Quasilocal energy conditions

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Abstract

The classical value of the Hamiltonian for a system with timelike boundary has been interpreted as a quasilocal energy. This quasilocal energy is not positive definite. However, we derive a ‘quasilocal dominant energy condition’ which is the natural consequence of the local dominant energy condition. We discuss some implications of this quasilocal energy condition. In particular, we find that it implies a ‘quasilocal weak energy condition’.

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1 Introduction

While the concept of energy plays a fundamental role in the classical dynamics of finite systems, it has not been clear how to extend this concept into the setting of general relativity. Traditionally, the relativist has had to choose between a ‘global’ energy—the Arnowitt–Deser–Misner (ADM) mass defined at spacelike infinity—and a ‘local’ energy—the energy density defined over an infinitesimal neighbourhood. The search for a ‘quasilocal’ definition of energy which bridges the gulf between these extremes has uncovered a number of different candidates. However, the issue of whether one of these has preferred status over the others has yet to be settled.

Recently, Brown and York have proposed that the quasilocal energy for a system (as measured by a given set of observers) be defined as the classical value of its Hamiltonian. This definition, generalizes the definition of energy which appears in mechanics and has a number of attractive features. Many of these have been outlined by Brown and York.

Here our purpose is not to review all the various features of this definition of quasilocal energy, nor to judge its merits by comparison to other candidates. An excellent list of references which samples the literature on local and quasilocal energy is given in the first citation of Ref. [1].
definitions. Quite apart from any interpretation as a quasilocal energy, the Hamiltonian plays a fundamental role in general relativity. It is natural to wonder what consequences the local energy conditions hold for the classical value of the Hamiltonian. Placing labels aside, this is the question we address.

We demonstrate that when the local dominant energy condition holds everywhere on a spacelike hypersurface, a ‘quasilocal dominant energy condition’ can be derived. Defining a ‘ground state’ as a state which extremizes the Hamiltonian over the class of vacuum states which all have a given boundary geometry, the quasilocal dominant energy condition implies that perturbations from a ground state which leave boundary conditions invariant cannot give rise to spacelike variations of the quasilocal momentum field. A consequence of this is that such perturbations cannot decrease the quasilocal energy below its ground state value.

While the details of this quasilocal energy condition will be specified below, it is important to stress at the outset that it does not imply positivity of the quasilocal energy. In fact, it is easy to show that the quasilocal energy is not positive definite. Explicit examples of negative energy states can be found in Ref. [2].
To see why the quasilocal energy cannot be positive definite, consider a system of metric and matter fields with support on a compact spacelike hypersurface without boundary. The extremal Hamiltonian on such a hypersurface vanishes by virtue of the constraint equations. Now let a closed 2–surface bifurcate the spacelike hypersurface into ‘inside’ and ‘outside’ regions. The Hamiltonians for the ‘inside’ and ‘outside’ regions will in general be non–vanishing (by virtue of boundary contributions to the Hamiltonian). Furthermore, since the Hamiltonian is additive, the values for the ‘inside’ and ‘outside’ regions must be equal in magnitude but opposite in sign. Thus, the Hamiltonian cannot be positive definite.

One might imagine that there is some way to define the ground state such that the difference between the quasilocal energy of a given state and the quasilocal energy of the ground state is positive definite. However, by an argument similar to that provided above, one can show that there is no definition of the ground state which would result in quasilocal energies which are both additive and positive definite relative to the ground.

These results are not so disastrous as they seem. It has been shown that for asymptotically flat spacetimes, the quasilocal energy (as evaluated at spacelike infinity and relative to the Minkowski vacuum) reduces to the
Arnowitt–Deser–Misner (ADM) energy \cite{ADM}. Also, in the local limit, the quasilocal energy (relative to the Minkowski vacuum) reduces to the three volume integral of the energy density \cite{quasilocal}. Hence, there is no obvious conflict with the positivity of the classical energy.

Before exploring matters further, let us review the Hamiltonian formulation of general relativity for systems with timelike boundary and the definition of quasilocal energy to which it gives rise.

2 Derivation of Hamiltonian and quasilocal energy

Let \( \{g^{ab},\mathcal{M}\} \) be a spacetime with topology \( \Sigma_t \times \bar{I}_t \), where the \( \Sigma_t \) are closed, orientable spacelike hypersurfaces and \( \bar{I}_t \) is a closed (timelike) interval. Let \( \mathcal{B} \) be the boundary of \( \mathcal{M} \) and let \( B_t \) be the (two dimensional) boundary of \( \Sigma_t \). By construction, \( \mathcal{B} \equiv \Sigma_{t_0} \cup \Sigma_{t_1} \cup B \) where \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \) are initial and final spacelike hypersurfaces and \( B \equiv B_t \times \bar{I}_t \), is the timelike boundary hypersurface. See Figure 1(a).

Let \( u^a \) be the future pointing unit normal to \( \Sigma_t \) and let \( n^a \) be the outward pointing normal to \( B_t \) tangent to \( \Sigma_t \). Similarly, let \( \bar{n}^a \) be the outward pointing normal to \( B \) and let \( \bar{u}^a \) be the future pointing normal to \( B_t \) tangent to \( B \).
Then, $h_{ab} = g_{ab} + u_a u_b$ is the induced metric on $\Sigma_t$, $\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b$ is the induced metric on $B$, and $\sigma_{ab} = \gamma_{ab} + \tilde{\nu}_a \tilde{\nu}_b = h_{ab} - n_a n_b$ is the induced metric on $B_t$. See Figure 1(b).

The action functional appropriate to fixed intrinsic geometry on $B$ is [5, 6, 7, 8]:

$$I = \frac{1}{16\pi} \int_M R g^{1/2} d^4 x - \frac{1}{8\pi} \int_{\Sigma_t} Kh^{1/2} d^3 x + \frac{1}{8\pi} \int_{\Sigma_t} K h^{1/2} d^3 x$$
$$+ \frac{1}{8\pi} \int_B \tilde{K} \gamma^{1/2} d^3 x + \frac{1}{8\pi} \int_{B_t} \eta \sigma^{1/2} d^2 x - \frac{1}{8\pi} \int_{B_t} \eta \sigma^{1/2} d^2 x + I_m + C. \quad (1)$$

where $K_{ab} \equiv h_{ae} \nabla^e u_b$ is the extrinsic curvature tensor on $\Sigma_t$ while $\tilde{K}_{ab} \equiv \gamma_{ab} \nabla^a \tilde{n}_b$ is the extrinsic curvature tensor on $B$ and $\eta \equiv \sinh^{-1} (u \cdot \tilde{n})$ is the local boost parameter on $B_t$. Also, $I_m[A_a, P^a]$ is the action associated with the matter fields and their conjugate momenta (which we label $A_a$ and $P^a$, respectively).

In (1), as with any action functional, it is possible to add a functional, $C$, of the fields which are held fixed on the boundary without affecting the equations of motion. The question of whether there exists a particularly useful choice for $C$ in the gravitational context has received some attention [6, 1]. However, here we are interested only in variations of the action between states satisfying the same boundary conditions. These variations are, of
course, independent of \( C \). For simplicity of presentation and without loss of
generality, we set \( C = 0 \).

To obtain the Hamiltonian formulation of \( I \), choose a time flow vector
field, \( t^a \), which satisfies \( t^a \nabla_a t = 1 \). Also suppose \( t^a \) is tangent to \( B \) on \( B_t \).
Define a lapse function, \( N = -u_a t^a \), and a shift vector, \( N^a = t^a - Nu^a \), on \( \Sigma_t \).
Also define a lapse function, \( \tilde{N} = -\tilde{u}_a t^a \), and a shift vector, \( \tilde{N}^a = t^a - \tilde{N} \tilde{u}^a \),
associated with the foliation of the intrinsic geometry of \( B \) into 2–surfaces,
\( B_t \). Further defining a momentum field, \( p^{ab} \), conjugate to \( h_{ab} \) in the usual
fashion, we obtain \([1, 7]\).

\[
I = \int_M \mathcal{L} \, d^3x \, dt + \int_B \mathcal{L}_B \, d^2x \, dt
\]
\[
= \int_M \left( p^{ab} \dot{h}_{ab} + p^a \dot{A}_a - \mathcal{H} - \dot{\mathcal{H}}_a - \dot{A}_a \mathcal{G}^a \right) \, d^3x \, dt
\]
\[
+ \int_B \left( \frac{\eta}{8\pi} \sqrt{\sigma - \tilde{N} \ddot{\mathcal{H}} - \tilde{N}^a \ddot{\mathcal{H}}_a - \ddot{A}_a \ddot{\mathcal{G}}^a} \right) \, d^2x \, dt
\]  

(2)

In the above, \( \mathcal{H} \) and \( \mathcal{H}_a \) correspond to the super–Hamiltonian and super–
momentum, respectively, while the \( \mathcal{G}^a \) are associated with any constraints on
the matter fields. Also, \( \ddot{\mathcal{H}} = -\frac{\delta I}{\delta N} \left|_B \right. \) and \( \ddot{\mathcal{H}}_a = -\frac{\delta I}{\delta N^a} \left|_B \right. \) are, respectively,
the ‘boundary super–Hamiltonian’ and ‘boundary super–momentum’ \([4, 7]\).

\[\text{Footnote:} \text{The treatment of Ref. [1] does not include the ‘kinetic’ term, } \frac{\eta}{8\pi} \sqrt{\sigma}, \text{ in the boundary’s}
\text{contribution to the Lagrangian. This is because the authors assume that the timelike and}
\text{spacelike portions of the boundary intersect each other orthogonally at } B_t \text{ and, hence,}
\text{that } \eta = 0 \text{ there.}\]
while \( \tilde{G}^a \) are associated with any conserved charge densities \[1\].

Perform Legendre transformations\[3\] on (2) to obtain the Hamiltonian,

\[
H = \int_{\Sigma_t} \left( \rho^{ab} \dot{h}_{ab} + P^a \dot{A}_a - \mathcal{L} \right) \, d^3x + \int_{B_t} \left( \frac{\eta}{8\pi} \sqrt{\sigma} - \mathcal{L}_B \right) \, d^2x \\
= \int_{\Sigma_t} \left( N \mathcal{H} + N^a \mathcal{H}_a + A_a \mathcal{G}^a \right) \, d^3x + \int_{B_t} \left( \tilde{N} \tilde{\mathcal{H}} + \tilde{N}^a \tilde{\mathcal{H}}_a + \tilde{A}_a \tilde{\mathcal{G}}^a \right) \, d^2x. \tag{3}
\]

When the geometry is allowed to vary freely everywhere on \( \Sigma_t \), we obtain the usual Hamilton’s equations,

\[
\dot{h}_{ab} = \frac{\delta H}{\delta \rho^{ab}} \\
\dot{\rho}^{ab} = -\frac{\delta H}{\delta h_{ab}} \tag{4}
\]

plus boundary counterparts,

\[
\frac{\sqrt{\sigma}}{8\pi} = \frac{\delta H}{\delta \eta} \bigg|_{B_t} \\
\frac{\dot{\eta}}{8\pi} = -\frac{\delta H}{\delta \sqrt{\sigma}} \bigg|_{B_t}. \tag{5}
\]

Define the quasilocal energy, \( E \), as the value of the Hamiltonian for the classical spacetime configuration as measured by observers for which \( t^a = \tilde{u}^a \) on \( B_t \) (e.g. as measured by observers which travel orthogonal to \( B_t \) on \( B \)).

\[3\] Since the treatment of Ref. \[1\] assumes \( \eta = 0 \), their definition of the Hamiltonian does not involve a Legendre transformation with respect to the boundary ‘kinetic’ term. Nonetheless, such a Legendre transformation is necessary if the extremal Hamiltonian is not to be depend on the gauge choice for the spacelike slicing.
This yields\footnote{Expression \((\ref{eq:quasilocal_energy})\) for the quasilocal energy was obtained in Ref’s \cite{4, 7}. This expression also agrees with the expression for the quasilocal energy obtained by Brown and York \cite{1} when \(\Sigma_t\) and \(B\) are orthogonal on \(B_t\) (i.e. \(n^a = \tilde{n}^a\)) but does not agree with their expression otherwise.}

\[
E = \int_{B_t} \tilde{\mathcal{H}} \big|_{\text{cl}} \ d^2x = -\frac{1}{8\pi} \int_{B_t} \sigma^b_a \tilde{n}_b \sqrt{\sigma} \ d^2x.
\]  

(6)

It is also possible to define the quasilocal energy and, more generally, a quasilocal momentum field in terms of a boundary surface stress tensor \cite{1}. This formulation employs a natural extension of classical Hamilton–Jacobi theory and is valuable to review because we shall find that the quasilocal dominant energy condition can be expressed in terms of variations of the boundary stress tensor.

Let \(S \equiv I|_{\text{extremum}}\) be the extremal action for a system subject to constrained geometry along \(B\). Define a boundary surface stress tensor, \(\tau^{ab}\), in terms of the variations of \(S\) induced by varying the constrained boundary geometry \cite{1},

\[
\tau^{ab} \equiv \frac{2}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma^{ab}}|_{B_t}.
\]  

(7)

Choose a time flow vector such that \(\tilde{N}\) is constant over \(B_t\). Let \(T\) be the lapse of proper time as measured from \(B_t\) to a neighboring slice \(B_{t+\Delta t}\) by observers traveling orthogonal to \(B_t\). Then, uniformly varying \(T\) over \(B_t\) and
taking the limit $\Delta t \to 0$, define the quasilocal energy by,

$$ E \equiv -\frac{\delta S}{\delta T} \bigg|_{B_t} = \int_{B_t} \tilde{H}_{cl} \, d^2 x = \int_{B_t} \tilde{u}^a \tilde{u}^b \tau_{ab} \sqrt{\sigma} \, d^2 x. \tag{8} $$

Thus, the quasilocal energy is the negative of the variation of the extremal action induced by uniformly varying the interval of proper time as measured by observers traveling orthogonal to $B_t$ on $B$. This definition naturally extends the definition of energy that arises in classical Hamilton–Jacobi theory.

More generally, define a quasilocal momentum field tangent to $B$ as follows. Let $e^a_{(i)}$ be a triad tangent to $B$ (for $i = 0, 1, 2$) such that $e^a_{(0)} \equiv \tilde{u}^a$ and $e^a_{(1)}$ and $e^a_{(2)}$ are tangent to $B_t$. We can then express $t^a$ as $t^a = t^i e^a_{(i)}$. Choosing $t^a$ such that $t^i$ are constant over $B_t$, define components of a quasilocal momentum field tangent to $B$ in the limit $\Delta t \to 0$ by,

$$ \mathcal{P}_t = \frac{\delta S}{\delta T^i} \bigg|_{B_t} = -\int_{B_t} e^a_{(i)} \tilde{u}^b \tau^a_{\cdot b} \sqrt{\sigma} \, d^2 x, \tag{9} $$

where $T^i = t^i \Delta t$.

### 3 Quasilocal Energy Conditions

In this section, we establish that the local dominant energy condition gives rise to a quasilocal counterpart. Let us begin by deriving an expression for
the variation of the quasilocal momentum field induced when we perturb a vacuum state to obtain a new state satisfying the same boundary conditions.

Consider a manifold $\mathcal{M}$ of the type discussed in the previous section. Constraining the geometry on its timelike boundary $B$. Let $\{g^{ab}\}$ be the class of all smooth metrics which cover $\mathcal{M}$ and have the appropriate geometry on $B$. Let $\{g^{ab}_{\text{vac}}\}$ be the sub-class of these metrics composed of all solutions which are vacuum states of the matter field (i.e. solutions to Einstein’s equations with $T^{ab} = 0$). Depending on the choice of the fixed geometry on $B$, $\{g^{ab}_{\text{vac}}\}$ may be a finite set, an infinite set, or the empty set.

Take variations of the Hamiltonian (3) around $g^{ab}_{\text{vac}}$ and impose Hamilton’s equations (4) and their boundary counterparts (5) to obtain,

$$
\int_{B_t} \left( \tilde{N} \delta \tilde{H} + \tilde{N}^{a} \delta \tilde{H}_a \right) d^2 x = \int_{\Sigma_t} \left\{ \tilde{h}_{ab} \delta \tilde{p}^{ab} - \tilde{p}^{ab} \delta h_{ab} - N \delta \mathcal{H}_{(m)} - N^a \delta \mathcal{H}_{a(m)} \right\} d^3 x
$$

$$
+ \frac{1}{8\pi} \int_{B_t} \left\{ \hat{\sigma} \delta \eta - \hat{\eta} \delta \sqrt{\sigma} \right\} d^2 x
$$

where $N \mathcal{H}_{(m)} + N^a \mathcal{H}_{a(m)} = u^a e^b T_{ab} \sqrt{h}$ is the matter’s contribution to the Hamiltonian.

Equation (10) is a general expression for the variation of the Hamiltonian around a vacuum solution. It can also be derived by taking variations of the microcanonical action [9] around $g^{ab}_{\text{vac}}$ and taking the limit $\Delta t \equiv t_1 - t_0 \to 0$. 

Now let us further refine the scope of our attention to variations of the Hamiltonian around an extremum of its vacuum values. Define \( \bar{g}^{ab} \) to be the set of solutions which extremize \( H \) over \( \{ g_{\text{vac}}^{ab} \} \). We will refer to any such states as ‘ground states’ compatible with the fixed geometry on \( B \). From (10) we obtain that the variation of \( H \) around a ground state is given by,

\[
\int_{B_t} \left( \bar{N} \delta \bar{H} + \bar{N}^a \delta \bar{H}_a \right) d^2 x = \int_{\Sigma_t} u^a t^b \delta T_{ab} \sqrt{h} d^3 x. \tag{11}
\]

Note that \( \bar{u}_a t_b \tau^{ab} \sqrt{\sigma} = \bar{N} \bar{H} + \bar{N}^a \bar{H}_a \), so equation (11) can also be expressed in the form,

\[
\int_{B_t} \bar{u}_a t_b \delta \tau_{ab} \sqrt{\sigma} d^2 x = \int_{\Sigma_t} u^a t^b \delta T_{ab} \sqrt{h} d^3 x. \tag{12}
\]

Finally, let us restrict the choice of the time flow vector on the boundary so that the quasilocal momentum field is well defined in the sense described in Section 2 (this is equivalent to restricting the gauge choice so that the boundary lapse and shift are constant over \( B_t \)). We obtain from (12),

\[
- t^i \delta \mathcal{P}_i = \int_{\Sigma_t} \left\{ u^a t^b \delta T_{ab} \sqrt{h} \right\} d^3 x, \tag{13}
\]

where the variation is conducted around a ground state compatible with a given fixed geometry on \( B \). When the local dominant energy condition is satisfied everywhere on \( \Sigma_t \), we have \( u^a t^b \delta T_{ab} \geq 0 \) (since variations are
performed around a $T^{ab} = 0$ solution). In this case, (13) implies,

$$t^i \delta P_i = t^i (P^i - \bar{P}^i) \leq 0. \quad (14)$$

We interpret equation (14) as a quasilocal dominant energy condition. It requires that perturbations from a ground state, $\bar{g}^{ab}$, which preserve the fixed geometry of $B$ cannot give rise to a spacelike variation of the quasilocal momentum vector $P^i$.

4 Discussion

Let us examine more closely the significance of the quasilocal energy condition (14).

First, it is important to stress that equation (14) in general only holds for variations conducted around a ground state (i.e. a vacuum state which extremizes the Hamiltonian over $\{g^{ab}_{\text{vac}}\}$). In fact, it is easy to see that equation (14) can be violated when variations are conducted around a vacuum solution which is not extremal. Perturbations from one vacuum solution to another would generate linear variations in the Hamiltonian and, hence, must in some cases result in a decrease of the quasilocal energy. [An explicit example which demonstrates this is given in the next section.]
Second, note that for a given fixed geometry along $B$, there is no guarantee that a ground state will exist. Nor is there any guarantee that when a ground state exists it is unique. In principle, there might even even be an infinite set of ground states compatible with a given boundary geometry.

Third, note that equation (14) holds only for local perturbations around a ground state. It does not necessarily imply that a ground state is a global minimum energy state, nor does it necessarily imply the existence of a finite least energy state (see example of next section). In some cases, a given fixed geometry on $B$ might give rise to an infinite class of vacuum solutions for which the energy could be arbitrarily negative.

Fourth, note that the quasilocal dominant energy condition yields a quasilocal weak energy condition as a consequence. Specifically, if we choose a time flow vector parallel to $\tilde{u}^a$, equation (14) yields

$$\delta E = E - \bar{E} \geq 0$$  \hspace{1cm} (15) $$

where $E$ is the quasilocal energy for the system with matter distribution and $\bar{E}$ is the quasilocal energy for the ground state.
5 Quasilocal energy condition for spacetimes with Schwarzschild geometry.

An explicit example will help clarify the discussion of the previous section.

Let \( \{g^{ab}, \mathcal{M}\} \) be a spacetime with Schwarzschild geometry,

\[
ds^2 = -(1 - 2M/r) \ dt^2 + (1 - 2M/r)^{-1} \ dr^2 + r^2 \ d\Omega^2. \tag{16}
\]

Further suppose the spacetime extends between timelike tubes at \( r = r_0 \) and \( r = r_1 \).

To evaluate the quasilocal energy as measured by observers on a given boundary \( 2 \)-surface, we must choose a time flow vector such that the lapse and shift are constant over the boundary \( 2 \)-surface. [Note that the lapse is not constant over \( B_i \) for \( \hat{t}^a = \left( \frac{\partial}{\partial \hat{t}} \right)^a \).] Let us take \( t^a \equiv \left( \frac{\partial}{\partial t} \right)^a \) where

\[
t - t_1 = (1 - 2M/r)^{1/2} \hat{t}
\]

and evaluate the quasilocal energy over the boundary \( 2 \)-surface at \( t = t_1 \). In this frame, the line element (16) becomes

\[
ds^2 = -dt^2 + \frac{2M(t-t_1)}{r^2(1 - 2M/r)} \ dr \ dt + \left[ \frac{1}{1 - 2M/r} - \frac{M^2(t-t_1)^2}{r^4(1 - 2M/r)^2} \right] \ dr^2 + r^2 \ d\Omega^2. \tag{18}
\]

Let us fix the geometry on \( B \) so that its intrinsic line element is

\[
ds^2 = -dt^2 + r^2 \ d\Omega^2 \tag{19}
\]
for \( r = r_0 \) and \( r = r_1 \). From (18) it is clear that the Schwarzschild solutions with \( 2M < r_0 \) provide isometric embeddings of the boundary geometry. Hence, \( \{ g^{ab}_{\text{vac}} \} \) consists of all Schwarzschild solutions with \( 2M < r_0 < r_1 \).

Evaluating expression (10) for variations in the Hamiltonian over this class of vacuum solutions yields,

\[
\delta E = (1 - 2M/r) \left. -1/2 \right|_{r_0}^{r_1} \delta M.
\]

Equation (20) implies that flat spacetime is the unique ground state compatible with these boundary conditions (i.e. \( \{ \bar{g}^{ab} \} \) consists of a unique element which is the flat metric). Thus, when the geometry on the timelike boundary is held fixed with line element given by (19), the quasilocal dominant energy condition implies \( \delta E \geq 0 \) for linear perturbations from flat spacetime so long as the local dominant energy condition is everywhere satisfied.

Note, however, that the quasilocal energy condition need not hold for variations around vacuum solutions which do not extremize the Hamiltonian. In particular, when both \( M \) and \( \delta M \) are greater than zero, equation (20) yields \( \delta E < 0 \).

Also note that even though \( \delta E \geq 0 \) for linear perturbations from flat
spacetime, the quasilocal energy associated with flat spacetime is neither a minimum nor a lower bound to the energies accessible with the boundary geometry (19). In fact, if we allow for vacuum solutions with $M < 0$, the quasilocal energy is not bounded from below: $E \to -\infty$ as $M \to -\infty$. On the other hand, if we confine attention to solutions with $M \geq 0$, the quasilocal energy achieves a lower bound for the vacuum solution with $M = r_0/2$.

6 Conclusions

The quasilocal energy (i.e. the classical value of the Hamiltonian) is not positive definite. Nor is there any way to define a ground state such that the quasilocal energy of a given state relative to the ground is both positive definite and additive. In fact, it may be (as in the example of Section (3)) that the set of quasilocal energies accessible to a system with a given boundary geometry is not bounded from below.

Despite all this, we have demonstrated that a quasilocal dominant energy condition holds for linear perturbations around a ground state so long as the local dominant energy condition is everywhere satisfied. The quasilocal dominant energy condition also implies a quasilocal weak energy condition.

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Figure 1: (a) A spacetime extending from spacelike hypersurface $\Sigma_{t_0}$ to spacelike hypersurface $\Sigma_{t_1}$ and out to timelike boundary $B$. The timelike and spacelike portions of the boundary intersect at $B_{t_0}$ and $B_{t_1}$. (b) The same spacetime with the various unit normals and boundary metrics indicated.
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