Carleman estimate for second order elliptic equations with Lipschitz leading coefficients and jumps at an interface

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Abstract

In this paper we prove a local Carleman estimate for second order elliptic equations with a general anisotropic Lipschitz coefficients having a jump at an interface. The argument we use is of microlocal nature. Yet, not relying on pseudodifferential calculus, our approach allows one to achieve almost optimal assumptions on the regularity of the coefficients and, consequently, of the interface.

Ce document prouve les estimations de Carleman locales pour les quations elliptiques de second ordre avec un coefficient de Lipschitz anisotrope en conditions de sauts linterface. Largument utilis est de nature microlocale. De plus, en vitant lutilisation du calcul pseudo-diffrentiel, notre approche permet dobtien des hypothses quasi optimales sur la rgularit du coefficient et par consequent sur linterface.

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1 Introduction

Since T. Carleman’s pioneer work [Car], Carleman estimates have been indispensable tools for proving the unique continuation property for partial differential equations. Recently, Carleman estimates have been successfully applied to study inverse problems, see for [Is], [KSU]. Most of Carleman estimates are proved under the assumption that the leading coefficients possess certain regularity. For example, for general second order elliptic operators, Carleman estimates were proved when the leading coefficients are at least Lipschitz [H], [H3]. The restriction of regularity on the leading coefficients also reflects the fact that the unique continuation may fail if the coefficients are only Hölder continuous in $\mathbb{R}^n$ with $n \geq 3$ (see examples constructed by Plis [P] and [M]). In $\mathbb{R}^2$, the unique continuation property holds for $W^{1,2}$ solutions of second elliptic equations in either non-divergence or divergence forms with essentially bounded coefficients [BJS], [BN], [AM], [S]. It should be noted that the unique continuation property for the second order elliptic equations in the plane with essentially bounded coefficients is deduced from the theory of quasiregular mappings. No Carleman estimates are derived in this situation.

From discussions above, Carleman estimates for second order elliptic operators with general discontinuous coefficients are not likely to hold. However, when the discontinuities occur as jumps at an interface with homogeneous or non-homogeneous transmission conditions, one can still derive useful Carleman estimates. This is the main theme of the paper. There are some excellent works on this subject. We mention several closely related papers including Le Rousseau-Robbiano [LR1], [LR2], and Le Rousseau-Lerner [LL]. For the development of the problem and other related results, we refer the reader to the papers cited above and references therein. Our result is close to that of [LL], where the elliptic coefficient is a general anisotropic matrix-valued function. To put our paper in perspective, we would like to point out that the interface is assumed to be a $C^\infty$ hypersurface in [LL] and the coefficients are $C^\infty$ away from the interface. Here we prove (Theorem 2.1) a local Carleman estimate for operator with leading coefficients which have a jump discontinuity at a flat interface and are Lipschitz continuous apart from such an interface. From this estimate, under
a standard change of coordinates, a Carleman estimate for the case of a more general $C^{1,1}$ interface follows. The obvious reason of assuming the interface being $C^{1,1}$ is that when we flatten the boundary by introducing a coordinates transform, the Jacobian matrix of this transform is Lipschitz and hence the coefficients in the new coordinates remain Lipschitz on both side of the interface (see Remark 2.2). The approach in [LL] is close to Calderón’s seminal work on the uniqueness of Cauchy problem [Cal] as an application of singular integral operators (or pseudodifferential operators). Therefore, the regularity assumptions of [LL] are due to the use of calculus of pseudodifferential operators and the microlocal analysis techniques.

The aim here is to derive the Carleman estimate using more elementary methods. Our approach does not rely on the techniques of pseudodifferential operators, but rather on the straightforward Fourier transform. Thus we are able to relax the regularity assumptions on the coefficients and the interface. We first consider the simple case where the coefficients depend only on the normal variable. Taking advantage of the simple structure of coefficients, we are able to derive a Carleman estimate by elementary computations with the help of the Fourier transform on the tangential variables. To handle the general coefficients, we rely on some type of partition of unity. In Section 2 after Theorem 2.1 we give a more detailed outline of our proof.

2 Notations and statement of the main theorem

Define $H_{\pm} = \chi_{\mathbb{R}^n_{\pm}}$ where $\mathbb{R}^n_{\pm} = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} | y \geq 0 \}$ and $\chi_{\mathbb{R}^n_{\pm}}$ is the characteristic function of $\mathbb{R}^n_{\pm}$. Let us stress that for a vector $(x, y)$ of $\mathbb{R}^n$, we mean $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. In places we will use equivalently the symbols $D, \nabla, \partial$ to denote the gradient of a function and we will add the index $x$ or $y$ to denote gradient in $\mathbb{R}^{n-1}$ and the derivative with respect to $y$ respectively.

Let $u_{\pm} \in C^\infty(\mathbb{R}^n)$. We define

$$u = H_+u_+ + H_-u_- = \sum_{\pm} H_{\pm}u_{\pm},$$

hereafter, we denote $\sum_{\pm} a_{\pm} = a_+ + a_-,$ and for $\mathbb{R}^{n-1} \times \mathbb{R}$

$$\mathcal{L}(x, y, \partial)u := \sum_{\pm} H_{\pm}\text{div}_{x,y}(A_{\pm}(x, y)\nabla_{x,y}u_{\pm}),$$

(2.1)

where

$$A_{\pm}(x, y) = \{a_{i,j}^{\pm}(x, y)\}_{i,j=1}^{n}, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R}$$

(2.2)

is a Lipschitz symmetric matrix-valued function satisfying, for given constants $\lambda_0 \in (0, 1], M_0 > 0$,

$$\lambda_0|z|^2 \leq A_{\pm}(x, y)z \cdot z \leq \lambda_0^{-1}|z|^2, \forall (x, y) \in \mathbb{R}^n, \forall z \in \mathbb{R}^n$$

(2.3)
and
\[ |A_\pm(x', y') - A_\pm(x, y)| \leq M_0(|x' - x| + |y' - y|). \] (2.4)

We define
\[ h_0(x) := u_+(x, 0) - u_-(x, 0), \quad \forall x \in \mathbb{R}^{n-1}, \] (2.5)
\[ h_1(x) := A_+(x, 0)\nabla_{x,y} u_+(x, 0) \cdot \nu - A_-(x, 0)\nabla_{x,y} u_-(x, 0) \cdot \nu, \quad \forall x \in \mathbb{R}^{n-1}, \] (2.6)
where \( \nu = -e_n \).

Let us now introduce the weight function. Let
\[ \varphi(y) = \begin{cases} \varphi_+(y) := \alpha_+ y + \beta y^2 / 2, & y \geq 0, \\ \varphi_-(y) := \alpha_- y + \beta y^2 / 2, & y < 0, \end{cases} \] (2.7)
where \( \alpha_+, \alpha_- \) and \( \beta \) are positive numbers which will be determined later. In what follows we denote by \( \varphi_+ \) and \( \varphi_- \) the restriction of the weight function \( \varphi \) to \( [0, +\infty) \) and to \( (-\infty, 0) \) respectively. We use similar notation for any other weight functions.

For any \( \varepsilon > 0 \) let
\[ \psi_\varepsilon(x, y) := \varphi(y) - \frac{\varepsilon}{2} |x|^2, \]
and let, for \( \delta > 0 \),
\[ \phi_\delta(x, y) := \psi_\delta (\delta^{-1} x, \delta^{-1} y). \] (2.8)

For a function \( h \in L^2(\mathbb{R}^n) \), we define
\[ \hat{h}(\xi, y) = \int_{\mathbb{R}^{n-1}} h(x, y) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^{n-1}. \]

As usual we denote by \( H^{1/2}(\mathbb{R}^{n-1}) \) the space of the functions \( f \in L^2(\mathbb{R}^{n-1}) \) satisfying
\[ \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi < \infty, \]
with the norm
\[ ||f||_{H^{1/2}(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} (1 + |\xi|^2)^{1/2} |\hat{f}(\xi)|^2 d\xi. \] (2.9)

Moreover we define
\[ [f]_{1/2, \mathbb{R}^{n-1}} = \left[ \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^n} dy dx \right]^{1/2}, \]
and recall that there is a positive constant \( C \), depending only on \( n \), such that
\[ C^{-1} \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi \leq [f]_{1/2, \mathbb{R}^{n-1}}^2 \leq C \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi, \]
so that the norm (2.9) is equivalent to the norm \(|f|_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}\). We use the letters \(C, C_0, C_1, \ldots\) to denote constants. The value of the constants may change from line to line, but it is always greater than 1.

We will denote by \(B_r(x)\) the ball centered at \(x \in \mathbb{R}^{n-1}\) with radius \(r > 0\). Whenever \(x = 0\) we denote \(B_r = B_r(0)\).

**Theorem 2.1** Let \(u\) and \(A_\pm(x, y)\) satisfy (2.1)-(2.6). There exist \(\alpha_+, \alpha_-, \beta, \delta_0, r_0\) and \(C\) depending on \(\lambda_0, M_0\) such that if \(\delta \leq \delta_0\) and \(\tau \geq C\), then

\[
\sum_{\pm} \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_+} |D^k u_\pm|^2 e^{2\tau \phi_\delta(x,y)} \, dx dy + \sum_{\pm} \sum_{k=0}^{1} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k u_\pm(x,0)|^2 e^{2\tau \phi_\delta(x,0)} \, dx
\]

\[
+ \sum_{\pm} \tau^2 [e^{\tau \phi_\delta(\cdot,0)} u_\pm(\cdot,0)]_{1/2,\mathbb{R}^{n-1}}^2 + \sum_{\pm} |D(e^{\tau \phi_\delta \pm} u_\pm)(\cdot,0)]_{1/2,\mathbb{R}^{n-1}}^2
\]

\[
\leq C \left( \sum_{\pm} \int_{\mathbb{R}^n_+} |u_\pm(x,\partial)|^2 e^{2\tau \phi_\delta(x,y)} \, dx dy + [e^{\tau \phi_\delta(\cdot,0)} h_1]_{1/2,\mathbb{R}^{n-1}}^2 \right)
\]

\[
+ \left[ \nabla_x (e^{\tau \phi_\delta} h_0)(\cdot,0) \right]_{1/2,\mathbb{R}^{n-1}}^2 + \frac{3}{\delta} \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau \phi_\delta(x,0)} \, dx + \frac{\tau}{\delta} \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau \phi_\delta(x,0)} \, dx \right)
\]

(2.10)

where \(u = H_+ u_+ + H_- u_-\), \(u_\pm \in C^\infty(\mathbb{R}^n)\) and \(u \subset B_{\delta/2} \times [-\delta r_0, \delta r_0]\), \(h_0\) and \(h_1\) are defined in (2.5) and (2.6), respectively, and \(\phi_\delta\) is given by (2.8).

**Remark 2.2** Estimate (2.10) is a local Carleman estimate near the flat interface \(y = 0\). As mentioned in the Introduction, one can derive from (2.10) a Carleman estimate for more general interfaces: if the interface is locally represented by the graph of a \(C^{1,1}\) function \(g(x)\), the map \((x, y) \to (x, y - g(x))\) flattens the interface and changes the operator preserving the Lipschitz character of the leading coefficients. Of course, the weight function in the new Carleman estimates will be changed accordingly.

On the other hand, an estimate like (2.10) is sufficient for some applications such as the inverse problem of estimating the size of an inclusion by one pair of boundary measurements [FLVW].

**Remark 2.3** Let us point out that the level sets

\[
\{(x, y) \in B_{\delta/2} \times (-\delta r_0, \delta r_0) : \phi_\delta(x, y) = t\}
\]

have approximately the shape of paraboloid and, in a neighborhood of \((0, 0)\), \(\partial_y \phi_\delta > 0\) so that the gradient of \(\phi\) points inward the halfspace \(\mathbb{R}^n_+\). These features are crucial to derive from the Carleman estimate (2.10) a Hölder type smallness propagation estimate across the interface \((x, 0) : x \in \mathbb{R}^{n-1}\) for weak solutions to the transmission problem

\[
\begin{cases}
\mathcal{L}(x, y, \partial) u = \sum_{\pm} H_\pm b_\pm \cdot \nabla_{x, y} u_\pm + c_\pm u_\pm, \\
u_+(x, 0) - u_+(x, 0) = 0, \\
A_+(x, 0) \nabla_{x, y} u_+(x, 0) \cdot \nu - A_-(x, 0) \nabla_{x, y} u_-(x, 0) \cdot \nu = 0,
\end{cases}
\]

(2.11)
where \( b_\pm \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \) and \( c_\pm \in L^\infty(\mathbb{R}^n) \). More precisely if the error of observation of \( u \) is known in an open set of \( \mathbb{R}^n_+ \), we can find a Hölder control of \( u \) in a bounded set of \( \mathbb{R}^n_- \). For more details about such type of estimate we refer to [LR1, Sect. 3.1].

The proof of Theorem 2.1 is divided into two steps as follows.

**Step 1.** We first consider the particular case of the leading matrices (2.2) independent of \( x \) and we prove (Theorem 3.1), for the corresponding operator \( \mathcal{L}(y, \partial) \), a Carleman estimate with the weight function \( \phi(x, y) = \varphi(y) + s\gamma \cdot x \), where \( s \) is a suitable small number and \( \gamma \) is an arbitrary unit vector of \( \mathbb{R}^{n-1} \). The features of the leading matrices and of the weight function \( \phi \) allow to factorize the Fourier transform of the conjugate of the operator \( \mathcal{L}(y, \partial)u \) with respect to \( \phi \). So that we can follow, roughly speaking, at an elementary level the strategy of [LL] for the operator \( \mathcal{L}(y, \partial) \). Nevertheless such an estimate has only a prepatory character to prove Theorem 2.1, because, due to the particular feature of the weight \( \phi \) (i.e. linear with respect to \( x \)), the Carleman estimate obtained in Theorem 3.1 cannot yield to any kind of significant smallness propagation estimate across the interface.

**Step 2.** In the second we adapt the method described in [Tr, Ch. 4.1] to an operator with jump discontinuity. More precisely, we localize the operator (2.1) with respect to the \( x \) variable and we linearize the weight function, again with respect the \( x \) variable, and by the Carleman estimate obtained in the Step 1 we derive some local Carleman estimates. Subsequently we put together such local estimates by mean of the unity partition introduced in [Tr].

### 3 Step 1 - A Carleman estimate for leading coefficients depending on \( y \) only

In this section we consider the simple case of the leading matrices (2.2) independent of \( x \). Moreover, the weight function that we consider is linear with respect to \( x \) variable, so that, as explained above, the Carleman estimates we get here are only preliminary to the one that we will get in the general case.

Assume that

\[
A_\pm(y) = \{a_{ij}(y)\}_{i,j=1}^n
\]  

(3.1)

are symmetric matrix-valued functions satisfying (2.3) and (2.4), i.e.,

\[
\lambda_0|z|^2 \leq A_\pm(y)z \cdot z \leq \lambda_0^{-1}|z|^2, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^n
\]  

(3.2)

\[
|A_\pm(y') - A_\pm(y'')| \leq M_0|y' - y''|, \forall y', y'' \in \mathbb{R}.
\]  

(3.3)

From (3.2), we have

\[
a_{nn}^\pm(y) \geq \lambda_0 \quad \forall y \in \mathbb{R}.
\]  

(3.4)
In the present case the the differential operator (2.1) became

$$\mathcal{L}(y, \partial)u := \sum_{\pm} H_{\pm} \text{div}_{x,y}(A_{\pm}(y) \nabla_{x,y} u_{\pm}),$$  \hspace{1cm} (3.5)$$

where $$u = \sum_{\pm} H_{\pm} u_{\pm}, u_{\pm} \in C^\infty(\mathbb{R}^n)$$

We also set, for any $$s \in [0, 1]$$ and $$\gamma \in \mathbb{R}^{n-1}$$ with $$|\gamma| \leq 1$$

$$\phi(x, y) = \varphi(y) + s \gamma \cdot x = H_+ \phi_+ + H_- \phi_-,$$  \hspace{1cm} (3.6)$$

where $$\varphi$$ is defined in (2.7).

Our aim here is to prove the following Carleman estimate.

**Theorem 3.1** There exist $$\tau_0$$, $$s_0$$, $$r_0$$, $$C$$ and $$\beta_0$$ depending only on $$\lambda_0$$, $$M_0$$, such that for $$\tau \geq \tau_0$$, $$0 < s \leq s_0 < 1$$, and for every $$w = \sum_{\pm} H_{\pm} w_{\pm}$$ with $$\text{supp} \, w \subset B_1 \times [-r_0, r_0]$$, we have that

$$\sum_{\pm} \sum_{k=0}^2 \tau^{3-2k} \int_{\mathbb{R}^n} |D^k w_{\pm}|^2 e^{2\tau \phi} \, dx \, dy$$

$$+ \sum_{\pm} \sum_{k=0}^1 \tau^{3-2k} \int_{\mathbb{R}^n} |D^k w(x, 0)|^2 e^{2\tau \phi(x, 0)} \, dx + \sum_{\pm} \tau^2 [(e^{\tau \phi} w_{\pm})(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2$$

$$+ \sum_{\pm} [\partial_y (e^{\tau \phi} w_{\pm})(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2 + \sum_{\pm} [\nabla_x (e^{\tau \phi} w_{\pm})(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2$$

$$\leq C \left( \sum_{\pm} \sum_{k=0}^2 \tau^{3-2k} \int_{\mathbb{R}^n} |\nabla (e^{\tau \phi(\cdot, 0)} (w_{\pm}(\cdot, 0) - w_{-}(\cdot, 0)))|^2_{1/2, \mathbb{R}^{n-1}} \right.$$

$$+ [e^{\tau \phi(\cdot, 0)} (A_+(0) \nabla_{x,y} w_{\pm}(x, 0) \cdot \nu - A_- (0) \nabla_{x,y} w_{\pm}(x, 0) \cdot \nu)]_{1/2, \mathbb{R}^{n-1}}^2$$

$$+ \tau \int_{\mathbb{R}^n} e^{2\tau \phi(x, 0)} [A_+(0) \nabla_{x,y} w_{\pm}(x, 0) \cdot \nu - A_- (0) \nabla_{x,y} w_{\pm}(x, 0) \cdot \nu]^2 \, dx$$

$$+ \tau^3 \int_{\mathbb{R}^{n-1}} e^{2\tau \phi(x, 0)} |w_{\pm}(x, 0) - w_{-}(x, 0)|^2 \, dx \bigg),$$

with $$\beta \geq \beta_0$$ and $$\alpha_{\pm}$$ properly chosen.

### 3.1 Fourier transform of the conjugate operator and its factorization

To proceed further, we introduce some operators and find their properties. We use the notation $$\partial_j = \partial_{x_j}$$ for $$1 \leq j \leq n-1$$. Let us denote $$B_{\pm}(y) = \{b_{jk}(y)\}_{j,k=1}^{n-1}$$, the symmetric matrix such that, for $$z = (z_1, \cdots, z_{n-1}, z_n) = (z', z_n)$$,

$$B_{\pm}(y) z' \cdot z' = A_{\pm}(y) z \cdot z$$ \hspace{1cm} (3.8)$$

$$z_n = -\sum_{j=1}^{n-1} a_{nj}(y) z_j \cdot z_n$$
In view of (3.2) we have
\[ \lambda_1|z'|^2 \leq B_\pm(y)z' \cdot z' \leq \lambda_1^{-1}|z'|^2, \quad \forall y \in \mathbb{R}, \ \forall z' \in \mathbb{R}^{n-1}, \] (3.9)

\[ \lambda_1 \leq \lambda_0 \text{ depends only on } \lambda_0. \]

Notice that
\[ b_{jk}^\pm(y) = a_{jk}^\pm(y) - \frac{a_{nj}^\pm(y)a_{nk}^\pm(y)}{a_{nn}^\pm(y)}, \quad j, k = 1, \ldots, n - 1. \] (3.10)

Now let us define the operator
\[ T_\pm(y, \partial_x)u_\pm := \sum_{j=1}^{n-1} \frac{a_{nj}^\pm(y)}{a_{nn}^\pm(y)} \partial_j u_\pm. \] (3.11)

It is easy to show, by direct calculations ([LL]), that
\[ \text{div}_{x,y}(A_\pm(y)\nabla_{x,y}w_\pm) = (\partial_y + T_\pm(y))a_{nn}^\pm(y)(\partial_y + T_\pm)u_\pm + \text{div}_x(B_\pm(y)\nabla_x u_\pm). \] (3.12)

Now, let \( w = \sum w_\pm \in C_0^\infty(\mathbb{R}^n) \). We set
\[ \theta_0(x) := w_+(x,0) - w_-(x,0) \quad \text{for } x \in \mathbb{R}^{n-1}, \]
and therefore
\[ e^{\tau \phi}L(y, \partial)(e^{-\tau \phi}v) = e^{\tau s_{\gamma} x}e^{\tau \varphi}L(y, \partial)(e^{-\tau \varphi}v). \]

It follows from (3.12) that
\[ e^{\tau \varphi}L(y, \partial)(e^{-\tau \varphi}v) = \sum_{\pm} H_\pm e^{-\tau \varphi}v_\pm \]
and therefore
\[ e^{\tau \phi}L(y, \partial)(e^{-\tau \phi}v) = e^{\tau s_{\gamma} x}e^{\tau \varphi}L(y, \partial)(e^{-\tau \varphi}v). \]
which leads to

\[
e^{\tau \phi} \mathcal{L}(y, \partial)(e^{-\tau \phi} v) = e^{\tau s \gamma \cdot x} e^{\tau \phi} \mathcal{L}(y, \partial)(e^{-\tau \phi} v)
\]

\[
= e^{\tau s \gamma \cdot x} \sum_{\pm} H_{\pm} \left[ (\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x))a_{n \pm}(y)(\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x))(e^{-\tau s \gamma \cdot x} v) \right]
\]

\[
+ e^{\tau s \gamma \cdot x} \sum_{\pm} H_{\pm} \text{div}_x \left( B_{\pm}(y) \nabla_x (e^{-\tau s \gamma \cdot x} v) \right).
\]

By the definition of \( T_{\pm}(y, \partial_x) \), we get that

\[
T_{\pm}(y, \partial_x)(e^{-\tau s \gamma \cdot x} v) = e^{-\tau s \gamma \cdot x} \sum_{j=1}^{n-1} \frac{a_{n \pm}(y)}{a_{m \pm}(y)} (\partial_j v - \tau s \gamma_j v) \]

\[
:= e^{-\tau s \gamma \cdot x} T_{\pm}(y, \partial_x - \tau s \gamma) v.
\]

To continue the computation, we observe that

\[
e^{\tau s \gamma \cdot x} \left[ (\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x))a_{n \pm}(y)(\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x))(e^{-\tau s \gamma \cdot x} v) \right]
\]

\[
= (\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x - \tau s \gamma))a_{n \pm}(y)(\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x - \tau s \gamma)) v
\] (3.17)

and

\[
e^{\tau s \gamma \cdot x} \text{div}_x \left( B_{\pm}(y) \nabla_x (e^{-\tau s \gamma \cdot x} v) \right)
\]

\[
= \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \partial_{jk}^2 v - 2s\tau \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \partial_j v \gamma_k + s^2 \tau^2 \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \gamma_j \gamma_k v.
\] (3.18)

Combining (3.16), (3.17) and (3.18) yields

\[
e^{\tau \phi} \mathcal{L}(y, \partial)(e^{-\tau \phi} v)
\]

\[
= \sum_{\pm} H_{\pm} \left[ (\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x - \tau s \gamma))a_{n \pm}(y)(\partial_y - \tau \varphi_y' + T_{\pm}(y, \partial_x - \tau s \gamma)) v \right]
\]

\[
+ \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \partial_{jk}^2 v - 2s\tau \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \partial_j v \gamma_k + s^2 \tau^2 \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \gamma_j \gamma_k v.
\] (3.19)

Now, we will focus on the analysis of \( e^{\tau \phi} \mathcal{L}(y, \partial)(e^{-\tau \phi} v) \). To simplify it, we introduce some notations:

\[
f(x, y) = e^{\tau \phi} \mathcal{L}(y, \partial)(e^{-\tau \phi} v),
\] (3.20)

\[
B_{\pm}(\xi, \gamma, y) = \sum_{j,k=1}^{n-1} b_{jk}^\pm(y) \xi_j \gamma_k, \quad \xi \in \mathbb{R}^{n-1},
\] (3.21)

\[
\zeta_{\pm}(\xi, y) = \frac{1}{a_{mn}(y)} [B_{\pm}(\xi, \gamma, y) + 2is\tau B_{\pm}(\xi, \gamma, y) - s^2 \tau^2 B_{\pm}(\gamma, \gamma, y)],
\] (3.22)
and

\[ t_\pm(\xi, y) = \sum_{j=1}^{n-1} \frac{a_{nj}^\pm(y)}{a_{nn}^\pm(y)} \xi_j. \quad (3.23) \]

By (3.19), we have

\[ \hat{f}(\xi, y) = \sum_\pm H_\pm P_\pm \hat{v}_\pm, \quad (3.24) \]

where

\[ P_\pm \hat{v}_\pm := (\partial_y - \tau \varphi_\pm' + it_\pm(\xi + i\tau s\gamma, y))a_{nn}^\pm(y)(\partial_y - \tau \varphi_\pm' + it_\pm(\xi + i\tau s\gamma, y))\hat{v}_\pm - a_{nn}^\pm(y)\xi_\pm(\xi, y)\hat{v}_\pm. \quad (3.25) \]

Our aim is to estimate \( f(x, y) \) or, equivalently, its Fourier transform \( \hat{f}(\xi, y) \). In order to do this, we want to factorize the operators \( P_\pm \).

For any \( z = a + ib \) with \((a, b) \neq (0, 0)\), we define the square root of \( z \),

\[ \sqrt{z} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i\sqrt{\frac{b}{\sqrt{2(a + \sqrt{a^2 + b^2})}}} \]

It should be noted that \( \Re \sqrt{z} \geq 0 \).

We define two operators

\[ E_\pm = \partial_y + it_\pm(\xi + i\tau s\gamma, y) - (\tau \varphi_\pm' + \sqrt{\xi_\pm}), \quad (3.26) \]

\[ F_\pm = \partial_y + it_\pm(\xi + i\tau s\gamma, y) - (\tau \varphi_\pm' - \sqrt{\xi_\pm}). \quad (3.27) \]

With all the definitions given above, we thus obtain that

\[ P_+ \hat{v}_+ = E_+ a_{nn}^+(y)F_+ \hat{v}_+ - \hat{v}_+ \partial_y(a_{nn}^+(y)\sqrt{\xi_+}), \quad (3.28) \]

\[ P_- \hat{v}_- = F_- a_{nn}^-(y)E_- \hat{v}_- + \hat{v}_- \partial_y(a_{nn}^-(y)\sqrt{\xi_-}). \quad (3.29) \]

Let us now introduce some other useful notations and estimates that will be intensively used in the sequel.

After taking the Fourier transform, the terms on the interface (3.13) and (3.15), become

\[ \eta_0(\xi) := \hat{v}_+(\xi, 0) - \hat{v}_-(\xi, 0) = e^{\tau \phi(x, 0)}\theta_0(x) \quad (3.30) \]

and

\[ \eta_1(\xi) := -e^{\tau \phi(x, 0)}\theta_1(x) \]

\[ = a_{nn}^+(0)\partial_y \hat{v}_+(\xi, 0) - \tau \alpha_+ \hat{v}_+(\xi, 0) + it_+(\xi + i\tau s\gamma, 0)\hat{v}_+(\xi, 0) \]

\[ - a_{nn}^-(0)\partial_y \hat{v}_-(\xi, 0) - \tau \alpha_- \hat{v}_-(\xi, 0) + it_-(\xi + i\tau s\gamma, 0)\hat{v}_-(\xi, 0). \quad (3.31) \]
For simplicity, we denote
\[ V_\pm(\xi) := a_{nn}^\pm(0) \left[ \partial_y \hat{v}_\pm(\xi, 0) - \tau \alpha_\pm \hat{v}_\pm(\xi, 0) + it_\pm(\xi + i\tau s\gamma, 0) \hat{v}_\pm(\xi, 0) \right], \] (3.32)
so that
\[ V_+(\xi) - V_-(\xi) = \eta_1(\xi). \] (3.33)

Moreover, we define
\[ m_\pm(\xi, y) := \sqrt{B_\pm(\xi, \xi, y) a_{nn}^\pm(y)}. \]

From (3.9) we have
\[ \lambda_1 |\xi|^2 \leq B_\pm(\xi, \xi, y) \leq \lambda_1^{-1} |\xi|^2, \quad \forall y \in \mathbb{R}, \forall \xi \in \mathbb{R}^{n-1}, \] (3.34)
and, from (3.3),
\[ |\partial_y B_\pm(\xi, \eta, y)| \leq M_1 |\xi| |\eta|, \quad \forall \xi, \eta \in \mathbb{R}^{n-1}, \] (3.35)
where \( M_1 \) depends only on \( \lambda_0 \) and \( M_0 \). In a similar way, we list here some useful bounds, that can be easily obtained from (3.9) and (3.3).

\[ \lambda_2 |\xi| \leq m_\pm(\xi, y) \leq \lambda_2^{-1} |\xi|, \] (3.36)
\[ |\partial_y m_\pm(\xi, y)| \leq M_2 |\xi|, \] (3.37)
\[ |t_\pm(\xi, y)| \leq \lambda_3^{-1} |\xi|, \] (3.38)
\[ |\partial_y t_\pm(\xi, y)| \leq M_3 |\xi|, \] (3.39)
\[ |\zeta_\pm(\xi, y)| \leq (\lambda_0 \lambda_1)^{-1} (|\xi|^2 + s^2 \tau^2), \] (3.40)
\[ |\partial_y \zeta_\pm(\xi, y)| \leq M_4 (|\xi|^2 + s^2 \tau^2). \] (3.41)

Here \( \lambda_2 = \sqrt{\lambda_0 \lambda_1} \), \( \lambda_3 \) depends only on \( \lambda_0 \), while \( M_2, M_3 \) and \( M_4 \) depends only on \( \lambda_0 \) and \( M_0 \).

### 3.2 Derivation of the Carleman estimate for the simple case

The derivation of the Carleman estimate (3.7) is a simple consequence of the auxiliary Proposition 3.1 stated below and proved in the following Section 3.3 via the inverse Fourier transform.

Let us define
\[ L := \sup_{\xi \in \mathbb{R}^{n-1} \setminus \{0\}} \frac{m_+(\xi, 0)}{m_-(\xi, 0)}. \]

Note that, by (3.36), \( \lambda_2^2 \leq L \leq \lambda_2^{-2} \). Now we introduce the fundamental assumption on the coefficients \( \alpha_\pm \) in the weight function. As in [LL], we choose positive \( \alpha_+ \) and \( \alpha_- \), such that
\[ L < \frac{\alpha_+}{\alpha_-}. \] (3.42)
This choice will only be conditioned by $\lambda_0$. These constants will be fixed.

Let us denote
\[ \Lambda = (|\xi|^2 + \tau^2)^{1/2}. \]

We now state our main tool.

**Proposition 3.1** There exist $\tau_0$, $s_0$, $\rho$, $\beta$ and $C$, depending only on $\lambda_0$ and $M_0$, such that for $\tau \geq \tau_0$, $\text{supp} \hat{\nu}_\pm(\xi, \cdot) \subset [-\rho, \rho]$, $s \leq s_0 < 1$, we have
\[
\frac{1}{\tau} \sum_{\pm} ||\partial_y^2 \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi, 0)|^2
\leq C \left( \sum_{\pm} ||P_- \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right). \tag{3.43}
\]

Here $\mathbb{R}_\pm = \{ y \in \mathbb{R} : y \geq 0 \}$.

**Proof of Theorem 3.1.** Substituting (3.24) and the definitions of $\eta_0$, $\eta_1$ (see (3.30), (3.31)) into the right hand side of (3.43) implies
\[
\frac{1}{\tau} \sum_{\pm} ||\partial_y^2 \hat{\nu}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{\nu}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{\nu}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{\nu}_\pm(\xi, 0)|^2
\leq C \left( \sum_{\pm} ||f(\xi, \cdot)||^2_{L^2(\mathbb{R})} + \Lambda |e^{\tau \phi_0(\cdot)} \theta_1(\cdot)|^2 + \Lambda^3 |e^{\tau \phi_0(\cdot)} \theta_0(\cdot)|^2 \right). \tag{3.44}
\]

Recalling (3.32), it is not hard to see that
\[
\Lambda \sum_{\pm} |\partial_y \hat{\nu}_\pm(\xi, 0)|^2 \leq C \left( \Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{\nu}_\pm(\xi, 0)|^2 \right). \tag{3.45}
\]

Since $\Lambda^4 \geq |\xi|^2 \tau^2 + |\xi|^4 + \tau^4$, $|\xi|^3 + |\xi|^2 \tau + |\xi| \tau^2 + |\tau|^3 \leq C \Lambda^3$, and $\Lambda^3 \leq C'(|\xi|^3 + \tau^3), \leq 12$
by integrating in $\xi$, we can deduce from (3.44) and (3.45) that

\[
\sum_{\pm} \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^{n}} |D^{k} v_{\pm}|^{2} + \sum_{\pm} \left[ \nabla_{x} v_{\pm}(\cdot, 0) \right]^{2}_{1/2, \mathbb{R}^{n-1}} + \sum_{\pm} \left[ \partial_{y} v_{\pm}(\cdot, 0) \right]^{2}_{1/2, \mathbb{R}^{n-1}}
+ \sum_{\pm} \tau^{2} \left[ v_{\pm}(\cdot, 0) \right]^{2}_{1/2, \mathbb{R}^{n-1}} + \sum_{\pm} \tau \int_{\mathbb{R}^{n-1}} |\nabla_{x} v_{\pm}(x, 0)|^{2} dx
+ \sum_{\pm} \tau \int_{\mathbb{R}^{n-1}} |\partial_{y} v_{\pm}(x, 0)|^{2} dx + \sum_{\pm} \tau^{3} \int_{\mathbb{R}^{n-1}} |v_{\pm}(x, 0)|^{2} dx
\leq C \left( ||f||^{2}_{L^{2}(\mathbb{R}^{n})} + \left[ e^{\tau \phi(x, 0)} \theta_{1}(\cdot) \right]^{2}_{1/2, \mathbb{R}^{n-1}} + \left[ \nabla_{x} (e^{\tau \phi(x, 0)} \theta_{0}(\cdot)) \right]^{2}_{1/2, \mathbb{R}^{n-1}} + \tau \int_{\mathbb{R}^{n-1}} e^{2\tau \phi(x, 0)} |\theta_{1}|^{2} dx + \tau^{3} \int_{\mathbb{R}^{n-1}} e^{2\tau \phi(x, 0)} |\theta_{0}|^{2} dx \right).
\] 

(3.46)

Replacing $v_{\pm} = e^{\tau \phi_{\pm}} w_{\pm}$ into (3.46) immediately leads to (3.7).

3.3 Proof of Proposition 3.1

Let $\kappa$ be the positive number

\[
\kappa = \frac{1}{2} \left( 1 - \frac{L \alpha_{-}}{\alpha_{+}} \right)
\]

(3.47)

depending only on $\lambda_{0}$ and $M_{0}$.

The proof of Proposition 3.1 will be divided into three cases

\[
\begin{cases}
\tau \geq \frac{\lambda_{2}^{2} |\xi|}{2s_{0}}, \\
m_{+}(\xi, 0) \left( 1 - \kappa \right) \alpha_{+} \leq \tau \leq \frac{\lambda_{2}^{2} |\xi|}{2s_{0}}, \\
\tau \leq \frac{m_{+}(\xi, 0)}{(1 - \kappa) \alpha_{+}}.
\end{cases}
\]

Recall that $\lambda_{2} = \sqrt{\lambda_{0} \lambda_{1}}$ (from (3.36)) depends only on $\lambda_{0}$. Of course, we first choose a small $s_{0} < 1$, depending on $\lambda_{0}$ and $M_{0}$ only, such that

\[
\frac{m_{+}(\xi, 0)}{(1 - \kappa) \alpha_{+}} \leq \frac{\lambda_{2}^{2} |\xi|}{2s_{0}}, \quad \forall \xi \in \mathbb{R}^{n}.
\]

A smaller value $s_{0}$ will be chosen later in the proof.

We need to introduce here some further notations. First of all, let us denote by $P^{0}_{\pm}$, $E^{0}_{\pm}$, and $F^{0}_{\pm}$
the operators defined by (3.25), (3.26) and (3.27), respectively, in the special case \( s = 0 \). We also give special names to these functions that will be used in the proof:
\[
\omega_+ (\xi, y) = a_{nn}^+(y) F_+ \hat{v}_+ (\xi, y), \quad \omega_- (\xi, y) = a_{nn}^-(y) E_- \hat{v}_- (\xi, y)
\] (3.48)
and, for the special case \( s = 0 \),
\[
\omega_0^+ (\xi, y) = a_{nn}^+(y) F_0^0 \hat{v}_+ (\xi, y), \quad \omega_0^- (\xi, y) = a_{nn}^-(y) E_0^0 \hat{v}_- (\xi, y).
\] (3.49)

Case 1:
\[
\tau \geq \frac{\lambda_2^2 |\xi|}{2s_0} 
\] (3.50)
Note that, in this case, we have \(|\xi| \leq 2\lambda_2^{-2}s_0\tau\), which implies
\[
\tau \leq \Lambda \leq \sqrt{5}\lambda_2^{-2}\tau.
\] (3.51)

We will need several lemmas. In the first one, we estimate the difference \( P_\pm \hat{v}_\pm - P_0^0 \hat{v}_\pm \).

**Lemma 3.2** Let \( \tau \geq 1 \) and assume (3.50), then we have
\[
|P_\pm \hat{v}_\pm (\xi, y) - P_0^0 \hat{v}_\pm (\xi, y)| \leq C_s\tau \left[ \tau (\alpha_\pm + 1 + |y|)|\hat{v}_\pm (\xi, y)| + |\partial_y \hat{v}_\pm (\xi, y)| \right],
\] (3.52)
where \( C \) depends only on \( \lambda_0 \) and \( M_0 \).

**Proof.** First, we point out that
\[
\zeta_\pm (\xi, y)|_{s=0} = \frac{B_\pm (\xi, y)}{a_{nn}^\pm (y)},
\]
By simple calculations, and dropping \( \pm \) for the sake of shortness, we can write
\[
P \hat{v} (\xi, y) - P^0 \hat{v} (\xi, y) = I_1 + I_2 + I_3,
\] (3.53)
where
\[
I_1 = (it(\xi + i\tau s\gamma, y) - it(\xi, y))a_{nn}(y)(\partial_y - \tau \varphi' + it(\xi + i\tau s\gamma, y))\hat{v},
\]
\[
I_2 = (\partial_y - \tau \varphi' + it(\xi, y))a_{nn}(y)(it(\xi + i\tau s\gamma, y) - it(\xi, y))\hat{v},
\]
and
\[
I_3 = a_{nn}^\pm (y)\zeta_\pm (\xi, y) - B_\pm (\xi, \xi, y).
\]
By linearity of \( t \) with respect to its first argument (see (3.23)) and by (3.38), we have
\[
|t(\xi + i\tau s\gamma, y) - t(\xi, y)| = |t(i\tau s\gamma, y)| \leq \lambda_3^{-1} s\tau,
\]
which, together with (3.2) and (3.50), gives the estimate
\[
|I_1| \leq \lambda_3^{-1} \lambda_0^{-1} s\tau \left[ |\partial_y \hat{v}| + \tau (\alpha_\pm + |y|)|\hat{v}| + \lambda_3^{-1} (|\xi| + s\tau)|\hat{v}| \right]
\]
\[
\leq C s\tau \left[ |\partial_y \hat{v}| + \tau (\alpha_\pm + |y|) + s\tau|\hat{v}| \right],
\] (3.54)
where $C$ depends on $\lambda_0$ only. On the other hand, by linearity of $t$ and by (3.39), we have
\[
|\partial_y (t(\xi + i\tau \gamma, y) - t(\xi, y))| = |\partial_y (t(i\tau s \gamma, y))| \leq M_3 s \tau,
\]
which, together with (3.2), (3.3) and (3.50), gives the estimate
\[
|I_2| \leq C s \tau \{ |\partial_y \hat{v}| + [\tau (\alpha_+ + \beta |y|) + s \tau]|\hat{v}| \},
\]
(3.55)
where $C$ depends on $\lambda_0$ and $M_0$ only.
Finally, by (3.22), (3.34) and (3.50),
\[
|I_3| = |2i s \tau B_\pm(\xi, \gamma, y) - s^2 \tau^2 B_\pm(\gamma, \gamma, y)| \leq C s \tau^2
\]
(3.56)
where $C$ depends only on $\lambda_0$. Putting together (3.53), (3.55), (3.54), and (3.56) gives (3.52).

Lemma 3.2 allows us to estimate $\|P_{\pm}^0 \hat{v}_\pm(\xi, \cdot)\|_{L^2(\mathbb{R}_\pm)}$ instead of $\|P_\pm \hat{v}_\pm(\xi, \cdot)\|_{L^2(\mathbb{R}_\pm)}$. Let us now go further and note that, similarly to (3.28) and (3.29), we have
\[
P_+^0 \hat{v}_+ = E_+^0 a_+^0(y)F_+^0 \hat{v}_+ - \hat{v}_+ \partial_y (a_+^0(y)m_+(\xi, y)),
\]
\[
P_-^0 \hat{v}_- = F_-^0 a_-^0(y)E_-^0 \hat{v}_- + \hat{v}_- \partial_y (a_-^0(y)m_-(\xi, y)).
\]
We can easily obtain, from (3.3) and (3.37), that
\[
|P_+^0 \hat{v}_+ - E_+^0 a_+^0(y)F_+^0 \hat{v}_+| \leq C|\xi||\hat{v}_+|
\]
(3.57)
and
\[
|P_-^0 \hat{v}_- - F_-^0 a_-^0(y)E_-^0 \hat{v}_-| \leq C|\xi||\hat{v}_-|
\]
(3.58)
where $C$ depends only on $\lambda_0$ and $M_0$.

**Lemma 3.3** Let $\tau \geq 1$ and assume (3.50). There exists a positive constant $C$ depending only on $\lambda_0$ and $M_0$ such that, if $s_0 \leq 1/C$ then we have
\[
\Lambda |a_+^0(0)F_+^0 \hat{v}_+(\xi, 0)|^2 + \Lambda^3 |\hat{v}_+(\xi, 0)|^2 + \Lambda^4 ||\hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda^2 ||\partial_y \hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)}
\leq C ||P_+ \hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)}
\]
(3.59)
and
\[
-\Lambda |a_-^0(0)E_-^0 \hat{v}_-(\xi, 0)|^2 - \Lambda^3 |\hat{v}_-(\xi, 0)|^2 + \Lambda^4 ||\hat{v}_-(\xi, \cdot)||^2_{L^2(\mathbb{R}_-)} + \Lambda^2 ||\partial_y \hat{v}_-(\xi, \cdot)||^2_{L^2(\mathbb{R}_-)}
\leq C ||P_- \hat{v}_-(\xi, \cdot)||^2_{L^2(\mathbb{R}_-)},
\]
(3.60)
where $\text{supp}(\hat{v}_+(\xi, \cdot)) \subset [0, \frac{1}{\beta}]$ and $\text{supp}(\hat{v}_-(\xi, \cdot)) \subset [-\frac{\alpha}{2\beta}, 0]$. 

Proof. Since supp \( \hat{v}_+(x,y) \) is compact, \( \hat{v}_+(\xi,y) \equiv 0 \) when \( |y| \) is large and the same holds for the function \( \omega_+^0(\xi,y) \) defined in (3.49). We now compute

\[
||E_+^0\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}^+)} = \int_0^\infty |\partial_y \omega_+^0(\xi,y)| dy + \int_0^\infty |\tau \alpha_+ + \tau \beta y + m_+(\xi,y)|^2 |\omega_+^0(\xi,y)|^2 dy - 2\text{Re} \int_0^\infty [\tau \alpha_+ + \tau \beta y + m_+(\xi,y)] \omega_+^0(\xi,y) [\partial_y \omega_+^0(\xi,y) + it(\xi,y) \omega_+^0(\xi,y)] dy.
\]

Integrating by parts, we easily get

\[
-2\text{Re} \int_0^\infty [\tau \alpha_+ + \tau \beta y + m_+(\xi,y)] \partial_y \omega_+^0(\xi,y) |\partial_y \omega_+^0(\xi,y)|^2 dy = [\tau \alpha_+ + m_+(\xi,0)] |\omega_+^0(\xi,0)|^2 + \int_0^\infty [\tau \beta y + m_+(\xi,0)] |\omega_+^0(\xi,y)|^2 dy.
\]

By (3.50) and (3.37), we have that

\[
\tau \beta + \partial_y m_+(\xi,y) \geq \tau \beta - M_2 |\xi| \geq \tau \beta - 2\tau s_0 \lambda_2^{-2} M_2 \geq \tau \beta / 2 \geq 0
\]

provided \( 0 < s_0 \leq \frac{\beta \lambda_2^2}{4M_2} \). Combining (3.51), (3.61), (3.62) and (3.63) yields

\[
||E_+^0\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}^+)} \geq \int_0^\infty [\tau \alpha_+ + \tau \beta y + m_+(\xi,y)]^2 |\omega_+^0(\xi,y)|^2 dy + [\tau \alpha_+ + m_+(\xi,0)] |\omega_+^0(\xi,0)|^2 \geq C^{-1} \Lambda_2 \int_0^\infty |\omega_+^0(\xi,y)|^2 dy + C^{-1} \Lambda_{\omega_+^0}(\xi,0)^2,
\]

where \( C \) depends only on \( \lambda_0 \).

Similarly, we have that

\[
\lambda_0^2 ||\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}^+)} \geq \int_0^\infty |\partial_y \hat{v}_+(\xi,y) + it(\xi,y) \hat{v}_+(\xi,y)|^2 dy + \int_0^\infty |\tau \alpha_+ + \tau \beta y - m_+(\xi,y)|^2 |\hat{v}_+(\xi,y)|^2 dy + [\tau \alpha_+ - m_+(\xi,0)] |\hat{v}_+(\xi,0)|^2 + \int_0^\infty |\tau \beta - \partial_y m_+(\xi,y)| |\hat{v}_+(\xi,y)|^2 dy.
\]

The assumption (3.50) and (3.36) imply

\[
\tau \alpha_+ + \tau \beta y - m_+(\xi,y) \geq \tau \alpha_+ - \lambda_2^{-1} |\xi| \geq \tau \alpha_+ - 2\lambda_2^{-3} \tau s_0 \geq \tau \alpha_+/2
\]

provided \( 0 < s_0 \leq \frac{\alpha_+ \lambda_2^3}{4} \). Thus, if we choose

\[
0 < s_0 \leq \min \left\{ 1, \frac{\beta \lambda_2^2}{4M_2}, \frac{\alpha_+ \lambda_2^3}{4} \right\},
\]

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we obtain from (3.63) and (3.65)

\[
C||\omega^0_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \geq \int_0^\infty |\partial_y \hat{v}_+(\xi, y) + it_+(\xi, y)\hat{v}_+(\xi, y)|^2 dy
+ \Lambda^2 \int_0^\infty |\hat{v}_+(\xi, y)|^2 dy + \Lambda |\hat{v}_+(\xi, 0)|^2. \tag{3.66}
\]

Additionally, we have that

\[
\int_0^\infty |\partial_y \hat{v}_+(\xi, y) + it_+(\xi, y)\hat{v}_+(\xi, y)|^2 dy \\
\geq \varepsilon \int_0^\infty \left( |\partial_y \hat{v}_+(\xi, y)|^2 - 2|\partial_y \hat{v}_+(\xi, y)||t_+(\xi, y)\hat{v}_+(\xi, y)| + |t_+(\xi, y)\hat{v}_+(\xi, y)|^2 \right) dy \\
\geq \varepsilon \int_0^\infty \left( \frac{1}{2} |\partial_y \hat{v}_+(\xi, y)|^2 - |t_+(\xi, y)|^2 |\hat{v}_+(\xi, y)|^2 \right) dy \\
\geq \frac{\varepsilon}{2} \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 dy - \lambda_0^{-1} \varepsilon |\xi|^2 \int_0^\infty |\hat{v}_+(\xi, y)|^2 dy, \tag{3.67}
\]

for any \(0 < \varepsilon < 1\). Choosing \(\varepsilon\) sufficiently small, we obtain, from (3.66) and (3.67),

\[
C||\omega^0_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \geq \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 dy + \Lambda^2 \int_0^\infty |\hat{v}_+(\xi, y)|^2 dy + \Lambda |\hat{v}_+(\xi, 0)|^2, \tag{3.68}
\]

where \(C\) depends only on \(\lambda_0\) and \(M_0\).

Combining (3.64) and (3.68) yields

\[
\Lambda^2 \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 + \Lambda^4 \int_0^\infty |\hat{v}_+(\xi, y)|^2 + \Lambda^3 |\hat{v}_+(\xi, 0)|^2 + \Lambda |\omega^0_+(\xi, 0)|^2 \\
\leq C||E^0_+\omega^0_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)}, \tag{3.69}
\]

where \(C\) depends only on \(\lambda_0\) and \(M_0\). From (3.52), since \(\text{supp}(\hat{v}_+(\xi, \cdot)) \subset [0, 1/\beta]\) and (3.50) holds, we have

\[
||P^0_+\hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \leq 2||P^0_+\hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \\
+ Cs^2_0 \left( \Lambda^2 \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 + \Lambda^4 \int_0^\infty |\hat{v}_+(\xi, y)|^2 \right) \tag{3.70}
\]

Moreover, by (3.57) and (3.50),

\[
||E^0_+\omega^0_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \leq 2||P^0_+\hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + Cs^2_0 \Lambda^2 \int_0^\infty |\hat{v}_+(\xi, y)|^2. \tag{3.71}
\]

Finally, by (3.69), (3.70) and (3.71) we get (3.59), provided \(s_0\) is small enough.
Now, we proceed to prove (3.60). Applying the same arguments leading to (3.62), we have that
\[
||F^0_0\omega^0_-(\xi, \cdot)||_{L^2(\mathbb{R}_-)}^2 \\
\geq \int_{-\infty}^0 [\tau \alpha_- + \tau \beta y - m_-(\xi, y)]^2|\omega^0_-(\xi, y)|^2 dy - [\tau \alpha_- - m_-(\xi, 0)]|\omega^0_-(\xi, 0)|^2 \\
+ \int_{-\infty}^0 [\tau \beta - \partial_y m_-(\xi; y)]|\omega^0_-(\xi, y)|^2 dy.
\] (3.72)

By (3.36) and (3.50) and since supp(\hat{v}_-(\xi, \cdot)) \subset [-\frac{\alpha}{2\beta}, 0], we can see that
\[
\tau \alpha_- + \tau \beta y - m_-(\xi, y) \geq \tau \alpha_-/2 - \lambda_2^{-1} |\xi| \geq \tau \alpha_-/2 - 2\lambda_2^{-3}\tau s_0 \geq \tau \alpha_-/4
\] (3.73)
provided 0 < s_0 \leq \frac{\alpha - \lambda_2^3}{8}. From (3.72) and (3.73), it follows
\[
||F^0_0\omega^0_-(\xi, \cdot)||_{L^2(\mathbb{R}_-)}^2 \geq \frac{\alpha^2}{16}\tau^2 \int_0^\infty |\omega^0_-(\xi, y)|^2 dy - \tau \alpha_-|\omega^0_-(\xi, 0)|^2 \\
\geq CA^2 \int_0^\infty |\omega^0_-(\xi, y)|^2 dy - CA|\omega^0_-(\xi, 0)|^2.
\] (3.74)

Arguing as before and recalling (3.51) we obtain (3.60). □

We now take into account the transmission conditions.

**Lemma 3.4** Let \( \tau \geq 1 \) and assume (3.50). There exists a positive constant \( C \) depending only on \( \lambda_0 \) and \( M_0 \) such that if \( s_0 \leq 1/C \) then
\[
\Lambda \sum_\pm |V_\pm(\xi)|^2 + \Lambda^3 \sum_\pm |\hat{v}_\pm(\xi, 0)|^2 + \Lambda^4 \sum_\pm ||\hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda^2 \sum_\pm ||\partial_y \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 \\
\leq C \sum_\pm ||P_\pm \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 + CA|\eta_1(\xi)|^2 + CA^3|\eta_0(\xi)|^2,
\] (3.75)
where supp(\hat{v}_+(\xi, \cdot)) \subset [-\frac{\alpha}{2\beta}, \frac{\alpha}{3}] with \( c_0 = \min(\alpha, 1) \).

**Proof.** It follows from (3.59) and (3.49) that, for some \( C \) depending only on \( \lambda_0 \) and \( M_0 \),
\[
\Lambda|\omega^+_0(\xi, 0)|^2 + \Lambda^3|\hat{v}_+(\xi, 0)|^2 \leq C||P_+ \hat{v}_+(\xi, \cdot)||_{L^2(\mathbb{R}_+)}^2.
\] (3.76)
By (3.32), (3.49), (3.36) and (3.38) we easily get
\[
V_+(\xi) = \omega^+_0(\xi, 0) - a^+_n(0)(\tau s t_+(\gamma, 0) + m_+(\xi, 0)) \hat{v}_+(\xi, 0),
\]
hence
\[
\Lambda|V_+(\xi)|^2 \leq 2\Lambda|\omega^+_0(\xi, 0)|^2 + CA^3|\hat{v}_+(\xi, 0)|^2 \leq C||P_+ \hat{v}_+(\xi, \cdot)||_{L^2(\mathbb{R}_+)}^2.
\] (3.77)
where $C$ depends only on $\lambda_0$ and $M_0$.

By (3.30) and (3.59), we have that
\[
\Lambda^3 |\hat{v}_-(\xi, 0)|^2 \leq 2\Lambda^3 |\hat{v}_+(\xi, 0)|^2 + 2\Lambda^3 |\eta_0(\xi)|^2 \leq C\|P_+ \hat{v}_+(\xi, \cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3 |\eta_0(\xi)|^2.
\] (3.78)

Using the definition of $\eta_1$ (see (3.31)) and (3.77), we also deduce that
\[
\Lambda |V_-(\xi)|^2 \leq 2\Lambda |V_+(\xi)|^2 + 2\Lambda |\eta_1(\xi)|^2 \leq C\|P_+ \hat{v}_+(\xi, \cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\Lambda |\eta_1(\xi)|^2.
\] (3.79)

Putting together (3.76), (3.77), (3.78) and (3.79), we then obtain
\[
\Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi, 0)|^2 + \Lambda \sum_{\pm} |V_\pm(\xi)|^2 \leq C\|P_+ \hat{v}_+(\xi, 0)\|^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3 |\eta_0(\xi)|^2 + 2\Lambda |\eta_1(\xi)|^2.
\] (3.80)

We now use (3.59) and (3.60) and get
\[
\Lambda^4 \sum_{\pm} ||\hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)} + \Lambda^2 \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}
\leq C \sum_{\pm} ||P_\pm \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)} + \Lambda |\omega_0^-(\xi, 0)|^2 + \Lambda^3 |\hat{v}_-(\xi, 0)|^2
\]

Arguing similarly as we did for (3.77) and using (3.79) and (3.80) we get
\[
\Lambda^4 \sum_{\pm} ||\hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)} + \Lambda^2 \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}
\leq C \sum_{\pm} ||P_\pm \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)} + 2\Lambda |V_-(\xi)|^2 + C\Lambda^3 |\hat{v}_-(\xi, 0)|^2
\]
\[
\leq C \left( \sum_{\pm} ||P_\pm \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right),
\] (3.81)

where $C$ depends on $\lambda_0$ and $M_0$ only. The proof is complete by combining (3.80) and (3.81). \hfill \Box

Since $\tau \geq 1$, it is easily seen that (3.75) implies
\[
\Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi, 0)|^2 + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2
\leq C \left( \sum_{\pm} ||P_\pm \hat{v}_\pm(\xi, \cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right),
\] (3.82)

where $C$ depends on $\lambda_0$ and $M_0$ only.

**Case 2:**
\[
\frac{m_\pm(\xi, 0)}{(1 - \kappa) \alpha_\pm} \leq \tau \leq \frac{\lambda^2_0 |\xi|}{2s_0}.
\] (3.83)
In this case, by (3.36) we have
\[ \frac{\lambda_2}{\alpha_+} |\xi| \leq \tau \leq \frac{\lambda_2^2 |\xi|}{2s_0}. \] (3.84)

In addition, in view of the definition of \( \zeta_\pm \), (3.34), (3.83), and recalling that \( \lambda_2 = \sqrt{\lambda_0 \lambda_1} \) and \( s \leq s_0 \), we have that
\[ |\zeta_\pm| \geq \frac{3}{4} \lambda_2^2 |\xi|^2. \] (3.85)

It is not hard to see from (3.40), (3.41), (3.84), (3.85) that
\[ |\partial_y \sqrt{\zeta_\pm}| \leq M_5 |\xi|, \] (3.86)
where \( M_5 \) depends only on \( \lambda_0 \) and \( M_0 \). Moreover, if we set \( R_\pm = \Re \sqrt{\zeta_\pm} \geq 0 \) and \( J_\pm = \Im \sqrt{\zeta_\pm} \), then (3.86) gives
\[ |\partial_y R_\pm| + |\partial_y J_\pm| \leq M_5 |\xi|. \] (3.87)

Using (3.86), we can easily obtain from (3.28), (3.29) that
\[ |P_+ \hat{v}_+(\xi, y) - E_+ a_+^{\pm}(y) F_+ \hat{v}_+(\xi, y)| \leq C |\xi| |\hat{v}_+(\xi, y)| \] (3.88)
and
\[ |P_\cdot \hat{v}_-(\xi, y) - F_- a_-^{\pm}(y) E_- \hat{v}_-(\xi, y)| \leq C |\xi| |\hat{v}_-(\xi, y)|, \] (3.89)
where \( C \) depends only on \( \lambda_0 \) and \( M_0 \).

We now prove the following lemma.

**Lemma 3.5** Assume (3.83). There exists a positive constant \( C \) depending only on \( \lambda_0 \) and \( M_0 \) such that, if \( 0 < s_0 \leq C^{-1}, \beta \geq C \) and \( \tau \geq C \), then we have
\[
\Lambda |V_+(\xi) + a_+^{\pm}(0) \sqrt{\zeta_+(\xi, 0)} \hat{v}_+(\xi, 0)|^2 + \Lambda^2 |a_+^{\pm}(y) F_+ \hat{v}_+(\xi, \cdot)|^2_{L^2(\mathbb{R}_+)} \leq C |E_+ a_+^{\pm}(y) F_+ \hat{v}_+(\xi, \cdot)|^2_{L^2(\mathbb{R}_+)}
\] (3.90)

and
\[
\Lambda |V_+(\xi) + a_+^{\pm}(0) \sqrt{\zeta_+(\xi, 0)} \hat{v}_+(\xi, 0)|^2 + \Lambda^3 |\hat{v}_+(\xi, 0)|^2 + \Lambda^4 \int_0^\infty |\hat{v}_+(\xi, y)|^2 dy \leq C |P_+ \hat{v}_+(\xi, \cdot)|^2_{L^2(\mathbb{R}_+)}
\] (3.91)

provided \( \text{supp}(\hat{v}_+(\xi, \cdot)) \subset [0, \frac{1}{\beta}] \).

**Proof.**

\[ E_+ \omega_+(\xi, y) = [\partial_y + i\tau \phi_+ - \sqrt{\zeta_+}] \omega_+(\xi, y) := I_3 - I_4, \]
where \( I_3 = \partial_y \omega_+ + it_+ (\xi + i\tau s_\gamma, y) \omega_+ - iJ_+ \omega_+ \) and \( I_4 = \tau \alpha_+ \omega_+ + \tau \beta y_\omega + R_+ \omega_+ \).

Our task now is to estimate
\[
|| E_+ \omega_+ (\xi, \cdot) ||^2_{L^2(\mathbb{R}^+)} = \int_0^\infty |I_3|^2 dy + \int_0^\infty [\tau \alpha_+ + \tau \beta y + R_+]^2 |\omega_+|^2 dy - 2\Re \int_0^\infty I_3 \overline{I}_4 dy.
\]

We first observe that
\[
-2\Re \int_0^\infty I_3 \overline{I}_4 = - \int_0^\infty [\tau \alpha_+ + \tau \beta y + R_+ (\xi, y)] \partial_y (|\omega_+ (\xi, y)|^2) dy
+ 2 \int_0^\infty [\tau \alpha_+ + \tau \beta y + R_+ (\xi, y)] t_+ (t s_\gamma, y) |\omega_+ (\xi, y)|^2 dy
= \int_0^\infty [\tau \beta + \partial_y R_+ (\xi, y)] |\omega_+ (\xi, y)|^2 dy + [\tau \alpha_+ + R_+ (\xi, 0)] |\omega_+ (\xi, 0)|^2
+ 2 \int_0^\infty [\tau \alpha_+ + \tau \beta y + R_+ (\xi, y)] t_+ (t s_\gamma, y) |\omega_+ (\xi, y)|^2 dy
\geq \int_0^\infty [\tau \beta + \partial_y R_+ (\xi, y) - \lambda_3^{-1} s \tau (\tau \alpha_+ + \tau \beta y + R_+)] |\omega_+ (\xi, y)|^2 dy
+ [\tau \alpha_+ + R_+ (\xi, 0)] |\omega_+ (\xi, 0)|^2,
\]

where in the last inequality we have used the fact that \( R_+ \geq 0 \). Combining (3.92) and (3.93) yields
\[
|| E_+ \omega_+ (\xi, \cdot) ||^2_{L^2(\mathbb{R}^+)}
\geq \int_0^\infty [ (\tau \alpha_+ + \tau \beta y + R_+ )^2 + \tau \beta + \partial_y R_+ (\xi, y) - \lambda_3^{-1} s \tau (\tau \alpha_+ + \tau \beta y + R_+)] |\omega_+ (\xi, y)|^2 dy
+ [\tau \alpha_+ + R_+ (\xi, 0)] |\omega_+ (\xi, 0)|^2
\geq \frac{\Lambda^2}{C} \int_0^\infty |\omega_+ (\xi, y)|^2 dy + \frac{\Lambda}{C} |\omega_+ (\xi, 0)|^2
\]

provided \( s_0 \) is small enough. Formulas (3.32) and (3.27) give
\[
\omega_+ (\xi, 0) = V_+ (\xi) + a_{an}^+ (0) \sqrt{\zeta_+ (\xi, 0)} \hat{v}_+ (\xi, 0),
\]

which leads to (3.90) by (3.94).

We now want to derive (3.91). Let us write
\[
F_+ \hat{v}_+ = [\partial_y + it_+ (\xi + i\tau s_\gamma, y) - \tau \varphi'_+ + \sqrt{\zeta_+}] \hat{v}_+ := I_5 - I_6,
\]

where \( I_5 = \partial_y \hat{v}_+ + it_+ (\xi + i\tau s_\gamma, y) \hat{v}_+ + iJ_+ \hat{v}_+ \) and \( I_6 = \tau \alpha_+ \hat{v}_+ + \tau \beta y \hat{v}_+ - R_+ \hat{v}_+ \).

Thus, we have
\[
|| F_+ \hat{v}_+ (\xi, \cdot) ||^2_{L^2(\mathbb{R}^+)}
= \int_0^\infty |I_5|^2 dy + \int_0^\infty [\tau \alpha_+ + \tau \beta y - R_+]^2 |\hat{v}_+ (\xi, y)|^2 dy - 2\Re \int_0^\infty I_5 \overline{I}_6 dy.
\]

(3.96)
Repeating the computations of (3.93) and (3.94) yields
\[
\|F_+\hat{v}_+(\xi, \cdot)\|_{L^2(\mathbb{R}^+)}^2 \\
\geq \int_0^\infty |I_5|^2 \, dy + \int_0^\infty [(\tau\alpha_+ + \tau\beta y - R_+)^2|\hat{v}_+(\xi,y)|^2 \, dy + \int_0^\infty (\tau\beta - \partial_y R_+)|\hat{v}_+(\xi,y)|^2 \, dy \\
- Cs\tau \int_0^\infty |\tau\alpha_+ + \tau\beta y - R_+||\hat{v}_+(\xi,y)|^2 \, dy + [\tau\alpha_+ - R_+(\xi,0)]|\hat{v}_+(\xi,0)|^2.
\]

(3.97)

We observe that
\[
R_+^2 = \frac{\Re \zeta_+ + |\zeta_+|}{2}
\]
and, by simple calculations,
\[
|\zeta_+| \leq -\Re \zeta_+ + 2\frac{B_+(\xi, \xi, y)}{a_{mn}^+(y)},
\]
which gives the estimate
\[
R_+(\xi, y) \leq \sqrt{\frac{B_+(\xi, \xi; y)}{a_{mn}^+(y)}} = m_+(\xi, y).
\]

(3.99)

From (3.83) and (3.99), we deduce that
\[
\tau\alpha_+ - R_+(\xi, 0) \geq \tau\alpha_+ - m_+(\xi, 0) \geq \tau\alpha_+ - (1 - \kappa)\tau\alpha_+ = \kappa\tau\alpha_+.
\]

(3.100)

On the other hand, using (3.100), (3.87) and (3.84), we can obtain that for \(y \geq 0\)
\[
\tau\alpha_+ + \tau\beta y - R_+(\xi, y) = \tau\alpha_+ - R_+(\xi, 0) + \tau\beta y - R_+(\xi, y) + R_+(\xi, 0) \\
\geq \kappa\tau\alpha_+ + y(\tau\beta - C\tau) \geq \kappa\tau\alpha_+
\]
provided \(\beta\) is large enough. Furthermore, if \(0 \leq y \leq 1/\beta\), then
\[
[\tau\alpha_+ + \tau\beta y - R_+]^2 + (\tau\beta - \partial_y R_+) - C\tau|\tau\alpha_+ + \tau\beta y - R_+| \geq (\kappa\tau\alpha_+)^2/4
\]
provided \(s_0\) is small enough and \(\tau\) is large enough. Now it follows from (3.97), (3.101), and (3.102) and arguing as in (3.67), that
\[
C\|F_+\hat{v}_+(\xi, \cdot)\|_{L^2(\mathbb{R}^+)}^2 \\
\geq \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 \, dy + \Lambda^2 \int_0^\infty |\hat{v}_+(\xi, y)|^2 \, dy + \Lambda|\hat{v}_+(\xi, 0)|^2.
\]

(3.102)

Finally, by (3.88), (3.90), and (3.102), we can easily derive (3.91) provided \(\beta \geq C\), \(\tau \geq C\) and \(s_0 \leq 1/C\) for some \(C\) depending on \(\lambda_0\) and \(M_0\).

\[\square\]

Similarly, we can prove that
Lemma 3.6 Assume (3.83). There exists a positive constant $C$ depending only on $\lambda_0$ and $M_0$ such that, if $0 < s_0 \leq C^{-1}$ and $\tau \geq C$ then we have

$$- \Lambda \langle V_-(\xi) - a_{nn}(0) \sqrt{\zeta_-} v_-(\xi, 0) \rangle^2 + \Lambda \langle |a_{nn}(y) E_- v_-(\xi, \cdot) | \rangle^2 \leq C \langle |F_0 a_{nn}(y) E_- v_-(\xi, \cdot) | \rangle^2 \quad (3.103)$$

and

$$- \Lambda \langle V_-(\xi) - a_{nn}(0) \sqrt{\zeta_-} v_-(\xi, 0) \rangle^2 \leq \Lambda^3 + \Lambda^3 \int_{\infty}^{0} |\bar{v}_-(\xi, y)|^2 dy \quad (3.104)$$

provided $\text{supp}(\hat{v}_-(\xi, \cdot)) \subset \left[ -\frac{\alpha}{2\beta}, 0 \right]$. 

Proof. Let $\omega_-(\xi, y) = a_{nn}(y) E_- v_-(\xi, y) = a_{nn}(y) [\partial_y + i t_-(\xi + i \tau s \gamma, y) - \tau \varphi_- - \sqrt{\zeta_-} v_-(\xi, y)$. If we write

$$F_\omega(\xi, y) = I_7 - I_8,$$

where

$$I_7 = \partial_y \omega_- + i t_-(\xi, y) \omega_- + i J_- \omega_-$$

$$I_8 = \tau \alpha_- \omega_- + \tau \beta y \omega_- + t_-(\tau s \gamma, y) \omega_- - R_- \omega_-,$$

we have

$$\langle |F_\omega(\xi, \cdot) | \rangle^2_{L^2(\mathbb{R})} \geq -2 \Re \int_{-\infty}^{0} I_7 I_8 dy$$

$$= - \int_{-\infty}^{0} [\tau \alpha_- + \tau \beta y + t_-(\tau s \gamma, y) - R_- (\xi, y)] \partial_y (\omega_-(\xi, y))^2 dy$$

$$= \int_{-\infty}^{0} [\tau \beta + \partial_y t_-(\tau s \gamma, y) - \partial_y R_-(\xi, y)] \omega_-(\xi, y)^2 dy$$

$$- [t_-(\tau s \gamma, 0) + \tau \alpha_- - R_-(\xi, 0)] \omega_-(\xi, 0)^2$$

$$\geq \int_{-\infty}^{0} \tau [\beta - M_3 s - 2 M_5 s_0 \lambda_-^2] \omega_-(\xi, y)^2 dy - (\lambda_3 s + \alpha_+ \tau) \omega_-(\xi, 0)^2,$$

hence, by (3.84),

$$\langle |F_\omega(\xi, \cdot) | \rangle^2_{L^2(\mathbb{R})} \geq C \Lambda \int_{-\infty}^{0} \omega_-(\xi, y)^2 dy - C \Lambda |\omega_-(\xi, 0)|^2, \quad (3.105)$$

provided $s_0$ is small enough. Since, by (3.32) and (3.26),

$$\omega_-(\xi, 0) = V_-(\xi) - a_{nn}(0) \sqrt{\zeta_-} v_-(\xi, 0),$$

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we get (3.103).

To derive (3.104), we denote

\[ E_-(\hat{v}_-(\xi, y)) = I_9 - I_{10}, \]

where

\[ I_9 = \partial_y \hat{v}_- + it_-(\xi, y)\hat{v}_- - i J_\hat{v}_-, \]
\[ I_{10} = \tau \alpha_\hat{v}_- + \tau \beta y \hat{v}_- + t_-(\tau s\gamma, y)\hat{v}_- + R_\hat{v}_-. \]

Observe that if \( -\frac{\alpha_-}{2\beta} \leq y \leq 0 \) then

\[ \tau \alpha_- + \tau \beta y + t_-(\tau s\gamma, y) + R_\hat{v}_- \geq \tau \alpha_- / 2 - \lambda_3^{-1}s\tau \geq \tau \alpha_- / 4 \quad (3.106) \]

provided \( s_0 \) is small. Furthermore, by choosing again \( s_0 \) small, we can make

\[ \tau \beta + \partial_y R_- + \partial_y t_-(\tau s\gamma, y) \geq \tau \left( \beta - 2M_5s_0\lambda_2^{-2} - M_3s_0 \right) \geq 0. \quad (3.107) \]

With the help of (3.106) and (3.107), and arguing as in (3.67) we get

\[ C \| E_-(\hat{v}_-(\xi, \cdot)) \|_{L^2(\mathbb{R}_-)}^2 \geq \int_{-\infty}^0 |\partial_y \hat{v}_- - (\xi, y)|^2 dy + \Lambda^2 \int_{-\infty}^0 |\hat{v}_-(\xi, y)|^2 dy - \Lambda |\hat{v}_-(\xi, 0)|^2. \quad (3.108) \]

Using (3.103), (3.108) and (3.89), we obtain (3.104) provided \( \tau \) is large.

\[ \square \]

**Lemma 3.7** Assume (3.83). There exists a positive constant \( C \), depending only on \( \lambda_0 \) and \( M_0 \), such that if \( s_0 \leq C^{-1} \), \( \beta \geq C \) and \( \tau \geq C \) then we have

\[ \Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi, 0)|^2 + \Lambda^3 \sum_{\pm} \|\hat{v}_\pm(\xi, \cdot)|_{L^2(\mathbb{R}_\pm)}^2 + \Lambda \sum_{\pm} \|\partial_y \hat{v}_\pm(\xi, \cdot)|_{L^2(\mathbb{R}_\pm)}^2 \leq C \left( \sum_{\pm} \|P_\pm \hat{v}_\pm(\xi, \cdot)|_{L^2(\mathbb{R}_\pm)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right), \quad (3.109) \]

provided \( \text{supp}(\hat{v}_\pm(\xi, \cdot)) \subset [-\frac{\alpha_-}{2\beta}, \frac{\alpha_-}{\beta}] \) with \( c_0 = \min (\alpha_-, 1) \).

**Proof.** We obtain from (3.91) that

\[ \Lambda |\omega_\pm(\xi, 0)|^2 + \Lambda^3 |\hat{v}_\pm(\xi, 0)|^2 \leq C \|P_\pm \hat{v}_\pm(\xi, \cdot)|_{L^2(\mathbb{R}_\pm)}^2. \quad (3.110) \]

On the other hand,

\[ \Lambda |V_+(\xi)|^2 \leq 2\Lambda |\omega_+(\xi, 0)|^2 + C\Lambda^3 |\hat{v}_+(\xi, 0)|^2 \leq C \|P_+ \hat{v}_+(\xi, \cdot)|_{L^2(\mathbb{R}_+)}^2. \quad (3.111) \]
Using the definition of \( \eta_0 \) and (3.110), we see that
\[
\Lambda^3 |\hat{v}_-(\xi, 0)|^2 \leq 2\Lambda^3 |\hat{v}_+ (\xi, 0)|^2 + 2\Lambda^3 |\eta_0 (\xi)|^2 \leq C ||P_+ \hat{v}_+ (\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3 |\eta_0 (\xi)|^2.
\]
(3.112)

Summing up (3.110) and (3.112) yields
\[
\Lambda^3 \sum_{\pm} |\hat{v}_\pm (\xi, 0)|^2 \leq C ||P_+ \hat{v}_+ (\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3 |\eta_0 (\xi)|^2.
\]
(3.113)

Likewise, the definition of \( \eta_1 \) and (3.111) lead to
\[
\Lambda |V_- (\xi)|^2 \leq C ||P_+ \hat{v}_+ (\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda |\eta_1 (\xi)|^2.
\]
(3.114)

Putting together (3.111), (3.113), and (3.114), we deduce that
\[
\Lambda^3 \sum_{\pm} |\hat{v}_\pm (\xi, 0)|^2 + \Lambda \sum_{\pm} |V_\pm (\xi)|^2 \leq C ||P_+ \hat{v}_+ (\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3 |\eta_0 (\xi)|^2 + 2\Lambda |\eta_1 (\xi)|^2.
\]
(3.115)

Finally, we first use (3.91) recalling that \( \Lambda \geq \tau \geq 1 \), (3.104), and then (3.114), (3.115) to get that
\[
\Lambda^3 \sum_{\pm} ||\hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_{\pm} ||\partial_y \hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)}
\leq C \sum_{\pm} ||P_\pm \hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\hat{V}_-(\xi) - a_{\pm}^- (0) R_- (\xi, 0) \hat{v}_-(\xi, 0)|^2 + \Lambda^3 |\hat{v}_-(\xi, 0)|^2
\leq C \left( \sum_{\pm} ||P_\pm \hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\eta_1 (\xi)|^2 + \Lambda^3 |\eta_0 (\xi)|^2 \right).
\]
(3.116)

The proof is complete by combining (3.115) and (3.116).

We conclude Case 2 by observing that we can write (3.109) in the form
\[
\Lambda \sum_{\pm} |V_\pm (\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm (\xi, 0)|^2 + \frac{1}{\tau} (\Lambda^4 \sum_{\pm} ||\hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2 \sum_{\pm} ||\partial_y \hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)})
\leq C \left( \sum_{\pm} ||P_\pm \hat{v}_\pm (\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\eta_1 (\xi)|^2 + \Lambda^3 |\eta_0 (\xi)|^2 \right),
\]
(3.117)

where \( C \) depends only on \( \lambda_0 \) and \( M_0 \).

**Case 3:**
\[
\tau \leq \frac{m_+ (\xi, 0)}{(1 - \kappa) \alpha_+}.
\]
(3.118)

In this case, we have
\[
\tau \leq \frac{2\lambda_+^{-1} |\xi|}{\alpha_+ + L\alpha_-} \quad \text{(from (3.36), (3.47)).}
\]

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From the definition of $\zeta_\pm$ (see (3.22)) and the inequality

$$B_\pm(\xi,\xi; y) - s^2\tau^2 B_\pm(\gamma,\gamma; y) \geq \lambda_1|\xi|^2 - \lambda_1^{-1}s^2\tau^2 \geq \frac{\lambda_1}{4}|\xi|^2,$$

that holds for $s_0$ is sufficiently small, we can derive the estimates

$$\begin{aligned}
\Re\zeta_\pm &\geq \frac{\lambda_2}{4}|\xi|^2, \\
R_\pm &\geq \frac{\lambda_2}{2}|\xi|, \\
|J_\pm| &\leq 4\lambda_2^{-3}s\tau, \\
|\partial_y \zeta_\pm| &\leq M_4 \left(1 + \frac{4s_0^2\lambda_2^{-2}}{(\alpha_+ + L\alpha_-)^2}\right)|\xi|^2 := M_6|\xi|^2, \\
|\partial_y \sqrt{\zeta_\pm}| &\leq \frac{M_6}{\lambda_2}|\xi| := M_7|\xi|.
\end{aligned}$$

(3.119)

**Lemma 3.8** Assume (3.118). There exist a positive constant $C$ such that, if $s_0 \leq C^{-1}$ and $\tau \geq C$, then we have

$$\Lambda|\omega_+(\xi,0)|^2 + A^2 \int_0^\infty |\omega_+(\xi,y)|^2 dy + \int_0^\infty |\partial_y \omega_+(\xi,y)|^2 dy \leq C||E_+\omega_+(\xi,\cdot)||^2_{L^2}.$$

(3.120)

Furthermore, if $\supp(\hat{v}_-(-\xi,\cdot)) \subset \left[-\frac{\alpha_-}{2\beta},0\right]$, then

$$\Lambda^2 \int_{-\infty}^0 |\hat{v}_-(\xi,y)|^2 dy + \int_{-\infty}^0 |\partial_y \hat{v}_-(\xi,y)|^2 dy \leq C||E_-\hat{v}_-(\xi,\cdot)||^2_{L^2} + C\Lambda|\hat{v}_-(\xi,0)|^2.$$

(3.121)

**Proof.** We write

$$E_+\omega_+ = I_{11} - I_{12},$$

where

$$I_{11} = \partial_y \omega_+ + it_+(\xi,y)\omega_+ - iJ_+\omega_+, \quad I_{12} = \tau\alpha_+\omega_+ + \tau\beta y\omega_+ + R_+\omega_+ + t_+(\tau s\gamma,y)\omega_+, $$

and thus

$$||E_+\omega_+(\xi,\cdot)||^2_{L^2} = \int_0^\infty |I_{11}|^2 dy + \int_0^\infty \left[\tau\alpha_+ + \tau\beta y + R_+ + t_+(\tau s\gamma,y)\right]^2|\omega_+(\xi,y)|^2 dy - 2\Re \int_0^\infty I_{11}\bar{I}_{12} dy.$$

(3.122)
We first estimate
\[
-2\Re \int_0^\infty I_{11} I_{12} \,
= \int_0^\infty \left[ \tau \beta + \partial_y R_+(\xi, y) + \partial_y t_+(\tau s \gamma, y) \right] |\omega_+(\xi, y)|^2 dy \\
+ [\tau \alpha_+ + R_+(\xi, 0) + t_+(\tau s \gamma, 0)] |\omega_+(\xi, 0)|^2 \\
\geq -(M_7 |\xi| - M_3 s \tau) \int_0^\infty |\omega_+(\xi, y)|^2 dy + \left( \tau \alpha_+ + \frac{\lambda_2}{\sqrt{2}} - \lambda_3^{-1} s \tau \right) |\omega_+(\xi, 0)|^2 \\
\geq -CA \int_0^\infty |\omega_+(\xi, y)|^2 dy + CA |\omega_+(\xi, 0)|^2,
\]
provided \( s_0 \) is small enough. Combining (3.122) and arguing as in (3.67), we get (3.120). Likewise, we obtain (3.121).

\( \square \)

**Lemma 3.9** Assume (3.118). There exists a positive constants \( C \), depending on \( \lambda_0, M_0 \), such that if \( s_0 \leq C^{-1}, \tau \geq C, \) and \( \beta \geq C \), then, for \( \text{supp}(\hat{v}_+(\xi, \cdot)) \subset [0, 1/\beta] \), we have that
\[
\frac{\Lambda^2}{\tau} \int_0^\infty |\hat{v}_+(\xi, y)|^2 dy + \int_0^\infty |\partial_y \hat{v}_+(\xi, y)|^2 dy \leq C \left( ||F_+ \hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda |\hat{v}_+(\xi, 0)|^2 \right).
\]

**Proof.** Write
\[
F_+ \hat{v}_+ = I_{13} - I_{14},
\]
where
\[
I_{13} = \partial_y \hat{v}_+ + it_+(\xi, y) \hat{v}_+ + iJ_+ \hat{v}_+ \\
I_{14} = \tau \alpha_+ \hat{v}_+ + \tau \beta y \hat{v}_+ - R_+ \hat{v}_+ + t_+(\tau s \gamma, y) \hat{v}_+.
\]

We have
\[
||F_+ \hat{v}_+(\xi, \cdot)||^2_{L^2(\mathbb{R}_+)} \\
= \int_0^\infty |I_{13}|^2 dy + \int_0^\infty p |\hat{v}_+(\xi, y)|^2 dy + \left[ \tau \alpha_+ - R_+(\xi, 0) + t_+(\tau s \gamma, 0) \right] |\hat{v}_+(\xi, 0)|^2,
\]
where \( p = [-\tau \alpha_+ - \tau \beta y + R_+ - t_+(\tau s \gamma, y)]^2 + (\tau \beta - \partial_y R_+ + \partial_y t_+(\tau s \gamma, y)) \).

We claim that
\[
p \geq C \frac{\Lambda^2}{\tau}.
\]

By (3.119) and (3.118), we deduce that for \( 0 \leq y \leq 1/\beta \)
\[
R_+ - \tau \alpha_+ - \tau \beta y - t_+(\tau s \gamma, y) \\
\geq \frac{\lambda_2}{2} |\xi| - \tau (\alpha_+ + 1 + \lambda_3^{-1} s_0) \geq \frac{\lambda_2}{4} |\xi|
\]

\( \square \)
provided $|\xi| \geq C_2 \tau = 4 \lambda_5^{-1}(\alpha_+ + 1 + \lambda_5^{-1}s_0)\tau$. By (3.127), we can easily obtain (3.126) in the case of $|\xi| \geq C_2 \tau$ with $\tau$ large. On the other hand, when $|\xi| \leq C_2 \tau$, we can estimate

$$p \geq \tau \beta - \partial_y R_+ + \partial_y t_+(\tau s\gamma, y) \geq \tau \beta - M_7 C_2 \tau - M_8 s \tau \geq \frac{\beta}{2} \Lambda^2 \frac{\tau}{\tau}$$

(3.128) required $\beta$ is big enough. The estimate (3.124) is an easy consequence of (3.125) and (3.126).

**Lemma 3.10** Assume (3.118). There exist positive constants $C$ and $\rho_1$, depending only $\lambda_0$ and $M_0$ such that if $\text{supp}(\hat{v}_-(\xi, \cdot)) \subset [-\rho_1, 0]$ then

$$\Lambda|\omega_-(\xi, 0)|^2 + \Lambda^2 ||\omega_-(\xi, \cdot)||_{L^2(R_-)}^2 \leq C||F_-(\xi, \cdot)||_{L^2(R_-)}^2.$$

(3.129)

**Proof.** From (3.48), we have

$$\text{supp}(\omega_-(\xi, \cdot)) \subseteq \text{supp}(\hat{v}_-(\xi, \cdot)).$$

We first compute

$$\Re \int_0^\infty |\xi|(F_-(\xi, \cdot))\bar{\omega}_- dy = \Re \int_0^\infty |\xi|\partial_y \omega_-- dy - \int_0^\infty |\xi|[\tau \alpha_- + \tau \beta y + t_-(\tau s\gamma, y) - R_-(\xi, y)]\omega_-^2 d y$$

$$= \frac{1}{2} |\xi|\omega_-(\xi, 0)^2 + \int_0^\infty |\xi|[R_-(\xi, y) - \tau \alpha_- - \tau \beta y - t_-(\tau s\gamma, y)]\omega_-^2 d y.$$

(3.130)

We want to show that

$$C_*|\xi| \leq R_-(\xi, y) - \tau \alpha_- - \tau \beta y - t_-(\tau s\gamma, y).$$

(3.131)

Assume that (3.131) is true. From (3.130) and (3.131), it follows that

$$\frac{1}{2} |\xi|\omega_-(\xi, 0)^2 + \int_0^\infty C_* |\xi|^2||\omega_-(\xi, y)||_{L^2}^2 dy$$

$$\leq \Re \int_0^\infty |\xi|(F_-(\xi, \cdot))\bar{\omega}_-$$

$$\leq C_* \frac{1}{2} \int_0^\infty |\xi|^2||\omega_-(\xi, y)||_{L^2}^2 dy + C \int_{-\infty}^0 |F_-(\xi, y)||_{L^2}^2 dy,$$

(3.132)

which implies (3.129).

To establish (3.131), we first note that, by simple calculations, we obtain

$$|m_-(\xi, 0) - R_-(\xi, 0)| \leq Cs|\xi|,$$
which can be used to derive for \( y \leq 0 \)
\[
R_-(\xi, y) - \tau a_+ - \tau \beta y - t_-(\tau s \gamma, y)
\]
\[
\geq m_-(\xi, 0) - |R_-(\xi, 0) - m_-(\xi, 0)| - |R_-(\xi, y) - R_-(\xi, 0)| - \tau a_+ - \lambda_3^{-1} \tau s \quad (3.133)
\]
\[
\geq m_-(\xi, 0) - \tau a_+ - C(s + |y|)|\xi|.
\]

On the other hand, by the definition of \( L \), (3.36) and (3.118), we can estimate
\[
m_-(\xi, 0) - \tau a_+ \geq \frac{m_+(\xi, 0)}{L} [1 - \frac{L \alpha_-}{(\overline{1 - \kappa}) \alpha_+}] \geq m_+(\xi, 0) \frac{\kappa}{L(1 - \kappa)} \geq \frac{\lambda_2 \kappa}{(1 - \kappa) L} |\xi|. \quad (3.134)
\]

Combining (3.133) and (3.134) yields (3.131) provided \( s \) and \(|y|\) are small.

\[\square\]

**Lemma 3.11** Assume (3.118). There exists \( C \), depending only on \( \lambda_0 \) and \( M_0 \), such that if \( s_0 \leq C^{-1} \), \( \tau \geq C \), \( \beta \geq C \), then for \( \text{supp}(\hat{v}_+(\xi, \cdot)) \subset \lbrack 0, \frac{1}{3} \rbrack \) we have
\[
\Lambda |V_+ (\xi) + a_{mn}^+(0) \sqrt{\zeta_+(\xi, 0) v_+(\xi, 0)}|^2 + \Lambda^2 |F_+ \hat{v}_+(\xi, \cdot)|^2 \leq C \left( |P_+ \hat{v}_+(\xi, \cdot)|^2 + \Lambda^2 |\hat{v}_+(\xi, \cdot)|^2 \right). \quad (3.135)
\]

Furthermore, if \( \text{supp}(\hat{v}_-(\xi, \cdot)) \subset [-\rho_1, 0] \), for \( \rho_1 \) as in Lemma 3.10, then
\[
\Lambda |V_- (\xi) - a_{mn}^- (0) \sqrt{\zeta_- (\xi, 0) v_- (\xi, 0)}|^2 + \Lambda^2 |E_- \hat{v}_-(\xi, \cdot)|^2 \leq C \left( |P_- \hat{v}_-(\xi, \cdot)|^2 + \Lambda^2 |\hat{v}_-(\xi, \cdot)|^2 \right). \quad (3.136)
\]

**Proof.** Inequality (3.135) follows from (3.120) and (3.88). Similarly, (3.136) follows from (3.129) and (3.89).

\[\square\]

**Lemma 3.12** There exist \( C, \rho_2 \), depending only on \( \lambda_0 \) and \( M_0 \), such that if \( s_0 \leq C^{-1} \), \( \tau \geq C \), \( \beta \geq C \) then for \( \text{supp}(\hat{v}_\pm (\xi, \cdot)) \subset [-\rho_2, \rho_2] \) we have that
\[
\Lambda \sum_\pm |V_\pm (\xi)|^2 + \Lambda^3 \sum_\pm |\hat{v}_\pm (\xi, 0)|^2 + \frac{\Lambda^4}{\tau^2} \sum_\pm |\hat{v}_\pm (\xi, \cdot)|^2 + \sum_\pm \frac{\Lambda^2}{\tau} \sum_\pm |\partial_y \hat{v}_\pm (\xi, \cdot)|^2 \leq C \left( \sum_\pm |P_\pm \hat{v}_\pm (\xi, \cdot)|^2 + \Lambda |\eta_\pm (\xi)|^2 + \Lambda^3 |\eta_0 (\xi)|^2 \right). \quad (3.137)
\]

**Proof.** By (3.119),
\[
\Lambda |a_{mn}^+(0) \sqrt{\zeta_+ (\xi, 0) v_+ (\xi, 0)} + a_{mn}^- (0) \sqrt{\zeta_- (\xi, 0) v_- (\xi, 0)}|^2 \geq \Lambda |a_{mn}^+(0) R_+ (\xi, 0) v_+ (\xi, 0) + a_{mn}^- (0) R_- (\xi, 0) v_- (\xi, 0)|^2 \geq \frac{1}{C} \Lambda^3 |\hat{v}_+(\xi, 0)|^2,
\]
hence, by (3.30) and (3.33) we have
\[
\frac{1}{C} \Lambda^3 |\hat{v}_+(\xi,0)|^2 \\
\leq |a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0) + a_{mn}^-(0)\sqrt{\zeta_-}(\hat{v}_-\xi,0) + \eta_0|^2 \\
= |V_+ + a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0) + a_{mn}^-(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0) - V_- - \eta_1 - a_{mn}^\pm\sqrt{\zeta_-}\eta_0|^2 \\
\leq 4 \left( |V_+ + a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + |V_- - a_{mn}^-(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 + |\eta_1|^2 + \Lambda^3 |\eta_0|^2 \right). \\
\tag{3.138}
\]

By (3.135), (3.136) and (3.30) we get
\[
\Lambda^3 \sum_{\pm} |\hat{v}_+(\xi,0)|^2 \\
\leq C \left( \sum_{\pm} ||P_{\pm}\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda^2 ||\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+,\mathbb{R}_-)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right). \\
\tag{3.139}
\]

Again from (3.135) and (3.136)
\[
\Lambda |V_+|^2 \\
\leq 2\Lambda |V_+ + a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + 2\Lambda |a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 \\
\leq 2\Lambda |V_+ + a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + C \Lambda^3 |\hat{v}_+(\xi,0)|^2 \\
\leq 2\Lambda |V_+ + a_{mn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + C \left( \Lambda |V_- - a_{mn}^-(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 + |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right) \\
\leq C \left( \sum_{\pm} ||P_{\pm}\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda^2 ||\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+,\mathbb{R}_-)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right). \\
\tag{3.140}
\]

By (3.33),
\[
\Lambda \sum_{\pm} |V_\pm(\xi)|^2 \leq C \left( \sum_{\pm} ||P_{\pm}\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda^2 ||\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+,\mathbb{R}_-)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right). \\
\tag{3.141}
\]

Combining (3.121), (3.124), (3.135) (3.136) and (3.139), we deduce that
\[
\frac{\Lambda^4}{r} \sum_{\pm} ||\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \frac{\Lambda^2}{r} \sum_{\pm} ||\partial_\theta \hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 \\
\leq C \left( \Lambda^2 ||E_+\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+,\mathbb{R}_-)}^2 + \Lambda^2 ||E_-\hat{v}_-(\xi,\cdot)||_{L^2(\mathbb{R}_-,\mathbb{R}_+)}^2 + \Lambda^3 \sum_{\pm} |\hat{v}_+(\xi,0)|^2 \right) \\
\leq C \left( \sum_{\pm} ||P_{\pm}\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}^2 + \Lambda^2 ||\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+,\mathbb{R}_-)}^2 + \Lambda |\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right) \\
\tag{3.142}
\]
Finally, putting together (3.139), (3.141) and (3.142) yields

\[ \Lambda \sum_{\pm} |V_{\pm}|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_{\pm}(\xi,0)|^2 + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \]

\[ \leq C \left( \sum_{\pm} ||P_{\pm} \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\eta_1|^2 + \Lambda^3 |\eta_0|^2 + \Lambda^2 \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \right) \]

that gives (3.137) if we take \( \tau \) large enough to absorb the term \( C \Lambda^2 \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}. \)

Now are ready to finish the proof of Theorem 3.1. Combining all cases (3.82), (3.117), (3.137), we conclude that

\[ \Lambda \sum_{\pm} |V_{\pm}|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_{\pm}(\xi,0)|^2 + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \]

\[ \leq C \left( \sum_{\pm} ||P_{\pm} \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda |\eta_1|^2 + \Lambda^3 |\eta_0|^2 \right). \]

(3.143)

Recall that

\[ P_{\pm} \hat{v}_{\pm} = (\partial_y - \tau \varphi'_{\pm} + it_{\pm}(\xi + it \tau s \gamma, y)) a_{mn}^\pm(y) (\partial_y - \tau \varphi'_{\pm} + it_{\pm}(\xi + it \tau s \gamma, y)) \hat{v}_{\pm} - a_{mn}^\pm(y) \xi_{\pm}(\xi, y) \hat{v}_{\pm}, \]

which implies

\[ |\partial_y^2 \hat{v}_{\pm}| \leq C (|P_{\pm} \hat{v}_{\pm}| + \Lambda |\partial_y \hat{v}_{\pm}| + \Lambda^2 |\hat{v}_{\pm}|) , \]

where \( C \) depends only on \( \lambda_0 \) and \( M_0. \)

Therefore, we can derive

\[ \frac{1}{\tau} \sum_{\pm} ||\partial_y^2 \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \]

\[ \leq C \left( \sum_{\pm} ||P_{\pm} \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \right). \]

(3.144)

The estimate (3.43) follows directly from (3.143) and (3.144). \( \square \)

4 Step 2 - The Carleman estimate for general coefficients

Having at disposal the Carleman estimate when \( A_{\pm} = A_{\pm}(y) \), we want to derive it for \( A_{\pm}(x, y) \). The main idea is to ”approximate” \( A_{\pm}(x, y) \) with coefficients depending on \( y \) only. For this purpose we will make use of a special kind of partition of unity introduced in the next section.
4.1 Partition of unity and auxiliary results

In this section we collect some results on a partition of unity that will be crucial in our proof. In particular we will carefully describe how this partition of unity behaves with respect to the function spaces that we use.

For any $r > 0$ and $x \in \mathbb{R}^{n-1}$, we define $Q_r(x) = \{y \in \mathbb{R}^{n-1} : |y_j - x_j| \leq r, j = 1, 2, \ldots, n-1\}$.

Let $\vartheta_0 \in C_0^\infty(\mathbb{R})$ such that $0 \leq \vartheta_0 \leq 1$, $\text{supp} \vartheta_0 \subset (-3/2, 3/2)$ and $\vartheta_0(t) = 1$ for $t \in [-1, 1]$.

Let $\vartheta(x) = \vartheta_0(x_1) \cdots \vartheta_0(x_{n-1})$, so that $\text{supp} \vartheta \subset \overset{\circ}{Q_{3/2}}(0)$ and $\vartheta_0(x) = 1$ for $x \in Q_1(0)$.

Given $\mu \geq 1$ and $g \in \mathbb{Z}^{n-1}$, we define $x_g = g/\mu$

and

$\vartheta_{g,\mu}(x) = \vartheta(\mu(x-x_g))$.

Thus, we can see that $\text{supp} \vartheta_{g,\mu} \subset \overset{\circ}{Q_{3/2\mu}}(x_g) \subset Q_{2\mu}(x_g)$

and

$|D^k \vartheta_{g,\mu}| \leq C_1 \mu^k (\chi_{Q_{3/2\mu}}(x_g) - \chi_{Q_{1/\mu}}(x_g)), \quad k = 0, 1, 2$, \hfill (4.1)

where $C_1 \geq 1$ depends only on $n$.

Notice that, for any $g \in \mathbb{Z}^{n-1}$,

$\text{card} \left\{ g' \in \mathbb{Z}^{n-1} : \text{supp} \vartheta_{g',\mu} \cap \text{supp} \vartheta_{g,\mu} \neq \emptyset \right\} = 5^{n-1}$. \hfill (4.2)

Thus, we can define

$\tilde{\vartheta}_\mu(x) := \sum_{g \in \mathbb{Z}^{n-1}} \vartheta_{g,\mu} \geq 1, \quad x \in \mathbb{R}^{n-1}$. \hfill (4.3)

By (4.1), we get that

$|D^k \tilde{\vartheta}_\mu| \leq C_2 \mu^k$, \hfill (4.4)

where $C_2 \geq 1$ depends on $n$.

Define

$\eta_{g,\mu}(x) = \vartheta_{g,\mu}(x)/\tilde{\vartheta}_\mu(x), \quad x \in \mathbb{R}^{n-1}$,
then we have that
\[
\begin{align*}
\sum_{g \in \mathbb{Z}^{n-1}} \eta_{g, \mu} &= 1, \quad x \in \mathbb{R}^{n-1}, \\
\text{supp } \eta_{g, \mu} &\subset Q_{3/2\mu}(x_g) \subset Q_{2/\mu}(x_g), \\
|D^k \eta_{g, \mu}| &\leq C_3 \mu^k \chi_{Q_{3/2\mu}(x_g)}, \quad k = 0, 1, 2,
\end{align*}
\]
where $C_3 \geq 1$ depends on $n$.

In Section 2 we have recalled the definition of $H^{1/2}(\mathbb{R}^{n-1})$ and its seminorm $[.]_{1/2, \mathbb{R}^{n-1}}$, in what follows we will also need the seminorm
\[
[f]_{1/2, Q_r} = \left[ \int_{Q_r} \int_{Q_r} |f(x) - f(y)|^2 \frac{dxdy}{|x - y|^n} \right]^{1/2},
\]
where $Q_r = Q_r(0)$.

**Lemma 4.1** Let $f \in C^\infty(\mathbb{R}^{n-1})$ and $\text{supp } f \subset Q_{3r/4}$ for some $r \leq 1$. There exists a positive constant $C$, depending only on $n$, such that
\[
[f]^2_{1/2, \mathbb{R}^{n-1}} \leq [f]^2_{1/2, \mathbb{R}^{n-1}} \leq [f]^2_{1/2, Q_r} + \frac{C}{r} \int_{Q_r} |f(x)|^2 dx.
\]

**Proof.** It follows easily from (2.9) and (4.6), that
\[
[f]^2_{1/2, \mathbb{R}^{n-1}} = I + [f]^2_{1/2, Q_r},
\]
where
\[
I = 2 \int_{\mathbb{R}^{n-1} \setminus Q_r} \int_{Q_r} |f(x) - f(y)|^2 \frac{dydx}{|x - y|^n} = 2 \int_{\mathbb{R}^{n-1} \setminus Q_r} \int_{Q_{3r/4}} |f(y)|^2 \frac{dydx}{|x - y|^n}.
\]
Note that there is a positive constant $C_n > 1$, depending only on $n$, such that, for $x \in \mathbb{R}^{n-1} \setminus Q_r$ and $y \in Q_{3r/4}$, we have
\[
C_n^{-1} |x| \leq |x - y| \leq C_n |x|,
\]
hence, by using Fubini theorem, there is a constant $C$ depending only on $n$, such that
\[
\frac{C}{r} \int_{Q_r} |f(y)|^2 dy \leq I \leq \frac{C}{r} \int_{Q_r} |f(y)|^2 dy,
\]
that, together with (4.8), gives (4.7). \qed

**Proposition 4.1** Let $\{\varsigma_g\}_{g \in \mathbb{Z}^{n-1}}$ be a family of smooth functions such that $\text{supp } \varsigma_g$ is contained in the interior of $Q_{3/2\mu}(x_g)$, then
\[
\left[ \sum_{g \in \mathbb{Z}^{n-1}} \varsigma_g \right]_{1/2, \mathbb{R}^{n-1}} \leq C \left( \sum_{g \in \mathbb{Z}^{n-1}} [\varsigma_g]^2_{1/2, Q_{\mu/2}(x_g)} + \sum_{g \in \mathbb{Z}^{n-1}} \mu \int_{Q_{\mu/2}(x_g)} |\varsigma_g|^2 \right),
\]
where $C$ depends only on $n$.  

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Proof. Let \( x' = \mu x \) and \( y' = \mu y \), then
\[
\sum_g s_g 1_{2, \mathbb{R}^{n-1}} = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} dy dx
= \mu^{2-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x'/\mu) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y'/\mu)|^2}{|x'-y'|^n} dy' dx'.
\]

In what follows we continue to denote the functions \( s_g(x/\mu) \) by \( s_g(x) \) and, for any \( x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \) we denote by \( ||x|| = \max\{|x_j| : j = 1, \cdots, n-1\} \). Note that \( \text{supp } s_g \subset Q_{3/2}(g) = \{ x \in \mathbb{R}^{n-1} : ||x - g|| \leq 3/2 \}. \)

We write
\[
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} dx dy = I_1 + I_2, \tag{4.10}
\]
where
\[
I_1 := \int_{\mathbb{R}^{2(n-1)}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} \chi(||x-y|| < 1) dx dy
I_2 := \int_{\mathbb{R}^{2(n-1)}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} \chi(||x-y|| \geq 1) dx dy.
\]

Let us first estimate \( I_2 \). It is not hard to see that
\[
I_2 \leq \int_{\mathbb{R}^{2(n-1)}} 2 \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x)|^2 + 2|\sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} \chi(||x-y|| \geq 1) dx dy
= 4 \int_{\mathbb{R}^{2(n-1)}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x)|^2}{|x-y|^n} \chi(||x-y|| \geq 1) dx dy \leq c_1 \int_{\mathbb{R}^{n-1}} |\sum_{g \in \mathbb{Z}^{n-1}} s_g(x)|^2 dx,
\]
where \( c_1 = 4 \int_{|y| \geq 1} |y|^{-n} dy \).

Now, since we have \( \text{card } \{ g' \in \mathbb{Z}^{n-1} : \text{supp } s_{g'} \cap \text{supp } s_g \neq \emptyset \} = 5^{n-1} \), we get
\[
|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x)|^2 \leq 5^{n-1} \sum_{g \in \mathbb{Z}^{n-1}} |s_g(x)|^2,
\]
so that
\[
I_2 \leq 5^{n-1} c_1 \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^{n-1}} |s_g(x)|^2 dx = 5^{n-1} c_1 \sum_{g \in \mathbb{Z}^{n-1}} \int_{Q_2(g)} |s_g(x)|^2 dx. \tag{4.11}
\]

Concerning \( I_1 \), we can see that
\[
I_1 = \int_{\mathbb{R}^{2(n-1)}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} \chi(||x-y|| < 1) dx dy
\leq \sum_{g \in \mathbb{Z}^{n-1}} \int_{Q_2(g)} \int_{\mathbb{R}^{n-1}} \frac{|\sum_{g \in \mathbb{Z}^{n-1}} s_g(x) - \sum_{g \in \mathbb{Z}^{n-1}} s_g(y)|^2}{|x-y|^n} \chi(||x-y|| < 1) dy dx.
\]
Let us note that for each \( x \in Q_2(g) \) we have

\[
\sum_{h \in \mathbb{Z}^{n-1}} \zeta_h(x) = \sum_{||h-g|| \leq 3} \zeta_h(x)
\]

and

\[
\text{dist}_{|| \cdot ||}(Q_2(g), Q_2(h)) \geq 1, \quad \text{for} \quad ||g - h|| \geq 5,
\]

where \( \text{dist}_{|| \cdot ||}(Q_2(g), Q_2(h)) = \min\{||z - w|| : z \in Q_2(g), w \in Q_2(h)\} \). Therefore, we have

\[
I_1 \leq \sum_{g \in \mathbb{Z}^{n-1}} \sum_{||h-g|| \leq 4} \int_{Q_2(g)} \int_{Q_2(h)} \left| \sum_{||g' - g|| \leq 3} \zeta_{g'}(x) - \sum_{g'' \in \mathbb{Z}^{n-1}} \zeta_{g''}(y) \right|^2 \frac{dy dx}{|x-y|^n}
\]

\[
= \sum_{g \in \mathbb{Z}^{n-1}} \sum_{||h-g|| \leq 4} \int_{Q_2(g)} \int_{Q_2(h)} \left| \sum_{||g' - g|| \leq 3} \zeta_{g'}(x) - \sum_{||g'' - h|| \leq 3} \zeta_{g''}(y) \right|^2 \frac{dy dx}{|x-y|^n}
\]

Now we note that if \( ||h - g|| \leq 4, y \in Q_2(h) \) and \( x \in Q_2(g) \) then we have

\[
\sum_{||g'' - h|| \leq 3} \zeta_{g''}(y) = \sum_{||g'' - g|| \leq 7} \zeta_{g''}(y)
\]

and

\[
\sum_{||g' - g|| \leq 3} \zeta_{g'}(x) = \sum_{||g' - g|| \leq 7} \zeta_{g'}(x).
\]

Thus

\[
I_1 \leq \sum_{g \in \mathbb{Z}^{n-1}} \sum_{||h-g|| \leq 4} \sum_{||g'' - g|| \leq 7} \int_{Q_2(g)} \int_{Q_2(h)} \left| \zeta_{g''}(x) - \zeta_{g''}(y) \right|^2 \frac{dy dx}{|x-y|^n}
\]

\[
\leq \left( 9 \cdot 15^2 \right)^{n-1} \sum_{g'' \in \mathbb{Z}^{n-1}} \sum_{||h-g''|| \leq 7} \int_{Q_2(g'' \cap Q_2(g))} \int_{Q_2(h)} \left| \zeta_{g''}(x) - \zeta_{g''}(y) \right|^2 \frac{dy dx}{|x-y|^n}
\]

\[
\leq C \sum_{g'' \in \mathbb{Z}^{n-1}} \left\{ \left| \zeta_{g''} \right|^2_{L^2(Q_2(g'' \cap Q_2(g)))} + \int_{Q_2(g'' \cap Q_2(g))} \left| \zeta_{g''}(x) \right|^2 dx \right\},
\]

(we used (4.7) in the last inequality) where \( C \) depends on \( n \) only. Combining (4.11) and (4.12), the proof is complete.
Proposition 4.2 Let $F \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$ with $\text{supp } F \subset Q_{3/2\mu}(x_y)$, and let $a$ be a function satisfying
\[ |a(z)| \leq E_a, \quad |a(x) - a(x')| \leq K_a |x - x'|, \tag{4.13} \]
for $z, x, x' \in \text{supp } \eta_{9,\mu} \cap \text{supp } F$ and $E_a, K_a$ positive constants. Then, there is a constant $C$ depending only on $n$ such that,
\[ |aF|^2_{1/2, Q_{3/2\mu}}(x_y) \leq C \left( E_a^2 [F]^2_{1/2, Q_{3/2\mu}}(x_y) + K_a^2 \mu^{-1} \int_{Q_{3/2\mu}(x_y)} |F(y)|^2 dy \right). \tag{4.14} \]

**Proof.** By (4.6),
\[
\begin{align*}
|aF|^2_{1/2, Q_{3/2\mu}}(x_y) &= \int_{Q_{3/2\mu}(x_y)} \int_{Q_{3/2\mu}(x_y)} \frac{|a(x)F(x) - a(y)F(y)|^2}{|x - y|^n} dxdy \\
&\leq 2 \int_{Q_{3/2\mu}(x_y)} \int_{Q_{3/2\mu}(x_y)} \left( \frac{|a(x)|^2 |F(x) - F(y)|^2}{|x - y|^n} + \frac{|F(y)|^2 |a(x) - a(y)|^2}{|x - y|^n} \right) dxdy, \\
&\leq C \left( E_a^2 [F]^2_{1/2, Q_{3/2\mu}}(x_y) + K_a^2 \mu^{-1} \int_{Q_{3/2\mu}(x_y)} |F(y)|^2 dy \right).
\end{align*}
\]

Proposition 4.3 Let $f \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$. Then
\[
\sum_{\rho \in \mathbb{Z}^{n-1}} |f \eta_{9,\mu}|^2_{1/2, Q_{3/2\mu}}(x_y) \leq C \left( [f]^2_{1/2, \mathbb{R}^{n-1}} + \mu \int_{\mathbb{R}^{n-1}} |f(y)|^2 dy \right). \tag{4.15} \]

**Proof.** By (4.6),
\[
[f \eta_{9,\mu}]^2_{1/2, Q_{3/2\mu}}(x_y) \leq I + 2 \int_{Q_{3/2\mu}(x_y)} \int_{Q_{3/2\mu}(x_y)} |\eta_{9,\mu}(x)|^2 \frac{|f(x) - f(y)|^2}{|x - y|^n} dxdy, \tag{4.16} \]
where
\[
I = 2 \int_{Q_{3/2\mu}(x_y)} \int_{Q_{3/2\mu}(x_y)} |f(y)|^2 |\eta_{9,\mu}(x) - \eta_{9,\mu}(y)|^2 |x - y|^{-n} dxdy.
\]
By (4.5),
\[
I \leq 2 C_3^2 \mu^2 \int_{Q_{3/2\mu}(x_y)} \int_{Q_{3/2\mu}(x_y)} \frac{|f(y)|^2}{|x - y|^{n-2}} dxdy \leq C \mu \int_{Q_{3/2\mu}(x_y)} |f(y)|^2 dy. \tag{4.17} \]
If we now use (4.2) and add up with respect to $g \in \mathbb{Z}^{n-1}$ we get (4.15). \qed

Proposition 4.4 Let $f \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$. Then
\[
\sum_{\rho \in \mathbb{Z}^{n-1}} |f \nabla x \eta_{9,\mu}|^2_{1/2, Q_{3/2\mu}}(x_y) \leq C \left( \mu^2 [f]^2_{1/2, \mathbb{R}^{n-1}} + \mu^3 \int_{\mathbb{R}^{n-1}} |f(y)|^2 dy \right). \tag{4.18} \]

We omit the proof that proceeds in the same way as that of Proposition 4.3.
4.2 Estimate of the left hand side of the Carleman estimate, I

We are ready to derive the Carleman estimate for general coefficients. In order to make clear the procedure that we follow let us introduce and recall some notation and some definitions. Let $0 < \delta \leq 1$ and define

$$A_\delta^\pm(x, y) := A_\pm(\delta x, \delta y),$$  \hspace{1cm} (4.19)$$

and the transmission conditions

$$L_\delta(x, y, \partial)w := \sum_\pm H_\pm \text{div}_{x,y}(A_\pm^\delta(x, y) \nabla_{x,y} w_\pm),$$  \hspace{1cm} (4.20)$$

Next, with $x_g = g/\mu g \in \mathbb{Z}^{n-1}$, we define

$$A_\delta^{g,\pm}(y) := A_\pm^\delta(x_g, y) = A_\pm(\delta x_g, \delta y),$$

$$L_\delta^{g,\pm}(y, \partial)w := \sum_\pm H_\pm \text{div}_{x,y}(A_\pm^{g,\delta}(y) \nabla_{x,y} w_\pm).$$

It is not hard to observe that

$$\lambda_0 |z|^2 \leq A_\pm^{g,\delta}(y)z \cdot z \leq \lambda_0^{-1} |z|^2, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^n$$

and

$$|A_\pm^{g,\delta}(y') - A_\pm^{g,\delta}(y)| \leq M_0 |y' - y|.$$  \hspace{1cm} (4.21)$$

Concerning the weight functions, let us introduce the following notation.

$$h_\varepsilon(x) := -\varepsilon |x|^2/2, \quad H_\varepsilon(x, x_g) := \varepsilon |x - x_g|^2/2,$$

$$\psi_\varepsilon(x, y) := \varphi(y) + h_\varepsilon(x), \quad \psi_{\varepsilon, g}(x, y) := \varphi(y) + \nabla_x h_\varepsilon(x_g) \cdot (x - x_g) + h_\varepsilon(x_g),$$

where $\varphi(y)$ is defined in (2.7). Moreover assume that $\alpha_+, \alpha_-, \beta$ are fixed positive numbers such that $\beta \geq \beta_0$ and $\lambda_2^{-1} < \frac{\alpha_0}{\alpha_-}$, in such a way that condition (3.42) is satisfied by the operator $L_\delta^{g,\pm}(y, \partial)$ and Theorem 3.1 holds true for such an operator.

Note that

$$\psi_{\varepsilon, g}(x, y) - \psi_\varepsilon(x, y) = H_\varepsilon(x, x_g),$$  \hspace{1cm} (4.21)$$

so that, trivially,

$$e^{\tau \psi_\varepsilon} \leq e^{\tau \psi_{\varepsilon, g}} \leq e^{2(n-1) \frac{\varepsilon}{\mu}} e^{\tau \psi_\varepsilon} \text{ in } Q_{\delta}^2(x_g).$$  \hspace{1cm} (4.22)$$
Let us define

\[
\Xi(w) := \sum_{\pm} \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_\pm} |D^k w_\pm|^2 e^{2\tau \psi_\varepsilon} dxdy
+ \sum_{\pm} \sum_{k=0}^{1} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k w_\pm(x,0)|^2 e^{2\tau \psi_\varepsilon(x,0)} dx
+ \sum_{\pm} \tau^2 [e^{\tau \psi_\varepsilon(\cdot,0)} w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}
+ \sum_{\pm} [\partial_y (e^{\tau \psi_\varepsilon \pm}(\cdot,0))]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm} [\nabla_x (e^{\tau \psi_\varepsilon w_\pm})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}},
\]

that will be used to estimate the left hand side of \((2.10)\).

In the present subsection we prove that if \(\text{supp } w \subset U := B_{1/2} \times [-r_0, r_0]\) and if we choose

\[
\tau \geq 1/\varepsilon \quad \text{and} \quad \mu = (\varepsilon \tau)^{1/2},
\]

then

\[
\Xi(w) \leq C \sum_{g \in \mathbb{Z}^{n-1}} \Xi(w_{\eta_g,\mu}) + CR_1,
\]

where

\[
R_1 := (\varepsilon \tau)^{1/2} \sum_{\pm} \int_{\mathbb{R}^{n-1}} e^{2\tau \psi_\varepsilon(x,0)} (|\partial_y w_\pm(x,0)|^2 + |\nabla_x w_\pm(x,0)|^2 + \tau^2 |w_\pm(x,0)|^2) dx
\]

and \(C\) depends only on \(\lambda_0, M_0\).

Now, in order to obtain \((4.25)\) we estimate from above each term in \((4.23)\). By \((4.5)\), we can write

\[
w_\pm(x, y) = \sum_{g \in \mathbb{Z}^{n-1}} w_\pm(x, y) \eta_{g,\mu}(x).
\]

From \((4.2), (4.21)\) and \((4.26)\), we can see that

\[
\sum_{\pm} \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_\pm} |D^k w_\pm|^2 e^{2\tau \psi_\varepsilon} dxdy \leq C \sum_{g \in \mathbb{Z}^{n-1}} \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_\pm} |D^k (w_{\pm \eta_g,\mu})|^2 e^{2\tau \psi_\varepsilon,\varepsilon} dxdy
\]

and

\[
\sum_{\pm} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k w_\pm(x,0)|^2 e^{2\tau \psi_\varepsilon(x,0)} dx \leq C \sum_{g \in \mathbb{Z}^{n-1}} \sum_{k=0}^{1} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k (w_{\pm \eta_g,\mu})(x,0)|^2 e^{2\tau \psi_\varepsilon,\varepsilon(x,0)} dx,
\]
where $C$ depends only on $n$.

Using (4.9), we obtain

$$
\left[ \nabla_x(e^{\tau \psi} w_{\pm})(\cdot, 0) \right]_{1/2, \mathbb{R}^{n-1}}^2 \leq C \sum_{g \in \mathbb{Z}^{n-1}} \left[ \nabla_x(e^{\tau \psi} \eta_{g, \mu} w_{\pm})(\cdot, 0) \right]_{1/2, Q_{g}^2(x_g)}^2 + \mu \int_{Q_{g}^2(x_g)} |\nabla_x(e^{\tau \psi} \eta_{g, \mu} w_{\pm})(x, 0)|^2 dx.
$$

(4.29)

Since

$$
\nabla_x(e^{\tau \psi} \eta_{g, \mu} w_{\pm})(x, 0) = e^{\tau \psi(x, 0)} \eta_{g, \mu} \nabla_x w_{\pm}(x, 0) + e^{\tau \psi(x, 0)} w_{\pm} \nabla_x \eta_{g, \mu}(x, 0) - (\varepsilon \tau x) e^{\tau \psi(x, 0)} \eta_{g, \mu} w_{\pm}(x, 0),
$$

by (4.2), we have that

$$
\sum_{g \in \mathbb{Z}^{n-1}} \mu \int_{Q_{g}^2(x_g)} |\nabla_x(e^{\tau \psi} \eta_{g, \mu} w_{\pm})(x, 0)|^2 dx \leq C \left( \mu \int_{\mathbb{R}^{n-1}} e^{2\tau \psi(x, 0)} |\nabla_x w_{\pm}(x, 0)|^2 dx + \mu^5 \int_{\mathbb{R}^{n-1}} e^{2\tau \psi(x, 0)} |w_{\pm}(x, 0)|^2 dx \right).
$$

(4.30)

Let us now state and prove two useful estimates.

**Lemma 4.2** If $\text{supp } f \subset Q_{3/2\mu}(x_g)$, then we have that

$$
[f e^{\tau \psi}(-, 0)]_{1/2, \mathbb{R}^{n-1}}^2 \leq C \left( [f e^{\tau \psi}(-, 0)]_{1/2, Q_{2/\mu}(x_g)}^2 + \mu \int_{Q_{2/\mu}(x_g)} |f(x)|^2 e^{2\tau \psi(x, 0)} dx \right),
$$

and

$$
[f e^{\tau \psi}(-, 0)]_{1/2, \mathbb{R}^{n-1}}^2 \leq C \left( [f e^{\tau \psi}(-, 0)]_{1/2, Q_{2/\mu}(x_g)}^2 + \mu \int_{Q_{2/\mu}(x_g)} |f(x)|^2 e^{2\tau \psi(x, 0)} dx \right),
$$

(4.31)

(4.32)

where $C$ depends only on $n$.

**Proof.** For sake of shortness, we only show the proof of (4.32). The proof of (4.31) is similar but slightly simpler.

Denote by

$$
F = f e^{\tau \psi}(-, 0), \quad a = e^{\tau(\psi_{\mu}-\psi)}(-, 0)
$$

so that $\text{supp } F \subset Q_{2/\mu}(x_g)$ and

$$
f e^{\tau \psi}(-, 0) = a F.
$$
Notice that, by (4.21) (and recalling that $\varepsilon\tau = \mu^2$), we have

$$|a(x)| \leq e^{2(n-1)} \text{ and } |\nabla a(x)| \leq 2\mu\sqrt{n-1}e^{2(n-1)} \text{ for every } x \in Q_{2/\mu}(x_g).$$

We can now apply Lemma 4.1 and, then, Proposition 4.2 (with $E_a = e^{2(n-1)}$ and $K_a = 2\mu\sqrt{n-1}e^{2(n-1)}$) and get

$$[fe^{\tau_x,(0)}]_{1/2,\mathbb{R}^n-1}^2 \leq [aF]_{1/2,\mathbb{R}^n-1}^2 \leq C \left( [aF]_{1/2,Q_{2/\mu}(x_g)}^2 + \mu \int_{Q_{2/\mu}(x_g)} |aF|^2 dx \right)$$

$$\leq C \left( [F]_{1/2,Q_{2/\mu}(x_g)}^2 + \mu \int_{Q_{2/\mu}(x_g)} |F|^2 dx \right),$$

that is (4.32).

**Lemma 4.3**

$$[xe^{\tau_x,(0)}\eta_{g,\mu}w_\pm]_{1/2,Q_{2/\mu}(x_g)}^2 \leq C \left( [e^{\tau_x,(0)}\eta_{g,\mu}w_\pm]_{1/2,Q_{2/\mu}(x_g)}^2 \right.$$

$$\quad + \frac{1}{\mu} \int_{Q_{2/\mu}(x_g)} |\eta_{g,\mu}w_\pm(x,0)|^2 e^{2\tau_x(x,0)} dx \right). \quad (4.33)$$

**Proof.** We apply Proposition 4.2 with $a(x) = x$ and $F(x) = e^{\tau_x(x,0)}\eta_{g,\mu}(x)w_\pm(x,0)$.

Since $\text{supp } w_\pm(\cdot,0) \subset B_{1/2}$ we have, with the notation of Proposition 4.2, $E_a = 1/2$, $K_a = 1$, so that (4.33) follows.

Let us now estimate $\sum_{\pm} \sum_{g \in \mathbb{Z}^{n-1}} \left[ \nabla_x(e^{\tau_x\eta_{g,\mu}w_\pm}(\cdot,0)) \right]_{1/2,Q_{2/\mu}(x_g)}^2$ from (4.29).

Since

$$\nabla_x(e^{\tau_x\eta_{g,\mu}w_\pm})(x,0) = e^{\tau_x} \nabla_x(\eta_{g,\mu}w_\pm)(x,0) - (\varepsilon\tau x)e^{\tau_x}\eta_{g,\mu}w_\pm(x,0),$$

we can deduce from (4.2), (4.31) and (4.33) that

$$\sum_{g \in \mathbb{Z}^{n-1}} \left[ \nabla_x(e^{\tau_x\eta_{g,\mu}w_\pm}(\cdot,0)) \right]_{1/2,Q_{2/\mu}(x_g)}^2 \leq 2 \sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau_x\eta_{g,\mu}w_\pm}(\cdot,0) \right]_{1/2,Q_{2/\mu}(x_g)}^2$$

$$+ 2(\varepsilon\tau)^2 \sum_{g \in \mathbb{Z}^{n-1}} \left[ xe^{\tau_x\eta_{g,\mu}w_\pm}(\cdot,0) \right]_{1/2,Q_{2/\mu}(x_g)}^2$$

$$\leq C \left( \sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau_x\eta_{g,\mu}w_\pm}(\cdot,0) \right]_{1/2,Q_{2/\mu}(x_g)}^2 + \mu \sum_{g \in \mathbb{Z}^{n-1}} \int_{Q_{2/\mu}(x_g)} |\nabla_x(\eta_{g,\mu}w_\pm)|^2 e^{2\tau_x(x,0)} dx \right).$$
The task now is to connect the estimate to the operator \( L \).

4.3 Estimate of the left hand side of the Carleman estimate, II

In this section, we will continue to estimate the upper bound of \( \Xi(w) \) using (4.25). The task now is to connect the estimate to the operator \( L(x, y, \partial) \) given in (2.1). To this aim we apply Theorem 3.1 to the function \( w\eta_{g, \mu} \) with the weight function \( \psi_{x, g} = \varphi(y) - \varepsilon x_g \cdot x + \varepsilon |x_g|^2/2 \). In order to do this we notice that if supp \( w \subset \Omega := B_{1/2} \times [-r_0, r_0] \) and \( \mu \geq 4 \) then either \( |x_g| \leq 1 \) or supp \( \eta_{g, \mu} \cap B_{1/2} = \emptyset \) so that, in both the cases, we can apply Theorem 3.1.

By applying (3.7) and by adding up with respect to \( g \in \mathbb{Z}^{n-1} \), we obtain that

\[
\sum_{g \in \mathbb{Z}^{n-1}} \Xi(w\eta_{g, \mu}) \leq C \sum_{g \in \mathbb{Z}^{n-1}} (d_{g, \mu}^{(1)} + d_{g, \mu}^{(2)} + d_{g, \mu}^{(3)}),
\]
where
\[ d_{g,\mu}^{(1)} = \int_{\mathbb{R}^n} |\mathcal{L}_{\delta,g}(y, \partial)(w \eta_{g,\mu})|^2 e^{2\tau \psi_{x,g}} dxdy, \]
\[ d_{g,\mu}^{(2)} = \tau^3 \int_{\mathbb{R}^n} |e^{\tau \psi_{x,g}(x,0)} \theta_{0:g,\mu}(x)|^2 dx + |\nabla_x (e^{\tau \psi_{x,g}} \theta_{0:g,\mu})(\cdot, 0)|^2_{1/2, \mathbb{R}^{n-1}}, \]
\[ d_{g,\mu}^{(3)} = \tau \int_{\mathbb{R}^n} |e^{\tau \psi_{x,g}(x,0)} \theta_{1:g,\mu}(x)|^2 dx + [e^{\tau \psi_{x,g}(\cdot,0)} \theta_{1:g,\mu}(\cdot)]^2_{1/2, \mathbb{R}^{n-1}}, \]
where we set
\[ \theta_{0:g,\mu}(x) := w_+(x, 0) \eta_{g,\mu}(x) - w_-(x, 0) \eta_{g,\mu}(x) = \theta_0(x) \eta_{g,\mu}, \quad (4.37) \]
\[ \theta_{1:g,\mu}(x) := A_{+}^\delta(0) \nabla_{x,y}(w_+ \eta_{g,\mu}) \cdot \nu - A_{-}^\delta(0) \nabla_{x,y}(w_- \eta_{g,\mu}) \cdot \nu. \quad (4.38) \]

We will estimate the three terms of (4.36) separately.

**Estimate of** \( \sum_{g \in \mathbb{Z}^{n-1}} d_{g,\mu}^{(1)} \).

By (2.3), (2.4), (4.5) and (4.19) we obtain that
\[
|\mathcal{L}_{\delta,g}(y, \partial)(w \eta_{g,\mu})| \\
\leq |\mathcal{L}_{\delta}(x, y, \partial)(w \eta_{g,\mu})| + |\mathcal{L}_{\delta}(x, y, \partial)(w \eta_{g,\mu}) - \mathcal{L}_{\delta,g}(y, \partial)(w \eta_{g,\mu})| \\
\leq \eta_{g,\mu} |\mathcal{L}_{\delta}(x, y, \partial)(w \eta_{g,\mu})| + C \eta_{g,\mu} |A_{\pm}^\delta(x, y) - A^\delta_{\pm}(x, y)||D^2 w_\pm| + C \chi \mathcal{Q}_{\mathbb{R}^n}(x_g) \left( \mu |D w_\pm| + \mu^2 |w_\pm| \right) \\
\leq \eta_{g,\mu} |\mathcal{L}_{\delta}(x, y, \partial)(w \eta_{g,\mu})| + C \chi \mathcal{Q}_{\mathbb{R}^n}(x_g) \left( \delta \mu^{-1} |D^2 w_\pm| + \mu |D w_\pm| + \mu^2 |w_\pm| \right),
\]

which, together with (4.2), (4.22) and (4.24), implies
\[ \sum_{g \in \mathbb{Z}^{n-1}} d_{g,\mu}^{(1)} \leq C \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^n} |\mathcal{L}_{\delta}(x, y, \partial)(w \eta_{g,\mu})|^2 e^{2\tau \psi_{x,g}} dxdy + CR_2, \quad (4.39) \]

where
\[ R_2 = \delta^2 \mu^{-2} \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^n} |D^2 w_\pm|^2 e^{2\tau \psi_{x,\pm}} dxdy + \mu^2 \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^n} |D w_\pm|^2 e^{2\tau \psi_{x,\pm}} dxdy \\
+ \mu^4 \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^n} |w_\pm|^2 e^{2\tau \psi_{x,\pm}} dxdy. \]

**Estimate of** \( \sum_{g \in \mathbb{Z}^{n-1}} d_{g,\mu}^{(2)} \).

By (4.2) and (4.22),
\[ \sum_{g \in \mathbb{Z}^{n-1}} \tau^3 \int_{\mathbb{R}^n} |e^{\tau \psi_{x,g}(x,0)} \theta_{0:g,\mu}(x)|^2 dx \leq C \tau^3 \int_{\mathbb{R}^n} e^{2\tau \psi_{x,0}(x,0)} |\theta_0(x)|^2 dx, \quad (4.40) \]
where $C$ depends only on $n$.

Next, we note that $\nabla_x (e^{\tau \psi_x} \theta_{g,\mu}) = e^{\tau \psi_x} \nabla_x \theta_{g,\mu} - \tau \epsilon x_g e^{\tau \psi_x} \theta_{g,\mu}$.

From (4.2), (4.15), (4.32), (4.33), and (4.37), it follows that

$$
\sum_{g \in \mathbb{Z}^n} \left| \nabla_x (e^{\tau \psi_x} \theta_{g,\mu})(\cdot, 0) \right|^2_{L^2, R^n}(x_0)
\leq C \left( \sum_{g \in \mathbb{Z}^n} \left| e^{\tau \psi_x \cdot 0} \nabla_x \theta_{g,\mu} \right|^2_{L^2, R^n}(x_0) + \tau \epsilon^2 \left| e^{\tau \psi_x \cdot 0} \theta_{g,\mu} \right|^2_{L^2, R^n}(x_0) \right) + \mu \int_{R^n} e^{2\tau \psi_x(x,0)} |\nabla_x \theta_0|^2 \, dx + \mu^5 \int_{R^n} e^{2\tau \psi_x(x,0)} |\theta_0|^2 \, dx. \quad (4.41)
$$

On the other hand, by (4.15), (4.18) and (4.33), we obtain that

$$
\sum_{g \in \mathbb{Z}^n} \left| e^{\tau \psi_x \cdot 0} \nabla_x \theta_{g,\mu} \right|^2_{L^2, R^n}(x_0)
\leq C \left( \left| \nabla_x (e^{\tau \psi_x \cdot 0} \theta_0) \right|^2_{L^2, R^n} + \mu^4 \left| e^{\tau \psi_x \cdot 0} \theta_0 \right|^2_{L^2, R^n} \right) + \mu \int_{R^n} e^{2\tau \psi_x(x,0)} |\nabla_x \theta_0|^2 \, dx + \mu^5 \int_{R^n} e^{2\tau \psi_x(x,0)} |\theta_0|^2 \, dx. \quad (4.42)
$$

Finally, putting together (4.40) and (4.42) yields

$$
\sum_{g \in \mathbb{Z}^n} d_{g,\mu}^{(2)} \leq C \left( \left| \nabla_x (e^{\tau \psi_x \cdot 0}) \right|^2_{L^2, R^n} + \tau^3 \int_{R^n} e^{2\tau \psi_x(x,0)} |\theta_0|^2 \, dx + R_3 \right), \quad (4.43)
$$

where

$$
R_3 = \sum_{\pm} \left( \mu^4 \left| e^{\tau \psi_x \cdot 0} w_{\pm}(\cdot, 0) \right|^2 + \mu \int_{R^n} e^{2\tau \psi_x(x,0)} |\nabla_x w_{\pm}(\cdot, 0)|^2 \, dx \right)
+ \mu^5 \int_{R^n} e^{2\tau \psi_x(x,0)} |w_{\pm}(\cdot, 0)|^2 \, dx.
$$

**Estimate of** $\sum_{g \in \mathbb{Z}^n} d_{g,\mu}^{(3)}$.

By (4.38) and by straightforward computations we can write $\theta_{1;g,\mu}$ as

$$
\theta_{1;g,\mu} = \theta_1 \eta_{g,\mu} + J_{g,\mu}^{(1)} + J_{g,\mu}^{(2)} + J_{g,\mu}^{(3)}, \quad (4.44)
$$

where

$$
J_{g,\mu}^{(1)} = w_+ A_+ (\delta x, 0) \nabla x_{y} \eta_{g,\mu} \cdot \nu - w_- A_- (\delta x, 0) \nabla x_{y} \eta_{g,\mu} \cdot \nu,
$$

$$
J_{g,\mu}^{(2)} = \eta_{g,\mu} (A_+ (\delta x_g, 0) - A_+ (\delta x, 0)) \nabla x_{y} w_+ \cdot \nu - \eta_{g,\mu} (A_- (\delta x_g, 0) - A_- (\delta x, 0)) \nabla x_{y} w_- \cdot \nu,
$$

$$
J_{g,\mu}^{(3)} = w_+ (A_+ (\delta x_g, 0) - A_+ (\delta x, 0)) \nabla x_{y} \eta_{g,\mu} \cdot \nu - w_- (A_- (\delta x_g, 0) - A_- (\delta x, 0)) \nabla x_{y} \eta_{g,\mu} \cdot \nu.
$$
By (2.3), (2.4) and (4.5),
\begin{align*}
|J_{g,\mu}^{(1)}| &\leq C\mu \sum_{\pm} |w_{\pm}(x,0)| \chi_{Q_{\frac{1}{2}}(x_0)}, \\
|J_{g,\mu}^{(2)}| &\leq C\delta \mu^{-1} \sum_{\pm} |\nabla_{x,y}w_{\pm}(x,0)| \eta_{g,\mu}, \\
|J_{g,\mu}^{(3)}| &\leq C\delta \mu^{-1} \sum_{\pm} |\nabla_{x,y}\eta_{g,\mu}| |w_{\pm}(x,0)|,
\end{align*}
where $C$ depends on $\lambda_0$, $M_0$ and $n$. From (4.2), (4.5), (4.22), (4.44) and (4.45), we have that
\begin{align*}
\sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau \psi_{\epsilon,0}}(x_0) \theta_{1,g,\mu}(x) \right]^2 dx &\leq C \left( \tau \int_{\mathbb{R}^n} |\theta_1|^2 e^{2\tau \psi_{\epsilon}(x,0)} dx ight. \\
&\quad + \delta^2 \varepsilon^{-1} \sum_{\pm} \int_{\mathbb{R}^n} |\nabla_{x,y}w_{\pm}(x,0)|^2 e^{2\tau \psi_{\epsilon}(x,0)} dx \\
&\quad \left. + (\delta^2 \tau + \tau^2 \varepsilon) \sum_{\pm} \int_{\mathbb{R}^n} |w_{\pm}(x,0)|^2 e^{2\tau \psi_{\epsilon}(x,0)} dx \right).
\end{align*}

We now turn to the second term of $d_{g,\mu}^{(3)}$. We first derive from (4.2), (4.15), and (4.32) that
\begin{align*}
\sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau \psi_{\epsilon,0}}(\cdot,0) \theta_{1,\eta_{g,\mu}} \right]^2 &\leq C \left[ e^{\tau \psi_{\epsilon}(\cdot,0)} \theta_1 \right]^2_{1/2,\mathbb{R}^{n-1}} + C\mu \int_{\mathbb{R}^n} |\theta_1|^2 e^{2\tau \psi_{\epsilon}(x,0)} dx.
\end{align*}
Again by (2.3), (2.4), (4.2), (4.14), (4.18), (4.32), and (4.45) we get
\begin{align*}
\sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau \psi_{\epsilon,0}}(\cdot,0) J_{g,\mu}^{(1)} \right]^2_{1/2,\mathbb{R}^{n-1}} &\leq C \left( \mu^3 \sum_{\pm} \int_{\mathbb{R}^n} |w_{\pm}(x,0)|^2 e^{2\tau \psi_{\epsilon}(x,0)} dx + \mu^2 \sum_{\pm} \left[ e^{\tau \psi_{\epsilon}(\cdot,0)} w_{\pm}(\cdot,0) \right]^2_{1/2,\mathbb{R}^{n-1}} \right).
\end{align*}

We now go to the next term $\sum_{g \in \mathbb{Z}^{n-1}} \left[ e^{\tau \psi_{\epsilon,0}}(\cdot,0) J_{g,\mu}^{(2)} \right]^2_{1/2,\mathbb{R}^{n-1}}$. By (2.3), (2.4), (4.2),
(4.5), (4.14), (4.32), (4.33), and (4.45) we have that

\[
\sum_{g \in \mathbb{Z}^{n-1}} [e^{\tau \psi_{g,0}} J_{g,\mu}^{(2)}]_{1/2,\mathbb{R}^{n-1}}^2
\]

\[
\leq C \sum_{g \in \mathbb{Z}^{n-1}} \sum_{\mu} \left| e^{\tau \psi_{g,0}} \eta_{g,\mu} (A_\pm (\delta x_g, 0) - A_\pm (\delta \delta x, 0)) \nabla_{x,y} w_\pm \cdot v \right|_{1/2,\mathbb{R}^{n-1}}^2
\]

\[
+ \mu \sum_{g \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}^{n-1}} \left| A_\pm (\delta x_g, 0) - A_\pm (\delta \delta x, 0) \right|^2 \left| \nabla_{x,y} w_\pm \right|^2 e^{2\tau \psi_{g,0}} dx
\]

\[
\leq C \sum_{\mu} \left( \delta^2 \mu_2 - \sum_{g \in \mathbb{Z}^{n-1}} \left| e^{\tau \psi_{g,0}} \eta_{g,\mu} \nabla_{x,y} w_\pm (\cdot, 0) \right|_{1/2,\mathbb{R}^{n-1}}^2 + \mu \right)
\]

\[
+ \delta^2 \mu_1 \int_{\mathbb{R}^{n-1}} \left| \nabla_{x,y} (w_\pm e^{\tau \psi}) (x, 0) \right|^2 dx + \delta^2 \mu_2 \int_{\mathbb{R}^{n-1}} \left| w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx
\]

\[
\leq C \sum_{\mu} \left( \delta^2 \mu_2 - \sum_{g \in \mathbb{Z}^{n-1}} \left| e^{\tau \psi_{g,0}} \eta_{g,\mu} \nabla_{x,y} w_\pm (\cdot, 0) \right|_{1/2,\mathbb{R}^{n-1}}^2 + \mu \right)
\]

\[
+ \delta^2 \mu_1 \int_{\mathbb{R}^{n-1}} \left| \nabla_{x,y} w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx + \delta^2 \mu_2 \int_{\mathbb{R}^{n-1}} \left| w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx
\]

(4.49)

Now we estimate \( \sum_{g \in \mathbb{Z}^{n-1}} [e^{\tau \psi_{g,0}} J_{g,\mu}^{(3)}]_{1/2,\mathbb{R}^{n-1}}^2 \). As in the previous estimates and using (4.14), (4.18), and (4.32), we obtain that

\[
\sum_{g \in \mathbb{Z}^{n-1}} [e^{\tau \psi_{g,0}} J_{g,\mu}^{(3)}]_{1/2,\mathbb{R}^{n-1}}^2
\]

\[
\leq C \left( \sum_{\mu} \left( \delta^2 \sum_{g \in \mathbb{Z}^{n-1}} \left| e^{\tau \psi_{g,0}} w_\pm (\cdot, 0) \right|_{1/2,\mathbb{R}^{n-1}}^2 + \delta^2 \mu \int_{\mathbb{R}^{n-1}} \left| w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx \right) \),
\]

(4.50)

where \( C \) depends on \( \lambda_0, M_0 \) and \( n \). Finally, combining (4.46), (4.47), (4.48), (4.49), and (4.50) implies

\[
\sum_{g \in \mathbb{Z}^{n-1}} d_{g,\mu}^{(3)} \leq C \left( \tau \int_{\mathbb{R}^{n-1}} \left| \theta_1 \right|^2 e^{2\tau \psi_{g,0}} dx + \left| e^{\tau \psi_{g,0}} \theta_1 \right|_{1/2,\mathbb{R}^{n-1}}^2 + R_4 \right),
\]

(4.51)

where

\[
R_4 = \delta^2 \mu_2 - \sum_{\mu} \left| \nabla_{x,y} (w_\pm e^{\tau \psi}) (\cdot, 0) \right|_{1/2,\mathbb{R}^{n-1}}^2 + \mu^2 + \delta^2 \mu_2 \int_{\mathbb{R}^{n-1}} \left| \nabla_{x,y} w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx
\]

\[
+ \delta^2 \int_{\mathbb{R}^{n-1}} \left| \nabla_{x,y} w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx
\]

\[
+ \tau \int_{\mathbb{R}^{n-1}} \left| w_\pm (x, 0) \right|^2 e^{2\tau \psi_{g,0}} dx.
\]
Consequently, we have from (4.25), (4.36), (4.39), (4.43) and (4.51) that

$$
\Xi(w) \leq C \left( \sum_{\pm} \int_{\mathbb{R}_+^n} |\mathcal{L}_\delta(x, y, \theta)(w_\pm)|^2 e^{2\tau\psi_\varepsilon \pm} dx dy + [e^{\tau\psi_\varepsilon \cdot (\cdot, 0)} \theta_1]_{1/2, \mathbb{R}^{n-1}}^2 
+ [\nabla_w (e^{\tau\psi_\varepsilon \cdot (\cdot, 0)}) \theta_0(x)]^2 dx + \tau^3 \int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon (x, 0)} |\theta_0(x)|^2 dx
+ \tau \int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon (x, 0)} |\theta_1(x)|^2 dx + R_5 \right),
$$

(4.52)

where

$$
R_5 = \delta^2 \mu^{-2} \sum_{\pm} \int_{\mathbb{R}_+^n} |D^2 w_\pm|^2 e^{2\tau\psi_\varepsilon} dx dy + \mu^2 \sum_{\pm} \int_{\mathbb{R}_+^n} |D w_\pm|^2 e^{2\tau\psi_\varepsilon} dx dy
+ \mu^2 \sum_{\pm} \int_{\mathbb{R}_+^n} |w_\pm|^2 e^{2\tau\psi_\varepsilon \pm} dx dy + (\mu + \delta^2 \varepsilon^{-1}) \sum_{\pm} \int_{\mathbb{R}^{n-1}} |D w_\pm (x, 0)|^2 e^{2\tau\psi_\varepsilon (x, 0)} dx
+ \mu^2 \sum_{\pm} \int_{\mathbb{R}^{n-1}} |w_\pm (x, 0)|^2 e^{2\tau\psi_\varepsilon (x, 0)} dx + (\mu^2 + \delta^2 \mu^{-2} \tau^2) \sum_{\pm} [e^{\tau\psi_\varepsilon \cdot (\cdot, 0)} w_\pm (\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2
+ \delta^2 \mu^{-2} \sum_{\pm} [D (w_{\pm} e^{\tau\psi_\varepsilon}) (\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2.
$$

We now set $\delta = \varepsilon$ and choose a sufficiently small $\delta_0$ and a sufficiently large $\tau_0$, both depending on $\lambda_0, M_0, n$ such that if $\varepsilon \leq \delta_0$ and $\tau \geq \tau_0$, then $R_5$ on the right hand side of (4.52) can be absorbed by $\Xi(w)$ (defined in (4.23)). In other words, we have proved that

$$
\sum_{\pm} \sum_{k=0}^2 \tau^{3-2k} \int_{\mathbb{R}_+^n} |D^k w_\pm|^2 e^{2\tau\psi_\varepsilon} dx dy + \sum_{\pm} \sum_{k=0}^1 \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k w_\pm (x, 0)|^2 e^{2\tau\psi_\varepsilon (x, 0)} dx
+ \sum_{\pm} \tau^2 [e^{\tau\psi_\varepsilon w_\pm (\cdot, 0)}]_{1/2, \mathbb{R}^{n-1}}^2 + \sum_{\pm} [\theta_y (e^{\tau\psi_\varepsilon \cdot w_\pm}) (\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2 + \sum_{\pm} [\nabla_x (e^{\tau\psi_\varepsilon w_\pm}) (\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2
\leq C \left( \sum_{\pm} \int_{\mathbb{R}_+^n} |\mathcal{L}_\delta(x, y, \theta)(w_\pm)|^2 e^{2\tau\psi_\varepsilon} dx dy + [e^{\tau\psi_\varepsilon \cdot (\cdot, 0)} \theta_1]_{1/2, \mathbb{R}^{n-1}}^2 + [\nabla_w (e^{\tau\psi_\varepsilon \cdot (\cdot, 0)}) \theta_0(x)]_{1/2, \mathbb{R}^{n-1}}^2
+ \tau \int_{\mathbb{R}^{n-1}} |\theta_0|^2 e^{2\tau\psi_\varepsilon (x, 0)} dx + \tau \int_{\mathbb{R}^{n-1}} |\theta_1|^2 e^{2\tau\psi_\varepsilon (x, 0)} dx \right).
$$

(4.53)

Now, applying (4.53) to the function $w(x, y) = u(\delta x, \delta y)$, by a standard change of
variable and multiplying by $\delta^{n-4}$, we have

$$
\sum_{\pm} \sum_{k=0}^{2} \tau^{3-2k} \delta^{2k-4} \int_{\mathbb{R}^n_{\pm}} |D^k u_{\pm}|^2 e^{2\tau \phi_\delta(x,y)} dx dy \\
+ \sum_{\pm} \sum_{k=0}^{1} \tau^{3-2k} \delta^{2k-3} \int_{\mathbb{R}^{n-1}} |D^k u_{\pm}(x,0)|^2 e^{2\tau \phi_\delta(x,0)} dx \\
+ \sum_{\pm} \tau^2 \delta^{-2} [e^{\tau \phi_\delta(\cdot,0)} u_{\pm}(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm} [D(e^{\tau \phi_\delta \pm u_{\pm}})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
\leq C \left( \sum_{\pm} \int_{\mathbb{R}^n_{\pm}} |\mathcal{L}(x,y,\partial)(u_{\pm})|^2 e^{2\tau \phi_\delta(x,y)} dx dy + [e^{\tau \phi_\delta(\cdot,0)} h_1]^2_{1/2,\mathbb{R}^{n-1}} \\
+ [\nabla_x (e^{\tau \phi_\delta h_0})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \frac{\tau^3}{\delta} \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau \phi_\delta(x,0)} dx + \frac{\tau}{\delta} \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau \phi_\delta(x,0)} dx \right),
$$

where $\phi_{\delta,\pm}$ is given by (2.8). Since $\delta \leq \delta_0$, estimate (2.10) follows.

**References**

[AM] G. Alessandrini and R. Magnanini, *Elliptic equations in divergence form, geometrical critical points of solutions and Stekloff eigenfunctions*, SIAM J. Math. Anal., 25 (1994), 1259-1268.

[BJS] L. Bers, F. John and M. Schechter, *Partial Differential Equations*, Interscience, 1964.

[BN] L. Bers and L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications*, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, 111-140. Edizioni Cremonese, Roma, 1955.

[Cal] A. Calderón, *Uniqueness in the Cauchy problem for partial differential equations*, Amer. J. Math., 80 (1958), 16-36.

[Car] T. Carleman, *Sur un problème d’unicité par les systèmes d’équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat., Astr. Fys., 26 (1939), no. 17, 9.

[FLVW] E. Francini, C.-L. Lin, S. Vessella, and J.-N. Wang, *Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate*, arXiv:1509.06001.

[H] L. Hörmander, *Uniqueness theorems for second order elliptic equations*, Comm. PDE, 8 (1983), 21-63.
[H3] L. Hörmander, *The Analysis of Linear Partial Differential Operators. Vol. III*, Springer-Verlag, New York, 1985.

[LL] J. Le Rousseau and N. Lerner, *Carleman estimates for anisotropic elliptic operators with jumps at an interface*, Analysis & PDE, 6 (2013), No. 7, 1601-1648.

[Is] V. Isakov, *Inverse problems for partial differential equations, volume 12 of Applied Mathematical Sciences*, Springer, New York, second edition, 2006.

[KSU] C. Kenig C, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data* Ann. Math. (2007) 165, 56791

[LR1] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, Arch. Rational Mech. Anal., 195 (2010), 953-990.

[LR2] J. Le Rousseau and L. Robbiano, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Inventiones Math., 183 (2011), 245-336.

[M] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal., 54 (1974), 105-117.

[P] A. Pliš, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 11 (1963), 95-100.

[S] F. Schulz, *On the unique continuation property of elliptic divergence form equations in the plane*, Math. Z., 228 (1998), 201-206.

[Tr] F. Trèves, *Linear partial differential equations*, Gordon and Breach, New York, 1970.