A “Domain Wall” Scenario for the AdS/QCD

E.Shuryak

Department of Physics and Astronomy, Stony Brook University, Stony Brook NY 11794-3800, USA

(Dated: February 2, 2008)

We propose a scenario for bottom-up gravity dual picture of QCD-like theories, which consists of two near-AdS$_5$ domains separated by the “domain wall” at which the effective coupling relatively rapidly switches from small perturbative value at its UV side to strong at its IR side. Its consequence are corresponding jumps in 5-d masses of the bulk fields, related to different anomalous dimensions of the corresponding operators. Inclusion of both weakly and strongly coupled parts of bulk wave functions allows for consistent inclusion of QCD hard processes. We further discuss how transitions from strong to weak coupling in hard observables should look like, exemplified by pion diffractive dissociation, which seems to show this transition experimentally. Then we propose a dynamical mechanism for this jump related to instantons, which are point-like bulk objects located at/near the domain wall. We further argue that in the limit of large number of colors $N_c \to \infty$ the “domain wall” is becoming a true singularity. Instanton-induced contributions to correlators and partition function of the instanton ensemble is reformulated in the AdS$_5$ language. Among other applications are lowest mesons as collective vibrations of the “domain wall”.

PACS numbers:

I. INTRODUCTION

A. The bottom-up approach to gravity dual

Discovery of the AdS/CFT correspondence $^1$ between $\mathcal{N}=4$ SYM in 4 dimensions and string theory in $AdS_5 \times S_5$ geometry had finally fulfilled the long-standing promise of exact gauge-string correspondence. However, $\mathcal{N}=4$ SYM theory is very far from QCD: it is non-confining and conformal.

Furthermore, the simplest and most useful form of the correspondence takes place in the large number of colors $N_c \to \infty$ limit, in which ’t Hooft gauge coupling $\lambda = g^2 N_c$ is large. One set of AdS/CFT applications is related to the finite-temperature QCD and recent discovery of Quark-Gluon Plasma at RHIC, apparently being strongly coupled $^2$. In this phase both QCD confinement and running coupling are of secondary importance, while AdS/CFT results are in agreement with phenomenology, see e.g. recent review $^3$.

One way to develop further those fascinating tools is to look for a correspondence including theories with running coupling, e.g. cascading theories $^4$. Another – known as top-down approach – is to invent brane constructions reproducing QCD-like theory as an effective theory in infrared. Important issues are implementation of flavor and chiral symmetry breaking in this context $^5$: currently the most widely used example of it is the Sakai-Sugimoto model $^7$.

This paper however is not about QGP or top-down models but about an alternative approach – known now as bottom-up or AdS/QCD – building a model based on known features of the QCD vacuum and hadronic physics, formulated using a holographic 5-d language.

From the onset of this paper we would like to emphasize the difference between (i) a holographic approach in general and (ii) classical gravity dual description in particular. The former (i) uses the 5-th coordinate as a general (energy) scale variable $z \sim 1/\mu$ in a sense in which it appears in the renormalization group, with $\mu$ being a normalization scale. A notion of a hadronic wave function $\psi(z)$ is introduced, as the probability amplitude for a hadron to have certain physical size. The more specific classical (super)gravity description – based on gauge-string duality and coupling inversion – may or may not be applicable at certain $z$, depending on whether the gauge coupling is or is not large. AdS/QCD assumes that gauge theory is strongly coupled in some region of scales. Even if so, some part of the $z$ domain corresponds to weak coupling, and the problem to be discussed below is how to unite both in the same framework. Solving this problem is key for such important applications as e.g. hard exclusive processes, for which existence of a single wave function $\psi(z)$ in the whole space, with single well-defined normalization procedure, is obviously required.

Let us now briefly review the papers which established AdS/QCD, pointing out where our approach would differ with what was used there.

Modifications at large $z$ – Infra Red (IR) were introduced to include the effect of confinement, present in QCD and absent in AdS/CFT. The first crude way to do so (introduced by Polchinski and Strassler) was done via the IR cutoff of the $AdS_5$ space above some $z > z_{IR}$. If this is done, the bulk fields propagating in the bulk get quantized, which generates a 4-d hadronic spectrum. It appears that this spectroscopy is not bad for a first-order model. One widely known good feature is Regge behavior of particle masses at large spins, see e.g. $^5$. The model can naturally be appended by quark masses/condensates, see $^9$. Good news about this simple model are not only related with hadronic masses: sizes of the states and related coupling constants to point operators are in some examples very nice as well. Very recently (after the first
version of the present paper was already posted) Schafer have demonstrated that the whole Euclidean vector and axial correlators $V(\tau), A(\tau)$ built out of these modes reproduce experimental data within 10-20% accuracy, at any Euclidean time $\tau$. (These results are extremely similar to what was obtained by Schafer and myself in much more sophisticated instanton model [11].) The remaining 10% deviations, at least at small distances, are clearly just neglected perturbative correction $1 + \alpha_s/\pi + ....$ We will however argue below that vector/axial channels are in fact exceptions rather than the rule, while generic correlators should not be well described by this simplest model.

Further improvements of Regge phenomenology to radial excitations motivated Karch et al [12] (below KKSS) to introduce the so called “soft wall” with a dilaton potential quadratic in $z$. Independent argument for quadratic wall was given in my paper [13] to which we return below. This soft wall was interpreted as a pure geometrical factor $\exp(Cz^2)$ in metric [13]. Although the applications discussed in these way are all interesting, it is hard to agree with pure geometric 5-d modeling of confinement. In particular, such metric acts in the same way on electric and magnetic (D1) strings, providing confinement to both. The dilaton field naturally has different coupling to electric and magnetic (D1) strings, which allows only the former to be confined and the latter screened, see “improved AdS/QCD” by Gursoy, Kiritsis and Nitti (GKN) [15, 16].

Modifications at small $z$ – Ultra Violet (UV) are also necessary since here QCD is weakly coupled. Evans et al [17] proposed to introduce an UV cutoff of the AdS space, simply removing some part of it $z < z_{UV}$ and reformulating the boundary conditions. Although that improves masses of the vector mesons, reconciling the lightest $n=0$ $\rho$ with a different trend for radial excitations $n = 1, 4$, it goes with a heavy price. As we will argue below in detail, simply cutting off the weakly coupled domain is not an acceptable option if one hope to address hard scattering.

In fact cutting off the space in IR, $z > z_{IR}$ is not acceptable as well. It may be enough for crude features like Regge trajectories, but would fail to address other important issues. I particular, QCD practitioners spent a lot time trying to understand why is it that quark masses changing from say $m_q, m_s \sim 100 \text{MeV}$ to $m_{\pi}, m_{\eta} \sim \text{few MeV}$ induce nontrivial changes in many observables, such as Dirac eigenvalues, nucleon structure functions, magnetic moments. In the gravity dual language it means that one has to accommodate those low mass scales, or $z$ of the order of several fm at least: we will return to this in section [V.B].

In the instanton context the weak coupling is obviously a dilaton-based potential related to the beta function: such potential was included in [13], and we return to this below. GKN [15, 16] have proposed a systematic approach toward building a gravity dual model to QCD, including the metric, dilaton and axion in the common Lagrangian. A clever tric allows to incorporate the beta function (which demands first order differential eqn) with Lagrangian formalism which demands second order ones. Solution of the resulting equations of motion are classified according to different dependence of these bulk fields on the 5-th holographic $z$ coordinate. A new issue raise by GKN (to be discussed in section [V.B]) is $z$-dependent axion field.

The main subject of the present paper is how coupling runs at intermediate $z$. While GKN simply suggested a smooth interpolation between IR and UV (the dashed line in Fig[1], we expect a very rapid growth – a jump – from weak to strong coupling, near some particular $z$ we call the domain wall $z_{dw}$, schematically shown by the solid line in Fig[1] We further suggest as an approximation two “domains”, the weakly and strongly coupled ones, separated by this “domain wall”. While we will argue that the wall is rather narrow already for real QCD with $N_c = 3$ colors, it should become a sharp jump in the large-$N_c$ limit.

B. Old phenomenology issues: the “chiral scale” versus $\Lambda_{QCD}$

Let me start by reminding some 30-year-old puzzles, containing the essence of the issues to be discussed. Basic chiral dynamics – chiral symmetry breaking by the quark-antiquark condensate, pions as Goldstone modes, their effective Lagrangian – were developed in 1960’s. Nambu-Iona-Lasinio model of 1961 introduced the “chiral scale”

$$\Lambda_{\chi} = 4\pi f_\pi \approx 1.2 \text{GeV}$$

as a boundary below which (at $Q < \Lambda_{\chi}$) the hypothetical 4-fermion interaction had to exist.

With the advent of QCD in 1970’s, observation of scaling violation and other effects the details of the running coupling were clarified. It turns out that that $\Lambda_{QCD} \sim 1 \text{fm}^{-1} \approx 200 \text{MeV}$ is quite small. (Precise definition depends on subtraction scheme, but basically it is the place of the Landau pole where the QCD coupling blows up if one would extrapolate the one-loop pQCD running to IR.) That lead to a puzzle: why don’t these
two scales match? In particular, at \( Q \sim \Lambda_\chi \) the perturbative coupling is still too small to account for the necessary strength of the NJL-type 4-fermion interaction to achieve chiral symmetry breaking. Thus, already 30 years ago it was clear that some important physics at the scale \( Q \sim \Lambda_\chi \) was missing.

The QCD sum rules have shown that non-perturbative splittings of the correlators for some channels (such as vector and axial ones) can be described by the operator product expansion (OPE). However, much stronger non-perturbative effects were found for all spin zero channels \[19\]. The largest are in gluonic spin zero channels, where the perturbative boundary scale is located as high as \( Q^2 > \Lambda^2_{\text{phys}} \approx 20 \text{GeV}^2 \). These puzzles were resolved by realization \[20\] that small-size instantons \( \rho \sim 1/3 \text{fm} \) are very abundant in the QCD vacuum. Some brief overview will be given in section IV, for review see see \[21\].

**C. Running coupling, from lattice and phenomenology**

The usual plots one finds in textbooks follow the coupling down to \( Q = 2 \text{GeV} \), where \( \alpha_s(Q) \approx 1/3 \). Extrapolation of these curves down to \( Q = 1 \text{GeV} \) can perhaps be still trusted, but what is beyond this scale remains unclear and depends on the definition used.

Having said that, let us still have a look at some results going well beyond this scale, looking for qualitative hints. One example, taken from \[23\], is shown in Fig. 2. It follows the gluon-ghost-ghost vertex function in Landau gauge, for 2-flavor QCD. The curves are different extrapolations based on pQCD. One can see that lattice data (points) rather rapidly depart from it already at \( q \sim 2 \text{GeV} \) upward. Quite remarkable feature of these results is that the coupling stays flat in some “conformal window” \( g = 3 - 1 \text{GeV} \). Note that the coupling becomes in IR rather strong indeed, especially if translated into the units of 't Hooft coupling

\[
\lambda_{q<1 \text{GeV}} = g^2 N_c \approx 80
\]

Another issue important for what follows is how the coupling runs for gauge theories with increasing number of colors \( N_c \). On general grounds one expects running to become more rapid: the question is whether some discontinuities may appear at finite or infinite \( N_c \). In recent paper by Allton et al \[22\] it was found that there exist a lattice bulk weak-strong phase transition starting from \( N_c = 5 \), well seen in the \( N_c = 8 \) example mainly discussed in this work. What is worth mentioning is that the lattice spacing corresponding to unstable (mixed phase) situation between weak and strong coupling is in the range:

\[
a \approx \frac{4.1}{\sqrt{\sigma}} \approx 0.2 - 0.5 \text{fm}
\]

**II. SPECTROSCOPY AND HARD PROCESSES IN THE TWO-DOMAIN PICTURE**

**A. From bulk to boundary spectroscopy**

General rules of “holography” connect operators on the boundary to fields \( \phi(z) \) propagating in the bulk. The conformal dimensions of boundary operators are related to 5-d masses of bulk fields. We will denote those \( M_5 \) to be distinguished from the usual 4-d masses denoted by \( m_4 \). The operator dimension \( \Delta \) matches the wave functions asymptotics at small \( z \) via \( \phi \sim z^\Delta \).

Before we go into specifics, let us first remind that all operators are divided into two classes: (i) The “protected” ones have zero anomalous dimensions while (ii) “unprotected” ones – a generic case – have nonzero “anomalous” dimensions resulting from field interaction and depending on the coupling. This distinction will be very important in what follows.

Bulk wave eqns for spin-S fields have a generic form:

\[
\partial_z e^{-B} \partial_z \phi(z) + (m_4^2 - M_5^2/z^2) e^{-B} \phi(z) = 0
\]

where

\[
B = z^2 + (2S - 1) \ln(z)
\]

We follow notations of KKSS \[12\], except that we have added extra 5-d mass term. Standard substitution \( \phi = e^{B/2} \tilde{\psi} \) transform this into a Schrodinger-like eqn without first order derivatives

\[
-\psi'' + V(z) \psi = m_4^2 \psi
\]
with
\[ V(z) = z^2 + 2(S - 1) + \frac{S^2 + M_5^2 - 1/4}{z^2} \] (7)

KKSS needed only bound state wave functions, given in terms of Laguerre polynomials. Quadratic IR potential was tuned by [12] to reproduce nice Regge trajectories: for absent bulk mass \( M_5 = 0 \) they are linear and \( m_4^2 = 4(n + S) \).

The absolute units we will use also follow from KKSS notations, can be fixed by calculate some physical mass. Rho meson is an example of the protected state, associated with conserved vector current: it has with \( M_5 = 0 \) and \( S = 1 \): a solution without nodes \((n=0)\) gives \( m_4^2 = 4 \). Using it as an input we fix our unit of length as
\[ \text{length unit} = 2/m_p = 0.51 \text{fm} \] (8)

The expected position of the domain wall at large \( N_c \) is
\[ z_{dw} \approx 0.4 \text{fm} = 0.777(\text{length unit}) \] (9)

Thus most of the wave function is in fact located in the strongly coupled domain and modifications due to weakly coupled domain at \( z < z_{dw} \) turns out to be very small, as far as the 4-d spectroscopy goes. However hard processes will be sensitive to this small tail of the wave function, and their relative normalization is crucial for what follows.

Because generic bulk objects are excited (closed) string states, there masses are non-trivial: in fact their evaluation is one of central subjects of current string theory. Interpolating anomalous dimensions (or \( M_5 \)) between weakly and strongly coupled regimes is a hot subject. Curiously it was not yet discussed in the AdS/QCD context: papers we know on both holographic spectroscopy and exclusive processes simply ignore them, using canonical (bare) dimensions. There is no conservation laws to protect them most of the operators – e.g. scalars or baryons – from acquiring anomalous dimensions, and one may ask if the usual form of AdS/QCD models can be used in those case.

So, what is actually known about the “bulk spectroscopy". While in weak coupling the magnitude of the anomalous dimensions is \( O(\lambda) \) small, in strong coupling they are in general large. As argued by Gubser et al. [25] generic or “vibrational" string states have anomalous dimensions of the order
\[ \Delta - \Delta_0 \sim (\lambda^{1/4}) \] (10)
but (to my knowledge) their theory is not yet developed. There is more progress for “rotational" string states, especially for operators with large flavor content (large angular momentum \( J \) in \( S_6 \)) which developed into exact spectroscopy using Bethe-ansatz methods.

Much less is done for the operators with the usual spin \( S \). In the case of large spin \( S \) anomalous dimension grow only logarithmically
\[ \Delta - S = f(\lambda)\ln(S/\sqrt{\lambda}) \] (11)
and \( f(\lambda) \) seems to be universal function for cusps: its large coupling limit \( f(\lambda) \rightarrow \sqrt{\lambda}/\pi \) correspond to folded spinning strings as shown in [24]. For orientation, if the strongest QCD coupling is \( \lambda \sim 80 \), and thus this function can reach \( f(\lambda) \sim 3 \) or so: thus expected anomalous dimensions may be of the size of several units or so.

Although we dont yet know appropriate \( M_5 \) values for all these operators, its quantitative role is easy to understand from the wave equation. In the effective potential \( M_5 \) simply adds to “angular momentum" coefficient of the \( 1/z^2 \) term. Thus increasing \( M_5 \) shift the wave function a bit more into IR (larger \( z \)), increasing the size of the state and its 4-d masses. More radical effect is the decrease of the tail of the wave function at small \( z \), important for hard processes. We will return to solutions of this eqn in the two-domain scenario, after we briefly review the formalism for hard processes.

B. Hard processes in AdS/QCD

Polchinski and Strassler pioneered treatment of hard elastic [28] and deep inelastic (DIS) [26] scattering in the holographic (strong coupling) setting. They placed the scattering process into the 5-d bulk, using for propagating bulk states string-based scattering amplitudes. As hadronic wave functions are depending on \( z \) as described above, hard probes emit virtual bulk fields as well. For example, electromagnetic formfactors can be thought in terms of bulk current-fields vertex
\[ FF(Q) \sim \int dz\psi(z)\partial_m\psi A_m(z) \] (12)
where bulk photon \( A_m(z) \) has imaginary 4-mass \( m_4 = iQ \) and thus for large \( Q \) it is exponentially decreasing at large \( z \) \( A_m \sim \exp(-Qz) \). Since hadrons have wave functions \( \phi \sim z^{\Delta} \) their convolution has a maximum at
\[ z_\ast = 2\Delta/Q \] (13)
It means that physical elastic (or exclusive) processes proceed via hadrons in a very compressed state, to a size \( \sim 1/Q \). (For deep inelastic process only some part of a hadron of such size is involved.)

Brodsky and Teramond [27] obtained the following nice expression for formfactors
\[ F(Q^2) = \Gamma(\tau)\frac{\Gamma(1 + Q^2/4)}{\Gamma(\tau + Q^2/4)} \] (14)
where \( \tau \) is the twist \( \tau = \Delta - S \) of the operator. They were able to describe available experimental data for pion and nucleon formfactors, assuming bare twist \( \tau = 2 \) for pions and \( \tau = 3 \) nucleons. While the former can indeed be produced by a conserved axial current with protected dimensions, there is no reason to think the nucleon’s conformal dimension is protected and remains integer. The absolute values of the pion formfactor does not match asymptotic
perturbative value: and for many other exclusive reactions in which the power laws seem to be working there are doubts that those are indeed the regime at asymptotically large $Q$. A generic qualitative behavior of all such processes will be discussed in the next section.

III. WAVE FUNCTIONS AND HARD PROCESSES IN THE TWO-DOMAIN SCENARIO

Let us first describe the two-domain wave functions we propose to use. To find them we will need more general solutions

$$\psi = \frac{C_W}{\sqrt{z}} W((m_4^2/4 + 1/2, M_5/2, z^2)) + \frac{C_M}{\sqrt{z}} M((m_4^2/4 + 1/2, M_5/2, z^2))$$

(15)

where $W, M$ are Whittaker functions and $C_W, C_M$ some constants.

Let me deviate from a well-trotted path here, and instead of canonical vector or pseudoscalar mesons have use an example more exotic states with larger operator dimension and $M_5$. For reasons to become clear in the next section, we will use objects with bare dimension 3. As we dont know its anomalous dimension, we will use some arbitrary value equal 4 (7 for total dimension at strong coupling). From spectroscopic point of view this can be e.g. a scalar meson ($\bar{q}q$)

As an approximate solution in spirit of tho-domain scenario, one can continuously connect solutions with two different bulk masses $M_5$ in the weak and strong coupling domains. The 4-d mass should of course be the same, derived from continuity of the logarithmic derivative at the domain wall

$$\frac{d}{dz} \log \psi(m_4, M_5^{\text{weak}}, z = z_{dw}) = \frac{d}{dz} \log \psi(m_4, M_5^{\text{strong}}, z = z_{dw})$$

(16)

Note that the Whittaker function appropriate for l.h.s. (small $z$) is $M$, while for r.h.s. (large $z$) it is $W$. Of course there should be a continuity of both the wave function and its derivative, with a jump in the second derivative defined by the jump in the bulk mass.

How it works in practice is shown in Fig. 3. Note that in this example the wave function modified tail is about 4 orders of magnitude down from its maximum. Although the value of the 4-d mass $m_4$ is only changed by a tiny amount, it is enough to influence the wave function behavior below $z = z_{dw}$ allowing to connect two functions smoothly, as shown in Fig. 3(b).

The reader may think that tiny tail of the wave function is completely irrelevant. It is indeed so for the mass and other bulk parameters of these states, but not for hard (exclusive) processes which is dominated by small $z$. As $Q$ grows, it moves from strongly to weakly coupled domain, and the behavior of all hard processes changes. One may ask a question, whether it is possible to even see this transition and thus locate the “domain wall”.

The formfactor is calculated from the combined wave function by

$$f(Q) = \int \frac{dz}{z} \psi^2(z)(QzK_1(Qz))$$

(17)
FIG. 4: (color online) The combination $Q^6 f(Q)$ where $f(Q)$ is elastic formfactor, versus $Q$ (in units $m_\rho/2$). The decreasing curve (boxes) is for wave function with the dimension 7. The curve stabilized at large $Q$ (circles) is for the combined wave function, which at $z < z_{dw}$ has the (original bare) dimension 3.

where the last bracket with Bessel function $K_1$ represent the bulk virtual photon. Schematically, there are three regions: (i) small $Q$ in which the integral is dominated by all $z$ and is close to the total charge, usually taken as 1, (ii) dominated by intermediate $z^* > z_{dw}$ in the strongly coupled domain, where one sees power behavior including anomalous dimension; (iii) small $z^* < z_{dw}$ in which the power law switches to bare operator dimension.

For the example we considered above, a dimension jumping from 7 in (ii) to 3 in (iii), the formfactor (times $(Q^2)^3$, bare dimension) is shown in Fig. 4. It displays falling behavior (ii) induced by anomalous dimension, which is then stopped at the r.h.s. of the picture as formfactor ends up in the weakly coupled regime (iii). Thus – at least in this example – transition from strong to weak coupling is clearly seen in the formfactor.

(In the next section we will discuss real-life experimental data for a process with bare dimension of the momentum 6, to which this figure should be compared. It is thus worth noticing that the value of $Q$ at which the formfactors levels off is as high as 6 GeV.)

A. Is the “domain wall” observed already?

How one can observe rapid jump in coupling/anomalous dimensions is already discussed above and displayed in Fig 4. The outlined procedure predicts uniquely a shape of the transition, from anomalous to bare dimensions seen in hard processes, as well as the magnitude of the processes. The transition happens when the peak of the convolution ($\text{III}$) crosses $z_{dw}$ as $Q$ increases. Ideally it should be reflected in the cross section of any hard process, as all QCD processes depend on the coupling.

The simplest object of the kind is the pion formfactor. Unfortunately experiment has not yet seen how transition to old perturbative prediction takes place, as it is very hard to measure it at large $Q$. All we know is that the observed $Q^2 F_\pi(Q)$ at $Q \sim 1 - 2$ GeV is about twice large than the asymptotic value, so some decrease and then leveling at new level should happen. As far as I know, nobody have seen it on the lattice either. The magnitude of the pion formfactor at $Q \sim 1 - 2$ GeV induced by instantons has been calculated in ref. [29].

FIG. 5: (upper) The Acceptance-corrected $u$ distributions of diffractive di-jets events from E791 [31]. (lower) The points are experimental $k_t$ distribution, shown as $k_t^2 d\sigma/dk_t$ (arbitrary units) vs $k_t$ (GeV). The curve is (rescaled) formfactor shown in Fig. 4 by circles: it is plotted to test the agreement of their shapes.
but unfortunately the approximation made are not good at large enough $Q_t$ so the transition to the perturbative regime has not been seen in this approach as well. More or less similar situation is with the nucleon formfactor and many other exclusive reactions.

Another hard reaction involving the pion is pion diffractive dissociation into two jets

$$\pi \rightarrow jet(k_1) + jet(k_2) \quad (18)$$

first theoretically discussed by Frankfurt et al. and studied experimentally by the Fermilab experiment E791. It seems to be showing a transition from the non-perturbative to the perturbative regime we are looking for. Large transverse momenta of the jets ensure that a pion is found in a small-size configuration, with $q$ and $\bar{q}$ at small distance $\sim 1/k_t$ from each other. Due to the so called color transparency effect, this small color dipole interacts weakly with the nuclei. In fact in the actual experiment the nuclear recoil to the nuclei used (Pt one of the targets) is very small, less or comparable to $1/R_{\text{nucleus}}$: the process is thus completely coherent over all nucleons, with the cross section $\sim A^2$. Literally, a pion squeezes through all small holes between the nucleons of the target, and makes a coherent wave after that.

Fig.5 displays two main findings of E791:

(i) In Fig.5 upper, one sees that the distribution over a jet momentum fraction $u$ is radically different in two momentum windows, the one at higher $k_t$ consistent with perturbative or “asymptotic” distribution $u(1-u)$.

(ii) Fig.5 lower) shows that two (somewhat shifted) windows also have radically different $k_t$ dependence. The one at higher $k_t > 1.9 \text{GeV}$ is consistent with bare dimension of momentum 6, while the lower one can be fitted with the power 10.

The lower figure should be compared with Fig.4 above, and the shapes are really very similar. Again, the interpretation is that the $k_t > 1.9 \text{GeV}$ regime corresponds to perturbative asymptotics domain, while the lower $k_t$ are affected by the strongly coupled region where anomalous dimension is high. The transition is quite clearly seen in the data, which we take as the first experimental evidence in favor of a rapid weak-strong transition. Note that in order to compare it with formfactor one should use $Q = (2k_t)$ which is $\sim 4 \text{GeV}$ at the transition point.

Many open questions to diffractive dissociation include: Can the dimension jump be indeed accounted for by some anomalous dimension? Does it correspond to expected jump of the coupling, from $\alpha_s = 1/2$ to 2? Is it related to instantons? Can “strongly coupled” distribution over fraction of the energy $u$ at $k_t < 1.5$ window Fig.5(left upper) be also explained? How similar diffractive data would look like for a kaon or photon beams, or for a proton dissociating into 3 jets?

### B. Beyond the two-domain scenario

Completing discussion of the wave functions and hard processes, let us briefly comment on how in principle one should get a more consistent solution of the problem, based not on a pictures with jumps but appropriate equations describing evolution of the coupling and anomalous dimensions.

The renormalizability of the theory implies that the equations depend on current coupling $\lambda(z)$ which itself changes due to first-order renormalization group equation

$$\frac{d\lambda}{dz} = \beta(\lambda(z)) \quad (19)$$

(Relation to second order eqn for the dilaton which follows from the Lagrangian is explained in ) Similar first order eqns are expected for the bulk (mixing) masses of a set of fields $\phi_A, A = 1..K$ with the same quantum numbers should be given by a similar evolution equations depending on the local coupling

$$\frac{dM^2_{AB}(z)}{dz} = \gamma_{AB}(\lambda(z)) \quad (20)$$

Since those are the first order eqns, one has to specify just their initial values, which are bare canonical dimensions at the boundary $M_{AB}(z = 0) = 0$. Combining these two eqns one gets a generic solution

$$M^2(\lambda) = M^2(0) + \int_0^\lambda d\lambda' \frac{\gamma(\lambda')}{\beta(\lambda')} \quad (21)$$

### IV. A “DOMAIN WALL” MADE OF INSTANTONS

#### A. Size distribution of QCD instantons

In the introduction we have mentioned puzzles related with the “chiral scale” and large non-perturbative corrections in all spin-zero channels, unaccounted by the OPE. These puzzles were resolved by realization that small-size instantons $\rho \sim 1/3 \text{fm}$ are in fact very abundant in the QCD vacuum. In spite of small coupling making the semiclassical exponent frighteningly small

$$\exp(-8\pi^2/g^2(\rho)) \sim \exp[-10] \quad (22)$$

large preexponent and attractive interactions increase it substantially, leading to actual instanton diluteness parameter

$$n_{I+A} \bar{\rho}^4 \sim 10^{-2} \quad (23)$$

(here $n_{I+A}, \bar{\rho}$ are the instanton (plus antiinstanton) density and mean radius, respectively.

Eventually, the so called Interacting Instanton Liquid Model (IILM) was developed, which included ’t Hooft
multi-fermion interaction to all orders. It provided quantitative deception of instanton effects including chiral symmetry breaking and various correlation function, for review see [21].

For the purpose of this paper we will not need to discuss its phenomenological consequences: we instead focus on one issue, the distribution over the instanton size \( \rho \). While asymptotic freedom at strongly suppresses instantons in UV (small \( \rho \)), there is no commonly accepted explanation of the suppression which is observed in IR: see e.g. lattice data shown in Fig.6. I suggested [33] that it is due to dual Higgs appearing in “dual superconductor” picture of confinement. As can be seen from the lower figure, both the quadratic dependence on \( \rho \) and the coefficient in the predicted correction factor

\[
\frac{dN}{d\rho} \frac{dN}{d\rho}_{\text{semiclassical}} = \exp(-2\pi \sigma \rho^2)
\]

(with \( \sigma \) being the QCD string tension) agree well with the available lattice data.

The main point we want to make by this point is that the instantons are rather well localized in \( \rho \) in the \( N_c = 3 \) case. For larger number of colors \( N_c \to \infty \) this tendency is further enhanced, as seen from Fig.7 which compares IILM calculation by T.Schafer [35] and lattice from M.Teper [36].

Since small size instantons are suppressed exponentially in \( N_c \), \( \exp(-8\pi^2/g^2) \sim (r\Lambda_{QCD})^{11/3}N_c \), the left side of these peaks are getting more and more steep at larger \( N_c \). There is clearly strong suppression from the right (IR) side as well, but its nature and \( N_c \) dependence remains unclear. The “dual superconductor” strength per color and thus the string tension remain finite at large \( N_c \), and eqn (24) predicts fixed suppression on the IR side. The Instanton liquid calculation in upper figure do not have a dual superconductor VEV but mutual repulsion of instantons, which work similarly: this a “triangular” shape of the curves at large \( N_c \). What is the shape of the lattice data (lower curve) is hard to tell: pragmatically it looks like a delta-function.

**B. Instantons in weakly coupled \( N = 4 \) theory**

Instantons in weakly coupled \( N = 4 \) theory were extensively studied by Dorey,Hollowood,Khoze and Mattis, see review [37]. The absence of running coupling decouples the tunneling amplitude from the size dependence. They have shown that in this theory instantons naturally live in \( AdS_5 \times S_5 \) space. The size \( \rho \) is simply identified with the 5-th coordinate \( z \), and the size distribution is given simply by \( dN/d\rho \sim d\rho/\rho^3 \) which is nothing but invariant volume element given by the \( AdS_5 \) metric.

This fits perfectly into string language in which instantons are point-like objects in 5d known as \( D_{-1} \) branes. They freely float in the bulk, filling it homogeneously like dust filling a room. Furthermore, on a quantum level one gets perfect fits as well. Instanton’s “hologram” on the boundary, readily calculated from the dilaton/axion bulk-to-brane propagators, gives the correct image \( G_{\mu\nu}(x) \sim \rho^4/(x^2+\rho^2)^4 \). A point object cannot be coupled to graviton: thus there is no holographic stress tensor. And indeed, instantons have it zero \( T_{\mu\nu} = 0 \) because of self-duality.

Furthermore, the remaining 8 supersymmetries rotating the instanton solution nicely relate fermionic zero modes to bosonic ones. The remaining \( S_5 \) space also nicely come out of mesonic variables associated with fermionic zero modes.
The effective charge is then defined as:

$$G_{\text{eff}}(\mu) = b \ln\left(\frac{\mu}{\Lambda_{\text{pert}}}\right) - \frac{4\pi^2}{(N_c^2 - 1)} \int_0^{1/\mu} dn(\rho) \rho^4 \left(\frac{8\pi^2}{g_{\text{eff}}^2(\rho)}\right)^2$$

where $b = 11N_c/3 - 2N_f/3$ is the usual one-loop coefficient of the beta function, and $dn(\rho)$ is the distribution of instantons (and anti-instantons) over size.

In it we compare QCD results (based on size distribution from the lattice already discussed) to both first order instanton correction and the exact results, for the $\mathcal{N}=2$ (Seiberg-Witten) theory. This is seen from the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{The figure on the left shows the instanton size distribution obtained from numerical simulations of the instanton ensemble in pure gauge QCD for different numbers of colors [33]. The figure on the right shows lattice results reported by Teper [33]. The \(* + \circ\) symbols correspond to $N_c = 2, 3, 4, 5$.}
\end{figure}

C. Instanton-induced corrections to coupling, in QCD and SUSY theories

So far we treated instantons as some point probes, while now we would ask What is the back reaction of instantons on the running coupling? This is in fact a very old question: Callan, Dashen and Gross 30 years ago derived its first order in instanton density, and even attempted to resum it nonlinearly. A decade ago the issue was looked at by Randall, Rattazzi and myself [38], from which we borrowed Fig. [8]

The external field is supposed to be normalized at some perturbative solution for the $\mathcal{N}=2$ SYM, the thick dashed line shows the one-instanton correction. Lines with symbols (as indicated on figure) stand for $N=0$ QCD-like theories, SU(2) and SU(3) pure gauge ones and QCD itself. Thin long-dashed and short-dashed lines are one and two-loop results.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8.png}
\caption{The effective charge $b g_{\text{eff}}^2(\mu)/8\pi^2$ (b is the coefficient of the one-loop beta function) versus normalization scale $\mu$ (in units of its value at which the one-loop charge blows up). The thick solid line correspond to exact Seiberg-Witten solution for the $\mathcal{N}=2$ SYM, the thick dashed line shows the one-instanton correction. Lines with symbols (as indicated on figure) stand for $N=0$ QCD-like theories, SU(2) and SU(3) pure gauge ones and QCD itself. Thin long-dashed and short-dashed lines are one and two-loop results.}
\end{figure}

The exact result for the effective coupling

$$\frac{8\pi}{g^2(u)} = \frac{K(\sqrt{1-k^2})}{K(k)}$$

where $K$ is the elliptic function and its argument has

$$k^2 = \frac{(u - \sqrt{u^2 - 4\Lambda^4})}{(u + \sqrt{u^2 - 4\Lambda^4})}$$

being the function of the gauge invariant vacuum expectation value of the squared scalar field

$$u = \frac{1}{2} <\phi^2> = \frac{a^2}{2} + \frac{\Lambda^4}{a^4} + \ldots$$

where $a$ is just its VEV. For large $a$ there is a weak coupling expansion which includes instanton effects

$$\frac{8\pi}{g^2(u)} = \frac{2}{\pi} \left( \log \left( \frac{2a^2}{\Lambda^2} \right) - \frac{3\Lambda^4}{a^4} + \ldots \right)$$

The behavior is shown in Fig. [8], where we have included both a curve which shows the full coupling (thick solid line), as well as a curve which illustrates only the one-instanton correction (thick dashed one). Because we will want to compare the running of the coupling in different theories, we have plotted $b g_{\text{eff}}^2/8\pi^2$ ($b=4$ in this case is the one-loop coefficient of the beta function) and measure all quantities in units of $\Lambda$, so that the one-loop charge blows out at 1. Note the very rapid change of the coupling induced by instantons. It is also of interest that
the full multi-instanton sum makes the rise in the coupling even more radical than with only the one-instanton correction incorporated.

One more argument in favor of rapid instanton-driven transition comes from analytic results on the $\mathcal{N}=4$ theory on a different 4-dim manifold called $K_3$ by Witten and Vafa [39], who were able to sum up all instanton contributions. Their results show that there is a rapid transition from a “dilute” to “dense” instanton phase, at the critical coupling $g^2/4\pi = 1$. Furthermore, it becomes a true phase transition at large $N_c$ [40]. This fact further suggests that the phenomenon is generic, presumably taking place for all gauge theories, supersymmetric or not.

D. Instantons in AdS and correlators

After this extended introduction, let us discuss how instantons can be incorporated into the AdS/QCD framework.

Two first steps are clear: (i) We would like to keep metric to be that of pure AdS$_{5}$ and (ii) identify the instanton with a bulk point, with its $z$ coordinate identified with the instanton size $\rho$.

Of course instanton distribution over $z$ in AdS/QCD should be completely different from that in AdS/CFT: instanton ensemble is no longer a kind of a dust, filling space homogeneously, but is instead located in a narrow region of $z$ shown in Figures above. This is of course caused by the running coupling, or in the AdS/QCD language, by the $z$-dependent dilaton potential. In IR it should be quadratic, in harmony with KKSS, and in UV it should match the perturbative beta function. In the large $N_c$ limit the instanton size distribution gets infinitely sharp.

Thus we come to the central idea of this paper: the “domain wall” is perhaps made of instantons.

Let us test this idea, starting with its simplest applications. Consider for example the instanton contributions to gluonic pseudoscalar (scalar) correlation functions

$$\Pi(x,y) = <O(x)O(y)>$$ (29)

where $O_{PS} = \frac{2\pi}{\rho}GG$ and $O_S$ has $GG$ without dual.

The “old-fashioned” way to calculate it is simply to include the semiclassical fields of the instantons in the correlator, and integrate over instantons. The answer is instructive to write for the Fourier transform of this (Euclidean) correlator

$$\Pi_{S,PS}(Q) = \pm \int \frac{d\rho d\kappa}{\rho^5} [Q^2\rho^2K_2(\rho\kappa)]^2$$ (30)

where $d(\rho)$ is the instanton size distribution and the bracket is Fourier distribution of instanton’s $GG(x)$.

This same result appears in the AdS/QCD model framework, where it comes from the diagram shown in Fig.9 and the square bracket in (30) is reinterpreted as a bulk-to-boundary propagator of the axion/dilaton. It was discussed by Katz and Schwartz [32] in the AdS/QCD model, which instead of the instanton density $d(\rho = z)$ have some “anomaly coupling” $\kappa$. These authors rightly noted that confinement-related IR cutoff of the AdS space automatically solves the divergence over large instanton sizes: but they did not mentioned that if their $\kappa$ be constant in their Lagrangian this very integral would power diverges at small $z$! Indeed, their version of AdS/QCD is conformal in UV and it does not know about the asymptotic freedom.

In our scenario (in the large $N_c$ limit) all instantons are at $z = z_{dw}$, and thus no power divergence in the correlators. In fact the delta-function distribution is exactly what I used for the simplest “random instanton liquid model” [20] in early 1980’s to understand correlators of this kind. Two parameters of the model were found to be the (4d) density $n_I = n_A \approx .5fm^{-4}$ and the size (now the domain wall position) $z_{dw} = 1/3fm$. They have quantitatively described a lot of phenomenology, including the $\eta'$ mass and mixing with glueball channels, which were among the puzzles pointed out in [19].

What about other correlation functions? As we already mentioned, vector and axial currents have been well described by Schäfer [10] in AdS/QCD models without instantons. Those are two exceptional cases, protected operators which indeed long known to ignore instanton scale $1/3fm$. In our two-domain scenario it means that corresponding bulk fields do not care for the domain wall as they are protected and have no anomalous dimensions on both sides of it.

Generic operators however should see the domain wall. An indeed, as was first documented in detail in review [48], all scalar/pseudoscalar operators split precisely at the instanton scale $z_{dw} = 1/3fm$. The AdS/QCD models which are conformal in UV – such as discussed by Schäfer [10] or Katz and Schwartz [32] – don’t even know about this scale and would not agree with the QCD phenomenology.
E. Domain wall made of interacting instantons

The partition function of ILM can be be written as

$$Z = \frac{1}{N_+!N_-!} \prod_i C(N_i)^{N_+ + N_-} \int \frac{d\Omega d^4x_i dz_i}{z_i^2} \exp \left[ -S_{\text{eff}}(z_i) - S_{\text{glue}}^f - S_{\text{weak}}^f - S_{\text{strong}}^f \right]$$

(31)

where we have identified summation over instantons and anti-instantons with proper integration over collective coordinates mentioned and orientations in color space \( \Omega \).

The action includes single-body effective action \( S_{\text{eff}}(z_i) \) which includes classical \( 8\pi^2/g^2 \) and quantum corrections, all depending on \( z \) because of running coupling (in UV) and confinement (in IR). The interaction we split into three parts, the gluonic one and two fermionic parts, which we explain subsequently. The bosonic action \( S_{\text{glue}}^f \) for a pair of two instantons (or anti-instantons) is the log of the moduli space metric (the overlaps of bosonic zero modes). For instanton-anti-instanton pair there are no relative zero modes and interaction is determined from the so called “streamline” equation: the details can be found in a review [21]. The only important point is that collective coordinates appear in two combinations

$$\cos(\theta_{IA}) = (1/N)Tr[\Omega_I \Omega_A^\dagger (\hat{R}_\mu \tau_\mu)] \quad d_{IA}^2 = \frac{\langle x_I - x_A \rangle^2}{z_I z_A}$$

(32)

The former is called the relative orientation factor, the orientation matrices \( \Omega \) live in \( SU(N) \) while the \( \tau_\mu \) is the usual 4-vector constructed from Pauli matrices: it is nonzero only in the \( 2^*2 \) corner of the \( N \times N \) color space. The \( \hat{R} \) is the unit vector in the direction of 4d inter-particle distance \( x_I - x_A \). The second combination of position and sizes appears due to conformal invariance of the classical YM theory.

The most complicated (nonlocal) part of the effective action is due to fermionic exchanges, originated from the determinants of the Dirac operator

$$\exp \left[ -S_{\text{weak}}^f - S_{\text{strong}}^f \right] = \Pi_f \det(\hat{\rho} i\mathcal{D} + im_f)$$

(33)

In the spirit of two-domain description, we have split it into two parts, which keep track of two different regimes. Their pictorial explanation in Fig.10 is also for clarity split into two parts.

The crucial part of this determinant includes a subspace spanned by the fermionic zero modes: thus instantons can be viewed as effective point-like vertices with 2\( N_f \) legs, known as ‘t Hooft vertex.

In the weakly coupled domain quarks propagate independently and the picture corresponds to Fig.10 (left). One can approximate quark propagation between instantons by free fermion propagators and right the corresponding part of the Dirac operator as

$$\exp(-S_{\text{weak}}^f) = \Pi_f \det \left( \frac{im_f}{T_{IA}} \frac{T_{IA}}{T_{IA}} \right)$$

(34)

with \( T_{IA} \) defined by

$$T_{IA} = \int d^4 x \psi_{0,I}^a(x - x_I)i\mathcal{D}\psi_{0,A}(x - x_A),$$

(35)

where \( \psi_{0,I} \) is the fermionic zero mode. The terms proportional to quark masses would be mentioned only in section ??: they have a similar overlap matrix element but without \( \mathcal{D} \) and for zero modes of \( \Pi \) or \( AA \) types. Note that each matrix element has the meaning of a “hopping amplitude” of a quark from one pseudoparticle to another. Furthermore, the determinant of the matrix (33) is also equal to a sum of all closed loop diagrams, and thus ILM sums all orders in ‘t Hooft effective 2\( N_f \)-fermion effective interaction. The matrix element can be written as a function of the same two variables (32) as the gauge action

$$\bar{\rho} T_{IA} = \cos(\theta_{IA})f(d_{IA}) \quad f \approx \frac{4d_{IA}}{(2 + d_{IA}^2)^2}$$

(36)

where the expression is an approximate simple parameterization of a more complicated exact result.

In the strongly coupled domain quarks propagate in form of (colorless) mesons, with picture shown in Fig.10 (right). The ‘t Hooft vertex can be rewritten in the Nambu-Jona-Lasinio form, which for two massless flavors is [21]

$$S_{\text{strong}}^f \sim \left( \frac{2N_c - 1}{2N_c} \right) [\langle \psi^+ \tau_a^- \psi \rangle^2 + \langle \psi^+ \gamma_5 \tau_a^- \psi \rangle^2] + \left( \frac{1}{N_c} \right) [\langle \psi^+ \sigma_{\mu\nu} \tau_a^- \psi \rangle^2$$

(37)

where 4-d flavor matrix \( \tau_a^- = (\tau_1, \tau_2, \tau_3, -i) \), including e.g. in the pseudoscalar term both the pion term and - with the opposite sign due to \( U(1)_A \) breaking - the \( \eta \). It only includes gamma matrices \( 1, \gamma_5, \sigma_{\mu\nu} \) which correspond to chiral flip: thus no vector/axial mesons. Ignoring subleading in \( N_c \) last term one may use only 4 spin-zero mesons, \( \pi, \eta, a_0, f_0 \).

This bosonized ‘t Hooft vertex is now quadratic for two flavors: thus the structure of lines in Fig.11 (right). The fact that this form should be used only below the domain wall \( z > z_m \) corresponds to the old UV cutoff of the NJL model at \( \Lambda_X \). The reason we show mesonic lines

![FIG. 10: Schematic structure of the “instanton wall”: black and open points indicate point-like instantons and anti-instantons. The left and right pictures show interaction between them, in weakly and strongly coupled domains. We assume two massless quark flavors, so the lines on the left – 4 per instanton – are exchanged zero mode quarks. Double lines in the right figure are mesons.](image-url)
propagating rather deep into large-$z$ IR region is based on the structure of the scalar propagator in the $AdS_5$ between two points with the same $z$

\[
D \sim \frac{z^4}{(z^2 + x^2)^4}
\]

which decreases with the 4-d distance $x$ much stronger than with the 5-th coordinate $z$. That is presumably the reason for the fact (mentioned in the introduction) that one cannot cut off $z$ till say 10 fm or so, without some observable consequences for QCD observables.

Summarizing this section: one can naturally rewrite instanton ensemble in the $AdS_5$ language. The classical measure $d^5x/z^{-5} = d^3x\sqrt{-g}$ is the right invariant measure in $AdS_5$. The variable $d_{IA}$ on which all the functions involved depend can be interpreted as the invariant distance between 2 points in the $AdS_5$ metric. So one can view all complicated interactions of the point-like bulk objects in $AdS_5$, without referring back to the boundary.

V. DISCUSSION

A. Mesons as vibrations of the wall

Top-down approaches introduce “matter branes” on which quarks live, starting with the $D_7$ branes by Karch and E. Katz $^5$ and then $D_8$ branes of the Sakai-Sugimoto model $^7$. In the latter the right and left parts of massless quarks on two separate sets of $N_f$ branes, but chiral symmetry breaking unite them into a single one, with some “connecting part” related right and left-handed components together. The position of this part fixes the “constituent quark” mass scale.

Our “domain wall” is a dynamically formed structure, but it is destined to play the same role as the “connecting part” of the matter brane. Indeed, the zero mode zone” quarks who lives near it are responsible for the chiral condensate and get constituent quark mass. Furthermore, as experience with instanton liquid calculations shows, the quarks belonging to “zero mode zone” dominantly contribute to the lowest hadrons, from pseudoscalar and vector mesons to even the nucleon. Lattice studies such as $^{11,12}$ confirmed that keeping only Dirac eigenstates with the lowest eigenvalues (and made of instanton zero modes) in the propagators is sufficient to reproduce hadronic correlation functions at large distances associated with those hadrons.

And yet brane constructions have provided a completely new view at the lowest pseudoscalar and vector mesons, identifying them with collective vibrations of the brane as a whole. This valuable idea can be applied to the domain wall as well. The effective action for mesons in brane constructions follow from the so called DBI action, which is some tension times the invariant area of the brane calculated in a background metric. A vibrating “domain wall” also generate mesons, and the corresponding effective action can include the area term as well. However it also should depend on the dilaton potential (which provides the minimum responsible for its fixed position $z_{dw}$)

\[
S_{eff} \sim \int d^4x d\tau e^{-V_{wall}(z-A_\tau(x,\tau))}\sqrt{-det(g_{wall})}
\]

The fact that “domain wall” is not a brane but has a finite width at finite $N_c$, and even in the infinite $N_c$ it is not a continuous object but just a 4d cloud of correlated point objects is not a problem: low lying oscillations with large wavelength are just collective vibrations of a wall as a whole. (AdS/QCD should not have true brane-like objects formed, as there is no supersymmetry to hold it together.)

In the leading quadratic order the effective action can only be a hidden-gauge-like action $S \sim \int d^4x(F_{\mu\nu})^2$, but the nonlinear terms should not be the same as there would be different contributions from the dilaton potential.

B. Other open issues

What is the geometry of other extra dimensions? The $\mathcal{N} = 4$ theory has them neatly put into a sphere $S_5$, which also nicely comes from mesons related to fermionic zero modes $^{37}$. One certainly do not expect in QCD any relation between flavor and these extra dimensions. There is however an important relation between instanton topology and extra dimensions, which is revealed by color “Higgsing” of the $\mathcal{N} = 4$ theory, namely starting with $N_c$ original $D_3$ branes put not into the same point but rather into certain $N_c$ points on a circle $S^1$. Thus color group is broken and monopoles can be introduced as open strings with ends on different branes. Instanton topology correspond to circles of $N_c$ segments, going around $S^1$. Although the radius of the circle can be later considered vanishing, this interpretation of the instantons as (small string) circles in extra dimensions is worth keeping.

What role of relative color orientation vectors? Can one get rid of them in a more generic formulation? Both bosonic and fermionic determinants in QCD partition function above contain some remnants of the color – the relative color orientation vectors. Pragmatically, those can be viewed as some kind of classical group-valued “spin variables” associated with otherwise point-like instantons in the $AdS_5$ bulk.

Perhaps a more natural formulation (which does not exist yet) is to combine the circle topology just mentioned with Kraan-van Baal picture of (finite-T) instantons as being made out of $N_c$ dyons. Not only can a classical solution be understood like that, but, the gauge part of the instanton measure itself can be exactly rewritten in terms of 4$N_c$ coordinates of $N_c$ dyons: see $^{43}$ in gauge theory and $^{44}$ in brane language. In the monopole language the ugly color angles are gone. One remaining task
tions should look like 4d “scalar Coulomb gas” advocated with plus/minus way, and resumming all axion interaction with the AdS bulk. It interacts with instantons and antiinstantons $N_c$ large.

AdS ensemble in the dynamics, which is orders of magnitude more difficult than the usual vacuum energy. In particular, for noninteracting instantons one gets the $\theta$-dependence due to potential induced by interaction with other bulk fields. However the physical meaning of “the running $\theta$ parameter”, as they call it, is not obvious and deserves to be discussed.

Standard rules of holography require that the UV boundary value $a(0)$ be identified with input UV value of the $\theta$ parameter, and the coefficient of $z^4$ with the topological charge density operator $G_{\mu\nu}G_{\mu\nu}$.

Although the QCD vacuum is known to have near-zero value of the $\theta$ parameter, discussion of ensembles with the nonzero topological charge / $\theta$ were done theoretically and numerically (on the lattice). Furthermore, Zahed and myself proposed charge asymmetry driven by electric field along the fireball angular momentum at nonzero $\theta$, which can do it. There are even preliminary indications from STAR experiment that this effect may in fact be already observed.

The nontrivial requirement put forward by GKN is that they must be related to each other, via a new condition

$$a(z_{IR}) = 0$$

at the infrared singularity. It is not clear why this is the case. Generally the fact that the (topological) charge density and $\theta$ ($\exp(i\theta)$ is the instanton fugacity) is expected. The relation should follow in a standard way from the free energy (axion action, or vacuum energy)

$$< Q > = \partial S/\partial (i\theta)$$

In particular, for noninteracting instantons one gets the usual vacuum energy $\sim \cos(\theta)$ and thus $< Q > \sim \sin(\theta)$, with zero charge at both $\theta = 0, \pi$.

The axion is a bulk dual to $\eta'$ in 4d. Naturally in the large $N_c$ limit it is massless both in 4d and in the AdS bulk. It interacts with instantons and antiinstantons with plus/minus way, and resumming all axion interactions should look like 4d “scalar Coulomb gas” advocated by Zhitnitsky for some time. If $N_c$ is not large then $\eta'$ meson has nonzero (and in fact large) mass: this corresponds to correlated instantons-antiinstantons and short screening length for $Q$.

Now, any consideration of a state with a nonzero average topological charge (or $\theta$-induced effects) must refer to the case when all quark masses are nonzero. If so, the ’tHooft vertices shown in Fig.10 should be complimented by those not shown there, with masses and chirality flips (described by the overlap integrals without the Dirac operator, $\sim m_f/x^2$). As a result, quark determinants remain nonzero even for different number of instantons and antiinstantons.

As a result, the difference in the total density between instantons and antiinstantons leads to a different their interaction as well. If also $N_c$ is finite, instantons and antiinstantons will not have the same distribution in sizes (their $z$-coordinates) and thus $z$-dependence of the axion field naturally appears. Thus in general the GKN’s introduction of $a(z)$ does make sense: but only if all quark masses are nonzero. How this important condition appears in their approach is unclear.

VI. SUMMARY

We argued that a good approximate geometry for AdS/QCD may be two $AdS_5$ domains, a weakly and a strongly coupled ones, separated by the “domain wall”. We identify its position with old “chiral scale” $\mu$. We further propose a dynamical explanation for the “domain wall”, relating it to mean instanton size. We further argued that the domain wall should become sharp in the large $N_c$ limit, in which all instantons get localized at the same value $z = z_{dw}$.

The gravity-dual description is only useful in the strongly coupled domain, while the usual weak-coupling perturbative/semiclassical methods should be used in the other. As we discussed, bulk properties such as bulk masses may jump across the domain wall. But important principle connecting them is continuity of all bulk fields required across the wall. Indeed, the wave equation with a jump in the bulk mass require only a jump in its second derivative.

Existence of one common normalized wave function of $z$ is very important for hard processes. Their asymptotics may have perturbative (bare) powers of momentum transfer, but its normalization would be still defined mostly by the wave functions in the strongly coupled domain.

The shape of the hard amplitude’s dependence on $Q$ is determined essentially by one number – the strong-coupling anomalous dimension/5-d mass. The fact that this nontrivial shape (circles in Fig.3) is seen experimentally (lower Fig.5), if not accidental, suggests two important lessons: (i) the anomalous dimensions may be rather large, 3-4 units; (ii) and they can indeed be about constant in the scale interval under consideration. These are
the best hints we have, that something like two distinct conformal regimes may in fact coexist in QCD.

A lot of work would be needed to clarify coupling behavior and the role of instantons in large-N, QCD and in the structure of the “domain wall”. Treatment of the lowest mesonic modes as collective vibrations of the “domain wall” is another exciting direction to study.

Acknowledgment

This paper was initiated at the Newton Institute (Cambridge) program “Strong Fields, Integrability and Strings”, Aug-Sept.07, so I thank its organizers and especially Elias Kiritsis, Misha Stephanov and Arkady Vainshtein for discussions. It is also pleasure to thank a long-time friend Stanley Brodsky, who helped to guide me through literature, and Dan Ashery for providing the E791 data.

This work is supported by the US-DOE grants DE-FG02-88ER40388 and DE-FG03-97ER4014.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].
[2] E.V. Shuryak, “Why does the quark gluon plasma at RHIC behave as a nearly ideal fluid?”, Prog. Part. Nucl. Phys. 53, 273 (2004) [hep-ph/0312227].
[3] E. Shuryak, arXiv:hep-ph/0703208.
[4] C. P. Herzog, Q. J. Ejaz and I. R. Klebanov, JHEP 0502, 009 (2005) [arXiv:hep-th/0412193].
[5] A. Karch and E. Katz, JHEP 0206, 043 (2002) [arXiv:hep-th/0205236].
[6] J. Babington, J. Erdmenger, Nick J. Evans, Z. Guralnik, I. Kirsch Phys.Rev.D69:066007,2004 [hep-th/0306018].
[7] T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113, 843 (2005) [arXiv:hep-th/0412141].
[8] S. J. Brodsky and G. F. de Teramond, Phys.Rev.Lett.96:201601,2006, arXiv:hep-th/0702205.
[9] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 95, 261602 (2005) [arXiv:hep-ph/0501128].
[10] T. Schaefer, arXiv:0711.0236 [hep-ph].
[11] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. 70, 323 (1998) [arXiv:hep-ph/9610451].
[12] C. Allton, A. Trivini, M. Teper and A. Trivini, arXiv:0710.1138 [hep-lat].
[13] S. Furui and H. Nakajima, Few Body Syst. 40, 101 (2006) arXiv:hep-lat/0503029.
[14] O. Andreev and V. I. Zakharov, Phys. Lett. B 636, 99 (2002) arXiv:hep-th/0204051.
[15] S. S. Gubser, T. R. Klebanov and A. M. Polyakov, Nucl. Phys. B 688, 69 (2004) [arXiv:hep-th/0311270].
[16] J. Polchinski and M. J. Strassler, JHEP 0305, 012 (2003) arXiv:hep-th/0209211.
[17] S. J. Brodsky and G. F. de Teramond, arXiv:0709.2072 [hep-ph].
[18] E. Katz and M. D. Schwartz, Phys. Rev. Lett. 88, 031601 (2002) [arXiv:hep-th/0109174].
[19] P. Faccioli, A. Schwenk and E. V. Shuryak, Phys. Rev. D 67, 113009 (2003) [arXiv:hep-ph/0202027].
[20] J. Polchinski, G. A. Miller and M. Strikman, Phys. Lett. B 304, 1 (1993) [arXiv:hep-ph/9305228].
[21] A. Trivini, M. Teper and A. Trivini, Phys.Rev.Lett.93,201601,2004, arXiv:hep-ph/0306018.
[22] T. Schafer and E. V. Shuryak, Nucl. Phys. B 203, 93 (1982).
[23] E. Shuryak, Nucl. Phys. B 503, 1 (1997) [arXiv:hep-ph/9610451].
[24] C. Allton, A. Trivini, M. Teper and A. Trivini, arXiv:0710.1138 [hep-lat].
[25] S. Furui and H. Nakajima, Few Body Syst. 40, 101 (2006) arXiv:hep-lat/0503029.
[26] S. S. Gubser, T. R. Klebanov and A. M. Polyakov, Nucl. Phys. B 636, 99 (2002) arXiv:hep-th/0204051.
[27] S. S. Gubser, T. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) arXiv:hep-th/9802109.
[28] J. Polchinski and M. J. Strassler, JHEP 0305, 012 (2003) arXiv:hep-th/0209211.
[29] S. J. Brodsky and G. F. de Teramond, arXiv:0709.2072 [hep-ph].
[30] J. Polchinski and M. J. Strassler, Phys. Rev. Lett. 88, 031601 (2002) [arXiv:hep-th/0109174].
[31] P. Faccioli, A. Schwenk and E. V. Shuryak, Phys. Rev. D 67, 113009 (2003) [arXiv:hep-ph/0202027].
[32] J. Polchinski, G. A. Miller and M. Strikman, Phys. Lett. B 304, 1 (1993) [arXiv:hep-ph/9305228].
[33] A. Trivini, M. Teper and A. Trivini, Phys.Rev.Lett.93,201601,2004, arXiv:hep-ph/0306018.
[34] T. Schafer and E. V. Shuryak, Nucl. Phys. B 203, 93 (1982).
[35] E. V. Shuryak, Nucl. Phys. B 203, 93 (1982).
[36] E. V. Shuryak, Nucl. Phys. B 203, 93 (1982).
[37] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Phys. Rev. D 67, 113009 (2003) [arXiv:hep-ph/0202027].
in $N=4$ sym on K3 at strong coupling. [hep-th/0510216]

[41] J. W. Negele, Nucl. Phys. Proc. Suppl. **73**, 92 (1999) [arXiv:hep-lat/9810053]. T. L. Ivanenko and J. W. Negele, Nucl. Phys. Proc. Suppl. **63**, 504 (1998) [arXiv:hep-lat/9709130].

[42] P. Faccioli and T. A. DeGrand, Phys. Rev. Lett. **91**, 182001 (2003) [arXiv:hep-ph/0304219].

[43] T. C. Kraan and P. van Baal, Phys. Lett. B **435**, 389 (1998) [arXiv:hep-th/9806034]. D. Diakonov and N. Gromov, Phys. Rev. D **72**, 025003 (2005) [arXiv:hep-th/0502132].

[44] K. M. Lee and P. Yi, Phys. Rev. D **56**, 3711 (1997) [arXiv:hep-th/9702107].

[45] E. Shuryak and I. Zahed, Phys. Rev. D **67**, 014006 (2003) [arXiv:hep-ph/0206022].

[46] D. Kharzeev and A. Zhitnitsky, [arXiv:0706.1026] [hep-ph].

[47] A. R. Zhitnitsky, Nucl. Phys. Proc. Suppl. **73**, 647 (1999).

[48] E. V. Shuryak, Rev. Mod. Phys. **65**, 1 (1993).