ON THE COEFFICIENTS IN AN ASYMPTOTIC EXPANSION OF
$(1 + 1/x)^x$

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Abstract. The function $g(x) = (1 + 1/x)^x$ has the well-known limit $e$ as $x \to \infty$. The coefficients $c_j$ in an asymptotic expansion for $g(x)$ are considered. A simple recursion formula is derived, and then using Cauchy’s integral formula the coefficients are approximated for large $j$. From this it is shown that $|c_j| \to 1$ as $j \to \infty$.

Key words. Series expansions, Asymptotic representations in the complex plane, Integrals of Cauchy type

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1. Introduction and main results. Chen and Choi [3] presented a method to estimate Euler’s constant $e$, accurate to as many decimal places as desired. Their starting point was the well known limit $\lim_{x \to \infty} g(x) = e$, where

$$(1.1) \quad g(x) = \left(1 + \frac{1}{x}\right)^x.$$                      

Their method was based on an asymptotic expansion of $g(x)$ for large values of $x$, namely

$$(1.2) \quad g(x) \sim \sum_{j=0}^{\infty} \frac{c_j}{x^j}.$$               

We shall show that this asymptotic expansion converges for $x > 1$.

In [3] they proved that $c_j = ea_j$, where $a_j$ are rational numbers which alternate in sign, and are given explicitly by

$$(1.3) \quad a_j = (-1)^j \sum_{k_1, k_2, \ldots, k_j} \frac{1}{k_1!k_2! \cdots k_j!} \left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \cdots \left(\frac{1}{j+1}\right)^{k_j}.$$                       

Here for each $j$ the sum is taken over all possible combinations of nonnegative integers $k_1, k_2, k_3, \ldots, k_j$ that satisfy the relation $\sum_{l=1}^{j} k_l = j$. We remark that Ponomorenko [7] gave a much simpler proof of (1.3) using Faà di Bruno’s formula generalizing the chain rule to higher derivatives.

The number of terms in the sum (1.3) is the partition function $P(j)$ [4, Sect. 27.14(i)]. Hardy and Ramanujan [5] gave the asymptotic formula

$$(1.4) \quad P(j) \sim \frac{\exp(\pi \sqrt{2j/3})}{4j^{3/2}}$$

as $j \to \infty$. It is therefore evident that the number of terms in the formula (1.3) grows exponentially in $\sqrt{j}$, and so is only practicable for small or moderate values of $j$.

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We derive a new way of computing the coefficients $c_j$, by a simple recursion formula. We also provide a simple asymptotic approximation for the coefficients as $j \to \infty$, and this shows that in absolute value they approach the value 1.

Our main results read as follows. In section 2 we prove (1.5), and in section 3 we prove (1.6).

**Theorem 1.1.** For $x > 1$ the expansion (1.2) converges, where the coefficients $c_j$ are given recursively by

\[
(1.5) \quad c_{j+1} = \frac{1}{j+1} \sum_{l=0}^{j} (-1)^{j-l+1} \frac{(j-l+1)}{j-l+2} c_l \quad (j = 0, 1, 2, \ldots).
\]

Moreover as $j \to \infty$

\[
(1.6) \quad c_j = (-1)^j \left(1 + \frac{1}{j}\right) + O\left(\frac{\ln(j)}{j^2}\right).
\]

Remark 1. Since $\lim_{x \to \infty} g(x) = e$ it is clear from (1.2) that $c_0 = e$.

2. Proof of the recursion formula (1.5). Define $z = 1/x$ and $f(z) = g(1/z)$ so that (1.2) is written in the new form

\[
(2.1) \quad f(z) = (1 + z)^{1/z} = \sum_{j=0}^{\infty} c_j z^j.
\]

and in this we consider $z$ complex. Moreover, on writing it as

\[
(2.2) \quad f(z) = \exp \left\{ z^{-1}\ln(1 + z) \right\},
\]

we note that it has a removable singularity at $z = 0$, and therefore can be considered analytic at $z = 0$ by assuming $f(0) = \lim_{z \to 0} f(z) = c_0 = e$. We also see it has one finite singularity (a logarithmic branch point) at $z = -1$. Therefore the radius of convergence of the series (2.1) is 1, i.e. it converges for $|z| < 1$. Thus (1.2) converges for $x > 1$, as asserted. By taking the principal branch of the logarithm in (2.2) we have that $f(z)$ is analytic on the cut plane $\mathbb{C} \setminus (-\infty, -1]$.

Next, on taking the natural logarithm of both sides of (2.1), we get

\[
(2.3) \quad \frac{1}{z} \ln(1 + z) = \ln \left\{ \sum_{j=0}^{\infty} c_j z^j \right\}.
\]

Then expand the $\ln(1 + z)$ term by its Maclaurin expansion valid for $|z| < 1$ and we arrive at

\[
(2.4) \quad \sum_{j=1}^{\infty} (-1)^j \frac{z^j}{j+1} = \ln \left\{ \sum_{j=0}^{\infty} c_j z^j \right\}.
\]

Next differentiate both sides with respect to $z$ to yield

\[
(2.5) \quad \sum_{j=1}^{\infty} (-1)^j \frac{j z^{j-1}}{j+1} = \sum_{j=1}^{\infty} j c_j z^{j-1} \left[ \sum_{j=0}^{\infty} c_j z^{j-1} \right]^{-1}.
\]
By shifting indices of the series starting at \( j = 1 \) to start at \( j = 0 \), and taking the series \( \sum_{j=0}^{\infty} c_j z^j \) to the left-hand side, we see this is equivalent to

\[
(2.6) \quad \sum_{j=0}^{\infty} c_j z^j \sum_{j=0}^{\infty} d_j z^j = \sum_{j=0}^{\infty} (j+1)c_{j+1} z^j,
\]

where \( d_j \) is given by

\[
(2.7) \quad d_j = (-1)^{j+1} \frac{j+1}{j+2}.
\]

We now use the Cauchy product, which is the discrete convolution of two infinite series. It is given by the formula \([2\), Sect. 73]\]

\[
(2.8) \quad \sum_{j=0}^{\infty} C_j \sum_{j=0}^{\infty} D_j = \sum_{j=0}^{\infty} \left[ \sum_{l=0}^{j} C_l D_{j-l} \right].
\]

Applying this to the left-hand side of (2.6) we combine both power series to the following single power series

\[
(2.9) \quad \sum_{j=0}^{\infty} c_j z^j \sum_{j=0}^{\infty} d_j z^j = \sum_{j=0}^{\infty} \left[ \sum_{l=0}^{j} c_l d_{j-l} \right] z^j.
\]

Finally substitute this into the left-hand side of (2.6), and equate coefficients of \( z^j \), to obtain

\[
(2.10) \quad \sum_{l=0}^{j} c_l d_{j-l} = (j+1)c_{j+1},
\]

and then using (2.7) this leads to (1.5).

3. **Proof of the asymptotic approximation (1.6).** We will use the famous Cauchy integral formula \([4\), Eq. 1.9.31]\] to obtain an integral representation for the coefficients \( c_j \). If \( C_r \) is the positively orientated circle \( \{ z : |z| = r \} \) for arbitrary \( r \in (0, 1) \) then from (2.1) and (2.2)

\[
(3.1) \quad c_j = \frac{1}{2\pi i} \oint_{C_r} \frac{(1+z)^{1/z}}{z^{j+1}} \, dz = \frac{1}{2\pi i} \oint_{C_r} \frac{\exp \left\{ z^{-1} \ln(1+z) \right\}}{z^{j+1}} \, dz.
\]

In the second integral of (3.1) we rewrite the exponential term using the geometric series

\[
(3.2) \quad \frac{1}{z} = -\frac{1}{1-(1+z)} = -(1+\delta + \delta^2 + \delta^3 + \cdots),
\]

where \( \delta = 1+z \), assuming \( 0 < |\delta| < 1 \). So from (2.2)

\[
(3.3) \quad f(z) = \exp \left\{ -\ln(\delta) \left( 1 + \sum_{j=1}^{\infty} \delta^j \right) \right\} = \frac{1}{\delta} \exp \left\{ -\ln(\delta) \sum_{j=1}^{\infty} \delta^j \right\}.
\]
Note that $\delta^j \ln(\delta) \to 0$ as $\delta \to 0$ for $j = 1, 2, 3, \cdots$ by L'Hopital's rule. So using the Maclaurin expansion of the exponential function along with (3.3) this function has the expansion

$$(3.4) \quad f(z) = \frac{1}{\delta} \left(1 - v + \frac{v^2}{2!} - \frac{v^3}{3!} + \cdots\right),$$

for $0 < |\delta| < 1$, where $v = \ln(\delta) \sum_{j=1}^{\infty} \delta^j$. From this one deduces for small $\delta$ that

$$(3.5) \quad v = -\delta \ln(\delta) - \delta^2 \ln(\delta) + O\{\delta^3 \ln(\delta)\},$$

and

$$(3.6) \quad v^j = O\{\delta^3 \ln^2(\delta)\} \quad (j = 3, 4, 5, \cdots).$$

Recalling $\delta = 1 + z$ we consequently have from (3.4) - (3.6)

$$(3.7) \quad f(z) = (1 + z)^{-1} - \ln (1 + z) + R(z),$$

where

$$R(z) = (1 + z)^{1/2} - (1 + z)^{-1} + \ln(1 + z) = \left[\frac{1}{2} \ln (1 + z)^2 - \ln (1 + z)\right] (1 + z) + O\left\{\ln (1 + z)^3 (1 + z)^2\right\},$$

as $z \to -1$.

Now substitute (3.7) into (3.1) to get

$$(3.9) \quad c_j = I_{1,j} + I_{2,j} + \eta_j,$$

where

$$(3.10) \quad I_{1,j} = \frac{1}{2\pi i} \oint_{C_r} \frac{1}{z^{j+1}(1 + z)} dz, \quad I_{2,j} = -\frac{1}{2\pi i} \oint_{C_r} \frac{\ln(1 + z)}{z^{j+1}} dz,$$

and

$$(3.11) \quad \eta_j = \frac{1}{2\pi i} \oint_{C_r} \frac{R(z)}{z^{j+1}} dz.$$

The integrals in (3.10) can readily be evaluated by residue theory. For the first we have by the geometric series expansion

$$(3.12) \quad I_{1,j} = \text{Res}_{z=0} \left\{\frac{1}{z^{j+1}(z + 1)}\right\} = \text{Res}_{z=0} \left\{\sum_{s=0}^{\infty} (-1)^s z^{s-j-1}\right\} = (-1)^j.$$

Likewise for $I_{2,j}$ one finds that

$$(3.13) \quad I_{2,j} = -\text{Res}_{z=0} \left\{\frac{\ln(1 + z)}{z^{j+1}}\right\} = \frac{(-1)^j}{j}.$$

For the integral (3.11) we make a change of variable $w = -(z + 1)$ to obtain the following

$$(3.14) \quad \eta_j = \frac{(-1)^{j+1}}{2\pi i} \oint_{C'_r} \frac{R(-1 - w)}{(1 + w)^{j+1}} dw.$$
The contour $C'_r$ in the $w$ plane is now the circle \{w : |w + 1| = r\} for $0 < r < 1$, and is positively orientated. This lies in the left half plane and encircles $w = -1$. The integrand of (3.14) has a branch point at $w = 0$ and a pole at $w = -1$, and is analytic elsewhere in the $w$ plane having a cut along the non-negative real axis. So we can deform the contour to a new one, called $\Gamma_{\epsilon, \rho}$, as seen in Figure 1.

This contour consists of circles $\gamma_{\epsilon}$ and $\gamma_{\rho}$ centered at $w = 0$, of radius $\epsilon$ and $\rho$, respectively, where $0 < \epsilon < 1 < \rho$, and horizontal line segments $l_1$ and $l_2$ with end points $w = \epsilon \pm i0$ and $w = \rho \pm i0$ above and below the cut.

Now from (3.8) we see that the integrand of (3.14) is $O\{w \ln(w)^2\}$ as $w \to 0$, and $O\{w^{-\epsilon-1} \ln(w)\}$ as $w \to \infty$. Hence the contributions of $\gamma_{\epsilon}$ and $\gamma_{\rho}$ vanish as $\epsilon \to 0$ and $\rho \to \infty$. Then the only contribution will be along $l_1 \cup l_2$, where now these lines extend from 0 to $\infty$. We are therefore left with

\begin{equation}
\eta_j = \frac{(-1)^{j+1}}{2\pi i} \int_{l_1 \cup l_2} \frac{R(-1 - w)}{(1 + w)^{j+1}} dw.
\end{equation}

Next, the contributions of real terms in the integrand of (3.15) cancel. Hence, using $\Im\{\ln(z)^2\} = 2 \arg(z) \ln(|z|)$ \((z \in \mathbb{C} \setminus \{0\})\), we have from (3.8) $\Im\{R(-1 - w)\} = \ldots$
\( \mathcal{O}\{w \ln(w)\} \) uniformly for \( w \in l_1 \cup l_2 \), and so from (3.15) for unbounded \( j \)

\[
\eta_j = \mathcal{O}\left\{ \int_0^\infty \frac{w \ln(w)}{(1+w)^{j+1}} \, dw \right\}.
\]

We then let \( 1 + w = e^t \), and consequently from (3.16) obtain

\[
\eta_j = \mathcal{O}\left\{ \int_0^\infty e^{-jt} (e^t - 1) \ln(e^t - 1) \, dt \right\}.
\]

Now split the integral in (3.17) into two integrals, one from \( t = 0 \) to \( t = 1 \) and the other from \( t = 1 \) to \( t = \infty \). For the first use \( e^t - 1 = \mathcal{O}(t) \) for \( 0 \leq t \leq 1 \), and for the second use \( \ln(e^t - 1) = \mathcal{O}(t) \) for \( 1 \leq t < \infty \). Thus as \( j \to \infty \) we deduce that

\[
\eta_j = \mathcal{O}\left\{ \int_0^1 e^{-jt} \ln(t) \, dt \right\} + \mathcal{O}\left\{ \int_1^\infty e^{-(j-1)t} \, dt \right\}
= \mathcal{O}\left( \frac{\ln(j)}{j^2} \right) + \mathcal{O}\left( \frac{1}{je^j} \right) = \mathcal{O}\left( \frac{\ln(j)}{j^2} \right),
\]

where the third \( \mathcal{O} \) term comes from [6, Chap. 9, Thm. 1.1], and the fourth \( \mathcal{O} \) term came from integration by parts. Finally, from (3.9), (3.12), (3.13) and (3.18) we arrive at (1.6).

**Addendum.** Christian Berg kindly made us aware of his recent joint paper [1]. In this they study a more general function \( h_\alpha(z) = (1 + 1/z)\alpha z \) where \( \alpha > 0 \). They obtain a number of results for the coefficients in a Maclaurin series in \( z \) for \( e^{-\alpha} h_\alpha(-1/z) \), which are polynomials in \( \alpha \). In particular they use properties of the exponential Bell partition polynomials to obtain a recursion relation, which for \( \alpha = 1 \) reduces to our formula (1.5). In comparison we only require elementary techniques to obtain (1.5).

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**REFERENCES**

[1] C. Berg, E. Massa, and A. P. Peron, A family of entire functions connecting the Bessel function \( J_1 \) and the Lambert \( W \) function, Constr. Approx., 53 (2021), pp. 121–154.

[2] J. W. Brown and R. V. Churchill, Complex variables and applications, Boston: McGraw-Hill Higher Education, 9 ed., 2014.

[3] C.-P. Chen and J. Choi, An asymptotic formula for \((1 + 1/x)^x\) based on the partition function, Amer. Math. Monthly, 44 (2014), pp. 338–343.

[4] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15, http://dlmf.nist.gov/. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[5] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatorial analysis, Proc. London Math. Soc., 17 (1918), pp. 75–115.

[6] F. W. J. Olver, Asymptotics and special functions, AKP Classics, A K Peters Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York].

[7] V. Ponomarenko, Asymptotic formula for \((1 + 1/x)^x\), revisited, Amer. Math. Monthly, 122 (2015), p. 587.