Statistical properties of the continuum Salerno model

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The statistical properties of the Salerno model is investigated. In particular, a comparison between the coherent and partially coherent wave modes is made for the case of a random phased wave packet.

It is found that the random phased induced spectral broadening gives rise to damping of instabilities, but also a broadening of the instability region in quasi-particle momentum space. The results can be of significance for condensation of magnetic moment bosons in deep optical lattices.

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Nonlinear wave propagation models are of importance in a wide variety of settings, such as in nonlinear optics [1, 2], in water wave propagation [3], and in plasma systems [4]. Moreover, also discrete systems show important nonlinear properties that give rise to solitonic structures in e.g. lattices (see, e.g. [5] and references therein for a review). The discrete Salerno model [6] combines onsite and intersite nonlinear terms in a lattice in order to encompass the different characteristics of nonlinear integrable and non-integrable lattice models. It has also been discovered as an asymptotic form of the Gross–Pitaevskii equation, with applications to Bose–Einstein condensation of atoms with magnetic moments in deep optical lattice traps [6, 7, 8]. Recently, the continuum limit of the Salerno model with focusing onsite and defocusing intersite nonlinearities was investigated [8] and soliton solutions were found.

In this Brief Report, we will investigate the modulational instability properties of the general continuum limit of the Salerno model. Furthermore, the effects of random phased wave packets on the stability properties will be studied using the Wigner formalism. Results regarding the effect of random pulse phases will be given.

The Salerno model was derived as a quantum modified discrete nonlinear Schrödinger equation, giving the time evolution of the field amplitude on the lattice. The relevant equation takes the form [6]

\[ i\partial_t \Phi_n + (\Phi_{n+1} + \Phi_{n-1})(1 + \mu |\Phi_n|^2) + 2\nu |\Phi_n|^2 \Phi_n = 0, \]

where \( \Phi_n = \Phi(t, x_n) \) is the field amplitude on the \( n \)-th lattice site located at \( x_n \) and \( \nu \) and \( \mu \) are the real parameters determining the strength of the onsite and intersite nonlinearity, respectively. The continuum approximation is introduced by the ansatz \( \Phi(t, x) = \Psi(t, x)e^{i2t} \) and using the expansion \( \Psi_{\mu \nu} \approx \Psi \pm \partial_x \Psi + \partial_x^2 \Psi/2 \) the continuum limit of the Salerno model is obtained from Eq. (1) according to [6]

\[ i\partial_t \Psi + 2(\nu + \mu) |\Psi|^2 \Psi + (1 + \mu |\Psi|^2) \partial_x^2 \Psi = 0, \]  

where \( \Psi \) is the continuum correspondence of the complex lattice field amplitude. We see that by letting \( \mu \to 0 \) we obtain the nonlinear Schrödinger equation, where the sign of \( \nu \) determines the character of the soliton solutions [1], while \( \nu \to 0 \) gives a continuum version of the Ablowitz–Ladik equation [8].

Next, we derive a wave-kinetic equation for the wave function \( \Psi \). We first introduce the Wigner function for the wave function \( \Psi \) according to [10, 11, 12]

\[ \rho(t, x, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{ik\xi} \langle \Psi^\dagger (t, x + \xi/2) \Psi(t, x - \xi/2) \rangle, \]

where the angular bracket denotes the ensemble average [13]. The Wigner function is a generalized distribution function for the quasi-particles representing \( \Psi \), and it satisfies the normalization

\[ I(t, x) \equiv \langle |\Psi(t, x)|^2 \rangle = \int_{-\infty}^{\infty} dk \rho(t, x, k). \]

Applying the time derivative to Eq. (3) and using Eq. (2) we obtain the wave-kinetic equation

\[ \partial_t \rho + 2k \partial_x \rho + 2I \left[ \sin \left( \frac{\sqrt{2\nu + \mu}}{k} k \right) \right] \left[ \left( \frac{2\nu + \mu}{k^2 + \partial_x^2} \right) \right] \rho = 0, \]

where the \( \sin \) and \( \cos \) operators are defined in terms of their respective Taylor expansion, and the arrows denote the direction of operation. Equation (5) describes the propagation of partially Taylor expansion, and the arrows denote the direction of operation. Equation (5) describes the propagation of partially Taylor expansion, and the arrows denote the direction of operation. Equation (5) describes the propagation of partially Taylor expansion, and the arrows denote the direction of operation.
Next, we look for unstable perturbations around the homogeneous solution $\rho_0(k)$. We let $\rho(t, x, k) = \rho_0(k) +\rho_1(k)\exp(iKx - i\Omega t)$, where $|\rho_1| \ll \rho_0$, and linearize (4) with respect to $\rho_1$. Equation (5) with the intensity (4) then gives the nonlinear dispersion relation

$$1 = \int dk \frac{2(\nu + \mu) + \mu K^2[\rho_0(k + K/2) - \rho_0(k - K/2)]}{\Omega - 2K(1 + \mu K)k}. \tag{6}$$

The dispersion relation (6) generalizes the results of previous modulational instability analysis of the nonlinear Schrödinger equation, where the Wigner formalism [14] as well as the mutual coherence method [15] has been used.

For a coherent background spectrum, i.e. $\rho_0(k) = I_0\delta(k)$, the dispersion relation (6) reduces to

$$\Omega = \pm \left\{ (1 + \mu I_0) \left[ (1 + \frac{1}{2} \mu I_0) K^4 - 4(\nu + \mu) I_0 K^2 \right] \right\}^{1/2}. \tag{7}$$

We see that for the nonlinear Schrödinger case ($\mu = 0$), we obtain the standard growth rate $\Gamma = -i\Omega = K[4\nu I_0 - K^2]^{1/2}$, showing the existence of a modulational instability for focusing ($\nu > 0$) nonlinearity.

For a random phase background wave function, we may represent the background quasi-particle distribution function by the Lorentz distribution

$$\rho_0(k) = \frac{I_0}{\pi} \frac{\Delta}{k^2 + \Delta^2}, \tag{8}$$

where $\Delta$ is the width of the distribution function. Using the Lorentz distribution in the nonlinear dispersion relation, we obtain from (6)

$$\Omega = -2i\Delta K \left( 1 + \frac{1}{2} \mu I_0 \right)$$

$$\pm \left\{ (1 + \mu I_0) \left[ (1 + \frac{1}{2} \mu I_0) K^4 - 4(\nu + \mu) I_0 K^2 \right] \right\}^{1/2} + 2\Delta^2 K^2 \mu I_0 \left( 1 + \frac{1}{2} \mu I_0 \right)^{1/2}. \tag{9}$$

We note that the dispersion relation (6) reduces to (7) when $\Delta \to 0$. Moreover, the case of a pure Kerr nonlinearity ($\mu = 0$) gives rise to the a linear damping term for the modulational growth rate. In general, the broadening of the background quasi-particle distribution is seen to give rise to a damping of the modulational instability growth rate through the first term in the dispersion relation (6), but through the intersite nonlinearity parameter $\mu$ we also have a new coupling due to the finite width of the distribution function.

Next, we numerically investigate the properties of the growth rate $\Gamma = \text{Im}(\Omega)$, deduced from (8). In Fig. 1 we have plotted the modulational instability growth rate for different parameter values. We have normalized the parameter $\nu$ to one, and used $\Delta = 0$ and $\Delta = 0.1$. For $\mu = \Delta = 0$ we retrieve the standard result for the modulational instability growth rate in the case of the nonlinear Schrödinger equation. This behavior is, however, affected by a finite width in the background wave spectrum.

All cases show a Landau-like damping of the growth rate, when a finite width of the background distribution $\rho_0$ is introduced. Thus, appropriate random phasing of the wave function $\Psi$ could act as a useful means of stabilizing, e.g. Bose–Einstein condensates in deep optical traps. It is interesting to note that there is an interplay between the onsite and intersite nonlinearities through the coefficients $\nu$ and $\mu$, respectively. As expected, the case of competing nonlinearities, i.e. $\mu \nu < 0$ (here represented by a repulsive intersite nonlinearity), gives rise to a reduced growth rate as compared to the case of a Kerr type nonlinearity ($\mu = 0$), since any perturbation growth in one nonlinearity can be decreased by the other nonlinear term. However, for $\mu, \nu > 0$ (the dotted curves) we note a significant increase in the growth rate.

To summarize, we have performed a wave-kinetic study of the Salerno model in the continuum limit. A Vlasov-like equation has been obtained for the quasi-particles, representing the dynamics of the wave function $\Psi$ in phase space. A modulational instability analysis has been carried out, and a comparison between the coherent and incoherent cases has been made. It was found that the interaction between the onsite and intersite nonlinearities and the finite width of the background distribution function gives rise to significant changes in the modulational instability growth rate.

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Physics at the Rutherford Appleton Laboratory, Chilton, Didcot, United Kingdom.

[1] Yu. Kivshar and G. P. Agrawal, Optical Solitons, From Fibers to Photonic Crystals (Academic, 2003).
[2] P. K. Shukla and J. J. Rasmussen, Opt. Lett. 11, 171 (1986).
[3] M. Onorato, A. R. Osborne, and M. Serio, Phys. Rev. Lett. 96, 014503 (2006).
[4] A. Hasegawa, Plasma Instabilities and Nonlinear Effects (Springer-Verlag, New York, 1975).
[5] D. Henning and G. P. Tsironis, Phys. Rep. 307, 333 (1999).
[6] M. Salerno, Phys. Rev. A 46, 6856 (1992).
[7] Z. D. Li, P. B. He, L. Li, J. Q. Liang, and W. M. Liu, Phys. Rev. A 71, 053611 (2005)
[8] J. Gomez-Gardeñes, B. A. Malomed, L. M. Floría, and A. R. Bishop, Phys. Rev. E 73, 036608 (2006).
[9] M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 17, 1011 (1976).
[10] E. P. Wigner, Phys. Rev. 40, 749 (1932);
[11] J. E. Moyal, Proc. Cambridge Philos. Soc. 45, 99 (1949).
[12] J. T. Mendonça, Theory of Photon Acceleration (IOP Publishing, Bristol, 2001).
[13] Yu. L. Klimontovich, The Statistical Theory of Nonequilibrium Processes in a Plasma (Pergamon Press, Oxford, 1967).
[14] D. Anderson, B. Hall, M. Lisak and M. Marklund, Phys. Rev. E 65, 046417 (2002); B. Hall, M. Lisak, D. Anderson, R. Fedele, and V. E. Semenov, ibid. 65, 035602 (2002); M. Marklund and P. K. Shukla, Rev. Mod. Phys. 78, No. 2, in press (2006).
[15] M. Soljacic, M. Segev, T. Coskun, D. N. Christodoulides, and A. Vishwanath, Phys. Rev. Lett. 84, 467 (2000).