SOME TYPES OF LINEAR COMBINATIONS OF WEYL-HEISENBERG WAVE PACKET FRAMES OVER LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. In this paper, we present necessary and sufficient conditions for some types of linear combination of frame elements (wave packet) to be a frame for $L^2(K)$, where $K$ is a local field of positive characteristic.

1. Introduction

Let $H$ be a separable real or complex Hilbert space (finite or infinite dimensional) with inner product $\langle ., . \rangle$. A countable sequence $\{f_k\}_{k \in I} \subset H$ is called a frame (or Hilbert frame) for $H$ if, there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 \text{ for all } f \in H. \quad (1.1)$$

The positive constants $A$ and $B$ are called the lower and upper frame bounds of the frame $\{f_k\}_{k \in I}$, respectively. They are not unique. The inequality in (1.1) is called the frame inequality of the frame. If $\{f_k\}_{k \in I}$ satisfies the upper inequality in (1.1) then we say that $\{f_k\}_{k \in I}$ is a Bessel sequence in $H$ with Bessel bound $B$.

Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [23] in 1952 in the context of non-harmonic Fourier series and popularized in 1986 by Daubechies, Grossmann, and Meyer [19]. Frames are basis-like building blocks that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, spherical codes, compressed sensing, signal processing, and wavelet analysis etc., see [9].

A type of sequence which is obtained by combined action of three operators and have frame properties, introduced first time by Cordoba and Fefferman [17] by applying certain collection of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. This sequence is called wave packet. Later, Labate et al. [29] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele [30, 31] gave applications of wave packet systems in boundedness of the Hilbert transforms. The wave packet systems have been studied by several authors, see [13, 14, 18, 21, 26, 27].

Recently many mathematicians and engineers generalized the frame and wavelet theory in various directions, see [9, [11, 12] and references therein. Benedetto and Benedetto [8] developed a wavelet theory for local fields and related groups. Behera and Jahan [2, 7, 14, 18] developed wavelets on local...
fields of positive characteristic. In this paper discuss some frame properties of wave packet system which is natural generalization of wavelets over local fields of positive characteristic.

1.1. Background on Local Fields: Let \( \mathbb{K} \) be a field and a topological space. Then \( \mathbb{K} \) is called a locally compact field or a local field if both \( \mathbb{K}^+ \) and \( \mathbb{K}^- \) are locally compact abelian groups, where \( \mathbb{K}^+ \) and \( \mathbb{K}^- \) denote the additive and multiplicative groups of \( \mathbb{K} \), respectively. Any field endowed with the discrete topology is a local field. The only connected local fields are \( \mathbb{R} \) and \( \mathbb{C} \). Disconnected local fields are, in fact, totally disconnected. Every local field comes equipped with a canonical absolute value which defines its topology. Let \( dx \) be the Haar measure for \( \mathbb{K}^+ \). For any non-zero \( \alpha \in \mathbb{K} \), \( d(\alpha x) \) is also a Haar measure. Let \( d(\alpha x) = |\alpha| dx \). We say that \( |\alpha| \) is the absolute value or valuation of \( |\alpha| \).

The map \( x \rightarrow |x| \) has the following properties:

(i) \( |x| = 0 \) if and only if \( x = 0 \).
(ii) \( |xy| = |x||y| \) for all \( x, y \in \mathbb{K} \).
(iii) \( |x + y| \leq \max\{|x|, |y|\} \) for all \( x, y \in \mathbb{K} \).

Note that \( |x + y| = \max\{|x|, |y|\} \), if \( |x| \neq |y| \). The set \( \mathcal{D} = \{x : |x| \leq 1\} \) is called the ring of integers in \( \mathbb{K} \). It is the unique maximal compact subring of \( K \). The set \( \mathcal{B} = \{x : |x| < 1\} \) is called the prime ideal in \( \mathbb{K} \). Note that the prime ideal in \( \mathbb{K} \) is the unique maximal ideal in \( \mathcal{D} \). It is principal and prime. The set of values \( |x| \) as \( x \) varies over \( \mathbb{K} \) is a discrete set of the form \( \{s^k : k \in \mathbb{Z}\} \cup \{0\} \) for some \( s > 0 \). So there is an element of \( \mathcal{B} \) of maximal absolute value. A prime element of \( \mathbb{K} \) an element of maximum absolute value in \( \mathcal{B} \). The ring of integers \( \mathcal{D} \) is compact and open, so \( \mathcal{B} \) is compact and open. Therefore, the residue space \( \mathcal{D}/\mathcal{B} \) is isomorphic to a finite field \( GF(q) \), where \( q = p^c \) for some prime \( p \) and \( c \in \mathbb{N} \). If \( K \) is a local field, then there is a nontrivial, unitary, continuous character \( \chi \) on \( \mathbb{K}^+ \). It can be proved that \( \mathbb{K}^+ \) is self dual. Let \( \chi \) be a fixed character on \( \mathbb{K}^+ \) that is trivial on \( \mathcal{D} \) but is nontrivial on \( \mathcal{B}^{-1} \). We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic.

For \( y \in \mathbb{K} \), define \( \chi_y(x) = \chi(yx), x \in \mathbb{K} \).

Let \( E \) be a measurable subset of \( \mathbb{K} \) and let \( |E| = \int_\mathbb{K} 1_E(x) \), where \( 1_E \) is the characteristic function of \( E \) and \( dx \) is the Haar measure of \( \mathbb{K} \) normalized so that \( |\mathcal{D}| = 1 \). Then, \( |\mathcal{B}| = q^{-1} \) and \( |p| = q^{-1} \). Therefore, if \( x \neq 0 \) and \( x \in \mathcal{K} \), then \( |x| = q^k \) for some \( k \in \mathbb{Z} \).

To impose a natural order on the sequence \( \{u(n)\}_{n=0}^{\infty} \), first we note that \( \Delta = \mathcal{D}/\mathcal{B} \cong GF(q) \) and \( GF(q) \) is a \( c \)-dimensional vector space over the field \( GF(p) \). Therefore, we can find a set \( \mathcal{A} = \{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \subset \mathcal{D}^* \) such that \( \text{span}\mathcal{A} \cong GF(q) \).

Every \( n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\} \) with \( 0 \leq n \leq q - 1 \) can be expressed as

\[
n = a_0 + a_1 p + \ldots + a_{c-1} p^{c-1}, \quad 0 \leq a_k \leq q - 1, \quad k = 0, 1, \ldots, c - 1.
\]

Let

\[
u(n) = a_0 + a_1 \epsilon_1 + \ldots + a_{c-1} \epsilon_{c-1} p^{-1}.
\]

Then, \( \{\nu(n) : n = 0, 1, \ldots, q - 1\} \) is a complete set of coset representatives of \( \mathcal{D} \) in \( \mathcal{B}^{-1} \). For a non-negative integer \( n \) we write

\[
n = b_0 + b_1 q + b_2 q^2 + \ldots + b_s q^s, \quad 0 \leq b_k \leq q - 1, \quad k = 0, 1, 2, \ldots, s,
\]

and define

\[
u(n) = u(b_0) + u(b_1) p^{-1} + \ldots + u(b_s) p^{-s}.
\]
The function $u(.)$ in general, not additive. But, for all $r \geq 0, k \geq 0$ and $0 \leq s \leq q^k - 1$, we have
$$u(rq^k + s) = u(r)p^{-k} + u(s).$$

**Definition 1.1.** Let $\mathbb{K}$ be a local field of characteristic $p > 0$. The character $\chi$ on $\mathbb{K}$ is defined as
$$\chi(\epsilon, \mu p^{-j}) = \begin{cases} \exp(2\pi i / p), \mu = 0 \text{ and } j = 1 \\ 1, \mu = 1, 2, \ldots, c - 1 \text{ or } j \neq 1, \end{cases}$$
where $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\}$ be same as above.

Note that $\chi$ is trivial on $D$ but nontrivial on $\mathfrak{B}^{-1}$. Since $K = \bigcup_{j \in \mathbb{Z}} p^{-j} \mathfrak{D}$, so we can regard $p^{-1}$ as the dilation (note that $|p^{-1}| = q$). Furthermore, $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in $\mathbb{K}$, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set.

Next, we recall operators which are used in wave packet system. Let $a, b \in \mathbb{K}$. Define operators $T_a, E_b, D_p$ on $L^2(\mathbb{K})$ by
$$T_a f(t) = f(t - a), \; f \in L^2(\mathbb{K}) \quad \text{(Translation by } a)$$
$$E_b f(t) = \chi(bt) f(t), \; f \in L^2(\mathbb{K}) \quad \text{(Modulation by } b)$$
$$D_p f(t) = q^2 f(p^{-1}t) \quad \text{(Dilation by prime } p),$$
As in case of standard wavelet and Gabor analysis these three class of operators play an important role in local field setting. For any $f \in L^2(\mathbb{K})$, we have
$$(D_p f)(\gamma) = q^{\frac{-j}{2}} \hat{f}(p\gamma), \quad (T_a f)(\gamma) = \chi_a(\gamma) \hat{f}(\gamma)$$

**1.2. Wave Packet System over local fields:** The wave packet system over local field of positive characteristic is natural generalization of wavelet structure on local fields. Let $\psi \in L^2(\mathbb{K})$. A system $\mathcal{W}(\psi) = \{D_p T_{u(k)a} E_{u(m)b} \psi : j, k, m \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0\}$ is called an Irregular Weyl-Heisenberg wave packet frame (in short, wave packet frame) for $L^2(\mathbb{K})$. A frame for $L^2(\mathbb{K})$ of the form $\mathcal{W}(\psi)$ is called an irregular Weyl-Heisenberg wave packet frame (or wave packet frame). That if there exists positive scalars $A_\psi \leq B_\psi$ such that
$$A_\psi \|f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} |\langle f, D_p T_{u(k)a} E_{u(m)b} \psi \rangle|^2 \leq B_\psi \|f\|^2 \text{ for all } f \in L^2(\mathbb{K}). \quad (1.2)$$

**1.3. Motivation for the work.** In his paper [1], an approach to determine all frames for separable Hilbert spaces is discussed. More precisely, starting from a frame $\{f_k\}_{k=1}^\infty$ for a separable Hilbert space $\mathcal{H}$, Aldroubi considered the question: “which conditions on the scalars $\{u_{l,k}\}_{k=1}^\infty$ will imply that the vectors
$$\phi_l = \sum_{k=1}^\infty u_{l,k} f_k, \quad l \in \mathbb{N} \quad (1.3)$$
are well defined and constitute a frame for $\mathcal{H}$? Note that by the frame decomposition, it is clear that all frames for $\mathcal{H}$ can be constructed this way. Aldroubi proved necessary and sufficient conditions for $\{\phi_l\}_{l=1}^\infty$ to be a frame for $\mathcal{H}$ (including in terms of boundedness of the operator $U = \{u_{l,k}\}_{l,k=1}^\infty$ on $l^2(\mathbb{N})$).
Kaushik, Singh and Virender [28] introduced the notion of \( WH \) packets with respect to a Gabor system \( \{ E_{mn}T_{\alpha \psi} \}_{m,n \in \mathbb{Z}}, \ (\psi \in L^2(\mathbb{R})) \), which is defined by a linear combination of the form

\[
\psi_{r,s} = \sum_{(j,k) \in I_{r,s}} \alpha_{j,k} E_{j} T_{\alpha \psi}, \ (r, s \in \mathbb{Z}),
\]

where \( \bigcup_{r,s \in \mathbb{Z}} I_{r,s} = \mathbb{Z} \times \mathbb{Z}, I_{r,s} \cap I_{r',s'} = \emptyset, \) for all \( r, s, r', s' \in \mathbb{Z} \) and \( \alpha_{j,k} \) are scalars. They proved necessary and sufficient conditions for \( WH \) packets \( \{ \psi_{r,s} \}_{r,s \in \mathbb{Z}} \) to be a frame for \( L^2(\mathbb{R}) \).

This idea of linear combination given in (1.4) was extended to wave packet setting in [33], where some similar results (given in [28]) are proved. In this paper, we discuss some frame properties of linear combination of wave packet system over local field of positive characteristic.

1.4. Outline of the paper. Let \( W(\psi) \equiv \{ D_{\varphi} T_{u(k)\alpha} E_{u(m)\psi} : j, k, m \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \} \) be a wave packet system in \( L^2(\mathbb{K}) \).

Let

\[
\psi_{r,s,t} = \sum_{(j,k,m) \in I_{r,s,t}} \alpha_{j,k,m} D_{\varphi} T_{u(k)\alpha} E_{u(m)\psi}, \ (r, s, t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}),
\]

where \( \bigcup_{r,s,t \in \mathbb{Z}} I_{r,s,t} = \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, I_{r,s,t} \cap I_{r',s',t'} = \emptyset, \) for all \( r, s, t, r', s', t' \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \) and \( \alpha_{j,k,m} \) are scalars. Then, in general, \( \psi_{r,s,t} (r, s, t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \) does not constitute a frame for \( L^2(\mathbb{R}) \) (even not well defined). We present some necessary and sufficient conditions for \( L(\psi) \equiv \{ \psi_{r,s,t} \}_{r,s,t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}} \) to be a frame for \( L^2(\mathbb{K}) \).

2. Necessary and Sufficient Conditions

The first result extends [28] Theorem 3.5.

**Theorem 2.1.** Let \( W(\psi) \) be a frame for \( L^2(\mathbb{K}) \). Let \( \Theta : \ell^2(\mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \to \ell^2(\mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \) defines a bounded linear operator such that

\[
\Theta : \{ (D_{\varphi} T_{u(k)\alpha} E_{u(m)\psi} \psi, f) \}_{j,k,m \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}} \to \{ (\psi_{r,s,t}, f) \}_{r,s,t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}}, \ f \in L^2(\mathbb{K}).
\]

Then, \( L(\psi) \) is a frame for \( L^2(\mathbb{K}) \) if and only if

\[
\sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{N}} |(\psi_{r,s,t}, f)|^2 \geq \gamma \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{N}} |(D_{\varphi} T_{u(k)\alpha} E_{u(m)\psi} \psi, f)|^2, \ \text{for all} \ f \in L^2(\mathbb{K}).
\]

**Proof.** Similar to proof of [33] Theorem 2.4. \( \square \)

The following theorem provided sufficient conditions on scalars which appear in \( L(\psi) \), to a frame for \( L^2(\mathbb{K}) \). This generalizes [28] Theorem 3.4] in wave packet setting.

**Theorem 2.2.** Assume that \( W(\psi) \) is a frame for \( L^2(\mathbb{K}) \). If \( \sup_{r,s,t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}} \| \psi_{r,s,t} \| < \infty \), and \( \{ \alpha_{j,k,m} \} \) is positively confined sequence and

\[
\inf_{r,s,t \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}} \left\{ \sum_{(j,k,m) \in I_{r,s,t}} |\alpha_{j,k,m}|^2 - \sum_{(j,k,m),(j',k',m') \in I_{r,s,t}} |\alpha_{j,k,m}| \ |\alpha_{j',k',m'}| \right\} > 0,
\]

then \( L(\psi) \) is a frame for \( L^2(\mathbb{R}) \).
Proof. Similar to proof of [33, Theorem 2.5].

Aldroubi proved sufficient conditions on scalars in his linear combination of frame vectors. This result was extended to wave packet setting in [33, Theorem 2.6]. This following theorem gives sufficient conditions on the scalars which similar to standard frames [11, Proposition 5.5.8] to wave packet frames on local fields.

**Theorem 2.3.** Suppose that \( \mathcal{W}(\psi) \) is a frame for \( L^2(\mathbb{K}) \) with frame bounds \( A, B \) and for a fixed \( s \in \mathbb{Z} \), let \( \{\alpha_{s,j,k,m}\} \) be a square summable sequence. If

\[
\mu := \inf_{j,k,m \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0} \left[ \sum_{s \in \mathbb{Z}} |\alpha_{s,j,k,m}|^2 - \sum_{(j',k',m') \in \mathbb{Z}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,j,k,m} \alpha_{s,j',k',m'} \right| \right] > 0,
\]

\[
\nu := \sup_{j,k,m \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0} \left[ \sum_{s \in \mathbb{Z}} \left| \sum_{s \in \mathbb{Z}} \alpha_{s,j,k,m} \alpha_{s,j',k',m'} \right| \right] < \infty.
\]

Then, the vectors \( \sum_{j,k,m \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0} \alpha_{s,j,k,m} D_p T_u(k)a E_{u(m)b} \psi \) constitute a frame for \( L^2(\mathbb{K}) \) with bounds \( \mu A, \nu B \).

**Proof.** Similar to proof of [33, Theorem 2.6].

It is well known that the finite sum of frames need not be a frame for the underlying space. Kaushik et al. [28] proved necessary and sufficient conditions for finite sum of Gabor frames. Similar results for finite sum of in wave packet setting can be found in [33]. The following result extends [28, Theorem 4.2] (in wave packet setting [33, Theorem 2.9]).

**Theorem 2.4.** Let \( \{\mathcal{W}(\psi_l) : l = 1, 2, ..., n\} \) be a finite family wave packet frames for \( L^2(\mathbb{K}) \). Then, \( \left\{ \sum_{l=1}^{n} \alpha_l \mathcal{W}(\psi_l) \right\} \) is a frame for \( L^2(\mathbb{K}) \) if and only if there exists \( M_0 > 0 \) and some \( p \) (\( 1 \leq p \leq n \)) such that

\[
M_0 \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} |\langle D_p T_u(k)a E_{u(m)b} \psi_l, f \rangle|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} \left| \left( \sum_{l=1}^{n} \alpha_l D_p T_u(k)a E_{u(m)b} \psi_l, f \right) \right|^2, f \in L^2(\mathbb{K})
\]

where \( \alpha_1, \alpha_2, ..., \alpha_n \) are scalars.

**Proof.** Similar to proof of [33, Theorem 2.9].

The following theorem extends [28, Theorem 4.3].

**Theorem 2.5.** Let \( \{\mathcal{W}(\psi_l) : l = 1, 2, ..., n\} \) be a finite family wave packet frames for \( L^2(\mathbb{K}) \) with frame bounds \( A_l, B_l \) (\( 1 \leq l \leq n \)). Let \( \alpha_1, \alpha_2, ..., \alpha_l \) be positive scalars. Then, \( \left\{ \sum_{l=1}^{n} \alpha_l \mathcal{W}(\psi_l) \right\} \) is a frame for \( L^2(\mathbb{K}) \) provided

\[
\alpha_p A_p > \sum_{l=1, l \neq p}^{n} B_l + 2 \sum_{l,j=1, l \neq p, l \neq j}^{n} \alpha_l \alpha_j \sqrt{B_l B_j}
\]

for some \( p \) (\( 1 \leq p \leq n \)).
Proof. Let \( f \in L^2(\mathbb{K}) \) be arbitrary.

Then

\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} \left| \left\langle \sum_{l=1}^{n} \alpha_n D_p T_{(k)a} E_{(m)b} \psi_l, f \right\rangle \right|^2 \leq n|\alpha_1|^2 B_1 \|f\|^2 + n|\alpha_2|^2 B_2 \|f\|^2 + \ldots + n|\alpha_n|^2 B_n \|f\|^2
\]

\[
= n \left( \sum_{l=1}^{n} |\alpha_l|^2 \right) \|f\|^2.
\]

This gives upper frame inequality.

By hypothesis (2.1), we can show that

\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} \left| \left\langle \sum_{l=1}^{n} \alpha_n D_p T_{(k)a} E_{(m)b} \psi_l, f \right\rangle \right|^2 \geq \left[ \alpha_p A_p - \sum_{l=1, l \neq p}^{n} B_l + 2 \sum_{l,j=1, l \neq j}^{n} \alpha_l \alpha_j \sqrt{B_l B_j} \right] \|f\|^2, \quad f \in L^2(\mathbb{K}).
\]

Hence \( \left\{ \sum_{l=1}^{n} \alpha_l \psi_l \right\} \) is a frame for \( L^2(\mathbb{K}) \).

\[ \square \]

Corollary 2.6. In Theorem 2.5 if (2.1) is replaced by

\[
\alpha_p A_p < \sum_{l=1}^{n} \alpha_l^2 A_l - \sum_{s,t=1, s \neq t} \alpha_s \alpha_t \sqrt{B_s B_t},
\]

for some \( 1 \leq p \leq n \), then \( \left\{ \sum_{l=1}^{n} \alpha_l \psi_l \right\} \) is a frame for \( L^2(\mathbb{K}) \).

Proof. Similar to proof of [33, Theorem 2.10].

Remark 2.7. Let \( \left\{ \psi_l : l = 1, 2, \ldots, n \right\} \) be a finite family wave packet frames for \( L^2(\mathbb{K}) \) with frame bounds \( A_l, B_l \) (1 \( \leq l \leq n \)). If \( \left\{ \sum_{l=1}^{n} \alpha_l \psi_l \right\} \) is a frame for \( L^2(\mathbb{K}) \) with frame bounds \( A_o \) and \( B_o \). Then, \( A_l, B_l \) and \( A_o, B_o \) are related in the following way:

\[
\sum_{l=1}^{n} \alpha_l^2 A_l - \sum_{l,t=1, l \neq t} \alpha_l \alpha_t \sqrt{B_l B_t} \leq B_o
\]

and

\[
\sum_{l=1}^{n} \alpha_l^2 B_l + \sum_{l,t=1, l \neq t} \alpha_l \alpha_t \sqrt{B_l B_t} \geq A_o,
\]

where square of scalars have standard meaning.

\[ \square \]
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