MODELLING OF THE DECAY OF ISOTROPIC TURBULENCE BY THE LES

U. S. Abdibekov\textsuperscript{1} and D. B. Zhakebaev\textsuperscript{1}
\textsuperscript{1}Al-Farabi Kazakh National University, Kazakhstan
E-mail: uali1@mail.ru, daurjaz@mail.ru

Abstract. This work deals with the modelling of degeneration of isotropic turbulence. To simulate the turbulent process the filtered three-dimensional nonstationary Navier-Stokes equation is used. The basic equation is closed with the dynamic model. The problem is solved numerically, and the equation of motion is solved by a modified method of fractional steps using compact schemes, the equation for pressure is solved by the Fourier method with a combination of matrix factorization. In the process of simulation changes of the kinetic energy of turbulence in the time, micro scale of turbulence and changes of inlongitudinal-transverse correlation functions are obtained, longitudinal and transverse one-dimensional spectra are defined.

1. Introduction

Investigation of the degeneracy of homogeneous isotropic turbulence is usually done on the basis of spectral equations and equations of correlation functions, for the closures of which to determination of the correlation functions is necessary. This fact makes this approach not effective, since it is not possible to obtain spectra for a large time interval and at the moment when the main dissipation take place \cite{1, 2}.

In this paper we can solve this problem by using the method of large eddy simulation. The idea is to specify in the phase space the initial conditions for the velocity field, which satisfies the condition of continuity. Thus the basic spectral equation is not solved, and the given initial condition with phase space transfers into the physical space using a Fourier transform. The obtained velocity field is used as an initial condition for the filtered Navier-Stokes equation. Next three-dimensional nonstationary Navier-Stokes equation is solved for modelling of degeneration of isotropic turbulence.

2. Problem Statement

Isotropic turbulized medium undergoes very fast homogeneous deformation, so that the density of the medium, all the characteristic dimensions and any averaged turbulence characteristics are constant, but variable in time. It is required to numerically simulate the changing of the characteristics of turbulence in time, i.e. degeneration of isotropic turbulence at different Reynolds numbers.

Numerical simulation of the problem is based on the nonstationary filtered Navier-Stokes
equations with the continuity equation in Cartesian coordinates:
\[
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j},
\]
where \( \bar{u}_i \) – velocity components, \( \bar{p} \) – pressure, \( t \) – time, \( \nu \) – kinematic viscosity coefficient, \( \tau_{ij} \) – subgrid tensor, responsible for small-scale structures, which ought to be modelled, \( i, j \) corresponds to \( -1, 2, 3 \).

To simulate the subgrid tensor viscous model is used, which is represented as:
\[
\tau_{ij} = \frac{\delta_{ij}}{3} \tau_{kk} = -2\nu_T \bar{S}_{ij},
\]
where \( \nu_T = C_S \Delta^2 (2\bar{S}_{ij} \bar{S}_{ij})^{1/2} \) – turbulent viscosity; \( C_S \) – empirical coefficient; \( \Delta = (\Delta_i \Delta_j \Delta_k)^{1/3} \) – width of the grid filter; \( \bar{S}_{ij} = \frac{1}{2} (\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}) \) – value of tensor velocity strain [3, 4].

Periodic boundary conditions are taken in all directions.

The initial values for each component are given in the following form in the phase space:
\[
\hat{u}_1 (k_1, 0) = k_1^{m-2} e^{-\frac{m}{4} \left( \frac{k_1}{k_{\text{max}}} \right)}, \quad \hat{u}_2 (k_2, 0) = k_2^{m-2} e^{-\frac{m}{4} \left( \frac{k_2}{k_{\text{max}}} \right)}, \quad \hat{u}_3 (k_3, 0) = k_3^{m-2} e^{-\frac{m}{4} \left( \frac{k_3}{k_{\text{max}}} \right)},
\]
where \( \hat{u}_1 \) – one-dimensional longitudinal spectrum, \( \hat{u}_2, \hat{u}_3 \) – one-dimensional cross-spectrum.

For this problem we choose variational parameter \( m \) and wave number \( k_{\text{max}} \) and values of parameters, which determine the kind of turbulence. In Fig.1 \( k_{\text{max}} = 20 \), parameter \( m \) varies. For the modelling of isotropic turbulence parameters \( k_{\text{max}} = 20 \) and \( m = 4 \) can be put, as their relationship corresponds to the experimental data [5].

The given initial condition with phase space transfers into the physical space using a Fourier transform.

**Figure 1.** Energy of the initial level of turbulence, depending on the fixed wave number \( k_{\text{max}} = 20 \) and the variational parameter \( m \):
1) \( m = 2 \); 2) \( m = 4 \); 3) \( m = 6 \); 4) \( m = 8 \).
3. Numerical method

For the solution of Navier-Stokes equation a splitting scheme by physical parameters is used, which consists of three stages. At the first stage Navier-Stokes equation without pressure is solved. To approximate the convective and diffusion terms in the equation a compact scheme of high order is used. At the second stage Poisson’s equation, obtained from the continuity equation, taking the velocity field from the first stage into account is solved. For the solution of three-dimensional Poisson equation an original algorithm for solving is developed - the spectral transformation in combination with a matrix factorization. The resulting pressure field at the third stage is used to recalculate the final velocity field [6, 7].

4. Correlation and spectral functions for determine action of isotropic turbulence characteristics

To find the turbulent characteristics in the physical space averaging over the volume of various quantities is required. Averaged quantities will be involved in finding the turbulent characteristics. The procedure for the calculation of the turbulent characteristics is presented in [8]. Averaged value, calculated for the entire area, which in this paper is rectangular, is calculated as follows:

\[
\langle u_i \rangle = \frac{1}{(N_1 + 1) (N_2 + 1) (N_3 + 1)} \sum_{n=1}^{N_1+1} \sum_{m=1}^{N_2+1} \sum_{q=1}^{N_3+1} (u_i)_{n,m,q}.
\]

Various correlation coefficients of velocities can be defined as shown in [8].

For isotropic turbulence, the longitudinal and transverse correlation looks as follows [9]:

\[
f(r, t) = \frac{R_{11}(r, t)}{R_{11}(0, t)}, \quad g_1(r, t) = \frac{R_{22}(r, t)}{R_{22}(0, t)}, \quad g_2(r, t) = \frac{R_{33}(r, t)}{R_{33}(0, t)}.
\]

Micro-length scale is determined with:

\[
\lambda_f = \left\{ \frac{2}{-f''(0)} \right\}^{1/2}, \quad \lambda_g = \left\{ \frac{2}{g''(0)} \right\}^{1/2},
\]

integral scale is expressed:

\[
\Lambda_f = \int_0^\infty f(r) \, dr, \quad \Lambda_g = \int_0^\infty g(r) \, dr.
\]

Longitudinal and transverse one-dimensional spectrum looks as follows [1, 2]:

\[
F_1(k_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_1r} R_{11}(r) \, dr = \frac{1}{\pi} \int_0^\infty \cos(k_1r) R_{11}(r) \, dr \geq 0
\]

and

\[
F_2(k_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_2r} R_{22}(r) \, dr = \frac{1}{\pi} \int_0^\infty \cos(k_2r) R_{22}(r) \, dr \geq 0,
\]

one-dimensional spectrum:
\[ E_1(k) = F_1(k) + 2F_2(k), \]

three-dimensional spectrum:

\[ E(k) = -k \frac{dE_1(k)}{dk}. \]

Dissipation of energy is calculated with the formula:

\[ \varepsilon(t) = 30 \cdot \nu \frac{R_{11}(0, t)}{\lambda_2(t)}. \]

Turbulent kinetic energy is found as follows

\[ E_{\text{kin}}(t) = \frac{1}{2} R_{ij}(0, t) = \int_0^\infty E_1(k) \, dk. \]

5. Results of modelling

As a result of modeling characteristics of isotropic turbulence are obtained. All the characteristics of isotropic turbulence have a physical meaning. For example, the physical meaning of the integral scale of turbulence is the size of large eddies. Alignment with the semi-empirical theory is that with time, the integral scale of turbulence must grow. As shown in Fig.2, the results illustrate the effect of viscosity on the internal structure of turbulence. The changing of coefficient of molecular viscosity leads to a proportional change in the integral scale. In Fig.3 - the changing of micro-scale – \( \lambda^2 \) is shown, calculated at different Reynolds numbers: 1 – \( Re = 5000 \); 2 – \( Re = 10000 \); 3 – \( Re = 20000 \). The results shown in Fig.5 illustrate the effect of viscosity on the change of the energy dissipation.

![Figure 2. Changing of the integral scale of turbulence, calculated at different Reynolds numbers: 1) \( Re = 5000 \); 2) \( Re = 10000 \); 3) \( Re = 20000 \); 4) \( Re = 30000 \); 5) \( Re = 50000 \).](image)

In Fig.4 the changing of energy dissipation at different Reynolds numbers is shown. The results shown in Fig.5 illustrate the effect of viscosity on the degeneration of kinetic energy of isotropic turbulence, calculated at different Reynolds numbers: 1 – \( Re = 5000 \); 2 – \( Re = 10000 \); 3 – \( Re = 20000 \). As can be seen from the Fig.4 and Fig.5 at high Reynolds numbers there are more energy-containing eddies and correspondingly the rate of energy dissipation is greater, which is natural in a physical experiment.

The correlation coefficients reflect the average, by volume, the correlation between the components of velocity at different points. The farther the dots between the various components of the velocity are, the smaller the correlation coefficients should be, i.e. they should be close
Figure 3. Modified micro-scale – $\lambda^2$, calculated at different Reynolds numbers: 1) $Re = 5000$; 2) $Re = 10000$; 3) $Re = 20000$.

Figure 4. Changing the energy dissipation, calculated at different Reynolds numbers: 1) $Re = 5000$; 2) $Re = 10000$; 3) $Re = 20000$.

Figure 5. Changing of the turbulent kinetic energy over time, calculated at different Reynolds numbers: 1) $Re = 5000$; 2) $Re = 10000$; 3) $Re = 20000$.

Figure 6. Changing of the longitudinal correlation function $f(r)$ in time, at $Re=20000$: 1) $t = 0$; 2) $t = 0.2$; 3) $t = 0.4$; 4) $t = 0.6$; 5) $t = 1$.

to zero. In Fig.6 the changing of the longitudinal correlation function $f(r)$ in time is shown, calculated at $Re = 50000$. It is seen that as r increases, they tend to zero.

In Fig.7 the variation of the longitudinal correlation function is shown, calculated at time $t = 0.4$ and at different Reynolds numbers: 1) $Re = 5000$; 2) $Re = 10000$; 3) $Re = 20000$.

In Fig.8 the velocity component in the physical space at $t = 0$ and $t = 0.6$ is shown. The
Figure 7. The component of speed $u$ at various moments of time: (a) $t = 0$; (b) $t = 0.6$.

Figure 8. Changing of the longitudinal spectrum calculated at $Re = 5000$.

Figure 9. Changing of cross-spectrum calculated at $Re = 5000$.

Figure 10. Changing of the three-dimensional spectrum calculated at $Re = 10000$ and time: 1) $t = 0$; 2) $t = 0.4$; 3) $t = 0.8$.

changing of the longitudinal and transverse spectrum, different times can be seen in Fig.8 and Fig.9 respectively, and three-dimensional spectrum is shown in Fig.10.
6. Conclusion

On the basis of large-scale eddies produced numerical simulations of the effect of viscosity on the degeneration of isotropic turbulence are produced.

Analyzing the results of modeling can make the following conclusion can be drawn: One dimensional spectra of the fields are non-negative and monotone, which corresponds to the requirements of the Khinchin’s theorem. The viscosity of the flow produces a significant effect on the turbulence and therefore can be used to control the turbulence. The obtained results allow us to calculate precisely the change in the characteristics of isotropic turbulence in time, at high Reynolds numbers. For finding the spectrum of turbulence there is no need in solving the spectral equations, the closure of which presents considerable difficulties.

Thus the numerical algorithm for solving non stationary three-dimensional Navier-Stokes equations for modelling of degeneration of isotropic turbulence at different Reynolds numbers is developed. The proposed method can be used to solve non-isotropic turbulence without any significant changes.

References

Monin, A. S., Yaglom, A. M. 1975 Statistical fluid mechanics, Vol. 2. In MIT Press., 886 p. Cambridge.

Hinze, J. O. 1959 Turbulence, an introduction to its mechanism and theory. In McGraw-Hill, 586 p. New York.

Ferziger, J. H. 1977 Large eddy simulation of turbulent flows. J. AIAA. 15, 1261-1267.

Sagaut, P. 1967 Large eddy simulation for incompressible flows. In Springer-Verl., 423 p. Heidelberg.

Sirovich, L., Smith, L., Yakhot, V. 1994 Energy spectrum of homogeneous and isotropic turbulence in far dissipation range. J. Phys. Rev. Letters. 72, 344-347.

Abdibekov, U. S., Zhumagulov, B. T., Surapbergenov, B. D. 2007 Numerical modelling of turbulent flow with large eddy simulation. Computational Technologies. 12, 4-9.

Danaev, N. T., Zhakebaev, D. B., Abdibekov, A. U. 2011 Algorithm for solving non-stationary three-dimensional Navier-Stokes equations with large Reynolds numbers on multiprocessor systems. Notes on Numerical Fluid Mechanics and Multidisciplinary Design. 115, 313–326.

Ievlev, V. M. 1990 Numerical modelling of turbulent. In Science Nauka, 216 p. Moscow.

Orszag, S. A., Patterson, G. S. 1972 Numerical Simulation of Three-Dimensional Homogeneous Isotropic Turbulence. J. Phys. Rev. Letters. 28, 76–79.