Two-dimensional QCD at infinite $N_c$ and finite $T$

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ABSTRACT

We analyze two-dimensional large $N_c$ QCD at finite temperature and show explicitly that the free energy has the correct $N_c$ dependence.

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The large $N_c$ expansion provides a qualitative explanation for many strong interaction phenomena.\[1\] It may well be that the problem of confinement is nothing more (or rather nothing less) than summing all planar graphs\[2\]. Also, the early speculations that these graphs correspond to an effective string theory has recently received new attention (see e.g. \[3\] and references therein). The large $N_c$ expansion together with the assumption of confinement implies that the number of degrees of freedom (d.o.f.) at low energy is vastly smaller than expected from perturbation theory. The low-ly ing spectrum consist of glueballs and $q\bar{q}$ mesons with degeneracy factors independent of $N_c$, while the perturbative number of d.o.f. are $N_c^2$ for gluons and $N_c N_f$ for fermions. As discussed by Thorn\[4\] and Pisarski,\[5\] this has important consequences for the finite temperature deconfinement phase transition. The basic observation is that the free energy in the low-temperature phase is $O(1)$, while in the high temperature (deconfined) phase it is $O(N_c)$. If we believe that solving QCD in the large $N_c$ limit corresponds to summing the planar graphs, we are faced with the following problem. Why is it that the graphs of fig. 1 do not give a contribution to the free energy to order $N_c^2$? In this paper we shall address a similar but much simpler question in the context of the 't Hooft model, i.e. large $N_c$ two-dimensional QCD. Here there are no dynamical d.o.f. associated with the gluon field so the diagrammatic expansion for the partition function always includes at least one quark line. The free energy is of the form ($\beta = 1/T$),

$$- T \ln Z = \beta F = N_c F_1(\beta, g^2) + F_0(\beta, g^2) + \frac{1}{N_c} F_{-1}(\beta, g^2) + \ldots \; ,$$

as illustrated in fig.2. Since the theory confines, and the spectrum is given by an infinite number of meson states, we expect the leading order term in $F$ to be of order $N_c^0$, that is a free meson gas. The relevant diagrams are shown on the second line of fig. 2, where it is also indicated how they can be resummed into meson graphs. Since the mesons interact weakly with a strength of order $1/\sqrt{N_c}$, we expect corrections to $F$ of order $1/N_c$ corresponding to the diagrams on line 3 of fig. 2 and so on.

Using the notation of ref. \[6\] the full mass-less planar quark propagator in light-cone gauge, $A_\perp = 0$, is

$$S(k) = \frac{-ik_-}{2k_+ k_- - k_- \Sigma(k_-) - i\epsilon} \; ,$$

where the full planar self energy part is

$$k_- \Sigma(k_-) = \frac{g^2}{\pi} \left( 1 - \frac{|k_-|}{\lambda} \right) \; .$$

Here $\lambda$ is an infrared cutoff that is to be taken to zero at the end of the calculation. As pointed out in \[7\], in the limit $\lambda \to 0$, $\lambda$ is nothing but a gauge parameter, so the dispersion relation for the fermion is explicitly gauge dependent. It is important to realize that even though the pole in the propagator moves to infinity in the $\lambda \to 0$ limit, this does not constitute a proof of confinement. Rather, since $\lambda$ is a gauge parameter, one must show that any calculated observable is independent of $\lambda$ in the $\lambda \to 0$ limit. We shall now prove that for any finite temperature and coupling constant, $F_1(\beta, g^2)$ in (1),
corresponding to the diagrams in the first line in fig.2, is identically zero in the limit \( \lambda \to 0 \).

The main idea is to relate the leading piece, \( \mathcal{F}_1 = F_1(\beta, g^2)/L \) (\( L \) is the volume), in the free energy density to the propagator via the relation,

\[
g^2 \frac{d^2}{dg^2} \mathcal{F}_1 = \frac{1}{2} \text{Tr}(S \Sigma)
\]  \( \text{(4)} \)

which we now prove using diagrammatic techniques.

Consider fig. 3a which shows all diagrams contributing to \( \mathcal{F}_1 \), and where the shadowed blob represent all the ways the \( n \) gluon lines can connect to each other to form a planar graph. (Note that there are no planar vertex corrections.) We have explicitly shown the power of \( g^2 \) and there is a crucial statistical weight factor \( 1/2n \) due to the \( 2n \)-fold symmetry of the diagram. In fig. 3b we show a similar set of diagrams contributing to the difference between the full and the free propagator - note the absence of the statistical weight factor. From the figure we immediately get,

\[
g^2 \frac{d^2}{dg^2} \mathcal{F}_1 = 1/2 \text{Tr} \left[ (S - S_0)S_0^{-1} \right], \tag{5}\]

and substituting the Dyson equation \( S = S_0 + S \Sigma S_0 \) we get \( \text{(4)} \). This relation can also be derived by noticing that in the light cone gauge the gluonic action is quadratic in \( A_+ \), and performing this Gaussian integral we get the following induced four Fermi interaction

\[
\mathcal{L} = \frac{g^2}{2} \int d^2x \, d^2y \, D(x - y) \overline{\psi} \lambda^a \gamma^\alpha \psi(x) \overline{\psi} \lambda^\alpha \gamma^\alpha \psi(y). \tag{6}\]

Now, the free energy can be expressed as,

\[
\frac{d^2}{dg^2} F = \frac{1}{2} \int d^2x \, d^2y \, D(x - y) \overline{\psi} \lambda^a \gamma^\alpha \psi(x) \overline{\psi} \lambda^\alpha \gamma^\alpha \psi(y), \tag{7}\]

where \( D(x - y) \) is the gluon propagator, and the expectation value is with respect to the induced (non-local) four Fermi interaction \( \text{(3)} \). In the large \( N_c \) limit \( \text{(7)} \) reduces to \( \text{(4)} \) using the Dyson equation for \( \Sigma \) given in \( \text{(6)} \).

Before substituting \( \text{(2)} \) and \( \text{(3)} \) in \( \text{(4)} \) we notice that the pole in \( \text{(2)} \) is given by,

\[
E_p = -k + \frac{g^2}{\sqrt{2} \pi \lambda} \text{sgn}(E - k), \tag{8}\]

where we have reintroduced the usual energy and momentum via \( \sqrt{2} k_\pm = E \pm k \). This dispersion relation, which is illustrated in fig. 4, will change at finite temperature, but in the limit \( \lambda \to 0 \), where the mass gap diverges, any finite temperature will not influence the spectrum. Thus, the only place where the temperature enters the calculation, is in the loop integration implied by the trace in the RHS of eq. \( \text{(4)} \). We get,

\[
\frac{d^2}{dg^2} (\mathcal{F}_1 - \mathcal{F}_{1_{\text{vac}}}) = \frac{1}{2\pi} \int \frac{dk}{2\pi} \int_{-i\infty}^{i\infty} \frac{dE}{2\pi i} n_\beta(E) E \nu \left( \frac{1 - |k_-|/\lambda}{2k_+k_- - k_- \Sigma(k_-) - i\epsilon} \right), \tag{9}\]
where \( n_\beta(E) \) is the Fermi distribution function and \( E_{\rm vf}(E) = f(E) + f(-E) \). The temperature independent vacuum part, \( \mathcal{F}^{\text{vac}}_1 \), is of no concern here. Doing the \( E \) integration by contours, we pick up the pole at \( E_p \) given by (8) and because of the mass gap, the integral (4) is suppressed by a factor \( n_\beta(E_p) \sim e^{-\beta/\lambda} \). Thus for any fixed temperature, the integrand goes to zero in the limit \( \lambda \to 0 \) and thus the whole integral equals zero. We have now proved that there is no \( g \)-dependence in the \( O(N_c) \) contribution to the free energy. We cannot logically exclude the possibility of a \( g^2 \) independent piece \( cT^2 \), where \( c \) is an \( g \) (and \( \lambda \)) independent constant. Such a behaviour is, however, expected for a gas of free, mass-less particles, and we find it extremely unlikely that such a contribution should be present in the \( g \to \infty \) limit. We thus conclude that the \( O(N_c) \) contribution to the free energy is identically zero for all \( g \).

Our argument breaks down at infinite temperature (or equivalently at vanishing \( g \), since what matters is the dimensionless quantity \( \beta g \)) where \( T \sim 1/\lambda \sim \infty \). In this regime, the thermal effects can overcome the string tension \( \sigma \sim g^2 \) allowing for a possible phase change. This can also be made plausible by recalling that the number of mesons in two dimensions grow quadratically with the energy \( (n \sim T^2/\sigma) \). As a result, the free energy at high temperature is of order \( nt \sim T^3/\sigma \). A phase change is expected when the latter is of order \( N_f N_c T_c \), \( i.e. \ T_c \sim g\sqrt{N_c} \to \infty \). Finally, if we think of \( \lambda \) as roughly the size of the system, we conclude that the large \( N_c \) limit does not commute with the thermodynamical limit (\( i.e. \) taking the box size to infinity at fixed \( T \)).

We believe that this result is illustrative to what happens in four dimensions, where we expect the low temperature limit of QCD to be dominated by pions. The strong infrared divergences in the quark and gluon sector at low energy will cause the order \( N_c^2 \) (gluonic) and \( N_c N_f \) (fermionic) terms in the free energy to vanish identically, just as in 2-dimensional QCD. In four dimensions, however, we expect a phase transition to take place at \( T_c \sim \Lambda \). As we approach this temperature from below, the mesonic interactions, although suppressed by \( 1/N_c \), will be dominant since the number of particles is likely to grow exponentially in a Hagedorn manner. Above the critical temperature, the quark - gluon description is believed to be more economical even though high temperature perturbative QCD is still plagued by infrared problems.

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1 Of course, at finite \( N_c \) there cannot be any finite \( T \) phase transition. What happens at infinite \( N_c \) is a tricky question and this argument is only suggestive.
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Figure Captions

Fig. 1: Leading order contribution to the free energy in QCD.

Fig. 2: Contribution to the free energy in two dimensional QCD.

Fig. 3: (a) Diagrams contribution to the leading part of the free energy where $g^2$ and statistical factors are shown explicitly.
(b) Corresponding graphs for the difference between the full planar propagator and the free one.

Fig. 4: The quark dispersion relation in light-cone gauge.