Holographic heat engines and static black holes: a general efficiency formula

Felipe Rosso

Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089-0484, U.S.A.

E-mail: felipero@usc.edu

ABSTRACT: Starting from simple observations regarding heat flows for static black holes (or any thermodynamic system with $C_V = 0$), we get inequalities which restrict their change in energy and adiabatic curves in the $p - T$ plane. From these observations we then derive an exact efficiency formula for virtually any holographic heat engine defined by a cycle in the $p - V$ plane, whose working substance is a static black hole. Moreover, we get an upper bound for its efficiency and show that for a certain class of black holes, this bound is universal and achieved by an “ideal gas” hole. Finally, we compute exact efficiencies for some particular and new engines.
Introduction

In recent times, the classic black hole thermodynamics [1–3] has been extended to include the cosmological constant Λ as a pressure through the relation \( p = -\Lambda/8\pi \)\(^1\). The fact that pressure is considered a dynamical variable, means that the mass of the black hole can no longer be identified with the energy, but has to be interpreted as the enthalpy [6]

\[
M = U + pV. \tag{0.1}
\]

The entropy and temperature are still defined in the usual way through \( S = A/4 \)\(^2\) and \( T = \kappa/2\pi \), where \( A \) is the area of the black hole and \( \kappa \) its surface gravity.

One of the consequences of considering pressure as a dynamical variable, is the appearance of its conjugate, the volume \( V \). For static black holes, this volume is simply given by the geometrical volume occupied in space. When considering non-static black holes, such as a Kerr black hole, this is no longer true and the volume becomes a more complicated quantity [7, 8].

In this context, the notion of holographic heat engines [9] is a natural thing to consider, by which a certain black hole is set in a thermodynamic cycle and produces work by changing its volume and pressure via \( dW = p\,dV \). Quite some effort has been made to compute the efficiencies of different holographic engines. For static black holes a couple of exact results stand out, for rectangular [10] and elliptical [11, 12] cycles in the \( p - V \) plane.

In this work we generalize these results by deriving an exact efficiency formula for static black holes which holds for virtually any holographic heat engine defined by a cycle in the \( p - V \) plane. We are able to do so by making some simple observations regarding heat flows along paths. We also get some inequalities which restrict their change in energy and adiabatic curves in the \( p - T \) plane. These results allow us to set an upper bound on the efficiency of heat engines which becomes universal and dimension independent when considering static black holes in the large volume limit or with vanishing pressure at zero temperature. Moreover, we show that for this class of black holes the bound is achieved by the so-called “ideal gas” black hole [13].

Outline This paper is organized as follows. We start in section 1 by remarking an important feature regarding heat flows along paths in the \( p - V \) plane when considering static black holes. To do so, we make use of the fact that the volume of static black holes has a simple expression. In section 2 we use the previous observations to find

\(^1\)See [4, 5] for reviews on the subject.

\(^2\)When considering modified theories of gravity, such as Gauss-Bonnet or Chern-Simons, there are other contributions to the entropy besides the area term [5].
bounds for their change in energy and adiabatic curves in the $p-T$ plane. We continue in section 3 by deriving an efficiency formula for practically any heat engine defined by a cycle in the $p-V$ plane. Later, we analyze the obtained formula and comment on it most remarkable features. We check its validity by calculating the efficiencies of previously known engines to then move ahead and apply it on some new examples, showing the scope of our results. Finally, in section 4 we review the work done and comment on the difficulties facing when trying to extend this procedure to non-static black holes.

1 Heat flow for static black holes

In this section we discuss a remarkable feature of static black holes that will play a crucial role in our calculations. As noted in previous work [9], since both the entropy and volume depend exclusively on $r_+$, the radial position of the event horizon, we have that an isochoric process $V = \text{const}$ is equivalent to an adiabatic $Q = 0$ ($C_V = 0$). Therefore, the adiabatic curves for any static black hole are given by vertical lines in the whole $p-V$ plane. We will take advantage of the simplicity and homogeneity of this result.

In this context, an interesting question arises; can we choose a reversible path in the $p-V$ plane so that the heat flow along that path has a definite sign? To answer this question, let's consider the diagram from figure 1 in which we have plotted several vertical adiabats and two paths which deviate from them, one to each side. It is clear that the heat flow along the deviated paths is of opposite sign and different from zero, because they are not vertical. Since in the whole $p-V$ plane the adiabats are given by vertical lines this means that any path that does not have a vertical section, by continuity cannot change the sign of its heat flow. The answer to our question is then the following; any path that can be described by a continuous function (which by definition cannot have a vertical section) will have a fixed sign for its heat flow.

![Figure 1](image)

**Figure 1.** Adiabatic curves in the $p-V$ diagram with two other paths which deviate from an adiabat to either side.
What about the precise sign of the heat flow in each direction? We can work this out by considering any specific path and checking its sign. Let’s take an isothermic process going to the right on the $p-V$ plane. The heat exchanged in this case can be written as

$$Q_{T_0} = T_0 \Delta S = \frac{T_0}{4} \left( A_2 - A_1 \right),$$

where $T_0$ is the temperature of the isotherm and we have used that the entropy is proportional to the area of the black hole. Now we notice that for static black holes, if the volume of the black hole increases so does its area, which means that if the isotherm is covered from left to right, $V_2 > V_1$ so that $A_2 > A_1$ and therefore $Q_{T_0} > 0$.

We can then conclude the following; any path in the $p-V$ plane described by a continuous function $f(V)$ and going from left to right, will have a positive heat flow into the system. If the path is covered in the opposite direction we will have the inverse behavior. This property will turn up having very deep and interesting consequences. It should be noted that we have strongly used the properties of static black holes to derive it.

Before moving on, we write an expression for the heat flow along a reversible path described by a function $f(V)$ and connecting two arbitrary states. Using the first law we have

$$Q = \Delta U + W,$$

where $W$ is the work done by the black hole. Since the path is reversible, we can calculate it by integrating $dW = f(V)dV$, so that we find

$$Q = \Delta U + \langle p \rangle \Delta V, \quad \langle p \rangle = \frac{1}{\Delta V} \int_{V_1}^{V_2} dV f(V), \quad (1.1)$$

where we have defined the average pressure along the path and $\Delta$ as the difference between the final and initial state. Writing the work contribution in this manner will prove to be very natural and useful in our following calculations. Note that in general we can write the internal energy in terms of the mass of the black hole using (0.1).

2 General constraints for static black holes

In this section we use the property previously discussed to find interesting and useful inequalities satisfied by static black holes.

---

3Some static black holes may have other contributions to the entropy besides the area term, so that the argument presented here is not clear. If for a specific black hole it was found that $Q_{T_0} < 0$, most of our results follow with some appropriate changes.
We start by considering a path described by a function between any two points in the \( p - V \) plane, so that the heat flow is given by (1.1). If we take the path from left to right, the heat flow will be non-negative, so that we find
\[
\frac{\Delta U}{\Delta V} \geq -\langle p \rangle, \tag{2.1}
\]
where we have allowed for the path to have vertical sections as long as it does not change its direction. Notice that the vertical sections do not contribute to the average pressure\(^4\). We can also consider the same situation but for a path going in the opposite direction. In that case, we will obtain the same relation because both \( \Delta U \) and \( \Delta V \) get a minus sign and \( \langle p \rangle \) is unchanged. This means that the inequality (2.1) holds for any two points in the \( p - V \) plane.

Notice that the right hand side is path dependent and the left side not. Since the inequality must hold for any path, we can choose a specific path which makes the most restrictive bound on the internal energy. Since the minimum pressure allowed by any thermodynamic system is be given by \( p(T = 0, V) \), it means that the critical path that will minimize the average pressure will be the one from figure 2.

![Figure 2](image)

**Figure 2.** Path between two arbitrary states which minimizes the average pressure. The dotted line corresponds to the pressure of the system at zero temperature \( p(T = 0, V) \). Notice that this will always be a non-increasing function of the volume for stable thermodynamic systems.

Since the vertical sections do not contribute to the average pressure, the most restrictive version of (2.1) is given by
\[
\frac{\Delta U}{\Delta V} \geq -\langle p \rangle_{T=0} = - \int_{V_1}^{V_2} dV \frac{p(0, V)}{\Delta V} \tag{2.2}
\]

\(^4\)We can understand this by recalling that \( \langle p \rangle \propto W \) and noticing that the work is zero for vertical paths.
We therefore get this interesting inequality which holds for any static black hole. Further on, it will turn out being a key ingredient to find an upper bound for the efficiency of holographic heat engines.

An interesting case of this inequality is when considering static black holes in the large volume limit or with \( p(0, V) = 0 \). For this class of black holes, the right hand side vanishes and (2.2) simplifies to

\[
\Delta U \Delta V \geq 0.
\]

(2.3)

We can now take an AdS black hole in the limit of large volume, usually referred as an “ideal gas” black hole [13]. This black hole has a mass given by \( M = pV \), so that from (0.1) its internal energy vanishes \( U = 0 \) and therefore (2.3) becomes an equality. Notice that this black hole has the particular property that \( Q = W \) for every thermodynamic process.

The bound (2.3) also becomes interesting when considering two states infinitesimally close to each other, so that we get a directional derivative instead. In particular, we can consider two points that have the same pressure and write the inequality in terms of the mass, so that (2.3) becomes

\[
\frac{\partial M}{\partial V} \bigg|_p \geq p.
\]

Remarkably, this inequality can be integrated exactly using some of the usual thermodynamic tricks. Writing the differential of the mass (0.1) in terms of the variables \((p, V)\) we find

\[
\frac{\partial M}{\partial V} \bigg|_p = T \frac{\partial S}{\partial V} \bigg|_p = T \frac{\partial p}{\partial T} \bigg|_S \geq p,
\]

where in the second equality we have used one of Maxwell’s thermodynamic relations. Noticing that \( S = \text{const} \) is equivalent to \( V = \text{const} \), we can integrate and find

\[
p \geq \left( \frac{p_0}{T_0} \right) T
\]

(2.4)

where this holds for adiabatic paths and static black holes in the large volume limit or with \( p(0, V) = 0 \).

To understand this relation we consider an “ideal gas” black hole where we get an equality instead. In this case, the adiabatic vertical lines in the \( p - V \) plane are directly mapped to linear curves in the \( p - T \) plane with slope \( p_0/T_0 \) and passing through the origin. If we instead consider any other system, inequality (2.4) provides a bound for the adiabatic curves in the \( p - T \) plane.
Notice that despite of the fact that the adiabatic curves in the $p − V$ plane are extremely simple, their shape in the $p − T$ usually are not, since the equation of state $p = p(T, V)$ might be complicated for an arbitrary static black hole.

3 Efficiency of holographic engines

The efficiency of an engine is defined as $\eta = W/Q_h$, where $W$ is the work done throughout a cycle and $Q_h$ the heat flow into the system. Usually, one of the difficulties when calculating $\eta$ for an arbitrary substance and cycle in the $p − V$ plane, is the fact that it is not easy to keep track of the signs of the heat flows along the paths. This is mainly because the adiabatic curves of the thermodynamic substance under consideration are usually non-trivial.

However, when considering static black holes, everything gets simplified since they have the simplest adiabatic curves of all; vertical lines in the whole $p − V$ plane. As discussed thoroughly in section 1 this means that paths going to the right have positive heat flow and to the left negative, meaning that it is trivial to keep track of the signs of the heat flow along the cycle.

In the following, we will derive an exact efficiency formula for practically any heat engine defined by a cycle in the $p − V$ plane. To do this, we consider a heat engine defined by the cycle of figure 3. The engine consists of two isochors (adiabats) and an upper and lower path given by any pair of functions $f_h$ and $f_c$ respectively.

![Figure 3](image-url)  

**Figure 3.** Heat engine considered. The process $2 \rightarrow 3$ and $4 \rightarrow 1$ are given by, isochors, equivalent to adiabats. The upper and lower paths are given by any pair of functions $f_h$ and $f_c$ respectively.

We are actually considering a family of engines, determined by the freedom to choose these functions
and the pressure and volume of the points 1–4. Practically any reasonable heat engine can be obtained from the cycle considered in figure 3.

Notice that there is a positive heat flow $Q_h$ into the engine only along the upper path, and a negative $Q_c$ only along the lower one. These heat flows can be written from (1.1) as

$$Q_h = (U_2 - U_1) + \langle p \rangle_h (V_2 - V_1), \quad (3.1)$$

$$Q_c = (U_4 - U_3) + \langle p \rangle_c (V_1 - V_2). \quad (3.2)$$

where the subscript $h$ and $c$ corresponds to the average value of $f_h$ and $f_c$ respectively.

To write the efficiency $\eta = W/Q_h$ we use that $W$ is given by the area enclosed by the cycle, which can be written as

$$W = \int_{V_1}^{V_2} dV \left( f_h(V) - f_c(V) \right) = \left( \langle p \rangle_h - \langle p \rangle_c \right) (V_2 - V_1).$$

Using this, together with (3.1) in $\eta = W/Q_h$, we find the following formula for the efficiency of our engine

$$\eta = \left( 1 - \frac{\langle p \rangle_c}{\langle p \rangle_h} \right) \left( \frac{\langle p \rangle_h + \Delta(M-pV)}{\Delta(V)} \right), \quad (3.3)$$

where $\Delta$ is taken as the difference between the states 1 and 2, and we have written the internal energy explicitly in terms of the mass of the black hole using (0.1). This formula holds for any static black hole and heat engine of the type given by figure 3.

**Analysis of the result**

In order to correctly understand the efficiency relation (3.3) it is important to keep in mind the difference between the specific working substance (black hole) under consideration and the characteristics of the heat engine, which is completely defined by the cycle in the $p-V$ plane. In this case, the heat engine from figure 3 is defined entirely by the pressure and volume of the points 1–4, and by the upper and lower paths.

The first interesting feature of (3.3) is the fact that its dependence on the particular black hole under consideration is extremely simple and only given by $\Delta M$. Every other factor depends on the characteristics of the engine. This means that if there are two distinct black holes that have the same value of $\Delta M$, then their efficiency will be the same.

An example in which this happens is when we considering an asymptotically AdS charged black hole and the same black hole but with a Gauss-Bonnet sector. If the space-time dimension is equal to five, their masses only differ in a constant value [13],
which means that they will have the same $\Delta M$ and therefore the same efficiency on any engine. It is curious that this behavior occurs exclusively for the five dimensional case.

Following the work previously done in [11, 14], the efficiency formula (3.3) also gives a very clear procedure for benchmarking black holes engines. It is enough to compare the value of $\Delta M$; black holes with lower $\Delta M$ will have larger efficiencies.

Now let us try to find an upper bound for the efficiency. From (3.3) we see that the efficiency is maximum, whenever $\Delta(M - pV) = \Delta U$ is minimum, which means that we can use inequality (2.2) and find

$$\eta \leq \left(1 - \frac{\langle p \rangle_c}{\langle p \rangle_h} \right) \left( \frac{\langle p \rangle_h}{\langle p \rangle_h - \langle p \rangle_{T=0}} \right).$$

This is an upper bound which holds for any engine and static black hole. It would be even more interesting to get an upper bound which is independent of the black hole under consideration. We can do this by restricting ourselves to black holes in the large volume limit or with $p(0, V) = 0$. In those cases, we get $\langle p \rangle_{T=0} = 0$ and the upper bound becomes

$$\eta \leq 1 - \frac{\langle p \rangle_c}{\langle p \rangle_h} = \eta_{h.g.}.$$  \hspace{1cm} (3.4)

which is simply the efficiency of an “ideal gas” black hole. We have then found a universal upper bound for the efficiency, independent of the spatial dimension, which holds for any heat engine defined by a cycle in the $p - V$ plane as described in figure 3.

Finally, let us analyze the dependence of (3.3) with the characteristics of the engine. Notice that apart from the coordinates of the four points, the formula only depends on the averages of the upper and lower paths $f_h$ and $f_c$. On top of being rather simple, this means that two very different engines will have the same efficiency as long as their averages along those paths is the same. In figure 4 we draw some examples of engines which have the same efficiency despite of being radically different. This is a non-trivial and remarkable behavior.

This result is unusual and seems to contradict the intuition we could have from usual thermodynamics. For instance, let us consider a fixed black hole and an engine $A$ in which the upper and lower paths are taken as isotherms. In this case, we know the efficiency is maximum and equal to Carnot’s $\eta_A = 1 - T_c/T_h$. Next, we consider another engine $B$ so that the mean pressure along the upper and lower path is the same as in engine $A$ but are not isotherms. In this case, our formula tells us that the efficiency of $B$ will be the same as $A$, $\eta_B = 1 - T_c/T_h$. Does this mean engine $B$ is as good as a Carnot engine without actually being one? This is something explicitly forbidden by the second law.
Three very different engines that have the same efficiency, since the average pressure along the upper and lower paths is always given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$.

To resolve this apparent problem, we notice that engine $B$ has to be in contact with more than two temperature reservoirs in order to perform the upper and lower paths. This means that the Carnot efficiency for engine $B$ will be given by $\eta_{\text{car}}^B = 1 - T'_c/T'_h$ where $T'_h$ and $T'_c$ are the hottest and lowest temperatures engine $B$ is in contact with. Since the average pressure along the paths has to be the same as in engine $A$ we have $T'_h > T_h$ and $T'_c < T_c$ so that $\eta_{\text{car}}^B > \eta_B$ and there is no contradiction with the second law.

### 3.1 Some applications

We can now apply the efficiency formula (3.3) to particular heat engines. Let’s start by considering some previously known examples, as a consistency check for our calculations.

1. **Rectangular engine:**

   We can get a rectangular engine from the cycle in figure 3 by taking $p_1 = p_2$, $p_3 = p_4$ and the upper and lower path as isobars, which means that the average pressures are given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$. Using this in the heat expressions (3.1) and (3.2) and writing everything in terms of the black hole mass we find

   $$\eta_{\text{rectangular}} = 1 - \frac{|Q_c|}{Q_h} = 1 - \left( \frac{M_3 - M_4}{M_2 - M_1} \right),$$

   which is the same result as in [10]. If we consider the “ideal gas” case in (3.4) we get

   $$\eta_{\text{rectangular}}^{\text{i.g.}} = 1 - \frac{p_4}{p_1},$$

   which is in agreement with the calculations in [9, 13].

2. **Elliptical engine:**

   To get an elliptical engine from the general cycle in figure 3, we consider $p_i = p$ for $i = 1, ..., 4$, and the upper and lower paths as half ellipses centered at $((V_1 + V_2)/2, p)$. 

---

**Figure 4.** Three very different engines that have the same efficiency, since the average pressure along the upper and lower paths is always given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$. 

To resolve this apparent problem, we notice that engine $B$ has to be in contact with more than two temperature reservoirs in order to perform the upper and lower paths. This means that the Carnot efficiency for engine $B$ will be given by $\eta_{\text{car}}^B = 1 - T'_c/T'_h$ where $T'_h$ and $T'_c$ are the hottest and lowest temperatures engine $B$ is in contact with. Since the average pressure along the paths has to be the same as in engine $A$ we have $T'_h > T_h$ and $T'_c < T_c$ so that $\eta_{\text{car}}^B > \eta_B$ and there is no contradiction with the second law.

### 3.1 Some applications

We can now apply the efficiency formula (3.3) to particular heat engines. Let’s start by considering some previously known examples, as a consistency check for our calculations.

1. **Rectangular engine:**

   We can get a rectangular engine from the cycle in figure 3 by taking $p_1 = p_2$, $p_3 = p_4$ and the upper and lower path as isobars, which means that the average pressures are given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$. Using this in the heat expressions (3.1) and (3.2) and writing everything in terms of the black hole mass we find

   $$\eta_{\text{rectangular}} = 1 - \frac{|Q_c|}{Q_h} = 1 - \left( \frac{M_3 - M_4}{M_2 - M_1} \right),$$

   which is the same result as in [10]. If we consider the “ideal gas” case in (3.4) we get

   $$\eta_{\text{rectangular}}^{\text{i.g.}} = 1 - \frac{p_4}{p_1},$$

   which is in agreement with the calculations in [9, 13].

2. **Elliptical engine:**

   To get an elliptical engine from the general cycle in figure 3, we consider $p_i = p$ for $i = 1, ..., 4$, and the upper and lower paths as half ellipses centered at $((V_1 + V_2)/2, p)$. 

---

**Figure 4.** Three very different engines that have the same efficiency, since the average pressure along the upper and lower paths is always given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$. 

To resolve this apparent problem, we notice that engine $B$ has to be in contact with more than two temperature reservoirs in order to perform the upper and lower paths. This means that the Carnot efficiency for engine $B$ will be given by $\eta_{\text{car}}^B = 1 - T'_c/T'_h$ where $T'_h$ and $T'_c$ are the hottest and lowest temperatures engine $B$ is in contact with. Since the average pressure along the paths has to be the same as in engine $A$ we have $T'_h > T_h$ and $T'_c < T_c$ so that $\eta_{\text{car}}^B > \eta_B$ and there is no contradiction with the second law.

### 3.1 Some applications

We can now apply the efficiency formula (3.3) to particular heat engines. Let’s start by considering some previously known examples, as a consistency check for our calculations.

1. **Rectangular engine:**

   We can get a rectangular engine from the cycle in figure 3 by taking $p_1 = p_2$, $p_3 = p_4$ and the upper and lower path as isobars, which means that the average pressures are given by $\langle p \rangle_h = p_1$ and $\langle p \rangle_c = p_4$. Using this in the heat expressions (3.1) and (3.2) and writing everything in terms of the black hole mass we find

   $$\eta_{\text{rectangular}} = 1 - \frac{|Q_c|}{Q_h} = 1 - \left( \frac{M_3 - M_4}{M_2 - M_1} \right),$$

   which is the same result as in [10]. If we consider the “ideal gas” case in (3.4) we get

   $$\eta_{\text{rectangular}}^{\text{i.g.}} = 1 - \frac{p_4}{p_1},$$

   which is in agreement with the calculations in [9, 13].

2. **Elliptical engine:**

   To get an elliptical engine from the general cycle in figure 3, we consider $p_i = p$ for $i = 1, ..., 4$, and the upper and lower paths as half ellipses centered at $((V_1 + V_2)/2, p)$.
We take the horizontal radius of the ellipses as \( R_v = (V_2 - V_1)/2 \) and the vertical one as \( R_p \). The average pressures can be computed easily so that we find

\[
\langle p \rangle_h = p + \frac{\pi}{4} R_p \\
\langle p \rangle_c = p - \frac{\pi}{4} R_p .
\]

Using this in equation (3.3) we get the efficiency of an elliptical engine

\[
\eta_{\text{elliptical}} = \frac{2}{1 + 2 \left( \frac{\Delta M}{\pi R_p R_v} \right)} ,
\]

which agrees with the relation found in [12], and in [11] for the ideal gas case. We can also compare this formula with the numerical calculations done in [11], where the efficiency of a circular engine was calculated for three different types of static black holes. We find agreement with their results to the third significant figure and a percentile difference smaller than 0.4%. Having checked the efficiency formula (3.3), we are now in a position to use it on new holographic heat engines.

3. Axially symmetric engine:

There is an interesting feature of the efficiency formula for an elliptical engine (3.5); it only depends on the ratio of \( \Delta M \) and the area enclosed by the engine, which is equal to the work. Are there any other types of engines which display this behavior? To answer this, we consider a cycle that is symmetric with respect to a horizontal axis passing through its center. In figure 5 we see an engine of this type.

![Figure 5](image)

**Figure 5.** Axially symmetric heat engine. The upper path can be given by any function \( f_h(V) \), while the lower one is fixed by \( f_c(V) = 2p - f_h(V) \), so that the cycle is axially symmetric.

Comparing with the general cycle of figure 3 we can get this engine by taking \( p_i = p \) for \( i = 1, ..., 4 \), and fixing the lower path by \( f_c(V) = 2p - f_h(V) \), so that the cycle is
axially symmetric. The upper path is given by any function $f_h$. A quick calculation shows that the average pressures are given by

$$\langle p \rangle_c = 2p - \langle p \rangle_h \quad \langle p \rangle_h = p + \frac{W}{2\Delta V}.$$ (3.6)

Using these relations in the efficiency formula (3.3) we find

$$\eta_{\text{symmetric}} = \frac{2}{1 + 2\left(\frac{\Delta M}{W}\right)},$$ (3.7)

which provides a nice generalization of equation (3.5).

4. Triangular type engine:

Another interesting and simple engine is given by considering the axially symmetric engine from figure 5, but taking either the upper or lower path as an isobar. When the isobar is taken along the lower path we will regard the triangular engine as being positive (+), while name it negative (−) when it is the other way about.

For a positive triangular engine the average pressures can be written as

$$\langle p \rangle_c = p \quad \langle p \rangle_h = p + \frac{W}{\Delta V}.$$ (3.8)

Notice that if we compare with the expression (3.6) for $\langle p \rangle_h$ there is only a difference in a factor of two because in this case the cycle is half the size. Using these relations in (3.3) we get the efficiency for a positive triangular engine

$$\eta_{\text{triangular}}^{(+)} = \frac{1}{1 + \left(\frac{\Delta M}{W}\right)}.$$ (3.9)

If we now consider a negative triangular engine the average pressures are given by

$$\langle p \rangle_c = p - \frac{W}{\Delta V} \quad \langle p \rangle_h = p,$$

so that the efficiency (3.3) is equal to

$$\eta_{\text{triangular}}^{(-)} = \frac{W}{\Delta M}.$$ (3.10)

This expression is particularly simple due to the fact that the upper path, which is the only part which contributes to $Q_h$, is an isobar.

We could continue calculating efficiencies for many other engines, but at this point we decide to stop, since the main features and techniques of the procedure have already been exposed.
4 Closing remarks

In this work, we have exploited the fact that static black holes have very simple adiabatic curves to derive interesting inequalities and an exact efficiency formula for holographic heat engines. Additionally, we obtained an upper bound for this efficiency and showed that it becomes universal and dimensional independent for black holes in the large volume limit or with $p(T = 0, V) = 0$. For this class of black holes, we obtained that the bound is achieved by an “ideal gas” black hole. Moreover, we found an equivalence relation between seemingly very different engines as far as their efficiency is concerned, since heat engines like the ones from figure 4, end up having the same efficiency. The derived inequality (2.2) is intriguing due to its generality and it would be interesting to continue studying its consequences.

A natural question that arises is whether these methods can be extended to include non-static black holes. The difficulty facing when wanting to make progress in this direction is the fact that there is no general formula for the adiabatic curves of non-static black holes. This means that the simple observation that paths going to the right have a positive heat inflow is not valid anymore. Then, if we try to calculate the efficiency of an engine like the one from figure 3, the upper and lower paths may have sections with positive and negative contributions of heat, so that we are unable to keep track of the directions of the heat flows and get expressions for $Q_h$ or $Q_c$. Nevertheless, it would be interesting to try to calculate the adiabatic curves for a simple non-static black hole and try to apply a similar procedure.

Acknowledgments

Felipe Rosso thanks Clifford V. Johnson, Avik Chakraborty and Scott MacDonald for comments on the manuscript.

References

[1] J. M. Bardeen, B. Carter and S. W. Hawking “The Four laws of black hole mechanics”, Commun. Math. Phys. 31 (1973) 161-170.
[2] J. D. Bekenstein “Black holes and entropy”, Phys. Rev. D7 (1973) 2333-2346.
[3] S. W. Hawking “Particle Creation by Black Holes”, Commun. Math. Phys. 43 (1975) 199-220.
[4] R. M. Wald “The thermodynamics of black holes”, Living Rev. Rel. 4 (2001) 6, [arXiv:9912119].
[5] D. Kubiznak, R. B. Mann and M. Teo “Black hole chemistry: thermodynamics with 
Lambda”, (2017) [arXiv:1608.06147v2].

[6] D. Kastor, R. Ray and J. Traschen “Enthalpy and Mechanics of AdS Black Holes”, 
(2009) [arXiv:0904.2765v2].

[7] M. Cvetiˇc, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and 
an Entropy Inequality for the Thermodynamic Volume”, (2011) [arXiv:1012.2888v2].

[8] S. MacDonald, “Thermodynamic Volume of Kerr-bolt-AdS Spacetime”, (2014) 
[arXiv:1406.1257v2].

[9] C. V. Johnson “Holographic heat engines”, (2014) [arXiv:1404.5982v2].

[10] C. V. Johnson, “An Exact Efficiency Formula for Holographic Heat Engines”, (2016) 
[arXiv:1602.02838v2].

[11] A. Chakraborty and C. V. Johnson “Benchmarking Black Hole Heat Engines”, (2016) 
[arXiv:1612.09272v1].

[12] R.A. Hennigar, F. McCarthy, A. Ballon and R.B. Mann, “Holographic heat engines: 
general considerations and rotating black holes”, (2017) [arXiv:1704.02314v1]

[13] C. V. Johnson “GaussBonnet Black Holes and Holographic Heat Engines Beyond 
Large N”, (2015) [arXiv:1511.08782v4].

[14] A. Chakraborty and C. V. Johnson “Benchmarking Black Hole Heat Engines, II”, 
(2017) [arXiv:1709.00088v1].