HAEFLIGER COHOMOLOGY OF RIEMANNIAN FOLIATIONS

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Abstract. In the first part, we compute the Haefliger cohomology of uniform complete Riemannian foliations by showing the duality between the Haefliger cohomology and the basic cohomology. In the second part, we will give a characterization of strongly tenseness condition of foliated manifolds in terms of the Haefliger cohomology. By combining the results of both parts, we prove that any uniform complete Riemannian foliation on any possibly noncompact manifold is strongly tense.

1. Introduction

1.1. Background: Theorems of Haefliger, Masa and Domínguez. A foliated manifold $(M, F)$ is called taut if $M$ admits a metric $g$ such that every leaf of $F$ is a minimal submanifold of $(M, g)$. Sullivan’s characterization [Su79] shows that tautness is a dynamical or homological property of compact foliated manifolds. Later Haefliger [Ha80] introduced Haefliger cohomology of compact foliated manifolds to give a cohomological characterization of tautness. As a corollary, Haefliger proved that tautness of compact foliated manifolds is a transverse property, namely, it is determined by the equivalence class of the holonomy pseudogroups.

Riemannian foliations are foliations whose holonomy pseudogroup admits an invariant Riemannian metric. Typical examples of Riemannian foliations are Lie foliations, namely, foliations whose holonomy pseudogroups are generated by the right $\Gamma$-action on a Lie group $G$, where $\Gamma$ is a finitely generated subgroup of $G$. Remarkably, tautness of Riemannian foliations on compact manifolds is topologically characterized: Carrièere conjectured [Ca84] that a transversely orientable Riemannian foliation on a compact manifold is taut if and only if the top degree component of the basic cohomology is nontrivial. After many partial results [KT83a, Ca84, Gh84, Ha85, MS85, EKAH84, AL90], Masa [Ma92] finally proved Carrièere’s conjecture for orientable Riemannian foliations. Álvarez López [AL92] removed the hypothesis of orientability.

Domínguez [Do98] proved that any Riemannian foliation on a compact manifold is tense based on Masa’s technique and the Álvarez class defined in [AL92]. Here, a foliated manifold $(M, F)$ is called tense if $M$ admits a metric such that the mean curvature form of $F$ is basic. Since Domínguez’s theorem implies Masa’s theorem, Domínguez’s theorem can be regarded as a generalization of Masa’s theorem. Theorems of Masa and Domínguez were proved essentially by computing Haefliger cohomology of Riemannian foliations on compact manifolds.

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1.2. Motivation. As asked in the realization problem of Haefliger [Ha84] (see also [Me97]), for a noncompact Lie group $G$ and a finitely generated subgroup $\Gamma$, it is a quite nontrivial problem to determine if the pseudogroup generated by the right $\Gamma$-action on $G$ is realized as the holonomy pseudogroup of a foliation on a compact manifold. Thus, in general, we do not know if characterizations of Haefliger and Masa are valid for such foliations. Our first motivation is to extend results of Haefliger, Masa and Domínguez to determine tautness of Lie foliations with given holonomy pseudogroup. In other words, our motivation is to remove mysterious hypothesis on compactness of manifolds from these preceding results, and to give characterizations purely in terms of holonomy pseudogroups. Theorems of Masa and Domínguez were generalized to Riemannian foliations on noncompact manifolds which are suitably embedded into Riemannian foliations on compact manifolds by Royo Prieto-Saralegi-Wolak [RPSAW08] by using Domínguez’s theorem. Masa [Ma09] generalized his characterization of tautness to Riemannian foliations such that the closures of the leaves define a compact foliation. In [NRPT12], tautness of transversally complete Riemannian foliation of dimension one was proved. But, unfortunately, the closures of leaves of foliations treated in all of these preceding works are compact, and hence the results are not sufficient to determine tautness of Lie foliations on possibly noncompact manifolds.

Our second motivation comes from the fact that tautness of foliations on noncompact manifolds has certain different aspects from the compact case (see Section 1.6).

1.3. Overview of the results. We call a foliated manifold uniform if the leaf space is compact. In terms of the holonomy pseudogroup, this condition is equivalent to the existence of relatively compact total transversal in a total transversal (see Lemma [1.3]), and it is weaker than compactness of $M$.

First we compute Haefliger cohomology of uniform complete Riemannian foliations (Theorems [1.1, 1.2] and Corollary [1.3]). Then we generalize Haefliger’s characterization of tautness of compact foliated manifold to characterization of strongly tenseness of possibly noncompact foliated manifolds (Theorem [1.6]). Here, strongly tenseness is a condition stronger than tenseness (see Section 1.6 for the definition of strongly tenseness of foliated manifolds and the reason why we consider strongly tenseness instead of tenseness). By using Corollary [1.3] and Theorem [1.6] we generalize Domínguez’s tenseness theorem to uniform complete Riemannian foliations (Theorem [1.8]).

One may conjecture that any complete Riemannian foliation is strongly tense as stated in [NRPT12]. Since general Riemannian foliation can be regarded as a family of Lie foliations with dense leaves in a sense by Molino’s structure theory, techniques in this article may be useful to prove it.

1.4. Main theorem: Haefliger cohomology of uniform complete Riemannian foliations. Let $M$ be a manifold with a foliation $F$. Let $(\mathcal{E}, \nabla)$ be a flat vector bundle over $M$. The Haefliger cohomology $H^\bullet_c(\text{Tr}F;\mathcal{E})$ of $(M,F)$ with values in $(\mathcal{E}, \nabla)$ (see Section 2.1), or its cochain complex $\Omega^\bullet_c(\text{Tr}F;\mathcal{E})$, is of dynamical nature in general as mentioned in [Ha80], and it is often difficult to compute them. For example, $\Omega^p_c(\text{Tr}F)$ is isomorphic to the $p$-th leafwise cohomology of $F$, where $p = \dim F$, by [Ha80] Corollary in Section 3.3. On the other hand, it is less difficult to compute the reduced cohomology $\tilde{H}^\bullet_c(\text{Tr}F;\mathcal{E})$ in the sense of Hector [He88], namely, the cohomology of the quotient of $\Omega^\bullet_c(\text{Tr}F;\mathcal{E})$ by the closure of zero with respect to the natural topology. There is a natural map $\tau : H^p_c(\text{Tr}F;\mathcal{E}) \to \tilde{H}^p_c(\text{Tr}F;\mathcal{E})$. For a closed manifold $M$ with a Riemannian foliation $F$, the map $\tau$ is an isomorphism due to Domínguez [Do98 Theorem 3.15], which is a generalization of Masa’s result [Ma92 Section 1] in the case where $(\mathcal{E}, \nabla)$
is trivial. Note that Hector [He88] Theorem 1.2] first proved that $\tau$ is an isomorphism in the case where the structural Lie algebra of $(M, F)$ is compact or nilpotent and $(E, \nabla)$ is trivial.

Our first result is a generalization of the isomorphism. Recall that a foliated manifold is called uniform if the leaf space is compact (see Lemma 3.5). In particular, Riemannian foliations with dense leaves are uniform.

**Theorem 1.1.** For a manifold $M$ with a uniform complete Riemannian foliation $F$ and a flat vector bundle $(E, \nabla)$, the natural map $\tau : H^\bullet_p(\text{Tr} F; E) \to \hat{H}^\bullet_p(\text{Tr} F; E)$ is an isomorphism.

Here, recall that a foliation is called complete if its holonomy pseudogroup is complete (see Definition 3.2). Our proof is based on a version of Sarkaria’s smoothing operator [Sa78] on transversals, which is a modification of the main step of [Ma92] and [Do98]. We will illustrate Theorem 1.1 with some examples of Lie foliations in Section 1.8.

1.5. **Duality of Haefliger cohomology of uniform complete Riemannian foliations.** We will apply Theorem 1.1 to show certain duality of Haefliger cohomology of uniform complete Riemannian foliations to extend duality, known in the case where $M$ is compact. For this purpose, we extend the well known duality between $\hat{H}^\bullet_p(\text{Tr} F; E)$ and the basic cohomology to complete Riemannian foliations with flat vector bundles.

**Theorem 1.2.** For a manifold $M$ with a complete Riemannian foliation $F$ and a flat vector bundle $(E, \nabla)$, we have $\hat{H}^\bullet_p(\text{Tr} F; E) \cong H^\bullet(\text{Tr} F; E)$, where $O_v$ is the orientation bundle of $TM/TF$ and $H^\bullet(M, F; E^* \otimes O_v)$ is the basic cohomology of $(M, F)$ with values in $E^*$. 

In the case where $E$ is trivial, it is already known [ALM08] p. 598]. In the case where $M$ is compact, more general duality for the whole spectral sequence associated to Riemannian foliations was proved by Álvarez López [AL89a]. We will illustrate Theorem 1.2 with some examples of Lie foliations in Section 1.8.

Combining Theorems 1.1 and 1.2 with the twisted Poincaré duality of the basic cohomology due to Sergiescu [Se85, Théorème I], we can compute $H^\bullet_p(\text{Tr} F; E)$ as follows. Let $P$ be the Sergiescu’s orientation sheaf of $(M, F)$, which was introduced in [Se85, Section 1].

**Corollary 1.3.** Let $M$ be a manifold with a codimension $n$ uniform complete Riemannian foliation $F$ and a flat vector bundle $(E, \nabla)$.

(i) We have $\text{dim } H^\bullet_p(\text{Tr} F; E) < \infty$.

(ii) If $M$ is connected and rank $E = 1$, then $H^0_p(\text{Tr} F; E)$ is isomorphic to $\mathbb{R}$ or $0$.

(iii) If $M$ is connected, then $H^0_p(\text{Tr} F; P) \cong \mathbb{R}$.

(iv) We have $H^\bullet_p(\text{Tr} F; E) \cong H^\bullet_p(\text{Tr} F; E^* \otimes P^*)$.

Corollary 1.3 was first proved in the case where $M$ is compact, $E$ is trivial and the structure Lie algebra of $F$ is compact or nilpotent by Hector [He88]. Masa [Ma92] Duality theorem in p. 26] proved it in the case where $M$ is compact and $E$ and $P$ are trivial. Domínguez [Do98, Theorem 5.2] proved it in the case where $M$ is compact and $E$ and $P$ may be nontrivial.

1.6. **Characterizations of strongly tenseness.** We briefly review strongly tenseness of foliated manifolds introduced in [NRPT12]. As seen in the work of Kamber-Tondeur [KT83a] and Domínguez [Do98], tenseness of compact manifolds with Riemannian foliations is a cohomological property which can be regarded as a twisted
version of tautness. The crucial fact [KT83b, Eq. 4.4] in their work is the following: On a compact manifold with a Riemannian foliation, the mean curvature form of any tense metric is closed. For more general foliated manifolds, tenseness is not a cohomological property. For example, it is known that there exists a Riemannian foliation on a noncompact manifold with tense metric whose mean curvature form is not closed due to Cairns-Escobales [CE97, Example 2.4]. For such foliations, it is difficult to relate tenseness with cohomology like in the work of Haefliger [Ha80]. Motivated by this fact, the following notion was introduced in [NRP12]: A foliated manifold \((M, \mathcal{F})\) is called strongly tense if \(M\) admits a Riemannian metric such that the mean curvature form of \(\mathcal{F}\) is basic and closed.

In the second part of this article, first we will show that strongly tenseness of compact foliated manifolds admits a Rummel-Sullivan type characterization (Proposition 7.5), which shows that strongly tenseness of foliated manifolds is a version of tautness twisted by a flat line bundle. Then we introduce the notion of strongly tenseness with a prescribed flat line bundle \((\mathcal{E}, \nabla)\) (Definition 7.7), and show its Haefliger type characterization. To state the result, we introduce the following terminologies.

**Definition 1.4.** Let \((\mathcal{E}, \nabla)\) be a flat line bundle over a manifold \(M\) with holonomy homomorphism \(\text{hol}(\nabla) : \pi_1 M \to \text{Aut}(\mathbb{R})\). Let \(r_1(\mathcal{E})\) denote the composite of
\[
\pi_1 M \xrightarrow{\text{hol}(\nabla)} \text{Aut}(\mathbb{R}) \cong \mathbb{R}^\times \xrightarrow{\log | \cdot |} \mathbb{R}.
\]
We will say that \((\mathcal{E}, \nabla)\) is basic if \(r_1(\mathcal{E})\) belongs to the basic cohomology \(H^1(M, \mathbb{R})\) when we regard \(r_1(\mathcal{E}) \in H^1(M; \mathbb{R}) \cong \text{Hom}(\pi_1 M, \mathbb{R})\) and \(H^1(M, \mathbb{R})\) as a subspace of \(H^1(M; \mathbb{R})\) by the canonical injection.

Note that the last terminology is not standard. By definition, if \((M, \mathcal{F})\) is strongly tense with \((\mathcal{E}, \nabla)\), then \((\mathcal{E}, \nabla)\) is basic.

**Definition 1.5.** Let \(\mathcal{E}\) be a trivial line bundle over a manifold \(T\). A section \(\xi\) of \(\mathcal{E}\) is called nonnegative if \(\xi\) is identified with a nonnegative function under a trivialization of \(\mathcal{E}\). For such a section \(\xi\), let
\[
\text{supp}_+ \xi := \{ x \in T \mid \xi(x) \neq 0 \}.
\]

Now the Haefliger type characterization of strongly tenseness is as follows.

**Theorem 1.6.** For a closed manifold \(M\) with an oriented foliation \(\mathcal{F}\) and a topologically trivial basic flat line bundle \((\mathcal{E}, \nabla)\) over \(M\), the following are equivalent:

(i) \((M, \mathcal{F})\) is strongly tense with \((\mathcal{E}, \nabla)\).

(ii) There exist a total transversal \(T\) of \((M, \mathcal{F})\) and a nonnegative section \(\xi \in \Omega^0_+(T; \mathcal{E})\) so that \(\text{supp}_+ \xi\) intersects every leaf of \(\mathcal{F}\) and \(d_\mathcal{F} \xi = 0\) in \(\Omega^1_{c}(\text{Tr} \mathcal{F}; \mathcal{E})\).

(iii) For any total transversal \(T\) of \((M, \mathcal{F})\), there exists a nonnegative section \(\xi \in \Omega^0_+(T; \mathcal{E})\) such that \(\text{supp}_+ \xi\) intersects every leaf of \(\mathcal{F}\) and \(d_\mathcal{F} \xi = 0\) in \(\Omega^1_{c}(\text{Tr} \mathcal{F}; \mathcal{E})\).

Clearly the conditions (iii) in the last theorem is invariant under equivalences of pseudogroups as Haefliger’s characterization of tautness [Ha80, Theorem 4.1]. Thus, as Haefliger proved that tautness is a transverse property of compact foliated manifolds as a consequence of his characterization [Ha80, Corollary 1 of Theorem 4.1], we get the following consequence of Theorem 1.6.

**Corollary 1.7.** Strongly tenseness of compact foliated manifolds is a transverse property; namely, it is determined by the equivalence class of the holonomy pseudogroup.
Moreover, we will extend the Haefliger type characterization of tautness and strongly tenseness to noncompact foliated manifolds (Theorem 9.9) by introducing a modified version of Haefliger cohomology.

1.7. Tenseness theorems. By combining consequences of Theorems 1.1 with the characterization of strongly tenseness in terms of Haefliger cohomology (Theorem 9.9), we will prove the following result.

**Theorem 1.8.** Let $M$ be a connected manifold with a uniform complete Riemannian foliation $\mathcal{F}$. If there exists a basic flat line bundle $(\mathcal{E}, \nabla)$ over $M$ such that $\hat{H}^0(\text{Tr} \mathcal{F}; \mathcal{E}) \cong \mathbb{R}$, then $(M, \mathcal{F})$ is strongly tense with $(\mathcal{E}, \nabla)$. In particular, $(M, \mathcal{F})$ is strongly tense.

See Definition 7.7 for the definition of strongly tenseness with $\mathcal{E}$. We will see that the nontriviality of $\hat{H}^0(\text{Tr} \mathcal{F}; \mathcal{E})$ is equivalent to $\hat{H}^0(\text{Tr} \mathcal{F}; \mathcal{E}) \cong \mathbb{R}$ (Theorems 6.2 and 6.7-(ii)).

Under the assumption of Theorem 1.8, it is not difficult to see that we have $\hat{H}^0(\text{Tr} \mathcal{F}; \mathcal{P}) \cong \mathbb{R}$ (see Theorems 6.2 and 6.7-(ii)), where $\mathcal{P}$ is the Sergiescu’s orientation sheaf of $(M, \mathcal{F})$. Since $\mathcal{P}$ is always basic, we get the following corollary of Theorem 1.8.

**Corollary 1.9.** Any uniform complete Riemannian foliation is strongly tense.

Remark 1.10. Theorem 1.8 generalizes Domínguez’s tenseness theorem [Do98], which says that any Riemannian foliation on a closed manifold is tense. In the case where $M$ is compact, Theorem 1.8 is essentially equivalent to Domínguez’s theorem (see Remark 8.4).

If $(M, \mathcal{F})$ is strongly tense with the trivial flat line bundle, then $(M, \mathcal{F})$ is taut by definition. Thus we get the following generalization of one direction of Masa’s characterization [Ma92, Minimality theorem] of tautness of Riemannian foliations on compact manifolds.

**Corollary 1.11.** A connected manifold $M$ with a uniform complete Riemannian foliation $\mathcal{F}$ is taut if $\hat{H}^0(\text{Tr} \mathcal{F})$ is nontrivial.

We will illustrate Corollaries 1.9 and 1.11 with some examples of Lie foliations in the next section.

Remark 1.12. There will be no direct generalization of the other direction of Masa’s characterization of tautness to Riemannian foliations on noncompact manifolds. It is since simple examples show that $\hat{H}^0(\text{Tr} \mathcal{F})$ may be trivial for taut Riemannian foliations $\mathcal{F}$ on noncompact manifolds $M$ (see Example 9.2).

1.8. Examples: Lie foliations on homogeneous spaces. We will illustrate the main results in this article by applying them to Lie foliations. Let $G$ be a connected Lie group. Recall that a foliation is called a $G$-Lie foliation if its holonomy pseudogroup is equivalent to the pseudogroup on $G$ generated by the right action of a subgroup of $G$. Clearly a $G$-Lie foliation is always Riemannian. We can construct typical $G$-Lie foliations with dense leaves as follows: Let $H$ be a connected Lie group. Let $\Gamma$ be a torsion-free lattice of $G \times H$. Let $p_1 : G \times H \to G$ be the first projection. We assume that $p_1(\Gamma)$ is dense in $G$. Let $M = \Gamma \backslash G \times H$ and $\mathcal{F}_\Gamma$ the foliation of $M$ whose lift to $G \times H$ is the foliation defined by the fibers of $p_1$. It is easy to see that holonomy pseudogroup of $(M, \mathcal{F}_\Gamma)$ is equivalent to the pseudogroup on $G$ generated by the right action of $p_1(\Gamma)$, namely, $(M, \mathcal{F}_\Gamma)$ is a $G$-Lie foliation. Here the leaves of $(M, \mathcal{F}_\Gamma)$ are dense by the density of $p_1(\Gamma)$ in $G$ (see [Mo88, Proposition 4.2]). A number of such examples are known, in particular, in the case
where $G$ and $H$ are semisimple. For example, we can construct an example from an irreducible lattice (see [Jo84]). In the case of $G = H = \text{SL}(2; \mathbb{R})$, we can take $\Gamma$ so that $\Gamma \cong p_1(\Gamma) \cong \text{SL}(2; \mathbb{Z}[\sqrt{2}])$.

The computation of the reduced Haefliger cohomology and the basic cohomology of $G$-Lie foliations with dense leaves is elementary (see Section 6.2). But, it was difficult to compute the Haefliger cohomology, even though the holonomy pseudogroups are described in a simple way. Masa and Domínguez used Sarkaria’s diffusion operator for this purpose, and no elementary computation is not known, as far as the author knows. The Molino’s structure theory says that the computation of the Haefliger cohomology can be reduced to the case of $G$-Lie foliations with dense leaves. So these are the essential cases.

Here we will apply main results in this article to Lie foliations. Let $\mathfrak{g}$ be the Lie algebra consisting of right invariant vector fields on $G$. Let $M$ be a manifold with a $G$-Lie foliation $\mathcal{F}$ with dense leaves and a flat vector bundle $(\mathcal{E}, \nabla)$. Since such $(M, \mathcal{F})$ is clearly complete and uniform, Corollary 1.2 is specialized to the following (see also Lemma 6.4).

**Corollary 1.13.** We have $H^\bullet_c(\text{Tr} \mathcal{F}; \mathcal{E}) \cong H^\bullet_\bullet(\mathfrak{g}; \mathcal{E}_0) \cong H^\bullet(\mathfrak{g}; \mathcal{E}_0^*)^*$, where $\mathcal{E}_0$ is the space of basic sections of $\mathcal{E}$ on $M$ which naturally admits the structure of a $\mathfrak{g}$-module.

Corollary 1.13 in this case directly follows from Corollary 1.13 and the Poincaré duality of Lie algebra cohomology (see [Kn88, Theorem 6.10]), where $\mathcal{P}$ corresponds to det $\mathfrak{g}$.

As a special case of Corollaries 1.9 and 1.11, we get the following.

**Corollary 1.14.** Any $G$-Lie foliation with dense leaves is strongly tense. Any $G$-Lie foliation with dense leaves is taut if $G$ is unimodular.

In the case where $M$ is compact, Corollaries 1.13 and 1.14 follow from the results of Masa and Domínguez. They are new in the case where $M$ is not compact.

**Organization of the article.** Sections 2 and 3 are devoted to recall fundamental notions and results. We will prove Theorem 1.1 for complete parallelizable pseudogroups in Section 4 and for general complete Riemannian pseudogroups in Section 5. Section 6 is devoted to prove Corollary 1.3. We will prove Theorem 1.6 in Section 7 which is independent of the preceding sections. In Section 8, we will prove Theorem 1.8 for compact foliated manifolds. In Section 9, Haefliger type characterization of strongly tenseness (Theorem 1.6) is generalized to noncompact foliated manifolds. In Section 10, we will prove Theorem 1.8.

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## 2. Definition of Haefliger cohomology

### 2.1. Haefliger cohomology of pseudogroups with values in equivariant flat vector bundles.**

Recall that a pseudogroup on a manifold $T$ is a collection of diffeomorphisms between two open sets of $T$ which contains the identities and is closed under composition, inversion, restriction and union. Let $\mathcal{H}$ be a pseudogroup on a manifold $T$. Let $(\mathcal{E}, \nabla)$ be a flat vector bundle over $T$. The following notion is a variant of the one used in [Ha85, Section 3.2].
Definition 2.1. An $\mathcal{H}$-action on $(\mathcal{E}, \nabla)$ is an assignment of each $h \in \mathcal{H}$ to an isomorphism $h_* : (\mathcal{E}|_{\text{Dom} h}, \nabla) \rightarrow (\mathcal{E}|_{\text{Im} h}, \nabla)$ which covers $h$ such that

(i) $h_*|_U = (h|_U)_*$ for any open set $U \subset \text{Dom} h$ and
(ii) $(h_1 \circ h_2)_* = h_1_* \circ h_2^*$ if $\text{Im} h^2 \subset \text{Dom} h^1$.

An $\mathcal{H}$-equivariant flat vector bundle over $T$ is defined to be a flat vector bundle over $T$ endowed with an $\mathcal{H}$-action.

Example 2.2. Let $G$ be a Lie group and $\Gamma$ a dense subgroup of $G$. Let $\gamma : G \rightarrow \text{GL}(V)$ be a representation of $G$ on a finite dimensional vector space $V$. Consider a trivial flat vector bundle $(\mathcal{E}, \nabla)$ over $G$ whose fiber is $V$, namely, $\mathcal{E} = G \times V \rightarrow G$. For $h \in \Gamma$, lift the right product $R_h : G \rightarrow G$ to $\mathcal{E} \rightarrow \mathcal{E}$ by $R_h(g, x) = (gh, \gamma(h^{-1})(x))$. Let $\mathcal{H}$ be the pseudogroup on $G$ generated by the right $\Gamma$-action. Then $(\mathcal{E}, \nabla)$ is an $\mathcal{H}$-equivariant flat vector bundle over $G$.

Let $\mathcal{O}_T$ be the space of differential forms on $T$, and $\Lambda^n_T$ the subspace of $\Omega^n (T; \mathcal{E})$ consisting of finite sums of elements of the form $h^* \alpha - \alpha$ for $h \in \mathcal{H}$ and $\alpha \in \Omega^n (T; \mathcal{E})$ such that $\text{supp} \alpha \subset \text{Im} h$. Let

$$\mathcal{O}^*_n (T; \mathcal{E}) := \mathcal{O}^*_n (T; \mathcal{E}) / \Lambda^n_T.$$

Since $\nabla$ is preserved by $h_*$ for any $h \in \mathcal{H}$, the covariant derivative $d_\nabla : \mathcal{O}^*_n (T; \mathcal{E}) \rightarrow \mathcal{O}^*_{n+1} (T; \mathcal{E})$ commutes with $h_*$. Thus we get $d_\nabla : \mathcal{O}^*_n (T; \mathcal{E}) \rightarrow \mathcal{O}^*_{n+1} (T; \mathcal{H}; \mathcal{E})$.

Definition 2.3. The cohomology of the complex $(\mathcal{O}^*_n (T; \mathcal{H}; \mathcal{E}), d_\nabla)$ is called the Haefliger cohomology of $\mathcal{H}$ with values in $(\mathcal{E}, \nabla)$ and denoted by $H^*_n (T; \mathcal{H}; \mathcal{E})$.

Remark 2.4. It is well known that the Haefliger cohomology is a part of the spectral sequence associated with $\mathcal{F}$ (see, for example, [ALM08, Section 28]). See [CM04, Section 7] for the explanation on the relations with other cohomologies associated with foliated manifolds.

We topologize $\mathcal{O}^*_n (T; \mathcal{E})$ with the natural topology (see [KRS89, Section 9]) called the LF-topology. In this topology, a sequence $\{\alpha_i\}_{i=1}^\infty$ in $\mathcal{O}^*_n (T; \mathcal{E})$ converges to $\alpha$ if and only if there exists a relatively compact subset $K$ of $T$ such that $\text{supp} \alpha_i \subset K$ for large enough $i$ and $\lim_{i \rightarrow \infty} ||\alpha - \alpha_i||_k = 0$ for any $k$, where $|| \cdot ||_k$ is a $C^k$-norm on $\mathcal{O}^*_n (T; \mathcal{E})$.

Since $\Lambda^n_T$ is not closed in $\mathcal{O}^*_n (T; \mathcal{E})$ in general, the complex $\mathcal{O}^*_n (T; \mathcal{H}; \mathcal{E})$ may not be Hausdorff with the quotient topology. The quotient of $\mathcal{O}^*_n (T; \mathcal{H}; \mathcal{E})$ by the closure of 0 is denoted by $\mathcal{O}^*_n (T; \mathcal{H}; \mathcal{E})$ and its cohomology is denoted by $\hat{H}^*_n (T; \mathcal{H}; \mathcal{E})$. Partly following [He88], $\hat{H}^*_n (T; \mathcal{H}; \mathcal{E})$ is called the reduced Haefliger cohomology. We have a canonical map

$$\tau : H^*_n (T; \mathcal{H}; \mathcal{E}) \rightarrow \hat{H}^*_n (T; \mathcal{H}; \mathcal{E}).$$

The maximal degree component of the Haefliger cohomology is easy to compute as follows.

Proposition 2.5. Let $\mathcal{H}$ be a pseudogroup on an $n$-dimensional manifold $T$. Let $\mathcal{O}_T$ be the orientation bundle of $T$. If the orbit space $T/\mathcal{H}$ is connected, then $H^*_n (T/\mathcal{H}; \mathcal{O}_T) \cong \hat{H}^*_n (T/\mathcal{H}; \mathcal{O}_T) \cong \mathbb{R}$.

Proof. Let $L$ be the subspace of $H^*_n (T; \mathcal{O}_T)$ defined by

$$L = \{ \sum_{j=1}^k [\alpha_j] - h_j^* [\alpha_{ij}] \mid \text{supp} \alpha_j \subset \text{Dom} h_j, 1 \leq j \leq k \}. $$

By the last part of the cohomology exact sequence of

$$0 \rightarrow \Lambda^n_T \rightarrow \mathcal{O}^*_n (T; \mathcal{O}_T) \rightarrow \mathcal{O}^*_n (T/\mathcal{H}; \mathcal{O}_T) \rightarrow 0,$$

we get that $H^*_n (T/\mathcal{H}; \mathcal{O}_T) \cong H^*_n (T; \mathcal{O}_T)/L$, which implies $H^*_n (T/\mathcal{H}; \mathcal{O}_T) \cong \mathbb{R}$ by assumption. Then $L$ is closed in $H^*_n (T; \mathcal{O}_T)$. So we have $\hat{H}^*_n (T/\mathcal{H}; \mathcal{O}_T) \cong H^*_n (T; \mathcal{O}_T)/L \cong \mathbb{R}$. $\square$
Proposition 2.7. The isomorphism classes of topological differential complexes

Definition 2.8. $T$ is a $T_1$ equivalence class of $\text{Hol}(T)$ in a way analogous to [Ha80, Corollary in Section 1.2], based on the fact that the $H$ equivalence of the groupoids obtained from the pseudogroups by taking germs. Twisted Haefliger cohomology is invariant under the equivalence in the following sense: If we have an equivalence from a pseudogroup $H_1$ on $T_1$ to a pseudogroup $H_2$ on $T_2$ and an $H_1$-equivariant flat vector bundle $E_1$ over $T_1$, there exists an $H_2$-equivariant flat vector bundle $E_2$ over $T_2$ and satisfies $H^*(T_1/H_1;E_1) \cong H^*(T_2/H_2;E_2)$.

2.2. Haefliger cohomology of foliated manifolds with values in flat vector bundles. Let $(M,F)$ be a foliated manifold. Let $(\mathcal{E},\nabla)$ be a flat vector bundle over $M$. A transversal of $(M,F)$ is an immersion $T \to M$ which is transverse to the leaves of $F$ by definition, but throughout this article we regard $T$ as a subset of $M$. Recall that a transversal $T$ of $(M,F)$ is called total if it intersects every leaf of $F$. For any total transversal $T$ of $(M,F)$, we have the holonomy pseudogroup $\text{Hol}(T)$ of $(M,F)$ on $T$. For any leaf path $\gamma$ which connects two points of $T$, we can consider the parallel transport along $\gamma$ on $(\mathcal{E},\nabla)$, which gives a $\text{Hol}(T)$-action on $E_T$. Throughout this article, such $\text{Hol}(T)$-equivariant flat vector bundle on $T$ obtained from $(\mathcal{E},\nabla)$ will be denoted by the same symbol $(\mathcal{E},\nabla)$. We topologize $\Omega^*(T;E)$ with the LF-topology like in the last section. The following result is proved in a way analogous to [Ha80 Corollary in Section 1.2], based on the fact that the equivalence class of $\text{Hol}(T)$ is independent of the choice of $T$.

Proposition 2.7. The isomorphism classes of topological differential complexes $(\Omega^*(T/\text{Hol}(T);E),d_\nabla)$ and $(\hat{\Omega}^*(T/\text{Hol}(T);E),d_\nabla)$ are independent of the choice of $T$.

Thus the following is well-defined.

Definition 2.8. For a foliated manifold $(M,F)$ with a total transversal $T$ and a flat vector bundle $(\mathcal{E},\nabla)$, let

$$\Omega^*_*(\text{Tr}F;\mathcal{E}) := \Omega^*_*(T/\mathcal{H};\mathcal{E}),$$

$$H^*_*(\text{Tr}F;\mathcal{E}) := H^*_*(\Omega^*_*(T/\mathcal{H};\mathcal{E}),d_\nabla)$$

and $H^*_*(\text{Tr}F;\mathcal{E})$ is called the Haefliger cohomology of $(M,F)$ with values in $(\mathcal{E},\nabla)$. The complex $\hat{\Omega}^*_*(\text{Tr}F;\mathcal{E})$ and the reduced Haefliger cohomology $\hat{H}^*_*(\text{Tr}F;\mathcal{E})$ are similarly defined.

3. Fundamental notions on pseudogroups

3.1. Riemannian pseudogroups and its structure theory. Let us recall the following fundamental notions.

Definition 3.1. A pseudogroup $\mathcal{H}$ on $T$ is called Riemannian if there exists a Riemannian metric $g$ on $T$ such that $h^*g = g$ on $\text{Dom}h$ for any $h \in \mathcal{H}$.

Definition 3.2. A pseudogroup $\mathcal{H}$ on $T$ is called complete if, for any two points $x, y \in T$, there exists open neighborhood $U_x$ (resp., $U_y$) of $x$ (resp., $y$) in $T$ such that, for every $h \in \mathcal{H}$ and $z \in U_x \cap \text{Dom}h$ with $h(z) \in U_y$, there exists some $\tilde{h} \in \mathcal{H}$ so that $U_x \subset \text{Dom}\tilde{h}$ and the germs of $\tilde{h}$ and $h$ at $z$ are equal.

If the holonomy pseudogroup of a foliated manifold $(M,F)$ is Riemannian (resp., complete), then $(M,F)$ is called Riemannian (resp., complete).

Let us recall the structure theory of complete Riemannian pseudogroups due to Salem [Sa88], which is analogous to the Molino’s structure theory of Riemannian
foliations on compact manifolds. A vector field \( X \) on \( T \) is called \( \mathcal{H} \)-invariant if \( h_*X = X \) on \( \text{Im} \, h \) for any \( h \in \mathcal{H} \).

**Definition 3.3.** A pseudogroup \( \mathcal{H} \) on \( T \) is called parallelizable if there exist \( \mathcal{H} \)-invariant vector fields \( X_1, \ldots, X_n \) on \( T \) such that \( \{X_1, \ldots, X_n\} \) gives a trivialization of the tangent bundle of \( T \). Such collection \( \{X_1, \ldots, X_n\} \) of \( \mathcal{H} \)-invariant vector fields is called a parallelism of \( \mathcal{H} \).

The following is an important class of parallelizable pseudogroups, which appears in the structure theory.

**Definition 3.4.** For a Lie group \( G \), a pseudogroup \( \mathcal{H} \) is called of \( G \)-Lie type if \( \mathcal{H} \) is equivalent to a pseudogroup on \( G \) generated by the right action of a subgroup of \( G \).

**Remark 3.5.** Note that this terminology is not common while the notion itself is common. Note also that the terminology \( \text{Lie pseudogroup} \) is commonly used for a different object.

We need the following part of the analog of the Molino theory for complete Riemannian pseudogroups due to Salem.

**Theorem 3.6** ([Sa88 Corollaires 1 and 2], see also [Moss, Appendix by Salem]). Let \( \mathcal{H} \) be a complete Riemannian pseudogroup on a manifold \( T \). Let \( T^\# \) be the orthonormal frame bundle over \( T \). The pseudogroup \( \mathcal{H}^\# \) on \( T^\# \) naturally induced from \( \mathcal{H} \) is complete parallelizable. In general, for a complete parallelizable pseudogroup \( \mathcal{H}^\# \) on \( T^\# \) whose orbit space is connected, the following holds:

(i) The space \( T^\# / \mathcal{H}^\# \) of the closures of the \( \mathcal{H}^\# \)-orbits in \( T^\# \) is a smooth manifold and the projection \( T^\# \to T^\# / \mathcal{H}^\# \) is a submersion.

(ii) There exists a connected and simply-connected Lie group \( G \) such that, for each point of \( x \in T^\# / \mathcal{H}^\# \), there exists an open neighborhood \( U \) of \( x \) in \( T^\# / \mathcal{H}^\# \) such that \( \mathcal{H}^\#|_{U} \) is equivalent to the pseudogroup on \( U \times G \) which is the product of the trivial pseudogroup on \( U \) and the pseudogroup on \( G \) generated by the right action of a dense subgroup of \( G \).

**3.2. Uniform pseudogroups.** We introduce the following notion according to the terminology for subgroups of Lie groups.

**Definition 3.7.** A pseudogroup \( \mathcal{H} \) over a manifold \( T \) is called uniform if there exists a relatively compact subset \( U \) of \( T \) whose \( \mathcal{H} \)-orbit is equal to \( T \).

The uniformness is characterized as follows.

**Lemma 3.8.** For a pseudogroup \( \mathcal{H} \) over a manifold \( T \), the following are equivalent:

(i) \( \mathcal{H} \) is uniform.

(ii) the space \( T/\mathcal{H} \) of \( \mathcal{H} \)-orbits is compact.

(iii) the space \( T/\overline{\mathcal{H}} \) of the closures of \( \mathcal{H} \)-orbits is compact.

**Proof.** Clearly (i) implies (ii), and (ii) implies (iii). Then it suffices to prove that (iii) implies (i). We take an open covering \( \{U_i\} \) of \( T \) so that \( U_i \) is relatively compact for each \( i \). By the compactness of \( T/\overline{\mathcal{H}} \), there exists a finite set \( I \) such that \( \bigcup_{i \in I} U_i \) intersects with the closure \( \overline{O} \) of any \( \mathcal{H} \)-orbit \( O \). It is easy to see that \( \overline{O} \cap \bigcup_{i \in I} U_i \neq \emptyset \) implies \( O \cap \bigcup_{i \in I} U_i \neq \emptyset \). Thus \( \bigcup_{i \in I} U_i \) is a relatively compact subset of \( T \) whose \( \mathcal{H} \)-orbit is \( T \).

\[ \square \]

For a foliated manifold \((M, \mathcal{F})\), the orbit space of the holonomy pseudogroup is homeomorphic to the leaf space. If \( M \) is compact, then the leaf space of \((M, \mathcal{F})\) is compact. Hence the holonomy pseudogroup of \((M, \mathcal{F})\) is uniform.
4. HAEFLIGER COHOMOLOGY OF UNIFORM PARALLELIZABLE PSEUDOGROUPS

4.1. The main result of this section. Section 4 is devoted to prove the following result, namely, Theorem 4.1 for parallelizable pseudogroups.

**Theorem 4.1.** For a uniform parallelizable pseudogroup $\mathcal{H}$ on a manifold $T$ and an $\mathcal{H}$-equivariant flat vector bundle $(\mathcal{E}, \nabla)$ over $T$, the natural map $\tau : H^*_\mathcal{H}(T/\mathcal{H}; \mathcal{E}) \to \hat{H}^*_\mathcal{H}(T/\mathcal{H}; \mathcal{E})$ is an isomorphism.

Note that the completeness is not assumed in this theorem.

4.2. Sarkaria’s smoothing operators on parallelizable pseudogroups. Let $\mathcal{H}$ be a pseudogroup on a manifold $T$. We assume that $\mathcal{H}$ is parallelizable with parallelism $\{X_1, \ldots, X_n\}$. Let $V$ be the real vector space generated by $X_1, \ldots, X_n$. We fix a compactly supported volume form $\mu$ on $V$. We assume that the time one map of the flow generates by any $X \in V$ is well-defined on $T$, which is denoted by $\phi_X$. The operator

$$s : \Omega^* \to \Omega^*_\mathcal{H}$$

is called a smoothing operator on $T$.

This operator $s$ is a transverse version of Sarkaria’s smoothing operator. Masa and Domínguez used Sarkaria’s smoothing operators on compact manifolds, which are Hilbert-Schmidt integration operators and hence compact with respect to the natural topology. Note that even if $\mathcal{H}$ is the holonomy pseudogroup of a compact foliated manifold, $T$ may be noncompact. In that case, $s$ may not be a compact operator in general. To avoid this problem of the noncompactness, we can use the restriction of the smoothing operators on relatively compact subsets. The following is well known. (It follows from Step 5 of the proof of Theorem 1 in Section 5.7 and Arzelà-Ascoli theorem.)

**Lemma 4.2.** For a relatively compact subset $U$ of $T$, the restriction of the smoothing operator $s : \Omega^*_\mathcal{H}(U; \mathcal{E}|_U) \to \Omega^*_\mathcal{H}(U; \mathcal{E})$ is compact.

4.3. Masa’s decomposition via smoothing operators. Let $\mathcal{H}$ be a uniform parallelizable pseudogroup on $T$. Let $(\mathcal{E}, \nabla)$ be a transverse flat vector bundle over $T$. Since $\mathcal{H}$ is uniform, there exists a relatively compact subset $U$ of $T$ whose $\mathcal{H}$-orbit is equal to $T$. It follows that the flow generated by any $\mathcal{H}$-invariant vector field is well-defined on $T$. Then we can consider the smoothing operator $s : \Omega^*_\mathcal{H}(T; \mathcal{E}) \to \Omega^*_\mathcal{H}(U; \mathcal{E}|_U)$ defined in (1). We take $\{h_i\}_{i=1}^\infty \subset \mathcal{H}$ so that $U = \{h_i(U)\}_{i=1}^\infty$ is a locally finite covering of $T$. Let $\{\rho_i\}$ be a partition of unity on $T$ subordinated to $U$. We get a map

$$\Phi_U : \Omega^*_\mathcal{H}(T; \mathcal{E}) \to \Omega^*_\mathcal{H}(U; \mathcal{E}|_U)$$

Let $\tilde{s}$ be the composite

$$\Omega^*_\mathcal{H}(T; \mathcal{E}) \xrightarrow{s} \Omega^*_\mathcal{H}(U; \mathcal{E}|_U) \xrightarrow{\Phi_U} \Omega^*_\mathcal{H}(U; \mathcal{E}|_U) \xrightarrow{s} \Omega^*_\mathcal{H}(T; \mathcal{E}) .$$

Here $s$ is compact by Lemma 4.2. Since the composite of any continuous operator with a compact operator is compact, $\tilde{s}$ is compact. We will apply the following fact.

**Theorem 4.3** ((see RRS Corollary 2 in Section VIII)). Let $B$ be a Hausdorff locally convex topological vector space and $\sigma : B \to B$ a compact operator. There exists $r \in \mathbb{Z}_{>0}$ such that

$$B = \ker(1 - \sigma)^r \oplus \text{Im}(1 - \sigma)^r ,$$
where $\ker(1-\sigma)^r$ is closed and of finite dimension, $\text{Im}(1-\sigma)^r$ is closed and
\[
\ker(1-\sigma)^r = \ker(1-\sigma)^{r+1}, \quad \text{Im}(1-\sigma)^r = \text{Im}(1-\sigma)^{r+1}.
\]

By this proposition, we get a positive integer $r$ and a decomposition
\[
\Omega^*_T(\mathcal{H}; \mathcal{E}) = K^\bullet \oplus I^\bullet,
\]
where $K^\bullet = \ker(1-\tilde{s})^r$ and $I^\bullet = \text{Im}(1-\tilde{s})^r$ so that the properties mentioned in Theorem 4.3 are satisfied. Since $\Lambda_H^\bullet$ is locally convex and invariant under $\tilde{s}$, Theorem 4.3 also yields another decomposition with similar properties
\[
\Lambda_H^\bullet = (\Lambda_H^\bullet \cap K^\bullet) \oplus (\Lambda_H^\bullet \cap I^\bullet).
\]
Thus we get a decomposition
\[
(2) \quad \Omega^*_T(\mathcal{H}/\mathcal{E}) = \Omega^*_T(\mathcal{H}/\mathcal{E})/\Lambda_H^\bullet = (K^\bullet/(\Lambda_H^\bullet \cap K^\bullet)) \oplus (I^\bullet/(\Lambda_H^\bullet \cap I^\bullet)).
\]
Note the following well known fact (see, for example, [Tr67, Theorem 9.1]).

**Proposition 4.4.** A finite dimensional Hausdorff topological vector space $B$ over $\mathbb{R}$ is isomorphic to $\mathbb{R}^{\dim B}$ with the product topology. In particular, every subspace of $B$ is closed.

By this proposition, $\Lambda_H^\bullet \cap K^\bullet$ is a closed subspace of $K^\bullet$. Then $K^\bullet/(\Lambda_H^\bullet \cap K^\bullet)$ is Hausdorff. Letting $\Xi^\bullet = K^\bullet/(\Lambda_H^\bullet \cap K^\bullet)$ and $\Upsilon^\bullet = I^\bullet/(\Lambda_H^\bullet \cap I^\bullet)$, we get the following.

**Proposition 4.5.** There is a decomposition as a differential complex
\[
(3) \quad \Omega^*_T(\mathcal{H}/\mathcal{E}) = \Xi^\bullet \oplus \Upsilon^\bullet
\]
such that
(i) $\Xi^\bullet$ is Hausdorff of finite dimension,
(ii) $(1-\tilde{s})^r$ is the second projection $\Omega^*_T(\mathcal{H}/\mathcal{E}) \to \Upsilon^\bullet$ and
(iii) $(1-\tilde{s})^{r+1}$ is a differential automorphism of $\Upsilon^\bullet$.

Let $\overline{\Xi}^\bullet$ be the closure of $0$ in $\Omega^*_T(\mathcal{H}/\mathcal{E})$. Since $\Xi^\bullet$ is Hausdorff by Proposition 4.4(i), $\overline{\Xi}^\bullet$ is a subcomplex of $\Upsilon^\bullet$. Thus, by taking the quotient of the both sides of (3) by $\overline{\Xi}^\bullet$, we get a decomposition of $\Omega^*_T(\mathcal{H}/\mathcal{E})$. Here $\tilde{s}$ induces a map $\hat{\Omega}^*_T(\mathcal{H}/\mathcal{E}) \to \hat{\Omega}^*_T(\mathcal{H}/\mathcal{E})$, which is denoted by the same symbol $\hat{s}$. The following proposition is a direct consequence of Proposition 4.4.

**Proposition 4.6.** There is a decomposition of a differential complex
\[
(4) \quad \hat{\Omega}^*_T(\mathcal{H}/\mathcal{E}) = \Xi^\bullet \oplus \Upsilon^\bullet/\overline{\Xi}^\bullet
\]
such that
(i) $(1-\hat{s})^r$ is the projection $\hat{\Omega}^*_T(\mathcal{H}/\mathcal{E}) \to \Upsilon^\bullet/\overline{\Xi}^\bullet$ and
(ii) $(1-\hat{s})^{r+1}$ is a differential automorphism of $\Upsilon^\bullet/\overline{\Xi}^\bullet$.

4.4. **Acyclicity of $\Upsilon^\bullet$ and $\Upsilon^\bullet/\overline{\Xi}^\bullet$**. We will use the notations $V$, $\phi_X$ and $\mu$ in Section 4.2. Let
\[
h : \quad \Omega^*_T(\mathcal{E}) \longrightarrow \Omega^{*-1}_T(\mathcal{E}) \quad \alpha \longmapsto - \int_V \left( \int_0^1 (t\phi_X\alpha)dt \right) \mu(X).
\]
We show

**Lemma 4.7.** For $\alpha \in \Omega^*_T(\mathcal{E})$, we have
\[
\left( \int_V \mu \right) \alpha - s \alpha = (d\nu h + hd\nu) \alpha.
\]
Proof. For a vector field $X$ on $T$, the inner product and the Lie derivative with respect to $X$ on $\Omega_c^\bullet(T;E)$ is denoted by $\iota(X)$ and $\theta(X)$, respectively. Then

\[
(\int_V \mu)\alpha - s\alpha = - \int_V (\phi_X^s \alpha - \alpha) \mu(X)
\]

\[
= - \int_V \left( \int_0^1 \left( \frac{d}{dt} \big|_{t=0} \phi_X^t \alpha \right) dt \right) \mu(X)
\]

\[
= - \int_V \left( \int_0^1 (\theta(tX)\phi_X^t \alpha) dt \right) \mu(X)
\]

\[
= - \int_V \left( \int_0^1 ((\partial V \iota(tX) + \iota(tX)\partial V)\phi_X^t \alpha) dt \right) \mu(X)
\]

\[
= (\partial V h + h\partial V)\alpha . \quad \Box
\]

Fix $\mu$ so that $\int_V \mu = 1$. Since $\iota(X)$ commutes with the $\mathcal{H}$-action, we get a continuous map $h : \Omega_c^\bullet(T/\mathcal{H};E) \to \Omega_c^\bullet(T/\mathcal{H};E)$.

**Proposition 4.8.** We have $H^\bullet(\Upsilon^\bullet) = 0$ and $H^\bullet(\Xi^\bullet) \cong H^\bullet_c(T/\mathcal{H};E)$.

**Proof.** Note that $s$ and $\iota$ induce the same map on $\Omega_c^\bullet(T/\mathcal{H};E)$ by definition. Let $\alpha$ be a cocycle of $\Upsilon^\bullet$. By Proposition 4.5(iii), there is a cocycle $\beta$ of $\Upsilon^\bullet$ such that $(1-s)^{r+1}\beta = \alpha$. By Lemma 4.7 we get

\[
\alpha = (1-s)^{r+1}\beta = (1-s)^r (\partial V h + h\partial V)\beta = \partial V (1-s)^r h\beta .
\]

Here, $(1-s)^r h\beta$ belongs to $\Upsilon^\bullet$ by Proposition 4.5(ii). Thus $\alpha$ is a cocycle in $\Upsilon^\bullet$, which implies $H^\bullet(\Upsilon^\bullet) = 0$. The second equation follows from $H^\bullet(\Upsilon^\bullet) = 0$ and Proposition 4.9.

Here $h$ induces a map $\hat{\Omega}_c^\bullet(T/\mathcal{H};E) \to \hat{\Omega}^\bullet_{c-1}(T/\mathcal{H};E)$ by its continuity. The following is proved in a way analogous to Proposition 4.9 by using Proposition 4.6.

**Proposition 4.9.** We have $H^\bullet(\Upsilon^\bullet/\mathcal{O}^\bullet) = 0$ and $H^\bullet(\Xi^\bullet) \cong H^\bullet_c(T/\mathcal{H};E)$.

Thus Theorem 4.1 follows from Propositions 4.8 and 4.9.

5. **Haefliger cohomology of uniform complete Riemannian pseudogroups**

In this section, we extend Theorem 4.1 from uniform parallelizable pseudogroups to uniform Riemannian pseudogroups, from which Theorem 4.1 follows. It will be done by a modification of a well known method based on equivariant cohomology (see [AL89], [Ma92] Section 2 and [Do98] Theorem 3.7]). First we will review the method. Let $K$ be a connected compact Lie group with Lie algebra $\mathfrak{k}$. Recall that the Cartan complex of a $\mathfrak{k}$-dga $(A^\bullet, d)$ is defined by

\[
C_i^\bullet(A^\bullet) := ((\bigwedge^i \mathfrak{k}^\ast) \otimes A^\bullet)_{\theta=0} ,
\]

where $\bigwedge^i \mathfrak{k}^\ast$ denotes the symmetric algebra of $\mathfrak{k}^\ast$ and the subscript $\theta=0$ means the subspace invariant under the diagonal $\mathfrak{k}$-action. The grading of $C_i^\bullet(A^\bullet)$ is given by $C_i^j(A^\bullet) = \bigoplus_{2k+i+j} ((\bigwedge^i \mathfrak{k}^\ast) \otimes A^j)_{\theta=0}$. Here $C_i^\bullet(A^\bullet)$ is a differential complex whose differential $d_\mathfrak{k}$ is defined by

\[
d_\mathfrak{k}\omega(X) := d(\omega(X)) - \iota(X)(\omega(X))
\]

for $X \in \mathfrak{k}$, where we regard $\omega \in C_i^\bullet(A^\bullet)$ as a $\mathfrak{k}$-equivariant polynomial map $\mathfrak{k} \to A^\bullet$. Let

\[
A^\bullet_{i=0,\theta=0} := \{ \omega \in A^\bullet \mid \iota(X)\omega = 0, \theta(X)\omega = 0, \forall X \in \mathfrak{k} \} ,
\]
which is a subcomplex of \((A^\bullet, d)\). We have a natural differential map

\[
\epsilon : (A^\bullet_{\alpha=0, \theta=0}, d) \rightarrow (C^\bullet_t(A^\bullet), d_t)
\]

induced from the canonical inclusion. We refer to [GHV76, Definition 8.1] for the definition of an algebraic connection. (Note that, in this reference, a \(\mathfrak{t}\)-dga module is called a graded vector space with an operation of \(\mathfrak{t}\), see [GHV76, Definition 7.1].) The following is well known (for example, see [GHV76, Theorem IV in Section 8.17]), which corresponds to the fact that the equivariant cohomology of a manifold with a free action of a connected Lie group is isomorphic to the cohomology of the quotient.

**Theorem 5.1.** If a \(\mathfrak{t}\)-dga \((A^\bullet, d)\) admits an algebraic connection, then \(\epsilon\) induces an isomorphism

\[
H^\bullet(A^\bullet_{\alpha=0, \theta=0}, d) \cong H^\bullet(C^\bullet_t(A^\bullet), d_t).
\]

The \(E^1\)-terms of the spectral sequence of \((C^\bullet_t(A^\bullet), d_t)\) with the filtration \(F^\ell\) given by

\[
F^\ell = \bigoplus_{j \leq \ell} (\bigvee j\mathfrak{t}^* \otimes A^\bullet)_{\theta=0}
\]

are \((\bigvee \mathfrak{t}^*)_{\theta=0} \otimes H^\bullet(A^\bullet, d)\), which are determined by \(H^\bullet(A^\bullet, d)\). Thus we get the following.

**Lemma 5.2.** Let \(f : (A^\bullet, d_A) \rightarrow (B^\bullet, d_B)\) be a homomorphism between \(\mathfrak{t}\)-dgas. Assume that \((B^\bullet, d_B)\) admits an algebraic connection. If \(f\) induces an isomorphism \(H^\bullet(A^\bullet, d_A) \cong H^\bullet(B^\bullet, d_B)\), then \(f\) induces an isomorphism \(H^\bullet(A^\bullet_{\alpha=0, \theta=0}, d_A) \rightarrow H^\bullet(B^\bullet_{\alpha=0, \theta=0}, d_B)\).

We will prove the following result, from which Theorem 5.1 directly follows.

**Theorem 5.3.** For a uniform Riemannian pseudogroup \(\mathcal{H}\) on a manifold \(T\) and an \(\mathcal{H}\)-equivariant flat vector bundle \((\mathcal{E}, \nabla)\) over \(T\), the natural map \(\tau : H^\bullet_\mathcal{H}(T/\mathcal{H}; \mathcal{E}) \rightarrow H^\bullet_\mathcal{H}(T/\mathcal{H}; \mathcal{E})\) is an isomorphism.

**Proof.** Let \(\pi' : T' \rightarrow T\) be the orientation covering of \(T\). Let \(\mathcal{H}'\) be the pseudogroup on \(T'\) induced from \(\mathcal{H}\), and \((\mathcal{E}', \nabla') = ((\pi')^*\mathcal{E}, (\pi')^*\nabla)\). Let \(\pi^\# : T^\# \rightarrow T'\) be one of the connected component of the frame bundle of \(T'\). Let \(\mathcal{H}^\#\) be the pseudogroup on \(T^\#\) naturally induced from \(\mathcal{H}'\). Let \((\mathcal{E}^\#, \nabla^\#) = ((\pi^\#)^*\mathcal{E}, (\pi^\#)^*\nabla)\). Let \(\mathfrak{t} = \mathfrak{so}(q)\). It is easy to see that \(\Omega^\bullet_\mathcal{H}(T^\#/\mathcal{H}^\#; \mathcal{E}^\#)\) and \(\Omega^\bullet_\mathcal{H}(T^\#/\mathcal{H}^\#; \mathcal{E}^\#)\) naturally admit the structures of \(\mathfrak{t}\)-dgas with algebraic connections.

Here \(\mathcal{H}^\#\) is uniform and parallelizable by Lemma 3.8 and Theorem 3.6. Then, by Theorem 5.1 the natural map \(H^\bullet_\mathcal{H}(T^\#/\mathcal{H}^\#; \mathcal{E}^\#) \rightarrow H^\bullet_\mathcal{H}(T^\#/\mathcal{H}^\#; \mathcal{E}^\#)\) is an isomorphism. By Lemma 5.2, so is \(H^\bullet_\mathcal{H}(T'/\mathcal{H}'; \mathcal{E}') \rightarrow H^\bullet_\mathcal{H}(T'/\mathcal{H}'; \mathcal{E}')\). Clearly \(H^\bullet_\mathcal{H}(T/\mathcal{H}; \mathcal{E})\) and \(H^\bullet_\mathcal{H}(T'/\mathcal{H}'; \mathcal{E}')\) are isomorphic to the \(\mathbb{Z}/2\mathbb{Z}\)-invariant subspace of \(\tilde{H}^\bullet_\mathcal{H}(T'/\mathcal{H}'; \mathcal{E}')\) and \(H^\bullet_\mathcal{H}(T'/\mathcal{H}'; \mathcal{E}')\), respectively. Hence \(H^\bullet_\mathcal{H}(T/\mathcal{H}; \mathcal{E}) \rightarrow H^\bullet_\mathcal{H}(T/\mathcal{H}; \mathcal{E})\) is an isomorphism.

\[\square\]

### 6. Reduced Haefliger cohomology

6.1. **Pairing with the invariant cohomology.** We recall the invariant cohomology of pseudogroups and the pairing between the Haefliger cohomology and the invariant cohomology. Let \(\mathcal{H}\) be a pseudogroup on an \(n\)-dimensional manifold \(T\). Let \((\mathcal{E}, \nabla)\) be an \(\mathcal{H}\)-equivariant flat vector bundle over \(T\). Let \(\mathcal{O}_T\) be the orientation bundle of \(T\). Recall that \(\alpha \in \Omega^\bullet(T; \mathcal{E})\) is said to be \(\mathcal{H}\)-invariant if \(h^\ast \alpha = \alpha\) on \(\text{Dom}~h\) for any \(h \in \mathcal{H}\). Let \(\Omega^\bullet(T; \mathcal{E})^\mathcal{H}\) be the subcomplex of \(\Omega^\bullet(T; \mathcal{E})\) consisting
of $\mathcal{H}$-invariant forms. We denote the cohomology of $\Omega^\bullet(T;\mathcal{E})^\mathcal{H}$ by $H^\bullet_{\text{inv}}(\mathcal{H};\mathcal{E})$. The wedge product
\begin{equation}
\Omega^\bullet(T;\mathcal{E}) \times \Omega^{n-\bullet}(T;\mathcal{E}^* \otimes \mathcal{O}_T) \rightarrow \Omega^n(T;\mathcal{O}_T)
\end{equation}
duces
\begin{equation}
\Omega^\bullet(T/H;\mathcal{E}) \times \Omega^{n-\bullet}(T;\mathcal{E}^* \otimes \mathcal{O}_T)^\mathcal{H} \rightarrow \Omega^n(T/H;\mathcal{O}_T)
\end{equation}
and hence
\begin{equation}
\tilde{\Omega}^\bullet(T/H;\mathcal{E}) \times \Omega^{n-\bullet}(T;\mathcal{E}^* \otimes \mathcal{O}_T)^\mathcal{H} \rightarrow \tilde{\Omega}^n(T/H;\mathcal{O}_T).
\end{equation}
If the orbit space $T/H$ is connected, then $\hat{H}^n(T/H;\mathcal{O}_T) \cong \mathbb{R}$ by Proposition 2.5. So, by decomposing $T/H$ into the connected components, we have a natural map
\begin{equation}
\Psi : \hat{H}^\bullet(T/H;\mathcal{E}) \rightarrow H^\bullet_{\text{inv}}(\mathcal{H};\mathcal{E}^* \otimes \mathcal{O}_T)^*.
\end{equation}

Remark 6.1. Note that the Haefliger cohomology does not admit a product in a natural way. This can be understood by the spectral sequence $E^{p,t}_2$ associated with $(M,\mathcal{F})$. Indeed, the $k$-th Haefliger cohomology is isomorphic to $E^{p,k}_2$ (see [ALM08 Section 28]), where $p = \dim \mathcal{F}$, while the natural product is $E^{p_1,k_1}_2 \times E^{p_2,k_2}_2 \rightarrow E^{p_1+p_2,k_1+k_2}_2$.

Sections 6.2 and 6.3 will be devoted to prove the following result, which was already known for the case of the trivial coefficient [ALM08 p. 598].

**Theorem 6.2.** For a complete Riemannian pseudogroup $\mathcal{H}$, the map $\Psi$ in (9) is an isomorphism.

Theorem 1.2 follows from this theorem, since the invariant cohomology of the holonomy pseudogroup of a foliated manifold $(M,\mathcal{F})$ is naturally isomorphic to the basic cohomology of $(M,\mathcal{F})$ by definition. Theorem 6.2 will be proved by an argument similar to the proof of the twisted Poincaré duality of basic cohomology of Riemannian foliations which are complete in the sense of [Se85, Definition 1.4] due to Sergiescu [Se85, Théorème I].

6.2. **Duality for complete parallelizable pseudogroups.** In this section, we show the following result, namely, Theorem 6.2 for complete parallelizable pseudogroups.

**Proposition 6.3.** For a complete parallelizable pseudogroup $\mathcal{H}$, the map $\Psi$ in (9) is an isomorphism.

Let $G$ be a connected and simply-connected Lie group and $\mathfrak{g}$ the Lie algebra consisting of right invariant vector fields on $G$. First we consider the case of pseudogroups of $G$-Lie type whose orbits are dense. The following lemma is due to Gilbert Hector.

**Lemma 6.4.** For a pseudogroup $\mathcal{H}$ of $G$-Lie type generated by the right action of a dense subgroup of $G$, the map $\Psi$ in (9) is an isomorphism and we have
\begin{equation}
\hat{H}^\bullet(G/\mathcal{H};\mathcal{E}) \cong H^\bullet_*(\mathfrak{g};\mathcal{E}_0) \cong H^\bullet_*(\mathfrak{g};\mathcal{E}_0^*)^*,
\end{equation}
where $\mathcal{E}_0 = \Omega^0(G;\mathcal{E})^\mathcal{H}$ which naturally admits the structure of a $\mathfrak{g}$-module.

**Proof.** Since $G$ is simply-connected, $(\mathcal{E},\nabla)$ is the trivial flat bundle over $G$. Since $\Omega^\bullet(G;\mathcal{E}^*)^\mathcal{H} = \Omega^\bullet(G;\mathcal{E}^*)^G$ by assumption, each element of $\Omega^\bullet(G;\mathcal{E}^*)^\mathcal{H}$ is determined by its value at the identity of $G$. So, letting $\mathcal{E}_0^* = \Omega^0(G;\mathcal{E}^*)^\mathcal{H}$, we have
\begin{equation}
\Omega^\bullet(G;\mathcal{E}^*)^\mathcal{H} \cong \Lambda^\bullet \mathfrak{g}^* \otimes \mathcal{E}_0^*.
\end{equation}
Here, the covariant derivative $d_H : E^r \rightarrow \mathfrak{g}^* \otimes E^r_0$ determines the structure of a \( g \)-module on \( E^r_0 \), and we have $H^*_\text{inv}(\mathcal{H}; E^*) \cong H^*(g; E^0_0)$. By (10), the composite of the pairing in (10) and the integration on \( G \) induces
\[
\Theta : \Omega^*_H(G; E) \rightarrow \bigwedge^* \mathfrak{g} \otimes E_0 .
\]
It is easy to see that \( \Theta \) is surjective. Clearly $\Lambda^*_H$ is contained in $\ker \Theta$, and hence so is $\overline{\Lambda}^*_H$. Take $\alpha \in \Omega^*_H(G; E)$ so that $\alpha \notin \overline{\Lambda}^*_H$. By the Hahn-Banach theorem for $\Omega^*(G; E)$ which is a Fréchet space, there exists $v \in \Omega^*_H(G; E)^*$ such that $v|\overline{\Lambda}^*_H = 0$ and $v(\alpha) \neq 0$. Here $v|\overline{\Lambda}^*_H = 0$ implies that $v$ is $\mathcal{H}$-invariant. Since $\mathcal{H}$ is generated by a dense subgroup of $G$, it follows that $v$ is $G$-invariant. Thus we have $v \in \bigwedge^* \mathfrak{g}^* \otimes E^*_0$ such that $v(\alpha) \neq 0$, which implies that $\ker \Theta \subset \overline{\Lambda}^*_H$. So we get
\[
\Omega^*_H(G/\mathcal{H}; E) \cong \bigwedge^* \mathfrak{g} \otimes E_0 .
\]
So $\Psi$ is an isomorphism for $\mathcal{H}$ by (10), (11) and the well known duality $H_*(g; E_0) \cong H^*(g; E^*_0)^*$ of Lie algebra homology and cohomology \cite{KriSS} Theorem 6.10]. We get the given formulas on $H^*_\text{inv}(G/\mathcal{H}; E)$ at the same time. 

**Lemma 6.5.** Let $T = \mathbb{R}^k \times G$ and assume that $\mathcal{H}$ is a pseudogroup on $T$ which is the product of the trivial pseudogroup on $\mathbb{R}^k$ and the pseudogroup on $G$ generated by the right action of a dense subgroup of $G$. Then $\Psi$ in (9) is an isomorphism for $\mathcal{H}$.

**Proof.** Let $G(0) = G \times \{0\}$. The Poincaré lemmas for $\mathbb{R}^k$ implies
\[
\tilde{H}^*_\text{inv}(T/\mathcal{H}; E) \cong \tilde{H}^*_\text{inv}(G(0)/\mathcal{H}(G(0)); E|G(0)) ,
\]
\[
\tilde{H}^*_\text{inv}(\mathcal{H}; E^*) \cong H^*_\text{inv}(\mathcal{H}(G(0)); E^*_0|G(0)) .
\]
Since $\mathcal{H}(G(0))$ is of $G$-Lie type, Lemma 6.4 implies that $\Psi$ in (9) is an isomorphism for $\mathcal{H}(G(0))$. Thus the claim follows. 

**Proof of Proposition 6.3.** By Theorem 3.1, $T/\overline{\mathcal{H}}$ is a smooth manifold and, for each point $x \in T/\overline{\mathcal{H}}$, there exists an open neighborhood $U$ of $x$ such that $U \cong \mathbb{R}^k$ and $\mathcal{H}|_{\pi^{-1}(U)}$ is equivalent to the pseudogroup on $U \times G$ which is the product of the trivial pseudogroup on $U$ and the pseudogroup of $G$-Lie type generated by the right action of a dense subgroup of $G$. Thus, Lemma 6.5 implies that $\Psi$ is an isomorphism for such $\mathcal{H}|_{\pi^{-1}(U)}$. Then, a well known argument using the Mayer-Vietoris sequences and the five lemma (see \cite{Sess} Proof of Proposition 2.4) implies that $\Psi$ is an isomorphism for $\mathcal{H}$. 

**6.3. Duality in the general case:** **Proof of Theorem 6.2.** Theorem 6.2 is deduced from Proposition 6.3 by the comparison of two spectral sequences similar to \cite[Démonstration du théorème I]{Sess} and \cite[Sections 4.7 and 4.8]{RPSAW}. Take $T^\#, \mathcal{H}^\#, E^\#, T', \mathcal{H}'$ and $E'$ like in the proof of Theorem 5.3. We similarly define $(E^*)^\#, (E'^*)^\prime, O_T^\#)$ and $O_T'$. Let $\mathfrak{g} = \mathfrak{so}(q)$. First, we show that $\Psi$ is an isomorphism for $\mathcal{H}'$, namely,
\[
\Psi : \tilde{H}_c^*(T'/\mathcal{H}'; E') \rightarrow H^*_\text{inv}(T'/\mathcal{H}', (E'^*)^\prime \otimes O_T') .
\]
We consider the spectral sequence $E^*_c$ of $\tilde{H}_c^*(T'/\mathcal{H}'; E^\#)$ given by
\[
\mathcal{F}^*\tilde{\Omega}^*_c(T^\#/\mathcal{H}^\#; E^\#) = \{ \alpha \mid \iota(X_0) \circ \cdots \circ \iota(X_k) \alpha = 0, X_i \in \mathfrak{f} \} .
\]
Here it is easy to see that $E^*_c$ converges to $\tilde{H}_c^*(T'/\mathcal{H}', E')$ and the $E_2$-terms are given by
\[
E^*_2 \cong \tilde{H}_c^*(T'/\mathcal{H}', E') \otimes H^1(\mathfrak{so}(q)) .
\]
We consider another spectral sequence. Define a filtration of \( \Omega^\bullet (T^\#; (\xi^*)^\# \otimes O_T^\#)^R^\# \) by

\[
F^m \Omega^{s+t}(T^\#; (\xi^*)^\# \otimes O_T^\#)^R^\# = \{ \alpha \mid \iota(X_0) \circ \cdots \circ \iota(X_r) \alpha = 0, X_i \in \xi \}. 
\]

Let \( C^*_{H^\#} = (\Omega^\bullet (T^\#; (\xi^*)^\# \otimes O_T^\#)^R^\#)^\ast \) be a complex whose differential is dual to \( d_C \). Let \( E_{r,t} \) be the spectral sequence of \( C^*_{H^\#} \) with the filtration dual to \( (14) \), which is defined by

\[
F^m C^{s+t}_{H^\#} = \{ v \mid v(\alpha) = 0, \forall \alpha \in F^{m-s} \Omega^{n-(s+t)}(T^\#; (\xi^*)^\# \otimes O_T^\#)^R^\# = 0 \}, 
\]

where \( m = \dim \mathfrak{g}(q) = q(q - 1)/2 \). Here it is easy to see that \( E_{r,t} \) converges to \( H^i_{\text{inv}}(\mathcal{H}^\#; (\xi^*)^\# \otimes O_T^\#)^\ast \) and the \( E_2 \)-terms are given by

\[
E_2^{r,t} \cong H^i_{\text{inv}}(\mathcal{H}^\#; (\xi^*)^\# \otimes O_T^\#)^\ast \otimes H^t(\text{SO}(q))^\ast. 
\]

Since \( \mathcal{H}^\# \) is complete and parallelizable, Proposition 6.3 implies that \( \Psi \) induces an isomorphism \( H^i_{\text{inv}}(\mathcal{H}^\#; (\xi^*)^\# \otimes O_T^\#)^\ast \) between the \( E_\infty \)-terms of the two spectral sequences. Then, by Moore’s comparison theorem (see, for example, [MC01]), \( \Psi \) induces an isomorphism \( E_2^{r,t} \cong E_2^{r,t} \). Then (12) follows from (13) and (15). The isomorphism \( H^i_{\text{inv}}(T/\mathcal{H}; \xi^\# \otimes \mathcal{O}_T^\#)^\ast \) follows from (12) like in the last part of the proof of Theorem 5.3.

6.4. Proof of Corollary 1.3. By using Theorems 1.1 and 6.2 we will prove the following pseudogroup version of Corollary 1.3.

**Corollary 6.6.** Let \( \mathcal{H} \) be a uniform complete Riemannian pseudogroup on an \( n \)-dimensional manifold on \( T \). Let \( \mathcal{E} \) be an \( \mathcal{H} \)-equivariant flat vector bundle \( \mathcal{E} \) over \( T \), and \( \mathcal{P} \) the Sergyescu’s orientation sheaf. Then we have the following.

(i) We have \( \dim H^i_{\text{inv}}(T/\mathcal{H}; \mathcal{E}) < \infty \).

(ii) If \( M \) is connected and \( \text{rank} \mathcal{E} = 1 \), then \( H^0_{\text{inv}}(T/\mathcal{H}; \mathcal{E}) \) is isomorphic to \( \mathbb{R} \) or 0.

(iii) If \( M \) is connected, then \( H^0_{\text{inv}}(T/\mathcal{H}; \mathcal{P}) \cong \mathbb{R} \).

(iv) We have \( H^i_{\text{inv}}(T/\mathcal{H}; \mathcal{E}) \cong H^0_{\text{inv}}(T/\mathcal{H}; \xi^\# \otimes \mathcal{P}^\#)^\ast \).

Corollary 1.3 follows from Corollary 6.6 since the invariant cohomology of the holonomy pseudogroup of a foliated manifold \( (M, \mathcal{F}) \) is naturally isomorphic to the basic cohomology of \( (M, \mathcal{F}) \) by definition. In turn, Corollary 6.6 follows from Theorems 1.1 and 6.2 and the following result on the invariant cohomology of complete Riemannian pseudogroups.

**Theorem 6.7 ([(i) and (iv)] Proposition 3.2.9.1)). Under the same assumption with Corollary 6.6, the following holds:**

(i) We have \( \dim H^i_{\text{inv}}(\mathcal{H}; \mathcal{E}) < \infty \).

(ii) If \( T/\mathcal{H} \) is connected and \( \text{rank} \mathcal{E} = 1 \), then \( H^0_{\text{inv}}(\mathcal{H}; \mathcal{E}) \) is isomorphic to \( \mathbb{R} \) or 0.

(iii) If \( T/\mathcal{H} \) is connected, then \( H^0_{\text{inv}}(\mathcal{H}; \mathcal{P}) \cong \mathbb{R} \).

(iv) There is an isomorphism \( H^i_{\text{inv}}(\mathcal{H}; \mathcal{E}) \cong H^0_{\text{inv}}(\mathcal{H}; \xi^\# \otimes \mathcal{P}^\#)^\ast \).

Theorem 6.7 (i) and (ii) are proved by the pseudogroup version of the argument due to El Kacimi-Alaoui-Sergiescu-Hector [EKASH83]. Haefliger’s proof of Theorem 6.7 (iii) and (iv) is the pseudogroup version of the proof of Sergiescu [So85 Théorème I].
7. Strongly tenseness of foliated manifolds

7.1. Definitions and Rummler’s formula. Recall that the mean curvature form of a foliated Riemannian manifold \((M, \mathcal{F}, g)\) is the one-form \(\kappa\) on \(M\) such that \(\kappa_x\) is the mean curvature form of the leaf of \(\mathcal{F}\) through \(x\) (see, for example, [CC00, Section 10.5] for formulas of \(\kappa\) in terms of \(g\)). We recall the following classical definition.

**Definition 7.1.** A foliated manifold \((M, \mathcal{F})\) is called taut (resp., tense) if \(M\) admits a metric \(g\) such that the mean curvature form of \((M, \mathcal{F}, g)\) is trivial (resp., basic).

The following notion was introduced in [NRP12] (see Section 1.3 for the motivation).

**Definition 7.2.** A foliated manifold \((M, \mathcal{F})\) is called strongly tense if \(M\) admits a metric \(g\) such that the mean curvature form of \((M, \mathcal{F}, g)\) is basic and closed.

Let \((M, g)\) be a Riemannian manifold with an oriented \(p\)-dimensional foliation \(\mathcal{F}\). Recall that the characteristic form \(\chi\) of \((M, \mathcal{F}, g)\) is defined to be the unique \(p\)-form on \(M\) such that \(\ker \chi = (T\mathcal{F})^\perp\) and \(\chi|_{\Lambda^p T\mathcal{F}}\) is the oriented leafwise volume form of norm one at each point on \(M\). We will say that \(\chi \in \Omega^p(M)\) is a characteristic form of \((M, \mathcal{F})\) if there exists a metric \(g\) on \(M\) such that \(\chi\) is the characteristic form of \((M, \mathcal{F}, g)\). By the Rummler’s formula [Ru79], the mean curvature form \(\kappa\) of \((M, \mathcal{F}, g)\) is given by

\[
\kappa(Y) = -d\chi(Y, E_1, \ldots, E_p),
\]

for any vector field \(Y\) on \(U\), where \(\{E_1, \ldots, E_p\}\) is a local oriented orthonormal frame of \(T\mathcal{F}|_U\).

7.2. Rummler-Sullivan type characterization of strongly tenseness. Let \(M\) be a manifold with a \(p\)-dimensional foliation \(\mathcal{F}\). Let \((\mathcal{E}, \nabla)\) be a flat vector bundle over \(M\). We recall the following terminologies.

**Definition 7.3.** We call \(\omega \in \Omega^*(M; \mathcal{E})\) an \(\mathcal{F}\)-trivial form if \(\iota(X_1) \circ \iota(X_2) \circ \cdots \circ \iota(X_p) \omega = 0\) for any vector fields \(X_1, X_2, \ldots, X_p\) tangent to \(\mathcal{F}\). We call \(\omega \in \Omega^*(M; \mathcal{E})\) relatively \(\mathcal{F}\)-closed if \(d\mathcal{F} \omega\) is \(\mathcal{F}\)-trivial.

In terms of \(\mathcal{F}\)-trivial forms, \((16)\) is expressed as follows:

**Lemma 7.4.** For a foliated Riemannian manifold \((M, \mathcal{F}, g)\) with characteristic form \(\chi\) and mean curvature form \(\kappa\), there exists an \(\mathcal{F}\)-trivial form \(\beta\) such that

\[
\mathcal{D}\chi = -\kappa \wedge \chi + \beta.
\]

In this section, we will use the following terminologies, which are not common: For a possibly non-orientable foliation \(\mathcal{F}\) on a Riemannian manifold \((M, g)\) and any connected simply-connected open subset \(U\) of \(M\), we have two characteristic forms \(\chi\) on \((U, \mathcal{F}|_U, g|_U)\) depending the choice of the orientations of \(\mathcal{F}|_U\). We call such \(\chi\) a local characteristic form of \((M, \mathcal{F}, g)\). For a line bundle \(\mathcal{E}\) over \(M\), a section of \((\mathcal{E} - 0)/\{\pm 1\}\) is called a nowhere vanishing double section of \(\mathcal{E}\).

**Proposition 7.5.** For a foliated Riemannian manifold \((M, \mathcal{F}, g)\), the following are equivalent:

(i) The mean curvature form \(\kappa\) of \((M, \mathcal{F}, g)\) is basic and closed.

(ii) There exists a topologically trivial flat line bundle \((\mathcal{E}, \nabla)\) over \(M\) with a nowhere vanishing section \(\sigma\) such that \(\sigma|_L\) is parallel for every leaf \(L\) of \(\mathcal{F}\) and \(\chi \otimes \sigma\) is relatively \(\mathcal{F}\)-closed for any local characteristic form \(\chi\) of \((M, \mathcal{F}, g)\).
(iii) There exists a flat line bundle \((E, \nabla)\) over \(M\) with a nowhere vanishing double section \(\sigma\) such that \(\sigma|_L\) is parallel for every leaf \(L\) of \(F\) and \(\chi \otimes \sigma\) is relatively \(F\)-closed for any local characteristic form \(\chi\) of \((M, F, g)\).

Proof. Let \((E, \nabla)\) be a flat line bundle over \(M\) with a nowhere vanishing double section \(\sigma\) such that \(\sigma|_L\) is parallel for every leaf \(L\) of \(F\). On a connected simply-connected open subset \(U_i\) of \(M\), we take a function \(f_i\) on \(U_i\) so that \(\frac{1}{f_i}\sigma\) is a parallel section of \((E|_{U_i}, \nabla|_{U_i})\). Note that \(f_i\) is basic. We get

\[
d\nabla \sigma = df_i \wedge \frac{1}{f_i} \sigma = d \log |f_i| \wedge \sigma.
\]

Let \(p = \dim F\). By Rummler’s formula (Lemma 7.4), for any of two local characteristic form \(\chi\) of \((M, F, g)\) defined on \(U_i\), there exists an \(F\)-trivial form \(\beta\) which satisfies (17). Then, by (18), we get

\[
d\nabla (\chi \otimes \sigma) = d\chi \otimes \sigma + (-1)^p \chi \wedge d\nabla \sigma
\]

\[
= (d\chi + (-1)^p \chi \wedge d \log |f_i|) \otimes \sigma
\]

\[
= (-\kappa + d \log |f_i|) \wedge \chi \otimes \sigma + \beta \otimes \sigma.
\]

Clearly (ii) implies (iii). Here (iii) implies (i) by (19). Indeed, if \(\chi \otimes \sigma\) is relatively \(F\)-closed, then, by (19), we get \(-\kappa + d \log |f_i| = 0\), which implies that \(\kappa\) is basic and closed.

We show (ii) by using (i). Assume that \(\kappa\) is basic and closed. We regard \(\kappa\) as a multiplicative homomorphism \(\kappa : \pi_1 M \to \mathbb{R}_+ : \gamma \mapsto \exp(f, \kappa)\). Let \((E, \nabla)\) be the flat line bundle over \(M\) whose holonomy homomorphism is \(\kappa\); namely, \(E\) is the quotient of \(M \times \mathbb{R}\) by the diagonal \(\pi_1 M\)-action, where \(u : \tilde{M} \to M\) is the universal cover of \(M\) and the \(\pi_1 M\)-action on \(\mathbb{R}\) is given by \(\kappa\). We consider a function \(\zeta : \tilde{M} \to \mathbb{R}\) such that \(d\zeta = u^* \kappa\). Then the graph of \(\zeta\) in \(\tilde{M} \times \mathbb{R}\) is invariant under the diagonal \(\pi_1 M\)-action, and its quotient in \(\tilde{E}\) is the image of a nowhere vanishing global section \(\sigma\) of \(\tilde{E}\). Since \(\kappa\) is basic, \(\sigma|_L\) is parallel for every leaf \(L\) of \(F\). We cover \(M\) with open sets \(U_i\) of \(M\) and take \(f_i \in C^\infty(U_i)\) like in the first paragraph of the proof. Then \(\{d \log |f_i|\}\) defines a basic closed one-form, which is cohomologous to \(\kappa\), and we get a basic function \(h\) such that \(dh = \kappa - \kappa'\). Then (19) gives

\[
d\nabla (\chi \otimes \sigma) = (dh \wedge \chi + \beta) \otimes \sigma.
\]

It follows that \(\chi \otimes e^h \sigma\) is relatively \(F\)-closed, which concludes the proof. \(\square\)

Remark 7.6. As described in the last proof, there exists a correspondence between closed one-forms on \(M\) and topologically trivial flat line bundles over \(M\) with nowhere vanishing sections. Here a basic closed one-form corresponds to a topologically trivial flat line bundles over \(M\) with a nowhere vanishing section \(\sigma\) such that \(\sigma|_L\) is parallel for each leaf \(L\) of \(F\).

Let us define the following.

**Definition 7.7.** We say that \((M, F)\) is strongly tense with \((E, \nabla)\) if the condition of Proposition 7.5 (iii) is satisfied.

The following is proved by an argument analogous to [AL92, Lemma 6.3].

**Proposition 7.8.** Let \((M, F) \to (M', F')\) be a finite covering map of foliated manifolds. Let \(\Gamma\) be the deck transformation group and \(E\) a \(\Gamma\)-equivariant flat line bundle over \(M\). Then \((M, F)\) is strongly tense with \((E, \nabla)\) if and only if \((M', F')\) is strongly tense with the quotient of \((E, \nabla)\) by \(\Gamma\).
Proof. The “if” part is trivial. We show the “only if” part. It suffices to prove the case where \( F \) is orientable. We assume that \( F \) is orientable and \((M,F)\) is strongly tense with \((\mathcal{E},\nabla)\). By Proposition 7.5, there exists a nowhere vanishing double section \( \sigma \) of \( \mathcal{E} \) and a characteristic form \( \chi \) which satisfies the condition of Proposition 7.5(ii). We define \( \varepsilon_1 : \Gamma \to \{-1,1\} \) by \( \varepsilon_1(\gamma) = 1 \) if \( \gamma \) preserves the orientations of the fibers of \( \mathcal{E} \) and \( \varepsilon_1(\gamma) = -1 \) otherwise. Take a nowhere vanishing double section \( \sigma_\gamma \) of \( \mathcal{E} \) by
\[
\sigma_\gamma = \sum_{\gamma \in \Gamma} \varepsilon_1(\gamma) \gamma^* \sigma,
\]
where the sum is taken in each branch of double sections. Then \( \sigma_\gamma \) is a nowhere vanishing double section of \( \mathcal{E} \) such that \( g^* \sigma_\gamma = \varepsilon_1(g) \sigma_\gamma \) and \( \sigma_\gamma|_L \) is parallel for each leaf \( L \) of \( F \). We have a positive basic function \( f \) on \((M,F)\) such that \( \sigma = f \sigma_\gamma \).

We define \( \varepsilon_2 : \Gamma \to \{-1,1\} \) by \( \varepsilon_2(\gamma) = 1 \) if \( \gamma \) preserves the orientations of \( F \) and \( \varepsilon_2(\gamma) = -1 \) otherwise. Let
\[
\chi' = \sum_{\gamma \in \Gamma} \varepsilon_2(\gamma) \gamma^* (f \chi).
\]
Then \( \chi' \otimes \sigma_\gamma \) is relatively \( F \)-closed. Here \( \chi' \) is positive along the foliation but it may not be a characteristic form in general. Thus we will take its Sullivan’s purification \cite{Su79}. Let \( \varpi : C^\infty(TM) \to C^\infty(TF) \) be the \( C^\infty \)-linear projection map determined by
\[
\varpi((\omega(X))(\chi')|_{\Lambda^p TF}) = (\omega(X)\chi')|_{\Lambda^{p-1} TF},
\]
which is well-defined by the fact that \( \chi'|_{\Lambda^p TF} \) is a leafwise volume form of \( F \). Let \( \chi_f = \varpi^* (\chi'|_{\Lambda^p TF}) \). By the naturality of the purification, we get \( \gamma^* \chi_f = \varepsilon_2(\gamma) \chi_\Gamma \).

Since \( \chi' \otimes \sigma_\gamma \) is relatively \( F \)-closed and it is easy to see that
\[
\varpi(X_1) \circ \varpi(X_2) \circ \cdots \circ \varpi(X_p) \chi_\Gamma = \varpi(X_1) \circ \varpi(X_2) \circ \cdots \circ \varpi(X_p) \chi_f
\]
for any vector fields \( X_1, X_2, \ldots, X_p \) tangent to \( F \), it follows that \( \chi_\Gamma \otimes \sigma_\gamma \) is also relatively \( F \)-closed. Let \( g \) be a Riemannian metric on \( M \) whose characteristic form is \( \chi_f \). Let \( \tilde{g} = \sum_{\gamma \in \Gamma} \gamma^* g \) and \( g_\Gamma = \frac{\tilde{g}}{|\chi_f|} \). Then \( g_\Gamma \) is a \( \Gamma \)-invariant metric whose characteristic form is \( \chi_\Gamma \). Thus \( g_\Gamma \) induces a strongly tense metric on \((M',F')\), which concludes the proof by Proposition 7.5. \( \square \)

Remark 7.9. It is not clear that, for a finite covering map \((M,F) \to (M',F')\) of foliated manifolds, if \((M,F)\) is tense, then \((M',F')\) is tense, while the inverse is trivial.

7.3. The integration along foliations. Let \( M \) be a manifold with an oriented foliation \( F \) of dimension \( p \). Let \((\mathcal{E},\nabla)\) be a flat vector bundle over \( M \). As in the case where \( F \) is trivial due to Haefliger \cite[Theorem 3.1]{Hae80}, the integration along \( F \) of differential forms with values in \( \mathcal{E} \),
\[
\int_F : \Omega_c^\bullet(M;\mathcal{E}) \to \Omega_c^{\bullet-p}(\text{Tr} F;\mathcal{E}), \tag{20}
\]
is defined as follows. Take a total transversal \( T \) and an open covering \( \{U_i\}_{i \in I} \) of \( M \) so that
(i) \( U_i \) is simply connected,
(ii) \( \{U_i\}_{i \in I} \) is locally finite,
(iii) \( T \cap U_i \) intersects the leaves of \( F|_{U_i} \) exactly once and
(iv) the projection \( \pi_i : U_i \to T \cap U_i \) whose fibers are the leaves of \( F|_{U_i} \) is a disk bundle.
Here $E|_{U_i}$ and $E|_{T\cap U_i}$ are trivial flat vector bundles for each $i$. So
\[ \Omega^k(U_i; E) \cong \Omega^0(U_i)^{\otimes k}, \quad \Omega^k(T \cap U_i; E|_{T \cap U_i}) \cong \Omega^0(T \cap U_i)^{\otimes k}, \]
where $k = \text{rank } E$. Then $f_{\pi_i}: \Omega^0(U_i; E) \to \Omega^0(U_i; F)$ is defined by the $k$-th direct sum of the integration $\Omega^k(U_i) \to \Omega^k(T \cap U_i)$ along $\pi_i$. Then (20) is defined by $f_{\pi_i} = \sum_{i \in I} f_{\pi_i}(\rho_i \alpha)$ for $\alpha \in \Omega^*(M; E)$, where $\{\rho_i\}_{i \in I}$ is a partition of unity subordinated to $\{U_i\}_{i \in I}$. We can show that $f_{\pi}$ is a well-defined surjective continuous homomorphism which commutes with $d\nu$ as the case where $E$ is trivial [Ha80, Theorem 3.1].

A twisted version of the proof of [Ha80, Theorem in Section 3.2] gives the following.

**Lemma 7.10.** The kernel of $f_{\pi}$ is equal to the subspace of $\Omega^*(M; E)$ generated by $F$-trivial forms and the differential of $F$-trivial forms.

Theorem 1.6 is proved by a twisted version of the proof of [Ha80, Theorem in Section 4.2] with Proposition 7.5 and Lemma 7.10 (See Theorem 9.9 for a similar argument).

**8. Domínguez’s tenseness theorem via Haefliger cohomology**

In this section, we will prove the following result.

**Theorem 8.1.** A compact connected manifold $M$ with a Riemannian foliation $F$ is strongly tense with a flat line bundle $(E, \nabla)$ over $M$ if and only if $\hat{H}^0(\text{Tr } F; E) \cong R$.

**Remark 8.2.** See Definition 1.4 for the definition of basic flat line bundles. Recall that if $(M, F)$ is strongly tense with $(E, \nabla)$, then $(E, \nabla)$ is basic by definition.

**Remark 8.3.** By Theorems 6.2 and 6.7-(iii), under the assumption of Theorem 8.1, $\hat{H}^0(\text{Tr } F; E) \cong R$ if and only if $\hat{H}^0(\text{Tr } F; E)$ is nontrivial.

As indicated in the following remark, this result is essentially equivalent to a theorem of Domínguez [Do98]. So our purpose in this section is to give an alternative proof of Domínguez’s theorem in terms of Haefliger cohomology. The “only if” part of Theorem 8.1 is not difficult. We will prove the “if” part of Theorem 8.1 by using Theorems 8.3 and 8.4. We will not use a theorem of Álvarez López [AL92, Corollary 3.5] on the closedness of the basic component of the mean curvature form of bundle-like metrics, which was used in the original proof due to Domínguez.

**Remark 8.4.** Let $M$ be a compact manifold with a Riemannian foliation $F$. By Theorems 6.2 and 6.7-(iii), the Sergiescu’s orientation sheaf $\mathcal{P}$ of $(M, F)$ satisfies $\hat{H}^0(\text{Tr } F; \mathcal{P}) \cong R$ and that any Riemannian foliation on a compact manifold is tense. But, by results of Álvarez López [AL92, Theorem 5.2] and [AL96, Theorem 1.1], if $\hat{H}^0(\text{Tr } F; E)$ and $\hat{H}^0(\text{Tr } F; \mathcal{P})$ are nontrivial, then the absolute value of the holonomy homomorphism $\pi_1 M \to \text{Aut}(R)$ of $(E, \nabla)$ is equal to those of the determinant line bundle of Molino’s commuting sheaf and Sergiescu’s orientation sheaf $\mathcal{P}$. In particular, there is essentially unique $E$ such that $(M, F)$ is strongly tense with $(E, \nabla)$. In this sense, Theorem 8.1 is essentially equivalent to Domínguez’s theorem.

First we will prove the “only if” part, which essentially follows only from Theorems 6.2 and 6.7-(iii). We note the following fact, which directly follows from the fact that $F$ is Riemannian and $E$ is topologically trivial as an $H$-equivariant line bundle over $T$. 


Lemma 8.5. Let $M$ be a manifold with a Riemannian foliation $F$ and a basic flat line bundle $(E, \nabla)$. Let $T$ be a total transversal of $(M, F)$. For any point $x \in T$, there exists a nonnegative $n$-form $\omega \in \Omega^n(T; \mathcal{E}^* \otimes \mathcal{O}_T)$ which is nonzero at $x$, where the nonnegativity of $\omega$ is defined in Definition 7.8 regarding $\omega$ as a global section of a topologically trivial line bundle over $T$.

Proof of the “only if” part of Theorem 8.7. By definition of strongly tenseness with $\mathcal{E}$, there exists a total transversal $T$ of $(M, F)$ and $\xi \in \Omega^0(T; \mathcal{E})$ so that the condition of Theorem 8.1(ii) is satisfied. Then $\xi$ generates a 0-cocycle in $H^0_0(\text{Tr } F; \mathcal{E})$ and hence in $H^0_c(\text{Tr } F; \mathcal{E})$. By Theorem 8.6, the canonical map $\tilde{H}^0_c(\text{Tr } F; \mathcal{E}) \to H^0_{\text{inv}}(\text{Hol}(T); \mathcal{E}^* \otimes \mathcal{O}_T)^*$ is an isomorphism. By Lemma 8.3 there exists $\omega \in \Omega^0(T; \mathcal{E}^* \otimes \mathcal{O}_T)^*$ such that $\Psi(\xi)(\omega) \neq 0$. Thus $\tilde{H}^0_c(\text{Tr } F; \mathcal{E})$ is nontrivial. Finally, by Theorems 8.2 and 8.7(iii), we get $\tilde{H}^0_c(\text{Tr } F; \mathcal{E}) \cong \mathbb{R}$. \hfill $\square$

We will use the following observation, which is proved by using Theorem 8.2.

Lemma 8.6. Let $\mathcal{H}$ be a complete Riemannian pseudogroup on a manifold $T$ whose orbit space is connected. Let $(\mathcal{E}, \nabla)$ be an $\mathcal{H}$-equivariant flat line bundle over $T$ which admits a nowhere vanishing $\mathcal{H}$-invariant section. Assume that $H^0_0(T/\mathcal{H}; \mathcal{E}) \cong \tilde{H}^0_0(T/\mathcal{H}; \mathcal{E}) \cong \mathbb{R}$. Then, for any relatively compact subset $K$ of $T$, there exists a nonnegative section $\xi \in \Omega^0_0(T; \mathcal{E})$ such that $K \subset \text{supp}_+ \xi$ and $d_\nabla \xi = 0$ in $\Omega^1_0(T/\mathcal{H}; \mathcal{E})$.

Proof. We fix a nowhere vanishing $\mathcal{H}$-invariant section $\sigma$ of $\mathcal{E}$ to identify $\mathcal{E}$ with the trivial line bundle over $T$. Similarly we identify $\Omega^n(T; \mathcal{E}^* \otimes \mathcal{O}_T)$ with functions on $T$ by using $\sigma$. Then, by the connectivity of $T/\mathcal{H}$, $H^0_{\text{inv}}(\mathcal{H}; \mathcal{E}^* \otimes \mathcal{O}_T)$ is identified with $\mathbb{R}$. By Theorem 8.2 and the hypothesis, $H^0_0(T/\mathcal{H}; \mathcal{E})$ and $\tilde{H}^0_0(T/\mathcal{H}; \mathcal{E})$ are identified with $\mathbb{R}$. For any section $\xi$ of $\mathcal{E}$, let $\text{supp}_+ \xi = \{x \in T \mid (\xi(x) > 0)\}$. Take $\xi \in \Omega^0_0(T; \mathcal{E})$ which represents a positive class in $H^0_0(T/\mathcal{H}; \mathcal{E})$ and $\tilde{H}^0_0(T/\mathcal{H}; \mathcal{E})$. By Theorem 8.2 for any $\omega \in \Omega^n(T; \mathcal{E}^* \otimes \mathcal{O}_T)^*$ which is positive, the product $\xi \omega$ is positive. By Lemma 8.3 and this fact, for any point $x \in T$, there exist a finite subset $\{h_{x,j}\}$ of $\mathcal{H}$ and a finite subset $\{\alpha_{x,j}\}_j$ of $\Omega^1_0(T; \mathcal{E})$ which satisfy the following conditions:

(i) $x \in \text{Dom } h_{x,j}$,

(ii) $\text{supp } \alpha_{x,j} \subset \text{Im } h_{x,j}$ and

(iii) $(\text{supp}_+ \xi) \cup \{x\} \subset \text{supp}_+ \xi + \sum_j (h_{x,j}^* \alpha_{x,j} - \alpha_{x,j})$.

For each $x \in T$, take an open neighborhood $V_x$ of $x$ in $T$ so that $V_x \subset \text{supp}_+ \xi + \sum_j (h_{x,j}^* \alpha_{x,j} - \alpha_{x,j})$.

By the compactness of $(\text{supp}_+ \xi \cup \text{supp}_+ \xi) \cup K$, we can take finite points $\{x(k)\}_{k=1}^n$ in $T$ so that $(\text{supp}_+ \xi \cup \text{supp}_+ \xi) \cup K \subset \bigcup_{k=1}^n V_{x(k)}$. Then we can inductively construct $\xi$ which satisfies the given conditions. \hfill $\square$
compact and noncompact foliated manifolds. We collect some of them in this section.

The tautness of compact foliated manifolds is a transverse property by [Ha80, Corollary 1 in Section 4.2]; namely, tautness is determined by the equivalence class of the holonomy pseudogroup. But it is not true for the noncompact case. Let $G$ be a Lie group.

**Proposition 9.1.** If the holonomy pseudogroup of any $G$-Lie foliation is finitely generated, then it is equivalent to a holonomy pseudogroup of a taut foliation on a noncompact manifold.

**Proof.** If the holonomy pseudogroup of any $G$-Lie foliation is finitely generated, then it is equivalent to the pseudogroup $H^\Gamma$ generated by the right action of a finitely generated subgroup $\Gamma$ of $G$ (see [Me97, Proposition 2.1]). In turn, $H^\Gamma$ is the holonomy pseudogroup of the suspension foliation of a $G$-bundle over a closed surface $\Sigma$. Indeed, if $\Gamma$ is generated by $m$ elements, then $\Sigma$ can be taken as the oriented closed surface of genus $2m$ and the $\pi_1 \Sigma$-action on $G$ is given by the composite of $\pi_1 \Sigma \to F_m \to \Gamma$. Since any suspension foliation is taut, the proof is concluded. □

So, the realization problem of pseudogroups with taut foliations up to equivalence is not well posed for noncompact foliated manifolds. Nevertheless we will see that tautness or strongly tenseness of noncompact foliated manifolds are invariant under a natural equivalence (see Remark 9.11).

By Corollary 1.13 for $G$-Lie foliations $F$ with dense leaves, we have $\hat{H}^0(\text{Tr}F) = 0$ if $G$ is not unimodular. Combining with Theorem 1.1 and Proposition 9.1, we get the following simple but remarkable examples.

**Example 9.2.** There are taut Riemannian foliations $F$ on noncompact manifolds $M$ such that $\hat{H}^0(\text{Tr}F) = 0$.

These examples imply that the easier direction of the Masa’s characterization of tautness of Riemannian foliations on compact manifolds [Ma92, Minimality theorem] cannot be directly generalized to noncompact manifolds.

As already mentioned in the introduction, a tense metric on a closed manifold with a Riemannian foliations is always strongly tense by [KT83b, Eq. 4.4], while the example of Cairns-Escobales [CE97, Example 2.4] shows that it is not true for the noncompact case. There is a further difference in this direction. The cohomology class of the mean curvature form of a tense metric on a Riemannian foliation on a compact manifold, or more generally, the basic component of the mean curvature form $\kappa$ of a Riemannian foliation on a compact manifold is unique by [AL92, Theorem 5.2]. But this result is not true in the noncompact case as the following example shows.

**Proposition 9.3.** Let $M = \mathbb{R} \times S^1$ and $F$ the foliation on $M$ defined by the trivial $\mathbb{R}$-bundle given by the second projection $\mathbb{R} \times S^1 \to S^1$. For any $t \in \mathbb{R}^\times$, there exists a strongly tense bundle-like metric on $(M, F)$ whose mean curvature form $\kappa$ satisfies $\int_{S^1} \kappa = t$.

**Proof.** For $t \in \mathbb{R}^\times$, let $\chi_t$ be a flat $\mathbb{R}$-connection form whose holonomy homomorphism is $h_t : \pi_1 S^1 \to \text{Aut}(\mathbb{R})$ determined by $h_t(\gamma) = t$, where $\gamma$ is a generator of $\pi_1 S^1$. Then, by Rummler’s formula (16), $\chi_t$ is the characteristic form of a strongly tense metric on $(M, F)$ such that the cohomology class of the mean curvature form $\kappa$ is given by $\int_{S^1} \kappa = \log |t|$. □

In [No12], it was proved that the Álvarez class is continuous under deformations of Riemannian foliations on closed manifolds. This is not true for noncompact manifolds as the following example shows.
The covering \(U\) of \((M,F)\) consisting of the pairs \((U_i,F|_{U_i})\) such that \(\text{supp}\,\beta_{ij} \subset \text{Im}\,h_{ij}\) is nontrivial only at \(t=0\).

The tautness of Riemannian foliations on closed manifolds is invariant under deformations under certain topological assumptions \([No12, \text{Corollary 4}]\). It is not known if a taut Riemannian foliation can be deformed to a nontaut Riemannian foliation on a closed manifold. The last example provides a simple example of deformation of a taut Riemannian foliation to a nontaut one on a noncompact manifold.

### 9.2. Modified Haefliger cohomology

In this section, we define a generalization of Haefliger cohomology to noncompact foliated manifolds, which coincides with the original version for compact foliated manifolds.

We consider the following refined version of total transversals.

**Definition 9.5.** Let \(T\) be a total transversal of a foliated manifold \((M,F)\). We call \(T\) fine if there exists a decomposition \(T = \bigcup_{i \in I} T_i\) to open closed subsets \(T_i\) and a locally finite open covering \(U = \{U_i\}_{i \in I}\) of \(M\) such that

1. \(U_i\) is relatively compact in \(M\),
2. If \(U_i \cap U_j\) is non-empty, then it is connected and simply connected.
3. \(T_i\) is a total transversal of \((U_i,F|_{U_i})\) which intersects each leaf of \((U_i,F|_{U_i})\) exactly once and
4. the map \(\pi_i: U_i \to T_i\) which collapses leaves of \(F|_{U_i}\) is a disk bundle.

The covering \(U\) is called a **covering compatible with** \(T\). For a fine total transversal \(T\) of \((M,F)\), an open subset \(K\) of \(T\) is called **exhausting** if \(T_i \cap K\) is relatively compact in \(T_i\) and \(\bigcup_{i \in I} \pi_i^{-1}(T_i \cap K) = M\).

It is easy to see that any foliations on possibly noncompact manifolds admits a fine total transversal.

We introduce the corresponding notion of Haefliger cohomology. Let \((M,F)\) be a foliated manifold. Let \(T = \bigcup_{i \in I} T_i\) be a fine total transversal of \((M,F)\) with compatible covering \(U = \{U_i\}_{i \in I}\). Let \((\mathcal{E},\nabla)\) be a flat vector bundle over \(M\). Let

\[
\Omega^\bullet_{cc}(T;\mathcal{E}) := \prod_{i \in I} \Omega^\bullet_{cc}(T_i;\mathcal{E}).
\]

We topologize \(\Omega^\bullet_{cc}(T;\mathcal{E})\) with the product of the LF-topology on each \(\Omega^\bullet_{cc}(T_i;\mathcal{E})\). Let \(J\) be the subset of \(I \times I\) consisting of the pairs \((i,j)\) such that \(U_i \cap U_j \neq \emptyset\). For \((i,j) \in J\), the holonomy map from an open set of \(T_i\) to an open set of \(T_j\) is denoted by \(h_{ij}\), which is unique by Definition \([Ca84, \text{Definition 4}(ii)]\). Let \(\Xi^\bullet_H\) be the subspace of \(\Omega^\bullet_{cc}(T;\mathcal{E})\) defined by

\[
(21) \quad \Xi^\bullet_H := \left\{ \sum_{(i,j) \in J} (h_{ij}\beta_{ji} - \beta_{ji}) \mid \beta_{ji} \in \Omega^\bullet_{cc}(T_j;\mathcal{E}) \text{ such that } \text{supp}\,\beta_{ji} \subset \text{Im}\,h_{ij} \right\}.
\]

As in the case of the Haefliger cohomology, we get the following.

**Lemma 9.6.** For a foliated manifold \((M,F)\) with a fine total transversal \(T\), the isomorphism class of \(\Omega^\bullet_{cc}(T;\mathcal{E})/\Xi_{\text{Hol}(T)}\) as a topological differential complex is independent of the choice of \(T\).
So, for any fine total transversal $T$ with compatible covering $U$, let
\[
\Omega^c_{cc}(\text{Tr } F; \mathcal{E}) := \Omega^c_{cc}(T; \mathcal{E})/\mathbb{Z}_{\text{Hol}(T)},
\]
\[
H^c_{cc}(\text{Tr } F; \mathcal{E}) := H^c\left(\Omega^c_{cc}(T; \mathcal{E})/\mathbb{Z}_{\text{Hol}(T)}, d\tau\right),
\]
which are well-defined by the last lemma.

**Remark 9.7.** Haefliger [Ha80] Corollary in Section 3.3] proved that $\Omega^p_{cc}(\text{Tr } F)$ is isomorphic to the compact supported leafwise cohomology $H^p_{cc}(F)$, where $p = \dim F$. Note that $\Omega^p(M)/\{F\text{-trivial forms}\} \cong C^\infty(\bigwedge^p T^* F)$ and that any $(p - 1)$-forms are $F$-trivial. Then, Lemma 9.8 below implies that $\Omega^p_{cc}(\text{Tr } F)$ is isomorphic to the leafwise cohomology $H^p(F)$.

9.3. **Haefliger type characterization in the noncompact case.** Let $M$ be a manifold with an oriented $p$-dimensional foliation $F$ and a flat vector bundle $(\mathcal{E}, \nabla)$. We will extend Theorem 1.6 to the case where $M$ is possibly noncompact. Let $T = \bigsqcup_{i \in I} T_i$ be a fine total transversal with compatible covering $\{U_i\}_{i \in I}$. Like in the compact case, the integration along $F$,
\[
\int_F : \Omega^c(M; \mathcal{E}) \to \Omega^c_{cc}(\text{Tr } F; \mathcal{E}),
\]
is well-defined by $\int_F \alpha = (\int_{T_i} \rho_i \alpha)_{i \in I}$, where $\pi_i : U_i \to T_i$ is the projection whose fibres are plaques of $F$ and $\{\rho_i\}_{i \in I}$ is a partition of unity on $M$ subordinated to $\{U_i\}_{i \in I}$. It is easy to see that $\int_F$ is a surjective map which commutes with $d\tau$.

**Lemma 9.8.** The kernel of $\int_F$ is equal to the subspace of $\Omega^c(M; \mathcal{E})$ generated by $F$-trivial forms and the differential of $F$-trivial forms.

The following result is proved by a twisted version of the proof of [Ha80 Theorem in Section 4.2] with Proposition 7.5 and Lemma 9.8.

**Theorem 9.9.** Let $M$ be a manifold with an oriented foliation $F$ and $(\mathcal{E}, \nabla)$ a topologically trivial basic flat line bundle over $M$ whose restriction to each leaf of $F$ is trivial. Then, the following are equivalent:

(i) $(M, F)$ is strongly tense with $(\mathcal{E}, \nabla)$.

(ii) There exists a fine total transversal $T$ of $(M, F)$ with exhausting subset $K$ and a nonnegative section $\xi \in \Omega^c_{cc}(T; \mathcal{E})$ such that $K \subset \text{supp}_+ \xi$ and $d\tau \xi = 0$ in $\Omega^c_{cc}(\text{Tr } F; \mathcal{E})$.

(iii) For any fine total transversal $T$ of $(M, F)$ with exhausting subset $K$, there exists a nonnegative section $\xi \in \Omega^c_{cc}(T; \mathcal{E})$ such that $K \subset \text{supp}_+ \xi$ and $d\tau \xi = 0$ in $\Omega^c_{cc}(\text{Tr } F; \mathcal{E})$.

**Proof.** Clearly (iii) implies (ii). We will show that (i) implies (iii). By Proposition 7.5, there exists a nowhere vanishing section $\sigma$ such that $\sigma|_L$ is parallel for every leaf $L$ of $F$ and $\chi \otimes \sigma$ is relatively $F$-closed. Let $T = \bigsqcup_{i \in I} T_i$ be a fine total transversal of $(M, F)$ with compatible covering $\{U_i\}_{i \in I}$. Let $K$ be the projection which collapses the leaves of $F|_{U_i}$. We take a partition of unity on $M$ subordinated to $\{U_i\}_{i \in I}$ so that $K \cap T_i \subset \pi_i(\text{supp}_+ \rho_i)$. Let $\xi = (\int_{T_i} \rho_i(\chi \otimes \sigma))_{i \in I}$. Since $d\tau|_U$ commutes with $f_x$, here $\xi$ satisfies the conditions in (iii). We will show (i) by using (ii). Let $T = \bigsqcup_{i \in I} T_i$ be the fine total transversal of $(M, F)$ satisfying (ii). Let $\{U_i\}_{i \in I}$ be the compatible covering and $\pi_i : U_i \to T_i$ be the projection which collapses the leaves of $F|_{U_i}$. For each $i$, we take a closed $p$-form $\eta_i$ on $U_i$ compatible with the orientation of $F$ so that $\int_{T_i} \eta_i = 1$ and $\bigcup_{i \in I} \text{supp}_+ \eta_i = M$, where $\text{supp}_+ \eta_i = \{x \in M \mid \eta_i(x) \neq 0\}$. Let $\eta = \sum_i (\eta_i \otimes \pi_i^\ast(\xi|_{T_i}))$. Then $\int_F d\tau \eta = d\tau \xi = 0$ in $\Omega^c_{cc}(\text{Tr } F; \mathcal{E})$. By Lemma 9.8, there exist two $F$-trivial forms $\beta_1$ and $\beta_2$ such that $d\tau(\eta + \beta_1) = \beta_2$. Since $(\mathcal{E}, \nabla)$ is
basic and topologically trivial, it admits a nowhere vanishing section \( \sigma \in \Omega^0(M; E) \) such that \( \sigma|_L \) is parallel for every leaf \( L \) of \( F \) (see Remark 7.6). Let \( \chi \) be the characteristic form of \((M,F)\) such that \( \chi \otimes \sigma = \eta + \beta_i \). Then \( \chi \) and \( \sigma \) satisfy the conditions in Proposition 7.8. So \((M,F)\) is strongly tense with \((E,\nabla)\). \(\square\)

Remark 9.10. Since tautness is strongly tenseness with the trivial flat line bundle, when \( E \) is trivial, Theorem 9.9(ii) or (iii) characterizes tautness of \((M,F)\).

Remark 9.11. Theorem 9.9 shows that strongly tenseness of noncompact foliated manifold is invariant under certain equivalences of the pseudogroup on fine total transversals finer than the usual equivalence, which induces a map between modified Haefliger cohomology.

10. **Strongly tenseness of Riemannian foliations on noncompact manifolds**

Here we will prove Theorem 1.8. Before the proof, we make the following remark on a difference from the compact case.

Remark 10.1. In the compact case, strongly tenseness implies the nontriviality of the Haefliger cohomology (the “only if” part of Theorem 8.1). But this result cannot be generalized to noncompact manifolds with Riemannian foliations. First, \( H^0_c(\text{Tr} F; E) \) can be trivial for taut Riemannian foliations \( F \) on noncompact manifolds (Example 9.2). Strongly tenseness with \( E \) may not imply the nontriviality of \( H^0_c(\text{Tr} F; E) \) for Riemannian foliations on noncompact manifolds, either. Since the cohomology class of nonnegative \( n \)-forms in \( \Omega^n_c(T; \partial T) \) may be trivial, the argument to show the nontriviality of \( H^0_c(\text{Tr} F; E) \) does not work for \( H^0_c(\text{Tr} F; E) \).

Proof of Theorem 1.8. By Proposition 7.8 it suffices to prove the case where \( F \) is oriented and \( E \) is topologically trivial.

Let \( T = \bigsqcup_{i \in I} T_i \) be a fine total transversal of \((M,F)\) with compatible covering \( \{U_i\}_{i \in I} \) and exhausting subset \( K \). Let \( K_i = K \cap T_i \). By the \( \sigma \)-compactness of \( M \), we can assume that \( I = \mathbb{Z}_{>0} \). Let \( \pi_i : U_i \to T_i \) be the disc bundle whose fibers are leaves of \( F|_{U_i} \).

Take another total transversal \( T_0 \) of \((M,F)\) so that \( T_0 \sqcup T \) is a fine total transversal of \((M,F)\). Since \((M,F)\) is uniform, there exists a relatively compact subset \( K_0 \) which intersects every leaf of \( F \). Let \( H = \text{Hol}(T_0 \sqcup T) \) and \( H_i = \text{Hol}(T_0 \sqcup T_i) \). For each \( i \in \mathbb{Z}_{>0} \), \( T_0 \sqcup T_i \) is a total transversal of \((M,F)\) and \( K_0 \sqcup K_i \) is a relatively open subset of \( T_0 \sqcup T_i \) which intersects every leaf of \( F \). Thus, Lemma 8.6 implies that, for each \( i \in \mathbb{Z}_{>0} \), there exists a nonnegative section \( \xi_i \in \Omega^0_c(T_0 \sqcup T_i; E) \) such that \( K_i \subset \text{supp}_e \xi_i \) and \( d\tau \xi_i = 0 \) in \( \Omega^1_c((T_0 \sqcup T_i)/H_i; E) \). We can take \( \xi_i \) so that there exists a relatively compact subset \( L_0 \) of \( T_0 \) such that \( \text{supp}(\xi_i|_{T_0}) \subset L_0 \) for any \( i \).

First we check if we can construct \( \xi \) satisfying the conditions of Theorem 9.9(ii) to show strongly tenseness with \( E \), by letting \( \xi = \sum_{i=1}^\infty \xi_i \). The first problem is that \( \sum_{i=1}^\infty \xi_i \) may not be well-defined in \( \Omega^0_c(T_0 \sqcup T; E) \), since \( \Omega^0_c(T_0 \sqcup T; E) \cong \prod_{i=1}^\infty \Omega^0_c(T_i; E) \) by definition and the sequence \( \{\sum_{i=1}^N \xi_i|_{T_0}\}_{N=1}^\infty \) may diverge in \( \Omega^0_c(T_0; E) \). To avoid this problem, we will slightly modify the construction of \( \xi \).

Let \( \|\cdot\|_f^\ell \) be a \( C^\ell \)-norm on \( \Omega^0_c(T; E) \), and take the sequence \( A = \{a_i\}_{i=1}^\infty \) of positive real numbers defined by

\[
a_i = \frac{1}{2^{i+1} \max_{0 \leq \ell \leq i} \|\xi_i|_{T_0}\|_f^\ell}.
\]

Then, since

\[
(i) \quad \{\|\sum_{i=1}^N a_i \xi_i|_{T_0}\|_f^\ell\}_{N=1}^\infty \text{ converges for every } \ell \text{ and}
\]

...
(ii) $T_0$ is compact and $\text{supp}(\xi^i|_{T_0}) \subset T_0$ for any $i$.

the sequence $\{\sum_{i=1}^{N} a_i \xi^i|_{T_0}\}_{N=1}^\infty$ converges in $\Omega^i_c(T_0; \mathcal{E})$ (see [dR84, Section 9]). Since $\xi^i|_{T_0} = 0$ for $j > 0$ and $i \neq j$, it follows that $\xi_A = \sum_{i=1}^\infty a_i \xi^i$ is well-defined in $\Omega^i_c(T_0 \cup T^i; \mathcal{E})$.

We will check if $\xi_A$ satisfies the conditions of Theorem 9.9 (ii). Like in Section 9.2, let $J = \{(j, k) \in (\mathbb{Z}_{\geq 0})^2 | U_j \cap U_k \neq \emptyset\}$. For $(j, k) \in J$, let $h_{jk}$ denote the holonomy map from an open set of $T_j$ to an open set of $T_k$, which is unique by Definition 9.5 (ii). Since $\{h_{jk}\}_{(j, k) \in J}$ generates $\mathcal{H}$ and $d_T \xi^i$ belongs to $\Lambda^i_{\mathcal{H}_i}$ (see Section 2.1 for the definition of $\Lambda^i_{\mathcal{H}_i}$), for each $i$, we have

$$d_T \xi^i = \sum_{(j, k) \in J} h_{jk}^* \beta_{kj} - \beta_{kj}$$

for some $\beta_{kj} \in \Omega^i_c(T_k; \mathcal{E})$ so that $\beta_{kj}$ is zero except a finite number of $(j, k) \in J$. If $\sum_{i=1}^\infty a_i \beta_{kj}$ is well-defined in $\Omega^i_c(T_k; \mathcal{E})$ for each $k \in \mathbb{Z}_{\geq 0}$, then we can multiply $22$ by $a_i$ and sum it up with respect to $i \in \mathbb{Z}_{\geq 0}$ to show that $\xi_A$ satisfies the conditions of $\xi$ in Theorem 9.9 (ii). But, in general, the sequence $\{\sum_{i=1}^N a_i \beta_{kj}\}_{N=1}^\infty$ in $\Omega^i_c(T_k; \mathcal{E})$ may diverge, because $a_i \beta_{kj}$ may be nontrivial for a fixed $k$ and infinitely many different $i$.

Then we modify the construction of $\xi_A$. First we modify a fine total transversal $T = \bigsqcup_{i=1}^\infty T_i$ so that there exists a fine total transversal $T' = \bigsqcup_{i=1}^\infty T'_i$ such that $T_i$ is a relatively compact subset of $T'_i$. We similarly modify the total transversal $T_0$ so that there exists a total transversal $T'_0$ such that $T_0$ is a relatively compact subset of $T'_0$. Let $\mathcal{H}' = \text{Hol}(T'_0 \cup T')$. We take $\xi'$, $a_i$, $\xi_A$ and $\beta_{kj}$ as above. Take a $C^\ell$-norm $|.|^\ell$ of $\Omega^i_c(T'_0 \cup T'; \mathcal{E})$. We take the sequence $B = \{b_i\}_{i=1}^\infty$ of positive real numbers defined by

$$b_i = \min \left\{ a_i, \frac{1}{2^{\max_{(j, k) \in J, 0 \leq \ell \leq 1} |\beta_{kj}|^\ell}} \right\}.$$ 

Let $\xi_B = \sum_{i=1}^\infty b_i \xi^i$. We consider $\xi_B$ as an element of $\Omega^i_C(T'_0 \cup T'; \mathcal{E})$. Since

(i) $\{|| \sum_{i=1}^N b_i \beta_{kj}||^\ell\}_{N=1}^\infty$ converges for every $\ell$ and

(ii) $\overline{T_k}$ is compact and $\text{supp}(\sum_{i=1}^N b_i \beta_{kj}) \subset \overline{T_k}$,

the sequence $\{\sum_{i=1}^N b_i \beta_{kj}\}$ converges in $\Omega^i_c(T'_k; \mathcal{E})$ (see [dR84, Section 9]). Then, by multiplying $22$ by $b_i$ and summing it up with respect to $i$, we get

$$d_T \xi_B = \sum_{(j, k) \in J} h_{jk}^* \left( \sum_{i=1}^\infty b_i \beta_{kj}^i \right) - \left( \sum_{i=1}^\infty b_i \beta_{kj}^i \right),$$

where $\sum_{i=1}^\infty b_i \beta_{kj}^i \in \Omega^i_c(T'_k; \mathcal{E})$. It follows that $d_T \xi_B$ belongs to $\Xi^i_{\mathcal{H}_i'}$ (see 21 for the definition of $\Xi^i_{\mathcal{H}_i'}$). By construction, $\mathcal{H}'$ and $\xi_B \in \Omega^i_c(T'_0 \cup T'; \mathcal{E})$ satisfy the conditions of Theorem 9.9 (ii). Thus, by Theorem 9.9 ($M, \mathcal{F}$) is strongly tense with $<\mathcal{E}, \nabla>$. 

\vspace{0.5cm}

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