 Dependence of the critical temperature on the Higgs field parametrization

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Abstract
We show that, despite the reparametrization symmetry of the Lagrangian describing the interaction between a scalar field and gauge vector bosons, the dynamics of the Higgs mechanism can be really affected by the representation gauge chosen for the Higgs field at temperatures \( T \neq 0 \). This can be the case when the physical system described by the Higgs field chooses to undergo a condensation with a given degree of freedom rather than with other possible ones, as apparently observed in exotic superconductor systems. Actually, we study in detail such peculiar conditions and find that, varying the parametrization (without changing any physical parameter) for the two degrees of freedom of the complex scalar field, we obtain different expressions for the Higgs mass at \( T \neq 0 \): in its turn this entails different expressions for the critical temperatures, ranging from zero to a maximum value, as well as different expressions for other basic thermodynamical quantities.

1. Introduction

The Higgs mechanism is the basic ingredient of many theories, ranging from superconductivity to elementary particle physics, where the phenomenon of spontaneous symmetry breaking (SSB) plays a key role. In fact, the original idea by Higgs (and others) \([1, 2]\) to give a non-vanishing mass to gauge vector bosons through the coupling with a scalar field \( \phi \), in the framework of the Nambu model of SSB \([3]\), was inspired by the fundamental works by Ginzburg and Landau \([4]\) aimed at accounting for the emergence of short-range electromagnetic interactions, mediated by massive-like photons, in superconductors and superfluids.
An undesired feature met in SSB theories is that, without the ‘condensation’ of some scalar degree of freedom, the scalar particles are always massless. Goldstone’s theorem indeed predicts that SSB in a relativistic field theory results in massless spin-zero bosons. However, Higgs showed that Goldstone bosons do not necessarily occur when a symmetry is locally spontaneously broken, since the Goldstone mode may provide the third polarization of a massive vector field, the other mode of the original complex scalar field remaining as a massive spin-zero particle, namely the ‘Higgs boson’. Actually, the just described phenomenon is the same as that occurring in superconductors: the so-called (non-relativistic) *Anderson mechanism*, where the local gauge symmetry is broken spontaneously, and the Goldstone (‘plasmon’) mode becomes massive due to the gauge field interaction, whereas the electromagnetic modes are massive despite the gauge invariance (*Meissner effect*). It is very striking that the only Higgs boson so far experimentally discovered is the massive collective mode [5] in superconductors (and in analogous condensed matter systems such as, e.g., superfluids).

One of the topical issues in present day experimental research on elementary particles is, indeed, just the search for the Higgs boson at LHC [6], whose effective discovery will be the keystone for the complete confirmation of the standard model of electroweak interaction by Glashow, Weinberg and Salam [7]. In the electroweak standard model, at temperatures (or energies) high enough so that the electroweak symmetry is unbroken, all elementary particles except the scalar Higgs boson are massless. At a critical temperature, the Higgs field spontaneously slides from the point of maximum energy in a randomly chosen direction. Once the symmetry is broken, the gauge boson particles, $W^\pm$ and $Z^0$ bosons, acquire masses. The mass can then be interpreted as a result of the interactions of the particles with the Higgs background field. In a similar way (although with some particular differences with respect to the case of gauge bosons) fermions, such as leptons and quarks, acquire mass as a result of their interaction with the Higgs field.

In a sense, the Higgs mechanism in particle physics can be considered as the superconductivity in the vacuum. It occurs when all of space is filled with a sea of particles which are charged (the so-called Higgs condensate) or, in field theoretic language, when a charged (then complex, not real) field has a non-zero vacuum expectation value. Interaction with a quantum fluid filling the space, the background Higgs field, prevents certain forces from propagating over long distances due to changes in the low-energy spectrum of the gauge fields that thus become short-ranged and the corresponding gauge bosons become massive.

Coming back to condensed matter physics, in superconductivity mean-field theory, the required scalar field describes the dynamics of the Cooper pairs in the superconductors, $\phi$ being interpreted as the wavefunction of the Cooper pairs in their center-of-mass frame.

Although in the present paper, for the sake of definiteness and without loss of generality, we will focus mainly on the standard Abelian Higgs mechanism, we shall often refer to superconductors, and then to the Ginzburg–Landau (GL) model, for an immediate and simple physical understanding, as actually happened in the historical development of this topic.

In the $U(1)$ Higgs mechanism the Lagrangian density describing a scalar field $\phi$ interacting with the electromagnetic field $A_\mu$ is given by (hereafter we assume $m^2 < 0$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi) \dagger (D^\mu \phi) - m^2 \phi \dagger \phi - \frac{\lambda}{4} (\phi \dagger \phi)^2,$$

(1)

where $\lambda$ is the self-interaction coupling giving, in superconductivity, the strength of the Cooper pair binding. $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and $D_\mu \equiv \partial_\mu + 2i e A_\mu$ is the covariant derivative (we choose, as for the Cooper pairs, the electric charge to be $2e$). The field $\phi$ is assumed to be a complex quantity, with *two* degrees of freedom, but only *one* non-vanishing vacuum expectation value (VEV), namely the condensation value of the field
below a critical temperature $T_c$. Usually, such two degrees of freedom are represented by the real and imaginary part of $\phi$ (see, for example, [2]), but other representations (for example, in terms of the modulus and phase of $\phi$) are of course possible. As is well known, the above Lagrangian is invariant under reparametrization of $\phi$, and the various representations of the complex field describe, at the classical (tree) level, the same physical reality. This is also true in two-Higgs (or more) models, where it holds the ‘rephasing’ invariance of the Higgs multiplet (see [8]). Already in 1984, however, some authors [9] pointed out that different representations of $\phi$ can lead to different expressions for the critical temperature $T_c$ where the condensation phenomenon takes place, since the dynamics ruled by different degrees of freedom could lead, in principle, to different predictions [10], when considering also the radiative, non-tree corrections. However, in [9] the key point was not yet fully realized (see, for example, the obscure and discussion at the end of appendix 2). Indeed, in the Abelian Higgs mechanism the physical system is described by means of Lagrangian density (1), which is in any case invariant under a reparametrization of the scalar field; yet, a key role is as well played by the condensation mechanism and in particular, for what is here considered, by the given choice of the scalar degree of freedom that condensates. In other words, different parametrizations of the field $\phi$ can effectively lead to different critical temperatures (and phases), if the physical system at hand chooses to undergo a condensation with a given degree of freedom rather than with other possible ones. The situation is, in a sense, analogous to what happens in the Aharonov–Bohm effect (or other similar effects), where physical consequences of the gauge chosen for the electromagnetic potential arise. Nevertheless, for the Abelian Higgs model, it should be emphasized that an expected difference in the critical temperatures of the system is purely formal if only one condensation occurs since, in this case, the different dependence of $T_c$ on the model parameters $m^2$, $\lambda$ is not observable, such parameters being not directly measurable (they can be deduced just by measurement of $T_c$ or other observables). The situation is, instead, just different if two (or more) condensations take place in a peculiar physical system such as, e.g., some non-standard superconductors. In this case, the difference between the critical temperatures leads to distinct superconductive phases, entailing two discontinuities in the specific heat and unusual magnetic properties. In a recent series of our papers [10–12] we have, indeed, studied in detail this possibility, and it is very remarkable that the two different representations mentioned above (real and imaginary parts versus modulus and phase) account, in a very simple manner, for the apparently exotic properties observed in the superconductivity of strontium ruthenate [12–14]. These observations evidently urge to consider accurately the problem of the field reparametrization in the Higgs mechanism; actually, in the present paper we shall expound a sufficiently general and comprehensive analysis of the physical implications of the field representation gauge.

2. General representation of a complex scalar field

We start our analysis from the very general assumption that the complex scalar field $\phi(x)$ appearing in (1) is endowed with two different degrees of freedom described by the real scalar fields $a(x), b(x)$. The only non-vanishing VEV of $\phi(x)$ is introduced by means of the (non-zero) real parameter $a_0$ as follows:

\[
\langle \phi(x) \rangle = \frac{a_0}{\sqrt{2}}
\]

Without loss of generality we assume that such VEV corresponds to $a = a_0, b = 0$, that is, the field $a(x)$ condenses for $a = a_0$, while the field $b(x)$ does not condense. Furthermore, we
also assume that the field $\phi(x)$ can be expanded in a Taylor series around such values; on very general grounds, we can then represent the field $\phi(x)$ with the following formula:

$$
\phi(x) = \sum_{n,m} a_0^{1-n-m} (c_{nm} + id_{nm})a^n b^m.
$$

(3)

Here $c_{nm}, d_{nm}$ are real, dimensionless coefficients that characterize the different representations, while the factor $a_0^{1-n-m}$ is introduced for dimensional reasons (both the fields $a, b$ have the same physical dimensions as $a_0$). Note that, in order to preserve the appearance of two degrees of freedom, at least one $c$-coefficient and one $d$-coefficient must be different from zero:

$$
\exists n' \in \mathbb{N}, \exists m' \in \mathbb{N}_0 : c_{n'm'} \neq 0, \quad \exists n'' \in \mathbb{N}_0, \exists m' \in \mathbb{N} : d_{n'm''} \neq 0.
$$

(4)

The most 'popular' parametrizations are the ‘Gauss representation’

$$
\phi = \frac{1}{\sqrt{2}} (a + ib),
$$

(5)

where we can observe a condensation of the real (or imaginary) part, or the ‘Euler representation’

$$
\phi = \frac{1}{\sqrt{2}} a e^{ib/a_0},
$$

(6)

where we can observe a condensation of the modulus (or phase). Correspondingly the $c_{nm}, d_{nm}$ coefficients take the values

$$
c_{nm} = \frac{1}{\sqrt{2}} \delta_{n1} \delta_{m0}, \quad d_{nm} = \frac{1}{\sqrt{2}} \delta_{n0} \delta_{m1}
$$

(7)

and

$$
c_{nm} = \frac{1}{\sqrt{2}} \frac{(-1)^k}{(2k)!} \delta_{n1} \delta_{m,2k}, \quad d_{nm} = \frac{1}{\sqrt{2}} \frac{(-1)^k}{(2k+1)!} \delta_{n1} \delta_{m,2k+1},
$$

(8)

respectively. In general, the coefficients $c_{nm}, d_{nm}$ are not completely arbitrary, but have to satisfy several constraints, that will be discussed in the following. First of all, we take into account that only one non-vanishing VEV exists, given by equation (2). By equating equations (2) and (3), for $a = a_0, b = 0$ we obtain

$$
\sum_n c_{n0} = \frac{1}{\sqrt{2}}, \quad \sum_n d_{n0} = 0.
$$

(9)

Let us now proceed with the Higgs mechanism, by expanding $\phi$ in equation (3) around its VEV,

$$
a \simeq a_0 + \tilde{a}, \quad b \simeq \tilde{b}
$$

(10)

($\tilde{a}, \tilde{b}$ are fluctuations fields), and inserting the resulting expression in Lagrangian (1) (we will consider only terms up to the second order in the fields). The kinetic terms for the fields $\tilde{a}$ and $\tilde{b}$ coming out from (1)

$$
\mathcal{L}_{\text{kin}} \simeq \left[ \left( \sum_n n c_{n0} \right)^2 + \left( \sum_n n d_{n0} \right)^2 \right] \partial_{\mu} \tilde{a} \partial^{\mu} \tilde{a} + \left[ \left( \sum_n c_{n1} \right)^2 + \left( \sum_n d_{n1} \right)^2 \right] \partial_{\mu} \tilde{b} \partial^{\mu} \tilde{b}
$$

$$
+ 2 \left[ \sum_n c_{n1} \sum_m mc_{m0} + \sum_n d_{n1} \sum_m md_{m0} \right] \partial_{\mu} \tilde{a} \partial^{\mu} \tilde{b}.
$$

(11)
In order to deal with well-defined energy eigenstates with easily identifiable kinematic terms, we will require the following constraints:

\[
\left( \sum_n n c_{n0} \right)^2 + \left( \sum_n n d_{n0} \right)^2 = \frac{1}{2},
\]

\[
\left( \sum_n c_{n1} \right)^2 + \left( \sum_n d_{n1} \right)^2 = \frac{1}{2},
\]

\[
\sum_n c_{n1} \sum_m m c_{m0} + \sum_n d_{n1} \sum_m m d_{m0} = 0.
\]

(12)

The potential terms for the scalar field

\[
\mathcal{L}_{\text{pot}} = -m^2 \phi^2 - b \phi^2
\]

(13)

can be analogously evaluated up to second-order terms. By taking into account the constraints in equations (9) and (12), we are able to write down the mass-potential terms in the following form:

\[
\mathcal{L}_{\text{pot}} \simeq -\left[ \frac{a_0^2}{2} \left( m^2 + \lambda a_0^2 \phi \right) \right] - \left[ \frac{a_0}{\sqrt{2}} \left( \frac{\lambda a_0^2}{4} \right) \sum_n n c_{n0} \right] \hat{a} \hat{\phi} - \left[ \frac{a_0}{\sqrt{2}} \left( \frac{\lambda a_0^2}{4} \right) \sum_n n c_{n1} \right] \hat{b} - \left[ \left( \frac{m^2 + \lambda a_0^2}{4} \right) \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \sum_n n(n-1)c_{n0} \right) \right]
\]

\[
+ \frac{\lambda a_0^2}{2} \left( \sum_n n c_{n0} \right)^2 \hat{a}^2 - \left[ \left( \frac{m^2 + \lambda a_0^2}{4} \right) \left( \frac{1}{2} + \frac{2}{\sqrt{2}} \sum_n c_{n2} \right) \right]
\]

\[
- \frac{\lambda a_0^2}{2} \left( \sum_n n c_{n0} \right)^2 + \frac{\lambda a_0^2}{4} \hat{b}^2.
\]

(14)

For a generic representation of the field \( \phi \), the terms in \( \hat{a}, \hat{b} \) and \( \hat{a} \hat{b} \) do not vanish, so that, in this case, the mass eigenstates of the system are not the fields \( \hat{a}, \hat{b} \) but a linear combination of them. However, as we will see below (see equation (24)), the quantity relevant for the calculation of the critical temperature, or of other observables, is the trace of the squared mass matrix [2] that is invariant under the mentioned transformation. Therefore, for simplicity, we will limit our attention to the following part of the potential term:

\[
\mathcal{L}_2 = -\frac{1}{2} m_a^2 \hat{a}^2 - \frac{1}{2} m_b^2 \hat{b}^2
\]

(15)

with

\[
m_a^2 = 2m^2 \left[ \frac{1}{2} + \frac{1}{\sqrt{2}} \sum_n n(n-1)c_{n0} \right]
\]

\[
+ \lambda a_0^2 \left[ \left( \sum_n n^2 c_{n0} \right)^2 \right] + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \sum_n n(n-1)c_{n0} \right) \right],
\]

(16)

\[
m_b^2 = 2m^2 \left[ \frac{1}{2} + \frac{2}{\sqrt{2}} \sum_n c_{n2} \right] + \lambda a_0^2 \left[ \left( \sum_n n^2 c_{n0} \right)^2 \right] + \frac{1}{2} \left( \frac{1}{2} + \frac{2}{\sqrt{2}} \sum_n c_{n2} \right).
The expectation value of the Higgs field is obtained by minimizing the potential in equation (13). For positive \(m^2\) only the solution \(\langle \phi \rangle = 0\) is possible, while for negative \(m^2\) a non-trivial solution arises:

\[
\langle \phi \rangle = \sqrt{-\frac{2m^2}{\lambda}} \equiv \frac{a_0}{\sqrt{2}}
\]

(17)

In such a case, at \(T = 0\), this implies that

\[
m^2 + \frac{\lambda a_0^2}{4} = 0,
\]

(18)

and the tadpole terms in \(\tilde{a}, \tilde{b}\) in equation (14) vanish. A straightforward calculation (from equations (15) and (16)) easily shows that, for \(T = 0\), the Higgs ‘mass’ reduces to \(M_H^2 = \lambda a_0^2\), thus being independent of the parametrization of the scalar field. The things change when radiative corrections at \(T \neq 0\) are taken into account since, in this case, a temperature-dependent effective mass \(m_{\text{eff}}\) rules the mechanism [2]. At tree level and in its minimum (2) the scalar potential is (see the first term in equation (14))

\[
U_0 = \frac{1}{2}m^2 a_0^2 + \frac{1}{16} \lambda a_0^4.
\]

(19)

For \(e^4 \ll \lambda\) we can neglect \(T = 0\) corrections to the potential; by assuming also that \(T^2 \gg -m^2, \lambda a_0^2, e^2 a_0^2\), the \(T\)-dependent quantum correction term is given by [2]

\[
U_1 \simeq -\frac{4\pi^2 T^4}{90} + \frac{T^2}{24} \left[ M_H^2 + 3M_V^2 \right],
\]

(20)

so that the total potential can be written as

\[
U \simeq \frac{1}{2}m_{\text{eff}}^2 a_0^2 + \frac{1}{16} \lambda a_0^4 - \frac{4\pi^2 T^4}{90},
\]

(21)

with

\[
m_{\text{eff}}^2 \simeq m^2 + \frac{T^2}{12a_0^2} \left[ M_H^2 + 3M_V^2 \right].
\]

(22)

The discussion above about the minimum of the potential can be repeated here with the only modification that the term \(m^2\) is now replaced by the effective one \(m_{\text{eff}}^2\). This leads to the important fact that the corrected tadpole terms vanish (as they should do at the minimum of the corrected potential) while, obviously, the tree-level ones in equation (14) do not, since equation (18) is in general no longer valid because of the additional \(T \neq 0\) terms. As a result, the expression for the Higgs mass changes.

Let us now introduce the following quantities that will directly enter in the expression for the critical temperature (in \(M_H^2\) the terms proportional to \(m^2\) are not included, but only the corrections proportional to \(a_0^2\) do appear):

\[
M_a^2 \equiv m_a^2 - m_a^2(a_0 = 0) = \lambda a_0^2 \left[ \left( \sum_n n^2 c_{n0} \right)^2 + \frac{1}{2} \left( \frac{1}{2} + \sum_n n^2 c_{n0} \right) \right],
\]

(23)

\[
M_b^2 \equiv m_b^2 - m_b^2(a_0 = 0) = \lambda a_0^2 \left[ \frac{1}{2} - \left( \sum_n n^2 c_{n0} \right)^2 + \frac{1}{2} \left( \frac{1}{2} + \sum_n n^2 c_{n2} \right) \right].
\]

By using the same notations as in [2], the Higgs mass is thus given by

\[
M_H^2 = \text{Tr} \left[ M_1^2 \right] = M_a^2 + M_b^2 = \lambda a_0^2 \left[ 1 + \frac{1}{2\sqrt{2}} \sum_n (n(n - 1)c_{n0} + 2c_{n2}) \right]
\]

(24)
and, evidently, depends on the representation chosen (through \(c_{n0}, c_{n2}\)). Such a difference in the behavior of the Higgs mass for \(T = 0\) and for \(T \neq 0\) comes from the fact that \(a_0\) and \(m^2\) (but not \(m^2_{\text{eff}}\)) are two independent bare parameters and no relation between them holds unless \(T = 0\), in which case equation (18) implies that only one independent parameter exists and the Higgs mass does not depend on the parametrization chosen. Note that, as naturally expected, the tadpole terms in \(\tilde{a}, \tilde{b}\) vanish only when all corrections are included, while non-vanishing tree-level terms do not affect the condition of a minimum for the complete potential. Instead, from the vector boson mass term

\[
L_{MV} = e^2 |\phi|^2 A_\mu A^\mu \simeq \frac{e^2 a_0^2}{2} A_\mu A^\mu
\]

we obtain the photon mass from

\[
M^2_V = e^2 a_0^2,
\]

that is, obviously independent of the representation chosen even at \(T \neq 0\).

### 3. Relation between field parametrization and broken phase thermodynamics

The critical temperature \(T_c\) of the system considered is defined as the temperature at which the system passes from the trivial expectation value \(\langle \phi \rangle = 0\) to the non-trivial one in equation (17) with \(m^2\) replaced by \(m^2_{\text{eff}}\): so that it is the solution of the equation

\[
m^2_{\text{eff}}(T_c) = 0,
\]

or

\[
T^2_c = \frac{-12m^2}{\lambda H + 3e^2},
\]

with

\[
H \equiv \frac{M^2_H}{\lambda a_0^2} = 1 + \frac{1}{2\sqrt{2}} \left[ \sum_n (n(n - 1)c_{n0} + 2c_{n2}) \right].
\]

The key result in equation (28) clearly shows that the observable \(T_c\) does depend on the representation chosen for the field \(\phi\) through the \(H\) term (or, what is the same, through the Higgs mass). As already pointed out above, however, such a dependence is only formal if the system possesses only one critical temperature, since the parameters \(\lambda\) and \(m^2\) appearing in the Lagrangian (1) are not directly observable. This is not the case, instead, for systems showing more than one critical temperature, as extensively discussed in [10, 11], so that it is quite relevant to discuss the possible consequences of equation (28). First of all, let us observe that, for the standard representations (5) and (6), we recover the known results [2, 9] namely

\[
T^2_c = \frac{-12m^2}{\lambda + 3e^2},
\]

and

\[
T^2_c = \frac{-16m^2}{\lambda + 4e^2},
\]

corresponding to Higgs masses \(M^2_H = \lambda a_0^2\) and \(M^2_H = \frac{3}{4} \lambda a_0^2\), respectively. It is noticeable that the above two results (and other similar values [15], cf also [9] for the Glashow–Weinberg–Salam model) can be directly obtained here as particular cases of a general theory, while in the literature they come out from different and very elaborate theoretical approaches. The latter value, for example, was derived using the real-time Green’s function approach. In general,
because of equation (29), the $H$ term parametrizes the relative strength between the self-interaction of the Cooper pairs (ruled by $\lambda$) and the electromagnetic interaction (ruled by $e$). It is quite remarkable that such a parameter, and thus $T_c$, depends on only ‘two’ coefficients, $c_{n0}$ and $c_{n2}$ (for all $n$). This fact leads to relevant consequences. Firstly, since the $d_{nm}$ coefficients do not contribute to the expression of $T_c$ (or $M_H$), representations of $\phi$ that differ only for the imaginary part $\text{Im}\{\phi\}$ give the same $T_c$ (and $M_H$). However, this fact does not at all imply that we can consider just real representations of $\phi$ (contrary to the assumption of two, not one, degrees of freedom); in fact, from the constraints (9), (12) it immediately follows that not all $d_{nm}$ coefficients can vanish. We have, then, a further limitation since, from equation (29), it is evident that $T_c$ (and $M_H$) depends only on the coefficients $c_{n0}$, $c_{n2}$. This means that only the terms $a^n$ and $a^n b^2$ in the expansions of $\phi$ contribute to $T_c$ (and $M_H$), so that the representations of $\phi$ whose real parts differ in their expansion around the VEV ($a = a_0$, $b = 0$) only for odd power terms in $b$ or for $O(b^4)$ even power terms give the same $T_c$ (and $M_H$). Another constraint on the coefficients $c_{n0}$, $c_{n2}$ (which can assume, of course, even negative values) comes from equation (24) by requiring that the Higgs mass squared (after the condensation) is a positive quantity

$$\sum_n (n(n-1)c_{n0} + 2c_{n2}) \geq -2\sqrt{2}. \quad (32)$$

Provided that such a condition is satisfied, from equation (28) we obtain the important result that, changing the representation of $\phi$, the critical temperature of the system cannot assume any arbitrarily large value, but is bounded in the interval

$$0 \leq T_c \leq T_c^{\text{max}}, \quad (33)$$

with

$$T_c^{\text{max}} = 2\sqrt{-\frac{m^2}{e^2}}, \quad (34)$$

corresponding to very large or extremely small values of the Higgs mass, respectively. Equation (28) can, then, be rewritten in the expressive form

$$T_c = T_c^{\text{max}} \sqrt{\frac{1}{1 + \frac{2H}{\sqrt{2}}}}, \quad (35)$$

and the dependence of $T_c$ on the ‘representation factor’ $H$ is depicted in figure 1.

Figure 1. Critical temperature versus the ‘representation factor’ $H$.

Note that, just from a mathematical point of view, $T_c \to 0$ (that is, no superconductivity) for $\lambda H/e^2 \to \infty$. Such a limit may have a correct physical meaning, provided that the
perturbation theory used throughout does not lose meaning (that is, for small coupling constants): this can be effectively the case when the Cooper pair self-interaction is stronger than the electromagnetic interaction among electrons. Instead, the critical temperature approaches its maximum value in (34) when the Higgs mass tends to zero ($H \to 0$) or $\lambda/e^2 \to 0$, that is for a Cooper pair self-interaction weaker in comparison to the electromagnetic interaction. A possible example of representation of $\phi$ that, independently of the value of $\lambda/e^2$, realizes such a case (that is, for which $M_H = 0$) is the following:

$$
\phi = \frac{a}{\sqrt{2}} \left(3 - 2 \frac{a}{a_0}\right) + i \frac{b}{\sqrt{2}} \left(1 - 2 \frac{b}{a_0}\right).
$$

(36)

(but, of course, many other representations are possible, according to what discussed above, provided that the constraints (9), (12) and (28) are satisfied). Interestingly enough, the maximum critical temperature in (34) has an impressive physical interpretation in terms of the entropy of the system. In fact, from the standard expression of the free energy $F$ in the GL model (evaluated in the VEV of the Higgs field) [11],

$$
F = F_0 + \frac{1}{2} a(T) a_0^2 + \frac{\lambda}{16} a_0^4
$$

(37)

with

$$
a(T) = -m^2 \left(1 - \frac{T^2}{T_c^2}\right),
$$

(38)

the entropy $S$ of the system can be easily computed from

$$
S = -\frac{\partial F}{\partial T}.
$$

(39)

obtaining

$$
S = \frac{a_0^2}{12} (\lambda H + 3 e^2) T = \left(\frac{M_H^2}{12} + \frac{M_V^2}{4}\right) T.
$$

(40)

For a given temperature $T$ (see figure 2), the entropy increases for increasing Higgs mass, starting from a minimum for $M_H = 0$. Thus, the maximum critical temperature in (34) corresponds, for a given temperature $T$, to the minimum of the entropy of the system (different from zero for a non-vanishing VEV of $\phi$) or, what is the same, to the maximum possible order of the system. This is what is expected, since higher temperature corresponds to a smaller Higgs mass, which in its turn advantages the transition to the more ordered broken phase.
4. Conclusions

Summing up, we have discussed how different reparametrizations of the scalar field ruling the Higgs mechanism (with two degrees of freedom and one non-vanishing VEV), as described by equation (3), affect the expression of the critical temperature of the system or, through the free energy (37), all the thermodynamical quantities of the standard GL models (applied, e.g., to superconductors and superfluids).

As is well known, in order to give mass to a gauge theory, within the Higgs mechanism the gauge invariance must be broken by a condensate. In the Higgs mechanism, the condensate is described by a quantum field with a non-vanishing expectation value, just as in the GL model. To make sure that the condensate value of the field does not pick out a preferred direction in spacetime, it must be a scalar field and, in order for the phase of the condensate to define a gauge, the field must be charged, that is complex valued. This means that it should contain two independent real-valued fields (for example, the real and imaginary parts) with a symmetry which rotates them into each other. The electromagnetic vector potential changes the phase of the Higgs quanta when they move from point to point, and defines how much to rotate the two real-valued fields into each other when comparing field values at nearby points. In the usual case, when the physical system (for example, standard superconductors) is able to support just one condensation (for example in the transition from metal to superconductor), the choice of the two real fields representing the charged Higgs scalar is merely formal since what is described above takes place in the same way with any parametrization chosen. The nature does not change physical results merely by changing a given parametrization. However, the situation is different if the physical system (for example, exotic superconductors [16]) is able to support more than one condensation (for example in the transitions from metal to superconductor of kind $\alpha$ to superconductor of kind $\beta$, etc, with decreasing temperature). In fact, in this case, different real fields can describe different condensates co-existing in the given physical system: by choosing a gauge where the phase of a given condensate is fixed, the potential energy for the fluctuations of the electromagnetic vector field is non-zero and depends on the given condensate. The gauge field always acquires a non-vanishing mass, but its value depends on the given condensate responsible for it.

Our study may then be relevant only for physical systems that exhibit more than one critical temperature, as the case, for example, of the superconductivity of strontium ruthenate [12]. Changing the possible representation of the scalar field $\phi$ results (with some interesting exceptions, discussed above in detail) in different values for the Higgs mass and, through this parameter, in different critical temperatures. From a physical viewpoint, this may occur in any system described by a scalar field that chooses to undergo a condensation with a given degree of freedom rather than with other possible ones in given temperature ranges, as is apparently observed in the mentioned exotic superconductor systems. In general, for such systems, the Higgs mass is not bounded, while the critical temperatures of the system may increase from zero (that is no superconductivity) up to a maximum value $T_{\text{max}}$, corresponding to the non-zero minimum of the entropy system (for given temperature). One possible representation for $\phi$ realizing such a limiting case is shown in equation (36), and can be regarded as a generalization of the standard representation (5) when the highest order terms in $a, b$ are included.

Although we have devoted our attention mainly to the Abelian Higgs mechanism applied to the standard GL model, usually referred to in condensed matter and solid state physics, nevertheless we expect that the consequences of our study will not be limited to that area of physics. Since the Higgs mechanism plays the same role in the Abelian as well as in the non-Abelian case, our model is straightforwardly generalizable to non-Abelian interactions as, for example, to electroweak theory where the scalar field is a representation of the $SU(2)$...
gauge group. There the scalar field is a representation of the chosen gauge group, and the gauge covariant derivative is defined by the rate of change of the field minus the rate of change from parallel transport using the gauge field as a connection [2]. A non-zero expectation value of the Higgs field defines a preferred gauge: the results are then expected to be similar to those obtained here for the Abelian Higgs mechanism.

Further studies (which are, however, beyond the aims of the present paper) on the elementary particle physics sector, with special reference to the unification of the fundamental forces and to the phase transitions in the early universe, can disclose interesting new phenomena affecting our understanding of the cosmological evolution.

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