The decomposition of global conformal invariants
II: The Fefferman-Graham ambient metric and
the nature of the decomposition.

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Abstract

This is the second in a series of papers where we prove a conjecture of
Deser and Schwimmer regarding the algebraic structure of “global confor-
mal invariants”; these are defined to be conformally invariant integrals of
geometric scalars. The conjecture asserts that the integrand of any such
integral can be expressed as a linear combination of a local conformal
invariant, a divergence and of the Chern-Gauss-Bonnet integrand.

The present paper addresses the hardest challenge in this series: It
shows how to separate the local conformal invariant from the divergence
term in the integrand; we make full use of the Fefferman-Graham ambient
metric to construct the necessary local conformal invariants, as well as all
the author’s prior work [1, 2, 3] to construct the necessary divergences.
This result combined with [3] completes the proof of the conjecture, sub-
ject to establishing a purely algebraic result which is proven in [6, 7, 8].

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project.
1 Introduction.

This is the second in a series of papers \cite{3,5,6,7,8} where we prove a conjecture of Deser-Schwimmer \cite{15} regarding the algebraic structure of global conformal invariants. We recall that a global conformal invariant is an integral of a natural scalar-valued function of Riemannian metrics, $\int_{M^n} P(g) dV_g$, with the property that this integral remains invariant under conformal re-scalings of the underlying metric. More precisely, $P(g)$ is assumed to be a linear combination, $P(g) = \sum_{l \in L} a_l C_l(g)$, where each $C_l(g)$ is a complete contraction in the form:

$$\text{contr}^l(\nabla^{(m_1)} R \otimes \cdots \otimes \nabla^{(m_s)} R);$$  \hfill (1.1)

here each factor $\nabla^{(m)} R$ stands for the $m$th iterated covariant derivative of the curvature tensor $R$. $\nabla$ is the Levi-Civita connection of the metric $g$ and $R$ is the curvature associated to this connection. The contractions are taken with respect to the quadratic form $g_{ij}$. In this series of papers we prove:

**Theorem 1.1** Assume that $P(g) = \sum_{l \in L} a_l C_l(g)$, where each $C_l(g)$ is a complete contraction in the form (1.1), with weight $-n$. Assume that for every closed Riemannian manifold $(M^n, g)$ and every $\phi \in C^\infty(M^n)$:

$$\int_{M^n} P(e^{2\phi} g) dV_{e^{2\phi} g} = \int_{M^n} P(g) dV_g;$$

Then $P(g)$ can then be expressed in the form:

$$P(g) = W(g) + \text{div}_i T^i(g) + \text{Pfaff}(R_{ijkl}).$$

Here $W(g)$ stands for a local conformal invariant of weight $-n$ (meaning that $W(e^{2\phi} g) = e^{-n\phi} W(g)$ for every $\phi \in C^\infty(M^n)$), $\text{div}_i T^i(g)$ is the divergence of a Riemannian vector field of weight $-n + 1$, and $\text{Pfaff}(R_{ijkl})$ is the Pfaffian of the curvature tensor.

We recall from the introduction in \cite{3} that this entire work can be naturally subdivided into two parts: Part I, consisting of \cite{3}, the present paper and \cite{5} prove Theorem 1.1 subject to deriving certain purely algebraic propositions, namely the “main algebraic Proposition” 5.2 in \cite{3} and Propositions 5.1 \& 5.2 in \cite{3}.
the present paper. Part II, consisting of papers [6, 7, 8] is devoted to proving these algebraic Propositions.

In [3] we explained that our proof of Theorem 1.1 relied on a main inductive step which asserts that given a \( P(g) \) as in Theorem 1.1 if the minimum number of factors among all complete contractions \( C^l(g) \) in \( P(g) \) is \( \sigma < \frac{n}{2} \), then we can subtract a divergence and a local conformal invariant from \( P(g) \) so as to cancel out the terms with \( \sigma \) factors in \( P(g) \), modulo introducing new terms \( \sigma + 1 \) factors. In conjunction with the results of [1, 2], this main inductive step implies Theorem 1.1.

This main inductive step consists of two sub-steps, the Propositions 3.1, 3.2 in section 3 in [3]. Proposition 3.1 was proven in [3] (subject to deriving the “main algebraic Proposition” 5.2 there). The present paper is devoted to proving the second (harder) Proposition 3.2.

We state this second Proposition again, after recalling two main pieces of notation:

**Conventions:** For any complete or partial contraction in \( \alpha \) factors \( T^1, \ldots, T^\alpha \), \( T^\alpha \) contr \((T^1 \otimes \cdots \otimes T^\alpha)\), an internal contraction is a pair of indices in a given factor \( T^j \) which contract against each other. For each complete contraction in the form (1.2), \( \delta_W \) stands for the total number of internal contractions. Also, for each complete or partial contraction, its “length” will be its number of factors.

Proposition 3.2 in [3] states:

**Proposition 1.1.** Consider any \( P(g) \). \( P(g) = \sum_{l \in L} a_l C^l(g) \) where each \( C^l(g) \) has length \( \geq \sigma \), and each \( C^l(g) \) of length \( \sigma \) is in the form:

\[
\text{contr}(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_\sigma)} W).
\]

Assume that \( \int_{M_n} P(g) dV_g \) is a global conformal invariant. Denote by \( L_\sigma \subset L \) the index set of terms with length \( \sigma \).

We claim that there is a local conformal invariant \( W(g) \) of weight \( -n \) and also a vector field \( T^i(g) \) as in the statement of Theorem 1.1 so that:

\[
\sum_{l \in L_\sigma} a_l C^l(g) - W(g) - \text{div}_i T^i(g) = 0,
\]

modulo complete contractions in the form (1.1) of length \( \geq \sigma + 1 \).

**Note:** As explained in [3], we prove this proposition for \( \sigma \geq 3 \). The special cases \( \sigma = 1, \sigma = 2 \) are treated in section 2 in [5].

**Digression:** A general discussion on local conformal invariants.

Since the present paper deals extensively with the issue of constructing local conformal invariants (as required in 1.3), we digress slightly to discuss some

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2See the first section in [1] for a very rigorous definition of complete and partial contractions.

3\( W \) below stands for the Weyl tensor (\( W_{ijkl} \) if we write out the indices), the trace-free part of the curvature tensor. \( \nabla^{(m)} W \) is the \( m^{th} \) covariant derivative of the Weyl tensor.
background material regarding the history and known constructions of such invariants:

The theory of local invariants of Riemannian structures (and indeed, of more general geometries, e.g. conformal, projective, or CR) has a long history. As stated above, the original foundations of this field were laid in the work of Hermann Weyl and Élie Cartan, see [23, 14]. The task of writing out local invariants of a given geometry is intimately connected with understanding polynomials in a space of tensors with given symmetries, which remain invariant under the action of a Lie group. In particular, the problem of writing down all local Riemannian invariants reduces to understanding the invariants of the orthogonal group.

In more recent times, a major program was laid out by C. Fefferman in [16] aimed at finding all local invariants in CR geometry. This was motivated by the problem of understanding the local invariants which appear in the asymptotic expansions of the Bergman and Szegö kernels of CR manifolds, in a similar way to which Riemannian invariants appear in the asymptotic expansion of the heat kernel; the study of the local invariants in the singularities of these kernels led to important breakthroughs in [11] and more recently by Hirachi in [20]. This program was later extended to conformal geometry in [17]. Both these geometries belong to a broader class of structures, the parabolic geometries; these admit a principal bundle with structure group a parabolic subgroup $P$ of a semi-simple Lie group $G$, and a Cartan connection on that principle bundle (see the introduction in [12]). An important question in the study of these structures is the problem of constructing all their local invariants, which can be thought of as the natural, intrinsic scalars of these structures.

In the context of conformal geometry, the first (modern) landmark in understanding local conformal invariants was the work of Fefferman and Graham in 1985 [17], where they introduced the ambient metric. This allows one to construct local conformal invariants of any order in odd dimensions, and up to order $\frac{n}{2}$ in even dimensions. The question is then whether all invariants arise via this construction.

The subsequent work of Bailey-Eastwood-Graham [11] proved that indeed in odd dimensions all conformal invariants arise via the Fefferman-Graham construction; in even dimensions, they proved that the same holds true when the weight in absolute value is bounded above by the dimension. The ambient metric construction in even dimensions was recently extended by Graham-Hirachi, [19]; this enables them to then identify in a satisfactory way all local conformal invariants even when the weight (in absolute value) exceeds the dimension.

An alternative construction of local conformal invariants can be obtained via the tractor calculus introduced by Bailey-Eastwood-Gover in [10]. This construction bears a strong resemblance to the Cartan conformal connection, and to the work of T.Y. Thomas, [22]. The tractor calculus has proven to be very universal; tractor bundles have been constructed [12] for an entire class of parabolic geometries. The relation between the conformal tractor calculus and the Fefferman-Graham ambient metric construction has been elucidated in [13].

The present series of papers [3–8], while pertaining to the question above (given that it ultimately deals with the algebraic form of local Riemannian and
conformal invariants), nonetheless addresses a different type of problem: We here consider Riemannian invariants $P(g)$ for which the integral $\int_{M^n} P(g) dV_g$ remains invariant under conformal changes of the underlying metric; we then seek to understand the possible algebraic form of the integrand $P(g)$, ultimately proving that it can be decomposed in the way that Deser and Schwimmer asserted. It is thus not surprising that the prior work on the construction and understanding of local conformal invariants plays a central role in this endeavor, in the present paper and in [5].

On the other hand, our resolution of the Deser-Schwimmer conjecture will also rely heavily on a deeper understanding of the algebraic properties of the classical local Riemannian invariants. The fundamental theorem of invariant theory (see Theorem B.4 in [11] and also Theorem 2 in [1]) is used extensively throughout this series of papers. However, the most important algebraic tool on which our method relies are certain “main algebraic Propositions” presented in [3] and in the present paper. These are purely algebraic propositions that deal with local Riemannian invariants. While the author was led to these Propositions out of the strategy that he felt was necessary to solve the Deser-Schwimmer conjecture, they can be thought of as results with an independent interest. The proof of these Propositions, presented in [6, 7, 8] is in fact not particularly intuitive. It is the author’s sincere hope that deeper insight will be obtained in the future as to why these algebraic Propositions hold.

**Local conformal invariants in our proof:** The first half of this paper deals largely with the problem of identifying a local conformal invariant in $P(g)$. As explained in more detail in the “outline” below, we construct three different kinds of local conformal invariants, each of which will “cancel out” a particular kind of terms in $P(g)$. The next challenge is to then cancel out the remaining piece in $P(g)|_\sigma := \sum_{l \in L_\sigma} a_l C^l(g)$ by subtracting only a divergence of a vector field. As explained above, one has a powerful tool in the challenge of constructing local conformal invariants; this is the Fefferman-Graham ambient metric, [17, 18]. A note is in order here: The roughest simplification of $P(g)|_\sigma$ is to “cancel out” terms $C^l(g)$ in $P(g)$ which do not involve internal contractions. For many reasons, the use of local conformal invariants to cancel out these particular terms in $P(g)$ is hardly surprising, in view (for example) of the proof of Proposition 3.2 in [11]. However, we believe that the use of local conformal invariants for the next two “simplifications” of $P(g)$ (in Lemmas 1.2, 1.3), is somewhat surprising: We prove that we can start with complete contractions in the form (1.2) which do contain internal contractions, and cancel out certain particular terms in this form in $P(g)$ by subtracting local conformal invariants and explicitly constructed divergences; we do this modulo introducing new terms which are “better” (from our point of view) than the terms we cancelled out. The construction of local conformal invariants for Lemmas 1.2, 1.3 depends

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4I.e. we cancel out the terms in the form (1.2) which contain no factor $\nabla^{(m)} W$ with two indices contracting against each other; we do this modulo introducing new terms in the form (1.2) which do contain such internal contractions. This is the content of Lemma 1.4.
essentially on explicitly constructed divergences in the ambient metric\(^5\) and the subsequent of cancellations that occur in these constructions.

1.1 Outline of the argument.

The proof of Proposition 1.1 addresses two main challenges: Firstly, how to separate the local conformal invariant “piece” \(W(g)\) in \(\sum_{i \in L_o} a_i C^i(g)\) from the divergence “piece” \(\text{div}_i T^i(g)\). Secondly, how to use the local equation (i.e. the “super divergence formula”) that we have derived in \([1]\) regarding the conformal variation of \(P(g)\) to construct the divergence \(\text{div}_i T^i(g)\) needed in \([1,3]\).

Our argument proceeds as follows: At a first step we explicitly construct a local conformal invariant \(W(g)\) and a divergence \(\text{div}_i T^i(g)\) such that:

\[
\sum_{i \in L_o} a_i C^i(g) = W(g) + \text{div}_i T^i(g) + \sum_{i \in L_{new}} C^i(g),
\]

where the complete contractions \(C^i(g), i \in L_{new}\) are in the form \([1,2]\), but have certain additional algebraic properties. (See Lemmas \([1,1]\) \([1,2]\) \([1,3]\) below).

Thus, the first step reduces matters to proving Proposition 1.1 with \(\sum_{i \in L_o} a_i C^i(g)\) replaced by \(\sum_{i \in L_{new}} a_i C^i(g)\). In step 2, we then revert to studying the new \(P(g)\) by focusing on the conformal variation \(I^j_1(\phi)\) and the super divergence formula for \(I^j_1(\phi)\). We prove that \(\sum_{i \in L_{new}} a_i C^i(g) = \text{div}_i T^i(g)\), for some vector field \(T^i(g)\). (This is proven in Lemma \([1,4]\) below).

A few remarks: It is not at all clear that the local conformal invariant we construct to prove \([1,3]\) is the unique local conformal invariant for which \([1,3]\) is true. It is also not clear that one cannot subtract further conformal invariants from \(\sum_{i \in L_{new}} a_i C^i(g)\) in order to simplify it even further. At any rate, (as discussed in section \([2]\) the local conformal invariants \(W(g)\) that one subtracts from \(P(g)\) in order to simplify it are all explicitly constructed using the Fefferman-Graham ambient metric. \([17,18]\). Our construction is elaborate and relies on the study of linear combinations of complete contractions in the ambient metric with specific algebraic properties. It is also worth noting that the algebraic properties of the terms in \(\sum_{i \in L_{new}} a_i C^i(g)\) are precisely what is needed in order to prove (by the methods in section \([3]\) that \(\sum_{i \in L_{new}} a_i C^i(g) = \text{div}_i T^i(g)\).

We now discuss in more detail the two main steps in the proof of Proposition 1.1. First Step: We show (in Lemmas \([1,1]\) \([1,2]\) \([1,3]\) below) that there exists a local conformal invariant \(W(g)\) and a divergence \(\text{div}_i T^i(g)\) so that:

\[
P(g) = W(g) + \text{div}_i T^i(g) + \sum_{f \in F} a_f C^f(g) + \sum_{j \in \text{Junk-Terms}} a_j C^j(g), \tag{1.4}
\]

\(^5\)These are local conformal invariants, by construction; they are (at least apriori) not divergences for the base metric \(g\).

\(^6\)Recall that \(I^j_1(\phi) = \frac{1}{4} \lim_{t \to 0} t^2 \int M \phi \pi^j_1(\phi) dV;\) we have then derived a useful local formula in \([3]\), which we called the “super divergence formula”.

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where the terms indexed in $\text{Junk} - \text{Terms}$ have at least $\sigma + 1$ factors, while the terms indexed in $F$ are in the form $\text{(1.2)}$ and have internal contractions in at least two different factors.

The second step is to show that there exists another divergence $\text{div}_i T'^i(g)$ so that (in the notation of (1.4)):

$$\sum_{f \in F} a_f C^f(g) = \text{div}_i T'^i(g) + \sum_{j \in \text{Junk} - \text{Terms}} a_j C^j(g). \quad (1.5)$$

The above two sub-steps combined prove Proposition 1.1.

An outline of the proof of (1.4): As explained, the proof of this step relies heavily on the ambient metric of Fefferman and Graham [17, 18]. This is a strong tool that allows one to explicitly construct all local conformal invariants of weight $-n$ in dimension $n$. By making detailed use of the precise form of the ambient metric (and in one instance of the super divergence formula), we are able to explicitly construct the local conformal invariant $W(g)$ and the divergence $\text{div}_i T'^i(g)$ needed for (1.4). The “main algebraic Propositions” 3.1 and 3.2 of the present paper are not used in deriving (1.4).

An outline of the proof of (1.5): (1.6) is proven by a new induction: At a rough level, the new induction can be described as follows: We denote by $j$ the minimum number of internal contractions among the complete contractions indexed in $F$. Denote the corresponding index set by $F_j \subset F$. We then prove that we can write:

$$\sum_{f \in F_j} a_f C^f(g) = \text{div}_i T'^i(g) + \sum_{j \in F_{j+1}} a_f C^f(g) + \sum_{t \in \text{Junk} - \text{Terms}} a_t C^t(g). \quad (1.6)$$

Here the terms indexed in $F_{j+1}$ are complete contractions in the form (1.5) with $j + 1$ internal contractions in total, of which at least two belong to different factors.

Observe that (1.5) clearly follows by iterative application of (1.6) \footnote{This is because there can be at most $\phi$ internal contractions in any non-zero complete contraction in the form (1.2) with weight $-n$.}

Now, there are a number of difficulties in proving (1.6):

How can one “recognize” the linear combination $\sum_{f \in F_j} a_f C^f(g)$ (which appears in $P(g)$) in $I_g^1(\phi)$? We observe that in this setting, $I_g^1(\phi)$ can be expressed as:

$$I_g^1(\phi) = \sum_{x \in X} a_x \text{contr}^* (\nabla^{(m_1)} R \otimes \cdots \otimes \nabla^{(m_2)} R \otimes \nabla^{(p)} \phi) + \sum_{j \in \text{Junk} - \text{Terms}} a_j C^j(g), \quad (1.7)$$
where the complete contractions indexed in $X$ have $\sigma + 1$ factors in total, while $
abla \sum_{j \in \text{Junk-Terms}} a_j C^j(g)$ stands for a generic linear combination of complete contractions with at least $\sigma + 2$ factors. We denote by $(I^1_g(\phi))\nabla \phi$ the “piece” in

\[ \sum_{x \in X} \ldots \]

which consists of terms with a factor $\nabla \phi$.

Observe that $(I^1_g(\phi))\nabla \phi$ can only arise from the terms in $P(g)$ with $\sigma$ factors in total.

In fact (as we will show in subsection 3.1), we can easily reconstruct

\[ \sum_{f \in F_j} a_f C^f(g) \]

in $P(g)$ if we are given the “piece” $(I^1_g(\phi))\nabla \phi$ in $I^1_g(\phi)$. Thus, our aim is to use the super divergence formula for $(I^1_g(\phi))\nabla \phi$ to express $(I^1_g(\phi))\nabla \phi$ as “essentially a divergence”.

The most important difficulty in deriving (1.6) appears when we apply the super divergence formula to $(I^1_g(\phi))$. To illustrate why this case is harder than the case $s > 0$ which was treated in [3], we will note that $(I^1_g(\phi))\nabla \phi$ can be expressed as follows:

\[ I^1_g(\phi) = (I^1_g(\phi))\nabla \phi + \sum_{q \in Q} a_q C^q(g) \cdot \Delta \phi + \sum_{z \in Z} a_z C^z(g) + \sum_{j \in \text{Junk-Terms}} a_j C^j(g), \]

where the terms $C^q(g) \cdot \Delta \phi$ are in the form:

\[ \text{contr}((\nabla^{(m_1)}R \otimes \cdots \otimes \nabla^{(m_s)}R \otimes \Delta \phi)), \]

(1.9)

(1.10)

where $p \geq 2$ and moreover if $p = 2$ then the two indices $a, b$ in $\nabla^{(2)} \phi$ are not contracting against each other in (1.10). Furthermore, it follows that the terms $C^q(g) \cdot \Delta \phi$ in (1.8) are both from the terms with $\sigma$ factors in $P(g)$ and from terms with $\sigma + 1$ factors in $P(g)$.

Now, to obtain a local formula for the expression $(I^1_g(\phi))\nabla \phi$, we consider the super divergence formula applied to $I^1_g(\phi)$ and pick out the terms which have $\sigma + 1$ factors in total, and furthermore have a factor $\nabla \phi$ (differentiated only once). It follows that the linear combination of those terms in $\text{supdiv}[I^1_g(\phi)]$ must vanish separately (modulo junk terms with more that $\sigma + 1$ factors): We thus obtain a new local equation, which we denote by:

\[ \text{supdiv}_+[I^1_g(\phi)] = \sum_{j \in \text{Junk-Terms}} a_j C^j(g). \]

(1.11)

Now, if one follows the algorithm for the super divergence formula, one observes that the terms in the RHS of (1.8) which can contribute to
Proposition 1.1. We claim that there is a conformally invariant scalar $L$ in $W$, but also the terms
\[ \sum_{q \in Q} a_q C^q(g) \cdot \Delta \phi \] in $W$ are the terms in \( (I^1_g(\phi))_{\nabla \phi} \) in $W$, but also the terms $\sum_{q \in Q} a_q C^q(g) \cdot \Delta \phi$ in $W$. Thus, whereas in $I^1_g(\phi)$ (before we consider the super divergence formula), the terms with a factor $\nabla \phi$ (and with $\sigma + 1$ factors in total) can only arise from the “worst piece” of $P(g)$ after we apply the super divergence formula we also obtain terms in $\text{supdiv}_g[I^1_g(\phi)]$ which arise from $\sum_{q \in Q} a_q C^q(g) \cdot \Delta \phi$. At this point the fact that the terms with length $\sigma$ in $P(g)$ are all in the form $I^2$ and have two internal contractions belonging to different factors is crucial. We prove $I^5$ (explaining how to overcome these difficulties) in section 3.

1.2 Divide Proposition 1.1 into four smaller claims.

The main assumption for all four of the Lemmas below is that $f_{\text{Ms}} P(g) dV_g$ is a global conformal invariant and $P(g) = \sum_{t \in L} a_t C^t(g)$, where each $C^t(g)$ has at least $\sigma$ factors. The contractions $C^t(g)$ that do have $\sigma$ factors will be indexed in $L_\sigma \subset L$ and will all be in the form $I^2$.

**First three Lemmas: The local conformal invariant “piece” in $P(g)$:**

We firstly focus on the complete contractions $C^t(g), t \in L_\sigma$ with no internal contractions. We index those complete contractions in the set $L_\sigma^0$ and consider the sublinear combination $P(g)|_{L_\sigma^0} := \sum_{t \in L_\sigma^0} a_t C^t(g))$. Our first claim is the following:

**Lemma 1.1** Assume that $P(g) = \sum_{t \in L} a_t C^t(g)$ is as in the assumption of Proposition 1.1. We claim that there is a conformally invariant scalar $W(g)$ of weight $-n$ such that:

\[ P(g) - W(g) = \sum_{t \in T} a_t C^t(g) + \sum_{u \in U} a_u C^u(g), \]

where all the complete contractions $C^t(g)$ are in the form $I^2$ with length $\sigma$ and $\delta_W \geq 1$. Each $C^u(g)$ has length $\geq \sigma + 1$.

Assuming we can prove the above Lemma, we are reduced to proving Proposition 1.1 under the extra assumption that all complete contractions $C^t(g), t \in L_\sigma$ have at least one internal contraction (in other words we may now assume that $L_\sigma^0 = \emptyset$).

We then denote by $L_\sigma^1 \subset L_\sigma$ the index set of the complete contractions $C^t(g), l \in L_\sigma^1$ with one internal contraction. We claim:

\[ \text{supdiv}_g[I^1_g(\phi)] \]

\[ (\text{Junk} - \text{Terms}) \]

\[ \text{can be distinguished from the contribution of the terms (I^1_g(\phi))_{\nabla \phi}.} \]
Lemma 1.2 Assume that \( P(g) = \sum_{l \in L} a_l C^l(g) \) satisfies the assumptions of Proposition [1.1]; additionally, assume that each \( C^l(g) \), \( l \in L_\sigma \), satisfies \( \delta_W \geq 1 \). We claim that there is a conformally invariant scalar \( W(g) \) and a vector field \( T^i(g) = \sum_{r \in R} a_r C^{r,i}(g) \) of weight \( -n+1 \), where each \( C^{r,i}(g) \) is in the form (1.3) so that:

\[
\sum_{l \in L_\uparrow} a_l C^l(g) - W(g) - \text{div}_i T^i(g) = \sum_{f \in F} a_f C^f(g),
\]

(1.12)

where each \( C^f(g) \) is in the form (1.3) and has \( \delta_W \geq 2 \). The above holds modulo complete contractions of length \( \geq \sigma + 1 \).

Assuming we can prove the above Lemma, we are reduced to proving Proposition [1.1] under the extra assumption that each \( C^l(g) \), \( l \in L_\sigma \), has \( \delta_W \geq 2 \).

Lemma 1.3 Assume that \( P(g) = \sum_{l \in L} a_l C^l(g) \) satisfies the assumptions of Proposition [1.1]; assume moreover that each \( C^l(g) \), \( l \in L_\sigma \), has \( \delta_W \geq 2 \). Consider the index set \( L_{\text{stack}}^n \subset L_\sigma \) that consists of complete contractions in the form (1.2) that have all their \( \delta_W \) internal contractions belonging to the same factor \( \nabla^{(m)} W_{ijkl} \).

We claim that there is a scalar conformal invariant \( W(g) \) and a vector field \( T^i(g) = \sum_{r \in R} a_r C^{r,i}(g) \) of weight \( -n+1 \) (where each \( C^{r,i}(g) \) is in the form (1.2) so that:

\[
\sum_{l \in L_{\text{stack}}} a_l C^l(g) - W(g) - \text{div}_i T^i(g) = \sum_{j \in J} a_j C^j(g),
\]

(1.13)

where each \( C^j(g) \) is in the form (1.2), has \( \delta_W \geq 2 \) and also at least two internal contractions belonging to different factors. The above holds modulo complete contractions of length \( \geq \sigma + 1 \).

Next Claims: The divergence “piece” in \( P(g) \).

For our next claim we will be assuming that for \( P(g) = \sum_{l \in L} a_l C^l(g) \) (which satisfies the assumptions of Proposition [1.1]), all complete contractions \( C^l(g) \), \( l \in L_\sigma \), are in the form (1.2) with \( \delta_W \geq 2 \) internal contractions, and at least two internal contractions belong to different factors.

Let \( j = \min_{l \in L_\sigma} \{ \delta_W | C^l(g) \} \). We define \( L^j_\sigma \subset L_\sigma \) to stand for the index set of the complete contractions with \( \delta_W = j \).

Lemma 1.4 Assume that \( P(g) = \sum_{l \in L} a_l C^l(g) \) satisfies the assumptions of Proposition [1.1]; assume additionally that all complete contractions \( C^l(g) \), \( l \in L_\sigma \), have \( \delta_W \geq 2 \) internal contractions and at least two of those internal contractions belong to different factors.

We claim that there is a linear combination of partial contractions, \( \sum_{h \in H} a_h C^{h,i}(g) \), where each \( C^{h,i}(g) \) is in the form (1.2) with weight \( -n+1 \) and \( \delta_W = j \), so that:

\[
\sum_{l \in L_\downarrow} a_l C^l(g) - \text{div}_i \sum_{h \in H} a_h C^{h,i}(g) = \sum_{v \in V} a_v C^v(g),
\]

(1.14)
where each $C_{v}(g)$ is in the form (1.2) with $\delta_{W} \geq j + 1$, and with at least two internal contractions in different factors. The above equation holds modulo complete contractions of length $\geq \sigma + 1$.

Clearly, if we can show the above four Lemmas, Proposition 1.1 will follow: First applying the first three Lemmas to $P(g)$ to derive that there exists a local conformal invariant $W(g)$ and a divergence $\text{div}_{i}T^{i}(g)$ as claimed in Proposition 1.1 such that:

$$P(g) = W(g) + \text{div}_{i}T^{i}(g) + \sum_{l \in L'} a_{l}C_{l}(g) + \sum_{j \in J} a_{j}C_{j}(g);$$

here each $C_{j}(g)$ has length $\geq \sigma + 1$; each $C_{l}(g), l \in L'$ has length $\sigma$ and is in the form (1.2) and moreover has $\delta_{W} \geq 2$ internal contractions, at least two of which belong to different factors. Then, iteratively applying (1.4) to $P'(g) := P(g) - W(g) - \text{div}_{i}T^{i}(g)$ we derive that there exists a divergence $\text{div}_{i}T^{i}(g)$ as required by Proposition 1.1 such that:

$$\sum_{l \in L'} a_{l}C_{l}(g) = \text{div}_{i}T^{i}(g) + \sum_{j \in J} a_{j}C_{j}(g),$$

where each $C_{j}(g)$ has length $\geq \sigma + 1$. Therefore, the above four Lemmas indeed imply Proposition 1.1. We present the proofs of these four Lemmas in the remainder of this paper.

2 The locally conformally invariant “piece” in $P(g)$: A proof of Lemmas 1.1, 1.2, 1.3.

2.1 The Fefferman-Graham construction of local conformal invariants and an algorithm for computations.

We start with a brief discussion of the ambient metric of Fefferman and Graham, \[17\] \[18\] which we use to construct local conformal invariants. The ambient metric construction provides a canonical embedding of a Riemannian manifold $(M^{n}, g)$ into an ambient Ricci-flat Lorentzian manifold $(\tilde{G}^{n+2}, \tilde{g}^{n+2})$. We refer the reader to the papers \[17\], \[18\] for a detailed exposition of this construction. We recall here a few features of this construction which will be useful to us: Recall that given coordinates $\{x^{1}, \ldots, x^{n}\}$ for $(M^{n}, g)$, then the ambient metric embedding provides a special coordinate system $\{x^{0}, x^{1}, \ldots, x^{n}, x^{n+1}\}$ for the ambient manifold $(\tilde{G}^{n+2}, \tilde{g}^{n+2})$. More precisely, any given point $x_{0} \in M^{n}$ is mapped to $\tilde{x}_{0}$ in $\tilde{G}^{n+2}$ with $\tilde{x}_{0}^{0} = 1$ and $x^{n+1} = 0$. We denote the vectors in $T\tilde{G}_{\tilde{x}_{0}}$ that correspond to the directions of $x^{0}, \ldots, x^{n}, x^{n+1}$ by $X^{0}, \ldots, X^{n}, X^{\infty}$ respectively. In what follows, we will often use the notion of “assigning of values” $0, 1, \ldots, n, \infty$ to the (lower) indices of tensors. By this we will mean that we evaluate those (covariant) tensors against the vectors $X^{0}, \ldots, X^{\infty}$.

\[14\] Notice that $\int_{M^{n}} P'(g)dV_{g}$ is also a global conformal invariant.
Now, let us furthermore recall that in the coordinate system \(\{x^0, \ldots, x^{n+1}\}\) the ambient metric at \(\tilde{x}_0\) is of the form:

\[
\tilde{g}^{n+2}_{IJ}dx^I dx^J = 2dx^0dx^{n+1} + g_{ij}dx^i dx^j,
\]

where \(0 \leq I, J \leq n + 1\) and \(1 \leq i, j \leq n\).

The Christoffel symbols of the ambient metric (evaluated with respect to this special coordinate system at a point with coordinates \((t, x^1, \ldots, x^n, 0)\)) are:

\[
\begin{align*}
\tilde{\Gamma}^0_{IJ} &= 
\begin{pmatrix}
0 & 0 & 0 \\
0 & -tP_{ij} & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\tilde{\Gamma}^k_{IJ} &= 
\begin{pmatrix}
0 & t^{-1}\delta^k_j & 0 \\
t^{-1}\delta^k_j & \Gamma^k_{ij} & g^{kl}P_{jl} \\
0 & g^{kl}P_{jl} & 0
\end{pmatrix}, \\
\tilde{\Gamma}^\infty_{IJ} &= 
\begin{pmatrix}
0 & 0 & t^{-1} \\
0 & -g_{ij} & 0 \\
t^{-1} & 0 & 0
\end{pmatrix}.
\end{align*}
\]

Now, as proven in [17], [18], one can construct local conformal invariants by considering complete contractions involving the ambient curvature tensor \(\tilde{R}\) and its ambient covariant derivatives (with respect to the Levi-Civita connection \(\tilde{\nabla}\) of the ambient metric). In other words, Fefferman and Graham proved that any linear combination of complete contractions (of weight \(-n\)) in the form:

\[
\text{contr}(\tilde{\nabla}^{(m_1)}_{r_1 \ldots r_{m_1}} \tilde{R}_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes \tilde{\nabla}^{(m_s)}_{r_s \ldots r_{m_s}} \tilde{R}_{i_s j_s k_s l_s}), \quad (2.3)
\]

say \(L(\tilde{g}) = \sum_{h \in H} a_h C^h(\tilde{g})\), is by construction a local conformal invariant of weight \(-n\). In particular, \(L(\tilde{g})\) can also be expressed a linear combination of complete contractions in the form \((1.2)\), involving covariant derivatives of the curvature tensor of the metric \(g\), so \(L(\tilde{g}) = F(g)\), where \(F(g) := \sum_{h \in H} a_h \text{contr}^h(\nabla^{(m)}R \otimes \cdots \otimes \nabla^{(m')}R)\). Furthermore, for any function \(\lambda > 0\), we will have \(F(\lambda^2 \cdot g) = \lambda^{-n} F(g)\).

In the rest of this subsection we will seek to understand how a given complete contraction in the form \((2.12)\) can be expressed as a linear combination of complete contractions of the form \((1.2)\), involving covariant derivatives of the curvature tensor of the metric \(g\).

From contractions in the ambient metric \(\tilde{g}\) to contractions in the base metric \(g\):

We do this in steps. Consider any complete contraction \(C(\tilde{g}^{n+2})\) in the form \((2.12)\), of weight \(-n\). We denote by \(\overline{\nabla}\) the Levi-Civita connection of \(\tilde{g}\) and by \(\nabla\) the Levi-Civita connection of \(g\).

\footnote{When \(n\) is even, the jet of the ambient metric at \(\tilde{x}^0\) is only defined up to order \(\frac{1}{2} - 1\). In our constructions, this restriction will always be fulfilled.}
Our aim is to write \( C(\tilde{g}^{n+2}) \) as a linear combination of complete contractions (with respect to the metric \( g \)) of the form:

\[
\text{contr}_g(T^\alpha_1 \otimes \cdots \otimes T^\alpha_r \otimes F^{\beta_1} \otimes F^{\beta_w}),
\]

(2.4)

where the factors \( T^\alpha \) are all in the form \( \nabla^{(m)} W \) (so each \( T^\alpha \) is an iterated covariant derivative of the Weyl tensor), all of whose \((m+4)\) indices are being contracted against another index in \( \text{contr}_g(\ldots) \). Each factor \( F^\beta \) is in the form \( \nabla^{(\delta)} \tilde{g}^{(\sigma)} \tilde{R} \) where \( \delta > 0 \) of its indices have a (fixed) value \( \infty \), \( \epsilon \geq 0 \) of its indices have a value \( 0 \) and the rest of its indices are being contracted (with respect to the metric \( g \)) against some other index in \( \text{contr}_g(\ldots) \).

We discuss how any complete contraction \( \text{contr}_g(\tilde{\nabla}^{(m_1)} \tilde{\nabla}^{(m_r)} \tilde{\nabla}^{(m_p)} \tilde{R}) \) can be written as a linear combination of contractions in the form \( \text{contr}_g(\ldots) \). This is a two step-procedure:

Consider any complete contraction \( C(\tilde{g}^{n+2}) \) in the ambient metric. Firstly we define the set of assignments, ASSIGN:

**Definition 2.1** An element \( \text{ass} \in \text{ASSIGN} \) is a rule that acts by picking out each particular contraction between indices \((\alpha, \beta)\) and assigning to that pair of indices either the values \((\infty, 0)\) or the values \((0, \infty)\) or repeatedly assigning any pair of numbers \( i, j \), \( 1 \leq i, j \leq n \) and then multiplying by \( g^{ij} \) and summing over all such pairs. For each assignment \( \text{ass} \in \text{ASSIGN} \) we obtain a complete contraction (in the metric \( g \)) involving tensors \( \tilde{\nabla}^{(m)} \tilde{R}; \) we denote this complete contraction \( \text{ass}[C(\tilde{g}^{n+2})] \).

Thus, for each element \( \text{ass} \in \text{ASSIGN} \) \( \text{ass}[C(\tilde{g}^{n+2})] \) is a complete contraction in the quadratic form \( g^{ij} \) (denote it by \( \text{contr}_g(T^1 \otimes \cdots \otimes T^r) \)) where the factors \( T^i \) are tensors \( \tilde{\nabla}^{(m)} \tilde{R} \) with some indices having the (fixed) value \( \infty \), some having the (fixed) value \( 0 \) and all the rest of the indices are being contracted against each other with respect to the metric \( g^{ij} \). (Note: when we separately consider a tensor \( T^i \) in the contraction \( \text{contr}_g(T^1 \otimes \cdots \otimes T^r) \) we will call the indices of the third kind above “free” indices; in \( \text{contr}_g(T^1 \otimes \cdots \otimes T^r) \) these can only be assigned values \( 1, \ldots, n \).) It follows that we can write:

\[
C(\tilde{g}^{n+2}) = \sum_{\text{ass} \in \text{ASSIGN}} \text{ass}[C(\tilde{g}^{n+2})].
\]

(2.5)

Now, the next step is to write out each \( \text{ass}[C(\tilde{g}^{n+2})] \) as a linear combination of contractions in the form \( \text{contr}_g(\ldots) \) (modulo a linear combination of contractions with length \( \geq \sigma + 1 \) which we do not care about). In order to do that, we will pick out each tensor \( T = \tilde{\nabla}^{(m)} \tilde{R}^{ijkl} \) in \( \text{ass}[C(\tilde{g}^{n+2})] \) (recall that each of the indices \( r_1, \ldots, r_m, i, j, k, l \) either has a (fixed) value \( 0 \) or \( \infty \), or is a free index that may take any values between \( 1, \ldots, n \)). We denote by \( \{u_1, \ldots, u_t\} \subset \{r_1, \ldots, r_m, i, j, k, l\} \) the set of free indices in \( T \) (i.e. the indices that may take 

\footnote{\( \nabla \) is the Levi-Civita connection of \( g \) and \( \tilde{\nabla} \) is the Levi-Civita connection of \( \tilde{g} \).}
values between $1, \ldots, n$. We will write $T_{u_1 \ldots u_t}$ to stress the fact that the free indices in $T$ that are being contracted (with respect to the metric $g^{ij}$) against other indices in $\text{ass}(C(g^{n+2}))$ are precisely the indices $u_1, \ldots, u_t$.

**Step 2:** We claim that $T_{u_1 \ldots u_t}$ can be expressed in the form:

$$T_{u_1 \ldots u_t} = \sum_{f=1}^{F} a_f T^f_{u_1 \ldots u_t}(g), \quad (2.6)$$

where each of the terms $T^f_{u_1 \ldots u_t}(g)$

is a tensor product with free indices $u_1, \ldots, u_t$ in one of two forms: Either $T^f_{u_1 \ldots u_t}(g)$ will be a tensor product

$$\nabla^{(b)}_{c_1 \ldots c_b} \nabla^{(q)}_{f_1 \ldots f_q} \tilde{R}_{zxcv} \otimes g_{ab} \otimes \cdots \otimes g_{a'b'} \quad (2.7)$$

for which at least one index in the tensor $\nabla^{(q)} \tilde{R}_{ijkl}$ has the value $\infty$, or it will just be a tensor of the form:

$$\nabla^{(d)}_{f_1 \ldots f_d} W_{zxcv}. \quad (2.8)$$

**Proof of (2.6):** Firstly, if $T_{u_1 \ldots u_t}$ has an index with value $\infty$, there is nothing to prove, since we are in the first of the two desired forms. Thus, we only have to study the case where all the indices $r_1, \ldots, r_{m+4}$ have values between $0$ and $n$. We will then show by induction how $T_{u_1 \ldots u_t}$ can be written as a linear combination in the form (2.6): We assume that we are able to write out any factor $T_{u_1 \ldots u_t} (= \nabla^{(m)} \tilde{R})$ as in (2.6) provided $m \leq d$. We now show how we can do this for $m = d + 1$.

We consider $T_{u_1 \ldots u_t} = \nabla^{(d+1)}_{r_1 \ldots r_{d+1}} \tilde{R}_{r_1, r_2, r_3, \ldots, r_{d+5}}$ and we distinguish cases based on the value of the index $r_1$: If $r_1$ has the value $0$, we then just denote by $d^*$ the number of the indices $r_2, \ldots, r_{d+5}$ that do not have the value $0$. It then follows that $T_{u_1 \ldots u_t} = (-d^* + d) \nabla^{(d)}_{r_2 \ldots r_d} \tilde{R}_{r_1, r_2, r_3, \ldots, r_{d+5}}$. Thus we are done, by our inductive hypothesis.

On the other hand, if the index $r_1$ is a free index (allowed to take values $1, \ldots, n$), we then denote by $\{a_1, \ldots, a_x\} \subset \{2, \ldots, d + 5\}$ the set of numbers for which the index $r_k$, $k \in \{a_1, \ldots, a_x\}$ has been assigned the value $0$. We also denote by $\{b_1, \ldots, b_x\} \subset \{2, \ldots, d + 5\}$ the set of numbers for which $r_k$, $k \in \{b_1, \ldots, b_x\}$ is a free index (taking values between $1, \ldots, n$). We then have:

$$\nabla^{(d+1)}_{r_1 \ldots r_{d+1}} \tilde{R}_{r_1, r_2, r_3, r_4, r_5} = \nabla^{(d)}_{r_2 \ldots r_{d+1}} \tilde{R}_{r_1, r_2, r_3, r_4, r_5}$$

$$- \sum_{k=1}^{x} \nabla^{(d)}_{r_2 \ldots r_{k-1} r_k \ldots r_{d+1}} \tilde{R}_{r_1, r_2, r_3, r_4, r_5}$$

$$+ \sum_{k=1}^{x} \nabla^{(d)}_{r_2 \ldots r_{k-1} r_k \ldots r_{d+1}} \tilde{R}_{r_1, r_2, r_3, r_4, r_5} \otimes g_{r_1 r_k} + Q(R). \quad (2.9)$$

Here $f$ is a label, while $u_1, \ldots, u_t$ are free indices.

Recall $\nabla$ is the Levi-Civita connection of $g$, and $\nabla$ is the Levi-Civita connection of $g^{n+2}$.
[2.9] follows virtue of the Christoffel symbols $\tilde{\Gamma}_{0i}^k = \delta_i^k$ and $\tilde{\Gamma}_{ij}^k = -g_{ij}$.

Now, observe that the tensors $\tilde{\nabla}^{(d)}_{\tau_2 \ldots \tau_{d+1}} \tilde{R}_{\tau_2 \ldots \tau_{d+2} \tau_{d+3} \ldots \tau_{d+5}}$ are in our desired form (2.4) because they contain an index $\infty$. Moreover, the tensors $\tilde{\nabla}^{(d)}_{\tau_2 \ldots \tau_{d+1}} \tilde{R}_{\tau_2 \ldots \tau_{d+2} \tau_{d+3} \ldots \tau_{d+5}}$ and $\tilde{\nabla}^{(d)}_{\tau_2 \ldots \tau_{d+1}} \tilde{R}_{\tau_2 \ldots \tau_{d+2} \tau_{d+3} \ldots \tau_{d+5}}$ fall under our inductive hypothesis. Therefore, expressing these tensors as linear combinations of tensors in the form (2.7), (2.8) we derive that (2.6) follows by induction. 

This analysis of each of the tensors $T_{\mu_1 \ldots \mu_\ell}$ allows us to analogously write each complete contraction $C(\tilde{g}^{n+2})$ as a sum of complete contractions in the form (2.4): We pick out each factor $T_\mu$ ($\mu = 1, \ldots, \sigma$) and replace it by one of the summands $T_\mu$ (times a coefficient) in (2.6). We then take the complete contraction (with respect to $g^{ij}$) of these new tensors $T_\mu$. The formal sum over all these substitutions will be denoted by $\sum_{\gamma \in BR_{ass}} \gamma \{C(\tilde{g}^{n+2})\}$. Therefore, using the above and (2.4) we derive a formula:

$$C(\tilde{g}^{n+2}) = \sum_{ass \in ASSIGN} \sum_{\gamma \in BR_{ass}} \gamma \{ass[C(\tilde{g}^{n+2})]\}; \quad (2.10)$$

(the symbol $BR_{ass}$ serves to illustrate that we have first picked a particular assignment $ass \in ASSIGN$).

Finally, we introduce a definition.

**Definition 2.2** For each complete contraction $C^\ell(g)$ in the form:

$$\text{contr}(\nabla^{(m_1)}_{\tau_1 \ldots \mu_1} W_{i_1 \ldots i_{k_1}} \otimes \cdots \otimes \nabla^{(m_\ell)}_{\tau_\ell \ldots \mu_\ell} W_{i_\ell \ldots i_{k_\ell}}), \quad (2.11)$$

with each $m_a \leq \frac{n-4}{2}$, we construct a complete contraction $\text{Amb}[C^\ell(g)]$ in the ambient metric:

$$\text{contr}(\nabla^{(m_1)}_{\tau_1 \ldots \mu_1} \tilde{R}_{i_1 \ldots i_{k_1}} \otimes \cdots \otimes \nabla^{(m_\ell)}_{\tau_\ell \ldots \mu_\ell} \tilde{R}_{i_\ell \ldots i_{k_\ell}}); \quad (2.12)$$

this is related to $C^\ell(g)$ in the following way: Consider any two factors $T_a = \nabla^{(m_a)}_{\tau_1 \ldots \tau_{m_a}} W_{i_1 \ldots i_{m_a}}$ and $T_b = \nabla^{(m_b)}_{\tau_1 \ldots \tau_{m_b}} W_{i_1 \ldots i_{m_b}}$ in $C^\ell(g)$ (the $a^{th}$ and $b^{th}$ factors) and suppose that the $k^{th}$ index in $T_1$ contracts against the $l^{th}$ index in $T_2$. Then, we require that in $\text{Amb}[C^\ell]$ we will have two factors $\tilde{T}_a = \nabla^{(m_a)}_{\tau_1 \ldots \tau_{m_a}} \tilde{R}_{i_1 \ldots i_{m_a}}$ and $\tilde{T}_b = \nabla^{(m_b)}_{\tau_1 \ldots \tau_{m_b}} \tilde{R}_{i_1 \ldots i_{m_b}}$ in $C^\ell(\tilde{g}^{n+2})$ and furthermore the $k^{th}$ index in $\tilde{T}_1$ will contract against the $l^{th}$ index in $\tilde{T}_2$.

We will call the local conformal invariant $\text{Amb}[C(g)]$ the ambient analogue of $C(g)$.

*Note:* Since we are assuming $m_a \leq \frac{n-4}{2}$, we know by [17], [18] that the above ambient complete contraction is well-defined.
2.2 Proof of Lemma 1.1

In order to prove this Lemma, let us consider the linear combination \( \sum_{l \in L_0} a_l C^l(g) \) of the local conformal invariant needed for the proof of Lemma 1.1 is precisely \( W(g) := \sum_{l \in L} a_l \text{Amb}[C^l(g)] \). This will follow by virtue of the following Lemma:

**Lemma 2.1** Given any \( C^l(g) \) in the form (1.2) with no internal contractions, we claim:

\[
C^l(g) - \text{Amb}[C^l] = \sum_{t \in T} a_t C^t(g),
\]

where each \( C^l(g) \) either has length \( \sigma + 1 \) (and is in the form (1.1)), or is in the form (1.2) and has length \( \sigma \) and \( \delta_W \geq 1 \).

**Proof:** Recall the algorithm from the previous subsection. Recall the equation (2.10). Let \( \text{ass}^* \in \text{ASSIGN} \) be the (unique) assignment where no index in the complete contraction \( \text{Amb}[C^l(g)] \) is assigned the value \( \infty \). Firstly we claim that for every \( \text{ass} \in \text{ASSIGN} \setminus \{ \text{ass}^* \} \):

\[
\sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma \{ \text{ass}[C(\tilde{g}^{n+2})] \} = \sum_{t \in T} a_t C^t(g), \tag{2.13}
\]

where the RHS is as in the claim of Lemma 2.1. To see this, recall the algorithm from the previous subsection. Observe that \( \text{ass}[C(\tilde{g}^{n+2})] \) stands for a complete contraction (with respect to the metric \( g^{ij} \)) of factors \( \tilde{\nabla}(m) R_{ijkl} \), and at least one of those factors has an index with the (fixed) value \( \infty \); it then follows from the higher-order Taylor expansion of the ambient metric \( \tilde{g} \) (see [17], [18]) that such a tensor \( T_{i_1...i_f} \), can be written out in terms of the metric \( g \) as follows:

\[
T_{i_1...i_f} = \sum \nabla^{(m)} W_{ijkl} + \sum T'_{i_1...i_f},
\]

where each tensor \( \nabla^{(m)} W_{ijkl} \) has \( f \) free lower indices, \( m + 4 > f \) and the rest of the indices are internally contracting. Moreover, there will be at least one internal contraction in \( \nabla^{(m)} W_{ijkl} \). \( \sum T'_{i_1...i_f} \) stands for a linear combination of partial contractions in the form:

\[
pcontr(\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_w)} R^{i'j'k'l'} \otimes g_{ab} \otimes \cdots \otimes g_{a'b'}) \tag{2.14}
\]

with \( w \geq 2 \) (i.e. with at least two factors \( \nabla^{(m)} R \)). Substituting this expression into \( \sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma \{ \text{ass}[C(\tilde{g}^{n+2})] \} \) we derive (2.13).

So we are reduced to considering \( \sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma \{ \text{ass}^*[C(\tilde{g}^{n+2})] \} \). That is, we are reduced to considering the case where no index in the complete contraction \( C^l(\tilde{g}^{n+2}) \) is assigned the value \( 0 \) or \( \infty \). But then recall the Christoffel symbols \( \tilde{\Gamma}^k_{ij} \) for the ambient metric \( \tilde{g}^{n+2} \):

\[\text{See Definition 2.2}\]

\[\text{Equivalently, } \text{ass}^* \text{ is the unique assignment where no particular contraction } (a, b) \text{ is assigned the values } (\infty, 0) \text{ or } (0, \infty).\]
If $1 \leq i, j, k \leq n$ we have $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}$. On the other hand, $\tilde{\Gamma}^0_{ij} = P_{ij}$ and $\tilde{\Gamma}^{n+1}_{ij} = -g_{ij}$. Thus we derive that for values $r_1, \ldots, l \in \{1, \ldots, n\}$:

$$\nabla^{(m)}_{r_1 \ldots r_m} \tilde{R}^{ijkl} = \nabla^{(m)}_{r_1 \ldots r_m} W^{ijkl} + T_{r_1 \ldots r_m ijkl} + T'_{r_1 \ldots r_m ijkl}, \quad (2.15)$$

where $T_{r_1 \ldots r_m ijkl}$ is a linear combination of tensor fields of the form $g_{ab} \otimes \cdots \otimes g_{cd} \otimes \nabla^{(p)} \tilde{R}_{ijkl}$, where the factor $\nabla^{(p)} \tilde{R}_{ijkl}$ contains at least one index with a value $\infty$. $T'_{r_1 \ldots r_m ijkl}$ stands for a linear combination of tensor fields of the form $(2.14)$ with $w \geq 2$.

This completes the proof of Lemma 2.1.

2.3 Proof of Lemma 1.2: Proof of half the Lemma.

We prove Lemma 1.2 in two steps. In order to explain this proof, we need one piece of notation:

**Definition 2.3** Consider a complete contraction $C(g)$ in the form $(1.2)$ with weight $-n + 2$. We define $\Delta_k [C(g)]$, for any $k$, $1 \leq k \leq \sigma$ to be the complete contraction that arises from $C(g)$ by picking out its $k$th factor $F_k$ and replacing it by $\Delta F_k$. Then, we define:

$$\Delta_k [C(g)] = \sum_{\ell=1}^{\sigma} \Delta_{k}^{\ell}[C(g)]$$

This operation extends to linear combinations.

Lemma 1.2 will follow by the next two claims:

**Lemma 2.2** Under the assumptions of Lemma 1.2 we claim that there is a vector field $T^i(g) = \sum_{r \in R} a_r C^{r,i}(g)$ (of the form required by Lemma 1.2) so that:

$$\sum_{l \in L_2} a_l C^l(g) - \text{div}_1 \sum_{r \in R} a_r C^{r,i}(g) = \Delta_r \left[ \sum_{v \in V} a_v C^v(g) \right] + \sum_{f \in F} a_f C^f(g), \quad (2.16)$$

modulo complete contractions of length $\geq \sigma + 1$. Here each $C^f(g)$ is in the form $(1.2)$ with length $\sigma$ and $\delta_W \geq 2$. On the other hand, each $C^v(g)$ is in the form $(1.2)$ with length $\sigma$, weight $-n + 2$ and with no internal contractions.

The next claim starts with the conclusion of the previous one:

**Lemma 2.3** Assume that $P(g)$ is as in the assumption of Lemma 1.2: assume furthermore that the sublinear combination $\sum_{l \in L_1} a_l C^l(g)$ of contractions with length $\sigma$ and precisely one internal contraction can be expressed in the form $\sum_{l \in L_1} a_l C^l(g) = \sum_{v \in V} a_v \Delta_r [C^v(g)]$.

We claim that there is a local conformal invariant $W(g)$ and a vector field $\sum_{r \in R} a_r C^{r,i}(g)$ of weight $-n + 1$ so that:
\[ \Delta_r \left[ \sum_{v \in V} a_v C^v(g) \right] - W(g) - \text{div}_l \sum_{r \in R} a_r C^{r,l}(g) = \sum_{f \in F} a_f C^f(g). \quad (2.17) \]

The above holds modulo complete contractions of length \( \sigma + 1 \). Here again each \( C^f(g) \) is in the form (1.2) with length \( \sigma \) and \( \delta_W \geq 2 \).

\[ \text{Proof of Lemma 2.2:} \] In order to prove this Lemma, we first slightly manipulate the sublinear combination \( \sum_{l \in L_2^1} a_l C^l(g) \) in \( P(g) \):

We first prove that by subtracting divergences from \( P(g) \), we may assume that every complete contraction \( C^l(g) \) with \( l \in L_2^1 \) will have its internal contraction in a factor of the form \( \nabla_l W_{ijkl} \):

This is done as follows: For each complete contraction \( C^l(g) \), \( l \in L_2^1 \), we isolate the one factor \( \nabla_{r_1 \ldots r_{\sigma - 1}} W_{ijkl} \) which contains the internal contraction (there can be only one such factor, by the definition of \( L_2^1 \)). Then, if the internal contraction is between two indices \( r_s, r_t \), we bring them to the positions \( r_1 \), \( r_{\sigma - 1} \) by repeatedly applying the curvature identity. That way we introduce correction terms that have length at least \( \sigma + 1 \). Then, we apply the identity

\[ \nabla_r W_{ijkl} + \nabla_i W_{rlk} + \nabla_j W_{rkl} = \sum (\nabla^s W_{srt} \otimes g), \quad (2.18) \]

(see subsection 2.3 – recall that the symbol \( \sum (\nabla^s W_{srt} \otimes g) \) stands for a linear combination of a tensor product of the three-tensor \( \nabla^s W_{srt} \) with an un-contracted metric tensor).

Thus (modulo introducing correction terms that are allowed in Lemma 2.2), we may assume that in each complete contraction \( C^l(g) \), \( l \in L_2^1 \) the internally contracting factor is of the form \( \nabla_{(m)} W_{ijkl} \). Finally, we subtract divergences from \( C^l(g) \) as in the proof of the silly divergence formula in [1], in order to arrange that all complete contractions \( C^l(g), l \in L_2^1 \) have the internal contraction in a factor of the form \( \nabla_l W_{ijkl} \) (this can be done modulo introducing complete contractions with more than one internal contractions—but these are allowed in the conclusion of our Lemma).

We note the transformation law for the factor \( \nabla^l W_{ijkl} \) under the re-scaling \( \hat{g} = e^{2\phi(x)} g \).

\[ (\nabla^l W_{ijkl})_g = (\nabla^l W_{ijkl})_{\hat{g}} + (n - 3) W_{ijkl} \nabla^l \phi |_{\hat{g}}. \quad (2.19) \]

Now, consider \( I^1_\phi[P(g)] := \text{Image}^1_\phi[P(g)] \). We firstly study the sublinear combination \( \text{Image}^1_\phi[P(g)] \).

Initially, we write \( \text{Image}^1_\phi[P(g)] \) out as a linear combination of complete contractions in the form:

\[ 21 \text{Recall that } \text{Image}^1_\phi[P(g)] := 4 \int_{t = 0}^{4 \pi} e^{\phi(x)} \phi \text{d}v_\phi \text{ and that } \int_{M^N} I^1_\phi(x) \text{d}v_\phi = 0. \]

\[ 22 \text{Recall that } P(g)(\phi) \text{ stands for the sublinear combination of terms in } P(g) \text{ with length } \sigma. \] Those terms are indexed in \( L_\sigma \subset L. \) Recall that \( \text{Image}^1_\phi[P(g)] := \sum_{l \in L_\sigma} a_l C^l(e^{2\phi(x)} g). \)
\[ \text{contr}(\nabla^{(m_1)} W_{ijkl} \otimes \cdots \otimes \nabla^{(m_m)} W_{i',j',k',l'} \otimes \nabla^{(\nu)} \phi). \tag{2.20} \]

Given that \( P(g) |_{\sigma} \) is a linear combination of complete contractions in the from (1.2), we immediately obtain such an expression for \( \text{Image}^{1}_{\phi}[P(g) |_{\sigma}] \), by applying the transformation laws from the Weyl tensor and the Levi-Civita connection, see the subsection 2.3 in [3]. Thus we write

\[ \text{Image}^{1}_{\phi}[P(g) |_{\sigma}] = \Sigma_{k \in K} a_k C^k_g(\phi), \]

where each \( C^k_g(\phi) \) is in the form (2.20) with length \( \sigma + 1 \).

Now, we re-express \( \text{Image}^{1}_{\phi}[P(g) |_{\sigma}] \) as a linear combination of complete contractions in the form:

\[ \text{contr}(\nabla^{(m_1)} R \otimes \cdots \otimes \nabla^{(m_m)} R \otimes \nabla^{(\nu)} \phi) \tag{2.21} \]

\((\nabla^{(m)} R \) above stands for the differentiated curvature tensor, where we do not write out the indices). This is done by picking out each complete contraction \( C^k_g(\phi) \) in the above equation and decomposing each Weyl tensor according to \( W_{ijkl} = R_{ijkl} + [P \wedge g] \) (see subsection 2.3 in [3]). Hence, we write out:

\[ \text{Image}^{1}_{\phi}[P(g) |_{\sigma}] = \Sigma_{k \in K} a_k \{ \Sigma_{w \in W} a_w C^{k,w}_g(\phi) \}. \]

Each complete contraction \( C^{k,w}_g(\phi) \) is in the form (2.21).

Now, we introduce some definitions regarding complete contractions \( C_g(\phi) \) of the form (2.21).

**Definition 2.4** For any complete contraction \( C_g(\phi) \), \( \delta \) will stand for the number of internal contractions among its factors \( \nabla^{(m)} R_{ijkl}, \nabla^{(p)} \text{Ric}, \nabla^{(\nu)} \phi \) (including the one (resp. two) internal contractions in the term \( \text{Ric}_{ij} = R^a_{\ iaj} \), \( R = R^a_{\ ab} \)). \( q \) will stand for the number of factors \( \nabla^{(p)} \text{Ric}, R \) (scalar curvature).

**Definition 2.5** Consider complete contractions \( C_g(\phi) \) in the form (2.21). \( \nu \) will stand for the number of derivatives on the factor \( \nabla^{(\nu)} \phi \).

Any such complete contraction with \( q = \delta = 0 \) and \( \nu = 1 \) will be called a target. Any complete contraction with \( \nu \geq 2 \) and \( \nabla^{(\nu)} \phi \neq \Delta \phi \) will be called irrelevant. Any complete contraction with \( \nu = 1 \) and \( q + \delta > 0 \) will be called a contributor.

Now, we consider complete contractions in the form (2.21) of length \( \sigma + 1 \) with a factor \( \Delta \phi \). If \( q = \delta = 0 \), we will call it dangerous. If \( \delta + q > 0 \) we will call it a contributor. Finally, any complete contraction of length \( \geq \sigma + 2 \) will be called irrelevant.

In the rest of this subsection, \( J_g(\phi) \) will stand for a generic linear combination of contributors and irrelevant complete contractions.

Our next aim is to understand \( \text{Image}^{1}_{\phi}[C^l(g)] \) for \( l \in L^1_{\sigma} \). We will need some definitions:
**Definition 2.6** Consider a complete contraction $C_g^l(\phi)$, $l \in L_g^1$ which is in the form (1.2) and has its internal contraction in a factor $\nabla^l W_{ijkl}$. We define the skeleton $C_g^{l\ast}(\phi)$ of $C^l(g)$ to be the complete contraction which is obtained from $C^l(g)$ by substituting the factor $\nabla^l W_{ijkl}$ by $\nabla^l R_{ijkl}$ and every other factor $\nabla^{(m)}_{r_1...r_{m}} W_{ijkl}$ by $\nabla^{(m)}_{r_1...r_{m}} R_{ijkl}$.

Also, consider any complete contraction $C^l(g)$ which is in the form (1.2) (not necessarily of weight $-\eta$) with $q = 0$ and $\delta = 0$. We then define its skeleton $C^{l\ast}(g)$ to be the complete contraction which is obtained from it by substituting each factor $\nabla^{(m)}_{r_1...r_{m}} W_{ijkl}$ by $\nabla^{(m)}_{r_1...r_{m}} R_{ijkl}$.

**Lemma 2.4** For any complete contraction $C^l(g)$ with $l \in L_g^1$, we claim that $\text{Image}_\phi^L[C^l(g)]$ can be expressed as follows:

$$\text{Image}_\phi^L[C^l(g)] = (n - 3)C_g^{l\ast}(\phi) + J_g(\phi).$$  \tag{2.22}$$

Proof: This is proven in two steps:

We first consider the complete contraction in $\text{Image}_\phi^L[C^l(g)]$ which arises as follows: We first substitute the factor $\nabla^l W_{ijkl}$ in $C^l(g)$ by $(n - 3)W_{ijkl}\nabla^l$. Let us denote the complete contraction that we obtain thus by $C_g^l(\phi)$. Then, we write $C_g^l(\phi)$ as a linear combination of complete contractions in the form (2.21):

$$C_g^l(\phi) = \sum_{t=1}^\infty a_t C_g^{t\ast}(\phi)$$

Let us assume that $C_g^{t\ast}(\phi)$ is obtained from $C_g^l(\phi)$ by substituting each factor $\nabla^{(m)}_{r_1...r_{m}} W_{ijkl}$ by a factor $\nabla^{(m)}_{r_1...r_{m}} R_{ijkl}$. The complete contractions $C_g^{t\ast}(\phi)$ with $t \geq 2$ arise by substituting at least one factor $\nabla^{(m)} W_{ijkl}$ in $C_g^l(\phi)$ by a factor $\nabla^{(m)}[\text{Ric} \otimes g]$ or $\nabla^{(m)}[R \otimes g \cdots \otimes g]$. Hence, each complete contraction $C_g^{t\ast}(\phi)$ with $t \geq 2$ will either have $q > 0$ or $\delta > 0$, so it will be a contributor. We see that the complete contraction $C_g^{t\ast}(\phi)$ above is $(n - 3) \cdot C_g^{t\ast}(\phi)$.

Let us now consider any complete contraction $C_g^{t\ast}(\phi)$ (in the form) (2.24) in $\text{Image}_\phi^L[C^l(g)]$ other than $C_g^{t\ast}(\phi)$. Then necessarily $C_g^{t\ast}(\phi)$ has arisen by applying the transformation laws for the Levi-Civita connection or the Weyl tensor to any indices in $C^l(g)$ other than the internal contraction in the factor $\nabla^l W_{ijkl}$. Hence $C_g^{t\ast}(\phi)$ will contain a factor $\nabla^l W_{ijkl}$, which still has an internal contraction. Therefore, writing $C_g^{t\ast}(\phi)$ as a linear combination of complete contractions in the form (2.21), as below:

$$C_g^{t\ast}(\phi) = \sum_{t=1}^\infty a_t C_g^{t\ast}(\phi)$$

we have that each $C_g^{t\ast}(\phi)$ will either have a factor $\nabla^{(m)} R_{ijkl}$ with an internal contraction or a factor $\nabla^{(p)} \text{Ric}$ or a factor $R$ (scalar curvature). Therefore, each such complete contraction $C_g^{t\ast}(\phi)$ is either a contributor or irrelevant. \hfill $\square$

By the same calculations we also derive:
Lemma 2.5 For any complete contraction \( C^l(g) \) with \( l \in L_\sigma \setminus L_\sigma^1 \):
\[
\text{Image}_1^g[C^l(g)] = J_g(\phi).
\] (2.23)

Proof: This fact follows by the same proof as for the previous Lemma. \( \square \)

Lemma 2.6 Consider the sublinear combination \( P(g)|_{\sigma+1} \) in \( P(g) \). Then:
\[
\text{Image}_1^g[P(g)|_{\sigma+1}] = \sum_{a \in U} a_u C_u^w(g) \Delta \phi + J_g(\phi),
\] (2.24)

where \( \sum_{a \in U} a_u C_u^w(g) \Delta \phi \) stands for a generic linear combination of dangerous complete contractions.

Proof: Straightforward from the transformation laws of the Levi-Civita connection and the curvature tensor under conformal re-scalings. (These can be found in subsection 2.3 in [3]). \( \square \)

We are now ready to get to the main part of proving Lemma 2.2. It will be useful to recall a few facts about the super divergence formula from [1]. This is the second and last instance in this series of papers where we make use of the super divergence formula in full strength. In all other instances we use the “main consequence” of the super divergence formula, as codified in subsection 2.2 in [3].

A few facts about the super divergence formula: We apply the super divergence formula to the operator \( I_1^g(\phi) = \sum_{l \in L} a_l C^l(g) \). Recall that \( \int_{M^n} I_1^g(\phi) dV_g = 0 \) for every compact \( (M^n, g), \phi \in C^\infty(M^n) \) and each \( C^l(g) \) is in the form (2.21). The super divergence formula applied to \( I_1^g(\phi) \) provides a local formula which expresses \( I_1^g(\phi) \) as a divergence of a vector field. We recall that there is a process by which each term \( C^l(g) \) in \( I_1^g(\phi) \) gives rise to divergences in the super divergence formula. In the end of [4] we summarized the conclusion of the super divergence formula as follows: Given \( I_1^g(\phi) = \sum_{l \in L} a_l C^l(g) \) we introduces the notion of “descendents” of each \( C^l(g), l \in L \); these are complete contractions involving factors \( \xi \) (or covariant derivatives thereof). Such complete contractions are divided up into categories (e.g. “good”, “hard”, “undecided”) and we then proceed by integrating by parts the factors \( \xi \) in those complete contractions. At each stage, we discard complete contractions which are “bad”, “hard” or “stigmatized”. In the end each complete contraction \( C^l_g(\phi) \) in \( I_1^g(\phi) \) contributes itself plus a linear combination of divergences to the super divergence formula; we have denoted this sum by \( \text{Tail}[C^l(g)] \). In other words, the super divergence formula can be summarized as:
\[
\sum_{l \in L} a_l \text{Tail}[C^l(g)] = 0.
\] (2.25)

The only further remarks we wish to make is that if \( C^l_g(\phi) \) has length \( \geq \sigma + 2 \) then \( \text{Tail}[C^l_g(\phi)] \) consists of complete contractions with length \( \geq \sigma + 2 \), and that
for every complete contraction $C_g^l(\phi)$ of length $\sigma + 1$ the only way that the factor $\nabla^{(\nu)} \phi$ in a complete contraction $C_g^l(\phi)$ can give rise to a factor $\xi$ is by replacing a pair of indices $r, r'$ in the factor $\nabla_{r_1...r_m}^{(\nu)} \phi$ that contract against each other by an expression $r_r \xi r'$. This factor $\xi r'$ will then be integrated by parts, giving rise to a divergence with respect to the index $r_r$.

Our Lemma will follow by picking out a particular “piece” in the super divergence formula which vanishes separately and then applying the operation Weylify (from subsection 5.1 in [4]) to the resulting equation:

**Lemma 2.7** Consider any irrelevant complete contraction $C_g^l(\phi)$. Then:

$$\text{Tail}[C_g^l(\phi)] = \sum_{w \in W} C_g^w(\phi),$$

where each complete contraction $C_g^w(\phi)$ has length $\geq \sigma + 1$, and if $C_g^w(\phi)$ does have length $\sigma + 1$ then it is irrelevant.

**Proof:** If $C_g^l(\phi)$ has length $\geq \sigma + 2$ then $\text{Tail}[C_g^l(\phi)]$ consists of terms with length $\geq \sigma + 2$, so we are done. Now, the case where $C_g^l(\phi)$ has length $\sigma + 1$: By our definition, a complete contraction of length $\sigma + 1$ is irrelevant if it has a factor $\nabla_{r_1...r_m}^{(m)} \phi \neq \Delta \phi$. But then, by virtue of Lemma 16 in [4] and the iterative integration by parts, it follows that each complete contraction in $\text{Tail}[C_g^l(\phi)]$ of length $\sigma + 1$ will have a factor $\nabla_{r_1...r_m}^{(p)} \phi, p \geq 2$. Hence each complete contraction of length $\sigma + 1$ in $\text{Tail}[C_g^l(\phi)]$ is irrelevant. □

We next consider $\text{Tail}[C_g^l(\phi)]$ when $C_g^l(\phi)$ is a contributor. (By definition if $C_g^l(\phi)$ is a contributor, then its factor $\nabla^{(p)} \phi$ is either of the form $\nabla \phi$ or $\Delta \phi$).

We claim the following:

**Lemma 2.8** Consider any contributor $C_g^l(\phi)$. Modulo complete contractions of length $\geq \sigma + 2$, we can write $\text{Tail}[C_g^l(\phi)]$ as a linear combination:

$$\text{Tail}[C_g^l(\phi)] = C_g^l(\phi) + \sum_{r \in R_l} a_r \text{div}_j C_g^{l,j}(\phi) + \sum_{w \in W} a_w C_g^w(\phi), \quad (2.26)$$

where each vector field $C_g^{l,j}(\phi)$ in the above equation is a partial contraction in the form (2.21) with length $\sigma + 1$, with one free index, and furthermore $\nu = 1$ (that is, it has a factor $\nabla \phi$), and the index $a$ in $\nabla_a \phi$ is not the free index, and furthermore $q = \delta = 0$; each complete contraction $C_g^w(\phi)$ is in the form (2.21) with either $\nu \geq 2$ or $q + \delta > 0$.

**Proof of Lemma 2.8** First consider the case where $C^l(g)$ contains a factor $\nabla \phi$: By virtue of the algorithm for the super divergence formula (see the concluding remarks in [4]), we derive that, modulo complete contractions of length $\geq \sigma + 2$:

$$\text{Tail}[C_g^l(\phi)] = C_g^l(\phi) - \sum f \in F \text{div}_j C_g^{l,j}(\phi),$$
where \( \Sigma_{f \in F} C^{i,j}_g(\phi) \) is a linear combination of vector fields in the form (2.21) with \( \nu \geq 1 \); if \( \nu = 1 \) then the index \( a \) in \( \nabla_a \phi \) is not the free index \( i \), because \( C^{i}_g(\phi) \) contains a factor \( \nabla \phi \), therefore no descendent of \( C^{i}_g(\phi) \) has a factor \( \tilde{\xi} \) contracting against \( \nabla \phi \). Now, if a vector field \( C^{i,j}_g(\phi) \) has \( \nu = 1 \) and no internal contractions, we index it in \( R_l \). Otherwise, we place \( \text{div}_i C^{i,j}_g(\phi) \) into the sum \( \Sigma_{w \in W} a_w C^{w}_g(\phi) \).

We now consider the case where \( C^{i}_g(\phi) \) has a factor \( \Delta \phi \). In that case, recall that by definition \( \delta + q > 0 \). We denote the set of good or undecided descendants of \( C^{i}_g(\phi) \) by \( \{C^{i,b}_g(\phi, \tilde{\xi})\}_{b \in B} \). Also recall Lemma 16 from [1]. Now, if \( C^{i,b}_g(\phi, \tilde{\xi}) \) contains a factor \( \Delta \phi \), it follows from Lemma 20 in [1] that, modulo complete contractions of length \( \geq \sigma + 2 \):

\[
\text{PO}[C^{i,b}_g(\phi, \tilde{\xi})] = \Sigma_{t \in T} a_t C^{t}_g(\phi),
\]

where each \( C^{t}_g(\phi) \) is in the form (2.21) and has a factor \( \nabla^{(\nu)} \phi, \nu \geq 2 \). We then place the complete contractions \( C^{t}_g(\phi) \) into the sum \( \Sigma_{w \in W} a_w C^{w}_g(\phi) \).

If the \( \tilde{\xi} \)-contraction \( C^{i,b}_g(\phi, \tilde{\xi}) \) has a factor \( \nabla_i \phi \), it follows from Lemma 16 in [1] that it will have an expression \( \nabla_i \phi \tilde{\xi}^i \). We then decree that the factor \( \tilde{\xi}_i \) will be the first to be integrated by parts. Notice the following: If \( \tilde{\xi}_i \) was the only \( \tilde{\xi} \)-factor in \( C^{i,b}_g(\phi, \tilde{\xi}) \) is \( \tilde{\xi}_i \), then \( C^{i,b}_g(\phi, \tilde{\xi}) \) is in the form (2.21) with at least one factor \( \nabla^{(p)} \text{Ric} \) or \( R \) (of the scalar curvature). Hence, in that case we place \( \text{PO}[C^{i,b}_g(\phi, \tilde{\xi})] \) into the sum \( \Sigma_{w \in W} a_w C^{w}_g(\phi) \).

If \( C^{i,b}_g(\phi, \tilde{\xi}) \) has more \( \tilde{\xi} \)-factors, then after the integration by parts of \( \tilde{\xi}_i \), we are reduced to the previous case. Thus our Lemma follows.

Finally, let us consider a dangerous complete contraction \( C^{i}_g(\phi) \). It will be in the form:

\[
\text{contr}(\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} R_{ijkl} \otimes \Delta \phi),
\]

where none of the factors \( \nabla^{(m)} R_{ijkl} \) have internal contractions. Hence, we derive:

\[
\text{Tail}[C^{i}_g(\phi)] = - \text{contr}(\nabla^{j}[\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} R_{ijkl}] \otimes \nabla_i \phi). \tag{2.27}
\]

We denote a linear combination as in the right hand side of the above by \( \Sigma_{h \in H} a_h \nabla^j[C^h(g)] \nabla_j \phi \).

In view of the above Lemmas, we see that by applying the super divergence formula to \( I^j_g(\phi) \), and pick out the sublinear combination with \( \sigma + 1 \) factors,

\[ \text{Note that by the remark made in the subsection “Conclusion” in [1] we are free to impose this restriction.} \]
with one factor $\nabla \phi$ and without internal contractions we derive a new local equation:

$$\sum_{l \in L^1} a_l C^l_i(g) + \sum_{r \in R} a_r \text{div}_j C^r_j(g) = \sum_{h \in H} a_h \nabla^j [C^h_i(g)] \nabla_j \phi \quad (2.28)$$

Now, applying the operation $Weylify$ to the above, we derive our Lemma 2.2 (by virtue of the discussion on the operation $Weylify$ in subsection 5.1 in [3]). $\square$

We will show Lemma 2.3 in the next section, after proving Lemma 1.3.

### 2.4 Proof of Lemmas 1.3 and 2.3 (the second half of Lemma 1.2).

This subsection contains elaborate constructions and calculations of local conformal invariants. All the divergences of vector fields that appear in the proofs of these two Lemmas are constructed explicitly; there is no recourse to the “super divergence formula”. The key to these constructions and to the calculations below is the ambient metric of Fefferman and Graham, see [17], [18]. We will be using the algorithm that was presented in subsection 2.1.

**Proof of Lemma 1.3** For this section, for each complete contraction $C^l_i(g), l \in L^\text{stack}$ we will call the factor to which all the internal contractions belong the important factor.

Firstly, we observe (easily) that by applying the fake second Bianchi identities from [3] (see (2.18) in this paper) we can write modulo terms of length $\geq \sigma + 1$:

$$\sum_{l \in L^\text{stack}} a_l C^l_i(g) = \sum_{l \in L^\text{stack}} a_l C^l_i(g) + \sum_{j \in J} a_j C^j_i(g),$$

where each $C^l_i(g), l \in L^\text{stack}$ is in the form (1.2) and has two of its internal contractions involving the indices $i, k$ in the important factor $\nabla l \cdot \nabla k W^{ijkl}$ also each $C^j_i(g)$ is in the form (1.2) and has at least two internal contractions belonging to different factors $\nabla^{(m)} W_{ijkl}$.

Thus we are reduced to proving our claim under the assumption that the important factor of each $C^l_i(g), l \in L^\text{stack}$ is of the form $\nabla^{l_1 \cdots l_{s-2} k} W_{ijkl}$.

We then observe that by integrating by parts the indices $r_1, \ldots, r_m$ in the factor $\nabla_{r_1 \cdots r_m} W_{ijkl}$ for each $l \in L^\text{stack}$ we can explicitly construct a vector field $T^i(g) = \sum_{h \in H} a_h C^{h,i}(g)$ so that:

$$C^l_i(g) - \text{div}_i T^i(g) = \sum C^* \sum_{j \in J} a_j C^j_i(g), \quad (2.29)$$

24Notice that this sublinear combination must vanish separately.

25In other words, we have two derivative indices $\nabla^i, \nabla^k$ contracting against the indices $i, k$ in the important factor
modulo complete contractions of length $\sigma + 1$. Here $\sum C^\ast (g)$ stands for a linear combination of complete contractions in the form (1.2) with $\delta_W = 2$, where both the internal contractions belong to a factor in the form $\nabla^i k W_{ijkl}$. Each $C^t (g)$ is a complete contraction in the form (1.2) with at least two internal contractions belonging to different factors. Now, abusing notation we will again denote $\sum_{t \in \mathcal{T}_{\text{stack}}} a_t C^t (g)$ by $\sum_{t \in L_{\text{stack}}} a_t C^t (g)$. Therefore, we are reduced to showing our Lemma in the case where each $C^t (g)$, $t \in L_{\text{stack}}$ has $\delta_W = 2$ and the two internal contractions belong to an important factor in the form $\nabla^i k W_{ijkl}$.

In order to state our claim, we recall the *ambient analogue $\text{Amb}[C(g)]$*, of any complete contraction in the from (1.2) (see Definition 2.2); we also introduce a new definition:

**Definition 2.7** Consider any $C^l (g)$, $l \in L_{\text{stack}}$ in the form (1.2), where there are precisely two internal contractions in $C^l (g)$, in some factor $\nabla^i k W_{ijkl}$. We let $C^{l, i_1 i_2} (g)$ be the tensor field that arises from $C^l (g)$ by making the two internal contractions into free indices.

We also let $C^{l, i_1 i_2} (g)$ be the vector field that arises from $C^l (g)$ by replacing the important factor $\nabla^i k W_{ijkl}$ by $\nabla^k W_{i_1 jkl}$ and $C^{l, i_1 i_2} (g)$ be the vector field that arises from $C^l (g)$ by replacing the important factor $\nabla^i k W_{ijkl}$ by $\nabla^i W_{ij, kl}$.

**Lemma 2.9** Consider the tensor field $C^{l, i_1 i_2} (g)$; consider $X \text{div}_{i_1} X \text{div}_{i_2} C^{l, i_1 i_2} (g)$ and construct $\text{Amb}[X \text{div}_{i_1} X \text{div}_{i_2} C^{l, i_1 i_2} (g)]$. We claim that modulo terms of length $\geq \sigma + 1$:

\[
\text{Amb}[X \text{div}_{i_1} X \text{div}_{i_2} C^{l, i_1 i_2} (g)] = X \text{div}_{i_1} X \text{div}_{i_2} C^{l, i_1 i_2} (g) + \frac{n - 4}{n - 3} X \text{div}_{i_1} C^{l, i_1} (g)
+ \frac{(n - 4)^2 - 2(n - 4) + 2}{(n - 3)(n - 4)} C^l (g) + \sum_{j \neq l} a_j C^j (g)
+ \sum_{t \in T^l} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_m} C^{t, i_1 \ldots i_m} (g) + \sum_{t \in T^m} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_n} C^{t, i_1 \ldots i_n} (g).
\]

Here each tensor field $C^{t, i_1 \ldots i_n} (g)$, $t \in T^l$ is in the form (1.2), has $\delta_W > 0$ and at least one of the free indices belongs to a factor $T_1$, and at least one internal contraction belongs to a factor $T_2$ with $T_1 \neq T_2$. Each tensor field $C^{t, i_1 \ldots i_n} (g)$, $t \in T^l$ also has $\delta_W > 0$ and in addition has a factor $T = \nabla^i W_{ijkl}$ with one internal contraction between a derivative index and an index $i$, or $k$ and then one of the free indices $i_1, \ldots, i_n$ is the index $j$ or $l$ in $T$, respectively. $\sum_{j \neq l} a_j \ldots$ is as in the conclusion of Lemma 1.3.

**Lemma 2.9** implies **Lemma 1.3**: Firstly, for each $t \in T^l$ we suppose with no loss of generality that $i_1$ belongs to $T_1$ (see the definition above). We then observe that for each $t \in T^l$: 25
\[
X \text{div}_1 \ldots X \text{div}_{i_1} C^{t, i_1, \ldots, i_t}(g) - \text{div}_1 X \text{div}_{i_2} \ldots X \text{div}_{i_1} C^{t, i_1, \ldots, i_t}(g) = \\
\sum_{j \in J} a_j C^j(g),
\]

(2.31)

modulo complete contractions of length \( \geq \sigma + 1 \).

For each \( t \in T^{\sharp \sharp} \) we assume with no loss of generality that \( i_1 \) is the index \( j \) or \( i \) (see the definition above). Then, for each \( t \in T^{\sharp \sharp} \):

\[
X \text{div}_1 \ldots X \text{div}_{i_1} C^{t, i_1, \ldots, i_t}(g) - \text{div}_1 X \text{div}_{i_2} \ldots X \text{div}_{i_1} C^{t, i_1, \ldots, i_t}(g) = 0,
\]

(2.32)

modulo complete contractions of length \( \geq \sigma + 1 \).

By the same reasoning we explicitly construct a linear combination of vector fields \( \sum_{r \in R} a_r C^{r, i}(g) \) so that:

\[
X \text{div}_1 X \text{div}_{i_2} C^{t, i_1, i_2}(g) + \frac{n-4}{n-3} X \text{div}_1 C^{i_1, i_2}(g) + \frac{n-4}{n-3} X \text{div}_{i_2} C^{i_1, i_2}(g) + \\
\frac{(n-4)^2 - 2(n-4) + 2}{(n-3)(n-4)} C^{l}(g) - \text{div}_i \sum_{r \in R} a_r C^{r, i}(g)
\]

(2.33)

\[
= \frac{(n-4)(n-3) - 2(n-4)^2 + (n-4)^2 - 2(n-4) + 2}{(n-3)(n-4)} C^{l}(g).
\]

We note that the constant \( C = \frac{-n+6}{(n-3)(n-4)} \) on the right hand side is not zero, since for \( n = 6 \) there can not be any complete contraction in the form:

\[
\text{contr}(\nabla^{(m_1)} W_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} W_{ij'k'l'})
\]

(2.34)

with a factor \( \nabla^{ik} W_{ijkl} \).

Therefore, if we can prove Lemma 2.9 our Lemma 1.3 will follow.

**Proof of Lemma 2.9**

Our proof relies on a careful calculation of \( \text{Amb}[X \text{div}_1, X \text{div}_{i_2} C^{t, i_1, i_2}(g)] \), based on the algorithm presented in subsection 2.1. We write \( X \text{div}_1, X \text{div}_{i_2} C^{t, i_1, i_2}(g) \) out as a sum in the obvious way, \( X \text{div}_1, X \text{div}_{i_2} C^{t, i_1, i_2}(g) = \sum_{t=1}^{(\sigma-1)^2} C^{t}(g) \); then we write:

\[
\text{Amb}[X \text{div}_1, X \text{div}_{i_2} C^{t, i_1, i_2}(g)] = \sum_{t=1}^{(\sigma-1)^2} \sum_{\text{ass} \in \text{ASSIGN}} \sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma \{\text{ass} \{\text{Amb}[C^{t}(g)]\}\}.
\]

(2.35)

Now, our Lemma will follow by a careful analysis of the right hand side of the above. To perform this analysis we must divide the right hand side into further sublinear combinations.
We call the factor $W_{i_1,i_2}$ in $C^{l,i_1i_2}(g)$ to which the two free indices belong the *important factor*. For the purposes of the discussion below, each complete contraction $C^l(g)$ above, the indices $i_1, i_2$ in the important factor (which are contracting against $\nabla^{i_1}, \nabla^{i_2}$) will still be called the *free indices*.

**Definition 2.8** Refer to (2.35). Consider any given term $\text{Amb}[C^t(g)]$. We will write $\text{ASSIGN}$ instead of $\text{ASSIGN}^t$ for simplicity.

Define $\text{ASSIGN}^- \subset \text{ASSIGN}$ to stand for the index set of assignments that assign the values $(\infty, 0)$ to two particular contractions $(a, b), (c, d)$ where $a, c$ belong to different factors.

Define $\text{ASSIGN}^1 \subset (\text{ASSIGN} \setminus \text{ASSIGN}^-)$ to stand for the index set of assignments in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ that assign to at least one particular contraction $(a, b)$ where $a$ belongs to the important factor, the values $(0, \infty)$.

Define $\text{ASSIGN}^2 \subset [\text{ASSIGN} \setminus (\text{ASSIGN}^- \cup \text{ASSIGN}^1)]$ to stand for the index set of assignments in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ that either do not assign the values $(\infty, 0)$ to any particular contraction, or do assign such values $(\infty, 0)$ but subject to the restriction that all the $\infty$’s are assigned to the same factor, which is not the important factor.

Define $\text{ASSIGN}^* \subset [\text{ASSIGN} \setminus (\text{ASSIGN}^- \cup \text{ASSIGN}^1)]$ to stand for the index set of assignments in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ for which the value $\infty$ is assigned to at least one index, and moreover all the indices $\infty$ are assigned to the important factor, but not to the free indices $i_1, i_2$.

Define $\text{ASSIGN}^+ \subset [\text{ASSIGN} \setminus (\text{ASSIGN}^- \cup \text{ASSIGN}^1)]$ to stand for the index set of assignments in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ for which all the values $\infty$ are assigned to indices in the important factor, and at least one of the free indices $i_1, i_2$ is assigned the value $\infty$.

Let us firstly observe that for each $t, 1 \leq t \leq (\sigma - 1)^2$:

\[
\text{ASSIGN} = \text{ASSIGN}^- \cup \text{ASSIGN}^1 \cup \text{ASSIGN}^2 \cup \text{ASSIGN}^* \cup \text{ASSIGN}^+;
\]

(2.36)

(in the above $\cup$ is a disjoint union). We define:

\[
\sum_{\text{ASSIGN}^-} \text{ass}\{\text{Amb}[X\text{div}_{i_1}X\text{div}_{i_2}C^{l,i_1i_2}(g)]\} = \sum_{t=1}^{(\sigma - 1)^2} \sum_{\text{ASSIGN}^-} \text{ass}\{\text{Amb}[C^t(g)]\},
\]

and also use the same definition for the other subsets of $\text{ASSIGN}$.

We will use (2.36), along with (2.35). Before doing so, let us recall a few more facts about the ambient metric:

A key observation is that any tensor in the form $\tilde{\nabla}_{f_1 \cdots f_d}^t \tilde{R}_{xycz}$ as in (2.7) (i.e. with at least one index have the value $\infty$) can be written out as a linear combination of tensors $\nabla^t W_{ijkl}(\otimes g \cdots \otimes g)$ with an internal contraction in $\nabla^t W$, modulo quadratic terms in the curvature. \footnote{\text{\textsuperscript{26}Here } (\otimes g \cdots \otimes g) \text{ means that there may be no such factors.}}
We only need to be more precise in the case of the factor $T = \tilde{R}_{ijkl}$: If one of the indices $i, j, k, l$ is given the value $0$ then $T = 0$. On the other hand, if $1 \leq j, k, l \leq n$ then $\tilde{R}_{ijkl} = -\frac{1}{n-2} \nabla^k W_{ijkl}$. If at least one of the pairs $i, j$ or $k, l$ are both assigned the value $\infty$ then $T = 0$. Finally, if $1 \leq j, l \leq n$ then $\tilde{R}_{ijkl} = \frac{1}{(n-3)(n-4)} \nabla^k W_{ijkl} + Q(R)$ (see \cite{17, 18}.

We will now calculate $\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1}_2(g)]$, $\ldots$, $\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]$. All equations below hold modulo terms of length $\sigma + 1$.

It immediately follows that:

$$\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)] = 0, \quad (2.37)$$

$$\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)] = \sum_{j \in J} a_j C^j(g). \quad (2.38)$$

We next seek to understand $\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]$. We claim that:

$$\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)] = X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g) + \sum_{t \in T} a_t X \div v_1 \ldots X \div v_t C^{i, i_1 \ldots i_t}_2(g) + \sum_{j \in J} a_j C^j(g). \quad (2.39)$$

**Proof of (2.39):** Consider the break-ups of each different $\text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]$:

$$\sum_{\text{assign}} \text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)] = \sum_{\text{assign}} \sum_{\gamma \in \text{BR}_{\gamma}} \gamma \{\text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]\} \quad (2.40)$$

Denote the RHS of the above by $\Lambda$. We will break $\Lambda$ into two sublinear combinations, $\Lambda^1, \Lambda^2$: A term $\gamma \{\text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]\}$ will belong to $\Lambda^1$ if it arises by applying one of the rules $\hat{\nabla}_i X_j \rightarrow \hat{\Gamma}_{ij}^k X_k = -g_{ij} X_k$ or $\hat{\nabla}_i X_0 \rightarrow \hat{\Gamma}_{i0}^k X_k = \delta^k_i X_k$ at least once (as in \cite{24}). (Here $\hat{\nabla}_i$ stands for a divergence index which has been assigned a value between $1, \ldots, n$, and $j$ is an original index which has been assigned a value $0, \ldots, n$). A term $\gamma \{\text{ass}[X \div v_i X \div v_2 C^{i, i_1 i_2}_2(g)]\}$ belongs to $\Lambda^2$ if it arises without applying any of the above two Christoffel symbols.

In the above notation, we will show that:

\footnote{Recall the break-ups from subsection 2.4.}

\footnote{This means that the index is not one of the indices in the vector field $\text{Amb}(C^{i, i_1 i_2}_2(g))$ but corresponds to an index from a divergence $\div v_1$ or $\div v_2$.}
\[ \Lambda^1 = 0, \] 

\[ \Lambda^2 = X \text{div}_1 X \text{div}_2 C^{l_1,i_1,i_2}(g) + \sum_{t \in T^2} a_t X \text{div}_1 \ldots X \text{div}_n C^{l_1,i_1 \ldots i_n} + \sum_{j \in J} a_j C^j(g). \] 

(2.41) 

(2.42)

In fact, (2.42) just follows by the definition of \( \Lambda^2 \): If we consider the complete contractions that belong to \( \Lambda^2 \) that arise without assigning any index the value \( \infty \), it follows that they add up to the term \( X \text{div}_1 X \text{div}_2 C^{l_1,i_1,i_2}(g) \); if we consider the complete contractions in \( \Lambda^2 \) that arise by assigning at least one index the value \( \infty \), it follows that they will add up to a linear combination 

\[ \sum_{t \in T^2} a_t X \text{div}_1 \ldots X \text{div}_n C^{l_1,i_1 \ldots i_n} + \sum_{j \in J} a_j C^j(g). \]

So, matters are reduced to showing (2.31). We will establish a bijection that will help us prove cancellation in \( \Lambda^1 \).

We arbitrarily pick out one of the free indices \( i_s \) (so \( s = 1 \) or \( s = 2 \)). For convenience we just set \( s = 1 \), but the same claim will be true if we just switch \( i_1 \) and \( i_2 \). We also arbitrarily pick out a particular contraction in the tensor field \( C^{l_1,i_1,i_2}(g) \), say \( \pi = (a,b) \) where \( a \) belongs to the factor \( T^k \) and \( b \) belongs to the factor \( T^l \) and \( k, l \neq 1 \). We also pick out a factor \( T^r, r > 1 \) arbitrarily. Then, we consider the complete contraction \( C^{l_1,i_1,i_2}[k,r](g) \) that arises in \( X \text{div}_1 X \text{div}_2 C^{l_1,i_1,i_2}(g) \) when \( \nabla^{i_1} \) hits the \( k^{th} \) factor \( T^k \) in \( C^{l_1,i_1,i_2}(g) \) and \( \nabla^{i_2} \) hits the \( r^{th} \) factor \( T^r \) in \( C^{l_1,i_1,i_2}(g) \). Accordingly, we consider the complete contraction \( C^{l_1,i_1,i_2}[l,r](g) \) that arises in \( X \text{div}_1 X \text{div}_2 C^{l_1,i_1,i_2}(g) \) when \( \nabla^{i_1} \) hits the \( l^{th} \) factor \( T^l \) in \( C^{l_1,i_1,i_2}(g) \) and \( \nabla^{i_2} \) hits the \( r^{th} \) factor \( T^r \) in \( C^{l_1,i_1,i_2}(g) \).

We now consider the ambient analogues \( \text{Amb}[C^{l_1,i_1,i_2}[k,r](g)] \), \( \text{Amb}[C^{l_1,i_1,i_2}[l,r](g)] \).

For \( \text{Amb}[C^{l_1,i_1,i_2}[l,r](g)] \) we define \( \text{ASSIGN}^{l_1,i_1,i_2}_{i_1,i_2,A} \subset \text{ASSIGN}^{\pi,A} \) to stand for the set of assignments that assign the particular contraction \( \pi = (a,b) \) the values \((0,\infty)\) and assign the pair \((i_1,\nabla^{i_1})\) base values (i.e. \( i_1 \in \{1,\ldots,n\} \)).

On the other hand, for \( \text{Amb}[C^{l_1,i_1,i_2}[l,r](g)] \) we define \( \text{ASSIGN}^{l_1,i_1,i_2}_{i_1,i_2,B} \) to stand for the set of assignments where \( \pi = (a,b) \) is assigned base values and the pair \((i_1,\nabla^{i_1})\) is also assigned base values.

Now, for each element \( a \in \text{ASSIGN}^{l_1,i_1,i_2}_{i_1,i_2,A} \), we define \( B_{a,s}^{l_1} \) to stand for the set of break-ups that replace the factor \( T^k = \tilde{\nabla}_{i_1}^{m+1} \tilde{R}_{ijkl} \) (or analogously when \( a = 0 \) is one of the internal indices) by a factor \(-\tilde{\nabla}_{i_1}^{(m)} \tilde{R}_{ijkl} \) (i.e. we apply \( \tilde{\Gamma}_{i_1,0}^{k} = \delta_{i_1}^{k} \) to the indices \( \tilde{\nabla}_{i_1} \), \( \tilde{\nabla}_{0} \), as in (2.9)). For \( C^{l_1,i_1,i_2}[l,r](g^{n+2}) \) and each element \( a \in \text{ASSIGN}^{l_1,i_1,i_2}_{i_1,i_2,B} \), we define \( B_{a,s}^{l_1} \) to stand for the set of break-ups that replace the factor \( T^l = \tilde{\nabla}_{i_1}^{(m+1)} \tilde{R}_{ijkl} \) (or analogously where \( b \) is one of the internal indices) by a factor \(-\tilde{\nabla}_{i_1}^{(m)} \tilde{R}_{ijkl} \) (i.e. we apply \( \tilde{\Gamma}_{i_1,b}^{k} = -g_{i_1,b} \)).

We then observe that in the above notation, for each \( \pi \in \Pi \), each \( r = 2,\ldots,\sigma \).
\[
\sum_{\text{ass} \in \text{ASSIGN}^A} \sum_{\gamma \in \text{BR}^1} \gamma\{\text{ass}[\text{Amb}[C^{l,i_1i_2|k,r}(g)]]\} \\
+ \sum_{\text{ass} \in \text{ASSIGN}^B} \sum_{\gamma \in \text{BR}^2} \gamma\{\text{Amb}[C^{l,i_1i_2|k,r}(g)]\} = 0.
\] (2.43)

In view of the above cancellation, when analyzing $\Lambda^1$ we may discard any contraction that involves using the symbol $\tilde{\Gamma}_i^j$ to a pair $\nabla_i X_k$ (where $i_*$ is a divergence index and $X_k$ is not contracting against the important factor). We may also discard any complete contraction that arises when we use the symbol $\tilde{\Gamma}_{i_0}^k$ to a pair $\nabla_i X_d$ where $X_d$ is an original index in $C^{l,i_1i_2}(g)$ that is not contracting the important factor.

So, to show (2.41), we only have to consider the terms in the LHS that arise by using one of the symbols $\tilde{\Gamma}_i^j$, $\tilde{\Gamma}_{i_0}^k$ to a pair $\nabla_i X_v$ where $i_*$ is a divergence index and $X_v$ is contracting against the important factor (and where in addition $v$ has been assigned a value $0, \ldots, n$).

Now, observe that there can be no contractions arising from an assignment $\text{ass} \in \text{ASSIGN}^A$ with an index $v$ having been assigned the value 0, and the same index $v$ contracting against an index in the important factor in $C^{l,i_1i_2}(g)$: In that case, the index $b_0$ would have been assigned a value $\infty$, which contradicts the fact that $\text{ass} \in \text{ASSIGN}^A$.

So we only need to observe that we may discard any complete contraction that arises when we use the symbol $\tilde{\Gamma}_{i_0}^k$ to a pair of indices $(\nabla_i, k)$ where $\nabla_i$ is the divergence index and $k$ is contracting against an index in the important factor-this follows because $\tilde{\Gamma}_{i_0}^k = -g_{i,k}$ and the Weyl tensor is trace-free. This proves our claim. □

Next, we wish to analyze
\[
\sum_{\text{ass} \in \text{ASSIGN}^*} \sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma\{\text{ass}[X_{\text{div}i_1} X_{\text{div}i_2} C^{l,i_1i_2}(g)]\}.
\]

Using the definition of $\text{ASSIGN}^*$ and the Christoffel symbols for $\tilde{g}$, we observe that:

\[
\sum_{\text{ass} \in \text{ASSIGN}^*} \sum_{\gamma \in \text{BR}_{\text{ass}}} \gamma\{\text{ass Amb}[X_{\text{div}i_1} X_{\text{div}i_2} C^{l,i_1i_2}(g)]\} = \\
- \frac{1}{n-3} X_{\text{div}i_1} C^{l,i_1}(g) - \frac{1}{n-3} X_{\text{div}i_2} C^{l,i_2}(g) \\
+ \frac{2}{(n-3)(n-4)} C^{l}(g) + \sum_{t \in T^*} b_t X_{\text{div}i_1} \ldots X_{\text{div}i_n} C^{l,i_1 \ldots i_n}(g) \\
+ \sum_{t \in T^{**}} b_t X_{\text{div}i_1} \ldots X_{\text{div}i_n} C^{l,i_1 \ldots i_n}(g) + \sum_{j \in J} a_j C^{l}(g).
\] (2.44)
Note: The first three terms in the RHS arise by applying the Christoffel symbol \( \tilde{\Gamma}^k_{i_0} = \delta^k_{i_0} \) at least once (we also use the first Bianchi identity here—the details are left to the reader).\(^{29}\) It is easy to observe that the sublinear combination that arises when the symbol \( \tilde{\Gamma}^k_{i_0} = \delta^k_{i_0} \) is never applied will equal \( \sum_{t \in T} \ldots \).

Finally, we must understand the sublinear combination
\[
\sum_{ass \in ASSIGN^+} \sum_{\gamma \in BR_{ass}} \gamma\{\text{ass}[X \text{div}_{i_1} X \text{div}_{i_2} C^l,i_1i_2(g^{n+2})]\}.
\]

We calculate that:
\[
\sum_{ass \in ASSIGN^+} \text{assAmb}[X \text{div}_{i_1} X \text{div}_{i_2} C^l,i_1i_2(g)] = \frac{(n-4)^2}{(n-3)(n-4)} C^l(g)
\]
\begin{align*}
&+ X \text{div}_{i_1} C^l,i_1(g) + X \text{div}_{i_2} C^l,i_2(g) - \frac{2}{(n-3)} C^l(g) \tag{2.45} \\
&+ \sum_{t \in T} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_{2i}} C^l,i_1\ldots i_{2i}(g) + \sum_{j \in J} a_j C^j(g).
\end{align*}

Note again, the first four terms in the RHS arise by applying the Christoffel symbol \( \tilde{\Gamma}^k_{i_0} = \delta^k_{i_0} \) at least once (we also again use the first Bianchi identity).

Thus, in view of (2.35), (2.36), by just adding up the equations (2.37), (2.38), (2.39), (2.44), (2.45) we derive Lemma 1.3.\(\Box\)

**Proof of Lemma 2.3**

Now, to show Lemma 2.3, we recall the notational conventions from the statement of the Lemma and introduce some additional ones.\(^{30}\) Recall that \( \sum_{f \in F} a_f C^f(g) \) stands for a generic linear combination of complete contractions in the form (1.2) with \( \delta_W \geq 2 \). Also, \( \sum_{t \in T} a_t C^l,i_1\ldots i_{n}(g) \) will now stand for a generic linear combination of tensor fields in the form (1.2) with \( a_t > 0 \) (i.e. with at least one free index) and with one internal contraction. All equations in this proof will hold modulo complete contractions of length \( \geq \sigma + 1 \).

Now, for each \( C^v(g) \) as in the statement of Lemma 2.3, we consider \( \nabla_i C^v(g) \) (which is thought of as the sum of \( \sigma \) partial contractions). Then, for each \( v \in V \) we consider the linear combination \( X \text{div}_i [\nabla_i C^v(g)] \) and we construct its ambient analogue (see the Definition 2.2):
\[
\text{Amb}\{X \text{div}_i [\nabla_i C^v(g)]\}.
\]

We then claim:

**Lemma 2.10** For each \( v \in V \):

\(\)

30The reader should note that the notational conventions here are different from the ones in the proof of the previous Lemma.
\[
\text{Amb}\{X \text{div}_i [\nabla_i C^\nu(g)]\} = 2\Delta_r C^\nu(g) + X \text{div}_i [\nabla_i C^\nu(g)] + \sum_{f \in F} a_f C^f(g) + \sum_{t \in T^t} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_{at}} C^{t,i_1 \ldots i_{at}}(g),
\]
(2.46)

modulo complete contractions of length \(\geq \sigma + 1\).

We observe that if we can prove the above, then Lemma 2.3 follows from two easy observations: Firstly:

\[
X \text{div}_i [\nabla_i C^\nu(g)] - \text{div}_i [\nabla_i C^\nu(g)] = -\Delta_r C^\nu(g).
\]

Secondly, for every \(t \in T^t\):

\[
X \text{div}_{i_1} \ldots X \text{div}_{i_{at}} C^{t,i_1 \ldots i_{at}}(g) - \text{div}_{i_1} X \text{div}_{i_2} \ldots X \text{div}_{i_{at}} C^{t,i_1 \ldots i_{at}}(g) = \sum_{f \in F} a_f C^f(g).
\]

Thus, it suffices to show Lemma 2.10.

Proof of Lemma 2.10: Firstly, we will number the factors in \(C^\nu(g)\) and denote them by \(F_1, \ldots, F_\sigma\). Then, we will denote by \(\nabla_i C^\nu(g)\) the vector field that arises from \(C^\nu(g)\) by replacing the factor \(F_\tau\) by \(\nabla_i F_\tau\). We also denote by \(m_\tau\) the number of derivatives in the factor \(F_\tau = \nabla_i^{m_\tau} W_{ijkl}\).

We will then show that for any \(\tau, 1 \leq \tau \leq \sigma\):

\[
\text{Amb}\{X \text{div}_i [\nabla_i C^\nu(g)]\} = \Delta_r C^\nu(g) + X \text{div}_i [\nabla_i C^\nu(g)] + \frac{m_\tau + 2}{n - 2} \Delta_r C^\nu(g) + \sum_{f \in F} a_f C^f(g) + \sum_{t \in T^t} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_{at}} C^{t,i_1 \ldots i_{at}}(g).
\]
(2.47)

In view of the equation \(\sum_{\tau=1}^\sigma (m_\tau + 2) = n - 2\), (2.47) implies Lemma 2.10.

Proof of (2.47): We will show this equation using the notions of assignment and break-ups from subsection 2.1.

We recall the following fact regarding the ambient metric: Consider any tensor \(T = \nabla^{(m)}_{r_2 \ldots r_m} R_{ijkl}\), where all the indices \(r_2, \ldots, r_m, i, j, k, l\) have values between 1 and \(n\). It then follows from the formula \(\partial \in g_{ab} = 2P_{ab}\) in (18) that:

\[
T = -\frac{1}{n - 2} \Delta_{r_2 \ldots r_m} W_{ijkl} + \sum_{h \in H} a_h [T^h(g)(\otimes g \ldots \otimes g)]_{r_2 \ldots r_m i j k l} + Q(R),
\]
(2.48)

where each tensor \(T^h(g)\) is in the form \(\nabla^{(m')} W_{ijkl}\) with at least two internal contractions.
Furthermore, it can be seen from [17, 18] that any component $T' = \tilde{\nabla}_T^{(m)} \tilde{R}_{ijkl}$ where at least two of the indices $r_1, \ldots, r_m$ having the value $\infty$ can be expressed as:

$$T' = \sum_{h \in H} a_h [T^h(g)(\otimes g \cdot \cdot \cdot \otimes g)]_{r_2 \ldots r_m} + Q(R),$$

(2.49)

with the same conventions as above.

Now, for the purposes of the next definition we write out $\text{Amb}\{X \text{div}_i [\nabla_i C^v(g)]\} = \sum_{t=1}^{\sigma-1} \text{Amb}[C^t(g)]$. We prove our Lemma by examining the right hand side of the equation:

$$\text{Amb}\{X \text{div}_i [\nabla_i C^v(g)]\} = \sum_{t=1}^{\sigma-1} \sum_{\text{ass} \in \text{ASSIGN}^t} \sum_{\gamma \in BR_{\text{ass}}} \gamma \{\text{ass}[\text{Amb}[C^t(g)]]\}.$$  

(2.50)

It will again be useful to break the right hand side of the above into sub-linear combinations. We will now call the factor $F_\tau$ (to which $\nabla_i$ belongs) the important factor. We will also call the index $\nabla_i$ in that factor the “free index”.

**Definition 2.9** Refer to (2.50). Consider any fixed $\text{Amb}[C^t(g)]$. We write $\text{ASSIGN}$ instead of $\text{ASSIGN}^t$ for simplicity.

We define $\text{ASSIGN}^- \subset \text{ASSIGN}$ to stand for the index set of assignments where at least two different indices in $\text{Amb}[C^t(g)]$ are assigned the value $\infty$.

We define $\text{ASSIGN}^1 \subset (\text{ASSIGN} \setminus \text{ASSIGN}^-)$ to stand for the index set of assignments where the pair of values $(\infty, 0)$ is assigned to at most one particular contraction, and moreover if such a pair is assigned to a particular contraction $(a, b)$ then neither $a$ nor $b$ belongs to the important factor.

We define $\text{ASSIGN}^* \subset (\text{ASSIGN} \setminus \text{ASSIGN}^-)$ to stand for the set of assignments in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ that assign the pair of values $(0, \infty)$ to exactly one particular contraction $(a, b)$ where in addition $a$ belongs to the important factor and is not the free index $\nabla_i$.

We define $\text{ASSIGN}^+ \subset (\text{ASSIGN} \setminus \text{ASSIGN}^-)$ to stand for the assignment in $(\text{ASSIGN} \setminus \text{ASSIGN}^-)$ that assigns the value $\infty$ to the free index $\nabla_i$, (and the value 0 to the index against which it contracts) and does not assign this value to any other index.

Clearly:

$$\text{ASSIGN} = \text{ASSIGN}^- \cup \text{ASSIGN}^1 \cup \text{ASSIGN}^* \cup \text{ASSIGN}^+;$$

(2.51)

(where in the above $\cup$ stands for a disjoint union).

Firstly, by virtue of (2.48), (2.49) we observe that:
\[ \sum_{\text{ass} \in \text{ASSIGN}} \text{ass}[\text{Amb}[X \text{div}_i \nabla_i C^\nu(g)]] = \sum_{f \in F} a_f C^f(g). \] (2.52)

By repeating the cancellation argument from equation (2.41) we derive that:

\[ \sum_{\text{ass} \in \text{ASSIGN}} \text{ass}[\text{Amb}[X \text{div}_i \nabla_i C^\nu(g)]] = X \text{div}_i [\nabla_i C^\nu(g)] + \frac{m_\tau + 2}{n - 2} \Delta_i C^\nu(g) \]
\[ + \sum_{t \in T} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_\sigma} C^{i_1 \ldots i_\sigma}(g) + \sum_{f \in F} a_f C^f(g). \] (2.53)

Note: The expression \( \frac{m_\tau + 2}{n - 2} \Delta_i C^\nu(g) \) arises from the assignments where no index is assigned a value \( \infty \), by applying \( \tilde{\Gamma}^k_{\infty} = -g^{\infty}_{ia} \) to the pairs of indices \( i, a \) (\( i \) is the free index) in the important factor.

Now, by virtue of the Christoffel symbol \( \tilde{\Gamma}^k_{0i} = \delta^k_i \), we straightforwardly derive:

\[ \sum_{\text{ass} \in \text{ASSIGN}} \text{ass}[\text{Amb}[X \text{div}_i \nabla_i C^\nu(g)]] = \]
\[ \sum_{k=1, k \neq \tau} \Delta^k_{\tau}[C^\nu(g)] + \sum_{t \in T} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_\sigma} C^{i_1 \ldots i_\sigma}(g) + \sum_{f \in F} a_f C^f(g). \] (2.54)

Furthermore, we derive:

\[ \sum_{\text{ass} \in \text{ASSIGN}} \text{ass}[\text{Amb}[X \text{div}_i \nabla_i C^\nu(g)]] = \]
\[ \frac{m_\tau + 2}{n - 2} \Delta_i C^\nu(g) + \sum_{t \in T} a_t X \text{div}_{i_1} \ldots X \text{div}_{i_\sigma} C^{i_1 \ldots i_\sigma}(g) + \sum_{f \in F} a_f C^f(g). \] (2.55)

Note: The sublinear combination \( \frac{m_\tau + 2}{n - 2} \Delta_i C^\nu(g) \) arises as follows: Recall that \( \text{ass} \in \text{ASSIGN}^\ast \) assigns one pair of values \((\infty, 0)\) to any particular contraction \((a, b)\) where \( a \) belongs to the important factor and \( a \) is not the free index \( i \). The coefficient then arises when we use the Christoffel symbol \( \tilde{\Gamma}^k_{0i} = \delta^k_i \) to the pair \( \nabla_i X_b \) (where \( X_b \) stands for the index that is contracting against the index \( a \) in the important factor—hence \( b \) has been assigned the value \( 0 \)). The second Bianchi identity is also employed here.

Finally we calculate:
\[
\sum_{\text{assign} \in \text{assign}^+} \sum_{t \in T^s} a_t X \text{div}_{i_t} \ldots X \text{div}_{i_1} C^{t,i_1 \ldots i_s}(g) + \sum_{f \in F} a_f C^f(g).
\]

The coefficient \(n - 2 - (m + 2)\) arises by assigning the pair \((\infty, 0)\) to the divergence pair \((i, \nabla^i)\). The coefficient \(n - 2 - (m + 2)\) arises by virtue of the formula \(\tilde{\Gamma}^k_i\) applied to any pair \((\nabla^i, X_k)\). (Recall that there the divergence index \(\nabla^i\) is assigned the value 0).

Plugging the equations (2.52)–(2.56) into (2.50) we obtain our Lemma 2.3.

\[2.3\]

Proof of Lemma 1.4: The divergence “piece” in \(P(g)\).

The aim of this section is to prove that when \(P(g)\) is as in the hypothesis of Lemma 1.4\(^\|\) then the sublinear combination \(\sum_{l \in L^j} a_l C^l(g)\) in \(P(g)\) can be cancelled out (modulo introducing new “better” terms as in the statement of Lemma 1.4\(^\|\)) by subtracting a divergence of a vector field, as allowed by the Deser-Schwimmer conjecture.

Again, we rely on the only tool we have at our disposal, which allows us to pass from the invariance under integration enjoyed by \(P(g)\) to a local equation: We consider the linear operator \(I^j_g(\phi) := \frac{1}{2M} \int_{[0, \infty)} \phi P(e^{2t\phi} g) dt\), for which \(\int_M I^j_g(\phi) dV_g = 0\). The aim is to invoke the super divergence formula for \(I^j_g(\phi)\) to derive the claim in the previous paragraph. The are two challenges one must address: Firstly, how can one recover \(\sum_{l \in L^j} a_l C^l(g)\) by examining \(I^j_g(\phi)\)? Secondly, how can one use the super divergence formula for \(I^j_g(\phi)\) to derive the claim on \(\sum_{l \in L^j} a_l C^l(g)\)?

The first challenge is not hard; it follows by studying the transformation law of covariant derivatives of the Weyl curvature, paying special attention to internal contractions.\(^\|\) This is carried out in subsection 3.1. The second challenge is not straightforward; one difficulty, already discussed in \(3\), is again that a direct application of the super divergence formula to \(I^j_g(\phi)\) does not imply Lemma 1.4\(^\|\). For that reason, we must formulate certain “main algebraic propositions”, see Propositions 3.1 and 3.2 in subsection 3.2 below, which will allow us to derive Lemma 1.4\(^\|\) from the super divergence formula applied to \(I^j_g(\phi)\). These propositions can be viewed as analogues of the “main algebraic propositions”.

\(31\) Recall that \(\sum_{l \in L^j} a_l C^l(g)\) stands for the sublinear combination of terms with length \(\sigma\) and \(j\) internal contractions; recall that \(j \geq 2\) and all other terms of length \(\sigma\) in \(P(g)\) are assumed to be in the form \(1.2\) with at least \(j - 1\) internal contractions.

\(32\) Which expresses \(I^j_g(\phi)\) as a divergence, \(I^j_g(\phi) = \text{div}_i T^i_g(\phi)\).

\(33\) See the definition above Proposition 1.1.
Proposition” in subsection 5.3 [3]; all three of these algebraic propositions will be proven in the series of papers [4, 7, 8].

However, there is an additional important challenge in this case, briefly discussed at the end of subsection 1.1. In a nutshell, the problem concerns a certain “loss of information” which occurs when we pass from \( I^I_g(\phi) \) (which satisfies the integral equation \( \int_M I^I_g(\phi)dV_g = 0 \)) to the super divergence formula applied to \( I^I_g(\phi) \), \( \text{supdiv}[I^I_g(\phi)] = 0 \). As explained at the end of subsection 1.1, upon examining the terms with \( \phi \) applied to \( \nabla \), this factor gives rise to a factor \( \Delta \phi \); this factor gives rise to a factor \( \nabla \phi \) in the super divergence formula, due to integrations by parts. Therefore the additional challenge in this case is that we must somehow “get rid” of the potentially harmful terms in \( \sum_{q \in Q} a_q C^q(g) \cdot \Delta \phi \) in \( I^I_g(\phi) \), without derivatives.

In order to do this, we will study the transformation laws of tensors \( T = \nabla^{r_1}{...}r_{n_\pi} \nabla^{m}{...}m W_{r_{m+1}{...}r_{m+2}r_{m+3}{...}r_{m+n_\pi}} \), where each of the indices \( r_{n_\pi} \) is contracting against an index \( r_{n_\pi} \) and all the other indices in \( T_h(g) \) are free. For each such tensor \( T_h(g) \), we define \( \text{Image}^1_{\phi}[T_h] \) to stand for the sublinear combination in

\[
\frac{d}{d\lambda}|_{\lambda=0} T_h(e^{2\lambda \phi} g)
\]

that involves factors \( \nabla(p)\phi \) with \( p > 0 \) (i.e. we are excluding the partial contractions with a factor \( \phi \), without derivatives).

Clearly, for each complete contraction \( C(g) \) in the form (1.2):

34 In [3] we denoted the sublinear combination of those terms by \( (I^I_g(\phi))_{\nabla \phi} \) – see (1.8).
35 Recall that we were able to reduce ourselves to this setting by virtue of our elaborate constructions in the ambient metric in the previous section.
36 Recall that \( \text{Image}^1_{\phi}[C(g)] = \frac{d}{d\tau}|_{\tau=0}[e^{\tau \phi} C^{2\tau \phi}(g)] \).
\[ \text{Image}_\phi^1[C(g)] = \sum_{h=1}^{\sigma} C_{\text{sub}(T_h)}(g), \]  

(3.1)

where \( C_{\text{sub}(T_h)}(g) \) stands for the complete contraction that arises from \( C(g) \) by replacing the factor \( T_h \) by \( \text{Image}_\phi^1[T_h] \) and then contracting indices according to the same pattern as for \( C(g) \).

**Transformation laws:** Recall that the Weyl tensor is conformally invariant, i.e. \( W^{ijkl}(e^{2\phi}g)e^{2\phi}W^{ijkl}(g) \). Recall also the transformation law of the Levi-Civita connection:

\[ \nabla_{k\ell}(e^{2\phi}g) = \nabla_{k\ell}(g) - \nabla_k\phi\eta_{\ell} - \nabla_\ell\phi\eta_k + \nabla_s\phi\eta_s g_{k\ell}. \]  

(3.2)

Now, in order to perform our calculations, we will have to introduce some notational conventions. For the purposes of this subsection, when we write \( \nabla^{(m)}R_{ijkl} \), \( \nabla^{(m)}\text{Ric} \) or \( \nabla^{(m)}R \) (\( R \) is the scalar curvature) we will mean a tensor in those forms, possibly with some internal contractions. Also as usual, whenever we write \( \nabla^{(m)}R_{ijkl} \) we will mean that no two of the indices \( i, j, k, l \) are contracting between themselves and, when we write \( \nabla^{(p)}\text{Ric}_{ij} \) we will mean that no two of the indices \( i, j \) are contracting between themselves.

**Notation:** We will denote by \( T^{\alpha}(\phi) \) a partial contraction of the form

\[ \nabla^{(m)}_{r_1...r_m}R_{r_m+1...r_m+2...r_m+3...r_m+4}\nabla^{(m)}\phi, \]  

(3.3)

where the factor \( \nabla\phi \) is contracting against one of the indices \( r_1, \cdots, r_{m+4} \) in the first factor. \( T^{\beta}(\phi) \) will stand for a partial contraction in the form:

\[ \nabla^{(m)}_{r_1...r_m}R_{ijkl}\Delta\phi. \]  

(3.4)

\( T^{\gamma}(\phi) \) will stand for a partial contraction in the form:

\[ \nabla^{(m)}_{r_1...r_m}R_{ijkl}\nabla_x\phi, \]  

(3.5)

where \( x \) is a free index. \( T^{\delta}(\phi) \) will stand for a partial contraction in the form:

\[ \nabla^{(m)}_{r_1...r_m}R_{ijkl}\nabla^{(2)}_{yx}\phi, \]  

(3.6)

where both \( y \) and \( x \) are free indices; finally, \( T^{\epsilon}(\phi) \) will be a generic tensor product in the form:

\[ \nabla^{(m)}_{r_1...r_m}R_{ijkl}\nabla^{(p)}_{s_1...s_p}\phi \]  

(3.7)

where either \( p \geq 3 \) or \( p = 2 \) and at least one of the indices \( s_1, s_2 \) is contracting against the factor \( \nabla^{(m)}R_{ijkl} \).

We also generically denote by \( T^{\eta}_{\text{Ric}}(\phi), T^{\alpha}_{\text{Ric}}(\phi), T^{\beta}_{\text{Ric}}(\phi), T^{\gamma}_{\text{Ric}}(\phi), T^{\delta}_{\text{Ric}}(\phi), T^{\epsilon}_{\text{Ric}}(\phi) \) tensor fields that are as above, but only with the factor \( \nabla^{(m)}R_{ijkl} \) formally replaced by an expression \( \nabla^{(m)}[\text{Ric} \otimes g]_{ijkl} \) (the indices \( i, j, k, l \) belonging to the factors \( \text{Ric}, g \)). We also denote by \( T^{\eta}_{R}(\phi), T^{\alpha}_{R}(\phi), T^{\beta}_{R}(\phi), T^{\gamma}_{R}(\phi), T^{\delta}_{R}(\phi), T^{\epsilon}_{R}(\phi) \),
$T^g_\delta(\phi)$ tensor fields that are as above, but only with the factor $\nabla^{(m)}R_{ijkl}$ replaced by an expression $\nabla^{(m)}R \otimes [g \otimes g]_{ijkl}$.

Now, for each factor in one of the 15 forms above, we will denote by $\delta$ the total number of internal contractions in the curvature term (i.e. in the factors $Ric_{ab} = R^b_{\ ab}$ and $R = R^b_{\ ab}$ we also count the internal contractions in $R$ itself) notice that in our definition any internal contractions in the factor $\nabla^{(p)}\phi$ do not count towards $\delta$. We will then denote by $\sum_{\delta \geq t} T$ a generic linear combination of tensor fields in any of the forms above other than (3.7) (and its analogues when we replace $\nabla^{(m)}R_{ijkl}$ by $\nabla^{(m)}Ric \otimes g$ or $\nabla^{(m)}R \otimes g$), with at least $t$ internal contractions in the curvature factor. We will also denote by $\sum_{\text{irrelevant}} T$ a generic linear combination of tensor fields in the form (3.7) or its analogues when we replace $\nabla^{(m)}R_{ijkl}$ by $\nabla^{(m)}Ric \otimes g$ or $\nabla^{(m)}R \otimes g \otimes g$.

We will first use the notation above in studying the transformation law of normalized factors $\nabla^{r_1 \ldots r_m} \nabla^{(m)}_{r_1 \ldots r_m} W_{ijkl}$, where each of the upper indices $r_{b \ldots h}$ contracts against a lower derivative index $r_{a \ldots b}$. Using the transformation law for the Weyl curvature and the Levi-Civita connection (see the subsection 2.3 in [3]), we calculate that modulo partial contractions of length $\geq 3$:

$$\text{Image}^{1}_{\phi}[\nabla^{r_1 \ldots r_m} \nabla^{(m)}_{r_1 \ldots r_m} W_{ijkl}] = (t \cdot (n - 2) - 4 \left(\frac{t}{2}\right)) \nabla^{a_2 \ldots a_t} \nabla^{(m)}_{r_1 \ldots r_m} R_{ijkl} \nabla^{a_1} \phi$$

$$+ \sum_{\delta = t - 1, \Delta} T + \sum_{\delta \geq t} T + \sum_{\text{irrelevant}} T.$$  

(3.8)

The sublinear combination $\sum_{\delta = t - 1, \Delta} T$ stands for a generic linear combination of partial contractions in the form (3.4) with $t - 1$ internal contractions.

We now consider a factor $T = \nabla^{r_1 \ldots r_m} \nabla^{(m)}_{r_1 \ldots r_m} W_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}}$ where exactly one of the indices $r_{a_1}, \ldots, r_{a_t}$ is contracting against one of the internal indices in the factor $W_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}}$. With no loss of generality we assume the index $r_{a_1}$ is contracting against the index $i = r_{m+4}$, and we write the factor $T$ in the form:

$$T = \nabla^{r_{a_2} \ldots r_{a_t}} \nabla^{(m)}_{r_1 \ldots r_m} W_{ijkl}.$$

In that case we apply the transformation law of the Weyl curvature and the Levi-Civita connection to derive that:

$$\text{Image}^{1}_{\phi}[\nabla^{r_{a_2} \ldots r_{a_t}} \nabla^{(m)}_{r_1 \ldots r_m} W_{ijkl}] = (n - 3) \nabla^{r_{a_2} \ldots r_{a_t}} \nabla^{(m)}_{r_1 \ldots r_m} R_{ijkl} \nabla^{i} \phi$$

$$+ [(t - 1)(n - 3) - 4 \left(\frac{t}{2}\right) \frac{n - 3}{n - 2}] \nabla^{r_{a_2} \ldots r_{a_t}} \nabla^{(m)}_{r_1 \ldots r_m} R_{ijkl} \nabla^{r_{a_2}} \phi +$$

$$\sum_{\delta = t - 1, \text{negligible}} T + \sum_{\delta = t - 1, \Delta} T + \sum_{\delta \geq t} T + \sum_{\text{irrelevant}} T,$$

(3.9)

Thus the above factor has $t$ internal contractions in total.
where the sublinear combination $\sum_{\delta=t-1} T$ stands for a generic linear combination of contractions in the form $\nabla^{(m)} W_{ijkl} \nabla^i \phi$ with $t-1$ internal contractions (notice that $\nabla \phi$ is contracting against the index $j$ and the index $i$ is also involved in an internal contraction). The sublinear combination $\sum_{\delta=t-1, \Delta} T$ stands for a generic linear combination of partial contractions in the form (3.4) with $t-1$ internal contractions.

Finally, we consider the transformation law of factors

$$T = \nabla^{(m)} W_{ijkl},$$

where more than one of the indices $r_{a1}, \ldots, r_{a\tau}$ are contracting against the internal indices in $W_{ijkl}$ (modulo introducing quadratic correction terms); moreover (for the same reasons), we may assume with no loss of generality that the indices $r_{a1}, r_{a2}$ are contracting against the indices $i, k$ and also that the tensor $T$ is symmetric with respect to the indices $j, l$ (this is because of the first Bianchi and the antisymmetry of $i, j$). Thus in this setting we write the factor $T$ in the form:

$$T = \nabla^{(m)} W_{ijkl}.$$

We then calculate:

$$\text{Image}_1 T = (n-4)\left(\frac{n-3}{n-2}\right)\nabla^{(m)} W_{ijkl} \nabla^i \phi + (n-4)\left(\frac{n-3}{n-2}\right)\nabla^{(m)} \nabla^i \phi + (t-2)(n-3)\nabla^{(m)} \nabla^i \phi - 4\left(\frac{n-3}{n-2}\right)\binom{\delta-2}{2} \nabla^{(m)} \nabla^i \phi + \sum_{\delta=t-1, \Delta} T + \sum_{\delta=t-1, \text{negligible}} T + \sum_{\delta=t-1, *} T + \sum_{\delta \geq t} T + \sum_{\delta \leq t} T,$$

(3.10)

Here $\sum_{\delta=t-1, *} T$ stands for a generic linear combination of contractions in the form $\nabla s_{a} \nabla^{(m+t+1)} R$ where $\nabla^{(m+t+1)} R$ is a factor of the scalar curvature, the factor $\nabla s_{a} \phi$ contracts against that factor and the total number of internal contractions in this term is $\delta - 1$. The other linear combinations in the last line of (3.10) follow the same notational conventions as for the RHS of (3.9).

Let us finally recall from the subsection “Technical tools” in [3] the formulas concerning the decomposition of iterated covariant derivatives of the Weyl tensor.

We make two notes regarding (3.8) and (6.9): Firstly the coefficients $(t \binom{n-2}{4})$, $(n-3)$, etc. are independent of $m$. Secondly, since $n > 4$ they are nonzero: this is clear because by the weight restriction we must have $t \leq \frac{n-4}{2}$.

These identities will be used later on in this section.

38 Including the two in $R = R^{ab} \omega_{ab}$ itself.
3.2 The main algebraic Propositions.

Propositions 3.1 and 3.2 are the main tools we will need for this section. (As explained in the introduction, these two Propositions, together with the main algebraic Proposition 5.1 in [3] will be proven in the series of papers [6, 7, 8]). We will be considering tensor fields $C_{\gamma_i...\gamma_{\alpha}}(\Omega_1, \ldots, \Omega_p, \phi)$ of length $\sigma + 1$ (with no internal contractions) in the form:

$$pcontr(\nabla^{(m_i)}R_{ijkl} \otimes \cdots \otimes \nabla^{(m_p)}R_{ijkl} \otimes \nabla^{(b_1)}\Omega_1 \otimes \cdots \otimes \nabla^{(b_p)}\Omega_p \otimes \nabla\phi). \quad (3.11)$$

Here $\sigma = s + p$ and $i_1, \ldots, i_{\alpha}$ are the free indices.

**Definition 3.1** A complete or partial contraction in the form (3.11) will be called acceptable if the following conditions hold:

1. Each $b_i \geq 2$, (in other forms each function $\Omega_h$ is differentiated at least twice).
2. No factors have internal contractions.\(^{39}\)
3. The factor $\nabla\phi$ is contracting against some other factor (in other words the index $a$ in $\nabla_a \phi$ is not a free index).

The above definition (and also a generalized version of it) will be used very often both in this paper and in the following ones. For the purposes of the next Proposition we will introduce two more definitions which will be used only in the present paper. (For the reader’s convenience we specify that the notion of type that we introduce below is a special case of the notion of weak character which we introduce in the paper [4]).

**Definition 3.2** A complete or partial contraction in the form (3.11) will have type $A$ if the factor $\nabla\phi$ is contracting against a factor $\nabla^{(m)}R_{ijkl}$. It will have type $B$ if the factor $\nabla\phi$ is contracting against a factor $\nabla^{(B)}\Omega_h$.

We will divide the type $A$ contractions into two subcategories: We will say a (complete or partial) contraction has type $A1$ if the factor $\nabla\phi$ is contracting against an internal index in a factor $\nabla^{(m)}R_{ijkl}$ has type $A2$ if it is contracting against a derivative index.

We will also say that a (complete or partial) contraction of type $A1$ has a removable index if $\sum m_i + \sum b_i > 2p$ in the notation of (3.11).

Given an acceptable partial contraction with $\sigma + 1$ factors and $a$ free indices, $C_{\gamma_i...\gamma_{\alpha}}(\Omega_1, \ldots, \Omega_p, \phi)$ in the form (3.11) and any of its free indices $i_s$, we recall that $div_iC_{\gamma_i...\gamma_{\alpha}}(\Omega_1, \ldots, \Omega_p, \phi)$ can be written as a sum of $\sigma + 1$ partial contractions with $(a - 1)$ free indices each; the $t^{th}$ summand corresponds to the $(a - 1)$-partial contraction that arises when the derivative $\nabla^{i_s}$ hits the $t^{th}$ factor.\(^{40}\)

---

\(^{39}\)I.e. no two indices in the same factor contract against each other.

\(^{40}\)(In other words, against one of the indices $i, j, k, i_s$)
Definition 3.3 Given $C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi)$ a partial contraction in the form (3.11) as above, we define $X \text{div}_{i_1} \ldots X \text{div}_{i_a} C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi)$ to stand for the sublinear combination in $\text{div}_{i_1} \ldots \text{div}_{i_a} C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi)$ which arises when the derivative $\nabla_{i_a}$ is not allowed to hit the factor $\nabla \phi$, nor the factor to which the index $i_a$ belongs.

Before stating our Proposition we make two notes regarding the notion of type, for the reader’s convenience. Firstly, observe that if a tensor field $C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi)$ in the form (3.11) is of type A or B then all the complete contractions in the sum $X \text{div}_{i_1} \ldots X \text{div}_{i_a} C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi)$ will also be of type A or B. Secondly: Consider a set of complete contractions, $\{C_g^l (\Omega_1, \ldots, \Omega_p)\}_{l \in L}$, where all the complete contractions are in the form (3.11) with a given number $\sigma_1$ of factors $\nabla^{(m)} R_{ijkl}$ and a given number $P$ or factors $\nabla^{(B)} \Omega_h$. Let $L^A \subset L$ stand for the index set of complete contractions of type A and $L^B \subset L$ be the index set of complete contractions of type B. Assume an equation:

$$\sum_{l \in L} a_l C_g^l (\Omega_1, \ldots, \Omega_p) = 0,$$

which is assumed to hold modulo complete contractions of length $\geq \sigma + 2$. Then, using the fact that the above holds formally, we can easily derive that:

$$\sum_{l \in L^A} a_l C_g^l (\Omega_1, \ldots, \Omega_p) = 0,$$
$$\sum_{l \in L^B} a_l C_g^l (\Omega_1, \ldots, \Omega_p) = 0.$$

(Both the above equations hold modulo complete contractions of length $\geq \sigma + 2$).

We are now ready to state our two main algebraic propositions:

Proposition 3.1 Consider two linear combinations of acceptable tensor fields in the form (3.11), either all in the form B or all in the form A2,

$$\sum_{l \in L_1} a_l C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi),$$
$$\sum_{l \in L_2} a_l C_g^{i_1 \ldots i_{\beta_1}} (\Omega_1, \ldots, \Omega_p, \phi),$$

where each tensor field above has length $\sigma + 1 \geq 4$ and a given number $\sigma_1$ of factors in the form $\nabla^{(m)} R_{ijkl}$. We assume that for each $l \in L_2$, $\beta_1 \geq \alpha + 1$. We assume that modulo complete contractions of length $\geq \sigma + 2$:

$$\sum_{l \in L_1} a_l X \text{div}_{i_1} \ldots X \text{div}_{i_n} C_g^{i_1 \ldots i_n} (\Omega_1, \ldots, \Omega_p, \phi) +$$
$$\sum_{l \in L_2} a_l X \text{div}_{i_1} \ldots X \text{div}_{i_{\beta_1}} C_g^{i_1 \ldots i_{\beta_1}} (\Omega_1, \ldots, \Omega_p, \phi) = 0. \quad (3.12)$$
We claim that there is a linear combination, \( \sum_{r \in R} a_r C_g^{r, i_1 \ldots i_{\alpha+1}}(\Omega_1, \ldots, \Omega_p) \), of acceptable \((\alpha + 1)\)-tensor fields in the form \((\ref{eq:3.11})\), all with length \(\sigma + 1\) and \(\sigma_1\) factors \(\nabla^{(m)} R_{ijkl}\) and all of type B or type A2 respectively so that

\[
\sum_{l \in \mathcal{L}_1} a_l C_g^{l, \ldots, i_\alpha}(\Omega_1, \ldots, \Omega_p) = \sum_{r \in R} a_r X_{\text{div}^i}^{i_\alpha+1} C_g^{r, (i_1 \ldots \Omega_\alpha)}(\Omega_1, \ldots, \Omega_p),
\]

modulo terms of length \(\geq \sigma + 2\).

Second main algebraic Proposition:

**Proposition 3.2** Consider two linear combinations of acceptable tensor fields in the form \((\ref{eq:3.11})\), all in the form A1,

\[
\sum_{l \in \mathcal{L}_1} a_l C_g^{l, \ldots, i_\alpha}(\Omega_1, \ldots, \Omega_p, \phi),
\]

\[
\sum_{l \in \mathcal{L}_2} a_l C_g^{l, \ldots, i_\beta}(\Omega_1, \ldots, \Omega_p, \phi),
\]

where each tensor field above has length \(\sigma + 1 \geq 4\) and a given number \(\sigma_1\) of factors in the form \(\nabla^{(m)} R_{ijkl}\). We assume that for each \(l \in \mathcal{L}_2\), \(\beta_1 \geq \alpha + 1\). Let \(\sum_{l \in \mathcal{L}_2} a_l C_g^{l, \ldots, i_\beta}(\Omega_1, \ldots, \Omega_p, \phi)\) stands for a generic linear combination of complete contractions in the form \((\ref{eq:3.11})\) of type A2.

We assume that modulo complete contractions of length \(\geq \sigma + 2\):

\[
\sum_{l \in \mathcal{L}_1} a_l X_{\text{div}}^{i_1} \ldots X_{\text{div}}^{i_\alpha} C_g^{l, \ldots, i_\alpha}(\Omega_1, \ldots, \Omega_p, \phi) + \sum_{l \in \mathcal{L}_2} a_l X_{\text{div}}^{i_1} \ldots X_{\text{div}}^{i_\beta} C_g^{l, \ldots, i_\beta}(\Omega_1, \ldots, \Omega_p, \phi) + \sum_{j \in J} a_j C_g^{l, \ldots, i_\beta}(\Omega_1, \ldots, \Omega_p, \phi) = 0.
\]

In the one case where \(\alpha = 1\) and the tensor fields indexed in \(\mathcal{L}_1\) have no removable free index\footnote{Recall that given a \(\beta\)-tensor field \(T^{i_1 \ldots i_\beta}\), \(T^{(i_1 \ldots i_\beta)}\) stands for a new tensor field that arises from \(T^{i_1 \ldots i_\beta}\) by symmetrizing over the indices \(i_1, \ldots, i_\beta\).} we impose the additional restriction that all the tensor fields indexed in \(\mathcal{L}_1\) must have the free index \(i_1\) not belonging to the factor against which \(\nabla^\phi\) is contracting.

Our claim is that there is a linear combination of acceptable \((\alpha + 1)\)-tensor fields in the form \((\ref{eq:3.11})\), all with length \(\sigma + 1\) and \(\sigma_1\) factors \(\nabla^{(m)} R_{ijkl}\) and all of type A1, \(\sum_{r \in R} a_r C_g^{r, i_1 \ldots i_{\alpha+1}}(\Omega_1, \ldots, \Omega_p)\), with length \(\sigma\) so that\footnote{Recall that given a \(\beta\)-tensor field \(T^{i_1 \ldots i_\beta}\), \(T^{i_1 \ldots i_\beta}\) stands for a new tensor field that arises from \(T^{i_1 \ldots i_\beta}\) by symmetrizing over the indices \(i_1, \ldots, i_\beta\).}
\[
\sum_{l \in L_1} a_l C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) + \sum_{r \in R} a_r X \text{div} i_{a+1} C^r_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) + \sum_{j \in J} a_j C^j_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) = 0
\]

modulo terms of length \(\geq \sigma + 2\). Here the tensor fields \(C^j_{i_1 \ldots i_n}\) are all acceptable and of type A2.

**Note:** The conclusions of the two Propositions above involve (symmetric) tensor fields of rank \(\alpha\). Clearly, if we just introduce new factors \(\nabla \upsilon\) and contract it against the indices \(i_1, \ldots, i_\alpha\) we see that equations (3.18) implies:

\[
\sum_{l \in L_1} a_l C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) \nabla i_1 \upsilon \ldots \nabla i_\alpha \upsilon = \sum_{r \in R} a_r X \text{div} i_{a+1} C^r_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) \nabla i_1 \upsilon \ldots \nabla i_\alpha \upsilon,
\]

modulo terms of length \(\geq \sigma + \alpha + 2\).\(^{45}\) Similarly for (3.15). In fact, (3.16) and (3.18) are equivalent, since (3.16) holds formally. We will find it more convenient below to use (3.16) instead of (3.18) (and similarly, we will use the equation that arises from (3.15) by contracting the free indices against factors \(\nabla \upsilon\)).

Let us also state a Corollary of these two Propositions:

**Corollary 1** Consider two linear combinations of acceptable tensor fields in the form (3.11), (with no restriction on the “type” of the tensor fields):

\[
\sum_{l \in L_1} a_l C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi),
\]

\[
\sum_{l \in L_2} a_l C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi),
\]

where each tensor field above has length \(\sigma + 1 \geq 4\) and a given number \(\sigma_1\) of factors in the form \(\nabla^{(m)} R_{ijkl}\). We assume that for each \(l \in L_2\), \(\beta_l \geq \alpha + 1\).

We assume that modulo complete contractions of length \(\geq \sigma + 2\):

\[
\sum_{l \in L_1} a_l X \text{div} i_1 \ldots X \text{div} i_n C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) + \sum_{l \in L_2} a_l X \text{div} i_1 \ldots X \text{div} j_l C^l_{i_1 \ldots i_n}(\Omega_1, \ldots, \Omega_p, \phi) = 0.
\]

\(^{44}\)Here \(\nabla \upsilon\) is some scalar-valued function.

\(^{45}\)Recall from \(\ref{3.11}\) that for a tensor field \(T^j_{\phi, \upsilon}(\phi, \upsilon)\) involving factors \(\nabla \phi, \nabla \upsilon, X \text{div} \upsilon T^j_{\phi, \upsilon}(\phi, \upsilon)\) stands for the sublinear combination in \(\text{div} j_l T^j_{\phi, \upsilon}(\phi, \upsilon)\) where the derivative \(\nabla j_l\) is not allowed to hit to which the index \(j_l\) belongs, nor any of the factors \(\nabla \upsilon, \nabla \phi\).
In the one case where \( \alpha = 1 \) and the tensor fields indexed in \( L_1 \) have no removable free index, we impose the additional restriction that all the tensor fields indexed in \( L_1 \) must have the free index \( i \) not belonging to the factor against which \( \nabla \phi \) is contracting if the tensor field \( C_g^{i_1 \ldots i_\sigma}(\Omega_1, \ldots, \Omega_p, \phi) \) is of type \( A_1 \).

Our claim is then that there is a linear combination of acceptable \((\alpha + 1)\)-tensor fields in the form \( \sum_{l \in R} c_l (\Omega_1, \ldots, \Omega_p, \phi) \), with length \( \sigma + 1 \) and \( \sigma_1 \) factors \( \nabla^{(n)} R_{ijkl} \), \( \sum_{r \in R} a_r C_g^{r^{i_1 \ldots i_{\sigma + 1}}}(\Omega_1, \ldots, \Omega_p) \), with length \( \sigma \) so that:

\[
\sum_{l \in L_1} a_l C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) = \sum_{r \in R} a_r X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{r^{i_1 \ldots i_{\sigma + 1}}} ,
\]  

(3.18)

modulo terms of length \( \geq \sigma + 2 \).

Corollary 1 follows from Propositions 3.1, 3.2. We prove this as follows: Divide the index sets \( L_1, L_2 \) into subsets \( L_1^A, \ldots, L_1^B \) and \( L_2^A, \ldots, L_2^B \), according to the following rule: We say \( l \in L_1^A / l \in L_2^A \) if the tensor field \( C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) / C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) \) is of type \( A \). We say \( l \in L_1^B / l \in L_2^B \) if the tensor field \( C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) / C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) \) (in the form \( 3.11 \)) is of type \( B \).

Observe that since \( 3.12 \) holds formally, then:

\[
\sum_{l \in L_1^A} a_l X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) +
\sum_{l \in L_2^A} a_l X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) = 0 ,
\]  

(3.19)

\[
\sum_{l \in L_1^B} a_l X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) +
\sum_{l \in L_2^B} a_l X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) = 0 ,
\]  

(3.20)

where both the above equations hold modulo terms of length \( \geq \sigma + 2 \). We will prove the claim of our Corollary separately for the two sublinear combinations indexed in \( L_1^A, L_2^A \). We first prove \( 3.18 \) for the sublinear combination indexed in \( L_1^A \).

We just apply Proposition 3.1 to \( 3.20 \) to derive that there exists a tensor field \( \sum_{r \in R} a_r C_g^{r^{i_1 \ldots i_{\sigma + 1}}}(\Omega_1, \ldots, \Omega_p, \phi) \), as required by our Corollary, so that:

\[
\sum_{l \in L_1^A} a_l C_g^{l^{i_1 \ldots i_\sigma}}(\Omega_1, \ldots, \Omega_p, \phi) = \sum_{r \in R} a_r X \text{div}_{i_{\sigma + 1} \ldots i_n} C_g^{r^{i_1 \ldots i_{\sigma + 1}}} ,
\]  

(3.21)

46See Definition 3.2.
modulo terms of length $\geq \sigma + 1$. This proves (3.18) for the sublinear combination $\sum_{l \in L^A_1} a_l C^{[(l_1, \ldots, l_\alpha)]}_g$.

Now, we prove (3.18) for the sublinear combination indexed in $L^A_1$. We claim:

**Lemma 3.1** Assume (3.19) and Proposition 3.2. Denote by $L^A_1, L^A_2 \subset L^A$ the index set of tensor fields of type $A_1$. We claim:

1. There exists a tensor field $\sum_{r \in R^A} a_r C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi)$, as required by Corollary 1, so that:

$$\sum_{l \in L^A_1} a_l C^{[(l_1, \ldots, l_\alpha)]}_g (\Omega_1, \ldots, \Omega, \phi) = \sum_{r \in R^A} a_r X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi) + \sum_{t \in T} a_t X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi)$$

(3.22)

where the partial contractions indexed in $T$ in the RHS are all acceptable in the form (3.27) with length $\sigma, \sigma_1$ factors $\nabla^{(m)} R_{ijkl}$ and have type $A_2$. The above holds modulo partial contractions of length $\geq \sigma + 2$.

2. Assume (3.12), with the additional assumption that $L^A_1 = \emptyset$ we claim that we can write:

$$\sum_{l \in L^A_2} a_l X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi) = \sum_{t \in T'} a_t X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} X \text{div}_{i_{\alpha + 1}} C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi)$$

(3.23)

where the complete contractions indexed in $T'$ in the RHS are all in the form (3.27) with length $\sigma, \sigma_1$ factors $\nabla^{(m)} R_{ijkl}$, rank $\beta_i \geq \alpha + 1$ and have type $A_2$. The above holds modulo complete contractions of length $\geq \sigma + 2$.

Observe that proving the above Lemma would imply Corollary 1.

We first prove (3.22); we invoke the last Lemma in the Appendix in (9) to derive:

$$\sum_{l \in L^A_1} a_l X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C^{(i_1, \ldots, i_\alpha)}_g (\Omega_1, \ldots, \Omega, \phi) =$$

$$\sum_{r \in R^A} a_r X \text{div}_{i_1} \cdots \text{div}_{i_\alpha} X \text{div}_{i_{\alpha + 1}} C^{(i_1, \ldots, i_{\alpha + 1})}_g (\Omega_1, \ldots, \Omega, \phi)$$

(3.24)
modulo complete contractions of length $\geq \sigma + 2$. Now, replacing the above into (3.12), we derive a new local equation:

$$
\sum_{l \in L_2^{A_*}} a_l \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^g_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{r \in R^A} a_r \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^r_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{t \in T} a_t \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^t_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{l \in L_2^{A_*}} a_l \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^l_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) = 0,
$$

which holds modulo complete contractions of length $\geq \sigma + 2$. Now, applying (3.23) to the above, we derive that we can write:

$$
\sum_{l \in L_2^{A_*}} a_l \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^g_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{r \in R^A} a_r \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^r_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{t \in T} a_t \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^t_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) = \sum_{r \in R^+} a_r \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^r_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi),
$$

which holds modulo complete contractions of length $\sigma + 2$. The complete contractions are as the ones indexed in $T'$ in the RHS of (3.23).

Now, replacing the above into (3.25) we derive a new local equation, where all the tensor fields of length $\sigma + 1$ are of type $A^2$. Thus, we are in a position to apply Proposition 3.1. We derive that there exists a linear combination of $(\alpha + 1)$-tensor fields as in the claim of Proposition 3.1, indexed in $R^+$ below, so that:

$$
\sum_{l \in L_2^{A_*}} a_l \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^g_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) + \sum_{t \in T} a_t \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^t_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi) = \sum_{r \in R^+} a_r \text{X div}_{i_1} \ldots \text{X div}_{i_h} C^r_{i_1 \ldots i_h} (\Omega_1, \ldots, \Omega_p, \phi),
$$

modulo terms of length $\geq \sigma + 2$. Now, adding (3.22) and (3.27) we derive Corollary (1). □

\footnote{We treat the sum of $X \text{div}$’s indexed in $T, L_1^A \setminus L_1^{A_*}$ as a generic linear combination $\sum_{j \in J}$ (see the statement of Proposition 3.2).}
Proof of Lemma 3.7. The first claim follows by applying Proposition 3.1 to (3.19). \(48\)

To derive the second claim, we proceed by induction: Let \(\beta_{\text{min}} > \alpha\) be the minimum rank among the tensor fields indexed in \(L_{1}^{A,+}\); denote the corresponding index set by \(L_{1}^{A,+} \alpha, \beta_{\text{min}}\). We apply Proposition 3.2 to (3.12) (where the minimum rank among the tensor fields indexed in \(L_{1}^{A,+}\) is now \(\beta_{\text{min}}\)); we derive that there exists a linear combination of partial contractions in the form (3.11) of type \(A_{1}\), each with length \(\sigma\), with \(\sigma_{1}\) factors \(\nabla^{(m)} R_{ijkl}\) and rank \(\beta_{\text{min}} + 1\), indexed in \(R^{A}\) below, such that:

\[
\sum_{t \in L_{1}^{A,+} \beta_{\text{min}}} a_{t} C_{g}^{t(1\ldots j)}(\Omega_{1}, \ldots, \Omega_{p}, \phi) = \sum_{r \in R^{A}} a_{r} X \text{div}_{\beta_{\text{min}} + 1} C_{g}^{r(1\ldots j)}(\beta_{\text{min}}) + \sum_{t \in T^{A}} a_{t} C_{g}^{t(1\ldots j)}(\Omega_{1}, \ldots, \Omega_{p}, \phi),
\]

(3.28)

where the terms indexed in \(T^{A}\) have all the features of the tensor fields indexed in \(L_{1}^{A,+} \beta_{\text{min}}\), but they are of type \(A_{2}\) rather than \(A_{1}\). Iteratively applying the above step, we derive (3.23). \(49\)

3.3 Proof of Lemma 1.4.

Recall that we are assuming that all the complete contractions \(C^{l}(g), l \in L_{2}\) in \(P(g)\) have at least two of their internal contractions belonging to different factors. Observe that since \(j \geq 2\) we must have \(\sigma \leq \frac{n}{2} - 2\).

We will again consider \(I_{g}^{j}(\phi) := \frac{d}{dt} \mid_{t=0} \{e^{nt\phi} P(e^{t\phi} g)\}\). Recall that using (3.1) and (3.2) we may express \(I_{g}^{j}(\phi)\) as a linear combination of complete contractions in the form:

\[
\text{contr}(\nabla^{(m_{1})} R_{ijkl} \cdots \nabla^{(m_{s})} R_{ijkl} \cdots \nabla^{(p_{1})} R_{ijkl} \cdots \nabla^{(p_{q})} R_{ijkl} \cdots \nabla^{(\nu)} R_{ijkl} \cdots \nabla^{(\nu)})
\]

Recall that for each \(C^{l}(g)\) of length \(\sigma\) in \(P(g)\), \(\text{Image}_{\phi}^{1}[C^{l}(g)]\) stands for the sublinear combinations in \(\text{Image}_{\phi}^{1}[C^{l}(g)]\) \(50\) of contractions (in the form (3.29)) with \(\nabla^{(\nu)} \phi = \nabla \phi\) or \(\nabla^{(\nu)} \phi = \Delta \phi\), respectively.

\(48\) We treat the sum of \(X \text{div}\)'s indexed in \(L_{2}\) as a sum \(\sum_{j \in L} \cdots\), as in the statement of (3.2).
\(49\) Notice that since all local equations we deal with involve complete contractions of a fixed weight \(-n\), the maximum possible number of free indexes among tensor fields in the form (3.11) is \(\frac{n}{2}\). Therefore we derive (3.23) after a finite number of iterations.
\(50\) Recall that \(L_{\sigma} \subset L_{\nu}\) stands for the index set of complete contractions in \(P(g)\) with (minimum) length \(\sigma\). Recall that \(L_{\nu} \subset L_{\sigma}\) stands for the index set of complete contractions in \(\sum_{t \in L_{\sigma}} a_{t} C^{l}(g)\) with the (minimum) number \(j\) of internal contractions.
\(51\) Recall that \(\text{Image}_{\phi}^{1}[C^{l}(g)] = e^{n\phi} C^{l}(e^{2\phi} g) - C^{l}(g)\).
By virtue of the formulas in subsection 3.1 (see especially (3.1), (3.8), (3.9), (3.10) and also the first two formulas from subsection “Technical Tools” in [3]) we observe that all the complete contractions in Image_{\phi}^\nu[\sum_{l \in L_\sigma} a_l C^l(g)] will have \( \delta_R \geq j - 1 \); furthermore, (for the same reasons) it follows that any complete contraction with \( \delta = j - 1 \) and with \( \beta > 0 \) factors \( \nabla^{(p)} \text{Ric}_{\phi} \), then the indices \( \alpha, \beta \) in each such factor must contract against each other. We calculate:

$$
\text{Image}_{\phi}^\nu[\sum_{l \in L_\sigma} a_l C^l(g)] = \text{Image}_{\phi}^\nu[\sum_{l \in L_\sigma} a_l C^l(g)] + \sum_{u \in U_{\Delta \phi}} a_u C^u_{g}(\phi) + \sum_{x \in X} a_x C^x_{g}(\phi),
$$

where \( \sum_{u \in U_{\Delta \phi}} a_u C^u_{g}(\phi) \) is a generic linear combination of terms in the form (3.29) with length \( \sigma + 1 \) and a factor \( \nabla^{(p)} \phi = \Delta \phi \). \( \sum_{x \in X} a_x C^x_{g}(\phi) \) stands for a generic linear combination of complete contractions in the form (3.29) with either length \( \geq \sigma + 2 \) or with length \( \sigma + 1 \) and a factor \( \nabla^{(p)} \phi \neq \Delta \phi \).

On the other hand, we see that for any complete contraction \( C^l(g) \) of length \( \geq \sigma + 1 \) in \( P(g) \):

$$
\text{Image}_{\phi}^\nu[C^l(g)] = \sum_{u \in U_{\Delta \phi}} a_u C^u_{g}(\phi) + \sum_{x \in X} a_x C^x_{g}(\phi),
$$

(3.31)

(with the same conventions as above).

For future reference we denote by \( \text{Image}_{\phi}^\nu[P(g)|_{\sigma}] \) the sublinear combinations in \( \text{Image}_{\phi}^\nu[P(g)|_{\sigma}] \) that consists of complete contractions with no factors \( \nabla^{(p)} \text{Ric} \), and with a factor \( \nabla \phi \).

Now, we need two main technical Lemmas for this subsection. We will be considering a linear combination \( Y_\phi(g) \) in the form:

$$
Y_\phi(g) = \sum_{\alpha} \{ \sum_{u \in U_{\Delta \phi}} a_u C^u_{g}(\phi) + \sum_{x \in X} a_x C^x_{g}(\phi) \} + \sum_{u \in U_{\Delta \phi}} a_u C^u_{g}(\phi). \tag{3.32}
$$

All the complete contractions \( C^u_{g}(\phi) \) here have \( \sigma + 1 \) factors. The linear combination \( \sum_{x \in X} a_x C^x_{g}(\phi) \) is a generic linear combination as defined earlier (see discussion under (3.30)). Each complete contraction indexed in \( U_{\phi, \nu}^\nu \), \( \nu > 0 \) has a factor \( \nabla \phi \) and \( \nu \) factors \( \nabla^{(p)} \text{Ric} \) and either has \( \delta_R \geq j \) or has \( \delta_R = j - 1 \) but then also has the property that in all its \( \nu \) factors \( \nabla^{(p)} \text{Ric}_{ab} \) the indices \( a, b \) are contracting against each other if \( \nu = 0 \) then \( \delta_R \geq j - 1 \). Each complete contraction indexed in \( U_{\Delta \phi}^\nu \) has a factor \( \Delta \phi \) and \( \nu \) factors \( \nabla^{(p)} \text{Ric} \) and there is no restriction on \( \delta_R \). Thus \( \alpha \) is an upper bound on the number of factors \( \nabla^{(p)} \text{Ric} \) in the complete contractions \( C^u_{g}(\phi) \) in \( Y_\phi(g) \).

Note: Observe that \( I_\phi^\nu(\phi) \) is of the form (3.32) above, where

$$
\sum_{u \in U_{\Delta \phi}^\nu} a_u C^u_{g}(\phi) = \text{Image}_{\phi}^\nu[P(g)|_{\sigma}]. \tag{3.33}
$$

---

52 Equivalently, there are \( \nu \) factors \( \nabla^{(p)} R \) of the differentiated scalar curvature.
Remark: We remark that in (3.33), each $C^u_g(\phi)$, $u \in U^\phi_{\Delta \phi}$ with $\beta > 0$ factors $R$ (of the scalar curvature) will satisfy $\delta \geq j - 1 + 2\beta$. This follows from the decomposition of the Weyl tensor, since each factor $R$ in $C^u_g(\phi)$ must have arisen from an (undifferentiated) factor $W_{ijkl}$ in $P(g)|_\sigma$; thus a term with no internal contractions gives rise to a term with two internal contractions.

We claim the following:

**Lemma 3.2** Consider $Y_g(\phi)$ as in (3.22). Assume that $\int_{M^n} Y_g(\phi) dV_g = 0$. Let $\delta_{\text{min}}$ stand for the minimum $\delta_R$ among the complete contractions indexed in $U^\alpha_{\Delta \phi} \cup U^\sigma_{\Delta \phi}$.

We denote the corresponding index sets by $U^\alpha_{\Delta \phi, \delta_{\text{min}}}, U^\sigma_{\Delta \phi, \delta_{\text{min}}}$.

In the case where $\delta_{\text{min}} < j - 1$ and in the case $\delta_{\text{min}} = j - 1$ under the additional assumption that $U^\alpha_{\nabla^\phi, j-1} = \emptyset$, we claim that there is a linear combination of vector fields, $\sum_{h \in H} a_h C^h_g(\Delta \phi)$, each in the form (3.22) with $\nabla^{(\nu)} \phi = \Delta \phi$ and with one free index, so that:

$$
\sum_{u \in U^\phi_{\Delta \phi, \delta_{\text{min}}, \beta = 1}} a_u C^u_g(\phi) - \text{div}_i \sum_{h \in H} a_h C^h_g(\Delta \phi) = \sum_{u \in U^\phi_{\Delta \phi, \delta_{\text{min}}, \beta = 1}} a_u C^u_g(\phi) +
\sum_{u \in U^\sigma_{\Delta \phi, \delta_{\text{min}}, \beta = 1}} a_u C^u_g(\phi) + \sum_{x \in X} a_x C_x^g(\phi),
$$

(3.34)

where the linear combinations in the RHS are generic linear combinations with the following properties: Each contraction indexed in $U^\alpha_{\Delta \phi, \delta_{\text{min}}, \beta = 1}$ is in the form (3.22), has $\alpha$ factors $\nabla^{(\nu)} \text{Ric}$ and a factor $\Delta \phi$ and $\delta_R = \delta_{\text{min}, \beta = 1}$. Each complete contraction indexed in $U^\alpha_{\Delta \phi, \delta_{\text{min}}, \beta = 1}$ has $\alpha - 1$ factors $\nabla^{(\nu)} \text{Ric}$, $\delta_R = \delta_{\text{min}, \beta = 1}$ and a factor $\Delta \phi$.

On the other hand, if $\delta_{\text{min}} \geq j - 1$, and $\alpha > 0$, we claim that there is a linear combination of vector fields, $\sum_{h \in H} a_h C^h_g(\phi)$, so that:

$$
\sum_{u \in U^\phi_{\Delta \phi, \delta_{\text{min}}, \beta = 1}} a_u C^u_g(\phi) - \text{div}_i \sum_{h \in H} a_h C^h_g(\Delta \phi) =
\sum_{u \in U^\phi_{\Delta \phi, \delta_{\text{min}}, \beta = 1}} a_u C^u_g(\phi) + \sum_{x \in X} a_x C_x^g(\phi),
$$

(3.35)

where the linear combinations in the RHS are generic linear combinations with the following properties: The terms indexed in $U^\alpha_{\Delta \phi, \delta_{\text{min}}, \beta = 1} \cup U^\sigma_{\Delta \phi, \delta_{\text{min}, \beta = 1}}$ still

---

\footnote{In other words $\delta_{\text{min}}$ stands for the minimum number of internal contractions in curvature factors (i.e., factors in the form $\nabla^{(m)} R_{ijkl}, \nabla^{(m)} \text{Ric}_{ijkl}$), among the complete contractions with the maximum number $\alpha$ of factors in the form $\nabla^{(\nu)} \text{Ric}$, and having a factor $\Delta \phi$ or $\nabla \phi$. Observe by definition that if $\delta_{\text{min}} < j - 1$ then $U^\alpha_{\nabla^\phi, j-1} = \emptyset$.}

\footnote{(In which case $U^\alpha_{\nabla^\phi, j-1} = \emptyset$ as noted in the previous footnote).}
have $\alpha$ factors $\nabla^{(p)} \text{Ric}$ and $\delta_{\min} + 1$ internal contractions in total, while the ones indexed in $U^{\alpha - 1}_{\Delta \phi}$ have $\alpha - 1$ factors $\nabla^{(p)} \text{Ric}$ and $\delta_R \geq j$ and the ones in $U^{\alpha - 1}_{\Delta \phi}$ have $\alpha - 1$ factors $\nabla^{(p)} \text{Ric}$ (and there is no restriction on $\delta_R$).

We will prove Lemma 3.2 further down. For the time being, we will show how it implies Lemma 1.4:

Lemma 3.2 implies Lemma 1.4:

Observe that iteratively applying the above Lemma, starting with $Y_\phi(\phi) = I^1_\phi(\phi)$, we derive that there is a vector field, $\sum_{h \in H} a_h C^i_{g} h(\phi)$ so that:

$$I^1_\phi(\phi) - \text{div}_V \sum_{h \in H} a_h C^h_{g} h(\phi) = \text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}] + \sum_{u \in U^h_{\Delta \phi, \delta \geq j - 1}} a_u C^u_{g} h(\phi) + \sum_{u \in U^h_{\Delta \phi, \delta \geq j}} a_u C^u_{g} h(\phi),$$

where we recall that $\text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}]$ stands for the sublinear combinations in $\text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}]$ that consists of complete contractions with no factors $\nabla^{(p)} \text{Ric}$, and with a factor $\nabla \phi$. Also, $\sum_{u \in U^h_{\Delta \phi, \delta \geq j - 1}} a_u C^u_{g} h(\phi), \sum_{u \in U^h_{\Delta \phi, \delta \geq j}} a_u C^u_{g} h(\phi)$ stand for generic linear combinations of complete contractions in the form (3.29) with a factor $\nabla \phi, \Delta \phi$ respectively $\delta_R \geq j - 1, \delta_R \geq j$ respectively.

Now, in order to prove Lemma 1.4 we examine the sublinear combination $\text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}]$. We recall that $L_\phi \subset L$ stands for the sublinear combination in $P(g)$ of complete contractions with length $\sigma$ and $j$ internal contractions. We recall that for each $C'_{g}(\phi), l \in L_\phi$ there must be at least two internal contractions belonging to different factors. For every $l \in L_\phi$ we denote by $\text{Image}^1_{Y,\phi,\star,+[P(g)_{|\sigma}]$ the sublinear combination in $\text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}]$ with $\delta_R = j - 1$.

Thus, we derive:

$$\text{Image}^1_{Y,\phi,\star}[P(g)_{|\sigma}] = \text{Image}^1_{Y,\phi,\star,+[P(g)_{|\sigma}] + \sum_{h \in H} a_h C^h_{g} h(\phi),$$

where each $C^h_{g} (\phi)$ is in the form (3.29), has $\delta \geq j$, no factors $\nabla^{(p)} \text{Ric}$ and a factor $\nabla \phi$ (notice this fits into the generic linear combination $\sum_{u \in U^h_{\Delta \phi, \delta \geq j - 1}} a_u C^u_{g} h(\phi)$).

In view of the above, we may integrate (3.26) to derive a global equation of the form:

$$\int_{M^n} \text{Image}^1_{Y,\phi,\star,+[P(g)_{|\sigma}] + \sum_{u \in U^h_{\Delta \phi, \delta \geq j - 1}} a_u C^u_{g} h(\phi) + \sum_{u \in U^h_{\Delta \phi, \delta \geq j}} a_u C^u_{g} h(\phi) d\nu_g = 0.$$
We now show how to derive Lemma 1.4 from the above. In order to do so, we will introduce some more notational conventions.

**Definition 3.4**  For each tensor field $C_{\sigma}^{i_{1} \cdots i_{n}}$ in the form

$$pconstr(\nabla^{(m)}W_{ijkl} \otimes \cdots \otimes \nabla^{(m_{\sigma})}W_{i'j'k'l'})$$

we will associate a list of numbers, which we will call the character of $C_{\sigma}^{i_{1} \cdots i_{n}}$.

We list out the factors $T_{1}, \ldots, T_{\sigma}$ of $C_{\sigma}^{i_{1} \cdots i_{n}}$ and we define $\text{List}(l) = (L_{1}, \ldots, L_{n})$ as follows: $L_{i}$ will stand for the number of free indices that belong to the factor $T_{i}$. We then define $\vec{\kappa}(l)$ to stand for the decreasing rearrangement of $\text{List}(l)$.

With a slight abuse of notation, we will also consider the complete contractions in the index set $L_{\sigma}^{l}$ in $P(g) = \sum_{l \in L} a_{L}C_{\sigma}^{l}(g)$ and define their character to be the character of the tensor field $C_{\sigma}^{l_{1} \cdots l_{j}}$, that arises from $C_{\sigma}^{l}$ by replacing all the internal contractions by free indices.

We can thus group up the different $j$-tensor fields indexed in $L_{\sigma}^{l}$ according to their double characters: Let $\vec{K}$ stand for the set of all the different characters appearing among the tensor fields indexed in $L_{\sigma}^{l}$. We write $L_{\sigma}^{l} = \bigcup_{\vec{K} \in \vec{K}} L_{\sigma}^{l, \vec{K}}$; here $L_{\sigma}^{l, \vec{K}} \subset L_{\sigma}^{l}$ stands for the index set of $j$-tensor fields with a character $\vec{K}$.

We also introduce a partial ordering among characters according to lexicographic comparison: a character $\vec{K}_{2}$ is subsequent to $\vec{K}_{1}$ if $\vec{K}_{1}$ is lexicographically greater than $\vec{K}_{2}$.

Notice that any two different characters are comparable, in the sense that one will be subsequent to the other.

Now, we return to the derivation of Lemma 1.4. We will again proceed by induction. We break up $L_{\sigma}^{l}$ into subsets $L_{\sigma}^{l, \vec{K}}$ that index complete contractions $C_{\sigma}^{l}(g)$ with the same character, $\vec{K}$.

There is a finite list of possible characters; we denote the set of characters of the complete contractions in $L_{\sigma}^{l}$ by $\vec{K}$.

If $\vec{K} = \emptyset$ there is nothing to prove. If $\vec{K} \neq \emptyset$, we pick out the maximal character $\vec{K}_{\text{max}} \in \vec{K}$. We will show the claim of Lemma 1.4 for the sublinear combination $\sum_{l \in L_{\sigma}^{l, \vec{K}_{\text{max}}}} a_{L}C_{\sigma}^{l}(g)$ in other words, we will prove that there exists a linear combination of partial contractions, $\sum_{h \in H} a_{h}C_{\sigma}^{h, i}(g)$, where each $C_{\sigma}^{h, i}(g)$ is in the form (1.3) with weight $-n + 1$ and $\delta W = j$, so that:

$$\sum_{l \in L_{\sigma}^{l, \vec{K}_{\text{max}}}} a_{L}C_{\sigma}^{l}(g) - \text{div}_{v} \sum_{h \in H} a_{h}C_{\sigma}^{h, i}(g) = \sum_{v \in V} a_{v}C_{\sigma}^{v}(g)$$

(3.39)

where each $C_{\sigma}^{v}(g)$ is in the form (1.2) with $\delta W \geq j + 1$. Moreover, each $C_{\sigma}^{v}(g)$ will have at least two internal contractions belonging to different factors. The above equation holds modulo complete contractions of length $\geq \sigma + 1$.

---

55Thus the character is ultimately a list of numbers. We can also refer to an abstract “character” $\vec{K}$, which does not necessarily need to correspond to a particular tensor field. We also note that two different tensor fields can have the same character.

56Recall that the character of a complete contraction $C_{\sigma}(g)$ (with factors $\nabla^{(m)}W_{ijkl}$) was defined to be the character of the tensor field that formally arises from $C_{\sigma}(g)$ by making all internal contractions into free indices.
Clearly, if we can show that claim then Lemma 1.4 will follow by induction.

Proof of (3.37): We introduce some notational normalizations. We write out \( \vec{\kappa}_{\text{max}} = (k_1, \ldots, k_a) \), where if \( i < j \) then \( k_i \geq k_j \). Note that \( k_1 \) is the maximum number of internal contractions that can belong to a given factor among all complete contractions in \( L_a^{\vec{\kappa}} \). This follows by the definition of ordering among different refined double characters. By the hypothesis of Lemma 1.4 we have that \( a \geq 2 \) (this reflects the fact that not all internal contractions can belong to the same factor). Now, let \( b \) be the largest number for which \( k_{a-b+1}, \ldots, k_a = k_{\min} \) have the same value. In other words, for each \( l \in L_a^{\vec{\kappa}_{\text{max}}} \) there are \( b \) different factors \( \nabla^{(m)}W_{ijkl} \), with \( k_{\min} \) internal contractions each. For notational convenience, we will assume that the last \( b \cdot k_{\min} \) free indices in \( C^{l,i_1 \ldots i_l}(g) \) correspond to those internal contractions (i.e. they arise from those internal contractions when we make the internal contractions in \( C^l(g) \) into free indices in \( C^{l,i_1 \ldots i_l}(g) \)).

Now, we will say that a \((j-1)\)-tensor field in the form
\[
\text{pcontr}(\nabla^{(m)}W_{ijkl} \otimes \cdots \otimes \nabla^{(m)}W_{i'j'k'l'} \otimes \nabla \phi)
\]
has double-character \( \vec{\kappa}''_{\text{max}} \) if its \( j-1 \) free indices are distributed among its factors according to the pattern: \((k_1, k_2, \ldots, k_{a-1}, k_{\min} - 1)\) (in other words one factor \( T_1 \) has \( k_1 \) free indices, the next factor \( T_2 \) has \( k_2 \) free indices etc) and the factor with \( k_a - 1 \) free indices is also contracting against the factor \( \nabla \phi \). As above, we will extend this definition to complete contractions in the form
\[
\text{contr}(\nabla^{(m)}W_{ijkl} \otimes \cdots \otimes \nabla^{(m)}W_{i'j'k'l'} \otimes \nabla \phi),
\]
with \( j-1 \) internal contractions: We will say that such a complete contraction \( C_g(\phi) \) has a double-character \( \vec{\kappa}''_{\text{max}} \) if the tensor field \( C_g^{l,i_1 \ldots i_l}(\phi) \) which arises from \( C_g(\phi) \) by making all the internal contractions into free indices has a double-character \( \vec{\kappa}''_{\text{max}} \).

We then examine the complete contractions in \( \text{Image}_1^{\vec{\kappa}_{\phi,+,+}[P(g)]|_\sigma} \); in particular we seek to understand the sublinear combination with double character \( \vec{\kappa}''_{\text{max}} \). Denote this sublinear combination by \( \{\text{Image}_1^{\vec{\kappa}_{\phi,+,+}[P(g)]|_\sigma}\}_{\vec{\kappa}''_{\text{max}}} \). Using formulas (3.8), (3.9), (3.10) we can straightforwardly describe \( \text{Image}_1^{\vec{\kappa}_{\phi,+,+}[P(g)]|_\sigma} \):

This sublinear combination arises exclusively from the sublinear combination \( \sum_{l \in L_a^{\vec{\kappa}_{\text{max}}} \cap C^l(g) \in P(g)} \) via the following process: Consider each \( C^l(g) \), \( l \in L_a^{\vec{\kappa}_{\text{max}}} \) and list out its \( \sigma \) factors \( T_1^l, \ldots, T_{\sigma}^l \). For each factor \( T_a^l \) we let \( \text{Subst}_a^{\vec{\kappa}_{\phi}[C^l(g)]} \) stand for the (linear combination of) complete contractions that arise from \( C^l(g) \) by replacing \( T_a^l \) by one of the terms explicitly written out in the RHSs of (3.8), (3.9), (3.10)\(^5\) and then replacing all the other factors \( T_b^l = \nabla^{(m)}W_{ijkl}, b = 1, \ldots, a-1, a+1, \ldots, \sigma \), by either \( \sum_{a} \nabla^{(m)}R_{ijkl} \) or

\(^5\)Recall that each number \( k_a \) stands for the number of internal contractions in some given factor in \( \vec{\kappa}_{\text{Max}} \). The second restriction reflects the fact that the list is taken in decreasing rearrangement.

\(^5\)Which of the equations we use depends on how many internal contractions in \( T_a^l \) involve internal indices.

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\( \nabla^{(m)} R_{ijkl} \) (depending on whether \( T_l \) has an internal contraction involving an internal index or not). Then, just by virtue of the formulas (3.8), (3.9), (3.10) and the first two formulas in the subsection 5.1 in [3], we calculate:

\[
\text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma = \sum_{l \in L^\sigma, \kappa_{\text{max}}^\sigma} a_l \sum_{a=1}^{\sigma} \text{Subst}_a \nabla \phi [C^l(g)] + \sum_{\text{negligible}} C_g(\nabla \phi),
\]

(3.40)

where \( \sum_{\text{negligible}} C_g(\nabla \phi) \) stands for a generic linear combination of complete contractions in the form contr(\( \nabla^{(m)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m)} R_{l'j'k'l'} \otimes \nabla \phi \)) with \( j - 1 \) internal contractions in total and with the factor \( \nabla \phi \) contracting against the index \( i \) in a factor \( T = \nabla^{(m)} R_{ijkl} \) for which the index \( i \) is internally contracting in \( T \).

Note: Notice that in the first linear combination in the RHS there can be at most \( k_1 \) internal contractions in any of the factors in any \( C_g(\nabla \phi) \).

Given (3.40) we are able to explicitly write out the sublinear combination \( \{\text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma\}_{\kappa_{\text{max}}^\sigma} \) in \( \text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma \) of terms with a double character \( \kappa_{\max}^\sigma \):

\[
\{\text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma\}_{\kappa_{\text{max}}^\sigma} = \sum_{l \in L^\sigma, \kappa_{\text{max}}^\sigma} a_l \sum_{f=\sigma-bk_{\text{min}}+1}^{\sigma} (\text{Const})_{l,f} C^{l,i_f} \nabla_{i_f} + \sum_{\text{negligible}} C_g(\nabla \phi).
\]

(3.41)

Here \( C^{l,i_f} \nabla_{i_f} \) stands for the vector field that arises from \( C(g) \) by formally replacing all factors \( \nabla^{(m)} W_{ijkl} \) by \( \nabla^{(m)} R_{ijkl} \) and then making the \( f \)th internal contraction into a free index \( i_f \).

Although the full description of the exact form of \( \{\text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma\}_{\kappa_{\text{max}}^\sigma} \) is rather messy, all that is really important here is to note that by applying the operation Weylify (see subsection 5.1 in [3]) to this sublinear combination, one can recover the sublinear combination \( \sum_{l \in L^\sigma, \kappa_{\text{max}}^\sigma} a_l C^l(g) \) in \( P(g) \):

Let us define a formal operation \( \text{Op} \) which acts on the complete contractions above by replacing all internal contractions by factors \( \nabla \nu \), setting \( \phi = \nu \) and then acting on the resulting complete contractions by the operation Weylify. It follows from the discussion above (3.40), (3.41) and the definition of the operation Weylify in [3] that:

\[
\text{Op}[\{\text{Image}_{\nabla \phi,*,+} [P(g)]_\sigma\}_{\kappa_{\text{max}}^\sigma}] = \sum_{l \in L^\sigma, \kappa_{\text{max}}^\sigma} a_l [b(k_{\text{min}}',(n-2)-4\left(\frac{k_{\text{min}}}{2}\right))] C^l(g).
\]

(3.42)
Two notes are in order: Firstly that the coefficient \(b \cdot (k_{\min} \cdot ((n - 2) - 4(\frac{k_{\min}}{2}))\) is universal, i.e. it depends only on the character \(\bar{\kappa}_{\max}\) (thus it can be factored out in the RHS of (3.42)), and secondly that it is non-zero: This follows from the simple observation that \(b \geq 1\) by definition and \(l \leq \frac{n}{2} - 3\), since the complete contractions \(C^l(g)\) have weight \(-n\) and involve factors \(\nabla^{(m)}W_{ijkl}\) and \(\sigma \geq 3\).

We now consider the linear combination \(\sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g(\phi)\). We denote by \(\sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g(\phi)\) the sublinear combination that consists of complete contractions with \(\delta_R = j - 1\).

Now, let us also denote by \(A = k_1\), where \(k_1\) is the first number on the list \((k_1, \ldots, k_a)\) for \(\bar{\kappa}_{\max}\) above. We will pay special attention to the complete contractions in \(\sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g(\phi)\) which have a factor \(\nabla^{(m)}R_{ijkl}\) with \(A\) internal contractions. We denote the index set of those complete contractions by \(SU^0_{\Delta \phi, \delta = j - 1} \subset U^0_{\Delta \phi, \delta = j - 1}\).

We claim that there is a linear combination of vector fields \(C^{h,i}(g)\Delta \phi\) (in the form (3.29) without factors \(\nabla^{(p)}Ric\) and with \(\nabla^{(r)}\phi = \Delta \phi\)), so that:

\[
\sum_{u \in SU^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g \Delta \phi - \text{div}_i \sum_{h \in H} a_h C^{h,i}(g) \Delta \phi = \sum_{u \in U^0_{\Delta \phi, \delta = j}} a_u C^u_g \Delta \phi + \sum_{x \in X} a_x C^x_g(\phi),
\]

(3.43)

where \(\sum_{u \in SU^0_{\Delta \phi, \delta = j}} a_u C^u_g \Delta \phi\) stands for a generic linear combination of complete contractions in the form (3.29) with \(\nabla^{(r)}\phi = \Delta \phi\) and with \(\delta_R = j\).

We prove (3.43) after showing that it implies Lemma 1.4.

(3.43) implies Lemma 1.4: Plugging (3.43) into (3.38), we may assume that no complete contractions \(C^u_g \cdot \Delta \phi\) in (3.38) with \(\delta_R = j - 1\) have a factor \(\nabla^{(m)}R_{ijkl}\) with \(A\) internal contractions. In other words (3.38) can now be rewritten in the form:

\[
\int_{M^n} \sum_{l \in L^0_{\Delta \phi, \delta = j - 1}} a_l \sum_{f = \sigma - b \cdot k_{\min} + 1} (\text{Const})_{l,f} C^{l,i}(g) \nabla_{i,j} + \sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} C^u_g(\nabla \phi) + \\
\sum_{l \in L^0_{\Delta \phi, \delta = j - 1}} a_l C^{l,i}(g) \nabla \phi + \sum_{h \in H} a_h C^h_g(\nabla \phi) + \sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g \cdot \Delta \phi + \\
\sum_{u \in U^0_{\Delta \phi, \delta = j - 1}} a_u C^u_g \cdot \Delta \phi + \sum_{x \in X} a_x C^x_g(\phi) dV_g = 0.
\]

(3.44)

Here each complete contraction indexed in \(L^0_{\sigma}\) has \(j - 1\) internal contractions and a factor \(\nabla \phi\), but its double character is not \(\bar{\kappa}'_{\max}\). The complete contractions

\[^{5}\text{See also the notational convention introduced in (3.41).}\]
indexed in $H$ are in the form $\supdiv$ with $\delta_R \geq j$ and no factors $\nabla^{(p)}Ric$. The complete contractions in $U^0_{\Delta\phi,\delta \geq j}$ have a factor $\Delta\phi$ and $\delta \geq j$, while the ones indexed in $U^0_{\Delta\phi,\delta = j-1}$ have $\delta = j-1$ but also have no factor $\nabla^{(m)}R_{ijkl}$ with $A$ internal contractions.

Integrating by parts with respect to the factor $\Delta\phi$ in the second line of (3.44) we derive:

$$
\int_{M^n} \sum_{l \in L^0_{k_{\text{max}}}} \sum_{f = \sigma - b_{k_{\text{min}}},+1} a_l \sum_{j = \sigma} C_{ij}^{l,i_1 \cdots i_j}(g) \nabla_{ij} + \sum_{\text{negligible}} C_g(\nabla\phi) + \\
\sum_{l \in L^j_1} \sum_{h \in H} a_l C_{ij}^{l,i_1 \cdots i_j}(g) \nabla_{ij} + \sum_{h \in H} a_h C_{ij}^{\phi^h}(\nabla\phi) - \sum_{u \in U^0_{\Delta\phi,\delta = j-1}} a_u \nabla[C^u] \cdot \nabla_i \phi - \\
\sum_{u \in U^0_{\Delta\phi,\delta \geq j}} a_u \nabla[C^u] \cdot \nabla_i \phi + \sum_{x \in X} a_x C_{ij}^x(\phi)dV_g = 0.
$$

(3.45)

(Not that the above equation has been derived independently of (3.43)).

We denote the integrand of the above by $M_g(\phi)$. We apply the "main conclusion" of the super divergence formula to the above integral equation (see subsection 2.2 in [3]) and we pick out the sublinear combination \supdiv+$M_g(\phi)$ of terms with length $\sigma + 1$, a factor $\nabla\phi$ and no internal contractions. Since the super divergence formula holds formally we derive that \supdiv+$M_g(\phi)$ = 0 modulo a linear combination of terms of length $\geq \sigma + 2$. Thus we derive an equation:

$$
\sum_{l \in L^0_{k_{\text{max}}}} a_l \sum_{f = \sigma - b_{k_{\text{min}}},+1} \sum_{j = \sigma} (\text{Const})_{l,f} X_{\text{div}i_1 \cdots \text{div}i_{j-1}} X_{\text{div}i_j} \cdot C_{ij}^{l,i_1 \cdots i_j}(g) \nabla_{ij} + \sum_{\text{negligible}} C_g(\nabla\phi) + \\
\sum_{l \in L^j_1} X_{\text{div}i_1 \cdots \text{div}i_{j-1}} C_{ij}^{\phi}(\nabla\phi) + \sum_{l \in L^j_1} X_{\text{div}i_1 \cdots \text{div}i_{j-1}} C_{ij}^{\phi}(\nabla\phi) + \\
\sum_{u \in U^0_{\Delta\phi,\delta = j-1}} a_u X_{\text{div}i_1 \cdots \text{div}i_{j-1}} \cdot \nabla[C^u] \cdot \nabla_i \phi + \sum_{u \in U^0_{\Delta\phi,\delta \geq j}} a_u X_{\text{div}i_1 \cdots \text{div}i_{j-1}} \cdot \nabla[C^u] \cdot \nabla_i \phi = 0.
$$

(3.46)

(the linear combination $\sum_{\text{negligible}} X_{\text{div}i_1 \cdots \text{div}i_{j-1}} C_{ij}^{l,i_1 \cdots i_j}(\nabla\phi)$ just arises from the linear combination of complete contractions $\sum_{\text{negligible}} C_g(\nabla\phi)$ by making all the internal contractions into free indices and then taking $X_{\text{div}}$ of the resulting free indices).
We now apply Corollary 1 to the above; we pick out the sublinear combination of $(j - 1)$-tensor fields with $A$ factors $\nabla v$ contracting against some factor $\nabla^{(m)} R_{ijkl}$ (this sublinear combination vanishes separately). We then pick out the sublinear combination of complete contractions with the property that the factors $\nabla v$ are contracting according to the following pattern: $k_1$ factors $\nabla v$ must contract against one factor $T_1$; $k_2$ factors $\nabla v$ must contract against some other factor $T_2$; $k_3$ factors $\nabla v$ must contract against a third factor $T_3$; \ldots; $k_a - 1$ factors $\nabla v$ must contract against an $a^{th}$ factor $T_a$ and, in this last case the factor $\nabla \phi$ must also contract against this factor $T_a$. Observe that the sublinear combination of those terms must vanish separately, since the equation to which this sublinear combination belongs holds formally. We thus obtain a new true equation which is in the form:

$$\sum_{l \in \mathcal{L}(j, \kappa_{\max})} \sum_{f=0}^{b \cdot k_{\min} - 1} a_l C_{i_1 \ldots i_j} (g) \nabla_{i_{j-f}} \phi \nabla_{i_1} v \ldots \nabla_{i_{j-f}} v \ldots \nabla_{i_j} v + \sum_{\text{negligible}} C_{i_1 \ldots i_j} (\nabla \phi) \nabla_{i_1} v \ldots \nabla_{i_j} v$$

(3.47)

Then, setting $v = \phi$ and then applying the operation $Weylify$ (see subsection 5.1 in [3] and also (3.42)), we derive that Lemma 1.4 follows from (3.43).

Proof of (3.43): We again refer to (3.45) (denote the integrand by $M_g(\phi)$) and apply the super divergence formula to this equation; denote the resulting local equation by $\text{supdiv}[M_g(\phi)] = 0$. We pick out the sublinear combination $\text{supdiv}[M_g(\phi)]_{\perp}$ of terms of length $\sigma + 1$ with a factor $\nabla \phi$ and with no internal contractions. This sublinear combination must vanish separately and we thus derive an equation:

Recall that the double-character $\vec{\nu}_{\max}$ was defined to be the character $(k_1, \ldots, k_2, k_3, \ldots, k_{a-1}, k_a - 1)$.

Recall the note after that equation, which shows that it has been derived independently of (3.43).

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\[
\sum_{l \in L_{j-1}} a_l X \div i_{1} \ldots X \div i_{j-1} C_{g}^{l i_{1} \ldots i_{j}} (g) \nabla_{i} \phi \\
+ \sum_{h \in H} a_{h} X \div i_{1} \ldots X \div i_{z_{h}} C_{g}^{h i_{1} \ldots i_{z_{h}}} (\nabla \phi) \\
- \sum_{u \in U_{A-1}^{*,*}} a_{u} X \div i_{1} \ldots X \div i_{j-1} \nabla^{i} \left[ C_{g}^{u i_{1} \ldots i_{j-1}} \right] \cdot \nabla_{i} \phi \\
- \sum_{u \in U_{A-1}^{*,*}} a_{u} X \div i_{1} \ldots X \div i_{z_{h}} \nabla^{i} \left[ C_{g}^{u i_{1} \ldots i_{z_{h}}} \right] \cdot \nabla_{i} \phi = 0; 
\]

(3.48)

Here the terms indexed in \( L_{j-1} \) are in the form (3.11), have rank \( j - 1 \) and also have at most \( A - 1 \) free indices belonging to the factor against which \( \nabla \phi \) contracts; they have arisen from the complete contractions in Image \( \nabla \phi, \ast, + \) \( P(g) \sigma \) in (3.38). The tensor fields \( C_{g}^{u i_{1} \ldots i_{z_{h}}} \), \( C_{g}^{u i_{1} \ldots i_{j-1}} \) arise from the complete contractions \( C_{g}^{h} (\phi), C_{g}^{u} (g) \) in (3.38) by replacing each internal contraction by a free index.

We denote by \( H_{*} \subset H \) the index set of tensor fields in the above equation where the factor \( \nabla \phi \) is contracting against an internal index. We also denote by \( L_{j-1,*} \subset L_{j-1} \) the index set of \( (j - 1) \)-tensor fields for which the factor \( \nabla \phi \) is contracting against an internal index in some factor \( \nabla^{(m)} R_{abcd} \). Now, by applying Lemma 3.1 to the above equation, we derive that we can write:

\[
\sum_{l \in L_{j-1,*}} a_{l} X \div i_{1} \ldots X \div i_{j-1} C_{g}^{l i_{1} \ldots i_{j}} (g) \nabla_{i} \phi \\
+ \sum_{h \in H_{*}} a_{h} X \div i_{1} \ldots X \div i_{z_{h}} C_{g}^{h i_{1} \ldots i_{z_{h}}} (\nabla \phi) = \\
\sum_{l \in L_{j-1}} a_{l} X \div i_{1} \ldots X \div i_{j-1} C_{g}^{l i_{1} \ldots i_{j}} (g) \nabla_{i} \phi \\
+ \sum_{h \in H} a_{h} X \div i_{1} \ldots X \div i_{z_{h}} C_{g}^{h i_{1} \ldots i_{z_{h}}} (\nabla \phi), 
\]

(3.49)

where the terms indexed in \( \tilde{L}_{j-1} \) the RHS stand for a generic linear combination of tensor fields in the form (3.11), each with rank \( j - 1 \) and with the factor \( \nabla \phi \) contracting against a derivative index and at most \( A - 1 \) free indices in any given factor; \( H_{*} \) indexes a generic linear combination of tensor fields in the form (3.11), each with rank \( > j - 1 \) and with the factor \( \nabla \phi \) contracting against a derivative index. Therefore, replacing the above into (3.48) we derive a new equation in the form:

\( ^{63} \) Recall that for each of these last tensor fields there can be at most \( A - 1 \) free indices in any given factor.
\[
\sum_{l \in \tilde{L}^{-1}} a_l X \text{div} v_1 \ldots X \text{div} v_{j-1} C^{d|1\ldots j} (\phi) \nabla_i \phi \\
+ \sum_{h \in H' C^{f|1\ldots j} (C^g_{h,i\ldots j} C) (\phi) - \\
\sum_{u \in U_{\Delta^0, \delta = j-1}} a_u X \text{div} v_1 \ldots X \text{div} v_{j-1} \nabla_i [C^u_{j\ldots i - 1}] \cdot \nabla_i \phi - \\
\sum_{u \in U_{\Delta^0, \delta \geq j}} a_u X \text{div} v_1 \ldots X \text{div} v_{u,i} \nabla_i [C^u_{j\ldots i}] \cdot \nabla_i \phi = 0,
\]

where the linear combinations indexed in \( \tilde{L}^{-1} \) and \( H' \) are generic linear combinations of the forms described in the above paragraph.

Now, for each \( u \in SU^{0}_{\Delta^0, \delta = j-1} \), we denote by \( \nabla_i^u [C^u_{j\ldots i - 1}] (g) \) the sublinear combination in \( \nabla_i [C^u_{j\ldots i - 1}] (g) \) \( \nabla_i \phi \) where \( \nabla_i \) is forced to hit a factor with \( A \) free indices.\(^{64}\) Now, we break up the index set \( SU^{0}_{\Delta^0, \delta = j-1} \) into subsets \( SU^{0, \kappa}_{\Delta^0, \delta = j-1} \) that index tensor fields with the same character.\(^{65}\) We denote by \( K \) the index set of those subsets. We claim that for every \( \kappa \in K \) there is a linear combination of \( (j+1) \)-vector fields \( \sum_{h \in H^\kappa} a_h C^{i_1 \ldots i_{j+1}} (g) \text{div} v_i \phi \) with a character \( \kappa \) (also recall that \( \nabla \phi \) is contracting against a derivative index) so that:

\[
\sum_{u \in SU^{0, \kappa}_{\Delta^0, \delta = j-1}} a_u \nabla_i^u [C^u_{j\ldots i - 1}] (g) \nabla_i \phi \nabla_i v_1 \ldots \nabla_i v = 0 \quad (3.51)
\]

Let us check how (3.51) implies (3.45).

We observe that for each \( \kappa \in K \), there is a nonzero combinatorial constant \( (\text{Const})_{\kappa} \) so that for any \( u \in SU^{0, \kappa}_{\Delta^0, \delta = j-1} \):

\[
Erase_{\phi} \nabla_i^u [C^u_{j\ldots i - 1}] (g) | \nabla_i \phi \right) = (\text{Const})_{\kappa} C^{i\ldots i} (g). 
\]

(In fact \( (\text{Const})_{\kappa} \) just stands for the number of factors in \( \kappa \) that have \( A \) free indices. To see this clearly observe that if \( \kappa = (s_1, \ldots, s_\ell) \) (where \( s_i \geq s_{i+1} \) and \( s_1 = A \) that means that each tensor field \( C^{i\ldots i - 1} (g) \) will have a factor \( F_s \) with \( s_1 = A \) free indices, a second factor \( F_2 \) with \( s_2 \) free indices, \ldots, and an \( a^{th} \) factor \( F_a \) with \( s_a \) free indices. Then if \( A = s_1 = s_2 = \ldots s_c \) and \( s_{c+1} < A \), \( \nabla_i^u [C^u_{j\ldots i - 1}] (g) \) stands for the sublinear combination in \( \nabla_i [C^u_{j\ldots i - 1}] (g) \) where \( \nabla_i \) is forced to hit one of the factors \( F_1, \ldots, F_c.\)

\(^{64}\)Recall that \( SU^{0}_{\Delta^0, \delta = j-1} \subset U^{0}_{\Delta^0, \delta = j-1} \) is exactly the index set of \( (j-1) \)-tensor fields which have at least one such factor.

\(^{65}\)In other words with the same pattern of distribution of free indices among the different factors in \( C^{i\ldots i - 1} (g) \).
Now, applying the operation \( \text{Erase}_\phi \) (see the Appendix in [3]) to the above and then making the factors \( \nabla \phi \) into internal contractions and finally multiplying by \( \Delta \phi \), we derive (3.43).

**Proof of (3.51):** We apply Proposition 3.1 to (3.50), deriving that there is a linear combination of tensor fields of type A1,

\[ \sum_{t \in T} a_t C_{ij}^{(i_1 \cdots i_{j-1})j}(g) \nabla_{ij} \phi, \]

(this means that the factors \( \nabla \phi \) is contracting against a derivative index in some factor \( \nabla^{(m)}R_{ijkl} \)) so that:

\[
\sum_{t \in L^1} a_t C_{ij}^{(i_1 \cdots i_{j-1})j}(g) \nabla_{ij} \phi \nabla_{i_1} \cdots \nabla_{i_{j-1}} u \\
- \sum_{u \in U_\alpha, \beta = j-1} a_u \nabla^{[i} [C_{g}^{u,i_1 \cdots i_{j-1}]} \cdot \nabla_i \phi \nabla_{i_1} \cdots \nabla_{i_{j-1}} u \\
= \sum_{t \in T} a_t X \text{div}_{ij} C_{ij}^{(i_1 \cdots i_{j-1}i_{j+1}}(g) \nabla_{ij} \phi \nabla_{i_1} \cdots \nabla_{i_{j-1}} u. \tag{3.53}
\]

Now, in the above equations we pick out the sublinear combination of terms where the factor \( \nabla \phi \) and the factors \( \nabla \upsilon \) are contracting according to the following pattern: The factor \( \nabla \phi \) must contract against a factor \( F_1 \) which is also contracting against another \( A \) factors \( \nabla \upsilon \). A second factor \( F_2 \) contracts against \( s_2 \) factors \( \nabla \upsilon, \ldots \) and an \( a \)th factor \( F_a \) contracts against \( s_a \) factors \( \nabla \upsilon \). This sublinear combination must vanish separately since (3.53) holds formally. Thus we derive (3.51).

We have proven (3.53). \( \square \)

**Proof of Lemma 3.2:**

We prove (3.34) and (3.35). We will start with (3.34), and we will first prove this equation under a simplifying assumption; our simplifying assumption is that no complete contraction in \( \sum_{u \in U_\alpha} a_u C_{g}^{u}(\phi, \Omega) \) has a factor \( R \) of the scalar curvature. After we complete this proof under the simplifying assumption, we will explain how the general case can be derived by fitting the argument below to a new downward induction on the maximum number of factors \( R \) of the scalar curvature, precisely as in subsection 5.4 in [3].

**Proof of (3.34) under the simplifying assumption:** Denote by \( C_{g}^{u}(\phi, \Omega) \) the complete contraction that arises from each \( C_{g}^{u}(\phi) \), \( u \in U_\alpha \), by formally replacing the \( \alpha \) factors \( \nabla^{(p)}Ric \) by factors \( \nabla^{(p+2)}\Omega \). Thus, \( C_{g}^{u}(\phi) \), \( u \in U_\alpha \nabla \Omega, u \in U_\Delta \Omega \) will be in one of the forms:

\[
\text{contr}(\nabla^{a_1 \cdots a_t} \nabla^{(m_1)}R_{ijkl}) \otimes \cdots \otimes \nabla^{b_1 \cdots b_v} \nabla^{(m_v)}R_{i'j'k'l'} \\
\otimes \nabla^{y_1 \cdots y_u} \nabla^{(p_1+2)}\Omega \otimes \cdots \otimes \nabla^{x_1 \cdots x_u} \nabla^{(p_u+2)}\Omega \otimes \nabla \phi, \tag{3.54}
\]

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\[
\text{contr}(\nabla^{a_1 \ldots a_t} \nabla^{(m)} R_{ijkl} \otimes \ldots \nabla^{b_1 \ldots b_s} \nabla^{(n)} R_{l'j'k'l'}) \\
\otimes \nabla^{y_1 \ldots y_u} \nabla^{(p_1+2)} \Omega \otimes \ldots \nabla^{x_1 \ldots x_v} \nabla^{(p_s+2)} \Omega \otimes \Delta \phi).
\]

(3.55)

For brevity, we write out:

\[
\sum_{u \in U_{\Delta \phi}} a_u C_u^g(\phi, \Omega^a) = \sum_{q \in Q} a_q C_q^g(\Omega^a) \Delta \phi,
\]

(3.56)

where each \(C_q^g(\Omega^a) \Delta \phi\) is in the form (3.55) with length \(\delta_R + \delta_\Omega + \alpha < j\).

We denote by \(C_q^g(\Omega^a)\) the tensor field that arises from each \(C_q^g(\Omega^a)\) by making all the internal contractions into free indices.

Applying the “main conclusion” of the super divergence formula (see [3]) to

\[
\int_M Y_g(\phi) dV_g = 0,
\]

and picking out the sublinear combination of length \(\sigma + 1\) with factor \(\nabla \phi\), we obtain a local equation in the form:

\[
\sum_{q \in Q} a_q X \text{div}_i \ldots X \text{div}_u \nabla^i [C_q^g \nabla^{i_1 \ldots i_{\delta_{\min}}} (\Omega^a)] \nabla_i \phi
\]

\[
+ \sum_{h \in H} a_h X \text{div}_i \ldots X \text{div}_u C_h^g \nabla^{i_1 \ldots i_{\delta_{\min} + 1}} (\Omega^a, \phi) = 0,
\]

(3.57)

modulo complete contractions of length \(\geq \sigma + 2\). All the tensor fields above are acceptable without internal contractions. Moreover, each \(b \geq j - 1\) (and \(b > j - 1\) if we are additionally have \(U_{\phi,j-1} = \emptyset\)) while each \(a \leq j - 1\).

We pick out the minimum rank \(a\) appearing above (we have denoted it by \(\delta_{\min}\), where \(\delta_{\min} < j - 1\), or \(\delta_{\min} = j - 1\) under the additional assumption that \(U_{\phi,j-1} = \emptyset\)) and denote by \(Q^{\delta_{\min}} \subset Q\) the corresponding index set. We will show that there is a linear combination of acceptable \((\delta_{\min} + 1)\)-tensor fields,

\[
\sum_{h \in H} a_h C_h^g \nabla^{i_1 \ldots i_{\delta_{\min} + 1}} (\Omega^a) \text{ so that:}
\]

\[
\sum_{q \in Q^{\delta_{\min}}} a_q C_q^g \nabla^{i_1 \ldots i_{\delta_{\min} + 1}} (\Omega^a) \nabla^i u_1 \ldots \nabla^{i_{\delta_{\min} + 1} u}
\]

\[
- \sum_{h \in H} a_h X \text{div}_{i_{\delta_{\min} + 1}} C_h^g \nabla^{i_1 \ldots i_{\delta_{\min} + 1}} (\Omega^a) \nabla^i u_1 \ldots \nabla^{i_{\delta_{\min} + 1} u} = 0,
\]

(3.58)

modulo complete contractions of length \(\geq \sigma + 1\). Let us assume we have proven this; we will then show how to derive (3.34).

Proof that (3.58) implies (3.34): We define an operation \(Op\) that replaces each factor \(\nabla^{(p+2)} \Omega\) by a factor \(\nabla^{(p)} Ric\) and replaces each factor \(\nabla \nu\) by an internal contraction and finally multiplies the contraction we obtain by \(\Delta \phi\) (notice that the operation \(Op\) is almost exactly the same as the operation Ricci_y.

Recall that \(X \text{div}_i[\ldots]\) in this context stands for the sublinear combination in \( \text{div}_i[\ldots] \) where the derivative \(\nabla^i\) is not allowed to hit \(\nabla \phi\), nor the factor to which the free index \(i\) belongs.

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defined in the appendix of [3]. Applying $Op$ to (3.58) and using the fact that (3.58) holds formally, we derive (3.34) (the proof of this fact is precisely the same as the operation “Riccify” in subsection 5.1 in [3]).

**Proof of (3.58):** Now, in order to derive (3.58) from (3.57), we will distinguish two subcases: Either $\alpha = \sigma$ or $\alpha < \sigma$.

We start with the second case. We pick out the sublinear combination in $\sum_{h \in H} \ldots$ in (3.57) where the factor $\nabla \phi$ is contracting against a factor $\nabla^m R_{ijkl}$ (the condition $\alpha < \sigma$ guarantees that there is such a factor). We denote by $H_* \subset H$ the index set of that sublinear combination. We also denote by

$$\nabla^i [C^q_{g,i^1 \ldots i^a} (\Omega^\alpha)] \nabla_i \phi$$

the sublinear combination in

$$\nabla^i [C^q_{g,i^1 \ldots i^a} (\Omega^\alpha)] \nabla_i \phi$$

where $\nabla^i$ is only allowed to hit factors $\nabla^m R_{ijkl}$.

Since (3.57) holds formally, we derive that:

$$\sum_{q \in Q} a_q X \text{div}_{i_1} \ldots X \text{div}_{i_a} \nabla^i [C^q_{g,i^1 \ldots i^a} (\Omega^\alpha)] \nabla_i \phi + \sum_{h \in H_*} a_h X \text{div}_{i_1} \ldots X \text{div}_{i_b} C^h_{g,i^1 \ldots i^b} (\Omega^\alpha, \phi) = 0. \quad (3.59)$$

We then denote by $H_* \subset H$ the index set of those tensor fields for which the factor $\nabla \phi$ is contracting against an internal index of the factor $\nabla^m R_{ijkl}$. By applying the second claim in Lemma 3.1 we deduce as before that we can write:

$$\sum_{h \in H_*} a_h X \text{div}_{i_1} \ldots X \text{div}_{i_b} C^h_{g,i^1 \ldots i^b} (\Omega^\alpha, \phi) \quad (3.60)$$

where each tensor field on the right hand side has $b > j - 1$ ($b > j$ under the additional assumption that $U^q_{\phi,j-1} = \emptyset$) and the factor $\nabla \phi$ is contracting against the derivative index of some factor $\nabla^m R_{ijkl}$. Substituting the above in (3.59) and applying Proposition 3.1 along with the operation $\text{Erase}_{\nabla \phi}$, we derive (3.58).

Now, the case $\alpha = \sigma$: We recall equation (3.57) where we polarize the function $\Omega$ into $\Omega_1, \ldots, \Omega_\sigma$, obtaining a new true equation. We denote the corresponding tensor fields by $C^q_{g,i^1 \ldots i^a} (\Omega_1, \ldots, \Omega_\sigma, \phi), C^h_{g,i^1 \ldots i^b} (\Omega_1, \ldots, \Omega_\sigma, \phi), 61$
We denote by $H_\ast \subset H$ the index set of those tensor fields for which $\nabla \phi$ is contracting against the factor $\nabla^{(n)} \Omega_1$. We also denote by $\nabla_\ast$ the sublinear combination in $\nabla^i$ where $\nabla^i$ is only allowed to hit the factor $\nabla^{(n)} \Omega_1$.

Thus, we derive that:

$$
\sum_{q \in Q} a_q X \text{div}_i X \text{div}_{i_2} \ldots X \text{div}_{i_n} \nabla_\ast [C_g^q,i_1\ldots i_n(\Omega_1,\ldots,\Omega_\sigma)] \nabla_i \phi + \sum_{h \in H_\ast} a_h X \text{div}_i X \text{div}_{i_2} C_g^{h,i_1\ldots i_n}(\Omega_1,\ldots,\Omega_\sigma, \phi) = 0.
$$

We denote by $H^\# \ast \subset H_\ast$ the index set of those tensor fields for which the factor $\nabla \phi$ is contracting against a factor $\nabla^{(2)} \Omega_1$ (with exactly two derivatives).

Then, by applying Lemma 4.6 in [6], we can derive:

$$
\sum_{h \in H^\# \ast} a_h X \text{div}_i X \text{div}_{i_2} C_g^{h,i_1\ldots i_n}(\Omega_\sigma, \phi) = \sum_{h \in H_{\ast\ast}} a_h X \text{div}_i X \text{div}_{i_2} C_g^{h,i_1\ldots i_n}(\Omega_\sigma, \phi),
$$

where each tensor field on the right hand side has $b > j - 1$ and the factor $\nabla \phi$ is contracting against a factor $\nabla^{(A)} \Omega$ with $A \geq 3$. Substituting the above in (3.59) and applying the operation $\text{Erase} \nabla \phi$ from the Appendix in [3], we notice that all resulting complete contractions are acceptable (in particular they have each factor $\nabla^{(a)} \Omega_a$ having $a_s \geq 2$); thus we derive our claim (3.34). 

**Proof of (3.35) under the simplifying assumption:**

The proof of this claim will be slightly more involved in the case where $\delta_{\min} = j - 1$. We first prove (3.35) in the case $\delta_{\min} > j - 1$:

**The case $\delta_{\min} > j - 1$:** For each $u \in U^\alpha_{\Delta \phi}$, we denote by $C_g^{u,i_1}(\phi)$ the complete contraction that arises from $C_g^u(\phi)$ by replacing $\Delta \phi$ by $\nabla_i \phi$. We observe that we can write:

$$
\sum_{u \in U^\alpha_{\Delta \phi}} a_u C_g^u(\phi) - \text{div}_i C_g^{u,i_1}(\phi) = \sum_{u \in U^\alpha_{\nabla \phi}} C_g^u(\phi)
$$

where the linear combination on the RHS stands for a generic linear combination of complete contractions with a factor $\nabla \phi$, $\alpha$ factors $\nabla^{(p)} \text{Ric}$ and $\delta > j - 1$.

---

67 As noted in [6], that Lemma is a consequence of the “fundamental Proposition”. There is no logical dependence of that Lemma on any of the work in this paper or in [3, 5].

68 We only need to check that the hypotheses of this Lemma are satisfied in the case where the minimum rank $b_{\min}$ appearing among the tensor fields in $H^\#_\ast$ is $b_{\min} = 1$, and in addition there are tensor fields indexed in $H^\#_\ast$ with rank 1 and the one free index $i_1$ belonging to an expression $\nabla^{(2)} \Omega_1 \nabla \phi$. In that case we see that the extra assumption needed for Lemma 4.6 is fulfilled by virtue of a weight restriction—this follows since $j \geq 2$ and all internal contractions in $P(g)_{\sigma_{\ast}}$ involve only derivative indices.
In view of this equation, it suffices to show (3.35) under the additional assumption that \( U_{\alpha} = \emptyset \).

Now, to show (3.35) under this extra assumption we consider the equation
\[
\int_M Y(\phi) dV_g = 0 \text{ (in the assumption of Lemma 3.2).}
\]
We apply the “main conclusion” of the super divergence formula (see [3]) and we pick out the sublinear combination of terms with length \( \sigma + 1 \) and a factor \( \nabla \phi \), thus deriving a new local equation:
\[
\sum_{u \in U_{\alpha}} a_u X \text{div} C_g^{u,i_1 \ldots i_{\min}}(\phi, \Omega^\alpha) + \sum_{u \in U_{\alpha} \setminus \delta_{\min}} a_u X \text{div} C_g^{u,i_1 \ldots i_{\min}}(\phi, \Omega^\alpha) = 0.
\]
(3.63)

Then (3.35) follows by applying Corollary 1 to (3.63); we derive that there is a linear combination of \((\beta + 1)\)-vector fields (indexed in \( U_{\alpha} \)) so that:
\[
\sum_{u \in U_{\alpha} \setminus \delta_{\min}} a_u X \text{div} C_g^{u,i_1 \ldots i_{\min}}(\phi, \Omega^\alpha) \nabla_{i_1} \ldots \nabla_{i_{\min}} v + \sum_{u \in U_{\alpha} \setminus \delta_{\min}} a_u X \text{div} C_g^{u,i_1 \ldots i_{\min+1}}(\phi, \Omega^\alpha) \nabla_{i_1} v \ldots \nabla_{i_{\min}} v = 0.
\]
(3.64)

By applying the operation \( \text{Ricci} \) to the above (see subsection 5.1 in [3]) we derive our claim, in the case \( \delta_{\min} > j - 1 \).

Now, the case \( \delta_{\min} = j - 1 \): We introduce some notational conventions. We break up the linear combination \( \sum_{u \in U_{\alpha}} a_u X \text{div} C_g^{u,i_1 \ldots i_{\min}}(\phi, \Omega^\alpha) \nabla_{i_1} \ldots \nabla_{i_{\min}} v \) into two pieces (by dividing the index set \( U_{\alpha} \) into two subsets): We will say that \( u \in U_{\alpha} \setminus \delta_{\min} \) if the complete contraction \( C_g^{u,i_1 \ldots i_{\min}}(\phi, \Omega^\alpha) \) has the property that all \( \alpha \) factors \( \nabla \phi \) have at least one internal contraction; we define \( U_{\alpha} \setminus \delta_{\min} = U_{\alpha} \setminus \delta_{\min} \). \( \text{Ricci} \) have at least one internal contraction.

By applying the second Bianchi identity we derive that we can write:
\[
\sum_{u \in U_{\alpha}} a_u C_g^{u,\phi}(\phi) = \sum_{u \in U_{\alpha} \setminus \delta_{\min}} a_u C_g^{u,\phi}(\phi) + \sum_{u \in U_{\alpha} \setminus \delta_{\min}} a_u C_g^{u,\phi}(\phi),
\]
(3.65)

where the complete contractions indexed in \( U_{\alpha} \setminus \delta_{\min} \) have all the generic properties of the complete contractions indexed in \( U_{\alpha} \setminus \delta_{\min} \) and have the feature that

\( \text{69} \)Since \( \delta_{\min} > j - 1 \geq 1 \) we do not have to worry about the extra restrictions in that corollary when \( \alpha = 1 \).

\( \text{70} \)Not counting the one in \( \text{Ricci} = R^{i}{}_{iab} \) itself.
the two indices \(a, b\) in every factor \(\nabla(p)R_{ab}\) contract against each other. The linear combination \(\sum_{u \in U_{\Delta \phi, j-1}^\alpha} a_u C^u(\phi)\) stands for a generic linear combination as allowed in the RHS of (3.33).

In view of the above, we may assume with no loss of generality that all the \(\alpha\) factors \(\nabla(p)R_{ab}\) in every \(C^u(\phi), u \in U_{\Delta \phi, j-1}^\alpha\) have the indices \(a, b\) contracting against each other. Now, for each \(u \in U_{\Delta \phi, j-1}^\alpha\), we denote by \(C^u_{a, i_1 ... i_\delta}(\phi)\) the vector field that arises from \(C^u(\phi)\) by replacing \(\Delta \phi\) by \(\nabla_i \phi\). We observe that we can write:

\[
\sum_{u \in U_{\Delta \phi, j-1}^\alpha} a_u C^u(\phi) - \text{div}_1 C^u_{a, i_1}(\phi) = \sum_{u \in U_{\Delta \phi, j-1}^\alpha} a_u C^u(\phi),
\]

where the linear combination on the RHS stands for a generic linear combination of complete contractions with a factor \(\nabla \phi, \alpha\) factors \(\nabla(p)Ric\) and \(\delta = j - 1\) and with the feature that all the \(\alpha\) factors \(\nabla(p)R_{ab}\) have the indices \(a, b\) contracting against each other. Thus, by virtue of (3.65), (3.66) we may assume with no loss of generality that \(U_{\Delta \phi, j-1}^\alpha\) is empty.

Now, we again consider the “main conclusion” of the super divergence formula applied to the integral equation \(\int_{M^n} Y_g(\phi)dV_g = 0\) (see the statement of Lemma 3.2) and we pick out the sublinear combination with no loss of generality that \(U_{\Delta \phi, j-1}^\alpha\) is empty.

Now, we apply Corollary 1 to (3.67) and pick out the sublinear combination of terms with length \(\sigma + 1\) and a factor \(\nabla \phi\), thus deriving a new local equation:

\[
\sum_{u \in U_{\phi, j-1}^\phi} a_u X\text{div}_{i_1} ... X\text{div}_{i_{j-1}} C^u_{g, i_1 ... i_\delta}(\phi, \Omega^\alpha)
+ \sum_{u \in U'} a_u X\text{div}_{i_1} ... X\text{div}_{i_{j-1}} C^u_{g, i_1 ... i_\delta}(\phi, \Omega^\alpha)
+ \sum_{u \in U_{\phi, h > j-1}} a_u X\text{div}_{i_1} ... X\text{div}_{i_2} C^u_{g, i_1 ... i_\delta}(\phi, \Omega^\alpha) = 0,
\]

where the tensor fields indexed in \(U'\) have rank \(j - 1\) but at least one of the \(\alpha\) factors \(\nabla^{(B)}\Omega\) does not contain a free index. The terms in \(U_{\phi, \delta > j-1}^\phi\) have rank \(> j - 1\).

Now, we apply Corollary 1 to (3.67) and pick out the sublinear combination of terms where all factors \(\nabla^{(p)}\Omega\) contract against at least one factor \(\nabla v\); this sublinear combination must vanish separately, thus we derive a new equation:

\[
\sum_{u \in U_{\phi, j-1}^\phi} a_u C^u_{g, i_1 ... i_\delta}(\phi, \Omega^\alpha)\nabla_{i_1} v ... \nabla_{i_{j-1}} v
- \sum_{\h \in H} a_h X\text{div}_{j} C^u_{g, i_1 ... i_\delta}(\phi, \Omega^\alpha)\nabla_{j} v ... \nabla_{i_{j-1}} v = 0,
\]

In other words that factor is of the form \(\nabla^{(p)} R\), where \(R\) stands for the scalar curvature.

These terms arise from the sublinear combination \(\sum_{u \in U_{\Delta \phi, j-1}^\alpha} a_u C^u(\phi)\).
where the tensor fields indexed in $H$ have each factor $\nabla^{(p)}\Omega$ contracting against at least one factor $\nabla v$. Now, recall that for each of the terms indexed in $U_{\nabla \phi,j-1}$ in the above, the last index $r_p$ in each factor $\nabla^{(p)}_{r_1...r_p}\Omega$ is contracting against a factor $\nabla v$. By just permuting indices we may assume that the same is true for each of the terms indexed in $H$. Then, since (3.68) holds formally we may assume that this last index in each of the factors $\nabla^{(p)}\Omega$ is not permuted in the formal permutations of indices by which (3.68) is made formally true. Then, applying the operation $\text{Ricci} f y$ to the above we derive that there is a vector field $\sum_{h\in H'} a_h C^h_\phi (\phi)$ so that:

$$\sum_{u\in U_{\nabla \phi,j-1}} a_u C^u_\phi (\phi) - \sum_{h\in H'} a_h C_{\phi,i}^u (\phi) = \sum_{u\in U_{\nabla \phi,j-1}} a_u C^u_\phi (\phi) + \sum_{x\in X} a_x C^x_\phi (\phi),$$

(3.69)

(in the notational conventions of (3.35)). Thus we are reduced to showing our claim in the case where $U_{\nabla \phi,j-1} = U_{\nabla \phi,j-1} = \emptyset$. Our claim then follows by appealing to (3.34), where we now have the additional assumption that $U_{\nabla \phi,j-1} = \emptyset$. □

Proof of (3.34), (3.35) without the simplifying assumption:

Now, we explain how to prove (3.34), (3.35) in the case where there are factors $R$ among the contractions indexed in $U_{\nabla \phi,j-1} \cup U_{\nabla \phi}$ (i.e. without the simplifying assumption). Let $M \epsilon \gamma$ be the maximum number of factors $R$ among the complete contractions indexed in $U_{\nabla \phi,j-1} \cup U_{\nabla \phi}$. We then reduce ourselves to the case where there are no such factors by a downward induction on $M \epsilon \gamma$. If $M \epsilon \gamma \leq \sigma - 3$ then we can prove (3.34), (3.35) by just repeating the proof under the simplifying assumption, as in the proof of Lemmas 5.4, 5.5 when $M \leq \sigma - 3$ in [3]: We prove the claims for the sublinear combinations in the LHS which have $M \epsilon \gamma$ factors $R$, allowing terms in the RHSs with $M \epsilon \gamma - 1$ factors $R$. The argument runs unobstructed when $M \epsilon \gamma \geq 3$, since (as in [3]) whenever we wish to invoke Corollary 11 or the first technical Proposition from [6], the equations to which we apply them have $\sigma \geq 3$.

The case $M \epsilon \gamma \leq 2$: We reduce ourselves to the case $M \epsilon \gamma \geq 3$ by an explicit construction of divergences, followed by an application of the “main conclusion” of the super divergence formula:

In the setting of equations (3.34), (3.35) we denote by $U_{\Delta \phi}, U_{\nabla \phi,*}$ the index sets of complete contractions $C^u(g) \Delta \phi, C^u_\phi (\phi)$ with at least $\sigma - 2$ factors $R$ (hence for those complete contractions $\alpha$ can be $\sigma, \sigma - 1$ or $\sigma - 2$). We claim that there exist a vector fields as required in (3.34), (3.35), indexed in $H, H'$ below, so that:
\[
\sum_{u \in U_{\Delta \phi, +}} a_u C_g^u(\phi) - \text{div}_i \sum_{h \in H} a_h C_g^{h,i}(\Delta \phi) = (\text{Const})_* C^*(g) \Delta \phi + \text{div}_i \sum_{h \in H} a_h C_g^{h,i}(\phi) + \sum_{x \in X} a_x C_g^x(\phi),
\]
\[
\sum_{u \in U_{\Delta \phi, -}} a_u C_g^u(\phi) = (\text{Const})_* C^*(g) \Delta \phi + \sum_{x \in X} a_x C_g^x(\phi),
\]
\[
\sum_{u \in U_{\nabla \phi, +}} a_u C_g^u(\phi) - \text{div}_i \sum_{h \in H'} a_h C_g^{h,i}(\phi) = (\text{Const})_{\sharp} C_{\sharp}^* (g) \Delta \phi + \sum_{x \in X} a_x C_g^x(\phi),
\]
\[
\sum_{u \in U_{\nabla \phi, \delta > j-1, -}} a_u C_g^u(\phi) + \sum_{x \in X} a_x C_g^x(\phi).
\]

Here the complete contractions \(C^*, C^\sharp\) are explicit complete contractions in forms and respectively (their form depends on the value of \(\alpha\)). Moreover the complete contractions indexed in \(U_{\Delta \phi, -}, U_{\nabla \phi, \delta > j-1, -}\) have at least \(j\) internal contractions and also strictly fewer than \(\sigma - 2\) factors \(R\), and a factor \(\Delta \phi, \nabla \phi\) respectively.

Finally, we claim that there exists a vector field indexed in \(H''\) below as required in (3.35) so that:

\[
-(\text{Const})_* \nabla_i [C^*(g)] \nabla_i \phi + (\text{Const})_{\sharp} C_{\sharp}^* (g) - \text{div}_i \sum_{h \in H''} a_h C_g^{h,i}(\phi) = \sum_{u \in U_{\nabla \phi, \delta > j-1, -}} a_u C_g^u(\phi) + \sum_{x \in X} a_x C_g^x(\phi),
\]

with the same conventions as above in the RHS.

Clearly, if we can show these three Lemmas we will have then reduced ourselves to showing Lemma 3.2 under the additional assumption that each contraction with a factor \(\nabla \phi\) or \(\Delta \phi\) can have at most \(\sigma - 3\) factors \(R\); this case has already been settled.

Mini-Proof of (3.70), (3.71): The divergences needed for these equations are constructed explicitly. The (simple) technique we use is explained in great detail in section 3 of [5]. Consider the two factors \(T_1, T_2\) which are not in the form \(R\) (of the scalar curvature). Consider any particular contractions between these two factors. If one of the indices is a derivative index, we explicitly subtract the divergence corresponding to that index. The correction terms we get are either allowed in the RHSs of our equations, or have increased the number of internal contractions \(^3\) One can see that we can perform this explicit construction repeatedly to obtain the terms \(C^*, C^\sharp\) in the RHSs.

Mini-Proof of (3.72): We perform the same explicit construction as described above for the term \(-(\text{Const})_* \nabla_i [C^*(g)] \nabla_i \phi\), to derive (3.72) with an

\(^3\) We also apply the second Bianchi identity whenever necessary to create particular contractions as described above, whenever possible.
additional term \(-2(\text{Const})_* + (\text{Const})_\# C_g(\phi)\) in the RHS. Then, substituting this into the integral equation \(\int_{\mathcal{M}} Y_g(\phi) dV_g = 0\) (see the assumption of Lemma 3.2) and applying the main conclusion of the super divergence formula, we derive that \(-2(\text{Const})_* + (\text{Const})_\# = 0\). \(\square\)

References

[1] S. Alexakis On the decomposition of Global Conformal Invariants I, Ann. of Math. 170 (2009), no. 3, 1241–1306.

[2] S. Alexakis On the decomposition of Global Conformal Invariants II, Adv. in Math. 206 (2006), 466-502.

[3] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer I, arXiv.

[4] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer II, arXiv.

[5] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer III, arXiv.

[6] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer IV, arXiv.

[7] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer V, arXiv.

[8] S. Alexakis The decomposition of Global Conformal Invariants: A Conjecture of Deser an Schwimmer VI, arXiv.

[9] M. Atiyah, R. Bott, V. K. Patodi, On the heat equation and the index theorem Invent. Math. 19 (1973), 279–330.

[10] T. N. Bailey, M. G. Eastwood, A. R. Gover Thomas’s structure bundle for conformal, projective and related structures Rocky Mountain J. Math. 24 (1994), no. 4, 1191–1217.

[11] T. N. Bailey, M. G. Eastwood, C. R. Graham Invariant Theory for Conformal and CR Geometry Ann. of Math. (2), 139 (1994), 491-552.

[12] A. Čap, A. R. Gover Tractor calculi for parabolic geometries Trans. Amer. Math. Soc. 354 (2002), no. 4, 1511–1548.

[13] A. Čap, A. R. Gover Standard tractors and the conformal ambient metric construction Ann. Global Anal. Geom. 24 (2003), no. 3, 231–259.

[14] É. Cartan Sur la réduction à sa forme canonique de la structure d’un groupe de transformations fini et continu, Oeuvres Complètes 1, Part 1, 293-355, Gauthier-Villars, Paris, 1952.
[15] S. Deser, A. Schwimmer *Geometric classification of conformal anomalies in arbitrary dimensions*, Phys. Lett. B309 (1993) 279-284.

[16] C. Fefferman *Monge-Ampère equations, the Bergman kernel and geometry of pseudo-convex domains*, Ann. of Math. 103 (1976), 395-416; *Erratum* 104 (1976), 393-394.

[17] C. Fefferman, C. R. Graham *Conformal Invariants* Élie Cartan et les mathématiques d’aujourd’hui, Astérisque numero hors serie, 1985, 95-116.

[18] C. Fefferman, C. R. Graham *The ambient metric*, arXiv:0710.0919.

[19] C. R. Graham, K. Hirachi *Inhomogeneous ambient metrics*, Symmetries and overdetermined systems of partial differential equations, 403–420, IMA Vol. Math. Appl., 144, Springer, New York, 2008.

[20] K. Hirachi *Construction of Boundary Invariants and the Logarithmic Singularity of the Bergman Kernel* Ann. of Math. (2) 151 (2000), no. 2 151-191.

[21] K. Hirachi *Logarithmic singularity of the Szegő kernel and a global invariant of strictly pseudoconvex domains* Ann. of Math. (2) 163 (2006), no. 2, 499–515.

[22] T. Y. Thomas *The differential invariants of generalized spaces*, Cambridge University Press, Cambridge 1934.

[23] H. Weyl *The classical groups*, Princeton University press.