Saved by the rook: a case of matchings and Hamiltonian cycles

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Abstract

The rook graph is a graph whose edges represent all the possible legal moves of the rook chess piece on a chessboard. The problem we consider is the following. Given any set $M$ containing pairs of cells such that each cell of the $m_1 \times m_2$ chessboard is in exactly one pair, we determine the values of the positive integers $m_1$ and $m_2$ for which it is possible to construct a closed tour of all the cells of the chessboard which uses all the pairs of cells in $M$ and some edges of the rook graph. This is an alternative formulation of a graph-theoretical problem presented in [Electron. J. Combin. 28(1) (2021), #P1.7] involving the Cartesian product $G$ of two complete graphs $K_{m_1}$ and $K_{m_2}$, which is, in fact, isomorphic to the $m_1 \times m_2$ rook graph. The problem revolves around determining the values of the parameters $m_1$ and $m_2$ that would allow any perfect matching of the complete graph on the same vertex set of $G$ to be extended to a Hamiltonian cycle by using only edges in $G$.

Keywords: Perfect matching, Hamiltonian cycle, Cartesian product of complete graphs, line graph, complete bipartite graph.

Math. Subj. Class.: 05C45, 05C70, 05C76.

1 Introduction

The rook chess piece is allowed to move in a horizontal and vertical manner only—no diagonal moves are permissible. The rook graph represents all the possible moves of a rook on a chessboard, with its vertices and edges corresponding to the cells of the chessboard, and the legal moves of the rook from one cell to the other, respectively. All the legal moves of a rook on a $m_1 \times m_2$ chessboard give rise to the $m_1 \times m_2$ rook graph. In what follows we consider the following problem.
Problem 1.1. Let $G$ be a $m_1 \times m_2$ chessboard and let $M$ be a set containing pairs of distinct cells of $G$ such that each cell of $G$ belongs to exactly one pair in $M$. Determine the values of $m_1$ and $m_2$ for which it is possible to construct a closed tour $H$ visiting all the cells of the chessboard $G$ exactly once, such that:

(i) consecutive cells in $H$ are either a pair of cells in $M$, or two cells in $G$ which can be joined by a legal rook move; and

(ii) $H$ contains all pairs of cells in $M$.

In other words, given any possible choice of a set $M$ as defined above, is a rook good enough to let one visit, exactly once, all the cells on a chessboard and finish at the starting cell, in such a way that each pair of cells in $M$ is allowed to and must be used once? We remark that $M$ can contain pairs of cells which are not joined by a legal rook move.

As many other mathematical chess problems, the above problem can be restated in graph theoretical terms (for a detailed exposition, we suggest the reader to [6]). We first give some definitions, and for definitions and notation not explicitly stated here, we refer the reader to [3]. All graphs considered in the sequel will be simple, that is, loops and multiple edges are not allowed. For any graph $G$ with vertex set $V(G)$ and edge set $E(G)$, we let $K_G$ denote the complete graph on the same vertex set $V(G)$ of $G$. Let $G$ be of even order, that is, having an even number of vertices. A Hamiltonian cycle of a graph $G$ is a cycle of $G$ which visits every vertex of $G$. A perfect matching $N$ of a graph $G$ is a set of edges of $G$ such that every vertex of $G$ belongs to exactly one edge in $N$. This means that no two edges in $N$ have a common vertex and that $N$ is a set of independent edges covering $V(G)$. Let $G$ be a graph of even order. A Hamiltonian cycle of $G$ can be considered as the disjoint union of two perfect matchings of $G$. A perfect matching of $K_G$ is said to be a pairing of $G$. In what follows we shall consider Hamiltonian cycles of $K_G$ (for some graph $G$ of even order) composed of a pairing of $G$ and a perfect matching of $G$. In order to distinguish between pairings of $G$, which may possibly contain edges not in $G$, and perfect matchings of $G$, we shall depict pairing edges as green, bold and dashed, and edges of a perfect matching of $G$ as black and bold. To emphasise that pairings can contain edges in $G$, we shall depict such edges with a black thin line underneath the green, bold and dashed edge described above. This can be clearly seen in Figure 2.

In 2015, the authors in [2] say that a graph $G$ has the Pairing-Hamiltonian property (the PH-property for short) if every pairing $M$ of $G$ can be extended to a Hamiltonian cycle $H$ of $K_G$ in which $E(H) - M \subseteq E(G)$. If a graph has the PH-property, for simplicity we shall sometimes say that the graph is PH. In order to provide the reader with some
examples of graphs having the PH-property, we remark that the authors in [2], amongst
other results, gave a complete characterisation of the cubic graphs, that is, graphs with all
vertices having degree 3, having the PH-property. There are only three: the complete graph
$K_4$, the complete bipartite graph $K_{3,3}$ and the 3-dimensional cube $Q_3$ (depicted in Figure
3). We note that in the first diagram of Figure 2, one of the green, bold and dashed edges
is not an edge of $Q_3$, and thus the diagram illustrates a possible pairing of $Q_3$ which is
not a perfect matching of $Q_3$. As shown in Figure 2, this pairing can be extended to a
Hamiltonian cycle of $Q_3$ by using edges of $Q_3$. The same argument can be repeated for all
pairings of the three graphs shown in Figure 3; hence why they have the PH-property. A
similar property to the PH-property is the PMH-property (see [1] for a more detailed introdution). A graph is said to have
the PMH-property, if every perfect matching $M$ of $G$ can be extended to a Hamiltonian
cycle $H$ of $K_G$ in which $E(H) - M \subseteq E(G)$. We note that in this case, $H$ would also
be a Hamiltonian cycle of $G$ itself. In other words, the PMH-property is equivalent to the
PH-property restricted to pairings of $G$ which are also perfect matchings of $G$. Thus, the
PMH-property is a somewhat weaker property than the PH-property.

The Cartesian product $G \square H$ of two graphs $G$ and $H$ is a graph whose vertex set is the
Cartesian product $V(G) \times V(H)$ of $V(G)$ and $V(H)$. Two vertices $(u_i, v_j)$ and $(u_k, v_l)$
are adjacent precisely if $u_i = u_k$ and $v_jv_l \in E(H)$ or $u_iu_k \in E(G)$ and $v_j = v_l$. Thus,

$$V(G \square H) = \{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\},$$
and

$$E(G \square H) = \{(u_i, v_j)(u_k, v_l) : u_i = u_k, v_jv_l \in E(H) \text{ or } u_iu_k \in E(G), v_j = v_l\}.$$ 

The $m_1 \times m_2$ rook graph is in fact isomorphic to the Cartesian product of the complete
graphs $K_{m_1}$ and $K_{m_2}$, denoted by $K_{m_1} \square K_{m_2}$.

Another result in [2] which we shall also be using later on is the following.

**Theorem 1.2** (Alahmadi et al. [2]). The Cartesian product of a complete graph $K_m$ ($m$
even and $m \geq 6$) and a path $P_q$ ($q \geq 1$) has the PH-property.
However, this was not the first time that pairings extending to Hamiltonian cycles were studied. In 2007, Fink [4] proved what we believe is one of the most significant results in this area so far: for every \( n \geq 2 \), the \( n \)-dimensional hypercube is \( \text{PH} \), thus answering a conjecture made by Kreweras (see [5]). The proof of this result, although technical, is very short and elegant.

With these notions in place, we can restate the above problem as follows.

**Problem 1.3** (Problem 1.1 restated). Let \( G \) be the \( m_1 \times m_2 \) rook graph, or equivalently \( K_{m_1} \Box K_{m_2} \). Determine for which values of \( m_1 \) and \( m_2 \) does \( G \) have the \( \text{PH} \)-property.

Clearly, in order for \( K_{m_1} \Box K_{m_2} \) to admit a pairing, at least one of \( m_1 \) and \( m_2 \) must be even, and without loss of generality, in the sequel we shall tacitly assume that \( m_1 \) is even.

We recall that the line graph \( L(G) \) of a graph \( G \) is the graph whose vertices correspond to the edges of \( G \), and two vertices of \( L(G) \) are adjacent if the corresponding edges in \( G \) are incident to a common vertex. The \( m_1 \times m_2 \) rook graph, or equivalently \( K_{m_1} \Box K_{m_2} \), can also be seen as the line graph of the complete bipartite graph \( K_{m_1,m_2} \). The authors in [1] give some sufficient conditions for a graph \( G \) in order to guarantee that its line graph \( L(G) \) has the \( \text{PMH} \)-property. Amongst other results, they show that the line graph of complete graphs \( K_n \), for \( n \equiv 0, 1 \pmod{4} \), has the \( \text{PMH} \)-property, and that, by a similar reasoning, \( L(K_{m,m}) \) has the \( \text{PMH} \)-property for every even \( m \geq 50 \). In Section 2, we determine for which values \( m_1 \) and \( m_2 \) (with \( m_1 \) not necessarily equal to \( m_2 \)) does \( L(K_{m_1,m_2}) \) admit not only the \( \text{PMH} \)-property, but also the \( \text{PH} \)-property. This gives a complete solution to Problem 1.3.

## 2 Main result

In this section we give a complete solution to Problem 1.3, summarised in the following theorem.

**Theorem 2.1.** Let \( m_1 \) be an even integer and let \( m_2 \geq 1 \). The \( m_1 \times m_2 \) rook graph does not have the \( \text{PH} \)-property if and only if \( m_1 = 2 \) and \( m_2 \) is odd.

**Proof.** When \( m_2 = 1 \), \( K_{m_1} \Box K_1 \) is \( K_{m_1} \) and the result clearly follows. Consequently, we shall assume that \( m_2 > 1 \). By Theorem 1.2, \( K_{m_1} \Box K_{m_2} \) is \( \text{PH} \) when \( m_1 \geq 6 \), since \( K_{m_1} \Box K_{m_2} \) contains \( K_{m_1} \Box P_{m_2} \), and, in general, if a graph contains a spanning subgraph which is \( \text{PH} \), the initial graph is itself \( \text{PH} \).

So consider the cases when \( m_1 = 2 \) or 4. If \( m_1 = 2 \), \( K_{m_1} \Box K_{m_2} \) is \( \text{PH} \) if and only if \( m_2 \equiv 0 \pmod{2} \). In fact, if \( m_2 \) is odd, the pairing consisting of the \( m_2 \)-edge-cut between the two copies of \( K_{m_2} \) cannot be extended to a Hamiltonian cycle, as can be seen in Figure 4. If \( m_2 \) is even, the result follows once again by Theorem 1.2 when \( m_2 \geq 6 \). If \( m_2 = 2 \),

![Figure 4: A pairing in \( K_2 \Box K_3 \) which cannot be extended to a Hamiltonian cycle](image)

the result easily follows, and when \( m_2 = 4 \), \( K_2 \Box K_4 \) is \( \text{PH} \) because the 3-dimensional cube
Since
we obtain two components
following technical lemma.

Proof. Let the 4 \times m rook graph be denoted by G. We let the vertex set of G be \{a_i, b_i, c_i, d_i : i \in [m]\}, such that for each i, the vertices a_i, b_i, c_i, d_i induce a complete graph on four vertices, denoted by K_4, and the vertices represented by the same letter induce a K_m. Let M be a pairing of G. We consider two cases:

Case 1. M does not induce a perfect matching in each K_4; and

Case 2. M induces a perfect matching in each K_4.

We start by considering Case 1, and without loss of generality assume that |M \cap E(K_4^2)| < 2. If we delete all the edges having exactly one end-vertex in K_4 from G, we obtain two components G_1 and G_2 isomorphic to K_4 and K_4 \Box K_{m-1}, respectively. Since G_1 is of even order and M \cap E(G_1) is not a perfect matching of this graph, G_1 has an even number (two or four) of vertices which are unmatched by M \cap E(G_1).

We pair these unmatched vertices such that M \cap E(G_1) is extended to a perfect matching M_1 of G_1. By a similar reasoning, M \cap E(G_2) does not induce a pairing of G_2 and the number of vertices in G_2 which are unmatched by M \cap E(G_2) is again two or four. Without loss of generality, let a_1, b_1 be two vertices in G_1 unmatched by M \cap E(G_1) such that a_1b_1 \in M_1, and let x, y be the two vertices in G_2 such that a_1x and b_1y are both edges in the pairing M of G. We extend M \cap E(G_2) to a pairing M_2 of G_2 by adding the edge xy to M \cap E(G_2), and we repeat this procedure until all vertices in G_2 are matched. Since m - 1 is even, G_2 has the PH-property and so M_2 can be extended to a Hamiltonian cycle H_2 of K_{G_2}. We extend H_2 to a Hamiltonian cycle of G containing M as follows. If c_1d_1 \in M \cap E(G_1), we replace the edge xy in H_2 by the edges xa_1, a_1d_1, d_1c_1, c_1b_1, b_1y, as in Figure 5. Otherwise, c_1d_1 \in M_1 - (M \cap E(G_1)), and so there exist two vertices u, v in G_2 such that c_1u and d_1v belong to belong to the initial pairing M, and uv belongs to M_2. In this case, we replace the edges xy and uv in H_2 by the edges xa_1, a_1b_1, b_1y, and uc_1, c_1d_1, d_1v, respectively. In either case, H_2 is extended to a Hamiltonian cycle of G containing the pairing M, as required.

Next, we move on to Case 2, that is, when M induces a perfect matching in each K_4. This case is true by Proposition 1 in [2], however, here we adopt a constructive and

\[ Q_3 \] is a subgraph of \( K_2 \Box K_4 \) and has the PH-property by Fink’s result in [4] (also referred to previously).

What remains to be considered is the case when \( m_1 = 4 \) and \( m_2 \geq 3 \). The graph \( K_4 \Box K_4 \) contains \( C_4 \Box C_4 \), the 4-dimensional hypercube \( Q_4 \), which is PH ([4]), and for \( m_2 \geq 6 \) and \( m_2 \) even, the result follows once again by Theorem 1.2. Therefore, what remains to be shown is the case when \( m_2 \geq 3 \) and \( m_2 \) is odd, which is settled in the following technical lemma. □

**Lemma 2.2.** For every odd \( m \geq 3 \), the \( 4 \times m \) rook graph has the PH-property.

**Proof.** Let the 4 \times m rook graph \( K_4 \Box K_m \) be denoted by G. We let the vertex set of G be \{a_i, b_i, c_i, d_i : i \in [m]\}, such that for each i, the vertices a_i, b_i, c_i, d_i induce a complete graph graph on four vertices, denoted by K_4, and the vertices represented by the same letter induce a K_m. Let M be a pairing of G. We consider two cases:

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We start by considering Case 1, and without loss of generality assume that |M \cap E(K_4^2)| < 2. If we delete all the edges in G having exactly one end-vertex in K_4 from G, we obtain two components G_1 and G_2 isomorphic to K_4 and K_4 \Box K_{m-1}, respectively. Since G_1 is of even order and M \cap E(G_1) is not a perfect matching of this graph, G_1 has an even number (two or four) of vertices which are unmatched by M \cap E(G_1).

We pair these unmatched vertices such that M \cap E(G_1) is extended to a perfect matching M_1 of G_1. By a similar reasoning, M \cap E(G_2) does not induce a pairing of G_2 and the number of vertices in G_2 which are unmatched by M \cap E(G_2) is again two or four. Without loss of generality, let a_1, b_1 be two vertices in G_1 unmatched by M \cap E(G_1) such that a_1b_1 \in M_1, and let x, y be the two vertices in G_2 such that a_1x and b_1y are both edges in the pairing M of G. We extend M \cap E(G_2) to a pairing M_2 of G_2 by adding the edge xy to M \cap E(G_2), and we repeat this procedure until all vertices in G_2 are matched. Since m - 1 is even, G_2 has the PH-property and so M_2 can be extended to a Hamiltonian cycle H_2 of K_{G_2}. We extend H_2 to a Hamiltonian cycle of G containing M as follows. If c_1d_1 \in M \cap E(G_1), we replace the edge xy in H_2 by the edges xa_1, a_1d_1, d_1c_1, c_1b_1, b_1y, as in Figure 5. Otherwise, c_1d_1 \in M_1 - (M \cap E(G_1)), and so there exist two vertices u, v in G_2 such that c_1u and d_1v belong to belong to the initial pairing M, and uv belongs to M_2. In this case, we replace the edges xy and uv in H_2 by the edges xa_1, a_1b_1, b_1y, and uc_1, c_1d_1, d_1v, respectively. In either case, H_2 is extended to a Hamiltonian cycle of G containing the pairing M, as required.

\[ \text{Figure 5: An illustration of the inductive step in Case 1 when } m_2 = 3 \]

Next, we move on to Case 2, that is, when M induces a perfect matching in each K_4. This case is true by Proposition 1 in [2], however, here we adopt a constructive and
more detailed approach highlighting the very useful technique used in [4]. There are three different ways how \( M \) can intersect the edges of \( K^*_4 \), namely \( M \cap E(K^*_4) \) can either be equal to \( \{a_i b_i, c_i d_i\}, \{a_i c_i, b_i d_i\}, \) or \( \{a_i d_i, b_i c_i\} \). The number of \( 4 \)-cliques intersected by \( M \) in \( \{a_i b_i, c_i d_i\} \) is denoted by \( \nu^\text{ab}_{cd} \), and we shall define \( \nu^\text{ac}_{bd} \) and \( \nu^\text{ad}_{bc} \) in a similar way. Without loss of generality, we shall assume that \( \nu^\text{ab}_{cd} \geq \nu^\text{ac}_{bd} \geq \nu^\text{ad}_{bc} \). We shall also assume that the first \( \nu^\text{ab}_{cd} \) \( 4 \)-cliques in \( \{K^*_4 : i \in [m]\} \) are the ones intersected by \( M \) in \( \{a_i b_i, c_i d_i\} \), and, if \( \nu^\text{ad}_{bc} \neq 0 \), the last \( \nu^\text{ad}_{bc} \) \( 4 \)-cliques are the ones intersected by \( M \) in \( \{a_i d_i, b_i c_i\} \). This can be seen in Figure 6, in which “unnecessary” curved edges of \( G \) are not drawn so as to render the figure more clear.

![Figure 6](image_url)

Figure 6: \( G \) when \( \nu^\text{ab}_{cd} = 2 \), \( \nu^\text{ac}_{bd} = 2 \) and \( \nu^\text{ad}_{bc} = 1 \)

When \( \nu^\text{cd}_{ab} = 1 \), we have that \( \nu^\text{bd}_{ac} = \nu^\text{ad}_{bc} = 1 \), and in this case it is easy to see that \( M \) can be extended to a Hamiltonian cycle of \( K_G \), for example \( (a_1, b_1, c_1, d_1, d_3, a_3, c_3, b_3, b_2, a_2, c_2, a_2) \). We remark that this is the only time when all the \( 4 \)-cliques are intersected differently by \( M \). Therefore, assume \( \nu^\text{cd}_{ab} \geq 2 \). First, let \( \nu^\text{cd}_{ab} = 2 \). If \( \nu^\text{bc}_{ac} = 0 \), then, \( \nu^\text{bd}_{ac} = 1 \) and it is easy to see that \( M \) can be extended to a Hamiltonian cycle of \( K_G \), for example \( (a_1, b_1, a_2, a_3, c_3, b_3, d_3, d_2, c_2, c_1, d_1) \). The only other possibility is to have \( \nu^\text{dc}_{bd} = 2 \) and \( \nu^\text{cd}_{ad} = 1 \), and once again \( M \) can be extended to a Hamiltonian cycle of \( K_G \), as Figure 6 shows.

Thus, we can assume that \( \nu^\text{cd}_{ab} \geq 3 \). Let \( r = \nu^\text{cd}_{ab} + \nu^\text{bd}_{ac} \) and let \( r' \) be the largest even integer less than or equal to \( r \). Moreover, let \( G_1 \) be the subgraph of \( G \) induced by the vertices \( \{b_i, c_i : i \in [m]\} \) (isomorphic to \( K_2 \square K_m \)) and let \( M_1 = \{b_1 b_2, \ldots, b_{r' - 1} b_{r'}, c_1 c_2, \ldots, c_{r' - 1} c_{r'}, b_{r' + 1} c_{r' + 1}, \ldots, b_m c_m\} \). Clearly, \( M_1 \) is a pairing of \( G_1 \) which contains \( M \cap E(G_1) \), and can be extended to a Hamiltonian cycle \( H_1 \) of \( K_{G_1} \) as follows: \( (b_1, b_2, \ldots, b_{r'}, b_{r' + 1}, c_{r' + 1}, c_{r' + 2}, b_{r' + 2}, \ldots, b_m c_m, c_{r'}, c_{r' - 1}, \ldots, c_1) \). This is depicted in Figure 7.

We note that if \( r' = m - 1 \), we do not consider the index \( r' + 2 \) in the last sequence of vertices forming \( H_1 \). Deleting the edges belonging to \( M_1 - M \) from \( H_1 \) gives a collection of \( r \) disjoint paths \( \mathcal{P} = \{P_i : i \in [r]\} \). We note that the union of all the end-vertices of the paths in \( \mathcal{P} \) give \( \{b_i, c_i : i \in [r]\} \). If we look at the example given in Figure 7, the only path in \( \mathcal{P} \) on more than two vertices is the path \( b_8 b_9 c_9 c_1 c_{10} b_{10} b_{11} c_{11} c_8 \).

Next, let \( G_2 \) be the subgraph of \( G \) induced by the vertices \( \{a_i, d_i : i \in [m]\} \), which is isomorphic to \( K_2 \square K_m \) as \( G_1 \). For every \( i \in [r] \), we let \( u_i \) and \( v_i \) be the two end-vertices of the path \( P_i \), and we let \( x_i \) and \( y_i \) be the two vertices in \( G_2 \) such that \( u_i x_i \) and \( v_i y_i \) both belong to \( M \). We remark that \( \{a_i, d_i : i \in [r]\} = \{x_i, y_i : i \in [r]\} \). Let \( M_2 = \{x_1 y_1, \ldots, x_r y_r\} \cup (M \cap E(G_2)) \). If \( r = m \), then \( M \cap E(G_2) \) is empty, otherwise it consists of \( \{a_{r+1} a_{r+1}, \ldots, a_m a_m\} \). If \( \nu^\text{cd}_{ab} \) is even (as in Figure 7), \( M_2 \) contains:

\[
\{a_1 d_1, a_2 a_3, \ldots, a_{\nu^\text{cd}_{ab} - 2} a_{\nu^\text{cd}_{ab} - 1}, a_{\nu^\text{cd}_{ab}} a_{\nu^\text{cd}_{ab} + 1}, d_2 d_3, \ldots, d_{\nu^\text{cd}_{ab} - 2} d_{\nu^\text{cd}_{ab} - 1}, d_{\nu^\text{cd}_{ab}} d_{\nu^\text{cd}_{ab} + 1}\}.
\]
Figure 7: $G_1$ and $G_2$ when $\nu_{ab}^{cd} = 4$, $r = r' = 8$, and $m = 11$ in Case 2

Otherwise, $M_2$ contains $\{a_1d_1, a_2a_3, \ldots, a_{\nu_{ab}^{cd} - 1}a_{\nu_{ab}^{cd}}^d, d_2d_3, \ldots, d_{\nu_{ab}^{cd} - 1}d_{\nu_{ab}^{cd}}^d\}$. Moreover, if $r$ is even, then $a_r, d_r \in M_2$. In either case, $M_2$ can be extended to a Hamiltonian cycle $H_2$ of $K_{G_2}$, as can be seen in Figure 7, which shows the case when $\nu_{cd}^{ab}$ and $r$ are both even. We remark that the green, bold and dashed edges in the figure are the ones in $M_1$ and $M_2$. If for each $i \in [r]$, we replace the edges $x_iy_i$ in $H_2$ by $x_iu_i$, the path $P_i$, and $v_iy_i$ (as in Figure 8), a Hamiltonian cycle of $G$ containing $M$ is obtained, proving our theorem.

Figure 8: Extending $H_1$ and $H_2$ from Fig. 7 to a Hamiltonian cycle of $K_G$ containing $M$

3 Bishop-on-a-rook graph

In the next theorem we present a rather simple proof to show that the complete bipartite graph having equal partite sets (otherwise it does not admit a perfect matching) is PH.

Theorem 3.1. For every $n \geq 2$, the complete bipartite graph $K_{n,n}$ has the PH-property.

Proof. Let $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$ be the partite sets of $K_{n,n}$. We proceed by induction on $n$. When $n = 2$, result holds since $K_{2,2} \simeq K_2 \Box K_2$. So assume $n > 2$ and let $M$ be a pairing of $K_{n,n}$. If $M = \{u_iw_i : i \in [n]\}$, then $M$ easily extends to a Hamiltonian cycle of the underlying complete graph on $2n$ vertices. Thus, assume there exists $j \in [n]$ such that $u_jw_j \notin M$. Without loss of generality, let $j$ be equal to $n$. Then, $M$ contains the edges $xu_n$ and $yw_n$, for some $x$ and $y$ belonging to the set $Z = \{u_i, w_i : i \in [n-1]\}$. We note that $Z$ induces the complete bipartite graph $K_{n-1,n-1}$ with partite sets $\{u_1, \ldots, u_{n-1}\}$ and $\{w_1, \ldots, w_{n-1}\}$, which we denote by $G'$. The set of
edges \( M' = M \cup xy - xu_n - yw_n \) is a pairing of \( G' \), and so, by induction on \( n \), \( M' \) can be extended to a Hamiltonian cycle \( H' \) of \( K_{G'} \). This Hamiltonian cycle can be extended to a Hamiltonian cycle \( H \) of the underlying complete graph of \( K_{n,n} \) by replacing the edge \( xy \) in \( H' \), by the edges \( xu_n, u_nw_n, w_ny \). The resulting Hamiltonian cycle \( H \) clearly contains \( M \), proving our theorem.

Although the statement and proof of Theorem 3.1 are quite easy, they may lead to another intriguing problem. From Theorem 2.1 we know that the rook is not good enough to solve our problem on a \( 2 \times m_2 \) chessboard when \( m_2 \) is odd. However, the above result shows that if the rook was somehow allowed to do only vertical and diagonal moves (instead of vertical and horizontal moves only), then it would always be possible to perform a closed tour on a \( 2 \times m_2 \) chessboard in such a way that each pair of cells in \( M \) is allowed to and must be used once, no matter the choice of \( M \). We shall call this new hybrid chess piece the \textit{bishop-on-a-rook}, and, as already stated, it is only allowed to move in a vertical and diagonal manner—no horizontal moves are permissible. As in the case of the rook, all the legal moves of a bishop-on-a-rook on a \( m_1 \times m_2 \) chessboard give rise to the \( m_1 \times m_2 \) bishop-on-a-rook graph, with \( m_1 \) corresponding to the vertical axis.

As before, for the \( m_1 \times m_2 \) bishop-on-a-rook graph to be PH, at least one of \( m_1 \) or \( m_2 \) must be even. Moreover, we remark that when \( m_2 \leq m_1 \), the \( m_1 \times m_2 \) bishop-on-a-rook graph contains \( K_{m_1} \Box K_{m_2} \) as a subgraph. Finally, we also observe that the \( m_1 \times m_2 \) bishop-on-a-rook graph is isomorphic to the co-normal product of \( K_{m_1} \) and \( K_{m_2} \), where the latter is the empty graph on \( m_2 \) vertices. The co-normal product \( G \ast H \) of two graphs \( G \) and \( H \) is a graph whose vertex set is the Cartesian product \( V(G) \times V(H) \) of \( V(G) \) and \( V(H) \), and two vertices \((u_i, v_j) \) and \((u_k, v_l) \) are adjacent precisely if \( u_iu_k \in E(G) \) or \( v_jv_l \in E(H) \). Thus,

\[
V(G \ast H) = \{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\}, \quad \text{and}
\]
\[
E(G \ast H) = \{((u_i, v_j)(u_k, v_l) : u_iu_k \in E(G) \text{ or } v_jv_l \in E(H)\}.
\]

We wonder for which values \( m_1 \) and \( m_2 \) is the \( m_1 \times m_2 \) bishop-on-a-rook graph PH.

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