Further Evidence for the Conformal Structure of a Schwarzschild Black Hole in an Algebraic Approach

Kumar S. Gupta
Saha Institute of Nuclear Physics
1/AF Bidhannagar
Calcutta - 700 064, India

Siddhartha Sen
School of Mathematics
Trinity College
Dublin, Ireland

and

Department of Theoretical Physics
Indian Association for the Cultivation of Science
Calcutta - 700032, India

Abstract

We study the excitations of a massive Schwarzschild black hole of mass $M$ resulting from the capture of infalling matter described by a massless scalar field. The near-horizon dynamics of this system is governed by a Hamiltonian which is related to the Virasoro algebra and admits a one-parameter family of self-adjoint extensions described by a parameter $z \in \mathbb{R}$. The density of states of the black hole can be expressed equivalently in terms of $z$ or $M$, leading to a consistent relation between these two parameters. The corresponding black hole entropy is obtained as $S = S(0) - \frac{3}{2} \log S(0) + C$, where $S(0)$ is the Bekenstein-Hawking entropy, $C$ is a constant with other subleading corrections exponentially suppressed. The appearance of this precise form for the black hole entropy within our formalism, which is expected on general grounds in any conformal field theoretic description, provides strong evidence for the near-horizon conformal structure in this system.

December 2001

PACS : 04.70.Dy, 04.60.-m

---

1Email: gupta@theory.saha.ernet.in
2Email: sen@maths.tcd.ie
1. Introduction

The notion of the existence of a conformal field theory in the near-horizon region of a black hole has recently led to interesting developments in quantum gravity [1, 2, 3, 4]. By imposing a suitable boundary condition at the horizon, it has been shown that the algebra of surface deformations contains a Virasoro subalgebra. This approach is based on extension of the idea of Brown and Henneaux [5] regarding asymptotic symmetries of three-dimensional anti-de-Sitter gravity.

In a previous paper [6] we proposed a novel algebraic approach for analyzing the near horizon conformal structure of a Schwarzschild black hole. Our method uses the time independent modes of a scalar field to probe the near-horizon geometry. The dynamics of the scalar field in the near-horizon region is described by a static Klein-Gordon (KG) operator. This operator, which is the Hamiltonian for the system, belongs to the representation space of the Virasoro algebra with central charge \( c = 1 \). In the quantum theory, it admits a one-parameter family of self-adjoint extensions labelled by \( e^{iz} \) where \( z \) is a real number. For a generic value of \( z \), the system admits an infinite number of bound states. When \( z \) is positive and satisfies the consistency condition \( z \sim 0 \), the bound states exhibit a scaling behaviour in the near-horizon region of the black hole. This property of the bound states reflects the existence of an underlying conformal structure in the near-horizon region.

In this Letter we provide further evidence for the existence of a near-horizon conformal structure of a Schwarzschild black hole. Our analysis is based on the identification of the bound states described above with the quantum excitations of the black hole resulting from the capture of the scalar field probe. These excitations will eventually decay through the emission of Hawking radiation. While the time-independent modes of the scalar field cannot describe the actual decay process, they do carry information about the static thermal properties of the system. Within our formalism, any such information must be obtained from the properties of the associated Hamiltonian. The spectrum of the Hamiltonian is determined by boundary conditions which are encoded in the choice of the corresponding domain. Since the self-adjoint parameter \( z \) labels the domains of the Hamiltonian, it is therefore directly related to the boundary conditions. At an operator level, the properties of the Hamiltonian are thus determined by the parameter \( z \). On the other hand, the only physical input in the problem is the mass \( M \) of the black hole which must also play a role in determining the spectrum. The parameters \( z \) and \( M \) thus play a conceptually similar role and they are likely to be related to each other. Establishing a relation between \( z \) and \( M \), which is consistent with the constraints of our model, is the first step in this approach.

The relationship between \( z \) and \( M \) naturally leads to the identification of black hole entropy
within this formalism. The density of states of a black hole is usually a function of the mass $M$. On the other hand, the density of states in our algebraic formalism has a smooth expression in terms of the self-adjoint parameter $z$. In view of the relation between $z$ and $M$, the density of states written as a function of $z$ can be reexpressed in terms of the variable $M$. This process leads to the identification of the black hole entropy consistent with the constraints of the model. The entropy so obtained is given by the Bekenstein-Hawking term together with a leading logarithmic correction whose coefficient is $-\frac{3}{2}$. The other subleading corrections terms are found to be exponentially suppressed. Such a logarithmic correction term to the Bekenstein-Hawking entropy with a $-\frac{3}{2}$ coefficient was first found in the quantum geometry formalism and has subsequently appeared in several other publications. In particular, using an exact convergent expansion for the partition of a number due to Rademacher, rather than the asymptotic formula due to Hardy and Ramanujan, it was shown in Ref. that within any conformal field theoretic description, the black hole entropy can be expressed in an exact and convergent series where the leading logarithmic correction to Bekenstein-Hawking term always appears with a universal coefficient of $-\frac{3}{2}$ with the other subleading terms exponentially suppressed. Here we show that the entropy of the Schwarzschild black hole contains correction terms precisely of this structure, which provides a strong evidence for the underlying near-horizon conformal structure of the system.

The organization of this paper is as follows. In Section 2 we briefly describe the near-horizon quantization indicating how the self-adjoint parameter $z$ appears. Section 3 provides the identification between $z$ and $M$ leading to the expression of the black hole entropy within this formalism. We conclude the paper in Section 4 with a discussion of the main results.

2. Near-horizon Quantization

The Klein-Gordon (KG) operator for the time-independent modes of a scalar field in the background of a Schwarzschild black hole of mass $M$ is given by

$$H = -\frac{d^2}{dx^2} - \frac{1}{4x^2},$$

(2.1)

where $x = r - 2M$ is the near horizon coordinate and $r$ is the radial variable. In what follows we shall assume that the black hole is massive, i.e. $M$ is large. The scalar field $\psi$ in this background obeys the eigenvalue equation

$$H\psi = \mathcal{E}\psi,$$

(2.2)

where $\psi \in L^2[R^+, dr]$. Eqn. (2.1) is an example of an unbounded linear operator on a Hilbert space. Below we shall first summarize some basic properties of these operators which would be useful for our analysis.
Let $T$ be an unbounded differential operator acting on a Hilbert space $\mathcal{H}$ and let $(\gamma, \delta)$ denote the inner product of the elements $\gamma, \delta \in \mathcal{H}$. By the Hellinger-Toeplitz theorem, $T$ has a well defined action only on a dense subset $D(T)$ of the Hilbert space $\mathcal{H}$. $D(T)$ is known as the domain of the operator $T$. Let $D(T^*)$ be the set of $\phi \in \mathcal{H}$ for which there is a unique $\eta \in \mathcal{H}$ with $(T\xi, \phi) = (\xi, \eta) \forall \xi \in D(T)$. For each such $\phi \in D(T^*)$ we define $T^*\phi = \eta$. $T^*$ is called the adjoint of the operator $T$ and $D(T^*)$ is the corresponding domain of the adjoint.

The operator $T$ is called symmetric or Hermitian if $T \subset T^*$, i.e. if $D(T) \subset D(T^*)$ and $T\phi = T^*\phi \forall \phi \in D(T)$. Equivalently, $T$ is symmetric iff $(T\phi, \eta) = (\phi, T\eta) \forall \phi, \eta \in D(T)$. The operator $T$ is called self-adjoint iff $T = T^*$ and $D(T) = D(T^*)$.

We now state the criterion to determine if a symmetric operator $T$ is self-adjoint. For this purpose let us define the deficiency subspaces $K_{\pm} \equiv \text{Ker}(i \mp T^*)$ and the deficiency indices $n_{\pm}(T) \equiv \dim[K_{\pm}]$. $T$ is (essentially) self-adjoint iff $(n_+, n_-) = (0, 0)$. $T$ has self-adjoint extensions iff $n_+ = n_-$. There is a one-to-one correspondence between self-adjoint extensions of $T$ and unitary maps from $K_+$ into $K_-$. Finally if $n_+ \neq n_-$, then $T$ has no self-adjoint extensions.

We now return to the discussion of the operator $H$. On a domain $D(H) \equiv \{\phi(0) = \phi'(0) = 0, \phi, \phi' \text{ absolutely continuous}\}$, $H$ is a symmetric operator with deficiency indices $(1,1)$. The corresponding deficiency subspaces $K_{\pm}$ are 1-dimensional and are spanned by

\[
\phi_+(r) = r^{1/2}H_0^{(1)}(re^{i\pi/4}),
\]

\[
\phi_-(r) = r^{1/2}H_0^{(2)}(re^{-i\pi/4})
\]

respectively, where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions. The operator $H$ is not self-adjoint on $D(H)$ but admits a one-parameter family of self-adjoint extensions. The are labelled by unitary maps from $K_+$ into $K_-$. The self-adjoint extensions of $H$ are thus labelled by $e^{iz}$ where $z \in \mathbb{R}$. Each value of the parameter $z$ defines a domain $D_z(H)$ on which $H$ is self-adjoint and thus corresponds to a particular choice of boundary condition.

The normalized bound state eigenfunctions and eigenvalues of Eqn. (2.1) are given by

\[
\psi_n(x) = \sqrt{2E_nx}K_0 \left( \sqrt{E_nx} \right)
\]

and

\[
E_n = -E_n = -\exp \left[ \frac{\pi}{2}(1 - 8n)\cot \frac{z}{2} \right]
\]

respectively, where $n$ is an integer and $K_0$ is the modified Bessel function. Thus for each value of $z$, the operator $H$ admits an infinite number of negative energy solutions. In our formalism, these solutions are interpreted as bound state excitations of the black hole due to the capture of the scalar field. As is obvious from Eqns. (2.5) and (2.6), different choices of $z$
leads to inequivalent quantization of the system. The physics of the system is thus encoded in the choice of the parameter $z$.

The self-adjoint parameter $z$ so far has been kept arbitrary. The requirement of near-horizon conformal symmetry places important constraint on $z$ [6]. To see this, consider a band-like region $\Delta = [x_0 - \delta/\sqrt{E_0}, x_0 + \delta/\sqrt{E_0}]$, where $x_0 \sim \sqrt{E_0}$ and $\delta \sim 0$ is real and positive. For $\Delta$ to belong near the horizon of the black hole, $z$ must be positive and satisfy the condition $z \sim 0$. When this condition is fulfilled, at any point $x \in \Delta$ the leading behaviour of the bound state wavefunction is given by

$$\psi_n = \sqrt{2E_n x \left(A + 2\pi n \cot \frac{z}{2}\right)}, \quad (2.7)$$

where $A = \ln 2 - \gamma$ and $\gamma$ is Euler’s constant. From Eqn. (2.7) we see that the bound state wavefunction exhibits a scaling behaviour near the black hole horizon which is a reflection of the underlying conformal structure in the quantum theory. It is important to note that the consistency condition on $z$ plays a crucial role in obtaining this result.

3. Density of States and Entropy

In the analysis presented above, the information about the spectrum of the Hamiltonian in the Schwarzschild background is coded in the parameter $z$. The wavefunctions and the energies of the bound states depends smoothly on $z$. Thus, within the near-horizon region $\Delta$, we propose to identify

$$\tilde{\rho}(z) \equiv \sum_{n=0}^{\infty} |\psi_n(z)|^2 \quad (3.1)$$

as the density of states for this system written in terms of the variable $z$. $\tilde{\rho}(z)dz$ counts the number of states when the self-adjoint parameter lies between $z$ and $z + dz$. As mentioned before, within the region $\Delta$ $z$ is positive and satisfies the consistency condition $z \sim 0$ [6]. From Eqns. (2.6) and (2.7) we therefore see that the term with $n = 0$ provides the dominant contribution to the sum in Eqn. (3.1). The contribution of the terms with $n \neq 0$ to the sum in Eqn. (3.1) is exponentially small for large $\cot \frac{z}{2}$. Physically this implies that the capture of the minimal probe excites only the lowest energy state in the near-horizon region of the massive black hole. The density of energy states of the black hole in the region $\Delta$ can therefore be written as

$$\tilde{\rho}(z) \approx |\psi_0|^2 = 2A^2 e^{\pi \cot \frac{z}{2}}. \quad (3.2)$$

As mentioned before, within the region $\Delta$, $\cot \frac{z}{2}$ is a large and positive number. We thus find that the density of states of a massive black hole is very large in the near-horizon region.

In order to proceed, we shall first provide a physical interpretation of the self-adjoint parameter $z$ using the Bekenstein-Hawking entropy formula. To this end, recall that in our formalism,
the capture of the scalar field probe gives rise to the excitations of the black hole which subsequently decay by emitting Hawking radiation. A method of deriving density of states and entropy for a black hole in a similar physical setting using quantum mechanical scattering theory has been suggested by ’t Hooft [7]. This simple and robust derivation uses the black hole mass and the Hawking temperature as the only physical inputs and is independent of the microscopic details of the system. The interaction of infalling matter with the black hole is assumed to be described by Schrödinger’s equation and the relevant emission and absorption cross sections are calculated using Fermi’s Golden Rule. Finally, time reversal invariance (which is equivalent to CPT invariance in this case) is used to relate the emission and absorption cross sections. The density of states for a massive black hole of mass \( M \) obtained from this scattering calculation is given by

\[
\rho(M) = e^{4\pi M^2 + C'} = e^S, \quad (3.3)
\]

where \( C' \) is a constant and \( S \) is the black hole entropy. It may be noted that for the purpose of deriving Eqn. (3.3) the infalling matter was described as particles. However, the above derivation of the density of states is independent of the microscopic details and is valid for a general class of infalling matter.

We are now ready to provide a physical interpretation of the parameter \( z \). First note that the density of states calculated in Eqns. (3.2) and (3.3) correspond to the same physical situation described in terms of different variables. In our picture, the near horizon dynamics of the scalar field probe contains information regarding the black hole background through the self-adjoint parameter \( z \). The same information in the formalism of ’t Hooft is contained in the black hole mass \( M \). It is thus meaningful to relate the density of states in our framework (cf. Eqn. 3.2) to that given by Eqn. (3.3). If these expression describe the same physical situation, we are led to the identification

\[
\frac{\pi}{4} \cot \frac{z}{2} = 4\pi M^2. \quad (3.4)
\]

Note that the analysis presented here is valid only for massive black holes. We have also seen that in the near-horizon region \( z \) must be positive and obey a consistency condition such that \( \cot \frac{z}{2} \) is large positive number. Thus the relation between \( z \) and \( M \) given by Eqn. (3.4) is consistent with the constraints of our formalism. We therefore conclude that the self-adjoint parameter \( z \) has a physical interpretation in terms of the mass of the black hole.

As stated above, the density of states of the black hole can be expressed either in terms of the variable \( z \) or in terms of \( M \). In view of the relation between \( z \) and \( M \), we can write

\[
\tilde{\rho}(z)dz \sim |J|\rho(M)dM, \quad (3.5)
\]

where \( J = \frac{dz}{dM} \) is the Jacobian of the transformation from the variable \( z \) to \( M \). When \( z \sim 0 \), using Eqns. (3.3) and (3.4) we get

\[
\tilde{\rho}(z)dz \sim e^{4\pi M^2} \frac{1}{M^3}dM \sim e^{4\pi M^2 - \frac{2\ln M^2}{2}}dM. \quad (3.6)
\]
The presence of the logarithmic correction term in the above equation is thus due to the effect of the Jacobian.

Finally, the entropy for the Schwarzschild black hole obtained from Eqn. (3.6) can be written as

\[ S = S(0) - \frac{3}{2} \ln S(0) + C, \]  

(3.7)

where \( S(0) = 4\pi M^2 \) is the Bekenstein-Hawking entropy and \( C \) is a constant. Thus the leading correction to the Bekenstein-Hawking entropy is provided by the logarithmic term in Eqn. (3.7) with a coefficient of \(-\frac{3}{2}\). The subleading corrections to the entropy coming from the \( n \neq 0 \) terms of Eqn. (3.1) are exponentially suppressed. As stated before, this is precisely the structure associated with the expression of black hole entropy whenever the same is calculated within a conformal field theoretic formalism \[12\]. We are thus led to conclude that the expression of the black hole entropy obtained in our formalism provides a strong indication of the underlying near horizon-conformal structure present in the system.

4. Conclusion

In this Letter we have extended the analysis based on the algebraic formalism developed in Ref. \[6\] to provide further evidence for the near-horizon conformal structure of a Schwarzschild black hole. Earlier, time independent modes of a scalar field have been used to probe the near-horizon geometry of the black hole. The near-horizon dynamics was described by a Hamiltonian that belongs to the representation space of the Virasoro algebra, which for the Schwarzschild background has a central charge \( c = 1 \). In the quantum theory, this Hamiltonian admits a one-parameter family of self-adjoint extensions giving rise to bound states in the corresponding spectrum. When \( z \) is positive and satisfies the consistency condition \( z \sim 0 \), the bound states exhibit a scaling behaviour in the near-horizon region of the black hole.

In order to investigate this idea further, we first observe that the self-adjoint parameter \( z \) describes the domain of the Hamiltonian and directly determines the spectrum within this formalism. On the other hand, the black hole mass \( M \) must also play a role in determining the spectrum. These two parameters thus play a conceptually similar role and it is expected that they will be related.

The next step of our analysis was based on the identification of these bound states with the excitations of the black hole resulting from the capture of the scalar field probe. These excitation would eventually decay through the emission of Hawking radiation. This process is described by quantum mechanical scattering theory in terms of the density of states of the

\[^{3}\text{It may be noted that the representation space for the } c = 1 \text{ conformal field theory obtained in Ref. } [6] \text{ is spanned by tensor densities of weight } \frac{1}{2}, \text{ i.e. spin } \frac{1}{2} \text{ [13].}\]
black hole which is a function of the variable $M$. On the other hand, the density of states following from our formalism is a smooth function of $z$. Identifying these two expressions of the density of states leads to a quantitative relation between $z$ and $M$ which is consistent with the constraints of the system. Such a relation also provides a physical interpretation of the self-adjoint parameter in terms of the mass of the black hole.

The relation between $z$ and $M$ naturally leads to the identification of black hole entropy within this formalism. The entropy thus obtained contains the usual Bekenstein-Hawking term together with a leading logarithmic correction which has $-\frac{3}{2}$ as the coefficient. Moreover, the subleading non-constant corrections are shown to be exponentially suppressed. It has been observed that the expression for the black hole entropy is expected to have precisely this structure whenever it is calculated within the conformal field theoretic formalism [12] and possibly even for the non-unitary case [20]. Thus the expression that we obtain for the black hole entropy provides strong support to the hypothesis of an underlying conformal structure in the near-horizon region of the Schwarzschild black hole.

Finally we would like to mention that the Hamiltonian appearing in Eqn. (2.1) can be used to describe the near-horizon dynamics in other black hole backgrounds with different coefficient of the inverse-square term [16]. It is thus plausible that the method developed in this Letter can be used to analyze the entropy for other black holes as well.

References

[1] A. Strominger, JHEP **9802** (1998) 009.
[2] S. Carlip, Phys. Rev. Lett. **82** (1999) 2828; Class. Quant. Grav. **16** (1999) 3327; Nucl. Phys. Proc. Suppl. **18** (2000) 10.
[3] S. N. Solodukhin, Phys. Lettt. B **454** (1999) 213.
[4] O. Dreyer, A. Ghosh and J. Wisniewski, Class. Quant. Grav. **18** (2001) 1929.
[5] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104** (1986) 207.
[6] Danny. Birmingham, Kumar S. Gupta and Siddhartha Sen, Phys. Lett. B**505** (2001) 191.
[7] G ’t Hooft, Int. Jour. Mod. Phys A**11** (1996) 4623; Talk at the International School of Subnuclear Physics, Erice, 1999, [hep-th/0003004](#).
[8] Romesh K. Kaul and Parthasarathi Majumdar, Phys. Rev. Lett. **84** (2000) 5255.
[9] S. Carlip, Class. Quant. Grav. **17** (2000) 4175.
[10] T. R. Govindarajan, Romesh K. Kaul and V. Suneeta, Class. Quant. Grav. 18 (2001) 2877.

[11] Saurya Das, Romesh K. Kaul and Parthasarathi Majumdar, Phys. Rev. D 63 (2001) 044019.

[12] Danny Birmingham and Siddhartha Sen, Phys. Rev. D 63, (2001) 04750.

[13] Saurya Das, Parthasarathi Majumdar and Rajat K. Bhaduri, hep-th/0111001.

[14] H. Rademacher, Topics in Algebraic Number Theory, Springer-Verlag, Berlin, 1973.

[15] G. H. Hardy and S. Ramanujan, Proc. Lond. Math. Soc. 2, (1918) 75.

[16] T. R. Govindarajan, V. Suneeta and S. Vaidya, Nucl. Phys. B583 (2000) 291.

[17] M. Reed and B. Simon, Methods of Modern Mathematical Physics, volume 1, Academic Press, New York, 1972; volume 2, Academic Press, New York, 1975.

[18] H. Narnhofer, Acta Physica Austriaca 40, 306 (1974).

[19] V. G. Kac and A. K. Raina, Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras, World Scientific, Singapore, 1987.

[20] J. A. Harvey, D. Kutasov, E. J. Martinec and G. Moore, hep-th/0111154.