A family of generalized $q$-Genocchi numbers and polynomials

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Abstract: In this paper we first consider the $q$-extension of the generating function for the higher-order generalized Genocchi numbers and polynomials attached to $\chi$. The purpose of this paper is to present a systemic study of some families of higher-order generalized $q$-Genocchi numbers and polynomials attached to $\chi$ by using the generating function of those numbers and polynomials.

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1. Introduction

As well known definition, the Genocchi polynomials are defined by

$$\left(\frac{2t}{e^t + 1}\right) e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi, \quad (1)$$

where we use the technical method’s notation by replacing $G_n(x)$ by $G_n(x)$, symbolically, (see [3,11]).

In the special case $x = 0$, $G_n = G_n(0)$ are called the $n$-th Genocchi numbers. From the definition of Genocchi numbers, we note that $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_{2n} = 2(1 - 2^n)B_{2n} = 2nE_{2n-1}(0)$, (see [8]), where $B_n$ is a Bernoulli number and $E_n(x)$ is an Euler polynomial. The first few Genocchi numbers for $2, 4, 6, \cdots$ are $-1, 1, -3, 17, -155, 2073, \cdots$. The first few prime Genocchi numbers are given by $G_6 = -3$ and $G_8 = 17$. It is known that there are no others prime Genocchi numbers with $n < 10^5$. For a real or complex parameter $\alpha$, the higher-order Genocchi polynomials are defined by

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (2)$$

In the special case $x = 0$, $G_n^{(\alpha)} = G_n^{(\alpha)}(0)$ are called the $n$-th Genocchi numbers of order $\alpha$. From (1) and (2), we note that $G_n = G_n^{(1)}$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let $\chi$ be the Dirichlet character with conductor $d$. It is known that the generalized Genocchi polynomials attached to $\chi$ are defined by

$$\left(\frac{2t}{e^t + 1}\right)^{\chi} e^{xt} = \sum_{n=0}^{\infty} G_n(x,\chi) \frac{t^n}{n!}, \quad (3)$$

In the special case $x = 0$, $G_n,\chi = G_n,\chi(0)$ are called the $n$-th generalized Genocchi numbers attached to $\chi$ (see [3, 4, 5, 6]).

For a real or complex parameter $\alpha$, the generalized higher-order Genocchi polynomials attached to $\chi$ are also defined by

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x,\chi) \frac{t^n}{n!}, \quad (4)$$
In the special case $x = 0$, $G_{n,\chi}^{(\alpha)} = G_{n,\chi}^{(\alpha)}(0)$ are called the $n$-th generalized Genocchi numbers attached to $\chi$ of order $\alpha$ (see [3, 4, 5, 6, 13, 14, 15]). From (3) and (4), we derive $G_{n,\chi} = G_{n,\chi}^{(1)}$.

Let us assume that $q \in \mathbb{C}$ with $|q| < 1$ as an indeterminate. Then we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}.$$  

The $q$-factorial is defined by

$$[n]_q! = [n]_q[n - 1]_q \cdots [2]_q[1]_q$$

and the Gaussian binomial coefficient is also defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} \quad \text{(see [2, 5]).}$$  

(5)

Note that

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$  

(5)

It is known that

$$\binom{n+1}{k}_q = \binom{n}{k}_q + q^k \binom{n}{k-1}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q, \quad \text{(see [2, 5]).}$$

(5)

The $q$-binomial formula are known that

$$(x - y)_q^n = (x - y)(x - qy) \cdots (x - q^{n-1}y) = \sum_{i=0}^{n} \binom{n}{i}_q q^{(i)}_{(1)} (1)_q (x^{n-i} y^i), \quad \text{(see [2, 9]),}$$

and

$$\frac{1}{(x - y)_{q}^{n}} = \frac{1}{(x - y)(x - qy) \cdots (x - q^{n-1}y)} = \sum_{l=0}^{\infty} \binom{n + l - 1}{l}_q (x^{n-l} y^l), \quad \text{(see [2, 9]).}$$

(6)

There is an unexpected connection with $q$-analysis and quantum groups, and thus with non-commutative geometry $q$-analysis is a sort of $q$-deformation of the ordinary analysis. Spherical functions on quantum groups are $q$-special functions. Recently, many authors have studied the $q$-extension in various area (see [1-15]). Govil and Gupta [2] has introduced a new type of $q$-integrated Meyer-König-Zeller-Durrmeyer operators and their results are closely related to study $q$-Bernstein polynomials and $q$-Genocchi polynomials, which are treated in this paper. In this paper, we first consider the $q$-extension of the generating function for the higher-order generalized Genocchi numbers and polynomials attached to $\chi$. The purpose of this paper is to present a systemic study of some families of higher-order generalized $q$-Genocchi numbers and polynomials attached to $\chi$ by using the generating function of those numbers and polynomials.

2. Generalized $q$-Genocchi numbers and polynomials

For $r \in \mathbb{N}$, let us consider the $q$-extension of the generalized Genocchi polynomials of order $r$ attached to $\chi$ as follows:

$$F^{(r)}_{q,\chi}(t, x) = 2^r t^r \sum_{m_1, \ldots, m_r = 0}^{\infty} \left( \prod_{j=1}^{r} \chi(m_j) \right) (-1)^{\sum_{j=1}^{r} m_j} e^{[x+m_1+\cdots+m_r]_q t} = \sum_{n=0}^{\infty} G^{(r)}_{n,\chi,q}(x) \frac{t^n}{n!}. \quad (7)$$

Note that

$$\lim_{q \to 1} F^{(r)}_{q,\chi}(t, x) = \left( \frac{2t \sum_{a=0}^{d-1} \chi(a)(-1)^n e^{at}}{e^{dt} + 1} \right)^r e^{xt}. \quad (7)$$
By (7) and (4), we can see that \( \lim_{q \to 1} G_{n, \chi, q}^{(r)}(x) = G_{n, \chi}^{(r)}(x) \). From (7), we note that
\[
G_{0, \chi, q}^{(r)}(x) = G_{1, \chi, q}^{(r)}(x) = \cdots = G_{r-1, \chi, q}^{(r)}(x) = 0,
\]
and
\[
G_{n+r, \chi, q}^{(r)}(x) = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{j=1}^{r} \chi(m_j) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + m_1 + \cdots + m_r]_q^n.
\]
In the special case \( x = 0 \), \( G_{n, \chi, q}^{(r)} = G_{n, \chi, q}^{(r)}(0) \) are called the \( n \)-th generalized \( q \)-Genocchi numbers of order \( r \) attached to \( \chi \). Therefore, we obtain the following theorem.

**Theorem 1.** For \( r \in \mathbb{N} \), we have
\[
G_{n+r, \chi, q}^{(r)} \left( \frac{n+r}{r} \right)^r = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{j=1}^{r} \chi(m_j) \right) (-1)^{\sum_{j=1}^{r} m_j} [m_1 + \cdots + m_r]_q^n.
\]

Note that
\[
2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [m_1 + \cdots + m_r]_q^n
= \frac{2^r}{(1-q)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) \left( \frac{-q^d}{1+q^d} \right)^r.
\]
Thus we obtain the following corollary.

**Corollary 2.** For \( r \in \mathbb{N} \), we have
\[
G_{n+r, \chi, q}^{(r)} \left( \frac{n+r}{r} \right)^r = 2^r \sum_{m=0}^{\infty} \binom{n+r-1}{m} (-1)^m \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) \left[ \sum_{i=1}^{r} a_i + md \right]_q^n.
\]

For \( h \in \mathbb{Z} \) and \( r \in \mathbb{N} \), we also consider the extended higher-order generalized \((h, q)\)-Genocchi polynomials as follows:
\[
F_{q, \chi}^{(h, r)}(t, x) = 2^r t^x \sum_{m_1, \ldots, m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j)m_j} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + m_1 + \cdots + m_r]_q^n
= \sum_{n=0}^{\infty} F_{q, \chi}^{(h, r)}(x) \frac{t^n}{n!}.
\]
From (8), we note that
\[
G_{0, \chi, q}^{(h, r)}(x) = G_{1, \chi, q}^{(h, r)}(x) = \cdots = G_{r-1, \chi, q}^{(h, r)}(x) = 0,
\]
and
\[
G_{n+r, \chi, q}^{(h, r)} \left( \frac{n+r}{r} \right)^r = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j)m_j} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + m_1 + \cdots + m_r]_q^n
= \frac{2^r}{(1-q)^n} \sum_{m=0}^{\infty} \binom{n}{m} q^{x+h-1} \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r} (h-j)a_j} (-1)^{a_1 + \cdots + a_r}
\times \sum_{m_1, \ldots, m_r=0}^{\infty} (-1)^{m_1 + \cdots + m_r} q^{d(m_1 + \cdots + m_r) + d(\sum_{j=1}^{r} (h-j)m_j)}
\times \frac{(-q^d)^{r+h}}{(1-q)^{r+h}}.
\]

where \((-x;q)_r = (1+x)(1+xq)\cdots(1+xq^{r-1})\).

Therefore, we obtain the following theorem.

**Theorem 3.** For \(h \in \mathbb{Z}, r \in \mathbb{N}\), we have
\[
\frac{G_{n+r,h+r,q}(x)}{(n+r)_r!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j)m_j} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{i=1}^{r} m_i} [x + m_1 + \cdots + m_r]_q^n
\]
\[
= \frac{2^r}{(1-q)^n} \sum_{l=0}^{\infty} \binom{n}{l} (-1)^l q^{l(x+\sum_{i=1}^{l} a_i)} \left( \prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r} (h-j)a_j} (-q^l)^{\sum_{i=1}^{l} a_i},
\]
and \(G_{0,h,r,q}(x) = G_{1,h,r,q}(x) = \cdots = G_{r-1,h,r,q}(x) = 0\).

Note that
\[
\frac{1}{(-q^d(h-r+l);q)_r} = \frac{1}{(1+q^d(h-r+l))} = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^{d(h-r+m)}.
\]

By (9), we see that
\[
\frac{1}{(1-q)^n} \sum_{l=0}^{\infty} \binom{n}{l} (-1)^l q^{l(x+\sum_{i=1}^{l} a_i)} \left( \prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r} (h-j)a_j} (-q^l)^{\sum_{i=1}^{l} a_i + dm} = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^{d(h-r+m)} [x + \sum_{i=1}^{r} a_i + dm]_q^n.
\]

By (9) and (10), we obtain the following corollary.

**Corollary 4.** For \(h \in \mathbb{Z}, r \in \mathbb{N}\), we have
\[
\frac{G_{n+r,h+r,q}(x)}{(n+r)_r!} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^{d(h-r)m} \sum_{a_1,\ldots,a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) q^{\sum_{j=1}^{r} (h-j)a_j} [x + \sum_{i=1}^{r} a_i + dm]_q^n.
\]

By (8), we can derive the following corollary.

**Corollary 5.** For \(h \in \mathbb{Z}, r, d \in \mathbb{N}\) with \(d \equiv 1\pmod{2}\), we have
\[
q^{d(h-1)} \frac{G_{n+r,h+r,q}(x+d)}{(n+r)_r!} + \frac{G_{n+r,h+r,q}(x)}{(n+r)_r!} = 2 \sum_{l=0}^{d-1} \chi(l)(-1)^l \frac{G_{n+r-1,h+r-l,q}(x)}{(n+r-1)_r!(r-1)!},
\]
and
\[
q^r \frac{G_{n+r+1,h+r,q}(x)}{(n+r)_r!} = (q-1) \frac{G_{n+r+1,h+r,q}(x)}{(n+r)_r!} + \frac{G_{n+r,h+r,q}(x)}{(n+r)_r!}.
\]

For \(h = r\) in Theorem 3, we obtain the following corollary.
Corollary 6. For \( r \in \mathbb{N} \), we have

\[
G_{n+r, \chi, q}^{(r,r)}(x) = 2^r \left( \frac{(n)_r}{r!} \right) \sum_{l=0}^{n} \left( \frac{(r)_l}{l!} \right) \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) \frac{q^{\sum_{j=1}^{r} ((r-j)a_j + l_j)} (-1)^{a_1 + \cdots + a_r} (-q^{a_1 + \cdots + a_r})}{\left( q^{d_1} q^{r_1} \right)_{r}}.
\]

Let \( x = r \) in Corollary 6. Then we have

\[
\frac{G_{n+r, \chi, q}^{-1}(r-x)}{(n+r-r)!} = (-1)^n q^{n+\left(\frac{r}{2}\right)} G_{n+r, \chi, q}^{(r,r)}(x) \left( \frac{(n+r)}{r!} \right).
\]

Let \( w_1, w_2, \ldots, w_r \in \mathbb{Q}_+ \). Then we define Barnes’ type generalized \( q \)-Genocchi polynomials attached to \( \chi \) as follows:

\[
F_{q, \chi}^{(r)}(t, x \mid w_1, w_2, \ldots, w_r) = 2^r t^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{m_1 + \cdots + m_r} e^{[x+\sum_{j=1}^{r} w_j m_j]} q^t
\]

\[
= \sum_{n=0}^{\infty} G_{n+r, \chi, q}^{(r)}(x \mid w_1, w_2, \ldots, w_r) \frac{t^n}{n!}.
\]

By (11), we see that

\[
\frac{G_{n+r, \chi, q}^{(r)}(x \mid w_1, \ldots, w_r)}{(n+r-r)!} = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + \sum_{j=1}^{r} w_j m_j] q^n.
\]

It is easy to see that

\[
2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + \sum_{j=1}^{r} w_j m_j] q^n
\]

\[
= \frac{2^r \left( \frac{(n)!}{r!} \right) (-q^x)^t \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) (-1)^{\sum_{j=1}^{r} a_j} a_j^t \sum_{i=1}^{r} w_i a_i}{\left( 1 + q^{d_1} q^{r_1} \right) \cdots (1 + q^{d_1} q^{r_1})}.
\]

Therefore, we obtain the following theorem.

Theorem 7. For \( r \in \mathbb{N} \), \( w_1, w_2, \ldots, w_r \in \mathbb{Q}_+ \), we have

\[
\frac{G_{n+r, \chi, q}^{(r,r)}(x \mid w_1, w_2, \ldots, w_r)}{(n+r-r)!} = 2^r \sum_{m_1, \ldots, m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{j=1}^{r} m_j} [x + \sum_{j=1}^{r} w_j m_j] q^n
\]

\[
= \frac{2^r \left( \frac{(n)!}{r!} \right) (-q^x)^t \sum_{a_1, \ldots, a_r=0}^{d-1} \left( \prod_{j=1}^{r} \chi(a_j) \right) (-1)^{\sum_{j=1}^{r} a_j} a_j^t \sum_{i=1}^{r} w_i a_i}{\left( 1 + q^{d_1} q^{r_1} \right) \cdots (1 + q^{d_1} q^{r_1})}.
\]
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