Stability of perturbed geodesics in \( n \)-dimensional axisymmetric spacetimes

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Abstract

The effect of self-gravity of a disk matter is evaluated by the simplest modes of oscillation frequencies for perturbed circular geodesics. We plotted the radial profiles of free oscillations of an equatorial circular geodesic perturbed within the orbital plane or in the vertical direction. The calculation is carried out to geodesics of an axisymmetric \( n \)-dimensional spacetime. The profiles are computed by examples of disks embedded in five-dimensional or six-dimensional spacetime, where we studied the motion of free test particles for three axisymmetric cases: (i) the Newtonian limit of a general proposed 5D and 6D axisymmetric spacetime; (ii) a simple Randall–Sundrum (RS) 5D spacetime; (iii) general 5D and 6D RS spacetime. The equation of motion of such particles is derived and the stability study is computed for both horizontal and vertical directions, to see how extra dimensions could affect the system. In particular, we investigate a disk constructed from Miyamoto–Nagai and Chazy–Curzon with a cut parameter to generate a disk potential. Those solutions have a simple extension for extra dimensions in case (i), and by solving vacuum Einstein field equations for a kind of RS–Weyl metric in cases (ii) and (iii). We find that it is possible to compute a range of possible solutions where such perturbed geodesics are stable. Basically, the stable solutions appear, for the radial direction, in special cases when the system has 5D and in all cases when the system has 6D and, for the axial direction, in all cases when the system has both 5D or 6D.

Keywords: stability study, axisymmetric solutions, extradimensions

(Some figures may appear in colour only in the online journal)
1. Introduction

A complete scheme to comprehend the dynamics of galaxies includes the study of many variables: the galactic shapes, the associated gravitational potentials and, as well, the galaxy main components such as gas, stars, dust, dark matter and maybe the central supermassive black hole. Particularly, the orbit behavior of the galaxy stars represents an important element to derive galactic gravitational potentials. In this aspect, the simplest scenario is based on a fundamental approximation: although galaxies are composed of stars, we shall neglect the forces from individual stars and consider only the large-scale forces from the overall mass distribution, neglecting small-scale irregularities due to individual stars or larger objects [1, 2].

The usual and most practical potentials to describe stellar circular orbits are the spherical, the axially symmetric and the bar-like form potentials. In the simplest case, the stars are moving in a static, spherically symmetric potential. This potential is the appropriate one for globular clusters, which are usually nearly spherical. However, few galaxies are even approximately spherical. Many real galaxies actually approximate figures of revolution and many of them have their stars confined to the equatorial plane of an axisymmetric configuration [3–5].

In the present contribution, we investigate configurations associated with axisymmetric potentials for general nD spacetimes with star orbits present only in the visible 3D space, focusing our calculations in the stability of perturbed stellar orbits. The motivation behind the nD consideration resides in the general introduction of spacetime extradimensions in theories like superstrings [6], braneworld gravity [7–12] and models of galaxies within a multi-dimensional universe [13–15]. In other words, we want to answer the question ‘could extradimensions affect the stability of the 3D orbits in the equatorial plane of an axisymmetric configuration’? In this aspect, compactified or warped extradimensions should represent perturbations that possibly could break the stability of the system. The possible presence of extra dimensions in the universe is one of the most astounding features of string theory. Despite the strong theory formalism, extra dimensions still remain unaccessible and obliterated to experiments. Since the presence of ten or more spacetime dimensions is one of the central conditions of string theory and M theory, it is not unrealistic to say that experimental observation or constraints on the extra dimensions properties would be a major advance in science. On the other hand, the lack of experimental evidence is usually explained by compactification which is the main geometric feature to explain why photons do not escape to the extra dimensions. Nevertheless, an alternative approach involves an extra dimension which is not compactified, as indicated by Randall–Sundrum (RS) [7–11]. This extra dimension implies deviations on Newton’s law of gravity at submillimetric scales, where objects may be indeed gravitating in more dimensions. The electromagnetic, weak and strong forces, as well as all the matter in the universe, would be trapped on a brane with three spatial dimensions. Only gravitons would be allowed to leave the surface and move into the full bulk, constituted by an anti-de Sitter—AdS5 spacetime, as prescribed by RS models [7, 8]. Here are the main motivations concerning the choice of RS as the metric to be tested in the present paper.

The stability of circular orbits in the equatorial plane can be studied both using an extension of the Rayleigh stability criterion [16] or a perturbative method based on inertial oscillations. In the Rayleigh criterion an orbit is perturbed by an infinitesimal change in the momentum of the test particle. Usually, the Rayleigh criterion is studied for a pure Newtonian circular orbit. On the other hand, oscillatory perturbative methods are based on the oscillations governed by the rotational restoring force and their frequencies. Such frequencies are characterized by the epicyclic frequency $\kappa$, defined by $\kappa^2 = 2\Omega(2\Omega + r\Omega'/dr)$, where $\Omega$ is
the angular velocity of the disk rotation. In this sense, the radial distribution of $\kappa$ is of importance in determining the behavior of oscillations. Concerning this radial distribution, general relativity has important roles. Namely, in general relativity the epicyclic frequency does not increase monotonically inward in the radial direction, but rather reaches a maximum at certain radius and then falls to zero at the radius of marginally stable circular geodesic [17, 18]. Oscillations outside the equatorial plane (vertical direction $z$) are also important and are decoupled from the radial oscillations.

Here, the stability will be investigated using the second method described above, namely the oscillatory perturbative method, for general axial configurations with $n$ dimensions. In this sense, the present work is divided as follows. Section 2 presents the equatorial circular geodesics in 4D axisymmetric fields, followed by the calculation of perturbations of such orbits, and the consequent investigation of their stability (section 3). Section 4 is devoted to the derivation of general equations that describe perturbed orbits of a $n$D axisymmetric configuration. Sections 5–8 present two examples to test the derived equation for the simplest case of a 5D configuration and two other more examples for 6D configurations. Explicitly, the following examples that will be treated here: (i) 3D orbits in the Newtonian limit case for a nDa xisi mmetry configuration with no compactification (section 5 for 5D and section 7 for 6D); (ii) 3D orbits for a pure RS metric, with Weyl axisymmetric terms, that has a volcano barrier potential to prevent matter getting out from the visible 3D space (that works as a compactification) (section 6.1); (iii) 3D orbits for a general RS–Weyl metric with the same assumption about compactification (section 6.2 for 5D and section 8 for 6D). In all cases, we introduce a cut method to generate a disk solution, such that the axial coordinate is transformed as $\xi = |z| + c$, where $c > 0$ is the cut parameter.

2. Equatorial circular geodesics in $(1 + 3)$ axisymmetric fields

As can be seen, e.g. in [18], rotating axisymmetric objects, in Weyl–Lewis–Papapetrou $1 + 3$ cylindrical coordinates $(t, \rho, \varphi, z)$, generate spacetimes described by the following metric [20]

$$ds^2 = -e^{2\xi}dt^2 + \rho^2B^2e^{-2\xi}(d\varphi - \omega dt)^2 + e^{2\lambda - 2\xi}(d\rho^2 + dz^2),$$

where $\xi, B, \omega$ and $\lambda$ are dependent functions of $\rho$ and $z$ only. The case where the source is an ideal fluid with zero pressure (dust) is the same as to fix $B = 1$, reducing the number of functions to three [21]. For a timelike worldline $x^\alpha(s)$ with four-velocity $u^\alpha = dx^\alpha/ds$ and angular velocity $\Omega = d\varphi/dt$, the specific angular momentum and the specific energy with respect to the rest frame (at spatial infinity) are given by

$$\ell = u_\varphi = u^t (g_{tt} + g_{\varphi\varphi} \Omega) = u^t \rho^2B^2e^{-2\xi}(\Omega - \omega),$$

$$E = -u_t = -u^t (g_{tt} + g_{\varphi\varphi} \Omega) = u^t e^{2\xi} + \omega \ell.$$  

Spatially circular orbits are the simplest type of worldline in a stationary axisymmetric field. It is also the most important worldline for the dynamics of astrophysical bodies. This particular case happens when $\rho = \text{const}, z = \text{const}$ and $\Omega = \text{const}$. The four-velocity is written as

$$u^\alpha = u^t(1, 0, \Omega, 0),$$

where

$$(u^t)^2 = -g_{tt} - 2g_{\varphi t} \Omega - g_{\varphi\varphi} \Omega^2 = e^{2\xi} - \rho^2B^2e^{-2\xi}(\Omega - \omega)^2 = (E - \Omega \ell)^2;$$
and the four-acceleration can be written as

\[ a_\alpha = -\frac{1}{2} g_{\beta\gamma,\alpha} u^\beta u^\gamma - \frac{u^\alpha t}{u^t} + u^\alpha \Omega_\alpha \ell = u^\alpha (E_\mu - \Omega L_\mu). \]

The equatorial symmetry is here defined as the reflectional symmetry with respect to a plane placed at \( z = 0 \). Naturally, only radial components of \( a_\alpha \) are non-null for circular orbits in the equatorial plane. Nevertheless, there are two particular cases of the orbital angular velocity where radial four-acceleration even vanishes:

\[ \Omega_\pm = -\frac{g_{t\varphi,\rho} \pm \sqrt{g_{t\varphi,\rho}^2 - g_{t\rho,\varphi} g_{\varphi\rho}}}{g_{\varphi\rho}}. \]  

(6)

Particularly, here the interest resides in static axisymmetric spacetimes, namely Weyl spacetimes, with no dragging \( \omega = 0 \) and null pressure \( B = 1 \), that reduces the number of unknown functions to only two, \( \xi (\rho, z) \) and \( \lambda (\rho, z) \). The metric is now

\[ ds^2 = -e^{2\xi}dt^2 + \rho^2 e^{-2\kappa}d\rho^2 + e^{2\lambda-2\xi}(d\rho^2 + dz^2). \]

(7)

In this case, the energy–momentum tensor satisfies \( T_\rho^\rho + T_z^z = 0 \) and the function \( \xi \) satisfies the Laplace equation and the Keplerian equatorial frequencies (6) read

\[ \Omega_\pm = \pm \frac{e^{2\xi}}{\rho} \sqrt{\rho \xi,\rho \over 1 - \rho \xi,\rho}. \]

(8)

with corresponding specific azimuthal angular momentum and specific energy respectively

\[ \ell = \pm \frac{\rho}{\varepsilon^2} \sqrt{\rho \xi,\rho \over 1 - 2 \rho \xi,\rho}, \quad E = \varepsilon \sqrt{1 - \rho \xi,\rho \over 1 - 2 \rho \xi,\rho}. \]

(9)

### 3. Stability of circular orbits in axisymmetric 4D fields

The stability of circular orbits in the disk plane can be studied using an extension of the Rayleigh stability criterion [16], or a perturbative method where we assume that the disk particles are describing equatorial circular geodesics in stationary axisymmetric fields. A general relativistic equivalent method to the Rayleigh criterion comes from a perturbative method based in radial or vertical oscillations of the test particle. In 4D, this method is derived in [14, 18]. Here we assume that a stable system is one in which the internal and external forces are such that any small perturbation results in forces that return the system to its prior state. In such a manner, we are interested in investigating the stability of perturbed geodesics for axisymmetric orbits. The geodesic equation in a 4D pattern is

\[ \dddot{x}^\alpha + \Gamma_\mu^\alpha \dddot{x}^\mu = 0. \]

(10)

Here the perturbation of the geodesic equation is done performing \( x^\alpha \rightarrow x^\alpha + \Delta^\alpha \)—where \( \Delta^\alpha = (b t, \delta \rho, b \varphi, b z) \). Substituting this map in equation (10), we have

\[ \dddot{x}^\alpha + \dddot{\Delta}^\alpha + \Gamma_\mu^\alpha \dddot{x}^\mu = 0, \]

(11)

\[ \dddot{x}^\alpha + \dddot{\Delta}^\alpha + \left[ \Gamma_\mu^\alpha (x) + \frac{\partial}{\partial x^\mu} \Gamma_\mu^\alpha \right] [\dddot{\Delta}^\nu + \dddot{\Delta}^\nu] = 0. \]

(12)
Using (10) we isolate only the perturbative part

$$\Delta^\alpha + \Gamma^\alpha_{\mu\nu}\Delta^\mu + \Gamma^\alpha_{\nu\mu}\Delta^\nu + \frac{\partial}{\partial x^\beta}\Gamma^\alpha_{\mu\nu}\Delta^\beta = 0,$$  \hspace{2cm} (13)

and how \(\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}\) we finally derive geodesic equations for perturbations

$$\Delta^\alpha + 2\Gamma^\alpha_{\mu\nu}\Delta^\mu + \Gamma^\alpha_{\mu\nu,\beta}\Delta^\beta = 0.$$  \hspace{2cm} (14)

Taking the general 4D case where the coordinates in the plane of the disk are given by \(x^\mu = (t, \rho, \varphi, z)\), and the axisymmetric metric is \(ds^2 = -e^{2\xi}dt^2 + \rho^2d\varphi^2 + e^{2\zeta}(d\rho^2 + dz^2)\). Making \(B = 1\) and \(\omega = 0\), as discussed in the previous section, we have, for this case, that the non-null Christoffel symbols are \(\Gamma^t_{\varphi\varphi}, \Gamma^t_{\varphi t}, \Gamma^z_{\varphi t}, \Gamma^z_{\varphi z}, \Gamma^t_{\varphi t}, \Gamma^t_{\varphi \varphi}, \Gamma^z_{\varphi t}, \Gamma^z_{\varphi \varphi}\), and four equations are derived

\[
\begin{align*}
(\delta t) + 2(\Gamma^t_{\rho t} + \Gamma^t_{\varphi \varphi})u'(\delta \rho) &= 0, \\
(\delta \rho) + 2(\Gamma^\rho_{\rho t} + \Gamma^\rho_{\varphi \varphi})u'(\delta t) + 2(\Gamma^\rho_{\varphi \varphi} + \Gamma^\rho_{\varphi \varphi})Ru' \delta \rho &= 0, \\
(\delta \varphi) + 2(\Gamma^\varphi_{\varphi t} + \Gamma^\varphi_{\varphi \varphi})u'(\delta \rho) &= 0, \\
(\delta z) + (\Gamma^z_{\varphi t} + 2\Gamma^z_{\varphi \varphi} + \Gamma^z_{\varphi \varphi})u'(\delta \rho) &= 0,
\end{align*}
\]

(15)–(18)

where \(\delta \xi = u^\mu = u'(1, 0, \Omega, 0)\), and \(u'\Omega = V_C\), \(\Omega = [\Omega_L]\), given by equation (8) in the Weyl axisymmetric configuration of section 2. From the proper frequency of the harmonic-oscillator equation (18) it is possible to write down the following angular frequency \(\tau_\perp\) with respect to radial infinity, provided that the harmonic-oscillator equation in the vertical (z) direction is decoupled from the others (and it is, see equation (18)):

$$\tau_\perp^2 = \Gamma^\varphi_{\varphi t} + 2\Gamma^\varphi_{\varphi \varphi} + \Gamma^\varphi_{\varphi \varphi} \Omega^2.$$  \hspace{2cm} (19)

For the axisymmetric metric presented in section 2, i.e. developing the Christoffel symbols from metric (7) and from the Keplerian frequency (8), oscillations in z direction read

$$\tau_\perp^2 = \frac{e^{4\xi - 2\lambda}}{1 - \rho e^{2\lambda}} \xi_{\varphi \varphi}. $$  \hspace{2cm} (20)

To evaluate oscillations at the radial directions, suppose that the solutions for \(\delta t, \delta \rho\) and \(\delta \varphi\) also have a form of harmonic oscillations proportional to \(e^{i\xi_t}\), where \(K\) is the proper angular frequency and \(s\) is the proper time. The condition for solvability of equations (15)–(17) is

$$\begin{vmatrix}
-K^2 & 2iK\Gamma^t_{\rho t}u' & 0 \\
2iK\Gamma^\rho_{\rho t}u' - K^2 + \Gamma^\rho_{\alpha \beta}u'^\alpha w^\beta & 2iK\Gamma^\rho_{\varphi \varphi}u' \Omega & K^2 \\
0 & 2iK\Gamma^\rho_{\varphi t}u' \Omega & -K^2 
\end{vmatrix} = 0.$$  \hspace{2cm} (21)

4. General perturbed motion equations in nD spacetimes

Now we need to work on calculations concerning nD spacetimes (with \(A, B, C, D = 0, \ldots, n\)), where the geodesic equations for perturbations should be written as
\[ \Delta^A + 2 \Gamma^A_{BC} \dot{x}^B \Delta^C + \Gamma^A_{BC.D} \Delta^D \dot{x}^C = 0, \]  
(22)

where \( \Gamma^A_{BC} \) are the Christoffel symbols and \( \dot{x}^A \) are proper time derivatives \( dx^A/ds \). To find effective equations for perturbations in terms only of visible fields, i.e. what are the effective expressions in terms of visible fields, it is necessary to analyze a general nD metric to derive a Lagrangian. The detailed steps on how to develop the motion equations can be seen in [19]. Here the novel results are the perturbed equations that will appear at the middle of the present section, with major posterior developments in sections 5–9. Also, at the end of this section, after deriving the perturbed motion equations, we subsequently calculate the epiciclic frequency and the orthogonal frequency.

The most general metric for such a universe is given by

\[
g(x^\alpha) = \begin{pmatrix}
g_{\alpha\beta} & g_{\alpha\beta} \\
g_{\alpha\beta} & g_{\alpha\beta}
\end{pmatrix},
\]

(23)

where \( \alpha, \beta = 0, \ldots, 3 \) and \( a, b = 4, \ldots, n \). Furthermore we consider the convention to make the metric as a function of only \( 3 + 1 \) coordinates: \( g_{AB} = g_{AB}(x^\alpha) \). These metric components \( g_{AB} \) contain the \( 3 + 1 \) universe metric terms \( g_{\alpha\beta} \) and the extra dimensional terms \( g_{ab} \), as well as the crossed components. Equation (23) can be rewritten for convenience as

\[
g_{AB} = g_{\alpha\beta} \delta^A_B + g_{ab} \delta^A_A \delta^B_B + g_{\alpha\beta} \delta^A_A \delta^B_B + g_{ab} \delta^A_A \delta^B_B,
\]

(24)

where \( \delta^A_B \) are the Kronecker symbols. The derivatives for such metric components are given by

\[
g_{AB,C} = g_{\alpha\beta,\gamma} \delta^A_B \delta^\gamma_C + g_{ab,\gamma} \delta^A_B \delta^\gamma_C + g_{\alpha\beta,\gamma} \delta^A_B \delta^\gamma_C + g_{ab,\gamma} \delta^A_B \delta^\gamma_C
\]

(25)

and the derivatives are straightforwardly provided by equation (24)

\[
g_{AB,C} = g_{\alpha\beta,\gamma} \delta^A_B \delta^\gamma_C + g_{ab,\gamma} \delta^A_B \delta^\gamma_C
\]

(26)

Assuming that the spacetime has a connection presenting no torsion, one yields the following Christoffel symbols

\[
\Gamma^A_{BC} = \frac{1}{2} g^{AM} (g_{BM,C} + g_{CM,B} - g_{BC,M}).
\]

SPLITTING this last expression by equations (25) and (26) it reads

\[
\Gamma^A_{BC} = \Gamma^n_{BC} \delta^n_A \delta^n_B \delta^n_C + \frac{1}{2} \left[ g^{AM} (g_{BM,C} + g_{CM,B} - g_{BC,M}) - g^{AM} g_{BC,M} \right].
\]

(27)

The Ricci tensor components are

\[
R_{AB} = \partial_B \Gamma^M_{AB} - \partial_B \Gamma^M_{AM} + \Gamma^N_{AB} \Gamma^M_{NM} - \Gamma^N_{AM} \Gamma^M_{NB},
\]

Taking into account that the terms of the metric depends solely on \( x^\alpha \), the equation above reads

\[
R_{AB} = R_{\alpha\beta} \delta^A_B \delta^B_A + \delta^B_A \delta^A_B.
\]

(28)

The stress tensor can be derived from the conventional definition \( T_{AB} = -\frac{\partial L_m}{\partial g_{AB}} + g_{AB} L_m \). Now, the Lagrangian for the gravitating test particles in a spacetime with extra dimensions, can be derived as [19]
\[ L = (g_{AB} \dot{x}^A \dot{x}^B)^{1/2} = (g_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta + g_{ab} \dot{x}^a \dot{x}^b)^{1/2}, \]  

(29)

where \( \dot{x}^A = dx^A/ds \). The motion equations come from the Euler–Lagrange expression \( \frac{d}{ds} \frac{\delta L}{\delta \dot{x}^C} - \frac{\partial L}{\partial x^C} = 0 \). As \( \partial_A = \partial_{\alpha} \delta_A^\alpha + \partial_{\beta} \delta_A^\beta \) and \( g_{ab} = g_{ab}(x^\nu) \) it follows that

\[
\frac{\partial L}{\partial x^C} = \frac{\partial L}{\partial x^C} \delta^C_C + \frac{\partial L}{\partial x^C} \delta^C_C,
\]

\[
\frac{\partial L}{\partial x^\nu} = \frac{1}{2} L^{-1}(g_{\alpha \beta, \gamma} \dot{x}^\alpha \dot{x}^\beta + g_{ab, \gamma} \dot{x}^a \dot{x}^b),
\]

and \( \frac{\partial L}{\partial x^C} = 0 \). It immediately yields

\[
\frac{\partial L}{\partial x^C} = \frac{1}{2} L^{-1}(g_{\alpha \beta, \gamma} \dot{x}^\alpha \dot{x}^\beta + g_{ab, \gamma} \dot{x}^a \dot{x}^b).
\]

Likewise, the term \( \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^C} \) can be developed:

\[
\frac{\partial L}{\partial \dot{x}^C} = \frac{1}{2} L^{-1}(g_{\alpha \beta, \gamma} \dot{x}^\alpha \dot{x}^\beta + g_{ab, \gamma} \dot{x}^a \dot{x}^b),
\]

\[
\frac{\partial L}{\partial \dot{x}^\nu} = \frac{1}{2} L^{-1}(g_{\alpha \beta, \gamma} \dot{x}^\alpha \dot{x}^\beta + g_{ab, \gamma} \dot{x}^a \dot{x}^b).
\]

Now

\[
\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}^C} \right) = L^{-1} \left[ \frac{\partial g_{\alpha \beta}}{\partial x^\nu} \dot{x}^\alpha \dot{x}^\beta + g_{\nu \gamma} \dot{x}^\nu \right].
\]

Also, one can write the integration constants

\[ g_{ab} \dot{x}^m = N_c, \]

(30)

since \( x^a \) are cyclic variables. Hence \( \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}^C} \right) = 0 \). Inserting the terms together, multiplying by \( L g^{\nu \gamma} \) and using (30) the equations of motion are derived:

\[
\dot{x}^\alpha + \Gamma^\alpha_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} g_{ab, \gamma} g^{\alpha \beta} N_c g^{ab} g^{bd}.
\]

(31)

Clearly a plausible interpretation is that the extra dimensions induce an external ‘force’ in the system, that depends only on \( g_{ab} \) and \( N_c \).

Up to this point we developed the major steps explained in [19] to obtain the equations of motion. From this moment we will derive the perturbed form of equation (31), highlighting the importance and the novelty of this new equation to evaluate the stability and the behavior of classical particles moving in, for example, axisymmetric orbits endowed with extra imprints. So, the perturbed form for equation (22) is properly split by doing \( x^\alpha \rightarrow x^\alpha + \Delta^\alpha \) in equation (31), yielding

\[
\dot{x}^\alpha + \Delta^\alpha + \Gamma^\alpha_{\mu \nu}(x + \Delta) [\dot{x}^\mu + \Delta^\mu][\dot{x}^\nu + \Delta^\nu] = V^\alpha(x + \Delta),
\]

(32)

\[
\dot{x}^\alpha + \Delta^\alpha + \Gamma^\alpha_{\mu \nu}(x) \left[ \dot{x}^\mu + \Delta^\mu \right][\dot{x}^\nu + \Delta^\nu] = V^\alpha + \frac{\partial}{\partial x^\nu} V^\nu \Delta^\nu,
\]

(33)

where \( V^\alpha = \frac{1}{2} g_{ab, \gamma} g^{\alpha \beta} N_c g^{ab} g^{bd} \) is the extradimensional signature. Using (31) we isolate only the perturbative part
\[ \Delta^\alpha + \Gamma^\alpha_{\mu \nu} \delta^\mu \delta^\nu + \Gamma^\alpha_{\mu \nu} \Delta^\mu \Delta^\nu + \frac{\partial}{\partial x^\beta} \Gamma^\alpha_{\mu \nu} \Delta^\beta \delta^\nu - \frac{\partial}{\partial x^\beta} V^\alpha \Delta^\beta = 0, \quad (34) \]

where we expanded the Christoffel symbols and the metric in terms of perturbations \( \Delta^\beta \) (desconsidering second order perturbations) and as \( \Gamma^\alpha_{\mu \nu} = \Gamma^\alpha_{\nu \mu} \) we finally derive the following equations for motion perturbations

\[ \Delta^\alpha + 2 \Gamma^\alpha_{\mu \nu} \delta^\mu \delta^\nu + \Gamma^\alpha_{\mu \nu, \beta} \Delta^\beta \delta^\nu - V^\alpha \Delta^\nu = 0. \quad (35) \]

Note that the term \( V^\alpha \Delta^\nu \) contains all the information about perturbations that can be carried out by extradimensions. Note also that for \( V^\alpha = 0 \), equation (35) recovers the original 4D harmonic-oscillator, i.e. equation (14). In complement, equation (35) actually contains four equations describing the perturbations of the visible field. Considering that perturbations have the form \( \delta x \sim e^{i k}, \) consequently the equations for \( \delta x^0, \delta x^1, \delta x^2 \) give the following condition for the solvability of the proper angular frequency \( K \):

\[
\begin{align*}
- K^2 & \quad 2iK [\Gamma_{01}^0 + \Gamma_{21}^0] u^0 \\
- K^2 & \quad 2iK [\Gamma_{01}^2 + \Gamma_{21}^2] u^2 \\
0 & \quad 2iK [\Gamma_{01}^1 + \Gamma_{21}^1] u^1
\end{align*}
\]

\[ \det \begin{pmatrix} - K^2 & 2iK [\Gamma_{01}^0 + \Gamma_{21}^0] u^0 & 0 \\
- K^2 & 2iK [\Gamma_{01}^2 + \Gamma_{21}^2] u^2 & 0 \\
0 & 2iK [\Gamma_{01}^1 + \Gamma_{21}^1] u^1 & - K^2 \end{pmatrix} = 0, \quad (36) \]

where

\[ V_{1,1} = \frac{1}{2} g_{ab,11} g_{cd} N_a N_b g_{cd} + \frac{1}{2} g_{ab,11} g_{cd} N_a N_d g_{bd} \]

The fourth perturbation equation is the same as equation (18). Note that if there are no crossed terms \( g_{02} \) or \( g_{20} \) of the metric (resulting \( \Gamma_{02}^1 = \Gamma_{20}^1 = \Gamma_{01}^1 = \Gamma_{10}^1 = 0 \) and if there are no extradimensions (resulting \( V_{1,1} = 0 \)), the expression (36) becomes the same as (21).

To evaluate oscillations at the radial directions, we calculate the determinant (36) supposing that the solutions for \( \delta t, \delta \rho, \) and \( \delta \phi \) also have a form of harmonic oscillations proportional to \( e^{i k}, \) where \( K \) is the proper angular frequency and \( s \) is the proper time. In this way, the epicyclic frequency \( \kappa^2 = K^2/(u^s)^2 \) and the perturbations \( \tau^2 \) in direction \( x^3 \) are respectively calculated as

\[ \kappa^2 = (\Gamma_{21}^2 - 4 \Gamma_{21}^1 \Gamma_{21}^0 + 4 \Gamma_{21}^0)^2 + (2 \Gamma_{20}^1 - 4 \Gamma_{21}^0 \Gamma_{21}^0 - 4 \Gamma_{21}^0)^2 \]

\[ - 4 \Gamma_{21}^0 \Gamma_{01}^0 - 4 \Gamma_{21}^0 \Gamma_{21}^0 \Omega + \Gamma_{00}^1 - 4 \Gamma_{01}^0 \Gamma_{01}^0 - 4 \Gamma_{01}^0 \Gamma_{01}^0 - V_{1,1}^1/(u^s)^2, \quad (38) \]

\[ \tau_1^2 = \Gamma_{00}^3 + 2 \Gamma_{02}^3 \Omega + \Gamma_{22}^3 \Omega^2. \quad (39) \]

5. Example 1: General perturbations for 5D metric in the Newtonian limit (without compactification)

The main aim now is to compute the gravitational potential in the Newtonian limit, since galaxies and clusters can be described physically as Newtonian objects—corresponding to the approximation in which gravity is weak. The weak limit is assumed uniquely in the four-dimensional spacetime: the deviation \( \gamma_{a3} \) of the four-dimensional metric \( g_{a3} = \eta_{a3} + \gamma_{a3} \) is small (\( \eta_{a3} \) denotes the Minkowski metric). Linearized gravity has a gauge freedom given by \( \gamma_{a3} \rightarrow \gamma_{a3} + \xi \eta_{a3} \), where \( \xi \) denotes the Lie derivative with respect to the generators \( \zeta^a \) of a differential diffeomorphism. To the first order, such transformation represents the same
physical transformation as $\gamma_{\alpha\beta}$. This gauge freedom is used to simplify the linearized Einstein equation. Solving the equation $\partial^\mu \partial_\mu \xi = - \partial^\mu \xi_{\alpha\beta}$ for $\xi_{\alpha\beta}$, a gauge transformation that leads to $\partial^\mu \xi_{\alpha\beta} = 0$—similar to the Lorentz gauge condition—can be elicited to obtain the simplified Einstein equation

$$T_{\alpha\beta} = - \frac{1}{4} \partial^\mu \partial_\mu \pi_{\alpha\beta},$$

(40)

and

$$\Sigma_{\alpha\beta} = \frac{1}{2} \left[ \left( g^{mn} \partial^\mu \partial_\mu g_{mn} \right) \eta_{\alpha\beta} - g^{mn} g_{mn,\alpha\beta} \right],$$

(41)

where $\Sigma_{\alpha\beta}$ refers to terms of the stress tensor dependent on extra terms $g_{ab}$ of the metric.

When gravity is weak, the linear approximation to GR should be valid. There exists a global inertial coordinate system of $h_{\alpha\beta}$ such that

$$T_{\alpha\beta} = T_{\alpha\beta} + \Sigma_{\alpha\beta} \approx \rho, t_3,$$

(42)

$$- \frac{1}{4} \partial^\mu \partial_\mu \pi_{\alpha\beta} + \frac{1}{2} \left[ \left( g^{mn} \partial^\mu \partial_\mu g_{mn} \right) \eta_{\alpha\beta} - g^{mn} g_{mn,\alpha\beta} \right] = \rho, t_3,$$

(43)

where $t_3$ is the time direction associated with this coordinate system. This equation can be interpreted as the modified Poisson equation considering a universe with more than 3 + 1 dimensions.

Define $\pi_{\alpha\beta} = -4\phi$, where $\phi = \phi(\vec{x})$ is a three-space scalar field. Furthermore consider a line element $ds^2 = \sum_{i=4}^{n} e^{\psi_i}dz_i^2$, where $ds^2$ is the world line for the extra sector, $z_i$ denotes the extra coordinates and $\psi_i = \psi_i(\vec{x})$ are potentials associated with extra dimensions.

If one asserts, as a first approximation the sigma model $g^{\mu\nu}(\sigma, \sigma^{-1})_\psi = 0$ for the extra part, where $\sigma$ denotes the diagonal matrix representing the metric associated to the system, we have

$$\partial^\mu \partial_\mu g_{ab} = 0,$$

(44)

yielding the following equation

$$- \frac{1}{4} \partial^\mu \partial_\mu \pi_{\alpha\beta} - \frac{1}{2} g^{mn} g_{mn,\alpha\beta} = \rho, t_3,$$

(45)

or in other words

$$\nabla^2 \phi = \rho.$$

(46)

It means that our visible matter density profile is provided uniquely by the four-dimensional field. In the 5D Newtonian limit one can write the following line element

$$d\sigma^2 = -(1 - 2\phi)d\tau^2 + d\vec{r} \cdot d\vec{r} + e^{-\phi} dy_1^2,$$

(47)

where $d\vec{r} \cdot d\vec{r}$ is the three-dimensional line element and $y_1$ is the extradimension. In cylindrical coordinates the 3D line element will be $d\vec{r} \cdot d\vec{r} = dr^2 + r^2 d\phi^2 + dz^2$.

To find a form for those functions $\phi$ and $\psi$, from (44) and (46) it yields

$$\nabla^2 \psi - \nabla \psi \cdot \nabla \psi = 0,$$

(48)
Nonlinear terms do not appear, since the $\sigma$ matrix is diagonal. In particular, equation (48) can be rewritten as

\[
\nabla^2 \phi = \rho.
\]

Figure 1. (a) The plots show $\kappa^2$ as a function of the $r$ coordinate. The system is stable if $\kappa^2 > 0$. When $N_{k1} \to 0$, the curves are stable. Larger the values of $N_{k1}$, more instabilities are present. (b) The plots show $\kappa^2$ as a function of the $r$ coordinate. The system is stable if $\kappa^2 > 0$. (c) The plot shows $\tau_+^2$ as a function of $r$ and $z$ for axisymmetric coordinates in the Newtonian limit in 5D and 6D to $a = 0$ and cut parameter is $c = 1$ (satisfying energy conditions). The system is stable when $\tau_+^2 > 0$. Nonlinear terms do not appear, since the $\sigma$ matrix is diagonal. In particular, equation (48) can be rewritten as
\[ \nabla^2 \chi = 0, \quad (50) \]

where the identification \( \chi = e^{-\psi} \) is accomplished.

Simple solutions for those functions are, for example

\[
\phi = -\frac{m}{\sqrt{r^2 + (\bar{z} + a)^2}}, \quad (51)
\]

and

\[
\chi = \frac{2m}{r}, \quad (52)
\]

where the coordinate \( \bar{z} = |z| + c \) introduces a cut method to generate a disk solution, where \( c > 0 \) is the cut parameter. Here, \( a \) is a general constant, and the above solutions are given for a particle of mass \( m \) in the position \( z_0 \). The solution (52) gives

The epicyclic frequency is calculated from equation (38) as

\[
\kappa^2(r) = \left[ 3 - \frac{1}{2} N_0^2 r^2 e^\psi \left( \frac{d^2 \psi}{dr^2} + \frac{d\psi}{dr} \right)^2 \right] \Omega^2 - \frac{4}{1 - 2\phi} \left( \frac{d\phi}{dr} \right)^2
- \frac{d^2 \phi}{dr^2} + \frac{1}{2} e^\psi (1 - 2\phi) N_0^2 \left[ \frac{d^2 \psi}{dr^2} + \left( \frac{d\psi}{dr} \right)^2 \right], \quad (54)
\]

where the squared angular velocity \( \Omega^2 \) is calculated from \( \Omega^2 (\psi^0)^2 = F(r) = (-g_{22}/g_{00})(\dot{\psi}^2/r^2) \) (see e.g. [14]), from (5) and (30) as

\[
\Omega^2 = \frac{F(r)(1 - 2\phi)}{1 + F(r)r^2}, \quad (55)
\]
with
\[
F(r) = \frac{r}{1 - \frac{2\phi}{r}} \left[ \left( e^\phi \partial_r \phi + \frac{(e^\phi \partial_r \psi)^2}{2} - \phi e^\psi \partial_r \psi - \partial_r \phi \right) N_{\parallel}^2 \right] \left( \frac{e^\psi \partial_r \psi}{2} - e^\psi + \frac{1}{N_{\parallel}^2} \right). \tag{56}
\]

The orthogonal perturbation $\tau^2_\perp$ is calculated simply from (39) as
\[
\tau^2_\perp = -\frac{\partial^2 \phi}{\partial z^2}, \tag{57}
\]
where in this last case $\tau^2_\perp$ is plotted for all $z > 0$ and $r > 0$. Figures 1(a) and (c) show respectively the curves for $\kappa^2$ and $\tau^2_\perp$ for some values of $N_{\parallel}$.

6. Example 2: 5D RS (compactified-like)

6.1. Pure RS

The RS metric is in general expressed as
\[
dx^2 = e^{-2\lambda \gamma} g_{\mu\nu} \, dx^\mu dx^\nu + dy^2, \tag{58}
\]
where $k^2 = 3/(2\ell^2)$, and the term $e^{-2\lambda \gamma}$ is called the warp factor [7, 8], which reflects the confinement role of an extradimensional anti-de Sitter bulk constant $\Lambda$ that prevents gravity from leaking into the extra dimension at low energies. The term $|y|$ provides the $Z_2$ symmetry of the three-brane at $y = 0$ and RS metric can be regarded as an alternative to compactification.

Here we will assume that $g_{\mu\nu}$ is the Weyl axisymmetric metric described in (7). The perturbed geodesics of this pure RS with Weyl coordinates is represented in figure 2 for $N_{\parallel} = 0$, that is a particular case of the second example below.

6.2. General RS

Another possibility is that one can assume a general RS metric with Weyl–Lewis–Papapetrou coordinates written as
\[
dx^2 = e^{-2\lambda \gamma} g_{\mu\nu} \, dx^\mu dx^\nu + e^{-\psi} dy^2, \tag{59}
\]
with the assumption discussed in section 4, i.e. that $g_{\nu m} x^m = N_{\gamma}$, with $g_{\gamma \gamma} = e^{-\psi}$. When $N_{\gamma} = 0$, it is recovered the original pure RS metric (58). With $g_{\mu\nu}$ given by (7) with solutions
\[
\lambda = \xi = -\frac{m}{\sqrt{r^2 + (\xi + a)^2}}, \tag{60}
\]
and
\[
\psi = \frac{2m}{r}. \tag{61}
\]

Here we have introduced the same cut method as before to generate a disk solution, such that $\xi = |z| + c$, where $c > 0$ is the cut parameter. The epicyclic $\kappa^2$ frequency calculated from equation (38) as
\[ \kappa^2(r) = \left[ 3 - 4r \left( \frac{dl}{dr} \right) + 2r^2 \left( \frac{dl}{dr} \right)^2 + r^4 \frac{d^2l}{dr^2} \right] e^{-2\lambda} \Omega^2 \]
\[ \quad - \left[ 2 \left( \frac{dl}{dr} \right)^2 - \frac{d^2l}{dr^2} \right] e^{2\lambda} = e^{2\lambda} V_{t,1}^2 + e^{-2\lambda} r^2 V_{r,1}^2 \Omega^2. \] (62)

The squared angular velocity \( \Omega \) is
\[ \Omega^2 = \frac{e^{2\lambda} F(r)}{1 + r^2 e^{-2\lambda} F(r)}. \] (63)

and
\[ F(r) = \frac{H(r)}{W(r)}. \]

with
\[ H(r) = r e^{-4\lambda}[2N_r e^\psi(\partial_r \lambda) - N_z e^\psi(\partial_z \lambda) - 2(\partial_r \lambda)], \] (64)
\[ W(r) = 2e^{-4\lambda} - 2rN_r e^{-4\lambda} e^\psi(\partial_r \lambda) + N_z r e^{-4\lambda} e^\psi(\partial_z \lambda), \] (65)
\[ + 2r e^{-4\lambda}(\partial_r \lambda) + 2N_z e^{-4\lambda} - 4rN_z (\partial_r \lambda) e^{-4\lambda} + 4r (\partial_r \lambda) e^{-4\lambda}, \] (66)

and
\[ V_{t,1}^2 = -\frac{1}{2} N_z^2 e^{-4\lambda} (\partial_z \psi)^2 - \frac{1}{2} N_z^2 e^{-4\lambda} (\partial_r \psi)^2 - 2N_z^2 e^\psi (\partial_r \lambda) (\partial_r \psi). \] (67)

The orthogonal perturbations are calculated from (39) as
\[ \tau_{\perp}^2 = [2\lambda_z - 2\xi_z + \xi_z^2 + 2(\xi_z^2)] e^{4\psi - 2\lambda} + [\xi_z^2 - 2\lambda_z \xi_z] e^{-2\lambda} \Omega^2. \] (68)

Figures 2 and 3 show respectively the curves for \( \kappa^2 \) and \( \tau_{\perp}^2 \) for some values of \( N_{t,1} \). The system is stable if \( \kappa^2 > 0 \) and \( \tau_{\perp} > 0 \). When \( N_{t,1} \to 0 \), the curves are stable both for the radial
and orthogonal perturbations for $r > 0.7$. Larger the values of $N_{y1}$, more instabilities are present.

### 7. Example 3: 6D Newtonian limit for an axial configuration

The metric for the Newtonian limit presented in section 5 can be expanded for the case of a 6D configuration as

$$d s^2 = - (1 - 2\phi) dr^2 + d\vec{x} \cdot d\vec{x} + e^{-\psi} dy_1^2 + e^{\psi} dy_2^2,$$

where $d\vec{x} \cdot d\vec{x}$ is the three-dimensional line element and $y_1$ and $y_2$ are the extradimensional coordinates. In cylindrical coordinates the 3D line element will be $d\vec{x} \cdot d\vec{x} = dr^2 + r^2 d\phi^2 + dz^2$.

From (44) and (46), simple solutions for those functions are (where, as stated before, $\zeta$ generates the disk solution)

$$\phi = - \frac{m}{\sqrt{r^2 + (\zeta + a)^2}},$$

and

$$\chi = \frac{2m}{r},$$

where $a$ is a constant, and both solutions are given for a particle of mass $m$ in the position $z \to 0$. In this case

$$\psi = -\ln \chi.$$

The epicyclic $\kappa^2$ frequency is calculated from equation (38) as

$$\kappa^2(r) = (3 + r^2 V_{11}^2) \Omega^2 - \frac{4}{1 - 2\phi} \left(\frac{d\phi}{dr}\right)^2 - \frac{d^2\phi}{dr^2} - V_{11}^2 (1 - 2\phi),$$

where $V_{11}^2$ and the squared angular velocity $\Omega^2$ are calculated as

$$V_{11}^2 = - \frac{1}{2} N_{y1}^2 e^{\psi}(\partial_r \psi)^2 - \frac{1}{2} N_{y2}^2 e^{-\psi}(\partial_r \psi)^2 - \frac{1}{2} N_{y1}^2 e^{\psi}(\partial_r^2 \psi) + \frac{1}{2} N_{y2}^2 e^{-\psi}(\partial_r^2 \psi),$$

$$\Omega^2 = \frac{(1 - 2\phi) F(r)}{1 + r^2 F(r)},$$

$$H(r) = N_{y1}^2 e^{\psi}[\partial_r (\partial_r \psi) - 2\phi (\partial_r \psi) - 2(\partial_r \phi)]$$

$$+ N_{y2}^2 e^{-\psi}[\partial_r (\partial_r \psi) - 2\phi (\partial_r \psi) + 2(\partial_r \phi)] - 2(\partial_r \phi)$$

with

$$F(r) = \frac{r}{1 - 2\phi} \left[ N_{y1}^2 e^{\psi}[r (\partial_r \psi) + 2] - N_{y2}^2 e^{-\psi}[r (\partial_r \psi) - 2] + 2 \right].$$

The orthogonal perturbation $\tau^2_\perp$ is calculated simply as

$$\tau^2_\perp = \frac{\partial^2 \phi}{\partial \zeta^2}.$$


where in this last case $\tau_1^2$ is plotted for all $z > 0$ and $r > 0$. Figures 1(b) and (c) show respectively the curves for $\kappa_2$ and $\tau_1^2$ for some values of $N_1$ and $N_2$. Note that in this case, there are no differences between 5D and 6D perpendicular perturbations.

8. Example 4: 6D RS

We propose a 6D general RS metric written as

$$\text{d}s^2 = e^{-2\lambda}\text{g}_{\mu\nu} \text{d}x^\mu \text{d}x^\nu + \text{e}^{\psi} \text{d}y_1^2 + \text{e}^{-\psi} \text{d}y_2^2. \quad (79)$$

Here it is assumed again that $g_{\mu\nu}$ is the Weyl axisymmetric metric described in (7), with the assumption discussed in section 4, i.e. that $g_{55}x^5 = N_1$ and $g_{66}x^6 = N_2$, with $g_{55} = \text{e}^\psi$ and $g_{66} = \text{e}^{-\psi}$. The term $|y|$ in the warp factor $e^{-2\lambda}|y|$ is $|y| = \sqrt{y_1^2 + y_2^2}$. The solutions are

$$\lambda = \xi = -\frac{m}{\sqrt{r^2 + (\xi + a)^2}}, \quad (80)$$

and

$$\psi = \frac{2m}{r}. \quad (81)$$

The epicyclic $\kappa^2$ frequency calculated from equation (38) is

$$\kappa^2(r) = \left[3 - 4r \left(\frac{d\lambda}{dr}\right) + 2r^2 \left(\frac{d\lambda}{dr}\right)^2 + r^2 \left(\frac{d^2\lambda}{dr^2}\right)^2\right] e^{-2\lambda} \Omega^2 \quad (82)$$

$$-\left[2 \left(\frac{d\lambda}{dr}\right)^2 - \frac{d^2\lambda}{dr^2}\right] e^{2\lambda} - e^{2\lambda} \psi_1^2 + e^{-2\lambda} \psi_1^2 \Omega^2, \quad (83)$$
with

\[ V^1_{1,1} = -\frac{1}{2} N_{12}^2 \mathcal{e}^{-\psi} (\partial_r \psi)^2 - \frac{1}{2} N_{12}^2 \mathcal{e}^{\psi} (\partial_r \psi)^2 \quad \text{and} \quad -\frac{1}{2} N_{12}^2 \mathcal{e}^{-\psi} (\partial_r \psi)^2. \] (84)

The squared angular velocity \( \Omega \) is

\[ \Omega^2 = \frac{e^{2\lambda} F(r)}{1 + r^2 e^{-2\lambda} F(r)}. \] (85)

and

\[ F(r) = \frac{H(r)}{W(r)}. \]

with

\[ H(r) = r e^{-4\lambda} [(2 N_{11} \mathcal{e}^{-\psi} + 2 N_{12} \mathcal{e}^{\psi}) (\partial_r \lambda) - (N_{11} \mathcal{e}^{\psi} - N_{12} \mathcal{e}^{-\psi}) (\partial_r \psi) - 2 (\partial_r \lambda)], \] (86)

\[ W(r) = 2 e^{-4\lambda} - 2 r (N_{11} \mathcal{e}^{\psi} + N_{12} \mathcal{e}^{-\psi}) (e^{-4\lambda} \partial_r \lambda) + (N_{11} \mathcal{e}^{\psi} - N_{12} \mathcal{e}^{-\psi}) r e^{-4\lambda} (\partial_r \psi) \]
\[ + 2 r e^{-4\lambda} (\partial_r \lambda) - 2 (N_{11} + N_{12}) e^{-4\lambda} - 4 r (N_{11} + N_{12}) (\partial_r \lambda) e^{-4\lambda} \]
\[ - 4 r (\partial_r \lambda) e^{-4\lambda}. \] (87)

The orthogonal perturbations are calculated from (39) as

\[ \tau_2^2 = [2 \lambda \gamma_2 - 2 \xi_{z\gamma} + \xi_{\gamma2} + 2 (\xi_{z\gamma})^2] e^{4\xi z} - 2 \lambda \gamma_2 \xi_{z\gamma} e^{-2\lambda} \Omega^2. \] (88)

Figures 4 and 5 show the behavior of \( \kappa^2 \) and \( \tau_2^2 \) for some values of \( N_{11} \) and \( N_{12} \).
9. Discussion and concluding remarks

In the present work perturbative terms were calculated explicitly in particle motion due to the presence of extradimensions (compactified or not) in an axially symmetric configuration. We showed that extradimensions add terms to the original perturbative equation of classical particle geodesics for any geometry. First of all, we calculated the equation of motion \( x^a \rightarrow x^a + \Delta x^a \), it is possible to find the equation (35) for perturbations in particle geodesic motion with the presence of extradimensions. The term \( V^a_{\mu} \Delta \xi^i \) arises and we test the epicyclic radial frequency \( \kappa^2 \) (38) and axial oscillations \( \tau^2 \) (39) for Weyl metric in cylindrical coordinates for 5D and 6D configurations. We showed that when \( \kappa^2 > 0 \) and \( \tau^2 > 0 \) the system is stable both in radial as in axial directions.

Five metrics were used to calculate \( \kappa^2 \) and \( \tau^2 \), namely (i) the Newtonian limit of a general proposed 5D (section 5, equations (54) and (57)) and 6D (section 7, equations (73) and (78)) axisymmetric spacetimes; (ii) a simple RS 5D spacetime (section 6.1); (iii) general 5D (section 6.2, equations (62) and (68)) and 6D (section 8, equations (82) and (88)) RS spacetimes.

In all cases, the solutions for the metric potentials that are used to compute the oscillations have a reflection symmetry in the axial coordinate to create an infinite thin disk of matter. This occurs because we have the coordinate \( \xi = |z| + c \) in the solutions, where \( c \) is the disk cut parameter. The matter in the disk comes from the discontinuity in the stress tensor when \( z \to 0 \), so \( \partial_z|z| = 2\vartheta(z) - 1 \) and \( \partial_z|z| = 2\delta(z) \), where \( \vartheta(z) \) and \( \delta(z) \) are, respectively, the Heaviside function and the Dirac distribution. Therefore, the Einstein field equations will be separated in two different pieces: one valid for \( z > 0 \) (the usual Einstein equations), and the other involving distributions with an associated energy–momentum tensor. Due to the discontinuous behavior of the derivatives of the metric tensor across the disk, the Riemann curvature tensor contains Dirac delta functions. The energy–momentum tensor can be obtained by the distributional approach due to Papapetrou and Hamounti [23], Lichnerowicz [24], and Taub [25]. It can be written as \( T^\alpha_\beta = \delta^\alpha_\beta \delta(z) \), where \( \delta \) is the Dirac function with support on the disk and \( |T^\alpha_\beta| \) is the distributional energy–momentum tensor, which yield the volume energy density and the principal stresses. The disk at \( z = 0 \) divides the space–time into two halves. The normal to the disk can be described by the co-vector \( n_\alpha = \partial_z/\partial_x^\alpha = (0, 0, 0, 1) \). Above the disk near \( z = 0 \), we can expand the metric as

\[
g_{\alpha\beta} = g_{\alpha\beta} + \frac{\partial g_{\alpha\beta}}{\partial z} \bigg|_{z=0} z + z^2 \frac{\partial^2 g_{\alpha\beta}}{\partial z^2} \bigg|_{z=0} + ..., \]

and below \( z = 0 \),

\[
g_{\alpha\beta} = g_{\alpha\beta} + \frac{\partial g_{\alpha\beta}}{\partial z} \bigg|_{z=0} z + z^2 \frac{\partial^2 g_{\alpha\beta}}{\partial z^2} \bigg|_{z=0} + .... \]

The quantity \( g_{\alpha\beta} \) means the value of \( g_{\alpha\beta} \) at \( z = 0 \). The discontinuities in the first derivatives of the metric tensor can be cast as \( h_{\alpha\beta} = g_{\alpha\beta}|_{z=0} - g_{\alpha\beta}|_{z=0} \) in such a manner that \( \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} (h^{\alpha}_{\gamma} h_{\beta\gamma} + h^{\alpha}_{\beta} h_{\gamma\gamma} - g^{\alpha\beta} h_{\gamma\gamma}) \) where \( \Gamma^{\alpha}_{\beta\gamma} = \Gamma^+_{\alpha\beta\gamma} - \Gamma^-_{\alpha\beta\gamma} \) at \( z = 0 \). In this way, we can identify the distributional energy–momentum tensor on the disk through Einstein equations as \( [R^\alpha_\beta] - \frac{1}{2} h^{\alpha}_{\beta} [R] = 8\pi [T^\alpha_\beta] \). Then the distributional energy–momentum tensor is given by \( [T^\alpha_\beta] = \frac{1}{16\pi} \{ h^{\alpha}_{\gamma} h_{\beta\gamma} - h^{\alpha}_{\beta} h_{\gamma\gamma} + g^{\alpha\beta} h_{\gamma\gamma} - g^{\alpha\beta} h_{\gamma\gamma} + h^\gamma_{\gamma} (g^{\alpha\beta} h_{\gamma\gamma} - g^{\alpha\beta} h_{\gamma\gamma}) \} \). The energy density \( \epsilon \) and pressures \( p_\alpha \), in the disk are calculated for the developed axisymmetric
configurations as \( \epsilon = -[T^r_\epsilon], \ p_2 = [T^r_{\epsilon z}], \ p_3 = [T^r_{\epsilon y}] = 0, \ p_4 = [T^r_{\epsilon z}] = 0. \) The energy conditions are always satisfied for the four examples below. Specifically, in examples 2 and 4 it is satisfied when the extra coordinate \( y \to 0, \) i.e. when we analyze the stability in the 3D disk.

Concerning the method to calculate the oscillations and the stability of the system, in general, it is verified that extra dimensions contribute to destabilize the disk, but stability is verified for some cases. In what follows, we present a summary of the most important points to be discussed from the mentioned examples.

Example 1—Newtonian limit of a general proposed 5D axial symmetry (section 5): figures 1(a) and (c) show respectively the curves for \( \kappa^2 \) and \( \tau^1_{ry} \) for some values of \( N_{y1} \) (integration constant thanks to extradimension) in the case where the 5D axisymmetric system is in the Newtonian limit. The system is stable if \( \kappa^2 > 0 \) and \( \tau^1_{ry} > 0. \) When \( N_{y1} \to 0, \) the curves are stable both for the radial and orthogonal perturbations. In this case, the 4D expected stability is recovered. The larger the values of \( N_{y1}, \) the more instabilities are present. It is possible to see this indeed in expression (54) since \( N_{y1} \) is associated with negative terms of \( \kappa^2. \) Nevertheless, there are indeed some values of \( N_{y1} \) where the perturbed axial system presents stability inside a region of the axial system between \( 0 < r < r_t. \) For example, figure 1(a) shows the other two cases. When \( N_{y1} = 0.20, \ r_t \approx 7. \) When \( N_{y1} > 0.30 \) the results show that \( \kappa^2 < 0 \) for all values of \( r, \) and therefore the system for such a case is unstable. The perpendicular perturbations of figure 1(c) show the system is stable for all \( r. \) In fact, from 5D axisymmetric Newtonian limit, it is possible to conclude that one extradimension indeed carries instabilities for the system in the radial direction and does not carry instabilities in the axial direction.

Example 2—general 5D RS with axial symmetry (section 6): figures 2 and 3 show respectively the curves for \( \kappa^2 \) and \( \tau^1_{ry} \) for some values of \( N_{y1} \) in the case where the 5D axisymmetric system is RS. The system is stable if \( \kappa^2 > 0 \) and \( \tau^1_{ry} > 0. \) When \( N_{y1} \to 0, \) the curves are stable both for the radial and orthogonal perturbations if \( r > 0.7 \) (this is the pure RS system). The larger the values of \( N_{y1}, \) the more instabilities are present. It is possible to see this indeed in expression (67) since \( N_{y1} \) is associated with negative terms of \( \kappa^2. \) Nevertheless, there are indeed some values of \( N_{y1} \) where the axial system presents stability. For example, for values \( N_{y1} \leq 0.10, \) the system is stable inside a region of the axial system, between \( r_1 < r < r_2. \) In the case of \( N_{y1} = 0.01, \ r_1 \approx 0.5 \) and \( r_2 \approx 0.8. \) When \( N_{y1} > 0.10 \) the results show that \( \kappa^2 < 0 \) for all values of \( r, \) and therefore the system for such case is unstable. The perpendicular perturbations of figure 3 are plotted for \( N_{y1} = 0.20 \) and indicate that in the disk plane \( (z \to 0) \) the system is stable only in the radial \( r > 2 \) range. When \( z \) acquires greater values (both negative or positive), this range of stable radial regions is also greater. From 5D axisymmetric RS, it is possible to conclude that one extradimension carries instabilities for the system in the radial direction and does not carry instabilities in the axial direction. Also, the 5D RS case presents more instabilities than the 5D Newtonian axisymmetric system.

Example 3—Newtonian limit of a general proposed 6D axial symmetry (section 7): figures 1(b) and (c) show respectively the curves for \( \kappa^2 \) and \( \tau^1_{ry} \) for some values of \( N_{y1} \) and \( N_{y2} \) in the case where the 6D axisymmetric system is in the Newtonian limit. The system is stable if \( \kappa^2 > 0 \) and \( \tau^1_{ry} > 0. \) When \( N_{y1} \to 0 \) and \( N_{y2} \to 0, \) the curves are stable both for the radial and orthogonal perturbations (if nevertheless \( r > 2.2). \) The results show that any values of \( N_{y1} \) and \( N_{y2} \) give stable results. In fact, there is a small range of \( r^2 \)s (between 0 and \( \sim 2.1 \)) that represents an unstable region. This is a central region of a Miyamoto–Nagai gravitational potential \( \phi = -\frac{m}{\sqrt{r^2 + (z + c)^2}} \) (with \( z = |z| + c), \) and such instabilities are indeed expected [22]. The perpendicular perturbations of figure 1(c) indicate that the system is always stable.
From 6D axisymmetric Newtonian limit, it is possible to conclude that two extradimensions does not carry instabilities for the system in the radial direction and also does not carry instabilities in the axial direction. There are only local instabilities represented in figure 1(b) e.g. by the peaks around \( r \approx 9 \) (\( N_{x1} = 0.20 \) and \( N_{x2} = 0.20 \)) and around \( r \approx 5 \) (\( N_{y1} = 0.30 \) and \( N_{y2} = 0.10 \)). Also it is important to highlight that \( \tau^2 \) is the same both for 5D and 6D Newtonian limit cases.

Example 4—general 6D RS with axial symmetry (section 8): figures 4 and 5 show respectively the curves for \( \kappa^2 \) and \( \tau^2 \) for some values of \( N_{y1} \) and \( N_{y2} \) in the case where the 6D axisymmetric system is RS. The system is stable if \( \kappa^2 > 0 \) and \( \tau^2 > 0 \). When \( N_{y1} = N_{y2} \rightarrow 0 \), the curves are stable both for the radial and orthogonal perturbations if \( r \gtrsim 0.6 \) (this is the pure RS system). The results show that any values of \( N_{y1} \) and \( N_{y2} \) give stable results. In fact, there is a small range of \( r \)'s (between 0 and \( \sim 0.5 \)) that represents an unstable region. This is a central region of a Miyamoto–Nagai gravitational potential \( \lambda = -\frac{m}{\sqrt{r^2 + (z+c)^2}} \) (with \( z = |z| + c \)), and such instabilities are indeed expected [22]. The perpendicular perturbations of figure 5 are plotted for this limit case when \( N_{y1} = 0.20 \) and \( N_{y2} = 0.20 \) and indicate that in the disk plane (\( z \rightarrow 0 \)) the system is stable for all \( r \). When \( z \) acquires greater values, this range of stable radial regions is also greater. In this sense, one can conclude that two extradimensions carry instabilities but fewer instabilities than the 5D RS case. From 6D axisymmetric RS configuration, it is possible to conclude that two extradimensions does not carry instabilities for the system in the radial direction and also does not carry instabilities in the axial direction. There are only local instabilities represented in figure 4 e.g. by the peaks around \( r \approx 0.9 \) (\( N_{y1} = 0.02 \) and \( N_{y2} = 0.02 \)), around \( r \approx 1.3 \) (\( N_{y1} = 0.05 \) and \( N_{y2} = 0.10 \)) and around \( r \approx 2.25 \) (\( N_{y1} = 0.20 \) and \( N_{y2} = 0.20 \)).

In all situations we have introduced a cut method to generate a disk solution and energy conditions are satisfied for cut parameters \( c \geq 1 \). In all stable examples cited above, the \( \kappa \) frequency radial distribution follows what is expected for general relativity, i.e. the epicyclic frequency does not increase monotonically inward in the radial direction, but rather reaches a maximum at certain radius and then falls to zero at the radius of marginally stable circular geodesic. The results of the present work are important to contribute to all astrophysical solutions that retrieve axisymmetric configurations living in an \( n \)D universe. For example, the impact of extra \( D \) perturbations at AGN disks, galaxies in general, accretion around stellar black holes, etc. Several complementary discussions about the implication of this both in astrophysics and in cosmology can be seen e.g. in [10–14].

An important final observation is that stable solutions were found only for the 6D cases and this actually coincides with some arguments in favor of an even \( n \)D, e.g., as has been emphasized by different authors that the Huygens principle does not hold for odd \( n \)D (see [26, 27]).

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