Resolution of overlapping branes

H. Lü a,1, J.F. Vázquez-Poritz b,2

a Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, MI 48109, USA
b Physique Théorique et Mathématique, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

Received 21 February 2002; received in revised form 18 March 2002; accepted 20 March 2002

Abstract

We obtain singularity resolutions for various overlapping brane configurations, including those of two heterotic 5-branes, type II 5-branes or D4-branes. In these solutions, the "harmonic" function \( H \) for each brane component depends only on the associated four-dimensional relative transverse space. The resolution is achieved by replacing these transverse spaces with Eguchi–Hanson or Taub-NUT spaces, both of which admit a normalisable self-dual (or anti-self-dual) harmonic 2-form. Due to the manner in which the interaction terms for the form fields modify their Bianchi identities or equations of motion, these normalisable harmonic 2-forms provide regular sources for the branes. We also obtain resolved 5-branes and D4-branes wrapped on \( S^1 \), which is fibred over the transverse Eguchi–Hanson or Taub-NUT spaces. The T-duality invariance of the NS–NS 5-brane is retained after the resolution. The resolved 5-branes and D4-branes provide regular supergravity duals of certain supersymmetric Yang–Mills theories in five and four dimensions.

© 2002 Elsevier Science B.V. Open access under CC BY license.

1. Introduction

BPS branes play an important role in string and M-theory. Certain supergravity solutions provide an explicit demonstration of various properties in dual gauge theories. However, in most cases such a solution exhibits a singularity at the origin of the brane, which imposes a severe restriction on the range of validity. It is expected that higher-order stringy terms or non-perturbative effects can resolve these singularities, while maintaining a fixed mass/charge ratio.

A typical \( p \)-brane or intersecting \( p \)-brane solution makes use of only the kinetic terms of the form fields, with zero contribution from the interacting terms. The interactions between form fields modify their Bianchi identities and/or equations of motion, e.g.,

\[
d F(n) = F(p) \wedge F(q) \\
\text{and/or} \quad d * F(n) = F(s) \wedge F(t).
\]

In many cases, the inclusion of such a contribution can resolve brane singularities at the level of supergravity. An early example of this is the heterotic 5-brane constructed in [1], where the 5-brane source is the matter SU(2) Yang–Mills instanton living in the four-dimensional Euclidean transverse space. This construction makes use of the Bianchi identity \( d F(3) = \)
In addition, we obtain regular 5-brane, D5-brane and D4-brane wrapped on an $S^1$, which is fibred over the transverse Eguchi–Hanson or Taub-NUT spaces. We conclude our Letter in Section 5. In Appendix A, we present certain properties of Eguchi–Hanson and Taub-NUT metrics that are used extensively throughout the Letter.

It should be emphasised that, although results are presented here mostly using the Eguchi–Hanson metric, the construction also works when it is replaced with the Taub-NUT metric of an appropriate orientation.

2. Overlapping heterotic 5-branes

The Lagrangian for the bosonic sector of ten-dimensional heterotic supergravity is given by

$$\mathcal{L}_\text{het} = R * \mathbb{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\phi} * F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{-\frac{1}{2}\phi} * F_{(2)}^i \wedge F_{(2)}^i, \quad (2)$$

where

$$F_{(3)} = dA_{(2)} + \frac{1}{2} A_{(1)}^j \wedge dA_{(1)}^j + \frac{1}{6} f_{ijk} A_{(1)}^i \wedge A_{(1)}^j \wedge A_{(1)}^k,$$

$$F_{(2)} = dA_{(1)}^i + \frac{1}{2} f_{ijk} A_{(1)}^i \wedge A_{(1)}^j \wedge A_{(1)}^k. \quad (3)$$

Consider the solution describing a non-standard intersection of two heterotic 5-branes [22,23] for the case in which the Yang–Mills fields $A_{(1)}^i$ are set to zero:

$$ds_{10} = (H \bar{H})^{-1/4} (-dt^2 + dw^2 + H d\gamma_1^2 + \bar{H} d\bar{\gamma}_1^2),$$

$$e^{-\phi} * F_{(3)} = \bar{H} dt \wedge d\gamma_1 \wedge d\gamma_\gamma \wedge dH^{-1} + H dt \wedge \bar{d}\gamma_1 \wedge d\gamma_\gamma \wedge d\bar{H}^{-1},$$

$$\phi = \frac{1}{2} \log(H \bar{H}), \quad (4)$$

where

$$\Box H = 0, \quad \Box \bar{H} = 0, \quad (5)$$

and $\Box (\widetilde{\Box})$ is taken over the $\gamma_1$ ($\bar{\gamma}_1$) directions. The solution can be represented diagrammatically (see Diagram 1).

The isotropic solution is given by $H = 1 + Q/r^2$ and $\bar{H} = 1 + \bar{Q}/r^2$, with $r^2 = y^i y^i$ and $\bar{r}^2 = \bar{y}^i \bar{y}^i$. 

In this Letter, we find that, for the overlap of two heterotic 5-branes as well as that of two D4-branes, the two relative transverse spaces are four-dimensional Ricci-flat spaces, and hence can be replaced by the Eguchi–Hanson or Taub-NUT metrics. Since the Eguchi–Hanson and Taub-NUT metrics both support one normalisable self-dual (or anti-self-dual, depending on the orientation) 2-form, we can resolve these two overlapping branes by making use of the corresponding Bianchi identities. We obtain resolved overlapping heterotic 5-brane and D4-brane solutions in Sections 2 and 3, respectively. In Section 4, by performing the T-duality on the D4/D4 system, we obtain a resolved 5-brane overlap in type II theories. In addition, we obtain regular 5-brane, D5-brane and
The solution has three singularities. The first one corresponds to $r \tilde{r} \to 0$, which is also a horizon. The other two are naked, corresponding to $r/\tilde{r} \to 0$ or $\tilde{r}/r \to 0$ while leaving $r \tilde{r}$ held fixed. These singularities can be resolved by introducing two sets of SU(2) Yang–Mills instantons living in the Euclidean 4-spaces $d\gamma_{1}^{2}$ and $d\gamma_{2}^{2}$ [26]. Here we demonstrate that it can also be resolved by a gravitational instanton together with matter $U(1)$ contributions. Since the solution (4) requires only that $d\gamma_{1}^{2}$ and $d\gamma_{2}^{2}$ are Ricci-flat, we can replace each of them with an Eguchi–Hanson metric, which we discuss in Appendix A. Since the Eguchi–Hanson metric admits a self-dual (or anti-self-dual) normalisable harmonic 2-form, we can turn on the matter $U(1)$ fields. The solution now becomes

$$ds_{10}^{2} = (H \tilde{H})^{-1/4} \times (-dt^{2} + dw^{2} + H ds_{EH}^{2} + \tilde{H} d\tilde{s}_{EH}^{2}),$$

e^{-\phi} F_{(3)} = \tilde{H} dt \wedge dw \wedge \tilde{G}_{(4)} \wedge dH^{-1} + H dt \wedge dw \wedge \Omega_{(4)} \wedge d\tilde{H}^{-1},

$$\phi = \frac{1}{2} \log(H \tilde{H}),$$

$$F_{(2)}^{1} = mL_{(2)}, \quad F_{(2)}^{2} = \tilde{m} \tilde{L}_{(2)},$$

where $\Omega_{(4)}$ and $\tilde{G}_{(4)}$ are the volume forms for the metric $ds_{EH}^{2}$ and $d\tilde{s}_{EH}^{2}$, respectively. Here we have turned on two $U(1)$ Cartan field strengths, labeled as $F_{(2)}^{1}$ and $F_{(2)}^{2}$, living on $ds_{EH}^{2}$ and $d\tilde{s}_{EH}^{2}$, respectively. Now the Bianchi identity $dF_{(3)} = \frac{1}{2} F_{(2)}^{1} \wedge F_{(2)}^{1} + \frac{1}{2} F_{(2)}^{2} \wedge F_{(2)}^{2}$ implies that

$$\Box H = -\frac{1}{4} \eta m^{2} L_{(2)}^{2}, \quad \Box \tilde{H} = -\frac{1}{4} \tilde{\eta} \tilde{m}^{2} \tilde{L}_{(2)}^{2},$$

where $\eta^{2} = 1 = \tilde{\eta}^{2}$ are the orientation parameters of the Eguchi–Hanson metrics (see Appendix A). The equations of motion for $F_{(3)}$ and $F_{(2)}$ are straightforwardly satisfied. The dilaton equation and the Einstein equation imply that $\eta = 1 = \tilde{\eta}$. In other words, although the equations of motion and Bianchi identity for the form fields imply that the $L_{(2)}^{2}$ and $\tilde{L}_{(2)}^{2}$ source terms can contribute both negatively or positively, depending on whether they are self-dual or anti-self-dual, they are restricted to a positive contribution due to the dilaton equation and Einstein equation. Thus, the Eguchi–Hanson instanton has to be such that its normalisable harmonic 2-form is self-dual. The generic solution for $H$ and $\tilde{H}$ has a logarithmic divergent term, which vanishes for appropriate integration constants [6], giving

$$H = 1 + \frac{m^{2}}{2a^{3}r^{2}}, \quad \tilde{H} = 1 + \frac{\tilde{m}^{2}}{2\tilde{a}^{3}\tilde{r}^{2}}.$$  

For vanishing $m$ or $\tilde{m}$, the solution reduces to the resolution of the heterotic 5-brane obtained in [6].

One can add a string along the common worldvolume of the above configuration. The corresponding “harmonic” function $K$ satisfies the equation [27]

$$\tilde{H} \Box K + H \Box \tilde{K} = 0,$$

which holds in our case as well. The general solution of $K$ depends on $H$ and $\tilde{H}$. A natural special solution is $K = h \tilde{h}$, where $h$ and $\tilde{h}$ are the harmonic functions of the two relative four-dimensional transverse spaces respectively. If the transverse spaces are Euclidean, for the limit in which the gravitational instanton sizes $a$ and $\tilde{a}$ vanish, then the three-component solution has the near-horizon structure $AdS_{3} \times S^{3} \times S^{3} \times E^{1}$ [27]. The non-vanishing entropy of this configuration disappears from the metric contribution once the gravita-
tional instanton is present. This suggests a phase transition associated with the vanishing gravitational instanton; this is analogous to the one associated with the Yang–Mills instantons for overlapping heterotic 5-branes, discussed in [26]. Note that it is also possible to add a pp-wave component [26].

The present resolution of overlapping 5-branes incorporates Yang–Mills fields, which are available only for heterotic string theory. On the other hand, unresolved overlapping 5-branes exist also in type II theories. While the resolution for the NS–NS and R–R overlapping 5-branes of type II theories is also possible, there are subtleties involved which will be discussed in Section 4.

3. Overlapping D4/D4 system

The D4-brane is a solution of type IIA supergravity, whose bosonic Lagrangian is given by

$$\mathcal{L}_{\text{IIA}} = R \ast \mathbb{I} - \frac{1}{2} d \phi \wedge d \phi - \frac{1}{2} \ast e^{\phi} \ast F(3) \wedge F(3)$$

$$- \frac{1}{2} e^{2 \phi} \ast F(4) \wedge F(4) - \frac{1}{2} e^{2 \phi} \ast F(2) \wedge F(2)$$

$$- \frac{1}{2} d A(3) \wedge d A(3) \wedge A(2),$$

where

$$F(4) = d A(3) - d A(2) \wedge A(1),$$

$$F(3) = d A(2), \quad F(2) = d A(1).$$

The solution for a non-standard intersection of two D4-branes is given by

$$ds_{10}^2 = -(H \tilde{H})^{-3/2}(d t^2 + H d \tilde{y}_1^2 + \tilde{H} d y_1^2 + H \tilde{H} d z^2),$$

$$e^{1/2 \phi} \ast F(4) = \tilde{H} d t \wedge d^4 \tilde{y} \wedge d H^{-1},$$

$$H d t \wedge d^4 y \wedge d \tilde{H}^{-1},$$

$$\phi = -\frac{1}{4} \log(H \tilde{H}),$$

where

$$\square H = 0, \quad \square \tilde{H} = 0,$$

and $\square$ (\tilde{\square}) is taken over the $y_i$ ($\tilde{y}_j$) directions. This solution can be illustrated by Diagram 2.

| Diagram 2 |
|---|
| The overlap of two D4-branes |
| $r$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $\tilde{y}_1$ | $\tilde{y}_2$ | $\tilde{y}_3$ | $\tilde{y}_4$ | $z$ |
| D4 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| D4 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

As with the heterotic 5-brane overlap, we replace the Euclidean 4-spaces $d y_1^2$ and $d \tilde{y}_1^2$ by Eguchi–Hanson metrics $d s_{EH}^2$ and $d \tilde{s}_{EH}^2$, respectively. By making use of the Bianchi identity, $d F(4) = F(3) \wedge F(2)$, we can consider the following ansatz

$$ds_{10}^2 = -(H \tilde{H})^{-3/2}(d t^2 + H d s_{EH}^2 + \tilde{H} d \tilde{s}_{EH}^2$$

$$+ H \tilde{H} d z^2),$$

$$e^{1/2 \phi} \ast F(4) = \tilde{H} d t \wedge \tilde{\Omega}(4) \wedge d H^{-1},$$

$$+ H d t \wedge \Omega(4) \wedge d \tilde{H}^{-1},$$

$$\phi = -\frac{1}{4} \log(H \tilde{H}),$$

$$F(3) = (m L_{(2)} - \tilde{m} \tilde{L}_{(2)}) \wedge dz,$$

$$F(2) = m L_{(2)} + \tilde{m} \tilde{L}_{(2)},$$

$$L_{(2)}$$ is a normalisable self-dual harmonic 2-form in $d s_{EH}^2$, which hence has positive orientation $\eta = 1$, and $\tilde{L}_{(2)}$ is a normalisable anti-self-dual harmonic 2-form in $d \tilde{s}_{EH}^2$, which hence has negative orientation $\eta = -1$. Since we have

$$F(4) = (\ast 4 d H \wedge \ast 4 d H) \wedge dz,$$

where $\ast 4$ and $\ast 4$ are the Hodge duals in $d s_{EH}^2$ and $d \tilde{s}_{EH}^2$, respectively, the Bianchi identity for $F(4)$ implies that

$$H = \frac{1}{2} m^2 L_{(2)}^2, \quad \tilde{H} = \frac{1}{2} \tilde{m}^2 \tilde{L}_{(2)}^2.$$
e^{2\phi} F_{(2)} = H^{-1} \tilde{H} dt \wedge \tilde{\Omega}_{(4)} \wedge dz \wedge \tilde{L}_{(2)}
- H \tilde{H}^{-1} dt \wedge \Omega_{(4)} \wedge dz \wedge \tilde{L}_{(2)}.
\ast d\phi = \frac{1}{4} H^{-1} \tilde{H} dt \wedge \tilde{\Omega}_{(4)} \wedge \ast_4 d\tilde{H} \wedge dz
+ \frac{1}{4} H \tilde{H}^{-1} dt \wedge \Omega_{(4)} \wedge \ast_4 d\tilde{H} \wedge dz. \quad (18)

We verify that all of the equations of motion for the form fields and the dilaton are satisfied. We did not verify the Einstein equation due to its complexity for these cases, although past experience leads us to believe that it is satisfied. The regular solution for (17) is given by

\begin{align}
H &= 1 + \frac{m^2}{a^2 r^2}, \quad \tilde{H} = 1 + \frac{\tilde{m}^2}{2a(r + \tilde{a})}, \quad (19)
\end{align}

Each of the Eguchi–Hanson metrics can also be replaced by a Taub-NUT metric with proper orientation such that it has the required normalisable self-dual or anti-self-dual harmonic 2-forms. The resulting regular solutions for $H$ and $\tilde{H}$ are given by

\begin{align}
H &= 1 + \frac{m^2}{2a(r + a)}, \quad \tilde{H} = 1 + \frac{\tilde{m}^2}{2a(r + \tilde{a})}. \quad (20)
\end{align}

If $m$ or $\tilde{m}$ vanishes, then the solution becomes that of a single resolved D4-brane with the transverse space $ds^2 = ds^2_4 + dz^2$, where $ds^2_4$ is a Ricci-flat manifold that admits normalisable self-dual or anti-self-dual harmonic 2-forms $L_{(2)}$. In this case, we have $F_{(2)} = mL_{(2)}$ and $F_{(3)} = m \ast_2 L_{(2)}$. Thus, we see that $F_{(3)} \wedge F_{(2)}$ always contributes positively to the Bianchi identity for the $F_{(4)}$. Both orientations of Eguchi–Hanson instanton or Taub-NUT can be used to resolve the D4-brane. This is very different from the case of the heterotic 5-brane, in which only one orientation can be used to resolve the brane. In order to cancel out the cross term in $F_{(3)} \wedge F_{(2)}$, we find that $ds^2_4$ and $d\tilde{s}^2_4$ must have opposite orientations such that they admit normalisable self-dual and anti-self-dual harmonic 2-forms, respectively.

4. Overlapping type II 5-branes

Overlaps of type II NS–NS or R–R 5-branes share the same metric structure as that of overlapping heterotic 5-branes, illustrated in Diagram 1. However, the resolution of the heterotic 5-brane is rather unique, since it makes use of multiple matter Yang–Mills fields which are absent in the type II theories. For this reason, the resolution of the type II 5-brane was previously unknown. A regular solution of the 5-brane wrapped around $S^2$ was obtained in [28] by lifting the four-dimensional $SU(2)$ gauged black hole [29]. This solution can apply for both type II and heterotic 5-branes. In this section, we obtain a resolved type II 5-brane overlap by performing T-duality on the D4/D4 system of the previous section. By turning off one component, we obtain a resolved 5-brane wrapped on $S^1$.

The resolved D4/D4 solution (18) has a $U(1)$ isometry in the $z$ direction. By performing T-duality on this direction, we obtain the overlap of two (R–R) D5-branes of type IIB supergravity:

\begin{align}
ds_{10}^2 &= (H \tilde{H})^{-1/4} (-dt^2 + dz^2 + A_{(1)}^2)^2 
+ H d\tilde{s}_{EH}^2 + \tilde{H} d\tilde{s}_{EH}^2),
F_{(3)}^{RR} &= \ast_4 dH + \ast_3 d\tilde{H}
- (mL_{(2)} + \tilde{m}\tilde{L}_{(2)}) \wedge (dz + A_{(1)}),
\phi &= -\frac{1}{2} \log(H \tilde{H}),
dA_{(1)} = mL_{(2)} - \tilde{m}\tilde{L}_{(2)}. \quad (21)
\end{align}

The resolved NS–NS overlapping 5-brane can be easily obtained by performing the S-duality of the type IIB theory, with the $F_{(3)}^{RR}$ replaced by $F_{(3)}^{NS}$ and the sign of the dilaton changed. Since the solution involves only the metric, dilaton and the 3-form field strength, it is also valid for resolving the 5-branes in type IIA or heterotic strings (without a Yang–Mills source). It is interesting to note that there is a twist along the direction $z$. The topology of a spatial slice of the solution can be viewed as a $U(1)$ fibration of the product space of two Eguchi–Hanson instantons with opposite orientations. The solution describes that a regular effective string, as common worldvolume of two 5-branes, wraps on the fibre circle $z$. Note that unlike the previous examples this resolution makes use of only the interaction between the gravity and the 3-form field strength, instead of the interactions associated with the Bianchi identities or the equations of motion of the form fields.

If $m$ or $\tilde{m}$ vanishes, then the solution becomes a resolved single 5-brane. Without the loss of generality,
we set $\tilde{m} = 0$. The metric solution becomes
\[ ds_{10}^2 = H^{-3/4} \left( -dt^2 + dx_1^2 + \cdots + dx_5^2 \right) + (dz + A_{(1)})^2 + H^{3/4} ds_{EH}^2, \]
with $dA_{(1)} = mL_{(2)}$. The expressions for the 3-form field strength and the dilaton can be easily read off from (21). The solution describes 5-branes wrapped around $S^1$, which is fibred over the transverse space $ds_{EH}^2$. It is regular everywhere, providing a well-behaved supergravity dual of a certain $D = 5$ super Yang–Mills theory.

It is important to note that the resolved NS–NS 5-brane as well as the resolved overlap of two NS–NS 5-branes are invariant under a T-duality transformation along the $z$ direction. This is rather different from the usual situation where T-duality would untwist the fibration [30]. (Analogous phenomenon was observed in the NS–NS dyonic string [31].) Invariance under T-duality is a property of NS–NS 5-branes. Since T-duality is a symmetry at all orders of perturbative Yang–Mills theory.

If instead the resolved 5-brane (22) carries the type IIB R–R $F_{(3)}^{RR}$ charge, we can perform T-duality on $x_4$ and obtain the resolved D4-brane of type IIA
\[ ds_{10}^2 = H^{-3/4} \left( -dt^2 + dx_2^2 + dx_3^2 \right) + (dz + A_{(1)})^2 + H^{3/4} ds_{EH}^2 + dz^2, \]
\[ F_{(4)} = (4dH - mL_{(2)} \wedge (dz + A_{(1)})) \wedge \tilde{z}, \]
\[ \phi = -\frac{1}{4} \log(H), \quad dA_{(1)} = mL_{(2)}. \]

Since this solution describes D4-brane wrapped on $S^1$, which is fibred over the transverse space $ds_{EH}^2 + dz^2$, it has an effective four-dimensional world-volume. Thus we find a new well-behaved supergravity solution dual to a certain $\mathcal{N} = 2$, $D = 4$ supersymmetric Yang–Mills theory.

5. Conclusions

The resolution of singularities in supergravity BPS brane solutions provides a convenient way of extending the validity of these solutions. In this Letter, we have considered the resolutions of two overlapping heterotic 5-branes, type II 5-branes or D4-branes. The relative transverse spaces in these overlapping solutions are all four-dimensional and hence can be replaced by either Eguchi–Hanson or Taub-NUT spaces, both of which admit normalisable self-dual or anti-self-dual harmonic 2-forms, depending on the orientation. Terms corresponding to interactions between form fields modify Bianchi identities and equations of motion. It follows that these normalisable harmonic 2-forms provide regular sources for the branes.

When each of the two relative transverse Euclidean 4-spaces is replaced by the Eguchi–Hanson or the Taub-NUT instanton, half of the supersymmetry is broken. Introducing a brane configuration will not break the supersymmetry any further. Thus our resolved overlapping brane solutions preserve $\frac{1}{4}$ of the supersymmetry.

We have also obtained resolved 5-branes and D4-branes wrapped on $S^1$, which is fibred over the transverse Eguchi–Hanson or Taub-NUT spaces. This provides a regular supergravity dual to a certain $D = 5$ and $D = 4$ super Yang–Mills theory.

The resolved solution are regular everywhere in the spacetime, with a stable dilaton and hence a stable string coupling constant. Furthermore, unlike a typical brane solution that requires a brane source term that is beyond supergravity, the resolved ones are complete purely within supergravity. Not all the BPS branes can be resolved at the level of supergravity. It is interesting to find those that can be resolved and study the special role they play in string and M-theory.

Acknowledgements

We would like to thank Xavier Bekaert, Nicolas Boulanger, Mirjam Cvetič and Chris Pope for useful conversations.

Appendix A. Normalisable harmonic forms in $D = 4$

A.1. Eguchi–Hanson instanton

Let us consider the Eguchi–Hanson solution [32]
\[ ds_4^2 = W^{-1} dr^2 + \frac{1}{4} r^2 W (d\psi + \eta \cos \theta d\phi)^2 \]
\[ W = 1 - \frac{a^4}{r^4}. \]  

(A.1)

The radial coordinate \( r \) lies in the range \( a \leq r \leq \infty \), and \( \psi \) has period \( 2\pi \) [33] to ensure regularity at \( r = a \). Thus, the metric is asymptotically locally Euclidean (ALE), with the periodicity condition on \( \psi \) implying that constant \( r \) surfaces are \( RP^3 = S^3/Z_2 \) [33]. The coordinate \( \psi \) can have two orientations of the fibration, corresponding to \( \eta = \pm 1 \). The metric is Kähler with the vielbein basis

\[
\begin{align*}
\epsilon^0 &= W^{-1/2} \, dr, \\
\epsilon^1 &= \frac{1}{2} r \, d\theta, \\
\epsilon^2 &= \frac{1}{r} \sin \theta \, d\phi, \\
\epsilon^3 &= \frac{1}{2} W^{1/2} (d\psi + \eta \cos \theta \, d\phi), \\
\end{align*}
\]

(A.2)

and with a Kähler form given by

\[
J = \epsilon^0 \wedge \epsilon^3 - \eta \epsilon^1 \wedge \epsilon^2. 
\]

(A.3)

It is anti-self-dual for \( \eta = 1 \) and self-dual for \( \eta = -1 \). The metric is also admits a harmonic 2-form, given by \[6\]

\[
L_{(2)} = \frac{1}{(r + a)^2} (\epsilon^0 \wedge \epsilon^3 - \eta \epsilon^1 \wedge \epsilon^2), 
\]

(A.4)

which is self-dual for \( \eta = 1 \) and anti-self-dual for \( \eta = -1 \). The square of \( L_{(2)} \) is given by

\[
L_{(2)}^2 = \frac{1}{16} \eta L_{(2)}^2 \Omega_{(4)}, 
\]

(A.5)

where \( \Omega_{(4)} \) is the volume form of the metric (A.1). It follows that the sign of \( \eta \) is not merely a choice of convention but rather it has a non-trivial physical consequence.

**A.2. Taub-NUT instanton**

The metric of the Taub-NUT instanton is given by [34]

\[
\begin{align*}
\frac{1}{4} r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).
\end{align*}
\]

where the radial coordinate runs from \( r = a \) to \( r = \infty \), and \( \psi \) has period \( 4\pi \) to guarantee that the solution is regular at \( r = a \). Note that \( \eta = \pm 1 \) are the two orientation choices. The Taub-NUT manifold has the topology \( \mathbb{R}^4 \) but, even though the metric at large \( r \) is asymptotically flat, it approaches the cylinder \( \mathbb{R}^2 \times S^1 \) rather than Euclidean space. The orthonormal frame is given by

\[
\begin{align*}
\epsilon^0 &= \left( \frac{r - a}{r + a} \right)^{1/2} \, dr, \\
\epsilon^1 &= \left( r^2 - a^2 \right)^{1/2} \, d\theta, \\
\epsilon^2 &= \left( r^2 - a^2 \right)^{1/2} \sin \theta \, d\phi, \\
\epsilon^3 &= 2a \left( \frac{r - a}{r + a} \right)^{1/2} (d\psi + \eta \cos \theta \, d\phi). \\
\end{align*}
\]

(A.7)

There is one normalisable harmonic 2-form, given by \[6\]

\[
L_{(2)} = \frac{1}{(r + a)^2} (\epsilon^0 \wedge \epsilon^3 - \eta \epsilon^1 \wedge \epsilon^2), 
\]

(A.8)

which is self-dual for \( \eta = -1 \) and anti-self-dual for \( \eta = +1 \). It is normalisable since

\[
L_{(2)}^2 = \frac{4}{(r + a)^4}, 
\]

(A.9)

**References**

[1] A. Strominger, Nucl. Phys. B 343 (1990) 167; A. Strominger, Nucl. Phys. B 353 (1991) 565, Erratum.
[2] J.M. Charap, M.J. Duff, Phys. Lett. B 69 (1977) 445.
[3] M.J. Duff, R. Minasian, E. Witten, Nucl. Phys. B 465 (1996) 413, hep-th/9601036.
[4] G.W. Gibbons, C.N. Pope, Commun. Math. Phys. 66 (1979) 267.
[5] D.N. Page, Phys. Lett. B 80 (1978) 55.
[6] M. Cvetič, H. Lü, C.N. Pope, Nucl. Phys. B 600 (2001) 103, hep-th/0011023.
[7] I.R. Klebanov, A.A. Tseytlin, Nucl. Phys. B 578 (2000) 123, hep-th/0002159.
[8] I.R. Klebanov, M.J. Strassler, JHEP 0008 (2000) 052, hep-th/0007191.
[9] M. Graña, J. Polchinski, Phys. Rev. D 63 (2001) 026001, hep-th/0009211.
[10] S. Gubser, hep-th/0001010.
[11] L.A. Pando Zayas, A.A. Tseytlin, JHEP 0011 (2000) 028, hep-th/0010088.
[12] M. Bertolini, V.L. Campos, G. Ferretti, P. Fré, P. Salomonson, M. Trigiante, Nucl. Phys. B 617 (2001) 3, hep-th/0010618.
[13] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, hep-th/0101011.
[14] C.P. Herzog, I.R. Klebanov, Phys. Rev. D 63 (2001) 126005, hep-th/0101020.
[15] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Nucl. Phys. B 606 (2001) 18, hep-th/0101096.
[16] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Nucl. Phys. B 617 (2001) 151, hep-th/0102185.
[17] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Nucl. Phys. B 620 (2002) 29, hep-th/0103155.
[18] M. Cvetič, G.W. Gibbons, J.T. Liu, H. Lü, C.N. Pope, hep-th/0106162.
[19] G. Papadopoulos, P.K. Townsend, Phys. Lett. B 380 (1996) 27, hep-th/9603087.
[20] A.A. Tseytlin, Nucl. Phys. B 475 (1996) 149, hep-th/9604035.
[21] R. Argurio, F. Englert, L. Houart, Phys. Lett. B 398 (1997) 61–68, hep-th/9701042.
[22] K. Behrndt, E. Bergshoeff, B. Janssen, Phys. Rev. D 55 (1997) 3785, hep-th/9704168.
[23] J.P. Gauntlett, D.A. Kastor, J. Traschen, Nucl. Phys. B 478 (1996) 544, hep-th/9604179.
[24] J.D. Edelstein, L. Tá triu, R. Tatar, JHEP 9806 (1998) 003, hep-th/9801049.
[25] R.R. Khuri, Phys. Rev. D 48 (1993) 2947, hep-th/9305143.
[26] E. Lima, H. Lü, B.A. Ovrut, C.N. Pope, Nucl. Phys. B 572 (2000) 112, hep-th/9909184.
[27] P.M. Cowdall, P.K. Townsend, Phys. Lett. B 429 (1998) 281; P.M. Cowdall, P.K. Townsend, Phys. Lett. B 434 (1998) 458, hep-th/9801165, Erratum.
[28] J.M. Maldacena, C. Núñez, Phys. Rev. Lett. 86 (2001) 588, hep-th/0008001.
[29] A.H. Chamseddine, M.S. Volkov, Phys. Rev. Lett. 79 (1997) 3343, hep-th/9707176.
[30] M.J. Duff, H. Lü, C.N. Pope, Nucl. Phys. B 532 (1998) 181, hep-th/9803061.
[31] M.J. Duff, H. Lü, C.N. Pope, Nucl. Phys. B 544 (1999) 145, hep-th/9807173.
[32] T. Eguchi, A.J. Hanson, Phys. Lett. B 74 (1978) 249.
[33] V.A. Belinsky, G.W. Gibbons, D.N. Page, C.N. Pope, Phys. Lett. B 76 (1978) 433.
[34] S.W. Hawking, Phys. Lett. A 60 (1977) 81.