Research Article

Bifurcations of a Fractional-Order Four-Neuron Recurrent Neural Network with Multiple Delays

Yu Fei,1 Rongli Li,1 Xiaofang Meng,1 and Zhouhong Li2

1School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
2Department of Mathematics, Yuxi Normal University, Yuxi, Yunnan 653100, China

Correspondence should be addressed to Zhouhong Li; zhouhli@yeah.net
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In recent years, many scholars have been interested in studying the dynamics of neural networks with such time delays [17–19]. It must be pointed out that exponential stabilization of memristor-based RNNs with disturbance and mixed time delays by periodically intermittent control has been considered by Wang et al. [20]. Using the appropriate Lyapunov–Krasovskii functionals and applying matrix inequality approach methods, Zhou [21] discussed the passivity of a class of recurrent neural networks with impulse and multiproportional delays. Zhou and Zhao [22] investigated the exponential synchronization and polynomial synchronization of recurrent neural networks with and without proportional delays. Robust stability analysis of recurrent neural networks is studied in Refs. [23, 24]. Furthermore, time delays are ubiquitous and unavoidable in the real world. Due to the existence of delays, the system can become unstable, and the dynamic behavior of nonlinear systems becomes more difficult. Moreover, since the solution space of the delay dynamical is infinite, it makes the systems more complex and bifurcation occurs. Hence, it is necessary to consider the properties and dynamics of neural networks via delays, such as time delay [25, 26], multiple delays [27, 28] time-varying delays [29, 30], and so on. In 2013, Zhang and Yang [31] studied a four-neuron recurrent neural network with multiple delays, described as follows:
Hopf bifurcation of fractional-order dynamical systems with multiple delays are reported, and therefore, the study of the fractional-order recurrent neural network with leakage delays. To the best of our knowledge, so far there are few results on the Hopf bifurcation of four-dimensional RNNs with two delays. Thus, for more recurrent neural network research results, see references [5, 10, 12, 20].

In more than three centuries, fractional calculus has developed into a classical mathematical concept. Nonlinear dynamics systems have shown that it has an exceptionally important role in generalizing ordinary differentiation and integration to arbitrary non-integer order. Therefore, if we study the effects of the memory and genetics factors, fractional neural networks have become a very important general than integer neural networks. In recent years, the application of fractional order neural networks has developed rapidly, and the complex dynamical behaviors of fractional neural networks has become a very important research hot points, such as stability or multistability, Hopf bifurcation, synchronization, chaos, and so on. For instance, in Ref. [32], the multistability of a fractional-order competitive neural networks with delay is investigated by using the fractional calculus and partitioning of state space. In Lu and Xue [33] study, adaptive synchronization is investigated for fractional delayed stochastic neural networks. Yuan and Huang [34] considered the quantitative analysis of fractional-order neural networks with time delay. Udhayakumar and Rajan [35] discussed Hopf bifurcation of a delayed fractional-order octonion-valued neural networks.

We also know that Hopf bifurcations, which include subcritical and supercritical ones, can be used to efficiently design biochemical oscillators. Furthermore, fractional-order neural networks with same delay cannot accurately describe the dynamical properties of real world neural networks compared with the ones with different delays. In recent years, some researchers have considered the dynamical behaviors by fractional models with time delay [36–48]. In 2019 [49], we also investigated the existence of Hopf bifurcation for four-neuron fractional neural networks with leakage delays. To the best of our knowledge, so far there are few results on the Hopf bifurcation of four-dimensional fractional-order recurrent neural network with multiple delays are reported, and therefore, the study of Hopf bifurcation of fractional-order dynamical systems with multiple delays remains an open problem.

Based on the above motivations, we are dedicated to presenting a theoretical exploration of stability and Hopf bifurcation for a four-neuron fractional-order recurrent neural network with multiple delays in this work. The main contributions can be highlighted as follows:

(i) A novel delayed fractional-order recurrent neural network with four-neuron and two different delays is studied

(ii) Double main dynamical properties of the fractional-order recurrent neural network with two delays are investigated: stability and oscillation

(iii) The Hopf bifurcation is discussed in terms of delays and order

In the article, we shall give some lemmas and definitions of fractional-order calculus in Section 2, and models description in Section 3. In Section 4, the local stability of the trivial steady state of delayed fractional-order RNNs is examined by applying the associated characteristic equation. In addition, the authors will care about the Hopf bifurcation of fractional-order RNNs with multiple delays. In Section 5, two numerical examples are provided to demonstrate the theoretical results. The last section gives some conclusions.

2. Preliminaries

This section we will give some Caputo definitions and lemma for fractional calculus as a basis for the theoretical analysis and simulation proofs.

Definition 1 (see [50]). The fractional integral of order \( \phi \) for a function \( f(x) \) is defined as follows:

\[
I^\phi f(x) = \frac{1}{\Gamma(\phi)} \int_{x_0}^{x} (x-s)^{\phi-1} f(s) ds,
\]

where \( \phi > 0 \), and \( \Gamma(\cdot) \) is the Gamma function satisfying \( \Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} dx \).

Definition 2 (see [50]). Caputo fractional derivative of order \( \phi \) for a function \( \psi(x) \in C^k[x_0, \infty), R \) is defined by

\[
D^\phi \psi(x) = \frac{1}{\Gamma(n-\phi)} \int_{x_0}^{x} \psi^{(k)}(s) (x-s)^{\phi-n+1} ds,
\]

where \( x \geq x_0 \) and \( k-1 \leq \phi < k, k \in N^+ \).

Moreover, when \( \phi \in (0, 1) \), then

\[
D^\phi \psi(x) = \frac{1}{\Gamma(1-\phi)} \int_{x_0}^{x} \psi'(s) (x-s)^{\phi-1} ds.
\]
Lemma 1 (see [51]). Consider the following fractional order autonomous model.

\[ D^\phi u = f(u, u(0)) = u_0, \]

in which \( 0 < \phi \leq 1, u \in \mathbb{R}^k, \) and \( J \in \mathbb{R}^{k \times k} \). Then the zero solution of the system (5) is asymptotically stable in the Lyapunov sense if all roots \( \lambda_i \) are the system (5) of character equation satisfy \( |\arg(\lambda_i)| > \phi \pi/2 \) and those critical eigenvalues that satisfy \( |\arg(\lambda_i)| = \phi \pi/2 \) have geometric multiplicity one.

3. Mathematics Model Elaboration

This article considers the following four-neuron fractional-order recurrent neural network with two delays:

\[
\begin{align*}
D^\phi x_1(t) &= -x_1(t) + f(x_2(t - \tau_1)), \\
D^\phi x_2(t) &= -x_2(t) + f(x_3(t - \tau_1)), \\
D^\phi x_3(t) &= -x_3(t) + f(x_4(t - \tau_1)), \\
D^\phi x_4(t) &= -x_4(t) + \omega_1 f(x_1(t - \tau_2)) + \omega_2 f(x_2(t - \tau_2)) + \omega_3 f(x_3(t - \tau_2)),
\end{align*}
\]

where \( \phi \in (0, 1] \) are fractional order; \( x_i(t) (i = 1, 2, 3, 4) \) stand for state variables; \( \omega_i (i = 1, 2, 3) \) denote the connection weights; the function of connecting neurons is denoted by \( f(x(-)) \); and \( \tau_1 \) and \( \tau_2 \) are the communication time delays.

Remark 1. In fact, if \( \phi = 1 \), the fractional delayed neural networks (6) changes into the general neural network (1).

Accordingly, the main purpose of this article is to investigate the stability and the application of Hopf bifurcations of the neural networks (6) taking different time delays \( \tau_1 \) and \( \tau_2 \) as the bifurcation parameters by the method of stability analysis [52]. In addition, the effects of the order on the creation of the Hopf bifurcation for the proposed fractional order neural network with multiple delays are also numerically discussed.

By applying Laplace transformation, its characteristic equation is given as

\[
\begin{vmatrix}
 s^\phi + 1 & -m_1 e^{\tau_1} & 0 & 0 \\
 0 & s^\phi + 1 & -m_2 e^{\tau_1} & 0 \\
 0 & 0 & s^\phi + 1 & -m_3 e^{\tau_1} \\
 -m_4 e^{-\tau_2} & -m_5 e^{-\tau_2} & -m_6 e^{-\tau_2} & s^\phi + 1
\end{vmatrix} = 0,
\]

where \( m_k = f'(0)(k = 1, 2, 3), m_k = \omega_j f'(0)(j = 1, 2, 3, k = 4, 5, 6). \)

From (8), we have

\[
K_1(s) + K_2(s)e^{-\tau_1} + K_3(s)e^{-2\tau_1} + K_4(s)e^{-3\tau_1} = 0,
\]

where

\[
K_1(s) = s^\phi + 4s^3 + 6s^2 + 4s^\phi + 1,
\]

\[
K_2(s) = -m_1m_2(s^\phi + 2s^2 + 1)e^{-\tau_1},
\]

\[
K_3(s) = -m_2m_3e^{\tau_1},
\]

\[
K_4(s) = -m_1m_2m_3e^{-\tau_2}.
\]

Multiplying \( e^{\tau_1} \) and \( e^{2\tau_1} \) on both sides of equation (9), respectively, we can obtain

Throughout of this paper, assume that the following condition holds true:

\((C1) f(\cdot) \in C(R, R), f(0) = 0, xf(x) > 0, \) for \( x \neq 0.\)

4. Main Results

This section chooses \( \tau_1 \) or \( \tau_2 \) as a bifurcation parameter to study the stability analysis and Hopf bifurcation for the fractional order RNNs (6) and to study the bifurcation points accurately.

4.1. Bifurcation Depending on \( \tau_1 \) in Equation (6). In this subsection, we first study the effects of \( \tau_1 \) on bifurcations of system (6) by establishing \( \tau_2 \).

Applying Taylor series formula, the following form of equation (6) at the origin is
\[
\begin{align*}
K_1(s)e^{2\tau_1} + & K_3(s)e^{\tau_1} + K_4(s)e^{-\tau_1} = 0, \\
K_1(s)e^{\tau_1} + & K_3(s)e^{-\tau_1} + K_4(s)e^{-2\tau_1} = 0.
\end{align*}
\]
\[\text{(11)}\]

\[
\begin{align*}
(A_1 + iB_1)e^{2\tau_1} + (A_2 + iB_2)e^{\tau_1} + (A_3 + iB_3)e^{-\tau_1} + (A_4 + iB_4)e^{-2\tau_1} = 0, \\
(A_1 + iB_1)e^{\tau_1} + (A_2 + iB_2)e^{-\tau_1} + (A_3 + iB_3)e^{-2\tau_1} + (A_4 + iB_4)e^{-3\tau_1} = 0.
\end{align*}
\]
\[\text{(12)}\]

Take \( s = i\omega = \omega (\cos \pi/2 + i \sin \pi/2) (\omega > 0) \) be a purely imaginary root of equation (11). Apply inserting \( s \) into equation (11) and separating the imaginary and real parts yields the following equations:

\[
\begin{align*}
A_1 \cos (2\omega \tau_1) - B_1 \sin (2\omega \tau_1) + (A_2 + A_4) \cos (\omega \tau_1) + (B_4 - B_2) \sin (\omega \tau_1) &= -A_3, \\
B_1 \cos (2\omega \tau_1) + A_1 \sin (2\omega \tau_1) + (B_2 + B_4) \cos (\omega \tau_1) + (A_2 - A_4) \sin (\omega \tau_1) &= -B_3, \\
A_1 \cos (2\omega \tau_1) - B_1 \sin (2\omega \tau_1) + (A_2 + A_4) \cos (\omega \tau_1) + (B_4 - B_2) \sin (\omega \tau_1) &= -A_3, \\
B_1 \cos (2\omega \tau_1) + A_1 \sin (2\omega \tau_1) + (B_2 + B_4) \cos (\omega \tau_1) + (A_2 - A_4) \sin (\omega \tau_1) &= -B_3.
\end{align*}
\]
\[\text{(13)}\]

Evidently,
\[
\begin{align*}
\cos (\omega \tau_1) &= \frac{F_{12}(\omega)}{F_{11}(\omega)} = F_{c1}(\omega), \\
\sin (\omega \tau_1) &= \frac{F_{12}(\omega)}{F_{21}(\omega)} = F_{s1}(\omega),
\end{align*}
\]
\[\text{(14)}\]

where \( A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, F_{11}, F_{12}, F_{21}, \) and \( F_{22} \) are given Appendix A. Obviously, from first to second equation of system (14), it can be implied that
\[
F_{c1}(\omega) + F_{s1}(\omega) = 1.
\]
\[\text{(15)}\]

From equation (13), one can obtain
\[
\tau^{(l)}_1 = \frac{1}{\omega} \arccos \left[ \frac{F_{12}(\omega)}{F_{11}(\omega)} \right] + 2\pi l, l = 0, 1, 2, \ldots
\]
\[\text{(16)}\]

\[
\begin{align*}
H_1(s) &= s^{4\delta} + 4s^{3\delta} + 6s^{2\delta} + 4s^{\delta} + 1, \\
H_2(s) &= -m_1m_6s^{2\delta} - 2m_3m_5s^{\delta} - m_2m_3s^{\delta} - m_2m_4s^{\delta} - m_1m_2m_3m_4.
\end{align*}
\]
\[\text{(19)}\]

If \( \tau_2 = 0 \), then the equation (18) becomes
\[
0 = s^{4\delta} + 4s^{3\delta} + 6s^{2\delta} + 4s^{\delta} + 1 - m_1m_6s^{2\delta} - 2m_3m_5s^{\delta} - m_2m_3s^{\delta},
\]
\[\text{(20)}\]

Suppose that all roots \( s \) of the equation (18) obey Lemma 1, then we get that both roots \( \lambda_i \) in equation (18) have negative real parts.

The imaginary and real parts of \( H_j(s) (j = 1, 2) \) can be denoted by \( H_j^R \) and \( H_j^I \), respectively. Multiplying \( e^{\tau_2} \) on both sides of equation (18), we can obtain
\[
H_1(s)e^{\tau_2} + H_2(s) = 0.
\]
\[\text{(21)}\]

Also, let \( s = i\nu = \nu (\cos \pi/2 + i \sin \pi/2) (\nu > 0) \) be a purely imaginary root of equation (11) if and only if
\[
\begin{align*}
H_1^R \cos (\nu \tau_2) - H_1^I \sin (\nu \tau_2) &= -H_2^R, \\
H_1^I \cos (\nu \tau_2) + H_1^R \sin (\nu \tau_2) &= -H_2^I.
\end{align*}
\]
\[\text{(22)}\]

This leads to form
\[
\begin{cases}
\cos (\nu r_2) = \frac{H^R_2 H^R_1 + H^I_1 H^I_2}{H^I_1 H^R_1 + H^R_1 H^I_2} = f_{c_1} (\nu), \\
\sin (\nu r_1) = \frac{-H^R_2 H^I_1 + H^I_1 H^R_1}{H^I_1 H^R_1 + H^R_1 H^I_2} = f_{s_1} (\nu).
\end{cases}
\] (23)

It is not difficult to see that
\[f^2_{c_1} (w) + f^2_{s_1} (w) = 1.\] (24)

Additionally, we will give the following assumptions which hold true.

(C2) The equation (24) has at least a positive real root.

From equation (24), the values of \(y\) can be obtained according to Mathematica software Mathematica 10.0, and then the Hopf bifurcation point \(\tau_{20}\) of fractional order recurrent neural network (6) with \(\tau_1 = 0\) can be derived. To demonstrate our main results, we further present the following hypothesis: (C3) \(Y_1 \Omega_1 + Y_2 \Omega_2 / \Omega_1^2 + \Omega_2^2 \neq 0\), where

\[
Y_1 = w_0 \left[ A_2 \sin w_0 \tau_{10} - B_2 \cos w_0 \tau_{10} + 2(A_3 \sin 2w_0 \tau_{10} - B_2 \cos 2w_0 \tau_{10}) + 3(A_4 \cos 3w_0 \tau_{10} + B_4 \sin 3w_0 \tau_{10}) \right],
\]

\[
Y_2 = w_0 \left[ A_2 \cos w_0 \tau_{10} + B_2 \sin w_0 \tau_{10} + 2(A_3 \cos 2w_0 \tau_{10} + B_2 \sin 2w_0 \tau_{10}) + 3(A_4 \cos 3w_0 \tau_{10} + B_4 \sin 3w_0 \tau_{10}) \right],
\]

\[
\Omega_1 = A'_1 + [A'_2 - \tau_1 A_2] \cos w_0 \tau_{10} + [B'_2 - \tau_1 B_2] \sin w_0 \tau_{10} - [A'_3 - 3\tau_1 A_3] \cos 2w_0 \tau_{10} - [B'_3 - 3\tau_1 B_3] \sin 2w_0 \tau_{10} + [A'_4 - 3\tau_1 A_4] \cos 3w_0 \tau_{10} - [B'_4 - 3\tau_1 B_4] \sin 3w_0 \tau_{10}.
\]

\[
\Omega_2 = B'_1 + [B'_2 - \tau_1 B_2] \cos w_0 \tau_{10} + [B'_3 - 3\tau_1 B_3] \sin w_0 \tau_{10} - [B'_4 - 3\tau_1 B_4] \cos 3w_0 \tau_{10} - [B'_5 - 3\tau_1 B_5] \sin 3w_0 \tau_{10}.
\]

Lemma 2. Let \(s(\tau_1) = \nu (\tau_{10}) + i w (\tau_1)\) be a root of equation (9) near \(\tau_1 = \tau_{10}\) satisfying \(\nu (\tau_{10}) = 0, w (\tau_1) = w_0\), then the following transversality condition is satisfied.

\[
\text{Re} \left[ \frac{ds}{d\tau_1} \right]_{(w = w_0, \tau_1 = \tau_{10})} \neq 0.
\] (26)

\[
0 = K'_1 (s) \frac{ds}{d\tau_1} + K'_2 (s) e^{-s \tau_1} \frac{ds}{d\tau_1} + K'_2 (s) e^{-2s \tau_1} \left( -\tau_1 \frac{ds}{d\tau_1} - s \right) + K'_3 (s) e^{-3s \tau_1} \frac{ds}{d\tau_1}
\]

\[
+ K'_4 (s) e^{-4s \tau_1} \left( -2 \tau_1 \frac{ds}{d\tau_1} - 2s \right) + K'_5 (s) e^{-5s \tau_1} \frac{ds}{d\tau_1} + K'_6 (s) e^{-6s \tau_1} \left( -3 \tau_1 \frac{ds}{d\tau_1} - 3s \right).
\] (27)

\[
\frac{ds}{d\tau_1} = \frac{Y (s)}{\Omega (s)},
\]

where

\[
Y (s) = s \left[ K'_2 (s) e^{-s \tau_1} + 2K'_3 (s) e^{-2s \tau_1} + 3K'_4 (s) e^{-3s \tau_1} \right],
\]

\[
\Omega (s) = K'_1 (s) + [K'_2 (s) - \tau_1 K'_2 (s)] e^{-s \tau_1} + [K'_3 (s) - 2\tau_1 K'_4 (s)] e^{-2s \tau_1}
\]

\[
+ [K'_4 (s) - 3\tau_1 K'_4 (s)] e^{-3s \tau_1}.
\] (28)

We further suppose that \(Y_1\) and \(Y_2\) are the real and imaginary parts of \(Y (s)\), respectively, and \(\Omega_1\) and \(\Omega_2\) are the real and imaginary parts of \(\Omega (s)\), respectively, then

\[
\text{Re} \left[ \frac{ds}{d\tau} \right]_{(\tau = \tau_{10}^*, w = w_0)} = \frac{Y_1 \Omega_1 + Y_2 \Omega_2}{\Omega_1^2 + \Omega_2^2}.
\] (29)
From (C3), we conclude that the transversality condition holds true. This completes the proof of Lemma 2.
From the above investigation, we can obtain the following results.

**Theorem 1.** assumptions (C1)–(C3) hold true, then the following results can be given:

(i) The zero equilibrium point of fractional order four-neuron recurrent neural network with multiple delays (6) is asymptotically stable when \( τ_1 \in [0, τ^*_{10}) \).

(ii) If \( τ_1 \in [0, τ^*_{10}) \), then fractional order four neurons recurrent neural network with multiple delays (6) causes Hopf bifurcation at the origin when \( τ_1 = τ^*_{10} \). That is, a branch of periodic solutions can bifurcate from the zero equilibrium point at \( τ_1 = τ^*_{10} \).

4.2. Bifurcation Depending on \( τ_1 \) in Equation (6). As in the previous subsection, next we change another delay \( τ_2 \) to the bifurcation parameter to account for the bifurcation of the model (6). It is hard to point out that equation (8) changes as follows:

\[
q_1(s) + q_2(s)e^{-τ_2s} = 0,
\]

where

\[
q_1(s) = 1 + 4s^2 + 6s^4 + 4s^6 + s^8,
q_2(s) = -m_2m_3(1 + 2s^2 + s^4)e^{-τ_2s} - m_1m_2m_3(1 + s^4)e^{-2τ_2s}.
\]

Multiplying \( e^{τ_2s} \) on both sides of equation (30), we can obtain

\[
q_1(s)e^{τ_2s} + q_2(s) = 0.
\]

Suppose \( q_1(s) = a_1 + ib_1 \) and \( q_2(s) = a_2 + ib_2 \), and from equation (32), we have

\[
(a_1 + ib_1)e^{τ_2s} + a_2 + ib_2 = 0,
\]

where \( a_1, a_2, b_1, b_2 \) are given in Appendix B.

Take \( s = i\bar{ω} = \bar{ω}(\cos π/2 + i \sin π/2)(\bar{ω} > 0) \) as a root of equation (33) if and only if

\[
\begin{aligned}
a_1 \cos(\bar{ω}τ_2) - b_1 \sin(\bar{ω}τ_2) &= -a_2, \\
b_1 \cos(\bar{ω}τ_2) + a_1 \sin(\bar{ω}τ_2) &= -b_2,
\end{aligned}
\]

that is,

\[
\begin{aligned}
\cos(\bar{ω}τ_2) &= \frac{-a_1a_2 + b_1b_2}{a_1^2 + b_1^2} = \rho(\bar{ω}), \\
\sin(\bar{ω}τ_2) &= \frac{-a_1b_1 + a_2b_2}{a_1^2 + b_1^2} = \varphi(\bar{ω}).
\end{aligned}
\]

It is simple to derive the following equation.

\[
ρ^2(\bar{ω}) + \varphi^2(\bar{ω}) = 1.
\]

From (35), one can obtain

\[
τ_2^0 = \frac{1}{\bar{ω}}[\arccos ρ(\bar{ω}) + 2lπ], l = 0, 1, 2, \ldots
\]

The bifurcation point is defined by \( \alpha_k(k = 1, 2, 3)(C3)(Y_1, Ω_1 + Y_2, Ω_2)(Ω_1^2 + Ω_2^2) \neq 0 \)

\[
τ^*_2 = \min\{τ^0_k\}, l = 0, 1, 2, \ldots
\]

C5(\( α_1β_1 + α_2β_2)/{(α_1^2 + α_2^2)} \neq 0 \) here \( τ^0_k \) is defined by equation (38)

If \( τ_2 = 0 \), then the equation (32) becomes

\[
M_1(s) + M_2(s)e^{-τ_2s} + M_3(s)e^{-2τ_2s} + M_4(s)e^{-3τ_2s} = 0,
\]

where

\[
M_1(s) = 1 + 4s^2 + 6s^4 + 4s^6 + s^8,
M_2(s) = -m_1m_2(1 + 2s^2 + s^4),
M_3(s) = -m_1m_2m_3(1 + s^4),
M_4(s) = -m_1m_2m_3m_4.
\]

Assume that all roots \( s \) of equation (39) observe Lemma 1, then we get that both roots of equation (39) have negative real parts.

The imaginary and real parts of \( M_i(s) \) \((i = 1, 2, 3, 4) \) can be expressed as \( M_i^R \) and \( M_i^I \), respectively. Multiplying both sides of the equation (39) by \( e^{τ_2s} \) and \( e^{-τ_2s} \) yields

\[
\begin{aligned}
M_1(s)e^{τ_2s} + M_2(s)e^{-τ_2s} + M_3(s)e^{-2τ_2s} + M_4(s)e^{-3τ_2s} &= 0, \\
M_1(s)e^{τ_2s} + M_2(s)e^{-2τ_2s} + M_3(s)e^{-3τ_2s} + M_4(s)e^{-4τ_2s} &= 0.
\end{aligned}
\]

Let \( s = i\bar{ν} = \bar{ν}(\cos π/2 + i \sin π/2)(\bar{ν} > 0) \) be a solution of equation (41). Substituting \( s \) into equation (41) and separating the imaginary and real units yields the following equations:
which lead to
\[
\begin{align*}
\cos \tau_1 &= \frac{E_{12}(\tau)}{E_{11}(\tau)} = \mathcal{C}(\tau)^2, \\
\sin \tau_1 &= \frac{E_{22}(\tau)}{E_{21}(\tau)} = \mathcal{S}(\tau)^2.
\end{align*}
\]  

(43)

Obviously, from first and second equation of system (43), we get
\[
\mathcal{C}(\tau)^2 + \mathcal{S}(\tau)^2 = 1.
\]  

(44)

To theoretically gain the sufficient conditions for the Hopf bifurcation, we assume that the following assumptions hold true:

(C4) Equation (36) has at least a positive real root.

By means of equation (36), the values of \( \bar{\omega} \) can be obtained according to mathematical software Mathematica 10.0, and then the bifurcation point \( \tau_{10} \) of recurrent fractional four-neuron neural networks (6) with \( \tau_2 = 0 \) can be derived. As a summary of our main results, we provide the following assumption: (C5) \( \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4 \neq 0 \), where
\[
\begin{align*}
\alpha_1 &= \alpha_1^0 + (\alpha_1^0 - \tau_2 \alpha_2) \cos \bar{\omega}_0 \tau_2 + (\beta_2^0 - \tau_2 \beta_2) \sin \bar{\omega}_0 \tau_2, \\
\alpha_2 &= \alpha_2^0 + (\alpha_2^0 - \tau_2 \beta_2) \cos \bar{\omega}_0 \tau_2 - (\alpha_1^0 - \tau_2 \alpha_2) \sin \bar{\omega}_0 \tau_2, \\
\beta_1 &= \bar{\omega}_0 (\alpha_2 \sin \bar{\omega}_0 \tau_2 - \beta_2 \cos \bar{\omega}_0 \tau_2), \\
\beta_2 &= \bar{\omega}_0 (\alpha_2 \cos \bar{\omega}_0 \tau_2 + \beta_2 \sin \bar{\omega}_0 \tau_2).
\end{align*}
\]  

(45)

Lemma 3. Let \( s(\tau_2) = \eta(\tau_2) + i\bar{\omega}(\tau_2) \) be a root of equation (9) near \( \tau_2 = \tau_{2j} \), satisfying \( \eta(\tau_{2j}) = 0, \bar{\omega}(\tau_{2j}) = \bar{\omega}_0 \), then we get the following transversality condition
\[
\operatorname{Re} \left[ \frac{d s}{d \tau_2} \right]_{\bar{\omega}=\bar{\omega}_0, \tau_2=\tau_{20}} \neq 0.
\]  

(46)

Proof. Similar to Lemma 2, by utilizing the implicit function theorem and differentiating (9) with respect to \( \tau_2 \), we get
\[
0 = q_1^*(s) \frac{d s}{d \tau_2} + q_2^*(s) e^{-s \tau_2} \frac{d s}{d \tau_2} + q_3^*(s) e^{-s \tau_2} \left( -\frac{d s}{d \tau_2} - s \right),
\]
\[
\frac{d s}{d \tau_2} = \frac{\beta(s)}{\alpha(s)}
\]  

(47)

where
\[
\begin{align*}
\beta(s) &= sq_2(s) e^{-s \tau_2}, \\
\alpha(s) &= q_1^*(s) + q_2^*(s) e^{-s \tau_2} - \tau_{2j} q_3(s) e^{-s \tau_2}.
\end{align*}
\]  

(48)

We further suppose that \( \alpha_1 \) and \( \alpha_2 \) are the real and imaginary parts of \( \beta(s) \), respectively, then we get
\[
\operatorname{Re} \left[ \frac{d s}{d \tau_2} \right]_{\bar{\omega}=\bar{\omega}_0, \tau_2=\tau_{20}} = \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_1^2 + \alpha_2^2}.
\]  

(49)

As a direct consequence of (C5), we can conclude that the transversality condition is satisfied. Then the proof of Lemma 3 is complete.

Based on the above analysis, the following conclusions can be drawn.

\[ \square \]

Theorem 2. By assuming that assumptions (C1), (C4), and (C5) are valid, the following conditions can be inferred:

(i) The zero equilibrium point of fractional order four-neuron recurrent neural network with multiple delays (6) is asymptotically stable when \( \tau_2 \in [0, \tau_{20}) \).

(ii) The fractional order four-neuron recurrent neural network with multiple delays (6) experiences a Hopf bifurcation at its origin when \( \tau_2 = \tau_{20}^* \); that is, a family of periodic solutions can bifurcate from the zero equilibrium point near \( \tau_2 = \tau_{20}^* \).

5. Numerical Examples

To demonstrate the validity and feasibility of the conclusions reached in this paper, we provide two examples. The simulations were based on a prediction and correction scheme [53] of Adam–Bashforth–Moulton and step-size \( h = 0.01 \).

5.1. Example 1. Consider the four-neuron fractional recurrent neural networks with multiple delays as
\[
\begin{align*}
D^\alpha x_1(t) &= -x_1(t) + f(x_2(t - \tau_1)), \\
D^\alpha x_2(t) &= -x_2(t) + f(x_3(t - \tau_1)), \\
D^\alpha x_3(t) &= -x_3(t) + f(x_4(t - \tau_1)), \\
D^\alpha x_4(t) &= -x_4(t) + \omega_1 f(x_1(t - \tau_2)) + \omega_2 f(x_2(t - \tau_2)) + \omega_3 f(x_3(t - \tau_2)).
\end{align*}
\]  

(50)
Choose parameters $\phi = 0.9$, $\omega_1 = 2$, $\omega_2 = \omega_3 = -2$, action function $f(\cdot) = \tanh(\cdot)$; therefore, $f(0) = \tanh(0) = 0$, $f'(0) = 1$.

Let the initial values be selected as $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.15, -0.14, 0.1, 0.2)$ for the system (50). First, taking fixed $\tau_2$ such that $\tau_2 = 0.6$ by complex computing, we get $\omega_{10} = 5.23599$, and then $\tau_{10} = 0.312709$. Obviously, it is easy to verify that the conditions in Theorem 1 are satisfied. The numerical simulations in Figures 1 and 2 that the zero equilibrium point of system (50) is locally asymptotically stable when $\tau_1 = 0.25 < \tau_{10} = 0.312709$. Moreover, Figures 3 and 4 simulates that the zero equilibrium point of system (50) is unstable, and Hopf bifurcation occurs when $\tau_1 = 0.35 > \tau_{10} = 0.312709$. The bifurcation diagrams are plotted in Figure 5, which illustrates the theoretical results.

5.2 Example 2. The same as example 1, let $\phi = 0.95$, and now we consider the following four-neurons fractional current network with double different delays:
Taking $\omega_1 = 1$, $\omega_2 = \omega_3 = -1.5$, $\phi = 0.95$, action function $f(\cdot) = \tanh(\cdot)$, then $f(0) = \tanh(0) = 0$, $f'(0) = 1$, and we first also set $\tau_1 = 0.8$, in the next step, we apply a complex calculation, and it obtains a $\bar{\omega}_{20} = 1.02089$ and $\tau_{20} = 0.329454$. Thus, Theorem 2 yields that the zero solution $(0,0,0,0)$ of the system (51) is locally asymptotically stable when $\tau_2 = 0.22 < \tau_{20}$, which is simulated in Figures 6 and 7 which describes the impact of fractional order on $\tau_{20}$. In
addition, the zero equilibrium point of the system (51) is unstable, and Hopf bifurcation occurs when $\tau_2 > \tau_{20}$, as shown in Figures 8 and 9. Moreover, the bifurcation diagrams are plotted in Figure 10, which illustrates the theoretical results.

**Remark 3.** In fact, in order to better reflect the influence of different time delays at the bifurcation point of the systems (50) and (51), the corresponding bifurcation point $\tau_{10}$ and $\tau_{20}$ and $\tau_{10}$ and $\tau_{20}$ can be determined by changing the order of $\phi$. This means that systems (50) and (51) involving...
Figure 7: Phase diagrams of system (51) with $\phi = 0.95$, $\tau_1 = 0.22 < \tau_20 = 0.329454$.

Figure 8: Time responses of system (51) with $\phi = 0.95$, $\tau_1 = 0.38 > \tau_20 = 0.329454$. 

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different two delays are prone to earlier Hopf bifurcation for some fixed fractional order $\phi$.

6. Conclusion

This paper examines the Hopf bifurcation problem of fractional recurrent neural networks with four neurons and two delays. Using time delay as the bifurcation parameter, several criteria are destabilized in order to ensure the Hopf bifurcation for the fractional four-neuron of recurrent neural networks. Based on our analysis, different communication time delays and order effects have quantitatively changed the dynamic behavior of the system (6). These results can contribute to our understanding of delayed fractional recurrent neural networks as a continuation of the previous work. The results of the simulations are illustrated by two numerical examples.
Appendix

A

\[ A_1 = 6\omega^2 \cos (\pi \phi) + 4\omega^{3\phi} \cos \left( \frac{3\pi \phi}{2} \right) + \omega^{4\phi} \cos (2\pi \phi) + 4\omega^\phi \cos \left( \frac{\pi \phi}{2} \right) + 1, \]

\[ B_1 = 6\omega^2 \sin (\pi \phi) + 4\omega^{3\phi} \sin \left( \frac{3\pi \phi}{2} \right) + \omega^{4\phi} \sin (2\pi \phi) + 4\omega^\phi \sin \left( \frac{\pi \phi}{2} \right), \]

\[ A_2 = -m_2 m_\phi \omega^2 \sin (\pi \phi) - m_2 m_\phi \omega^2 \sin (\pi \phi) \cos (\pi \phi), \]

\[ -2m_2 m_\phi \omega^3 \sin \left( \frac{\pi \phi}{2} \right) \sin (\omega \tau) - 2m_2 m_\phi \omega^3 \cos \left( \frac{\pi \phi}{2} \right) \cos (\omega \tau) - m_2 m_\phi \sin (\omega \tau), \]

\[ B_2 = -m_2 m_\phi \omega^2 \sin (\pi \phi) \cos (\omega \tau) + m_2 m_\phi \omega^2 \cos (\pi \phi) \sin (\omega \tau), \]

\[ -2m_2 m_\phi \omega^3 \sin \left( \frac{\pi \phi}{2} \right) \cos (\omega \tau) + 2m_2 m_\phi \omega^3 \cos \left( \frac{\pi \phi}{2} \right) \sin (\omega \tau) + m_2 m_\phi \sin (\omega \tau), \]

\[ A_3 = -m_2 m_3 m_\phi \omega^3 \sin \left( \frac{\pi \phi}{2} \right) \sin (\omega \tau) - m_2 m_3 m_\phi \omega^3 \cos \left( \frac{\pi \phi}{2} \right) \cos (\omega \tau) - m_2 m_3 m_\phi \cos (\omega \tau), \]

\[ B_3 = -m_2 m_3 m_\phi \omega^3 \sin \left( \frac{\pi \phi}{2} \right) \cos (\omega \tau) + m_2 m_3 m_\phi \omega^3 \cos \left( \frac{\pi \phi}{2} \right) \sin (\omega \tau) + m_2 m_3 m_\phi \sin (\omega \tau), \]

\[ A_4 = -m_2 m_3 m_4 \cos (\omega \tau), \]

\[ B_4 = m_2 m_3 m_4 \sin (\omega \tau), \]

\[ F_{11} = -2\mathcal{B}_i (A_i B_i + A_i B_i) (A_i^+ - A_i^- (A_i^+ + 2A_i^+ - 2B_i^+ + B_i^+ + 2B_i^+)) + 2A_i (A_i A_i + A_i A_i) \]

\[ + A_i A_i B_i - A_i B_i A_i - A_i B_i B_i - A_i B_i B_i) - A_i (A_i^+ + B_i^+) + 2A_i B_i (A_i B_i - A_i B_i) \]

\[ - A_i^+ (A_i^+ + B_i^+) + 2A_i B_i (A_i B_i - A_i B_i) - A_i^+ (A_i^+ + B_i^+) + 2A_i B_i (A_i B_i - A_i B_i) \]

\[ - 2A_i^+ B_i^+ + 2B_i B_i B_i - B_i^+ (B_i^+ - B_i^+), \]

\[ F_{12} = B_i (A_i^+ + B_i^+) (A_i^+ + B_i^+) - (A_i B_i + A_i B_i) (A_i A_i + B_i B_i) (A_i^+ + B_i^+) \]

\[ + A_i (A_i B_i - A_i B_i) + B_i (A_i A_i - A_i B_i) (A_i^+ + B_i^+) - B_i (A_i A_i + B_i B_i) \]

\[ + A_i B_i (A_i B_i - A_i B_i) - B_i (A_i B_i + A_i B_i) (A_i^+ + B_i^+), \]

\[ F_{21} = A_i^+ - A_i^- (A_i^+ + 2A_i^+ - 2B_i^+ + B_i^+ + 2B_i^+) + 2A_i (A_i A_i + A_i B_i B_i + A_i B_i B_i) \]

\[ - A_i B_i B_i - A_i^+ (A_i^+ + B_i^+) + 2A_i B_i (A_i B_i - A_i B_i) - A_i^+ B_i^+ + 2A_i B_i B_i \]

\[ + A_i^+ - A_i^+ B_i^+ + 2A_i B_i^+ + B_i^+ - B_i^+ B_i^+ - 2B_i^+ B_i^+ + 2B_i B_i B_i - B_i^+ B_i^+ + B_i^+, \]

\[ F_{22} = A_i^+ - B_i^+ + A_i^+ (A_i B_i + B_i) + A_i (B_i - B_i) - A_i B_i) + A_i (A_i^+ - B_i^+) + 2A_i A_i B_i \]

\[ + A_i^+ B_i^+ - 2A_i A_i B_i + A_i^+ B_i^+ - B_i^+ B_i^+ + B_i^+ B_i^+ - B_i^+ B_i^+ - A_i^+ A_i B_i \]

\[ + A_i (A_i^+ B_i - B_i^+) + B_i^+ - B_i^+ B_i^+ B_i^+ + B_i^+ B_i^+ B_i^+ + B_i^+ B_i^+ B_i^+ - B_i^+ B_i^+ + B_i^+, \]

\[ + A_i^+ B_i - A_i^+ B_i^+ + A_i B_i B_i^+ + A_i B_i B_i^+ + 2A_i B_i B_i - A_i B_i B_i^+ - A_i^+ B_i \]

\[ + A_i^+ B_i - A_i^+ B_i^+ + A_i B_i B_i^+ + A_i B_i B_i^+ + 2A_i B_i B_i - A_i B_i B_i^+ - A_i^+ B_i \]

\[ + A_i^+ B_i - A_i^+ B_i^+ + A_i B_i B_i^+ + A_i B_i B_i^+ + 2A_i B_i B_i - A_i B_i B_i^+ - A_i^+ B_i. \]
\[ a_1 = 6\omega^2\phi \cos (\pi\phi) + 4\omega^4 \cos \left( \frac{3\pi\phi}{2} \right) + \omega^6 \cos (2\pi\phi) + 4\omega^8 \cos \left( \frac{\pi\phi}{2} \right) + 1, \]
\[ b_1 = 6\omega^2\phi \sin (\pi\phi) + 4\omega^4 \sin \left( \frac{3\pi\phi}{2} \right) + \omega^6 \sin (2\pi\phi) + 4\omega^8 \sin \left( \frac{\pi\phi}{2} \right) \).
\[ a_2 = -m_2m_3 \left( m_1m_4 \cos (3\omega t_1) + m_5\omega^6 \sin \left( \frac{\pi\phi}{2} \right) \sin (2\omega t_1) \right) \]
\[ -m_2m_3m_5 \left( \omega^6 \cos \left( \frac{\pi\phi}{2} \right) \cos (2\omega t_1) + \cos (2\omega t_1) \right) \]
\[ -m_3m_6 \left( \omega^2\phi \sin (\pi\phi) \sin (\omega t_1) + \omega^4 \phi \cos (\pi\phi) \cos (\omega t_1) \right) \]
\[ +2\omega^6 \sin \left( \frac{\pi\phi}{2} \right) \sin (\omega t_1) + 2\omega^8 \cos \left( \frac{\pi\phi}{2} \right) \cos (\omega t_1) + \cos (\omega t_1) \right) \]
\[ b_2 = m_2m_3 \left( m_1m_4 \sin (3\omega t_1) + m_5\omega^6 \sin \left( \frac{\pi\phi}{2} \right) \cos (2\omega t_1) \right) + m_2m_3m_5 \left( \omega^6 \cos \left( \frac{\pi\phi}{2} \right) \right) \]
\[ \times \sin (2\omega t_1) + \sin (2\omega t_1) - m_3m_6 \left( \omega^2\phi \sin (\pi\phi) \cos (\omega t_1) - \omega^4 \phi \cos (\pi\phi) \sin (\omega t_1) \right) \]
\[ +2\omega^6 \sin \left( \frac{\pi\phi}{2} \right) \cos (\omega t_1) - 2\omega^8 \cos \left( \frac{\pi\phi}{2} \right) \sin (\omega t_1) - \sin (\omega t_1) \right). \]

**Data Availability**

Data sharing not is applicable in this article as no datasets were generated or analysed during the current paper.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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