Existence of Erdős-Burgess constant in commutative rings

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Abstract

Let $R$ be a commutative unitary ring. An idempotent in $R$ is an element $e \in R$ with $e^2 = e$. The Erdős-Burgess constant associated with the ring $R$ is the smallest positive integer $\ell$ (if exists) such that for any given $\ell$ elements (not necessarily distinct) of $R$, say $a_1, \ldots, a_\ell \in R$, there must exist a nonempty subset $J \subset \{1, 2, \ldots, \ell\}$ with $\prod_{j \in J} a_j$ being an idempotent. In this paper, we prove that except for an infinite commutative ring with a very special form, the Erdős-Burgess constant of the ring $R$ exists if and only if $R$ is finite.

Key Words: Zero-sum; Davenport constant; Erdős-Burgess constant; Idempotents; Jacobson radical; Noetherian rings; Multiplicative semigroups of rings

MSC: 11B75; 05E40; 20M13

1 Introduction

Let $S$ be a nonempty commutative semigroup, endowed with a binary associative operation $\ast$. Let $E(S)$ be the set of idempotents of $S$, where $e \in S$ is said to be an idempotent if $e \ast e = e$. Idempotent is one of central notions in Semigroup Theory and Algebra, also connects closely with other fields, see [8, 11] for the idempotent theorem in harmonic analysis, see [13] for the application in coding theory. One of our interest to combinatorial properties concerning idempotents in semigroups comes from a question of P. Erdős to D.A. Burgess (see [3] and [10]), which can be restated as follows.

Let $S$ be a finite nonempty semigroup of order $n$. A sequence of terms from $S$ of length $n$ must contain one or more terms whose product, in some order, is idempotent?
Burgess [3] in 1969 gave an answer to this question in the case when $S$ is commutative or contains only one idempotent. D.W.H. Gillam, T.E. Hall and N.H. Williams [10] proved that a sequence $T$ over any finite semigroup $S$ of length at least $|S \setminus E(S)| + 1$ must contain one or more terms whose product, in the order induced from the sequence $T$, is an idempotent, and therefore, completely answered Erdős’ question. The Gillam-Hall-Williams Theorem was extended to infinite semigroups by the author [19]. It was also remarked that the bound $|S \setminus E(S)| + 1$, although is optimal for general semigroups $S$, can be improved, at least in principle, for specific classes of semigroups. Naturally, one combinatorial invariant was aroused by Erdős’ question with respect to idempotents of semigroups. Since we deal with the multiplicative semigroup of a commutative ring in this paper, we introduce only the definition of this invariant for commutative semigroups here.

**Definition.** ([19], Definition 4.1) For a commutative semigroup $S$, we define the Erdős-Burgess constant of $S$, denoted by $I(S)$, to be the smallest positive integer $\ell$ (if exists) such that every sequence $T$ of terms from $S$ and of length $\ell$ must contain one or more terms whose product is an idempotent. If no such integer $\ell$ exists, we let $I(S) = \infty$.

Note that if the commutative semigroup $S$ is finite, Gillam-Hall-Williams Theorem definitely tells us that the Erdős-Burgess constant of $S$ exists (i.e., $I(S)$ is finite) and bounded above by $|S \setminus E(S)| + 1$. In particular, when the semigroup $S$ happens to be a finite abelian group, the Erdős-Burgess constant reduces to a classical combinatorial invariant, the Davenport constant. The Davenport constant of a finite abelian group $G$, denoted $D(G)$, is defined as the smallest positive integer $\ell$ such that every sequence of terms from $G$ of length at least $\ell$ contains one or more terms with the product being the identity element of $G$. This invariant was popularized by H. Davenport in the 1960’s, notably for its link with algebraic number theory (as reported in [16]). Davenport constant has been investigated extensively in the past over 50 years, and found applications in other areas, including Factorization Theory of Algebra [8, 9], Classical Number Theory [1], Graph Theory [2], and Coding Theory [14]. What is more important, a lot of researches were motivated by the Davenport constant together with the celebrated EGZ Theorem obtained by P. Erdős, A. Ginzburg and A. Ziv [4] in 1961 on additive properties of sequences in groups, which have been developed into a branch, called zero-sum theory (see e.g. [7], and [12] for a survey), in Combinatorial Number Theory. Recently some zero-sum type problems were also investigated in the setting of commutative semigroups (see e.g. [17, 18, 20, 21]).

To investigate the Erdős-Burgess constant associated with commutative rings, one fundamental question remains:

**When does the Erdős-Burgess constant exist for a commutative ring?**

In this paper, we shall answer this question by proving the following theorem.
**Theorem 1.1.** Let $R$ be a commutative unitary ring, and let $S_R$ be the multiplicative semigroup of $R$. If $I(S_R)$ is finite, then one of the following two conditions holds:

(i) The ring $R$ is finite;

(ii) The Jacobson radical $J(R)$ is finite and $R/J(R) \cong B \times \prod_{i=1}^{t} \mathbb{F}_{q_i}$, where $B$ is an infinite Boolean unitary ring, and $\mathbb{F}_{q_1}, \ldots, \mathbb{F}_{q_t}$ are finite fields with $0 \leq t \leq I(S_R) - 1$ and prime powers $q_1, \ldots, q_t > 2$.

**Remark.** Recall that by Gillam-Hall-Williams Theorem, if the ring $R$ is finite then $I(S_R)$ exists. Hence, Theorem 1.1 asserts that the Erdős-Burgess constant exists only for finite commutative rings except for an infinite commutative rings with a very special form given as (ii). That is, to study this invariant in the realm of commutative rings, we may consider it only for finite commutative rings.

## 2 The Proof of Theorem 1.1

For integers $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. Let $(R, +, \cdot)$ be a commutative unitary ring, and let $T$ be a sequence of terms from $R$. By $|T|$ we denote the length of the sequence $T$. We call $T$ an idempotent-product free sequence provided that no idempotent of $R$ can be represented as a product of one or more terms from $T$. By the definition, we have immediately that the Erdős-Burgess constant $I(S_R)$ exists if and only if sup $\{|T| : T$ is taken over all idempotent-product free sequences over $R\}$ is finite.

**Lemma 2.1.** Let $G$ be an abelian group. Then $I(G)$ is finite if and only if $G$ is finite.

**Proof.** Since the identity element is the unique idempotent in $G$, the sufficiency of the lemma is well-known in zero-sum theory and follows from a simple application of the pigeonhole principle. Now we show the necessity. Suppose $|G|$ is infinite. Let $T = (a_1, \ldots, a_n)$ be an arbitrary idempotent-product free sequence over $G$. By the infinity of $|G|$, we can find a nonidentity element $g \in G$ such that $g^{-1}$, the inverse of $g$, can not be represented as the product of one or more terms from $T$. We see that the sequence $(a_1, \ldots, a_n, g)$ obtained by adjoining the element $g$ to $T$ is idempotent-product free. By the arbitrariness of $T$, we conclude that $I(G)$ is infinite, completing the proof.

**Lemma 2.2.** Let $S$ be a commutative semigroup and $S'$ a subsemigroup of $S$. If $I(S)$ is finite, then $I(S')$ is finite and $I(S') \leq I(S)$.

**Proof.** The conclusion follows immediately from the fact that any idempotent-product free sequence of terms from $S'$ is also an idempotent-product free sequence of terms from $S$.  

\[3\]
Lemma 2.3. Let \( S \) and \( S' \) be commutative semigroups. If there is an epimorphism \( \varphi \) of \( S \) onto \( S' \), then \( I(S') \leq I(S) \).

Proof. Let \( T' = (b_1, \ldots, b_\ell) \) be an arbitrary idempotent-product free sequence of terms from \( S' \). We can take a sequence \( T = (a_1, \ldots, a_\ell) \) of terms from \( S \) such that \( \varphi(a_i) = b_i \) for each \( i \in [1, \ell] \). Since the epimorphism \( \varphi \) always maps an idempotent of \( S \) to an idempotent of \( S' \), we have that the sequence \( T \) is also idempotent-product free in \( S \). By the arbitrariness of \( T' \), we derive that \( I(S') \leq I(S) \).  

\[ \square \]

Lemma 2.4. (see [13], Theorem 3.9) A ring \( R \) has a representation as a subdirect sum of rings \( S_i, i \in \mathcal{A} \), if and only if for each \( i \in \mathcal{A} \) there exists in \( R \) a two-sided ideal \( K_i \) such that \( R/K_i \cong S_i \) and, moreover, \( \bigcap_{i \in \mathcal{A}} K_i = (0_R) \).

Lemma 2.5. (see [13], Theorem 3.16) A ring is isomorphic to a subdirect sum of fields \( \mathbb{F}_2 \) is and only it is a Boolean ring.

Lemma 2.6. Let \( R \) be a commutative unitary ring. Let \( \{M_i : i \in \mathcal{A}\} \) be a family (nonempty) of maximal ideals of \( R \) with index two. Then \( R/\bigcap_{i \in \mathcal{A}} M_i \) is a Boolean unitary ring.

Proof. Let

\[ N = \bigcap_{i \in \mathcal{A}} M_i. \]  

We see that \( M_i/N \) are distinct maximal ideals of \( R/N \) with index \( [R/N : M_i/N] = [R : M_i] = 2 \), and so

\[ \frac{R/N}{M_i/N} \cong \mathbb{F}_2, \]  

where \( i \in \mathcal{A} \). By (1), we derive that

\[ \bigcap_{i \in \mathcal{A}} (M_i/N) = (0_{R/N}). \]  

By (2), (3), Lemma 2.4 and Lemma 2.5 we derive that \( R/N \) is a Boolean unitary ring.  

\[ \square \]

Proof of Theorem 1.1. Suppose

\[ I(S_R) = n \]  

is finite and

\[ |R| = \infty. \]  

It suffices to prove (ii) holds. Since the group \( U(R) \) is a subsemigroup of \( S_R \) where \( U(R) \) denotes the group of units of the ring \( R \), it follows from (4) and Lemma 2.2 that \( I(U(R)) \leq n \). By Lemma 2.1 we derive that \( |U(R)| < \infty \). Since \( 1_R + J(R) \subset U(R) \), it follows that

\[ |J(R)| < \infty. \]
Claim A. The index of each maximal ideal in \( R \) is finite.

Proof of Claim A. Assume to the contrary that there exists some maximal ideal \( M \) such that the index of \( M \) in \( R \) is infinite, i.e., \( R/M \) is an infinite field. Since the group \( U(R/M) \) is a subsemigroup of \( S_{R/M} \) and there is a canonic epimorphism of the semigroup \( S_R \) onto \( S_{R/M} \) with rings’ multiplication of \( R \) and \( R/M \), it follows from Lemma 2.2 and Lemma 2.3 that \( I(U(R/M)) \leq I(S_{R/M}) \leq I(S_R) = n \). Combined with Lemma 2.4 we have that \( |U(R/M)| < \infty \) and so \( |R/M| = |U(R/M)| + 1 < \infty \), a contradiction. This proves Claim A. \( \square \)

Claim B. The ring \( R \) has at most \( n - 1 \) maximal ideals with index greater than two.

Proof of Claim B. Assume to the contrary that there exists at least \( n \) distinct maximal ideals, say \( M_1, \ldots, M_n \), of \( R \) with index greater than two. Combined with Claim A, we see that \( R/M_i \) is a finite field of order \( |R/M_i| > 2 \), which implies that \( |U(R/M_i)| \geq 2 \) and so the group \( U(R/M_i) \) contains at least one non-idempotent element, for each \( i \in [1, n] \). Therefore, there are \( b_1, b_2, \ldots, b_n \) (not necessarily distinct) of \( R \) such that \( b_i^2 \not\equiv b_i \pmod{M_i} \) for each \( i \in [1, n] \). By the Chinese Remainder Theorem, we can find \( a_1, \ldots, a_n \) of \( R \) such that \( a_i \equiv b_i \pmod{M_i} \) and \( a_i \equiv 1_k \pmod{M_j} \) for \( j \in [1, n] \setminus \{i\} \), where \( i \in [1, n] \). Let \( L \) be the sequence consisting of exactly all these terms \( a_1, \ldots, a_n \). We check that the sequence \( L \) is idempotent-product free, which implies that \( I(S_R) \geq |L| + 1 = n + 1 \), a contradiction with (4). This proves Claim B. \( \square \)

Claim C. The ring \( R \) has infinitely many maximal ideals with index two.

Proof of Claim C. Assume to the contrary that there exists only finitely many maximal ideals with index two. Combined with Claim A and Claim B, we derive that \( R \) has only finitely many maximal ideals. Since \( J(R) = \bigcap_{M \text{ ranges over all maximal ideals}} M \), it follows from the Chinese Remainder Theorem that \( R/J(R) \cong \prod_{M \text{ ranges over all maximal ideals}} R/M \). Combined with Claim A, we derive that \( |R/J(R)| \) is finite. By (6), we derive that \( R \) is finite, which is a contradiction with (5). This proves Claim C. \( \square \)

Let \( N = \bigcap_{i \in A} M_i \) where \( \{M_i : i \in A\} \) is the set of all maximal ideals of \( R \) of index two. Take a representation

\[
J(R) = N \cap K_1 \cap \cdots \cap K_t
\]  

such that \( t \geq 0 \) is minimal, where \( K_1, \ldots, K_t \) are distinct maximal ideals of \( R \) of index greater than two. By the minimality of \( t \), we conclude that \( N \nsubseteq K_i \) for each \( i \in [1, t] \) and so \( N, K_1, \ldots, K_t \) are pairwise coprime. By (7) and the Chinese Remainder Theorem, we derive that \( R/J(R) \cong (R/N) \times (\prod_{i=1}^t R/K_i) \). By Claim A, we derive that there exists primes powers \( q_1, \ldots, q_t > 2 \) such that \( R/K_i \cong \mathbb{F}_{q_i} \) for each \( i \in [1, t] \). i.e.,

\[
R/J(R) \cong (R/N) \times (\prod_{i=1}^t \mathbb{F}_{q_i}).
\]  

(8)
By Lemma 2.6, we have $R/N$ is a Boolean unitary ring. By (5) and (6), we see $|R/J(R)|$ is infinite. Combined with (8), we derive that $|R/N|$ is infinite. Combined with (4) and Claim B, $t \leq I(S_R) - 1$ and (ii) holds readily. This completes the proof of the theorem. □

As a consequence of Theorem 1.1, we have the following.

**Corollary 2.7.** If $R$ is a commutative Noetherian unitary ring, then $I(S_R)$ is finite if and only if $R$ is finite.

**Proof.** Since any infinite Boolean ring is not Noetherian (see [4], Proposition 9.6), we could derive that the ring $R$ meeting Condition (ii) of Theorem 1.1 is not Noetherian. Then the conclusion follows immediately. □

**Proposition 2.8.** Let $R$ be an infinite commutative unitary ring. If $I(S_R)$ is finite, then $R$ has infinitely many maximal ideals of index two and has at most $I(S_R) - 1$ maximal ideals with index greater than two and has no maximal ideals of infinite index.

We remark that Proposition 2.8 can be derived from the arguments of Theorem 1.1. However, to show that Theorem 1.1 itself implies Proposition 2.8 we give a short proof here.

**Proof.** By Theorem 1.1, $R/J(R) \cong B \times \prod_{i=1}^{t} F_{q_i}$, where $B$ is an infinite Boolean unitary ring, $F_{q_1}, \ldots, F_{q_t}$ ($t \geq 0$, $q_1, \ldots, q_t > 2$) are finite fields. Note that infinite Boolean unitary ring $B$ has infinitely many maximal ideals, and each of the maximal ideals has index two (see [4], Proposition 9.4 and Proposition 9.6). Since $B$ has an identity, any ideal $K \triangleleft R/J(R)$ must be of the form $K = K_0 \times K_1 \times \cdots \times K_t$ where $K_0 \triangleleft B$, $K_1 \triangleleft F_{q_1}$, \ldots, $K_t \triangleleft F_{q_t}$. We derive that $R/J(R)$ has infinitely many maximal ideals of index two and has exactly $t \leq I(S_R) - 1$ maximal ideals with index greater than two, in precise, with indices $q_1, \ldots, q_t > 2$ respectively, and has no maximal ideals of infinite index, thus, so does the ring $R$, since $J(R)$ is the intersection of all maximal ideals of $R$. □

From Proposition 2.8 we have the following immediately.

**Corollary 2.9.** If $R$ is a commutative semi-local unitary ring, i.e., $R$ has only finitely many maximal ideals. Then $I(S_R)$ is finite if and only if $R$ is finite.

We conjecture that the conditions of Theorem 1.1 should be also sufficient. Hence, we close this paper with the following conjecture.

**Conjecture 2.10.** Let $R$ be a commutative unitary ring with $R/J(R) \cong B \times \prod_{i=1}^{t} F_{q_i}$, where $B$ is an infinite Boolean unitary ring, $F_{q_1}, \ldots, F_{q_t}$ ($t \geq 0$) are finite fields, and the Jacobson radical $J(R)$ is finite. Then $I(S_R)$ is finite.
Remark 2.11. Note that if the ring $R$ has a zero Jacobson Radical $J(R) = (0_R)$ ($R$ is called Jacobson-semisimple), then $R \cong B \times \prod_{i=1}^{t} F_{q_i}$, it is not hard to show that $I(S_R) = I(S \prod_{i=1}^{t} F_{q_i})$ which is finite and Conjecture 2.10 holds true.

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