Two-step PR-scheme for recovering signals in detectable union of cones by magnitude measurements

Youfa Li, Deguang Han

Abstract—Motivated by the research on sampling problems for a union of subspaces (UoS), we investigate in this paper the phase-retrieval problem for the signals that are residing in a union of (finitely generated) cones (UoC for short) in $\mathbb{R}^n$. We propose a two-step PR-scheme: PR = detection + recovery. We first establish a sufficient and necessary condition for the detectability of a UoC, and then design a detection algorithm that allows us to determine the cone where the target signal is residing. The phase-retrieval will then be performed within the detected cone, which can be achieved by using at most $\Gamma$-number of measurements and with very low complexity, where $\Gamma(\leq n)$ is the maximum of the ranks of the generators for the UoC. Numerical experiments are provided to demonstrate the efficiency of our approach, and to exhibit comparisons with some existing phase-retrieval methods.

Index Terms—phase retrieval, union of cones, circulant matrix, FFT, computational complexity, the amount of measurements.

I. INTRODUCTION

Phase-retrieval is a nonlinear problem that seeks to recover a signal $\mathbf{x}$, up to a global phase ambiguity, from the magnitudes of its linear measurements

$$b_i := |\langle \mathbf{x}, a_i \rangle|, \ i = 1, \ldots, m.$$  

Phase-retrieval has been widely applied in many applications such as X-ray crystallography (11), quantum tomography (2), audio processing (3) and frame theory (4, 5, 6, 7).

Besides phase-retrieval, the sampling theory for a union of subspaces (UoS for short) is another important sampling problem (c.f. 8, 9, 10). In signal processing, while traditionally we work on signals in a single linear space or subspace, there are practical demands requiring us to deal with the signals that lie in a UoS. A typical example is the sparse signal recovering or compressed sensing (c.f. 11) where the signals are sitting in the finite union of (small dimensional) subspaces. M. Mishali, Y. Eldar and A. Elron 10 established Xampling for recovering signals in the UoS of $L^2(\mathbb{R})$. By Xampling, the target subspace where the signal sits is detected before recovery. As mention in 10, the detection can considerably reduce the computational complexity and measurement cost (the sampling rate). Note that the phase information of the measurements in 10 is assumed known. Motivated by 10, we will study the phase-retrieval problem for the union of cones (UoC for short). In order to introduce the main problems and discuss our main contributions, we need to recall and establish some notations and definitions.

A. Notations and definitions

We use boldface letters to denote column vectors, e.g., $\mathbf{x}$, calligraphic and upper-case letters to denote matrices (operator), e.g., $\mathcal{X}$, and underlined letters to denote a random variable, e.g., $\underline{X}$. For a matrix $\mathcal{X}$, its Hermitian transpose and transpose are denoted by $\mathcal{X}^*$ and $\mathcal{X}^T$, respectively. For a linear operator $\mathcal{Q}$ from a vector space $H_1$ to another vector space $H_2$, we denote by $\mathcal{R}(\mathcal{Q})$ and $\mathcal{N}(\mathcal{Q})$ the range and null spaces of $\mathcal{Q}$, respectively. Moreover, for a set $S \subset \mathcal{R}(\mathcal{Q})$, denote by $\text{invim}(S)$ the inverse image of $S$. For a vector $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \succ 0$ ($\mathbf{x} \prec 0$) implies every coordinate of $\mathbf{x}$ is strictly larger (smaller) than 0. Denote $\mathbb{R}^{+,m} := \{ \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \succ 0 \}$. The standard orthonormal basis for $\mathbb{R}^n$ is denoted by $\{e_1, \ldots, e_n\}$.

For a matrix $\mathcal{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{n \times m}$, we denote by $\text{cone}(\mathcal{X})$ the cone generated from its column vectors, i.e.,

$$\text{cone}(\mathcal{X}) := \{ \theta_1 \mathbf{x}_1 + \cdots + \theta_m \mathbf{x}_m | \theta_i \geq 0, i = 1, \ldots, m \}.$$  

(1.1)
A finite set of vectors \( \{ \mathbf{x}_1, \ldots, \mathbf{x}_m \} \subseteq \mathbb{R}^n \) is called a frame for \( \mathbb{R}^n \) if there exist two constants \( 0 < C_1 \leq C_2 \) such that

\[
C_1 \| \mathbf{z} \|^2 \leq \sum_{i=1}^{m} |\langle \mathbf{z}, \mathbf{x}_i \rangle|^2 \leq C_2 \| \mathbf{z} \|^2 \quad (1.2)
\]

holds for every \( \mathbf{z} \in \mathbb{R}^n \). Equivalently, a finite set is a frame for \( \mathbb{R}^n \) if and only if it is a spanning set of \( \mathbb{R}^n \). The cone in \((1.1)\) is called a frame cone if \( \{ \mathbf{x}_1, \ldots, \mathbf{x}_m \} \) is a frame for \( \mathbb{R}^n \).

Based on the above denotations, in what follows we propose the definition of a detectable UoC.

**Definition 1.1:** We say that \( \bigcup_{k=1}^{L} \text{cone}(X_k) \) is detectable, if there exists a so-called detector \( \mathcal{G} := [g_1, \ldots, g_n] \in \mathbb{R}^{n \times \kappa} \) such that for any nonzero target signal \( \mathbf{z} \in \bigcup_{k=1}^{L} \text{cone}(X_i) \), the unique index \( l \) can be determined by the detection measurements \( \{ \langle g_1, \mathbf{z} \rangle, \ldots, \langle g_n, \mathbf{z} \rangle \} \) so that \( \mathbf{z} \in \text{cone}(X_l) \).

**B. Our goals, schemes and problems in the present paper**

The union of cones (UoC) is an important type of subset of \( \mathbb{R}^n \) which has been widely considered in many areas of research such as operations research (c.f. [12], [13], [14]), signal processing (c.f. [11], [15], [16], [17]), and representation theory (c.f. [18]). Incidentally, since a linear frame is a type of cone (e.g. \( \mathbb{R}^2 \) can be regarded as the cone generated from \( \{ \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2 \} \)), a union of linear spaces can be regarded as a UoC.

It is well-known that the computational complexity and the amount of measurements are two important considerations for the performance of any phase-retrieval method (c.f. [14], [19], [20]). The goal of this paper is to establish a phase retrieval method in a UoC having low computational complexity and requiring very few measurements. Motivated by [10], this goal will be achieved by establishing the following two-step PR-scheme:

\[
\text{phase retrieval} = \text{detection} + \text{recovery} \quad (1.3)
\]

Naturally, we need to address the issues in the following problem:

**Problem 1.2:** Under what conditions, is a UoC detectable, namely, the target cone can be detected by magnitude measurements? Is it possible to utilize the detectability to reduce the amount of phase-retrievable measurement vectors and computational complexity (e.g. it can be \( O(n) \) or the FFT complexity \( O(n \log n) \))?

**C. Existing results and our contributions**

In what follows we introduce our main contribution in this paper from the aspects of cost of measurements and computational complexity (computational cost).

1) **Cost of measurements:** For the detection, we establish the necessary and sufficient condition on the detectability of \( \bigcup_{k=1}^{L} \text{cone}(X_k) \). Based on the condition we design a detector \( \mathcal{G} := [g_1, \ldots, g_{n-L-1}] \in \mathbb{R}^{n \times (L-1)} \) and the detection algorithm to detect the target cone. The detection can be achieved by using only \((L-1)\)-number of measurements. Once the detection is completed, we then perform the phase-retrieval on a single cone. As will be discussed in Remark 2.2, there are at least \( L-1 \) cones, e.g. \( \text{cone}(X_i), i = 1, \ldots, L-1 \) in the detectable \( \bigcup_{k=1}^{L} \text{cone}(X_k) \) which satisfy the following overlap property

\[
\mathcal{R}(X_i^T) \cap \mathbb{R}^{+m_i}, \neq \emptyset. \quad (1.4)
\]

For the target cone \( \text{cone}(X_i) \) in \((1.4)\), we will design \( \text{rank}(X_i) \)-number of measurement vectors for the phase retrieval. Our contribution on the amount of measurements is that if all the \( L \) cones satisfy \((1.4)\), then \((L-1+1)\)-number of measurement vectors are sufficient for the two-step PR-scheme \((1.3)\), where \( \Gamma = \max_k \{ \text{rank}(\text{cone}(X_k)) \} \).

We emphasize two features of this approach. (i) By the complement property for phase retrievable frames (c.f. [4], [5], [6]), we know that any phase-retrieval method that applies to the signals in \( \mathbb{R}^n \) requires at least \( 2n-1 \) measurement vectors. Obviously, for many detectable UoCs, the scheme \((1.3)\) requires much fewer than \( 2n-1 \) measurements \((L-1+\Gamma < 2n-1)\). (ii) It is well known that the amount of measurement vectors can be significantly reduced for sparse signals (e.g. [21], [22]). In our case, \( \Gamma \) being small does not necessarily imply that the signals in \( \bigcup_{k=1}^{L} \text{cone}(X_k) \) are sparse. So the reduction strategy for the amount of measurements by scheme \((1.3)\) is different from the treatment of sparse signals.

2) **Computational complexity:** By using the i.i.d Gaussian measurement vectors, E. Candes, Y. Eldar, T. Strohmer and V. Voroninski [23] proposed the well-known PhaseLift method to recover \( \mathbf{z} \) in \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)). Since then, based on the random measurements, many other efficient phase-retrieval methods such as Wirtinger Flow [24], Alternating Minimization [25], PhaseCut [26] and BlockPR [20] have been proposed. Among the above methods, the BlockPR, which holds for flat signals, has the lowest computational complexity \( O(n \log^2 n) \). The signals in a cone may not be necessarily flat, and so they do not necessarily satisfy the condition required for the BlockPR method. However, by exploiting the structure of the detectable UoC, the goal of significantly reducing the computational complexity can also be achieved. Our Algorithm 1 for detection costs \( O(Ln) \)-number of operations. Theorem 2.5 shows that if the target signal lies in the cone \( \text{cone}(X_i) \) satisfying \((1.4)\), then after detection, the phase-retrieval of the target signal can be

\[
\ldots
\]
completed by \( O(\gamma \log \gamma) \)-operations, where \( \gamma = \text{rank}(X_t) \).

Our contribution on the computational complexity is that the proposed phase-retrieval scheme \( \text{(1.3)} \) for a detectable union of \( L \)-cones all satisfying \( \text{(1.4)} \) has the computational complexity \( O(\Gamma \log \Gamma) + O(Ln) \), which can be \( O(n) \) or \( O(n \log n) \) for many cases of \( L \) and \( \Gamma \).

II. TWO-STEP SCHEME FOR RECOVERING SIGNALS IN DETECTABLE UNION OF CONES

Our PR-scheme \( \text{(1.3)} \) consists of detection and recovery. In Subsection II-A we establish the sufficient and necessary condition for the detectability of a UoC. The algorithm for this detection is presented in Algorithm 1. Following this we discuss in Subsection II-B (Remark 2.2) the cone structure derived from the above condition that is crucial to help achieve our goal. We also establish that a union of linear subspaces (or spaces) is not detectable (Remark 2.3). The main results on the recovery will be presented in Subsection II-C

A. Detection

This subsection aims at establishing the sufficient and necessary condition for the detectability of a UoC, and presenting a detection algorithm for the target cone.

**Theorem 2.1:** A UoC \( \bigcup_{k=1}^L \text{cone}(X_k) \), where \( X_k = [x_{k,1}, \ldots, x_{k,m_k}] \subseteq \mathbb{R}^{n \times m_k} \), is detectable if and only if for every \( k \geq 2 \) we have either

\[
\text{invim}(R(X_t^T) \cap \mathbb{R}^{+m_t}) \cap N(X_k^T) \neq \emptyset
\]

or

\[
\text{invim}(R(X_k^T) \cap \mathbb{R}^{+m_k}) \cap N(X_k^T) \neq \emptyset,
\]

where \( l = 1, \ldots, k-1 \).

**Proof:** The proof is given in Subsection IV-A. Suppose that \( z \in \bigcup_{k=1}^L \text{cone}(X_k) \). If, for example, the first equation in \( (2.5) \) holds, pick \( g \in \text{invim}(R(X_t^T) \cap \mathbb{R}^{+m_t}) \cap N(X_k^T) \), then we determine that \( z \notin \text{cone}(X_l) \) when \( |\langle z, g \rangle| = 0 \), and \( z \notin \text{cone}(X_k) \) when \( |\langle z, g \rangle| > 0 \). It is easy to see that based on \( (2.5) \), we can use the \( L-1 \) exclusions similar to the above to detect the target cone. The detection can be completed by using Algorithm 1.

B. Remarks on the detectable union of cones

**Remark 2.2:** (i) By Algorithm 1 the source of any \( f \in \bigcup_{k=1}^L \text{cone}(X_k) \) can be detected through \( L-1 \) exclusions if condition in \( (2.5) \) is satisfied. Only one measurement vector is required for every exclusion. Therefore we need \( (L-1) \)-number of measurement vectors for the target cone detection. Moreover the detection requires \( O(Ln) \)-number of operations. (ii) The condition \( (2.5) \) implies that the overlap property

\[
R(X_t^T) \cap \mathbb{R}^{+m_t} \neq \emptyset
\]

holds for at least \( L-1 \) number of cones.

As mentioned in Section II, a linear space (subspace) is a special type of cone. An interesting problem is: can the union of linear spaces (subspaces) be detectable? The following remark tells us that a detectable UoC has at most one of the cones that is a linear subspace (This can be easily proved by Remark 2.2 (ii) and the fact that a cone satisfying \( (2.6) \) is not a linear subspace). That is for any signal in the union of linear spaces (subspaces), the target cone where the signal is residing cannot be detected by magnitude measurements.

**Remark 2.3:** Suppose that the UoC \( \bigcup_{k=1}^L \text{cone}(X_k) \) is detectable. Consequently, there exist at least \( L-1 \) cones satisfying the overlap property \( (2.6) \), and none of the \( L-1 \) cones is a linear space (subspace). If there exists a linear space (subspace) among the \( L \) cones, then it is the unique one and does not have the overlap property \( (2.6) \). In other words, a union of linear spaces (subspaces) is not detectable, and it does not satisfy the requirement for the proposed approach.

In the following remark we discuss how to check \( (2.5) \) and \( (2.6) \).

**Remark 2.4:** The condition \( (2.6) \) is equivalent to that the system of linear inequalities

\[
X_t^T x > 0
\]

has a solution. There exist many methods (e.g. in \([27], [28], [29]\)) in the literature that can be used to deter-
mine whether the (2.7) has a solution. The condition
\(\text{invm}(R(X_k^T) \cap \mathbb{R}^{+m_k}) \cap \mathcal{N}(X_k^T) \neq \emptyset\) in (2.5) is
equivalent to that the optimum of the following quadratic
programming problem
\[
\begin{aligned}
&\min \|X_k^T x\|^2_2 \\
&\text{s.t. } X_k^T x \succ 0,
\end{aligned}
\] (2.8)
is zero.

C. Recovery

After the detection by the procedures outlined in Algo-
rithm 1 we can detect the cone that contains the target
signal. What left is to perform phase retrieval on the
target cone but not on the entire set UoC. As discussed
in Section I applying some of the existing methods to a
finitely generated cone is either too expensive in terms
of computational complexity and measurements or not
even applicable due to the restriction of the methods. For
example, the recently proposed fast method BlockPR by
M. A. Iwen, A. Viswanathan, and Y. Wang [20] applies to
flat vectors, but does not necessarily applies to vectors
in a cone. In this subsection we establish a fast PR
method for the cone in a detectable UoC with relatively
fewer measurements and low computational complexity.
The main results are outlined in Theorem 2.5, Theorem
2.6 and Proposition 2.7.

**Theorem 2.5:** Let \(\text{cone}(\mathcal{X})\) be a cone with \(\mathcal{X} = \{x_1, \ldots, x_m\} \in \mathbb{R}^{n \times m}\) such that the overlap property
(2.6) holds. Then there exist \(\gamma\)-vectors \(\{k\}_{k=1}^n\) such that
\(\{|z, f_k\}_{k=1}^n\) determines \(z\) (up to a unimodular scalar)
for any \(z \in \text{cone}(\mathcal{X})\), where \(\gamma = \text{rank}(\mathcal{X})\). Moreover, \(\{k\}_{k=1}^n\) can be designed in such a way that the recovery
of \(z\) requires only \(O(\gamma \log \gamma)\)-number of operations, i.e.,
the computational cost is FFT-time.

**Proof:** The proof is given in Section [III].

Theorem 2.5 implies that the property (2.6) is crucial for
reducing the amount of measurements and computational
complexity for the PR in a cone. By Remark 2.2 (ii)
there are at least \(L - 1\) cones in the detectable UoC
\(\bigcup_{k=1}^L \text{cone}(\mathcal{X}_k)\) which satisfy (2.6). We have the following
result for the case when all the \(L\) cones in \(\bigcup_{k=1}^L \text{cone}(\mathcal{X}_k)\)
satisfy (2.6).

**Theorem 2.6:** Suppose that \(\bigcup_{k=1}^L \text{cone}(\mathcal{X}_k)\) is
detectable, and all the \(L\) cones satisfy the overlap property
(2.6). Then, by using the two-step PR-scheme (1.3),
any target signal in the UoC can be determined by at
most \(L - 1 + \Gamma\) magnitude measurements, where \(\Gamma = \max_k \text{rank}(\mathcal{X}_k)\}. Moreover, our scheme costs at most
\(O(Ln) + O(L \log \Gamma)\)-number of operations.

**Proof:** By Remark 2.2 (i), the detection strategy in Algo-
rithm 1 needs \(L - 1\) magnitude measurements. After
the detection step, the phase-retrieval is performed on the
target cone. Since all the cones satisfy the overlap property
(2.6), by Theorem 2.5 the phase-retrieval on the target cone
needs at most \(\Gamma\) magnitude measurements. Then \(L - 1 + \Gamma\)
measurements are sufficient for the two-step PR-scheme.
The rest of the proof can be concluded by Remark 2.2 (i)
and Theorem 2.5.

The following proposition states that for many cases of
\(L\) and \(\Gamma\), the scheme (1.3) requires very few measurements
and has very low computational complexity.

**Proposition 2.7:** (i) The smaller \(L + \Gamma\), the fewer
measurements we need for our PR scheme (1.3). In
particular, when \(L + \Gamma < 2n\) we can use less then \(2n - 1\)
measurements (the critical amount related to complement
property) to complete our PR scheme.

(ii) As for the computational complexity, if \(\log \Gamma \lesssim n\)
and \(L\) is a constant independent of \(n\), then our scheme can be performed by \(O(n)\)-number of operations. If \(\Gamma \approx n\),
then our scheme can be done by \(O(n \log n)\)-number of operations,
the FFT time.

**Remark 2.8:** (i) Suppose that \(X_k = [x_{k,1}, \ldots, x_{k,m_k}]\) satisfies
(2.6), i.e., \(R(X_k^T) \cap \mathbb{R}^{+m_k} \neq \emptyset\). Then \(\text{cone}(X_k)\)
ever contains the unit ball of \(\mathbb{R}^n\). (ii) Suppose that \(\bigcup_{k=1}^L \text{cone}(X_k)\) is detectable and all the \(L\) cone generators
satisfy (2.6). Then \(\bigcup_{k=1}^L \text{cone}(X_k)\) does not contain the
unit ball of \(\mathbb{R}^n\) if \(L + \Gamma < 2n\).

**Proof:** We first prove Part (i). By Theorem 2.5 there exist
\(n\) phase retrievable vectors for \(\text{cone}(X_k)\). If the unit ball
\(B \subseteq \text{cone}(X_k)\), then the \(n\) vectors above can also do PR
for \(B\) and for \(\mathbb{R}^n\). By the complement property in (3),
however, it requires at least \(2n - 1\) vectors to do PR for
\(\mathbb{R}^n\) and also for the unit ball. This is a contradiction, and
the proof is concluded. Part (ii) can be proved similarly by
Theorem 2.6 and the complement property.

III. Proof of Theorem 2.5 Algorithm for the
Phase-retrievable Measurement Vectors, and
The Recovery Formula

Before proving Theorem 2.5 and presenting an algo-
rithm for \(\{k\}_{k=1}^n\) therein, we need some preparations. Recall that \(\text{cone}(\mathcal{X})\) in Theorem 2.5 may not be a frame
cone. However, the cone in Lemma 3.1 or Lemma 3.2
will be required to be a frame-type. In order to avoid
notation confusions, we will use \(\text{cone}(Y)\) instead of
\(\text{cone}(\mathcal{X})\) before we present the proof of Theorem 2.5
where \(Y \in \mathbb{R}^{n \times m}\).

Suppose that the column vectors of \(Y = [y_1, \ldots, y_m]\)
constitute a frame of \(\mathbb{R}^n\), and the overlap property (2.6)
holds for \( \mathcal{Y} \). For any \( z := (z_1, \ldots, z_m) \in \mathcal{Y} \), it is easy to check by the frame property (1.2) that
\[
p := (\mathcal{Y}^T)^{-1}yz
\]is the unique solution to the following equation with respect to the variable \( x \in \mathbb{R}^n \),
\[
((x, y_1), (x, y_2), \ldots, (x, y_m))^T = z.
\]Since the measurements \( \langle p, y_1 \rangle, \ldots, \langle p, y_m \rangle \) are all positive, we will call \( p \) an anchor vector.

A. Two auxiliary lemmas and design of special anchor vector

**Lemma 3.1:** Let \( \mathcal{Y} := [y_1, \ldots, y_m] \in \mathbb{R}^{n \times m} \) and \( \text{cone}(\mathcal{Y}) \) be a frame cone of \( \mathbb{R}^n \) such that (2.6) holds, i.e., \( \mathcal{Y}^T \cap \mathbb{R}^+ \neq \emptyset \). Then \( \mathcal{Y}^T \cap \mathbb{R}^+ \) contains \( n \)-linearly independent vectors.

**Proof:** If \( m = n \), then the \( n \)-column vectors of \( \mathcal{Y} \) are a basis of \( \mathbb{R}^n \). Naturally, in this case, \( \mathcal{Y}^T = \mathbb{R}^n \) and the result holds. We next prove the lemma for the case of \( m > n \). Without losing generality, we assume that the first \( n \)-column vectors \( \{y_1, \ldots, y_n\} \) of \( \mathcal{Y} \) form a basis of \( \mathbb{R}^n \). Let \( z_1 := (z_1, \ldots, z_m)^T \in (\mathcal{Y}^T)^{-1}y_1 \). Denote
\[
a_1 := (a_{11}, \ldots, a_{1m})^T = (\mathcal{Y}^T)^{-1}y_1.
\]By (3.9), \( a_1 \) is the solution to (3.10) with \( z \) being replaced by \( z_1 \). Recall that \( \{y_1, \ldots, y_n\} \) of \( \mathcal{Y} \) is a basis of \( \mathbb{R}^n \). Then \( a_1 \) can be also expressed as \( \{y_1, \ldots, y_n\}^T(z_1, \ldots, z_n)^T \). Since the set of all the \( n \times n \) invertible matrices is dense in \( \mathbb{R}^{n \times n} \), there exist \( g_k := (g_{1k}, \ldots, g_{nk})^T \in \mathbb{R}^{n \times n} \) for \( k = 2, \ldots, n \) such that
\[
A_n := \begin{bmatrix}
   z_{11} & z_{12} & \cdots & z_{1n} \\
   z_{21} & z_{22} & \cdots & z_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   z_{n1} & z_{n2} & \cdots & z_{nn}
\end{bmatrix}
\]is invertible and
\[
\max\{\|y_1, \ldots, y_n\|_\infty : k = 2, \ldots, n\} < \min\{\|z_{11}, \ldots, z_{1n}\|_\infty : k = 2, \ldots, n\}.
\]For \( k = 2, \ldots, n \), define
\[
a_k := [y_1, \ldots, y_n]^T((z_{11}, \ldots, z_{1n})^T + g_k).
\]Now it follows from (3.11), (3.12) and (3.13) that
\[
[z_{11}, \ldots, z_{1n}] := (\mathcal{Y}^T)\{a_1, \ldots, a_n\} \in \mathbb{R}^+ \times \mathbb{R}^+.
\]That is, \( z_1 \in \mathcal{Y}^+ \cap \mathbb{R}^+ \). Using (3.14) again, the invertible matrix \( A_n \) consist of the first \( n \) rows of
\[
[z_1, \ldots, z_n].
\]Thus \( \text{rank}([z_1, \ldots, z_n]) = n \), and the proof is concluded.

We also need circulant matrices that ensure fast computation (More details about this topic can be referred to [Wu]). For a vector \( p := (p_0, \ldots, p_{n-1})^T \in \mathbb{C}^n \), its discrete Fourier transform (DFT) \( \hat{p} = (\hat{p}_0, \ldots, \hat{p}_{n-1})^T \) is defined by \( \hat{p}_k = \sum_{l=0}^{n-1} p_l e^{-2\pi ik/n} \). For the row vector \( p^T \), we denote its generating circulant matrix by \( \text{circ}(p^T) \), namely,
\[
\text{circ}(p^T) = \begin{bmatrix}
p_0 & p_1 & p_2 & \cdots & p_{n-1} \\
p_{n-1} & p_0 & p_1 & \cdots & p_{n-2} \\
p_{n-2} & p_{n-1} & p_0 & \cdots & p_{n-3} \\
p_{n-3} & p_{n-2} & p_{n-1} & \cdots & p_{n-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_1 & p_2 & \cdots & p_{n-1} & p_0
\end{bmatrix}
\]
The circulant matrix \( \text{circ}(p^T) \) can be decomposed by DFT via
\[
\text{circ}(p^T) = nF\text{diag}(\hat{p}_0, \ldots, \hat{p}_{n-1})F^*,
\]where \( F \) is the scaled DFT matrix
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & W & W^2 & \cdots & W^{n-1} \\
1 & W^2 & W^4 & \cdots & W^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & W^{n-1} & W^{(n-1)2} & \cdots & W^{(n-1)(n-1)}
\end{bmatrix},
\]with \( W = e^{-2\pi i/n} \). For any \( x \in \mathbb{R}^n \), by the fast Fourier transform (FFT), the computation of \( \text{circ}(p^T)x \) only costs \( O(n\log n) \)-number of operations. The \( l_0 \)-norm \( \|x\|_0 \) of any vector \( x \) is defined as the number of its nonzero coordinates. By (3.15), the circulant matrix \( \text{circ}(p^T) \) is invertible if and only if \( \|\hat{p}\|_0 = n \).

The following lemma tells us how to explicitly construct a special anchor vector \( p \) of \( \mathcal{Y} \) in Lemma 3.1 such that \( \|\hat{p}\|_0 = n \). It will be seen in the proof of Theorem 2.5 that such an anchor vector is crucial for explicitly constructing a special class of measurement vectors that will satisfy the requirements of Theorem 2.5.

**Lemma 3.2:** Let the frame cone \( \text{cone}(\mathcal{Y}) \) of \( \mathbb{R}^n \) be as in Lemma 3.1 such that \( \mathcal{Y}^T \cap \mathbb{R}^+ \neq \emptyset \). Then there exists an anchor vector \( p \in \text{invim}(\mathcal{Y}^T \cap \mathbb{R}^+) \) such that \( \|\hat{p}\|_0 = n \).

**Proof:** As in the proof of Lemma 3.1, we assume that the first \( n \)-column vectors \( \{y_1, \ldots, y_n\} \) of \( \mathcal{Y} \) form a basis of \( \mathbb{R}^n \). For convenient notation, denote \( \mathcal{Y}_n := [y_1, \ldots, y_n] \). By Lemma 3.1, there exist \( n \)-linearly independent vectors \( z_k := (z_{1k}, \ldots, z_{nk})^T \in \mathcal{Y}^T \cap \mathbb{R}^+ \), where \( k = 1, \ldots, n \). Define \( K_n := [z_1, z_2, \ldots, z_n] \), and as in (3.13), \( a_k := \mathcal{Y}_n^T(z_{1k}, \ldots, z_{nk})^T \). Then
rank([a_1, \ldots, a_n]) = n. Moreover, by (3.19), \[ a_1, \ldots, a_n \]  = (Y^T)^{-1} Y K_n. Therefore, rank(Y K_n) = n. Now for any fixed \( l \in \{1, 2, \ldots, n\} \), there exists a column vector \( \tilde{a}_l := (\tilde{a}_{l,1}, \ldots, \tilde{a}_{l,n})^T \) of 

\[
[a_1, \ldots, a_n] = F[a_1, \ldots, a_n] = F(Y^T)^{-1} Y K_n
\]
such that

\[
\tilde{a}_{l, \ell} \neq 0. \tag{3.16}
\]

If not, then it is easy to conclude that \( A(l,:) Y K_n = 0 \), where \( A(l,:) \) is the \( l \)-th row of \( A := (Y^T)^{-1} \). From the invertibility of \( Y K_n \), we deduce that \( A(l,:) = 0 \), which is a contradiction with the invertibility of \( A \).

Pick a vector \( \tilde{a}_l \in \{a_1, \ldots, a_n\} \). If \( \|\tilde{a}_l\|_0 = n \), then the proof is completed by letting \( p := F^* \tilde{a}_l \). Otherwise, by the property (3.16), there exists \( \tilde{a}_j \in \{a_1, \ldots, a_n\} \) such that \((\text{supp}(\tilde{a}_j))_c \cap \text{supp}(\tilde{a}_l) \neq \emptyset \), where \( \text{supp}(\tilde{a}_j) \) is the support of \( \tilde{a}_j \), and \((\text{supp}(\tilde{a}_j))_c \cap \text{supp}(\tilde{a}_l) = \{1, 2, \ldots, n\} \setminus \text{supp}(\tilde{a}_j) \). It is easy to prove that \( \|r_{\tilde{a}_l} + \tilde{a}_j\|_0 \geq \|\tilde{a}_l\|_0 + 1 \), where \( \nu > \max_{\ell \in \text{supp}(\tilde{a}_l)} \left| \tilde{a}_{l, \ell} \right| \).

On the other hand, it is obvious that \( \nu \tilde{a}_l + \tilde{a}_j \in F(Y^T)^{-1} Y (R(Y^T) \cap R^{+m}) \). Thus by at most \( n \)-procedures discussed above, we will be able to get a vector \( \tilde{a} \in F(Y^T)^{-1} Y (R(Y^T) \cap R^{+m}) \) such that \( \|\tilde{a}\|_0 = n \). Therefore

\[
p := F^* \tilde{a} \tag{3.17}
\]
is an anchor vector satisfying \( \|p\|_0 = n \).

Next based on the proofs of Lemmas 3.1 and 3.2 we establish Algorithm 2 for designing an anchor vector \( p \in \text{invim}(R(Y^T) \cap R^{+m}) \) such that \( \|p\|_0 = n \).

**B. Proof of Theorem 2.5**

The proof will be concluded from two cases: frame cone and non-frame cone.

1) **cone(\( X \)) is a frame cone**: Obviously, \( \gamma = \text{rank}(X) = n \). By Algorithm 2 we can construct an anchor vector \( p_1 \in \text{invim}(R(Y^T) \cap R^{+m}) \) such that

\[
\|p_1\|_0 = n. \tag{3.18}
\]

Thus the circulant matrix \( \text{circ}(p_1^T) \) is invertible. Denote \( \text{circ}(p_1^T) = [p_1, p_2, \ldots, p_n]^T \). Let \( f_1 := p_1 \) and design \( \{f_k\}_{k=2}^n \) by

\[
f_k = \delta_k p_1 + p_k, k \geq 2, \tag{3.19}
\]

where \( \delta_k > 0 \) is selected in such a way that any \( x_t \in \{x_1, \ldots, x_m\} \) satisfies

\[
(x_t, f_k) > 0. \tag{3.20}
\]

**Algorithm 2**: Based on \( q_1 \in \text{invim}(R(Y^T) \cap R^{+m}) \), design \( p \in \text{invim}(R(Y^T) \cap R^{+m}) \) such that \( \|p\|_0 = n \).

**Input**: \( Y = [y_1, \ldots, y_m] \in \mathbb{R}^{n \times m} \), \( q_1 \in \text{invim}(R(Y^T) \cap R^{+m}) \), \( z_1 = Y^T q_1 \), \( \bar{q}_1 = F q_1 \).

1. if \( \|\bar{q}_1\|_0 < n \) then
2. Extend \( z_1 \) to linearly independent vectors \( \{z_k\}_{k=1}^n \subseteq R(Y^T) \cap R^{+m} \) by using (3.13) and (3.14); \( q_2, \ldots, q_n \leftarrow F(Y^T)^{-1} Y [z_2, \ldots, z_n]; \)
3. for \( j = 2 : n \) do
4. Find \( \tilde{q}_j \in \{q_2, \ldots, q_n\} \) such that \( (\text{supp}(q_j))_c \cap \text{supp}(\tilde{q}_j) \neq \emptyset \). Pick \( \nu > \max_{\ell \in \text{supp}(q_j)} \left| \tilde{q}_{j, \ell} \right| \), \( \tilde{q}_j \leftarrow \nu \tilde{q}_j + \tilde{q}_j; \)
5. if \( \|q_j\|_0 = n \) then
6. \( \text{break; } \)
7. end
8. end
9. end

**Output**: \( p = F^* \tilde{q}_1 \).

It follows from (3.20) that \( \text{sgn}(z, f_k) \geq 0 \) for any \( z \in \text{cone}(X) \) and \( k = 2, \ldots, n \). On the other hand, it is easy to follow from

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\delta_2 & 1 & 0 & \cdots & 0 \\
\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix}
= 
\begin{bmatrix}
p_1^T \\
p_2^T \\
p_3^T \\
\vdots \\
p_n^T
\end{bmatrix}
= 
\begin{bmatrix}
f_1^T \\
f_2^T \\
f_3^T \\
\vdots \\
f_n^T
\end{bmatrix} \tag{3.21}
\]

that \( \{f_k\}_{k=1}^n \) is a basis of \( C^n \). Thus the target signal \( z \in \text{cone}(X) \) can be determined, up to a global sign, by the following linear system of equations

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\delta_2 & 1 & 0 & \cdots & 0 \\
\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
p_1^T \\
p_2^T \\
p_3^T \\
\vdots \\
p_n^T
\end{bmatrix} = 
\begin{bmatrix}
|z, f_1| \\
|z, f_2| \\
|z, f_3| \\
\vdots \\
|z, f_n|
\end{bmatrix}.
By (3.15), the above system can be rewritten as
\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\delta_2 & 1 & 0 & \cdots & 0 \\
\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix} 
\cdot 
F \text{diag}(\tilde{p}_0, \ldots, \tilde{p}_{n-1}) F^* \mathbf{z} 
= 
\begin{bmatrix}
|\langle \mathbf{z}, \mathbf{f}_1 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_2 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_3 \rangle| \\
\vdots \\
|\langle \mathbf{z}, \mathbf{f}_n \rangle|
\end{bmatrix}.
\]

That is, up to a global sign, \( \mathbf{z} \) can be recovered by
\[
\mathbf{z} = 
\text{FFT} \left( \text{diag}^{-1} (\text{FFT}(\mathbf{p}_1^T)) \text{IFFT} \right) 
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\delta_2 & 1 & 0 & \cdots & 0 \\
-\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix} 
\begin{bmatrix}
|\langle \mathbf{z}, \mathbf{f}_1 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_2 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_3 \rangle| \\
\vdots \\
|\langle \mathbf{z}, \mathbf{f}_n \rangle|
\end{bmatrix}.
\tag{3.22}
\]

It is easy to see that the computational complexity of (3.22) is \( O(n \log n) \).

2) \textit{cone}(\( \mathcal{X} \)) \textit{is not a frame cone}: Denote \( \gamma := \text{rank}(\mathcal{X}) \). Then \( \gamma < n \). Define an isometry \( \mathcal{P} : \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \rightarrow \mathbb{R}^\gamma \). Specifically,
\[
\mathcal{P} (\mathbf{e}_k) = \mathbf{e}_k, k = 1, \ldots, \gamma,
\tag{3.23}
\]
where \( \{\mathbf{e}_k\}_{k=1}^\gamma \) and \( \{\mathbf{e}_k\}_{k=1}^\gamma \) are the orthonormal basis and the standard orthonormal basis of \( \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \) and \( \mathbb{R}^\gamma \), respectively. Denote \( \mathcal{Y} := \mathcal{P} \mathcal{X} \). By the linear and isometry property, \( \mathcal{P} (\text{cone}(\mathcal{X})) = \text{cone}(\mathcal{Y}) \), and \( \mathcal{Y} \) also satisfies the overlap property (2.6). By Algorithm 2, we can design an anchor vector \( \mathbf{\tilde{p}}_1 \in \mathbb{R}^\gamma \) of \( \mathcal{Y} \) such that \( \mathcal{Y}^T \mathbf{\tilde{p}}_1 > 0 \) and \( ||\mathbf{\tilde{p}}_1||_2 = \gamma \).

Denote \( \mathbf{\tilde{f}}_1 := \mathbf{\tilde{p}}_1 \in \mathbb{R}^\gamma \). Invoking Case III-B1 for \( n = \gamma \), we can additionally design \( (\gamma - 1) \) vectors \( \{\mathbf{f}_k\}_{k=2}^\gamma \) such that \( \{\mathbf{f}_k\}_{k=1}^\gamma \) are phase retrievable for \( \text{cone}(\mathcal{Y}) \). That is, any signal \( \mathbf{z} \in \text{cone}(\mathcal{Y}) \) can be determined by the \( \gamma \) magnitude measurements \( \{|\langle \mathbf{z}, \mathbf{f}_k \rangle|\}_{k=1}^\gamma \), and the corresponding complexity is \( O(\gamma \log \gamma) \). Particularly, for the target \( \mathbf{z} \), its projection \( \mathcal{P} \mathbf{z} \) can be recovered by invoking (3.22), namely,
\[
\mathcal{P} \mathbf{z} = \text{FFT} \left( \text{diag}^{-1} (\text{FFT}(\mathbf{\tilde{p}}_1^T)) \right) 
\times \text{IFFT} \left( 
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\delta_2 & 1 & 0 & \cdots & 0 \\
-\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix} 
\cdot 
\begin{bmatrix}
|\langle \mathbf{z}, \mathbf{f}_1 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_2 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_3 \rangle| \\
\vdots \\
|\langle \mathbf{z}, \mathbf{f}_n \rangle|
\end{bmatrix} \right)
\],
\tag{3.24}

where the constants \( \{\delta_k\}_{k=2}^\gamma \) satisfy (3.20) with \( n, \mathcal{X} \) and \( \mathbf{p}_1 \) being replaced by \( \gamma, \mathcal{Y} \) and \( \mathbf{\tilde{p}}_1 \), respectively. Now define \( \mathbf{f}_k := \mathcal{P}^{-1} \mathbf{f}_k, k = 1, \ldots, \gamma \). By the isometry property, we have \( |\langle \mathcal{P} \mathbf{z}, \mathbf{f}_k \rangle| = |\langle \mathbf{z}, \mathbf{f}_k \rangle| \). Then the recovery formula (3.22) can be rewritten as
\[
\mathcal{P} \mathbf{z} = \text{FFT} \left( \text{diag}^{-1} (\text{FFT}((\mathcal{P} \mathbf{f}_1)^T)) \right) 
\times \text{IFFT} \left( 
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\delta_2 & 1 & 0 & \cdots & 0 \\
-\delta_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_n & 0 & 0 & \cdots & 1
\end{bmatrix} 
\cdot 
\begin{bmatrix}
|\langle \mathbf{z}, \mathbf{f}_1 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_2 \rangle| \\
|\langle \mathbf{z}, \mathbf{f}_3 \rangle| \\
\vdots \\
|\langle \mathbf{z}, \mathbf{f}_n \rangle|
\end{bmatrix} \right)
\],
\tag{3.25}

Denote \( \mathcal{P} \mathbf{z} = \sum_{k=1}^\gamma c_k \mathbf{e}_k \). Then
\[
\mathbf{z} = \mathcal{P}^{-1} \mathcal{P} \mathbf{z} = \sum_{k=1}^\gamma c_k \mathbf{e}_k
\]
which costs \( O(\gamma) \) operations. Then the total complexity is \( O(\gamma \log \gamma) + O(\gamma) = O(\gamma \log \gamma) \). Integrating Subsection III-B1 and III-B2 the proof is concluded.

C. Algorithm for designing measurement vectors for a cone satisfying the overlap property (2.6)

Based on Algorithm 2 and Subsection III-B1 (the proof of Theorem 2.3), we propose the following Algorithm 3 for explicitly constructing rank(\( \mathcal{X} \))-vectors that can be used to perform the fast phase-retrieval for \( \text{cone}(\mathcal{X}) \).

The existence of \( \{\delta_k\}_{k=2}^\gamma \) in Algorithm 2 is guaranteed by the following remark.

Remark 3.3: There are many choices for the sequence \( \{\delta_k\}_{k=2}^\gamma \) in (3.26). For example, for any \( k \in \{2, \ldots, m\} \), if
\[
\delta_k > \frac{||\mathbf{\tilde{p}}_k||_2}{\max\{||\mathbf{y}_i||_2 : i = 1, \ldots, m\}} \cdot \kappa_{\min},
\tag{3.28}
\]


\begin{algorithm}
\caption{Designing rank($\chi$)-vectors for the fast phase-retrieval of cone($\chi$) satisfying the isometry property (2.6).}
\begin{itemize}
    \item \textbf{Input:} $\chi = [x_1, \ldots, x_m] \in \mathbb{R}^{n \times m}$, $q_k \in \text{invim}(R(\chi^T) \cap \mathbb{R}^{+m})$, $\gamma = \text{rank}(\chi)$, and an isometry (an $\gamma \times n$ matrix) $\Psi : \text{span}\{x_1, \ldots, x_m\} \rightarrow \mathbb{R}^\gamma$. If $\gamma = n$, then we just pick $\Psi$ as the identity matrix.
    \item Suppose that the target $z$ can be recovered by the following two procedures:
    \begin{enumerate}
        \item $Y \leftarrow \Psi^T \chi$: $q_1 \leftarrow \Psi q_1$.
        \item If $|q_k| < \gamma$, then using $q_k$, design an anchor vector $p_1 \in \text{invim}(R(Y^T) \cap \mathbb{R}^{+m})$ by Algorithm 2 such that $|\langle p_1 \rangle| = \gamma$.
    \end{enumerate}
\end{itemize}
\begin{itemize}
    \item \textbf{Output:} $f_k \leftarrow \Psi^{-1} f_k$, $k = 1, \ldots, \gamma$.
\end{itemize}
\end{algorithm}

where $\gamma_{\min} = \min\{{|\langle x_i, p_1 \rangle|, \ldots, |\langle x_m, p_1 \rangle|}\}$, then for any $x_i \in \{x_1, \ldots, x_m\}$ it follows from (3.26) and (3.28) that

$$
\langle x_i, f_k \rangle = |\langle y_i, f_k \rangle| - |\langle y_i, p_k \rangle| = |\langle x_i, p_1 \rangle| - |\langle x_i, p_k \rangle| \geq \delta_k \gamma_{\min} - \|p_k\|_2 \max\{|\|y_i\|_2 : i = 1, \ldots, m\} \geq 0.
$$

\subsection{D. Recovery formula}

In this subsection we abstract the recovery formula from Subsection III-B. Suppose that the target $z$ lies in the detectable UoC $\bigcup_{k=1}^L \text{cone}(\chi_k)$. After the detection we find that $z \in \text{cone}(\chi_k)$. If $\text{cone}(\chi_k)$ satisfies (2.6), then $z$ can be recovered by the following two procedures:

\begin{enumerate}
    \item \textbf{P1:}
    \begin{align}
    \Phi z &= \sum_{k=1}^{\gamma} c_k e_k \\
    &= \text{FFT}^{-1} \left( \text{diag}^{-1} \left( \text{FFT}(\Phi^T f_k) \right) \right) \\
    &\times \text{FFT} \left( \begin{bmatrix}
0 & 0 & \cdots & 0 \\
-\delta_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\delta_\gamma & 0 & \cdots & 1
\end{bmatrix} \right)
\end{align}
\end{enumerate}

\begin{enumerate}
    \item \textbf{P2:}
    \begin{align}
    z &= \Psi^{-1} \Psi f = \sum_{k=1}^{\gamma} c_k e_k.
    \end{align}
\end{enumerate}

The following note is helpful for conducting (3.29) and (3.30).

Note 3.4: (i) $\gamma = \text{rank}(\chi_k)$. (ii) $\{\bar{e}_k\}_k$ and $\{e_k\}_k$ are the orthonormal basis and the standard orthonormal basis of $\text{span}\{x_{k,1}, \ldots, x_{k,m_k}\}$ and $\mathbb{R}^\gamma$, respectively. The map $\Psi : \text{span}\{x_{k,1}, \ldots, x_{k,m_k}\} \rightarrow \mathbb{R}^\gamma$ is an isometry (an $\gamma \times n$ matrix). As in Algorithm 3 we just set $\Psi$ to the identity matrix when $\gamma = n$. (iii) The measurement vectors $\{f_k\}_{k=2}^\gamma$ are designed by Algorithm 3 with $\chi$ therein being replaced by $\chi_k$, and the sequence $\{\tilde{e}_k\}_{k=2}^\gamma$ satisfies the requirement in (3.26).

\subsection{E. Stability of the recovery formula (3.29) and (3.30)}

Since the measurements are often contaminated by noise in practice, we need to establish the stability for the recovery in Subsection III-D (3.29) and (3.30) in the noisy setting. We will consider the model for observing a measurement in the noisy setting:

$$
|\langle q, z \rangle| = |\langle q, z \rangle| + n.
$$

where $q$ represents any measurement vector and the additive noise $n$ obeys the Gaussian distribution, namely,

$$
n \sim N(0, \sigma^2).
$$

The chi-square distribution $\chi^2(s)$ with $s$ degrees of freedom will be useful for probability estimation. Its density function is

$$
\rho_s(x) = \begin{cases} 
\frac{1}{2^s G(s)} x^{s/2-1} e^{-x/2}, & x > 0, \\
0, & x \leq 0,
\end{cases}
$$

with the Gamma function $G(t) := \int_0^\infty x^{t-1} e^{-x} dx$. Denote the distribution function by $\Phi_s(x) := \int_0^x \rho_s(t) dt$.

\textbf{Theorem 3.5:} Let the target $z \in \text{cone}(\chi_k)$ be as in Subsection III-D. Consequently, it can be recovered by (3.29) and (3.30). Suppose that the measurements $|f_k, z|$ used for (3.29) is contaminated by the noise $n_k$ obeying the Gaussian distribution in (3.32), $k = 1, \ldots, \gamma$. Then for any fixed $\epsilon > 0$, with at least the following probability

$$
P_{\gamma-1} \left( \frac{\epsilon}{\gamma \sigma^2} \right) \\
= -1 + \Phi_{\gamma-1}(\gamma - 1 - \frac{\epsilon}{\gamma \sigma^2}) + \Phi_1(1 + \frac{\epsilon}{\gamma \sigma^2}) - \Phi_1(1 - \frac{\epsilon}{\gamma \sigma^2}),
$$

the recovery error is bounded by

$$
\min \{||z - z_0||_2, ||z + z_0||_2\} \\
\leq \sqrt{2||z||^2 + \max\{|\delta_1, \ldots, \delta_k\} (\gamma - 1)\epsilon + \frac{\epsilon}{\gamma \sigma^2} ||z||^2},
$$

(3.34)
where \( \mathbf{n} = (n_1, \ldots, n_r) \) and \( \overline{\mathbf{z}}_r \) is the recovery result from (3.29) and (3.30) in the noisy setting.

**Proof:** The proof is given in the Appendix section. \( \square \)

The graphs of \( P_{\gamma^{-1}}(\frac{\gamma}{\sigma^2}) \) in (3.33) corresponding to \( \gamma = 81, 801 \) and 8001 are plotted in Fig. III.1. It is observed in Fig. III.1 that as \( \gamma = \text{rank}(\mathbf{X}) \) increases, the behavior of \( P_{\gamma^{-1}}(\frac{\gamma}{\sigma^2}) \) changes very mildly.

**IV. NUMERICAL SIMULATION**

We have established in Section III the two-step PR-scheme for detectable UoCs (Algorithm 1) for detection while (3.29) and (3.30) for recovery. As mentioned in Proposition 2.7, it requires very few measurements and has low computational complexity. On the other hand, as introduced in Section I some efficient phase retrieval methods are available in the literature. As an iterative method, Alternating Minimization [25] converges geometrically to the target, and shows good performance on recovery accuracy. BlockPR [20] performs well on the aspect of computational speed. The task of this section is to present some numerical simulations demonstrating the efficiency of the two-step PR-scheme, and to compare with Alternating Minimization and BlockPR on the aspects of time cost, measurement cost (the amount of measurements) and relative error (recovery accuracy).

**A. Two-step PR-scheme for random signals in the noiseless setting**

Let \( \mathbf{X}_1 := [\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,2n-1}] \in \mathbb{R}^{n \times (2n-1)} \). Herein

\[
\mathbf{x}_{1,1}^T = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\frac{1}{1+\gamma} & \frac{1}{1+\gamma} & \cdots & \frac{1}{1+\gamma} \\
\frac{1}{1+\gamma} & \frac{1}{1+\gamma} & \cdots & \frac{1}{1+\gamma} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{1+\gamma} & \frac{1}{1+\gamma} & \cdots & \frac{1}{1+\gamma}
\end{bmatrix}_{n \times (n-1)},
\]

and for \( 2 \leq k \leq n \),

\[
\mathbf{x}_{1,k}^T = \mathbf{x}_{1,1}^T \odot \mathbf{i}_k,
\]

where \( \mathbf{i}_k := [1, \ldots, 1, -1, 1, \ldots, 1] \) with \( -1 \) being the \( k \)-th element, \( \odot \) is the element-wise product of two vectors, and

\[
\begin{bmatrix}
\mathbf{x}_{1,n+1}, \ldots, \mathbf{x}_{1,2n-1}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{y}_1 & \mathbf{y}_1 & \cdots & \mathbf{y}_1 \\
\mathbf{y}_2 & \mathbf{y}_2 & \cdots & \mathbf{y}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{y}_n & \mathbf{y}_n & \cdots & \mathbf{y}_n
\end{bmatrix}_{n \times (n-1)},
\]

with \( a = 0.115, b = 0.8850 \). Furthermore define

\[
\mathbf{X}_2 := [\mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2,n}] = 
\begin{bmatrix}
\mathbf{2} & \mathbf{2} & \cdots & \mathbf{2} \\
\mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\
(-1)^{1+3} & (-1)^{1+2} & \cdots & (-1)^{1+n} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{n+1} & (-1)^{n+2} & \cdots & (-1)^{n+n}
\end{bmatrix}_{n \times n}.
\]

Pick \( \mathbf{g} := [1, 2, 0, \ldots, 0]^T \). By the direct computation, we found that

\[
\mathbf{X}_1^T \mathbf{g} \succ 0, \mathbf{X}_2^T \mathbf{g} = 0. \quad (4.36)
\]

That is, \( \bigcup_{k=1}^2 \text{cone}(\mathbf{X}_k) \) is detectable and \( \mathbf{g} \) is eligible as a detector. Moreover it is easy to check that both cone(\( \mathbf{X}_1 \)) and cone(\( \mathbf{X}_2 \)) satisfy the property (2.6).

Any signal \( \mathbf{z} = \sum_{k=1}^2 \mathbf{x}_{1,k} \in \text{cone}(\mathbf{X}_1) \) (4.37) as an example to check the efficiency of (4.3), where the random variable \( \mathbf{z} \) obeys the uniform distribution on the interval \( (0, 1/100) \). In this case, \( \gamma = \text{rank}(\mathbf{X}_1) = n \) and as mentioned in Note 5.4 the isometry \( \mathcal{F} \) in (5.29) and (5.30) is set to the identity matrix. There are many choices of \( \mathbf{q}_1 \in \text{inv}(\mathbb{R}(\mathbf{X}_1^T) \cap \mathbb{R}_+^{2n-1}) \) in Algorithm 5. For example choose \( \mathbf{q}_1 = (1, 0, 0, \ldots, 0)^T \). Based on \( \mathbf{q}_1 \), we design \( \{\mathbf{f}_k\}_{k=1}^n \) by using Algorithm 3 such that \( \mathbf{f}_1 = \mathbf{q}_1 \) and \( \{\mathbf{f}_k\}_{k=1}^n \) are given by (5.26) with \( \delta_k = 0.0542 \). By direct computation, we can check that both (3.18) and (3.20) hold with \( \mathbf{p}_1 \) therein being replaced by \( \mathbf{f}_1 \). Therefore the \( n+1 \) vectors \( \{\mathbf{g}, \mathbf{f}_1, \ldots, \mathbf{f}_n\} \) are phase retrievable for the target signal. Specifically, Algorithm 1 is conducted by using the detector \( \mathbf{g} \). After the detection we found \( \mathbf{z} \in \text{cone}(\mathbf{X}_1) \), and the recovery formula (3.29) is conducted by the magnitude measurements \( \{\|\mathbf{z}, \mathbf{f}_1\|, \ldots, \|\mathbf{z}, \mathbf{f}_n\|\} \).

We include the simulation results of Alternating Minimization and BlockPR for comparison. Incidentally, we
use the matlab software available in [33] to conduct the BlockPR. The relative recovery error is defined by

\[
\text{error} := 10 \log_{10} \left[ \min \{ \|z - z_r\|_2 / \|z\|_2, \|z + z_r\|_2 / \|z\|_2 \} \right],
\]

and it is reported in dB, where \(z_r\) is the recovery result. Here we say that a target is successfully recovered if the error is smaller than \(-30\) dB \((\min \{ \|z - z_r\|_2 / \|z\|_2, \|z + z_r\|_2 / \|z\|_2 \} \leq 0.1\%\)). Real-valued standard Gaussian measurements are used for Alternating Minimization. We found in this simulation that the BlockPR with the Fourier-like measurements performs better than that with random measurements. Therefore, we use the Fourier-like measurements for BlockPR. We next compare the measurement cost, relative recovery error and time cost of the three methods.

Recall that the two-step PR-scheme just requires \(n + 1\) measurements, and the BlockPR software requires at least \(3n\) measurements. Therefore for comparing the computing time at the same amount of measurements, we first recover \(z\) in \((4.37)\) by the two-step PR-scheme and Alternating Minimization, respectively. Both the two methods are conducted by \(n + 1\) measurements for 100 trials, and their average time costs and errors are recorded. In Fig. IV.2 and Fig. IV.3 we plotted the numerical results on time cost and errors. It is observed from the black solid curve in Fig. IV.2 that Alternating Minimization has the computational complexity which essentially scales squarely with the problem size \(n\). Actually it follows from [25] that the theoretic computational complexity of Alternating Minimization is \(O(n^2 \log^2 n (\log n + \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}))\), where \(\epsilon\) is the computing accuracy. By Proposition [27] the two-step PR-scheme has the FFT computational complexity \(O(n \log n)\) instead. Obviously, in this simulation the two-step PR-scheme costs much less time than Alternating Minimization.

The curve (in red) in Fig. IV.3 affirms that, just requiring \(n + 1\) measurements, \(z\) can be perfectly recovered by the two-step PR-scheme. By direct observation on the solid black curve in Fig. IV.3, \(n + 1\) measurements are obviously not sufficient for Alternating Minimization, which is in accordance with [25]. That is, for successfully recovering \(z\), more measurements are necessary.

Next we continued the simulation for recovering \(z\), where \(4n\) measurements are used for Alternating Minimization, and \(3n\) measurements for BlockPR. We observed from Fig. IV.2 that BlockPR has essentially the FFT computational complexity, but the two-step PR-scheme has a much smaller constant than BlockPR. Although Alternating Minimization and BlockPR cost much more measurements and computing time, it is observed from Fig. IV.3 that the error of the two-step PR-scheme is much smaller than theirs. On the other hand, the error of Alternating Minimization is the second smallest (black and dash curve in Fig. IV.3). Recall that Alternating Minimization is an iterative method. Besides on the amount of random measurements (the more measurements are used,
the better performance it shows with higher probability), the recovery error also depends on the convergence to the target. By [25], Alternating Minimization converges geometrically to the target $z$. Recall that our scheme (1.3) is conducted by Algorithm 1 and recovery formula (3.29). Neither of the two steps is iterative instead, and consequently our scheme is free from the convergence problem in the computation. Therefore our recovery error mainly depends on the computing round-off error. We observed from Fig. IV.3 that the round-off error is very small.

B. Two-step PR-scheme for the random signals in the noisy setting

In this subsection we check the stability to noise of our scheme (1.3) in the previous simulation, where any measurement $|\langle q, z \rangle|$ was contaminated by the Gaussian noise

$\tilde{n} \sim N(0, \sigma^2)$.

That is, what we observed is

$|\langle q, z \rangle| = |\langle q, z \rangle| + \tilde{n}$.  \hspace{1cm} (4.39)

Since the stability for the recovery formula (3.29) has been given in Theorem 3.5 we just need to establish the stability for detection before conducting the numerical simulation in the noisy setting. As already shown in Algorithm 1 (steps 5-7, steps 11-13), the threshold technique (the threshold value therein is 0) was substantially used in the detection strategy in the noiseless setting. Based on (4.36), for the detection in the noisy setting we need to modify the threshold technique in Algorithm 1 as follows.

For $\cup_{k=1}^2 \text{cone}(X_k)$ and a given threshold value $T$, if the noisy measurement $|\langle q, z \rangle| \geq T$, then the target $z$ is regarded as not in $\text{cone}(X_2)$, or else not in $\text{cone}(X_1)$.
A natural problem is how to choose the threshold value $T$ such that the target cone can be detected successfully by the detection strategy associated with the above technique. We give an answer in the following proposition where the lower bound of $\theta_1 + \cdots + \theta_{m_k}$ is regarded as the prior information (The similar information was also necessary for the stability of the phase-retrieval in shift-invariant space (Q. Sun et. al \cite{11}, \cite{12}).

**Proposition 4.1:** Suppose that the target $z = \theta_1 x_{k,1} + \cdots + \theta_{m_k} x_{k,m_k} \in \text{cone}(A_k)$ where $k \in \{1, 2\}$, $m_1 = 2n - 1$ and $m_2 = n$. If $\theta_1 + \cdots + \theta_{m_k} \geq r > 0$, then choosing the threshold value $T := \frac{r}{\sqrt{\text{det}(A_k)^2}}$, with at least the probability $\int_{-\infty}^{T} \min \lambda_i^T g$ for at least the probability $\int_{-\infty}^{T} \min \lambda_i^T g$, the target cone can be successfully detected by the modified strategy in the previous square frame, where $\min \lambda_i^T g$ is the minimum of the $2n - 1$ coordinates of the vector $\lambda_i^T g$.

**Proof:** The proposition is proved in the Appendix section.

To check the stability of the scheme in (1.3), we conduct the simulation in Subsection [V-A] by adding the Gaussian noise to the magnitude measurements. Following [20], the variance is chosen such that the desired signal to noise ratio (SNR) is expressed by

$$\text{SNR} = 10 \log_{10} \left( \frac{\|Mf\|^2}{m\sigma^2} \right),$$

where $M$ is the measurement matrix having $m$ column vectors. SNR is also reported in dB. For the scheme (1.3) in the simulation, $M = [g, f_1, \ldots, f_n]$. We conducted the two-step PR scheme, Alternating Minimization and BlockPR on the random signal $z$ in (4.37) for 100 trials, where $50 \leq n \leq 500$. We plotted the average error to the noise level in Fig. IV.4 ($n = 50$) and Fig. IV.5 ($n = 500$), and the average computing time to the dimension size $n$ in Fig. IV.6.

It was observed from Fig. IV.4-5 that for successfully recovering the target $z$ (i.e. the error is smaller than $-30$ dB), the requirement on the noise level of the two-step PR scheme is weakest. As SNR being large sufficiently, the two-step PR scheme has the smallest error among the three methods, which coincides with the results in noiseless setting as shown in Fig. IV.3. Moreover, Fig. IV.6 confirms again that the two-step PR scheme required the less computing time.

V. APPENDIX

A. Proof of Theorem 2.7

The proof will be concluded for the cases of $L = 2$ and $L > 2$, respectively.

**Case of $L = 2.$** For this case, (2.5) is equivalent to that either

$$\text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_i}) \cap \mathcal{N}(X_i^T) \neq \emptyset, \quad (5.40)$$

or

$$\text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_2}) \cap \mathcal{N}(X_i^T) \neq \emptyset \quad (5.41)$$

holds.

**Sufficiency:** If, for example, $\text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_1}) \cap \mathcal{N}(X_i^T) \neq \emptyset$, then we can use a measurement vector $g \in \text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_1}) \cap \mathcal{N}(X_i^T)$ as an detector to complete the detection. Specifically, for any fixed target nonzero vector $z \in \bigcup_{k=1}^{g} \text{cone}(A_k)$, if $|\langle z, g \rangle| = 0$, then $z \notin \text{cone}(A_2)$ but $z \in \text{cone}(A_1)$. If $|\langle z, g \rangle| = 0$, then $z \notin \text{cone}(A_1)$ but $z \in \text{cone}(A_2)$.

**Necessity:** Suppose that we can use an detector $g \in \mathbb{R}^n$ to detect the source of any $z \in \bigcup_{k=1}^{g} \text{cone}(A_k)$. Then

$$\{\langle g, y \rangle : y \in \text{cone}(A_1) \} \cap \{\langle g, y \rangle : y \in \text{cone}(A_2) \} = \emptyset. \quad (5.42)$$

If $\{\langle g, y \rangle : y \in \text{cone}(A_1) \} \cap \mathbb{R}^+ = \emptyset$, then it is straightforward to check that $\{\langle g, y \rangle : y \in \text{cone}(A_1) \} \supseteq \mathbb{R}^+$. Therefore (5.42) is equivalent to the condition that one of the two sets therein is $\mathbb{R}^+$ while the other is $\emptyset$. Without losing generality, we can assume that $\{\langle g, y \rangle : y \in \text{cone}(A_1) \} = \mathbb{R}^+$. This implies that $\lambda_i^T g \in \mathbb{R}^{+ \cdot m_i}$. In fact, if not, then there exist $(\theta_1, \ldots, \theta_{m_i}) > 0$ such that $(\sum_{k=1}^{m_i} \theta_k x_{1,k}, g) = 0$, which leads to a contradiction with the assumption.

**Case of $L > 2.$** Invoking the result of the case of $L = 2$, the condition in (2.5) is equivalent to that each sub-union $\text{cone}(A_k) \cup \text{cone}(A_l)$ is detectable.

**Necessity:** If the UoC $\bigcup_{k=1}^{n} \text{cone}(A_k)$ is detectable, then by Definition 1.1 each sub-union $\text{cone}(A_k) \cup \text{cone}(A_l)$ is detectable.

**Sufficiency:** When (5.40) holds for $k = 2$, for example, $\text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_1}) \cap \mathcal{N}(X_i^T) \neq \emptyset$. We pick vector $g \in \text{invim}(\mathcal{R}(X_i^T) \cap \mathbb{R}^{+ \cdot m_1}) \cap \mathcal{N}(X_i^T)$. If $|\langle g, z \rangle| > 0$, then $z \notin \text{cone}(A_2)$. Conversely, if $|\langle g, z \rangle| = 0$, then $z \notin \text{cone}(A_1)$. Now there are two cases: (a) If $z \notin \text{cone}(A_1)$, then similarly we next determine whether $z \notin \text{cone}(A_2)$ or $z \notin \text{cone}(A_3)$. (b) If $z \notin \text{cone}(A_2)$, then we next need to determine whether $z \notin \text{cone}(A_1)$ or $z \notin \text{cone}(A_3)$. The exclusion procedures can go forward due to (5.40). After $L - 1$ exclusions, we can detect the target cone where $z$ lies.

B. The proof Theorem 3.3

The measurements for the recovery (3.29) are contaminated by $n = [\tilde{\mathbf{f}}_1, \ldots, \tilde{\mathbf{f}}_n]$, namely, the measurements we obtained are

$$\{\langle \tilde{\mathbf{f}}_k, z \rangle \}_{k=1}^{n} = \{\langle \mathbf{f}_k, z \rangle + \mathbf{n}_k \}_{k=1}^{n}.$$
In the procedure of recovering $\mathfrak{P}f$, the emerging error is
\[
\text{Error} = \text{FFT} \left( \text{diag}^{-1}(\text{FFT}(\mathfrak{P}h_1)^T)) \times \text{IFFT} \left( \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\delta_2 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_\gamma & 0 & 0 & \cdots & 1 
\end{bmatrix} \right) \right).
\]
(5.43)

For any vector $z \in \mathbb{R}^\gamma$, it is easy to check that $\|\text{FFTz}\|_2 = \sqrt{n}|z|_2$ and $\|\text{IFFTz}\|_2 = \sqrt{n}|z|_2$. By this property, $\text{Error}$ in (5.43) is estimated as follows,
\[
\|\text{Error}\|_2 \leq \sqrt{\frac{2\|\text{FFT}\|_2 + \text{max}(\delta_2, \ldots, \delta_\gamma) (\gamma-1)|\mathfrak{P}|^2}{\text{min FFT}(\mathfrak{P}h_1)}}.
\]
(5.44)

We next estimate the probability
\[
P_r((\frac{n^2 - 2}{\gamma} - \frac{n^2}{\gamma}) > \epsilon)
\leq P_r((\frac{n^2 - 2}{\gamma} - \frac{n^2}{\gamma}) > \frac{\gamma-\epsilon}{\gamma-1})
\leq P_r((\frac{n^2 - 2}{\gamma} - \frac{n^2}{\gamma}) > \frac{\gamma-\epsilon}{\gamma-1})
\]
\[
= 1 - \Phi_\gamma - 1(\gamma - 1 - \frac{2\epsilon}{\gamma}) + \Phi_1(1 - \frac{\epsilon}{2})
\]
(5.45)

Therefore with the probability at least
\[
1 + \phi_{\gamma - 1}(\gamma - 1 + \frac{2\epsilon}{\gamma}) + \Phi_1(1 - \frac{\epsilon}{2(\gamma-1))})
\]
\[
-\Phi_{\gamma - 1}(\gamma - 1 - \frac{2\epsilon}{\gamma}) - \Phi_1(1 - \frac{\epsilon}{2(\gamma-1))})
\]

it holds that $P_r((\frac{n^2 - 2}{\gamma} - \frac{n^2}{\gamma}) \leq \epsilon)$. By (5.44) and (5.45), with at least the above probability, it holds that
\[
\|\text{Error}\|_2 \leq \sqrt{\frac{2\|\text{FFT}\|_2 + \text{max}(\delta_2, \ldots, \delta_\gamma) (\gamma-1)|\mathfrak{P}|^2}{\text{min FFT}(\mathfrak{P}h_1)}}.
\]
(5.46)

C. The proof Theorem 4.1

Suppose that the following event
\[
\mathfrak{n} < \frac{\epsilon}{2} \min \lambda_1^T g
\]
(5.47)
holds. If $z \in \text{cone}(\mathfrak{X}_1)$, then
\[
\langle \mathfrak{g}, z \rangle = \|g\| \Theta_1 + \cdots + \Theta_m) \min \lambda_1^T g - |\mathfrak{n}|
\geq \frac{\epsilon}{2} \min \lambda_1^T g
\]
(5.48)

On the other hand, if $z \in \text{cone}(\mathfrak{X}_2)$, then
\[
\langle \mathfrak{g}, z \rangle = n < \frac{\epsilon}{2} \min \lambda_1^T g.
\]
(5.49)

It follows from (5.48) and (5.49) that the target cone can be successfully detected by the modified detection strategy. On the other hand, (5.47) holds with the probability $\int_{\text{min} \lambda_1^T g}^{\text{max} \lambda_1^T g} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$. The proof is concluded.

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