Recurrent random walks on \( \mathbb{Z} \) with infinite variance: transition probabilities of them killed on a finite set

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Abstract

In this paper we consider an irreducible random walk on the integer lattice \( \mathbb{Z} \) that is in the domain of normal attraction of a strictly stable process with index \( \alpha \in (1, 2) \) and obtain the asymptotic form of the distribution of the hitting time of the origin and that of the transition probability for the walk killed when it hits a finite set. The asymptotic forms obtained are valid uniformly in the natural domain of the space and time variables.

1 Introduction

Let \( S_n = X_1 + \cdots + X_n \) be a random walk on the integer lattice \( \mathbb{Z} \) started at \( S_0 \equiv 0 \), where the increments \( X_1, X_2, \ldots \) are independent and identically distributed random variables defined on some probability space \((\Omega, \mathcal{F}, P)\) and taking values in \( \mathbb{Z} \). Let \( E \) indicate the expectation under \( P \) as usual and \( X \) be a random variable having the same law as \( X_1 \). We suppose throughout the paper that the walk \( S_n \) is

1) in the domain of normal attraction of a strictly stable law of index \( 1 < \alpha < 2 \) or, what amounts to the same thing (cf [9]), if \( \phi(\theta) := E e^{i\theta X} \), then

\[
\lim_{\theta \to \pm 0} \frac{1 - \phi(\theta)}{|\theta|^\alpha} = c_o e^{\pm i\pi \gamma/2}
\]

with some real numbers \( c_o \) and \( \gamma \) such that \( c_o > 0 \) and \( |\gamma| \leq 2 - \alpha \).

For simplicity we also suppose (except in Theorem 7) that

2) the walk is strongly aperiodic in the sense of Spitzer [16], namely for any \( x \in \mathbb{Z} \), \( P[S_n = x] > 0 \) for all sufficiently large \( n \).

The condition 1) entails \( EX = 0 \) so that the walk is recurrent. (See Section 7.2 for an equivalent condition in terms of the tails of distribution function of \( X \) and some related facts.) The condition 2) gives rise to no loss of generality (see Remark 6.1(b)).

Under these assumptions we obtain in this paper precise asymptotic forms of the distribution of the hitting time of the origin and of the transition probability for the walk killed when it hits the origin. The estimates obtained are uniform for the space variables within the natural space-time regime \( x = O(n^{1/\alpha}) \). We extend the results to the case when the walk is
killed on hitting a finite set instead of the origin. The corresponding results are obtained for
the walks with finite variance by the present author [17], [19]. In a classical paper [11] Kesten
studied similar problems and obtained an exact asymptotic result for the ratio of transition
probability and hitting time ‘density’ under a mild assumption on the walk where, however,
the space variables are fixed (cf. Remark 2.4 at the end of the next section). Although we
consider the problem for all admissible $\gamma$, our main interest is in the extreme case $\gamma = |2 - \alpha|$ 
when the limiting stable process has jumps only in one direction: the other case is much
simpler and the asymptotic forms obtained are quite different between the two cases for large
space variables. The condition 1) is restrictive and it is desirable to replace it by a weaker
one, to that end however we encounter a serious difficulty for the present approach. In any
case it must be worth to reveal what kind of behaviour of the transition probability of the
killed process even under such a restrictive condition.

2 Statements of results

We first introduce fundamental objects that appear in the description of our results and state
some well known facts concerning them. Put $p^n(x) = P[S_n = x]$, $p(x) = p^1(x)$ ($x \in \mathbb{Z}$) and
define the potential function

$$a(x) = \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)];$$

the series on the RHS is convergent and $a(x)/|x| \to 1/\sigma^2$ and $a(x + y) - a(x) \to \pm y/\sigma^2$
as $x \to \pm \infty$ (cf. Spitzer [16]:Sections 28 and 29). To make expressions concise we use the
notation

$$a^1(x) = 1(x = 0) + a(x),$$

where $1(S)$ equals 1 or 0 according as a statement $S$ is true or false. The condition 3) in
Introduction entails that $a(x) > 0$ whenever $x \neq 0$, whereas if $S$ is left-continuous (i.e.,
p$(x) = 0$ for $x \leq 2$), then $a(x) = 0$ for all $x \geq 0$ (under $\sigma = \infty$), and similarly for
right-continuous walks. (See Section 8.3 for additional facts related to $a$.)

We write $S^x$ for $x + S_n$, the walk started at $x \in \mathbb{Z}$. For a subset $B \subset \mathbb{R}$, put

$$\sigma^x_B = \inf\{n \geq 1 : S^x_n \in B\},$$

the time of the first entrance of the walk $S^x$ into $B$. To avoid the overburdening of notation
we write $S^x_{\sigma^x_B}$ for $S^x_{\sigma^x_B}$ and $S^x_{\sigma_B}$ for $S^x_{\sigma_B}$; sometimes $\sigma B$ is written for $\sigma_B$, e.g., $S^x_{\sigma_B}$ for $S^x_{\sigma_B}$.

When the spatial variables become indefinitely large the asymptotic results are naturally
expressed by means of the stable process appearing in the scaling limit and we need to introduce
relevant quantities. Let $Y_t$ be a stable process started at zero with characteristic exponent

$$\psi(\theta) = e^{i(\text{sgn } \theta)\pi \gamma/2|\theta|^\alpha} \quad (|\gamma| \leq 2 - \alpha, \gamma \text{ is real})$$

so that $E e^{i\psi Y_t} = e^{-t \psi(\theta)}$, where $\text{sgn } \theta = 1$ if $\theta > 0$, 0 if $\theta = 0$ and
$-1$ if $\theta < 0$. (\gamma has the

same sign as the skewness parameter so that the extremal case $\gamma = 2 - \alpha$ corresponds to the
spectrally positive case.) Denote by $p_t(x)$ and $f^x(t)$ the density of the distribution of $Y_t$ and
of the first hitting time to the origin by $Y_t^x := x + Y_t$, respectively:

$$p_t(x) = P[Y_t \in dx]/dx, \quad f^x(t) = (d/dt)P[\exists s \leq t, Y_s^x = 0];$$
there exist the jointly continuous versions of these densities (for \( t > 0 \)) and we shall always choose such ones. It follows that \( S_{nt}/n^{1/\alpha} \Rightarrow Y_{c,t} \) (weak convergence of distribution) and by Gnedenko’s local limit theorem [10]

\[
\limsup_{n \to \infty} |n^{1/\alpha}p_n(x) - p_{c_0}(x/n^{1/\alpha})| = 0,
\]

where \([b]\) denotes the integer part of a real number \( b \). For real numbers \( s, t, s \vee t = \max\{s, t\} \) and \( s \wedge t = \max\{s, t\} \), \( t_+ = t \vee 0 \), \( t_- = -t_+ \) and \([t]\) denotes the smallest integer that does not less than \( t \); for positive sequences \( (s_n) \) and \( (t_n) \), \( s_n \sim t_n \) and \( s_n \sim t_n \) mean, respectively, that the ratio \( s_n/t_n \) approaches unity and that \( s_n/t_n \) is bounded away from zero and infinity. We use the letters \( x, y, z \) and \( w \) to represent integers which indicate points assumed by the walk when discussing matters on the random walk, while the same letters may stand for real numbers when the stable process is dealt with; we shall sometimes use the Greek letters \( \xi, \eta \) etc. to denote the real variables the stable process may assume.

### 2.1. Hitting time distribution.

Let \( f^x(n) \) denote the probability that the walk started at \( x \) visits the origin at \( n \) for the first time:

\[
f^x(n) = P[\sigma^x_{\{0\}} = n].
\]

Put

\[
\kappa_{\alpha, \gamma} = \kappa_{\alpha, -\gamma} = \frac{(\alpha - 1) \sin \frac{\pi}{\alpha}}{\Gamma(\frac{1}{\alpha}) \sin \frac{\pi(\alpha - \gamma)}{2\alpha}} = \frac{(1 - \frac{1}{\alpha}) \sin \frac{\pi}{\alpha}}{\Gamma(1/\alpha) \pi};
\]

in particular if \( \gamma = |2 - \alpha| \), \( \kappa_{\alpha, \gamma} = (\alpha - 1)/\Gamma(1/\alpha) \).

**Theorem 1.** For any admissible \( \gamma \), as \( n \to \infty \)

\[
f^0(n) \sim \kappa_{\alpha, \gamma} c_0^{1/\alpha}/n^{2-1/\alpha}.
\]

When \( \gamma = 0 \) (i.e., the limit stable process is symmetric), the above asymptotic form of \( f^0(n) \) is derived by Kesten [11] in which an asymptotic form for \( \alpha = 1 \) is also obtained, which reads \( f^0(n) \sim \pi c_0/n(\log n)^2 \).

We write \( x_n \) for \( x/n^{1/\alpha} \).

**Theorem 2.** Let \(|\gamma| < 2 - \alpha \). Then, for each \( M > 1 \), as \( n \to \infty \)

\[
f^x(n) \sim \begin{cases}
a^\dagger(x)f^0(n) & (x_n \to 0), \\
c_0 f^x(c_0)/n & (\text{uniformly for } 1/M \leq |x_n| < M).
\end{cases}
\]

**Theorem 3.** Let \(|\gamma| = 2 - \alpha \). Then as \( n \to \infty \) (2.2) holds if \( x\gamma \leq 0 \), and uniformly for \( 0 < \gamma x_n < M \),

\[
f^x(n) \sim a^\dagger(x)f^0(n) + \frac{|x_n|c_0(-x_n)}{n}.
\]

In case \(|x_n| \to \infty \) an upper bound is provided by the following proposition, where we include a reduced version of that for the case \(|x_n| < 1 \) given above.

**Proposition 2.1.** There exists a constant \( C \) such that for all \( \gamma \) and \( x \),

\[
f^x(n) \leq C(|x_n|^{\alpha - 1} \wedge |x_n|^{-\alpha})/n.
\]
Remark 2.1. (a) We shall see (cf. Lemma 3.1(i)) that as $|x| \to \infty$

$$c_0 a(x) = \begin{cases} o(|x|^{\alpha-1}) & \text{if } \gamma x \to +\infty, |\gamma| = 2 - \alpha \\ \kappa_{\alpha, \gamma, \text{sgn } x} |x|^{\alpha-1} \{1 + o(1)\} & \text{otherwise} \end{cases}$$ (2.4)

where $\kappa_{\alpha, \gamma, \text{sgn } x}$ is a constant (depending on $\alpha$, $\gamma$ and $\text{sgn } x$) which is positive if $\gamma \text{sgn } x \neq 2 - \alpha$ and equals $1/\Gamma(\alpha)$ if $\gamma \text{sgn } x = -2 + \alpha$. In particular

$$f^x(n) \sim \kappa_{\alpha, \gamma} c_0^{1/\alpha} a(x)/n^{2-1/\alpha} \sim c_0 f^x(c_0 n)$$

as $x \to \infty$ and $x/n^{1/\alpha} \to 0$ (Lemma 7.1), so that the two expressions on the RHS of (2.2) are asymptotically equivalent to each other in this regime. This is contrasted with the first half of (2.3) which implies that if $\gamma = 2 - \alpha$, as $x \to +\infty$ under $x < Mn^{1/\alpha}$

$$f^x(n) \sim \begin{cases} \kappa_{\alpha, \gamma} c_0^{1/\alpha} a(x)/n^{2-1/\alpha} & (a(x)/x \gg n^{1-2/\alpha}), \\
p_{\alpha}(x_0) x/n^{1+1/\alpha} & (a(x)/x \ll n^{1-2/\alpha}), \end{cases}$$ (2.6)

where $s \ll t$ means $t > 0$ and $s/t \to 0$. It is noted that $a(x), x > 0$ is positive if $P[X \geq 2] > 0$ and possibly bounded (see (2.21)).

(b) Whenever $|x_n| \to 0$ (2.3) is valid for all (admissible) $\gamma$, for if either $|\gamma| < 2 - \alpha$ or $x\gamma < 0$, then in view of (2.4) the second term on the RHS of (2.3) is negligible as $x_n \to 0$ in comparison to the first so that it reduces to the first case of (2.2).

(c) If $\gamma = 2 - \alpha$ (when the limiting stable process has no negative jumps), then it holds that

$$f^x(t) = xt^{-1}p_\gamma(-x) \quad \text{for } x > 0$$ (2.7)

(cf., e.g., [1 Corollary 7.3]), which shows that in the regime $1/M < |x_n| < M$ the asymptotic forms of $f^x(n)$ given in Theorems 3 and 2 are equivalent to each other in view of the scaling relation (2.9) below. Thus for all $|\gamma| \leq 2 - \alpha$, as $n \to \infty$

$$f^x(n) \sim c_0 f^x(c_0 n) \quad \text{uniformly for } 1/M \leq |x_n| \leq M.$$ (2.8)

(d) It seems hard to improve the estimate for $|x_n| > 1$ given in Proposition 2.1 under (1.1) only. However, if we assume some additional regularity condition on $p(x)$ as $x \to -\infty$ (resp. $+\infty$) the upper bound of $f^x(n)$ for $x_n > 1$ (resp. $x_n < -1$) is improved to $|x_n|^{-\alpha-1}/n$.

The density function $f^x(t)$ satisfies the scaling relation

$$f^x(c_0 t) = f^x/t^{1/\alpha}(c_0)/t = f^1(c_0 t/x^{\alpha})/x^{\alpha}.$$ (2.9)

In case $\gamma = |2 - \alpha|$, expansions of $f^x(t)t$ into power series of $x/t^{1/\alpha}$ are known. Indeed, if $\gamma = 2 - \alpha$, owing to (2.7) the power series expansion for $x > 0$ is obtained from that of $t^{1/\alpha}p_\gamma(-x)$ which is found in [9], while for $x < 0$, the series expansion is recently derived by Peskir [12]. Peskir’s result implies

$$f^x(t) = [\Gamma(\alpha - 1)\Gamma(1/\alpha)]^{-1}(-x)^{\alpha-1}t^{-2+1/\alpha}\{1 + O([-x/t^{1/\alpha}]^{2-\alpha})\} \quad (x < 0)$$ (2.10)

for $x = O(t^{1/\alpha})$. For $|\gamma| < 2 - \alpha$ a corresponding asymptotic form is obtained as a by-product of the proof of Theorem 2. As a consequence we have the following corollary.
Corollary 1. As $t \to \infty$

$$f^\dagger(t) \sim \begin{cases} 
\frac{[-1/\Gamma(1-1/\alpha)]/t}{\kappa^{\dagger}_{\alpha,\gamma}/t^{2-1/\alpha}} & \text{if } \gamma = 2 - \alpha, \\
\kappa^{\dagger}_{\alpha,\gamma}/t^{2-1/\alpha} & \text{if } \gamma \neq 2 - \alpha,
\end{cases}$$

where

$$\kappa^{\dagger}_{\alpha,\gamma} = \frac{\sin \pi/\alpha}{\pi p_1(0)} \int_0^\infty u^{1-\alpha} p'_1(-u) du = \frac{\Gamma(2-\alpha) \sin(\pi/\alpha) \sin\left(\frac{\pi}{\alpha} (\alpha + \gamma)\right)}{\alpha^2 \pi^2 p_1(0)}.$$  

(The last expression shows that $\kappa^{\dagger}_{\alpha,\gamma}$ is positive if $\gamma < 2 - \alpha$ and zero if $\gamma = 2 - \alpha$ (cf. Lemma 7.1). In case $\gamma = -2 + \alpha$ the formula above yields the leading term in (2.10).)

2.2. Transition probability of the walk killed on \{0\}.

For a non-empty subset $B \subset \mathbb{Z}$ put

$$p^n_B(x, y) = P[S^n = y, \sigma_B > n]$$

(2.11)

(in particular $p_0^B(x, y) = 1(x = y)$ and $p^n_B(x, y) = 0$ whenever $n \geq 1, y \in B$) and similarly for a closed set $\Delta \subset \mathbb{R}$

$$p_\Delta^\xi(\xi, \eta) = P[Y^\xi_t \in d\eta, \sigma^\Delta_\xi > t]/d\eta.$$  

($Y^\xi_t = \xi + Y_t$ and $\sigma^\Delta_\xi$ is the first entrance time of $Y^\xi$ into $\Delta$.) By the scaling law for stable processes we have

$$p_\Delta^n(x, y) = n^{-1/\alpha} p^\Delta/n^{1/\alpha}(x_n, y_n).$$

In this subsection we give the results for the special case $B = \{0\}$. The results in the general case of finite sets closely parallel to them and are given in the last subsection 2.4.

We write $x_n$ (resp. $y_n$) for $x/n^{1/\alpha}$ (resp. $y/n^{1/\alpha}$) as before. From Theorems 2 and 3 it follows that for all $\gamma$

$$f^x(n) \sim a^\dagger(x) f^0(n) + \frac{|x_n| p^n_{c_0}(x_n)}{n} \quad (n \to 0),$$

where the second term on the RHS is redundant unless $|\gamma| = 2 - \alpha$ and $\gamma x > 0$ as noted in Remark 2.1(b).

Theorem 4. Let $|\gamma| < 2 - \alpha$. For any $M > 1$, uniformly for $|x_n| \vee |y_n| < M$, as $n \to \infty$

$$p^n_{\{0\}}(x, y) \sim \begin{cases} 
f^x(n)a(-y) & (y_n \to 0), \\
a^\dagger(x) f^{-y(n)} & (x_n \to 0, y_n \neq 0), \\
p^n_{c_0}(x, y) & (|x_n| \wedge |y_n| \geq 1/M). 
\end{cases}$$  

(2.12)

(The first two formulae on the RHS are asymptotically equivalent to each other as $x_n \vee y_n \to 0$ but not if $x_n \vee y_n > 1/M$.)

Recalling Remark 2.1(a) it follows that if $|\gamma| < 2 - \alpha$, then for any $M > 1$,

$$p^n_{\{0\}}(x, y) \approx |x_n y_n|^{-1/\alpha} n^{1/\alpha} \quad (|x_n \vee |y_n| < M)$$

(2.13)
Theorem 5. Let \( \gamma = 2 - \alpha \). For any \( M > 1 \), uniformly for \( |x_n| < M \) and \( 0 < y \leq Mn^{1/\alpha} \), as \( n \to \infty \)

\[
p_{\{0\}}^n(x, y) \sim \begin{cases} 
  f^n(x)a(-y) & (y_n \to 0), \\
  a^\dagger(x)f^{-y}(n) + \frac{(x_n) + K_{\{\gamma\}}(y_n)}{n^{1/\alpha}} & (x_n \to 0), \\
  p_{\{0\}}^{\{0\}}(x, y) & (|x_n| \wedge y_n \geq 1/M).
\end{cases}
\]

where \( K_t(\eta) = 0 \) (\( \eta \leq 0 \)) and

\[
K_t(\eta) = \lim_{\xi \downarrow 0} \frac{1}{p(t)} p^{(-\infty, 0]}(\xi, \eta) \quad (\eta > 0).
\]

The duality relation \( p_{\{0\}}^n(x, y) = p_{\{0\}}^n(-y, -x) \) \((xy \neq 0)\) gives another apparently different statement of Theorem 5. Specializing to the case \( |x_n| \wedge |y_n| \to 0 \) and incorporating Theorems 2 and 3 we here write down it as the following corollary for convenience of later citations.

Corollary 2. If \( \gamma = 2 - \alpha \), uniformly for \(-Mn^{1/\alpha} \leq x \leq 0\) and \( |y_n| < M \), as \( n \to \infty \)

\[
p_{\{0\}}^n(x, y) \sim \begin{cases} 
  a^\dagger(x)c_0f_{-y}(c_0)/n & (x_n \uparrow 0, y_n > 1/M), \\
  a^\dagger(x)[f^0(x)a(-y) + |y_n| p_{c_0}(y_n)n^{-1}] & (x_n \uparrow 0, y < 0), \\
  a(-y)f^0(n) + \frac{(y_n) - K_{c_0}(-x_n)}{n^{1/\alpha}} & (y_n \to 0).
\end{cases}
\]

Note that (2.16) includes the case \( y < 0, x < 0 \) that is excluded from (2.14). The case \( x > 0 \) and \( y < 0 \) excluded from the both will be discussed after Remark 2.2 below. If the walk is left-continuous in particular, namely if \( P[X \leq -2] = 0 \) (possible for \( \gamma = 2 - \alpha \)), then in case \( y < 0, a(-y) = 0 \) and (2.14) cannot hold, its right side vanishing while the left side being positive for \( x \leq 0 \). This case however is included in (2.16). Similarly (2.16) for the case \( a(x) = 0 \) is complemented by (2.14). If the walk is not left-continuous, (2.14) (resp. (2.16)) is extended to the case \(-M < y < 0\) (resp. \(0 < x < M\)). The extension can be trivially made in the course of the proof, although we shall not mention it. The same comment applies to several places in the sequel where analogous situations occur.

For \( \gamma = -2 + \alpha \), the result specialized to the case \( y > 0 \) and \( |x_n| \wedge |y_n| \to 0 \), is given as follows: uniformly for \( x \geq 0 \) and \( |x_n| \vee |y_n| < M \), as \( n \to \infty \)

\[
p_{\{0\}}^n(x, y) \sim \begin{cases} 
  f^y(n)a(x) & (x_n \to 0), \\
  a^\dagger(x)a(-y)f^0(n) + \frac{(y_n) + K_{c_0}(x_n)}{n^{1/\alpha}} & (y_n \to 0),
\end{cases}
\]

which is immediately deduced from (2.14) by using duality relations: \( p_{\{0\}}^n(x, y) = \hat{p}_{\{0\}}^n(y, x), \hat{a}(x) = a(-x) \) and \( \hat{f}^x(n) = f^{-x}(n) \) and

\[
\hat{K}_t(\eta) = \lim_{\xi \downarrow 0} \frac{p_{\{0\}}^{(-\infty, 0]}(\eta, \xi)}{\xi} = \lim_{\xi \downarrow 0} \frac{p_{\{0\}}^{[0, \infty)}(-\xi, -\eta)}{\xi} \quad (\eta > 0),
\]

where \( \hat{\ } \) indicates the corresponding functions for the dual walk.

Remark 2.2. (a) The same crossover as described in Remark 2.1(a) takes place in (2.14) plainly for the first case of it but also in the second case: in the both the crossover occurs around \( a(x)/x \approx n^{1-2/\alpha} \) as in (2.6), and similarly in (2.16) around \( a(-y)/y \approx n^{1-2/\alpha} \).
(b) If $\gamma = 2 - \alpha$, then $p_{i}^{(0)}(x,y) = p_{i}^{(-\infty,0)}(x,y)$ $(x,y > 0)$. (Cf. e.g., [1].)

(c) Paralleling Remark 2.1(c) concerning $f^{x}(n)$ it holds that for all admissible $\gamma$ and for $|x_n|, |y_n| \in [1/M, M]$, $p_{i}^{(0)}(x,y) \sim n^{-1/\alpha}p_{c_{0}}^{(0)}(x_n,y_n)$, whenever $xy > 0$. This remains true in case $xy < 0$ if either $|\gamma| < 2 - \alpha$ or $x\gamma < 0$, but does not anymore otherwise, namely if either $x > 0, y < 0$ and $\gamma = 2 - \alpha$ or $x < 0, y > 0$ and $\gamma = -2 + \alpha$ (see Theorem 6 and (2.22)).

(d) Let $\gamma = 2 - \alpha$. The first formula (2.14) implies that if one takes the successive limit as first $x_n \to \xi > 0$, $y_n \to \eta > 0$ as well as $n \to \infty$ and then $\xi \vee \eta \to 0$, then

$$\frac{p_{i}^{(0)}(x,y)}{f^{x}(n)a(-y)} \to \frac{p_{c_{0}}^{(0)}(x,y)}{p_{c_{0}}^{(0)}(x,y)} \rightarrow 1.$$ 

Since the limit of the first ratio of the middle member equals 1 by virtue of the second relation of (2.14), it therefore follows from Theorem 3 that as $\xi \vee \eta \to 0$ and $n \to \infty$

$$\frac{p_{c_{0}}^{(0)}(\xi,\eta)n^{-1/\alpha}}{\xi p_{c_{0}}^{(0)}(-\xi)a(-y)} \to 1. \quad (2.17)$$

On noting $p_{c_{0}}^{(0)}(-\xi, -\eta) = p_{c_{0}}^{(0)}(\eta, \xi)$ and using (2.5) this shows that

$$p_{c_{0}}^{(0)}(\xi, \eta) \sim \frac{p_{c_{0}}^{(0)}(0)}{c_{0}\Gamma(\alpha)} \times \begin{cases} \xi \eta^{\alpha-1} & (\xi \downarrow 0, \eta \downarrow 0), \\ -\eta(-\xi)^{\alpha-1} & (\xi \uparrow 0, \eta \uparrow 0). \end{cases} \quad (2.18)$$

(e) In the same way as in (d) we deduce from Theorem 4 that if $|\gamma| < 2 - \alpha$,

$$p_{1}^{(0)}(\xi, \eta) \sim \kappa_{\alpha,\gamma}^{\prime}\{\sin[\frac{1}{2}\pi(\alpha + (\text{sgn} \xi)\gamma)]\} \{\sin[\frac{1}{2}\pi(\alpha - (\text{sgn} \eta)\gamma)]\} |\xi\eta|^{\alpha-1} \quad (|\xi| \vee |\eta| \to 0),$$

where $\kappa_{\alpha,\gamma}^{\prime} = \kappa_{\alpha,\gamma}\Gamma(1 - \alpha)/\pi$. Similarly, with $\xi > 0$ fixed and $\eta$ tending to zero, noting (2.7) we see that if $\gamma < 2 - \alpha$ or $y > 0$ (i.e., when $a(-y)/y^{\alpha-1}$ tends to a positive constant),

$$\frac{p_{c_{0}}^{(0)}(\xi, y_n)}{c_{0}\kappa^{2}(c_{0})|a(-y)/n^{1-\alpha}|} \to 1 \quad (y_n \to 0, y \to \infty, \xi > 0).$$

(f) If $\gamma = 2 - \alpha$, then $p_{c_{0}}(0) = 1/c_{0}^{\alpha/\alpha}\alpha\Gamma(1 - 1/\alpha)$ (see Lemma 3.2) and by (2.18)

$$K_{c_{0}}(\eta) \sim \frac{p_{c_{0}}^{(0)}(0)}{c_{0}\Gamma(\alpha)} \eta^{\alpha-1} \quad (\eta \downarrow 0), \quad (2.19)$$

with the help of which we deduce from Theorem 5 and its corollary that for $|x_n| \vee |y_n| < M$,

$$p_{i}^{(0)}(x,y) \sim \begin{cases} f^{x}(n)a(-y) \times |y_n|^{\alpha-1}\{a^{\uparrow}(x)n^{-1} + |x_n|n^{-1/\alpha}\} & (y > 0), \\ (|x_n|^{\alpha-1} \vee 1)\{a(-y)n^{-1} + |y_n|n^{-1/\alpha}\} & (x \leq 0). \end{cases} \quad (2.20)$$

From Theorem 3 is excluded the regime $x > 0, y < 0, x \wedge (-y) \to +\infty$ (as noted previously), where there arises a difficulty in estimating $p_{i}^{(0)}(x,y)$ in general; in below we give a result under an extra assumption on the tail as $t \to -\infty$ of the distribution function

$$F(t) := P[X \leq t].$$
In [21 Theorem 2(iii)] a criterion for the limit

\[ C^+ := \lim_{x \to +\infty} a(x) \leq \infty \]

(which exists) to be finite is obtained. Under the present assumption on \( F \) it say that

\[ \int_0^{\infty} F(-t)t^{2\alpha-2}dt < \infty \quad \text{and} \quad F(-2) > 0 \quad (2.21) \]

is necessary and sufficient for \( 0 < C^+ < \infty \). Note that (2.21) entails \( \gamma = 2 - \alpha \) and the walk is not left-continuous.

**Theorem 6.** Let (2.21) hold. Then, given \( M > 1 \), uniformly for \( -M < y_n < 0 < x_n < M \),

\[ (i) \quad p^n_{(0)}(x,y) \sim a^+(x)a(-y)f^0(n) + \frac{a^+(x)|y_n|p_{c_0}(y_n) + a(-y)x_np_{c_0}(-x_n)}{n} \quad (x_n \wedge (-y_n) \to 0), \]

\[ (ii) \quad p^n_{(0)}(x,y) \sim C^+(x_n - y_n)p_{c_0}(y_n - x_n) \quad \frac{C^+c_0(f^{x-y}(c_0n)}{n} \quad \text{as } n \to \infty. \]

An application of Theorem 6 leads to the next result which exhibits a way the condition \( C^+ < \infty \) is reflected in the behaviour of the walk \( S^x \), \( x > 0 \): conditioned on \( S^x_n = -x \) it enters \((-\infty,-1] \) without visiting the origin ‘continuously’ or by a very long jump for large \( x \) according as \( C^+ \) is finite or not. Exactly the same behaviour of the pinned walk is observed in [19] in the case \( E[X]^2 < \infty \) but with the condition (2.21) replaced by \( E[|X|^{\gamma}; X < 0] < \infty \) which is equivalent to \( \lim_{x \to -\infty}[a(x) - x/\sigma^2] < \infty \).

**Proposition 2.2.** For each \( M \geq 1 \), under the constraint \( -M \sqrt{\gamma} < y < 0 < x < M \sqrt{\gamma} \)

\[ \Pr[S^x_{\sigma_{\{0\}}(-\infty,0]} < -R | \sigma^x_{\{0\}} > n, S^x_n = y] \]

\[ \longrightarrow \begin{cases} 0 & \text{as } R \to \infty \quad \text{uniformly for } x, y \text{ if } C^+ < \infty, \\ 1 & \text{as } x \wedge (-y) \to \infty \text{ for each } R > 0 \text{ if } C^+ = \infty. \end{cases} \]

We state the following upper bound as a proposition, a unified but partly reduced version obtained by combining theorems above and the results in Section 5.

**Proposition 2.3.** (i) For all admissible \( \gamma \) and \( M > 1 \), there exists a constant \( C_M \) such that for all \( n \geq 1 \) and \( x \in \mathbb{Z}, \)

\[ p^n_{(0)}(x,y) \leq C_M((|x_n| \vee 1)^{\alpha-1} \wedge |x_n|^{-\alpha})|y|^{\alpha-1} \quad \text{if } |y_n| < M. \]

(ii) If \( \gamma = 2 - \alpha \), there exists a constant \( C \) such that for all \( x, y \in \mathbb{Z}, \)

\[ p^n_{(0)}(x,y) \leq C \left[ \frac{a^+(x)a(-y)}{n^{2-1/\alpha}} + \frac{a^+(x)(y_n-1) + a(-y)(x_n+1)}{n} \right] \quad (2.22) \]

(i) is the same as Lemma 5.2. (ii) follows from Theorems 4 and 5 in case \( |x_n| \vee |y_n| < 1 \), from Proposition 5.2 in case \( |x_n| \vee |y_n| \geq 1 \) with \( xy < 0 \), from Lemmas 5.1 and 5.2 in case
\( |x_n| \land |y_n| \leq 1 \leq |x_n| \lor |y_n| \) with \( xy \geq 0 \) and the bound \( p^n(x) \leq Cn^{1/\alpha} \) (entailed by the local limit theorem) in case \( |x_n| \land |y_n| \geq 1 \) with \( xy \geq 0 \).

### 2.3. Comparing \( p^n_{(0)}(x,y) \) and \( p^n_{(-\infty,0)}(x,y) \).

Let \( V_{as} \) (resp. \( U_{ds} \)) denote the renewal function of weakly ascending (resp. strictly descending) ladder height process of the walk \( S \) and \( Q_t(\eta) \) and \( Q_t(\eta), \eta \geq 0 \) the distribution functions of the stable meander of length \( t \) at time \( t \) for \( Y \) and \( -Y \), respectively (see (7.2) for the definition). Doney [3] obtains an elegant asymptotic formulae of \( p^n_{(0,x,\infty)}(0,y) \) \( (x \geq 0) \), which under the present assumption and with our notation may be rewritten as

\[
\begin{align*}
\frac{U_{ds}(x)V_{as}(y)p_{c_0}(0)}{n^{1+1/\alpha}} & \quad (x_n \lor y_n \to 0), \\
\frac{V_{as}(y)P[\sigma^0_{[0,\infty)} > n]\hat{Q}'_{c_0}(x_n)}{n^{1/\alpha}} & \quad (y_n \to 0, x_n > 1/M), \\
\frac{U_{ds}(x)P[\sigma^0_{(-\infty,-1)} > n]\hat{Q}'_{c_0}(y_n)}{n^{1/\alpha}} & \quad (x_n \to 0, y_n > 1/M), \\
\frac{p_{c_0}^{(-\infty,0)}(x,y)}{n^{1/\alpha}} & \quad (x_n \land y_n \geq 1/M)
\end{align*}
\]

by using the duality relation.

In the regime \( x \asymp y \asymp n^{1/\alpha} \), where \( p_{c_0}^{(0)}(x,y)/p_{c_0}^{(-\infty,0)}(x,y) = p_{c_0}^{(0)}(x_n,y_n)/p_{c_0}^{(-\infty,0)}(x_n,y_n) \asymp 1 \), we have \( p^n_{(0)}(x,y) \asymp p^n_{(-\infty,0)}(x,y) \) for all values of \( \gamma \).

If \(|\gamma| < 2 - \alpha \), then \( V_{as} \) and \( U_{ds} \) vary regularly with exponents which are both larger than \( \alpha - 1 \) and whose sum equals \( \alpha \) (cf. [3]) and each of the products \( V_{as}(n^{1/\alpha})P[\sigma^0_{[0,\infty)} > n] \) and \( U_{ds}(n^{1/\alpha})P[\sigma^0_{(-\infty,-1)} > n] \) approaches to a positive constant as \( n \to \infty \). These lead to an estimate for \( p^n_{(-\infty,0)}(x,y) \) analogous to that for \( p^n_{(0)}(x,y) \) given in (2.13) and comparing them yields

\[
\frac{p^n_{(-\infty,0)}(x,y)/p^n_{(0)}(x,y)}{n^{1/\alpha}} \to 0 \quad as \quad x_n \land y_n \to 0.
\]

In case \(|\gamma| = |2 - \alpha| \) we need to take a closer look at the situation that turns out to be precisely parallels the crossover mentioned in Remark 2.2(a) as given by (2.27) below for \( \gamma = 2 - \alpha \).

Let \( \gamma = 2 - \alpha \). Then \( p^n_{(0)}(x,y) \sim p^n_{(-\infty,0)}(x,y) \) for \( x_n \land y_n > 1/M \) in view of (2.14) and (2.23) (see also Remark 2.2(b)). According to [15, Theorems 2 and 9]

\[
U_{ds}(x) \sim xL(x) \quad and \quad V_{as}(x) \sim k^x x^{\alpha-1}/L(x) \quad (x \to \infty)
\]

with a positive constant \( k^x \) and a slowly varying \( L(x) \) that tends to zero as \( x \to \infty \). (More information is found in Remark 2.3(b) given below. A condition sufficient for \( L \) to be asymptotic to a positive constant is considered in Remark 3.1)

Let \( \hat{Z} \) be the first (strictly) descending ladder height, namely \( \hat{Z} = S_{\sigma(-\infty,0)} \), and suppose

\[
E|\hat{Z}| < \infty.
\]

Then \( L \) may be taken to be the constant \( 1/E|\hat{Z}| \) owing to the renewal theorem and letting first \( x_n \to \xi > 0 \) and \( y_n \to \eta > 0 \) and then \( \xi \downarrow 0 \) or \( \eta \downarrow 0 \) in (2.14) and (2.23) we see that uniformly for \( x, y > 0 \) and \( x \lor y \leq Mn^{1/\alpha} \) as \( x \land y \to \infty \)

\[
p^n_{(-\infty,0)}(x,y) \sim \begin{cases} y_n^{\alpha-1}x_n p_{c_0}(-x_n)/n^{1/\alpha} c_0(\alpha) & (y_n \to 0), \\
x_n K_{c_0}(y_n)/n^{1/\alpha} & (x_n \to 0),
\end{cases}
\]

(2.26)
(this is confirmed by making an elementary computation (see Remark 2.3(c)) which however
is not needed) and compare this with (2.14) to deduce that uniformly for \(0 \leq x, y < Mn^{1/\alpha}\),
as \(n \to \infty\)
\[
P^n_{[0]}(x, y) \sim p^n_{(-\infty, 0)}(x, y) + a^\dagger(x)p^0(n)a(-y).
\]  
(2.27)

According to Kesten [11] (see Remark 2.4 of the next subsection) this asymptotic relation with \(x, y\) fixed (when the first term on the RHS is superfluous) is valid for every recurrent walk that is strongly aperiodic and having \(\sigma^2 = \infty\). It is quite plausible that (2.27) holds for \(x, y\) subject to the same constraint as above for every such random walk on \(Z\) with \(E|\hat{Z}| < \infty\). If \(E|\hat{Z}| = \infty\), \(p^n_{(-\infty, 0)}(x, y)\) may depend on \(L\) in the regime \(x_n \wedge y_n \to 0\) while \(p^n_{[0]}(x, y)\) does not, and (2.27) must be violated.

**Remark 2.3.** (a) Let \(\gamma = 2 - \alpha\) and \(K_t\) be given in (2.15). Then for \(x, y > 0\)
\[
x_p(-x) = tf^x(t) = t^{1/\alpha}\hat{Q}'_t(x)/\Gamma(1/\alpha) \quad \text{and} \quad K_t(y) = \alpha p_t(0)Q'_t(y)
\]  
(2.28)

(see Lemma 7.4), and on re-writing the first equality and using (2.19)
\[
\hat{Q}'_t(\eta) = t^{-1/\alpha}\Gamma(1/\alpha)p_t(-\eta)\eta \quad (\eta > 0) \quad \text{and} \quad Q'_t(\eta) \sim \eta^{\alpha-1}/t^\alpha\Gamma(\alpha) \quad (\eta \downarrow 0).
\]  
(2.29)

Note that \(Q_t(\eta) = Q_1(\eta/t^{1/\alpha})\), entailing \(Q'_n(y) = Q'_1(y_n)/n^{1/\alpha}\).

(b) It is known [22, Eq(15), Eq(31)] that for some positive constant \(b_x\),
\[
P[\sigma^0_{(-\infty, -1)} > n] \sim b/U_{[0]}(n^{1/\alpha}), \quad \text{and}
\]  
\[
P[\sigma^0_{[0, +\infty)} > n] \sim \beta/n \quad \text{with} \quad \beta := 1/\Gamma(\rho)\Gamma(1 - \rho),
\]
where \(\rho := \lim_n n^{-1}\sum_{k=1}^n P[S_k > 0] = 1/2(1 - \gamma/\alpha)\) (cf. (8.2)). We derive in below that if \(\gamma = 2 - \alpha\), then
\[
b = \frac{1}{c_0^{1/\alpha}\Gamma(1 - 1/\alpha)} \quad \text{and} \quad \kappa^V = \frac{1}{c_0\Gamma(\alpha)}
\]  
(2.30)

\((\kappa^V\) appears in (2.24)), the former identity (together with \(\rho = 1 - 1/\alpha\) and (2.24) entails
\[
P[\sigma^0_{(-\infty, -1)} > n] \sim \frac{1/\Gamma(1 - 1/\alpha)}{\Gamma(1 - 1/\alpha)L(n^{1/\alpha})} \quad \text{and} \quad P[\sigma^0_{[0, +\infty)} > n] \sim \frac{c_0^{1/\alpha}L(n^{1/\alpha})}{n^{1-1/\alpha}\Gamma(1/\alpha)}.
\]  
(2.31)

(2.30) as well as what are stated prior to it is valid even if \(X\) belongs to a domain of attraction
(instead of a normal domain). For the derivation, employing (7.3) that says \(p^0_{cs}(\xi, \eta) \sim \alpha p_{cs}(0)Q'_{cs}(\eta)|\xi| (\xi \downarrow 0, \eta > 0)\) we deduce from the third and fourth cases of (2.23) that
\[
b U_{ds}(x)/U_{ds}(n^{1/\alpha}) \sim \alpha p_{cs}(0)x_n \quad (\xi = x_n \downarrow 0),
\]
which immediately leads to \(b = \alpha p_{cs}(0) = 1/c_0^{1/\alpha}\Gamma(1 - 1/\alpha)\) (see Lemma 3.2 for the second equality). In a similar way employing (7.4) and the second formula of (2.23) we derive \(\kappa^V = 1/c_0\Gamma(\alpha)\) as required.

(c) By (2.28) and (2.31) we observe that as \(n \to \infty\)
\[
P[\sigma^0_{[0, +\infty)} > n]Q'_cs(x_n) \sim L(n^{1/\alpha})x_np_{cs}(-x_n)/n^{1-1/\alpha} \quad (x > 0),
\]  
(2.32)

which together directly derive (2.26) from (2.23) as noted before.
2.4. Extension to the process killed on a finite set

Let $A$ be a finite subset of $\mathbb{Z}$. Suppose for simplicity that for some $M > 1$

$$g_A(x, y) > 0 \quad \text{if} \quad |x| \land |y| > M,$$  \hspace{1cm} (2.33)

where for a non-empty $B \subset \mathbb{Z}$, $g_B$ denotes the Green function for the walk killed on $B$:

$$g_B(x, y) = \sum_{n=0}^{\infty} p^n_B(x, y).$$  \hspace{1cm} (2.34)

Under the condition $\sigma^2 = \infty$ there exists

$$u_A(x) = \lim_{|y| \to \infty} g_A(x, y)$$ \hspace{1cm} (2.35)

[16] T30.1. $u_A$ is positive and harmonic for the killed walk: $u_A(x) = \sum_{z \notin A} p(z - x)u_A(z) > 0$ for all $x \in \mathbb{Z}$. Put

$$f^\pm_A(n) = P[\sigma^A_x = n].$$

In order to obtain the asymptotic form of $f^\pm_A(n)$ we may simply replace $f^x(n)$ and $a^\dagger(\cdot)$ by $f^\pm_A(n)$ and $u_A(\cdot)$, respectively, in Theorems 2 and 3, the resulting formula being valid in the same range of variables so that uniformly for $|x| < Mn^{1/\alpha}$, as $n \to \infty$

$$f^\pm_A(n) \sim \begin{cases} \frac{u_A(x)f^0(n)}{u_A(x)f^0(n) + \frac{(x_n)\pm p_{c, x}(\pm x_n)}{n}} & \text{if } x = o(n^{1/\alpha}) \text{ and } |\gamma| < 2 - \alpha, \\ \frac{u_A(x)f^0(n)}{u_A(x)f^0(n) + \frac{(x_n)\pm p_{c, x}(\pm x_n)}{n}} & \text{if } x = o(n^{1/\alpha}) \text{ and } |\gamma| = 2 - \alpha, \end{cases}$$  \hspace{1cm} (2.36)

where in the symbols $\pm$ and $\mp$ the upper (resp. lower) sign prevails if $\gamma > 0$ (resp. $\gamma < 0$).

After virtually the same replacement [replace $f(n)$ by $f^\pm_A(n)$, $C^+$ by $C^+_x := \lim_{x \to \infty} u_A(x)$ and $a(\cdot)$ by $u_A(\cdot)1(y \notin A)$] Theorems 4 to 6 with $p^n_A(x, y)$ in place of $p^n_{\{0\}}(x, y)$ remain true. In case $\gamma = 2 - \alpha$ in particular, the result corresponding to Theorem 5 read

$$p^n_A(x, y) \sim \begin{cases} f^\pm_A(n)u_A(-y) & \text{if } x \to 0, \\ u_A(x)f^\pm_A(n) + \frac{(x_n)\pm K_{c, x}(y_n)}{n^{1/\alpha}} & \text{if } x \to 0, \\ p^0_{\{0\}}(x, y) & \text{if } |x_n| \land y_n \geq 1/M. \end{cases}$$  \hspace{1cm} (2.37)

uniformly for $|x_n| < M$ and $-M < y < Mn^{1/\alpha}$.

Note that from the definition of $g_A(x, y)$ it follows that $u_A$ is the probability distribution of the hitting place of $A$ by the dual walk ‘started at infinity’. From (2.36) we therefore infer

$$\sum_{z \in A} f^\pm_A(n) \sim f^0(n),$$

which relation is observed in [11] when $\gamma = 0$. By a similar consideration or by the identity

$$P[\sigma^A_x = n, S^x_n = y] = \sum_{z \notin A} p^n_A(x, z)p(y - z) \quad (y \in A)$$  \hspace{1cm} (2.38)

one deduces from (2.37) the following asymptotic form of space-time hitting distribution.
Corollary 3. Uniformly for $|x_n| < M$, as $n \to \infty$

$$P[\sigma_A^* = n, S_n^* = y] \sim f^*_A(n)u_A(-y) \quad (y \in A).$$

Remark 2.4. As mentioned in Introduction Kesten [11] obtained asymptotic formulae of $p^n_A(x, y)$ with $x, y$ fixed for a large class of random walks on multidimensional lattices $\mathbb{Z}^d$, which if specialized to one-dimensional recurrent walk may read in the present notation

$$\lim_{n \to \infty} p^n_A(x, y) / \sum_{z \in A} f^*_A(n) = u_A(x)u_{-A}(-y) \quad (y \notin A),$$

provided the walk is strongly aperiodic and having infinite variance.

The rest of the paper is organized as follows. The proofs of Theorems 2 and 3 are given in Section 3 and those of Theorems 4 and 5 in Section 4. In Section 5 some estimations of $p^n_{\{0\}}(x, y)$ are made in case $xy < 0$ and, for this purpose, beyond the regime $|x| \vee |y| = O(n^{1/\alpha})$: Propositions 5.1 and 5.2 give there a lower and upper bound, respectively and Theorem 6 and Proposition 2.2 are proved after them; Proposition 2.1 is proved at the end of Section 5 where we prove to this end an upper bound in case $|y| = O(n^{1/\alpha})$ and $|x_n| \to \infty$. In Section 6 the results are extended to those for an arbitrary finite set instead of the single point set $\{0\}$. In Section 7 we deal with the limit stable process and present some properties of $f^x(t)$ and $p^x_{\{0\}}(x, y)$. In the last section we give miscellaneous consequences of the present assumption on the random walk that are derived from the general theory: they are (1) condition (1.1) expressed in terms of the tails of $F$ and some related facts, (2) some upper bounds of $p^n(x)$ for $|x| > n^{1/\alpha}$ and (3) ‘escape probabilities’ from the origine.

3 Estimation of $f^x(n)$

3.1. In several places in this subsection we shall apply following identity

$$\int_0^\infty \left\{ \cos u \over \sin u \right\} {du \over u^\nu} = \left\{ \begin{array}{ll} \Gamma(1 - \nu) \sin {\pi \over 2} \nu & 0 < \nu < 1 \\
\Gamma(1 - \nu) \cos {\pi \over 2} \nu & 0 < \nu < 2, \end{array} \right. \quad (3.1)$$

where for $\nu = 1$, $\Gamma(1 - \nu) \cos {\pi \over 2} \nu$ is understood to be $\frac{1}{2}\pi$, its limit value (cf. [9] pp.10, 68, [23] p.260)).

Lemma 3.1. Put $\kappa^a_{\alpha, \gamma, \pm} = -\Gamma(1 - \alpha)\pi^{-1} \sin[\frac{\pi}{2}\pi(\alpha \pm \gamma)]$. Then

(i) $\lim_{x \to \pm \infty} c_a(x) / |x|^{\alpha-1} = -\kappa^a_{\alpha, \gamma, \pm}$,

(ii) $\lim_{x \to \pm \infty} c_a(a(x + 1) - a(x)) / |x|^{2-\alpha} = \pm(\alpha - 1)\kappa^a_{\alpha, \gamma, \pm};$

$\kappa^a_{\alpha, \gamma, \pm} > 0$ if $|\gamma| < 2 - \alpha$, and $\kappa^a_{\alpha, \gamma, \pm} = 0$ or $1/\Gamma(\alpha)$ according as $\pm \gamma > 0$ or $< 0$ if $|\gamma| = 2 - \alpha$; in particular if $\gamma = 2 - \alpha$, then $c_a(x) \sim (-x)^{\alpha-2}/\Gamma(\alpha)$ as $x \to -\infty$ and $c_a(x) \sim o(|x|^\alpha)$ as $x \to +\infty$.

Proof. Although a little more general result of (i) is given in [20] Section 6.1], we give its proof, which is partly used in the proof of (ii). From the second of formula (3.1) one obtains

$$\int_0^\infty \left\{ 1 - \cos u \over \sin u \right\} {du \over u^\nu} = \left\{ \begin{array}{ll} -\Gamma(1 - \alpha) \sin {\pi \over 2} \alpha, & \\
\Gamma(1 - \alpha) \cos {\pi \over 2} \alpha, & \end{array} \right. \quad (3.2)$$
where for the first formula one takes \( \nu = \alpha - 1 \) and performs integration by parts. In the representation \( a(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - e^{ix\theta})(1 - \phi(\theta)) d\theta \) we replace \( 1 - \phi(\theta) \) by \( c_{\phi}; \psi(\theta) \), its principal part about zero, and compute the resulting integral. Changing a variable we have

\[
\int_{-\pi}^{\pi} \frac{1 - e^{ix\theta}}{c_{\phi}\psi(\theta)} \frac{1 - \cos u + i \sin u}{\cos \frac{\pi\gamma}{2} i u + \sin \frac{\pi\gamma}{2} u} |u|^{-\alpha} du \quad (\pm = x/|x|),
\]

which an easy computation with the help of (3.2) shows to be asymptotically equivalent as \( |x| \to \infty \) to

\[
-\frac{2\Gamma(1 - \alpha)}{c_{\phi}} |x|^{\alpha - 1} \left[ \cos \frac{\pi\gamma}{2} \sin \frac{\pi\alpha}{2} + \sin \frac{\pi\gamma}{2} \cos \frac{\pi\alpha}{2} \right].
\]

The combination of the sine’s and cosine’s in the square brackets being equal to \( \sin[\frac{1}{2} \pi (\alpha \pm \gamma)] \) we find the equality (i), provided that the replacement mentioned at the beginning causes only a negligible term of the magnitude \( o(|x|^{\alpha - 1}) \), but this is assured from the way of computation carried out above since the integrand in the RHS integral in (3.3) is summable on \( \mathbb{R} \).

For the proof of (ii) it suffices to show that

\[
\int_{-\pi}^{\pi} e^{ix\theta} (1 - e^{i\theta}) \left[ \frac{1}{1 - \phi(\theta)} - \frac{1}{\psi(\theta)} \right] d\theta = o(|x|^{\alpha - 2}),
\]

since \( a(x + 1) - a(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\theta} (1 - e^{i\theta})(1 - \phi(\theta))^{-1} d\theta \) and this integral with \( \psi(\theta) \) replacing \( 1 - \phi(\theta) \) is asymptotically equivalent to \( \pm |(\alpha - 1) c_{\phi}; \gamma|/c_{\phi} |x|^{\alpha - 2} \) as one sees by looking at the increment of the RHS of (3.3). Because of the fact that if \( \psi(\theta) = \{1 - \phi(\theta)\}(1 + \delta(\theta)) \) then \( \delta'(\theta) \to 0 \quad (\theta \to 0) \) (cf. (3.2)), the relation (3.4) is shown in a usual way.

**3.2.** For evaluation of \( f^{\wedge}(n) \) we follow [18], although therein the walk is assumed to have finite variance. Set

\[
\pi_{x}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\phi(\theta)} d\theta \quad (\tau \neq 0, x \in \mathbb{Z})
\]

and

\[
f^{\wedge}_x(\tau) = \sum_{n=1}^{\infty} f(x) e^{in\tau}.
\]

Since \( |\phi(\theta)| < 1 \) for \( 0 < |\theta| \leq \pi \) by aperiodicity of the walk, the function \( 1 - e^{i\tau} \phi(\theta) \) does not vanish in \( \mathbb{R} \times [\pi, \pi] \) except at \( \tau = \theta = 0 \). We have the following identities

\[
\pi_{x}(\tau) = \lim_{r \uparrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\phi(\theta)} d\theta = \lim_{r \uparrow} \sum_{n=0}^{\infty} p^{n}(x) e^{in\tau} r^{n};
\]

\[
f^{\wedge}_x(\tau) = \frac{1(x = 0)}{\pi_{0}(\tau)} + \frac{\pi_{-x}(\tau)}{\pi_{0}(\tau)}, \quad \text{especially} \quad f^{\wedge}_0(\tau) = 1 - \frac{1}{\pi_{0}(\tau)}.
\]

Note that \( \text{Re} \pi_{x}(\tau) \) and \( \text{Re} f^{\wedge}_x(\tau) \) are even, while \( \text{Im} \pi_{x}, \text{Im} f^{\wedge}_x(\tau) \) are odd. Obviously \( \text{Re} f^{\wedge}_x(\tau) \) equals the cosine series \( \sum_{n=1}^{\infty} f(x) \cos nx \) and we shall use the following inversion formulae

\[
f^{\wedge}_x(\tau) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^{\wedge}_x(\sigma) \cos n\tau d\tau = \frac{2}{\pi} \int_{0}^{\pi} \text{Re} f^{\wedge}_x(\tau) \cos n\tau d\tau.
\]

Note that \( \pi_{0}(\tau) \) vanishes nowhere on \([\pi, \pi]\) and is smooth off the origin, and that

\[
1 - e^{i\tau} \phi(\theta) = -i\tau + c_{\phi}\psi(\theta) + O(\tau^2 + |\tau|^\alpha) + o(|\theta|^\alpha)
\]
and for some constant $C > 0$

$$|1 - e^{i\tau \phi(\theta)}| \geq C^{-1}(|\tau| + |\eta|^\alpha) \quad (-\pi < \tau < \pi, -\pi < \theta < \pi).$$

We shall compare $\pi_x(\theta)$ with the corresponding function for the limit stable process function given by

$$\pi_x^\infty(\tau) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau \theta} d\theta$$

$$(-i\tau + c_0\psi(\theta)) = \int_0^\infty E[e^{i\theta\gamma}] e^{i\tau t} dt (\theta \neq 0)$$

**Lemma 3.2.** As $\tau \to 0$

$$1/\pi_0(\tau) \sim 1/\pi_0^\infty(\tau), \quad [1/\pi_0]'(\tau) \sim [1/\pi_0^\infty]'(\tau) \quad \text{and} \quad [1/\pi_0]''(\tau) \sim [1/\pi_0^\infty]''(\tau)$$

(3.9)

where the indications the differentiation; and for $\tau \neq 0$,

$$\pi_0^\infty(\tau) = p_1(0)c^{-1/\alpha}_0 \Gamma(1 - 1/\alpha) i e^{-i\pi/2\alpha} |\tau|^{1/\alpha} \tau, \quad p_1(0) = \frac{\Gamma(1/\alpha)}{\pi^\alpha} \sin \frac{\pi(\alpha - \gamma)}{2\alpha;}$$

in particular, $(d/d\tau)^j[1/\pi_0(\tau)] = O(|\tau|^{-1/\alpha + 1 - j}) (\tau \to 0, j = 0, 1, 2)$ and if $|\gamma| = 2 - \alpha$, $p_1(0) = 1/\alpha \Gamma(1 - 1/\alpha)$.  

**Proof.** In view of [3.8] it is easy to deduce from the defining expression of $\pi_0(\tau)$ that $\pi_0(\tau) \sim \pi_0^\infty(\tau)$, $\pi_0'(\tau) \sim \pi_0^\infty'(\tau)$ and $\pi_0''(\tau) \sim \pi_0^\infty''(\tau)$, which show [3.9]. The expression of $p_1(0)$ is obtained by specializing the series expansion of $p_1(x)$ as found in e.g. [3] Lemma 17.6.1.

Direct computation of $\pi_0^\infty(\tau)$ is not hard at all but here we apply the fact that $\pi_0^\infty$ is the Fourier transform of $p_{c_0 t}(0) = p_{c_0}(0)/t^{1/\alpha} (t \geq 0)$ (verified by using the analogue of (3.5)) so that

$$\pi_0^\infty(\tau) = p_{c_0}(0) \int_0^\infty t^{-1/\alpha} e^{i\tau t} dt.$$  

This integral is written as $|\tau|^{1/\alpha} \tau^{-1} \int_0^\infty t^{-1/\alpha} e^{i\tau t} dt$ and is evaluated by applying (3.1), giving the formula of the lemma.  

From the expression of $\pi_0^\infty(\tau)$ given in Lemma 3.2 we have

$$\frac{1}{\pi_0^\infty(\tau)} = \frac{c^{1/\alpha}_0 \alpha \pi [\sin(\pi/2\alpha) - i \cos(\pi/2\alpha)]}{\Gamma(1/\alpha) \Gamma(1 - 1/\alpha) \sin[\pi(\alpha - \gamma)/2\alpha]} \tau^{1-1/\alpha} (\tau > 0)$$

(3.10)

with which we compute the integral arising in (3.7) to show Theorem 1.

**Proof of Theorem 1.** The formula to be shown is $f^0(n) \sim \kappa_{c_0} c_0^{1/\alpha}/n^{2-1/\alpha}$. In case $\gamma = 0$ (i.e., the walk is centered) this is obtained by Kesten [11] and the same proof applies. Here we proceeds somewhat differently as follows. On making trivial decomposition $1/\pi_0 = 1/\pi_0^\infty + [1/\pi_0 - 1/\pi_0^\infty]$ an integration by parts transforms $f^0(n) = \frac{2}{\pi} \int_0^\pi \Re [-1/\pi_0(\tau)] \cos n\tau d\tau$ into

$$\frac{2}{\pi n} \int_0^\pi \Re [1/\pi_0^\infty]'(\tau) \sin n\tau d\tau + \frac{2}{\pi n} \int_0^\pi \Re [1/\pi_0 - 1/\pi_0^\infty]'(\tau) \sin n\tau d\tau.$$  

The first term, easily evaluated by (3.1) owing to (3.10), gives the asymptotic form asserted by the lemma. The second integral restricted on $[0, 1/n]$ is shown to be $o(1/n^{1-1/\alpha})$ by using $[1/\pi_0 - 1/\pi_0^\infty]'(\tau) = o(|\tau|^{-1/\alpha})$ and that on $(1/n, \pi)$ is dealt with by integrating by parts once more. Further details are omitted.
Lemma 3.3. For any $M > 1$, uniformly for $M^{-1} < x/n^{1/\alpha} < M$, as $n \to \infty$

$$f^x(n) \sim f^r(n).$$

Proof. Bring in the functions $R_1(\tau, \theta)$ and $R_2(\tau, \theta)$ by

$$R_1 = \frac{1}{1 - e^{i\tau} \phi(\theta)} - \frac{1}{-i\tau + 1 - \phi(\theta)} \quad \text{and} \quad R_2 = \frac{1}{-i\tau + 1 - \phi(\theta)} - \frac{1}{-i\tau + c_0 \psi(\theta)}$$

so that

$$\frac{1}{1 - e^{i\tau} \phi(\theta)} = \frac{1}{-i\tau + c_0 \psi(\theta)} + R_1 + R_2.$$ (3.11)

It is easily observed that

$$f^x(n) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\pi^{\ast}(\tau)}{\pi_0(\tau)} \cos n\tau \, d\tau + \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \frac{\cos n\tau \, d\tau}{\pi_0(\tau)} \int_{-\pi}^{\pi} (R_1 + R_2)e^{ix\theta} \, d\theta.$$ 

Using Lemma 3.2 we can readily deduce that the first term on the RHS is asymptotically equivalent to $f^r(n)$ in the same sense as in the lemma and that the second term is $o(1/n)$, which shows the assertion of the lemma since $f^r(t)$ is positive (because of a Huygens-like property) and continuous on $t > 0$ and hence $f^x(n) = f^r(n/x^{1/\alpha})/x^{1/\alpha} \geq c_M/n$ for some $c_M > 0$ for the range of $x$ specified in the lemma.

3.3. In this subsection we prove Theorems 2 and 3 and Corollary 1. Recalling $f^x(\tau) = [-1(x = 0) + \pi_{-x}(\tau)]/\pi_0(\tau) \ (x \neq 0)$ we introduce as in [18]

$$e_x(\tau) := \pi_{-x}(\tau) - \pi_0(\tau) + a(x) \quad (3.12)$$

so that

$$f^x(\tau) = \frac{e_x(\tau)}{\pi_0(\tau)} + 1 - \frac{a^l(x)}{\pi_0(\tau)}.$$ 

The integral representation $a(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} (1 - \phi)^{-1}(1 - e^{ix\theta}) \, d\theta$ yields

$$e_x(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{1 - e^{i\tau} \phi(\theta)} - \frac{1}{1 - \phi(\theta)} \right) (e^{ix\theta} - 1) \, d\theta.$$ 

We make the decomposition $e_x(\tau) = c_x(\tau)/2\pi + i s_x(\tau)/2\pi$, where

$$c_x(\tau) = \int_{-\pi}^{\pi} \left( \frac{1}{1 - e^{i\tau} \phi(\theta)} - \frac{1}{1 - \phi(\theta)} \right) \cos x\theta \, d\theta$$

$$s_x(\tau) = \int_{-\pi}^{\pi} \left( \frac{1}{1 - e^{i\tau} \phi(\theta)} - \frac{1}{1 - \phi(\theta)} \right) \sin x\theta \, d\theta.$$ 

The computations the present approach necessitates are carried out in the proofs of the succeeding two lemmas.

Lemma 3.4. For some constants $C_1$ and $C_2$

$$\left| \int_{0}^{\pi} \frac{s_x(\tau)}{\pi_0(\tau)} \cos n\tau \, d\tau \right| \leq \frac{C_1 |x|}{n^{1+1/\alpha}} \quad \text{and} \quad \left| \int_{0}^{\pi} \frac{c_x(\tau)}{\pi_0(\tau)} \cos n\tau \, d\tau \right| \leq \frac{C_2 |x|^{1+\varepsilon}}{n^{1+(1+\varepsilon)/\alpha}},$$

where in the second bound $\varepsilon$ is any constant not larger than unity such that $0 \leq \varepsilon < 2\alpha - 2$ and $C_2$ may depend on $\varepsilon$.  

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Proof. First we claim

\[ |(d/d\tau)^j s_x(\tau)| \leq C|x|^2/\alpha - 1 - j \quad (j = 0, 1, 2, 3). \]  (3.13)

Writing

\[ s_x(\tau) = \int_{-\pi}^{\pi} \frac{(e^{i\tau} - 1)\phi(\theta)}{1 - e^{i\tau}\phi(\theta)} \cdot \frac{\sin x\theta}{1 - \phi(\theta)} d\theta \]  (3.14)

and using \(|1 - e^{i\tau}\phi(\theta)| \geq C^{-1}(|\tau| + |\theta|^\alpha)|\) we see that

\[ |s_x(\tau)| \leq C' \int_{0}^{\pi} \frac{|x\theta|^{1 - \alpha}}{|\tau| + |\theta|^\alpha} d\theta \leq C' |x|^2/\alpha - 1 \int_{0}^{\infty} \frac{u^{1 - \alpha}}{1 + u^\alpha} du, \]  (3.15)

hence the claimed bound of \(s_x\). Differentiating the defining expression of \(s_x\) we have

\[ s_x'(\tau) = i e^{i\tau} \int_{-\pi}^{\pi} \frac{\phi(\theta) \sin x\theta}{\{1 - e^{i\tau}\phi(\theta)} d\theta, \]  (3.16)

which yields the claimed bound for \(j = 1\) in the same way as above. Those for \(j = 2, 3\) are similar and the claim has been verified.

Now integrating by parts gives

\[ \int_{0}^{\pi} \frac{s_x(\tau)}{\pi_0(\tau)} \cos n\tau d\tau = -\frac{1}{n} \int_{0}^{\pi} \frac{s_x}{\pi_0}'(\tau) \sin n\tau d\tau. \]  (3.17)

By Lemma 3.2 \(|(d/d\tau)^j(1/\pi_0(\tau))| \leq |\tau|^{1 - 1/\alpha - j}\), which together with (3.13) shows that the integral restricted to \(\tau < 1/n\) is \(O(xn^{-1/\alpha})\). On integrating by parts once more the remaining integral admits the same bound, showing the first one of the lemma.

Following the proof of (3.13) performed above but by using the bound \(1 - \cos x\theta \leq |x\theta|^{1+\varepsilon}\) in place of \(|\sin x\theta| \leq |x\theta|\) (so that the integral corresponding to the last one in (3.15) is finite) we obtain

\[ |(d/d\tau)^j c_x(\tau)| \leq C|x|^{1+\varepsilon}|\tau|^{(2+\varepsilon)/\alpha - 1 - j} \quad (j = 0, 1, 2, 3). \]  (3.18)

The rest of the proof is the same as above.

By the same computation as in the preceding proof we obtain the following bounds

\[ \left| \int_{0}^{\pi} s_x(\tau) \cos n\tau d\tau \right| \leq \frac{C_1|x_n|}{n^{1/\alpha}}, \quad \text{and} \quad \left| \int_{0}^{\pi} c_x(\tau) \cos n\tau d\tau \right| \leq \frac{C_2|x_n|^{1+\varepsilon}}{n^{1/\alpha}}; \]  (3.19)

expanding \(e_x(\tau)e_{-y}(\tau) = (-s_x s_{-y} + is_x c_y + ic_x s_{-y} + c_x c_{-y})(\tau)/4\pi^2\) we also have

\[ \left| \int_{0}^{\pi} \frac{e_x(\tau)e_{-y}(\tau)}{\pi_0(\tau)} \cos n\tau d\tau \right| \leq \frac{C_3}{n^{1/\alpha}} \{ |x_n| \vee |x_n|^{1+\varepsilon} \} \{ |y_n| \vee |y_n|^{1+\varepsilon} \}, \]  (3.20)

which are used not in this but in the next section. Here \(\varepsilon\) is chosen as in Lemma 3.4.

Lemma 3.5. There exists a constant \(\Lambda\) such that for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[ \left| \frac{2}{\pi} \int_{0}^{\pi} \text{Re} \frac{is_x(\tau)}{\pi_0(\tau)} \cos n\tau d\tau - \frac{\Lambda x_n}{n} \right| < \varepsilon \frac{|x_n|}{n} \text{ if } |x_n| < \delta, |x| \wedge n > 1/\delta, \]

where \(x_n = x/n^{1/\alpha}\).
Proof. We evaluate the RHS of (3.17). Take $M > 1$ such that $\cos M = 0$. Then, on integrating by parts and applying $|s_x/\pi_0|''(\tau)| \leq C|x|^{1/\alpha-2}$

$$
\frac{1}{n} \left| \int_{M/n}^{\pi} \Re \left[ is_x/\pi_0 \right]'(\tau) \sin n\tau \, d\tau \right| = \frac{1}{n^2} \left| \int_{M/n}^{\pi} \Re \left[ is_x/\pi_0 \right]''(\tau) \cos n\tau \, d\tau \right| \\
\leq C(1 - 1/\alpha)^{-1} M^{1/\alpha-1} |x_n/n|.
$$

(3.21)

Here we have applied the fact that $\Re \left[ is_x/\pi_0 \right]'(\tau)$ vanishes at $\pi$ since it is odd and periodic with period $2\pi$, hence attains the same value for $\tau = \pm \pi$. [To see that $\Re \left[ is_x/\pi_0 \right]'(\tau)$ is odd, it suffices to show that $is_x$ (as well as $\pi_x(\tau)$) has the even real and odd imaginary parts, which may be verified, e.g., by observing that $\int_{-\pi}^{\pi} (1 - re^{i\tau} \phi(\theta))^{-1} \sin x\theta \, d\theta$ is represented by a Fourier series (with real coefficients).]

For the integral over $0 < \tau < M/n$ let $c_0 = 1$ for simplicity. We replace $1 - e^{i\tau} \phi(\theta)$ by $-i\tau + \psi(\theta)$ and $1 - \phi(\theta)$ by $\psi(\theta)$ in the integral defining $s_x(\tau)$ as in the proof of Lemma 3.3, the replacement being justified without difficulty in view of (3.8). We further replace $\sin x\theta$ by $x\theta$ and extend the range of integration to the whole real line, which we shall show to cause only a negligible error (see the end of this proof). In any case these modifications of $s_x(\tau)$ together result in the function

$$
\psi_x(\tau) := \int_{-\infty}^{\infty} \left( \frac{1}{-i\tau + \psi(\theta)} - \frac{1}{\psi(\theta)} \right) \sin \theta \, d\theta = \int_{-\infty}^{\infty} \frac{i\tau \theta}{\{ -i\tau + c_0 \psi(\theta) \}} \psi(\theta) \, d\theta.
$$

In view of (3.21) it will suffice to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $M$, $|x|$ and $n$ large enough, if $|x_n| < \delta$, then

$$
\left| \int_{0}^{M/n} \Re \left[ is_x/\pi_0 \right]'(\tau) \sin n\tau \, d\tau - \Lambda x_n \right| < \varepsilon |x_n|.
$$

(3.22)

After substitution of $\psi(\theta) = e^{\pm i\pi/2} |\theta|^\alpha$ and the change of variable $u = \theta / |\tau|^{1/\alpha}$ we have

$$
-is_x(\tau) = \frac{x|\tau|^{2/\alpha}}{\tau} \int_{-\infty}^{\infty} \frac{u}{\{ -ie^{\pm i\pi/2} \text{sgn} \tau + e^{\pm i\pi/2} |u|^{\alpha} \}} |u|^{\alpha} \, du,
$$

(3.23)

where the upper or lower sign in $\pm$ prevails according as $u > 0$ or $u < 0$. By Lemma 3.2 or (3.10)

$$
-\Re \left[ is_x/\pi_0 \right]'(\tau) = \Lambda_1 x\tau^{1/\alpha-1} \quad \text{for} \quad \tau > 0
$$

for a constant $\Lambda_1$, and with the help of $(d/d\tau)^j[s_x(\tau)(1/\pi_0 - 1/\pi_0^\infty)(\tau)] = o(|\tau|^{1/\alpha-j}) \quad (j = 1, 2)$ we deduce that

$$
-\int_{0}^{M/n} \Re \left[ is_x/\pi_0 \right]'(\tau) \sin n\tau \, d\tau \sim x_n \Lambda_1 \int_{0}^{M} s^{1/\alpha-1} \sin s \, ds
$$

and on using (3.11) we conclude that (3.22) holds with $\Lambda = 2\pi^{-1} \Lambda_1 \Gamma(1/\alpha) \sin(\pi/2\alpha)$.

It remains to show that the error caused by the replacement of $s_x$ by $s_x^0$ is negligible. The range $|\theta| > 1/x$ in the integral defining $s_x$, which corresponds to $u > 1/x\tau^\alpha$ in the integral on the RHS of (3.10) that (absolutely) converges, is negligible since for $|\tau| < M/n$, $x\tau^\alpha \to 0$ as $x \to 0$. The same is true for the derivative

$$
(s^0_\tau)'(\tau) = i \int_{-\infty}^{\infty} \{ -i\tau + \psi(\theta) \}^{-2} \sin x\theta \, d\theta = i \int_{-\infty}^{\infty} \{ -i + \psi(u) \}^{-2} \sin(x\tau^{1/\alpha} u) \, du,
$$

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the same integral as obtained by replacing \{1 - e^{i\tau \phi(\theta)}\} and \(e^{i\tau \phi(\theta)}\) by \{-i\tau + \psi(\theta)\} and 1, respectively, in the RHS of (3.16), so that \((s_{2}^{-})'(\tau) \sim s'_{2}(\tau)\) uniformly for \(|\tau| < M/n\) as \(x_n \to 0\).

This finishes the proof of Lemma 3.3.

**Proof of Theorem 2.** Let \(c_0 = 1\) for simplicity. According to the decomposition (3.12) we have

\[
f^x(n) = \frac{1}{\pi^2} \int_{0}^{\pi} \text{Re} \left[ \frac{c_x(\tau) + is_x(\tau)}{\pi_0(\tau)} \right] \cos n\tau d\tau + \frac{2}{\pi} \int_{0}^{\pi} \text{Re} \left[ \frac{1}{\pi_0(\tau)} \right] \cos n\tau d\tau. \tag{3.24}
\]

By Lemmas 3.4 and 3.5 it follows that as \(x_n \to 0\)

\[
f^x(n) = a^+(x)f^0(n) + \frac{\Lambda x_n}{\pi^2 n}\{1 + o(1)\}. \tag{3.25}
\]

The first term on the RHS is the leading term and we have the first formula of (2.2). Indeed this is evident from Theorem 1 when \(x\) remains in a bounded set since \(a^+(x) > 0\), while applying and Lemma 3.11) in addition we have \(a^+(x)f^0(n) \sim \kappa_{\pm} |x_n|^{-\alpha-1}/n\) as \(x_{\pm} \wedge n \to \infty\) with some \(\kappa_{\pm} > 0\), showing that the second term of (3.25) is negligible as \(x_n \to 0\). The second formula of (2.2) follows from Lemma 3.3.

**Proof of Theorem 3.** Let \(\gamma = 2 - \alpha\). First note that for the regime \(1/M \leq |x_n| \leq M\) the result follows from Lemma 3.3. In case \(x_n \to 0\) we apply relation (3.25) (valid for all \(\gamma\)). For \(x < 0\) \(a(x)\) behave in a similar way to the case \(|\gamma| = 2 - \alpha\), so that the preceding proof works well. For \(x > 0\), it follows that \(a(x) = o(x^{\alpha-1})\) as \(x \to \infty\), hence, on the one hand, taking limit in (3.25) we obtain

\[
nf^x(n)/x_n \to \Lambda/\pi^2 \quad \text{as} \quad x_n \to \xi > 0 \quad \text{and} \quad \xi \downarrow 0 \quad \text{in this order}.
\]

On the other hand, owing to the identity \(c_0 f^x(c_0 n) = x_n p_{c_0}(-x_n)/n\) it follows that

\[
n c_0 f^x(c_0 n)/x_n \to p_{c_0}(0)
\]

in the same way of taking the limit as above. By the result for the case \(x_n \sim 1\) this leads to \(\Lambda/\pi^2 = p_{c_0}(0)\), which allows us to replace the second term on the RHS of (3.25) by \(x_n p_{c_0}(0)/n\), thus concludes the proof, the case \(\gamma = -2 + \alpha\) being dealt with in the same way.

**Remark 3.1.** In view of (3.24)—recall \(c_x + is_x = 2\pi e_x\)—what is shown in the proofs above is paraphrased as follows: If \(\gamma = 2 - \alpha\), then uniformly for \(|x_n| < M\), as \(n \to \infty\)

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{c_x(\tau)}{\pi_0(\tau)} \cos n\tau d\tau = \begin{cases} x_n p_{c_0}(-x_n)/n + o(a(x)/n^{2-1/\alpha}) & (x > 0), \\
o(a(x)/n^{2-1/\alpha}) & (x < 0, x_n \to 0), \end{cases} \tag{3.26}
\]

and if \(|\gamma| < 2 - \alpha\), the integral on the LHS is \(o(a(x)/n^{2-1/\alpha})\) as \(x_n \to 0\).

**Proof of Corollary 1** The first expression of \(\kappa_{\alpha,\gamma}^f\) as well as the equivalence relation in case \(\gamma \neq 2 - \alpha\) follows from Lemma 3.1. For \(\gamma = 2 - \alpha\) the equivalence relation follows from what is mentioned in the paragraph preceding the corollary. As in the last part of the proof of Theorem 3 given above, by Lemma 3.3 (with \(c_0 = 1\)) and scaling relation of \(f^x(t)\) it follows that

\[
nf^x(n) \sim nf^x(n) \sim f^1(1/x_n^\alpha)/x_n^\alpha,
\]

which together with Theorems 2 and 3 shows that if \(\gamma < 2 - \alpha\), then \(f^1(t) \sim \kappa_{\alpha,\gamma}^f t^{2-1/\alpha}\) with \(\kappa_{\alpha,\gamma}^f\) determined by \(\kappa_{\alpha,\gamma} a(x)/n^{1-1/\alpha} \sim \kappa_{\alpha,\gamma} x_n^{\alpha-1}\). By Lemma 3.1(i) and the expression defining \(\kappa_{\alpha,\gamma}\), this leads to the second expression of \(\kappa_{\alpha,\gamma}^f\).

Because of the similarity of the proof to that of Lemma 3.5 we here give the following lemma that is used in the next section.
Lemma 3.6. There exists a function \( D_n(y), y \in \mathbb{Z} \) such that \( |D_n(y)| \leq C |y_n| \) for \( |y_n| < M \) and for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |x_n| < \delta, |x| \wedge n > 1/\delta, |y_n| < 1/\varepsilon \),

\[
\left| \frac{2}{\pi} \int_0^\pi \text{Re} \frac{e_x(\tau) e_y(\tau)}{\pi_0(\tau)} \cos n \tau \, d\tau - \frac{D_n(y) x_n}{n^{1/\alpha}} \right| < \varepsilon \frac{|x_n y_n|}{n^{1/\alpha}},
\]

Proof. On recalling the derivation of (3.20) the terms \( |x| \) and \( |x|^{1+\varepsilon} \) on the RHS of it correspond to \( s_x \) and \( c_x \), respectively and similarly for \( |y| \) and \( |y|^{1+\varepsilon} \), and one sees it suffices to show that

\[
\left| \frac{2}{\pi} \int_0^\pi \text{Re} \frac{is_x(\tau) e_y(\tau)}{\pi_0(\tau)} \cos n \tau \, d\tau - \frac{D_n(y) x_n}{n^{1/\alpha}} \right| < \varepsilon \frac{|x_n y_n|}{n^{1/\alpha}} \tag{3.27}
\]

provided \( |x_n| < \delta, |x| \wedge n > 1/\delta, |y_n| < 1/\varepsilon \). First suppose \( 3/2 < \alpha < 2 \) so that \(-1 < 3/\alpha - 2 < 0 \). By (3.9), (3.13), (3.18) it follows that \( |[s_x e_y/\pi_0](\tau)| \leq C |xy| |\tau|^{3/\alpha - 2} \), and on integrating by parts

\[
\frac{2}{\pi} \int_0^\pi \text{Re} \frac{is_x(\tau) e_y(\tau)}{\pi_0(\tau)} \cos n \tau \, d\tau = \frac{2}{\pi n} \int_0^\pi \text{Re} \left[ -is_x e_y/\pi_0 \right]'(\tau) \sin n \tau \, d\tau. \tag{3.28}
\]

Integrating by parts once more we observe that the contribution from \( |\tau| > M/n \) to the integral on the RHS becomes negligibly small as \( M \) is taken large and then \( s_x^2(\tau) \) may be replaced by \( s_x^0(\tau) \) as in the proof of Lemma 3.5. [Here we have applied the fact that \( \text{Re} [-is_x e_y/\pi_0]'(\tau) \) is odd, hence vanishes at \( \tau = \pi \).] Now define

\[
\omega(\tau) = -is_x^0(\tau)/x
\]

and

\[
D_n(y) := \frac{2n^{2/\alpha - 1}}{\pi} \int_0^\pi \text{Re} \left[ \frac{\omega e_y}{\pi_0} \right]'(\tau) \sin n \tau \, d\tau. \tag{3.29}
\]

By the same reason as above the contribution from \( \tau > M/n \) to the integral on the RHS becomes negligible as \( M \) gets large, and we see that (3.27) is satisfied. Noting \( |[\omega e_y/\pi_0]'(\tau)| \leq C_1 |y| |\tau|^{3/\alpha - 2} \) we also deduce that \( |D_n(y)| = O(y_n) \).

In case \( 1 < \alpha < 3/2 \) we can further integrate the RHS of (3.28) by parts to have

\[
\int_0^\pi \text{Re} \frac{is_x(\tau) e_y(\tau)}{\pi_0(\tau)} \cos n \tau \, d\tau = \frac{1}{n^2} \int_0^\pi \text{Re} \left[ -is_x e_y/\pi_0 \right]''(\tau) \cos n \tau \, d\tau \tag{3.30}
\]

and accordingly putting \( D_n(y) = \frac{2n^{2/\alpha - 2}}{\pi} \int_0^\pi \text{Re} \left[ \frac{\omega e_y}{\pi_0} \right]''(\tau) \cos n \tau \, d\tau \) and making a similar argument to the above we obtain (3.27).

Let \( \alpha = 3/2 \). This is a critical case when \( [s_x e_y/\pi_0]'(\tau) \) tends to a constant multiple of \( xy \) as \( \tau \to 0 \), and by the very this fact we have \( [s_x e_y/\pi_0]''(\tau) = o(1/\tau) \). Split the range of integral on the LHS of (3.28) at \( \tau = 2\pi N/n \) with a positive integer \( N \) and denote by \( I \) and \( II \) the integrals over \((0, M/n] \) and \([M/n, \pi] \), respectively, where \( M = 2\pi N \). Then on integrating by parts once more

\[
I = \frac{1}{n^2} \int_0^{M/n} \text{Re} \left[ -is_x e_y/\pi_0 \right]''(\tau)(\cos n \tau - 1) \, d\tau, \quad \text{and}
\]

\[
II = \frac{1}{n^2} \text{Re} \left[ -is_x e_y/\pi_0 \right]'(M/n) + \frac{1}{n^2} \int_{M/n}^\pi \text{Re} \left[ -is_x e_y/\pi_0 \right]''(\tau) \cos n \tau \, d\tau.
\]
Since the integrand of the first integral above is at most \( o(1/\tau) \times n\tau \), we see that \( I = o(1/n^2) \). The second integral which we further integrate by parts is dominated by a constant multiple of \( |xy|/Mn \), thus negligible since \( M \) can be chosen arbitrarily large, while \( \text{Re}[-i\xi e_{-\xi}/\pi_0] = \kappa xy (n \to \infty) \) with some \( \kappa \in \mathbb{R} \). Finally recalling \( n^2 = n^{3/\alpha} \), we find that \( (3.27) \) holds with \( D_n(y) = (2/\pi)\kappa y_n \), and hence conclude the proof of the lemma. \( \square \)

4 Estimates of \( p_{\{0\}}^n(x, y) \)

In this section we prove Theorems 4 and 5, the proofs being given at the end of the section. We continue to use the notation \( \pi_x(\tau) \) introduced in the preceding section.

The arguments that follow are based on the representation

\[
p_{\{0\}}^n(x, y) = p^n(y - x) - \sum_{k=1}^n f^x(n - k)p^k(y) \tag{4.1}
\]

or, to say more exactly, its Fourier version: from \( (3.3) \) one can easily deduce that \( p^n(x) = (1/2\pi)\int_{-\pi}^\pi \pi_x(\tau)e^{-i\tau}d\tau \) and, on combining this with \( (3.6), (4.1) \) may be written as

\[
p_{\{0\}}^n(x, y) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{\pi_y(x) - \pi_y(n\tau - x)}{\pi_0(\tau)} e^{-i\tau}d\tau \quad (x \neq 0) \tag{4.2}
\]

and

\[
p_{\{0\}}^n(0, y) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{\pi_y(\tau)}{\pi_0(\tau)} e^{-i\tau}d\tau. \tag{4.3}
\]

Note that for \( y \neq 0 \), \( p_{\{0\}}^n(0, y) = f^{-y}(n) \) by duality (or by coincidence of the Fourier coefficients), so that in the case \( x = 0 \) the required estimate is immediate from Theorems 1 to 3 that have been verified in the preceding section.

Lemma 4.1. Uniformly for \( x, y \in \mathbb{Z} \), as \( n \to \infty \)

\[
\begin{align*}
&\quad (i) \quad p^n(y - x) - p^n(-x) - p^n(y) + p^n(0) \\
&= p_{\{0\}}(y - x) - p_{\{0\}}(-x) - p_{\{0\}}(y) + p_{\{0\}}(0) + o(xy/n^{3/\alpha}), \text{and} \\
&\quad (ii) \quad p^n(y - x) - p^n(-x) = p_{\{0\}}(y - x) - p_{\{0\}}(-x) + o(y/n^{2/\alpha}).
\end{align*}
\]

Proof. Put \( c = c_0 \cos(\pi\gamma/2) (> 0) \), choose a positive constant \( \varepsilon \) so that \( 1 - |\phi(\theta)| \geq |\theta|^\alpha c/2 \) for \( |\theta| < \varepsilon \) and put \( \eta = \sup_{\varepsilon \leq |\theta| \leq \varepsilon} |\phi(\theta)| < 1 \). Then the error in the first relation (i) is written as

\[
\begin{align*}
\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \left[ \phi(\theta) \right]^n - e^{-nc_0\psi(\theta)} K_{x,y}(\theta)d\theta + O(\eta^n + e^{-nc_0^\alpha}),
\end{align*}
\]

where and \( K_{x,y}(\theta) = e^{-i(y-x)x} - e^{ix} - e^{-iy} + 1 \). By \( (1.1) \) \( \log[\phi(\theta)e^{c_0\psi(\theta)}] = o(|\theta|^\alpha) \) as \( \theta \to 0 \).

Since \( K_{x,y}(\theta) = (e^{ix} - 1)(e^{-iy} - 1) \), we have \( |K_{x,y}(\theta)| \leq |xy|^2 \) and, scaling \( \theta \) by \( n^{1/\alpha} \) and applying the dominated convergence theorem, we deduce that the integral above is \( o(xy/n^{3/\alpha}) \), showing (i).

The proof of (ii) is similar, rather simpler. One may only to use \( |e^{-i(y-x)x} - e^{ix}| \leq |y\theta| \) in place of the bound of \( K_{x,y}(\theta) \). \( \square \)
Lemma 4.2. Given $M > 1$, if either $\gamma = 2 - \alpha$ and $y > 0$ or $|\gamma| < 2 - \alpha$, then as $n \to \infty$ and $|x_n| \lor y_n \to 0$

$$p_{(0)}^n (x, y) \sim a(-y) f^x(n).$$

(4.4)

If either $\gamma = 2 - \alpha$ and $x_+/a^t(x) = o(n^{2/\alpha - 1})$ or $|\gamma| < 2 - \alpha$, then the expression on the RHS is asymptotically equivalent to $\kappa_{\alpha, \gamma} c_o^{1/\alpha} a^t(x) a(-y) / n^{2 - 1/\alpha}$ as $n \to \infty$ and $|x_n| \to 0$.

Proof. Of the integrand in (4.2) we make the decomposition

$$\pi_{y-x}(\tau) - \pi_{-x}(\tau) \pi_y(\tau) / \pi_0(\tau) = \pi_{y-x} - \pi_{-x} - \pi_y + \pi_0 - a(x) a(-y) / \pi_0 + [-e_x e_{-y} + a(x) e_{-y} + a(-y) e_x] / \pi_0,$$

(4.5)

where we recall $e_x = e_x(\tau) = \pi_{-x}(\tau) - \pi_0(\tau) + a(x) = [c_x(\tau) + is_x(\tau)]/2\pi$. Noting that

$$p_t(y-x) - p_t(-x) - p_t(y) + p_t(0) = -xy p''_t(0) t^{-3/\alpha} \{1 + o(1)\} \quad \text{as} \quad |x_n| \lor |y_n| \to 0,$$

we apply Theorem 1 and Lemma 4.1(i) to see

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \pi_{y-x} - \pi_{-x} - \pi_y + \pi_0 - \frac{a(x) a(-y)}{\pi_0} \right] e^{-int} d\tau$$

$$= p^n(y-x) - p^n(-x) - p^n(y) + p^n(0) + a(x) a(-y) f^0(n)$$

$$= \kappa_{\alpha, \gamma} c_o^{2}(x) a(a(-y)) / (c_o n)^{2-1/\alpha} - \frac{p''_t(0) xy}{(c_o n)^{3/\alpha}} + o \left( \frac{xy}{n^{3/\alpha}} \right).$$

(4.6)

By Lemma 3.4 and (3.20) the integral $\int_{-\pi}^{\pi} [-e_x e_{-y} + a(x) e_{-y} + a(-y) e_x] \cos n\tau d\tau / \pi_0$ is dominated in absolute value by a constant multiple of

$$\left[ \kappa_{\alpha, \gamma} c_o^{2}(x) a(a(-y)) / (c_o n)^{2-1/\alpha} - \frac{p''_t(0) xy}{(c_o n)^{3/\alpha}} + o \left( \frac{xy}{n^{3/\alpha}} \right) \right].$$

(4.7)

If either $\gamma = 2 - \alpha$, $x < M$ and $y > 0$ or $\gamma < 2 - \alpha$, both ratios in (4.7) are $o(a(x) a(-y) / n^{2-1/\alpha})$ (as $|x_n| \lor y_n \to 0$), so that the first term on the right most member of (4.6) is dominant over the others and in view of Theorem 2 formula (4.3) follows.

In the other case $\gamma = 2 - \alpha$, $x \geq M$ and $y > 0$, the term $a(x) |y_n| / n$ in (4.7) is negligible while we have to take account of $a(-y) |x_n| / n$ and turn back to the integral $\pi^{-1} \int_{-\pi}^{\pi} -a(-y) e_x(\tau) \cos n\tau d\tau / \pi_0(\tau)$ which is asymptotic to $a(-y)p_{c_o}(0) x_n / n$ as we have noted in Remark 3.1 (after the proof of Theorem 3) so that the terms of order at most $|x_n y_n| / n^{1/\alpha}$ are negligible, and we see that the combination

$$\kappa_{\alpha, \gamma} c_o^{2}(x) a(a(-y)) / (c_o n)^{2-1/\alpha} + \frac{a(-y)p_{c_o}(0) x_n}{n} \sim a(-y) f^x(n)$$

constitutes the leading term. See (2.6) for the second half of the theorem.

From (4.6) and (4.7) (with a simple amplification for the case $|x_n| \lor |y_n| > 1/M$) we have the following upper bound: For some constant $C$ depending only on $F$,

$$p_{(0)}^n (x, y) \leq C \left[ \frac{a^t(x) a(-y)}{n^{2-1/\alpha}} + \frac{a^t(x) |y| + a(-y) |x|}{n^{1+1/\alpha}} \right] \quad \text{if} \quad |x| \lor |y| < n^{1/\alpha}.$$

In the next section we shall remove the restriction $|x| \lor |y| < n^{1/\alpha}$ and improve the estimate in case $|\gamma| = 2 - \alpha$ and $\gamma x < 0, xy < 0$ (cf. Proposition 5.2).
Lemma 4.3. Uniformly for $|x_n|, |y_n| \in [1/M, M]$, as $n \to \infty$

$$p_{[0]}^n(x, y) \sim p_{[0]}^{[0]}(x, y) = p_{c_0}^{(0)}(x_n, y_n)/n^{1/\alpha}.$$  

Proof. Let $c_0 = 1$ for simplicity. In view of identity (4.1) It suffices to show that for $\varepsilon > 0$,

$$\sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} f^x(n-k)p^k(y) \sim \frac{1}{n^{1/\alpha}} \int_{\varepsilon}^{(1-\varepsilon)} f^x(t-s)p_\varepsilon(y_n)ds \ (n \to \infty)$$

and the sum over $k \in [0, \varepsilon n] \cup [(1-\varepsilon)n, n]$ and the corresponding integral are both negligible as $n \to \infty$ and $\varepsilon \downarrow 0$ in this order. The first requirement is easily deduced from the asymptotic form of $f^x(k)$ given in Theorems 2 and 3 (see also (2.7) in case $\gamma = 2 - \alpha$) and the local limit theorem 10, according to which uniformly for $y \in \mathbb{Z}$, as $k \to \infty$

$$p^k(y) = p_k(y) + o(1/k^{1/\alpha}). \quad (4.8)$$

For the second one, we address only the sum, the integral being similarly treated. Denoting the sums over $k \in [0, \varepsilon n]$ and $[(1-\varepsilon)n, n]$ by $\Sigma_{<\varepsilon}$ and $\Sigma_{>(1-\varepsilon)}$, respectively, we must show that

$$\lim_{\varepsilon, n \to \infty} \left( \Sigma_{<\varepsilon} + \Sigma_{>(1-\varepsilon)} \right)n^{1/\alpha} = 0.$$  

The sum $\Sigma_{>(1-\varepsilon)}$ is immediately disposed of by the fact that $p^k(y) = O(n^{-1/\alpha})$ for $k > (1-\varepsilon)n$ and

$$\sum_{k > (1-\varepsilon)n} f^x(n-k) = P[\sigma^x_0 \leq \varepsilon n] \to 0$$

in the present scheme of passing to the limit. As for $\Sigma_{<\varepsilon}$ we use the bound $f^x(n-k) \leq O(1/n)$ ($k < \varepsilon n$) as well as (4.8) to see that

$$\Sigma_{<\varepsilon} \leq \frac{C}{n} \sum_{k<n} p_k(y) = \frac{C}{n} \sum_{k<n} p_{k/y^n}(1) \sim \frac{C}{n^{1/\alpha}} \int_0^{\varepsilon/y^n} p_t(1) dt,$$

showing $n^{1/\alpha} \Sigma_{<\varepsilon} \to 0$ as required.  

Theorems 4 and 5 follow from Lemmas 4.2 and 4.3 when either $|x_n| \lor |y_n| \to 0$ or $|x_n| \land |y_n|$ is bounded away from zero. We need to deal with the case when $|x_n| \lor |y_n|$ is bounded away from zero and $|x_n| \land |y_n| \to 0$.

Lemma 4.4. For any $M > 1$, uniformly for $1/M < |x_n| \lor |y_n| < M$, it holds that if $\gamma = 2 - \alpha$, then as $n \to \infty$ and $|x_n| \lor y_n \to 0$ under $y > 0$  

$$p_{[0]}^n(x, y) \sim \begin{cases} a(-y)f^x(n) & y_n \to 0, \\ a^+(x)f^{-y}(n) + \frac{x+K_{c_0}(y_n)}{n^{2/\alpha}} & x_n \to 0, \end{cases} \quad (4.9)$$

where $K_t(\eta)$ is given by (2.13); and if $|\gamma| < 2 - \alpha$, then $p_{[0]}^n(x, y) \sim a(-y)f^x(n)$ or $a^+(x)f^{-y}(n)$ according as $y_n \to 0$ or $x_n \to 0$, $y \neq 0$.

Proof. As before the proof rests on the Fourier representation (4.2). Let $\gamma > -2 + \alpha$.

First consider the case $y_n \to 0$. This time we employ the decomposition

$$\pi_{y-x} - \pi_{-x}\pi_y/\pi_0 = \pi_{y-x} - \pi_{-x} + a(-y)\pi_{-x}/\pi_0 + \pi_{-x}e_{-y}/\pi_0.$$  

Owing to Lemma 4.1(ii) and the present assumption on $x, y$,

$$p^n(y-x) - p^n(-x) = n^{-1/\alpha}[p_{c_0}^n(-x_n)y_n + o(y_n)].$$
Hence, by (3.6)

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \pi_{y-x} - \pi_{-x} + \frac{a(-y) \pi_{-x}}{\pi_0} \right] \cos n\tau \, d\tau = a(-y)f^x(n) + O(y_n/n^{1/\alpha}).
\]

Now suppose $1/M \leq |x_n| \leq M$ and $y > 0$, which imply $f^x(n) \asymp 1/n$ and $a(-y) \asymp |y|^{\alpha-1}$, respectively. Using Lemma 3.4 (3.19) (both with $y$ in place of $x$), (3.20), and the identity $\pi_{-x} = e_x + \pi_0 - a(x)$ we then deduce

\[
\int_{0}^{\pi} \frac{\pi_{-x}(\tau) e^{-y(\tau)}}{\pi_0(\tau)} \cos n\tau \, d\tau = O(y_n/n^{1/\alpha}).
\]

By Theorems 2 and 3, $y_n/n^{1/\alpha}$ is negligible in comparison to $a(-y)f^x(n) \asymp a(-y)/n$ as $y_n \to 0$, hence the first relation in (4.9) follows.

If $|\gamma| < 2 - \alpha$, the first case of it is proved by the arguments above which are valid without the condition $y > -M$, while the second case follows from the first by duality.

Let $\gamma = 2 - \alpha$, $1/M \leq |y_n| \leq M$ and $x \neq 0$. We follow the proof of Lemma 4.2 employing the decomposition (4.3) and applying the estimates given there. On the one hand by the first equality of (4.6)

\[
\frac{2}{\pi} \int_{-\pi}^{\pi} \left[ \pi_{y-x} - \pi_{-x} - \pi_y + \pi_0 \right] \cos n\tau \, d\tau = -[p'_c(y_n) - p'_c(0)]n^{-\alpha/2}\{1 + o(1)\}
\]

as $x_n \to 0$ (uniformly for $y_n \in [1/M, M]$) and by $[e_{-y}(\tau) - a(-y)]/\pi_0(\tau) = f^y(\gamma) - 1$,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \frac{a(x)e_{-y}(\tau)}{\pi_0(\tau)} + \frac{a(x)a(-y)}{\pi_0(\tau)} \right] \cos n\tau \, d\tau = a(x)f^{-y}(n)
\]

On the other hand by Lemmas 3.4 and 3.5

\[
\frac{2}{\pi} \int_{0}^{\pi} \Re \frac{a(-y)e_x(\tau)}{\pi_0(\tau)} \cos n\tau \, d\tau = \frac{a(-y)x_n(\Lambda + o(1))}{n}
\]

which together with Lemma 3.6 shows that uniformly for $y_n \in [1/M, M]$, as $x_n \to \xi > 0$ and $\xi \to 0$ in this order

\[
\frac{2}{\pi} \int_{0}^{\pi} \Re \frac{-ae_xe_{-y}(\tau) + a(-y)e_x(\tau)}{\pi_0(\tau)} \cos n\tau \, d\tau = \frac{[D_n(y) + \Lambda' y_n^{\alpha-1}]n^{-\alpha/2}x_n + o(x_n)}{n^{1/\alpha}}
\]

with $\Lambda' = \Lambda/c_\alpha \Gamma(\alpha)$ ($\Lambda$ and $D_n(y)$ are given in the proof of Lemmas 3.5 and 3.6, respectively). These together yield

\[
n^{1/\alpha}p^{n}_{0}(x, y) = a(x)f^{-y}(n) + \{p'_c(0) - p'_c(y_n) - D_n(y) + \Lambda y_n^{\alpha-1}\}x_n + o(x_n).
\]

Since $a(x) = o(x^{\alpha-1})$, $f^0(n) = O(1/n^{2-1/\alpha})$ and $R_n(y) = O(y_n/n)$, the second term on the RHS of (4.11) tends to zero as $x_n \to \xi > 0$, and hence in view of Lemma 4.3 letting $x_n \to \xi > 0$ yield

\[
p^{0}_{c}(\xi, y_n) = \{p'_c(0) - p'_c(y_n) - D_n(y) + \Lambda' y_n^{\alpha-1}\} \xi + o(\xi) \quad (\xi \to 0).
\]

Thus dividing both sides by $\xi$ and passing to the limit we find

\[
K_c(y_n) = p'_c(0) - p'_c(y_n) - D_n(y) + \Lambda' y_n^{\alpha-1} + o(1)
\]
(n \to \infty) \text{ uniformly for } y_n \in [1/M, M], \text{ which together with (1.11) shows the second relation of (4.9), the term } O(x_n/n^{1/n}) \text{ being negligible as compared with } a^\tau(x)f^{-y}(n) \text{ for } x < 0. \text{ Lemma 4.4 has been proved.} \]

Proof of Theorems 4 and 5. Both Theorems 4 and 5 follow from Lemmas 4.2, 4.3 and 4.4. Note that the case \( x = 0 \) is dealt with by (4.3). (Note that \(|x_n| \ll a(x)/n^{1-1/\alpha} \) as \( x_n \uparrow 0 \).)

5 Estimates of \( p_{\{0\}}^n(x, y) \) in case \( xy < 0 \) and proof of Theorem 6

Here we derive estimates of \( p_{\{0\}}^n(x, y) \) for \( x, y \) not necessarily confined in \( 0 < |x_n|, |y_n| < M \), that lead to Proposition 2.3 and are useful for the proof of Theorem 6. We assume \( \gamma = 2 - \alpha \) throughout this section except for Lemma 5.2, the case \( xy < 0 \) being dealt with by duality. Sometimes we suppose \( E \) has been proved. (Note that \(|x| << a(x)/n^{1-1/\alpha} \) as \( x_n \uparrow 0 \).)

Proposition 5.1. Suppose \( E|\hat{Z}| < \infty \). Then, given \( M \geq 1 \), for \(-M < y_n < 0 < x_n < M \)

\[
p_{\{0\}}^n(x, y) \geq c_M \left( \sum_{w=2}^{x\land|y|} p(-w)w^{2\alpha-1} \right) \left[ a(x) + a(-y) \frac{n}{n^{2-1/\alpha}} + \frac{x_n + |y_n|}{n} \right] \tag{5.1}
\]

where \( c_M \) is a positive constant (depending on \( M \) and \( F \)).

Proof. The walk is supposed to be not left-continuous, otherwise the result being trivial. This proof employs the obvious lower bound

\[
p_{\{0\}}^n(x, y) \geq \sum_{\delta x^\alpha \leq k \leq n/2} \sum_{0 \leq w \leq (k/\delta)^{1/\alpha}} \sum_{z=1}^{x} p_{(-\infty, 0)}^k(x, w)p(-z - w)p_{\{0\}}^{n-k}(z, y)
\]

valid for any constant \( \delta > 0 \). We may and do suppose \( x \leq -y < Mn^{1/\alpha} \), the case \(-y < x \) being dealt with by duality. \( \delta \) needs to be chosen so small that \( \delta x^\alpha > \eta n \) for some \( \eta < 1/2 \). To this end we take, e.g.,

\[
\delta = 1/2(2M^2)^{\alpha/2},
\]

entailing \( n/2[(x - y)x]^{\alpha/2} > \delta \), which after substituting from \( x - y \geq 2x \) and multiplying by \( x^\alpha \) reduces to \( \delta x^\alpha < n/2^{1+\alpha/2} \).

For \( k, w, z \) taken from the range of summation above, we have by Theorem 5 (see (2.20))

\[
p_{\{0\}}^{n-k}(-z, y) \asymp a(-z)f^{-y}(n - k) \asymp z^{\alpha-1}\{a(-y) + |y_n|n^{1-1/\alpha}\}/n^{2-1/\alpha},
\]

and by (2.23)

\[
p_{(-\infty, 0)}^k(x, w) \asymp U_{ds}(x)V_{as}(w)/k^{1+1/\alpha};
\]

by \( k \geq \delta x^\alpha \) it also follows that \( x \leq (k/\delta)^{1/\alpha} \). Hence, putting

\[
m(x) = \sum_{z=0}^{x} \sum_{w=0}^{x} p(-z - w)V_{as}(w)z^{\alpha-1} \tag{5.2}
\]
we have
\[ p^n_{0}(x, y) \geq c^n m(x) U_{ds}(x) \left( \frac{a(-y)}{n^{2-1/\alpha}} + \frac{|y_n|}{n} \right) \sum_{\delta x^\alpha \leq k \leq n/2} \frac{1}{k^{1+1/\alpha}}. \]

Since \( \delta x^\alpha \leq n/2^{1+1/\alpha} \), the last sum is bounded below by a positive multiple of \( 1/x \). In the double sum in (5.2) restricting the inner summation to \( w \leq x - z \), making change of the variable \( w = j - z \) and interchanging the order of summation we obtain
\[ m(x) \geq \sum_{j=0}^{x} \sum_{z=0}^{j} p(-j) V_{as}(j - z) z^{\alpha - 1}. \] (5.3)

Now suppose \( E|\hat{Z}| < \infty \). Then \( V_{as}(w) \approx w^{\alpha - 1} \) and \( U_{ds}(x) \approx x \), and we see that \( m(x) \geq c'' \sum_{j=0}^{x} p(-j) j^{2\alpha - 1} \) and the required lower bound follows.

**Remark 5.1.** (a) Even in case \( E|\hat{Z}| = \infty \) we know that if \( \gamma = 2 - \alpha \), \( V_{as}(w) \) varies regularly of index \( \alpha - 1 \) and \( V_{as}(x)U_{ds}(x) \sim Cx^{\alpha} \) as \( x \to \infty \) with a positive constant \( C \) [13]; hence from (5.3) we have \( m(x) \geq c_1 \sum_{w=1}^{x} p(-w) V_{as}(w) w^{\alpha} \) and instead of (5.1)
\[ p^n_{0}(x, y) \geq c_2 \left( \sum_{w=1}^{x} p(-w) V_{as}(w) w^{\alpha} \right) \frac{D_n(x, -y) \vee D_n(-y, x)}{n^{2-1/\alpha}} \] (5.4)

where \( D_n(x, z) = U_{ds}(x)x^{-1}\{zn^{1-2/\alpha} + a(z)\} \) for \( x, z > 0 \).

(b) Let \( \gamma = 2 - \alpha \) and \( E|\hat{Z}| < \infty \) so that \( U_{ds}(x) \approx x \). Suppose that \( F(x) \) is regularly varying as \( x \to -\infty \) of index \( -\beta \) (necessarily \( \beta \geq \alpha \)). Then \( a(x) \approx \sum_{w=1}^{x} \sum_{z=1}^{\infty} p(-w - z)[V_{as}(z)]^2 \) ([21] Theorem 2(i), (iii)) and we deduce \( \sum_{w=1}^{x} p(-w) w^{2\alpha - 1} \approx a(x) \) so that in view of Proposition 5.2(ii) (given shortly) the lower bound (5.1) is exact.

**Lemma 5.1.** Suppose \( \gamma = 2 - \alpha \). For each \( M > 1 \) there exists a constant \( C_M \) such that
\[ (i) \quad p^n_{0}(x, y) \leq C_M \left( \frac{a^\dagger(x)}{n} + \frac{(x_n) + (x_n)_{1/\alpha}}{n^{1/\alpha}} \right) (y_n^{\alpha - 1} \wedge y_n^{1 - \alpha}) \quad (|x| \leq Mn^{1/\alpha}, y > 0), \]

and that if \( E|\hat{Z}| < \infty \),
\[ (ii) \quad p^n_{(-\infty, 0]}(x, y) \leq C_M n^{-1/\alpha} x_n [y_n^{\alpha - 1} \wedge y_n^{1 - \alpha}] \quad \text{for} \ 0 \leq x \leq Mn^{1/\alpha} \text{ and } y > 0. \]

**Proof.** Let \( |x_n| < M \). By Theorem 5 (cf. (2.20)) as before we have
\[ p^n_{0}(x, y) \approx a(-y) f^x(n) \approx [a^\dagger(x) + (x_n)_{1/\alpha}] n^{-1/\alpha} y_n^{\alpha - 1} \quad \text{for} \ 0 < y_n \leq 3M \]
and for the proof of (i) it therefore suffices to show that for some constant \( C \),
\[ p^n_{0}(x, y) \leq C[a(x) + (x_n)_{1/\alpha}] n^{-1/\alpha} y_n^{\alpha - 1} \quad \text{for} \ y_n > 3M. \] (5.5)

Putting \( R = \lceil y/2 \rceil + 1, N = \lfloor n/2 \rfloor \) we make the decomposition.
\[ p^n_{0}(x, y) = \sum_{k=1}^{N} \sum_{z \geq R} P[S_k^x = z, \sigma_{R, \infty}^x = k > \sigma_{0,1}^x] p_{0}^{n-k}(z, y) + \sum_{z < R} P[\sigma_{R, \infty}^x \wedge \sigma_{0,1}^x > N, S_N^x = z] p_{0}^{n-N}(z, y) = J_1 + J_2 \quad (\text{say}). \] (5.6)
By the bound \( p^n(w) = O(n^{-1/\alpha}) \) (valid for all \( w \in \mathbb{Z} \)) it follows that

\[
J_1 \leq C P[\sigma_{[R,\infty)}^x < \sigma_{[0]}^x]/n^{1/\alpha}.
\]  
\[ (5.7) \]

On using Lemma \[8.3\]

\[
P[\sigma_{[R,\infty)}^x < \sigma_{[0]}^x] \leq C P[\sigma_{[R)}^x < \sigma_{[0]}^x] = \frac{a^\dagger(x) + a(-R) - a(x - R)}{a(R) + a(-R)} \leq C\left[ a^\dagger(x)R^{-\alpha+1} + x_+R^{-1} \right],
\]  
\[ (5.8) \]

where Lemma \[3.1(ii)\] is applied to estimate the increment of \( a \) for the inequality (as for the equality see \[8.9\]). These together lead to

\[
J_1 \leq C'\left[ a^\dagger(x)y^{-\alpha+1}/n^{1/\alpha} + (x_n)_+ / y \right] \leq C\left[ a^\dagger(x) + (x_n)_+ n^{-1/\alpha} \right]/ny_n^{\alpha-1}.
\]

On the other hand by employing the bound \( p^n(x) \leq C n^{-1/\alpha} / |x|^\alpha \) (see Lemma \[8.1\])

\[
J_2 \leq P[\sigma_{[0]}^x > 1/n] \sup_{z \leq R} p_n^{-N}(z, y) \leq C\left[ n^f(x) \right] n^{-1/\alpha} / y^\alpha \leq C'\left[ a^\dagger(x) + (x_n)_+ n^{-1/\alpha} \right]/y^\alpha,
\]

and hence \((5.5)\) is obtained. Thus (i) has been proved.

(ii) is derived in a similar way; we define \( J_1 \) and \( J_2 \) analogously. From \( \gamma = 2 - \alpha \) we have \( \lim P[S_n^0 > 0] = 1/\alpha \) which together with \( E|\tilde{Z}| < \infty \) entails \( P[\sigma_{(-\infty, 0)}^x] \geq 1/2n] \leq C x n^{-1/\alpha} \) so that \( J_2 \leq C' n^{-1/\alpha} x n / y_n^\alpha \). For the estimation of \( J_1 \) we use, instead of \( (5.8) \),

\[
P[\sigma_{[0]}^{\infty} < \sigma_{(-\infty, 0)}^x] \leq C[V_{as}(R) - V_{as}(R - x)] / V_{as}(R)
\]

as \( R \to \infty \) uniformly for \( 0 < x < R \) (cf. \[21\] Remark 5.1) for the first relation). With this we take an average in the bound corresponding to \( (5.7) \) to see

\[
J_1 \leq C' \int_{R}^{2R} \frac{V_{as}(r) - V_{as}(r - x)}{n^\alpha V_{as}(r)} \cdot \frac{dr}{R} \leq \frac{C'\left[ \int_{2R-x}^{2R} + \int_{R-x}^{R} \right]}{n^\alpha R^{\alpha-1}} \leq \frac{C'\left[ r^{\alpha-1} \right]}{n^\alpha y},
\]

showing the bound of (ii).

In the next lemma \( \gamma \) may be any admissible constant.

**Lemma 5.2.** For each \( M > 1 \), there exists a constant \( C_M \) such that for all \( n \geq 1, x \in \mathbb{Z} \),

\[
p_{(0)}^n(x, y) \leq C_M |y|^{\alpha-1}/|x|^\alpha \quad \text{if} \quad |x_n| > 1 \quad \text{and} \quad |y_n| < M.
\]

**Proof.** We prove the bound of the lemma in the dual form which is given as

\[
p_{(0)}^n(x, y) \leq C_M |x|^{\alpha-1}/y^\alpha \quad \text{for} \quad |x_n| < M, |y_n| > 1.
\]  
\[ (5.9) \]

The proof is carried out by examining the proof of Lemma \[5.1\]. We can suppose \( y_n > 3M \) by symmetry and Theorems \[2\] and \[3\] and let \( R = \lfloor y/2 \rfloor, N = \lceil n/2 \rceil \), and \( J_1 \) and \( J_2 \) be defined as in the proof of Lemma \[5.1\]. We have shown that \( J_2 \) admits the same upper bound as required for \( p_{(0)}^n(x, y) \) in \( (5.9) \) which though presented in case \( \gamma = 2 - \alpha \) applies to the other case too. As for \( J_1 \) we first recall

\[
J_1 = \sum_{k=1}^{N} \sum_{z \geq R} P[S_k^x = z, \sigma_{[R,\infty)}^x = k > \sigma_{[0]}^x] p_{(0)}^{n-k}(z, y).
\]
The double sum with the additional restriction $|z - y| > \frac{1}{2}R$ to the inner sum is dominated by a constant multiple of

$$P[\sigma_{[R, \infty)} > \sigma_{(0)}^n] \leq C n^{1-1/\alpha} \frac{1}{y^\alpha} \leq C n^{1-1/\alpha} \frac{|x|^\alpha}{y^\alpha},$$

where we have used Lemmas 8.3 and 8.4. On writing down the probability under the double summation sign by means of transition probabilities it suffices to show that

$$\sum_{k=0}^{N-1} \sum_{w<R} \sum_{\frac{3}{2}R \leq z \leq 3R} p_k^n(x, w)p(z - w)p^n(y - z) \leq C|x|^{\alpha-1}/y^\alpha. \tag{5.10}$$

By the trivial bound $\sum_{k=0}^{N} p_k^n(x, w) \leq g_0(x, w) \leq C|x|^{\alpha-1}$ and $\sum_{w<R} p(z - w) \leq CR^{-\alpha}$ for $z > \frac{3}{2}R$ the above triple sum is bounded by

$$C'|x|^{\alpha-1}/y^\alpha \sum_{\frac{3}{2}R \leq z \leq 3R} p^n(y - z)$$

On using Lemma 8.3 again the sum above is bounded by a constant multiple of

$$\sum_{z : |y - z| \leq n^{1/\alpha}} n^{1-1/\alpha} + \sum_{z : |y - z| > n^{1/\alpha}} n^{1-1/\alpha} \leq 2n\int_{n^{1/\alpha}}^\infty u^{-\alpha} du \leq C', \tag{5.11}$$

showing (5.10) as required.

**Lemma 5.3.** Suppose $\gamma = \alpha - 2$ and define $\omega_{n, x, y}$ for $x \neq 0$ and $y > 0$ via

$$p_{(0)}^n(x, y) = a(-y)f^\gamma(n)\omega_{n, x, y}. \tag{5.11}$$

Then, $\omega_{n, x, y}$ is dominated by a constant multiple of $1 \land y_{\gamma n^{-2\alpha+1}}$ (in particular uniformly bounded), and tends to unity as $y_n \to 0$ and $n \to \infty$ uniformly for $0 < x < M n^{1/\alpha}$ for each $M > 1$.

**Proof.** The convergence of $\omega_{n, x, y}$ to zero follows from Theorems 4 and 5 (the first case) and the stated of $\omega_{n, x, y}$ is derived from Lemma 5.1(i) with a simple manipulation. □

In the sequel we use the notation $H_B^x(y) = P[S_{\sigma_B}^x = y], B \subset \mathbb{Z}$. It follows that

$$H_B^x(y) = \sum_{z \notin B} g_B(x, z)p(y - z) \quad \text{for} \quad y \in B. \tag{5.12}$$

**Proposition 5.2.** Suppose $\gamma = 2 - \alpha$. Then for some constant $C$

(i) $p_{(0)}^n(x, y) \leq C \left[ a^1(x)a(-y) \frac{n^2}{n^{1/\alpha}} + a^1(x)(|y_n| \land 1) + a(-y)(x_n \land 1) \right] \quad (x \geq 0, y \leq -1),$

(ii) $p_{(0)}^n(x, y) \leq Cn^{-1} \left[ a(x)(y_n^{\alpha-1} \land 1) + a(-y)(|x_n|^{\alpha-1} \land 1) \right] \quad (x \leq -1, y \geq 1).$

**Proof.** First we prove (ii). The proof is based on Lemmas 5.1(i) and 5.2 that entail for $k \leq n/2$

$$p_{(0)}^{n-k}(z, y) \leq C[a(z)/n + z/n^{2/\alpha}](y_n^{\alpha-1} \land y_n^{1-\alpha}) \quad (0 < z < n^{1/\alpha}, y > 0) \quad \text{and} \tag{5.13}$$

$$p_{(0)}^{n-k}(z, y) \leq Cy^{\alpha-1}/z^\alpha \quad (z > n^{1/\alpha}, 0 < y < n^{1/\alpha}), \tag{5.14}$$
respectively. Let \( x \leq -1 \) and \( y \geq 1 \) and consider the RHS of the trivial inequality

\[
P[\sigma_x^{\infty} \leq n/2, \sigma_{\{0\}}^x > n, S_n^x = y] \leq \sum_{z > 0} H_{[0,\infty)}^x(z) \sup_{k \leq n} p_{\{0\}}^{n-k}(z, y). \tag{5.15}
\]

By \( g_{[0,\infty)}(x, z) \leq g_{\{0\}}(x, x) \) it follows that \( H_{[0,\infty)}^x(z) \leq C a(x) P[X \geq z] \) (cf. (8.10)), and hence

\[
\sum_{0 < z < n^{1/\alpha}} H_{[0,\infty)}^x(z) z \leq C' a(x) \sum_{1 \leq z < n^{1/\alpha}} z^{1-\alpha} \leq C_1 a(x) n^{-2+2/\alpha}.
\]

Since \( H_{[0,\infty)}^x\{a\} = O(a(x)) \) (cf. [21]), this together with (5.13) shows

\[
\sum_{0 < z < n^{1/\alpha}} H_{[0,\infty)}^x(z) p_{\{0\}}^{n-k}(z, y) \leq C'' n^{-1} a(x)(y_n^{\alpha-1} \wedge y_n^{1-\alpha}).
\]

In a similar way

\[
\sum_{z \geq n^{1/\alpha}} H_{[0,\infty)}^x(z) z^{-\alpha} < C' a(x) n^{-2+1/\alpha} \quad \text{and} \quad \sum_{z \geq n^{1/\alpha}} H_{[0,\infty)}^x(z) < C' a(x) n^{-1+1/\alpha}
\]

and by (5.14) and the bound \( p_{\{0\}}^{n-k}(z, y) \leq C/n^{1/\alpha} \) (following the local limit theorem)

\[
\sum_{z \geq n^{1/\alpha}} H_{[0,\infty)}^x(z) \sup_{k \leq n/2} p_{\{0\}}^{n-k}(z, y) \leq C'' n^{-1} a(x)(y_n^{\alpha-1} \wedge n^{-1}) = C'' n^{-1} a(x)(y_n^{\alpha-1} \wedge 1).
\]

Thus the RHS of (5.15) is bounded by a constant multiple of \( n^{-1} a(x)(y_n^{\alpha-1} \wedge 1) \).

Let \( \hat{S}^x \) and \( \hat{\sigma}_{\{0\}}^x \) denote the dual walk and its hitting time, respectively. It then follows that

\[
p_{\{0\}}^y(x, y) - P[\sigma_x^{\infty} \leq n/2, \sigma_{\{0\}}^x > n, S_n^x = y] \\
\leq P[\hat{\sigma}_{\{\infty,0\}}^y \leq n/2, \hat{\sigma}_{\{0\}}^y > n, \hat{S}_n^y = x]. \tag{5.16}
\]

By duality relation the probability on the RHS is the same as what we have just estimated but with \( x \) and \( y \) replaced by \( -y \) and \( -x \), respectively, and hence dominated by a constant multiple of \( n^{-1} a(-y)(|x_n|^{-\alpha} \wedge 1) \). This concludes (ii).

For the proof of (i) we apply (5.14) with \( z, y \) replaced by \( -y, -z \), in which we may replace \((-z_n)^{\alpha-1} \wedge (-z_n)^{1-\alpha}\) by \( a(z)/n^{1-1/\alpha} \) \((z < 0)\) to obtain

\[
p_{\{0\}}^{n-k}(z, y) = p_{\{0\}}^{n-k}(-y, -z) \leq C a(z) \frac{a(-y) + (|y_n| \wedge 1)n^{1-1/\alpha}}{n^{2-1/\alpha}} \quad (z < 0, k \leq n/2). \tag{5.17}
\]

This is valid at least for all \(-n^{1/\alpha} < y < 0\) and can be extended to \( y \leq -n^{1/\alpha} \). For the proof of the extension we have only to observe that if \( y \leq -n^{1/\alpha} \), then the RHS is not less than \( c/n^{1/\alpha} \) if \( z < -n^{1/\alpha} \) with some \( c > 0 \) while for \( z \leq -n^{1/\alpha} \), (5.17) follows from Lemma 5.2 (note that \( 1/y^\alpha = O(1/n) \)). Since \( E[S_{\{1,\infty\}}] = \infty \) we have \( H_{\{\infty,0\}}^x\{a\} = a(x) \) (cf. [21] or (6.19)), and from (5.17) we deduce

\[
P[\sigma_{\{\infty,0\}}^x \leq n/2, \sigma_{\{0\}}^x > n, S_n^x = y] \leq \sum_{z < 0} H_{\{\infty,0\}}^x(z) \sup_{k \leq n/2} p_{\{0\}}^{n-k}(z, y) \\
\leq C a(x) \frac{a(-y) + (|y_n| \wedge 1)n^{1-1/\alpha}}{n^{2-1/\alpha}}.
\]

By the analogue of (5.16) we conclude (i) by duality relation as above.

\[\square\]

The next lemma concerns the hitting distribution of the negative half line defined by

\[h^x(n, y) = P[\sigma_{\{\infty,0\}}^x = n, S_n^x = y].\]
Lemma 5.4. Suppose $E|\hat{Z}| < \infty$. Then,
(i) for $M > 1$ and $\varepsilon > 0$, uniformly for $0 \leq x_n < M$ and $y \leq 0$

$$h^x(n, y) = \frac{x_n p_{c_\varepsilon}(-x_n)}{n} \left[ H^\infty_{(-\infty,0]}(y)\{1 + o_\varepsilon(1)\} + r(n, y) \right]$$

where $o_\varepsilon(1)$ is bounded and tend to zero as $n \to \infty$ and $\varepsilon \to 0$ in this order and

$$|r(n, y)| \leq C_M \sum_{z > \varepsilon n^{1/\alpha}} z^{\alpha - 1} p(y - z)$$

for a constant $C_M$ depending only on $M$ and $F$; and

(ii) there exists a constant $C$ such that for all $x \geq 1, y < 0$ and $n \geq 1$,

$$h^x(n, y) \leq C(n^{-1} \wedge x^{-\alpha})x_n H^\infty_{(-\infty,0]}(y).$$

Proof. Let $\varepsilon > 0$ and in the expression

$$h^x(n + 1, y) = \sum_{z = 1}^{\infty} p_{-\infty,0}^n(x, z)p(y - z)$$

(5.18)

we divide the sum into two parts, the sum on $z < \varepsilon n^{1/\alpha}$ and the remainder which are denoted by $\Sigma_{< \varepsilon n^{1/\alpha}}$ and $\Sigma_{\geq \varepsilon n^{1/\alpha}}$, respectively. By Doney’s result (2.23) and (2.32) it follows that

$$p_{-\infty,0}^n(x, z) = (E|\hat{Z}|)^{-1} V_{as}(z) x_n p_{c_\varepsilon}(-x_n) n^{-1}\{1 + o_\varepsilon(1)\}$$

uniformly for $z \leq \varepsilon n^{1/\alpha}$,

(note $L(n^{1/\alpha}) \to 1/E|\hat{Z}|$ in (2.32)) and substituting this and using

$$\frac{1}{E|\hat{Z}|} \sum_{z = 1}^{\infty} V_{as}(z - 1)p(y - z) = H^+_{(-\infty,0]}(y) \quad (y \leq 0),$$

(5.19)

(cf. [21] Eq(2.7)) we deduce that

$$\Sigma_{< \varepsilon n^{1/\alpha}} = \frac{x_n p_{c_\varepsilon}(-x_n)}{n} \left[ H^\infty_{(-\infty,0]}(y) - \frac{1}{E|\hat{Z}|} \sum_{z > \varepsilon n^{1/\alpha}} V_{as}(z)p(y - z) \right] \{1 + o_\varepsilon(1)\}.$$ 

By Lemma 5.1(ii)

$$\Sigma_{\geq \varepsilon n^{1/\alpha}} \leq C \frac{x_n}{n} \sum_{z > \varepsilon n^{1/\alpha}} z^{\alpha - 1} p(y - z),$$

and on noting $V_{as}(z) \simeq z^{\alpha - 1}$ the assertion (i) follows. It in particular follows that

$$h^x(n, y) \leq C n^{-1} x_n H^\infty_{(-\infty,0]}(y) \quad \text{for} \quad 0 \leq x \leq M n^{1/\alpha}, y \leq 0,$$

(5.20)

since $|r(n, y)| \leq C'M H^+_{(-\infty,0]}(y)$ in view of (5.19).

For the proof of (ii) we replace $p_{-\infty,0}^n(x, z)$ by $p_{10}^n(x, z)$ in (5.18) to have an upper bound and verify that for $y \leq 0$ and $x \geq 2n^{1/\alpha}$,

$$\sum_{z = 1}^{\infty} p_{10}^n(x, z)p(y - z) \leq C x^{2\alpha} H^+_{(-\infty,0]}(y) + \frac{C}{n^{1/\alpha}} F(y - \frac{1}{2}x).$$

(5.21)
For verification of (5.21) we break the range of summation into three parts $0 < z \leq n^{1/\alpha}$, $n^{1/\alpha} < z \leq x/2$ and $z > x/2$, and denote the corresponding sums by $J_1$, $J_2$ and $J_3$, respectively. It is immediate from Lemma 5.2 and (5.19) that $J_1 \leq C x^{-\alpha} H_{(-\infty,0)}(y)$. By the bound $p_n^n(z) \leq C n^{1-1/\alpha} / |z|^\alpha$ (cf. Lemma 8.1) it follows that $p_{(0)}^n(x, z) \leq C n^{1-1/\alpha} x^{-\alpha}$ for $n^{1/\alpha} < z \leq x/2$, which combined with the bound

$$\sum_{z > n^{1/\alpha}} p(y - z) \leq \sum_{z = 1}^{\infty} \left( \frac{z}{n^{1/\alpha}} \right)^{\alpha-1} p(y - z) \leq \frac{2 E [\hat{Z}]}{n^{1-1/\alpha}} \left[ \sup_{z \geq 1} \frac{z^{\alpha-1}}{V_{as}(z)} \right] H_{(-\infty,0)}^{+\infty}(y) \quad (5.22)$$

yields $J_2 = C' x^{-\alpha} H_{(-\infty,0)}^{+\infty}(y)$. Finally $J_3 \leq C n^{-1/\alpha} F(y - x/2)$. These estimates together verify (5.21). As in (5.22) we derive $F(y - \frac{1}{2} x) \leq C_1 H_{(-\infty,0)}^{+\infty}(y) / x^{\alpha-1}$. Hence

$$h^x(n, y) \leq C_1 H_{(-\infty,0)}^{+\infty}(y) / x^{\alpha-1} n^{1/\alpha} \quad \text{for} \quad x > 2 n^{1/\alpha}, y < 0, n \geq 1,$$

which combined with (5.20) shows the bound in (ii). The proof of Lemma 5.4 is complete. \qed

**Proof of Theorem 6.** If either $x$ or $y$ remains in a bounded set, the formula (i) of Theorem 6 agrees with that of Theorem 5, so that we may and do suppose both $x$ and $-y$ tend to infinity. Note that the second ratio on the RHS of (i) is then asymptotically equivalent to the ratio in (ii), hence (i) and (ii) of Theorem 6 is written as a single formula. Let $c_0 = 1$ for simplicity and put

$$\Phi(t; \xi) = t^{-1} \xi p_t(-\xi).$$

Then what is to be shown may be stated as follows: as $n \to \infty$ and $x \vee (-y) \to \infty$

$$p_{(0)}^n(x, y) \sim k_{\alpha, \gamma} a(x) a(-y) n^{-2 + 1/\alpha} + C^+ \Phi(n; x - y) \quad (5.23)$$

uniformly for $-M < y_n < 0 < x_n < M$, provided $0 < C^+ = \lim_{z \to -\infty} a(z) < \infty$.

We follow the proof in [17] to the corresponding result. We employ the representation

$$p_{(0)}^n(x, y) = \sum_{k=1}^{n} \sum_{z \leq 0} h^x(k, z) p_{(0)}^{n-k}(z, y). \quad (5.24)$$

Break the RHS into three parts by partitioning the range of the first summation as follows

$$1 \leq k < \varepsilon n; \quad \varepsilon n \leq k \leq (1 - \varepsilon) n; \quad (1 - \varepsilon) n < k \leq n \quad (5.25)$$

and call the corresponding sums $I$, $II$ and $III$, respectively. Here $\varepsilon$ is a positive constant that will be chosen small. The proof is divided into two cases corresponding to (i) and (ii).

*Case $x_n \wedge y_n \to 0$: By duality one may suppose that $x_n \to 0$. From $EZ = \infty$ and (2.21) it follows [21] Theorems 1 and 2] that

$$\sum_{z \leq 0} H_{(-\infty,0)}^{x}(z) a(z) = a(x) \quad \text{and} \quad C^+ = \sum_{z \leq 0} H_{(-\infty,0)}^{+\infty}(z) a(z) < \infty, \quad (5.26)$$

respectively. From the latter bound above and Lemma 5.4 (or (5.20)) one deduces,

$$(*) \quad \sum_{k \geq \varepsilon n} \sum_{z < 0} h^x(k, z) a(z) \leq C \varepsilon x_n \quad (5.27)$$
with a constant $C_\varepsilon$ depending on $\varepsilon$. As the dual of (5.11) of Lemma 5.3 we have

$$p_{n(0)}^n(z, y) = a(z) f^{-y}(n)\{1 + r_{n,z,y}\} \quad (z < 0, -M n^{1/\alpha} < y < 0) \quad (5.28)$$

where $r_{n,z,y}$ is uniformly bounded and tends to zero as $z/n^{1/\alpha} \to 0$ and $n \to \infty$ uniformly for $y$, which together with (5.27) shows

$$II \leq C_{\varepsilon,M} x_n f^{-y}(n).$$

Similarly on using (5.28) above

$$I = \sum_{1 \leq k < \varepsilon n} \sum_{z = -\infty}^{-1} h^x(k, z) a(z) f^{-y}(n - k)\{1 + r_{n-k,z,y}\}.$$  

For the evaluation of the last double sum we may replace $f^{-y}(n - k)$ by $f^{-y}(n)(1 + O(\varepsilon))$, and the contribution to it of $r_{n-k,z,y}$ is negligible since $\sum_{z > N} H_{-\infty,0}^\varepsilon(x) a(z) \to 0 (N \to \infty)$ uniformly in $x$ in view of the second relation of (5.20). By (5.27) the summation over $z$ may be extended to the whole half line $k \geq 1$. Now applying the first relation of (5.26) we find

$$I = a(x) f^{-y}(n)\{1 + O(\varepsilon) + o(1)\}.$$  

As for $III$ first observe that by (5.28) and Theorem 3

$$\sum_{k=1}^{\varepsilon n} p_{k(0)}^n(z, y) = g(z, y) - r_n \leq C(a(z) \land a(y)) \quad 0 \leq r_n \leq C_\varepsilon a(z)f^{-y}(n)n$$

(y, z < 0). If $y_n$ is bounded away from zero so that $x/y \to 0$, then $III = O(x_n/n) = o(y_n/n)$. On the other hand, applying Lemma 5.4 we see that if $y_n \to 0$,

$$III = x_n p_1(x_n)n^{-1} \sum_{z < 0} H_{-\infty,0}^\varepsilon(z, y) g(z, y)(1 + O(\varepsilon)) + O(x_nf^{-y}(n)),$$

whereas by (5.26) and the subadditivity of $a$ we infer that $\sum_{z \leq 0} H_{-\infty,0}^\varepsilon(z, y) g(z, y) \to C^+$ as $y \to \infty$. Hence

$$III = x_n p_1(x_n)n^{-1}(C^+ + o(1) + O(\varepsilon)) + O(x_nf^{-y}(n)).$$

Adding these expressions of $I$, $II$ and $III$ yields the desired formula, because of arbitrariness of $\varepsilon$ as well as the identity $x_n p_1(x_n)/n = \Phi(n; x)$.

CASE $x_n \land (-y_n) \geq 1/M$. By Lemma 5.4(ii) and (5.28) it follows that in this regime

$$I \leq \sum_{1 \leq k < \varepsilon n} \frac{C}{k^{1/\alpha} x^{\alpha-1}} \sum_{z < 0} H_{-\infty,0}^\varepsilon(z) a(z)f^{-y}(n) \leq C' \varepsilon^{-1/\alpha} n.$$  

For evaluation of $III$ change the variable $k$ into $n-k$ and apply Lemma 5.2 to $p_{k(0)}^k(-y, -z)$ (with $(-y, z)$ in place of $(x, y)$) to see that for any $\delta > 0$

$$\sum_{k=1}^{\varepsilon n} p_{k(0)}^k(z, y) \leq C \sum_{k \leq \delta \varepsilon n} k^{-1/\alpha} + C_\delta \sum_{\delta \varepsilon n < k < \varepsilon n} |z|^{\alpha-1}/|y|^\alpha$$

$$\leq C(\delta |z|^{\alpha})^{1-1/\alpha} + C_\delta (\varepsilon n)|z|^{\alpha-1}/|y|^\alpha,$$
where \( C_\delta \) may depend on \( \delta \) but \( C \) does not. Then by Lemma 5.4(ii)

\[
III \leq C' n^{-1}[C\delta^{1-1/\alpha} + C_\delta \varepsilon] \sum_{z < 0} H_{(-\infty,0]}^{+}(z)|z|^{\alpha-1} \leq C''[C\delta^{1-1/\alpha} + C_\delta \varepsilon]/n,
\]
hence for any \( \varepsilon' > 0 \) we can choose \( \varepsilon > 0 \) and \( \delta > 0 \) so that \( III \leq \varepsilon'/n \).

By Lemma 5.4(i), (5.28) and (5.26)

\[
II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k p_1(-x_k)}{k} \sum_{z = -x}^{-1} H_{(-\infty,0]}^{+}(z)p_{0}^{n-k}(z,y)(1 + o_{\varepsilon}(1)) + o\left(\frac{f-y(n)}{n^{1/\alpha}}\right).
\]

Here (and in the rest of the proof) the estimate indicated by \( o_{\varepsilon} \) may depend on \( \varepsilon \) but is uniform in the passage to the limit under consideration once \( \varepsilon \) is fixed.

Since \(-y_n\) is bounded away from zero as well as from infinity, we may replace \( p_{0}^{n-k}(z,y) \) by \( a(z)y_{n-k}p_1(y_{n-k})/(n-k) \) to see that

\[
II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k |y_{n-k}|p_1(-x_k)p_1(y_{n-k})}{k(n-k)} \sum_{z = -x}^{-1} H_{(-\infty,0]}^{+}(z)a(z)(1 + o_{\varepsilon}(1)) + \frac{o(1/n)}{n^{1/\alpha}}.
\]

noting the identity \( x_k p_1(-x_k) = x_n p_{k/n}(-x_n) = \Phi(k/n; x_n) k/n \) and similarly for \( y_{n-k}p_1(y_{n-k}) \) and

\[
\sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k |y_{n-k}|p_1(-x_k)p_1(y_{n-k})}{k(n-k)} = \frac{1 + o(1)}{n} \int_{0}^{1} \Phi(t; x_n)\Phi(1-t; y_n)dt + O\left(\frac{\varepsilon}{n}\right).
\]

Here we have used the fact that \( \int_{0}^{1} p_1(\xi)dt/t = \int_{\xi^{1/\alpha}}^{\infty} = O(\xi^{1/\alpha}) \). Since for \( \xi > 0 \), \( \Phi(dt; \xi)dt \) is the distribution of the hitting-time to zero by \( \xi + Y \), we have

\[
\int_{0}^{1} \Phi(t; x_n)\Phi(1-t; -y_n)dt = \Phi(1; x_n - y_n).
\]

Hence

\[
II = \frac{1}{n} C^+ \Phi(1; x_n - y_n)\{1 + o(1)\} + \frac{o(1/n)}{n^{1/\alpha}}.
\]

(5.29)

(as well as \( nI + nIII \to 0 \)) as \( n \to \infty \) and \( \varepsilon \to 0 \) in this order. Thus (5.23) is obtained, the first term on the RHS of it being negligible.

**Proof of Proposition 2.2**. The case \( C^+ = 0 \) is trivial. If \( 0 < C^+ < \infty \), by noting that Theorem 6 and Lemma 5.1(i) (in the dual form (5.17)) together yield

\[
\frac{p_{0}^{n-k}(z,y)}{p_{0}^{n}(x,y)} \leq C \leq a(z)[1 + |y|n^{1-2/n}] \leq Ca(z) (z < 0, k < n/2)
\]

and that \( H_{(-\infty,0]}^{+}(z) \leq CH_{(-\infty,0]}^{\infty}(z) \), we deduce that \( \sum_{k < n/2} \sum_{z < -R} h^x(k,z)p_{0}^{n-k}(z,y) \) is at most a constant multiple of \( \sum_{z < -R} H_{(-\infty,0]}^{\infty}(z)a(z) \) which approaches zero as \( R \to \infty \); for the sum over \( n/2 \leq k \leq n \), one uses the bound \( \sum_{n/2 \leq k \leq n} p_{0}^{n-k}(z,y) \leq Ca(z) \) as well as Lemma 5.4(ii) to obtain the same bound in a similar way. This verifies the first half of the asserted formula.
The second half obviously follows if $E|\hat{Z}| = \infty$ so that $H^{\infty}_{(-\infty,0]}$ vanishes. Let $E|\hat{Z}| < \infty$. Then we can apply Lemma 5.4(ii) as well as Theorem 3 (in the dual form (2.16)) to see that the contribution to the sum (5.24) from $-R \leq z \leq 0$ is dominated by a positive multiple of

\[
\sum_{-R \leq z \leq 0} \sup_{k<n/2} \left[ H^z_{(-\infty,0]}(z)p^n_{\{0\}}(z,y) + h^z(k,z)g_{\{0\}}(z,y) \right] \leq CR^{\alpha-1} \left[ a(-y) \frac{x}{n^{2-1/\alpha}} + x \vee |y| \right],
\]

which is negligible as compared with the lower bound of $p^n_{\{0\}}(x,y)$ given by Proposition 5.1 provided that $C^+=\infty$ or, equivalently, $\sum_{w\geq1} w^{2\alpha-1}p(-w) = \infty$. \hfill \Box

**Proof of Proposition 2.1.** In case $|x_n| \leq 3$ the assertion follows from Theorems 2 and 3. We let $x_n > 3$, the case $x_n < -3$ being treated in the same way. In the obvious identity

\[
f^x(n) = \sum_{y} p^n_{\{0\}}(x,y)p(-y)
\]

the sum on the RHS over $|y| \leq n^{1/\alpha}$ is bounded by a constant multiple of $x^{-\alpha}$ by virtue of Lemma 5.2. Since $p^n_{\{0\}}(x,y) \leq p^n(y-x)$, it suffices to show that

\[
\sum_{|y|>n^{1/\alpha}} p^n(y-x)p(-y) \leq Cn^{-1/\alpha}x^{-\alpha}.
\]

We break the sum into three parts by splitting the range of summation at $y = x \pm n^{1/\alpha}$ and denote them by $\Sigma_{|x-y|<n^{1/\alpha}}$, $\Sigma_{n^{1/\alpha}<y\leq x-n^{1/\alpha}}$ and $\Sigma_{y\geq x+n^{1/\alpha}}$. The first sum is estimated as follows:

\[
\Sigma_{|x-y|<n^{1/\alpha}} \leq Cn^{-1/\alpha} \sum_{|y-x|<n^{1/\alpha}} p(y) = n^{-1/\alpha}x^{-\alpha} \times o(1).
\]

For the second sum we further split its range of summation at $y = x/2$ and apply Lemma 5.1 to see that $\Sigma_{n^{1/\alpha}<y\leq x-n^{1/\alpha}}$ is at most a constant multiple of

\[
\sum_{n^{1/\alpha}<y<x/2} p(-y) \frac{n^{1-1/\alpha}}{x^{\alpha}} + \sum_{x/2<y\leq x-n^{1/\alpha}} p(-y) \frac{n^{1-1/\alpha}}{(x-y)^{\alpha}} \leq Cn^{-1/\alpha}.
\]

The third sum is evaluated to be $o(1/x^{\alpha})$ in a similar way. Thus (5.31) and hence Proposition 2.1 has been verified. \hfill \Box

### 6 Extension to an arbitrary finite set

Let $A$ be a finite non-empty subset of $\mathbb{Z}$. The function $u_A(x), x \in \mathbb{Z}$ defined in (2.35) may be given by

\[
u_A(x) = g_A(x,y) + a(x-y) - E_x[a(S_{\sigma(A)} - y)]
\]

(whether (2.33) is assumed or not), the RHS being independent of $y \in \mathbb{Z}$ (cf. [19, Lemma 3.1], [14]) and the difference of the last two terms in it tending to zero as $|y| \to \infty$. Taking an arbitrary $w_0 \in A$ for $y$ it in particular follows that

\[
u_A(x) = a^t(x-w_0) - E[a(S_{\sigma(A)}^x - w_0)]
\]

so that $u_A(x) \sim a(x)$ as $x \to \pm \infty$ if $a(x) \to \infty$ as $x \to \pm \infty$. The function $u_A$ is harmonic for the semi-group $p^n_A$ as noted previously, and $u_A(S_n^x)1(n < \sigma_A^x)$ is accordingly a martingale.
for each \(x \in \mathbb{Z}\). Put \(f^x_A(n) = P[\sigma_A = n]\). We state only the extensions corresponding to those given in Theorem 3 (restricted to the case \(\gamma = 2 - \alpha\)) and Theorem 5. In the following theorem we include the case of periodic walks (i.e., the condition 2) stated in Section 1 may be violated). What is stated about (6.1) also holds for the periodic walk.

**Theorem 7.** Let \(\nu \geq 1\) denote the period of the walk, which amount to assume (in addition to (6.7)) that \(p^{\nu n}_{\nu}(0) > 0\) and \(p^{\nu n+j}_{\nu}(0) = 0\) (\(1 \leq j < \nu\)) for all sufficiently large \(n\). Let \(\gamma = 2 - \alpha\) and \(M\) be any number greater than 1. Then,

(i) for \(x\) with \(P[S^x_n \in A] > 0\), as \(n \to \infty\)

\[
f^x_A(n) \sim \begin{cases} 
  u_A(x)f^x_A(n) + \nu x_n p_{\nu}(0) & (0 \leq x_n < M), \\
  u_A(x)f^x_A(n) & (x < 0, x_n \to 0), \\
  \nu c_0 \nu^{x} & (-1/M \leq x_n < -M)
\end{cases}
\]

and

\[
f^0_A(n) \sim f^0(\lfloor n/\nu \rfloor) \sim \nu_{\alpha, \gamma} c_0^{1/\nu} n^{2-1/\nu};
\]

(ii) uniformly for \(|x| < Mn^{1/\nu}\) and \(-M < y < Mn^{1/\nu}\) with \(u_A(-y) > 0\), \(p^0(y-x) > 0\), as \(n \to \infty\)

\[
p^0_A(x,y) \sim \begin{cases} 
  f^x_A(n)u_A(-y) & (y_n \to 0), \\
  u_A(x)f^y_A(n) + \nu (x_n + K_{\nu}(y_n)) n^{1/\nu} & (x_n \to 0), \\
  \nu p^0_{\nu}(x,y) & (|x_n| \wedge y_n \geq 1/M).
\end{cases}
\]

**Remark 6.1.** (a) If (2.33) is violated, then \(u_A(-y) = 0\) for \(y \leq \min A\) and (6.5) says nothing about \(p^x_A(x,y)\) which is positive for \(x \leq \min A\) and whose asymptotic form is found in the dual version of (6.5) deduced by using \(p^0_A(x,y) = p^0_A(y-x)\).

(b) The results for the periodic walks are derived from those of the aperiodic ones. The process \(\tilde{S}_n = S_{\nu n}/\nu\), \(n = 0, 1, \ldots\) is a strongly aperiodic walk on \(\mathbb{Z}\) such that \(1 - E[e^{i\theta \tilde{S}_n}] \sim \nu^{1-\alpha}[1 - \phi(\theta)]\); hence in case \(A = \{0\}\), the results restricted on \(\nu \mathbb{Z}\) follow immediately from those of the aperiodic walks and the extension to \(\mathbb{Z}\) is then readily performed by using \(E[a(S^x_n)] = a^{\dagger}(x)\). The general case is reduced to the case \(A = \{0\}\) in the same way as for aperiodic walks as is described below.

The basic idea of proof is the same as in [19], the details are rather simpler because of uniqueness of positive harmonic function for the killed walk. In the sequel we may and do assume \(\nu = 1\) (see Remark 6.1(b)).

Take an integer \(R > M\) and let \(\tau^*_R\) be the first exit time of \(S^x\) from the interval \((-R, R)\):

\[
\tau^*_R = \sigma^x_{(-\infty, -R \cup R, \infty)} = \inf\{n \geq 1 : |S^x_n| \geq R\}.
\]

Put \(N = mR^m [(1 + \log n)]\) with a positive integer \(m\) determined shortly and decompose

\[
p^x_A(x,y) = \sum_{k=1}^{N-1} \sum_{|z| \geq R} P[\tau^*_R = k < \sigma^x_A, S^x_k = z]p^x_{A}^{n-k}(z,y) \tag{6.6}
\]

provided that \(1 < N < n/2\). Here

\[
\varepsilon(x,y; R) = \sum_{z} P[\tau^*_R \wedge \sigma^x_A \geq N, S^x_N = z]p^x_{A}^{n-N}(z,y). \tag{6.7}
\]
Using the fact that the process $Y^n_t := S_{nt}/n^{1/\alpha}$ converges to a stable process we deduce that there exists a constant $\lambda > 0$ such that $\sup_{x, y} P_x[\tau^x_R \geq R^\alpha] < e^{-\lambda}$ for all sufficiently large $R$, by which we deduce (cf. [9, (XI.3.14)]) that for all sufficiently large $k$

$$P[\tau^x_R > k] \leq e^{-\lambda k/R^\alpha}. \quad (6.8)$$

Hence

$$\varepsilon(x, y; R) \leq CE^{-\lambda N/R^\alpha}/n^{1/\alpha} = O(n^{-\lambda n/n^{1/\alpha}}), \quad (6.9)$$

so that $\varepsilon(x, y; R)$ is negligible if $m > 2/\lambda$ and our task reduces to the evaluation of the double sum in (6.6) with an appropriate choice of $R = R_n$.

It is easily seen that at least within $|x_n| \vee |y_n| < M$

$$0 \leq p^n_{[0]}(x, y) - p^n_A(x, y) \leq C \sup_{k \leq n/2} [p^n_{[0]}(0, y) + f^x_A(n - k)]$$

$$\leq C'[f^{-y}(n) + f^x(n)] \quad (6.10)$$

(cf. the proof of [19, Lemma 5.1] for the first inequality and Theorem 5 for the second). If $C^+ = \lim_{x \to \infty} a(x) = \infty$, it therefore follows that within $|x| \vee |y| < Mn^{1/\alpha}$

$$p^n_{[0]}(x, y) \sim p^n_A(x, y) \quad \text{as} \quad |x| \wedge |y| \wedge n \to \infty \quad (6.11)$$

and hence both (6.3) and (6.5) hold for $\gamma = 2 - \alpha$ if $|x| \wedge y \to \infty$ in view of Theorems 1 through 5. In the sequel we suppose $\gamma = 2 - \alpha$ (entailing $u_A(-y) \sim y^{\alpha - 1}/\Gamma(\alpha)$; the other case being similarly dealt with) and verify that the restriction $|x| \wedge y \to \infty$ can be removed in the above. In case $C^+ < \infty$ the situation is not much different and rather simpler. At the end of the section we shall advance certain remarks about the extension of Theorem 6.

In the sequel we shall tacitly suppose $|x| \vee |y| = O(n^{1/\alpha})$.

Let $C^+ = \infty$ so that (6.11) holds. First of all we observe that in the case $x_n \to 0$ of (2.14) the second term on its RHS is negligible relative to the first so that $p^n_{[0]}(x, y) \sim a^\dagger(x)f^{-y}(n)$, if $|x| = o(n^{2/\alpha - 1})$ (since $f^0(n) \asymp 1/n^{2-1/\alpha}$). This together with (6.11) and (2.2) implies that as $|x| \wedge y \to \infty$

$$p^n_A(x, y) \sim \begin{cases} u_A(x)f^0(n)u_A(-y) & (y_n \to 0) \\ u_A(x)c_\alpha f^{-y_n}(c_\alpha)/n & (1/M \leq y_n < M) \end{cases}$$

under $x = o(n^{2/\alpha - 1}), \quad (6.12)$

and, in view of duality, for the proof of (6.3) it suffices to show that (6.12) remains true for each $x$ fixed. To this end we prove two lemmas, Lemmas 6.1 and 6.2, the proof of (6.12) will be given after that of Lemma 6.2.

**Lemma 6.1.** (i) For $|x| < R$

$$u_A(x) = E[u_A(S^x_{\tau^x_R}; \tau^x_R < \sigma_A)]. \quad (6.13)$$

(ii) Let $\gamma = 2 - \alpha$ and $b > 1$. Then uniformly for $|x| < R$, as $R \to \infty$

$$E[u_A(S^x_{\tau^x_R}; \tau^x_R = \sigma^x_{(-bR,-R)} < \sigma^x_A] = u_A(x) + \{a^\dagger(x) + a(-x) + x/R^{2-\alpha}\} \times o(1).$$

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Lemma 6.2. The process \( m = u_A(S^n_x) \mathbf{1}(n < \sigma^x_A) \) is a martingale for each \( x \in \mathbb{Z} \). Noting that \( m_n = u_A(S^n_x) \mathbf{1}(S^n_x \notin A) \) for \( n \geq 1 \) and using the optional stopping theorem we see

\[
u_A(x) = E[u_A(S^n_x); n < \sigma^x_A \wedge \tau^x_R] + E[u_A(S^n_{\tau^x_R}); \tau^x_R < n \wedge \sigma^x_A].
\]

The first expectation approaches zero as \( n \to \infty \) since \( u_A \) is bounded on \((-R, R)\) so that the monotone convergence shows (6.13). Turning to the proof of (ii) let \( \gamma = 2 - \alpha \) and \( B(R) = (-\infty, -R] \cup A \cup [R, \infty) \). We may suppose \( 0 \in A \) for simplicity. Putting \( \alpha(x) := [\alpha^x(x) + a(-x)]/2 \) so that \( g_B(x, \cdot) \leq g_B(x, x) = 2\alpha(x) \) we see

\[
P[S^n_{\tau^x_R} \leq z, \tau^x_R < \sigma^x_A] = \sum_{w \notin B(R)} g_B(x, w) F(z - w)
\]

and making summation by parts we deduce that for any \( b > 1 \),

\[
E[u_A(S^n_{\tau^x_R}); \tau^x_R = \sigma^x_{[-\infty,-bR]} < \sigma^x_A] \\
\leq C \sum_{z \leq -bR} |\gamma|^{\alpha - 1} P[S^n_{\tau^x_R} = z, \tau^x_R < \sigma^x_A] \\
\leq C' \sum_{z \leq -bR} (bR) o(\alpha) F(-bR + R) + \sum_{z \leq -bR} |\gamma|^{\alpha - 2} F(z + R),
\]

of which the last member divided by \( \alpha(x) \) tends to zero since \( F(z) = o(|z|^{-\alpha}) \) as \( z \to -\infty \). By virtually the same way we derive a bound analogous to (6.14) and make summation by parts again to obtain

\[
E[u_A(S^n_{\tau^x_R}); \tau^x_R = \sigma^x_{[R, \infty)}] < \sigma^x_A] \leq C \sum_{z \geq bR} \gamma z^{\alpha - 1} P[S^n_{\tau^x_R} = z, \tau^x_R < \sigma^x_A] \times o(1) \leq \alpha(x) \times o(1),
\]

where we have \( o(1) \) since \( u_A(z) = o(z^{\alpha - 1}) \) as \( z \to \infty \). It holds that for \( |x| < R \),

\[
P[\tau^x_R = \sigma^x_{[R, \infty)} < \sigma^x_A] \leq P[\tau^x_R = \sigma^x_{[R, \infty)} < \sigma^x_A] \leq C' \alpha(x)/R^{\alpha - 1} + x/R\}
\]

(see Lemma 6.4), which together with \( u_A(z) = o(z^{\alpha - 1}) \) shows

\[
E[u_A(S^n_{\tau^x_R}); \tau^x_R = \sigma^x_{[R, \infty)}] < \sigma_A = (\alpha(x) + x + R^{\alpha - 2}) \times o(1)
\]

uniformly for \( |x| < R \). Now the assertion of Lemma 6.1 is easy to verify. \( \square \)

Lemma 6.2. Suppose \( C^+ = \infty \). For each \( x \), as \( |y| \wedge n \to \infty \) under \( |y| < M_n^{1/\alpha} \)

\[
p_A^n(x, y) \sim u_A(x) f^{-y}(n).
\]

Proof. In (6.6) take \( R = R_n \sim n^{2/\alpha - 1}/\log n \). Then, by virtue of (6.10) and Corollary 2 uniformly for \(-2R < z < -R, k \leq N \) and \( |y| \leq M_n^{1/\alpha} \) as \( n \wedge |y| \to \infty \)

\[
p_A^n(z, y) \sim p_0^n(z, y) \sim a(z) f^{-y}(n).
\]

We can replace \( a(z) \) by \( u_A(z) \) in the right-most member for obvious reason. Then by the exponential bound (6.8) and (6.9)

\[
p_A^n(x, y) = E[p_A^{n-\tau^x_R}(S^n_{\tau^x_R}); \tau^x_R < \sigma^x_A \wedge N] + o(1/n^{2-1/\alpha}) \\
\geq E[u_A(S^n_{\sigma_{[R, \infty)}})], \tau^x_R = \sigma^x_{[R, \infty)} < \sigma^x_A \wedge N] f^{-y}(n) \{1 + o(1)\}
\]

\[
= u_A(x) f^{-y}(n) \{1 + o(1)\},
\]

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where the last equality follows from Lemma 6.1. Comparing this applied with \(A = \{0\}\) and the formula of Corollary 2 we see that the inequality sign above must be replaced by the equality sign for \(A = \{0\}\), and hence
\[
E[p_{[0]}^{n-\tau} (S_{\tau_R}, y); \tau_R^x \neq \sigma_{(-2R,-R]}^x; \tau_R^x < \sigma_{\{0\}}^x \land N] = o(f^{-y}(n)),
\]
(6.16)
which shows that the same replacement of the inequality sign is valid for \(A\) itself, concluding the asserted relation.

Note that the case \(|x| \to \infty\) and \(|y| < M\) is included in Lemma 6.2 by duality relation so that for each \(y\) with \(u_{-A}(-y) \neq 0\),
\[
p^n_{A}(x,y) = p^n_{-A}(-y, -x) \sim f^x(n)u_{-A}(-y).
\]
(6.17)
It remains to deal with the case \(|x| \land |y| < M\), but now having (6.17) available we may replace the RHS of (6.15) by \(f^x(n)u_{-A}(-y) \sim u_A(z)f^0(n)u_{-A}(-y)\) for \(|y| < M\) and repeating the same argument made after (6.15) leads to the required relation. Thus we have shown the formula for \(p^n_A(x,y)\) of Theorem 7 in case \(C^+ = \infty\). That for \(f^n_A(n)\) follows from it in view of the expression of \(P[\sigma_{A}^x = n, S_{\tau_R}^x = y]\) given in (2.38) with the help of \(
\sum_{z \notin A} u_{-A}(-z)p(y-z) = u_{-A}(-y) \land \sum_{y \in A} u_{-A}(-y) = 1.
\)

Case \(C_+ < \infty\). Recalling (6.10), namely \(0 \leq p_{[0]}(x,y) - p^n_{A}(x,y) \leq C[f^{-y}(n) + f^x(n)]\) we infer from Theorem 4 (see also Corollary 2) that
\[
p^n_{A}(x,y) \sim p^n_{[0]}(x,y) \quad \text{as } (-x) \land y \to \infty,
\]
(6.18)
which allows us to follow the same arguments made above to verify (6.5) except for the case \(x > -M, |y| < M\). The completion of the proof is performed as follows. By duality we may consider the case \(|x| < M, y < M\). By virtue of (6.18) (applied with \(S_{\tau_R}^x \leq -R\) in place of \(x\)) the argument deriving (6.16) is valid for such \(x, y\) and hence using Lemma 6.1 we deduce that for \(|x| < M, y < M\) as \(R \land (-y) \to \infty\),
\[
p^n_{A}(x,y) = E[p^n_{A}^{-\tau} (S_{\tau_R}^x, y); \tau_R^x < N \land \sigma_{A}]\{1 + o(1)\}
\]
\[
= E[\sigma_{(-\infty,-R]}^x; \tau_R^x < \sigma_{\{0\}}^x < \sigma_{\{0\}}^x + o(f^{-y}(n)),
\]
\[
= u_{A}(x)f^{-y}(n)\{1 + o(1)\},
\]
hence as before we have (6.17) as \(x \to \infty\) for \(|y| < M\) with \(u_{-A}(-y) \neq 0\), and can repeat the same argument to conclude the formula asserted in Theorem 7. \(f^n_A(n)\) is dealt with as in the case \(C^+ = \infty\). The proof of Theorem 7 is complete.

We conclude this section with a short remark about the extension of Theorem 6. On letting \(x \to \infty\) in (6.2) it follows that
\[
C^+_A := \lim_{x \to +\infty} u_{A}(x) = C_+ - \sum_{z \in A} H_{A}^{+\infty}(z)u_{A}(z - w_0) \leq \infty
\]
indpendently of the choice of \(w_0 \in A\), where \(H_{A}^{+\infty}(z) := \lim_{x \to +\infty} H_{A}^{+\infty}(z)\) (cf. [16] Theorem 30.1) for existence of the limit). We may suppose \(A \subset (-\infty, 0]\). Then for \(x \geq 1\) and \(y < \min A\), we have \(g_A(x,y) = \sum_{z \notin A, z \leq 0} H_{(-\infty,0]}^{x}(z)g_A(z,y)\) and noting \(g_A(z,y) \leq g_{[0]}(z,z)\) let first \(y \to -\infty\) and then \(x \to +\infty\) to see that
\[
u_{A}(x) = \sum_{z \notin A, z \leq 0} H_{(-\infty,0]}^{x}(z)u_{A}(z) \quad \text{and} \quad C_A^+ = \sum_{z \notin A, z \leq 0} H_{(-\infty,0]}^{+\infty}(z)u_{A}(z).
\]
(6.19)
With these identities we can follow the proof of Theorem 6 word for word except for trivial modifications to obtain the corresponding formula for \(p^n_A(x,y)\).
7 Some properties of $f_t^ε$ and $p_t^{(0)}$}

We have stated the asymptotic form of $f_t^ε(t)$ in Corollary 1. In the following lemma we obtain it for $\gamma \neq 2 - \alpha$ by direct computation concerning the limit stable process.

Lemma 7.1. As $t \to \infty$

$$f_t^1(t) \sim \left[ \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \right] \left[ \int_0^\infty u^{-\alpha} p_1'(u)du \right] \frac{1}{t^{2-1/\alpha}}.$$  

Proof. According to [1, Lemma 8.13]

$$\int_0^t f_t^1(s)ds = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \int_0^t (t-s)^{-1+1/\alpha} p_s(-1)ds.$$  

Substitution from $p_s(-1) = s^{-1/\alpha} p_1(-s^{-1/\alpha})$ and the change of variable $u = s/t$ transform the integral on the RHS into

$$\int_0^1 (1-u)^{-1+1/\alpha} u^{-\alpha} p_1(-tu^{-1/\alpha})du.$$  

On noting that $\int_0^1 u^{-1/\alpha-1} |p_1'(-u^{-1/\alpha})|du = \alpha \int_1^\infty |p_1'(-s)|ds < \infty$ differentiation leads to

$$f_t^1(t) = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \cdot \frac{1}{\alpha t^{1+1/\alpha}} \int_0^1 (1-u)^{-1+1/\alpha} u^{-2/\alpha} p_1'(-tu^{-1/\alpha})du, \quad (7.1)$$

After the change of variable $u = 1/ts^\alpha$ this becomes

$$f_t^1(t) = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \cdot \frac{1}{s^{2-1/\alpha}} \int_1^\infty (1 - \frac{1}{ts^\alpha})^{-1+1/\alpha} s^{-\alpha} p_1'(-s)ds,$$

which shows the relation of the lemma, the integral above being asymptotically equivalent to $\int_1^\infty s^{-\alpha} p_1'(-s)ds$ as $t \to \infty$. \hfill \Box

Lemma 7.2. If $\varphi(t)$ is a continuous function on $t \geq 0$, then for $T > 0$

$$\lim_{y \to \pm0} \int_0^T \frac{p_t(y) - p_t(0)}{|y|^{\alpha-1}} \varphi(t)dt = b_{\alpha,\gamma}^\pm \varphi(0) \quad \text{with} \quad b_{\alpha,\gamma}^\pm = \alpha \int_0^\infty \frac{p_1(\pm u) - p_1(0)}{u^{\alpha}} du.$$  

Proof. Let $w_y(t) = |p_t(y) - p_t(0)|/|y|^{\alpha-1}$. For any $\varepsilon > 0$,

$$\int_0^\varepsilon w_y(t)dt = \int_0^\varepsilon |p_1(y/t^{1/\alpha}) - p_1(0)| |y| dt/|y|^{\alpha+1/\alpha} = \alpha \int_0^\varepsilon \frac{p_1(\pm u) - p_1(0)}{|u|^{\alpha}} du,$$

where $\pm$ accords to the sign of $y$. The last member converges to $b_{\alpha,\gamma}^\pm$ and $w_y(t) = O(|y|^{2-\alpha}) \to 0$ ($y \to 0$) uniformly for $t > \varepsilon$ (since $p_1'$ is bounded), and hence the result follows. \hfill \Box

Lemma 7.3. Let $b_{\alpha,\gamma}^\pm$ be given as in Lemma 7.2. Then for $x > 0$

$$\lim_{y \to \pm0} p_t^{(0)}(x, y)/|y|^{\alpha-1} = b_{\alpha,\gamma}^\pm f_t^x(t);$$

and

$$b_{\alpha,\gamma}^\pm = -\pi^{-1}\Gamma(1-\alpha) \sin[\pi(\alpha + \gamma)/2],$$

in particular if $\gamma = 2 - \alpha$, $b_{\alpha,\gamma}^- = 0$ (the trivial case) and $b_{\alpha,\gamma}^+ = 1/\Gamma(\alpha)$.  

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Proof. Although the result follows from Theorems 4 and 5 we use them only for the identification of $b_{\alpha\gamma}^\pm$ in this proof that is based on the identity

$$p_t^{(0)}(x,y) = p_t(y-x) - \int_0^t f^\gamma(t-s)p_s(y)ds.$$ 

On subtracting from this equality that for $y = 0$ when the LHS vanishes, and then dividing by $|y|^{\alpha-1}$

$$\frac{p_t^{(0)}(x,y)}{|y|^{\alpha-1}} = \frac{p_t(y-x) - p_t(-x)}{|y|^{\alpha-1}} - \int_0^t \frac{p_s(y) - p_s(0)}{|y|^{\alpha-1}}f^\gamma(t-s)ds.$$ 

As $y \to 0$, the first term on the RHS tends to zero and Lemma 7.2 applied to the the second term yields the equality of the lemma. By applying Theorems 4 and 5 with $c_0 = 1$ it follows that

$$b_{\alpha\gamma}^\pm = \frac{1}{\Gamma(1)} \lim_{y_n \to 0} \frac{p_t^{(0)}(x,y_n)}{|y|^{\alpha-1/n^{1-1/\alpha}}} = \frac{\alpha(-y)}{\Gamma(\alpha)\Gamma(1/\alpha)t^{1-1/\alpha}}$$

(see Remark 2.2(e)), of which the last limit is evaluated in Lemma 3.2(i) as asserted.

Let $Q_t(y)$ denote the distribution function of a stable meander, which may be expressed as

$$Q_t(y) = \lim_{\varepsilon \to 0} P[Y_t \leq y \mid \sigma_{(-\infty,-\varepsilon]} > t]$$

(cf. [1, Theorem 18]) and satisfies the scaling relation $Q_t(y) = Q_1(y/t^{1/\alpha})$.

**Lemma 7.4.** If $\gamma = 2 - \alpha$, then for $y > 0$

$$K_t(y) := \lim_{x \to 0} p_t^{(0)}(x,y)/x = \alpha p_t(0)Q_t'(y)$$

and

$$\lim_{y \to 0} \frac{p_t^{(0)}(x,y)}{y^{\alpha-1}} = \frac{f^\gamma(t)}{\Gamma(\alpha)} = \frac{Q_t'(x)}{\Gamma(\alpha)\Gamma(1/\alpha)t^{1-1/\alpha}}.$$  

**Proof.** First we show (7.3). In the proof of Lemma 4.4 (that if adapted to the stable process is much simplified) it is in effect shown that the convergence in (7.3) is locally uniform in $y > 0$, and our task is to identify the limit, for which it suffices to show

$$\lim_{x \to 0} \frac{1}{x} \int_0^y p_t^{(0)}(x,z)dz = \alpha p_t(0)[Q_t(y) - Q_t(\delta)]$$

for any $0 < \delta < y$. For $\gamma = 2 - \alpha$, $\sigma_{(-\infty,-\varepsilon]}^Y$ agrees with $\sigma_{(-\varepsilon]}^Y$ a.s. Hence for $x > 0$, the integral in (7.3) which equals $P[\delta - x < Y_t \leq y - x, \sigma_{(-\varepsilon]}^Y > t]$ (since $\sigma_{(-\varepsilon]}^{Y+Y} = \sigma_{(-\varepsilon]}^Y$) is expressed as

$$P[\delta - x < Y_t \leq y - x, \sigma_{(-\varepsilon,-\varepsilon]}^Y > t]P[\sigma_{(-\varepsilon,-\varepsilon]} > t].$$

The first factor converges to $Q_t(y) - Q_t(\delta)$ as $x \downarrow 0$. For the second one, recalling $f^\gamma(s) = xs^{-1}p_s(x) = xs^{-1-1/\alpha}p_1(xs^{-1/\alpha})$ and making a change of variable we have

$$P[\sigma_{(-\varepsilon,-\varepsilon]} > t] = \int_t^\infty f^\gamma(s)ds = \alpha \int_0^{x^{1/\alpha}} p_1(u)du.$$ 

Thus dividing by $x$ and passing to the limit conclude the required formula (7.5) since $p_1(0)t^{-1/\alpha} = p_t(0)$. 

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As for (7.4) we make use of the duality relation and write (7.2) as
\[ \hat{Q}_t(x) = \lim_{\varepsilon \downarrow 0} \frac{\int_0^{x+\varepsilon} p_t^{(0,\infty)}(-\varepsilon, -\xi) d\xi}{P[\sigma_{[e,\infty)} > t]} = \lim_{\varepsilon \downarrow 0} \frac{\int_0^{x+\varepsilon} p_t^{(0,\infty)}(\xi, \varepsilon) d\xi}{P[\sigma_{[e,\infty)} > t]} \]
The first equality of (7.4) follows from the preceding lemma and is written as
\[ p_t^{(0,\infty)}(\xi, \varepsilon) = p_t^{(0,\infty)}(\xi, \varepsilon) \sim f^t(\xi) e^{\alpha - 1}/\Gamma(\alpha) (\xi > 0). \]
By \( \gamma = 2 - \alpha \) we have \( P[Y_t > 0] = 1 - 1/\alpha \) (cf. [8, p.313 (18)]) which entails \( P[\sigma_{[e,\infty)} > t] = P[\sigma_{[1,\infty)} > t/\varepsilon^\alpha] \sim C\varepsilon^{-1+1/\alpha} \) \[\text{[1] Proposition VIII.2}\] and accordingly obtain
\[ \hat{Q}_t(x) = \frac{t^{1-1/\alpha}}{C\varepsilon^{\Gamma(\alpha)}} \int_0^x f^t(d\xi). \]

We derive \( C = 1/\Gamma(\alpha)\Gamma(1/\alpha) \) from \( \hat{Q}_t(+\infty) = 1 \) with the help of the next lemma (cf. Remark 7.1). Finally differentiation concludes the second equality of (7.4).

**Lemma 7.5.**
\[ \int_{-\infty}^{\infty} p_t(x)|x|dx = \frac{2t^{1/\alpha}}{\pi} \Gamma(1 - 1/\alpha) \sin\left[\frac{\pi}{\alpha} (\alpha - \gamma)/\alpha\right], \]
in particular if \( \gamma = 2 - \alpha \), \( \int_0^{\infty} f^t(t)dt = t^{-1} \int_0^{\infty} p_t(-x)dx = t^{-1+1/\alpha}/\Gamma(1/\alpha) \).

**Proof.** Put \( \chi_\lambda(x) = |x|e^{-\lambda x} \) (\( \lambda > 0, -\infty < x < \infty \)). By Parseval equality
\[ \int_{-\infty}^{\infty} p_t(x)|x|dx = \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} p_t(x)\chi_\lambda(x)dx = \frac{1}{\pi} \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} e^{-t\psi(\theta)} C_\lambda(\theta) d\theta, \]
where \( C_\lambda(\theta) = \int_0^{\infty} \chi_\lambda(x) \cos \theta x dx \), or explicitly
\[ C_\lambda(\theta) = \frac{\lambda^2 - \theta^2}{(\lambda^2 + \theta^2)^2}. \]
Observing \( \int_0^{\infty} C_\lambda(\theta) d\theta = 0 \), we infer that as \( \lambda \downarrow 0 \)
\[ \int_{0}^{\infty} e^{-t\psi(\theta)} C_\lambda(\theta) d\theta = \int_{0}^{\infty} [e^{-t\psi(\theta)} - 1] C_\lambda(\theta) d\theta \rightarrow \int_{0}^{\infty} \frac{1 - \exp\{\frac{-te^{i\gamma\pi/2}e^{i\alpha}}{\theta^2}\}}{\theta^2} d\theta. \]
The last integral equals \( (te^{i\gamma\pi/2}e^{1/\alpha})^{1/\alpha} \Gamma(1 - 1/\alpha) \) \[\text{[8] p.313 (18)}, \] and we find the first formula of the lemma obtained. If \( \gamma = 2 - \alpha \), then \( \Gamma(1 - 1/\alpha) \sin\left[\frac{\pi}{\alpha} (\alpha - \gamma)/\alpha\right] = \pi/\Gamma(1/\alpha) \), which together with \( f^t(t) = xt^{-1}p_t(-x) \) and \( \int_0^{\infty} p_t(x)dx = 0 \) shows the second formula.

**Remark 7.1.** We have used Lemma 7.5 for identification of the constant factor in (7.4). Alternatively we could have applied the exact formula for \( P[\sup_{s \leq t} Y_s \in d\xi]/d\xi \) obtained in [2] (cf. also [6]).

### 8 Auxiliaries

Here we give miscellaneous consequences of the assumptions 1) and 2) stated in Section 1 that are derived from the general theory.
8.1 Condition (1.1) in terms of the tails of $F$

The assumption (1.1) on the characteristic function $\phi(\theta)$ is equivalent to the condition

$$P[X > x] \sim q^+ B x^{-\alpha} \quad \text{and} \quad P[X < -x] \sim q^- B x^{-\alpha}$$  \hspace{1cm} (8.1)

as $x \to \infty$ with some positive constant $B$ and two non-negative constants $q^+$ and $q^-$ such that $q^+ + q^- = 1$. The Lévy measure $M\{dx\}$ is then given by

$$M\{(-y, x]\} = \frac{\alpha B}{2 - \alpha} (q^- y^{2-\alpha} + q^+ x^{2-\alpha}) \quad (x \geq 0, y \geq 0)$$

and the characteristic exponent of the limiting stable process by

$$c_0 \psi(\theta) = |\theta|^\alpha B \Gamma(1 - \alpha) \{ \cos \frac{1}{2} \alpha \pi - i (\text{sgn} \theta)(q^+ - q^-) \sin \frac{1}{2} \alpha \pi \}$$

(cf. [3] (XVII.3.18)). From this we read off

$$c_0 = B \Gamma(1 - \alpha) [(\cos \frac{1}{2} \alpha \pi)/(\cos \frac{1}{2} \gamma \pi)] \quad \text{and} \quad \tan \frac{1}{2} \gamma \pi = (q^+ - q^-)(- \tan \frac{1}{2} \alpha \pi)$$

(which reduce to $c_0 = -B \Gamma(1 - \alpha)$ and $q^+ = 1$, respectively, if $\gamma = 2 - \alpha$) and hence

$$\psi(\theta) = |\theta|^{\alpha} (\cos \frac{1}{2} \gamma \pi) \{1 + i (\text{sgn} \theta)(q^+ - q^-)(- \tan \frac{1}{2} \alpha \pi)\};$$

According to Zolotarev [24] (cf. [3] Section 8.9.2, [1] Section VIII.1) Spitzer’s constant $\rho := \lim_{n \to \infty} n^{-1} \sum_{k=1}^n P[S_k > 0]$ is given by

$$\rho = \frac{1}{2} (1 - \gamma/\alpha).$$  \hspace{1cm} (8.2)

From (8.1) it follows that

$$\phi'(\theta) \sim -\psi'(\theta) = \mp \alpha c_0 e^{\pm i \gamma \pi/2} |\theta|^{\alpha-1} \quad \text{as} \quad \theta \to \pm 0.$$  \hspace{1cm} (8.3)

Indeed, on writing $\phi'(\theta) = i \int_{-\infty}^{\infty} (e^{i \theta t} - 1) t dF(t)$ the integration by parts yields

$$\phi'(\theta) = i \int_{-\infty}^{\infty} \{ e^{i \theta t} - 1 + i \theta t e^{i \theta t} \}[-F(t) 1(t < 0) + (1 - F(t)) 1(t > 0)] dt,$$

and scaling by the factor $1/|\theta|$ we find that $\phi'(\theta) \sim \pm \zeta |\theta|^{\alpha-1}$, where

$$\zeta = i B \int_{-\infty}^{\infty} \{1 - e^{\pm i u} = i u e^{\pm i u} \} \frac{q^- 1(u < 0) - q^+ 1(u > 0)}{|u|^{\alpha}} du.$$ 

Since $\zeta$ depends on the regularity of tails of $F$ only and $-\psi'(\theta)$ is given by the above integral with $dF$ replaced by the Levy measure associated with $\psi$, $\pm \zeta |\theta|^{\alpha-1}$ must be equal to $-\psi'(\theta)$.

Remark 8.1. Put $\beta_+ = \frac{1}{2}(\alpha + \gamma)$. If $q^+ < q^-$, then $\alpha - 1 \leq \beta_+ < \beta_- \leq 1$, where the equality in each extremity holds if and only if $q^+ = 0$. In order to consider the behaviour of $U_{ds}$ and $V_{as}$ we rewrite condition (8.1) as

$$P[X > x] = B x^{-\alpha} (q^+ + r_+(x)) \quad \text{and} \quad P[X < -x] = B x^{-\alpha} (q^- + r_-(x)),$$

(8.5)
where \( r_\pm(x) \to 0 \) as \( x \to 0 \). If \( \int_1^\infty (|r_+(x)| + |r_-(x)|) x^{-1} dx < \infty \), then, as \( x \to \infty \)

\[
U_{\text{ds}}(x) \sim C'' x^{\beta_+} \quad \text{and} \quad V_{\text{as}}(x) \sim C''' x^{\beta_-}
\]  

(8.6)

with some positive constants \( C' \) and \( C'' \) such that \( 1/C'' C''' = c_0 \Gamma(1 + \beta_+) \Gamma(1 + \beta_-) \). The proof is carried out by computation based on Spitzer’s expressions of the generating functions of \( v^- \) and \( v^+ \) given in [16, P18.7], the computation being somewhat involved and omitted. In order that \( E_Z < \infty \) (entailing \( 8.6 \) with \( \beta_+ = \alpha - 1 \) and \( \beta_- = 1 \)), it is necessary and sufficient that \( q^+ = 0 \) and \( \int_1^\infty r_+(x) x^{-1} dx < \infty \) according to Chow’s criterion [4]. If \( q^+ = 0 \) with \( \int_1^\infty r_+(x) x^{-1} dx = \infty \), then \( U_{\text{ds}}(x) = o(x) \) so that \( L(x) \to 0 \) in (2.24) and hence \( U_{\text{ds}}(x) x^{\alpha+1} \to \infty \) (\( x \to \infty \)).

### 8.2 An upper bound of \( p^n(x) \) for \( |x| > n^{1/\alpha} \)

Let \( \lambda(\theta) = \sum_{x \in \mathbb{Z}} w_x e^{i\theta x} \), the sum of the trigonometric series with the coefficient such that \( M := \sum |w_x| < \infty \) and put \( m(r) = \int_0^r dt \sum_{|x| > t} |w_x| \). Then

\[
|\lambda(\theta) - \lambda(\theta')| \leq 2|\theta - \theta'| m(1/|\theta - \theta'|) \quad (\theta \neq \theta'),
\]

(Erickson [7, Lemma 5]). Applied with \( w_x = x p(x) \), \( m(r) = \int_0^r E[|X|; |X| > t] dt = O(r^{2-\alpha}) \) this yields

\[
|\phi'(\theta) - \phi'(\theta')| \leq C|\theta - \theta'|^{\alpha-1}
\]

(8.7)

under (8.1). This same bound is satisfied by \( \psi'(\theta) \) as is verified directly (or by a similar reasoning). The following lemma is based on this bound.

**Lemma 8.1.** If \( p \) satisfies (1.1), then for some constant \( C \)

\[
p^n(x) \leq C n^{-1/\alpha} (1 \wedge |x_n|^{-\alpha}).
\]

**Proof.** For \( |x_n| \leq 1 \), the bound follows from the local limit theorem. Let \( |x_n| > 1 \). The integration by parts and the change of variable \( \theta = t/n^{1/\alpha} \) transforms the expression

\[
p^n(x) = \frac{1}{2\pi} \int_{-\pi}^\pi \phi(t) e^{-ix\theta} d\theta
\]

into

\[
p^n(x) = \frac{1}{2\pi i x} \int_{-\pi/n^{1/\alpha}}^{\pi/n^{1/\alpha}} \frac{n^{n-1}}{n^{n-1}} \phi'(t/n^{1/\alpha}) e^{(n-1) \log \phi(t/n^{1/\alpha})} e^{-ix\theta} d\theta.
\]  

(8.8)

Put

\[
Q_n(t) = -(n-1) \log \phi(t/n^{1/\alpha}), \quad R_n(t) = n^{-1/\alpha} \phi'(t/n^{1/\alpha}).
\]

By (1.1), (8.3) and (8.7) it follows that

\[
Q_n(t) = \psi(t) \{1 + o(1)\}, \quad R_n(t) = -\psi'(t) \{1 + o(1)\}
\]

where \( o(1) \) is bounded for \( |t| < \pi n^{1/\alpha} \) and \( o(1) \to 0 \) as \( t/n^{1/\alpha} \to 0 \), and

\[
R_n(t) - R_n(t') = O(|t - t'|^{\alpha-1})
\]

uniformly for \( n \). By periodicity of \( \phi \), \( p^n(x) = -\frac{1}{2\pi} \int_{-\pi}^\pi \phi(\theta + \pi/x_n) \phi'(t/n^{1/\alpha}) e^{-ix\theta} d\theta \), we accordingly obtain that \( p^n(x) = (I + J)/4\pi ix \), where

\[
I = \int_{-\pi/n^{1/\alpha}}^{\pi/n^{1/\alpha}} [R_n(t) - R_n(t + \pi/x_n)] e^{-Q_n(t)} e^{-ix\theta} d\theta
\]

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and
\[
J = \int_{-\pi/n^\lambda}^{\pi/n^\lambda} R_n(t + \pi/x_n)[e^{-Q_n(t)} - e^{-Q_n(t+\pi/x_n)}]e^{-ix_n^t}dt.
\]
Noting $|\phi(\theta)| < 1$ for $0 < |\theta| \leq \pi$ and $\text{Re} \psi(\theta)/|\theta|^\alpha = \cos \frac{1}{2} \gamma \pi > 0$, we can choose a constant $\lambda > 0$ so that $\text{Re}Q_n(t) > \lambda|t|^{\alpha}$ for $|t| < \pi/n^\lambda + \pi$ for $n$ large enough. Hence if $f_n(t) = [R_n(t) - R_n(t + \pi/x_n)]e^{-Q_n(t)|x_n|^{\alpha-1}}$, then $f_n(t)$ is dominated in absolute value by $(e^{-|t|^{\alpha}/2})$, and we deduce that $I = |x_n|^{1-\alpha} \int_{-\pi/n^\lambda}^{\pi/n^\lambda} f_n(t)e^{-ix_n^t}dt = O(|x_n|^{1-\alpha})$. In a similar way we obtain $J = O(1/|x_n|)$. Since $|x_n|^{1-\alpha}/|x| = n^{1/\alpha}|x_n|^{-\alpha}$, this concludes the proof. \(\square\)

### 8.3 Escape probabilities from the origin

By [16] Proposition 29.4
\[
g_{(0)}(x, y) = a^t(x) + a(-y) - a(x - y),
\]
which entails the subadditivity $a(x + y) \leq a(x) + a(y)$ and
\[
P[\sigma^x_{(y)} < \sigma^x_{(0)}] = \frac{g_{(0)}(x, y)}{g_{(0)}(y, y)} = \frac{a^t(x) + a(-y) - a(x - y)}{a(y) + a(-y)}.
\]
(8.9)

Put $\omega(r) = g_{(-\infty, 0)}(r, r)$ ($r = 1, 2, \ldots$). Note that $\omega(r) = g_{(0, \infty)}(-r, -r)$ and $\omega(r) \leq g_{(0)}(r, r) = a(r) + a(-r)$.

**Lemma 8.2.** Given a positive integer $R$, put $\tau^x_R = \inf\{n \geq 1 : S^x_n \notin (-R, R)\}$. There exists a constant $C$ depending only on $F$ such that for any $R > 1$, $z \geq 0$ and $|x| < R$,
\[
P[S^x_{\tau^x_R} = R + z] \leq \omega(R - |x|)P[X > z].
\]

**Proof.** Since $g_{[R, \infty)}(x, w) = g_{(0, \infty)}(x - R, w - R) \leq \omega(R - |x|)$ and similarly for $g_{(-\infty, -R]}(x, w)$, it follows that $g_{(-\infty, -R) \cup [R, \infty)}(x, w) \leq \omega(R - |x|)$, and hence
\[
P[S^x_{\tau^x_R} = R + z] \leq \omega(R - |x|) \sum_{w : |w| < R} p(R + z - w) \leq \omega(R - |x|)P[X > z],
\]
showing the inequality of the lemma. \(\square\)

The same proof as above shows
\[
H^x_{(0, \infty)}(z) := P[S^x_{\sigma^x_{(0, \infty)}} = z] \leq \omega(-x)P[X \geq z] \quad (x \leq 0, z > 0).
\]
(8.10)

**Lemma 8.3.** If $\gamma > \alpha - 2$,
\[
\lim\inf_{R \to \infty} \inf_{x \in \mathbb{Z}} P[\sigma^x_{(R)} < \sigma^x_{(0)} | \sigma^x_{(R, \infty)} < \sigma^x_{(0)}] =: q > 0;
\]
(8.11)

with $q = 1$ for $\gamma = 2 - \alpha$.

**Proof.** In view of (8.9) and the decomposition
\[
P[\sigma^x_{(R)} < \sigma^x_{(0)} | \sigma^x_{(R, \infty)} < \sigma^x_{(0)}] = \sum_{z \geq R} P[S^x_{\sigma^x_{(R, \infty)}} = z | \sigma^x_{(R, \infty)} < \sigma^x_{(0)}]P[\sigma^x_{(R)} < \sigma^x_{(0)}],
\]

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where $\bar{\sigma}_{(0,R]}$ is defined to be zero if $z = N$ and agree with $\sigma_{(0,R]}^z$ otherwise, for the first half of the lemma it suffices to show that

$$\lim_{R \to \infty} \inf_{z \geq R} \frac{a(z) + a(-R) - a(z - R)}{a(R) + a(-R)} = \frac{\kappa_{\alpha,\gamma,-}}{\kappa_{\alpha,\gamma,-} + \kappa_{\alpha,\gamma,+}},$$

the last ratio being positive if $\gamma > \alpha - 2$ and equals unity if $\gamma = 2 - \alpha$. If $|\gamma| < 2 - \alpha$, by Lemma 3.1(ii) $a(z) - a(z - R) > 0$ for $R$ large enough and the equality above follows immediately from Lemma 3.1(i). The case $\gamma = 2 - \alpha$ also follows from Lemma 3.1(i) and (ii), the latter showing $\sup_{z \geq R} |a(z) - a(z - R)| = o(R^{\alpha-1})$.

**Lemma 8.4.** For any $\gamma$ there exists a constant $C$ such that for $R > 1$,

$$P[\sigma_{(0,R]}^z < \sigma_{(0]}^z] \leq C[a^1(x)R^{-\alpha+1} + x_+/R] \quad (x \leq R).$$

**Proof.** For $\gamma > \alpha - 2$, on using Lemma 3.1(ii) the result is deduced from the preceding lemma. The case $\gamma = \alpha - 2$ follows by [20, Lemma 5.5] (use the fact that $m_a = a(S_{(0]}^z \cup (R,\infty))$ is a martingale).

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