Projective symmetries and induced electromagnetism in metric-affine gravity

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ABSTRACT

We present a framework in which the projective symmetry of the Einstein-Hilbert action in metric-affine gravity is used to induce an effective coupling between the Dirac lagrangian and the Maxwell field. The effective U(1) gauge potential arises as the trace of the non-metricity tensor $Q_{\mu a}^a$ and couples in the appropriate way to the Dirac fields to allow for local phase shifts. On shell, the obtained theory is equivalent to Einstein-Cartan-Maxwell theory in presence of Dirac spinors.

1. Introduction

Metric-affine gravity is an extension of the usual (metric) theories of gravity, where the affine connection $\Gamma_{\mu\nu}^\rho$ is considered as an independent (in general, dynamical) degree of freedom, whose expression is ultimately determined by its own equation of motion. In particular, it has been shown \cite{1,2} that for the $N$-dimensional Einstein-Hilbert action, $S = \frac{1}{2\kappa} \int d^N x \sqrt{|g|} R(\Gamma)$ ($N > 2$), the most general expression for the affine connection is of the form

$$\tilde{\Gamma}_{\mu\nu}^\rho = \hat{\Gamma}_{\mu\nu}^\rho + V_\mu \delta_\rho^\nu,$$

where $\hat{\Gamma}_{\mu\nu}^\rho$ is the Levi-Civita connection and $V_\mu$ an arbitrary vector field. It turns out, however, that this vector field $V_\mu$ does not have any physically measurable influence: both the Einstein equation and the geodesic equation turn out to be identical to the equations in the metric formalism. In this sense, the metric and the Palatini formalism are equivalent for the Einstein-Hilbert action.

In \cite{2} it was argued that the vector field $V_\mu$ is related to the reparametrisation freedom of geodesics: affine geodesics of the connection $\hat{\Gamma}_{\mu\nu}^\rho$ turn out to be pre-geodesics of the Levi-Civita ones, through the reparametrisation

$$\frac{d\tau}{d\lambda}(\lambda) = \exp\left[\int_0^\lambda d\xi V_\rho d\xi'\right],$$

where $\lambda$ is the affine parameter for the $\tilde{\Gamma}_{\mu\nu}^\rho$ geodesics and $\tau$ the proper time along the Levi-Civita ones.

In a certain way, the solution (1) reflects the projective symmetry $\Gamma_{\mu\nu}^\rho \rightarrow \Gamma_{\mu\nu}^\rho + V_\mu \delta_\rho^\nu$, under which the Einstein-Hilbert is known to be invariant \cite{3,4}. Indeed, the Riemann tensor transforms under the projective transformation as $R_{\mu\nu\rho}^\lambda \rightarrow R_{\mu\nu\rho}^\lambda + F_{\mu\nu}^\rho (V) \delta_\rho^\lambda$ and the Ricci scalar $R = g^{\mu\nu} \delta_\nu^\lambda R_{\mu\nu\rho}^\lambda$ is easily seen to be invariant.

In more general theories, which include matter terms that couple to the connection, the Palatini connection (1) might not be a solution and the metric and the Palatini formalism will in general...
no longer be equivalent. However, as we will show, there is a way to restore the symmetry and at the same time give a physical meaning to the Palatini vector $V_\mu$. The previously arbitrary vector field will start to play the role of the electromagnetic potential of Maxwell theory and the projective symmetry of the entire action will be related to local $U(1)$ transformations.

We will present our model in the context of Einstein-Dirac theory in the metric-affine set-up, but it can equally well be done in presence of complex scalar fields. Since we will be working with Dirac spinors, we will switch to the tangent space description, where the metric degrees of freedom are represented by the *Vielbeins* $e^a_\mu$, which are the components of a local orthonormal coframe. The metric in this basis is then Minkowski, $g_{\mu \nu} = e^\mu_a e^\nu_b g_{ab}$, in any point of the manifold. Additionally, the affine connection is substituted by the components of the connection one-form, $\omega^{ab}_\mu$, through the appropriate basis transformation (sometimes called Vielbein Postulate). We will refer to $\omega^{ab}_\mu$ simply as *connection* from now on. Note that no antisymmetry in the last two indices is assumed, as the affine connection is not necessarily metric-compatible.

2. Spinor covariant derivatives

The first issue to address is the construction of the covariant derivative for the Dirac spinor in the metric-affine context. A popular choice is the natural extension of the Lorentz covariant derivative to arbitrary (non-symmetric) connections:

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \psi.$$  

(3)

However, as the Lorentz generator $\frac{1}{2} \gamma^{ab}$ is antisymmetric, only the antisymmetric part of the connection, $\omega^{[ab]}_\mu$, will couple to the spinors. Another possibility is to enhance the Lorentz generators to the general product of two gamma matrices and write a covariant derivative of the form

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \psi.$$  

(4)

Notice however that, even though $\omega_{\mu ab} \gamma^{ab}_\mu$ contains more terms than $\omega_{\mu ab} \gamma^{ab}$, most of the symmetric terms are projected out, due to the anti-commutation relations of the gamma matrices, $\{ \gamma^a, \gamma^b \} = 2\eta^{ab}$. The difference between the two covariant derivatives therefore reduces to the trace $2\omega_{\mu ab} \eta^{ab} \psi$, and the expression (4) can be written without loss of generality as

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \psi - \frac{1}{8} Q_{\mu a}^a \psi,$$  

(5)

where we have used the definition of the non-metricity tensor $Q_{\mu ab} \equiv -\nabla_\mu \eta_{ab}$ to relate its trace to the trace of the connection, $\omega_{\mu a}^a = \frac{1}{4} Q_{\mu a}^a$. There is actually no way the spinor can couple to the traceless symmetric part of the connection. Inspired by [5,6] we will write down a generalised expression for the spinor covariant derivative,

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \psi - k Q_{\mu a}^a \psi,$$  

(6)

(with $k$ in principle an arbitrary complex parameter), try to identify the gauge group this connection corresponds to and determine its dynamics through the first-order formalism.

It is not difficult to see that if $k$ is real-valued, the extra term in (6) is the gauge term for local rescalings $\psi \rightarrow e^{i\Omega} \psi$, provided that the trace of the non-metricity transforms as $Q_{\mu a}^a \rightarrow Q_{\mu a}^a + \partial_\mu \Omega$ [7]. On the other hand, for purely imaginary values of $k = i\epsilon$, (6) is a covariant derivative for local $U(1)$ transformations, $\psi \rightarrow e^{i\epsilon\Lambda} \psi$, provided that $Q$ transforms as $Q_{\mu a}^a \rightarrow Q_{\mu a}^a + \partial_\mu \Lambda$. Of course the trace of the non-metricity is not an independent field, but a part of the full connection $\omega_{\mu a}^b$. We should therefore embed the transformation of $Q_{\mu a}^a$ in a general transformation rule for $\omega_{\mu a}^b$. The simplest way of doing this are the transformation rules, respectively for a dilatation and a $U(1)$ transformation,

$$\omega_{\mu a}^b \rightarrow \omega_{\mu a}^b + \frac{1}{2N} \partial_\mu \Omega \delta_a^b, \quad \omega_{\mu a}^b \rightarrow \omega_{\mu a}^b + \frac{1}{2N} \partial_\mu \Lambda \delta_a^b,$$  

(7)

\[\text{We use the notation } \gamma^{ab...c} \equiv [\gamma^a \gamma^b...\gamma^c].\]
substituting \( (\delta \kappa) \) with \( \bar{\omega} \) Maxwell potential, Dirac current as source term, where the trace of the non-metricity is playing the role of the \( \delta \kappa \) independent field and there is no a priori reason to think of \( A \) role of the Maxwell potential together with the transformation rule (7b).

3. Induced electromagnetism from non-metricity

We are now in the position to write down an action for Einstein-Dirac theory that is invariant under the combined local \( U(1) \) phase shifts and projective transformation (7b):

\[
S = \int d^4x |e| \left[ \frac{1}{2\kappa} \mathcal{R}(\omega) - \frac{i}{2} F_{\mu\nu}(Q) F^{\mu\nu}(Q) + \frac{i\hbar}{2} (\bar{\psi} \gamma^\mu \nabla_\mu \psi + \nabla_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi \right].
\]

Here \( \mathcal{R}(\omega) \) is the Ricci scalar of the full connection, \( \mathcal{R}(\omega) = \varepsilon^\mu_\nu \varepsilon^\nu_\alpha \mathcal{R}_{\mu\alpha \beta}(\omega) \) with \( \mathcal{R}_{\mu\alpha \beta}(\omega) = 2 \partial_\beta \omega_{\mu\alpha}^a - 2 \omega_{[\mu\alpha}^c \omega_{\nu]\beta]_c \) and \( F_{\mu\nu}(Q) = 2 \partial_\mu Q_{\nu\alpha}^a \). Note that we have added explicitly a gauge invariant kinetic term for the trace of the non-metricity, since these degrees of freedom do not appear in \( \mathcal{R}(\omega) \), as a consequence of the projective symmetry. This term was also introduced in [7], though without explicitly relating it to the \( U(1) \) symmetry of the matter action.\(^3\)

It is important to realise that the only degrees of freedom of the action (9) are the Vielbeins \( e^a_\mu \), the connection \( \omega_{\mu\alpha}^b \) and the Dirac spinor \( \psi \). In particular, at this stage \( Q_{\mu\alpha}^a \) is not an independent field and there is no a priori reason to think of \( Q_{\mu\alpha}^a \) as the Maxwell potential. The dynamics of \( Q_{\mu\alpha}^a \) should in fact be fully derived from the dynamics of \( \omega_{\mu\alpha}^b \).

The equation of motion of the connection,

\[
0 = \frac{\kappa}{|e|} e^b_\sigma e^\sigma_a \frac{\delta S}{\delta \omega_{\mu\alpha}^b} = \frac{i}{2} T_{\lambda\sigma}^\mu g^{\rho\lambda} - \delta_\sigma^\mu Q_{\lambda}^\lambda \rho - \delta_\sigma^\mu g^{\lambda\rho} \left( \frac{i}{2} Q_{\lambda\tau} - T_{\lambda\tau} \right) \\
+ \kappa \left[ 2 \left( \bar{\nabla}_\nu F^{\nu\mu}(Q) - i\hbar \bar{\psi} \gamma^\mu \psi \right) \delta_\rho^\nu - \frac{i\hbar}{4} \bar{\psi} \gamma^{\mu\rho} \sigma \psi \right],
\]

can be solved in full generality. Indeed, taking the \( \delta_\rho^\nu \) trace, the equation reduces to

\[
\bar{\nabla}_\nu F^{\nu\mu}(Q) = i\hbar \bar{\psi} \gamma^\mu \psi,
\]

with \( \bar{\nabla} \) the Levi-Civita covariant derivative. This is clearly the Maxwell equation with the vector Dirac current as source term, where the trace of the non-metricity is playing the role of the Maxwell potential, \( Q_{\mu\alpha}^a \equiv A_\mu \) (see also [7]). Once (11) is taken into account, the equation of motion (10) simplifies considerably: if we define \( S_{\mu\alpha}^b \equiv \frac{i\hbar}{2} \bar{\psi} \gamma_{\mu\alpha}^b \psi \), then the most general solution of the traceless part of (10) is given by \([1,9]\)

\[
\omega_{\mu\alpha}^b = \bar{\omega}_{\mu\alpha}^b + V_\mu \delta_\alpha^b + \kappa S_{\mu\alpha}^b,
\]

where \( \bar{\omega}_{\mu\alpha}^b \) is the Levi-Civita connection in the anholonomic frame and \( V_\mu \) is a one-form, which is basically the vector field encountered in (1). While for the Einstein-Hilbert action the vector field \( V_\mu \) was completely arbitrary and void of physical content, in this context it takes up the role of the Maxwell potential \( A_\mu \). Indeed, it is straightforward to see that \( V_\mu \) represents the trace \( Q_{\mu\alpha}^a = 2N V_\mu \) of the non-metricity \( Q_{\mu\alpha\beta} = 2N_{\alpha\beta} \) of the solution (12) and hence not only is its dynamics dictated by the Maxwell equation (11), but also the projective symmetry (7) makes it behave as a \( U(1) \) gauge field. For future reference, we note that the torsion of the solution (12) is given by \( T_{\mu\nu}^\rho = 2V_\mu \delta_\nu^\rho + 2\kappa S_{\mu\nu}^\rho \).

On the other hand, the equations of motion for Dirac spinor \( \psi \) can be easily computed. Upon substituting (12), its takes the form of the inhomogeneous Dirac equation coupled to the electromagnetic field, with \( S_{\mu\alpha}^b \) acting as its source term:

\[
\frac{i\hbar}{4} \bar{\psi} \gamma^{\mu\rho} S_{\mu\rho} \psi.
\]

\(^3\)Including this term is equivalent to adding the quadratic curvature term \( \mathcal{R}_{\mu\nu\sigma}^a \mathcal{R}_{\mu\nu\sigma}^b \) (see for example [8]).
Finally, after taking into account the solution (12), the Vielbein equation of motion splits into its symmetric and antisymmetric parts,

\[
\hat{R}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \hat{R} = \frac{1}{4} \kappa^2 g_{\mu \nu} S^{\rho \lambda \sigma} S_{\rho \lambda \sigma} + \kappa \left[ F_{\mu \lambda} (A) F_{\nu \lambda} (A) - \frac{1}{4} g_{\mu \nu} F_{\lambda \sigma} (A) F^{\lambda \sigma} (A) \right] \\
- \frac{i \kappa}{2} \left[ \bar{\psi} \gamma_{\nu} \left( \hat{\nabla}_{\mu} - \text{i} e A_{\mu} \right) \psi - \left( \hat{\nabla}_{\mu} + \text{i} e A_{\mu} \right) \bar{\psi} \gamma_{\nu} \psi \right],
\]

\[
\hat{\nabla}_{\lambda} S_{\mu \nu}^{\lambda} = \frac{i \kappa}{2} \left[ \bar{\psi} \gamma_{\nu} \left( \hat{\nabla}_{\mu} - \text{i} e A_{\mu} \right) \psi - \left( \hat{\nabla}_{\mu} + \text{i} e A_{\mu} \right) \bar{\psi} \gamma_{\nu} \psi \right],
\]

which play the role of the Einstein equation and an equation that renders the torsion dynamical. Note that the total energy-momentum tensor on the right-hand side of the Einstein equation does not only contain the standard contributions of the electromagnetic and the Dirac field, but also an additional contribution of \( S_{\mu a} b \), quadratic in \( \kappa \). The set of equations (11), (13) and (14) are the usual equations of motion of Einstein-Cartan gravity, coupled to the electromagnetic field and to Dirac spinors.

4. Discussion

We have shown that in the framework of metric-affine gravity, the Maxwell field can be geometrised and be interpreted as part of a more general connection \( \omega_{\mu a} b \). In particular, the Maxwell equation is encoded in (the trace of) the equation of motion of the connection. In order to obtain this it is necessary to relate the \( U(1) \) transformation of the matter fields with the projective transformation of the connection that leaves the Einstein-Hilbert term, and hence the entire action, invariant. The equations of motion, after substituting the solution for the connection, turn out to be equivalent to the equations of motion of the Einstein-Cartan theory (i.e. the Einstein theory with a metric-compatible torsionful connection) in the presence of electromagnetism and a Dirac field.

Actually, it is not that surprising that the action (9) is on-shell equivalent to Einstein-Cartan gravity, coupled to the Maxwell and Dirac fields. Consider the decomposition of the general connection \( \omega_{\mu a} b \) into its antisymmetric (and hence metric-compatible), its traceless symmetric and its trace parts,

\[
\omega_{\mu a} b = \tilde{\omega}_{\mu a} b + V_{\mu} \delta^{b}_{a} + \tilde{Q}_{\mu a} b,
\]

where \( \tilde{\omega}_{\mu (ab)} = \tilde{Q}_{\mu [ab]} = \tilde{Q}_{\mu a c} = 0 \). Under this decomposition, the Ricci scalar \( \mathcal{R}(\omega) \) takes the form

\[
\mathcal{R}(\omega) = \tilde{\mathcal{R}}(\tilde{\omega}) + \tilde{Q}_{\mu a b} \tilde{Q}^{\mu a b} - \tilde{Q}_{\mu}^{\mu \lambda \nu} \tilde{Q}_{\nu}^{\nu \lambda},
\]

where \( \tilde{\mathcal{R}}(\tilde{\omega}) \) is the Ricci scalar of \( \tilde{\omega}_{\mu a b} \), which is a connection in its own right. Note that the trace \( V_{\mu} \) does not contribute to the Ricci scalar, due to the projective symmetry.

With this decomposition, the action (9) can be written as

\[
S = \int d^{N} x |e| \left[ \frac{1}{2 \kappa} \tilde{\mathcal{R}}(\tilde{\omega}) + \frac{1}{2 \kappa} \left( \tilde{Q}_{\mu \lambda \nu} \tilde{Q}^{\mu \lambda \nu} - \tilde{Q}_{\mu}^{\mu \lambda \nu} \tilde{Q}_{\nu}^{\nu \lambda} \right) - \frac{i}{4} F_{\mu \nu} (V) F^{\mu \nu} (V) + \mathcal{L}_{\text{Dirac}} \right].
\]

In this set-up it is clear from its kinetic term and its coupling to the Dirac spinors that the trace of the connection \( V_{\mu} \) already plays the role of the Maxwell potential. However the action (17) is not yet equivalent to the standard Einstein-Cartan-Maxwell-Dirac theory, due to the presence of the traceless symmetric part of the connection \( \tilde{Q}_{\mu \nu \rho} \).

The equation of motion (10) of the original connection \( \omega_{\mu a} b \) must be equivalent to the combined equations of motion of its three parts: on the one hand, the equation for the metric-compatible connection \( \tilde{\omega}_{\mu a} b \) yields the standard Einstein-Cartan-Dirac solution \( \tilde{\omega}_{\mu a} b = \tilde{\omega}_{\mu a} b + \kappa S_{\mu a} b \) and the equation for \( V_{\mu} \) gives rise directly to the Maxwell equation coupled to the spinors. On the other hand, the traceless symmetric part \( \tilde{Q}_{\mu \nu \rho} \) appears quadratically, implying its own triviality,

\[
\tilde{Q}_{\mu \nu \rho} = 0.
\]

\[1\]
So, with $\hat{Q}_{\mu\nu\rho}$ on-shell, we easily see that the action (17) becomes the action of the Einstein-Cartan theory coupled to electromagnetism and a Dirac field. Consequently, the tensor $S_{\mu a}^{\ b}$ turns out to be the spin-density current associated to the Lorentz connection $\tilde{\omega}_{\mu a}^{\ b}$.

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