Improving Convergence of Belief Propagation Decoding

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Abstract—The decoding of Low-Density Parity-Check codes by the Belief Propagation (BP) algorithm is revisited. We check the iterative algorithm for its convergence to a codeword (termination), we run Monte Carlo simulations to find the probability distribution function of the termination time, \( n_{\text{it}} \). Tested on an example [155, 64, 20] code, this termination curve shows a maximum and an extended algebraic tail at the highest values of \( n_{\text{it}} \). Aiming to reduce the tail of the termination curve we consider a family of iterative algorithms modifying the standard BP by means of a simple relaxation. The relaxation parameter controls the convergence of the modified BP algorithm to a minimum of the Bethe free energy. The improvement is experimentally demonstrated for Additive-White-Gaussian-Noise channel in some range of the signal-to-noise ratios. We also discuss the trade-off between the relaxation parameter of the improved iterative scheme and the number of iterations.

Low-Density Parity-Check (LDPC) codes [1], [2] are the best linear block error-correction codes known today [3]. In addition to being good codes, i.e. capable of decoding without errors in the thermodynamic limit of an infinitely long block length, these codes can also be decoded efficiently. The main idea of Belief Propagation (BP) decoding is in approximating the actual graphical model, formulated for solving statistical inference Maximum Likelihood (ML) or Maximum-A-Posteriori (MAP) problems, by a tree-like structure without loops. Being efficient but suboptimal the BP algorithm fails on certain configurations of the channel noise when close to optimal (but inefficient) MAP decoding would be successful.

BP decoding allows a certain duality in interpretation. First of all, and following the so-called Bethe-free energy variational approach [5], BP can be understood as a set of equations for beliefs (BP-equations) solving a constrained minimization problem. On the other hand, a more traditional approach is to interpret BP in terms of an iterative procedure — so-called BP iterative algorithm [1], [5], [2]. Being identical on a tree (as then BP equations are solved explicitly by iterations from leaves to the tree center) the two approaches are however distinct for a graphical problem with loops. In case of their convergence, BP algorithms find a minimum of the Bethe free energy [4], [6], [7], however in a general case convergence of the standard iterative BP is not guaranteed. It is also understood that BP fails to converge primarily due to circling of messages in the process of iterations over the loopy graph.

To enforce convergence of the iterative algorithm to a minimum of the Bethe Free energy some number of modifications of the standard iterative BP were discussed in recent years. The tree-based re-parametrization framework of [8] suggests to limit communication on the loopy graph, cutting some edges in a dynamical fashion so that the undesirable effects of circles are suppressed. Another, so-called concave-convex procedure, introduced in [9] and generalized in [10], suggests to decompose the Bethe free energy into concave and convex parts thus splitting the iterations into two sequential sub-steps.

Noticing that convergence of the standard BP fails mainly due to overshooting of iterations, we develop in this paper a tunable relaxation (damping) that cures the problem. Compared with the aforementioned alternative methods, this approach can be practically more advantageous due to its simplicity and tunability. In its simplest form our modification of the BP iterative procedure is given by

\[
\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta} \sum_{\beta \neq \alpha} \eta_{i\beta}^{(n+1)} = h_i + \sum_{\beta \neq \alpha} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \eta_{j\beta}^{(n)} \right) + \frac{1}{\Delta} \sum_{\beta \neq \alpha} \eta_{i\beta}^{(n)},
\]

where Latin and Greek indexes stand for bits and checks and the bit/check relations, e.g. \( \eta \) expresses the LDPC code considered; \( h_i \) is the channel noise-dependent value of log-likelihoods; and \( \eta_{i\alpha}^{(n)} \) is the message associated at the \( n \)-th iteration with the edge (of the respective Tanner graph) connecting \( i \)-th bit and \( \alpha \)-th check. \( \Delta \) is a tunable parameter. By choosing a sufficiently small \( \Delta \) one can guarantee convergence of the iterative procedure to a minimum of the Bethe free energy. On the other hand \( \Delta = +\infty \) corresponds exactly to the standard iterative BP. In the sequel we derive and explain the modified iterative procedure [1] in detail.

The manuscript is organized as follows. We introduce the Bethe free energy, the BP equation and the standard iterative BP in Section I. Performance of standard iterative BP, analyzed with a termination curve, is discussed in Section II. Section III describes continuous and sequentially discrete (iterative) versions of our relaxation method. We discuss performance of the modified iterative scheme in Section IV where Bit-Error-Rate and the termination curve for an LDPC code performed over Additive-White-Gaussian-Noise (AWGN) channel are discussed for a range of interesting values of the Signal-to-Noise-Ratios (SNR). We also discuss here the trade-off between convergence and number of it-
erations aiming to find an optimal strategy for selection of the model’s parameters. The last Section \[\text{[1]}\] is reserved for conclusions and discussions.

I. BETHE FREE ENERGY AND BELIEF PROPAGATION

Consider a generic factor model [11], [12], [13] with a binary configurational space, \(\sigma_i = \pm 1, i = 1, \ldots, n\), which is factorized so that the probability \(p(\sigma_i)\) to find the system in the state \(\{\sigma_i\}\) and the partition function \(Z\) are

\[
p(\sigma_i) = Z^{-1} \prod_{\alpha} f_{\alpha}(\sigma_{\alpha}), \quad Z = \sum_{\{\sigma_i\}} \prod_{\alpha} f_{\alpha}(\sigma_{\alpha}),
\]

where \(\alpha\) labels non-negative and finite factor-functions \(f_{\alpha}\) with \(\alpha = 1, \ldots, m\) and \(\sigma_{\alpha}\) represents a subset of \(\sigma_i\) variables. Relations between factor functions (checks) and elementary discrete variables (bits), expressed as \(i \in \alpha\) and \(\alpha \ni i\), can be conveniently represented in terms of the system-specific factor (Tanner) graph. If \(i \in \alpha\) we say that the bit and the check are neighbors. Any spin (a-posteriori log-likelihood) correlation function can be calculated using the partition function, \(Z\), defined by Eq. \[\text{[2]}\]. General expression for the factor functions of an LDPC code is

\[
f_{\alpha}(\sigma_{\alpha}) \equiv \exp \left( \sum_{i \in \alpha} b_i(\sigma_i) / q_i \right) \delta \left( \left\{ \prod \sigma_i \right\}, 1 \right).
\]

Let us now reproduce the derivation of the Belief Propagation equation based on the Bethe Free energy variational principle, following closely the description of [4]. (See also the Appendix of [16].) In this approach trial probability distributions, called beliefs, are introduced both for bits and checks \(b_i\) and \(b_{\alpha}\), respectively, where \(i = 1, \ldots, N, \alpha = 1, \ldots, M\). A belief is defined for given configuration of the binary variables over the code. Thus, a belief at a bit actually consists of two probabilities, \(b_i(+)\) and \(b_i(-)\), and we use a natural notation \(b_i(\sigma_i)\). There are \(2^k\) beliefs defined at a check, \(k\) being the number of bits connected to the check, and we introduce vector notation \(\sigma_{\alpha} = (\sigma_{i_1}, \ldots, \sigma_{i_k})\) where \(i_1, \ldots, i_k \in \alpha\) and \(\sigma_i = \pm 1\). Beliefs satisfy the following inequality constraints

\[
0 \leq b_i(\sigma_i), b_{\alpha}(\sigma_{\alpha}) \leq 1,
\]

the normalization constraints

\[
\sum_{\sigma_i} b_i(\sigma_i) = \sum_{\sigma_{\alpha}} b_{\alpha}(\sigma_{\alpha}) = 1,
\]

as well as the consistency (between bits and checks) constraints

\[
\sum_{\sigma_{\alpha} \ni \sigma_i} b_{\alpha}(\sigma_{\alpha}) = b_i(\sigma_i),
\]

where \(\sigma_{\alpha} \ni \sigma_i\) stands for the set of \(\sigma_j\) with \(j \in \alpha, j \neq i\).

The Bethe Free energy is defined as a difference of the Bethe self-energy and the Bethe entropy,

\[
F_{\text{Bethe}} = U_{\text{Bethe}} - H_{\text{Bethe}},
\]

\[
U_{\text{Bethe}} = - \sum_{\alpha} \sum_{\sigma_{\alpha}} b_{\alpha}(\sigma_{\alpha}) \ln f_{\alpha}(\sigma_{\alpha}),
\]

\[
H_{\text{Bethe}} = - \sum_{\alpha} \sum_{\sigma_{\alpha}} b_{\alpha}(\sigma_{\alpha}) \ln b_{\alpha}(\sigma_{\alpha})
\]

\[
+ \sum_i (q_i - 1) \sum_{\sigma_i} b_i(\sigma_i) \ln b_i(\sigma_i),
\]

where \(\sigma_{\alpha} = (\sigma_{i_1}, \ldots, \sigma_{i_k}), i_1, \ldots, i_k \in \alpha\) and \(\sigma_i = \pm 1\). The entropy term for a bit enters Eq. \[\text{[7]}\] with the coefficient \(1 - q_i\) to account for the right counting of the number of configurations for a bit: all entries for a bit (e.g. through the check term) should give \(+1\) in total.

Optimal configurations of beliefs are the ones that minimize the Bethe Free energy \[\text{[7]}\] subject to the constraints \[\text{[5]}\]. Introducing these constraints into the effective Lagrangian through Lagrange multiplier terms

\[
L = F_{\text{Bethe}} + \sum_{\alpha} \gamma_{\alpha} \left( \sum_{\sigma_{\alpha}} b_{\alpha}(\sigma_{\alpha}) - 1 \right)
\]

\[
+ \sum_i \gamma_i \left( \sum_{\sigma_i} b_i(\sigma_i) - 1 \right)
\]

\[
+ \sum_i \sum_{\alpha \ni \sigma_i} \lambda_{i\alpha}(\sigma_i) \left( b_i(\sigma_i) - \sum_{\sigma_\alpha \ni \sigma_i} b_{\alpha}(\sigma_{\alpha}) \right),
\]

and looking for the extremum with respect to all possible beliefs leads to

\[
\frac{\delta L}{\delta b_i(\sigma_i)} = 0
\]

\[
\Rightarrow b_i(\sigma_i) = f_{\alpha}(\sigma_{\alpha}) \exp \left[ -\gamma_{\alpha} - 1 + \sum_{i \in \alpha} \lambda_{i\alpha}(\sigma_i) \right],
\]

\[
\frac{\delta L}{\delta b_{\alpha}(\sigma_{\alpha})} = 0
\]

\[
\Rightarrow b_{\alpha}(\sigma_{\alpha}) = \exp \left[ \frac{1}{q_i - 1} \left( \gamma_i + \sum_{\alpha \ni \sigma_i} \lambda_{i\alpha}(\sigma_i) \right) - 1 \right].
\]

Substituting \(\lambda_{i\alpha}(\sigma_i) \equiv \ln \prod_{\beta \ni \alpha; \beta \neq \alpha} \mu_{i\beta}(\sigma_i)\) into Eqs. \[\text{[11]}\] we arrive at

\[
b_{\alpha}(\sigma_{\alpha}) \propto f_{\alpha}(\sigma_{\alpha}) \prod_{\beta \neq \alpha} \prod_{i \in \beta} \mu_{i\beta}(\sigma_i)
\]

\[
= f_{\alpha}(\sigma_{\alpha}) \prod_{i \in \alpha} \exp \left( \lambda_{i\alpha}(\sigma_i) \right),
\]

\[
b_i(\sigma_i) \propto \prod_{\alpha \ni \sigma_i} \mu_{i\alpha}(\sigma_i) = \exp \left( \frac{\sum_{\alpha \ni \sigma_i} \lambda_{i\alpha}(\sigma_i)}{q_i - 1} \right),
\]

where \(\propto\) is used to indicate that we should use the normalization conditions \[\text{[5]}\] to guarantee that the beliefs sum up to one. Applying the consistency constraint \[\text{[6]}\] to Eqs. \[\text{[13]}\],
making summation over all spins but the given \( \sigma_i \), and also using Eq. (14) we derive the following BP equations

\[
\prod_{\alpha \neq i} \mu_{i\alpha}(\sigma_i) \propto b_i(\sigma_i)
\]

\[
\propto \left( \prod_{\beta \neq i} \mu_{i\beta}(\sigma_i) \right) \sum_{\sigma_i} f_\alpha(\sigma_i) \prod_{j \neq i} \mu_{j\beta}(\sigma_j) \prod_{j \neq i, \beta \neq i} \mu_{j\alpha}(\sigma_j).
\]

The right hand side of Eq. (15) rewritten for the LDPC case becomes

\[
b_i(\sigma_i) \propto \exp(\eta_{i\alpha}) \prod_{\beta \neq i} \mu_{i\beta}(\sigma_i)
\]

\[
\times \left( \prod_{j \in \beta} (\mu_{j\alpha}(+)+\mu_{j\alpha}(-)) + \prod_{j \in \alpha} (\mu_{j\alpha}(+) - \mu_{j\alpha}(-)) \right).
\]

Thus constructing \( b_i(+) / b_i(-) \) for the LDPC case in two different ways (correspondent to left and right relations in Eq. (15)), equating the results and introducing the \( \eta_{i\alpha} \) field

\[
\exp(2\eta_{i\alpha}) = \frac{\mu_{i\alpha}(+) - \mu_{i\alpha}(-)}{\mu_{i\alpha}(+) + \mu_{i\alpha}(-)}
\]

one arrives at the following BP equations for the \( \eta_{i\alpha} \) fields:

\[
\eta_{i\alpha} = h_i + \sum_{\beta \neq i} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \eta_{j\beta} \right).
\]

Iterative solution of this equation corresponding to Eq. (11) with \( \Delta = +\infty \) is just a standard iterative BP (which can also be called sum-product) used for the decoding of an LDPC code.

A simplified min-sum version of Eq. (11) is

\[
\eta_{i\alpha}^{(n+1)} + \frac{1}{\Delta} \sum_{\beta \neq i} \eta_{j\beta}^{(n+1)} = h_i + \sum_{\beta \neq i} \prod_{j \in \beta} \min_{\eta_{j\beta}} |\eta_{j\beta}|^{n+1} + \frac{1}{\Delta} \sum_{\beta \neq i} \eta_{j\beta}^{(n)}.
\]

II. TERMINATION CURVE FOR STANDARD ITERATIVE BP

To illustrate the standard BP iterative decoding, given by Eqs. (11,19) with \( \Delta = +\infty \), we consider the example of the [15, 64, 20] code of Tanner [14] performing over AWGN channel characterized by the transition probability for a bit, \( p(x|\sigma) = \exp(-s^2(x - \sigma)^2/2\sqrt{2\pi}s^2) \), where \( \sigma \) and \( x \) are the input and output values at a bit and \( s^2 \) is the SNR. Launching a fixed codeword into the channel, emulating the channel by means of a standard Monte-Carlo simulations and then decoding the channel output constitutes our “experimental” procedure.

We analyze the probability distribution function of the iteration number \( n_{\text{it}} \) at which the decoding terminates. The termination probability curve for the standard min-sum, described by Eq. (19) with \( \Delta = +\infty \), is shown in Fig. 1 for SNR = 1, 2, 3.

The result of decoding is also verified at each iteration step for compliance with a codeword: iteration is terminated if a codeword is recovered. This termination strategy still give an error, although the probability to confuse actual and a distant codewords is much less than the probability not to recover a codeword for many iterations. If one neglects the very low probability of the codewords’ confusion, then the probability of still having a failure after \( n_{\text{it}} \) iterations is equal to the integral/sum over the termination curve from \( n_{\text{it}} \) and up. Note also that the probability that even infinite number of iterations will not result in a codeword can actually be finite.

Discussing Fig. 1 one observes two distinct features of the termination probability curve. First, in all cases the curve reaches its maximum at some relatively small number of iterations. Second, each curve crosses over to an algebraic-like decay which gets steeper with the SNR increase.

The emergence of an algebraically extended tail (that is a tail which does not decay fast) is not encouraging, as it suggests that increasing the number of iteration will not bring much of an improvement in the iterative procedure. It also motivates us to look for possibilities of accelerating convergence of the BP algorithm to a minimum of the Bethe free energy.

Note also the wiggling of termination curves for SNR = 2, 3 near the crossover point (see Fig. 1). It is possibly related to the cycling of the BP dynamics (and thus the inability of BP to converge).

III. RELAXATION TO A MINIMUM OF THE BETHE FREE ENERGY

The idea is to introduce relaxational dynamics (damping) in an auxiliary time, \( t \), thus enforcing convergence to a minimum of the Bethe Free energy. One chooses \( b_i(\sigma_i) \) as the main variational field and considers relaxing variational equations Eqs. (12) according to

\[
\frac{\partial}{\partial t} b_i(\sigma_i) = -\frac{1}{\tau_0} \frac{\delta L}{\delta b_i(\sigma_i)},
\]

Fig. 1. The termination probability curve for SNR = 1, 2, 3. Notice that the probability of termination (successful decoding) without any iterations is always finite. Few points on the right part of the plot correspond to the case when the decoding was not terminated even at the maximum number of iterations, 16384 (decoding fails to converge to a codeword).
while keeping the set of remaining variational equations Eqs. (5, 11, 16) intact. Here positive parameters $\tau_i$ have the physical meaning of correlation/relaxation times. Performing calculations, that are completely equivalent to the ones described in Section I, we arrive at the following modified BP equations

$$
\eta_{i\alpha} + (q_i - 1)Q_i = h_i + \sum_{\beta \neq \alpha} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \eta_{j\beta} \right), \quad (21)
$$

$$
Q_i = \tau_i \partial_t \tanh \left( \sum_{\alpha \neq i} \eta_{i\alpha} + (q_i - 1)Q_i \right). \quad (22)
$$

We are interested to approach (find) a solution of the original BP Eq. (18). One assumes $Q_i \ll \sum_{\alpha \neq i} \eta_{i\alpha}$, thus ignoring the second term under $\tanh$ in Eq. (22). The resulting continuous equation is

$$
\eta_{i\alpha} + \frac{(q_i - 1)\tau_i}{\cosh^2 \left( \sum_{\alpha \neq i} \eta_{i\alpha} \right)} \partial_t \left( \sum_{\alpha \neq i} \eta_{i\alpha} \right)
= h_i + \sum_{\beta \neq i} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \eta_{j\beta} \right). \quad (23)
$$

Eq. (1) represents a simple discretized version of the Eq. (23) where the correlation coefficients $\tau_i$ are chosen to make the coefficient in front of the second term on the left hand side of Eq. (23) independent of the bit index, $i$. Then the resulting time dependent coefficient can be rescaled to one by an appropriate choice of the temporal unit; $t_n$ is the uniform discrete time, $n$ is positive integer, $t_{n+1} - t_n = \Delta > 0$; the left hand side (right hand side) of Eq. (23) is taken at $t_{n+1}$ ($t_n$) and the temporal derivative is discretized in a standard retarded way, $\partial_t \eta_{i\alpha} \rightarrow (\eta_{i\alpha}^{(n+1)} - \eta_{i\alpha}^{(n)})/\Delta$. This choice of relaxation coefficients and discretization, resulted in Eq. (1), was taken out of consideration in the final formula for simplicity, realizability at all positive $\Delta$ and also its equivalence to the standard iterative BP at $\Delta \rightarrow +\infty$.

IV. MODIFIED ITERATIVE BP: TEST OF PERFORMANCE

We test the min-sum version (19) of the modified iterative BP with the Monte Carlo simulations of the [155, 64, 20] code at few values of SNRs. The resulting termination probability curves are shown in Fig. 2 for SNR = 1, 2, 3.

The simulations show a shift of the probability curve maximum to the right (towards larger number of iterations) with the damping parameter decrease however once the maximum is achieved, the decay of the curve at a finite $\Delta$ is faster with the number of iterations than in the standard BP case. The decay rate actually increases as $\Delta$ decreases.

We conclude that at the largest $n_{it}$, the performance of a modified iterative BP is strictly better. However to optimize the modified iterative BP, thus aiming at better performance than given by the standard iterative BP, one needs to account for the trade-off between decreasing $\Delta$ leading to a faster decay of the termination probability curve at the largest $n_{it}$, but on the other side it comes with the price in the actual number of iteration necessary to achieve the asymptotic decay regime.

The last point is illustrated by Fig. 3 where the decoding error probability depends non-monotonically on $\Delta$. One can also see that the modification of BP could improve the decoding performance; e.g., at SNR = 3 and maximally allowed $n_{it} = 32$ (after which the decoding unconditionally stops) the decoding error probability is reduced by factor of about 40 by choosing $\Delta = 1$ (see the bottom curve at Fig. 3b)).
V. CONCLUSIONS AND DISCUSSIONS

We presented a simple extension of the iterative BP which allows (with proper optimization in the $\Delta - n_{it}$ plane) to guarantee not only an asymptotic convergence of BP to a local minimum of the Bethe free energy but also a serious gain in decoding performance at finite $n_{it}$.

In addition to their own utility, these results should also be useful for systematic improvement of the BP approximation. Indeed, as it was recently shown in [15], [16] solution of the BP equation can be used to express the full partition function (or a-posteriori log-likehoods calculated within MAP) in terms of the so-called loop series, where each term is associated with a generalized loop on the factor graph. This loop calculus/series offers a remarkable opportunity for constructing a sequence of efficient approximate and systematically improvable algorithms. Thus we anticipate that the improved iterative BP discussed in the present manuscript will become an important building block in this future approximate algorithm construction.

We already mentioned in the introduction that our algorithm can be advantageous over other BP-based algorithms converging to a minimum of the Bethe free energy mainly due to its simplicity and tunability. In particular, the concave-convex algorithms of [9], [10], as well as related linear programming decoding algorithms [17], are formulated in terms of beliefs. On the contrary our modification of the iterative BP can be extensively simplified and stated in terms of the fewer number of $\eta$ fields each associated with an edge of the factor graph rather than with much bigger family of local code-words. Thus in the case of a regular LDPC code with $M$ checks of the connectivity degree $k$ one finds that the number of variables taken at each step of the iterative procedure is $k*M$ and $2^{k-1}M$ in our iterative scheme and in the concave-convex scheme respectively. Having a tunable correlation parameter $\tau$ in the problem is also advantageous as it allows generalizations (e.g. by turning to a individual bit dependent relaxation rate). This flexibility is particularly desirable in the degenerate case with multiple minima of the Bethe free energy, as it allows a painless implementation of annealing as well as other more sophisticated relaxation techniques speeding up and/or improving convergence.

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![Fig. 3. Decoding error probability as a function of $\Delta$ for SNR = 2 (a) and SNR = 3 (b). Different curves correspond to different maximally allowed $n_{it}$, starting from $n_{it} = 1$ (top curve) and increasing $n_{it}$ by factor of 2 with each next lower curve. The points on the right correspond to the standard BP ($\Delta = \infty$).](image-url)