of CHURCH’s thesis:

For concrete diophantine polynomials $D = D(\vec{x}) : \mathbb{Z}^m \rightarrow \mathbb{Z}$:

For quantified arithmetical choice theories $Q + \text{ACC}$ like $\text{ZFC}$ and already $\text{PA} + \text{ACC}$,

$Q^+ = Q + \text{Con}_Q$ derives:

TURING machine $\text{TM}_{\nabla_Q}$ corresponding—CHURCH—to totally defined $\mu$-recursive decision map

$$\nabla_Q : \text{DIO} \rightarrow \{0, 1\},$$

when written coefficient list $\downarrow D \downarrow$ of a diophantine polynomial $D$ on its (initial) TAPE, eventually reaches HALT state, leaves result 0 (as its final TAPE) iff $D$ has a zero $\vec{z} : D(\vec{z}) = 0$, and result 1 iff $D$ is overall non-null:

$$(\forall \vec{x} \in \mathbb{Z}^n) [D(\vec{x}) \neq 0].$$

This contradicts Matiyasevich’s THEOREM unsolving Hilbert’s 10th problem, within theory $Q^+$ which strengthens his framework of Peano Arithmetic $\text{PA} + \text{ACC}$ with countable axiom of choice. Whence

Conclusion:

- $\text{ZFC}^+ = \text{ZFC} + \text{Con}_\text{ZFC}$ is contradictory, so
- $\text{ZFC} \vdash \neg \text{Con}_\text{ZFC} : \text{ZFC}$ is internally inconsistent,
- same for theory $\text{PA} + \text{ACC}$:
  
  Peano-Arithmetic with axiom of countable choice is internally inconsistent
• **Question:** is already Peano Arithmetic $\text{PA}$ by itself internally inconsistent? It would be if axiom $\text{ACC}$ of countable choice were derivable within $\text{PA}$ or independent from $\text{PA}$, as is axiom of choice $\text{AC}$ from set theory. This would mean that formal existential quantification is incompatible with free-variables Primitive Recursive Arithmetic $\text{PR}$.

**Discussion**

• After his talk at Humboldt University Berlin, I have mailed to Matiyasevich the question, if his unsolving of Hilbert’s 10th problem is really constructive: it depends heavily on formal existential quantification. No reply: may be he considers this question when present paper will be brought to his attention.

• I have submitted the 200? version of present work, claiming self-inconsistency $\text{PA} \vdash \neg \text{Con}_{\text{PA}}$, to the *Journal of Symbolic Logic*. The (anonymous) referee:

  ... this is certainly false. ... Robert 'Rob' Goldblatt ed.: under these circumstances etc.

**What is such editorial policy good for?**

9 Hilbert 10 constructively

In this section we show that the local version $\nabla[D] : 1 \rightarrow 2$ of the $\mu$-recursive decision algorithm $\nabla = \nabla_{\text{DIO}}(d) : \text{DIO} \rightarrow 2$ irreductably decides each (single) diophantine equation—correctly—when placed in p.r. non-infinite-descent theory $\pi \text{R} = \text{PR}+(\pi)$ of op.cit. in the References.

This will give a positive solution to Hilbert’s 10th problem in that constructive framework, at least when stated in its original form quoted in first section above.
Formally, this problem allows for solution by a separate decision algorithm ("process") for each diophantine polynomial. By localisation at a given polynomial, we extract such a decision-family from the forgoing sections, and formalise it within $\pi R$.

We index that family (externally) by the diophantine constants $\delta : 1 \rightarrow \text{DIO} \subset \mathbb{N}$, among which the diophantine polynomials

$$D = D(\vec{x}) = D(x_1, \ldots, x_m) : \mathbb{Z}^m \rightarrow \mathbb{Z}$$

are represented by their coefficient list codes $\downarrow D \uparrow : 1 \rightarrow \text{DIO}$.

**Definition:** For PR predicates $\varphi_0, \varphi_1 : A \times \mathbb{N} \rightarrow 2$ we define the *race winner predicate*

$$\mu_\lor[\varphi_0, \varphi_1] : A \rightarrow 2$$

between $\varphi_0$ and $\varphi_1$ slightly assymmetrically by

$$\mu_\lor[\varphi_0, \varphi_1] = \mu_\lor[\varphi_0, \varphi_1](a)$$

$$\overset{\text{def}}{=} (dc \circ (\varphi_0, \varphi_1)) \circ (A \times \mu[\varphi_0 \lor \varphi_1]) \circ \Delta_A :$$

$$A \rightarrow A \times A \rightarrow A \times \mathbb{N} \rightarrow 2 \times 2 \overset{dc}{\rightarrow} 2,$$

with

$dc = dc(u, v) : 2 \times 2 \rightarrow 2$ defined by

$$dc(u, v) = \overset{\text{def}}{=} \begin{cases} 0 & \text{if } u = 1, \\ 1 & \text{if } u = 0 \land v = 1, \\ \text{definably undefined if } u = v = 0. \end{cases}$$

This (partial) race winner predicate $\mu_\lor[\varphi_0, \varphi_1](a) : A \rightarrow 2$ is characterised—within $S = \text{PR}$ as well as in $S = \pi R$—by

$$S \vdash [\varphi_0(a, n) \land \bigwedge_{i < n} \neg \varphi_1(a, n) \implies \mu_\lor[\varphi_0, \varphi_1](a) = 0]$$

$$\land [\varphi_1(a, n) \land \bigwedge_{i \leq n} \neg \varphi_0(a, n) \implies \mu_\lor[\varphi_0, \varphi_1](a) = 1].$$

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We allow us to write for this intuitively—in classical terms of a (partial) case-distinction:

\[
\mu_{\lor} [\varphi_0, \varphi_1](a) = \begin{cases} 
0 & \text{if } \mu \varphi_0(a) < \infty \land \mu \varphi_0(a) \leq \mu \varphi_1(a), \\
1 & \text{if } \mu \varphi_1(a) < \infty \land \mu \varphi_1(a) < \mu \varphi_0(a).
\end{cases}
\]

Our decision family

\[\nabla[\delta] : 1 \rightarrow 2, \ \delta : 1 \rightarrow \text{DIO} \subset \mathbb{N}\]

now is defined in the present \(\mu\)-recursive frame as this type of race winning, of PR search for a zero (in the evaluation) of \(\delta\) against PR search for a (first) internal non-nullity proof for (the evaluation) of \(\delta\), namely by

\[
\nabla[\delta] = \text{def} \ \mu_{\lor} [\varphi_0[\delta], \varphi_1[\delta]] : 1 \rightarrow 2, \ \text{with}
\]

\[
\varphi_0[\delta](k) = \text{def} \ [\text{ev}(\delta, \text{ct}_* (k)) = 0] : \mathbb{N} \rightarrow 2,
\]

\[
\varphi_1[\delta](k) = \text{def} \ \text{Prov}_S(k, \Gamma(\vec{x}) \text{ev}(\delta, \vec{x}) \neq 0^\top).
\]

Here

\[\text{ev} = \text{ev}(d, x) : \mathbb{N} \times \mathbb{N} \supset \text{DIO} \times \mathbb{Z}^* \rightarrow \mathbb{Z}\]

is evaluation with the characteristic evaluation property

\[\text{ev} (\bot D, (x_1, \ldots, x_m)) = D(x_1, \ldots, x_m) : \mathbb{Z}^m \rightarrow \mathbb{Z},\]

realised by (iterated) Horner’s schema (each application reduces the number of remaining variables by 1), or by “brute force” evaluation of monomials.

9.1 Decision Correctness

Soundness Recall: Main result of op. cit. in the References is (logical) soundness of theory \(\pi \mathbf{R}\):

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• For a (p. r.) predicate $\chi = \chi(a) : A \rightarrow 2$ we have

$$\pi R \vdash \text{Prov}_{\pi R}(k, \Gamma \chi^\top) \implies \chi(a) : N \times A \rightarrow 2,$$

$a \in A$ free, meaning here for all $a \in A$, and $k \in N$ free, meaning here exists $k \in N$. This entails

• **PR soundness of $\pi R$** : For a p. r. predicate $\chi = \chi(a) : A \rightarrow 2$,

$$\pi R \vdash \text{Prov}_{\pi R}(k, \Gamma \chi^\top) \implies \chi(a) : N \times A \rightarrow 2,$$

as well as in particular

• **Diophantine soundness of $\pi R$** : for a diophantine polynomial $D = D(\vec{x}) : Z^* \rightarrow 2$

$$\pi R \vdash \text{Prov}_{\pi R}(k, \Gamma (\vec{x})D(\vec{x}) \neq 0^\top) \implies D(\vec{x}) \neq 0,$$

$k \in N$, $\vec{x} \in Z^*$ free.

• Already $\text{PR}^+ = \text{PR} + \text{Con}_{\text{PR}}$ is diophantine sound. This needs an extra Proof.

We consider here frame $S = \pi R$,

$$\pi R^+ = \pi R + \text{Con}_{\pi R} = \pi R,$$

the latter by op. cit. equivalent to soundness of theory $\pi R$.

Namely from PR Soundness we get the

**Local Correctness-Lemma** for $\nabla[\delta]$ in $\pi R$ : The partial $\text{PR}$-map $\nabla[\delta] : 1 \rightarrow 2$ has the following correctness properties:

$\pi R \vdash$

• $\delta$ does not fall in both of the two defined-cases stated for $\nabla[\delta]$. 


• \( \nabla[\delta] = 0 \implies \text{ev}(\delta, \text{ct}_* \circ \mu \varphi_0[\delta]) = 0 \) : \( \delta \) is implied to have available a zero in its evaluation,

• \( \nabla[\delta] = 1 \implies \text{ev}(\delta, \vec{x}) \notin \mathbb{Z}, \vec{x} \) free in \( \mathbb{Z}^* \): \( \delta \) is implied to be evaluated globally non-null, in particular:

• By diophantine evaluation for \( D = D(x_1, \ldots, x_m) : \mathbb{Z}^* \to \mathbb{Z} \) diophantine:
  - \( \nabla[D] := \nabla[\downarrow D \downarrow] = 0 \implies D(\text{ct}_* (\mu \varphi_0[\downarrow D \downarrow])) = 0 : D \) is implied to have a zero, as well as
  - \( \nabla[D] = 1 \implies [D(\vec{x}) \neq 0], \) here again \( \vec{x} \) free over \( \mathbb{Z}^* : D \) is implied to be globally non-null q.e.d.

### 9.2 Decision Termination

The final question to treat for this—canonical—family

\[ \nabla = \nabla_{\text{DIO}}[\delta] : 1 \to 2, \delta : 1 \to \text{DIO} \subset \mathbb{N} \]

of local—\( \mu \)-recursive—decision algorithms, is termination, for each \( \delta \), in particular for \( \delta = \downarrow D \downarrow, D = D(\vec{x}) \) diophantine.

**Assume** \( \nabla[d_0] \) *not* to terminate for a particular constant \( d_0 : 1 \to \text{DIO} \), in particular \( d_0 \) of form \( D_0 = D_0(\vec{x}) \).

Since we argue here purely syntactically—within the theory \( \hat{S} \supset S = \text{PR} + \text{(abstr)} \) of partial p.r. maps—no modelling in mind except some primitive recursive Metamathematics (these in turn g"odelised within \( S \))—we discuss the stronger assumption

\( \nabla[d_0] \) \( T \)-derivably does *not* terminate for a given diophantine constant \( d_0 : 1 \to \text{DIO}, T \) an extension of \( S \).

This **assumption** reads:

\[ T \vdash (k) \psi[d_0](k) : \]
here $k$ is free over $\mathbb{N}$, and the PR predicate $\psi[d_0](k) : \mathbb{N} \to 2$ is defined by

$$
\psi[d_0](k) = \psi_0[d_0](k) \land \psi_1[d_0](k) \quad \text{with} \\
\psi_0[d_0](k) = [\text{ev}(d_0, c_t(k)) \neq 0], \text{and} \\
\psi_1[d_0](k) = \neg \text{Prov}_T(k, \neg \text{ev}(d_0, \vec{x}) \neq 0^\dagger).
$$

So the assumption (“of the contrary”) reads:

$$
T \vdash [\text{ev}(d_0, c_t(k)) \neq 0] \\
\land \neg \text{Prov}_T(k, \neg (\vec{x})\text{ev}(d_0, \vec{x}) \neq 0^\dagger).
$$

Here $k \in \mathbb{N}$ is the only free variable in the accessible level, $\vec{x}$ is free over $\mathbb{Z}^*$, but encapsulated within gödelisation, not visible on the object language level.

The derivably-non-termination assumption

$$
T \vdash \psi[d_0](k), \text{ } k \text{ free,}
$$

would entail in particular (first conjunct $\psi_0[d_0]$):

$$
T \vdash \text{ev}(d_0, c_t(k)) \neq 0 : \mathbb{N} \to 2.
$$

Internalising (formalising) this metamathematical statement, we (would) get by Proof-Internalisation—cf. Smoryński 1977—a constant $p_0 : 1 \to \text{Proof}_T \subset \mathbb{N}$ guilty for this last statement:

$$
T \vdash \text{Prov}_T(p_0, \neg \text{ev}(d_0, \vec{x}) \neq 0^\dagger);
$$

this would give, by definition of $\nabla[d_0]$:

$$
T \vdash \nabla[d_0] = 1,
$$

a contradiction to our assumption that $d_0$ be derivably not decided by $\nabla_{\text{DIO}}$, i.e. to $T \vdash \psi[d_0]$.

Conclusion:
• $\pi R = \pi R + \text{Con}_{\pi R}$ derives the alleged decision algorithm (family) $\nabla = \nabla_{\text{DIO}}[D] : 1 \rightarrow 2$ to be correct for each diophantine polynomial (if defined).

• no diophantine polynomial $D = D(\vec{x})$ can come with a $T$-proof (i. p. a $\pi R$-proof) showing $\nabla [D]$ to be undefined, not to terminate, in other words:

• correct termination of the $\mu$-recursive decision family $\nabla = \nabla_{\text{DIO}}[D]$ at each diophantine polynomial is $\pi R$-irrefutable, in the sense that otherwise—refutation—

$$\pi R \vdash \text{Prov}_{\pi R}(q, \text{"false"})$$,

$q : 1 \rightarrow \mathbb{N}$ a suitable PR point,

inconsistency of (self-consistent) theory $\pi R$ would be the consequence.

**Outlook**

Irrefutable correct termination of uniform decision algorithm

$$\nabla_{\text{DIO}} = \nabla_{\text{DIO}}(d) : \text{DIO} \rightarrow 2, \ d \in \text{DIO free}$$

is treated within the general framework of

*Arithmetical Decision* to come.

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