ON THE BEHAVIOR OF KNESER SOLUTIONS OF NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We obtain priory estimates and sufficient conditions for Kneser solutions of ordinary differential equations to vanish in a neighborhood of infinity.

1. Introduction

We study solutions of the differential equations

\[ w^{(m)} = Q(r, w, \ldots, w^{(m-1)}), \quad r \geq a, \]

of order \( m \geq 2 \) satisfying the conditions

\[ (-1)^i w^{(i)}(r) \geq 0, \quad r \geq a, \quad i = 0, \ldots, m - 1, \]

where \( Q \) belongs to the Caratheodory class \( K_{loc}([a, \infty) \times \mathbb{R}^m), a > 0 \). Throughout the paper, it is assumed that

\[ (-1)^m Q(r, t_0, \ldots, t_{m-1}) \geq q(r) h(t_0) - \sum_{i=1}^{m-1} b_i(r) t_i \]

on the set \( \{ (r, t_0, \ldots, t_{m-1}) : r \geq a, t_0 > 0, (-1)^i t_i \geq 0, i = 1, \ldots, m - 1 \} \), where \( q : [a, \infty) \to [0, \infty) \) and \( b_i : [a, \infty) \to [0, \infty), i = 1, \ldots, m - 1 \), belong to the space \( L_{\infty, loc}([a, \infty)) \) and \( h : (0, \infty) \to (0, \infty) \) is a continuous function.

As an example of (1.1), one can take the equation

\[ w^{(m)} + z_{m-1}(r) w^{(m-1)} + \ldots + z_1(r) w' = z(r, w) \]

in which the coefficients of lower derivatives and the right-hand side are continuous functions with \( (-1)^m z(r, t) \geq q(r) h(t) \) for all \( r \geq a \) and \( t > 0 \). Inequality (1.3) should obviously be fulfilled, if we put

\[ Q(r, t_0, \ldots, t_{m-1}) = z(r, t_0) - \sum_{i=1}^{m-1} z_i(r) t_i \]

and \( b_i = |z_i|, i = 1, \ldots, m - 1 \).

As is customary, a function \( w : [a, \infty) \to \mathbb{R} \) is called a solution of (1.1), (1.2) if its derivatives \( w^{(i)}, i = 0, \ldots, m - 1 \), are locally absolutely continuous on the interval \([a, \infty)\), equation (1.1) holds for almost all \( r \in [a, \infty) \), and conditions (1.2) hold for all \( r \in [a, \infty) \).

By a non-trivial solution of (1.1), (1.2) we mean a solution that does not vanish on the whole interval \([a, \infty)\).

1991 Mathematics Subject Classification. 34A34, 34A40, 34C11, 34C41.

Key words and phrases. Nonlinear ordinary differential equations; Kneser solutions; Singular solutions of the first kind.

The research was supported by RFBR, grant 11-01-12018-ofi-m-2011.
Definition 1.1 (6). A non-trivial solution of (1.1), (1.2) is singular of the first kind if it vanishes in a neighborhood of infinity; otherwise this solution is called regular (proper).

In the literature, solutions of problem (1.1), (1.2) are also known as Kneser solutions. Starting from the pioneering paper of A. Kneser [7], they attract the attention of many mathematicians [1–10]. Our aim is to obtain priori estimates and sufficient conditions for any solution of (1.1), (1.2) to be singular of the first kind. In particular, we generalize results of [9], where the case of $b_1 = \ldots = b_{m-1} = 0$ was considered.

2. Main Results

We denote

$$g(t) = \inf_{(t/\theta, \theta t)} h, \quad t \in (0, \infty),$$

$$f(r) \frac{q(r)}{1 + \sum_{i=1}^{m-1} r^{m-i} \operatorname{ess sup}_{(r/\sigma, r\sigma) \cap [a, \infty)} b_i}, \quad r \in [a, \infty)$$

and

$$\mu(r) = 1 + r^m \operatorname{ess sup}_{(r/\sigma, r\sigma) \cap [a, \infty)} f, \quad r \in [a, \infty),$$

where $\theta > 1$ and $\sigma > 1$ are some real numbers which can be chosen arbitrary.

Theorem 2.1. Let

$$\int_0^1 g^{-1/m}(t)t^{1/m-1} dt < \infty \quad (2.1)$$

and

$$\int_a^\infty \xi^{m-1} f(\xi) \mu^{1/m-1}(\xi) d\xi = \infty. \quad (2.2)$$

Then any non-trivial solution of (1.1), (1.2) is singular of the first kind.

Theorem 2.2. Suppose that condition (2.1) is valid,

$$\int_a^\infty \xi^{m-1} f(\xi) d\xi = \infty \quad (2.3)$$

and, moreover,

$$\limsup_{r \to \infty} \frac{r^m f(r)}{\int_a^\infty \xi^{m-1} f(\xi) d\xi} < \infty. \quad (2.4)$$

Then any non-trivial solution of (1.1), (1.2) is singular of the first kind.

Theorems 2.1 and 2.2 are proved in Section 3. Now, we demonstrate their exactness.

Example 2.1. Consider the problem

$$w'' + b(r)w' = p(r)w^\lambda, \quad r \geq a, \quad (2.5)$$

$$(-1)^i w^{(i)}(r) \geq 0, \quad r \geq a, \quad i = 0, 1, \quad (2.6)$$

where $\lambda < 1$ and, moreover, $b: [a, \infty) \to \mathbb{R}$ and $p: [a, \infty) \to [0, \infty)$ are locally bounded measurable functions such that

$$|b(r)| \leq Br^a, \quad B = \text{const} > 0, \quad (2.7)$$
for all sufficiently large \( r \) and

\[ p(r) \sim r^l \quad \text{as} \quad r \to \infty, \quad (2.8) \]
i.e.

\[ c_1 r^l \leq p(r) \leq c_2 r^l \]

with some constants \( c_1 > 0 \) and \( c_2 > 0 \) for almost all \( r \) in a neighborhood of infinity.

At first, let \( s > -1 \). By Theorem 2.2 if

\[ l \geq s - 1, \quad (2.9) \]

then any non-trivial solution of (2.5), (2.6) is singular of the first kind. At the same time, in case of \( l < s - 1 \), it does not present any particular problem to verify that

\[ w(r) = r^{(l-s+1)/(1-\lambda)} \]
is a regular solution of (2.5), (2.6), where \( b(r) = -r^s \) and \( p \) is a non-negative continuous function satisfying relation (2.8). Therefore, condition (2.9) is exact.

Now, assume that \( s \leq -1 \). If

\[ l \geq -2, \quad (2.10) \]
then Theorem 2.2 implies that any non-trivial solution of (2.5), (2.6) is singular of the first kind. This condition is exact too. Namely, if \( l < -2 \), then, putting

\[ w(r) = r^{(l+2)/(1-\lambda)}, \]
we obviously obtain a regular solution of (2.5), (2.6), where \( b \equiv 0 \) and \( p \) is a non-negative continuous function for which (2.8) holds.

**Example 2.2.** In equation (2.5), let \( b : [a, \infty) \to \mathbb{R} \) and \( p : [a, \infty) \to [0, \infty) \) are locally bounded measurable functions such that (2.7) is valid with \( s > -1 \) and, moreover,

\[ p(r) \sim r^{s-1} \log^\nu r \quad \text{as} \quad r \to \infty. \quad (2.11) \]
In the case of \( \nu = 0 \), the last relation obviously takes the form (2.8) with the critical exponent \( l = s - 1 \). Also assume that \( \lambda < 1 \).

According to Theorem 2.2 if

\[ \nu \geq -1, \quad (2.12) \]
then any non-trivial solution of (2.5), (2.6) is singular of the first kind. In so doing, if \( \nu < -1 \), then there exists a real number \( \varepsilon > 0 \) such that

\[ w(r) = \log^{(\nu+1)/(1-\lambda)}(r/\varepsilon) \]
is a regular solution of (2.5), (2.6), where \( b(r) = -r^s \) and \( p \in C([a, \infty)) \) is a non-negative function satisfying relation (2.11). This demonstrates the exactness of (2.12).

Now, let (2.7) is valid with \( s \leq -1 \). We examine the critical exponent \( l = -2 \) in formula (2.8). Assume that

\[ p(r) \sim r^{-2} \log^\gamma r \quad \text{as} \quad r \to \infty. \quad (2.13) \]

By Theorem 2.2 if

\[ \gamma \geq -1, \]
then any non-trivial solution of (2.5), (2.6) is singular of the first kind. The above condition is exact. Really, in the case of \( \gamma < -1 \), it can be verified that

\[ w(r) = \log^{(\gamma+1)/(1-\lambda)}(r/\varepsilon) \]
is a regular solution of (2.5), (2.6) for some sufficiently small \( \varepsilon > 0 \), where \( b \equiv 0 \) and \( p \in C([a, \infty)) \) is a non-negative function for which (2.13) holds.

**Example 2.3.** Consider the equation

\[
 w'' + b(r)w' = p(r)w \ln^\lambda \left(1 + \frac{1}{w} \right), \quad r \geq a,
\]

(2.14)

where \( \lambda > 2 \) and, moreover, \( b : [a, \infty) \to \mathbb{R} \) and \( p : [a, \infty) \to [0, \infty) \) are locally bounded measurable functions such that (2.7) and (2.8) are fulfilled.

If \( s > -1 \), then in accordance with Theorem 2.2 inequality (2.9) guarantees that any non-trivial solution of (2.14), (2.6) is singular of the first kind. Assume that (2.9) does not hold or, in other words, \( l < s - 1 \). Then, putting

\[
 w(r) = 1 + r^{l-s+1},
\]

we obtain a regular solution of (2.14), (2.6), where \( b(r) = -r^s \) and \( p \) is a non-negative continuous function satisfying relation (2.8).

For \( s \leq -1 \), by Theorem 2.2 any non-trivial solution of (2.14), (2.6) is singular of the first kind if inequality (2.10) is valid. In turn, if (2.10) is not valid, then

\[
 w(r) = 1 + r^{l+2}
\]

is a regular solution of (2.14), (2.6), where \( b \equiv 0 \) and \( p \) is a non-negative continuous function for which (2.8) holds.

**Theorem 2.3.** In the hypotheses of Theorem 2.2, let the condition

\[
 \int_0^1 g^{-1/m}(t) t^{1/m-1} dt = \infty \quad (2.15)
\]

be fulfilled instead of (2.1). Then any solution of (1.1), (1.2) satisfies the estimate

\[
 w(r) \leq G_0^{-1} \left( C \left( \int_a^r \xi^{m-1} f(\xi) d\xi \right)^{1/m} \right) \quad (2.16)
\]

for all sufficiently large \( r \), where \( G_0^{-1} \) is the function inverse to

\[
 G_0(\xi) = \int_1^\xi g^{-1/m}(t) t^{1/m-1} dt,
\]

and the constant \( C > 0 \) depends only on \( m, \theta, \sigma, \) and on the value of the limit in the left-hand side of (2.4).

**Theorem 2.4.** Let (2.1) be valid,

\[
 \int_a^\infty \xi^{m-1} f(\xi) d\xi < \infty
\]

and, moreover,

\[
 \limsup_{r \to \infty} \frac{r^m f(r)}{\int_r^\infty \xi^{m-1} f(\xi) d\xi} < \infty. \quad (2.17)
\]

Then any regular solution of (1.1), (1.2) satisfies the estimate

\[
 w(r) \geq G_\infty^{-1} \left( C \left( \int_r^\infty \xi^{m-1} f(\xi) d\xi \right)^{1/m} \right) \quad (2.18)
\]
for all sufficiently large \( r \), where \( G^{-1}_\infty \) is the function inverse to
\[
G_\infty(\xi) = \int_0^\xi g^{-1/m}(t)^{1/m-1} dt,
\]
and the constant \( C > 0 \) depends only on \( m, \theta, \sigma \), and on the value of the limit in the left-hand side of (2.17).

Theorems 2.3 and 2.4 are proved in Section 3.

Example 2.4. Consider equation (2.5), where \( b : [a, \infty) \to \mathbb{R} \) and \( p : [a, \infty) \to [0, \infty) \) are locally bounded measurable functions such that (2.7) and (2.8) are valid.

At first, let \( s > -1 \). If \( \lambda > 1 \) and \( l > s - 1 \), then in accordance with Theorem 2.3 any solution of (2.5), (2.6) satisfies the estimate
\[
w(r) \leq C r^{(l-s+1)/(1-\lambda)}
\]
for all sufficiently large \( r \), where the constant \( C > 0 \) does not depend on \( w \). In the case that \( \lambda < 1 \) and \( l < s - 1 \), using Theorem 2.4, we obtain
\[
w(r) \geq C r^{(l-s+1)/(1-\lambda)}
\]
for any regular solution of (2.5), (2.6), where \( r \) runs through a neighborhood of infinity and \( C > 0 \) is a constant independent of \( w \).

Now, assume that \( s \leq -1 \). By Theorem 2.3 if \( \lambda > 1 \) and \( l > -2 \), then any solution of (2.5), (2.6) satisfies the estimate
\[
w(r) \leq C r^{(l+2)/(1-\lambda)}
\]
for all sufficiently large \( r \), where the constant \( C > 0 \) does not depend on \( w \). In turn, if \( \lambda < 1 \) and \( l < -2 \), then Theorem 2.4 implies the inequality
\[
w(r) \geq C r^{(l+2)/(1-\lambda)}
\]
for any regular solution of (2.5), (2.6), where \( r \) runs through a neighborhood of infinity and \( C > 0 \) is a constant independent of \( w \).

Example 2.5. In (2.5), let the locally bounded measurable functions \( b : [a, \infty) \to \mathbb{R} \) and \( p : [a, \infty) \to [0, \infty) \) satisfy relations (2.7) and (2.11), where \( s > -1 \). If \( \lambda > 1 \) and \( \nu > -1 \), then in accordance with Theorem 2.3 we have
\[
w(r) \leq C \log^{(\nu+1)/(1-\lambda)} r
\]
for any solution of (2.5), (2.6), where \( r \) runs through a neighborhood of infinity and \( C > 0 \) is a constant independent of \( w \). In the case that \( \lambda < 1 \) and \( \nu < -1 \), by Theorem 2.4 any regular solution of (2.5), (2.6) satisfies the estimate
\[
w(r) \geq C \log^{(\nu+1)/(1-\lambda)} r
\]
for all sufficiently large \( r \), where the constant \( C > 0 \) does not depend on \( w \).

Now, assume that \( s \leq -1 \) in (2.7) and, moreover, relation (2.13) is fulfilled instead of (2.11). By Theorem 2.3 if \( \lambda > 1 \) and \( \gamma > -1 \), then
\[
w(r) \leq C \log^{(\gamma+1)/(1-\lambda)} r
\]
for any solution of (2.5), (2.6), where \( r \) runs through a neighborhood of infinity and \( C > 0 \) is a constant independent of \( w \). In turn, if \( \lambda < 1 \) and \( \gamma < -1 \), then in accordance with Theorem 2.4 one can claim that
\[
w(r) \geq C \log^{(\gamma+1)/(1-\lambda)} r
\]
for any regular solution of (2.5), (2.6), where \( r \) runs through a neighborhood of infinity and \( C > 0 \) is a constant independent of \( w \).

It does not present any particular problem to show that all estimates given in Examples 2.4 and 2.5 are exact.

3. Proof of Theorems 2.1–2.4

Agree on the following notation. In this section, by \( c \) we denote various positive constants that can depend only on \( m, \theta, \) and \( \sigma \). For Lemma 3.7 and Theorem 2.3 these constants can also depend on the value of the limit in the left-hand side of (2.4). Analogously, in the case of Theorem 2.4, the constants \( c \) can depend on the value of the limit in the left-hand side of (2.17). We put

\[
\eta(t) = \inf_{(\theta^{-1/2}t, \theta^{1/2}t)} h, \quad t \in (0, \infty)
\]

and

\[
\varphi(\xi) = \xi^m \text{ess sup}_{(\xi/\sigma, \xi] \cap [a, \infty)} f, \quad \xi \in [a, \infty).
\]

**Lemma 3.1.** Let \( w \) be a solution of (1.1), (1.2) and, moreover, \( a \leq r_1 < r_2 \) be real numbers such that \( \sigma r_1 \geq r_2 \) and \( \theta^{1/2}w(r_2) \geq w(r_1) > 0 \). Then \( w(r_1) - w(r_2) \geq c \sup_{[w(r_2), w(r_1)]} \eta \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) d\xi. \)

**Proof.** Integrating by parts, we obtain

\[
w(r_1) - w(r_2) = \sum_{1 \leq k \leq i-1} \frac{(-1)^k w^{(k)}(r_2)(r_2 - r_1)^k}{k!} + \frac{1}{(i-1)!} \int_{r_1}^{r_2} (\xi - r_1)^{i-1} (-1)^i w^{(i)}(\xi) d\xi, \quad i = 1, \ldots, m.
\]

By (1.2), this implies the inequality

\[
w(r_1) - w(r_2) \geq \frac{1}{(i-1)!} \int_{r_1}^{r_2} (\xi - r_1)^{i-1} (-1)^i w^{(i)}(\xi) d\xi, \quad i = 1, \ldots, m. \tag{3.1}
\]

We denote

\[
\Omega_i = \left\{ \xi \in [r_1, r_2] : (-1)^i w^{(i)}(\xi) b_i(\xi) \geq \frac{1}{m} q(\xi) h(w) \right\}, \quad i = 1, \ldots, m - 1.
\]

In addition, let

\[
\Omega_m = \left\{ \xi \in [r_1, r_2] : (m-w^{(m)}(\xi) \geq \frac{1}{m} q(\xi) h(w) \right\}.
\]

From (1.1)–(1.3), it follows that

\[\text{mes } [r_1, r_2] \setminus \bigcup_{i=1}^{m} \Omega_i = 0.\]

For all \( i \in \{1, \ldots, m\} \) we have

\[
(-1)^i w^{(i)}(\xi) \geq \frac{1}{m} r^{m-i} f(\xi) h(w) \tag{3.2}
\]

almost everywhere on \( \Omega_i \). Hence, condition (1.2) allows us to assert that

\[
(-1)^i w^{(i)}(\xi) \geq \frac{1}{m} \chi_{\Omega_i}(\xi) \xi^{m-i} f(\xi) h(w)
\]
for almost all $\xi \in [r_1, r_2]$ and for all $i \in \{1, \ldots, m-1\}$, where $\chi_{\Omega_i}$ is the characteristic function of the set $\Omega_i$, i.e.

$$\chi_{\Omega_i}(\xi) = \begin{cases} 1, & \xi \in \Omega_i, \\ 0, & \xi \not\in \Omega_i. \end{cases}$$

Combining this with (3.1), we obtain

$$w(r_1) - w(r_2) \geq c \int_{\Omega_i} (\xi - r_1)^{m-1} f(\xi) h(w) \, d\xi, \quad i = 1, \ldots, m - 1. \quad (3.3)$$

Let us also establish the validity of the inequality

$$w(r_1) - w(r_2) \geq c \int_{\Omega_m} (\xi - r_1)^{m-1} f(\xi) h(w) \, d\xi. \quad (3.4)$$

Really, taking (3.1) into account, we have

$$w(r_1) - w(r_2) \geq \frac{1}{(m-1)!} \int_{\Omega^+} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi - \frac{1}{(m-1)!} \int_{\Omega^-} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi, \quad (3.5)$$

where $\Omega^+ = \{\xi \in [r_1, r_2] : (-1)^m w^{(m)}(\xi) \geq 0\}$ and $\Omega^- = [r_1, r_2] \setminus \Omega^+$. In the case that

$$\frac{1}{2} \int_{\Omega^+} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi \geq \int_{\Omega^-} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi,$$

formula (3.5) implies the estimate

$$w(r_1) - w(r_2) \geq \frac{1}{2(m-1)!} \int_{\Omega^+} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi$$

from which, taking into account (3.2) and the evident inclusion $\Omega_m \subset \Omega^+$, we immediately obtain (3.4).

Now, let

$$\frac{1}{2} \int_{\Omega^+} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi < \int_{\Omega^-} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi. \quad (3.6)$$

We put

$$\omega_i = \left\{ \xi \in \Omega_i : (-1)^i w^{(i)}(\xi) b_i(\xi) \geq \frac{1}{m-1} |w^{(m)}(\xi)| \right\}, \quad i = 1, \ldots, m - 1.$$

According to (1.1)-(1.3), for almost all $\xi \in \Omega^-$ there exists $i \in \{1, \ldots, m-1\}$ such that

$$(-1)^i w^{(i)}(\xi) b_i(\xi) \geq \frac{|w^{(m)}(\xi)| + q(\xi) h(w)}{m-1}.$$

Consequently,

$$\text{mes} \Omega^- \setminus \bigcup_{i=1}^{m-1} \omega_i = 0.$$

For all $i \in \{1, \ldots, m-1\}$ we obviously have

$$(-1)^i w^{(i)}(\xi) b_i(\xi) \geq \frac{\xi^{m-i} |w^{(m)}(\xi)|}{(m-1)(1 + \xi^{m-i} b_i(\xi))}.$$
almost everywhere on \( \omega_i \). Combining the last inequality with (3.1), one can conclude that
\[
w(r_1) - w(r_2) \geq \frac{c}{1 + r_2^{m-i} \text{ess sup } b_i} \int_{\omega_i} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi, \quad i = 1, \ldots, m - 1.
\]
This implies the estimate
\[
w(r_1) - w(r_2) \geq \frac{c}{1 + \sum_{i=1}^{m-1} r_2^{m-i} \text{ess sup } b_i} \int_{\Omega} (\xi - r_1)^{m-1} |w^{(m)}(\xi)| \, d\xi
\]
from which, using (3.6), we obtain
\[
w(r_1) - w(r_2) \geq \frac{c}{1 + \sum_{i=1}^{m-1} r_2^{m-i} \text{ess sup } b_i} \int_{\Omega} (\xi - r_1)^{m-1} q(\xi) h(w) \, d\xi,
\]
whence (3.4) follows at once.

Inequalities (3.3) and (3.4) enable us to assert that
\[
w(r_1) - w(r_2) \geq c \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) h(w) \, d\xi.
\]
Since
\[
\inf_{\xi \in (r_1, r_2)} h(w(\xi)) \geq \sup_{[w(r_2), w(r_1)]} \eta,
\]
this completes the proof. \( \square \)

**Lemma 3.2.** Let \( r_1 < r_2 \) and \( 0 < \alpha < 1 \) be some real numbers, then
\[
\left( \int_{r_1}^{r_2} \psi(\xi) \, d\xi \right)^{\alpha} \geq A \int_{r_1}^{r_2} \psi(\xi) \zeta^{\alpha-1}(\xi) \, d\xi
\]
for any non-negative function \( \psi \in L([r_1, r_2]) \), where
\[
\zeta(\xi) = \int_{r_1}^{\xi} \psi(\zeta) \, d\zeta
\]
and \( A > 0 \) is a constant depending only on \( \alpha \).

**Remark 3.1.** If \( \zeta(\xi) = 0 \) for some \( \xi \in (r_1, r_2) \), then \( \psi = 0 \) almost everywhere on the interval \((r_1, \xi)\). In this case, we assume by definition that \( \psi(\xi) \zeta^{\alpha-1}(\xi) = 0 \).

**Lemma 3.2** is proved in [9, Lemma 2.1].

**Lemma 3.3.** Let \( w \) be a solution of (1.1), (1.2) and, moreover, \( a \leq r_1 < r_2 \) be real numbers such that \( \sigma r_1 \geq r_2, w(r_1) > 0, \) and \( b^{1/2} w(r_2) \geq w(r_1) \geq b^{1/4} w(r_2) \). Then
\[
\int_{w(r_2)}^{w(r_1)} \eta^{-1/m}(t) t^{1/m-1} \, dt \geq c \int_{r_1}^{r_2} \zeta^{m-1} f(\xi) \varphi^{1/m-1}(\xi) \, d\xi. \tag{3.7}
\]

**Proof.** From Lemma 3.1 it follows that
\[
w^{1/m}(r_1) \left( \inf_{[w(r_2), w(r_1)]} \eta^{-1/m} \right) \geq c \left( \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) \, d\xi \right)^{1/m}.
\]
In so doing, by Lemma 3.2 we have
\[
\left( \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) \, d\xi \right)^{1/m} \geq c \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) \chi_{1/m}^{1/(m-1)}(\xi) \, d\xi,
\]
where
\[
\chi(\xi) = \int_{r_1}^{\xi} (\zeta - r_1)^{m-1} f(\zeta) \, d\zeta.
\]
Therefore,
\[
w^{1/m}(r_1) \inf_{w(r_2) \geq w(r_1)} \eta^{-1/m} \geq c \int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) \chi_{1/m}^{1/(m-1)}(\xi) \, d\xi. \tag{3.8}
\]
Let us estimate the right-hand side of the last inequality. Since
\[
\frac{1}{(\xi - r_1)^m} \int_{r_1}^{\xi} (\xi - r_1)^{m-1} f(\zeta) \, d\zeta \leq \text{ess sup} f \leq \xi^{-m} \varphi(\xi)
\]
for all \( \xi \in (r_1, r_2) \), we obtain
\[
(\xi - r_1)^{m-1} \chi_{1/m}^{1/(m-1)}(\xi) = \left( \frac{1}{(\xi - r_1)^m} \int_{r_1}^{\xi} (\xi - r_1)^{m-1} f(\zeta) \, d\zeta \right)^{1/m-1} \geq \xi^{m-1} \varphi_{1/m}^{1/m-1}
\]
for all \( \xi \in (r_1, r_2) \). Hence, one can claim that
\[
\int_{r_1}^{r_2} (\xi - r_1)^{m-1} f(\xi) \chi_{1/m}^{1/(m-1)}(\xi) \, d\xi \geq \int_{r_1}^{r_2} \xi^{m-1} f(\xi) \varphi_{1/m}^{1/m-1} \, d\xi.
\]
At the same time, for the left-hand side of (3.8) we have
\[
\int_{w(r_1)}^{w(r_2)} \eta^{-1/m}(t) t^{1/m-1} \, dt \geq c w^{1/m}(r_1) \inf_{w(r_2) \geq w(r_1)} \eta^{-1/m}.
\]
Thus, inequality (3.8) implies (3.7).

The proof is completed. \( \square \)

**Lemma 3.4.** Let \( u : (0, \infty) \to (0, \infty) \) be a continuous function and, moreover,
\[
v(t) = \inf_{(t, \lambda, \mu)} u, \quad t \in (0, \infty),
\]
where \( \lambda > 1 \) is some real number. Then
\[
\left( \int_{t_1}^{t_2} v^{-1/m}(t) t^{1/m-1} \, dt \right)^m \geq A \int_{t_1}^{t_2} \frac{dt}{u(t)}
\]
for all non-negative real numbers \( t_1 \) and \( t_2 \) such that \( t_2 \geq \lambda t_1 \), where the constant \( A > 0 \) depends only on \( m \) and \( \lambda \).

Lemma 3.4 is proved in [9, Lemma 2.3].

**Proof of Theorem 2.1.** Assume to the contrary that \( w \) is a regular solution of problem (1.1), (1.2). In particular, \( w \) is a positive function on the whole interval \([a, \infty)\). This simple fact follows immediately from (1.2).

Construct a sequence of real numbers \( \{r_i\}_{i=0}^{\infty} \). We take \( r_0 = a \). Assume further that \( r_i \) is already known. If \( \theta_{1/4} w(\sigma_{1/2} r_i) \geq w(r_i) \), then we put \( r_{i+1} = \sigma_{1/2} r_i \); otherwise we take \( r_{i+1} \in (r_i, \sigma_{1/2} r_i) \) such that \( \theta_{1/4} w(r_{i+1}) = w(r_i) \).
Let $\Xi_1$ be the set of positive integers $i$ satisfying the condition $\sigma^{1/2}r_{i-1} > r_i$ and $\Xi_2$ be the set of all other positive integers. By Lemma 3.3, the inequality
\[
\int_{w(r_{i-1})}^{w(r_{i+1})} \eta^{-1/m}(t)t^{1/m-1} \, dt \geq c \int_{r_{i-1}}^{r_{i+1}} \xi^{m-1} f(\xi) \varphi^{1/m-1}(\xi) \, d\xi
\] (3.9)
is fulfilled for all $i \in \Xi_1$. In turn, if $i \in \Xi_2$, then
\[
w(r_{i-1}) - w(r_{i+1}) \geq c \sup_{w(r_{i+1}), w(r_{i-1})} \eta \int_{r_{i-1}}^{r_{i+1}} (\xi - r_{i-1})^{m-1} f(\xi) \, d\xi
\] according to Lemma 3.1. Combining this with the evident inequalities
\[
\int_{w(r_{i-1})}^{w(r_{i+1})} \frac{dt}{\eta(t)} \geq (w(r_{i-1}) - w(r_{i+1})) \frac{1}{\inf_{[w(r_{i+1}), w(r_{i-1})]} \eta}
\]
and
\[
\int_{r_{i-1}}^{r_{i+1}} (\xi - r_{i-1})^{m-1} f(\xi) \, d\xi \geq c \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi,
\]
we conclude that
\[
\int_{w(r_{i-1})}^{w(r_{i+1})} \frac{dt}{\eta(t)} \geq c \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi. \tag{3.10}
\]
By (2.2), at least one of the following two relations is valid:
\[
\sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \mu^{1/m-1}(\xi) \, d\xi = \infty, \tag{3.11}
\]
\[
\sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \mu^{1/m-1}(\xi) \, d\xi = \infty. \tag{3.12}
\]
In the case that (3.11) holds, summing (3.9) over all $i \in \Xi_1$, we obtain
\[
\int_{0}^{w(a)} \eta^{-1/m}(t)t^{1/m-1} \, dt = \infty.
\]
This contradicts condition (2.1).

Assume that (3.12) is fulfilled. In this case, summing (3.10) over all $i \in \Xi_2$, we have
\[
\int_{0}^{w(a)} \frac{dt}{\eta(t)} = \infty. \tag{3.13}
\]
At the same time, from Lemma 3.1, it follows that
\[
\left( \int_{0}^{1} g^{-1/m}(t)t^{1/m-1} \, dt \right)^m \geq c \int_{0}^{1} \frac{dt}{\eta(t)}.
\]
The last inequality and (3.13) imply (2.11). Thus, we again arrive at a contradiction with (2.11).

The proof is completed. \hfill \Box

**Lemma 3.5.** Let (2.4) hold and, moreover, $\lambda > 1$ be some real number. Then
\[
\int_{0}^{r} \xi^{m-1} f(\xi) \, d\xi \leq A \int_{0}^{r} \xi^{m-1} f(\xi) \, d\xi
\]
for all sufficiently large $r$, where the constant $A > 0$ depends only on $\lambda$ and on the value of the limit in the left-hand side of (2.4).
Lemma 3.6. In the hypotheses of Lemma 3.5, let \(2.17\) be fulfilled instead of \(2.4\), then
\[
\int_{r}^{\infty} \xi^{m-1} f(\xi) \, d\xi \leq A \int_{r}^{\infty} \xi^{m-1} f(\xi) \, d\xi
\]
for all sufficiently large \(r\), where the constant \(A > 0\) depends only on \(\lambda\) and on the value of the limit in the left-hand side of \(2.17\).

Lemmas 3.5 and 3.6 are proved in [9, Lemmas 2.7 and 2.8].

Lemma 3.7. Let \(w\) be a regular solution of \((1.1)\), \((1.2)\). If relations \((2.3)\) and \((2.4)\) are valid, then
\[
\int_{w(a)}^{w(r)} g^{-1/m}(t) t^{1/m-1} \, dt \geq c \left( \int_{r}^{a} \xi^{m-1} f(\xi) \, d\xi \right)^{1/m}
\]
for all sufficiently large \(r\).

Proof. Consider the sequence of real numbers \(\{r_i\}_{i=0}^{\infty}\) and the sets \(\Xi_1\) and \(\Xi_2\) constructed in the proof of Theorem 2.1. It can be seen that
\[
\lim_{n \to \infty} w(r_n) = 0.
\]

(3.15)

In fact, if
\[
\lim_{n \to \infty} w(r_n) = w(\infty) > 0,
\]
then \(i \in \Xi_2\) for all sufficiently large \(i\). Summing \((3.10)\) over all \(i \in \Xi_2\), we obtain
\[
\int_{w(\infty)}^{w(a)} \frac{dt}{\eta(t)} \geq c \sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi.
\]

From \((2.3)\), it follows that the right-hand side of the last expression is equal to infinity, whereas the left-hand side is bounded. This contradiction proves \((3.15)\).

According to \((2.4)\), there exists \(j \geq 1\) such that
\[
\varphi(\xi) = \xi^{m-1} \text{ ess sup}_{(\xi/\sigma, \xi]} f(\xi) \leq c \int_{a}^{\xi} \xi^{m-1} f(\xi) \, d\xi
\]
for all \(\xi \geq r_j\).

We denote \(\Xi_{1,n} = \{1 \leq i \leq n : i \in \Xi_1\}\) and \(\Xi_{2,n} = \{1 \leq i \leq n : i \in \Xi_2\}\), \(n = 1, 2, \ldots\). Also take an integer \(l \geq 1\) such that \(\theta^{1/2} w(r_{n+1}) \leq w(a)\) and
\[
\frac{1}{4} \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi \geq \int_{a}^{r_{j+1}} \xi^{m-1} f(\xi) \, d\xi
\]
for all \(n \geq l\). In view of \((3.15)\) and \((2.3)\), such an integer \(l\) obviously exists.

At first, let
\[
\sum_{i \in \Xi_{1,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi \geq \frac{1}{2} \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi
\]
for some \(n \geq l\). Summing \((3.9)\) over all \(i \in \Xi_{1,n}\), we obtain
\[
\int_{w(r_{n+1})}^{w(a)} \eta^{-1/m}(t) t^{1/m-1} \, dt \geq c \sum_{i \in \Xi_{1,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \varphi^{1/m-1}(\xi) \, d\xi.
\]
By (3.16), this implies the inequality
\[
\int_{w(r_{n+1})}^{w(a)} \eta^{-1/m}(t) t^{1/m-1} \, dt \geq c \left( \int_a^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi \right)^{1/m}.
\]

It is easy to see that
\[
\sum_{i \in \Xi_{1,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi \leq \int_{a}^{r_{j+1}} \xi^{m-1} f(\xi) \, d\xi;
\]
therefore, taking into account (3.17) and (3.18), we have
\[
\sum_{i \in \Xi_{1,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi = \sum_{i \in \Xi_{1,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi - \sum_{i \in \Xi_{1,j}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi
\geq \frac{1}{4} \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi.
\]

The last relation and (3.19) imply that
\[
\int_{w(r_{n+1})}^{w(a)} \eta^{-1/m}(t) t^{1/m-1} \, dt \geq c \left( \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi \right)^{1/m}.
\]

Since \( g(t) \leq \eta(t) \) for all \( t \in (0, \infty) \), this yields the estimate
\[
\int_{w(r_{n+1})}^{w(a)} g^{-1/m}(t) t^{1/m-1} \, dt \geq c \left( \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi \right)^{1/m}.
\]

Now, assume that
\[
\sum_{i \in \Xi_{2,n}} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) \, d\xi \geq \frac{1}{2} \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi
\]
for some \( n \geq l \). Summing (3.10) over all \( i \in \Xi_{2,n} \), we obtain
\[
\int_{w(r_{n+1})}^{w(a)} \frac{dt}{\eta(t)} \geq c \int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi.
\]

Combining this with the inequality
\[
\left( \int_{w(r_{n+1})}^{w(a)} g^{-1/m}(t) t^{1/m-1} \, dt \right)^m \geq c \int_{w(r_{n+1})}^{w(a)} \frac{dt}{\eta(t)}
\]
which follows from Lemma 3.4, we arrive at (3.20) once more.

For any \( r \geq r_{l+1} \) there exists \( n \geq l \) such that \( r_{n+1} \leq r \leq r_{n+2} \). From Lemma 3.5 we have
\[
\int_{a}^{r_{n+1}} \xi^{m-1} f(\xi) \, d\xi \geq c \int_{a}^{r} \xi^{m-1} f(\xi) \, d\xi.
\]

It can also be seen that
\[
\int_{w(r)}^{w(a)} g^{-1/m}(t) t^{1/m-1} \, dt \geq \int_{w(r_{n+1})}^{w(a)} g^{-1/m}(t) t^{1/m-1} \, dt.
\]

Combining the last two estimates with (3.20), we obtain (3.14).

The proof is completed.
Proof of Theorem 2.2. If \( w \) is a regular solution of (1.1), (1.2), then in accordance with Lemma 3.7, inequality (3.14) is fulfilled for all \( r \) in a neighborhood of infinity. Passing in this inequality to the limit as \( r \to \infty \), we arrive at the contradiction with (2.1) and (2.3).

The proof is completed. \( \square \)

Proof of Theorem 2.3. If \( w \) vanishes in a neighborhood of infinity, then (2.16) is obvious. In turn, if \( w \) is a regular solution of (1.1), (1.2), then condition (2.3) and estimate (3.14) of Lemma 3.7 imply the relation

\[
\lim_{r \to \infty} w(r) \to 0.
\]

Hence, (2.15) allows one to claim that

\[
\int_{w(r)}^{1} g^{-1/m}(t) t^{1/m-1} dt \geq \frac{1}{2} \int_{w(r)}^{w(a)} g^{-1/m}(t) t^{1/m-1} dt
\]

for all sufficiently large \( r \). Combining this with (3.14), we have

\[
\int_{w(r)}^{1} g^{-1/m}(t) t^{1/m-1} dt \geq c \left( \int_{a}^{r} \xi^{m-1} f(\xi) d\xi \right)^{1/m}
\]

for all sufficiently large \( r \). To complete the proof, it remains to note that the last expression is equivalent to (2.16). \( \square \)

Proof of Theorem 2.4. As mentioned above, every regular solution of (1.1), (1.2) is a positive function on the whole interval \([a, \infty)\). This is obvious in view of (1.2). By Lemma 3.6 and relation (2.17), there exists a real number \( \rho \geq a \) such that

\[
\int_{\xi}^{\infty} \zeta^{-m_{1}} f(\xi) d\zeta \leq c \int_{\xi}^{\infty} \zeta^{-m_{1}} f(\xi) d\zeta
\]

and

\[
\varphi(\xi) = \zeta^{-m_{1}} \sup_{(\xi, \zeta) \in [a, \infty)} f \leq c \int_{\xi}^{\infty} \zeta^{-m_{1}} f(\xi) d\zeta
\]

for all \( \xi \geq \rho \). Let \( r \in [\rho, \infty) \). Consider a sequence of real numbers \( \{r_{i}\}_{i=0}^{\infty} \) defined as follows. We take \( r_{0} = r \). Assume that \( r_{i} \) is already known. If \( \theta^{1/4} w(\sigma^{1/2} r_{i}) \geq w(r_{i}) \), then we put \( r_{i+1} = \sigma^{1/2} r_{i} \); otherwise we take \( r_{i+1} \in (r_{i}, \sigma^{1/2} r_{i}) \) such that \( \theta^{1/4} w(r_{i+1}) = w(r_{i}) \).

As in the proof of Theorem 2.1 by \( \Xi_{1} \) we mean the set of positive integers \( i \) satisfying the condition \( \sigma^{1/2} r_{i-1} > r_{i} \). Also let \( \Xi_{2} \) be the set of all other positive integers. Thus, inequality (3.9) is valid for all \( i \in \Xi_{1} \), whereas (3.10) holds for all \( i \in \Xi_{2} \). Summing (3.9) over all \( i \in \Xi_{1} \), we obtain

\[
\int_{0}^{w(r)} \eta^{-1/m}(t) t^{1/m-1} dt \geq c \sum_{i \in \Xi_{1}} \int_{r_{i}}^{r_{i+1}} \zeta^{m_{1}} f(\xi) \varphi^{1/m_{1}}(\xi) d\xi.
\]

(3.23)

Analogously, from (3.10), it follows that

\[
\int_{0}^{w(r)} \frac{dt}{\eta(t)} \geq c \sum_{i \in \Xi_{2}} \int_{r_{i}}^{r_{i+1}} \zeta^{m_{1}} f(\xi) d\xi.
\]

(3.24)

If

\[
\sum_{i \in \Xi_{1}} \int_{r_{i}}^{r_{i+1}} \zeta^{m_{1}} f(\xi) d\xi \geq \frac{1}{2} \int_{r_{1}}^{\infty} \zeta^{m_{1}} f(\xi) d\xi,
\]

(3.25)
then in accordance with (3.22) and (3.23) we have
\[\int_0^{w(r)} \eta^{-1/m}(t)t^{1/m-1} dt \geq c \left( \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi \right)^{1/m} \sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) d\xi \]
\[\geq c \left( \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi \right)^{1/m}.\]

This implies the estimate
\[\int_0^{w(r)} g^{-1/m}(t)t^{1/m-1} dt \geq c \left( \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi \right)^{1/m}.\] (3.26)

In turn, if (3.25) is not valid, then
\[\sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} \xi^{m-1} f(\xi) d\xi \geq \frac{1}{2} \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi.\]

Therefore, (3.24) allows one to claim that
\[\int_0^{w(r)} \frac{dt}{\eta(t)} \geq c \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi.\]

Since
\[\left( \int_0^{w(r)} g^{-1/m}(t)t^{1/m-1} dt \right)^m \geq c \int_0^{w(r)} \frac{dt}{\eta(t)},\]

according to Lemma 3.4, we again obtain (3.26).

Finally, by relation (3.21), estimate (3.26) implies that
\[\int_0^{w(r)} g^{-1/m}(t)t^{1/m-1} dt \geq c \left( \int_{r_1}^{\infty} \xi^{m-1} f(\xi) d\xi \right)^{1/m},\]

whence (2.18) follows immediately.

The proof is completed. \(\square\)

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