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Three lectures on
“Fifty Years of KdV: An Integrable System”

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The goal in the first two Coxeter lectures was to give an answer to the question
“What is an integrable system?”
and to describe some of the tools that are available to identify and integrate such systems. The goal of the third lecture was to describe the role of integrable systems in certain numerical computations, particularly the computation of the eigenvalues of a matrix. This paper closely follows these three Coxeter lectures, and is written in an informal style with an abbreviated list of references. Detailed and more extensive references are readily available on the web. The list of authors mentioned is not meant in any way to be a detailed historical account of the development of the field and I ask the reader for his’r indulgence on this score.

The notion of an integrable system originates in the attempts in the 17\textsuperscript{th} and 18\textsuperscript{th} centuries to integrate certain specific dynamical systems in some explicit way. Implicit in the notion is that the integration reveals the long-time behavior of the system at hand. The seminal event in these developments was Newton’s solution of the two-body problem, which verified Kepler’s laws, and by the end of the 19\textsuperscript{th} century many dynamical systems of great interest had been integrated, including classical spinning tops, geodesic flow on an ellipsoid, the Neumann problem for constrained harmonic oscillators, and perhaps most spectacularly, Kowalewski’s spinning top. In the 19\textsuperscript{th} century, the general and very useful notion of Liouville integrability for Hamiltonian systems, was introduced: If a Hamiltonian system with Hamiltonian $H$ and $n$ degrees of freedom has $n$ independent, Poisson commuting integrals, $I_1, \ldots, I_n$, then the flow $t \mapsto z(t)$ generated by $H$ can be integrated explicitly by quadrature,

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or symbolically,

\[
\begin{align*}
I_k(z(t)) &= \text{const}, \ 1 \leq k \leq n, \ \text{rank} (dI_1, \ldots, dI_n) = n, \ \{I_k, I_j\} = 0, \ 1 \leq j, k \leq n \\
\Rightarrow \text{ explicit integration.}
\end{align*}
\]

Around the same time the Hamilton-Jacobi equation was introduced, which proved to be equally useful in integrating systems.

The modern theory of integrable systems began in 1967 with the discovery by Gardner, Greene, Kruskal and Miura \[GGKM\] of a method to solve the Korteweg de Vries (KdV) equation

\[
\begin{align*}
q_t + 6qq_x - q_{xxx} &= 0 \\
q(x, t)_{t=0} &= q_0(x) \to 0 \text{ as } |x| \to \infty.
\end{align*}
\]

The method was very remarkable and highly original and expressed the solution of KdV in terms of the spectral and scattering theory of the Schrödinger operator \(L(t) = -\partial_x^2 + q(x, t)\), acting in \(L^2(-\infty < x < \infty)\) for each \(t\). In 1968 Peter Lax \[Lax\] reformulated \[GGKM\] in the following way. For \(L(t) = -\partial_x^2 + q(x, t)\) and \(B(t) = 4\partial_x^3 - 6q\partial_x - 3q_x\).

\[
\begin{align*}
\text{KdV} &\equiv \partial_t L = [B, L] = BL - LB \\
&\equiv \text{isospectral deformation of } L(t) \\
\Rightarrow \text{spec } (L(t)) &= \text{spec } (L(0)) \Rightarrow \text{integrals of the motion for KdV.}
\end{align*}
\]

\(L, B\) are called \textit{Lax pairs}: By the 1970’s, Lax pairs for the Nonlinear Schrödinger Equation (NLS), the Sine-Gordon equation, the Toda lattice, . . . , had been found, and these systems had been integrated as in the case of KdV in terms of the spectral and scattering theory of their associated “L” operators.

Over the years there have been many ideas and much discussion of what it means for a system to be integrable, i.e. explicitly solvable. Is a Hamiltonian system with \(n\) degrees of freedom integrable if and only if the system is Liouville integrable, i.e. the system has \(n\) independent, commuting integrals? Certainly as explained above, Liouville integrability implies explicit solvability. But is the converse true? If we can solve the system in some explicit
fashion, is it necessarily Liouville integrable? We will say more about this matter further on. Is a system integrable if and only if it has a Lax pair representation as in (3)? There is, however, a problem with the Lax-pair approach from the get-go. For example, if we are investigating a flow on \( n \times n \) matrices, then a Lax-pair would guarantee at most \( n \) integrals, viz., the eigenvalues, whereas an \( n \times n \) system has \( O(n^2) \) degrees of freedom — too little, a priori, for Liouville integrability. The situation is in fact even more complicated. Indeed, suppose we are investigating a flow on real skew-symmetric \( n \times n \) matrices \( A \) — i.e. a flow for a generalized top. Such matrices constitute the dual Lie algebra of the orthogonal group \( O_n \), and so carry a natural Lie-Poisson structure. The symplectic leaves of this structure are the co-adjoint orbits of \( O_n \)

\[
A = A_A = \{ O \ A \ O^T : O \in O_n \}
\]

Thus any Hamiltonian flow \( t \to A(t) \) on \( A \), \( A(t = 0) = A \), must have the form

\[
A(t) = O(t) \ A \ O(t)^T
\]

for some \( O(t) \in A \) and hence has Lax-pair form

\[
\frac{dA}{dt} = \dot{O} \ A \ O^T + O \ A \ \dot{O}^T = [B, A]
\]

where

\[
B = \dot{O} \ O^T = -B^T
\]

The Lax-pair form guarantees that the eigenvalues \( \{\lambda_i\} \) of \( A \) are constants of the motion. But we see from (4) that the co-adjoint orbit through \( A \) is simply specified by the eigenvalues of \( A \). In other words the eigenvalues of \( A \) are just parameters for the symplectic leaves under considerations: They are of no help in integrating the system: Indeed \( d\lambda_i|_{A_A} = 0 \) for all \( i \). So for a Lax-pair formulation to be useful, we need

\[
\text{Lax pair + “something”}
\]

So, what is the “something”? A Lax-pair is a proclamation, a marker, as it were, on a treasure map that says “Look here!” The real challenge in each case is to turn the Lax-pair,
if possible, into an effective tool to solve the equation. In other words, the real task is to find the “something” to dig up the treasure! Perhaps the best description of Lax-pairs is a restatement of Yogi Berra’s famous dictum “If you come to a fork in the road, take it”. So if you come upon a Lax-pair, take it!

Over the years, with ideas going back and forth, Liouville integrability, Lax-pairs, “algebraic integrability”, “monodromy”, the discussion of what is an integrable system has been at times, shall we say, quite lively. There is, for example, the story of Henry McKean and Herman Flashka discussing integrability, when one of them, and I’m not sure which one, said to the other: “So you want to know what is an integrable system? I’ll tell you! You didn’t think I could solve it. But I can!”

In this “wild west” spirit, many developments were taking place in integrable systems. What was not at all clear at the time, however, was that these developments would provide tools to analyze mathematical and physical problems in areas far removed from their original dynamical origin. These tools constitute what may now be viewed as an integrable method (IM).

There is a picture that I like that illustrates, very schematically, the intersection of IM with different areas of mathematics. Imagine some high dimensional space, the “space of problems”. The space contains a large number of “parallel” planes, stacked one on top of the other and separated. The planes are labeled as follows: dynamical systems, probability theory and statistical mechanics, geometry, combinatorics, statistical mechanics, classical analysis, numerical analysis, representation theory, algebraic geometry, transportation theory, .... In addition, there is another plane in the space labeled “the integrable method (IM)”: Any problem lying on IM can be solved/integrated by tools taken from the integrable method. Now the fact of the matter is that the IM-plane intersects all of the parallel planes described above: Problems lying on the intersection of any one of these planes with the IM-plane are thus solvable by the integrable method.
Figure 1: Intersections of the Integrable Method

For each parallel plane we have, for example, the following intersection points:

- dynamical systems: Korteweg-de Vries (KdV), Nonlinear Schrödinger (NLS), Toda, Sine-Gordon, . . .

- probability theory and statistics: Random matrix theory (RMT), Integrable probability theory, Principal component analysis (PCA), . . .

- geometry: spaces of constant negative curvature $R$, general relativity in $1 + 1$ dimensions, . . .

- combinatorics: Ulam’s increasing subsequence problem, tiling problems, (Aztec diamond, hexagon tiling, . . .), random particle systems (TASEP, . . .), . . .

- statistical mechanics: Ising model, XXZ spin chain, 6 vertex model, . . .

- classical analysis: Riemann-Hilbert problems, orthogonal polynomials, (modern) special function theory (Painlevé equations), . . .
• numerical analysis: QR, Toda eigenvalue algorithm, Singular value decomposition, \dots

• representation theory: representation theory of large groups ($S_\infty$, $U_\infty \ldots$), symmetric function theory, \dots

• algebraic geometry: Schottky problem, infinite genus Riemann surfaces, \dots

• transportation theory: Bus problem in Cuernavaca, Mexico, airline boarding, \ldots

The list of such intersections is long and constantly growing.

The singular significance of KdV is just that the first intersection that was observed and understood as such, was the junction of IM with dynamical systems, and that was at the point of KdV.

How do we come to such a picture? First we will give a precise definition of what we mean by an integrable system. Consider a simple harmonic oscillator:

$$\dot{x} = y \quad , \quad \dot{y} = -\omega^2 x$$

$$x(t)|_{t=0} = x_0 \quad , \quad y(t)|_{t=0} = y_0$$

The solution of (9) has the following form:

$$\begin{align}
    x(t; x_0, y_0) &= \frac{1}{\omega} \sqrt{\omega^2 x_0^2 + y_0^2} \sin \left( wt + \sin^{-1} \left( \frac{\omega x_0}{\sqrt{\omega^2 x_0^2 + y_0^2}} \right) \right) \\
    y(t; x_0, y_0) &= \sqrt{\omega^2 x_0^2 + y_0^2} \cos \left( wt + \sin^{-1} \left( \frac{\omega x_0}{\sqrt{\omega^2 x_0^2 + y_0^2}} \right) \right)
\end{align}$$

Note the following features of (10): Let $\varphi: \mathbb{R}^2 \to \mathbb{R}_+ \times (\mathbb{R}/2\pi \mathbb{Z})$

$$(\alpha, \beta) \mapsto A = \frac{1}{\omega} \sqrt{\omega^2 \alpha^2 + \beta^2}, \ \theta = \sin^{-1} \left( \frac{\omega \alpha}{\sqrt{\omega^2 \alpha^2 + \beta^2}} \right)$$

Then

$$\varphi^{-1}: \mathbb{R}_+ \times (\mathbb{R}/2\pi \mathbb{Z}) \to \mathbb{R}^2$$

has the form

$$\varphi^{-1}(A, \theta) = (A \sin \theta, \ \omega A \cos \theta)$$

Thus (10) implies

$$\begin{align}
    \eta(t; \eta_0) &= \varphi^{-1} (\varphi(\eta_0) + \vec{\omega} t) \\
    \text{where} \quad &\eta(t) = (x(t), y(t)), \ \eta_0 = (x_0, y_0), \ \vec{\omega} = (0, \omega)
\end{align}$$
In other words:

(12a) There exists a bijective change of variables \( \eta \mapsto \varphi(\eta) \) such that

(12b) \( \eta(t; \eta_0) \) evolves according to (9) \( \Rightarrow \)

\[ \varphi(\eta(t); \eta_0) = \varphi(\eta_0) + t \vec{\omega} \]

i.e., in the variables \((A, \theta) = \varphi(\alpha, \beta)\), solutions of (9) move linearly.

(12c)

\( \eta(t, \eta_0) \) is recovered from formula (11) via a map

\[ \varphi^{-1}(A, \theta) = (A \sin \theta, \omega A \cos \theta) \]

in which the behavior of \( \sin \theta, \cos \theta \) is very well understood.

The same is true for \( \varphi \). What we learn, in particular, based on this knowledge of \( \varphi \) and \( \varphi^{-1} \), is that

\( \eta(t; \eta_0) \) evolves periodically in time with period \( 2\pi/\omega \)

We are led to the following:

We say that a dynamical system \( t \mapsto \eta(t) \) is **integrable** if

\[ \left\{ \begin{array}{l}
\text{There exists a bijective map } \varphi : \eta \mapsto \varphi(\eta) \equiv \zeta \\
\text{such that } \varphi \text{ linearizes the system}
\end{array} \right. \]

(13a)

\[ \varphi(\eta(t)) = \varphi(\eta(t = 0)) + \vec{\omega} t \]

and so

\[ \eta(t; \eta(t = 0)) = \varphi^{-1}(\varphi(\eta(t = 0)) + \vec{\omega} t) \]

AND

(13b)

\[ \left\{ \begin{array}{l}
The behavior of \( \varphi, \varphi^{-1} \) are well enough understood so that
\end{array} \right. \]

the behavior of \( \eta(t; \eta(t = 0)) \) as \( t \to \infty \) is clearly revealed.

More generally, we say a system \( \eta \) which depends on some parameters \( \eta = \eta(a, b, \ldots) \) is
integrable if

\begin{align*}
\text{(14a) } & \quad \exists \text{ a bijective change of variables } \eta \to \zeta = \varphi(\eta) \text{ such that the dependence of } \zeta \text{ on } a, b, \ldots. \\
& \quad \zeta(a, b, \ldots) = \varphi(\eta(a, b, \ldots)) \\
& \quad \text{is simple/well-understood}
\end{align*}

and

\begin{align*}
\text{(14b) } & \quad \exists \text{ the behavior of the function theory } \\
& \quad \eta \mapsto \zeta \equiv \varphi(\eta), \quad \zeta \mapsto \eta = \varphi^{-1}(\zeta) \\
& \quad \text{is well-enough understood so that the behavior of} \\
& \quad \eta(a, b, \ldots) = \varphi^{-1}(\zeta(a, b, \ldots))
\end{align*}

is revealed in an explicit form as \(a, b, \ldots\) vary, becoming, in particular, large or small.

Notice that in this definition of an integrable system, various sufficient conditions for integrability such as commuting integrals, Lax-pairs, \ldots, are conspicuously absent. A system is integrable, if you can solve it, but subject to certain strict guidelines. This is a return to McKean and Flaschka, an institutionalization, as it were, of the “Wild West”.

According to this definition, progress in the theory of integrable systems is made

**EITHER**

by discovering how to linearize a new system

\[ \eta \to \zeta = \varphi(\eta) \]

using a **known** function theory \(\varphi\). For example: Newton’s problem of two gravitating bodies, is solved in terms of trigonometric functions/ellipses/parabolas—mathematical objects already well-known to the Greeks. In the 19\textsuperscript{th} century, Jacobi solved geodesic flow on an ellipsoid using newly minted hyperelliptic function theory, and so on, \ldots

**OR**

by discovering/inventing a new function theory which linearizes the given problem at hand. For example: To facilitate numerical calculations in spherical geometry, Napier, in the early
1700’s, realized that what he needed to do was to linearize multiplication

\[ \eta \tilde{\eta} \rightarrow \varphi(\eta \tilde{\eta}) = \varphi(\eta) + \varphi(\tilde{\eta}) \]

which introduced a new function theory — the logarithm. Historically, no integrable system has had greater impact on mathematics and science, than multiplication! There is a similar story for all the classical special functions, Bessel, Airy, . . . , each of which was introduced to address a particular problem.

The following aspect of the above evolving integrability process is crucial and gets to the heart of the Integrable Method (IM): Once a new function theory has been discovered and developed, it enters the toolkit of IM, finding application in problems far removed from the original discovery.

Certain philosophical points are in order here.

(i) There is no difference in spirit, philosophically, between our definition of an integrable system and what we do in ordinary life. We try to address problems by rephrasing them (read “change of variables”) so we can recognize them as something we know. After all, what else is a “precedent” in a law case? We introduce new words — a new “function theory” — to capture new developments and so extend and deepen our understanding. Recall that Adam’s first cognitive act in Genesis was to give the animals names. The only difference between this progression in ordinary life versus mathematics, is one of degree and precision.

(ii) This definition presents “integrability” not as a predetermined property of a system frozen in time. Rather, in this view the status of a system as integrable depends on the technology/function theory available at the time. If an appropriate new function theory is developed, the status of the system may change to integrable.

How does one determine if a system is integrable and how do you integrate it? Let me say at the outset, and categorically, that I believe there is no systematic answer to this question. Showing a system is integrable is always a matter of luck and intuition.

We do, however, have a toolkit which one can bring to a problem at hand.

At this point in time, the toolkit contains, amongst others, the following components:
(a) a broad and powerful set of functions/transforms/constructions

\[ \eta \rightarrow \zeta = \varphi(\eta) \]

that can be used to convert a broad class of problems of interest in mathematics/physics, into “known” problems: In the simplest cases \( \eta \rightarrow \varphi(\eta) \) linearizes the problem.

(b) powerful techniques to analyze \( \varphi, \varphi^{-1} \) such that the asymptotic behavior of the original \( \eta \)-system can be inferred explicitly from the known asymptotic behavior of the \( \zeta \)-system, as relevant parameters, e.g. time, become large.

(c) a particular, versatile class of functions, the Painlevé functions, which play the same role in modern (nonlinear) theoretical physics that classical special functions played in (linear) 19th century physics. Painlevé functions form the core of modern special function, and their behavior is known with the same precision as in the case of the classical special functions. We note that the Painlevé equations are themselves integrable in the sense of Definition (14a).

(d) a class of “integrable” stochastic models — random matrix theory (RMT). Instead of modeling a stochastic system by the roll of a die, say, we now have the possibility to model a whole new class of systems by the eigenvalues of a random matrix. This RMT plays the role of a stochastic special function theory. RMT is “integrable” in the sense that key statistics such as the gap probability, or edge statistics, for example, are given by functions, e.g. Painlevé functions, that describe (deterministic) integrable problems as above. We will say more about this later.

We will now show how all this works in concrete situations. Note, however, by no means all known integrable systems can be solved using tools from the IM-toolkit. For example the beautiful system that Patrick Gérard et al. have been investigating recently (see e.g. [GeLe]), seems to be something completely different. We will consider various examples. The first example is taken from dynamics, viz., the NLS equation.

To show that NLS is integrable, we first extract a particular tool from the toolkit — the Riemann-Hilbert Problem (RHP): Let \( \Sigma \subset \mathbb{C} \) be an oriented contour and let \( v : \Sigma \rightarrow \)
$G\ell(n,\mathbb{C})$ be a map (the “jump matrix”) from $\Sigma$ to the invertible $n \times n$ matrices, $v, v^{-1} \in L^\infty(\Sigma)$. By convention, at a point $z \in \Sigma$, the (+) side (respectively (−) side) lies to the left (respectively right) as one traverse $\Sigma$ in the direction of the orientation, as indicated in Figure 2. Then the (normalized) RHP $(\Sigma, v)$ consists in finding an $n \times n$ matrix-valued function $m = m(z)$ such that

1. $m(z)$ is analytic in $\mathbb{C}/\Sigma$
2. $m_+(z) = m_-(z) v(z)$, $z \in \Sigma$
   where $m_\pm(z) = \lim_{z' \to z\pm} m(z')$
3. $m(z) \to I_n$ as $z \to \infty$

Here “$z' \to z\pm$” denotes the limit as $z' \in \mathbb{C}/\Sigma$ approaches $z \in \Sigma$ from the ($\pm$)-side, respectively. The particular contour $\Sigma$ and the jump matrix $v$ are tailored to the problem at hand.

There are many technicalities involved here: Does such an $m(z)$ exist? In what sense do the limits $m_\pm$ exist? And so on . . . . Here we leave such issues aside. RHP’s play an analogous role in modern physics that integral representations play for classical special functions, such as the Airy function $Ai(z)$, Bessel function $J_n(z)$, etc. For example, $Ai(z) = \frac{1}{2\pi i} \int_{C} \exp \left( \frac{t^3}{3} - zt \right) dt$ for some appropriate contour $C \subset \mathbb{C}$, which makes it possible to analyze the behavior of $Ai(z)$ as $z \to \infty$, using the classical steepest descent method.
Now consider the defocusing NLS equation on $\mathbb{R}$

\[
\begin{cases}
  i u_t + u_{xx} - 2|u|^2 u = 0 \\
  u(x, t) \bigg|_{t=0} = u_0(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases}
\]

In 1972, Zakharov and Shabat [ZaSh] showed that NLS has a Lax-pair formulation, as follows: Let

\[
L(t) = (i \sigma)^{-1} (\partial_x - Q(t))
\]

where

\[
\sigma = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(t) = \begin{pmatrix} 0 & u(x,t) \\ -u(x,t) & 0 \end{pmatrix}.
\]

For each $t$, $L(t)$ is a self-adjoint operator acting on vector valued function in $(L^2(\mathbb{R}))^2$. Then for some explicit $B(t)$, constructed from $u(x,t)$ and $u_x(x,t)$,

\[
\text{(16)} \quad u(x,t) \text{ solves NLS} \iff \frac{dL(t)}{dt} = [B(t), L(t)].
\]

This the second tool we extract from our toolkit. So the Lax operator $L(t)$ marks a point, as it were, on our treasure map. How can one use $L(t)$ to solve the system?

One proceeds as follows: This crucial step was first taken by Shabat [Sha] in the mid-1970’s in the case of KdV and developed into a general scheme for ordinary differential operators by Beals and Coifman [BC] in the early 1980’s.

The map $\varphi$ in 13(a) above for NLS is the scattering map constructed as follows: Suppose $u = u(x)$ is given, $u(x) \to 0$ sufficiently rapidly as $|x| \to \infty$. For fixed $z \in \mathbb{C}/\mathbb{R}$, there exists a unique $2 \times 2$ solution of the scattering problem

\[
\text{(17)} \quad L\psi = z\psi,
\]

where

\[
\text{(18)} \quad m = m(x, z; u) \equiv \psi(x, z) e^{-iz,x\sigma}
\]

is bounded on $\mathbb{R}$ and

\[
m(x, z; u) \to I \quad \text{as} \quad x \to -\infty.
\]
For fixed $x \in \mathbb{R}$, such so-called Beals-Coifman solutions also have the following properties:

\[(19)\]
\[m(x, z; u) \text{ is analytic in } z \text{ for } z \in \mathbb{C}/\mathbb{R}
\[\text{and continuous in } \mathbb{C}_+ \text{ and in } \mathbb{C}_-\]

\[(20)\]
\[m(x, z; u) \to I \text{ as } z \to \infty \text{ in } \mathbb{C}/\mathbb{R}.
\]

Now both $\psi_\pm(x, z; u) = \lim_{z' \to z} \psi(x, z; u)$, $z \in \mathbb{R}$ clearly solve $L \psi_\pm = z \psi_\pm$ which implies that there exists $v = v(z) = v(z; u)$ independent of $x$, such that for all $x \in \mathbb{R}$

\[(21)\]
\[\psi_+(x, z) = \psi_-(x, z) v(z) \quad z \in \mathbb{R}
\]

or in terms of

\[(22)\]
\[m_\pm = \psi_\pm(x, z) e^{-ixz\sigma}
\]

we have

\[(23)\]
\[m_+(x, z) = m_-(x, z) v_x(z) \quad z \in \mathbb{R}
\]

where

\[(24)\]
\[v_x(z) = e^{ixz\sigma} v(z) e^{-ixz\sigma}
\]

Said differently, for each $x \in \mathbb{R}$, $m(x, z)$ solves the normalized RHP $(\Sigma, v_x)$ where $\Sigma = \mathbb{R}$, oriented from $-\infty$ to $+\infty$, and $v$ is as above. In this way, a RHP enters naturally into the picture introduced by the Lax operator $L$.

It turns out that $v$ has a special form

\[(25)\]
\[
\begin{align*}
v(z) &= \begin{pmatrix} 1 - |r(z)|^2 & r(z) \\ -\bar{r}(z) & 1 \end{pmatrix} \\
v_x(z) &= \begin{pmatrix} 1 - |r|^2 & re^{ixz} \\ -\bar{r}e^{-ixz} & 1 \end{pmatrix}
\end{align*}
\]

where $r(z)$, the reflection coefficient, satisfies $\|r\|_\infty < 1$. We define the map $\varphi$ for NLS as follows:

\[(26)\]
\[u \mapsto \varphi(u) \equiv r\]
Suppose \( r \) is given and \( x \) fixed. To construct \( \varphi^{-1}(r) \) we must solve the RHP \((\mathbb{R}, v_x)\) with \( v_x \) as in (25). If \( m = m(x, z) \) is the solution of the RHP, then expanding at \( z = \infty \), we have
\[
m(x, z) = I + \frac{(m_1(x))}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty.
\]
A simple calculation then shows that
\[
(27) \quad u(x) = \varphi^{-1}(r) = -i(m_1(x))_{12}.
\]
Thus
\[
\varphi \leftrightarrow \text{scattering map} ; \quad \varphi^{-1} \leftrightarrow \text{RHP}.
\]

Now the key fact of the matter is that
\[
(28) \quad u \text{ linearizes NLS}.
\]
Indeed if \( u(t) = u(x, t) \) solves NLS with \( u(x, t)|_{t=0} = u_0(x) \), then
\[
(29) \quad r(t) = \varphi(u(t)) = r(z; u_0) e^{-it z^2} = \varphi(u_0)(z) e^{-it z^2}
\]
or
\[
\log r(t) = \log r(z, u_0) - it z^2
\]
which is linear motion!

This leads to the celebrated solution procedure
\[
(30) \quad u(t) = \varphi^{-1}\left(\varphi(u_0)(\cdot) e^{-it(\cdot)^2}\right).
\]
Thus condition (13a) for the integrability of NLS is established.

But condition (13b) is also satisfied. Indeed the analysis of the scattering map \( u \to r = \varphi(u) \) is classical and well-understood. The inverse scattering map is also well-understood because of the nonlinear steepest descent method for RHP’s introduced by Deift and Zhou in 1993 [DZ1]. This is the third tool we extract from our toolkit. One finds, for example, that as \( t \to \infty \)
\[
(31) \quad u(x, t) = \frac{1}{\sqrt{t}} \alpha(z_0) e^{ix^2/4t - i\nu(z_0) \log 2t}
\]
\[
+ O\left(\frac{\ln t}{t}\right)
\]
where

\[(32)\quad z_0 = x/2t \quad \text{and} \quad \nu(z) = -\frac{1}{2\pi} \log \left(1 - |r(z)|^2\right)\]

and

\[\alpha(z) \text{ is an explicit function of } r.\]

We see, in particular, that the long-time behavior of \(u(x, t)\) is given with the same accuracy and detail as the solution of the linear Schroödinger equation \(i u_t^0 + u_{xx}^0 = 0\) which can be obtained by applying the classical steepest descent method to the Fourier representation of \(u^0(x, t)\)

\[u^0(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}^0(z) e^{i(xz - tz^2)} dz\]

where \(\hat{u}^0\) is the Fourier transform of \(u^0(x, t = 0)\). As \(t \to \infty\), one finds

\[(33)\quad u^0(x, t) = \hat{u}^0(z_0) \sqrt{\frac{2t}{\pi}} e^{ix^2/4t} + o\left(\frac{1}{\sqrt{t}}\right).\]

We see that NLS is an integrable system in the sense advertised in (13ab).

We note that the asymptotic formula (31) (32) for NLS was first obtained by Zakharov and Manakov in 1976 [ZaMa] using inverse scattering techniques, also taken from the IM toolbox, but without the rigor of the nonlinear steepest descent method.

The next example, taken from Statistical Mechanics, utilizes another tool from the toolkit, viz. the theory of integrable operators, IO’s.

IO’s were first singled out as a distinguished class of operators by Sakhnovich in the 60’s, 70’s and the theory of such operators was then fully developed by Fokas, Its and Kitaev [FIK] in the 1990’s. Let \(\Sigma\) be an oriented contour in \(\mathbb{C}\). We say an operator \(K\) acting on measurable functions \(h\) on \(\Sigma\) is integrable if it has a kernel of the form

\[(34)\quad K(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y) \frac{1}{x - y}, \quad n < \infty, \quad x, y \in \Sigma,\]

where

\[(35)\quad f_i, g_i \in L^\infty(\Sigma), \quad \text{and} \quad Kh(x) = \int_{\Sigma} K(x, y) h(y) dy.\]
If $\Sigma$ is a “good” contour (i.e. $\Sigma$ is a Carleson curve), $K$ is bounded in $L^p(\Sigma)$ for $1 < p < \infty$.

Integral operators have many remarkable properties. In particular the integrable operators form an algebra and $(I + K)^{-1}$, if it exists, is also integrable if $K$ is integrable. But most remarkably, $(I + K)^{-1}$ can be computed in terms of a naturally associated RHP on $\Sigma$. It works like this. If $K(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y)/x - y$, then

$$ (I + K)^{-1} = I + R $$

where $R(x, y) = \sum_{i=1}^{n} F_i(x) G_i(y)/x - y$

for suitable $F_i, G_i$. Now assume for simplicity that $\sum_{i=1}^{n} f_i(x) g_i(x) = 0$ and let

$$ v(z) = I - 2\pi f(z) g(z)^T, \quad z \in \Sigma, $$

where $f = (f_1, \ldots, f_n)^T$, $g = (g_1, \ldots, g_n)^T$ and suppose $m(z)$ solves the normalized RHP $(\Sigma, v)$. Then

$$ F(z) = m_+(z) f(z) = m_-(z) f(z) $$

and

$$ G(z) = (m_+^{-1})^T g(z) = (m_-^{-1})^T g(z) $$

Here is an example how integrable operators arise. Consider the spin–$\frac{1}{2}$ XY model in a magnetic field with Hamiltonian

$$ H = -\frac{1}{2} \sum_{\ell \in \mathbb{Z}} (\sigma_{\ell}^x \sigma_{\ell+1}^x + \sigma_{\ell}^z) $$

where $\sigma_{\ell}^x, \sigma_{\ell}^z$ are the standard Pauli matrices at the $\ell^{th}$ site of a 1-d lattice.

As shown by McCoy, Perk and Schrock [McPS] in 1983, the auto-correlation function $X(t)$

$$ X(t) = \langle \sigma_0^z(t) \sigma_0^z \rangle_T = \frac{\text{tr} \left( e^{-\beta H} \left( e^{-iHt} \sigma_0^x e^{iHt}\right) | \sigma_0^x \right)}{\text{tr} e^{-\beta H}} $$

where $\beta = \frac{1}{T}$, can be expressed as follows:

$$ X(t) = e^{-t^2/2} \det (1 - K_t) $$
Here $K_t$ is the operator on $L^2(-1,1)$ with kernel

$$K_t(z, z') = \varphi(z) \frac{\sin it(z-z')}{\pi(z-z')} , \quad -1 \leq z, z' \leq 1,$$

and

$$\varphi(z) = \tanh \left( \beta \sqrt{1-z^2} \right) , \quad -1 < z < 1.$$

Observe that

$$K_t(z, z') = \sum_{i=1}^{2} f_i(z) g_i(z')$$

where

$$f = (f_1, f_2)^T = \left( \frac{-e^{tz} \varphi(z)}{2\pi i} , \frac{-e^{-tz} \varphi(z)}{2\pi i} \right)^T$$

$$g = (g_1, g_2)^T = \left( e^{-tz} , -e^{tz} \right)^T$$

so that $K_t$ is an integrable operator. We have

$$v = v_t = I - 2\pi i f g^T = \begin{pmatrix} 1 + \varphi(z) & -\varphi(z) e^{2zt} \\ \varphi(z) e^{-2zt} & 1 - \varphi(z) \end{pmatrix} , \quad z \in (-1,1).$$

As

$$\frac{d}{dt} \log \det (1 - K_t) = \frac{d}{dt} \text{tr} \log(1 - K_t)$$

$$= -\text{tr} \left( \frac{1}{1 - K_t} \dot{K}_t \right)$$

we see that $\frac{d}{dt} \log \det(1 - K_t)$, and ultimately $X(t)$, can be expressed via (36) (38) (39) in terms of the solution $m_t$ of the RHP $(\sum = (-1,1), v_t)$

Applying the nonlinear steepest descent method to this RHP as $t \to \infty$, one finds (Deift-Zhou (1994) [DZ2]) that

$$X(t) = \exp \left( \frac{t}{\pi} \int_{-1}^{1} \log |\tanh \beta s| \, ds + o(t) \right)$$

This shows that $H$ in (40) is integrable in the sense that key statistics for $H$ such as the autocorrelation function $X(t)$ for the spin $\sigma_0^x$ is integrable in the sense of (14ab)

$$X(t) \mapsto K_t \in \text{integrable operators}$$
and \( \varphi^{-1} \) is computed with any desired precision using RH-steepest descent methods to obtain (45). Note that the appearance of the terms \( \varphi(z) e^{\pm 2zt} \) in the jump matrix \( v_t \) for \( K_t = \varphi(X(t)) \), makes explicit the linearizing property of the map \( \varphi \).

Another famous integrable operator appears in the bulk scaling limit for the gap probability for invariant Hermitian ensembles in random matrix theory. More precisely, consider the ensemble of \( N \times N \) Hermitian matrix \( \{M\} \) with invariant distribution

\[
P_N(M) \, dM = \frac{e^{-N \operatorname{tr} V(M)} \, dM}{\int e^{-N \operatorname{tr} V(M)} \, dM},
\]

where \( V(x) \to +\infty \) as \( |x| \to \infty \) and \( dM \) is Lebesgue measure on the algebraically independent entries of \( M \).

Set \( P_N([\alpha, \beta]) = \text{gap probability} \equiv \operatorname{Prob} \{M \text{ has no eigenvalues in } [\alpha, \beta]\}, \quad \alpha < \beta. \)

We are interested in the scaling limit of \( P_N([\alpha, \beta]) \) i.e.

\[
P(\alpha, \beta) = \lim_{N \to \infty} P_N \left( \left[ \frac{\alpha}{\rho_N}, \frac{\beta}{\rho_N} \right] \right)
\]

for some appropriate scaling \( \rho_N \sim N \). One finds (and here RH techniques play a key role) that

\[
P(\alpha, \beta) = \det(1 - K_s) \quad , \quad s = \beta - \alpha
\]

where \( K_s \) has a kernel

\[
K_s(x, y) = \frac{\sin(x - y)}{\pi(x - y)} \quad \text{acting on } L^2(0, 2s).
\]

Clearly \( K_s(x, y) = \frac{e^{ix} e^{-iy} - e^{-ix} e^{iy}}{2\pi i(x - y)} \) is an integrable operator. The asymptotics of \( P(\alpha, \beta) \) can then be evaluated asymptotically with great precision as \( s \to \infty \), by applying the nonlinear steepest descent method for RHP’s to the RHP associated with the integrable operator \( K_s \), as in the case for \( K_t \) in (44) et seq.

Thus RMT is integrable in the sense that a key statistic, the gap probability in the bulk scaling limit, is an integrable system in the sense of (14ab):
\[ \varphi^{-1} \] is then evaluated via the formula \[ \det (1 - K_s) \]
which can be controlled precisely as \( s \to \infty \).

The situation is similar for many other key statistics in RMT. It turns out that \( P_{(\alpha, \beta)} \) solves the Painlevé V equation as a function of \( s = \beta - \alpha \) (this is a famous, result of Jimbo, Miwa and Môri and Sato, 1980 [JMMS]). But the Painlevé V equation is a classically integrable Hamiltonian system which is also integrable in the sense of (14ab). Indeed it is a consequence of the seminal work of the Japanese School of Sato et al. that all the Painlevé equations can be solved via associated RHP’s (the RHP for Painlevé II in particular was also found independently by Flaschka and Newell), and hence are integrable in the sense of (14ab) and amenable to nonlinear steepest descent asymptotic analysis, as described, for example, in the text, Painlevé Transcendents by Fokas, Its, Kapaev and Novokshenov (2006) [FIKN].

There is another perspective one can take on RMT as an integrable system. The above point of view is that RMT is integrable because key statistics are described by functions which are solutions of classically integrable Hamiltonian systems. But this point of view is unsatisfactory in that it attributes integrability in one area (RMT) to integrability in another (Hamiltonian systems). Is there a notion of integrability for stochastic systems that is intrinsic? In dynamics the simplest integrable system is free motion

\[ \dot{x} = y, \quad \dot{y} = 0 \implies x(t) = x_0 + y_0 t, \quad y(t) = y_0. \]

Perhaps the simplest stochastic system is a collection of coins flipped independently. Now, we suggest, just as an integrable Hamiltonian system becomes (47) in new variables, the analogous property for a stochastic system should be that, in the appropriate variables, it is integrable if it just a product of independent spin flips.

Consider the scaled gap probability,

\[ P_{(\alpha, \beta)} = \text{Prob} \{ \text{no eigenvalues in } (\alpha, \beta) \} = \det (1 - K_s) \]

But as the operator \( K_s \) is trace-class and \( 0 \leq K_s < 1 \), it follows that

\[ P_{\alpha, \beta} = \prod_{i=1}^{\infty} (1 - \lambda_i) \]
where \(0 \leq \lambda_i < 1\) are the eigenvalues of \(K_s\). Now imagine we have a collection of boxes, \(B_1, B_2, \ldots\). With each box we have a coin: With probability \(\lambda_i\) a ball is placed in box \(B_i\), or equivalently, with probability \(1 - \lambda_i\) there is no ball placed in \(B_i\). The coins are independent. Thus we see that the probability that there are no eigenvalue in \((\alpha, \beta)\), is the same as the probability of no balls being placed in all the boxes!

This is an intrinsic probabilistic view of RMT integrability. It applies to many other stochastic systems. For example, consider Ulam’s longest increasing subsequence problem:

Let \(\pi = \pi(1) \pi(2) \ldots \pi(N)\) be a permutation in the symmetric group \(S_N\). If

\[
(50) \quad i_1 < i_2 < \cdots < i_k \quad \text{and} \quad \pi(i_1) < \cdots < \pi(i_k)
\]

we say that

\[
(51) \quad \pi(i_1) \pi(i_2) \ldots \pi(i_k)
\]

is an increasing subsequence for \(\pi\) of length \(k\). Let \(\ell_N(\pi)\) denote the greatest length of any increasing subsequence for \(\pi\), e.g. for \(N = 6\), \(\pi = 315624 \in S_6\) has \(\ell_6(\pi) = 3\) and 356, 254 and 156 are all longest increasing subsequences for \(\pi\). Equip \(S_N\) with uniform measure. Thus for \(n \leq N\).

\[
(52) \quad q_{n,N} \equiv \text{Prob} (\ell_N \leq n) = \frac{\# \{\pi : \ell_N(\pi) \leq n\}}{N!}
\]

**Question.** How does \(q_{n,N}\) behave as \(n, N \to \infty\)?

**Theorem** (Baik-Deift-Johansson, 1999 [BDJ]). Let \(t \in \mathbb{R}\) be given. Then

\[
(53) \quad F(t) \equiv \lim_{N \to \infty} \text{Prob} \left( \ell_N \leq 2\sqrt{N + t N^{1/6}} \right)
\]

exists and is given by \(e^{-\int_{-\infty}^{\infty} \frac{(x-t)^2}{4} dx}\) where \(u(x)\) is the (unique) Hastings-McLeod solution of the Painlevé II equation

\[
(54) \quad u'' = 2u^3 + xu
\]

normalized such that

\[
u(x) \sim Ai(x) = \text{Airy function, as } x \to +\infty
\]
(The original proof of this Theorem used RHP/steepest descent methods. The proof was later simplified by Borodin, Olshanski and Okounkov using the so-called Borodin-Okounkov-Case-Geronimo formula.)

Some observations:

(i) As Painlevé II is classically integrable, we see that the map

\[ q_{n,N} \xrightarrow{\varphi} u^2(t) = -\frac{d^2}{dx^2} \log F(x) \]

transforms Ulam’s longest increasing subsequence problem into an integrable system whose behavior is known with precision. There are many other classical integrable systems associated with \( q_{n,N} \) but that is another story (see Baik, Deift, Suidan (2016) [BDS]).

(ii) The distribution \( F(t) = e^{-\int_t^\infty (x,t) u^2(x) dx} \) is the famous Tracy-Widom distribution for the largest eigenvalue \( \lambda_{\text{max}} \) of a random Hermitian matrix in the edge-scaling limit. In other words, the length of the longest increasing subsequence behaves like the largest eigenvalue of a random Hermitian matrix. More broadly, what we are seeing here is an example of how RMT plays the role of a stochastic special function theory describing a stochastic problem from some other a priori unrelated area. This is no different, in principle, from the way the trigonometric functions describe the behavior of the simple harmonic oscillator. RMT is a very versatile tool in our IM toolbox — tiling problems, random particle systems, random growth models, the Riemann zeta function, . . . , all the way back to Wigner, who introduced RMT as a model for the scattering resonances of neutrons off a large nucleus, are all problems whose solution can be expressed in terms of RMT.

(iii) \( F(t) \) can also be written as

\[ F(t) = \det (1 - A_t) \]

where \( A_t \) is a particular trace class integrable operator, the Airy operator, with \( 0 \leq A_t < 1 \). Thus \( F(t) = \prod_{i=1}^\infty \left( 1 - \hat{\lambda}_i(t) \right) \) where \( \{\hat{\lambda}_i(t)\} \) are the eigenvalues of \( A_t \). We conclude that \( F(t) \), the (limiting) distribution for the length \( \ell_N \) of the longest increasing
subsequence, corresponds to an integrable system in the above intrinsic probabilistic sense.

(iv) It is of considerable interest to note that in recent work Gavrylenko and Lisovyy \cite{GaLi} have shown that the isomonodromic tau function for general Fuchsian systems can be expressed, up to an explicit elementary function, as a Fredholm determinant of the form $\det (1 - K)$ for some suitable trace class operator $K$. Expanding the determinant as a product of eigenvalues, we see that the general Fuchsian system, too, is integrable in the above intrinsic stochastic sense.

Another tool in our toolbox concerns the notion of a scattering system. Consider the Toda lattice in $\left( \mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i \right)$ with Hamiltonian

\begin{equation}
H_T(x, y) = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n-1} e^{(x_i - x_{i+1})}
\end{equation}

(56)

giving rise to Hamilton’s equations

\begin{equation}
\dot{x} = (H_T)_y, \quad \dot{y} = -(H_T)_x.
\end{equation}

(57)

The scattering map for a dynamical system maps the behavior of the system in the distant past onto the behavior of the system in the distant future. In my PhD I worked on abstract scattering theory in Hilbert space addressing questions of asymptotic completeness for quantum systems and classical wave systems. When I came to Courant I started to study the Toda system and I was amazed to learn that for this multi-particle system the scattering map could be computed explicitly. When I expressed my astonishment to Jürgen Moser, he said to me, “But every scattering system is integrable!” It took me some time to understand what he meant. It goes like this:

Suppose that you have a Hamiltonian system in $\left( \mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i \right)$ with Hamiltonian $H$, and suppose that the solution

$$z(t) = (x(t), y(t)) \quad z(0) = (x(0), y(0)) = (x_0, y_0)$$

of the flow generated by $H$ behaves asymptotically like the solutions $\hat{z}(t)$ of free motion with Hamiltonian

$$\hat{H}(x, y) = \frac{1}{2} y^2$$
for which
\[ \dot{x} = y, \quad \dot{y} = 0 \quad \text{with} \quad \dot{z}(0) = (\dot{x}_0, \dot{y}_0), \]
yielding
\[ \hat{z}(t) = (\dot{x}_0 + \dot{y}_0 t, \dot{y}_0). \]
As \( z(t) \sim \dot{z}(t) \) by assumption, we have as \( t \to \infty \),

\begin{align*}
(58a) \quad & x(t) = ty^\# + x^\# + o(1) \\
(58b) \quad & y(t) = y^\# + o(1)
\end{align*}

for some \( x^\#, \ y^\# \).

Write
\[ z(t) = U_t(z(0)), \quad \dot{z}(t) = \hat{U}_t(\dot{z}(0)). \]
Then, provided \( o(1) = o \left( \frac{1}{t} \right) \) in (58b),
\[ W_t(z_0) \equiv \hat{U}_{-t} \circ U_t(z_0) \]
\[ = \hat{U}_{-t} \left( ty^\# + x^\# + o(1), \quad y^\# + o \left( \frac{1}{t} \right) \right) \]
\[ = \left( ty^\# + x^\# + o(1) - t \left( y^\# + o \left( \frac{1}{t} \right) \right), \quad y^\# + o \left( \frac{1}{t} \right) \right) \]
\[ = \left( x^\# + o(1), \quad y^\# + o(1) \right). \]
Thus
\[ W_\infty(z_0) = \lim_{t \to \infty} W_t(z_0) \]
exists. Now
\[ W_t \circ U_s = \hat{U}_{-t} \circ U_{t+s} \]
\[ = \hat{U}_s \circ W_{t+s}, \]
and letting \( t \to \infty \), we obtain

\begin{equation}
(59) \quad W_\infty \circ U_s = \hat{U}_s \circ W_\infty
\end{equation}
so that \( W_\infty \) is an intertwining operator between \( U_s \) and \( \hat{U}_s \).
But clearly $W_t$ is the composition of symplectic maps, and so is symplectic, and hence $W_\infty$ is a symplectic map and hence $W^{-1}_\infty$ is symplectic. Thus from (59) we see that
\begin{equation}
U_s = W^{-1}_\infty \circ \hat{U}_s \circ W_\infty
\end{equation}
is symplectically equivalent to free motion, and hence is integrable. In particular if $\{\hat{\lambda}_k\}$ are the Poisson commuting integrals for $\hat{H}$, then $\{\lambda_k = \hat{\lambda}_k \circ W_\infty\}$ are the (Poisson commuting) integrals for $H$.

What this computation is telling us is that if a system is scattering, or more generally, if the solution of one system looks asymptotically like some other system, then it is in fact (equivalent to) that system. Remember the famous story of Roy Cohn during the McCarthy hearings, when he was trying to convince the panel that a particular person was a Communist? He said: “If it looks like a duck, walks like a duck, and quacks like a duck, then it’s a duck!”

Now direct computations, due originally to Moser, show that the Toda lattice is scattering in the sense of (58(a)(b)). And so what Moser was saying is that the system is necessarily integrable. The Toda lattice is a rich and wonderful system and I spent much of the 1980’s analyzing the lattice and its various generalizations together with Carlos Tomei, Luen-Chau Li and Tara Nanda. I will say much more about this system below. It was a great discovery of Flaschka [Fla] (and later independently, Manakov [Man]) that the Toda system indeed had a Lax pair formulation (see (74) below).

The idea of a scattering system can be applied to PDE’s. Some 15–20 years ago Xin Zhou and I [DZ3] began to consider perturbations of the defocussing NLS equation on the line,
\begin{equation}
i u_t + u_{xx} - 2|u|^2 u - \epsilon |u|^\ell u = 0, \quad \ell > 2
\end{equation}
with
\[
u(x, t = 0) = u_0(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]
In the spatially periodic case, $u(x, t) = u(x + 1, t)$, solutions of NLS (the integrable case: $\epsilon = 0$) move linearly on a (generically infinite dimensional) torus. In the perturbed case ($\epsilon \neq 0$), KAM methods can be (extended and ) applied (with great technical virtuosity) to
show (here Kuksin, Pöschel, Kappeler have played the key role) that, as in the familiar finite
dimensional case, some finite dimensional tori persist for (61) under perturbation. However,
on the whole line with \( u_0(x) \to 0 \) as \( |x| \to \infty \), the situation, as we now describe, is very
different.

In the spirit of it “walks like a duck”, what is the “duck” for solutions of (61)? The
“duck” here is a solution \( u^\#(x,t) \) of the NLS equation.

\[
i u_t^\# + u_{xx}^\# - 2|u^\#|^2 u^\# = 0
\]
\[
u^\#(x,0) = u_0^\#(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]

Recall the following calculations from classical KAM theory in \( \mathbb{R}^{2n} \), say: Suppose that
the flow with Hamiltonian \( H_0 \) is integrable and \( H_\epsilon = H_0 + \epsilon \hat{H} \) is a perturbation of \( H_0 \).

Hamilton’s equation for \( H_\epsilon \) has the form

\[
z_t = J \nabla H_\epsilon = J \nabla H_0 + \epsilon J \nabla \hat{H}, \quad z(0) = z_0
\]

with \( J = ( O \quad I_n \quad -I_n \quad 0 ) \). If \( J \nabla H_0 \) is linear in \( z \), say \( J \nabla H_0 = Az \), then we can solve (63) by
D’Alembert’s principle to obtain

\[
z(t) = e^{At} z_0 + \epsilon \int_0^t e^{A(t-s)} J \nabla \hat{H}(s) \, ds
\]

to which an iteration procedure can be applied. If \( J \nabla H_0 \) is not linear, however, no such
D’Alembert formula exist, and this is the reason that the starting point for any KAM investi
gation is to first write (63) in action-angle variables \( z \mapsto \zeta \) for \( H_0 \): Then \( J \nabla H_0 \) is linear
and (64) applies.

With this in mind, we used the linearizing map for NLS described in (26)

\[
u(x,t) = u(t) \mapsto \varphi(u(t)) = r(t) = r(t;z)
\]
as a change of variables for the perturbed equation (61). And although the map \( \varphi \) no longer
linearizes the equation, it does transform the equation into the form

\[
\frac{\partial r}{\partial t}(t,z) = -i z^2 r(t,z) + \epsilon F(z,t;r(t))
\]

to which D’Alembert’s principle can be applied

\[
\hat{r}(t,z) = r_0(z) + \epsilon \int_0^t F(z,s;\hat{r}e^{-i\epsilon s^2}) \, ds
\]
where \( r_0(z) = \varphi(z) \) and \( \hat{r}(t, z) = r(t, s) e^{itz^2} \). The functional \( F \) depends on \( \varphi \) and \( \varphi^{-1} \), and so, in particular, involves the RHP \( (\Sigma = \mathbb{R}, v_t) \). Fortunately this RHP can be evaluated with sufficient accuracy using steepest descent methods in order to obtain the asymptotics of \( \hat{r}(t, z) \) as \( t \to \infty \), and hence of \( u(x, t) = \varphi^{-1} \left( \hat{r}(t) e^{-itx^2} \right) \).

Let \( U_t^\epsilon(u_0) \) be the solution of (61) and \( U_t^{NLS}(u_0^\#) \) be the solution of NLS (62) with \( u_0, u_0^\# \) in \( H^{1,1} = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}), \quad xf \in L^2(\mathbb{R}) \} \), respectively. Then the upshot of this analysis is, in particular, that

\[
W^\pm(u_0) = \lim_{t \to \pm \infty} U_{-t}^{NLS} \circ U_t^\epsilon(u_0)
\]

exist strongly which shows that as \( t \to \pm \infty \),

\[
U_t^\epsilon(u_0) \sim U_t^{NLS}(W^\pm(u_0))
\]

and much more. In particular, there are commuting integrals for (61), \ldots,

Three observations:

(a) As opposed to KAM where integrability is preserved on sets of high measure, here integrability is preserved on open subsets of full measure.

(b) As a tool in our IM toolbox, integrability makes it possible to analyze perturbations of integrable systems, via a D’Alembert principle.

(c) There is a Catch 22 in the whole story. Suppose you say, “my goal is describe the evolution of solutions of the perturbed equation (61) as \( t \to \infty \)”. To do this one must have in mind what the solutions should look like as \( t \to \infty \): Do they look like solutions of NLS, or perhaps like solutions of the free Schrödinger equation \( iu_t + uu_{xx} = 0 \)? Now suppose you disregard any thoughts on integrability and utilize any method you can think of, dynamical systems ideas, etc., to analyze the system and you find in the end that the solution indeed behaves like NLS. But here’s the catch; if it looks like NLS, then the wave operators \( W_\pm \) in (67) exist, and hence the system is integrable! It looks like a duck, walks like a duck and quacks like a duck, and so it’s a duck! In other words, whatever methods you used, they would not have succeeded unless the system was integrable in the first place!
Finally, I would like to discuss briefly an extremely useful algebraic tool in the IM toolbox, viz., Darboux transforms/Backlund transformations. These are explicit transforms that convert solutions of one (nonlinear) equation into solutions of another equation, or into different solutions of the same equation. For example, the famous Miura transform, a particular Darboux/Backlund transform,

$$v(x, t) \rightarrow u(x, t) = v_x(x, t) + v^2(x, t)$$

converts solutions $v(x, t)$ of the modified KdV equation

$$v_t + 6v^2v_x + v_{xxx} = 0$$

into solutions of the KdV equation

$$u_t + 6uu_x = u_{xxx} = 0.$$  

Darboux transforms can be used to turn a solution of KdV without solitons into one with solitons, etc. Darboux/Backlund transforms also turn certain spectral problems into other spectral problems with (essentially) the same spectrum, for example,

$$H = - \frac{d^2}{dx^2} + q(x) \quad \rightarrow \quad \tilde{H} = - \frac{d^2}{dx^2} + \tilde{q}(x)$$

where

$$\tilde{q} = q - 2 \frac{d^2}{dx^2} \log \varphi,$$

and $\varphi$ is any solution of $H\varphi = 0$,

constructs $\tilde{H}$ with (essentially) the same spectrum as $H$. Thus a Darboux/Backlund transform is a basic isospectral action. The literature on Darboux transforms is vast, and I just want to discuss one application to PDE’s which is perhaps not too well known.

Consider the Gross-Pitaevskii equation in one-dimension,

$$i u_t + \frac{1}{2} u_{xx} + V(x) u + |u|^2 u = 0$$

$$u(x, 0) = u_0(x).$$

For general $V$ this equation is very hard to analyze. A case of particular interest is where

$$V(x) = q \delta(x), \quad q \in \mathbb{R} \quad \text{and} \quad \delta \text{ is the delta function.}$$
For such \( V \), (68) has a particular solution

\[
(70) \quad u_\lambda(x, t) = \lambda e^{i\lambda^2 t/2} \text{sech} \left( \lambda |x| + \tanh^{-1} \left( \frac{q(x)}{\lambda} \right) \right)
\]

for any \( \lambda > |q| \). This solution is called the Bose-Einstein condensate for the system.

**Question.** Is \( u_\lambda \) asymptotically stable? In particular, if

\[
(71) \quad u(x, t = 0) = u_\lambda(x, t = 0) + \epsilon w(x) \quad , \quad \epsilon \quad \text{small},
\]

does

\[
u(x, t) \rightarrow u_\lambda(x, t) \quad \text{as} \quad t \to \infty?
\]

In the case where \( w(x) \) is even, one easily sees that the initial value problem (IVP) (68) with initial value given by (71) is equivalent to the initial boundary value problem (IBVP)

\[
(72) \quad \begin{cases}
iu_t + \frac{1}{2} u_{xx} + |u|^2 u = 0, & x > 0, \quad t > 0 \\
u(x, t = 0) = (71) \quad \text{for} \quad x > 0 \\
u_x(0, t) + qu(0, t) = 0.
\end{cases}
\]

Now NLS on \( \mathbb{R} \) is integrable, but is NLS on \( \{x > 0\} \) with boundary conditions as in (72) integrable? Remember that the origin of the boundary condition is the physical potential (read “force”!) \( V(x) \). So we are looking at a dynamical system, which is integrable on \( \mathbb{R} \), interacting with a new “force” \( V \). It is not at all clear, a priori, that the combined system is integrable in the sense of (13ab).

The stability question for \( u_\lambda \) was first consider by Holmer and Zworski (2009) \[HoZ\], and using dynamical systems methods, they showed asymptotic stability of \( u_\lambda \) for times of order \( |q|^{-2/7} \). But what about times larger than \( |q|^{-2/7} \)? Following on the work of Holmer and Zworski, Jungwoon Park and I \[DeP\] begin in 2009 to consider this question. Central to our approach was to try to show that the IBVP for NLS as in (72) was integrable, and then use RH/steepest-descent methods. In the linear case, a standard approach is to use the method of images: for Dirichlet and Neumann boundary conditions, one just reflects, \( u(x) = -u(-x) \) or \( u(x) = +u(-x) \) for \( x < 0 \), respectively.
For the Robin boundary condition in the linear case, the reflection is a little more complicated, but still standard. In this way one then gets an IVP on the line that can be solved by familiar methods. In the non-linear case, similar methods work for the Dirichlet and Neumann boundary conditions, but for the Robin boundary condition case, \( q \neq 0 \), how should one reflect across \( x = 0 \)? It turns out that there is a beautiful method due to Bikbaev and Tarasov where they construct a particular Darboux transform version \( b(x) \) of the initial data \( u(x, t = 0), x > 0 \), and then define

\[
\begin{align*}
    v(x) &= b(-x) \quad x < 0 \\
    &= u(x, t = 0) \quad x > 0.
\end{align*}
\]

If \( v(x, t) \) is the solution of (the integral equation) of NLS on \( \mathbb{R} \) with initial conditions (73), then \( v(x, t) \big|_{x>0} \) is a solution of the IBVP (72) for \( t \geq 0 \). In other words, the Darboux transform can function as a tool in our toolkits to show that a system is integrable.

Applying RH/steepest descent methods to \( v(x, t) \), one finds that \( u_\lambda \) is asymptotically stable if \( q > 0 \), but for \( q < 0 \), generically, \( u_\lambda \) is not asymptotically stable: In particular, for times \( t >> |q|^{-2} \), as \( t \to \infty \), a second “soliton” emerges and one has a “two soliton” condensate.

We note that (72) can also be analyzed using Fokas’ unified integration method instead of the Bikbaev-Tarasov transform, as in Its-Shepelsky (2012) [ISh].

**Algorithms**

As discussed above, the Toda lattice is generated by the Hamiltonian

\[
H_T(x, y) = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n-1} e^{(x_i-x_{i+1})}.
\]

The key step in analyzing the Toda lattice was the discovery by Flaschka [Fla], and later independently by Manakov [Man], that the Toda equations have a Lax-pair formulation

\[
\begin{align*}
    \frac{dx}{dt} &= H_{T,y}, \\
    \frac{dy}{dt} &= -H_{T,x} \\
    \frac{dM}{dt} &= [M, B(M)]
\end{align*}
\]
where
\[
M = \begin{pmatrix}
a_1 & b_1 \\
b_1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & b_{n-1} \\
b_{n-1} & \cdots & a_n
\end{pmatrix}, \quad B(M) = \begin{pmatrix}
0 & -b_1 \\
b_1 & 0 & -b_2 & 0 \\
& b_2 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & 0 & -b_{n-1} \\
& & & & b_{n-1} & 0
\end{pmatrix} = M - M^T
\]
and
\[
a_k = -y_k/2, \quad 1 \leq k \leq n \\
b_k = \frac{1}{2} e^{\frac{1}{2}(x_k-x_{k+1})}, \quad 1 \leq k \leq n - 1.
\]  
(75)

In particular, the eigenvalues \( \{\lambda_n\} \) of \( M \) are constants of the motion for Toda, \( \{\lambda_n(t) = \lambda_n, t \geq 0\} \). Direct calculation shows that they are independent and Poisson commute, so that Toda is Liouville integrable. Now as \( t \to \infty \), one can show, following Moser (1975), that the off diagonal entries \( b_k(t) \to 0 \) as \( t \to \infty \). As noted by Deift, Nanda and Tomei (1983) [DNT], what this means is that Toda gives rise to an eigenvalue algorithm:

Let \( M_0 \) be given and let \( M(t) \) solve the Toda equations (74) with \( M(0) = M_0 \). Then
\[
\begin{align*}
\bullet & \quad t \mapsto M(t) \text{ is isospectral, } \text{spec} \, M(t) = \text{spec} \, M_0. \\
\bullet & \quad M(t) \to \text{diag} \, (\lambda_1, \ldots, \lambda_n) \quad \text{as} \quad t \to \infty.
\end{align*}
\]  
(76)

Hence \( \lambda_1, \ldots, \lambda_n \) must be the eigenvalues of \( M_0 \).

Note that \( H_T(M) = \frac{1}{2} \, \text{tr} \, M^2 \).

Now the default algorithm for eigenvalue computation is the QR algorithm. The algorithm without “shifts” works in the following way. Let \( M_0 = M_0^T \), \( \det M_0 \neq 0 \), be given, where \( M_0 \) is \( n \times n \).

Then \( M_0 \) has a unique QR-factorization
\[
M_0 = Q_0 \, R_0, \quad Q_0 \text{ orthog, } R_0 \text{ upper triangular}
\]  
(77)

with \( (R_0)_{ii} > 0 \), \( i = 1, \ldots, n \).

Set
\[
M_1 \equiv R_0 \, Q_0 = Q_0^T \, M_0 \, Q_0
\]

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from which we see that 
\[ \text{spec } (M_1) = \text{spec } M_0. \]

Now \(M_1\) has its own QR-factorization 
\[ M_1 = Q_1 R_1 \]

Set 
\[ M_2 = R_1 Q_1 \]
\[ = Q_1^T M_1 Q_1 \]

so that again \(\text{spec } M_2 = \text{spec } M_1 = \text{spec } M_0.\)

Continuing, we obtain a sequence of isospectral matrices 
\[ \text{spec } M_k = \text{spec } M_0 \ , \ k > 0, \]

and as \(k \to \infty\), generically, 
\[ M_k \to \text{diag } (\lambda_1, \ldots, \lambda_n) \]

and again \(\lambda_1, \ldots, \lambda_n\) must be the eigenvalues of \(M_0\). If \(M_0\) is tridiagonal, one verifies that 
\(M_k\) is tridiagonal for all \(k\).

There is the following Stroboscope Theorem for the QR algorithm (Deift, Nanda, Tomei (1983) [DNT]), which is motivated by earlier work of Bill Symes [Sym]:

(78) Theorem (QR: tridiagonal)

Let \(M_0 = M_0^T\) be tri-diagonal. Then there exists a Hamiltonian flow \(t \mapsto M_{QR}(t)\), 
\(M_{QR}(0) = M_0\) with Hamiltonian 
\[ H_{QR}(M) = \text{tr } (M \log M - M) \]

with the properties

(i) the flow is completely integrable

(ii) (Stroboscope property) \(M_{QR}(k) = M_k, \ k \geq 0, \) where \(M_k\) are the QR iterates 
starting at \(M_0\), \(\det M_0 \neq 0\)
(iii) $M_{QR}(t)$ commutes with the Toda flow

(iv) $\frac{dM}{dt} = [B(\log M), \ M]$, $B(\log M) = (\log M)_- - (\log M)_T$.

More generally, for any $G : \mathbb{R} \to \mathbb{R}$

$$H_G(M) = \text{tr} \ G(M)$$

$$\to \dot{M}_G = [M, B(\text{g}(M))] \text{ , } \text{g}(M) = G'(M)$$

generates an eigenvalue algorithm, so in a concrete sense, we can say, at least in the tri-diagonal case, that eigenvalue computation is an integrable process.

Now the Lax equation (74) for Toda clearly generates a global flow $t \mapsto M(t)$ for all full symmetric matrices $M_0 = M_0^T$.

Question. (i) Is the Toda flow for general symmetric matrices $M_0$ Hamiltonian?

(ii) Is it integrable?

(iii) Does it constitute an eigenvalue algorithm i.e. $\text{spec } (M(t)) = \text{spec } (M_0)$, $M(t) \to \text{diagonal as } t \to \infty$?

(iv) Is there a stroboscope theorem for general $M_0$?

As shown in [DLNT], the answer to all these questions is in the affirmative. Property (ii) is particularly novel. The Lax-pair for Toda only gives $n$ integrals, viz. the eigenvalues of $M(t)$, but the dimension of the symplectic space for the full Toda is generically of dimension $2 \left\lfloor \frac{n^2}{4} \right\rfloor$, so one needs of order $\frac{n^2}{4} >> n$ Poisson commuting integrals. These are obtained in the following way: consider, for example, the case $n = 4$. Then $\left\lfloor \frac{n^2}{4} \right\rfloor = 4$

- $\det (M - z) = 0$ has 4 roots $\lambda_{01}, \lambda_{02}, \lambda_{03}, \lambda_{04}$
- $\det (M - z)_1 = 0$ has 2 roots $\lambda_{11}, \lambda_{12}$

where $(M - z)_1$ is obtained by chopping off the 1st row and last column of $M - z$
Now \[ \lambda_{-1} + \lambda_{02} + \lambda_{03} + \lambda_{04} = \text{trace } M \]
and \[ \lambda_{11} + \lambda_{12} = \text{“trace” of } M_1 \]

are the co-adjoint invariants that specify the \( 8 = 10 - 2 = 2 \left[ \frac{2^2}{2} \right] \) dimensional symplectic leaf \( \mathcal{L}_{c_1, c_2} = \{ M : \text{tr } M = c_1, \text{tr } M_1 = c_2 \} \) on which the Toda flow is generically defined. The four independent integrals needed for integrability are then \( \lambda_{01}, \lambda_{02}, \lambda_{03}, \lambda_{11} \). For general \( n \), we keep chopping: \((M - z)I_2\) is obtained by chopping off the first two rows and last two columns, etc. The existence of these “chopped” integrals, and their Poisson commutativity follows from the invariance properties of \( M \) under the actions of a tower of groups, \( G_1 \subset G_2 \subset \ldots \). This shows that group theory is also a tool in the IM toolbox. This is spectacularly true in the work of Borodin and Okshanski on “big” groups like \( S_\infty \) and \( U_\infty \), and related matters.

Thus we conclude that eigenvalue computation in the full symmetric case is again an integrable process.

Remark. The answer to Questions (i) . . . (iv) is again in the affirmative for general, not necessarily symmetric matrices \( M \in M(n, \mathbb{R}) \). Here we need \( \sim \frac{n^2}{2} \) integrals . . . , but this is a whole other story (Deift, Li, Tomei (1989) [DLT]).

The question that will occupy us in the remainder of this paper is the following: We have discussed two notions of integrability naturally associated with matrices: Eigenvalue algorithms and random matrix theory. What happens if we try to combine these two notions? In particular,

\[ (80) \quad \text{“What happens if we try to compute the eigenvalues of a random matrix?”} \]

Let \( \Sigma_N \) denote the set of real \( N \times N \) symmetric matrices. Associated with each algorithm \( \mathcal{A} \), there is, in the discrete case, such as \( QR \), a map \( \varphi = \varphi_\mathcal{A} : \Sigma_N \rightarrow \Sigma_N \) with the properties

- isospectrality: \( \text{spec } (\varphi_\mathcal{A}(H)) = \text{spec } (H) \)
- convergence: the iterates \( X_{k+1} = \varphi_\mathcal{A}(X_k), \quad k \geq 0, \quad X_0 = H \) given, converge to a diagonal matrix \( X_\infty, \quad X_k \rightarrow X_\infty \) as \( k \rightarrow \infty \)
and in the continuum case, such as Toda, there exists a flow $t \mapsto X(t) \in \Sigma_N$ with the properties

- isospectrality: $\text{spec } (X(t)) = \text{spec } (X(0))$
- convergence: $X(t)$ converges to a diag. matrix $X_\infty$ as $t \to \infty$.

In both cases, necessarily, the diagonal entries of $X_\infty$ are the eigenvalues of $H$.

Given $\epsilon > 0$, it follows, in the discrete case, that for some $m$ the off-diagonal entries of $X_m$ are $O(\epsilon)$ and hence the diagonal entries of $X_m$ give the eigenvalues of $H$ to $O(\epsilon)$. The situation is similar for continuous flows $t \mapsto X(t)$. Rather than running the algorithm until all the off-diagonal entries are $O(\epsilon)$, it is customary to run the algorithm with deflations as follows: For an $N \times N$ matrix $Y$ in block form

$$ Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} $$

with $Y_{11}$ of size $k \times k$ and $Y_{22}$ of size $(N - k) \times (N - k)$ for some $k \in \{1, 2, \ldots, N - 1\}$, the process of projecting

$$ Y \mapsto \text{diag } (Y_{11}, Y_{22}) $$

is called deflation. For a

(81) \quad \text{given } \epsilon > 0, \text{ algorithm } \mathcal{A}, \text{ and matrix } H \in \Sigma_N

define the $k$-deflation time.

(82) \quad T^{(k)}(H) = T_{\epsilon, \mathcal{A}}^{(k)}(H) \quad , \quad 1 \leq k \leq N - 1 ,

to be the smallest value of $m$ such that $X_m$, the $m^{th}$ iterate of $\mathcal{A}$ with $X_0 = H$, has block form

$$ X_m = \begin{pmatrix} X_{11}^{(k)} & X_{12}^{(k)} \\ X_{21}^{(k)} & X_{22}^{(k)} \end{pmatrix} $$

$X_{11}^{(k)}$ is $k \times k$, $X_{22}^{(k)}$ is $(N - k) \times (N - k)$ with

(83) \quad \|X_{12}^{(k)}\| = \|X_{22}^{(k)}\| < \epsilon.
The deflation time \( T(H) \) is then defined as

\[
T(H) = T_{\epsilon, \mathcal{A}}(H) = \min_{1 \leq k \leq N-1} T_{\epsilon, \mathcal{A}}^{(k)}(H)
\]

(84)

If \( \hat{k} \in \{1, 2, \ldots, N-1\} \) is such that

\[
T(H) = T_{\epsilon, \mathcal{A}}^{(\hat{k})}(H)
\]

it follows that the eigenvalues of \( H \) are given by the eigenvalues of the block diagonal matrix 
\[
\text{diag} \left( X_{11}^{(\hat{k})}, X_{22}^{(\hat{k})} \right)
\]
to \( O(\epsilon) \). After, running the algorithm to time \( T(H) \), the algorithm restarts by applying the basic algorithm (in parallel) to the smaller matrices \( X_{11}^{(\hat{k})} \) and \( X_{22}^{(\hat{k})} \) until the next deflation time, and so on.

In 2009, Deift, Menon, Pfrang [DMP] considered the deflation time \( T = T_{\epsilon, \mathcal{A}} \) for \( N \times N \) matrices chosen from an ensemble \( \mathcal{E} \). For a given algorithm \( \mathcal{A} \) and ensemble \( \mathcal{E} \) the authors computed \( T(H) \) for 5000–to 15000 samples of matrices \( H \) chosen from \( \mathcal{E} \) and recorded the normalized deflation time

\[
\tilde{T}(H) \equiv \frac{T(H) - \langle T \rangle}{\sigma}
\]

(85)

where \( \langle T \rangle \) is the sample average and \( \sigma^2 \) is the sample variance for \( T(H) \) for the 5,000 to 15,000 above samples. Surprisingly, the authors found that

\[
\begin{cases}
\text{for a given } \epsilon \text{ and } N, \text{ in a suitable scaling regime (} \epsilon \text{ small, } N \text{ large),} \\
\text{the histogram of } \tilde{T} \text{ was universal,} \\
\text{independent of the ensemble } \mathcal{E}.
\end{cases}
\]

(86)

In other words the fluctuations in the deflation time \( T \), suitably scaled, were universal independent of \( \mathcal{E} \).
Here are some typical results of their calculations (displayed in a form slightly different from [DMP])

Figure 3: Universality for $\tilde{T}$ when (a) $A$ is the QR eigenvalue algorithm and when (b) $A$ is the Toda algorithm. Panel (a) displays the overlay of two histograms for $\tilde{T}$ in the case of QR, one for each of the two ensembles $\mathcal{E} = BE$, consisting of iid mean-zero Bernoulli random variables and $\mathcal{E} = GOE$, consisting of iid mean-zero normal random variables. Here $\epsilon = 10^{-10}$ and $N = 100$. Panel (b) displays the overlay of two histograms for $\tilde{T}$ in the case of the Toda algorithm, and again $\mathcal{E} = BE$ or GOE. And here $\epsilon = 10^{-8}$ and $N = 100$.

Subsequently in 2014, Deift, Menon, Olver, Trogdon [DMOT] raised the question of whether the universality results of [DMP] were limited to eigenvalue algorithms for real symmetric matrices or whether they were present more generally in numerical computation. And indeed, the authors in [DMOT], found similar universality results for a wide variety of numerical algorithms, including

- other eigenvalue algorithms such as $QR$ with shifts, the Jacobi eigenvalue algorithm, and also algorithms applied to complex Hermitian ensembles
- conjugate gradient (CG) and GMRES (Generalized minimal residual algorithm) algorithms to solve linear $N \times N$ systems $Hx = b$ where $b = (b_1, \ldots, b_N)$ is i.i.d, and

$$H = XX^T, \quad X \text{ is } N \times m \text{ and random for CG}$$
and

\[ H = I + X, \quad X \text{ is } N \times N \text{ is random for GMRES} \]

- an iterative algorithm to solve the Dirichlet problem \( \Delta u = 0 \) on a random star shaped region \( \Omega \subset \mathbb{R}^2 \) with random boundary data \( f \) on \( \partial \Omega \). (Here the solution is constructed via the double layer potential method.)

- a genetic algorithm to compute the equilibrium measure for orthogonal polynomials on the line

- a decision process investigated by Bakhtin and Correll [BaCo] in experiments using live participants.

All of the above results were numerical/experimental. In order to establish universality in numerical computation as a bona fide phenomenon, and not just an artifact suggested, however strongly, by certain computations as above, it was necessary to prove universality rigorously for an algorithm of interest. In 2016 Deift and Trogdon [DT1] considered the 1-deflation time \( T(1) \) for the Toda algorithm. Thus one runs Toda \( t \mapsto X(t), X(0) = H \), until time \( t = T(1) \) for which

\[
E \left( T^{(1)} \right) = \sum_{j=2}^{N} \left| X_{1j} \left( T^{(1)} \right) \right|^2 < \epsilon^2.
\]

Then \( X_{11} \left( T^{(1)} \right) \) is an eigenvalue of \( H \) to \( O(\epsilon) \). As Toda is a sorting algorithm, almost surely

(87) \[
| X_{11} \left( T^{(1)} \right) - \lambda_{\text{max}} | < \epsilon
\]

where \( \lambda_{\text{max}} \) is the largest eigenvalue of \( H \). Thus the Toda algorithm with stopping time given by the 1-deflation time is an algorithm to compute the largest eigenvalue of a real symmetric (or Hermitian) matrix.

Here is the result in [DT1] for \( \beta = 1 \) (real symmetric case) and \( \beta = 2 \) (Hermitian case). Order the eigenvalues of a real symmetric or Hermitian matrix by \( \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_N \). Then

(88) \[
F_{\beta}^{\text{gap}}(t) \equiv \lim_{N \to \infty} \text{Prob} \left( \frac{1}{C_V 2^{-2/3} N^{2/3} (\lambda_N - \lambda_{N-1})} \leq t \right), \quad t \geq 0
\]
exists and is universal for a wide range of invariant and Wigner ensembles. $F^{\text{gap}}_\beta(t)$ is clearly the distribution function of the inverse of the top gap $\lambda_N - \lambda_{N-1}$ in the eigenvalues. Here $C_V$ is an ensemble dependent constant.

**Theorem** (Universality for $T^{(1)}$). Let $\sigma > 0$ be fixed and let $(\epsilon, N)$ be in the scaling region

$$L \equiv \frac{\log^{-1}\epsilon}{\log N} \geq \frac{5}{3} + \frac{\sigma}{2}$$

Then if $H$ is distributed according to any real ($\beta = 1$) or complex ($\beta = 2$) invariant or Wigner ensemble, we have

$$\lim_{N\to\infty} \mathbb{P}(\frac{T^{(1)}}{C_V^{2/3} 2^{-2/3} N^{2/3} (\log^{-1} \epsilon - 2/3 \log N)} \leq t) = F^{\text{gap}}_\beta(t).$$

Thus $T^{(1)}$ behaves statistically like the inverse gap $(\lambda_N - \lambda_{N-1})^{-1}$ of a random matrix.

Now for $(\epsilon, N)$ in the scaling region, $N^{2/3} (\log^{-1} \epsilon - 2/3 \log N) = N^{2/3} \log N (\alpha - 2/3)$, and it follows that $\text{Exp} (T^{(1)}) \sim N^{2/3} \log N$. This is the first such precise estimate for the stopping time for an eigenvalue algorithm: Mostly estimates are in the form of upper bounds, which are often too big because the bounds must take worst case scenarios into account.

**Notes.**

- The proof of this theorem uses the most recent results on the eigenvalues and eigenvalues of invariant and Wigner ensembles by (Yau, Erdös, Schlein, Bourgade . . . , and others (see e.g. [EY]).

- Similar universality results have now been proved (Deift and Trogdon (2017) [DT2]) for QR acting on positive definite matrices, the power method and the inverse power method.

- The theorem is relevant in that the theorem describes what is happening for “real life” values of $\epsilon$ and $N$. For example, for $\epsilon = 10^{-16}$ and $N \leq 10^9$, we have $\frac{\log^{-1}\epsilon}{\log N} \geq \frac{16}{9} > \frac{5}{3}$.

- Once again RMT provides a stochastic function theory to describe an integrable stochastic process, viz., 1-deflation. But the reverse is also true. Numerical algorithms with random data, raise new problems and challenges within RMT!
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