A remark on Dickey’s stabilizing chain

Andrei K. Svinin

Institute for System Dynamics and Control Theory, Siberian Branch of Russian Academy of Sciences, P.O. Box 292, 664033 Irkutsk, Russia

Abstract

We observe that Dickey’s stabilizing chain can be naturally included into two-dimensional chain of infinitely many copies of equations of KP hierarchy.

Keywords: Stabilizing chain, KP hierarchy

1. Introduction

In his article [1], Dickey introduced the so-called stabilizing chain of truncated Kadomtsev-Petviashvili (KP) hierarchies. The latter can be formulated in the language of formal pseudo-differential dressing operators (dressing ΨDO’s)

\[ W_i = 1 + \sum_{m=1}^{\infty} w_m \partial^{-m} \]

which are forced to be connected by relations

\[ (\partial + u_i) W_i = (\partial + v_{i+1}) W_{i+1} \]

(1)

for \( i \geq 0 \). One requires that solution of (1) such that \( w_{ii} \) is not identically equal to zero. Evolution equations of KP hierarchy are given by

\[ \partial_s W_i = (Q^i)_s W_i - W_i \partial^i, \]

(2)

for \( s \geq 2 \), where \( Q_i \equiv W_i \partial W_i^{-1} \) and \( \partial_s \equiv \partial/\partial t_s \). Remember that the subscript \( + \) means taking only nonnegative powers of \( \partial \) in pseudo-differential operator under consideration. To complete the description of stabilizing chain we need to add evolution equations for “gluing” fields \( u_i \) and \( v_i \):

\[ \partial_s u_i = -\text{res}_0 \left( \left( \partial + u_i \right) (Q^i)_s \left( \partial + u_i \right)^{-1} \right), \]

(3)

\[ \partial_s v_i = -\text{res}_0 \left( \left( \partial + v_i \right) (Q^i)_s \left( \partial + v_i \right)^{-1} \right). \]

(4)

Remember that \( \text{res}_0 (\sum a_m \partial^m) \equiv a_{-1} \). It was shown in [1] that equations (1,2) are well defined. Moreover, general solution of stabilizing chain is shown to be given in terms of Wronskians

\[ W_i = \frac{1}{\tau_{i+1}} \]

where \( \tau_i = \text{Wr}[y_{0i}, ..., y_{i-1}1, y_{i+1}] \) and \( \tau_{i+1} = \text{Wr}[y_{0i}, ..., y_{i-1}, y_{i+1}] \). By definition, the set \( \{y_{0i}, ..., y_{i-1}, y_{i+1}\} \) is the basis of the kernel for differential operator \( P_i \equiv W_i \partial^i \). In what follows, we set \( y_{1i} \equiv y'_{i+1} \). Functions \( y_{1i} \) are forced to be solutions of hierarchy evolution equations \( \partial_s y = \partial^i y \). As is known any analytic solution of this hierarchy can be presented as series over Schur polynomials

\[ y = \sum_{m \geq 0} c_m P_m(x, t_2, t_3, ...) \]

Let us remember that Schur polynomials are defined through the relation

\[ \exp \left( \sum_{s \geq 0} t_s C^s \right) = \sum_{m \geq 0} p_m x^m, \]

where \( t_1 = x \) and have, in virtue of their definition, following easily verified properties:

\[ p_m(x, 0, 0, ...) = x^m/m! \quad \text{and} \quad \partial_s p_m = \partial^i p_m = p_{m-s}. \]

As was shown in [1], the sequence \( \{\tau_{i+1}\} \) has the property of stabilization with respect to gradation which is defined by the rule: \( [\tau_i] = k \) . Namely, if one choose

\[ y_{ki} = (-1)^k \left( p_{i-k+1} + c_{1}^{(k)} p_{i-k} + c_{2}^{(k)} p_{i-k+1} + \cdots \right), \]

for \( k = 0, \ldots, i - 1 \), then any term of weight \( l \) do not depend on \( i \) when \( i \geq l \). In this case, one says that the sequence \( \{\tau_{i+1}\} \) has the stable limit. Moreover, with special choice of constants \( c_m^{(k)} \) this stable limit yields expression for Kontsevich integral [2].

In the next two sections we present our observation that it is quite natural to put equations (1,2,4) into two-dimensional chain of KP hierarchies.

2. Two-dimensional chain of KP hierarchies

2.1. Two-dimensional chain of dressing ΨDO’s

Here we construct two-dimensional chain of truncated dressing ΨDO’s \( \{W_{ij}\} \) related with each other by some suitable relations.
With infinite set of suitable constants \( c_{kl} : k, l \in \mathbb{Z}, k \geq 0 \) we define collection of analytic functions
\[
\sum_{m=0}^{\infty} c_{kl-m} \frac{x^m}{m!}.
\]
\( (5) \)

Obviously, by definition, \( \Psi_{kl} = \Psi_{kj-l} \). Let us define
\[
\tau_{ij} \equiv W_{ij}[\Psi_{0,i+1-j}, \Psi_{i+1-j}]
\]
and an infinite set of differential operators
\[
P_{ij} = \partial^i + \sum_{m=0}^{i} w_{im} \partial^{-m} = \frac{1}{\tau_{ij}}
\]
\[\begin{bmatrix}
\Psi_{0,i+1-j} & \cdots & \Psi_{i+1-j-1-j} & 1 \\
\Psi_{0,i+1-j} \partial & \cdots & \Psi_{i+1-j-1-j} \partial & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\Psi_{0,i+1-j} \partial^i & \cdots & \Psi_{i+1-j-1-j} \partial^i & \cdots & \ddots & \ddots \\
\end{bmatrix}
\]

for \( i, j \in \mathbb{Z}, i \geq 0 \). Require that \( w_{ij} \) is not identically equal to zero for any values of \( i \) and \( j \). This is equivalent to the fact that \( P_{ij} \) has not \( y = const \) as a solution. In what follows, it will be useful following technical proposition.

**Lemma 1.** In virtue of their definition, operators \( P_{ij} \) satisfy equations
\[
(\partial + v_{i+1,j})P_{i+1,j} = P_{i+1,j+1} \partial, \quad (\partial + u_{ij})P_{ij} = P_{i+1,j+1} \partial
\]
with
\[
v_{ij} = -\partial \ln \left( \frac{\tau_{i+1,j}}{\tau_{ij}} \right), \quad u_{ij} = -\partial \ln \left( \frac{\tau_{i+1,j+1}}{\tau_{ij}} \right).
\]
\( (6) \)

Proof. The first relation in (6) follows from the fact that \( kerP_{i+1,j+1} \) is defined as a linear spanning of derivatives of the functions which belong to \( kerP_{i+1,j} \). Functions \( \Psi_{0,i+2-j}, \cdots, \Psi_{i+2-j} \) are linearly independent. Otherwise, \( y = const \) will belong to \( kerP_{i+1,j} \). Moreover, we have the relation
\[
(\partial + v_{i+1,j})P_{i+1,j}(1) = 0
\]
hold. From
\[
P_{i+1,j}(1) = \frac{W_{ij}[\Psi_{0,i+2-j}, \cdots, \Psi_{i+2-j}, 1]}{\tau_{i+1,j}} = (-1)^{i+1} \frac{\tau_{i+1,j+1}}{\tau_{i+1,j}}
\]
we derive expression for \( v_{ij} \) in (7). The second relation in (6) follows from the fact that all basic functions of \( kerP_{i+1,j+1} \) except for \( \Psi_{i+1-j} \) belong to \( kerP_{ij} \). In addition, we have
\[
(\partial + u_{ij})P_{ij}(\Psi_{i+1-j}) = (\partial + u_{ij}) \left( \frac{\tau_{i+1,j+1}}{\tau_{ij}} \right) = 0.
\]
The latter gives corresponding expression for \( u_{ij} \) in (7). Therefore lemma is proved.

As a consequence of (5), we have two equations
\[
(\partial + v_{i+1,j}) W_{i+1,j} = W_{i+1,j+1} \partial, \quad (\partial + u_{ij}) W_{ij} = W_{i+1,j+1} \partial
\]
\( (8) \)

for \( \PsiDO \)’s \( W_{ij} \equiv P_{ij} \partial^{-i} \). So, we can think of \( u_{ij} \) and \( v_{ij} \) as “gluing” variables which relate \( \PsiDO \)’s of special truncated form on two-dimensional chain. Two formulas in (8) define shifts \((i, j) \rightarrow (i, j + 1) \) and \((i, j) \rightarrow (i + 1, j + 1) \), respectively. As a consequence, of these relations we see that \( W_{ij} \) also satisfies the relation
\[
(\partial + u_{ij}) W_{ij} = (\partial + v_{i+1,j}) W_{i+1,j}.
\]
\( (9) \)

which manages the shift \((i, j) \rightarrow (i + 1, j) \). We see that this equation is nothing else but (1). The only difference is that dressing operators \( W_{ij} \) in (7) are parameterized by additional discrete variable \( j \).

### 2.2. Two-dimensional chain of KP hierarchies

We know that if one replaces the basis \( \{x^n/m!\} \) by that of Schur polynomials \( \{p_m(x_0, x_1, x_2, \ldots)\} \) in (5), that is,
\[
\Psi_{kl} \rightarrow y_{kl} = \sum_{m=0}^{\infty} c_{kl-m} p_m,
\]
then each \( W_{ij} \) automatically will be solution of KP hierarchy (2) (see, for example [3]), while the sequence of dressing operators along shifts \((i, j) \rightarrow (i + 1, j + 1) \) and \((i, j) \rightarrow (i + 1, j + 1) \), due to (8) is nothing else but the semi-infinite 1-Toda lattice (the discrete KP hierarchy) with initial condition \( W_{0j} = 1 \). Then “gluing” variables \( u_{ij} \) and \( v_{ij} \), by their construction, automatically satisfy equations (1)
\[
\partial_{j} u_{ij} = -\text{res}_{\partial} \left( (\partial + u_{ij})(\Psi^{i}_{j}), (\partial + u_{ij})^{-1} \right),
\]
\[
\partial_{i} v_{ij} = -\text{res}_{\partial} \left( (\partial + v_{ij})(\Psi^{i}_{j}), (\partial + v_{ij})^{-1} \right).
\]

### 3. Conclusion

In this brief note we have shown how Dickey’s stabilizing reference is that dress- ing chain (14) can be included into two-dimensional lattice of Korteweg-De Vries hierarchies. One learns from this presentation, that, the latter in a sense can be viewed as a superposition of two compatible discrete KP hierarchies.

### References

[1] L.A. Dickey, J. Appl. Math. 1 (2001) 175.
[2] C. Itzikson, J.-B. Zuber, Internat. J. Modern Phys. A 7 (1992) 5661.
[3] Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, Prog. Theor. Phys. Suppl. 94 (1988) 210.