Asymptotic behavior of positive solutions of semilinear elliptic equations in $\mathbb{R}^n$ *

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Abstract

We will investigate the asymptotic behavior of positive solutions of the elliptic equation

$$\Delta u + |x|^{l_1}u^p + |x|^{l_2}u^q = 0 \text{ in } \mathbb{R}^n. \quad (0.1)$$

We establish that for $n \geq 3$ and $q > p > 1$, any positive radial solution of (0.1) has the following property: $\lim_{r \to \infty} r^{2 + l_1/p - 1} u$ and $\lim_{r \to 0} r^{2 + l_2/q - 1} u$ always exist if $n + l_1/n - 2 < p < q$, $p \neq n + 2 + 2l_1/n$, $q \neq n + 2 + 2l_2/n$. In addition, we prove that the singular solution of (0.1) is unique under a certain condition.

1. Introduction

In this paper we will study the asymptotic behavior of positive solutions of the following equation

$$\Delta u + K_1(|x|)u^p + K_2(|x|)u^q = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3, \quad (1.1)$$

and in particular, of positive radial solutions of

$$\Delta u + |x|^{l_1}u^p + |x|^{l_2}u^q = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3, \quad (1.2)$$

where $-2 < l_2 < l_1 \leq 0, 1 < p < q$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and $K_1, K_2$ are a locally Hölder continuous in $\mathbb{R}^n \setminus \{0\}$. By an entire solution of (1.1), we mean a positive weak solution of (1.1) in $\mathbb{R}^n$ satisfying (1.1) pointwise in $\mathbb{R}^n \setminus \{0\}$.

When $K_1(|x|) = K_2(|x|), p = q$, then (1.2) reduces to, by scaling

$$\Delta u + K(|x|)u^p = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3. \quad (1.3)$$

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Equation (1.3) has their roots from many mathematical and physical fields, e.g., the well-known scalar curvature equation in the study of Riemannian geometry, the scalar field equation for the standing wave of nonlinear Schrödinger and the Klein-Göder equations, the Matukuma equation describing the dynamics of globular cluster of star, etc. We refer the interested readers to [16, 18, 20, 21, 23] and the references therein. There have been many works devoted to studying the existence of positive solutions of (1.3) in $\mathbb{R}^n$ after the first contribution by Ni [20] in 1982, see [2, 6, 7, 8, 11, 12, 23] and the references therein. One of the features of the equation is that (1.3) can possess infinitely many solutions as long as the exponent $p$ and the dimension $n$ are large enough. Recent studies in [1, 3, 4, 9, 10] paid special attention to this phenomenon. The purpose of this paper is to study the asymptotic behavior of positive entire solutions and to present the uniqueness result. In such perspective, we review related works as follows. In the fast decay case, i.e. $|K(r)| \leq Cr^l, l < -2$, Ni [20] showed that equation (1.3) possess infinitely many positive solutions which are bounded from below by positive constants. Li and Ni [18] showed that, for positive solution of (1.3), the limit $u_\infty = \lim_{x \to \infty} u(x)$ always exists. Furthermore, if $u_\infty = 0$, then, for any $\varepsilon > 0$, 

$$u(x) \leq \begin{cases} 
C|x|^{2-n} & \text{if } p > \frac{n+l}{n-2}, \\
C_\varepsilon |x|^{(1-\varepsilon)(l+2)/1-p} & \text{if } p \leq \frac{n+l}{n-2},
\end{cases}$$

where $C_\varepsilon$ is a constant depending on $\varepsilon$, and if $u_\infty > 0$, then 

$$|u(x) - u_\infty| \leq \begin{cases} 
C|x|^{2-n} & \text{if } l < -n, \\
C|x|^{2-n} \log |x| & \text{if } l = -n, \\
C|x|^{2+l} & \text{if } -n < l < -2,
\end{cases}$$

at $\infty$.

For the slow decay case, i.e. $K(r) \geq Cr^l$, for some $l > -2$ and $r$ large enough. And also $K(r)$ satisfy:

(K.1) $K(r) > 0$ in $r > 0$ and $\lim_{r \to \infty} r^{-l}K(r) = k_\infty > 0$,
(K.2) $K(r)$ is differentiable and $\left[\frac{d}{dr}(r^{-l}K(r))^+\right] \in L^1, r > 0$,
(K.2) $K(r)$ is differentiable and $\left[\frac{d}{dr}(r^{-l}K(r))^-\right] \in L^1, r > 0$.

Li [17] gave an accurate description on the asymptotic behavior of positive solutions of (1.3), which is stated as follows

Theorem A (Li [17, Theorem 1]) Let $u$ be a positive radial solution of (1.3). Assume that $K$ satisfies 

(i) (K.1) and (K.2), if $0 < \frac{2+l}{p-1} < \frac{n-2}{2}$ or 
(ii) (K.1) and (K.3), if $\frac{n-2}{2} < \frac{2+l}{p-1} < n - 2$.

Then, 

$$\lim_{r \to \infty} r^m u(r) = u_\infty \equiv \begin{cases} 
\left[\frac{2+l}{p-1}(n - 2 - \frac{2+l}{p-1})\right]^{1/(p-1)} / k_\infty^{1/(p-1)} & \text{or} \\
0.
\end{cases}$$
Furthermore, if \( u_\infty = 0 \), then \( \lim r^{n-2}u(r) \) exists and is finite and positive.

**Remark 1.1.** When \( l = -2 \), a result similar to Theorem A holds (see [16,17]).

The equation (1.2), if \( l_1 = l_2 = 0 \), reduces to the following famous Lane-Emden equation

\[
\Delta u + u^p + u^q = 0 \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3.
\] (1.4)

The equation (1.4) has been paid much attention recently. When \( p \) and \( q \) are in the range \( \frac{n}{n-2} < p < \frac{n+2}{n-2} < q \), the mixed growth structure (supercritical for \( u \) large and subcritical for \( u \) small) has a profound impact on the existence and non-existence theory, and changes the outcome for the Lane-Emden equation, the analysis is surprisingly difficult. Recently Bamon etc in [5] proved that if \( q \) is fixed and let \( p \) approach \( \frac{n}{n-2} \) from below, then (1.4) has a large number of radial solutions. A similar result holds in [5] for \( p > \frac{n}{n-2} \) and \( p \) is fixed, while letting \( q \) approach \( \frac{n+2}{n-2} \). In addition, they proved that (1.4) don’t possess any solution if \( q \) is fixed and then let \( p \) be close enough to \( \frac{n}{n-2} \).

For physical reasons, we consider the positive radial solutions of equation (1.2), it reduces to

\[
u'' + \frac{n-1}{r}u' + r^{l_1}u^p + r^{l_2}u^q = 0 \quad n \geq 3.
\] (1.5)

where \( r = |x| \).

In order to state our main results concerning this question, we need some definitions, which will be used throughout this paper:

\[
\alpha_1 = \frac{2 + l_1}{p-1}, \quad \alpha_2 = \frac{2 + l_2}{q-1},
\]

\[
\lambda_1^{-1} = \alpha_1(n - 2 - \alpha_1), \quad \lambda_2^{-1} = \alpha_2(n - 2 - \alpha_2),
\]

\[
v_1(r) = r^{\alpha_1}u(r), \quad v_2(r) = r^{\alpha_2}u(r),
\]

(note that \( \alpha_1 > \alpha_2, n - 2 - \alpha_1 > 0 \)).

We sometimes suppose

\[
p \neq \frac{n + 2 + 2l_1}{n - 2}, \quad q \neq \frac{n + 2 + 2l_2}{n - 2}.
\] (1.6)

Now we give the following definitions:

\( u(r) \) is said to be singular solution of (1.2) at infinity if \( \limsup_{r \to \infty} r^{\alpha_1}u(r) > 0 \), and is said to be regular at infinitely if \( \lim_{r \to \infty} r^{n-2}u \) exists. Similarly, \( u(r) \) is said to be singular solution of (1.2) at 0 if \( \limsup_{r \to 0} r^{\alpha_2}u > 0 \), is said to be regular at 0 if \( \lim_{r \to 0} u \) exists.

One of the main results is as follows.

**Theorem 1.** If \( -2 < l_2 < l_1 \leq 0, \frac{n+1}{n-2} < p < q \), and \( p, q \) satisfy (1.6) Then, for any solution of (1.3), we have:

\[
\lim_{r \to \infty} r^{\alpha_1}u = \lambda_1, \quad \text{or} \quad \lim_{r \to 0} r^{n-2}u = c_1
\] (1.7)
for some constant $c_1 > 0$. Moreover,
\[ \lim_{r \to 0} r^{\alpha_2} u = \lambda_2, \quad \text{or} \quad \lim_{r \to 0} u = c_2 \]
for some constant $c_2 > 0$.

**Remark 1.2.** If, say, the origin is a singularity, the term $|x|^l_2 u^q$ is dominant and the term $|x|^l_1 u^p$ is thus a small perturbation. At infinity, the situation is just reversed, i.e. the term $|x|^l_1 u^p$ is dominant and the term $|x|^l_2 u^q$ is a small perturbation.

The device in the proof of Theorem 1 is an energy function which plays a central role in the convergence of (1.7) and (1.8). If $p = \frac{n+2+2l}{n-2}$, then the coefficient of the energy function is 0 and $r^{2+l_1} u$ could oscillates endlessly near the $\infty$. As the same reason, if $q = \frac{n+2+2l}{n-2}$, then $r^{2+l_2} u$ could oscillates endlessly near the origin.

**Theorem 2.** If $-2 < l_2 < l_1 \leq 0$, $\frac{n+2}{n-2} < p < q$, let $u$ be a positive radial solution of (1.2), then

(i) if $q = \frac{n+2+2l_2}{n-2}$, then either (1.8) holds, or $v_2(r)$ oscillates endlessly near the origin between two sequences $\mu_{1,i}$ and $\mu_{2,i}$ satisfying $0 < \mu_{1,i} < \mu_{2,i}$ and

\[ \lim_{i \to \infty} \mu_{1,i} = \mu_1, \quad \lim_{i \to \infty} \mu_{2,i} = \mu_2 \]

\[ \mu_1 = \liminf_{r \to 0} r^{\frac{n-2}{2}} u(r) < \limsup_{r \to 0} r^{\frac{n-2}{2}} u(r) = \mu_2, \]

where $\mu_1$ and $\mu_2$ are fixed values satisfying
\[ 0 < \mu_1 \leq \lambda_2 \leq \mu_2, \quad b(\mu_1) = b(\mu_2) = 0, \quad \text{with} \quad b(v) = \frac{1}{q+1} v^{q+1} - \frac{\lambda_2^{q-1}}{2} v^2; \quad (1.9) \]

(ii) if $p = \frac{n+2+2l_1}{n-2}$, then either (1.7) holds or $v_1(r)$ oscillates endlessly near $\infty$ between two sequences $\mu'_{1,i}$ and $\mu'_{2,i}$ satisfying $0 < \mu'_{1,i} < \mu'_{2,i}$ and

\[ \lim_{i \to \infty} \mu'_{1,i} = \mu'_1, \quad \lim_{i \to \infty} \mu'_{2,i} = \mu'_2 \]

\[ \mu'_1 = \liminf_{r \to \infty} r^{\frac{n-2}{2}} u(r) < \limsup_{r \to \infty} r^{\frac{n-2}{2}} u(r) = \mu'_2, \]

where $\mu'_1$ and $\mu'_2$ are fixed values satisfying
\[ 0 < \mu'_1 \leq \lambda_1 \leq \mu'_2, \quad b_1(\mu'_1) = b_1(\mu'_2), \quad \text{with} \quad b_1(v) = \frac{1}{p+1} v^{p+1} - \frac{\lambda_1^{p-1}}{2} v^2. \quad (1.10) \]

For equation (1.3), when $K \equiv 1$, it becomes into the simplest model which is the generalized Lan-Emden equation or Emden-Fowler equation in astrophysics
\[ \Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3. \quad (1.11) \]

For Equation (1.11), we have the following uniqueness result.
Theorem B. If $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$, then (1.11) admits exactly one solution $\frac{2}{p-1}(n-2 - \frac{2}{p-1})r^{-\frac{2}{p-1}}$, singular at the infinity. If $p > \frac{n+2}{n-2}$ then (1.11) admits exactly one solution $\frac{2}{p-1}(n-2 - \frac{2}{p-1})r^{-\frac{2}{p-1}}$, singular at the origin.

Remark 1.3. From the Theorem B, we know that (1.11) has only one solution that is singular at infinity for $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$ and at the origin for $p > \frac{n+2}{n-2}$. Inspired by the Theorem B, we derive the following result.

Theorem 3. (i) if $-2 < l_2 < l_1 \leq 0, 0 < \frac{n+l_1}{n-2} < p < \frac{n+2+2l_2}{n-2}$, then (1.2) admits exactly one solution singular at infinity. This solution has the following exact limits:

$$\lim_{r \to \infty} r^{\alpha_1}u = \lambda_1; \quad \lim_{r \to 0} r^{\alpha_2}u = \lambda_2,$$

(ii) if $-2 < l_2 < l_1 \leq 0, 0 < \frac{n+2+2l_2}{n-2} < p$, then (1.2) admits exactly one solution singular at the origin. This solution has the exact limits:

$$\lim_{r \to \infty} r^{\alpha_1}u = \lambda_1, \quad \lim_{r \to 0} r^{\alpha_2}u = \lambda_2.$$

This paper is organized as follows. In Section 2, we make some basic observations and fundamental estimates of positive solutions. In Section 3, the asymptotic behaviors at $\infty$ and 0 of positive solutions are studied. Finally, the uniqueness result is established.

Throughout this paper, unless otherwise stated, the letter $C$ will always denote various generic constant which is independent of $u$ and change from line to line.

2. Preliminaries

In this section we present some preliminary results for radial solutions $u(r)$ of (1.5), where $r = |x|$ is the radius.

First we will prove the following priori estimates, which are inspired by the work of Ni [20].

Lemma 2.1. If $-2 < l_2 < l_1 \leq 0, 0 < \frac{n+l_1}{n-2} < p < q$, let $u(r)$ be a positive radial solution of (1.5) for $r \in (0, \infty)$, then we have for some positive constant $C$

(i) If $u$ tends to $\infty$ as $r \to 0$, then $u(r) \leq Cr^{-\frac{2+l_2}{n-2}}$;

(ii) If $u$ tends to 0 as $r \to \infty$, then $u(r) \leq Cr^{-\frac{2+l_1}{n-2}}$.

Proof. For a radial solution $u = u(r)$, we rewrite (1.2) in the following form:

$$ (r^{n-1}u')' + r^{n-1}(r^{l_1}u^p + r^{l_2}u^q) = 0. \tag{2.1} $$

Since $u \to \infty$ as $r \to 0$, there exists small $r_0 > 0$, such that $u' \not< 0$ in $(0, r_0)$.

Integrating (2.1) from $\bar{r}$ to $r$ ($\bar{r} < r < r_0$), we obtain

$$ r^{n-1}u'(r) = r^{n-1}u'() - \int_\bar{r}^r s^{n-1}(s^{l_1}u^p + s^{l_2}u^q)ds. $$
Therefore, $r^{n-1}u'(r) \leq -\int_{\bar{r}}^{r} s^{n-1}(s^1u^p + s^2u^q)ds$, for all $0 < \bar{r} < r$. Then letting $\bar{r} \to 0$, we obtain

$$r^{n-1}u'(r) \leq -\int_{0}^{r} s^{n-1}(s^1u^p + s^2u^q)ds \leq -\int_{0}^{r} s^{n-1}s^2u^qds.$$ 

Since $u$ is decreasing near $r = 0$, we find that

$$r^{n-1}u'(r) < -u^q(r) \int_{0}^{r} s^{n-1}s^2ds = -\frac{1}{n + l_2}r^{n+l_2}u^q(r),$$

which in turn leads to

$$\frac{u'(r)}{u^q(r)} \leq -r^{l_2+1}. \quad (2.2)$$

Integrating (2.2) over $(\bar{r}, r)$, we have

$$\int_{\bar{r}}^{r} \frac{u'(s)}{u^q(s)}ds \leq -\int_{\bar{r}}^{r} s^{l_2+1}ds. \quad (2.3)$$

It follows from (2.3) that

$$u^{1-q}(r) \geq u(\bar{r})^{1-q} + \frac{q - 1}{l_2 + 2}(r^{l_2+2} - \bar{r}^{l_2+2}).$$

Letting $\bar{r} \to 0$, we have

$$u^{1-q}(r) \geq Cr^{2+l_2}.$$ 

So we have $u(r) \leq Cr^{\frac{2+l_2}{1-q}}$ for $0 < r < r_0$, and the proof is completed.

Part (ii) of Lemma 2.1 may be handled in a similar way. Similar to the proof of (i) there exists a large number $R > 0$ such that, for all $r > R$,

$$r^{n-1}u'(r) < -\int_{R}^{r} s^{n-1}s^1u^pds.$$ 

By a similar computation, we have

$$\frac{u'(r)}{u^p} \leq C(\frac{R^{n+l_1}}{n + l_1}r^{1-n} - r^{1+l_1}). \quad (2.4)$$

Integrating (2.4) over $(R, r)$, we have

$$u^{1-p}(r) \geq Cr^{2+l_1}, \quad \text{at} \quad r = \infty.$$ 

So we have $u(r) \leq Cr^{\frac{2+l_1}{1-p}}$ as $r \to \infty$, and the proof of part (ii) is completed.

**Lemma 2.2.** Let $u$ be a positive radially symmetric solution of (1.2), then there exist two positive number $\bar{r}$ and $\bar{\bar{r}}$ such that

(i) $|u'(r)| \leq Cr^{-\left(\frac{2+l_1}{n+l_1}+1\right)}$;

(ii) $|u''(r)| \leq Cr^{-\left(\frac{2+l_1}{n+l_1}+2\right)}$ for $0 < r < \bar{r}$. 

Lemma 2.1 and Green's identity, we obtain
\[ \alpha \]
Let \( R \)
where
\[ \alpha \]
and the proof is over.

Proof. (i) Integrate (1.2) in a small ball \( B_r \) with radius \( r \) centered at 0. From the Lemma 2.1 and Green's identity, we obtain
\[ \begin{align*}
- \omega_n r^{n-1} u'(r) &= - \int_{B_r} \Delta u = \int_{B_r} (|x|^l_1 u^p + |x|^l_2 u^q) dx \\
&\leq C \int_{B_r} s^{l_1} u^p s^{n-1} ds \leq C \int_{\omega} s^{2+\frac{l_1}{q-1}l_2+n-1} ds \\
&= Cr^{-\frac{2+\frac{l_1}{q-1}l_2+n}{q-1}}.
\end{align*} \]

So as \( r \to 0 \), we have
\[ -u'(r) = |u'(r)| \leq Cr^{-\frac{2+\frac{l_1}{q-1}l_2+1}{q-1}} = Cr^{-\frac{2+\frac{l_2}{q-1}}{q-1}} \]
\[ |u''| \leq \frac{n-1}{r} |u'| + r^{l_1} u^p + r^{l_2} u^q \]
\[ \leq C[r^{-\frac{2+\frac{l_2}{q-1}+2}{q-1}} + r^{l_2-\frac{2+\frac{l_2}{q-1}}{q-1}}] \leq Cr^{-\frac{2+\frac{l_2}{q-1}+2}{q-1}}. \]

And the proof of (i) is completed.

Part (ii) of Lemma 2.2 may be handled in a similar fashion. As (i), for large \( r \) we have
\[ \omega_n r^{n-1} (-u'(r)) = - \int_{B_r} \Delta u = \int_{B_r} s^{l_1} u^p + s^{l_2} u^q dx \]
\[ \leq C + C \int_{R_0} s^{2+\frac{l_1}{p-1}l_1+n-1} ds, \]
where \( R_0 \) is a large positive number. By a simple computation, we obtain
\[ |u'(r)| \leq Cr^{-\frac{2+\frac{l_1}{p-1}+1}{p-1}}, \quad |u''(r)| \leq Cr^{-\frac{2+\frac{l_1}{p-1}+2}{p-1}} \quad \text{at} \quad r = \infty, \]
and the proof is over.

Lemma 2.3. Suppose that \( u \) is a positive solution of (1.5). Let \( v(r) = r^\alpha u(r) \), then \( v \) satisfies
\[ v'' + \frac{n-1-2\alpha}{r} v' - \frac{(n-2-\alpha)\alpha}{r^2} v + r^{l_1-(p-1)\alpha} u^p + r^{l_2-(q-1)\alpha} u^q = 0. \]
Let \( \alpha = \alpha_1 \), then we have \( (v_1 = r^{\alpha_1} u) \)
\[ v_1'' + \frac{n-1-2\alpha_1}{r} v_1' - \frac{(n-2-\alpha_1)\alpha_1}{r^2} v_1 + \frac{v_1^p}{r^2} + r^{l_2-(q-1)\alpha_1} v_1^q = 0. \] (2.5)
Let \( \alpha = \alpha_2 \), then we have \( (v_2 = r^{\alpha_2} u) \)
\[ v_2'' + \frac{n-1-2\alpha_2}{r} v_2' - \frac{(n-2-\alpha_2)\alpha_2}{r^2} v_2 + \frac{v_2^p}{r^2} + r^{l_2-(q-1)\alpha_2} v_2^p = 0 \] (2.6)

This lemma can be proved by straightforward calculations, thus we omit it here.
Lemma 2.4. we have
\[ v_1^2 r \in L^1(R, \infty), \quad v_2^2 r \in L^1(0, R), \] (2.7)
where \( R \) is a large positive number.

Proof. Multiplying (2.5) by \( v_1' r^2 \) and integrating from \( R \) to \( r > R \), we obtain
\[
\frac{v_1'^2 s^2}{2} \bigg|_R^r + c_1 \int_R^r v_1'^2 s ds - \frac{\lambda_1^{-1}}{2} v_1^2 \bigg|_R^r + \frac{1}{p+1} v_1^{p+1} \bigg|_R^r + \int_R^r s^{\lambda_2 - \alpha_1(q-1)+2} v_1'^2 ds = 0, \tag{2.8}
\]
where \( c_1 = n - 2 - 2\alpha_1 \).

From (ii) of Lemma 2.2, we obtain
\[ |v_1'| \leq \frac{C}{r}; \quad |v_1''| \leq \frac{C}{r^2}, \quad \text{at} \quad r = \infty, \tag{2.9}\]
so \( v_1^{p+1} \bigg|_R^r \), \( \frac{v_1'^2 s^2}{2} \bigg|_R^r \), and \( \int_R^r s^{\lambda_2 - \alpha_1(q-1)+2} v_1'^2 ds \) are bounded at \( r = \infty \), and from that we have
\[ \int_R^r v_1'^2 ds \leq C, \quad \text{for all} \quad r > R \]
since \( c_1 \neq 0 \) by (1.6), and (2.7)_1 follows. (2.7)_2 is handled by the similar way, we omit it here.

Lemma 2.5. We have
\[ \lim_{r \to \infty} rv_1' = 0; \quad \lim_{r \to 0} rv_2' = 0. \tag{2.10}\]

Proof. Now we prove the (2.10)_1. Suppose for contradiction that it is not true, then there exist a sequence \( r_k \to +\infty \) such that
\[ |v_1'(r_k)r_k| \geq C. \]
From (2.9), one obviously has, near \( \infty \),
\[ |v_1'^2 r^2| \leq \frac{M}{r}, \]
for some \( M > 0 \). Combining the above two inequalities yields
\[ |v_1'^2 (r)r^2 - v_1'(r_k)^2 r_k^2| \leq M |r - r_k| \max\left(\frac{1}{r}, \frac{1}{r_k}\right), \]
and so
\[ v_1'^2 (r) r^2 \geq \frac{C^2}{2}, \quad r \in [(1 + \varepsilon)^{-1} r_k, (1 + \varepsilon) r_k], \quad \varepsilon(1 + \varepsilon) = \frac{C^2}{2M}. \]
This contradicts (2.7)_1.

The proof of (2.10)_2 is handled by the same way, as the previous proof, we have, near 0,
\[ |(v_2'^2 r^2)| \leq \frac{M}{r}. \]
for some $M > 0$, and by a similar caculation, we will obtain a contradiction and the proof is completed.

**Lemma 2.6** Let $u$ be a positive superharmonic function near $\infty$ and $\bar{u}$ its spherical mean. Then, $r^{n-2}\bar{u}$ is increasing as $r \to \infty$.

**Proof.** Put $f(t) := r^{n-2}\bar{u}(r), t = \log r$. Then $f$ satisfies

$$f'' - (n-2)f' \leq 0$$

and $f'(t) \leq e^{(n-2)(t-T)}f'(T)$ on $[T,t]$ for $T$ large. Because $f$ is positive, $f$ must be increasing near $\infty$. It implies that $(r^{n-2}\bar{u}(r))_r > 0$ near $\infty$.

### 3. Asymptotic behavior

In this section we investigate the asymptotic behavior at $\infty$ and 0 of positive radial solutions of (1.2). We will prove any positive radial solution of (1.2) must behave either like $r^{-\alpha_1}$ or $r^{2-n}$ at $\infty$. In addition, if any positive radial solution of (1.2) is singular at the origin, then it must behave like $r^{-\alpha_2}$. Now we give the proof of Theorem 1.

**The proof of Theorem 1.** Consider the function $a(r) = \frac{v_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1}v_1^2}{2}$. By Lemma 2.4 and Lemma 2.5, we have, for fixed $R > 0$,

$$\frac{v_1^2 r^2}{2} \to 0, \quad \int_R^r v_1^2 s \to c_2, \quad \int_R^r r^{2\frac{p+1}{p-1}+2}v_1^p \to c_3, \quad \text{as } r \to \infty$$

for some constant $c_2$ and $c_3$. This implies, by (2.8), that $a(r)$ must tend to a finite constant $c_4$ as $t \to \infty$. We claim $v_1$ approaches a finite limit as $r \to \infty$. If not, we may choose two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $\infty$ as $i \to \infty$ such that

$$\begin{align*}
\{\{\eta_i\}\text{ are local minima of } v_1, \{\xi_i\}\text{ are local maxima.} \\
\eta_i < \xi_i < \eta_{i+1}, i = 1, 2, \ldots
\end{align*}$$

And we have

$$v_1(\eta_i) \to m_1, \quad v_1(\xi_i) \to m_2 \quad \text{as } i \to \infty,$$

for some positive constants $m_1, m_2$ with $m_1 < m_2$. Since $a(r)$ tend to a finite constant as $r \to \infty$, we have

$$\frac{m_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1}m_1^2}{2} = \frac{m_2^{p+1}}{p+1} - \frac{\lambda_1^{p-1}m_2^2}{2} = c_4,$$

where $c_4$ is a constant. However, the intermediate value theorem shows that there exists $r_i \in (\eta_i, \xi_i)$ such that

$$v(r_i) = m_0, \quad m_1 < m_0 < m_2 \text{ and } \frac{da(v)}{dv}(r_i) = 0.$$ 

Furthermore, we have $\frac{m_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1}m_1^2}{2} \neq c_4$, since $\frac{v_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1}v_1^2}{2}$ has only one minima for $v_1 \in [0, +\infty]$. A contradiction is obtained. Similarly, we conclude that $r^{a_2}u(r)$ approaches a finite limit as $r \to 0$. 

The characteristic equation of (3.11) has the two characteristic values $D$ where

\[ \frac{d^2v_1}{dt^2} + (n - 2 - 2\alpha_1)\frac{dv_1}{dt} - (n - 2 - \alpha_1)v_1 + v_1^p + e^{(l_2-(q-1)\alpha_1+2)t}v_1^q = 0, \]

where $t = \log r$. From Lemma 2.5, we have $v'(t) \to 0$ as $t \to \infty$. So $\lim_{t \to \infty} v''(t)$ exists and must be 0. Immediately, we have $v_\infty = \lambda_1$ or 0.

If $\lim_{r \to \infty} v_1(r) = 0$ or $\lim_{r \to 0} v_2(r) = 0$, (2.5) and (2.6) suggests that $v_i(i = 1, 2)$ tends to zero at an algebraic rate. Indeed, since $p, q > 1$ and $v_i(i = 1, 2)$ is expected to satisfy asymptotically the following equations

\[ v''_1 + \frac{n-1-2\alpha_1}{r}v'_1 - \frac{\lambda_1^{p-1}}{r^2}v_1 = 0 \quad r \to \infty, \]
\[ v''_2 + \frac{n-1-2\alpha_2}{r}v'_2 - \frac{\lambda_2^{q-1}}{r^2}v_2 = 0 \quad r \to 0. \]

Therefore $v_i$ should satisfy the following asymptotical behaviors

\[ v_1 \approx r^{-(n-2-\alpha_1)} \text{ at } \infty, \quad v_2 \approx r^{\alpha_2} \text{ at } 0. \]

Now we claim: (i) If $\lim_{r \to \infty} v_1 = 0$, then $\lim_{r \to \infty} r^{n-2}u = c > 0$;

(ii) If $\lim_{r \to 0} v_2 = 0$, then $\lim_{r \to 0} u = c_1 > 0$.

First, we prove (i), by assumption, for any $\varepsilon > 0$ there exists a positive number $r_\varepsilon$ such that $v_1$ satisfies

\[ v''_1 + \frac{n-1-2\alpha_1}{r}v'_1 - \frac{\lambda_1^{p-1}}{r^2}v_1 \geq 0, \quad r > r_\varepsilon. \quad (3.11) \]

The characteristic equation of (3.11) has the two characteristic values

\[ a_1 = \alpha_1 - \frac{n-2-\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha_1 + O(\varepsilon) \]
\[ a_2 = \alpha_1 - \frac{n-2+\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha_1 + 2 - n + O(\varepsilon) \]

Rewrite (3.11)

\[ (D - \frac{a_1 - 1}{r})(D - \frac{a_2}{r})v_1 \geq 0, \]

where $D := \frac{d}{dr}, D^2 := \frac{d^2}{dr^2}$.
Let $(D - \frac{a_2}{r})v_1 = U_1$, so we have

\[ U'_1 + \frac{1 - a_1}{r}U_1 \geq 0, \]

from which, we have

\[ [r^{1-\alpha_1+O(\varepsilon)}(D - \frac{a_2}{r})v_1]' \geq 0. \quad (3.12) \]
Observe that, for $\varepsilon$ small enough,
\[
\lim_{r \to \infty} r^{1-\alpha_1+O(\varepsilon)} \left( D - \frac{\alpha_2}{r} \right) v_1 = 0 \quad \text{by (2.9)},
\]
since $1 - \alpha_1 + O(\varepsilon) < 1$. It follows from (3.12) that
\[
(D - \frac{\alpha_2}{r}) v_1 \leq 0, \quad r > r_\varepsilon.
\]
Integrating once from $r_\varepsilon$ to $r$ yields
\[
v_1 \leq c_\varepsilon r^{\alpha_2} = c_\varepsilon r^{\alpha_1 + 2 - n + O(\varepsilon)}, \quad r > r_\varepsilon.
\]
Thus for $\varepsilon$ sufficiently small,
\[
v''_1 + \frac{n-1-2\alpha_1}{r} v'_1 - \frac{\lambda_{p-1}}{r^2} v_1 = g(r) \tag{3.13}
\]
with
\[
g(r) = \frac{v_1^p}{r^2} + r^{j_2 - (q-1)\alpha_1} v_1^q = O(r^{-2-\delta}) \quad \text{near } \infty,
\tag{3.14}
\]
where $\delta$ is a positive constant.
Applying the method of variation of parameters to (3.13), $v_1$ is represented by
\[
v_1(r) = C_1(R) r^{\alpha_1} + C_2(R) r^{\alpha_1 + 2 - n} + \frac{r^{\alpha_1 + 2 - n}}{2 - n} \int_R^r s^{n-1-\alpha_1} g(s) ds - \frac{r^{\alpha_1}}{2 - n} \int_R^r s^{1-\alpha_1} g(s) ds.
\]
By a similar computation, we have, from (3.14) and the bounded of $v_1(r)$,
\[
v_1(r) = C r^{\alpha_1 + 2 - n} + o(r^{\alpha_1 + 2 - n}),
\]
hence,
\[
r^{n-2} u(r) \leq C \quad \text{at } r = \infty.
\]
By Lemma 2.6, there exists a constant $c$ such that $r^{n-2} u(r) \to c$ as $r \to \infty$.
The proof of (ii) is handled by the same way. As the proof of (i), for any $\varepsilon' > 0$ there exists a positive number $r_{\varepsilon'}$ such that
\[
\lim_{r \to 0} r^{n-1-\alpha_2+O(\varepsilon')} (D - \frac{\alpha_2}{r}) v_2 = 0,
\]
and
\[
(D - \frac{\alpha_2}{r}) v_2 \geq 0 \quad 0 < r < r_{\varepsilon'}.
\]
Integrating once from $r$ to $r_{\varepsilon'}$ yields
\[
v_2 \leq c_\varepsilon' r^{\alpha_2 + O(\varepsilon')}.
\]
Applying the method of variation of parameters to (2.6), we immediately that $v_2$ is bounded by $r^{\alpha_2}$, and in turn $u$ is bounded, by standard theory, we have $\lim_{r \to 0} u(r) = c_1$.  

When \( p = \frac{n+2+2\ell}{n-2} \) or \( q = \frac{n+2+2\ell}{n-2} \), there is another possibility; \( r^{\frac{n-2}{2}} u \) could oscillate endlessly near \( \infty \) or oscillate endlessly near the origin.

**The proof of Theorem 2.** (i) The transformed function \( v_2(t) := r^{\alpha_2} u, t = \log r \), satisfies
\[
v''_2 - \lambda^q_2 v_2 + v^q_2 + e^{\delta_1 t} v^p_2 = 0,
\]
where \( \delta_1 = (\alpha_1 - \alpha_2)(p - 1) > 0 \). Note that by Lemma 2.1 (i), \( v_2(t) \) is bounded near \(-\infty \). Define an energy function
\[
E(t) := \frac{1}{2} v'^2_2 - \lambda^q_2 \frac{1}{2} v^2_2 + \frac{1}{q+1} v^{q+1}_2.
\]
As in (2.8), we have
\[
E(t) = C(T) + \int_t^T e^{\delta_1 s} v^p_2 v'_2 ds,
\]
where \( T \) is a fixed number.

In case that \( v_2 \) oscillates near \(-\infty \), then we may suppose that
\[
0 \leq \mu_1 = \lim_{t \to -\infty} \inf v_2(t) < \lim_{t \to -\infty} \sup v_2(t) = \mu_2 < \infty.
\]
Then, there exists two sequences \( \eta_i \) and \( \varepsilon_i \) going to \(-\infty \) as \( i \to \infty \) such that \( \eta_i \) and \( \varepsilon_i \) are local minima and local maxima of \( v_2 \), respectively, satisfying \( \eta_i < \varepsilon_i < \eta_{i+1}, i = 1, 2, \ldots \). From (2.8), we know \( \lim_{t \to -\infty} \int_t^T e^{\delta_1 s} v^p_2 v'_2 ds \) exists, so \( E = \lim_{t \to -\infty} E(t) \) exists, and from that we have
\[
\lim_{i \to \infty} E(\eta_i) = -\lambda^q_2 \frac{1}{2} \mu_1^2 + \frac{1}{q+1} \mu_1^{q+1} = -\lambda^q_2 \frac{1}{2} \mu_2^2 + \frac{1}{q+1} \mu_2^{q+1} = \lim_{i \to \infty} E(\varepsilon_i),
\]
which implies \( b(\mu_1) = b(\mu_2) \).

Observe that for each \( i > 1 \),
\[
0 \leq v''_2(\eta_i) = \lambda^q_2 v_2(\eta_i) - v^q_2(\eta_i) - v^p_2(\eta_i)e^{\delta_1 \eta_i}, \quad (3.16)
\]
while
\[
0 \geq v''_2(\varepsilon_i) = \lambda^q_2 v_2(\varepsilon_i) - v^q_2(\varepsilon_i) - v^p_2(\varepsilon_i)e^{\delta_1 \varepsilon_i}. \quad (3.17)
\]
Putting \( v_2(\eta_i) = \mu_{1,i}, v_2(\varepsilon_i) = \mu_{2,i} \), we have
\[
\lim_{i \to \infty} \mu_{1,i} = \mu_1, \quad \lim_{i \to \infty} \mu_{2,i} = \mu_2.
\]
From (3.16) and (3.17) we obtain
\[
0 < \mu_1 \leq \lambda_2 \leq \mu_2.
\]

The proof of (ii) is handled by the similar way, we omit it here.

**4. A uniqueness result**

In this section, we shall prove a uniqueness result for positive radial singular solutions of (1.2) when \( p > \frac{n+2+2\ell}{n-2} \) and \( q < \frac{n+2+2\ell}{n-2} \). More precisely, we require that solutions be singular at infinity for \( q < \frac{n+2+2\ell}{n-2} \), and at the origin for \( p > \frac{n+2+2\ell}{n-2} \). In
order to prove the Theorem 3, we need a finer asymptotic behavior of solution of (1.2) near 0 and ∞.

Let \( w(t) = v_1 - \lambda_1, t = \log r \). If \( \lim_{t \to +\infty} v(t) = \lambda_1 \), then it satisfy \( w(t) \to 0 \) as \( t \to +\infty \) and

\[
 w'' + (n - 2 - 2\alpha_1)w' + (2 + l_1)(n - 2 - \alpha_1)w + f(t) + (w + \lambda_1)^q e^{bt} = 0, \tag{4.1}
\]

where \( \delta = \frac{(2 + l_1)(1-q)}{p-1} + 2 + l_2 < 0 \), and

\[
 f(w) = (\lambda_1 + w)^p - \lambda_1^p - p\lambda_1^{p-1}w = \lambda_1^p \sum_{k=2}^{\infty} \frac{(p-k+1)}{k!} \left( \frac{w}{\lambda_1} \right)^k
 = \frac{(p-1)\lambda_1^{p-2}}{2} w^2 + o(w^2) \quad \text{for } w \text{ near 0}.
\]

**Theorem 4.1.** Let \( u(r) \) be a singular solution of (1.2) at infinity and \( \frac{n+l_1}{n-2} < p < \frac{n+2+2\delta}{n-2} \), then for any \( \varepsilon \in (0, -\delta) \) we have

\[
 w(t) = O(e^{-\varepsilon t}), \quad w'(t) = O(e^{-\varepsilon t}), \quad \text{as } t \to +\infty. \tag{4.2}
\]

**Proof.** For \( T > 0 \) and \( t \in (0, T) \), multiply (4.1) by \( 2w'(t) \) and integrate from \( t \) to \( T \) to obtain

\[
 [w^2 + (2 + l_1)(n - 2 - \alpha_1)w^3]|^T_t + 2(n - 2 - 2\alpha_1) \int_T^t w'^2 ds
 + 2 \int_T^t f(t)w' + 2 \int_T^t (w + \lambda_1)^q e^{bt} w'ds = 0. \tag{4.3}
\]

By (2.10), it is easy to see that for large \( t \)

\[
 w^2(T) \to 0, \quad w^3(T) \to 0, \quad \int_T^t f(s)w'ds = -\frac{(p-1)\lambda_1^{p-2}}{6} w^3(t) + o(w^3(t)) \quad \text{as } T \to +\infty.
\]

It follows from Lemma 2.4 and by letting \( T \to \infty \) in (4.3) that for large \( t \)

\[
 w^2(t) + (2 + l_1)(n - 2 - \alpha_1)w^2(t) \leq C|w|^3 + \int_t^\infty (w + \lambda_1)^q e^{\delta s} w'ds,
\]

since \( n - 2 - 2\alpha_1 < 0 \). It follows that for large \( t \)

\[
 w^2(t) + (2 + l_1)(n - 2 - \alpha_1)w^2(t) \leq C \int_t^\infty (w + \lambda_1)^q e^{\delta s} w'ds \leq C e^{\delta t}. \tag{4.4}
\]

Hence by (4.4)

\[
 |w'(t)| + |w(t)| \leq Ce^{\delta t}.
\]

Therefore

\[
 \left| \int_t^T (w + \lambda_1)^q e^{\delta s} w'ds \right| \leq C \int_t^T e^{\frac{\delta s}{2}} \leq C e^{\frac{\delta}{4} t},
\]

and so by (4.4)

\[
 |w'(t)| + |w(t)| \leq Ce^{\frac{\delta}{4} t}.
\]
Thus for any $m > 0$, using a simple iteration of $m$-step in (4.4) yields

$$|w'(t)| + |w(t)| \leq C e^{(\frac{2m-1}{2} t)},$$

and (4.2) follows by taking $m$ large.

**Remark 4.1.** As a matter of fact, apply by the method of variation of parameters to (4.1), the $\varepsilon$ can reach at $-\delta$. Since from (4.2), we have $f(t) = O(w^2(t)) = O(e^{\delta t})$ and

$$w(t) = -\frac{1}{\omega} \int_t^\infty e^{\frac{n-2-2\alpha}{2}(s-t)} \sin \omega(s-t)[f(s) + e^{\delta t}(v_1 + \lambda_1)^p]ds = O(e^{\delta t}),$$

where $\omega = ((2 + l_1)(n - 2 - \alpha_1) - \frac{1}{4}(n - 2 - 2\alpha_1)^2)^\frac{1}{2}$.

**Remark 4.2.** If $u(r)$ is a singular solution of (1.2) at 0, and $\frac{n+l_1}{n-2} < q < \frac{n+2+2l_1}{n-2}$, then we have a similar result

$$v_2(r) - \lambda_2 = O(r^\epsilon), \quad (v_2(r) - \lambda_2)' = O(r^{\epsilon+1}) \quad \text{as} \quad r \to 0,$$

where $\epsilon \in (0, \frac{(2+1)(1-p)}{q-1} + 2 + l_1)$.

Now we are ready to prove Theorem 3. Let $u_1(r)$ and $u_2(r)$ be two different singular solutions of (1.2) at infinity. We introduce the function

$$\bar{w}(t) = r^{\alpha_1}u_1(r) - r^{\alpha_1}u_2(r) = w_1(t) - w_2(t), \quad t = \log r,$$

and show that $\bar{w}(t)$ is identically zero.

**Proof of Theorem 3.** Clearly $\bar{w}(t)$ satisfies

$$\bar{w}''(t) + (n - 2 - 2\alpha_1)\bar{w}'(t) + (2 + l_1)(n - 2 - \alpha_1)\bar{w} + (f(w_1) - f(w_2)) + [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta t} = 0. \quad (4.5)$$

For $T > 0$ and $t \in (0, T)$, multiply (4.5) $2\bar{w}'$ and integrate from $t$ to $T$ to obtain

$$[\bar{w}'^2 + (2 + l_1)(n - 2 - \alpha_1)\bar{w}^2]_t^T + (n - 2 - 2\alpha_1) \int_t^T \bar{w}'^2$$

$$+ 2 \int_t^T [f(w_1) - f(w_2)]\bar{w}' + 2 \int_t^T [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta s} \bar{w}' = 0. \quad (4.6)$$

Thus we have the following estimates:

$$\left| \int_t^T [f(w_1) - f(w_2)]\bar{w}' \right| \leq C \int_t^T |\bar{w}'\bar{w}|e^{-\varepsilon s} \leq C \int_t^T e^{-\varepsilon s}(\bar{w}^2 + \bar{w}'^2),$$

and

$$\int_t^T [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta s} \bar{w}' \leq C \int_t^T e^{-\delta s}(\bar{w}^2 + \bar{w}'^2).$$

For large $t$, it follows, by letting $T \to +\infty$ and using Theorem 4.1, that

$$\bar{w}'^2(t) + (2 + l_1)(n - 2 - \alpha_1)\bar{w}^2(t) \leq C \int_t^\infty e^{-\varepsilon s}(\bar{w}^2 + \bar{w}'^2),$$

and

$$\int_t^\infty [w_1(t) + \lambda_1]^q - (w_2(t) + \lambda_1)^q]e^{\delta s} \bar{w}' \leq C \int_t^\infty e^{-\delta s}(\bar{w}^2 + \bar{w}'^2).$$
since $n - 2 - 2\alpha_1 \leq 0$. From which, we have

$$\tilde{w}'^2(t) + \tilde{w}^2(t) \leq C \int_t^\infty e^{-\varepsilon s}(\tilde{w}'^2 + \tilde{w}^2),$$

since $(2 + l_1)(n - 2 - \alpha_1) > 0$. Hence $\tilde{w}'^2(t) + \tilde{w}^2(t) \equiv 0$ for all sufficiently large $t$ by the Gronwall inequality. According to the uniqueness of ordinary differential equation, $w$ is identically zero for all $t$.

When $p > \frac{n+2+2l_1}{n-2}$, we use the transformation

$$\tilde{w}_1(t) = r^{\alpha_2}u_1(r) - r^{\alpha_2}u_2(r) = z_1(t) - z_2(t), \quad t = \log r,$$

where $z(t) = r^{\alpha_2}u(r) - \lambda_2 \to 0$ as $t \to -\infty$. Then $\tilde{w}_1(t)$ satisfies

$$\tilde{w}_1''(t) + (n - 2 - 2\alpha_2)\tilde{w}_1'(t) + (2 + l_2)(n - 2 - \alpha_2)\tilde{w}_1 + (f(z_1) - \bar{f}(z_2)) + [(z_1(t) + \lambda_2)^p - (z_2(t) + \lambda_2)^p]e^{\delta_2 t} = 0 \quad (4.7)$$

where $\delta_2 = (P - 1)(\alpha_1 - \alpha_2) > 0$, and

$$\bar{f}(z) = \lambda_2^q - \lambda_2^q - q\lambda_2^{q-1}z = \lambda_2^q \sum_{k=2}^{\infty} \frac{q-k+1}{k!} \left(\frac{z}{\lambda_2}\right)^k$$

$$= \frac{(q-1)\lambda_2^{q-2}}{2} z^2 + o(z^2) \quad \text{for} \ t \nearrow -\infty.$$

The equation (4.7) is the same as (4.5). On the other hand, one easily sees from previous proof that the key ingredient of the proof for the case $p > \frac{n+2+2l_1}{n-2}$ is that the coefficient of the term $\tilde{w}_1'(t)$ is not less than 0. Fortunately, we have $n - 2 - 2\alpha_2 > 0$ when $p > \frac{n+2+2l_1}{n-2}$. Hence the proof above carries over immediately.

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