NONHOMOGENEOUS DISTRIBUTIONS AND OPTIMAL QUANTIZERS
FOR SIERPIŃSKI CARPETS

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Abstract. The purpose of quantization of a probability distribution is to estimate the probability by a discrete probability with finite support. In this paper, a nonhomogeneous probability measure $P$ on $\mathbb{R}^2$ which has support the Sierpiński carpet generated by a set of four contractive similarity mappings with equal similarity ratios has been considered. For this probability measure, the optimal sets of $n$-means and the $n$th quantization errors are investigated for all $n \geq 2$.

1. Introduction

Quantization is a destructive process. Its purpose is to reduce the cardinality of the representation space, in particular when the input data is real-valued. It is a fundamental problem in signal processing, data compression and information theory. We refer to [GG, GN, Z] for surveys on the subject and comprehensive lists of references to the literature, see also [AW, GKL, GL1, GL2]. Let $\mathbb{R}^d$ denote the $d$-dimensional Euclidean space, $\| \cdot \|$ denote the Euclidean norm on $\mathbb{R}^d$ for any $d \geq 1$, and $n \in \mathbb{N}$. Then the $n$th quantization error for a Borel probability measure $P$ on $\mathbb{R}^d$ is defined by

$$V_n := V_n(P) = \inf \left\{ \int_{a \in \alpha} \| x - a \|^2 dP(x) : \alpha \subset \mathbb{R}^d, \ \text{card}(\alpha) \leq n \right\},$$

where the infimum is taken over all subsets $\alpha$ of $\mathbb{R}^d$ with $\text{card}(\alpha) \leq n$. If $\int \| x \|^2 dP(x) < \infty$ then there is some set $\alpha$ for which the infimum is achieved (see [AW, GKL, GL1, GL2]). Such a set $\alpha$ for which the infimum occurs and contains no more than $n$ points is called an optimal set of $n$-means, or optimal set of $n$-quantizers. The collection of all optimal sets of $n$-means for a probability measure $P$ on $\mathbb{R}^d$ is denoted by $C_n := C_n(P)$. It is known that for a continuous probability measure an optimal set of $n$-means always has exactly $n$-elements (see [GL2]). Given a finite subset $\alpha \subset \mathbb{R}^d$, the Voronoi region generated by $a \in \alpha$ is defined by

$$M(a|\alpha) = \{ x \in \mathbb{R}^d : \| x - a \| = \min_{b \in \alpha} \| x - b \| \},$$

i.e., the Voronoi region generated by $a \in \alpha$ is the set of all points in $\mathbb{R}^d$ which are closest to $a \in \alpha$, and the set $\{ M(a|\alpha) : a \in \alpha \}$ is called the Voronoi diagram or Voronoi tessellation of $\mathbb{R}^d$ with respect to $\alpha$. A Borel measurable partition $\{ A_a : a \in \alpha \}$ of $\mathbb{R}^d$ is called a Voronoi partition of $\mathbb{R}^d$ with respect to $\alpha$ (and $P$) if $A_a \subset M(a|\alpha)$ ($P$-a.e.) for every $a \in \alpha$. Given a Voronoi tessellation $\{ M_i \}_{i=1}^k$ generated by a set of points $\{ z_i \}_{i=1}^k$ (called sites or generators), the mass centroid $c_i$ of $M_i$ with respect to the probability measure $P$ is given by

$$c_i = \frac{1}{P(M_i)} \int_{M_i} xdP = \frac{\int_{M_i} xdP}{\int_{M_i} dP}.$$
The Voronoi tessellation is called the centroidal Voronoi tessellation (CVT) if \( z_i = c_i \) for \( i = 1, 2, \ldots , k \), that is, if the generators are also the centroids of the corresponding Voronoi regions.

Let us now state the following proposition (see [GG, GL2]):

**Proposition 1.1.** Let \( \alpha \) be an optimal set of \( n \)-means and \( a \in \alpha \). Then,

(i) \( P(M(a|\alpha)) > 0 \),

(ii) \( P(\partial M(a|\alpha)) = 0 \),

(iii) \( a = E(X : X \in M(a|\alpha)) \), and

(iv) \( P \)-almost surely the set \( \{ M(a|\alpha) : a \in \alpha \} \) forms a Voronoi partition of \( \mathbb{R}^d \).

Let \( \alpha \) be an optimal set of \( n \)-means and \( a \in \alpha \), then by Proposition 1.1 we have

\[
a = \frac{1}{P(M(a|\alpha))} \int_{M(a|\alpha)} xdP = \frac{\int_{M(a|\alpha)} xdP}{\int_{M(a|\alpha)} dP},
\]

which implies that \( a \) is the centroid of the Voronoi region \( M(a|\alpha) \) associated with the probability measure \( P \) (see also [DFC, RU]).

A transformation \( f : X \rightarrow X \) on a metric space \((X, d)\) is called contractive or a contraction mapping if there is a constant \( 0 < c < 1 \) such that \( d(f(x), f(y)) \leq cd(x, y) \) for all \( x, y \in X \). On the other hand, \( f \) is called a similarity mapping or a similitude if there exists a constant \( s > 0 \) such that \( d(f(x), f(y)) = sd(x, y) \) for all \( x, y \in X \). Here \( s \) is called the similarity ratio of the similarity mapping \( f \). Let \( C \) be the Cantor set generated by the two contractive similarity mappings \( S_1 \) and \( S_2 \) on \( \mathbb{R} \) given by \( S_1(x) = r_1x \) and \( S_2(x) = r_2x + (1 - r_2) \) where \( 0 < r_1, r_2 < 1 \) and \( r_1 + r_2 < \frac{1}{2} \). Let \( P = p_1 P \circ S_1^{-1} + p_2 P \circ S_2^{-1} \), where \( P \circ S_i^{-1} \) denotes the image measure of \( P \) with respect to \( S_i \) for \( i = 1, 2 \) and \((p_1, p_2)\) is a probability vector with \( 0 < p_1, p_2 < 1 \). Then, \( P \) is a singular continuous probability measure on \( \mathbb{R} \) with support the Cantor set \( C \) (see [H]). For \( r_1 = r_2 = \frac{1}{3} \) and \( p_1 = p_2 = \frac{1}{2} \), Graf and Luschgy gave a closed formula to determine the optimal sets of \( n \)-means for the probability distribution \( P \) for any \( n \geq 2 \) (see [GL3]). For \( r_1 = \frac{1}{4}, r_2 = \frac{1}{2} \), \( p_1 = \frac{3}{4} \) and \( p_2 = \frac{1}{4} \), L. Roychowdhury gave an induction formula to determine the optimal sets of \( n \)-means and the \( n \)th quantization error for the probability distribution \( P \) for any \( n \geq 2 \) (see [R2]). Let us now consider the Sierpiński carpet which is generated by the four contractive similarity mappings \( S_1, S_2, S_3 \) and \( S_4 \) on \( \mathbb{R}^2 \) such that \( S_1(x_1, x_2) = \frac{1}{3}(x_1, x_2), S_2(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, 0), S_3(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (0, \frac{2}{3}) \), and \( S_4(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, \frac{2}{3}) \) for all \((x_1, x_2) \in \mathbb{R}^2 \). If \( P \) is a Borel probability measure on \( \mathbb{R}^2 \) such that \( P = \frac{1}{4} P \circ S_1^{-1} + \frac{1}{4} P \circ S_2^{-1} + \frac{1}{4} P \circ S_3^{-1} + \frac{1}{4} P \circ S_4^{-1} \), then \( P \) has support the Sierpiński carpet. For this probability measure, Cömez and Roychowdhury gave closed formulas to determine the optimal sets of \( n \)-means and the \( n \)th quantization error for any \( n \geq 2 \) (see [CR]).

In this paper, we have considered the probability distribution \( P \) given by \( P = \frac{1}{8} P \circ S_1^{-1} + \frac{1}{8} P \circ S_2^{-1} + \frac{1}{8} P \circ S_3^{-1} + \frac{1}{8} P \circ S_4^{-1} \) which has support the Sierpiński carpet generated by the four contractive similarity mappings given by \( S_1(x_1, x_2) = \frac{1}{3}(x_1, x_2), S_2(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, 0), S_3(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (0, \frac{2}{3}) \), and \( S_4(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, \frac{2}{3}) \) for all \((x_1, x_2) \in \mathbb{R}^2 \). The probability distribution \( P \) considered in this paper is called ‘nonhomogeneous’ to mean that the probabilities associated with the mappings \( S_1, S_2, S_3 \) and \( S_4 \) are not equal. For this probability distribution in Proposition 3.1, Proposition 3.3 and Proposition 3.4, first we have determined the optimal sets of \( n \)-means and the \( n \)th quantization errors for \( n = 2, 3 \), and 4. Then, in Theorem 3.9 we state and prove an induction formula to determine the optimal sets of \( n \)-means for all \( n \geq 2 \). We also give some figures to illustrate the location of the optimal points (see Figure 1). In addition, running the induction formula in computer algorithm, we obtain some results and observations about the optimal sets of \( n \)-means which are given in Section 4; a tree diagram of the optimal sets of \( n \)-means for a certain range of \( n \) is also given (see Figure 2).
2. Preliminaries

In this section, we give the basic definitions and lemmas that will be instrumental in our analysis. For $k \geq 1$, by a word $\omega$ of length $k$ over the alphabet $I := \{1, 2, 3, 4\}$ it is meant that $\omega := \omega_1 \omega_2 \cdots \omega_k$, i.e., $\omega$ is a finite sequence of symbols over the alphabet $I$. Here $k$ is called the length of the word $\omega$. If $k = 0$, i.e., if $\omega$ is a word of length zero, we call it the empty word and is denoted by $\emptyset$. Length of a word $\omega$ is denoted by $|\omega|$. $I^*$ denotes the set of all words over the alphabet $I$ including the empty word $\emptyset$. By $\omega \tau := \omega_1 \cdots \omega_k \tau_1 \cdots \tau_\ell$ it is meant that the word obtained from the concatenations of the words $\omega := \omega_1 \omega_2 \cdots \omega_k$ and $\tau := \tau_1 \tau_2 \cdots \tau_\ell$ for $k, \ell \geq 0$. The maps $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, $1 \leq i \leq 4$, will be the generating maps of the Sierpiński carpet defined as before. For $\omega = \omega_1 \omega_2 \cdots \omega_k \in I^k$, set $S_\omega = S_{\omega_1} \circ \cdots \circ S_{\omega_k}$ and $J_\omega = S_\omega([0, 1] \times [0, 1])$. For the empty word $\emptyset$, by $S_\emptyset$ we mean the identity mapping on $\mathbb{R}^2$, and write $J = J_\emptyset = S_\emptyset([0, 1] \times [0, 1]) = [0, 1] \times [0, 1]$. The sets $\{J_\omega \cdot \omega \in \{1, 2, 3, 4\}^k\}$ are just the $4^k$ squares in the $k$th level in the construction of the Sierpiński carpet. The squares $J_{\omega_1}$, $J_{\omega_2}$, $J_{\omega_3}$ and $J_{\omega_4}$ into which $J_\omega$ is split up at the $(k + 1)$th level are called the basic squares of $J_\omega$. The set $S = \cap_{k \in \mathbb{N}} \cup_{\omega \in \{1, 2, 3, 4\}^k} J_\omega$ is the Sierpiński carpet and equals the support of the probability measure $P$ given by $P = \frac{1}{8} P \circ S_1^{-1} + \frac{1}{8} P \circ S_2^{-1} + \frac{3}{8} P \circ S_3^{-1} + \frac{3}{8} P \circ S_4^{-1}$. Set $s_1 = s_2 = s_3 = s_4 = \frac{1}{3}$, $p_1 = p_2 = \frac{1}{8}$ and $p_3 = p_4 = \frac{3}{8}$, and for $\omega = \omega_1 \omega_2 \cdots \omega_k \in I^k$, write $c(\omega) := \text{card}\{i : \omega_i = 3 \text{ or } 4, 1 \leq i \leq k\}$, where card$(A)$ of a set $A$ represents the number of elements in the set $A$. Then, for $\omega = \omega_1 \omega_2 \cdots \omega_k \in I^k$, $k \geq 1$, we have $s_\omega = \frac{1}{3^k}$ and $p_\omega = p_{\omega_1} p_{\omega_2} \cdots p_{\omega_k} = \frac{c(\omega)}{8^k}$.

Let us now give the following lemma.

**Lemma 2.1.** Let $f : \mathbb{R} \to \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,

$$
\int f \, dP = \sum_{\omega \in I^k} p_\omega \int f \circ S_\omega \, dP.
$$

**Proof.** We know $P = p_1 P \circ S_1^{-1} + p_2 P \circ S_2^{-1} + p_3 P \circ S_3^{-1} + p_4 P \circ S_4^{-1}$, and so by induction $P = \sum_{\omega \in I^k} p_\omega P \circ S_\omega^{-1}$, and thus the lemma is yielded. \[\square\]

Let $S_{(i1)}$, $S_{(i2)}$ be the horizontal and vertical components of the transformation $S_i$ for $i = 1, 2, 3, 4$. Then for any $(x_1, x_2) \in \mathbb{R}^2$ we have $S_{(11)}(x_1) = \frac{1}{3} x_1$, $S_{(12)}(x_2) = \frac{1}{3} x_2$, $S_{(21)}(x_1) = \frac{1}{3} x_1 + \frac{2}{3}$, $S_{(22)}(x_2) = \frac{1}{3} x_2$, $S_{(31)}(x_1) = \frac{1}{3} x_1$, $S_{(32)}(x_2) = \frac{1}{3} x_2 + \frac{2}{3}$, and $S_{(41)}(x_1) = \frac{1}{3} x_1 + \frac{2}{3}$, $S_{(42)}(x_2) = \frac{1}{3} x_2 + \frac{2}{3}$. Let $X = (X_1, X_2)$ be a bivariate random variable with distribution $P$. Let $P_1, P_2$ be the marginal distributions of $P$, i.e., $P_1(A) = P(A \times \mathbb{R})$ for all $A \in \mathcal{B}$, and $P_2(B) = P(\mathbb{R} \times B)$ for all $B \in \mathcal{B}$. Here $\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}$. Then $X_1$ has distribution $P_1$ and $X_2$ has distribution $P_2$.

Let us now state the following lemma. The proof is similar to Lemma 2.2 in [CR].

**Lemma 2.2.** Let $P_1$ and $P_2$ be the marginal distributions of the probability measure $P$. Then,

$$
P_1 = \frac{1}{8} P_1 \circ S_{(11)}^{-1} + \frac{1}{8} P_1 \circ S_{(21)}^{-1} + \frac{3}{8} P_1 \circ S_{(31)}^{-1} + \frac{3}{8} P_1 \circ S_{(41)}^{-1} \quad \text{and}
$$

$$
P_2 = \frac{1}{8} P_2 \circ S_{(12)}^{-1} + \frac{1}{8} P_2 \circ S_{(22)}^{-1} + \frac{3}{8} P_2 \circ S_{(32)}^{-1} + \frac{3}{8} P_2 \circ S_{(42)}^{-1}.
$$

Let us now give the following lemma.

**Lemma 2.3.** Let $E(X)$ and $V(X)$ denote the the expectation and the variance of the random variable $X$. Then,

$$
E(X) = (E(X_1), E(X_2)) = \left(\frac{1}{2}, \frac{3}{4}\right) \quad \text{and} \quad V := V(X) = E\|X - \left(\frac{1}{2}, \frac{3}{4}\right)\|^2 = \frac{7}{32}.
$$
Proof. We have

\[
E(X_1) = \int x dP = \frac{1}{8} \int x dP \circ S^{-1}_{(11)} + \frac{1}{8} \int x dP \circ S^{-1}_{(21)} + \frac{3}{8} \int x dP \circ S^{-1}_{(31)} + \frac{3}{8} \int x dP \circ S^{-1}_{(41)}
\]

\[
= \frac{1}{8} \int \frac{1}{3} x dP + \frac{1}{8} \int \left( \frac{1}{3} x + \frac{2}{3} \right) dP + \frac{3}{8} \int \frac{1}{3} x dP + \frac{3}{8} \int \left( \frac{1}{3} x + \frac{2}{3} \right) dP,
\]

which after simplification yields \( E(X_1) = \frac{1}{2} \), and similarly \( E(X_2) = \frac{3}{4} \). Now,

\[
E(X_1^2) = \int x^2 dP
\]

\[
= \frac{1}{8} \int x^2 dP \circ S^{-1}_{(11)} + \frac{1}{8} \int x^2 dP \circ S^{-1}_{(21)} + \frac{3}{8} \int x^2 dP \circ S^{-1}_{(31)} + \frac{3}{8} \int x^2 dP \circ S^{-1}_{(41)}
\]

\[
= \frac{1}{8} \int \frac{1}{9} x^2 dP + \frac{1}{8} \int \left( \frac{1}{9} x^2 + \frac{4}{9} x + \frac{4}{9} \right) dP
\]

\[
= \frac{1}{18} E(X_1^2) + \frac{1}{18} E(X_1^2) + \frac{4}{18} E(X_1) + \frac{4}{18}
\]

\[
= \frac{1}{9} E(X_1^2) + \frac{1}{3}.
\]

This implies \( E(X_1^2) = \frac{3}{4} \). Similarly, we can show \( E(X_2^2) = \frac{21}{32} \). Thus, \( V(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8} \), and similarly \( V(X_2) = \frac{3}{32} \). Hence,

\[
E\|X - (\frac{1}{2}, \frac{3}{4})\|^2 = E(X_1 - \frac{1}{2})^2 + E(X_2 - \frac{3}{4})^2 = V(X_1) + V(X_2) = \frac{7}{32}.
\]

Thus, the proof of the lemma follows. \( \square \)

Let us now give the following note.

Note 2.4. From Lemma 2.3 it follows that the optimal set of one-mean is the expected value and the corresponding quantization error is the variance \( V \) of the random variable \( X \). For words \( \beta, \gamma, \cdots, \delta \) in \( I^k \), by \( a(\beta, \gamma, \cdots, \delta) \) we mean the conditional expectation of the random variable \( X \) given \( J_\beta \cup J_\gamma \cup \cdots \cup J_\delta \), i.e.,

\[
(1) \quad a(\beta, \gamma, \cdots, \delta) = E(X|X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta \cup \cdots \cup J_\delta} x dP.
\]

For \( \omega \in I^k \), \( k \geq 1 \), since \( a(\omega) = E(X : X \in J_\omega) \), using Lemma 2.1, we have

\[
a(\omega) = \frac{1}{P(J_\omega)} \int_{J_\omega} x dP(x) = \int_{J_\omega} x dP \circ S^{-1}_\omega(x) = \int S_\omega(x) dP(x) = E(S_\omega(X)) = S_\omega(\frac{1}{2}, \frac{3}{4}).
\]

For any \((a, b) \in \mathbb{R}^2\), \( E\|X - (a, b)\|^2 = V + \|(\frac{1}{2}, \frac{3}{4}) - (a, b)\|^2 \). In fact, for any \( \omega \in I^k \), \( k \geq 1 \), we have \( \int_{J_\omega} \|x - (a, b)\|^2 dP = p_\omega \int \|x_1, x_2 - (a, b)\|^2 dP \circ S^{-1}_\omega \), which implies

\[
(2) \quad \int_{J_\omega} \|x - (a, b)\|^2 dP = p_\omega \left(s_\omega V + \|a(\omega) - (a, b)\|^2 \right).
\]

The expressions (1) and (2) are useful to obtain the optimal sets and the corresponding quantization errors with respect to the probability distribution \( P \).
In this section we determine the optimal sets of \( n \)-means for all \( n \geq 2 \). First, prove the following proposition.

**Proposition 3.1.** The set \( \alpha = \{a(1,3), a(2,4)\} \), where \( a(1,3) = (\frac{1}{6}, \frac{3}{4}) \) and \( a(2,4) = (\frac{5}{6}, \frac{3}{4}) \), is an optimal set of two-means with quantization error \( V_2 = \frac{31}{288} = 0.107639 \).

**Proof.** With respect to the vertical line passing through the centroid \( (\frac{1}{2}, \frac{3}{4}) \), the Sierpiński carpet has the maximum symmetry, i.e., with respect to the line \( x_1 = \frac{1}{2} \) the Sierpiński carpet is geometrically symmetric. Also, observe that, if the two basic rectangles of similar geometrical shape lie in the opposite sides of the line \( x_1 = \frac{1}{2} \), and are equidistant from the line \( x_1 = \frac{1}{2} \), then they have the same probability; hence, they are symmetric with respect to the probability distribution \( P \) as well. Due to this, among all the pairs of two points which have the boundaries of the Voronoi regions oblique lines passing through the point \( (\frac{1}{2}, \frac{3}{4}) \), the two points which have the boundary of the Voronoi regions the line \( x_1 = \frac{1}{2} \) will give the smallest distortion error. Again, we know that the two points which give the smallest distortion error are the centroids of their own Voronoi regions. Let \( (a_1, b_1) \) and \( (a_2, b_2) \) be the centroids of the left half and the right half of the Sierpiński carpet with respect to the line \( x_1 = \frac{1}{2} \) respectively. Then using \( (\square) \),

![Figure 1. Configuration of the points in an optimal set of n-means for 1 ≤ n ≤ 16.](image)
we have 
\[(a_1, b_1) = E(X : X \in J_1 \cup J_3) = \frac{1}{P(J_1) + P(J_3)} \left( P(J_1) \int_{J_1} xdP + P(J_3) \int_{J_3} xdP \right) = \left( \frac{1}{6}, \frac{3}{4} \right) \]

and 
\[(a_2, b_2) = E(X : X \in J_2 \cup J_4) = \frac{1}{P(J_2) + P(J_4)} \left( P(J_2) \int_{J_2} xdP + P(J_4) \int_{J_4} xdP \right) = \left( \frac{5}{6}, \frac{3}{4} \right). \]

Write \( \alpha := \{(\frac{1}{6}, \frac{3}{4}), (\frac{5}{6}, \frac{3}{4})\} \). Then, the distortion error is obtained as 
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP = \int_{J_1 \cup J_3} \|x - \left( \frac{1}{6}, \frac{3}{4} \right)\|^2 dP + \int_{J_2 \cup J_4} \|x - \left( \frac{5}{6}, \frac{3}{4} \right)\|^2 dP = \frac{31}{288} = 0.107639.
\]

Since \( V_2 \) is the quantization error for two-means, we have \( 0.107639 \geq V_2 \). Suppose that the points in an optimal set of two-means lie on a vertical line. Then, we can assume that \( \beta = \{(p, a), (p, b)\} \) is an optimal set of two-means with \( a \leq b \). Then, by the properties of centroids we have
\[
(p, a)P(M((p, a)\mid \beta)) + (p, b)P(M((p, b)\mid \beta)) = \left( \frac{1}{2}, \frac{3}{4} \right),
\]
which implies \( pP(M((p, a)\mid \beta)) + pP(M((p, b)\mid \beta)) = \frac{1}{2} \) and \( aP(M((p, a)\mid \beta)) + bP(M((p, b)\mid \beta)) = \frac{3}{4} \). Thus, we see that \( p = \frac{1}{2} \), and the two points \((p, a)\) and \((p, b)\) lie on the opposite sides of the point \( \left( \frac{1}{2}, \frac{3}{4} \right) \). Since the optimal points are the centroids of their own Voronoi regions, we have \( 0 \leq a \leq b \leq 1 \). Then, notice that \( J_{33} \cup J_{34} \cup J_{43} \cup J_{44} \subset M(\left( \frac{1}{2}, \frac{3}{4} \right)\mid \beta) \) and \( J_1 \cup J_2 \subset M(\left( \frac{1}{2}, a \right)\mid \beta) \). Suppose that \( a \geq \frac{5}{12} \). Then as \( a(33, 34, 43, 44) = E(X : X \in J_{33} \cup J_{34} \cup J_{43} \cup J_{44}) = (\frac{1}{2}, \frac{35}{36}) \), we have 
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq \int_{J_1 \cup J_2} \|x - \left( \frac{1}{2}, \frac{5}{12} \right)\|^2 dP + \int_{J_{33} \cup J_{34} \cup J_{43} \cup J_{44}} \|x - \left( \frac{1}{2}, \frac{35}{36} \right)\|^2 dP = \frac{515}{4608} = 0.111762,
\]
which is a contradiction, as \( 0.111762 > 0.107639 \geq V_2 \) and \( \alpha \) is an optimal set of two-means. Thus, we can assume that \( a < \frac{5}{12} \). Since \( a < \frac{5}{12} \) and \( b \leq 1 \), we have \( \frac{1}{2}(a + b) \leq \frac{1}{2}(\frac{5}{12} + 1) = \frac{17}{24} \), which yields that \( B \subset M(\left( \frac{1}{2}, b \right)\alpha) \) where \( B = J_{33} \cup J_{34} \cup J_{43} \cup J_{44} \cup J_{313} \cup J_{314} \cup J_{323} \cup J_{324} \cup J_{413} \cup J_{414} \cup J_{423} \cup J_{424} \). Using \( [I] \), we have \( E(X : X \in B) = (\frac{3}{2}, \frac{503}{540}) \) which implies that \( b \leq \frac{503}{540} \).

Now if \( a \geq \frac{1}{3} \), we have 
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq \int_{J_1 \cup J_2} \|x - \left( \frac{1}{2}, \frac{1}{3} \right)\|^2 dP + \int_{B} \|x - \left( \frac{1}{2}, \frac{503}{540} \right)\|^2 dP = \frac{106847}{829440} = 0.128818 > V_2,
\]
which is a contradiction. So, we can assume that \( a < \frac{1}{3} \). Then, \( J_1 \cup J_1 \subset M(\left( \frac{1}{2}, a \right)\alpha) \) and \( J_3 \cup J_4 \subset M(\left( \frac{1}{2}, b \right)\alpha) \), and so \( (\frac{1}{2}, a) = E(X : X \in J_1 \cup J_2) = (\frac{1}{2}, \frac{1}{4}) \) and \( (\frac{1}{2}, b) = E(X : X \in J_3 \cup J_4) = (\frac{1}{2}, \frac{11}{12}) \), and 
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP = \int_{J_1 \cup J_2} \|x - \left( \frac{1}{2}, \frac{1}{4} \right)\|^2 dP + \int_{J_3 \cup J_4} \|x - \left( \frac{1}{2}, \frac{11}{12} \right)\|^2 dP = \frac{13}{96} = 0.135417 > V_2,
\]
which leads to another contradiction. Therefore, we can assume that the points in an optimal set of two-means can not lie on a vertical line. Hence, \( \alpha = \{(\frac{1}{6}, \frac{3}{4}), (\frac{5}{6}, \frac{3}{4})\} \) forms an optimal set of two-means with quantization error \( V_2 = \frac{31}{288} = 0.107639 \). \qed

Remark 3.2. The set \( \alpha \) in Proposition 3.1 forms a unique optimal set of two-means.
Proposition 3.3. The set \( \alpha = \{a(1, 2), a(3), a(4)\} \), where \( a(1, 2) = E(X : X \in J_1 \cup J_2) = (\frac{1}{2}, \frac{1}{4}) \), \( a(3) = E(X : X \in J_3) = (\frac{1}{6}, \frac{11}{12}) \) and \( a(4) = E(X : X \in J_4) = (\frac{5}{6}, \frac{11}{12}) \), forms an optimal set of three-means with quantization error \( V_3 = \frac{5}{96} = 0.0520833 \).

Proof. Let us first consider the three-point set \( \beta \) given by \( \beta = \{a(1, 2), a(3), a(4)\} \). Then, the distortion error is obtained as

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP = \int_{J_1 \cup J_2} \|x - a(1, 2)\|^2 dP + \int_{J_3} \|x - a(3)\|^2 dP + \int_{J_2} \|x - a(4)\|^2 dP = 0.0520833.
\]

Since \( V_3 \) is the quantization error for an optimal set of three-means, we have \( 0.0520833 \geq V_3 \). Let \( \alpha := \{(a_i, b_i) : 1 \leq i \leq 3\} \) be an optimal set of three-means. Since the optimal points are the centroids of their own Voronoi regions, we have \( \alpha \subset [0, 1] \times [0, 1] \). Then, by the definition of centroid, we have

\[
\sum_{(a_i, b_i) \in \alpha} (a_i, b_i) P(M((a_i, b_i)|\alpha)) = \left(\frac{1}{2}, \frac{3}{4}\right),
\]

which implies \( \sum_{(a_i, b_i) \in \alpha} a_i P(M((a_i, b_i)|\alpha)) = \frac{1}{2} \) and \( \sum_{(a_i, b_i) \in \alpha} b_i P(M((a_i, b_i)|\alpha)) = \frac{3}{4} \). Thus, we conclude that all the optimal points can not lie in one side of the vertical line \( x_1 = \frac{1}{2} \) or in one side of the horizontal line \( x_2 = \frac{3}{4} \). Without any loss of generality, due to symmetry we can assume that one of the optimal points, say \( (a_1, b_1) \), lies on the vertical line \( x_1 = \frac{1}{2} \), i.e., \( a_1 = \frac{1}{2} \), and the optimal points \( (a_2, b_2) \) and \( (a_3, b_3) \) lie on a horizontal line and are equidistant from the vertical line \( x_1 = \frac{1}{2} \). Further, due to symmetry we can assume that \( (a_2, b_2) \) and \( (a_3, b_3) \) lie on the vertical lines \( x_1 = \frac{1}{6} \) and \( x_1 = \frac{5}{6} \) respectively, i.e., \( a_2 = \frac{1}{6} \) and \( a_3 = \frac{5}{6} \).

Suppose that \( (\frac{1}{2}, b_1) \) lies on or above the horizontal line \( x_2 = \frac{3}{4} \), and so \( (\frac{1}{6}, b_2) \) and \( (\frac{5}{6}, b_3) \) lie on or below the line \( x_2 = \frac{3}{4} \). Then, if \( \frac{2}{3} \leq b_2, b_3 \leq \frac{2}{3} \), we have

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_1 \cup J_3} \min_{\frac{2}{3} \leq b \leq \frac{2}{3}} \|x - (\frac{1}{6}, b)\|^2 dP = 0.0820313 > V_3,
\]

which is a contradiction. If \( \frac{1}{2} \leq b_2, b_3 \leq \frac{2}{3} \),

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_1 \cup J_3} \min_{\frac{2}{3} \leq b \leq \frac{2}{3}} \|x - (\frac{1}{6}, b)\|^2 dP + \int_{J_3} \|x - (\frac{1}{2}, \frac{2}{3})\|^2 dP + \int_{J_3} \min_{\frac{2}{3} \leq b \leq 1} \|x - (\frac{1}{2}, b)\|^2 dP
\]

\[
= 2\left(\frac{6521}{442368} + \frac{281}{18432} + \frac{277}{110592}\right) = 0.0649821 > V_3,
\]

which leads to a contradiction. If \( \frac{1}{3} \leq b_2, b_3 \leq \frac{1}{2} \), then

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_3 \cup J_2} \|x - (\frac{1}{6}, \frac{1}{2})\|^2 dP + \int_{J_3} \|x - (\frac{1}{6}, \frac{1}{3})\|^2 dP + \int_{J_3} \min_{\frac{2}{3} \leq b \leq 1} \|x - (\frac{1}{2}, b)\|^2 dP
\]

\[
= 2\left(\frac{811}{110592} + \frac{1}{256} + \frac{78373}{486048}\right) = 0.0546912 > V_3,
\]
which gives a contradiction. If $0 \leq b_2, b_3 \leq \frac{1}{3}$, then
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \left( \int_{J_1} \|x - a(1)\|^2 dP + \int_{J_{31} \cup J_{34}} \min_{\frac{3}{4} \leq b \leq 1} \|x - (\frac{1}{2}, b)\|^2 dP \right)
\]
\[
= 2 \left( \frac{7}{2304} + \frac{109}{3072} \right) = 0.0770399 > V_3
\]
which leads to another contradiction. Therefore, we can assume that $(\frac{1}{2}, b_1)$ lies on or below the horizontal line $x_2 = \frac{3}{4}$, and $(\frac{1}{6}, b_2)$ and $(\frac{5}{6}, b_3)$ lie on or above the line $x_2 = \frac{3}{4}$. Notice that for any position of $(\frac{1}{2}, b_1)$ on or below the line $x_2 = \frac{3}{4}$, always $J_{31} \cup J_{33} \cup J_{34} \subset M((\frac{1}{6}, b_2)|\alpha)$ which implies that $b_2 \leq \frac{70}{84}$. Similarly, $b_3 \leq \frac{70}{84}$. Suppose that $\frac{1}{2} \leq b_1 \leq \frac{3}{4}$. Then, writing $A = J_{33} \cup J_{321} \cup J_{324}$ and $B = J_{11} \cup J_{12} \cup J_{14} \cup J_{132}$, we have
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \left( \int_{J_{31} \cup J_{33} \cup J_{34} \cup J_{323}} \min_{\frac{3}{4} \leq b \leq \frac{70}{84}} \|x - (\frac{1}{6}, b)\|^2 dP + \int_{A} \|x - (\frac{1}{6}, \frac{3}{4})\|^2 dP + \int_{B} \|x - (\frac{1}{2}, \frac{1}{2})\|^2 dP \right)
\]
\[
= 2 \left( \frac{588517}{78299136} + \frac{5347}{1327104} + \frac{6601}{442368} \right) = 0.0529346 > V_3,
\]
which is a contradiction. So, we can assume that $b_1 < \frac{1}{2}$. Suppose that $\frac{1}{3} \leq b_1 < \frac{1}{2}$. Then, as $\frac{3}{4} \leq b_2 \leq \frac{70}{84}$, we see that $J_{31} \cup J_{33} \cup J_{34} \cup J_{321} \cup J_{323} \cup J_{324} \subset M((\frac{1}{6}, b_2)|\alpha)$. Then, writing $A_1 := J_{31} \cup J_{33} \cup J_{34} \cup J_{321} \cup J_{323} \cup J_{324}$ and $A_2 := J_{322} \cup J_{331} \cup J_{1333} \cup J_{1334} \cup J_{1332} \cup J_{1324}$ and $A_3 := J_{11} \cup J_{12} \cup J_{14} \cup J_{131} \cup J_{132} \cup J_{134} \cup J_{1332}$, we have
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \left( \int_{A_1} \min_{\frac{3}{4} \leq b \leq \frac{70}{84}} \|x - (\frac{1}{6}, b)\|^2 dP + \int_{A_2} \|x - (\frac{1}{6}, \frac{3}{4})\|^2 dP + \int_{A_3} \|x - (\frac{1}{2}, \frac{1}{3})\|^2 dP \right)
\]
\[
= 2 \left( \frac{242191}{27869184} + \frac{4135547}{1146617856} + \frac{31584803}{2293235712} \right) = 0.0521401 > V_3,
\]
which gives a contradiction. So, we can assume that $b_1 \leq \frac{1}{3}$. Then, notice that $J_{11} \cup J_{12} \cup J_{132} \cup J_{141} \cup J_{142} \cup J_{144} \cup J_{21} \cup J_{22} \cup J_{241} \cup J_{231} \cup J_{232} \cup J_{233} \subset M((\frac{1}{6}, b_1)|\alpha)$ which implies that $b_1 \geq \frac{13}{68}$. Thus, we have $\frac{13}{68} \leq b_1 \leq \frac{1}{3}$. Suppose that $\frac{3}{4} \leq b_2, b_3 \leq \frac{5}{6}$. Then,
\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \left( \int_{J_3} \min_{\frac{3}{4} \leq b \leq \frac{5}{6}} \|x - (\frac{1}{6}, b)\|^2 dP + \int_{J_{11} \cup J_{12} \cup J_{14} \cup J_{131} \cup J_{132}} \min_{\frac{3}{4} \leq b \leq \frac{3}{4}} \|x - (\frac{1}{2}, b)\|^2 dP \right)
\]
\[
+ \int_{J_{331} \cup J_{332} \cup J_{334}} \|x - (\frac{1}{6}, \frac{3}{4})\|^2 dP + \int_{J_{34}} \|x - (\frac{1}{2}, \frac{1}{3})\|^2 dP \right)
\]
\[
= 2 \left( \frac{3}{256} + \frac{147359}{15261696} + \frac{32969}{10616832} + \frac{3881}{1327104} \right) = 0.054808 > V_3,
\]
which leads to a contradiction. So, we can assume that $\frac{5}{6} < b_2, b_3 \leq 1$. Then, we have $J_1 \cup J_2 \subset M((\frac{1}{2}, b_1)|\alpha)$, $J_3 \subset M((\frac{1}{6}, b_2)|\alpha)$ and $J_4 \subset M((\frac{5}{6}, b_3)|\alpha)$ which yield that $(\frac{1}{2}, b_1) = a(1, 2)$, $(\frac{1}{6}, b_2) = a(3)$ and $(\frac{5}{6}, b_3) = a(4)$, and the quantization error is $V_3 = \frac{5}{96} = 0.0520833$. Thus, the proof of the proposition is complete.

\[\Box\]
**Proposition 3.4.** The set \( \alpha = \{a(1), a(2), a(3), a(4)\} \) forms an optimal set of four-means with quantization error \( V_4 = \frac{7}{288} = 0.0243056 \).

**Proof.** Let us consider the four-point set \( \beta := \{a(1), a(2), a(3), a(4)\} \). Then, the distortion error is given by

\[
\int \min_{c \in \beta} \|x - c\|^2 dP = 4 \int_{J} \|x - a(i)\|^2 dP = \frac{7}{288} = 0.0243056.
\]

Since, \( V_4 \) is the quantization error for four-means, we have \( 0.0243056 \geq V_4 \). As the optimal points are the centroids of their own Voronoi regions, \( \alpha \subset J \). Let \( \alpha \) be an optimal set of \( n \)-means for \( n = 4 \). By the definition of centroid, we know

\[
\sum_{(a,b) \in \alpha} (a, b) P(M((a,b)|\alpha)) = \left( \frac{1}{2}, \frac{3}{4} \right).
\]

If all the points of \( \alpha \) are below the line \( x = \frac{3}{4} \), i.e., if \( b < \frac{3}{4} \) for all \((a, b) \in \alpha \), then by (3), we see that \( \frac{3}{4} = \sum_{(a,b) \in \alpha} bP(M((a,b)|\alpha)) < \sum_{(a,b) \in \alpha} \frac{3}{4} P(M((a,b)|\alpha)) = \frac{3}{4} \), which is a contradiction. Similarly, it follows that if all the points of \( \alpha \) are above the line \( x = \frac{3}{4} \), or left of the line \( x_1 = \frac{1}{2} \), or right of the line \( x_1 = \frac{1}{2} \), a contradiction will arise. Suppose that all the points of \( \alpha \) are on the line \( x_2 = \frac{3}{4} \). Then, for \( (x_1, x_2) \in \cup_{i,j=3} J_{ij} \), we have \( \min_{c \in \alpha} \|x_1, x_2 - c\| = \frac{5}{36} \), and for \( (x_1, x_2) \in \cup_{i,j=1} J_{ij} \), we have \( \min_{c \in \alpha} \|x_1, x_2 - c\| = \frac{23}{36} \), which implies that

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP = 4 \int_{J} \min_{c \in \alpha} \|x_1, x_2 - c\|^2 dP + 4 \int_{J} \min_{c \in \alpha} \|x_1, x_2 - c\|^2 dP
\]

\[
= 4 \left( \frac{5}{36} \right)^2 P(J_{33}) + 4 \left( \frac{23}{36} \right)^2 P(J_{11}) = \frac{377}{10368} = 0.0363619 > V_4,
\]

which is a contradiction. Thus, we see that all the points of \( \alpha \) cannot lie on \( x_2 = \frac{3}{4} \). Similarly, all the points of \( \alpha \) cannot lie on \( x_1 = \frac{1}{2} \). Recall that the Sierpiński carpet has maximum symmetry with respect to the line \( x_1 = \frac{1}{2} \). As all the points of \( \alpha \) cannot lie on the line \( x_1 = \frac{1}{2} \), due to symmetry we can assume that the points of \( \alpha \) lie either on the three lines \( x_1 = \frac{1}{6}, x_1 = \frac{5}{6} \) and \( x_1 = \frac{1}{2} \), or on the two lines \( x_1 = \frac{1}{6} \) and \( x_1 = \frac{5}{6} \).

Suppose \( \alpha \) contains points from the line \( x_1 = \frac{1}{2} \). As \( \alpha \) cannot contain all the points from \( x_1 = \frac{1}{2} \), we can assume that \( \alpha \) contains two points, say \( (\frac{1}{2}, b_1) \) and \( (\frac{1}{2}, b_2) \) with \( b_1 < b_2 \), from the line \( x_1 = \frac{1}{2} \) which are in the opposite sides of the centroid \( \frac{1}{2}, \frac{3}{4} \), and the other two points, say \( (\frac{1}{6}, a_1) \) and \( (\frac{5}{6}, a_2) \), from the lines \( x_1 = \frac{1}{6} \) and \( x_1 = \frac{5}{6} \). Then, if \( \alpha \) does not contain any point from \( J_3 \cup J_4 \), we have

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_{31} \cup J_{33}} \|x - (\frac{1}{6}, \frac{2}{3})\|^2 dP = \frac{25}{768} = 0.0325521 > V_4,
\]

which leads to a contradiction. So, we can assume that \( (\frac{1}{6}, a_1) \in J_3 \) and \( (\frac{5}{6}, a_2) \in J_4 \). Suppose \( \frac{2}{3} \leq a_1, a_2 \leq \frac{5}{6} \). Then, notice that \( J_{31} \cup J_{33} \cup J_{321} \cup J_{323} \subset M((\frac{1}{6}, a_1)\alpha) \) and similar is the expression for the point \( (\frac{5}{6}, a_2) \). Further, notice that \( J_{11} \cup J_{12} \cup J_{14} \cup J_{21} \cup J_{22} \cup J_{23} \subset M((\frac{1}{2}, \frac{1}{2})|\alpha) \). Therefore, under the assumption \( \frac{2}{3} \leq a_1, a_2 \leq \frac{5}{6} \), writing \( A_1 := J_{31} \cup J_{33} \cup J_{321} \cup J_{323} \) and \( A_2 := J_{11} \cup J_{12} \cup J_{14} \), we have the distortion error as

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{A_1} \min_{\frac{2}{3} \leq b \leq \frac{5}{6}} \|x - (\frac{1}{6}, b)\|^2 dP + \int_{A_2} \min_{0 \leq b \leq \frac{1}{2}} \|x - (\frac{1}{2}, b)\|^2 dP
\]

\[
= 2 \left( \frac{2051}{331776} + \frac{2021}{276480} \right) = 0.0269833 > V_4,
\]
which leads to a contradiction. So, we can assume that \( \frac{5}{6} < a_1, a_2 \leq 1 \). Then, we see that 
\[ J_1 \cup J_2 \subset M((\frac{1}{2}, b)|\alpha) \] 
for \( b = \frac{1}{2} \), and so the distortion error is

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_1} \min_{0 \leq k \leq \frac{3}{2}} \|x - (\frac{1}{2}, b)\|^2 dP = \frac{13}{384} = 0.0338542 > V_4
\]

which is a contradiction. All these contradictions arise due to our assumption that \( \alpha \) contains points from the line \( x_1 = \frac{1}{2} \). So, we can assume that \( \alpha \) can not contain any point from the line \( x_1 = \frac{1}{2} \), i.e., we can assert that \( \alpha \) contains two points from the line \( x_1 = \frac{5}{6} \) and two points from the line \( x_1 = \frac{5}{6} \). Thus, we can take \( \alpha := \{(\frac{5}{6}, a_1), (\frac{1}{2}, b_1), (\frac{5}{6}, a_2), (\frac{5}{6}, b_2)\} \) where \( a_1 \leq \frac{3}{4} \leq b_1 \) and \( a_2 \leq \frac{3}{4} \leq b_2 \). Notice that the Voronoi region of \((\frac{1}{2}, a_1)\) contains \( J_1 \) and the Voronoi region of \((\frac{5}{6}, a_2)\) contains \( J_2 \). If the Voronoi region of \((\frac{1}{2}, a_1)\) contains points from \( J_3 \), we must have \( \frac{1}{2}(a_1 + b_1) \geq \frac{2}{3} \) which yields \( a_1 \geq \frac{4}{3} - b_1 \geq \frac{4}{3} - \frac{3}{4} = \frac{7}{12} \), and similarly if the Voronoi region of \((\frac{5}{6}, a_2)\) contains points from \( J_4 \), we must have \( a_2 \geq \frac{7}{12} \). But, then

\[
\int \min_{c \in \alpha} \|x - c\|^2 dP \geq 2 \int_{J_1} \|x - (\frac{1}{6}, \frac{7}{12})\|^2 dP + 2 \int_{J_3 \cup J_4} \|x - a(33, 34)\|^2 dP
\]

which is a contradiction. So, we can assume that the Voronoi regions of \((\frac{1}{2}, a_1)\) and \((\frac{5}{6}, a_2)\) do not contain any point from \( J_3 \cup J_4 \). Thus, we have \((\frac{1}{2}, a_1) = a(1) = (\frac{1}{6}, \frac{1}{2}), (\frac{5}{6}, a_2) = a(2) = (\frac{5}{6}, \frac{1}{2}), (\frac{5}{6}, b_1) = a(3) = (\frac{5}{6}, \frac{11}{12}), \) and \((\frac{5}{6}, b_2) = a(4) = (\frac{5}{6}, \frac{11}{12})\), and the quantization error is \( V_4 = \frac{7}{288} = 0.0243056 \). Thus, the proof of the proposition is complete. \( \square \)

**Note 3.5.** Let \( \lambda \) be an optimal set of \( n \)-means for some \( n \geq 2 \). Then, for \( a \in \alpha \), we have \( a = a(\omega), a = a(\omega_1, \omega_3), \) or \( a = a(\omega_2, \omega_4) \) for some \( \omega \in I^* \). Moreover, if \( a \in \alpha \), then \( P \)-almost surely \( M(a|\alpha) = J_\omega \) if \( a = a(\omega) \), \( M(a|\alpha) = J_{\omega_1} \cup J_{\omega_3} \) if \( a = a(\omega_1, \omega_3) \), and \( M(a|\alpha) = J_{\omega_2} \cup J_{\omega_4} \) if \( a = a(\omega_2, \omega_4) \). For \( \omega \in I^* \), \((i = 1 \text{ and } j = 3), (i = 2 \text{ and } j = 4), \) or \((i = 1, j = 2)\) write

\[
E(\omega) := \int_{J_\omega} \|x - a(\omega)\|^2 dP, \quad \text{and} \quad E(\omega_i, \omega_j) := \int_{J_{\omega_i} \cup J_{\omega_j}} \|x - a(\omega_i, \omega_j)\|^2 dP.
\]

Let us now give the following lemma.

**Lemma 3.6.** For any \( \omega \in I^* \), let \( E(\omega), E(\omega_1, \omega_3), \) \( E(\omega_2, \omega_4) \), and \( E(\omega_1, \omega_2) \) be defined by \(4\). Then, \( E(\omega_1, \omega_3) = E(\omega_2, \omega_4) = \frac{31}{126} E(\omega), E(\omega_1, \omega_2) = \frac{13}{84} E(\omega), E(\omega_1) = E(\omega_2) = \frac{1}{72} E(\omega), \) and \( E(\omega_3) = E(\omega_4) = \frac{1}{21} E(\omega). \)

**Proof.** By \(2\), we have

\[
E(\omega_1, \omega_3) = \int_{J_{\omega_1} \cup J_{\omega_3}} \|x - a(\omega_1, \omega_3)\|^2 dP = \int_{J_{\omega_1}} \|x - a(\omega_1, \omega_3)\|^2 dP + \int_{J_{\omega_3}} \|x - a(\omega_1, \omega_3)\|^2 dP
\]

\[
= p_{\omega_1}(s_{\omega_1}^2 V + \|a(\omega_1) - a(\omega_1, \omega_3)\|^2) + p_{\omega_3}(s_{\omega_3}^2 V + \|a(\omega_3) - a(\omega_1, \omega_3)\|^2).
\]

Notice that

\[
a(\omega_1, \omega_3) = \frac{1}{p_{\omega_1} + p_{\omega_3}} \left( p_{\omega_1} S_{\omega_1}(\frac{1}{2}, \frac{3}{4}) + p_{\omega_3} S_{\omega_3}(\frac{1}{2}, \frac{3}{4}) \right) = \frac{1}{8 + \frac{3}{8}} \left( \frac{1}{8} S_{\omega_1}(\frac{1}{2}, \frac{3}{4}) + \frac{3}{8} S_{\omega_3}(\frac{1}{2}, \frac{3}{4}) \right),
\]

which implies \( a(\omega_1, \omega_3) = \frac{1}{2} S_{\omega_1}(\frac{1}{2}, \frac{3}{4}) + \frac{3}{4} S_{\omega_3}(\frac{1}{2}, \frac{3}{4}) \). Thus, we have

\[
\|a(\omega_1) - a(\omega_1, \omega_3)\|^2 = \|S_{\omega_1}(\frac{1}{2}, \frac{3}{4}) - \frac{1}{4} S_{\omega_1}(\frac{1}{2}, \frac{3}{4}) - \frac{3}{4} S_{\omega_3}(\frac{1}{2}, \frac{3}{4})\|^2 = \frac{9}{16} s_{\omega}^2 \|0, \frac{2}{3}\|^2 = \frac{1}{4} s_{\omega}^2,
\]

\[
\|a(\omega_3) - a(\omega_1, \omega_3)\|^2 = \|S_{\omega_3}(\frac{1}{2}, \frac{3}{4}) - \frac{1}{4} S_{\omega_3}(\frac{1}{2}, \frac{3}{4}) - \frac{3}{4} S_{\omega_1}(\frac{1}{2}, \frac{3}{4})\|^2 = \frac{9}{16} s_{\omega}^2 \|0, \frac{2}{3}\|^2 = \frac{1}{4} s_{\omega}^2.
\]
and similarly, \(\|a(\omega_3) - a(\omega_1, \omega_3)\|^2 = \frac{1}{16} s_3^2 \| (0, \frac{2}{3}) \|^2 = \frac{1}{36} s_3^2\). Thus, we obtain,

\[
E(\omega_1, \omega_3) = p_\omega(s_1^2 V + \frac{1}{4} s_3^2) + p_\omega(s_3^2 V + \frac{1}{36} s_3^2) = p_\omega s_3^2 V(p_1 s_1^2 + p_3 s_3^2) + p_\omega s_3^2 (\frac{1}{4} p_1 + \frac{1}{36} p_3);
\]

and similarly, we can prove the rest of the lemma. Thus, the proof of the lemma is complete. \(\square\)

**Remark 3.7.** From the above lemma it follows that \(E(\omega_1, \omega_3) = E(\omega_2, \omega_4) > E(\omega_1, \omega_2) > E(\omega_3) = E(\omega_4) > E(\omega_1) = E(\omega_2)\).

The following lemma gives some important properties about the distortion error.

**Lemma 3.8.** Let \(\omega, \tau \in I^*\). Then

(i) \(E(\omega) > E(\tau)\) if and only if \(E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau) < E(\omega) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4)\);

(ii) \(E(\omega) > E(\tau_1, \tau_3)(= E(\tau_2, \tau_4))\) if and only if \(E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4) < E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau_1, \tau_2) + E(\tau_3) + E(\tau_4)\);

(iii) \(E(\omega_1, \omega_3)(= E(\omega_2, \omega_4)) > E(\tau_1, \tau_3)(= E(\tau_2, \tau_4))\) if and only if \(E(\omega_1, \omega_2) + E(\omega_3) + E(\omega_4) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4) < E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau_1, \tau_2) + E(\tau_3) + E(\tau_4)\);

(iv) \(E(\omega_1, \omega_3)(= E(\omega_2, \omega_4)) > E(\tau)\) if and only if \(E(\omega_1, \omega_2) + E(\omega_3) + E(\omega_4) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4)\);

(v) \(E(\omega_1, \omega_2) > E(\tau)\) if and only if \(E(\omega_1) + E(\omega_2) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4)\);

(vi) \(E(\omega_1, \omega_2) > E(\tau_1, \tau_3)(= E(\tau_2, \tau_4))\) if and only if \(E(\omega_1) + E(\omega_2) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4) < E(\omega_1, \omega_2) + E(\tau_1, \tau_2) + E(\tau_3) + E(\tau_4)\);

(vii) \(E(\omega_1, \omega_2) > E(\tau_1, \tau_2)\) if and only if \(E(\omega_1) + E(\omega_2) + E(\tau_1, \tau_2) < E(\omega_1, \omega_2) + E(\tau_1, \tau_2)\);

(viii) \(E(\omega) > E(\tau_1, \tau_2)\) if and only if \(E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau_1, \tau_2) < E(\omega) + E(\tau_1, \tau_2)\).

**Proof.** Let us first prove (iii). Using Lemma 3.6 we see that

\[
LHS = E(\omega_1, \omega_2) + E(\omega_3) + E(\omega_4) + E(\tau_1, \tau_3) + E(\tau_2, \tau_4) = \frac{5}{21} E(\omega) + \frac{31}{63} E(\tau),
\]

\[
RHS = E(\omega_1, \omega_3) + E(\omega_2, \omega_4) + E(\tau_1, \tau_2) + E(\tau_3) + E(\tau_4) = \frac{31}{63} E(\omega) + \frac{5}{21} E(\tau).
\]

Thus, \(LHS < RHS\) if and only if \(\frac{5}{21} E(\omega) + \frac{31}{63} E(\tau) < \frac{31}{63} E(\omega) + \frac{5}{21} E(\tau)\), which yields \(E(\omega) > E(\tau)\), i.e., \(E(\omega_1, \omega_3) > E(\tau_1, \tau_3)\). Thus (iii) is proved. The other parts of the lemma can similarly be proved. Thus, the lemma follows. \(\square\)

In the following theorem, we give the induction formula to determine the optimal sets of n-means for any \(n \geq 2\).

**Theorem 3.9.** For any \(n \geq 2\), let \(\alpha_n := \{a(i) : 1 \leq i \leq n\}\) be an optimal set of n-means, i.e., \(\alpha_n \subset \mathcal{C}_n := \mathcal{C}_n(P)\). For \(\omega \in I^*\), let \(E(\omega)\), \(E(\omega_1, \omega_3)\) and \(E(\omega_2, \omega_4)\) be defined by (4). Set

\[
\tilde{E}(a(i)) := \begin{cases} E(\omega) \text{ if } a(i) = a(\omega) \text{ for some } \omega \in I^*, \\ E(\omega_k, \omega_\ell) \text{ if } a(i) = a(\omega_k, \omega_\ell) \text{ for some } \omega \in I^*, \end{cases}
\]

where \((k = 1, \ell = 3)\), or \((k = 2, \ell = 4)\), or \((k = 1, \ell = 2)\), and \(W(\alpha_n) := \{a(j) : a(j) \in \alpha_n \text{ and } \tilde{E}(a(j)) \geq \tilde{E}(a(i)) \text{ for all } 1 \leq i \leq n\}\). Take any \(a(j) \in W(\alpha_n)\), and write

\[
\alpha_{n+1}(a(j)) := \begin{cases} (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega_1, \omega_3), a(\omega_2, \omega_4)\} \text{ if } a(j) = a(\omega), \\ (\alpha_n \setminus \{a(\omega_1, \omega_3), a(\omega_2, \omega_4)\}) \cup \{a(\omega_1, \omega_2), a(\omega_3), a(\omega_4)\} \text{ if } a(j) = a(\omega_1, \omega_3) \text{ or } a(\omega_2, \omega_4), \\ (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega_1), a(\omega_2)\} \text{ if } a(j) = a(\omega_1, \omega_2), \\ \end{cases}
\]
Then $\alpha_{n+1}(a(j))$ is an optimal set of $(n+1)$-means, and the number of such sets is given by
\[
\text{card} \left( \bigcup_{\alpha_n \in C_n} \{\alpha_{n+1}(a(j)) : a(j) \in W(\alpha_n)\} \right).
\]

Proof. By Proposition 3.1, Proposition 3.3 and Proposition 3.4, we know that the optimal sets of two-, three-, and four-means are respectively $\{a(1,3), a(2,4)\}$, $\{a(1,2), a(3), a(4)\}$, and

**Figure 2.** Tree diagram of the optimal sets from $\alpha_8$ to $\alpha_{21}$. 
that a obtained in this case is strictly greater than the corresponding error obtained in the case when \( \sum \). Thus, the lemma is true for \( n = 2 \) and \( n = 3 \). For any \( n \geq 3 \), let us now assume that \( \alpha_n \) is an optimal set of \( n \)-means. Let \( \alpha_n := \{ a(i) : 1 \leq i \leq n \} \). Let \( \tilde{E}(a(i)) \) and \( W(\alpha_n) \) be defined as in the hypothesis. If \( a(j) \notin W(\alpha_n) \), i.e., if \( a(j) \in \alpha_n \setminus W(\alpha_n) \), then by Lemma 3.8 the error

\[
\sum_{a(i) \in (\alpha_n \setminus \{ a(j) \})} \tilde{E}(a(i)) + E(\omega_1, \omega_3) + E(\omega_2, \omega_4) \quad \text{if} \quad a(j) = a(\omega),
\]

\[
\sum_{a(i) \in (\alpha_n \setminus \{ a(\omega_1, \omega_3), a(\omega_2, \omega_4) \})} \tilde{E}(a(i)) + E(\omega_1, \omega_2) + E(\omega_3) + E(\omega_4) \quad \text{if} \quad a(j) = a(\omega_1, \omega_3) \text{ or } a(\omega_2, \omega_4),
\]

\[
\sum_{a(i) \in (\alpha_n \setminus \{ a(j) \})} \tilde{E}(a(i)) + E(\omega_1) + E(\omega_2) \quad \text{if} \quad a(j) = a(\omega_1, \omega_2),
\]

obtained in this case is strictly greater than the corresponding error obtained in the case when \( a(j) \in W(\alpha_n) \). Hence for any \( a(j) \in W(\alpha_n) \), the set \( \alpha_{n+1}(a(j)) \), where

\[
\alpha_{n+1}(a(j)) := \begin{cases} 
(\alpha_n \setminus \{ a(j) \}) \cup \{ a(\omega_1, \omega_3), a(\omega_2, \omega_4) \} & \text{if} \quad a(j) = a(\omega), \\
(\alpha_n \setminus \{ a(\omega_1, \omega_3), a(\omega_2, \omega_4) \}) \cup \{ a(\omega_1, \omega_2), a(\omega_3), a(\omega_4) \} & \text{if} \quad a(j) = a(\omega_1, \omega_3) \text{ or } a(\omega_2, \omega_4), \\
(\alpha_n \setminus \{ a(j) \}) \cup \{ a(\omega_1), a(\omega_2) \} & \text{if} \quad a(j) = a(\omega_1, \omega_2),
\end{cases}
\]

is an optimal set of \((n + 1)\)-means, and the number of such sets is

\[
\text{card}\left( \bigcup_{\alpha_n \in \mathcal{C}_n} \{ \alpha_{n+1}(a(j)) : a(j) \in W(\alpha_n) \} \right).
\]

Thus the proof of the theorem is complete (also see Note 4.1).

**Remark 3.10.** Once an optimal set of \( n \)-means is known, by using (2), the corresponding quantization error can easily be calculated.

**Remark 3.11.** By Theorem 3.9 we note that to obtain an optimal set of \((n + 1)\)-means one needs to know an optimal set of \( n \)-means. We conjecture that unlike the homogeneous probability distribution, i.e., when the probability measures on the basic rectangles at each level of the Sierpiński carpet construction are equal, for the nonhomogeneous probability distribution considered in this paper, to obtain the optimal sets of \( n \)-means a closed formula can not be obtained.

Running the induction formula given by Theorem 3.9 in computer algorithm, we obtain some results and observations about the optimal sets of \( n \)-means, which are given in the following section.

### 4. Some results and observations

First, we explain about some notations that we are going to use in this section. Recall that the optimal set of one-mean consists of the expected value of the random variable \( X \), and the corresponding quantization error is its variance. Let \( \alpha_n \) be an optimal set of \( n \)-means, i.e., \( \alpha_n \in \mathcal{C}_n \), and then for any \( a \in \alpha_n \), we have \( a = a(\omega) \), or \( a = a(\omega_i, \omega_j) \) for some \( \omega \in I^*, \) where \((i = 1, j = 3), (i = 2, j = 4), \) or \((i = 1, j = 2)\). For \( \omega = \omega_1 \omega_2 \cdots \omega_k \in I^k, \) \( k \geq 1 \), in the sequel, we will identify the elements \( a(\omega) \) and \( a(\omega_i, \omega_j) \) by the set \( \{ \omega_1, \omega_2, \cdots, \omega_k \} \) and \( \{ \omega_1, \omega_2, \cdots, \omega_k, i \}, \{ \omega_1, \omega_2, \cdots, \omega_k, j \} \) respectively. Thus, we can write

\[
\alpha_2 = \{ \{ 1 \}, \{ 3 \}, \{ 2 \}, \{ 4 \} \}, \quad \alpha_3 = \{ \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \} \},
\]

\[
\alpha_4 = \{ \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \} \},
\]
and so on. For any \( n \geq 2 \), if \( \text{card}(\mathcal{C}_n) = k \), we write

\[
\mathcal{C}_n = \begin{cases} 
\{\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,k}\} & \text{if } k \geq 2, \\
\{\alpha_n\} & \text{if } k = 1.
\end{cases}
\]

If \( \text{card}(\mathcal{C}_n) = k \) and \( \text{card}(\mathcal{C}_{n+1}) = m \), then either \( 1 \leq k \leq m \), or \( 1 \leq m \leq k \) (see Table 1). Moreover, by Theorem 3.9, an optimal set at stage \( n \) can contribute multiple distinct optimal sets at stage \( n + 1 \), and multiple distinct optimal sets at stage \( n \) can contribute one common optimal set at stage \( n + 1 \); for example from Table 1 one can see that the number of \( \alpha_{21} = 8 \), the number of \( \alpha_{22} = 28 \), the number of \( \alpha_{23} = 56 \), the number of \( \alpha_{24} = 70 \), and the number of \( \alpha_{25} = 56 \).

By \( \alpha_{n,i} \rightarrow \alpha_{n+1,j} \), it means that the optimal set \( \alpha_{n+1,j} \) at stage \( n + 1 \) is obtained from the optimal set \( \alpha_{n,i} \) at stage \( n \), similar is the meaning for the notations \( \alpha_n \rightarrow \alpha_{n+1,j} \), or \( \alpha_{n,j} \rightarrow \alpha_{n+1} \), for example from Figure 3

\[
\begin{align*}
\{\alpha_{16} \rightarrow \alpha_{17,1}, \alpha_{16} \rightarrow \alpha_{17,2}, \alpha_{16} \rightarrow \alpha_{17,3}, \alpha_{16} \rightarrow \alpha_{17,4}\}; \\
\{\{\alpha_{17,1} \rightarrow \alpha_{18,1}, \alpha_{17,1} \rightarrow \alpha_{18,2}, \alpha_{17,1} \rightarrow \alpha_{18,4}\},\{\alpha_{17,2} \rightarrow \alpha_{18,1}, \alpha_{17,2} \rightarrow \alpha_{18,3}, \alpha_{17,2} \rightarrow \alpha_{18,5}\}, \\
\{\alpha_{17,3} \rightarrow \alpha_{18,2}, \alpha_{17,3} \rightarrow \alpha_{18,3}, \alpha_{17,3} \rightarrow \alpha_{18,6}\},\{\alpha_{17,4} \rightarrow \alpha_{18,4}, \alpha_{17,4} \rightarrow \alpha_{18,5}, \alpha_{17,4} \rightarrow \alpha_{18,6}\}\}; \\
\{\{\alpha_{18,1} \rightarrow \alpha_{19,1}, \alpha_{18,1} \rightarrow \alpha_{19,2}\},\{\alpha_{18,2} \rightarrow \alpha_{19,1}, \alpha_{18,2} \rightarrow \alpha_{19,3}\},\{\alpha_{18,3} \rightarrow \alpha_{19,1}, \alpha_{18,3} \rightarrow \alpha_{19,4}\}, \\
\{\alpha_{18,4} \rightarrow \alpha_{19,3}, \alpha_{18,4} \rightarrow \alpha_{19,3}\},\{\alpha_{18,5} \rightarrow \alpha_{19,2}, \alpha_{18,5} \rightarrow \alpha_{19,4}\},\{\alpha_{18,6} \rightarrow \alpha_{19,3}, \alpha_{18,6} \rightarrow \alpha_{19,4}\}\}; \\
\{\alpha_{19,1} \rightarrow \alpha_{20}, \alpha_{19,2} \rightarrow \alpha_{20}, \alpha_{19,3} \rightarrow \alpha_{20}, \alpha_{19,4} \rightarrow \alpha_{20}\}.
\end{align*}
\]

Moreover, one can see that

\[
\begin{align*}
\alpha_8 &= \left\{\{1, 1\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 1\}, \{2, 3\}, \{2, 2\}, \{2, 4\}, \{3, 1\}, \{3, 3\}, \{3, 2\}, \{3, 4\}, \{4, 1\}, \{4, 3\}, \{4, 2\}, \{4, 4\}\right\} \text{ with } V_8 = \frac{31}{2592} = 0.0119599; \\
\alpha_{9,1} &= \left\{\{3, 3\}, \{3, 4\}, \{1, 1\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 1\}, \{2, 3\}, \{2, 2\}, \{2, 4\}, \{3, 1\}, \{3, 2\}, \{4, 1\}, \{4, 3\}, \{4, 2\}, \{4, 4\}\right\}, \\
\alpha_{9,2} &= \left\{\{4, 3\}, \{4, 4\}, \{1, 1\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 1\}, \{2, 3\}, \{2, 2\}, \{2, 4\}, \{3, 1\}, \{3, 3\}, \{3, 2\}, \{3, 4\}, \{4, 1\}, \{4, 2\}\right\} \text{ with } V_9 = \frac{25}{2592} = 0.00964506; \\
\alpha_{10} &= \left\{\{3, 3\}, \{3, 4\}, \{4, 3\}, \{4, 4\}, \{1, 1\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 1\}, \{2, 3\}, \{2, 2\}, \{2, 4\}, \{3, 1\}, \{3, 2\}, \{4, 1\}, \{4, 2\}\right\} \text{ with } V_{10} = \frac{19}{2592} = 0.00733025,
\end{align*}
\]

and so on.

**Note 4.1.** Notice that there is only one optimal set of \( n \)-means for \( n = 72 \). By the notations used in Theorem 3.9, we can write \( \alpha_{72} = \{a(i) : 1 \leq i \leq 72\} \). Then,

\[
W(\alpha_{72}) = \{\{1, 3, 3\}, \{1, 3, 4\}, \{1, 4, 3\}, \{1, 4, 4\}, \{1, 4, 3\}, \{2, 3, 3\}, \{2, 3, 4\}, \{2, 4, 3\}, \{2, 4, 4\}, \{3, 1, 3\}, \{3, 1, 4\}, \{3, 2, 3\}, \{3, 2, 4\}, \{3, 3, 1\}, \{3, 3, 2\}, \{3, 4, 1\}, \{3, 4, 2\}, \{3, 4, 3\}, \{3, 4, 4\}, \{4, 1, 3\}, \{4, 1, 4\}, \{4, 2, 3\}, \{4, 2, 4\}, \{4, 3, 1\}, \{4, 3, 2\}, \{4, 4, 1\}, \{4, 4, 2\}\}.
\]

Since \( \text{card}(W(\alpha_{72})) = 24 \), by the theorem, we have \( \text{card}(\mathcal{C}_{73}) = \binom{24}{1} = 24 \), \( \text{card}(\mathcal{C}_{74}) = \binom{24}{2} = 276 \), \( \text{card}(\mathcal{C}_{75}) = \binom{24}{3} = 2024 \), \( \text{card}(\mathcal{C}_{76}) = \binom{24}{4} = 10626 \), etc., for details see Table 1.
Let us now conclude the paper with the following remark:

**Remark 4.2.** Consider a set of four contractive affine transformations \( S_{(i,j)} \) on \( \mathbb{R}^2 \), such that 
\[
S_{(1,1)}(x_1, x_2) = \left( \frac{1}{2} x_1, \frac{1}{4} x_2 \right), \quad S_{(2,1)}(x_1, x_2) = \left( \frac{1}{2} x_1 + \frac{1}{2}, \frac{1}{4} x_2 \right), \quad S_{(1,2)}(x_1, x_2) = \left( \frac{3}{4} x_1, \frac{1}{2} x_2 + \frac{1}{4} \right), \quad \text{and} \quad S_{(2,2)}(x_1, x_2) = \left( \frac{3}{4} x_1 + \frac{1}{2}, \frac{1}{2} x_2 + \frac{1}{4} \right)
\]
for all \((x_1, x_2) \in \mathbb{R}^2\). Let \( S \) be the limit set of these contractive mappings. Then, \( S \) is called the Sierpiński carpet generated by \( S_{(i,j)} \) for all \( 1 \leq i, j \leq 2 \). Let \( P \) be the Borel probability measure on \( \mathbb{R}^2 \) such that 
\[
P = \frac{1}{16} P \circ S_{(1,1)}^{-1} + \frac{3}{16} P \circ S_{(2,1)}^{-1} + \frac{3}{16} P \circ S_{(1,2)}^{-1} + \frac{9}{16} P \circ S_{(2,2)}^{-1}.
\]
Then, \( P \) has support the Sierpiński carpet \( S \). For this probability measure, the optimal sets of \( n \)-means and the \( n \)th quantization error are not known yet for all \( n \geq 2 \).

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