Submodularity on a tree: 
Unifying $L^2$-convex and bisubmodular functions

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Abstract

We introduce a new class of functions that can be minimized in polynomial time in the value oracle model. These are functions $f$ satisfying $f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y)$ where the domain of each variable $x_i$ corresponds to nodes of a rooted binary tree, and operations $\sqcap, \sqcup$ are defined with respect to this tree. Special cases include previously studied $L^\natural$-convex and bisubmodular functions, which can be obtained with particular choices of trees. We present a polynomial-time algorithm for minimizing functions in the new class. It combines Murota’s steepest descent algorithm for $L^\natural$-convex functions with bisubmodular minimization algorithms.

1 Introduction

Let $f: D \to \mathbb{R}$ be a function of $n$ variables $x = (x_1, \ldots, x_n)$ where $x_i \in D_i$; thus $D = D_1 \times \ldots \times D_n$. We call elements of $D_i$ labels, and the argument of $f$ a labeling. Denote $V = \{1, \ldots, n\}$ to be the set of nodes. We will consider functions $f$ satisfying

$$f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y) \quad \forall x, y \in D$$

where binary operations $\sqcap, \sqcup : D \times D \to D$ (expressed component-wise via operations $\sqcap, \sqcup : D_i \times D_i \to D_i$) are defined below.

There are several known cases in which function $f$ can be minimized in polynomial time in the value oracle model. The following two cases will be of particular relevance:

- **$L^2$-convex function**: $D_i = \{0, 1, \ldots, K_i\}$ where $K_i \geq 0$ is integer, $a \sqcap b = \lfloor \frac{a+b}{2} \rfloor$, $a \sqcup b = \lceil \frac{a+b}{2} \rceil$. Property (1) is then called discrete midpoint convexity [32].

- **Bisubmodular functions**: $D_i = \{-1, 0, +1\}$, $a \sqcup b = \text{sign}(a + b)$, $a \sqcap b = |ab| \text{sign}(a + b)$.

In this paper we introduce a new class of functions which includes the two classes above as special cases. We assume that labels in each set $D_i$ are nodes of a tree $T_i$ with a designated root $r_i \in D_i$. Define a partial order $\preceq$ on $D_i$ as follows: $a \preceq b$ if $a$ is an ancestor of $b$, i.e. $a$ lies on the path from $b$ to $r_i$ ($a, b \in D_i$). For two labels $a, b \in D_i$ let $P[a \to b]$ be unique path from $a$ to $b$ in $T_i$, $\rho(a, b)$ be the number of edges in this path, and $P[a \to b, d]$ for integer $d \geq 0$ be the $d$-th node of this path so that $P[a \to b, 0] = a$ and $P[a \to b, \rho(a, b)] = b$. If $d > \rho(a, b)$ then we set by definition $P[a \to b, d] = b$.

1 Pronounced as “L-natural convex”.
Figure 1: Examples of trees. Roots are always at the bottom. (a) Illustration of the definition of \(a \sqcap b\), \(a \sqcup b\), \(a \wedge b\) and \(a \vee b\). (b) A tree for \(L^2\)-convex functions. (c) A tree for bisubmodular functions. (d) A tree for which a weakly tree-submodular function can be minimized efficiently (see section 4).

With this notation, we can now define \(a \sqcap b\), \(a \sqcup b\) as the unique pair of labels satisfying the following two conditions: (1) \(\{a \sqcap b, a \sqcup b\} = \{P[a \to b, \lfloor \frac{d}{2} \rfloor], P[a \to b, \lceil \frac{d}{2} \rceil] \}\) where \(d = \rho(a, b)\), and (2) \(a \sqcap b \preceq a \sqcup b\) (Figure 1(a)). We call functions \(f\) satisfying condition (1) with such choice of \((D, \sqcap, \sqcup)\) strongly tree-submodular. Clearly, if each \(T_i\) is a chain with nodes 0, 1, ..., \(K\) and 0 being the root (Figure 1(b)) then strong tree-submodularity is equivalent to \(L^2\)-convexity. Furthermore, if each \(T_i\) is the tree shown in Figure 1(c) then strong tree-submodularity is equivalent to bisubmodularity.

The main result of this paper is the following

**Theorem 1.** If each tree \(T_i\) is binary, i.e. each node has at most two children, then a strongly tree-submodular function \(f\) can be minimized in time polynomial in \(n\) and \(\max_i |D_i|\).

**Weak tree-submodularity**  We will also study alternative operations on trees, which we denote as \(\wedge\) and \(\lor\). For labels \(a, b \in D_i\) we define \(a \wedge b\) as their highest common ancestor, i.e. the unique node on the path \(P[a \to b]\) which is an ancestor of both \(a\) and \(b\). The label \(a \lor b\) is defined as the unique label on the path \(P[a \to b]\) such that the distance between \(a\) and \(a \lor b\) is the same as the distance between \(a \wedge b\) and \(b\) (Figure 1(a)).

We say that function \(f\) is weakly tree-submodular if it satisfies

\[
f(x) + f(y) \geq f(x \wedge y) + f(x \lor y) \quad \forall x, y \in D\]

We will show that strong tree-submodularity (1) implies weak tree-submodularity (2), which justifies the terminology. If all trees are chains shown in Figure 1(b) \((D_i = \{0, 1, ..., K\}\) with 0 being the root) then \(\wedge\) and \(\lor\) correspond to the standard operations “meet” and ”join” (min and max) on an integer lattice. It is well-known that in this case weakly tree-submodular functions can be minimized in time polynomial in \(n\) and \(K\) [39, 32]. In section 4 we give a slight generalization of this result; namely, we allow trees shown in Figure 1(d).

1.1 Related work

Studying operations \((\sqcap, \sqcup)\) that give rise to tractable optimization problems received a considerable attention in the literature. Some known examples of such operations are reviewed below. For
simplicity, we assume that domains \( D_i \) (and operations \( \{\cap, \cup\} \)) are the same for all nodes: \( D_i = D \) for some finite set \( D \).

**Submodular functions on lattices** The first example that we mention is the case when \( D \) is a distributive lattice and \( \cap, \cup \) are the meet and join operations on this lattice. Functions that satisfy (1) for this choice of \( D \) and \( \cap, \cup \) are called **submodular functions** on \( D \); it is well-known that they can be minimized in strongly polynomial time [18, 37, 19].

Recently, researchers considered submodular functions on non-distributive lattices. It is known that a lattice is non-distributive if it contains as a sublattice either the pentagon \( \mathcal{N}_5 \) or the diamond \( \mathcal{M}_3 \). Krokhin and Larose [27] proved tractability for the pentagon case, using nested applications of a submodular minimization algorithm. The case of the diamond was considered by Kuivinen [28], who proved pseudo-polynomiality of the problem. The case of general non-distributive lattices is still open.

**\( L^2 \)-convex functions** The concept of \( L^2 \)-convexity was introduced by Fujishige and Murota [16] as a variant of \( L \)-convexity by Murota [30]. \( L^2 \)-convexity is equivalent to the combination of submodularity and integral convexity [13] (see [32] for details).

The fastest known algorithm for minimizing \( L^2 \)-convex functions is the **steepest descent** algorithm of Murota [31, 32, 33]. Murota proved in [33] that algorithm’s complexity is \( O(n \min\{K, n \log K\} \cdot \text{SFM}(n)) \) where \( K = \max_i |D_i| \) and \( \text{SFM}(n) \) is the complexity of a submodular minimization algorithm for a function with \( n \) variables. The analysis of Kolmogorov and Shioura [22] improved the bound to \( O(\min\{K, n \log K\} \cdot \text{SFM}(n)) \). In section 2 we review Murota’s algorithm (or rather its version without scaling that has complexity \( O(K \cdot \text{SFM}(n)) \)).

Note, the class of \( L^2 \)-convex functions is a subclass of submodular functions on a totally ordered set \( D = \{0, 1, \ldots, K\} \).

**Bisubmodular functions** Bisubmodular functions were introduced by Chandrasekaran and Kabadi as rank functions of \( (poly-)pseudomatroids \) [7, 21]. Independently, Bouchet [3] introduced the concept of \( \Delta \)-matroids which is equivalent to pseudomatroids. Bisubmodular functions and their generalizations have also been considered by Qi [35], Nakamura [34], Bouchet and Cunningham [4] and Fujishige [15].

It has been shown that some submodular minimization algorithms can be generalized to bisubmodular functions. Qi [35] showed the applicability of the ellipsoid method. Fujishige and Iwata [17] developed a weakly polynomial combinatorial algorithm for minimizing bisubmodular functions with complexity \( O(n^2 \text{EO} \log M) \) where \( \text{EO} \) is the number of calls to the evaluation oracle and \( M \) is an upper bound on function values. McCormick and Fujishige [29] presented a strongly combinatorial version with complexity \( O(n^7 \text{EO} \log n) \), as well as a \( O(n^9 \text{EO} \log^2 n) \) fully combinatorial variant that does not use divisions. The algorithms in [29] can also be applied for minimizing a bisubmodular function over a **signed ring family**, i.e. a subset \( R \subseteq D \) closed under \( \cap \) and \( \cup \).

**Valued constraint satisfaction and multimorphisms** Our paper also fits into the framework of **Valued Constraint Satisfaction Problems (VCSPs)** [11]. In this framework we are given a **language** \( \Gamma \), i.e. a set of cost functions \( f : D^m \to \mathbb{R}_+ \cup \{\infty\} \) where \( D \) is a fixed discrete domain and \( f \) is a function of arity \( m \) (different functions \( f \in \Gamma \) may have different arities). A \( \Gamma \)-**instance** is any function \( f : D^n \to \mathbb{R}_+ \cup \{\infty\} \) that can be expressed as a finite sum of functions from \( \Gamma \):

\[
    f(x_1, \ldots, x_n) = \sum_{t \in T} f_t(x_{i(t,1)}, \ldots, x_{i(t,m_t)})
\]

where \( T \) is a finite set of terms, \( f_t \in \Gamma \) is a function of arity \( m_t \), and \( i(t,k) \) are indexes in \( \{1, \ldots, n\} \). A finite language \( \Gamma \) is called **tractable** if any \( \Gamma \)-instance can be minimized in polynomial
time, and NP-hard if this minimization problem is NP-hard. These definitions are extended to infinite languages $\Gamma$ as follows: $\Gamma$ is called tractable if any finite subset $\Gamma' \subset \Gamma$ is tractable, and NP-hard if there exists a finite subset $\Gamma' \subset \Gamma$ which is NP-hard.

Classifying the complexity of different languages has been an active research area. A major open question in this line of research is the Dichotomy Conjecture of Feder and Vardi (formulated for the crisp case), which states that every constraint language is either tractable or NP-hard [14]. So far such dichotomy results have been obtained for some special cases, as described below.

A significant progress has been made in the crisp case, i.e. when $\Gamma$ only contains functions $f : D^m \to \{0, +\infty\}$. The problem is then called Constraint Satisfaction (CSP). The dichotomy is known to hold for languages with a 2-element domain (Schaefer [36]), languages with a 3-element domain (Bulatov [6]), conservative languages [5] (Bulatov [5]), and languages containing a single relation without sources and sinks (Barto et al. [1]). All dichotomy theorems above have the following form: if all functions in $\Gamma$ satisfy a certain condition given by one or more polymorphisms then the language is tractable, otherwise it is NP-hard.

For general VCSPs the dichotomy has been shown to hold for Boolean languages, i.e. languages with a 2-element domain (Cohen et al. [11], conservative languages (Kolmogorov and Živný [23, 24, 25]), who generalized previous results by Deineko et al. [12] and Takhanov [38]), and $\{0, 1\}$-valued languages with a 4-element domain (Jonsson et al. [20]). In these examples tractable subclasses are characterized by one or more multimorphisms, which are generalizations of polymorphisms. A multimorphism of arity $k$ over $D$ is a tuple $\langle \text{OP}_1, \ldots, \text{OP}_k \rangle$ where $\text{OP}_i : D^k \to D$. Language $\Gamma$ is said to admit multimorphism $\langle \text{OP}_1, \ldots, \text{OP}_k \rangle$ if every function $f \in \Gamma$ satisfies

$$f(x_1) + \ldots + f(x_k) \geq f(\text{OP}_1(x_1, \ldots, x_k)) + \ldots + f(\text{OP}_k(x_1, \ldots, x_k))$$

for all labelings $x_1, \ldots, x_k$ with $f(x_1) < +\infty, \ldots, f(x_k) < +\infty$. (The pair of operations $\langle \cap, \cup \rangle$ used in [1] is an example of a binary multimorphism.) The tractable classes mentioned above (for $|D| > 2$) are characterized by complementary pairs of STP and MJN multimorphisms [24] (that generalized symmetric tournament pair (STP) multimorphisms [10], and 1-defect chain multimorphisms [20] (that generalized tractable weak-tree submodular functions in section 4 originally introduced in [26]).

To make further progress on classifying complexity of VCSPs, it is important to study which multimorphisms lead to tractable optimisation problems. Operations $\langle \cap, \cup \rangle$ and $\langle \wedge, \vee \rangle$ introduced in this paper represent new classes of such multimorphisms: to our knowledge, previously researchers have not considered multimorphisms defined on trees.

Combining multimorphisms Finally, we mention that some constructions, namely Cartesian products and Malt’sev products, can be used for obtaining new tractable classes of binary multimorphisms from existing ones [27]. Note, Krokhin and Larose [27] formulated these constructions only for lattice multimorphisms $\langle \cap, \cup \rangle$, but the proof in [27] actually applies to arbitrary binary multimorphisms $\langle \cap, \cup \rangle$.

2 Steepest descent algorithm

It is known that for $L^1$-convex functions local optimality implies global optimality [32]. We start by generalizing this result to strongly tree-submodular functions. Let us define the following “local”

$$f(x_1) + \ldots + f(x_k) \geq f(\text{OP}_1(x_1, \ldots, x_k)) + \ldots + f(\text{OP}_k(x_1, \ldots, x_k))$$

A crisp language $\Gamma$ is called conservative if it contains all unary cost functions $f : D \to \{0, +\infty\}$ [5]. A general-valued language is called conservative if it contains all unary cost functions $f : D \to \mathbb{R}_+$ [23, 24, 25].
Strong tree-submodularity and the fact that $f$ is first, let us prove that $y$. Clearly, the restriction of $f$ to $\text{INWARD}(x)$ is a submodular function, and the restriction of $f$ to $\text{OUTWARD}(x)$ is bisubmodular assuming that each tree $T_i$ is binary.\footnote{If label $x_i$ has less than two children in $T_i$, then variable's domain after restriction will be a strict subset of $\{-1, 0, +1\}$. Therefore, we may need to use a bisubmodular minimization algorithm over a signed ring family $\mathcal{R} \subseteq \{-1, 0, +1\}^n$.}

**Proposition 2.** Suppose that $f(x) = \min\{f(y) \mid y \in \text{INWARD}(x)\} = \min\{f(y) \mid y \in \text{OUTWARD}(x)\}$. Then $x$ is a global minimum of $f$.

**Proof.** First, let us prove that $f(x) = \min\{f(y) \mid y \in \text{NEIB}(x)\}$. Let $x^*$ be a minimizer of $f$ in $\text{NEIB}(x)$, and denote $\mathcal{D}^* = \{y \in \mathcal{D} \mid y_i \in D_i^* = \{x_i, x_i^\prime\}\} \subseteq \text{NEIB}(x)$. We treat set $D_i^*$ as a tree with root $x_i \cap x_i^\prime$. Clearly, the restriction of $f$ to $\mathcal{D}^*$ is an $L^2$-convex function under the induced operations $\cap, \cup$. It is known that for $L^2$-convex functions optimality of $x$ in sets $\{y \in \mathcal{D}^* \mid y \leq x\}$ and $\{y \in \mathcal{D}^* \mid y \geq x\}$ suffices for optimality of $x$ in $\mathcal{D}^*$ \cite[Theorem 7.14]{32}, therefore $f(x) \leq f(x^*)$. This proves that $f(x) = \min\{f(y) \mid y \in \text{NEIB}(x)\}$.

Let us now prove that $x$ is optimal in $\mathcal{D}$. Suppose not, then there exists $y \in \mathcal{D}$ with $f(y) < f(x)$. Among such labelings, let us choose $y$ with the minimum distance $\rho(x, y)$. We must have $y \notin \text{NEIB}(x)$, so $\rho(x, y) \geq 2$. Clearly, $\rho(x, x \cup y) \leq \rho(x, y) - 1$ and $\rho(x, x \cap y) \leq \rho(x, y) - 1$. Strong tree-submodularity and the fact that $f(y) < f(x)$ imply that the cost of at least one of the labelings $x \cup y, x \cap y$ is smaller than $f(x)$. This contradicts to the choice of $y$. \hfill \Box

Suppose that each tree $T_i$ is binary. The proposition shows that a greedy technique for computing a minimizer of $f$ would work. We can start with an arbitrary labeling $x \in \mathcal{D}$, and then apply iteratively the following two steps in some order:

1. Compute minimizer $x^{\text{in}} \in \arg\min\{f(y) \mid y \in \text{INWARD}(x)\}$ by invoking a submodular minimization algorithm, replace $x$ with $x^{\text{in}}$ if $f(x^{\text{in}}) < f(x)$.
2. Compute minimizer $x^{\text{out}} \in \arg\min\{f(y) \mid y \in \text{OUTWARD}(x)\}$ by invoking a bisubmodular minimization algorithm, replace $x$ with $x^{\text{out}}$ if $f(x^{\text{out}}) < f(x)$.

The algorithm stops if neither step can decrease the cost. Clearly, it terminates in a finite number of steps and produces an optimal solution. We will now discuss how to obtain a polynomial number of steps. We denote $K = \max_i |D_i|$.

### 2.1 $L^2$-convex case

For $L^2$-convex functions the **steepest descent** algorithm described above was first proposed by Murota \cite{31,32,33}, except that in step 2 a submodular minimization algorithm was used. Murota’s algorithm actually computes both of $x^{\text{in}}$ and $x^{\text{out}}$ for the same $x$ and then chooses a better one by comparing costs $f(x^{\text{in}})$ and $f(x^{\text{out}})$. A slight variation was proposed by Kolmogorov and Shioura \cite{22}, who allowed an arbitrary order of steps. Kolmogorov and Shioura also established a tight bound on the number of steps of the algorithm by proving the following theorem.
Theorem 3 \[22\]. Suppose that each tree $T_i$ is a chain. For a labeling $x \in \mathcal{D}$ define

$$
\begin{align*}
\rho^-(x) &= \min\{\rho(x, y) \mid y \in \text{OPT}^- [x]\}, \quad \text{OPT}^- [x] = \arg\min\{f(y) \mid y \in \mathcal{D}, y \preceq x\} \\
\rho^+(x) &= \min\{\rho(x, y) \mid y \in \text{OPT}^+ [x]\}, \quad \text{OPT}^+ [x] = \arg\min\{f(y) \mid y \in \mathcal{D}, y \succeq x\}
\end{align*}
$$

(a) Applying step (1) or (2) to labeling $x \in \mathcal{D}$ does not increase $\rho^-(x)$ and $\rho^+(x)$.

(b) If $\rho^-(x) \geq 1$ then applying step (1) to $x$ will decrease $\rho^-(x)$ by 1.

(c) If $\rho^+(x) \geq 1$ then applying step (2) to $x$ will decrease $\rho^+(x)$ by 1.

In the beginning of the algorithm we have $\rho^-(x) \leq K$ and $\rho^+(x) \leq K$, so the theorem implies that after at most $K$ calls to step (1) and $K$ calls to step (2) we get $\rho^-(x) = \rho^+(x) = 0$. The latter condition means that $f(x) = \min\{f(y) \mid y \in \text{INWARD}(x)\} = \min\{f(y) \mid y \in \text{OUTWARD}(x)\}$, and thus, by proposition 2, $x$ is a global minimum of $f$.

2.2 General case

We now show that the bound $O(K)$ on the number of steps is also achievable for general strongly tree-submodular functions. We will establish it for the following version of the steepest descent algorithm:

**S0** Choose an arbitrary labeling $x^0 \in \mathcal{D}$ and set $x := x^0$.

**S1** Compute minimizer $x^{\text{in}} \in \arg\min\{f(y) \mid y \in \text{INWARD}(x)\}$. If $f(x^{\text{in}}) < f(x)$ then set $x := x^{\text{in}}$ and repeat step S1, otherwise go to step S2.

**S2** Compute minimizer $x^{\text{out}} \in \arg\min\{f(y) \mid y \in \text{OUTWARD}(x)\}$. If $f(x^{\text{out}}) < f(x)$ then set $x := x^{\text{out}}$ and repeat step S2, otherwise terminate.

Note, one could choose $x^i_\rho$ to be the root of tree $T_i$ for each node $i \in V$, then step S1 would be redundant.

**Theorem 4.** (a) Step S1 is performed at most $K$ times. (b) Each step S2 preserves the following property:

$$
f(x) = \min\{f(y) \mid y \in \text{INWARD}(x)\}
$$

(c) Step S2 is performed at most $K$ times. (d) Labeling $x$ produced upon termination of the algorithm is a minimizer of $f$.

**Proof.** For a labeling $x \in \mathcal{D}$ denote $\mathcal{D}^- [x] = \{y \in \mathcal{D} \mid y \preceq x\}$. We will treat domain $\mathcal{D}^- [x]$ as the collection of chains with roots $r_i$ and leaves $x_i$. Let $\rho^-(x)$ be the quantity defined in (3a). There holds

$$
f(x) = \min\{f(y) \mid y \in \text{INWARD}(x)\} \iff \rho^-(x) = 0
$$

Indeed, this equivalence can be obtained by applying proposition 2 to function $f$ restricted to $\mathcal{D}^- [x]$.

(a) When analyzing the first stage of the algorithm, we can assume without loss of generality that $\mathcal{D} = \mathcal{D}^- [x^0]$, i.e. each tree $T_i$ is a chain with the root $r_i$ and the leaf $x^0_i$. Indeed, removing the rest of the tree will not affect the behaviour of steps S1. With such assumption, function $f$ becomes $L^2$-convex. By theorem (3b), steps S1 will terminate after at most $K$ steps.

(b,c) Property (4) (or equivalently $\rho^-(x) = 0$) clearly holds after termination of steps S1. Let $z$ be the labeling upon termination of steps S2. When analyzing the second stage of the algorithm,
we can assume without loss of generality that \( \mathcal{D} = \mathcal{D}^{-}[z] \), i.e. each tree \( T_i \) is a chain with the root \( r_i \) and the leaf \( z_i \). Indeed, removing the rest of the tree will not affect the behaviour of steps \( S2 \). Furthermore, restricting \( f \) to \( \mathcal{D}^{-}[z] \) does not affect the definition of \( \rho^{-}(x) \) for \( x \in \mathcal{D}^{-}[z] \).

By theorem \([3]a\), steps \( S2 \) preserve \( \rho^{-}(x) = 0 \); this proves part \( (b) \). Part \( (c) \) follows from theorem \([3]c\).

\( (d) \) When steps \( S2 \) terminate, we have \( f(x) = \min\{f(y) \mid y \in \text{OUTWARD}(x)\} \). Combining this fact with condition \([1] \) and using proposition \([2] \) gives that upon algorithm’s termination \( x \) is a minimizer of \( f \).

\[ \square \]

3 Translation submodularity

In this section we derive an alternative definition of strongly tree-submodular functions. As a corollary, we will obtain that strong tree submodularity \([1] \) implies weak tree submodularity \([2] \).

Let us introduce another pair of operations on trees. Given labels \( a, b \in D_i \) and an integer \( d \geq 0 \), we define

\[
\begin{align*}
    a \uparrow^d b &= \mathcal{P}[a \rightarrow b, d] \land b & a \downarrow_d b &= \mathcal{P}[a \rightarrow b, \rho(a \uparrow^d b, b)]
\end{align*}
\]

In words, \( a \uparrow^d b \) is obtained as follows: (1) move from \( a \) towards \( b \) by \( d \) steps, stopping if \( b \) is reached earlier; (2) keep moving until the current label becomes an ancestor of \( b \). \( a \downarrow_d b \) is the label on the path \( \mathcal{P}[a \rightarrow b] \) such that the distances \( \rho(a, a \downarrow_d b) \) and \( \rho(a \uparrow^d b, b) \) are the same, as well as distances \( \rho(a, a \uparrow^d b) \) and \( \rho(a \downarrow_d b, b) \). Note, binary operations \( \uparrow^d, \downarrow_d : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D} \) (and corresponding operations \( \uparrow^d, \downarrow_d : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D} \)) are in general non-commutative. One exception is \( d = 0 \), in which case \( \uparrow^d, \downarrow_d \) reduce to the commutative operations defined in the introduction: \( x \uparrow^0 y = x \land y \) and \( x \downarrow_0 y = x \lor y \).

For fixed labels \( a, b \in D_i \) it will often be convenient to rename nodes in \( \mathcal{P}[a \rightarrow b] \) to be consecutive integers so that \( a \land b = 0 \) and \( a \leq 0 \leq b \). Then we have \( a = -\rho(a, a \land b), b = \rho(a \land b, b) \) and

\[
\begin{align*}
    a \uparrow^d b &= \max\{0, \min\{a + d, b\}\} & a \downarrow_d b &= a + b - (a \uparrow^d b)
\end{align*}
\]

**Theorem 5.** \( (a) \) If \( f \) is strongly tree-submodular then for any \( x, y \in \mathcal{D} \) and integer \( d \geq 0 \) there holds

\[
    f(x) + f(y) \geq f(x \uparrow^d y) + f(x \downarrow_d y)
\]

\( (b) \) If \([6] \) holds for any \( x, y \in \mathcal{D} \) and \( d \geq 0 \) then \( f \) is strongly tree-submodular.

Note, this result is well-known for \( L^2 \)-convex functions \([32] \) section 7.1], i.e. when all trees are chains shown in Figure \([11]b\); inequality \([6] \) was then written as \( f(x) + f(y) \geq f((x + d \cdot 1) \land y) + f(x \lor (y - d \cdot 1)) \), and was called translation submodularity. In fact, translation submodularity is one of the key properties of \( L^2 \)-convex functions, and was heavily used, for example, in \([22] \) for proving theorem \([6] \).

Setting \( d = 0 \) in theorem \([5]a \) gives

**Corollary 6.** A strongly tree-submodular function \( f \) is also weakly tree-submodular, i.e. \([1] \) implies \([2] \).

A proof of parts \( (b) \) and \( (a) \) of theorem \([6] \) is given in sections \([3.1] \) and \([3.2] \) respectively. In both proofs we always implicitly assume that for each \( i \in V \) labels in \( \mathcal{P}[x_i \rightarrow y_i] \) are renamed to be consecutive integers with \( x_i \land y_i = 0 \) and \( x_i \leq 0 \leq y_i \).

7
3.1 Proof of theorem 5(b)

We prove inequality (1) for \(x, y \in D\) using induction on \(\rho_1(x, y) = \sum_{i \in V} \rho(x_i, y_i)\). The base case \(\rho_1(x, y) = 0\), or \(x = y\), is trivial; suppose that \(\rho_1(x, y) \geq 1\). Denote \(d_{\text{max}} = \rho(x, y) \geq 1\) and \(d = \lfloor d_{\text{max}}/2 \rfloor \geq 0\). Two cases are possible.

**Case 1** \(d_{\text{max}}\) is even. We can assume without loss of generality that there exists \(k \in V\) such that \(y_k - x_k = d_{\text{max}}\) and \(|x_k| \geq y_k\). (If there is no such \(k\), we can simply swap \(x\) and \(y\); inequality (1) will be unaffected since operations \(\cap, \cup\) are commutative, and \(\rho(x, y), \rho_1(x, y)\) will not change.) Consider labelings \(x', y' \in D\) defined as follows:

\[
y'_i = \begin{cases} 
  y_i - 1 & \text{if } y_i - x_i = d_{\text{max}}, \ |x_i| \geq y_i \\
  y_i & \text{otherwise}
\end{cases} \quad x'_i = x_i \sqcup y'_i
\]

for each \(i \in V\). We claim that

(a) \(x \sqcap y' = x \sqcap y\) \quad (b) \(x \sqcup y' = x'\)
(c) \(x' \uparrow^d y = y'\) \quad (d) \(x' \downarrow^d y = x \sqcup y\)

Indeed, for each node \(i \in V\) one of the following holds:

- \(y_i - x_i \leq d_{\text{max}} - 1\). Then \(y'_i = y_i, x'_i = x_i \sqcup y_i\), so (a) and (b) hold for node \(i\). We also have \(y_i - x'_i = y_i - (x_i \sqcup y_i) = 0 \leq \lceil (y_i - x_i)/2 \rceil \leq \lceil (d_{\text{max}} - 1)/2 \rceil \leq d\), which implies (c) and (d).

- \(y_i - x_i = d_{\text{max}}\) and \(|x_i| < y_i\). Then \(y'_i = y_i, x'_i = x_i \sqcup y_i = (y_i + x_i)/2, y_i - x'_i = d\); as above, this implies (a)-(d).

- \(y_i - x_i = d_{\text{max}}\) and \(|x_i| \geq y_i\). Then \(y'_i = y_i - 1, x'_i = x_i \sqcup y'_i = [x_i + y_i - 1]/2 = (x_i + y_i)/2 - 1 = y'_i - d\). Checking that (a)-(d) hold is straightforward.

We have \(y'_k = y_k - 1\), and so \(\rho_1(x, y') < \rho_1(x, y)\). Therefore,

\[
f(x) + f(y') \geq f(x \sqcap y') + f(x \sqcup y') \quad f(x') + f(y) \geq f(x' \uparrow^d y) + f(x' \downarrow^d y)
\]

where the first inequality follows from the induction hypothesis and the second one follows from (1). Summing these inequalities and subtracting \(f(x') + f(y')\) from both sides using (a)-(d) gives (1).

**Case 2** \(d_{\text{max}}\) is odd. By swapping \(x\) and \(y\), if necessary, we can assume without loss of generality that there exists \(k \in V\) such that \(y_k - x_k = d_{\text{max}}\) and \(|x_k| < y_k\). (Note, we cannot have \(y_i - x_i = d_{\text{max}}\) and \(|x_i| = y_i\) since \(d_{\text{max}}\) is odd). Consider labelings \(x', y' \in D\) defined as follows:

\[
x'_i = \begin{cases} 
  x_i + 1 & \text{if } y_i - x_i = d_{\text{max}}, \ |x_i| < y_i \\
  x_i & \text{otherwise}
\end{cases} \quad y'_i = x'_i \sqcap y_i
\]

for each \(i \in V\). We claim that

(a) \(x' \sqcap y = y'\) \quad (b) \(x \sqcup y = x \sqcup y\)
(c) \(x \uparrow^d y' = x \sqcap y\) \quad (d) \(x \downarrow^d y' = x'\)

Indeed, for each node \(i \in V\) one of the following holds:

- \(y'_i - x_i \leq d_{\text{max}} - 1\). Then \(x'_i = x_i, y'_i = x_i \sqcup y_i\), so (a) and (b) hold for node \(i\). We also have \(y'_i - x_i = (x_i \sqcup y_i) - x_i \leq \lceil (y_i - x_i)/2 \rceil \leq \lceil (d_{\text{max}} - 1)/2 \rceil \leq d\), which implies (c) and (d).
• \( y_i - x_i = d_{\text{max}} \) and \( |x_i| > y_i \). Then \( x'_i = x_i, \ y'_i = x_i \lor y_i = \lceil (x_i + y_i)/2 \rceil \leq 0 \), so (a) and (b) hold for node \( i \). (c) and (d) hold since \( y'_i \leq 0 \).

• \( y_i - x_i = d_{\text{max}} \) and \( |x_i| < y_i \). Then \( x'_i = x_i + 1, \ y'_i = x'_i \lor y_i = \lceil (x_i + y_i - 1)/2 \rceil \). Checking that (a)-(d) hold is straightforward.

We have \( x'_k = x_k + 1 \), and so \( \rho_1(x', y) < \rho_1(x, y) \). Therefore,

\[
\begin{align*}
 f(x'') + f(y) & \geq f(x' \cap y) + f(x' \lor y) & f(x) + f(y') & \geq f(x \uparrow^d y') + f(x \downarrow^d y')
\end{align*}
\]

where the first inequality follows from the induction hypothesis and the second one follows from (6). Summing these inequalities and subtracting \( f(x'') + f(y') \) from both sides using (a)-(d) gives (7).

### 3.2 Proof of theorem 5(a)

We say that the triplet \((x, y, d)\) is valid if \( x, y \in \mathcal{D} \) and \( d \in [0, \rho(x, y)] \). We denote \( z = x \uparrow^d y \); we have \( x_i \leq 0 \leq z_i \leq y_i \). Let us introduce a partial order \( \leq \) over valid triplets as the lexicographical order with variables \((y_1 - x_1, \ldots, y_n - x_n, -d)\). Note, the last component \(-d\) is the least significant. We use induction on this partial order. The induction base is trivial: if the first \( n \) components are zeros then \( x = y \) so (8) is an equality, and if the last component is minimal (i.e. \( d = \rho(x, y) \)) then \( x \uparrow^d y = y \) and \( x \downarrow^d y = x \), so (9) is again an equality. Suppose that \( x \neq y \) and \( d \leq \rho(x, y) - 1 \).

Consider integer \( d' \geq d \), and denote \( y' = x \uparrow^{d+1} y \) and \( \delta_i = y'_i - z_i \). Let us describe cases when the argument above can be applied; such cases can be eliminated from consideration. First, suppose that \( y'_i - z_i \geq 2 \) for some node \( i \in V \), then there exists \( d' \geq d \)
such that the labeling $y' = x \uparrow^{d+1} y$ has at least one node $j \in V$ with $y'_j \in [z_j + 1, y_j - 1]$.  Let us choose the minimum integer $d'$ that has this property.  Then $\delta_i \in \{0, 1\}$ for all nodes $i \in V$, since $\delta_i \geq 2$ would contradict to the minimality of chosen $d'$.  We also have $y'_j \neq y_j$ and $x'_j \neq x_j$ (since $x'_j - x_j = \delta_j = 1$), so the conditions above are satisfied.  Therefore, from now on we assume without loss of generality that $y_i - z_i \in \{0, 1\}$ for all nodes $i \in V$.

We can also take $d' = d$.  Condition $\delta_i \in \{0, 1\}$ is then satisfied for all nodes.  Therefore, we can assume without loss of generality that either $x' = x$ or $y' = y$ where $y' = x \uparrow^{d+1} y$, $x' = x \downarrow d y'$, otherwise the induction argument above could be applied.  Suppose that $x' = x$.  This is equivalent to $x \uparrow^d y' = y'$, or to the following condition for all nodes $i \in V$: either $x_i + d < 0$ or $y_i - x_i \geq d$.  It can be checked that $x \uparrow^{d+1} y = x \uparrow^d y$ and $x \downarrow_d y = x \downarrow_d y$.  Furthermore, $(x, y, d + 1) \Rightarrow (x, y, d)$, so (6) follows by the induction hypothesis.  We thus assume from now on that $y' = y$.

Equations below summarize definitions and assumptions made so far:

$$z_i = x_i \uparrow^d y_i$$  \hspace{1cm} (8a)

$$y'_i = x_i \uparrow^{d+1} y_i = y_i$$  \hspace{1cm} (8b)

$$x'_i = x_i \downarrow_d y_i$$  \hspace{1cm} (8c)

$$\delta_i = y_i - z_i = x'_i - x_i \in \{0, 1\}$$  \hspace{1cm} (8d)

Let $S$ be the set of nodes $i \in V$ with $\delta_i = 1$.  It is straightforward to check that

$$i \in S \quad \Rightarrow \quad x_i + d = z_i = y_i - 1$$  \hspace{1cm} (8e)

$$i \in V - S \quad \Rightarrow \quad x_i \uparrow^d y_i = y_i \quad \text{and} \quad x_i \downarrow_d y_i = x_i = x'_i$$  \hspace{1cm} (8f)

If $S$ is empty then $x \uparrow^d y = y$, $x \downarrow_d y = x$, so inequality (8b) is trivial.  Thus, we can assume that $S$ is non-empty.  Suppose that $S$ contains two distinct nodes $i$ and $j$.  Let us modify labelings $x'$ and $y'$ as follows: for node $j$ set $x'_j = x_j$, $y'_j = z_j$.  It is straightforward to check that conditions (8) for $d' = d$ still hold.  Furthermore, $x'_i > x_i$, $y'_i < y_j$, so $(x, y', d) < (x, y, d)$ and $(x', y, d) < (x, y, d)$.  Applying the argument described above gives (8).

We are left with the case when $S$ contains a single node $j$.  We will consider 5 possible subcases.  In 4 of them, we will do the following: (i) specify new labelings $x'$ and $y'$ with $x'_i, y'_i \in [x_i, y_i]$ for each node $i$; (ii) specify four identities involving $x, y, x', y'$ such that the right-hand sides contain expressions $x', y', x \uparrow^d y, x \downarrow_d y$, and the left-hand sides contain expressions of the form $x \circ_1 y', x \circ_2 y, x \circ_3 y, x \circ_4 y$ where $\circ_k$ is one of the operations $\uparrow, \downarrow, \uparrow^d, \downarrow_d$ and $\circ_k$ is the corresponding “symmetric” operation.  This will describe how to prove (6): we would need to sum two inequalities

$$f(x) + f(y') \geq f(x \circ_1 y') + f(x \circ_2 y') \quad f(x') + f(y) \geq f(x \circ_2 y') + f(x \circ_2 y')$$

that hold either by strong tree-submodularity or by the induction hypothesis, then use provided identities to prove (6).  Checking the identities and the applicability of the induction hypothesis in the case of operations $\uparrow^d, \downarrow_d$ is mechanical, and we omit it.

**Case 1** $z_j = x_j + d \geq 1$ (implying $d \geq 1$).  The identities are

(a) $x \uparrow^{d-1} y' = x'$

(b) $x \downarrow_{d-1} y' = x \downarrow_d y$

(c) $x' \cup y = x \uparrow^d y$

(d) $x' \cap y = y'$

and labelings $x'$, $y'$ are defined as follows:

- if $i = j$ set $x'_j = y_j - 2$, $y'_j = y_j - 1$;
• otherwise if \( x_i + d = y_i > 0 \) set \( x'_i = y'_i = y_i - 1 \).

• otherwise (if \( y_i = 0 \) or \( x_i + d > y_i \)) set \( x'_i = y'_i = y_i \).

The remainder is devoted to the case \( z_j = x_j + d = 0 \). Note that we must have \( y_j = 1 \).

**Case 2** \( d \geq 1, z_j = x_j + d = 0 \) and there exists node \( k \in V - \{j\} \) with \( x_k = 0, y_k > 0 \). Then

\[
\begin{align*}
(a) & \quad x' \uparrow^d y = x \uparrow^d y \quad (b) & \quad x' \downarrow_d y = y' \\
(c) & \quad x \sqcup y' = x' \quad (d) & \quad x \sqcap y' = x \downarrow_d y
\end{align*}
\]

\( x', y' \) are defined as follows:

• if \( i = j \) set \( x'_j = x_j, y'_j = x_j + 1 \);

• otherwise if \( i = k \) set \( x'_k = y'_k = x_k + 1 = 1 \);

• otherwise set \( x'_i = y'_i = x_i \).

**Case 3** \( d \geq 1, z_j = x_j + d = 0 \) and there is no node \( k \in V - \{j\} \) with \( x_k = 0, y_k > 0 \). The identities are

\[
\begin{align*}
(a) & \quad x' \uparrow^{d-1} y = x \uparrow^{d} y \quad (b) & \quad x' \downarrow_{d-1} y = y' \\
(c) & \quad x \sqcup y' = x \downarrow_d y \quad (d) & \quad x \sqcap y' = x'
\end{align*}
\]

\( x', y' \) are defined as follows:

• if \( i = j \) set \( x'_j = x_i + 1, y'_j = x_i + 2 \);

• otherwise if \( x_i < 0 \) set \( x'_i = y'_i = x_i + 1 \);

• otherwise (if \( x_i = y_i = 0 \)) set \( x'_i = y'_i = 0 \).

Note, to verify identities (11) for node \( j \), one should consider cases \( d = 1 \) and \( d \geq 2 \) separately.

**Case 4** \( d = 0 \) (implying \( x_j = 0, y_j = 1 \)) and there exists node \( k \in V - \{j\} \) with \( x_k < 0 \). Then

\[
\begin{align*}
(a) & \quad x \uparrow^0 y' = x' \quad (b) & \quad x \downarrow_0 y' = x \downarrow_0 y \\
(c) & \quad x' \sqcup y = y' \quad (d) & \quad x' \sqcap y = x \uparrow^0 y
\end{align*}
\]

\( x', y' \) are defined as follows:

• if \( i = j \) set \( x'_j = 0, y'_j = 1 \);

• otherwise if \( x_i < 0, y_i = 0 \) set \( x'_i = y'_i = -1 \);

• otherwise (if \( x_i = y_i = 0 \)) set \( x'_i = y'_i = 0 \).

**Case 5** \( d = 0 \) (implying \( x_j = 0, y_j = 1 \)) and there is no node \( k \in V - \{j\} \) with \( x_k < 0 \). Thus, \( x_i = y_i = 0 \) for all \( i \in V - \{j\} \). There holds \( x \uparrow^0 y = x, x \downarrow_0 y = y \), so inequality (6) is trivial.

11
4 Weakly tree-submodular functions

In this section we consider functions \( f \) that satisfy condition (2), but not necessarily condition (1). It is well-known [39, 32] that such functions can be minimized efficiently if all trees \( T_i \) are chains rooted at an endpoint and \( \max_i |D_i| \) is polynomially bounded. The algorithm utilizes Birkhoff’s representation theorem [2] which says that there exists a ring family \( \mathcal{R} \) such that there is an isomorphism between sets \( \mathcal{D} \) and \( \mathcal{R} \) that preserves operations \( \land \) and \( \lor \). (A subset \( \mathcal{R} \subseteq \{0,1\}^m \) is a ring family if it is closed under operations \( \land \) and \( \lor \).) It is known that submodular functions over a ring family can be minimized in polynomial time, which implies the result. Note that the number of variables will be \( m = O(\sum_i |D_i|) \).

Another case when \( f \) satisfying (2) can be minimized efficiently is when \( f \) is bisubmodular, i.e. all trees are as shown in Figure 1(c). Indeed, in this case the pairs of operations \( \langle \land, \lor \rangle \) and \( \langle \land, \lor \rangle \) coincide.

An interesting question is whether there exist other classes of weakly tree-submodular functions that can be minimized efficiently. In this section we provide one rather special example. We consider the tree shown in Figure 1(d). Each \( T_i \) has nodes \( \{0,1,\ldots,K,K-1,K+1\} \) such that 0 is the root, the parent of \( k \) for \( k = 1,\ldots,K \) is \( k-1 \), and the parent of \( K-1 \) and \( K+1 \) is \( K \).

In order to minimize function \( f \) for such choice of trees, we create \( K+1 \) variables \( y_{i0}, y_{i1},\ldots,y_{iK} \) for each original variable \( x_i \in D_i \). The domains of these variables are as follows: \( D_{i0} = \ldots = D_{iK-1} = \{0,1\}, D_{iK} = \{-1,0,+1\} \). Each domain is treated as a tree with root 0 and other nodes being the children of 0; this defines operations \( \land \) and \( \lor \) for domains \( D_{i0},\ldots,D_{iK-1},D_{iK} \). The domain \( \tilde{D} \) is set as the Cartesian product of individual domains over all nodes \( i \in V \). Note, a vector \( y \in \tilde{D} \) has \( n(K+1) \) components.

For a labeling \( x \in \mathcal{D} \) let us define labeling \( y = \psi(x) \in \tilde{D} \) as follows:

\[
\begin{align*}
    x_i = k \in \{0,1,\ldots,K\} & \quad \Rightarrow \quad y_{i0} = \ldots = y_{iK-1} = 1, \; y_{iK} = \ldots = y_{iK} = 0 \\
    x_i = K-1 & \quad \Rightarrow \quad y_{i0} = \ldots = y_{iK-1} = 1, \; y_{iK} = -1 \\
    x_i = K+1 & \quad \Rightarrow \quad y_{i0} = \ldots = y_{iK-1} = 1, \; y_{iK} = +1
\end{align*}
\]

It is easy to check that mapping \( \psi : \mathcal{D} \to \tilde{D} \) is injective and preserves operations \( \land \) and \( \lor \). Therefore, \( \mathcal{R} = \text{Im} \psi \) is a signed ring family, i.e. a subset of \( \tilde{D} \) closed under operations \( \land \) and \( \lor \). It is known [29] that bisubmodular functions over ring families can be minimized in polynomial time, leading to

**Proposition 7.** Functions that are weakly tree-submodular with respect to trees shown in Figure 1(d) can be minimized in time polynomial in \( n \) and \( \max_i |D_i| \).

5 Conclusions and discussion

We introduced two classes of functions (strongly tree-submodular and weakly tree-submodular) that generalize several previously studied classes. For each class, we gave new examples of trees for which the minimization problem is tractable.

Our work leaves a natural open question: what is the complexity of the problem for more general trees? In particular, can we minimize efficiently strongly tree-submodular functions if trees are non-binary, i.e. if some nodes have three or more children? Note that the algorithm in section 2 and its analysis are still valid, but it is not clear whether the minimization procedure in step S2 can be implemented efficiently. Also, are there trees besides the one shown in Figure 1(d) for which weakly tree-submodular functions can be minimized efficiently?
More generally, can one characterize for which operations \( \langle \cap, \cup \rangle \) the minimization problem is tractable? Currently known tractable examples are distributive lattices, some non-distributive lattices \cite{27, 28}, operations on trees introduced in this paper, and combinations of the above operations obtained via Cartesian product and Malt’sev product \cite{27}. Are there tractable cases that cannot be obtained via lattice and tree-based operations?

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