A bootstrap for the number of $\mathbb{F}_q^r$-rational points
on a curve over $\mathbb{F}_q$

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Abstract

In this note we present a fast algorithm that finds for any $r$ the number $N_r$ of $\mathbb{F}_q^r$ rational points on a smooth absolutely irreducible curve $C$ defined over $\mathbb{F}_q$, assuming that we know $N_1, \ldots, N_g$, where $g$ is the genus of $C$. The proof of its validity is given in detail and its working are illustrated with several examples. In an Appendix we list the Python function in which we have implemented the algorithm together with other routines used in the examples.

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Glossary

In this note, curve over $\mathbb{F}_q$ means (unless explicitly stated otherwise) a smooth absolutely irreducible projective curve over the finite field $\mathbb{F}_q$ of cardinal $q$.

The Hasse-Weil-Serre upper bound for the number of $\mathbb{F}_q$-rational points on a curve $C/\mathbb{F}_q$ is $N_q(g) = q + 1 + g[2\sqrt{q}]$ \[5, 6, 7\].

A curve $C/\mathbb{F}_q$ is said to be maximal if the number of its $\mathbb{F}_q$ points equals $N_q(g)$.

For historical aspects and background, we refer to the excellent surveys \[10, 11\], and the many references provided there. The general context provided by the Weil conjectures is outlined in \[1\], Appendix C.

1. Ingredients

The zeta function of a projective variety $X/\mathbb{F}_q$ is the power series

$$Z(T) = Z(X/\mathbb{F}_q) = \exp \left( \sum_{r=1}^{\infty} N_r(X) \frac{T^r}{r} \right), \quad (1)$$
where \( N_r = N_r(X) \) denotes the number of \( \mathbb{F}_q \)-rational points of \( X \). This function generates all the numbers \( N_r \) according to the relation

\[
N_r = \frac{1}{(r-1)!} \frac{d^r}{dT} \log Z(T)|_{T=0}.
\]

(2)

The needed information about \( Z(T) \) is provided by the Weil conjectures (see [1] for the history of work on them and in particular about their proofs). For the case of curves \( C/\mathbb{F}_q \), which is the one we need in this note, they were actually proved by Weil himself [13] and can be summarized as follows (see [3] for proofs in present day algebraic geometry language):

- **Rationality.** \( Z(T) = P(T) \frac{(1-T)(1-qT)}{(1-T)^g} \), with \( P(T) \in \mathbb{Z}[T] \).
- **Functional equation.** If \( g \) is the genus of \( C \), \( P(T) = q^{g}T^{2g} P(1/qT) \). In particular, \( \deg(P) = 2g \).
- **Analogue of the Riemann hypothesis.** \( P(T) = \prod_{j=1}^{2g} (1 - \alpha_j T) \), with \( \alpha_j \in \mathbb{C} \) such that \( |\alpha_j| = \sqrt{q} \).

2. **Basic algorithm**

Using (2), it is easy to conclude that

\[
N_r(C) = q^r + 1 - N_r,
\]

(3)

Thus we see that knowing \( N_r \) is equivalent to knowing \( S_r \).

Now we can describe a procedure for computing the \( N_r \) for \( r > 2g \) assuming that we know \( N_1, \ldots, N_{2g} \). Since the Newton sums \( S_r = \sum_{j=1}^{2g} \alpha_j^r \) are symmetric polynomials of the \( \alpha_i \), they are polynomial expressions in the (signed) elementary symmetric polynomials \( c_1, \ldots, c_{2g} \) of \( \alpha_1, \ldots, \alpha_g, \alpha_{g+1}, \ldots, \alpha_{2g} \) (in other words, \( c_j = (-1)^j \sigma_j \), where \( \sigma_j \) is the standard elementary symmetric polynomial of degree \( j \) in \( \alpha_1, \ldots, \alpha_{2g} \)). Even though these expressions were essentially derived in the XVII century (first by Girard and later by Newton), for convenience we include their statement and a proof in the Appendix A.

For our purposes here, the net result is that we can proceed as follows:

1. For \( j = 1, \ldots, 2g \), set \( S_r = q^r + 1 - N_r \).
2. Use the formula (A.2) to recursively compute \( c_1, \ldots, c_{2g} \):
   \[
   c_j = -(S_j + c_1S_{j-1} + \cdots + c_{j-1}S_1)/j.
   \]
3. Use the formula (A.1) to successively get \( S_{2g+1}, \ldots, S_r \).
4. Set \( N_i = q^i + 1 - S_i \) for \( i = 2g + 1, \ldots, r \).
3. An improved algorithm

Since $P$ has real coefficients, if $\alpha_j$ is a root, then so is $\bar{\alpha}_j = q/\alpha_j$. The possible real roots are $\pm \sqrt{q}$, and there is an even number of them because the degree of $P$ is even. In fact, there must be an even number of $-\sqrt{q}$ (and hence an even number of $\sqrt{q}$) as otherwise the coefficient of $T^{2g}$ (namely $q^g$) would be negative. This implies that we can index the roots of $P$ by $\alpha$ hence an even number of $\sqrt{q}$.

An improved algorithm in such a way that $\alpha_{2g-j+1} = \bar{\alpha}_j = q/\alpha_j$, $j = 1, \ldots, g$. Therefore, $P$ has the following form:

$$P(T) = c_{2g}T^{2g} + c_{2g-1}T^{2g-1} + \cdots + c_1T + c_0, \quad c_{2g} = q^g, \quad c_0 = 1. \quad (4)$$

3.1 Proposition. We have that $c_{g+l} = q^l c_{g-l}$ for $l = 1, \ldots, g$.

Proof. Since $\alpha_j \mapsto q/\alpha_j$ exchanges $\alpha_1, \ldots, \alpha_g$ and $\alpha_{2g}, \ldots, \alpha_{g+1}$, if we set $f(T) = \prod_{j=1}^{2g}(T-\alpha_j) = c_0T^{2g} + c_1T^{2g-1} + \cdots + c_{2g-1}T + c_{2g}$, then $T^{2g}f(q/T)$ has the same roots as $f(T)$ and therefore $T^{2g}f(q/T) = c_{2g}f(T) = q^g f(T)$. Now the claim follows by equating the coefficients of $T^{g+1}$ on both sides: on the right we get $q^g c_{g-l}$ and on the left $q^{g-l} c_{g+l}$. \hfill \Box

This proposition gives the bootstrap at the root of our improved algorithm: after computing $c_1, \ldots, c_g$ from $N_1, \ldots, N_g$ as in the basic algorithm, we automatically get $c_{g+1}, \ldots, c_{2g}$, namely $qc_{g-1}, \ldots, q^{g-1}c_1, q^gc_0$, and so we have the following improved procedure:

1. For $j = 1, \ldots, g$, set $S_j = q^j + 1 - N_j$.
2. Use the formula (A.2) to get $c_j$ for $j = 1, \ldots, g$:
   \[ c_j = -(S_j + c_1S_{j-1} + \cdots + c_{j-1}S_1)/j. \]
3. For $j = g+1, \ldots, 2g$, set $c_j = q^{j-g}c_{2g-j}$, get $S_j$ with equation (A.2),
   \[ S_j = -(c_1S_{j-1} + \cdots + c_{j-1}S_1 + jc_j), \]
   and set $N_j = q^j + 1 - S_j$.
4. For $j > 2g$, use the formula (A.1) to recursively compute the
   \[ S_j = -(c_1S_{j-1} + \cdots + c_{2g}S_{j-2g}) \]
   and set $N_j = q^j + 1 - S_j$.

We include the listing of our Python implementation of this procedure in Appendix B (the function \texttt{XN}).

3.2 Remark. Thus the infinite sequence $\{N_j(C)\}_{j \geq 1}$ only depends on $q$ and the list $[N_1, \ldots, N_g]$. One interesting consequence is that given a positive integer $s$, the subsequence $\{N_{sj}(C)\}_{j \geq 1}$ must be the result of computing the long sequence for $q^s$ and the list $[N_s, \ldots, N_{sg}]$. 

3
4. Elliptic curves revisited

Over \( \mathbb{F}_2 = \mathbb{Z}_2 \) there are 32 cubic polynomials in normal form (cf. \cite{2} or \cite{9} for notations and terminology)

\[ E = y^2 + a_1 xy + a_3 + x^3 + a_2 x^2 + a_4 x + a_6 \]

of which precisely 16 are non-singular. For these cases, \( g = 1 \), the HWS bound is \( m = \lfloor 2\sqrt{2} \rfloor = 2 \) and all the integers in the HWS interval \([1, 5]\) occur as \( N_1(E) \) for some \( E \) (this can be be checked with Deuring’s algorithm, which is explained, and implemented, in the subsection “The Deuring function” of Appendix B). Now a straightforward computation yields the following distribution:

\[
\begin{align*}
N & \quad E \\
1 & \quad y^2 + y + x^3 + x + 1, \ y^2 + y + x^3 + x^2 + 1 \\
2 & \quad y^2 + xy + x^3 + x^2 + 1, \ y^2 + xy + x^3 + x^2 + x, \\
 & \quad y^2 + (x + 1)y + x^3 + 1, \ y^2 + (x + 1)y + x^3 + x + 1 \\
3 & \quad y^2 + y + x^3, \ y^2 + y + x^3 + 1 \\
 & \quad y^2 + y + x^3 + x + x, \ y^2 + y + x^3 + x^2 + x + 1 \\
4 & \quad y^2 + xy + x^3 + 1, \ y^2 + xy + x^3 + x \\
 & \quad y^2 + (x + 1)y + x^3 + x, \ y^2 + (x + 1)y + x^3 + x^2 + x \\
5 & \quad y^2 + y + x^3 + x, \ y^2 + y + x^3 + x^2
\end{align*}
\]

Computing the sequences of values returned by \( XN \) with inputs \( q = 2 \) and \( [N_1] \), for \( N_1 = 1, \ldots, 5 \), and \( k = 20 \) we get the following data (the top row is the maximum value \( N_q(1) \) of \( \#E(\mathbb{F}_q) \) supplied by “Serre’s procedure”, as described in Appendix B):

\[
\begin{array}{cccccccccccc}
S & 5 & 9 & 14 & 25 & 44 & 81 & 150 & 289 & 558 & 1089 \\
N & & & & & & & & & & \\
1 & 1 & 5 & 13 & 25 & 41 & 65 & 113 & 225 & 481 & 1025 \\
2 & 2 & 8 & 14 & 22 & 56 & 142 & 288 & 518 & 968 \\
3 & 3 & 9 & 9 & 33 & 81 & 129 & 225 & 513 & 1089 \\
4 & 4 & 8 & 16 & 44 & 56 & 116 & 288 & 508 & 968 \\
5 & 5 & 5 & 25 & 25 & 65 & 145 & 225 & 545 & 1025 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
S & 2139 & 4225 & 8374 & 16641 & 33131 & 66049 & 131797 & 263169 & 525737 & 1050625 \\
N & & & & & & & & & & \\
1982 & 4144 & 8374 & 16472 & 32494 & 65088 & 131174 & 263144 & 525086 & 1047376 \\
2049 & 3969 & 8193 & 16641 & 32769 & 65025 & 13073 & 263169 & 524289 & 1046529 \\
2116 & 4144 & 8012 & 16472 & 33044 & 65088 & 130972 & 263144 & 523492 & 1047376 \\
1985 & 4225 & 8065 & 16385 & 33025 & 65025 & 131585 & 262145 & 523265 & 1050625 \\
\end{array}
\]
The red entries are maximal values. The blue values of row $S$ indicate that no elliptic curve of the five defined over $\mathbb{F}_2$ achieves them, and in this case the yellow entries indicate the curve (or two curves in two cases) that yield the highest value. Here we remark that the tables agree with the conclusions in Table 1 of [14] (page 305), except for the $k = 8$, which is classified there as maximal (provided by $E_4$), but in the date above we see that $S = 289$ and that the maximum achieved by our five elliptic curves is 288 (two of them, $E_2$ and $E_4$). This means that there is an elliptic curve defined over $\mathbb{F}_{2^8}$ that has 289 rational points, one more than the maximum of the number of $\mathbb{F}_{2^8}$-rational points for our five curves. In fact, since in the first six columns the maximum is achieved, Remark 3.2 tells us that that curve cannot be defined over $\mathbb{F}_{2^k}$ for $k = 2, 4$.

4.1 Remark. Of course, the algorithm expects that $N_1$ is known, for a given $q$. A different question is finding the $N_1$ points explicitly, which is fundamental in applications such as in coding theory. For small $q$, this can often be computed in a straightforward manner, but otherwise the problem of finding fast algorithms is quite subtle and appears to be quite involved (cf. Schoof’s [4]).

5. On the Klein quartic

The Klein quartic $C/\mathbb{F}_2$ is given by the homogeneous equation

$$F(x, y, z) = x^3y + y^3z + z^3x.$$  (5)

It is non-singular, absolutely irreducible and has genus 3. Let us compute $N_1, N_2, N_3$. First notice that the three points (in homogeneous coordinates) $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are the only ones that satisfy $xyz = 0$. In particular, there are two points at infinity. If $xyz \neq 0$, then we can look at the affine curve $C_z = x^3y + y^3 + x$. Over $\mathbb{F}_2$ it is clear that there are no more points, hence $N_1 = 3$. Over $\mathbb{F}_4$, there are two more points: $(\alpha, \alpha^2, 1)$ and $(\alpha^2, \alpha, 1)$, where $\alpha^2 = \alpha + 1$, and so $N_2 = 5$. To get $N_3$, let $\mathbb{F}_8$ be generated by $\beta$ with $\beta^3 = \beta + 1$. Since $y^3 = y^{10}$, on dividing $C_z$ by $y^3$ we get $(x/y^3)^3 + 1 + x/y^3 = 0$. Since $\xi^3 + \xi + 1 = 0$ has three solutions in $\mathbb{F}_8$ ($\beta, \beta^2, \beta^4$), we conclude that $C_z$ has $7 \times 3 = 21$ points other than $(0, 0)$ that are $\mathbb{F}_8$-rational and therefore $N_3 = 24$. With this, the values for $N_k$ supplied by $\text{XN}$ (for $k \leq 12$) are the following:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| $N_k$ | 3 | 5 | 24 | 17 | 33 | 38 | 129 | 257 | 528 | 1025 | 2049 | 4238 |

Over $\mathbb{F}_5$, one finds that $N_1 = 6$, $N_2 = 26$ and $N_3 = 126$. With this, we can find a similar table (for $k = 1, \ldots, 9$):
Appendix A. Newton sums

Let $\alpha_1, \ldots, \alpha_n$ be variables. For $j = 1, \ldots, n$, let

$$c_j(\alpha_1, \ldots, \alpha_n) = (-1)^j \sigma_j(\alpha_1, \ldots, \alpha_n),$$

where $\sigma_j$ is the degree $j$ symmetric polynomial in $\alpha_1, \ldots, \alpha_n$. Finally, let $S_j = \alpha_1^j + \cdots + \alpha_n^j$ for all $j \geq 0$, with the convention $S_0 = n$.

**Appendix A.1 Proposition** (Giard-Newton identities). (1) If $j \geq n$, then

$$S_j + c_1 S_{j-1} + \cdots + c_{n-1} S_{j-(n-1)} + c_n S_{j-n} = 0. \quad (A.1)$$

(2) If $1 \leq j \leq n$, then

$$S_j + c_1 S_{j-1} + \cdots + c_{j-1} S_1 + j c_j = 0. \quad (A.2)$$

**Proof.** (1) Let $X$ be a new variable. From the definitions it follows that

$$\sum_{k=0}^n c_k X^{n-k} = (X - \alpha_1) \cdots (X - \alpha_n).$$

Therefore $\sum_{k=0}^n c_k \alpha_i^{n-k} = 0$ for $i = 1, \ldots, n$ (Vieta’s formulas). If we multiply this relation by $\alpha_i^{j-n}$ ($j \geq n$) and sum for $i = 1, \ldots, n$, we get $\sum_{k=0}^n c_k S_{j-k} = 0$, which is the stated equation.

(2) We will proceed by induction on $n$. For $n = 1$, the statement, namely $S_1 + c_1 = 0$, is tautologically true. Assume now that the statement is true for $n - 1$:

$$S_j' + c_1' S_{j-1}' + \cdots + c_{j-1}' S_1' + j c_j' = 0$$

for $1 \leq j \leq n - 1$, where the $S_j'$ and $c_j'$ have the same meaning as $S_j$ and $c_j$, but with respect to the variables $\alpha_1, \ldots, \alpha_{n-1}$. The key observation is that $S_j'$ and $c_j'$ coincide with the result of setting $\alpha_n = 0$ in $S_j$ and $c_j$. This implies that the polynomials $S_j + c_1 S_{j-1} + \cdots + c_{j-1} S_1 + j c_j$ (for $1 \leq j \leq n - 1$) are divisible by $\alpha_n$. Since they are symmetric in $\alpha_1, \ldots, \alpha_n$, they are divisible by $\alpha_1 \cdots \alpha_n$. Therefore they vanish, as their degrees are $< n$. Finally note that the equality for $j = n$ has been established in (1).

Appendix B. Python code

The listings in this appendix are available for downloading at Listings-MSX.
The function $XN$. This function implements our main algorithm (Section 3). Besides the standard Python facilities, only a working implementation of the rational numbers (here denoted $\mathbb{Q}$) is needed.

```python
def XN(q,X,k):
g = len(X)  # the genus of the curve
if k<=g: return X[:k]  # only $k>g$ gives something new
X = [0]+X  # trick so that $X[j]$ refers to $F_{q^j}$
X = [x>>Q for x in X]  # consider $X$ as a list of $\mathbb{Q}$
S = [q**(j)+1-X[j] for j in range(1,g+1)]  # First $g$ Newton sums
S = [0]+S  # similar trick as for $X$
# Computation of $c_1,...,c_g$; set $c0=1$
c = [1>>Q]  # to compute in the rational field
for j in range(1,g+1):
cj = S[j]
   for i in range(1,j):
      cj += c[i]*S[j-i]
      c += [-cj/j]  # Add $c_{g+i}$, for $i=1,...,g$
for i in range(1,g+1):
c += [q**i*c[g-i]]
# Find $S_j$ for $j = g+1,...,k$
for j in range(g+1,k+1):
   if j>2*g:
      Sj=0
   else:
      Sj = j*c[j]
      for i in range(1,j):
         if i>2*g: break
         Sj += c[i]*S[j-i]
      S += [-Sj]
# Find $X[i]$ for $i = g+1,...,k$
for i in range(g+1,k+1):
   X += [q**i+1-S[i]]
return X[1:]
```

The Deuring function. This function implements Deuring’s algorithm to list the possible cardinals #$E(\mathbb{F}_q)$ of the elliptic curves $E/\mathbb{F}_q$. Our main reference here has been [12]. We have split the computation in two parts: the function Deuring_offsets($q$) (which computes the list of integers $t$ in the segment $[-m,m]$, $m = \lfloor 2\sqrt{q} \rfloor$, such that #$E(\mathbb{F}_q) = q + 1 - t$ for some $E$), and Deuring_set($q$), that outputs the list in question.
def deuring_offsets(q):
    P = prime_factors(q)  # prime_factors(12) => [2, 2, 3]
    p = P[0]; n = len(P)
    m = int(2*sqrt(q))
    D = [t for t in range(-m,m+1) if gcd(p,t) == 1]
    if n/2 == 0:
        r = p**(n//2)
        D += [-2*r,2*r]
        if p%3 != 1:
            D += [-r,r]
    if n%2 and (p==2 or p==3):
        r = p**((n+1)//2)
        D += [-r,r]
    if n%2 or (n%2==0 and p%4!=1):
        D += [0]
    return sorted([t for t in D])

#
def Deuring_set(q):
    D = deuring_offsets(q)
    return [t+q+1 for t in D]

Examples: Here are the Deuring lists for the first 8 prime powers q

q=2 (m=2): [1, 2, 3, 4, 5]
q=3 (m=3): [1, 2, 3, 4, 5, 6, 7]
q=4 (m=4): [1, 2, 3, 4, 5, 6, 7, 8, 9]
q=5 (m=4): [2, 3, 4, 5, 6, 7, 8, 9, 10]
q=7 (m=5): [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]
q=8 (m=5): [4, 5, 6, 8, 9, 10, 12, 13, 14]
q=9 (m=6): [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]
q=11 (m=6): [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]

The Serre procedure for the function $N_q(g)$, $g = 1, 2, 3$. For the function Serre we have followed [5].

def Serre(q, g=1):
    D = ifactor(q)  # ifactor(12) => {2: 2, 3: 1}
    if len(D)>1:
        return 'Serre: {} is not a prime power'.format(q)
    p = list(D)[0]
    e = D[p]  # q = p^e
    m = int(2*sqrt(q))  # g·m is the HWS bound
    if g==1:
        if e%2 and e>=3 and m%p==0: return q+m

8
else: return q+m+1
if g==2:
    if q==4: return 10
    if q==9: return 20
    if e%2==0: return q+1+2*m
def special(s):
    if m%p==0 or is_square(s-1) or
       is_square(4*s-3) or
       is_square(4*s-7):
        return True
    else: return False
if special(q):
    if 2*sqrt(q)-m > (sqrt(5)-1)/2: return q+2*m
    else: return q+2*m-1
return q+1+2*m
if g==3:
P = [2,3,4,5,7,8,9]
T={2:7,3:10,4:14,5:16,7:20,8:24,9:28}
if P.count(q): return T[q]
return "Serre> I do not know the value of N({},{}).format(q,g)

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