Quasilocal energy and naked black holes

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Abstract.
We extend the Brown and York notion of quasilocal energy to include coupled electromagnetic and dilaton fields and also allow for spatial boundaries that are not orthogonal to the foliation of the spacetime. We investigate how the quasilocal quantities measured by sets of observers transform with respect to boosts. As a natural application of this work we investigate the naked black holes of Horowitz and Ross calculating the quasilocal energies measured by static versus infalling observers.

I INTRODUCTION

The definition of energy in general relativity continues to be an area of active research. It is widely accepted that while one cannot localize gravitational energy and therefore define a gravitational stress energy tensor one can define a notion of the total energy in a spacetime (for example the ADM or Bondi masses). In between these two extremes one can define energy quasilocally – that is define the amount of energy contained in a finite volume of spacetime.

One popular definition of quasilocal energy (QLE) was proposed by Brown and York in 1993 [1]. As we shall see in the next section their approach derives a Hamiltonian from the standard Einstein-Hilbert action and a notion of quasilocal energy from that Hamiltonian. Since its proposal this quasilocal energy has found application in gravitational thermodynamics and the study of the production of pairs of black holes. It has been shown to reduce to the ADM and Bondi masses in the appropriate limits as well as a Newtonian notion of gravitational energy for certain specific examples. References for these may be found in [2].

In this paper, we extend this notion of quasilocal energy to include coupled electromagnetic and dilaton fields and also allow for spatial boundaries that are not orthogonal to the foliation of the spacetime. Using the second generalization we can calculate the quasilocal energies measured by sets of observers who are moving around in a spacetime. We see that the QLE transforms in a Lorentzian way under boosts of the observers. In the last section we find a natural application for this work in the naked black holes first studied by Horowitz and Ross [3]. Such black holes have small curvature constants yet observers falling into them experience massive tidal forces as they approach the event horizon. We calculate the energies measured by these observers.

II QUASILOCAL ENERGY

In classical mechanics the action $I$ of a point particle is the time integral of its kinetic energy minus its potential energy. To wit,

$$I = \int dt (\dot{p} \dot{q} - H)$$

where $p$ is the particle momentum, $q$ its position, and $H$ its Hamiltonian/potential energy. Taking the first variation of this action we obtain the equations of motion of the particle (plus certain boundary conditions that must be satisfied so that the variation will vanish).

Now the action for gravity is well known so it is natural to extend and reverse this procedure to define a Hamiltonian for gravity. We follow the procedure of Brown and York [1]. Given a region of spacetime $M$ (figure 1) the role of time is played by a foliation $\Sigma_t$ of the region and an accompanying timelike vector field $T^\alpha$ such that $T^\alpha \partial_\alpha t = 1$. With respect to the foliation we can write this vector field in terms of a lapse function $N$ and spacelike shift vector field $V^\alpha$. Namely, $T^\alpha = Nu^\alpha + V^\alpha$ where $u^\alpha$ is the unit normal to $\Sigma_t$. The region is finite and bounded by a timelike three surface $B$ with unit normal $n^\alpha$ and two spacelike surfaces $\Sigma_1$ and $\Sigma_2$. $\Sigma_t$ induces a foliation $\Omega_t$ of $B$ which...
FIGURE 1. A three dimensional schematic of the region $M$, its assorted normal vector fields, and a typical element of the foliation.

has normals $\tilde{u}^{\alpha}$ and $\tilde{n}^{\alpha}$ when viewed as embedded in $B$ and $\Sigma_t$ respectively. Brown and York assumed $u^{\alpha} = \tilde{u}^{\alpha}$ and $n^{\alpha} = \tilde{n}^{\alpha}$ (equivalently $\eta = n^{\alpha}u_{\alpha} = 0$). We drop that assumption here and allow for nonorthogonal foliations of $M$ (as we did in [4]).

Define $K_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ as the extrinsic curvatures of the surfaces $\Sigma_t$ and $B$ in $M$ respectively. $k_{\alpha\beta}$ is the extrinsic curvature of $\Omega_t$ in $\Sigma_t$. $g_{\alpha\beta}$, $\gamma_{\alpha\beta}$, $h_{\alpha\beta}$, and $\sigma_{\alpha\beta}$ are the metrics on $M$, $B$, $\Sigma_t$, and $\Omega_t$ respectively. We assume that the boundary $B$ is generated by Lie dragging an $\Omega_t$ along $T^\alpha$. Physically we view $B$ as being the history of a set of observers evolving according to the vector field $T^\alpha$.

Allowing electromagnetic $F_{\alpha\beta}$ and dilaton $\phi$ fields (whose coupling if governed by the constant $a$) the regular Hilbert action for general relativity over the region $M$ is

$$I = \frac{1}{2\kappa} \int_M d^4\sqrt{-g} \left( R - 2\Lambda - 2(\nabla_\alpha \phi)(\nabla^\alpha \phi) - \epsilon^{2\alpha\beta} F_{\alpha\beta} F_{\alpha\beta} \right)$$

$$+ 1 \int_\Sigma d^3\sqrt{h} K - \frac{1}{\kappa} \int_B d^3x \sqrt{-\gamma} \Theta + \frac{1}{\kappa} \int_\Omega d^2x \sqrt{-\sigma} \sinh^{-1}(\eta) - L,$$

$\kappa = 8\pi$ in units where $G = c = 1$. The boundary terms ensure that if we keep metrics and certain matter terms fixed on the boundaries then the variational principle is properly defined. The $I$ term is any functional defined with respect to the boundary metric $\gamma_{\alpha\beta}$. Since that metric is kept constant during the variation, $\delta I = 0$ and so doesn’t affect the equations of motion. This term is called the reference term and as we shall see a bit later, its form determines the zero of the action.

We break up this action with respect to the foliation to obtain

$$I = \int dt \int_{\Sigma_t} d^3x \left\{ P_{h}^{\alpha\beta} \mathcal{L}_{T} h_{\alpha\beta} + P_{\phi} \mathcal{L}_{T} \phi + P_{A}^{\alpha} \mathcal{L}_{T} (\tilde{A}_\alpha) \right\} + \int dt \int_{\Omega_t} d^2x \left\{ P_{\sqrt{\sigma}} \mathcal{L}_{T} \sqrt{\sigma} \right\}$$

$$- \int dt \int_{\Sigma_t} d^3x \left\{ N \mathcal{H}^m + V^\alpha \mathcal{H}^m_\alpha + (T^\alpha A_\alpha) \mathcal{Q} \right\} - \int dt \int_{\Omega_t} d^2x \sqrt{\sigma} \left\{ \tilde{N}(\tilde{e}^m + \tilde{e}^m) - \hat{V}^\alpha (\tilde{j}_\alpha + \tilde{j}_\alpha) \right\} - L.$$
The $P$'s are momentum terms, $\mathcal{L}_T$ is the Lie derivative in the direction $T$. $\mathcal{H}^m$ and $\mathcal{H}_a^m$ are the regular gravitational Hamiltonian constraint equations and $Q$ is an electromagnetic constraint. $A_a$ is the electromagnetic vector potential, $\tilde{A}_a$ is the projection of that potential into $\Sigma_t$, and $\tilde{N}$ and $\tilde{V}^\alpha$ are the lapse and shift for the foliation of the boundary $B$. Then by analogy with the point particle action, we can define a Hamiltonian

$$
\mathcal{H}^m = \int_{\Omega_t} d^2 x \sqrt{\sigma} \left\{ \tilde{N}(\tilde{\varepsilon} + \tilde{\varepsilon}^m) - \tilde{V}^\alpha (j^\alpha + \tilde{j}^\alpha_m) \right\} - H,
$$

where we have assumed that the constraint equations are satisfied. The $\tilde{\varepsilon}$ terms are QLE densities and the $j^\alpha$ terms are angular momentum densities. $H$ is the reference term calculated from $I$. We define the quasilocal energy to be the Hamiltonian for observers measuring proper time ($\tilde{N} = 1$) who are at rest with respect to the leaf $\Omega_t$ ($\tilde{V}^\alpha = 0$).

**A Calculating the quasilocal energy**

We now consider the exact form of the quasilocal energy densities.

$$
\tilde{\varepsilon} = \frac{1}{8\pi} \tilde{k} = -\frac{1}{8\pi} \sigma^{\alpha\beta} \nabla_\alpha n_\beta = -\frac{1}{16\pi} \sigma^{\alpha\beta} \mathcal{L}_n \sigma_{\alpha\beta}
$$

and so is the extrinsic curvature of $\Omega_t$ with respect to a surface (locally) defined by the tangent vectors of $\Omega_t$ and the normal vector $n^\alpha$. Geometrically it measures how the area of $\Omega_t$ changes in the direction $n^\alpha$.

For asymptotically flat spacetimes we define the reference term $\mathcal{E}$ so that $E$ will vanish for flat space $\mathcal{M}$. The simplest way to do this is to (locally) embed $B$ in $\mathcal{M}$ and then define $\tilde{\varepsilon}$ in the same way as $\varepsilon$. Specifically we embed the two surface $\Omega_t$ and then define a vector field $T^\alpha$ over $\Omega_t$ such that $T^\alpha T_\alpha = T^\alpha T_\alpha$, $\nabla_\alpha \sigma_{\alpha\beta} = T^\alpha \sigma_{\alpha\beta}$, and $T^\alpha \sigma_{\alpha} = T^\alpha \sigma_{\alpha}$. Then we define a reference quasilocal energy density as

$$
\tilde{\varepsilon} = \frac{1}{8\pi} \tilde{k} = -\frac{1}{8\pi} \sigma^{\alpha\beta} \nabla_\alpha n_\beta = -\frac{1}{16\pi} \sigma^{\alpha\beta} \mathcal{L}_n \sigma_{\alpha\beta}.
$$

Finally, the matter term is

$$
\tilde{\varepsilon}^m = -\frac{1}{4\pi} (n_\alpha \tilde{E}^\alpha)(u^\beta A_\beta),
$$

where $\tilde{E}^\alpha \equiv e^{-2\phi} F_{\alpha\beta} \tilde{u}^\beta$ is the electric field. $\tilde{\varepsilon}^m$, may be thought of as a charge times a Coulomb potential. Roughly it represents the potential energy of the region $\mathcal{M}$ with respect to the EM potential $A_\alpha$. Note that it is gauge dependent. Three natural gauge choices set $\tilde{\varepsilon}^m = 0$ at infinity, the quasilocal surface $\Omega_t$, or the black hole horizon.

In the following we consider two quasilocal energies. The first is $E_{tot} - E$ with the gauge chosen so that $\tilde{\varepsilon}^m = 0$ on the black hole horizon, and the second is the geometrical energy $E_{Geo} - E$ where we have chosen the gauge so $\tilde{\varepsilon}^m = 0$ on $\Omega_t$. Then,

$$
E_{Geo} = \int_{\Omega_t} d^2 x \sqrt{\sigma} \tilde{\varepsilon}, \quad E_{tot} = \int_{\Omega_t} d^2 x \sqrt{\sigma} (\tilde{\varepsilon} + \tilde{\varepsilon}^m), \quad E_t = \int_{\Omega_t} d^2 x \sqrt{\sigma} \tilde{E}^m.
$$

**B Transformation laws for the boosted QLE**

Next we investigate how the QLE transforms with respect to motion of the observers. Consider two sets of observers who instantaneously coincide on a surface $\Omega_t$. Here we take them as a “static” set being evolved by the timelike unit vector $T^\alpha = u^\alpha$ with normal vector $n^\alpha$ from $\Omega_t$ and a “moving” set being evolved by $T^\alpha = u^\alpha$. Then the moving set are seen to have velocity $v = -\frac{T^\alpha}{\nabla_\alpha n_\alpha}$ in the direction $n^\alpha$ by the static set. Defining $\gamma = (1 - v^2)^{-1/2}$, it is not hard to show that

$$
\varepsilon^* = \gamma (\varepsilon + v j^\gamma) \quad \text{and} \quad \varepsilon^{*m} = \gamma (\varepsilon + v j^{m*}).
$$

$j^\gamma = -\frac{1}{16\pi} \sigma^{\alpha\beta} \mathcal{L}_u \sigma_{\alpha\beta}$ and so represents the (local) rate of change of the area of $\Omega_t$ as measured by the observers with respect to proper time. It can be thought of as a momentum flow through the surface. $j^{m*}$ is a matter term that can be set to zero by an appropriate gauge choice (which we’ll make here for simplicity). Thus, if the $j^\gamma$ terms are zero, the transformation laws are very similar to those for energy in special relativity.
Things are slightly complicated because the reference terms transform with respect to a different velocity. Looking back at the defining conditions for the reference term we see that by construction $j_\perp = j_\perp$. Then, it is not surprising that the surface of observers would have to travel at a different speed in the reference spacetime than they do in the original one to keep this rate of change the same. Thus,

$$\varepsilon^\tau = \frac{\gamma(\xi + vj_\perp)}{}$$

where $\gamma$ is defined in an analogous way to $v$.

With these transformation laws it is also easy to see that $\varepsilon^* - j^*\varepsilon$ is a constant, independent of the boost. This is analogous to the special relativity relation $E^2 - p^2c^2 = m^2$ (which is a constant) and will be of use in the later calculations.

### III NAKED BLACK HOLES

Naked black holes are a subclass of the low-energy-limit string theory solutions with metric

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + R(r)^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $F(r) = (r - r_\ast)/(r - r_\perp)$ and $R(r) = r (1 - r_\perp)^{a^2/(1 + a^2)}$. $r_\ast$ is the location of the black hole horizon and (for the coupling constant $a \neq 0$) $r_\perp$ gives the position of the singularity behind the horizon. There are also dilaton and Maxwell fields defined by

$$\phi = -\frac{a}{1 + a^2} \ln \left(1 - \frac{r_\perp}{r}\right) \quad \text{and} \quad F = G_0 \sin \theta d\theta \wedge d\phi.$$  

The (ADM) mass and magnetic charge of these solutions are given by $M = \frac{r_\ast}{2} + \frac{1 - a^2}{1 + a^2} r_\perp$ and $G_0 = \left(\frac{r_\ast - r_\perp}{1 + a^2}\right)^{1/2}$. If $a = 0$ this solution reduces to a magnetically charged Reissner-Nordstöm (RN) black hole. For the purposes of this short paper we shall also assume that if $a \neq 0$ then $a \approx 1$. Very small values of $a$ cause complications; we deal with these elsewhere [2].

If $R_+ = R(r_+) \gg 1$ (that is $R_+$ is much larger than the Planck length) then these black holes have a very large surface area and a correspondingly large mass (in the corresponding Planck units). All of the curvature invariants are small outside of the horizon, and static ($r = \text{constant}$, $t = \text{constant}$) observers measure very small curvature components. Members of a spherical set of these observers will naturally carry an orthonormal tetrad $\{u^a, \theta^a, n^a, \hat{\phi}^a\}$, where with $N = 1$ and $V^\alpha = 0$, $T^\alpha = u^\alpha$ defines the evolution of the observers, $n^\alpha$ is the normal to the spherical surface, and the other two components point along that surface. Then a typical curvature component measured by a set of these observers “hovering” near to the horizon is $\mathcal{R}_{\alpha\beta\gamma\delta} \propto R_+^{-2}$ $\ll 1$. All other measures of the curvature (including curvature invariants) measured by such static observers are similarly small.

If, however, these holes are also extremely close to being extreme with $\delta \equiv (1 - r_\perp/r_+) \ll a/R_+$, then observers who are falling into these holes tell a very different story about the curvature components. Consider a spherical set of observers who started out with velocity zero at some very large $r$ and then fell towards the black hole along a radial geodesic. Then they naturally carry a tetrad $\{\tilde{u}^\alpha = \gamma(u^\alpha + vx^\alpha), \tilde{\theta}^\alpha = R^{-1}\partial_\theta, \tilde{n}^\alpha = \gamma(\tilde{u}^\alpha + vv^\alpha), \tilde{\phi}^\alpha = (R \sin \theta)^{-1}\partial_\phi\}$, where again $\tilde{T}^\alpha = \tilde{u}^\alpha$ describes their evolution and $\tilde{\phi}^\alpha$ and $\tilde{\theta}^\alpha$ point along the spherical surfaces. $v = -(1 - F)^{1/2}$ is the radial velocity of the infalling observers as seen by the static ones, and $\gamma = (1 - v^2)^{-1/2}$ is the standard Lorentz factor from special relativity. Then, a typical curvature component seen by such observers as they cross the black hole horizon is $\mathcal{R}_{\tilde{\phi}\tilde{n}\tilde{\phi}\tilde{\theta}} \propto a^2(R_+\delta)^{-2} \gg 1$. That is they see extremely large, Planck scale curvatures. The resulting huge geodesic deviation laterally crushes them. Horowitz and Ross [3] dubbed this subclass of Maxwell-dilaton holes *naked* because Planck scale curvature components could be seen outside of their horizons.

### A QLE of naked black holes

These black holes seem almost tailor-made to be investigated by our method of defining boosted quasilocal energies. A static ($r = \text{constant}$, $t = \text{constant}$) set of observers measure
where $\dot{R} = \frac{dR}{dt}$. Note that both of these go to $R_+ \gg 1$ at the horizon. This is not surprising since both measure the quasilocal energy and large $R_+$ corresponds to a large $M$ black hole. Keep in mind that for a near extreme magnetic RN hole $R_+ \approx 2m$.

Next consider the infalling measurements. The radial velocity of the infalling observers in the original spacetime is $v = -(1 - F)^{1/2}$ with respect to the static observers. By contrast, the shell of observers have to travel at $v = -\dot{R}(1 - F)^{1/2}/(1 + \dot{R}^2(1 - F))^{1/2}$ in the reference spacetime if they want their surface area to change at the same rate. Therefore at the horizon $r_+$, 

$$E_{\text{Geo}} - E = R - \sqrt{(r - r_+)(r - r_-)} \dot{R} \quad \text{and} \quad E_{\text{Tot}} - E = R \left(1 - \sqrt{\frac{r - r_+}{r - r_-}}\right),$$

where $C_1$ and $C_2$ are constants that are on the order of unity. Thus we see that our extension of the QLE formalism detects the difference between regular “clothed” and naked black holes. Note however that while static/infalling observers see small/large curvatures they measure large/small geometric QLE’s.

**B Why do naked black holes behave this way?**

These small/large measurements can be understood physically in the following manner. As was noted by Horowitz and Ross for a naked black hole $R_+ \delta$ is more-or-less the time left to an infalling observer before she reaches the singularity at $r_-$. Less rigorously, for near extreme black holes $r_+ \approx r_-$ and so in some sense for naked black holes the singularity is “just behind” the horizon.

At the singularity the surface area of an $r = \text{constant}$ shell of observers goes to zero. Intuitively this means that for the naked black holes we expect the magnitude of $j_r$ (basically the rate of change of the area) to be very large since the area is very large but will soon be zero. By contrast for an RN black hole, the area only goes to zero at $r = 0$ which is not so “close” to the horizon. Thus $j_r$ need not be so large. These expectations are borne out by the calculations.

Thus, we can see why observers falling into a naked black hole experience the huge lateral crushing forces. As a shell of them travelling on geodesics cross the horizon the surface area of that shell is rapidly decreasing and so the crushing lateral forces are to be expected. By contrast for “clothed” black holes the rate of change of the area is much smaller and so are the corresponding lateral forces.

The relative sizes of $j_r$ in the two cases also explain the geometrical QLE observations. As we saw earlier, $\varepsilon^2 - j_r^2$ is a constant independent of boosts. Therefore if $\varepsilon$ and $\varepsilon^*$ are the geometric and reference QLE densities for the static observers, $\varepsilon^*$ and $\varepsilon^*$ are their boosted counterparts, and recalling that $j_\varpi = j_r$,

$$\varepsilon^* - \varepsilon^* = \sqrt{j_r^2 + \varepsilon^2} - \sqrt{j_r^2 + \varepsilon^2} \approx \frac{\varepsilon^2 - \varepsilon^2}{2j_r}$$

for $j_r$ much larger than $\varepsilon$ and $\varepsilon^*$. Thus as $j_r$ becomes larger and larger, $\varepsilon^* - \varepsilon^*$ becomes smaller and smaller. Physically, though $\varepsilon^*$ and $\varepsilon^*$ are boosted to be very large, at the same time the difference between them becomes smaller and smaller.

By contrast $E_{\text{Tot}}^* - E^*$ includes matter terms. These terms are also boosted to be very large but there is no corresponding term in the reference spacetime to cancel them out (as there is for the geometrical energy). Therefore in $E_{\text{Tot}}^* - E^*$ the matter terms dominate over the geometrical ones which are small and so the total infalling quasilocal energy is large.

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