Supergravity background
of $\lambda$-deformed model for $AdS_2 \times S^2$ supercoset

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Abstract

Starting with the $\hat{F}/G$ supercoset model corresponding to the $AdS_n \times S^n$ superstring one can define the $\lambda$-model of arXiv:1409.1538 either as a deformation of the $\hat{F}/\hat{F}$ gauged WZW model or as an integrable one-parameter generalization of the non-abelian T-dual of the $AdS_n \times S^n$ superstring sigma model with respect to the whole supergroup $\hat{F}$. Here we consider the case of $n = 2$ and find the explicit form of the 4d target space background for the $\lambda$-model for the $PSU(1,1|2)/SO(1,1) \times SO(2)$ supercoset. We show that this background represents a solution of type IIB 10d supergravity compactified on a 6-torus with only metric, dilaton $\Phi$ and the RR 5-form (represented by a 2-form $F$ in 4d) being non-trivial. This implies that the $\lambda$-model is Weyl invariant at the quantum level and thus defines a consistent superstring sigma model. The supergravity solution we find is different from the one in arXiv:1410.1886 which should correspond to a version of the $\lambda$-model where only the bosonic subgroup of $\hat{F}$ is gauged. Still, the two solutions have equivalent scaling limit of arXiv:1504.07213 leading to the isometric background for the metric and $e^\Phi F$ which is related to the $\eta$-deformed $AdS_2 \times S^2$ sigma model of arXiv:1309.5850. Similar results are expected in the $AdS_3 \times S^3$ and $AdS_5 \times S^5$ cases.

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1 Introduction

There are two special integrable models that are closely associated with the superstring sigma model on $AdS_n \times S^n$. One is the $\eta$-model [1] – a particular integrable deformation of the $AdS_n \times S^n$ supercoset model generalising the bosonic Yang-Baxter sigma model of [2]. The other one is the $\lambda$-model [3, 4] generalising the bosonic model of [5] (see also [6]). The $\lambda$-model is based on the $\hat{F}/\hat{F}$ gauged WZW model closely related to the $AdS_n \times S^n$ supercoset and may be interpreted as an integrable deformation of the non-abelian T-dual of the $AdS_n \times S^n$ supercoset action.

While for the $\eta$-model the corresponding target space background was found in [7, 8, 9] (but turns out not to be a supergravity solution [18]), in the case of the $\lambda$-model the GS sigma model action was so far not determined directly apart from the metric [11, 12] and the dilaton [4, 13]. Our aim below will be to find the full $\lambda$-model background (metric, dilaton and the RR field strength) from the $\lambda$-model action and also as a solution of the type II supergravity equations.

We shall consider the simplest example of the $AdS_2 \times S^2$ model. The resulting background
differs from the supergravity solution based on the metric and dilaton of the bosonic model that was found in [11].

Let us start with a brief review of the $\lambda$-model [4] (see also [13]). The $\lambda$-model may be interpreted as a unique integrable deformation of the first-order action that interpolates between the supercoset $AdS_n \times S^n$ superstring model and its non-abelian T-dual with respect to the full supergroup symmetry. In general, one may consider a model based on the supercoset $\tilde{G}/G \supset F_1$ where $\tilde{G}$ is a supergroup (e.g. $PSU(2,2|4)$ in the $AdS_5 \times S^5$ case or $PSU(1,1|2)$ in the $AdS_2 \times S^2$ case) and $F_1$ and $G_i$ are bosonic subgroups. The $\lambda$-model is defined by the action

$$\hat{I}_{k,\lambda}(f, A) = \frac{k}{4\pi} \left( \int d^2x \ STp \left[ \frac{1}{2} f^{-1} \partial_+ f f^{-1} \partial_- f + A_+ \partial_- f f^{-1} \right] - A_- f^{-1} \partial_+ f - f^{-1} A_+ f A_- + A_+ A_- \right)$$

$$- \frac{1}{3} \int d^3x \epsilon^{abc} STp \left[ f^{-1} \partial_a f f^{-1} \partial_b f f^{-1} \partial_c f \right] + (\lambda^{-2} - 1) \int d^2x \ STp \left[ A_+ P_\lambda A_- \right] \right), \quad (1.1)$$

where $f \in \tilde{G}$, $A_\pm \in \mathfrak{f} = \text{alg}(\tilde{G})$ and $P_\lambda$ is a combination of $Z_4$ projectors

$$P_\lambda = P^{(2)} + \frac{1}{\lambda^{-1} + 1} \left( P^{(1)} - \lambda P^{(3)} \right). \quad (1.2)$$

All but the last term in (1.1) correspond to the $\tilde{G}/\tilde{G}$ gauged WZW model with integer coupling (level) $k$ and $\lambda$ is a “deformation” parameter. This action has no global symmetry but there is a local fermionic symmetry and a $G_1 \times G_2$ gauge symmetry which will be fixed by a condition on $f$ after integrating out the gauge fields.

The direct limit $\lambda \to 1$ for fixed $k$ leaves one with $\tilde{G}/\tilde{G}$ gauged WZW model. One can also consider another special limit of $\lambda \to 1$ combined with sending $k \to \infty$ and scaling the supergroup field $f \to 1$ as in [5]

$$f = \exp(-\frac{4\pi}{k} v) = 1 - \frac{4\pi}{k} v + \mathcal{O}(k^{-2}) , \quad \lambda = 1 - \frac{\pi h}{k} + \mathcal{O}(k^{-2}) , \quad k \to \infty , \quad (1.3)$$

where $v \in \mathfrak{f}$ and $h$ are kept fixed. This leads to the following action

$$\hat{I}_{k \to \infty, \lambda \to 1}(f \to 1, A) = \int d^2x \ STp \left[ v \left( \partial_- A_+ - \partial_+ A_- + [A_-, A_+] \right) \right] + \frac{1}{2} h \int d^2x \ STp \left( A_+ PA_- \right), \quad (1.4)$$

where $P = P_\lambda|_{\lambda=1} = P^{(2)} + \frac{1}{2} \left( P^{(1)} - P^{(3)} \right)$ is the projector appearing in the standard $AdS_n \times S^n$ superstring Lagrangian $L = STp \left[ J_+^1 J_-^2 + \frac{1}{2} (J_+^{[3]} J_-^{[1]} - J_+^{[1]} J_-^{[3]}) \right]$ [15, 16]. Eq. (1.4) may be interpreted as a first-order action “interpolating” between the $AdS_n \times S^n$ supercoset action (if one first integrates out $v$ getting $A_\pm = g^{-1} \partial_\pm g$) and its non-abelian T-dual with respect to the full supergroup (if one first integrates out $A_\pm$).\(^2\) Thus the $\lambda$-model (1.1) may be interpreted as a deformation of the first-order interpolating action (1.4). If one first integrates out $A_\pm$ in (1.1)

\(^1\)Equivalently, $(\lambda^{-2} - 1) A_+ P_\lambda A_- = A_+ (\Omega - 1) A_-$, where $\Omega = P^{(0)} + \lambda^{-2} P^{(2)} + \lambda^{-1} P^{(1)} + \lambda P^{(3)}$.

\(^2\)Here the non-abelian duality contains both the standard bosonic and also the fermionic transformations like in the abelian fermionic T-duality in [14].
and gauge-fixes the supergroup field $f$ the resulting sigma model may be viewed as a deformation of the non-abelian T-dual of the original $AdS_n \times S^m$ supercoset model.  

Next, let us recall the relations between parameters of the $\eta$-model and $\lambda$-model [1, 4]. In terms of the Poisson algebra deformation parameter $\epsilon$ the parameter $\eta$ of [1] (or $\propto$ introduced in [7]) is

$$\epsilon^2 = \frac{4\eta^2}{(1 + \eta^2)^2} = \frac{\chi^2}{1 + \chi^2}, \quad \epsilon^2 \in [0, 1], \quad \eta^2 \in [0, 1], \quad \chi^2 \in [0, \infty], \quad (1.5)$$

The parameter $\lambda$ in the action (1.1) of [4] is related to $\epsilon^2$ by

$$\epsilon^2 = -\frac{(1 - \lambda^2)^2}{4\lambda^2} = -\frac{1}{4b^2(1 + b^2)}, \quad \epsilon^2 \in [-\infty, 0], \quad \lambda^2 \in [0, 1], \quad b^2 \in [0, \infty]. \quad (1.6)$$

Here $\chi = \frac{2\eta}{1 - \eta^2}$ and $b^2 = \frac{\lambda^2}{1 - \lambda^2}$ are natural deformation parameters in the bosonic parts of the two models. Comparing (1.5) and (1.6) the parameters of the two deformed models may be related by an analytic continuation [13]

$$\eta = i\frac{1 - \lambda}{1 + \lambda}, \quad \lambda = i\frac{1 - \eta}{i + \eta}, \quad \chi = i\kappa, \quad \kappa \equiv \frac{1}{1 + 2b^2} = \frac{1 - \lambda^2}{1 + \lambda^2}, \quad \lambda = \sqrt{\frac{1 - \kappa}{1 + \kappa}}, \quad (1.7)$$

where $\kappa \in [0, 1]$ (for $\lambda^2 \in [0, 1]$) is the parameter that we will often use below instead of $\lambda$. Also, the overall couplings of the two models are related by ($h$ is the tension of the $\eta$-model)$^4$

$$k = i\frac{h}{\pi}, \quad \text{i.e.} \quad h = k\pi. \quad (1.9)$$

As was found in [13], the relations (1.7), (1.9) allow one to obtain the metric of the $\eta$-model as a special limit of the $\lambda$-model metric (1.1). More precisely, this singular limit (that generates isometries corresponding to the bosonic Cartan directions) leads to an abelian T-dual of the $\eta$-model metric [13].

Starting with the bosonic version of the $\lambda$-model in (1.1) corresponding to the $AdS_n \times S^m$ supercoset and integrating out the gauge fields $A^{(2)}_{\pm}$ one can find the corresponding metric and dilaton $\Phi_B$ field [11, 12]. In [11, 12] this background was embedded as a solution into type II supergravity by finding the corresponding RR field strength. The limit [13] of this supergravity background was shown [17] to give a type II supergravity solution which has the metric and RR field $F = e^{\Phi_B}F$ which are related by the standard T-duality rule to the metric and $\mathcal{F}$ extracted from the $\eta$-model action in [7, 9]. However, this scaling limit leaves a term in the dilaton which is

$^3$Another special limit of (1.1) is $\lambda \to 0$ in which $A_+ (\Omega - 1)A_- \to A_+ (\lambda^{-2}P_2 + \lambda^{-1}P_1 - P_3)A_-$ implying that we should set $A^{(2)}_\pm = 0, A^{(1)}_\pm = 0, A^{(3)}_\pm = 0$ so that the remaining gauge fields are from the bosonic subalgebra and "half" of the fermionic directions (reflecting the presence of $\propto$-symmetry in the model resulting upon integrating out the remaining $A^{(0)}_\pm, A^{(1)}_\pm, A^{(3)}_\pm$ fields). Integrating out gauge fields and fixing gauge on $f$ will still lead to a non-trivial background discussed below.

$^4$The relation between the quantum deformation parameters $q$ for the two models (cf. [1, 7, 3, 4, 13]) is $q = e^{-\frac{\pi}{2\kappa}} \leftrightarrow q = e^{-\frac{\pi}{2\chi}},$ with the real $q$ corresponding to the $\eta$-model and the root of unity $q$ to the $\lambda$-model.
linear in isometric coordinates thus obstructing the application of the standard T-duality to the full background. This is an explanation for why the η-model background does not correspond to a solution of supergravity. Indeed, it was found to solve only the weaker one-loop scale invariance conditions but not the Weyl invariance conditions (equivalent to the supergravity equations) for the corresponding superstring sigma model [18].

Starting instead with the full supercoset λ-model in (1.1) and integrating out both the bosonic and fermionic components of the gauge fields $A_\pm$ one gets the same effective sigma model metric as in the bosonic model case [11] but the expression for the dilaton turns out to be different from the “bosonic” one in [11, 12] containing an extra “numerator” factor from integration over the fermionic components of the gauge fields [4, 13]. As we shall show below on the $AdS_2 \times S^2$ supercoset example, the resulting metric-dilaton background also solves the type II supergravity equations when supplemented by a proper RR field strength $F$ which is different from the one in [11]. Thus the same λ-model metric can be embedded into type II supergravity using at least two different $(\Phi, F)$ pairs. Similar non-uniqueness of the supergravity solutions was observed in [20] in the η-model context.

Furthermore, we shall show that it is the combination $\mathcal{F} = e^{\Phi} F$ of this RR field strength with the λ-model dilaton [13] that indeed directly corresponds to the sigma model that originates from the λ-model (1.1), i.e. this background is the one that corresponds to the λ-model of [4]. The fact that this background solves the supergravity equations confirms that the λ-model is not only scale-invariant [19] but (in contrast to the η-model [18]) is also Weyl-invariant as a quantum sigma model on a curved 2d background and thus it defines a consistent superstring theory.

The scaling limit [13] applied to this new solution leads to an equivalent background to the one found in [13] from the scaling limit of the “bosonic” solution, in agreement with the expected relation between the λ-model and η-model.\(^5\)

The structure of this paper is as follows. We shall start in section 2 with reviewing the form of the 4d metric and dilaton corresponding to the λ-model for the $AdS_2 \times S^2$ supercoset. We shall then present the solution of type IIB supergravity compactified on a 6-torus that supports this metric and dilaton background by a RR 5-form background.

In section 3 we shall explain how to extract the metric, dilaton and this RR background (1.1) by integrating out the gauge fields $A_\pm$ in the λ-model action (1.1), writing the resulting quadratic fermionic action in the GS superstring sigma model form and using the κ-symmetry invariance [4] of the resulting action. We shall use a short-cut method based on studying the structure of the κ-symmetry variation of the world-sheet metric.

Section 4 will contain some concluding remarks. In Appendix A we will check that the λ-model

\(^5\)The equivalence between the scaling limits of the “bosonic” background that should correspond to gauging just bosonic generators and the full η-model background may be understood by noting that the scaling limit “blows up” the bosonic Cartan directions, and thus the gauging of the fermionic directions should not be important.
background is a solution of type IIB supergravity directly in 10 dimensions starting with the type II equations of motion written in bispinor notation for the RR field strengths. Appendix B will present the realisation of $\mathfrak{su}(1,1|2)$ used in section 3. Appendices C and B will contain some technical details on $\kappa$-symmetry variations and representation of Dirac matrices.

2 $\lambda$-model background for the $\text{AdS}_2 \times S^2$ supercoset

2.1 Metric

To find the target space metric it is sufficient to set fermions to zero, i.e. consider just the bosonic version of the $\lambda$-model [11]. In the case of $\text{AdS}_2 \times S^2$ the relevant bosonic coset space is $SO(2,1) \times SO(3)$. Starting with the $\lambda$-model action (1.1), integrating out the gauge field and imposing a gauge-fixing condition on the $SO(2,1) \times SO(3)$ field $f$ by choosing it as [13]

$$f = \begin{bmatrix} \exp(it\sigma_3) \exp(\xi\sigma_1) \\ \exp(i\varphi\sigma_3) \exp(i\zeta\sigma_1) \end{bmatrix},$$

we find the following metric

$$T^{-1}ds^2 = \kappa \left[ -dt^2 + \cot^2 t d\xi^2 - (\kappa^{-2} - 1)(\cosh \xi dt - \cot t \sinh \xi d\xi)^2 ight.$$  
$$+ d\varphi^2 + \cot^2 \varphi d\zeta^2 + (\kappa^{-2} - 1)(\cos \zeta d\varphi + \cot \varphi \sin \zeta d\zeta)^2 \right].$$

(2.2)

In [11] a different coordinate patch was used where the metric is related to the one in (2.2) by the analytic continuation $(t,\xi) \to (\tilde{t},\tilde{\xi})$ with $t = -i\tilde{\xi}, \xi = \tilde{t}$. Explicitly, choosing

$$f = \begin{bmatrix} \exp(\tilde{\xi}\sigma_2) \exp(\tilde{\sigma_1}) \\ \exp(i\varphi\sigma_3) \exp(\tilde{\zeta}\sigma_1) \end{bmatrix},$$

(2.3)

leads to

$$T^{-1}\tilde{ds}^2 = \kappa \left[ d\tilde{\xi}^2 - \coth^2 \tilde{\xi} d\tilde{\tilde{t}}^2 + (\kappa^{-2} - 1)(\cosh \tilde{t} d\tilde{\xi} + \coth \tilde{\xi} \sinh \tilde{\xi} d\tilde{t})^2 ight.$$  
$$+ d\varphi^2 + \cot^2 \varphi d\zeta^2 + (\kappa^{-2} - 1)(\cos \zeta d\varphi + \cot \varphi \sin \zeta d\zeta)^2 \right].$$

(2.4)

The metric (2.4) (times 6-torus) can be embedded [11] into type II 10d supergravity if supplemented with a real dilaton and RR 5-form flux $F$ while a similar embedding of (2.2) requires an imaginary RR flux.

The “real” patch choice of (2.3) is more natural in the context of the full supercoset $\lambda$-model and as we shall see below the corresponding RR background will again be real for (2.4) and imaginary for (2.2). Note that the metric (2.2) or (2.4) has no isometries. The reason why (2.2) was preferred in [13] is that it admits a special singular coordinate redefinition in which the first 2d part of the 4d metric develops a time-like (rather than space-like as for (2.4)) isometry and thus is related to the metric corresponding to the $\eta$-deformed $\text{AdS}_2 \times S^2$ model of [10, 1].

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\*We shall use similar notation for the bosonic part of the action as in [13]: $I = \frac{1}{2} \int d^2x g_{mn}(X) \partial_+ X^m \partial_+ X^n$ with $ds^2 = g_{mn}(X)dX^m dX^n$, i.e. the tension $T = \frac{k}{\pi}$ in the $\lambda$-model or $T = h$ in the $\eta$-model will be included in the metric.
The two metrics (2.2) and (2.4) look essentially the same when written in the algebraic coordinates \((x, y; p, q)\) defined for (2.2) by [13]:

\[
\begin{align*}
    t &= \arccos \sqrt{\kappa x^2 - \kappa^{-1} y^2} , \\
    \xi &= \operatorname{arccosh} \frac{\kappa^{1/2} x}{\sqrt{\kappa x^2 - \kappa^{-1} y^2}} , \\
    \kappa x^2 - \kappa^{-1} y^2 &\leq 1 , \\
    \varphi &= \arccos \sqrt{\kappa p^2 + \kappa^{-1} q^2} , \\
    \zeta &= \arccos \frac{\kappa^{1/2} p}{\sqrt{\kappa p^2 + \kappa^{-1} q^2}} , \\
    \kappa p^2 + \kappa^{-1} q^2 &\leq 1 .
\end{align*}
\]  

(2.5)

(2.6)

Then (2.2) takes a simple diagonal form:

\[
T^{-1} ds^2 = \frac{1}{1 - \kappa x^2 + \kappa^{-1} y^2}(-dx^2 + dy^2) + \frac{1}{1 - \kappa p^2 - \kappa^{-1} q^2}(dp^2 + dq^2) .
\]  

(2.7)

This metric has an asymptotically flat region and no isometries. The metric (2.4) is also given by (2.7) in the coordinate patch where:

\[
\kappa x^2 - \kappa^{-1} y^2 \geq 1 ,
\]  

(2.8)

i.e. when \(y\) is time-like and \(x\) is space-like.\(^8\)

In what follows we shall formally use the metric (2.7) with an understanding that one can always consider the physical region for the \(\lambda\)-model (2.8) where (2.7) will be supported by a real dilaton and RR background.

### 2.2 Dilaton

Assuming that the \(\lambda\)-model defined by (1.1) has no "bare" dilaton term, the dilaton should be generated (as in the standard T-duality transformation or gWZW models) upon integrating out the gauge fields \(A_{\pm}\). In the purely bosonic \(\lambda\)-model one then gets for (2.1), i.e. for the metric (2.2),(2.7) \((e^{\Phi_0} = T\kappa e^{\overline{\Phi}_0})\)

\[
e^{\Phi_B} = e^{\overline{\Phi}_0}(\sqrt{g})^{1/2} = \frac{e^{\Phi_0}}{\sin t \sin \varphi} = \frac{e^{\Phi_0}}{\sqrt{(1 - \kappa x^2 + \kappa^{-1} y^2)(1 - \kappa p^2 - \kappa^{-1} q^2)}} .
\]  

(2.9)

The dilaton corresponding to (2.3), i.e. for the metric (2.4) or (2.7) in the region \(1 - \kappa x^2 + \kappa^{-1} y^2 < 0\) is found by the obvious analytic continuation leading to a factor of \(i\) that can be absorbed into a shift of \(\Phi_0\). This dilaton

\[
e^{\Phi_B} = \frac{e^{\Phi_0}}{\sqrt{-(1 - \kappa x^2 + \kappa^{-1} y^2)(1 - \kappa p^2 - \kappa^{-1} q^2)}}
\]  

(2.10)

was assumed as a starting point for constructing a supergravity embedding for the metric (2.4) in [11].

\(^7\)Compared to [13] we rescaled these coordinates by factors of \(\kappa\) to make the metric manifestly conformally flat. Explicitly, \(x = \kappa^{-1/2}\cos t \cosh \xi, \ y = \kappa^{1/2}\cos t \sinh \xi, \ p = \kappa^{-1/2}\cos \varphi \cos \zeta, \ q = \kappa^{1/2}\cos \varphi \sin \zeta\). For the coordinate patch used in [11] we have \(x = \kappa^{-1/2}\cosh \tilde{\xi} \cosh \tilde{t}, \ y = \kappa^{1/2}\cos t \sinh \tilde{t}, \) so that instead of \(\kappa x^2 - \kappa^{-1} y^2 \leq 1\) we have \(\kappa x^2 - \kappa^{-1} y^2 \geq 1\), i.e. \(y\) rather than \(x\) is playing the role of a time-like direction.

\(^8\)The first 2d part of the metric (2.4) written in similar conformal coordinates appeared also in eq. (5.19) in [11]. Note that the parameter \(\lambda\) used in [11] is the square of \(\lambda\) used in [4, 13] and here.
At the same time, integrating out the full superalgebra gauge field in (1.1) leads [4] to an extra fermionic $A_{\neq}^{(1)}, A_{\parallel}^{(2)}$ contribution to the dilaton which in the present $AdS_2 \times S^2$ supercoset case is [13] (for the group field $f$ having only the bosonic part (2.1))

\[
e^{\Phi} = e^{\Phi_0} M'/(\sqrt{\eta})^{1/2} = \frac{e^{\Phi_0} M'}{\sin t \sin \phi}, \tag{2.11}
\]

\[
M' = -(1 - \lambda^2)^2 + (1 + \lambda^4 + 2\lambda^2 \cosh 2\xi) \cos^2 t + (1 + \lambda^4 + 2\lambda^2 \cos 2\xi) \cos^2 \phi - 4\lambda(1 + \lambda^2) \cos t \cos \phi \cosh \xi \cos \zeta. \tag{2.12}
\]

In terms of the algebraic coordinates and $\kappa = \frac{1 - \lambda^2}{1 + \lambda^2}$ in (1.8) the fermionic contribution $M'$ is

\[
M' = c_0 M, \quad M \equiv \kappa - x^2 + y^2 - p^2 - q^2 + 2\sqrt{1 - \kappa^2} xp, \quad c_0 = -\frac{4\kappa}{(1 + \kappa)^2}, \tag{2.13}
\]

where $c_0$ can be absorbed into $\Phi_0$.

Thus the dilaton expected to be part of the $\lambda$-model target space background in the “real” patch (2.3) where the metric is given by (2.7) is in the region (2.8) may be written as (cf. (2.10))

\[
e^{\Phi} = e^{\Phi_0} M = e^{\Phi_0} \frac{\kappa - x^2 + y^2 - p^2 - q^2 + 2\sqrt{1 - \kappa^2} xp}{\sqrt{-(1 - \kappa x^2 + \kappa^{-1} y^2)(1 - \kappa p^2 - \kappa^{-1} q^2)}}. \tag{2.14}
\]

Like (2.10) this expression is real if $\Phi_0$ is real and $1 - \kappa x^2 + \kappa^{-1} y^2 \leq 0$, $1 - \kappa p^2 - \kappa^{-1} q^2 \geq 0.9$

### 2.3 RR background

The metric (2.7) and the “bosonic” dilaton (2.10) were promoted in [11] to an exact type IIB supergravity solution by supplementing them with an $F_5$ RR field strength background. Let us show that one can also find an $F_5$ background that extends the metric (2.7) and the full dilaton (2.14) to a different 10d supergravity solution. The resulting background will correspond to the GS superstring action resulting from the $\kappa$-symmetric [4] $\lambda$-model (1.1) as we will show below in section 3. That implies that (in contrast to what happens in the $\eta$-model [18]) the $\lambda$-model represents not only a scale-invariant [19], but also a Weyl-invariant sigma model and thus defines a consistent superstring theory.

First, let us recall how one can embed a 6-torus compactified background $M^4 \times T^6$ (e.g. the undeformed $AdS_2 \times S^2$ solution [22]) into type IIB 10d supergravity (see Appendix A in [20]).

We shall assume that the $B$-field and the RR scalar are vanishing from the start and choose the following ansatz for the metric and the RR $F_3$ and $F_5$ field strengths ($z_i$ are 3 complex coordinates of the 6-torus)

\[
ds_{10}^2 = g_{mn}(x) dx^m dx^n + e^{W(x)} dz_i d\bar{z}_i, \tag{2.15}
\]

\[
F_3 = \frac{1}{2} dC(x) \wedge J_2 + \frac{1}{12} \ast (dC(x) \wedge J_2 \wedge J_2 \wedge J_2), \tag{2.16}
\]

\[
F_5 = \frac{1}{2} (F \wedge Re \Omega_3 - F^* \wedge Im \Omega_3), \quad F \equiv \frac{1}{2} F_{mn}(x) dx^m \wedge dx^n, \tag{2.17}
\]

\[
J_2 \equiv \frac{1}{2} dz_k \wedge d\bar{z}_k, \quad \Omega_3 \equiv dz_1 \wedge dz_2 \wedge dz_3. \tag{2.18}
\]

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\[9\] The fermionic factor in (2.12) remains real under the analytic continuation (cf. (2.12)) with $i$ coming just from the bosonic square root term $\sqrt{1 - \kappa x^2 + \kappa^{-1} y^2}$.
Here the 4d scalar C parametrises the RR 3-form and $F_{mn}$ is the 4d vector field strength (with $F^*$ being its 4d dual) representing the 4d reduction of the RR 5-form. To write an effective 4d action one may formally relax the self-duality condition on $F_5$ replacing (2.17) with $F_5 = \frac{1}{\sqrt{2}} F \wedge \text{Re } \Omega_3$ which solves the same equations of motion (has same stress tensor expressed in terms of $F$).

Then the dimensional reduction of the relevant bosonic part of the 10d type IIB supergravity action

$$S_{10} = \int d^{10}x \sqrt{-g_{10}} \left[ e^{-2\Phi_{10}} (R + 4(\partial \Phi_{10})^2) - \frac{1}{12} F_{\mu \nu \lambda} F^{\mu \nu \lambda} - \frac{1}{360} F_{\mu \nu \lambda \rho \sigma} F^{\mu \nu \lambda \rho \sigma} \right]$$

(2.19)
gives the following 4d action ($\Phi = \Phi_{10} - \frac{3}{2} W$)

$$S = \int d^{4}x \sqrt{-g} \left[ e^{-2\Phi} (R + 4(\partial \Phi)^2 - \frac{3}{2}(\partial W)^2) - \frac{1}{4} F_{mn} F^{mn} - \frac{1}{8}(3eW + e^{-3W})(\partial C)^2 \right].$$

(2.20)

This action always admits a solution with $W = 0$, $C = 0$ which we will assume in what follows. The resulting effective 4d action for the metric $g$, dilaton $\Phi$ and the RR field strength $F_{mn}$ becomes simply

$$S = \int d^{4}x \sqrt{-g} \left[ e^{-2\Phi} (R + 4(\nabla \Phi)^2) - \frac{1}{4} F_{mn} F^{mn} \right].$$

(2.21)

The corresponding equations of motion are\(^10\)

$$R_{mn} + 2\nabla_m \nabla_n \Phi = \frac{1}{2} e^{2\Phi} (F_{mp} F_{n}^{\ p} - \frac{1}{2} g_{mn} F_{kl} F^{kl}) ,$$

(2.22)

$$R + 4\nabla^2 \Phi - 4(\nabla \Phi)^2 = 0 ,$$

(2.23)

$$\partial_n (\sqrt{-g} F^{mn}) = 0 , \quad \partial_{[m} F_{nk]} = 0 .$$

(2.24)

As follows from (2.22),(2.23) the dilaton should also satisfy

$$R + 2\nabla^2 \Phi = 0 , \quad \nabla^2 e^{-2\Phi} = 0 .$$

(2.25)

As was found in [20] on the example of the $\eta$-deformed $AdS_2 \times S^2$ metric there may be several solutions for the dilaton and the $F$-form that solve (2.22),(2.23) for the same 4d metric. Thus given the metric (2.7) the solution for $\Phi, F$ need not be unique. Indeed, both the dilatons $\Phi_B$ in (2.10) and $\Phi$ in (2.14) satisfy, as one can check, each of the two equations in (2.25).\(^11\)

Ref. [11] found a real “bosonic” solution for $F = dA$ that supports the metric (2.4) and the associated “bosonic” dilaton. Written as a solution for the metric (2.7) and the dilaton (2.10) in the algebraic coordinate patch (2.8) the corresponding vector potential $A \equiv A_m dx^m$ takes the following simple form [13]\(^12\)

$$A_B = \frac{1}{2} c \sqrt{1 - \kappa^2} \ p \ dy , \quad c = 4 (\kappa T)^{-1/2} e^{-\phi_0} ,$$

(2.26)
i.e. only one component of the field strength $F_B$ is non-vanishing. This solution becomes imaginary in the coordinate patch (2.1), i.e. for the metric (2.7) with $1 - \kappa x^2 + \kappa^{-1} y^2 > 0$ where the dilaton (2.10) contains an extra factor of $i$ that can be absorbed into $e^{\Phi_0}$ so that it reappears in $A_B$ in (2.26) or, equivalently, the RR background $e^\Phi F$ that supports the metric (2.2) is imaginary [13].

The “bosonic” solution (2.7),(2.10),(2.26) may be related to a yet to be investigated alternative version of the $\lambda$-model in which one gauges only the bosonic subgroup of $\hat{F}$, i.e. where the gauge fields do not have fermionic components and thus the dilaton is given just by the “bosonic” one in (2.10).

One may wonder why the background of the bosonic $\lambda$-model should have a real embedding into supergravity given that it is a deformation of a non-abelian T-dual of $AdS_2 \times S^2$ background in all directions including time. Indeed, it is known that standard abelian T-duality applied in a time-like direction maps a real RR background to an imaginary one (cf. [23]) and non-abelian T-duality should be generalising the abelian one. However, there is a subtlety in this argument which can be understood in our particular case as follows: to be able to apply abelian T-duality one needs first to take a limit [13] that “enhances” the Cartan directions and thus generates isometries. It turns out [13] that to generate a time-like isometry one needs to start with the coordinate patch (2.1), i.e. the metric (2.2), while taking the limit of the metric (2.4) gives a space-like isometry. Thus in the case of the metric (2.4) there is actually no reason to expect that embedding to supergravity should lead to a complex solution, while the RR flux needed to support the metric (2.2) is indeed purely imaginary.

Like the above “bosonic” solution of [11], our new solution of (2.22),(2.23) for $F = dA$ that supports the metric (2.4) or, equivalently, (2.7) in the “physical” region (2.8), and the full $\lambda$-model dilaton (2.11) or (2.14) turns out to be real. This is consistent with the reality property of the supercoset $\lambda$-model action (1.1).

The solution we found is similar in structure to the one discussed in [20] for the $\eta$-deformed $AdS_2 \times S^2$ metric (though here we have no isometries): (i) the dilaton (2.14) contains a factor $M = M(x,y,p,q)$ that (in contrast to the metric (2.7) and the “bosonic” $\Phi_B$ in (2.10)) does not factorise into two separate 2d parts, and (ii) the same function $M$ enters also the vector

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\[13\] To construct such a model one may start with the $AdS_2 \times S^2$ GS action, split the supergroup element into bosonic and fermionic parts, then write down the first-order action with respect to the bosonic currents only, and finally deform this interpolating action by replacing the bosonic analog of the $v$-term in (1.4) by the gWZW model with the group element $f$ being the bosonic one (the original GS fermions will thus play the role of “spectators” only). If such a model will still represent an integrable deformation of the bosonic non-abelian T-dual of the $AdS_2 \times S^2$ model, the supergroup invariance (that is apparently broken by this “bosonic” construction) may be recovered at the level of a hidden integrable structure.

\[14\] We thank B. Hoare for clarifying discussions on this issue.
potential (cf. (2.26))\(^{15}\)
\[
A = \frac{1}{2} c \, M^{-1} \left[ y \, dx + (\sqrt{1 - \kappa^2} \, p - x) \, dy \right], \quad c = 4 (\kappa \, T)^{-1/2} e^{-\Phi_0}. \tag{2.27}
\]
The field strength \(F_{mn} = \partial_m A_n - \partial_n A_m\) that corresponds to (2.27) has the following components\(^{16}\)
\[
F_{01} = c \, M^{-2} \kappa \left( 1 - \kappa p^2 - \kappa^{-1} q^2 \right), \\
F_{02} = c \, M^{-2} \, y \left( p - \sqrt{1 - \kappa^2} \, x \right), \quad F_{03} = c \, M^{-2} \, y \, q, \\
F_{12} = c \, M^{-2} \left[ -xp + \frac{1}{2} \sqrt{1 - \kappa^2} \left( \kappa + x^2 + y^2 + p^2 - q^2 \right) \right], \\
F_{13} = c \, M^{-2} \, q \left( \sqrt{1 - \kappa^2} \, p - x \right), \quad F_{23} = 0. \tag{2.28}
\]
The combination that enters the GS superstring action is the field strength times the dilaton with the components taken in the vielbein basis (i.e. \(F_{ab} \equiv e^\Phi F_{\alpha \beta}\), or, equivalently, the RR bi-spinor \(F_{ab}^{\alpha \beta} \Gamma^\beta\) (cf. (3.1),(3.2),(A.9)). Multiplying (2.28) by the dilaton (2.14) and the inverse vielbein factors corresponding to the metric (2.7) \(g_{mn} = \eta_{ab} E_m^a E_n^b\) we find for \(F_{ab}\)
\[
F_{01} = c \Phi_0 \, M^{-1} \kappa \sqrt{1 - \kappa x^2 + \kappa^{-1} y^2} (1 - \kappa p^2 - \kappa^{-1} q^2), \\
F_{02} = c \Phi_0 \, M^{-1} \, y \left( p - \sqrt{1 - \kappa^2} \, x \right), \quad F_{03} = c \Phi_0 \, M^{-1} \, y \, q, \\
F_{12} = c \Phi_0 \, M^{-1} \left[ -xp + \frac{1}{2} \sqrt{1 - \kappa^2} \left( \kappa + x^2 + y^2 + p^2 - q^2 \right) \right], \\
F_{13} = c \Phi_0 \, M^{-1} \, q \left( \sqrt{1 - \kappa^2} \, p - x \right), \quad F_{23} = 0. \tag{2.29}
\]
Once again, this background (2.7),(2.14),(2.27) is real in the region with \(1 - \kappa x^2 + \kappa^{-1} y^2 \leq 0\) corresponding to (2.3),(2.4). In Appendix A we will rederive the embedding of the solution (2.7),(2.14),(2.28) into type IIB supergravity directly in 10d.

As we shall show in section 3, this solution (2.7),(2.14),(2.27) is exactly the background that appears in the GS sigma model that comes out of the supercoset \(\lambda\)-model (1.1) upon integrating out the gauge field \(A_\pm\) in the gauge (2.3). This effectively demonstrates that the \(\lambda\)-model constructed in \([4]\) leads to a Weyl-invariant GS superstring action.

2.4 Special cases

Let us now discuss some special cases of the solution (2.7),(2.14),(2.27). The first is \(\lambda = 0\) or \(\kappa = 1\) (see (1.8)).\(^{17}\) In this case the metric (2.7) is that of a direct product of two 2d spaces corresponding to the bosonic gWZW models \(SO(2,1)/SO(1,1)\) and \(SO(3)/SO(2)\). It has two “rotational” \(SO(1,1)\) and \(SO(2)\) isometries (corresponding to translations in \(\xi\) or \(\tilde{t}\) and \(\zeta\) in

\(^{15}\)Remarkably, the corresponding \(F_{mn}\) still solves the Maxwell equations in (2.24) despite the fact that the metric (2.7) does not contain any dependence on \(M\).

\(^{16}\)Here \((x^0, x^1, x^2, x^3) = (x, y, p, q)\), i.e. \(A_0 = \frac{1}{2} c M^{-1} y, A_1 = \frac{1}{2} c M^{-1} (\sqrt{1 - \kappa^2} p - x)\).

\(^{17}\)In the \(\eta\)-model this corresponds to the \(\kappa = i\) point related to the Pohlmeyer reduced theory \([8, 13]\).
(2.2) or (2.4)). We then get from (2.7),(2.14),(2.27) (assuming $1 - x^2 + y^2 \leq 0$, $1 - p^2 - q^2 \geq 0$

$$\kappa = 1 : \quad T^{-1}ds^2 = \frac{1}{1 - x^2 + y^2}(-dx^2 + dy^2) + \frac{1}{1 - p^2 - q^2}(dp^2 + dq^2),$$

$$e^\Phi = e^{\phi_0} \frac{1 - x^2 + y^2 - p^2 - q^2}{\sqrt{- (1 - x^2 + y^2)(1 - p^2 - q^2)}},$$

$$A = 2T^{-1/2} e^{-\phi_0} \frac{1}{1 - x^2 + y^2 - p^2 - q^2} (y\,dx - x\,dy).$$

In contrast, the “bosonic” solution (2.7),(2.10),(2.26) for $\kappa = 1$ becomes just the standard metric-dilaton gWZW background [24] ($A_B$ in (2.26) vanishes for $\kappa = 1$).

Another special case is when $\lambda = 1$ or $\kappa = 0$ and $k$ in (1.1) sent to infinity with the coordinates $(x,y,p,q)$ and the rescaled tension $h$ in (1.9) kept fixed.\(^{18}\) To define this limit we need to start with the analytic continuation of the solution to the region with $1 - \kappa x^2 + \kappa^{-1} y^2 \geq 0$, $1 - \kappa p^2 - \kappa^{-1} q^2 \leq 0$ where the dilaton (2.14) is still real. As a result, we get the following solution

$$\kappa = 0 : \quad h^{-1}ds^2 = \frac{-dx^2 + dy^2}{y^2} - \frac{dp^2 + dq^2}{q^2}, \quad h = \kappa T = \kappa \frac{k}{\pi},$$

$$e^\Phi = e^{\phi_0} \frac{y^2 - q^2 - (x-p)^2}{y^2 - q^2},$$

$$A = 2h^{-1/2} e^{-\phi_0} \frac{1}{y^2 - q^2 - (x-p)^2} [y\,dx - (x-p)\,dy].$$

The metric in (2.33) is a product of $AdS_2$ and $H^2$ (with $--$ signature). In contrast to the (analytic continuation of) the standard $AdS_2 \times S^2$ solution here it is supported by a non-trivial dilaton and non-constant RR flux.\(^{19}\)

There is also another way of taking the $\kappa = 0$ limit (in which the coordinates $(x,y,p,q)$ are no longer fixed but are scaled in a special way) that leads to the metric of the non-abelian T-dual of $AdS_2 \times S^2$ (cf. Appendix A in [13]). Consider first the second (“sphere”) part of the metric in (2.2),(2.6) and define $z \equiv \cos \varphi = \sqrt{\kappa p^2 + \kappa^{-1} q^2}$, $w \equiv \cos \zeta = \frac{\kappa^{1/2} p}{\sqrt{\kappa p^2 + \kappa^{-1} q^2}}$ so that $p = \kappa^{-1/2} zw$, $q = \kappa^{1/2} z \sqrt{1 - w^2}$. The standard form of the metric of the non-abelian T-dual of $S^2$ is found by setting $z = 1 - \frac{\kappa^2}{2(1-\kappa^2)} Z^2$, $w = 1 - \frac{\kappa^2}{2(1-\kappa^2)} W^2$ and taking the limit $\kappa \to 0$ in (2.2) for fixed $Z$ and $W$ [25]

$$h^{-1}ds^2 = Z^{-2}(dW^2 + \frac{1}{4}[d(Z^2 + W^2)]^2) = \frac{dU^2 + dW^2}{2U - W^2}, \quad U \equiv \frac{1}{2}(Z^2 + W^2),$$

where, as in (2.33), the rescaled tension $h$ is again fixed in the limit. The coordinates $(p,q)$ used in (2.7) are thus no longer fixed in this $\kappa \to 0$ limit. Let us set

$$x = \kappa^{-1/2}(1 + \kappa^2 V), \quad y = \kappa^{3/2} Y, \quad p = \kappa^{-1/2}(1 - \kappa^2 U), \quad q = \kappa^{3/2} W,$$

where to satisfy the “physical” patch (2.4) conditions $1 - \kappa x^2 + \kappa^{-1} y^2 < 0, 1 - \kappa p^2 - \kappa^{-1} q^2 > 0$ we assume that $Y^2 < 2V$, $W^2 < 2U$. Then the $\kappa \to 0$, $h$-fixed limit of the metric (2.7) with

\(^{18}\)In the $\eta$-model this corresponds to $\kappa = 0$ or the undeformed $AdS_2 \times S^2$ theory.

\(^{19}\)The isometries of the metric are broken by the $(\Phi,A)$ background to just two: simultaneous rescaling of all 4 coordinates and simultaneous shifts of $x$ and $p$. 

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fixed new coordinates \((V, Y, U, W)\) becomes the metric of the non-abelian T-dual of \(AdS_2 \times S^2\)
\[
h^{-1} ds^2 = \frac{dV^2 - dY^2}{2V - Y^2} + \frac{dU^2 + dW^2}{2U - W^2} . \tag{2.38}
\]
\(\Phi\) in (2.14) and \(A\) in (2.27) take the following form
\[
e^\Phi = e^{\Phi_0} \frac{2(U - V) - 1}{\sqrt{(2V - Y^2)(2U - W^2)}}, \quad A = 2h^{1/2}e^{-\Phi_0} \frac{1}{2(U - V) - 1} YdV , \tag{2.39}
\]
where in \(A\) we have dropped a pure gauge term \(\sim dY.\)

### 2.5 Scaling limits

As was found in [13], making the formal coordinate redefinition \((t, \xi; \varphi, \zeta) \rightarrow (t, \rho; \varphi, r)\) in (2.2) combined with infinite imaginary shifts of the \((t, \varphi)\) directions and setting \(\kappa = i\kappa\)
\[
t \rightarrow t + \frac{i}{2} \log \frac{1 - \kappa^2 \rho^2}{1 + \rho^2} + i \log \gamma_1 , \quad \xi \rightarrow \frac{1}{2} \log \frac{-1 + \kappa \rho}{1 + \kappa \rho} ,
\]
\[
\varphi \rightarrow \varphi + \frac{i}{2} \log \frac{1 + \kappa^2 r^2}{1 - r^2} + i \log \gamma_2 , \quad \zeta \rightarrow \frac{i}{2} \log \frac{1 + i\kappa r}{1 - i\kappa r} , \quad \gamma_1, \gamma_2 \rightarrow \infty . \tag{2.40}
\]
transforms the metric (2.2) into
\[
h^{-1} ds^2 = \frac{1}{1 - \kappa^2 \rho^2} \left[ -(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \frac{1}{1 + \kappa^2 r^2} \left[ (1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] , \tag{2.41}
\]
which is the \(\eta\)-deformed \(AdS_2 \times S^2\) metric \([1, 7, 8, 21]\) with \(h \equiv \frac{k}{\pi} \kappa\) as string tension. This limiting procedure becomes more transparent in the algebraic coordinates (2.7). Here we will consider the patch where \(1 - \kappa x^2 + \kappa^{-1} y^2 > 0, 1 - \kappa p^2 - \kappa^{-1} q^2 < 0\). Performing independent infinite rescalings of the coordinates \((x, y)\) and \((p, q)\)
\[
(x, y) \rightarrow \gamma_1(x, y) , \quad (p, q) \rightarrow \gamma_2(p, q) , \quad \gamma_1, \gamma_2 \rightarrow \infty \tag{2.42}
\]
generates scaling isometries in each of the two factors of the metric (2.7):
\[
ds^2 = \frac{1}{-\kappa x^2 + \kappa^{-1} y^2} (-dx^2 + dy^2) + \frac{1}{-\kappa p^2 - \kappa^{-1} q^2} (dp^2 + dq^2) . \tag{2.43}
\]
Doing analytic continuation of coordinates, setting \(\kappa = -i\kappa\) (cf. (1.8)) and absorbing the overall factor of \(\kappa\) into \(h\) in (1.9) converts this metric into [13]
\[
h^{-1} ds^2 = \frac{1}{y^2 - \kappa x^2 x^2} (dy^2 + dx^2) + \frac{1}{q^2 + \kappa p^2} (-dq^2 + dp^2) . \tag{2.44}
\]
This may be interpreted as the metric of \(\eta\)-deformed \(H^2 \times dS_2\) background which is related to \(AdS_2 \times S^2\) by analytic continuation. In this scaling limit (2.42) (combined with a shift of the constant part of the dilaton) the “bosonic” solution (2.7), (2.10), (2.26) thus reduces to the metric (2.43) and
\[
e^{\Phi_B} = \frac{e^{\Phi_0}}{\sqrt{(\kappa x^2 - \kappa^{-1} y^2)(-\kappa p^2 - \kappa^{-1} q^2)}} , \quad A_B = \frac{1}{2} c e^{-\Phi_0} \sqrt{1 - \kappa^2} p \, dy . \tag{2.45}
\]

Note that in this limit the “bosonic” solution of [11] leads to the same metric (2.38) supported by a different combination of fields: \(e^{\Phi_B} = \frac{e^{\Phi_0}}{\sqrt{(2V - Y^2)(2U - W^2)}} , \quad A_B = 2h^{1/2}e^{-\Phi_0} U dY.\)
As was shown in [13, 17], written in angular coordinates and after an analytic continuation the metric (2.43) and $e^{\Phi} F_B$ (but not the dilaton, cf. [18]) of this background is found to be T-dual to those of the $\eta$-deformed AdS$_2 \times S^2$ background.

In the case of the new solution (2.7),(2.14),(2.27) the infinite rescaling (2.42) with fixed $\frac{2\lambda}{\gamma_2}$ gives again the metric (2.43) while the dilaton and the RR gauge field become

$$e^{\Phi} = \frac{e^{\Phi_0} \tilde{M}}{\sqrt{(\kappa x^2 - \kappa^{-1} y^2)(-\kappa p^2 - \kappa^{-1} q^2)}} ,$$
$$\tilde{M} \equiv (\gamma_1 \gamma_2)^{-1} M = -\frac{\gamma_2}{\gamma_1} (p^2 + q^2) + \frac{\gamma_1}{\gamma_2} (-x^2 + y^2) + 2 \sqrt{1 - \kappa^2} xp ,$$
$$A = \frac{1}{2} e^{-\Phi_0} \tilde{M}^{-1} \left[ \sqrt{1 - \kappa^2} p \, dy + \frac{\gamma_1}{\gamma_2} (y \, dx - x \, dy) \right] .$$

While the metric (2.43) has two scaling isometries, $\Phi$ and $A$ have only one isometry under the same simultaneous rescaling of all the coordinates.\footnote{In contrast, the bosonic dilaton in (2.45) is invariant only under the opposite scaling of $(x, y)$ and $(p, q)$.} This limiting background (2.46),(2.48) (which is obviously different from (2.45)) is of course still a solution of the supergravity equations (2.22),(2.23) and thus defines a consistent GS sigma model which is classically T-dual to the GS action for the $\eta$-deformed AdS$_2 \times S^2$ model. Absence of the two isometries in the dilaton does not allow one to perform the T-duality on the whole background (i.e. at the quantum level).\footnote{Let us note that this solution is also different from a class of solutions with $\eta$-deformed AdS$_2 \times S^2$ metric found in [20] as there the existence of two separate $U(1)$ isometries was assumed from the start.}

Let us now consider a particular “asymmetric” case of the scaling limit (2.42) with $\frac{2\lambda}{\gamma_2} \to 0$.

Then $\tilde{M} \to -\frac{2\gamma_2}{\gamma_1} (p^2 + q^2)$ and (2.46),(2.48) become\footnote{Note that the alternative limit $\frac{2\lambda}{\gamma_2} \to \infty$ leads to a different background that has to do with our particular choice of the vector potential on the orbit of duality transformations that breaks symmetry between $(x, y)$ and $(p, q)$ pairs of coordinates.}

$$e^{\Phi} = \frac{e^{\Phi_0} (p^2 + q^2)}{\sqrt{(-\kappa x^2 + \kappa^{-1} y^2)(-\kappa p^2 + \kappa^{-1} q^2)}} ,$$
$$e^{\Phi_0} = -\frac{\gamma_2}{\gamma_1} e^{\Phi_0} ,$$
$$A = \frac{1}{2} e^{-\Phi_0} (p^2 + q^2)^{-1} \sqrt{1 - \kappa^2} p \, dy .$$

The fact that the dilaton becomes factorizable and that $A$ looks similar to $A_B$ in (2.45) is not accidental. Doing the coordinate transformation $(p, q) \to (p', q')$

$$p' = \frac{p}{p^2 + q^2} , \quad q' = \frac{q}{p^2 + q^2} ,$$

that preserves the form of the metric (2.43) we find that (2.43),(2.49),(2.50) take the same simple form as in (2.45)

$$ds^2 = \frac{1}{-\kappa x^2 + \kappa^{-1} y^2} (-dx^2 + dy^2) + \frac{1}{-\kappa p^2 + \kappa^{-1} q^2} (dp'^2 + dq'^2) ,$$
$$e^{\Phi} = \frac{e^{\Phi_0}}{\sqrt{(-\kappa x^2 + \kappa^{-1} y^2)(-\kappa p^2 + \kappa^{-1} q^2)}} , \quad A = \frac{1}{2} e^{-\Phi_0} \sqrt{1 - \kappa^2} p' \, dy .$$

Thus the “asymmetric” scaling limit (2.42) with $\gamma_1, \gamma_2 \to \infty$ and $\frac{2\lambda}{\gamma_2} \to 0$ of the $\lambda$-model background (2.7),(2.14),(2.27) is equivalent to the scaling limit (2.45) of the “bosonic” solution of
We conclude that the proposal of [13] that a scaling limit of the $\lambda$-model should give a background which is classically T-dual to the $\eta$-model background is now confirmed for the RR background as well. The underlying reason why the above “asymmetric” scaling limit of the $\lambda$-model is required to recover the T-dual of the $\eta$-model and why this limit is the same as the limit of the “bosonic” solution should be related to the fact that such a limit “enhances” the bosonic Cartan directions suppressing the effect of gauging of the fermionic directions and thus ameliorating the distinction between gauging the full supergroup in the $\lambda$-model and just its bosonic part as in the model that should correspond to the “bosonic” background of [11].

3 RR background from supercoset $\lambda$-model

Let us now show that the RR background (2.17), (2.28) appears in the GS superstring sigma model that emerges from the $\lambda$-model action (1.1) upon integrating out the gauge fields $A_\pm$. As was shown in [4], the $\lambda$-model action has a local fermionic symmetry that may be interpreted as $\kappa$-symmetry of the resulting GS action. Expressing the supergroup field $f$ in terms of the bosonic and fermionic coordinates and expanding to quadratic order in fermions one may be able to put the resulting quadratic fermionic action into the standard type IIB GS form

$$ S_2 = \int d\sigma d\tau \, i \Theta_1 \Pi_+^{IJ} \varepsilon_{IJ}^a \Gamma_a \, D_\beta^M \Theta_M, \quad \Pi_+^{IJ} = \frac{1}{2} \left( \gamma^{IJ} \delta^{IJ} \pm \epsilon^{IJ} \sigma^{IJ} \right), \quad (3.1) $$

where $\Theta_1, \Theta_2$ are 32-component Majorana-Weyl spinors of positive chirality ($\Gamma_{11} \Theta = \Theta$), $E_a^\alpha = E^a_\mu \partial_\alpha X^\mu$ is the pullback of the vielbein for the metric $g_{\mu\nu} = \eta_{ab} E_a^\mu E_b^\nu$ in the bosonic part of the $\sigma$-model, and $\gamma^{\alpha\beta} \equiv \sqrt{-h} \, h^{\alpha\beta}$ where $h$ is the world-sheet metric. For a generic IIB supergravity background, the operator $D_\alpha^{IJ}$ in (3.1) takes the form (see, e.g., [28, 27])

$$ D_\alpha^{IJ} = \delta^{IJ} (\partial_\alpha - \frac{1}{4} \omega^{ab}_{\alpha} \Gamma_{ab}) + \frac{i}{8} \sigma^{IJ}_3 E_a^\alpha H_{abc} \Gamma^{bc} $$

$$ \quad - \frac{1}{8} \epsilon^{IJ} \left( \epsilon^{I} \Gamma^\alpha F_a + \frac{1}{3} \sigma_1^{IJ} \Gamma^{abc} F_{abc} + \frac{1}{54} \epsilon^{IJ} \Gamma^{abcdde} F_{abcdde} \right) E_a^\alpha \Gamma_h. \quad (3.2) $$

Assuming that one finds the action in the form (3.1) one can then extract the combination $\mathcal{F} = e^\Phi F$ of the RR field strengths and dilaton from the the operator $D_\alpha^{IJ}$ (cf. [9] in the case of the $\eta$-model). In the present case of the $\lambda$-model there is a natural candidate [4] for the dilaton expressed in terms of the superdeterminant of the matrix in the quadratic $A_+ A_-$ term in (1.1); that should allow to extract the RR flux $F$ itself ($F$ should then satisfy the Bianchi identities if the whole construction is consistent).

For the standard GS action in type IIB supergravity background [29] the sum of the bosonic and quadratic fermionic action is invariant (to leading order in $\Theta$) under the $\kappa$-symmetry variations for the $\sigma$-model fields and the 2d metric (here $\kappa_{IJ}$ are the Grassmann 32-component
spinor \(\kappa\)-symmetry parameters)

\[
\delta X^\mu = \frac{i}{2} \bar{\Theta} I^\mu \delta \Theta_I + O(\Theta^3), \quad \Gamma^\mu = E^\mu a _a , \quad (3.3)
\]

\[
\delta \Theta_I = \frac{1}{2} \Pi^J_{\beta J} K_{\alpha J} + O(\Theta^3), \quad \Gamma_\beta = E^\beta _a _a , \quad (3.4)
\]

\[
\delta \gamma^{\alpha \beta} = -2i \Pi^J_{\alpha J} \Pi^N_{\beta J} K_{\alpha N} D_{\beta N} \Theta_L + O(\Theta^3). \quad (3.5)
\]

Thus if \(\kappa\)-symmetry is in place, we may then extract the operator \(D_{IJ}^I\) containing the information about the background RR fields not from the action (3.1) directly but rather from the expected form of the \(\kappa\)-symmetry variation of the world-sheet metric in (3.5) which is linear in \(\Theta_I\).

Let us note that a generic choice of the coordinates \((X^\mu, \Theta_I)\) in the \(\lambda\)-model action will not lead to the standard GS form (3.3)–(3.5) of the \(\kappa\)-symmetry variation for the world-sheet metric. A natural way to find the right coordinates is to study the \(\kappa\)-symmetry variations of \((X^\mu, \Theta_I)\) first and find the proper field redefinition that puts them into the form (3.3)–(3.5). This will be the strategy that we will use here.

The invariance of the \(\lambda\)-model (1.1) under the fermionic symmetry was proved in [4] in the conformal gauge (i.e. using the Virasoro constraints). The action was found to be invariant under the two independent variations \(\delta_1, \delta_2\) of \(f\) defined by

\[
O_+^{-1}(f^{-1} \delta_1 f) = A^{(2)}_+ a + \tilde{a} A^{(2)}_-, \quad \tilde{a} \in \mathfrak{f}^{(1)},
\]

\[
O_-^{-1}(f^{-1} \delta_2 f) = A^{(2)}_- a + \tilde{a} A^{(2)}_+, \quad \tilde{a} \in \mathfrak{f}^{(3)},
\]

where one needs to substitute the solutions for the gauge fields in (1.1)

\[
A_\pm = \mp O_\pm^{-1}(f^{-1} \partial_\pm f) \quad (3.7)
\]

projected to the coset part of the superalgebra. Here the linear operators \(O_\pm\) act on a generic element \(M\) of the superalgebra as

\[
O_+(M) = f^{-1} M f - \tilde{\Omega}_+(M), \quad O_-(M) = M - f^{-1} \Omega(M) f, \quad (3.8)
\]

where \(f\) is an element of the supergroup \(\hat{F}\) and (cf. (1.2))

\[
\Omega = P^{(0)} + \lambda^{-1} P^{(1)} + \lambda^{-2} P^{(2)} + \lambda P^{(3)}, \quad \Omega^T = P^{(0)} + \lambda P^{(1)} + \lambda^{-2} P^{(2)} + \lambda^{-1} P^{(3)} . \quad (3.9)
\]

While in [4] where the conformal gauge was assumed the 2d metric was not transformed and instead the Virasoro constraints were used, we can also deduce a “conformal-gauge” version \(\delta \gamma|_{c.g.}\) of the \(\kappa\)-symmetry variation of the 2d metric obtained by formally imposing the conformal

---

\(^{26}\)Deriving the explicit form of the quadratic fermionic action from the \(\lambda\)-model and comparing to the expected form (3.1) is an involved calculation. One difficulty lies in the fact that a random choice of the bosonic and fermionic coordinates \((X^\mu, \Theta_I)\) is likely not be the right one to get the action in the standard GS form (3.1), i.e. one would need to find a proper field redefinition to match (3.1). In general, this would involve rotating the fermions by an \((X^\mu\)-dependent\) matrix, and also shifting the bosons by terms quadratic in \(\Theta_I\), generating extra \(\Theta^2\) terms from the change of the bosonic action (see below).
gauge on the r.h.s. of (3.5).\footnote{It would be interesting to repeat the derivation of \cite{[4]} without imposing the conformal gauge, but for us \delta\gamma_{\alpha\beta}|_{c.g.} will be enough in order to extract the operator \hat{D}^{IJ}_\alpha unambiguously.} The two independent \kappa-symmetry variations that we will need to compute are \cite{[4]}
\begin{align}
\delta_1\gamma^{-+}|_{c.g.} &= -2\lambda^2 \text{STr}(W[\tilde{a}, A_1^{(1)})], \\
\delta_2\gamma^{++}|_{c.g.} &= -2\lambda \text{STr}(W[\tilde{a}, A_1^{(3)})],
\end{align}
(3.10)
where \(W\) is defined in (B.12). This will allow us to extract the RR field background.

A crucial step in the derivation turns out to be the inversion of the operators \(O_\pm\) in (3.8) as their action is quite involved, especially on the odd part of the superalgebra. Here we will consider only the simplest case of the \(\lambda\)-model for the \(AdS_2 \times S^2\) supercoset. We shall use the explicit representation of \(su(1,1|2)\) superalgebra in terms of \(4 \times 4\) matrices described in Appendix B.

### 3.1 Choice of group element and \(\kappa\)-symmetry variations of coordinates

We shall choose the gauge-fixed group element \(f \in PSU(1,1|2)\) as a product of an element \(f_B\) corresponding to the bosonic subalgebra, and the fermionic part \(f_F\)
\begin{align}
f &= f_B f_F, \\
f_B &= \tilde{f}_B \oplus \hat{f}_B, \\
f_F &= \exp(Q_I \theta_I),
\end{align}
(3.11)
where we found it convenient to choose
\begin{align}
\tilde{f}_B &= e^{\frac{i}{2} \xi \sigma_1 e^{i \sigma_3} \xi \sigma_1}, \\
\hat{f}_B &= e^{\frac{i}{2} \zeta \sigma_3 e^{i \sigma_3} \zeta \sigma_1}.
\end{align}
(3.12)
This parametrisation of bosonic coordinates is related to (2.1) by a gauge transformation. The expressions in the “real” patch (2.3) may be obtained by a simple analytic continuation. While in the case of the \(AdS_2 \times S^2\) supercoset action the choice (3.11) for the group element\footnote{In the supercoset construction, the choice (3.11) should be accompanied by a proper parameterisation of the group element, compatible with gauge transformations that act only from the right.} directly leads to the standard GS type quadratic fermionic action, in the \(\lambda\)-model case one would need an additional \(\lambda\)-dependent redefinition of the fermionic and bosonic coordinates in order to put the action in the GS form.

The bosonic part of the action of the \(\lambda\)-model contains the metric \(g_{mn}\) (2.2),(2.7) and no \(B\)-field \cite{[11],[4]}. There are two natural equivalent ways to define a vielbein \(E^a = E^a_m dX^m\) corresponding to \(g_{mn}\)
\begin{align}
E^{(\pm)a} &= 2c \text{STr} \left[ P^a((O_\pm^{(0)})^{-1}(f^{-1}df) \right], \\
\text{c} &= \frac{\sqrt{1-\lambda^4}}{2\lambda^2} = \frac{\sqrt{\kappa}}{1-\kappa},
\end{align}
(3.13)
where the superscript \((\pm)\) on \(E^a\) indicates which linear operator \(O_\pm^{(0)}\) is entering its definition. The superscript \((0)\) means that we switch off the fermions in \(O_\pm\) in (3.8) (see Appendix B for notation). Then using (3.11),(3.12) we find (see Appendix C for the details)
\begin{align}
E^{(\pm)0} &= \pm \frac{2\lambda^2 c}{\lambda^2-1} \left( \cosh \xi dt - \sinh \xi \cot t d\xi \right), \\
E^{(\pm)1} &= \frac{2\lambda^2 c}{\lambda^2+1} \left( \sinh \xi dt - \cosh \xi \cot t d\xi \right), \\
E^{(\pm)2} &= \pm \frac{2\lambda^2 c}{\lambda^2-1} \left( \cos \zeta d\varphi + \sin \zeta \csc \varphi d\zeta \right), \\
E^{(\pm)3} &= \frac{2\lambda^2 c}{\lambda^2+1} \left( \sin \zeta d\varphi - \cos \zeta \csc \varphi d\zeta \right),
\end{align}
(3.14)
so that \( E^{(+)a} \) and \( E^{(-)a} \) are related by a Lorentz transformation\(^{29}\)
\[
E^{(-)a} = \Lambda^a_b E^{(+)b}, \quad \Lambda^a_b = \text{diag} \ (1, 1, 1, -1) .
\] (3.15)

The corresponding metric is the one in (2.2) (in this section we set tension \( T = 1 \)). In the algebraic coordinates (2.5),(2.6) where the metric is given by (2.7) the \( E^{(+)a} \) take the obvious diagonal form. Below we will use the \( E^a = E^{(+)a} \) choice.

Let us now turn to the \( \kappa \)-symmetry variations of bosonic and fermionic coordinates found by computing (3.6) explicitly in the parameterisation (3.11),(3.12) (see Appendix C). When we project (3.6) on odd generators and expand to leading order in fermions, we find the \( \kappa \)-symmetry transformation for the fermions \( \delta \theta_I \) in terms of the corresponding parameter \( \kappa_I \) defined in (C.10).

To put it into the standard GS form we need to redefine \( \theta_I \) and the parameters \( \kappa_I \) as
\[
\theta \rightarrow c^{-1/2} k_{+,F} \cdot (\lambda^{-1}U \oplus 1_4) \theta, \quad \kappa \rightarrow 2c^{1/2} \left( (\lambda^{-1}U) \oplus 1_4 \right) \kappa ,
\] (3.16)
where we have collected \( \theta_I \) and \( \kappa_I \) into the 2-vectors \( \theta \) and \( \kappa \). Here \( k_{+,F} \) is the 8 \times 8 matrix which encodes the action of the operator \( O_{+}^{(0)} \) on the odd generators of the algebra (see eq. (C.8)), while \( U \) is the 4 \times 4 matrix (acting only on spinor indices of \( \theta_1 \) and \( \kappa_1 \)) which implements the Lorentz transformation \( \Lambda \) in (3.15) on the fermions. Writing the result in 10d notation we get for the non-vanishing \( \kappa \)-symmetry variations\(^{30}\)
\[
\delta \Theta_1 = E^a_a \Gamma_a K_1, \quad \delta \Theta_2 = E^a_{+} \Gamma_a K_2 .
\] (3.17)

Here \( \Gamma_a \) are 32 \times 32 gamma matrices (defined in Appendix D), and we embedded the 4-component spinors \( \theta_I \) and \( \kappa_I \) into the 32-component spinors \( \Theta_I \) and \( K_I \) in (3.4) as
\[
\Theta_I = \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \theta_I \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad K_I = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \kappa_I \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\] (3.18)
so that \( \Gamma_{11} \Theta_I = + \Theta_I \) and \( \Gamma_{11} K_I = - K_I \).

Next, if we project (3.6) on the bosonic coset generators of the superalgebra and keep only the leading order term in fermions, we find an equation for the \( \kappa \)-symmetry variation of the bosonic coordinates \( \delta X^m \). To put it into the standard form (3.3), i.e.
\[
\delta X^m = \frac{i}{2} \bar{\Theta}_I \Gamma_m \delta \Theta_I , \quad \Gamma_m = E^a_a \Gamma_a ,
\] (3.19)
we need, in addition to the redefinition of the fermions (3.16), to do a redefinition of the bosons\(^{31}\)
\[
X^m \rightarrow X^m + \bar{\Theta}_I \Delta^m_{ij} \Theta_J ,
\] (3.20)

\(^{29}\)This Lorentz transformation has a simple form because of the gauge choice for the bosonic group element (3.12). For a generic gauge choice the Lorentz transformation will depend also on the bosonic coordinates.

\(^{30}\)These transformations should be compared to (3.4) after we fix conformal gauge. We follow the conventions of [4] and take \( \sigma^\pm = \tau \pm \sigma \) and \( \gamma^\tau = - \gamma^{\pi} = \epsilon^\pi = - \epsilon^\tau = 1 \), so that \( \gamma^{+} = \gamma^{-} = \epsilon^{-} = - \epsilon^+ = 2 \). It is assumed that here the index \( a \) runs only from 0 to 3.

\(^{31}\)Note that an attempt to put \( \delta X^m \) into the standard form by shifting the bosons as in (3.20) could, in principle, fail. Indeed, \( \Delta^m_{ij} \) must have a definite symmetry property in order for the shift of \( X^m \) not to vanish because of the Grassmann nature of \( \Theta_I \). If the terms that we want to cancel by redefinition do not have the same symmetry property, then (3.20) is not enough to get to the standard form.
where $\Delta m_{IJ}$ is defined in (C.13). It should be noted that for our present purpose of extracting the RR background, i.e. the information about the derivative (3.2) from (3.5), the study of the $\varkappa$-symmetry variations of the bosons is not required and is just a check. In fact, the redefinition (3.20) does not modify the result for the $\varkappa$-variation of the world-sheet metric (3.10),(3.5) at the leading order in fermions. However, one would need to know the explicit form of (3.20) in order to find the RR background directly from the action as that would require putting the quadratic fermionic term in the $\lambda$-model action into the standard GS form (3.1).

It is worth mentioning that our results prove that the action at quadratic order in fermions will be of the standard form of GS. In fact, the full action is invariant under a local fermionic symmetry which we found to be the standard $\varkappa$-symmetry at the leading order in fermions.

3.2 RR background from $\varkappa$-symmetry variation of the world-sheet metric

Let us now consider the variation of the world-sheet metric. Starting with (3.10) we first need to compute $A_+^{(1)}$ and $A_-^{(3)}$ at linear order in fermions, do the field redefinitions (3.16) and then compare to (3.5), where one formally needs to impose the conformal gauge on the right-hand side

$$
\delta \gamma^{--}_{\text{c.g.}} = -8i \, K_1 D_+^{IJ} \Theta_J, \quad \delta \gamma^{++}_{\text{c.g.}} = -8i \, K_2 D_-^{IJ} \Theta_J.
$$

(3.21)

We then isolate the contributions in $D^{IJ}$ depending on the tensors $\delta^{IJ}, \sigma_1^{IJ}, \epsilon^{IJ}, \sigma_3^{IJ}$ and compare to (3.2). The $\delta^{IJ}$ structures are found to be given by the standard derivatives of the fermions (with no unwanted matrix rotation of the spinor indices) plus terms with the spin connection $\omega^{ab}$ constructed from the vielbein $E^a = E^{(+)+}$ in (3.14), in agreement with (3.2). Another consistency check is the absence of a contribution proportional to $\sigma_3^{IJ}$ reflecting the vanishing of the $H$-field background (absent in the bosonic part of the action [4, 11]).

Thus from (3.10) we get

$$
D_+^{IJ} = \delta^{IJ} \left( \partial_\alpha - \frac{1}{4} \omega^{ab}_\alpha \Gamma_{ab} \right) + \frac{1}{8} S^{IJ} E^a \Gamma_a,
$$

(3.22)

where $S^{IJ}$ is off-diagonal in $I, J$ and should thus represent the contribution of the RR fields in (3.2). Since we can compute the $\varkappa$-symmetry variations $\delta_1$ and $\delta_2$ independently, we can easily check that $S^{IJ}$ is proportional to $\epsilon^{IJ}$. The absence of $\sigma_3^{IJ}$ term implies, according to (3.2), the vanishing of the RR 3-form.\(^{32}\)

\(^{32}\)Knowing that the $\varkappa$-symmetry variations of both the bosonic and fermionic coordinates are put in the standard form, it is then not surprising that we find the expected values of $\omega^{ab}$ and $H$: the action is invariant under $\varkappa$-symmetry, and at leading order this symmetry is indeed relating $\omega^{ab}$ and $H$ to the metric $g_{mn}$ and the $B_{mn}$ field appearing in the bosonic action.

\(^{33}\)This argument illustrates the power of our method of extracting the RR background from the structure of the $\varkappa$-symmetry transformations: if one would try to extract this information from the computation of the action one would first need to project $S^{IJ}$ on the product of 3 gamma matrices. The central point is that the action is quadratic in the same $\Theta_I$, while the $\varkappa$-symmetry variation of the world-sheet metric contains two different Grassmann spinors $\Theta_I$ and $K_I$.  

19
Next, let us make an assumption that also the RR 1-form term in (3.2) vanishes, i.e. that we should have
\[
S^{IJ} = -\frac{1}{2\pi} \epsilon^{IJ} e^\Phi \Gamma^{abcd} F_{abcd},
\]
(3.23)
where \(F_{abcd}\) are the 10d vielbein components of \(F_5\) to be found. To extract the dilaton factor we assume that the dilaton originates from integrating out the gauge fields \(A_+, A_-\) in (1.1) and is thus given by [4] (here \(\mathcal{M} \in \mathfrak{psu}(1,1|2), \text{ cf. } (3.8))
\[
\Phi = -\frac{1}{2} \text{STr log } \hat{O}, \quad \hat{O}(\mathcal{M}) = fO_-(\mathcal{M})f^{-1} = f\mathcal{M}f^{-1} - \Omega(\mathcal{M}).
\]
(3.24)
Setting fermions in \(f\) to zero we may split the contributions to the \(X^m\)-dependent part of \(\Phi\) into those of the bosonic directions and the fermionic directions in the algebra, \(e^\Phi = e^{\Phi_B}e^{\Phi_F}\). Then \(\Phi_B\) is given by (2.9) (or its analytic continuation (2.10) [11]) with a particular \(e^{\Phi_0}\) while \(e^{\Phi_F}\) is proportional to \(M\) or \(M'\) defined in (2.12) [13]
\[
e^{\Phi_B} = -\frac{(1-\kappa)^2}{8\kappa} \frac{1}{\sin t \sin \varphi}, \quad e^{\Phi_F} = -\frac{4\kappa}{1-\kappa^2} M.
\]
(3.25)
While the \(\lambda\)-model for the \(AdS_2 \times S^2\) supercoset is effectively defined in 4d target space we shall assume that it corresponds to a \(T^6\) compactification of 10d superstring theory, i.e. that the corresponding quadratic fermionic action can be obtained from the 10d GS one by embedding both the fermions and RR fluxes into the 10d theory. Thus to extract the components of the RR 5-form we shall assume that we can write it as in (2.17), i.e.
\[
F_5 = \frac{1}{2}(1 + \ast) F \wedge \text{Re } \Omega_3,
\]
(3.26)
where \(F = \frac{1}{2} F_{mn} dx^m \wedge dx^n\) and \(\Omega_3\) is defined in (2.18). The matrix \(S^{IJ}\) in (3.23) can then be rewritten as
\[
S^{IJ} = -\epsilon^{IJ} e^\Phi F_{ab} \Gamma^{abcd} \Gamma^{468} \mathcal{P}_4,
\]
(3.27)
where \(\mathcal{P}_4 \equiv \frac{1}{2} \left( 1 - \Gamma^{4567} - \Gamma^{4589} - \Gamma^{6789} \right)\) is the same projector as in (A.18),(A.19) (here \(a,b = 0,1,2,3\) and \(4,\ldots,9\) are the torus directions). The resulting tangent-space components of the RR field \(F_{ab}\) that we find from (3.21),(3.22) are then
\[
F_{01} = K \left( \lambda^2 - 1 \right) \sin^2 t \sin^2 \varphi, \quad K \equiv e^{-2\Phi_F} 8i \ c^{-1}\lambda^{-8} \left( \lambda^4 - 1 \right)^2,
F_{02} = \frac{1}{2} K \sin 2t \sinh \xi \sin \left( \lambda^2 + 1 \right) \cos \varphi \cos \zeta - 2 \lambda \cos t \cosh \xi,
F_{03} = \frac{1}{2} K \left( \lambda^2 - 1 \right) \sin 2t \sinh \xi \sin 2\varphi \sin \zeta,
F_{12} = (\lambda^4 - 1)^{-1} K \sin t \sin \left[ \lambda \left( \lambda^4 + 1 \right) \cos^2 \varphi \cos 2\zeta + 2 \cos t \cosh \xi \right]
- \left( \lambda^2 + 1 \right)^3 \cos t \cosh \xi \cos \varphi \cos \zeta + \lambda \left( \left( \lambda^4 + 1 \right) \cos^2 t \cosh 2\zeta + \lambda^4 + \lambda^2 (\cos 2t + \cos 2\varphi) + 1 \right),
F_{13} = \frac{1}{2} K \sin t \sin 2\varphi \sin \zeta \left[ 2 \lambda \cos \varphi \cos \zeta - (\lambda^2 + 1) \cos t \cosh \xi \right].
\]
(3.28)
Translated to the algebraic coordinates and analytically continued to the “physical” patch (2.3),(2.8) that leads to the same background as in (2.28) or (2.29) which solves the supergravity equations.
4 Concluding remarks

In this paper we have found the RR background corresponding to the $\lambda$-model for the $AdS_2 \times S^2$ supercoset. We demonstrated that this background (supplemented by 6-torus directions) solves the type II supergravity equations, implying that the $\lambda$-model is Weyl invariant at the quantum level and thus defines a consistent superstring theory.

It would be interesting to generalize these results to the case of the $\lambda$-model for $AdS_3 \times S^3$ and $AdS_5 \times S^5$ supercosets. This is technically challenging (given the lack of isometries) but should be possible with some guidance from the supergravity solutions [11, 12] that should correspond to the “bosonic” version of the $\lambda$-models associated with these higher-dimensional supercosets.

We also confirmed the suggestion of [13] that there exists a singular scaling limit of the $\lambda$-model background that is closely related (classically T-dual) to the analytic continuation of the $\eta$-model. The $\eta$-model itself fails to be Weyl invariant, i.e. does not correspond to a standard supergravity solution [18]. It thus appears that the $\lambda$-model is more general and better defined than the $\eta$-model at the quantum level. One reason is that the $\lambda$-model has a natural “first-order” form, i.e. is naturally defined on a bigger space including the gauge fields $A_\pm$ where the Weyl invariance should be manifest (with no need for an extra dilaton). A similar “uplifting” may be possible for the $\eta$-model (cf. [13]) and the resulting model should be Weyl invariant too (reflecting the fact that the classical T-dual of the $\eta$-model, which is also a limit of the $\lambda$-model, represents a consistent supergravity solution [18]).

One open question is about the possible interpretation of the $\lambda$-model background (2.7), (2.14), (2.27). The (analytic continuation of) 4d metric (2.7) interpolates between the metric of $SO(1,2)/SO(2) \times SO(3)/SO(2)$ gWZW model (for $\kappa = 1$) and that of a symmetric space $AdS_2 \times H^2$ (for $\kappa = 0$). There are curvature singularities on the lines $34 \kappa x^2 - \kappa^{-1} y^2 = 1$ and $\kappa p^2 + \kappa^{-1} q^2 = 1$. Restricted to these curves the “fermionic” factor $M$ in the dilaton (2.14) is equal to $-\kappa - (1 - \kappa^2)(x^2 + p^2) + 2\sqrt{1 - \kappa^2}xp$, i.e. the dilaton is also singular if $M \neq 0$. The RR field strength (2.28) or (2.29) is singular only when $M = 0.35$ The metric (2.7) has no isometries (for $\kappa \neq 0, 1$) but the corresponding geodesics should be integrable (as the underlying sigma model is integrable). The existence of hidden conserved charges should aid the construction of physical observables corresponding to this geometry. Moreover, like in the gWZW case, the singularity of the metric seen by point-like probes may not be visible in correlation functions for vertex operators constructed in terms of fields in the original $\lambda$-model action in (1.1).

34Some comments on the global structure of the first 2d part of the metric (2.7) appeared in [11]. Like in the gWZW context [26] here it should be more appropriate to analyse the geometric structure in terms of the original group variables $f$ rather than local coordinates.

35That means the singularities of the metric and dilaton terms in the l.h.s. of the Einstein equation (2.22) cancel each other.
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A Type II supergravity equations in terms of RR bispinor and $\lambda$-model background as 10d solution

Here we shall rederive the embedding of the $\lambda$-model background (2.7),(2.14),(2.27) into the space of solutions of type IIB 10d supergravity. We shall use the bispinor notation for the RR field strengths.

Let us first present the type II supergravity equations of motion in spinor notation starting from the superspace constraints as given in [27]. Let us consider first the type IIA case. From eqs. (C.20) and (C.18) of [27] we find (here $a, b, ... = 0, 1, ..., 9$ are 10d tangent space indices)

$$\Gamma^b U_{ab} + 2i \nabla_a \nabla^a \chi \big|_{\Theta=0} - \frac{i}{4} H_{abc} \Gamma^{bc} \Gamma^1 \nabla_a \chi \big|_{\Theta=0} = 0 ,$$  \hspace{1cm} (A.1)

where $\chi$ is the dilatino superfield, $H_{abc}$ is the NSNS three-form field strength and $U_{ab}$ contains also the curvature tensor and the RR bispinor $S$ which in the IIA case is given by $S = e^\Phi \left( \frac{1}{4} \Gamma^{ab} \Gamma^1 \Gamma^a \Gamma^b + \frac{1}{32} \Gamma^{abcd} F_{abcd} \right)$. Using the expression for the torsion and for the spinor derivative of the dilatino eq.(A.1) becomes

$$\Gamma^b U_{ab} + 2i \nabla_a T - \frac{i}{4} T G_a - \frac{i}{4} H_{abc} \Gamma^{bc} \Gamma^1 T = 0 ,$$ \hspace{1cm} (A.3)

where $T$ contains also the derivative of the dilatron

$$T = \frac{i}{2} \nabla_a \Phi \Gamma^a + \frac{i}{24} H_{abc} \Gamma^{abc} \Gamma^1 + \frac{i}{16} \Gamma^a \Gamma^a \Gamma^1 .$$  \hspace{1cm} (A.4)

The matrices $U_{ab}$ and $T$ appear in the integrability condition for the Killing spinor equation and in the dilatino equation respectively. Combining (A.3) with the same equation multiplied from left and right by $\Gamma^1$ gives

$$\Gamma^b \nabla_b \Gamma^1 S_a - \nabla_a \Phi \Gamma^b \nabla_b \Gamma^1 S_a - \frac{1}{8} H_{bcd} \Gamma^b \nabla_c \Gamma^1 S_a \Gamma^{cd} \Gamma^1 - \frac{1}{2} H_{abd} \Gamma^d \Gamma^1 \Gamma^1 S_b + \frac{1}{24} H_{bcd} \Gamma^{bcd} \Gamma^1 \Gamma^1 S_a = 0 .$$  \hspace{1cm} (A.5)

Multiplying this by $\Gamma^a$ from the right gives the RR sector equations of motion and the Bianchi identities in the form

$$\Gamma^b \nabla_b S - \nabla_a \Phi \Gamma^b S + \frac{1}{8} H_{abc} \Gamma^a \Gamma^1 S_{bc} + \frac{1}{24} H_{abc} \Gamma^{abc} \Gamma^1 S = 0 .$$  \hspace{1cm} (A.6)
The remaining equations give the Einstein equations and NSNS three-form equation of motion and Bianchi identity

\[ R_{ab}^{bc} \Gamma_c + 2 \nabla_a \nabla_b \Phi \Gamma^b + \frac{1}{2} \nabla^b H_{abc} \Gamma_{c11} - \nabla^b \Phi H_{abc} \Gamma_c \Gamma_{11} - \frac{1}{4} H_{abc} H^{bcd} \Gamma_d - \frac{1}{32} \Gamma^b \Gamma_d \Gamma_d = 0. \]  

(A.7)

The dilaton equation arises from eq. (C.19) of [27] upon using (A.6), (A.7)

\[ 0 = -i \Gamma^a \nabla_a T + \frac{i}{5} \Gamma^a T \Gamma a + 2i \nabla_a \Phi \Gamma^a T - \frac{i}{21} H_{abc} \Gamma_{11} T - \frac{1}{4} ST \]

\[ = \nabla^a \nabla_a \Phi - \nabla^a \Phi \nabla_a \Phi - \frac{1}{4} R_{ab} \nabla^b - \frac{1}{32} H_{abc} H^{abc}. \]  

(A.8)

The type IIB supergravity equations take the same form but with the 32 × 32 matrices \( \Gamma_a \) projected down to 16 × 16 blocks using \( \frac{1}{2} (1 \pm \Gamma_{11}) \) and \( \Gamma_{11} \) replaced by \( \sigma_3 \times I \) where \( \sigma_3 \) acts on the \( SO(2) \) indices \( I, J = 1, 2 \) of the two MW spinors. The RR bispinor here takes the form \( (i \sigma_{21}) = \varepsilon_{11} \); see [27] for further details

\[ S = - (i \sigma_2 \Gamma^a F_a + \frac{1}{3} \sigma_1 \Gamma^{abc} F_{abc} + \frac{1}{32} i \sigma_2 \Gamma^{abde} F_{abde}) \frac{1}{2} (1 - \Gamma_{11}) . \]  

(A.9)

Let us now consider the 10d metric corresponding to (2.7) (cf. (2.15); here we set \( T = 1 \) for notational simplicity)

\[ ds^2 = \frac{-dx^2 + dy^2}{H_1(x, y)} + \frac{dp^2 + dq^2}{H_2(p, q)} + dz^i dz^i, \]  

(A.10)

\[ H_1 = 1 - \kappa x^2 + \kappa^{-1} y^2; \quad H_2 = 1 - \kappa p^2 - \kappa^{-1} q^2. \]  

(A.11)

The corresponding spin connection and curvature in terms of the zweibein 1-forms read

\[ \omega^{01} = \frac{y}{\kappa \sqrt{H_1}} e^0 - \frac{\kappa x}{\sqrt{H_1}} e^1, \quad \omega^{23} = -\frac{q}{\kappa \sqrt{H_2}} e^2 + \frac{\kappa p}{\sqrt{H_2}} e^3, \]  

(A.12)

\[ R^{01} = \frac{1}{H_1} \left[ \kappa + \kappa^{-1} - (1 - \kappa^2) x^2 + (1 - \kappa^{-2}) y^2 \right] e^0 \wedge e^1, \]  

(A.13)

\[ R^{23} = -\frac{1}{H_2} \left[ \kappa + \kappa^{-1} - (1 - \kappa^2) p^2 - (1 - \kappa^{-2}) q^2 \right] e^2 \wedge e^3. \]  

(A.14)

The metric (A.10) should be supplemented by the dilaton in (2.14) (we again consider the “real” patch (2.3) where \( H_1(x, y) < 1 \))

\[ e^{2\Phi} = -e^{2\Phi_0} \frac{M^2}{H_1 H_2}, \quad M = \kappa - x^2 + y^2 - p^2 - q^2 + 2 \sqrt{1 - \kappa^2} x p, \]  

(A.15)

which solves (A.8) and (2.25). Our aim is then to show that the type IIB supergravity equations are solved provided the above metric and dilaton are supplemented by the RR five-form field strength given by (with all other fields being zero)

\[ F_5 = \frac{1}{4} (1 + \ast) \left( \frac{2 dx \wedge dy + \sqrt{1 - \kappa^{-2}} dy \wedge dp - \partial_y M dx + \partial_x M dy}{M} \right) \wedge \text{Re} \, \Omega_3 \]

\[ = \frac{1}{2} (1 + \ast) M^{-2} \left[ \kappa H_2 dx \wedge dy + y(p - \sqrt{1 - \kappa^{-2}} x) dx \wedge dp + y q dx \wedge dq \right. \]

\[ + \left. \left[ \frac{1}{2} \sqrt{1 - \kappa^2} (x^2 + y^2 + p^2 - q^2) - x p \right] dy \wedge dp + q(\sqrt{1 - \kappa^2} p - x) dy \wedge dq \right] \wedge \text{Re} \, \Omega_3. \]  

(A.16)
Here $\Omega_3$ is the holomorphic three-form on $T^6$ defined in (2.18). This expression for $F_3$ is the same as (2.17) with $F_{mn}$ given by (2.28).

The corresponding RR bispinor (A.9) is then given by (here projection from the right by $\frac{1}{2}(1 - \Gamma_{11})$ as in (A.9) is understood)

$$S = i\sigma^2 \tilde{S} , \quad \tilde{S} = -\frac{1}{2\delta} e^\Phi F_{abde} \Gamma^{abde} ,$$  \hspace{1cm} (A.17)

$$\tilde{S} = e^{\Phi_0} \left( \sqrt{1 - \kappa^2} \Gamma^{12} + 2\sqrt{-H_1/H_2} \Gamma^{01} - \frac{\partial_1 M \partial_a \Gamma^{0a} + \partial_0 M \partial_a \Gamma^{1a}}{2M \sqrt{-H_1/H_2}} \right) \Gamma^{4689} \mathcal{P}_4 .$$  \hspace{1cm} (A.18)

We have defined the projector

$$\mathcal{P}_4 = \frac{1}{4}(1 - i\mathcal{J} \Gamma^{(7)}) = \frac{1}{4}(1 - \Gamma^{56789} - \Gamma^{4589} - \Gamma^{6789}) .$$  \hspace{1cm} (A.19)

Here $\mathcal{J} = \Gamma^{45} + \Gamma^{67} + \Gamma^{89}$ is the Kähler form on $T^6$ contracted with gamma matrices and $\Gamma^{(7)} = i\Gamma^{456789}$. The fact that $(i\mathcal{J} \Gamma^{(7)})^2 + 2i\mathcal{J} \Gamma^{(7)} = 0$ means that $i\mathcal{J} \Gamma^{(7)} \frac{1}{2}(1 \pm \Gamma_{11})$ has 12 eigenvalues equal to 1 and four equal to $-3$. Note also that $\mathcal{P}_4 \Gamma_a \mathcal{P}_4 = 0$ where $a' = 4,\ldots,9$.

One can then check that the supergravity equations and Bianchi identities for the RR fields (A.6) are satisfied, namely,

$$\Gamma^a \nabla_a S - \nabla_a \Phi \Gamma^a S = \Gamma^a \partial_a S - \frac{1}{2}\Gamma^a \omega^{bc} [\Gamma_{bc}, S] - \partial_a \Phi \Gamma^a S = 0 .$$  \hspace{1cm} (A.20)

The Einstein equations (A.7) for $H_{abc} = 0$ simplify to (projection from the right by $\frac{1}{2}(1 + \Gamma_{11})$ is suppressed)

$$R_{ab}^{\hspace{2mm}bc} \Gamma_c + 2\nabla_a \Phi \Gamma_b - \frac{1}{12} \Gamma^b \mathcal{S}_a \mathcal{S}_b = 0 .$$  \hspace{1cm} (A.21)

These are also satisfied provided

$$e^2 = 16\kappa^{-1} e^{-2\Phi_0} ,$$  \hspace{1cm} (A.22)

which is in agreement with (2.27),(2.28) (in this Appendix $T = 1$).

**B Realisation of $\mathfrak{su}(1,1|2)$ superalgebra**

The superalgebra $\hat{\mathfrak{f}} = \mathfrak{su}(1,1|2)$ is represented by $4 \times 4$ matrices $\mathcal{M}$ which satisfy $\text{STr} \mathcal{M} = 0$ and the reality condition $\mathcal{M}^\dagger H + H \mathcal{M} = 0$, with $H = \text{diag}(\sigma_3, 1_2)$. The $\mathbb{Z}_4$ automorphism $\Upsilon$

$$\Upsilon(\mathcal{M}) = \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right) \left( \begin{array}{cc} m_{11}^t & -m_{21}^t \\ m_{12}^t & m_{22}^t \end{array} \right) \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right) , \quad \mathcal{M} = \left( \begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array} \right) ,$$  \hspace{1cm} (B.1)

identifies four subspaces $\hat{\mathfrak{f}}^{(k)}$ labeled by $k = 0,\ldots,3$ depending on the eigenvalue of $\Upsilon$ on an element $\mathcal{M} \in \hat{\mathfrak{f}}^{(k)}$, $\Upsilon(\mathcal{M}) = i^k \mathcal{M}$. We define the supertrace as $\text{STr}(\mathcal{M}) = \sum_{j=1}^2 \mathcal{M}_{jj} - \sum_{j=3}^4 \mathcal{M}_{jj}$.

We will realise the $\mathfrak{su}(1,1|2)$ algebra in terms of explicit $4 \times 4$ matrices. In the upper-left $2 \times 2$ block we place generators of $\text{AdS}_2$, while we put generators of $S^2$ in the lower-right one. The off-diagonal blocks contain the odd generators of the algebra. We denote by $P_a, J_{ab}$ the bosonic
generators of the algebra, where indices $a = 0, 1$ are used for $\text{AdS}_2$ and $a, b = 2, 3$ for $S^2$. We define them as
\[
P_a = \begin{pmatrix} -\frac{i}{2} \hat{\gamma}_a & 0 \\ 0 & 0 \end{pmatrix}, \quad a = 0, 1,
\]
\[
J_{ab} = \begin{pmatrix} \frac{1}{2} \gamma_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad a, b = 0, 1,
\]
\[
P_a = \begin{pmatrix} 0 & 0 \\ 0 & i \hat{\gamma}_a \end{pmatrix}, \quad a = 2, 3,
\]
\[
J_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \hat{\gamma}_{ab} \end{pmatrix}, \quad a, b = 2, 3,
\] (B.2)

where
\[
\hat{\gamma}_0 = i \sigma_3, \quad \hat{\gamma}_1 = \sigma_2, \quad \hat{\gamma}_2 = -\sigma_3, \quad \hat{\gamma}_3 = -\sigma_2,
\]
\[
\hat{\gamma}_{ab} = \frac{1}{2} [\hat{\gamma}_a, \hat{\gamma}_b], \quad \hat{\gamma}_{ab} = \frac{1}{2} [\hat{\gamma}_a, \hat{\gamma}_b].
\] (B.3)

To define the odd generators we use the matrices $Q^{Ia\hat{a}}$, where $I = 1, 2$ and $a, \hat{a} = 1, 2$ are spinor indices in $\text{AdS}_2$ and $S^2$ respectively
\[
Q^{Ia\hat{a}} = \frac{e^{+i\pi/4}}{\sqrt{2}} \begin{pmatrix} 0 & m^{Ia\hat{a}} \\ -\sigma_3 (m^{I\hat{a}a})^\dagger \sigma_3 & 0 \end{pmatrix},
\]
\[
m^{I\hat{a}a} = -e^{+i\pi/4} (-1)^{\hat{a}} u^{\hat{a}a}, \quad m^{2\hat{a}a} = e^{-i\pi/4} (-1)^{\hat{a}} u^{\hat{a}a},
\] (B.4)

where $u^{\hat{a}a}$ is the $2 \times 2$ matrix with zero everywhere except the element 1 at position $(\hat{a}, \hat{a})$.

Considering the Grassmann envelope of the superalgebra and demanding that $Q^{Ia\hat{a}}\theta_{Ia\hat{a}}$ satisfies the reality condition of $\mathfrak{su}(1, 1|2)$ we find that the fermions $\theta_I$ satisfy the Majorana condition\(^{36}\)
\[
\bar{\theta}_I = \theta_I^\dagger (\hat{\gamma}_0 \otimes 1_2) = \theta_I^\dagger (\sigma_3 \otimes \sigma_3).
\] (B.5)

Then $J_{ab}$ and $P_a$ belong to the subspaces of grading 0 and 2 respectively, while $Q^{Ia\hat{a}}$ and $Q^{2a\hat{a}}$ to the ones of grading 1 and 3. The commutation relations can be read off by computing explicitly the matrix multiplications
\[
\text{AdS}_2: \quad [P_a, P_b] = J_{ab}, \quad [P_a, J_{bc}] = \eta_{a[b} P_{c]}, \quad (B.6)
\]
\[
S^2: \quad [P_a, P_b] = -J_{ab}, \quad [P_a, J_{bc}] = \eta_{a[b} P_{c]}, \quad (B.7)
\]
\[
[Q^I \theta_I, P_a] = -\frac{i}{2} e^{J_I} Q^I \gamma_a \theta_I, \quad [Q^I \theta_I, J_{ab}] = -\frac{1}{2} \delta^{IJ} Q^I \gamma_{ab} \theta_I, \quad (B.8)
\]
\[
[Q^I \lambda_I, Q^J \psi_J] = i \delta^{IJ} \lambda_I \gamma^a \psi_J P_a - e^{IJ} \lambda_I (\gamma^{01} J_{01} - \gamma^{23} J_{23}) \psi_J - \frac{1}{2} \delta^{IJ} \lambda_I \psi_J 1,
\] (B.9)

where it was convenient to introduce the $4 \times 4$ matrices\(^{37}\)
\[
\gamma_a = \hat{\gamma}_a \otimes 1_2, \quad a = 0, 1,
\]
\[
\gamma_{ab} = \hat{\gamma}_{ab} \otimes 1_2, \quad a, b = 0, 1,
\]
\[
\gamma_a = 1_2 \otimes i \hat{\gamma}_a, \quad a = 2, 3,
\]
\[
\gamma_{ab} = 1_2 \otimes \hat{\gamma}_{ab}, \quad a, b = 2, 3.
\] (B.10)

To get $\mathfrak{psu}(1, 1|2)$ from $\mathfrak{su}(1, 1|2)$ one simply needs to project out the generator proportional to the identity 1.

---

\(^{36}\)We will be omitting spinor indices most of the time. When we need to reintroduce them we assume that gamma matrices for $\text{AdS}_2 (\gamma_a)_{\alpha}^{\hat{\beta}}$ are acting only on checked indices of the fermions $\theta_{Ia\hat{a}}$, while gamma matrices for $S^2 (\gamma_a)_{\alpha}^{\hat{\beta}}$ are acting only on their hatted indices.

\(^{37}\)These matrices $\gamma_a$ are not gamma matrices since they do not satisfy the Clifford algebra relations when we mix indices from $\text{AdS}_2$ and $S^2$. However, they appear naturally in the supercoset construction and they have a natural embedding in the $32 \times 32$ gamma matrices, see Appendix D.
In the above basis for $\mathfrak{su}(1,1|2)$ we find the following bilinear form induced by the supertrace
\[
\text{Str}[J_{01}J_{01}] = \frac{1}{2}, \quad \text{Str}[P_a P_b] = \frac{1}{2} \eta_{ab}, \quad \text{Str}[Q^I \lambda_I Q^J \psi_J] = -\epsilon^{IJ} \tilde{\lambda}_I \psi_J = -\epsilon^{IJ} \bar{\psi}_J \lambda_I.
\] (B.11)
We define the matrix
\[
W = \text{diag}(1,1,-1,-1),
\] (B.12)
which is not an element of $\mathfrak{psu}(1,1|2)$, but plays an important role in the computation of the kappa-symmetry variation of the world-sheet metric in (3.10).

With a group element of the form (3.11), we find that the Maurer-Cartan form is
\[
f^{-1} df = (e^a - \frac{1}{2} \bar{\theta}_I \gamma^a D^{IJ} \theta_J) P_a + Q^I D^{IJ} \theta_J \\
- \frac{1}{2} \omega^{ab} J_{ab} + \frac{i}{4} \epsilon^{IJ} \bar{\theta}_I (\gamma^{01} J_{01} - \gamma^{23} J_{23}) D^{JK} \theta_K + \mathcal{O}(\theta^3).
\] (B.13)
Here the operator $D^{IJ}$ defined on fermions $\theta$ is
\[
D^{IJ} = \delta^{IJ} \left( d - \frac{1}{4} \omega^{ab} \gamma_{ab} \right) - \frac{i}{2} \epsilon^{IJ} e^a \gamma_a,
\] (B.14)
where $e^a, \omega^{ab}$ are the vielbein and the spin-connection of the undeformed $\text{AdS}_2 \times \text{S}^2$ supercoset.

The explicit form of the Maurer-Cartan form is necessary to derive most of the ingredients needed to construct the $\lambda$-model, from the solution of the gauge fields $A_{\pm}$ to the $\kappa$-symmetry transformations upon the formal substitution of the derivative $d$ with the variation $\delta$.

### C Relations for $O_{\pm}$ operators in $\kappa$-symmetry variations

Here we collect some results on the linear operators $O_{\pm}$ defined in (3.8) which are needed for the explicit calculations in section 3. To start, we expand $O_{\pm}$ and its inverse $O_{\pm}^{-1}$ in powers of fermions as
\[
O_{\pm} = O_{\pm}^{(0)} + O_{\pm}^{(1)} + O_{\pm}^{(2)} + \ldots, \quad O_{\pm}^{-1} = O_{\pm}^{\text{inv.}(0)} + O_{\pm}^{\text{inv.}(1)} + O_{\pm}^{\text{inv.}(2)} + \ldots, \quad (C.1)
\]
where one has the obvious relations\(^{38}\)
\[
O_{\pm}^{\text{inv.}(0)} = (O_{\pm}^{(0)})^{-1}, \quad O_{\pm}^{\text{inv.}(1)} = -(O_{\pm}^{(0)})^{-1} \circ (O_{\pm}^{(1)}) \circ (O_{\pm}^{(0)})^{-1}, \quad \ldots
\] (C.2)
Let us define the matrices $(k_{\pm})^j_i$ as
\[
O_{\pm}^{(0)}(T_i) = (k_{\pm})^j_i T_j,
\] (C.3)
where $T_i$ denotes any (bosonic or fermionic) generator of the superalgebra. It is easy to see that the matrices $k_{\pm}$ can be put into block form where each of the three blocks mixes only the generators of $\text{AdS}_2$, or of $\text{S}^2$, or odd generators, respectively and all blocks are invertible. Then
\[
(O_{\pm}^{(0)})^{-1}(T_i) = (k_{\pm})^{-1}_i T_j, \quad (k_{\pm})^{-1}_i (k_{\pm})^j_k = \delta^j_k.
\] (C.4)

\(^{38}\)For the computation of the $\kappa$-symmetry variations in section 3 it will be enough to stop at linear order in fermions but to determine the quadratic fermionic action would require to go to quadratic order.
Let us present the expression for $k_\pm$ written in block form

$$k_{\pm,A} \oplus k_{\pm,S} \oplus k_{\pm,F}$$  \hspace{1cm} (C.5)$$

in algebraic coordinates where they take a more compact form.\(^{39}\) In the basis \{\textbf{P}_0, \textbf{P}_1, \textbf{J}_{01}\} for the generators of AdS\(_2\) we find

$$k_{\pm,A} = \begin{pmatrix}
\frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & -\frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & -\frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} \\
\frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & \frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} \\
\pm \frac{2y\sqrt{\kappa}}{\kappa} & \pm 2x \sqrt{\kappa + H_1} \end{pmatrix}, \hspace{1cm} H_1 = 1 - \kappa x^2 + \kappa^{-1} y^2 \hspace{1cm} (C.6)$$

In the basis \{\textbf{P}_2, \textbf{P}_3, \textbf{J}_{23}\} for the generators of S\(^2\) we find

$$k_{\pm,S} = \begin{pmatrix}
\frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & -\frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & -\frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} \\
\frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} & \frac{2\kappa}{\kappa - 1} - \frac{2y(\kappa + 1)}{\kappa(\pm 1 + \kappa)} \\
\pm \frac{2y\sqrt{\kappa}}{\kappa} & \pm 2y \sqrt{\kappa + H_2} \end{pmatrix}, \hspace{1cm} H_2 = 1 - \kappa p^2 - \kappa^{-1} q^2 \hspace{1cm} (C.7)$$

Let us order the odd generators in (B.4) as \{\textbf{Q}_{111}, \textbf{Q}_{112}, \textbf{Q}_{121}, \textbf{Q}_{122}, \textbf{Q}_{211}, \textbf{Q}_{212}, \textbf{Q}_{221}, \textbf{Q}_{222}\} and decompose the matrices\(^{40}\) $k_{\pm,F}$ as

$$k_{\pm,F} = \sum_{\mu=0}^{3} \sum_{a=0}^{3} \sum_{\tilde{a}=0}^{3} c_{\pm}^F(\mu, a, \tilde{a}) \textbf{s}_\mu \otimes \textbf{g}_a \otimes \textbf{g}_{\tilde{a}}, \hspace{1cm} (C.8)$$

where we have defined

$$\textbf{s}_\mu = \{1_2, \sigma_1, i\sigma_2, \sigma_3\}, \hspace{0.5cm} \textbf{g}_a = \{1_2, \gamma_0^I, \gamma_1^I, -\gamma_0^I\}, \hspace{0.5cm} \textbf{g}_{\tilde{a}} = \{1_2, \gamma_2^I, \gamma_3^I, -\gamma_2^I\}. \hspace{1cm} (C.9)$$

The coefficients $c_{\pm}^F$ are different for the two operators and are given by

$$c_{\pm}^F(0, 0, 0) = \kappa px - \frac{1}{\sqrt{1 - \kappa^{-1}}}, \hspace{1cm} c_{\pm}^F(0, 0, 3) = -qx, \hspace{1cm} c_{\pm}^F(0, 1, 1) = i\sqrt{H_1} \sqrt{H_2},$$

$$c_{\pm}^F(0, 3, 0) = py, \hspace{1cm} c_{\pm}^F(0, 3, 3) = -\frac{qy}{\kappa}, \hspace{1cm} c_{\pm}^F(2, 0, 1) = -\kappa px \sqrt{H_2},$$

$$c_{\pm}^F(2, 1, 0) = i\sqrt{\kappa p} \sqrt{H_1}, \hspace{1cm} c_{\pm}^F(2, 1, 3) = -\frac{qy\sqrt{1 + \kappa}}{\sqrt{\kappa}}, \hspace{1cm} c_{\pm}^F(2, 3, 1) = -\frac{qy\sqrt{1 + \kappa}}{\kappa},$$

$$c_{\pm}^F(3, 0, 0) = \frac{\kappa px}{\sqrt{1 - \kappa^{-1}}}, \hspace{1cm} c_{\pm}^F(3, 0, 3) = \frac{qy}{\sqrt{1 - \kappa^{-1}}}, \hspace{1cm} c_{\pm}^F(3, 1, 0) = -\frac{qy}{\sqrt{1 - \kappa^{-1}}},$$

$$c_{\pm}^F(3, 1, 1) = -\frac{qy}{\sqrt{1 - \kappa^{-1}}}, \hspace{1cm} c_{\pm}^F(3, 2, 0) = -\frac{qy}{\sqrt{1 - \kappa^{-1}}}, \hspace{1cm} c_{\pm}^F(3, 2, 3) = -\frac{qy}{\sqrt{1 - \kappa^{-1}}}.$$

\(^{39}\)The results of this appendix have been simplified by assuming $y > 0, q > 0.$

\(^{40}\)It is assumed that their action with explicit indices is as $C_{\pm}^{(ij)}(Q^{i\hat{a}}) = (k_{\pm,F})_{j\hat{b}}^{i\hat{a}} Q^{j\hat{b}}.$
Let us now demonstrate how one can put the $\kappa$-symmetry variations into the standard form. We need to compute (3.6), where $f^{-1}\delta f$ is obtained from (B.13) by formally substituting the derivative $d$ with the variation $\delta$, and by assigning weights 0 and 1 to the variations $\delta \theta_I$ and $\delta X^m$ at leading order in expansion in fermions. When we project on odd generators we get equations for $\delta \theta_I$. At leading order in fermions the l.h.s. of these equations is just $(\mathcal{O}^{(0)}_\kappa)^{-1}(\mathbf{Q}\delta \theta) = \mathbf{Q}(k_{+p}^{-1})t\delta \theta$, where $t$ is transposition. From the r.h.s. of (3.6) we find

\begin{align}
A_{-}^{(2)}\hat{\alpha} + \hat{\alpha}A_{-}^{(2)} &= -\frac{1}{2}c^{-1}\mathbf{Q}^1(E_{-}^{(-)}\gamma_a - E_{-}^{(-)}\gamma_\bar{a})\kappa_1, \\
A_{+}^{(2)}\hat{\alpha} + \hat{\alpha}A_{+}^{(2)} &= +\frac{1}{2}c^{-1}\mathbf{Q}^2(E_{+}^{(+)}\gamma_a - E_{+}^{(+)}\gamma_\bar{a})\kappa_2,
\end{align}

(C.10)

where $\hat{\alpha}$ are AdS indices and $\hat{\alpha}$ are sphere ones. We used that (as can be shown with our realisation of the superalgebra)

\begin{align}
\mathbf{Q}^I\mathbf{P}_{\hat{\alpha}} + \mathbf{P}_{\hat{\alpha}}\mathbf{Q}^I &= -\frac{1}{2}\mathbf{Q}^I\gamma_\bar{a}, \\
\mathbf{Q}^I\mathbf{P}_{\hat{\alpha}} + \mathbf{P}_{\hat{\alpha}}\mathbf{Q}^I &= +\frac{1}{2}\mathbf{Q}^I\gamma_a.
\end{align}

(C.11)

We first redefine $\theta \rightarrow c^{-1/2}k_{+p}^{I}\theta$, and $\kappa \rightarrow 2c^{1/2}\kappa$, finding that the $\kappa$-symmetry transformations for the fermions become $\delta \theta_1 = -(E_{-}^{(-)}\gamma_a - E_{-}^{(-)}\gamma_\bar{a})\kappa_1$, and $\delta \theta_2 = (E_{+}^{(+)}\gamma_a - E_{+}^{(+)}\gamma_\bar{a})\kappa_2$. These variations differ only in the choice of the vielbein, meaning that it is enough to redefine $\theta_1 \rightarrow U\theta_1$ and $\kappa_1 \rightarrow -U\kappa_1$ with $U = -\sigma_2 \otimes \sigma_2$ such that $U^{-1}\gamma_a U = \Lambda_a^\alpha \gamma_\alpha$ to obtain

\begin{align}
\delta \theta_1 = (E_{-}^{a}\gamma_a - E_{-}^{\bar{a}}\gamma_\bar{a})\kappa_1, \\
\delta \theta_2 = (E_{+}^{a}\gamma_a - E_{+}^{\bar{a}}\gamma_\bar{a})\kappa_2,
\end{align}

(C.12)

where $E = E^{(+)}. This is the desired standard form of the $\kappa$ symmetry variations which can be rewritten also in the 10d notation as in (3.17).

There is still a freedom to rescale $\theta_I \rightarrow c_I\theta_I$ and $\kappa_I \rightarrow c_I\kappa_I$. Then $c_1 = \lambda$, $c_2 = 1$ are fixed by requiring that the $\kappa$-symmetry transformations for the bosons are also of the standard form. These are obtained by projecting (3.6) on generators of grading 2, keeping only the leading order contributions. After taking into account the previous redefinitions for $\theta$ we find

\begin{align}
E^a_\alpha\delta X^m &= \frac{i}{2}\theta_I\gamma^a\delta \theta_I + 2\bar{\theta}_I\Delta^a_I\delta \theta_J, \\
\Delta^a_I &= \delta^{IJ}\left(\epsilon_m^{a}012 + \epsilon_m^{a}023\right) + \epsilon^IJ\left(\epsilon_m^{a}02 + \epsilon_m^{a}03 + \epsilon_m^{a}12\right)^{12} + \epsilon_m^{a}13\gamma^{13} + \epsilon_m^{a}0123\gamma^{0123}.
\end{align}

(C.13)

The last term in (C.13) can be canceled by a shift $X^m \rightarrow X^m + E^m_\alpha\bar{\theta}_I\Delta^a_I\theta_J$. This is possible thanks to the symmetry property of $\Delta^a_I$ under transposition of the spinor indices and labels $I, J$, which makes this shift non-vanishing.\(^{42}\)

The $\kappa$-symmetry transformations of the world-sheet metric are obtained from (3.10). One needs to compute $A_\pm$ by implementing the above redefinitions of the fermions. The redefinitions of the bosons do not modify the result at the leading order. Notice that the non-vanishing contribution comes from the last term in the commutation relation (B.9), similarly to what happens in the undeformed supercoset case.

\(^{41}\)Here we choose to omit the rather long explicit expressions for the coefficients $c_{a_1\ldots a_n}^{a_1\ldots a_n}$.

\(^{42}\)Notice that in general the symmetry property of $\gamma_{m_1\ldots m_n}$ does not need to be the same as of $\Gamma_{m_1\ldots m_n}$, see end of Appendix D for definitions.
D Gamma matrices

The basis for the $32 \times 32$ gamma matrices that we use is

\[
\Gamma_0 = \sigma_1 \otimes \sigma_3 \otimes 1_2 \otimes 1_2, \\
\Gamma_1 = \sigma_1 \otimes \sigma_2 \otimes 1_2 \otimes 1_2, \\
\Gamma_2 = \sigma_2 \otimes 1_2 \otimes \sigma_3 \otimes 1_2, \\
\Gamma_3 = \sigma_2 \otimes 1_2 \otimes \sigma_2 \otimes 1_2, \\
\Gamma_4 = \sigma_2 \otimes 1_2 \otimes \sigma_1 \otimes 1_2, \\
\Gamma_5 = \sigma_1 \otimes \sigma_1 \otimes 1_2 \otimes 1_2, \\
\Gamma_6 = \sigma_2 \otimes 1_2 \otimes \sigma_1 \otimes 1_2, \\
\Gamma_7 = -\sigma_1 \otimes \sigma_1 \otimes 1_2 \otimes 1_2, \\
\Gamma_8 = \sigma_2 \otimes 1_2 \otimes \sigma_1 \otimes 1_2, \\
\Gamma_9 = \sigma_1 \otimes \sigma_1 \otimes 1_2 \otimes 1_2.
\]

(D.1)

These $\Gamma_a$ satisfy

\[
\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}1_{32}, \quad \text{Tr}(\Gamma_a) = 0, \quad (\mathcal{C} \Gamma_m)^t = +\mathcal{C} \Gamma_m,
\]

(D.2)

with $\mathcal{C} = i\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_2$.

The gamma matrices corresponding to AdS$_2$ and S$_2$ are

\[
\Gamma_a = \sigma_1 \otimes \gamma_a \otimes 1_2 \otimes 1_2, \quad a = 0, 1; \\
\Gamma_a = \sigma_2 \otimes 1_2 \otimes \gamma_a \otimes 1_2, \quad a = 2, 3.
\]

(D.3)

With this definition we have

\[
\bar{\Theta}_I \Gamma_a \Theta_J = \bar{\theta}_I \gamma_a \theta_J,
\]

(D.4)

where $\gamma_a$ are defined in (B.10) and the 32-component spinors $\Theta_I$ are related to the 4-component $\theta_I$ as in (3.18)

\[
\Theta_I = \left( \begin{array}{c} 1 \\ 0 \\ \theta_I \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right) \).
\]

(D.5)

The conjugate fermions are defined by $\bar{\Theta}_I = \Theta_J^t \mathcal{C}$ and $\bar{\theta}_I = \theta_J^t (\sigma_3 \otimes \sigma_3)$. More generally, we define $\gamma_{m_1 ... m_n}$ with $n$ indices by requiring that $\bar{\Theta}_I \Gamma_{m_1 ... m_n} \Theta_J = \bar{\theta}_I \gamma_{m_1 ... m_n} \theta_J$ when $n$ is odd, and $\bar{K}_I \Gamma_{m_1 ... m_n} \Theta_J = \bar{\kappa}_I \gamma_{m_1 ... m_n} \theta_J$ when $n$ is even.

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