Self-oscillating geometric pump in a dissipative atom-cavity system

Davide Dreon, Alexander Baumgärtner, Xiangliang Li, Simon Hertlein, Tilman Esslinger and Tobias Donner
Institute for Quantum Electronics, Eidgenössische Technische Hochschule Zürich, Otto-Stern-Weg 1, 8093 Zurich, Switzerland
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Pumps are transport mechanisms in which direct currents result from a cyclic evolution of the potential [1, 2]. As Thouless has shown, the pumping process can have topological origins, when considering the motion of quantum particles in spatially and temporally periodic potentials [3]. However, the periodic evolution that drives these pumps has always been assumed to be imparted from outside, as was the case in the experimental systems studied so far [4–12]. Here we report on an emergent mechanism for geometric pumping in a quantum gas coupled to an optical resonator, where we observe a particle current without applying a periodic drive. The pumping potential experienced by the atoms is formed by the self-consistent cavity field interfering with the static laser field driving the atoms. Due to dissipation, the cavity field evolves between its two quadratures [13], each corresponding to a different centrosymmetric crystal configuration [14]. This self-oscillation results in a time-periodic potential analogous to that in a Rice-Mele pump [15]. In the experiment, we directly follow the evolution by measuring the phase winding of the cavity field with respect to the driving field and observing the atomic motion in-situ. The discovered mechanism combines the dynamics of topological and open systems.

Models for geometric pumping show singularities in their parameter space, which lead to pumping as soon as these are encircled [1, 10], with the transported charge equaling the winding number for insulating systems. Such singularities appear in lattices with bipartite unit cells, as in the Rice-Mele model. This is the basis for topological pumps realized in cold atoms, where a one-dimensional optical lattice is overlaid by a second lattice with twice the spatial period [9, 10, 17–19]. The latter is then slowly moved relative to the former along the joint lattice axis. The pumping follows a cyclic evolution after which the potential landscape resembles itself with the centre of the Wannier functions being shifted by a unit vector.

The location of the Wannier centre in the unit cell defines the polarisation of the system, which is the real space manifestation of the band topology [20–22]. The pumping current can thus be understood as the periodic evolution of the polarisation due to the motion of the Wannier centre [21]. In the case of a centrosymmetric crystal, the Wannier center may either sit on or in-between the inversion center, defining two different $Z_2$ classes. When evolving continuously between the different centrosymmetric classes without closing the band gap, a current must occur. This argument can be extended to pumping mechanism in two-dimensional environments [23].

![FIG. 1. Non-stationary lattice in a dissipative atom-cavity system.](image)

a. A synthetic crystal is realised by a $^{87}\text{Rb}$ BEC self-organizing in an optical cavity. Photons scatter in the cavity from a pair of imbalanced transverse beams (with Rabi rates $\Omega_{1,2}$) and leak from the cavity at a rate $\kappa$. A heterodyne setup records in real time the coherent cavity field $\alpha = |\alpha|e^{i\phi}$. b. The interfering light fields create an optical lattice potential that can be decomposed into a component with a short lattice period ($\lambda/2$) and a large period ($\lambda$), with $\lambda$ being the wavelength of light. The spatial position of the long period lattice is determined by the time phase $\phi$ of the scattered field. c. A finite photon number in the real ($Q$) or imaginary ($P$) quadrature of the cavity field breaks each a $Z_2$ symmetry, corresponding to different centrosymmetric crystals in real space. In absence of dissipation, increasing the coupling strength results in a structural phase transition between the symmetry broken states. A strong dissipative coupling mixes the optical quadratures, and a third regime appears in between these states, where the cavity field undergoes persistent oscillations. d. In this regime, the chiral evolution of $\phi$ drags the atoms through the $Z_2$ crystal configurations. The dashed rectangle on top of the lattice potentials highlights the unit cell. After a full revolution, the wavefunction’s centre of mass (yellow crosses) has been shifted by one unit cell, closing a pump cycle.

*esslinger@phys.ethz.ch
Emergent dynamics of the intracavity field. a, Time evolution of the intracavity field phase $\phi$, recorded for at $|\Delta_c/\kappa| = 6.94(2)$. The coupling $V_0$ ramps linearly from 12(1) to 49(5) $E_r$. The insets show the calculated real space potentials associated to a purely real or imaginary steady-state field quadrature ($Q$ and $P$, respectively). b, Same as a, but for different dissipative coupling strengths. From top to bottom $|\Delta_c/\kappa| = 8.50(2), 5.99(2), 5.31(2)$. c, Phase diagram. Each horizontal line shows the mean phase $\bar{\phi}$ obtained from averaging over 50 experimental runs. The polar angle is folded into the first quadrant, $\phi \in [0, \pi/2]$, such that an increasing phase appears as $\pi/4$ on average. d, Polar plot of the cavity field $\alpha$ in quadrature space, same data as in a. The radii indicate the modulus of $|\alpha|$ and the colormap follows the time axis.

In this work, we show that geometric pumping can emerge self-consistently without applying any external time-dependent drive. We demonstrate this employing a Bose-Einstein condensate coupled to a dissipative cavity, where the dynamics is governed by the self-consistent interplay between matter and light-fields. The geometric pumping process is set in motion by dissipation after crossing a dynamical instability point of a self-organized phase, which marks a transition from a steady-state to a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24]. Coupling internal degrees of freedom to a cavity, a self-oscillation between density and a self-oscillating state [24].

In our experimental setup, a Bose-Einstein condensate (BEC) of $^{87}$Rb is placed within an optical resonator. We illuminate the atoms with a pair of counter-propagating laser beams, having different Rabi rates $\Omega_{1,2}$, and that intersect the cavity mode at a 60° angle (Fig. 1a). Collective light scattering from the transverse beams into the cavity is accompanied by a spatial symmetry breaking in the BEC [27] [28], which forms a density wave by self-organising in the optical lattice generated by the interference of the cavity field with the transverse beams. The resulting potential $V_{\text{lattice}} = V_{\text{short}} + V_{\text{long}}(\phi)$ consists of a short-spaced two-dimensional lattice $V_{\text{short}}$, fixed in space, and a lattice $V_{\text{long}}$ with doubled periodicity, whose position depends on the time phase $\phi$ between the transverse beam and the cavity field (Fig. 1b). The total lattice has two $\mathbb{Z}_2$ symmetry classes, which are set by $\phi$: for 0 or $\pi$ the scattered field is in-phase with the coupling beams and for $\pm \pi/2$ it out of phase, i.e. the atoms couple either to the real or the imaginary quadrature of the cavity field, respectively (Fig. 1c). Hence, the quadratures $Q$ and $P$ of the cavity field directly reflect the unit cell polarisation of the lattice [14]. Which of the two symmetry classes is realized can be controlled by the strength of the transverse beams.

Yet, in the open system the steady state response of the cavity field results in a phase shift $\phi_c = \tan^{-1}(\Delta_c/\kappa)$, that depends on the ratio between the detuning of the laser frequency from cavity resonance, $\Delta_c$, and the photon dissipation rate $\kappa$, as expected for an oscillator driven close to resonance. This phase shift leads to a local distortion of the effective potential minima and the equilibrium position of the atoms is shifted with respect to the situation with $\Delta_c \gg \kappa$. This distortion reflects the dissipative coupling between the cavity quadratures. Sufficiently strong dissipative coupling eventually drives a dynamical instability [13] [29] [30], if the system is prepared in vicinity to the transition point between the two $\mathbb{Z}_2$ symmetry classes. In our case, the system is dragged in a chiral motion in quadrature space around a singularity and simultaneously through the $\mathbb{Z}_2$ classes of the crystals.

We detect the emergence of this oscillation in the real-time recording the quadratures of the cavity field. The cavity detuning $\Delta_c$ and the power of the transverse beams, measured in terms of the standing wave potential depth $V_0$, are used as experimental control parameters. In our experimental protocol we increase the power of the transverse beams linearly in time while keeping the detuning $\Delta_c$ (i.e. $\delta_c$) at a fixed value. Fig. 2a shows a trace of a heterodyne measurement of the cavity field phase $\phi$. At the beginning of the ramp, the system is strongly coupled to the real quadrature $Q$, while the imaginary quadrature $P$ is prevailing at the end. In these limiting cases, the phase takes on a defined steady-state value that is determined by the dominant coupling. In between these steady states, the dynamical instability is entered,
and we observe a monotonic increase of the phase $\phi$ in a step-like fashion. This region of self-oscillation widens and the windings increases as we repeat the experiment at progressively smaller $|\Delta_c|$. (Fig. 2) This is consistent with dissipative effects becoming stronger as $\Delta_c/\kappa \to 0$. In Fig. 2, we have merged the data obtained for a wide range of detunings $\Delta_c$, to a phase diagram. The heterodyne data plotted in optical quadrature space (Fig. 2b), highlights the chiral dynamics of the intracavity coherent field $\alpha$. The ellipsoidal shape, as much as the step-like behaviour of the time evolution of $\phi$, is a direct consequence of the self-consistency, as we will detail in the following.

An explicit connection between quadratures and crystal structures comes from a microscopic picture of the optical potentials. The short period lattice is generated by the standing waves

$$V_{\text{short}} = V_0 \cos^2(k_x r) + hU_0 \hat{a}^\dagger \hat{a} \cos^2(k_c r),$$

where the momenta $k_x$ and $k_c$ (with $|k| = k = 2\pi/\lambda$) are oriented along the transverse coupling beams and the cavity axis, respectively. The period of the potential is set by the wavelength $\lambda = 780$ nm of the light, which we detune by $\Delta_\alpha = +2\pi \times 70.3(1)$ GHz with respect to the $D_2$ transition frequency of $^{87}$Rb. At this detuning, spontaneous scattering is largely suppressed and the potential depths can be approximated by off-resonant two-photon scattering, yielding $V_0 = \hbar \Omega_1 \Omega_2 / \Delta_\alpha$ for the lattice formed by the coupling beams and $\hat{a}^\dagger \hat{a} U_0 = \hat{a}^\dagger \hat{a} \eta^2 / \Delta_\alpha$ for the intra-cavity lattice. The latter is proportional to the intra-cavity photon number $\hat{a}^\dagger \hat{a}$, with $\hat{a}$ ($\hat{a}^\dagger$) the photon annihilation (creation) operator. Here, $\eta$ is the single photon vacuum Rabi rate of the cavity. The interference lattice is given by

$$V_{\text{long}} = \hbar \eta_0 \hat{q} \cos(k_x r) \cos(k_c r) + \hbar \eta_p \sin(k_x r) \cos(k_c r),$$

where $\eta_0,\eta_p = g(\Omega_1 \pm \Omega_2) / \Delta_\alpha$ are the two-photon scattering rates between the transverse beams and the cavity mode. The operators of the quadratures of the cavity field are given by $\hat{q} = (\hat{a} + \hat{a}^\dagger) / 2$ and $\hat{p} = (\hat{a} - \hat{a}^\dagger) / 2i$. A finite cavity field $\langle \hat{a} \rangle \neq 0$ is the signature of a self-organised phase [27], in which the atomic density is periodically modulated in space, such that the Bragg condition for the scattered light is fulfilled. This density wave breaks the discrete $\mathbb{Z}_2$ symmetry of the Hamiltonian [22], which is invariant under the transformation $(\hat{a}, x) \rightarrow (\hat{a} e^{i\pi}, x + \lambda)$. The phase transition happens above a critical coupling strength of $\Omega_{1,2}$.

From Eq. 2, it is evident that $\langle \hat{q} \rangle = Q = 0$ or $\langle \hat{p} \rangle = P = 0$ result in different lattice structures. Couplings to both optical quadratures is facilitated by imbalanced Rabi rates $\Omega_1 \neq \Omega_2$ [14, 13]. The resulting asymmetry between $\eta_q,\eta_p$ explains the different cavity field amplitude in the two crystalline phases, leading to the ellipsoidal trajectory in Fig. 2a. For the chosen atomic detuning $\Delta_c > 0$, the self-organised phase corresponding to the real quadrature $\langle \hat{q} \rangle$ vanishes for increasing transverse lattice depth $\Delta_c$. Therefore, increasing the coupling strength results in lowering $\langle \hat{q} \rangle$ and leads to a competition with the crystal structure with $\langle \hat{p} \rangle \neq 0$.

![FIG. 3. Self-consistent atomic pump.](image)

**a.** Quadrature space and corresponding unit cell polarisation (along the $y$ direction). The singularity is at $|\alpha| = 0$. A $2\pi$ rotation of $\alpha$ adds a polarisation quantum to the system, displacing the atoms (right blue arrow). **b.** Centre of mass displacement as a function of phase winding numbers. Triangles and circles indicate movement along $x$ and $y$, i.e. parallel and orthogonal to the transverse beams. Solid lines indicate theoretical displacements obtained from a numerical simulation. Orange points refer to a second cavity. For the first cavity $|\Delta_c/\kappa| = 4.08(8)$ and a linear ramp from $V_0 = 0$ to $82.8$ $E_r$, for the second cavity $|\Delta_c/\kappa| = 3.00(2)$ and a $V_0$ ramp from 0 to 11(1) $E_r$. Error bars are standard deviations. **c.** The second cavity mirrors the potential in real space, inverting the polarisation and thus the direction of the current (orange points in b). **d.** Results from numerical simulation at $|\Delta_c/\kappa| = 12$ and a linear ramp from 0 to 24 $E_r$ in 10 ms. From top to bottom: the photon phase $\phi$, expectation values of the overlap integrals $\Theta_q$ and $\Theta_p$, particle current along $y$.

Photons leaking out of the cavity provide real-time access to the atomic distribution, probing the phase transitions [35] and the atomic currents [36]. In our experiment, the cavity field phase is given by (see Methods)

$$\phi = \phi_\kappa + \phi_{qp} = \tan^{-1} \left( \frac{\Delta_c \eta_p \Theta_p - \kappa \eta_0 \Theta_q}{\kappa \eta_p \Theta_p + \Delta_c \eta_0 \Theta_q} \right),$$

containing the information on the atomic distribution via the expectation values of the overlap integrals $\Theta_q = \langle \cos(k_x r) \cos(k_c r) \rangle$ and $\Theta_p = \langle \sin(k_x r) \cos(k_c r) \rangle$. Here, the bare cavity detuning $\Delta_c$ has been replaced by $\Delta_c$, which is the dispersively shifted resonant frequency of
the cavity (see Methods). Therefore, the data in Fig. 2 provide an indirect measurement of the atomic displacement via the overlap integrals in Eq. (3), i.e. a change in the phase of the light field signals that the position of the atomic cloud has shifted in real space.

The chiral evolution of $\alpha$ (e.g. Fig. 2l) demonstrates the presence of a pumping mechanism, since it implies a monotonous displacement of the atoms in a non-vanishing optical potential. As further confirmation, we perform in-situ measurements of the center-of-mass position of the atomic cloud. We observe that the cloud moves in real space, mainly in the direction orthogonal to the transverse beams, with a displacement that correlates with the observed windings of the photon phase $\phi$ (Fig. 3). We observe a displacement of less than one lattice site per pump cycle. We attribute this to the inhomogeneous filling of the lowest band and reduced adiabaticity for increasing pump speeds.

The angle between the transverse beams and the cavity is crucial for pumping. For deep $V_0$, the atomic distribution resembles a collection of 1D bipartite chains along the $y$ direction, which reminds of the one-dimensional Rice-Mele pump, although the measured lattice depths do not fully justify a tight binding approximation. In this analogy, the angle between the beams breaks the inversion symmetry of the unit cell, contrary to the case of orthogonal interference.

The direction of the atomic current can in principle be inverted by winding around the singularity with opposite chirality in parameter space. In our system, the chiral evolution of $\phi$ is fixed by the intrinsic non-Hermitian origin of the dynamics. We can however change the sign of the chirality by inverting the system in real space. We use a second cavity which is mounted realising a continuos dissipative topological time crystal. Coupling the cavity field to a coalescence of its eigenmodes (Fig. 4b). The appearance of an imaginary part in the spectrum is a feature of optical systems with gain and loss channels and, in general, of non-Hermitian models showing exceptional appearance of an imaginary part in the spectrum is a feature of optical systems with gain and loss channels and, in general, of non-Hermitian models showing exceptional abundance of finitely many eigenvalues, emerging features from the photonic space to a pumping potential, thus breaking simultaneously time and space translation symmetry.

The observation of a self-consistent pump rises interesting questions that call for future investigations. A natural extension of this work would be a generalisation to insulating systems. Coupling the cavity field to a self-organised Mott insulator [39] or Fermi gas [40], the transported charge would be quantised and the cavity phase would directly reflect the topological index [41], realising a continuos dissipative topological time crystal. In addition, applying our scheme to spin-dependent self-organisation [42] may generate dissipative $Z_2$ spin pumps.

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Author contributions

D.D, A.B. and X.L. prepared the experiment, D.D., A.B., X.L. and S.H. took and analysed the data, D.D. performed the numerical simulations. T.D. and T.E. supervised the work. All authors contributed to discussions of the manuscript.

Author information

The authors declare no competing financial interests. Correspondence and requests for materials should be addressed to T.E. (esslinger@phys.ethz.ch).

METHODS

Preparation of the Bose Einstein Condensate (BEC)

We prepare a gas of $^{87}\text{Rb}$ in an optical dipole trap. We reach degeneracy of the bosonic gas via optical evaporation, preparing an almost pure BEC of $N = 2.5(4) \times 10^5$ $^{87}\text{Rb}$ atoms in the hyperfine state $|F = 1, m_F = -1\rangle$. The final crossed optical dipole trap consists of two laser beams at a far-detuned wavelength of 1064 nm that create an ellipsoid-shaped trap with the three trap frequencies $\omega_x, \omega_y, \omega_z = 2\pi \times [89(3), 74(1), 224(2)]$Hz. These trap frequencies set a time frame of around 10 ms in which we can observe displacements of the cloud without relaxation due to the harmonic confinement.

A magnetic offset field of $\sim 25$ G is applied to avoid spin-dependent effects on self-organization as e.g. superradiance to the orthogonally polarized cavity mode that is significantly detuned due to the birefringence of our cavity as we studied in [14].

Imbalanced Transverse Beams

To engineer the couplings $\eta_{q,p}$ to the two different quadratures of the light field we use imbalanced transverse beams as explained and utilized in [14]. The BEC is placed at a distance of around 4 mm from an in-vacuo retro-reflecting mirror. Tuning the imbalance is achieved by mechanically shifting the position of the focusing lens along the optical axis of the pump beam. We extract the imbalance from the experimental data by comparing the threshold of the two distinct self-organized phases (coupling either to the field quadrature $P$ or $Q$) with numerical calculations [14]. The imbalance is quantified by the imbalance parameter

$$\gamma = \sqrt{\frac{\Omega_1}{\Omega_2}},$$

with $\Omega_{1,2}$ being the Rabi frequencies of the two plane waves counterpropagating along $x$. The lattice depth $V_0$ is calibrated using Raman-Nath diffraction at the standing-wave component (that couples to the $P$ quadrature of the cavity field) together with the extracted imbalance. For the data in Fig. 2,3 we use an imbalance parameter of $\gamma = 1.36(5)$. 

Heterodyne measurement

Balanced optical heterodyne detection is used to record the amplitude of the intracavity field and its phase with respect to the transverse beam lattice. A local oscillator (LO) laser field is combined on a beam splitter with the light field leaking from one cavity mirror and guided together onto the two photodiodes of a balanced photodetector. The signal is afterwards mixed down to a moderate radio frequency of 300 kHz with a home build IQ filter device. The LO is generated by shifting the frequency of the same laser source as the transverse beam with an acousto-optical modulator by 50 MHz. To compensate for phase drifts introduced by sending the light fields through separate optical fibers, we lock their relative phase after passing the fibers. The 300 kHz signal data is recorded with a computer interfaced oscilloscope (PicoScope 5444b). The signal is then processed by a software which extracts the quadratures via a digital IQ mixer and applies a low-pass filter using a binning window of $1 \times 10^{-4}$ s. All related radio frequency signal sources are phase locked to a 10 MHz GPS frequency standard which has a fractional stability better than $10^{-12}$ to minimize the technical phase noise in the heterodyne detection system. More technical details and design considerations are described in detail in [14].

The measured absolute value of the phase $\phi$ of the intracavity field is dominated by technical fluctuations in-between experimental repetitions like e.g. fiber length drifts, air density changes and unsynchronized start of experimental sequences. Nevertheless, the value relative to the pump field is fixed: When the system enters the superradiant phases coupled to the quadrature $Q$ ($P$), the phase $\phi$ is locked to either 0 or $\pi$ (or $\pi/2$ or $-\pi/2$). However, since the lines in the phase diagrams consist of independent measurements, we plot $\phi$ modulo $\pi/2$ in Fig. 5. In the main text we focus on the dynamics between the quadratures $P$ and $Q$, where we set the starting point for the physical processes to the positive $Q$ quadrature.

To calibrate the amplitude of the intracavity light field, we carry out Raman-Nath diffraction analogous to the
calibration of the transverse beam by applying to the cavity a short coherent on-axis probe.

Cavities and Detunings

We can reverse the displacement of the atomic cloud by performing the experiment with two physically distinct cavities that set different directions for the cavity wave vector (see Fig. 3) and also show different rates of dissipation. Both cavities are near-planar with a vacuum Rabi coupling of $|g_1, g_2| = 2\pi \times [1.95, 1.77]$ MHz and a mode waist of $[48.7, 50.4] \mu m$ which is much larger than the atomic cloud itself. This also justifies to model the system with a transversely constant cavity lattice field within the atomic cloud. Cavity 1 with dissipation rate $\kappa_1 = 2\pi \times 147(4)$ kHz is used for most experiments demonstrated in the main text (see Fig. 2, 3). Cavity 2 is used for the orange data points in Fig. 3 and has a higher dissipation rate of $\kappa_2 = 2\pi \times 800(11)$ kHz.

All measurements are done far in the dispersive regime where the cavity resonance is detuned by $\Delta_c = +2\pi \times 70$ GHz with respect to the Rubidium D$_2$-line. The detuning between the frequency of the transverse beams and the cavity is tuneable within tens of MHz via an electro-optical modulator. The length of the cavity is stabilised using a low amplitude, far-detuned additional laser field at 830 nm, allowing a constant feedback on the cavity length while having a negligible effect on the atomic cloud.

Imaging of the atomic cloud

Besides the detection of the light field leaking out of the cavities we use resonant absorption imaging to measure the atom number, temperature, momentum distribution, and centre-of-mass position of the atomic cloud. The physical imaging is done through the viewport below the vacuum chamber by an imaging objective built of two spherical lenses with an effective numerical aperture of $NA = 0.15$. This limits the resolution at our imaging wavelength of $\lambda_{D_2} = 780$ nm to $\Delta X_{\text{res}} = 2.5 \mu m$. The final image of the atoms is recorded on a CCD camera (Pointgrey Grasqopper CCD) with a pixel size of 4.50 $\mu m$ and 1452 $\times$ 1932 pixels. This results together with the magnification of the imaging system in an effective pixel size of 2.25 $\mu m$, which allows for precise centre-of-mass detection in situ, while still allowing for time-of-flight measurements with the same imaging system.

Temperature, number of atoms, and the momentum state populations are extracted from absorption images taken after 25 ms of ballistic expansion of the atomic cloud. The displacement measurements of the cloud’s centre-of-mass position shown in Fig. 3 are instead extracted by fitting a 2D-Gaussian to in situ absorption images taken within the dipole trap. Shot-to-shot fluctuations of the cloud position (standard error of 4.54 $\mu m$) are dominated by position fluctuations of the dipole trap. However, by independently imaging the position of the dipole beam and correlating the trap position with the cloud position, we reduce the error by a factor of $\approx 4$. This procedure, together with the large number of measurements, results in the error bars displayed in Fig. 3.

Measurement protocol and data selection

The data shown is taken by setting the cavity detuning $\Delta_c$ to a fixed value and ramping the transverse beam within 5 ms linearly from zero to a desired transverse beam lattice depth $V_0$.

While Fig. 2 shows a phase diagram resulting from averaging 50 experimental repetitions, in Fig. 3 we display a phase diagram constructed from a single experimental run at each $\Delta_c$. Fig. 4 shows the consecutive repetitions of such experimental runs at a fixed detuning $\Delta_c$, from which the averaged phase diagram can be derived. We attribute the observed variations between the different runs to fluctuations in the atom number and in the relative alignment between the cavity mode and the transverse beams.

The displacement data in Fig. 3 shows the correlations between the number of phase windings and the displacement of the cloud. Since the duration of a single phase winding is short ($< 0.1$ ms) compared to the whole ramp time, the fluctuations shown in Fig. 3 make it impossible to deterministically choose a number of phase windings. Instead, we choose a final transverse beam lattice depth such that the ramp ends within the dynamic phase. After recording a large number of repetitions we select a posteriori the desired number of phase windings.

FIG. 5. Phase diagrams. a. Phase diagram using the phase data $\phi(t)$ from the heterodyne detector by varying cavity detuning $\Delta_c$ of cavity 1. The two different self-organised phases can be well observed for values around $\phi = 0$ and $\phi = \pi/2$. At low transverse beam fields the system shows no self-organisation and $\phi$ is not well defined. Between the two self-organised phases the dynamical phase with varying $\phi(t)$ is visible. b. Many repetitions of the same trace with constant $\Delta_c = -1.1$ MHz. The extent of the instability region varies slightly at each repetition.
Many-body Hamiltonian

In the following we derive a model that describes the self-organisation within the quadratures \( P \) and \( Q \) and shows how by including the dissipation channel \( \kappa \) the non-Hermitian dynamics occurs. With the two counterpropagating plane waves (with the strength \( \Omega_1 \) and \( \Omega_2 \)) of the transverse beam we define all effective two-photon Rabi frequencies in the dispersive coupling regime and rotating-wave approximation: The transverse beam lattice \( V_0 = \hbar \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \) is the dispersive shift per cavity photon \( \eta = \frac{\Omega_1^2 + \Omega_2^2}{2 \Delta_\kappa} \), the coupling to the quadrature \( Q \), \( \eta_q = \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \) and the coupling to the quadrature \( P \), \( \eta_p = \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \). The Hamiltonian in the standard form of [14] is:

\[
\hat{H} = -\hbar \Delta_\kappa \hat{a}^\dagger \hat{a} + \hat{H}_0 + \hbar U_0 \hat{a}^\dagger \hat{a} \hat{B} + \hbar \eta \hat{B} \hat{\Theta}_q - i \hbar \eta_p \hat{B} \hat{\Theta}_p \tag{5}
\]

where \( \hat{a}^\dagger (\hat{a}) \) are the creation (destruction) operators for the cavity photon field and having defined:

\[
\begin{align*}
\hat{H}_0 &= \int dr \hat{\Psi}(r) \left( \frac{\nabla^2}{2m} + V_{\text{trap}}(r) \right) \\
+ \frac{g}{2} |\hat{\Psi}(r)|^2 + V_0 \cos(k_x r)^2 |\hat{\Psi}(r)| \\
\hat{B} &= \int dr \hat{\Psi}^\dagger(r) \cos(k_x r)^2 \hat{\Psi}(r) \\
\hat{\Theta}_q &= \int dr \hat{\Psi}^\dagger(r) \cos(k_x r) \hat{\Psi}(r) \\
\hat{\Theta}_p &= \int dr \hat{\Psi}^\dagger(r) \sin(k_x r) \cos(k_x r) \hat{\Psi}(r)
\end{align*}
\]

where \( \hat{\Psi} \) is the atomic field operator from eigenstates \( \psi_0, \psi_p, \psi_q \) and \( \kappa \) the non-Hermitian dynamics occurs. With the two patterns coupling to the different quadratures \( \kappa \kappa \) the non-Hermitian dynamics occurs. With the two counterpropagating plane waves (with the strength \( \Omega_1 \) and \( \Omega_2 \)) of the transverse beam we define all effective two-photon Rabi frequencies in the dispersive coupling regime and rotating-wave approximation: The transverse beam lattice \( V_0 = \hbar \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \) is the dispersive shift per cavity photon \( \eta = \frac{\Omega_1^2 + \Omega_2^2}{2 \Delta_\kappa} \), the coupling to the quadrature \( Q \), \( \eta_q = \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \) and the coupling to the quadrature \( P \), \( \eta_p = \frac{\Omega_1 \Omega_2}{\Delta_\kappa} \). The Hamiltonian in the standard form of [14] is:

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+ \frac{g}{2} |\hat{\Psi}(r)|^2 + V_0 \cos(k_x r)^2 |\hat{\Psi}(r)| \\
\hat{B} &= \int dr \hat{\Psi}(r) \cos(k_x r)^2 \hat{\Psi}(r) \\
\hat{\Theta}_q &= \int dr \hat{\Psi}(r) \cos(k_x r) \hat{\Psi}(r) \\
\hat{\Theta}_p &= \int dr \hat{\Psi}(r) \sin(k_x r) \cos(k_x r) \hat{\Psi}(r)
\end{align*}
\]

\( k_x \) and \( k_c \) are the pump and cavity wave vectors respectively (with \( |k| = k = 2\pi/\lambda \)). The direction of the transverse beam is chosen such that the field with strength \( \Omega_1 \) is along the x-axis and the field with strength \( \Omega_2 \) is counterpropagating. The atomic cloud is small enough to neglect the transverse spatial component of the electromagnetic modes. The physical interpretation of the integrals is: \( \hat{H}_0 \) is describing the BEC with collisional interaction \( g \) and trapped in the harmonic potential \( V_{\text{trap}} \) and in the optical lattice of the transverse beam, \( \hat{B} \) is the overlap with the cavity mode, and \( \hat{\Theta}_q \) and \( \hat{\Theta}_p \) the overlaps with the two patterns coupling to the different quadratures. These integrals are our choice of order parameter in a phenomenological theory of the self-organisation phase transition. Note that changing sign \( \langle \Omega_1 - \Omega_2 \rangle \) from an Hamiltonian point of view corresponds to flipping \( x \rightarrow -x \) and the same is valid for changing the direction of \( k_c \).

Heisenberg equation of motion and dissipation

To observe the dynamics of the photonic part of the Hamiltonian we look at the Heisenberg equation of motion of the field operator of the light, incorporating also the cavity dissipation \( \kappa \):

\[
\frac{\partial \langle \hat{a} \rangle}{\partial t} = -\frac{i}{\hbar} \langle [\hat{a}, \hat{H}] \rangle - \kappa \langle \hat{a} \rangle, \tag{10}
\]

where we omitted quantum fluctuations since they equal zero at the mean field level for \( \alpha = \langle \hat{a} \rangle \). We assume a quasi-stationary light field \( \partial_t \alpha = 0 \) since the time scales of the light field separate from the motional time scales of the atomic wavefunction. With this we get for the light field:

\[
\alpha = e^{i \phi} \sqrt{\frac{\eta_q^2 \Theta_q^2 + \eta_p^2 \Theta_p^2}{\Delta_c^2 + \kappa^2}}
\]

\[
= e^{i \phi + i \phi_{qp}} \sqrt{\frac{\eta_q^2 \Theta_q^2 + \eta_p^2 \Theta_p^2}{\Delta_c^2 + \kappa^2}}. \tag{11}
\]

The complex light phase \( \phi \) is defined in equation [3] of the main text and can be separated into \( \phi_{\infty} = \tan^{-1}(\frac{\Delta_\kappa}{\kappa}) \) and \( \phi_{qp} = \tan^{-1}(-\frac{\Delta_\kappa}{\omega_p \exp}) \). For simpler notation we define \( \Theta_i = \langle \hat{\Theta}_i \rangle \), \( B = \langle \hat{B} \rangle \) and \( \Delta_c = \Delta_c - U_0 B \), which is the dynamical dispersive cavity detuning due to the changing overlap of the atomic wavefunction with the cavity field.

Equation of motion and low-energy momentum expansion

To get further intuition for the dynamics of the photon field and the atomic wavefunction, we can construct an atomic field operator from eigenstates \( \psi_i \) of the momentum operator:

\[
\hat{\Psi} = \sum_i \hat{c}_i \psi_i, \tag{12}
\]
that only uses the three momentum modes momentum families separated by $k$ expectation values of the operators in Eq. (6)-(9). The integral defined in Eq. 19 only connect the BEC to momentum families separated by $k_c$, $k_x$, and linear combinations. In a low-energy approximation it is then appropriate to use the Ansatz

$$\hat{Ψ} = \hat{c}_0 \psi_0 + \hat{c}_q \psi_q + \hat{c}_p \psi_p,$$

(14)

that only uses the three momentum modes $\psi_q = \frac{e^{i k_c q}}{\sqrt{N}} \cos(k_c r) \cos(k_r r)$, $\psi_p = \frac{e^{i k_c p}}{\sqrt{N}} \sin(k_c r) \cos(k_r r)$, and $\psi_0 = \frac{1}{\sqrt{N}}$, where $A$ defines the area of the Wigner Seitz cell. In the definition of Eq. (14), each mode is normalised to unity, $\int \psi_i^\dagger \psi_i = 1$ and the field operator is normalised to the total particle number $N$, $\int \hat{Ψ}^\dagger \hat{Ψ} = \hat{c}_0^\dagger \hat{c}_0 + \hat{c}_q^\dagger \hat{c}_q + \hat{c}_p^\dagger \hat{c}_p = N$. Under this approximation, and neglecting the harmonic trapping potential the collisional interactions, we arrive at the following mean-field equations of motion:

$$\begin{align*}
\dot{\hat{c}}_0 &= i \hat{D}_c \alpha - \kappa \hat{c}_0 - \frac{i}{2} \left( \hat{c}_q^\dagger \hat{c}_q + \hat{c}_p^\dagger \hat{c}_p \right) + \frac{\eta}{2} \left( \hat{c}_q^\dagger \hat{c}_p + \hat{c}_p^\dagger \hat{c}_q \right) \\
\dot{\hat{c}}_q &= -i (\omega_q + \frac{V_0}{4\hbar}) \hat{c}_q - i \frac{\eta}{2} \Re(\alpha) \hat{c}_p \\
\dot{\hat{c}}_p &= -i (\omega_p - \frac{V_0}{4\hbar}) \hat{c}_p - i \frac{\eta}{2} \Im(\alpha) \hat{c}_q
\end{align*}$$

(15)

Minimal model and non-Hermitian dynamics

To demonstrate the instability, it is sufficient to reduce the equations of motion further by keeping the $c_0$ population constant ($c_0 \sim \sqrt{N}$). This is an appropriate approximation since the relative depletion of the zero momentum state is small. As a result of this approximation, the number of coupled equations is further reduced. Applying another time derivative to $c_{q,p}$, and using the steady state value for $\alpha$, the differential equations for the populations of $\psi_q, \psi_p$ can be written as

$$\begin{pmatrix}
\dot{c}_q \\
\dot{c}_p
\end{pmatrix} =
\begin{pmatrix}
\Omega_q^2 & -K_q \\
K_p & \Omega_p^2
\end{pmatrix}
\begin{pmatrix}
c_q \\
c_p
\end{pmatrix}.$$

(16)

In the above matrix notation, we have defined

$$\begin{align*}
\Omega_i^2 &= -\left( \omega_i \pm \frac{V_0}{4\hbar} \right)^2 - N \eta_i^2 \left( \omega_i \pm \frac{V_0}{4\hbar} \right) \frac{\Delta_r}{\Delta_r^2 + \kappa^2} \\
K_i &= N \eta_i \eta_p \left( \omega_i \pm \frac{V_0}{4\hbar} \right) \frac{\kappa}{\Delta_r^2 + \kappa^2}
\end{align*}$$

(17)

(18)

with $i \in \{ q, p \}$ and $\pm$ is positive for $i = q$ and negative for $i = p$. The eigenenergies $\omega_i$ of the q and p mode can be obtained from the free particle dispersion as $\omega_{q,p} = \pm 2 \omega_r$, with $\omega_r$ being the recoil frequency. The two-photon coupling strengths $\eta_{q,p}$ are defined in the main text via the Rabi rates of the two transverse beams and can be rewritten as $\eta_{q,p} = \frac{2}{\sqrt{2}} \left( \Omega_i \pm \Omega_j \right) = \left( \gamma \pm \frac{1}{2} \right) \sqrt{U_0 V_0}$ to express the explicit dependences on the imbalance parameter $\gamma$, transverse beam lattice strengths $V_0$, and dispersive shift per cavity photon $U_0$.

Note that in Eq. (17) the coupling to the Q quadrature becomes weaker for strong enough transverse beam lattice strengths $V_0$, which makes the energies of the P and Q modes cross eventually. For more details of the differences of the two couplings see Ref. 31.

The diagonal elements of the matrix describe the soft mode energies. Above a critical pump strength, these matrix elements give rise to growing populations $c_{q,p}$. The off-diagonal elements describe the coupling rate of the two modes via the dissipation.

From diagonalising the non-Hermitian matrix in Eq. (16), we get an analytical expression of the eigenvalues

$$\epsilon_\pm^2 = \frac{\Omega_q^2 + \Omega_p^2}{2} \pm \frac{1}{2} \sqrt{(\Omega_q^2 - \Omega_p^2)^2 - 4 K_q K_p}.$$

(19)

Real and imaginary part of $\epsilon_\pm$ describe growth rate and oscillation frequency of the decoupled eigenvectors $c_\pm$, from which the evolution of $c_{q,p}$ can be derived. The populations start oscillating between each other when $(\Omega_q^2 - \Omega_p^2)^2 < 4 K_q K_p$, i.e. the dynamics emerges once the dissipative coupling strength $K_i \propto \kappa$ is larger than the energy gap between the two energies. Note that the term $4 K_q K_p \propto \Delta_r^{-4}$ while $\Omega^4 \propto \Delta_r^{-2}$, such that the dissipation can only overcome the energy gap in the regime of $\kappa/\Delta_r \approx 1$. This can be seen in Fig. 1, when the real part of the eigenvalues coalesce as they are close enough together.

Numerical simulations

Alternatively to the simplified analytical model described above, we numerically integrate the self-consistent many-body Hamiltonian Eq. (5) using a standard symmetrised split-step Fourier method. This allows to propagate the equations in time and calculate the dynamics of the system, as well as to obtain the ground state by first performing a Wick rotation. The cavity field is assumed in its steady state at each integration time-step, so it can be numerically calculated from the wave function using Eq. (11). All theoretical simulations reproduced in the figures throughout the paper include the trap and inter-particle collisional interactions. In the calculations, the transverse curvature of the coupling beams and the cavity mode are neglected, such that the initial harmonic confinement is not modified. We checked the validity of the simulation by comparing the numerical
FIG. 7. Comparison of experimental and numerical phase diagrams for cavity 2. Figures show the amplitude of the intra-cavity light field dependent on $V_0$ and the cavity detuning $\Delta_c$. a, Dataset showing in each row a trace of single experimental realisation for the given cavity detuning $\Delta_c$. The transverse beam lattice is linearly ramped to the final transverse lattice strength $V_0 = 40 E_r$ within 5 ms. b, Corresponding simulation of the experimental results with GPE simulation.

Phase diagram to the experimental one (Fig. 7), which show qualitative agreement. The current shown in Fig. 3 is calculated from the probability current

$$j(t) = -i \frac{\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla^* \psi)$$

(20)

integrated over the whole system.

The polarisation of Fig. 3a-c is calculated by obtaining the ground state wave function in the unit cell at different choices of $\alpha$. Since this calculation was performed in periodic boundary conditions (PBC), the centre of mass has to be extracted using [20]. The polarisation

$$P_x = \frac{e}{2\pi L^2} \text{Im} \log \langle \psi | e^{-i \frac{\pi}{2} \hat{X}} \psi \rangle$$

(21)

is the phase of the expectation value of $\hat{X}$ (the position operator) in PBC, which, in the words of reference [20], is ‘just a Berry’s phase in disguise’. Here, $L$ is the linear system size and the charge is set to $e = 1$ for neutral atoms.

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