An Ergodic Theorem for PSPACE functions

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December 22, 2020

Abstract

We initiate the study of effective pointwise ergodic theorems in resource-bounded settings. Classically, the convergence to the ergodic averages for integrable functions can in general, be arbitrarily slow [11]. However, we show that for PSPACE L¹ functions and a class of PSPACE computable measure-preserving ergodic transformations, the ergodic average exists and is equal to the space average on every EXPSPACE random. Further, we show that the class of EXPSPACE randoms is a strict subset of the class of PSPACE randoms studied by Huang and Stull [9].

1 Introduction

In Kolmogorov’s program to found information theory on the theory of algorithms, we investigate whether individual “random” objects obey probabilistic laws, i.e., properties which hold in sample spaces with probability 1. Indeed, a vast and growing literature establishes that every Martin-Löf random sequence (see for example, [3] or [13]) obeys the Strong Law of Large Numbers [17], the Law of Iterated Logarithm [18], and surprisingly, the Birkhoff Ergodic Theorem [19], [12], [7], [1] and the Shannon-McMillan-Breiman theorem [5], [6]. In effective settings, the theorem for Martin-Löf random points implies the classical theorem since the set of Martin-Löf randoms has Lebesgue measure 1, and hence is stronger. It is also known that other well-studied notions of randomness like Schnorr randomness also satisfy ergodic theorems [14].

In this work, we initiate the study of ergodic theorems in resource-bounded settings. This is a difficult question, since classically, the convergence speed in ergodic theorems is known to be arbitrarily slow (e.g. see Krengel [11], and V’yugin [19]). We establish ergodic theorems in resource-bounded settings which hold on every resource-bounded random object of a particular class. The main insight in our work is that convergence of subsequences of ergodic averages determine the class of randoms on which they converge. The main technical hurdle is the lack of sharp tail bounds. The only general tail bound in ergodic settings is the maximal ergodic inequality, which yields only an inverse linear bound in the number of sample points, in contrast to the inverse exponential bounds in the Chernoff and the Azuma-Hoeffding inequalities.

Our proofs necessarily involve new techniques. In particular, we study ergodic phenomena for PSPACE computable families of functions.

We first establish that for PSPACE L¹ computable functions and a suitable class of PSPACE-computable measure-preserving ergodic transformations, Birkhoff’s ergodic theorem holds for every
EXPSPACE random. The techniques employed in this theorem are new, and shed new light into the convergence phenomena involved in powerful classical theorems such as the ergodic theorem. The techniques are major adaptations of ideas from Huang and Stull [9], Rute [14], Ko [10] and Hoyrup and Rojas [4] together with new quantitative bounds on the convergence of ergodic averages. We also show that the space of EXPSPACE randoms defined in terms of EXPSPACE tests is a strict subset of PSPACE randoms as studied in Huang and Stull [16]. Our main contribution is the identification of the exact setting and notions of convergence which are amenable to proving convergence of ergodic averages on resource-bounded randoms, and these may be useful in investigating further dynamical systems in resource-bounded settings.

2 Preliminaries

Let $\Sigma = \{0, 1\}$ be the binary alphabet, and denote the set of all finite binary strings by $\Sigma^*$ and the set of infinite binary strings by $\Sigma^\infty$. For $\sigma \in \Sigma^*$ and $y \in \Sigma^* \cup \Sigma^\infty$, we write $\sigma \subseteq y$ if $\sigma$ is a prefix of $y$.

As is typical in resource-bounded settings, some integer parameters are given in unary representation. The set of unary strings is represented as $1^*$, and the representation of $n \in \mathbb{N}$ in unary is $1^n$, a string consisting of $n$ ones.

For any finite string $\sigma$, the cylinder $[\sigma]$ is the set of all infinite sequences with $\sigma$ as a prefix. This notation can be extended to sets of strings in a natural way. Let the Borel $\sigma$-algebra generated by the set of all cylinders be denoted by $B(\Sigma^\infty)$.

We denote finite strings using small Greek letters like $\sigma$, $\alpha$ etc. The length of a finite binary string $\sigma$ is denoted by $|\sigma|$.

Throughout the paper we take into account the number of cells used in the output tape and the working tape when calculating the space complexity of functions. We assume a finite representation for the set of rational numbers $\mathbb{Q}$. Following the works of Hoyrup, and Rojas [8], we introduce the notion of a PSPACE-computable probability space on the Cantor space by endowing it with a PSPACE-computable probability measure.

**Definition 2.1.** Consider the probability space $(\Sigma^\infty, B(\Sigma^\infty))$. A Borel probability measure $\mu : B(\Sigma^\infty) \rightarrow [0, 1]$, is a **PSPACE-probability measure** if there exists a PSPACE machine $M : \Sigma^* \times 1^* \rightarrow \mathbb{Q}$ such that for every $\sigma \in \Sigma^*$, and $n \in \mathbb{N}$, we have that $|M(\sigma, 1^n) - \mu([\sigma])| \leq 2^{-n}$.

A **PSPACE-probability Cantor space** is a pair $(\Sigma^\infty, \mu)$ where $\Sigma^\infty$ is the Cantor space, and $\mu$ is a PSPACE probability measure.

In order to define PSPACE (EXPSPACE) randomness using PSPACE (EXPSPACE) tests we require the following method for approximating sequences of open sets in $\Sigma^\infty$ in polynomial (exponential) space.

**Definition 2.2** (PSPACE/EXPSPACE sequence of open sets [9]). A sequence of open sets $\langle U_n \rangle_{n=1}^\infty$ is a PSPACE sequence of open sets if there exists a sequence of sets $\langle S_n^k \rangle_{k,n \in \mathbb{N}}$, where $S_n^k \subseteq \Sigma^*$ such that

1. $U_n = \bigcup_{k=1}^\infty [S_n^k]$,

2. For any $m > 0$, $\mu \left(U_n - \bigcup_{k=1}^m [S_n^k] \right) \leq \frac{1}{2^m}$.

3. There exists a polynomial $p$ such that $\max\{|\sigma| : \sigma \in \bigcup_{k=1}^m S_n^k\} \leq p(n + m)$. 

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4. The function $g : \Sigma^* \times 1^* \times 1^* \rightarrow \{0, 1\}$ such that

$$g(\sigma, 1^n, 1^m) = \begin{cases} 1 & \text{if } \sigma \in S_n^m \\ 0 & \text{otherwise,} \end{cases}$$

is decidable by a PSPACE machine.

The definition of EXPSPACE sequence of open sets is similar but the bound in condition 3 is replaced with $2^{p(n+m)}$ and the machine in condition 4 can be an EXPSPACE machine.

Henceforth, we study the notion of randomness on $(\Sigma^\infty, \mu)$. The following is the definition of PSPACE randomness from [16]. We also define its EXPSPACE analogue.

**Definition 2.3** (PSPACE/EXPSPACE randomness [16]). A sequence of open sets $\langle U_n \rangle_{n=1}^\infty$ is a PSPACE test if it is a PSPACE sequence of open sets and for all $n \in \mathbb{N}$, $\mu(U_n) \leq \frac{1}{2^n}$.

A set $A \subseteq \Sigma^\infty$ is PSPACE null if there is a PSPACE test $\langle U_n \rangle_{n=1}^\infty$ such that $A \subseteq \bigcap_{n=1}^\infty U_n$. A set $A \subseteq \Sigma^\infty$ is PSPACE random if $A$ is not PSPACE null.

The EXPSPACE analogues of the above concepts are defined similarly except that $\langle U_n \rangle_{n=1}^\infty$ can be an EXPSPACE sequence of open sets.

By considering the sequence $\left( \bigcup_{i=1}^k S_n^i \right)_{k, n \in \mathbb{N}}$ instead of $\langle S_n^k \rangle_{k, n \in \mathbb{N}}$, without loss of generality, we can assume that for each $n$, $\langle S_n^k \rangle_{k=1}^\infty$ is an increasing sequence of sets.

Next, we introduce a PSPACE version of Solovay tests, where the relaxation is that the measures of the sets $U_n$ can be any sufficiently fast convergent sequence. We later show that this captures the same set of randoms as PSPACE tests.

**Definition 2.4** (PSPACE Solovay test). A sequence of open sets $\langle U_n \rangle_{n=1}^\infty$ is a PSPACE Solovay test if it is a PSPACE sequence of open sets and

1. $\sum_{n=1}^\infty \mu(U_n) < \infty$.
2. There exists a polynomial $p$ such that $\forall m \geq 0$, $\sum_{n=p(m)+1}^\infty \mu(U_n) \leq \frac{1}{2^m}$.

A set $A \subseteq \Sigma^\infty$ is PSPACE Solovay null if there exist a PSPACE Solovay test $\langle U_n \rangle_{n=1}^\infty$ such that $A \subseteq \bigcap_{n=1}^\infty U_n$. A set $A \subseteq \Sigma^\infty$ is PSPACE Solovay random if $A$ is not PSPACE Solovay null.

**Lemma 2.5.** A set $A \subseteq \Sigma^\infty$ is PSPACE null if and only if $A$ is PSPACE Solovay null.

A proof of the above lemma can be found in the appendix.

The set of PSPACE Solovay randoms and PSPACE randoms are equal, hence to prove PSPACE randomness results, it suffices to form Solovay tests.\(^1\)

\(^{1}\)This equivalence may not hold between EXPSPACE tests and EXPSPACE Solovay tests, but we do not require this in the present work.
3 PSPACE $L^1$ computability

In this section, we briefly recall standard definitions for PSPACE computability of real numbers, sequences of real numbers, $L^1$ computable functions, and measure-preserving transformations. The justifications and proofs of equivalences of various notions are present in Stull’s thesis [16].

**Definition 3.1** (PSPACE computable number [16]). A real number $a$ is PSPACE computable if there exists a PSPACE machine $M$ such that for each $m \geq 1$, $M(1^m) \in \mathbb{Q}$ satisfying $|M(1^m) - a| \leq \frac{1}{2^m}$.

We investigate the ergodic theorem in the setting of $L^1$-computable functions and PSPACE simple transformations. Let us initially define PSPACE sequence of simple functions.

**Definition 3.2** (PSPACE sequence of simple functions [16]). A sequence of simple functions $\langle f_n \rangle_{n=1}^\infty$ where each $f_n : \Sigma^\infty \to \mathbb{Q}$ is a PSPACE sequence of simple functions if

1. There is a controlling polynomial $p$ such that for each $n$, there exists $\{d_1, d_2 \ldots d_k\} \subseteq \mathbb{Q}$ and $\{\sigma_1, \sigma_2 \ldots \sigma_k\} \subseteq \Sigma^{p(n)}$ such that

   $$f_n = \sum_{i=1}^{k} d_i \chi_{[\sigma_i]}$$

   where $\chi_{[\sigma]}$ is the characteristic function of the cylinder $[\sigma]$.

2. There is a PSPACE machine $M$ such that for each $n \in \mathbb{N}, \sigma \in \Sigma^*$

   $$M(1^n, \sigma) = \begin{cases} f_n(\sigma0^n) & \text{if } |\sigma| \geq p(n) \\ ? & \text{otherwise.} \end{cases}$$

Now, we define PSPACE $L^1$-computable functions in terms of limits of convergent PSPACE sequence of simple functions, with a PSPACE computable rate of convergence.

**Definition 3.3** (PSPACE $L^1$-computable functions [16]). A function $f \in L^1(\Sigma^\infty, \mu)$ is PSPACE $L^1$-computable if there exists a PSPACE sequence of simple functions $\langle f_n \rangle_{n=1}^\infty$ such that for every $n \in \mathbb{N}$, $\|f - f_n\| \leq 2^{-n}$.

A sequence of $L^1$ functions $\langle f_n \rangle_{n=1}^\infty$ converging to $f$ in the $L^1$-norm need not have pointwise limits. Hence the following concept ([14]) is important in studying the pointwise ergodic theorem in the setting of $L^1$-computability.

**Definition 3.4** ($\tilde{f}$ for PSPACE $L^1$-computable $f$). Let $f \in L^1(\Sigma^\infty, \mu)$ be PSPACE $L^1$-computable and let $\langle f_n \rangle_{n=1}^\infty$ be any PSPACE sequence of simple functions in $L^1(\Sigma^\infty, \mu)$ approximating $f$ (as in Definition 3.3). Define $\tilde{f} : \Sigma^\infty \to \mathbb{R} \cup \{\text{undefined}\}$ as

$$\tilde{f}(x) = \begin{cases} \lim_{n \to \infty} f_n(x) & \text{if the limit exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The following is a crucial property which we use in the proof of the effective PSPACE ergodic theorem.

**Lemma 3.5** (Integral of PSPACE $L^1$-computable functions). Let $f \in L^1(\Sigma^\infty, \mu)$ be PSPACE $L^1$-computable. Then, $\int f \, d\mu$ and $\|f\|_1$ are PSPACE computable real numbers.
A proof of the above lemma can be easily obtained by choosing a sufficiently close approximation $f_n$ of $f$ from the approximating PSPACE sequence of simple functions and then computing $\int f_n d\mu$ (or $\|f_n\|_1$) in polynomial space. To define ergodic averages, we restrict ourselves to the following class of transformations.

**Definition 3.6 (PSPACE simple transformation).** A measurable function $T : (\Sigma^\infty, \mu) \rightarrow (\Sigma^\infty, \mu)$ is a PSPACE simple transformation if there is a constant $c$ and a PSPACE machine $M$ such that such that for any $\sigma \in \Sigma^*$

$$T^{-1}([\sigma]) = \bigcup_{i=1}^k [\sigma_i]$$

such that:

1. $\{\sigma_i\}_{i=1}^k$ is a prefix free set and for all $1 \leq i \leq k$, $|\sigma_i| \leq |\sigma| + c$
2. For each $\sigma, \alpha \in \Sigma^*$,

$$M(\sigma, \alpha) = \begin{cases} 
1 & \text{if } |\alpha| \geq |\sigma| + c \text{ and } \alpha 0^\infty \in T^{-1}([\sigma]) \\
0 & \text{if } |\alpha| \geq |\sigma| + c \text{ and } \alpha 0^\infty \notin T^{-1}([\sigma]) \\
? & \text{otherwise}
\end{cases}$$

It is easy to verify that if $T$ is a PSPACE simple transformation then for any $k \geq 2$, $T^k$ is also a PSPACE simple transformation.

PSPACE computability as defined above, relates convergence of $L^1$ norms in a natural fashion. The pointwise ergodic theorem deals with pointwise almost everywhere convergence. We introduce the modes of convergence we deal with in the present work.

**Definition 3.7 (P-rapid/EXP-rapid limit point).** A real number $a$ is a P-rapid limit point of the real number sequence $\langle a_n \rangle_{n=1}^\infty$ if there exists a polynomial $p$ such that for all $m \in \mathbb{N}$, $\exists k \leq p(m)$ such that $|a_k - a| \leq \frac{1}{2^m}$.

The definition of EXP-rapid limit point is similar but the upper bound for $k$ is replaced with $2^p(m)$.

Note that this requires rapid convergence only on a subsequence. The following definition is the $L^1$ version of the above.

**Definition 3.8 (P-rapid/EXP-rapid $L^1$-limit point).** A function $f \in L^1(\Sigma^\infty, \mu)$ is a P-rapid (EXP-rapid) $L^1$-limit point of a sequence $\langle f_n \rangle_{n=1}^\infty$ of functions in $L^1(\Sigma^\infty, \mu)$ if $0$ is a P-rapid (EXP-rapid) limit point of $\|f_n - f\|_1$.

Now we define P-rapid/EXP-rapid analogue of almost everywhere convergence ([14]). Convergence in several well studied theorems like Law of Iterated Logarithm and the Strong Law of Large Numbers satisfy EXP-rapidity, hence the notion is sufficiently general.

**Definition 3.9 (P-rapid/EXP-rapid almost everywhere convergence).** A sequence of measurable functions $\langle f_n \rangle_{n=1}^\infty$ is P-rapid almost everywhere convergent to a measurable function $f$ if there exists a polynomial $p$ such that for all $m_1$ and $m_2$,

$$\mu \left( \left\{ x : \sup_{n \geq p(m_1+m_2)} |f_n(x) - f(x)| \geq \frac{1}{2^{m_1}} \right\} \right) \leq \frac{1}{2^{m_2}}.$$ 

The definition of EXP-rapid almost everywhere convergence is similar but the lower bound for $n$ in the above inequality is replaced with $2^{p(m_1+m_2)}$. 
4 P-rapid/EXP-rapid almost everywhere convergence of ergodic averages

Now we present PSPACE versions of Theorem 2 and Proposition 5 from [4]. Let \( A_n^f = \frac{f + f \circ T + f \circ T^2 + \ldots + f \circ T^{n-1}}{n} \).

The main estimate which we require in this section is the maximal ergodic inequality, which we now recall.

**Lemma 4.1 (Maximal ergodic inequality [2]).** If \( f \in L^1(\Sigma^\infty, \mu) \) and \( \delta > 0 \) then,

\[
\mu \left( \left\{ x : \sup_{n \geq 1} |A_n^f(x)| > \delta \right\} \right) \leq \frac{\|f\|_1}{\delta}.
\]

Using this lemma, we now prove the almost everywhere convergence of ergodic averages. In contrast to [4], we give a direct proof of the theorem for \( L^1 \) functions with possibly infinite essential supremum using Markov’s inequality.

**Theorem 4.2.** Let \( f \) be any function in \( L^1(\Sigma^\infty, \mu) \) and let \( T \) be an ergodic measure preserving transformation. If \( \int f \, d\mu \) is a P-rapid (EXP-rapid) \( L^1 \)-limit point of \( A_n^f \) then \( A_n^f \) is P-rapid (EXP-rapid) almost everywhere convergent to \( \int f \, d\mu \).

**Proof.** We only prove the case when \( \int f \, d\mu \) is an EXP-rapid \( L^1 \)-limit point of \( A_n^f \). The proof for the P-rapid case is similar. By replacing \( f \) with \( f - \int f \, d\mu \) we can assume without loss of generality that \( \int f \, d\mu = 0 \).

We construct a polynomial \( q \) such that for any \( m_1 \) and \( m_2 \),

\[
\mu \left( \left\{ x : \sup_{n \geq 2^q(m_1+m_2)} |A_n^f(x)| > 2^{-m_1} \right\} \right) \leq 2^{-m_2}.
\]

Since \( \int f \, d\mu = 0 \) is a EXP-rapid \( L^1 \)-limit point of \( A_n^f \) there is a polynomial \( p \) such that for any \( \exists k \leq 2^{p(m_1+m_2+2)} \) with \( \|A_k^f\|_1 \leq \frac{1}{2^{m_1+m_2+2}} \).

Applying the maximal ergodic inequality to \( g = A_k^f \), we get

\[
\mu \left( \left\{ x : \sup_{n \geq 1} |A_n^g(x)| > \frac{1}{2^{m_1+1}} \right\} \right) \leq \frac{1}{2^{m_2+1}}.
\]

Expanding \( A_n^g \),

\[
A_n^g = A_n^f + \frac{u \circ T^n - u}{nk},
\]

where \( u = (k-1)f + (k-2)f \circ T + \ldots + f \circ T^{k-2} \). Note that

\[
\|u\|_1 \leq \frac{k(k-1)}{2} \|f\|_1.
\]

Let \( M \) be any upper bound for \( \|f\|_1 \). And let \( n_0 = (2^{p(m_1+m_2+2)} - 1)M2^{m_1+m_2+2} \). From the above, we get \( \|A_n^g - A_n^f\|_1 \leq \frac{1}{2^{m_1+m_2+2}} \) for any \( n \geq n_0 \). Now from Markov’s inequality it follows that,

\[
\mu \left( \left\{ x : \sup_{n \geq n_0} |A_n^f(x) - A_n^g(x)| > \frac{1}{2^{m_1+1}} \right\} \right) \leq \frac{\|A_n^g - A_n^f\|_1}{2^{m_1+1}} \leq \frac{1}{2^{m_2+1}}
\]

Hence, from 1 and 2, we get

\[
\mu \left( \left\{ x : \sup_{n \geq n_0} |A_n^f(x)| > \frac{1}{2^{m_1}} \right\} \right) \leq \frac{1}{2^{m_2}}.
\]

Since \( n_0 \) is upper bounded by a term of the form \( 2^q(m_1+m_2) \) for a polynomial \( q \), the claim holds. \( \square \)
5 An ergodic theorem for PSPACE $L^1$ functions

We now establish the main theorem in our work, namely, that for PSPACE $L^1$ computable functions, the ergodic average exists, and is equal to the space average, on every EXPSPACE random. We utilize the almost everywhere convergence results proved in the previous section, to prove the convergence on every PSPACE/EXPSPACE random. The following fact was shown in [9]. However, for our ergodic theorem we require an alternate proof of this fact using techniques from [14].

Lemma 5.1. Let $\langle f_n \rangle_{n=1}^\infty$ be a PSPACE sequence of simple functions which converges P-rapid (EXP-rapid) almost everywhere to $f \in L^1(\Sigma^\infty, \mu)$. Then,

1. $\lim_{n \to \infty} f_n(x)$ exists for all PSPACE (EXPSPACE) random $x$.

2. Given a PSPACE sequence of simple functions $\langle g_n \rangle_{n=1}^\infty$ which is P-rapid (EXP-rapid) almost everywhere convergent to $f$, $\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} f_n(x)$ for all PSPACE (EXPSPACE) random $x$.

Proof. We consider the case when the convergence is EXP-rapid. The proof for P-rapid case is similar. We initially show 1. For each $k \geq 0$, since $\langle f_n \rangle_{n=1}^\infty$ is EXP-rapid almost everywhere convergent to $f$, we have a polynomial $q$ such that

$$\mu(\{x : \sup_{n \geq 2^{q(k)}} |f_n(x) - f(x)| \geq \frac{1}{2^{k+2}}\}) \leq \frac{1}{2^{k+2}}.$$ 

It is easy to verify that

$$\mu(\{x : \sup_{n \geq 2^{q(k)}} |f_n(x) - f_{2^{q(k)}}(x)| \geq \frac{1}{2^{k+1}}\}) \leq \frac{1}{2^{k+1}}.$$ 

Define

$$U_k = \{x : \max_{2^{q(k)} \leq n < 2^{q(k+1)}} |f_n(x) - f_{2^{q(k)}}(x)| > \frac{1}{2^{k+1}}\}.$$ 

Observe that

$$\mu(U_k) \leq \mu(\{x : \sup_{n \geq 2^{q(k)}} |f_n(x) - f_{2^{q(k)}}(x)| \geq \frac{1}{2^{k+1}}\}) \leq \frac{1}{2^{k+1}}.$$ 

Let $r$ be the controlling polynomial and let $M$ be the PSPACE machine witnessing the fact that $\langle f_n \rangle_{n=1}^\infty$ is a PSPACE sequence of simple functions. $U_k$ is hence a union of cylinders of length at most $r(2^{q(k+1)})$. The machine $M$ can be used to construct a machine $N$ that on input $(\sigma, 1^k)$ outputs 1 if $[\sigma] \subseteq U_k$ and outputs 0 otherwise. Define

$$V_k = \bigcup_{i=k+1}^\infty U_i.$$ 

From the above observations it can be verified that $\langle V_k \rangle_{k=1}^\infty$ is an EXPSPACE test.

If $x \in \Sigma^\infty$ is an EXPSPACE random then $x$ is in at most finitely many $V_k$ and hence in only finitely many $U_k$. Hence, for some $k \geq k_0$ and for all $n \geq 2^{q(k)}$ we have

$$|f_n(x) - f_{2^{q(k)}}(x)| \leq \sum_{j=k}^\infty \frac{1}{2^{j+1}} \leq \frac{1}{2^k}.$$
This shows that \( f_n(x) \) is a Cauchy sequence. This completes the proof of 1.

Given \( (g_n)_{n=1}^\infty \), which is EXP-rapid almost everywhere convergent to \( f \), the interleaved sequence \( f_1, g_1, f_2, g_2, f_3, g_3 \ldots \) can be easily verified to be EXP-rapid almost everywhere convergent to \( f \). 2 now follows directly from 1.

Let \( f \) be a PSPACE \( L^1 \)-computable function with an \( L^1 \) approximating PSPACE sequence of simple functions \( \langle f_n \rangle_{n=1}^\infty \). For any \( m_1, m_2 \geq 0 \),

\[
\mu \left( \left\{ x : \sup_{n \geq m_1 + m_2 + 1} |f_n(x) - f(x)| \geq \frac{1}{2^{m_1}} \right\} \right) \leq \sum_{n=m_1+m_2+1}^\infty \mu \left( \left\{ x : |f_n(x) - f(x)| \geq \frac{1}{2^{m_1}} \right\} \right) \leq \sum_{n=m_1+m_2+1}^\infty \|f_n - f\|_1 2^{m_1} \leq \sum_{i=1}^\infty \frac{1}{2^{m_2+i}} = 2^{-m_2}.
\]

Hence, \( \langle f_n \rangle_{n=1}^\infty \) is P-rapid almost everywhere convergent to \( f \) and we get the following corollary.

**Corollary 5.2.** Let \( f \in L^1(\Sigma^\infty, \mu) \) be a PSPACE \( L^1 \)-computable function with an \( L^1 \) approximating PSPACE sequence of simple functions \( \langle f_n \rangle_{n=1}^\infty \). Then,

1. \( \lim_{n \to \infty} f_n(x) \) exists for all PSPACE random \( x \).
2. Given a PSPACE sequence of simple functions \( \langle g_n \rangle_{n=1}^\infty \) \( L^1 \) approximating \( f \), \( \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} f_n(x) \) for all PSPACE random \( x \).

The following lemma establishes an important closure property satisfied by PSPACE simple transformations. This is crucial in our proof of the main theorem.

**Lemma 5.3.** Let \( f \) be a PSPACE \( L^1 \)-computable function with an \( L^1 \) approximating PSPACE sequence of simple functions \( \langle f_n \rangle_{n=1}^\infty \). Let \( T \) be a PSPACE simple transformation and \( p \) be a polynomial. Then, \( \langle A_n^{f(p)} \rangle_{n=1}^\infty \) is a PSPACE sequence of simple functions.

Now, we prove the ergodic theorem for PSPACE \( L^1 \) functions, which is our main result. The proof involves adaptations of techniques from Rute [14], together with new quantitative bounds which yield the result within prescribed resource bounds.

**Theorem 5.4.** Let \( f \in L^1(\Sigma^\infty, \mu) \) be a PSPACE \( L^1 \)-computable function and let \( T \) be a PSPACE simple ergodic measure preserving transformation. If \( \int fd\mu \) is an EXP-rapid \( L^1 \)-limit point of \( A_n^f \) then, \( \lim_{n \to \infty} A_n^f = \int fd\mu \) on EXPSPACE randoms.

**Proof.** Let \( \langle f_m \rangle_{m=1}^\infty \) be any PSPACE sequence of simple functions \( L^1 \) approximating \( f \). We initially approximate \( A_n^f \) with a PSPACE sequence of simple functions \( \langle g_n \rangle_{n=1}^\infty \) which converges to \( \int fd\mu \) on EXPSPACE randoms. Then we show that \( \bar{A}_n^f \) has the same limit as \( g_n \) on PSPACE randoms, hence on EXPSPACE randoms.

For each \( n \), it is easy to verify that \( \langle A_n^f \rangle_{n=1}^\infty \) is a PSPACE sequence of simple functions \( L^1 \) approximating \( A_n^f \) with the same rate of convergence. Using techniques similar to those in Lemma 5.1 and Corollary 5.2, we can obtain a polynomial \( p \) such that

\[
\mu \left( \left\{ x : \sup_{m \geq p(n+i)} |A_n^f(x) - A_n^{f(p(n+i)}(x)| \geq \frac{1}{2^{n+i+1}} \right\} \right) \leq \frac{1}{2^{n+i+1}}.
\]
For every $n > 0$, let $g_n = A_{n}^{f_{p(n)}}$. We initially show that $(g_n)_{n=1}^{\infty}$ converges to $\int f \, d\mu$ on EXPSPACE randoms. Let $m_1, m_2 \geq 0$. From Theorem 4.2, $A_{n}^{f}$ is EXP-rapid almost everywhere convergent to $\int f \, d\mu$. Hence there is a polynomial $q$ such that

$$
\mu \left( \left\{ x : \sup_{n \geq 2^{j(m_1 + m_2)}} \left| A_{n}^{f}(x) - \int f \, d\mu \right| \geq \frac{1}{2^{m_1 + 1}} \right\} \right) \leq \frac{1}{2^{m_2 + 1}}.
$$

Let $N(m_1, m_2) = \max\{2m_1, 2m_2, 2^{q(m_1 + m_2)}\}$. Then,

$$
\sum_{n \geq N(m_1, m_2)} \frac{1}{2^{k+1}} = \frac{1}{2^{N(m_1, m_2)}} \leq \min \left\{ \frac{1}{2^{m_1 + 1}}, \frac{1}{2^{m_2 + 1}} \right\}.
$$

Now, we have

$$
\mu \left( \left\{ x : \sup_{n \geq N(m_1, m_2)} \left| g_n - \int f \, d\mu \right| > \frac{1}{2^{m_1}} \right\} \right) \leq \sum_{n \geq N(m_1, m_2)} \mu \left( \left\{ x : \left| g_n - A_{n}^{f}(x) \right| > \frac{1}{2^{m_1 + 1}} \right\} \right)
$$

$$
+ \mu \left( \left\{ x : \sup_{n \geq 2^{j(m_1 + m_2)}} \left| A_{n}^{f}(x) - \int f \, d\mu \right| \geq \frac{1}{2^{m_1 + 1}} \right\} \right)
$$

$$
\leq \sum_{n \geq N(m_1, m_2)} \frac{1}{2^{n+1}} + \frac{1}{2^{m_2 + 1}}
$$

$$
\leq \frac{1}{2^{m_2}}.
$$

Note that $N(m_1, m_2)$ is exponentially bounded in $m_1 + m_2$. Hence, $g_n$ is EXP-rapid almost everywhere convergent to $\int f \, d\mu$. From Lemma 5.3 it follows that $(g_n)_{n=1}^{\infty} = (A_{n}^{f_{p(n)}})_{n=1}^{\infty}$ is a PSPACE sequence of simple functions (in parameter $n$). From these observations and Lemma 5.1 we get that $\lim_{n \to \infty} g_n(x) = \int f \, d\mu$ for any $x$ which is EXPSPACE random.

We now show that $\lim_{n \to \infty} A_{n}^{f} = \lim_{n \to \infty} g_n$ on PSPACE randoms. Define

$$
U_{n,i} = \left\{ x : \max_{p(n+i) \leq m \leq p(n+i+1)} \left| A_{n}^{f_{m}}(x) - A_{n}^{f_{p(n+i+1)}}(x) \right| \geq \frac{1}{2^{n+i+1}} \right\}.
$$

We already know $\mu(U_{n,i}) \leq \frac{1}{2^{n+i+1}}$. As in Lemma 5.1, $U_{n,i}$ can be shown to be a union of polynomial length cylinders and $U_{n,i}$ is polynomial space approximable in parameters $n$ and $i$. Define

$$
V_m = \bigcup_{n,i \geq 0, n+i=m} U_{n,i}.
$$

From the polynomial space approximability of $U_{n,i}$, it follows that $(V_m)_{m=1}^{\infty}$ is a PSPACE approximable sequence of sets. Now, we show that $(V_m)_{m=1}^{\infty}$ is a PSPACE Solovay test. Note that

$$
\mu(V_m) \leq \frac{m}{2^m}.
$$

It can be shown that for any $j$,

$$
\sum_{n > j} \frac{m}{2^{n}} = \frac{1}{2^{j-1}} + \frac{j}{2^j}.
$$
Given any \( k \geq 0 \), let \( p(k) = 3(k + 1) \). Hence, we have

\[
\sum_{n=p(k)+1}^{\infty} \frac{m}{2^m} = \frac{1}{2^{3(k+1)}} + \frac{3(k + 1)}{2^{3(k+1)}} < \frac{1}{2^{k+1}} + \frac{3(k + 1)}{2^{2(k+1)}} < \frac{2}{2k+1} = \frac{1}{2^k}.
\]

The last inequality holds since \( 3(k + 1) < 2^{k+1} \) for all \( k \geq 0 \). Hence, \( \langle V_m \rangle_{m=1}^{\infty} \) is a PSPACE Solovay test. Now, let \( x \) be a PSPACE random. \( x \) is in at most finitely many \( V_m \) and hence in at most finitely many \( U_{n,i} \). Hence for some large enough \( N \) for all \( n \geq N, i \geq 0 \) and for all \( m \) such that \( p(n+i) \leq m \leq p(n+i+1) \), we have \(|A^m_{n}(x) - A^{f_{p(n)}}_{n}(x)| < \frac{1}{2^{n+i+1}} \). It follows that for all \( n \geq N \) and for all \( m \geq p(n) \) that,

\[
|A^m_{n}(x) - g_{n}(x)| = |A^m_{n}(x) - A^{f_{p(n)}}_{n}(x)| \leq \sum_{i=0}^{\infty} \frac{1}{2^{n+i+1}} \leq 2^{-n}.
\]

Therefore, \( \lim_{n \to \infty} \bar{A}^f_n(x) = \lim_{n \to \infty} g_n(x) \) on all PSPACE random \( x \) and hence on all \( x \) which is EXPSPACE random.

Hence, we have shown that \( \lim_{n \to \infty} \bar{A}^f_n = \int f \, d\mu \) on EXPSPACE randoms which completes the proof of the theorem. \( \square \)

6 A PSPACE random which is EXPSPACE non-random

In the concluding section, we show that EXPSPACE randoms are a proper subset of PSPACE randoms. We demonstrate this in the setting of \( (\Sigma^\infty, \mathcal{B}(\Sigma^\infty), \mu) \) where \( \mu \) is the Bernoulli measure \( \mu([\sigma]) = \frac{1}{2^{|\sigma|}} \). The technique in the proof may be more generally applicable for randomness hierarchies and more general computable measures. Our strategy to establish this result is to show that there is no universal PSPACE test which itself is a PSPACE test (similarly for EXPSPACE tests), and there is a universal test for PSPACE which is an EXPSPACE test. These facts yield our desired result.

**Lemma 6.1.** Let \( \langle U_n \rangle_{n=1}^{\infty} \) be a PSPACE test. Then, there exists a PSPACE test \( \langle V_n \rangle_{n=1}^{\infty} \) such that

\[
\left( \bigcap_{n=1}^{\infty} V_n \right) \bigcap \left( \bigcap_{n=1}^{\infty} U_n \right) ^c \neq \emptyset.
\]

**Proof.** Let \( \langle S_k \rangle_{n,k \in \mathbb{N}} \) be the PSPACE approximating sequence of sets for \( \langle U_n \rangle_{n=1}^{\infty} \) such that \( \langle S_k \rangle_{k=1}^{\infty} \) is increasing for each \( n \). Let \( p \) be the controlling polynomial for \( \langle U_n \rangle_{n=1}^{\infty} \). We define \( \langle V_n \rangle_{n=1}^{\infty} \) inductively.

Let \( n = 1 \). Let \( T_1 = S_1^1 \). We know that \( \mu(U_1 - [T_1]) \leq \frac{1}{10} < \frac{1}{8} \). Consider \( E_1 = \{ \sigma \in \Sigma^{p(5)} : [\sigma] \nsubseteq [T_1] \} \). Let \( \langle \sigma_i \rangle_{i=1}^{[E_1]} \) be the enumeration of \( E_1 \) in the lexicographic order. Now define \( V_1 = \{ \langle \sigma_i \rangle_{i=1}^{1} : 1 \leq i \leq 2^{p(5)-1} \} \). Observe that

\[
\mu(V_1) = \frac{2^{p(5)-1}}{2^{p(5)}} = \frac{1}{2}.
\]

We know that \( V_1 \) is a union of certain cylinders of length \( p(5) \). Let \( C_1 = V_1 - U_1 \). Since \( \mu(U_1 - [T_1]) \leq \frac{1}{10} < \frac{1}{8} \) we have \( \mu(C_1) > \frac{1}{4} \).
Let $n > 1$. Inductively assume that $V_{n-1}$ is defined as the union of cylinders in $\Sigma^{p(2(n-1)+3)}$ such that $\mu(V_{n-1}) \leq \frac{1}{2^{n-1}}$ and $\mu(V_{n-1} \cap \bigcup_{i=1}^{n-1} U_i) \leq \frac{1}{2^{n-1}+2}$. Hence, setting $C_{n-1} = V_{n-1} - \bigcup_{i=1}^{n-1} U_i$ we get that $\mu(C_{n-1}) \geq \frac{1}{2^{n-1}+1}$. Now, we define $V_n$. Let

$$T_n = \bigcup_{i=1}^{n-1} U_i.$$ 

Let $E_n = \{\sigma \in \Sigma^{p(2n+2)} : [\sigma] \subseteq V_{n-1} \text{ and } [\sigma] \not\subseteq [T_n]\}$. It is easy to see that

$$\mu([E_n]) \geq \mu(C_{n-1}) > \frac{1}{2n}.$$ 

Let $\langle \sigma^n_i \rangle_{i=1}^n$ be the enumeration of $E_n$ in the lexicographic order. Define $V_n = \{[\sigma^n_i] : 1 \leq i \leq 2^{p(2n+3) - n}\}$. Observe that

$$\mu(V_i) = \frac{2^{p(2n+3) - n}}{2^{p(2n+3)}} = \frac{1}{2n}.$$ 

Clearly, $V_n$ is a union of cylinders in $\Sigma^{p(2n+3)}$ and $V_n \subseteq V_{n-1}$. From the definition of $T_n$, we obtain

$$\mu\left(V_n \cap \bigcup_{i=1}^{n} U_i\right) \leq \sum_{i=1}^{n} \frac{1}{2^{n+i+2}} < \frac{1}{2^{n+2}}.$$ 

Defining $C_n = V_n - \bigcup_{i=1}^{n} U_i$ we get

$$\mu(C_n) \geq \mu(V_n) - \mu\left(V_n \cap \bigcup_{i=1}^{n} U_i\right) = \frac{1}{2n} - \frac{1}{2^{n+2}} > \frac{1}{2^{n+1}}.$$ 

Conditions 1, 2 and 3 in Definition 2.3 are easily verified for $\langle V_n \rangle_{n=1}^{\infty}$. Using the output of the machine computing $\langle S^k_n \rangle_{n,k \in \mathbb{N}}$ as in condition 4 of Definition 2.3, the first $2^{p(2n+3) - n}$ strings in lexicographic order in $E_n$ can be found in polynomial space. Hence, $\langle V_n \rangle_{n=1}^{\infty}$ is a PSPACE test.

Since $V_n \subseteq V_{n-1}$, it readily follows that $C_n \subseteq C_{n-1}$. Since $\langle V_n \rangle_{n=1}^{\infty}$ is a decreasing sequence of closed sets, the sets $C_n = V_n - \bigcup_{i=1}^{n} U_i$ forms a decreasing sequence of non-empty closed sets (since each $C_n$ has non-zero measure). From the compactness of the Cantor space, it follows that $\bigcap_{n=1}^{\infty} C_n$ is non-empty. The lemma now follows by observing that $\bigcap_{n=1}^{\infty} C_n \subseteq \bigcap_{n=1}^{\infty} V_n$ and $\left(\bigcap_{n=1}^{\infty} C_n\right) \cap \left(\bigcap_{n=1}^{\infty} U_n\right) = \phi$. \hfill $\square$

The following is the EXPSPACE version of the above lemma.

**Lemma 6.2.** Let $\langle U_n \rangle_{n=1}^{\infty}$ be an EXPSPACE test. Then, there exists an EXPSPACE test such that

$$\left(\bigcap_{n=1}^{\infty} V_n\right) \cap \left(\bigcap_{n=1}^{\infty} U_n\right) \cap C \neq \phi.$$ 

(3)
The proof is almost identical to that of Lemma 6.1. The following is the most important consequence of the above results.

**Corollary 6.3.**

1. There is no universal PSPACE test for PSPACE non-rands.
2. There is no universal EXPSPACE test for EXPSPACE non-rands.

**Proof.** The claim easily follows by assuming that a PSPACE (EXPSPACE) universal test \( \langle U_n \rangle_{n=1}^{\infty} \) exists and then applying Lemma 6.1 (Lemma 6.2).

However, there is an EXPSPACE universal test for PSPACE non-rands.

**Lemma 6.4.** There exists an EXPSPACE test \( \langle U_n \rangle_{n=1}^{\infty} \) such that for any PSPACE test \( \langle V_n \rangle_{n=1}^{\infty} \), we have \( \cap_{n=1}^{\infty} V_n \subseteq \cap_{n=1}^{\infty} U_n \).

Now, we can complete the construction of a PSPACE random which is EXPSPACE non-random.

**Theorem 6.5.** There is a PSPACE random which is EXPSPACE non-random.

**Proof.** Consider the EXPSPACE universal test \( \langle U_n \rangle_{n=1}^{\infty} \) for PSPACE non-rands from Lemma 6.4. Now, using Lemma 6.2, we can construct an EXPSPACE test \( \langle V_n \rangle_{n=1}^{\infty} \) such that (3) holds. Hence, \( \cap_{n=1}^{\infty} V_n \) contains some PSPACE random and the theorem follows.

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Appendix

Proof. (Proof of Lemma 2.5) It is easy to see that if $A$ is PSPACE null then $A$ is PSPACE Solovay null. Conversely, let $A$ be PSPACE Solovay null and let $\langle U_n \rangle_{n=1}^\infty$ be any Solovay test which witnesses this fact. Let $V_n = \bigcup_{i=p(n)}^\infty U_n$. We show that $\langle V_n \rangle_{n=1}^\infty$ is a PSPACE test. Let $\langle S_k^n \rangle_{n,k \in \mathbb{N}}$ be any sequence of sets approximating $\langle U_n \rangle_{n=1}^\infty$ as in definition 2.2 such that $\langle S_k^n \rangle_{n=1}^\infty$ is increasing for each $n$. We define a sequence of sets $\langle T_k^n \rangle_{n,k \in \mathbb{N}}$ approximating $V_n$ as follows.

Let $r(n,k) = \max\{p(k), p(n) + 2\}$. Define

$$T_k^n = \bigcup_{i=p(n)+1}^{r(n,k)} S_i^{r(n,k) - p(n) + k}.$$  

We can easily verify the first three conditions in definition 2.2. Using the machines witnessing that $\langle U_n \rangle_{n=1}^\infty$ is a PSPACE sequence of open sets, we can construct the corresponding machines for $\langle V_n \rangle_{n=1}^\infty$ in a straightforward manner. $\square$

Proof. (Proof of Lemma 5.3) Let $q$ be a controlling polynomial and $M_F$ be a machine witnessing the fact that $\langle f_{p(n)} \rangle_{n=1}^\infty$ is a PSPACE sequence of simple functions. Let $c_T$ be a constant and $M_T$ be a machine witnessing the fact that $T$ is a PSPACE simple transformation. For any $n \geq 1$, we have

$$A_n^{f_{p(n)}} = \frac{f_{p(n)} + f_{p(n)} \circ T + f_{p(n)} \circ T^2 + \ldots f_{p(n)} \circ T^n}{n}.$$  

The functions $\langle f_{p(n)} \circ T^i \rangle_{i=1}^n$ are simple functions defined on cylinders of length at most $q(p(n)) + c_T n$. Hence, the polynomial $r(n) = q(p(n)) + c_T n$ is a controlling polynomial for the sequence of functions $\langle A_n^{f_{p(n)}} \rangle_{n=1}^\infty$ as in condition 1 of Definition 3.2. Now, let us verify condition 2 of Definition 3.2. We construct a machine $N$ such that for each $n \in \mathbb{N}$ and $\sigma \in \Sigma^*$,

$$N(1^n, \sigma) = \begin{cases} A_n^{f_{p(n)}}(\sigma0^\infty) & \text{if } |\sigma| \geq r(n) \\ ? & \text{otherwise.} \end{cases}$$  

On input $(1^n, \sigma)$ if $|\sigma| < r(n)$ then $N$ outputs $?$ else it operates as follows. $f_{p(n)}(\sigma0^\infty)$ can be computed easily by running $M_F$. For any $i \in \{1,2,3,\ldots n\}$, $f_{p(n)} \circ T^i$ is computed as follows. $f_{p(n)}$ is a simple function defined on cylinders of length at most $q(p(n))$. The $q(p(n))$ length cylinder $\sigma'$ such that $\sigma0^\infty \in T^{-i}([\sigma'])$ can be found by running the machine $M_T$ on all cylinders of length $q(p(n))$ for $i$ times successively and hence $f_{p(n)} \circ T^i$ can be computed. At each successive run the cylinder lengths go up only by a constant factor $c_T$. If $T$ is a polynomial upper bound for the space complexity of $M_T$, $f_{p(n)} \circ T^i$ can be computed in $O(nt(q(p(n))) + c_T n)$ space for any $i \leq n$.

The results of computations of $f_{p(n)} \circ T^i$ for $i \in \{1,2,3,\ldots n\}$ can be added up and divided by $n$ easily in polynomial space. $N$ outputs the result of this computation. Since $N$ is a PSPACE machine, the proof is complete. $\square$

Proof. (Proof of Lemma 6.4) The proof is an adaptation of the standard construction of a universal Martin-Löf test (see [13]) into the resource bounded setting. We define $\langle U_n \rangle_{n=1}^\infty$ by constructing the approximating sequence of sets $\langle S_k^n \rangle_{n,k \in \mathbb{N}}$ and an EXPSPACE machine $M$ computing it (as in Definition 2.3) simultaneously. Intuitively, the machine computing $\langle U_n \rangle_{n=1}^\infty$ does the following. The machine assumes that the each string $e_i$ in the standard enumeration $\langle e_i \rangle_{i=1}^\infty$ of finite length strings
is an encoding of Turing machines computing a PSPACE test \( \langle V^n \rangle \infty_{n=1} \) (as in Definition 2.3). Our machine checks if this assumption is true at every stage of its operation. When any unexpected output is obtained which contradicts the assumption that a particular string \( e_i \) is a valid program computing some PSPACE test, our machine corrects the error and suspends further simulation on \( e_i \). The proof also uses techniques similar to those in the proof of the space hierarchy theorem ([15]) to ensure that every PSPACE test is eventually captured by the EXPSPACE test \( \langle U^n \rangle \infty_{n=1} \).

The working of machine \( M \) on input \((\sigma, 1^n, 1^m)\) is as follows. If \(|\sigma| \neq 2^{n+m}\) then \( M \) outputs 0. On inputs with \(|\sigma| = 2^{n+m}, M \) does the following operations.

Let \( (e_i)_{i=1}^\infty \) denote the standard lexicographic ordering of strings in \( \Sigma^* \). \( M \) runs the universal Turing machine on \((e_i, (\sigma', 1^n+i, 1^j)) \) for \( i \in \{1, 2, 3, \ldots m + 1\}, j \in \{1, 2, 3, \ldots 3m\} \) and \( \sigma' \in \bigcup_{k=1}^{m} \Sigma^k \) for at most \( 2^{n+m+|\sigma|} \) steps as long as the simulation uses at most \( 2^{n+m+|\sigma|} \) tape cells. The simulations are ordered in lexicographic ordering of the tuples \((i, j, \sigma')\). We use the simulation to define sets \( T[i]^j_{n+i} \) for \( i \in \{1, 2, 3, \ldots m + 1\} \) and \( j \in \{1, 2, 3, \ldots 3m\} \).

Consider the simulations on \((e_i, (\sigma', 1^n+i, 1^j)) \) for a fixed \( i \). If any output other than 0 or 1 is obtained or if the simulation does not terminate then we do not include \( \sigma' \) into \( T[i]^j_{n+i} \). If for a particular \( j \) and \( \sigma' \), the simulation obtains output 1, then all \( 2^{n+i+j} \) extensions of \( \sigma' \) are added into \( T[i]^j_{n+i} \). For every \( i \), \( M \) keeps a running sum of the measures of cylinders associated with strings that are currently in \( \bigcup_{j=1}^{3m} T[i]^j_{n+i} \). If the renewed running sum after obtaining output 1 for some \( j \) and \( \sigma' \) is greater than \( \frac{1}{2^n+m} \) then \( \sigma' \) is not included into \( T[i]^j_{n+i} \) and the simulation of \((e_i, (\sigma', 1^n+i, 1^j)) \) for all \( j' \) greater than the current \( j \) is skipped by the machine \( M \) and it resumes the simulations with \( i' = i + 1 \). In this case, we define \( T[i]^j_{n+i} = \phi \) for all \( j' > j \). \( M \) also keeps a running sum of the measures of cylinders associated with strings that are currently in \( \bigcup_{j=1}^{3m} T[i]^j_{n+i} \) for each \((i, j)\) pair. If for any fixed \( i, j \) after the simulation of \((e_i, (\sigma', 1^n+i, 1^j)) \), \( M \) finds that this running sum exceeds \( \frac{1}{2^n+m} \) then this situation is handled as before by not adding \( \sigma' \) into \( T[i]^j_{n+i} \) and resuming the simulations with \( j' = j + 1 \).

The working of \( M \) defines \( T[i]^j_{n+i} \) for \( i \in \{1, 2, 3, \ldots m + 1\} \) and \( j \in \{1, 2, 3, \ldots 3m\} \). Let

\[
S_n^m = \bigcup_{1 \leq i \leq m+1, 1 \leq j \leq 3m} T[i]^j_{n+i},
\]

(4)

Finally, \( M \) checks if for the given \( \sigma \in \Sigma^{2^{n+m}} \) there exists some \( \sigma' \in S_n^m \) such that \( \sigma' \subseteq \sigma \). If so \( M \) outputs 1 else it outputs 0. The description of the working of \( M \) on input \((\sigma, 1^n, 1^m)\) is complete.

\( S_n^m \) contains only strings from \( \Sigma^{2^{n+m}} \) and from the description of \( M \) it can be verified that \( M \) is an EXPSPACE machine computing \( \langle S_n^m \rangle \infty_{n,m \in \mathbb{N}} \) as in condition 4 of definition 2.3. For a fixed \( n \), on inspecting the operation of \( M \) on \((\sigma, 1^n, 1^m)\) it can be seen that the sets \( T[i]^j_{n+i} \) are consistently defined. For any \( E_{i,n} = \bigcup_{j=1}^{\infty} T[i]^j_{n+i} \). Now

\[
\mu(E_{i,n}) \leq \frac{1}{2^n+i}.
\]

(5)

follows from the check done using the running sum for each \( i \) in the working of \( M \). And, due to
the check done using the second running sum, it can be seen that for any $i, j$

$$\mu \left( E_{i,n} - \bigcup_{j=1}^{k} [T[i]_{n+i}] \right) \leq \frac{1}{2^k}. \quad (6)$$

Now, define

$$U_n = \bigcup_{m=1}^{\infty} [S_n^m]$$

From 4 and 5, we get that

$$\mu(U_n) \leq \sum_{i=1}^{\infty} \frac{1}{2^{n+i}} = \frac{1}{2^n}. \quad (7)$$

Now, for any $m > 0$,

$$\mu(U_n - [S_n^m]) \leq \sum_{i=1}^{m+1} \mu \left( E_{i,n} - \bigcup_{j=1}^{3m} [T[i]_{n+i}] \right) + \sum_{i=m+2}^{\infty} \mu(E_{i,n})$$

$$\leq \sum_{i=1}^{m+1} \frac{1}{2^{3m}} + \sum_{i=m+2}^{\infty} \frac{1}{2^{n+i}}$$

$$< m \frac{1}{2^{3m}} + \frac{1}{2^{m+1}}$$

$$= m \frac{1}{2^{2m-1}} \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}}$$

$$< \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} = \frac{1}{2^m}. \quad (8)$$

The second inequality follows from 5 and 6. The last inequality follows since $m < 2^{2m-1}$ for all $m \geq 0$. From the above, we get that $\langle U_n \rangle_{n=1}^{\infty}$ is an EXPSPACE test.

Let $\langle V_n \rangle_{n=1}^{\infty}$ be a PSPACE test with machine $N$ computing its approximating sequence of sets as in condition 4 of Definition 2.3. Let $p$ be the controlling polynomial of $\langle V_n \rangle_{n=1}^{\infty}$ and let $q$ be a polynomial upper bound for the space complexity of $N$. Choose any $i$ such that $e_i$ is an encoding of $N$. Let $K > 0$ be large enough so that for all $n \geq K$ and any $m$ and $\sigma$ we have

$$2^{n+m} \geq p(n+m),$$

and

$$2^{n+m+|\sigma|} \geq q(n+m+|\sigma|).$$

From the construction of $\langle U_n \rangle_{n=1}^{\infty}$ it can be seen that for any $n > K$,

$$V_{n+i} \subseteq U_n.$$ 

Hence, it follows that $\bigcap_{n=1}^{\infty} V_n \subseteq \bigcap_{n=1}^{\infty} U_n. \quad \Box$