DGP Cosmological model with generalized Ricci dark energy

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The braneworld model proposed by Dvali, Gabadadze and Porrati (DGP) leads to an accelerated universe without cosmological constant or other form of dark energy. Nevertheless, we have investigated the consequences of this model when a Ricci-like holographic $\alpha H^2 + \beta \dot{H}$ is included as IR cutoff for the dark energy. Using the SNIa and $H(z)$ data sets, we estimate and constrain the holographic parameters ($\alpha, \beta$) for the late-time universe in the two branches of the DGP, as well as the scale $r_c$ at which the gravity leaks out into the bulk. We find a good fit to the data from the model, where best estimated values computed for the positive branch are: $\alpha = 0.88^{+7.55}_{-0.886}$, $\beta = 0.638^{+3.9}_{-0.63}$ and $r_c = 9.99^{+568.12}_{-9.9}$. And for the negative branch: $\alpha = 1.23^{+12.1}_{-1.2}$, $\beta = 0.813^{+6.31}_{-0.81}$ and $r_c = 6.00^{+332.97}_{-6.0}$. We find a strong correlation between the three parameters. On the other hand, we perform an analytical study of the model in the late and future time dark energy dominated epoch, where we obtain analytical solutions and find conditions in both branches of the holographic parameters and $r_c$ in order to the weak energy condition holds. We find that the best estimated values lies in the permitted region found in our analytical study of the model.
I. INTRODUCTION

The acceleration in the expansion of the universe during recent cosmological times, first indicated by supernova observations [1] and also supported by the astrophysical data obtained from WMAP, indicates the existence of a dark fluid with negative pressure, which have been identified as dark energy due to its unknown nature. Other non conventional approaches have advocated extra dimensions inspired by string and superstring theories. One of these models that have lead to an accelerated universe without cosmological constant or other form of dark energy is the braneworld model proposed by Dvali, Gabadadze, and Porrati (DGP) [2], [3], [4] (for reviews, see [5] and [6]). In a cosmological scenario, this approach leads to a late-time acceleration as a result of the gravitational leakage from a 3-dimensional surface (3-brane) to a fifth extra dimension on Hubble distances.

It is a well known fact that the DGP model has two branches of solutions: the self-accelerating branch and the normal one. The self accelerating branch leads to an accelerating universe without invoking any exotic fluid, but present problems like ghost [7]. Nevertheless, the normal branch requires a dark energy component to accommodate the current observations [8], [9]. Extend models of gravity on the brane with f(R) terms have been investigated to obtain self acceleration in the normal branch [10]. Solutions for a DGP brane-world cosmology with a k-essence field were found in [11] showing big rip scenarios and asymptotically de Sitter phase in the future.

In the present work we explore, in the framework of the holographic dark energy models [12], [13], [14], based on the holographic principle [15], which is believed to be a fundamental principle for the quantum theory of gravity, a DGP cosmology. Based on the validity of the effective quantum field theory, Cohen et al [12] suggested that the total energy in a region of size $L$ should not exceed the mass of a black hole of the same size, which means $\rho_A \leq L^{-2} M_p^2$. The largest $L$ is chosen by saturating the this bound so that we obtain

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the holographic dark energy (HDE) density

$$\rho_\Lambda = 3c^2 M_p^2 L^{-2},$$

(1)

where $c$ is a free dimensionless $O(1)$ parameter that can be determined by observations. Taking $L$ as the Hubble radius $H = H_0^{-1}$ this $\rho_\Lambda$ is comparable to the observed dark energy density, but gives wrong EoS for the dark energy [13].

For higher dimensional space-times, the holographic principle in cosmological scenarios has been formulated considering the maximal uncompactified space of the model, i.e. in the bulk, leading to a crossing of phantom divide for the holographic dark energy, in 5D two-brane models [16]. Other investigation shows that when IR cut-off is the event horizon the vacuum energy would end up with a phantom phase with an inevitable Big Rip singularity [17].

Recently, a modified holographic dark energy model has been formulated using the mass of black holes in higher dimensions and the Hubble scale as IR cutoff [18]. Using the future event horizon as IR cutoff, it was found in that the EoS of the holographic dark energy can cross phantom divide [19]. The inclusion of a Gauss-Bonnet term in the bulk and an holographic energy density have been explored in [20], obtaining a late time acceleration consistent with observations. In the same approach, but using a Ricci dark energy, scenarios free of future singularities were found in [21].

Our aim in this work is to investigate a DGP model of a flat universe filled with an holographic Ricci dark energy [22] and dark matter. Using the SNIa data set and the Hubble parameters for different redshifts we constraint the holographic parameters and also for the parameter $r_c H_0$, where $r_c$ is the characteristic scale of the DGP model given by

$$r_c = \frac{1}{2} \frac{M_{(4)}^2}{M_{(5)}^2},$$

(2)

which sets a length beyond which gravity starts to leak out into the bulk.

In the next section we treat the DGP model with null curvature and a barotropic equation of state. In section III we work in the late-time phase universe and solve numerically a differential equation for $E = H/H_0$, where $H$ is the Hubble parameter and we show the table with the best estimates for the holographic parameters and figures of the confidence regions which was obtained marginalizing some variables for each branches. In section IV we expose the main calculation to use the SNIa data set and the Hubble parameters for
different redshifts and we display the covariance matrix for each branches. In section V we change the time treat and now consider a dominating dark energy density, the cosmic evolution is then driving by only one fluid and we present how the future evolution will behave in terms of the redshift obtaining analytical expressions for the Hubble parameter and the scale factor, observing a “Big Rip” type singularity for positive branch. Finally we make the conclusions in the section VI.

II. DGP MODEL

For an homogeneous and isotropic universe described by the FLRW metric the field equation is given by \[3\], \[4\] (with \(8\pi G = c = 1\))

\[
3 \left( H^2 - \frac{\epsilon}{r_c} \sqrt{H^2 + \frac{k}{a^2}} \right) = \rho - \frac{3k}{a^2},
\]

where \(a\) is the cosmic scale factor, \(\rho\) is the total cosmic fluid energy density on the brane. The parameter \(\epsilon = \pm 1\) represents the two branches of the DGP model. It is well known that the solution with \(\epsilon = +1\) represent the self-accelerating branch, since even without dark energy the expansion of the universe accelerates, and for late times the Hubble parameter approaches a constant, \(H = 1/r_c\). In the previous investigation, \(\epsilon = -1\) has been named as the normal branch, where acceleration only appears if a dark energy component is included. By considering the null curvature case, the Eq.(3) becomes

\[
H^2 - \frac{\epsilon}{r_c} H = \rho,
\]

and the weak energy condition (WEC) implies \(r_c H \geq \epsilon\). If cosmic fluid satisfy a barotropic equation of state \(p = \omega \rho\), the conservation equation is given by

\[
\dot{\rho} + 3H (1 + \omega) \rho = 0.
\]

From Eqs. (4) and (5), we obtain an expression for equation of state parameter \(\omega\) in terms of the Hubble parameter which is given by

\[
1 + \omega = -\frac{1}{3} \left( \frac{2 - \epsilon (r_c H)^{-1}}{1 - \epsilon (r_c H)^{-1}} \right) \frac{\dot{H}}{H^2}.
\]

According to Eq. (6), \(1+\omega < 0\) implies that \(\dot{H} > 0\), since \((2 - \epsilon (r_c H)^{-1}) / (1 - \epsilon (r_c H)^{-1}) > 0\) for both cases \((r_c H)^{-1} \leq 1\) and we notice now that the WEC is strict. By using the
deceleration parameter defined by
\[ q = -\left(1 + \frac{\dot{H}}{H^2}\right), \quad (7) \]
we can write
\[ \frac{1 + \omega}{1 + q} = \frac{1}{3} \left(\frac{2 - \epsilon (r_c H)^{-1}}{1 - \epsilon (r_c H)^{-1}}\right), \quad (8) \]
From this above equation we notice that \( r_c H = 1 \) implies a divergence, so we only consider the case \( r_c H \neq 1 \) in the rest of the work. For today we obtain that \( r_c H_0 \) has the following expression
\[ r_c H_0 = \epsilon \left[\frac{3 (1 + \omega (0)) - (1 + q (0))}{3 (1 + \omega (0)) - 2 (1 + q (0))}\right] \neq 1. \quad (9) \]
In the next section, the condition \( r_c H_0 > 1 \) is used from the beginning when the this free parameter is constrained from observational data.

III. THE HOLOGRAPHIC DARK ENERGY AND MATTER COMPONENT

A. Dynamics of the model

We shall consider an holographic Ricci dark energy, so the holographic energy density takes the form
\[ \rho_h = \frac{3}{8\pi G} (\alpha H^2 + \beta \dot{H}), \quad (10) \]
where \( \alpha \) and \( \beta \) are positive constants. This type of holographic dark energy works fairly well in fitting the observational data. Nevertheless, a global fitting on the parameters of this model using a combined cosmic observations from type Ia supernovae, baryon acoustic oscillations, Cosmic Microwave Background and the observational Hubble data do not favor the holographic Ricci dark energy model over the \( \Lambda \)CDM model \[23\]. For the far future, the EoS behaves like a quintom model, crossing the phantom barrier \[24\], \[25\]. The statefinder diagnostic of this model, in the framework of general relativity, indicates that interactions in the dark sector are favored \[26\]. It was found that without giving a priori some specific model for the interaction function, this can experience a change of sign during the cosmic evolution \[27\].

For a spatially flat FRW universe composed by the holographic dark energy as well as a matter component (dark and baryon matter), the Friedmann equation \[4\] in the DGP
cosmology has the form (with units)

\[ H^2 - \epsilon \frac{H}{r_c} = \frac{8\pi G}{3}(\rho_h + \rho_m), \]  

(11)

where the subscripts “h” and “m” stand for holographic dark energy and matter respectively. The pressureless matter scale in the usual way, so \( \rho_m = \rho_{m0}a^{-3} \), where \( \rho_{m0} \) is the present-day value of the matter density in the Universe.

Inserting the expressions for \( \rho_h \) and \( \rho_m \) at eq. (11) and reorganizing terms, we have

\[ \beta \dot{H} - H^2(1 - \alpha) + \epsilon \frac{H}{r_c} + \left(\frac{8\pi G}{3}\right) \frac{\rho_{m0}}{a^3} = 0. \]

(12)

We change the derivative of \( H \) with respect to time to the scale factor as \( \dot{H} = (dH/da)\dot{a} = (dH/da)aH \), to obtain the differential equation

\[ \beta a \frac{dH(a)}{da} - (1 - \alpha)H(a) + \epsilon \frac{H}{r_c} + \left(\frac{8\pi G}{3}\right) \frac{\rho_{m0}}{a^3H(a)} = 0. \]

(13)

Dividing the eq. (13) by the Hubble constant \( H_0 \), defining the parameter density \( \Omega_{m0} \equiv \rho_{m0}/\rho_{\text{crit}}^0 \) where \( \rho_{\text{crit}}^0 \equiv 3H_0^2/(8\pi G) \), changing of variable from the scale factor to the redshift, and defining the dimensionless Hubble parameter as \( E \equiv H/H_0 \), the differential equation (13) becomes

\[ \beta(1 + z)\frac{dE(z)}{dz} + (1 - \alpha)E(z) - \frac{\epsilon}{r_cH_0} - \Omega_{m0} \frac{(1 + z)^3}{E(z)} = 0. \]

(14)

We solve numerically this differential equation with the initial condition \( E(z = 0) = 1 \), and for both branches, \( \epsilon = \pm 1 \). We consider \( \Omega_{m0} = 0.27 \), and the values of \( (\alpha, \beta, r_cH_0) \) are estimated and constrained using the cosmological observations of type Ia Supernovae and the Hubble parameter.
FIG. 1: Marginal credible regions for the parameters $(\alpha, \beta, r_c H_0, H_0)$, for the positive branch $\epsilon = +1$. They were computed using the combined SNe + $H(z)$ data sets. Each panel was calculated marginalizing over the other two free parameters. The best estimated values for the parameters are shown in table I. The credible regions shown correspond to 68.3\%(1\sigma), 95.4\%(2\sigma) and 99.73\%(3\sigma). For the three bottom panels it is also shown the 99.99\%(4\sigma) region. We notice a correlation among the parameters $(\alpha, \beta, r_c H_0)$; we calculated the following linear relationships: $0.527 \alpha + 0.2 = \beta$, $75 \alpha - 55 = r_c H_0$ and $142 \beta - 75 = r_c H_0$, for the top left, middle and right panels respectively. It means that as any of these three parameters increases its value, the other two will linearly increase too. The three bottom panels indicates that the values of $(\alpha, \beta, r_c H_0)$ are almost uncorrelated with $H_0$, i.e., the value that we may consider for $H_0$ will have a negligible effect on the constraints on $(\alpha, \beta, r_c H_0)$. Nevertheless, from the combined joint credible regions of $H_0$ with $\alpha$ and $\beta$ we obtain the constrains: $0 < \alpha < 32.3$ and $0 < \beta < 18.4$ at 4\sigma of confidence level (see bottom left and middle panels). There is a very large dispersion on the parameter $r_c H_0$, and given that $H_0$ alone is well constrained (see the three bottom panels), then the dispersion is due to the leakage length scale $r_c$. So, the SNe + $H(z)$ observations are not able to set more useful constraints on $r_c$. 
FIG. 2: Marginal credible regions for the parameters $(\alpha, \beta, r_c H_0, H_0)$, for the negative branch $\epsilon = -1$. They were computed using the combined SNe $+H(z)$ data sets. Each panel was calculated marginalizing over the other two free parameters. The credible regions shown correspond to $68.3\% (1\sigma)$, $95.4\% (2\sigma)$ and $99.73\% (3\sigma)$. For the three bottom panels it is also shown the $99.99\% (4\sigma)$ region. In a similar way to the case of positive branch, there is a correlation among the parameters $(\alpha, \beta, r_c H_0)$; we find that they follow the linear relationships: $0.517 \alpha + 0.2 = \beta$, $39 - 27 \alpha = r_c H_0$ and $49 - 51 \beta = r_c H_0$, for the top left, middle and right panels respectively. $\alpha$ and $\beta$ have a positive correlation. However, there are a negative correlation of $r_c H_0$ with $(\alpha, \beta)$, i.e., as $\alpha$ or $\beta$ increase their values then $r_c H_0$ decreases until becoming zero. Negative values of $r_c H_0$ are unphysical. Given this negative correlation of $r_c H_0$, and in particular with $\alpha$, it allows us to set the constraint $0 < r_c H_0 < 57$ at $3\sigma$ (see top middle panel); the maximum value is obtained in the limit situation of $\alpha = 0$, and the minimum value of zero when $\alpha \simeq 2.05$ or $\beta \simeq 1.55$, at $3\sigma$. The dispersion on $r_c H_0$, as for the case $\epsilon = +1$, is due again to the length scale $r_c$ only, since $H_0$ is tightly constrained (see the bottom panels). The three bottom panels indicates that the values of $(\alpha, \beta, r_c H_0)$ are almost uncorrelated with $H_0$. From the combined joint credible regions of $H_0$ with $\alpha$ and $\beta$ we obtain the constrains: $0 < \alpha < 55$ and $0 < \beta < 29$ at $4\sigma$ of confidence level (see bottom left and middle panels).
Best estimates

| $\alpha$           | $\beta$           | $r_c H_0$ | Branch ($\epsilon$) | $\chi^2_{\text{min}}$ | $\chi^2_{\text{d.o.f.}}$ |
|---------------------|-------------------|-----------|----------------------|-------------------------|---------------------------|
| $0.886^{+7.55}_{-0.886}$ | $0.638^{+3.99}_{-0.638}$ | $9.99^{+568.12}_{-9.9}$ | 1                     | 568.46                  | 0.967                     |
| $1.232^{+12.156}_{-1.23}$ | $0.813^{+6.31}_{-0.81}$ | $6.00^{+332.97}_{-6.0}$ | $-1$                 | 568.42                  | 0.967                     |

TABLE I: Best estimated values for the parameters ($\alpha, \beta, r_c H_0$), computed using the combined SNe + $H(z)$ data sets. The first three columns show the best estimated values. The fourth column indicates the assumed branch of the DGP brane cosmology, and the fifth and sixth columns show the minimum of the $\chi^2$ function and its corresponding $\chi^2$ function by degrees of freedom, $\chi^2_{\text{d.o.f.}}$. Negative values for ($\alpha, \beta, r_c H_0$) are not considered because they are unphysical or not allowed. The errors in the estimations are given to 68.3 % (1σ) of confidence level. Figures 1 and 2 and show the confidence intervals. The Hubble constant $H_0$ was marginalized assuming a flat prior distribution.

IV. COSMOLOGICAL PROBES

We test the viability of the model and constrain its free parameters ($\alpha, \beta, r_c H_0$) using the type Ia Supernovae (SNe Ia) observations and the Hubble parameter $H(z)$ measured at different redshifts. We compute their best estimated values, the goodness-of-fit of the model to the data and the credible regions by a $\chi^2$ function minimization, to constrain their possible values with levels of statistical confidence.

A. Type Ia Supernovae

We use the “Union2.1” SNe Ia data set (2012) from “The Supernova Cosmology Project” (SCP) composed by 580 type Ia supernovae [28]. Let us consider the definition of luminosity distance $d_L$ in a flat FRW cosmology

$$d_L(z, \alpha, \beta, r_c H_0) = \frac{c(1 + z)}{H_0} \int_0^z \frac{dz'}{E(z', \alpha, \beta, r_c H_0)},$$

(15)

where $E(z, \alpha, \beta, r_c H_0)$ is given by the numerical solution of the differential equation (14) and ‘c’ is the speed of light given in units of km/sec. The theoretical distance moduli for the $k$-th supernova with redshift $z_k$ is defined as

$$\mu^k(z_k, \alpha, \beta, r_c H_0) \equiv m - M = 5 \log_{10} \left[ \frac{d_L(z_k, \alpha, \beta, r_c H_0)}{\text{Mpc}} \right] + 25,$$

(16)
where \( m \) and \( M \) are the apparent and absolute magnitudes of the SNe Ia respectively, the superscript ‘t’ stands for “theoretical”. We construct the statistical \( \chi^2 \) function as

\[
\chi^2_{\text{SNe}}(\alpha, \beta) \equiv \sum_{k=1}^{n} \frac{[\mu^t(z_k, \alpha, \beta, r_c H_0) - \mu_k]^2}{\sigma_k^2},
\]

(17)

where \( \mu_k \) is the observational distance moduli for the \( k \)-th supernova, \( \sigma_k^2 \) is the variance of the measurement and \( n \) is the amount of supernova in the data set (\( n = 580 \)). The Hubble constant \( H_0 \) is marginalized assuming a constant prior distribution function.

### B. Hubble expansion rate

For the Hubble parameter \( H(z) \) measured at different redshifts, we use the 12 data listed in table 2 of Busca et al. (2012)\(^{29}\), where 11 data come from references \([30–32] \). We assumed \( H_0 = 70 \) km s\(^{-1}\) Mpc\(^{-1}\) for the data of Blake et al. (2011)\(^{30}\) as Busca et al. suggest. The \( \chi^2 \) function is defined as

\[
\chi^2_H(\alpha, \beta) = \sum_{i}^{12} \left( \frac{H(z_i, \alpha, \beta, r_c H_0) - H_i^{\text{obs}}}{\sigma_{H_i}} \right)^2,
\]

(18)

where \( H_i^{\text{obs}} \) and \( H(z_i, \alpha, \beta, r_c H_0) = H_0 \cdot E(z_i, \alpha, \beta, r_c H_0) \) are the observed and theoretical values of \( H(z) \) respectively. In this part, we assume \( H_0 = 73.8 \) km s\(^{-1}\) Mpc\(^{-1}\) as found by Riess et al. (2011)\(^{32}\); \( E(z, \alpha, \beta, r_c H_0) \) is given by the numerical solution of the differential equation \(^{14}\) and \( \sigma_{H_i} \) is the standard deviation of each \( H_i^{\text{obs}} \) datum.

We construct the total \( \chi^2_t \) function that combine the SNe and \( H(z) \) data sets together, as

\[
\chi^2_t = \chi^2_{\text{SNe}} + \chi^2_H,
\]

(19)

where \( \chi^2_{\text{SNe}} \) and \( \chi^2_H \) are given by expressions (17) and (18) respectively. We numerically minimize it to compute the “best estimates” for \((\alpha, \beta)\). The minimum value of the \( \chi^2 \) function gives the best estimated values and measures the goodness-of-fit of the model to data. We use also the definition of “\( \chi^2 \) function by degrees of freedom”, \( \chi^2_{\text{d.o.f.}} \), defined as

\[
\chi^2_{\text{d.o.f.}} \equiv \chi^2_{\text{min}}/(n - p)
\]

where \( n \) is the number of total combined data used and \( p \) the number of free parameters estimated.

Once computed the best estimated values for \((\alpha, \beta, r_c H_0)\) through the minimization of the \( \chi^2_t \) function (19), we compute also the covariance matrices \( C_+ \) and \( C_- \) for the positive
and negative branches, $\epsilon \pm 1$ respectively, for the parameters $(\alpha, \beta, r_cH_0)$, given in this order for the rows and columns in the matrices,

$$C_+ = \begin{pmatrix} 57.0846 & 30.1379 & 4291.13 \\ 30.1379 & 15.919 & 2264.72 \\ 4291.13 & 2264.72 & 322679 \end{pmatrix}, \quad C_- = \begin{pmatrix} 147.768 & 76.673 & -4047.28 \\ 76.673 & 39.793 & -2099.65 \\ -4047.28 & -2099.65 & 110871 \end{pmatrix}. \tag{20}$$

V. DARK ENERGY DOMINATION PHASE

In what follows we shall consider the late time evolution of our model, where the holographic dark energy density $\rho_h$ dominates. So, neglecting $\rho_m$ in (11) we can solve the equation for $r_cH(z)$

$$r_cH(z) = \frac{\epsilon}{1 - \alpha} + \left[r_cH_0 - \frac{\epsilon}{1 - \alpha}\right] (1 + z) - \frac{1 - \alpha}{\beta}, \quad \text{for } \alpha \neq 1. \tag{21}$$

The solution for the scale factor yields

$$a(t) = a_0 \left[ \frac{\epsilon}{1 - \alpha} e^{\frac{\epsilon}{1 - \alpha} (t - t_0)} \left[ \frac{\epsilon}{r_cH_0 - \left[r_cH_0 - \frac{\epsilon}{1 - \alpha}\right] e^{\frac{\epsilon}{1 - \alpha} (t - t_0)} \right] \right]^{\frac{1}{1 - \alpha}}. \tag{22}$$

Notice that $a(t)$ have a singularity at the time $t_s$ given by

$$t_s = t_0 + \frac{\beta r_c}{\epsilon} \ln \left(1 + \frac{\epsilon}{r_cH_0 - \frac{\epsilon}{1 - \alpha}}\right), \tag{23}$$

if the exponent $\beta/(1 - \alpha)$ is positive. From the expression for the acceleration

$$\frac{\ddot{a}}{a} = \frac{H^2}{\beta} \left(1 - \alpha + \beta - \frac{\epsilon}{r_cH}\right), \tag{24}$$

we can obtain the conditions upon the parameter $\alpha$ and $\beta$ in order to have an accelerated late time expansion. These conditions also ensuring do not violate WEC. The following results are obtained

Since the values found for the parameter with the SNIa Data plus the Cosmological Probes are $\alpha = 0.886$ and $\beta = 0.638$ for the branch $\epsilon = +1$, the conditions to have an
accelerated expansion at the holographic dark energy dominated phase are satisfied, and $\beta/(1-\alpha)$ is positive. Then for this branch the universe evolves to a Big Rip singularity. For this case, when $t \to t_s$ then $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ so the singularity is classified as Type I, a “Big Rip” singularity \cite{33}.

For the other branch, $\epsilon = -1$, the fitted values for $\alpha$ and $\beta$ also satisfy the conditions to have accelerated expansion at the dark energy domination. There is no future Big rip and the solution behaves represents a de Sitter like expansion for the very far future.

VI. DISCUSSION AND CONCLUSIONS

From the comparison of the model with the late time SNe Ia and $H(z)$ cosmological observations, we find that for the positive branch $\epsilon = +1$, there is a positive covariance among the three parameters ($\alpha, \beta, r_c H_0$), it means that as one of them increases its value the other two will increase too. The linear relationships that we computed from the credible regions shown in figure \ref{fig:credible_regions} are: $0.527\alpha + 0.2 = \beta$, $75\alpha - 55 = r_c H_0$ and $142\beta - 75 = r_c H_0$. Also, we find that the value of the Hubble constant $H_0$ has a negligible effect on the estimated values of $(\alpha, \beta, r_c H_0)$ given that the covariance of $H_0$ with respect to the other three parameters is very small (see matrix $C_+$ and figure \ref{fig:credible_regions}). Despite the almost negligible covariance between $H_0$ with the other parameters, we find that it allows us to set constraints on the magnitudes of the holographic parameters $(\alpha, \beta)$: the joint credible regions of $H_0$ with $\alpha$ and $\beta$ constrain them to be $0 < \alpha < 32.3$ and $0 < \beta < 18.4$ at $4\sigma$ of confidence level. Negative values of $(\alpha, \beta, r_c H_0)$ are not considered.

For the parameter $r_c H_0$, we find a very large dispersion on their possible values, even when it is constrained jointly with $H_0$. Since $H_0$ alone is well constrained in our results, then it turns out that the dispersion is due to the leakage length scale $r_c$ only. So, the SNe + $H(z)$ observations are not able to set more useful constraints on $r_c$ in the present work.
The only thing that we can realize about $r_c$ is the linear relationship with $(\alpha, \beta)$ as explained above.

For the negative branch, $\epsilon = -1$, we find that $\alpha$ and $\beta$ have a positive covariance but both of them have a negative covariance with $r_cH_0$. This means that as $\alpha$ increases its value, $\beta$ increases too, but $r_cH_0$ decreases. We find the following linear relationship among the parameters: $0.517\alpha + 0.2 = \beta$, $39 - 27\alpha = r_cH_0$ and $49 - 51\beta = r_cH_0$. The negative covariance between $\alpha$ and $r_cH_0$, as well as the fact that they can have positive values only, gives the constraint $0 < r_cH_0 < 57$ at $3\sigma$, that is a much better constraint than in the case of $\epsilon = +1$.

We find also a negligible covariance of $H_0$ with the other three parameters (see the covariance matrix $C_-$ and figure 2), indicating a negligible effect of the value of $H_0$ on the estimation and inference about the values of $(\alpha, \beta, r_cH_0)$. Also, from the combined joint credible regions of $H_0$ with $\alpha$ and $\beta$ we obtain the constrains: $0 < \alpha < 55$ and $0 < \beta < 29$ at $4\sigma$ of confidence level. Given that $\epsilon = -1$ is the normal branch, i. e., it is not a self-accelerated branch like $\epsilon = +1$, then the values of $(\alpha, \beta)$ are larger in this branch than in the positive one. This is expected because larger values of $(\alpha, \beta)$ implies a larger contribution of the holographic dark energy density, that is needed in the negative branch to accelerate the universe.

From the values for the parameters $\alpha$ and $\beta$ which best fit the cosmological data, we have found that for the late time phase, where the holographic dark energy density dominates, the normal branch with $\epsilon = -1$ leads to a de Sitter like expansion in the far future. On the other hand, the cosmological scenario in the positive branch presents a Big Rip singularity of Type I, according to the classification given in [33].

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