COUNTING INTEGRAL POINTS OF BOUNDED HEIGHT ON VARIETIES WITH LARGE FUNDAMENTAL GROUP

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Abstract. The present note is devoted to an amendment to a recent paper of Ellenberg, Lawrence and Venkatesh ([ELV21]). Roughly speaking, the main result here states the subpolynomial growth of the number of integral points with bounded height of a variety over a number field whose fundamental group is large. Such an improvement, i.e. requiring large fundamental group as opposed to the existence of a geometric variation of pure Hodge structures, was already asked in op.cit.

1. Introduction

In order to state our theorem and cast it in a general framework, let us review some background material.

1.1. Growth of integral points.

1.1.1. Notation. Let $K$ be a number field with ring of integers $\mathcal{O}_K$, $X$ a projective variety over $K$, $L$ an ample line bundle on $X$, and $h$ an (absolute logarithmic) height function relative to $L$. (Further details can be found in section 3.1.)

For an open subset $U$ of $X$, a subring $R$ of $K$ containing $\mathcal{O}_K$, a projective flat $\mathcal{O}_K$-scheme $\mathcal{X}$ with generic fiber $X$, and a real number $c$, set

$$\nu(X, U, L, h, R; c) := \log^+ \# \{ x \in U(R) : h(x) \leq c \},$$

where $U$ is the complement of the Zariski closure of $X \setminus U$ in $X$, and, for a real number $t$, $\log^+ t = \log \max\{1, t\}$. When $U = X$, the redundant specification of the open subset will be discarded.

1.1.2. Asymptotic behaviour. The precise value of the function $\nu(X, U, L, h, K; c)$ is not much of interest. Instead, its asymptotic behaviour for $c \to \infty$ ought to reflect the usual ‘trichotomy’ of algebraic varieties into the Fano, the Calabi-Yau, and the general type classes (and their logarithmic variants):\footnote{The word ‘trichotomy’ here is improper—rather, according to the Minimal Model Program, any integral variety should be obtained as iterated fibration with generic fibre belonging to one of the preceding three classes.}

- When $X$ is smooth Fano and $L$ is the anti-canonical bundle $\omega_X$, Manin conjectured (assuming the Zariski density of $X(K)$) the existence of an open subset $U$ of $X$ for which

$$\nu(X, \omega_X, U, K; c) \sim [K : \mathbb{Q}]c + (\rho - 1) \log c + C,$$

where $\rho$ is the Picard rank of $X$ and $C$ is a real number.\footnote{As of nowadays, Manin’s conjecture is known not to hold in such a form ([BT96]) and alternative hypothetical statements have been suggested. This does not affect the main theme of our paper, and the interested reader may consult for example [Pey03].}

As an instance of the conjecture, Schanuel ([Sch79]) earlier proved the existence of an explicit real number $C = C(N, K)$ for which

$$\nu(\mathbb{P}^N_K, O(1), K; c) \sim [K : \mathbb{Q}](N + 1)c + C.$$
• Rather vaguely, logarithmic growth is expected for Calabi-Yau varieties. For example, let $A$ be an abelian variety and $r := \dim_{\mathbb{Q}} A(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Q}$ its Mordell-Weil rank. Then, for any ample line bundle $L$ on $A$, there is a positive real number $C = C(A, \mathbb{K}, L) > 0$ such that

$$\nu(A, L, \mathbb{K}; c) \sim \frac{1}{2} \log c + C;$$

see [HS00, Theorem B.6.3].

• As soon as one enters the general type realm, the Lang-Vojta conjecture in the projective case predicts the existence of a non-empty open subset for which $K$-rational points are only finitely many for any finite extension $K'$ of $K$—in other words, the counting function of rational points for such an open subset is eventually constant. This is the case for curves by a celebrated theorem of Faltings ([Fal83], [Fal84]). Reformulating the Lang-Vojta conjecture in the non-compact case in terms of counting functions is less eloquent. Nonetheless, recall the finiteness of $S$-integral points for affine curves—a theorem of Siegel ([Sie14])—and for (fine) moduli spaces of curves and of abelian varieties—the Shafarevič conjecture, proved by Faltings in op.cit.

Above, with an abuse of notation, the chosen model $X$ and height $h$ have been dropped, for the asymptotic behaviour is independent from them. Also, the factor $[K : \mathbb{Q}]$ appears because of the normalization of height adopted in this paper.

In another direction, Pila ([Pil95]) proves that, given integers $N, d \geq 1$ and a positive real number $\varepsilon > 0$, there is a real number $C = C(N, d, \varepsilon)$ such that, for each closed integral subvariety $X$ of $\mathbb{P}^N_{\mathbb{Q}}$ of degree $d$,

$$\nu(X, \mathcal{O}(1), \mathbb{Q}; c) \leq (\dim X + \frac{1}{2} + \varepsilon)c + C.$$ 

See also Browning, Heath-Brown and Salberger ([BHBS06]), and Castryck, Cluckers, Dittmann and Nguyen ([CCDN20]).

1.1.3. Linear growth. Here we will content ourselves with much cruder estimates—namely, the slope of the function $c \mapsto \nu(X, L, U, h, \mathbb{K}; c)$,

$$\text{gr. rat}_K(X, L, U) := \limsup_{c \to \infty} \frac{\nu(X, L, U, h, \mathbb{K}; c)}{c},$$

thereafter called the (linear) growth rate of rational points. Of course, as the notation suggests, the real number $\text{gr. rat}_K(X, L, U)$ does not depend on the choice of the model $X$ nor on that of the height function $h$.

Similarly, for a finite set $S$ of places of $K$ including the Archimedean ones, the real number

$$\limsup_{c \to \infty} \frac{\nu(X, L, U, h, \mathcal{O}_K, S; c)}{c}$$

measures the presence of the $S$-integral points on $U$. Again this does not depend on $X$ and $h$, but a priori does depend on the set $S$. Taking the supremum ranging over all such finite sets of places $S$ allows to get rid of such a dependence and gives rise to an invariant

$$\text{gr. int}_K(X, L, U)$$

called the (linear) growth rate of integral points of $U$ with respect to $X$ and $L$.

When $U = X$, rational and integral points coincide, thus their growth rate do. As above, in such a case the redundant repetition of $X$ is discarded from notation.

These growth rates furnish invariants way rougher than the asymptotic behaviour. To stress this, note that the following holds:

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3To the extent of our knowledge there is no precise conjecture on the behaviour of the growth of rational points for Calabi-Yau varieties.
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- \( \text{gr} \cdot \text{rat}_K (\mathbb{P}^N_K, O(1)) = (N + 1)[K : \mathbb{Q}] \) for an integer \( N \geq 1 \);
- \( \text{gr} \cdot \text{rat}_K (A, L) = 0 \) for an abelian variety \( A \) over \( K \) and an ample line bundle \( L \) on \( A \);
- \( \text{gr} \cdot \text{int}_K (X, L, U) = 0 \) for a smooth projective curve \( X \) over \( K \), an ample line bundle \( L \) on \( X \) and an open subset \( U \) of \( X \), provided that \( U \) is not isomorphic to the projective or the affine line over \( K \).

In particular, linear growth rates cannot distinguish Calabi-Yau varieties from those of general type. Needless to say, zero linear growth rate is a much weaker condition than logarithmic growth or no growth at all.

1.2. Large fundamental groups. Abandon the notation above and let \( X \) be a normal integral variety over an algebraically closed field \( k \) of characteristic 0.

1.2.1. Definition. Adopting Kollár’s terminology ([Kol93], [Kol95]), the étale fundamental group of \( X \) is large if, for any closed positive-dimensional integral subvariety \( Y \) of \( X \) with normalization \( f : \tilde{Y} \to Y \), the image of the induced map\(^4\) \( \pi^\text{ét}_1(\tilde{Y}) \to \pi^\text{ét}_1(X) \) is infinite. (See also [Cam95].)

Remark 1.1. Thanks to [Kol93, Proposition 2.9.1], the étale fundamental group of \( X \) is large if and only if, for any non-constant morphism \( f : Y \to X \) between integral normal varieties, the image of the induced map \( \pi^\text{ét}_1(Y) \to \pi^\text{ét}_1(X) \) is infinite. Furthermore, it is sufficient (thus equivalent) to test the latter condition only on smooth connected curves \( Y \).

1.2.2. Basic properties. The following are direct consequences of the definition and the above remark:

(1) The class of (integral, normal) algebraic varieties with large étale fundamental group is closed under products.
(2) Given a quasi-finite morphism \( f : X' \to X \) between integral normal algebraic varieties, if \( X \) has a large étale fundamental group, then \( X' \) has a large étale fundamental group too. The converse holds as soon as \( f \) is finite étale.
(3) The étale fundamental group of an isotrivial fibration whose base and fiber have large étale fundamental group is large.\(^5\)
(4) Let \( k' \) be an algebraically closed field extension of \( k \). If the étale fundamental group of the variety \( X' \) deduced from \( X \) by extending scalars to \( k' \) is large, then so is the one of \( X \).\(^6\)
(5) All smooth connected curves, except the affine and the projective line, have large étale fundamental group.

Remark 1.2. Taking products of a suitable number of copies of an elliptic and of a curve of genus \( \geq 2 \) yields examples of smooth projective (connected) varieties \( X \) with large étale fundamental group and any possible Kodaira dimension between 0 and \( \dim X \).

\(^4\)We abusively drop the choice of a base-point for (étale) fundamental groups.
\(^5\)Namely, let \( f : X' \to X \) be a surjective morphism between integral normal varieties whose geometric fibers are all isomorphic to some fixed integral normal variety \( F \). If \( X \) and \( F \) have large étale fundamental group, then \( X' \) has.
\(^6\)Applying the specialization map of étale fundamental groups to an exhaustive family of normal cycles (see 2.2) permits to show that the converse holds when \( X \) is proper. In the non-compact case we ignore whether having large étale fundamental group is a property compatible with extension of scalars.
1.2.3. Comparison with the topological fundamental group. Over the complex numbers, the property of having large étale fundamental group can be read off the usual topological fundamental group.

For, upon fixing a point \( x \in X(\mathbb{C}) \), the natural group homomorphism
\[
i_X : \pi_1^{\text{top}}(X(\mathbb{C}), x) \to \pi_1^{\text{ét}}(X, x)
\]
identifies the étale fundamental group with the profinite completion of the topological one. Let \( \hat{X} \) be the topological cover of \( X(\mathbb{C}) \) corresponding to the normal subgroup \( \text{Ker} \ i_X \) of \( \pi_1^{\text{top}}(X(\mathbb{C}), x) \).

**Proposition 1.3** ([Kol93, Proposition 2.12.3]). Suppose \( X \) proper.\(^7\) Then, the étale fundamental group of \( X \) is large if and only if the complex analytic space \( \hat{X} \) does not contain positive-dimensional compact complex analytic subspaces.

The latter condition is satisfied, for instance, if the complex space \( \hat{X} \) is holomorphically separable;\(^8\) examples of holomorphically separable spaces are open subsets of \( \mathbb{C}^n \) (or, more generally, open subsets of Stein spaces).

**Remark 1.4.** In view of the above characterization, it is useful to be able to determine \( \hat{X} \). One idle (but useful) case is when \( \hat{X} \) is itself a universal cover of \( X(\mathbb{C}) \), which boils to down to saying that \( i_X \) is injective.\(^9\) Recall that a group \( \Gamma \) is said to be:

- **linear** if it admits a faithful representation \( \rho : \Gamma \to \text{GL}(V) \), where \( V \) is a finite-dimensional vector space over some field;
- **residually finite** if the natural map \( \Gamma \to \hat{\Gamma} \), where \( \hat{\Gamma} \) is its profinite completion, is injective.

A classical result of Malcev ([Mal40]) states that a finitely generated linear group is residually finite. Gathering the above considerations, if the topological fundamental group \( \pi_1^{\text{top}}(X(\mathbb{C}), x) \) is linear, then \( \hat{X} \) is a universal cover of \( X(\mathbb{C}) \).

**Example 1.5.** The above remark can be applied to say that the étale fundamental group of \( X \) is large when \( X \) is

- an abelian variety, for it is the quotient of \( \mathbb{C}^n \) by a lattice;
- a quotient of a bounded symmetric domain in \( \mathbb{C}^n \) by a torsion-free cocompact lattice of its biholomorphism group.

It follows that semi-abelian varieties, being a fibration over an abelian variety by a torus, have large étale fundamental group.

**Remark 1.6.** Having large étale fundamental group is not a biholomorphic invariant of non-compact varieties, preventing Proposition 1.3 to hold true in such a case.

For instance, let \( G \) be the universal vector extension of a complex elliptic curve \( E \). The exponential map \( \text{Lie} \ G \to G(\mathbb{C}) \) is a universal cover and its kernel \( \Lambda \) is a rank 2 free abelian group. The topological fundamental group of \( G(\mathbb{C}) \) is naturally identified with \( \Lambda \), and therefore residually finite. Despite \( \hat{G} = \text{Lie} \ G \) being a Stein space, the étale fundamental group of \( G \) is not large: the kernel of the natural projection \( G \to E \) is the affine line, which is simply connected.

On the other hand, the discrete subgroup \( \Lambda \) generates \( \text{Lie} \ G \) as a complex vector space, so that \( G(\mathbb{C}) = \text{Lie} \ G / \Lambda \) is biholomorphic to \( (\mathbb{C}/\mathbb{Z})^2 \cong (\mathbb{C}^\times)^2 \). The algebraic group \( \mathbb{G}_m^2 \) has large fundamental group, whence the sought-after example.

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\(^7\)The properness assumption is missing in the statement of loc.cit., although crucially invoked in the proof—in the non-compact case the statement is false; see Remark 1.6.

\(^8\)That is, given two distinct points \( x, y \), there is a global holomorphic function \( f \) such that \( f(x) \neq f(y) \).

\(^9\)An instance where this does not occur has been first observed by Toledo ([To93]).
1.2.4. The role of local systems. Still under the assumption $k = \mathbb{C}$, a local system $\mathcal{L}$ on $X(\mathbb{C})$ with coefficients in some field is large if, given a non-constant morphism $f : Y \to X$ with $Y$ a normal irreducible complex variety, the local system $f^* \mathcal{L}$ has infinite monodromy. The étale fundamental group of a complex variety carrying a large local system is large.

Local systems underlying variations of Hodge structures are the prominent example of large local systems. More precisely, let $\mathcal{L}$ be a real local system on $X(\mathbb{C})$ underlying a polarizable variation of pure real Hodge structures (or, more generally, an admissible graded-polarizable variation of mixed real Hodge structures, cf. [SZ85]). Assume further that the associated period mapping has discrete fibers. Then, the theorem of the fixed part (see [Gri70], [Del71], [Sch73] in the pure case, and [SZ85] in the mixed case) implies that $\mathcal{L}$ is large.

**Example 1.7.** Mixed Shimura varieties carry a large local system coming from their interpretation as period spaces for (graded polarized) integral mixed Hodge structures.

**Example 1.8.** A full range of (fine) moduli spaces of polarized varieties (e.g. smooth projective curves, Calabi-Yau varieties, most complete intersections) admit a large local system—namely, the one whose fiber at a point is the middle cohomology of the corresponding variety. Indeed, such a local system is large when the ‘infinitesimal Torelli theorem’ is satisfied, whence the above list; see [Bee87].

1.3. Main result. Time has come to state the main result of the present note:

**Main theorem.** Let $K$ be a number field, $X$ a projective variety over $K$, $L$ an ample line bundle over $X$, and $U$ an open subset of $X$ which is geometrically integral, normal, and whose geometric étale fundamental group is large. Then,

$$\text{gr. int}_K(X, L, U) = 0.$$ 

Such a statement can be applied to varieties appearing in examples 1.5, 1.7 and 1.8; in particular, this replies positively to a question raised in [ELV21, p.3-4, end of §1.1].

Its proof relies on two independent ingredients. The first is of arithmetic nature and is a quite formal consequence of work on uniform bounds initiated by the determinant method of Heath-Brown ([HB02]), and pursued by many authors including Broberg ([Bro04]), Salberger ([Sal07]), and Chen ([Che12a], [Che12b]):

**Theorem A.** Let $X$ be an integral projective variety of dimension $n$ over a number field $K$, and $L$ an ample line bundle over $X$. Assume there is a closed subvariety $Z$ of $X$ and an integer $d \geq 1$ such that any positive-dimensional integral closed subvariety $Y$ of $X$ not contained in $Z$ satisfies \( \deg(Y, L|_Y) \geq d \dim Y \). Then,

$$\text{gr. rat}_K(X, L, X \setminus Z) \leq \frac{n(n + 3)}{2d} [K : \mathbb{Q}].$$

On the other hand, the second one is purely geometric and already appears in arguments of the first-named author ([Bru20]):

**Theorem B.** Let $X$ be a normal integral projective variety over an algebraically closed field of characteristic $0$, $L$ an ample line bundle over $X$, and $U$ a non-empty open subset of $X$ whose étale fundamental group is large.

Then, given an integer $d \geq 1$, there is a finite surjective map $\pi : X' \to X$ with $X'$ normal integral such that $\pi$ is étale over $U$ and, for each positive-dimensional integral closed subvariety $Y'$ of $X'$ meeting $\pi^{-1}(U)$, \( \deg(Y', \pi^* L|_{Y'}) \geq d \).

\(^{10}\)That is, for an algebraic closure $\bar{K}$ of $K$, the variety $\bar{U}$ deduced from $U$ extending the scalars to $\bar{K}$ has large étale fundamental group.
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Organization of the paper. Introduction left aside, the paper has three sections, respectively dealing with the proof of Theorem A, Theorem B and the Main theorem.

Conventions. A variety over a field $k$ is a separated finite type $k$-scheme.

2. Geometry

2.1. Degree of the singular locus.

2.1.1. Degree. For a proper variety $X$ over a field $k$ together with an ample line bundle $L$, let

$$\deg(X, L) := (\dim X)! \lim_{i \to \infty} \frac{\dim_k \Gamma(X, L^\otimes i)}{i^{\dim X}}.$$  

The theory of Hilbert polynomials shows that such a limit exists and is a positive rational number. When $X$ is integral, the asymptotic version of Riemann-Roch states

$$\deg(X, L) = L^n,$$

where $n = \dim X$ and $L^n$ is the top self-intersection of $L$. In particular, $\deg(X, L)$ is a positive integer.

Lemma 2.1. Let $\pi: Y \to X$ be a finite morphism between proper varieties over $k$ and $L$ an ample line bundle over $X$. If there exists a scheme-theoretically dense open subset $U$ of $X$ such that $\pi^{-1}(U) \to U$ is an isomorphism, then

$$\deg(Y, \pi^* L) = \deg(X, L).$$

Proof. Since the open subset $U$ is scheme-theoretically dense, the natural map $\phi: \mathcal{O}_X \to \pi^* \mathcal{O}_Y$ is injective. The support of the coherent $\mathcal{O}_X$-module $F = \text{Coker} \phi$ is contained in the closed subset $X \setminus U$, thus in particular of dimension $< \dim X$. For $i \geq 1$ big enough, the cohomology group $H^1(X, L^\otimes i)$ vanishes, yielding a short exact sequence

$$0 \to \Gamma(X, L^\otimes i) \to \Gamma(Y, \pi^* L^\otimes i) \to \Gamma(X, F \otimes L^\otimes i) \to 0.$$

Since the support of $F$ has dimension $< \dim X$,

$$\lim_{i \to \infty} \frac{\dim \Gamma(X, F \otimes L^\otimes i)}{i^{\dim X}} = 0.$$

The dimensions of $X$ and $Y$ being the same, the result follows.

Lemma 2.2. Let $X$ be an equidimensional proper variety over $k$, $L$ an ample line bundle. Then,

$$\deg(X, L) \geq \sum_{X'} \deg(X', L_{X'}),$$

the sum ranging on the irreducible components of $X$ endowed with the reduced structure. Moreover, equality holds if $X$ is reduced.

Proof. One reduces first to $X$ reduced, as for $i \geq 0$ big enough, the restriction map $\Gamma(X, L^\otimes i) \to \Gamma(X_{\text{red}}, L^\otimes i)$ is surjective. Under the assumption of $X$ being reduced, let $X_1, \ldots, X_n$ be the irreducible components of $X$. Lemma 2.1, applied to the finite morphism $X_1 \sqcup \cdots \sqcup X_n \to X$, gives the desired result.
2.1.2. Degree in families. Let \( f : X \to S \) be a proper morphism between algebraic varieties over a field \( k \) and \( L \) a relatively ample line bundle on \( X \). Consider:

- the function \( \delta_{X/S,L} : S \to \mathbb{N} \),
- \( s \mapsto \max\{\deg(Z, L_{|Z}) : Z \text{ irreducible component of } X_s\} \),

where \( Z \) is endowed with its reduced structure, and

- the function \( \iota_{X/S} : S \to \mathbb{N} \) associating to \( s \in S \) the number of irreducible components of \( X_s \) where \( s \) is a geometric point over \( S \).

**Lemma 2.3.** Let \( f : X \to S \) be a proper morphism between varieties over \( k \) and \( L \) relatively ample line bundle on \( X \).

Then, the functions \( \delta_{X/S,L}, \iota_{X/S} : S \to \mathbb{N} \) are bounded above.

**Proof.** The statement only depends on the reduced structure of \( S \). Moreover, by treating separately each irreducible component, the scheme \( S \) may be supposed to be integral. By Noetherian induction, it suffices to show the statement on a nonempty open subset of \( S \).

Let \( \eta \) be the generic point of \( S \) and \( X_{\eta,1}, \ldots, X_{\eta,r} \) the irreducible components of \( X_\eta \). For \( i = 1, \ldots, r \), let \( X_i \) be the closure of \( X_{\eta,i} \) in \( X \). According to [Stacks, Lemma 054Y], there is an open subset \( S' \) of \( S \) such that

\[
f^{-1}(S') \subset X_1 \cup \cdots \cup X_r.
\]

Up to treating each of the \( X_i \) separately and up to replacing \( S \) by \( S' \), one may assume \( X \) to be integral. Upon shrinking \( S \), the morphism \( f \) may be assumed to be flat ([Stacks, Proposition 052B]). By flatness ([Stacks, Lemma 02JS]), the fibers of \( f \) are then pure of dimension \( d := \dim X - \dim S \).

The relative ampleness of \( L \) implies the existence of an integer \( i_0 \geq 1 \) such that, for \( q \geq 1 \) and \( i \geq i_0 \), the higher direct image \( R^q f_* L^{\otimes i} \) vanishes. The semi-continuity theorem ([Har77, Theorem III.12.8]) can be applied to say, for \( i, q \in \mathbb{N} \), that the function \( s \mapsto \dim_{\kappa(s)} H^q(X_s, L^{\otimes i}) \) is upper semi-continuous. Thus, up to replacing \( L \) by \( L^{\otimes i_0} \) and \( S \) by a nonempty open subset, one may assume, for \( i \geq 1, q \geq 1 \) and \( s \in S \),

\[
H^q(X_s, L^{\otimes i}) = 0.
\]

On the other hand, the Hilbert polynomial of \( L_{|X_s} \) does not depend on \( s \), for the morphism \( f \) is flat ([FGI 05, Theorem 5.10]). Combined with the vanishing of higher cohomology, this implies that, for \( i \in \mathbb{N} \), the function \( s \mapsto \dim_{\kappa(s)} \Gamma(X_s, L^{\otimes i}) \) is constant. As a consequence, so is the function

\[
s \mapsto \deg(X_s, L_{|X_s}) = d! \lim_{i \to \infty} \frac{\dim_{\kappa(s)} \Gamma(X_s, L^{\otimes i})}{i^d}.
\]

For \( s \in S \), the inequality

\[
\deg(X_s, L_{|X_s}) \geq \sum_Z \deg(Z, L_{|Z}),
\]

where the sum ranges on the irreducible components of \( X_s \) endowed with the reduced structure, shows that the function \( \delta_{X/S,L} \) is bounded above.

On the other hand, the degree is invariant under extension of scalars, thus

\[
\deg(X_s, L_{|X_s}) \geq \sum_Z \deg(Z, L_{|Z}),
\]

where the sum ranges on the irreducible components of \( X_s \) endowed with the reduced structure, where \( s \) is a geometric point over \( S \). Since \( \deg(Z, L_{|Z}) \) is a positive integer, the right-hand side of the above inequality is bounded below by \( \iota_{X/S}(s) \). This shows that the function \( \iota_{X/S} \) is bounded above. \( \square \)
Proposition 2.4. For integers $N, D \geq 1$, there is an integer $R_D = R_D(N)$ such that, for a field $k$, a closed subvariety $X$ of $\mathbb{P}^N_k$ of degree $\leq D$, the following statements hold:

1. the number of irreducible components of $X$ is $\leq R_D$;
2. any irreducible component $Z$ of its singular locus $X^{\text{sing}}$ (endowed with the reduced structure) has degree $\deg(Z, \mathcal{O}_{\mathbb{P}^N_k}(1)) \leq R_D$;
3. for an algebraic closure $k$ of $k$, the number of irreducible components of $X^k_{\text{sing}}$ is $\leq R_D$.

Proof. According to [SGA 6, Exp. XIII, Corollaire 6.11 (ii)], it suffices to prove the statement when the subvarieties in question have a fixed Hilbert polynomial $P \in \mathbb{Q}[z]$. For, consider the Hilbert scheme

$$S = \text{Hilb}_{\mathbb{P}^N_k, \mathcal{O}(1)}$$

of the closed subschemes of $\mathbb{P}^N_k$ with Hilbert polynomial $P$ with respect to $\mathcal{O}(1)$. Let $X \subseteq \mathbb{P}^N_k$ be the universal family. The morphism $\pi: X \to S$ induced by the second projection is proper and flat. Let $n$ be its relative dimension.

For (1), apply Lemma 2.3 to the morphism $\pi: X \to S$ and the relatively ample line bundle $L := \mathcal{O}_{\mathbb{P}^N_k}(1)|_X$.

For (2) and (3), let $X^{\text{sing}}$ be the closed subset in $X$ where the coherent $\mathcal{O}_X$-module $\Omega_{X/S}$ has rank $\geq n + 1$. One concludes by applying Lemma 2.3 to the morphism $\pi: X^{\text{sing}} \to S$ and the relatively ample line bundle $L := \mathcal{O}_{\mathbb{P}^N_k}(1)|_{X^{\text{sing}}}$. □

2.2. Families of normal cycles. Let $k$ be a field of characteristic $0$ and $X$ a geometrically integral normal variety over $k$.

2.2.1. Definition. Following Kollár ([Kol93]), a normal cycle on $X$ is a finite morphism $f: Z \to X$ which is birational onto its image, where $Z$ is a geometrically integral normal variety.

A family of normal cycles on $X$ is the datum of morphism of $k$-schemes $\pi: Z \to S$ and $f: Z \to X$ where

- $S$ is reduced and a countable disjoint union of varieties over $k$;
- the morphism $\pi$ is separated, of finite type, flat and with geometrically integral normal fibers;
- for $s \in S$, the map $f_s: Z_s \to X_{\kappa(s)}$ is a normal cycle, where $\kappa(s)$ is the residue field at $s$.

2.2.2. Exhaustive families. A family of normal cycles $(\pi: Z \to S, f: Z \to X)$ on $X$ is exhaustive\footnote{Kollár uses the name ‘weakly complete’} if, given a field extension $k'$ of $k$ and a normal cycle $g: Z \to X_{k'}$, there is a unique $s \in S(k')$ such that $g = f_s$.

Given an exhaustive family of normal cycles $(\pi: Z \to S, f: Z \to X)$ on $X$ and an open immersion $j: U \to X$, the couple

$$(\pi: Z \times_X U \to \pi(Z \times_X U), f: Z \times_X U \to U)$$

is an exhaustive family of normal cycles on $U$.

Proposition 2.5. If $X$ is projective, then there exists an exhaustive family of normal cycles $(\pi: Z \to S, f: Z \to X)$ on $X$ such that, for any ample line bundle $L$ on $X$ and any integer $d \geq 1$, the set

$$S_{L, d} := \{ s \in S : \deg(Z_s, f_s^* L) \leq d \}$$

is a finite union of connected components of $S$ (thus of finite type).
Lemma 2.6. Let \( \pi: Y \to S \) be a morphism between varieties over \( k \) with integral generic fibers and \( S \) reduced. Let \( \nu: \tilde{Y} \to Y \) the normalization of \( Y \).

Then, there is a dense open subset \( S' \) of \( S \) such that, for \( s \in S' \), the induced morphism \( \tilde{Y}_s \to Y_s \) is the normalization.

Proof. First, one may assume \( Y \) to be reduced; for, the ideal sheaf of nilpotent elements of \( Y \) is supported on a closed subset meeting no generic fiber of \( \pi \).

Let \( S_0 \) be the open subset of \( S \) such that the fibers of \( \tilde{\pi} : \tilde{Y} \to S \) are (geometrically) normal ([EGA IV, Corollaire 9.9.5]). Note that \( S_0 \) contains each of the generic points of \( S \) because the generic fibers of \( \tilde{\pi} \) are normal.

The morphism \( \nu \) is finite birational, the latter meaning that \( \nu \) is an open immersion on a dense open subset \( U \) of \( \tilde{Y} \). For each \( s \in S \), the induced map \( \nu_s : \tilde{Y}_s \to Y_s \) is finite, whereas this will not be the case in the general for the ‘birational’ property. To remedy that, remark that the image \( \tilde{\pi}(U) \) is constructible and contains all of the generic points of \( S \). Therefore, the subset \( S_0 \cap \tilde{\pi}(U) \) contains a dense open subset \( S' \).

Lemma 2.7. Let \( f: Y \to S \) be a morphism between varieties over \( k \) with geometrically integral fibers and \( S \) reduced.

Then, there is a surjective immersion of varieties \( \iota : S' \to S \) and a finite morphism \( \nu : Y' \to Y \times_S S' \) such that the composite map \( Y' \to S' \) is flat and, for each \( s' \in S' \), the induced map \( Y'_{s'} \to Y_{\iota(s')} \) is the normalization.

Note that the morphism \( \iota \) being a surjective immersion simply means that \( S' \) is the disjoint union of finitely many pairwise disjoint locally closed subsets of \( S \) whose union is the whole \( S \).

Proof. Arguing by Noetherian induction, it suffices to construct a dense open subset \( S_0 \) of \( S \) and a finite morphism \( Y_0 \to Y \times_S S_0 \) with the properties in the statement. This is obtained combining Lemma 2.6 with the openness of the locus where a morphism is flat.

Proof of Proposition 2.5. Let \( H \) be a connected component of the Hilbert scheme \( \text{Hilb}(X) \) of \( X \) endowed with the reduced structure. Let \( u : \mathcal{U} \to H \) be the base-change to \( H \) of the universal family of closed subschemes of \( X \) parameterized by \( \text{Hilb}(X) \).

The locus where the fibers of the map \( u \) are geometrically integral is constructible ([EGA IV, Théorème 9.7.7]); let \( H' \) be disjoint union of the irreducible components of such a constructible subset, again endowed with the reduced structure. The base-change \( u' : \mathcal{U}' \to H' \) of \( u : \mathcal{U} \to H \) along the immersion \( H' \to H \) satisfies the hypotheses of Lemma 2.7. Applying it yields a surjective immersion \( \iota : \mathcal{S}' \to H' \) and finite morphism \( \nu : \mathcal{Z}_H \to \mathcal{U}' \times_{H'} \mathcal{S}'_H \) such that the composite map \( \pi_H : \mathcal{Z}_H \to \mathcal{S}_H \) is flat with normal geometrically integral fibers and, for each \( s \in \mathcal{S}_H \), the induced morphism \( \nu_s : \mathcal{Z}_{H,s} \to \mathcal{U}'_{\iota(s)} \) is the normalization.

The sought-for exhaustive family of normal cycles on \( X \) is

\[
\pi = \bigsqcup_H \pi_H : \mathcal{Z} := \bigsqcup_H \mathcal{Z}_H \longrightarrow S := \bigsqcup_H \mathcal{S}_H,
\]

the disjoint unions running over all the connected components of the Hilbert scheme of \( X \) endowed with the reduced structure.

For an ample line bundle \( L \) and an integer \( d \geq 1 \) there are only finitely many connected components of the Hilbert scheme parameterizing varieties of degree \( \leq d \), whence the result. \( \square \)
Proposition 2.8. Suppose $k = \mathcal{C}$ and $X$ projective. Let $(\pi : \mathcal{Z} \to S, f : \mathcal{Z} \to X)$ be an exhaustive family of normal cycles on $X$.

Then, there is a surjective immersion of finite type $\iota : S' \to S$ of $\mathcal{C}$-schemes such that, for any connected component $T$ of $S$, the induced map $(\mathcal{Z'} \times_S T')(\mathcal{C}) \to T'(\mathcal{C})$ is a topological fiber bundle, where $T' = \iota^{-1}(T)$ and $\mathcal{Z'} = \mathcal{Z} \times_S S'$.

Proof. See the proof of [Kol93, Proposition 2.4].

Proposition 2.9. Suppose $k$ is an algebraically closed subfield of $\mathcal{C}$. Let $L$ be an ample line bundle over $X$, $U$ a nonempty open subset of $X$ and $d \geq 1$ an integer.

Then, up to conjugation, there are only finitely many subgroups of $\pi_1^{et}(U)$ obtained as the image of $\pi_1^{et}(f^{-1}(U)) \to \pi_1^{et}(U)$ with $f : Z \to X$ a normal cycle such that

$$\deg(Z, f^*L) \leq d.$$ 

Proof. Since the geometric étale fundamental group is insensible to extension of scalars, and since there are more normal cycles on $X_{\mathcal{C}}$ than on $X$, there is no loss of generality in assuming $k = \mathcal{C}$.

Let $(\pi : \mathcal{Z} \to S, f : \mathcal{Z} \to X)$ be an exhaustive family of normal cycles satisfying the property in the statement of Proposition 2.5. According to Proposition 2.8, up to taking a surjective immersion of finite type, one may suppose that, for each connected component $T$ of $S$, the induced map $\pi^{-1}(T)(\mathcal{C}) \to T(\mathcal{C})$ is a topological fiber bundle.

Let $S_U$ be the image of $\mathcal{Z}_U := f^{-1}(U)$ in $S$ via $\pi$. Then

$$(\pi : \mathcal{Z}_U \to S_U, f : \mathcal{Z}_U \to U)$$

is an exhaustive family of normal cycles on $U$; moreover, for each connected component $T$ of $S_U$, the induced map $\pi^{-1}(T)(\mathcal{C}) \to T(\mathcal{C})$ is again a topological fiber bundle. It follows that, for $t, t' \in T(\mathcal{C})$, the images of $\pi_1^{top}(\mathcal{Z}_U, t)(\mathcal{C})$ and $\pi_1^{top}(\mathcal{Z}_U, t')(\mathcal{C})$ in $\pi_1^{top}(U(\mathcal{C}))$ are conjugated subgroups. Passing to profinite completions, the corresponding affirmation holds for étale fundamental groups.

Let $C_{L,d}$ be the set of conjugacy classes of subgroups of $\pi_1^{et}(U)$ obtained as the image of $\pi_1^{et}(g^{-1}(U)) \to \pi_1^{et}(U)$ with $g : Z \to X$ a normal cycle such that

$$\deg(Z, g^*L) \leq d.$$ 

With the notation of Proposition 2.5, the argument above shows the inequality

$$\#C_{L,d} \leq \#(\text{connected components of } S_{L,d})$$

and the right-hand side is $< \infty$ by the cited result.

2.3. Proof of Theorem B. Let us begin with two easy facts:

Lemma 2.10. Let $\pi : X' \to X$ be a finite surjective morphism between integral normal varieties over $k$ which is Galois of group $G$ and étale over a non-empty open subset $U$ of $X$. Let $Y'$ an integral closed subvariety of $X'$ and $Y := \pi(Y')$. Then, the map $\pi|_{Y'} : Y' \to Y$ has degree

$$\# \text{Im}(\nu_1^{et}(\nu^{-1}(U \cap Y)) \to G),$$

where $\nu : \tilde{Y} \to Y$ is the normalization.

Proof. Since the degree is computed on the generic point, there is no harm in replacing $X$ by $U$ and $X'$ by $\pi^{-1}(U)$, so that the morphism $\pi$ is finite étale.

The induced morphism $\tilde{Y} \times_X X' \to \tilde{Y}$ is étale, thus the fibered product $\tilde{Y} \times_X X'$ is normal ([SGA 1, Exp. I, Théorème 9.5 (i)]). The connected components of $\tilde{Y} \times_X X'$ are therefore integral ([Stacks, Lemma 033M]), and the normalization $\tilde{Y}'$ of $Y'$ is one of them.
The usual dictionary between connected covers and subgroups of the fundamental group implies that the degree of the induced map \( \tilde{\pi} : \tilde{Y} \to \tilde{Y} \) is \( \# \text{Im}(\pi_1^\text{et}(\tilde{Y}) \to G) \). The normalization being a birational map, the degree of \( \pi : Y' \to Y \) coincides with that of \( \tilde{\pi} \), whence the statement. \( \square \)

**Lemma 2.11.** Let \( \Gamma \) be a residually finite group and \( F \) a finite subset of \( \Gamma \). Then, there is a finite-index normal subgroup \( N \) of \( \Gamma \) such that the map \( F \to \Gamma/N \) is injective.

**Proof.** Consider the finite subset \( F' := \{ \gamma \delta^{-1} : \gamma, \delta \in F \} \setminus \{ e \} \) where \( e \) is the neutral element of \( \Gamma \).

Saying that the group \( \Gamma \) is residually finite amounts to the fact that, for each \( \gamma \in \Gamma \setminus \{ e \} \), there is a finite-index normal subgroup \( N_\gamma \) to which \( \gamma \) does not belong. Then, the finite-index normal subgroup \( N := \bigcap_{\gamma \in F'} N_\gamma \) does the job. \( \square \)

We are now in position to proceed with the proof.

**Proof of Theorem B.** To begin with, recall the setup: let \( X \) be a normal irreducible projective variety over an algebraically closed field \( k \) of characteristic 0, \( L \) an ample line bundle, \( U \) an non-empty open subset of \( X \) such that the étale fundamental group of \( U \) is large, and \( d \geq 1 \) an integer.

By Lefschetz’s principle, the varieties \( X, U \) and the line bundle \( L \) can be defined over a subfield of \( k \) finitely generated over \( \mathbb{Q} \). Since having large étale fundamental group is a property that passes to algebraically closed subfields, there is no loss of generality in assuming that \( k \) is a subfield of \( \mathbb{C} \).

Then, according to Proposition 2.9, there are finitely many subgroups \( H_1, \ldots, H_r \) of \( \pi_1^\text{et}(U) \) such that, given a normal cycle \( f : Z \to X \) such that \( \text{deg}(Z, f^*L) \leq d \), the image of the group homomorphism \( \pi_1^\text{et}(f^{-1}(U)) \to \pi_1^\text{et}(U) \) is conjugated to some of the subgroups \( H_i \).

Needless to say, by definition of large étale fundamental group, each of the subgroups \( H_i \) is infinite. By design, the étale fundamental group is profinite (in particular, residually finite), thus each finite subset injects into some finite quotient: Lemma 2.11 implies that there exists a normal subgroup \( N \) of \( \pi_1^\text{et}(U) \) such that, for each \( i = 1, \ldots, r \), the image of \( H_i \to G := \pi_1^\text{et}(U)/N \) has cardinality \( \geq d \).

Let \( U' \to U \) be the finite étale cover of \( U \) associated with the subgroup \( N \). As argued in the proof of Lemma 2.10, the normality of \( U \) implies that of \( U' \), thus the variety \( U' \), a priori just connected, is integral. Let \( \pi : X' \to X \) be the normalization of \( X \) in \( U' \). By construction, the variety \( X' \) is integral normal, and the morphism \( \pi \) is Galois of group \( G \) and étale over \( U \).

To see that such a cover fulfills the requirements, let \( Y' \) be a positive-dimensional integral closed subvariety of \( X' \) meeting \( \pi^{-1}(U) \) and set \( Y := \pi(Y') \). The projection formula reads

\[
\text{deg}(Y', \pi^*L_{|Y'}) = \text{deg}(\pi_{1|Y'}) \text{deg}(Y, L_{|Y}),
\]
where \( \text{deg}(\pi_{1|Y'}) \) is the degree of the finite map \( Y' \to Y \) induced by \( \pi \). Of course, if \( \text{deg}(Y, L_{|Y}) > d \), then \( \text{deg}(Y', \pi^*L_{|Y'}) \geq d \).

Suppose instead \( \text{deg}(Y, L_{|Y}) \leq d \) and let \( \nu : \tilde{Y} \to Y \) be the normalization. By the projection formula (or Lemma 2.1),

\[
\text{deg}(\tilde{Y}, \nu^*L_{|\tilde{Y}}) = \text{deg}(Y, L_{|Y}) \leq d.
\]
According to Lemma 2.10, the map \( \pi_{1|Y'} \) has degree \( \# \text{Im}(\pi_1^\text{et}(U \cap Y)) \to G \). On the other hand, by construction, the image of \( \pi_1^\text{et}(\nu^{-1}(U \cap Y)) \to \pi_1^\text{et}(U) \) is conjugated to some of the subgroups \( H_i \), thus

\[
\# \text{Im}(\pi_1^\text{et}(\nu^{-1}(U \cap Y)) \to G) = \# \text{Im}(H_i \to G) \geq d.
\]
By ampleness of $L$, the degree $\deg(Y, L|_Y)$ is a positive integer, thus the projection formula implies $\deg(Y', \pi^*L|_{Y'}) \geq d$ as desired. 

\section{Arithmetic}

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers.

\subsection{Growth rates.}

\subsubsection{Hermitian line bundles.}

A Hermitian line bundle $\mathcal{L}$ on a proper and flat $\mathcal{O}_K$-scheme $X$ is the datum of a line bundle $L$ on $X$ and, for every embedding $\sigma : K \to \mathbb{C}$, a continuous metric $\| \cdot \|_\sigma$ on the holomorphic line bundle $L|_{X_\sigma}(\mathbb{C})$. Moreover, the collection of metrics $\{\| \cdot \|_\sigma\}$ is supposed to be compatible with complex conjugation.

For a morphism of $\mathcal{O}_K$-schemes $f : Y \to X$, where the $\mathcal{O}_K$-scheme $Y$ is also proper and flat, the pull-back $f^* \mathcal{L}$ is defined in the evident way.

\subsubsection{Degree.}

The (Arakelov) degree of a Hermitian line bundle $\mathcal{L}$ on $\text{Spec} \mathcal{O}_K$ is

$$\deg \mathcal{L} = \log \#(L/s) - \sum_{\sigma : K \to \mathbb{C}} \log \|s\|_\sigma,$$

where $s$ is a non-zero element of the $\mathcal{O}_K$-module $L$; the quantity above does not depend on its choice because of the product formula.

\subsubsection{Extension of scalars.}

Let $K'$ be a finite extension of $K$ and

$$\pi : \text{Spec} \mathcal{O}_{K'} \to \text{Spec} \mathcal{O}_K$$

the morphism induced by the inclusion of $\mathcal{O}_K$ in $\mathcal{O}_{K'}$. Given a Hermitian line bundle $\mathcal{L}$ over $\mathcal{O}_K$, the Hermitian line bundle $\pi^* \mathcal{L}$ over $\mathcal{O}_{K'}$ has degree

$$\deg \pi^* \mathcal{L} = [K' : K] \deg \mathcal{L}. \quad (3.1)$$

\subsubsection{Height.}

Let $\mathcal{L}$ be a Hermitian line bundle on a proper and flat $\mathcal{O}_K$-scheme $X$. By the valuative criterion of properness, a $K$-rational point $P$ of $X$ extends uniquely to an $\mathcal{O}_K$-valued point $P$ of $X$. The height of the point $P$ with respect to the Hermitian line bundle $\mathcal{L}$ is

$$h_{\mathcal{L}}(P) = \deg P^* \mathcal{L} / [K : \mathbb{Q}].$$

Replacing the number field $K$ by a finite extension permits to define the height for any point of $X$ with values in an algebraic closure $\bar{K}$ of $K$ (equation (3.1) implies that the height is well-defined). A routine variation of the proof of [HS00, Theorem B.3.2 (d)] yields:

\begin{proposition}
Let $X$ and $X'$ be proper and flat $\mathcal{O}_K$-schemes endowed respectively with Hermitian line bundles $\mathcal{L}$ and $\mathcal{L}'$.

Suppose that there exists an isomorphism $f : X_K \to X'_K$ between the generic fibers of $X$ and $X'$, and an isomorphism $\mathcal{L}|_{X_K} \cong f^* \mathcal{L}'|_{X'_K}$ of line bundles over $X_K$. Then,

$$\sup_{P \in X(\mathbb{K})} |h_{\mathcal{L}}(P) - h_{\mathcal{L}'}(f(P))| < +\infty.$$

A line bundle $\mathcal{L}$ on a proper and flat $\mathcal{O}_K$-scheme is generically ample if its restriction to the generic fiber $X_K$ is ample. In this framework the Northcott property can be stated as follows (see for instance [HS00, Theorem B.2.3]):

\begin{proposition}
Let $X$ be a proper and flat $\mathcal{O}_K$-scheme together with a Hermitian line bundle $\mathcal{L}$ on $X$. If $\mathcal{L}$ is generically ample, then, for a real number $c$, the set

$$\{P \in X(\mathbb{K}) : h_{\mathcal{L}}(P) \leq c\}$$

is finite.
\end{proposition}
3.1.5. Counting function. Let $X$ be a proper and flat $\mathcal{O}_K$-scheme, $\mathcal{L}$ a generically ample Hermitian line bundle on $X$ and $U$ an open subset of $X$. For a subring $R$ of $K$ containing $\mathcal{O}_K$ and a real number $c$, Proposition 3.2 permits to define

$$\nu(X, \mathcal{L}, U, R, c) := \log^+ \# \{ P \in U(R) : h_{\mathcal{L}}(P) \leq c \},$$

where $\log^+ z := \log \max\{1, z\}$ for $z \in \mathbb{R}$. The growth rate of $R$-points of $U$ is

$$\text{growth}(X, \mathcal{L}, U, R) := \limsup_{c \to +\infty} \frac{\nu(X, \mathcal{L}, U, R, c)}{c}.$$

Clearly, such a function is non-decreasing in the variable $R$ (with respect to inclusion).

3.1.6. Growth rates. Let $X$ be a proper $K$-scheme, $L$ an ample line bundle on $X$ and $U$ an open subset of $X$. Choose a proper and flat $\mathcal{O}_K$-scheme $\mathcal{X}$, a Hermitian line bundle $\mathcal{L}$ on $\mathcal{X}$ and an open subset $U$ of $\mathcal{X}$ whose generic fibers are respectively $X$, $L$ and $U$. By Proposition 3.1, the real numbers

$$\text{gr. rat}_K(X, L, U) = \text{growth}(X, \mathcal{L}, U, K),$$

$$\text{gr. int}_K(X, L, U) = \sup_S \text{growth}(X, \mathcal{L}, U, \mathcal{O}_K[S]),$$

where the supremum ranges on the finite set of places $S$ of $K$ containing the Archimedean ones, do not depend on the chosen $X$, $\mathcal{L}$ and $U$. They are called the growth rate respectively of rational and integral points of $U$ (with respect to $X$ and $L$). Clearly,

$$\text{gr. int}_K(X, L, U) \leq \text{gr. rat}_K(X, L, U).$$

Remark 3.3. Some considerations:

1. The growth rate of rational and integral points differ in general when $U$ is not proper. For instance, take $X = \mathbb{P}_K^1$, $L = \mathcal{O}(1)$ and $U = \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$. Then,

$$\text{gr. rat}_K(X, L, U) = \text{gr. rat}_K(X, L, X) = 2[K : \mathbb{Q}],$$

$$\text{gr. int}_K(X, L, U) = 0,$$

because of the finiteness of solutions to the $S$-unit equation (see, for instance, [BG06, Chapter 5]).

2. Let $K'$ be a finite extension of $K$, $X'$ and $U'$ the $K'$-schemes deduced respectively from $X$ and $U$ by extending scalars to $K'$, and $L'$ the line bundle on $X'$ deduced from $L$. Then,

$$\text{gr. rat}_K(X, L, U) \leq \text{gr. rat}_{K'}(X', L', U'),$$

and similarly for the growth rate of integral points. However, as the example above shows, in general the real numbers $\text{gr. rat}_{K'}(X', L', U')$ tend to $\infty$ as soon as the degree of $K'$ does.

3. For an integer $n \geq 1$,

$$\text{gr. rat}_K(X, L, U) = n \text{gr. rat}_K(X, L^\otimes n, U)$$

and similarly for the growth rate of integral points. In particular, these growth rates really depend on the line bundle $L$ and not only on its restriction to $U$.

3.2. Proof of Theorem A.
3.2.1. **Statement.** Let $N \geq 1$ be an integer. For a locally closed subvariety $U$ of $\mathbb{P}^N_K$ and a real number $c$, with the notation of section 3.1.5, let

$$\nu_K(U, c) := \nu(X, \mathcal{L}, U, K, c),$$

where $X$ is the Zariski closure of $U$ in $\mathbb{P}^N_{O_K}$, $Z$ is the Zariski closure of $X_K \setminus U$ in $\mathbb{P}^N_{O_K}$, $U = X \setminus Z$, and $\mathcal{L}$ the Hermitian line bundle on $\mathbb{P}^N_{O_K}$ obtained by endowing the line bundle $O_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N_{O_K}$ with metrics, for an embedding $\sigma : K \to \mathbb{C}$,

$$\|s\|_\sigma(x) = \frac{|s(x)|_\sigma}{\max_{i=0, \ldots, N} |x_i|_\sigma},$$

where $s$ is a local section of $O(1)$ and $x_0, \ldots, x_N$ are the homogeneous coordinates of a point $x$ in $\mathbb{P}^N(\mathbb{C})$.

The height function associated with the Hermitian line bundle $\mathcal{L}$ is, for a $K$-rational point $x$ of $\mathbb{P}^N_K$,

$$h(x) := h_{\mathcal{L}}(x) = \sum_{v \in V^0_K} \log \max_{i=0, \ldots, N} |x_i|_v + \sum_{\sigma : K \to \mathbb{C}} \log \max_{i=0, \ldots, N} |x_i|_\sigma,$$

where $V^0_K$ is the set of finite places of $K$. In the formula above, for a $p$-adic place $v$, the absolute value $| \cdot |_v$ is normalized as $|p|_v = p^{-[K_v : \mathbb{Q}_p]}$ where $K_v$ is the completion of $K$ with respect to $v$.

**Theorem 3.4.** Let $Z$ be a closed subvariety of $\mathbb{P}^N_K$, $\varepsilon > 0$ a real number and $n \geq 0$, $D \geq 1$ integers.

Then, there is a real number $C_{n, D} = C_{n, D}(N, K, Z, \varepsilon)$ with the following property: for an integral $n$-dimensional closed subvariety $X$ of $\mathbb{P}^N_K$, of degree $\leq D$ such that each positive-dimensional integral closed subvariety in $X$ not contained in $Z$ has degree $\geq d^{\dim Z}$ for some integer $d \geq 1$, and a real number $c \geq [K : \mathbb{Q}]\varepsilon$, the following inequality holds:

$$\nu_K(X \setminus Z, c) \leq c[K : \mathbb{Q}](1 + \varepsilon)^{n(n + 3)}{2d} + C_{n, D}.$$ 

Note that, for $n = 0$, the above inequality reads $\nu_K(X \setminus Z, c) \leq C_{n, D}$. Before going into the proof of the preceding statement, let us see how it permits to prove Theorem A.

**Proof of Theorem A.** First of all, and rather crucially, notice that the hypotheses and the conclusions are insensitive to taking powers of $L$. Therefore, up to replacing $L$ with a multiple big enough, one may assume that $L$ is very ample. Via the associated embedding $X \to \mathbb{P}(\Gamma(X, L)^\vee)$, Theorem 3.4 can be applied to give

$$\text{gr. rat}_K(X, L, X \setminus Z) \leq [K : \mathbb{Q}]\frac{n(n + 3)}{2d},$$

as desired. \hfill \Box

3.2.2. **Proofs.** The statement will be deduced by induction from the following:

**Theorem 3.5 ([Che12b, Theorem A]).** Let $\varepsilon > 0$ be a real number and $D \geq 0$ an integer.

Then, there are positive integers $A_D = A_D(N, K, \varepsilon)$ and $B_D = B_D(N, K, \varepsilon)$ with the following property: for an integral closed subvariety $X$ of $\mathbb{P}^N_K$ of degree $D' \leq D$, and a real number $c \geq [K : \mathbb{Q}]\varepsilon$, the set

$$\{x \in X^\text{reg}(K) : h(x) \leq c\}$$
can be covered by no more than
\[ A_D \exp \left( \frac{n+1}{D^{1/n}} (1+\varepsilon) [K : \mathbb{Q}] c \right) \]
hypersurfaces in \( \mathbb{P}_K^n \) of degree \( \leq B_D \) not containing \( X \), where \( n = \dim X \).

Proof of Theorem 3.4. The proof goes by induction on \( n \). For \( n = 0 \), there is nothing to do, as \( X \) is a singleton.

Suppose \( n \geq 1 \) and the result true in dimension \( < n \). Let \( A_D \) and \( B_D \) be as in the statement of Theorem 3.5, and \( R_D \) as in that of Proposition 2.4.

Let \( X \) be an integral \( n \)-dimensional closed subvariety of \( \mathbb{P}_K^n \) of degree \( D' \leq D \) such that all positive-dimensional integral closed subvarieties \( Y \) of \( X \) not contained in \( Z \) have degree \( \geq d^{\dim Y} \). Quite trivially, such a property is inherited by integral subvarieties \( X' \) of \( X \) not contained in \( Z \): any positive-dimensional integral closed subvariety \( Y' \) of \( X' \) not contained in \( Z \) has degree \( \geq d^{\dim Y'} \).

Now, by Theorem 3.5, there exist \( r \) hypersurfaces \( H_1, \ldots, H_r \) of \( \mathbb{P}_K^n \) of degree \( \leq B_D \) with
\[ r \leq A_D \exp \left( \frac{n+1}{D^{1/n}} (1+\varepsilon) [K : \mathbb{Q}] c \right) \]
not containing \( X \) such that the set \{ \( x \in X^{\text{reg}}(K) : h(x) \leq c \} \) is contained in the union of \( H_1, \ldots, H_r \). For \( i = 1, \ldots, r \), the hyperplane section \( X_i := H_i \cap X \) of \( X \) is pure of dimension \( n - 1 \) and has degree \( \leq DB_D \).

Each irreducible component \( X'_i \) of \( X_i \) has degree \( \text{deg}(X'_i, \mathcal{O}(1)) \leq DB_D \) by Lemma 2.2. Therefore, it is possible to apply the induction hypothesis to such an \( X'_i \) and obtain
\[ \nu_K(X'_i \setminus Z, c) \leq c[K : \mathbb{Q}] (1+\varepsilon) \frac{(n-1)(n+2)}{2d} + C_{n-1,DB_D}, \]
because \( d \leq DB_D \).

Since the hyperplane section \( X_i \) has at most \( R_{DB_D} \) irreducible components by Proposition 2.4,
\[ \nu_K(X_i \setminus Z, c) \leq c[K : \mathbb{Q}] (1+\varepsilon) \frac{(n-1)(n+2)}{2d} + C_{n-1,DB_D} + \log R_{DB_D}. \]
Taking into account (3.2), and recalling \( D' \geq d^n \), the preceding inequality yields
\[ \nu_K(X^{\text{reg}}_i \setminus Z, c) \leq c[K : \mathbb{Q}] (1+\varepsilon) \frac{n(n+3)}{2d} + C'_{n,D}, \]
where \( C'_{n,D} := C_{n-1,DB_D} + \log R_{DB_D} + \log A_D \) and the following identity has been noticed:
\[ (n + 1) + \frac{(n-1)(n+2)}{2} = (n + 1) + \sum_{i=1}^{n-1} (i+1) = \sum_{i=1}^{n} (i+1) = \frac{n(n+3)}{2}. \]

Let \( Y_1, \ldots, Y_s \) be the irreducible components of the singular locus \( X^{\text{sing}} \) of \( X \). According to Proposition 2.4, the subvariety \( Y_i \) has degree \( \leq RD \) and \( s \leq RD \). The induction hypothesis, applied to an irreducible component \( Y_i \) not contained in \( Z \), gives
\[ \nu_K(Y_i \setminus Z, c) \leq c[K : \mathbb{Q}] (1+\varepsilon) \frac{(n-1)(n+2)}{2d} + \tilde{C}_{n-1,RD}, \]
where \( \tilde{C}_{n-1,RD} := \max\{C_{0,RD}, \ldots, C_{n-1,RD}\} \), because the irreducible component \( Y_i \) has degree \( \leq RD \) and dimension \( \leq n - 1 \). In particular,
\[ \nu_K(X^{\text{sing}} \setminus Z, c) \leq c[K : \mathbb{Q}] (1+\varepsilon) \frac{(n-1)(n+2)}{2d} + C'_{n,D}, \]
where \( C'_{n,D} := \tilde{C}_{n-1,RD} + \log RD \).
Combining the inequalities (3.3) and (3.4) yields the bound in the statement with \( C_{n,D} := \log(\exp(C'_{n,D}) + \exp(C''_{n,D})) \). □

3.3. Growth rates on covers. The last ingredient for the proof of the main theorem is a version of the Chevalley-Weil theorem for growth rates of integral points (Proposition 3.11). Its proof is a variation of the classical one (see [Ser89, 4.2] or Theorem 2.3 in Corvaja’s contribution to [CCD+21]).

3.3.1. Reminder on twists. Let \( S \) be a Noetherian scheme and \( G \) a finite group. The constant group \( S \)-scheme with value \( G \) is still denoted \( G \).

A principal \( G \)-bundle is a faithfully flat finitely presented (a posteriori étale) \( S \)-scheme \( P \) endowed with an action of \( G \) such that the morphism \( G \times_S P \to P \times_S P \), \((g,p) \mapsto (gp,p)\) is an isomorphism. The set of isomorphism classes of principal \( G \)-bundles over \( S \) is denoted \( \text{H}^1(S,G) \) or simply \( \text{H}^1(A,G) \) if \( S = \text{Spec} \, A \) is affine.

**Proposition 3.6.** Let \( K \) be a number field, \( S \) a finite set of places of \( K \) containing the Archimedean ones. Then,

\[ \#\text{H}^1(\mathcal{O}_{K,S},G) < +\infty. \]

**Proof.** A principal \( G \)-bundle \( P \) over \( \mathcal{O}_{K,S} \) is finite étale as an \( \mathcal{O}_{K,S} \)-scheme. Thus its generic fiber \( P_K \) is the spectrum of a \( K \)-algebra \( A \) whose dimension as a \( K \)-vector space is \( \#G \) and is the product of (finitely many) finite extensions of \( K \), all of which are unramified outside \( S \). The Hermite-Minkowski bound implies that, for any integer \( D \geq 1 \), there are only finitely many isomorphism classes of finite extensions of \( K \) of degree \( \leq D \) unramified outside \( S \) ([Ser89, 4.1]). The statement follows. □

**Definition 3.7.** Let \( X \) be an \( S \)-scheme endowed with an action of \( G \) and \( P \) a principal \( G \)-bundle over \( S \). The twist of \( X \) by \( P \), if it exists, is the categorical quotient of \( X \times_S P \) by the diagonal action \( g(x,p) = (gx, gp) \) of \( G \).

Clearly, isomorphic principal \( G \)-bundles give rise to isomorphic twists. With an abuse of notation, for \( t \in \text{H}^1(S,G) \), let \( X_t \) denote the twist of \( X \) by a principal \( G \)-bundle in the isomorphism class \( t \). For a scheme \( X \) over \( P \), the datum of an equivariant formation of \( G \) is equivalent to a descent datum. Therefore, the theory of faithfully flat (actually, étale) descent shows the following existence result:

**Proposition 3.8 ([Ols16, Proposition 4.4.9]).** Let \( X \) be an affine \( S \)-scheme endowed with an action of \( G \) and \( P \) a principal \( G \)-bundle over \( S \). Then, the twist of \( X \) by \( P \) exists.

**Remark 3.9.** The construction of twists is functorial. Namely, let \( X, Y \) be \( S \)-schemes endowed with an action of \( G \), and \( P \) a principal \( G \)-bundle over \( S \) for which the twists of \( X \) and \( Y \) by \( P \) exist. Then, a \( G \)-equivariant morphism \( f : X \to Y \) induces a morphism \( f_P : X_P \to Y_P \) between twists.

3.3.2. Lifting points. A useful construction consists in twisting schemes to lift points. More precisely, let \( X \) be an \( S \)-scheme and \( \pi : Y \to X \) a principal \( G \)-bundle over \( X \). For an \( S \)-valued point \( x \) of \( X \), the scheme-theoretic fiber of \( \pi \) at \( x \),

\[ P := Y \times_X S, \]

is a principal \( G \)-bundle over \( S \). The twist of \( Y \) by \( P \) exists as X-scheme\(^{12}\) and the \( G \)-invariant map \( \pi : Y \to X \) induces a morphism \( \pi_P : Y_P \to X \).

**Lemma 3.10.** There is an \( S \)-valued point \( y \) of \( Y_P \) such that \( \pi_P(y) = x \).

\(^{12}\)Indeed, it is identified with the twist of \( Y \) by the \( G \)-principal bundle \( P \times_S X \), and the latter exists because \( \pi \) is a finite morphism (in particular affine) according to the finiteness of \( G \).
Proof. The scheme-theoretic fiber $X_P \times X$ of $\pi_P$ at $x$ is isomorphic to the twist $P_P$ of $P$ by itself. The diagonal embedding $\Delta : P \to P \times S$ is $G$-equivariant and, taking its quotient by $G$, defines the wanted $S$-valued point $y : S \to P_P$. \hfill $\Box$

3.3.3. Statement. A finite surjective morphism $f : Y \to X$ between algebraic varieties over $K$ is Galois if, for each geometric point $\tilde{x}$ of $X$, the group $\text{Aut}(f)$ acts transitively on the geometric fiber $Y_{\tilde{x}}$.

The group $\text{Aut}(f)$ acts by definition on $Y$. For $t \in H^1(K, \text{Aut}(f))$, let $Y_t$ be the twist of $Y$ and $f_t : Y_t \to X$ the twist of the $\text{Aut}(f)$-invariant morphism $f$.

Proposition 3.11. Let $f : Y \to X$ be a finite surjective morphism between integral projective varieties over $K$, $U$ an open subset of $X$ over which $f$ is étale, and $L$ an ample line bundle on $X$. If the morphism $f$ is Galois, then

$$\text{gr. int}_K(X, L, U) \leq \max_{t \in H^1(K, \text{Aut}(f))} \text{gr. int}_K(Y_t, f_t^* L, f_t^{-1}(U)).$$

Proof. Let $G := \text{Aut}(f)$. Up to taking a power of $L$—an operation that does not affect the statement—the variety $X$ can be realized as the generic fiber of a projective flat $\mathcal{O}_K$-scheme $\tilde{X}$ over which the line bundle $L$ extends to an ample line bundle $\mathcal{L}$. Let $\mathcal{Y}$ be the normalization of $\tilde{X}$ in $Y$. The induced morphism $\mathcal{Y} \to \tilde{X}$, still denoted $f$, is finite surjective and Galois of group $G$.

Pick continuous metrics $\{ \| \cdot \|_{L, \sigma} \}_{\sigma : K \to \mathbb{C}}$ on $\mathcal{L}$ so that $\mathcal{L} = (\mathcal{L}, \| \cdot \|_{L, \sigma})$ is a metrized line bundle. Let $\mathcal{Z} := \mathcal{X} \setminus U$, $\mathcal{Z}$ its Zariski closure in $\mathcal{X}$, $\mathcal{U} := \mathcal{X} \setminus \mathcal{Z}$ and $\mathcal{V} = f^{-1}(\mathcal{U})$.

Let $S$ be a finite set of places of $K$ containing the Archimedean ones. Up to enlarging $S$, one may assume that the morphism $\mathcal{V} \times_{\mathcal{O}_K} \mathcal{O}_{K, S} \to \mathcal{U} \times_{\mathcal{O}_K} \mathcal{O}_{K, S}$ induced by $f$ is étale. For an isomorphism class of $G$-principal bundles $t \in H^1(\mathcal{O}_{K, S}, G)$, let $\mathcal{Y}_S, t$ be the twist of $\mathcal{Y}_S$ by $t$ and $f_{S, t} : \mathcal{Y}_S, t \to \mathcal{X} \times_{\mathcal{O}_K} \text{Spec} \mathcal{O}_{K, S}$ the induced morphism. Consider the relative normalization $f_{t} : \mathcal{Y}_t \to \mathcal{X}$ in $\mathcal{Y}_{S, t}$ (see [Stacks, Definition 035H]) and set $\mathcal{V}_t := f_{t}^{-1}(\mathcal{U})$.

Claim 3.12. With the notation above,

$$(3.5) \quad \text{growth}(\mathcal{X}, \mathcal{L}, \mathcal{U}, \mathcal{O}_{K, S}) \leq \max_{t \in H^1(\mathcal{O}_{K, S}, G)} \text{growth}(\mathcal{Y}_t, f_t^* \mathcal{L}, \mathcal{V}_t, \mathcal{O}_{K, S}).$$

Proof of the Claim. According to Proposition 3.6, the set $H^1(\mathcal{O}_{K, S}, G)$ is finite. Therefore, the $\mathcal{O}_K$-scheme

$$\tilde{\mathcal{Y}} := \bigsqcup_{t \in H^1(\mathcal{O}_{K, S}, G)} \mathcal{Y}_t$$

is proper. Let $\tilde{f} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ the morphism induced by the various twists of $f$. Given an $\mathcal{O}_{K, S}$-point $x$ of $\mathcal{U}$, there is an $\mathcal{O}_{K, S}$-point $\tilde{x}$ of $\tilde{\mathcal{U}}$ mapping to $x$. Indeed, the scheme-theoretic fiber

$$\mathcal{P} := \mathcal{Y} \times_{\mathcal{X}} \text{Spec} \mathcal{O}_{K, S}$$

of $f$ at $x$ is a principal $G$-bundle over $\mathcal{O}_{K, S}$. Letting $t$ denote the isomorphism class of $\mathcal{P}$, Lemma 3.10 states that there is an $\mathcal{O}_{K, S}$-point $y$ of $\mathcal{V}_t$ such that $f_t(y) = x$. By definition,

$$h_{f_t} \mathcal{L}(y) = h_{\mathcal{L}}(x).$$

In particular,

$$\text{growth}(\mathcal{X}, \mathcal{L}, \mathcal{U}, \mathcal{O}_{K, S}) \leq \text{growth}(\tilde{\mathcal{Y}}, \tilde{f}^* \mathcal{L}, \tilde{f}^{-1}(\mathcal{U}), \mathcal{O}_{K, S}),$$

whence the claim. \hfill $\Box$
One concludes by bounding the right-hand side of (3.5) by
\[ \max_{t \in H^1(K, G)} \text{gr. int}_K(Y_t, f_t^*L, V_t), \]
where \( V_t = f_t^{-1}(U) \), so that the right-hand side of the so-obtained inequality
\[ \text{growth}(X, \bar{\mathcal{L}}, U, \mathcal{O}_K, S) \leq \max_{t \in H^1(K, G)} \text{gr. int}_K(Y_t, f_t^*L, V_t), \]
is independent of \( S \). As \( S \) is arbitrary, taking the supremum over all finite sets of places of \( K \) (containing the Archimedean ones) finishes the proof. \( \square \)

4. Proof of the main theorem

4.1. Setup. Let \( \bar{K} \) be an algebraic closure of \( K \). For a variety \( Y \) over \( K \), let \( \bar{Y} \) denote the variety over \( \bar{K} \) obtained by extending scalars.

Keep the notation from the statement of the main theorem. There is no loss of generality in assuming \( X \) geometrically integral and normal, as \( U \) is so. Also, one may assume that \( U \) has a \( K \)-rational point, for the main theorem is trivial otherwise. Let \( \varepsilon > 0 \) be a real number and pick an integer \( d \geq 1 \) such that
\[ \frac{n(n+3)}{2d} [K : \mathbb{Q}] \leq \varepsilon, \]
where \( n = \dim X \).

4.2. Geometric input. Thanks to Theorem B, there exists a finite surjective morphism \( f: X' \to X \) of algebraic varieties over \( \bar{K} \) with \( X' \) integral, \( f \) étale over \( \bar{U} \) and, for a positive-dimensional integral closed subvariety \( Y' \) of \( X' \) not contained in \( Z' := f^{-1}(Z) \),
\[ \deg(Y', f^*L_{|Y'}) \geq d^n. \]

Claim 4.1. There exists an integral variety \( X'' \) over \( K \) and a Galois finite surjective morphism \( g: X'' \to X \) of algebraic varieties over \( K \) such that \( g \) is étale over \( U \) and the morphism \( \bar{g}: \bar{X}'' \to \bar{X} \) factors through the morphism \( f: X' \to \bar{X} \).

In particular, a positive-dimensional integral closed subvariety \( Y'' \) of \( X'' \) not contained in \( g^{-1}(Z) \) satisfies
\[ \deg(\bar{Y}'', g^*L_{|Y''}) \geq d^n \geq d^{\dim Y''}. \]
(Note that \( Y'' \) is not necessarily geometrically irreducible, but no irreducible component of \( \bar{Y}'' \) is contained in \( g^{-1}(Z) \).)

Proof of the Claim. Argue as in the proof of the implication (iii) \( \Rightarrow \) (iv) in [Poo17, Lemma 3.5.57]. First, the map \( X' \to \bar{X} \) may be assumed to be Galois, for it suffices to replace the function field \( K(X') \) of \( X' \) by its Galois closure \( F \) over the function field \( K(\bar{X}) \) of \( \bar{X} \), and \( X' \) by its normalization in \( F \).

Let us show that \( X' \) comes by extension of scalars from some cover \( X'' \to X \) defined over \( K \) with \( X'' \) geometrically integral and normal. For, pick a \( K \)-rational point of \( U \) (which exists by assumption) and a \( K \)-point \( x' \) in \( f^{-1}(x) \). Let \( G \) be the stabilizer of \( x' \) in the Galois group \( \text{Gal}(K(X')/K(X)) \). The sought-for \( X'' \) is obtained as the normalization of \( X \) in the finite extension \( K(X')^G \) of \( K(X) \). (See loc. cit. for details.) \( \square \)
4.3. Arithmetic input. Let $G = \text{Aut}(g)$ be the Galois group of the cover $g$. For an isomorphism class $t$ of principal $G$-bundles over $K$, let $X''_t$ be the twist of the variety $X''$ by $t$ and $g_t : X''_t \to X$ that of the morphism $g$. According to Proposition 3.11,
\[ \text{gr. int}_K(X, L, U) \leq \max_{t \in \text{Hom}(K, G)} \text{gr. int}_K(X''_t, g_t^*L, g_t^{-1}(U)). \]
For an irreducible subvariety $Y''$ of $X''_t$ not contained in $g_t^{-1}(Z)$,
\[ \deg(Y'', g_t^*L|_{Y''}) \geq d \dim Y'', \]
because, after extending scalars to $K$, the cover $g_t$ is isomorphic to $g$.

Theorem A can therefore be applied to the integral projective variety $X''_t$, the closed subvariety $g_t^{-1}(Z)$, and the ample line bundle $g_t^*L$, to give
\[ \text{gr. rat}_K(X'', g_t^*L, g_t^{-1}(U)) \leq \frac{n(n+3)}{2d} [K : \mathbb{Q}] \leq \varepsilon. \]
Combining these two inequalities yields $\text{gr. int}_K(X, L, U) \leq \varepsilon$, as the growth rate of integral points is lesser than that of rational points. As $\varepsilon > 0$ is arbitrary, this concludes the proof. \qed

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