1. Introduction

1.1. Uniformisation of Riemann surfaces. One of the main reasons for the beauty and elegance of Riemann surface theory is the fact that there is a very short list of simply-connected Riemann surfaces, given by the uniformisation theorem of Koebe and Poincaré.

**Theorem 1.1** (Uniformisation theorem for Riemann surfaces, 1907). Let $X$ be a simply-connected Riemann surface. Then, $X$ is biholomorphic to exactly one of the following Riemann surfaces:

- the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,
- the complex plane $\mathbb{C}$,
- the unit disk $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.
The history of this result is rich, begins with the Riemann mapping theorem in the 1850s, and involves many of the most important figures of mathematics at the end of the 19th and the beginning of the 20th century. It is surveyed for example in [Gra94].

One of the main consequences of the uniformisation theorem is the “great trichotomy” seen in the geometry of compact Riemann surfaces, cf. [Rei02]: they fall into three classes that can be defined and characterised in topological1, numerical2, algebro-geometric3, and differential-geometric4 terms.

Some of the equivalences contained in the previous enumeration can actually be proven using the uniformisation theorem: The existence of constant curvature metrics can be deduced from the existence of constant curvature metrics on the universal covers that are invariant under the respective deck transformation groups: the Fubini-Study metric on \( \hat{\mathbb{C}} = \mathbb{P}^1 \), the standard flat metric on \( \mathbb{C} \), and the Poincaré metric on \( \mathbb{D} \).

One further aspect that will appear again later is the fact that while the theorem of Gauß-Bonnet tells us that the integral over the curvature of any Riemannian metric on a compact Riemann surface \( X \) equals \( 2 - 2g(X) \), there always exists a distinguished metric of constant curvature, whose sign is dictated by the topology. Moreover, we can determine the universal cover of a given compact Riemann surface up to biholomorphism just by computing the degree of the canonical bundle on the surface itself.

In some sense, the uniformisation theorem reduces the study of the geometry of (compact) Riemann surfaces to the investigation of the \( \pi_1 \)-equivariant geometry of the universal cover. The corresponding function theory is the study of theta functions in the genus zero case, and of modular forms in the higher genus case. One result that can be obtained using the study of these special functions is that every compact Riemann surface is in fact projective, see for example [Sha94, Chapter IX].

1.2. Problems in higher dimensions. Moving on to higher dimensions, one quickly realises that a statement similar to Koebe’s and Poincaré’s result is not possible, as many new phenomena appear. In some sense, there are just too many simply-connected complex manifolds in any given dimension, as exemplified by the following.

a) The only simply-connected compact Riemann surface is \( \mathbb{P}^1 \), defining the class of the trichotomy with negative canonical bundle. On the other hand, Lefschetz’ theorem implies that for example any smooth hypersurface of degree greater than or equal to five in \( \mathbb{P}^3 \) is simply-connected. The canonical bundle of such a surface is ample and it therefore belongs to the opposite end of the spectrum. At the same time, this yields non-trivial families of (compact) simply-connected non-biholomorphic manifolds, which also do not exist in dimension one.

b) Every complex manifold with ample anti-canonical bundle, i.e., every Fano manifold, is rationally chain-connected by work of Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92], and hence simply-connected, see for example [Cam91, Thm. 3.5]. Moreover, Fano manifolds of fixed dimension form a bounded family, see again [KMM92]. While classification up to deformation was achieved in dimension three by Iskovskih [Isk77, Isk78] and

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1Euler characteristic positive, zero or negative; genus 0, 1 or \( \geq 2 \); fundamental group trivial, Abelian or nonabelian
2degree of canonical line bundle negative, zero or positive
3canonical line bundle anti-ample, trivial or ample
4existence of a metric of constant positive, zero or positive curvature
Mori-Mukai [MM82, MM03] at the beginning of the 1980’s building on work of Shokurov [Sho79], it is impossible in higher dimensions. 

c) Calabi-Eckmann manifolds are complex manifolds whose underlying real-differentiable manifold is isomorphic to the product $S^{2n+1} \times S^{2m+1}$ of to odd-dimensional spheres, see [CES3]. They contain open subsets diffeomorphic to $\mathbb{R}^{2(n+m)+2}$ which do not admit any non-constant holomorphic function. In particular, the corresponding complex structure on $\mathbb{R}^{2(n+m)+2}$ cannot be described using a single coordinate chart.

d) On the other hand, looking at basins of attractions for (the iteration of) certain holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^n$, one finds open subsets of $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$, the so-called Fatou-Bieberbach domains; see [MNTU00, Chapter 6.3]. These examples stand in sharp contrast to the statement of the Riemann mapping theorem in dimension one.

e) The unit disk is the only Riemann surface in the list that admits non-constant bounded holomorphic functions. In higher dimensions, there are many more examples: First shown by Poincaré, the unit ball $B^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 < 1\}$

and the unit polydisk $D \times \cdots \times D$ are not biholomorphically equivalent for $n \geq 2$, see for example [Nar95, Chapter 5, Proposition 4]. In fact, every bounded homogeneous domain is contractible by [VGP63] and hence simply-connected. Bounded symmetric domains were classified by Cartan, see for example [Hel01, Chapter X], but starting in dimension four not every bounded homogeneous domain is symmetric, as shown by a famous example of Piatetski-Shapiro, [PS69]. Starting in dimension 7 there are infinite families of bounded homogeneous domains that are not symmetric.

f) Given one simply-connected complex manifold $X$ of dimension greater than one, one can produce infinitely many new ones by blowing-up points in $X$. Hence, to achieve some understanding one certainly has to impose some minimality condition. This point will reappear in Section 1.6.

One solution to the issue raised by the enumeration above is to modify the question and ask for a characterisation of those compact complex manifolds/smooth projective varieties whose universal cover is biholomorphic to some fixed simply-connected model space having well-understood geometry. Motivated by the uniformisation theorem for Riemann surfaces, in the following discussion we hence concentrate on the following

Goal. Characterise those compact complex manifolds whose universal cover is biholomorphic to $\mathbb{P}^n$, $\mathbb{C}^n$, or $B^n$.

In fact, as in dimension one, every holomorphic automorphism of $\mathbb{P}^n$ has a fixed point, and so for this part one is left with the task of characterising $\mathbb{P}^n$. The techniques used in various approaches to this problem are mostly based on studying rational curves and are hence different in spirit from the other two cases. We refer the reader to [Mor79, SY80, Wah83, AW01, CMSB02, Keb02, ADK08] and from Section 1.4 onwards restrict ourselves to studying quotients of $\mathbb{C}^n$ and $B^n$. We note that as in the case of Riemann surfaces, the geometry of these manifolds can be studied using $\pi_1$-equivariant objects on the universal cover, theta functions and automorphic forms.

1.3. Metric characterisations. From the differential-geometric formulation of the great trichotomy, one derives the idea that searching for special metrics is one approach to the uniformisation problem also in higher dimensions. And indeed,
one has the following result; cf. [KN96, Chapter IX, Theorem. 7.9], where it is credited to Hawley and Igusa:

**Theorem 1.2.** If a projective manifold admits a Kähler metric of constant holomorphic sectional curvature, the universal cover of $X$ is biholomorphic to $\mathbb{P}^n$, $\mathbb{C}^n$, or $\mathbb{B}^n$ (depending on the sign of the curvature).

However, determining whether a given projective manifold admits a Kähler metric of constant curvature is a difficult task, and \textit{a priori} not an algebro-geometric (or even topological) condition.

1.4. Deriving necessary algebro-geometric conditions. Let $X$ be a projective manifold whose universal cover is biholomorphic to $\mathbb{C}^n$. It was conjectured by Iitaka and proven by Nakayama in dimension less than or equal to three (and in all dimensions assuming the Abundance Conjecture) that in this situation there exists an Abelian variety $A$ and a finite group $G$ of fixed-point free holomorphic automorphisms of $A$ such that $X \cong A/G$, [Nak99, Theorem 1.4]. Consequently, the tangent bundle of $X$ is flat, and therefore we obtain the intersection-theoretic conditions $c_1(X) = 0 \in H^2(X, \mathbb{R})$ and $c_2(X) \cdot [H]^{n-2} = 0$, where $H$ is any ample divisor on $X$. In particular, we have

$$\left( c_2(X) - \frac{n}{2(n+1)} \cdot c_1^2(X) \right) \cdot [H]^{n-2} = 0. \quad (1.2.1)$$

If $X$ is a projective manifold whose universal cover is biholomorphic to $\mathbb{B}^n$, then the Bergman metric of $\mathbb{B}^n$, which has constant negative holomorphic sectional curvature, is invariant under the deck transformation group. It induces a Kähler metric $g$ on $X$ whose associated $(1,1)$-form is the curvature form of a metric in the canonical bundle of $X$, which is therefore ample by Kodaira’s theorem. Note that ampleness can be detected using intersection theory by the Nakai-Moishezon criterion. Additionally, using the fact that $g$ has constant holomorphic sectional curvature and that we can compute the Chern classes of $X$ from $g$, or using the Hirzebruch proportionality principle [Hir95, Appendix 1] one concludes that

$$\left( c_2(X) - \frac{n}{2(n+1)} \cdot c_1^2(X) \right) \cdot [K_X]^{n-2} = 0. \quad (1.2.2)$$

1.5. The Miyaoka-Yau inequality and uniformisation for higher-dimensional manifolds. In fact, the two equations (1.2.1) and (1.2.2) represent the extremal case of an inequality and they characterise exactly those projective manifolds whose universal cover is isomorphic to $\mathbb{C}^n$ and $\mathbb{B}^n$ by the following result of Yau [Yau77].

**Theorem 1.3** (Miyaoka-Yau inequality). Let $X$ be an $n$-dimensional projective manifold whose canonical class is numerically trivial or ample, respectively. Then, we have

$$\left( c_2(X) - \frac{n}{2(n+1)} \cdot c_1^2(X) \right) \cdot [A]^{n-2} \geq 0, \quad (1.3.1)$$

where $A$ is either an arbitrary ample divisor on $X$ or equal to $K_X$, respectively. We call (1.3.1) the Miyaoka-Yau inequality. In case of equality, the universal cover of $X$ is biholomorphic to $\mathbb{C}^n$ or $\mathbb{B}^n$, respectively.

**Sketch of proof.** The proof is based on Yau’s solution of the Calabi conjecture, which in the situation at hand produces a Kähler-Einstein metric on $X$ whose associated $(1,1)$-form represents the class of $A$ in $H^{1,1}(X, \mathbb{R})$. Using this metric to compute differential forms representing the Chern classes of the tangent bundle and exploiting the symmetries of the curvature tensor imposed by the Kähler-Einstein condition, one sees that the desired equality holds \textit{pointwise} for the chosen differential forms. The inequality (1.3.1) follows by integration. Yau credits this part of the argument to Guggenheimer.
In case of equality, the pointwise computations done before yield enough restrictions on the metric to see that $X$ has constant holomorphic sectional curvature; the complete computation can for example be found in [Zhe00, pp. 225f]. We conclude using Theorem 1.2.

Hence, after Yau’s result, the question of the existence of a constant curvature metric in a sense is intersection-theoretic, as it is a posteriori guaranteed by numerical triviality/ampleness of the canonical bundle and equality in Theorem 1.3. The result is very close in spirit to the one-dimensional case: again, one can a priori use any metric on $X$ to check whether the Chern classes of $X$ satisfy equality in (1.3.1). However, if this is the case, there exists a distinguished metric having constant curvature, whose sign again depends on the sign of the canonical class.

Generalisations of the Miyaoka-Yau inequality, the question whether there is an algebro-geometric proof, and the problem of uniformisation in case of equality have attracted considerable interest in the last few decades, see Section 1.8 for a discussion. Here, we only mention that one important approach to the problem that avoids the construction of Kähler-Einstein metrics is based on results of Donaldson [Don85], Uhlenbeck-Yau [UY86], and Simpson [Sim88] concerning the existence of Hermitian Yang-Mills connections in stable holomorphic (Higgs) bundles. These metrics, although a priori less directly connected to the geometry of $X$, are then used to conclude that in case equality is attained in (1.3.1) the tangent bundle is flat (in the numerically trivial case) or that $\mathcal{T}_X \oplus \mathcal{O}_X$ is projectively flat (in the case of ample canonical bundle). We will see later that this second approach generalises to the singular setup in a natural way.

1.6. Relation to the minimal model program. In a sense, Theorem 1.3 gives a satisfactory answer to the uniformisation question for projective manifolds in higher dimensions. As it can be applied to projective manifolds with numerically trivial or ample canonical bundle, it is natural to look for a way of producing such varieties. At this point the minimal model program comes into play.

Let $X$ be a projective $n$-dimensional manifold of Kodaira dimension $n$. In general, though the canonical divisor is rather positive, it will not be ample. However, by [BCHM10], the variety $X$ admits a minimal model $X_{\text{min}}$ with terminal singularities and nef canonical divisor, which is moreover semiample by the basepoint-free theorem, [KM98, Theorem 3.3]. The corresponding morphism $\varphi : X_{\text{min}} \rightarrow X_{\text{can}}$ maps $X_{\text{min}}$ birationally onto the canonical model $X_{\text{can}}$ of $X$, which has canonical singularities and ample canonical divisor. A variety with at worst terminal singularities and nef canonical divisor will be called minimal, cf. Reminder 2.3 on page 8.

At least conjecturally, the picture is the same in the case of projective manifolds $X$ of Kodaira dimension zero: we expect $X$ to have a minimal model $X_{\text{min}}$ with terminal singularities and numerically trivial canonical divisor, which then in fact is torsion, due to a theorem of Kawamata [Kaw85, Theorem 8.2].

In both cases, the fact that we made the canonical divisor of $X$ to have definite sign on the minimal/canonical model came at the cost of introducing terminal/canonical singularities$^5$. As a result, Yau’s Theorem cannot be applied to outcomes of the minimal model program. While existence of singular Kähler-Einstein structures on varieties with klt singularities and trivial/ample canonical bundle has been established in [EGZ09], the asymptotics of the metric near the singularities is currently not understood well-enough to argue as in the proof of Theorem 1.3 sketched above.

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$^5$In fact, for technical reasons it is very often necessary to work in the slightly bigger class of klt singularities.
1.7. The Miyaoka-Yau inequality and uniformisation for higher-dimensional minimal varieties: recent results. In order to formulate the main result discussed in this note, we start with the following observations concerning the singularities of minimal and canonical models that we have to deal with:

If $X$ is a variety with terminal singularities\(^6\), then the singular locus of $X$ has codimension at least three. When this is the case, we say that $X$ is smooth in codimension two. As a consequence, the localisation sequence for Chow groups, [Ful98, Chapter 1, Proposition 1.8], allows to define first and second Chern classes of coherent sheaves, as in the non-singular situation. Furthermore, every 2-dimensional klt singularity is analytically equivalent to a quotient singularity $\mathbb{C}^2/G$, where $G$ is a finite subgroup of $\text{GL}_2(\mathbb{C})$. Consequently, for every klt variety $X$ there exists a closed subvariety $Z$ of codimension at least three such that $X \setminus Z$ has at worst quotient singularities. In this case, we say that $X$ has quotient singularities in codimension two. This allows to define first and second orbifold Chern classes\(^7\) of reflexive sheaves, written as $\widehat{c}_1(\mathscr{E}), \widehat{c}_2(\mathscr{E})$. In particular, rational intersection numbers of $\widehat{c}_2(\mathscr{E})$ with $(n-2)$-tuples of Cartier divisors exist.

We can now formulate the main results discussed in these notes. These show that the fundamental Chern class inequalities continue to hold in the singular setting, characterise singular torus- and ball-quotients in terms of Chern classes, and give purely numerical criteria to guarantee that a space with klt singularities has in fact only quotient singularities.

**Theorem 1.4** (Characterisation of singular quotients of Abelian varieties, cf. [GKP16, Theorem 1.17]). Let $X$ be a normal, complex, projective variety of dimension $n$ with at worst canonical singularities. Assume that $X$ is smooth in codimension two and that the canonical divisor is numerically trivial, $K_X \equiv 0$. Further, assume that there exists an ample Cartier divisor $H \in \text{Div}(X)$ such that $c_2(\mathcal{T}_X) \cdot [H]^{n-2} = 0$. Then, there exists an Abelian variety $A$ and a finite, surjective, Galois morphism $A \to X$ that is étale in codimension two.

In other words, once the assumptions of Theorem 1.4 are fulfilled for $X$, we can realise it as the quotient of an Abelian variety by a finite group whose fixed points lie in codimension three or higher. In particular, in this case $X$ has at worst quotient singularities. In a sense, the map $A \to X$ provides a singular uniformisation of $X$, cf. Nakayama’s result discussed in Section 1.4. Generalisations to klt spaces have been obtained in [LT14]. The proof of Theorem 1.4 presented in Section 7.1 uses the inequality $c_2(\mathcal{T}_X) \cdot [H]^{n-2} \geq 0$, proven by Miyaoka [Miy87], that holds for any canonical variety $Z$ that is smooth in codimension two and whose divisor is numerically trivial.

**Theorem 1.5** (Q-Miyaoka-Yau inequality, [GKPT15, Theorem 1.1]). Let $X$ be an $n$-dimensional, projective, klt variety of general type whose canonical divisor $K_X$ is nef. Then,

$$
(2(n+1) \cdot \widehat{c}_2(\mathcal{T}_X) - n \cdot \widehat{c}_1(\mathcal{T}_X)^2) \cdot [K_X]^{n-2} \geq 0.
$$

**Theorem 1.6** (Characterisation of singular ball quotients, [GKPT15, Theorem 1.2]). Let $X$ be an $n$-dimensional minimal variety of general type. If equality holds in the Q-Miyaoka-Yau inequality (1.5.1), then the canonical model $X_{\text{can}}$ is smooth in codimension two, there exists a ball quotient $Y$ and a finite, Galois, quasi-étale morphism $f: Y \to X_{\text{can}}$. In particular, $X_{\text{can}}$ has only quotient singularities.

Here, a ball quotient is a projective manifold whose universal cover is the unit ball. In fact, it can be shown that in the situation of Theorem 1.6, the canonical

\(^6\)for example the minimal model of a projective manifold of Kodaira dimension zero

\(^7\)also called “Q-Chern classes”
model $X_{can}$ can be realised as the quotient of $B^n$ by a properly discontinuous action of $\Gamma = \pi_1(X_{can, \text{reg}})$ that is free in codimension two, cf. [GKPT15, Theorem 1.3]. The variety $X_{can}$ in this sense admits a singular uniformisation by the unit ball. This motivates the term singular ball quotients. We emphasise at this point that the theory of automorphic forms does not require the discrete group $\Gamma$ to act freely on the unit ball, and can therefore be applied to study the geometry of $X_{can} = B^n/\Gamma$, see for example [Kol95, Part II].

Our approach to the proof of the above results is based on stability properties of (Higgs) sheaves and is motivated by Simpson’s approach to the uniformisation problem alluded to at the end of Section 1.5. We generalise flatness criteria and relevant results of nonabelian Hodge theory to the singular setting. In particular, we develop a theory of Higgs sheaves on singular spaces. We refer the reader to Section 1.9 below, where the contents of this article are described in detail.

1.8. Earlier work. Generalisations of the Miyaoka-Yau inequality and uniformisation in case of equality have attracted considerable interest in the last few decades.

Inequality (1.3.1) and the uniformisation result were extended to the context of compact Kähler varieties with only quotient singularities by Cheng-Yau [CY86] using orbifold Kähler-Einstein metrics. Tsuji established Inequality (1.3.1) for smooth minimal models of general type in [Tsu88]. Enoki’s result on the semistability of tangent sheaf of minimal models, [Eno88], was used by Sugiyama [Sug90] to establish the Bogomolov-Gieseker inequality for the tangent sheaf of any resolution of a given minimal model of general type with only canonical singularities, the polarisation given by the pullback of the canonical bundle on the minimal model. By using a strategy very similar to ours, that is via results of Simpson [Sim88], Langer in [Lan02, Thm. 5.2] established the Miyaoka-Yau inequality in this context. He recently also gave the first purely algebraic proof of the Bogomolov inequality for semistable Higgs sheaves on smooth projective varieties over fields of arbitrary characteristic, cf. [Lan15]. A strong uniformisation result, together with the Miyaoka-Yau inequality, was established by Kobayashi [Kob85] in the case of open orbifold surfaces.

After the work of Tsuji, the past few years have witnessed significant developments in the theory of singular Kähler-Einstein metrics and Kähler-Ricci flow. These are evident, for example, in the works of Tian-Zhang [TZ06], Eyssidieux-Guedj-Zeriahi [EGZ09], and Zhang [Zha06]. In particular, Inequality (1.3.1) together with a uniformisation result for smooth minimal models of general type have been successfully established by Zhang [Zha09].

1.9. Outline of the paper. After introducing some basic notions and definitions in Sections 2 and 3, an important construction is recalled in Section 4: Maximally quasi-étale covers of mildly singular spaces over which global, flat, analytic sheaves extend across the singular locus. Later on, in Sections 7 and 9, these covers turn out to be extremely useful for the uniformisation problems. In Section 5, Simpson’s work on nonabelian Hodge theory is briefly recalled in a setting that is specifically useful for dealing with the ball-quotient problem in Section 9. In Section 6 we introduce the notion of Higgs sheaves over singular spaces and briefly discuss their various fundamental properties. The material of Sections 5 and 6 is used in Sections 8 and 9, where we establish the Miyaoka-Yau inequality and uniformisation by the ball, so the reader who is only interested in Theorem 1.4 can safely skip them.

In Section 7 we work out the sketch of the proof of the uniformisation by Euclidean space. Section 8 is devoted to establishing the Miyaoka-Yau inequality.
The main ingredients here are the stability result of [Gue15] and the Restriction Theorem 6.10. The concluding Section 9 discusses the proof of Theorem 1.6.

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This overview article summarises the content of several research articles, including [GKP11, GKKP11, GKP14, GKP16, GKPT15], which are joint work with Thomas Peternell. The results presented here are therefore not new. The exposition frequently follows the original articles. There exists some overlap with [KP14].

2. Notation

2.1. Global conventions. Throughout this paper, all schemes, varieties and morphisms will be defined over the complex number field. We follow the notation and conventions of Hartshorne’s book [Har77]. In particular, varieties are always assumed to be irreducible. For all notation around Mori theory, such as klt spaces and klt pairs, we refer the reader to [KM98].

2.2. Varieties. Once in a while, we need to switch between algebraic and analytic categories. The following notation is then useful.

Definition 2.2 (Minimal varieties). A normal, projective variety $X$ is called minimal if $X$ has at worst terminal singularities and if $K_X$ is nef.

Reminder 2.3 (Basepoint-free theorem and canonical models). If $X$ is a projective, klt variety of general type whose canonical divisor $K_X$ is nef, the basepoint-free theorem asserts that $K_X$ is semiample, [KM98, Theorem 3.3]. A sufficiently high multiple of $K_X$ thus defines a birational morphism $\phi: X \to Z$ to a normal projective variety with at worst klt singularities whose canonical divisor $K_Z$ is ample, cf. [KM98, Lemma 2.30]. There exists a $Q$-linear equivalence $K_X \sim_Q \phi^* K_Z$. If $X$ is a minimal variety of general type, then $Z$ has at worst canonical singularities. In this case, we set $Z = X_{can}$, and call it the canonical model of $X$.

Definition 2.4 (Ball quotient). A smooth projective variety $X$ of dimension $n$ is a ball quotient if the universal cover of $X^{an}$ is biholomorphic to the unit ball $B^n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1 \}$. Equivalently, there exists a discrete subgroup $\Gamma < \text{Aut}_c(B^n)$ of the holomorphic automorphism group of $B^n$ such that the action of $\Gamma$ on $B^n$ is cocompact and fixed-point free, and such that $X$ is isomorphic to $B^n/\Gamma$.

The following will be used for notational convenience.

Notation 2.5 (Big and small subsets). Let $X$ be a normal, quasi-projective variety. A closed subset $Z \subset X$ is called small if $\text{codim}_X Z \geq 2$. An open subset $U \subseteq X$ is called big if $X \setminus U$ is small.

Fundamental groups are basic objects in our arguments. We will use the following notation.
Definition 2.6 (Fundamental group and étale fundamental group). If $X$ is a complex, quasi-projective variety, we set $\pi_1(X) := \pi_1(X^{an})$, and call it the fundamental group of $X$. The étale fundamental group of $X$ will be denoted by $\tilde{\pi}_1(X)$.

Remark 2.7. Recall that $\tilde{\pi}_1(X)$ is isomorphic to the profinite completion of $\pi_1(X)$, cf. [Mil80, §5 and references given there].

2.3. Morphisms. Galois morphisms appear prominently in the literature, but their precise definition is not consistent. We will use the following definition, which does not ask Galois morphisms to be étale.

Definition 2.8 (Covers and covering maps, Galois morphisms). A cover or covering map is a finite, surjective morphism $\gamma : X \to Y$ of normal, quasi-projective varieties. The covering map $\gamma$ is called Galois if there exists a finite group $G \subset \text{Aut}(X)$ such that $\gamma$ is isomorphic to the quotient map.

Definition 2.9 (Quasi-étale morphisms). A morphism $f : X \to Y$ between normal varieties is called quasi-étale if $f$ is of relative dimension zero and étale in codimension one. In other words, $f$ is quasi-étale if $\dim X = \dim Y$ and if there exists a closed, subset $Z \subseteq X$ of codimension $\leq 2$ such that $f|_{X \setminus Z} : X \setminus Z \to Y$ is étale.

2.4. Intersection and slope. Given a normal, $n$-dimensional projective variety $X$ and a Cartier divisor $H \in \text{Div}(X)$, we write $[H]$ for its numerical class, ditto with $\text{Q-Cartier}$ $\mathbb{Q}$-divisors. If $H$ is Cartier and $D$ is a Weil-divisor on $X$, there is a well-defined intersection number between $D$ and $[H]^{n-1}$, which we denote by $[D] \cdot [H]^{n-1} \in \mathbb{Z}$. The construction is found in Fulton’s book [Ful98] and is reviewed in [GKPT15, Section 2.6]. In particular, if $\mathcal{E}$ is any coherent sheaf, we can associated a Weil divisor to $\text{det} \mathcal{E}$ and compute its intersection number with $[H]^{n-1}$. The result of this operation is written as $[\mathcal{E}] \cdot [H]^{n-1} \in \mathbb{Z}$.

Definition 2.10 (Slope with respect to a nef divisor). Let $X$ be a normal, projective variety and $H$ be a nef $\mathbb{Q}$-Cartier divisor on $X$. If $\mathcal{E} \neq 0$ is any torsion free, coherent sheaf on $X$, define the slope of $\mathcal{E}$ with respect to $H$ as

$$\mu_H(\mathcal{E}) := \frac{[\mathcal{E}] \cdot [H]^{\dim X - 1}}{\text{rank } \mathcal{E}}.$$ 

Part I. Techniques

3. Reflexive differentials

Kähler differentials are among the most fundamental objects of algebraic geometry. Defined by universal properties, they behave will with respect to pull-back and form a presheaf on the category of schemes. Given a singular space $X$ the sheaves $\Omega^p_X$ of Kähler differentials are however generally hard to deal with. Even in the simplest of settings, these sheaves have torsion as well as cotorsion; we refer the reader to the paper [GR11] for a discussion and for a series of elementary examples.

To obtain a more manageable sheaf, we will often consider the double dual of $\Omega^p_X$. The resulting sheaf of reflexive differentials is reflexive, and thus much better behaved geometrically. On the downside, reflexive differentials do not possibly have the universal properties known from Kähler differentials: since the latter are defined by universal properties, any other construction that satisfies the same universal properties necessarily gives us the sheaf of Kähler differentials back! Once we restrict ourselves to spaces with klt singularities, however, there is more we can say. It has been observed in a series of papers by Greb-Kebekus-Kovács [GKK10] and Greb-Kebekus-Kovács-Peternell [GKKP11] that reflexive differentials do have
some universal properties once we restrict ourselves to (morphisms between) klt spaces. This allows to study reflexive differentials in the context of the minimal model program. These results have been applied to a variety of settings, including a study of hyperbolicity of moduli spaces, [KK10], a partial generalisation of the Beauville–Bogomolov decomposition theorem [GKP11], and deformations of Calabi–Yau varieties [Kol15].

3.1. Definitions and main results. We briefly recall the relevant definitions and results below. Since reflexive differentials have already been discussed in a few other surveys, we restrict ourselves to the smallest amount of material required in our applications. There are more general results for dlt and log canonical pairs, including the existence of residue maps, for which we refer the reader to the references listed in Section 3.4 below.

**Definition 3.1** (Reflexive differentials). Given a normal, complex variety $X$, a reflexive differential on $X$ is a differential form defined only on the smooth locus, without imposing any boundary condition near the singularities. Equivalently, a reflexive differential is a section in the double dual of the sheaf of Kähler differentials. Denoting the sheaf of reflexive differentials by $\Omega^{[p]}_X$, we have $\Omega^{[p]}_X = (\Omega^p_X)^{**} = \iota_* (\Omega^p_{X_{\text{reg}}})$, where $\iota : X_{\text{reg}} \to X$ denotes the inclusion of the smooth locus. More generally, given a quasi-projective variety $X$ and a coherent sheaf $\mathcal{E}$ on $X$, write $\Omega^{[p]}_X = (\Omega^p_X)^{**}$, $\mathcal{E}^{[m]} := (\mathcal{E}^{\otimes m})^{**}$ and $\det \mathcal{E} := (\wedge^{\text{rank} \mathcal{E}} \mathcal{E})^{**}$. Given any morphism $f : Y \to X$, write $f^! \mathcal{E} := (f^* \mathcal{E})^{**}$, etc.

The following result asserts the existence of a useful pull-back morphism for reflexive differentials in the klt setting.

**Theorem 3.2** (Existence of pull-back morphisms in general, [Keb13b, Theorems 1.3 and 5.2]). Let $f : X \to Y$ be any morphism between normal, complex varieties. Assume that there exists a Weil divisor $D$ on $Y$ such that the pair $(Y, D)$ is klt. Then there exists a pull-back morphism $d_{\text{refl}} f : f^* \Omega^{[p]}_Y \to \Omega^{[p]}_X$, uniquely determined by natural universal properties.

3.2. Discussion. The “natural universal properties” mentioned in Theorem 3.2 are a little awkward to formulate. Precise statements are given in [Keb13b, Section 5.3]. In essence, it is required that the pull-back morphisms agree with the pull-back of Kähler differentials wherever this makes sense, and that pull-back is functorial in composition of morphisms. The following theorem, which appeared first, is thus a special case, but also forms a main ingredient in the proof of Theorem 3.2.

**Theorem 3.3** (Extension theorem, [GKKP11, Theorem 1.4]). Let $Y$ be a normal variety and $f : X \to Y$ a resolution of singularities. Assume that there exists a Weil $\mathbb{Q}$-divisor $D$ on $Y$ such that the pair $(Y, D)$ is klt. If $\sigma \in H^0(Y, \Omega^{[p]}_Y) = H^0(Y_{\text{reg}}, \Omega^p_{Y_{\text{reg}}})$ is any reflexive differential form on $Y$, then there exists a differential form $\tau \in H^0(X, \Omega^p_X)$ that agrees on the complement of the $f$-exceptional set with the usual pull-back of the Kähler differential $\sigma |_{Y_{\text{reg}}}$.
It should be noted that Theorem 3.2 does not require the image of \( f \) to intersect the smooth locus of \( Y_{\text{reg}} \). One particularly relevant setting to which Theorem 3.2 applies is that of a klt space \( Y \), and the inclusion (or normalisation) of the singular locus, say \( f : X = Y_{\text{sing}} \to Y \). It might seem surprising that a pull-back morphism exists in this context, because reflexive differential forms on \( Y \) are, by definition, differential forms defined on the complement of \( Y_{\text{sing}} \), and no boundary conditions are imposed that would govern the behaviour of those forms near the singularities.

### 3.3. Immediate consequences

It had been known for a long time that the existence of a pull-back functor for reflexive forms will give partial answers to the Lipman-Zariski conjecture. The following corollary is perhaps not obvious, but follows in fact rather quickly using an argument going back to Steenbrink and van Straten.

**Theorem 3.4** (The Lipman-Zariski conjecture for klt spaces, [GKKP11, Theorem 6.1]). Let \( X \) be a normal, projective, klt variety. If the tangent sheaf \( \mathcal{T}_X \) is locally free, then \( X \) is smooth. \( \square \)

We refer the reader to [KP14, Section 6], which sketches a proof of Theorem 3.4 as a consequence of Theorem 3.3. There are generalisations as well as newer proofs that do not rely on the extension theorem; cf. [Dru14, GKK10, Jø14, Gra15].

### 3.4. References

The universal properties of reflexive differentials on klt and log canonical spaces were first established in the papers [GK10, GKKP11]. The formulation presented here comes from the subsequent paper [Keb13b]. The interested reader will definitively also want to look at [HJ14] for a different take on the same circle of ideas. The papers [GKP14, GKKP11] as well as the surveys [Keb13a, KP14] discuss reflexive differentials and their applications in greater detail, see [Hub15] for a different perspective. Kollár’s book on the singularities of the minimal model program also reviews the basic results, [Kol13, Section 8.5].

4. **Existence of maximally quasi-étale covers**

Representations of fundamental groups feature prominently in nonabelian Hodge theory, and are one of the recurring themes in this survey, cf. Section 5.2 below. If \( X \) is smooth, projective and of dimension \( n := \dim X \geq 3 \), the classical Lefschetz hyperplane theorem allows to reduce complexity by cutting down. If \( \mathcal{L} \in \text{Pic}(X) \) is very ample and \( H_1, \ldots, H_{n-2} \in |\mathcal{L}| \) are general hypersurfaces with associated complete intersection \( S := H_1 \cap \cdots \cap H_{n-2} \), it asserts that the group morphism induced by the inclusion, \( \pi_1(S) \to \pi_1(X) \), is isomorphic. We refer to [LaZ04, Theorem 3.1.21] for a discussion.

The situation is substantially more involved when \( X \) is singular, even in the simple case where \( X \) has isolated singularities, or somewhat more general, where \( X \) is smooth in codimension two —this will be our most relevant setting, since spaces with terminal singularities always have this property. Under these assumptions, the general complete intersection surface \( S \) is still smooth and contained in the smooth locus \( X_{\text{reg}} \), but the appropriate generalisation of the Lefschetz hyperplane theorem, [GM88, Theorem in Section II.1.2], only gives an isomorphism between \( \pi_1(S) \) and \( \pi_1(X_{\text{reg}}) \), rather than between \( \pi_1(S) \) and \( \pi_1(X) \).

In summary, we see that to use the cutting-down method successfully, we need to compare \( \pi_1(X_{\text{reg}}) \) and \( \pi_1(X) \). Since we are chiefly interested in representations of fundamental groups rather than fundamental groups themselves, the following theorem of Grothendieck simplifies the problem somewhat.

**Theorem 4.1** (Profinite completions dictate representations, [Gro70, Theorem 1.2b]). Let \( \alpha : G \to H \) be a morphism of between finitely generated groups, and
let $\alpha^* : \text{Rep}_C(H) \to \text{Rep}_C(G)$ be the associated pull-back functor of finite-dimensional representation. If the associated morphism $\tilde{\alpha} : \tilde{G} \to \tilde{H}$ between profinite completions is bijective, then $\alpha^*$ induces an equivalence of categories.

For spaces with klt singularities, we have shown that the difference between profinite completions $\tilde{\pi}_1(X)$ and $\tilde{\pi}_1(X_{\text{reg}})$ can be made to vanish.

**Theorem 4.2.** (Existence of maximally quasi-étale covers, [GKP16, Theorem 1.4]). Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a Weil $Q$-divisor $\Delta$ such that $(X, \Delta)$ is klt. Then, there exists a normal variety $\tilde{X}$ and a quasi-étale, Galois morphism $\gamma : \tilde{X} \to X$, such that the following, equivalent conditions hold.

1. Any finite, étale cover of $\tilde{X}_{\text{reg}}$ extends to a finite, étale cover of $\tilde{X}$.
2. The natural map $\pi_* : \tilde{\pi}_1(\tilde{X}_{\text{reg}}) \to \tilde{\pi}_1(\tilde{X})$ of étale fundamental groups induced by the inclusion of the smooth locus, $\iota : \tilde{X}_{\text{reg}} \to \tilde{X}$, is an isomorphism.

The proof of Theorem 4.2 builds on work of Chenyang Xu who proved that local étale fundamental groups vanish for spaces with isolated klt singularities, [Xu14]. We want to emphasise that Xu’s result is by no means elementary, and uses many of the recent advances in higher-dimensional birational geometry, such as boundedness results for log Fano manifolds.

### 4.1. Application to flatness

We aim to apply Theorem 4.2 to the study of flat sheaves on klt spaces. Since we are dealing with singular spaces, we do not attempt to define flat sheaves via connections. Instead, a flat sheaf $\mathcal{F}$ will always be an analytic, locally free sheaf, given by a representation of the fundamental group.

**Definition 4.3.** If $Y$ is any complex space, and $\mathcal{F}$ is any locally free sheaf on $Y$, we call $\mathcal{F}$ flat if it is defined by a representation of the fundamental group. A locally free, algebraic sheaf on a complex algebraic variety $Y$ is called flat if and only if the associated analytic sheaf on the underlying complex space $Y^{\text{an}}$ is flat.

We obtain the following consequences of Theorems 4.1 and 4.2.

**Theorem 4.4.** (Étale fundamental groups dictate flatness, [GKP16, Section 11.1]). Let $X$ be a normal, complex, quasi-projective variety, and assume that the natural inclusion map between étale fundamental groups, $\iota_* : \tilde{\pi}_1(X_{\text{reg}}) \to \tilde{\pi}_1(X)$, is isomorphic. If $\mathcal{F}^o$ is any flat, locally free, analytic sheaf defined on the complex manifold $X_{\text{reg}}^{\text{an}}$, then there exists a flat, locally free, analytic sheaf $\mathcal{F}$ on $X^{\text{an}}$ such that $\mathcal{F}^o = \mathcal{F}|_{X_{\text{reg}}^{\text{an}}}$.

**Theorem 4.5.** (Flat sheaves on maximally quasi-étale covers, [GKP16, Theorem 1.13]). Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a Weil $Q$-divisor $\Delta$ such that $(X, \Delta)$ is klt. Then, there exists a normal variety $\tilde{X}$ and a quasi-étale, Galois morphism $\gamma : \tilde{X} \to X$, such that the following holds. If $\mathcal{G}^o$ is any flat, locally free, analytic sheaf on the complex space $X_{\text{reg}}^{\text{an}}$, there exists a flat, locally free, algebraic sheaf $\mathcal{G}$ on $X$ such that $\mathcal{G}^o = \mathcal{G}|_{X_{\text{reg}}^{\text{an}}}$.

Given a normal variety $X$ and a flat, locally free, analytic sheaf $\mathcal{F}^o$ on $X_{\text{reg}}^{\text{an}}$, Deligne has shown in [Del70, II.5, Corollary 5.8 and Theorem 5.9] that $\mathcal{F}^o$ is algebraic, and thus extends to a coherent, reflexive, algebraic sheaf $\mathcal{F}$ on $X$. The above theorems hence provide criteria to guarantee that Deligne’s extended sheaves are in fact locally free.

### 4.2. References

The existence of maximally quasi-étale covers has been shown in [GKP16, Theorem 1.4]. The paper contains more general results, discusses the relation to flatness in details and gives applications. The survey paper [KP14] covers these results in greater detail.
5. Nonabelian Hodge theory

The proof of our main result makes heavy use of Simpson’s nonabelian Hodge correspondence, which relates representations of the fundamental group to Higgs bundles. We will also use Simpson’s construction of variations of Hodge structures from special Higgs bundles. Before recalling these results in more detail below, we begin with the definition of a Higgs bundle and present a few examples.

**Definition 5.1 (Higgs bundle).** Let \( X \) be a complex manifold. A Higgs bundle is a pair \((\mathcal{E}, \theta)\) consisting of a holomorphic vector bundle \(\mathcal{E}\), together with an \(\mathcal{O}_X\)-linear morphism \(\theta: \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X\), called Higgs field, such that the composed morphism

\[
\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\theta \otimes \text{Id}} \mathcal{E} \otimes \Omega^2_X \xrightarrow{\text{Id} \otimes \wedge^2} \mathcal{E} \otimes \Omega^2_X
\]

vanishes. Following tradition, the composed morphism will be denoted by \(\theta \land \theta\). A coherent subsheaf \(\mathcal{F} \subseteq \mathcal{E}\) is said to be \(\theta\)-invariant if \(\theta(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega^1_X\).

**Definition 5.2 (System of Hodge bundles).** Let \( X \) be a complex manifold. A system of Hodge bundles is a Higgs bundle \((\mathcal{E}, \theta)\) on \( X \), together with a number \( n \in \mathbb{N} \) and a direct sum decomposition

\[
\mathcal{E} = \bigoplus_{p+q=n} \mathcal{E}^{p,q}
\]

such that for all indices \((p, q)\), the restriction \(\theta|_{\mathcal{E}^{p,q}}\) takes its image in \(\mathcal{E}^{p-1,q+1} \otimes \Omega^1_X\). The restricted maps are traditionally written as \(\theta^{p,q}: \mathcal{E}^{p,q} \to \mathcal{E}^{p-1,q+1} \otimes \Omega^1_X\).

**Example 5.3 (Higgs sheaves with trivial field).** Let \( X \) be a complex manifold. Let \(\mathcal{E}\) be any holomorphic vector bundle. Then, together with and consider the zero morphism \(\theta: \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X\). In this example, any subsheaf of \(\mathcal{E}\) is \(\theta\)-invariant.

**Example 5.4 (A natural Higgs sheaf attached to a complex manifold).** Let \( X \) be a complex manifold. Set \(\mathcal{E} := \Omega^1_X \oplus \mathcal{O}_X\) and define an operator \(\theta\) as follows,

\[
\theta: \Omega^1_X \oplus \mathcal{O}_X \longrightarrow \left(\Omega^1_X \oplus \mathcal{O}_X\right) \otimes \Omega^1_X
\]

\[
a + b \mapsto (0 + 1) \otimes a.
\]

An elementary computation shows that \(\theta \land \theta = 0\), so that \((\mathcal{E}, \theta)\) forms a Higgs bundle. Observe that the direct summand \(\mathcal{O}_X \subseteq \mathcal{E}\) is \(\theta\)-invariant. On the other hand, non-zero subsheaves of the direct summand \(\Omega^1_X\) are never invariant. In fact, \((\mathcal{E}, \theta)\) a system of Hodge bundles. Indeed, the corresponding direct sum decomposition is given by

\[
\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}, \quad \text{with} \quad \mathcal{E}^{1,0} = \Omega^1_X \quad \text{and} \quad \mathcal{E}^{0,1} = \mathcal{O}_X.
\]

5.1. **Elementary constructions.** The Higgs bundles on a given complex manifold form a category, with the obvious definition for a morphism. The following additional constructions allow for direct sums, tensor products, duals, and pullback.

**Construction 5.5 (Direct sum and tensor product).** Let \( X \) be a complex manifold, and let \((\mathcal{E}_1, \theta_1)\) and \((\mathcal{E}_2, \theta_2)\) be two Higgs bundles on \( X \). Then, there are natural Higgs fields on the direct sum and tensor product,

\[
(\mathcal{E}_1 \oplus \mathcal{E}_2, \theta_1 \oplus \theta_2) \quad \text{and} \quad (\mathcal{E}_1 \otimes \mathcal{E}_2, \theta_1 \otimes \text{Id}_{\mathcal{E}_2} + \text{Id}_{\mathcal{E}_1} \oplus \theta_2).
\]

**Construction 5.6 (Dual and endomorphisms).** Let \( X \) be a complex manifold, and let \((\mathcal{E}, \theta)\) be a Higgs bundle on \( X \). The Higgs field can be seen as a section of the sheaf \(\mathcal{E}^* \otimes \mathcal{E} \otimes \Omega^1_X\), which is naturally isomorphic to \(\mathcal{E} \otimes \mathcal{E}^* \otimes \Omega^1_X\). This allows to equip \(\mathcal{E}^*\), and then also \(\text{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}\) with natural Higgs fields.
Theorem 5.9 (Nonabelian Hodge correspondence). The following major result of Simpson, known as the nonabelian Hodge correspondence relates Higgs bundles to representations of the fundamental group. We define the appropriate notion of stability first.

Definition 5.8 (Higgs stability). Let $X$ be a projective manifold and $H \in \text{Div}(X)$ be a nef divisor. We say that $(E, \theta)$ is semistable with respect to $H$ if the inequality $\mu_H(\mathcal{F}) \leq \mu_H(E)$ holds for all $\theta$-invariant subsheaves with $0 < \text{rank} \mathcal{F} < \text{rank} E$. The Higgs sheaf is called stable if the inequality is always strict. Direct sums of stable Higgs bundles are called polystable.

Theorem 5.9 (Nonabelian Hodge correspondence, [Sim91, Theorem 1]). Let $X$ be a projective manifold and $H \in \text{Div}(X)$ be an ample divisor. Then, there exists an equivalence between the categories of all representations of $\pi_1(X)$, and of all $H$-semistable Higgs bundles with vanishing Chern classes. □

5.3. Higgs bundles induced by variations of Hodge Structures. As mentioned in the introduction to this chapter, Simpson constructed variations of Hodge structures via Higgs bundles. We briefly recall the most relevant definitions and results.

Definition 5.10 (Polarised, complex variation of Hodge structures, [Sim88, Section 8]). Let $X$ be a complex manifold, and $w \in \mathbb{N}$ a natural number. A polarised, complex variation of Hodge structures of weight $w$, or pCVHS in short, is a $C^\infty$-vector bundle $V$ with a direct sum decomposition $V = \oplus_{r+s=w} V^{r,s}$, a flat connection $D$ that decomposes as follows

$$
(5.10.1) \quad D|_{V^{r,s}} : V^{r,s} \to A^{0,1}(V^{r+s-1}) \oplus A^{1,0}(V^{r,s}) \oplus A^{0,1}(V^{r,s}) \oplus A^{1,0}(V^{r-1,s+1}),
$$

and a $D$-parallel Hermitian metric on $V$ that makes the direct sum decomposition orthogonal and that on $V^{r,s}$ is positive definite if $r$ is even and negative definite if $r$ is odd.

Given a pCVHS, one constructs an associated Higgs bundle as follows.

Construction 5.11 (Higgs sheaves induced by a pCVHS). Given a pCVHS as in Definition 5.10, use (5.10.1) to decompose $D$ as $D = \overline{\partial} \oplus \bar{\partial} \oplus \bar{\partial} \oplus \theta$. The operators $\overline{\partial}$ equip the $C^\infty$-bundles $V^{r,s}$ with complex structures. We write $\mathcal{E}^{r,s}$ for the associated locally free sheaves of $\mathcal{O}_X$-modules, and set $\mathcal{E} := \oplus \mathcal{E}^{r,s}$. The operators $\theta$ then define an $\mathcal{O}_X$-linear morphism $\mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$. As $D$ is flat, this is a Higgs field.

Definition 5.12 (Higgs bundles induced by a pCVHS). Let $X$ be a complex manifold and $(E, \theta)$ a Higgs bundle on $X$. We say that $(E, \theta)$ is induced by a pCVHS if there exists a pCVHS on $X$ such that $(E, \theta)$ is isomorphic to the Higgs bundle obtained from it via Construction 5.11.

Scaling the Higgs field induces an action of $\mathbb{C}^*$ on the set of isomorphism classes of Higgs bundles. Under suitable assumptions, Simpson shows that Higgs bundles induced by a pCVHS correspond exactly to $\mathbb{C}^*$-fixed points. The following theorem summarises his results.
Theorem 5.13 (Higgs bundles induced by a pCVHS, I, [Sim92, Corollary 4.2]). Let $X$ be a complex, projective manifold of dimension $n$ and $H \in \text{Div}(X)$ be an ample divisor. Let $(\mathcal{E}, \theta)$ be a Higgs bundle on $X$. Then, $(\mathcal{E}, \theta)$ comes from a variation of Hodge structures in the sense of Definition 5.12 if and only if the following three conditions hold.

(5.13.1) The Higgs bundle $(\mathcal{E}, \theta)$ is $H$-polystable.

(5.13.2) The intersection numbers $c_1(\mathcal{E}) \cdot |H|^{n-1}$ and $c_2(\mathcal{E}) \cdot |H|^{n-2}$ both vanish.

(5.13.3) For any $t \in \mathbb{C}^*$, the Higgs bundles $(\mathcal{E}, \theta)$ and $(\mathcal{E}, t \cdot \theta)$ are isomorphic.

□

Remark 5.14. With $X$ and $H$ as in Theorem 5.13, any Higgs bundle $(\mathcal{E}, \theta)$ that satisfies (5.13.1) and (5.13.2) carries a flat $\mathcal{C}^\infty$-connection, [Sim92, Theorem 1(2) and Corollary 1.3]. In particular, all its Chern classes vanish.

As one immediate consequence of Theorem 5.13, we obtain the following minor strengthening of [Sim92, Corollary 4.3].

Corollary 5.15 (Higgs bundles induced by a pCVHS, II, [GKPT15, Corollary 6.36]). Let $X$ be a projective manifold, and $H \in \text{Div}(X)$ be an ample divisor. Let $i : S \to X$ be a submanifold. The push-forward map $i_* : \pi_1(S) \to \pi_1(X)$ induces a restriction map

$$r : \left\{ \begin{array}{l}
\text{Isomorphism classes of } H\text{-semistable Higgs bundles } (\mathcal{E}, \theta) \text{ on } X \\
\text{with vanishing Chern classes.}
\end{array} \right\} \to \left\{ \begin{array}{l}
\text{Isomorphism classes of } H\text{-semistable Higgs bundles } (\mathcal{E}, \theta) \text{ on } S \\
\text{with vanishing Chern classes.}
\end{array} \right\}$$

\[(\mathcal{E}, \theta) \mapsto (\mathcal{E}, \theta)|_S.\]

In particular, if $(\mathcal{E}, \theta)$ is any $H$-semistable Higgs bundle $(\mathcal{E}, \theta)$ on $X$ with vanishing Chern classes, then $(\mathcal{E}, \theta)|_S$ is again $H$-semistable. The map $r$ has the following properties.

(5.15.1) If $i_*$ is surjective, then $r$ is injective. In particular, if $(\mathcal{E}, \theta)$ is a Higgs bundle on $X$ such that $(\mathcal{E}, \theta)|_S$ comes from a pCVHS, then $(\mathcal{E}, \theta)$ comes from a pCVHS.

(5.15.2) If in addition the induced push-forward map $i_* : \pi_1(S) \to \pi_1(X)$ of algebraic fundamental groups is isomorphic, then $r$ is surjective.

Proof. Simpson’s nonabelian Hodge correspondence, Theorem 5.11, gives an equivalence between the categories of representations of the fundamental group $\pi_1(X)$ (resp. $\pi_1(S)$) and $H$-semistable Higgs bundles on $X$ (resp. $S$) with vanishing Chern classes. The correspondence is functorial in morphisms between manifolds, and pull-back of Higgs bundles corresponds to the push-forward of fundamental groups, [Sim92, Remark 1 on Page 36]. In particular, we see that the restriction of an $H$-semistable Higgs bundle with vanishing Chern classes is again $H$-semistable.

In the setting of (5.15.1) where the push-forward map $\pi_1(S) \to \pi_1(X)$ is surjective, this immediately implies that the restriction $r$ is injective. The restriction map $r$ is clearly equivariant with respect to the actions of $\mathbb{C}^*$ obtained by scaling the Higgs fields. Injectivity therefore implies that the isomorphism class of a Higgs bundle $(\mathcal{E}, \theta)$ is $\mathbb{C}^*$-fixed if and only if the same is true for $(\mathcal{E}, \theta)|_S$. Theorem 5.13 thus proves the second clause of (5.15.1).

Now assume that we are in the setting of (5.15.2), where in addition the push-forward map $\pi_1(S) \to \pi_1(X)$ is assumed to be isomorphic. Since fundamental groups of algebraic varieties are finitely generated, Theorem 4.1 implies that every representation of $\pi_1(S)$ comes from a representation of $\pi_1(X)$. The claim thus again follows from Simpson’s nonabelian Hodge correspondence. □

The following proposition links Higgs bundles coming from variations of Hodge structures to minimal model theory. It is crucial for all that follows.

Proposition 5.16 (Higgs bundles and minimal model theory, [GKPT15, Corollary 6.39]). Let $Y$ be a normal, projective variety with at worst canonical singularities and let $\pi : \tilde{Y} \to Y$ be a resolution of singularities. Let $(\mathcal{F}_Y, \theta_Y)$ be a Higgs bundle on $\tilde{Y}$.
that is induced by a pCVHS. Then, $\mathcal{F}_Y$ comes from $Y$. More precisely, there exists a locally free sheaf $\mathcal{F}_Y$ on $Y$ such that $\mathcal{F}_Y = \pi^*\mathcal{F}_Y$. Necessarily, we then have $\mathcal{F}_Y \cong \pi_*(\mathcal{F}_Y)^{**}$.

Proof. It suffices to construct $\mathcal{F}_Y$ locally in the analytic topology, near any given point $y \in Y$. Recall from [Tak03, Page 827] that there exists a contractible, open neighbourhood $U = U(y) \subseteq Y^{an}$ whose preimage $\tilde{U} := \pi^{-1}(U)$ is simply connected. By assumption, $(\mathcal{F}_Y, \theta_Y)$ is induced from a pCVHS, say $\mathcal{V}$. Let $\rho : \tilde{U} \to D$ be the corresponding period map.

We claim that $\rho$ factors through the resolution,

$$
\tilde{U} \xrightarrow{\rho} \mathcal{D}.
$$

Since $U$ is normal, this will follows once we show that $\rho$ is constant on fibres of $\pi$. The fibres of $\pi$, however, are known to be rationally chain-connected, [HM07, Corollary 1.5]. In summary, $\rho$ will factor as soon as we show that for any morphism $\eta : \mathbb{P}^1 \to \tilde{U}$, the composed map $\rho \circ \eta : \mathbb{P}^1 \to D$ is constant. Given one such $\eta$, we obtain a pCVHS on $\mathbb{P}^1$ whose associated period map equals $\rho \circ \eta$, simply by pulling back $\mathcal{V}$ via $\eta$. However, due to hyperbolicity properties of the period domain $D$, this map has to be constant, [CMS03, Application 13.4.3].

It is known that $\mathcal{F}_Y|_U \cong \rho^*(\mathcal{F}_D)$ for some vector bundle $\mathcal{F}_D$ on the period domain $D$, cf. [GKPT, Proposition 6.38]. If $\rho_U : U \to D$ is the holomorphic map whose existence was shown in the previous paragraph, the vector bundle $\mathcal{F}_U := \rho_U^*(\mathcal{F}_D)$ hence fulfils $\pi_*\mathcal{F}_U \cong \mathcal{F}_Y|_U$, as desired. $\square$

5.4. References. Higgs operators appeared in [Hit87] where Hitchin studied Yang-Mills equations with the aim of finding conditions for existence of flat connections on a compact Riemann surface. In analogy to results of Narasimhan-Seshadri, Hitchin observed that solutions to Yang-Mills equations impose additional holomorphic data on the given holomorphic bundle, a condition that is nowadays referred to as Higgs stability. Higgs fields were also introduced in the theory of variation of Hodge structures in smooth families of projective varieties, where they are encoded in the Griffiths transversality and holomorphicity properties of the Gauss-Manin connection. A fundamental result of Griffiths, cf. [Gri68], then showed that the existence of variation of Hodge structures gives rise to a holomorphic map, the period map, from the universal cover to the classifying space of Hodge structures.

This result of Griffiths led Simpson to study uniformisation problems via variations of Hodge structures. He aimed to find holomorphic and numerical conditions on a suitable Higgs bundle for it to define a complex variation of Hodge structures whose associated period map would then provide an identification of the universal cover. This was famously achieved in [Sim88]. The arguments are parallel to earlier work of Hitchin, Donaldson, and Uhlenbeck-Yau, [Hit87, Don85, UY86], in tracing a correspondence between stable Higgs bundles with vanishing Chern classes and flat connections.

There are many overview papers on the subject, including Simpson’s ICM talk [Sim91]. The reader might also want to look at the excellent survey [Eys11], or at the short note [BGPG07].

6. HIGGS SHEAVES ON SINGULAR SPACES

6.1. Fundamentals. On a singular variety, the correct definition of a “Higgs sheaf” is perhaps not obvious. As we will note below, the following generalisation of Definition 5.1 turns out to have just enough universal properties to make
the strategy of our proof work. In the converse direction, it seems that Definition 6.1 and our notion of stability are in essence uniquely dictated if we ask all these universal properties to hold.

**Definition 6.1 (Higgs sheaf, generalisation of Definition 5.1).** Let X be a normal variety. A Higgs sheaf is a pair \((\mathcal{E}, \theta)\) consisting of a coherent sheaf \(\mathcal{E}\) of \(\mathcal{O}_X\)-modules, together with an \(\mathcal{O}_X\)-linear morphism \(\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X\), called Higgs field, such that the composed morphism

\[
\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\theta \otimes \text{Id}} \mathcal{E} \otimes \Omega^1_X \otimes \Omega^1_X \xrightarrow{\text{Id} \otimes [\lambda]} \mathcal{E} \otimes \Omega^2_X
\]

vanishes. Following tradition, the composed morphism will be denoted by \(\theta \wedge \theta\). The definition of systems of Hodge sheaves carries over verbatim.

**Warning 6.2.** There exists an obvious notion of \textit{morphism of Higgs sheaves}, but there is generally no way to equip kernels or cokernels with Higgs fields. Higgs sheaves hence do not form an Abelian category.

**Definition 6.3 (Invariant and generically invariant subsheaves).** Setting as in Definition 6.1. A coherent subsheaf \(\mathcal{F} \subseteq \mathcal{E}\) is called \(\theta\)-invariant if \(\theta|_{\mathcal{F}}\) is contained in the image of the natural map

\[
\mathcal{F} \otimes \Omega^1_X \to \mathcal{E} \otimes \Omega^1_X.
\]

Call \(\mathcal{F}\) generically invariant if the restriction \(\mathcal{F}|_{\text{reg}}\) is invariant with respect to \(\theta|_{\text{reg}}\).

**Warning 6.4.** As \(\Omega^1_X\) is not locally free, in Definition 6.3 the sheaf \(\mathcal{F} \otimes \Omega^1_X\) is generally not a subsheaf of \(\mathcal{E} \otimes \Omega^1_X\). As a consequence, there is generally no induced Higgs field on invariant or generically invariant subsheaves. At this point, our setting differs substantially from the smooth case. Even though we will later define (semi-)stability for Higgs sheaves, this will make it impossible to easily construct an analogue of the Harder-Narasimhan filtration.

**6.2. Explanation and examples.** At first sight, it might seem most natural and functorial to define Higgs fields as morphisms to \(\mathcal{E} \otimes \Omega^1_X\). However, in our application to uniformisation for varieties of general type, the naturally induced sheaf of geometric origin is \(\mathcal{E} := \Omega^1_X \oplus \mathcal{O}_X\), as discussed in Example 6.5 below. Looking at \(\Omega^1_X \oplus \mathcal{O}_X\) instead would render any discussion of semistability moot, as semistability requires torsion freeness and even the most simple klt singularities lead to torsion in \(\Omega^1_X\), see [GR11] for examples.

On the other hand, the reader might wonder why \(\theta\) takes its values in \(\mathcal{E} \otimes \Omega^1_X\) and not in its reflexive hull. The advantages of our choice will become apparent when pull-back functors are defined. None of the constructions there will work for reflexive hulls.

**Example 6.5 (A natural Higgs sheaf attached to a normal variety, generalising Example 5.4).** Let X be a normal variety. Set \(\mathcal{E} := \Omega^1_X \oplus \mathcal{O}_X\) and define a Higgs field

\[
\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X, \quad a + b \mapsto (0 + 1) \otimes a.
\]

As before, the direct summand \(\mathcal{O}_X \subseteq \mathcal{E}\) is generically \(\theta\)-invariant, and subsheaves of the direct summand \(\Omega^1_X\) are never generically \(\theta\)-invariant, unless they are zero.

**Construction 6.6 (Direct sum, tensor product, dual and endomorphisms).** Construction 5.5 of Higgs fields on the direct sum and tensor product of two Higgs bundles carries over to Higgs sheaves. If the Higgs sheaf is locally free, an immediate analogue of Construction 5.6 defines natural Higgs fields on the dual sheaf.
and on the endomorphism sheaf. We refer to [GKPT15, Sections 5.1 and 6.1] for
details, and for further constructions.

6.3. Pull-back. One of the most basic properties of Higgs bundles is the existence
of a pull-back functor. For Higgs sheaves on singular spaces, we do not believe
that a reasonable notion of pull-back exists in general. In fact, to pull back Higgs
sheaves is at least as difficult as to pull-back reflexive differentials, and examples
abound which show that there is generally no notion of pulling-back for reflexive
differentials. Worse still, even in settings where pull-back morphisms happen to
exist, the pull-back may fail to be functorial. For spaces with klt singularities,
however, we have seen in Section 3 that functorial pull-back functor does exist.
For these spaces, the following construction will then give a functorial pull-back of
Higgs sheaves.

Construction 6.7 (Pull-back of Higgs sheaves, generalisation of Construction 5.7).
Let \((X, D)\) be a klt pair and let \((\mathcal{E}, \theta)\) be a Higgs sheaf on \(X\). Given a normal
variety \(Y\) and a morphism \(f : Y \to X\), consider the sheaf morphism \(\theta'\), defined as
the composition of the following maps,

\[
f^* \mathcal{E} \xrightarrow{f^* \theta} f^* (\mathcal{E} \otimes \Omega^1_X) = f^* \mathcal{E} \otimes f^* \Omega^1_X \xrightarrow{\text{Id} \otimes d \circ \text{refl}} f^* \mathcal{E} \otimes \Omega^1_Y.
\]

One verifies that \(\theta' \wedge \theta' = 0\), so that \(\theta'\) equips \(f^* \mathcal{E}\) with the structure of a Higgs
sheaf. By minor abuse of notation, this Higgs sheaf will be denoted as \(f^*(\mathcal{E}, \theta)\)
or \((f^* \mathcal{E}, f^* \theta)\). If \(f\) is a closed or open immersion, we will also write \(\mathcal{E}, \theta|_Y\) or
\((\mathcal{E}|_Y, \theta|_Y)\).

If the space \(Y\) of Construction 6.7 is smooth, the construction can be generalised
further, to define a Higgs field on the reflexive pull-back \(f^!(\mathcal{E}) := (f^* \mathcal{E})^{**}\). The
resulting notion of “reflexive pull-back” is important, but fails to have any form of
functoriality, cf. [GKPT15, Sect. 6.4].

6.4. Stability. We close this section generalising the notion of stability from Higgs
bundles to Higgs sheaves. Again, it might not be obvious at first sight that the fol-
lowing definition, which considers slopes of subsheaves that are only generically
injective, is the “right” one. It has the advantage that it behaves well with respect
to the reflexive pull-back discussed above. The paper [GKPT15] uses this to com-
pare stability of the Higgs sheaf \((\mathcal{E}, \theta)\) with that of its reflexive pull-back.

Definition 6.8 (Stability of Higgs sheaves). Let \(X\) be a normal, projective variety and
\(H\) be any nef, Q-Cartier \(Q\)-divisor on \(X\). Let \((\mathcal{E}, \theta)\) be a Higgs sheaf on \(X\), were \(\mathcal{E}\)
is torsion free. We say that \((\mathcal{E}, \theta)\) is semistable with respect to \(H\) if the inequality
\(\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})\) holds for all generically \(\theta\)-invariant subsheaves \(\mathcal{F} \subseteq \mathcal{E}\) with
\(0 < \text{rank} \mathcal{F} < \text{rank} \mathcal{E}\). The Higgs sheaf is called stable with respect to \(H\) if the inequality
is always strict. Direct sums of stable Higgs sheaves are called polystable.

Remark 6.9. For Higgs bundles, Definition 6.8 reproduces the earlier notion of sta-
tility, as introduced in Definition 5.8 above. We refer to [GKPT15, Sect. 6.6] for
details.

6.4.1. The restriction theorem. We conclude with a restriction theorem of Mehta-
Ramanathan type, which will be crucial for the proof of our main results. Its
(rather long and protracted) proof relies on Langer’s generalised Bogomolov-
Gieseker inequalities for sheaves with operators, resolving singularities and cut-
ting down in order to reduce to a setting where Langer’s results apply. The func-
torial properties of Higgs sheaves play a pivotal role in this.
Theorem 6.10 (Restriction theorem for stable Higgs sheaves, [GKPT15, Theorem 6.22]). Let $(X, \Delta)$ be a projective klt pair of dimension $n \geq 2$, let $H \in \text{Div}(X)$ be an ample, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and let $(\mathcal{E}, \theta)$ be a torsion free Higgs sheaf on $X$ of positive rank. Assume that $(\mathcal{E}, \theta)$ is stable with respect to $H$. If $m \gg 0$ is sufficiently large and divisible, then there exists a dense open set $U \subseteq |m \cdot H|$ such that the following holds for any hyperplane $D \in U$ with associated inclusion map $\iota: D \to X$.

(6.10.1) The hyperplane $D$ is normal, connected and not contained in $\text{supp} \Delta$. The pair $(D, \Delta|_D)$ is klt.
(6.10.2) The sheaf $\mathcal{E}|_D$ is torsion free. The Higgs sheaf $\iota^*(\mathcal{E}, \theta)$ is stable with respect to $H|_D$. □

For Higgs bundles on manifolds with ample polarisation, the theorem appears in Simpson’s work, [Sim92, Lemma 3.7]. Langer proves a similar theorem for sheaves on projective manifolds, polarised by tuples of divisors that need not be ample, [Lan15, Theorem 10]. He works in positive characteristic but says that mutatis mutandis, his arguments will also work in characteristic zero, cf. [Lan15, Page 906].

Part II. Proof of the main results

7. CHARACTERISATION OF TORUS QUOTIENTS

In this section we will very briefly sketch the proof of Theorem 1.4 on the uniformisation of singular varieties with vanishing Chern classes by the Euclidean space. There are various similarities and some crucial differences between the methods required for the proof of the two uniformisation results, Theorems 1.4 and 1.6. Our hope is that a comparison between the two proofs would prove useful in clarifying the main ideas and techniques behind both results. We have therefore chosen to present an outline of the proof following the strategy of [GKP16], even though this is covered in at least one other survey, [KP14, Section 9]. We remark that the case of canonical threefolds with vanishing Chern classes was achieved by Shepherd-Barron and Wilson in [SBW94]. Theorem 1.4 has been generalised to klt spaces in [LT14], providing a complete numerical characterisation of quotients of Abelian varieties by finite groups acting freely in codimension one. Both of these latter results require working with orbifold Chern classes which would require a rather lengthy preparation and technical details that, for the sake of simplicity, we have decided to avoid in the current article.

7.1. Outline of the proof of Theorem 1.4. The proof consists of two main steps. Our aim in the first step, which is modelled on the strategy of [MR84], is to construct a reflexive sheaf $\mathcal{F}$ on $X$, formed as the coherent extension of a flat, locally-free, analytic sheaf on $X_{\text{reg}}$, that verifies the isomorphism $\mathcal{F}|_S \cong \mathcal{T}_X|_S$, for a complete intersection surface $S$ cut out by general members of linear systems of sufficiently large multiples of $H$. In the second step we use the aforementioned sheaf isomorphism on $S$ to find a global isomorphism $\mathcal{F} \cong \mathcal{T}_X$. Of course when $X$ is smooth, this already implies that $\mathcal{T}_X$ is flat. When $X$ is singular, one then needs a method to extend the flatness of $\mathcal{T}_X|_{X_{\text{reg}}}$ across the singular locus. According to Theorem 4.5 this can be achieved when the singularities are mild, at least up to a suitable cover. This is the main ingredient of the second step.

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8or “Q-Chern classes”
**Step 1: Construction of a flat sheaf on** $X_{\text{reg}}$. We first notice that owing to the celebrated generic semipositivity result of Miyaoka [Miy87] we know that $\mathcal{F}_X$ is slope-semistable with respect to $H$. Next, choose a sufficiently large and divisible integer $m \gg 0$, and choose a general tuple of hyperplanes $D_1, \ldots, D_{n-2} \in \langle m \cdot H \rangle$, with general complete intersection surface $S := D_1 \cap \cdots \cap D_{n-2}$. The following items will then hold.

1. The intersection $S$ is a smooth surface, and entirely contained in $X_{\text{reg}}$. This is because $X$ is smooth in codimension two by assumption.
2. The restriction $\mathcal{F}_X|_S$ is semistable with respect to $H|_S$. This follows from Flenner’s Mehta-Ramanathan theorem for normal varieties, [Fle84, Theorem 1.2].
3. The natural morphism $\iota : \pi_1(S) \to \pi_1(X_{\text{reg}})$, induced by the inclusion $\iota : S \hookrightarrow X_{\text{reg}}$, is isomorphic. This is the content of Goresky-MacPherson’s Lefschetz hyperplane theorem for homotopy groups, [GM89, Theorem in Section II.1.2].
4. Let $\mathcal{F}^\circ$ be any locally free, flat, analytic sheaf on $X_{\text{reg}}$ with rank $\mathcal{F} = n$. Then, $\mathcal{F}^\circ$ is isomorphic to $\mathcal{F}_X|_{X_{\text{reg}}}$ if and only if the restrictions $\mathcal{F}^\circ|_S$ and $\mathcal{F}_X|_S$ are isomorphic. This follows because flat sheaves of fixed rank form a bounded family, [GKP16, Proposition 9.1], and because of the Bertini-type theorem for isomorphism classes in bounded families, [GKP16, Corollary 5.3].

Now according to [Sim92, Corollary 3.10] the semistability of $\mathcal{F}_X|_S$ together with the vanishing condition on its Chern classes imply that $\mathcal{F}_X|_S$ comes from a representation of $\pi_1(S)$. Item (7.1.3) now allows to extend this to a representation of $\pi_1(X_{\text{reg}})$. In other words, we find a locally-free, flat bundle $\mathcal{F}^\circ$ on $X_{\text{reg}}$ such that $\mathcal{F}^\circ|_S \cong \mathcal{F}_X|_S$. Define $\mathcal{F} := \iota_* \mathcal{F}^\circ$, where $\iota : X_{\text{reg}} \to X$ is the inclusion map. The sheaf $\mathcal{F}$ is then coherent, and in fact reflexive on $X$.

**Step 2: Reduction to the smooth case.** As $\mathcal{F}^\circ$ is flat, Item (7.1.4) applies and we find that $\mathcal{F}^\circ \cong \mathcal{F}_X|_{X_{\text{reg}}}$. Now, let $\gamma : \bar{X} \to X$ be a maximal quasi-étale cover, as given by Theorem 4.5. Since $\gamma$ is unramified in codimension one, $\mathcal{F}_{\bar{X}} \cong \gamma^*(\mathcal{F}_X)$. As one consequence, we see that $\mathcal{F}_{\bar{X}}$ is flat over $\bar{X}^\circ := \gamma^{-1}(X_{\text{reg}})$, which is a big subset of $\bar{X}$. We also see that

$$K_{\bar{X}} \equiv 0 \quad \text{and} \quad c_2(\mathcal{F}_{\bar{X}}) \cdot [\gamma^*H]^{n-2} = 0,$$

where the last equality is a consequence of the projection formula. According to Theorem 4.5, the sheaf $\mathcal{F}_{\bar{X}}$ comes from a representation of $\pi_1(\bar{X})$. In particular, it is locally-free. Now thanks to the solution to the Lipman-Zariski conjecture, Theorem 3.4, we find that $\bar{X}$ is smooth. Theorem 1.4 now follows from the original result of Yau, Theorem 1.3. \hfill \Box

### 8. Proof of the Miyaoka-Yau Inequality

8.1. **Proof of Theorem 1.5 in a simplified setting.** For the purposes of this survey, we prove Theorem 1.5 only under the following simplifying assumptions. Section 8.2 briefly discusses the missing pieces for a proof in the general case.

**Assumption 8.1.** The canonical bundle $K_X$ is ample, and $X$ is smooth in codimension two. In particular, Chern classes $c_1$ and $c_2$ exist.
**Step 1: Setup.** The natural Higgs sheaf on $X$. We begin by considering the natural Higgs sheaf $(\mathcal{E}, \theta)$, as given in Example 6.5, where $\mathcal{E} = \Omega_X^1 \oplus \mathcal{O}_X$ and $\theta(a + b) = (0 + 1) \otimes a$. The main reason for our interest in $(\mathcal{E}, \theta)$ is the observation that the Bogomolov-Gieseker discriminant $\Delta(\mathcal{E})$ computes the Miyaoka-Yau discriminant of $\mathcal{T}_X$. Indeed, we have

$$\Delta(\mathcal{E}) \cdot [K_X^{n-2}] := (2(\text{rank} \mathcal{E}) \cdot c_2(\mathcal{O}_X) - (\text{rank} \mathcal{E}) - 1) \cdot c_1^2(\mathcal{T}_X) \cdot [K_X^{n-2}]$$

$$= (2(n + 1) \cdot c_2(\mathcal{T}_X) - n \cdot c_1^2(\mathcal{T}_X)) \cdot [K_X^{n-2}].$$

To establish the Miyaoka-Yau inequality for $\mathcal{T}_X$, it will therefore suffice to show that $\mathcal{E}$ verifies the Bogomolov-Gieseker inequality, $\Delta(\mathcal{E}) \geq 0$. This will follow from a major result of Simpson, who verified the Bogomolov-Gieseker inequality for Higgs bundles that are stable with respect to an ample polarisation, [Sim88, Theorem 1 and Proposition 3.4]. To apply Simpson’s result, we need to show that $(\mathcal{E}, \theta)$ is stable with respect to $K_X$ and then cut down to reduce to the case of a Higgs bundle (rather than a mere sheaf) on a smooth surface.

**Step 2: Stability.** Generalising a classical result of Enoki, [Eno88, Corollary 1.2], Guenancia [Gue15, Theorem A] has shown that the tangent sheaf of a klt projective variety with ample canonical sheaf is necessarily polystable. Projecting a potentially destabilising, generically $\theta$-invariant subsheaf of $(\mathcal{E}, \theta)$ to the $\mathcal{E}_X$-summand of $\mathcal{E}$ and recalling from Example 6.5 that no subsheaf of the direct summand $\Omega_X^1$ is ever generically $\theta$-invariant, we deduce the following result, see [GKPT15, Corollary 8.2].

**Lemma 8.2.** The Higgs sheaf $(\mathcal{E}, \theta)$ is stable with respect to $K_X$. □

**Step 3: End of proof.** Choose a sufficiently large and divisible integer $m \gg 0$, and choose a general tuple of hyperplanes $H_1, \ldots, H_{n-2} \in |m \cdot K_X|$, with general complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2}$. Using the assumption that $X$ is smooth in codimension two, the surface $S$ is smooth, and entirely contained in the smooth locus of $X$. In particular, $(\mathcal{E}, \theta)|_S$ is a Higgs bundle. A repeated application of the restriction theorem for stable Higgs sheaves, Theorem 6.10, shows that $(\mathcal{E}, \theta)|_S$ is stable with respect to $K_S$, and Simpson’s result [Sim88, Theorem 1 and Proposition 3.4] applies to give that

$$0 \leq \Delta(\mathcal{E}|_S) = \frac{\Delta(\mathcal{E}|_S) \cdot [K_S^{n-2}]}{m^{n-2}}.$$

As we have seen in Step 1, this finishes the proof of Theorem 1.5 in the simplified setting of Assumption 8.1. □

8.2. **Proof of Theorem 1.5 in the general case.** The proof in the general case works along the same lines as the proof presented above. However, there are two problems that need to be overcome.

8.2.1. **The canonical sheaf might not be ample.** By assumption, the canonical divisor $K_X$ is nef and not necessarily ample. It can, however, be approximated by ample divisors. This adds an additional layer of complexity but causes no fundamental problems, because Simpson’s theory works with arbitrary ample divisors, which may or may not equal $K_X$.

8.2.2. **The variety is not necessarily smooth in codimension two.** The proof presented above used that assumption that $X_{\text{reg}}$ is a big set. That is not necessarily true in the general setting. It follows as a consequence of the classification of klt surface singularities, however, that there exists a big set $X^0 \subseteq X$ where $X^0$ has only quotient singularities, [GKKP11, Proposition 9.3]. The full proof of Theorem 1.5 uses
9. Characterisation of singular ball quotients

9.1. Smoothness criterion. The following smoothness criterion is the centrepiece in our proof of the uniformisation result, Theorem 1.6. Before returning to the proof of Theorem 1.6 in Section 9.2 below, we will therefore discuss its proof in some detail.

Proposition 9.1 (Smoothness criterion, [GKPT15, Proposition 9.3]). Let $Y$ be a projective variety of dimension $n$ that is smooth in codimension two and has at worst canonical singularities. Assume furthermore that the étale fundamental groups of $Y$ and of its smooth locus agree, $\pi_1(Y_{\text{reg}}) \cong \pi_1(Y)$. If $K_Y$ is ample and if equality holds in the Miyaoka-Yau inequality (1.5.1), then $Y$ is smooth.

Here are the main steps of the proof, which is taken almost verbatim from [GKPT15, Section 9.2]. The main object of study is the canonical Higgs sheaf $(\mathcal{F}_\gamma, \theta_Y)$ on $Y$. In Step 1 we consider this system. In analogy to Section 7, we fix a complete intersection surface $S$ that verifies various properties required in the next steps. This includes satisfying the property that $(\mathcal{F}_\gamma, \theta_Y)|_S$ is stable and that a Lefschetz hyperplane theorem holds. In Step 2 we construct a pC VHS on $S$ out of this data, whose induced Higgs bundle is $\mathcal{E} \text{nd}(\mathcal{F}_\gamma)|_S$. It goes without saying that Simpson’s result on the existence of Hermitian-Yang-Mills metrics for stable Higgs bundles is the key ingredient here. In Step 3 we extend this pC VHS to a Higgs bundle $(\mathcal{F}_\gamma, \theta_Y)$ on a resolution $\tilde{Y}$ of $Y$ and consider local period maps

$$ \rho : \{\text{1-connected subset of } \tilde{Y}\} \to \{\text{period domain}\} $$

Thanks to a factorisation via the period domain, Proposition 5.16, we know that $\mathcal{F}_Y$ comes from a locally free sheaf $\mathcal{F}_\gamma$ on $Y$. In the final step we prove that $\mathcal{E} \text{nd}(\mathcal{F}_\gamma) \cong \mathcal{F}_\gamma$. It follows that $\mathcal{E} \text{nd}(\mathcal{F}_\gamma)$ is locally free and then so is $\mathcal{F}_\gamma$. Proposition 9.1 thus follows from the Lipman-Zariski conjecture for varieties with canonical singularities, Theorem 3.4.

We will now go through the steps in more detail. We aim to present the proof in a way such that the parallels to Section 7 become obvious.

Step 1: Setup. We begin by considering the natural Higgs sheaf $(\mathcal{F}_\gamma, \theta_Y)$, as given in Example 6.5, where $\mathcal{F}_\gamma = \Omega^{[1]}_Y \oplus \mathcal{O}_Y$ and $\theta(a+b) = (0+1) \otimes a$. By Lemma 8.2 the Higgs sheaf $(\mathcal{F}_\gamma, \theta_Y)$ is stable with respect to the ample bundle $K_Y$.

Choose a strong log resolution of singularities, $\pi : \tilde{Y} \to Y$, such that there exists a $\pi$-ample Cartier divisor supported on the exceptional locus of $\pi$.

Claim 9.2. Write $r := (n+1)^2$. Let $B_r$ denote the set of locally free sheaves $\mathcal{F}$ on $X$ that have rank $r$, satisfy $\chi(K_\gamma^\text{max}(\mathcal{F})) = \chi(K_\gamma^\text{max}(\mathcal{E} \text{nd}(\mathcal{F}_\gamma)))$, and have Chern classes $c_i(\pi^*\mathcal{F}) = 0$ for all $0 < i < r$. Then, $B_r$ is bounded.
Proof of Claim 9.2. Since $X$ has rational singularities, the Euler characteristics $\chi_X(\mathcal{G})$ and $\chi_Y(\pi^*\mathcal{G})$ agree for all locally free sheaves $\mathcal{G}$ on $Y$. The assumption on Chern classes thus guarantees that the Hilbert polynomials of the members $\mathcal{F} \in B_r$ are constant, cf. [Ful98, Corollary 15.2.1]. Boundedness thus follows from [HL10, Theorem 3.3.7]. This ends the proof of Claim 9.2.

Next, choose a sufficiently large and divisible integer $m \gg 0$, and choose a general tuple of hyperplanes $H_1, \ldots, H_{n-2} \in |m \cdot K_X|$, with general complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2}$. The following items will then hold.

(9.2.1) The intersection $S$ is a smooth surface, and entirely contained in $Y_{\text{reg}}$. This is because $Y$ is smooth in codimension two by assumption.

(9.2.2) The restriction $(\mathcal{E}_Y, \theta_Y)|_S$ is stable with respect to $K_Y|_S$. This follows from the Restriction Theorem 6.10.

(9.2.3) The natural morphism $\iota_S : \pi_1(S) \to \pi_1(Y_{\text{reg}})$, induced by the inclusion $\iota : S \hookrightarrow Y_{\text{reg}}$, is isomorphic. This is the content of Goresky-MacPherson’s Lefschetz hyperplane theorem for homotopy groups, [GM88, Theorem in Section II.1.2].

(9.2.4) Let $\mathcal{F} \in B_r$. Then, $\mathcal{F}$ is isomorphic to $\text{End} \mathcal{E}_Y$ if and only if the restrictions $\mathcal{F}|_S$ and $(\text{End} \mathcal{E}_Y)|_S$ are isomorphic. This is a consequence of the boundedness statement in Claim 9.2, and of a Bertini-type theorem for isomorphism classes in bounded families [GKP16, Corollary 5.3].

Claim 9.3. The natural morphism $\pi_1(S) \to \pi_1(Y)$ is surjective and induces an isomorphism of profinite completions.

Proof. The natural morphism $\pi_1(Y_{\text{reg}}) \to \pi_1(Y)$ is surjective, [FL81, 0.7.B on Page 33], and induces an isomorphism of profinite completions by assumption. Composed with the inclusion $S \hookrightarrow Y_{\text{reg}}$, Claim 9.3 follows from Item (9.2.3) above. □

Step 2: Construction of a pCVHS on $S$. Since $S$ is entirely contained in the smooth locus of $Y$, the restricted Higgs sheaf $(\mathcal{E}_Y, \theta_Y)|_S$ is actually a Higgs bundle, and Construction 5.6 allows to equip the corresponding endomorphism bundle with a Higgs field. For brevity of notation, set $\mathcal{F}_S := \text{End} \mathcal{E}_Y|_S$ and write $(\mathcal{F}_S, \Theta_S)$ for associated Higgs bundle, constructed as in 5.6. The rank of $\mathcal{F}_S$ equals $r = (n + 1)^2$.

Claim 9.4. The Higgs bundle $(\mathcal{F}_S, \Theta_S)$ is induced by a pCVHS, in the sense of Definition 5.12.

Proof of Claim 9.4. We need to check the properties listed in Theorem 5.13.

Item (5.13.1): polystability with respect to $K_Y|_S$. By Theorem 6.10, we know that both $(\mathcal{E}_Y, \theta_Y)|_S$ and its dual are $K_Y|_S$-stable Higgs bundles on the smooth surface $S$. In particular, it follows from [Sim92, Theorem 1(2)] that both bundles carry a Hermitian-Yang-Mills metric with respect to $K_X|_S$, and thus so does $(\mathcal{F}_S, \Theta_S)$. Hence it follows from [Sim92, Theorem 1] that $(\mathcal{F}_S, \Theta_S)$ is polystable with respect to $K_Y|_S$.

Item (5.13.2): vanishing of Chern classes. As $\mathcal{F}_S$ is the endomorphism bundle of the locally free sheaf $\mathcal{E}_Y|_S$, its first Chern class vanishes. Vanishing of $c_2(\mathcal{F}_S)$ is then an immediate consequence of the assumed equality in (1.5.1). Together with polystability, this implies that $\mathcal{F}_S$ is flat, [Sim92, Theorem 1], and hence all its Chern classes vanish.

Item (5.13.3): we have seen in Example 5.4 that $\mathcal{E}_Y$ has the structure of a system of Hodge bundles. Its isomorphism class is therefore fixed under the action of $C^*$, [Sim92, Page 45]. Observing that the same holds for its dual and its endomorphism bundle, this ends the proof of Claim 9.4. □
Step 3: Extension of the $\mathfrak{pCVHS}$ to a resolution. Since $S$ is entirely contained in the smooth locus of $Y$, it is canonically isomorphic to its preimage $\tilde{S} := \pi^{-1}(S)$ in the resolution $\tilde{X}$. Let $(\mathcal{F}_\tilde{S}, \mathcal{O}_\tilde{S})$ be the Higgs bundle on $\tilde{S}$ that corresponds to $(\mathcal{F}_S, \mathcal{O}_S)$ under this isomorphism.

There exists a $Q$-divisor $E \in \text{Q Div}(\tilde{Y})$, supported entirely on the $\pi$-exceptional locus, such that $H := \pi^*(K_Y) + E$ is ample. Since $\tilde{S}$ and supp $E$ are disjoint, the Higgs bundle $(\mathcal{F}_\tilde{S}, \mathcal{O}_\tilde{S})$ is clearly semistable with respect to $H|_{\tilde{S}}$.

Recall from [Tak05, Theorem 1.1] that the natural map of fundamental groups, $\pi_1(\tilde{Y}) \to \pi_1(Y)$ is isomorphic. Together with Claim 9.3, this implies that $\pi_1(\tilde{S}) \to \pi_1(Y)$ is surjective, and induces an isomorphism of profinite completions. Items (5.15.2) and (5.15.1) of Corollary 5.15 therefore allow to find a Higgs bundle $(\mathcal{F}_\tilde{Y}, \mathcal{O}_\tilde{Y})$ on $\tilde{Y}$ that restricts to $(\mathcal{F}_S, \mathcal{O}_S)$, and is induced by $\mathfrak{pCVHS}$. We have seen in Remark 5.14 that all Chern classes of $\mathcal{F}_\tilde{Y}$ vanish.

Step 4: Identification of the $\mathfrak{pCVHS}$. We have seen in Proposition 5.16 that $\mathcal{F}_\tilde{Y}$ comes from $Y$. More precisely, there exists a locally free sheaf $\mathcal{F}_Y$ on $Y$ such that $\mathcal{F}_\tilde{Y} = \pi^*(\mathcal{F}_Y)$. First notice that $\mathcal{F}_Y$ is a member of the family $\mathcal{B}_r$ that was introduced in Claim 9.2 on page 22. Item (9.2.4) thus gives an isomorphism $\text{End} \mathcal{E}_Y \cong \mathcal{F}_Y$, showing that $\text{End} \mathcal{E}_Y$ is locally free. But $\text{End} \mathcal{E}_Y$ contains $\mathcal{F}_Y$ as a direct summand. It follows that $\mathcal{F}_Y$ is locally-free and thus $Y$ is smooth by the partial solution of the Zariski-Lipman problem for spaces with canonical singularities, Theorem 3.4.

This finishes the proof of the smoothness criterion, Proposition 9.1. \qed

9.2. Proof of Theorem 1.6 in a simplified setting. For the purposes of this survey, we prove Theorem 1.6 only under the following simplifying assumptions. Section 9.3 briefly discusses the missing pieces for a proof in the general case.

Assumption 9.5. The canonical bundle $K_X$ is ample, and $X$ is therefore equal to its canonical model.

Recalling from Definition 2.2 that minimal varieties have terminal singularities, we infer that $X$ is smooth in codimension two. In particular, Chern classes $c_1$ and $c_2$ exist.

Now consider a maximally quasi-étale cover $f : Y \to X$, as given by Theorem 4.2. Since $f$ is unramified in codimension two we find that $K_Y = f^*(K_X)$ is also ample and that $Y$ again has terminal singularities, cf. [Laz04, Proposition 1.2.13] and [KM98, Proposition 5.20]. Since $\mathcal{F}_Y$ and $f^*\mathcal{F}_X$ differ only along a set of codimension three, the projection formula for Chern classes yields that

$$
(9.5.1) \quad \left(2(n + 1) \cdot c_2(\mathcal{F}_Y) - n \cdot c_1(\mathcal{F}_Y)^2\right) \cdot [K_Y]^{n-2} = 0.
$$

In other words, equality holds in the Miyaoka-Yau inequality for $Y$. In particular, the smoothness criterion of Proposition 9.1 applies, showing that $Y$ is smooth. So, $Y$ is uniformised by the ball, thanks to the original result of Yau, Theorem 1.3. This finishes the proof of Theorem 1.6 in the simplified setting of Assumption 9.5. \qed

9.3. Proof in the general case. To prove Theorem 1.6 in general, we show that the tangent sheaf of the canonical model satisfies the equality in Miyaoka-Yau inequality, and that it is smooth in codimension two. This is a consequence of two computations with orbifold Chern classes:

Let $\pi : X \to X_{can}$ be the morphism from $X$ to its canonical model.
The Miyaoka-Yau equality for $\mathcal{X}_{can}$. We claim that $\mathcal{X}_{can}$ verifies the Miyaoka-Yau equality. Reason: on the one hand we know from Theorem 1.5 that $\mathcal{X}_{can}$ verifies the Miyaoka-Yau inequality. On the other hand, Chern classes calculations similar to [SBW94, Proposition 1.1] show that

\begin{equation}
(9.6.1) \quad c_2(\mathcal{X}) \cdot [K]\cdot c_2(\mathcal{X}_{can}) \cdot [K_{can}]^{n-2} = c_2(\mathcal{Y}) - c_2(\mathcal{Z}) \geq 0,
\end{equation}

where $\tilde{S}$ is the birational transform of a complete intersection surface $S = D_1 \cap \cdots \cap D_{n-2}$, for sufficiently general members $D_i$ of $|m \cdot K_{can}|$, for $m$ sufficiently large and divisible. But this implies that the MY discriminant for $\mathcal{X}_{can}$ is bounded from the above by the one for $\mathcal{X}$. In other words, the MY discriminant of $\mathcal{X}_{can}$ is at most zero.

The singularities of the canonical model. As the MY discriminant of $\mathcal{X}_{can}$ is equal to zero, (9.6.1) implies that

\begin{equation}
(9.6.2) \quad c_2(\mathcal{Y}) = c_2(\mathcal{Z}).
\end{equation}

But as $\mathcal{X}_{can}$ has only canonical singularities, every connected exceptional divisor of $\pi|\tilde{S} : \tilde{S} \to S$ is a tree of $\mathbb{P}^1$'s. The fact that $\tilde{c}_2$ computes the orbifold Euler characteristic of $S$ implies that the equality can only hold if $S$ is smooth. But if general complete intersections surfaces are smooth, then $\mathcal{X}_{can}$ needs to be smooth in codimension two.

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