A HYPERGEOMETRIC APPROACH, 
VIA LINEAR FORMS INVOLVING LOGARITHMS, 
TO CRITERIA FOR IRRATIONALITY OF EULER’S CONSTANT

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WITH AN APPENDIX BY SERGEY ZLOBIN

Abstract. Using an integral of a hypergeometric function, we give necessary and sufficient conditions for irrationality of Euler’s constant $\gamma$. The proof is by reduction to known irrationality criteria for $\gamma$ involving a Beukers-type double integral. We show that the hypergeometric and double integrals are equal by evaluating them. To do this, we introduce a construction of linear forms in 1, $\gamma$, and logarithms from Nesterenko-type series of rational functions. In the Appendix, S. Zlobin gives a change-of-variables proof that the series and the double integral are equal.

1. Introduction

In [12] we gave criteria for irrationality of Euler’s constant $\gamma$, which is defined by the limit

$$\gamma := \lim_{N \to \infty} (H_N - \log N),$$

where

$$H_N := \sum_{k=1}^{N} \frac{1}{k}$$

is the Nth harmonic number. The criteria involve a double integral $I_n$ modeled on Beukers’ integrals [2] for $\zeta(2)$ and $\zeta(3)$, and the main step in the proof was to show that

$$d_{2n}I_n \in \mathbb{Z} + \mathbb{Z}\gamma + \mathbb{Z}\log(n+1) + \mathbb{Z}\log(n+2) + \cdots + \mathbb{Z}\log(2n),$$

where

$$d_n := \text{LCM}(1, 2, \ldots, n)$$

denotes the least common multiple of the first $n$ natural numbers.

Here we define $I_n$ instead as an integral involving a hypergeometric function; we prove that the same criteria hold with this new $I_n$. The proof is by showing that the old and new definitions of $I_n$ are equivalent. (Alternatively, one could give a self-contained proof along the lines of [12]; the required inequality $I_n < 2^{-4n}$ follows easily from Lemma 1 below.) To show the equivalence, we introduce a series modeled on the one Nesterenko used in [8] to give a new proof of Apéry’s theorem that $\zeta(3)$ is irrational. (We modify Nesterenko’s rational function, and where he differentiates in order to go “up” from $\zeta(2)$ to $\zeta(3)$, we integrate to go “down” to $\gamma$, which one may think of as “$\zeta(1)$.”) We prove that both versions of $I_n$ are equal to the sum of our series, by evaluating them. In the Appendix, Sergey Zlobin gives a change-of-variables proof that the double integral and the series are equal, without evaluating them.

The chronology of discovery, different from what one might expect from the above, was as follows: After reading Nesterenko’s paper [8], we constructed the series and derived irrationality criteria for $\gamma$ from it. Later, Huylebrouck’s survey [6] of multiple integrals in irrationality proofs led us to find
the double integral, and using it we rederived the criteria. Zudilin’s work [18] gave us the idea to express the series in hypergeometric form. Here we use Thomae’s transformation to simplify the hypergeometric function (compare [11]). We hope that the variety of expressions for $I_n$ will turn out to be useful in determining the arithmetic nature of $\gamma$.

After seeing [11], the Hessami Pilehroods [5] extended our non-hypergeometric results to generalized Euler constants. In particular, they used our construction (in Section 4 below) of linear forms involving $\gamma$ and logarithms from series.

For an approach to irrationality criteria for $\gamma$ using Padé approximations, see Prévost’s preprint [10].

Recently, Zudilin and the author obtained results [17] analogous to those in [12], but using $q$-logarithms instead of ordinary logarithms.

2. Hypergeometric Irrationality Criteria for Euler’s constant

We state the criteria. First recall that the hypergeometric function $3F_2$ is defined by the series

$$3F_2\left(\begin{array}{c} a, b, c \\ d, e \end{array} \right| z \right) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!(d)_k(e)_k} z^k,$$

where $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k > 0$. The only case we need is when $a, b, c, d, e$ are positive real numbers and $z = 1$. In that case, the series converges if $a+b+c < d+e$. Note that a permutation of either the upper parameters $a, b, c$ or the lower parameters $d, e$ does not change the value of the sum.

Now for $n > 0$, let $S_n$ be the positive integer

$$S_n := \prod_{m=1}^{n} \prod_{k=0}^{\min(m-1,n-m)} \prod_{j=k+1}^{n-k} (n+m)(n+k)^{2d_{2n}}$$

and let $I_n$ be the “hypergeometric integral”

$$I_n := \int_{n+1}^{\infty} \frac{(n!)^2 \Gamma(t)}{(2n+1)\Gamma(2n+1+t)} \cdot 3F_2\left(\begin{array}{c} n+1, n+1, 2n+1 \\ 2n+2, 2n+1+t \end{array} \right| 1 \right) dt,$$

whose convergence follows immediately from the proof of Lemma [11].

Theorem 1 (Hypergeometric (ir)rationality criteria for $\gamma$). The following statements are equivalent:

1. The fractional part of $\log S_n$ is equal to $d_{2n}I_n$, for some $n > 0$.
2. The assertion is true for all $n$ sufficiently large.
3. Euler’s constant is a rational number.

After establishing some preliminary results, we give the proof in Section [6].

3. A Series for $I_n$

We express the integral $I_n$ as a series.

Lemma 1. If $n > 0$, then

$$I_n = \sum_{v=n+1}^{\infty} \int_{v}^{\infty} \left( \frac{n!}{t(t+1)\cdots(t+n)} \right)^2 dt.$$
Proof. Fix $n > 0$. We claim that
\[
I_n = \int_{n+1}^{\infty} \frac{(n!)^2 \Gamma(t)^2}{\Gamma(t+n+1)^2} 3F_2 \left( \begin{array}{c} t,1,t \\ t+n+1,t+n+1 \end{array} \right) dt.
\]
To show this, we apply Thomae’s transformation \[^{[1]} p. 14, \] [4 p. 104], \[^{[7]}\]
\[
\frac{\Gamma(a)}{\Gamma(d)\Gamma(e)} 3F_2 \left( \begin{array}{c} a,b,c \\ d,e \end{array} \right) = \frac{\Gamma(s)}{\Gamma(s+b)\Gamma(s+c)} 3F_2 \left( \begin{array}{c} s,d-a,e-a \\ s+b,s+c \end{array} \right),
\]
where $s := d + e - a - b - c$. After permuting the upper parameters in the resulting function $3F_2$, we obtain the integrand in the definition of $I_n$, proving the claim.

Now since
\[
3F_2 \left( \begin{array}{c} t,1,t \\ t+n+1,t+n+1 \end{array} \right) = 1 + \frac{t^2}{(t+n+1)^2} + \frac{t^2(t+1)^2}{(t+n+1)^2(t+n+2)^2} + \ldots
\]
we can use the identity $\Gamma(x+1) = x\Gamma(x)$ to write
\[
I_n = \int_{n+1}^{\infty} \sum_{\nu=0}^{\infty} \frac{(n!)^2 \Gamma(t+\nu)^2}{\Gamma(t+\nu+n+1)^2} dt = \int_{n+1}^{\infty} \sum_{\nu=0}^{\infty} R_n(t+\nu) dt,
\]
where
\[
R_n(t) := \left( \frac{n!}{(t(t+1)\cdots(t+n))} \right)^2
\]
is the rational function in \[^{[2]}\]. Interchanging integral and summation, and replacing $t$ with $t - \nu$, and $\nu$ by $\nu - n - 1$, we arrive at \[^{[2]}\].

\section{4. Constructing Linear Forms in $1$, $\gamma$ and Logarithms From Series}

We give a method for constructing linear forms involving $\gamma$ and logarithms from certain series of rational functions.

\textbf{Proposition 1.} Fix $n > 0$ and let $R(t)$ be a rational function over $\mathbb{C}$ of the form
\[
R(t) = \sum_{k=0}^{n} \left( \frac{B_{k2}}{(t+k)^2} + \frac{B_{k1}}{t+k} \right).
\]
If $R(t) = O(t^{-3})$ as $t \to \infty$, then
\[
\sum_{\nu=n+1}^{\infty} \int_{\nu}^{\infty} R(t) dt = B\gamma + L - A,
\]
where
\[
B := \sum_{k=0}^{n} B_{k2}, \quad L := \sum_{m=1}^{n} \sum_{k=m}^{n} B_{k1} \log(n+m), \quad A := \sum_{k=0}^{n} B_{k2}H_{n+k}.
\]

\textbf{Proof.} From \[^{[3]}\], for $|t|$ large we obtain an expansion $R(t) = \sum_{i=1}^{\infty} b_i t^{-i}$, with $b_1 = \sum_{k=0}^{n} B_{k1}$ and $b_2 = \sum_{k=0}^{n}(B_{k2} - kB_{k1})$. The asymptotic hypothesis implies that $b_1 = b_2 = 0$, so we have the relations
\[
\sum_{k=0}^{n} B_{k1} = 0, \quad \sum_{k=0}^{n}(B_{k2} - kB_{k1}) = 0.
\]
In view of (5.1), the sums \( \sum_{k=0}^{n} B_{k1} \log(t+k) \) and \( \sum_{k=0}^{n} B_{k1} \log(1+kt^{-1}) \) are equal. Hence for \( N>n \) we have

\[
\sum_{\nu=n+1}^{N} \int_{\nu}^{\infty} R(t) \, dt = \sum_{\nu=n+1}^{N} \sum_{k=0}^{n} \left( \frac{B_{k2}}{\nu+k} - B_{k1} \log(\nu+k) \right) .
\]

Define \( B, L, A \) by (6), and rewrite \( L \) as

\[
L = \sum_{n=1}^{N} \sum_{m=n+1}^{N+k} \left( \frac{B_{k2}}{m} - B_{k1} \log m \right)
\]

by the quantity \( \sum_{k=0}^{n} \sum_{m=n+1}^{N} B_{k1} \log m \), which vanishes by (5.1). Since \( n \) is fixed and \( k \leq n \), the double sum in (7) equals \( -\sum_{n=1}^{N} \sum_{k=0}^{n} B_{k1} \log(\nu+k) \) as \( \nu \to \infty \). Using (5.2), it follows that the left-hand side of (6) is equal to \( B(H_n - \log N) + L - A + O(N^{-1}) \), and we obtain the required formula by letting \( N \) tend to infinity. \( \square \)

5. Summing the Series for \( I_n \)

Applying Proposition 1 to the rational function \( R(t) \), we sum series (2) for \( I_n \).

Lemma 2. If \( n>0 \), then

\[
I_n = \left( \begin{array}{c} 2n \\ n \end{array} \right) \gamma + L_n - A_n,
\]

where \( L_n \) is the linear form in logarithms

\[
L_n := \sum_{m=1}^{n} \sum_{k=0}^{n-m} \sum_{j=k+1}^{n-k} \left( \begin{array}{c} n \\ k \end{array} \right)^2 \frac{2}{j} \log(n+m)
\]

and \( A_n \) is the rational number

\[
A_n := \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)^2 H_{n+k}.
\]

Moreover, inclusion (1) and \( d2nL_n = \log S_n \).

Proof. The partial fraction decomposition of the integrand in (2) is given by the right-hand side of (3) with

\[
B_{k2} = (t+k)^2 R_n(t)|_{t=-k} = \left( \begin{array}{c} n \\ k \end{array} \right)^2
\]

and

\[
B_{k1} = \frac{d}{dt} \left( (t+k)^2 R_n(t) \right)|_{t=-k} = 2 \left( \begin{array}{c} n \\ k \end{array} \right)^2 \left( H_k - H_{n-k} \right),
\]

where \( H_0 = 0 \). Using the relations \( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) = \left( \begin{array}{c} 2n \\ n \end{array} \right) \) and \( B_{n-k,1} = -B_{k1} \), the result follows from Proposition 1 and the definition of \( S_n \). \( \square \)
6. A Double Integral For $I_n$ and Proof of the Criteria

We obtain another representation of $I_n$, as a double integral, and prove Theorem \[1\]

**Lemma 3.** For $n > 0$, the following equality holds:

\[
\sum_{\nu=n+1}^{\infty} \int_{\nu}^{\infty} \left( \frac{n!}{t(t+1)\cdots(t+n)} \right)^2 dt = \int_{[0,1]^2} \frac{(x(1-x)y(1-y))^n}{(1-xy)(-\log xy)} dx dy.
\]

**Proof.** By Lemmas \[1, 2\] and \[12, (8)\], the series and the double integral, respectively, are both equal to \((\frac{2n}{n}) \gamma + L_n - A_n\). \[\square\]

**Proof of Theorem \[1\].** This follows immediately from Lemmas \[1\] and \[3\] together with the main result of \[12\], which is the same as Theorem \[1\] except that in \[12\] we defined $I_n$ to be the double integral in \[8\]. \[\square\]

**Remark.** There exist representations of many constants as double integrals of the same shape as the one in \[8\] — see \[3, 14, 15, 16\].

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**Appendix by Sergey Zlobin**

In this appendix we prove the statement of Lemma \[1\] without expanding integrals to linear forms. First, we develop $1/(1 - xy)$ in a geometric series

\[
J_n := \int_{[0,1]^2} \frac{x^n(1-x)^ny^n(1-y)^n}{(1-xy)(-\log xy)} dx dy
\]

\[
= \sum_{k=0}^{\infty} \int_{[0,1]^2} \frac{x^{n+k}(1-x)^ny^{n+k}(1-y)^n}{-\log xy} dx dy.
\]

(To justify the interchange of the sum and the double integral, one can expand $1/(1 - xy)$ in a finite sum with remainder and make the same estimations as in \[12\].) Further, we substitute

\[
-\frac{(xy)^k}{\log xy} = \int_{k}^{\infty} (xy)^t dt
\]

and obtain

\[
J_n = \sum_{k=0}^{\infty} \int_{[0,1]^2} \left( \int_{k}^{\infty} x^{n+t}(1-x)^ny^{n+t}(1-y)^n dt \right) dx dy
\]

\[
= \sum_{k=0}^{\infty} \int_{[0,1]^2} \left( \int_{k}^{\infty} x^{n+t}(1-x)^ny^{n+t}(1-y)^n dx dy \right) dt,
\]

where we can change the order of integration because the integrand is nonnegative and all the integrals converge. Since

\[
\int_{0}^{1} u^{n+t}(1-u)^n du = \frac{n!}{(t + n + 1)(t + n + 2)\cdots(t + 2n + 1)}
\]
we have

\[ J_n = \sum_{k=0}^{\infty} \int_k^\infty \left( \frac{n!}{(t + n + 1)(t + n + 2) \cdots (t + 2n + 1)^2} \right)^2 dt \]

\[ = \sum_{k=n+1}^{\infty} \int_k^\infty \left( \frac{n!}{t(t+1) \cdots (t+n)} \right)^2 dt, \]

and we get the desired identity.

The same method can be applied to prove that (minus) the series Nesterenko used in [8] is equal to Beukers’ triple integral in [2]. Another proof of that fact is given in [9] and uses an identity with a complex integral.
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