Contour deformation trick in hybrid NLIE

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Abstract

The hybrid NLIE of AdS$_5 \times$ S$^5$ is applied to a wider class of states. We find that the Konishi state of the orbifold AdS$_5 \times$ S$^5$/Z$_2$ satisfies $A_1$ NLIE with the source terms which are derived from contour deformation trick. Conversely, we show that for general states the analyticity of Q-functions are constrained by demanding that the contour deformation trick yields the correct source terms.
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1 Introduction and Summary

The primary example of AdS/CFT correspondence is the one between four-dimensional $\mathcal{N} = 4$ super Yang-Mills and AdS$_5 \times S^5$ string theory [1]. The spectrum of string states on AdS$_5 \times S^5$ can be computed by the mirror Thermodynamic Bethe Ansatz (TBA) equations [2, 3, 4] based on string hypothesis in the mirror model [5, 6]; or equivalently the extended Y-system on $\mathfrak{psu}(2,2|4)$-hook [7, 8, 9]. It is believed that these methods give the exact answer, because they capture all finite-size corrections [10, 11].

The numerical study of the mirror TBA has made progress [12, 13, 14, 15]. However, it suffers from the problem of critical coupling constants [16]. The analyticity of the unknown variables called Y-functions changes around certain values of 't Hooft coupling constant, and the explicit form of the TBA equations changes there discontinuously. As a result, it is difficult to solve the equation with high precision around the critical values, and to judge if the exact energy is a smooth function around the critical points.

The author has recently applied the method of hybrid nonlinear integral equations (hybrid NLIE) [17] to the mirror TBA for AdS$_5 \times S^5$ [18]. This method replaces the horizontal part of the mirror TBA equations by $A_1$ NLIE. The hybrid NLIE consists of a smaller set of unknown variables than the mirror TBA, and we expect that it suffers less often from the problem of critical coupling constants. We exemplify our expectation in a way similar to [16].

For this purpose the mirror TBA for the twisted AdS$_5 \times S^5$ offers a desired playground, because all Y-functions have intricate analytic properties, depending on the twist angle $\alpha$ and 't Hooft coupling constant $g = \sqrt{\lambda/2\pi}$. In fact, the mirror TBA for $Y_{M|w}$ in the untwisted model do not have critical coupling constants asymptotically. The orbifold Konishi state is the simplest nontrivial example that exhibits critical behavior in the mirror TBA for $Y_{M|w}$. For this state, we find that the hybrid NLIE also exhibits critical behavior, that is, its source terms change discontinuously across certain values of coupling constant.

The orbifold Konishi state is a two-particle state in the $\mathfrak{sl}(2)$ sector of AdS$_5 \times S^5/\mathbb{Z}_S$, where the $\mathbb{Z}_S$ acts on $\mathfrak{su}(2)^2 \subset [\mathfrak{su}(2|2)^2 \cap \mathfrak{su}(4)]$. This is also a special state in the twisted AdS$_5 \times S^5$, $\beta$- or $\gamma$-deformed AdS$_5 \times S^5$ models. The orbifold and $\gamma$-deformed models are another important example of AdS/CFT correspondence, realized in gauge theory [24, 25, 26, 27, 28, 29, 30] and in string theory [31, 32, 33, 34, 35, 36, 37, 38]. Finite-size corrections of deformed theories have been studied in gauge theory [39, 40, 41, 42, 43, 44, 45, 46, 47], in string theory [48, 49, 50, 51, 52], by Lüscher formula [53, 54, 55, 56, 57, 58], and by the mirror TBA or Y-system [58, 59, 60, 61, 62]. It is not clear if the corresponding sigma model on twisted AdS$_5 \times S^5$ possesses integrability (see [63] for review),

\footnote{This equation is called Klümper-Batchelor-Pearce or Destri-de Vega equation in the literature [19, 20, 21, 22, 23, 24, 25]. We call it $A_1$ NLIE, since it can be derived from $A_1$ TQ-relations and analyticity conditions as shown in [18].}

\footnote{We checked this claim for several four particle states.}
though integrable twists exist mathematically.

Next, we notice that such discontinuous change of the NLIE can be explained by the contour deformation trick. The contour deformation is an idea that the TBA for excited states follows from the TBA for the ground state by analytic continuation of coupling constant \[54, 55\]. In practice, however, it is hard to keep track of analytic continuation of coupling constant. It is much easier to study the equations whose integration contour is deformed. This idea is called contour deformation trick. The contour deformation trick is a guideline to study various states in the mirror TBA \[16\] including boundstates \[56\]. The \( A_1 \) NLIE with source terms has been studied in various examples \[57, 58, 59, 60, 61, 62, 63\], and the contour deformation trick was used in \[64, 65, 66\].

With successful examples of the contour deformation in mind, we ask what the most general possible source terms are, and if they are obtained by the contour deformation trick. In principle, \( A_1 \) NLIE can be derived even when the Q-functions are meromorphic, rather than analytic, in the upper or lower half plane. Then the isolated singularities of Q-functions provide extra source terms to \( A_1 \) NLIE. It is a nontrivial question whether such source terms can be explained by the contour deformation trick, particularly with the same contour as in the orbifold Konishi state. Indeed, mismatch is found between the two results, and we interpret this as the constraints on the analyticity of Q-functions: *The contour deformation trick with the orbifold Konishi’s contour yields the correct source terms of \( A_1 \) NLIE, if and only if there exists a gauge such that \( Q(v), L(v + \frac{2i}{g}) \) are analytic and nonzero in the upper half plane, \( \overline{Q}(v), \overline{L}(v - \frac{2i}{g}) \) are analytic and nonzero in the lower half plane.* The details of this statement is explained in Section 3.

The contour deformation trick illustrates the difference between hybrid NLIE and FiNLIE \[67\]. In the latter the integrals run over the gap discontinuity of dynamical variables, which is not something to be deformed. In contrast, hybrid NLIE is written in terms of gauge-invariant (but frame-dependent) variables, allowing us to handle the equations similar to that of the mirror TBA.

This paper is organized as follows. In Section 2, we study the orbifold Konishi state from the mirror TBA and hybrid NLIE, and clarify the critical behavior in the asymptotic limit. In Section 3, we discuss the source terms of hybrid NLIE in view of contour deformation trick. Section 4 is for conclusion. In appendices, we introduce our notation, derive the asymptotic transfer matrix in the form of Wronskian, and the results in Section 3.

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3See the discussion at the end of Appendix B for the frame dependence.

4Regarding the analyticity of Q-functions, a similar but different statement has also been proven in \[67\].
2 TBA and NLIE for twisted $\text{AdS}_5 \times S^5$

We study the critical behavior of hybrid NLIE for the orbifold Konishi state as a specific example. We briefly review the mirror TBA in twisted $\text{AdS}_5 \times S^5$ and their critical behavior.

2.1 Orbifold Konishi state

The orbifold Konishi state can be defined in two equivalent ways. The first is to consider the $\mathfrak{sl}(2)$ Konishi descendant on the orbifold $\text{AdS}_5 \times S^5$, where the $\mathbb{Z}_S$ action is chosen as follows (see [49]). We decompose the transverse $8+8$ fields of $\text{AdS}_5 \times S^5$ into $(2|2) \otimes (2|2)$ representation of $\mathfrak{su}(2|2)_\mathbb{L} \times \mathfrak{su}(2|2)_\mathbb{R}$, as

$$(\Phi^I , D_\mu Z , \Psi , \overline{\Psi}) \leftrightarrow (Y_{bb} , Y_{\beta\dot{\beta}} , Y_{b\beta} , Y_{\beta b}) \equiv (y_b , y_\beta , \eta_\beta , \eta_{\dot{\beta}} , y_\dot{\beta} , \eta_\beta y_b), \quad (2.1)$$

where $b, \dot{b} = 1, 2$ refer to the $S^5$ part, and $\beta, \dot{\beta} = 3, 4$ refer to the $\text{AdS}_5$ part of $\mathfrak{su}(2|2)^2$. The boundary conditions of $y_b$ are twisted by $\mathbb{Z}_S$ as

$$(y_1(\sigma = 2\pi) , y_2(\sigma = 2\pi)) = \left( e^{\alpha_L} , 0 \right) \left( e^{-\alpha_L} , 0 \right) \begin{pmatrix} y_1(\sigma = 0) \\ y_2(\sigma = 0) \end{pmatrix}, \quad \alpha_L = \frac{2\pi n_L}{S} \quad (n_L \in \mathbb{Z}). \quad (2.2)$$

Similarly, the boundary conditions of $y_\beta$ are twisted by $\alpha_R = \frac{2\pi n_R}{S}$. The orbifold action affects only the auxiliary part of the asymptotic Bethe Ansatz equations. Thus, if we set the total momentum to zero as in the ordinary Konishi state, the asymptotic Bethe roots remain unchanged before and after orbifolding. This is called orbifold Konishi state.

The second is to introduce integrable twisted boundary conditions to the transfer matrix of $\text{AdS}_5 \times S^5$. To preserve the integrability, the twist operator must commute with the S-matrix. When the twist operator belongs to $[\mathfrak{su}(2|2)^2 \cap \mathfrak{su}(4)]$ and the twist angle is equal to a multiple of $2\pi/S$, Konishi state of the twisted $\text{AdS}_5 \times S^5$ is equivalent to the orbifold Konishi state.

The second point of view is useful to construct the twisted transfer matrix, as defined by

$$T_{Q,1}^L = \text{str}_Q [g_0 S_{01} S_{02} \ldots S_{0N}], \quad g_0 = \text{diag} \left( e^{+i\alpha_L} , e^{-i\alpha_L} , 1, 1 \right). \quad (2.3)$$

and similarly for $T_{Q,1}^R$. The $S_{0i}$ is the S-matrix between the mirror particle and the $i$-th particle in string theory. We can diagonalize (2.3) by algebraic Bethe Ansatz [68]. In practice, it is easier to twist the generating function for the eigenvalues of transfer matrices [69, 49, 48]. Later we will need the transfer matrices in the form of Wronskian. This construction will be discussed in Appendix [3]. In what follows we set $\alpha_L = \alpha_R \equiv \alpha$ for simplicity.

The twisted mirror TBA is obtained as follows. The twist angle $\alpha$ in string theory corresponds to the insertion of defect operator in mirror theory [46]. In particular, the same mirror string hypothesis is used both twisted and untwisted models. In the case of orbifold, the defect
operator can be identified as an extra chemical potential, and it changes the \(|v| \to \infty\) asymptotics of \(Y\)-functions \([50]\). The mirror TBA equations for twisted \(\text{AdS}_5 \times S^5\) are solved by the twisted transfer matrices in the asymptotic limit \([10]\).

2.2 TBA and NLIE in horizontal strips

We compare mirror TBA and hybrid NLIE in the horizontal part of the \(\mathfrak{psu}(2, 2|4)\)-hook for the twisted \(\text{AdS}_5 \times S^5\). We will consider only the states which are invariant under the interchange \(s \to -s\) of the \(\mathfrak{psu}(2, 2|4)\)-hook.

The simplified TBA equation for \(Y_{1|w}\) and \(Y_{M|w} (M \geq 2)\) can be written as

\[
\log Y_{1|w} = -V_{1|w} + \log(1 + Y_{2|w}) \ast s_K + \log \frac{1 - \frac{1}{Y_{1|w}}}{1 + \frac{1}{Y_{1|w}}} \ast s_K, \tag{2.4}
\]

\[
\log Y_{M|w} = -V_{M|w} + \log(1 + Y_{M+1|w}) \ast s_K + \log(1 + Y_{M-1|w}) \ast s_K, \tag{2.5}
\]

where \(V_{M|w}\) is the source term, which depends on the state and the values of \((\alpha, g)\) under consideration. In the hybrid NLIE, the \(1 + Y_{M+1|w}\) on the right hand side is replaced by

\[
1 + Y_{2|w} = (1 + a^{\nu[+\gamma]}_3)(1 + \overline{a}^{\nu[-\gamma]}_3), \quad 1 + Y_{M+1|w} = \log(1 + a^{\nu[+\gamma]}_{M+2})(1 + \overline{a}^{\nu[-\gamma]}_{M+2}). \tag{2.6}
\]

The pair of parameters \(\{a^{\nu}_s, \overline{a}^{\nu}_s\} (s \geq 3)\) are determined by \(A_1\) NLIE,

\[
\log a^{\nu}_s = -J^{\nu}_s + \log(1 + a^{\nu}_s) \ast K_f - \log(1 + \overline{a}^{\nu}_s) \ast K^{[-2+2\gamma]}_s + \log(1 + Y_{s-2|w}) \ast s_{K}^{-\gamma}, \tag{2.7}
\]

\[
\log \overline{a}^{\nu}_s = -J^{\nu}_s + \log(1 + \overline{a}^{\nu}_s) \ast K_f - \log(1 + a^{\nu}_s) \ast K^{[-2+2\gamma]}_f + \log(1 + Y_{s-2|w}) \ast s_{K}^{[+\gamma]}, \tag{2.8}
\]

where \(\nu\) is I or II, and \(\gamma (0 < \gamma < 1)\) is a regularization parameter. We leave \(s \in \mathbb{Z}_{\geq 3}\) unspecified, though one can substitute \(s = 3\) at any time.

The case of \(\nu = I\) is simpler than \(\nu = II\), because the source terms \(\{J^I_3, J^I_3\}\) vanishes in the Konishi state of the untwisted \(\text{AdS}_5 \times S^5\) model, at least asymptotically. Below we consider the case \(a^I_s, \overline{a}^I_s\) only, and omit \(\nu = I\)\(\] \(\] \(\]

Critical lines and analyticity. The source terms in TBA or NLIE change discontinuously as we vary the parameters \((\alpha, g)\). We divide the \((\alpha, g)\) plane into subregions according to different from of the source terms. The boundary of subregions is called critical lines. We denote the critical lines by \(\alpha = a^{(i)}_{\text{cr}}(g)\) or \(g = g^{(i)}_{\text{cr}}(\alpha)\).

\[\]Since the variables \(\{a^{\nu}_s, \overline{a}^{\nu}_s\}\) depend on the choice of frame for Wronskian Q-functions, it makes little sense to consider the equations for general \(\nu\). See the discussion at the end of Appendix \(\] \(\]
The critical lines are different for different integral equations of TBA or NLIE. So the phase space of a given state in the twisted $AdS_5 \times S^5$ is divided into infinitely many tiny regions as

$$g^{(I)}_{cr}(\alpha) = \bigcup_{(a,s) \in T-hook} g^{(i)}_{cr}(\alpha)[Y_{a,s}]$$

for TBA, (2.9)

$$g^{(I)}_{cr}(\alpha) = \bigcup_{(a,|s|\leq 2) \in T-hook} g^{(i)}_{cr}(\alpha)[Y_{a,s}] \cup \left\{g^{(j)}_{cr}(\alpha)[a_3, \bar{a}_3]\right\}$$

for hybrid NLIE. (2.10)

The critical lines, or discontinuous changes of source terms, come from the change of the analyticity of unknown variables in a given integral equation. This statement holds true for both simplified TBA and NLIE. The TBA for the orbifold Konishi has already been studied in detail [50], so we will make this statement more precise for the NLIE.

2.3 Source terms of $A_1$ NLIE

We determine the source terms in $A_1$ NLIE ($J_s, \bar{J}_s$), by taking examples of the twisted ground state and orbifold Konishi state.

Source term of twisted ground state. The ground state of the twisted $AdS_5 \times S^5$ satisfies the mirror TBA with $V_{M|w} = 0$ [30]. It also satisfies the hybrid NLIE with the chemical potential

$$J_s = -i\alpha, \quad \bar{J}_s = +i\alpha.$$ (2.11)

This result follows immediately from the asymptotic solution discussed in Appendix B. Even for excited states, each term in the $A_1$ NLIE approaches its ground state value in the limit $v \rightarrow \pm \infty$, just like TBA. Furthermore, the orbifold Konishi state satisfies the same equation at small $\alpha \neq 0$ and small $g$. For general $(\alpha, g)$ we should add logarithms of S-matrix to the source term.

Main strip of hybrid NLIE. Before studying source terms at general $(\alpha, g)$, let us discuss the main strip of the mirror TBA or the hybrid NLIE. The main strip is defined by the region of complex plane in which the respective equation remains valid without modification. It is helpful to identify the main strip in advance, because the critical lines are often related to the movement of extra zeroes going in or out of this strip.

The main strip of the simplified TBA for $Y_{M|w}$ (2.4), (2.5) is $A_{-1,1}$, because we encounter the singularity of $s_K$ along the boundary of $A_{-1,1}$. Analytic continuation of the simplified TBA beyond $A_{-1,1}$ requires us to add an extra term $\sim \log(1 + Y^\pm)$ for some $Y$.

The main strip for the hybrid NLIE is smaller than that of the simplified TBA. Consider the holomorphic part of $A_1$ NLIE (2.7), which contains the kernels $K_f, K_f^{+2-2\gamma}, s_K^{-\gamma}$. Since
these kernels are singular at \( K_f(\pm 2i/g) \) and \( s_K(\pm i/g) \), the main strip of (2.7) is

\[
\text{Im} \, v \in \left( -\frac{1 - \gamma}{g}, \frac{2\gamma}{g} \right) \quad (0 \leq \gamma \leq 1).
\]

(2.12)

The main strip of the anti-holomorphic part of \( A_1 \) NLIE (2.8) is the complex conjugate of the above result.

**Source terms of orbifold Konishi.** We describe the source terms of hybrid NLIE for \((a_s, \bar{a}_s)\) describing the asymptotic orbifold Konishi state at general \((\alpha, g)\). One can check all these results explicitly by using the formulae in Appendix [3].

The holomorphic part of \( A_1 \) NLIE (2.7) consists of the dynamical variables

\[
\begin{align*}
    b_s &= \frac{Q[s+1]}{\bar{Q}^{1-s}} T_{1,s-1}^{1-s} L[s+1], \\
    1 + b_s &= \frac{Q[s-1]}{\bar{Q}^{1-s}} T_{1,s-1}^{1-s} L[s+1], \\
    1 + \bar{b}_s &= \frac{\bar{Q}[s-1]}{Q^{1-s}} T_{1,s}^{-1} L[-s-1], \\
    1 + Y_{s-2|w} &= \frac{T_{1,s-1}^{-1} T_{1,s-1}^{1} T_{2,s-1}^{-1} T_{0,s-1}^{-} T_{2,s-1}^{1} T_{0,s-1}^{1}}{T_{1,s-1}^{1-s} L[s-1]}.
\end{align*}
\]

(2.13)

The variables \( a_s, \bar{a}_s \) and \( b_s, \bar{b}_s \) are related by

\[
\begin{align*}
    a_s(v) &= b_s \left( v - \frac{i\gamma}{g} \right), \\
    \bar{a}_s(v) &= \bar{b}_s \left( v + \frac{i\gamma}{g} \right),
\end{align*}
\]

(0 < \( \gamma < 1 \)).

(2.14)

As long as the asymptotic orbifold Konishi state is concerned, neither \( Q \)- nor \( L \)-functions have singularities around the real axis, and all critical behaviors come from the extra zeroes of \( T \)-functions, \( T_{1,s-1} \) and \( T_{1,s} \), inside the main strip (2.12). Since the location of extra zeroes is determined by the values of \((\alpha, g)\), the critical lines \( \alpha_{cr}(g) \) are defined by

\[
\begin{align*}
    T_{1,s-1} \left( -\frac{i}{g} \right) = 0 \quad \text{or} \quad T_{1,s} \left( -\frac{i(1 - \gamma)}{g} \right) = 0 \quad \text{at} \quad \alpha = \alpha_{cr}(g).
\end{align*}
\]

(2.15)

The solution to the equations \( T_{1,Q}(\frac{-i}{g}) = 0 \) also defines the critical lines of the mirror TBA for the twisted AdS\( _5 \times S^5 \), and their asymptotic solutions have been studied in [50]. The first equation of (2.13) has \( s-1 \) solutions and the second has \( s \) solutions for \( 0 < g \lesssim 1 \). We denote them by \( \alpha_{s-1,i}(g) \), \( \alpha_{s,i}(g, \gamma) \) with the ordering

\[
\begin{align*}
    0 < \alpha_{s-1,1}(g) &< \frac{\pi}{s - 1} < \alpha_{s-1,2}(g) < \frac{2\pi}{s - 1} < \cdots < \frac{(s - 2)\pi}{s - 1} < \alpha_{s-1,s-1}(g) < \pi, \\
    0 < \alpha_{s,1}(g, \gamma) &< \frac{\pi}{s} < \alpha_{s,2}(g, \gamma) < \frac{2\pi}{s} < \cdots < \frac{(s - 1)\pi}{s} < \alpha_{s,s}(g, \gamma) < \pi.
\end{align*}
\]

(2.16)

\( ^6 \)The equation \( T_{1,Q}(\frac{-1}{g}) = 0 \) has more asymptotic solutions for \( g > 1 \), which are called Type II and Type III critical behaviors in [50].
It is instructive to keep track of the zeroes of \( T_{1,Q} \) in detail, as they behave in an interesting way when \( \alpha \) is around \( \frac{n\pi}{Q} \) for \( n \in \mathbb{Z}, \ 1 \leq n \leq Q - 1 \). If \( \alpha \) is slightly less than \( \frac{n\pi}{Q} \), \( T_{1,Q} \) has no zeroes around the real axis. Let us increase \( \alpha \). When \( \alpha \) reaches \( \frac{n\pi}{Q} \), then \( T_{1,Q} \) acquires a pair of real zeroes at \( \pm \infty \). The pair of zeroes run toward the origin along the real axis as \( \alpha \) increases, and collide at the origin. After the collision, they run along the imaginary axis in the opposite directions towards \( \pm i\infty \). They cross \( \pm \frac{i}{g} \) at \( \alpha = \alpha_{\text{cr}}^{(i)} \). There are exceptions at \( \alpha = 0, \pi \).

In the limit \( \alpha \to 0 \), a pair of zeroes of \( T_{1,Q} \) run to \( \pm \infty \) along the real axis. Nothing happens around \( \alpha = \pi \). As for \( \alpha \in (\pi, 2\pi) \) the movement of zeroes is symmetric with respect to the flip \( \alpha \to \pi - \alpha \).

Let us define the interval

\[
I_{s-1}(g) \equiv \bigcup_{n=1}^{s-1} \left( \frac{(n-1)\pi}{s}, \alpha_{s-1,n}(g) \right), \quad I_s(g, \gamma) \equiv \bigcup_{n=1}^{s} \left( \frac{(n-1)\pi}{s}, \alpha_{s,n}(g, \gamma) \right). \tag{2.17}
\]

Whenever \( \alpha \) crosses the boundary of the interval \( I_{s-1}(g) \cup I_s(g, \gamma) \), the source terms of hybrid NLIE \( (J_s, \bar{J}_s) \) change discontinuously. The \( (J_s, \bar{J}_s) \) at fixed \( (\alpha, g) \) are given explicitly as follows. Start from the source terms for the grounds state (2.11). If \( \alpha \in I_{s-1}(g) \), add \( (j_B, \bar{j}_B) \) to \( (J_s, \bar{J}_s) \); and then if \( \alpha \in I_s(g, \gamma) \), add \( (j_C, \bar{j}_C) \) to \( (J_s, \bar{J}_s) \), where \( j_B, \bar{j}_B, j_C, \bar{j}_C \) are defined by

\[
j_B(v) = \sum_j \log S_f \left( v - b_j + \frac{i(1-\gamma)}{g} \right), \quad \bar{j}_B(v) = -\sum_j \log S_f \left( v - b_j - \frac{i(1-\gamma)}{g} \right), \tag{2.18}
\]

\[
j_C(v) = \sum_j \log S \left( v - c_j + \frac{i(1-\gamma)}{g} \right), \quad \bar{j}_C(v) = -\sum_j \log S \left( v - c_j - \frac{i(1-\gamma)}{g} \right), \tag{2.19}
\]

where \( b_j, c_j \) are defined as the zeroes of dynamical variables:

\[
1 + a_s \left( b_j - \frac{i(1-\gamma)}{g} \right) = 1 + \bar{a}_s \left( b_j + \frac{i(1-\gamma)}{g} \right) = 0, \quad b_j \in \mathcal{A}_{-1+\gamma,1-\gamma}, \tag{2.20}
\]

\[
1 + Y_{s-2|w} \left( c_j - \frac{i}{g} \right) = 0, \quad c_j \in \mathcal{A}_{-1,1}. \tag{2.21}
\]

One can derive the critical lines of (2.15) from these results, by recalling that \( (1 + a_s), (1 + \bar{a}_s) \) are related to \( T_{1,s} \), and \( 1 + Y_{s-2|w} \) is related to \( T_{1,s-1} \). It will turn out in Section 3 that each term of (2.18), (2.19) can be explained by the contour deformation trick of the NLIE (2.7), where the deformed contour runs through the lower half plane.

One remark is needed to evaluate the integrals in TBA and NLIE correctly in a numerical way. Consider the convolutions \( \log(1 + a_s) * K_f - \log(1 + \bar{a}_s) * K^{[1+2-2\gamma]}_f \) in (2.7). If \( (1 + a_s) \)

\[7\]Recall that \( s = 3 \) is the minimum choice of hybrid NLIE. In contrast, the phase space \( (\alpha, g) \) of the mirror TBA for orbifold Konishi state is classified partially by \( \cup_{s=1}^{\infty} I_s(g) \), which consists of infinitely many segments of the width \( \sim \frac{\pi}{s} \) for each \( s \).
crosses the branch cut of logarithm running the negative real axis, then the integrand changes discontinuously. Suppose there exists \( v_d \in \mathbb{R} \) such that
\[
\text{Im} [1 + a_s(v_d)] = 0 \quad \text{with} \quad \text{Re} [1 + a_s(v_d)] < 0.
\] (2.22)
Then we need to integrate \( \log(-1) = \pm \pi i \) over \((v_d, \infty)\) or \((-\infty, v_d)\), which provides extra source terms. As for asymptotic Konishi state, whenever \((1 + a_s)\) crosses the branch cut of logarithm, then \((1 + \bar{a}_s)\) crosses the branch cut at the same point. Thus we get
\[
\Delta J_s = -\log \left[ S_f(v - v_d)S_f(v - v_d + \frac{2i(1 - \gamma)}{g}) \right] - 2\pi i, \quad (2.23)
\]
\[
\Delta \bar{J}_s = +\log \left[ S_f(v - v_d)S_f(v - v_d - \frac{2i(1 - \gamma)}{g}) \right] + 2\pi i. \quad (2.24)
\]
The discontinuity of logarithm can in principle happen for the integral with \( \log(1 + Y_{s-2|w}) \).

3 Contour deformation trick for TBA and NLIE

In the last section we studied the ground and orbifold Konishi states in the twisted AdS_5 \times S^5, in which the hybrid NLIE acquires source terms. In this section, we turn our attention to the structure of the source term for general states. It is known that the origin of the source term in the simplified TBA for general states can be explained by both integration of Y-system and contour deformation trick. This is no longer trivial in hybrid NLIE, as we shall see below.

3.1 General source terms in the simplified TBA

Take the simplified TBA for \( Y_{1|w} \) as an example, and the following discussion applies to other simplified TBA equations as long as the Y-system exists at that node. We will derive the source terms by integration of Y-system and contour deformation trick.

The explanation by integration of Y-system goes as follows\(^8\). Consider the logarithmic derivative of Y-system for \( Y_{1|w} \)
\[
dl \left[ Y_{1|w}^- Y_{1|w}^+ \right] = \ndl \left[ (1 + Y_{2|w}) \left( \frac{1 - \frac{1}{y}}{1 - \frac{1}{y_v}} \right) \right], \quad \ndl f(v) \equiv \frac{\partial}{\partial v} \log f(v). \tag{3.1}
\]
Suppose \( Y_{1|w}(v) \) has a set of single zeroes \( r_j \) inside the strip \( \mathcal{A}_{-1,1} \). If we take the convolution of (3.1) with \( s_K \), the left hand side becomes
\[
\int_{\mathbb{R}} dt \frac{\partial}{\partial t} \log \left[ Y_{1|w}(t^-)Y_{1|w}(t^+) \right] s_K(v - t) = \ndl Y_{1|w}(v) + 2\pi i \sum_j s_K \left( v - r_j - \frac{i}{g} \right). \tag{3.2}
\]
\(^8\)This explanation is also called TBA lemma in the literature.
If we integrate both sides with respect to $v$, we obtain the simplified TBA equation (2.4) with

$$V_{1|w} = v_{1|w} - \sum_j \log S \left( v - r_j - \frac{i}{g} \right),$$

(3.3)

where $v_{1|w}$ is an integration constant fixed by the behavior $v \to \pm \infty$, where all $Y$-functions approach the ground state value.

The explanation by contour deformation trick goes as follows. We start from the simplified TBA equation (2.4) for the ground state, $V_{1|w} = v_{1|w}$. To obtain the TBA equation for excited states, we regard the contour of integration in the right hand side of (2.4) as running somewhere far below in the complex plane. When we pull the deformed contour back to the real axis, we obtain additional terms by picking up the residues.

Let $\{\rho_n\}$ be a set of roots $Y_{1|w}(\rho_n) = 0$, where $\rho_n \in A_{n-1,n}$ for $n \geq 1$ and $\rho_n \in A_{n,n+1}$ for $n \leq -1$. From the $Y$-system (3.1) it follows that

$$1 + Y_{2|w}(\rho_n^\pm) = 0 \quad \text{or} \quad 1 - \frac{1}{Y_{-}(\rho_n^\pm)} = 0 \quad \text{or} \quad 1 - \frac{1}{Y_{+}(\rho_n^\pm)} = \infty, \quad n \in \mathbb{Z} \neq 0.$$  

(3.4)

When we straighten the deformed contours of (2.4) running through the lower half plane, the source term $V_{1|w}$ becomes

$$V_{1|w} = v_{1|w} + \log S \left( v - \rho_1^{-} \right) + \log S \left( v - \rho_{-1}^{-} \right).$$

(3.5)

where the contributions from $\rho_{-n}$ ($n \geq 2$) vanish owing to $S^{-}S^{+} = 1$. This result agrees perfectly with (3.3).

In short, there is no constraint on T- or Y-functions from the mirror TBA.

### 3.2 General source terms in $A_1$ NLIE

#### 3.2.1 Fourier transform method

The $A_1$ NLIE was derived from the assumptions that $Q^{[s-2]} , L^{[s]}$ are analytic in the upper half plane, and $\overline{Q}^{[2-s]} , \overline{L}^{[-s]}$ are analytic in the lower half plane [18]. This derivation can be generalized to the case where dynamical variables have zeroes or poles in the complex plane:

$$T_{1,s}(t_{s,n}) = T_{1,s}(t_{s,-n}) = Q(q_n) = \overline{Q}(\overline{q}_n) = L(\ell_n) = \overline{L}(\overline{\ell}_n) = 0,$$

$$\{t_{s,n} , q_n , \ell_n\} \in A_{n-1,n}, \quad \{t_{s,-n} , \overline{q}_n , \overline{\ell}_n\} \in A_{-n,-n+1}, \quad (n \geq 1).$$

(3.6)

---

9Note that $\log \frac{1 - \frac{1}{v-i0}}{1 - \frac{1}{v+i0}} \ast s_K = \log \frac{1 - \frac{1}{v-i0}}{1 - \frac{1}{v+i0}} \ast s_K$ owing to $Y_{-}(v-i0) = Y_{+}(v+i0)$ for $v \in (-\infty, -2) \cup (+2, +\infty)$.

10There can be multiple roots as well as poles inside the same strip of the complex plane. It is straightforward to generalize the whole argument for such cases.

11The Fourier transform of logarithmic derivative diverges if these functions have zeroes on the line $g \Im v \in \mathbb{Z}$. We should regularize this by shifting the zeroes slightly upward or downward.
In general, these functions can have multiple zeroes or poles in the complex plane. The generalization for such case is straightforward; if they have poles, the logarithmic derivative have the residue with the opposite sign. For simplicity we do not discuss poles.

The whole derivation is discussed in Appendix \[C.1]\). Eventually we obtain the derivative of the source terms \(J_s\) appearing in the hybrid NLIE \((2.7)\) as

\[
J'_s = J'_{s,T} + J'_{s,L} + J'_{s,Q} + J'_{s,q},
\]

where

\[
\begin{align*}
\left. \frac{J'_s}{2\pi i} \right|_T &= -K_f(v-t_{s,1}^-) - K_f(v-t_{s,-1}^-) - s_K(v-t_{s-1,1}^-) - s_K(v-t_{s-1,-1}), \\
\left. \frac{J'_s}{2\pi i} \right|_L &= -\sum_{n=-\infty}^{\infty} \left\{ K_f(v-\ell_{s+n+1}^{[s-1]}) + s_K(v-\ell_{s+n}^{[s-2]}) \right\}, \\
\left. \frac{J'_s}{2\pi i} \right|_Q &= \sum_{n=1}^{\infty} K_1(v-q_{s+n-1}^{[s-2]}), \\
\left. \frac{J'_s}{2\pi i} \right|_Q &= \sum_{n=1}^{\infty} K_1(v-q_{s+n-1}^{[s-1]}) - \delta(v-q_{s+n-1}^{[s-1]}).
\end{align*}
\]

### 3.2.2 Contour deformation trick in \(A_1\) NLIE

We start from the \(A_1\) NLIE for the ground state with constant source terms \((J_s, J_s) = (j_s, j_s)\). Then we apply the contour deformation trick to obtain extra source terms. We use the deformed contour as that of the orbifold Konishi state, depicted in Figure 1. For the NLIE of \(a_s\), it runs slightly above the line \(Im\ v = (-s + 1 + \gamma)/g\), and run down along the imaginary axis. Note that the integrands have branch cut discontinuity along the line \(Im\ v = (1 - s + \gamma)/g\).

Again we throw the details of computation in Appendix \[C.2]\). After straightening the contour we obtain the following result:

\[
J'^{\text{CDT}}_s = j_s - \log \left( S_{f}(v-t_{s,1}^-) S_{f}(v-t_{s,-1}^-) \right) - \log \left( S(v-t_{s-1,1}^-) S(v-t_{s-1,-1}^-) \right) - \log \left( \prod_{j=s+2}^{2s} S_{f}(v-\ell_{j}^{[s-1]}) \prod_{j=s+1}^{2s-2} S(v-\ell_{j}^{[s+1]}) \right) + \log \left( \prod_{j=3}^{s+1} S_{f}(v-\ell_{j}^{[s-1]}) \prod_{j=3}^{s} S(v-\ell_{j}^{[s-1]}) \right) - \log \prod_{j=1}^{s-1} S_{1}(v-q_{j}^{[s-1]}) + \log \prod_{j=s}^{2s-2} S_{1}(v-q_{j}^{[s-2]}).
\]  

(3.11)
Figure 1: The deformed contour used in the NLIE for $b_s$ for the orbifold Konishi state. (Left) the contour in $z$-torus, where the vertical and horizontal axes are normalized by the period of the rapidity torus with moduli $k = -4g^2/Q^2$, with $Q = s - 1$ for $(1 + b_s)$, $(1 + \overline{b_s})$ and $Q = s - 2$ for $(1 + Y_{s-2|w})$. The real line in $z$-torus corresponds to the real axis of the mirror $v$-plane, and the line $\text{Im} z = -1$ corresponds to the real axis of the string $v$-plane. We assumed that there are no singularities like Bethe roots along the string real axis. (Right) the contour in $v$-plane, where the orange region corresponds to the region surrounded by the deformed contour and the mirror real axis.

3.2.3 Comparison

Let us compare the Fourier transform of the derivative of the source terms (C.19) (Fourier source terms), with the source terms predicted by the contour deformation trick (3.11) (CDT source terms). We can make a similar argument for the NLIE of $\overline{b_s}$. Since this is complex conjugate to $b_s$, we just have to impose the complex-conjugate constraints in addition.

It turns out that there are mismatches in two results. Let us have a closer look for each of the T, L, Q-functions.

**T-functions.** The Fourier source terms (3.8) agree with the first line of the CDT source terms (3.11).

This agreement forbids further *ad hoc* modification to the path of the deformed contour of the NLIE for $b_s$. If part of the deformed contour collects the residues of T-functions in the upper half plane, it spoils the above nice agreement and contradicts with the lessons from the orbifold Konishi.
L-functions. The Fourier source terms \((3.9)\) partially agree with the second line of the CDT source terms \((3.11)\).

The terms with \(\{\ell_m\}\) agree with each other if \(\ell_{m \geq 2s-1}\) lie along the imaginary axis in the lower half plane, so that all of them are picked up by the deformed contour.

The terms with \(\{\ell_m\}\) do not agree, because they have the opposite signs. Moreover, the roots \(\{\ell_m\}\) in \((3.9)\) lie in the upper half plane, while those in \((3.11)\) lie in the lower half plane.

Q-functions. The Fourier source terms \((3.10)\) partially agree with the third line of the CDT source terms \((3.11)\).

If the deformed contour pick up all \(\{q_m\}\), then the terms with \(\{q_m\}\) perfectly agree with each other.

The terms with \(\{q_m\}\) disagree. The roots \(\{q_{\pm s+n-1}\}\) \((n \geq 2)\) in \((3.10)\) lie in the upper half plane, while those in \((3.11)\) lie in the lower half plane. The corresponding source terms have the opposite signs. One exception is \(q_{s-1}\) in the Fourier source term \((3.10)\). It lies in the lower half plane, but this term is not present in the CDT source term \((3.11)\).

The mismatch of the source terms can be explained by different analyticity conditions used in two methods, as summarized in Table 1. In particular, the extra zeroes of \(Q(v)\) at \(v \in A_{0,s-2}\) and those of \(L(v)\) at \(v \in A_{2,s}\) modify only the CDT source terms.

| FT  | \(Q^{[s-2]}\), \(L^{[s]}\) are meromorphic in the upper half plane. |
|-----|---------------------------------------------------------------------|
| CDT | \(Q\), \(L^{[s]}\) are meromorphic in the upper half plane. |

Table 1: Analyticity conditions used in the Fourier transformation method and the contour deformation trick. The complex conjugate conditions for \(\overline{Q}, \overline{L}\) are also used. We make no assumptions about \(Q\), \(L^{[s]}\) in the lower half plane, \(\overline{Q}, \overline{L}^{[s]}\) in the upper half plane.

Here is a summary of the results. Concerning the source terms in the \(A_1\) NLIE for \(b_s\), we find agreement for the extra terms from \(T, \overline{T}, \overline{Q}\), disagreements for those from \(L, Q\). Therefore, if we demand that the CDT source terms agree with the Fourier source terms, then \(L^{[s]}\), \(Q\) should not have any zeroes or poles in the upper half plane. We can make similar arguments for the NLIE for \(\overline{b}_s\). Since this is complex conjugate to the NLIE for \(b_s\), we find the complex-conjugate constraints: \(\overline{T}^{[s]}\), \(\overline{Q}\) should be analytic and nonzero in the lower half plane.

Strictly speaking, the \(T, L, Q\)-functions may have singularities which can be simultaneously removed by gauge transformation. We forbid such gauge artifacts, and assume that the roots

---

12 We may neglect the delta function \(\delta(v - \ell_{s+1}^{[-s-1]})\), because it just changes integration constant after integration.

13 We neglect the delta function as before.
\{t_{s,n}, \ell_n, q_n, \bar{\ell}_n, \bar{q}_n\} are independent. In other words, the contour deformation trick works fine as long as one can choose a gauge such that all zeroes or poles can be associated to the T-functions rather than the L- and Q-functions.

It is actually possible to rescue the disagreement of source terms by using the most complicated deformed contour, which yields weaker constraints. This contour has to pick up the residues only from $L^{[+2]}, Q$ in the upper half plane, and the residues only from $T, \overline{L}^{[-2]}, \overline{Q}$ in the lower half plane. We should not collect any source terms from $T, \overline{L}^{[-2]}, \overline{Q}$ in the upper half plane, nor from $L^{[+2]}, Q$ in the lower half plane, because it spoils the nice agreement as shown above. This prescription works as long as the zeroes of poles of $T, \{L^{[+2]}, Q\}, \{\overline{T}^{[-2]}, \overline{Q}\}$ do not collide for any value of the coupling constant. If the singularities are found in the same position, one cannot separate one residue from others. Such situations are forbidden if we demand the validity of contour deformation trick.

4 Conclusion

In this paper we generalized the hybrid NLIE of \cite{18} and applied it to a wider class of states.

First, we studied the ground and the orbifold Konishi states of twisted AdS$_5 \times S^5$. In the mirror TBA, the orbifold Konishi states have infinitely many asymptotic critical lines from $Y_{M|w}$ nodes. In the hybrid NLIE, the number of critical lines is indeed reduced to a finite number\footnote{As long as the sl(2) sector is concerned, this conclusion is expected because the exact truncation method of \cite{70} can be applied without modification.} The quantization condition for the extra zeroes is written in terms of NLIE variables $(a_s, \overline{a}_s, Y_{1,s-1})$.

Second, we derived the source terms of hybrid NLIE for general states in two ways, Fourier transform method and contour deformation trick. By demanding that the contour deformation trick gives the correct answer, we obtained constraints on the analyticity of Q-functions.

It is interesting to generalize the gauge-invariant NLIE to $A_n$ cases. The $SU(N)$ principal chiral models contain boundstate spectrum for $N \geq 3$, and its NLIE has been studied in \cite{71}. We should be able to reproduce their results by $A_2$ NLIE and contour deformation trick.

While this paper is in preparation, hybrid NLIE of AdS$_5 \times S^5$ made out of $A_1$ and $A_3$ NLIE coupled to the quasi-local formulation of the mirror TBA \cite{72} has appeared in \cite{73}. We expect that the contour deformation trick will also work to obtain this new NLIE for excited states.

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A Notation

We follow the notation of [16] [18]:

\[ x_s(v) = \frac{v}{2} \left( 1 + \sqrt{1 - \frac{4}{v^2}} \right), \quad x(v) = \frac{1}{2} \left( v - i\sqrt{4 - v^2} \right). \]

\[ R(\pm)(v) = \prod_{j=1}^{K} \frac{x(v) - x_{s,j}^\pm}{\sqrt{x_{s,j}^\pm}}, \quad B(\pm)(v) = \prod_{j=1}^{K} \frac{\frac{1}{2} x(v) - x_{s,j}^\pm}{\sqrt{x_{s,j}^\pm}}, \tag{A.1} \]

together with \( f^{[\pm m]} = f(v \pm \frac{im}{g}) \) and \( f(v)^\pm = f(v)^{[\pm 1]} \). The complex rapidity plane are divided into the strips,

\[ \mathcal{A}_{m,n} = \left\{ v \in \mathbb{C} \mid \text{Im} v \in \left( \frac{m}{g}, \frac{n}{g} \right) \right\}. \tag{A.2} \]

We use the following kernels and S-matrices:

\[ s_K(v) = \frac{1}{2\pi i} \frac{d}{dv} \log S(v), \quad S(v) = -\tanh[\frac{\pi}{4}(vg - i)], \]

\[ K_Q(v) = \frac{1}{2\pi i} \frac{d}{du} \log S_Q(v), \quad S_Q(v) = \frac{v - i\frac{Q}{g}}{u + \frac{iQ}{g}}. \tag{A.3} \]

\[ K_f(v) = \frac{1}{2\pi i} \frac{\partial}{\partial v} \log S_f(v), \quad S_f(v) = \frac{\Gamma \left( \frac{Q}{4\pi} (v + \frac{2i}{g}) \right) \Gamma \left( -\frac{4}{4\pi} (v) \right)}{\Gamma \left( \frac{Q}{4\pi} \right) \Gamma \left( -\frac{4}{4\pi} (v - \frac{2i}{g}) \right)}. \tag{A.4} \]

One can check the properties \( S^+S^- = 1 \) and \( S^-S_f^+ = K_1 \).

The convolutions are defined by

\[ F \ast K(v) = \int_{-\infty}^{\infty} dt f(t) K(v - t), \quad F \ast K(v) = \int_{-2}^{2} dt f(t) K(v - t). \tag{A.5} \]

The logarithmic derivative and its Fourier transform are defined by

\[ d\ell X(v) \equiv \frac{\partial}{\partial v} \log X(v), \quad \widehat{d\ell X}(k) \equiv \int_{-\infty}^{+\infty} dv e^{ikv} \frac{\partial}{\partial v} \log X(v). \tag{A.6} \]

We also use \( D_k = e^{k/g} \) and \( s_K = 1/(D_k + D_k^{-1}) \). It is useful to keep in mind that the operator \( D_k \) shifts the location of zeroes,

\[ D^n_k e^{ikq} = e^{ikq^{[-n]}} = e^{ikq^{[-n]}} , \quad D^{-n}_k e^{ik\bar{q}} = e^{ik\bar{q}^{[+n]}} . \tag{A.7} \]

Another useful formulae are\[^{15}\]

\[ \text{FT}^{-1} \left[ \theta(-k) D_k^{\pm n} \frac{D_k - D_k^{-1}}{D_k + D_k^{-1}} e^{ikq} \right] = -K_f(v - q^{[-n]}) - s_k(v - q^{[1-n]}) , \]

\[ \text{FT}^{-1} \left[ \theta(-k) D_k^{-\pm n} \frac{D_k - D_k^{-1}}{D_k + D_k^{-1}} e^{ikq} \right] = +K_f(v - q^{[+n]}) + s_k(v - q^{[1-n]}) . \tag{A.8} \]

\[^{15}\text{The symbol } \text{FT}^{-1} \text{ means the inverse Fourier transform, } \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikv}.\]
The q-number is defined by
\[
[s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}, \quad q = e^{i\alpha}.
\] (A.9)

## B Twisted asymptotic data

Below we summarize the data to solve the mirror TBA and hybrid NLIE for twisted AdS$_5 \times$S$^5$ in the asymptotic limit. In particular, we need the twisted transfer matrices written in the form of Wronskian to solve the hybrid NLIE asymptotically.

The twisted transfer matrices of su$(2|2)$ symmetry can be constructed by the generating functional called quantum characteristic function \[69, 48\]. In particular, the quantum characteristic function $D_0$ generates $T_{1,s}$ through
\[
D_0 = (1 - U_0 T_1 U_0) (1 - U_0 T_2 U_0)^{-1} (1 - U_0 T_3 U_0)^{-1} (1 - U_0 T_4 U_0),
\]
\[
\equiv \sum_{s=0}^{\infty} (-1)^s U_0^s T_{1,s}(x_0^{[\pm s]}) U_0^s,
\] (B.1)
where $U_0$ is the shift operator acting on the mirror rapidity,
\[
U^s f(v) U^{-s} \equiv f\left(v + \frac{is}{g}\right) = f^{[s]}.
\] (B.2)

The $T_n$ are the components of the fundamental transfer matrix, $T_{1,1} = T_1 - T_2 - T_3 + T_4$, and they can be written as $[74, 16]$
\[
T_n = S_0 \tilde{T}_n, \quad S_0 = \prod_{j=1}^{K_{II}} \frac{y_j - x_0^-}{y_j - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}}, \quad \prod_{i=1}^{K_{I}} \frac{x_i^+ - x_i^-}{x_i^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}},
\] (B.3)
with
\[
\tilde{T}_1 = \prod_{j=1}^{K_{II}} \frac{\nu_j - v - \frac{i}{g}}{\nu_j - v + \frac{i}{g}} \prod_{i=1}^{K_{I}} \frac{1 - \frac{1}{x_0 x_i}}{1 - \frac{1}{x_0 x_i}} \sqrt{\frac{x_i^+}{x_i^-}}, \quad \tilde{T}_2 = e^{+i\alpha} \prod_{j=1}^{K_{II}} \frac{\nu_j - v - \frac{i}{g}}{\nu_j - v + \frac{i}{g}} \prod_{k=1}^{K_{III}} \frac{w_k - v + \frac{2i}{g}}{w_k - v},
\]
\[
\tilde{T}_3 = e^{-i\alpha} \prod_{k=1}^{K_{III}} \frac{w_k - v - \frac{2i}{g}}{w_k - v}, \quad \tilde{T}_4 = \prod_{i=1}^{K_{I}} \frac{x_i^+ - x_i^-}{x_i^+ - x_i^-} \sqrt{\frac{x_i^+}{x_i^-}},
\] (B.4)

where we used $x_0 = x(v), x_i = x_s(u_i), \nu_j = y_j + 1/y_j$, and introduced the twist by$^7$
\[
T_2 \to e^{i\alpha} T_2, \quad T_3 \to e^{-i\alpha} T_3.
\] (B.5)

$^1$We introduce $S_0$ since the transfer matrix is defined modulo overall scalar factor.

$^1$We rearranged the index $n = 1, 2, 3, 4$ from the one used in Section 2.1.
By expanding (B.1), we obtain
\[
T_{1,s} = \prod_{m=1}^{s} \left( -S_0^{[-s-1+2m]} \right) \cdot \left[ \tilde{\rho}_{s+1} - \tilde{T}_1^{[-s+1]} \tilde{\rho}_s^+ - \tilde{\rho}_s \tilde{T}_4^{[s+1]} + \tilde{T}_1^{-[-s+1]} \tilde{\rho}_{s-1} \tilde{T}_4^{[-s+1]} \right], \quad (B.6)
\]
\[
\tilde{\rho}_s = \prod_{m=1}^{s-1} \tilde{T}_2^{[-s+2m]} + \sum_{k=1}^{s-2} \left( \prod_{m=1}^{k} \tilde{T}_2^{[-s+2m]} \prod_{n=k+1}^{s-1} \tilde{T}_3^{[-s+2n]} \right) + \prod_{n=1}^{s-1} \tilde{T}_3^{[-s+2n]} \quad (s \geq 2). \quad (B.7)
\]

Together with \( S_0 \tilde{\rho}_1 = 1, \tilde{\rho}_0 = 0 \). Note that
\[
\prod_{m=1}^{s} \left( -S_0^{[-s-1+2m]} \right) = \prod_{j=1}^{K_1} y_j - x_0^{[-s]} \cdot \prod_{m=1}^{s} \left( -\frac{R_0^{[-s-2+2m]}}{R_0^{[-2}}} \right). \quad (B.8)
\]

The transfer matrices \( T_{1,s} \) (B.6) can be expressed as the Wronskian of \( Q \)-functions in the following way. Let us rewrite \( \tilde{\rho}_{s \geq 1} \) as
\[
\tilde{\rho}_s = \frac{U_3^{[-s-1]}}{U_2^{[-s]}} \sum_{k=0}^{s-1} \varrho^{[-s+1+2k]}, \quad \varrho \equiv \frac{U_2}{U_3}, \quad \tilde{T}_2 \equiv \frac{U_2^+}{U_3}, \quad \tilde{T}_3 \equiv \frac{U_3^+}{U_3}, \quad (B.9)
\]
and “differencize” the summation
\[
M_\rho^+ - M_\rho^- = \varrho \Rightarrow M_\rho^{[s]} - M_\rho^{[-s]} = \sum_{k=0}^{s-1} \varrho^{[-s+1+2k]}, \quad \tilde{\rho}_s = \frac{U_3^{[-s-1]}}{U_2^{[-s]}} \left( M_\rho^{[s]} - M_\rho^{[-s]} \right). \quad (B.10)
\]

After a little algebra, (B.6) becomes
\[
T_{1,s} = \prod_{m=1}^{s} \left( -S_0^{[-s-1+2m]} \right) \cdot \frac{U_3^{[-s-2]}}{U_2^{[-s-2]}} T_{1,s}, \quad T_{1,s} = \det \begin{pmatrix} Q^{[s]} & P^{[s]} \\ \overline{Q}^{[-s]} & \overline{P}^{[-s]} \end{pmatrix}, \quad (B.11)
\]
where
\[
Q^{[s]} = \tilde{T}_4^{[s-1]} - \tilde{T}_3^{[-s-1]} = \prod_{i=1}^{K_1} \left[ \frac{x_0^{[s]} - x_i^{-}}{x_0^{[s]} + x_i^{-}} - e^{-2\alpha} \prod_{k=1}^{K_1} \frac{w_i - v - i(s+1)g}{g} \right],
\]
\[
\overline{Q}^{[-s]} = \tilde{T}_4^{[1-s]} - \tilde{T}_3^{[-1-s]} = \prod_{j=1}^{K_1} \left[ \frac{v_j - v + i(s-1)g}{g} - e^{2\alpha} \prod_{k=1}^{K_1} \frac{w_i - v + i(s-1)g}{g} \right],
\]
\[
P^{[s]} = +\varrho^{[s]} \tilde{T}_4^{[s-1]} - Q^{[s]} M_\rho^{[s+1]},
\]
\[
\overline{P}^{[-s]} = -\varrho^{[-s]} \tilde{T}_4^{[-1-s]} - Q^{[-s]} M_\rho^{[-s-1]}.
\]

It follows that
\[
L^{[s]} \equiv \det \begin{pmatrix} Q^{[s]} & Q^{[s-2]} \\ P^{[s]} & P^{[s-2]} \end{pmatrix} = \frac{\varrho^{[s]} T_4^{[s-1]} T_3^{[-1-s]} Q^{[s]} - T_2^{[s-1]} Q^{[s-2]}}{T_3^{[s-1]} Q^{[s]} - T_3^{[-1-s]} Q^{[s-2]}},
\]
\[
\overline{L}^{[-s]} \equiv \det \begin{pmatrix} \overline{Q}^{[-s-2]} & \overline{Q}^{[-s]} \\ \overline{P}^{[-s-2]} & \overline{P}^{[-s]} \end{pmatrix} = \frac{\varrho^{[-s]} T_4^{[-1-s]} T_3^{[s-1]} Q^{[-s]} - T_2^{[s-1]} Q^{[-s-2]}}{T_3^{[s-1]} Q^{[-s]} - T_3^{[-1-s]} Q^{[-2-s]}}. \quad (B.13)
\]
A few remarks are in order. First, since our twist (B.5) affects $T_{1,s}$ only through $\rho_s$, the results (B.12) should formally agree with [75] modulo gauge transformation. Second, if wants to solve a couple of difference equations (B.9) explicitly for specific states, it is important to choose a good gauge for $T$-functions. Third, for the purpose of getting the asymptotic solution of the hybrid NLIE, we do not have to compute the second set of $Q$-functions ($P, \overline{P}$). Once we know $T_{1,s}, Q, \overline{Q}$, we obtain $L, \overline{L}$ by the $A_1$ $TQ$-relations, and they provide sufficient data to construct the gauge-invariant variables ($b_s, \overline{b}_s$). Fourth, as will be discussed in (B.22), there exists a gauge transformation of $T$-system which brings the first (or second) set of $Q$-functions to unity.

In the main text, we argued that there should always exist a gauge such that $Q^{[s-1]}, L^{[s]}$ are analytic and nonzero in the upper half plane, and $\overline{Q}^{[1-s]}, \overline{L}^{[-s]}$ are analytic and nonzero in the lower half plane. It is easy to check that $(Q, \overline{Q})$ in (B.12) satisfy this condition for the level-matched states with real $(u_i, w_k, \alpha)$, by using $\text{Im} x(v) < 0$ for $v \in \mathbb{C}$.

To see it, let us first remove the prefactor $\prod_{j=1}^{K_{II}} \frac{\nu_j - v + \frac{i(s-2)}{g}}{\nu_j - v + \frac{i}{g}}$ from $\overline{Q}$ by a gauge transformation, and consider the quantities inside the square brackets in (B.12). We find that $(Q, \overline{Q})$ do not have zeroes from the following observations:

$$\left| \frac{x(v) - x_i}{x(v) - x_i} \right| < 1 \quad (v \in \mathbb{C}), \quad \left| \frac{e^{-i\alpha} \prod_{k=1}^{K_{II}} (w_k - v - \frac{i}{g})}{w_k - v + \frac{i}{g}} \right| > 1 \quad (\text{Im } v > 0),$$

$$\left| \frac{x(v) - x_i}{x(v) - x_i} \right| < 1 \quad (v \in \mathbb{C}), \quad \left| \frac{e^{i\alpha} \prod_{k=1}^{K_{II}} (w_k - v + \frac{i}{g})}{w_k - v - \frac{i}{g}} \right| > 1 \quad (\text{Im } v < 0). \quad (B.14)$$

Their apparent poles at $v = w_k^\pm$ can be removed again by a gauge transformation.

**Transfer matrix for orbifold Konishi.** Consider the orbifold Konishi state. Since $K_{II} = K_{III} = 0$, it satisfies

$$\tilde{\rho}_s = \sum_{k=1}^{s} e^{i\alpha(s+1-2k)} = e^{i\alpha s} - e^{-i\alpha s} = [s]_q, \quad (B.15)$$

where $[s]_q$ is the q-number (A.9). The difference equations (B.9), (B.10) have the solution\(^{18}\)

$$U_2 = \frac{1}{U_3} = e^{\alpha gv/2}, \quad M_\rho = e^{\alpha gv} - \frac{1}{2i \sin \alpha}, \quad (\alpha \neq \pi \mathbb{Z}). \quad (B.16)$$

\(^{18}\)Linear difference equations can be solved by e.g. Fourier transform.
We added a constant to $M_\rho$ to keep the limit $\alpha \to 0$ non-singular. In summary, the asymptotic $Q$-functions for the orbifold Konishi state are given by

\[ Q^{[+s]} = \frac{\mathcal{R}^{[+s]}(-)}{\mathcal{R}^{[+s]}(+)} - e^{-i\alpha}, \quad \overline{Q}^{-[s]} = \frac{\mathcal{B}^{-[s]}(-)}{\mathcal{B}^{-[s]}(+)} - e^{+i\alpha}, \]

\[ P^{[+s]} = e^{\alpha(gv+is)} \frac{\mathcal{R}^{[+s]}(-)}{\mathcal{R}^{[+s]}(+)} - Q^{[+s]} M^{[+s+1]}_\rho, \quad \overline{P}^{-[s]} = -e^{\alpha(gv-is)} \frac{\mathcal{B}^{-[s]}(-)}{\mathcal{B}^{-[s]}(+)} - \overline{Q}^{-[s]} M^{[-s-1]}_\rho, \quad (B.17) \]

and the corresponding $T_{1,s}$ are

\[ T_{1,s} = e^{\alpha gv} \left( [s+1]_q - [s]_q \frac{\mathcal{R}^{[+s]}(-)}{\mathcal{R}^{[+s]}(+)} - [s]_q \frac{\mathcal{B}^{-[s]}(-)}{\mathcal{B}^{-[s]}(+)} + [s-1]_q \frac{\mathcal{R}^{[+s]}(+)}{\mathcal{B}^{-[s]}(-)} \right), \quad (B.18) \]

Finally, we define the $L$-functions as the solution of the $A_1$ TQ-relations,

\[
\begin{align*}
Q^{\nu-}_{1,s-1} T_{1,s} - Q^{\nu}_{1,s} T^{-}_{1,s-1} &= \overline{Q}^{\nu-}_{1,s-1} L_{1,s}, \quad \overline{Q}^{\nu+}_{1,s} T_{1,s} - Q^{\nu+}_{1,s} T^{-}_{1,s-1} = Q^{\nu+}_{1,s-1} \overline{L}_{1,s}, \\
(T_{1,s}, Q^{I}_{1,s}, Q^{II}_{1,s}, \overline{Q}^{I}_{1,s}, \overline{Q}^{II}_{1,s}, L_{1,s}, \overline{L}_{1,s}) &= (T_{1,s}, Q^{[+s]}, P^{[+s]}, \overline{Q}^{-[s]}, \overline{P}^{-[s]}, L^{[+s]}, \overline{L}^{-[s]}). \quad (B.19)
\end{align*}
\]

which yields

\[
\begin{align*}
L^{[+s]} &= e^{\alpha g(v+i\frac{(s-2)}{g})} \left( 1 + \frac{\mathcal{R}^{[+s]}(-)}{\mathcal{R}^{[+s]}(+)} \frac{\mathcal{R}^{[s-2]}(-)}{\mathcal{R}^{[s-2]}(+)} - 2 \cos \alpha \frac{\mathcal{R}^{[s-2]}(-)}{\mathcal{R}^{[s-2]}(+)} \right), \\
\overline{L}^{-[s]} &= e^{\alpha g(v-i\frac{(s-2)}{g})} \left( 1 + \frac{\mathcal{B}^{-[s]}(-)}{\mathcal{B}^{-[s]}(+)} \frac{\mathcal{B}^{[2-s]}(-)}{\mathcal{B}^{[2-s]}(+)} - 2 \cos \alpha \frac{\mathcal{B}^{[2-s]}(-)}{\mathcal{B}^{[2-s]}(+)} \right). \quad (B.20)
\end{align*}
\]

It also follows that

\[ T_{0,s} T_{2,s} = T^{+}_{1,s} T^{-}_{1,s} - T_{1,s-1} T_{1,s+1} = L^{+}_{1,s} \overline{L}^{-}_{1,s}. \quad (B.21) \]

Here is a caution for numerical computation. The Wronskian formulae can be numerically unstable at large $|v|$ due to the cancellation of two vectors $(Q, P) \sim (\overline{Q}, \overline{P})$. To avoid this problem we should use the analytic expression like $(B.18)$ instead of the Wronskian form $(B.11)$. This remark also applies to the $L$-functions $(B.20)$.

**Symmetry in $A_1$ TQ-relations.** The first line of $(B.19)$ is invariant under the holomorphic gauge transformation,

\[ T_{1,s} \rightarrow g^{[+s]}_1 g^{[-s]}_2 T_{1,s}, \quad Q^{\nu}_{1,s} \rightarrow g^{[+s]}_1 Q^{\nu}_{1,s}, \quad \overline{Q}^{\nu}_{1,s} \rightarrow g^{[-s]}_2 \overline{Q}^{\nu}_{1,s}, \quad (B.22) \]

provided that the $L$-functions transform as

\[ L^{+}_{1,s} \rightarrow g^{[s+1]}_1 g^{[s-1]}_1 L^{+}_{1,s}, \quad \overline{L}^{-}_{1,s} \rightarrow g^{[-s+1]}_2 g^{[-s-1]}_2 \overline{L}^{-}_{1,s}. \quad (B.23) \]
The TQ-relations are also invariant under the anti-holomorphic transformation,

\[ T_{1,s} \rightarrow g_{1}^{[+s]} g_{2}^{[-s]} T_{1,s}, \quad Q_{1,s}^{\nu} \rightarrow g_{2}^{[-s]} Q_{1,s}^{\nu}, \quad \overline{Q}_{1,s}^{\nu} \rightarrow g_{1}^{[+s]} \overline{Q}_{1,s}^{\nu}, \quad (B.24) \]

although it spoils the translational invariance of Q-functions, namely the second line of (B.19). The combination of two transformations (B.22), (B.24) generates a symmetry group larger than the usual gauge transformation of T-functions.

The Y-functions and the variables \((b_{s}, \overline{b}_{s})\) are invariant under both transformations:

\[
1 + b_{s} = \frac{Q_{1,s-1}^{\nu} T_{1,s}^{+}}{Q_{1,s-1}^{\nu} L_{1,s}^{+}}, \quad 1 + \overline{b}_{s} = \frac{\overline{Q}_{1,s-1}^{\nu} T_{1,s}^{-}}{\overline{Q}_{1,s-1}^{\nu} \overline{L}_{1,s}^{-}}. \quad (B.25)
\]

However \((b_{s}, \overline{b}_{s})\) are not invariant under the frame rotation \([67]\),

\[
\left( \begin{array}{c} Q' \\ P' \end{array} \right) = G \left( \begin{array}{c} Q \\ P \end{array} \right), \quad \left( \begin{array}{c} \overline{Q}' \\ \overline{P}' \end{array} \right) = G \left( \begin{array}{c} \overline{Q} \\ \overline{P} \end{array} \right), \quad G^+ = G^-, \quad G \in SL(2, \mathbb{C}). \quad (B.26)
\]

This transformation do not change Wronskians \(T, L, \overline{T}\), but it acts on \((b_{s}, \overline{b}_{s})\) in a non-linear way. As a result, the \(A_{1}\) NLIEs before and after the transformation are related in a complicated way.

### C Derivations

We derive our claims in Section 3.

#### C.1 Derivation of \(A_{1}\) NLIE with source terms

Below we generalize the derivation of \(A_{1}\) NLIE \([18]\) assuming that \(T, L, Q\)-functions have zeroes in the complex plane as

\[
T_{1,s}(t_{s,n}) = T_{1,s}(t_{s,-n}) = Q(q_{n}) = \overline{Q}(\overline{q}_{n}) = L(\ell_{n}) = \overline{L}(\overline{\ell}_{n}) = 0, \quad \{t_{s,n}, q_{n}, \ell_{n}\} \in \mathcal{A}_{n-1,n}, \quad \{t_{s,-n}, \overline{q}_{n}, \overline{\ell}_{n}\} \in \mathcal{A}_{-n,-n+1}, \quad (n \geq 1). \quad (C.1)
\]

The \(A_{1}\) TQ-relations (B.19) suggest to study the following two variables:

\[
1 + b_{s} = \frac{Q_{1,s-1}^{[s-1]} T_{1,s}^{+}}{Q_{1,s-1}^{[s-1]} L_{1,s}^{[s+1]}}, \quad b_{s} = \frac{Q_{1,s-1}^{[s+1]} T_{1,s}^{+}}{Q_{1,s-1}^{[s-1]} L_{1,s}^{[s+1]}},
\]

\[
1 + \overline{b}_{s} = \frac{\overline{Q}_{1,s-1}^{[1-s]} T_{1,s}^{-}}{\overline{Q}_{1,s-1}^{[1-s]} \overline{L}_{1,s}^{-}}, \quad \overline{b}_{s} = \frac{\overline{Q}_{1,s-1}^{[s-1]} T_{1,s}^{-}}{\overline{Q}_{1,s-1}^{[s-1]} \overline{L}_{1,s}^{-}}. \quad (C.2)
\]

Our goal is to deduce the equation of the form \(\log b_{s} = \log(1 + b_{s}) \ast K_{f} + \ldots\) by taking Fourier transform of the logarithmic derivative of these equations. See Appendix \([3]\) for notation.
As a warm-up, consider the T-system at \((1, s - 1)\),
\[
\hat{d}l \left[ T_{1,s-1}^+ T_{1,s-1}^- \right] = \hat{d}l \left[ (1 + Y_{1,s-1}) L^{[+s]} \tilde{L}^{[-s]} \right]. \tag{C.3}
\]
When \(T_{1,s-1}(v)\) has zeroes inside the strip \(\mathcal{A}_{-1,1}\), we find the relations\(^{19}\)
\[
\begin{align*}
\hat{d}l T_{1,s-1}^+ &= \int_{\mathbb{R} + \frac{i}{g}} dv' e^{ik(v' - \frac{i}{g})} \partial_{v'} \log T_{1,s-1}(v') = D_k \left\{ \hat{d}l T_{1,s-1} - 2\pi i e^{ik_{t,s-1,1}} \right\}, \quad D_k \equiv e^{+k/g}, \\
\hat{d}l T_{1,s-1}^- &= \int_{\mathbb{R} - \frac{i}{g}} dv' e^{ik(v' + \frac{i}{g})} \partial_{v'} \log T_{1,s-1}(v') = D_k^{-1} \left\{ \hat{d}l T_{1,s-1} + 2\pi i e^{ik_{t,s-1,1}} \right\}. \tag{C.4}
\end{align*}
\]
The equation \((C.3)\) becomes
\[
\hat{d}l T_{1,s-1} = \hat{d}l \left[ (1 + Y_{1,s-1}) L^{[+s]} \tilde{L}^{[-s]} \right] \hat{s}_K + 2\pi i \left[ D_k e^{ik_{t,s-1,1}} - D_k^{-1} e^{ik_{t,s-1,1}} \right] \hat{s}_K. \tag{C.5}
\]
where \(\hat{s}_K \equiv 1/(D_k + D_k^{-1})\).

The relations \((C.4)\) can be generalized to the Q- and L-functions (see Figure 2):
\[
\begin{align*}
\hat{d}l Q^{[r+n]} &= D_k^n \hat{d}l Q^{[r]} - 2\pi i D_k^{r+n} \sum_{j=1}^{n} e^{ik_{r+j,n}}, \\
\hat{d}l Q^{[r-n]} &= D_k^n \hat{d}l Q^{[r]} + 2\pi i D_k^{r-n} \sum_{j=1}^{n} e^{ik_{r-n,j}}, \\
\hat{d}l Q^{[-r-n]} &= D_k^{-n} \hat{d}l Q^{[-r]} + 2\pi i D_k^{-r-n} \sum_{j=1}^{n} e^{ik_{r+n,j}}, \\
\hat{d}l Q^{[-r+n]} &= D_k^{-n} \hat{d}l Q^{[-r]} - 2\pi i D_k^{-r+n} \sum_{j=1}^{n} e^{ik_{r-n,j}}, \tag{C.6}
\end{align*}
\]
with \(r, n \in \mathbb{Z}_{\geq 1}\). By taking the limit \(n \to \infty\), we find\(^{20}\)
\[
\begin{align*}
\hat{d}l Q^{[s]} &= +2\pi i D_k^s \sum_{n=1}^{\infty} e^{ik_{s+n}} \quad \text{for } \text{Re } k > 0, \quad \left( \text{if } \lim_{n \to \infty} D_k^{-n} \hat{d}l Q^{[r+n]} \to 0 \right) \\
\hat{d}l \tilde{Q}^{[-s]} &= -2\pi i D_k^{-s} \sum_{n=1}^{\infty} e^{ik_{s+n}} \quad \text{for } \text{Re } k < 0, \quad \left( \text{if } \lim_{n \to \infty} D_k^{r} \hat{d}l \tilde{Q}^{[-r+n]} \to 0 \right). \tag{C.7}
\end{align*}
\]

**Important lemma.** In order to derive the NLIE of gauge-invariant variables, it is important to look for a combination of \(1 + b_s, 1 + \tilde{b}_s\) which do not depend on \(T_{1,s}\). The answer is
\[
\mathcal{X}_s \equiv \frac{1 + b_s^-}{1 + \tilde{b}_s^+} = \frac{Q^{[s-2]} Q^{[s]} \tilde{L}^{[-s]}}{Q^{-[s]} \tilde{Q}^{-[2-s]} L^{[+s]}}, \tag{C.8}
\]
\(^{19}\)\(T_{1,s-1}\) should not have branch cuts on the real axis, which is asymptotically true for twisted AdS\(_5 \times S^5\).
\(^{20}\)We can derive \((C.7)\) also by assuming that \(Q \text{ or } \tilde{Q}\) are meromorphic in the upper or lower half plane.
Figure 2: Zeros of \( Q^{[+s]}(v) \) and \( Q^{[-s]}(v) \). Notice that when \( Q(v) \) has a zero at \( v = q_{s+1} \in \mathcal{A}_{s,s+1} \) as in (3.6), the shifted function \( Q^{[+s]}(v) \) has a zero at \( v = q_{s+1} - s \in \mathcal{A}_{0,1} \).

We then assume that

\[
Q^{[s-2]} \text{ and } L^{[+s]} \text{ are meromorphic in the upper half plane,}
\]

\[
\overline{Q}^{[2-s]} \text{ and } \overline{L}^{[-s]} \text{ are meromorphic in the lower half plane.} \quad (C.9)
\]

These assumptions are realistic, because \( Q(v), L(v + \frac{2\pi}{g}) \) do not have branch cuts for \( \text{Im} v > 0 \) and \( s \geq 3 \) in our setup. By applying \( \hat{d}l \) on both sides of (C.8), we obtain

\[
\hat{d}l \mathcal{X}_s = 2\pi i \text{ Res}_{\text{UHP}} \hat{d}l \frac{Q^{[s-2]}[s][+s]}{L^{[+s]}} + \hat{d}l \frac{\overline{L}^{[-s]}[s]}{Q^{[-s]}[2-s]} \quad (\text{Re } k > 0),
\]

\[
\hat{d}l \mathcal{X}_s = \hat{d}l \frac{Q^{[s-2]}[s][+s]}{L^{[+s]}} + 2\pi i \text{ Res}_{\text{LHP}} \hat{d}l \frac{\overline{L}^{[-s]}[s]}{Q^{[-s]}[2-s]} \quad (\text{Re } k < 0). \quad (C.10)
\]

where \( \text{Res}_{\text{UHP}} \) and \( \text{Res}_{\text{LHP}} \) collect the residues in the upper and lower half planes, respectively.

By using (C.10) and \( \hat{d}lf = \theta(+k) \hat{d}lf + \theta(-k) \hat{d}lf \), we obtain

\[
\hat{d}l \frac{Q^{[s-2]}[s]}{L^{[+s]}} = +\theta(-k) \hat{d}l \mathcal{X}_s + 2\pi i \text{ Res} \hat{d}l \mathcal{X}_s,
\]

\[
\hat{d}l \frac{\overline{Q}^{[2-s]}[s]}{L^{[-s]}} = -\theta(+k) \hat{d}l \mathcal{X}_s + 2\pi i \text{ Res} \hat{d}l \mathcal{X}_s, \quad (C.11)
\]

\[
\text{Res} \hat{d}l \mathcal{X}_s \equiv \theta(+k) \text{ Res}_{\text{UHP}} \hat{d}l \frac{Q^{[s-2]}[s][+s]}{L^{[+s]}} + \theta(-k) \text{ Res}_{\text{LHP}} \hat{d}l \frac{\overline{Q}^{[-s]}[s][2-s]}{\overline{L}^{[-s]}}, \quad (C.12)
\]

The last term can be computed explicitly with the help of (C.7) as

\[
\text{Res} \hat{d}l \mathcal{X}_s = \theta(+k) \left\{ D_k \sum_{n=1}^{\infty} e^{ikq_{s+n}} + D_k^{s-2} \sum_{n=1}^{\infty} e^{ikq_{s-2+n}} - D_k \sum_{n=1}^{\infty} e^{ik\ell_{s+n}} \right\}
\]

\[
+ \theta(-k) \left\{ -D_k^{s-2} \sum_{n=1}^{\infty} e^{ik\ell_{s+n}} - D_k^{s-2} \sum_{n=1}^{\infty} e^{ik\ell_{s-2+n}} + D_k \sum_{n=1}^{\infty} e^{ik\ell_{s+n}} \right\}. \quad (C.13)
\]
NLIE for $b_s$. In order to derive the $A_1$ NLIE with source terms, consider \( \hat{d}l \mathcal{b}_s \) in (C.2),

\[
\begin{align*}
\hat{d}l \mathcal{b}_s &= \hat{d}l Q^{[s+1]} + \hat{d}l T_{1,s-1} - \hat{d}l Q^{[1-s]} - \hat{d}l L^{[s+1]}, \\
&= D_k^2 \left\{ \hat{d}l Q^{[s-1]} - \hat{d}l L^{[s]} \hat{s}_K \right\} - \left\{ \hat{d}l Q^{[1-s]} - \hat{d}l L^{[-s]} \hat{s}_K \right\} + \hat{d}l (1 + Y_{1,s-1}) \hat{s}_K \\
&\quad - 2\pi i D_k^{s+1} \left[ e^{ikq_{s+1}} + e^{ikq_s} - e^{ik\ell_{s+1}} \right] + 2\pi i \left[ D_k e^{ikt_{s+1} - 1} - D_k^{-1} e^{ikt_{s+1} - 1} \right] \hat{s}_K. \quad \text{(C.14)}
\end{align*}
\]

To rewrite the quantities in the curly brackets, we use $\mathcal{X}_s$ in (C.8). With the help of the formulae (C.11) and

\[
\begin{align*}
\hat{d}l \left[ Q^{[s-2]} Q^{[s]} \right] &= \left( D_k + D_k^{-1} \right) \hat{d}l Q^{[s-1]} + 2\pi i \left[ D_k^{s-2} e^{ikq_{s-1}} - D_k^s e^{ikq_s} \right], \\
\hat{d}l \left[ Q^{[2-s]} Q^{[-s]} \right] &= (D_k + D_k^{-1}) \hat{d}l Q^{[1-s]} - 2\pi i \left[ D_k^{2-s} e^{ik\tau_{s-1}} - D_k^{-s} e^{ik\tau_s} \right], \quad \text{(C.15)}
\end{align*}
\]

we obtain

\[
\hat{d}l \mathcal{b}_s = \left\{ D_k^2 \theta(-k) + \theta(k) \right\} \hat{s}_K \hat{d}l \mathcal{X}_s + \hat{d}l (1 + Y_{1,s-1}) \hat{s}_K + 2\pi i \left( D_k^2 - 1 \right) \hat{s}_K \text{Res} \hat{d}l \mathcal{X}_s \\
&\quad + 2\pi i \left[ -D_k e^{ik\tau_{s+1}} - D_k^s e^{ikq_s} - D_k^{2-s} e^{ik\tau_{s-1}} + D_k^{-s} e^{ik\tau_s} \right] \hat{s}_K \\
&\quad - 2\pi i D_k^{s+1} \left[ e^{ikq_{s+1}} - e^{ik\ell_{s+1}} \right] + 2\pi i \left[ D_k e^{ikt_{s+1} - 1} - D_k^{-1} e^{ikt_{s+1} - 1} \right] \hat{s}_K. \quad \text{(C.16)}
\]

Since we want an equation of the form $\hat{d}l \mathcal{b}_s = \hat{d}l (1 + b_s) \hat{K}_f + \ldots$, we rewrite $\hat{d}l \mathcal{X}_s$ as

\[
\hat{d}l \mathcal{X}_s = D_k^{-1} \hat{d}l (1 + b_s) - D_k \hat{d}l (1 + \overline{b}_s) + 2\pi i \text{Res} \hat{d}l \frac{1 + b_s^{-}}{1 + \overline{b}_s^{+}}, \quad \text{(C.17)}
\]

\[
\text{Res} \hat{d}l \frac{1 + b_s^{-}}{1 + \overline{b}_s^{+}} = e^{ik\tau_{s+1} + e^{ikq_{s+1} - s + 2} - e^{ik\tau_{s+1}} - e^{ik\ell_{s+1} + e^{ik\tau_{s-1} + e^{ikq_{s-1} - s + 2} - e^{ik\ell_{s+1}}}},
\]

The last line is the collection of the residues of $\hat{d}l (1 + b_s)$ inside $\mathcal{A}_{-1,0}$ and $\hat{d}l (1 + \overline{b}_s)$ inside $\mathcal{A}_{0,1}$ with appropriate shift.

In summary, Fourier transform of the derivative of $A_1$ NLIE with the source term is

\[
\hat{d}l \mathcal{b}_s = -\text{FT} (J'_f) + \hat{d}l (1 + b_s) \hat{K}_f - \hat{d}l (1 + \overline{b}_s) \hat{K}_f^{[+2]} + \hat{d}l (1 + Y_{1,s-1}) \hat{s}_K, \quad \text{(C.18)}
\]

where $\hat{K}_f = \left\{ D_k \theta(-k) + D_k^{-1} \theta(k) \right\} \hat{s}_K$ is the Fourier transform of the kernel $K_f$, and

\[
- \frac{\text{FT} (J'_f)}{2\pi i} = D_k \hat{K}_f \text{Res} \hat{d}l \frac{1 + b_s^{-}}{1 + \overline{b}_s^{+}} + (D_k^2 - 1) \hat{s}_K \text{Res} \hat{d}l \mathcal{X}_s \\
+ \left[ -D_k e^{ik\tau_{s+1}} - D_k^s e^{ikq_s} - D_k^{2-s} e^{ik\tau_{s-1}} + D_k^{-s} e^{ik\tau_s} \right] \hat{s}_K \\
- D_k^{s+1} \left[ e^{ikq_{s+1}} - e^{ik\ell_{s+1}} \right] + \left[ D_k e^{ikt_{s+1} - 1} - D_k^{-1} e^{ikt_{s+1} - 1} \right] \hat{s}_K. \quad \text{(C.19)}
\]

Here Res $\hat{d}l \mathcal{X}_s$ is given in (C.13), and it consists of infinitely many terms. To obtain (2.7), we have to apply the inverse Fourier transform and integrate with respect to $v$. The inverse Fourier transform of (C.19) is remarkably simple and given by (3.7). The integration constants can be fixed by consideration of the limit $v \to \pm \infty$. 

\footnote{The formulae (A.8) are useful for this computation.}
Case of orbifold Konishi state. Let us check if the above results are consistent with the source terms of $A_1$ NLIE for orbifold Konishi state discussed in Section 2.3. As for the asymptotic orbifold Konishi state, $Q^{[s-2]}, L^{[s]}$ are analytic in the upper half plane and $\overline{Q}^{[2-s]}, \overline{L}^{[-s]}$ are analytic in the lower half plane. We have to take care of the extra zeroes of $T$-functions only.

Since the $A_1$ NLIE is written in terms of $(a_s, \overline{a}_s) = (b_s^{[-\gamma]}, \overline{b}_s^{[\gamma]})$ we have to modify slightly the derivation. In (C.14) we applied $\hat{dl}$ to the definition of $b_s$. If we use $a_s = b_s^{[-\gamma]}$, we obtain

$$\hat{dl} a_s = \hat{dl} b_s^{[-\gamma]} = D_k^{-\gamma}\left[\hat{dl} b_s + 2\pi i e^{ikts_{-1,\gamma}}\right] \quad (C.20)$$

Actually we may neglect the residue term. After the inverse Fourier transform, it becomes a $\delta$-function, whose integration is just a constant. There is another reason why we do not have to take care of the extra zeroes of $T_{1,s-1}$: the rapidity of $Y_{s-1|w}$ in (2.7), (2.8) is not shifted at all.

An important modification occurs at the equation (C.17), which changes as

$$\hat{dl} \mathcal{X}_s \equiv D_k^{-1+\gamma}\hat{dl}(1+a_s) - D_k^{1-\gamma}\hat{dl}(1+\overline{a}_s) + 2\pi i \text{Res} \frac{1+a_s^{[1+\gamma]}}{1+\overline{a}_s^{[1-\gamma]}} \hat{dl} \quad (C.21)$$

Now the last term is the collection of the residues of $\hat{dl}(1+a_s)$ inside $\mathcal{A}_{-1+\gamma,0}$ and $\hat{dl}(1+\overline{a}_s)$ inside $\mathcal{A}_{0,1-\gamma}$ with appropriate shift. Since both $(1+a_s^{[1+\gamma]})$ and $(1+\overline{a}_s^{[1-\gamma]})$ are proportional to $T_{1,s}$, this means that the extra zeroes of $T_{1,s}$ inside the strip $\mathcal{A}_{-1+\gamma,1-\gamma}$ contribute to the source term (C.19). The rest of the derivation goes without any change.

One can see that this conclusion is consistent with the critical behavior observed in (2.27), (2.21).

C.2 Contour deformation for $A_1$ NLIE

We collect the residues when we straighten the deformed contour of the NLIE in the presence of extra zeroes (3.4). To simplify the discussion we remove the regulator $\gamma$ by taking the limit $\gamma \ll 1$. Then, the holomorphic part of $A_1$ NLIE for the ground state $(J_s = j_s)$ takes the form

$$\log b_s = -J_s + \log(1+b_s) \ast K_f - \log(1+\overline{b}_s) \ast K_f^{[1+2-0]} + \log(1+Y_{s-2|w}) \ast s_K \quad (C.22)$$

where the variables in the right hand side are defined by

$$1 + b_s = \frac{Q^{[s-1]} T_{1,s}^{+}}{Q^{-[1-s]} L^{[s+1]}}, \quad 1 + \overline{b}_s = \frac{\overline{Q}^{-[s-1]} T_{1,s}^{-}}{Q^{-[s-1]} L^{[-s-1]}}, \quad 1 + Y_{1,s-1} = \frac{T_{1,s-1}^{-} T_{1,s-1}^{+}}{L^{[s-1]} L^{[-s-1]}} \quad (C.23)$$

Asymptotically $Q, L^{[+2]}$ have no branch cuts in the upper half plane, $\overline{Q}, L^{[-2]}$ have no branch cuts in the lower half plane, excluding the real axis. Thus, at best we can pull the integration
contour of \((1 + b_s), (1 + \overline{b_s})\) up to \(\text{Im} \, v = \pm(s - 1)/g\) and that of \((1 + Y_{s-2}w)\) up to \(\pm(s - 2)/g\). Around the imaginary axis we can further deform them toward \(\pm \infty\). Our choice of deformed contour is depicted in Figure 1. This contour can pick up all zeroes of \(T, L, Q\)-functions inside the strip \(\mathcal{A}_{-s+1,0}\) or \(\mathcal{A}_{-s+2}\). Recalling our notation (3.6), we find\(^{22}\)

\[
\log(1 + b_s) \times K_f \rightarrow + \log \left[ \frac{S_f(v - t_{s,1}) \prod_{j=1}^{s-2} S_f(v - t_{s,-j}) \prod_{j=1}^{s-1} S_f(v - q_j^{[1-s]})}{\prod_{j=s}^{2s-2} S_f(v - q_j^{[s-1]}) \prod_{j=s+1}^{s} S_f(v - \ell_j^{[-s-1]})} \right],
\]

\[
- \log(1 + \overline{b_s}) \times K_f^{[+2]} \rightarrow - \log \left[ \frac{\prod_{j=2}^{s} S_f(v - t_{s,-j}) \prod_{j=s+1}^{2s-2} S_f(v - q_j^{[-s-1]})}{\prod_{j=1}^{s} S_f(v - q_j^{[-s-1]}) \prod_{j=s+1}^{2s} S_f(v - \ell_j^{[s-1]})} \right],
\]

\[
\log(1 + Y_{1,s-1}) \times s_K \rightarrow + \log \left[ \frac{\prod_{j=2}^{s} S(v - t_{s,-1,-j}) \cdot S(v - t_{s,-1,1}) \prod_{j=1}^{s-3} S(v - t_{s-1,-j})}{\prod_{j=3}^{s} S(v - \ell_j^{[-s-1]}) \prod_{j=s+1}^{2s-2} S(v - \ell_j^{[s-1]})} \right].
\]

We assume that all roots \(t_{s,-n}(n \geq 1)\) lie along the imaginary axis, as they do for the orbifold Konishi state at weak coupling. Since the deformed contour pick up the corresponding residues, we can replace the upper bound of the product of \(S\)-matrices with \(t_{s,-n}, t_{s-1,-n}\) by \(\infty\).

After straightening the contour and using \(S^+ S^- = 1\) and \(S^2_f S^2_f = S_1\), the source term \(J_s\) in (C.22) becomes

\[
J_s^{CDT} = j_s - \log \left[ S_f(v - t_{s,1}) S_f(v - t_{s,-1}) \right] - \log \left[ S(v - t_{s-1,1}) S(v - t_{s-1,-1}) \right]
\]

\[
- \log \left[ \frac{\prod_{j=1}^{s-1} S(v - q_j^{[-s]})}{\prod_{j=1}^{2s-2} S_1(v - q_j^{[s-2]})} \cdot \frac{\prod_{j=s+1}^{s} S_f(v - \ell_j^{[-s-1]})}{\prod_{j=s+1}^{2s-2} S_1(v - \ell_j^{[s-2]})} \right].
\]

\[
\text{References}
\]

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\(^{22}\)Use \(S_f(v^{[+2]} - t) = S_f(v - t^{[-2]})\) to compute the extra terms from \(\log(1 + \overline{b_s}) \times K_f^{[+2]}\).
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