Matching Queues, Flexibility and Incentives

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Problem definition: Agents in online marketplaces (such as ridesharing and freelancing platforms) are often strategic, and heterogeneous in their compatibility with different types of jobs: fully flexible agents can fulfill any job, whereas specialized agents can only complete specific subsets of jobs. Convention wisdom suggests reserving agents that are more flexible whenever possible, however this may incentivize agents to pretend to be more specialized, leading to loss in matches. We focus on designing a practical matching policy that performs well in a strategic environment. Methodology/results: We model the allocation of jobs to agents as a matching queue, and analyze the equilibrium performance of various matching policies when agents are strategic and report their own types. We show that reserving flexibility naively can backfire, to the extent that the equilibrium throughput can be arbitrarily bad compared to a policy which simply dispatches jobs to agents at random. To balance matching efficiency with agents’ strategic considerations, we propose a new policy dubbed flexibility reservation with fallback and show that it enjoys robust performance. Managerial implications: Our work highlights the importance of considering agent strategic behavior when designing matching policies in online platforms and service systems. The robust performance guarantee, along with the parameter-free nature of our proposed policy makes it easy to implement in practice. We illustrate how this policy is implemented in the driver destination product of major ridesharing platforms.

Key words: online platforms, flexibility, matching queues, ridesharing.

History: This version, 1/2024.

1. Introduction

Matching markets play critical roles in business and society, facilitating the coordination of demands and supplies. A key challenge that emerges in designing matching policies is that they must account for the heterogeneous preferences of strategic agents while maintaining operational efficiency. In a typical matching market, agents need to be matched with jobs with which they are compatible. Different agents can be heterogeneous in their flexibility: the more flexible agents may be compatible
Motivating Applications. Ridesharing platforms such as Uber and Lyft, offer drivers the option to specify a destination area toward which they are willing to take trips. Figure 1a and Figure 1b illustrate how a driver specifies a desired destination area using Uber’s “driver destinations” feature, opting out of trips heading the opposite directions (Uber 2022). For drivers who need to head home or towards social or vocational obligations, this feature allows them to continue to earn on the platform instead of going offline. This provides substantial benefits for both sides: drivers earn more, and a larger driver pool implies shorter pick-up times and better reliability for riders.

If the platform knew exactly drivers’ flexibility over trip destinations, conventional wisdom would implement a matching policy that reserves flexible drivers as much as possible. In other words, a trip toward a specific destination would first be dispatched to a specialized driver who could only serve that destination. If there is no such driver, the trip would then be dispatched to flexible drivers who can serve more destinations. As a consequence of such aggressive reservation of flexible supply, if specialized drivers towards certain destinations experience shorter waiting times to receive a request,
flexible drivers may be incentivized to pretend to be specialized. Such strategic behavior results in a “loss of flexibility”, and could degrade overall system performance relative to not reserving flexibility.

A second application is of gig-economy workers opting in for different types of jobs on the same platform. For example, Uber’s driver partners can decide to receive UberEats (on-demand food delivery) jobs, or Uber rides jobs, or both (see Figure 1c). Drivers often are capable of doing both types of jobs, though private information such as whether a driver is accompanied by a friend in her car on a particular day may prevent them from doing a certain job. To ensure higher service level, it is preferable if the flexible drivers make themselves dispatchable for both types of jobs. However, flexible drivers may have an incentive to restrict themselves to only one type of job, if doing so gives them higher priority for those jobs and reduces their waiting times. We refer interested readers to popular posts from online forums (see e.g., Reddit 2019, UberPeople 2020) where experienced drivers discuss strategies of misreporting their flexibility to gain priority of certain jobs.

The use of monetary incentives can potentially alleviate these strategic concerns, but in many practical settings, monetary incentives are restricted by regulation or business constraints. For example, in 2019 Uber experimented with a 30% earning penalties on drivers that are using destination mode, and evenly distribute these reduced earnings to drivers that are not on destination mode (RideGuru 2019b). The driver feedbacks was negative (see driver discussions in RideGuru 2019a). In addition, paying drivers differently based on their levels of flexibility does not align with the changing regulatory environment (see California Assembly Bill 5 and Proposition 22). Ridesharing platforms are moving towards giving driver partners more flexibility and transparency including sharing trip information upfront, as well as providing drivers the options to accept or decline any trips without any penalties.

**Research Question.** In this paper, we investigate matching markets wherein arriving agents wait (with limited patience) if not matched promptly, while jobs are lost if not matched upon arrival. We focus on settings with heterogeneous agents who have different compatibilities over jobs and may misreport their preferences if doing so leads to a quicker match. Our goal is to understand how matching policies impact the behavior of strategic agents, and to design policies that improve overall system throughput without using monetary incentives. For example, how does a platform that reserves flexibility compare to one that does not offer a feature for agents to express preferences over jobs and the different levels of agent flexibility are simply ignored? In the aforementioned ridesharing examples, this is a baseline policy where the platform always assigns a rider to her nearest driver regardless of the driver’s job preference. If reserving flexibility can potentially backfire, are we able to
design simple, practical and easily implementable matching policies that deliver robust performance in a strategic environment?

**Model.** We study a game-theoretic queueing model, where jobs (e.g., trips in the context of ridesharing) and agents (e.g., drivers) arrive over time according to general counting processes. Each job is associated with one of a finite number of types (e.g., trip destinations, or delivery jobs versus rides jobs), and each agent type is associated with a subset of job types that she is able to fulfill. The platform organizes a set of queues. Agent type is private information and agents report their types by joining corresponding queues. Jobs leave the platform if they are not matched upon arrival, while agents have exponentially distributed patience levels, and abandon the system if they have not been matched when their patience exhausts. Jobs are assigned to agents in a queue at random, while agents are able to see the types of the jobs dispatched to them, and are free to decline jobs without penalty. As we mentioned in the introduction, this is consistent with the current practice and regulation, where agents are independent contractors of the platforms who are given the freedom to choose what jobs to accept (Rideshare Guy 2019, California Proposition 22 2020). However, jobs are impatient to agent rejections and they might be lost after excessive rejections. We study the impact of matching policies on the system’s throughput, i.e., the number of matches per unit of time.

**Contributions.** We now summarize our key contributions.

- We first confirm our intuition by showing that when agents’ types are known, the full-information first-best policy for throughput is achieved by reserving flexibility whenever possible (Proposition 1) as flexible agents are more “valuable” for future matches. We call policies satisfying such properties flexibility reservation (FR) policies.

- When agents are strategic and their types are private information, we show that such flexibility reservation may lead to longer waiting time for a match for flexible agents. This incentivizes them to under-report the set of jobs they are able to fulfill, and this loss of flexibility may lower the system throughput. Indeed, in Proposition 3, we show that the equilibrium throughput under the flexibility reservation policy can become arbitrarily bad compared to a naïve policy which simply dispatches jobs to agents uniformly at random.

- To balance matching efficiency with agents’ strategic considerations, we propose a new policy dubbed flexibility reservation with fallback (FRfb). Intuitively, the FRfb policy retains a similar structure to the FR policies (i.e., reserving more flexible agents when possible), but offers additional seemingly incompatible edges along which jobs can be dispatched. In particular, when there is no available compatible agents to match, the job will be sent to a pool of seemingly incompatible agents since some of them might be under-reporting and are, in fact, compatible with the job. In contrast
to the potential fragility of the FR policy, we show that, for a natural family of nested compatibility graphs (that are common in the aforementioned ridesharing examples) the proposed FRfb policy enjoys robust performance improvement. In particular, under any market conditions, and regardless of the strategy profile taken by the agents, the FRfb policy always achieves higher throughput than the random policy (Theorem 1). We further demonstrate its performance via extensive simulations over general compatibility graphs. This robust performance guarantee, along with its parameter-free nature, makes our FRfb policy easy to implement in practice. In particular, we illustrate how this policy is implemented in the driver destination product of a major ridesharing platform.

Organization of the paper. In Section 2, we discuss related work. In Section 3, we introduce the main model elements and analyze the full-information and non-strategic setting. In Section 4, we consider strategic agents and study the performance of flexibility reservation in this setting. In Section 5, we propose and analyze the flexibility reservation policy with fallback. We conduct extensive simulation experiments in Section 6 to demonstrate the performance of various policies. We conclude and discuss a real-world implementation in Section 7. All proofs and auxiliary technical results are presented in the appendix as well as the online supplement.

2. Literature Review

In this section, we briefly review related work.

Skill-based queueing systems. Our queueing-theoretic modeling framework generally falls under the skill-based server models. In these models, servers are flexible in the types of customers they can serve, and customers are flexible in the servers at which they can be processed. These models are motivated by problems such as call centers where service representatives may speak different languages. Under the first-come first-served (FCFS) service discipline, various variants of the model exhibit product-form steady-state distributions. Notable works include Adan et al. (2009), Visschers et al. (2012) and Adan and Weiss (2014). We also refer readers to a recent excellent overview by Gardner and Righter (2020) which synthesizes various related technical results in the field. Our modeling framework differs from this stream of literature in that we focus on random queueing discipline instead of FCFS, motivated by quality-driven dispatch used in, for example, ride-sharing platforms. As a consequence, existing product-form results do not apply (see, e.g., Castro et al. 2020 for a setting similar to ours under FCFS). Our focus is on designing a performant dispatching policy, while this stream of works mainly concerns about performance evaluation under particular dispatching policies.

Queueing games. Our work also belongs to the literature on managing queues with strategic agents. In his pioneering work, Naor (1969) shows that agents’ selfish joining behaviors can lead to
inefficient system outcomes. He argues that to address such inefficiencies, the system provider can rely on monetary transfers to restore first-best outcome. Related to our setting, the role of monetary transfers have also been studied in strategic queues with priority. There are two perspectives about this type of queues. First, there are works that analyze priority schemes within queues, see e.g., Kleinrock (1967), Dolan (1978), Hassin (1995), Afeche and Mendelson (2004) and Yang et al. (2017). Essentially, in this line of research, customers pay an amount to participate in an auction that determines (or trades) their position (priority) in the queue. The second perspective, which is more aligned with our work, corresponds to priorities between classes of customers (see e.g., Cobham 1954). In this type of model there is a single server who serves customers with higher priority first (preemptive or non-preemptive regimes can be accommodated), and then continues with customers of lower priorities. In the present work, motivated by regulatory and business constraints outlined in Section 1, we do not consider designing monetary incentives. As a consequence, attaining the most efficient outcome might no longer be possible.

In non-monetary settings, there are problems where strategic agents have diverse options when joining a system, and the system provider is constrained to use matching or scheduling policies to optimize the system performance. One of the main challenges in managing these systems is that matching policies alone are, typically, not enough to incentivize agents to make system-wide optimal choices. Parlaktürk and Kumar (2004), for example, consider a system with two queues in which strategic agents choose to start their service in one of the queues (and then continue on the other) to minimize their sojourn time. Their focus is on the design of state-dependent scheduling rules of server resources that have a performance close to the first best. In the context of call centers, Armony and Maglaras (2004) study a model in which heterogeneous, time-sensitive customers strategically choose between two modes of service (online or call back option). The system’s manager designs how to allocate servers between the two modes to maximize quality of service subject to certain operational constraints; see Hassin (2009) for a related framework with two queues. The key features that make our study distinct from these previous works are that in our setting the number of queues is a design choice, not all servers are equal and there is an underlying compatibility graph.

Finally, our study is also related to the work on allocating heterogeneous items. In such models, items with heterogeneous values are assigned to agents in specific order and agents are free to accept or reject them. In the context of kidney transplants, Su and Zenios (2006) propose a market design that is similar to ours. However, their system admits a decomposition of the queues because the matching polices they analyze are not state-dependent. In our case, such a decomposition is not possible because our state-dependent policies couple the queues. Su and Zenios (2004) also study
the role of queueing discipline on patient acceptance and kidney allocation efficiency where they show the last-come first-served (LCFS) discipline can mitigate the impact of patient rejection and propose a variation of the first-come first-served (FCFS) discipline to achieve most of the benefits. In a ridesharing setting concerning dispatching jobs with heterogeneous earnings to drivers in an airport virtual queue, Castro et al. (2021) propose a family of randomized FCFS queueing disciplines to reduce drivers’ incentives of cherry-picking high earning jobs. Hassin and Nathaniel (2021) consider a cyclic setting where, upon accepting an item, the agent returns to the end of the queue. They fully characterize the equilibrium behavior of the proposed queueing discipline and demonstrate its improvement over FCFS. In our model, the queueing discipline is fixed while we study the effect of matching policies. Leshno (2022) studies waiting lists for allocating two types of stochastically arriving items to an infinite collection of infinitely patient agents with private heterogeneous preferences over item types. The work focuses on minimizing welfare loss, which occurs when a waiting list grows too long, causing an agent to accept a less preferred item rather than wait for their most preferred item. In our setting, short-lived agents arrive stochastically and we focus on maximizing throughput.

**Non-monetary mechanism design.** In addition to the queueing literature above, there is also a stream of work stemming from scheduling and algorithmic mechanism design that considers systems with strategic agents and no monetary transfers. In the prototypical setting, strategic agents decide in which shared resource or machine to complete a task to minimize its finish time (see e.g., Koutsoupia and Papadimitriou 1999). The common theme of these works is to analyze the resulting equilibrium and, in some cases, the price of anarchy. For examples, Christodoulou et al. (2004) consider the case of coordination mechanisms, Ashlagi et al. (2010) study a setting with competing schedulers, Ashlagi et al. (2013) consider a related problem in a queueing setting, and Koutsoupia (2014) studies a setting in which machines can lie about the time it takes them to complete a task. A main difference of our work with the aforementioned papers is that, in our setting, tasks have different types which makes them compatible only with a subset of machines (agents). There are other works, however, that do incorporate a compatibility graph among agents and jobs. For example, Dughmi and Ghosh (2010) consider a version of the generalized assignment problem in which agents can misreport their compatibility with tasks (or, more generally, their value for tasks) for a given matching design. Motivated by kidney exchange programs, Ashlagi et al. (2015) consider the problem of designing a matching mechanism that makes it incentive compatible for the hospital to reveal its compatibility graph. Like this past work, we consider the design of matching policies without payments. Significantly and unlike this past work, our matching policies critically depend on the dynamically evolving state of the system.
3. Preliminaries

In this section, we introduce our model and characterize the optimal matching policy in the full information setting, where agent types are observed by the platform. Later, in Section 4, we extend the model to the private information setting.

3.1. Model

We study a matching platform where jobs and agents arrive randomly over time. Jobs and agents are of different types. Let $\mathcal{L} = \{1, \ldots, \ell\}$ be the set of job types and $\mathcal{A} = \{1, \ldots, d\}$ be the set of agent types. Agents’ preferences over jobs are determined by a compatibility graph. For each job $j \in \mathcal{L}$, $\mathcal{A}(j)$ is the set of agent types that are compatible with job type $j$ and similarly for each agent type $i \in \mathcal{A}$, $\mathcal{L}(i)$ is the set of job types agent type $i$ can serve. For two types of agents $i \neq i' \in \mathcal{A}$, we say that agent type $i$ is more flexible than agent type $i'$ (or agent type $i'$ is more specialized than agent type $i$) if $\mathcal{L}(i) \supset \mathcal{L}(i')$. Figure 2 below depicts an example of the compatibility graph.

![Compatibility Graph](image)

**Figure 2** The compatibility between agents and jobs.

Jobs and agents of different types arrive according to some general arrival processes with arbitrary distributions of inter-arrival times that do not have to be independently or identically distributed. We assume that a job is lost if it is not matched upon arrival, while agents have exponentially-distributed patience levels and those that have not been matched abandon the platform at rate $\theta > 0$. Let $A^{(t)} = (A_1^{(t)}, A_2^{(t)}, \ldots, A_d^{(t)})$ denote the state of the system given by the number of agents of each type on the platform at time $t \geq 0$, where $A_i^{(t)}$ is the number of agents of type $i$ at time $t$. Sometimes, we drop the time index $t$ to represent static values.

A policy determines how the platform matches incoming jobs to agents on the platform. We consider a space of policies $\Phi$, where each policy $\phi \in \Phi$ is a mapping from the current state (i.e., the number of agents of each type) and the type of the arriving job to a distribution over (compatible) agent types. To ease the presentation, we defer the formal definition of the set of feasible policies to Appendix A. In other words, an arriving job is assigned to an agent type depending on the state of the system...
and the job type. Once a particular type is selected, the policy then dispatches the job to any agent with that type. We call a policy non-idling if it always assigns a job to an agent if there are available compatible agents. Our goal is to investigate how various policies affect the system’s performance in terms of throughput, i.e., the long-run average number of matches. Let $M^\phi(T)$ denote the total number of matches made under policy $\phi$, up to time $T > 0$ and $\mathbb{E}[M^\phi(T)|A^{(0)} = A]$ is the expected number of matches up to time $T$ given that the initial number of agents on the platform is $A \in \mathbb{Z}_{\geq 0}$. The expectation is taken over all the randomness in the system, i.e., the arrival of agents and jobs, agents’ abandonment, and the policy $\phi$. The throughput under a policy $\phi$, $TP(\phi)$, is of the form:

$$TP(\phi) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[M^\phi(T)|A^{(0)} = A]. \quad (1)$$

Note that this long-run average is invariant to the initial condition $A^{(0)}$. We call the matching policy that maximizes throughput under full information the first-best policy (FB).

### 3.2. Observable Agent Types: Flexibility Reservation is Optimal

In comparison to specialized agents, flexible agents are more valuable for the platform because they can fulfill more job types. Hence, a good matching policy should prioritize reserving flexible agents as much as possible.

**Definition 1** (Flexibility Reservation). A matching policy satisfies the Flexibility Reservation (FR) property if it dispatches a job of type $j$ to a type $i \in A(j)$ agent before a different type $i' \in A(j)$ agent as long as $L(i) \subset L(i')$.

The following result formally shows that when agents’ types are known to the platform, FR is satisfied by an optimal policy.

**Proposition 1.** Among all policies in $\Phi$, there exists an optimal non-idling policy achieving the highest throughput that satisfies the FR property.

**Remark 1.** Proposition 1 leads to a complete characterization of an optimal matching policy under nested compatibility graphs. A compatibility graph is nested if for each type of job $j \in \mathcal{L}$, the set of compatible agents $A(j) = \{i, i', i'', \ldots\}$ can be ordered in a way such that $\mathcal{L}(i) \subset \mathcal{L}(i') \subset \mathcal{L}(i'') \subset \cdots$. Proposition 1 then implies that a policy that dispatches any job of type $j \in \mathcal{L}$ in this order is optimal. Figure 3 below gives an example of such nested compatibility graphs where $A = \mathcal{L} = \{1, \ldots, \ell\}$, and there is one flexible agent (type 1) that can serve all types of jobs (i.e., $\mathcal{L}(1) = \mathcal{L}$) and the rest of the agents can can only serve their corresponding types of jobs (i.e., $\mathcal{L}(i) = \{i\}, \forall i \neq 1$). This compatibility graph appears in the ride-sharing examples mentioned in Section 1: flexible agents
represent drivers who can serve rides to all locations or that can serve both passengers and deliveries, and the specialized agent represents drivers who can serve rides to a particular location or that can serve either passengers or deliveries but not both.

![Diagram of nested compatibility graphs](image)

**Figure 3** An example of nested compatibility graphs where there is one flexible agent (type 1) that can serve all job types, and the rest of the agents are specialized that can only serve its corresponding type of jobs.

To prove Proposition 1, we build on the following result, which states that under an optimal matching policy, having an additional more flexible agent always results in (weakly) higher expected throughput than having one extra more specialized agent of any type, all else being equal. Moreover, there is no value of idling — it is (weakly) better off to always match the job when there are compatible agents available. Let $e_i \in \mathbb{Z}_{\geq 0}^\ell$ be the unit vector which is zero except for the $i^{th}$ element.

**Lemma 1 (Value of a More Flexible Agent).** The following holds

$$1 + \sup_{\phi \in \Phi} \mathbb{E}[M^{\phi}(T) \mid A^{(0)} = A] \geq \sup_{\phi \in \Phi} \mathbb{E}[M^{\phi}(T) \mid A^{(0)} = A + e_i] \geq \sup_{\phi \in \Phi} \mathbb{E}[M^{\phi}(T) \mid A^{(0)} = A + e_i']$$

for any time horizon $T \geq 0$, any $A \in \mathbb{Z}_{\geq 0}^\ell$ and any agent types $i \neq i' \in A$ such that $\mathcal{L}(i) \supset \mathcal{L}(i')$.

To see why Lemma 1 implies Proposition 1, suppose that there exists an optimal policy $\phi'$ that does not satisfy the FR property. Then at some moment $t$, for two types of agents $i \neq i' \in A$ such that $\mathcal{L}(i) \supset \mathcal{L}(i')$ with $A_i > 0$ and $A_{i'} > 0$, policy $\phi'$ assigns an incoming job to a compatible type $i$ agent instead of a compatible type $i'$ agent. Now consider an alternative action that assigns the same job to a type $i'$ agent. By the second inequality of Lemma 1, continuing running the optimal policy from this moment onward weakly improves throughput. This suggests that any optimal policy that does not satisfy the FR property can be modified to satisfy such a property without decreasing its throughput. Similar arguments can be made on the sub-optimality of any idling policies using the first inequality in Lemma 1.
4. Strategic Agents

The previous section assumed that the platform can observe agents’ true types. In practice, however, such information is usually private to the agents. Consider ridesharing, for example, where a driver can signal that she is of a specialized type by turning on the “driver destination” feature and accept only trips towards a specific destination, or opt to take delivery jobs only. Whether the driver indeed can perform only the jobs indicated cannot be observed by the platform. This section studies matching policies where agent types are private and agents are strategic.

4.1. Priority Matching Queues

We implement matching policies for strategic agents via priorities queues. The agents choose queues that signal their types. Then, the platform dispatches jobs to queues based on the job type and a priority list. Each queue has a different job-type-dependent dispatching priority, and we allow two queues to have the same priority for a given job type. Agents are told the type of job offered and may decline it, at which point the platform dispatches to another agent.

When a job is dispatched to a queue (or multiple queues with the same priority), we assume that it will be assigned uniformly at random among agents in that queue (or those queues). Under the assumption of a spatially uniform distribution of drivers and request locations (assumed by, e.g., Castillo et al. 2017, Yan et al. 2020 and Besbes et al. 2022), random allocation is equivalent to the closest-match or first-dispatch protocol widely used by ridesharing platforms, which dispatches a rider request to its closest driver.

We assume that an agent is free to reject jobs of a type that she is not able or willing to serve. In our setting, since all jobs have the same reward, agents accept any compatible job (and reject incompatible ones). However, they can strategize to reduce their waiting time by choosing the queue to join that minimizes their equilibrium expected waiting time; see Section 4.2 for a formal discussion.

We now introduce notation for formally defining the matching policies that we study in the strategic setting. A matching policy $\pi \triangleq (Q, \rho)$ consists of a set of queues $Q$ and a tuple $\rho = (\rho_1, \rho_2, \ldots, \rho_\ell)$. The set $Q$ can be of any size. Let $Q \in \mathbb{Z}_{\geq 0}^{|Q|}$ be the number of agents in each queue. For each job type $j \in \mathcal{L}$, $\rho_j$ specifies an ordered list of subsets of $Q$ that are mutually exclusive. A type $j$ job would first be distributed uniformly at random without replacement to the agents in the first subset of queues in the ordered list $\rho_j$ until someone accepts it, the job is lost or there are no more agents in the subset of queues to offer the job to. We assume that jobs have limited patience for rejections — the job will be lost after too many rejections. If the job remains unmatched (either because the first subset of queues are all empty or because all the agents in the first subset of queues rejected the job) and has not yet been lost due to rejection, it is sent to an agent in the next subset of queues, selected again
uniformly at random without replacement. The job is eventually lost if it remains unmatched after exhausting all queues in the ordered list.

We model the patience of a job using a generic function \( \beta(a, b) \in [0, 1] \) that indicates the probability of a successful match if the job is sent to a subset of queues with a total of \( b \) agents among which \( a \) are compatible. We make the following assumptions regarding \( \beta(a, b) \).

**Assumption 1.** We assume that

1. \( \beta(a, b) \) is non-increasing in \( b \) and \( \beta(a, a + c) \) is non-decreasing in \( a \) for any fixed \( c \geq 0 \);
2. \( \beta(a, a + c) \) is convex in \( c \);
3. \( \beta(0, b) = 0, \forall b \geq 0 \) and \( \beta(b, b) = 1, \forall b > 0 \).

The first bullet point says increasing the number of agents while keeping the number of compatible agents constant leads to a lower probability of success, and increasing the number of compatible agents while maintaining the incompatible constant leads to a higher probability of success. Note that these two conditions imply that \( \beta(a, b) \) is non-decreasing in \( a \). The second bullet point states that additional incompatible agents have a diminishing negative effect in the probability of success. For example, if we assume that each rejection by an agent results in an i.i.d. event in which the job survives with some probability, one can show that the resulting \( \beta(a, b) \) satisfies Assumption 1.

We now discuss how various policies can be described using this notation. Consider the FR policy under the nested compatibility graph in Figure 3. There is one flexible type of agent that can serve all types of jobs, and the rest are all specialized agents that can only serve one particular type of job. Under this compatibility graph, as indicated by Remark 1, the FR property fully characterizes the optimal matching policy. This particular FR policy \( \pi^{FR} \equiv (Q^{FR}, \rho^{FR}) \), as illustrated in Figure 4a, can be interpreted as offering \( \ell \) queues \( (Q^{FR} = \{1, 2, \ldots, \ell\}) \), and the ordered list is specified by \( \rho^{FR}_1 = (\{1\}) \) and \( \rho^{FR}_j = (\{j\}, \{1\}) \) for all \( j \in L \setminus \{1\} \). For this particular example, we sometimes refer to queue 1 as the *flexible queue*, and call the queues 2 through \( \ell \) the *specialized queues*. Alternatively, consider a random policy \( \pi^{RND} \) that simply assigns jobs to agents uniformly at random regardless of whether they are compatible or not. This policy \( \pi^{RND} \equiv (Q^{RND}, \rho^{RND}) \) offers a single queue \( (Q^{RND} = \{1\}) \), and the ordered lists contain only queue 1 for all types of jobs, i.e., \( \rho^{RND}_j = (\{1\}) \) for all \( j \in L \). See Figure 4b. In the ridesharing examples, this corresponds to not offering the driver destination feature, or not allowing drivers to specify in advance whether they will accept dispatches for Uber rides, food delivery, or both, and simply dispatch the job to its closest driver.

### 4.2. Strategic Behavior of Agents

We now describe the strategic behavior of agents. Upon arrival, each agent chooses which queue to join with the goal of minimizing the expected waiting time until a successful match. We denote the
(a) Flexibility Reservation (FR) under a particular nested compatibility graph with one type of fully flexible agent that is compatible to all types of jobs and $\ell - 1$ types of specialized agents each being compatible to only one type of job. The solid arrows from jobs to queues represent the first priority, and the dash-dotted arrows represent the second priority.

(b) A random policy (RND) where jobs are sent to agents randomly regardless of their compatibility.

Figure 4 Examples of the FR and the random policies, when agent types are not observed by the platform.

strategy of agents of type $i$ as $\sigma_i = [0, 1]^{|Q|}$, where $\sigma_{i,q}$ for each $q \in Q$ is the probability with which a type $i$ agent joins queue $q$ and $\sum_{q \in Q} \sigma_{i,q} = 1$.

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d)$ denote the strategy profile of all agent types. We assume that upon arrival, agents decide on which queue to join based on the steady-state expected waiting times of each queue. We assume that agents know the expected waiting times till a match through repeatedly interacting with the platform (see, e.g., Reddit 2019 for a discussion among experienced Uber’s drivers regarding the waiting time comparison of opting in different types of dispatches), and do not switch queues once they join. We note that under the random policy there is only one possible strategy profile — all types of agents join queue 1. In what follows it will be convenient to denote the set of all possible
strategy profiles as $\Sigma(Q) \triangleq \prod_{i \in A} \Sigma_i(Q)$, where $\Sigma_i(Q)$ is the set of probability measures over the set of queues $Q$.

Consider a strategy profile $\sigma \in \Sigma(Q)$ under some policy $\pi$. Let $W^\pi_{i,q}(\sigma)$ be the steady-state time a type $i \in A$ agent would wait until matched given strategy profile $\sigma$, if she decided to join queue $q \in Q$. This is also termed virtual waiting time in the queueing literature (De Kok and Tijms 1985). $\mathbb{E}[W^\pi_{i,q}(\sigma)]$ is the expected steady-state virtual waiting time where the expectation is taken over all randomness including the state of the system when the agent joins. We say that $\sigma$ forms a Nash equilibrium if and only if for all $i \in A$ and all $q \in Q$:

$$\sigma_{i,q} > 0 \Rightarrow \mathbb{E}[W^\pi_{i,q}(\sigma)] \leq \mathbb{E}[W^\pi_{i,q'}(\sigma)], \forall q' \in Q.$$ (NE)

Intuitively, eq. (NE) requires that if type $i$ agents join queue $q$ with non-zero probability, the expected (virtual) waiting time from joining queue $q$ must be the smallest among any queue.

The following result guarantees the existence of a Nash equilibrium for any policy and model primitives such that the expected steady-state virtual waiting times, $\mathbb{E}[W^\pi_{i,q}(\sigma)]$, are continuous with respect to the strategy profile $\sigma$. Its proof consists of writing eq. (NE) as an equivalent variational inequality. We then cast it as a fixed point equation which is guaranteed to have a solution in virtue of Brouwer’s fixed-point theorem.

**Proposition 2 (Existence of Nash Equilibrium).** For any matching policy $\pi$ such that the expected waiting times $\mathbb{E}[W^\pi_{i,q}(\sigma)]$ is continuous in agents strategies $\sigma$ for all $i \in A$ and all $q \in Q$, there exists a strategy profile $\sigma$ that forms a Nash equilibrium.

### 4.3. Perils of Flexibility Reservation

In contrast to Proposition 1 showing the optimality of the FR policy under perfect information, when agent types are private, there might not exist a policy satisfying FR, even under a nested compatibility graph, that achieves the maximum throughput. Consider, the example of Figure 4a and a scenario where the arrival rate of type 1 job is low. Under the FR policy, the flexible agents may have an incentive to pretend to be specialized in order to reduce their waiting times. This loss of flexibility in turn leads to the loss of type 1 jobs that could have been fulfilled, and might result in an overall decrease of throughput. In fact, we show in this subsection that it can achieve arbitrarily lower throughput than the naive random policy $\pi^{RND}$.

We now provide an example and its corresponding analysis using the nested compatibility graph in Figure 3. Let $\sigma^{FR}$ be an equilibrium strategy profile under the FR policy. Let $\text{TP}(\pi; \sigma)$ denote the throughput under policy $\pi$ and strategy profile $\sigma$, defined as $\limsup_{T \to \infty} (1/T) \mathbb{E}[M^\pi(T)]$ where $M^\pi(T)$ is the number of matches up to time $T$ under policy $\pi$. Note that since there is only one
possible strategy profile $\sigma^{RND}$ under the random policy (all drivers joining the same queue), we use $\text{TP}(\pi^{RND})$ to denote its throughput.

**Proposition 3 (FR Can Be Arbitrarily Bad).** Consider the aforementioned compatibility graph of Figure 3 with two types of agents and jobs, $\mathcal{L} = \{1, 2\}$ and $\mathcal{A} = \{1, 2\}$. For any $\beta(a, b)$ satisfying Assumption 1, there exists instances such that for any $\delta > 0$

\[
\text{TP}(\pi^{\text{FR}}; \sigma^{\text{FR}}) \leq \delta \cdot \text{TP}(\pi^{\text{RND}}).
\]

As mentioned above, FR performs poorly when the flexible agents have an incentive to pretend to be specialized and switch to queue 2. This happens when the rate of type 1 job is low so that queue 1 is less attractive. On the other hand, only a high rate of type 1 jobs makes reserving flexibility matter — in the extreme case where the rate of type 1 jobs is zero, all policies effectively become the same. In turn, the proposition identifies settings in which such trade-off resolves against FR, leading flexible agents to switch to queue 2 and the platform to lose significant throughput at the same time.

Consider a situation in which type 1 jobs have a high arrival rate only during certain times and a low arrival rate most of the time, while type 2 jobs arrive more consistently. In this case, under FR, flexible agents might prefer to exclusively serve jobs that arrive at a more consistent rate. This, in turn, limits the throughput of the system to that of the jobs with a more consistent arrival rate. However, by sending all jobs to the entire pool of agents, type 1 jobs can be matched during the period they have a high arrival rate, potentially leading to a significant throughput difference between the two policies. One real-world example comes from Uber drivers choosing between food delivery and rides. UberEats requests arrive at a high rate only during certain times of the day (e.g., dinner time) while UberRides requests are more consistent.

To formalize this intuition, we let agents of type 1 and type 2 arrive at the platform according to two Poisson processes with the corresponding rates $\lambda_1$ and $\lambda_2$. Similarly, jobs of type 2 arrive according to a Poisson process with rate $\mu_2$. However, we assume that jobs of type 1 arrive according to a Markov-modulated process (MMP) with two states $\{L, H\}$ — $H$ as a high state with higher arrival rate $\mu_{1,H}$ and $L$ as a low state with lower arrival rate $\mu_{1,L}$. Denote by $\kappa_{L \rightarrow H}$ the transition rate from state $L$ to $H$, and $\kappa_{H \rightarrow L}$ the transition rate from state $H$ to $L$.

Specifically, suppose that the MMP is such that the arrival rate of jobs is zero, $\mu_{1,L} = 0$, in the low state and $\mu_{1,H} \geq \lambda_1$, to be determined, in the high state. Additionally, assume that the time it takes from low to high states has mean $1/\varepsilon$ for $\varepsilon > 0$ (that we will take small) while the time it takes to go from high to low state has mean 1. In turn, under $\pi^{\text{FR}}$, a type 1 agent that joins the first queue would have to wait $\Omega(1/\varepsilon)$. However, if all agents join the second queue, we can approximate the
virtual wait time of a type 1 agent by \((\lambda_1 + \lambda_2 - \mu_2^+)/\theta \). This is because, under \(\pi^{FR}\), when all agents join the second queue, the fluid equilibrium queue length is \(Q^* = (\lambda_1 + \lambda_2 - \mu_2^+)/\theta\). Since the jobs are sent at random to agents in the queue, a back of the envelope calculation gives that the virtual waiting time, \(W_{1,2}\), is \(Q^*/\mu_2\) — it solves \(W_{1,2} = 1/\mu_2 + (1 - 1/Q^*)W_{1,2}\). All type 1 agents would then prefer to join the second queue if \((\lambda_1 + \lambda_2 - \mu_2^+)/\theta \mu_2 = o(1/\varepsilon)\). If the latter is satisfied, the throughput under \(\pi^{FR}\) is bounded by

\[
\text{TP}(\pi^{FR}; \sigma^{FR}) \leq \mu_2, \tag{3}
\]
as all type 1 jobs will be lost. In contrast, under \(\pi^{RND}\), there is only one queue where all jobs are sent to. In particular, type 1 jobs are sent to this queue. Consider a cycle of the MMP (a cycle goes through a complete high state and a low state), it takes an average of \(1/\varepsilon + 1\) to complete the cycle. If we only consider the potential matches between type 1 agents and type 1 jobs during the high state of the MMP, we could potentially have at least a throughput of \(\min\{\lambda_1, \mu_{1,H}\}/(1/\varepsilon + 1)\).

However, this throughput is not fully attainable due to two issues: (1) some of the type 1 agents will abandon the system at rate \(\theta\) and, (2) because the assignment is random, we will lose some of the type 1 jobs when attempting a match if there are too many type 2 agents in the queue. To address the first, we can consider the type 1 agents that simultaneously satisfy three conditions: (i) those who have patience larger than \(\Delta > 0\) (the value of \(\Delta\) will be determined later), (ii) those for which a type 1 job arrives to the system within \(\Delta\) of their arrival to the queue, and (iii) those for which the next type 1 agent arrives \(\Delta\) after their arrival to the queue. These arrivals give a number of matches that is (approximately) at least \(\lambda_1 e^{-\Delta \theta} (1 - e^{-\mu_{1,H} \Delta}) e^{-\lambda_1 \Delta}\). For the second issue, we note that the number of type 2 agents can never grow without bounds because of the reneging. In turn, there is effectively a maximum number of total agents (on average), say \(N\), whenever the system attempts to match a type 1 job to a type 1 agent. In turn, the number of matches is (approximately) at least \(\lambda_1 e^{-\Delta \theta} (1 - e^{-\mu_{1,H} \Delta}) \beta(1, N) > 0\). Hence, under \(\pi^{RND}\) the throughput can be approximately lower bounded by

\[
\text{TP}(\pi^{RND}) \geq \frac{\lambda_1 e^{-\Delta \theta} (1 - e^{-\mu_{1,H} \Delta}) e^{-\lambda_1 \Delta} \beta(1, N)}{1/\varepsilon + 1}.
\]

Now, note that for \(\text{TP}(\pi^{RND})\) to be large or grow without bound, the values of \(\Delta, \theta\) and \(\mu_{1,H}\) must be such that the probability of reneging within \(\Delta\) is small and that the probability of a type 1 job arriving within \(\Delta\) is large enough. One sufficient condition is to have \(\Delta \mu_{1,H} \uparrow \infty\) and \(\Delta \theta \downarrow 0\). In addition, we need to have enough type 1 agent arrivals: \(\varepsilon \lambda_1 \uparrow \infty\). To satisfy these conditions, we can set \(\Delta = \varepsilon^2\), \(\theta = \log_{10}(1/\varepsilon)/\varepsilon\), \(\mu_{1,H} = \log_{10}(1/\varepsilon)/\varepsilon^2\) and \(\lambda_1 = \log_{10}(1/\varepsilon)/\varepsilon\). With this choice, we have

\[
\text{TP}(\pi^{RND}) \geq \varepsilon \frac{\log_{10}(1/\varepsilon)}{\varepsilon} e^{-\varepsilon^2 \log_{10}(1/\varepsilon)} (1 - e^{-\varepsilon^2 \log_{10}(1/\varepsilon)}) e^{-\log_{10}(1/\varepsilon)^2/\varepsilon^2} \beta(1, N),
\]
which grows to $\infty$ as $\varepsilon \downarrow 0$. Moreover, under this scaling, we also have $(\lambda_1 + \lambda_2 - \mu_2)^+/(\theta \mu_2) \approx \log_{10}(1/\varepsilon) = o(1/\varepsilon)$ which suggests that all type 1 agents are incentivized to join queue 2 and all type 1 jobs are lost. Thus the throughput of the FR policy $\pi^{\text{FR}}$ is bounded above by (3) as we are not scaling $\mu_2$. This suggests that the throughput under $\pi^{\text{RND}}$ is arbitrarily larger than the throughput under $\pi^{\text{FR}}$. In the proof of Proposition 3, we provide a rigorous treatment for these arguments. It is worth mentioning that we can show Proposition 3 still holds if jobs are assigned to agents in queues with an FCFS manner. For the sake of space, we omit the proof here.

5. Flexibility Reservation with Fallback

The possibility for policies satisfying the FR property to achieve arbitrarily lower throughput than a random policy is intriguing but also unsatisfactory — in ridesharing, for example, this means that the platform may sometimes be better off ignoring the job preferences drivers set when making matching decisions or, more drastically, not offering the driver destination feature altogether. In this section, we introduce the Flexibility Reservation with Fallback (FRfb) policy, which has a more versatile matching process.

**Definition 2 (Flexibility Reservation with Fallback).** The Flexibility Reservation with Fallback (FRfb) policy organizes a set of queues, one for each type of agent, $Q^{\text{FRfb}} = A$. It dispatches a job of type $j \in L$ following an order of $\rho^{\text{FRfb}}_j = (Q_{j,1}, Q_{j,2}, \cdots, Q_{j,K})$ where $Q_{j,1}, Q_{j,2}, \cdots, Q_{j,K}$ form a partition of $Q^{\text{FRfb}}$. In particular, $Q_{j,1}, Q_{j,2}, \cdots, Q_{j,K-1}$ form a partition of the set of compatible queues $A(j)$ where $Q_{j,k} = \{i \in A(j) : |L(i)| = k\}, \forall k \in \{1, \cdots, K - 1\}$, i.e., the set of compatible queues that can serve exactly $k$ types of jobs. On the other hand, the last subset $Q_{j,K}$ is constructed as $Q_{j,K} = Q^{\text{FRfb}} \setminus A(j)$, i.e., the subset of queues that are incompatible with serving type $j$ jobs.

It can be seen from the construction that FRfb still reserves flexibility and satisfies Definition 1. It first attempts to match a job $j$ with agents that can only serve jobs of type $j$, and then it tries to match the job with agents of increasing flexibility (that can serve more than one job). On the other hand, a distinguishing feature of the FRfb policy is that if there is no successful match after exhausting all agents in the compatible queues, the job will be dispatched to the rest of the agents in the incompatible queues uniformly at random until the job is matched or lost. This exploits the fact that some compatible agents might “hide” in incompatible queues. We now give a couple of examples to illustrate the construction of the FRfb policy.

**Example 1.** Consider two types of agents and jobs, $L = \{1, 2\}$ and $A = \{1, 2\}$, with type 1 agents being flexible and type 2 agents being specialized. Under the nested compatibility graph of Figure 3,
the FRfb policy is specified by $Q^{FRfb} = \{1, 2\}$, $\rho_1^{FRfb} = (\{1\}, \{2\})$, and $\rho_2^{FRfb} = (\{2\}, \{1\})$. In comparison to the FR policy, the novelty of the FRfb policy is to add a seemingly incompatible edge from type 1 jobs to the specialized queue (see the blue dash-dotted arrow in Figure 5). Note that if agents are not strategic and each joins their designated queues, this step in the FRfb policy would not add any value. However, when flexible agents are strategic, this allows type 1 jobs to be dispatched to the specialized queue when the flexible queue is empty. Hence, some of these jobs that would have been lost would be completed by the flexible agents who joined the specialized queue.

**Figure 5** An example of FRfb policy under a nested compatibility graph with two types of jobs and agents. The FRfb policy adds a seemingly incompatible edge (in blue) from type 1 jobs to the specialized queue 2. The solid arrows from jobs to queues represent the first priority, and the dash-dotted arrows represent the second priority.

**Example 2.** Consider a general compatibility graph with $L = 3$ types of jobs and the set of agents (queues) $A$ serving jobs $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}$. We call them agent types (and queue numbers) 1 to 7 respectively. In this case,

$$\begin{align*}
\rho_1^{FRfb} &= (\{1\}, \{4, 6\}, \{7\}, \{2, 3, 5\}), \\
\rho_2^{FRfb} &= (\{2\}, \{4, 5\}, \{7\}, \{1, 3, 6\}), \\
\rho_3^{FRfb} &= (\{3\}, \{5, 6\}, \{7\}, \{1, 2, 4\}).
\end{align*}$$

We now give some insights regarding the performance of the FRfb policy. At first glance, the FRfb policy makes better use of flexible agents — jobs that cannot be fulfilled by agents in the compatible queues will be dispatched to the rest of the agents, and can potentially be fulfilled by the compatible agents therein. This benefit, however, does come with a cost since the resulting higher service rate for the incompatible queues may incentivize more flexible agents to pretend to be less flexible. The following result formalizes this observation with an example.

**Proposition 4.** Consider the nested compatibility graph in Figure 3 with two types of agents and jobs, $\mathcal{L} = \{1, 2\}$ and $A = \{1, 2\}$. For any model primitives, there exists equilibrium strategies $\sigma^{FRfb}$ and $\sigma^{FR}$ such that the fraction of flexible agents joining the specialized queue 2 in equilibrium is weakly higher under the FR policy than that under the FRfb policy, i.e., $\sigma_{1,2}^{FRfb} \geq \sigma_{1,2}^{FR}$. 


Proposition 4 relates to Braess’ paradox (see e.g., Braess 1968): the addition of a seemingly beneficial edge may lead to an undesired outcome in equilibrium. In our setting, this proposition shows that more flexible agents may choose to join queue 2, and as a consequence, the platform may end up using flexible agents in an inefficient manner by matching more type 2 jobs to flexible agents.

Fortunately, despite this potential drawback of the FRfb policy, we show a result that is in sharp contrast to Proposition 3. While the FR policy can perform arbitrarily bad in equilibrium compared to a random policy under the nested compatibility graph of Figure 3 with only two types of agents, we show in the following theorem that the FRfb policy’s performance is always better than the randomized policy for the same type of the compatibility graph under any number of agent types and any nondominated strategy profile, in equilibrium or not.

**Theorem 1 (Robust Performance of the FRfb Policy).** Consider the aforementioned compatibility graph of Figure 3 with any number of agent and job types, under FRfb, it is a dominant strategy for specialized agents to stay their corresponding queues. For any such strategy profile, i.e., \( \sigma \in \Sigma(Q^{FRfb}) \) such that \( \sigma_{i,i} = 1 : \forall i \neq 1 \), the throughput achieved by the FRfb policy is always higher than or equal to that under the random policy:

\[
TP(\pi^{FRfb}; \sigma) \geq TP(\pi^{RND}).
\]

Theorem 1 establishes that the FRfb policy outperforms the random policy under any strategy played by the type 1 flexible agent. In turn, this result provides supporting evidence for the good performance of FRfb in strategic environments. Although Theorem 1 is restricted to the type of compatibility graphs of Figure 3, later in Section 6, we will see that FRfb attains good performance in a variety of compatibility graphs and problem instances.

Under the nested compatibility graph of Figure 4a with two types of agents and jobs, \( L = \{1, 2\} \) and \( A = \{1, 2\} \), the FRfb policy becomes exactly the same as the random policy when all flexible agents switch to queue 2. As the switching fraction decreases, FRfb improves over random. When there are more than two types, it is interesting to note that there is a strict separation between FRfb and random, i.e., FRfb is strictly better than random because, regardless of how flexible agents pretend to be specialized, the two policies can never become the same.

The proof of Theorem 1 relies on a key observation: fixing the number of agents of each type on the platform, for any type \( j \neq 1 \) job, the probability matching a job to a fully flexible agent under FRfb is always (weakly) lower than that under the random policy. Moreover, this always holds regardless of how many fully flexible agents pretend being specialized by joining the specialized queues under FRfb. To see this, suppose that at time \( t \), the number of agents of each type on the platform is
\( A^{(t)} = (A_1^{(t)}, A_2^{(t)}, \ldots, A_k^{(t)}) \). Conditional on the job being matched, the random policy matches a type \( j \neq 1 \) job to a flexible agent with probability \( A_1^{(t)}/(A_1^{(t)} + A_j^{(t)}) \), whereas the FRfb policy randomizes among \( A_j^{(t)} \) type \( j \) agents and those flexible agents who hide in queue \( j \neq 1 \), which is upper bounded by \( A_1^{(t)} \). This reveals that the FRfb policy is better than the random policy in terms of preserving flexible agents. The proof formalizes this intuition.

In addition to Theorem 1, Proposition 5 below gives a \( 1/2 \) performance guarantee of the throughput under FRfb compared to the full information first-best throughput under any compatibility graphs and any strategy profile when agent rejection has no penalty. This further sets apart the FR and the FRfb policies as Proposition 3 indicates that FR has no constant factor performance guarantee.

**Proposition 5 (Half Approximation).** When \( \beta(a, b) = 1, \forall a > 0 \), policy FRfb achieves at least a half of the full-information first-best throughput under any strategy profile \( \sigma \in \Sigma(Q^{\text{FRfb}}) \):

\[
\frac{\text{TP}(\pi^{\text{FRfb}}; \sigma)}{\sup_{\phi \in \Phi} \text{TP}(\phi)} \geq \frac{1}{2}.
\]

**A numerical example.** We now give a numerical example to illustrate our theoretical results. In particular, we consider the nested compatibility graph shown in Figure 3 with two types of agents and jobs \( L = \{1, 2\} \) and \( A = \{1, 2\} \) (type 1 agent is flexible and type 2 agent is specialized). Following the example used in Proposition 3, agents of types 1 and 2 arrive according to Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \). Similarly, jobs of type 2 arrive according to a Poisson process with rate \( \mu_2 \); on the other hand, jobs of type 1 arrive according to a Markov-modulated process with two states \( \{L, H\} \) — \( H \) as a high state with higher arrival rate \( \mu_{1,H} \) and \( L \) as a low state with lower arrival rate \( \mu_{1,L} \). Denote \( \kappa_{L \rightarrow H} \) as the transition rate from state \( L \) to \( H \), and \( \kappa_{H \rightarrow L} \) as the rate vice versa. We consider a situation where these rates scale with a parameter \( \varepsilon \) in a similar fashion as those in Proposition 3 (see the caption of Figure 6 for the specific scaling). We model the patience of the job to rejections and the success probability of a match \( \beta(a, b) \) in the following way. We assume that each rejection by an agent results in an independent event in which the job survives with probability \( p \). It is not difficult to see that the resulting \( \beta(a, b) \) satisfies Assumption 1. We consider two scenarios, one in which jobs have infinite patience, \( p = 1 \), and another in which jobs will be lost (on average) if declined by five agents, \( p = 0.8 \). We note that ride-sharing drivers typically have at most 15 seconds to accept or decline a job, so this gives a rider patience for about one minute, which is quite conservative.

We compare three policies: (1) the FR policy whose performance is independent of the value of \( p \) as it never sends jobs to incompatible agents; (2) the random policy (RND) that sends jobs to agents uniformly at random whose performance depends on \( p \); (3) our proposed FRfb policy whose performance also depends on \( p \) as type 1 jobs will be sent to possibly type 2 agents if there is no agent
in queue 1. We report their performance in Figure 6 as $\varepsilon$ decreases. Figure 6a shows the throughput as a fraction of the first-best, the maximum throughput under full information. Figure 6b reports the fraction of flexible agents that pretend to be specialized in equilibrium.

Corroborating Proposition 3, Figure 6a shows that the performance of the FR policy (red curves) continues to deteriorate as $\varepsilon \downarrow 0$ as more and more flexible agents join queue 2, see Figure 6b, and we lose the opportunity to match type 1 jobs. In contrast, the proposed FRfb policy consistently delivers near-optimal throughput. FRfb is slightly outperformed by FR when $\varepsilon$ is relatively large, reflecting the Braess’ paradox result (the comparison of the equilibrium fraction corroborates Proposition 4). In fact, as shown in Figure 6b, when $p = 1.0$ and $\varepsilon$ is relatively large, all flexible agents join queue 2. This is because when $\varepsilon$ is relatively large, almost all jobs arrive to queue 2. A flexible agent therefore waits less being in queue 2, because she now gets higher priority for type 2 jobs while still getting dispatched the type 1 jobs. But in contrast to the FR policy, this does not imply that type 1 jobs will be lost, as the FRfb policy will attempt to match them with the flexible agents in queue 2. Under the FR policy, however, as long as the waiting time for type 1 job is low enough, it is to the best interest of flexible agents to stay in queue 1 because they no longer get type 1 jobs once they switch to queue 2. In turn, in equilibrium, FRfb may operate exactly as RND, while FR may achieve the first best, which explains the throughput gap between FR and FRfb ($p = 1.0$) when $\varepsilon$ is relatively large.

It is interesting to observe that the throughput of the FRfb policy does not necessarily decrease as the survival probability $p$ decreases. On the one hand, a low value of $p$ makes type 1 jobs harder to match successfully; on the other hand, it also makes queue 2 less attractive as it effectively increases the waiting time of flexible agents in it, resulting in fewer flexible agents switching to queue 2 (see Figure 6b). In fact, in the extreme case of $p = 0$, a job is immediately lost after one agent rejection, FR and FRfb become the same in equilibrium.

Finally, note that the throughput comparison of the RND and FRfb policies in Figure 6a under the same values of $p$ confirms Theorem 1. Perhaps more interestingly, in this case, the throughput of FRfb with $p = 0.8$ is also higher than the random policy with $p = 1.0$, further highlighting the robustness of the FRfb policy.

6. Simulation Experiments

In this section, we conduct extensive simulation experiments to demonstrate the performance of various policies under general compatibility graphs. We consider three types of compatibility graphs: (1) $G_1$ is the nested compatibility graph depicted in Figure 3 with $\mathcal{A} = \mathcal{L} = \{1, \cdots, \ell\}$ where the fully flexible type 1 agent can serve all types of jobs ($\mathcal{L}(1) = \mathcal{L}$) and type $i \neq 1$ agents are specialized
**Figure 6** Performance of various policies. Figure 6a depicts throughput as a fraction of the first-best throughput for policies FR, RND and FRfb under two values of job survival probabilities $p \in \{0.8, 1.0\}$ after each agent rejection. Figure 6b shows the fraction of flexible agents that join queue 2 in equilibrium for policies FR and FRfb under two values of job survival probabilities $p \in \{0.8, 1.0\}$. The model primitives are $\lambda_2 = 30$, $\mu_2 = 10$, $\mu_1, L = 12$, $\lambda_1 = \log_{10} (1/\varepsilon)^2/\varepsilon$ and $\mu_1, H = \log_{10} (1/\varepsilon)^2/\varepsilon$.

to serve only type $i$ jobs ($\mathcal{L}(i) = \{i\}$, $\forall i \neq 1$); (2) $G_2$ is another nested compatibility graph with $\mathcal{A} = \mathcal{L} = \{1, \cdots, \ell\}$ and $\mathcal{L}(i) = \{i, i+1, \cdots, \ell\}$, i.e., all agent types are completely nested with $\mathcal{L}(1) \supset \mathcal{L}(2) \supset \cdots \supset \mathcal{L}(\ell)$; and finally (3) $G_3$ is a complete graph with $\mathcal{A} = 2^\mathcal{L}$, i.e., each agent type can serve a particular subset of job types. We compare the following policies.

1. **The (near) optimal matching policy under full information (FB).** The exact optimal policy under full information for compatibility graphs $G_1$ and $G_2$ can be uniquely pinned down using the optimality of flexibility reservation (Definition 1 and Proposition 1) as they are nested. On the other hand, for the complete compatibility graph $G_3$, the optimal policy is intractable to compute due to the curse of dimensionality. We thus construct a near-optimal policy using an affine value function approximation and solving a corresponding approximate linear program to obtain weights for each queue (de Farias and Van Roy 2003). This effectively yields a priority list $\rho_j$ for each job type $j$ where $\rho_j$ is a strict order of all compatible queues. The order of the queues is determined by the order of the affine weights. The details of how these weights are computed are provided in Appendix B. As before, we denote by FR this policy evaluated under agents’ equilibrium joining strategies.

2. **Random policy (RND).** Jobs are sent to agents uniformly at random without replacement.

3. **Flexibility reservation with fallback policy (FRfb).** This is the proposed policy constructed according to Definition 2.
4. An iterative procedure to approximate a Stackelberg equilibrium (SB). This is a popular heuristic (see, e.g., Marcotte and Marquis 1992) to approximate the solution of a Stackelberg game (in our case, the leader is the platform who sets up a matching policy and the followers are agents reporting their types) or the so-called second-best, the matching policy maximizing the throughput when agents are strategic. The platform organizes a set of queues, one for each agent type. The iterative procedure starts by assuming that all agents truthfully report their types by joining designated queues. It optimizes the matching policy based on the rates of reported agent types. Agents then form a new equilibrium joining strategy and the matching policy is re-optimized according to the rates of the reported types under the new equilibrium. This iterative procedure terminates when neither the platform wants to deviate from the matching policy nor the agents want to deviate from their joining strategies. The converging policy does not necessarily correspond to a Stackelberg equilibrium (a subgame perfect equilibrium) but rather to a Nash equilibrium at which neither the leader nor the followers can unilaterally improve their prospects. We report the performance of the matching policy that has the highest equilibrium throughput during this iterative process. To optimize the matching policy given the rates of reported agent types for the complete compatibility graph $G_3$, we use the same affine value function approximation technique described in Appendix B. For compatibility graphs $G_1$ and $G_2$, on the other hand, the optimal matching policy does not depend on the rates of reported agent types. As a consequence, the iterative procedure will converge after one iteration, and its performance is the same as that of the FR policy.

**Computing equilibrium joining strategy.** Computing the joining equilibrium exactly with a large number of agent types can be a daunting and computationally intensive task. Instead, we use an evolutionary dynamics technique called replicator dynamics to approximate the agents’ equilibrium strategies. These dynamics are initialized with a strategy profile $\sigma^0 = \{\sigma_1^0, \sigma_2^0, \ldots, \sigma_d^0\}$. Denote by $u^\pi_{i,q}(\sigma) = \exp(-\beta E[W_{i,q}^\pi])$ the payoff of an agent of type $i$ joining queue $q$ under policy $\pi$ if all other agents adopt the strategy profile $\sigma$ and $\gamma$ is some parameter that represents a learning rate. In the experiment, we choose $\gamma = 10$. Maximizing this payoff is equivalent to minimizing virtual waiting time, which is aligned with eq. (NE). Let $\bar{u}_{i}^\pi(\sigma) = \sum_{q \in Q} \sigma_{i,q} u^\pi_{i,q}(\sigma)$ be the average payoff of an agent type $i$. We use the following updating rule:

$$
\sigma_{i,q}^{t+1} - \sigma_{i,q}^t = \sigma_{i,q}^t \cdot \frac{u^\pi_{i,q}(\sigma^t) - \bar{u}_{i}^\pi(\sigma^t)}{\bar{u}_{i}^\pi(\sigma^t)}, \quad \forall i \in \mathcal{A}, q \in \mathcal{Q}.
$$

(4)

It is not hard to see that as long as $\sum_{q \in Q} \sigma_{i,q}^t = 1$, we have $\sum_{q \in Q} \sigma_{i,q}^{t+1} = 1$, i.e., $\sigma^{t+1}$ remains a valid strategy profile. Moreover, if the replicator dynamics (4) converges, i.e., $\sigma^{t+1} = \sigma^t = \sigma^*$ and $\sigma^*$ is stable in the sense that there exists a neighborhood of $\sigma^*$ such that starting from any $\sigma^0$
in this neighborhood, the replicator dynamics (4) approaches $\sigma^*$, then $\sigma^*$ is a Nash equilibrium strategy (Taylor and Jonker 1978, Cressman 2003). For each iteration $t$, we estimate $u^\pi_i(q)(\sigma^t)$ using Monte-Carlo simulation and terminate the dynamics as long as the weighted variance of agent payoff
$$\sum_{q \in Q} \sigma^t_{i,q}(u^\pi_i(q)(\sigma^t) - \bar{u}^\pi_i(\sigma^t))^2$$
is small enough for each agent type $i \in A$ — when the weighted variance
$$\sum_{q \in Q} \sigma^t_{i,q}(u^\pi_i(q)(\sigma^t) - \bar{u}^\pi_i(\sigma^t))^2 = 0,$$
an equilibrium is reached.

**Simulation setup and results.** We set the number of job types $\ell = 4$. We assume that agents and jobs arrive to the platform according to Poisson processes and the total arrival rates of jobs and agents are $\sum_{j \in L} \mu_j = 4\ell$ and $\sum_{i \in A} \lambda_i = 5\ell$, respectively. The total arrival rate of agents is made slightly higher than the rate of jobs to balance overall demand and supply as agents renege in the system. The reneging rate of all agents is assumed to be $\theta = 1$. Given the total arrival rates of agents, the proportion of each type is sampled from a symmetric Dirichlet distribution with parameter $\alpha = \{\alpha_1, \cdots, \alpha_d\}$. We consider two types of distributions. One is a symmetric Dirichlet distribution with $\alpha_1 = \cdots = \alpha_d = 1$, i.e., on average each type of agent has the same proportion; the other one has $\alpha_i = |L(i)|, \forall i \in A$, i.e., on average type $i$ agent has a proportion of $|L(i)|/(|\sum_{i \in A} |L(i)||)$ — more flexible agents on average have a larger presence. On the other hand, the proportion of each job type is always sampled from a symmetric Dirichlet distribution with all parameters equal to one. We average the performance over 100 random draws of $[\lambda_i]_{i \in A}$ and $[\mu_j]_{j \in L}$. Similarly to the example in Figure 6, we assume that each rejection by an agent results in an independent event in which the job survives with probability $p \in \{0.8, 1.0\}$.

For each matching policy, we compute the equilibrium joining strategy using the aforementioned replicator dynamics with Monte-Carlo simulation. The length of each simulation run is $T = 10,000$. After each run, the payoff is calculated and the strategy profile is updated according to eq. (4). This procedure continues until convergence. Table 1 below provides detail of the computational results. The first four rows report the throughput of different policies as a ratio over the throughput under FB, the (approximate) full-information first-best. It can be observed that our FRfb policy consistently delivers the highest throughput. Under the complete compatibility graph $G_3$, FRfb even outperforms the approximate first-best policy with ratios over 1. It is also interesting to see that the FRfb policy under rejection penalty ($p = 0.8$) outperforms the random policy without any rejection penalty ($p = 1.0$), strengthening the message of Theorem 1. The performance of the FR policy degrades especially when flexible agents possess larger proportion in the system (the case of $\alpha_i = |L(i)|, \forall i \in A$) as the incentive of under-reporting their types increases. In these cases, their performance is worse than those of the random policy when rejection has no penalty, corroborating Proposition 3. Under the complete compatibility graph $G_3$, the SB policy improves upon the FR
policy, though not by much. The four rows in the middle of Table 1 show the fraction of jobs lost due to excessive agent rejection, out of the total number of jobs lost. The FRfb policy has a range of 1% – 8% when \( p = 0.8 \), which is significantly lower than those of the random policy. This is reassuring as the FRfb policy only sends the jobs to potentially incompatible agents as a last resort, leading to relatively low rejection probabilities. The last four rows report the fraction of agents who misreport their types in equilibrium. Corroborating Proposition 4, more agents misreport their types in FRfb compared to FR. On the other hand, SB seems to slightly better incentivize agents to report their true types under the compatibility graph \( G_3 \), which might explain the throughput improvement over FR.

| Agent Distribution | \( \alpha_i = 1, \forall i \in A \) | \( \alpha_i = |\mathcal{L}(i)|, \forall i \in A \) |
|--------------------|-----------------|-----------------|
| **Compatibility Graphs** | \( G_1 \) | \( G_2 \) | \( G_3 \) | \( G_1 \) | \( G_2 \) | \( G_3 \) |
| **Survival Probability (\( p \))** | 0.8 | 1.0 | 0.8 | 1.0 | 0.8 | 1.0 | 0.8 | 1.0 | 0.8 | 1.0 | 0.8 | 1.0 |
| **Throughput (ratio over FB)** | FR | 0.986 | 0.986 | 0.958 | 0.958 | 0.969 | 0.969 | 0.956 | 0.956 | 0.945 | 0.945 | 0.958 | 0.958 |
| | RND | 0.821 | 0.942 | 0.891 | 0.954 | 0.864 | 0.954 | 0.898 | 0.957 | 0.926 | 0.962 | 0.898 | 0.964 |
| | FRfb | 0.990 | 0.996 | 0.981 | 0.992 | 1.041 | 1.059 | 0.969 | 0.976 | 0.979 | 0.989 | 1.046 | 1.061 |
| | SB | 0.986 | 0.986 | 0.958 | 0.958 | 0.978 | 0.978 | 0.956 | 0.956 | 0.945 | 0.945 | 0.964 | 0.964 |
| **Jobs Lost due to Rejection** | (fraction of total lost jobs) | FR | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| | RND | 0.245 | 0.000 | 0.216 | 0.000 | 0.252 | 0.000 | 0.309 | 0.000 | 0.219 | 0.000 | 0.237 | 0.000 |
| | FRfb | 0.017 | 0.000 | 0.043 | 0.000 | 0.071 | 0.000 | 0.041 | 0.000 | 0.066 | 0.000 | 0.085 | 0.000 |
| | SB | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| **Agents Deviated** | (fraction of total agents) | FR | 0.051 | 0.051 | 0.211 | 0.211 | 0.511 | 0.511 | 0.191 | 0.191 | 0.347 | 0.347 | 0.650 | 0.650 |
| | RND | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| | FRfb | 0.095 | 0.142 | 0.278 | 0.313 | 0.692 | 0.707 | 0.297 | 0.325 | 0.434 | 0.469 | 0.819 | 0.831 |
| | SB | 0.051 | 0.051 | 0.211 | 0.211 | 0.483 | 0.483 | 0.191 | 0.191 | 0.347 | 0.347 | 0.636 | 0.636 |

Table 1 Simulation results.

Finally, we report in Figure 7 the average virtual waiting times aggregated by different priority group where the priority group of type \( i \) agent is calculated as \( |\mathcal{L}(i)| \), the number of job types it can serve. Roughly speaking, agents with a smaller priority group will be prioritized for matching under FR and FRfb. We consider a rejection penalty \( p = 0.8 \) and compare policies FR, FRfb as well as the random policy under no rejection penalty to give it more advantage. The virtual waiting time distributions over priority groups under FR and RND represent two extremes, more flexible agents enjoy relatively low waiting times under RND as they serve more job types, while less flexible agents benefit under FR as they get prioritized. The FRfb policy strikes a balance between the two extremes: flexible agents have lower waiting time compared to those under the FR policy because they get more dispatches even if they pretend to be less flexible; on the other hand, more specialized agents experience longer waiting time compared to those under the FR policy as they face more competition from flexible agents as there are more of them under-reporting their types.
7. Conclusion and Discussion

One may wonder if there exists a matching policy that induces completely truthful reporting as
the FRfb policy potentially encourages more agents to under-report their types. For example, using
the nested compatibility graph of Figure 3 with two types, it is tempting to think about a policy
that partially prioritizes flexible agents e.g., a type 2 job can be sent to queue 2 first with a given
probability, and this probability can be fine tuned to make sure no type 1 agent wants to misreport
their types. However, calculating this probability would require precise knowledge of the arrival rates
as well as assumptions on agent strategic behaviors, and can be computational intensive with multiple
agent types. On the other hand, Figure 7 seems to suggest that the equilibrium outcome under the
FRfb policy has a similar feature — compared to FR, it reserves flexibility to a lesser extent, but it
is completely parameter-free.

From a practical point of view, the simplicity of the FRfb policy makes it a good candidate for
implementation. Consider our ridesharing example. Figure 8 depicts how FRfb is implemented in
the actual driver destination product at Uber. Figure 8a shows the dispatch screen when a driver
who opts in the destination mode, that reports to be specialized, receives a job towards her chosen
destination, which represents a dispatch of job type \( j \neq 1 \) to queue \( j \) in the model (see Figure 3 for
the corresponding compatibility graph). Figure 8b shows the dispatch screen when the same driver
is being dispatched a job which is away from her chosen destination, this represents a dispatch of
job type 1 to queue \( j \neq 1 \). In both cases, the driver is free to accept (by clicking the “tap to accept”
button in the figure) or reject (by clicking the “no thanks” button in the figure) the dispatch based on
her underlying true preferences. For the case in Figure 8b, a “true” specialized driver will reject the
dispatch whereas a “pretended” specialized but flexible driver will accept the dispatch. According to
FRfb, such away-from-destination dispatches will only occur when the platform runs out of drivers who reported being flexible.

(a) A driver with destination mode receives a ride toward her destination
(b) A driver with destination mode receives a ride away from her destination

Figure 8 An implementation of the flexibility reservation with fallback (FRfb) policy in the driver destination product on the Uber’s driver app.

In conclusion, while finding optimal policies that partially prioritizes flexible agents can potentially lead to improved throughput, their performance would be dictated by the proper calibration of the system parameters and agents strategic behaviors. In contrast, the FRfb policy, as illustrated in the ridesharing example with Uber, stands out for its simplicity and practicality. The policy’s parameter-free nature and robust performance guarantee make it a practical choice for implementation.

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Appendix A: Formal Definition of a Matching Policy under Full Information

Here we formally define the set of matching policies we consider under the full information setting. The policy acts each time a job arrives. Let \( j \in \mathcal{L} \) be an arbitrary type of arriving job. Let \( A \) indicate the state at the time when this job arrives where \( A = (A_1, A_2, \ldots, A_d) \in \mathbb{Z}_{\geq 0}^d \) contains the number of waiting agents of each type. An assignment policy is a function \( \phi: \mathbb{Z}_{\geq 0}^d \times \mathcal{L} \rightarrow [0, 1]^{d+1} \), where \( \phi_i(A, j) \) denotes the probability of assigning a type \( j \) job to a type \( i \) agent, when the number of agents of different types is \( A \). In addition, \( \phi_0(A, j) \) is the probability of losing the job, which means that the incoming job request is rejected either intentionally or because there is no available compatible agent. This allows us to consider idling policies as well where the platform can choose to reject jobs even if there are compatible agents available. The set of admissible policy \( \Phi \) is defined as:

\[
\Phi \triangleq \left\{ \phi : \begin{array}{l}
\sum_{i \in \{0, \ldots, d\}} \phi_i(A, j) = 1, \\
\forall j \in \mathcal{L}, \forall A \in \mathbb{Z}_{\geq 0}^d \\
\phi_i(A, j) = 0, \\
\forall j \in \mathcal{L}, \forall i \in \mathcal{A}, \forall A: A_i = 0 \\
\phi_i(A, j) = 0, \\
\forall j \in \mathcal{L}, \forall i \neq \mathcal{A}(j)
\end{array} \right\}. \tag{5}
\]

The second row of eq. (5) restricts the admissible policy to be a probability measure over all actions — there are \( d + 1 \) actions in total including \( d \) types of agents to be assigned and option 0 representing rejection of the job; the third row prevents assigning jobs to an agent type without available agents; the last row ensures that jobs are only assigned to compatible agents.
Appendix B: Computing Near-Optimal Matching Policy under Full Information

We develop an approximate dynamic programming approach to obtain a near-optimal matching policy under full information and general compatibility graphs. We assume that the arrival processes of agents and jobs follow Poisson processes so that the underlying dynamic optimization problem can be cast as a Markov decision process. We denote by $\lambda_i$ the arrival rate of agent type $i \in \mathcal{A}$ and $\mu_j$ the arrival rate of job type $j \in \mathcal{L}$. Using the definitions of state and policy in Section A, we define the (relative) value function $V(A)$ for each possible state $A \in \mathbb{Z}^d_{\geq 0}$. It is not hard to check that the resulting Markov chain under any matching policy is a unichain. The maximum throughput rate $g$ therefore satisfies the following optimality equation (Puterman 2014):

$$g = \sum_{i \in \mathcal{A}} \lambda_i (V(A + e_i) - V(A)) + \sum_{i \in \mathcal{A}} \theta A_i 1_{\{A_i > 0\}} (V(A - e_i) - V(A)) + \sum_{j \in \mathcal{L}} \mu_j \Gamma_{A,j}, \quad \forall A \in \mathbb{Z}^d_{\geq 0},$$

This can be reformulated as the following countably infinite linear program.

$$\min_{g,V,\Gamma} g$$

s.t. $g \geq \sum_{i \in \mathcal{A}} \lambda_i (V(A + e_i) - V(A)) + \sum_{i \in \mathcal{A}} \theta A_i 1_{\{A_i > 0\}} (V(A - e_i) - V(A)) + \sum_{j \in \mathcal{L}} \mu_j \Gamma_{A,j}, \quad \forall A \in \mathbb{Z}^d_{\geq 0},$

$$\Gamma_{A,j} \geq 0, \quad \forall j \in \mathcal{L}, \forall A \in \mathbb{Z}^d_{\geq 0},$$

$$\Gamma_{A,j} \geq 1 + V(A - e_i) - V(A), \quad \forall j \in \mathcal{L}, \forall A \in \mathbb{Z}^d_{\geq 0}, \forall i \in \mathcal{A} : A_i > 0.$$

To reduce the number of decision variables (noting that there are countably-infinitely many), we use an affine value function approximation $V(A) \approx \alpha_0 + \sum_{i \in \mathcal{A}} \alpha_i A_i$ and the affine weights $[\alpha_i]_{i \in \mathcal{A} \cup \{0\}}$ are to be optimized. Replacing $V(A)$ with its affine approximation in the above linear program yields the following approximate linear program (de Farias and Van Roy 2003),

$$\min_{\beta,\alpha,\Gamma} g$$

s.t. $g \geq \sum_{i \in \mathcal{A}} \lambda_i \alpha_i - \sum_{i \in \mathcal{A}} \theta A_i 1_{\{A_i > 0\}} \alpha_i + \sum_{j \in \mathcal{L}} \mu_j \Gamma_{A,j}, \quad \forall A \in \mathbb{Z}^d_{\geq 0},$

$$\Gamma_{A,j} \geq 0, \quad \forall j \in \mathcal{L}, \forall A \in \mathbb{Z}^d_{\geq 0},$$

$$\Gamma_{A,j} \geq 1 - \alpha_i, \quad \forall j \in \mathcal{L}, \forall A \in \mathbb{Z}^d_{\geq 0}, \forall i \in \mathcal{A} : A_i > 0.$$

A practical technique to solve the above approximate linear program is through constraint sampling on the state space $\mathbb{Z}^d_{\geq 0}$. In particular, only “important” states are included by simulating the induced Markov chain under some baseline policies. de Farias and Van Roy (2004) show that only a polynomial
number of states in the number of agent types \(d\) are needed to get a near-optimal policy with high probability if the sampling distribution is chosen properly. In our experiments, states are sampled by simulating the random policy.

Once the optimal weights \([\alpha_i^*]_{i \in A}\) are obtained from solving the approximate linear program, the resulting matching policy simply follows a priority list structure: a job of type \(j\) will be sent to a type \(i \in A(j)\) agent in an increasing order of \([\alpha_i^*]_{i \in A(j)}\) depending on which type of agent is available. If all agents with types within \(A(j)\) are not available, the job is lost.

**Appendix C: Additional Results**

**Lemma 2.** For any compatibility graph, under the random policy,

\[
1 + \mathbb{E}[M^{RND}_k(T) | A^{(0)} = A] \geq \mathbb{E}[M^{RND}_k(T) | A^{(0)} = A + e_i],
\]

for all \(i \in \mathcal{L}\), any time horizon \(T \geq 0\) and any \(A\).

**Proof.** As we fix the policy to be the random policy, we suppress the policy notation in the rest of the proof. Let \(M_k(A, t)\) denote the expected number of matches starting with state \(A\) from time \(t\) up until the time horizon \(T\) expires or until we reach a total of \(k\) events, whichever occurs first. Events include job arrivals, agent arrivals, and reneging. As the proof of Lemma 1,

For any finite \(k\) and any state \(A\), \(|M_\infty(A, t) - M_k(A, t)|\) is bounded by the expected number of events (job arrival events in particular) in excess of \(k\) over \([t, T]\). This upper bound vanishes as \(k\) grows large. Thus, \(\lim_{k \to \infty} M_k(A, t) = M_\infty(A, t)\) for each \(A\) and \(t\). Thus, to show our claim, it is sufficient to show that \(1 + M_k(A, t) \geq M_k(A + e_i, t)\) for all finite \(k\) and all \(i \in \mathcal{L}\). We now show this via induction on \(k\).

This induction hypothesis is trivially true for \(k = 0\) and so we turn our attention to showing it is true for \(k > 0\) assuming that it is true for \(k - 1\). Consider two systems at time \(t\), one with state \(A\), and the other with state \(A + e_i\). We will refer to the two systems as \(S_0\) and \(S_1\) respectively.

We fix a sequence of agent and job arrivals within \([t, T]\). Also consider a collection of independent exponential random variables representing the times until each of several possible reneging events occur in the system. Then the next event occurs randomly within the following categories:

1. reneging by an agent counted in \(A\), from each type with a non-zero component in \(A\);
2. an agent arrival, from each of the \(\ell\) types;
3. a job arrival, from each of the \(\ell\) types;
4. reneging by the agent not counted in \(A\), which is of type \(i\) in system \(S_1\).
We let random variable $t'$ represent the sum of $t$ and the minimum of these times. The random variable achieving this minimum determines the event that occurs next. If $T$ occurs before any of these times, then the time horizon expires before the next event occurs.

For either system $S_j$, $j = 0, 1$, let random variable $m(j)$ represent whether a match results from the next event, let $A'(j)$ represent the system state after the event occurs. Let $\Delta$ indicate whether the next event occurs in both systems ($\Delta = 1$) or just system $S_1$ ($\Delta = 0$, which occurs on event type 4). Thus,

$$M_k(A, t) = \mathbb{E}[m(0) + M_{k-\Delta}(A'(0), t')] \geq \mathbb{E}[m(0) + M_{k-1}(A'(0), t')]$$

$$M_k(A + e_i, t) = \mathbb{E}[m(1) + M_{k-1}(A'(1), t')]$$

The expectation is taken over $m(j), A'(j)$ and $t'$. To show our result, we will condition on the next event on a case-by-case basis to show that

$$1 + m(0) + M_{k-1}(A'(0), t') \geq m(1) + M_{k-1}(A'(1), t'),$$

(7)

where the comparison represents first-order stochastic dominance. If $t' > T$, then $M_{k-1}(A'(j), t') = 0$, $m(j) = 0$ for all $j$, automatically verifying this expression. Thus, it is sufficient to focus on $t' \leq T$.

**Case 1:** The next event is reneging by an agent counted in $A$ or an agent arrival. In this case, let $A''$ represent $A$ modified by this event, so that $A'(0) = A''$ and $A'(1) = A'' + e_i$. Also, $m(0) = m(1) = 0$. Then, by the induction hypothesis, $1 + M_{k-1}(A'', t') \geq M_{k-1}(A'' + e_i, t')$, showing (7).

**Case 2:** The next event is a job of type $j \notin A(i)$. In this case, $m(0)$ is a Bernoulli random variable with probability $\beta(\sum_{v' \in A(i)} A_{v'}, \sum_{v' \in \mathcal{L}} A_{v'})$ being one, and $m(1)$ is a Bernoulli random variable with probability $\beta(\sum_{v' \in A(i)} A_{v'}, \sum_{v' \in \mathcal{L}} A_{v'} + 1)$ being one. We have $\beta(\sum_{v' \in A(i)} A_{v'}, \sum_{v' \in \mathcal{L}} A_{v'}) \geq \beta(\sum_{v' \in A(i)} A_{v'}, \sum_{v' \in \mathcal{L}} A_{v'} + 1)$. Thus, there exists a coupling of the randomness of $m(0)$ and $m(1)$ such that: (1) $m(0) = m(1) = 1$, both systems match to the same agent; (2) $m(0) = 1, m(1) = 0$, $S_0$ matches the job to an agent in $A$ while $S_1$ fails to match; (3) $m(0) = m(1) = 0$, both systems fail to match the job. We now discuss case by case:

1. when $m(0) = m(1) = 1$, and suppose $i' \in A(i)$ is the agent type both systems match this type $j$ job to. We have $A'' = A - e_{i'}$ representing $A$ modified by this event so that $A'(0) = A''$ and $A'(1) = A'' + e_i$. Then, by the induction hypothesis, $2 + M_{k-1}(A'', t') \geq 1 + M_{k-1}(A'' + e_i, t')$, showing (7);

2. when $m(0) = 1, m(1) = 0$, $S_0$ matches this job to a type $i' \in A(i)$ agent counted in $A$. We have $A'' = A - e_{i'}$ representing $A$ modified by this event so that $A'(0) = A''$ and $A'(1) = A + e_{i'} + e_i$. Then, by the induction hypothesis, $2 + M_{k-1}(A'', t') \geq M_{k-1}(A'' + e_{i'} + e_i, t')$, showing (7);

3. when $m(0) = m(1) = 0$, we have $A'(0) = A$ and $A'(1) = A + e_i$. Then, by the induction hypothesis, $1 + M_{k-1}(A, t') \geq M_{k-1}(A + e_i, t')$, showing (7).
**Case 3:** The next event is a job of type \( j \in \mathcal{A}(i) \). In this case, \( m(0) \) is a Bernoulli random variable with probability \( \beta(\sum_{i' \in \mathcal{A}(j)} A_{i'} + \sum_{i' \in \mathcal{L} A_{i'}}) \) being one, \( m(1) \) is a Bernoulli random variable with probability \( \beta(\sum_{i' \in \mathcal{A}(j)} A_{i'} + 1, \sum_{i' \in \mathcal{L} A_{i'}} + 1) \) being one. We have \( \beta(\sum_{i' \in \mathcal{A}(j)} A_{i'} + 1, \sum_{i' \in \mathcal{L} A_{i'}} + 1) \geq \beta(\sum_{i' \in \mathcal{A}(j)} A_{i'}, \sum_{i' \in \mathcal{L} A_{i'}}) \). Thus, there exists a coupling of the randomness of \( m(0), m(1) \) such that:

1. when \( m(0) = m(1) = 1 \), both systems match to the same agent, or \( S_1 \) matches the job to the additional type 0 agent \( e_i \) not counted in \( A \), while \( S_0 \) matches an agent in \( A \); (2) \( m(0) = 0, m(1) = 1 \), \( S_1 \) matches the job to the additional type \( i \) agent not counted in \( A \) while \( S_0 \) fails to match the job; (3) both systems fail to match the job. We now discuss all these outcomes case by case:

1. when \( m(0) = m(1) = 1 \), and both systems match to the same agent. Suppose \( i' \in \mathcal{A}(j) \) is the agent type both systems match this type \( j \) job to. We have \( A'' = A - e_{i'} \) represent \( A \) modified by this event so that \( A'(0) = A'', A'(1) = A'' + e_i \). Then, by the induction hypothesis, \( 2 + M_{k-1}(A'', t') \geq 1 + M_{k-1}(A'' + e_i, t') \), showing (7);

2. when \( m(0) = m(1) = 1 \), and \( S_1 \) matches the job to the additional type \( i \) agent not counted in \( A \), \( S_0 \) matches an agent of type \( i' \) in \( A \). We have \( A'(0) = A - e_i, A'(1) = A \). Then, by the induction hypothesis, \( 2 + M_{k-1}(A - e_i, t') \geq 1 + M_{k-1}(A, t') \), showing (7);

3. when \( m(0) = 0, m(1) = 1 \), \( S_1 \) matches the job to the additional type \( i \) agent not counted in \( A \), while \( S_0 \) fails to match. We have \( A'(0) = A', A'(1) = A \). Then, (7) can be directly shown;

4. when \( m(0) = m(1) = 0 \), we have \( A'(0) = A \) and \( A'(1) = A + e_i \). Then, by the induction hypothesis, \( 1 + M_{k-1}(A, t') \geq M_{k-1}(A + e_i, t') \), showing (7).

**Case 4:** The next event is reneging by the agent not counted in \( A \). In this case, \( A'(0) = A'(1) = A \) and \( m(0) = m(1) = 0 \). Thus, (7) is verified directly, with \( 1 + M_{k-1}(A, t') \geq M_{k-1}(A, t') \).

This concludes the proof of Lemma 2. □
Online Appendix to Matching Queues, Flexibility and Incentives

Appendix EC.1: Proofs

**Proof of Lemma 1.** Let $M^\phi_A(t, T)$ denote the expected number of matches under policy $\phi$ starting with state $A$ from time $t$ up until the time horizon $T$ expires or until we reach a total of $k$ events, whichever occurs first. Events include job arrivals, agent arrivals, and reneging.

Denote by $M^\phi_i(t, T)$ the expected number of matches under policy $\phi$ within $[0, T]$ without a constraint on the total number of events. We will show that $1 + \sup_{\phi, \Phi} M^\phi_k(A, t) \geq \sup_{\phi, \Phi} M^\phi_\infty(A + e_i, t)$ for all $i \neq i' \in \Phi$ such that $\mathcal{L}(i) \geq \mathcal{L}(i')$ and all $t \in [0, T]$.

For any finite $k$ and any state $A$, $|\sup_{\phi, \Phi} M^\phi_k(A, t) - \sup_{\phi, \Phi} M^\phi_k(A, t)|$ is bounded by the expected number of events (job arrival events in particular) in excess of $k$ over $[t, T]$. This upper bound vanishes as $k$ grows large. Thus, $\lim_{k \to \infty} \sup_{\phi, \Phi} M^\phi_k(A, t) = \sup_{\phi, \Phi} M^\phi_\infty(A, t)$ for each $A$ and $t$. Thus, to show our claim, it is sufficient to show that $1 + \sup_{\phi, \Phi} M^\phi_k(A, t) \geq \sup_{\phi, \Phi} M^\phi_k(A + e_i, t)$ for all finite $k$. We now show this via induction on $k$.

This induction hypothesis is trivially true for $k = 0$ and so we turn our attention to show it is true for $k > 0$ assuming that it is true for $k - 1$. Consider three states at time $t$, one with state $A$, another with state $A + e_i$, and the third with state $A + e_{i'}$. We will refer to the three systems as $S_0$, $S_1$, and $S_2$ respectively.

We fix a sequence of agent and job arrivals within $[t, T]$. Also consider a collection of independent exponential random variables representing the times until each of several possible reneging events occur in the system. Then the next event occurs randomly within the following categories:

1. reneging by an agent counted in $A$, from a type with a non-zero component in $A$;
2. an agent arrival, from one of the $d$ types;
3. a job arrival, from one of the $\ell$ types;
4. reneging by the agent not counted in $A$, which is of type $i$ in system $S_1$ and type $i'$ in system $S_2$.

We let random variable $t'$ represent the sum of $t$ and the minimum of these times. The event achieving this minimum determines the event that occurs next. If $T$ occurs before any of these times, then the time horizon expires before the next event occurs. Regarding reneging of the agents not counted in $A$, note that reneging occurs at the same rate regardless of agent type, so it is valid to couple them with a single event.

For each system $S_j$, $j = 0, 1, 2$, let random variable $m(j)$ represent whether a match results from the next event, let $A'(j)$ represent the system state after the event occurs, and let $\Delta$ indicate whether the next event occurred in all three systems ($\Delta = 1$) or just systems 1 and 2 ($\Delta = 0$, which occurs on event type 4). Thus,

$$\sup_{\phi, \Phi} M^\phi_k(A, t) = \mathbb{E} \left[ m(0) + \sup_{\phi, \Phi} M^\phi_{k-\Delta}(A'(0), t') \right] \geq \mathbb{E} \left[ m(0) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(0), t') \right],$$

$$\sup_{\phi, \Phi} M^\phi_k(A + e_i, t) = \mathbb{E} \left[ m(1) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(1), t') \right],$$

$$\sup_{\phi, \Phi} M^\phi_k(A + e_{i'}, t) = \mathbb{E} \left[ m(2) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(2), t') \right].$$

The expectations are taken over $m(j), A'(j)$ and $t'$. To show our result, we will consider the next event on a case-by-case basis to show that with probability one,

$$1 + m(0) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(0), t') \geq m(1) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(1), t') \geq m(2) + \sup_{\phi, \Phi} M^\phi_{k-1}(A'(2), t'). \quad (EC.1)$$

If $t' > T$, then $\sup_{\phi, \Phi} M^\phi_{k-1}(A'(j), t') = 0$, $m(j) = 0$ for all $j$, automatically verifying this expression. Thus, it is sufficient to focus on $t' \leq T$.

**Case 1:** The next event is reneging by an agent counted in $A$ or an agent arrival. In this case, let $A''$ represent $A$ modified by this event, so that $A'(0) = A'', A'(1) = A'' + e_i$, and $A'(2) = A'' + e_{i'}$. Also, $m(j) = 0$ for all $j$. Then, by the induction hypothesis, $1 + \sup_{\phi, \Phi} M^\phi_{k-1}(A'', t') \geq \sup_{\phi, \Phi} M^\phi_{k-1}(A'' + e_i, t') \geq \sup_{\phi, \Phi} M^\phi_{k-1}(A'' + e_{i'}, t')$, showing (EC.1).
Case 3: The next event is reneging by an agent not counted in $A$. In this case, $A'(0) = A'(1) = A'(2) = A$ and $m(0) = m(1) = m(2) = 0$. Thus, (EC.1) is verified directly, with $1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t') \geq \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t')$.

Case 4: The next event is a job type that is not compatible with both agents of types $i$ and $i'$. In this case, we can couple the matching decision so that all three systems match the job with an agent in $A$ or all three systems choose not to match. In either case, the induction step holds.

Case 4: The next event is a job type that is compatible with agent type $i'$ thus compatible with type $i$ as well. We consider the following subcases.

(i) If $S_0$ and $S_1$ (or $S_1$ and $S_2$) make the same matching decision, then the inequality involving the comparison of $S_0$ and $S_1$ (or $S_1$ and $S_2$) in (EC.1) holds. In the next two subcases we consider $S_0$ and $S_1$, $S_1$ and $S_2$ make different matching decisions respectively.

(ii) If $S_0$ and $S_1$ make different matching decisions, consider the first scenario where $S_0$ decides to match an agent of type $j$ and $S_1$ decides to match an agent of type $i$ and $j \neq i$. The first part of the inequality in (EC.1) can be verified,

$$1 + m(0) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(0), t') = 1 + 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A - e_j, t')$$

$$\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t')$$

$$= 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A + e_i, t') = m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(1), t').$$

Now consider the second scenario where $S_0$ decides to match an agent of type $j$ and $S_1$ decides to match an agent of type $j' \neq j \neq i$.

$$1 + m(0) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(0), t') = 1 + 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A - e_j, t')$$

$$\geq 1 + 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A - e_j, t')$$

$$\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A + e_i - e_j, t')$$

(induction hypothesis at $k - 1$)

$$= m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(1), t').$$

For the third scenario where $S_0$ decides not to match but $S_1$ decides to match with a type $i$ agent, trivially,

$$1 + m(0) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(0), t') = 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t')$$

$$\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A + e_i, t')$$

(induction hypothesis at $k - 1$)

$$= m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(1), t').$$

For a fourth scenario where $S_0$ decides not to match but $S_1$ decides to match with a type $j \neq i$ agent,

$$1 + m(0) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(0), t') = 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t')$$

$$\geq 1 + 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A - e_j, t')$$

(indoptimality of sup_{\phi \in \Phi} M_{k-1}^\phi(A, t))

$$\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A + e_i - e_j, t')$$

(induction hypothesis at $k - 1$)

$$= m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(1), t').$$

For the last scenario where $S_0$ decides to match with an agent of type $j$ but $S_1$ decides not to match,

$$1 + m(0) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(0), t') = 1 + 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A - e_j, t')$$

$$\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi(A, t')$$

(indoptimality of sup_{\phi \in \Phi} M_{k-1}^\phi(A, t))

$$\geq \sup_{\phi \in \Phi} M_{k-1}^\phi(A + e_i, t')$$

(induction hypothesis at $k - 1$)

$$= m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi(A'(1), t').$$
(iii) If $S_1$ and $S_2$ make different matching decisions, consider the first scenario where $S_1$ decides to match with an agent of type $j$ while $S_2$ decides to match to an agent of type $j' \neq j$, and assumes that $S_1$ contains a type $j'$ agent. The second part of the inequality in (EC.1) can be verified,
\[
m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(1), t') = 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(optimality of } \sup_{\phi \in \Phi} M_{k}^\phi (A, t)) \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(induction hypothesis at } k - 1) \\
= m(2) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(2), t').
\]

Consider the second scenario where $S_1$ decides to match with an agent of type $j$ while $S_2$ decides to match to an agent of type $j' \neq j$, and assumes that $S_1$ does not contain a type $j'$ agent. Then it must be that $j' = \hat{t}'$. This suggests that the incoming job is compatible with a type $i$ agent as well because $L(i) \supset L(i')$.
\[
m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(1), t') = 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(optimality of } \sup_{\phi \in \Phi} M_{k}^\phi (A, t)) \\
= 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \\
= m(2) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(2), t').
\]

For the third scenario where $S_1$ decides to match with an agent of type $j$ while $S_2$ decides not to match, 
\[
m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(1), t') = 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \\
\geq \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i, t') \quad \text{(optimality of } \sup_{\phi \in \Phi} M_{k}^\phi (A, t)) \\
\geq \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i, t') \quad \text{(induction hypothesis at } k - 1) \\
= m(2) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(2), t').
\]

For the fourth scenario where $S_1$ decides not to match while $S_2$ decides to match with an agent of type $j'$ and $S_1$ contains a type $j'$ agent,
\[
m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(1), t') = \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i, t') \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(optimality of } \sup_{\phi \in \Phi} M_{k}^\phi (A, t)) \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(induction hypothesis at } k - 1) \\
= m(2) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(2), t').
\]

For the last scenario where $S_1$ decides not to match while $S_2$ decides to match with an agent of type $j'$ and $S_1$ does not contain a type $j'$ agent, this indicates that $j' = \hat{t}'$. This further suggests that the incoming job is compatible with a type $i$ agent as well because $L(i) \supset L(i')$.
\[
m(1) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(1), t') = \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i, t') \\
\geq 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \quad \text{(optimality of } \sup_{\phi \in \Phi} M_{k}^\phi (A, t)) \\
= 1 + \sup_{\phi \in \Phi} M_{k-1}^\phi (A + e_i - e_j, t') \\
= m(2) + \sup_{\phi \in \Phi} M_{k-1}^\phi (A'(2), t').
\]
For this direction, we prove it by contradiction. Suppose that there exists a point theorem, we deduce that an equilibrium always exists. This completes the proof.

Proof of Proposition 1. Suppose that there exists an optimal policy \( \phi' \) that does not satisfy the FR property. Then at some moment \( t \), for two types of agents \( i \neq i' \in \mathcal{A} \) such that \( \mathcal{L}(i) \supset \mathcal{L}(i') \) with \( A_i > 0 \) and \( A_{i'} > 0 \), policy \( \phi' \) assigns an incoming job to a compatible type \( i \) agent instead of a compatible type \( i' \) agent. Now consider an alternative action that assigns the same job to a type \( i' \) agent. By the second inequality of Lemma 1, continuing running the optimal policy from this moment onward for any time duration \( T \) weakly improves throughput in expectation. This suggests that any optimal policy that does not satisfy the FR property can be modified to satisfy such a property without decreasing its throughput over any time horizon. Similarly, suppose that there exists another optimal policy \( \phi'' \) that does not match an incoming job even if there are compatible agents available. Suppose that this event occurs at some time \( t \), and consider an alternative action that assigns the job to any compatible agent. Similarly, by the first inequality of Lemma 1, continuing running the optimal policy from this moment onward for any time duration \( T \) does not decrease the throughput in expectation. This indicates that any idling optimal policy can be made non-idling without deteriorating its performance. This concludes the proof.

Proof of Proposition 2. The proof follows similar steps as Theorem 1 in Smith (1979) for proving the existence of Wardrop equilibrium. Fix a policy \( \pi \), we first give an equivalent definition of \( \sigma \) being an equilibrium, aside from eq. (NE).

\[
\sum_{i \in \mathcal{L}, q \in \mathcal{Q}} \mathbb{E}[W_i^\pi(q)](\sigma_{i,q} - \tilde{\sigma}_{i,q}) \leq 0, \quad \forall \tilde{\sigma} \in \Sigma(\mathcal{Q}) \iff \sigma_{i,q} > 0 \Rightarrow \mathbb{E}[W_i^\pi(q)] \leq \mathbb{E}[W_{i,q'}^\pi(q)], \forall q' \in \mathcal{Q}.
\]

We prove this equivalence:

• “\( \iff \)”

\[
\begin{align*}
\sum_{i \in \mathcal{L}, q \in \mathcal{Q}} \mathbb{E}[W_i^\pi(q)]\sigma_{i,q} & = \sum_{i \in \mathcal{L}, q \in \mathcal{Q}, \sigma_{i,q} > 0} \mathbb{E}[W_i^\pi(q)]\sigma_{i,q} \\
& = \sum_{i \in \mathcal{L}, q \in \mathcal{Q}, \sigma_{i,q} > 0} \min_k \mathbb{E}[W_i^\pi(q)]\sigma_{i,q} \\
& = \sum_{i \in \mathcal{L}, q \in \mathcal{Q}, \sigma_{i,q} > 0} \min_k \mathbb{E}[W_i^\pi(q)]\tilde{\sigma}_{i,q} \\
& \leq \sum_{i \in \mathcal{L}, q \in \mathcal{Q}, \sigma_{i,q} > 0} \mathbb{E}[W_i^\pi(q)]\tilde{\sigma}_{i,q} \\
& = \sum_{i \in \mathcal{L}, q \in \mathcal{Q}} \mathbb{E}[W_i^\pi(q)]\tilde{\sigma}_{i,q}.
\end{align*}
\]

• “\( \implies \)”

For this direction, we prove it by contradiction. Suppose that there exists a \( \sigma_{i,q}^* > 0 \) and \( \mathbb{E}[W_i^{\pi^*}(q)] > \mathbb{E}[W_{i,q'}^{\pi^*}(q)], \forall q \in \mathcal{Q} \). Now consider a \( \tilde{\sigma} \) which is constructed as follows:

\[
\tilde{\sigma}_{i,q} = \begin{cases} 
\sigma_{i,q}, & i \neq i^*, \\
0, & i = i^*, q = q^*, \\
\sigma_{i,q} + \frac{\sigma_{i,q}^* - \sigma_{i,q}}{|\mathcal{Q}|}, & i = i^*, q \neq q^*.
\end{cases}
\]

It can be seen that \( \sum_{i \in \mathcal{L}, q \in \mathcal{Q}} \mathbb{E}[W_i^\pi(q)](\sigma_{i,q} - \tilde{\sigma}_{i,q}) > 0 \), which reaches a contradiction.

This equivalent condition says that the vector \( \mathbb{E}[W^\pi(q)] \) with components \( \mathbb{E}[W_i^\pi(q)] \) is normal to the simplex \( \Sigma(\mathcal{Q}) \) at \( \sigma \). Let \( P_\Sigma(\cdot) \) be the projection operator onto \( \Sigma(\mathcal{Q}) \). Define the mapping \( f : \Sigma(\mathcal{Q}) \to \Sigma(\mathcal{Q}) \) by \( f(\sigma) = P_\Sigma(\sigma - \mathbb{E}[W_i^\pi(q)]) \), then the equilibrium condition can be cast as the fixed point equation \( f(\sigma) = \sigma \). Since, by assumption, \( W^\pi(\sigma) \) is continuous and projection operator onto bounded convex set is continuous, then \( f \) is a continuous function as composition of continuous functions is continuous. By Brouwer's fixed point theorem, we deduce that an equilibrium always exists. This completes the proof.
Proof of Proposition 3. Consider the following rates for some $\epsilon > 0$:

- $\mu_{1, L} = 0$,
- $\kappa_{H \to L} = 1$,
- $\kappa_{L \to H} = \frac{\epsilon}{\lambda}$,
- $\lambda_1 = \log(1/\epsilon)^2/\epsilon$,
- $\mu_1, H = \log(1/\epsilon)/\epsilon^2$,
- $\mu_2$ and $\lambda_2$ are kept as constants,
- $\theta = \log(1/\epsilon)/\epsilon$.

We analyze and compare the performance of different policies as $\epsilon$ approaches zero. First, let us consider the system under $\pi^{FR}$. We will first establish in Step 1 that all type 1 agents choose the second queue. That is, consider an alternative system in which all agents choose to join the second queue. In this system, let $W_{1,1}$ be the time a type 1 agent would have to wait until matched assuming she does not renege in the first queue when she joins the first queue in steady-state, and similarly let $W_{1,2}$ be the time a type 1 agent would have to wait until matched assuming she does not renege in the second queue when she joins the second queue in steady-state. Then all type 1 agents joining the second queue is implied by $E[W_{1,1}] > E[W_{1,2}]$. After showing this, in Step 2 we will bound the throughput of the resulting system under $\pi^{RND}$ in which all agents join the second queue and compare it to the throughput of the system under the $\pi^{FR}$ policy.

Step 1. Let $T$ be the time until the next arrival of a type 1 job from a state in steady-state. Let $Y$ be the time until the next time a job of type 2 observes that the second queue is empty from a state in steady-state. Conditional on any state (in steady-state), we have that

$$W_{1,1} \geq \min\{T, Y\}.$$  

This inequality holds because an agent that joins the first queue (which is empty under the assumption that nobody joins this queue) has to wait until a job of type 1 arrives or until a job of type 2 is not sent to the second queue which only happens when the second queue is empty upon the arrival of such job.

Now, consider a type 2 agent, which we refer to as agent $\hat{a}$, who joins the second queue. Consider two systems that consist only of the second queue and they start with the same number of agents in steady-state. System $S_1$ has agent $\hat{a}$ in it while system $S_2$ does not. We couple these two systems as follows. Whenever there is an agent arrival or an agent reneging (note that $\hat{a}$ doesn’t renege) from system $S_1$, the same agent arrives or reneges from system $S_2$. Upon the arrival of a job to system $S_1$, if that job is not assigned to agent $\hat{a}$ then the job is assigned to the same agent in system $S_2$. If the job is assigned to agent $\hat{a}$, we remove agent $\hat{a}$ from the system and we also choose another agent to remove at random who we remove from both systems. Note that when this happens both systems continue to evolve in the same fashion. Furthermore, let $X$ be the type 2 job inter-arrival time (which is an exponential random variable with rate $\mu_2$), note that

$$Y \geq W_{1,2} + X,$$

when $\hat{a}$ is matched while there are other agents in the queue. Indeed, in this case, we still need to wait for at least an extra arrival of a type 2 for a job of type 2 to observe the second queue empty. Let $E$ denote the event “$\hat{a}$ is matched while there are other agents in the queue.” Note also that in $E^c$ agent $\hat{a}$ is matched alone and, therefore, $Y$ and $W_{1,2}$ coincide. Then,

$$E[W_{1,1}] \geq E[\min\{T, Y\}]$$

$$= E[\min\{T, Y\} | E]P(E) + E[\min\{T, Y\} | E^c]P(E^c)$$

$$\geq E[\min\{T, W_{1,2} + X\} | E]P(E) + E[\min\{T, W_{1,2}\} | E^c]P(E^c)$$

$$= E[\min\{T, W_{1,2} + X\} | E]P(E) + E[\min\{T, W_{1,2} + X - X\} | E^c]P(E^c)$$

$$\geq E[\min\{T, W_{1,2} + X\} | E]P(E) + E[\min\{T, W_{1,2} + X\} - X | E^c]P(E^c)$$

$$= E[\min\{T, W_{1,2} + X\}] - E[X | E^c]P(E^c)$$

$$= E[\min\{T, W_{1,2} + X\}] - E[X]P(E^c)$$

$$= E[\min\{T, W_{1,2} + X\}] - (1/\mu_2)E[E^c].$$

Next, we will provide bounds for the two terms in the last expression above. In particular, we will establish that $E[\min\{T, W_{1,2} + X\}]$ is bounded below by $E[W_{1,2}]$ plus a strictly positive term and that $P(E^c)$ converges to zero under the right scaling, leading to $E[W_{1,1}] > E[W_{1,2}]$. We will show the latter in Step (1.b).
We first provide a bound for $\mathbb{P}(T \leq t)$. Let $S$ denote the state of the Markov-modulated process, recall that $S \in \{L, H\}$, and let $\pi_L$ and $\pi_H$ be its steady-state probabilities which are (Fischer and Meier-Hellstern 1993)

$$\pi_L = \frac{\kappa_{H \to L}}{\kappa_{L \to H} + \kappa_{H \to L}} \quad \text{and} \quad \pi_H = \frac{\kappa_{L \to H}}{\kappa_{L \to H} + \kappa_{H \to L}},$$

(EC.3)

and we have

$$\mathbb{P}(T \leq t) = \mathbb{P}(T \leq t \mid S = L) \pi_L + \mathbb{P}(T \leq t \mid S = H) \pi_H$$

$$\leq (1 - e^{-\theta \kappa_{L \to H} t}) \pi_L + \pi_H$$

$$= (1 - e^{-\theta \varepsilon}) \cdot \frac{1}{1 + \varepsilon} + \frac{\varepsilon}{1 + \varepsilon},$$

(EC.4)

where in the inequality we have used that $\mathbb{P}(T \leq t \mid S = H) \leq 1$ and that when $S = L$, we first need to switch to the $H$ state for a job of type 1 to arrive. In the second equality, we have replaced $\kappa_{H \to L} = 1$ and $\kappa_{L \to H} = \varepsilon$. This yields,

$$\mathbb{E}[\min\{T, W_{1.2} + X\}] \geq \mathbb{E}[(W_{1.2} + X)1\{T \geq W_{1.2} + X\}]$$

$$= \mathbb{E}[(W_{1.2} + X)1\{T \geq W_{1.2} + X\} \mid W_{1.2}, X]$$

$$= \mathbb{E}[(W_{1.2} + X) \mathbb{P}(T \geq W_{1.2} + X \mid W_{1.2}, X)]$$

$$\geq \mathbb{E}[(W_{1.2} + X) \frac{e^{-\varepsilon(W_{1.2} + X)}}{1 + \varepsilon}]$$

(by eq. (EC.4))

$$\geq \mathbb{E}[(W_{1.2} + X) \frac{(1 - \varepsilon(W_{1.2} + X))}{1 + \varepsilon}]$$

$$= \mathbb{E}[(W_{1.2} + X) \frac{(1 + \varepsilon - \varepsilon(1 + (W_{1.2} + X)))}{1 + \varepsilon}]$$

$$= \mathbb{E}[W_{1.2}] + \frac{1}{\mu_2} - \frac{\varepsilon}{1 + \varepsilon} \mathbb{E}[(W_{1.2} + X)(1 + (W_{1.2} + X))].$$

(EC.5)

In Step (1.a), we will show that $\varepsilon \mathbb{E}[W_{1.2}]$ and $\varepsilon \mathbb{E}[W_{1.2}^2]$ converge to zero as $\varepsilon \downarrow 0$.

**Step (1.a).** To show that $\varepsilon \mathbb{E}[W_{1.2}]$ and $\varepsilon \mathbb{E}[W_{1.2}^2]$ converge to zero as $\varepsilon \downarrow 0$, we consider an alternative system with waiting time $W \geq W_{1.2}$ (after coupling). In this system, we have an arrival rate of agents $\lambda = \lambda_1 + \lambda_2$, a reneging rate $\theta$ and arrival rate of jobs $\mu_2$. Here $W$ represents the steady-state waiting time of a focal agent who doesn’t renege after joining. The system is different from the original system in that whenever a job arrives and it is not matched to the focal agent, we don’t match it to any other agent. That is, agents other than the focal agent can only leave through reneging.

Consider the new system. Let $T_{n+1,n}$ be the time it takes for the system to have $n$ agents (other than the focal agent) starting with $n+1$ agents (other than the focal). Let $W_n$ be the waiting time for the focal agent to get a match when there are $n$ other agents, let $A$, $R_n$ and $J$ be the time until the next agent arrival, reneging, and job arrival when there are $n$ other agents, respectively. Let $M_n = \min\{A, R_n, J\}$, then

$$W_n = M_n + 1\{A \leq M_n\} W_{n+1} + 1\{R_n \leq M_n\} W_{n-1} + 1\{J \leq M_n\} \left(1 - \frac{1}{n+1}\right) W_n$$

$$\leq M_n + 1\{A \leq M_n\} (T_{n+1,n} + W_n) + 1\{R_n \leq M_n\} W_n + 1\{J \leq M_n\} \left(1 - \frac{1}{n+1}\right) W_n$$

$$= M_n + 1\{A \leq M_n\} T_{n+1,n} + W_n \left(1\{A \leq M_n\} + 1\{R_n \leq M_n\} + 1\{J \leq M_n\} \left(1 - \frac{1}{n+1}\right)\right)$$

$$= M_n + 1\{A \leq M_n\} T_{n+1,n} + W_n \left(1 - \frac{1\{J \leq M_n\}}{n+1}\right),$$

(EC.6)

where in the inequality we have used that $W_{n-1} \leq W_n$ (the waiting time would be longer when there are more other agents), and we have also used that $W_{n+1} \leq T_{n+1,n} + W_n$. The latter holds because when the system has $n+1$ other agents, the focal agent may get matched before the system goes back to $n$ other agents; however if we ignore a potential match during the time it takes to get back to $n$ from $n+1$, then
the focal agent will wait even longer. After the system goes back to $n$, the focal agent will have to wait $W_n$. Hence, from eq. (EC.6),

$$W_n \frac{1}{n+1} \leq M_n + 1 \{ A \leq M_n \} T_{n+1,n}. \quad (EC.7)$$

Next, we will bound $T_{n+1,n}$. This random variable is the absorption time to state $n$ starting from state $n + 1$. To simplify the notation, let us use $T_k := T_{k,n}$ for $k \geq n$ and $T_n = 0$ to denote this absorption time. We have the following recursion:

$$T_k = \min \{ R_k, A \} + 1 \{ R_k \geq A \} T_{k+1} + 1 \{ A \geq R_k \} T_{k-1}, \quad \forall k \geq n + 1. \quad (EC.8)$$

Let $x_k := E[T_k]$ then

$$x_k = \frac{1}{k\theta + \lambda} + \frac{\lambda}{k\theta + \lambda} x_{k+1} + \frac{k\theta}{k\theta + \lambda} x_{k-1}, \quad \forall k \geq n + 1.$$ 

Rearranging the above yields

$$(x_{k+1} - x_k) = -\frac{1}{\lambda} + \frac{k\theta}{\lambda} (x_k - x_{k-1}), \quad \forall k \geq n + 1.$$ 

Iterating this recursion, we obtain

$$(x_{k+1} - x_k) = \frac{k!}{n!} \left( \frac{\theta}{\lambda} \right)^{k-n} x_{n+1} - \frac{1}{\lambda} \sum_{j=0}^{k-(n+1)} \left( \frac{\theta}{\lambda} \right)^j \frac{k!}{(k-j)!}.$$ 

rearranging terms and taking $k \uparrow \infty$, we obtain

$$x_{n+1} = \left( \frac{1}{\lambda} \sum_{n=0}^{n+1} \frac{(\frac{\theta}{\lambda})^n}{n!} \right) \frac{1}{p_n} \sum_{j \geq n+1} p_j, \quad \text{with } p_j = e^{-\lambda/\theta} (\lambda/\theta)^j / j!.$$ 

Plugging this bound in (EC.7) and taking expectation yields:

$$\mathbb{E}[W_n] \leq \frac{1}{n+1} \mathbb{P}(J \leq M_n) \left( \mathbb{E}[M_n] + \mathbb{P}(A \leq M_n) \mathbb{E}[T_{n+1,n}] \right)$$

$$= \frac{(n+1)(n\theta + \lambda + \mu_2)}{\mu_2} \left( \frac{1}{n\theta + \lambda + \mu_2} + \frac{\lambda}{n\theta + \lambda + \mu_2} x_{n+1} \right)$$

$$= \frac{(n+1)}{\mu_2} \left( 1 + \sum_{j \geq n+1} p_j \right).$$

Note that the steady-state probabilities of the Markov chain associated with the “other agents” in the alternative system are given exactly by $\{p_j\}_{j \in \mathbb{N}}$. Hence,

$$\mathbb{E}[W] = \sum_{n \geq 0} \mathbb{E}[W_n] p_n$$

$$\leq \sum_{n \geq 0} \frac{(n+1)}{\mu_2} \left( 1 + \frac{1}{p_n} \sum_{j \geq n+1} p_j \right) p_n$$

$$= \frac{1}{\mu_2} + \frac{\lambda}{\theta \mu_2} + \frac{1}{\mu_2} \sum_{n \geq 0} (n+1) \sum_{j \geq n+1} p_j$$

$$= \frac{1}{\mu_2} + \frac{\lambda}{\theta \mu_2} + \frac{1}{\mu_2} \sum_{j \geq 1} \sum_{n=0}^{j-1} (n+1)p_j$$

$$= \frac{1}{\mu_2} + \frac{\lambda}{\theta \mu_2} + \frac{1}{2\mu_2} \sum_{j \geq 1} j(j+1)p_j$$

$$\leq \frac{1}{\mu_2} + \frac{\lambda}{\theta \mu_2} + \frac{1}{\mu_2} \sum_{j \geq 0} j^2p_j$$

$$= \frac{1}{\mu_2} + \frac{\lambda}{\theta \mu_2} + \frac{1}{\mu_2} \left( \left( \frac{\lambda}{\theta} \right)^2 + \left( \frac{\lambda}{\theta} \right) \right).$$
Next, we will bound $\mathbb{E}[W^2]$. We first use (EC.8) to obtain a recursion for $\mathbb{E}[T_k^2]$. Squaring (EC.8) yields

$$T_k^2 - 2T_k \min \{R_k, A\} + \min \{R_k, A\} = 1 \{R_k \geq A\} T_{k+1}^2 + 1 \{A \geq R_k\} T_{k-1}^2, \quad k \geq n + 1,$$

and taking expectation and letting $y_k := \mathbb{E}[T_k^2]$ yields

$$y_k = \frac{1}{k \theta + \lambda} + \frac{2}{(k \theta + \lambda)^2} = \frac{\lambda}{k \theta + \lambda} y_{k+1} + \frac{k \theta}{k \theta + \lambda} y_{k-1}, \quad k \geq n + 1.$$ 

We obtain the recursion:

$$(y_{k+1} - y_k) = \frac{1}{\lambda} \left( k \theta (y_k - y_{k-1}) - 2x_k + \frac{2}{(k \theta + \lambda)} \right), \quad k \geq n + 1.$$

Iterating this recursion, we obtain

$$(y_{k+1} - y_k) = \frac{k!}{n!} \left( \frac{\theta}{\lambda} \right)^{k-n} y_{n+1} - \frac{2}{\lambda} \sum_{j=0}^{k-(n+1)} \left( \frac{\theta}{\lambda} \right)^j \frac{k!}{(k-j)!} \left( x_{k-j} - \frac{1}{(k-j) \theta + \lambda} \right).$$

Noting that $\lim_{k \to \infty} y_{k+1} - y_k = 0$. Hence, rearranging terms and taking $k \uparrow \infty$ gives

$$y_{n+1} = \frac{2}{\sum_{j \geq n+1} (\frac{\lambda}{\theta})^j} \sum_{j \geq n+1} p_j \left( x_j - \frac{1}{j \theta + \lambda} \right).$$

Squaring (EC.7) and taking expectation delivers

$$\mathbb{E}[W_n^2] \leq \frac{(n+1)^2}{\mathbb{P}(J \leq M_n)} \left( \mathbb{E}[(M_n^2) + 2 \mathbb{E}[M_n \{A \leq M_n\}] x_{n+1} + \mathbb{P}(A \leq M_n) y_{n+1} \right)$$

$$= \frac{(n+1)^2(n \theta + \lambda + \mu_2)}{\mu_2} \left( \frac{2}{(n \theta + \lambda + \mu_2)^2} + \frac{2}{(n \theta + \lambda + \mu_2)^2} x_{n+1} + \frac{\lambda}{n \theta + \lambda + \mu_2} y_{n+1} \right)$$

$$= \frac{(n+1)^2}{\mu_2} \left( \frac{2}{(n \theta + \lambda + \mu_2)^2} + \frac{2}{(n \theta + \lambda + \mu_2)^2} \sum_{j \geq n+1} p_j \left( x_j - \frac{1}{j \theta + \lambda} \right) \right) \left( \sum_{j \geq n+1} p_j \left( x_j - \frac{1}{j \theta + \lambda} \right) \right).$$

Now we take expectation over $n$ and bound the two terms (*) and (**) above separately:

$$\mathbb{E} \left[ \frac{(n+1)^2}{\mu_2} \right] \left( * \right) = \sum_{n \geq 0} \frac{(n+1)^2}{\mu_2} \left( \frac{2}{(n \theta + \lambda + \mu_2)^2} + \frac{2}{(n \theta + \lambda + \mu_2)^2} \sum_{j \geq n+1} p_j \left( x_j - \frac{1}{j \theta + \lambda} \right) \right) \right) \right) \right) \right).$$

$$\leq \frac{2}{\mu_2 (\lambda + \mu_2)} \sum_{n \geq 0} (n+1)^2 \left( \frac{1 + \frac{1}{p_n} \sum_{j \geq n+1} p_j}{p_n} \right)$$

$$= \frac{2}{\mu_2 (\lambda + \mu_2)} \left( 1 + 3 \left( \frac{\lambda}{\theta} \right)^2 + \sum_{n \geq 0} (n+1)^2 \sum_{j \geq n+1} p_j \right)$$

$$= \frac{2}{\mu_2 (\lambda + \mu_2)} \left( 1 + 3 \left( \frac{\lambda}{\theta} \right)^2 + \frac{1}{6} \left( 6 \lambda \theta + 9 \left( \frac{\lambda}{\theta} \right)^2 + 2 \left( \frac{\lambda}{\theta} \right)^3 \right) \right).$$

Similarly,

$$\mathbb{E} \left[ \frac{(n+1)^2}{\mu_2} \right] \left( ** \right) = \frac{2}{\mu_2} \sum_{n \geq 0} (n+1)^2 \sum_{j \geq n+1} p_j \left( x_j - \frac{1}{j \theta + \lambda} \right) \right) \right) \right) \right) \right).$$

$$\leq \frac{2}{\mu_2} \sum_{n \geq 0} (n+1)^2 \sum_{j \geq n+1} p_j x_j.$$
\[
\begin{align*}
\frac{2}{\mu_2 \lambda} \sum_{n \geq 0} (n+1)^2 \sum_{j \geq n+1} \frac{p_{j-1}}{\mu_2} \sum_{t \geq j} p_t \\
= \frac{2}{\mu_2 \lambda} \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} (n+1)^2 \frac{\lambda}{j^{\theta}} \sum_{t \geq j} p_t \\
= \frac{2}{6 \mu_2 \theta} \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} (1+j)(1+2j) \sum_{t \geq j} p_t \\
\leq \frac{2}{\mu_2 \theta} \sum_{j=1}^{\infty} \sum_{t \geq j} p_t \\
= \frac{2}{\mu_2 \theta} \left( \sum_{j=1}^{4[(\lambda/\theta)k]} j^2 \sum_{t \geq j} p_t + \sum_{j=4[(\lambda/\theta)k]+1}^{\infty} \sum_{t \geq j} j^2 \sum_{t \geq j} p_t \right) \\
\leq \frac{2}{\mu_2 \theta} \left( 4[(\lambda/\theta)k]^3 + \left( C_1 + C_2 \left( \frac{\lambda}{\theta} \right) + C_3 \left( \frac{\lambda}{\theta} \right)^2 \right) \right),
\end{align*}
\]
where in (a) we have used the following bound for \( N \sim \text{Poisson}(\lambda/\theta) \):
\[
\sum_{t \geq j} e^{\lambda/\theta}(\lambda/\theta)^t \frac{t!}{e^{1+\delta}} = P(N \geq j) \leq e^{-j} P(N \geq j) \leq e^{-j} e^{\lambda/\theta(1-\frac{1}{e})} \leq e^{-j/2}, \quad (j \geq 4\lambda/\theta).
\]
The first inequality above holds by Markov’s inequality and \( C_1, C_2 \) and \( C_3 \) are constants independent of everything else.

In order to conclude this part of the proof, we need to argue that \( \varepsilon E[W] \) and \( \varepsilon E[W^2] \) (so that \( \varepsilon E[W_{1,2}] \) and \( \varepsilon E[W_{1,2}^2] \) as \( W \geq W_{1,2} \)) converge to zero as \( \varepsilon \downarrow 0 \). From the bounds we just developed it is enough to show that \( \varepsilon(\lambda/\theta)^k \) and \( \varepsilon(\lambda_1/\theta)^k \) converge to zero as \( \varepsilon \downarrow 0 \) for any fixed \( k \). We have
\[
\varepsilon(\lambda/\theta)^k = \varepsilon \left( \frac{\log(1/\varepsilon)^2}{\log(1/\varepsilon)} \right)^k = \varepsilon (\log(1/\varepsilon))^k \rightarrow 0, \quad \text{as } \varepsilon \downarrow 0.
\]

**Step (1.b).** Now, let us consider \( P(E^c) \). In system \( S_2 \), \( E^c \) corresponds to a job of type 2 finding the queue empty. The latter happens with probability \( \pi_0 \) where \( \pi_0 \) is the steady-state probability of a queue with reneging having 0 agents. This queue has agent arrival rate \( \lambda_1 + \lambda_2 \), reneging rate \( \theta \) and job arrival rate \( \mu_2 \). Hence,
\[
P(E^c) = \pi_0 = \left( 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_1 + \lambda_2}{\mu_2 + i\theta} \right) \leq \frac{\mu_2 + \theta}{\lambda_1 + \lambda_2} = \frac{\mu_2/\theta + 1}{\lambda_1/\theta + \lambda_2/\theta},
\]
which converges to zero as \( \lambda_1/\theta = \log(1/\varepsilon) \) and \( \theta = \log(1/\varepsilon)/\varepsilon \) go to infinity (\( \varepsilon \downarrow 0 \)). In turn, from (EC.2), (EC.5) and steps (1.a) and (1.b), we conclude that
\[
\forall \delta > 0, \exists \varepsilon(\delta) > 0, \forall \varepsilon \leq \varepsilon(\delta) : E[W_{1,1}] \geq E[W_{1,2}] + \frac{1}{\mu_2} - \delta. \tag{EC.9}
\]
That is, we can always find \( \varepsilon > 0 \) small enough such that \( E[W_{1,1}] > E[W_{1,2}] \), as desired. Note that the throughput in this case is bounded by \( \mu_2 \) as *all* agents join the second queue and *no* type 1 job is sent to the second queue.

**Step 2.** Now let us consider the system under \( \pi_{\text{RND}} \). That is, there is a single queue where the jobs are dispatched to at random. Our goal is to lower bound \( TP(\pi_{\text{RND}}) \). Let \( \lambda = \lambda_1 + \lambda_2 \) be the arrival rate to the system. First, we will provide a lower bound for the number of matches that can be done in the time it takes the Markov-modulated process (MMP) to go from the low to the high state and back.

Let \( t_L \sim \exp(\varepsilon) \) be a random variable for the time the MMP spends in the low state and let \( t_H \sim \exp(1) \) be a random variable for the time the MMP spends in the high state. Note that the MMP takes \( t_L + t_H \) to
go from low state to low state. For each interval of the form \([0, t_L + t_H]\), we will count a subset of the jobs that will be matched to agents. In particular, we consider matches that occur in \([t_L, t_L + t_H - \Delta]\) between type 1 agents and type 1 jobs for some small \(\Delta = \varepsilon^2\) while ignoring type 2 jobs. This gives a lower bound on the total number of matches. Let \(N_1\) be the number of type 1 agents that arrive in \([t_L, t_L + t_H - \Delta]\) and let \(\{R_i, T_i\}_{i=1}^{N_1}\) denote their patience and their arrival times. We consider specific type of matches that occur within \(\Delta = \varepsilon^2\) of the arrival of a type 1 agent. In particular, these matches occur if an agent has patience greater than \(\Delta\), if a type 1 job arrives in \([T_i, T_i + \Delta]\), and if the job arrives before the arrival of the next agent at \(T_{i+1}\). Now, note that every match successfully occurs with certain probability depending on the number of compatible jobs available within the mix. We consider a worst-case scenario in which we start the interval \([t_L, t_L + t_H]\) with the largest possible number of incompatible agents — all type 2 agents that have not abandoned by time \(t_L\) plus the type 2 agents that would arrive in \([t_L, t_L + t_H]\) without any abandonment.

Let \(N_2\) be the initial number of type 2 in the queue at time 0 and \(N_{L.2}\) the number of these agents plus the ones that arrive in \([0, t_L]\) who have not abandoned by time \(t_L\) yet, and \(N_{H.2}\) the number of type 2 agents that would arrive in \([t_L, t_L + t_H]\) without any abandonment. Given this, and conditioning on \(N_1, N_2, N_{L.2}\) and \(N_{H.2}\), we can lower bound the number of matches in \([0, t_L + t_H]\) by

\[
\Pr[\text{matches} \mid t_H \geq 1] \geq e^{-\theta \Delta} (1-e^{-\mu_1 H \Delta}) e^{-\lambda_1 \Delta} \sum_{i=1}^{N_1} \Pr[R_i \geq \Delta \mid t_H \geq 1] e^{-1}
\]

where in the last line we have used that, conditional on \(N_1\), \(S_i\) has the same distribution as the order statistics of \(N_1\) random variables uniformly distributed in \([0, t_L - \Delta]\). Hence, taking the expectation of the number of matches over \(t_L, t_H, N_1, N_{L.2}\) and \(N_{H.2}\), we have

\[
\mathbb{E}[\# \text{ matches} \mid t_H \geq 1] \geq \mathbb{E}[\mathbb{E}[\mathbb{E}[\text{matches} \mid t_H \geq 1] \mid t_H \geq 1] \mid t_H \geq 1] \mathbb{E}[t_H \geq 1] e^{-1}
\]

Now, because \(\beta(a, a + c)\) is convex in \(c\), we have that

\[
\mathbb{E}[\beta(1, N_{L.2} + N_{H.2} + 1) \mid t_H \geq 1] \geq \mathbb{E}[\mathbb{E}[\beta(N_{L.2} + N_{H.2} + 1) \mid t_H \geq 1] \mid t_H \geq 1] \geq \beta(1, N_2 + \lambda_2/\theta + 2\lambda_2 + 1), \tag{EC.10}
\]
where we have used that, conditional on $t_L$, $\mathbb{E}[N_{L,2} + N_{H,2} | t_H \geq 1] = N_2 e^{-\theta t_L} + (\lambda_2/\theta) (1 - e^{-\theta t_L}) + \lambda_2 \mathbb{E}[t_H | t_H \geq 1] \leq N_2 + (\lambda_2/\theta) + \lambda_2 \mathbb{E}[t_H | t_H \geq 1]$ and that $\beta(a,a+c)$ is non-increasing in $c$, and $\mathbb{E}[t_H | t_H \geq 1] = 2$. In turn, we can conclude that

$$\frac{\mathbb{E}[\# \text{ matches}]}{\mathbb{E}[t_L + t_H]} \geq \frac{\lambda_2 \cdot e^{-\theta L} \cdot (1 - e^{-\mu_1 H \Delta}) \cdot e^{-\lambda_1 L \cdot (1 - 2\Delta)} \cdot \beta(1, N_2 + (\lambda_2/\theta) + 2\lambda_2 + 1) e^{-1}}{1/\varepsilon + 1}. \quad \text{(EC.11)}$$

For our choice of parameters, the above becomes

$$\frac{\log(1/\varepsilon)^2 \cdot e^{-\log(1/\varepsilon)} \cdot e^2 \cdot (1 - e^{-\log(1/\varepsilon)}) \cdot e^{-\log(1/\varepsilon)^2} \cdot (1 - 2\varepsilon^2) \cdot \beta(1, N_2 + (\lambda_2/(\log(1/\varepsilon))) + 2\lambda_2 + 1) e^{-1}}{1/\varepsilon + 1}, \quad \text{(EC.12)}$$

which $\uparrow \infty$ as $\varepsilon \downarrow 0$.

To conclude the proof, we need to analyze the long run behavior. To do so, let us consider a sequence of intervals $[0,t_L^k]$ and $[t_L^k,t_H^k]$ for $k \in \mathbb{N}$ with $t_L^k$ and $t_H^k$ IID random variables similar to $t_L$ and $t_H$, respectively. We start the system with $N_2$ type 2 agents, and then at the beginning of all subsequent intervals $k$, we start with the number of type 2 agents who did not abandon after $t_L^k$ and those type 2 agents who arrive during $[t_L^k, t_L^k + t_H^k]$ without any abandonment. Note that this will only lower the probability of a successful match (between type 1 agent and type 1 job) in each interval and, thus, lower bound the long-run throughput. To be specific, for the 0th interval we start with $N_0 = N_2$ type 2 agents, for the 1st interval we start with $N_1 = N_2^0 + N_2^1$, where $N_2^1$ is the number of initial agents plus those who arrive during $[0, t_L^0]$ that did not renege by time $t_L^0$ and $N_2^0$ is the number of new arrivals of type 2 agents in the interval $[t_L^0, t_L^0 + t_H^0]$ without any abandonment. In general, for interval $k \geq 1$, we start with $N_k = N_{k-1}^1 + N_{k-1}^2$ type 2 agents, where $N_{k-1}^1$ and $N_{k-1}^2$ are defined in an analogous fashion to $N_{L,2}^1$ and $N_{H,2}^1$, respectively.

Now, note that using $\{R_j\}$ to denote a sequence of IID $\sim \exp(\theta)$ random variables, and noting that $t_L^k$ and $t_H^k$ are IID $\sim \exp(\varepsilon)$ and $\sim \exp(1)$, respectively, we can compute the expectation of $N_k$ as follows:

$$\mathbb{E}[N^k | t_{H}^{k-1} \geq 1] = \mathbb{E}[N_{L,2}^k | t_{H}^{k-1} \geq 1] + \mathbb{E}[N_{H,2}^k | t_{H}^{k-1} \geq 1]$$

$$= \mathbb{E} \left[ \sum_{j=1}^{N_{k-1}^1} 1 \{ R_j \geq t_{L}^{k-1} \} \right] + \mathbb{E} \left[ \lambda_2 \theta (1 - e^{-\theta t_{L}^{k-1}}) \right] + \mathbb{E} \left[ \lambda_2 t_{H}^{k-1} | t_{H}^{k-1} \geq 1 \right]$$

$$= \mathbb{E} \left[ N_{k-1}^1 | t_{H}^{k-2} \geq 1 \right] \mathbb{E} \left[ R_j \geq t_{L}^{k-1} \right] + \lambda_2 \theta \left( 1 - \mathbb{E} \left[ e^{-\theta t_{L}^{k-1}} \right] \right) + 2\lambda_2$$

$$= \mathbb{E} \left[ N_{k-1}^2 | t_{H}^{k-2} \geq 1 \right] \mathbb{E} \left[ e^{-\theta t_{L}^{k-1}} \right] + \lambda_2 \theta \left( 1 - \mathbb{E} \left[ e^{-\theta t_{L}^{k-1}} \right] \right) + 2\lambda_2$$

$$\leq \mathbb{E} \left[ N_{k-1}^1 | t_{H}^{k-2} \geq 1 \right] \frac{1}{1 + \log(1/\varepsilon)} + \frac{\lambda_2}{\theta} + 2\lambda_2,$$

where in the last inequality we take $\varepsilon < 1$. Using the recursion above, we have

$$\mathbb{E}[N^k | t_{H}^{k-1} \geq 1] \leq \frac{N_2}{(1 + \log(1/\varepsilon))} + \left( \frac{\lambda_2}{\theta} + 2\lambda_2 \right) \sum_{j=0}^{k-1} \frac{1}{(1 + \log(1/\varepsilon))}$$

$$= \frac{N_2}{(1 + \log(1/\varepsilon))} + \left( \frac{\lambda_2}{\theta} + 2\lambda_2 \right) \frac{(1 + \log(1/\varepsilon))^{k-1} - 1}{(1 + \log(1/\varepsilon))^{k-1} \log(1/\varepsilon)}$$

$$\leq N_2 + \left( \frac{\lambda_2}{\theta} + 2\lambda_2 \right) \left( 1 + \frac{1}{\log(1/\varepsilon)} \right), \quad k \geq 1.$$

To conclude the proof, we lower bound the long run throughput by the number of matches that can be done in each 4th interval. For each interval, we lower bound the number of matches as we did in (EC.12). The only difference is that instead of using (EC.10), we use the following bound for the probability of success:

$$\mathbb{E} [\beta(1, N^k + 1) | t_H \geq 1/2] \geq \mathbb{E} [\beta(1, \mathbb{E}[N^k + 1 | t_H \geq 1/2]) \geq \mathbb{E} \left[ \beta \left( 1, N_2 + \left( \frac{\lambda_2}{\theta} + \frac{3\lambda_2}{2} \right) \left( 1 + \frac{1}{\log(1/\varepsilon)} \right) + 1 \right) \right].$$
The throughput is then lower bounded by

\[
\text{TP}(\pi^{\text{RND}}) = \lim_{N \to \infty} \frac{\sum_{k=1}^{N} E[\# \text{ matches in the } k^{th} \text{ interval}]}{\sum_{k=1}^{N} E[L + H^k]}
\]

\[
\geq \frac{\lambda_1 \cdot e^{-\mu \Delta} \cdot (1 - e^{-\mu_1, H \Delta}) \cdot e^{-\lambda_1 \Delta} (1 - 2\Delta) \cdot \beta (1, N_2 + (\frac{\lambda_2}{\beta} + \frac{3\Delta}{2}) (1 + \frac{1}{\log(1/\sigma)}) + 1)}{1/\varepsilon + 1}
\]

which \(\uparrow \infty = \varepsilon \downarrow 0\) for any \(N_2 \geq 0\). This completes the proof. \(\square\)

**Proof of Proposition 4.** Recall that \(W_{\sigma_0,q}^{\text{FRfb}}(\sigma)\) and \(W_{\sigma_0,q}^{\text{FR}}(\sigma)\) are the steady state waiting time of type 0 agent in queue \(q\) under the FRfb and the FR policy, respectively, when the queue-joining strategy is \(\sigma\). Since we restrict ourselves to two types \(\mathcal{L} = \{0, 1\}\), we can effectively use \(\sigma_{0,1}\) to represent the strategy profile \(\sigma\). In the rest of the proof, we abuse the notation a little by using \(W_{\sigma_0,q}(\sigma_{0,1})\) to denote \(W_{\sigma_0,q}(\sigma)\).

Note that \(W_{\sigma_0,1}^{\text{FR}}(\sigma_{0,1})\) is increasing in \(\sigma_{0,1}\) because under the ACR policy, larger \(\sigma_{0,1}\) increases the arrival rate of agents to queue 1 but it does not change the arrival rate of jobs to this queue, and queue 1 behaves as an M/M/1+M queue. Also, for any \(\sigma_{0,1} \in [0, 1]\) we must have \(W_{\sigma_1,0}^{\text{FRfb}}(\sigma_{0,1}) \geq W_{\sigma_1,0}^{\text{FR}}(\sigma_{0,1})\). To see why this is true, consider a sample path argument where we tag a type 0 agent arriving to queue 1 in two systems — one runs under the FRfb policy and the other runs under the FR policy — that encounter the same queue length in queue 1. Let us further consider the time it takes the tagged agent to be matched. Under the FR policy, the tagged agent can only be matched to some job of type 1 but not type 0. In contrast, under the FRfb policy, the tagged agent has chances to be matched to a job of type 0 as long as queue 0 is empty, on top of being possibly matched to type 1 job. In turn, the tagged agent on any sample path will wait less under the FRfb policy than under the FR policy. That is, \(W_{\sigma_1,0}^{\text{FRfb}}(\sigma_{0,1}) \geq W_{\sigma_1,0}^{\text{FR}}(\sigma_{0,1})\) for any \(\sigma_{0,1}\).

Now we argue that \(W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1}) = W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1})\) for all \(\sigma_{0,1} \in [0, 1]\). Similar to the sample path argument in the previous paragraph, consider a tagged type 0 agent arriving to queue 0, we show that this arrival experiences the same waiting time in systems under two policies for each sample path. Indeed, simply notice that the allocation of jobs to agents in queue 0, when starting from the same state, is the same under both policies. The only difference merges when queue 0 is empty, but at that moment the tagged agent has already left the system.

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**Figure EC.1** Illustration for the Proof of Proposition 4

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Figure EC.1 above illustrates these waiting times which reflect the relationship we just proved above. \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1})\) collapses with \(W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\) for all \(\sigma_{0,1}\) (blue curve), while \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1}) \geq W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\) for all \(\sigma_{0,1}\) (black solid and dashed curves). To conclude the proof, we consider three cases below based on the relative value of \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1})\) (and \(W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\)), illustrated by the three different blue curves in Figure EC.1.

**Case 1:** \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1}) = W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\)

**Case 2:** \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1}) > W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\)

**Case 3:** \(W_{\sigma_0,0}^{\text{FRf}}(\sigma_{0,1}) > W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\)

In such a case, since \(W_{\sigma_0,0}^{\text{FRfb}}(\sigma_{0,1}) \geq W_{\sigma_0,0}^{\text{FR}}(\sigma_{0,1})\), \(\forall \sigma_{0,1}\), we can deduce that the only equilibrium under the FR and the FRfb policies are \(\sigma_{0,1} = \sigma_{0,1}^{\text{FRfb}} = 1\), respectively.
The next event is reneging by an agent counted in $W_{0,0}^{\text{FR}}(0)$, there exists $\sigma_{0,1}^* \in [0,1]$ such that $W_{0,0}^{\text{FR}}(\sigma_{0,1}^*) = W_{0,1}^{\text{FR}}(\sigma_{0,1}^*)$.

In such case, $\sigma_{0,1}^* = \sigma_{0,1}^*$, while clearly $\sigma_{0,1}^* \geq \sigma_{0,1}^*$ as for all $\sigma_{0,1} < \sigma_{0,1}^*$ we have $W_{0,0}^{\text{FR}}(\sigma_{0,1}) < W_{0,1}^{\text{FR}}(\sigma_{0,1})$.

Case 2: $W_{0,0}^{\text{FR}}(0) = W_{0,0}^{\text{FRh}}(0) > W_{0,1}^{\text{FR}}(0)$, there exists $\sigma_{0,1}^* \in [0,1]$ such that $W_{0,0}^{\text{FRh}}(\sigma_{0,1}^*) = W_{0,1}^{\text{FR}}(\sigma_{0,1}^*)$.

In such case, $\sigma_{0,1}^* = \sigma_{0,1}^*$, while clearly $\sigma_{0,1}^* \geq \sigma_{0,1}^*$ as for all $\sigma_{0,1} < \sigma_{0,1}^*$ we have $W_{0,0}^{\text{FRh}}(\sigma_{0,1}) < W_{0,1}^{\text{FR}}(\sigma_{0,1})$.

Proof of Theorem 1. Fix $i \neq 1$. Under the FRfb policy, type $i$ agents can only be matched in queue $i$ or queue $1$ with type $i$ jobs. In any other queue their waiting time would be infinity. Since the FRfb policy only attempts to match type $i$ jobs with agents in queue $1$ when queue $i$ becomes empty, a type $i$ agent would wait less in queue $i$ than in queue $1$. This holds regardless of the strategy that flexible agents play.

We now define two sets of state representations for the FRfb policy and the random policy respectively.

- The FRfb policy: $A \in \{A_i\}_{i \in \mathcal{L}}$, number of agents of each type as all agents join the same queue.
- The FRfb policy: $Q = \{(Q_{i,t}), \{Q_i\}_{i \in \mathcal{L}\{1\}}\}$ where $Q_{i,t}$ is the number of type $i$ agents in queue $i$ and $Q_i$ is the number of type $i \neq 1$ agents in queue $i$.

We first prove the result for a fixed time horizon $T > 0$. We want to prove the following inequality holds for all $T \geq 0, \sigma \in \Sigma(\pi^{\text{FR}})$:

$$
E[M^{\text{FRh}}(T; \sigma) | Q(0) = Q] \geq E[M^{\text{RND}}(T) | A(0) = A], \forall Q, A : A_1 = \sum_{i \in \mathcal{L}} Q_{1,i}, A_i = Q_i, \forall i \in \mathcal{L} \setminus \{1\}.
$$

Let $M_k^{\text{FRh}}(Q, t)$ and $M_k^{\text{RND}}(A, t)$ be the expected number of matches starting with states $Q$ and $A$ from time $t$ up until the time horizon $T$ expires or until we reach a total of $k$ events, whichever occurs first, under the RCR policy and the random policy respectively. Events include job arrivals, agent arrivals, and reneging. It is sufficient to show that under any strategy profile $\sigma \in \Sigma(\pi^{\text{FRh}})$, $M_k^{\text{FRh}}(Q, t) \geq M_k^{\text{RND}}(A, t)$ for any finite $k$, any $t \in [0, T]$, any $Q, A$ such that $A_1 = \sum_{i \in \mathcal{L}} Q_{1,i}, A_i = Q_i, \forall i \in \mathcal{L} \setminus \{1\}$. We show this via induction on $k$. It can be seen that $k = 0$ holds trivially, so we proceed to show it is true for $k > 0$ assuming that it is true for $k - 1$.

We fix a sequence of agent and job arrivals within $[t, T]$. Also consider a collection of independent exponential random variables representing the times until each of several possible reneging events occur in the system. Then the next event occurs randomly within the following categories:

1. Reneging by an agent counted in $A$, from each type with a non-zero component in $A$;
2. An agent arrival, from each of the $\ell$ types;
3. A job arrival, from each of the $\ell$ types.

We let random variable $t'$ represent the sum of $t$ and the minimum of these times. The random variable achieving this minimum determines the event that occurs next. If $T$ occurs before any of these times, then the time horizon expires before the next event occurs.

Under the FRfb policy and the random policy, let random variables $m(\pi^{\text{FRh}})$ and $m(\pi^{\text{RND}})$ represent whether a match results from the next event, let $Q'(\pi^{\text{FRh}})$ and $A'(\pi^{\text{RND}})$ represent the system state after the event occurs. Thus,

$$
M_k^{\text{FRh}}(Q, t) = E[m(\pi^{\text{FRh}}) + M_{k-1}^{\text{FRh}}(Q'(\pi^{\text{FRh}}), t')],
M_k^{\text{RND}}(A, t) = E[m(\pi^{\text{RND}}) + M_{k-1}^{\text{RND}}(A'(\pi^{\text{RND}}), t')].
$$

The expectation is taken over $m(\cdot), A'(\cdot), Q'(\cdot)$ and $t'$. To show our result, we will consider the next event on a case-by-case basis to show that with probability one,

$$
m(\pi^{\text{FRh}}) + M_{k-1}^{\text{FRh}}(Q'(\pi^{\text{FRh}}), t') \geq m(\pi^{\text{RND}}) + M_{k-1}^{\text{RND}}(A'(\pi^{\text{RND}}), t').
$$

If $t' > T$, then $m(\pi^{\text{FRh}}) = m(\pi^{\text{RND}}) = 0$ and $M_{k-1}^{\text{FRh}}(Q'(\pi^{\text{FRh}}), t') = M_{k-1}^{\text{RND}}(A'(\pi^{\text{RND}}), t') = 0$, automatically verifying this expression. Thus, it is sufficient to focus on $t' \leq T$.

Case 1: The next event is reneging by an agent counted in $A$ or an agent arrival. In this case, we have $Q'(\pi^{\text{FRh}}) = A'(\pi^{\text{RND}})$ and $m(\pi^{\text{FRh}}) = m(\pi^{\text{RND}}) = 0$. Then, the induction hypothesis shows (EC.13).

Case 2: The next event is a job of type $j$ and $A_j > 0$. When $j \neq 1$, this implies that $Q_j > 0$ and,

$$
m(\pi^{\text{FRh}}) + M_{k-1}^{\text{FRh}}(Q'(\pi^{\text{FRh}}), t')
$$
\[ = 1 + \frac{Q_j}{Q_j + Q_{1,j}} M_{k-1}^{\text{FRh}} (Q - e_j, t') + \frac{Q_{1,j}}{Q_j + Q_{1,j}} M_{k-1}^{\text{FRh}} (Q - e_{1,j}, t') \]

\[ \geq 1 + \frac{A_j}{A_1 + A_j} M_{k-1}^{\text{FRh}} (Q - e_j, t') + \frac{A_1}{A_1 + A_j} M_{k-1}^{\text{FRh}} (Q - e_{1,j}, t') \]

(EC.14)

\[ \geq 1 + \frac{A_j}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_j, t') + \frac{A_1}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_{1,j}, t') \]

(induction hypothesis)

\[ = \beta \left( A_1 + A_j, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + \frac{A_j}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_j, t') + \frac{A_1}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_{1,j}, t') \right) \]

\[ \geq \beta \left( A_1 + A_j, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + \frac{A_j}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_j, t') + \frac{A_1}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_{1,j}, t') \right) \]

(Lemma 2)

\[ = \beta \left( A_1 + A_j, \sum_{i \in \mathcal{L}} A_i \right) + \beta \left( A_1 + A_j, \sum_{i \in \mathcal{L}} A_i \right) \left( \frac{A_j}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_j, t') + \frac{A_1}{A_1 + A_j} M_{k-1}^{\text{RND}} (A - e_{1,j}, t') \right) \]

\[ = \beta \left( A_1 + A_j, \sum_{i \in \mathcal{L}} A_i \right) M_{k-1}^{\text{RND}} (A, t') \]

\[ = m(\pi^{\text{RND}}) + M_{k-1}^{\text{RND}} (A(\pi^{\text{RND}}), t'), \]

showing (EC.13). Inequality (EC.14) holds because: (1) \( Q_{1,j} < Q_1 = A_j \) which leads to \( Q_{1,j}/(Q_j + Q_{1,j}) \geq A_j/(A_1 + A_j) \), \( Q_{1,j}/(Q_j + Q_{1,j}) \leq A_1/(A_1 + A_j) \); (2) \( M_{k-1}^{\text{FRh}} (Q - e_j, t') \geq M_{k-1}^{\text{FRh}} (Q - e_{1,j}, t') \) by Lemma 2 which we state and prove in Appendix C.

When \( j = 1 \), we have \( \sum_{i \in \mathcal{L}} Q_{1,i} > 0 \). Under the case that \( Q_{1,1} > 0 \),

\[ m(\pi^{\text{FRh}}) + M_{k-1}^{\text{FRh}} (Q(\pi^{\text{FRh}}), t') \]

\[ = 1 + M_{k-1}^{\text{FRh}} (Q - e_{1,1}, t') \]

(induction hypothesis)

\[ \geq \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{\text{RND}} (A - e_1, t') \right) \]

\[ \geq \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{\text{RND}} (A - e_j, t') \right) \]

(Lemma 2)

\[ = m(\pi^{\text{RND}}) + M_{k-1}^{\text{RND}} (A(\pi^{\text{RND}}), t'), \]

showing (EC.13).

Under the case that \( Q_{1,1} = 0 \), let random variable \( X \in \{1, \cdots, \ell\} \) denote the queue number from which the job is matched to a type 1 agent conditioning on the job being successfully matched,

\[ m(\pi^{\text{FRh}}) + M_{k-1}^{\text{FRh}} (Q(\pi^{\text{FRh}}), t') \]

\[ = \beta \left( \sum_{i \in \mathcal{L}} Q_{1,i} \sum_{i \in \mathcal{L}} Q_{1,i} + \sum_{i \in \mathcal{L} \setminus \{1\}} Q_i \right) \left[ 1 + M_{k-1}^{\text{FRh}} (Q - e_{1,X}, t') \right] \]
+ \left(1 - \beta \left( \sum_{i \in \mathcal{L}} Q_{1,i}, \sum_{i \in \mathcal{L}} Q_{1,i} + \sum_{i \in \mathcal{L} \setminus \{1\}} Q_i \right) \right) \mu_{k-1}^{FRb} (Q, t')

= \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \mathbb{E}_X \left[ 1 + M_{k-1}^{FRb} (Q - e_{1,X}, t') \right] + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) \mu_{k-1}^{FRb} (Q, t')

\geq \beta \left( A_0, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{RND} (A - e_{1,t'}) \right) + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) M_{k-1}^{RND} (A, t') \quad \text{(induction hypothesis)}

= m(\pi_{RND}) + M_{k-1}^{RND} \left( A'(\pi_{RND}), t' \right),

showing (EC.13).

\textbf{Case 2:} The next event is a job of type \( j \in \mathcal{L} \) and \( A_j = 0 \) as well as \( A_i = 0 \). This implies that \( Q_j = 0 \) and \( \sum_{i \in \mathcal{L}} Q_{1,i} = 0 \). In this case, (EC.13) holds simply by the induction hypothesis.

\textbf{Case 3:} The next event is a job of type \( j \neq 1 \) and \( A_j = 0 \) but \( A_1 > 0 \). This implies that \( Q_j = 0 \) and \( \sum_{i \in \mathcal{L}} Q_{1,i} > 0 \). Under the case that \( Q_{1,j} = 0 \) and \( Q_{1,1} > 0 \), let random variable \( X \in \{1, \cdots, t\} \), \( X \neq j \) denote the queue number from which the job is matched to a type 0 agent conditioning on the job being successfully matched,

\[ m(\pi_{FRb}) + M_{k-1}^{FRb} (Q'(\pi_{FRb}), t') \]

\[ = \beta \left( \sum_{i \in \mathcal{L}} Q_{1,i}, \sum_{i \in \mathcal{L}} Q_{1,i} + \sum_{i \in \mathcal{L} \setminus \{1\}} Q_i \right) \mathbb{E}_X \left[ 1 + M_{k-1}^{FRb} (Q - e_{1,X}, t') \right] + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) \mu_{k-1}^{FRb} (Q, t') \]

\[ \geq \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{RND} (A - e_{1,t'}) \right) + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) M_{k-1}^{RND} (A, t') \quad \text{(induction hypothesis)}

= m(\pi_{RND}) + M_{k-1}^{RND} \left( A'(\pi_{RND}), t' \right),

showing (EC.13).

Under the case that \( Q_{1,j} > 0 \),

\[ m(\pi_{FRb}) + M_{k-1}^{FRb} (Q'(\pi_{FRb}), t') \]

\[ = 1 + M_{k-1}^{FRb} (Q - e_{1,j}, t') \]

\[ = \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{FRb} (Q - e_{1,j}, t') \right) + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) M_{k-1}^{FRb} (Q - e_{1,j}, t') \]

\[ \geq \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{RND} (A - e_{1,t'}) \right) + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) \mu_{k-1}^{FRb} (Q - e_{1,j}, t') \quad \text{(induction hypothesis)} \]

\[ \geq \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \left( 1 + M_{k-1}^{RND} (A - e_{1,t'}) \right) + \left(1 - \beta \left( A_1, \sum_{i \in \mathcal{L}} A_i \right) \right) M_{k-1}^{RND} (A, t') \quad \text{(Lemma 2)}

= m(\pi_{RND}) + M_{k-1}^{RND} \left( A'(\pi_{RND}), t' \right),
showing (EC.13). Finally, under the case that $Q_{i,j} = 0$ but $Q_{1,1} > 0$,

$$m(\pi^{\text{FRfb}}) + M_{k-1}^{\pi^{\text{FRfb}}} (Q(\pi^{\text{FRfb}}), t')$$

$$= 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t')$$

$$= \beta \left( A_1, \sum_{i \in L} A_i \right) \left( 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t') \right) + \left( 1 - \beta \left( A_1, \sum_{i \in L} A_i \right) \right) \left( 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t') \right)$$

$$\geq \beta \left( A_1, \sum_{i \in L} A_i \right) \left( 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t') \right) + \left( 1 - \beta \left( A_1, \sum_{i \in L} A_i \right) \right) \left( 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t') \right)$$

$$\geq \beta \left( A_1, \sum_{i \in L} A_i \right) \left( 1 + M_{k-1}^{\pi^{\text{FRfb}}} (Q - e_{1,1}, t') \right) + \left( 1 - \beta \left( A_1, \sum_{i \in L} A_i \right) \right) M_{k-1}^{\pi^{\text{FRfb}}} (A, t')$$

$$= m(\pi^{\text{RND}}) + M_{k-1}^{\pi^{\text{RND}}} (A'(\pi^{\text{RND}}), t'),$$

showing (EC.13). This completes the proof. □

**Proof of Proposition 5.** Let $A_T$ be the set of agents that arrive in sample path $\omega_T$ during time $[0, T]$. For any matching policy $\pi$ define

$$A_T^\pi = \{ a \in A_T : a \text{ is matched by } \pi \}, \quad A_T^\pi = \{ a \in A_T : a \text{ reneges in } \pi \},$$

where in $A_T^\pi$ we include those agents did not renege in $[0, T]$ but were not matched as well. With a bit of abuse of notation, let $\pi(\omega_T)$ be the total number of matches under sample path $\omega_T$ and policy $\pi$. Let OPT and OPT($\omega_T$) be the offline optimal policy and $\pi(\omega_T)$ be the offline optimal policy and its number of matches. We have that

$$\text{OPT}(\omega_T) - \pi(\omega_T) = |A_1^{\text{OPT}}| - |A_2^{\pi}|$$

$$= |A_1^{\text{OPT}} \cap A_2^{\pi}| + |A_1^{\text{OPT}} \cap A_2^\pi| - |A_1^{\text{OPT}} \cap A_2^\pi| - |A_1^\pi \cap A_2^{\text{OPT}}|$$

$$= |A_1^{\text{OPT}} \cap A_2^{\pi}| - |A_1^\pi \cap A_2^{\text{OPT}}|$$

$$\leq |A_1^{\text{OPT}} \cap A_2^{\pi}|.$$

Consider an agent $a$ in $A_1^{\text{OPT}} \cap A_2^\pi$, this agent is matched by OPT at time, say, $t$. At this time $a$ is in both systems, the one run by OPT and $\pi$. Also, since OPT is matching $a$ there must be a job arrival at $t$. Policy $\pi$ is not matching this incoming job arrival with $a$ (because $a$ reneges in $\pi$); but since $a$ is in the system run by $\pi$ and $\pi$ is non-idling then it must be that $\pi$ is matching the job arrival to some other agent $a'$. That is, for every $a \in A_1^{\text{OPT}} \cap A_2^\pi$ there exists another $a' \in A_T$ that is matched by $\pi$. Therefore $|A_1^{\text{OPT}} \cap A_2^\pi| \leq \pi(\omega_T)$. This concludes the proof. □