Formal Groups, Elliptic Curves, and Some Theorems of Couveignes

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Abstract. The formal group law of an elliptic curve has seen recent applications to computational algebraic geometry in the work of Couveignes to compute the order of an elliptic curve over finite fields of small characteristic ([2], [6]). The purpose of this paper is to explain in an elementary way how to associate a formal group law to an elliptic curve and to expand on some theorems of Couveignes. In addition, the paper serves as background for [1]. We treat curves defined over arbitrary fields, including fields of characteristic two or three. The author wishes to thank Al Laing for a careful reading of an earlier version of the manuscript and for many useful suggestions.

1 Definition and construction of formal group laws

Let $R$ be a commutative ring with a multiplicative identity 1 and let $R[[X]]$ denote the ring of formal power series of $R$. In general it is not possible to compose two power series in a meaningful way. For example, if we tried to form the composition $f \circ g$ with $f = 1 + \tau + \tau^2 + \tau^3 + \cdots$ and $g = 1 + \tau$ we would get

$$f \circ g = 1 + (1 + \tau) + (1 + \tau)^2 + (1 + \tau)^3 + \cdots$$

The constant term is $1 + 1 + 1 + \cdots$, which makes no sense. But there are some cases where $f \circ g$ does make sense, namely when $f$ is a polynomial or when the constant term of $g$ is zero. Let $R[[X,Y]] = R[[X]][[Y]]$, the ring of formal power series in two variables. If $F \in R[[X,Y]]$ and $g,h \in \tau R[[\tau]]$ then

$$F(g,h) \quad \text{makes sense and belongs to } R[[\tau]].$$

If in addition $F$ has a zero constant term, then $F(g,h) \in \tau R[[\tau]]$.

A one dimensional (commutative) formal group law over $R$ is a power series $F \in R[[X,Y]]$ with zero constant term such that the “addition” rule on $\tau R[[\tau]]$ given by

$$g \oplus_F h = F(g,h)$$

makes $\tau R[[\tau]]$ into an abelian group with identity 0. In other words, for every $g, h$ we must have $(f \oplus_F g) \oplus_F h = f \oplus_F (g \oplus_F h)$ (associative law), $f \oplus_F g = g \oplus_F f$ (commutative law), $f \oplus_F 0 = f$ (0 is identity), and for each $f \in \tau R[[\tau]]$ there exists $g \in \tau R[[\tau]]$ such that $f \oplus_F g = 0$ (inverses). Denote this group by $C(F)$.

An equivalent and more widely known definition is the following: a formal group
law over \( R \) is a power series \( F(X,Y) \in R[[X,Y]] \) such that

\[
\begin{align*}
(i) \quad F(X,0) &= X; & \text{(Additive Identity)} \\
(ii) \quad F(X,Y) &= F(Y,X) & \text{(Commutative Law)} \quad (1.1) \\
(iii) \quad F(F(X,Y),Z) &= F(X,F(Y,Z)) & \text{(Associative Law)}.
\end{align*}
\]

The first property implies that \( F \) has the form \( X + YH(X,Y). \) By symmetry in \( X \) and \( Y, \) it must therefore be of the form

\[ F(X,Y) = X + Y + XYZG(X,Y), \quad G \in R[[X,Y]]. \quad (1.2) \]

**Proposition 1.1** Let \( F \) be a power series in two variables with coefficients in \( R \) such that \( F(0,0) = 0. \) The following are equivalent.

1. The three conditions in (1.1) hold;
2. The binary operation on \( \tau R[[\tau]] \) defined by \( f \oplus_F g = F(f,g) \) makes \( \tau R[[\tau]] \) into an abelian group with identity \( 0; \)
3. The binary operation on \( \tau R[[\tau]] \) defined by \( f \oplus_F g = F(f,g) \) makes \( \tau R[[\tau]] \) into an abelian semigroup with identity \( 0. \)

**Proof.** We will show (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1). Assume (1) holds. Define a binary operation on \( \tau R[[\tau]] \) by \( f \oplus_F g = F(f,g) \) for \( f, g \in \tau R[[\tau]]. \) The three conditions immediately imply \( f \oplus_F 0 = f, \) \( f \oplus_F g = g \oplus_F f, \) and \( f \oplus_F (g + h) = f \oplus_F g + f \oplus_F h \) for \( f, g, h \in \tau R[[\tau]]. \) It remains only to prove the existence of inverses. For this, it suffices to prove there is a power series \( \iota \in \tau R[[\tau]] \) such that \( F(\iota \circ g) = 0 \) for all \( g \in \tau R[[\tau]]. \) Let \( \iota^{(1)} = -\iota. \) By (1.2) \( F(\tau, \iota^{(1)}) \equiv \tau - \iota \equiv 0 \pmod{\tau^2}. \) Now assume inductively that \( \iota^{(N)} \in \tau R[[\tau]] \) satisfies \( F(\tau, \iota^{(N)}) \equiv 0 \pmod{\tau^{N+1}} \) and \( \iota^{(N)} \equiv \iota^{(N-1)} \pmod{\tau^N}. \) Then there is \( a \in R \) such that \( F(\tau, \iota^{(N)}) \equiv a\tau^{N+1} \pmod{\tau^{N+2}}. \) Let \( \iota^{(N+1)} = \iota^{(N)} - a\tau^{N+1} \). By (1.2)

\[ F(\iota^{(N)}, -a\tau^{N+1}) \equiv \iota^{(N)} - a\tau^{N+1} = \iota^{(N+1)} \pmod{\tau^{N+2}}. \]

Thus

\[
F(\tau, \iota^{(N+1)}) \equiv F(\tau, F(\iota^{(N)}, -a\tau^{N+1})) = F(F(\tau, \iota^{(N)}), -a\tau^{N+1}) \\
\equiv F(\tau, \iota^{(N)}) - a\tau^{N+1} \equiv 0 \pmod{\tau^{N+2}}.
\]

This completes the induction. Let \( \iota \in \tau R[[\tau]] \) be the power series such that \( \iota \equiv \iota^{(N)} \pmod{\tau^{N+1}} \) for all \( N. \) Then \( F(\tau, \iota(\tau)) = 0, \) and hence \( F(x, \iota(x)) = 0 \) for all \( x \in \tau R[[\tau]]. \) This proves (1) \( \Rightarrow \) (2). It is obvious that (2) \( \Rightarrow \) (3).

Now assume (3) holds. We will prove condition (iii) of (1.1) holds; the other conditions in (1.1) can be proved similarly. Let \( G(X,Y,Z) = F(F(X,Y),Z) - F(X,F(Y,Z)). \) We must show \( G = 0. \) By hypothesis, if \( a, b, c \) are any positive integers then

\[ G(\tau^a, \tau^b, \tau^c) = (\tau^a \oplus_F \tau^b) \oplus_F \tau^c - \tau^a \oplus_F (\tau^b \oplus_F \tau^c) = 0 \]

as an element of \( R[[\tau]]. \) We must show that every coefficient of \( G \) is zero. Write
\[ G = \sum_{i,j,k \geq 0} g_{ijk} X^i Y^j Z^k. \]

Since the \( N \)th coefficient of \( G(\tau^a, \tau^b, \tau^c) \) is zero we have
\[
\sum_{\{ i,j,k \in \mathbb{Z}_0 | (a,b,c) \cdot (i,j,k) = N \}} g_{ijk} = 0 \tag{1.3}
\]
for all positive integers \( a, b, c, N \). We need to show each \( g_{ijk} = 0 \). Suppose not. Among all \( i, j, k \) for which \( g_{ijk} \) is nonzero, consider those for which \( N_1 = i+j+k \) is minimal. Among all \( i, j, k \) with \( g_{ijk} \neq 0 \) and \( i+j+k = N_1 \), consider those for which \( N_2 = i+j \) is minimal. Finally, among all \( i, j, k \) with \( g_{ijk} \neq 0 \), \( i+j+k = N_1 \), and \( i+j = N_2 \) select the one for which \( N_3 = i \) is minimal. Call this triple \((i_0, j_0, k_0)\); that is, \( i_0 + j_0 + k_0 = N_1 \), \( i_0 + j_0 = N_2 \), \( i_0 = N_3 \). Choose integers \( M_1, M_2, M_3 \) such that
\[ M_3 \geq 1, \quad M_2 > M_3 N_3, \quad M_1 > M_2 N_2 + M_3 N_3. \]

Let
\[ (a, b, c) = (M_1 + M_2 + M_3, M_1 + M_2, M_1), \quad N = M_1 N_1 + M_2 N_2 + M_3 N_3. \]

We will obtain a contradiction by showing that
\[
\sum_{\{ i,j,k \in \mathbb{Z}_0 | (a,b,c) \cdot (i,j,k) = N \}} g_{ijk} = g_{i_0,j_0,k_0} \neq 0. \tag{1.4}
\]
Suppose \( g_{ijk} \neq 0 \) and \( (a, b, c) \cdot (i, j, k) = N \). The equality can be written
\[ M_1(i+j+k) + M_2(i+j) + M_3 i = N. \tag{1.5} \]
Now \( i+j+k \geq N_1 \) by the minimality of \( N_1 \). Strict inequality cannot hold, since otherwise
\[ N = M_1(i+j+k) + M_2(i+j) + M_3 i \geq M_1(N_1 + 1) > M_1 N_1 + M_2 N_2 + M_3 N_3 = N. \]
Thus \( i+j+k = N_1 \). By minimality of \( N_2 \) we know \( i+j \geq N_2 \). Again strict inequality cannot hold, since otherwise
\[ N = M_1(i+j+k) + M_2(i+j) + M_3 i \geq M_1 N_1 + M_2 (N_2 + 1) > M_1 N_1 + M_2 N_2 + M_3 N_3 = N. \]
Thus \( i+j = N_2 \). Now the equality (1.5) shows \( i = N_3 \). This establishes (1.4) and completes the proof. \( \square \)

The following proposition gives a general method to construct formal group laws.

**Proposition 1.2** Let \( G \) be an abelian group, \( 0_G \) its identity element, and write its multiplication law additively. Suppose there is a one-to-one map \( T : \tau R[[\tau]] \to G \).
such that \( T(0) = 0_G \), and a power series \( F \in R[[X,Y]] \) with zero constant term such that
\[
T(g) + T(h) = T(F(g,h))
\]
for all \( g, h \in \tau R[[\tau]] \). Then \( F \) defines a formal group law.

Some easy examples of the above proposition are: (1) \( G = R[[\tau]] \) under addition, \( T = \text{inclusion} \), \( F(X,Y) = X + Y \) (called the additive group law), and (2) \( G = R[[\tau]]^\times \) under multiplication, \( T(g) = 1 + g \), \( F(X,Y) = X + Y + XY \) (called the multiplicative group law). A less trivial example is the construction of the group law associated to an elliptic curve, which will be given in §4.

**Proof of Proposition 1.2.** The hypothesis is that there is an injective map \( T \) from \( \tau R[[\tau]] \) into an abelian group \( G \) such that \( T(0) = 0_G \), and there is a power series \( F \in \tau R[[X,Y]] \) with zero constant term such that
\[
T(g) + T(h) = T(F(g,h))
\]
for all \( g, h \in \tau R[[\tau]] \). We need to show that \( F \) gives an abelian group law on \( \tau R[[\tau]] \). By Prop. 1.1, it suffices to show \( F \) makes \( \tau R[[\tau]] \) into an abelian semi-group with identity 0; that is, if \( f, g, h \in \tau R[[\tau]] \) then
\[
f \oplus_F (g \oplus_F h) = (f \oplus_F g) \oplus_F h, \quad f \oplus_F g = g \oplus_F f, \quad f \oplus_F 0 = f.
\]
Now \( T(f \oplus_F (g \oplus_F h)) = T(f) + T(g \oplus_F h) = T(f) + T(g) + T(h) \) and similarly \( T((f \oplus_F g) \oplus_F h) = T(f) + T(g) + T(h) \). This proves the first identity, since \( T \) is one-to-one. The other two identities are proved similarly. □

## 2 Homomorphisms of formal group laws

If \( F \) is a formal group law then write \( \mathcal{C}(F) \) for the group it determines. That is, \( \mathcal{C}(F) = \tau R[[\tau]] \) as a set, and the group law is given by \( g \oplus_F h = F(g,h) \). If \( F, F' \) are two formal group laws then a **homomorphism** from \( F \) to \( F' \) is defined as a power series \( U(\tau) \in \tau R[[\tau]] \) with zero constant term such that \( g \mapsto U(g) \) defines a homomorphism from \( \mathcal{C}(F) \) into \( \mathcal{C}(F') \). Explicitly,
\[
U \circ (x \oplus_F y) = (U \circ x) \oplus_{F'} (U \circ y)
\]
for all \( x, y \in \tau R[[\tau]] \). In terms of power series this can be written
\[
U(F(X,Y)) = F'(U(X), U(Y)).
\]
(2.1)

The reason that \( U \) has zero constant term is that \( U \) must take \( \tau R[[\tau]] \) into itself. An example of a homomorphism from \( F \) to itself is the multiplication by \( n \) map, denoted \([n] \) or \([n]_F \), which is defined by the rules:
\[
[n] = 0, \quad [1] = \tau, \quad [n+1] \tau = [n] \tau \oplus_F \tau = F([n] \tau, \tau) \text{ if } n > 0,
\]
\[
[n] = i \circ [-n] \text{ if } n < 0.
\]
(2.2)
Let $G_1, G_2$ be abelian groups, and let $T_i : \tau R[[\tau]] \to G_i$ ($i = 1, 2$) be one-to-one maps such that $T_i(0)$ is the identity element of $G_i$. Let $F_i$ be power series with zero constant term such that

$$T_i(g \oplus G_i T_i(h) = T_i(g \oplus F_i h), \quad i = 1, 2,$$

where $\oplus_{G_i}$ denotes addition on the group $G_i$ and $g \oplus F_i h = F_i(g, h)$. We showed that $F_i$ is a formal group law, and the above equation simply states that $T_i$ is a group homomorphism from $C(F_i)$ into $G_i$.

**Lemma 2.1** Let $G_i, T_i, F_i, C(F_i)$ be as above. Suppose there is a group homomorphism $\psi : G_1 \to G_2$ and a power series $U$ with zero constant term such that

$$\psi(T_1(g)) = T_2(U(g)) \quad (2.3)$$

for all $g \in \tau R[[\tau]]$. Then $U$ is a homomorphism between the formal group laws defined by $F_1$ and $F_2$.

**Proof.** It suffices to show that $U$ is a homomorphism from $C(F_1)$ to $C(F_2)$. By hypothesis there is a commutative diagram

$$\begin{array}{ccc}
C(F_1) & \xrightarrow{T_1} & G_1 \\
U & \downarrow & \downarrow \psi \\
C(F_2) & \xleftarrow{T_2} & G_2
\end{array}$$

Here $T_1, T_2, \psi$ are homomorphisms and $T_1, T_2$ are injective. It follows by diagram chasing that $U$ is a homomorphism, as claimed. \qed

As a special case, let $G_1 = G_2 = G, T_1 = T_2 = T, F_1 = F_2 = F$, and $\psi(g) = ng$, where $n \in \mathbb{Z}$. Then $U = [n]$, which was defined by (2.2). The power series for $[n]$ may either be computed from the recursion (2.2) or from the formula (2.3), which in this context reads

$$nT(g) = T([n](g)) \quad \text{for} \quad g \in \tau R[[\tau]]. \quad (2.4)$$

For the additive formal group law we have $T = \text{inclusion of } \tau R[[\tau]] \text{ into } R[[\tau]]$ and the formula reads $ng = [n](g)$. So in that case,

$$[n](\tau) = n\tau \quad \text{(Additive Formal Group)}$$

For the multiplicative formal group law we have $G = R[[\tau]]^\times$ and $T(g) = 1 + g$, so the formula reads $(1 + g)^n = 1 + [n](g)$. In the special case where $n = p = \text{the characteristic of } R \text{ with } p > 0$ we have $(1 + g)^p = 1 + g^p$, and therefore

$$[p](\tau) = \tau^p \quad \text{(Multiplicative Formal Group in Char. p).}$$
3 Height

If $R$ has characteristic $p$ then the height of a homomorphism $U$, written $\text{ht}(U)$, is the largest integer $h$ such that $U(\tau) = V(\tau^p)$ for some power series $V$, or $\infty$ if $U = 0$. The height of the formal group law is defined as the height of the homomorphism $[p]$. For the additive formal group law defined by $F(X,Y) = X + Y$ we have $[p](\tau) = p\tau = 0$, so the height of $F$ is $\infty$. For the multiplicative formal group law defined by $F(X,Y) = X + Y + XY$ we have $[p](\tau) = \tau^p$, therefore the multiplicative formal group law has height one.

Example 3.1 Let $F = \sum f_{ij}X^iY^j$ be a formal group law over an integral domain $R$ of characteristic $p \neq 0$. Let $F^{(p)} = \sum f_{ij}^pX^iY^j$. We claim that $F^{(p)}$ is a formal group law, and $\phi = \tau^p$ is a homomorphism (evidently of height 1) from $F$ to $F^{(p)}$. For the first assertion, replace $X, Y, Z$ by $X^{1/p}, Y^{1/p}, Z^{1/p}$ in the relation (1.1) then take the $p$th power. This yields the corresponding relations for $F^{(p)}$. For the second assertion, note that

$$F^{(p)}(\phi(X), \phi(Y)) = F(X,Y)^p = \phi(F(X,Y)).$$

Observe that $\phi^k : F \to F^{(p^k)}$. \hfill $\square$

Proposition 3.2 Let $F_1, F_2$ be formal group laws over an integral domain $R$ of characteristic $p$. Let $U(\tau) = \sum u_i\tau^i$ be a homomorphism from $F_1$ to $F_2$ of height $k$. Then the first nonzero coefficient of $U$ is $u_{k^i}$. Moreover, there is a homomorphism $V : F_1^{(p^k)} \to F_2$ such that $U = V \circ \phi^k$.

Proof. If $k = 0$ then $u_j \neq 0$ for some $j$ which is prime to $p$, therefore $U'(\tau) = \sum m_i u_i \tau^{m_i-1}$ is nonzero. Differentiate the equation $U(F_1(X,Y)) = F_2(U(X), U(Y))$ with respect to $Y$ and then set $Y = 0$. We obtain

$$U'(F_1(X,0)) \frac{\partial F_1}{\partial Y}(X,0) = \frac{\partial F_2}{\partial Y}(U(X), U(0)) U'(0).$$

Since $F_i(X,Y) = X + Y + XY G_i(X,Y)$ for $i = 1,2$, this becomes

$$U'(X)(1 + XG_1(X,0)) = (1 + U(X)G_2(U(X),0)) u_1.$$  

The left side is nonzero, therefore $u_1 \neq 0$.

Now let $k \geq 1$ and set $q = p^k$. By definition of height, there is a power series $V(\tau) \in \tau R[[\tau]]$ such that $U(\tau) = V(\tau^q)$. Now $V'$ is nonzero, since otherwise $V$ would be a function of $\tau^p$, so that $q$ could be replaced by $pq$. We claim $V$ is a homomorphism from $F_1^{(q^k)}$ to $F_2$. We have to show $V(F_1^{(q^k)}(X,Y)) = F_2(V(X), V(Y))$. The left side is $V(F_1(X^{1/q}, Y^{1/q})) = U(F_1(X^{1/q}, Y^{1/q}))$. The right side is $F_2(U(X^{1/q}), U(Y^{1/q}))$. These two are equal because $U$ is a homomorphism from $F_1$ to $F_2$. Since $V' \neq 0$, $V$ has height zero. It follows from the case $k = 0$ that the first coefficient of $V$ is nonzero. Thus the coefficient of $\tau^q$ in $U$ is nonzero. \hfill $\square$
Proposition 3.3 Let $F,F',F''$ be formal group laws over an integral domain $R$ of characteristic $p$. In parts (a), (b), (d) and (e) assume $p > 0$.

(a) If $U : F \to F'$, and $V : F' \to F''$, then $\text{ht}(V \circ U) = \text{ht}(V) + \text{ht}(U)$.

(b) If there is a nonzero homomorphism $U$ from $F$ to $F'$ then $F$ and $F'$ have the same height.

(c) For $n \in \mathbb{Z}$, $[n]_F = n\tau + \tau^2(\cdots)$.

(d) Every formal group $F$ over a ring of characteristic $p$ has height at least one.

(e) If $n = ap^t$ with $(a,p) = 1$ then $\text{ht}([n]_F) = t \text{ht}(F)$.

Proof. (a) Define the degree of a nonzero power series $\sum a_i \tau^i$ to be the smallest $i$ such that $a_i \neq 0$. Prop. 3.2 asserts that if $U$ is a nonzero homomorphism of formal group laws then $\text{deg}(U) = p^{\text{ht}(U)}$. The degrees of power series multiply when they are composed, therefore $p^{\text{ht}(V \circ U)} = p^{\text{ht}(V)}p^{\text{ht}(U)} = p^{\text{ht}(V) + \text{ht}(U)}$. (b) Certainly $[p]_{F'} \circ U = U \circ [p]_F$, so $[p]_F$ and $[p]_{F'}$ have the same height by (a). (c) can easily be shown by induction, using (2.2). (d) is immediate from (c) and Prop. 3.2. (e) $\text{ht}([n]_{p^t}) = \text{ht}([a]_F) + t \text{ht}([p]_F)$ by (a). The height of $[a]_F$ is zero by (c), and $\text{ht}([p]_F) = \text{ht}(F)$ by definition.

If $F,F'$ are formal group laws over an integral domain $R$ and $U_1, U_2 : F \to F'$, define $U_1 \oplus_{F'} U_2 = F'(U_1, U_2)$, $U_1 \oplus_{F'} U_2$ is a homomorphism from $F$ to $F'$. This composition rule makes $\text{Hom}(F,F')$ into an abelian group. In particular, it is a $\mathbb{Z}$-module. Suppose that $R$ has characteristic $p > 0$. We put a topology on $\text{Hom}(F,F')$ by decreeing that $U$ and $V$ are close iff $U \ominus_{F'} V$ has a large height. In other words, the topology on $\text{Hom}(F,F')$ is induced from the height metric $|U| = e^{\text{ht}(U)}$, where $0 < c < 1$.

Proposition 3.4 Let $F,F'$ be formal groups over an integral domain $R$ of characteristic $p > 0$.

(a) $\text{ht}(U_1 \oplus_{F'} U_2) \geq \inf\{\text{ht}(U_1), \text{ht}(U_2)\}$. If $\text{ht}(U_1) < \text{ht}(U_2)$ then $\text{ht}(U_1 \oplus_{F'} U_2) = \text{ht}(U_1)$. Hence, the height metric is nonarchimedean.

(b) The map $\mathbb{Z} \times \text{Hom}(F,F') \to \text{Hom}(F,F')$ given by $(n,U) \mapsto [n]_{F'} \circ U$ is continuous with respect to the $p$-adic metric on $\mathbb{Z}$ and the height metric on $\text{Hom}(F,F')$. Hence, $\text{Hom}(F,F')$ is naturally a $\mathbb{Z}_p$-module.

(c) If $\text{ht}(F) < \infty$ then $\text{Hom}(F,F')$ is a faithful $\mathbb{Z}_p$-module.

Proof. (a) Write $F'(X,Y) = X + Y + XYG'(X,Y)$. Then $U_1 \oplus_{F'} U_2 = F'(U_1, U_2) = U_1 + U_2 + U_1U_2G'(U_1, U_2)$. Part (a) is therefore true when the word “degree” is substituted for the word “height”. Since $\text{ht}(U_1) = \log_p(\text{deg}(U_1))$, (a) follows. (b) We must show that if $n = m + ap^t$ with $t$ large and if $U,V \in \text{Hom}(F,F')$ are close then $n \cdot U$ is close to $m \cdot V$. But

$$n \cdot U \oplus_{F'} V = [n]_{F'} \circ (U \oplus_{F'} V) \oplus_{F'} [ap^t]_{F'} \circ V.$$  

The height of $[n]_{F'} \circ (U \oplus_{F'} V)$ is $\geq \text{ht}(U \oplus_{F'} V)$. The height of $[ap^t]_{F'} \circ V$ is $\geq t$. Both these heights are large, so the height of the sum is large by (a). (c) We must show that if $a \in \mathbb{Z}_p$ and $0 \neq U \in \text{Hom}(F,F')$ then $a \cdot U = 0$ if and only if $a = 0$. Write $a = pk^b$, where $b \in \mathbb{Z}_p^\ast$. We have $a \cdot U = [p^k] \circ b \cdot U$. Certainly $b \cdot U \neq 0,$
since $b$ is invertible, and $[p^k]$ is nonzero since it has finite height. Thus $a \cdot U$ is the composition of two nonzero formal power series over $R$, and since $R$ is an integral domain, this composition is nonzero.

It is a theorem of M. Lazard ([3], [4]) that if $R$ is a separably closed field of characteristic $p$ then two formal group laws $F, F'$ defined over $R$ are isomorphic if and only if they have the same height. This gives a partial converse to Prop. 3.3(b). We will see that the height of the formal group law associated to an elliptic curve $E$ defined over a field $R$ of characteristic $p$ is one or two according as $E$ is ordinary or supersingular. Thus Lazard’s Theorem implies that the formal group laws of any two ordinary elliptic curves (or any two supersingular elliptic curves) are isomorphic over the algebraic closure of $R$. On the other hand, the condition that two elliptic curves over $R$ be isomorphic is much more restrictive (the two curves must have the same $j$-invariant; see [7], p. 47-50) This means that isomorphisms of formal group laws are far more abundant than isomorphisms of elliptic curves.

4 Constructing the formal group law of an elliptic curve

Let $E$ be an elliptic curve over a field $K$ determined by a nonsingular Weierstrass equation

$$W(X, Y, Z) = Y^2 Z + a_1 X Y Z + a_3 Y Z^2 - (X^3 + a_2 X^2 Z + a_4 X Z^2 + a_6 Z^3), \tag{4.1}$$

$a_i \in K$. Let $L$ be the quotient field of $K[[\tau]]$. Since $K \subset L$, we can consider the points in $E(L)$. Let $R$ be a subring of $K$ (possibly $R = K$) containing 1 and all the Weierstrass coefficients $a_i$. We will construct a formal group law by embedding $\tau R[[\tau]]$ into $E(L)$ and “stealing” the group law from $E(L)$.

Consider points of the form $(t, -1, s)$ in $E(K)$. Then $t$ can be regarded as the function $-X/Y \in K(E)$, where $K(E)$ denotes the function field of $E$ over $K$, and $t$ is a uniformizer at the identity $O = (0, 1, 0)$. Also $s$ can be regarded as the function $-Z/Y$, and $s$ has a triple zero at $O$. Let $\Omega$ be the ring of functions in $K(E)$ which are defined at $O$ and $M$ the ideal of functions in $\Omega$ which vanish at $O$. Then $M$ is principal, generated by $t$, and $\Omega/M \cong K$ by the map $f + M \mapsto f(O)$. $\Omega$ has a metric induced by $M$, namely $|f| = c^n$, where $0 < c < 1$ and $n$ is the largest integer such that $f \in M^n$. The uniformizer $t$ determines an isometry $\Psi : \Omega \to K[[\tau]]$ (where $K[[\tau]]$ has the $\tau$-adic topology) as follows: $f \mapsto \sum_{i=0}^{\infty} a_i \tau^i$ (where $a_i \in K$) if for each $N$, $f - \sum_{i=0}^{N} a_i t^i \in M^{N+1}$. The image of $\Psi$ is dense in $K[[\tau]]$, since it contains all polynomials.

Let $S(\tau) = \Psi(s) = \sum_{i=3}^{\infty} s_i \tau^i$. We will prove below that if $f \in \tau R[[\tau]]$ then $(f, -1, S(f)) \in E(L)$, so there is an embedding $T : \tau R[[\tau]] \to E(L)$ given by

$$T(f) = (f, -1, S(f)). \tag{4.2}$$

The formal group law of $E$ will be the power series $F \in \tau R[[\tau]]$ such that $T(f) + T(h) = T(F(f, h))$. All we need to do is to prove this power series $F$ exists; it will automatically be a formal group law because of Prop. 1.2.
By dividing through the Weierstrass equation by $Y^3$ we see that $s$ and $t$ satisfy the equation
\[ s = t^3 + a_1 ts + a_2 t^2 s + a_3 s^2 + a_4 t s^2 + a_6 s^3. \] (4.3)

The series $S$ can be computed by recursively substituting approximations for $s$ into the right hand side of (4.3) and expanding to get improved approximations. We start with the approximation $s = O(t^3)$ to obtain
\[ s = t^3 + a_1 t O(t^3) + a_2 t^2 O(t^3) + a_3 (O(t^3))^2 + a_4 t (O(t^3))^2 + a_6 (O(t^3))^3 = t^3 + O(t^4). \]

On the next round substitute $t^3 + O(t^4)$ for $s$ in the right side of the equation to obtain $s = t^3 + a_1 t^4 + O(t^5)$. This procedure yields the general rule:
\[ s_0 = s_1 = s_2 = 0, \quad s_3 = 1, \quad \text{and if } n \geq 4 \text{ then} \]
\[ s_n = a_1 s_{n-1} + a_2 s_{n-2} + a_3 \sum_{i+j=n} s_i s_j + a_4 \sum_{i+j=n-1} s_i s_j + a_6 \sum_{i+j+k=n} s_i s_j s_k. \] (4.4)

**Lemma 4.1** Let $W$ be the Weierstrass equation (4.1), where $a_i \in R$ and $R$ is an integral domain. Let $s_i \in R$ be defined by the recursion (4.4) and let $S = \sum s_i \tau^i \in \tau R[[\tau]]$. Then $W(\tau, -1, S) = 0$ in $R[[\tau]]$. If $f, g \in \tau R[[\tau]]$ and $W(f, -1, g) = 0$ then $g = S \circ f$.

**Remark.** Since the Weierstrass equation is cubic in the variable $Z$, it follows that for fixed $f \in \tau R[[\tau]]$, the equation $W(f, -1, g) = 0$ has three solutions for $g$ in the algebraic closure of the quotient field of $R[[\tau]]$. The lemma asserts that exactly one of these solutions lies in $\tau R[[\tau]]$.

**Proof.** Let $K$ be the quotient ring of $R$ and let $E$ be the elliptic curve over $K$ with equation $W$. Let $t = -X/Y$, $s = -Z/Y \in K(E)$, and $\Psi : \Omega \rightarrow K[[\tau]]$ be as described in the beginning of this section. Then $\psi(t) = \tau$, $\Psi(s) = S$. Now $W(t, -1, s) = 0$, so
\[ 0 = \Psi(W(t, -1, s)) = W(\tau, -1, S). \]

From this it follows that $W(f, -1, S \circ f) = 0$ for any $f \in \tau K[[\tau]]$.

Now suppose $f, g \in \tau R[[\tau]]$ and $W(f, -1, g) = 0$. Let $h = S \circ f$. Then
\[ 0 = W(f, -1, h) - W(f, -1, g) = (g - h) \left( -1 + a_1 f + a_2 f^2 + a_3 (g + h) + a_4 f (g + h) + a_6 (g^2 + gh + h^2) \right). \]

Since $-1 + a_1 f + \cdots$ is a unit in $R[[\tau]]$, $g - h$ must be zero. □

The above lemma establishes that the map $T : \tau K[[\tau]] \rightarrow E(L)$ is well-defined, furthermore it is obviously one-to-one. Recall Prop. 1.2, which guarantees that if we can find a power series $F$ in two variables with the properties that
Proof. (b) Suppose \( t_1 \neq t_2 \) and let \( m = (s_1 - s_2)/(t_1 - t_2) \), \( b = s_1 - mt_1 \), \( A = 1 + a_2m + a_4m^2 + a_6m^3 \). If \( A \neq 0 \) then

\[
P_1 + P_2 = -(t_3, -1, mt_3 + b),
\]

\[
t_3 = -t_1 - t_2 - \frac{a_1m + a_2b + a_3m^2 + 2a_4mb + 3a_5m^2b}{A}.
\] (4.6)

Proposition 4.2 Let \( P_i = (t_i, -1, s_i) \) for \( i = 1, 2 \) be points on the elliptic curve with Weierstrass equation (4.1).

(a) Suppose \( t_1 \neq 0 \) and let \( m = s_1/t_1 \). If \( 1 + a_2m + a_4m^2 + a_6m^3 \neq 0 \) then

\[
-P_1 = \left( \frac{-t_1}{1 - a_1t_1 - a_3s_1}, -1, \frac{-s_1}{1 - a_1t_1 - a_3s_1} \right).
\] (4.5)

(b) Suppose \( t_1 \neq t_2 \) and let \( m = (s_1 - s_2)/(t_1 - t_2) \), \( b = s_1 - mt_1 \), \( A = 1 + a_2m + a_4m^2 + a_6m^3 \). If \( A \neq 0 \) then

The left side is of the form

\[
(1 + a_2m + a_4m^2 + a_6m^3)t_3^2 + (a_1m + a_3m^2 + a_2b + 2a_4mb + 3a_5m^2b)t_3^2
\]

and the right side is of the form \( At^3 - A(t_1 + t_2 + t_3)^2 + \cdots \). Now (b) follows immediately.

(a) Let \( P_2 = (0, 1, 0), m = s_1/t_1 \), \( A = 1 + a_2m + a_4m^2 + a_6m^3 \). Since \( A \neq 0 \), (b) implies that \( P_1 + (0, 1, 0) + (t_3, -1, mt_3) = (0, 1, 0) \), where \( t_3 = -t_1 - (a_1m + a_3m^2)/A \). Thus \(-P_1 = (t_3, -1, mt_3) \). Now

\[
t_3^3A = t_1^3 + a_2t_1^2s_1 + a_4t_1s_1^2 + a_6s_1^3 = s_1 - a_1t_1s_1 - a_3s_1^2,
\]

thus

\[
t_3 = -t_1 - \frac{a_1m + a_3m^2}{A} = \frac{-t_1(t_1^3A) - (a_1t_1^2s_1 + a_3t_1s_1^2)}{t_1^3A}
\]

\[
= \frac{-t_1s_1}{s_1 - a_1t_1s_1 - a_3s_1^2} = \frac{-t_1}{1 - a_1t_1 - a_3s_1}.
\]
**Theorem 4.3** There is a power series \( F(t_1, t_2) \in R[[X, Y]] \) with zero constant term such that for \( f, g \in \tau R[[\tau]] \),

\[
T(f) + T(g) = T(F(f, g)).
\]

(4.7)

Therefore \( F \) is a formal group law.

**Proof.** Consider Prop. 4.2, but treat \( t_1, t_2 \) as indeterminates and substitute \( S(t_1), S(t_2) \) for \( s_1, s_2 \). In other words, we are working over the field \( L' \) = the quotient field of \( R[[t_1, t_2]] \). We need to show \( t_3 \) of equation (4.6) is a power series in \( t_1, t_2 \).

Let \( M \) be the ideal of \( R[[t_1, t_2]] \) generated by \( t_1 \) and \( t_2 \). That is, \( M \) is the set of elements \( \mu \in R[[t_1, t_2]] \) for which \( \mu(0, 0) = 0 \). If \( \mu \in M \) and \( u \) is a unit of \( R \) then \( u + \mu \) is a unit in \( R[[t_1, t_2]] \). Now

\[
m = \frac{S(t_1) - S(t_2)}{t_1 - t_2} = \sum_{i=3}^{\infty} s_i(t_1^i - t_2^i) \]

\[
= \sum_{i=3}^{\infty} s_i(t_1^{i-1} + t_1^{i-2}t_2 + \cdots + t_1t_2^{i-1} + t_2^{i-1})
\]

so \( m \) belongs to \( M^2 \). Then \( A = 1 + a_2m + a_4m^2 + a_6m^3 \) is a unit in \( R[[t_1, t_2]] \), since \( A \) is the sum of a unit in \( R \) and an element of \( M \). In particular, \( A \neq 0 \), so Prop. 4.2(b) applies. Also \( b = S(t_1) - mt_1 \in M^3 \). Now (4.6) shows that \( t_3 \in M \). Thus we can write \( t_3 = G(t_1, t_2), \) \( G \in M \). Certainly \( t_3 \neq 0 \), because \( G \equiv -t_1 - t_2 \text{ mod } M^2 \). We have \( (t_1, -1, S(t_1)) + (t_2, -1, S(t_2)) = (t_3, -1, s_3) \) in \( E(L') \), where \( s_3 = mt_3 + b \in M^3 \). By Prop. 4.2(a), the right side is

\[
\left( \frac{-t_3}{1 - a_1t_3 - a_3s_3}, -1, \frac{-s_3}{1 - a_1t_3 - a_3s_3} \right).
\]

Let

\[
F(t_1, t_2) = \frac{-t_3}{1 - a_1t_3 - a_3s_3} \in M, \quad H(t_1, t_2) = \frac{-s_3}{1 - a_1t_3 - a_3s_3} \in M^3.
\]

If we substitute \( t_1 = f(\tau), t_2 = g(\tau) \) for \( f, g \in \tau R[[\tau]] \) we get a homomorphism \( R[[t_1, t_2]] \to R[[\tau]] \), which induces a homomorphism \( E(L') \to E(L) \). It follows that

\[
(f, -1, S(f)) + (g, -1, S(g)) = (F(f, g), -1, H(f, g)).
\]

By Lemma 4.1 \( H(f, g) = S(F(f, g)) \). This proves (4.7). The fact that \( F \) is a formal group law follows from Prop. 1.2. \( \square \)

The first few terms of \( F \) are:

\[
F(X, Y) = X + Y - a_1XY - a_2(X^2Y + XY^2)
- (2a_3X^3Y + (3a_3 - a_1a_2)X^2Y^2 + 2a_3XY^3) + \cdots
\]
5 Homomorphisms of formal group laws arising from isogenies

Let $E, E'$ be two elliptic curves defined over the same field $K$. An algebraic map from $E$ to $E'$ is a function $\alpha : E(K) \to E'(K)$ such that for each $P \in E$ there exist homogeneous polynomials $f_1, f_2, f_3$ of the same degree and not all vanishing at $P$ such that for all but finitely many $Q \in E(K)$,

$$\alpha(Q) = (f_1(Q), f_2(Q), f_3(Q)).$$

An example of an algebraic map from $E$ to itself is the translation by $P$ map $\tau_P(Q) = P + Q$ for $P, Q \in E$. The algebraic map is said to be defined over a field $K$ if $E, E'$ are defined over $K$ and if all the coefficients of $f_1, f_2, f_3$ can be chosen to belong to $K$. It is a theorem ([7], p. 75) that every nonconstant algebraic map from $E$ into $E'$ which takes the origin to the origin is a group homomorphism. Such an algebraic map is called an isogeny. If $\tau : E \to E'$ and $-Q = \tau(0, 1, 0) \in E'$ then $\tau \circ \tau$ takes the origin of $E$ into the origin of $E'$. Thus every nonconstant algebraic map is the composition of an isogeny with a translation. Two curves $E, E'$ are called isogenous over $K$ if there exists an isogeny defined over $K$ from $E$ into $E'$. The endomorphism ring of $E$, written $\text{End}_K(E)$, is the set of isogenies over $K$ from $E$ to itself, together with the constant zero map, with the addition and multiplication laws:

$$(\alpha + \beta)(P) = \alpha(P) + \beta(P), \quad \alpha \beta = \alpha \circ \beta.$$  

Note that $\mathbb{Z} \subset \text{End}_K(E)$. If $K$ is the finite field with $q$ elements then the Frobenius endomorphism $\varphi_q$ defined by $\varphi_q(X, Y, Z) = (X^q, Y^q, Z^q)$. Since $\varphi_q$ coincides with the Galois action, it commutes with any endomorphism of $E$ which is defined over $K$. In particular, $\varphi_q$ commutes with $\mathbb{Z}$.

We claim that an isogeny of elliptic curves over $K$ gives rise to a homomorphism of the corresponding formal group laws over $K$. Indeed, let

$$I(X, Y, Z) = (f_1(X, Y, Z), f_2(X, Y, Z), f_3(X, Y, Z))$$

be an isogeny between elliptic curves $E, E'$ over $K$. Here $f_1, f_2, f_3$ are homogeneous polynomials of the same degree, say $d$, and $f_1, f_2, f_3$ do not simultaneously vanish at the origin. Since the origin of $E$ is carried to the origin of $E'$, $f_1$ and $f_3$ vanish at $O = (0, 1, 0)$ but $f_2(O) \neq 0$. Thus $f_1/Y^d \in M$ and $f_2/Y^d \in \Omega^\times$. Now $f_1/Y^d = f_1(X/Y, 1, Z/Y) = f_1(-t, 1, -s) = (-1)^d f_1(t, -1, s) \in M$ and similarly $f_2/Y^d = (-1)^d f_2(t, -1, s) \in \Omega^\times$. Thus

$$f_1(X, Y, Z)/f_2(X, Y, Z) = f_1(t, -1, s)/f_2(t, -1, s) \in M.$$  

Let $U(\tau) = \sum_{i=1}^{\infty} u_i \tau^i$ denote the expansion of $f_1/f_2$ with respect to $t$. Practically speaking, $U$ can be obtained by expanding $s$ as a power series $S$ and then computing

$$f_1(\tau, -1, S(\tau))/f_2(\tau, -1, S(\tau))$$
in the ring $K[[\tau]]$. Note that $f_2(\tau, -1, S(\tau))$ is invertible since its constant term is nonzero.

**Proposition 5.1** Let $E, E', E''$ be elliptic curves over $K$ and let $F, F', F''$ denote the associated formal group laws over $K$. If $I : E \to E'$ is an isogeny then the power series $U$ constructed above belongs to $\text{Hom}(F, F')$. The map $I \mapsto U$ is a one-to-one group homomorphism from $\text{Isog}(E, E') \to \text{Hom}(F, F')$. If $I' : E' \to E''$ and $I''$ corresponds to $U' \in \text{Hom}(F', F'')$ then $I' \circ I$ corresponds to $U' \circ U \in \text{Hom}(F, F'')$.

**Proof.** Let $L$ be the quotient field of $K[[\tau]]$. Since $I$ is defined over $K$, it is a priori defined over $L$. The discussion above shows that $I$ can be written in a neighborhood of the origin as

$$I(X, Y, Z) = \left(\frac{f_1(t, -1, s)}{f_2(t, -1, s)}, -1, \frac{f_3(t, -1, s)}{f_2(t, -1, s)}\right).$$

Let $T : \tau K[[\tau]] \to E(L)$ and $T' : \tau K[[\tau]] \to E'(L)$ be the embeddings (4.2). Substitute $(X, Y, Z) = T(f) = (f, -1, S(f)) \in E(L)$, where $f \in \tau K[[\tau]]$. Then $t = -X/Y$ changes to $f$ and $s = -Z/Y$ changes to $S \circ f$. Thus $I(T(f)) = (U(f), -1, V(f))$, where $U(\tau) = f_1(\tau, -1, S(\tau))/f_2(\tau, -1, S(\tau)) \in \tau K[[\tau]]$ and $V(\tau) = f_3(\tau, -1, S(\tau))/f_2(\tau, -1, S(\tau)) \in \tau K[[\tau]]$. By Lemma 4.1, $V = S' \circ U$, where $S'(t)$ is the power series expansion for $-Z/Y$ in the curve $E'$. Thus

$$I(T(f)) = T'(U(f)). \quad (5.1)$$

By Lemma 2.1, this equation proves that $U$ is a homomorphism of formal group laws.

If $I_1, I_2 \in \text{Isog}(E, E')$, and if $U_1, U_2 \in \text{Hom}(F, F')$ are the corresponding homomorphisms of formal group laws then on the elliptic curve $E(L)$,

$$(I_1 + I_2)(\tau, -1, S(\tau))$$

$$= I_1(\tau, -1, S(\tau)) + I_2(\tau, -1, S(\tau)) \quad \text{by definition of } I_1 + I_2$$

$$= T'(U_1) + T'(U_2) \quad \text{by (5.1)}$$

$$= T'(F'(U_1, U_2)) \quad \text{by (4.7).}$$

On the other hand, if $I_1 + I_2$ corresponds to $U_3$ then

$$(I_1 + I_2)(\tau, -1, S(\tau)) = T'(U_3).$$

Since $T'$ is one-to-one, $U_3 = F'(U_1, U_2) = U_1 \oplus F' U_2$. This shows that the map $I \mapsto U$ is a group homomorphism.

Finally, if $I : E \to E', I' : E' \to E''$ correspond to $U, U'$, respectively, then since $U$ is the unique solution in $\tau K[[\tau]]$ to $I \circ T = T' \circ U$,

$$I' \circ I \circ T = I' \circ T' \circ U = T'' \circ U' \circ U,$$

whence $I' \circ I$ corresponds to $U' \circ U$. \qed
Example 5.2 Let $F$ be the formal group law over $R$ associated to an elliptic curve $E$ with Weierstrass equation (4.1), where the coefficients $a_i \in R$, and $R$ is an integral domain. We will compute $[-1]_F$. Let $g \in \tau R[[\tau]]$. By Proposition 4.2(a),

$$[-1]_E T(g) = [-1]_E (g, -1, S \circ g) = \left(\frac{-g}{1 - a_1 g - a_3 S \circ g}, -1, \frac{-S \circ g}{1 - a_1 g - a_3 S \circ g}\right)$$

The right side is $T(-g/(1 - a_1 g - a_3 S \circ g))$ by Lemma 4.1. Now Lemma 2.1 implies

$$[-1]_F = \frac{-\tau}{1 - a_1 \tau - a_3 S} = -\tau \sum_{n=0}^{\infty} (a_1 \tau + a_3 S)^n.$$

This definition does not depend on the choice of uniformizer $t$. This definition does not depend on the choice of uniformizer $t$. An isogeny which is not separable is called inseparable. In characteristic zero, all isogenies are separable. In characteristic $p$, the Frobenius is not separable, since it carries uniformizers into $p$th powers of uniformizers. It is a theorem ([7], II.2.12) that every isogeny can be factored as $\varphi^k_p$ from $E$ into $E^{(q)}$ ($q = p^k$) composed with a separable isogeny from $E^{(q)}$ into $E'$.

Lemma 5.3 Let $I$ be an isogeny from $E$ to $E'$ and let $U(\tau) = \sum u_i \tau^i$ be the corresponding homomorphism between the formal group laws. $I$ is separable iff $u_1 \neq 0$.

Proof. Let $t'$ be the function $-X/Y \in K(E')$. $U$ is the power series expansion of $t' \circ I$ with respect to the uniformizer $t = -X/Y \in K(E)$. Thus $t' \circ I$ is not a uniformizer at the identity of $E$ iff $t' \circ I \in M_{\mathbb{F}_p}(0, 1, 0)$ iff $u_1 = 0$. □

Example 5.4 Let $E$ be an elliptic curve whose Weierstrass coefficients $a_i$ belong to a field $K$ of characteristic $p > 0$, and let $F$ be its associated formal group law. Let $E^{(p)}$ be the elliptic curve with Weierstrass coefficients $a_i^p$. Then the Frobenius map $\varphi_p : E \to E^{(p)}$ defined by $\varphi(X, Y, Z) = (X^p, Y^p, Z^p)$ corresponds to the homomorphism of formal group laws $\phi = \tau^p : F \to F^{(p)}$. □

6 Height of an elliptic curve

We begin this section with some facts about elliptic curves over finite fields. If $\alpha : E \to E'$ is an isogeny, define $\alpha^* K(E') = \{ f \circ \alpha \mid f \in K(E') \}$; this is a subfield of $K(E)$. The degree of an isogeny $\alpha : E \to E'$ is the index of $\alpha^* K(E')$ in $K(E)$. This number is finite because both fields have transcendence degree 1 and $\alpha$ is a nonconstant map. If $K$ has characteristic $p$ then the Frobenius isogeny $\varphi_p(X, Y, Z) = (X^p, Y^p, Z^p)$ from $E$ into $E^{(p)}$ has degree $p$. Here $E^{(p)}$ is the curve
whose Weierstrass equation is obtained from that of $E$ by raising the coefficients to the $p$th power.

Every isogeny $\alpha : E \to E'$ has a dual isogeny $\hat{\alpha} : E' \to E$. The dual isogeny is characterized by the property that $\alpha \circ \hat{\alpha} = [\deg(\alpha)]_{E'}$ and $\hat{\alpha} \circ \alpha = [\deg(\alpha)]_E$, where $[n]_E$ denotes multiplication by $n$. If $E = E'$, then there is an integer $a(\alpha)$, called the trace of $\alpha$, such that $\alpha + \hat{\alpha} = [a(\alpha)]_E$. The endomorphism $\alpha$ satisfies the quadratic equation

$$\alpha^2 - [a(\alpha)] \alpha + [\deg(\alpha)] = 0 \quad \text{in End}(E).$$

In particular, if $K$ has $q$ elements then there is $t \in \mathbb{Z}$ such that

$$\varphi_q^2 - [t] \varphi_q + [q] = 0.$$

The integer $t$ is called the trace of Frobenius. It is well known ([7], Ch. 5) that $|t| \leq 2\sqrt{q}$ and the cardinality of $E(K)$ is $q + 1 - t$.

The height of a formal group law was defined in §3. Naturally, the height of an elliptic curve is defined to be the height of the associated formal group law.

**Proposition 6.1** An elliptic curve over a field of characteristic $p$, where $p > 0$, has height one or two.

*Proof.* Let $\varphi_p : E \to E^{(p)}$ be the $p$th power Frobenius and $\hat{\varphi}_p : E^{(p)} \to E$ its dual. Let $F$ be the formal group law associated to $E$, and let $V(\tau) = \sum v_i \tau^i : F^{(p)} \to F$ be the homomorphism of formal group laws associated to $\hat{\varphi}_p$. Then $[p]_F = V(\tau^p)$. If $\hat{\varphi}_p$ is separable then $v_1 \neq 0$, so $E$ has height one. If $\hat{\varphi}_p$ is inseparable, it can be written as a composition of a power of $\varphi_p$ and a separable isogeny ([7], Corollary II.2.12). Since the degree of $\hat{\varphi}_p$ equals the degree of $\varphi_p$, only one power of $\varphi_p$ can occur in this decomposition. Thus $\hat{\varphi}_p = \alpha \circ \varphi_p$ with $\alpha$ an isomorphism. Let $A = \sum a_i \tau^i$ be the power series corresponding to $\alpha$ and let $A'$ be the power series corresponding to $\alpha^{-1}$. Then $[p]_E = A(\tau^p) = a_1 \tau^{p^2} + \cdots$, and $a_1 \neq 0$ because $A \circ A'(\tau) = \tau$. In this case $E$ has height two. \qed

An elliptic curve in characteristic $p$ of height one is called ordinary. An elliptic curve in characteristic $p$ of height 2 is called supersingular. The next lemma gives another characterization of supersingular and ordinary curves when the underlying field is finite.

**Proposition 6.2** An elliptic curve $E$ over a finite field $K$ with $q = p^n$ elements is supersingular iff $p$ divides the trace of Frobenius iff $|E(K)| \equiv 1 \mod p$. If $E$ is supersingular and $n$ is even then $|E(K)| = q + 1 + m\sqrt{q}$, $m \in \{-2, -1, 0, 1, 2\}$. If $E$ is supersingular, $n$ is odd, and $p \geq 5$, then $|E(K)| = q + 1$. If $E$ is supersingular, $n$ is odd, and $p \leq 3$ then $|E(K)| = q + 1 + m\sqrt{pq}$, where $m \in \{-1, 0, 1\}$.

For a more precise statement about which values of $|E(K)|$ can occur, the reader may consult [8], Theorem 4.1.
Proof. As above, let $F$ be the formal group law corresponding to $E$ and $V : F(p) \rightarrow F$ the homomorphism of formal group laws corresponding to $\hat{\varphi}_p$. In other words, $V$ is defined by $[p]_F = V(\tau^p)$. Recall that $E(p)$ denotes the elliptic curve whose Weierstrass equation is obtained by taking the $p$th powers of the Weierstrass coefficients for $E$, and we use similar notation for isogenies. Now $\hat{\phi}(p^k) : E(p^{k+1}) \rightarrow E(p^k)$ is the dual of the map $\varphi_p : E(p^k) \rightarrow E(p^{k+1})$, so

$$\hat{\varphi}_p \circ \varphi_p^{(p)} \circ \cdots \circ \varphi(p^{n-1})$$

is the dual of $\varphi_p^n$. The corresponding formal group law homomorphism is

$$N(V) = V \circ V^{(p)} \circ \cdots \circ V^{(p^n-1)}.$$

Let $t$ be the trace of Frobenius, so that $|E(K)| = q + 1 - t$. Since $[t]_E$ is the sum of $\varphi_p^n$ and its dual in $\text{End}(E)$, it follows that

$$[t]_F = N(V) \oplus_F \tau^p = F(N(V), \tau^p).$$

If $E$ is supersingular then $V$ has height one, so $N(V)$ has height $n$. In that case, $[t]_F$ has height at least $n$, so $[t^2]_F$ has height at least $2n$. Since the height of $F$ is two in this case, Prop. 3.3(e) implies $t^2$ is divisible by $p^n$. Since $|t| \leq 2\sqrt{q}$ and $q|t^2$, we deduce that $t^2 \in \{0, q, 2q, 3q, 4q\}$. Since $t \in \mathbb{Z}$, we find $t \in \{0, \pm q^{1/2}, \pm 2q^{1/2}\}$ if $n$ is even; $t = 0$ if $n$ is odd and $p > 3$; $t \in \{0, \pm \sqrt{2q}\}$ if $n$ is odd and $p = 2$, $t \in \{0, \pm \sqrt{3q}\}$ if $n$ is odd and $p = 3$. Since $|E(K)| = q + 1 - t$, the cardinality of $E(K)$ must be of the form stated.

Next suppose $E$ is ordinary. Then $N(V)$ has height zero, so $[t]_F$ has height zero. In that case Prop. 3.3(e) implies $t$ is prime to $p$. \hfill \square

**Proposition 6.3** If $E$ is an ordinary elliptic curve defined over a field $K$ of cardinality $p^n$, and $F$ is its associated formal group law then the trace of the Frobenius endomorphism is equal mod $p$ to the norm from $K$ to $\mathbb{F}_p$ of the first nonzero coefficient of $[p]_F$.

**Proof.** Let $|K| = p^n = q$. The homomorphism of $F$ associated to $\varphi_q^2 + [-t]_E \varphi_q + [q]_E$ is zero, thus each of its coefficients is zero. Now $\varphi_q$ corresponds to the power series $\tau^q$, and $[-t]_E$ corresponds to a power series of the form $-t\tau + \tau^q(\cdots)$, therefore $\varphi_q^2 + [-t]_E \circ \varphi_q$ corresponds to $F(\tau^q, -t\tau + \tau^q(\cdots))$, which is of the form $-t\tau^q + \tau^q(\cdots)$. Finally, we evaluate $[q]_F$. Let $\phi = \tau^p$. Since $\phi \circ V = V^{(p)} \circ \phi$,

$$[q]_F = (V \circ \phi)^n = V \circ V^{(p)} \circ \cdots \circ V^{(p^n-1)} \circ \phi^n = (N_{K/\mathbb{F}_p}(v)\tau + (\cdots)\tau^q) \circ \tau^q,$$

so $[q]_F = N_{K/\mathbb{F}_p}(v)\tau^q + (\tau^q(\cdots))$. Thus

$$0 = F( -t\tau^q + \tau^q(\cdots), N_{K/\mathbb{F}_p}(v)\tau^q + \tau^q(\cdots)) = (-t + N_{K/\mathbb{F}_p}(v))\tau^q + \tau^q(\cdots).$$

\hfill \square
7 Some theorems of Couveignes

Let $R$ be an integral domain of characteristic $p$. Let $F_p \subset R$ be the field with $p$ elements if $p$ is prime, and $F_p = \mathbb{Z}$ if $p = 0$. Let

$$F = \sum_{i,j} f_{ij} X^i Y^j, \quad F' = \sum_{i,j} f'_{ij} X^i Y^j$$

be two formal group laws over $R$, and let $U(\tau) = \sum_{i=1}^{\infty} u_i \tau^i \in \tau R[[\tau]]$ be a homomorphism from $F$ to $F'$. Couveignes proved with an elementary argument in his PhD thesis that the coefficients $u_i$ satisfy some simple relations over $R$. He used these relations to compute the orders of elliptic curves over finite fields of small characteristic (see [2] and [6]). In [1] it is shown that Couveignes’ method is closely related to the modified Schoof algorithm which was developed by Atkins and Elkies; see [5] and its bibliography. In this section we state and prove Couveignes’ theorems. In the next section we prove related results which are used in [1].

**Theorem 7.1** Let $i$ be a positive integer which is not a power of $p$. If $p = 0$ assume $(i \choose m)$ is a unit in $R$ for some $1 \leq m < i$. There is a polynomial $C_i$ in several variables with coefficients in $F_p$ such that for each $F,F',U$ as above we have

$$u_i = C_i(u_j, f_{k\ell}, f'_{k\ell} | 1 \leq j < i, 1 \leq k + \ell \leq i).$$

**Proof.** Let $A$ be transcendental and work in the integral domain $R[A]$. Since $U$ is a homomorphism,

$$U(F(\tau, A\tau)) = F'(U(\tau), U(A\tau)).$$

By (1.2) there are power series $G, G' \in R[[X, Y]]$ such that $F(X, Y) = X + Y + XYG(X, Y)$ and $F'(X, Y) = X + Y + XYG'(X, Y)$. Therefore

$$\sum u_j (\tau + A\tau + A^2 G(\tau, A\tau))^j = \sum u_j \tau^j + \sum u_j (A\tau)^j + U(\tau)U(A\tau)G'(U(\tau), U(A\tau)).$$

This can be rewritten

$$0 = \sum u_j \tau^j \{(1 + A + A^2)G(\tau, A\tau)^j \} - (1 + A^i)$$

$$\quad - A^2 \sum_{j=0}^{\infty} u_{j+1} \tau^j (\sum_{j=0}^{\infty} u_{j+1} (A\tau)^j) G'(\sum_{j=1}^{\infty} u_j \tau^j, \sum_{j=1}^{\infty} u_j (A\tau)^j).$$

The coefficient of $\tau^i$ is of the form $u_i \{(1 + A)^i \} - (1 + A^i) + M_i$, where $M_i$ is a polynomial in $A, u_1, u_2, \ldots, u_{i-1}$ and in some of the coefficients of $G, G'$. This gives the relation

$$u_i \{(1 + A)^i \} - (1 + A^i) - M_i = 0.$$
The hypothesis that \( i \) is not a power of \( p \) implies \((1 + A)^i \neq 1 + A^i \). If \( p = 0 \) choose \( m \) such that \( \binom{1}{m} \) is a unit in \( R \), and if \( p > 0 \) let \( m \) be a positive integer such that the coefficient of \( A^m \) is nonzero in the polynomial \((1 + A)^i - (1 + A^i) \).

In characteristic \( p \) this coefficient is a unit in \( R \) because it is a nonzero element of the prime field \( \mathbb{F}_p \). Since \( A \) is transcendental, the coefficient of \( A^m \) in our relation must be identically zero. This coefficient gives our desired formula for \( u_i \) in terms of the \( u_j \) and the coefficients of \( F \) and \( F' \).

The next theorem accounts for the \( u_i \) when \( i \) is a power of \( p \). It was proved by Couveignes for formal group laws associated to ordinary elliptic curves, but his argument generalizes easily to formal group laws of any height.

**Theorem 7.2** Let \( i \) be a power of a prime \( p \) and let \( h > 0 \). There is a polynomial \( C_i \) in several variables with coefficients in \( \mathbb{F}_p \) such that: if \( F = \sum f_{k\ell}X^kY^\ell \) and \( F' = \sum f'_{k\ell}X^jY^\ell \) are formal group laws of height \( h \) over a domain \( R \) of characteristic \( p \) and \( U = \sum u_j\tau^j : F \to F' \) a homomorphism then

\[
v_i' u_i^q - v_i^i u_i = C_i(u_j, f_{k\ell}, f'_{k\ell} | j < i, k + \ell \leq qi)
\]

where \( q = p^h \) and \( v_1, v_1' \) are the first nonzero coefficients of the power series \([p]_F, [p]_{F'}\), respectively.

**Proof.** By Prop. 3.2 we can write \([p]_F(\tau) = V \circ \phi^h(\tau) = V(\tau^q)\), where \( V(\tau) = \sum v_j\tau^j \) is a homomorphism of height zero from \( F(\tau) \) to \( F' \). It is easy to show by induction on \( n \) that for \( n > 0 \) the \( j \)th coefficient of \([n]_F \) is a polynomial in the \( f_{k\ell} \) with \( k + \ell \leq j \). Since \( v_j \) is the \( j \)th coefficient of \([p]_F \), \( v_j \) is a polynomial in the \( f_{k\ell} \) with \( k + \ell \leq jq \). Similarly \([p]_{F'} = V' \circ \phi^h, V'(\tau) = \sum v'_j\tau^j\), and \( v'_j \) is a polynomial in the \( f'_{k\ell} \) with \( k + \ell \leq jq \). Since \([p]_{F'} \circ U = U \circ [p]_F \),

\[
V'(U(\tau)^q) = U(V(\tau^q)).
\]

Let \( \sigma = \tau^q \). The left side is

\[
v'_1(\sum_{j=1}^{\infty} u_j^q \sigma^j) + v'_2(\sum_{j=1}^{\infty} u_j^q \sigma^j)^2 + \cdots,
\]

and the coefficient of \( \sigma^j \) is of the form \( v'_j u_j^q \) plus terms involving \( u_j \) for \( j < i \) and \( v'_j \) for \( j \leq i \). The right side is

\[
u_1(\sum_{j} v_j \sigma^j) + u_2(\sum_{j} v_j \sigma^j)^2 + \cdots + u_i(\sum_{j} v_j \sigma^j)^i + \cdots.
\]

This time the coefficient of \( \sigma^i \) is of the form \( u_i(v_1)^i \) plus terms involving \( u_j \) for \( j < i \) and \( v_j \) for \( j \leq i \). By equating the two sides we get \( v'_j u_j^q - v_j u_i \), equals a polynomial in the \( u_j \) for \( 1 \leq j < i \) and the \( v_j, v'_j \) for \( 1 \leq j \leq i \). \( \square \)
8 Further results relating to Couveignes’ theorems

Fix the following notation throughout this section. Let $R$ be an integral domain of characteristic $p > 0$, $F$ and $F'$ formal group laws of height $h$ over $R$, and $q = p^h$. Let $C_1, C_2, \ldots$ denote Couveignes’ relations given in §7 evaluated at the coefficients of $F, F'$ but leaving the $u_i$ as indeterminates; thus $C_i \in R[X_1, \ldots, X_i]$ and $C_i = X_i + a$ certain polynomial in $X_1, \ldots, X_i-1$ if $i$ is not a power of $p$; $C_i = v_i^t X_i^q - v_i^t X_i$ a certain polynomial in $X_1, \ldots, X_i-1$ if $i$ is a power of $p$. Here the $v_i$ and $v_i'$ lie in $R$, since they are polynomials in the coefficients of $F$ and $F'$, respectively. Couveignes’ theorems assert that if $\sum u_i \tau^i \in \text{Hom}(F, F')$ then $C_i(u_1, \ldots, u_i) = 0$ for all $i$. Let $K$ denote the separable algebraic closure of the quotient field of $R$.

Lemma 8.1 There are exactly $q^n$ solutions $(u_1, \ldots, u_{p^n-1})$ with $u_i \in K$ to the first $p^n - 1$ of Couveignes’ relations.

Proof. For each solution $(w_1, \ldots, w_{q-1})$ to the first $i - 1$ of Couveignes’ equations over $K$ there are $q$ values or 1 value of $w_i$ such that $(w_1, \ldots, w_i)$ is a solution to the $i$th relation, according as $i$ is or is not a power of $p$. (To see that the $q$ solutions for $w_i$ are distinct when $i$ is a power of $p$, note that the derivative with respect to $X_i$ of $C_i$ is $v_i'$, which is nonzero.) The lemma now follows easily by induction.

Theorem 8.2 If $u_1, u_2, \ldots$ is a solution to Couveignes’ relations then $\sum u_i \tau^i \in \text{Hom}(F, F')$.

Proof. Without loss of generality we can replace $R$ by $K$. In Chapter III, §2 of [3] it is shown that $\text{Hom}(F, F')$ is free over $\mathbb{Z}_p$ of rank $h^2$ and $p^n \text{Hom}(F, F')$ is the set of homomorphisms with height $\geq nh$. (In fact, it is shown that $\text{Hom}(F, F')$ is the maximal order of a central division algebra over $\mathbb{Q}_p$ of rank $h^2$ and invariant $1/h$, but we do not need this here.) It follows that a complete set of $\mathbb{Z}_p$-module generators $U_1, \ldots, U_{h^2}$ can be found such that the height of each generator is less than $h$, and if $\sum c_i U_i$ has height $\geq nh$ for some $c_i \in \mathbb{Z}_p$ then each $c_i$ is divisible by $p^n$. If $U, U' \in \text{Hom}(F, F')$ and $U \equiv U' \mod q^n$ (meaning that the $i$th coefficient of $U$ and $U'$ coincide for all $i \leq q^n$) then

$$0 = F'(U', [-1]_{F'} \circ U') \equiv F'(U, [-1]_{F'} \circ U') = U \otimes_{F'} U' \mod q^n,$$

so $U \otimes_{F'} U'$ has height $\geq nh$, and it is therefore divisible by $p^n$. Thus $\sum c_i U_i \equiv \sum c'_i U_i \mod q^n \implies c_i \equiv c'_i \mod q^n$. This shows that the number of distinct elements $\sum_{i=1}^{q^n-1} u_i \tau^i$ which are truncations of power series in $\text{Hom}(F, F')$ is the cardinality of $(\mathbb{Z}/p^n\mathbb{Z})^{h^2}$, which is $q^{nh}$. Each truncation gives rise to a solution $(u_1, \ldots, u_{p^n-1})$ of the first $q^n - 1$ of Couveignes’ relations. Since this coincides with the total number of solutions, each solution of Couveignes’ relation arises from $\text{Hom}(F, F')$. \qed
Corollary 8.3 If \( h = 1 \) and if \( \text{Hom}(F, F') \) contains a homomorphism (with coefficients in \( R \)) of height \( k \) then all the solutions \((v_1, v_2, \ldots)\) in \( K \) to Couveignes’ relations for which \( v_i = 0 \) for \( i < p^k \) actually lie in \( R \).

Proof. Let \( U \) be the homomorphism of height \( k \) and \( \mathbb{Z}_p \cdot U = \{ c \cdot U \mid c \in \mathbb{Z}_p \} \). As mentioned in the previous proof, \( \text{Hom}(F, F') \cong \mathbb{Z}_p \), and it is generated by a homomorphism \( U_0 \) of height zero. Find \( a \in \mathbb{Z}_p \) such that \( U = a \cdot U_0 \). Since \( \text{ht}(a \cdot U_0) = v_p(a), v_p(a) = k \). Thus \( \mathbb{Z}_p \cdot U = \mathbb{Z}_p \cdot U_0 = p^k \mathbb{Z}_p \cdot U_0 \). Since \( U \) is defined over \( R \), so is \( c \cdot U \) for each \( c \in \mathbb{Z}_p \). Thus every element of \( p^k \mathbb{Z}_p \cdot U_0 \) has coefficients in \( R \). The coefficients of such elements are precisely the solutions \((v_1, v_2, \ldots)\) to Couveignes’ relations which have \( v_i = 0 \) for all \( i < p^k - 1 \). \( \square \)

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