**SU-bordism: structure results and geometric representatives**

I. Yu. Limonchenko, T. E. Panov, and G. S. Chernykh

**Abstract.** The first part of this survey gives a modernised exposition of the structure of the special unitary bordism ring, by combining the classical geometric methods of Conner–Floyd, Wall, and Stong with the Adams–Novikov spectral sequence and formal group law techniques that emerged after the fundamental 1967 paper of Novikov. In the second part toric topology is used to describe geometric representatives in $SU$-bordism classes, including toric, quasi-toric, and Calabi–Yau manifolds.

Bibliography: 56 titles.

**Keywords:** special unitary bordism, $SU$-manifolds, Chern classes, toric varieties, quasi-toric manifolds, Calabi–Yau manifolds.

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Introduction

SU-bordism is the bordism theory of smooth manifolds with a special unitary structure in the stable tangent bundle. Geometrically, an SU-structure on a manifold \( M \) is defined by a reduction of the structure group of the stable tangent bundle of \( M \) to the group \( SU(N) \). Homotopically, an SU-structure is the homotopy class of a lift of the map \( M \to BO(2N) \) classifying the stable tangent bundle to a map \( M \to BSU(N) \). A manifold \( M \) admits an SU-structure whenever it admits a stably complex structure with \( c_1(TM) = 0 \).

The theory of bordism and cobordism experienced spectacular development in the beginning of the 1960s. Most leading topologists of the time contributed to this development. The idea of bordism was first explicitly formulated by Pontryagin [43] who related the theory of framed bordism to the stable homotopy groups of spheres. In early papers such as [47] by Rokhlin, bordism was called ‘intrinsic homology’, referring to Poincaré’s original idea of homological cycles. The most basic of bordism theories, unoriented bordism, was the subject of the fundamental paper [51] of Thom, who completely calculated the unoriented bordism ring \( \Omega^O \).

The description of the oriented bordism ring \( \Omega^{SO} \) was completed by the end of the 1950s in papers of Novikov [38], [39] (the ring structure modulo torsion) and Wall [53] (products of torsion elements), with important earlier contributions made by Thom [51] (description of the ring \( \Omega^{SO} \otimes \mathbb{Q} \), Averbuch [4] (absence of odd torsion), Milnor [33] (the additive structure modulo torsion), and Rokhlin [47].

The theory culminated in the calculation of the complex (or unitary) bordism ring \( \Omega^U \) in papers of Milnor [33] and Novikov [38], [39]. The ring \( \Omega^U \) was shown to be isomorphic to a graded integral polynomial ring \( \mathbb{Z}[a_i : i \geq 1] \) on infinitely many generators, with one generator in every even degree, \( \deg a_i = 2i \). This result has since found numerous applications in algebraic topology and beyond. We review the unitary bordism theory in §1, since it is instrumental in the subsequent description of the structure of the SU-bordism ring.

The study of SU-bordism in the 1960s outlined the limits of applicability of methods of algebraic topology. The coefficient ring \( \Omega^{SU} \) is considered to be known. It is not a polynomial ring, although it becomes so after inverting 2. The main contributors here are Novikov [39] (description of the ring \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \)), Conner and Floyd [22] (products of torsion elements), Wall [54], and Stong [50] (the multiplicative structure of \( \Omega^{SU} / \text{Tors} \)). Nevertheless, as noted by Stong (see [50], p. 266), “an intrinsic description of \( \Omega^{SU} / \text{Tors} \) is extremely complicated”. The best-known description of the ring \( \Omega^{SU} / \text{Tors} \) is a subtly embedded subring in the polynomial ring \( \mathfrak{d} \), the coefficient ring of the Conner–Floyd theory of \( c_1 \)-spherical manifolds (see the details in §6).

The Adams–Novikov spectral sequence and the formal group law techniques brought into topology by the fundamental paper [40] of Novikov led to a new systematic approach to earlier geometric calculations with the SU-bordism ring. In particular, the exact sequence (0.1) of Conner and Floyd relating the graded components of the rings \( \Omega^{SU} \) and \( \mathfrak{d} \) admits an intrinsic description in terms of non-trivial
differentials in the Adams–Novikov spectral sequence for the $MSU$ spectrum (see §5). This approach was further developed in the context of bordism of manifolds with singularities in the works of Mironov [34], Botvinnik [9], and Vershinin [52]. The main purpose was to describe the coefficient ring $\Omega^{Sp}$ of the next classical bordism theory, symplectic bordism (nowadays also known as quaternionic bordism), which still remains unknown and mysterious. See [12], §3, for an account of results on $\Omega^{Sp}$ known by 1975. The Adams–Novikov spectral sequence has also become the main computational tool for the stable homotopy groups of spheres [45].

There is also the classical problem of finding geometric representatives of bordism classes in different bordism theories, in particular, for the unitary and special unitary bordism rings. The importance of this problem was emphasised in the original works such as the monograph [22] of Conner and Floyd.

Over the rationals, the bordism rings are generated by projective spaces, but the integral generators are more subtle, since they involve divisibility conditions on their characteristic numbers. One of the few general results known from the early 1960s on geometric representatives for bordism classes is that the complex bordism ring $\Omega^{U}$, which is an integral polynomial ring, can be generated by the so-called Milnor hypersurfaces $H(n_1, n_2)$. These are hyperplane sections of the Segre embeddings of products $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$ of complex projective spaces. Similar generators exist for unoriented and oriented bordism rings.

The early progress in this problem was impeded by the lack of examples of higher-dimensional (stably) complex manifolds for which the characteristic numbers can be calculated explicitly. With the appearance of toric varieties in the late 1970s and the subsequent development of toric topology in the beginning of this century [15], a number of explicitly constructed concrete examples of stably complex and $SU$-manifolds with a large torus symmetry have been produced. The characteristic numbers of these manifolds can be effectively calculated using combinatorial-geometric techniques. These developments enriched bordism and cobordism theory with new geometric methods.

In [18], Buchstaber and Ray constructed a set of generators for $\Omega^{U}$ consisting entirely of complex projective toric manifolds $B(n_1, n_2)$ which are projectivisations of sums of line bundles over bounded flag manifolds. Another toric family $\{L(n_1, n_2)\}$ with the same property is presented in §8. Wilfong [55] identified low-dimensional complex bordism classes which contain projective toric manifolds (there is a complete description in dimensions up to 6 and partial results in dimension 8). Furthermore, by a result due to Solomadin and Ustinovskiy [49], polynomial generators of the ring $\Omega^{U}$ can be chosen among projective toric manifolds (a partial result of this sort was obtained earlier in [56]). Quasi-toric manifolds enjoy more flexibility: it was shown by Buchstaber, Panov, and Ray [16] that one can get a geometric representative in every complex bordism class if toric manifolds are relaxed to quasi-toric ones; the latter still have a large torus action, but are only stably complex instead of being complex. In part II of this survey we review similar results in the context of $SU$-bordism.

A renewed interest in $SU$-manifolds has been stimulated by the study of mirror symmetry and other geometric constructions motivated by theoretical physics; the notion of a Calabi–Yau manifold plays a central role here. By a Calabi–Yau manifold one usually understands a Kähler $SU$-manifold; it has a Ricci flat metric by
a theorem of Yau. The relationship between Calabi–Yau manifolds and SU-bordism is discussed in §§11–13 of this survey.

Part I contains structure results on the SU-bordism ring $\Omega^{SU}$. We combine geometric methods of Conner–Floyd, Wall, and Stong with the Adams–Novikov spectral sequence and formal group law techniques in this description.

Section 1 is a summary of complex bordism theory. By a theorem of Milnor and Novikov,

$$\Omega^U \cong \mathbb{Z}[a_i : i \geq 1], \quad \deg a_i = 2i,$$

and two stably complex manifolds are bordant if and only if they have identical Chern characteristic numbers. Polynomial generators are detected by a special characteristic number $s_i$ (sometimes called the Milnor number). For any integer $i \geq 1$, let

$$m_i = \begin{cases} 1 & \text{if } i + 1 \neq p^k \text{ for any prime } p, \\ p & \text{if } i + 1 = p^k \text{ for some prime } p \text{ and integer } k > 0. \end{cases}$$

Then the bordism class of a stably complex manifold $M^{2i}$ may be taken to be the $2i$-dimensional generator $a_i$ if and only if $s_i[M^{2i}] = \pm m_i$.

SU-manifolds and SU-bordism are introduced in §2. By a theorem of Novikov, $\Omega^{SU} \otimes \mathbb{Z} \left[ \frac{1}{2} \right]$ is a polynomial algebra with one generator in every even degree $\geq 4$:

$$\Omega^{SU} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \cong \mathbb{Z} \left[ \frac{1}{2} \right] [y_i : i \geq 2], \quad \deg y_i = 2i.$$ 

The bordism class of an SU-manifold $M^{2i}$ may be taken to be the $2i$-dimensional generator $y_i$ if and only if $s_i[M^{2i}] = \pm m_i m_{i-1}$ up to a power of 2. The extra divisibility in dimensions $2p^k$ comes from the simple observation that the $s_i$-number of an SU-manifold $M^{2i}$ of dimension $2i = 2p^k$ is divisible by $p$ (Proposition 2.2).

The algebra of operations $A^U$ in complex cobordism and the Adams–Novikov spectral sequence are considered in §3.

The $A^U$-module structure on $U^*(MSU)$ needed for calculations with the Adams–Novikov spectral sequence is determined in §4. Two geometric operations are introduced here. The boundary homomorphism $\partial : \Omega^U_{2n} \to \Omega^U_{2n-2}$ sends a bordism class $[M^{2n}]$ to the bordism class $[N^{2n-2}]$ dual to $c_1(M) = c_1(\det \mathcal{F} M)$. The restriction of $\det \mathcal{F} M$ to $N$ is the normal bundle $\nu(N \subset M)$. The stable complex structure on $N$ is defined via the isomorphism $\mathcal{F} M|_N \cong \mathcal{F} N \oplus \nu(N \subset M)$. Then $c_1(N) = 0$, so $N$ is an SU-manifold. This implies that $\partial^2 = 0$.

Similarly, the homomorphism $\Delta : \Omega^U_{2n} \to \Omega^U_{2n-4}$ takes a bordism class $[M^{2n}]$ to the bordism class of the submanifold $L^{2n-4}$ dual to

$$-c_1^2(M) = c_1(\det \mathcal{F} M)c_1(\overline{\det \mathcal{F} M}),$$

with the restriction of $\det \mathcal{F} M \oplus \overline{\det \mathcal{F} M}$ giving the complex structure in the normal bundle.

The $A^U$-module $U^*(MSU)$ is then identified with the quotient $A^U/(A^U \Delta + A^U \partial)$ (Theorem 4.5).

The Adams–Novikov spectral sequence for the MSU spectrum is calculated in §5, and the consequences are drawn for the structure of the SU-bordism ring $\Omega^{SU}$. 
It is proved in Theorem 5.8 that the kernel of the forgetful homomorphism $\Omega^SU \to \Omega^U$ consists of torsion elements, and every torsion element in $\Omega^SU$ has order 2.

To describe the torsion part of $\Omega^SU$, Conner and Floyd [22] introduced the group

$$\mathcal{W}_{2n} = \text{Ker}(\Delta: \Omega^U_{2n} \to \Omega^U_{2n-4})$$

and identified it with the subgroup of $\Omega^U_{2n}$ consisting of the bordism classes $[M^{2n}]$ such that every Chern number of $M^{2n}$ with $c_1^2$ as a factor vanishes (see Theorem 6.3). The forgetful homomorphism decomposes as $\Omega^SU_{2n} \to \mathcal{W}_{2n} \to \Omega^U_{2n}$, and the restriction of the boundary homomorphism $\partial: \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$ is defined. (A similar approach was previously used by Wall [53] to identify the torsion of the oriented bordism ring $\Omega^SO$.)

The relationship between the groups $\Omega^*_{SU}$ and $\mathcal{W}_*$ is described by the following exact sequence of Conner and Floyd:

$$0 \to \Omega^SU_{2n-1} \xrightarrow{\theta} \Omega^SU_{2n} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\beta} \Omega^SU_{2n-2} \xrightarrow{\theta} \Omega^SU_{2n-1} \to 0,$$

(0.1)

where $\theta$ is the multiplication by the generator $\theta \in \Omega^SU_1 \cong \mathbb{Z}_2$, $\alpha$ is the forgetful homomorphism, and $\alpha \beta = -\partial: \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$. This exact sequence has the form of an exact couple, whose derived couple can be identified with the $E_2$ term of the Adams–Novikov spectral sequence for $MSU$ (see Lemma 5.9).

The homology of $(\mathcal{W}_*, \partial)$ was described by Conner and Floyd (see [22], Theorem 11.8) as a polynomial algebra over $\mathbb{Z}_2$ on the following generators:

$$H(\mathcal{W}_*, \partial) \cong \mathbb{Z}_2[\omega_2, \omega_{4k} : k \geq 2], \quad \deg \omega_2 = 4, \quad \deg \omega_{4k} = 8.$$

This leads to the following description of the free and torsion parts of $\Omega^SU$ (Theorem 5.11):

(a) Tors $\Omega^SU_n = 0$ unless $n = 8k + 1$ or $8k + 2$, in which case Tors $\Omega^SU_n$ is a $\mathbb{Z}_2$-vector space of rank equal to the number of partitions of $k$;

(b) $\Omega^SU_{2i}/\text{Tors}$ is isomorphic to $\text{Ker}(\partial: \mathcal{W} \to \mathcal{W})$ if $2i \not\equiv 4 \text{ mod } 8$ and is isomorphic to $\text{Im}(\partial: \mathcal{W} \to \mathcal{W})$ if $2i \equiv 4 \text{ mod } 8$;

(c) there exist $SU$-bordism classes $w_{4k} \in \Omega^SU_{8k}$, $k \geq 1$, such that every torsion element of $\Omega^SU$ is uniquely expressible in the form $P \cdot \theta$ or $P \cdot \theta^2$, where $P$ is a polynomial in the variables $w_{4k}$ with coefficients 0 or 1, and an element $w_{4k} \in \Omega^SU_{8k}$ is determined by the condition that it represents a polynomial generator $\omega_{4k}$ in $H_{8k}(\mathcal{W}_*, \partial)$ for $k \geq 2$, and $w_4 \in \Omega^SU_8$ represents $\omega_2^2$.

The direct sum $\mathcal{W} = \bigoplus_{i \geq 0} \mathcal{W}_{2i}$ is not a subring of $\Omega^U$: one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \not\in \mathcal{W}_4$. However, $\mathcal{W}$ becomes a commutative ring with unity with respect to the twisted product

$$a \ast b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b,$$

where $\cdot$ denotes the product in $\Omega^U$ and $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$. This leads to a complex-oriented multiplicative cohomology theory introduced and studied by Buchstaber in [11].

The ring structure of $\mathcal{W}$ is described in Theorem 6.10: $\mathcal{W}$ is an integral polynomial ring with one generator in each even degree except 4:

$$\mathcal{W} \cong \mathbb{Z}[x_1, x_i : i \geq 3], \quad x_1 = [\mathbb{C}P^1], \quad \deg x_i = 2i,$$
where \( s_i(x_i) = m_im_{i-1} \) for \( i \geq 3 \). The boundary operator \( \partial : \mathcal{W} \to \mathcal{W}, \partial^2 = 0 \), satisfies the equality
\[
\partial(a \ast b) = a \ast \partial b + \partial a \ast b - x_1 \ast \partial a \ast b,
\]
and the polynomial generators of \( \mathcal{W} \) can be chosen so as to satisfy the relations
\[
\partial x_1 = 2, \quad \partial x_{2i} = x_{2i-1}.
\]

The ring structure of \( \Omega^{SU} \) is described in §7. The forgetful map \( \alpha : \Omega^{SU} \to \mathcal{W} \) is a ring homomorphism. Therefore, the ring \( \Omega^{SU} / \text{Tors} \) can be described as a subring of \( \mathcal{W} \).

We have
\[
\mathcal{W} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \cong \mathbb{Z} \left[ \frac{1}{2} \right] [x_1, x_{2k-1}, 2x_{2k} - x_1x_{2k-1} : k \geq 2],
\]
where \( x_1^2 = x_1 \ast x_1 \) is a \( \partial \)-cycle, and each of the elements \( x_{2k-1} \) and \( 2x_{2k} - x_1x_{2k-1} \) with \( k \geq 2 \) is a \( \partial \)-cycle.

It follows from the description of the ring \( \mathcal{W} \) that there exist indecomposable elements \( y_i \in \Omega^{SU}_{2i}, i \geq 2 \), such that \( s_i(y_i) = m_im_{i-1} \) if \( i \) is odd, \( s_2(y_2) = -48 \), and \( s_i(y_i) = 2m_im_{i-1} \) if \( i \) is even and \( i > 2 \). These elements are mapped as follows under the forgetful homomorphism \( \alpha : \Omega^{SU} \to \mathcal{W} \):
\[
y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1x_{2k-1}, \quad k \geq 2.
\]

In particular, \( \Omega^{SU} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \cong \mathbb{Z} \left[ \frac{1}{2} \right] [y_i : i \geq 2] \) embeds in \( \mathcal{W} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \) as the polynomial subring generated by \( x_1^2, x_{2k-1} \), and \( 2x_{2k} - x_1x_{2k-1} \).

In Part II we describe geometric representatives for \( SU \)-bordism classes arising from toric topology.

In §8 we collect the necessary facts about toric varieties and quasi-toric manifolds, their cohomology rings and characteristic classes.

In §9 we provide explicitly constructed families of quasi-toric manifolds that admit an \( SU \)-structure, following Lü and Panov [31]. Quasi-toric \( SU \)-manifolds can be constructed by taking iterated complex projectivisations (which are projective toric manifolds) and then modifying the stably complex structure so that the first Chern class becomes zero. The underlying smooth manifold of the result is still toric, but the stably complex structure is not the standard one. Nevertheless, the resulting \( SU \)-structures on quasi-toric manifolds are invariant under the torus actions. The first examples of this sort were obtained by Lü and Wang in [32].

In §10 we describe quasi-toric generators for the \( SU \)-bordism ring. According to a result of [31] (which we include as Theorem 10.8), there exist quasi-toric \( SU \)-manifolds \( M^{2i} \) of dimension \( 2i \geq 10 \) with \( s_i(M^{2i}) = m_im_{i-1} \) if \( i \) is odd and \( s_i(M^{2i}) = 2m_im_{i-1} \) if \( i \) is even. These quasi-toric manifolds represent indecomposable elements \( y_i \in \Omega^{SU} \) which are polynomial generators of \( \Omega^{SU} \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \). In low dimensions \( 2i < 10 \), it is known that quasi-toric \( SU \)-manifolds \( M^{2i} \) are null-bordant. It is therefore interesting to ask which \( SU \)-bordism classes of dimension \( > 8 \) can be represented by quasi-toric manifolds.
As we have seen from the description of the ring $\Omega^{SU}$ above, characteristic numbers of $SU$-manifolds satisfy intricate divisibility conditions. Ochanine’s theorem [42] asserting that the signature of an $(8k+4)$-dimensional $SU$-manifold is divisible by 16 is one of the most famous examples. We therefore find it quite miraculous that polynomial generators for the $SU$-bordism ring $\Omega^{SU} \otimes \mathbb{Z}[^1_2]$ occur within the most basic families of examples that one can produce using toric methods: 2-stage complex projectivisations, and 3-stage projectivisations with the first stage being just $\mathbb{C}P^1$. The proof of Theorem 10.8 involves calculating the characteristic numbers and checking divisibility conditions. Some interesting results on binomial coefficients modulo a prime are obtained as a byproduct.

In §11 we review Batyrev’s construction [6] of Calabi–Yau manifolds arising from toric geometry. In its most basic form, this construction gives an algebraic hypersurface representing the $SU$-bordism class $\partial[V]$ for a smooth toric Fano variety $V$. A more general construction produces (smooth) Calabi–Yau manifolds from hypersurfaces in toric Fano varieties with Gorenstein singularities, using a special desingularisation. Gorenstein toric Fano varieties correspond to so-called reflexive polytopes, and there are finitely many of them in each dimension. Four-dimensional reflexive polytopes and Calabi–Yau threefolds arising from them are completely classified [28], [1]; there are also classification results for five-dimensional reflexive polytopes and Calabi–Yau fourfolds.

The $SU$-bordism classes of Calabi–Yau hypersurfaces in smooth toric Fano varieties generate the $SU$-bordism ring $\Omega^{SU} \otimes \mathbb{Z}[^1_2]$. More precisely, the indecomposable elements $y_i \in \Omega^{SU}$ defined above can be represented by integral linear combinations of the bordism classes of Calabi–Yau hypersurfaces. This result, proved in [30], is reviewed in §12 (unlike the situation with quasi-toric manifolds, there is no restriction on the dimension of $y_i$ here).

It is interesting to ask which bordism classes in $\Omega^{SU}$ can be represented by Calabi–Yau manifolds. This question is an $SU$-analogue of the following well-known open problem of Hirzebruch: which bordism classes in $\Omega^U$ contain connected (irreducible) non-singular algebraic varieties? If one drops the connectedness assumption, then any $U$-bordism class of positive dimension can be represented by an algebraic variety in view of a theorem of Milnor (see [50], p. 130). Since a product and a positive integral linear combination of algebraic classes are also algebraic classes (possibly disconnected), one only needs to find in each dimension $i$ algebraic varieties $M$ and $N$ with $s_i(M) = m_i$ and $s_i(N) = -m_i$. For $SU$-bordism, the situation is different: if a class $a \in \Omega^{SU}$ can be represented by a Calabi–Yau manifold, then $-a$ does not necessarily have this property.

This issue already occurs in complex dimension 2: the class $y_2 \in \Omega^U_4$ can be represented by a Calabi–Yau surface (a $K3$-surface), while $-y_2$ cannot be represented by a smooth complex surface. The situation is different in dimension 3, where both generators $y_3$ and $-y_3$ can be represented by Calabi–Yau threefolds. The same holds in complex dimension 4, as shown by Theorem 13.5.

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Part I. Structure results

1. Complex bordism

We briefly summarise the basic definitions and constructions of complex bordism (also known as unitary bordism or $U$-bordism). More details can be found in [22], [50], [13], and [15].

Let $\eta_n$ denote the universal (tautological) complex $n$-plane bundle over the infinite-dimensional Grassmannian $BU(n)$. Let $\zeta$ be a real $2n$-plane bundle over a cellular space (a $CW$-complex) $X$. A complex structure on $\zeta$ can be defined in one of the following equivalent ways:

1) an equivalence class of real vector bundle isomorphisms $\zeta \to \xi$, where $\xi$ is a complex $n$-plane bundle over $X$, and two such isomorphisms are equivalent if they differ by composing with an isomorphism of complex vector bundles;
2) a homotopy class of real $2n$-plane bundle maps $\zeta \to \eta_n$ which are isomorphisms on each fibre;
3) a homotopy class of a lift of the map $X \to BO(2n)$ classifying the bundle $\zeta$ to a map $X \to BU(n)$.

All manifolds are smooth, compact, and without boundary (unless otherwise specified). A stably complex structure (that is, a unitary structure or $U$-structure) on a manifold $M$ (possibly with boundary) is an equivalence class of complex structures on the stable tangent bundle of $M$, that is, an equivalence class of real bundle isomorphisms

$$c_\mathcal{T}: \mathcal{T} M \oplus \mathbb{R}^k \xrightarrow{\cong} \xi,$$

where $\xi$ is a complex vector bundle and $\mathbb{R}^k$ denotes the trivial real $k$-plane bundle over $M$. Two such complex structures are said to be equivalent if they differ by adding trivial complex summands and composing with isomorphisms of complex vector bundles. An isomorphism (1.1) defines a lift of the map $M \to BO(2l)$ classifying the bundle $\mathcal{T} M \oplus \mathbb{R}^k$ to a map $M \to BU(l)$; here $2l = \dim \mathbb{R} \xi = \dim M + k$. Composing $c_\mathcal{T}$ with an isomorphism of complex bundles results in a homotopy of the lift, and adding a trivial complex summand $\mathbb{C}^m$ to (1.1) results in composing the lift with the canonical map $BU(l) \to BU(l+m)$. Therefore, stably complex structures on $M$ correspond naturally and bijectively to the homotopy classes of lifts of the classifying map $M \to BO$ to a map $M \to BU$.

Remark. Instead of defining a stably complex structure as an equivalence class of isomorphisms (1.1), one can define it by fixing a single isomorphism for sufficiently large $k$. The reason is that adding trivial complex summands induces a canonical one-to-one correspondence between complex structures on the bundles $\mathcal{T} M \oplus \mathbb{R}^k$ with different $k$ if $k \geq 2$ (see [22], Theorem 2.3).

A stably complex manifold (a unitary manifold or $U$-manifold) is a pair $(M, c_\mathcal{T})$ consisting of a manifold and a stably complex structure on it.

Complex (co)bordism is a generalised (co)homology theory which arises from $U$-manifolds. It can be defined either geometrically or homotopically.

In the geometric approach, the bordism group $U_n(X)$ is defined as the set of bordism classes of continuous maps $M \to X$, where $M$ is an $n$-dimensional $U$-manifold. The details of the geometric approach are described in [22], §1 (see also [15], Appendix D). We briefly recall the key points here.
Construction 1.1 (geometric $U$-bordism). A stably complex manifold $M$ {bords (or is null-bordant)} if there is a stably complex manifold $W$ with boundary such that $\partial W = M$ and the stably complex structure induced on the boundary of $W$ coincides with that of $M$. The induced stably complex structure on $\partial W$ is defined via the isomorphism $\mathcal{F}W|_{\partial W} \cong \mathcal{F}M \oplus \mathbb{R}$. This isomorphism depends on whether we choose an inward or outward pointing normal vector to $M$ in $W$ as a basis for $\mathbb{R}$, and whether we place this normal vector at the beginning or at the end of the tangent frame of $M$. Our choice is to use the outward pointing normal and place it at the end. Then using the stably complex structure on $W$, we obtain a stably complex structure on $M = \partial W$ by means of the isomorphism

$$\mathcal{F}M \oplus \mathbb{R}^{k+1} \cong \mathcal{F}W|_{\partial W} \oplus \mathbb{R}^k \cong \xi.$$  

If we chose the inward pointing normal instead, then the resulting stably complex structure on $M = \partial W$ would be different. Namely, if $c_\mathcal{F}: \mathcal{F}M \oplus \mathbb{R}^{k+1} \to \xi$ is the stably complex structure on $M$ described above, then it can be seen that the stably complex structure resulting from the inward pointing normal is equivalent to the following:

$$\mathcal{F}M \oplus \mathbb{R}^{k+1} \oplus \mathbb{C} \xrightarrow{\ c_\mathcal{F} \oplus \tau \ } \xi \oplus \mathbb{C},$$  

(1.2)

where $\tau: \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

Given a stably complex manifold $(M, c_\mathcal{F})$, we refer to the stably complex structure defined by (1.2) as {opposite} to $c_\mathcal{F}$ and denote it by $-c_\mathcal{F}$. When $c_\mathcal{F}$ is clear from the context, then we write $M$ instead of $(M, c_\mathcal{F})$ and $-M$ instead of $(M, -c_\mathcal{F})$.

For a fixed topological pair $(X, A)$ and a non-negative integer $n$, we consider pairs $(M, f)$, where $M$ is a compact $n$-dimensional $U$-manifold with boundary and $f: (M, \partial M) \to (X, A)$ is a continuous map. Such a pair $(M, f)$ {bords} (or is null-bordant) if there exist a compact $(n+1)$-dimensional $U$-manifold $W$ with boundary and a continuous map $F: W \to X$ such that:

(a) $M$ is a regularly embedded submanifold of $\partial W$, and the $U$-structure on $M$ is obtained by restricting the $U$-structure to $\partial W$;

(b) $F|_M = f$ and $F(\partial W \setminus M) \subset A$.

The pairs $(M_1, f_1)$ and $(M_2, f_2)$ are {bordant} if the disjoint union $(M_1, f_1) \sqcup (-M_2, f_2)$ bordes. Bordism is an equivalence relation: reflexivity follows by considering the stably complex structure on $M \times I$ such that $\partial (M \times I) = M \sqcup (-M)$, and transitivity uses the angle straightening procedure. The resulting equivalence class is referred to as the {bordism class} of $(M, f)$, denoted by $[M, f]$.

The bordism classes $[M, f]$ form an Abelian group with respect to disjoint union, and we denote it by $U^a_*(X, A)$ for the moment, and refer to it as the (geometric) {unitary bordism group} of $(X, A)$. Geometric $U$-bordism is a generalised homology theory, satisfying the Eilenberg–Steenrod axioms except for the dimension axiom.

The homotopic approach to defining complex (co)bordism is based on the notion of $MU$-spectrum, which we also briefly recall.

Construction 1.2 (homotopic $U$-bordism). The Thom space of the universal complex $n$-plane bundle $\eta_n$ over $BU(n)$ is denoted by $MU(n)$. The Thom spectrum $MU = \{ Y_i, \Sigma Y_i \to Y_{i+1} : i \geq 0 \}$ has $Y_{2k} = MU(k)$ and $Y_{2k+1} = \Sigma Y_{2k}$, the map $\Sigma Y_{2k} \to Y_{2k+1}$ is the identity, and $\Sigma Y_{2k+1} \to Y_{2k+2}$ is defined as the map
\[ \Sigma^2 MU(k) = S^2 \wedge MU(k) \to MU(k+1) \] of Thom spaces that corresponds to the bundle map \( \eta_k \oplus \mathbb{C} \to \eta_{k+1} \) classifying \( \eta_k \oplus \mathbb{C} \). The \( MU \)-spectrum defines a generalised (co)homology theory, known as (homotopic) unitary (co)bordism, with bordism and cobordism groups of a cellular pair \((X, A)\) given by

\[
U_n(X, A) = \lim_{k \to \infty} \pi_{2k+n}((X/A) \wedge MU(k)), \\
U^n(X, A) = \lim_{k \to \infty} [\Sigma^{2k-n}(X/A), MU(k)].
\]

The bordism groups of a single space \( X \) are defined as \( U_n(X) := U_n(X, \emptyset) \). We shall use the notation \( X_+ \) for \( X/\emptyset \), which is \( X \) with a disjoint basepoint added.

When \((X, A)\) is a finite cellular pair, the bordism group \( U_n(X, A) \) is isomorphic to \( \pi_{2k+n}((X/A) \wedge MU(k)) \) for sufficiently large \( k \), and similarly for \( U^n(X, A) \).

By definition, the homotopic bordism and cobordism groups of a point satisfy the equality

\[ U_n(pt) = U^{-n}(pt) = \pi_{2k+n}(MU(k)) \]

for sufficiently large \( k \), and \( U_n(pt) = 0 \) for \( n < 0 \).

The equivalence of the geometric and homotopic approaches to complex bordism is established by the following result of Conner and Floyd.

**Theorem 1.3** ([22], (3.1)). The generalised homology theory \( U'_*(\cdot) \) is isomorphic, over the category of cellular pairs and continuous maps, to the generalised homology theory \( U_*(\cdot) \).

**Sketch of proof.** The proof follows the original ideas of Thom [51] in the oriented case (see also [21], Chap. 1). We define a functor \( \varphi : U'_n(X, A) \to U_n(X, A) \) between the homology theories and prove that it induces an isomorphism on the homology of a point.

For a cellular pair \((X, A)\), there is an isomorphism \( U'_n(X, A) \cong U'_n(X/A, pt) \), so we can restrict attention to the case \( A = \emptyset \) and define only the maps \( \varphi : U'_n(X) \to U_n(X) \).

Take a geometric bordism class \([M, f] \in U'_n(X)\) represented by a continuous map \( f : M \to X \) from a \( U \)-manifold \( M \). We embed \( M \) into some Euclidean space \( \mathbb{R}^{n+2k} \) and denote the normal bundle of this embedding by \( \nu \). The real bundle isomorphism \( \mathcal{TM} \oplus \nu \cong \mathbb{R}^{n+2k} \) allows us to convert stably complex structures on \( M \) to complex structures on the normal bundle \( \nu \). (This can be understood in the most naive sense by working with tangent and normal frames, but one needs to check that this conversion procedure is compatible with the appropriate stabilisations; see also [22], (2.3).)

The Pontryagin–Thom map

\[ S^{2k+n} \to \text{Th}(\nu) \]

identifies a closed tubular neighbourhood of \( M \) in \( \mathbb{R}^{2k+n} \subset S^{2k+n} \) with the total space \( D(\nu) \) of the disk bundle of \( \nu \), and collapses the closure of the complement of the tubular neighbourhood to the basepoint of the Thom space \( \text{Th}(\nu) = D(\nu)/S(\nu) \).
Now we define a map \( D(\nu) \to X \times D(\eta_k) \) in which the first component is the composite \( D(\nu) \to M \xrightarrow{f} X \) and the second component is the disk bundle map corresponding to the classifying map \( \nu \to \eta_k \) for the above-defined complex structure on \( \nu \). Doing the same for the sphere bundles, we obtain a map of pairs

\[
(D(\nu), S(\nu)) \to (X \times D(\eta_k), X \times S(\eta_k))
\]

and therefore a map of Thom spaces

\[
\text{Th}(\nu) \to (X/\varnothing) \land MU(k).
\]

Composing with the Pontryagin–Thom map, we obtain a map \( S^{2k+n} \to (X/\varnothing) \land MU(k) \) representing a homotopic bordism class in the group \( U_n(X) \) (see (1.3)). One needs only to check that the maps resulting from bordant pairs \((M, f)\) are homotopic, therefore defining a functor \( \varphi : U'_s(\cdot) \to U_s(\cdot) \).

To show that \( \varphi : U'_s(pt) \to U_s(pt) \) is an isomorphism, we construct an inverse map \( U_s(pt) \to U'_s(pt) \) as follows. Take a homotopy class of maps \( g : S^{2k+n} \to MU(k) \) representing an element in the homotopic bordism group \( U_n(pt) \). By changing \( g \) within its homotopy class if necessary we may assume that \( g \) is smooth and transverse along the zero section \( BU(k) \subset MU(k) \). Then \( M := g^{-1}(BU(k)) \) is an \( n \)-dimensional submanifold in \( S^{2k+n} \). Furthermore, there is a complex bundle map from the normal bundle \( \nu \) of \( M \) in \( S^{2k+n} \) to the normal bundle of \( BU(k) \) in \( MU(k) \), which is \( \eta_k \). We therefore obtain a complex structure on \( \nu \), which can be converted into a stably complex structure on \( M \). The result is a geometric bordism class in \( U'_s(pt) \), giving an inverse map to \( \varphi \). \( \square \)

Hereafter we denote both the geometric and the homotopic unitary bordism groups by \( U_s(\cdot) \).

**Construction 1.4** (products). For the product bundle \( \eta_m \times \eta_n \), there exist the corresponding classifying map \( BU(m) \times BU(n) \to BU(m+n) \) (unique up to a homotopy) and the bundle map \( \eta_m \times \eta_n \to \eta_{m+n} \). The latter induces a map of Thom spaces

\[
MU(m) \land MU(n) \to MU(n+m),
\]

which is associative and commutative up to homotopy. These maps are used to define product operations in complex (co)homology theory. Namely, there exist a canonical pairing (the *Kronecker product*)

\[
\langle \cdot, \cdot \rangle : U^m(X) \otimes U^n(X) \to \Omega^U_{n-m},
\]

the \( \sim \)-product

\[
\sim : U^m(X) \otimes U^n(X) \to U_{n-m}(X)
\]

and the \( \sim \)-product (or simply product)

\[
\sim : U^m(X) \otimes U^n(X) \to U^{m+n}(X),
\]

defined as follows. Assume given a cobordism class \( x \in U^m(X) \) represented by a map \( \Sigma^{2l-m}X_+ \to MU(l) \) and a bordism class \( \alpha \in U_n(X) \) represented by a map \( S^{2k+n} \to X_+ \land MU(k) \). Then \( \langle x, \alpha \rangle \in \Omega^U_{n-m} \) is represented by the composite

\[
S^{2k+2l-n-m} \xrightarrow{\Sigma^{2l-m}\alpha} \Sigma^{2l-m}X_+ \land MU(k) \xrightarrow{x \land \text{id}} MU(l) \land MU(k) \to MU(l+k).
\]
If $\Delta: X_+ \to (X \times X)_+ = X_+ \wedge X_+$ is the diagonal map, then $x \sim \alpha \in U_{n-m}(X)$ is represented by the composite map

$$S^{2k+2l+n-m} \xrightarrow{\Sigma^{2l-m} \alpha} \Sigma^{2l-m} X_+ \wedge MU(k) \xrightarrow{\Sigma^{2l-m} \Delta \wedge \text{id}} X_+ \wedge \Sigma^{2l-m} X_+ \wedge MU(k) \xrightarrow{\text{id} \wedge x \wedge \text{id}} X_+ \wedge MU(l) \wedge MU(k) \to X_+ \wedge MU(l+k).$$

The $\sim$-product is defined similarly; it transforms $U^*(X) = \bigoplus_{n \in \mathbb{Z}} U^n(X)$ into a graded commutative ring, called the complex cobordism ring of $X$. The direct sum

$$\Omega_U := U^*(\text{pt}) = \bigoplus_n U^n(\text{pt})$$

is often called simply the complex cobordism ring. It is graded by non-positive integers. We also use the notation $\Omega^U$ for the non-negatively graded ring $U_*(\text{pt}) = \bigoplus_n U_n(\text{pt})$, the complex bordism ring, where $U_n(\text{pt}) = U^{-n}(\text{pt})$. Each ring $U^*(X)$ is a module over $\Omega_U$.

A stably complex $n$-manifold $M$ has the fundamental bordism class $[M] \in U_n(M)$, which is defined geometrically as the bordism class of the identity map $M \to M$. In this case there are the Poincaré–Atiyah duality isomorphisms [3] (see also [15], Construction D.3.4):

$$D_U: U^k(M) \overset{\cong}{\to} U_{n-k}(M), \quad x \mapsto x \sim [M].$$

We have

$$H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n], \quad \text{deg } c_i = 2i,$$

where the $c_i$ are the universal Chern characteristic classes. Given a partition $\omega = (i_1, \ldots, i_k)$ of a number $n = |\omega| = i_1 + \cdots + i_k$ by positive integers, we define the monomial $c_\omega = c_{i_1} \cdots c_{i_k}$ of degree $2|\omega|$ and the corresponding characteristic class $c_\omega(\xi)$ of a complex $n$-plane bundle $\xi$. The corresponding tangential Chern characteristic number of a stably complex manifold $M$ is defined by

$$c_\omega[M] := \langle c_\omega(\mathcal{F}M), [M] \rangle.$$

Here $[M]$ is the fundamental homology class of $M$, and $\mathcal{F}M$ is regarded as a complex bundle via the isomorphism (1.1). We often write $c_\omega(M)$ instead of $c_\omega(\mathcal{F}M)$ for a stably complex manifold $M$. The number $c_\omega[M]$ is assumed to be zero when $2|\omega| \neq \dim M$.

One important characteristic class is $s_n$. It is defined as the polynomial in $c_1, \ldots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $i_1(x_1, \ldots, x_n)$ and then replacing each $i_1$ by $c_i$. We define the corresponding characteristic number by

$$s_n[M] := \langle s_n(\mathcal{F}M), [M] \rangle.$$

It is known as the $s$-number or the Milnor number of $M$.

For any integer $i \geq 1$ let

$$m_i = \begin{cases} 1 & \text{if } i + 1 \neq p^k \text{ for any prime } p, \\ p & \text{if } i + 1 = p^k \text{ for some prime } p \text{ and some integer } k > 0. \end{cases} \quad (1.4)$$
The structure of the $U$-bordism ring $\Omega^U$ is described by the following fundamental result of Milnor and Novikov.

**Theorem 1.5** (Milnor, Novikov). (a) The complex bordism ring $\Omega^U$ is a polynomial ring over $\mathbb{Z}$ with one generator in every positive even dimension:

$$\Omega^U \cong \mathbb{Z}[a_i : i \geq 1], \quad \text{deg } a_i = 2i.$$

(b) The bordism class of a stably complex manifold $M^{2i}$ may be taken to be the $2i$-dimensional polynomial generator $a_i$ if and only if

$$s_i[M^{2i}] = \pm m_i.$$

(c) Two stably complex manifolds are bordant if and only if they have identical sets of Chern characteristic numbers.

Part (c) of Theorem 1.5 can be restated by saying that the universal characteristic numbers homomorphism $e: \Omega^U_{2n} \to H_{2n}(BU)$ is a monomorphism in each dimension. The latter homomorphism (for the normal characteristic numbers) can be identified with the composite

$$\Omega^U_{2n} = \pi_{2n+2N}(MU(N)) \to H_{2n+2N}(MU(N)) \to H_{2n}(BU(N))$$

of the Hurewicz homomorphism and the Thom isomorphism. By Serre’s Theorem, the Hurewicz homomorphism in this case is an isomorphism modulo the class of finite groups. The injectivity of $e: \Omega^U_{2n} \to H_{2n}(BU)$ then follows from the absence of torsion in $\Omega^U$.

The ring isomorphism $\Omega^U \cong \mathbb{Z}[a_i : i \geq 1], \text{deg } a_i = 2i$, was proved in 1960 by Novikov [38] using the Adams spectral sequence and the structure theory of Hopf algebras. A more detailed account of his argument was given in [39]. Milnor’s paper [33] contained only the proof of the additive isomorphism (including the absence of torsion in $\Omega^U$ and the ranks calculation); the ring structure of $\Omega^U$ was intended to be included in the second part of [33], which was not published. Another geometric proof for the ring isomorphism was given by Stong in 1965 and included in his monograph [50]. All these results preceded the introduction of formal group law techniques in cobordism by Novikov [40]. Quillen [44] used formal group laws and tom Dieck’s power operations to prove that the classifying map from Lazard’s universal formal group law to the formal group law in complex cobordism induces the ring isomorphism $\mathbb{Z}[a_i : i \geq 1] \cong \Omega^U$.

**Construction 1.6** (formal group law of geometric cobordisms). Let $X$ be a cellular space. Since $\mathbb{C}P^\infty \cong MU(1)$, the cohomology group $H^2(X) = [X, \mathbb{C}P^\infty]$ is a subset (not a subgroup!) of the cobordism group $U^2(X)$. That is, every element $x \in H^2(X)$ determines a cobordism class $u_x \in U^2(X)$. The elements of $U^2(X)$ obtained in this way are called geometric cobordisms of $X$.

When $X = X^k$ is a manifold, a class $x \in H^2(X)$ is Poincaré dual to a submanifold $M \subset X$ of codimension 2 with a fixed complex structure on the normal bundle. Furthermore, if $X$ is a stably complex manifold representing a bordism class $[X] \in \Omega^U_k$, then we have

$$[M] = \varepsilon D_U(u_x) \in \Omega^U_{k-2},$$
where $D_U : U^2(X) \to U_{k-2}(X)$ is the Poincaré–Atiyah duality map and where $\varepsilon : U_{k-2}(X) \to \Omega^2_{k-2}$ is the augmentation. By definition, $\varepsilon D_U$ is the Kronecker product with $[X]$.

Given two geometric cobordisms $u, v \in U^2(X)$ corresponding to elements $x, y \in H^2(X)$, respectively, we denote by $u +_U v$ the geometric cobordism corresponding to the cohomology class $x + y$. Then the following relation holds in $U^2(X)$:

$$u +_U v = F_U(u, v) = u + v + \sum_{k \geq 1, \ell \geq 1} \alpha_{kl} u^k v^\ell,$$

(1.5)

where the coefficients $\alpha_{kl} \in \Omega^{2(k+l-1)}$ do not depend on $u$, $v$, or $X$. The series $F_U(u, v)$ given by (1.5) is a (commutative one-dimensional) formal group law over the complex cobordism ring $\Omega_U$. It was introduced by Novikov in [40], §5, Appendix 1, and called the formal group law of geometric cobordisms. More details of this construction can be found in [13] and [15], Appendix E.

We have

$$U^*(BU) = \Omega_U[[c^U_1, c^U_2, \ldots, c^U_i, \ldots]],$$

where $c^U_i$ is the $i$th universal Conner–Floyd characteristic class, and the identity above is understood as an isomorphism between the graded components. For a complex $i$-dimensional vector bundle $\xi$ over a cellular space $X$, the Conner–Floyd characteristic class $c^U_i(\xi) \in U^{2i}(X)$ is defined to be the pullback $f^*(c^U_i)$ along the map $f : X \to BU$ classifying $\xi$.

Let $\eta$ be the tautological line bundle over $\mathbb{C}P^\infty$ and let $\bar{\eta}$ be its conjugate (the line bundle of a hyperplane). The class $u = c^U_1(\bar{\eta}) \in U^2(\mathbb{C}P^\infty)$ is the cobordism class corresponding to the inclusion $\mathbb{C}P^\infty = BU(1) \to MU(1)$, which is a homotopy equivalence. In other words, $c^U_1(\bar{\eta})$ is the geometric cobordism corresponding to the first Chern class $c_1(\bar{\eta}) \in H^2(\mathbb{C}P^\infty)$. Then $c^U_1(\eta) \in U^2(\mathbb{C}P^\infty)$ is the power series inverse to $u = c^U_1(\bar{\eta})$ in the formal group law $F_U$; we denote this series by $\bar{u}$.

Similarly, for a complex line bundle $\xi$ over a cellular space $X$ the first Conner–Floyd class $c^U_1(\xi) \in U^2(X)$ coincides with the geometric cobordism corresponding to $c_1(\xi) \in H^2(X)$. The formal group law of geometric cobordisms gives the expression of the first Conner–Floyd class of the tensor product $\xi \otimes \zeta$ of line bundles over $X$ in terms of the classes $u = c^U_1(\xi)$ and $v = c^U_1(\zeta)$:

$$c^U_1(\xi \otimes \zeta) = F_U(u, v).$$

If $\xi$ is a complex vector bundle of arbitrary dimension over $X$, then the geometric cobordism corresponding to $c_1(\xi) \in H^2(X)$ is $c^U_1(\det \xi) \in U^2(X)$ (it is defined by the map $X \to \mathbb{C}P^\infty$ classifying the determinant line bundle $\det \xi$). In general, $c^U_1(\det \xi) \neq c^U_1(\xi)$. Consider the determinant homomorphism $\det : U \to U(1)$ and the corresponding map $\det : BU \to BU(1) = \mathbb{C}P^\infty$. We define the universal characteristic class $d^U = \det^* u \in U^2(BU)$. Then we have $d^U(\xi) = c^U_1(\det \xi)$.

**2. SU-manifolds and the SU-spectrum**

A special unitary structure (an SU-structure) on a manifold $M$ is a stably complex structure $c_{\mathcal{F}}$ (see (1.1)), with a choice of an SU-structure on the complex
vector bundle $\xi$. Equivalently, an $SU$-structure is the homotopy class of a lift of the map $M \to BU$ classifying $\xi$ to a map $M \to BSU$. A stably complex manifold $(M, c, \varphi)$ admits an $SU$-structure if and only if the first (integral) Chern class of $\xi$ vanishes: $c_1(\xi) = 0$. Furthermore, such an $SU$-structure is unique if $H^1(M; \mathbb{Z}) = 0$ (the latter follows by considering the homotopy fibration sequence corresponding to the fibration $BSU \to BU$ with fibre $S^1$). An $SU$-manifold is a stably complex manifold with a fixed $SU$-structure. By some abuse of notation, we often refer to a stably complex manifold $M$ with $c_1(M) = 0$ as an $SU$-manifold, meaning that such a manifold admits an $SU$-structure.

There is a generalised homology theory resulting from $SU$-structures, known as $SU$-bordism. As in the case of $U$-bordism, it can be defined either geometrically or homotopically.

In the geometric approach, the bordism group $SU_n(X)$ is defined to be the set of bordism classes of continuous maps $M \to X$, where $M$ is an $n$-dimensional $SU$-manifold. The homotopic approach is based on the notion of the $MSU$-spectrum. Let $\tilde{\eta}_n$ denote the universal (tautological) complex $n$-plane bundle over $BSU(n)$. The Thom space of $\eta_n$ is denoted by $MSU(n)$. The Thom spectrum $MSU = \{Z_i, \Sigma Z_i \to Z_{i+1} : i \geq 0\}$ has $Z_{2k} = MSU(k)$ and $Z_{2k+1} = \Sigma Z_{2k}$. The $SU$-bordism and cobordism groups of a cellular pair $(X, A)$ are given by

$$SU_n(X, A) = \lim_{k \to \infty} \pi_{2k+n}((X/A) \wedge MSU(k))$$

and

$$SU^n(X, A) = \lim_{k \to \infty} [\Sigma^{2k-n}(X/A), MSU(k)].$$

These define a multiplicative generalised (co)homology theory, as in the case of $U$-bordism.

The $SU$-bordism ring is defined by $\Omega^{SU} = SU_*(pt)$.

Unlike $\Omega^U$, the ring $\Omega^{SU}$ has torsion. The first torsion element appears already in dimension 1: the fact that $MSU(k)$ has no cells in dimensions $2k + 1$ through $2k + 3$ implies that $\Omega^{SU}_1 = \pi^+_1 = \mathbb{Z}_2$. The generator $\theta$ of $\Omega^{SU}_1$ is represented by a circle with a non-trivial framing inducing a non-trivial $SU$-structure.

The first structure result on the ring $\Omega^{SU}$ was a theorem of Novikov from 1962, showing that $\Omega^{SU}$ becomes a polynomial ring if we invert 2 (otherwise it is not a polynomial ring, even modulo torsion). Recall from Theorem 1.5 that a bordism class $[M^{2i}] \in \Omega^{SU}_{2i}$ is a polynomial generator of $\Omega^U$ whenever $s_i[M^{2i}] = \pm m_i$, where the numbers $m_i$ are defined in (1.4). More intricate divisibility conditions on the $s_i$-number are required to identify polynomial generators in the ring $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$.

**Theorem 2.1** (Novikov [39], Appendix 1). $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra with one generator in every even degree $\geq 4$:

$$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_i : i \geq 2], \quad \deg y_i = 2i.$$
The bordism class of an $SU$-manifold $M^{2i}$ may be taken to be the $2i$-dimensional generator $y_i$ if and only if

$$s_i[M^{2i}] = \pm m_im_{i-1} \text{ up to a power of 2.}$$

Note that up to a power of 2 we have

$$m_im_{i-1} = \begin{cases} 1 & \text{if } i \neq p^k, i \neq p^k - 1 \text{ for an odd prime } p, \\ p & \text{if } i = p^k \text{ or } i = p^k - 1 \text{ for some odd prime } p. \end{cases}$$

The extra divisibility in dimensions $2i = 2p^k$ comes from the following simple observation.

**Proposition 2.2.** If $M^{2n}$ is an $SU$-manifold of dimension $2n = 2p^k$ for a prime $p$, then

$$s_n[M^{2n}] = 0 \mod p.$$  

**Proof.** For $n = p^k$ we have

$$s_n(M^{2n}) = x_1^n + \cdots + x_n^n \equiv (x_1 + \cdots + x_n)^n = c_n^n(M^{2n}) = 0 \mod p. \quad \square$$

As in the case of unitary bordism, Theorem 2.1 implies that the $SU$-bordism class of an $SU$-manifold is determined modulo 2-primary torsion by its characteristic numbers. By a result of Anderson, Brown, and Peterson [2], $KO$-theory characteristic numbers together with the ordinary characteristic numbers completely determine the $SU$-bordism class.

### 3. Operations in complex cobordism and the Adams–Novikov spectral sequence

A (stable) operation $\theta$ of degree $n$ in complex cobordism is a family of additive maps

$$\theta: U^k(X, A) \to U^{k+n}(X, A),$$

defined for all cellular pairs $(X, A)$, which are functorial with respect to $(X, A)$ and commute with the suspension isomorphisms. The set of all operations is a ring with respect to addition and composition; furthermore, there is an algebra structure over the ring $\Omega_U$. This algebra is denoted by $A^U$; it was described in the papers of Landweber [29] and Novikov [40], §5.

**Construction 3.1** (operations and characteristic classes). There is an isomorphism of $\Omega_U$-modules

$$A^U \cong U^*(MU) = \lim_{\leftarrow} U^{*+2N}(MU(N)).$$

Given an element $a \in U^n(MU)$ of $A^U$ represented by a map of spectra $a: MU \to \Sigma^n MU$, we denote the corresponding operation by

$$a^*: U^*(X) \to U^{*+n}(X),$$
where $X$ is the cellular space. The operation $a^*$ is described as follows. For a given element $x \in U^m(X)$ represented by a map $x: X \to \Sigma^m MU$, the element $a^*x \in U^{m+n}(X)$ is represented by the composite

$$X \xrightarrow{x} \Sigma^m MU \xrightarrow{\Sigma^m a} \Sigma^{m+n} MU.$$ 

This defines a left action of $A^U$ on the cobordism groups of $X$, and turns $U^*$ into a functor to the category of graded left $A^U$-modules.

There is a similarly defined action

$$a_*: U_*(X) \to U_{*-n}(X)$$

of $A^U$ on the bordism groups. For a given element $x \in U_m(X)$ represented by a map $x: \Sigma^m S \to X \wedge MU$, the element $a_*x \in U_{m-n}(X)$ is represented by the composite

$$\Sigma^{m-n} S \xrightarrow{\Sigma^{-n} x} \Sigma^{-n}(X \wedge MU) \xrightarrow{\Sigma^{-n} (1 \wedge a)} X \wedge MU.$$ 

There are natural Thom isomorphisms

$$\varphi^N: U_{n+2N}(MU(N)) \to U_n(BU(N)) \quad \text{and} \quad \varphi_*^*: U^n(BU(N)) \to U^{n+2N}(MU(N)).$$

Since $U_n(BU)$ is the direct limit of $U_n(BU(N))$ and $U^n(BU)$ is the inverse limit of $U^n(BU(N))$, and similarly for $MU$, we also have the stable Thom isomorphisms

$$\varphi_*: U_n(MU) \to U_n(BU) \quad \text{and} \quad \varphi^*: U^n(BU) \to U^n(MU).$$

It follows that every universal characteristic class $\alpha \in U^n(BU)$ defines an operation $a = \varphi^*(\alpha) \in U^n(MU)$, and vice versa.

If $x \in U_m(X)$ is represented by a singular manifold $M^m \xrightarrow{f} X$, then $a_*x$ can be interpreted geometrically as follows. Let $\alpha = (\varphi^*)^{-1}a$ be the characteristic class corresponding to $a$. Consider $\alpha(-\mathcal{T} M) \in U^n(M^m)$, where $\mathcal{T} M$ is the tangent bundle and $-\mathcal{T} M$ is the stable normal bundle of $M$. Applying the Poincaré–Atiyah duality operator

$$D_U: U^n(M^m) \to U_{m-n}(M^m),$$

we obtain the element $D_U \alpha(-\mathcal{T} M) \in U_{m-n}(M)$ represented by $Y_{\alpha} \xrightarrow{f_{\alpha}} M$. Then $a_*x \in U_{m-n}(X)$ is represented by the composite $Y_{\alpha} \xrightarrow{f_{\alpha}} M \xrightarrow{f} X$.

There is an isomorphism of left $\Omega_U$-modules

$$A^U = U^*(MU) \cong \Omega_U \hat{\otimes} S,$$

where $\hat{\otimes}$ is the completed tensor product, and $S$ is the Landweber–Novikov algebra, generated by the operations $S_\omega = \varphi^*(s_\omega^U)$ corresponding to universal characteristic classes $s_\omega^U \in U^*(BU)$ defined by symmetrising monomials $t_1^i \cdots t_k^i$ indexed by partitions $\omega = (i_1, \ldots, i_k)$. Therefore, any element $a \in A^U$ can be written uniquely as an infinite series $a = \sum_\omega \lambda_\omega S_\omega$ where $\lambda_\omega \in \Omega_U$. The Hopf algebra structure of $S$ is described in [29] and [40], §5.

Restricting to the case $X = pt$, we obtain representations of $A^U$ on $\Omega_U = U^*(pt)$ and $\Omega^U = U_*(pt)$. Unlike the situation with ordinary (co)homology, we have the following.
Lemma 3.2 ([40], Lemmas 3.1 and 5.2). The representations of $A^{U}$ on $\Omega_{U} = U^{*}(pt)$ and $\Omega^{U} = U_{*}(pt)$ are faithful.

Remark. More generally, for given spectra $E$ and $F$ of finite type, the natural homomorphism $F^{*}(E) \rightarrow \text{Hom}^{*}(\pi_{*}(E), \pi_{*}(F))$ is injective when $\pi_{*}(F)$ and $H_{*}(E)$ do not have torsion (see [48] for details).

Along with the representation of $A^{U}$ on the bordism $U_{*}(X)$ of any $X$, there is another representation of $A^{U}$ on $U_{*}(BU)$ defined as follows.

Construction 3.3 (representation of $A^{U}$ on $U_{*}(BU)$; $a \mapsto \tilde{a}$). Let $a \in U^{n}(MU)$ be an element of $A^{U}$. We define

$$\tilde{a} := \varphi_{*}a_{*}\varphi_{*}^{-1} : U_{m}(BU) \rightarrow U_{m-n}(BU).$$

The geometrical meaning of this operation is described as follows. Let $[M, \xi] \in U_{m}(BU)$ be a bordism class, where $\xi$ is the pullback of the (stable) tautological bundle over $BU$ along a singular manifold $M \rightarrow BU$. The element $a \in U^{n}(MU)$ defines a universal characteristic class $\alpha = (\varphi^{*})^{-1}a \in U^{n}(BU)$ and hence a class $\alpha(\xi) \in U^{n}(M)$. Consider the Poincaré–Atiyah dual class $D_{U}(\alpha(\xi)) = [Y_{a}, f_{a}] \in U_{m-n}(M)$, where $Y_{a} \xrightarrow{f_{a}} M$ is a singular manifold in $M$. Then

$$\tilde{a}[M, \xi] = [Y_{a}, f_{a}^{*}(\xi + T M) - SY_{a}] \in U_{m-n}(BU).$$

Applying the augmentation $\varepsilon : U_{*}(BU) \rightarrow \Omega^{U}$, we obtain

$$\varepsilon(\tilde{a}[M, \xi]) = [Y_{a}] = \langle (\varphi^{*})^{-1}a, [M, \xi] \rangle \in U_{m-n}(pt) = \Omega^{U}_{m-n}.$$  \hspace{1cm} (3.1)

where $\langle \cdot, \cdot \rangle$ denotes the Kronecker product in (co)bordism of $BU$.

Lemma 3.4. The representation of $A^{U}$ on $U_{*}(BU)$ given by $a \mapsto \tilde{a}$ is faithful.

Proof. Setting $\xi = -T M$ in Construction 3.3, we find that

$$\tilde{a}[M, -T M] = [Y_{a}, -SY_{a}].$$

This implies that we can consider the representation $a \mapsto a_{*}$ on $U_{*}(pt)$ as a sub-representation of the representation $a \mapsto \tilde{a}$ on $U_{*}(BU)$. Since $a \mapsto a_{*}$ is faithful by Lemma 3.2, the representation $a \mapsto \tilde{a}$ is also faithful. \hfill \Box

The main properties of the cohomological Adams–Novikov spectral sequence for complex cobordism are summarised next. Details can be found in [40] (see also [35], [5], and [9]).

Theorem 3.5 (Adams–Novikov spectral sequence for complex cobordism). Let $X$ be a connected spectrum whose ordinary homology with $\mathbb{Z}$-coefficients is torsion-free and finitely generated in each dimension. Then there exists a spectral sequence

$$\{E^{p,q}_{r}, d_{r} : E^{p,q}_{r} \rightarrow E^{p+r,q+r-1}_{r}, \ r \geq 2\}$$

with the following properties:

(a) $E^{p,q}_{2} = \text{Ext}_{A_{*}^{U}(U^{*}(X), U^{*}(pt))}^{p,q}$, where $U^{*}$ is the complex cobordism theory and $A^{U} = U^{*}(MU)$ is the algebra of operations;
(b) there exists a filtration

$$\pi_n(X) = F^{0,n} \supset F^{1,n+1} \supset F^{2,n+2} \supset \cdots, \quad \bigcap_{s \geq 0} F^{s,n+s} = 0,$$

whose adjoint bigraded module coincides with the infinity term of the spectral sequence, $E^\infty_{p,q} \cong F^p,q/F^{p+1,q+1}$.

(c) the edge homomorphism

$$\pi_n(X) = F^{0,n} \to E^0_\infty \to E^0_2 = \text{Hom}^n_{A^U}(U^*(X), U^*(pt))$$

coincides with the naturally defined map.

Furthermore, if $X$ is a ring spectrum, then the spectral sequence is multiplicative.

Remark. The natural map

$$h: \pi_n(X) \to \text{Hom}^n_{A^U}(U^*(X), U^*(pt))$$

in Theorem 3.5, (c), is defined as follows. For a given element $\alpha \in \pi_n(X)$ represented by a map $f: \Sigma^n S \to X$ and an element $\beta \in U^p(X)$ represented by a map $g: X \to \Sigma^p MU$, the element $h(\alpha)(\beta) \in U^{p-n}(pt)$ is represented by the composite

$$\Sigma^n S \xrightarrow{f} X \xrightarrow{g} \Sigma^p MU.$$

4. The $A^U$-module structure of $U^*(MSU)$

In order to apply Theorem 3.5 to the special unitary bordism spectrum $MSU$ we need to describe the $A^U$-module structure on $U^*(MSU)$. The main result here (Theorem 4.5) is due to Novikov. We provide a complete proof by filling in some details missing in [40].

Consider the universal characteristic class $d^U \in U^2(BU)$ introduced at the end of §1, $d^U(\xi) = c^U_1(\det \xi)$. We also set $\bar{d}^U = c^U_1(\det \xi)$. The spectral sequence of the fibration $BU \to BU \xrightarrow{\det} BU(1)$ implies that the homomorphism $U^*(BU) \to U^*(BSU)$ is surjective and its kernel is the ideal $I(d^U)$ generated by $d^U$. Using the Thom isomorphisms

$$\varphi^*: U^*(BSU) \to U^*(MSU) \quad \text{and} \quad \varphi^*: U^*(BU) \to U^*(MU),$$

we get that the natural map $MSU \to MU$ induces an epimorphism $U^*(MU) \to U^*(MSU)$ with kernel $\varphi^*(I(d^U))$. Since $U^*(MU) \to U^*(MSU)$ is an $A^U$-module map, we get that

$$U^*(MSU) = A^U / \varphi^*(I(d^U))$$

as an $A^U$-module. (4.1)

This is the first description of the required $A^U$-module structure.

Next we define some important operations in $A^U$. Recall that every characteristic class $\alpha \in U^*(BU)$ defines an operation $\varphi^*(\alpha) \in A^U = U^*(MU)$.

Construction 4.1 (the operations $\Delta_{(k_1,k_2)}$). Given positive integers $k_1$ and $k_2$, define

$$\Delta_{(k_1,k_2)} = \varphi^*((\bar{d}^U)^{k_1}(d^U)^{k_2}) \in (A^U)^{2k_1+2k_2}.$$
The corresponding operation \( \widetilde{\Delta}_{(k_1,k_2)} : U_*(BU) \to U_{*-2k_1-2k_2}(BU) \) (see Construction 3.3) can be described geometrically as follows. Let \([M, \xi] \in U_n(BU)\), and let \(i_1 : Y_1 \hookrightarrow M\) and \(i_2 : Y_2 \hookrightarrow M\) be codimension-2 submanifolds Poincaré dual to \(-c_1(\xi)\) and \(c_1(\xi)\), respectively. We have
\[
\nu(Y_1 \subset M) = (\det \xi)|_{Y_1} \quad \text{and} \quad \nu(Y_2 \subset M) = (\det \xi)|_{Y_2}.
\]
These submanifolds are Poincaré–Atiyah dual to the classes \(c_1^U(\det \xi) = \tilde{d}^U(\xi)\) and \(c_1^U(\xi) = d^U(\xi)\), respectively. The Poincaré–Atiyah dual to \((\tilde{d}^U(\xi))^{k_1}(d^U(\xi))^{k_2} \in U^{2k_1+2k_2}(M)\) is given by the transverse intersection
\[
Y_{k_1,k_2} = \underbrace{Y_1 \cdots Y_1}_{k_1} \cdot \underbrace{Y_2 \cdots Y_2}_{k_2}
\]
with the complex structure in the normal bundle \(\nu = \nu(Y_{k_1,k_2} \subset M) = (\det \xi)^{\oplus k_1} \oplus (\det \xi)^{\oplus k_2}|_{Y_{k_1,k_2}}\). Then we have
\[
\widetilde{\Delta}_{(k_1,k_2)}[M, \xi] = [Y_{k_1,k_2}, \xi|_{Y_{k_1,k_2}} + \nu] \in U_{n-2k_1-2k_2}(BU).
\]
In the case when \(\xi = -\mathcal{F}M\) we obtain \((\Delta_{(k_1,k_2)})_*[M] = [M_{k_1,k_2}]\), where \(M_{k_1,k_2}\) is the submanifold dual to \((\det \mathcal{F}M)^{\oplus k_1} \oplus (\det \mathcal{F}M)^{\oplus k_2}\).

**Construction 4.2** (the operations \(\Psi_{(k_1,k_2)}\)). For non-negative integers \(k_1\) and \(k_2\), let \(k = k_1 + k_2\). Let \(\xi\) be a complex line bundle over \(\mathbb{C}P^n\). Consider the projectivisation bundle \(p : \mathbb{C}P(\xi \oplus \mathbb{C}^k) \to \mathbb{C}P^n\), where \(\mathbb{C}^k\) denotes the trivial bundle of rank \(k\). The tangent bundle of \(\mathbb{C}P(\xi \oplus \mathbb{C}^k)\) splits stably as
\[
\mathcal{F}\mathbb{C}P(\xi \oplus \mathbb{C}^k) \cong p^* \mathcal{F}\mathbb{C}P^n \oplus (\eta \otimes p^* (\xi \oplus \mathbb{C}^k)) = p^* \mathcal{F}\mathbb{C}P^n \oplus (\eta \otimes p^* \xi) \oplus \eta \oplus \eta \oplus k,
\]
where \(\eta\) denotes the tautological line bundle over \(\mathbb{C}P(\xi \oplus \mathbb{C}^k)\) (see [15], Theorem D.4.1). We change the stably complex structure on \(\mathbb{C}P(\xi \oplus \mathbb{C}^k)\) to a new one, determined by the isomorphism of real vector bundles
\[
\mathbb{F}\mathbb{C}P(\xi \oplus \mathbb{C}^k) \oplus \mathbb{R}^2 \cong p^* \mathbb{F}\mathbb{C}P^n \oplus (\eta \otimes p^* \xi) \oplus \eta \oplus k_1 \oplus \eta \oplus k_2,
\]
and we denote the resulting stably complex manifold by \(P^{(k_1,k_2)}(\xi)\).

We obtain a bordism class \([P^{(k_1,k_2)}(\xi), p] \in U_{2n+2k}(\mathbb{C}P^n)\). Its dual cobordism class
\[
\chi_{(k_1,k_2)}(\xi) := (DU)^{-1}[P^{(k_1,k_2)}(\xi), p] \in U^{-2k}(\mathbb{C}P^n)
\]
defines a universal cobordism characteristic class of line bundles, which we denote by \(\chi_{(k_1,k_2)} \in U^{-2k}(\mathbb{C}P^\infty)\).

Now we can extend the definition of \(\chi_{(k_1,k_2)}\) to complex vector bundles of arbitrary rank by setting \(\chi_{(k_1,k_2)}(\xi) := \chi_{(k_1,k_2)}(\det \xi)\). As a result, we obtain a universal characteristic class \(\chi_{(k_1,k_2)} \in U^{-2k}(BU)\) and the corresponding operation
\[
\Psi_{(k_1,k_2)} = p^* \chi_{(k_1,k_2)} \in U^{-2(k_1+k_2)}(MU) = (AU)^{-2(k_1+k_2)}.
\]
Geometrically, \((\Psi_{(k_1,k_2)})_*[M^{2n}]\) is the class of the manifold \([\mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}^{k_1+k_2})]\) of dimension \(2n + 2k_1 + 2k_2\) with the stably complex structure
\[
p^*(\mathcal{F}M) \oplus (\eta \otimes p^*(\det \mathcal{F}M)) \oplus \eta \oplus k_1 \oplus \eta \oplus k_2.
\]
We use the following notation for particular operations:

\[ \partial = \Delta_{(1,0)}, \quad \Delta = \Delta_{(1,1)}, \quad \chi = \Psi_{(1,0)}, \quad \text{and} \quad \Psi = \Psi_{(1,1)}. \]

Geometrically, \( \partial_* [M] \) is represented by a submanifold dual to \( c_1(\det \mathcal{F}M) = c_1(M) \), and \( \chi_* [M] \) is represented by the manifold \( \mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}) \) with the standard stably complex structure. The operations \( \partial_* \) and \( \Delta_* \) were studied in detail by Conner and Floyd in [22], where they denoted them simply by \( \partial \) and \( \Delta \).

The operations introduced above satisfy the algebraic relations described next.

**Lemma 4.3.**

\[ \partial^2 = \Delta \partial = 0, \quad \Delta \Psi = \text{id}, \quad \partial \Psi = 0, \quad \chi \partial = [\mathbb{C}P^1] \partial, \quad \text{and} \quad \partial \chi \partial = 2 \partial. \]

**Proof.** By Lemma 3.2, it suffices to check the relations on \( O^U \), the bordism of a point. Recall that \( \partial_* [M] \) is represented by a submanifold dual to \( c_1(M) \), which is an \( SU \)-manifold. Therefore, \( (\Delta_{(k_1,k_2)})_* \partial_* = 0 \). In particular, \( \partial^2_* = \Delta_* \partial_* = 0 \).

The identity \( \Delta_\Psi \Psi_* = \text{id} \) is proved in [22], Theorem 8.1. The equality \( \partial_\Psi \Psi_* = 0 \) is stated in [22], Theorem 8.2, but its proof contains an inaccuracy in the calculation of characteristic classes, so we give a correct argument below.

Take \( [M^{2n}] \in O^U_{2n} \). Then \( \Psi_* [M^{2n}] \) is represented by the manifold \( \mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}^2) \) with the stably complex structure given by the isomorphism

\[ \mathcal{F} \mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}^2) \oplus \mathbb{R}^2 \cong p^* \mathcal{F}M \oplus (\bar{\eta} \otimes p^* \det \mathcal{F}M) \oplus \bar{\eta} \oplus \eta. \]

We denote this stably complex manifold by \( P^{2n+4} \). Now \( \partial_* \Psi_* [M^{2n}] = \partial_* [P^{2n+4}] \) is represented by a submanifold \( N^{2n+2} \subset P^{2n+4} \) dual to \( c_1(P^{2n+4}) = c_1(\bar{\eta}) \). We can take as \( N^{2n+2} \) the submanifold \( \mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}) \) with the stably complex structure given by the isomorphism

\[ \mathcal{F} \mathbb{C}P(\det \mathcal{F}M \oplus \mathbb{C}) \oplus \mathbb{R}^2 \cong p^* \mathcal{F}M \oplus (\bar{\eta} \otimes p^* \det \mathcal{F}M) \oplus \eta. \]

Note that \( [N^{2n+2}] \) is precisely \( (\Psi_{(0,1)}))*[M^{2n}] \). To see that \( N^{2n+2} \) is null-bordant, we calculate its total Chern class. Let \( c_i = c_i(M) \) and \( d = c_1(\bar{\eta}) \). Then \( d^2 = p^* c_1 \cdot d \).

We have

\[ c(N^{2n+2}) = (1 + p^* c_1 + \cdots + p^* c_n)(1 + d - p^* c_1)(1 - d) \]
\[ = (1 + p^* c_1 + \cdots + p^* c_n)(1 - p^* c_1) \]
\[ = 1 + p^* (c_2 - c_1^2) + p^* (c_3 - c_1 c_2) + \cdots + p^* (c_n - c_1 c_{n-1}) \]

(this calculation was performed incorrectly in [22], pp. 36–37). Hence \( c_\omega(N^{2n+2}) = p^* c_\omega(M^{2n}) \), where \( c_\omega' = c_i - c_1 c_{i-1} \), and all the characteristic numbers \( c_\omega[N^{2n+2}] \) vanish by a dimension argument.

The equality \( \partial \Psi = \Psi_{(0,1)} = 0 \) can also be obtained geometrically, by observing that the stably complex structure on \( N^{2n+2} \) is trivial on each fibre \( \mathbb{C}P^1 = S^2 \) of the projectivisation, so it extends over the associated 3-disk bundle.

To verify the equality \( \chi_* \partial_* = [\mathbb{C}P^1] \partial_* \), observe that \( \partial_* [M^{2n}] = [ Y^{2n-2} ] \), where \( Y^{2n-2} \) is an \( SU \)-manifold, so that \( \det \mathcal{F}Y \) is trivial. Then \( \chi_* \partial_* [M^{2n}] \) is represented by \( \mathbb{C}P(\det \mathcal{F}Y \oplus \mathbb{C}) = \mathbb{C}P^1 \times Y \), which implies the required equality.
The last equality is obtained by applying \( \partial_* \) to both sides of the equality \( \chi_* \partial_* = [\mathbb{C}P^1] \partial_* \). In the notation of the previous paragraph, we need to verify that \( \partial_* (\mathbb{C}P^1 \times Y) = 2Y \), which follows by observing that \( 2Y \subset \mathbb{C}P^1 \times Y \) represents the homology class dual to \( c_1(\mathbb{C}P^1 \times Y) = c_1(\mathbb{C}P^1) \otimes 1 \). \( \square \)

**Remark.** In §5 of [40] the equality \( [\partial, \chi] = 2 \) is asserted instead of \( \partial \chi \partial = 2 \partial \). However, \( [\partial, \chi] = 2 \) cannot hold. Indeed, applying \( \partial \) from the right, we get that \( \partial \chi \partial = 2 \partial \), but applying it from the left, we get that \( -\partial \chi \partial = 2 \partial \), which implies that \( \partial = 0 \). On the other hand, \( \partial[\mathbb{C}P^1] = 2 \).

**Corollary 4.4.** If \( a \partial + b \Delta = 0 \) for some \( a, b \in A^U \), then \( b = 0 \).

**Proof.** Applying \( \Psi \) from the right, we get that \( b = 0 \). \( \square \)

We can now formulate the key result about \( U^*(MSU) \), which will be used in the calculation of the corresponding Adams–Novikov spectral sequence.

**Theorem 4.5** ([40], Theorem 6.1). (a) The left \( A^U \)-module \( U^*(MSU) \) is isomorphic to \( A^U / (A^U \Delta + A^U \partial) \). The kernel of the natural homomorphism \( A^U = U^*(MU) \to U^*(MSU) \) is identified with \( A^U \Delta + A^U \partial \).

(b) The left annihilator of \( \partial \) is equal to \( A^U \Delta + A^U \partial \).

**Proof.** The original proof in [40] is quite sketchy. Filling in the details has required a good bit of technical work. The proof consists of three parts.

I. We show that \( \tilde{\partial}(U_* (BU)) = U_* (BSU) \). In other words, a bordism class \( [X, \xi] \in U_m(BU) \) lies in the image of \( \tilde{\partial} \) if and only if it is represented by a pair \( (X, \xi) \) where \( \xi \) is an \( SU \)-bundle, that is, \( c_1(\xi) = 0 \).

To prove the inclusion \( \tilde{\partial}(U_* (BU)) \supset U_* (BSU) \), take \( [X, \xi] \in U_m(BU) \) with \( c_1(\xi) = 0 \). Consider the bordism class \( [X \times \mathbb{C}P^1, \xi \times \eta] \in U_{m+2}(BU) \), where \( \eta \) is the tautological line bundle over \( \mathbb{C}P^1 \). By the definition of \( \tilde{\partial} \) (Construction 3.3),

\[ \tilde{\partial}[X \times \mathbb{C}P^1, \xi \times \eta] = [Y, \zeta] \]

where \( Y \subset X \times \mathbb{C}P^1 \) is a codimension-2 submanifold dual to \( c_1(\xi \times \eta) = 1 \otimes c_1(\eta) \), so we can take \( Y = X \), and then

\[ \zeta = \xi \times \eta \mid_X + \mathcal{J}(X \times \mathbb{C}P^1) \mid_X - \mathcal{J}X = \xi \]

as stable bundles. Therefore, \( [X, \xi] = \tilde{\partial}[X \times \mathbb{C}P^1, \xi \times \eta] \).

To prove the opposite inclusion \( \tilde{\partial}(U_* (BU)) \subset U_* (BSU) \), take \( [Y, \zeta] = \tilde{\partial}[X, \xi] \). We need to show that \( \zeta \) is represented by an \( SU \)-bundle. But by Construction 3.3,

\[ \tilde{\partial}[X, \xi] = [Y, \xi \mid_Y + \mathcal{J}X \mid_Y - \mathcal{J}Y] \in U_{m-2}(BU), \]

where \( Y \subset X \) is a codimension-2 submanifold with the normal bundle \( \nu(Y \subset X) = \det \xi \mid_Y \). Then

\[ c_1(\zeta) = c_1(\xi \mid_Y + \mathcal{J}X \mid_Y - \mathcal{J}Y) = c_1(\xi \mid_Y) + c_1(\nu) = c_1(\det \xi \mid_Y) + c_1(\det \xi \mid_Y) = 0, \]

so \( \zeta \) is an \( SU \)-bundle.
II. We show that \( \text{Ann}_L \partial = \varphi^*(I(d^U)) \), where \( \text{Ann}_L \) denotes the left annihilator of \( \partial \) in \( A^U \). Let \( a\partial = 0 \) for some \( a \in A^U \). Then \( a\tilde{\partial} = 0 \), which is equivalent to \( \tilde{a}|_{U_*(BSU)} = 0 \) by part I. In other words,

\[
\tilde{a}[X, \xi] = [Y_a, f_a^*(\xi + \mathcal{T}X) - \mathcal{T}Y_a] = 0
\]

for any \( SU \)-bundle \( \xi \). In particular \( [Y_a] = 0 \) in \( \Omega_U \). On the other hand, \( [A] = ((\varphi^*)^{-1}a, [X, \xi]) \) by (3.1). It follows that

\[
(\varphi^*)^{-1}a \in U^*(BU) = \text{Hom}_{\Omega^U}(U_*(BU), \Omega^U)
\]

lies in the ideal \( I(d^U) \), because the latter consists precisely of the homomorphisms \( U_*(BU) \to \Omega^U \) vanishing on bordism classes of \( SU \)-bundles. Thus, \( a \in \varphi^*(I(d^U)) \) and \( \text{Ann}_L(\partial) \subset \varphi^*(I(d^U)) \). For the reverse inclusion note that \( a \in \varphi^*(I(d^U)) \) implies that \( \tilde{a}|_{U_*(BSU)} = 0 \). By Part I, \( \tilde{a}\tilde{\partial} = 0 \). Now Lemma 3.4 gives us that \( a\tilde{\partial} = 0 \), so \( a \in \text{Ann}_L(\partial) \).

III. We show that \( \varphi^*(I(d^U)) = A^U \Delta + A^U \partial \).

Corollary 4.4 implies that \( A^U \Delta + A^U \partial \) is a direct sum, and hence we write it as \( A^U \Delta \oplus A^U \partial \).

Lemma 4.3 and part II give us the inclusion \( A^U \Delta \oplus A^U \partial \subset \text{Ann}_L \partial = \varphi^*(I(d^U)) \).

Consider the short exact sequence

\[
0 \to A^U \Delta \oplus A^U \partial \to \varphi^*(I(d^U)) \to \varphi^*(I(d^U))/(A^U \Delta \oplus A^U \partial) \to 0 \quad (4.3)
\]

of graded \( \Omega_U \)-modules. Let

\[
N = \varphi^*(I(d^U))/(A^U \Delta \oplus A^U \partial).
\]

We need to show that \( N = 0 \).

First we show that \( N \) has no \( \Omega_U \)-torsion. Suppose that \( \lambda n = 0 \) for a non-zero \( \lambda \in \Omega_U \) and \( n = x + (A^U \Delta + A^U \partial) \in N \), \( x \in \varphi^*(I(d^U)) \). That is,

\[
\lambda x = a\Delta + b\partial \quad \text{for some} \quad a, b \in A^U.
\]

Multiplying by \( \Psi \) from the right and using Lemma 4.3, we obtain

\[
a = \lambda x\Psi \quad \text{and} \quad b\partial = \lambda x - \lambda x\Psi \Delta = \lambda y.
\]

Therefore, \( \tilde{b}\tilde{\partial} = \tilde{\lambda}\tilde{y} \). For a bordism class \( [Y, \zeta] \in U_*(BSU) \) we now have

\[
\langle (\varphi^*)^{-1}b, [Y, \zeta] \rangle = \langle (\varphi^*)^{-1}b, \tilde{\partial}[X, \xi] \rangle = \varepsilon(\tilde{\lambda}\tilde{y}[X, \xi]) = \lambda\varepsilon(\tilde{y}[X, \xi]),
\]

where the first equality follows from part I, and the second from (3.1). Consider the natural projection \( p: U^*(BU) \to U^*(BSU) \), which is Kronecker dual to the natural inclusion \( U_*(BSU) \leftarrow U_*(BU) \). Then the above equality implies that \( p((\varphi^*)^{-1}b) = \lambda w \) for some \( w \in U^*(BSU) \). We have \( w = p(t) \) for some \( t \in U^*(BU) \), hence \( p((\varphi^*)^{-1}b - \lambda t) = 0 \) and we get that \( (\varphi^*)^{-1}b - \lambda t \in \text{Ker} p = I(d^U) \). Therefore, \( b - \lambda \varphi^*(t) \in \varphi^*(I(d^U)) \) and \( b\partial = \lambda \varphi^*(t)\partial \) by part II. It follows that

\[
\lambda x = a\Delta + b\partial = \lambda(x\Psi \Delta + \varphi^*(t)\partial).
\]
Since $A^U$ has no $\Omega_U$-torsion, we conclude that

$$x = x\varPsi \Delta + \varphi^*(t)\partial \in A^U \Delta \oplus A^U \partial,$$

and hence $n = 0$, as required.

Now consider the following $A_U$-linear maps:

$$p_\Delta: A^U \to A^U \Delta, \quad p_\partial: A^U \to A^U \partial,$$

$$a \mapsto 2a\Psi \Delta, \quad a \mapsto a(1 - \Psi \Delta)\chi \partial.$$

These maps behave like mutually orthogonal projections. Namely, they satisfy the equalities

$$p_\Delta|_{A^U \Delta} = 2\text{id}_{A^U \Delta}, \quad p_\partial|_{A^U \partial} = 2\text{id}_{A^U \partial}, \quad p_\partial|_{A^U \Delta} = 0.$$

This is a direct calculation using Lemma 4.3:

$$p_\Delta(a\Delta) = 2a\Delta \Psi \Delta = 2a\Delta, \quad p_\Delta(b\partial) = 2b\partial \Psi \Delta = 0,$$

$$p_\partial(a\Delta) = a\Delta(1 - \Psi \Delta)\chi \partial = (a\Delta - a\Delta \Psi \Delta)\chi \partial = 0,$$

and

$$p_\partial(b\partial) = b\partial(1 - \Psi \Delta)\chi \partial = (b\partial - b\partial \Psi \Delta)\chi \partial = b\partial \chi \partial = 2b\partial.$$

We therefore have an $A^U$-linear map

$$p = p_\Delta + p_\partial: A^U \to A^U \Delta \oplus A^U \partial$$

satisfying the equality $p|_{A^U \Delta \oplus A^U \partial} = 2\text{id}_{A^U \Delta \oplus A^U \partial}$. We use the following purely algebraic fact.

**Lemma 4.6.** Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence of Abelian groups. Assume that $A$ does not have $n$-torsion for some fixed $n \in \mathbb{Z}$ and there exists a homomorphism $p: B \to A$ such that $p \circ i = n\text{id}_A$. Then there exists an injective homomorphism $s: nC \hookrightarrow B$.

If one starts with a short exact sequence of $R$-modules for some commutative ring $R$, then $s$ is also an $R$-module homomorphism.

**Proof.** Let $nc \in nC$. If $nc = \pi(nb)$, then

$$nc = \pi(nb - i(p(b))) \quad \text{and} \quad p(nb - i(p(b))) = np(b) - np(b) = 0.$$

Hence, there is an element $x := nb - i(p(b)) \in B$ satisfying $\pi(x) = nc$ and $p(x) = 0$. If $x'$ is another such element, then $\pi(x-x') = 0$, so $x-x' = i(y)$ and $0 = p(x-x') = p(i(y)) = ny$. Since $A$ has no $n$-torsion, we have $y = 0$ and $x = x'$. Therefore, $x$ is uniquely defined and there is a well-defined homomorphism $s: nC \to B$, $nc \mapsto x$, satisfying the equalities $p \circ s = 0$ and $\pi \circ s = \text{id}_{nC}$. The latter implies that $s$ is injective. \(\square\)
Applying Lemma 4.6 to the short exact sequence (4.3) and the map \( p = p_\Delta + p_\partial \) restricted to \( \varphi^*I(d^U) \), we conclude that \( 2N \) injects into \( \varphi^*I(d^U) \subset A^U \). Since \( N \) has no 2-torsion, \( N \) itself also injects into \( \varphi^*I(d^U) \subset A^U \). Furthermore, applying \( \otimes_{\Omega_U} \mathbb{Z} \) to (4.3), we obtain a short exact sequence of graded Abelian groups
\[
0 \to ((A^U \Delta) \otimes_{\Omega_U} \mathbb{Z}) \oplus ((A^U \partial) \otimes_{\Omega_U} \mathbb{Z}) \xrightarrow{i \otimes_{\Omega_U} \mathbb{Z}} \varphi^*(I(d^U)) \otimes_{\Omega_U} \mathbb{Z} \to N \otimes_{\Omega_U} \mathbb{Z} \to 0.
\]
(4.4)
The injectivity of the second map follows from the equality
\[
(p \otimes_{\Omega_U} \mathbb{Z})(i \otimes_{\Omega_U} \mathbb{Z}) = 2 \text{id}
\]
and the absence of torsion in \(((A^U \Delta) \otimes_{\Omega_U} \mathbb{Z}) \oplus ((A^U \partial) \otimes_{\Omega_U} \mathbb{Z})\) (the latter group is described below). Note that \( M \otimes_{\Omega_U} \mathbb{Z} = M/(\Omega_U^+ M) \) for any \( \Omega_U \)-module \( M \), where \( \Omega_U^+ \) denotes the ideal of non-zero (negatively) graded elements in \( \Omega_U \).

Next, we show that \( N \otimes_{\Omega_U} \mathbb{Z} \) is a finite group in each degree using a dimension counting argument.

Since \( \Delta \) has right inverse \( \Psi \), the \( A^U \)-module \( A^U \Delta \) is free on a single 4-dimensional generator. That is, \((A^U \Delta)^{2k} = U^{2k-4}(MU)\). Thus,
\[
((A^U \Delta) \otimes_{\Omega_U} \mathbb{Z})^{2k} = (U^{*-4}(MU) \otimes_{\Omega_U} \mathbb{Z})^{2k} = H^{2k-4}(MU; \mathbb{Z}) \cong \mathbb{Z}^p(k-2),
\]
where \( p(k) \) denotes the number of integer partitions of \( k \). Moreover,
\[
(A^U \partial)^{2k} = (A^U)^{2k-2} \partial \cong (A^U)^{2k-2}/(\text{Ann}_{L^2} \partial)^{2k-2} = (A^U)^{2k-2}/(\varphi^*I(d^U))^{2k-2} = U^{2k-2}(MSU),
\]
where the third equality follows from part II of this proof and the last one is (4.1). It follows that
\[
((A^U \partial) \otimes_{\Omega_U} \mathbb{Z})^{2k} \cong H^{2k-2}(MSU; \mathbb{Z}) = \mathbb{Z}^{\tilde{p}(k-1)},
\]
where \( \tilde{p}(k) \) is the number of integer partitions of \( k \) without 1. Finally,
\[
(\varphi^*I(d^U)) \otimes_{\Omega_U} \mathbb{Z} = \varphi^*_H I(c_1),
\]
where \( \varphi^*_H : H^*(BU; \mathbb{Z}) \to H^*(MU; \mathbb{Z}) \) is the Thom isomorphism in ordinary cohomology, and \( I(c_1) \) is the ideal in \( H^*(BU; \mathbb{Z}) \) generated by the universal first Chern class \( c_1 \). Therefore,
\[
((\varphi^*I(d^U)) \otimes_{\Omega_U} \mathbb{Z})^{2k} = (\varphi^*_H I(c_1))^{2k} = \mathbb{Z}^{p(k-1)}.
\]
Applying the equalities obtained to the \( 2k \) homogeneous part of (4.4), we obtain
\[
0 \to \mathbb{Z}^{p(k-2)+\tilde{p}(k-1)} \to \mathbb{Z}^{p(k-1)} \to (N \otimes_{\Omega_U} \mathbb{Z})^{2k} \to 0.
\]
The equality \( p(k-1) = p(k-2) + \tilde{p}(k-1) \) now implies that \((N \otimes_{\Omega_U} \mathbb{Z})^{2k}\) is a finite group.

Thus, \( N \) is a graded \( \Omega_U \)-submodule of \( A^U \) such that \((N \otimes_{\Omega_U} \mathbb{Z})^{2k}\) is a finite group for any \( k \). We need to show that \( N = 0 \). Consider the \( \Omega_U \)-linear projection \( p_\omega : A^U \to \Omega_U \) mapping an element \( a \in A^U \) to its coefficient \( \lambda_\omega \) in the power
series expansion \( a = \sum_\omega \lambda_\omega S_\omega \), where the \( S_\omega \in A^U \) are the Landweber–Novikov operations. Since \( N \otimes_{\Omega^U} \mathbb{Z} = N/(\Omega^U_1 N) \) is finite in each dimension, the group \( p_\omega(N)/(\Omega^U_1 p_\omega(N)) \) is also finite in each dimension. We must show that \( p_\omega(N) = 0 \). The general algebraic setting is as follows. Suppose that \( R \) is a non-negatively (or non-positively) graded ring without torsion, and let \( I \subset R \) be an ideal such that the group \( I/(R^+I) \) is finite in each dimension. Then \( I = 0 \). Indeed, let \( x \in I \) be an element of minimal degree. Then \( nx \in R^+I \) for some non-zero integer \( n \). Since \( \deg x \) is minimal in \( I \), every non-zero element of \( R^+I \) has degree greater then \( \deg x \). Hence \( nx = 0 \). And since \( R \) has no torsion, we conclude that \( x = 0 \) and \( I = 0 \). Returning to our situation, we obtain \( p_\omega(N) = 0 \) for any \( \omega \). Thus, \( N = 0 \) as claimed.

We have therefore proved that \( \varphi^*(I(a^U)) = A^U \Delta + A^U \partial \). Combining this equality with \((4.1)\), we obtain the statement \((a)\) of the theorem, and combining it with the equalities in part \( \Pi \) of the proof, we obtain \( \text{Ann}_L \partial = A^U \Delta + A^U \partial \), the statement \((b)\). Theorem 4.5 is proved. \( \square \)

5. Calculation with the spectral sequence

Here we apply the Adams–Novikov spectral sequence (Theorem 3.5) to the \( SU \)-bordism spectrum \( X = MSU \). As a result, we obtain a multiplicative spectral sequence with the \( E_2 \)-term

\[
E_2^{p,q} = \text{Ext}_{A^U}^{p,q}(U^*(MSU), U^*(pt))
\]

converging to \( \pi_*(MSU) = \Omega_*^{SU} \).

Theorem 4.5 implies that there is a free resolution of left \( A^U \)-modules:

\[
0 \leftarrow U^*(MSU) \cong A^U/(A^U \partial + A^U \Delta) \leftarrow A^U \xleftarrow{f_0} A^U \oplus A^U \xleftarrow{f_1} A^U \oplus A^U \xleftarrow{f_2} \cdots ,
\]

where \( A^U \twoheadrightarrow A^U/(A^U \partial + A^U) \) is the quotient projection, and

\[
f_0(a, b) = a\partial + b\Delta, \quad f_i(a, b) = (a\partial + b\Delta, 0), \quad i \geq 1.
\]

We rewrite it more formally as follows.

Proposition 5.1. There is a free resolution of left \( A^U \)-modules

\[
0 \leftarrow U^*(MSU) \leftarrow R^0 \xleftarrow{f_0} R^1 \xleftarrow{f_1} R^2 \xleftarrow{f_2} \cdots ,
\]

where \( R^0 = A^U(u_0) \) is a free module on a single generator of degree 0, \( R^i = A^U(u_i, v_i) \) is a free module on two generators with \( \deg u_i = 2i \) and \( \deg v_i = 2i + 2 \) for \( i \geq 1 \), and \( f_{i-1}(u_i) = \partial u_{i-1} \) and \( f_{i-1}(v_i) = \Delta u_{i-1} \).

Proof. We have \( f_{i-1}f_i = 0 \) because \( \partial^2 = \Delta \partial = 0 \). The exactness at \( R^0 \) is Theorem 4.5. To prove the exactness at \( R^i \) for \( i \geq 1 \), suppose that

\[
0 = f_{i-1}(au_i + bv_i) = (a\partial + b\Delta)u_{i-1}.
\]

Then \( a\partial + b\Delta = 0 \), which implies that \( b = 0 \) and \( a\partial = 0 \) by Corollary 4.4. Hence, \( a \in \text{Ann}_L \partial \), so \( a = a'\partial + b'\Delta \) by Theorem 4.5, \( b' \). Thus, \( au_i + bv_i = au_i = f_i(a'u_{i+1} + b'v_{i+1}) \), as needed. \( \square \)
Applying the functor $\text{Hom}^q_{A^e}(-, U^*(pt))$ to the resolution in Proposition 5.1 and using the isomorphism $\Omega^U_q = \Omega^U_q$, we obtain a complex whose homology is given by the terms $E^{*,q}_2$ of the spectral sequence

$$0 \rightarrow \Omega^U_q \xrightarrow{d^0} \Omega^U_{q-2} \oplus \Omega^U_{q-4} \xrightarrow{d^1} \Omega^U_{q-4} \oplus \Omega^U_{q-6} \xrightarrow{d^2} \cdots.$$  

(5.1)

The differentials are given by $d^0(a) = (\partial a, \Delta a)$ and $d^i(a, b) = (\partial a, \Delta a)$ for $i \geq 1$. Here we denote by $\partial$ and $\Delta$ the action of the corresponding operations on $\Omega^U$. We also use this notation below.

Conner and Floyd [22] defined the groups

$$\mathcal{U}_q = \text{Ker}(\Delta: \Omega^U_q \rightarrow \Omega^U_{q-4}).$$

The identities $\partial^2 = \Delta \partial = 0$ imply that the restriction of the differential $\partial: \mathcal{U}_k \rightarrow \mathcal{U}_{k-2}$ is defined.

**Proposition 5.2.** The complex (5.1) is quasi-isomorphic to its subcomplex

$$0 \rightarrow \mathcal{U}_q \xrightarrow{\partial} \mathcal{U}_{q-2} \xrightarrow{\partial} \mathcal{U}_{q-4} \xrightarrow{\partial} \cdots.$$  

**Proof.** Consider the inclusion

$$i: \mathcal{U}_k \rightarrow \Omega^U_k \oplus \Omega^U_{k-2}, \quad w \mapsto (w, 0), \quad \text{where} \ w \in \text{Ker} \Delta.$$

This is a map of chain complexes, since

$$i(\partial w) = (\partial w, 0) = (\partial w, \Delta w) = d(w, 0) = di(w).$$

The induced map in homology is injective, because $i(w) = d(a, b)$ implies that $(w, 0) = (\partial a, \Delta a)$, and hence

$$w = \partial a, \quad \text{with} \ a \in \text{Ker} \Delta = \mathcal{U}_*.$$

To prove surjectivity, take a cycle $(a, b) \in \Omega^U_k \oplus \Omega^U_{k-2}$. Then $0 = d(a, b) = (\partial a, \Delta a)$. Since the map $\Delta: \Omega^U_{k+2} \rightarrow \Omega^U_{k-2}$ is surjective (it has a right inverse $\Psi$), there is a $b' \in \Omega^U_{k+2}$ such that $\Delta b' = b$. Then $a - \partial b' \in \text{Ker} \Delta$ is a $\partial$-cycle, and

$$(a, b) - i(a - \partial b') = (a, b) - (a - \partial b', 0) = (\partial b', b) = d(b', 0),$$

so $i(a - \partial b')$ represents the same homology class as $(a, b)$. \square

**Proposition 5.3.** The following assertions hold for the $E_2$-term of the spectral sequence:

(a) $E^{0,q}_2 = \text{Ker} (\partial: \mathcal{U}_q \rightarrow \mathcal{U}_{q-2}) = (\text{Ker} \partial) \cap (\text{Ker} \Delta) \subset \Omega^U_q$;

(b) $E^{p,q}_2 = H_{q-2p}(\mathcal{U}_*, \partial)$ for $p > 0$;

(c) the edge homomorphism $h: \Omega^S_q \rightarrow E^{0,q}_2$ coincides with the forgetful homomorphism $\Omega^S_q \rightarrow \mathcal{U}_q$.

Therefore, the spectral sequence is concentrated in the first quadrant (that is, $E^{p,q}_r = 0$ for $p < 0$ or $q < 0$), $E^{p,q}_r = 0$ for odd $q$ and for $q < 2p$, and the differentials $d_r: E^{p,q}_r \rightarrow E^{p+r,q+r-1}_r$ are trivial for even $r$. 
Proof. Statements (a) and (b) follow from Proposition 5.2. To prove (c), recall that the edge homomorphism
\[ h : \Omega^0_{q} \to E^0_{2,q} = \text{Hom}^q_{\mathcal{A}^U}(U^*(MU), \Omega_U) \]
is defined as follows. For an element \( \alpha \in \Omega^0_{q} \) represented by a map \( f : S^q \to MU \) and an element \( \beta \in U^p(MU) \) represented by a map \( g : MU \to \Sigma^p MU \), the element \( h(\alpha)(\beta) \in \Omega^p_{U} \) is represented by the composite \( g \circ f : S^q \to \Sigma^p MU \).

Through the identification of \( E^0_{2,q} \) with \( \text{Ker}(\partial : \mathcal{W}_q \to \mathcal{W}_{q-2}) \), an \( A^U \)-homomorphism \( \varphi : U^*(MU) \to \Omega^0_{U} \) is mapped to the element \( \varphi(\iota) \), where \( \iota \in U^0(MU) \) is the class represented by the canonical map of spectra \( MU \to MU \). The edge homomorphism therefore becomes \( \Omega^0_{q} \to \Omega^U_q \), \( \iota \mapsto h(\alpha)(\iota) \), which is precisely the forgetful homomorphism, proving (c). The rest follows from the fact that \( \mathcal{W}_* \) is concentrated in non-negative even degrees. □

In particular, \( d_2 = 0 \) and \( E_2 = E_3 \). We shall denote this term simply by \( E \).

We have \( E^{1,2} = H_0(\mathcal{W}_*, \partial) = \mathbb{Z}_2 \), because \( \mathcal{W}_0 = \Omega^U_0 = \mathbb{Z} \), \( \mathcal{W}_2 = \Omega^U_2 = \mathbb{Z} \) with generator \( [\mathbb{C}P^1] \), and \( \partial[\mathbb{C}P^1] = 2 \). Let \( \theta \in E^{1,2} \) be the generator. By dimensional reasons, it is an infinite cycle because it lies on the ‘border line’ \( q = 2p \).

**Proposition 5.4.** Multiplication by \( \theta \) defines an isomorphism
\[ E^{p,q} \to E^{p+1,q+2} \quad \text{for} \ p > 0 \]
and an epimorphism \( E^{0,q} \to E^{1,q+2} \) with kernel \( \text{Im} \partial \).

Proof. For \( p > 0 \) the map \( E^{p,q} \xrightarrow{\partial} E^{p+1,q+2} \) is the identity isomorphism \( H_{q-2p}(\mathcal{W}_*) \to H_{q-2p}(\mathcal{W}_*) \). For \( p = 0 \) the homomorphism \( E^{0,q} \to E^{1,q+2} \) maps \( \text{Ker}(\partial : \mathcal{W}_q \to \mathcal{W}_{q-2}) \) onto \( H_q(\mathcal{W}_*) \), so its kernel is \( \text{Im} \partial \). □

This implies that \( E^{p,q} = \theta E^{p-1,q-2} \) for \( p \geq 1 \). In particular, \( E^{k,2k} = \mathbb{Z}_2 \) is generated by \( \theta^k \), so the only non-trivial elements on the border line \( q = 2p \) are \( 1, \theta, \theta^2, \theta^3, \ldots \).

Now consider \( E^{0,4} = \text{Ker}(\partial : \mathcal{W}_4 \to \mathcal{W}_2) \). Note that \( \partial|_{\Omega^U_4} = 0 \), because \( c_1 \) is the only Chern number in \( \Omega^U_4 \). Hence \( E^{0,4} = \mathcal{W}_4 \). Furthermore, \( \mathcal{W}_4 \cong \mathbb{Z} \) is generated by \( K = 9[\mathbb{C}P^1]^2 - 8[\mathbb{C}P^2] \)

(this bordism class has characteristic numbers \( c_1^2 = 0 \) and \( c_2 = 12 \)). Therefore, \( K \) represents a generator of \( E^{0,4} = \mathbb{Z} \).

We have a potentially non-trivial differential \( d_3 : E^{0,4} \to E^{3,6} \) (see Fig. 1).

**Proposition 5.5.** \( d_3(K) = \theta^3 \).

Proof. Suppose that \( d_3(K) = 0 \). We also have \( d_i(K) = 0 \) for \( i > 3 \), because \( d_i(K) \in E^{i,i+3}_i \) is below the border line \( p = 2q \). This implies that \( K \) is an infinite cycle, so it represents an element in \( E^{\infty}_\infty \). We obtain \( E^{0,4} = E^{0,4}_\infty \), which implies that the edge homomorphism \( \Omega^{SU}_4 \to E^{0,4}_2 \) is surjective. But it coincides with the forgetful homomorphism \( \Omega^{SU}_4 \to \mathcal{W}_4 \) by Proposition 5.3, and the forgetful homomorphism is not surjective, since \( \text{td}(K) = 1 \), while the Todd genus of a 4-dimensional \( SU \)-manifold is even (this follows from Rokhlin’s signature theorem [46]), a contradiction. □
Proposition 5.6. $E_{p,q}^p = 0$ for $p \geq 3$ and $E_4 = E_\infty$.

Proof. Take a $d_3$-cycle $x \in E_{p,q}^p$ with $p \geq 3$. We have $x = \theta^3 y$ for some $y \in E_{p-3,q-6}^p$ and $0 = d_3 x = \theta^3 d_3 y$. Now $d_3 y \in E_{p,q-4}^p$, and multiplication by $\theta^3$ is an isomorphism in this dimension by Proposition 5.4, hence $d_3 y = 0$. This implies that $x = \theta^3 y = d_3(Ky)$, that is, $x$ is a boundary, and $E_{p,q}^p = 0$ for $p \geq 3$. For dimension reasons this implies that $d_i = 0$ for $i > 4$, and $E_\infty = E_4$. □

It follows that the infinite term of the spectral sequence consists of three columns only, and it is easy to see that $E_{\infty}^{1,*} = \theta E_0^{0,*}$ and $E_{\infty}^{2,*} = \theta E_1^{1,*}$. Moreover, in the first three columns we have $E_\infty = \text{Ker} d_3$ for dimension reasons, and multiplication by $\theta$ is injective on $E_\infty^{1,*}$. In particular, $E_{\infty}^{k,2k} = E_{\infty}^{k,2k}$ is $\mathbb{Z}_2$ with generator $\theta^k$ for $0 \leq k \leq 2$, and $E_{\infty}^{k,2k} = 0$ for $k \geq 3$.

By Proposition 5.6, the Adams–Novikov filtration in $\Omega^{SU}$ satisfies the equality $F_{p,q}^p = 0$ for $p \geq 3$, that is, the filtration consists of three terms only:

$$\Omega_n^{SU} = F^{0,n} \supset F^{1,n+1} \supset F^{2,n+2} = E_\infty^{2,n+2}.$$

If $n = 2k + 1$ is odd, then

$$F^{0,2k+1}/F^{1,2k+2} = E_\infty^{0,2k+1} = 0 \quad \text{and} \quad F^{2,2k+3} = E_\infty^{2,2k+3} = 0$$

by Proposition 5.3. Therefore,

$$\Omega_{2k+1}^{SU} = E_\infty^{1,2k+2}.$$ (5.2)

If $n = 2k$ is even, then $F^{1,2k+1}/F^{2,2k+2} = E_\infty^{1,2k+1} = 0$, so we obtain a short exact sequence

$$0 \to E_\infty^{2,2k+2} \to \Omega_{2k}^{SU} \to E_\infty^{0,2k} \to 0.$$ (5.3)

Example 5.7. In low dimensions we have:

1) $\Omega_0^{SU} = E_0^{0,0} = E_\infty^{0,0} \cong \mathbb{Z}$, because $E_\infty^{2,2} = 0$;
2) \( \Omega^{SU}_1 = E^{1,2}_\infty \cong E^{1,2} \cong \mathbb{Z}_2 \) with generator \( \theta \);
3) \( \Omega^{SU}_2 = E^{2,4}_\infty \cong \mathbb{Z}_2 \) with generator \( \theta^3 \), because \( 0 = E^{0,2} = \text{Ker} \partial \subset \mathcal{W}_2 \) (recall that \( \mathcal{W}_2 \) is generated by \([CP^1]\), and \( \partial[CP^2] = 2 \);
4) \( \Omega^{SU}_3 = E^{1,4}_\infty = \theta E^{0,2}_\infty = 0 \);
5) \( \Omega^{SU}_4 = E^{0,4}_\infty \cong \mathbb{Z} \) with generator \( 2K \), the equality \( \Omega^{SU}_4 = E^{0,4}_\infty \) follows from (5.3), because \( E^{2,6}_\infty = \theta^2 E^{0,2}_\infty = 0 \), and a generator of \( E^{0,4}_\infty = \text{Ker} d_3 \) is \( 2K \), because \( d_3(K) = \theta^3 \);
6) \( \Omega^{SU}_5 = E^{1,6}_\infty = \theta E^{0,4}_\infty = 0 \), because \( \theta \cdot 2K = 0 \).

**Theorem 5.8.** (a) The kernel of the forgetful homomorphism \( \Omega^{SU} \to \Omega^U \) consists of torsion elements.

(b) Every torsion element in \( \Omega^{SU} \) has order 2. More precisely,

\[
\Omega^{SU}_{2k+1} = \theta \Omega^{SU}_{2k}, \quad \text{Tors } \Omega^{SU}_{2k} = \theta^2 \Omega^{SU}_{2k-2}.
\]

**Proof.** We have \( \Omega^{SU}_{2k+1} = E^{1,2k+2}_\infty = \theta E^{0,2k}_\infty = \theta \Omega^{SU}_{2k} \), because \( \Omega^{SU}_{2k} \to E^{0,2k} \) is surjective. This also implies that \( \Omega^{SU}_{2k+1} \) consists of 2-torsion elements, proving (a) and (b) in odd dimensions.

In even dimensions we use the exact sequence (5.3). Since \( E^{0,2k}_\infty \subset E^{0,2k}_\mathcal{W} \subset \mathcal{U} \) is torsion-free and \( E^{2,2k+2}_\infty = \theta^2 E^{0,2k-2}_\infty \) consists of 2-torsion elements, we obtain

\[
\text{Tors } \Omega^{SU}_{2k} = E^{2,2k+2}_\infty = \theta^2 E^{0,2k-2}_\infty = \theta^2 \Omega^{SU}_{2k-2},
\]

proving (b). To finish the proof of (a), it remains to note that the kernel of \( \Omega^{SU} \to \Omega^U \) coincides with the kernel of \( \Omega^{SU}_{2k} \to E^{0,2k}_\infty \) by Proposition 5.3, (c), and the latter kernel is the torsion in \( \Omega^{SU}_{2k} \) by the above. \( \square \)

The next lemma gives a short exact sequence, originally due to Conner and Floyd [22], which is the key ingredient in the calculation of the torsion in \( \Omega^{SU} \).

**Lemma 5.9.** There is a short exact sequence of \( \mathbb{Z}/2\)-modules

\[
0 \to \Omega^{SU}_{2k-1} \to H_{2k-2}(\mathcal{W}, \partial) \to \Omega^{SU}_{2k-5} \to 0.
\]

**Proof.** Consider the commutative diagram

\[
\begin{array}{c}
0 \to \Omega^{SU}_{2k-1} = E^{1,2k}_\infty \to E^{1,2k}_\infty \to E^{4,2k+2}_\infty \to E^{7,2k+4}_\infty \to 0 \\
0 \to \Omega^{SU}_{2k-5} \to E^{1,2k-4}_\infty \to E^{4,2k-2}_\infty \to 0
\end{array}
\]

The rows are exact by Proposition 5.6 and (5.2). By the commutativity of the diagram, \( \text{Im } d^{1,2k}_3 = \text{Ker } d^{4,2k+2}_3 \cong \text{Ker } d^{1,2k-4}_3 = \Omega^{SU}_{2k-5} \). From this we obtain a short exact sequence

\[
0 \to \Omega^{SU}_{2k-1} \to E^{1,2k}_\infty \to \Omega^{SU}_{2k-5} \to 0.
\]

It remains to note that \( E^{1,2k}_\infty = H_{2k-2}(\mathcal{W}, \partial) \). \( \square \)
Remark. The exact sequence in Lemma 5.9 is the derived exact sequence of the 5-term exact sequence (0.1) in the Introduction.

The homology of the complex \((\mathcal{W}_*, \partial)\) was described by Conner and Floyd. For the relation of this calculation to the Adams–Novikov spectral sequence, see §5 in [8].

Theorem 5.10 ([22], Theorem 11.8). \(H(\mathcal{W}_*, \partial)\) is the following polynomial algebra over \(\mathbb{Z}_2\):

\[
H(\mathcal{W}_*, \partial) \cong \mathbb{Z}_2[\omega_2, \omega_{4k} : k \geq 2], \quad \deg \omega_2 = 4, \quad \deg \omega_{4k} = 8k.
\]

Remark. The multiplication in \(H(\mathcal{W}_*, \partial)\) is induced by the multiplication in \(\Omega^U\) (see §6). It coincides with the multiplication in the \(E_2\) term of the Adams–Novikov spectral sequence.

We finally obtain the following information about the free and torsion parts of \(\Omega^{SU}\).

Theorem 5.11. (a) Tors \(\Omega^{SU}_{n} = 0\) unless \(n = 8k + 1\) or \(8k + 2\), in which case Tors \(\Omega^{SU}_{n}\) is a \(\mathbb{Z}_2\)-vector space of rank equal to the number of partitions of \(k\).

(b) The quotient \(\Omega^{SU}_{2i}/\text{Tors}\) is isomorphic to the image of the forgetful homomorphism \(\alpha : \Omega^{SU}_{2i} \rightarrow \Omega^U\), which is \(\text{Ker}(\partial : \mathcal{W}_{2i} \rightarrow \mathcal{W}_{2i-2})\) if \(2i \neq 4 \mod 8\) and \(\text{Im}(\partial : \mathcal{W}_{2i} \rightarrow \mathcal{W}_{2i-2})\) if \(2i \equiv 4 \mod 8\).

(c) There exist SU-bordism classes \(w_{4k} \in \Omega^{SU}_{8k}\), \(k \geq 1\), such that every torsion element of \(\Omega^{SU}\) is uniquely expressible in the form \(P \cdot \theta\) or \(P \cdot \theta^2\) where \(P\) is a polynomial in the variables \(w_{4k}\) with coefficients 0 or 1. An element \(w_{4k} \in \Omega^{SU}_{8k}\) is determined by the condition that it represents a polynomial generator \(\omega_{4k}\) in \(H_{8k}(\mathcal{W}_*, \partial)\) for \(k \geq 2\) and \(w_{4} \in \Omega^{SU}_{8}\) represents \(\omega_{2}^2\).

Remark. The only indeterminacy in the definition of \(w_{4k}\) is the choice of a \(\partial\)-cycle in \(\mathcal{W}_{8k}\) representing a polynomial generator \(\omega_{4k}\) or \(\omega_{2}^2\) in Theorem 5.10. Once we have fixed a representative \(w_{4k} \in \mathcal{W}_{8k}\), it lifts uniquely to \(w_{4k} \in \Omega^{SU}_{8k}\), since the forgetful homomorphism \(\alpha : \Omega^{SU}_{8k} \rightarrow \mathcal{W}_{8k}\) is injective onto \(\text{Ker} \partial\) in dimension \(8k\) by statements (a) and (b).  

Proof of Theorem 5.11. We prove (a). Theorem 5.10 gives us that \(H_{q-2p}(\mathcal{W}_*) = 0\) unless \(q - 2p = 8k\) or \(q - 2p = 8k + 4\). First consider the case of odd \(n\). Lemma 5.9 provides an exact sequence

\[
0 \rightarrow \Omega^{SU}_{8k-1} \rightarrow H_{8k-2}(\mathcal{W}_*) \rightarrow \Omega^{SU}_{8k-5} \rightarrow 0,
\]

which implies that \(\Omega^{SU}_{8k-1} = \Omega^{SU}_{8k-5} = 0\). We also have an exact sequence

\[
0 \rightarrow \Omega^{SU}_{8k+1} \rightarrow H_{8k}(\mathcal{W}_*) \rightarrow \Omega^{SU}_{8k-3} \rightarrow 0,
\]

which splits, because \(H(\mathcal{W}_*)\) is a \(\mathbb{Z}_2\)-module. Therefore,

\[
\Omega^{SU}_{8k+1} \oplus \Omega^{SU}_{8k-3} \cong H_{8k}(\mathcal{W}_*) \cong H_{8k+4}(\mathcal{W}_*) \cong \Omega^{SU}_{8k+5} \oplus \Omega^{SU}_{8k+1}.
\]

Hence \(\Omega^{SU}_{8k-3} = \Omega^{SU}_{8k+5}\). Since this is valid for all \(k\), we see that \(\Omega^{SU}_{8k+5} = 0\). Therefore, the only non-trivial group \(\Omega^{SU}_n\) with odd \(n\) is \(\Omega^{SU}_{8k+1}\), and Lemma 5.9 gives an
isomorphism $\Omega_{SU}^{8k+1} \cong H_{8k}(\mathcal{W}_*)$. It now follows from Theorem 5.10 that $\Omega_{SU}^{8k+1}$ is a $\mathbb{Z}_2$-vector space of rank equal to the number of partitions of $k$.

For even $n = 2m$ Theorem 5.8 gives us that $\text{Tors} \Omega_{SU}^{2m} = \theta \Omega_{SU}^{2m-1}$, which is non-zero only for $2m = 8k + 2$ by the previous paragraph. Multiplication by $\theta$ defines an isomorphism

$$\Omega_{SU}^{8k+1} = E_{\infty}^{1,8k+2} \xrightarrow{\cdot \theta} E_{\infty}^{2,8k+4} = \text{Tors} \Omega_{SU}^{8k+2}.$$ 

This finishes the proof of (a).

To prove (b) recall that $\text{Tors} \Omega_q^{SU}$ is the kernel of the forgetful homomorphism $\Omega_q^{SU} \to \mathcal{W}_q$ by Theorem 5.8, (a), and the forgetful homomorphism coincides with the edge homomorphism $h: \Omega_q^{SU} \to E_2^{0,q}$ by Proposition 5.3, (c). Thus, $\Omega^{SU}/\text{Tors} = \text{Im} h$. Furthermore, $\text{Im} h = \text{Ker}(d_3: E_3^{0,*} \to E_3^{3,*+2})$ by Proposition 5.6.

Now if $2i \neq 8k, 8k + 4$, then

$$d_3(E_i^{0,2i}) = \theta^{-1} d_3(\theta E_i^{0,2i}) = \theta^{-1} d_3(E_i^{1,2i+2}) = 0,$$

because $E_i^{1,2i+2} = H_{2i}(\mathcal{W}_*) = 0$ by Theorem 5.10. Therefore, $\Omega_{SU}/\text{Tors} \cong \text{Ker} d_3 = E_i^{0,2i} = \partial$ in this case.

For $2i = 8k$ we observe that

$$0 = \Omega_{SU}^{8k-3} = E_{\infty}^{1,8k-2} = \text{Ker} d_3^{1,8k-2} \subset E^{1,8k-2}.$$

This implies that

$$0 = \text{Ker}(d_3^{1,8k-2}\theta^{-2}) = \text{Ker}(\theta^{-2}d_3^{3,8k+2}) = \text{Ker} d_3^{3,8k+2}.$$ 

(5.4)

Hence, $\text{Im} d_3^{0,8k} \subset \text{Ker} d_3^{3,8k+2} = 0$ and $\Omega_{SU}^{8k}/\text{Tors} \cong \text{Ker} d_3^{0,8k} = E^{0,8k} = \partial$.

It remains to consider the case $2i = 8k + 4$. The exact sequence (5.3) gives us that $\Omega_{SU}^{8k+2} = E_{\infty}^{0,8k+4}$ because $E_{\infty}^{2,8k+6} \subset E^{2,8k+6} = H_{8k+2}(\mathcal{W}_*) = 0$. Consider the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \to & \Omega_{SU}^{8k+4} = E_{\infty}^{0,8k+4} & \to & E^{0,8k+4} & \xrightarrow{d_3^{0,8k+4}} & E^{3,8k+6} \\
& & \downarrow{\cdot \theta^3} & & \Downarrow{\cong} & \downarrow{\cdot \theta^3} & \\
0 & \to & E^{3,8k+10} & \xrightarrow{d_3^{3,8k+10}} & E^{6,8k+12} \\
\end{array}
$$

The lower row is exact by (5.4). This diagram implies that

$$\Omega_{SU}^{8k+4} \cong \text{Ker} d_3^{0,8k+4} = \text{Ker}(E^{0,8k+4} \xrightarrow{\cdot \theta^3} E^{3,8k+10}) = \text{Ker}(E^{0,8k+4} \xrightarrow{\cdot \theta} E^{1,8k+6}) = \text{Im} \partial,$$

where the last two equalities follow from Proposition 5.4. This finishes the proof of (b).

It remains to prove (c). Using statement (b) and Theorem 5.8, (b), we identify the homomorphism $\Omega_{SU}^{8n} \xrightarrow{\theta} \Omega_{SU}^{8n+1}$ with the projection $\text{Ker} \partial \to \text{Ker} \partial/\text{Im} \partial = \mathcal{W}_n \to \mathcal{W}_{n+1}$. 


$H_{8k}(\mathcal{W}_*)$. Now take an element $\alpha \in \Omega^{SU}_{8k+1}$ and write it as a polynomial $P(\omega_{4k})$ in $\omega_{4k}$ with $\mathbb{Z}_2$-coefficients, using Theorem 5.10. (To simplify the notation, we use $\omega_2^2$ for the missing generator $\omega_4$ in this argument.) We choose some lifts $w_{4k} \in \Omega^{SU}_{8k} = \text{Ker} \partial \subset \mathcal{W}_{4k}$ of the classes $\omega_{4k}$, and then $\alpha = P(w_{4k})$ maps to $\alpha$. In other words, $\alpha = P(w_{4k}) \cdot \theta$, where $P$ is now considered as a polynomial with coefficients 0 and 1. If $\alpha = Q(w_{4k}) \cdot \theta$ for another such polynomial $Q$, then $P(\omega_{4k}) = Q(\omega_{4k})$, which implies that $P = Q$, because the $\omega_{4k}$ are polynomial generators and both $P$ and $Q$ have coefficients 0 and 1. Therefore, any element of $\Omega^{SU}_{8k+1}$ is uniquely represented as $P \cdot \theta$, as needed. For elements of $\text{Tors} \Omega^{SU}_{8k+2}$, recall that $\Omega^{SU}_{8k+1} \xrightarrow{\theta} \text{Tors} \Omega^{SU}_{8k+2}$ is an isomorphism. This finishes the proof of Theorem 5.11. □

6. The ring $\mathcal{W}$

Theorem 5.11, (b), relates the group $\Omega^{SU}/\text{Tors}$ to the subgroup

$$\text{Ker}(\partial: \mathcal{W} \to \mathcal{W}) = (\text{Ker} \partial) \cap (\text{Ker} \Delta)$$

of $\Omega^{U}$. Although $\mathcal{W} = \text{Ker} \Delta$ is not a subring of $\Omega^{U}$, there is a product structure in $\mathcal{W}$ such that $\Omega^{SU}/\text{Tors} \subset \mathcal{W}$ is a subring. This leads to a description of the ring structure in $\Omega^{SU}/\text{Tors}$. We review this approach here, following [22], [54], and [50].

We recall the geometric operations $\partial: \Omega^{U}_{2n} \to \Omega^{U}_{2n-2}$ and $\Delta: \Omega^{U}_{2n} \to \Omega^{U}_{2n-4}$ (see (4.2)).

**Construction 6.1** (\(\partial\) and $\Delta$ revisited). Consider a stably complex manifold $M = M^{2n}$ with the fundamental class $[M^{2n}] \in H_{2n}(M; \mathbb{Z})$. Let $N = N^{2n-2}$ be a stably complex submanifold dual to the cohomology class $c_1(M) = c_1(\det \mathcal{F}M)$. That is, we have an inclusion

$$i: N^{2n-2} \hookrightarrow M^{2n} \quad \text{such that} \quad i_*(\lbrack N \rbrack) = c_1(M) \smile \lbrack M \rbrack \quad \text{in} \quad H_*(M; \mathbb{Z}).$$

The restriction of $\det \mathcal{F}M$ to $N$ is the normal bundle $\nu(N \subset M)$. The stably complex structure on $N$ is defined via the isomorphism $\mathcal{F}M|_N \cong \mathcal{F}N \oplus \nu(N \subset M)$. Then $c_1(N) = 0$, so $N$ is an $SU$-manifold.

The homomorphism $\partial = \Delta_{(1,0)}: \Omega^{U}_{2n} \to \Omega^{U}_{2n-2}$ sends a bordism class $[M]$ to the bordism class $[N]$ dual to $c_1(M)$ as described above. This operation is well defined on bordism classes, because $[N] = \varepsilon D_U(c_1^U(\det \mathcal{F}M))$, where $D_U: U^2(M) \to U_{2n-2}(M)$ is the Poincaré–Atiyah duality homomorphism and $\varepsilon: U_{2n-2}(M) \to \Omega^{U}_{2n-2}$ is the augmentation. We have $\partial^2 = 0$ because $N$ is an $SU$-manifold.

Similarly, the homomorphism $\Delta = \Delta_{(1,1)}: \Omega^{U}_{2n} \to \Omega^{U}_{2n-4}$ takes a bordism class $[M]$ to the bordism class of the submanifold $L = L^{2n-4}$ dual to $\det \mathcal{F}M \oplus \det \mathcal{F}M$. That is, we have an inclusion

$$j: L^{2n-4} \hookrightarrow M^{2n} \quad \text{such that} \quad j_*(\lbrack L \rbrack) = -c_1^2(M) \smile \lbrack M \rbrack \quad \text{in} \quad H_*(M; \mathbb{Z}).$$

We also introduce the homomorphism $\partial_k = \Delta_{(k,0)}: \Omega^{U}_{2n} \to \Omega^{U}_{2n-2k}$ taking a bordism class $[M]$ to the bordism class of the submanifold $[P]$ dual to $(\det \mathcal{F}M)^{\oplus k}$. We have $[P] = \varepsilon D_U(u^k)$, where $u = c_1^U(\det \mathcal{F}M)$.

**Lemma 6.2.** Let $[M] \in \Omega^{U}$ be a bordism class such that every Chern number of $M$ with $c_1^k$ as a factor vanishes. Then $\partial_k[M] = 0$. 


Proof. We have $\partial_k[M] = [P]$, where $j : P \hookrightarrow M$ is a submanifold such that

$$\mathcal{T}P \oplus j^*(\det \mathcal{T}M)^{\otimes k} = j^*(\mathcal{T}M).$$

Assume that $c_1^k c_{\omega}[M] = 0$ for any $\omega$. We need to prove that $c_{\omega}[P] = 0$. Calculating the total Chern classes for the bundles above, we get that

$$c(P)(1 + j^*c_1(M))^k = j^*c(M)$$

or

$$c(P) = j^*\left(\frac{c(M)}{1 + c_1(M))^{k}}\right) = j^*\bar{c}(M),$$

where $\bar{c}(M)$ is a polynomial in Chern classes of $M$. Then for any $\omega = (i_1, \ldots, i_p)$ we have

$$\langle c_{\omega}(P), [P] \rangle = \langle j^*\bar{c}_{\omega}(M), [P] \rangle = \langle \bar{c}_{\omega}(M), c_1^k(M) \sim [M] \rangle = \langle c_1^k\bar{c}_{\omega}(M), [M] \rangle = 0. \quad \Box$$

The group $\mathcal{W}_{2n}$ has been defined by

$$\mathcal{W}_{2n} = \text{Ker}(\Delta : \Omega^U_{2n} \to \Omega^U_{2n-4}).$$

The same group can also be defined in terms of characteristic numbers and geometrically, as described next. A cohomology class $x \in H^2(M)$ is spherical if $x = f^*(u)$ for a map $f : M \to \mathbb{C}P^1$, where $u = c_1(\eta)$ and $\eta$ is the tautological line bundle over $\mathbb{C}P^1$.

**Theorem 6.3.** The following three groups are identical:

(a) the group $\mathcal{W} = \text{Ker} \Delta$;

(b) the subgroup of $\Omega^U$ consisting of the bordism classes $[M]$ such that every Chern number of $M$ with $c_1^2$ as a factor vanishes;

(c) the subgroup of $\Omega^U$ consisting of the bordism classes $[M]$ for which $c_1(M)$ is a spherical class.

Proof. The equivalence of (a) and (b) was proved in [22], (6.4). We give a more direct argument below. By definition $\Delta[M] = [L]$, where $j : L \hookrightarrow M$ is a submanifold such that

$$\mathcal{T}L \oplus j^*(\det \mathcal{T}M \oplus \det \mathcal{T}M) = j^*(\mathcal{T}M).$$

Calculating the Chern classes, we get that

$$c(L)(1 + j^*c_1(M))(1 - j^*c_1(M)) = j^*c(M),$$

$$c_1(L) - c_1(M) j^*c_1^2(M) = j^*c_1(M).$$

In particular, for $i = 1$ we obtain $c_1(L) = j^*c_1(M)$, so we can rewrite the formula above as

$$(c_i - c_1^2c_i-2)(L) = j^*c_i(M).$$

Given a partition $\omega = (i_1, \ldots, i_p)$ and the corresponding Chern class $c_{\omega} = c_{i_1} \cdots c_{i_p}$, we obtain the following relation for characteristic numbers:

$$\langle (c_{i_1} - c_1^2c_{i_1-2}) \cdots (c_{i_p} - c_1^2c_{i_p-2})(L), [L] \rangle = \langle j^*c_{\omega}(M), [L] \rangle = \langle -c_1^2c_{\omega}(M), [M] \rangle.$$
If $\Delta[M] = [L] = 0$, then the left-hand side above vanishes, and from the right-hand side we see that every Chern number of $M$ with $c_1^2$ as a factor vanishes.

For the reverse implication, assume that $-c_1^2c_\omega[M] = 0$ for any $\omega$. We need to prove that $c_\omega[L] = 0$. This is done in the same way as in the proof of Lemma 6.2.

The equivalence of (a) and (c) was proved in [50], Chap. VIII. □

**Corollary 6.4.** If $[M] \in \mathcal{W}$, then $\partial_k[M] = 0$ for any $k \geq 2$.

**Proof.** By Theorem 6.3, $[M] \in \mathcal{W}$ implies that every Chern number of $M$ with $c_1^2$ as a factor vanishes. But then every Chern number of $M$ with $c_1^k$ as a factor vanishes (since $k \geq 2$). Thus, $\partial_k[M] = 0$ by Lemma 6.2. □

**Remark.** For the operation $\partial = \partial_1$, there is no analogue of the equivalence between (a) and (b) in Theorem 6.3. More precisely, by Lemma 6.2 the group $\ker \partial$ contains the subgroup of $\Omega^U$ consisting of the bordism classes $[M]$ such that every Chern number of $M$ with $c_1$ as a factor vanishes. However, there is no reverse inclusion. For example, any element of $\Omega^U_4$ lies in $\ker \partial$, but $c_1^2[CP^2] \neq 0$. In fact, the subgroup of $\Omega^U$ consisting of the bordism classes $[M]$ such that every Chern number of $M$ with $c_1$ as a factor vanishes coincides with the intersection $\ker \partial \cap \ker \Delta$.

It follows from either of the descriptions of the group $\mathcal{W}_{2n}$ that we have forgetful homomorphisms $\Omega^U_{2n} \to \mathcal{W}_{2n} \to \Omega^U_{2n}$, and the restriction of the boundary homomorphism $\partial : \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$ is defined.

**Lemma 6.5.** For any elements $a, b \in \mathcal{W}$,

$$\partial(a \cdot b) = a \cdot \partial b + \partial a \cdot b - [CP^1] \cdot \partial a \cdot \partial b$$

and

$$\Delta(a \cdot b) = -2\partial a \cdot \partial b,$$

where $a \cdot b$ denotes the product in $\Omega^U$.

**Proof.** Let $a = [M^{2m}]$ and $b = [N^{2n}]$ for some stably complex manifolds $M$ and $N$. Then $\partial(a \cdot b) \in \Omega^U_{2m+2n-2}$ is represented by a submanifold $X \subset M \times N$ dual to $c_1(M \times N) = x + y$, where $x = p_1^*c_1(M)$, $y = p_2^*c_1(M)$, and $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are the projection maps. Let $u, v \in U^2(M \times N)$ be the geometric cobordisms corresponding to $x, y$, respectively (see Construction 1.6). Then we have

$$\partial(a \cdot b) = [X] = \varepsilon D_U(u +_\mu v).$$

On the other hand,

$$u +_\mu v = F_U(u, v) = u + v + \sum_{k \geq 1, \ l \geq 1} \alpha_{kl}u^k v^l.$$

To identify $\partial(a \cdot b) = [X]$, we apply $\varepsilon D_U$ to both sides of this equality. We have $\varepsilon D_U(u) = \partial a \cdot b$ (the submanifold dual to $p_1^*c_1(M)$ in $M \times N$ is the product with $N$ of the submanifold dual to $c_1(M)$ in $M$). Similarly, $\varepsilon D_U(v) = a \cdot \partial b$ and $\varepsilon D_U((uv) = \partial a \cdot \partial b$. We claim that $\varepsilon D_U(u^k v^l) = 0$ if $k \geq 2$ or $l \geq 2$. Indeed, $\varepsilon D_U(u^k v^l)$ is the bordism class of the submanifold in $M \times N$ dual to $p_1^*(\det \mathcal{F} M)^{\oplus k} \oplus p_2^*(\det \mathcal{F} N)^{\oplus l}$. 

This bordism class is $\partial ka \cdot \partial b$. Since $a, b \in \mathcal{W}$, Corollary 6.4 implies that either $\partial ka = 0$ or $\partial b = 0$.

The first equality in the lemma follows by noting that $\alpha_{11} = -[\mathbb{C}P^1]$ (see, for example, [15], Theorem E.2.3).

For the second equality, $\Delta(a \cdot b) \in \Omega^U_{2m+2n-4}$ is represented by a submanifold $L \subset M \times N$ dual to $-c_1^2(M \times N) = (x+y)(-x-y)$. Arguing as above, we get that

$$\Delta(a \cdot b) = [L] = \varepsilon D_U \left( F_U(u,v)F_U(v,u) \right) = \varepsilon D_U(-2uv) = -2\partial a \cdot \partial b. \quad \square$$

The direct sum $\mathcal{W} = \bigoplus_{i \geq 0} \mathcal{W}_{2i}$ is not a subring of $\mathcal{O}^U$: one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \not\in \mathcal{W}_4$.

The ring structure in $\mathcal{W}$ will be defined using a projection operator $\rho: \mathcal{O}^U \to \mathcal{O}^U$ which is described next. We recall the operation $\Psi: \mathcal{O}^U_{2n} \to \mathcal{O}^U_{2n+4}$ defined in Construction 4.2, the right inverse of $\Delta$.

**Proposition 6.6.** The homomorphism $\rho = \id - \Psi \Delta: \mathcal{O}^U \to \mathcal{O}^U$ is a projection operator, with $\im \rho = \mathcal{W}$, $\ker \rho = \Psi(\Omega^U)$, and $\partial \rho = \rho \partial = \partial$.

**Proof.** The relation $\Delta \Psi = \id$ in Lemma 4.3 implies that $(\id - \Psi \Delta)^2 = \id - \Psi \Delta$, so $\rho$ is a projection. The same relation implies that $\Delta \rho = 0$, that is, $\im \rho \subset \ker \Delta = \mathcal{W}$. The inclusion $\im \rho \supset \ker \Delta$ is obvious. The equality $\ker \rho = \im \Psi$ is proved similarly. Finally, $\partial (\id - \Psi \Delta) = \partial - \partial \Psi \Delta = \partial$ because $\partial \rho = 0$, and $(\id - \Psi \Delta) \partial = \partial - \Psi \Delta \partial = \partial$ because $\Delta \partial = 0$. $\square$

**Corollary 6.7.** $\text{rank } \mathcal{W}_{2n} = \text{rank } \mathcal{O}^U_{2n} - \text{rank } \mathcal{O}^U_{2n-4}$.

**Proof.** Proposition 6.6 implies that $\mathcal{O}^U = \ker \rho \oplus \im \rho$. We have $(\im \rho)_{2n} = \mathcal{W}_{2n}$ and $(\ker \rho)_{2n} = \Psi(\mathcal{O}^U_{2n-4}) \cong \mathcal{O}^U_{2n-4}$ because $\Psi$ is injective (having the left inverse $\Delta$).

Using the projection $\rho = \id - \Psi \Delta$, we define the twisted product of elements $a, b \in \mathcal{W}$ by

$$a \ast b = \rho(a \cdot b),$$

where $\cdot$ denotes the product in $\mathcal{O}^U$. A geometric description of this multiplication is given next.

**Proposition 6.8.** $a \ast b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b$, where $V^4$ is the manifold $\mathbb{C}P^2$ with the stably complex structure defined by the isomorphism $\mathcal{T} \mathbb{C}P^2 \oplus \mathbb{R}^2 \cong \bar{\eta} \oplus \bar{\eta} \oplus \eta$.

**Proof.** We need to verify that $\Psi \Delta(a \cdot b) = -2[V^4] \cdot \partial a \cdot \partial b$. By Lemma 6.5, $\Delta(a \cdot b) = -2\partial a \cdot \partial b$. Recall from Construction 4.2 that $\Psi[M]$ is represented by the manifold $\mathbb{C}P(\det \mathcal{T}M \oplus \mathbb{C}^2)$ with the stably complex structure $p^* \mathcal{T}M \oplus (\bar{\eta} \otimes p^* \det \mathcal{T}M) \oplus \bar{\eta} \oplus \eta$. In our case $[M] = -2\partial a \cdot \partial b$, and thus $\det \mathcal{T}M$ is a trivial bundle. We get that the bordism class $\Psi \Delta(a \cdot b) = \Psi[M]$ is represented by the total space of a trivial bundle over $M$ whose fibre is $\mathbb{C}P^2$ with the stably complex structure $\bar{\eta} \oplus \bar{\eta} \oplus \eta$. The latter bordism class is $[V^4] \cdot [M] = -2[V^4] \cdot \partial a \cdot \partial b$, as asserted. $\square$

**Remark.** We can also take $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$ with the standard complex structure, since this manifold is bordant to the one described in Proposition 6.8.

**Theorem 6.9.** The direct sum $\mathcal{W} = \bigoplus_{i \geq 0} \mathcal{W}_{2i}$ is a commutative associative unital ring with respect to the product $\ast$. 

Proof. We need to verify that the product \( * \) is associative. This is a direct calculation using the formula from Proposition 6.8. \( \square \)

The projection \( \rho = \id - \Psi \Delta \) was defined by Conner and Floyd in [22], (8.4), and used by Novikov (see [40], Remark 5.3). Stong (see [50], Chap. VIII) introduced another projection \( \pi : \Omega^U \to \Omega^U \) with image \( \mathcal{W} \), defined geometrically as follows. Take \([M] \in \Omega^U\). Then \( \pi[M] \) is the bordism class \([N]\) of the submanifold \( N \subset \mathbb{CP}^1 \times M \) dual to \( \eta \otimes \det \mathcal{F}M \). It follows easily from this geometric definition that \( c_1(\pi[M]) \) is a spherical class. This is the way the equivalence of (a) and (c) in Theorem 6.3 is proved.

Buchstaber [11] used Stong’s projection \( \pi : \Omega^U \to \mathcal{W} \) (under the name of ‘a projection of Conner–Floyd type’) to define a complex-oriented cohomology theory with the coefficient ring \( \mathcal{W} \), and he then studied the corresponding formal group law. A general algebraic theory of projections of Conner–Floyd type was developed in [10], and then used to classify stable associative multiplications in complex cobordism.

Both projection operators \( \rho \) and \( \pi \) have the same image \( \mathcal{W} \) and coincide on elements of the form \( a \cdot b \) with \( a, b \in \mathcal{W} \). Therefore, they define the same product in \( \mathcal{W} \). Nevertheless, the projections \( \rho \) and \( \pi \) are different, since they have different kernels. Indeed, take \([M^6] = \Psi[\mathbb{CP}^1]\). Then \( \rho[M^6] = 0 \) because \([M^6] \in \text{Im} \Psi = \text{Ker} \rho \). On the other hand, \( \pi[M^6] \neq 0 \), because one can check that \( c_2^3[M^6] = -2 \), \( c_3[M^6] = 2 \), and \( c_3(\pi[M^6]) = (-c_3^3 + c_3)[M^6] = 4 \), which is non-zero. Also, \( c_3(\rho[\mathbb{CP}^3]) = 68 \), while \( c_3(\pi[\mathbb{CP}^3]) = -60 \).

Recall from Theorem 1.5 that a bordism class \([M^{2i}] \in \Omega^{2i}_U\) represents a polynomial generator of the ring \( \Omega^U \) whenever \( s_i[M^{2i}] = \pm m_i \), where the numbers \( m_i \) are defined in (1.4). A similar description for the ring \( \mathcal{W} \) is given next.

**Theorem 6.10.** \( \mathcal{W} \) is a polynomial ring on generators in every positive even degree but 4:
\[
\mathcal{W} \cong \mathbb{Z}[x_1, x_i : i \geq 3], \quad x_1 = [\mathbb{CP}^1], \quad \deg x_i = 2i.
\]
The polynomial generators \( x_i \) are specified by the condition \( s_i(x_i) = \pm m_i m_{i-1} \) for \( i \geq 3 \). The boundary operator \( \partial : \mathcal{W} \to \mathcal{W}, \partial^2 = 0 \), satisfies the equality
\[
\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b, \quad (6.1)
\]
and the polynomial generators of \( \mathcal{W} \) can be chosen so that
\[
\partial x_1 = 2, \quad \partial x_{2i} = x_{2i-1}.
\]

**Proof.** We start by checking the equality (6.1):
\[
\partial(a * b) = \partial \rho(ab) = \partial(ab) = a \partial b + b \partial a - [\mathbb{CP}^1] \partial a \partial b = a * \partial b + b * \partial a - [\mathbb{CP}^1] * \partial a * \partial b.
\]
Here the second equality is by Proposition 6.6, the third equality is in Lemma 6.5, and the last equality also follows from Lemma 6.5, since the equality \( \Delta(ab) = -2 \partial a \partial b \) for \( a, b \in \mathcal{W} \) implies that \( a * b = ab \) whenever \( a \in \text{Im} \partial \) or \( b \in \text{Im} \partial \).

In the rest of this proof we denote the product of elements in \( \mathcal{W} \) by \( a * b \) only when it differs from the product in \( \Omega^U \); otherwise we denote it by \( a \cdot b \) or simply by \( ab \).
We start by defining bordism classes \( b_i \in \mathcal{W}_{2i} \) for each \( i \geq 1 \) except \( i = 2 \). Let

\[
b_i = \begin{cases} 
[C_P^1] & \text{if } i = 1, \\
\pi[C_P^{2p} \times C_P^{2p+1}] & \text{if } i = 2^p(2q + 1), \ p \geq 1, \ q \geq 1, \\
\pi[C_P^{2p}] & \text{if } i = 2p^p + 1, \ p \geq 1, \\
\partial b_{i+1} & \text{if } i \text{ is odd and } i \geq 3,
\end{cases}
\]

where \( \pi : \Omega^U \to \mathcal{W} \) is Stong’s projection defined above. One can check that

\[
s_i(b_i) = 1 \mod 2 \quad \text{if } i \neq 2^k - 1, \ i \neq 2^k,
\]

\[
s_i(b_i) = 2 \mod 4 \quad \text{if } i = 2^k - 1,
\]

\[
s_i(b_i) = 2 \mod 4 \quad \text{if } i = 2p+1,
\]

\[
s_{(2p,2p)}(b_{2p+1}) = 1 \mod 2.
\]

(6.2)

Consider the inclusion \( \iota : \mathcal{W} \otimes \mathbb{Z}_2 \to \Omega^U \otimes \mathbb{Z}_2 \). The formula in Proposition 6.8 for the product in \( \mathcal{W} \) implies that \( \iota \) is a ring homomorphism. The relations (6.2) imply that there are polynomial generators \( a_i \) of the ring \( \Omega^U \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[a_i : i \geq 1] \) such that

\[
\iota(b_i) = a_i \quad \text{for } i \neq 2p+1 \quad \text{and} \quad \iota(b_{2p+1}) = (a_{2p})^2 + \cdots,
\]

where \( \cdots \) denotes decomposable elements corresponding to partitions which are strictly less than \( (2^p, 2^p) \) in the lexicographic order. From this it follows that the elements \( \iota(b_i) \) are algebraically independent in the polynomial ring \( \Omega^U \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[a_i : i \geq 1] \). Therefore, \( \mathcal{W} \otimes \mathbb{Z}_2 \) contains the polynomial subring \( \mathbb{Z}_2[b_1, b_i : i \geq 3] \).

By comparing the ranks using Corollary 6.7 we conclude that

\[
\mathcal{W} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[b_1, b_i : i \geq 3].
\]

Next we observe that \( s_i(b_i) \) is an odd multiple of \( m_i m_{i-1} \) for \( i \geq 3 \), that is,

\[
s_i(b_i) = (2q_i + 1)m_i m_{i-1}, \quad i \geq 3.
\]

(6.3)

For even \( i \) this follows from (6.2) and the fact that \( s_i(b_i) \) is a multiple of \( m_i \) (see Theorem 1.5, (b)). For odd \( i \) we have \( b_i = \partial b_{i+1} \), so \( b_i \) is represented by an \( SU \)-manifold, and (6.3) follows from (6.2) and Proposition 2.2.

By Theorem 2.1 there exist elements \( y_i \in \Omega^{SU}_{2i}, i \geq 2 \), such that

\[
s_i(y_i) = 2^{k_i} m_i m_{i-1}, \quad k_i \geq 0.
\]

(6.4)

For the integers \( q_i \) in (6.3) and \( k_i \) in (6.4) we find integers \( \beta_i \) and \( \gamma_i \) such that

\[
\beta_i 2^{k_i+1} + \gamma_i (2q_i + 1) = 1.
\]

Then \( \gamma_i \) is odd, that is, \( \gamma_i = 2\alpha_i + 1 \) for some integer \( \alpha_i \). Now let \( x_1 = [C_P^1] \) and

\[
x_i' = (2\alpha_i + 1)b_i + 2\beta_i y_i, \quad i \geq 3.
\]

Then the equalities above imply that \( s_i(x_i') = m_i m_{i-1} \). The required elements \( x_i \) are obtained by modifying the \( x_i' \) as follows:

\[
x_{2i-1} = x_{2i-1}', \quad x_{2i} = x_{2i}' - x_1((\alpha_{2i} - \alpha_{2i-1})b_{2i-1} - \beta_{2i-1}y_{2i-1}).
\]
Then as before we have
\[ s_i(x_i) = m_im_{i-1} \]
because the elements \( x_i - x'_i \) are decomposable. The new elements \( x_{2i} \) still belong to \( \mathcal{W} \). To verify this we use the second equality in Lemma 6.5:
\[ \Delta x_{2i} = \Delta x'_{2i} + 2 \partial x_1 \partial ((\alpha_{2i} - \alpha_{2i-1})b_{2i-1} - \beta_{2i-1}y_{2i-1}) = 0 \]
because \( x'_{2i} \in \mathcal{W} = \text{Ker} \Delta, \partial b_{2i-1} = \partial^2 b_{2i} = 0, \) and \( \partial y_{2i-1} = 0 \) since \( y_{2i-1} \in \Omega^{SU} \).
To verify the equality \( \partial x_{2i} = x_{2i-1} \) we use the first equality in Lemma 6.5:
\[
\begin{align*}
\partial x_{2i} &= \partial x'_{2i} - \partial x_1 \cdot ((\alpha_{2i} - \alpha_{2i-1})b_{2i-1} - \beta_{2i-1}y_{2i-1}) \\
&= (2\alpha_{2i} + 1)\partial b_{2i} - 2((\alpha_{2i} - \alpha_{2i-1})b_{2i-1} - \beta_{2i-1}y_{2i-1}) \\
&= (2\alpha_{2i-1} + 1)b_{2i-1} + 2\beta_{2i-1}y_{2i-1} = x_{2i-1}.
\end{align*}
\]
We now define a homomorphism
\[ \varphi: \mathcal{R} = \mathbb{Z}[x_1, x_i: i \geq 3] \to \mathcal{W} \]
sending the polynomial generator \( x_i \) to the corresponding element of \( \mathcal{W} \) defined above. Observe that \( \varphi \otimes \mathbb{Z}_2 \) sends \( x_i \) to \( b_i \) modulo decomposable elements. As we have seen, \( \mathcal{W} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[b_1, b_i: i \geq 3] \), which implies that \( \varphi \otimes \mathbb{Z}_2 \) is an isomorphism. Since \( \mathcal{R} \) and \( \mathcal{W} \) are torsion-free, \( \varphi \) is injective and \( \varphi(\mathcal{R}_n) \subset \mathcal{W}_n \) is a subgroup of odd index in each dimension.
We will show that \( \varphi: \mathcal{R} \to \mathcal{W} \) becomes surjective after tensoring with \( \mathbb{Z}[\frac{1}{2}] \). This will imply that \( \varphi \) is an isomorphism.
Note that for any \( \alpha \in \mathcal{W} \) we have
\[
\partial(x_1 * \alpha) = \partial x_1 \cdot \alpha + x_1 \cdot \partial \alpha - x_1 \cdot \partial x_1 \cdot \partial \alpha = 2\alpha - x_1 \cdot \partial \alpha.
\]
Hence \( \alpha = \frac{1}{2} \partial(x_1 * \alpha) + \frac{1}{2} x_1 \partial \alpha \) in \( \mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] \). It follows that \( \mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] \) is generated by \( 1 \) and \( x_1 \) as a module over \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \subset \mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] \) (note that \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \) is a subring of \( \mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] \) by the formula in Proposition 6.8). Furthermore, this module is free because the equality \( 0 = a + x_1 b \) with \( a, b \in \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \) implies that \( 0 = \partial(a + x_1 b) = \partial x_1 \cdot b = 2b \), and therefore \( b = 0 \) and \( a = 0 \). Thus,
\[
\mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] = \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] (1, x_1).
\]
We now consider the following elements in \( \varphi(\mathcal{R}) \subset \mathcal{W} \):
\[
\begin{align*}
y_2 &= 2x_1 \ast x_1 = \partial(x_1 * x_1 * x_1), \\
y_{2i} &= \partial(x_1 * x_{2i}) = 2x_{2i} - x_1 x_{2i-1}, \quad i \geq 2, \\
y_{2i-1} &= x_{2i-1} = \partial x_{2i}, \quad i \geq 2.
\end{align*}
\]
These elements actually lie in \( \Omega^{SU} \), because they belong to \( \text{Im} \partial \). Moreover,
\[
\begin{align*}
s_2(y_2) &= 2s_2(x_1 \cdot x_1 + 8[V^4]) = -16s_2(\mathbb{C}P^2) = -48 = -8m_2m_1, \\
s_{2i}(y_{2i}) &= 2s_{2i}(x_{2i}) = 2m_{2i}m_{2i-1}, \quad i \geq 2, \\
s_{2i-1}(y_{2i-1}) &= s_{2i-1}(x_{2i-1}) = m_{2i-1}m_{2i-2}, \quad i \geq 2.
\end{align*}
\]
and therefore the \( y_i \) are polynomial generators of the ring \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \) by Theorem 2.1. It follows that \( \mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] = \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \langle 1, x_1 \rangle \subset \varphi(\mathcal{R} \otimes \mathbb{Z}[\frac{1}{2}]) \). Thus, \( \varphi \otimes \mathbb{Z}[\frac{1}{2}] \) is an epimorphism, and this completes the proof of Theorem 6.10. □

7. The ring structure of \( \Omega^{SU} \)

The forgetful map \( \alpha: \Omega^{SU} \to \mathcal{W} \) is a ring homomorphism. This follows from Proposition 6.8 because \( \partial \alpha(x) = 0 \) for any \( x \in \Omega^{SU} \). Therefore, the ring \( \Omega^{SU} / \text{Tors} \) can be described as a subring of \( \mathcal{W} \).

Note that by Theorem 6.10
\[
\mathcal{W} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}] [x_1, x_{2k-1}, 2x_{2k} - x_1 x_{2k-1} : k \geq 2],
\]
where \( x_1^2 = x_1 \ast x_1 \) is a -cycle, and each of the elements \( x_{2k-1} \) and \( 2x_{2k} - x_1 x_{2k-1} \) for \( k \geq 2 \) is a \( \partial \)-cycle.

For any integer \( n \geq 3 \) define
\[
g(n) = \begin{cases} 2m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is odd}, \\ m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is even}, \\ -48 & \text{if } n = 3. \end{cases}
\]
These numbers appear in (6.6). For example, \( g(4) = 6 \) and \( g(5) = 20 \). For \( n > 3 \), the number \( g(n) \) can take the following values: \( 1, 2, 4, p, 2p, \) and \( 4p \), where \( p \) is an odd prime.

**Theorem 7.1.** There exist indecomposable elements \( y_i \in \Omega^{SU}_{2i} \), \( i \geq 2 \), with minimal \( s \)-numbers given by \( s_i(y_i) = g(i + 1) \). These elements are mapped as follows under the forgetful homomorphism \( \alpha: \Omega^{SU} \to \mathcal{W} \):
\[
y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1 x_{2k-1}, \quad k \geq 2,
\]
where the \( x_i \) are polynomial generators of \( \mathcal{W} \). In particular, the ring \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}] [y_i : i \geq 2] \) embeds into (7.1) as the polynomial subring generated by the elements \( x_1^2, x_{2k-1} \), and \( 2x_{2k} - x_1 x_{2k-1} \).

*Proof.* The elements \( y_i \in \Omega^{SU}_{2i} \) were defined in (6.5), and their \( s \)-numbers were given by (6.6). We only need to check that the \( s \)-number of \( y_i \) is the minimum possible in \( \Omega^{SU}_{2i} \).

For \( y_{2k-1} \) the number \( m_{2k-1}m_{2k-2} \) is minimum possible for all the elements in \( \mathcal{W}_{4k-2} \) by Theorem 6.10, and therefore it is also minimum possible in \( \Omega^{SU}_{4k-2} \subset \mathcal{W}_{4k-2} \). (Note that indecomposability in \( \mathcal{W} \) with respect to the product \( \ast \) is the same as indecomposability in \( \Omega^U \) in dimensions \( > 4 \); this follows from Proposition 6.8.)

For \( y_2 = 2x_1^2 \), we have \( \Omega^U_4 = \text{Im} \partial = \mathbb{Z}[y_2] \), where \( y_2 = 2K \) in the notation of Example 5.7.

Now consider \( y_{2k} \) for \( k \geq 2 \). We have \( s_{2k}(y_{2k}) = 2m_{2k}m_{2k-1} \). Take any element \( a \in \Omega^{SU}_{4k} \subset (\text{Ker} \partial)_{4k} \). It follows from (7.1) that \( \text{Ker}(\partial): \mathcal{W} \to \mathcal{W}' \) consists of \( \mathbb{Z}[\frac{1}{2}] \)-polynomials in \( x_{2i}^2 \), \( x_{2i-1} \), and \( 2x_{2i} - x_1 x_{2i-1} \) which have integral coefficients as polynomials in the variables \( x_i \). We write
\[
a = \lambda(2x_{2k} - x_1 x_{2k-1}) + b,
\]
where $\lambda \in \mathbb{Z}[\frac{1}{2}]$ and $b$ is a decomposable element in $\mathbb{Z}[\frac{1}{2}][x_1^2, x_{2i-1}, 2x_{2i} - x_1x_{2i-1}]$. Then $b$ does not contain the term $x_1x_{2k-1}$, and hence $\lambda \in \mathbb{Z}$. Therefore, $s_{2k}(a) = 2\lambda s_{2k}(x_{2k}) = \lambda \cdot 2m_{2k}m_{2k-1}$, so $2m_{2k}m_{2k-1}$ is the minimal possible $s$-number in $\Omega_{4k}^SU$. □

Recall that the image of the forgetful homomorphism $\alpha: \Omega^SU \to \mathcal{W}$ is isomorphic to $\Omega^SU / \text{Tors}$ by Theorem 5.8, (a). Furthermore, by Theorem 5.11, (b), $\Omega_{2i}^SU / \text{Tors}$ is isomorphic to $\text{Ker}(\partial: \mathcal{W} \to \mathcal{W})$ if $2i \not\equiv 4 \mod 8$ and to $\text{Im}(\partial: \mathcal{W} \to \mathcal{W})$ if $2i \equiv 4 \mod 8$. Combining this with Theorem 7.1, we obtain a description of $\Omega^SU / \text{Tors}$ as a subring in $\mathcal{W}$. Finally, the multiplicative structure of the torsion elements is described by Theorem 5.11, (c). Collecting these pieces of information together, we obtain in principle a complete description of the ring $\Omega^SU$. However, as Stong noted at the end of Chap. X in [50], an intrinsic description of this ring is extremely complicated. For example, the non-trivial graded components of $\Omega^SU$ in dimension $\leq 10$ are described in terms of the elements $x_i$ and $y_i$ in Theorem 7.1 as follows:

$$
\begin{align*}
\Omega^SU_0 &= \mathbb{Z}; \\
\Omega^SU_1 &= \mathbb{Z}[\theta]; \\
\Omega^SU_2 &= \mathbb{Z}[\theta^2]; \\
\Omega^SU_4 &= \mathbb{Z}[y_2], \quad y_2 = 2x_1^2; \\
\Omega^SU_6 &= \mathbb{Z}[y_3], \quad y_3 = x_3; \\
\Omega^SU_8 &= \mathbb{Z}\left\langle \frac{1}{4}y_2^2, y_4 \right\rangle, \quad y_4 = 2x_4 - x_1x_3; \\
\Omega^SU_9 &= \mathbb{Z}[\theta x_4]; \\
\Omega^SU_{10} &= \mathbb{Z}\left\langle \frac{1}{2}y_2y_3, y_5 \right\rangle \oplus \mathbb{Z}[\theta^2x_1^4], \quad y_5 = x_5.
\end{align*}
$$

We have

$$y_2 = 2x_1^2 = 2(9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2])$$

as a $U$-bordism class. In dimension 8 we have

$$\frac{1}{4}x_1^4 = (9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2]) \times (9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2])$$

as a $U$-bordism class, because $x_1^2 = 9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2]$ is a $\partial$-cycle. Also, $y_2^2/4 = x_4^4$ can be chosen as $w_4$ in Theorem 5.11, (c). We see that 8 is the first dimension where $\Omega^SU / \text{Tors}$ differs from a polynomial ring, since the square of the 4-dimensional generator $y_2$ is divisible by 4. Moreover, the product of the 4- and 6-dimensional generators is divisible by 2.

Part II. Geometric representatives

8. Toric varieties and quasi-toric manifolds

Here we collect the necessary information about toric varieties and quasi-toric manifolds. Standard references on toric geometry include Danilov’s survey [24] and books by Oda [41], Fulton [26], and Cox, Little, and Schenck [23]. More information about quasi-toric manifolds can be found in [15], Chap. 6.

A toric variety is a normal complex algebraic variety $V$ containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on $V$. 
There is a fundamental correspondence of toric geometry between the isomorphism classes of complex $n$-dimensional toric varieties and rational fans in $\mathbb{R}^n$. Under this correspondence,

- cones $\leftrightarrow$ affine toric varieties,
- complete fans $\leftrightarrow$ complete (compact) toric varieties,
- normal fans of polytopes $\leftrightarrow$ projective toric varieties,
- non-singular fans $\leftrightarrow$ non-singular toric varieties,
- simplicial fans $\leftrightarrow$ toric orbifolds.

A fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of strongly convex cones $\sigma_j$ in $\mathbb{R}^n$ such that every face of a cone in $\Sigma$ belongs to $\Sigma$ and the intersection of any two cones in $\Sigma$ is a face of each. A fan is rational (with respect to the standard integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$) if each of its cones is generated by rational (or lattice) vectors. In particular, each one-dimensional cone of a rational fan $\Sigma$ is generated by a primitive vector $a_j \in \mathbb{Z}^n$. A fan $\Sigma$ is simplicial if each of its cones $\sigma_j$ is generated by part of a basis of $\mathbb{R}^n$ (such a cone is also said to be simplicial). A fan $\Sigma$ is non-singular if each of its cones $\sigma_j$ is generated by part of a basis of the lattice $\mathbb{Z}^n$. A fan $\Sigma$ is complete if the union of its cones is the whole of $\mathbb{R}^n$.

Projective toric varieties are particularly important for us. A projective toric variety $V$ is defined by a lattice polytope, that is, a convex $n$-dimensional polytope $P$ with vertices in $\mathbb{Z}^n$. The normal fan $\Sigma_P$ is the fan whose $n$-dimensional cones $\sigma_v$ correspond to the vertices $v$ of $P$, and each $\sigma_v$ is generated by the primitive inside-pointing normals to the facets of $P$ meeting at $v$. The fan $\Sigma_P$ defines a projective toric variety $V_P$. Different lattice polytopes with the same normal fan produce different projective embeddings of the same toric variety.

A polytope $P$ is said to be non-singular or Delzant when its normal fan $\Sigma_P$ is non-singular. Non-singular projective toric manifolds correspond to non-singular lattice polytopes. Note that a non-singular $n$-dimensional polytope $P$ is necessarily simple, that is, there are precisely $n$ facets meeting at every vertex of $P$.

Irreducible torus-invariant divisors on $V$ are the toric algebraic subvarieties of complex codimension 1 corresponding to the one-dimensional cones of $\Sigma$. When $V$ is projective, they also correspond to facets of $P$. Below we assume that there are $m$ one-dimensional cones (or facets), we let $a_1, \ldots, a_m$ be the corresponding primitive vectors, and we let $D_1, \ldots, D_m$ be the corresponding codimension-1 subvarieties (irreducible divisors).

**Theorem 8.1** (Danilov–Jurkiewicz). Let $V$ be a non-singular toric manifold of complex dimension $n$, with the corresponding complete non-singular fan $\Sigma$. Then the cohomology ring $H^*(V; \mathbb{Z})$ is generated by the degree-two classes $v_i$ dual to the invariant submanifolds $D_i$, and is given by

$$H^*(V; \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ such that $a_{i_1}, \ldots, a_{i_k}$ do not span a cone in $\Sigma$;

(b) $\sum_{i=1}^m \langle a_i, x \rangle v_i$ for any vector $x \in \mathbb{Z}^n$. 

There is the same description of the cohomology ring for complete toric orbifolds with coefficients in $\mathbb{Q}$.

It is convenient to consider the integer $n \times m$ matrix

$$\Lambda = \begin{pmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nm}
\end{pmatrix}, \tag{8.1}$$

whose columns are the vectors $a_i$ written in the standard basis of $\mathbb{Z}^n$. Then the ideal $\mathcal{I}$ in part (b) of Theorem 8.1 is generated by the $n$ linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of $\Lambda$.

Below, by a toric manifold we mean a non-singular complete (compact) toric variety.

**Theorem 8.2.** The following isomorphism of complex vector bundles holds for a toric manifold $V$:

$$\mathcal{T}V \oplus \mathbb{C}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\mathbb{C}^{m-n}$ is the trivial $(m-n)$-plane bundle, and $\rho_i$ is the line bundle corresponding to the divisor $D_i$, with $c_1(\rho_i) = v_i$. In particular, the total Chern class of $V$ is given by

$$c(V) = (1 + v_1) \cdots (1 + v_m).$$

**Example 8.3.** A basic example of a toric manifold is the complex projective space $\mathbb{C}P^n$. The cones of the corresponding fan are generated by proper subsets of the set of $m = n + 1$ vectors $e_1, \ldots, e_n, -e_1 - \cdots - e_n$, where $e_i \in \mathbb{Z}^n$ is the $i$th standard basis vector. This is the normal fan of the lattice simplex $\Delta^n$ with vertices at $0$ and $e_1, \ldots, e_n$. The matrix (8.1) is given by

$$\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & \ddots & 0 & \vdots \\
0 & \cdots & 1 & -1
\end{pmatrix}.$$  

Theorem 8.1 gives the cohomology of $\mathbb{C}P^n$ as

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[v_1, \ldots, v_{n+1}]/(v_1 \cdots v_{n+1}, v_1 - v_{n+1}, \ldots, v_n - v_{n+1}) \cong \mathbb{Z}[v]/(v^{n+1}),$$

where $v$ is any of the vectors $v_i$. Theorem 8.2 gives the standard decomposition

$$\mathcal{T}\mathbb{C}P^n \oplus \mathbb{C} \cong \eta \oplus \cdots \oplus \eta \quad (n + 1 \text{ summands}),$$

where $\eta = \mathcal{O}(-1)$ is the tautological (Hopf) line bundle over $\mathbb{C}P^n$, and $\bar{\eta} = \mathcal{O}(1)$ is its conjugate, or the line bundle corresponding to a hyperplane section $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

**Example 8.4.** The complex projectivisation of a sum of line bundles over a projective space is a toric manifold. This example will feature in several subsequent constructions.
Given two positive integers \( n_1 \) and \( n_2 \) and a sequence of integers \( (i_1, \ldots, i_{n_2}) \), consider the projectivisation

\[
V = \mathbb{C}P(\eta^\otimes i_1 \oplus \cdots \oplus \eta^\otimes i_{n_2} \oplus \mathbb{C}),
\]

where \( \eta^\otimes i \) denotes the \( i \)th tensor power of \( \eta \) over \( \mathbb{C}P^{n_1} \) when \( i \geq 0 \) and the \( i \)th tensor power of \( \bar{\eta} \) otherwise. The manifold \( V \) is the total space of a bundle over \( \mathbb{C}P^{n_1} \) with fibre \( \mathbb{C}P^{n_2} \). It is also a projective toric manifold with the corresponding matrix (8.1) given by

\[
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & \ddots & 0 & \vdots \\
0 & 0 & 1 & -1 \\
& & & \\
\end{pmatrix}
\]

The polytope \( P \) here is combinatorially equivalent to a product \( \Delta^{n_1} \times \Delta^{n_2} \) of two simplices. Theorem 8.1 describes the cohomology of \( V \) as

\[
H^*(V) \cong \mathbb{Z}[v_1, \ldots, v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2+2}]/\mathcal{I},
\]

where the ideal \( \mathcal{I} \) is generated by the elements

\[
v_1 \cdot \cdots v_{n_1+1}, \quad v_{n_1+2} \cdot \cdots v_{n_1+n_2+2}, \quad v_1 - v_{n_1+1}, \quad \ldots, \quad v_{n_1} - v_{n_1+1},
\]

\[
i_1 v_{n_1+1} + v_{n_1+2} - v_{n_1+n_2+2}, \quad \ldots, \quad i_{n_2} v_{n_1+1} + v_{n_1+n_2+1} - v_{n_1+n_2+2}.
\]

In other words,

\[
H^*(V) \cong \mathbb{Z}[u, v]/(u^{n_1+1}, v(v - i_1 u) \cdots (v - i_{n_2} u)),
\]

where \( u = v_1 = \cdots = v_{n_1+1} \) and \( v = v_{n_1+n_2+2} \). Theorem 8.2 gives us that

\[
c(V) = (1 + u)^{n_1+1}(1 + v - i_1 u) \cdots (1 + v - i_{n_2} u)(1 + v).
\]

If \( i_1 = \cdots = i_{n_2} = 0 \), then we obtain \( V = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \).

The same information can be retrieved from the following well-known description of the tangent bundle and the cohomology ring of a complex projectivisation.

**Theorem 8.5** (Borel and Hirzebruch [7], §15). Let \( p : \mathbb{C}P(\xi) \to X \) be the projectivisation of a complex \( n \)-plane bundle \( \xi \) over a complex manifold \( X \), and let \( \gamma \) be the tautological line bundle over \( \mathbb{C}P(\xi) \). Then there is an isomorphism of vector bundles

\[
\mathcal{T}\mathbb{C}P(\xi) \oplus \mathbb{C} \cong p^* \mathcal{T}X \oplus (\gamma \otimes p^* \xi).
\]

Moreover, the integral cohomology ring of \( \mathbb{C}P(\xi) \) is the quotient of the polynomial ring \( H^*(X)[v] \) on one generator \( v = c_1(\bar{\gamma}) \) with coefficients in \( H^*(X) \) by the single relation

\[
v^n + c_1(\xi)v^{n-1} + \cdots + c_n(\xi) = 0.
\]
The relation above is just $c_n(\tilde{\gamma} \otimes p^*\xi) = 0$.

In Example 8.4 we have the bundle $\xi = \eta^{G_{i_1}} \oplus \cdots \oplus \eta^{G_{i_{n_2}}} \oplus \mathbb{C}$ over $X = \mathbb{C}P^{n_1}$, and $H^*(X) = \mathbb{Z}[u]/(u^{n_1+1})$, where $u = c_1(\tilde{\eta})$, so that (8.4) becomes $\nu(v - i_1u) \cdots (v - i_{n_2}u) = 0$, and the ring $H^*(\mathbb{C}P(\xi))$ given by Theorem 8.5 is precisely (8.2). Moreover, the total Chern class of the bundle $p^*\mathcal{J}X \oplus (\tilde{\gamma} \otimes p^*\xi)$ is given by (8.3).

The quotient of the projective toric manifold $V_P$ by the action of the compact torus $T^n \subset (\mathbb{C}^\times)^n$ is the simple polytope $P$. Davis and Januszkiewicz [25] introduced the following topological generalisation of projective toric manifolds.

A quasi-toric manifold over a simple $n$-dimensional polytope $P$ is a smooth manifold $M$ of dimension $2n$ with a locally standard action of the torus $T^n$ and a continuous projection $\pi: M \to P$ whose fibres are $T^n$-orbits. (An action of $T^n$ on $M^{2n}$ is locally standard if every point $x \in M^{2n}$ is contained in a $T^n$-invariant neighbourhood which is equivariantly homeomorphic to an open subset of $\mathbb{C}^n$ with the standard coordinatewise action of $T^n$ twisted by an automorphism of the torus.) The orbit space of a locally standard action is a manifold with corners. The quotient $M/T^n$ of a quasi-toric manifold is homeomorphic to $P$ as a manifold with corners.

Not every simple polytope can be the quotient of a quasi-toric manifold. Nevertheless, quasi-toric manifolds constitute a much larger family than projective toric manifolds and enjoy much more flexibility for topological applications.

If $F_1, \ldots, F_m$ are the facets of $P$, then each $M_i = \pi^{-1}(F_i)$ is a quasi-toric submanifold of $M$ of codimension 2, called a characteristic submanifold. The characteristic submanifolds $M_i \subset M$ are analogues of the invariant divisors $D_i$ on a toric manifold $V$. Each $M_i$ is fixed pointwise by a closed 1-dimensional subgroup (a sub-circle) $T_i \subset T^n$ and therefore corresponds to a primitive vector $\lambda_i \in \mathbb{Z}^n$ defined up to a sign. Choosing a direction of $\lambda_i$ is equivalent to choosing an orientation of the normal bundle $\nu(M_i \subset M)$ or, equivalently, choosing an orientation for $M_i$, provided that $M$ itself is oriented. An omniorientation of a quasi-toric manifold $M$ consists of a choice of orientation of $M$ and each characteristic submanifold $M_i$, $1 \leq i \leq m$.

The vectors $\lambda_i$ play the role of the generators $a_i$ of the one-dimensional cones of the fan corresponding to a toric manifold $V$ (or the normal vectors to facets of $P$ when $V$ is projective). However, the $\lambda_i$ need not be the normal vectors to the facets of $P$ in general.

There is an analogue of Theorem 8.1 for quasi-toric manifolds.

**Theorem 8.6.** If $M$ is an omnioriented quasi-toric manifold of dimension $2n$ over a polytope $P$, then the cohomology ring $H^*(M; \mathbb{Z})$ is generated by the degree-two classes $v_i$ dual to the oriented characteristic submanifolds $M_i$, and is given by the isomorphism

$$H^*(M; \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ such that $F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset$ in $P$;
(b) $\sum_{i=1}^m (\lambda_i, x)v_i$ for all $x \in \mathbb{Z}^n$. 

By analogy with (8.1), we consider the integer \( n \times m \) matrix

\[
\Lambda = \begin{pmatrix}
\lambda_{11} & \cdots & \lambda_{1m} \\
\vdots & \ddots & \vdots \\
\lambda_{n1} & \cdots & \lambda_{nm}
\end{pmatrix}
\]  

(8.5)

whose columns are the vectors \( \lambda_i \) written in the standard basis of \( \mathbb{Z}^n \). Changing a basis in the lattice results in multiplying \( \Lambda \) on the left by a matrix in \( \text{GL}(n, \mathbb{Z}) \). The ideal (b) of Theorem 8.6 is generated by the \( n \) linear forms \( \lambda_j v_1 + \cdots + \lambda_j m v_m \) corresponding to the rows of \( \Lambda \). Moreover, \( \Lambda \) has the property that \( \det(\lambda_1, \ldots, \lambda_n) = \pm 1 \) whenever the corresponding facets \( F_1, \ldots, F_n \) intersect in a vertex of \( P \).

There is also an analogue of Theorem 8.2.

**Theorem 8.7.** For a quasi-toric manifold \( M \) of dimension \( 2n \), there is an isomorphism of real vector bundles

\[
\mathcal{T} M \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m,
\]

(8.6)

where \( \rho_i \) is the real 2-plane bundle corresponding to the orientation of the characteristic submanifold \( M_i \subset M \), so that \( \rho_i|_{M_i} = \nu(M_i \subset M) \).

Buchstaber and Ray introduced a family \( \{ B(n_1, n_2) \} \) of projective toric manifolds that generate the unitary bordism ring \( \Omega^U \) multiplicatively [18]. The details of this construction can be found in [15], §9.1. We proceed to describe another family of toric generators for \( \Omega^U \).

**Construction 8.8.** Given two positive integers \( n_1 \) and \( n_2 \), we define the manifold \( L(n_1, n_2) \) as the projectivisation \( \mathbb{C}P(\eta \oplus \mathbb{C}^{n_2}) \), where \( \eta \) is the tautological line bundle over \( \mathbb{C}P^{n_1} \). This \( L(n_1, n_2) \) is a particular case of the manifolds in Example 8.4, so it is a projective toric manifold, and the corresponding matrix (8.1) has the form

\[
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & \ddots & 0 & \vdots \\
0 & 0 & 1 & -1 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\]

(8.7)

The cohomology ring is given by

\[
H^*(L(n_1, n_2)) \cong \mathbb{Z}[u, v]/(u^{n_1+1}, v^{n_2+1} - uv^{n_2})
\]

(8.8)

with \( u^{n_1} v^{n_2} (L(n_1, n_2)) = 1 \). There is an isomorphism of complex bundles

\[
\mathcal{T} L(n_1, n_2) \oplus \mathbb{C}^2 \cong \bigoplus_{n_1+1} p^* \eta \oplus \cdots \oplus \bigoplus_{n_1+1} p^* \eta \oplus (\gamma \otimes p^* \eta) \oplus \bigoplus_{n_2} \gamma \oplus \cdots \oplus \gamma,
\]

(8.9)
where $\gamma$ is the tautological line bundle over $L(n_1, n_2) = \mathbb{C}P(\eta \oplus \mathbb{C}^{n_2})$. The total Chern class is

$$c(L(n_1, n_2)) = (1 + u)^{n_1+1}(1 + v - u)(1 + v)^{n_2}$$

with $u = c_1(p^*\eta)$ and $v = c_1(\tilde{\eta})$. We also let $L(n_1, 0) = \mathbb{C}P^{n_1}$ and $L(0, n_2) = \mathbb{C}P^{n_2}$, and then the equalities (8.8)–(8.10) still hold.

**Theorem 8.9** ([31], Theorem 3.8). The bordism classes $[L(n_1, n_2)] \in \Omega^U_{2(n_1+n_2)}$ generate the unitary bordism ring $\Omega^U$ multiplicatively.

Theorem 8.9 implies that every unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, since toric manifolds are connected. In bordism theory a disjoint union may be replaced by a connected sum representing the same bordism class. However, taking a connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic. This can be remedied by appealing to quasi-toric manifolds, as explained next. Recall that an omnioriented quasi-toric manifold has an intrinsic stably complex structure, arising from the isomorphism in Theorem 8.7. One can form the equivariant connected sum of quasi-toric manifolds, as explained by Davis and Januszkiewicz in [25], but the resulting invariant stably complex structure does not represent the sum of the bordism classes of the two original manifolds. A more intricate connected sum construction is needed, as outlined below. The details can be found in [16] or [15], §9.1.

**Construction 8.10.** This procedure applies to two omnioriented $2n$-dimensional quasi-toric manifolds $M$ and $M'$ over $n$-polytopes $P$ and $P'$, respectively. The connected sum will be taken at fixed points of $M$ and $M'$ corresponding to vertices $v \in P$ and $v' \in P'$. We need to assume that $v$ is the intersection of the first $n$ facets of $P$, that is, $v = F_1 \cap \cdots \cap F_n$, and the corresponding characteristic matrix (8.5) of $M$ is in the refined form, that is,

$$A = (I \mid A_*) = \begin{pmatrix} 1 & 0 & 0 & \cdots & \lambda_{1,n+1} & \cdots & \lambda_{1,m} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & \lambda_{n,n+1} & \cdots & \lambda_{n,m} \end{pmatrix},$$

where $I$ is the identity matrix and $A_*$ is an $n \times (m - n)$ matrix. The same assumptions are made for $M'$, $P'$, $v'$, and $A'$.

The next step depends on the **signs** of the fixed points, $\omega(v)$ and $\omega(v')$. The sign of $v$ is determined by the omniorientation data; it is $+1$ when the orientation of $T_vM$ induced from the global orientation of $M$ coincides with the orientation arising from $p_1 \oplus \cdots \oplus p_n|_v$, and it is $-1$ otherwise.

If $\omega(v) = -\omega(v')$, then we take the connected sum $M \# M'$ at $v$ and $v'$. It is a quasi-toric manifold over $P \# P'$ with the characteristic matrix $(A_* \mid I \mid A'_*)$.

If $\omega(v) = \omega(v')$, then we need an additional connected summand. Consider the quasi-toric manifold $S = S^2 \times \cdots \times S^2$ over the $n$-cube $I^n$, where each $S^2$ is the quasi-toric manifold over the segment $I$ with the characteristic matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$. It represents zero in $\Omega^U$, and may be thought of as $\mathbb{C}P^1$ with the stably complex
structure given by the isomorphism \( T \mathcal{CP}^1 \oplus \mathbb{R}^2 \cong \tilde{\eta} \oplus \eta \). The characteristic matrix of \( S \) is therefore \( (I \mid I) \). Now consider the connected sum \( M \# S \# M' \). This is a quasi-toric manifold over \( P \# I^n \# P' \) with characteristic matrix \( (\Lambda \mid I \mid I \mid \Lambda') \).

In either case, the resulting omnioriented quasi-toric manifold \( M \# M' \) or \( M \# S \# M' \) with the canonical stably complex structure represents the sum of bordism classes \([M] + [M'] \in \Omega^U_{2n}\).

The conclusion which can be derived from the above construction and either of the toric generating sets \( \{B(n_1, n_2)\} \) or \( \{L(n_1, n_2)\} \) for the ring \( \Omega^U \) is as follows.

**Theorem 8.11** (see [16]). In dimensions \( > 2 \), every unitary bordism class contains a quasi-toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.

### 9. Quasi-toric SU-manifolds

Omnioriented quasi-toric manifolds with \( SU \)-structures as stably complex structures can be detected using the following simple criterion.

**Proposition 9.1** (see [17]). An omnioriented quasi-toric manifold \( M \) has \( c_1(M) = 0 \) if and only if there exists a linear function \( \varphi: \mathbb{Z}^n \to \mathbb{Z} \) such that \( \varphi(\lambda_i) = 1 \) for \( i = 1, \ldots, m \). Here the \( \lambda_i \) are the columns of the matrix (8.5). In particular, if some \( n \) vectors among \( \lambda_1, \ldots, \lambda_m \) form the standard basis \( e_1, \ldots, e_n \), then \( M \) is an \( SU \)-manifold if and only if all the column sums of \( \Lambda \) are equal to 1.

**Proof.** By Theorem 8.7, \( c_1(M) = v_1 + \cdots + v_m \). By Theorem 8.6, \( v_1 + \cdots + v_m \) is equal to zero in \( H^2(M) \) if and only if \( v_1 + \cdots + v_m = \sum_i \varphi(\lambda_i)v_i \) for some linear function \( \varphi: \mathbb{Z}^n \to \mathbb{Z} \), whence the result follows. \( \square \)

**Proposition 9.2.** A toric manifold \( V \) cannot be an \( SU \)-manifold.

**Proof.** If \( \varphi(\lambda_i) = 1 \) for all \( i \), then the vectors \( \lambda_i \) lie in the positive half-space with respect to \( \varphi \), so they cannot span a complete fan. \( \square \)

The following more subtle result also rules out low-dimensional quasi-toric manifolds.

**Theorem 9.3** ([17], Theorem 6.13). A quasi-toric \( SU \)-manifold \( M^{2n} \) represents 0 in \( \Omega^U_{2n} \) if \( n < 5 \).

The reason for this is that the Krichever genus \( \varphi_K: \Omega^U \to R_K \) (see [15], §E.5) vanishes on quasi-toric \( SU \)-manifolds, but \( \varphi_K \) is an isomorphism in dimension \( < 10 \).

The first examples of quasi-toric \( SU \)-manifolds representing non-zero bordism classes in \( \Omega^U_{2n} \) for all \( n \geq 5 \) except \( n = 6 \) were constructed in [32]. Subsequently, two general series of quasi-toric \( SU \)-manifolds representing non-zero bordism classes in \( \Omega^U_{2n} \) (and therefore in \( \Omega^{SU}_{2n} \)) were constructed in [31] for all \( n \geq 5 \), including for \( n = 6 \). These series are presented next. They will be used below to provide geometric representatives for multiplicative generators in the \( SU \)-bordism ring.
Construction 9.4. Assume now that $n_1 = 2k_1$ is a positive even integer and $n_2 = 2k_2 + 1$ is positive odd. We change the stably complex structure (8.9) to

$$TL(n_1, n_2) \oplus \mathbb{R}^4 \cong p^*\bar{\eta} \oplus p^*\eta \oplus \cdots \oplus p^*\bar{\eta} \oplus p^*\eta \oplus (\bar{\gamma} \otimes p^*\eta) \oplus \bar{\gamma} \oplus \gamma \oplus \cdots \oplus \bar{\gamma} \oplus \gamma \oplus \gamma,$$

and we denote the resulting stably complex manifold by $\tilde{L}(n_1, n_2)$. Its cohomology ring is given by the same formula (8.8), but

$$c(\tilde{L}(n_1, n_2)) = (1 - u^2)^{k_1}(1 + u)(1 + v - u)(1 - v^2)^{k_2}(1 - v), \quad (9.1)$$

so $\tilde{L}(n_1, n_2)$ is an $SU$-manifold of dimension $2(n_1 + n_2) = 4(k_1 + k_2) + 2$.

Viewing $L(n_1, n_2)$ as a quasi-toric manifold with the omniorientation coming from the complex structure, we see that changing a line bundle $\rho_i$ in (8.6) to its conjugate results in changing $\lambda_i$ to $-\lambda_i$ in (8.5). Applying this operation to the corresponding columns of (8.7) and then multiplying on the left by a suitable matrix in $GL(n, \mathbb{Z})$, we find that $\tilde{L}(n_1, n_2)$ is the omnioriented quasi-toric manifold over $\Delta^{n_1} \times \Delta^{n_2}$ corresponding to the characteristic matrix

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & -1 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
0 \\
\end{pmatrix}.
$$

The column sums of this matrix are obviously 1.

Construction 9.5. The previous construction can be iterated by considering projectivisations of sums of line bundles over $L(n_1, n_2)$. We shall need just one particular family of this sort.
Lemma 10.1. For given positive even \( n_1 = 2k_1 \) and odd \( n_2 = 2k_2 + 1 \), consider the omnioriented quasi-toric manifold \( \tilde{N}(n_1, n_2) \) over \( \Delta^1 \times \Delta^{n_1} \times \Delta^{n_2} \) with the characteristic matrix

\[
\begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

The column sums are again obviously 1, so \( \tilde{N}(n_1, n_2) \) is a quasi-toric \( SU \)-manifold of dimension \( 2(1 + n_1 + n_2) = 4(k_1 + k_2) + 4 \).

It can be seen that \( \tilde{N}(n_1, n_2) \) is the projectivisation of a sum of \( n_2 + 1 \) line bundles over \( \mathbb{C}P^1 \times \mathbb{C}P^{n_2} \) with an amended stably complex structure.

The cohomology ring in Theorem 8.6 is

\[ H^*(\tilde{N}(n_1, n_2)) \cong \mathbb{Z}[u, v, w]/(u^2, v^{n_1+1}, (w - u)^2(v + w)w^{n_2-2}) \]  

(9.2)

with \( vw^{n_1}w^{n_2}(\tilde{N}(n_1, n_2)) = 1 \). The total Chern class is

\[ c(\tilde{N}(n_1, n_2)) = (1 - v^2)^k_1(1 + v)(1 - (w - u)^2)(1 - v - w)(1 - w^2)^{k_2 - 1}(1 + w). \]  

(9.3)

10. Quasi-toric generators for the \( SU \)-bordism ring

As shown in [31], the elements \( y_i \in \Omega^{SU}_{2i} \) described in Theorem 7.1 can be represented by quasi-toric \( SU \)-manifolds when \( i \geq 5 \). We outline the proof here, emphasising some interesting divisibility properties for binomial coefficients. These divisibility properties arise from an analysis of the characteristic numbers of the quasi-toric \( SU \)-manifolds \( \tilde{L}(n_1, n_2) \) and \( \tilde{N}(n_1, n_2) \) introduced in the previous section.

Lemma 10.1. For \( n_1 = 2k_1 > 0 \) and \( n_2 = 2k_2 + 1 > 0 \),

\[ s_{n_1+n_2}[\tilde{L}(n_1, n_2)] = -\binom{n_1 + n_2}{1} + \binom{n_1 + n_2}{2} - \cdots - \binom{n_1 + n_2}{n_1 - 1} + \binom{n_1 + n_2}{n_1}. \]

Proof. Using (9.1) and (8.8), we calculate

\[
s_{n_1+n_2}(\tilde{L}(n_1, n_2)) = (v - u)^{n_1+n_2} + (k_2 + 1)(-1)^{n_1+n_2}v^{n_1+n_2} + k_2v^{n_1+n_2} \\
= (v - u)^{n_1+n_2} - v^{n_1+n_2} \\
= \left(-\binom{n_1 + n_2}{1} + \binom{n_1 + n_2}{2} - \cdots - \binom{n_1 + n_2}{n_1 - 1} + \binom{n_1 + n_2}{n_1}\right)u^{n_1}v^{n_2},
\]

and the result follows by evaluation on the fundamental class of \( \tilde{L}(n_1, n_2) \). \( \square \)
Note that $s_3(\tilde{L}(2, 1)) = 0$ in accordance with Theorem 9.3. On the other hand, $s_{2+n_2}(\tilde{L}(2, n_2)) \neq 0$ for $n_2 > 1$, providing an example of a non-bounding quasi-toric $SU$-manifold in each dimension $4k + 2$ with $k > 1$.

**Lemma 10.2.** For $k > 1$, there is a linear combination $y_{2k+1}$ of $SU$-bordism classes $[\tilde{L}(n_1, n_2)]$ with $n_1 + n_2 = 2k + 1$ such that $s_{2k+1}(y_{2k+1}) = m_{2k+1}m_{2k}$.

**Proof.** By the previous lemma,

$$s_{n_1+n_2}[\tilde{L}(n_1, n_2) - \tilde{L}(n_1 - 2, n_2 + 2)] = \left( \begin{array}{c} n_1 + n_2 \\ n_1 \end{array} \right) - \left( \begin{array}{c} n_1 + n_2 \\ n_1 - 1 \end{array} \right).$$

The result follows from the next lemma. □

**Lemma 10.3** ([31], Lemma 4.14). For any integer $k > 1$,

$$\gcd\left\{ \left( \frac{2k + 1}{2i} \right) - \left( \frac{2k + 1}{2i - 1} \right), \ 0 < i \leq k \right\} = m_{2k+1}m_{2k}.$$

Lemma 10.3 also follows from the results of Buchstaber and Ustinov on the coefficient rings of universal formal group laws ([19], §9).

Now we turn our attention to the manifolds $\tilde{N}(n_1, n_2)$ in Construction 9.5.

**Lemma 10.4.** For $n_1 = 2k_1 > 0$ and $n_2 = 2k_2 + 1 > 0$, let $n = n_1 + n_2 + 1$, so that $\dim \tilde{N}(n_1, n_2) = 2n = 4(k_1 + k_2 + 1)$. Then

$$s_n[\tilde{N}(n_1, n_2)] = 2\left( -\left( \begin{array}{c} n \\ 1 \end{array} \right) + \left( \begin{array}{c} n \\ 2 \end{array} \right) - \cdots - \left( \begin{array}{c} n \\ n_1 - 1 \end{array} \right) + \left( \begin{array}{c} n \\ n_1 \end{array} \right) - n_1 \right).$$

**Proof.** Using (9.3) and (9.2), we calculate

$$s_n(\tilde{N}(n_1, n_2)) = 2(w - u)^n + (v + w)^n + (2k_2 - 1)w^n$$

$$= 2w^n - 2nuw^{n-1} + w^n + \left( \begin{array}{c} n \\ 1 \end{array} \right) vw^{n-1} + \cdots + \left( \begin{array}{c} n \\ 2k_1 \end{array} \right) v^{2k_1}w^{2k_2+2} + (2k_2 - 1)w^n$$

$$= -2nuw^{n-1} + (n - n_1)w^n + \left( \begin{array}{c} n \\ 1 \end{array} \right) vw^{n-1} + \cdots + \left( \begin{array}{c} n \\ n_1 \end{array} \right) v^{n_1}w^{n-n_1}.$$

(10.1)

Now we have to express each monomial above in terms of $uw^{n_1}w^{n_2}$, using the equalities in (9.2), namely,

$$u^2 = 0, \quad v^{n_1+1} = 0, \quad w^{n_2+1} = 2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}. \quad \text{(10.2)}$$

We have

$$uw^{n-1} = uw^{n_1-1}w^{n_2+1} = uw^{n_1-1}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1})$$

$$= -uvw^{n-2} = \cdots = (-1)^j uv^j w^{n-j-1} = \cdots = uw^{n_1}w^{n_2}. \quad \text{(10.3)}$$
We also show that

\[ v^j w^{n-j} = (-1)^j \cdot 2uv^{n_1}w^{n_2}, \quad 0 \leq j \leq n_1, \tag{10.4} \]

by verifying this equality successively for \( j = n_1, n_1 - 1, \ldots, 0 \). Indeed, \( v^{n_1}w^{n-n_1} = v^{n_1}w^{n_2+1} = 2uv^{n_1}w^{n_2} \) by (10.2). We have

\[ v^{j-1}w^{n-j+1} = v^{j-1}w^{n_1+1-j}w^{n_2+1} = v^{j-1}w^{n_1+1-j}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}) \]

\[ = 2uv^{j-1}w^{n-j} - v^j w^{n-j} + 2uv^j w^{n-1-j} = -v^j w^{n-j}, \]

where the last equality holds because of (10.3). The equality (10.4) is therefore completely verified. Using (10.3) and (10.4) in (10.1), we obtain

\[ s_n(\tilde{N}(n_1, n_2)) \]

\[ = (-2n + 2(n - n_1) - 2\left(\frac{n}{1}\right) + 2\left(\frac{n}{2}\right) - \cdots - 2\left(\frac{n}{n_1 - 1}\right) + 2\left(\frac{n}{n_1}\right))uv^{n_1}w^{n_2}. \]

The result follows by evaluation on the fundamental class \( \langle \tilde{N}(n_1, n_2) \rangle \). \( \square \)

Note that \( s_4(\tilde{N}(2, 1)) = 0 \) by Theorem 9.3. On the other hand, \( s_n(\tilde{N}(2, n_2)) = n^2 - 3n - 4 > 0 \) for \( n > 4 \), providing an example of a non-bounding quasi-toric \( SU \)-manifold in each dimension \( 4k \) with \( k > 2 \). This includes a 12-dimensional quasi-toric \( SU \)-manifold \( \tilde{N}(2, 3) \), which was missing in [32].

**Lemma 10.5.** For \( k > 2 \), there is a linear combination \( y_{2k} \) of \( SU \)-bordism classes \( [\tilde{N}(n_1, n_2)] \) with \( n_1 + n_2 + 1 = 2k \) such that \( s_{2k}(y_{2k}) = 2m_{2k}m_{2k-1} \).

**Proof.** This follows from Lemma 10.4 and Lemmas 10.6 and 10.7 below. \( \square \)

**Lemma 10.6** ([31], Lemma 4.17). For \( k > 2 \) the largest power of 2 that divides each number

\[ a_i = -\left(\frac{2k}{1}\right) + \left(\frac{2k}{2}\right) - \cdots - \left(\frac{2k}{2i-1}\right) + \left(\frac{2k}{2i}\right) - 2i, \quad 0 < i < k, \]

is 2 for \( 2k = 2^s \) and is 1 otherwise.

**Lemma 10.7** ([31], Lemma 4.18). For \( k > 2 \), the largest power of an odd prime \( p \) that divides each number

\[ a_i = -\left(\frac{2k}{1}\right) + \left(\frac{2k}{2}\right) + \cdots - \left(\frac{2k}{2i-1}\right) + \left(\frac{2k}{2i}\right) - 2i, \quad 0 < i < k, \]

is \( p \) for \( 2k + 1 = p^s \) and is 1 otherwise.

We now get a result about quasi-toric representatives in the \( SU \)-bordism ring.

**Theorem 10.8.** There are quasi-toric \( SU \)-manifolds \( M^{2i} \), \( i \geq 5 \), with \( s_i(M^{2i}) = m_i m_{i-1} \) for odd \( i \) and \( s_i(M^{2i}) = 2m_i m_{i-1} \) for even \( i \), and they have the minimum possible characteristic numbers \( s_i \) and represent polynomial generators of \( \Omega^*SU \otimes \mathbb{Z}[\frac{1}{2}] \).
Proof. It follows from Lemmas 10.2 and 10.5 that there exist linear combinations of $SU$-bordism classes represented by quasi-toric $SU$-manifolds with the required properties. By applying Construction 8.10 to two quasi-toric $SU$-manifolds $M$ and $M'$ we get a quasi-toric $SU$-manifold representing their bordism sum. Also, the $SU$-bordism class $-[M]$ can be represented by the omnioriented quasi-toric $SU$-manifold obtained by reversing the global orientation of $M$. Therefore, we can replace the linear combinations obtained using Lemmas 10.2 and 10.5 by suitable connected sums which are quasi-toric $SU$-manifolds.

By analogy with Theorem 8.11, we may ask the following.

**Question 10.9.** Which $SU$-bordism classes of dimension $> 8$ can be represented by quasi-toric $SU$-manifolds?

### 11. $SU$-manifolds arising in toric geometry

We refer to a compact Kähler manifold $M$ with $c_1(M) = 0$ as a *Calabi–Yau manifold*. (Apparently, this is the most standard definition, though there are other definitions of Calabi–Yau manifolds in the literature, sometimes not equivalent to this one.) According to a theorem of Yau conjectured by Calabi, a Calabi–Yau manifold admits a Kähler metric with Ricci curvature zero (only the vanishing of the first real Chern class is required for this). By definition, a Calabi–Yau manifold is an $SU$-manifold.

The standard complex structure on a toric manifold is never an $SU$-structure (Proposition 9.2), so there are no toric Calabi–Yau manifolds. However, the following construction gives Calabi–Yau hypersurfaces in special toric manifolds.

**Construction 11.1** (Batyrev [6]). A toric manifold $V$ is a Fano manifold if its anticanonical class $D_1 + \cdots + D_m$ (representing $c_1(V)$) is very ample. In geometric terms, the projective embedding $V \hookrightarrow \mathbb{CP}^s$ corresponding to $D_1 + \cdots + D_m$ comes from a lattice polytope $P$ in which the lattice distance from 0 to each hyperplane containing a facet is 1. Such a lattice polytope $P$ is said to be reflexive; its polar polytope $P^*$ is also a lattice polytope.

The submanifold $N$ dual to $c_1(V)$ (see Construction 6.1) is given by the hyperplane section of the embedding $V \hookrightarrow \mathbb{CP}^s$ defined by the divisor $D_1 + \cdots + D_m$. Therefore, $N \subset V$ is a smooth algebraic hypersurface in $V$, so $N$ is a Calabi–Yau manifold of complex dimension $n - 1$.

Thus, any $n$-dimensional toric Fano manifold $V$ (or equivalently, any non-singular reflexive $n$-dimensional polytope $P$) gives rise to a canonical $(n - 1)$-dimensional Calabi–Yau manifold $N_P$.

Batyrev [6] also extended this construction to some singular toric Fano varieties. A complex normal irreducible $n$-dimensional projective algebraic variety $W$ with only Gorenstein canonical singularities is called a *Calabi–Yau variety* if $W$ has a trivial canonical bundle and $H^i(W, \mathcal{O}_W) = 0$ for $0 < i < n$.

Suppose that $f$ is a Laurent polynomial in $n$ variables, and let $P = P(f)$ be its Newton polytope (the convex hull of the lattice points corresponding to the non-zero coefficients of $f$). Then $f$ defines an affine hypersurface $Z_f$ in the algebraic torus $(\mathbb{C}^\times)^n$, and its Zariski closure $\overline{Z}_{f,P}$ is a hypersurface in the projective toric variety $V_P$. The hypersurface $\overline{Z}_{f,P}$ is said to be $P$-regular if it intersects each
The multinomial coefficient $\binom{\omega}{\alpha_1, \ldots, \alpha_k}$ is defined for each $\omega = (\alpha_1, \ldots, \alpha_k)$. Let $\alpha(\omega) = \binom{n}{\omega} (i_1 + 1)^{i_1} \cdots (i_k + 1)^{i_k}$. (12.1)

**Lemma 12.1.** For any $\omega \in \hat{P}(n)$

$$s_{n-1}(N_\omega) = -\alpha(\omega).$$
Proof. The cohomology ring of $\mathbb{C}P^\omega = \mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}$ is given by

$$H^*(\mathbb{C}P^\omega; \mathbb{Z}) \cong \mathbb{Z}[u_1, \ldots, u_k]/(u_1^{i_1+1}, \ldots, u_k^{i_k+1}),$$

where

$$u_1 := v_1 = \cdots = v_{i_1+1}, \quad u_2 := v_{i_1+2} = \cdots = v_{i_1+i_2+2}, \ldots,$$

$$u_k := v_{i_1+\cdots+i_{k-1}+k} = \cdots = v_{i_1+\cdots+i_k+k} = v_m.$$

Since $\omega \in \hat{P}(n)$, we have $v_i^{n-1} = 0$ in the ring $H^*(\mathbb{C}P^\omega; \mathbb{Z})$ for any $i$. The formula in Lemma 11.2 gives us that

$$s_{n-1}(N_\omega) = -((v_1 + \cdots + v_m)^n, [\mathbb{C}P^\omega]) = - ((i_1 + 1)u_1 + \cdots + (i_k + 1)u_k)^n, [\mathbb{C}P^\omega]).$$

The value on the fundamental class $[\mathbb{C}P^\omega]$ equals the coefficient of the monomial $u_1^{i_1} \cdots u_k^{i_k}$ in the above polynomial, whence the result follows. □

Lemma 12.2 ([30], Lemma 2.3). For $n \geq 3$,

$$\gcd_{\omega \in \hat{P}(n)} \alpha(\omega) = g(n),$$

where the numbers $g(n)$ and $\alpha(\omega)$ are given by (7.2) and (12.1) respectively.

The proof of this lemma given in [30] uses results due to Mosley [36] on the divisibility of multinomial coefficients.

Theorem 12.3. The $SU$-bordism classes of Calabi–Yau hypersurfaces $N_\omega$ in the toric manifolds $\mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}$ with $\omega \in \hat{P}(n)$, $n \geq 3$, multiplicatively generate the $SU$-bordism ring $\Omega^{SU}[\frac{1}{2}]$.

Proof. For any $n \geq 3$ we use Lemmas 12.2 and 12.1 to find a linear combination of the bordism classes $[N_\omega] \in \Omega^{SU}_{2n-2}$ whose $s$-number is precisely $g(n)$. This linear combination is the polynomial generator $y_{n-1}$ in $\Omega^{SU}[\frac{1}{2}]$ described in Theorem 7.1. □

We actually prove an integral result: the elements $y_i \in \Omega^{SU}$ described in Theorem 7.1 can be represented by integral linear combinations of the bordism classes of Calabi–Yau manifolds $N_\omega$. The element $y_i$ is part of a basis of the Abelian group $\Omega^{SU}_{2i}$. There arises the following related question.

Question 12.4. Which bordism classes in $\Omega^{SU}$ can be represented by Calabi–Yau manifolds?
note that it uses hypersurfaces in $CP^n$ and a calculation similar to Lemma 11.2. For $SU$-bordism, the situation is different: if a class $a \in \Omega^{SU}$ can be represented by a Calabi–Yau manifold, then $-a$ does not necessarily have this property. Therefore, the next step towards answering the question above is whether $y_i$ and $-y_i$ can simultaneously be represented by Calabi–Yau manifolds. We elaborate on this in the next section.

13. Low-dimensional generators in the $SU$-bordism ring

Here we describe geometric Calabi–Yau representatives for the generators $y_i$ of the $SU$-bordism ring (see Theorem 7.1) in complex dimension $i \leq 4$. For $i \geq 5$, each generator $y_i \in \Omega^{SU}_{2i}$ can be represented by a quasi-toric manifold, by Theorem 10.8. On the other hand, every quasi-toric $SU$-manifold of real dimension $\leq 8$ is null-bordant by Theorem 9.3.

Recall from §7 that

$$\Omega^S_4 = \mathbb{Z}\langle y_2 \rangle, \quad \Omega^S_6 = \mathbb{Z}\langle y_3 \rangle, \quad \Omega^S_8 = \mathbb{Z}\langle \frac{1}{4} y_2^2, y_4 \rangle,$$

with the values of the $s$-numbers given by

$$s_2(y_2) = -48, \quad s_3(y_3) = m_3m_2 = 6, \quad \text{and} \quad s_4(y_4) = 2m_4m_3 = 20.$$

**Example 13.1.** Consider the Calabi–Yau hypersurface $N_{(3)} \subset CP^3$ corresponding to the partition $\omega = (3)$. We have $c_1(CP^3) = 4u$, where $u \in H^2(CP^3; \mathbb{Z})$ is the canonical generator dual to a hyperplane section. Therefore, $N_{(3)}$ can be given by a generic quartic equation in homogeneous coordinates on $CP^3$. The standard example is the quartic given by $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, which is a $K3$-surface. Lemma 11.2 gives us that

$$s_3(N_{(3)}) = \langle 4u^2 \cdot 4u - (4u)^3, [CP^3] \rangle = -48,$$

so $N_{(3)}$ represents the generator $y_2 \in \Omega^S_{4SU}$.

We note that Theorem 12.3 gives us another representative for the same generator $y_2$. Namely, the only partition of $n = 3$ that belongs to $\tilde{P}(n)$ is $(1, 1, 1)$. The corresponding Calabi–Yau surface is $N_{(1,1,1)} \subset CP^1 \times CP^1 \times CP^1$. We have

$$c_1(CP^1 \times CP^1 \times CP^1) = 2u_1 + 2u_2 + 2u_3,$$

so $N_{(1,1,1)}$ is a surface of multidegree $(2, 2, 2)$ in $CP^1 \times CP^1 \times CP^1$. Lemma 11.2 gives $s_3(N_{(1,1,1)}) = -\alpha(1, 1, 1) = -48$, and hence $N_{(1,1,1)}$ also represents $y_2$.

On the other hand, the additive generator $-y_2 \in \Omega^S_{4SU}$ cannot be represented by a compact complex surface. This was proved in [37], Theorem 3.2.5, by analysing the classification results on complex surfaces. It is easy to see that a complex surface $S$ with $H^1(S; \mathbb{Z}) = 0$ (which holds for Calabi–Yau surfaces arising from toric Fano varieties) cannot represent $-y_2$. Indeed, such an $S$ has Euler characteristic $c_2(S) = \chi(S) \geq 2$, while $s_2(-y_2) = 48 = -2c_2(-y_2)$, so $c_2(-y_2) = -24$ is negative.

**Example 13.2.** The 6-dimensional sphere $S^6$ has a $T^2$-invariant almost complex structure arising from its identification with the homogeneous space $G_2/SU(3)$ of the exceptional Lie group $G_2$ (see [7], §13). Therefore, $S^6$ is an $SU$-manifold with $s_3[S^6] = 3c_3[S^6] = 6$. Hence, the $SU$-bordism class $[S^6]$ can be taken as $y_3$. 

Example 13.3. Here we show that the generator $-y_4 \in \Omega_8^{SU}$ can be represented by the Grassmannian $\text{Gr}_2(\mathbb{C}^4)$ of 2-planes in $\mathbb{C}^4$ with an amended stably complex structure.

Let $\gamma$ be the tautological 2-plane bundle over $\text{Gr}_2(\mathbb{C}^4)$, and let $\gamma^\perp$ be the orthogonal 2-plane bundle. Then $\mathcal{T} \text{Gr}_2(\mathbb{C}^4) \cong \text{Hom}(\gamma, \gamma^\perp)$ and

$$\mathcal{T} \text{Gr}_2(\mathbb{C}^4) \oplus \text{Hom}(\gamma, \gamma) \cong \text{Hom}(\gamma, \gamma^\perp \oplus \gamma) \cong \text{Hom}(\gamma, \mathbb{C}^4) \cong \gamma \oplus \bar{\gamma} \oplus \bar{\gamma} \oplus \gamma.$$

The standard complex structure on $\text{Gr}_2(\mathbb{C}^4)$ is therefore given by the stable bundle isomorphism

$$\mathcal{T} \text{Gr}_2(\mathbb{C}^4) \cong 4\bar{\gamma} - \gamma \gamma,$$

where we write $4\bar{\gamma} = \bar{\gamma} \oplus \bar{\gamma} \oplus \bar{\gamma} \oplus \gamma$ and $\gamma \gamma = \gamma \otimes \gamma = \text{Hom}(\gamma, \gamma)$. We change the stable complex structure to

$$\mathcal{T} \text{Gr}_2(\mathbb{C}^4) \cong 2\bar{\gamma} + 2\gamma - \gamma \gamma$$

and we let $\widetilde{\text{Gr}}_2(\mathbb{C}^4)$ be the resulting stably complex manifold. Since $c_1(\widetilde{\text{Gr}}_2(\mathbb{C}^4)) = 0$, $\widetilde{\text{Gr}}_2(\mathbb{C}^4)$ is an $SU$-manifold. It has the same cohomology ring as the Grassmannian,

$$H^*(\text{Gr}_2(\mathbb{C}^4)) \cong \mathbb{Z}[c_1, c_2]/(c_1^3 = 2c_1c_2, c_2^2 = c_1^2c_2),$$

where $c_i = c_i(\gamma)$. The top-degree cohomology group $H^8(\text{Gr}_2(\mathbb{C}^4)) \cong \mathbb{Z}$ is generated by $c_1^2c_2$.

Now let us calculate the class $s_4(\widetilde{\text{Gr}}_2(\mathbb{C}^4)) = 2s_4(\gamma) + 2s_4(\gamma^\perp) - s_4(\gamma \gamma)$. We have

$$s_4 = c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4,$$

so that

$$s_4(\gamma) = s_4(\gamma) = c_1^4 - 4c_1^2c_2 + 2c_2^2 = 2c_1^2c_2 - 4c_1^2c_2 + 2c_1^2c_2 = 0.$$ 

It remains to calculate $s_4(\gamma \otimes \gamma)$. Using the splitting principle, we write $\gamma = \eta_1 + \eta_2$ for line bundles $\eta_1$ and $\eta_2$, and we calculate

$$c(\gamma \gamma) = c(\eta_1 + \eta_2)(\eta_1 + \eta_2) = c(\eta_1\eta_2 + \eta_2\eta_1) = c(\eta_1\eta_2)c(\eta_2\eta_1) = (1 - c_1(\eta_1) + c_1(\eta_2))(1 - c_1(\eta_2) + c_1(\eta_1)) = 1 - c_1(\eta_1)^2 - c_1(\eta_2)^2 + 2c_1(\eta_1)c_1(\eta_2) = 1 - c_1(\eta_1)^2 + 4c_1(\eta_1)c_1(\eta_2) = 1 - c_1(\gamma)^2 + 4c_2(\gamma).$$

Hence $c_1(\gamma \gamma) = c_3(\gamma \gamma) = c_4(\gamma \gamma) = 0$, and

$$s_4(\gamma \gamma) = 2c_2(\gamma \gamma)^2 = 2(4c_2 - c_1^2)^2 = 2(16c_2^2 - 8c_1^2c_2 + c_1^4) = 20c_1^2c_2.$$ 

It follows that $s_4[\widetilde{\text{Gr}}_2(\mathbb{C}^4)] = -20$ and $[\widetilde{\text{Gr}}_2(\mathbb{C}^4)] = -y_4 \in \Omega_8^{SU}$.

Example 13.4. Theorem 12.3 gives us the following representatives for the generators $y_3 \in \Omega_6^{SU}$ and $y_4 \in \Omega_8^{SU}$:

$$y_3 = 15N_{(2,2)} - 19N_{(1,1,1,1)} \quad \text{and} \quad y_4 = 56N_{(1,1,3)} - 59N_{(1,2,2)}.$$
Unlike the situation in complex dimension 2, both \( y_3 \) and \(-y_3\) can be represented by Calabi–Yau manifolds. The same holds in complex dimension 4, as the next theorem shows.

**Theorem 13.5.** The following statements hold.

(a) In complex dimension 2 the class \(-y_2 \in \Omega^2_{SU}\) can be represented by a Calabi–Yau surface \( M \). One can take as \( M \) any \( K3\)-surface different from a torus; it has Euler characteristic \( \chi(M) = 24 \) and

\[
h^{1,1}(M) = 20.
\]

The class \( y_2 \in \Omega^2_{SU} \) cannot be represented by a Calabi–Yau surface.

(b) In complex dimension 3 both the \( SU\)-bordism classes \( y_3 \) and \(-y_3\) can be represented by Calabi–Yau 3-folds. These 3-folds \( M \) can be obtained using Batyrev’s construction from Fano toric varieties over reflexive 4-polytopes. Such an \( M \) represents the class \( y_3 \in \Omega^3_{SU} \) if \( \chi(M) = 2 \) or, equivalently,

\[
h^{1,1}(M) - h^{2,1}(M) = 1.
\]

Similarly, \( M \) represents the class \(-y_3 \in \Omega^3_{SU} \) if \( \chi(M) = -2 \) or, equivalently,

\[
h^{1,1}(M) - h^{2,1}(M) = -1.
\]

(c) In complex dimension 4, both \( SU\)-bordism classes \( y_4 \) and \(-y_4\) can be represented by Calabi–Yau 4-folds. These 4-folds \( M \) can be obtained using Batyrev’s construction from Fano toric varieties over reflexive 5-polytopes. Such an \( M \) represents \( y_4 \in \Omega^4_{SU} \) if \( \chi(M) = 282 \) or, equivalently,

\[
h^{1,1}(M) - h^{2,1}(M) + h^{3,1}(M) = 39.
\]

Similarly, \( M \) represents \(-y_4 \in \Omega^4_{SU} \) if \( \chi(M) = 294 \) or, equivalently,

\[
h^{1,1}(M) - h^{2,1}(M) + h^{3,1}(M) = 41.
\]

**Proof.** We denote both the Chern characteristic classes and the Chern characteristic numbers of \( M \) by \( c_i \) throughout this proof, we denote the Hodge numbers by \( h^{i,j} \), and we denote the (real) Betti numbers by \( b^i \) for \( i = 0, \ldots, \dim_{\mathbb{C}} M \). For a Kähler \( n\)-manifold \( M \) we have \( h^{p,q} = h^{q,p} \) (Hodge duality), \( b^i = \sum_{p+q=i} h^{p,q} \), and

\[
\chi(M) = \sum_{i=0}^{2n} (-1)^i b^i = \sum_{p+q=0} h^{p,q}.
\]

Furthermore, a Calabi–Yau manifold \( M \) obtained from Batyrev’s construction is projective algebraic, so it satisfies the equality \( h^{p,q} = h^{n-p,n-q} \) (Serre duality). Finally, the holonomy group of such a Calabi–Yau manifold \( M \) is the full group \( SU(n) \), and therefore \( h^{n,0} = 1 \) and \( h^{i,0} = 0 \) for \( 0 < i < n \) (see [6], Theorem 4.1.9).

Statement (a) is a summary of Example 13.1.

We prove (b). For the generator \( y_3 \in \Omega^3_{SU} \) we have \( 6 = s_3(y_3) = 3c_3(y_3) \), so the Euler characteristic of a complex \( SU\)-manifold \( M \) representing \( y_3 \) satisfies the
equality $\chi(N) = c_3(N) = 2$. For a Calabi–Yau 3-fold $M$ obtained from Batyrev’s construction we have

$$b^1 = 2h^{1,0} = 0, \quad b^2 = 2h^{2,0} + h^{1,1} = h^{1,1}, \quad b^3 = 2h^{3,0} + 2h^{2,1} = 2 + 2h^{2,1},$$

and

$$\chi(M) = 2b^0 - 2b^1 + 2b^2 - b^3 = 2(h^{1,1} - h^{2,1}).$$

It follows that $M$ represents $y_3$ if and only if $h^{1,1} - h^{2,1} = 1$. Similarly, $M$ represents $-y_3$ if and only if $h^{1,1} - h^{2,1} = -1$.

The fact that such $M$ exist follows by analysing the database [28] (see also [1]) of reflexive polytopes and Calabi–Yau hypersurfaces in their corresponding toric Fano varieties. This database contains the complete list of 473 800 776 reflexive polytopes and Calabi–Yau hypersurfaces in their corresponding toric Fano 3-folds. From there one can see that for each $h^{1,1}$ with $16 \leq h^{1,1} \leq 90$ there exists a reflexive 4-polytope with the corresponding Calabi–Yau 3-fold satisfying $h^{1,1} - h^{2,1} = 1$. But if $h^{1,1}$ is not in this interval, then there is no Calabi–Yau 3-fold with $h^{1,1} - h^{2,1} = 1$ coming from a toric Fano variety. In the case of the equality $h^{1,1} - h^{2,1} = -1$ the possible interval is $15 \leq h^{1,1} \leq 89$.

We note also that the Calabi–Yau 3-folds $M$ and $M^*$ representing $y_3$ and $-y_3$ can be chosen to be mirror dual in the sense of [6], that is, to satisfy the relations

$$h^{1,1}(M) = h^{2,1}(M^*) \quad \text{and} \quad h^{2,1}(M) = h^{1,1}(M^*).$$

We prove (c). It is convenient to use the partial Euler characteristics $\chi_k = \sum_{i=0}^{4} (-1)^i h^{i,k}$, for $0 \leq k \leq 4$. In particular, $\chi_0$ is the Todd genus of a complex manifold. For a Calabi–Yau 4-fold $M$ obtained from Batyrev’s construction we have

$$\chi_0 = h^{0,0} - h^{1,0} + h^{2,0} - h^{3,0} + h^{4,0} = 2,$$

$$\chi_1 = h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} + h^{4,1} = -h^{1,1} + h^{2,1} - h^{3,1},$$

$$\chi_2 = h^{0,2} - h^{1,2} + h^{2,2} - h^{3,2} + h^{4,2} = -2h^{2,1} + h^{2,2}.$$ 

Therefore,

$$\chi(M) = \chi_0 - \chi_1 + \chi_2 - \chi_3 + \chi_4 = 2\chi_0 - 2\chi_1 + \chi_2 = 2(2 + h^{1,1} - 2h^{2,1} + h^{3,1}) + h^{2,2}. \quad (13.1)$$

On the other hand, the Hirzebruch–Riemann–Roch theorem (Theorem 21.1.1 in [27]) implies the following equalities in terms of the Chern numbers of $M$:

$$720\chi_0 = -c_4 + 3c_2^2, \quad 180\chi_1 = -31c_4 + 3c_2^2, \quad 120\chi_2 = 79c_4 + 3c_2^2.$$ 

For the generator $y_4 \in \Omega^4_{SU}$, we have $s_4 = 2c_2^2 - 4c_4 = 20$. Since $\chi_0 = 2$, the equality $2c_2^2 - 4c_4 = 20$ is equivalent to either of the conditions

$$\chi(M) = c_4 = 282 \quad \text{or} \quad -\chi_1 = h^{1,1}(M) - h^{2,1}(M) + h^{3,1}(M) = 39,$$

as asserted.
Similarly, for \(-y_4\), the condition \(s_4 = 2c_2^2 - 4c_4 = -20\) is equivalent to either of the conditions
\[
\chi(M) = c_4 = 294 \quad \text{or} \quad -\chi_1 = h^{1,1}(M) - h^{2,1}(M) + h^{3,1}(M) = 41.
\]

The existence of \(M\) follows by an analysis of the database [28] as in (b). In particular, there exist Calabi–Yau 4-folds with \(h^{1,1} = 16\), \(h^{2,1} = 30\), and \(h^{3,1} = 53\) representing the class \(y_4\), and also Calabi–Yau 4-folds with \(h^{1,1} = 17\), \(h^{2,1} = 45\), and \(h^{3,1} = 69\) representing \(-y_4\).

The class \(-y_4 \in \Omega_{SU}^4\) can also be represented by a Calabi–Yau manifold \(Z_S\) of Borcea–Voisin type constructed in [20] as a crepant resolution of singularities of the quotient of a hyperkähler manifold by a non-symplectic involution. This follows by comparing the formula in Theorem 13.5, (c), with the calculation of the Hodge numbers in [20], §5.2.

The generator \(y_2^2/4 = x_4 = w_4\) of the group \(\Omega_{SU}^4 = \mathbb{Z}(y_2^2/4, y_4)\) cannot be represented by a Calabi–Yau 4-fold with the full holonomy group \(SU(4)\). Indeed, as we noted at the end of §7,
\[
\frac{1}{4}y_2^2 = x_4 = (9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2]) \times (9[\mathbb{C}P^1] \times [\mathbb{C}P^1] - 8[\mathbb{C}P^2]),
\]
so the Todd genus of a manifold representing the class \(\frac{1}{4}y_2^2\) is 1. On the other hand, a Calabi–Yau 4-fold with the full holonomy group \(SU(4)\) has Hodge numbers \(h^{0,1} = h^{0,2} = h^{0,3} = 0\), so its Todd genus is equal to \(h^{0,0} + h^{0,4} = 2\).

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