Extension with log-canonical measures and an improvement to the plt extension of Demailly–Hacon–Păun

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Abstract

With a view to proving the conjecture of “dlt extension” related to the abundance conjecture, a sequence of potential candidates for replacing the Ohsawa measure in the Ohsawa–Takegoshi $L^2$ extension theorem, called the “lc-measures”, which hopefully could provide the $L^2$ estimate of a holomorphic extension of any suitable holomorphic section on a subvariety with singular locus, are introduced in the first half of the paper. Based on the version of $L^2$ extension theorem proved by Demailly, a proof is provided to show that the lc-measure can replace the Ohsawa measure in the case where the classical Ohsawa–Takegoshi $L^2$ extension works, with some improvements on the assumptions on the metrics involved. The second half of the paper provides a simplified proof of the result of Demailly–Hacon–Păun on the “plt extension” with the superfluous assumption “$\text{supp} D \subset \text{supp} (S + B)$” in their result removed. Most arguments in the proof are readily adopted to the “dlt extension” once the $L^2$ estimates with respect to the lc-measures of holomorphic extensions of sections on subvarieties with singular locus are ready.

Dedicated to Prof. Fabrizio Catanese on the occasion of his 71st birthday.

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1 Introduction

This work is the first step towards generalising the result in [15], namely the extension theorem on purely log-terminal (plt) pairs, to an extension theorem on divisorially log-terminal (dlt) pairs. The latter extension theorem is essential in proving the Abundance Conjecture in algebraic geometry (see, for example, [15–18]).

There are two main results in this paper. The first one is an Ohsawa–Takegoshi-type $L^2$ extension theorem which replaces the Ohsawa measure in the estimate by a measure supported on the log-canonical (lc) centres of a given subvariety (Theorems 1.4.5 and 3.4.1). Such measure (called “lc-measure”, see Definition 1.4.3) seems to be well-suited to the use in birational geometry and can possibly provide the best possible estimates for minimal holomorphic extension with universal constant (see Example 2.3.2). The current result, following the line of thought and formulation given in [12], essentially recovers the classical Ohsawa–Takegoshi $L^2$ extension theorem from codimension-1 subvarieties (in which the section to be extended should vanish on the singular locus), with some relaxation on the assumptions on the given metrics and auxiliary functions in the formulation and improvement in the estimate.

The second result is an improvement to the result of the plt extension of Demailly–Hacon–Păun in [15] in the sense that a superfluous assumption is removed in their theorem with a simplified proof (see Theorems 1.6.1 or 4.6.1). Although the proof presented in this paper makes use of the $L^2$ extension with respect to the lc-measure, one may also use the version of $L^2$ extension with respect to the Ohsawa measure (for example, the version in [12]) together with the suitable slight improvement to the setup as presented in Sect. 1.3. However, it is the authors’ belief that the lc-measures will play a role in the future proof of the “dlt extension”. That’s why the two independent results are presented together to emphasise their linkage.

1.1 Background

Readers are referred to the survey by Varolin ([48]) for a quick outlook of the development of the celebrated Ohsawa–Takegoshi extension theorem since the paper [42] by Ohsawa and Takegoshi. They are also referred to [4,21] and [2] for some later development on the extension with optimal estimates.

For the background on the abundance conjecture, and the relevant non-vanishing conjecture as well as the conjecture on dlt extension, readers are referred to [15–18]. Among them, the work of Gongyo and Matsumura in [18] provides a proof to the dlt extension with a strong assumption via the $L^2$ injectivity theorem [37], while that of Demailly, Hacon and Păun in [15] proves the extension theorem for plt pairs via the Ohsawa–Takegoshi $L^2$ extension theorem, whose technique is followed closely in this paper.

By the time when this paper is finished, the authors are notified that Mihai Păun and Junyan Cao are finishing their version of $L^2$ extension theorem [6] which is aiming for...
the dlt extension [15, Conj. 1.3]. The authors are also aware of the work of Chen-Yu Chi on the quantitative extension of holomorphic sections from unions of strata of divisors [7].

The present work takes off from the work of Demailly in [12]. Let $X$ be a weakly pseudoconvex Kähler manifold, $K_X$ its canonical bundle and $(L, e^{-\varphi L})$ a hermitian line bundle on $X$ equipped with a hermitian metric $e^{-\varphi L}$ which is possibly singular. In [12], Demailly proves a new version of the Ohsawa–Takegoshi $L^2$ extension theorem applicable to the questions on extending $(K_X \otimes L)|_S$-valued holomorphic sections on possibly non-reduced subvarieties $S$ defined via multiplier ideal sheaves (a feature which can be considered as a far-reaching generalisation to the result in the work of Dano Kim, [30,31], in which an $L^2$ extension theorem for extending holomorphic sections on maximal log-canonical centres of some log-canonical pairs $(X, D)$ is proved). An interesting new input of this version of $L^2$ extension theorem is that, if the ambient manifold $X$ is compact (or if it is holomorphically convex, see [5]), via a brilliant use of the Hausdorff-ness of the topology on the relevant cohomology groups, a holomorphic extension of a $(K_X \otimes L)|_S$-valued section $f$ on a subvariety $S$ can be assured without the need of any $L^2$ assumption on the section $f$ with respect to the Ohsawa measure (provided that the suitable weak positivity assumption involving $\varphi_L$ and $\psi$ still holds true), although $f$ is still required to be locally extendible to some holomorphic section in some multiplier ideal sheaf constructed from $\varphi_L$ (namely, the multiplier ideal sheaf of $\varphi_L + m_0 \psi$ in the notation in Sect. 1.3).

Note that the Ohsawa measure in the estimates given by all different versions of the Ohsawa–Takegoshi extension theorem diverges to infinity around the singular points of the subvariety from which the given holomorphic section is extended. That’s why, although the extension theorem of Demailly without the $L^2$ assumption loses the estimate on the extended section, it was considered advantageous since the sections to be extended would not have to vanish along the singular locus of the subvariety. It was hoped that, with this feature in the new version of the extension theorem, one could follow the arguments as in [15] to construct a suitable psh potential in order to prove the so called “dlt extension” (see [15, Conj. 1.3]).

Unfortunately, in the course of proving the dlt extension, in order to show that the given $(\mu (K_X + S + B))|_S$-valued section on the subvariety $S$ (see Theorem 1.6.1 or Sect. 4.1 for the notations $\mu$, $S$ and $B$) has local extensions lying in the suitable multiplier ideal sheaf, one has to prove via an approximation of the metric on $K_X + S + B$ (where each approximating metric is constructed from some “algebraic” metric on $K_X + S + B + \frac{1}{k}A$ for some ample divisor $A$ and positive integer $k$) and make use of the estimates provided by the Ohsawa–Takegoshi $L^2$ extension theorem to prove convergence as in [15] (see also Theorem 4.5.4 for a relevant statement). It follows that, in order to prove the dlt extension via the argument in [15], the estimate in the $L^2$ extension theorem is indispensable.

In view of this, the goal of the present work is to resume the estimate of the Ohsawa–Takegoshi $L^2$ extension theorem under Demailly’s setting by replacing the (generalised) Ohsawa measure by the “measure on log-canonical centres”, or the “lc-measure” for short, which is defined in Definition 1.4.3. The latter measure, instead of diverging to infinity around the singular locus of the subvariety $S$, can indeed
be supported in the singular locus of $S$ (or on some lc centres of $(X, S)$ if $S$ is a divisor). This provides the means to get some sort of control over the $L^2$ norm of the holomorphic extensions, and eventually can be useful in proving the dlt extension.

In the remaining of Sect. 1, the main results (Theorems 1.4.5 and 1.6.1) are presented. A discussion on the lc-measures can be found in Sect. 2. Example 2.3.1 shows that the lc-measures can filter out Ohsawa’s example ([41, after Prop. 5.4]), while Example 2.3.2 is a computation by Bo Berndtsson of a concrete estimate of the minimal holomorphic extensions in a fundamental example which, it turns out, can be expressed in terms of lc-measures. These give evidence that the lc-measures could be used for providing $L^2$ estimates of holomorphic extensions in the general situations. Section 3 is devoted to the proof of the $L^2$ extension theorem with respect to the lc-measure supported on lc centres of codimension 1, while Sect. 4 is devoted to the proof of the improvement to the plt extension of [15].

1.2 Notation

In this paper, the following notations are used throughout.

Notation 1.2.1 Set $i := \frac{\sqrt{-1}}{2\pi}$.  

Notation 1.2.2 Each potential $\varphi$ (of the curvature of a metric) on a holomorphic line bundle $L$ in the following represents a collection of local functions $\{\varphi_\gamma\}_\gamma$ with respect to some fixed local coordinates and trivialisation of $L$ on each open set $V_\gamma$ in a fixed open cover $\{V_\gamma\}_\gamma$ of $X$. The functions are related by the rule $\varphi_\gamma = \varphi_{\gamma'} + 2\text{Re} h_{\gamma\gamma'}$ on $V_\gamma \cap V_{\gamma'}$ where $e^{h_{\gamma\gamma'}}$ is a (holomorphic) transition function of $L$ on $V_\gamma \cap V_{\gamma'}$ (such that $s_{\gamma} = s_{\gamma'} e^{h_{\gamma\gamma'}}$, where $s_{\gamma}$ and $s_{\gamma'}$ are the local representatives of a section $s$ of $L$ under the trivialisations on $V_\gamma$ and $V_{\gamma'}$ respectively). Inequalities between potentials is meant to be the inequalities under the chosen trivialisations over open sets in the fixed open cover $\{V_\gamma\}_\gamma$.

Notation 1.2.3 For any prime (Cartier) divisor $E$, let

- $\phi_E := \log |s_E|^2$, representing the collection $\left\{\log |s_{E, \gamma}|^2\right\}_\gamma$, denote a potential (of the curvature of the metric) on the line bundle associated to $E$ given by the collection of local representations $\{s_{E, \gamma}\}_\gamma$ of some canonical section $s_E$ (thus $\phi_E$ is uniquely defined up to an additive constant);
- $\varphi_E^{\text{sm}}$ denote a smooth potential on the line bundle associated to $E$;
- $\psi_E := \phi_E - \varphi_E^{\text{sm}}$, which is a global function on $X$, when both $\phi_E$ and $\varphi_E^{\text{sm}}$ are fixed.

All the above definitions are extended to any $\mathbb{R}$-divisor $E$ by linearity. For notational convenience, the notations for a $\mathbb{R}$-divisor and its associated $\mathbb{R}$-line bundle are used interchangeably.

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1 The notation is chosen by mimicking the reduced Planck constant $h = \frac{\hbar}{2\pi}$.
Notation 1.2.4 For any \((n, 0)\)-form (or \(K_X\)-valued section) \(f\), define \(\|f\|^2 := c_n f \wedge \bar{f}\), where \(c_n := (-1)^{n(n-1)}/(n!)^2\). For any Kähler metric \(\omega = \pi \sum_{1 \leq j, k \leq n} h_{jk} dz_j \wedge d\bar{z}_k\) on \(X\), set \(d \text{vol}_{X, \omega} := \omega^n/n!\). Set also \(\|f\|^2_{\omega} \text{vol}_{X, \omega} = |f|^2\).

Notation 1.2.5 For any two non-negative functions \(u\) and \(v\), write \(u \preceq v\) (equivalently, \(v \succeq u\)) to mean that there exists some constant \(C > 0\) such that \(u \leq C v\), and \(u \sim v\) to mean that both \(u \preceq v\) and \(u \succeq v\) hold true. For any functions \(\eta\) and \(\phi\), write \(\eta \preceq \log \phi\) if \(e^\eta \preceq e^\phi\). Define \(\succeq \log\) and \(\sim \log\) accordingly.

1.3 Basic setup

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\), and let \(\mathcal{I}(\phi) := \mathcal{I}_X(\phi)\) be the multiplier ideal sheaf of the potential \(\phi\) on \(X\) given at each \(x \in X\) by

\[
\mathcal{I}(\phi)_x := \mathcal{I}_X(\phi)_x := \left\{ f \in O_{X,x} \mid \begin{array}{l}
 f \text{ is defined on a coord. neighbourhood } V_x \ni x \\
 \text{and } \int_{V_x} |f|^2 e^{-\phi} d\lambda < +\infty
\end{array} \right\},
\]

where \(d\lambda\) is the Lebesgue measure on \(V_x\). Throughout this paper, the following are assumed on \(X\):

1. \((L, e^{-\phi_L})\) is a hermitian line bundle with an analytically singular metrics \(e^{-\phi_L}\), where \(\phi_L\) is locally equal to \(\phi_1 - \phi_2\), where each of the \(\phi_i\)'s is a quasi-psh local function with neat analytic singularities, i.e. locally

\[
\phi_i \equiv c_i \log \left( \sum_j |g_{ij}|^2 \right) \mod \mathcal{C}^\infty,
\]

where \(c_i \in \mathbb{R}_{\geq 0}\) and \(g_{ij} \in O_X\);

2. \(\psi\) is a global function on \(X\) such that it can also be expressed locally as a difference of two quasi-psh functions with neat analytic singularities;

3. \(\sup_X \psi \leq 0\) (which implies that \(\psi\) is quasi-psh after some blow-ups as it has only neat analytic singularities);

4. there exist numbers \(m_0, m_1 \in \mathbb{R}_{\geq 0}\) with \(m_0 < m_1\) such that

\[
\mathcal{I}(\phi_L + m_0 \psi) = \mathcal{I}(\phi_L + m \psi) \supset \mathcal{I}(\phi_L + m_1 \psi) \quad \text{for all } m \in [m_0, m_1),
\]

i.e. \(m_1\) is a jumping number of the family \(\{\mathcal{I}(\phi_L + m \psi)\}_{m \in \mathbb{R}_{\geq 0}}\) (such numbers exist on compact \(X\) as \(\psi\) is quasi-psh after suitable blow-ups and thus it follows from the openness property of multiplier ideal sheaves and (2.1.1));

5. \(S := S^{(m_1)}\) is a reduced subvariety defined by the annihilator

\[
\text{Ann}_{O_X} \left( \frac{\mathcal{I}(\phi_L + m_0 \psi)}{\mathcal{I}(\phi_L + m_1 \psi)} \right); \]

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in particular, \( S \subset (\psi)^{-1} (-\infty) \).

**Remark 1.3.1** If \( \varphi_L + (m_1 + \beta)\psi \) is a quasi-psh potential for all \( \beta \in [0, \delta] \) for some \( \delta > 0 \) (which holds true in all the main theorems in this paper), openness property guarantees that, on every compact subset \( K \subset X \), there exists \( \varepsilon > 0 \) such that \( \mathcal{I}(\varphi_L + (m_1 + \varepsilon)\psi)|_K = \mathcal{I}(\varphi_L + m_1\psi)|_K \) (see [26, Main Thm. (ii)] or [36, Thm. 1.1]; see also [22]). This then implies that the analytic subspace defined by \( \text{Ann}_{\mathcal{O}_X}(\mathcal{I}(\varphi_L + m_0\psi)\mathcal{I}(\varphi_L + m_1\psi)) \) is automatically reduced, following the arguments in [12, Lemma 4.2].

**Remark 1.3.2** Most of the arguments in Sects. 2 and 3 of this paper can be adapted to the case when \( X \) is just a weakly pseudoconvex (non-compact) Kähler manifold (after passing to a relatively compact exhaustion) provided that the upper-boundedness on \( \psi \) in (3) still holds true on \( X \). As this is not automatic on a non-compact manifold, and the most interesting applications which the authors concern about are on compact manifolds, the background manifold is assumed to be compact in this paper for the sake of clarity.

**Definition 1.3.3** Suppose that \( \varphi \) is a potential or a global function on \( X \) such that it is locally a difference \( \varphi_1 - \varphi_2 \) of quasi-psh local functions with neat analytic singularities as in (1) above. The **polar ideal sheaf** \( \mathcal{P}_{\varphi} \) of \( \varphi \) is defined to be the ideal sheaf generated by the local holomorphic functions \( g_{ij} \) for all \( j \)'s and \( i = 1, 2 \).

**Notation 1.3.4** Given a set \( V \subset X \), a section \( f \) of \( \mathcal{I}(\varphi_L + m_0\psi)\mathcal{I}(\varphi_L + m_1\psi) \) on \( V \) (which is supported in \( S \cap V \)), and a section \( F \) of \( \mathcal{I}(\varphi_L + m_0\psi) \) on \( V \), the notation

\[
F \equiv f \mod \mathcal{I}(\varphi_L + m_1\psi) \text{ on } V
\]

is set to mean that, for all \( x \in V \), if \( (F)_x \) and \( (f)_x \) denote the germs of \( F \) and \( f \) at \( x \) respectively, one has

\[
((F)_x \mod \mathcal{I}(\varphi_L + m_1\psi))_x = (f)_x.
\]

If such a relation between \( F \) and \( f \) holds, \( F \) is said to be an **extension** of \( f \) on \( V \). If the set \( V \) is not specified, it is assumed to be the whole space \( X \). Such notation is also applied to cases with a slight variation of the sheaf \( \mathcal{I}(\varphi_L + m_1\psi) \) (for example, with \( \mathcal{I}(\varphi_L + m_1\psi) \) replaced by \( \mathcal{C}_X^\infty \otimes \mathcal{I}(\varphi_L + m_1\psi) \)).

### 1.4 Lc-measure and extension theorem

As explained above, the first goal of this paper is to replace the generalised Ohsawa measure \( |f|_{i\partial}^2 d \text{vol}_{S,\omega,\varphi_L}[\psi] \) in the previous versions of \( L^2 \) extension theorem (as in [12]) by the measure on log-canonical (lc) centres given as follows.

**Definition 1.4.1** If \( S \) given in Sect. 1.3 is a reduced divisor with snc on \( X \), define \( \text{lc}_X^\sigma(S) \) to be the union of all lc centres of \( (X, S) \) of codimension \( \sigma \) in \( X \) (see [35, ...Springer]}
Def. 4.15] for the definition of lc centres when $S$ is a divisor). For a general reduced subvariety $S$ in $X$ given in Sect. 1.3, define $\text{lc}^\sigma_X(S)$ as

$$\text{lc}^\sigma_X(S) := \pi \left( \text{lc}^\sigma_X(\tilde{S}) \right),$$

where $\pi : \tilde{X} \to X$ is a log-resolution of $(X, \varphi_L, \psi)$ and $\tilde{S}$ is the reduced divisor with snc described in Sect. 2.1 (which satisfies $\pi(\tilde{S}) = S$). Moreover, an lc centre of $(X, S)$ (or, more precisely, lc centre of $(X, \varphi_L, \psi, m_1)$) of codimension $\sigma$ is meant to be the image under $\pi$ of an lc centre of $(\tilde{X}, \tilde{S})$ of codimension $\sigma$ in $\tilde{S}$.

**Remark 1.4.2** Admittedly, it is confusing to talk about the “codimension” of an lc centre of “$(X, S)$” when $S$ is not a divisor. For example, with a suitable choice of $\varphi_L$ and $\psi$ such that $S = \{p\} \subset X$ (a point), the lc centre of $(X, \{p\})$ has codimension 1 (see Example 3.5.1). The choice of language here is just to favour the case when $S$ is an snc divisor.

**Definition 1.4.3** The lc-measure supported on the lc centres of $(X, S)$ of codimension $\sigma$ in $X$ (or $\sigma$-lc-measure for short) with respect to $f \in H^0 \left( J(\varphi_L + m_0 \psi) \right)$, denoted as $|f|^2_\omega \, \text{lcv}_{\omega,\varphi_L}^{\sigma,m_1}[\psi]$, is defined by

$$\varphi^{\infty}_0(S) \ni g \mapsto \int_{\text{lc}^\sigma_X(S)} g \, |f|^2_\omega \, \text{lcv}_{\omega,\varphi_L}^{\sigma,m_1}[\psi] := \lim_{\varepsilon \to 0^+} \varepsilon \int_X \tilde{g} \left| \tilde{f} \right|^2_\omega \frac{e^{-\varphi_L - m_1 \psi}}{|\psi|^{\sigma + \varepsilon}} \, d\text{vol}_{X,\omega},$$

where

- $\tilde{f}$ is a smooth extension of $f$ to a section on $X$ such that $\tilde{f} \in \mathcal{C}^\infty \otimes J(\varphi_L + m_0 \psi)$;
- $\tilde{g}$ is any smooth extension of $g$ to a function on $X$;
- $\sigma$ is a non-negative integer and the measure $|f|^2_\omega \, \text{lcv}_{\omega,\varphi_L}^{\sigma,m_1}[\psi]$ is supported (if finite) on a (reduced) subvariety $\text{lc}^\sigma_X(S)$ of $S$ (see Definition 1.4.1).

Note that, as the given section $f$ takes values in $K_X \otimes L$, the lc-measure defined above does not depend on $\omega$.

The $\sigma$-lc-measure vanishes when $\sigma$ is large and diverges when $\sigma$ is small (see Sect. 2.2), and can be finite and non-zero only at one particular value of $\sigma$ depending on the given section $f$. Here an ad hoc definition of such special value of $\sigma$ is given. Another definition can be found in Definition 2.2.5.

**Definition 1.4.4** Given the setting above, the *codimension of minimal lc centres (mlc)* of $(X, S)$ (or of $(X, \varphi_L, \psi, m_1)$) with respect to $f$, denoted by $\sigma_f = \sigma_{f,\varphi_L+m_1\psi}$, is the smallest integer $\sigma$ such that

$$\int_{\text{lc}^\sigma_X(S)} |f|^2_\omega \, \text{lcv}_{\omega,\varphi_L}^{\sigma,m_1}[\psi] < \infty.$$
From the calculation in Sect. 2.2, $\sigma_f$ is ranging between 0 and the codimension of mlc of $(X, S)$ (when $S$ is an snc divisor). If $\mathcal{J}(\varphi_L + m_0\psi) = \mathcal{O}_X$ and if $\sigma_f \geq 1$, then $f$ vanishes on all lc centres of $(X, S)$ with codimension $< \sigma_f$ in $X$ but is non-trivial on at least one lc centre of codimension $\sigma_f$. Moreover, from the discussion in Sect. 2.1, if $\pi: \tilde{X} \to X$ is a log-resolution of $(X, \varphi_L, \psi)$, the codimension $\sigma_f$ coincides with $\sigma_{\pi^*f \otimes s_E}$ (see Sect. 2.1 for the meaning of $s_E$ and log-resolution of $(X, \varphi_L, \psi)$).

The authors would like to mention that the use of such lc-measure was inspired by the study of residue currents in [3,43] and [1]. In their works, the kind of current-valued function (in 1-variable case)

$$R > 0 \ni \varepsilon \mapsto \varepsilon \bar{\omega} \frac{idz \wedge d\bar{z}}{|z|^{2(1-\varepsilon)}}$$

is studied. Such function gives a holomorphic family (so $\varepsilon \in \mathbb{C}$) of currents for $\Re\varepsilon > 0$ and can be analytically continued across $\varepsilon = 0$. Its value at $\varepsilon = 0$ is a residue measure on $\{z = 0\}$. The lc-measure considered in this paper is essentially given by the value at $\varepsilon = 0$ of the current-valued function

$$R > 0 \ni \varepsilon \mapsto \varepsilon \frac{\bigwedge_{j=1}^{\sigma} (idz_j \wedge d\bar{z}_j)}{\prod_{j=1}^{\sigma} |z_j|^2 \cdot \sum_{j=1}^{\sigma} \log |z_j|^2 |\sigma + \varepsilon|}$$

after analytically continued across $\varepsilon = 0$.

It happens that the lc-measure above can be fitted into the Ohsawa–Takegoshi-type $L^2$ extension theorem, at least in the codimension-1 case. The first main result of this paper can be stated as follows.

**Theorem 1.4.5** (Theorem 3.3.4, see also Theorem 3.4.1 for a more general statement). Suppose that

(1) there exists $\delta > 0$ such that

$$i\partial \bar{\partial} \varphi_L + (m_1 + \beta) i\partial \bar{\partial} \psi \geq 0 \text{ on } X \text{ for all } \beta \in [0, \delta], \text{ and}$$

(2) for any given constant $\ell > 0$, the function $\psi$ is normalised (by adding to it a suitable constant) such that

$$\psi < -\frac{e}{\ell}, \text{ and } \frac{1}{|\psi|} + \frac{2}{|\psi| \log |\psi|} \leq \delta.$$ 

Then, for any holomorphic section $f \in H^0 \left( S, K_X \otimes L \otimes \mathcal{J}(\varphi_L + m_0\psi) \right)$, if one has

$$\int_S |f|_{\omega}^2 d\text{lc}_{\omega, \varphi_L}^{1,(m_1)} [\psi] < \infty$$
(which holds true when either the mlc of \((X, S)\) or the mlc of \((X, S)\) with respect to \(f\) has codimension 1, see Definitions 1.4.4 and 2.2.5), then there exists a holomorphic section \(F \in H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_0\psi))\) such that

\[ F \equiv f \mod \mathcal{I}(\varphi_L + m_1\psi) \]

with the estimate

\[ \int_X |F|^2 e^{-\varphi_L - m_1\psi} \frac{1}{|\ell \psi|} \frac{1}{((\log |\ell \psi|)^2 + 1)} \leq \int_S |f|^2 d\text{lc}v_{\omega, \varphi_L}[\psi]. \]

See Remark 3.3.5 for the purpose of the number \(\ell\) in the estimate. Notice that the weight in the estimate of \(F\) above is pointwisely dominating (up to a multiple constant) the weight in the estimate in [12] (which is in the magnitude of \(e^{\varphi_L + m_1\psi}\)). Therefore, the above estimate includes the estimate in [12] up to a constant multiple.

For the proof, it is first argued in Sect. 2.1 that it suffices to consider the case where the polar ideal sheaves of \(\varphi_L\) and \(\psi\) (see Definition 1.3.3) are the defining ideal sheaves of some snc divisors (and thus \(S\) is an snc divisor in particular). The proof then goes along the lines of arguments in [12].

For the sake of simplicity, the proofs in Sect. 3 are given for the case where \(m_0 = 0\) and \(m_1 = 1\). The result for the general \(m_0\) and \(m_1\) can be obtained by replacing \(\varphi_L\) by \(\varphi_L + m_0\psi\) and \(\varphi_L + \psi\) by \(\varphi_L + m_1\psi\) in the arguments.

As in the classical cases, the problem is reduced to solve for a weak solution of a \(\overline{\partial}\)-equation with “error” (derived from the smooth extension of \(f\), and depending on the \(\varepsilon\) in the definition of the lc-measure) using the twisted Bochner–Kodaira inequality (3.2.1) with suitably chosen auxiliary functions (see Theorems 3.3.1), at least on the complement of the polar sets of \(\varphi_L\) and \(\psi\). (When \(\sigma_f > 1\) (see Definition 1.4.4 or 2.2.5), the curvature term in the twisted Bochner–Kodaira formula (see Lemma 3.2.1) has a negative summand in the curvature term, which is the obstacle in obtaining the result on extending sections from lc centres of higher codimensions.)

The weak solution with “error” can be continued across the polar sets of \(\varphi_L\) and \(\psi\) via the \(L^2\) Riemann continuation theorem\(^3\) [8, Lemme 6.9]. The required holomorphic extension of \(f\) is then constructed from the above solution with the “error” and letting \(\varepsilon \to 0^+\). The required estimate is also obtained after taking the necessary limits. The regularity of the limit is assured by following a similar argument as in [12].

The above theorem is applicable when \(\varphi_L + m_1\psi\) and \(\psi\) has neat analytic singularities. For the potential with arbitrary singularities, an approximation of the potential is needed and is handled in Theorem 3.4.1.

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\(^3\) This is usually named as the “Riemann extension theorem”. The current naming “Riemann continuation theorem” is used just to distinguish this theorem from the Ohsawa–Takegoshi-type extension theorem which is studied in this paper. The use of “continuation” is found in Grauert–Remmert’s book [20, Section A.3.8] (English translation by Huckleberry), but “extension” is used in [19, Section 7.1], a later publication of the same authors.
1.5 Further questions

There are several questions that the authors would still like to understand.

(1) As stated in Remark 3.3.7, it is not yet clear to the authors whether the current result (if allowing $X$ to be non-compact) includes the results on optimal constant in [4] and [21]. Moreover, is it possible to determine the “optimal constant” in this Ohsawa–Takegoshi-type extension theorem with lc-measure?

(2) The lc-measure is inspired by the residue currents studied in [3,43] and [1], obtained by replacing their residue currents (i.e. limit of $\varepsilon e^{-\varphi_L - (1-\varepsilon)m_1 \psi} d \text{vol}_{X,\omega}$ in the notation of this paper) by a current with “Poincaré-growth” singularities (i.e. limit of $e^{-\varphi_L - m_1 \psi} d \text{vol}_{X,\omega}$). Is it possible to replace the lc-measure in the main theorem by some measure which is defined by currents which diverge to infinity even faster, like the limits of

$$e^{-\varphi_L - m_1 \psi} d \text{vol}_{X,\omega} \quad \text{and so on (where } \log^{\circ j} \text{ denotes the composition of } j \text{ copies of log functions)}.$$ It seems to the authors that this could be related to the question stated in Remark 3.3.6, which is asking for the estimates with some better weights given in [38].

(3) The first author started to consider the lc-measure during the study of analytic adjoint ideal sheaves with Chen-Yu Chi from National Taiwan University. The lc-measure on various lc centres can possibly be the means to generalise the works of Guenancia [23] and Dano Kim [32] on this subject. Furthermore, these lc-measures provide a way to characterise the lc centres which can be defined by the multiplier ideal sheaves of quasi-psh functions. Considering such linkage, it would be of interest to see more of their applications in analytic and algebraic geometry.

1.6 Improved plt extension of Demailly–Hacon–Păun

Another main result of this paper is the following improvement to the result of plt extension of Demailly–Hacon–Păun in [15], which removes the superfluous assumption on the support of the given $\mathbb{Q}$-divisor $D$.

**Theorem 1.6.1** (Theorem 4.6.1) Let $X$ be a projective manifold and $(X, S + B)$ be a purely log-terminal (plt) and log-smooth pair with $B$ being a $\mathbb{Q}$-divisor such that $S = [S + B]$. Let $\mu \in \mathbb{N}$ be such that $\mu(K_X + S + B)$ is a $\mathbb{Z}$-(Cartier)-divisor. Assume that $\mu \geq 2$ and

- $K_X + S + B$ is pseudo-effective (pseff);
- $K_X + S + B \sim_{\mathbb{Q}} D$, where $D$ is an effective $\mathbb{Q}$-divisor with snc support;
- $\text{supp } S \subset \text{supp } D$ (without the assumption that $\text{supp } D \subset \text{supp } (S + B)$);
• no irreducible components of $S$ lies in the diminished stable base locus $B_{-} (K_{X} + S + B)$ (see, for example, [15, §2.1] for the definition).

Then, every $u \in H^{0} (S, \mathcal{O}_{S} (\mu (K_{X} + S + B)) \otimes \mathcal{F}_{S}^{\chi})$ extends to a holomorphic section in $H^{0} (X, \mu (K_{X} + S + B))$ (see Sect. 4.6 and [15] for the definitions of the extension obstruction ideal sheaf $\mathcal{F}_{S}^{\chi}$ and the corresponding extension obstruction divisor $\Xi$).

In particular, when $K_{X} + S + B$ is nef, the restriction map $H^{0} (X, \mu (K_{X} + S + B)) \rightarrow H^{0} (S, \mathcal{O}_{S} (\mu (K_{X} + S + B)))$ is surjective.

The assumption $\text{supp } D \subset \text{supp } (S + B)$ in [15, Thm. 1.7] is used there to “remove” the logarithmic singularities in the denominator in the estimate obtained from the Ohsawa–Takegoshi theorem [15, Thm. 4.3] so that an estimate in the unweighted $L^{2}$ norm, and hence the sup-norm, of the auxiliary extended holomorphic sections can be obtained. The superfluous assumption is needed as the logarithmic poles are estimated in sup-norm.

In Lemma 4.4.1, the logarithmic poles are estimated in $L^{p}$ norm via Hölder’s inequality, avoiding the use of the superfluous assumption. Moreover, the choice of the sequence of auxiliary potentials (which are denoted as $\varphi_{m}$ in [15, §5]) is replaced by the sequence of potentials constructed from Bergman kernels of spaces of global holomorphic sections (see Sects. 4.2 and 4.3), which is a priori uniformly bounded from above (see (4.2.5b)), thus avoiding the complicated inductive construction of the $\varphi_{m}$ in [15, §5] as well as simplifying the proof of uniform boundedness of such sequence when restricted to the subvariety $S$ (see Theorem 4.5.4).

Apart from the technicalities, the argument in the proof of the theorem above essentially follows that in [15, §5].

2 The measures on lc centres

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ equipped with a (smooth) Kähler metric $\omega$. All notations follow those given in Section 1.3.

2.1 Effects of log-resolutions

As $\varphi_{L}$ and $\psi$ are locally differences of quasi-psh functions with neat analytic singularities, by the result of [27,28] (see also [34] or [39]), there is a log-resolution $\pi: \tilde{X} \rightarrow X$ of $(X, \varphi_{L}, \psi)$, which is the composition of a sequence of blow-ups at smooth centres such that the inverse image ideal sheaves $\mathcal{P}_{\varphi_{L}} \cdot \mathcal{O}_{\tilde{X}}$ and $\mathcal{P}_{\psi} \cdot \mathcal{O}_{\tilde{X}}$ of the polar ideal sheaves $\mathcal{P}_{\varphi_{L}}$ and $\mathcal{P}_{\psi}$ of $\varphi_{L}$ and $\psi$ respectively (see Definition 1.3.3) are principal ideal sheaves given by some divisors, and the sum of these divisors together with the exceptional divisors of $\pi$ (i.e. components of the relative canonical divisor $K_{\tilde{X}/X} := K_{\tilde{X}}/\pi^{*} K_{X}$) has only snc.

Assume that $\varphi_{L} + m_{1} \psi$ is psh. Without further assumption, one can decompose $K_{\tilde{X}/X}$ into two effective $\mathbb{Z}$-divisors $E$ and $R$ (with the corresponding canonical holo-
morphic sections denoted by $s_E$ and $s_R$) such that $R$ is the maximal divisor satisfying

$$\pi^*\varphi_L + m_1\pi^*\psi - \phi_R := \pi^*\varphi_L + m_1\pi^*\psi - \log |s_R|^2$$

being psh.

Suppose that $\pi$ is restricted to $\pi^{-1}(V)$ where $(V, z)$ is a coordinate neighbourhood in $X$, and let $\{U_\gamma\}_\gamma$ be a covering of $\pi^{-1}(V) = \bigcup_\gamma U_\gamma$ by coordinate neighbourhoods $(U_\gamma, w_\gamma)$ in $\tilde{X}$. Let also $\{Q_\gamma\}_\gamma$ be a partition of unity subordinated to $\{U_\gamma\}_\gamma$. Then, with $s_E, s_R, z$ and $w_\gamma$’s suitably chosen, one has, for any $f \in H^0(V, K_X \otimes L)$ which is viewed as an $(n, 0)$-form $f = f_V \ dz^1 \wedge \cdots \wedge dz^n$,

$$\int_V |f|^2 e^{-\psi_L - m\psi} = \int_V |f_V|^2 e^{-\psi_L - m\psi} c_n dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$$

$$= \sum_\gamma \int_{U_\gamma} Q_\gamma \pi^*|f_V|^2 e^{-\pi^*\psi_L - m\pi^*\psi} |s_E, \gamma \otimes s_R, \gamma|^2 c_n dw^1_\gamma \wedge \cdots \wedge dw^n_\gamma + \bar{d}w^1_\gamma \wedge \cdots \wedge \bar{d}w^n_\gamma,$$

where $c_n := (-1)^{\binom{n-1}{2}} \left(\sqrt{\frac{1}{2}}\right)^n$. This shows that, if $f \in I_X(\varphi_L + (m - \varepsilon)\psi) \setminus I_X(\varphi_L + m\psi)$ for all sufficiently small $\varepsilon > 0$, then $\pi^* f \otimes s_E \in \tilde{I}_X(\pi^*\varphi_L + (m - \varepsilon)\pi^*\psi - \phi_R) \setminus \tilde{I}_X(\pi^*\varphi_L + m\pi^*\psi - \phi_R)$ also for those $\varepsilon > 0$.

In other words, if $m_p$ is a jumping number of the family $\{I_X(\varphi_L + m\psi)\}_{m \in \mathbb{R}_{>0}}$, it is also a jumping number of the family $\{\tilde{I}_X(\pi^*\varphi_L + m\pi^*\psi - \phi_R)\}_{m \in \mathbb{R}_{>0}}$. In particular, there exists $m_0' \in [m_0', m_1]$ such that the ideal sheaf $\tilde{I}_X(\pi^*\varphi_L + m\pi^*\psi - \phi_R)$ stays unchanged for $m \in [m_0', m_1]$ and shrinks when $m = m_1$.

Moreover, by the choice of $R$, if $E$ is non-trivial, the weight $e^{-\pi^*\varphi_L - m_1\pi^*\psi + \phi_R}$ is then integrable around a general point of $E$. Since the weight has only snc singularities, it can be easily seen from Fubini’s theorem that, for any $m \in [m_0', m_1]$,

$$\pi^* f_V \otimes s_E \in \tilde{I}_X(\pi^*\varphi_L + m\pi^*\psi - \phi_R) \iff \pi^* f_V \in \tilde{I}_X(\pi^*\varphi_L + m\pi^*\psi - \phi_R).$$

Since $\pi$ is bimeromorphic (birational) and thus the image of the exceptional locus are of codimension at least 2 in $X$, it can be seen from the Riemann continuation theorem and the identity theorem for holomorphic functions that any holomorphic

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4 Thanks to Chen-Yu Chi for pointing out to the authors that the converse is not true. An example is provided by taking $\psi = \log \left(\left|x^3\right|^2 + \left|y^2\right|^2\right) - 1$ and $\varphi_L = 0$ on $X = \Delta^2 \subset \mathbb{C}^2_{(x, y)}$, the unit 2-disc centred at the origin, and considering the standard principalisation of the ideal $(x^3, y^2)$ given by blowing-up 3 times. In this case, $\frac{5}{6}$ and $\frac{7}{6}$ are consecutive jumping numbers of $I_X(m\psi)$ but $\tilde{I}_X(m\pi^*\psi - \phi_R)$ also has 1 as a jumping number.

5 See footnote 3.
function $\tilde{f}_{\pi^{-1}(V)}$ on $\pi^{-1}(V)$ for an open set $V \subset X$ can be expressed as $\pi^* f_V$ for some holomorphic function $f_V$ on $V$. It then follows from the above equality between integrals on $V$ and $\pi^{-1}(V)$ that, for any $m \in [m'_0, m_1]$,

$$\mathcal{I}_X (\varphi_L + m \psi) \cdot \mathcal{O}_X \subset \mathcal{I}_X (\pi^* \varphi_L + m \pi^* \psi - \phi_R) \quad \text{and} \quad \mathcal{I}_X (\varphi_L + m \psi) = \pi_* \mathcal{I}_X (\pi^* \varphi_L + m \pi^* \psi - \phi_R). \quad (2.1.1)$$

Furthermore, suppose that $\varphi_L + (m_1 + \beta) \psi$ is quasi-psh for all $\beta \in [0, \delta]$ for some $\delta > 0$, and if $\tilde{S}$ is the reduced divisor defined by $\text{Ann}_{\tilde{\mathcal{O}}_{\tilde{X}}} (\mathcal{I}_X (\pi^* \varphi_L - \phi_R + m'_0 \pi^* \psi))$, it then follows from (2.1.1) that

$$\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I}_X (\varphi_L + m'_0 \psi)}{\mathcal{I}_X (\varphi_L + m_1 \psi)} \right) = \pi_* \text{Ann}_{\tilde{\mathcal{O}}_{\tilde{X}}} \left( \frac{\mathcal{I}_X (\pi^* \varphi_L - \phi_R + m'_0 \pi^* \psi)}{\mathcal{I}_X (\pi^* \varphi_L - \phi_R + m_1 \pi^* \psi)} \right).$$

One therefore has $\pi (\tilde{S}) = S$.

The above discussion can be concluded as follows.

**Snc assumption 2.1.1** When it helps in the computation, by replacing $\pi^* \psi$ by $\psi$, $\pi^* \varphi_L - \phi_R$ by $\varphi_L$, $m'_0$ by $m_0$, and $\pi^* f_V \otimes \mathcal{E}$ by $f_V$ (thus replacing $\pi^* L \otimes R^{-1}$ by $L$, and therefore $\pi^* K_X \otimes \pi^* L \otimes E = K_{\tilde{X}} \otimes \pi^* L \otimes R^{-1}$ by $K_{\tilde{X}} \otimes L$), it can be assumed that

- $S$ is a reduced divisor, and
- the polar ideal sheaves $\mathcal{P}_{\varphi_L}$ and $\mathcal{P}_\psi$ of $\varphi_L$ and $\psi$ respectively are principal and the corresponding divisors have only snc with each other.

Moreover, the estimates on the holomorphic extension obtained in the main theorems in the following sections are valid even before blowing up.

### 2.2 Computation with lc-measures

In this section, $\varphi_L$ and $\psi$ are assumed to satisfy the snc Assumption 2.1.1.

The well-defined-ness of the measure on lc centres of $(X, S)$ of codimension $\sigma$ (called the “lc-measure” or the “$\sigma$-lc-measure” for short) is justified below. Define $\tilde{\varphi}_L$ by

$$\tilde{\varphi}_L + \psi_S := \varphi_L + m_1 \psi,$$

where $\psi_S := \phi_S - \varphi_S^{sm} < 0$ (see Notation 1.2.3 for the meaning of $\phi_S$ and $\varphi_S^{sm}$).

A potential $\varphi$ is said to have Kawamata log-terminal (klt) singularities if $\mathcal{I}(\varphi) = \mathcal{O}_X$.

**Proposition 2.2.1** Given the snc Assumption 2.1.1 on $\varphi_L$ and $\psi$, suppose further that

$(\dagger)$ $\tilde{\varphi}_L$ has only klt singularities and $\tilde{\varphi}_L^{-1}(\infty) \cup \tilde{\varphi}_L^{-1}(-\infty)$ does not contain any lc centres of $(X, S)$. 
Suppose also that $V$ is an open coordinate neighbourhood on which

$$\psi|_V = \sum_{j=1}^{\sigma_V} v_j \log |z_j|^2 + \sum_{k=\sigma_V+1}^n c_k \log |z_k|^2 + \alpha \quad \text{and}$$

$$\varphi_L|_V = \sum_{k=\sigma_V+1}^n \ell_k \log |z_k|^2 + \beta,$$

where

- each $z_j$ is a holomorphic coordinate and $(r_j, \theta_j)$ its corresponding polar coordinates on $V$ for $j = 1, \ldots, n$,
- $S \cap V = \{z_1 \cdots z_{\sigma_V} = 0\}$,
- $\alpha$ and $\beta$ are smooth functions such that $\sup_V \alpha < 0$,
- $\sup_V \frac{\partial}{\partial r_j} \alpha > -1$ (i.e. $\sup_V r_j$ is sufficiently small) for $j = 1, \ldots, \sigma_f$,
- $\sup_V \log |z_k|^2 < 0$ for $k = \sigma_f + 1, \ldots, n$,
- $v_j$’s are constants such that $v_j > 0$ for $j = 1, \ldots, \sigma_V$, and
- $c_k$’s are constants such that $c_k \geq 0$ for $k = \sigma_V + 1, \ldots, n$,
- $\ell_k$’s are constants such that $\ell_k < 1$ (due to the klt assumption, possibly negative) for $k = \sigma_V + 1, \ldots, n$.

Then, for any compactly supported smooth function $f$ on $V$ such that

$$f = \prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{1+a_k} \cdot g \quad \text{with} \quad g|_{S^{\sigma_f}} \neq 0,$$

where $S^{\sigma_f} := \{z_1 = \cdots = z_{\sigma_f} = 0\}$, $\sigma_f$ is some non-negative integer $\leq \sigma_V$, $a_{\sigma_f+1}, \ldots, a_{\sigma_V}$ are some non-negative integers (with the convention $\prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{1+a_k} = 1$ when $\sigma_f = \sigma_V$) and $g$ is some compactly supported bounded function on $V$ with $|g|^2$ being smooth, one has

$$\int_{k^+_X(S) \cap V} |f|^2 d \psi_{\varphi_L}^{(\sigma_f)}[\psi] := \lim_{\varepsilon \rightarrow 0^+} \int_V \frac{|f|^2 e^{-\varphi_L \psi - \psi_{S^\sigma_f}^m} d \text{vol}_{X,\omega}}{|\psi|^\sigma + \varepsilon}$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_V \prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{2a_k} \cdot |g|^2 e^{-\varphi_L \psi_{S^\sigma_f}^m} d \text{vol}_{X,\omega}$$

$$\cdot \frac{|z_1 \cdots z_{\sigma_f}|^2}{|\psi|^\sigma + \varepsilon}$$

$$= \begin{cases} 0 & \text{when } \sigma > \sigma_f \text{ or } \sigma_f = 0, \\ \frac{\pi^{\sigma_f}}{(\sigma_f - 1)!} \prod_{j=1}^{\sigma_f} v_j \prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{2a_k} \cdot |g|^2 \omega & \text{when } \sigma = \sigma_f \geq 1, \\ \infty & \text{otherwise}, \end{cases}$$

where $(\cdot)_\omega$ denotes the contraction of a section with the metric on $K_{S^{\sigma_f}} \otimes K_X^{-1}|_{S^{\sigma_f}}$ induced from $\omega$. 

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**Remark 2.2.2** With the snc assumption on $\varphi_L$ and $\psi$, it is easy to see that $X$ can be covered by the kind of open coordinate neighbourhoods described in the proposition.

**Proof** Writing $\omega$ locally as $\sum_{1 \leq j, k \leq n} h_{j,k}dz_j \wedge d\bar{z}_k$ and choosing the canonical section defining $\phi_S$ suitably, it follows that

\[
\varepsilon \int_V \frac{|f|^2 e^{-\varphi_L-m\varphi} d\text{vol}_X}{|\psi|^{\sigma+\varepsilon}} = \varepsilon \int_V \frac{|f|^2 e^{-\beta+\varphi_S} \det(h_{j,k})}{|\psi|^{\sigma+\varepsilon}} \left( \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j \right) \wedge^n f_0 := \frac{1}{|\psi|^{\sigma+\varepsilon}} \left( \frac{\sqrt{-1}}{2\pi} dz_k \wedge d\bar{z}_k \right),
\]

where $\ell_{\sigma}+1$, $\ell_{\sigma_2}+2$, $\ldots$, $\ell_{\sigma}$ and $a_{\sigma}+1$, $a_{\sigma_2}+2$, $\ldots$, $a_n$ are all defined to be 0. In view of Fubini’s Theorem, integrations with respect to the variables $z_{\sigma_1}, \ldots, z_n$ are done at the last step. Since all $(\ell_k-a_k)$’s are < 1, the integral with respect to all variables is convergent as soon as the integral with respect to variables $z_1, \ldots, z_{\sigma}$ is bounded. The differentials corresponding to $z_{\sigma_1}, \ldots, z_n$ are made implicit in what follows. Notice that $F_0$ is a smooth function.

Observe that, if $\sigma = 0$, the integral above is convergent and bounded above by $O(\varepsilon)$. Therefore, it goes to 0 when $\varepsilon \to 0^+$.

Assume that $\sigma \geq 1$ in what follows. Set

\[
t_j := v_j \log r_j^2 = v_j \log |z_j|^2.
\]

The integrand is integrated with respect to each $r_j$ over [0, 1] (thus to each $t_j$ over $(-\infty, 0]$) and to each $\theta_j$ over [0, 2$\pi$]. Write also $\partial r_j$ for $\partial_{r_j}$. The integral in question then becomes

\[
(\star) \quad \frac{\varepsilon}{\prod_{j=1}^{\sigma} \nu_j} \int \frac{F_0}{|\psi|^{\sigma+\varepsilon}} \prod_{j=1}^{\sigma} dt_j \cdot \prod_{j=1}^{\sigma} \frac{d\theta_j}{2} = \frac{\varepsilon}{\nu} \int \frac{F_0}{1 + \frac{\nu_1}{2\nu_1} \partial r_1} \frac{d|\psi|}{|\psi|^{\sigma+\varepsilon}} \prod_{j=2}^{\sigma} dt_j \cdot d\theta.
\]

Note that the integral above is treated as an iterated integral instead of an integral of differential form, and $1 + \frac{\nu_1}{2\nu_1} \partial r_1 > 0$ on supp $F_0$ by assumption.
If $\sigma \geq \sigma_f$, one can apply integration by parts and induction to yield

\[
(*) = \left[\frac{-\varepsilon}{\nu (\sigma - 1 + \varepsilon)} \int \partial r_1 \left( \frac{F_0}{1 + \frac{r_1}{2\nu} \partial r_1 \alpha} \right) dr_1 \int \frac{dt_2}{|\psi|^{\sigma-1+\varepsilon}} \prod_{j=3}^{\sigma_f} dt_j \cdot d\theta \right. = F_i
\]

\[
= \left. \frac{-\varepsilon}{\nu (\sigma - 1 + \varepsilon) (\sigma - 2 + \varepsilon)} \int \frac{F_1}{1 + \frac{r_2}{2\nu} \partial r_2 \alpha} d \left( \frac{1}{|\psi|^{\sigma-2+\varepsilon}} \right) dr_1 \prod_{j=3}^{\sigma_f} dt_j \cdot d\theta \right.
\]

\[
= \ldots = \left. \frac{(-1)^{\sigma_f}}{\nu \prod_{j=1}^{\sigma_f} (\sigma - j + \varepsilon)} \int \frac{F_{\sigma_f}}{|\psi|^{\sigma-\sigma_f+\varepsilon}} \prod_{j=1}^{\sigma_f} dr_j \cdot d\theta \right.
\]

Note that the $F_j$’s are defined inductively by

\[
F_j := \partial r_j \left( \frac{F_{j-1}}{1 + \frac{r_j}{2\nu} \partial r_j \alpha} \right)
\]

and all of them are smooth functions. When $\sigma > \sigma_f$, the last expression in (**) is bounded above by $O(\varepsilon)$. Therefore, the integral tends to 0 again as $\varepsilon \to 0^+$. When $\sigma = \sigma_f$, the last expression in (**) is bounded above, but the multiple constant in front of the integral does not converge to 0 as $\varepsilon \to 0^+$. After letting $\varepsilon \to 0^+$, the dominated convergence theorem and the fundamental theorem of calculus gives

\[
(**) = \left( -1 \right)^{\sigma_f} \left( \sigma_f - 1 \right)! \frac{1}{\nu} \int F_{\sigma_f} \prod_{j=1}^{\sigma_f} dr_j \cdot d\theta = \left( -1 \right)^{\sigma_f-1} \left( \sigma_f - 1 \right)! \frac{1}{\nu} \int F_{\sigma_f-1} \big|_{r_{\sigma_f}=0} \prod_{j=1}^{\sigma_f-1} dr_j \cdot d\theta
\]

\[
= \ldots = \frac{\pi^{\sigma_f}}{\left( \sigma_f - 1 \right)! \nu} \int f^{S^{\sigma_f}} \left( \prod_{k=\sigma_f+1}^{\sigma_f} |z_k|^{2a_k} \cdot |g|^2 \right) e^{-\tilde{\phi}_2 + \tilde{\phi}_m} d \text{vol}_{S^{\sigma_f}, \omega},
\]

which is the desired result.

It remains to check for the case $\sigma < \sigma_f$ but $\sigma_f \geq 1$. By assumption, $g$ is not identically 0 on $S^{\sigma_f}$. As the integrand in question is non-negative, by shrinking $V$ (in such a way that $V \cap S^{\sigma_f} \neq \emptyset$ after shrinking) if necessary, one can assume without loss of generality that $|g|^2 > 0$ on $V$, and thus $F_0 > 0$ on $V$. Consider a further change of variables

\[
|\psi| = |\psi|, \quad q_j := \frac{t_j}{|\psi|} = \frac{|t_j|}{|\psi|} \quad \text{for} \quad j = 2, \ldots, \sigma_f,
\]
where each $q_j$ varies within $[0, 1]$ on $V$. The expression in (*) then becomes
\[
(*) = \frac{(-1)^{\sigma_f}}{\nu} \int \frac{F_0}{1 + \frac{\nu}{2v_1} \partial_1 \alpha} \frac{|\psi|^{\sigma_f - 1} d |\psi|}{|\psi|^{\sigma + \epsilon}} \prod_{j=2}^{\sigma_f} dq_j \cdot d\theta
\]

where
\[
\frac{F_0}{1 + \frac{\nu}{2v_1} \partial_1 \alpha} > 0 \text{ on } V \text{ and } d \left( |\psi|^{\sigma_f - \sigma - \epsilon} \right) \text{ is non-integrable on } V \text{ when } \epsilon \text{ is sufficiently small, the expression above tends to } \infty \text{ as } \epsilon \to 0^+.
\]

**Remark 2.2.3** Having the Taylor expansion in mind, for a general compactly supported smooth function $f$ on $X$, on every local coordinate neighbourhood $V$ where $f|_{S \cap V} \neq 0$ and $S \cap V = \{z_1 \ldots z_{\sigma_V} = 0\}$, there is an integer $\sigma_f$ (dependent on $V$) such that
\[
f|_V = \sum_{p \in \mathcal{S}_{\sigma_V}/(\mathcal{S}_{\sigma_V - \sigma_f} \times \mathcal{S}_{\sigma_f})} \prod_{\substack{k=1 \atop \nu_k = \sigma_f}}^{\sigma_V} |z_{p(k)}|^{1+\mu(k)} \cdot g_p \quad \text{with} \quad g_{p'}|_{\nu'\sigma^{\sigma_f}} \neq 0 \text{ for some } p',
\]

where every $p$ is a choice of $\sigma_V - \sigma_f$ elements from the set $\{1, 2, \ldots, \sigma_V\}$ (which is abused to mean a corresponding permutation of $\{1, 2, \ldots, \sigma_V\}$), each $g_p$ is a bounded function on $V$ with $|g_p|^2$ being smooth, and $\nu'\sigma^{\sigma_f} := \{z_{p'(1)} = \cdots = z_{p'(\sigma_f)} = 0\}$.

The same kind of calculation in the proof of the proposition shows that summands of the sum over $p \in \mathcal{S}_{\sigma_V}/(\mathcal{S}_{\sigma_V - \sigma_f} \times \mathcal{S}_{\sigma_f})$ are mutually orthogonal (by considering only the monomials in $|z_j|^{\nu}$’s with constant coefficients) with respect to the inner product induced from $d \operatorname{lc}_\omega[\psi_L][\psi]$ when $\sigma > \sigma_f - 2$. Therefore, using a partition of unity, the results in the proposition still hold for $f|_V$, except that the integral in the case $\sigma = \sigma_f$ is now the sum of integrals over all lc centres in $V$ of codimension $\sigma_f$, i.e.
\[
\int_{\operatorname{lc}_\omega(S) \cap V} |f|^2 d \operatorname{lc}_\omega[\psi_L][\psi] = \begin{cases} 0 & \text{when } \sigma > \sigma_f \text{ or } \sigma_f = 0, \\ \infty & \text{when } \sigma < \sigma_f \text{ and } \sigma_f \geq 1, \end{cases}
\]

(that the integral diverges when $\sigma \leq \sigma_f - 2$ follows from the inequality $\frac{1}{|\psi|^{\sigma_f - 1 + \theta}} \leq \frac{1}{|\psi|^{\sigma_f - 1 + \theta}}$ on a neighbourhood of any $\nu'\sigma^{\sigma_f}$ for $\sigma \leq \sigma_f - 2$ and the fact that the integral diverges when $\sigma = \sigma_f - 1$) and
\[
\int_{\operatorname{lc}_\omega(S) \cap V} |f|^2 d \operatorname{lc}_\omega[\psi_L][\psi]
\]

\[
= \sum_p \frac{\pi^{\sigma_f}}{(\sigma_f - 1)! \prod_{j=1}^{\sigma_f} v_{p(j)}^{\nu_p(j)}} \int_{\nu'\sigma_f} \left( \prod_{k=1}^{\sigma_V} \int_{\nu_k = \sigma_f + 1} \cdots |z_{p(k)}|^{2\mu(k)} \cdot |g_p|^2 \right) e^{-\bar{\psi}_L + \psi_L^{im}} \, d \nu_g|_{\nu_f, \omega}
\]
when $\sigma = \sigma_f$.

Note that the largest $\sigma_f$ among all different local neighbourhoods $V$ covering $X$ is the codimension of mlc of $(X, S)$ with respect to $f$ (see Definition 1.4.4 or 2.2.5). Considering all such $V$‘s, the proposition also holds true for $f$ with $\sigma_f$ being the codimension of mlc of $(X, S)$ with respect to $f$ in all cases of $\sigma$ (after the modification for the case $\sigma = \sigma_f$).

**Remark 2.2.4** If $f$ vanishes to suitable orders along the polar subspaces of $\tilde{\varphi}_L$ and $\psi$, the assumption (†) is not necessary in the proposition. Suppose that, on an open set $V \subset X$, $\psi|_V$ is given as in the proposition while

$$\tilde{\varphi}_L|_V := \sum_{j=1}^n \ell_j \log |z_j|^2 + \beta,$$

where $\beta$ is a smooth function on $V$ and $\ell_j \geq 0$ are arbitrary constants for $j = 1, \ldots, n$ such that the $m_1$ in $\tilde{\varphi}_L + \psi_S = \varphi_L + m_1 \psi$ is a jumping number and $\text{Ann}_X(\frac{\frac{1}{2}(\varphi_L + m_0 \psi)}{\frac{1}{2}(\varphi_L + m_0 \psi)})$ defines $S$ as in Section 1.3 with $S \cap V = \{z_1 \cdots z_{\sigma_V} = 0\}$ (i.e. no upper bound on $\ell_j$’s and $\ell_j > 0$ for any $j = 1, \ldots, \sigma_V$ is allowed). If $f \in \mathcal{C}^\infty \otimes \mathcal{I}(\varphi_L + m_0 \psi)$, it implies that $f \in \mathcal{C}^\infty \otimes \mathcal{I}(\tilde{\varphi}_L)$ as

$$G := |f|^2 e^{-\tilde{\varphi}_L} = |f|^2 e^{-\varphi_L - m_1 \psi - \psi_S^\text{sm} + \psi_S} = |f \otimes s_S|^2 e^{-\varphi_L - m_1 \psi - \psi_S^\text{sm}}.$$

Note that $G^{-1}(\infty)$ does not contain any lc centres of $(X, S)$ (if $G^{-1}(\infty)$ contains an lc centre of $(X, S)$, then it can be seen that, as $Ge^{-\psi_S} \, d\text{vol}_{X, \omega} = |f|^2 \, e^{-\varphi_L - m_1 \psi} \, d\text{vol}_{X, \omega}$ is not integrable, there exists some $m_0 < m' < m_1$ such that $|f|^2 \, e^{-\varphi_L - m' \psi} \, d\text{vol}_{X, \omega}$ is also not integrable, contradicting the fact that $m_1$ is the only jumping number in the interval $(m_0, m_1)$). That means that $G|_V$ cannot be decomposed locally into a quotient $g \, h$ of continuous functions where $g|_{\{z_j=0\}} \not\equiv 0$ while $|z_j|^{2a_j}$ divides $h$ (in the ring of continuous functions) for some positive number $a_j > 0$ for any $j = 1, \ldots, \sigma_V$. Let $\sigma_f$ be the maximal codimension in $X$ of all lc centres of $(X, S)$ not contained in $G^{-1}(0)$. Then, by expanding $f$ locally on any suitable open set $V$ into a sum as in Remark 2.2.3 and cancelling the factors of $|z_j|$’s suitably, one obtains

$$G|_V = |f|^2 \, e^{-\varphi_L - m_1 \psi + \psi_S} \bigg|_V = \left| \sum_{\rho \in \mathcal{S}_{\sigma_V} / \mathcal{S}_{\sigma_f} \atop r := \sigma_V - \sigma_f} \prod_{k=\sigma_f+1}^{\sigma_V} |z_{p(k)}|^{1+a_{p(k)}} \cdot g_p \right|^{2} \frac{1}{\prod_{k=\sigma_V+1}^n |z_k|^{2\gamma_k}},$$

where $\gamma_k$’s are constants such that $\gamma_k < 1$ for $k = \sigma_V + 1, \ldots, n$ (as $G$ is integrable on $X$), and each $g_p$ is a bounded function on $V$ with $|g_p|^2$ being smooth. Moreover, there
exist an open set $V$ and some $p$ such that the corresponding $g_p$ on $V$ satisfies $g_p|_{π^*f} ≠ 0$. Following the discussion in Remark 2.2.3, the trichotomy in the conclusion of the proposition ($σ > σ_f$, $σ = σ_f$ and $σ < σ_f$) still holds even without the assumption ($†$).

The following definition of $σ_f$ is given after the discussion in Remark 2.2.4.

**Definition 2.2.5** Given any function or vector-bundle-valued section $f$ on $S$ such that $f ∈ L_∞ \otimes J(ϕ_L + m_0 ψ)$ with $ϕ ∈ L_∞ \otimes J(ϕ_L + m_1 ψ)$ denoting any local lifting of $f$, define the **codimension of mlc of $(X, ϕ_L, ψ, m_1)$ with respect to $f$**, denoted by $σ_f,ϕ_L,m_1$, to be the maximal codimension of all the lc centres of $(X, ϕ_L, ψ, m_1)$ which are not contained in the zero locus of $G := |f|^2 e^{−ϕ_L}$. When it is understood that $S$ is defined from the data $ϕ_L, ψ$ and $m_1$, the quantity is also called the **codimension of mlc of $(X, S)$ with respect to $f$** and denoted by $σ_f$. An lc centre of $(X, S)$ with such codimension which does not lie in $G^{−1}(0)$ is called an **mlc of $(X, S)$ with respect to $f$**.

It can be seen from Proposition 2.2.1 and the subsequent remarks that this definition coincides with Definition 1.4.4 when the snc assumption on $ϕ_L$ and $ψ$ holds.

### 2.3 Illustration

The following examples are to show that the lc-measures are potentially good replacement of the Ohsawa measure.

**Example 2.3.1** In [41, after Prop. 5.4], Ohsawa provides the following example (with a slight modification, which the authors owe Bo Berndtsson for). On the unit bi-disc $Δ ⊂ \mathbb{C}^2$ centred at the origin with coordinates $(z, w)$, let $ϕ^{(k)} := \log (|z − w|^2 + \frac{1}{k})$ (or any decreasing sequence which converges to $ϕ := \log |z − w|^2$) and $ω$ be the Euclidean metric. Then, there is no universal constant $C > 0$ such that, for any holomorphic function $f$ on $S := \{(z, w) ∈ Δ | zw = 0\}$ with $ʃ_Δ |f|^2 e^{−ϕ} \, dvol ≤ C$, there exist holomorphic functions $F^{(k)}$ on $Δ$ such that $F^{(k)}|_S = f$ and

$$(*) \int_Δ |F^{(k)}|^2 e^{−ϕ^{(k)}} \, dvol ≤ C \int_Δ |f|^2 e^{−ϕ} \, dvol ≤ C \int_Δ |f|^2 e^{−ϕ} \, dvol.$$  

Indeed, if

$$f = \begin{cases} z & \text{on } \{w = 0\} \\ 0 & \text{on } \{z = 0\} \end{cases},$$

then, the existence of $F^{(k)}$ and the estimate $(*)$ imply the existence of a holomorphic extension $F$ of $f$ such that the estimate $(*)$ holds with $F$ replacing $F^{(k)}$. This in turn implies that $F = zG$ for some holomorphic function $G$ with $G|_{z=w} = 0$, which is impossible since it means that $F$ vanishes to order 2 but $f$ vanishes only up to order 1 on $\{w = 0\}$. 

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Set $\psi := \phi_S = \log |zw|^2$. The lc-measures can filter out Ohsawa’s example (so does the Ohsawa measure $\int_S |f|^2 \ d \nu_S$ \cite{Ohsawa90} \cite{Demailly92} such that the estimates with universal constant could still be possible. Note that $S$ is defined by the annihilator $\text{Ann}_{\Delta} \left( \mathcal{J}(\phi(k) + \frac{1}{2} \psi) - \mathcal{J}(\phi(k) + \psi) \right)$ (the coefficient $\frac{1}{2}$ of $\psi$ is chosen such that the annihilator still defines $S$ scheme theoretically when $\phi(k)$’s are replaced by $\phi$), and the mlc of $(X, S)$ is of codimension 2. Taking $f$ as above and letting $\tilde{f} = z$ (noting that $\mathcal{J}(\phi(k) + \frac{1}{2} \psi) = \mathcal{O}_\Delta$), computation in Proposition 2.2.1 shows that

$$\int_{\{z=0=w\}} |f|^2 \ \text{lcv}_{\omega,\phi(k)}^2[\psi] = 0$$

and

$$\int_{\{z=w=0\} \cap \Delta} |f|^2 \ \text{lcv}_{\omega,\phi(k)}^2[\psi] = \int_{\{w=0, \ |z| < 1\}} e^{-2\log(|z|^{2} + 1)} \pi d\nu = \pi^2 \log (1 + k)$$

(so $\sigma_{f,\phi(k) + \psi} = 1$), where the latter integral diverges to $\infty$ as $k \to \infty$. This shows that the function $f$ in Ohsawa’s example is ruled out by the $L^2$ extension theorem in the first place, provided that lc-measures are used.

Indeed, this can also be seen without taking any approximating sequence of $\varphi = \log |z-w|^2$. Note that $f$ has no local lifting to $\mathscr{E} \otimes \mathcal{J}(\varphi + \frac{1}{2} \psi)$ on $\Delta$ as shown by the similar argument using vanishing order above. This already shows that the lc-measures $|f|^2 \ \text{lcv}_{\omega,\phi(k)}^2[\psi]$ are not well-defined for $\sigma = 1$ and 2. As a comparison with the computation above, let $\pi : \tilde{\Delta} \to \Delta$ be the blow-up of $\Delta$ at the origin with exceptional divisor $E$, which is a log-resolution of $(\Delta, \varphi, \psi)$. Consider a neighbourhood $\tilde{U} \subset \tilde{\Delta}$ of $\pi^{-1} \{w = 0\}$ with coordinates $(s_E, w_1)$ such that $\pi^* w = s_E w_1$, $\pi^* z = s_E$ and $|w_1| < \frac{1}{2}$, where $E \cap \tilde{U} = \{s_E = 0\}$ and $\tilde{S} \cap \tilde{U} = \{s_E w = 0\}$ (where $\tilde{S}$ is defined as in Sect. 2.1). Take a smooth extension $\tilde{f}^\pi$ of $\pi^* f$ on $\tilde{\Delta}$ in $\mathcal{J}_{\tilde{\Delta}}(\pi^* \varphi - \varphi_E + \frac{1}{2} \pi^* \psi)$ such that $\tilde{f}^\pi|_{\tilde{U}} = s_E$ and vanishes outside of a larger neighbourhood $\tilde{U}_1 := \{|w_1| < 1\}$ of $\tilde{U}$. It then follows that

$$\int_{\pi^* \tilde{U}_1} |\pi^* f|^2 \ \text{lcv}_{\omega,\pi^* \varphi - \varphi_E}^2[\pi^* \psi] := \lim_{\varepsilon \to 0^+} \varepsilon \int_{\tilde{U}_1} \frac{d \sigma_{\pi^* \psi}^2}{|\pi^* \psi|^\sigma + \varepsilon} e^{-\pi^* \varphi + \varphi_E - \pi^* \psi} \ d \nu_{\tilde{\Delta}}$$

$$= \lim_{\varepsilon \to 0^+} \varepsilon \int_{\tilde{U}_1} \frac{d \sigma_{\pi^* \psi}^2}{|s_E|^2 |w_1|^2 |1 - w_1|^2 |\pi^* \psi|^\sigma + \varepsilon}$$

$$\overset{\text{Prop. 2.2.1}}{=} \begin{cases} \frac{\pi^2}{2} & \text{when } \sigma = 2, \\ \infty & \text{when } \sigma = 1. \end{cases}$$

This again shows that Ohsawa’s example will be excluded for the consideration of $L^2$ extension if the $L^2$ extension theorem with respect to the lc-measures
is proved. This also provides an example that $|\pi^* f|^2 d \text{lcv}_{\omega, \pi^* \varphi - \varphi_E}^2 [\pi^* \psi] \neq \lim_{k \to \infty} |\pi^* f|^2 d \text{lcv}_{\omega, \pi^* (k\varphi_E - \varphi_E)}^2 [\pi^* \psi]$, even though both sides are finite.

**Example 2.3.2** (from a private note by Bo Berndtsson; see also [6, Appendix A]). Berndtsson computes a concrete estimate for holomorphic functions on the unit bidisc $\Delta \subset \mathbb{C}^2$ extended from holomorphic functions on $S := \{(z, w) \in \Delta \mid zw = 0\}$ with minimal $L^2$ norm with singular weight $e^{-\varphi}$ on $\Delta$. It turns out that the estimate can be expressed in terms of lc-measures, thus giving a hint on how the estimate looks like in general.

Assume that $\varphi^{-1}(-\infty)$ does not contain any lc centre of $(\Delta, S)$ and set $\psi := \phi_S = \log |zw|^2$. Assume also that $\varphi$ is psh and has only neat analytic singularities for simplicity. Let $H := H^0 (\Delta, \mathcal{I}(\varphi)) \cap L^2 (\Delta; e^{-\varphi})$ be the space of $L^2$ holomorphic functions $F$ on $\Delta$ with respect to the norm-square $\|F\|_\varphi^2 := \int_\Delta |F|^2 e^{-\varphi} d \text{vol}_\Delta$ and consider the filtration

$$H =: H_3 \supset H_2 \supset H_1 \supset H_0 = \{0\},$$

where $H_\sigma$ is the closed subspace of functions which vanish on $\text{lcv}_{\Delta}^\sigma (S)$ for $\sigma = 1, 2$.

Note that

$$\text{lcv}_{\Delta}^1 (S) = S \quad \text{and} \quad \text{lcv}_{\Delta}^2 (S) = \{z = w = 0\} = \{(0, 0)\}.$$

Let $A_\sigma$ be the orthogonal complement such that $H_{\sigma+1} = H_\sigma \oplus A_\sigma$ for $\sigma = 0, 1, 2$, and thus $H = A_2 \oplus A_1 \oplus A_0$.

Suppose $F \in H$ is the minimal holomorphic extension with respect to the norm $\|\cdot\|_\varphi$ of some $L^2$ holomorphic function $f$ on $S$ (with respect to the potential $\varphi|_S$). Then $F$ is orthogonal to $H_1 = A_0$. Write $F = F_2 + F_1$ such that $F_\sigma \in A_\sigma$ for $\sigma = 1, 2$, so $F_1$ vanishes on $\text{lcv}_{\Delta}^2 (S)$ but is non-trivial on $\text{lcv}_{\Delta}^1 (S)$ (if $F_1 \neq 0$), which implies $\sigma_{F_1, \varphi + \psi} = 1$, while $F_2$ is non-trivial on $\text{lcv}_{\Delta}^2 (S)$ (if $F_2 \neq 0$), which implies $\sigma_{F_2, \varphi + \psi} = 2$. Therefore, $f = (F_2 + F_1)|_S$ and $f_0 := f(0, 0) = F_2(0, 0)$.

To compute $\|F_2\|_\varphi^2$, let $B(\cdot, \cdot)$ be the Bergman kernel of $H$ with respect to the norm $\|\cdot\|_\varphi$ and write $\hat{0} := (0, 0)$ and $\hat{z} := (z, w)$ when necessary. By considering an orthonormal basis $\{e_0, e_1, \ldots\}$ of $H$ such that $B(\hat{0}, \hat{0}) = |e_0(\hat{0})|^2$, one sees that $F_2(\hat{z}) = cB(\hat{z}, \hat{0})$ for some constant $c$ and thus

$$F_2(\hat{z}) = \frac{f_0B(\hat{z}, \hat{0})}{B(\hat{0}, \hat{0})} \implies \|F_2\|_\varphi^2 = \frac{|f_0|^2}{B(\hat{0}, \hat{0})}.$$
Next is to compute \( \| F_1 \|_{\varphi}^2 \). Since \( F_1 (0) = 0 \), there exist holomorphic functions \( h_1 \) and \( h_2 \) such that \( F_1 = zh_1 + wh_2 \). Notice that \( F_1 \) is the minimal holomorphic extension of \( f - F_2|_S \) with respect to the norm \( \| \cdot \|_{\varphi} \). If \( h_1 \) (resp. \( h_2 \)) is replaced by the minimal extension \( \tilde{h}_1 \) (resp. \( \tilde{h}_2 \)) of \( h_1|_{w=0} \) (resp. \( h_2|_{z=0} \)) with respect to \( \| \cdot \|_{\varphi} \), the sum \( z\tilde{h}_1 + w\tilde{h}_2 \) is still an extension of \( f - F_2|_S \). The classical Ohsawa–Takegoshi extension theorem provides estimates for minimal holomorphic extensions on Stein manifold which are extended from smooth hypersurfaces (note that \( \varphi|_S \) is well-defined on each irreducible component of \( S \) by assumption). Therefore, one has

\[
\| F_1 \|_{\varphi}^2 \leq \| \tilde{h}_1 + \tilde{h}_2 \|_{\varphi}^2 \\
\leq \pi^2 \int_{|z| < 1, |w| = 0} |h_1|^2 e^{-\varphi} i d\bar{z} + \pi^2 \int_{|w| < 1, |z| = 0} |h_2|^2 e^{-\varphi} i d\bar{w}
\]

by Prop. 2.2.1

\[
\int_S |F_1|^2 d \text{lcv}_{\omega, \varphi}[\psi] = \int_{\text{lc}^1_{\Delta}(S)} |f - F_2|^2 d \text{lcv}^1_{\omega, \psi}[\psi],
\]

where the constant involved in \( \lesssim \) is “universal”, i.e. it does not depend on \( \varphi \) or any functions appearing in the integrands of the integrals on either side of the inequality.

As a result, one has the estimate

\[
\| F \|_{\varphi}^2 = \| F_2 \|_{\varphi}^2 + \| F_1 \|_{\varphi}^2 \lesssim \int_{\text{lc}^2_{\Delta}(S)} |f|^2 d \text{lcv}^2_{\omega, \varphi}[\psi] + \int_{\text{lc}^1_{\Delta}(S)} |f - F_2|^2 d \text{lcv}^1_{\omega, \varphi}[\psi],
\]

where the constant involved in \( \lesssim \) is universal.

**Remark 2.3.3** The estimate, though essentially the best one could expect in general, may look unsatisfactory in the sense that one seems to have lost control of the estimate due to the integral of \( f - F_2|_{\text{lc}^1_{\Delta}(S)} \) on the right-hand-side. In practice, one may need to manipulate the estimate on \( \| F_2 \|_{\varphi}^2 \) in order to obtain some control of \( \int_{\text{lc}^1_{\Delta}(S)} |f - F_2|^2 d \text{lcv}^1_{\omega, \varphi}[\psi] \).

### 3 Extension with estimates with respect to lc-measures on codimension-1 lc centres

For simplicity, suppose that \( m_0 = 0 \) and \( m_1 = 1 \). The arguments remain the same for the case of general jumping numbers.

As discussed in Sect. 2.1, one can assume that \( S \) is a reduced divisor in \( X \) and that \((X, S)\) is a log-smooth and log-canonical (lc) pair.
3.1 Setup for the extension theorem

The goal of the following is to replace the generalised Ohsawa measure in the Ohsawa–Takegoshi $L^2$ extension theorem by the lc-measure given by

$$|f|^2_\omega \, 	ext{d cv}_{\omega, \varphi L} [\psi] := \lim_{\varepsilon \to 0^+} \varepsilon\left|\hat{f}\right|^2_\omega \frac{e^{-\varphi L_{-\psi}}}{|\psi|^{\sigma+\varepsilon}} \, \text{d} \text{vol}_{X, \omega},$$  \hspace{1cm} (3.1.1)$$

where $\hat{f}$ is any smooth extension of $f$ on $X$ such that $|\hat{f}|^2 e^{-\varphi L}$ is locally integrable, for the case $\sigma = 1$. The behaviour of such measure is discussed in Sect. 2.

Set

- $P_{\varphi L} := \varphi_L^{-1}(-\infty)$ and $P_\psi := \psi^{-1}(-\infty)$ (only the negative poles), which are closed analytic subsets of $X$ such that $P_{\varphi L} \cup P_\psi$ has only snc by the assumptions on $\varphi_L$ and $\psi$ (Sects. 1.3, 2.1);
- $X^\circ := X \setminus (P_{\varphi L} \cup P_\psi)$, which has the structure of a complete Kähler manifold;
- $\varphi := \varphi_L + \psi + \nu$, which is a potential (of the curvature of a hermitian metric) on $L$, where $\nu$ is a real-valued smooth function on $X^\circ$;

3.2 Bochner–Kodaira formula

The key tool for proving this version of extension theorem is still the twisted Bochner–Kodaira formula (see [38, Eq. (8)], also [11, Ch. VIII] or [12, §3.C]). The following notations are used:

- $\Theta^\omega (\zeta, \zeta)_\varphi$ denotes, for any real $(1, 1)$-form $\Theta$ (usually in the form $i \partial \bar{\partial} \varphi$) and any $K_X \otimes L$-valued $(0, q)$-form $\zeta$, the trace of the contraction between $i^{-1} \Theta$ and $e^{-\varphi} \zeta \wedge \bar{\zeta}$ with respect to the hermitian metric on $X$ given by $\omega$ (in the convention such that $\Theta^\omega (\zeta, \zeta)_\varphi \geq 0$ whenever $\Theta \geq 0$);
- $\vartheta$ denotes the formal adjoint of $\bar{\partial} \psi$ with respect to the global $L^2$-norm on $X^\circ$;
- $(\partial \psi)^\omega \cdot$ denotes the adjoint of $\bar{\partial} \psi \wedge \cdot$ with respect to $(\cdot, \cdot)_{X^\circ, \omega, \varphi}$ on $X^\circ$.

Lemma 3.2.1 Let $\sigma \geq 1$ be a positive integer. With the auxiliary functions defined for every $\varepsilon \in \mathbb{R}$ as

$$\nu := - \log \log \left|\frac{\ell \psi}{e}\right|, \quad \eta_\varepsilon := |\psi|^{\sigma(1-\varepsilon)} e^{-\nu} = |\psi|^{\sigma(1-\varepsilon)} \log \left|\frac{\ell \psi}{e}\right|$$

and

$$\lambda_\varepsilon := \eta_\varepsilon \left( \sigma (1 - \varepsilon) \log \left|\frac{\ell \psi}{e}\right| + 1 \right)^2,$$
and letting $L$ be endowed with the metric with potential $\varphi := \varphi_L + \psi + \nu$, the Bochner–Kodaira formula becomes

$$
\int_{X^\circ} |\bar{\omega}|^2 \eta_\nu + \int_{X^\circ} |\partial \xi|_\varphi^2 (\eta_\nu + \lambda_\varphi) - \varepsilon \int_{X^\circ} \left( \frac{\sigma (1 - \varepsilon)}{|\psi|^2} + \frac{2}{|\psi|^2 \log \frac{|\psi|}{\varepsilon}} \right) |(\partial \psi) \circ \xi|_\varphi^2 \eta_\nu
$$

$$
= \int_{X^\circ} |\nabla^{(1,0)} \xi|_{\omega, \varphi}^2 \eta_\nu + \int_{X^\circ} |\partial \xi|_\varphi^2 \eta_\nu - \varepsilon \int_{X^\circ} (\partial \xi, (\partial \log \eta_\nu) \circ \xi) \eta_\nu + \int_{X^\circ} (\partial \log \eta_\nu) \circ \xi |^2 \eta_\nu.
$$

for any compactly supported $K_X \otimes L$-valued smooth $(0, 1)$-forms $\xi \in \Omega^{0,1}_{0,X} (X^\circ; K_X \otimes L)$ on $X^\circ$.

**Proof** From [44, §1.3] or [38, Eq. (8)], it follows that

$$
\int_{X^\circ} |\bar{\omega}|^2 \eta_\nu + \int_{X^\circ} |\partial \xi|_\varphi^2 \eta_\nu - \varepsilon \int_{X^\circ} (\partial \xi, (\partial \log \eta_\nu) \circ \xi) \eta_\nu + \int_{X^\circ} (\partial \log \eta_\nu) \circ \xi |^2 \eta_\nu.
$$

A direct computation with the choices of $\nu$ and $\eta_\nu$ yields

$$
\int_{X^\circ} |\bar{\omega}|^2 \eta_\nu + \int_{X^\circ} |\partial \xi|_\varphi^2 \eta_\nu + \int_{X^\circ} \left( \frac{\sigma^2 (1 - \varepsilon)^2}{|\psi|^2} + \frac{2 \sigma (1 - \varepsilon)}{|\psi|^2 \log \frac{|\psi|}{\varepsilon}} \right) |(\partial \psi) \circ \xi|_\varphi^2 \eta_\nu
$$

$$
+ \int_{X^\circ} \frac{1}{|\psi|^2 \left( \log \frac{|\psi|}{\varepsilon} \right)^2} |(\partial \psi) \circ \xi|_\varphi^2 \eta_\nu
$$

$$
= \int_{X^\circ} |\nabla^{(1,0)} \xi|_{\omega, \varphi}^2 \eta_\nu + \int_{X^\circ} (i \partial \bar{\partial} (\varphi_L + \psi) + \frac{\sigma (1 - \varepsilon)}{|\psi|} + \frac{2}{|\psi|^2 \left( \log \frac{|\psi|}{\varepsilon} \right)^2}) \eta_\nu + \int_{X^\circ} \left( \frac{\sigma (1 - \varepsilon)}{|\psi|^2} + \frac{2}{|\psi|^2 \log \frac{|\psi|}{\varepsilon}} \right) |(\partial \psi) \circ \xi|_\varphi^2 \eta_\nu
$$

$$
+ 2 \varepsilon \int_{X^\circ} (\partial \xi, (\partial \log \eta_\nu) \circ \xi) \eta_\nu.
$$

It follows from the choices of $\lambda_\varphi$ that

$$
\int_{X^\circ} (\partial \log \eta_\nu) \circ \xi |^2 \eta_\nu = \int_{X^\circ} \frac{1}{|\psi|^2 \left( \log \frac{|\psi|}{\varepsilon} \right)^2} |(\partial \psi) \circ \xi|_\varphi^2 \eta_\nu.
$$

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As a result, the acclaimed formula is obtained after completing the square for the inner-product terms by adding \( \int_{X^o} |\partial \zeta|^2_\varphi \lambda_\epsilon \) to both sides, and collecting terms of \( |(\partial \psi)^\omega \partial \zeta|^2_\varphi \) (in \text{VioletRed}) suitably. \( \square \)

It follows from Lemma 3.2.1 that, when \( \sigma = 1 \) and the remaining terms on the right-hand-side (in \text{NavyBlue}) are semi-positive, one has

\[
\int_{X^o} |\bar{\partial} \zeta|^2_{\omega, \varphi} \eta_\epsilon + \int_{X^o} |\partial \zeta|^2_\varphi (\eta_\epsilon + \lambda_\epsilon) \geq \epsilon \int_{X^o} \frac{1 - \epsilon}{|\psi|^2} |(\partial \psi)^\omega \partial \zeta|^2_\varphi \eta_\epsilon \tag{3.2.1}
\]

for all compactly supported \( \zeta \). Positivity of the terms in \text{NavyBlue} is provided by suitable curvature assumption.

The completeness of \( X^o \) guarantees that \( \omega \) can be modified to a complete metric, and, in that case, the inequality (3.2.1) holds true also for all (weighted) \( L^2(0, 1) \)-forms \( \zeta \) in both of the domains of \( \bar{\partial} \) and its Hilbert space adjoint \( \bar{\partial}^* \) (see, for example, [11, Ch. VIII, §3]), and thus the Riesz Representation Theorem can be invoked.

### 3.3 Proof of the extension theorem with 1-lc-measure

Let \( \theta : [0, \infty) \to [0, 1] \) be a smooth non-increasing function such that \( \theta \equiv 1 \) on \([0, \frac{1}{A}]\) and \( \equiv 0 \) on \([\frac{1}{B}, \infty)\), where \( 1 < B < A \), and \( |\theta'| \leq \frac{AB}{A - B} + \epsilon_0 \) on \([0, \infty)\) for some positive constant \( \epsilon_0 \). Define also that \( \theta_\epsilon := \theta \circ |\psi|^{-\epsilon} \) and \( \theta'_\epsilon := \theta' \circ |\psi|^{-\epsilon} \) for convenience.

It is shown below (Theorem 3.3.4) that the Ohsawa measure in the Ohsawa–Takegoshi extension theorem can be replaced by the lc-measure (3.1.1) in the classical case, i.e. when \text{mlc} of \( (X, S) \) are of codimension 1 (and \( S \) is smooth as \( (X, S) \) is log-smooth), or when the holomorphic section \( f \) to be extended vanishes on the singular locus of \( S \), or more precisely, when the \text{mlc} of \( (X, S) \) with respect to \( f \) (see Definitions 1.4.4 or 2.2.5) is of codimension 1.

**Theorem 3.3.1** Suppose that

(1) there exists \( \delta > 0 \) such that

\[
i \partial \bar{\partial} (\varphi_L + \psi) + \beta \partial \bar{\partial} \psi \geq 0 \quad \text{on} \ X \quad \text{for all} \ \beta \in [0, \delta) , \ \text{and}
\]

(2) for any given constant \( \ell > 0 \), the function \( \psi \) is normalised (by adding to it a suitable constant) such that

\[
\psi < -\frac{\epsilon}{\ell} \quad \text{and} \quad \frac{1}{|\psi|} + \frac{2}{|\psi| \log \left| \frac{\ell \psi}{\epsilon} \right|} \leq \delta .
\]

(See Remark 3.3.5 for the use of the constant \( \ell \).)
Let $\widetilde{f}$ be an $K_X \otimes L$-valued smooth section on $X$ such that $|\partial \overline{\partial} \widetilde{f}|^2 e^{-\varphi_L-\psi} \log \frac{|\ell \psi|}{e}$ is integrable over $X$. Then, for any numbers $\varepsilon, \varepsilon' > 0$, the $\partial$-equation

$$\partial u_{\varepsilon} = v_{\varepsilon} := \partial \left( \frac{1}{|\psi|^\varepsilon} \right) \widetilde{f} = \frac{\varepsilon \theta' \varepsilon \wedge \widetilde{f}}{|\psi|^{1+\varepsilon}} + \frac{\theta \partial \overline{\partial} \widetilde{f}}{\varepsilon} =: v_{\varepsilon}^{(1)}$$

can be solved with an $\varepsilon'$-error, in the sense that there are a smooth $K_X \otimes L$-valued $(0,1)$-form $w_{\varepsilon', \varepsilon}$ and a smooth section $u_{\varepsilon', \varepsilon}$ on $X^\circ$ such that

$$\partial u_{\varepsilon', \varepsilon} + w_{\varepsilon', \varepsilon} = v_{\varepsilon} \text{ on } X^\circ,$$

with the estimates

$$\int_{X^\circ} |u_{\varepsilon', \varepsilon}|^2 e^{-\varphi_L-\psi} \log \frac{|\ell \psi|}{e} + \frac{1}{\varepsilon'} \int_{X^\circ} |w_{\varepsilon', \varepsilon}|^2 e^{-\varphi_L-\psi} \log \frac{|\ell \psi|}{e}$$

$$\leq \frac{1}{\varepsilon'} \int_X |\theta' \partial \overline{\partial} \widetilde{f}|^2 + \varepsilon \int_X |\theta' \partial \overline{\partial} \widetilde{f}|^2 e^{-\varphi_L-\psi} \log \frac{|\ell \psi|}{e} \leq \frac{1}{1-\varepsilon} \int_X |\theta' \partial \overline{\partial} \widetilde{f}|^2 e^{-\varphi_L-\psi} \log \frac{|\ell \psi|}{e}.$$

**Remark 3.3.2** It is well known that a locally $L^1$ function $f$, which satisfies

$$i \partial \overline{\partial} f \geq 0 \text{ as a current,}$$

coincides with a uniquely determined psh function almost everywhere (see, for example, [29, Thm. 1.6.10, Thm. 1.6.11]). Since $\varphi_L$ and $\psi$ are locally differences of quasi-psh functions, a simple argument shows that $(\varphi_L + \psi) + \beta \psi$ is a psh potential on $X$ for every $\beta \in [0, \delta]$ by the assumption (1).

**Proof** Let $L$ be endowed with a metric with potential $\varphi := \varphi_L + \psi + \nu$ and choose the auxiliary functions $\nu, \eta_{\varepsilon}$ and $\lambda_{\varepsilon}$ as in Lemma 3.2.1 with $\sigma = 1$. The curvature assumption (1) and the normalisation assumption (2) assure that the terms on the right-hand-side (in NavyBlue) in Lemma 3.2.1 is semi-positive, and thus the twisted Bochner–Kodaira inequality (3.2.1) holds true. Write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X^\circ, \omega, \varphi}$ as the global inner product on $X^\circ$ induced by the potential $\varphi := \varphi_L + \psi + \nu$ and $\| \cdot \| := \| \cdot \|_{X^\circ, \omega, \varphi}$ the corresponding norm. Although $\omega$ is not assumed to be complete in the statement, the standard argument (see, for example, [11, Ch. VIII, §6]) reduces the problem to the case where $\omega$ is complete on $X^\circ$, which is assumed to be the case in what follows.

Assuming that $u_{\varepsilon}^{(2)} = 0$ on $X$, the usual argument with the Cauchy–Schwarz inequality and the twisted Bochner–Kodaira inequality (3.2.1) yields, for any com-

---

6 Note that $\varphi_L + \psi$ is, being psh by Remark 3.3.2, locally bounded from above, so the weight in the norm $\| \cdot \|_{X^\circ, \omega, \varphi}$ is everywhere positive on $X^\circ$ even though $\varphi_L$ itself may go to $+\infty$. 
pactly supported smooth $K_X \otimes L$-valued $(0, 1)$-form $\zeta$ on $X^\circ$, that

$$\langle \zeta, v_{\epsilon} \rangle = \left\| (\partial_{\psi})_{\kappa} \zeta, v_{\epsilon}^{(1)} \right\| = \left\| (\partial_{\psi})_{\kappa} \zeta, v_{\epsilon}^{(1)} \right\| = \left\| (\partial_{\psi})_{\kappa} \zeta, \frac{\epsilon_\theta f}{|\psi|^{1+\epsilon}} \right\|$$

$$\leq \left( \epsilon \int_{X^\circ} \left| (\partial_{\psi})_{\kappa} \zeta, \left( \frac{2}{|\psi|^2} \eta_{\epsilon} \right) \right| \right)^{1/2} \left( \int_{\text{supp} \theta_{\epsilon}} \left| \theta_{\epsilon} \right|^2 \left| f \right|^2 e^{-\phi_L - \psi} \right)^{1/2},$$

by (3.2.1) for any $\epsilon > 0$, one can handle the situation using the argument as in [12, after (5.20)] or the following slight variation of that. For any compactly supported smooth $K_X \otimes L$-valued $(0, 1)$-form $\zeta$ on $X^\circ$, applying the Cauchy–Schwarz inequality directly yields

$$\langle \zeta, v_{\epsilon} \rangle \leq \left\| (\zeta, v_{\epsilon}^{(1)}) \right\| + \left\| (\zeta, v_{\epsilon}^{(2)}) \right\|$$

$$\leq \mathcal{N}_1 \left( \partial_{\psi} \right) \mathcal{N}_2 \left( \phi \right) + \| \zeta \| \left\| \theta_{\epsilon} \phi \right\|$$

$$\leq \left( \left( \mathcal{N}_1 \left( \partial_{\psi} \right) \right)^2 + \eta_\epsilon \| \zeta \|^2 \right)^{1/2} \left( \left( \mathcal{N}_2 \left( \phi \right) \right)^2 + \frac{1}{\epsilon} \left\| \theta_{\epsilon} \phi \right\|^2 \right)^{1/2},$$

for any $\epsilon'> 0$. Note that the norm-square $\left\| \theta_{\epsilon} \phi \right\|^2 = \int_X \left| \theta_{\epsilon} \phi \right|^2 e^{-\phi_L - \psi}$ converges on $X$ by assumption (given the choice of $\psi$ in Lemma 3.2.1). The Riesz representation theorem then assures the acclaimed existence of solution $(\eta_\epsilon, \epsilon_{\psi}, w_{\epsilon'})$ and estimate, with the fact that $(\eta_\epsilon + \lambda_\epsilon) e^{\nu} \leq |\psi|^{-\epsilon} \left( (\log |\epsilon\psi|)^2 + 1 \right)$.

When $v_{\epsilon}^{(2)} \neq 0$, one can handle the situation using the argument as in [12, after (5.20)] or the following slight variation of that. For any compactly supported smooth $K_X \otimes L$-valued $(0, 1)$-form $\zeta$ on $X^\circ$, applying the Cauchy–Schwarz inequality directly yields

$$\langle \zeta, v_{\epsilon} \rangle \leq \left\| (\zeta, v_{\epsilon}^{(1)}) \right\| + \left\| (\zeta, v_{\epsilon}^{(2)}) \right\|$$

$$\leq \mathcal{N}_1 \left( \partial_{\psi} \right) \mathcal{N}_2 \left( \phi \right) + \| \zeta \| \left\| \theta_{\epsilon} \phi \right\|$$

$$\leq \left( \left( \mathcal{N}_1 \left( \partial_{\psi} \right) \right)^2 + \eta_\epsilon \| \zeta \|^2 \right)^{1/2} \left( \left( \mathcal{N}_2 \left( \phi \right) \right)^2 + \frac{1}{\epsilon} \left\| \theta_{\epsilon} \phi \right\|^2 \right)^{1/2},$$

for any $\epsilon' > 0$. Note that the norm-square $\left\| \theta_{\epsilon} \phi \right\|^2 = \int_X \left| \theta_{\epsilon} \phi \right|^2 e^{-\phi_L - \psi}$ converges on $X$ by assumption (given the choice of $\psi$ in Lemma 3.2.1). The Riesz representation theorem then assures the acclaimed existence of solution $(\eta_\epsilon, \epsilon_{\psi}, w_{\epsilon'})$ and estimate, with the fact that $(\eta_\epsilon + \lambda_\epsilon) e^{\nu} \leq |\psi|^{-\epsilon} \left( (\log |\epsilon\psi|)^2 + 1 \right)$.

Note also that the smoothness of $(\eta_\epsilon, \epsilon_{\psi}, w_{\epsilon'})$ follows from the smoothness of $v_{\epsilon}$ and the regularity of the $\phi$ operator. This completes the proof. □

Theorem 3.3.1 holds true irrespective of the codimension of mlc of $(X, S)$. The required extension of $f$ with estimate given in terms of the measure in (3.1.1) can be obtained by letting $\epsilon \to 0^+$ (after estimating $|\theta_{\epsilon}^2|$ by a constant and followed by
\( \varepsilon' \to 0^+ \), provided that the right-hand-side of the estimate converges. However, before starting the limit process, the solutions of the \( \overline{\partial} \)-equation (3.3.1) should be continued to the whole of \( X \).

**Proposition 3.3.3** Under the assumptions (1) and (2) in Theorem 3.3.1, there exists solution \((u_{\varepsilon',\varepsilon}, w_{\varepsilon',\varepsilon})\) to the \( \overline{\partial} \)-equation (3.3.1), namely \( \overline{\partial}u_{\varepsilon',\varepsilon} + w_{\varepsilon',\varepsilon} = v_{\varepsilon} \), with the estimate given in Theorem 3.3.1, which holds true on the whole of \( X \) (not only on \( X^\circ \)).

**Proof** First, for fixed \( \varepsilon \) and \( \varepsilon' \), apply Theorem 3.3.1 with \( \varphi = \varphi_L + \psi + \nu \), where \( 0 < r \ll 1 \), and obtain \( u_{\varepsilon',\varepsilon,r} \) and \( w_{\varepsilon',\varepsilon,r} \) satisfying the \( \overline{\partial} \)-equation (3.3.1) with the estimate in the Theorem. The number \( r \) is chosen sufficiently small (which depends on \( \varepsilon \)) such that the assumptions (1) and (2) in Theorem 3.3.1 imply that, with \( \sigma = 1 \), the terms on the right-hand-side (in NavyBlue) in Lemma 3.2.1 (after \( r \) is inserted) are semi-positive, so that Theorem 3.3.1 can be invoked.

Notice that \( v_{\varepsilon} \) is smooth on \( X \). In view of [8, Lemme 6.9], it suffices to show that both \( u_{\varepsilon',\varepsilon,r} \) and \( w_{\varepsilon',\varepsilon,r} \) are in \( L^2_{\text{loc}}(X) \) in order to show that the \( \overline{\partial} \)-equation (3.3.1) with solution \((u_{\varepsilon',\varepsilon,r}, w_{\varepsilon',\varepsilon,r})\) holds true on the whole of \( X \). The claim is then proved after letting \( r \to 0^+ \).

The curvature assumption (1) in Theorem 3.3.1 infers that \( \varphi_L + (1 + r) \psi \) is psh on \( X \) (see Remark 3.3.2), thus locally bounded above by some constant. Since \( \frac{\ell \psi}{\varepsilon} \) is also bounded above from 0 by assumption (2) in Theorem 3.3.1, it follows that \( e^{-\varphi_L - (1 + r) \psi} \log \left| \frac{\ell \psi}{\varepsilon} \right| \) is bounded from below by some positive constant. From the estimate provided by Theorem 3.3.1, \( w_{\varepsilon',\varepsilon,r} \) is in \( L^2_{\text{loc}}(X) \).

From the fact that
\[
x^{\varepsilon} |\log x|^s \leq \frac{s^s}{e^{\varepsilon x^s}}
\]
for all \( x \in [0, 1) \), \( \varepsilon > 0 \) and \( s \geq 0 \) (if \( 0^0 \) is treated as 1), it can be seen easily that
\[
e^{-r \psi} = \frac{e^{-r |\psi|} |\psi|^{2 - \varepsilon} \ell (|\ell \psi|^2 + 1)}{(2 - \varepsilon \ell \psi)^{2 - \varepsilon} \ell (2 \psi)^2 + (1 - \varepsilon \ell \psi)^{1 - \varepsilon}}.
\]

Together with the fact that \( \varphi_L + \psi \) being locally bounded from above, it yields \( u_{\varepsilon',\varepsilon,r} \in L^2_{\text{loc}}(X) \).

It follows from [8, Lemme 6.9] that \((u_{\varepsilon',\varepsilon,r}, w_{\varepsilon',\varepsilon,r})\) satisfies the \( \overline{\partial} \)-equation (3.3.1) on the whole of \( X \). It follows from the estimate in Theorem 3.3.1 (with \( -\varphi_L - \psi \) replaced by \( -\varphi_L - (1 + r) \psi \) that one can let \( r \to 0^+ \) and obtain weak limits \( u_{\varepsilon',\varepsilon,r} \to u_{\varepsilon',\varepsilon} \) and \( w_{\varepsilon',\varepsilon,r} \to w_{\varepsilon',\varepsilon} \) in their respective weighted \( L^2 \) spaces (after possibly passing to convergent subsequences). The estimate in Theorem 3.3.1 still holds true for the limits. Since the \( \overline{\partial} \)-Eq. (3.3.1) holds true for \((u_{\varepsilon',\varepsilon,r}, w_{\varepsilon',\varepsilon,r})\) on \( X \) in the sense of currents, it also holds true for \((u_{\varepsilon',\varepsilon}, w_{\varepsilon',\varepsilon})\) on \( X \) in the same sense. This completes the proof. \( \square \)

The theorem of holomorphic extension from the codimension-1 lc centres of \((X, S)\) is summarised in the following theorem.
Theorem 3.3.4 (Theorem 1.4.5). Assume the assumptions (1) and (2) in Theorem 3.3.1. Let \( f \) be any holomorphic section in \( H^0 \left( S, K_X \otimes L \otimes \mathcal{F}(\varphi_L + \psi) \right) \). If one has

\[
\int_S |f|^2_o d \lc_{o,\varphi_L} \psi] < \infty
\]

(which holds true when either the mlc of \((X, S)\) or the mlc of \((X, S)\) with respect to \( f \) has codimension 1, see Definitions 1.4.4 and 2.2.5), then there exists a holomorphic section \( F \in H^0 \left( X, K_X \otimes L \otimes \mathcal{F}(\varphi_L) \right) \) such that

\[
F \equiv f \mod \mathcal{F}(\varphi_L + \psi)
\]

with the estimate

\[
\int_X \frac{|F|^2 e^{-\varphi_L - \psi}}{|\psi| \left( (\log |\ell \psi|)^2 + 1 \right)} \leq \int_S |f|^2_o d \lc_{o,\varphi_L} \psi\big].
\]

Proof Given any local holomorphic liftings \( \{\tilde{f}_\gamma\}_\gamma \) of \( f \) (i.e. \( \tilde{f}_\gamma \in \mathcal{F}(\varphi_L) \)) on some open set \( V_\gamma \) in \( X \) and \( \tilde{f}_\gamma \equiv f \mod \mathcal{F}(\varphi_L + \psi) \) on \( V_\gamma \) for each \( \gamma \) and a partition of unity \( \{\chi_\gamma\}_\gamma \) subordinated to an open cover \( \{V_\gamma\}_\gamma \) of \( X \), the smooth section \( \tilde{f} := \sum_\gamma \chi_\gamma \tilde{f}_\gamma \) of the coherent sheaf \( K_X \otimes L \otimes \mathcal{F}(\varphi_L) \) satisfies the properties

\[
f \equiv \tilde{f} \mod \mathcal{O}_X^\infty \otimes \mathcal{F}(\varphi_L + \psi) \quad \text{and} \quad \overline{\partial} \tilde{f} \equiv 0 \mod \mathcal{O}_X^\infty \otimes \mathcal{F}(\varphi_L + \psi)
\]

as shown in [12, Proof of Thm. 2.8]. Notice that one has the inequality \( \log \left| \frac{\ell \psi}{e} \right| \leq \frac{\ell}{e^{\delta \psi}} \) using (3.3.2) for any \( \delta > 0 \), and the assumption (1) in Theorem 3.3.1 infers that \( \varphi_L + (1 + \delta') \psi \) is psh for all \( \delta' \in [0, \delta] \). Upper-boundedness of \( \psi \) also implies that \( \varphi_L + (1 + \delta') \psi \leq \varphi_L + \psi \). Therefore, by the strong effective openness property of multiplier ideal sheaves of psh functions (see [26, Main Thm.], also [22]), it follows that

\[
\overline{\partial} \tilde{f} \in \mathcal{O}_X^\infty \otimes \mathcal{F}(\varphi_L + (1 + \delta') \psi) \quad \text{for} \ 0 < \delta' \ll 1,
\]

which in turn implies that

\[
\left| \overline{\partial} \tilde{f} \right|_o^2 e^{-\varphi_L - \psi} \log \left| \frac{\ell \psi}{e} \right|
\]

is integrable over \( X \).

Theorem 3.3.1 and Proposition 3.3.3 can then be invoked to provide the sections \( u_{\ell, \psi} \) and \( v_{\ell, \psi} \) with the estimate as stated in the Theorem such that they satisfy the \( \overline{\partial} \)-equation (3.3.1), namely \( \overline{\partial} u_{\ell, \psi} + v_{\ell, \psi} = v_{\psi} \), on the whole of \( X \). Both \( u_{\ell, \psi} \) and \( v_{\ell, \psi} \) are smooth on \( X \) by the regularity of the \( \overline{\partial} \) operator and the smoothness of \( v_{\psi} \).

Notice that \( \frac{e^{\varphi_L - \psi}}{|\psi|^{\ell - (\log |\ell \psi|)^2 + 1}} \) is not integrable at any point of \( S \) for any \( \varepsilon > 0 \), the finiteness of the integral of \( u_{\ell, \psi} \) implies that, around every point in \( X \), there exists a local function \( g \in \mathcal{F}(\varphi_L + \psi) \) (a monomial in local coordinates under the snc
assumption on $\varphi_L$ and $\psi$) such that $|u_{e',\varepsilon}| \leq C |g|$ for some constant $C > 0$, which in turn implies that $u_{e',\varepsilon} \in C^\infty_0 \otimes \mathcal{F}(\varphi_L + \psi)$.

Recall that $|\theta_e| \leq \frac{A-B}{AB} + \varepsilon_0$ on $X$ by the choice of $\theta_e$. Setting $F_{e',\varepsilon} := \theta_e \tilde{f} - u_{e',\varepsilon}$ (which is an extension of $f$) and using the inequality

$$|F_{e',\varepsilon}|^2 \leq (1 + \alpha^{-1}) |\theta_e \tilde{f}|^2 + (1 + \alpha) |u_{e',\varepsilon}|^2$$

for any positive real number $\alpha$, one obtains the estimate

$$\int_X \frac{|F_{e',\varepsilon}|^2 e^{-\varphi_L - \psi}}{|\psi| (\log |\ell \psi|)^2 + 1} + \frac{1}{\varepsilon'} \int_X |w_{e',\varepsilon}|^2 e^{-\varphi_L - \psi} \log \frac{|\ell \psi|}{\varepsilon'} \leq \left(1 + \alpha^{-1}\right) \int_X \frac{|\theta_e \tilde{f}|^2 e^{-\varphi_L - \psi}}{|\psi| (\log |\ell \psi|)^2 + 1} + (1 + \alpha) \int_X \frac{|u_{e',\varepsilon}|^2 e^{-\varphi_L - \psi}}{|\psi|^{1-\varepsilon} (\log |\ell \psi|)^2 + 1}$$

$$+ \frac{1}{\varepsilon'} \int_X |w_{e',\varepsilon}|^2 e^{-\varphi_L - \psi} \log \frac{|\ell \psi|}{\varepsilon'} \leq \left(1 + \alpha^{-1}\right) \int_X \frac{|\theta_e \tilde{f}|^2 e^{-\varphi_L - \psi}}{|\psi| (\log |\ell \psi|)^2 + 1} + \frac{1 + \alpha}{\varepsilon'} \int_X |\theta_e \tilde{f}|^2 e^{-\varphi_L - \psi} \log \frac{|\ell \psi|}{\varepsilon'}$$

$$+ (1 + \alpha) \left(\frac{A-B}{AB} + \varepsilon_0\right) \frac{2}{1-\varepsilon} \int_X |\tilde{f}|^2 e^{-\varphi_L - \psi} \frac{|\ell \psi|}{|\psi|^{1+\varepsilon}}.$$
\( F_{\varepsilon k} \rightarrow F \) weakly in the weighted \( L^2 \) space on \( X \) as \( \varepsilon_k \rightarrow 0^+ \). Moreover, \( F_{\varepsilon k} - s_{\varepsilon k} \) is a holomorphic extension of \( f \) on \( V \) with both norms-squares

\[
\int_V \frac{|F_{\varepsilon k} - s_{\varepsilon k}|^2}{|\psi|((\log|\ell \psi|)^2 + 1)} \leq \int_V \frac{|F_{\varepsilon k} - s_{\varepsilon k}|^2 e^{-\varphi_{L-}\psi}}{|\psi|((\log|\ell \psi|)^2 + 1)}
\]

being bounded above uniformly in \( \varepsilon_k \). As \( |\psi|((\log|\ell \psi|)^2 + 1) \) belongs to \( L^1(V) \) (or \( L^1(X) \)), the Cauchy–Schwarz inequality applied to the norm-square on the left-hand-side above assures that \( F_{\varepsilon k} - s_{\varepsilon k} \) is also bounded above in \( L^1(V) \) uniformly in \( \varepsilon_k \).

Being holomorphic, Cauchy’s estimate and the above boundedness guarantee that the sequence \( \{F_{\varepsilon_k} - s_{\varepsilon_k}\}_k \) is locally bounded above in \( V \). Montel’s theorem then assures that there is a subsequence which converges locally uniformly in \( V \) to a holomorphic function \( F_V \) on \( V \). Notice that, if \( V \cap S \neq \emptyset \), then \( F_V \equiv f \) mod \( \mathcal{J}(\varphi_L + \psi) \) on \( V \), as can be seen, under the snc assumption 2.1.1, from the facts that \( F_{\varepsilon_k} - s_{\varepsilon_k} \equiv f \) mod \( \mathcal{J}(\varphi_L + \psi) \) for all \( \varepsilon_k \) and that all Taylor coefficients of \( F_{\varepsilon_k} - s_{\varepsilon_k} \) around any point have to converge to the corresponding Taylor coefficients of \( F_V \).

As a result, there is a subsequence of \( \{F_{\varepsilon_k}\}_k \) which converges pointwisely a.e. on \( V \) to the holomorphic extension \( F_V \) of \( f \). It turns out that \( F = F_V \) a.e. on \( V \). By considering all open sets \( V \) in a cover of \( X \), it follows that \( F \) is indeed a holomorphic extension of \( f \) on \( X \), after possibly altering its values on a measure 0 set.

Finally, to obtain the acclaimed estimate for \( F \), noting that \( F \) comes with the estimate

\[
\int_X \frac{|F|^2 e^{-\varphi_{L-}\psi}}{|\psi|((\log|\ell \psi|)^2 + 1)} \leq (1 + \alpha) \left(\frac{A - B}{AB} + \varepsilon_0\right)^2 \int_S |f|^2 d \omega,_{\psi,\varphi_L} \leq \delta
\]

and letting \( \alpha \rightarrow 0^+, A \rightarrow +\infty, B \rightarrow 1^+ \) and \( \omega_0 \rightarrow 0^+ \) (and choosing the limit of \( F \) suitably such that it converges locally uniformly) yield the desired result.

**Remark 3.3.5** In some applications, it is necessary to control how fast the estimate grows when the constant \( \delta \) in the normalisation of \( \psi \) shrinks. The constant \( \ell \) in the estimate is there to give a more precise control. Choose \( \ell := \delta \) and write

\[
\psi =: \psi_0 - \frac{a}{\delta},
\]

where \( a > 0 \) is a constant and \( \sup_X \psi_0 = 0 \). Then \( a \) can be chosen independent of \( \delta \) such that the assumption (2) in Theorem 3.3.1 is satisfied. Indeed, choosing \( a \) such that

\[
a > e \quad \text{and} \quad \frac{1}{a} + \frac{2}{a \log \frac{a}{e}} = 1
\]

suffices (thus \( a \approx 4.6805 \)). In this case, the estimate obtained is

\[
\int_X \frac{|F|^2 e^{-\varphi_{L-}\psi_0}}{|\delta \psi_0 - a|((\log|\delta \psi_0 - a|)^2 + 1)} \leq \frac{1}{\delta} \int_S |f|^2 d \omega,_{\psi_0,\varphi_L} \leq \delta.
\]
Note that \( \frac{e^{-\psi}}{|\delta\psi_0 - a|^2 + 1} \) is bounded below by a positive constant independent of \( \delta \) when \( \delta < a \) (which can be seen easily by applying (3.3.2) suitably).

**Remark 3.3.6** Concerning the weight in the norm of the extension \( F \), McNeal and Varolin prove in [38] some estimates with better weights. More precisely, for the case \( \psi := \psi_S = \phi_S - \varphi_S^{\text{sm}} \) (which is suitably normalised for each of the weights below), they obtain holomorphic extension with an estimate in the norm with any of the following weights:

\[
\frac{\delta' e^{-\psi_S}}{|\psi_S|} , \quad \frac{\delta' e^{-\psi_S}}{|\psi_S| \log |\psi_S|^{1+\delta'}} , \quad \frac{\delta' e^{-\psi_S}}{|\psi_S| \log |\psi_S| \cdots \log^p N - 1 |\psi_S| \cdot (\log^N |\psi_S|)^{1+\delta'}} ,
\]

where \( \delta' \in (0, 1] \) is a fixed number in each case, and \( \log^j \) denotes the composition of \( j \) copies of log functions here. It would be interesting to see if it is possible to obtain these weights in the setting of this paper.

**Remark 3.3.7** It is not clear to the authors whether Theorem 3.3.4, if allowing \( X \) to be non-compact, does include the results in [4] and [21] on the optimal constant for the estimate, although the constant in the current estimate looks “optimal”.

### 3.4 Extension theorem with a sequence of potentials

In applications it is often necessary to deal with a sequence of potentials \( \{ \varphi_L^{(k)} + m_1 \psi^{(k)} \}_{k \in \mathbb{N}} \) rather than just a single one. Following the idea of J.-P. Demailly, it is advantageous to allow the curvatures of such sequence possessing slight negativity which diminishes in the limit. It is the purpose of this section to handle such cases.

Assume that

1. there are sequences \( \{ \varphi_L^{(k)} \}_{k \in \mathbb{N}} \) and \( \{ \psi^{(k)} \}_{k \in \mathbb{N}} \) which satisfy all the assumptions in Sect. 1.3 in place of \( \varphi_L \) and \( \psi \) respectively,
2. both \( \varphi_L^{(k)} + m_1 \psi^{(k)} \) and \( \psi^{(k)} \) converge in \( L^1 \) to \( \varphi_L^{(\infty)} + m_1 \psi^{(\infty)} \) and \( \psi^{(\infty)} \) respectively, with the property that

\[
\varphi_L^{(\infty)} + m_1 \psi^{(\infty)} \lesssim_{\log} \varphi_L^{(k)} + m_1 \psi^{(k)} \quad \text{and} \quad |\psi^{(\infty)}| \lesssim_{\log} |\psi^{(k)}|
\]

for every \( k \in \mathbb{N} \) (where the constants involved in \( \lesssim_{\log} \)'s may depend on \( k \)), and
3. the multiplier ideal sheaf of \( \varphi_L^{(k)} + m_1 \psi^{(k)} \) decreases as \( k \) increases, i.e.

\[
\mathcal{J} \left( \varphi_L^{(k+1)} + m_1 \psi^{(k+1)} \right) \subset \mathcal{J} \left( \varphi_L^{(k)} + m_1 \psi^{(k)} \right) \quad \text{for all} \ k \in \mathbb{N}.
\]

In particular, all \( \varphi_L^{(k)} \)'s and \( \psi^{(k)} \)'s are assumed to have only neat analytic singularities. All families \( \{ \mathcal{J} \left( \varphi_L^{(k)} + m_\psi^{(k)} \right) \}_{m \in \mathbb{R}_{\geq 0}} \) have the same jumping numbers \( m_0 \) and \( m_1 \).
and the annihilator $\text{Ann}_X \left( \mathcal{I} \left( \varphi_L^{(k)} + m_0 \psi^{(k)} \right) \right)$ defines the same reduced subvariety $S$ for all $k$ by assumption. However, the snc Assumption 2.1.1 is not assumed unless explicitly mentioned, as there may not be simultaneous resolution for all the potentials in general.

**Theorem 3.4.1** Suppose that

1. there exists $\delta > 0$ (independent of $k$) such that, for any $k \in \mathbb{N}$,

   $$i \bar{\partial} \bar{\partial} \left( \varphi_L^{(k)} + m_1 \psi^{(k)} \right) + \beta i \bar{\partial} \psi^{(k)} \geq -\frac{1}{k} \omega \text{ on } X \text{ for all } \beta \in [0, \delta],$$

2. for any given constant $\ell > 0$ and for each $k \in \mathbb{N}$, the function $\psi^{(k)}$ is normalised (by adding a suitable constant for each $k$ without affecting convergence) such that

   $$\psi^{(k)} < -\frac{e}{\ell} \text{ and } \frac{1}{|\psi^{(k)}|} + \frac{2}{|\psi^{(k)}| \log |\ell \psi^{(k)}|} \leq \delta.$$

Let $f$ be any holomorphic section in $H^0 \left( S, K_X \otimes L \otimes \bigcap_k \mathcal{I} \left( \varphi_L^{(k)} + m_0 \psi^{(k)} \right) \right)$ such that

$$\lim_{k \to \infty} \int_S |f|^2 \omega^{\text{lcv}_{1, (m_1)} \left( \varphi^{(k)} \right)}, \psi^{(k)} \rangle < \infty.$$

Then, there exists a holomorphic section $F \in H^0 \left( X, K_X \otimes L \otimes \bigcap_k \mathcal{I} \left( \varphi_L^{(k)} + m_0 \psi^{(k)} \right) \right)$ such that

$$F \equiv f \mod \bigcap_k \mathcal{I} \left( \varphi_L^{(k)} + m_1 \psi^{(k)} \right)$$

with the estimate

$$\int_X \frac{|F|^2 e^{-\psi^{\infty} - m_1 \psi^{\infty}}}{|\psi^{\infty}| \left( (\log |\ell \psi^{\infty}|)^2 + 1 \right)} \leq \lim_{k \to \infty} \int_S |f|^2 \omega^{\text{lcv}_{1, (m_1)} \left( \varphi^{(k)} \right)}, \psi^{(k)} \rangle.$$

**Remark 3.4.2** In general, $\bigcap_k \mathcal{I} \left( \varphi_L^{(k)} + m_1 \psi^{(k)} \right) \neq \mathcal{I} \left( \varphi_L^{\infty} + m_1 \psi^{\infty} \right)$, as the example in [13, Example 1.7] shows (see also [33, Example 3.5], or Example 2.3.1).

**Proof** For simplicity, assume that $m_0 = 0$ and $m_1 = 1$ as before. The proof goes with the standard technique applied in, for example, [12] (which applies [12, Prop. 3.12] to handle the diminishing negative curvature).

For each $k \in \mathbb{N}$, applying the curvature assumption $(1_k)$ and the normalisation assumption $(2_k)$ to the curvature term (in NavyBlue) of the twisted Bochner–Kodaira
formula in Lemma 3.2.1 with \( \sigma = 1 \) yields the inequality
\[
\int_{X} |\overline{\partial} \xi |_{\omega, \varphi} \eta_{\epsilon}^{(k)} + \int_{X} |\partial \xi |_{\varphi}^{2} \left( \eta_{\epsilon}^{(k)} + \lambda_{\epsilon}^{(k)} \right) + \frac{1}{K} \int_{X} |\xi |_{\omega, \varphi} \eta_{\epsilon}^{(k)} \\
\geq \varepsilon \int_{X} \frac{1 - \varepsilon}{\varphi(k)} \left( \partial \psi(k) \right)^{\omega} |\xi |_{\varphi}^{2} \eta_{\epsilon}^{(k)}
\]
for all compactly supported \( K_{X} \otimes L \)-valued \((0, 1)\)-forms \( \xi \) (in which \( \varphi = \varphi_{L}^{(k)} + \psi^{(k)} + \nu^{(k)} \) and the formal adjoint \( \partial \) both depend on \( k \)). Using the notation in the proof of Theorem 3.3.1 and with the same argument there, one obtains
\[
\langle \xi, v_{\epsilon} \rangle \leq \left| \left( (\xi)_{\ker \overline{\partial}}, v_{\epsilon}^{(1)} \right) \right| + \left| \left( (\xi)_{\ker \overline{\partial}}, v_{\epsilon}^{(2)} \right) \right| \\
\leq \left( \left( \mathcal{N}_{1} (\partial \xi) \right)^{2} + \left( \frac{1}{k} + \varepsilon' \right) \| \xi \|^{2} \right)^{\frac{1}{2}} \left( \left( \mathcal{N}_{2} (\tilde{f}) \right)^{2} + \frac{1}{\varepsilon'} \| \theta_{\epsilon} \tilde{\theta} \tilde{f} \|^{2} \right)^{\frac{1}{2}}
\]
for any \( \varepsilon' > 0 \) and \( k \in \mathbb{N} \) (here, \( v_{\epsilon} \) also depends on \( k \), as \( \theta_{\epsilon} = \theta \circ |\psi(k)|_{\epsilon} \) does). The Riesz representation theorem, together with the argument in Proposition 3.3.3, provides a solution \( \left( u_{e', \epsilon}^{(k)}, w_{e', \epsilon}^{(k)} \right) \) to the \( \overline{\partial} \)-equation \( \overline{\partial} u_{e', \epsilon}^{(k)} + w_{e', \epsilon}^{(k)} = v_{\epsilon} \) on the whole of \( X \). Setting \( F_{e', \epsilon}^{(k)} := \theta_{\epsilon} \tilde{f} - u_{e', \epsilon}^{(k)} \), the argument in the proof of Theorem 3.3.4 then yields
\[
\int_{X} \frac{|F_{e', \epsilon}^{(k)}|^{2} e^{-\varphi_{L}^{(k)} - \psi^{(k)}}}{|\psi(k)| \left( (\log |\ell \psi(k)|)^{2} + 1 \right)} + \frac{1}{k + \varepsilon'} \int_{X} |u_{e', \epsilon}^{(k)}|^{2} e^{-\varphi_{L}^{(k)} - \psi^{(k)}} \log |\ell \psi(k)| \frac{1}{\varepsilon} \\
\leq \left( 1 + \alpha^{-1} \right) \int_{X} \frac{|\theta_{\epsilon} \tilde{f} |^{2} e^{-\varphi_{L}^{(k)} - \psi^{(k)}}}{|\psi(k)| \left( (\log |\ell \psi(k)|)^{2} + 1 \right)} \\
+ \frac{1 + \alpha}{\varepsilon'} \int_{X} |\theta_{\epsilon} \tilde{f} |^{2} e^{-\varphi_{L}^{(k)} - \psi^{(k)}} \log |\ell \psi(k)| \frac{1}{\varepsilon} \\
+ \left( 1 + \alpha \right) \left( \frac{A - B}{AB} + \varepsilon_{0} \right)^{2} \frac{\varepsilon}{1 - \varepsilon} \int_{X} \frac{|\tilde{f} |^{2} e^{-\varphi_{L}^{(k)} - \psi^{(k)}}}{|\psi(k)|^{1+\varepsilon}}
\]
for some \( \alpha > 0 \), where
\[
F_{e', \epsilon}^{(k)} \equiv f \mod \mathcal{C}X_{\infty} \otimes \mathcal{J} \left( \varphi_{L}^{(k)} + \psi^{(k)} \right).
\]
7 Here \( f \) is abused to mean its image under the map
\[
\cap_{k'} \mathcal{J} \left( \varphi_{L}^{(k')} + m_{0} \psi^{(k')} \right) \rightarrow \mathcal{J} \left( \varphi_{L}^{(k)} + m_{0} \psi^{(k)} \right)
\]
Notice that the assumption of \( f \) being \( L^2 \) with respect to the limit of lc-measures implies that \( f \) is \( L^2 \) with respect to \( d \text{lcv}^1_{\omega,\psi_L^{(k)}}[\psi^{(k)}] \) for every \( k \gg 0 \), which in turn implies that \( \int_X \frac{|\tilde{f}|^2 e^{-\psi_L^{(k)}}}{|\psi^{(k)}|(\log|\ell_{\psi^{(k)}}|)^2 + 1} \) is finite for each \( k \gg 0 \) (see the proof of Theorem 3.3.4).

Choose \( \epsilon' := \left( \int_X \left| \partial_\omega \tilde{\partial} \| \frac{2}{\epsilon} e^{-\psi_L^{(k)}} \log \left| \frac{\ell_{\psi^{(k)}}}{\epsilon} \right| \right|^2 \right)^{\frac{1}{2}} \) and omit all subscripts “\( \epsilon' \)” as before. Notice that the right-hand-side of the above estimate is bounded above uniformly, thanks to the assumption that \( f \) being \( L^2 \) with respect to the lc-measure, when the limits are taken in the order \( \epsilon \to 0^+ \) followed by \( k \to \infty \). The required section \( F \) is then obtained after first taking the limit \( \epsilon \to 0^+ \) (obtaining the weak limits \( F^{(k)} \) of \( F^{(k)}_{\epsilon} \) and \( w^{(k)}_{\epsilon} \) of \( w^{(k)}_{\epsilon} \) in their respective \( L^2 \) spaces), then \( k \to \infty \) (obtaining the weak limit \( F \) of \( F^{(k)} \) while \( w^{(k)} \to 0 \) strongly), and followed by \( \alpha \to 0^+, A \to +\infty, B \to 1^+ \) and \( \epsilon_0 \to 0^+ \). The acclaimed estimate also follows.

To justify that \( F \) is the required holomorphic extension of \( f \), consider any polydisc \( V \) in the given open cover of \( X \) and solve \( \partial s^{(k)}_\epsilon = w^{(k)}_\epsilon \) for \( s^{(k)}_\epsilon \) on \( V \) with the \( L^2 \) Hörmander estimate

\[
\| s^{(k)}_\epsilon \|_{\psi_L^{(k)} + \psi^{(k)}}^2 \leq C \left\| w^{(k)}_\epsilon \right\|_{\psi_L^{(k)} + \psi^{(k)}}^2 ,
\]

where the constant \( C \) is independent of \( k \) and \( \epsilon \). This assures that one can extract weak limit \( s^{(k)}_\epsilon \) of \( s^{(k)}_\epsilon \) as \( \epsilon \to 0^+ \). As \( F^{(k)}_\epsilon - s^{(k)}_\epsilon \) is holomorphic on \( V \) with the unweighted \( L^1 \) norm bounded from above uniformly in \( \epsilon \) (followed from the same argument as in Theorem 3.3.4), it converges locally uniformly on \( V \) to the holomorphic section \( F^{(k)} - s^{(k)} \) after passing to a subsequence. This also implies that

\[
F^{(k)} - s^{(k)} \equiv f \mod \mathcal{I}(\psi_L^{(k)} + \psi^{(k)}) \quad \text{on} \ V
\]

(which can be seen by temporary taking a log-resolution of \( (X, \psi_L^{(k)} + \psi^{(k)}) \) and arguing as in Theorem 3.3.4).

Notice that the unweighted \( L^1 \) norm of \( F^{(k)} - s^{(k)} \) may not be bounded above uniformly in \( k \) since \( \psi^{(k)} \) depends on \( k \). To get around that, notice that

\[
\frac{1}{|\psi^{(k)}|(\log(\ell^{(k)})^2 + 1)}
\]

via inequality (3.3.2) with the constant in \( \lesssim \) independent of \( k \). Hölder’s inequality infers that

\[
\int_V \left| F^{(k)} - s^{(k)} \right|^3 \leq \left( \int_V \left| F^{(k)} - s^{(k)} \right|^2 \right)^{\frac{3}{2}} \left( \int_V |\psi^{(k)}| \right)^{\frac{3}{2}} .
\]

As \( \psi^{(k)} \to \psi^{(\infty)} \) in the \( L^1 \) norm, this assures that \( F^{(k)} - s^{(k)} \) is bounded above uniformly in \( k \) in the \( L^{\frac{2}{3}} \) norm. This retains the local uniform boundedness of \( F^{(k)} - s^{(k)} \) in the sup-norm via the use of the Harnack inequality for psh functions (see, for example,
Since (**) also implies that
\[ F^{(k)} - s^{(k)} \equiv f \mod \mathcal{S}\left(\varphi^{(k)}_L + \psi^{(k)}\right) \text{ on } V \]
for all \( k' \leq k \), as followed from the assumption (2) stated at the beginning of Sect. 3.4. One then sees that
\[ F \equiv f \mod \bigcap_k \mathcal{S}\left(\varphi^{(k)}_L + \psi^{(k)}\right). \]

To see that \( F \) is in \( \bigcap_k \mathcal{S}\left(\varphi^{(k)}_L\right) \), notice that
\[ \left| \psi^{(k)} \right| + C \leq \left| \psi^{(\infty)} \right| + C \]
and therefore
\[ e^{-\psi_L^{(k)} - \psi^{(k)}} \leq e^{-\psi^{(\infty)} - \psi^{(k)}} \]
by the assumption (1) stated at the beginning of Sect. 3.4. Then \( F \) belonging to \( \bigcap_k \mathcal{S}\left(\varphi^{(k)}_L\right) \) can be seen from the estimate. This completes the proof.

3.5 Illustration

The following example illustrates how Theorem 3.4.1 can be applied to obtain the classical result on prescribing value at a point to a holomorphic section with estimate.

**Example 3.5.1** (Extension from a point). Let \( X \) be a projective \( n \)-fold and \( A \) an ample line bundle on \( X \) endowed with a potential \( \varphi_A^{sm} \). Set \( \omega := i \partial \overline{\partial} \varphi_A^{sm} \). Suppose that \( L \) is a pseudo-effective line bundle equipped with a psh potential \( \varphi_L \) (with arbitrary singularities). The goal is to obtain a global section \( F \) of the line bundle \( K_X \otimes L^\otimes \mu \otimes A^\otimes \mu \) for some sufficiently large \( \mu \in \mathbb{N} \) with the prescribed value \( a \) at a point \( p \in X \setminus (\varphi_L)^{-1}(-\infty) \) with estimate.

Let \( \left\{ \varphi^{(k)}_L \right\}_{k \in \mathbb{N}} \) be a sequence of quasi-psh potentials with neat analytic singularities which approximates \( \varphi_L \) and satisfies the properties
\[ \varphi_L \leq \varphi^{(k+1)}_L \leq \log \varphi^{(k)}_L \quad \text{and} \quad i \partial \overline{\partial} \varphi^{(k)}_L \geq -\frac{1}{k} i \partial \overline{\partial} \varphi_A^{sm} = -\frac{1}{k} \omega \text{ on } X \]
for all \( k \in \mathbb{N} \) (for example, \( \left\{ \varphi^{(k)}_L \right\}_{k \in \mathbb{N}} \) can be the approximation of \( \varphi_L \) constructed in [9]).

Let \( \theta : [0, 1] \to [0, 1] \) be a smooth cut-off function such that \( \theta \equiv 1 \) on \([0, \frac{1}{2}]\) and is compactly supported on \([0, 1]\). For any \( p \in X \setminus (\varphi_L)^{-1}(-\infty) \) which lies in a coordinate chart \((V, \mathbf{z})\) with coordinates \( \mathbf{z} = (z_1, \ldots, z_n) \) such that \( z(p) = 0 \) and
\[ |z|^2 = \sum_{j=1}^n |z_j|^2 < 1 \text{ on } V \text{ and such that both } L \text{ and } A \text{ are trivialised, define} \]
\[
\psi := \theta \left( |z|^2 \right) \log |z|^{2n} - e ,
\]

which is a global function on \( X \). It can be seen that \( \psi \leq -e \) on \( X \), and \( m = 1 \) is the first jumping number of the family \( \{ \mathcal{I} \left( \mu \varphi_{L+A}^{(k)} + m \psi \right) \}_{m \in \mathbb{R}_{\geq 0}} \) for any \( k \in \mathbb{N} \) and \( \mu \in \mathbb{N} \), where \( \varphi_{L+A}^{(k)} := \varphi_L^{(k)} + \varphi_A^{sm} \) is set for convenience. Indeed, the annihilator
\[
\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I} \left( \mu \varphi_{L+A}^{(k)} \right)}{\mathcal{I} \left( \mu \varphi_{L+A}^{(k)} + \psi \right)} \right)
\]
defines the set \( \{ p \} \) with reduced structure.

It can be seen that, for any \( k \geq 2 \) and a sufficiently large integer \( \mu \in \mathbb{N} \), one has
\[
i \partial \bar{\partial} \left( \mu \varphi_{L+A}^{(k)} + \psi \right) + \beta i \partial \bar{\partial} \psi \geq \mu \left( 1 - \frac{1}{k} \right) \omega + (1 + \beta) i \partial \bar{\partial} \left( \theta \left( |z|^2 \right) \log |z|^{2n} \right) \geq 0
\]
for \( \beta \in [0, 1] \). The curvature assumption \((1_k)\) and the normalisation assumption \((2_k)\) of Theorem 3.4.1 are satisfied.

Set \( \tilde{f} := \theta \left( |z|^2 \right) dz_1 \wedge \cdots \wedge dz_n \) and \( f := \tilde{f}(p) \). Given any constant \( a \), in order to obtain a holomorphic extension of \( af \) with estimate, it remains to check that the limit of 1-lc-measures \( \lim_{k \to \infty} |f|_{\omega}^2 d \ell \psi_{L+A}^{(k)} \left[ \psi \right] \) is finite at \( p \).

Let \( \pi : \tilde{X} \to X \) be the blow-up of \( X \) at \( p \) with exceptional divisor \( E \). Then, \( \pi^* \psi = n \phi_E - \psi_{nE}^{sm} \) for some smooth potential \( \psi_{nE}^{sm} \) (on \( E^{\otimes n} \) and \( K_{\tilde{X}/X} = E^{\otimes (n-1)} \)). Let \( \tilde{U} \) be a neighbourhood in \( \tilde{X} \) covering a dense subset of \( E \) with coordinates \((w, s_E) = (w_1, w_2, \ldots, w_{n-1}, s_E) \) given by \( \pi^* z_j = s_E w_j \) for \( j = 1, \ldots, n-1 \) and \( \pi^* z_n = s_E \) such that \( E \cap \tilde{U} = \{ s_E = 0 \} \). It follows from Proposition 2.2.1 that
\[
\int_{\{ p \}} |af|^2_{\omega} d \ell \psi_{L+A}^{(k)} \left[ \psi \right] = \lim_{\varepsilon \to 0^+} \varepsilon \int_{\tilde{X}} \frac{|a \pi^* \theta|^2}{\mu \pi^* \psi} e^{-\mu \pi^* \varphi_{L+A}^{(k)} - \varphi_{nE}^{sm}} d \text{vol}_E \wedge \pi i d s_E \wedge d \overline{s_E}
\]
\[
= \frac{\pi}{n} \int_{\tilde{E}} |a|^2 e^{-\mu \pi^* \varphi_{L+A}^{(k)} + \varphi_{nE}^{sm}} d \text{vol}_E
\]
\[
= |a|^2 e^{-\mu \varphi_{L+A}^{(k)}(p)} \frac{\pi}{n} \int_{\tilde{U} \cap E} \frac{e^e}{\left( 1 + |w|^2 \right)^n} \bigwedge_{j=1}^{n-1} (\pi i dw_j \wedge d \overline{w_j})
\]
\[
= |a|^2 e^{-\mu \varphi_{L+A}^{(k)}(p)} \frac{\pi^{n-1}}{n} \frac{e^e}{n!} \to |a|^2 e^{-\mu \varphi_L - \mu \varphi_A^{sm} (p)} \frac{\pi^n}{n!} e^e
\]
as \( k \to \infty \), which is definitely finite. Theorem 3.4.1 can now be invoked to obtain the required \( F \) with estimate.
4 Improvement to the result of Demailly–Hacon–Păun on plt extension

Divisors are treated as line bundles without further mention in this section.

4.1 Setup for the plt extension

Let \( X \) be projective. Consider a pair \( (X, S + B) \) which is plt and log-smooth with \( B \) being a \( \mathbb{Q} \)-divisor and \( S = \lfloor S + B \rfloor = \sum_{j \in I_S} S_j \) where each \( S_j \) is reduced and irreducible (thus \( S \) and \( B \) have no common irreducible component and have only snc, and the irreducible components \( S_j \) of \( S \) are mutually disjoint). Let \( \mu \in \mathbb{N} \) be such that \( \mu (K_X + S + B) \) is a \( \mathbb{Z} \)-divisor and write

\[
K_X + L := K_X + S + F := \mu (K_X + S + B) \quad \text{(i.e. } F := (\mu - 1) (K_X + S + B) + B \text{)}.
\]

\( (F \) is defined for the convenience of readers when referred to \([15]\).) Assume that \( \mu \geq 2 \) and

- \( K_X + S + B \) is pseudo-effective (pseff);
- \( K_X + S + B \sim_\mathbb{Q} D \), where

\[
D = \sum_{j \in I_S} v_j S_j + D_2 := v_S \cdot S + D_2
\]

is an effective \( \mathbb{Q} \)-divisor with snc support, and \( S \) and \( D_2 \) have no common components;
- \( \text{supp } S \subset \text{supp } D \) (i.e. \( v_j \neq 0 \) for all \( j \in I_S \)).
- no irreducible components of \( S \) lies in the diminished stable base locus \( B_- (K_X + S + B) \) (see, for example, \([15, \S 2.1]\) for the definition).

Let \( \rho := K_X + S + B - D \) be the \( \mathbb{Q} \)-line bundle in \( \text{Pic}^0(X) \otimes \mathbb{Q} \) which admits a smooth pluriharmonic potential \( \varphi^\rho_{\text{sm}} \), i.e. its curvature form is \( i \partial \bar{\partial} \varphi^\rho_{\text{sm}} = 0 \). The potentials \( \phi_S, \phi_{v_S}, \phi_B \) and \( \phi_{D_2} \), which are defined from canonical sections of their respective \( \mathbb{Q} \)-line bundles as shown in their subscripts, are fixed such that they are negative under the given choice of trivialisations.

Moreover, choose a sufficiently ample divisor \( A \) on \( X \) such that it is globally generated. Let \( \{s_{A,i}\}_{i \in I_A} \) be a basis of \( H^0(X, A) \) and endowed \( A \) with a smooth psh potential \( \varphi^A_{\text{sm}} = \log \left( \sum_{i \in I_A} |s_{A,i}|^2 \right) \), which in turn provides a Kähler form \( \omega := i \partial \bar{\partial} \varphi^A_{\text{sm}} \) on \( X \), and induces a smooth potential \( \varphi^K_{\text{sm}} \) on \( K_X \). Fix also smooth potentials \( \varphi^\text{sm} := \varphi^K_{\text{sm}} + S + B \) on \( K_X + S + B \) and \( \varphi^B_{\text{sm}} \) on \( B \). All the smooth potentials are chosen such that they are negative under the given choice of trivialisations for convenience.
4.2 Bergman kernel potentials

Let \( \varphi_{\text{min}} \leq \varphi_{KX+S+B}^{\text{sm}} \) be a psh potential with minimal singularities on the pseff \( \mathbb{Q} \)-line bundle \( K_X + S + B \) (ref. [14, Thm. 1.5]). Since \( \varphi_{vS} + \varphi_{D_2} + \varphi_{\rho}^{\text{sm}} \) is also a psh potential of \( K_X + S + B \), after adding suitable constants to the potentials, one can assume that

\[
\varphi_{vS} + \varphi_{D_2} + \varphi_{\rho}^{\text{sm}} \leq \varphi_{\text{min}} \leq \varphi_{KX+S+B}^{\text{sm}} \leq 0 .
\] (4.2.1)

The following construction of an approximation of \( \varphi_{\text{min}} \) is almost a paraphrase of the discussion on the “algebraic version of the super-canonical metric” in [10, §20.6] with the generalisation in [10, §20.13] taken into account.

Let \( B_{\ell,k} := B_{\ell,k,A} \) be the Bergman kernel of \( H^0(X, \ell k \mu (K_X + S + B) + \ell A) \) with respect to the potential \( \ell k \mu \varphi_{KX}^{\text{sm}} + S + B + \ell \varphi_A^{\text{sm}} \). Note that, for all \( j \in I_S \), \( S_j \not\subset B_- (K_X + S + B) \) implies that

\[
S_j \not\subset (B_{\ell,k})^{-1}(0) \quad \text{when } k \in \mathbb{N} \text{ and } \ell \gg 0 .
\] (4.2.2)

Now, for every \( k \in \mathbb{N} \), fix an \( \ell := \ell(k) \gg 0 \) and define the Bergman kernel potentials \( \varphi^{(k)} \) by

\[
\varphi^{(k)} := \varphi_B^{(k)} := \varphi_B^{(\ell,k)} := \varphi_B^{(\ell,k,A)} := \frac{1}{\ell k \mu} \log B_{\ell,k} - \frac{1}{k \mu} \varphi_A^{\text{sm}} .
\] (4.2.3)

The integer \( \ell \) is chosen such that the polar set of \( \varphi_{B_{\ell,k,A}}^{(\ell,k)} \) is precisely the stable base locus of the linear system of \( k \mu (K_X + S + B) + A \) (see Lemma 4.2.1 for the existence of such \( \ell \)). These \( \varphi^{(k)} \)’s have only neat analytic singularities.

Choose \( \ell \gg 0 \) such that the Ohsawa–Takegoshi extension theorem with respect to the potential \( \ell k \mu \varphi_{\text{min}} + \ell \varphi_A^{\text{sm}} \) can be applied to obtain global sections of \( \ell k \mu (K_X + S + B) + \ell A \) with prescribed value at any point on the projective manifold \( X \) outside of \( (\varphi_{\text{min}})^{-1}(-\infty) \) for every \( k \geq 1 \) (for example, one can use the version of the extension theorem in [10, Thm. 13.6], or Example 3.5.1, together with the analytically singular approximation of psh functions in [9, Prop. 3.7]). Following the arguments in [9, Prop. 3.1], since \( \varphi_{\text{min}} \leq \varphi_{KX+S+B}^{\text{sm}} =: \varphi^{\text{sm}} \), after adding a suitable constant to \( \varphi_{\text{min}} \) if necessary (where the constant is independent of \( k \)), one obtains

\[
\varphi_{\text{min}}(z) \leq \varphi_B^{(\ell,k)}(z) \leq \sup_{\zeta \in \Delta(z;r)} \left( \varphi_A^{\text{sm}}(\zeta) + \frac{1}{k \mu} \varphi_A^{\text{sm}}(\zeta) \right) - \frac{1}{k \mu} \varphi_A^{\text{sm}}(z) + \frac{C - 2 n \log r}{\ell k \mu}
\] (4.2.4)

for all \( z \in X \), where \( \Delta(z;r) \) is the polydisc of radius \( r \) centred at \( z \) in some coordinate chart, and \( C \) is some constant independent of \( k \) and \( r \). It is emphasised here that the inequalities are valid under the fixed trivialisations of \( K_X + S + B \) and \( A \) on each open subset in a fixed cover of \( X \).
The properties of $\varphi^{(k)} = \varphi^{(\ell,k)}_B$ necessary for the present purpose are collected as follows:

$$\varphi^{(k)} \geq \phi_{V,S} + \phi_{D_2} + \varphi_{p}^{\text{sm}} \text{ (from (4.2.1) and (4.2.4)),} \quad (4.2.5a)$$

$\varphi^{(k)}$ is locally bounded above uniformly in $k$ (from (4.2.4)),

$$\varphi^{(k)} \geq -\frac{1}{k\mu} \varphi_{A} \text{ (from (4.2.3)),} \quad (4.2.5b)$$

$\varphi^{(k)}$ has only neat analytic singularities (from (4.2.3)), and

$$\varphi^{(k)} \not\subset \left(\varphi^{(k)}\right)^{-1} (-\infty) \quad \forall \, j \in I_S \text{ and } \forall \, k \in \mathbb{N} \text{ (from (4.2.2)).} \quad (4.2.5c)$$

The following lemma justifies the definition, in particular, the existence of a suitable $\ell$.

**Lemma 4.2.1** There exists an $\ell_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and for all $\ell' \geq \ell \geq \ell_0$, $\varphi^{(\ell,k)}_B \sim \log \varphi^{(\ell',k)}_B$, and the constants involved in $\sim \log$ are independent of $k$, $\ell$ and $\ell'$.\end{quote}

**Proof** The integer $\ell_0 > 0$ is chosen sufficiently large such that, for any integers $\ell' > \ell \geq \ell_0$ and for any $p \in X \setminus \left(\varphi^{(\ell',k)}_B\right)^{-1} (-\infty)$ being the centre of a polydisc $(\Delta, z)$ with coordinates $z$ in some coordinate chart, one has

$$i\partial i\overline{\partial} \varphi^{(k)} \geq \left(\ell - \frac{\ell k \mu}{\ell' k \mu}\right) i\partial i\overline{\partial} \varphi^{(k)} + i\partial i\overline{\partial} \left(-\varphi_{KX}^{\text{sm}} + (1 + \beta) \theta \left(|z|^2 \log |z|^{2n}\right)^{\frac{1}{2}}\right)$$

$$\geq 0$$

for every $\beta \in [0, \delta]$ for some constant $\delta > 0$. Here $\theta : [0, 1] \to [0, 1]$ is a smooth cut-off function which is identically equal to 1 on a neighbourhood of 0 and vanishes outside of a larger neighbourhood. It can be seen that $\ell_0$ can be chosen independent of $k$, $\ell$ and $\ell'$, even the point $p$ (as $X$ is compact).

With almost the same proof as in the proof of the first inequality in (4.2.4), namely, applying the Ohsawa–Takegoshi extension theorem (or Example 3.5.1) with respect to...
the potential $\ell k \mu \varphi_B^{(\ell', k)} + \ell \varphi_A^{sm}$ to obtain a global section $f$ of $\ell k \mu (K_X + S + B) + \ell A$ with prescribed value at any point $p \in X \setminus \{(\varphi_B^{(\ell', k)})^{-1}(\infty)\}$ such that

$$\|f\|_{\ell, k}^2 := \int_X |f|^2 e^{-\ell k \mu \varphi^{sm} - \ell \varphi_A^{sm}}$$

$$\lesssim \int_X |f|^2 \left( (\log |\delta \psi|)^2 + 1 \right) e^{-\ell k \mu \varphi^{(\ell', k)} + \ell \varphi_A^{sm} - \psi}$$

$$\lesssim \left( |f|^2 e^{-\ell k \mu \varphi_B^{(\ell', k)} - \ell \varphi_A^{sm}} \right) (p)$$

$$\Rightarrow e^{\ell k \mu \varphi_B^{(\ell', k)}} (p) \lesssim \left( \frac{|f|^2}{\|f\|_{\ell, k}^2} e^{-\ell \varphi_A^{sm}} \right) (p) \leq \left( B_{\ell, k} e^{-\ell \varphi_A^{sm}} \right) (p),$$

where constant in the first $\lesssim$ is independent of $k$, $\ell$ and $\ell'$ thanks to the fact that $\varphi_B^{(\ell', k)}$ is bounded above uniformly in $k$ and $\ell'$ (see (4.2.4)) and the use of inequality (3.3.2) to estimate the terms with $\psi$, while the constant in the second $\lesssim$ is independent of $k$, $\ell$ and $\ell'$ thanks to the universality of the constant in the Ohsawa–Takegoshi extension. As a result, one sees that

\[(*) \quad \varphi_B^{(\ell', k)} \lesssim_{\log} \varphi_B^{(\ell, k)}, \]

where the constant involved in $\lesssim_{\log}$ is independent of $k$, $\ell$ and $\ell'$.

For the reverse inequality, it follows easily by means of mean-value-inequality that, for any $m \in \mathbb{N}$,

\[(**) \quad \varphi_B^{(\ell, k)} \lesssim_{\log} \varphi_B^{(m\ell, k)} \]

with the constant in $\lesssim_{\log}$ being independent of $k$, $\ell$ and $m$. Indeed, for any fixed $x \in X$, take $h \in H^0 (X, \ell k \mu (K_X + S + B) + \ell A)$ with $\|h\|_{\ell, k}^2 := \|h\|_{\ell k \mu \varphi^{sm} + \ell \varphi_A^{sm}} = 1$ and $|h(x)|^2 = B_{\ell, k} (x)$. Then, one has

$$|h^m (x)|^2 \leq B_{m\ell, k} (x) \|h^m\|_{m\ell, k} \leq B_{m\ell, k} (x) \|h\|_{\ell, k}^2 \left( \sup_X |h|^2 e^{-\ell k \mu \varphi^{sm} - \ell \varphi_A^{sm}} \right)^{m-1}$$

\[\text{mean-value-ineq} \leq B_{m\ell, k} (x) \left( \frac{C}{\pi r^2} \right)^n e^{\sup_X (\ell k \mu \varphi^{sm} + \ell \varphi_A^{sm}) - \inf_X (\ell k \mu \varphi^{sm} + \ell \varphi_A^{sm})} \left( \frac{1}{m k \mu} \right)^{m-1} \]

for some constant $C > 0$ and small $r > 0$ which are independent of $k$, $\ell$ and $m$. Note that $\sup_X (\ell k \mu \varphi^{sm} + \ell \varphi_A^{sm})$ means the maximum of $\sup_{V_\gamma} (\ell k \mu \varphi^{sm} + \ell \varphi_A^{sm})$ among all $V_\gamma$ in a finite cover $\{V_\gamma\}_\gamma$ (and the same interpretation applies to the term with $\inf_X$). The claim for $\lesssim_{\log}$ follows after applying $\frac{1}{m k \mu} \log$ on and subtracting $\frac{1}{k \mu} \varphi_A^{sm}$ from both sides.

When $m\ell > \ell'$, one can apply $(*)$ to obtain $\varphi_B^{(\ell, k)} \lesssim_{\log} \varphi_B^{(m\ell, k)} \lesssim_{\log} \varphi_B^{(\ell', k)}$. This completes the proof.
The choice of the ample divisor $A$ does not play a role as soon as only the asymptotic behaviour is concerned.

**Lemma 4.2.2** Let $A$ and $A'$ be two arbitrary ample divisors. It follows that

$$\varphi_{B,A}^{(k')} \lesssim \log \varphi_{B,A}^{(k)} \lesssim \log \varphi_{B,A'}^{(k)}$$

for any $k, k', k'' \in \mathbb{N}$ whenever $k'A' - kA$ and $k''A - k'A'$ are ample, where the constants involved in $\lesssim \log$ are independent of $k, k'$ and $k''$.

In particular, when $A = A'$, it follows that $\varphi^{(k)} = \varphi_{B,A}^{(k)}$ is getting more singular as $k$ increases, i.e.

$$\varphi_{B,A}^{(k'')} \lesssim \log \varphi_{B,A}^{(k')} \lesssim \log \varphi_{B,A}^{(k)}$$

for any $k \leq k' \leq k''$.

**Proof** Take $\ell \gg 0$ such that $\ell k'A' - \ell kA$ is globally generated, and choose $\varphi_{A}^{\mathrm{sm}}$ and $\varphi_{A'}^{\mathrm{sm}}$ such that $k' \varphi_{A}^{\mathrm{sm}} = k \varphi_{A}^{\mathrm{sm}} + \varphi_{k'A'-kA}^{\mathrm{sm}}$, where $\varphi_{k'A'-kA}^{\mathrm{sm}}$ is constructed from a basis of $H^0(X, \ell k'A' - \ell kA)$. It follows that $B_{\ell,k',k,kA} \geq B_{\ell,k,k',kA'}$ for all $s \in \ell k'A' - \ell kA$ in the basis of $H^0(X, \ell k'A' - \ell kA)$. One gets, after summing up the inequalities for the whole basis of $H^0(X, \ell k'A' - \ell kA)$ (with dimension being bounded above by $O(\ell^n (k + k')^n)$),

$$\varphi_{B,A}^{(\ell,k,k')} = \varphi_{B,k,kA}^{(\ell,k,k')} \lesssim \log \varphi_{B,k',kA'}^{(\ell,k,k)} = \varphi_{B,A'}^{(\ell,k,k)} ,$$

hence the inequality on the right-hand-side in the claim after taking Lemma 4.2.1 into account, with the constant involved in $\lesssim \log$ being independent of $\ell, k$ and $k'$. The other inequality follows by interchanging the role of $A$ and $A'$. $\square$

By passing to a subsequence when necessary, one can assume that $\{\varphi^{(k)}\}_{k \in \mathbb{N}}$ converges in $L^1_{\mathrm{loc}}$ (thanks to (4.2.5b) and the fact that $\varphi^{(k)} \geq \varphi_{\min} \neq -\infty$) to a psh (thanks to (4.2.5c)) potential $\varphi^{(\infty)}$, which is given pointwisely by the upper regularised limit

$$\varphi^{(\infty)}(z) := \lim_{k \to +\infty} \varphi^{(k)}(z) := \lim_{\xi \to z} \lim_{k \to +\infty} \varphi^{(k)}(\xi) \quad \text{for all } z \in X .$$

By the minimality of $\varphi_{\min}$, it follows that $\varphi^{(\infty)} \sim_{\log} \varphi_{\min}$. Indeed, it follows from (4.2.4) that, by letting $r \to 0^+$ (after $k \to \infty$), one has $\varphi^{(\infty)} = \varphi_{\min}$.
4.3 The choice of $\varphi_L^{(k)}$ and $\psi^{(k)}$

Set $\nu_{\text{max}} := \max_{j \in I_S} \nu_j$. For each $k \in \mathbb{N}$, define the global function $\psi^{(k)}$ on $X$ and the potential $\varphi_L^{(k)}$ on $L$ such that

$$\psi^{(k)} := \frac{1}{\nu_{\text{max}}} (\varphi_{v_S} + \phi_D + \varphi_{\rho}^{\text{sm}} - \varphi^{(k)}) \quad \text{and} \quad (4.3.1)$$

$$\varphi_L^{(k)} + \psi^{(k)} := (\mu - 1) \varphi^{(k)} + \phi_S + \phi_B . \quad (4.3.2)$$

For the convenience of readers who would like to compare the current choices with those in [15], define also $\varphi_{\tau_k}, \varphi_{F,k}$ and $\psi_{v_S}^{\tau_k}, k$ (which may not follow the convention in Notation 1.2.3) by

$$\varphi_{\tau_k} := \varphi^{(k)} , \quad \varphi_{F,k} := (\mu - 1) \varphi^{(k)} + \phi_B \quad \text{and} \quad \psi_{v_S}^{\tau_k}, k := \varphi^{(k)} - \phi_D - \varphi_{\rho}^{\text{sm}} .$$

It follows from (4.2.5a) that $\psi^{(k)} \leq 0$ on $X$, so, for any $m \leq m'$, it follows that

$$\mathcal{I}\left(\varphi_L^{(k)} + m' \psi^{(k)}\right) \subset \mathcal{I}\left(\varphi_L^{(k)} + m \psi^{(k)}\right) ,$$

i.e. the family $\left\{ \mathcal{I}\left(\varphi_L^{(k)} + m \psi^{(k)}\right) \right\}_{m \in \mathbb{R}}$ is decreasing. As

$$\varphi_L^{(k)} + m \psi^{(k)} = \left(\mu - 1 + \frac{1 - m}{\nu_{\text{max}}} \right) \varphi^{(k)} - \frac{1 - m}{\nu_{\text{max}}} \left(\phi_D + \varphi_{\rho}^{\text{sm}}\right) + \phi_S - \frac{1 - m}{\nu_{\text{max}}} \varphi_{v_S} + \phi_B . \quad (4.3.3)$$

it can be seen that $m = 1$ is a jumping number of the family by considering the coefficients of $\phi_S$'s, after taking (4.2.5e) into account. Since the coefficient of $\varphi^{(k)}$ is decreasing as $m$ grows and that of $\phi_D$ is negative as $m$ varies within $[0, 1)$, the decreasing family has to remain unchanged for $m \in [0, 1)$. In the context of Theorem 1.4.5, one has $m_0 = 0$ and $m_1 = 1$ for all $k \in \mathbb{N}$. The ideal $\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I}(\varphi_L^{(k)})}{\mathcal{I}(\varphi_L^{(k)} + \psi^{(k)})} \right)$ obviously defines the reduced subvariety $S$, which is already an snc divisor.

Each $\varphi_L^{(k)} + \psi^{(k)}$ is locally bounded from above uniformly in $k$ since so is $\varphi^{(k)}$ (see (4.3.2)).

Applying $i \partial \overline{\partial}$ to (4.3.3) and putting $m = 1 + \beta := 1 + \nu_{\text{max}} \lambda$, where $\lambda \in [0, \mu - 1]$, it follows from the Poincaré–Lelong formula $[E] = i \partial \overline{\partial} \phi_E$ that

$$i \partial \overline{\partial} \varphi_L^{(k)} + (1 + \beta) i \partial \overline{\partial} \psi^{(k)} = \left(\mu - 1 - \lambda\right) i \partial \overline{\partial} \varphi^{(k)} + \lambda [D_2 + v_S \cdot S] + [S + B] \quad \text{by (4.2.5c)}$$

$$\geq - \frac{1}{k} \omega \quad \text{on } X . \quad (4.3.4)$$
Setting $\delta_0 := (\mu - 1)\nu_{\text{max}}$ such that the above inequality holds true when $\beta$ varies within $[0, \delta_0]$, this gives the curvature assumption $(1_k)$ in Theorem 3.4.1. As $\delta_0$ is independent of $k$ and $\psi^{(k)}$s are bounded above uniformly in $k$, the normalisation assumption $(2_k)$ in Theorem 3.4.1 can be made satisfied by adding a suitable constant (independent of $k$) to each $\psi^{(k)}$.

It remains to verify the $L^2$-ness of the given section to be extended with respect to the $1$-lc-measure under the above choice of metrics in order to invoke Theorem 3.4.1.

4.4 The main technical lemma

Lemma 4.4.1 Suppose that $\delta > 0$ is a constant independent of $k$ and $U$ is a section in $H^0(X, k\mu (K_X + S + B) + A)$ such that

$$\|U\|_{\mathcal{E}, s}^{2(1+\delta)} := \int_X \frac{|U|^{2(1+\delta)} e^{-\delta \varphi^{(k)} - \varphi_S - \varphi_B - \frac{1}{k\mu} (1+\delta) \varphi_A}}{\psi^{(k)}}^s \leq M$$

for some numbers $s > 0$ and $M > 0$. Then, one has

$$\int_X |U|^\frac{1}{k\mu} (1+\delta) e^{-\frac{1}{2} (1+\delta) \left( \varphi^{\text{sm}} + \frac{1}{k\mu} \varphi^\text{sm}_A \right)} d\text{vol}_{X, \omega} \lesssim \left( M \int_X \left| \psi^{(k)} \right|^s d\text{vol}_{X, \omega} \right)^{\frac{1}{2}},$$

where the constant involved in $\lesssim$ is independent of $k$. This in turn implies that

$$|U|^\frac{1}{k\mu} (1+\delta) e^{-(1+\delta) \left( \varphi^{\text{sm}} + \frac{1}{k\mu} \varphi^\text{sm}_A \right)} \lesssim M \quad \text{on } X,$$

where the constant involved in $\lesssim$ is independent of $k$.

Proof Notice that

$$\delta \varphi^{(k)} + \varphi_S + \varphi_B + \frac{1}{k\mu} (1+\delta) \varphi^\text{sm}_A \lesssim_{\text{log}} (1+\delta) \left( \varphi^{\text{sm}} + \frac{1}{k\mu} \varphi^\text{sm}_A \right) - \varphi^{\text{sm}}_{K_X}$$

on $X$, with the constant involved in $\lesssim_{\text{log}}$ being independent of $k$ since $\varphi^{(k)}$ is locally bounded above uniformly in $k$ (see (4.2.5b)) and $\delta$ is independent of $k$. The first claim then follows immediately from this inequality and the Cauchy–Schwarz inequality.

An argument with the Harnack inequality for plurisubharmonic functions (see, for example, [11, Ch. I, Prop. 4.22(b)]) then yields

$$|U|^\frac{1}{k\mu} (1+\delta) e^{-\frac{1}{2} (1+\delta) \left( \varphi^{\text{sm}} + \frac{1}{k\mu} \varphi^\text{sm}_A \right)} \lesssim \int_X |U|^\frac{1}{k\mu} (1+\delta) e^{-\frac{1}{2} (1+\delta) \left( \varphi^{\text{sm}} + \frac{1}{k\mu} \varphi^\text{sm}_A \right)} d\text{vol}_{X, \omega} \lesssim \left( M \int_X \left| \psi^{(k)} \right|^s d\text{vol}_{X, \omega} \right)^{\frac{1}{2}}$$

on $X$, where the constants in both $\lesssim$’s are independent of $k$. 

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It remains to show that \( \int_X |\psi^{(k)}|^s \, \text{vol}_{X,\omega} \) is bounded above uniformly in \( k \). Since \( \psi^{(k)} \leq 0 \) and \( s > 0 \), it follows that
\[
\left| \psi^{(k)} \right|^s = \left( -\psi^{(k)} \right)^s \leq \frac{1}{\nu_{\max}^s} \left( \phi^{(k)} - \phi_{V_S} - \phi_{D_2} - \phi_{\nu}^{sm} \right)^s \leq \frac{1}{\nu_{\max}^s} \left( C + \phi^{sm} - \phi_{V_S} - \phi_{D_2} - \phi_{\nu}^{sm} \right)^s
\]
where the constant \( C \) is independent of \( k \) by (4.2.5b). The far right-hand-side is independent of \( k \) and is \( L^1 \) since \( \phi_{V_S} + \phi_{D_2} \) has only logarithmic poles and \( \nu S \cdot S + D_2 \) has only snc. This completes the proof.

\[\square\]

### 4.5 A lower bound for \( \phi^{(k)} \)

For any \( v_{k,A} \in H^0(S, \mathcal{O}(k\mu (K_X + S + B) + A)) \), consider the set
\[
\mathcal{E} := \mathcal{E}(v_{k,A}) := \left\{ U \in H^0(X, k\mu (K_X + S + B) + A) \bigg| U |_S = v_{k,A} \right\}. \tag{4.5.1}
\]
(The section \( v_{k,A} \) will in fact be \( u^k \otimes s_A, i|_S \) in application.) For a given number \( \delta > 0 \), define an \( L^{2 \pi} (1+\delta) \)-norm \( \| \cdot \|_\mathcal{E} \) on \( \mathcal{E} \) such that
\[
\| U \|_\mathcal{E}^{2 \pi (1+\delta)} := \int_X \left| U \right|^{\frac{2 \pi}{\delta}} \text{vol}_{X,\omega} e^{-\delta \phi^{(k)} - \phi_S - \phi_B - \frac{1}{\delta} \frac{1}{\nu} (1+\delta) \phi^{sm}_{\nu}}. \tag{4.5.2}
\]
Note that indeed \( \| U \|_\mathcal{E}^{2 \pi (1+\delta)} < \infty \) for all \( U \in H^0(X, k\mu (K_X + S + B) + A) \) since

- \( \| U \|_\mathcal{E}^{2 \pi (1+\delta)} e^{-\delta \phi^{(k)}} \leq \left( \| U \|_\mathcal{E} \right)^2 \left( \frac{2 \pi}{\delta} \right)^{\frac{1}{\nu}} < \infty \) as \( \phi^{(k)} \) is a Bergman kernel potential (although the bound may depend on \( k \)),
- \( e^{-\phi_B} \) is integrable on the compact \( X \),
- the exponent on \( |\psi^{(k)}| \) is \( > 1 \) while \( S \subset (\psi^{(k)})^{-1}(-\infty) \) (see (4.3.1) and (4.2.5e)), and
- \( S \) and \( B \) having only snc.

For convenience, define also
\[
\| v_{k,A} \|_{\mathcal{E},S}^{2 \pi (1+\delta)} := \int_S \left| v_{k,A} \right|^{\frac{2 \pi}{\delta}} \text{vol}_{S,\omega} e^{-\delta \phi^{(k)} - \phi_B - \frac{1}{\delta} \frac{1}{\nu} (1+\delta) \phi^{sm}_{\nu}} \, \text{vol}_{S,\omega}, \tag{4.5.3}
\]
where \( \text{vol}_{S,\omega} = \sum_{j \in I_S} \text{vol}_{S_j,\omega} \). This norm of \( v_{k,A} \) is also finite as \( e^{-\phi_B} |_S \) is integrable on \( S \), and \( \left( |v_{k,A}|^{\frac{2 \pi}{\delta}} e^{-\delta \phi^{(k)}} \right) |_S = \left( |U|^{\frac{2 \pi}{\delta}} e^{-\delta \phi^{(k)}} \right) |_S \) for any \( U \in \mathcal{E} \) and is thus bounded from above. In order to have better control on the dependence of its...
upper bound on \( k \), [15, Lemma 5.5] (which is due to Hörmander ([29, Thm. 4.4.5]) in its original form; the form in [15, Lemma 5.5] is due to Tian ([47, Prop. 2.1])) is invoked to obtain the following lemma.

**Lemma 4.5.1** There exist constants \( \delta > 0 \) and \( C > 0 \) which are independent of \( k \), \( v_k, A \) and \( \varphi^{(k)} \) such that

\[
\| v_{k, A} \|_{\mathcal{E}, S}^{\frac{2}{k\mu} (1+\delta)} \leq C \sup_S \left( |v_{k, A}|_\omega^{\frac{2}{k\mu} (1+\delta)} e^{-\delta \varphi^{sm} - \varphi_B^{sm} - \frac{1}{k\mu} (1+\delta) \varphi_A^{sm}} \right) =: C \left\| v_{k, A} \right\|_{\infty}^{\frac{2}{k\mu} (1+\delta)}.
\]

**Proof** It follows readily from Hölder’s inequalities that

\[
\| v_{k, A} \|_{\mathcal{E}, S}^{\frac{2}{k\mu} (1+\delta)} \leq \| v_{k, A} \|_{\infty}^{\frac{2}{k\mu} (1+\delta)} \int_S e^{-\delta (\varphi^{(k)} - \varphi^{sm}) - (\varphi_B - \varphi_B^{sm})} \, d\text{vol}_{S, \omega} \leq \| v_{k, A} \|_{\infty}^{\frac{2}{k\mu} (1+\delta)} \left( \int_S e^{-\frac{q}{q-1} (\varphi^{(k)} - \varphi^{sm})} \, d\text{vol}_{S, \omega} \right)^{1 - \frac{1}{q}} \left( \int_S e^{-\varphi_B - \varphi_B^{sm}} \, d\text{vol}_{S, \omega} \right)^{\frac{1}{q}},
\]

and \( q > 1 \) is chosen sufficiently close to 1 such that the last integral on the right-hand-side converges.

[15, Lemma 5.5] is applied to assure that there exist constants \( \delta > 0 \) and \( C' > 0 \), which depend only on the cohomology class of \( \frac{q}{q-1} i \partial \bar{\partial} \varphi^{sm} \), such that

\[
\int_S e^{-\delta \frac{q}{q-1} (\varphi^{(k)} - \varphi^{sm})} \, d\text{vol}_{S, \omega} \leq C'.
\]

This completes the proof. \( \square \)

From now on, the constant \( \delta \) is chosen to be the one provided by Lemma 4.5.1.

**Assume that \( \mathcal{E} \) is non-empty.** There exists, by Lemma 4.4.1 and Montel’s Theorem, an element \( U_{k, A}^{\min} \in \mathcal{E} \) whose \( \| \cdot \|_{\mathcal{E}} \)-norm attains the minimum on \( \mathcal{E} \). Define

\[
\varphi_{U_{k, A}^{\min}}^{(k)} := \frac{1}{k\mu} \left( \log \left| U_{k, A}^{\min} \right|_{k, A}^2 - \varphi_A^{sm} \right)
\]

as a potential on \( K_X + S + B \), which, of course, depends on the choice of \( v_{k, A} \).

**Lemma 4.5.2** Suppose a sequence \( \{ v_{k, A} \}_{k \in \mathbb{N}} \) of sections as described above is given. If \( \mathcal{E} (v_{k, A}) \) is non-empty and \( U_{k, A}^{\min} \) is chosen as above for every \( k \in \mathbb{N} \), then one has

\[
\| U_{k, A}^{\min} \|_{\mathcal{E}}^{\frac{2}{k\mu} (1+\delta)} \lesssim \| v_{k, A} \|_{\mathcal{E}, S}^{\frac{2}{k\mu} (1+\delta)}
\]

for every \( k \in \mathbb{N} \), where the constant involved in \( \lesssim \) is independent of \( k \).
The strategy is to invoke Ohsawa–Takegoshi extension theorem using $\varphi_{U_{\min}}^{(k)}$ as a potential to obtain an extension of $v_{k,A}$ with the required estimate, and then argue by minimality.

Write

$$k\mu (K_X + S + B) + A = K_X + (k\mu - 1) (K_X + S + B) + S + B + A =: L_k$$

and endow the line bundle $L_k$ with the potential $\varphi_{L_k}^{(k)}$ defined such that

$$\varphi_{L_k}^{(k)} + \psi^{(k)} := (k\mu - 1 - \delta) \varphi_{U_{\min}}^{(k)} + \delta \varphi^{(k)} + \phi_S + \phi_B + \varphi_{sm}^{A},$$

(4.5.5)

where $\psi^{(k)}$ is the function defined in (4.3.1), and $\delta > 0$ is the one provided by Lemma 4.5.1 (which can be replaced by a smaller one if $k\mu - 1 - \delta$ is negative). Notice that both $\varphi_{U_{\min}}^{(k)}$ and $\varphi_{L_k}^{(k)}$ are well-defined potentials on $S$. It can be seen that the family $\{\mathcal{I}(\varphi_{L_k}^{(k)} + m\psi^{(k)})\}_{m \in \mathbb{R}}$ jumps along $S$ when $m = 1$. When $m$ increases, the coefficient of $\varphi_{U_{\min}}^{(k)}$ in $\varphi_{L_k}^{(k)} + m\psi^{(k)}$ is unchanged while that of $\varphi^{(k)}$ decreases, one can then see that $\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I}(\varphi_{L_k}^{(k)} + m_0\psi^{(k)})}{\mathcal{I}(\varphi_{L_k}^{(k)} + \psi^{(k)})} \right)$ defines exactly $S$ (with no other subvarieties in $D_2$ or $\text{div} \left( U_{\min_{k,A}} \right)$) for some number $m_0 \in [0, 1)$.

It follows from the Poincaré–Lelong formula $[E] = i \partial \overline{\partial} \phi_E$ that, for $\beta = v_{\max} \lambda$,

$$i \partial \overline{\partial} \varphi_{L_k}^{(k)} + (1 + \beta) i \partial \overline{\partial} \psi^{(k)} = \left( 1 - \frac{1 + \delta}{k\mu} \right) [U_{\min_{k,A}}^{\text{lin}}] + [S] + [B] + \frac{1 + \delta}{k\mu} i \partial \overline{\partial} \varphi_{sm}^{A} + (\delta - \lambda) i \partial \overline{\partial} \varphi^{(k)} + \lambda [D_2] + \lambda [v_S \cdot S] \geq 1 + \frac{\delta}{k\mu} i \partial \overline{\partial} \varphi_{sm}^{A} - \frac{\delta - \lambda}{k\mu} i \partial \overline{\partial} \varphi_{A}^{\text{sm}} + \lambda [D_2] + \lambda [v_S \cdot S] \geq 0$$

for all $\lambda \in [0, \delta]$.\(^8\)

According to Proposition 2.2.1, as the minimal lc centres of $(X, S)$ are of codimension 1 (as the irreducible components $S_j$ of $S$ are disjoint), the norm of $v_{k,A}$ under the 1-lc-measure $d\text{lcv}^1_{\omega,\varphi_{L_k}^{(k)}}[\psi^{(k)}]$ (on codimension-1 lc centres of $(X, S)$)

\(^8\) This is precisely the place where $\delta > 0$ is needed, and thus the use of [15, Lemma 5.5] cannot be avoided.
where the constant involved in \( \lesssim \) is independent of \( k \).

Since \( \delta \) is independent of \( k \), by adding a suitable constant independent of \( k \) to \( \psi^{(k)} \), the normalisation assumption (2) in Theorem 3.3.1 (with \( \ell = \delta \)) can also be fulfilled (see Remark 3.3.5).

By the Ohsawa–Takegoshi extension theorem with lc-measure in Theorem 3.3.4 (or via Demailly’s version in [12, Thm. 2.8]), one obtains a holomorphic extension \( V_k \in \mathcal{E} \) of \( v_k \) on \( X \) with estimate

\[
\left| \int_X \frac{|V_k|^2}{\psi^{(k)}} \right| \left( \frac{1}{\psi^{(k)}} \left( (\log |\delta \psi^{(k)}|)^2 + 1 \right) \right) \lesssim \left\| v_k \right\|_{\mathcal{E},S}^2
\]

where the constant involved in the last \( \lesssim \) is independent of \( k \) (and \( \delta \)). As \( \frac{1}{\psi^{(k)}} \lesssim \frac{1}{(\log |\delta \psi^{(k)}|)^2 + 1} \) with the constant involved in \( \lesssim \) being independent of \( k \) via a use of (3.3.2), it follows that

\[
\left\| V_k \right\|_{\psi^{(k)}_{L_k}}^2 := \int_X \left| \frac{|V_k|^2}{\psi^{(k)}} \right| \frac{1}{\psi^{(k)}} \lesssim \left\| v_k \right\|_{\mathcal{E},S}^2 \left( \frac{2}{|\psi^{(k)}_{L_k}|} \right)^{1+\delta} \]

where the constant in \( \lesssim \) is independent of \( k \).

Considering the definitions of the potentials \( \varphi^{(k)}_{\min} \) in (4.5.4) and \( \varphi^{(k)}_{L_k} \) in (4.5.5), as well as the definition of the norm \( \| \cdot \|_{\mathcal{E}} \) in (4.5.2), Hölder’s inequality, together with the minimality of \( U^{\min}_{k,A} \) in the norm \( \| \cdot \|_{\mathcal{E}} \), yields

\[
\left\| U^{\min}_{k,A} \right\|_{\mathcal{E}}^2 \lesssim \left\| V_k \right\|_{\mathcal{E}}^2 \left( \left\| U^{\min}_{k,A} \right\|_{\mathcal{E}}^2 \right)^{1-\frac{1}{k^{\mu}}(1+\delta)}.
\]

Therefore, it follows that

\[
\left\| U^{\min}_{k,A} \right\|_{\mathcal{E}}^2 \lesssim \left\| V_k \right\|_{\mathcal{E}}^2 \lesssim \left\| v_k \right\|_{\mathcal{E},S}^2,
\]

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where the constant involved in \( \lesssim \) is independent of \( k \).

Combining the Lemmata 4.4.1, 4.5.2 and 4.5.1, one obtains that, if \( E \) is non-empty, there exists a minimal element \( U_{k,A}^{\min} \in E \) such that

\[
U_{k,A}^{\min} \geq \frac{2}{\mu} (1+\delta) e^{-(1+\delta) \varphi^m} - \frac{1}{\mu} (1+\delta) \varphi_A^m (p) \lesssim \| u_{k,A} \|_{\tau, \infty}^2 (1+\delta) \tag{4.5.6}
\]

for any \( p \in X \), where the constant involved in \( \lesssim \) is independent of \( k \) and \( p \).

Now the following improvement to [15, Thm. 6.1] can be proved.

**Theorem 4.5.3** (cf. [15, Thm. 6.1]). **Under the setup in Sect. 4.1 (without the assumptions \( \text{supp } D_2 \subset \text{supp } B \) and the existence of \( u \in H^0 (S, \mathcal{O}_S (\mu (K_X + S + B))) \)), the potential \( \varphi_{\min} \in S \) is well-defined on every component of \( S \).**

**Proof** Since \( S_j \not\subset B_\infty (K_X + S + B) \) for every \( j \in I_S \), using the Ohsawa–Takegoshi extension theorem, one can find that, for every \( j \in I_S \) and for every \( k \in \mathbb{N} \), there exist \( \ell \gg 0 \) and \( W_{\ell, k, \ell A} \in H^0 (X, \ell k \mu (K_X + S + B) + \ell A) \) such that \( W_{\ell, k, \ell A} |_{S_j} \neq 0 \) but \( W_{\ell, k, \ell A} |_{S_j'} = 0 \) for all \( j' \in I_S \setminus \{ j \} \). Set \( v_{\ell, k, \ell A} := W_{\ell, k, \ell A} |_S \) and renormalise it such that

\[
\| v_{\ell, k, \ell A} \|_{\tau, \infty} = \sup_S \left( \left| v_{\ell, k, \ell A} \right|_{\omega} \right) = 1.
\]

Recall that \( \delta > 0 \) is chosen as in Lemma 4.5.1, so it is independent of \( k \) in particular.

Define the set \( E := E \left( v_{\ell, k, \ell A} \right) \) as in (4.5.1). This set is obviously non-empty, which implies the existence of the minimal element \( U_{\ell, k, \ell A}^{\min} \in E \) such that the estimate (4.5.6) holds.

Since \( \varphi^{(k)} = \varphi^{(\ell, k)} \) is a Bergman kernel potential constructed from holomorphic sections in \( H^0 (X, \ell k \mu (K_X + S + B) + \ell A) \), it follows that

\[
\frac{1}{\ell k \mu} \left( \log \left| U_{\ell, k, \ell A}^{\min} \right|^2 - \ell \varphi_A^m \right) \leq \varphi^{(\ell, k)} + \frac{1}{\ell k \mu} \log \| U_{\ell, k, \ell A}^{\min} \|_{\ell k \mu \varphi_A^m + \ell \varphi_A^m},
\]

where

\[
\left( \| U_{\ell, k, \ell A}^{\min} \|_{\ell k \mu \varphi_A^m + \ell \varphi_A^m}^2 \right)^{\frac{1}{\tau \mu}} \leq \left( \int_X \left| U_{\ell, k, \ell A}^{\min} \right|^2 e^{-\ell k \mu \varphi_A^m - \ell \varphi_A^m} d \nu X, \omega \right)^{\frac{1}{\tau \mu}} \sim \| v_{\ell, k, \ell A} \|_{\tau, \infty}^2 = 1.
\]

Therefore,

\[
\frac{1}{\ell k \mu} \left( \log \left| U_{\ell, k, \ell A}^{\min} \right|^2 - \ell \varphi_A^m \right) \lesssim \log \varphi^{(\ell, k)}
\]

on \( X \). Notice that \( v_{\ell, k, \ell A} = U_{\ell, k, \ell A}^{\min} \big|_S \) and the sup-norm of \( v_{\ell, k, \ell A} \) on \( S \) is the same as the one on \( S_j \). It follows that, after restricting to \( S_j \) and adding suitable potentials
(which are uniformly bounded in \(k\)) on both sides, one has

\[
0 = \log \sup_{S_j} \left( |v_{\ell k, \ell A}^{2/\omega} e^{\frac{\delta}{(1+\delta)} \phi_{sm} - \frac{1}{(1+\delta) \omega B}} - \frac{1}{(1+\delta) \omega A} \right) \leq \log \sup_{S_j} \phi^{(k)},
\]

where the constant involved in \(\lesssim\) is independent of \(k\). Taking the upper regularised limit yields

\[
0 \lesssim \log \sup_{S_j} \phi^{(\infty)} = \sup_{S_j} \phi_{\min}.
\]

This implies that \(\phi_{\min}\) is well-defined on \(S_j\). Since this holds true for every \(j \in I_S\), \(\phi_{\min}\) is therefore well-defined on every component of \(S\).

Recall that \(A\) is chosen to be globally generated and \(\{s_{A,i}\}_{i \in I_A}\) is a basis of \(H^0(X, A)\) such that \(\phi_{sm}^A = \log \left( \sum_{i \in I_A} |s_{A,i}|^2 \right)\). The following is the key result of this section.

**Theorem 4.5.4** Under the setup given in Sect. 4.1 (without the assumption \(\text{supp } D_2 \subseteq \text{supp } B\)), suppose that there is a section \(u \in H^0(S, \mathcal{O}_S(\mu(K_X + S + B)))\) and that the sets \(E \left( u^k \otimes s_{A,i}, S \right) \) constructed as in (4.5.1) are non-empty for all \(k \in \mathbb{N}, i \in I_A\). Then, one has

\[
\log |u|^2 \lesssim_{\log} \phi^{(k)} \quad \text{on } S
\]

for all \(k\) with the constant involved in \(\lesssim_{\log}\) being independent of \(k\), and therefore

\[
\log |u|^2 \lesssim_{\log} \phi_{\min} \quad \text{on } S.
\]

**Proof** The proof goes almost as in the proof of Theorem 4.5.3.

For fixed \(k \in \mathbb{N}\) and \(i \in I_A\), the set \(E := E \left( u^k \otimes s_{A,i}, S \right) \) being non-empty implies the existence of the minimal element \(U_{k,A,i}^{\min} \in E\) such that the estimate (4.5.6) holds.

The potential \(\phi^{(k)} = \phi^{(\ell, k)}\) being a Bergman kernel potential implies that, for \(\ell \gg 0\),

\[
\frac{1}{k \mu} \left( \log |U_{k,A,i}^{\min}|^2 - \phi_{sm}^A \right) \leq \phi^{(k)} + \frac{1}{\ell k \mu} \log \left( U_{k,A,i}^{\min} \right)^{\ell} \leq \frac{1}{\ell k \mu} \|U_{k,A,i}^{\min}\|^2 + \ell \phi_{sm}^A + \ell \phi_{A}^{sm},
\]

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where
\[
\left( \left\| \left( U_{k,A,i}^{\min} \right) \right\|_r^2 \ell_k \mu \varphi^{sm} + \ell \varphi_A^{sm} \right)^{1/k \mu} \lesssim \left( \int_X \left| U_{k,A,i}^{\min} \right|^{2e^{-\ell_k \mu \varphi^{sm} - \ell \varphi_A^{sm}}} d \text{vol}_{X,\omega} \right)^{1/k \mu} \leq \left\| u^k \otimes s_{A,i} \right\|_{L^\infty} \leq 1
\]

with the constants involved in both \( \lesssim \)'s being independent of \( k \). Therefore,
\[
\frac{1}{k \mu} \left( \log \left| U_{k,A,i}^{\min} \right| - \varphi_A^{sm} \right) \lesssim_{\log} \varphi^{(k)} \quad \text{on } X
\]
\[
\Rightarrow \quad \log |u|^2 + \frac{1}{k \mu} \log \left( |s_{A,i}|^2 e^{-\varphi_A^{sm}} \right) \lesssim_{\log} \varphi^{(k)} \quad \text{on } S
\]
\[
\Rightarrow \quad |u|^{2k} \left( |s_{A,i}|^2 e^{-\varphi_A^{sm}} \right) \lesssim e^{k \mu \varphi^{(k)}} \quad \text{on } S,
\]

where the constants involved in \( \lesssim_{\log} \)'s and \( \lesssim \) are independent of \( k \). Summing up the last inequality over \( i \in I_A \) yields the inequality of the first claim.

The second claim follows from taking the upper regularised limit of \( \varphi^{(k)} \) as \( k \to \infty \) together with the fact that \( \varphi^{(\infty)} = \varphi_{\min} \).

\[ \square \]

### 4.6 Proof of the main theorem

Let \( \pi : \tilde{X} \to X \) be a log-resolution of \((X, S + B)\) such that
\[
K_{\tilde{X}} + \tilde{S} + \tilde{B} = \pi^* (K_X + S + B) + \tilde{E},
\]

where \( \tilde{S}, \tilde{B} \) and \( \tilde{E} \) are effective \( \mathbb{Q} \)-divisors with no common components such that \( \tilde{S} = \lfloor \tilde{S} + \tilde{B} \rfloor \) and \( \pi(\tilde{S}) = S \). Let \( \mathcal{E} := (N_\sigma (\| K_{\tilde{X}} + \tilde{S} + \tilde{B} \|_{\tilde{S}}) \wedge \tilde{B}_{\tilde{S}}) \) be the extension obstruction divisor introduced in [15] (see [15, §2.1] for the definition of \( N_\sigma (\| K_{\tilde{X}} + \tilde{S} + \tilde{B} \|_{\tilde{S}}) \)). The corresponding extension obstruction ideal sheaf \( \mathcal{I}_{\mathcal{E}} \) is defined as
\[
\mathcal{I}_{\mathcal{E}} := \left\{ w \in \mathcal{O}_S \mid \exists \text{ log-resolution } \pi : \tilde{X} \to X \text{ s.t. irred. comp. of } \tilde{B} \right. \text{ are disjoint and } \text{div } (\pi^* w) + \mu \tilde{E}_{\tilde{S}} \geq \mu \mathcal{E} \}
\]

Notice that, when \( K_X + S + B \) is nef, one has \( \mathcal{E} = 0 \), and thus \( \mathcal{I}_{\mathcal{E}} = \mathcal{O}_S \).

**Theorem 4.6.1** Under the setup in Sect. 4.1 (without the assumption \( \text{supp } D_2 \subset \text{supp } B \)), every \( u \in H^0 (S, \mathcal{O}_S (\mu (K_X + S + B)) \otimes \mathcal{I}_{\mathcal{E}}) \) extends to a holomorphic section in \( H^0 (X, \mu (K_X + S + B)) \).
In particular, when $K_X + S + B$ is nef, the restriction map $H^0(X, \mu (K_X + S + B)) \to H^0(S, \mathcal{O}_S (\mu (K_X + S + B)))$ is surjective.

**Proof** Let $u \in H^0(S, \mathcal{O}_S (\mu (K_X + S + B)) \otimes \mathcal{I}_S)$ and define $E_{k,i} := E(u^k \otimes s_{A,i} |_S)$ as in (4.5.1).

By the result of Hacon and McKernan in [25, Thm. 6.3] (see also [24, Thm. 3.16]) on the extension of pluricanonical sections, in which the technique was originated in the work of Siu ([46]), it follows that, on the plt pair $(X, S + B)$, the set $E_{k,i} = E(u^k \otimes s_{A,i} |_S)$ are non-empty for every $k \in \mathbb{N}$ and $i \in I_A$ when $u$ is a section to the ideal $\mathcal{I}_S^k$.

Then, given the choice of $\psi^{(k)}$ and $\varphi_L^{(k)}$ in (4.3.1) and (4.3.2), Theorem 4.5.4 assures that

$$
\int_S |u|^2 \, d\text{cv}^1_{\omega, \varphi_L^{(k)}[\psi^{(k)}]} = \sum_{j \in I_S} \frac{\pi v_{\text{max}}}{v_j} \int_{S_j} |u|_{\omega}^2 \, e^{-(\mu - 1)\varphi_{\text{min}} - \varphi_B} \, d\text{vol}_{S_j, \omega}
$$

and its limit as $k \to \infty$, which is

$$
\sum_{j \in I_S} \frac{\pi v_{\text{max}}}{v_j} \int_{S_j} |u|_{\omega}^2 \, e^{-(\mu - 1)\varphi_{\text{min}} - \varphi_B} \, d\text{vol}_{S_j, \omega}
$$

up to a constant multiple, are finite (as $e^{-\varphi_B}$ is integrable), which verifies the $L^2$ assumption in Theorem 3.4.1.

The inequality (4.3.4) verifies the curvature assumption $(1_k)$ in Theorem 3.4.1 and provides a $\delta_0 := (\mu - 1) v_{\text{max}}$ which is independent of $k$. Considering the definition of $\psi^{(k)}$ in (4.3.1) and the fact that $\varphi^{(\infty)} = \varphi_{\text{min}}$, which is bounded from above, one sees that a uniform constant can be added to $\psi^{(k)}$ such that the normalisation assumption $(2_k)$ in Theorem 3.4.1 is satisfied. Therefore, it follows from Theorem 3.4.1 that there exists a holomorphic extension $U$ of $u$ with the estimate

$$
\int_X |U|^2 \, e^{-(\mu - 1)\varphi_{\text{min}} - \varphi_S - \varphi_B} \leq \sum_{j \in I_S} \frac{\pi v_{\text{max}}}{v_j} \int_{S_j} |u|_{\omega}^2 \, e^{-(\mu - 1)\varphi_{\text{min}} - \varphi_B} \, d\text{vol}_{S_j, \omega}.
$$

This completes the proof. \qed

**Compliance with ethical standards**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
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