One sided invertibility of matrices over commutative rings, corona problems, and Toeplitz operators with matrix symbols✩

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Abstract

Conditions are established under which Fredholmness, Coburn’s property and one- or two-sided invertibility are shared by a Toeplitz operator with matrix symbol $G$ and the Toeplitz operator with scalar symbol $\det G$. These results are based on one-sided invertibility criteria for rectangular matrices over appropriate commutative rings and related scalar corona type problems.

Keywords: Toeplitz operator, Corona problem, Wiener-Hopf factorization, one-sided invertibility, Coburn’s property

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1. Introduction

To outline the main topics of this paper, we need first to agree on some standard notation and introduce some terminology. For any set $X$, we will

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denote by $X^{n \times m}$ the set of $n \times m$ matrices with entries in $X$, abbreviating $X^{n \times 1}$ to $X^n$. If $X$ is a Banach space (a ring, a (Banach) algebra), then $X^{n \times m}$ (resp., $X^{n \times n}$) is also supplied with the Banach space (resp., ring, (Banach) algebra) structure. A diagonal matrix from $X^{n \times n}$ with the diagonal entries $x_1, \ldots, x_n$ will be denoted $\text{diag}[x_1, \ldots, x_n]$.

Some important examples of $X$ include the Lebesgue spaces $L_p(\mathbb{R})$ of functions defined on the real line $\mathbb{R}$, and their subspaces $H^\pm_p$ of the traces on $\mathbb{R}$ of the functions from the Hardy spaces over the half-planes $\mathbb{C}^\pm := \{z \in \mathbb{C}: \pm \text{Im} z > 0\}$. We also denote by $M^\pm_p$, $1 \leq p \leq \infty$, the linear sums of $H^\pm_p$ with the algebra $\mathcal{R}$ of all rational functions in $L_\infty(\mathbb{R})$: $$M^\pm_p = H^\pm_p + \mathcal{R}.$$ The closure of $\mathcal{R}$ in the uniform norm is the algebra $C$ of all functions continuous on the one point compactification $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$, while the closure of $M^\pm_\infty$ coincides with $H^\pm_\infty + C$. The latter is thus a (closed) subalgebra of $L_\infty(\mathbb{R})$. Finally, for any ring $\mathcal{A}$, we let $\mathcal{G} \mathcal{A}$ stand for the set of its invertible elements.

Recall that a bounded linear operator $A: X \to Y$ acting between Banach spaces $X$ and $Y$ is Fredholm if its kernel $\ker A$ and cokernel $\text{coker} A = Y/\text{Im} A$ are finite dimensional. Note that then $\dim \text{coker} A = \dim \ker A^*$; the difference $$\text{Ind} A = \dim \ker A - \dim \text{coker} A$$ is the (Fredholm) index of $A$. We say that two operators $A: X \to Y$ and $\tilde{A}: \tilde{X} \to \tilde{Y}$ are Fredholm equivalent if either they are both Fredholm, with the same Fredholm index, or they are both non-Fredholm. Further, $A$ and $\tilde{A}$ are nearly Fredholm equivalent if they are both Fredholm (with no relation imposed on their indices) or both non-Fredholm, and strictly Fredholm equivalent if they are Fredholm equivalent and, in case they are both Fredholm operators, $\dim \ker A = \dim \ker \tilde{A}$ and $\dim \text{coker} A = \dim \text{coker} \tilde{A}$ their kernels have the same dimensions, and their cokernels have the same dimension.

We are ultimately interested in Fredholm properties of Toeplitz operators $T_G$ with matrix symbols $G \in L^n_\infty$ acting on $(H^\pm_p)^n$, $1 < p < \infty$, and in particular their relations with those of $T_{\det G}$.

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1$H^\pm_p$ consist of all functions analytic and bounded in $\mathbb{C}^\pm$; see Section 4 for the precise definition of $H^\pm_p$ for $p < \infty$. 

2
Observe first of all that for $G \in (H^\pm_\infty + C)^{n \times n}$, the operators $T_G$ and $T_{\text{det} G}$ are Fredholm equivalent. This follows directly from the Fredholmness criterion and index formula from [14], see also [15] or [23, Section 5.1]. Indeed, $T_G$ is Fredholm if and only if $G \in \mathcal{G}(H^\pm_\infty + C)^{n \times n}$, which in turn happens if and only if $\det G \in \mathcal{G}(H^\pm_\infty + C)$. Under this condition, $\text{Ind} T_G$ coincides, up to the sign, with the winding number of the harmonic extension of $\det G$ into $\mathbb{C}^\pm$ along horizontal lines sufficiently close to $\mathbb{R}$. The particular case of $G \in C^{n \times n}$ is of course simpler [25]: $T_G$ is then Fredholm if and only if $G \in \mathcal{G}C$, while $\text{Ind} T_G = \text{Ind} T_{\text{det} G}$ is the opposite of the winding number of $\det G$ over $\mathbb{R}$. On the other hand, already for piecewise continuous $G$ with just one point of discontinuity, starting with $n = 2$, there are examples of both nearly Fredholm (but not Fredholm) and not even nearly Fredholm equivalent operators $T_G, T_{\text{det} G}$. These examples can be easily constructed, based on the Fredholm criterion and index formula for Toeplitz operators with (matrix) piecewise continuous symbols, see e.g. [4, 11, 23].

Now suppose that

$$G = M_- G_0 M_+^{-1}, \quad (1.1)$$

where $M_\pm \in \mathcal{G}(H^\pm_\infty + C)^{n \times n}$. Then the Toeplitz operators $T_G$ and $T_{G_0}$ are nearly Fredholm equivalent [23, Theorem 5.5]. These two operators are strictly Fredholm equivalent if $M_\pm \in \mathcal{G}(H^\pm_\infty)^{n \times n}$ [23, 25].

In particular, if $G_0 = I$ in (1.1), we conclude from here that the operator $T_G$ is Fredholm whenever

$$GM_+ = M_- \quad (1.2)$$

for some $M_\pm \in \mathcal{G}(M_\pm_\infty)^{n \times n}$. We remark that, in this case, $\det G$ admits an analogous (scalar) representation

$$\det G = (\det M_-)(\det M_+)^{-1}, \quad (1.3)$$

and $T_{\text{det} G}$ is also Fredholm.

Note also that each of the $n$ columns of $M_+$, together with the corresponding column of $M_-$, due to (1.2) yield a solution to the Riemann-Hilbert problem

$$G \Phi_+ = \Phi_-, \quad \Phi_\pm \in (M_\pm_\infty)^n. \quad (1.4)$$

In the $2 \times 2$ case it was shown in [6] that only one solution to (1.4) is needed to conclude that $T_G$ is Fredholm equivalent to $T_{\text{det} G}$ — as it happens e.g. in the case of continuous symbols, — as long as $\Phi_\pm$ are corona-type pairs, left invertible over $M_\pm_\infty$. Moreover, the Fredholm equivalence is strict.
if $\Phi_{\pm}$ are left invertible over $H^\pm_\infty$, i.e., satisfy the corona condition in the corresponding half-planes $\mathbb{C}^\pm$. Thus it is possible to reduce the study of the Fredholm properties of Toeplitz operators with a $2 \times 2$ matrix symbol to the study of analogous properties for a Toeplitz operator with a scalar symbol. The following question is then natural to ask: is it possible to generalize the results of [6] to $n \times n$ matrix symbols if, instead of $n$ solutions to (1.4), we have $n - 1$ solutions satisfying some form of a left invertibility condition? Or if we have an $n \times n$ symbol whose elements are continuous on $\hat{\mathbb{R}}$ except for one column or a row?

Left invertibility of $n \times m$ ($m \leq n$) matrix functions over $H^\pm_\infty$ was studied in [18, Theorem 3.1] where a generalization of the Carleson corona theorem to the case of matrix valued analytic functions was obtained by reduction to the scalar corona theorem via determinants. There it was shown, in particular, that if the determinants of all $m \times m$ submatrices satisfy a (scalar) corona condition, then the $n \times m$ matrix function is left invertible over $H^\pm_\infty$. We may therefore ask: is this also a necessary condition? Can we deduce analogous results in the more abstract context of a unital commutative ring, allowing a broader range of applications, and obtain expressions for the left inverses in terms of the solutions to an associated scalar corona-type problem?

We now turn to a different question concerning Toeplitz operators. In the scalar case, they possess what is known as Coburn’s property, first observed in the Hilbert space (that is, for $p = 2$) setting in the proof of Theorem 4.1 in [12]: for any Toeplitz operator with non-zero symbol $g \in L_\infty(\mathbb{R})$, $T_g$ or $T^*_g$ has a zero kernel. It follows, in particular, that Fredholmness of $T_g$ implies its one-sided invertibility.

In this respect the situation is quite different when the symbol is matricial. The latter case presents much greater difficulties, some of which are naturally due to the non-commutativity of multiplication and the impossibility of division by vectorial functions. The degree of difficulty increases with the order of the matrix symbols involved, as reflected by the overwhelmingly greater number of results and papers concerning Toeplitz operators and related problems for $2 \times 2$ symbols, as compared with the general $n \times n$ case.

Coburn’s property is among many familiar properties holding in the scalar setting but not, in general, in the matricial setting, even in the simplest case when the symbol is diagonal. A natural question thus arises: what classes of Toeplitz operators with matricial symbols satisfy Coburn’s property?

In this paper we address and relate all these apparently different questions, taking an algebraic point of view that enables us to unify and tackle
different problems in different settings. This approach provides moreover a
good illustration of how the study of Toeplitz operators knits together dif-
ferent areas of mathematics such as operator theory, complex analysis, and
algebra.

The paper is organized in the following way. Section 2 contains one-sid ed
invertibility criteria for rectangular matrices with elements from an abstract
commutative ring $\mathcal{A}$, along with formulas for the respective inverses. In
Section 3 these results are recast for $\mathcal{A}$ being $H_\infty^\pm$ or $M_\infty^\pm$
with the help of corresponding corona theorems. These results are used in the main Section 5,
where conditions are established on matrix functions $G$ guaranteeing that $T_G$
and $T_{\det G}$ are (nearly or strongly) Fredholm equivalent. It is preceded by a
short Section 4 containing the necessary background information on the rela-
tions between Fredholmness of $T_G$ and factorization of $G$. Some special cases
(unitary or orthogonal matrices, and matrix functions continuous except for
one row or column) are considered in Section 6. Finally, in Section 7 we dea l
with almost periodic symbols $G$.

2. One sided invertibility of matrices over commutative rings

In this section, $\mathcal{A}$ is a unital commutative ring. We say that an element
$a \in \mathcal{A}^{n \times k}$, $k \leq n$, is \textit{left invertible over} $\mathcal{A}$ if there exists $b \in \mathcal{A}^{k \times n}$ such that
$ba = I_k$, the identity matrix in $\mathcal{A}^{k \times k}$. The notion of right invertibility over
$\mathcal{A}$ is introduced in a similar way. The treatment of (one sided) invertibil-
ity of square matrices with elements in $\mathcal{A}$ can be found in [22, Chapter I]. We
are interested in the case of $\mathcal{A}^{n \times k}$ with $k \neq n$.

For any matrix $\Phi \in \mathcal{A}^{n \times n}$ and $I \subset \{1, \ldots, n\}$, $\Phi_I$ will stand for its
submatrix obtained by keeping the $i$-th rows, $i \in I$ while deleting all other
rows. If $m \leq n$, label by $I_1, I_2, \ldots, I_N$ all $N = \binom{n}{m}$ subsets of $\{1, \ldots, n\}$ with
$m$ elements, and denote $\Phi^\Phi_{I_k} := \det \Phi_{I_k}$.

\textbf{Lemma 1.} Let $\Phi \in \mathcal{A}^{n \times m}$ with $m \leq n$, and let $\Phi_I$ be some $m \times m$
submatrix of $\Phi$. Denote by $\Delta^\Phi_{pq}$ the determinant of the matrix obtained from $\Phi_I$
by deleting its $p$-th row and $q$-th column. Define $\Phi^*_I \in \mathcal{A}^{m \times n}$ by setting its
$(q,p)$-entry according to the formula

$$
\Phi^*_{qp} = \begin{cases}
(-1)^{p+q} \Delta^\Phi_{pq} & \text{if } p \in I, \\
0 & \text{otherwise}.
\end{cases}
$$

(2.1)
Then
\[ \Phi^* \Phi = \det \Phi \diag[(-1)^q]_{q=1,\ldots,m}. \]

**Proof.** By construction,
\[ \sum_{p=1}^{m} \Phi^*_{qp} \Phi_{pq} = (-1)^q \det \Phi, \quad q = 1, \ldots, m, \]
while \( \sum_{p=1}^{m} \Phi^*_{qp} \Phi_{pl} = 0 \) for \( q \neq l \) as the determinant of a matrix with two coinciding columns. \( \square \)

**Theorem 2.** (i) An element \( \Phi \) of \( \mathcal{A}^{n \times m} \) is left invertible over \( \mathcal{A} \) if and only if the column
\[
\Delta := \begin{bmatrix}
\Phi^* & \\
\Phi_{1}^* & \\
\Phi_{2}^* & \\
\vdots & \\
\Phi_{N}^* & 
\end{bmatrix}
\]
is left invertible in \( \mathcal{A} \).

(ii) If \( \Phi \in \mathcal{A}^{n \times m} \) is left invertible over \( \mathcal{A} \) with left inverse \( \Psi \in \mathcal{A}^{m \times n} \), then the row
\[ \Delta^* = [d_{1}^{\Psi}, d_{2}^{\Psi}, \ldots, d_{N}^{\Psi}] \]
is a left inverse of \( \Delta \) over \( \mathcal{A} \).

(iii) If \( \Delta \) is left invertible in \( \mathcal{A} \) with a left inverse \( \Delta^* = [\Delta_{1}^*, \Delta_{2}^*, \ldots, \Delta_{N}^*] \), then
\[ \Psi = \diag[(-1)^q]_{q=1,\ldots,n} \sum_{k=1}^{N} \Delta_{k}^* \Phi^*_{Ik}, \]
where \( \Phi^*_{Ik} \) are defined in accordance with (2.1) with \( I = I_k \), is a left inverse of \( \Phi \).

**Proof.** (ii) If \( \Psi \Phi = I_m \), then the Cauchy-Binet formulas show that \( \Delta^* \Delta = 1 \).

(iii) We have
\[ \Psi \Phi = \diag[(-1)^q]_{q=1,\ldots,n} \sum_{k=1}^{N} \Delta_{k}^* \Phi^*_{Ik} \Phi \]
which, by Lemma 1, is equal to
\[ \diag[(-1)^q]_{q=1,\ldots,n} \sum_{k=1}^{N} (\Delta_{k}^* d_{Ik}^\Phi) \diag[(-1)^q]_{q=1,\ldots,n} = I_m. \]
(i) is an immediate consequence of (ii) and (iii).

The “if” part of Theorem 2 is an abstract version of its particular case when $\mathcal{A}$ is the algebra of bounded analytic functions on the unit disc contained in the proof of [18, Theorem 3.1]; the “only if” part shows moreover that the converse is true.

In what follows, we adapt the notation to the special case $m = n - 1$ which is of particular relevance to the main results of the paper.

Given $\Phi \in \mathcal{A}^{n \times (n - 1)}$ we denote by $\Delta_{p;\cdot}(\Phi)$ the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by omitting the row $p$ in $\Phi$; we denote by $\Delta_{p,s,j}(\Phi)$ the determinant of the $(n - 2) \times (n - 2)$ submatrix of $\Phi$ obtained by omitting the rows $p$ and $s$ ($p \neq s$) and column $j$ (we take $p, s \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n - 1\}$). Analogously, for $\Psi \in \mathcal{A}^{(n - 1) \times n}$, we use the notation $\Delta_{\cdot;p}(\Psi)$ for the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by omitting the column $p$ in $\Psi$; and $\Delta_{j;p,s}(\Phi)$ stands for the determinant of the $(n - 2) \times (n - 2)$ submatrix of $\Psi$ obtained by omitting the columns $p$ and $s$ ($p \neq s$) and row $j$.

**Corollary 1.** An element $\Phi \in \mathcal{A}^{n \times (n - 1)}$ is left invertible over $\mathcal{A}$ if and only if the column

$$\begin{bmatrix}
\Delta_{1;\cdot}(\Phi) \\
\vdots \\
\Delta_{n;\cdot}(\Phi)
\end{bmatrix}$$

is left invertible over $\mathcal{A}$.

Moreover, in this case a left inverse of $\Phi$ is given by

$$\Psi = \begin{bmatrix}
\Psi_1 \\
\Psi_2 \\
\vdots \\
\Psi_{n-1}
\end{bmatrix}, \quad \Psi_j \in \mathcal{A}^{1 \times n}, \quad j = 1, 2, \ldots, n - 1,$$

with

$$\Psi_j = (-1)^j \Delta^* \begin{bmatrix}
0 & \Delta_{1,2;j} & \Delta_{1,3;j} & \cdots & \Delta_{1,n;j} \\
-\Delta_{1,2;j} & 0 & \Delta_{2,3;j} & \cdots & \Delta_{2,n;j} \\
-\Delta_{1,3;j} & -\Delta_{2,3;j} & 0 & \cdots & \Delta_{3,n;j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\Delta_{1,n;j} & -\Delta_{2,n;j} & -\Delta_{3,n;j} & \cdots & 0
\end{bmatrix} \cdot \tilde{I}_n,$$
for \( j = 1, 2, \ldots, n - 1 \), where \( \Delta_{p,s,j} := \Delta_{p,s,j}(\Phi) \),
\[
\Delta^* = [\Delta_{1,:}(\Psi^T), \ldots, \Delta_{n,:}(\Psi^T)]
\]  
(2.5)
is a left inverse of \( (2.2) \) over \( \mathcal{A} \), and
\[
\tilde{I}_n = \text{diag}[1, -1, 1, \ldots, (-1)^{n+1}].
\]  
(2.6)

The following result will be crucial in establishing relations between left invertibility of some matrix functions and Fredholmness of Toeplitz operators.

**Theorem 3.** Let \( \Phi \in \mathcal{A}^{n \times (n-1)} \) be left invertible over \( \mathcal{A} \), and let \( \Psi \in \mathcal{A}^{(n-1) \times n} \) be its left inverse:
\[
\Psi \Phi = I_{n-1}.
\]  
(2.7)

Let moreover
\[
\Phi_e = [\Phi \ N], \quad \Psi_e = \begin{bmatrix} \Psi \\ \tilde{N} \end{bmatrix}, \quad \text{with} \quad N \in \mathcal{A}^{n \times 1}, \quad \tilde{N} \in \mathcal{A}^{1 \times n}.
\]  
(2.8)

Then:

(i) if
\[
N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}, \quad N_j = (-1)^{j-1} \Delta_{.,j}(\Psi),
\]  
then
\[
\Psi_e \Phi_e = \begin{bmatrix} I_{n-1} & 0_{(n-1) \times 1} \\ \tilde{N} \Phi & \tilde{N} \tilde{N} \end{bmatrix}.
\]  
(2.10)

(ii) if
\[
\tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 & \ldots & \tilde{N}_n \end{bmatrix} \quad \text{with} \quad \tilde{N}_j = (-1)^{j-1} \Delta_{.,j}(\Phi),
\]  
then
\[
\Psi_e \Phi_e = \begin{bmatrix} I_{n-1} & \Psi \tilde{N} \\ 0_{1 \times (n-1)} & \tilde{N} \tilde{N} \end{bmatrix}.
\]  
(2.12)
(iii) if $N$ and $\tilde{N}$ satisfy (2.9) and (2.11), respectively, then

$$\Psi_e \Phi_e = \Phi_e \Psi_e = I_n \quad \text{and} \quad \det \Phi_e = \det \Psi_e = (-1)^{n-1}. \quad (2.13)$$

Proof. (i) Since $\Psi$ is right invertible, $\Psi^T$ is left invertible. Therefore, by Corollary [1],

$$\left[ \begin{array}{c} \Delta_1,(\Psi^T) \\ \vdots \\ \Delta_n,(\Psi^T) \end{array} \right]$$

is left invertible over $\mathcal{A}$. Let $c_1, \ldots, c_n \in \mathcal{A}$ be such that $\sum_{j=1}^n c_j \Delta_{j;(\Psi^T)} = 1$. Then, setting

$$\Psi_1 = [c_1, -c_2, \ldots, (-1)^n c_n]$$

and using cofactor expansion across the first row, we see that $\det \Psi_0 = 1$, where

$$\Psi_0 := \left[ \begin{array}{c} \Psi_1 \\ \Psi \end{array} \right].$$

Thus, $\Psi_0$ is invertible, $N$ is the first column of $\Psi_0^{-1}$, and the equality $\Psi N = 0$ follows. Part (ii) is proved analogously. For Part (iii) note that the Cauchy-Binet formula yields

$$\sum_{j=1}^n \Delta_{j;(\Psi)} \Delta_{j;(\Phi)} = 1.$$ 

Thus, taking into account parts (i) and (ii), we have $\Phi_e \Psi_e = I_n$. Then also $\Psi_e \Phi_e = I_n$ (this is a general property of matrices with elements in unital commutative rings, see e.g. [2]). Finally, expanding $\det \Phi_e$ along the last column, we obtain

$$(-1)^{n-1} \det \Phi_e = \sum_{j=1}^n \Delta_{.j,(\Psi)} \Delta_{.j,(\Phi)}, \quad (2.14)$$

which is equal to the $(n, n)$ entry of the product $(\text{adj } \Phi_e) \cdot (\text{adj } \Psi_e)$, where we denote by $\text{adj } X \in \mathcal{A}^{n \times n}$ the algebraic adjoint (adjugate) of a matrix $X \in \mathcal{A}^{n \times n}$. Since $\Psi_e$ and $\Phi_e$ are inverses of each other, then so are $\text{adj } \Phi_e$ and $\text{adj } \Psi_e$, and (2.14) is equal to 1, as claimed. \hfill $\square$
3. Corona tuples and one sided invertibility in $H^\pm_\infty$ and $M^\pm_\infty$

Having the results of Section 2 in mind, we now establish necessary and sufficient conditions for the left invertibility of $n$-tuples in some concrete unital algebras of interest: $H^\pm_\infty$ and $M^\pm_\infty$.

The corona tuples, with respect to these algebras, are defined as follows:

$$HCT_n^\pm := \left\{ [h_1^\pm, h_2^\pm, \ldots, h_n^\pm] : h_j^\pm \in H^\pm_\infty \text{ and } \inf_{z \in \mathbb{C}} \left| \sum_{j=1}^{n} h_j^\pm(z) \right| > 0 \right\},$$

$$MCT_n^\pm := \left\{ [r_1 h_1^\pm, r_2 h_2^\pm, \ldots, r_n h_n^\pm] : [h_1^\pm, \ldots, h_n^\pm] \in HCT_n^\pm \text{ and } r_1, \ldots, r_n \in \mathcal{GR} \right\}.$$

**Theorem 4.** (a) Let $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in H^\pm_\infty$. Then $[h_1^\pm, h_2^\pm, \ldots, h_n^\pm] \in HCT_n^\pm$ if and only if

$$\begin{bmatrix}
  h_1^\pm \\
  \vdots \\
  h_n^\pm
\end{bmatrix}$$

is left invertible over $H^\pm_\infty$, i.e. there exist $g_j \in H^\pm_\infty$, $j = 1, 2, \ldots, n$, such that $\sum_{j=1}^{n} g_j h_j = 1$.

(b) The following statements are equivalent for $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in M^\pm_\infty$:

1. $[h_1^\pm, h_2^\pm, \ldots, h_n^\pm] \in MCT_n^\pm$;

2. There exist $r \in \mathcal{GR}$ and $[g_1, \ldots, g_n] \in HCT_n^\pm$ such that $h_j^\pm = r g_j$, $j = 1, 2, \ldots, n$;

3. $\begin{bmatrix}
  h_1^\pm \\
  \vdots \\
  h_n^\pm
\end{bmatrix}$ is left invertible over $M^\pm_\infty$.

Part (a) is the classical corona theorem, going back to Carleson [10]. When proving (b), the case $p = \infty$ of the following simple observation is needed:

$$M^\pm_p = \{ s \phi : s \in \mathcal{GR}, \phi \in H^\pm_p \}. \quad (3.1)$$

For a proof see [6, Proposition 2.3].

*Proof of Part (b).* We follow here the logic of [6, Theorem 2.6], where the case $n = 2$ was considered.

(1) implies (2): Let (1) hold, that is, $h_j = s_j \phi_j$, where $s_j \in \mathcal{GR}$ and $\{ \phi_1, \ldots, \phi_n \}$ is a corona $n$-tuple in $H^\pm_\infty$ (the case of $H^-_\infty$ can be treated along
the same lines). Denoting by \( \{z_1, \ldots, z_N\} \) the set of all zeros and poles of \( s_1, \ldots, s_n \) in \( \mathbb{C}^+ \) and by \( U_\epsilon \) its \( \epsilon \)-neighborhood, observe that \( h_1, \ldots, h_n \) are analytic, bounded, and satisfy the corona condition on \( \mathbb{C}^+ \setminus U_\epsilon \) for every \( \epsilon > 0 \). On the other hand, each of the functions \( h_i \) has either a zero or a pole at \( z_j, i = 1, \ldots, n; j = 1, \ldots, N \). Let \( \ell_j \) be the minimum of the orders of all \( h_i \) at the given \( z_j \) (recall that the order of \( \phi \) at \( z_0 \) is \( k \) (respectively, \( -k \)) if \( z_0 \) is a zero (respectively, pole) of \( \phi \) with multiplicity \( k \).) Introduce

\[
s(z) = (z + i)^{-\sum_{i=1}^{N} \ell_j} \prod_{j=1}^{N} (z - z_j)^{\ell_j}.
\]

Then the functions \( g_i = s^{-1}h_i \) are analytic, bounded and satisfy the corona condition on \( \mathbb{C}^+ \setminus U_\epsilon \) simultaneously with \( h_i \), because \( s \) is analytic, bounded and bounded away from zero on this set. Due to the choice of \( \ell_j \), we also have that all functions \( g_i \) are analytic on \( U_\epsilon \) and for each \( j \) at least one of them assumes a non-zero value at \( z_j \). Consequently, \( [g_1, \ldots, g_n] \in HCT_n^+ \).

(2) implies (3): Let \( h_j^+ = \chi g_j, j = 1, 2, \ldots, n, \) for some \( \chi \in \mathcal{GR} \) and \( [g_1, \ldots, g_n] \in HCT_n^+ \). Using part (a), we have \( \sum_{j=1}^{n} \ell_j g_j = 1 \) for some \( \ell_j \in H_\infty^+ \). Now \( \sum_{j=1}^{n} (\chi^{-1} \ell_j) h_j^+ = 1 \), where \( \chi^{-1} \ell_j \in M_\infty^+ \) by (3.1), and (3) holds.

(3) implies (1): We have \( \sum_{j=1}^{n} g_j h_j^+ = 1 \) for some \( g_j \in M_\infty^+ \). By (3.1), \( h_j^+ = r_j \phi_j \) and \( g_j = s_j \psi_j \) for some \( r_j, s_j \in \mathcal{GR} \) and \( \phi_j, \psi_j \in H_\infty^+ \). Consequently,

\[
\sum_{j=1}^{n} r_j s_j \phi_j \psi_j = 1 \text{ on } \mathbb{C}_\pm.
\]

Without loss of generality we may suppose that the functions \( \phi_j \) do not all vanish simultaneously at any point in some open set \( \Omega \subseteq \mathbb{C}_\pm \) containing all the poles of \( r_j s_j, j = 1, \ldots, n, \) in the upper half plane, since otherwise a respective rational factor could be moved from \( \phi_j \) to \( r_j \). Then the \( n \)-tuple \( [\phi_1, \ldots, \phi_n] \) satisfies the corona condition on \( \Omega \).

Since \( r_j s_j (j = 1, 2, \ldots, n) \) are bounded on \( \mathbb{C}_\pm \setminus \Omega \), the corona condition for \( (\phi_1, \ldots, \phi_n) \) follows from (3.2). Thus, \( [\phi_1, \ldots, \phi_n] \in HCT_n^+ \).

Note that (2) implies (1) in a trivial way.

Corollary 1 admits therefore the following interpretation.

**Theorem 5.** (a) Let \( \Phi \in (H_\infty^+)_{n \times (n-1)} \). Then \( \Phi \) is left invertible over \( H_\infty^+ \) if and only if \( [\Delta_1.(\Phi), \ldots, \Delta_n.(\Phi)] \in HCT_n^+ \).
(b) Let $\Phi \in (M^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $M^\pm_\infty$ if and only if $[\Delta_1(\Phi), \ldots, \Delta_n(\Phi)] \in MCT^\pm_n$. In both cases formula (2.3) applies, provided $\Phi$ is left invertible over the respective algebra.

According to [6, Theorem 2.7], $\Phi \in (M^\pm_\infty)^{2 \times 1}$ is left invertible in $M^\pm_\infty$ if and only if there exists a matrix function $R \in \mathcal{GR}^{2 \times 2}$ and $f^+ \in HCT^+_1$ such that $\Phi = Rf^+$. We here extend this result to include $\Phi \in (M^\pm_\infty)^{n \times (n-1)}$ with arbitrary $n \in \mathbb{N}$.

**Theorem 6.** Let $\Phi \in (M^\pm_\infty)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $M^\pm_\infty$ if and only if there exist $R \in \mathcal{GR}^{n \times n}$, $Q \in \mathcal{GR}^{(n-1) \times (n-1)}$ and $F \in (H^\pm_\infty)^{n \times (n-1)}$, the latter being left invertible over $H^\pm_\infty$, such that

$$\Phi = RFQ.$$  \hfill (3.3)

**Proof.** The sufficiency is obvious. When proving necessity, let us consider the case of invertibility over $M^+_\infty$; the case of $M^-_\infty$ can of course be treated in a similar way.

So, let $\Phi$ be left invertible over $M^+_\infty$. Denote by $\Phi_j$ the $j$-th column of $\Phi$:

$$\Phi = [\Phi_1 \Phi_2 \ldots \Phi_{n-1}], \quad \Phi_j \in (M^+_\infty)^{n \times 1}.$$  

Then $\Phi_j \in MCT^+_n$ and, by Proposition 4(c), there exist $\tilde{r}_j \in \mathcal{GR}$ and $\tilde{g}_j^+ \in HCT^+_n$ such that $\Phi_j = \tilde{r}_j \tilde{g}_j^+$. Thus,

$$\Phi = \tilde{G}^+Q,$$  \hfill (3.4)

$$Q = \text{diag}([\tilde{r}_1, \ldots, \tilde{r}_{n-1}],$$  

$$\tilde{G}^+ = [\tilde{g}_1^+ \tilde{g}_2^+ \ldots \tilde{g}_{n-1}^+] \in (H^+_\infty)^{n \times (n-1)}$$  \hfill (3.5)

and

$$\Delta_k(\tilde{G}^+) = \left(\prod_{j=1}^{n-1} r_j^{-1}\right) \Delta_{jk}(\Phi), \quad k = 1, 2, \ldots, n.$$  

On the other hand, by Theorem 4 (or 3(c))

$$\Delta(\Phi) := [\Delta_{11}(\Phi), \Delta_{12}(\Phi), \ldots, \Delta_{nn}(\Phi)] \in MCT^+_n$$  

so that, by Proposition 4(c), there exist $r \in \mathcal{GR}$ and $g^+ \in HCT^+_n$ such that

$$\Delta(\Phi) = rg^+.$$

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Therefore, if \( g_i^+ \in (H_\infty^+)^{1\times n} \) is the left inverse of \( g^+ \) over \( H_\infty^+ \), using the notation
\[
\Delta(\tilde{G}^+) := [\Delta_{;1}(\tilde{G}^+), \Delta_{;2}(\tilde{G}^+), \ldots, \Delta_{;n}(\tilde{G}^+)],
\]
we have
\[
g_i^+ \cdot \Delta(\tilde{G}^+) = \left( \sum_{j=1}^{n-1} r_j^{-1} \right) r g_i^+ g^+ = \left( \sum_{j=1}^{n-1} r_j^{-1} \right) r \in H_\infty^+ \cap \mathcal{G}\mathcal{R}. \tag{3.6}
\]

Let now, in the notation of (3.5),
\[
g_i^+ = [(g_i^+)_1, (g_i^+)_2, \ldots, (g_i^+)_n]^T \in (H_\infty^+)^{n \times 1}, \quad h_k^+ := (-1)^{k+1}(g_i^+)_k, \quad k = 1, 2, \ldots, n,
\]
and
\[
h^+ = [h_1^+, h_2^+, \ldots, h_n^+]^T \in (H_\infty^+)^{1 \times n}.
\]

Let moreover
\[
\tilde{M}^+ = [h^+ \tilde{g}_1^+ \tilde{g}_2^+ \ldots \tilde{g}_{n-1}^+] = [h^+ | \tilde{G}^+]. \tag{3.7}
\]
Then \( \tilde{M}^+ \in (H_\infty^+)^{n \times n} \) and
\[
\det \tilde{M}^+ = g_i^+ \Delta(\tilde{G}^+) \in H_\infty^+ \cap \mathcal{G}\mathcal{R}
\]
by (3.6). Following Theorem 3.4 in [9] or Lemma 2.1 in [23], we see that there exist \( R \in \mathcal{G}\mathcal{R}^{n \times n} \) and \( M^+ \in \mathcal{G}(H_\infty^+)^{n \times n} \) such that \( R^{-1} \tilde{M}^+ = M^+ \). From here and (3.7),
\[
R^{-1} \tilde{G}^+ = G^+, \tag{3.8}
\]
where \( G^+ \) is left invertible over \( H_\infty^+ \); its left inverse equals \((M^+)^{-1}\) with the first row deleted.

It follows from (3.4) and (3.8) that (3.3) holds, with \( F = G^+ \). \qed

**Remark 1.** The obvious analogues of Theorems 5 and 6 for right invertible matrices over \( \mathcal{A} \) are also valid. We will not explicitly state these analogues, but use them as needed in the sequel.
4. Fredholmness of Toeplitz operators and factorization

Let $L^p(\mathbb{R})$, $1 < p \leq \infty$, be the standard Lebesgue spaces of functions on the real line $\mathbb{R}$ with respect to the Lebesgue measure, while $H^p_\pm$ denote the Hardy spaces $H^p(\mathbb{C}^\pm)$ in the open upper (resp. lower) halfplane $\mathbb{C}^+$ (resp. $\mathbb{C}^-$). For $1 < p < \infty$, $H^p_\pm$ consists of all functions $f$ holomorphic in $\mathbb{C}^\pm$ for which

$$\sup_{\pm y > 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty.$$ 

This definition is standard in many sources, see, e.g., [16, 19, 21, 28] for basics on $H^p_\pm$ and associated singular integral operators.

For $p \in ]1, \infty[$, the space $L^p(\mathbb{R})$ splits into the direct sum of $H^p_+$ and $H^p_-$. We denote by $P^\pm$ the projection of $L^p(\mathbb{R})$ onto $H^p_\pm$ parallel to $H^p_-$. We will also need the modified projections $\tilde{P}^\pm$, acting from $L^\infty(\mathbb{R})$ into $L^p_\pm := (\xi \pm i)H^p_\pm$, $p \in ]1, \infty[$, by

$$\tilde{P}^\pm \phi = (\xi + i)P^\pm \left( \frac{\phi}{\xi + i} \right). \quad (4.1)$$

Toeplitz operators with matrix symbol $G \in (L^\infty(\mathbb{R}))^{n \times n}$ are defined as follows:

$$T_G : (H^p_+)^n \longrightarrow (H^p_+)^n, \quad T_G \phi^+ = P^+ G \phi^+ \quad (p \in ]1, +\infty[). \quad (4.2)$$

There is a close relation between properties of Toeplitz operators and factorization of their symbols. Thus, we remind now the basic definitions and properties concerning the latter.

Given $p \in (1, \infty)$, an $L^p$-factorization of a function $G \in (L^\infty(\mathbb{R}))^{n \times n}$ is defined as a representation

$$G = G_- D G_+, \quad (4.3)$$

where $D$ is a diagonal rational matrix of the form

$$D = \text{diag} \ (r^{k_j})_{j=1,2,\ldots,n}, \quad k_j \in \mathbb{Z} \text{ for all } j = 1, 2, \ldots n, \quad (4.4)$$

$$r(\xi) = \frac{\xi - i}{\xi + i}, \quad \text{for } \xi \in \mathbb{R}, \quad (4.5)$$

and the factors $G_\pm$ are such that, for

$$p' = \frac{p}{p - 1}, \quad \lambda_\pm(\xi) = \xi \pm i \ (\xi \in \mathbb{R}), \quad (4.6)$$
we have
\[ \lambda_+^{-1}G_+^{-1} \in (H_p^+)^{n \times n}, \quad \lambda_-^{-1}G_- \in (H_p^-)^{n \times n}. \] (4.7)
\[ \lambda_-^{-1}G_- \in (H_p^-)^{n \times n}, \quad \lambda_-^{1}G_-^{-1} \in (H_p^-)^{n \times n}. \] (4.8)

Under conditions (4.7), (4.8), \( G_-P^+G_-^{-1} \) can be considered as a closable operator on \((L_p(\mathbb{R}))^n\) defined on a dense linear set \( \lambda_+^{-1}G_+R^n \). If, in addition, \( G_-P^+G_-^{-1} \) is bounded in the metric of \((L_p(\mathbb{R}))^n\) (and therefore extends onto \((L_p(\mathbb{R}))^n\) by continuity), we say that (4.3) is a \( \text{Wiener-Hopf (WH) } p \)-factorization of \( G \).

For each \( p \), the diagonal middle factor in (4.3) is unique up to the order of its diagonal elements, and the integers \( k_j \) are called the partial indices of \( G \), its sum \( \text{Ind}_p(G) \) being the (total) \( p \)-index of \( G \). In the case of a scalar symbol possessing a \( WH \) \( p \)-factorization, the partial and the total indices coincide and will be simply called the \( p \)-index of \( G \).

The factorization (4.3) is said to be bounded if
\[ G_+ \in \mathcal{G}(R^+)^{n \times n}, \quad G_- \in \mathcal{G}(R^-)^{n \times n}. \] (4.10)

Clearly, a bounded factorization is a \( WH \) \( p \)-factorization for all \( p \in ]1, +\infty[ \).

Any matrix function in \( \mathcal{G}R^{n \times n} \) admits a factorization (4.3) with \( G_+ \in \mathcal{G}(R^+)^{n \times n} \), \( G_- \in \mathcal{G}(R^-)^{n \times n} \), where \( R^\pm := R \cap H^\pm \) is the subalgebra of \( R \) consisting of all rational functions without poles in \( \mathbb{C}^\pm \cup \{\infty\} \).

In particular, every scalar function in \( \mathcal{G}R \) is the product of functions in \( \mathcal{G}R^+, \mathcal{G}R^- \), and some integer power of the function \( r \) defined by (4.5). Thus, without loss of generality condition \( s \in \mathcal{G}R \) in (3.1) may be substituted by \( s = s_\pm r^j \), where \( s_\pm \in \mathcal{G}R^\pm \) and \( j \in \mathbb{Z} \).

The relation between Fredholm properties of \( T_G \) and factorization (4.3) is well known; see e.g. [25, Theorem 5.2]. For convenience of reference, we give the precise statement here (as it was done also in [1]).

**Theorem 7.** Let \( G \in (L_\infty(\mathbb{R}))^n \), \( p \in ]1, +\infty[ \). Then \( T_G \) is Fredholm on \((H_p^+)^n\) if and only if \( G \) admits a \( WH \) \( p \)-factorization.

The partial indices are related to the dimension of the kernel and the cokernel of \( T_G \) by
\[ \dim \ker T_G = \sum_{k_j \leq 0} |k_j|, \quad \dim \text{coker } T_G = \sum_{k_j \geq 0} k_j. \] (4.11)
Thus, the index of $T_G$, $\text{Ind } T_G$, is given by (see Theorem 7)

$$\text{Ind } T_G := \dim (\ker T_G) - \dim (\coker T_G) = -\text{Ind}_p G.$$ 

We see thus that the existence of a canonical $p$-factorization for $G$ is particularly interesting, since it is equivalent to invertibility for $T_G$. Moreover, the inverse operator can then be defined in terms of $G_\pm$ by

$$T_G^{-1} = G_+^{-1}P^+G_+^{-1}I.$$ (4.12)

5. One sided invertibility and Fredholness of Toeplitz operators

In this section we show that one-sided invertibility over the algebras $H^\pm_\infty + C$ or $H^\pm_\infty$ of certain submatrices of the $n \times n$ matrix function $G$ implies that the Toeplitz operators $T_G$ and $T_{\det G}$ are at least nearly, and in some cases strictly, Fredholm equivalent. In particular, in the latter case $T_G$ possesses Coburn’s property.

Given $\Phi^\pm \in (H^\pm_\infty + C)^{(n-1) \times n}$, $\Psi^\pm \in (H^\pm_\infty + C)^{(n-1) \times n}$ such that $\Psi^\pm \Phi^\pm = I_{n-1}$, let moreover $\Phi^\pm_e, \Psi^\pm_e$ be defined by

$$\Phi^\pm_e = [\Phi^\pm \ N^\pm], \quad \Psi^\pm_e = \left[ \begin{array}{c} \Psi^\pm \\ \tilde{N}^\pm \end{array} \right],$$ (5.1)

where

$$N^\pm = \begin{bmatrix} \Delta_{-1}(\Psi^\pm) \\ -\Delta_{-2}(\Psi^\pm) \\ \vdots \\ (-1)^{n-1}\Delta_{-n}(\Psi^\pm) \end{bmatrix} \in (H^\pm_\infty + C)^{n \times 1},$$ (5.2)

$$\tilde{N}^\pm = \begin{bmatrix} \Delta_1(\Phi^\pm), -\Delta_2(\Phi^\pm), \ldots, (-1)^{n-1}\Delta_n(\Phi^\pm) \end{bmatrix} \in (H^\pm_\infty + C)^{1 \times n}.$$(5.3)

Theorem 8. Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, and let $\Psi$ be an $(n-1) \times n$ submatrix of $G$ obtained by omitting one row in $G$.

(a) If $\Psi \in (H^\pm_\infty + C)^{(n-1) \times n}$, and if $\Psi$ is right invertible over $H^\pm_\infty + C$, then $T_G$ is nearly Fredholm equivalent to $T_{\det G}$, for every fixed $p \in ]1, \infty[$.

(b) If moreover $\Psi \in (H^\pm_\infty)^{(n-1) \times n}$, and if $\Psi$ is right invertible over $H^\pm_\infty$, then, for any fixed $p \in ]1, \infty[$, $\ker T_G = \{0\}$ or $\ker T^*_G = \{0\}$, and $T_G$ is strictly Fredholm equivalent to $T_{\det G}$. In particular, $T_G$ is one- or two- sided invertible simultaneously with $T_{\det G}$.
(c) If, in the setting of (b), in addition \( \text{Ind} T_{\det G} \geq 0 \) and the omitted row \( \hat{G}_n \) of \( G \) is its last one, then a WH \( p \)-factorization of \( G \) is given by \((4.3)\) with

\[
G_+ = \begin{bmatrix} I_{n-1} & 0 \\
\gamma_+ & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} I_{n-1} & 0 \\
0 & r^k \end{bmatrix}.
\]

Here

\[
\det G = \gamma_r k \gamma_+ \tag{5.7}
\]

is a WH \( p \)-factorization of \( \det G \) and \( \Phi^+ \) is a right inverse of \( \Psi \).

Note that in \((5.7)\) \( k \leq 0 \) since it is opposite to \( \text{Ind} T_{\det G} \). This condition is essential for the statement (c) to be valid, while of course omitting the \( n \)-th row (as opposed to some other row) is just to simplify the notation.

Proof. (a) Since \( \Psi \) is right invertible over \( H_+^{\infty} + C \), by Theorem 3 (taking \((2.13)\) into account) \( \Phi_e^+ \in \mathcal{G}(H_+^{\infty} + C)^{n \times n} \) and — see \((2.10)\), where \( G \) takes the place of \( \Psi_e^+ \) — we have

\[
G = \tilde{G}(\Phi_e^+)^{-1}, \quad \text{where} \quad \tilde{G} = \begin{bmatrix} I_{n-1} & 0_{(n-1) \times 1} \\
\hat{G}_n \Phi_e^+ & (-1)^{n-1} \det G \end{bmatrix}.
\]

By \([23, \text{Theorem 5.5}]\) (or rather its version for the right factorization) \( G \) is WH \( p \)-factorable only simultaneously with \( \tilde{G} \). In its turn, this matrix function is block triangular, with one of the blocks (the identity matrix) obviously WH \( p \)-factorable. According to \([23, \text{Corollary 4.1}]\), \( \tilde{G} \) itself is WH \( p \)-factorable only simultaneously with its other diagonal block, that is, the function \( \det G \). In the language of Toeplitz operators this means that \( T_G \) and \( T_{\det G} \) are nearly Fredholm equivalent.

(b) If moreover \( \Psi^+ \in (H_+^{\infty})^{(n-1) \times n} \) is right invertible over \( H_+^{\infty} \), then \( (\Phi_e^+)^{-1} \) (which is equal to \( \Psi_e^+ \) by Lemma 3) is invertible in \((H_+^{\infty})^{n \times n} \). Consequently, from \((5.8)\), \( \ker T_G = \{0\} \Leftrightarrow \ker T_{\tilde{G}} = \{0\} \Leftrightarrow \ker T_{\det G} = \{0\} \), and analogously \( \ker T_G^* = \ker T_{\tilde{G}^*} = \{0\} \Leftrightarrow \ker T_{\det G}^* = \ker T_{\det G}^* = \{0\} \). Since, by Coburn’s property, \( \ker T_{\det G}^* \) or \( \ker T_{\det G}^* \) is \( \{0\} \), the same is true.
regarding \( \ker T_G \) and \( \ker T_G^* \). On the other hand, since in (5.8) we have \((\Phi^e)^{\pm 1} \in (H^+)_{n \times n} \), not only \( G \) and \( \tilde{G} \) admit WH \( p \)-factorizations along with \( \det G \), but also their partial indices coincide. Due to the triangular structure of \( \tilde{G} \), the set of its partial indices is majorized by the set of the indices of its diagonal entries \([26]\), see also \([23, \text{Theorem 4.7}]\). Without going into details of the majorization relation and its properties, we note here only the following pertinent piece of information: since the indices of the diagonal entries of \( \tilde{G} \) are \( 0, \ldots, 0 \) \((n-1) \) times) and \( k \), all its partial indices are of the same sign as \( k \), and their sum amounts to \( k \). According to (4.11), the defect numbers of \( T_G \) are the same as those of \( T_{\det G} \). In particular, one of them is zero, and the other coincides in absolute value with \( \text{Ind} T_{\det G} \). This guarantees one sided invertibility (which becomes two sided if and only if \( k = 0 \)).

(c) Under the condition \( k \leq 0 \), the matrix functions \( G^{\pm} \) defined by (5.4)-(5.6) satisfy (4.7), (4.8). Since \( G \) is WH \( p \)-factorable and (4.3) holds, condition (4.9) is satisfied automatically \([23, \text{Theorem 3.8}]\), and (5.4)-(5.6) deliver the desired WH \( p \)-factorization. In particular, the partial indices of \( G \) are \( 0, \ldots, 0 \) \((n-1) \) times) and \( k \), so they coincide with the indices of the diagonal entries of \( \tilde{G} \).

The next result is a dual version of Theorem 8.

**Theorem 9.** Let \( G \in (L_\infty(\mathbb{R}))_{n \times n} \), and let \( \Phi \) be an \( n \times (n-1) \) submatrix of \( G \) obtained by omitting one column in \( G \) (it will be assumed that the \( n \)th column is omitted, essentially without loss of generality).

(a) If \( \Phi \in (H^{-}_\infty + C)^{n \times (n-1)} \), and if \( \Phi \) is left invertible over \( H^{-}_\infty + C \), then \( T_G \) and \( T_{\det G} \) are nearly Fredholm equivalent, for every fixed \( p \in ]1, \infty[ \).

(b) If moreover \( \Phi \in (H^{-}_\infty)^{n \times (n-1)} \) and \( \Phi \) is left invertible over \( H^{-}_\infty \), then, for any fixed \( p \in (1, \infty) \), \( \ker T_G = \{0\} \) or \( \ker T_G^* = \{0\} \), and the operator \( T_G \) is strictly Fredholm equivalent to \( T_{\det G} \). In particular, \( T_G \) is invertible if and only if so is \( T_{\det G} \).

(c) If, in the setting of (b), in addition \( \text{Ind} \det G \leq 0 \) and the omitted column of \( G \) is \( \tilde{G}_n \), its last one, then a WH \( p \)-factorization of \( G \) is given by (4.3) with

\[
G_- = \Phi^e \begin{bmatrix} I_{n-1} & r^{-k} \bar{P}_-(\gamma^*_+\Psi_- \tilde{G}_n) \\ 0 & (-1)^{n-1} \gamma_- \end{bmatrix}, \quad D = \begin{bmatrix} I_{n-1} & 0 \\ 0 & r^k \end{bmatrix},
\]

\[
G_+ = \begin{bmatrix} I_{n-1} & \gamma_+ \bar{P}_+(\gamma^*_+\Psi_- \tilde{G}_n) \\ 0 & \gamma_+ \end{bmatrix}.
\]
Here \( \det G = \gamma_-^{-1}k \gamma_+ \) is a WH \( p \)-factorization of \( \det G \), \( \Psi_- \) is a left inverse of \( \Phi \), and \( \Phi^-_e = (\Psi^-_e)^{-1} \) is given by (5.1)–(5.3).

Of course, formulas similar to those given in Theorem 9(c) hold when the removed column is not the last one.

In the previous results we have used the one sided invertibility of a submatrix of \( G \) to study the Fredholmness, and other associated properties, of the Toeplitz operator \( T_G \). Now we turn to the study of the same properties of \( T_G \) based on one sided invertibility of a solution to a Riemann-Hilbert problem with coefficient \( G \).

**Theorem 10.** Let \( G \in (L_{\infty}(\mathbb{R}))^{n\times n} \), and let

\[
G \Phi^+ = \Phi^-, \quad \Phi^\pm \in (H^\pm_{\infty} + C)^{n\times(n-1)},
\]

where \( \Phi^\pm \) are left invertible over \( H^\pm_{\infty} + C \). Then:

(i) \( T_G \) is nearly Fredholm equivalent to \( T_{\det G} \);

(ii) If moreover \( \Phi^\pm \) are left invertible over \( H^\pm_{\infty} \), with left inverses \( \Psi^\pm \in (H^\pm_{\infty})^{(n-1)\times n} \) for \( \Phi^\pm \), respectively, then \( \ker T_G = \{0\} \) or \( \ker T_G^* = \{0\} \), and \( T_G \) is strictly Fredholm equivalent to \( T_{\det G} \). In particular, \( T_G \) is invertible if and only if \( T_{\det G} \) is invertible.

(iii) Assuming that

\[
\det G = \gamma_-^{-1}k \gamma_+ \quad \text{with } k \geq 0
\]

is a WH \( p \)-factorization for \( \det G \), a WH \( p \)-factorization for \( G \) is given by (4.3) with

\[
G^- = \Phi^-_e \cdot \begin{bmatrix}
I_{n-1} & 0 \\
0 & \gamma_-
\end{bmatrix} \begin{bmatrix}
I_{n-1} & \alpha_- \\
0 & 1
\end{bmatrix},
\]

\[
\Psi^+_e \cdot \begin{bmatrix}
I_{n-1} & 0_{(n-1)\times1} \\
0_{1\times(n-1)} & 1
\end{bmatrix} \begin{bmatrix}
I_{n-1} & 0 \\
0 & \gamma_+
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
I_{n-1} & 0_{(n-1)\times1} \\
0_{1\times(n-1)} & r^k
\end{bmatrix}
\]

where \( \Phi^-_e, \Psi^+_e \) are given by (5.1)–(5.3),

\[
\alpha_+ = \tilde{D}^+(Q) \in (L^+_p)^{(n-1)\times1},
\]

and

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\[ \alpha_- = r^{-k} \bar{P}^-(Q) \in (L_p^-(n-1) \times 1), \]  
\[ \text{and where} \]  
\[ Q := \Psi^- G N^+ \in (L_\infty(\mathbb{R}))^{(n-1) \times 1}, \]  
\[ \text{with } N^+ \text{ as in (5.2)}. \]  

Proof. (i) Let \( \Phi_{e}^\pm, \Psi_{e}^\pm \) be defined as in (5.1), (5.3), where \( \Psi_{e}^\pm \in (H_{\infty}^\pm + C)^{(n-1) \times n} \) is a left inverse of \( \Phi_{e}^\pm \) over \( H_{\infty}^\pm \). From Theorem 3 it follows that \( \Psi_{e}^\pm \in G(H_{\infty}^\pm + C)^{n \times n} \) and \( (\Psi_{e}^\pm)^{-1} = \Phi_{e}^\pm. \)

Defining
\[ G_0 = \Psi_e^- G \Phi_e^+, \]  
we can rewrite (5.9) as
\[ G_0 \Phi_e^+ = \Psi_e^- \Phi^- . \]  
On the other hand, it also follows from Theorem 3 (see (2.12) or (2.13), taking (2.8) into account) that
\[ \Psi_e^\pm \Phi_e^\pm = \begin{bmatrix} I_{n-1} \\ 0_{1 \times (n-1)} \end{bmatrix}, \]  
therefore (5.18) implies that \( G_0 \) has the form
\[ G_0 = \begin{bmatrix} I_{n-1} & Q \\ 0_{1 \times (n-1)} & \det G \end{bmatrix}. \]  

In particular, \( \det G = \det G_0. \)

From (5.17) it follows according to [23, Theorem 5.5] that \( T_G \) is Fredholm if and only if \( T_{G_0} \) is Fredholm, and this in turn is equivalent to \( T_{\det G} \) being Fredholm (by (5.20)).

(ii) If \( \Phi_{e}^\pm \in (H_{\infty}^\pm)^{n \times (n-1)} \) and \( \Phi_{e}^\pm \) is left invertible over \( H_{\infty}^\pm \), with a left inverse \( \Psi_{e}^\pm \), then
\[ \Psi_{e}^- \in G((H_{\infty}^-)^{n \times n}), \quad \Phi_{e}^+ \in G((H_{\infty}^+)^{n \times n}), \]  
and it follows that \( T_G \) is strictly Fredholm equivalent to \( T_{\det G} \) and that \( \ker T_G = \{0\} \) or \( \ker T_G^* = \{0\} \) (see a similar reasoning in the proof of Theorem 8).

(iii) The formulas for \( G_{\pm} \) and \( D \) follow from
\[ G = \Phi_e^- G_0 \Psi_e^+, \]  

together with (5.20) and (5.10). \( \square \)
6. Special cases

Let \( G \in L_{\infty}^{n \times n} \) with all rows but one having elements in \( M_{\infty}^{+} \) (the case of all columns but one having elements in \( M_{\infty}^{-} \) can be treated analogously). Assume for simplicity that

\[
G = \begin{bmatrix} \Psi \\ g_n \end{bmatrix} \quad \text{with } \Psi \in (M_{\infty}^{+})^{(n-1)\times n}, \ g_n \in L_{\infty}^{1\times n}. \tag{6.1}
\]

Then the following results hold.

**Theorem 11.** (i) If \( G \) is unitary with constant determinant and \( g_n^T \in MCT_{-}^{n} \), then \( T_G \) is Fredholm for all \( p \in (1, \infty) \). If moreover \( \Psi \in (H_{\infty}^{+})^{(n-1)\times n} \) and \( g_n^T \in HCT_{-}^{n} \), then \( T_G \) is invertible.

(ii) If \( G \) is (complex) orthogonal with constant determinant and \( g_n^T \in MCT_{+}^{n} \), then \( T_G \) is Fredholm for all \( p \in (1, \infty) \). If moreover \( \Psi \in (H_{\infty}^{+})^{(n-1)\times n} \) and \( g_n^T \in HCT_{+}^{n} \), then \( T_G \) is invertible.

(iii) If one of the \((n-1)\times(n-1)\) minors of \( \Psi \) is invertible in \( M_{\infty}^{+} \), then \( T_G \) is nearly Fredholm equivalent to \( T_{\det G} \). If the above mentioned minor is in fact invertible in \( H_{\infty}^{+} \), then \( T_G \) is strictly Fredholm equivalent to \( T_{\det G} \) and \( \ker T_G = \{0\} \) or \( \ker T_G^* = \{0\} \).

Of course \(|\det G| = 1\) in case (i) and \(\det G = \pm 1\) in case (ii).

**Proof.** (i) Observe that the \(k\)-th entry \( g_{nk} \) of \( g_n \) coincides with \((-1)^n k \det G \Delta_{,k}(\Psi)\), \( k = 1, \ldots, n \). Thus, condition \( g_n^T \in MCT_{-}^{n} \) can be rewritten equivalently as

\[
(\Delta_{,k}(\Psi))_{k=1,\ldots,n} \in MCT_{+}^{n}. \tag{6.2}
\]

By the right invertibility analogue of Theorem 3 it follows that \( \Psi \) is right invertible over \( M_{\infty}^{+} \), and Theorem 8(a) implies that \( T_G \) is Fredholm. The second part of (i) follows analogously from Theorem 8(b).

(ii) If \( G \) is orthogonal, then \( g_{nk} = (-1)^{n+k} \det G \Delta_{,k}(\Psi) \), so that now \( g_n^T \in MCT_{+}^{n} \) can be rewritten as (6.2). The rest of the proof goes as in (i).

(iii) The invertibility of any \((n-1)\times(n-1)\) minor of \( \Psi \in (M_{\infty}^{+})^{(n-1)\times n} \) implies (6.2). So, Theorems 5 and 8 again do the job. \(\square\)

**Remark 2.** In the case of orthogonal (6.1) we automatically have \( g_n \in (M_{\infty}^{+})^{1\times n} \), so that the relation \( G G^T = I \) immediately provides the right inverse of \( \Psi \) over \( M_{\infty}^{+} \). It can then be used in factorization formulas (5.4) – (5.6) of Theorem 8(c).
A factorization of unitary matrices $G$ with $\det G = 1$, $\Psi \in (H^+_{\infty})^{(n-1) \times n}$ and $g^T_n \in HCT_n^{-}$ as in Theorem 11, was by different methods considered earlier in [17].

We will now show that the one sided invertibility requirement in part (a) of Theorems 8, 9 can be lifted if the submatrix in question is continuous. First we will dispose of the case when it is rational.

**Lemma 12.** Let $G$ be of the form (6.1) with $\Psi \in \mathcal{R}^{(n-1) \times n}$. Then $T_G$ is nearly Fredholm equivalent to $T_{\det G}$.

**Proof.** If the determinants $\Delta_{.,k}(\Psi)$, $k = 1, \ldots, n$ (which are rational functions in $\mathcal{R}$) have at least one common zero in $\dot{\mathbb{R}}$, then $\det G$ has the same zero and thus neither $T_{\det G}$ nor $T_G$ is Fredholm.

Suppose now there are no common zeros of $\Delta_{.,k}(\Psi)$ in $\dot{\mathbb{R}}$. Since there are at most finitely many such zeros in $\mathbb{C}^\pm$, then (6.2) holds again. By Theorems 5, $\Psi$ is right invertible over $M^+_{\infty}$. The statement now follows from Theorem 8. \qed

**Theorem 13.** Let $G \in L_{\infty}^{n \times n}$ be such that all its elements except maybe for those located in one row or one column are continuous on $\dot{\mathbb{R}}$. Then $T_G$ is nearly Fredholm equivalent to $T_{\det G}$.

**Proof.** Without loss of generality, $G$ is of the form (6.1) with $\Psi \in C^{(n-1) \times n}$.

**Necessity.** Suppose $T_G$ is Fredholm. Then $\det G$ is invertible in $L_{\infty}$. Expanding $\det G$ across the last row, represent it as

$$\det G = \sum_{j=1}^{n} f_j g_{n,j},$$

where the cofactors $f_j$ are continuous due to the continuity of $\Psi$. Let us approximate $\Psi$ by a rational matrix function $\tilde{\Psi}$ so closely that the Toeplitz operator with the modified symbol $G_1 = \left[ \begin{array}{c} \tilde{\Psi} \\ g_n \end{array} \right]$ remains Fredholm. In particular, $\det G_1 = \sum_{j=1}^{n} \tilde{f}_j g_{n,j}$ is still invertible.

Now let

$$\tilde{g}_{n,j} = g_{n,j} \det G / \det G_1, \quad \tilde{g}_n = [\tilde{g}_{n,1} \ldots \tilde{g}_{n,n}] \quad \text{and} \quad \tilde{G} = \left[ \begin{array}{c} \tilde{\Psi} \\ \tilde{g}_n \end{array} \right].$$

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The matrix function $\tilde{G}$ can be made arbitrarily close to $G$, so that we may suppose $T\tilde{G}$ to be Fredholm. By Lemma 12 the operator $T_{\det \tilde{G}}$ is Fredholm. It remains to observe that

$$\det \tilde{G} = \sum_{j=1}^{n} \tilde{f}_j \tilde{g}_{n,j} = \det G.$$  

Sufficiency. Along with $T_G$, let us consider $T_{\text{adj} G}$, where $\text{adj} G$ stands for the transposed matrix of the cofactors of $G$. Recall that

$$G \text{ adj} G = \text{ adj} GG = (\det G) I_n, \quad (6.3)$$

and let $I_+$, $I_+^n$ denote the identity operators on $H_p^+$, $(H_p^+)^n$, respectively. Since the first $n - 1$ rows of $G$ and the last column of $\text{adj} G$ are continuous on $\mathbb{R} \cup \{\infty\}$, the operator

$$k_\ell := T_{\text{adj} G} T_G - T_G \text{ adj} G$$

is compact (Corollary 3.5 in [25]). Taking (6.3) into account, we conclude that

$$T_{\text{adj} G} T_G = (\det G) I_+^n + k_\ell$$

is Fredholm and therefore $T_G$ has a left regularizer (that is, a left inverse modulo the ideal of compact operators).

To show that $T_G$ has also a right regularizer — and therefore $T_G$ is Fredholm — we consider $T_G T_{\text{adj} G}$. In this case, the difference $T_G T_{\text{adj} G} - T_{G \text{ adj} G}$ may not be compact, so we have to use different (and somewhat more involved) arguments. Let $[T_{ij}]$, $(i, j \in \{1, 2\})$, be the block representation of the operator

$$T_G T_{\text{adj} G} - T_{G \text{ adj} G} = T_G T_{\text{adj} G} - (\det G) I_+^n,$$

corresponding to the decomposition $(H_p^+)^n = (H_p^+)^{n-1} \oplus H_p^+$. The operators $T_{11}$, $T_{12}$, and $T_{22}$ are compact (by [25, Corollary 7.5]), and we can write

$$T_G T_{\text{adj} G} = (\det G) I_+^n + \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} (\det G) I_+ & 0 \\ T_{21} & (\det G) I_+ \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}.$$

Thus $T_G T_{\text{adj} G}$ is a compact perturbation of a block triangular operator which is Fredholm since its diagonal elements are Fredholm (see, e.g., Corollary 1.3 in [25]). Thus $T_G T_{\text{adj} G}$ is Fredholm which implies that $T_G$ has a right regularizer as well.

$\square$
Note that some relations between semi-Fredholmness of $T_G$ and $T_{\det G}$ in the setting of Theorem 13 can be extracted from Markus-Feldman results ([24], see also [22, Chapter 1]) on the one sided invertibility of matrices over some non-commutative ring but with entries from different rows or columns pairwise commuting.

7. Toeplitz operators with almost periodic symbols

The difference between the scalar and matrix settings becomes even more profound for Toeplitz operators with almost periodic symbols. To define the latter, first we introduce $APP$, the (non-closed) algebra of *almost periodic polynomials*, that is, linear combinations of the functions $e_\lambda(t) := e^{i\lambda t}$, $\lambda \in \mathbb{R}$.

The Banach algebra $AP$ of almost periodic functions by definition is the closure of $APP$ in $L^\infty(\mathbb{R})$. We will also need $APW$, the closure of $APP$ in the Wiener norm

$$\| \sum c_je_{\lambda_j} \| = \sum |c_j|,$$

with no repetitions in the set $\{ \lambda_j \}$.

Let further $AP^\pm$ ($APW^\pm$) denote the closure in $AP$ (respectively, $APW$) of all $f = \sum c_je_{\lambda_j} \in APP$ with $\pm \lambda_j \geq 0$. Note that $AP^\pm$ (respectively, $APW^\pm$) consist of all functions $f \in AP$ (respectively, $APW$) that admit holomorphic continuation into $\mathbb{C}^\pm$. Of course, $AP^\pm$ and $APW^\pm$ are unital Banach subalgebras of $AP$ and $APW$, respectively.

For any $f \in GAP$ there exists a unique $\kappa \in \mathbb{R}$ such that a continuous branch of $\log(e^{-\kappa}f)$ lies in $AP$. This $\kappa$ is called the *mean motion* of $f$, and is sometimes denoted $\kappa(f)$.

Operators $T_f$ with scalar $f \in AP$ were treated by Coburn-Douglas [13] and Gohberg-Feldman [20], and the situation with them is as follows: the operator $T_f$ is semi-Fredholm if $f \in GAP$ and has a non-closed range otherwise. Moreover, for $f \in GAP$ with $\kappa(f) = 0$, $T_f$ is invertible while in the case of non-zero $\kappa(f)$ one of its defect numbers is infinite. In particular, $T_f$ is Fredholm only if it is invertible.

The latter property persists for matrix $AP$ symbols, see [3, Chapter 18]. However, it is no longer true that the invertibility of $G \in AP^{n \times n}$, or even $APW^{n \times n}$, implies the semi Fredholmness of $T_G$. Moreover, there exist

$$G = \begin{bmatrix} e^{-\lambda} & 0 \\ f & e_{\lambda} \end{bmatrix} \quad (7.1)$$
(so that \( \det G \equiv 1 \)) with \( \lambda > 0 \) and \( f \in APP \) for which the range of \( T_G \) is not closed \([27]\).

To describe the situation further, we introduce the notion of \( AP \) and \( APW \) factorization.

Representation \((4.3)\) in which \( G_\pm \) satisfy
\[
G_+ \in \mathcal{G}(AP^+)_{n \times n}, \quad G_- \in \mathcal{G}(AP^-)_{n \times n}
\]
and the diagonal elements of \( D \) have the form \( e^\mu_j \), as opposed to \((4.4)\), is called a (right) \( AP \) factorization of \( G \). An \( AP \) factorization of \( G \) is by definition its \( APW \) factorization if conditions \((7.2)\) are strengthened to
\[
G_+ \in \mathcal{G}(APW^+)_{n \times n}, \quad G_- \in \mathcal{G}(APW^-)_{n \times n}.
\]
The real parameters \( \mu_j \) are defined uniquely, provided that an \( AP \) (or \( APW \)) factorization of \( G \) exists, and are called its \emph{partial \( AP \) indices}. Of course, a canonical (that is, satisfying \( \mu_1 = \ldots = \mu_n = 0 \)) \( AP \) factorization of \( G \) is at the same time a bounded canonical factorization.

In line with Theorem 7 (though requiring a rather involved independent proof), Toeplitz operators \( T_G \) with \( G \in APW_{n \times n} \) are invertible if and only if \( G \) admits a canonical \( AP \) (equivalently, \( APW \)) factorization \([3, \text{Section 9.4}]\). However, the necessary and sufficient conditions for \( AP \) factorization, canonical or not, to exist are presently not known. The question is open even for already mentioned triangular \( 2 \times 2 \) matrix functions \((7.1)\). Quite a few partial results were obtained in this direction, showing that the problem is indeed intriguing and complicated. An interested reader may consult \([3]\) for a coherent description of the state of affairs as of about ten years ago, and \([5, 7, 8]\) for some more current results.

Because of these reasons, statements relating the Fredholm properties of \( T_G \) and \( T_{\det G} \), as well as factorization formulas for \( G \), are of special interest in the \( AP \) setting.

The \( AP \) version of Theorem 8 is as follows.

**Theorem 14.** Let \( G \in AP_{n \times n} \) be invertible, and suppose that it contains a submatrix \( \Psi \in (AP^+)_{(n-1) \times n} \) which is right invertible over \( AP^+ \). Then the operator \( T_G \) is invertible (resp. right invertible, or left invertible) on \( (H_p^+)_{n} \) for any (equivalently, all) \( p \in (1, \infty) \) if and only if \( \det G \) has zero (resp. non-positive, or non-negative) mean motion \( \kappa \). If in addition \( G \in APW_{n \times n} \) and \( \kappa \geq 0 \), then \( G \) is \( APW \) factorable, and its partial \( AP \) indices are \( 0, \ldots, 0 \) \((n - 1 \text{ times})\) and \( \kappa \).
The proof runs along the same lines as that of Theorem 8, taking into consideration that \( \det G \) is an invertible AP function and thus the operator \( T_{\det G} \) is automatically one sided invertible. To construct the APW factorization, one can still use formulas (5.4)–(5.6) substituting \( r_k \) by \( e_{\kappa(\det G)} \) and \( \tilde{P}^\pm \) by the projections of APW onto \( APW^\pm \).

The analogue of Theorem 10 also holds.

**Theorem 15.** Let \( G \in APW^{n \times n} \) be invertible, with \( \kappa(\det G) \geq 0 \). Moreover, let there exist \( \Phi^\pm \in (APW^\pm)^{n-1 \times n} \) left invertible over \( APW^\pm \) and such that \( G\Phi^+ = \Phi^- \). Then \( G \) is APW factorable, with the partial AP indices equal \( 0, \ldots, 0 \) \((n - 1 \text{ times})\) and \( \kappa(\det G) \).

To state the analogue of Theorem 11, let us introduce the notion of the AP corona tuple as

\[
APCT_n^\pm := \left\{ [h_1^\pm, h_2^\pm, \ldots, h_n^\pm] : h_j^\pm \in AP^\pm \quad \text{and} \quad \inf_{z \in \mathbb{C}^\pm} \left( \sum_{j=1}^{n} |h_j^\pm(z)| \right) > 0 \right\}.
\]

The almost periodic version of the corona theorem, in principle contained already in [1] and stated explicitly in [29], reads:

Let \( h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in AP^\pm \). Then \( [h_1^\pm, h_2^\pm, \ldots, h_n^\pm] \in APCT_n^\pm \) if and only if

\[
\begin{bmatrix}
  h_1^\pm \\
  \vdots \\
  h_n^\pm
\end{bmatrix}
\]

is left invertible over \( AP^\pm \).

Consequently, the AP analogue of Theorem 5 holds.

**Theorem 16.** Let \( \Phi \in (AP^\pm)^{n \times (n-1)} \). Then \( \Phi \) is left invertible over \( AP^\pm \) if and only if \( [\Delta_{1,\cdots,n}(\Phi)] \in APCT_n^\pm \).

Let now \( G \in L_{\infty}^{n \times n} \) with all rows but one having elements in \( AP^+ \) (the case of all columns but one having elements in \( AP^- \) can be treated analogously). Assume for simplicity that

\[
G = \begin{bmatrix} \Psi \\ g_n \end{bmatrix} \quad \text{with} \quad \Psi \in (AP^+)^{(n-1) \times n}, \quad g_n \in L_{\infty}^{1 \times n}.
\]

Invoking Theorem 16 we immediately obtain

**Theorem 17.** (i) If \( G \) is unitary with constant determinant and \( g_n^T \in APCT_n^- \), then \( T_G \) is invertible for all \( p \in (1, \infty) \).

(ii) If \( G \) is (complex) orthogonal with constant determinant and \( g_n^T \in APCT_n^+ \), then \( T_G \) is invertible for all \( p \in (1, \infty) \).
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