Curvature in conformal mappings of two-dimensional lattices and foam structure

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The elegant properties of conformal mappings, when applied to two-dimensional lattices, find interesting applications in two-dimensional foams and other cellular or close-packed structures. In particular, the two-dimensional honeycomb (whose dual is the triangular lattice) may be transformed into various conformal patterns, which compare approximately to experimentally realizable two-dimensional foams. We review and extend the mathematical analysis of such transformations, with several illustrative examples. New results are adduced for the local curvature generated by the transformation.

Keywords: conformal crystals; foams; curvature

1. Introduction

The relationship $w = f(z)$, where $f$ is any analytical function, can be viewed as a mapping that sets up a correspondence between the points of the $z$- and $w$-planes. Such mappings are known as conformal mappings. The geometrical operations of inversion, reflection, translation and magnification are all examples of conformal transformations in Euclidean space. Conformal mappings have a number of interesting properties, the most important being isogonality: any two curves that intersect are transformed into curves that intersect at the same angle.

Consider a discrete two-dimensional set of points in the $z$-plane generated by two primitive vectors: the resulting structure in the image plane, due to a conformal mapping, is known as a conformal lattice. It is therefore a purely geometrical object. A strictly conformal crystal is a physical system consisting of particles located on the sites of a conformal lattice. A conformal crystal is a physical system, in which the arrangement of particles approximates a conformal lattice (Rothen & Pierański 1996).

There are numerous examples of conformal crystals occurring in both nature and the laboratory (e.g. Rothen et al. 1993; Rothen & Pierański 1996); despite this, the geometric properties of conformal crystals are at present poorly understood. In the

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present paper, we examine some of the factors that determine the local curvature in these conformal patterns. We begin our analysis by considering the equation describing the curvature induced by a conformal mapping, which was derived by Needham (1997) and also by Mancini & Oguey (2005a, b). For a given line in the $z$-plane, we see that whereas the first derivative of the transforming function relates the direction of a line to its transformed counterpart, the induced curvature involves the second derivative. Here, we use the equation for the induced curvature to compute the mean and mean square curvature of the conformal lattice.

One of the most easily recognized examples of a naturally occurring conformal crystal is the phyllotactic design of a sunflower (Rothen et al. 1993). Another example is the so-called ‘gravity’s rainbow’ structure, which is the name given to the striking arrangement of arches formed by a cluster of magnetized steel balls in an external force field (Rothen et al. 1993). More recently, conformal lattices have been shown to have a connection with disclinations in two-dimensional crystalline structures (Riviera et al. 2005; Mughal & Moore 2007).

Conformal crystals can be physically realized by sandwiching an ordered, quasi-two-dimensional, foam in a Hele–Shaw cell with non-parallel plates (e.g. Drenckhan et al. 2004). Yet another method involves the use of ferrofluid foams in magnetic fields (Elias et al. 1999). The advantage of these foam-based methods is that a variety of conformal crystals can be realized by tuning the geometry of the experiment. However, the use of foams to approximate conformal lattices involves two complications: first, as was shown by Mancini & Oguey (2005a, b), the curvature of the soap films perpendicular to the glass plates has to be taken into account and, second, the total curvature of a soap film must always be constant. The limitations that these conditions impose on realizing a given conformal crystal, using foams, will be discussed in detail below.

The paper is organized as follows. In §2, we introduce the equation for curvature and give some properties of conformal transformations. The relationship between ordered two-dimensional soap froths and conformal transformations is detailed in §3. In §4, we calculate the mean and mean square curvature of the edges connected to a given vertex of the conformal lattice when the original lattice in the $z$-plane is free of curvature. We illustrate these results with some examples that include the case of complex inversion. In §5, we generalize our results and include the case where the original lattice in the $z$-plane has a curvature. In §6, we compute the higher order terms in the expression for the induced curvature. Finally, a brief conclusion is presented in §7.

2. Some properties of conformal transformations

(a) Scaling of areas

Although a conformal mapping $w = f(z)$ preserves angles (the isogonal property), it does not preserve areas. If $\text{d} s_z = \left(\text{d}x^2 + \text{d}y^2\right)^{1/2}$ is a small element of line in the $(x, y)$-plane, upon being mapped to the $w$-plane it will be magnified and has a length given by

$$\text{d} s_w = \left| \frac{\text{d} w}{\text{d} z} \right| \text{d} s_z = |f^{(1)}(z)| \text{d} s_z,$$

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where $f^{(n)}(z)$ is the $n$th derivative of the function $f(z)$. Hence, a small element of area in the $z$-plane, denoted by $dA_z$, upon being mapped to the $w$-plane will have an area

$$dA_w = \left| \frac{dw}{dz} \right|^2 dA_z = |f^{(1)}(z)|^2 dA_z. \quad (2.2)$$

(b) **Induced curvature**

Consider a curve $K$ in the $z$-plane. If we apply an analytical mapping $f$ to this curve, then it will transform into another curve in the image plane, which we denote by $\tilde{K}$. Let us now choose some arbitrary point on $K$, which we denote by $p$. The unit tangent vector to the curve at point $p$ is given by

$$d\xi = e^{i\phi},$$

where $\phi$ is the angle the tangent vector makes with the $x$-axis (for an illustration see figure 1a). Upon applying an analytical transformation, point $p$ is mapped to a new point in the image plane, which has coordinates $f(p) = u + iv = \Re e^{i\theta}$. It has been shown (Needham 1997) that if the instantaneous curvature of $K$ at the point $p$ is given by $\kappa$, then the instantaneous curvature at $f(p)$ is given by

$$\tilde{\kappa} = \frac{1}{|f^{(1)}(p)|} \Im \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \overline{\xi(p)} \right] + \frac{\kappa}{|f^{(1)}(p)|}, \quad (2.3)$$

where ‘Im’ is the imaginary component (figure 1b). The first term in equation (2.3) is the curvature in the image plane if $K$ is a straight line. If, however, $K$ has a curvature $\kappa$, then, in the image plane, this additional curvature is scaled by a factor of $1/|f^{(1)}(p)|$ (again, see Needham (1997) for a beautiful derivation of these and other results).

The usefulness of equation (2.3) can be demonstrated by a short example. Consider the mapping

$$w = f(z) = z^\alpha, \quad (2.4)$$

where

$$f^{(1)}(z) = \alpha z^{\alpha - 1} \quad \text{and} \quad f^{(2)}(z) = \alpha(\alpha - 1)z^{\alpha - 2} \quad (2.5)$$
and let us represent the $z$-plane and the image plane in terms of complex polar coordinates, so that

$$z = re^{i\theta} \quad \text{and} \quad w = \Re e^{i\lambda}.$$ 

Upon substituting equation (2.5) into equation (2.3), we find that the effect of mapping a straight line in the $z$-plane (i.e. $\kappa = 0$) is to yield a curve in the $w$-plane with curvature

$$\tilde{\kappa} = \frac{1}{\alpha r^{a-1}} \text{Im} \left[ \frac{\alpha - 1}{z} \tilde{\xi}(\phi) \right].$$

Writing out $\tilde{\xi}(\phi)$ explicitly and simplifying, this becomes

$$\tilde{\kappa} = \frac{1}{\alpha r^{a-1}} \text{Im} \left[ \frac{\alpha - 1}{r} e^{i\phi} \right] = \frac{\alpha - 1}{\alpha} \frac{1}{r^{a-1}} \text{Im} \left[ \frac{e^{i(\phi - \theta)}}{r} \right] = \frac{\alpha - 1}{\alpha} \frac{1}{r^a} \sin(\phi - \theta).$$

Note, the curvature is expressed in terms of the coordinates of the $z$-plane. To obtain the curvature in the $w$-plane, we must use the relationship $R = r\alpha$, which gives

$$\tilde{\kappa} = \frac{\alpha - 1}{\alpha} \frac{1}{R} \sin(\phi - \theta).$$

Thus, the maximum curvature occurs when the tangent vector is perpendicular to the vector connecting the origin to the point $p$ and is given by

$$|\tilde{\kappa}_{\max}| = \frac{\alpha - 1}{\alpha} \frac{1}{R}. \quad (2.6)$$

This means that lines, in the $z$-plane, drawn perpendicular to the vector connecting the origin with point $p$ will acquire the greatest curvature.

### 3. Properties of two-dimensional soap foams and their relationship to conformal transformations

A dry two-dimensional foam consists of two-dimensional bubbles separated by lines that meet at vertices. Such a foam can be represented by a network consisting of two-dimensional cells separated by one-dimensional edges. Equilibrium conditions impose strict restrictions on the topology and geometry of such a foam network (see Weaire & Hutzler 2001): Plateau’s law stipulates that the edges can intersect only three at a time and must do so at an angle of $2\pi/3$; the edges themselves (in the absence of gravity or other external fields) have a constant curvature (circular arcs) and this curvature is related to the corresponding pressure difference between the adjacent bubbles by the Laplace–Young relation. It follows that the sum of the curvatures of the three edges at a given vertex vanishes or, equivalently, that the mean curvature of the adjoining edges vanishes.

All of the above conditions are automatically satisfied by complex inversion, $f(z) = 1/z$, or more generally a bilinear conformal transformation, which has the form

$$f(z) = \frac{az + b}{cz + d}.$$
It can be decomposed into four sequential transformations: translation; inversion; expansion and rotation; and a final translation (Needham 1997). Of these, only inversion generates curvature. Conformality ensures that the mean curvature of the edges connected to each vertex vanishes, while only complex inversion (or a bilinear transformation) has the special property that it will map a circular arc into another circular arc. Given a dry two-dimensional foam structure at equilibrium, inversion will therefore produce a new equilibrium structure; this was discussed by Weaire (1999), who used it to provide a neat proof of the decoration theorem.

A quasi-two-dimensional foam can be realized by sandwiching a single layer of bubbles between a pair of narrowly separated glass plates (Hele–Shaw cell). In the ideal case, if all the bubbles trapped in the Hele–Shaw cell have the same volume (monodisperse), then an ordered quasi-two-dimensional foam can be realized. Upon viewing the Hele–Shaw cell from above (i.e. from the direction perpendicular to the plates), the bubbles are observed to form a honeycomb structure.

Let us now assume that the bottom plate of the Hele–Shaw cell is flat while the upper plate is slightly angled or curved; if the bubbles are monodisperse, then this imposes a specific variation in the area of the bubbles. The variation in the bubble area can be made to closely match the variation required by a given conformal transformation, as stipulated by equation (2.2). Note that although the area of the bubbles (as observed from the direction perpendicular to the bottom plate) may change their volume remains constant, thus the height of the upper surface is related to the analytical function \( h(w) = 1/dA_w \). This fact has been used to transform the (straight-edged) honeycomb structure, in a variety of ways, to generate approximations of conformal lattices (i.e. conformal crystals) (Drenckhan et al. 2004; figure 2).

In the case of a quasi-two-dimensional foam, it is important to remember that the boundaries between bubbles are in fact two-dimensional soap films and not one-dimensional edges, as they are often approximated. As noted by Mancini & Oguey (2005a,b), the mean curvature of the soap film \( H \) is the average of the transverse curvature \( \kappa_t \) (i.e. the curvature of the soap film in the direction perpendicular to the glass plates) and the longitudinal curvature \( \kappa \) (i.e. the curvature parallel to the glass plates),

\[
H = \frac{1}{2} (\kappa + \kappa_t).
\]

Mancini and Oguey considered two cases. The first is the trivial case when the internal pressure is the same for all bubbles in the conformal crystal, and thus \( \kappa_t = -\kappa \) and \( H = 0 \). In other words, the longitudinal and transverse curvatures vary in such a way as to make the total curvature vanish. From an experimental perspective, this case is somewhat artificial. Of more direct relevance to experimental situations is the second case where the volume of all the bubbles is the same. In this case, it is found that \( \kappa_t = -2\kappa \). Again, the value of the transverse curvature depends on the longitudinal curvature, but now the average curvature does not vanish but is in fact given by \( H = -(1/2)\kappa \).

The results derived in §6 will be concerned with this second non-trivial case. Since in this case the mean curvature \( H \) is not constant for a given film, it is important to know by how much the total curvature deviates from being a constant.
(as required by the conditions of equilibrium). This discrepancy sets a limit on the applicability of conformal transformations in the context of foams. In §6, we calculate the magnitude of this discrepancy to the lowest order.

4. Lattice curvature and conformal transformations

In this section, we examine the curvature that a lattice in the $z$-plane acquires upon being mapped to the $w$-plane. Although the triangular lattice is of primary importance, the results derived are general enough to include other structures. This includes the square lattice and the honeycomb structure. To keep things simple, we consider only the case where the original edges in the $z$-plane, connected to a given vertex, are free of curvature. This condition will be relaxed in §5.

Consider a vertex in the $z$-plane located at $p = x + iy$. Connected to this vertex are $m \geq 3$ straight edges labelled $\xi_n$, with $n = 0, 1, 2, ..., m$ (figure 3a). Note, in the case of $m = 6$ or $m = 4$, it is possible to define two edges, which are connected to the vertex, as primitive vectors and thus tessellate the $z$-plane forming a triangular or square Bravais lattice, respectively. We cannot, however, generate the honeycomb structure in the same way (using a vertex with $n = 3$) since the honeycomb structure is not a Bravais lattice. Nevertheless, we can tessellate the $z$-plane with a honeycomb structure, in a unique way, by making a Voronoi construction of the points that define a triangular lattice. This is because the triangular lattice and the honeycomb structure together define a Voronoi/Delaunay dual.
From the point $p$, we can draw tangent vectors along each edge, which we denote by $b_x^0, b_x^1, \ldots, b_x^m$. The angular separation between the tangent vectors is given by $2\pi/m$. We define an angle $g$, which is the angle between the $x$-axis and the first vector $\xi_0$. Thus, the angle that the $n$th vector makes with the $x$-axis is $(\phi_{n,m} + g)$, where $\phi_{n,m} = 2\pi n/m$. Clearly, the edges are invariant under a rotation of $2\pi/m$ and so we restrict $g$ to the range $0 \leq g \leq 2\pi/m$. This is shown schematically in figure 3a.

From the point $p$, we can draw tangent vectors along each edge, which we denote by $\xi_0, \xi_1, \ldots, \xi_{m-1}$. The angular separation between the tangent vectors is given by $2\pi/m$. We define an angle $\gamma$, which is the angle between the $x$-axis and the first vector $\xi_0$. Thus, the angle that the $n$th vector makes with the $x$-axis is $(\phi_{n,m} + \gamma)$, where $\phi_{n,m} = 2\pi n/m$. Clearly, the edges are invariant under a rotation of $2\pi/m$ and so we restrict $\gamma$ to the range $0 \leq \gamma \leq 2\pi/m$. This is shown schematically in figure 3a.

**Figure 3.** (a) A vertex in the $z$-plane at a point $p$ (which in this case is part of a hexagonal lattice); tangent vectors to the edges are denoted by $\xi_n$. (b) The effect of the transformation $f(z)$ on the hexagonal lattice cell. In the $w$-plane, the lattice cell becomes deformed and its centre is now to be found at $f(p)$.

(a) The induced curvature of an edge upon being mapped to the image plane

Upon applying an analytical mapping, the vertex will now be located at a point $f(p) = u + iv$ in the $w$-plane. Since all the edges are straight, using equation (2.3), it is found that the $m$ edges have a new curvature in the image plane given by

$$\kappa_n = \frac{1}{|f^{(1)}(p)|} \text{Im} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \xi_n \right] = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \text{Im} \left[ e^{i\phi_{n,m}} \xi_n \right], \quad (4.1)$$

where we have set $\kappa_n = 0$ (here, $\kappa_n$ is the curvature of the edges in the $z$-plane). We also have

$$\Theta = \theta_2 - \theta_1 = \text{Arg} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \right],$$

where Arg stands for the complex argument. From here on we shall adopt the notation that

$$\theta_n = \text{Arg}[f^{(n)}(p)]$$

and that the $n$th tangent vector is given by

$$\xi_n = e^{i(\phi_{n,m} + \gamma)},$$
equation (4.1) can therefore be written as
\[
\tilde{k}_n = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \sin(\Theta + \phi_{n,m} + \gamma) = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \sin(v + \phi_{n,m}),
\] (4.2)
where \(v = \gamma + \Theta\).

(b) The mean induced edge curvature of the edges for a given vertex in the image plane

The mean edge curvature is defined as the sum of the curvatures of the one-dimensional edges connected to a given vertex, divided by the number of edges. Thus, in the \(w\)-plane, the mean edge curvature is
\[
\tilde{C}_m = \frac{1}{m} \sum_{n=0}^{m-1} \tilde{k}_n = \frac{1}{m} \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \sum_{n=0}^{m-1} \sin(v + \phi_{n,m}) = 0,
\] (4.3)
where we have substituted equation (4.2) into equation (4.3). As expected the mean edge curvature in the \(w\)-plane vanishes.

(c) The mean square value of the induced curvature of the edges, connected to a given vertex, in the image plane

The mean square curvature of the one-dimensional edges, connected to a given vertex, in the \(w\)-plane can be found by performing the following steps: (i) compute the difference between the induced curvature of the edges and the mean curvature (which due to conformality is zero), (ii) compute the square of the difference, and (iii) sum up all the squares and divide by the number of edges; this gives
\[
\tilde{Q}_m = \frac{1}{m} \sum_{n=0}^{m-1} \tilde{k}_n^2 = \frac{1}{m} \left( \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \right)^2 \sum_{n=0}^{m-1} \sin^2(v + \phi_{n,m}) = \frac{1}{2} \left( \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \right)^2,
\] (4.4)
where we have substituted equation (4.2) into equation (4.4). Thus, the mean square curvature of the edges, connected to a given vertex in the image plane, is independent with respect to the orientation of the vertex in the image or \(z\)-planes. In effect, the mean square curvature is a function which gives us an indication of how ‘bent’ the conformal lattice is in a given region.

(d) Examples

To illustrate the results derived for the mean edge curvature and the mean square curvature of the edges for a given vertex, as derived above, we now examine two conformal lattices that have been generated by applying a conformal transformation to a disc containing a triangular lattice. Just such a disc is shown in figure 4a; it can be seen that the lattice is free of curvature. Applying the mapping \(f(z) = z^{1/2}\) to the triangular lattice in the \(z\)-plane yields a conformal lattice in the image plane (figure 4b). From equation (4.4), the mean square curvature is found to be
\[
\tilde{Q}_m = \frac{1}{2r} = \frac{1}{2R^2},
\] (4.5)
where we have transformed from the coordinates in the $z$-plane ($r, \theta$) to the $w$-plane coordinates ($R, A$). Similarly, the effect of inversion is shown in figure 4c. To compute the mean square curvature, we substitute $f(z) = 1/z$ into equation (4.4), which yields

$$
\tilde{Q}_m = \frac{r^2}{2} = \frac{2}{R^2}.
$$

(4.6)

To verify our results, we estimate the mean square curvature directly from the transformed lattice. To do so, note that any given hexagonal cell in the image plane can be decomposed into three arcs that cross the centre of the cell at $f(p)$, as shown in figure 3b. Each of these three arcs can be further decomposed into three points that can be used to fit the equation of a circle, from which the curvature of the arc can be estimated. Let us denote the three directly computed curvatures as $\chi_1$, $\chi_2$ and $\chi_3$; where we use the symbol $\chi_n$ instead of $\kappa_n$, since fitting circles to arcs can only give an estimation of the curvature (except in the case of complex inversion where the fitting gives an exact match); note we quantify this discrepancy in §6. Thus, we can define the estimated mean square curvature as

$$
\langle \chi \rangle^2 = \frac{1}{3} (\chi_1^2 + \chi_2^2 + \chi_3^2).
$$

(4.7)

For the two conformal lattices shown in figure 4, we can estimate the mean square curvature for a given vertex using equation (4.7). The results are shown by the circles in figure 5 along with the theoretical results (equations (4.5) and (4.6)). In both cases, the mean edge curvature is found to vanish at every edge.

In the case of $w = z^{1/2}$, the conformal lattice has been produced by allowing the range of the $z$-plane to extend to $\phi < 4\pi$. In both cases, the mean edge curvature vanishes everywhere but the mean square curvature does not. Also, the mean square curvature diverges at the origin—note that some authors (e.g. Needham 1997) contend that the origin is a critical point of the transformation while others say it is a pole—nevertheless, at the central point, the angle-preserving property of the transformation breaks down; this can be easily seen in the case of the $w = z^{1/2}$ transformation. Here, instead of being preserved, the angle $\pi/3$ between rays (heavy lines) emanating from the origin is halved.

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In §4, we considered a vertex, in the $z$-plane, to which there are connected a finite number of straight edges. We found that when the vertex is mapped to the image plane the mean edge curvature vanishes and that the mean square curvature is independent with respect to the orientation of the vertex.

Let us now consider the case where the edges connected to a vertex, in the $z$-plane, have some initial curvature. Let us also assume that the number of edges connected to the vertex is infinite—i.e. we form the isotropic mean over the local directions. As the number of edges tends to infinity, the tangent vectors together describe a circle of unit radius centred on the vertex. Upon averaging over the length of the circle, the mean edge curvature in the $z$-plane can be written as

$$C = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\phi) \, d\phi,$$

where $\kappa(\phi)$ is now a continuous function over the range $0 \leq \phi < 2\pi$ and describes the curvature of the edges in the $z$-plane. Similarly, in the $w$-plane, the mean edge curvature is given by

$$\tilde{C} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\kappa}(\phi) \, d\phi,$$

where

$$\tilde{\kappa}(\phi) = \frac{1}{|f^{(1)}(p)|} \text{Im} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \overline{\xi(\phi)} \right] + \frac{\kappa(\phi)}{|f^{(1)}(p)|},$$

equation (5.3) can be written as

$$\tilde{\kappa}(\phi) = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \text{Im} \left[ e^{i\Theta} \overline{\xi(\phi)} \right] + \frac{\kappa(\phi)}{|f^{(1)}(p)|} = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \sin(\Theta + \phi) + \frac{\kappa(\phi)}{|f^{(1)}(p)|},$$

where $\Theta$ has the same definitions as given in §4. Substituting equation (5.4) into equation (5.2) yields

$$\tilde{C} = \frac{1}{2\pi} \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \int_0^{2\pi} \sin(\Theta + \phi) \, d\phi + \frac{C}{|f^{(1)}(p)|} = \frac{C}{|f^{(1)}(p)|}.$$

Figure 5. Mean square curvature for the mapping (a) $f(z) = z^{1/2}$ and (b) $f(z) = 1/z$. Circle, $\langle \chi \rangle^2$; solid line, theoretical result.

5. Some generalizations

In §4, we considered a vertex, in the $z$-plane, to which there are connected a finite number of straight edges. We found that when the vertex is mapped to the image plane the mean edge curvature vanishes and that the mean square curvature is independent with respect to the orientation of the vertex.

Let us now consider the case where the edges connected to a vertex, in the $z$-plane, have some initial curvature. Let us also assume that the number of edges connected to the vertex is infinite—i.e. we form the isotropic mean over the local directions. As the number of edges tends to infinity, the tangent vectors together describe a circle of unit radius centred on the vertex. Upon averaging over the length of the circle, the mean edge curvature in the $z$-plane can be written as

$$C = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\phi) \, d\phi,$$

where $\kappa(\phi)$ is now a continuous function over the range $0 \leq \phi < 2\pi$ and describes the curvature of the edges in the $z$-plane. Similarly, in the $w$-plane, the mean edge curvature is given by

$$\tilde{C} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\kappa}(\phi) \, d\phi,$$

where

$$\tilde{\kappa}(\phi) = \frac{1}{|f^{(1)}(p)|} \text{Im} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \overline{\xi(\phi)} \right] + \frac{\kappa(\phi)}{|f^{(1)}(p)|},$$

equation (5.3) can be written as

$$\tilde{\kappa}(\phi) = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \text{Im} \left[ e^{i\Theta} \overline{\xi(\phi)} \right] + \frac{\kappa(\phi)}{|f^{(1)}(p)|} = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \sin(\Theta + \phi) + \frac{\kappa(\phi)}{|f^{(1)}(p)|},$$

where $\Theta$ has the same definitions as given in §4. Substituting equation (5.4) into equation (5.2) yields

$$\tilde{C} = \frac{1}{2\pi} \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \int_0^{2\pi} \sin(\Theta + \phi) \, d\phi + \frac{C}{|f^{(1)}(p)|} = \frac{C}{|f^{(1)}(p)|}.$$
Thus, the conformal transformation scales the original mean edge curvature by the magnification factor $1/|f^{(1)}(p)|$ of the transformation (see equation (2.1)).

Since the mean edge curvature does not necessarily vanish, we define the mean square curvature in the $z$-plane as

$$Q = \frac{1}{2\pi} \int_0^{2\pi} (\kappa(\phi) - C)^2 d\phi. \quad (5.5)$$

Similarly, we define the mean square curvature in the $w$-plane as

$$\tilde{Q} = \frac{1}{2\pi} \int_0^{2\pi} (\tilde{\kappa}(\phi) - \tilde{C})^2 d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left( \tilde{\kappa}(\phi)^2 - 2\tilde{C} \tilde{\kappa}(\phi) + \tilde{C}^2 \right) d\phi, \quad (5.6)$$

substituting equation (5.4) into equation (5.6) and performing the resulting integrals gives

$$\tilde{Q} = \frac{1}{2} \left( \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \right)^2 + \frac{Q}{|f^{(1)}(p)|^2} + \frac{1}{\pi} \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^3} \int_0^{2\pi} \kappa(\phi) \sin(\Theta + \phi) d\phi. \quad (5.7)$$

The first term is the mean square curvature of the edges in the image plane if the edges in the $z$-plane have no curvature. The second term simply states that the original mean square curvature of the edges is scaled by a factor of $1/|f^{(1)}(p)|^2$. The third term is a coupling between the original curvature of the edges and the transformation $f(z)$. To get a better understanding of the third term, let us label it as $\tilde{A}$. It can be written as

$$\tilde{A} = \frac{1}{\pi} \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^3} \int_0^{2\pi} \kappa(\phi) \sin \Theta \cos \phi + \cos \Theta \sin \phi d\phi. \quad (5.8)$$

Since $\kappa(\phi)$ is an arbitrary function that exists over the interval $0 \leq \phi \leq 2\pi$, it is natural to express it in terms of a Fourier series, so that

$$\kappa(\phi) = \frac{1}{2} a_0 + \sum_{j=1}^{\infty} a_j \cos(j\phi) + b_j \sin(j\phi). \quad (5.9)$$

Substituting equation (5.9) into equation (5.8), we find

$$\tilde{A} = \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^3} (a_1 \sin \Theta + b_1 \cos \Theta), \quad (5.10)$$

thus, the only contribution to the mean square curvature is from the mode $j=1$. The contributions from all other modes cancel out, so that an increase in the mean square curvature for a given edge is cancelled out by a decrease in the mean square curvature for some other edge. Furthermore, equation (5.10) depends on the orientation of the vertex in the $z$-plane through the angle $\Theta$. The turning points (i.e. the maximum and minimum mean square curvatures) depend on the values of the coefficients $a_1$ and $b_1$ and are located at $\Theta = \tan^{-1}(a_1/b_1)$ and $\Theta = \tan^{-1}(a_1/b_1) + \pi$.

Thus, when the edges in the $z$-plane are not free of curvature, the effect of the conformal mapping is to scale the mean edge curvature, while the mean square curvature is found to be no longer independent with respect to the orientation of the vertex.
6. Higher order terms in the induced curvature

For a given two-dimensional curve, its curvature is defined as the rate of change of the angle of the tangent vector with respect to the distance along the curve. We can think of this as a limiting process: take two tangent vectors a short distance apart on the curve; measure the angle they make with respect to a fixed axis; and take the limit in which the separation between the tangent vectors vanishes. This then gives the instantaneous value of the curvature.

Consider a curve in the image plane, which has been generated by applying a complex mapping to some curve in the $z$-plane. We can choose a pair of points on the image curve a finite distance apart and for both points construct a tangent vector. Thus, we can ask: what is the rule which gives the total difference in angle between the two tangent vectors?

In this section, we derive an expression for the average induced curvature, which we define as the total change in the angle of the tangent vector over a segment of a curve averaged over the length of the segment. The result is a series expansion in which equation (2.3) is the lowest order term. This series expansion will lead us to an expression that can be used to measure the degree to which a given image arc differs from being perfectly circular. Here, we consider only the effect of the mapping $w = f(z)$ on a straight line in the $z$-plane.

Consider a straight line $K$ in the $z$-plane, which joins point $p$ to $q$, and let us assume it has a length $|\xi| = L$. This is shown in figure 6a. At $p$, we have drawn the unit tangent vector $\hat{\xi} = e^{i\phi}$, so that the vector joining $p$ to $q$ is given by

$$\xi = |\xi| \hat{\xi}(\phi).$$

(6.1)

The line $K$ is described by the equation

$$K = z(t) = p + t\xi = re^{i\theta} + te^{i\phi},$$

(6.2)

where $0 \leq t \leq |\xi|$. Note that, at $t=0$, we have $z=p$ and, at $t=|\xi|$, we have

$$z(|\xi|) = q = p + \xi = re^{i\theta} + |\xi|e^{i\phi}.$$

Thus, $K$ has been parametrized by the free parameter $t$, which measures the distance along $K$ from the point $p$. The line $K$ has the real and imaginary components

$$x(t) = r \cos \theta + t \cos \phi \quad \text{and} \quad y(t) = r \sin \theta + t \sin \phi,$$

respectively.

Now consider what happens if we apply an analytical mapping $w = f(z)$ to $K$; the result is a new curve $\tilde{K}$ in the image plane, which starts at the point labelled $\tilde{f}(p)$ and ends at $f(q)$, as shown in figure 6b. This curve now has some new length, which we call $l$ (not to be confused with the chord distance between $f(p)$ and $f(q)$, which we denote by $|\tilde{\xi}|$). We define the average induced curvature of the image curve $\tilde{K}$ as

$$\tilde{\psi} = \frac{\Delta \tilde{\phi}}{l} = \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{l},$$

(6.3)

where the tangent vectors at $f(p)$ and $f(q)$ are labelled $\tilde{\xi}_1$ and $\tilde{\xi}_2$, and make an angle of $\tilde{\phi}_1$ and $\tilde{\phi}_2$, respectively, with the $u$-axis. Thus, $\tilde{\psi}$ gives the total change in the angle of the tangent vector in the image plane, as it traverses the image curve, averaged over the length of the curve.
If, for equation (6.3), we keep the point \( p \) fixed and take the limit in which \( jxj \) goes to zero, we recover the instantaneous curvature of the curve \( \tilde{K} \) at the point \( f(p) \), that is
\[
\tilde{k}(p) = \lim_{jxj \to 0} \tilde{\psi}.
\]

Let us now define the difference between the instantaneous curvature of the image arc at its starting point and the average curvature of the arc, which we denote as
\[
\Delta \tilde{\psi} = \tilde{\psi} - \tilde{k}(p).
\]
We note that if the image arc is a perfect circle, then the instantaneous curvature has a constant value at every point on the arc and therefore \( |\Delta \tilde{\psi}| = 0 \). This fact can be used to quantify the degree to which the image curve deviates from perfect circularity (assuming that the curvature of the edge does not change sign).

In the context of two-dimensional and quasi-two-dimensional foams, it is important to know the degree to which a given arc, generated by a conformal mapping, deviates from perfect circularity. Laplace’s law requires that the curvature of a soap film is constant; therefore, for a two-dimensional soap film, a large value of \( |\Delta \tilde{\psi}| \) implies a large error in how well the soap film approximates the conformal lattice. The same holds true for quasi-two-dimensional foams (in the constant volume case), except that we have to remember that the total curvature of the soap film, i.e. the sum of the longitudinal \( \kappa = \kappa(p) \) and transverse \( \kappa_t \) curvatures, has an instantaneous value of
\[
H = -\frac{1}{2} \tilde{k}(p).
\]
In either case, Laplace’s law is satisfied only if the soap film has a constant longitudinal curvature, i.e. it describes a circular arc when viewed from a direction perpendicular to the bottom plate of the Hele–Shaw cell.

In the following, we decompose the task of finding \( \tilde{\psi} \) into two parts: first, we compute \( l \) and, second, we compute \( \Delta \phi \).

\((a)\) Arc length in the image plane

As stated above, the real and imaginary components of the line \( K \) are a function of the free parameter \( t \). Upon mapping \( K \) to the image plane (using
the analytical function \( w = f(z) \), the result is the image curve \( \overline{K} \)—the real and imaginary components of which are also functions of the free parameter \( t \) and denoted by \( u(t) \) and \( v(t) \), respectively. We can write the real and imaginary components of \( \overline{K} \) in the form of a series expansion about the point \( p \) (i.e. \( t = 0 \)), giving

\[
u(t) = u + t \frac{u^{(t)}}{1!} + t^2 \frac{u^{(tt)}}{2!} + \cdots + \text{higher order terms}
\]

and also

\[
v(t) = v + t \frac{v^{(t)}}{1!} + t^2 \frac{v^{(tt)}}{2!} + \cdots + \text{higher order terms},
\]

where

\[
u = u(t)\big|_{t=0} = u(0),
\]

\[
u^{(t)} = \frac{du(t)}{dt}\big|_{t=0} \quad \text{and} \quad \nu^{(tt)} = \frac{d^2 u(t)}{dt^2}\big|_{t=0} \text{ etc.}
\]

We also define \( \nu^{(t)} \) and \( \nu^{(tt)} \) (and higher order terms) in a similar manner.

The total length of the arc \( \overline{K} \) is given by

\[
l = \int_0^{\xi} \sqrt{\left( \frac{du(t)}{dt} \right)^2 + \left( \frac{dv(t)}{dt} \right)^2} \, dt.
\]

We can substitute equations (6.4) and (6.5) into equation (6.8), then expand the integrand in powers of \( t \), and integrate each term with respect to \( t \) to give

\[
l = l_0|\xi| + l_1|\xi|^2 + l_2|\xi|^3 + \cdots + \text{higher order terms},
\]

where we assume that \( |\xi| \) is small enough to guarantee convergence of the series. The coefficients in equation (6.9) are defined as

\[
l_0 = \sqrt{u^{(t)^2} + v^{(t)^2}},
\]

\[
l_1 = \frac{1}{2} \frac{u^{(t)} v^{(tt)} + v^{(t)} u^{(tt)}}{\sqrt{u^{(t)^2} + v^{(t)^2}}},
\]

\[
l_2 = \frac{1}{6} \frac{u^{(tt)} u^{(tt)} + v^{(tt)} v^{(tt)} + v^{(t)^2}}{\sqrt{u^{(t)^2} + v^{(t)^2}}} - \frac{(u^{(t)} u^{(tt)} + v^{(t)} v^{(tt)})^2}{2(u^{(t)^2} + v^{(t)^2})^2}
\]

We note that the coefficients given by equations (6.10)–(6.12) (and higher order terms) are defined with respect to the parameter \( t \). In order to express the coefficients in terms of the complex number \( z \), we use equation (6.2) from which we have the relationship \( \frac{dz}{dt} = e^{i\phi} \); this can be rearranged to give \( dt = e^{-i\phi} \, dz \). Thus, equations (6.6) and (6.7) (and higher order derivatives) can

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be written as
\[
\frac{d^n u}{dt^n} \Bigg|_{t=0} = \text{Re} \left[ \frac{d^n w}{dt^n} \Bigg|_{t=0} \right] = \text{Re} \left[ \frac{d^n w}{d^nz} \right]_{z=p} = \text{Re} \left[ \frac{d^n w}{d^nz} \right]_{z=p} e^{in\phi} 
\]
where we have made a change of variable from \( t \) to \( z \) and used the fact that, at \( t=0 \), we have \( z=p \); also, we used
\[
\frac{d^n w}{d^nz} \Bigg|_{z=p} = |f^{(n)}(z)| e^{i \text{Arg}[\xi(z)]} \Bigg|_{z=p} = |f^{(n)}(z)| e^{i\theta_n}.
\]
Similarly, for the derivatives \( v_t \) and \( v_{tt} \) (and higher order terms), we have
\[
\frac{d^n v}{dt^n} \Bigg|_{t=0} = |f^{(n)}(p)| \sin(\theta_n + n\phi). \quad (6.14)
\]
Finally, the coefficients given by equations (6.10)--(6.12) can be expressed in terms of equations (6.13) and (6.14) to give
\[
l_0 = |f^{(1)}(p)|, \quad (6.15)
\]
\[
l_1 = \frac{|f^{(1)}(p)|}{2} \text{Re} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \tilde{\xi}(\phi) \right], \quad (6.16)
\]
\[
l_2 = \frac{|f^{(1)}(p)|}{12} \left( \frac{|f^{(2)}(p)|^2}{|f^{(1)}(p)|^2} - \text{Re} \left[ \left( \frac{f^{(2)}(p)}{f^{(1)}(p)} \tilde{\xi}(\phi) \right)^2 - 2 \left( \frac{f^{(3)}(p)}{f^{(1)}(p)} \tilde{\xi}(\phi)^2 \right) \right] \right). \quad (6.17)
\]

Upon substituting equations (6.15)--(6.17) into equation (6.9), we have a series expansion in powers of \( |\xi| \). We note that the lowest order term in equation (6.9) is identical to equation (2.1).

(b) Change in the angle of the tangent vector in the image plane

We now compute the difference in angle, \( \Delta \phi = \tilde{\phi}_2 - \tilde{\phi}_1 \), of the tangent vectors in the image plane. This derivation is adapted from the one used by Needham (1997) to compute the instantaneous curvature. The difference here is that we endeavour to retain all details that may yield higher order terms in the expression for \( \Delta \phi \).

Consider again the straight line \( K \) in the \( z \)-plane, as shown in figure 6. Note that the tangent vector at \( p \) upon being mapped by the function \( w=f(z) \) is, after being translated to \( f(p) \), magnified by a length \( |f^{(1)}(p)| \) and rotated by an angle \( \text{Arg}[f^{(1)}(p)] \). Similarly, upon applying the mapping \( w=f(z) \), the tangent vector at \( q \) is, after being translated to \( f(q) \), magnified by a length \( |f^{(1)}(p)| \) and rotated by an angle \( \text{Arg}[f^{(1)}(q)] \). However, the rotation of the tangent vector at \( f(p) \) will differ very slightly by \( \Delta \phi \) from the rotation suffered by the tangent vector at \( f(p) \).

To compute the difference in angle between the tangent vectors, consider the two points \( p \) and \( q \) as shown in figure 7a; upon applying the mapping \( f^{(1)}(z) \), the points are now located at \( f^{(1)}(p) \) and \( f^{(1)}(q) \), respectively. Here, the vector
connecting $f^{(1)}(p)$ to $f^{(1)}(q)$ is labelled $\mathbf{\chi}$ and is given by

$$\mathbf{\chi} = f^{(1)}(q) - f^{(1)}(p) = f^{(2)}(p)\hat{\xi}|\xi| + \frac{1}{2!}f^{(3)}(p)\hat{\xi}^2|\xi|^2 + \frac{1}{3!}f^{(4)}(p)\hat{\xi}^3|\xi|^3 + \ldots$$

+ higher order terms,

where we have made use of equation (6.1).

Note that, with reference to figure 7b, to transform the vector $f^{(1)}(p)$ into the vector $f^{(1)}(q)$, it is necessary to expand the length of $f^{(1)}(p)$ by some factor and to rotate $f^{(1)}(p)$ by the angle denoted by $\sigma$. This triangle is the result of dividing all the vectors shown in (b) by the complex number $f^{(1)}(p)$. Dividing by $f^{(1)}(p)$ has two effects: first, the triangle is rotated so that one of its sides is now parallel to the real axis and, second, all the sides of the triangle are uniformly scaled. This second effect means that the new triangle is similar to the triangle shown in (b). Adapted from Needham (1997).

Figure 7. (a) Two points $p$ and $q$ in the $z$-plane. (b) Upon applying the mapping $f^{(1)}(z) = df(z)/dz$, the points are mapped to $f^{(1)}(p)$ and $f^{(1)}(q)$, which are connected by the vector $\mathbf{\chi}$. Note that to transform $f^{(1)}(p)$ into $f^{(1)}(q)$ it is necessary to magnify $f^{(1)}(p)$ by some factor and to rotate $f^{(1)}(p)$ by the angle denoted by $\sigma$. (c) This triangle is the result of dividing all the vectors shown in (b) by the complex number $f^{(1)}(p)$. Dividing by $f^{(1)}(p)$ has two effects: first, the triangle is rotated so that one of its sides is now parallel to the real axis and, second, all the sides of the triangle are uniformly scaled. This second effect means that the new triangle is similar to the triangle shown in (b). Adapted from Needham (1997).

By a simple application of trigonometry, we have

$$\Delta \tilde{\phi} = \sigma = \tan^{-1}\left(\frac{\text{Im}[v]}{1 + \text{Re}[v]}\right).$$

Upon substituting equation (6.18) into equation (6.19), we can expand equation (6.19) in powers of $|\hat{\xi}|$ (we assume that $|\hat{\xi}|$ is small enough to allow convergence) to give

$$\Delta \tilde{\phi} = \sigma = \sigma_0|\hat{\xi}| + \sigma_1|\hat{\xi}|^2 + \sigma_2|\hat{\xi}|^3 + \ldots + \text{higher order terms},$$

(6.20)
where we have made use of equation (6.18), we find

\[ \sigma_0 = \text{Im} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \xi(\phi) \right], \]  
\[ \sigma_1 = \frac{1}{2} \text{Im} \left[ \left( \frac{f^{(3)}(p)}{f^{(1)}(p)} \right)^2 - \frac{1}{2} \left( \frac{f^{(2)}(p) f^{(3)}(p)}{(f^{(1)}(p))^2} \right)^2 \right], \]  
\[ \sigma_2 = \text{Im} \left[ \frac{1}{3} \left( \frac{f^{(2)}(p)}{f^{(1)}(p)} \right)^3 - \frac{1}{2} \left( \frac{f^{(2)}(p) f^{(3)}(p)}{(f^{(1)}(p))^2} \right) + \frac{1}{6} \left( \frac{f^{(4)}(p)}{(f^{(1)}(p))^2} \right) \right] \xi(\phi)^3. \]  

(c) The average induced curvature

Substituting equations (6.9) and (6.20) into equation (6.3) and expanding in powers of \(|\xi|\) yields

\[ \tilde{\psi} = \tilde{\psi}_0 + \tilde{\psi}_1 |\xi| + \tilde{\psi}_2 |\xi|^2 + \cdots + \text{higher order terms}, \]  
where we find

\[ \tilde{\psi}_0 = \tilde{\kappa} = \frac{1}{|f^{(1)}(p)|} \text{Im} \left[ \frac{f^{(2)}(p)}{f^{(1)}(p)} \overline{\xi}(\phi) \right], \]

\[ \tilde{\psi}_1 = \frac{1}{2|f^{(1)}(p)|} \left( \text{Im} \left[ \frac{f^{(3)}(p)}{f^{(1)}(p)} \overline{\xi}(\phi)^2 \right] - \frac{3}{2} \text{Im} \left[ \left( \frac{f^{(2)}(p) \overline{\xi}(\phi)}{f^{(1)}(p)} \right)^2 \right] \right), \]

\[ \tilde{\psi}_2 = \frac{1}{48|f^{(1)}(p)|} \left( \frac{3 |f^{(2)}(p)|^2}{|f^{(1)}(p)|} \text{Im} \left[ \frac{f^{(2)}(p) \overline{\xi}(\phi)}{f^{(1)}(p)} \right] + 27 \text{Im} \left[ \left( \frac{f^{(2)}(p) \overline{\xi}(\phi)}{f^{(1)}(p)} \right)^3 \right] \right. \]

\[ \left. - \frac{2 |f^{(2)}(p)|^2}{|f^{(1)}(p)|^2} \text{Im} \left[ \frac{f^{(3)}(p) \overline{\xi}(\phi)}{f^{(2)}(p)} \right] - 34 \text{Im} \left[ \frac{f^{(2)}(p) f^{(3)}(p) \overline{\xi}(\phi)}{(f^{(1)}(p))^2} \right] \right) + 8 \text{Im} \left[ \frac{f^{(4)}(p)}{f^{(1)}(p)} \overline{\xi}(\phi)^3 \right]. \]

We observe that, as \(|\xi| \to 0\), then \(\tilde{\psi} \to \tilde{\psi}_0 = \tilde{\kappa}\), giving the instantaneous curvature of the image curve at the point \(f(p)\).

(i) Example \(f(z) = z^3\)

Let us apply equation (6.24) to a simple example; we choose the complex mapping \(f(z) = z^3\). Consider again a straight line in the \(z\)-plane given by equation (6.2), and let us set \(r = 1, \theta = 0, \phi = \pi/2\) and \(|\xi| = 0.1\), so that we have

\[ K = z(t) = 1 + t e^{i(\pi/2)} \]  
and \(0 \leq t \leq 0.1\). Upon applying the conformal transformation \(f(z) = z^3\), the result is the curve in the \(w\)-plane.

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The derivatives for the function \( f(z) = z^3 \) are given by
\[ f^{(1)}(z) = 3z^2, \quad f^{(2)}(z) = 6z \]
and \( f^{(3)}(z) = 6 \), with all higher derivatives being equal to zero. Upon evaluating these derivatives at the point \( p = r e^{i\theta} = 1 e^{i0} = 1 \), we have \( f^{(1)}(p) = 3, \quad f^{(2)}(p) = 6 \)
and \( f^{(3)}(p) = 6 \); together with the angle of the tangent vector to the straight line at the point \( p \) (i.e. \( \phi = \pi/2 \)) and the length of the straight line in the \( z \)-plane (\(|\xi| = 0.1\)), we find
\[
\tilde{\psi}_0 = \frac{1}{3} \text{Im} \left[ \frac{6}{3} e^{i\pi/2} \right] = \frac{2}{3} \sin \pi/2 = \frac{2}{3}
\]
and, similarly, \( \tilde{\psi}_1 = 0, \quad \tilde{\psi}_2 = -1/2.25, \quad \tilde{\psi}_3 = 0 \) and \( \tilde{\psi}_4 = 0.281481 \); thus, up to the fourth order in \(|\xi|\), we have \( \tilde{\psi} \approx 0.66225037 \).

This value can be compared with a numerically computed value. To do so, for the curve in the image plane, we need to compute the length of the curve and the total change in the angle of the tangent vector. This can be done by taking a large number of equally spaced points on the curve. By drawing a straight line between any two adjacent points, we get a series of segments that approximate the curve; the approximation improves as more points are taken. It is then a simple matter for a computer to calculate the length of each segment and to find the total length of the curve by summing up the lengths of all the segments. To compute the total change in the angle of the tangent vector, we first compute the angle that the first segment makes with the \( u \)-axis, and then we compute the angle that the final segment makes with the \( u \)-axis, by taking the difference between the two we find the change in the angle of the tangent vector between the start and the end of the curve. Upon taking \( 1 \times 10^5 \) such equally spaced points on the image curve, we have an numerically computed value for the average curvature, giving \( \tilde{\psi}_{\text{Num}} = 0.66224 \). By taking more points, it can be shown that the first four significant figures of this value have converged. Within these limits, it is clear that equation (6.24) and \( \tilde{\psi}_{\text{Num}} \) are in agreement.

\((d)\) Mean square of the average induced curvature

For the instantaneous curvature, we found it useful to compute the mean square curvature (see equation (4.4)), which is independent of the orientation of the vertex, and therefore serves as a useful measure of how strongly the conformal lattice is curved at any given point in the image plane. Similarly, we can compute a mean square value of the average induced curvature. We find that this leads to a series expansion in which the lowest order term is the mean square curvature given by equation (4.4). When the higher order terms in this expansion vanish, it means that all the image edges, at that particular point in the image plane, have a constant curvature. If, on the other hand, the higher order terms do not vanish, this means that Laplace’s law, for a two-dimensional or quasi-two-dimensional foam (in the constant volume case) used to realize the conformal lattice, cannot be perfectly satisfied for every edge. The magnitude of the higher order correction serves as an estimate of the discrepancy between the foam and the conformal map; in most practical cases, the discrepancy can be assumed to be negligible if the leading order correction in the expansion is small.

Consider again a vertex \( p \) in the \( z \)-plane to which there are connected a number of straight edges of length \(|\xi|\) separated by a constant angular separation, such as that shown in figure 3. Upon being mapped to the image plane, the point

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p is now located at $f(p)$ and the straight edges are transformed into arcs. If we were to sum up the average induced curvature, for each of the arcs emanating from the vertex at $f(p)$, we would find that it vanishes (this follows from writing all the terms in equation (6.24) explicitly and integrating over $\phi$). If, however, for each arc we square the average induced curvature and sum up the squares, we have a quantity that does not vanish. Thus, we can compute the mean square value of the average induced curvature $\tilde{\omega}$, which we denote by $\tilde{\omega}$; this can be done, most easily, by integrating equation (6.24) with respect to $\phi$ over the range $0 \leq \phi \leq 2\pi$, giving

$$\tilde{\omega} = \frac{1}{2\pi} \int_0^{2\pi} \psi(\phi)^2 \, d\phi = \tilde{\omega}_0 + \tilde{\omega}_1 |\xi|^2 + O(|\xi|^4),$$

where we find

$$\tilde{\omega}_0 = \frac{1}{2} \left( \frac{|f^{(2)}(p)|}{|f^{(1)}(p)|^2} \right)^2,$$

and

$$\tilde{\omega}_1 = \frac{1}{|f^{(1)}(p)|^2} \left( \frac{|f^{(2)}(p)|^4}{|f^{(1)}(p)|^4} \left( \frac{11}{32} \frac{5}{12} \text{Re} \left( \frac{f^{(1)}(p)f^{(3)}(p)}{(f^{(2)}(p))^2} \right) \right) + \frac{1}{8} \frac{|f^{(3)}(p)|^2}{|f^{(1)}(p)|^2} \right).$$

There are two circumstances under which $\tilde{\omega} = \tilde{\omega}_0 = \tilde{Q}$. The first, which is the trivial case, is when the lattice spacing $|\xi|$ of the original lattice in the $z$-plane vanishes. The second case is complex inversion $f(z) = 1/z$ (or, more generally, a bilinear transformation), for which we find $\tilde{\omega}_1 = \tilde{\omega}_2 = \tilde{\omega}_3 \ldots = 0$. The vanishing of all the higher order corrections means that all the curves emanating from the vertex at $f(p)$ are circular arcs. It follows that, for a two-dimensional or quasi-two-dimensional foam, Laplace’s law is satisfied by each of the arcs.

Let us now turn our attention to the mapping $f(z) = z^{1/2}$; in this case, we do not expect Laplace’s law to hold for each of the arcs. We have already computed the lowest order term in equation (6.26) and found that (see equation (4.5)).

$$\tilde{\omega}_0 = \tilde{Q} = \frac{1}{2R^2}.$$

Upon substituting $f(z) = z^{1/2}$ into equation (6.26), we find

$$\tilde{\omega}_1 = \frac{7}{128} \frac{1}{r^3} = \frac{7}{128} \frac{1}{R^6},$$

where we have transformed from the coordinates in the $z$-plane $(r, \lambda)$ to the $w$-plane coordinates $(R, \Lambda)$ by using the relationship $\sqrt{r} = R$. Assuming that the original lattice has a constant lattice spacing $|\xi|$, we can say that a two-dimensional foam, or a quasi-two-dimensional foam, is capable of producing a good approximation of a conformal lattice when

$$\tilde{\omega}_1 |\xi|^2 = \frac{7}{128} \frac{|\xi|^2}{R^6}$$

is small. We see that this happens at regions a large distance from the central (critical) point of the conformal lattice, i.e. when $R \gg (7/128)^{1/6}|\xi|^{1/3}$.

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7. Conclusion

Since its inception in the nineteenth century, the theory of two-dimensional foams has been the subject of elegant mathematical analysis, and it has been realized from time to time that conformal transformations are relevant to it—for example, they have been used by Weaire (1999) to prove the important decoration theorem—and doubtless conformal transformations can be used for other proofs as well. Since the hexagonal honeycomb plays a central role in the theory of foams, its transformations are particularly interesting, and many of them are experimentally realizable. The present paper is intended to provide a reasonably complete foundation for the future use of these transformations, especially as regards their effect upon the local curvature.

This work may also serve to stimulate further experimental investigations of two-dimensional foam structures of the kind investigated by Drenckhan et al. (2004). For this, the results of §6 are relevant and provide a limit to which conformal transformations can be realized using constant volume foams in Hele–Shaw cells.

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