1. Introduction

Which triangles share circumcircle and pedal-circle? Which one share circumcircle and negative-pedal circle? Are these triangles related? Is it possible to draw all of them?

In this paper we give a poristic answer to this problem and provide functorial recipes to construct all these triangles.

The ground-case obtains when pedal-point is the i-center: to find all triangles that share in-circle and circumcircle. Of course, nowadays anyone knows that if two circles are, respectively the circumcircle and the i-circle of a reference triangle, then they are so for infinitely many other triangles; as a matter a fact, any point of the circumcircle is a vertex of one (and only one) triangle circumscribed to the i-circle. But till midst 1700, this phenomenon was not so well captured. The algebraic relation between the radius of the i-circle and circumcircle of a given triangle was proved by Chapple in 1746 (see e.g. [OW] or [W]), enforced the fact that not any two circles are meant to be i-circle and circumcircle of some triangle.

Nevertheless, the first who understood the poristic nature of this formula was Collin MacLaurin (1698-1746), a scottish mathematician, who proved a special case of Poncelet’s porism: a porism for triangles and a pair of conics. For a self-contained proof of MacLaurin theorem, which use systems of triangles auto-polar w.r to a conic, see [GSO], section 9.5. An elementary geometric proof of Poncelet porism, in its full generality is in [A]; see [P] for Poncelet’s original proof.

When pedal-point is the orthocenter, the pedal circle is the (classic) Euler circle (or the nine-point circle); as a matter a fact its proprieties were first proved by Poncelet and Brianchon in [BP] in relation with a problem of construction of a i-conic (the Apollonius hyperbola)! The problem of finding all triangles that find all triangles that shares the Euler-circle and circumcircle, was studied, with different methods in [Ox], [We] and quite recently in [Pa], who answered these questions in the acute case; in [G], the obtuse case is solved. The approach we adopt here differs from those cited above. The key observation is that the i-conic of a triangle, which have a prescribed focus into pedal point \( D \) is precisely the negative-pedal (curve)
Figure 1. If $\Gamma$ (orange) is any circumconic of $\triangle ABC$, $\mathcal{E}_D$ the pedal-circle (purple) w.r. to $D$ and $\gamma_D$, (doted orange hyperbola) is the negative-pedal of $\mathcal{E}_D$, then $(\Gamma, \gamma_D)$ form a poristic pair for $n = 3$. All triangles (e.g. green and blue) inscribed in $\Gamma$ and circumscribed to $\gamma_D$ share the same pedal-circle $\mathcal{E}_D$. The arc $D_1D_2$ of $\Gamma$, situated inside the conic $\gamma_D$ is infertile: it cannot contain vertices of such triangles, since there is no tangent line to $\gamma_D$ passing through these points.

of the pedal-circle w.r. to $D$. This fact enables a poristic approach that embedded a recipe for the construction of all poristic triangles, as well.

The straightforwardness of the proofs is due to a systematic use of inversive methods, polar duality and above all, to poristic virtues of the negative-pedal curve and to Poncelet’s porism.

The reader not acquainted with inversive methods (circle in version, dual curves, negative polar curve), may consult either the classic [P],[S],[Ch1],[Ch], or the beautiful books [A], [GSO]. For a very quick review, see [W] and the references therein.

2. A PEDAL PORISM

Let $\mathcal{T}$ a triangle, which we shall call reference triangle and let a point $D$ neither on its sides, nor on its circumcircle, which we shall call pedal-point.

**Definition 1.** The triangle whose vertices are the projections of the pedal point $D$ on the sides of $\mathcal{T}$ is the pedal triangle; we call its circumcircle, $\mathcal{E}_D$, the pedal-circle.

This definitions naturally extend those of classic Euler circle: if the pedal point $D$ is either the circumcenter, or the orthocenter, then pedal-circle is the (classic) Euler circle (or the nine-point circle). Pedal-circles w.r. to i-centers are simply i-circles (inscribed or exinscribed circle). For further notable pedal-circles and a comprehensive study of their proprieties, see [PS].

The following definition are classic.
**Definition 2.** The negative-pedal of a curve $\gamma$, w.r. to a pedal point $D$ is the envelope of the perpendiculars through point $P \in \gamma$, to line $PD$, as point $P$ sweeps $\gamma$. We denoted it by $N(\gamma)$.

Negative pedal curves were studied with some intensity by the end of the 1800; a interested reader might see e.g. [Am1], [Am2]; for a quick acquaintance, see also [W]. The following are two equivalent definitions of a polar dual of a curve.

**Definition 3.** The polar dual (or reciprocal) of a (regular) curve $\gamma$ w.r. to an inversion circle is the envelope of the polars of points $P$, as $P$ sweeps $\gamma$.

**Definition 4.** The polar dual (or reciprocal) of a (regular) curve $\gamma$ w.r. to an inversion circle, denoted by $R(\gamma)$ is the loci of the poles of the tangents $t_P$ to $\gamma$, as $P$ sweeps $\gamma$.

Dual curves and the method of polar reciprocals are due to Poncelet; see [P]; see also [Ch], [Ch1]; see also [S]. The following useful description of the negative-pedal, as a loci of points is classic.

**Proposition 1.** The negative-pedal of a curve $\gamma$, denoted by $N(\gamma)$, is the polar dual of its inverse: $N(\gamma) = R(\gamma')$, where $\gamma'$ is the inverse of $\gamma$ w.r. to an inversion circle centred at the pedal point.

Using Proposition 1 and known facts on polar-dual of a circle, we get the following key result (see e.g. [La] and [Lo] for alternative approach)

**Proposition 2.** The negative-pedal of a circle w.r. to pedal point $D$ not on the refereed circle, is the conic centred at the center of the circle, whose focus is $D$ and whose main axis is (precisely) its diameter through $D$.

The negative-pedal of a circle is either an ellipse (when the pedal point is inside the circle) or a hyperbola (when $D$ is outside). If $D$ is on the circle, the pedal curve reduces to a point. The negative-pedal curve of a circle is never a parabola.

Before proceed, let us point out the key-fact, that ties-up i-conics, pedal-circles and negative-pedal curves.

**Proposition 3.** The i-conic of $T$ focused in $D$ is the negative-pedal (curve) of its pedal-circle $E_D$.

**Proof.** The proof is a direct consequence of the definition of the negative-pedal and pedal-circle

For a proof, see e.g. Chapter 7.5 [GSO]; for a nice construction of an i-conic with a prescribed focus, see e.g. [Ch1]; see also [GSO], Example 7.2.2.

Now we may prove the key-ingredient of a poristic approach.

**Lemma 1.** Let $E$ any circle, $D$ any point not on $E$ and $\gamma_D$ the negative-pedal of $E$.

Then a triangle have pedal-circle $E$ if and only if is circumscribed to $\gamma_D$ (its sides 1 tangents the conic $\gamma_D$).

**Proof.** Refer to figure 1

$\Rightarrow$ If $\triangle ABC$ have pedal-circle $E$, then necessarily the feet of the perpendiculars from $D$ to the sides of this triangle are on $E$. Thus, by the definition of a negative-pedal curve as an envelope of lines (see 2) these sides tangents the negative-pedal of circle $E$, i.e. the conic $\gamma_D$.

$\Leftarrow$ By hypothesis, the conic $\gamma_D$ has a focus in $D$ and is inscribed in $\triangle ABC$; the feet of the perpendiculars from $D$ to the sides of triangle determines the pedal-circle $E_D$ of $\triangle ABC$. On the other side, these feet of the perpendicular from the focus of

1by "side" we mean the line that pass through two vertices of a triangle
the conic to the tangents to that conic, belong to one and the same circle, \( E \), (the circle) whose negative-pedal is \( \gamma_D \). Thus, the pedal-circle \( E_D \) of \( \triangle ABC \) is precisely \( E \).

In other words, three arbitrary tangents to a given conic \( \gamma_D \) determine triangles whose pedal-circle is the (unique) circle which diameter is the main axis of the conic \( \gamma_D \).

This led to a first poristic result.

**Proposition 4.** (the pedal porism) Let \( \gamma_D \) the i-conic of \( T \) focused in \( D \); then \((C, \gamma_D)\) form a poristic pair.

All triangles inscribed in \( C \) share the same pedal-circle \( E_D \) if and only if are circumscribed to \( \gamma_D \).

And now the construction of all these triangles.

**Construction 1.** Refer to figure 1. Let \( C \) a point located on \( C \). Let the circle of diameter \( |CD| \) intercept \( E_D \) in \( A_D \) and \( B_D \). The lines \( CA_D \) and \( CB_D \) intercept (again) \( C \) in \( B \) and \( A \).

Then the line \( AB \) is a tangent to \( \gamma_D \) and the feet of \( D \) over \( AB \) is on circle \( E_D \).

All poristic triangles are obtainable in this manner.

**Proof.** By construction, \( A_D \) and \( B_D \) are the feet of the perpendiculars from \( D \) to \( CA_D \) and \( CB_D \), respectively. Since \( E_D \) is the negative-pedal of \( \gamma_D \), \( CA_D \) and \( CB_D \) are two tangents to \( \gamma_D \). By Poncelet’s (MacLauren) porism, \( AB \) is tangent to \( \gamma_D \) since \((C, \gamma_D)\) form a poristic pair. The second assertion is now a consequence of the fact that \( E_D \) is the negative-pedal of \( \gamma_D \). 

**Remark 1.** The proof above give an insight on infertile arcs of \( C \). A necessary condition for this construction to work is the existence of tangents from \( C \) to the i-conic \( \gamma_D \); and this happens iff \( C \) is located on \( C \) and outside the i-conic (not on the arc \( D_1D_2 \); see again figure 1): the arc \( D_1D_2 \) delimited by the intersection of the circumcircle with the i-conic focused in \( O \) contain no vertices of admissible triangle.

**Remark 2.** This construction still hold when the pedal point is \( H \), which corresponds to classic Euler circles. This construction is different from those in [G] and also makes clear what happens in the obtuse case of an Euler circle. In this case, the pedal point \( D \) lie outside the triangle yet inside the circumcircle. The i-conic \( \gamma_D \) is a hyperbola and this causes the infertile arcs. When the triangle is acute, its orthocenter is inside the triangle and the i-conic is an ellipses. In this case, the i-conic is inside the circumcircle, and any initial point \( C \) is admissible.

3. A POLAR PORISM

In this section, we shall provide another construction of triangles sharing the same circumcircle and Euler negative-pedal circle, based on an ad-hoc polar-porism.

When not specified otherwise, we subextend that poles and polars, as well as the dual polars of curves or inversion are wrt to inversion circle \( I \) centred at \( D \).

**Proposition 5.** (a polar-porism) Let \( T \) a triangle and \( C \) its circumcircle and let an inversion circle \( I \) centered at a point \( D \), located neither on the sides of \( T \), nor on its circumcircle. The polars of the vertices of \( T \), wrt to \( I \), determines a new triangle \( T_p \); let \( C_p \) its circumcircle. Let \( \gamma_D \) the i-conic of \( T \) focused in \( D \).

Then \([C_p, C] \) form a polar-poristic pair of circles, in the following sense: if from (any) point \( A \) of circle \( C \) we let the polar of \( A \) intercept circle \( C_p \) in two distinct points \( B_p \) and \( C_p \), and subsequently the polars of \( B_p \) and \( C_p \) intercept \( C \) in \( A, B \) and \( A, C \), respectively.
Finally, let $A_p$ be the pole of $BC$; then

i) $\triangle ABC$ and $\triangle A_pB_pC_p$ are polars: the vertices of the former are the poles of the later and vice-verse;

ii) $A_p$ is on circle $C_p$.

**Proof.** First note that, by construction and by the fundamental theorem on pole-polars, triangles $\mathcal{T}$ and $\mathcal{T}_p$ are mutually polars

Let $\gamma_D$ and $\Gamma_D$ be, respectively, the duals of circles $C_p$ and $C$.

Since $C_p$ is the circumcircle of $\mathcal{T}_p$, then its dual $\gamma_D$ is the i-conic focused in $D$ of triangle $\mathcal{T}$.

Similarly, $\Gamma_D$, the dual of $C$ is the i-conic focused in $D$ of $\mathcal{T}_p$.

By Poncelet’s porism, $(C, \gamma_D)$ form a poristic pair for $n = 3$, since triangle $\mathcal{T}$ is inscribed into the former and circumscribed to the later. Similarly, since by hypotheses $\mathcal{T}_p$ is inscribed in $C_p$ and circumscribed to $\Gamma_D$, the $(C_p, \Gamma_D)$ form a poristic pair for $n = 3$.

Thus, if $A \in \mathcal{C}$ is any point and $B_pC_p$ is its polar ($B_p, C_p \in C_p$) then $B_pC_p$ is a tangent to the conic $\Gamma_p$, since the later is, by hypothesis, the dual of $C$, hence the envelope of the polars of (all) points in $\mathcal{C}$.

Similarly, since by construction, $B_p$ and $C_p$ are on the circle $C_p$, their polars are the tangents from $A$ to $\gamma_D$, the dual of $\mathcal{C}_p$, again by the definition of a dual curve and by the fundamental theorem on pole-polars.

So, if these polars intercept circle $\mathcal{C}$ in $A, C$ and $A, B$ respectively, then $AB$ and $AC$ tangent $\gamma_D$ and $A, B, C$ are all on $\mathcal{C}$. Thus, by Poncelet’s porism, the line $BC$ necessarily tangent the $\gamma_D$. Therefore, its pole, which is, by hypothesis, the point $A_p$ is on the dual of $\Gamma_p$, the circle $C_p$.

Thus, triangles $\triangle ABC$ and $\triangle A_pB_pC_p$ are mutually polar and their vertices are located on $\mathcal{C}$ and $\mathcal{C}_p$ respectively. $\square$

The polar-porism revealed in Proposition 5 implicitly prove the following.

**Corollary 1.** i) The poles $A_p, B_p, C_p$ of the sides of triangles $\triangle ABC$ (which are) inscribed into a circle $\mathcal{C}$ describes a given circle, $\mathcal{C}_p$, if and only if the sides of (those) $\triangle ABC$ tangents a (one and the same) conic, $\gamma_D$.

ii) In this case, the sides of the respective polar triangle $\triangle A_pB_pC_p$ tangents a caustic conic $\Gamma_p$.

The following result relates pedal-circle of the original (triangle) with the circumcircle of its polar triangle.

**Proposition 6.** The pedal-circle of a triangle is the inverse of the circumcircle of its polar triangle, w.r. to an inversion circle centered into the pedal-point.

**Proof.** Refer to figure 3. The i-conic $\gamma_D$ inscribed in $\triangle ABC$ is the negative-pedal of pedal-circle of $\triangle ABC$; as such, $\gamma_D$ is the dual of the inverse of $\mathcal{E}_D$:

$$\gamma_D = \mathcal{R}[\mathcal{I}(\mathcal{E}_D)];$$

performing the dual and using the fact that the duality is an involution, we get

$$\mathcal{R}(\gamma_D) = \mathcal{I}(\mathcal{E}_D).$$

The dual of $\gamma_D$, (a conic focused in $D$), w.r. to $\mathcal{I}$, (an inversion circle centred in $D$), is a circle. This circle is the loci of the poles of the tangents at $\gamma_D$. Further, since $\gamma_p$ is, by construction, the i-conic of $\triangle ABC$, the dual of $\gamma_D$ also is the circumcircle of the polar triangle $\triangle A_pB_pC_p$, which is what we needed to proof. $\square$

**Corollary 2.** The lines joining the pedal-point $D$ with the vertices of the polar triangle $\mathcal{T}_p$, intersects the sides of triangle $\mathcal{T}$ in points located on the pedal-circle.
Figure 2. \( C \) is the circumcircle of \( \triangle ABC \) (violet circle) and \( C_p \) (green circle), the circumcircle of the polar triangle \( \triangle A_pB_pC_p \). \( C_p \) is fixed, does not depend on point \( A \) on \( C \). The dual of \( C \) w.r. to inversion circle centred in \( D \) is \( \Gamma_D \) (violet hyperbola) and the dual of \( C_p \) is \( \gamma_D \) (green ellipse). Then \( \gamma_D \) is the i-conic of \( \triangle ABC \) and \( \Gamma_D \) is the i-conic of \( \triangle A_pB_pC_p \). There are three poristic pairs: i) \((C, \gamma_D)\); the fertile arc \( D_1D_2 \) of \( C \) lies outside the i-conic; ii) \((C_p, \Gamma_D)\); the fertile arc \( L_1L_2 \) of \( C_p \) lies outside the i-conic \( \Gamma_D \); iii) the polar-poristic \((C, C_p)\); (shown \( \triangle ABC \) and its polar \( A_pB_pC_p \)). The poristic pairs \((C, \gamma_D)\) and \((C_p, \Gamma_D)\) are dual images of each other.

Proof. Refer to figure 3. The i-conics \( \gamma_D \) and \( \Gamma_D \) inscribed in \( \triangle ABC \) and \( \triangle A_pB_pC_p \), respectively, are the negative-pedals of their pedal-circles. Since the negative-pedal is the reciprocal of the inverse, their pedal-circles \( E_D \) and \( E_D' \) are, respectively, the inverses of \( C_p \) and \( C \). The perpendiculars from pedal-point \( D \) to the sides of \( \triangle ABC \) intercepts it on points located on pedal-circle \( E_D \) and pass through the vertices of \( \triangle A_pB_pC_p \). The same for \( \triangle A_pB_pC_p \).

Corollary 3. The poles of the sides of any triangle are located on the inverse of its pedal-circle.

We are almost ready to give a recipe for the construction of all triangles sharing the circumcircle and pedal-circle, using polar triangles.

We only have to pay attention to a phenomenon which may occur, whenever the pedal point \( D \) lies outside \( T \).

Refer to figure 2.

Lemma 2. (the lemma of the infertile arcs) Let \( T \) a triangle, \( C \) its circumcircle, \( D \) a pedal point neither on its sides, nor on its circumcircle and \( I \) an inversion circle centred in \( D \). Let \( \gamma_D \) the i-conic of \( T \) focused in \( D \). There are two cases.

i) If \( D \) is inside \( T \), then \( \gamma_D \) is inside the circumcircle and tangents internally the sides of triangle \( T \). In this case, for all \( A \) in \( C \) the polar of \( A \) intercepts \( T_p \).
Figure 3. $\mathcal{E}_D$ the pedal-circle of $\triangle ABC$ (purple) coincides with the inverse of $\mathcal{C}_p$, the circumcircle of the polar triangle $\triangle A_pB_pC_p$ (blue). Similarly, the pedal of the polar triangle (solid violet) is the inverse of $C$, the circumcircle of the original. The perpendiculars from $D$ to the sides of $\triangle ABC$ intercept them on points located on pedal-circle $E_D$ and pass through the vertices of $\triangle A_pB_pC_p$.

ii) If $D$ is outside $\Delta T$, then the $i$-conic $\gamma_D$ tangents externally $\mathcal{T}$ and intercepts $\mathcal{C}$ in two distinct points $D_1$ and $D_2$; let $D_1D_2$ the arc of $\mathcal{C}$ that contain no vertex of $\mathcal{T}$; then a polar of a point $A$ in $\mathcal{C}$ intercepts $\mathcal{C}_p$ if and only if point $D$ is located outside $\mathcal{T}$.

\textbf{Proof.} The proof uses the fact that $\mathcal{C}_p$, the circumcircle of the polar triangle is the dual of $\gamma_D$. Thus, the polar of points located on on $\gamma_D$ are the tangents to $\mathcal{C}_p$, while, by a continuity argument, the polar of points located into interior of $\Gamma^2$ are external to $\mathcal{C}_p$. Therefore no polar of points that lie on the arc $D_1D_2$ of $\mathcal{C}$ intercepts $\mathcal{C}_p$. \hfill $\square$

Finally, a recipe. Refer to figure 2.

\textbf{Construction 2.} Let $\mathcal{C}$ be the circumcircle and $\mathcal{E}_D$ be the pedal-circle of a triangle $\mathcal{T}$ and let $\mathcal{E}_D'$ the inverse of $\mathcal{E}_D$, w.r.t. to inversion circle $I$ centred in $D$.

Let $A$ be any point on $\mathcal{C}$ and let $a$ be its polar; if $a$ is external to $\mathcal{E}_D'$, then there is no triangle with vertex in $A$ sharing the same circumcircle and pedal-circle with $\mathcal{T}$. Otherwise, let the polar of $A$ intercept $\mathcal{E}_D'$ in $B_p, C_p$; the polars of $B_p$ and $C_p$ intercepts (again) $\mathcal{C}$ in $C$ and $A$ and $B$, respectively. Then, the pedal-circle of $\triangle ABC$ thus construct is $\mathcal{E}_D$. All triangles that are inscribed in $\mathcal{C}$ and which shares the same pedal-circle are obtainable in this manner.

\textsuperscript{2}by internal points of a conic we mean points located into the same (conex) component of the plane as those containing one of its foci.
 Remark 3. As a matter of fact, when perform this construction, we obtain (concomitantly) two systems of triangles, one, poristic w.r. to \((C, \gamma_D)\), and the other one, poristic w.r. to \((C_p, \Gamma_D)\) and both respectively sharing the same circumcircle and pedal-circle. Since we are interested in the former, the later was discarded.

4. A negative-pedal porism

Definition 5. (the negative-pedal triangle and the negative-pedal circle) Let \(T\) a triangle, \(C\) its circumcircle and \(D\) a pedal point neither on its sides, nor on its circumcircle. The negative-pedal triangle (or negative-pedal triangle, or the negative-pedal triangle) of \(T\) w.r. to pedal point \(D\), whose sides \(a', b'\) and \(c'\) are the perpendiculars through the vertices of \(T\) to lines that join the pedal-point \(D\) to the vertices of \(T\).

The circumcircle of the negative-pedal triangle, denoted by \(C_D\), is the negative-pedal (or negative-pedal or negative-pedal) circle.

Many classic circles (or triangles) may be looked upon as negative-pedal circles (or triangles) w.r. to pedal points that are notorious centres of the reference triangle. The negative-pedal triangle the circumcenter is the tangential triangle. The negative-pedal triangle of the orthocenter is the anti-complementary triangle.

Negative-pedal triangle is not defined for the points that belong to the sides of the given triangle.

Negative-pedal circle, pedal-circle and circumcircle are closely related.

Refer to figure 4

Proposition 7. The pedal-circle of \(T'\), the negative-pedal triangle of \(T\) is \(C\), the circumcircle of \(T\).

The proof of this fact is straightforward and we omit it. The results above update Proposition 4.

While the poristic tie between circumcircle and pedal-circle requires the mediation of an i-conic, there exists a straightforward poristic bound between the negative-pedal circle and circumcircle.

Proposition 8. (A negative-pedal porism). Let \(T\) a triangle, \(T_D\) its negative-pedal triangle and \(C, C_D\), their circumcircles.

Then \((C, C_D)\) form a negative-pedal poristic pair of circles, in the following sense: from any point \(A\) on \(C\) let the perpendicular through \(A\) to \(DA\) intercept \(C_D\) in two distinct points \(B'\) and \(C'\). The circles of diameters \([B'D]\) and \([C'D]\) intercept (again) circle \(C\) in \(C\) and \(B\), respectively. Finally, let \(A'\) the intersection of \(BC'\) and \(CB'\). Then:

i) \(A'\) is on circle \(C_D\);

ii) \(\triangle A'B'C'\) share with \(T'\) the same circumcircle and pedal-circle \(C\).

If the perpendicular through \(A\) to \(DA\) does not intercept \(C_D\) in two distinct points, there is no such triangle with a vertex in \(A\). This can only occur iff \(A\) is located on an infertile arc \(D_1D_2\) of \(C\) located inside \(\gamma_D\), the i-conic of \(\triangle ABC\). (see figure 4).

Proof. Refer to figure 5. Let \(A'\) the intersection of \(C'B\) with the circle \(C_D\). Then a (classic) poristic pair is \((C_D, \Gamma_D)\) formed by the negative-pedal circle \(C_D\) and the negative-pedal of the circumcircle \(C\), the in-ellipse \(\Gamma_D\). The construction performed above is equivalent with Poncelet porism for this pair of conics \((C_D, \Gamma_D)\) taking as an initial point \(A\). We left the details to the reader. ☐

Corollary 4. Two triangles inscribed into the same circle, share the negative-pedal circle if and only if (they) share the same pedal-circle.
The negative-pedal porism above allow a simple construction of all triangles sharing circumcircle and negative-pedal circle. Refer to figure 5.

**Construction 3.** Let $\mathcal{T}$ a triangle, $\mathcal{T}_D$ its negative-pedal triangle and $\mathcal{C}, \mathcal{C}_D$, their circumcircles. Let $A$ a point in $\mathcal{C}$. Let $a'$ the perpendicular in $A$ to $DA$, and let $B', C'$ the intersection of $a'$ and $\mathcal{C}_D$. The circles of diameters $[B'D]$ and $[C'D]$ intercept (again) the circumcircle $\mathcal{C}$ in $C$ and $B$, respectively. Then

I i) the negative-pedal triangle of $\Delta ABC$ is $\Delta A'B'C'$; equivalently, $\Delta ABC$ is the pedal triangle of $\Delta A'B'C'$;

I ii) the circumcircle of $\Delta A'B'C'$ is $\mathcal{C}_D$; equivalently, the negative-pedal circle of $\Delta ABC$ is $\mathcal{C}_D$; the pedal-circle of $\Delta A'B'C'$ is $\mathcal{C}$.

II) If any of these intersections is empty, the process stops. This happens if the initial point $A$ was located on an "infertile" arc $D_1D_2$ of $\mathcal{C}$ (see figure 4).

Finally, let us show how these two circles relate.

**Proposition 9.** The negative-pedal circle and the inverse of the pedal-circle of a triangle are homothetic. Therefore, two triangles inscribed in $\mathcal{C}$ share the pedal-circle $\mathcal{E}_D$ if and only if they share the negative-pedal circle $\mathcal{C}_D$. 
The negative-pedal triangle of \( \triangle ABC \) w.r. to pedal-point \( D \) is \( \triangle A'B'C' \), whose sides are the perpendiculars in \( A, B, \) and \( C \) to \( DA, DB \) and \( DC \) respectively. Its circumcircle, \( C_D \) (orange circle) is the negative-pedal circle of \( \triangle ABC \) w.r. to \( D \). \( \Gamma_D \), the i-conic of \( \triangle A'B'C' \) focused in \( D \) (orange ellipse) is the negative-pedal of circumcircle \( C \). \((C_D, \Gamma_D)\) form a poristic pair for \( n = 3 \).

Proof. Referring to figure 6. The poles of \( BC, AC, \) and \( AB \) w.r. to an inversion circle \( I \) centered in \( D \), are, respectively, the inverses of the feet of \( D \) on the sides of \( \triangle ABC \); denote them by \( A_p, B_p \) and \( C_p \). Therefore, the lines \( DC, DB \) and \( DA \) are perpendicular on \( A_pB_p, A_pC_p \) and \( B_pC_p \) respectively.

On the other hand, by the definition of a negative-pedal triangle, \( DC, DB \) and \( DA \) are also perpendicular on \( A'B', A'C' \) and \( B'C' \) respectively. Therefore, \( \triangle A'B'C' \) and \( \triangle A_pB_pC_p \) are homothetic. Their homothety center, \( H \) is also the homothety center of their circumcircles. Thus \( C_D \), the circumcircle of the negative-pedal triangle is fixed if and only if \( \mathcal{E}_D' \), the circumcircle of the polar is fixed. Since the later is the inverse of the Euler circle, it is fixed if and only if \( \mathcal{E}_D \) is fixed.

5. References

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Figure 6. $A_p, B_p, C_p$ are, respectively, the poles of $BC, AC,$ and $AB$ w.r. to $I$ (dotted black). $\mathcal{E}_D'$ (dotted purple), the circumcircle of $\triangle A_pB_pC_p$ is the inverse of the pedal-circle $\mathcal{E}_D$ (purple). The negative-pedal $\triangle A'B'C'$ (blue) and $\triangle A_pB_pC_p$ are homothetic and so are their circumcircles $\mathcal{C}_D$ and $\mathcal{E}_D'$. Therefore $\mathcal{C}_D$ is fixed iff $\mathcal{E}_D'$ is fixed iff $\mathcal{E}_D$ is fixed.

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