UNBOUNDED ABSOLUTE WEAK* CONVERGENCE
IN DUAL BANACH LATTICES

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Abstract. Several recent papers investigated unbounded versions of order norm and weak convergences in Banach lattices. In this paper, we study the unbounded variant of weak* convergence and its relationship with other convergences. In particular, we characterize order continuous Banach lattices, KB space and reflexive Banach lattices in terms of this convergence.

1. Introduction

Throughout this paper, $E$ stands for a Banach lattice. In [3,4,5,6] a net $(x_\alpha)$ in $E$ is said to be unbounded order convergent ($uo$-convergent) to $x$ if for every $u \in E_+$ the net $(|x_\alpha - x| \land u)$ converges to zero in order. In [7,8,9], it is called unbounded norm convergent ($un$-convergent) to $x$ if $\| |x_\alpha - x| \land u \| \to 0$. In [10,11], it is called unbounded absolute weakly convergent to $x$ if $|x_\alpha - x| \land u \overset{w}{\to} 0$. These concepts were investigated in [xxx]. We consider the unbounded version of weakly star convergence in $E'$. We say that $(x'_\alpha)$ is unbounded absolute weak* convergent ($uaw^*$-convergent) to $x'$ if $|x'_\alpha - x| \land u' \overset{w^*}{\to} 0$ for every $u' \in E'_+$; we write $x'_\alpha \overset{uaw^*}{\longrightarrow} x'$. For Banach lattices, we refer the reader to [1,2].

2. Results

Lemma 2.1. (1) $uaw^*$-limits are unique;
(2) $x'_\alpha \overset{uaw^*}{\longrightarrow} x'$ and $y'_\beta \overset{uaw^*}{\longrightarrow} y'$, then $ax'_\alpha + by'_\beta \overset{uaw^*}{\longrightarrow} ax' + by'$, for any scalars $a, b$;
(3) $x'_\alpha \overset{uaw^*}{\longrightarrow} x'$ iff $(x'_\alpha - x') \overset{uaw^*}{\longrightarrow} 0$;
(4) if $x'_\alpha \overset{uaw^*}{\longrightarrow} x'$, then $|x'_\alpha| \overset{uaw^*}{\longrightarrow} |x'|$;
(5) if $x'_\alpha \overset{uaw^*}{\longrightarrow} x'$, then $y'_\beta \overset{uaw^*}{\longrightarrow} x'$ for any subnet $(y'_\beta)$ of $(x'_\alpha)$.

Proof. The proof is obviously.

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Lemma 2.2. Every disjoint net in $E'$ is $\text{uaw}^*$-null.

Proof. Since every disjoint net in $E'$ is $\text{uaw}$-null by [10, Lemma 2], it is easy to see every disjoint net in $E'$ is $\text{uaw}^*$-null. \qed

Theorem 2.3. The following are equivalent:
1. $E$ is order continuous;
2. $\text{uo}$-convergence implies $\text{un}$-convergence;
3. order bounded disjoint sequence is norm-null.

Proof. (1) $\Rightarrow$ (2) is obviously by [7, Proposition 2.5].
(2) $\Rightarrow$ (3): for any order bounded disjoint sequence $(x_n)$ is $\text{uo}$-null by [5, Corollary 3.6], so $(x_n)$ is $\text{un}$-null. Since $(x_n)$ is order bounded sequence, so $(x_n)$ converges to 0.
(3) $\Rightarrow$ (1) is easy to see by [1, Theorem 4.14 and 2, Theorem 2.4.2]. \qed

Theorem 2.4. The following are equivalent:
1. $E$ is KB space;
2. every norm bounded $\text{uo}$-Cauchy net in $E$ is $\text{un}$-convergent;
3. every norm bounded $\text{uo}$-Cauchy sequence in $E$ is $\text{un}$-convergent;

Proof. (1) $\Rightarrow$ (2) According to [3, Theorem 4.7], we have that every norm bounded $\text{uo}$-Cauchy net in $E$ is $\text{uo}$-convergent. Since $E$ is order continuous, then we have (2).
(2) $\Rightarrow$ (3) is obviously.
(3) $\Rightarrow$ (1): If $E$ is not KB space, then $E$ contain a sublattice isomorphic to $c_0$. So, for $x_n = \sum e_n = (1, 1, ..., 1, 0, 0, ...)$ is norm bounded $\text{uo}$-Cauchy sequence, but it is not $\text{un}$-convergent, that is impossible. \qed

It was observed in [7, 8, 9, 10] that $\text{un}$-convergence and $\text{uaw}$-convergence are given by a topology, and sets of the forms $V_{u, \epsilon} = \{x \in E, \| |x| \wedge u \| < \epsilon \}$, $V_{u, \epsilon, f} = \{x \in E, f(|x| \wedge u) < \epsilon \}$, form a base of zero neighborhoods for this topology. Using a similar argument, one can show that sets of the form

$$V_{u', \epsilon, x} = \{x' \in E', (|x'| \wedge u)(x) < \epsilon \},$$

where $u' \in E'_{+, \epsilon} > 0$, and $x \in E_+$ form a base of zero neighborhoods for a Hausdorff topology, and the convergence in this topology is exactly the $\text{uaw}^*$-convergence.

Similarly to [8, 10], for every $u' \in E'_{+, \epsilon} > 0$, and $x \in E_+$, either $V_{u', \epsilon, x}$ is contained in $[-\epsilon', \epsilon']$, in which case $\epsilon'$ is a strong unit, or $V_{u', \epsilon, x}$ contains a non-trivial ideal. So, we have a question:

Problem 2.5. Does every $\text{uaw}^*$ neighbourhood of zero contain a non-trivial ideal?
Corollary 2.6. Let $E$ be a Banach lattice. $E'$ has a strong order unit when one of the following conditions is valid:

(1) $uaw$-topology agrees with norm topology.
(2) $uaw$-topology agrees with weak topology.
(3) $uaw$-topology agrees with weak* topology.

Proof. Suppose those are agree. It follows that $V_{u',\varepsilon,x}$ is contained in $B_{E'}$ for some $u' \in E'_+, \varepsilon > 0$, and $x \in E_+$, but it does not contain a non-trivial ideal, so we conclude that $V_{u',\varepsilon,x}$ is contained in $[-e',e']$; hence $e'$ is a strong order unit in $E'$ by [1, page 194 and 2, page 18].

By [3, 7, 10], in general, norm convergence implies $u$-convergence, $u$-convergence implies $uaw$-convergence, $uaw$-convergence implies $uaw^*$-convergence, we also find that norm convergence implies absolute weakly star convergence and absolute weakly star convergence implies $uaw^*$-convergence, for order bounded nets, the last convergences is same.

When $E'$ has $w$-lattice operation continuous property, then norm convergence implies $u$-convergence, $u$-convergence implies $uaw$-convergence, $uaw$-convergence implies $uaw^*$-convergence and norm convergence implies weak convergence. $u$-convergence implies $uaw$-convergence, $uaw$-convergence implies $uaw^*$-convergence.

When $E'$ has $w^*$-lattice operation continuous property, then norm convergence implies $u$-convergence implies $uaw^*$-convergence implies $uaw^*$-convergence and norm convergence implies weakly convergence implies $uaw$-convergence implies $uaw^*$-convergence.

When $E'$ is KB space (order continuous), then order convergence implies norm convergence implies $u$-convergence, $u$-convergence implies $uaw$-convergence, $uaw$-convergence implies $uaw^*$-convergence and order convergence implies $u$-convergence, $u$-convergence implies $uaw$-convergence, $uaw$-convergence implies $uaw^*$-convergence.

Next, we have some conclusions.

Proposition 2.7. Let $E$ be a Banach lattice:

(1) $x_{\alpha} uaw x$, then $|x_{\alpha}| \wedge |x| \rightarrow |x|$ and $|x'(x)| \leq \liminf_{\alpha} f_{\alpha}(x') x_{\alpha}(x' \in E'_+)$;

(2) $x'_{\alpha} uaw x$, then $|x'_{\alpha}| \wedge |x'| \rightarrow |x'|$ and $|x'(x)| \leq \liminf_{\alpha} f_{\alpha}|x'_{\alpha}|(x)(x \in E_+)$.

Proof. We prove (2) Clearly, $|x'_{\alpha}| \wedge |x'| - |x| \wedge |x| \leq |x'_{\alpha} - x'| \wedge |x'| \rightarrow 0$. Thus $|x'_{\alpha}| \wedge |x'| \rightarrow |x'|$. Since $|x'| = |x' - x'_{\alpha}| \wedge |x| + |x'_{\alpha}| \wedge |x'|$, it follows that $|x'(x)| \leq \liminf_{\alpha} (|x'_{\alpha}| \wedge |x'|)(x) \leq \liminf_{\alpha} |x'_{\alpha}|(x)$.
Proposition 2.8. Let $E$ be an Banach lattice:

1. $x_\alpha \xrightarrow{uaw} x$ and $(x_\alpha) \subset [-u, u] + \epsilon B_{\rho_z} (\forall x' \in E')$, then $x_\alpha \xrightarrow{\sigma(E',E)} x$.

2. $x'_\alpha \xrightarrow{uaw} x'$ and $(x'_\alpha) \subset [-u', u'] + \epsilon B_{\rho_z} (\forall x \in E)$, then $x'_\alpha \xrightarrow{\sigma(E',E)} x'$.

Proof. We prove (2), since $|x'_\alpha| = |x'_\alpha| \wedge u' + (|x'_\alpha| - u')^+$, so that $|x'_\alpha|(x) = (|x'_\alpha| \wedge u')(x) + ((|x'_\alpha| - u')^+)(x)$, and since $(x'_\alpha)$ is $uaw^*$-convergent net and $(x'_\alpha) \subset [u', -u'] + \epsilon B_{\rho_z}$, so $x'_\alpha \xrightarrow{\sigma(E',E)} x'$.

Proposition 2.9. Let $E$ be an Banach lattice and $e' \in E'_+$ is a quasi interior point, then $x'_\alpha \xrightarrow{uaw^*} 0$ if and only if $|x'_\alpha| \wedge e' \xrightarrow{w^*} 0$.

Proof. The forward implication is immediate. For the reverse implication, note that $|x'_\alpha| \wedge u' \leq |x'_\alpha| \wedge (u' - u' \wedge ne') + |x'_\alpha| \wedge (u' \wedge ne') \leq (u' - u' \wedge ne') + n(|x'_\alpha| \wedge e')$, therefore

$$
(|x'_\alpha| \wedge u')(x) \leq (u' - u' \wedge ne')(x) + n(|x'_\alpha| \wedge e')(x)
$$

for $\forall x \in E, n \in N$. Since $e'$ is quasi-interior and $x'_\alpha \xrightarrow{uaw^*} 0$, so $|x'_\alpha| \wedge e' \xrightarrow{w^*} 0$.

Corollary 2.10. $x'_\alpha \xrightarrow{uaw^*} 0$ iff $|x'_\alpha| \wedge (e' + u') \xrightarrow{\sigma(E',E)} 0$.

Corollary 2.11. If $E'$ is KB space(order continuous) with a weak order unit, then $x'_\alpha \xrightarrow{uaw^*} 0$ iff $|x'_\alpha| \wedge e' \xrightarrow{w^*} 0$.

For some special spaces, we describe $uaw^*$-convergence.

For $1 < p < \infty$ in $L_p(\mu)$, $uaw^*$-convergent iff $uaw$-convergent iff un-convergent iff un-convergent by measure. In $l_p$, $uaw^*$-convergent iff $uaw$-convergent iff un-convergent iff $uo$-convergent iff coordinate-wise convergence.

Problem 2.12. How to describe $uaw^*$-convergence in $L_1(\mu)$ and $L_\infty(\mu)$?

Theorem 2.13. Let $E$ be an Banach lattice. The following are equivalent:

1. $E$ has order continuous norm;
2. for any norm bounded net $(x'_\alpha)$ in $E'$, if $x'_\alpha \xrightarrow{uaw^*} 0$, then $x'_\alpha \xrightarrow{\sigma(E',E)} 0$;
3. for any norm bounded net $(x'_\alpha)$ in $E'$, if $x'_\alpha \xrightarrow{uaw^*} 0$, then $x'_\alpha \xrightarrow{\sigma(E',E)} 0$;
(4) For any norm bounded sequence \((x'_n)\) in \(E'\), if \(x'_n \xrightarrow{uaw^*} 0\), then \(x'_n \xrightarrow{\sigma(E',E)} 0\);

(5) For any norm bounded sequence \((x'_n)\) in \(E'\), if \(x'_n \xrightarrow{uaw^*} 0\), then \(x'_n \xrightarrow{\sigma(E',E)} 0\).

Proof. Since \(x'_\alpha \xrightarrow{uaw^*} 0\) iff \(|x'_\alpha| \xrightarrow{uaw^*} 0\), it is clear that (2) ⇔ (3) and (4) ⇔ (5).

Let \(E\) be order continuous Banach lattice. Suppose \((x'_\alpha) \subset E'\) is norm bounded and \(x'_\alpha \xrightarrow{uaw^*} 0\). Pick any \(x \in E_+\). For any \(\epsilon > 0\), by [1, Theorem 4.18], there exists \(u' \in E_+\) such that \((|x'_\alpha| - u')^+(x) < \epsilon\) for all \(\alpha\). Since \((|x'_\alpha| - u')^+(x) = |x'_\alpha|(x) - |x'_\alpha| \land u'(x) < \epsilon\) and \(|x'_\alpha| \land u' \xrightarrow{w^*} 0\), so we have \(x'_\alpha \xrightarrow{\sigma(E',E)} 0\). This proves (1) ⇒ (3).

The implication (3) ⇒ (5) is clear. Now we prove (5) ⇒ (1). Assume that (5) holds, take any norm bounded disjoint sequence \((x'_n)\) in \(E'\). By lemma 2.2, \(x'_n \xrightarrow{uaw^*} 0\). Thus, \(x'_n \xrightarrow{w^*} 0\) by assumption. It follows from [2, Corollary 2.4.3] that \(E\) is order continuous. □

Problem 2.14. When the closed unit ball is \(uaw^*\)-compact?

When a subset is \(uaw^*\)-compact iff it is \(uaw^*\)-sequence compact?

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