INTRODUCTION TO QUANTUM MECHANICS AND THE QUANTUM-CLASSICAL TRANSITION

JOSÉ F. CARIÑENA, JESÚS CLEMENTE-GALLARDO, AND GIUSEPPE MARMO

Abstract. In this paper we present a survey of the use of differential geometric formalisms to describe Quantum Mechanics. We analyze Schrödinger and Heisenberg frameworks from this perspective and discuss how the momentum map associated to the action of the unitary group on the Hilbert space allows to relate both approaches. We also study Weyl-Wigner approach to Quantum Mechanics and discuss the implications of bi-Hamiltonian structures at the quantum level.

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1. Introduction

1.1. The need for a quantum theory and relevant mathematical structures. Interference phenomena of material particles (say, electrons, neutrons, etc) provide us with the most convincing evidence for the need to elaborate a new mechanics which goes beyond and encompasses classical mechanics. At the same time, ‘corpuscular’ behaviour of radiation, light, as exhibited in phenomena like photoelectric and Compton effects shows that also the description of radiation has to undergo deep changes. The relation between the corpuscular-like and the wave-like behaviour is fully captured by the following equation that we may call the Einstein–de Broglie relation

\[ p_j \, dx^j - E \, dt = \hbar (k_j \, dx^j - \omega \, dt). \quad (1) \]

This relation between the Poincaré 1-form on the phase-space over space-time and the optical 1-form on the optical phase-space establishes a relation between momentum and energy of the ‘corpuscular’ behaviour and the frequency of the ‘wave’ behaviour. The proportionality coefficient is the Planck constant.

The way we use this relation is to predict under which experimental conditions light of a given wave length and frequency would be detected as a corpuscle with a corresponding momentum and energy and vice-versa (i.e. when an electron would be detected as a wave in the appropriate experimental conditions).

If we examine more closely an interference experiment, like the double slit one, we find some peculiar aspects for which we do not have a simple interpretation in the classical setting.
If we perform the experiment in such a way that we make sure that, at each time, only one electron is present between the source and the screen, we find that the electron impinges on the screen at ‘given points’.

After few hundred electrons have passed, we find a picture of random spots distributed on the screen. However, with several thousands electrons, we get a very clear typical interference figure.

The same situation happens again if we experiment with photons (light quanta), with an experimental arrangement that makes sure that only one photon is present at each time.

This experiment suggests that the new theory must have a wave character (to take into account the interference aspects), statistical-probabilistic character along with an intrinsically discrete aspect. All this is quite counter-intuitive for particles, but it is even more startling for light. Within the classical setting we have to accept that it is not so simple to provide a single model capable of capturing these various aspects at the same time.

From the historical point of view, things developed differently because inconsistencies arose already in the derivation of the law for the spectral distribution of energy density of a black-body. Moreover, it was not possible to account for the stability of atoms and molecules along with the detected atomic spectra. We refer to [32] for an account of the experimental foundations of quantum theory and for other background material.

The efforts of theoreticians gave rise to two alternative, but equivalent formulations of quantum mechanics. They are usually called the Schrödinger picture and the Heisenberg picture. As we are going to see in the coming sections, the first one uses as a primary object the carrier space of states, while the latter uses as carrier space the space of observables.
Schrödinger equation has the form

\[ i\hbar \frac{d}{dt}\psi = H\psi \]  

The complex valued function \( \psi \) is called the **wave function**, it is defined on the configuration space of the system we are considering, and it is interpreted as a probabilistic amplitude. This interpretation requires that

\[ \int_D \psi^* \psi \, d\mu = 1; \]

i.e. because of the probabilistic interpretation \( \psi^* \psi \, d\mu \) must be a probability density and therefore \( \psi \) must be required to be square-integrable. Thus wave functions must be elements of a Hilbert space of square integrable functions.

The operator \( H \), acting on wave functions, is the infinitesimal generator of a one-parameter group of unitary transformations describing the evolution of the system under consideration.

These are the basic ingredients appearing in the Schrödinger evolution equation. The presence of the new fundamental constant \( \hbar \) within the new class of phenomena implies some fundamental aspects completely new from the previous classical ones. It is clear that any measurement process requires an exchange of energy (or information) between the object we are measuring and the measuring apparatus. The existence of \( \hbar \) requires that these exchanges cannot be made arbitrarily small and therefore idealized to be negligible. Thus the presence of \( \hbar \) in the quantum theory means that in the measurement process we cannot conceive of a sharp separation between the ‘object’ and the ‘apparatus’ so that we may forget of the apparatus altogether.

We should remark that even if the apparatus may be described classically, it is to be considered as a quantum system with a quantum interaction with the object to be measured. Moreover, in the measuring process, there is an inherent ambiguity in the ‘cut’ between what we identify as the object and what we identify as apparatus \[41\] [42].

The problem of measurement in quantum theory is a very deep one and goes beyond the scope of these notes. We may simply mention that within the von Neumann formulation of Quantum Mechanics (see [80]) the measurement problem gives rise to the so called ‘wave-function collapse’. The state vector of the system we are considering, when we measure some real dynamical variable \( A \), is projected onto one of the eigenspaces of \( A \) with some probability that can be computed. As the scope of these notes is only to highlight the various mathematical structures present in the different formulations of quantum mechanics we shall adhere to the von Neumann projection prescription.

To avoid technicalities we shall mainly work within a finite dimensional framework, i.e. with finite dimensional Hilbert spaces. In this setting we are going to deal with the Schrödinger and Heisenberg pictures and we shall also provide a geometrical unifying version of the two pictures. However,
to be able to consider the quantum-classical transition in a meaningful way, we shall consider the Weyl–Wigner formalism in infinite dimensional Hilbert spaces.

Before closing this introduction and to better put into perspective the Schrödinger and the Heisenberg pictures we are going to make a few general considerations on the minimal mathematical structure required for the description of a physical system.

From a general point of view, we need three ingredients:

- a space of states, that we denote as $\mathcal{S}$,
- a space of observables, that we denote as $\mathcal{O}$ and
- a real valued pairing $\mathcal{O} \times \mathcal{S} \to \mathbb{R}$. This pairing, which produces a real number out of an observable and a state, represents the measuring operation.

In Quantum Mechanics, we have two main pictures.

- in the Schrödinger picture, $\mathcal{S}$ is associated with a Hilbert space $\mathcal{H}$ and the set of dynamical variables (the observables) is a derived concept. Observables are identified with self-adjoint bounded operators on $\mathcal{H}$.
- in the Heisenberg picture the situation is complementary: the set of dynamical variables (the observables) is the primary concept. They are assumed to be (the real part) of a $\mathbb{C}^*$-algebra $\mathcal{A}$. The states, on the other hand, are a derived concept defined as a proper subset of the set of linear functionals on $\mathcal{A}$.

It should be stressed, however, that a physical system requires, in addition to either one of the two primary carrier spaces, a concrete realization of it to allow us to identify the physical variables. This last requirement is often overlooked in the literature. We can clarify this last point with a specific example taken from classical mechanics but that applies equally well within Quantum Mechanics.

**Example 1.** Let us consider the carrier space for a classical system to be a phase space $(\mathbb{R}^3 - \{\vec{0}\}) \times \mathbb{R}^3$ equipped with a Poisson bracket. Considering coordinates $(\vec{\xi}, \vec{\eta})$ we define the Poisson structure in the form

$$\{\xi_j, \xi_k\} = 0, \quad \{\eta_j, \eta_k\} = \lambda \epsilon_{jkl} \frac{\xi_l}{\|\xi\|^3}, \quad \{\xi_j, \eta_k\} = \delta_{jk}. $$

This carrier space is appropriate to describe an electron-monopole system or a massless particle with helicity. Indeed if we set

$$\xi_j = x_j \text{ (position)} \quad \eta_j = p_j \text{ (momentum)},$$

the resulting Poisson brackets take the form required in the electron-monopole system. The brackets of the momenta are thus proportional to the magnetic field of the monopole.

If we set $\xi_j = p_j$ and $\eta_j = x_j$, on the other hand, we endow the carrier space with the Poisson structure required to model the dynamical behaviour of a massless spinning particle. The cubic term in the denominator of the
bracket of two position coordinates accounts then for the fact that a zero-
rest-mass particle cannot be reduced to rest. And the non-vanishing of these
brackets is taking into account that massless particles cannot be localized in
space.

A very similar situation prevails in the corresponding quantum situation.
For a more detailed description of these problems, the interested reader is
addressed to [4].

In conclusion, the description of a physical system requires not only an
abstract mathematical model (a Poisson manifold, a Hilbert space, a \(\mathbb{C}^*\) -
algebra,…) but also a specific realization with an identification of the
physical variables.

For further reading see [50, 79, 73, 72, 52, 38, 11].

2. Two formulations of Quantum Mechanics

Our goal in this section is to present the very basic ingredients of Quantum
Mechanics, just to establish the departure point of the analysis we will carry
on in the following sections. We will just mention the two most familiar
formulations of Quantum Mechanics, our aim being to identify the relevant
mathematical structures required for their definition. Once we know them,
they will be studied in much more detail and from the point of view of
Geometry in the following sections. For more details see [26, 25, 46, 61, 82,
34].

2.1. The Schrödinger formalism. In this formalism the carrier space is
the Hilbert space of states of the system \(\mathcal{H}\), very often the space of com-
plex square integrable functions defined on some spatial domain \(D \subset \mathbb{R}^n\),
identified with the configuration space. This is the set of pure states \(S\) of
our system represented by the wave-functions we mentioned in the intro-
duction. Observables are then defined as self-adjoint operators acting on this
Hilbert space. Thus, the set of operators \(\mathcal{O}\) depends, for its definition, on
the definition of the set of states. The pairing is defined in terms of the
Hermitian structure of the Hilbert space associating a real value to the pair
(pure state, observable) as

\[(\psi, A) \mapsto \langle A \rangle = \langle \psi, A\psi \rangle \in \mathbb{R}.\]

Dynamics is defined on this space by means of the Schrödinger equation

\[
i\hbar \frac{d}{dt} \psi = H \psi, \quad \psi \in \mathcal{H},
\]

where \(H\) is the Hamiltonian operator of the system and is assumed to be
Hermitian. In more technical terms, we can consider thus a vector field
corresponding to the equations of motion

\[
\frac{d}{dt} \psi = \frac{1}{i\hbar} H \psi,
\]
which becomes the infinitesimal generator of a one-parameter group of unitary transformations. As we would like to concentrate our attention on the geometrical aspects, in this paper we will assume, for the sake of simplicity, that the Hilbert space is finite dimensional.

In the particular case of a one-level system, we can introduce two real variables $q$ and $p$ to represent $\psi$ (its real and imaginary parts respectively $\psi = \frac{1}{\sqrt{2}}(q + ip)$) and the Schrödinger equation takes the form (see [28, 67, 62]):

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{\hbar} \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$ 

We conclude thus that the description of the dynamics in terms of real coordinates is represented by a Hamiltonian vector field. Actually this is a general property, also valid for infinite dimensional systems: Schrödinger equations of motion will be a particular Hamiltonian dynamics on some infinite dimensional symplectic vector space.

We can elaborate a little further on this statement. If $\mathcal{H}$ denotes a complex Hilbert space we can decompose the Hermitian product $\langle \cdot, \cdot \rangle$ into real and imaginary parts as follows:

$$\langle \psi, \phi \rangle = g(\psi, \phi) + i\omega(\psi, \phi), \quad \forall \psi, \phi \in \mathcal{H},$$

where the real and imaginary parts represent an Euclidean and a symplectic product respectively. On the associated realification of $\mathcal{H}$ (say $\mathcal{H}_R$) we have a complex structure $J : \mathcal{H}_R \to \mathcal{H}_R$ satisfying $J^2 = -\mathbb{I}$. Thus the carrier space is endowed with a Kähler structure. Vector fields associated with the Schrödinger equation are not only symplectic, they are also Killing vector fields or, more specifically, they are Kählerian vector fields, i.e. they preserve the Kähler structure.

2.2. **Heisenberg formalism.** In this picture observables are associated with Hermitian operators. They encode the measurable information of the system and the dynamics must now be defined as a flow on this space. States are thus defined as normalized positive functionals on Hermitian operators.

Hermitian operators do not carry an associative algebra structure (i.e. the product of two Hermitian operators will not be, in general, Hermitian). However, it is possible to endow the set with one scalar product and two binary products:

- The scalar product is the restriction to the set $\mathcal{O}$ of Hermitian operators of the scalar product of two complex matrices defined as

  $$\langle A, B \rangle = \text{Tr}(A^*B).$$

  In the case of Hermitian matrices this becomes

  $$\langle A, B \rangle = \text{Tr}(AB) \quad \forall A, B \in \mathcal{O}. \quad (4)$$

- The first binary operation is the Abelian real Jordan algebra product

  $$A \circ B := \frac{1}{2}[A, B]_+ = \frac{1}{2}(AB + BA) = \frac{1}{4}((A + B)^2 - (A - B)^2).$$
Let us recall, for completeness, the definition of Jordan algebra:

**Definition 1.** A (non-associative) algebra \((A, \cdot)\) is called a **Jordan algebra** if the composition law is commutative and for any two arbitrary elements \(A, B \in A\), \((AB)A^2 = A(AB^2)\).

With this definition we can conclude

**Lemma 1.** \((O, \circ)\) is a Jordan algebra.

**Proof.** The commutativity is obvious. The second condition follows from the associativity of the original product:

\[
[[A, B]_+, A^2]_+ = (AB + BA)A^2 + A^2(AB + BA) = (ABA^2 + BA^3 + A^3B + A^2BA)
\]

\[
[A, [B, A^2]_+]_+ = A(ABA^2 + A^2B) + (BA^2 + A^2B)A = ABA^2 + A^3B + A^3B + A^2BA
\]

The map \(\{A, B, C\} = (A \circ B) \circ C - A \circ (B \circ C)\) is called the **associator** of the algebra, and the algebra is associative if and only if the associator is identically zero.

The second binary structure is a Lie algebra structure

\[
[A, B]_\pm = \frac{1}{i\hbar} (AB - BA),
\]

which comes from the fact that for any Hermitian operator \(A\), \(-iA\) is an infinitesimal generator of the unitary group.

Therefore, multiplying each element in the set by the imaginary unit we get the Lie algebra of the unitary group.

**Proposition 1.** The scalar product (4) is also invariant with respect to this new product, and we have:

\[
\langle [A, B]_-, C \rangle = \langle A, [B, C]_- \rangle, \quad \langle [A, B]_+, C \rangle = \langle A, [B, C]_+ \rangle.
\]

Moreover we also have the compatibility relation

\[
[A, B \circ C]_- = [A, B]_- \circ C + B \circ [A, C]_-
\]

i.e. \(ad_A\) is a derivation of the Jordan algebra for any \(A \in O\).

**Proof.** These properties follow directly from the definitions.

Actually these two structures can be combined together in the notion of Lie–Jordan algebra (see [29] for details).

Now we can proceed to define dynamics on this algebra. It is introduced by means of the **Heisenberg equation**, which makes use of the skew-symmetric structure of the algebra:

\[
\frac{d}{dt} A = [A, H]_- \quad A \in A
\]

where \(H\) is the Hamiltonian of the system.
Remark 1. The equations of motion written in this form are necessarily derivations of the two products (i.e. a derivation of the Lie–Jordan product) and can be considered hence ‘intrinsically Hamiltonian’. In the Schrödinger picture, if the vector field is not anti-Hermitian, the equation still makes sense, but the dynamics is not Kählerian.

3. Geometric Quantum Mechanics I: The Schrödinger picture

Our purpose in this section is to present Quantum Mechanics from a geometric perspective. We choose to do it in the case of finite dimensional Hilbert spaces (i.e. systems with a finite number of energy levels, for instance) because of their simplicity, although all the objects that we are going to introduce make sense in general for infinite dimensional Hilbert spaces as well. Further details can be found in [31, 15, 16, 51, 23, 22, 8, 43, 74, 33, 10, 77, 9, 12, 13, 14, 53, 20, 2, 1, 7].

3.1. The Hilbert space as a real differentiable manifold. Thinking in terms of the Schrödinger picture, we know that the set of pure states $S$ is associated with a Hilbert space. Let us study in detail the geometrical objects which play a role in the definition of the dynamics within the Schrödinger picture.

We want to consider the space $S$ as a differentiable manifold instead of a linear space. We can consider the complex vector space as a real vector space (i.e. a ‘realification’) if we consider the natural complex structure $J$ ($J^2 = -1$) of the Hilbert space. But to consider a differentiable structure implies that we have to associate tensorial objects with the vectors and linear maps which we have studied so far.

3.1.1. The tensors.

- The first task is the association of vectors of $\mathcal{H}$ with vector fields on the manifold. Being a linear space, $\mathcal{H}$ can be identified with the tangent space at any point, and hence we can write $T\mathcal{H} \sim \mathcal{H} \times \mathcal{H}$. Thus it makes sense to consider, for an element $\eta \in \mathcal{H}$ the vector field $X_\eta$ defined as a section of $T\mathcal{H}$:

$$X_\eta : \psi \mapsto (\psi, \eta).$$

This vector field acts on a function $f$ as:

$$X_\eta(f)(\psi) = \frac{d}{dt}f(\psi + t\eta)|_{t=0}.$$ 

Besides, these constant sections define a separating set in the Hilbert space.

- Let us consider the Kähler structure on the Hilbert space. Let $\langle \psi_1, \psi_2 \rangle \in \mathbb{C}$ denote the scalar product of two vectors $\psi_1$ and $\psi_2$, and consider the structure of real manifold. The scalar product above is written as $\langle \psi_1, \psi_2 \rangle = g(X_{\psi_1}, X_{\psi_2}) + i \omega(X_{\psi_1}, X_{\psi_2})$, where $g$ is now
a symmetric tensor and $\omega$ a skew-symmetric one. The properties of the Hermitian product ensure that:

- the symmetric tensor is positive definite and non-degenerate, and hence defines a Riemannian structure on the real vector space.
- the skew-symmetric tensor is also non degenerate, and is closed with respect to the natural differential structure of the vector space. Hence, the tensor is a symplectic form.

As the inner product is sesquilinear, it satisfies

$$\langle \psi_1, i\psi_2 \rangle = i\langle \psi_1, \psi_2 \rangle, \quad \langle i\psi_1, \psi_2 \rangle = -i\langle \psi_1, \psi_2 \rangle.$$ 

This implies

$$g(X\psi_1, X\psi_2) = \omega(JX\psi_1, X\psi_2),$$

or, equivalently, that the triple $(J, g, \omega)$ defines a Kähler structure.

These two tensors $g$ and $\omega$ are in a covariant form. We can also define their contravariant forms by considering the dual vector space $\mathcal{H}^*$ identified with $\mathcal{H}_\mathbb{R}$, for instance via the metric $g$ (which is non-degenerate). The association of vectors of $\mathcal{H}$ with vector fields can be extended to associate also 1-forms with the elements of $\mathcal{H}^*$. We will have then an assignment $\mathcal{H}^* \ni \tilde{\psi} \mapsto \langle \tilde{\psi}, \psi \rangle$ for any $\tilde{\psi} \in \mathcal{H}^*$, i.e. we write $T^*\mathcal{H} \sim \mathcal{H} \times \mathcal{H}^*$. In this way we define the contravariant tensors $G$ and $\Omega$, which allow us to define a scalar product on $\mathcal{H}^*$ as:

$$\langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle = G(\alpha_{\psi_1}, \alpha_{\psi_2}) + i\Omega(\alpha_{\psi_1}, \alpha_{\psi_2}) \quad \forall \tilde{\psi}_1, \tilde{\psi}_2 \in \mathcal{H}^*.$$ 

If we select an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$, we may define coordinates by setting $\langle e_k | \psi \rangle = z_k(\psi) = \frac{1}{2}(q_k + i p_k)(\psi)$, and we have used Dirac’s notation for bras and kets.

In these coordinates we have a contra-variant version of the Euclidean structure given by $G = \sum_{k=1}^n \left( \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial p_k} \right)$ and the Poisson tensor $\Omega = \sum_{k=1}^n \left( \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k} \right)$ while the complex structure has the form $J = \sum_{k=1}^n \left( \frac{\partial}{\partial p_k} \otimes dq_k + \frac{\partial}{\partial q_k} \otimes dp_k \right)$.

In terms of complex coordinates the Hermitian structure has the form $h = \sum_{k=1}^n d\bar{z}_k \otimes dz_k$. The corresponding contra-variant form is given by

$$G + i \Omega = \sum_{k=1}^n \left( \frac{\partial}{\partial q_k} - i \frac{\partial}{\partial q_k} \right) \otimes \left( \frac{\partial}{\partial q_k} + i \frac{\partial}{\partial q_k} \right) = 4 \sum_{k=1}^n \frac{\partial}{\partial z_k} \otimes \frac{\partial}{\partial \bar{z}_k}.$$
We may now define binary products on functions by setting
\[
\{f_1, f_2\} = \sum_{k=1}^{n} \left( \frac{\partial f_1}{\partial q_k} \frac{\partial f_2}{\partial p_k} - \frac{\partial f_1}{\partial p_k} \frac{\partial f_2}{\partial q_k} \right),
\]
\[
\{f_1, f_2\}^+ = \sum_{k=1}^{n} \left( \frac{\partial f_1}{\partial q_k} \frac{\partial f_2}{\partial q_k} + \frac{\partial f_1}{\partial p_k} \frac{\partial f_2}{\partial p_k} \right),
\]
\[
\langle f_1 | f_2 \rangle = 4 \sum_{k=1}^{n} \frac{\partial f_1}{\partial z_k} \frac{\partial f_2}{\partial \bar{z}_k}.
\]

3.1.2. Additional tensor fields. In addition, we can consider the linear structure of the Hilbert space and associate with it the Liouville vector field:
\[
\Delta : \mathcal{H} \rightarrow T\mathcal{H}, \quad \psi \mapsto (\psi, \psi),
\]
which, as usual, allows us to define homogeneous polynomial functions: a function \(f \in C^\infty(\mathcal{H})\) is homogeneous of degree \(k\) if \(\Delta(f) = kf\). Combining this tensor with the complex structure, it is possible to define a new vector field
\[
\Gamma = J(\Delta).
\]
\(\Gamma\) and \(\Delta\) commute and therefore generate an integrable distribution which defines a foliation on the Hilbert space.

These two tensors also help us to define a way of associating a tensor to any operator acting on \(\mathcal{H}\). There are several ways to do it, some are more immediate than others:
- We can associate a quadratic function \(f_t\) to any constant symmetric 2-tensor field \(t\) in the form:
  \[
  f_t(\psi) = \frac{1}{2} t(\Delta, \Delta)(\psi) = \frac{1}{2} t(\psi, \psi);
  \]
  and similarly for higher order tensors.
- Skew-symmetric 2-tensors \(\gamma\) are transformed into functions in a similar way, using also the complex structure:
  \[
  f_\gamma(\psi) = \frac{1}{2} \gamma(\Delta, \Gamma)(\psi).
  \]
When we consider as skew-symmetric tensor the symplectic form \(\omega\), the resulting function is the Hamiltonian function generating the one-parameter group of unitary transformations which defines the multiplication by a phase.
- Any linear operator \(A : \mathcal{H} \rightarrow \mathcal{H}\) can be identified with:
  - a (1:1) tensor field
  \[
  T_A : T\mathcal{H} \rightarrow T\mathcal{H} \quad T_A : (\phi, \psi) \mapsto (\phi, A\psi),
  \]
  - or two different vector fields
  \[
  X_A = T_A(\Delta) : \mathcal{H} \rightarrow T\mathcal{H} \quad X_A : \psi \mapsto (\psi, A\psi),
  \]
and
\[ Y_A = T_A(J(\Delta)) : \mathcal{H} \rightarrow T\mathcal{H}, \quad Y_A : \psi \mapsto (\psi, JA\psi). \]  
(14)

If \( A \) is Hermitian, then \( X_A \) corresponds to a gradient vector field with respect to the Kähler structure, while \( Y_A \) corresponds to the Hamiltonian vector field associated to the evaluation function of the operator (i.e. \( \psi \mapsto \langle \psi, A\psi \rangle \)). These associations have different properties:

* the mapping \( A \mapsto T_A \) is an isomorphism of associative algebras (and as a result also with respect to the Lie algebra structure). As we are interested in the ‘realification’ of operators acting on the complex vector space, we shall restrict our considerations to tensors satisfying \( T_A J = J T_A \).
* the mappings \( A \mapsto X_A \) or \( A \mapsto Y_A \) on the other hand are only isomorphisms of Lie structures, and the properties of the associative product of operators is lost.

• Occasionally, to make easier the comparison with the usual formalism, we consider the space \( S \) as a real manifold but, at each point \( \psi \), we may consider \( T_\psi S \) as a complex vector space. In this case, vector fields would have a real and an imaginary part. However, even when this notation might be misleading, we shall always be considering the derivations on \( S \) in the real sense. Hence, we shall not be considering derivations with respect to complex variables and, as a result, we do not need to consider complex analyticity for our functions. We will have, though, complex valued functions arising as the contraction of complex valued vector field with complex values one forms.

By using the ‘mixed’ point of view, it is also possible to associate a complex valued quadratic function on \( \mathcal{H} \) to any linear transformation \( A : \mathcal{H} \rightarrow \mathcal{H} \) by defining
\[ 2f_A(\psi) = g(\Delta, T_A(\Delta))(\psi) + i\omega(\Delta, T_A(\Delta))(\psi) = \langle \psi, A\psi \rangle. \]  
(15)

Given the quadratic function \( F \in \mathcal{F}(\mathcal{H}) \), associated to the operator \( \hat{F} \), the 1-form \( dF \) acts on a vector field \( X_\eta \) as
\[ dF(X_\eta)(\psi) = X_\eta(F)(\psi) = \frac{1}{2} \frac{d}{dt} \langle \psi + t\eta, \hat{F}(\psi + t\eta) \rangle |_{t=0}. \]

The Hamiltonian vector field corresponding to \( F \) by \( \omega \):
\[ dF(X_\eta)(\psi) = \frac{1}{2} \langle \psi, \hat{F}\eta \rangle + \frac{1}{2} \langle \eta, \hat{F}\psi \rangle = g(X_{\hat{F}}, X_\eta)(\psi) \]
\[ = \omega(Y_{\hat{F}}, X_\eta)(\psi) = (iY_{\hat{F}}\omega)(X_\eta)(\psi), \]
where we used the relation between the Riemannian and the symplectic Kähler forms and the definition of the vector fields.

In conclusion tensor fields on the real manifold \( S \) will be considered as modules over complex-valued functions on \( S \).
Please notice that the above definitions are intrinsic and can be applied whenever the tensors used are available. As a result, it is also possible to define these objects at the level of infinite dimensional Hilbert spaces.

3.1.3. Observables as quadratic functions. Our geometrization procedure has allowed to replace operators with complex valued functions $A \mapsto f_A$. This association is clearly injective, but it is not onto, i.e. there are functions on $\mathcal{H}$ which are not quadratic. The association is obviously linear but the image is not closed under the pointwise product (the product of two quadratic functions is not quadratic but quartic). Therefore the product cannot be the image of an operator. Thus the pointwise product does not allow to transfer the associative product of operators to the set of quadratic functions. However, we might consider a non-local product, inner in the space of quadratic functions, and defined as

$$ (f_A \star f_B)(\psi) = (f_{AB})(\psi). $$

This product is not commutative and non-local. It requires, however, that we start with operators, or their tensorial versions as $(1,1)$ tensor fields. It seems advisable to describe this product only in terms of the tensors already available on the real differentiable manifold $\mathcal{S}$.

We notice first that if $A$ and $B$ are Hermitian, with associated real valued quadratic functions $f_A$ and $f_B$, the product $f_A \star f_B$ need not be a real valued function (for the product of two Hermitian operators is not Hermitian, in general). Using the fact that these functions are quadratic, we may consider the quantity $G(df_A, df_B)$, where $G$ is the contravariant form of the metric tensor. This combination is clearly a quadratic function, because we know that $G$ satisfies $L_\Delta G = -2G$. By straightforward computations, we can obtain that

$$ G(df_A, df_B) = f_{AB+BA}. $$

In a similar way, by using the skew-symmetric tensor $\Omega$, we obtain

$$ \Omega(df_A, df_B) = -if_{[A,B]} = f_{[A,B]}^-. $$

Thus the Lie–Jordan algebra structure on the space of real quadratic functions can be extracted from the Hermitian tensor. For later use (when we consider the complex projective space) it is convenient to characterize these functions defining the Lie–Jordan structure without using the notion of quadratic function (for it does not make sense on nonlinear spaces). It is possible to show that:

**Lemma 2.** Given a function $f$, the Hamiltonian vector field $X_f = \Omega(df, \cdot)$ preserves the metric tensor $G$, i.e. $L_{X_f}G = 0$ if and only if $f$ is a quadratic function associated with some Hermitian operator.

Thus, a subset of functions in $\mathcal{F}(\mathcal{H})$ can be selected and defines a Lie–Jordan algebra with the tensors $G$ and $\Omega$ if and only if they are real quadratic. Moreover, if we consider this subset of functions we get

$$ G(df_A, df_B) + i\Omega(df_A, df_B) = f_A \star f_B. $$
By linearity, we can extend these operations to complex valued functions. Then, this operation defines a $C^*$-algebra structure on the space of complex valued functions whose real and imaginary parts are associated with Hermitian operators. The norm of this $C^*$-algebra is given by the usual sup norm, i.e. the supremum of the values that the operator takes on normalized states.

In this way, our geometrization procedure has reproduced the algebra of observables in terms of real valued functions on $S$.

3.1.4. **Transformations.** As we have already remarked, the evolution of a quantum system defines a one-parameter group of transformations of $S$ which preserve the Kähler structure. All transformations preserving the Kähler structure form the set of unitary transformations of $\mathcal{H}$, and in the case of finite dimensional Hilbert spaces ($\dim_\mathbb{C} \mathcal{H} = N$), they provide a realization of the unitary group $U(N)$.

In order to represent these operators in our setting, we can consider the set of Hermitian operators and use the vector field association (13). The one parameter group of unitary transformations associated with the Hermitian operator $A$ is

$$U(\alpha) = e^{-i\alpha A/\hbar}.$$  \hfill (17)

3.2. **The complex projective space.** The probabilistic interpretation of states assigns a physical meaning only to 'normalized wave-functions by means of the probability densities $\psi^*\psi d\mu$'. Thus the meaningful physical space is the complex projective space associated to the Hilbert space $\mathcal{H}$. Then, the next step is to induce the geometrical structures we have considered above onto the complex projective space.

The usual way to define the complex projective space is by means of equivalence classes. The complex vectors $\psi_1$ and $\psi_2$ are considered to be equivalent if there exists a nonzero complex number $\lambda$ such that $\psi_2 = \lambda \psi_1$. We can denote the equivalence class by one of its representatives, say $[\psi]$. If we write $\lambda = \rho e^{i\alpha}$ ($0 < \rho \in \mathbb{R}, \alpha \in \mathbb{R}$) we see immediately that equivalence classes are orbits of the group $S^1 \times \mathbb{R}_+$. Thus, removing the zero vector, the complex projective space is the set of orbits of that group acting on $\mathcal{H} - \{0\}$. The infinitesimal generators of the action can be easily determined: $\Delta$ is the generator of the modulus part (dilations) while $\Gamma = J(\Delta)$ is the generator of the phase change. If we consider the vector space endowed with its canonical Kähler structure $(J, g, \omega)$, we can further characterize $\Gamma$ as being the Hamiltonian vector field corresponding to the function $\frac{1}{2}g(\Delta, \Delta)$:

**Lemma 3.** $\Gamma$ is the Hamiltonian vector field corresponding to the quadratic function associated with the identity operator.

**Proof.** Having considered the function $\frac{1}{2}g(\Delta, \Delta)$, we have

$$d \left( \frac{1}{2}g(\Delta, \Delta) \right) (X_\eta) = g(\Delta, X_\eta) = \omega(J\Delta, X_\eta) = i_\Gamma \omega(X_\eta).$$
The vector fields $\Delta$ and $\Gamma$ commute and therefore define an involutive distribution, hence we can consider the foliation defined by such vector fields. The corresponding quotient manifold, is again the complex projective space $\mathbb{P}\mathcal{H}$.

This remark allows us to consider projectable tensorial quantities. For instance, we have:

**Lemma 4.** For each Hermitian operator $A$, the expectation value function, defined as

$$e_A(x) = \frac{\langle x, Ax \rangle}{\langle x, x \rangle},$$

is invariant under $\Delta$ and $\Gamma$.

**Proof.** The invariance under the Liouville vector field is immediate. The invariance under $\Gamma$ follows from the fact that

$$\Gamma(e_A) = \{e^I, e_A\} = e_{[I,A]} = 0.$$

But hence we proved:

**Theorem 1.** The space of the expectation values functions of Hermitian operators projects onto the quotient space defined by the foliation generated by $\Delta$ and $\Gamma$.

A general operator, dynamical variable, can be decomposed into real and imaginary parts, both parts given by Hermitian operators. Therefore we can say that both parts project onto the complex projective space.

### 3.2.1. Eigenvalues and eigenstates.

In the ‘geometrized’ Hilbert space description we have to recover now the description of the ‘eigenvectors’ and ‘eigenvalues’.

To this aim is appropriate to introduce expectation values associated with Hermitian operators, $A = A^+$:

$$e_A(\psi) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}.$$

We find that:

1. Critical points of $\text{de}_A$ correspond to the eigenvectors of $A$.
2. Values of $e_A$ at critical points are the corresponding eigenvalues of $A$.

**Remark 2.** Critical points of $\text{de}_A$ coincide with the critical points of the corresponding Hamiltonian vector field $\Omega(\text{de}_A)$ or of the corresponding gradient vector field $G(\text{de}_A)$. 
3.2.2. **Observables on the complex projective space.** We have seen that in our geometrization the algebra of observables can be recovered in terms of functions with the help of the contravariant tensors $G$ and $Ω$. We noticed that expectation value functions are projectable and therefore it makes sense to consider the binary products $G(de_A, de_B)$ and $Ω(de_A, de_B)$. Unfortunately, because of $L_∆G = -2G$ and $L_∆Ω = -2Ω$ the result of those operations on projectable functions will not be projectable. Thus, in order to make them inner operations, we may use a conformal factor for both tensors, as, for instance $G_P = ⟨ψ, ψ⟩G$ and $Ω_P = ⟨ψ, ψ⟩Ω$. In this way we would define projectable tensors, and hence inner products of projectable functions. A new problem arises though: if we define the bracket corresponding to the skew-symmetric part $Ω_P(df, dh)(ψ) = {f, h}_P(ψ) = ⟨ψ, ψ⟩{f, h}(ψ)$, this new bracket does not satisfy the Jacobi identity. Indeed, to make it to satisfy the Jacobi identity we have to consider a Jacobi bracket instead of a Poisson one, in the form

$$[f, h] = {f, h}_P + fL_Xh - hL_Xf,$$

(19)

where $X = Ω(d⟨ψ, ψ⟩)$ is the Hamiltonian vector field associated with the function $⟨ψ, ψ⟩$. Now, for the expectation values function we find that

$$[e_A, e_B] = {e_A, e_B}_P, \quad ∀A, B,$$

because the function $⟨ψ, ψ⟩$ is a central element for the subalgebra generated by the expectation value functions. Therefore, the use of the conformal tensors $G_P$ and $Ω_P$ allows us to define a $C^*$-algebra structure on the space of expectation value functions, which are projectable onto the complex projective space.

By explicit computation it is possible to show that

$$e_A ⋆ e_B = e_{AB} = G_P(de_A, de_B) + iΩ_P(de_A, de_B) + e_Ae_B.$$

If we consider the projection $π : ℋ_0 = ℋ - {0} → ℋ$ we may identify $π^*(F(ℋ))$ with the subalgebra of $F(ℋ)$ satisfying the conditions

$$df(∆) = 0; \quad df(J(∆)) = 0.$$

Within this subalgebra we may further restrict to those functions $f$ such that $Y_f = Ω(df)$ satisfy $L_YG = 0$. This subset of functions gives rise to a $C^*$-algebra.

**Remark 3.** The symmetric product of the expectation value function associated with a given Hermitian operator is

$$G_P(de_A, de_A) = \frac{⟨ψ, A^2ψ⟩}{⟨ψ, ψ⟩} - \frac{⟨ψ, Aψ⟩⟨ψ, Aψ⟩}{⟨ψ, ψ⟩^2}.$$

It represents the dispersion of the main value of the observable corresponding to the operator $A$ when measured in the pure state $ψ$. Therefore the
square of the corresponding Hamiltonian vector field is strictly related to the uncertainty in the measurement of $A$ in the pure state $\psi$.

### 3.3. Quantum dynamics

Let us consider now the dynamics. On the Hilbert space considered as a real differential manifold it is possible to rewrite Schrödinger equation by using the tensors introduced above so as to become

$$\dot{\psi} = -J\hat{H}\psi,$$

where we took $\hbar = 1$, and $J$ is the complex structure.

The solutions of this equation corresponds to the flow of the vector field

$$Y_H = T_H(J(\Delta)).$$

Having introduced a vector field to describe the dynamics we may now state a few properties:

i) $Y_H$ is Hamiltonian with Hamiltonian function $f_H = \frac{1}{2}\langle \psi, H\psi \rangle$. It is projectable onto the complex projective space.

ii) The vector field associated with the Hamiltonian function $e_H = \frac{\langle \psi, H\psi \rangle}{\langle \psi, \psi \rangle}$ projects onto the complex projective space. It projects onto the same vector field associated with $Y_H$.

iii) Critical points of $e_H$ (projected) on $PH$ correspond to the eigenvectors of $H$ and the values of $e_H$ at those points correspond to the eigenvalues.

iv) The Lie algebra of symmetries for the dynamics is generated by the expectation-value-functions associated to Hermitian operators commuting with $\hat{H}$.

v) The dynamics can be written in terms of Poisson bracket and defines a derivation for the $\star$-product.

### 4. Geometric Quantum Mechanics II: the Heisenberg picture

#### 4.1. Introduction

Following the algebraic approach advocated by Segal [75] and Haag and Kastler [40], we consider the space of observables as the collection of all the self-adjoint elements of a $C^*$-algebra with identity element [78].

States are identified as the elements of the convex body $S = \{ \phi \in A^* | \phi(A^*A) \geq 0, \forall A \in A; \phi(1) = 1 \}$. A state is pure if it cannot be written as a convex combination of other two states.

It is not difficult to show in the finite-dimensional case, for the infinite-dimensional case it is a theorem by Gleason [35], that any state can be written in the form $\phi(A) = \text{Tr} \rho_\phi A$. When $\dim \mathcal{H} = \infty$ density states are characterized by the property of being normal states on the von Neumann algebra of bounded operators (i.e. completely additive states).

Moreover, $\rho_\phi$ is a no-negatively defined operator of $gl(\mathcal{H})$, i.e. those $\rho_\phi \in gl(\mathcal{H})$ which can be written in the form $\rho_\phi = T^+T$ for some $T \in gl(\mathcal{H})$ and, in addition, satisfy $\text{Tr} \rho_\phi = 1$. Thus, $S$ is a convex body in the affine hyperplane in $u^*(\mathcal{H})$, determined by the equation $\text{Tr} \rho_\phi = 1$. The tangent
space to this affine hyperplane at a point is therefore identified with the space of traceless Hermitian operators, and it is in a one-to-one correspondence with the Lie algebra of the group $SU(H)$.

4.2. The geometrical description of the Heisenberg picture. Now, we are going to see how it is possible to obtain the Heisenberg description in geometrical terms.

A specific way to ‘geometrize’ the Lie algebra structure of $u(H)$ is to associate with it a linear Poisson tensor on the dual vector space $u^*(H)$ as follows. As $H$ is assumed to be finite-dimensional, we can identify $u(H)$ with the space of real valued linear functions on its dual space, i.e. $u(H) = \text{Lin}(u^*(H), \mathbb{R})$, and we set, for any pair of linear functions on $u^*(H)$ defined by the two elements $u, v \in u(H)$:

$$\{\hat{u}, \hat{v}\} = \hat{[u, v]}$$

(20)

where the commutator on the right hand side is computed by thinking of $u, v$ as elements of the Lie algebra $u(H)$, and the left hand side is to be read as a linear function on $u^*(H)$. We will use the ‘hat’ to denote the elements of $u(H)$ seen as linear functions on the dual $u^*(H)$. We are implicitly using here the property that the vector space $u(H)$ is isomorphic to its bi-dual, which holds for vector spaces which are reflexive, in particular finite dimensional ones. Then, we have:

Proposition 2. Let $O$ be the space of observables of a finite level quantum system. Then, $O^*$ can be endowed with a Poisson structure.

Having replaced the Lie algebra structure with the Poisson tensor associated with the Poisson bracket on $u^*(H)$, we are now able to perform also nonlinear transformations on the Poisson manifold. In this sense we speak of the ‘geometrization’ of the algebra structure of the vector space $u(H)$.

If we denote by $\hat{A}$ and $\hat{B}$ the linear functions on $u^*(H)$ corresponding to elements $A, B \in u(H)$, we can define the Poisson bi-vector $\Lambda$ as:

$$\Lambda(d\hat{A}, d\hat{B})(\xi) = \{\hat{A}, \hat{B}\}(\xi) = \xi([A, B]) = \frac{i}{2} \xi(AB - BA), \quad \xi \in u^*(H),$$

(21)

where we used the scalar product of the Lie algebra.

Hence we recover the well-known Kirillov–Konstant–Souriau Poisson tensor on the dual of any Lie algebra.

By using a similar procedure we may also ‘geometrize’ the Jordan algebra structure on the space of observables. Again we set:

$$\mathcal{R}(\xi)(d\hat{A}, d\hat{B}) = \xi([A, B]_+) = \frac{i}{2} \text{Tr}(\xi(AB + BA)), \quad \xi \in u^*(H),$$

(22)

where use is made of the relation $\xi(A) = \frac{i}{2} \text{Tr}(\xi A)$.

These two tensor fields can be put together to form a complex tensor field:

$$(\mathcal{R} + i\Lambda)(\xi)(d\hat{A}, d\hat{B}) = (\hat{A}\hat{B})(\xi) = \xi(AB) = \text{Tr}(\xi AB), \quad \xi \in u^*(H).$$

(23)
By using this tensor field we can define a \(*\)-product in the form
\[
(\hat{A} \ast \hat{B})(\xi) = \xi(AB) = \mathcal{R} + i\Lambda)(\xi)(d\hat{A}, d\hat{B}).
\]

In this context, the compatibility condition of the Lie and the Jordan structures can be simply stated by saying

**Proposition 3.** The Hamiltonian vector fields associated with observables (i.e. real linear functions on \(u^*(\mathcal{H})\)) are infinitesimal symmetries for the tensor field \(\mathcal{R}\) associated with the Jordan structure, and therefore derivations for the \(*\)-product.

**Proof.** It is a direct consequence of the compatibility between both brackets, summarized in (6).

**Remark 4.** Our ‘geometrization’ carries along the possibility of performing nonlinear transformations because we have replaced the algebraic structures on the linear space \(u(\mathcal{H})\) with tensorial objects on the manifold \(u^*(\mathcal{H})\). It should be remarked, however, that now we have the possibility of two different associative products on linear functions on \(u^*(\mathcal{H})\):

- the point-wise product \((\hat{A} \cdot \hat{B})(\xi) = \hat{A}(\xi)\hat{B}(\xi)\), which gives a quadratic function out of two linear ones and
- a non-local product \((\hat{A} \ast \hat{B})(\xi) = \hat{A}\hat{B}(\xi)\). In this case we obtain a linear function as the product of other two linear ones, but in general it will be a complex valued function even if the factors were real ones. This result has to do with the fact that the product of two Hermitian operators is not Hermitian and therefore it gives rise to real and imaginary parts.

**Example 2.** At this point it may be adequate to give a simple example of the objects introduced so far. Let us consider the Lie algebra \(u(2)\) of \(2 \times 2\) Hermitian matrices corresponding to a spin 1/2 physical system. We introduce an orthonormal basis with respect to the scalar product on the algebra. We set thus:

\[
U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and also the associated linear functions

\[
\hat{X} = x, \quad \hat{Y} = y, \quad \hat{Z} = z, \quad \hat{U} = u
\]

where the functions are to be understood as \(z(A) = \frac{1}{2}\text{Tr}(ZA)\), and so on, for any \(A \in u(2)\). In these coordinates, the Poisson tensor field is given by

\[
\Lambda = 2 \left( x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right),
\]

while the tensor associated to the Jordan structure becomes:

\[
\mathcal{R} = 2 \frac{\partial}{\partial u} \otimes_z \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)
\]
\[ + 2u \left( \frac{\partial}{\partial u} \otimes_s \frac{\partial}{\partial u} + \frac{\partial}{\partial x} \otimes_s \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes_s \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes_s \frac{\partial}{\partial z} \right) \].

It is immediately seen that \( R \) is invariant under the action of the vector fields provided by the linear Hamiltonian functions with respect to the Poisson tensor \( \Lambda \). We can even consider the non-local product, for instance we get

\[ \hat{Z} \star \hat{Y} = -i\hat{X}, \quad \hat{X} \star \hat{Y} = i\hat{Z}, \quad \hat{Z} \star \hat{X} = i\hat{Y}. \]

It is also easy to see that the Hamiltonian vector fields associated with linear functions provide derivations both for the point-wise product and for the non-local product. Thus, the associated equations of motion do not carry a quantum or a classical behaviour, it is the product what distinguishes the commutative or the non-commutative nature of the space along with the locality or non-locality of the operation. And therefore distinguishes Classical from Quantum Mechanics.

4.3. Dynamics. It is now possible to write equations of motion on the space of observables. In the Heisenberg picture it is defined as

\[ \frac{d}{dt} \hat{A} = \frac{1}{\hbar} [\hat{H}, \hat{A}]_. \]

By using the ‘geometrization’, i.e. by thinking in terms of the dual space \( u^*(\mathcal{H}) \), we find

\[ \frac{d}{dt} \hat{\tilde{A}} = \frac{1}{\hbar} \{\hat{\tilde{H}}, \hat{\tilde{A}}\}_. \]

As there are different algebra structures on \( u^*(\mathcal{H}) \) it is important to study the compatibility of differential equations with such algebra structures.

Lemma 5. The linear differential equations which preserve both products correspond to the infinitesimal generators of unitary transformations.

5. The momentum map: relating Schrödinger and Heisenberg pictures

Having geometrized both the Schrödinger and the Heisenberg pictures of Quantum Mechanics, we are going to show now how they are related. For more details see [37, 68, 76].

We have already stressed that both \( \mathcal{H} \) and \( \mathcal{PH} \) carry, among other structures, a symplectic one. The unitary group acts on both of them and the two actions are related by the projection map \( \pi : \mathcal{H}_0 \to \mathcal{PH} \), where \( \mathcal{H}_0 = \mathcal{H} - \{0\} \). These actions are strongly symplectic and therefore with
associated equivariant momentum maps:

\[ \begin{array}{c}
\mathcal{H}_0 \\
\pi \\
\downarrow \\
\mathcal{P}H \\
\mu \\
\downarrow \\
\mathcal{H}^\ast \\
\tilde{\mu} \\
\end{array} \]

It is not difficult to see that for any Hermitian operator \( A \) we have

\[ \mu(\psi)(A) = \langle \psi, A\psi \rangle = \rho_\psi(A), \]

while

\[ \tilde{\mu}([\psi])(A) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle} = \tilde{\rho}[\psi](A), \quad \psi \in [\psi]. \]

It is possible to rewrite these expressions in a different form, to find:

\[ \rho_\psi(A) = \text{Tr}(A|\psi\rangle\langle \psi|); \quad \tilde{\rho}[\psi](A) = \text{Tr} \left( A \frac{|\psi\rangle\langle \psi|}{\langle \psi, \psi \rangle} \right), \]

where \( |\psi\rangle\langle \psi| : \mathcal{H} \to \mathcal{H} \) denotes the rank one linear map defined as \( |\psi\rangle\langle \psi| : \phi \mapsto \langle \phi, \psi \rangle \psi \).

This form shows that \( \tilde{\rho}_\psi \) can be written as a rank-one projector with the help of the scalar product defined by the trace.

The association between the equivalence class \([\psi]\) (element of the complex projective space) and \( \tilde{\rho}_\psi \) (rank one projector) is clearly one-to-one and onto. This shows that the complex projective space, along with the symplectic structure and the Riemannian tensor, may be identified with the minimal orbit of the coadjoint action of the unitary group in \( \mathcal{H}^\ast \), which passes through \( \tilde{\rho}_\psi \).

The main properties of the momentum map can be collected in the following proposition:

**Proposition 4.**

i) The momentum map is equivariant with respect to the action of \( U(\mathcal{H}) \) on \( \mathcal{H}_0 \) and the coadjoint action of \( U(\mathcal{H}) \) on \( \mathcal{H}^\ast \). In particular, this says that the Schrödinger equation of motion on \( \mathcal{H} \) is \( \mu \)-related with the Heisenberg equation of motion on \( \mathcal{H} \) (the space \( \mathcal{H} \) is identified with the dual by means of the scalar product defined by the trace). Moreover,

ii) \( \mu^*(\hat{A}) = f_A, \tilde{\mu}^*(\hat{A}) = e_A \).

iii) \( \mu^*(\{\hat{A}, \hat{B}\}) = \{f_A, f_B\} \) and \( \tilde{\mu}^*(\{\hat{A}, \hat{B}\}) = \{e_A, e_B\} \).

iv) \( \mu^*(R(d\hat{A}, d\hat{B})) = G(\mu^*(d\hat{A}), \mu^*(d\hat{B})) \) and for the other mapping \( \tilde{\mu}^*(R(d\hat{A}, d\hat{B})) = G_P(\tilde{\mu}^*(d\hat{A}), \tilde{\mu}^*(d\hat{B})) + e_A e_B \).

**Proof.** Direct computation.
Remark 5. Had we started with the Heisenberg picture, we would be able to reconstruct the Hilbert space description by means of the Gelfand–Naimark–Segal (GNS) construction \cite{39,29}. This requires the choice of a state (a functional on the algebra of observables). When the chosen state is pure, the representation will be irreducible. The Hilbert space associated with a pure state would play exactly the same role that our Hilbert space \( H \) has played for the Schrödinger picture. In this case the corresponding momentum map \( \mu \) would provide us with a symplectic realization of the Poisson manifold \( u^*(\mathcal{H}) \) (with the Lie-Poisson structure). We recall what a symplectic realization is:

Definition 2. A symplectic realization of a Poisson manifold \((N,\{\cdot,\cdot\})\) is a Poisson map \( \Phi : M \to N \), where \((M,\omega)\) is a symplectic manifold. When \( M \) is a symplectic vector space we have a special situation and \( \Phi \) is called a classical Jordan–Schwinger map \cite{63}.

Remark 6. We would like to emphasize that the GNS construction brings in the quantum theory an entirely new framework that the traditional Schrödinger formalism is lacking of. This is due to the fact that the Hilbert space on which the observables act as operators is not a perennial feature of the theory, nor of the model to be constructed, but it is dependent on the state or preparation of the system under consideration. In other terms, as the state is prepared by the observer, the corresponding Hilbert space with the associated representation of the \( \mathbb{C}^* \)-algebra is ‘selected’ by the observer.

In concluding this section we notice that the geometrical version we have presented allows us to put the Schrödinger and the Heisenberg pictures within an unified framework of Hilbert spaces, actions of the unitary group and its associated momentum maps.

The GNS construction may be given the nice geometrical description of the construction of a symplectic realization of the Poisson manifold \( u^*(\mathcal{H}) \), which in turn can be considered as a generalization to arbitrary dimension of the Jordan–Schwinger map.

For completeness, let us expose a little more our considerations on the space of states:

5.1. States: Density states. We have seen how the momentum map \( \tilde{\mu} \) allows us to embed the complex projective space \( \mathcal{P} \mathcal{H} \) on the dual of the Lie algebra \( u(N) \). The resulting elements represent the set of pure states of the quantum system. But in many physical situations we have more general states, i.e. density states which are convex combinations of pure states. They are represented by a family \( \rho = \{\rho_1, \cdots, \rho_k\} \), each element satisfying

\[
\rho_k^2 = \rho_k, \quad \rho_k^\dagger = \rho_k, \quad \text{Tr} \rho_k = 1,
\]

along with a probability vector, namely \( \vec{p} = (p_1, p_2, \cdots, p_k) \) with \( \sum_j p_j = 1 \) and \( p_j \geq 0 \ \forall j \). Out of these we construct a density state \( \rho = \sum_j p_j \rho_j \).
The evaluation of this state on some observable $A$ is given by
\[ \rho(A) = \sum_j p_j \text{Tr} \rho_j A = \text{Tr} \rho A. \quad (24) \]

We shall call **density states** to the set $\mathcal{D}(\mathcal{H})$ of all convex combinations of pure states [36].

As any of the elements in $\rho$ can be embedded into $u^*(N)$, it makes perfect sense to consider $\rho$ also as an element in the dual of the unitary algebra. And we hence consider the geometric structure we defined on $u^*(N)$ as the Poisson or the Jordan brackets
\[ \{ f_A, f_B \}(\rho) = \sum_k p_k \{ f_A, f_B \}(\psi_k) \]
\[ (f_A, f_B)(\rho) = \sum_k p_k (f_A, f_B)(\psi_k) \quad (25) \]

where $f_A(\rho) = \sum_k p_k f_A(\psi)$.

As for the geometric structures on $\mathcal{D}(\mathcal{H})$ we shall consider it as a real manifold with boundary embedded into the real vector space $u^*(N)$. On this space the two structures above (25), define a Poisson and a Riemannian structure. The Poisson structure is degenerate. However it is also possible to define a generalized complex structure satisfying
\[ J^3 = -J \quad (26) \]

The boundary is a stratified manifold, corresponding to the union of symplectic orbits of $U(N)$ of different dimensions, passing through density matrices of not maximal rank. For further information see [36] [37].

### 6. Quantum mechanics on phase space

The phase-space formulation of Quantum Mechanics has a long history. See [21] [70] [18] [19] [56] [57] for further details. As it stands nowadays, we may identify two basic independent ideas behind this formulation:

- The first one, due to Weyl, emerged from the desire to ‘quantize’ classical systems by using bounded operators (one-parameter groups of unitary operators instead of their infinitesimal generators which would create domain problems due to their unboundness, see for instance Wintner’s theorem in [32]. By a clever use of the Fourier transform, Weyl [81] [82] was able to set up a rule that maps a classical dynamical variable (a function on phase-space) onto a corresponding operator for the quantum system in a linear manner.

- The second idea is due to Wigner [84] who associated a phase-space distribution with each quantum state. This was motivated by the statistical properties of the states, in the way we mentioned in the introduction.
It was Moyal who discovered [69] that the Weyl correspondence rule can be inverted by the Wigner map and therefore that the two approaches were exactly inverse of one another. As a result, the quantum expectation value of an operator can be represented in a classical-looking form as a statistical average of the corresponding phase-space function. In this way, Quantum Mechanics can be represented as a ‘statistical theory’ on the classical phase-space. It should be mentioned, though, that as a function on the classical phase-space the Wigner distribution (associated with Hermitian operators) is real but not necessarily point-wise non-negative, and thus it cannot be interpreted as a true probability distribution. The occurrence of negative values for the Wigner function associated with states is closely related to the impossibility of simultaneously measuring conjugated variables (as position and momenta).

6.1. Weyl systems. We consider a phase-space defined by a symplectic vector space \((E, \omega)\), with symplectic structure \(\omega\).

**Definition 3.** A Weyl map is a (strongly) continuous map from \(E\) to the set of unitary operators on some Hilbert space \(\mathcal{H}\):

\[
W : E \to U(\mathcal{H}),
\]

such that

\[
W(e_1)W(e_2)W^+(e_1)W^+(e_2) = e^{i\omega(e_1, e_2)} I,
\]

for any pair of vectors \(e_1, e_2 \in E\). The symbol \(I\) stands for the identity operator on the Hilbert space \(\mathcal{H}\).

A theorem by von Neumann [80] asserts that such a map exists for any finite dimensional symplectic vector space. As a matter of fact the Hilbert space \(\mathcal{H}\) can be realized as the space of square integrable functions on any Lagrangian subspace of \(E\). If we choose a Lagrangian subspace \(L\), it is possible to ‘decompose’ \(E\) into:

\[
E \sim L \oplus L^* = T^*L \sim L^* \oplus (L^*)^* = T^*L^*.
\]

The Lebesgue measure is a translational invariant measure on \(L\) and we can construct a specific realization of the Weyl map \(W\). We define

\[
U = W|_{L^*}, \quad V = W|_{L},
\]

and the action on \(L^2(L, d^n x)\) is then given by

\[
(V(y)\psi)(x) = \psi(x + y), \quad (U(y)\psi)(x) = e^{i\alpha(x)}\psi(x),
\]

for any \(x, y \in L\), \(\alpha \in L^*\) and \(\psi \in L^2(L, d^n x)\).

Out of these two operators \(U\) and \(V\) we can also recover \(W\) by setting

\[
W = U \circ V.
\]

But other ways of ‘reconstructing’ \(W\) are also possible.
The strong continuity condition we put in the definition of $W$ allows us to use Stone’s theorem to obtain

$$W(v) = e^{iR(v)}, \quad v \in E,$$

with $R(v)$ being the infinitesimal generator of the one parameter group of unitary transformations $W(tv)$, for $t \in \mathbb{R}$. We have $R(v_1 + v_2) = R(v_1) + R(v_2)$. When we select a complex structure on the space $E$, say by a tensor $J : E \to E$ with $J^2 = -1$, it is possible to define what are known as ‘creation’ and ‘annihilation’ operators:

$$a(v) = \frac{1}{\sqrt{2}} (R(v) + iR(Jv)) \quad a^+(v) = \frac{1}{\sqrt{2}} (R(v) - iR(Jv)).$$

This complex structure allows to define a ‘Hermitian inner product’ on $E$ by setting

$$\langle v_1, v_2 \rangle = \omega(v_1, Jv_2) - i \omega(v_1, v_2),$$

where we must choose $J$ in such a way that $g(v_1, v_2) = \omega(Jv_1, v_2)$ defines an Euclidean inner product on $E$.

By selecting a decomposition of $E$ into $L \oplus L^*$, we may write $W$ in a explicit form. Indeed, if we take $(x, \alpha) \in L \oplus L^*$ we set

$$W(x, \alpha) = \exp \left( \frac{i}{\hbar} (x \hat{p} + \alpha \hat{q}) \right).$$

Where $\hat{p}$ and $\hat{q}$ are the infinitesimal generators, as from Stone’s theorem, associated with vectors $(0, 1)$ and $(1, 0)$ in $L \oplus L^*$ respectively.

It is now possible to associate an operator with any function $g$ on $E$ admitting a Fourier transform $\tilde{g}$. Consider thus one such function $\tilde{g}$ admitting as a Fourier transform

$$g(p, q) = \frac{1}{(2\pi)^n} \int d^n x \, d^n \alpha \, \tilde{g}(x, \alpha) e^{i(x \hat{p} + \alpha \hat{q})}.$$

We can associate to $g$ the operator

$$W(g) = \frac{1}{(2\pi)^n} \int d^n x \, d^n \alpha \, \tilde{g}(x, \alpha) \exp \left( \frac{i}{\hbar} (x \hat{p} + \alpha \hat{q}) \right),$$

i.e. we have replaced the Weyl exponential $e^{i(x \hat{p} + \alpha \hat{q})}$ with the corresponding exponential Weyl operator.

This association defines a unitary isomorphism between Hilbert–Schmidt operators on $L$ and square integrable functions on $L \oplus L^*$.

**Remark 7.** To consider the transformation properties under the symplectic linear group, it is often more convenient to use the symplectic Fourier transform, written as

$$g(q, p) = \frac{1}{(2\pi)^n} \int d^n x \, d^n \alpha \, \tilde{g}(x, \alpha) \exp \left( \frac{i}{\hbar} (\alpha q - xp) \right).$$
The Weyl map can be then given three equivalent expressions:

- \( W_1(x, \alpha) = \exp \left( \frac{i}{\hbar}(\alpha \hat{q} - x \hat{p}) \right) \)
- \( W_2(x, \alpha) = \exp \left( \frac{i}{\hbar}(\alpha \hat{q}) \right) \exp \left( -\frac{i}{\hbar}(x \hat{p}) \right) \)
- \( W_3(x, \alpha) = \exp \left( -\frac{i}{\hbar}(x \hat{p}) \right) \exp \left( \frac{i}{\hbar}(\alpha \hat{q}) \right) \)

They are related by

\[
\exp \left( \frac{i}{\hbar}(\alpha \hat{q} - x \hat{p}) \right) = \exp \left( -\frac{i}{2\hbar}(\alpha(x)) \right) \exp \left( -\frac{i}{\hbar}(x \hat{p}) \right) \exp \left( \frac{i}{\hbar}(\alpha \hat{q}) \right) = \exp \left( -\frac{i}{\hbar}(x \hat{p}) \right) \exp \left( \frac{i}{\hbar}(\alpha \hat{q}) \right) \exp \left( -\frac{i}{\hbar}(x \hat{p}) \right)
\]

These relations follow from recalling that \( \hat{A} + \hat{B} = \hat{A} \hat{B} \hat{A}^{-1} \hat{B}^{-1} \) whenever \( \hat{A} \) and \( \hat{B} \) commute with \([\hat{A}, \hat{B}]\).

The last two versions of the Weyl map are encountered in the theory of pseudo-differential operators where one deals with symbols and operators, the symbols being functions on phase-space corresponding to quantum mechanical operators \([44, 45]\).

6.2. Wigner’s construction. The second basic idea was due to Wigner.

We shall give here a rather abstract presentation of this idea. It relies on the construction of two maps, that we can denote as

\[
U : E \to \mathcal{A}; \quad T : E \to \mathcal{A}^*.
\]

Thus we associate an operator in \( \mathcal{A} \) and an operator in \( \mathcal{A}^* \) to any vector in \( E \). We impose the condition:

\[
T(e')(U(e)) = \delta(e' - e),
\]

where by \( \delta(e' - e) \) we denote the Dirac distribution. We also ask that

\[
\int_E \text{d}e T(e) \otimes U(e) = \mathbb{I}.
\]

Hence we may construct a resolution of the identity from \( \mathcal{A} \) to \( \mathcal{A} \) or from \( \mathcal{A}^* \) to \( \mathcal{A}^* \). Now, if we consider an operator \( A \in \mathcal{A} \), we can define a function on \( E \) by setting:

\[
W_A(e) = T(e)(A)
\]

and reconstruct an operator on \( \mathcal{A} \) from an element \( f \) of the set of functions on \( E \) as

\[
\hat{A}_f = \int_E \text{d}e f(e) U(e)
\]
By evaluating $T(e')$ on $\hat{A}_f$ we get:

$$T(e')(\hat{A}_f) = \int_E df(e) T(e')(U(e)) = \int_E df(e) \delta(e' - e) = f(e').$$

On the other hand:

$$\int_E df(T(A)(U(e)) = I A = A$$

Thus by constructing two maps endowed with the previous properties we are able to build a one-to-one map from the set of operators onto the set of functions of the space $E$. It is clear that the existence of a vector space structure on $E$ plays no role and hence we can generalize this construction to an arbitrary manifold.

The construction of these two maps requires some ingenuity and it is not a trivial task. For this reason very often in the literature specific maps are associated with the names of those who first constructed them.

Assuming that we are able to define the above maps with the required properties, it is simple to induce additional structures on the set of functions of $E$. Consider for instance the following operation:

$$T(e)(A_1 \cdot A_2) =: T(e)(A_1) \ast T(e)(A_2) = (f_{A_1} \ast f_{A_2})(e).$$

It is quite obvious from the definition that this induces a product on the set of functions which inherits all the properties from the operator algebra structure. In particular, we can transfer the equation of motion by writing:

$$i\hbar \frac{df}{dt} = f_H \ast f - f \ast f_H,$$

where $f_H$ is the function of $E$ associated to the Hamiltonian operator in $A$.

The difference between classical and quantum mechanics may now emerge more neatly because both theories are written in terms of the same vector space of functions ($\mathcal{F}(E)$). The difference lies on the product we consider on that set: the point-wise product is appropriate for classical mechanics, while this new product $\ast$ is appropriate for quantum mechanics.

We shall now consider more specifically the construction of these two maps $T$ and $U$ for the symplectic vector space $(E, \omega)$. The origin of our approach can be traced back to Dirac (see [24]). Let us try to explain it in simple terms.

For vector spaces $V$ admitting a numerable basis, we can define a set

$$S = \mathbb{N} \times \mathbb{N},$$

and define a map

$$T : S \to \text{Lin}(V, V),$$

by means of a basis $\{e_j\}$ in the vector space

$$(j, k) \mapsto T(j, k) = |e_j\rangle\langle e_k|.$$

Out of a linear map $A : V \to V$ we find a function on $S$ by setting

$$f_A(j, k) = \langle e_k, Ae_j \rangle.$$
i.e. the function is given by the matrix elements of the operator $A$ with respect to the basis we have chosen. Equivalently we can write:

\[ f_A(j, k) = \text{Tr}(T(j, k)A). \]

We can also get the operator from the function as

\[ A = \sum_{jk} f_A(j, k) |e_k\rangle\langle e_j|. \]

The scalar product induced on the set of operators by the trace

\[ \langle M, N \rangle = \text{Tr}M^+N \quad M, N \in \text{Lin}(V, V), \]

allows us to associate a dual element to $T(j, k)$. Thus we define $U(m, n)$ to be:

\[ \text{Tr}T(j, k)U(m, n) = \text{Tr}|e_k\rangle\langle e_j, e_m\rangle\langle e_n| = \delta_{jm}\delta_{kn}. \]

Moreover, the orthonormality of the basis $\{|e_j\rangle\}$ implies the orthonormality of the elements $T(j, k)$. However, orthonormality is far less important than the property of completeness.

To deal with functions on phase space we need eigenvectors of the position and of the momentum operators, which we shall denote as $|q\rangle$ and $|p\rangle$, respectively. We should mention, though, that we are using these elements with the usual abuse of notation made by physicists, for these vectors are not normalizable in the usual sense:

- They are indeed the eigenvectors of the position and momentum operators:
  \[ \hat{Q}|q\rangle = q|q\rangle, \quad \hat{P}|p\rangle = p|p\rangle. \]

- They form complete sets:
  \[ \int_{-\infty}^{\infty} |q\rangle dq\langle q| = \mathbb{I} = \int_{-\infty}^{\infty} |p\rangle dp\langle p|. \]

- Both sets are related to each other:
  \[ \langle q, p \rangle = \frac{1}{\sqrt{(2\pi\hbar)}} e^{i\frac{p}{\hbar}q}; \quad |p\rangle = \int_{-\infty}^{\infty} dq|q\rangle\langle q|p\rangle \]

- The norm of each element defines the Dirac delta distribution, i.e.
  \[ \langle q, q' \rangle = \delta(q - q') \quad \langle p, p' \rangle = \delta(p - p') \]

We can also state the property as

\[ \delta(\hat{Q} - q) = |q\rangle\langle q| \quad \delta(\hat{P} - p) = |p\rangle\langle p|. \]

Any Hermitian operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ can be described as a function on $Q \times Q$ (the Cartesian product of two copies of the configuration space), by assigning its integral kernel to it, i.e. the ‘matrix elements’ in the continuous basis $|q\rangle$,

\[ \hat{A} \mapsto f_A(q, q') = \langle q', \hat{A}q \rangle = \langle q, \hat{A}q' \rangle^*. \]
To provide a representation on a phase-space (say $T^*Q$), we can use the mixed matrix elements $\langle q, A|p \rangle$, which, regarded as a function of the phase-space variables, also describes $\hat{A}$ completely. We have two options though (see [70])

$$f_A(q, p) = \langle q, \hat{A}|p \rangle, \quad f'_A = \langle p, \hat{A}|q \rangle.$$  

For Hermitian operators both functions differ by a conjugation:

$$A \text{ Hermitian} \Rightarrow f_A = (f'_A)^*.$$  

By introducing

$$\hat{T}(q, p) = \frac{1}{2\pi \hbar} \int dq' |q + \frac{1}{2} q'| e^{i \hbar pq'},$$

we find that

$$A(q, p) = 2\pi \hbar \text{Tr} \hat{A} \hat{T}(q, p).$$

These operators are closely related to the Weyl operators we introduced above. Indeed, it can be proved that

$$\hat{T}(q, p) = \int_{E = L \oplus L^*} \frac{d^n x d^n \alpha}{(2\pi \hbar)^{2n}} e^{-i(xp - \alpha q)} e^{\frac{i}{\hbar} (xp - \alpha q)},$$

i.e. the operators we need for the one-to-one correspondence are the symplectic Fourier transform of the Weyl operators $W_1(x, \alpha)$.

Summarizing, we have that for any state $\hat{\rho}$ of a quantum system, we can define the function

$$\rho(q, p) = \text{Tr} \hat{\rho} \hat{T}(q, p),$$

and the values of an operator $\hat{A}$ on that state will be written as

$$\text{Tr} \hat{\rho} \hat{A} = \int_E d^n q d^n p \rho(q, p) A(q, p),$$

where the function $A(q, p)$ is associated with the operator $\hat{A}$ satisfying

$$\int dp A(q, p) = 2\pi \hbar \langle q, \hat{A}|q \rangle \quad \int dq A(q, p) = 2\pi \hbar \langle p, \hat{A}|p \rangle.$$  

Associated to the observables we construct functions

$$\hat{A} \mapsto f_A(p, q) = \int_{-\infty}^{\infty} dp' e^{i \hbar q p'} \langle p + \frac{p'}{2}|\hat{A} - \frac{p'}{2}\rangle,$$

and associated to the functions we recover operators by using:

$$f_A(q, p) \mapsto \hat{A} = \int_E \frac{d^n p}{(2\pi \hbar)^n} f_A(q, p) |q + \frac{p}{2}\rangle|q - \frac{p}{2}| e^{i \hbar pq'}.$$  

This last formula can also be read as a decomposition of the operator $\hat{A}$ in the basis provided by $\hat{T}$.
Remark 8. By writing

$$|q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{d^n\alpha}{\sqrt{(2\pi\hbar)^n}} \exp \left( \frac{i}{\hbar} \alpha(q - Q) \right) = \delta(q - Q)$$

we can rewrite the expression of $\hat{T}$ as:

$$\hat{T}(p,q) = \int \frac{d^n x d^n\alpha}{(2\pi\hbar)^n} \exp \left( \frac{i}{\hbar} \alpha(q - Q) + x(p - P) \right)$$

In this way, it is simple to prove the following properties:

- $\text{Tr}\hat{T}(p,q) = 1$
- $\text{Tr}\hat{T}(p_1,q_1)\hat{T}(p_2,q_2) = 2\pi\hbar\delta(p_1 - p_2)\delta(q_1 - q_2)$
- And

$$\text{Tr}\hat{T}(p_1,q_1)\hat{T}(p_2,q_2)\hat{T}(p_3,q_3) = 4\exp \left( \frac{2i}{\hbar} (q_1 - q_3)(p_2 - p_3) - (q_2 - q_3)(p_1 - p_3) \right)$$

With these formulae, we can study how the algebraic structures of the set of operators are transferred to the set of functions. In particular we can compute the product $\hat{A}\hat{B}$. We can obtain it from the factors, or consider the function corresponding to it as an element of the set of operators:

$$\hat{A}\hat{B} = \int_{E \times E} \frac{d^n p_1 d^n q_1 d^n p_2 d^n q_2}{(2\pi\hbar^{2n})} f_A(p_1,q_1) f_B(p_2,q_2) \hat{T}(p_1,q_1)\hat{T}(p_2,q_2)$$

$$= \int_{E} \frac{d^n p d^n q}{(2\pi\hbar^n)} f(p,q) \hat{T}(p,q),$$

where $f(p,q) = \text{Tr}\hat{A}\hat{B}\hat{T}(p,q)$.

It is possible to prove that it is possible to write the function $f(p,q)$ in terms of the functions $f_A$ and $f_B$ and an operation defined by bi-differential operators:

$$f(p,q) = f_A(p,q) \exp \left[ \frac{\hbar}{2i} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right] f_B(p,q).$$

This expression is written normally using the $*$-symbol for the product:

$$f(p,q) = f_A(p,q) * f_B(p,q)$$

where

$$* = \exp \left[ \frac{\hbar}{2i} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right]$$
The set of functions endowed with this operation becomes an associative algebra (convergence problems and formal series). We can consider also the symmetric and the skew-symmetric parts of it:

\[ f_{[A,B]} = 2i f_A(p,q) \sin \left( \frac{\hbar}{2i} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right) f_B(p,q) \]

\[ f_{(A,B)} = 2f_A(p,q) \cos \left( \frac{\hbar}{2i} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right) f_B(p,q) \]

Thus it is simple to prove that the limit defined by \( \hbar \to 0 \) leads to

\[ \lim_{\hbar \to 0} \frac{1}{i \hbar} f_{[A,B]} = \frac{\partial f_A}{\partial p} \frac{\partial f_B}{\partial q} - \frac{\partial f_A}{\partial q} \frac{\partial f_B}{\partial p} \]

\[ \lim_{\hbar \to 0} \frac{1}{2} f_{(A,B)} = (f_A f_B)(p,q). \]

Hence, we see how this formalism of Quantum Mechanics turns out to be much better adapted to deal with the classical limit of quantum mechanical systems. For further details on the quantum-classical transition see [48, 49, 66, 30, 54, 24].

7. Quantum dynamics on phase space

It is now possible to write the equations of motion of a quantum dynamical system on phase space. Consider thus a quantum system evolving on the space of density matrices, thus defining a curve

\[ \rho(t) = \frac{\langle \psi(t) | \psi(t) \rangle}{\langle \psi(t), \psi(t) \rangle}. \]

The quadratic function corresponding to an operator \( A \) in the evolution of the state \( \psi(t) \) defines a curve on the space of quadratic functions

\[ e_A(\psi(t)) = \frac{\langle \psi(t), A \psi(t) \rangle}{\langle \psi(t), \psi(t) \rangle}. \]

By using the corresponding Wigner function, we can write:

\[ e_A(\psi(t)) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi \hbar)} f_A(p,q) W(p,q);t). \]

Here we denote by \( W(p,q);t) \) the Wigner function corresponding to an arbitrary time \( t \). This raises the question on the definition of these ‘time-dependent’ Wigner functions. We can sumarize their properties as follows:

- \( W(p,q);t) = \int_{-\infty}^{\infty} \frac{d^3x}{(2\pi \hbar)^3} e^{\frac{i}{\hbar} p x} \psi(q - \frac{x}{2};t) \psi^*(q + \frac{x}{2};t) \)
  This can also be written as \( = \int_{-\infty}^{\infty} \frac{d^3x}{(2\pi \hbar)^3} e^{\frac{i}{\hbar} p x} \phi(p + \frac{x}{2};t) \phi^*(p - \frac{x}{2};t) \).
We have
\[ W(p, q; t) = 2\pi \hbar \frac{\delta^2}{\delta p \delta q} \text{Tr}(\rho(t)\delta(q - \hat{Q})\delta(p - \hat{P}) = \sqrt{2\pi} \hbar e^{\frac{\hbar}{2i} \int_0^t \frac{d^2}{dq dq} W(ho; t)} \phi(p; t) \psi^*(q; t) \]

And
\[ |W(p, q; t)|^2 \leq \int_{-\infty}^{\infty} \frac{d^n x}{(2\pi \hbar)^n} \psi(q - \frac{x}{2})^2 \int_{-\infty}^{\infty} \frac{d^n x'}{(2\pi \hbar)^n} \psi(q + \frac{x'}{2})^2 = \frac{1}{(\pi \hbar)^2} \]

This inequality captures the uncertainty relations.

The equations of motion for the Wigner function follow from von Neumann equation on states
\[ \frac{d}{dt} \rho(t) = i[H, \rho(t)]. \tag{28} \]

In Weyl–Wigner representation we find thus:
\[ \frac{d}{dt} W(p, q; t) = \frac{2}{\hbar} H(p, q) \sin \frac{\hbar}{2} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) W(p, q; t) \]

where \( H(p, q) \) is the Weyl transform of the Hamiltonian operator.

Then if we write the ‘inner derivation’ associated with this Hamiltonian ‘function’ we obtain:
\[ D(p, q) = \frac{2i}{\hbar} H(p, q) \sin \frac{\hbar}{i} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \]

We can thus write:
\[ \frac{d}{dt} W(p, q; t) = -iD(p, q)W(p, q; t) \]

A formal solution for this equation can always be written in the exponential form
\[ W(p, q; t) = e^{-iD(p, q)(t-t_0)}W(p, q; t_0) \]

**Example 3.** Consider for instance a Hamiltonian of mechanical type, i.e.
\[ H(p, q) = \frac{p^2}{2m} + V(q) \]

The equation above becomes
\[ \left( \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial q} - \frac{dV}{dq} \frac{\partial}{\partial p} \right) W(p, q; t) = \sum_k \frac{(-1)^k}{(2k + 1)!} \left( \frac{\hbar}{2} \right)^{2k+1} \frac{d^{2k+1}V(q)}{dq^{2k+1}} \frac{\partial^{2k+1}}{\partial p^{2k+1}} W(p, q; t) \]
Thus if we consider the classical limit by considering \( \hbar \to 0 \) we obtain that the limit of the von Neumann equation written in terms of the Wigner function becomes:

\[
\left( \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial q} - \frac{dV}{dq} \frac{\partial}{\partial p} \right) W(p,q;t) = 0;
\]

i.e. the classical Hamilton equations.

8. Alternative Hamiltonian descriptions

Let us recall very briefly what a classical bi-Hamiltonian system is. For more details the interested reader is addressed to [17, 65]. On the space of functions (observables) \( \mathcal{F}(E) \), a dynamical system \( \Gamma \) is bi-Hamiltonian if there exist two Poisson brackets and two Hamiltonian functions such that the corresponding Hamilton vector fields coincide with \( \Gamma \), i.e.

\[
\frac{d}{dt} f = \{H_1, f\}_1 = \{H_2, f\}_2
\]

In terms of Poisson bi-vector fields we have

\[
\Lambda_1(dH_1) = \Gamma = \Lambda_2(dH_2)
\]

When besides this \( \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2 \) is a skew-symmetric tensor defining a new Poisson bracket, the two Poisson structures are said to be compatible.

Going over to Quantum Mechanics, it seems natural to try to imitate the same definition, taking into account that this time one has, in addition to the Poisson structure \( \Lambda \), a Riemannian tensor \( G \). We define thus:

**Definition 4.** Consider two Kähler structures \( (G_1, \Lambda_1, J_1) \) and \( (G_2, \Lambda_2, J_2) \). A dynamical system \( \Gamma \) is said to be bi-Kählerian if it preserves all the tensors:

\[
L_\Gamma G_1 = L_\Gamma G_2 = L_\Gamma \Lambda_1 = L_\Gamma \Lambda_2 = 0
\]

This already implies that \( L_\Gamma J_1 = L_\Gamma J_2 = 0 \).

It is not difficult to prove that these two admissible (i.e. invariant) Hermitian tensors are compatible if they give rise to the same complex structure, i.e. \( J_1 = J = J_2 \)

From these two inner products on the Hilbert space, we find that the ‘row-by-column’ product (i.e. the corresponding coordinate expression) will change by means of the insertion of a positive matrix \( K \):

\[
(x_1^*, \cdots, x_n^*)(x_1, \cdots, x_n)^T = \sum_j x_j^* x_j \mapsto (x_1^*, \cdots, x_n^*) \cdot_K (x_1, \cdots, x_n)^T = \sum_j x_j^* K^{jm} x_m
\]

Thus on the space of matrices (operators) the induced alternative product becomes:

\[
AB \mapsto A \cdot_K B = AKB.
\]
This implies that the Heisenberg picture we get two alternative descriptions of the dynamics:

\[
\frac{i}{\hbar} \frac{d}{dt} A = [A, H_1] = [A, H_K]_K.
\]

Thus:

\[
AH_1 - H_1 A = AKH_K - H_KKA, \quad \forall A.
\]

Thus we obtain an obvious solution

\[
H_1 = KH_K.
\]

For it to be admissible we need

\[
[H_1, K] = 0.
\]

For general considerations, see [17]. From here it is possible to carry on the analysis of bi-Kählerian dynamics (quantum dynamics) along the same lines of the classical situation.

Assuming that both Poisson structures give rise to the corresponding symplectic structures (i.e. both are non-degenerate), we can construct the corresponding Weyl systems and the corresponding Wigner–Weyl formalisms. The corresponding quantum dynamics:

\[
i\hbar \frac{d}{dt} f_A = f_{H \ast f_A} - f_A \ast f_H = f_{H_K \ast_K f_A} - f_A \ast_K f_{H_K}.
\]

By considering the deformed associative product in the form \( A \cdot_N B = AB \) we find

\[
f_A \ast_K f_B = f_A \ast f_K \ast f_B.
\]

This implies that in the ‘classical limit’ the symmetric bracket becomes \( f_A f_K f_B \) while the skew-symmetric part becomes:

\[
\lim_{\hbar \to 0} \hbar^{-1} (f_A \ast_K f_B - f_{B \ast_K f_A}) = f_K \{f_A, f_B\} + f_A L_X f_B - f_B L_X f_A.
\]

Thus the classical limit is not a Poisson bracket but a Jacobi bracket.

We may also remark that linearly related alternative symplectic structures give rise to Poisson brackets which are always compatible. Therefore, to obtain classical limits (of alternative products on the space of operators) which are not compatible we have to consider nonlinear transformations also at the quantum level [31]. Therefore, in the ‘Heisenberg cut’ when we describe a quantum system; it may be necessary to associate the Hilbert space structure or the associative product structure of the space of operators with the ‘apparatus’ rather than with the ‘object’.

9. Conclusions

In this survey we have presented a brief description of the different tools that Differential Geometry offers to describe quantum mechanical systems. We show, first, the necessity of going beyond Classical Mechanics to describe microscopic physical systems; and, at the same time, we obtain a
series of properties which the new description must provide. We have studied the three most common approaches to Quantum Mechanics, Schrödinger, Heisenberg and Wigner-Weyl from a geometrical perspective. In the two first cases we have seen that a Kähler structure and a Lie-Jordan structure arise naturally and play a decisive role in the description of the dynamics. We proved also that the Schrödinger and the Heisenberg pictures are related via the momentum map associated to the symplectic action of the unitary group on the set of states of our system. In what regards the Weyl-Wigner formalism, we provided an abstract description of its construction, and proved why it is the most suitable approach to study the quantum-classical transition. Finally, in a very concise way, we discussed how the identification of the geometric structures that we just mentioned above allows us to generalize the concept of bi-Hamiltonian classical structures to the quantum domain, by considering more than one Kähler structure (in the Schrödinger representation) or different associative products on the algebra of observables (in the Heisenberg picture).

We provide also a quite extensive list of references in order to allow the interested reader to complete the topics presented in these lectures and extend them if necessary.

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**Departamento de Física Teórica, Universidad de Zaragoza, Ciudad Universitaria, 50009 Zaragoza (SPAIN)**

**BIFI-Universidad de Zaragoza, Corona de Aragón 42, 50009 Zaragoza (SPAIN)**

**Dipartamento de Scienze Fisiche, Università Federico II and INFN Naples, Via Cintia I, 80126 Naples (ITALY)**