Approximation Theorem for Principle Eigenvalue of Discrete $p$-Laplacian

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Abstract
For the principle eigenvalue of discrete weighted $p$-Laplacian on the set of nonnegative integers, the convergence of an approximation procedure and the inverse iteration is proved. Meanwhile, in the proof of the convergence, the monotonicity of an approximation sequence is also checked. To illustrate these results, some examples are presented.

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1 Introduction

In order to compute the principle eigenvalue of discrete weighted $p$-Laplacian, a series of results were presented in [6]. In fact, the results are more or less complete, including the variational formulas in different formulations, explicit lower and upper estimates, a criterion for positivity and an approximating procedure for the principle eigenvalue. The approximating procedure is very much practical in computation. It is regretted that the convergence is still missed in that setup. This is the aim of this paper.

We need some notation to state our main results. Here, the notation and boundary conditions are the same as those used in [6 §2], unless otherwise stated. Let $E := \{k \in \mathbb{Z}_+ : 0 \leq k < N + 1\}$ with $N \leq \infty$ and $\{\nu_k : k \in E\}$ is a positive sequence with boundary condition $\nu_{-1} = 0$. First, we define an operator $\Pi$ as follows

$$\Pi_i(f) = \frac{1}{f_i^{p-1}} \left[ \sum_{j=i}^N \nu_j \left( \sum_{k=0}^j \mu_k f_k^{p-1} \right)^{p^*-1} \right]^{p-1}, \quad i \in E,$$

where $f$ is a positive function on $E$, $p^*$ is the conjugate number of $p$ (i.e., $1/p + 1/p^* = 1$), $\{\mu_k : k \in E\}$ is a positive sequence and $\nu_j = \nu_j^{1-p^*}$. In fact, the operator $\Pi$ in linear case was first introduced in [2] by Chen in 2001. Next, let $\partial_k(f) = f_{k+1} - f_k$, then the weighted $p$-Laplacian $\Omega_p$ ($p > 1$) can be rewritten as

$$\Omega_p(f) = \nu_k | \partial_k(f) |^{p-1} \text{sgn}(\partial_k(f)) - \nu_{k-1} | \partial_{k-1}(f) |^{p-1} \text{sgn}(\partial_{k-1}(f)). \quad (1)$$
Let $\lambda_p$ denote the principle eigenvalue of the weighted $p$-Laplacian $\Omega_p$ and set
$$\sigma_p = \sup_{n \in E} \mu[0, n] \hat{v}[n, N]^{p-1},$$
where we write $\mu[n, n] = \sum_{j=m}^n \mu_j$ for a measure $\mu$. In what follows, let $f_k^{(n)}$ denote the kth component of the vector $f^{(n)}$ and $D_p(f) = \sum_{k \in E} \nu_k |f_k - f_{k+1}|^p$. When $N \leq \infty$, the approximation procedure for $\lambda_p$ is presented in Theorem 1.

**Theorem 1** Assume that $N \leq \infty$ and $\sigma_p < \infty$.

(1) Define
$$f^{(1)} = \hat{v}[\cdot, N]^{1/p^*}, \quad f^{(n)} = f^{(n-1)} \left( \Pi \left( f^{(n-1)} \right) \right)^{p^* - 1}, \quad n \geq 2,$$
and $\delta_n = \sup_{i \in E} \Pi_i \left( f^{(n)} \right)$. Then $\delta_n$ is decreasing in $n$ (denote its limit by $\delta_\infty$) and
$$\lambda_p \geq \delta_\infty^{-1} \geq \cdots \geq \delta_1^{-1} > 0.$$

(2) For fixed $\ell, m \in E$, $\ell < m$, define
$$f^{(1, \ell, m)} = \hat{v}[\cdot, \ell, m] \mathcal{I}_{\leq m},$$
$$f^{(n, \ell, m)} = f^{(n-1, \ell, m)} \left( \Pi \left( f^{(n-1, \ell, m)} \right) \right)^{p^* - 1} \mathcal{I}_{\leq m}, \quad n \geq 2,$$
where $\mathcal{I}_{\leq m}$ is the indicator of the set $\{0, 1, \cdots, m\}$ and then define
$$\delta'_n = \sup_{\ell, m; \ell < m} \min_{i \leq m} \Pi_i \left( f^{(n, \ell, m)} \right).$$
Then $\delta'_n$ is increasing in $n$ (denote its limit by $\delta'_\infty$) and
$$\sigma_p \geq \delta'_\infty^{-1} \geq \cdots \geq \delta'_1^{-1} \geq \lambda_p.$$

Next, define
$$\tilde{\delta}_n = \sup_{\ell, m; \ell < m} \frac{\mu(f^{(n, \ell, m)})^p}{D_p(f^{(n, \ell, m)})}, \quad n \geq 1.$$
Then $\tilde{\delta}_n^{-1} \geq \lambda_p$ and $\tilde{\delta}_{n+1} \geq \delta'_n$ for $n \geq 1$.

In fact, as an extension of [3], Theorem 1 was first presented in [6, Theorem 2.4], where the monotonicity of $\{\delta_n\}$ and $\{\delta'_n\}$ was proved, besides, the comparison property of $\{\delta_n\}$ and $\{\delta'_n\}$ was also proved there. Here, we remark that three questions are open in Theorem 1: (1) $\delta_\infty^{-1} \geq \lambda_p$; (2) $\delta_\infty^{-1} \geq \lambda_p$; (3) Does $\{\delta_n\}$ have monotonicity in $n$? The main aim of this paper is to answer these three questions in the case of finite $N$ and the answers are stated in Theorem 2 below.
**Theorem 2**  Assume that $N < \infty$. Let $f^{(1)} > 0$ be given.

1. Define successively functions $f^{(n)}$ on $E$:
   \[
   f^{(n)} = f^{(n-1)} \left( \Pi \left( f^{(n-1)} \right) \right)^{p^*-1}, \quad n \geq 2.
   \]

2. For $n \geq 1$, define three sequences as follows
   \[
   \delta_n = \sup_{i \in E} \left( f^{(n)} \right), \quad \delta'_n = \inf_{i \in E} \left( f^{(n)} \right), \quad \overline{\delta}_n = \frac{\mu \left( f^{(n)} \right)}{D_p \left( f^{(n)} \right)}.
   \]

   Then $\{\delta_n\}$ is decreasing in $n$, both $\{\delta'_n\}$ and $\{\overline{\delta}_n\}$ are increasing in $n$ and
   \[
   0 < \delta_n^{-1} \leq \lambda_p \leq \delta_n^{-1} \leq \delta_n^{-1} \leq \sigma_n^{-1} < \infty, \quad n \geq 1.
   \]

   Furthermore,
   \[
   \lim_{n \to \infty} \delta_n^{-1} = \lim_{n \to \infty} \overline{\delta}_n^{-1} = \lim_{n \to \infty} \delta'_n^{-1} = \lambda_p.
   \]

   The final assertion of Theorem 2 is the main addition to [6; Theorem 2.4].

The next result is the inverse iteration for obtaining the principle eigenpair of $\Omega_p$ on finite space.

**Algorithm 3**  (Inverse iteration)  Assume that $N < \infty$. Given a positive $v^{(0)}$ such that $\|v^{(0)}\|_{\mu, p} = 1$, at the $n$th ($n \geq 1$) step, solve the nonlinear equation in $w^{(n)}$

\[
\begin{align*}
-\Omega_p w^{(n)}(k) &= \mu_k |v_k^{(n-1)}|^{p-2} v_k^{(n-1)}, \quad k \in E, \\
w^{(n)}_{N+1} &= 0.
\end{align*}
\]

and define

\[
v^{(n)} = \|w^{(n)}\|_{\mu, p}^{-1} w^{(n)}, \quad z_n = D_p(v^{(n)}),
\]

where $v_k^{(n)}$ denotes the $k$th component of the vector $v^{(n)}$. $\| \cdot \|_{\mu, p}$ denotes the $L^p(\mu)$-norm. Then $\{v^{(n)}\}$ converges to the eigenfunction of $\lambda_p$. Next, $\{z_n\}$ is decreasing in $n$ and

\[
\lim_{n \to \infty} z_n = \lambda_p.
\]

As far as we know, the inverse iteration method in numerical algebraic computation was extended to nonlinear eigenproblems by Hein and Bhler in 2010. For $p$-Laplacian, the inverse iteration method was introduced in 2009 by Biezuner, Ercole and Martins. In 2015, the inverse iteration for obtaining $q$-eigenpairs was given in [7] by Ercole. In 2016, the inverse iteration was introduced in [9] by Hynd and Lindgren for $p$-ground states. Apart from the ordinary $p$-Laplacian operator, we are studying the weighted one, as shown by [2] including two measures $\mu$ and $\nu$. It is quite natural and similar to the generalization from the Laplacian operator to the elliptic one.

Here, we mention that this paper is based on the author’s master thesis [10] with some improvements. Besides, further work about shifted inverse iteration was presented in Chen [5] where the work related to this paper was cited.
2 Examples

To illustrate the convergence of Theorem 2 and Algorithm 3, we consider two examples taken from [6; Example 2.6 and 2.7]. Using the above two different methods to compute the maximal eigenvalue with MATLAB-R2013a. Here, unless otherwise stated, the initial we take is \( w_0 = f^{(1)} = \hat{\nu}[\cdot, N]^{1/p^*} \) as introduced in [6, §2] and \( v(0) = \|w(0)\|_{\mu^*_p}^{-1}w(0) \).

**Example 4** Assume that \( E = \{0, 1, \ldots, N\} \). Let \( \mu_k = 20^k, \nu_k = 20^{k-1} \) for \( k \in E \). Then \( \lambda_p \approx 0.782379 \) when \( p = 4.5 \) and \( N = 80 \). Partial results with the inverse iteration and approximation procedure in Theorem 2 are given in Tables 1 and 2. Here and in what follows, we stop our computation once the six precisely significant digits is achieved.

| \( k \) | \( z_k \) | \( k \) | \( z_k \) | \( k \) | \( z_k \) |
|----|----|----|----|----|----|
| 0  | 0.828685 | 11 | 0.785009 | 90 | 0.782417 |
| 1  | 0.796974 | 12 | 0.784798 | 100 | 0.782403 |
| 2  | 0.791961 | 13 | 0.784613 | 120 | 0.782388 |
| 3  | 0.789715 | 14 | 0.784449 | 140 | 0.782383 |
| 4  | 0.788379 | 15 | 0.784303 | 160 | 0.782381 |
| 5  | 0.787472 | 16 | 0.784172 | 180 | 0.782380 |
| 6  | 0.786805 | 17 | 0.783759 | 200 | 0.782380 |
| 7  | 0.786291 | 18 | 0.783155 | 210 | 0.782380 |
| 8  | 0.785878 | 19 | 0.782839 | \( \geq 217 \) | 0.782379 |
| 9  | 0.785539 | 20 | 0.782656 |
| 10 | 0.785253 | 21 | 0.782482 |

Table 1 and Figure 1 show that \( \{z_k\} \) decreases faster for smaller \( k \), then goes down slowly and finally becomes stable, which is similar to the linear case in [4].

**Table 2. Partial outputs of \( (n, 1/\delta_{n+1}) \)**
Figure 1  The figure of $z_k$ for $k = 0, 1, \ldots, 350.$

| $n$ | $1/\delta_{n+1}$ | $1/\delta_{n+1}$ | $n$ | $1/\delta_{n+1}$ | $1/\delta_{n+1}$ |
|-----|-----------------|-----------------|-----|-----------------|-----------------|
| 0   | 0.778535        | 4.44787         | 16  | 0.778549        | 0.818055        |
| 1   | 0.778535        | 1.59105         | 20  | 0.778579        | 0.809951        |
| 2   | 0.778535        | 1.14480         | 30  | 0.778796        | 0.799288        |
| 3   | 0.778535        | 1.01136         | 40  | 0.779214        | 0.794022        |
| 4   | 0.778535        | 0.948633        | 50  | 0.779722        | 0.790885        |
| 5   | 0.778535        | 0.912311        | 70  | 0.780660        | 0.787332        |
| 6   | 0.778535        | 0.888671        | 90  | 0.781319        | 0.785410        |
| 7   | 0.778535        | 0.872080        | 100 | 0.781552        | 0.784767        |
| 8   | 0.778535        | 0.859805        | 120 | 0.781877        | 0.783870        |
| 9   | 0.778536        | 0.850360        | 140 | 0.782074        | 0.783315        |
| 10  | 0.778536        | 0.842871        | 160 | 0.782194        | 0.782970        |
| 11  | 0.778537        | 0.836789        | 180 | 0.782266        | 0.782555        |
| 12  | 0.778538        | 0.831752        | 200 | 0.782309        | 0.782621        |
| 13  | 0.778539        | 0.827513        | 210 | 0.782324        | 0.782574        |
| 14  | 0.778542        | 0.823897        | 217 | 0.782332        | 0.782548        |
| 15  | 0.778545        | 0.820776        | 218 | 0.782333        | 0.782544        |

From Tables 1 and 2, it follows that $1/\delta_n \leq 1/\delta_{n+1} \leq 1/\delta'_{n+1}$. Besides, it also shows that the sequence $\{z_n = 1/\delta_{n+1}\}$ converges faster than the sequences $\{1/\delta_n\}$ and $\{1/\delta'_{n}\}$.

To see the serious influence of different initials for inverse iteration, we simply take

$$w^{(0)} = \bar{\nu}[M, N]^{1/p^*},$$

$$w^{(0)} = 1,$$

as initials, respectively. The comparing results are given in Table 3.

Table 3. Outputs when $N = 20, p = 3$ for different initials
Table 3 shows that a good initial for the inverse iteration is essential.

Example 5 [6; Example 2.7] Assume that $E = \{0, 1, \cdots, N\}$. Let $\mu_k = \nu_k = 1$ for each $k \in E$. When $p = 2.5$ and $N = 40$, the outputs of $z_n$, $1/\delta_{n+1}$, $1/\delta'_{n+1}$, and $1/\delta''_{n+1}$ are given in Table 6.

Table 6. Outputs of $(n, z_n, 1/\delta_{n+1}, 1/\delta'_{n+1}, 1/\delta''_{n+1})$

| n | $z_n$          | $\delta_{n+1}$^-1 | $\delta'_{n+1}$^-1 | $\delta''_{n+1}$^-1 |
|---|----------------|-------------------|---------------------|---------------------|
| 0 | 0.000458009    | 0.000255829       | 0.00160159          | 0.000458009         |
| 1 | 0.000271491    | 0.000269664       | 0.000283578         | 0.000271491         |
| 2 | 0.000271279    | 0.000271048       | 0.000272059         | 0.000271279         |
| 3 | 0.000271277    | 0.000271250       | 0.000271349         | 0.000271277         |
| 4 |                |                   | 0.000271285         |                    |
| 5 |                |                   | 0.000271278         |                    |
| 6 |                |                   | 0.000271277         |                    |

Here take $z_n$ for instance, when $n \geq 3$, the outputs are the same and so are omitted.

3 Proofs of the main results in Section 1

To prove Algorithm 3, we need to solve equation (2). Fortunately, the solution can be expressed explicitly in terms of operator $II$, as shown in Lemma 6 below. Actually, equation (2) is a particular case of Poisson Equation (5):

$$- \Omega_p g(k) = \mu_k |f_k|^{p-1} \text{sgn}(f_k), \quad k \in E,$$

where $g, f \in \{g : E \to \mathbb{R} | g_{-1} = g_0, g_{N+1} = 0\}$. In this section, we assume that $N < \infty$. 

Lemma 6  Given $f > 0$, then the unique solution to (5) is

$$g(k) = f_k (\Pi_k(f))^{p^{*}-1}, \quad k \in E.$$  

Proof. With $f > 0$ at hand, by (1), making a summation on both sides of equation (5) over $0 \sim k$, we get

$$-\nu_k |\tilde{\varphi}_k(g)|^{p-1} \text{sgn}(\tilde{\varphi}_k(g)) = \sum_{j=0}^{k} \mu_j f_j^{p-1} > 0, \quad k \in E. \quad (6)$$

Thus $\tilde{\varphi}_k(g) < 0$. Combining with the boundary condition $g_{N+1} = 0$, we have $g_k > 0$ for each $k \in E$. Besides, by (6) and the boundary condition at $N$, we have $g_N = \tilde{\varphi}_N \left( \sum_{j=0}^{N} \mu_j f_j^{p-1} \right)^{p^{*}-1}$. Now with $g > 0$ and $\tilde{\varphi}(g) < 0$ at hand, by (6) and induction on $k$, it follows that

$$g_k = \sum_{i=k}^{N} \tilde{\nu}_i \left( \sum_{j=0}^{i} \mu_j f_j^{p-1} \right)^{p^{*}-1}, \quad k \in E.$$  

Therefore, we conclude $g(k) = f_k (\Pi_k(f))^{p^{*}-1}$, it is easy to verify the uniqueness of the solution, which proves the lemma. \qed

Now in our setup, $\lambda_p > 0$ by [6; Theorem 2.3], and the eigenfunction corresponding to $\lambda_p$ is strictly monotone by [6; Proposition 3.2]. Note that in Algorithm [3, we choose the positive initial $v^{(0)}$. Now, by induction on $n$, both $\{w^{(n)}\}$ and $\{v^{(n)}\}$ are positive and decreasing functions according to Lemma 6. Hence, as the mimic eigenfunctions, we are happy to get the monotone sequences $\{w^{(n)}\}$ and $\{v^{(n)}\}$. Therefore, in what follows, the modulus can be discarded. Besides, noting that $\Pi(f) = \Pi(cf)$ for each constant $c > 0$, we have Remark 7 by induction on $n$.

Remark 7  Assume that $v^{(0)} = cf^{(1)}$ for a constant $c > 0$, where $v^{(0)}$ and $f^{(1)}$ are given in Algorithm [3 and Theorem 2 respectively. Then

$$z_n = \delta_{n+1}^{-1}, \quad n \geq 0.$$  

Noticing that $\{v^{(n)}\}$ in Algorithm [3 is an approximation sequence of the eigenfunction of $\lambda_p$, we have for large enough $n$,

$$-\Omega_p v^{(n)} \approx \lambda_p \text{diag}(\mu)(v^{(n)})^{p-1}.$$  

Furthermore,

$$\lambda_p \approx \frac{-\Omega_p v^{(n)}(v^{(n)})}{(\text{diag}(\mu)(v^{(n)})^{p-1}, v^{(n)})} = \frac{D_p(v^{(n)})}{\|v^{(n)}\|_{\mu,p}^{p-1}} = \delta_{n}^{-1}.$$
Hence the convergence of Algorithm 3 could be expected. Define
\[ \xi_{n-1} = \frac{1}{\|w^{(n)}\|_{\mu,p}^{p-1}}, \quad n \geq 1. \]
(7)
Recall that \( v^{(n)} = \|w^{(n)}\|_{\mu,p}^{-1} w^{(n)} \). Inserting this into (2), it follows that
\[
\begin{align*}
&\left\{ -\Omega_p v^{(n)}(k) = \xi_{n-1} \mu_k |v_k^{(n-1)}|^{p-2} v_k^{(n-1)}, \quad k \in E, \\
&v^{(n)}_{N+1} = 0,
\end{align*}
\]
(8)
where \( v^{(n)} \) and \( v^{(n-1)} \) are given in Algorithm 3.

Now we start to prove Algorithm 3. Inspired by the method used in [9; Proposition 2.4], we have the following proposition for the convergence of \( \{z_n\} \).

**Proposition 8** The sequences
\[ z_n = D_p(v^{(n)}) \quad \text{and} \quad \xi_n = \|w^{(n+1)}\|_{\mu,p}^{1-p} \]
defined in Algorithm 3 and (7), respectively, are decreasing in \( n \) and satisfy
\[ \lambda_p \leq z_{n+1} \leq \xi_n \leq z_n \leq \xi_{n-1}. \]
Therefore
\[ \lambda_p \leq \lim_{n \to \infty} z_n = \lim_{n \to \infty} \xi_n =: \xi. \]
(9)

**Proof.** Since \( \nu_{-1} = 0 \) and \( f_{N+1} = 0 \), we have for arbitrary \( \{H_k\} \)
\[
\sum_{k=0}^{N} \nu_{k-1} f_k H_{k-1} = \sum_{k=1}^{N} \nu_k f_{k+1} H_k = \sum_{k=0}^{N} \nu_k f_{k+1} H_k.
\]
Combining this with (4), we have
\[
(-\Omega_p g, f)
= -\sum_{k=0}^{N} \nu_k f_k |\hat{\nu}_k(g)|^{p-1} \text{sgn} (\hat{\nu}_k(g)) + \sum_{k=0}^{N} \nu_{k-1} f_k |\hat{\nu}_{k-1}(g)|^{p-1} \text{sgn} (\hat{\nu}_{k-1}(g))
= \sum_{k=0}^{N} \nu_k |\hat{\nu}_k(g)|^{p-1} \text{sgn} (\hat{\nu}_k(g)) \hat{\nu}_k(f).
\]
(10)
In particular,
\[ (-\Omega_p g, g) = \sum_{k=0}^{N} \nu_k |\hat{\nu}_k(g)|^{p} = D_p(g). \]
If \( (g, f) \) satisfy Poisson equation (5) and \( f \neq 0 \), then for each \( h \in \{g : E \to \mathbb{R} | g_{-1} = g_0, g_{N+1} = 0\} \), we have
\[ |(-\Omega_p g, h)| \leq \|f\|_{\mu,p}^{-1} \text{sgn}(f), h \|_\mu \leq \|f\|_{\mu,p}^{-1} \|h\|_{\mu,p} \quad (\text{Hölder inequality}). \]
Dividing both sides by \( \|g\|_{\mu,p}^p \), it follows that

\[
\frac{|(-\Omega_p, g)|}{\|g\|_{\mu,p}^p} \leq \frac{\|f\|_{\mu,p}^{p-1}}{\|g\|_{\mu,p}^{p-1}} \|h\|_{\mu,p}.
\]

In particular, when \( h = g \), we have

\[
\frac{D_p(g)}{\|g\|_{\mu,p}^p} \leq \frac{\|f\|_{\mu,p}^{p-1}}{\|g\|_{\mu,p}^{p-1}} = \frac{\|f\|_{\mu,p}^p}{\|g\|_{\mu,p}^{p-1}}. \tag{11}
\]

Note that here we use the norm \( \| \cdot \|_{\mu,p} \) to control the ordinary inner product (in particular, control \( D_p \)). Now, we use \( D_p \) to control the norm \( \| \cdot \|_{\mu,p} \). Again, let \( (g, f) \) satisfy Poisson equation (5) and \( f \neq 0 \), then

\[
\|f\|_{\mu,p}^p = (|f|^{p-1} \text{sgn}(f), f)_{\mu} (-\Omega_p g, f)
\]

\[
= \sum_{k=0}^{N} \nu_k |\partial_k(g)|^{p-1} \text{sgn}(\partial_k(g)) \partial_k(f)
\]

\[
= (|\partial(g)|^{p-1} \text{sgn}(\partial(g)), \partial(f))_\nu, \tag{12}
\]

Hölder inequality

\[
\leq D_p(g) \frac{p-1}{p} D_p(f) \frac{1}{p}.
\]

Combining (11) with (12), it follows that

\[
\frac{D_p(g)}{\|g\|_{\mu,p}^p} \leq \frac{\frac{p-1}{p}}{\|g\|_{\mu,p}^{p-1}} D_p(f) \frac{1}{p}. \tag{13}
\]

By a cancellation procedure, we get

\[
\frac{D_p(g)}{\|g\|_{\mu,p}^p} \leq \frac{D_p(f)}{\|f\|_{\mu,p}^p}.
\]

Now letting \( f = v^{(n)} \), \( g = u^{(n+1)} \) and noting that \( \|v^{(n)}\|_{\mu,p} = 1 \), we have

\[
\frac{z_{n+1}}{D_p(u^{(n+1)})} \leq \frac{1}{\|u^{(n+1)}\|_{\mu,p}^{p-1} \|v^{(n)}\|_{\mu,p}} = 1
\]

\[
\leq \frac{1}{\|v^{(n)}\|_{\mu,p}^{p-1} \|v^{(n)}\|_{\mu,p}} = \frac{1}{\|v^{(n)}\|_{\mu,p}^{p-1}} = \xi_n
\]

\[
\leq \frac{D_p(u^{(n+1)}) \frac{p-1}{p}}{\|u^{(n+1)}\|_{\mu,p}^{p-1} \|v^{(n)}\|_{\mu,p}^{p-1}} = \frac{z_{n+1}^{(p-1)/p} z_{n+1}^{1/p}}{z_{n+1}^{(p-1)/p} z_{n+1}^{1/p}}.
\]

By a simple cancellation procedure, we get

\[
z_{n+1} \leq \xi_n \leq z_n \leq \xi_{n-1}.
\]

This implies not only the decreasing property of \{z_n\} and \{\xi_n\} in Algorithm 3 and (7), respectively, but also (9).

Next, we prove the convergence of the mimic eigenfunction sequence \{v^{(n)}\).
Proposition 9 \hspace{1em} The sequence \( \{v^{(n)}\} \) converges to an eigenfunction of \( \Omega_\rho \) corresponding to \( \xi \).

**Proof.** To prove the result, we adopt two steps. First, we prove that there exists a subsequence of \( \{v^{(n)}\} \) converging to an eigenfunction of \( \Omega_\rho \) corresponding to \( \xi \). Next, we prove that any convergent subsequence of \( \{v^{(n)}\} \) converges to the same \( v \).

(a) Prove that there exists a subsequence of \( \{v^{(n)}\} \) converging to an eigenfunction of \( \xi \).

Since \( \{v^{(n)}\} \) is on the unit sphere of \( L^p(\mu) \), there exists a subsequence \( \{v^{(n_k)}\} \) of \( \{v^{(n)}\} \) and a function \( v \) on \( E \) such that
\[
v^{(n_k)} \rightarrow v \quad \text{pointwise}.
\]
Thus it suffices to prove that \( v \) is an eigenfunction of \( \xi \). By Lemma 6, \( v^{(n)} \) \( (n \geq 1) \) is positive and strictly decreasing, therefore, \( v \) is nonnegative and decreasing. According to (8), we get
\[
-\Omega_\rho v^{(n_k+1)}(i) = \xi_{n_k}\mu_i \left( v_i^{(n_k)} \right)^{p-1}, \quad i \in E.
\]

Again, by Lemma 6
\[
v^{(n_k+1)}_\ell = \sum_{i=\ell}^N \hat{\nu}_i \left( \sum_{j=0}^i \xi_{n_k}\mu_j \left( v_j^{(n_k)} \right)^{p-1} \right)^{p^*-1}.
\]

By Proposition 8, we have \( \xi_{n_k} \rightarrow \xi \). Combing this with \( v^{(n_k)} \rightarrow v \) (pointwise), it follows that
\[
v^{(n_k+1)} \rightarrow \bar{v} \quad \text{pointwise},
\]
where
\[
\bar{v}_\ell = \sum_{i=\ell}^N \hat{\nu}_i \left( \sum_{j=0}^i \xi\mu_j v_j^{p-1} \right)^{p^*-1}.
\]

Since
\[
\lim_{k \rightarrow \infty} z_{n_k} = \lim_{k \rightarrow \infty} D_\rho \left( v^{(n_k)} \right) = \xi = \lim_{k \rightarrow \infty} z_{n_k+1} = \lim_{k \rightarrow \infty} D_\rho \left( v^{(n_k+1)} \right),
\]
and
\[
\|v^{(n_k)}\|_{\mu,p} = \|v^{(n_k+1)}\|_{\mu,p} = 1,
\]
we get \( D_\rho(\bar{v}) = D_\rho(v) = \xi, \|\bar{v}\|_{\mu,p} = \|v\|_{\mu,p} = 1 \). Combining with (13), it follows that \( \bar{v} \) is positive and decreasing. Besides, it can easily get that
\[
-\Omega_\rho \bar{v}(\ell) = \xi\mu\bar{v}_\ell^{p-1}.
\]
Furthermore, we have

$$\xi = \xi \|v\|_{p,\mu}^p = (\xi v^{p-1}, v)_{\mu} \quad (|\partial(\bar{v})|^{p-1} \text{sgn}(\partial\bar{v}), \partial v)_{\nu},$$

by Hölder inequality, we have

$$((|\partial(\bar{v})|^{p-1} \text{sgn}(\partial\bar{v}), \partial v)_{\nu} \leq D_p(\bar{v}) \frac{p-1}{p} D_p(v) \frac{1}{p} = \xi,$$ (15)

which means the equality in (15) holds. Thus there exist constants $c_1$ and $c_2$ satisfying $c_1 c_2 \neq 0$ such that $c_1 |\partial(\bar{v})|^{p-1} p/\text{sgn}(\partial\bar{v}) + c_2 |\partial v|^p = 0$. Combining this with $D_p(v) = D_p(\bar{v}) = \xi$, we get $|\partial_j \bar{v}| = |\partial_j v|$. Since both $v$ and $\bar{v}$ are decreasing, it follows that $\partial_j \bar{v} = \partial_j v$. By the boundary condition at $N+1$, we get $\bar{v}_N = v_N$. With positive $\bar{v}$ at hand, by induction on $k$, we conclude that

$$v_k = \bar{v}_k > 0, \quad k \in E.$$

Combining the positivity and decreasing property of $\bar{v}$ with (14), it follows that $v$ is a positive and decreasing eigenfunction of $\xi$. Hence

$$-\Omega_p v(k) = \xi \mu_k v_k^{p-1}.$$

(b) Prove that any convergent subsequence of $\{v^{(n)}\}$ converges to the same $v$.

In fact, the proof in (a) means that $v$ is a solution to

$$\begin{cases}
-\Omega_p f = \xi \text{diag}(\mu) f^{p-1}, \\
\sum_{j=1}^{N+1} f_j = 0.
\end{cases}$$ (16)

It is easy to verify that equation (16) has a unique positive and decreasing solution $f$ with $\|f\|_{\mu, p} = 1$. Thus every convergent subsequence of $\{v^{(n)}\}$ converges to the same $v$.

The proofs (a) and (b) above show that the sequence $\{v^{(n)}\}$ converges to an eigenfunction of $\xi$ and hence $(\xi, v)$ is an eigenpair of $-\Omega_p$. □

By Propositions 8 and 9, the sequence $\{(v^{(n)}, z_n)\}$ is an approximation one for eigenpair of $-\Omega_p$. As the mimic eigenfunctions, $\{v^{(n)}\}$ is a positive and decreasing sequence which has the same property with the principle eigenfunction of $-\Omega_p$. Thus, it is expected to prove that $\xi = \lambda_p$. In fact, the double summation form of variational formulas in (6) plays a typical role to prove that $\{(v^{(n)}, z_n)\}$ is an approximation sequence for the principle eigenpair of $-\Omega_p$.

**Proposition 10** Each of the sequences $\{\delta_n\}, \{\delta'_n\}$ and $\{\bar{\delta}_n\}$ in Theorem 2 converges to $\lambda_p$.

$$\lim_{n \to \infty} \delta_n^{-1} = \lim_{n \to \infty} \delta'_n^{-1} = \lim_{n \to \infty} \bar{\delta}_n^{-1} = \lambda_p = \xi.$$
**Proof.** (a) First, we prove

\[
\lim_{n \to \infty} \frac{w(n)}{w(n+1)} = 1 \quad \text{(pointwise)}.
\]

By Proposition 9, \(\{v^{(n)}\}\) converges to an eigenfunction (denoted by \(v\)) of \(\xi\) and \(v\) is positive and decreasing. Thus by Proposition 8, we have

\[
\lim_{n \to \infty} \frac{w^{(n)}}{w^{(n+1)}} = 1 \quad \text{(pointwise)}.
\]

(b) Prove

\[
\lim_{n \to \infty} \frac{w^{(n)}}{w^{(n+1)}} = \lim_{n \to \infty} \frac{\sup_{k \in E} II_k(v^{(n)})}{\sup_{k \in E} II_k(v^{(n+1)})} = \xi^{-1}.
\]

In fact, since

\[
w^{(n+1)} = v^{(n)} \left( II(v^{(n)}) \right)^{p^*-1} = \left( w^{(n)} \right)^{p^*-1} v^{(n)} \left( II(v^{(n)}) \right)^{p^*-1},
\]

we get

\[
\lim_{n \to \infty} II(v^{(n)}) = \lim_{n \to \infty} \left( w^{(n+1)}/w^{(n)} \right)^{p-1} \left( w^{(n)} \right)^{p-1} = \xi^{-1}.
\]

(c) Prove

\[
\xi = \lambda_p.
\]

In fact, (b) means that

\[
\lim_{n \to \infty} \frac{w^{(n)}}{w^{(n+1)}} = \lim_{n \to \infty} \frac{\sup_{k \in E} II_k(v^{(n-1)})}{\sup_{k \in E} II_k(v^{(n)})}.
\]

Since \(II\) is homogeneous with respect to a constant, let \(v^{(0)}\) coincide with \(f^{(1)}\) up to a fixed constant, then

\[
\lim_{n \to \infty} \sup_{k \in E} II_k(v^{(n-1)}) = \lim_{n \to \infty} \sup_{k \in E} II_k(f^{(n)}) = \lim_{n \to \infty} \delta_n,
\]

which means \(\lim_{n \to \infty} \delta_n = \xi^{-1}\). For similar reason,

\[
\xi^{-1} = \lim_{n \to \infty} \inf_{k \in E} II_k(v^{(n-1)}) = \lim_{n \to \infty} \inf_{k \in E} II_k(f^{(n)}) = \lim_{n \to \infty} \delta'_n.
\]

By [6] Theorem 2.1, we know that

\[
\delta_n^{-1} \leq \lambda_p \leq \delta'_n^{-1}.
\]

Taking limit on both sides of (17) and noting that

\[
\lim_{n \to \infty} \delta_n^{-1} = \lim_{n \to \infty} \delta'_n^{-1} = \xi,
\]

we get \(\xi = \lambda_p\). Combing with Remark 7, we get \(\lim_{n \to \infty} \delta_n = 1/\lambda_p\). □
Now, we have proved Algorithm 3 in terms of Propositions 8 ~ 10. For Theorem 2, the remained results can be obtained easily from the dual variation of $\lambda_p$ in [6], refer to [6] Theorem 2.1 and Theorem 2.4]. Hence, we have also completed the proof of Theorem 2. Here, we mention that in Theorem 2 and Algorithm 3 we need only $v^{(0)} > 0$, which can be seen from the linear case according to Perron-Frobenius theorem. The linear case can be proved beautifully using spectrum decomposition.

Diagram 1  Here $\uparrow$ (\downarrow) denotes increasing (decreasing).

To conclude this section, we discuss the infinite case: $N = \infty$. Roughly speaking, what we know is shown by Diagram 1. To explain the meaning of the diagram, noticing (a) by [6 Theorem 2.4(1)], for fixed $N$, $\{\delta_n^{(N)}\}$ is decreasing in $n$; (b) by Proposition 10, $\lim_{n \to \infty} \delta_n^{(N)} = 1/\lambda_p^{(N)}$, (c) by [6 Remark 3.3(2)], $\lim_{N \to \infty} \lambda_p^{(N)} = \lambda_p$. Combining these three facts together, we obtain

$$\lim_{N \to \infty} \lim_{n \to \infty} \delta_n^{(N)} = 1/\lambda_p.$$ 

On the other hand, by Theorem 1(1), we have a sequence $\{f^{(n)}\}_{n \geq 1}$ on $E = \{k \in \mathbb{Z}_+ : 0 \leq k < \infty\}$. Hence, we can define

$$\bar{\delta}_n^{(N)} = \sup_{0 \leq i \leq N} \Pi_i(f^{(n)})_{\leq N}.$$ 

Then it is easy to check that $\{\bar{\delta}_n^{(N)}\}$ is increasing in $N$, and its limit is $\delta_n$ defined in Theorem 1(1) when $N = \infty$. Furthermore, by Theorem 1(1), $\{\delta_n\}$
is decreasing in \( n \), denoting its limits by \( \delta_x \):

\[
\delta_x = \lim_{n \to \infty} \lim_{N \to \infty} \tilde{\delta}_n^{(N)}.
\]

Therefore, the main open question here is \( \delta_x = 1/\lambda_p \)? It is regretted that at the moment we do not know how to solve this problem.

For the dual boundary, similar results hold, which are stated in the next section.

4 Results on the dual boundary condition (DN-case)

In this section, the notation are the same as those used in [6, §4], unless otherwise stated. Let \( E := \{ k \in \mathbb{Z}_+ : 0 \leq k < N + 1 \} \) with \( N \leq \infty \), \( \tilde{c}_k^- = f_{k-1} - f_k \), where \( \tilde{c}_k^- \) denotes the left difference, then \( \Omega_p \) can be rewritten as

\[
\Omega_p f(k) = -\nu_{k+1} |\tilde{c}_{k+1}^- (f)|^{p-1} \text{sgn}(\tilde{c}_{k+1}^- (f)) + \nu_k |\tilde{c}_k^- (f)|^{p-1} \text{sgn}(\tilde{c}_k^- (f)).
\]

Under the DN-boundary condition, the operator \( II \) is defined as follows

\[
II_i(f) = \frac{1}{f_i^{p-1}} \left[ \sum_{j=1}^i \hat{\nu}_j \left( \sum_{k=j}^N \mu_k f_k^{p-1} \right)^{p-1} \right], \quad i \in E,
\]

and \( \sigma_p \) is defined below

\[
\sigma_p = \sup_{n \in E} (\mu[n, N] \hat{\nu}[1, n])^{p-1}.
\]

Here we mention that the positive sequence \( \{ \nu_k : k \in E \} \) satisfies the boundary condition that \( \nu_{N+1} := 0 \) if \( N < \infty \). When \( N \leq \infty \), the approximation procedure for \( \lambda_p \) is presented in Theorem [11].

**Theorem 11** Assume that \( N \leq \infty \) and \( \sigma_p < \infty \).

(1) Define

\[
 f^{(1)} = \hat{\nu}[1, \cdot]^{1/p^*}, \quad f^{(n)} = f^{(n-1)} \left( II \left( f^{(n-1)} \right) \right)^{p^* - 1}, \quad (n \geq 2),
\]

and \( \delta_n = \sup_{i \in E} II_i (f^{(n)}) \). Then \( \delta_n \) is decreasing in \( n \) (denote its limit by \( \delta_x \) and

\[
\lambda_p \geq \delta_x^{-1} \geq \cdots \geq \delta_1^{-1} > 0.
\]
(2) For fixed \( m \in E \), define

\[
f^{(1,m)} = \nu[1, \cdot \wedge m], \quad f^{(n,m)} = f^{(n-1,m)} \left( II \left( f^{(n-1,m)} \right) \right)^\nu - 1, \quad n \geq 2,
\]

and

\[
\delta_n' = \sup_{m \in E} \inf_{i \in E} \left( f^{(n,m)} \right).
\]

Then \( \delta_n' \) is increasing in \( n \) (denote its limit by \( \delta'_\infty \)) and

\[
\sigma_p^{-1} \geq \delta_1' \geq \cdots \geq \delta'_\infty \geq \lambda_p.
\]

Next, define

\[
\tilde{\delta}_n = \sup_{m \in E} \frac{\mu \left( f^{(n,m)} \right)}{D_p \left( f^{(n,m)} \right)}, \quad n \geq 1.
\]

Then \( \tilde{\delta}_n^{-1} \geq \lambda_p \) and \( \tilde{\delta}_{n+1} \geq \tilde{\delta}_n \) for every \( n \geq 1 \).

Theorem 11 was first presented in [6; Theorem 4.3]. There also exist three similar open questions in Theorem 11 and the answers for finite \( N \) are stated in Theorem 12 below.

**Theorem 12**  Assume that \( N < \infty \). Let \( f^{(1)} > 0 \) be given.

1. Define successively functions on \( E \):

\[
f^{(n)} = f^{(n-1)} \left( II \left( f^{(n-1)} \right) \right)^\nu - 1, \quad n \geq 2.
\]

2. For \( n \geq 1 \), define three sequences as follows

\[
\delta_n = \sup_{i \in E} II_i \left( f^{(n)} \right), \quad \delta_n' = \inf_{i \in E} II_i \left( f^{(n)} \right), \quad \tilde{\delta}_n = \frac{\mu \left( f^{(n)} \right)}{D_p \left( f^{(n)} \right)}.
\]

Then \( \{\delta_n\} \) is decreasing in \( n \), both \( \{\delta'_n\} \) and \( \{\tilde{\delta}_n\} \) are increasing in \( n \) and

\[
0 < \delta_n^{-1} \leq \lambda_p \leq \tilde{\delta}_{n-1}^{-1} \leq \delta_n'^{-1} \leq \sigma_p^{-1} < \infty, \quad \text{for any} \quad n \geq 1.
\]

Furthermore,

\[
\lim_{n \to \infty} \delta_n^{-1} = \lim_{n \to \infty} \tilde{\delta}_n^{-1} = \lim_{n \to \infty} \delta_n'^{-1} = \lambda_p.
\]

The final assertion of Theorem 12 is the main addition to [6; Theorem 4.3]. The next result is the inverse iteration for obtaining the principle eigenpair of \( \Omega_p \) corresponding to DN-boundary condition.
Algorithm 13 (Inverse iteration) Assume that $N < \infty$. Given a positive $w^{(0)}$, at the $(n+1)$th step, let $w^{(n+1)}$ be the solution to the following equation

$$\begin{cases}
-\Omega_p w(k) = \mu_k |w_k^{(n)}|^{p-2}w_k^{(n)}, & k \in E, \\
w_0 = 0.
\end{cases}$$

(18)

Let

$$z_{n+1} = \frac{D_p(w^{(n+1)})}{\|w^{(n+1)}\|_{\mu,p}}.$$ 

Then the sequence $\{w^{(n)}/\|w^{(n)}\|_{\mu,p}\}$ converges to the eigenfunction of $\lambda_p$. Next, $\{z_n\}$ is decreasing in $n$ and

$$\lim_{n \to \infty} z_n = \lambda_p.$$

Since the proof here is similar to the one given in Section 3, the only different point is that the sum should be the tail summation due to the boundary condition. Let $v^{(n)} = \|w^{(n)}\|_{\mu,p}^{-1}w^{(n)}$, the equivalent form of (18) is

$$\begin{cases}
-\Omega_p v^{(n)}(k) = \varsigma_{n-1}\mu_k \left(v_k^{(n-1)}\right)^{p-1}, & k \in E, \\
v_0^{(n)} = 0,
\end{cases}$$

where $\varsigma_{n-1} = \|w^{(n)}\|_{\mu,p}^{-1} \|w^{(n-1)}\|_{\mu,p}^{-1}$, and

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \varsigma_n = \varsigma.$$

Here we omit the details of the proof.

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