Future Boundaries and the Black Hole Information Paradox.

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Abstract: The black hole information paradox is the incompatibility of quantum mechanics with the semi-classical picture of Hawking radiation. Hawking radiation appears thermal and eventually leads to the complete disappearance of a black hole. However, black holes could be formed from a pure quantum state. The transition from such an initial state to the final state of pure Hawking radiation cannot be described by unitary time evolution. In this paper, we present an analysis in quantum gravity that shows how boundary conditions in the future prevent a loss of quantum mechanical information from the spacetime. In classical physics, the future boundary of the spacetime in the black hole interior is a singularity. Realistic gravitational collapse results in a BKL type of approach to the singularity. But, solving the Wheeler-DeWitt equation reveals that the singularity does not form and can be replaced by specifying a final state density matrix. Such a condition is natural within the context of consistent histories version of quantum mechanics. We provide a self-contained treatment of these issues. How information escapes from the black hole will be treated elsewhere.  

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1If I have omitted a reference to relevant work, please let me know and I will incorporate it into this paper in its final form.
1 Introduction

The information paradox has been with us for almost fifty years now, following on from Hawking’s discovery [1], [2] that black holes appear to radiate thermally at a temperature $T_H$. The first law of black hole mechanics [3] expresses the change in the mass of a black hole in an asymptotically flat spacetime as its angular momentum and electric charge change infinitesimally when a black hole transitions between two equilibrium states. In such a process,

$$dM = \frac{\kappa}{8\pi} dA + \Omega \cdot dJ + \Phi dQ$$

(1.1)

where $M$ is the mass of the black hole, $\kappa$ its surface gravity, $A$ the area of the event horizon, $\Omega$ its angular velocity, $J$ its angular momentum, $\Phi$ its electrostatic potential and $Q$ its electric charge $^1$. The first law is a theorem in classical general relativity for black holes in an asymptotically flat spacetime. The first law also reflects the no-hair theorems which tell us that the geometry of a stationary black hole, is completely described by $M$, $J$ and $Q$ and that the metric is given by the Kerr-Newman solutions, [4–9]. For a recent review of the uniqueness theorems, see [10].

The temperature of Hawking radiation, $T_H$ is given by

$$T_H = \frac{\kappa}{2\pi}$$

(1.2)

$^1$We use natural units such that $G = c = \hbar = k = 1$. The signature of spacetime is taken to be $(-+++).$ The Riemann tensor is defined by $(\nabla_a \nabla_b - \nabla_b \nabla_a) V_c = R_{abc}^d V_d.$
which allowed a reinterpretation of the first law of black hole mechanics as the first law of thermodynamics [2] and led to the identification of black hole entropy $s_H$ in terms of the horizon area as

$$s_H = \frac{A}{4}$$  \hspace{1cm} (1.3)

The idea that area of the event horizon could be identified with entropy had previously been suggested by Bekenstein, [11–13]. His arguments were based on an examination of the laws of thermodynamics for a black hole interacting with matter. He argued that if the second law of thermodynamics for such a system was true, then the black hole must have an entropy proportional to the area of the horizon. However, he was unable to fix the constant of proportionality. We will refer to $s_H$ as the Bekenstein-Hawking entropy.

The identification of the horizon area as black hole entropy is supported by another result in classical general relativity, the area theorem. Any change to a black hole that results in a change of its horizon area by $\Delta A$ is such that

$$\Delta A \geq 0$$  \hspace{1cm} (1.4)

provided any matter involved obeys the weak energy condition, [14, 15]. The identification of the horizon area as entropy means that the area theorem can be reinterpreted as the second law of thermodynamics.

Since black holes produce thermal radiation, they will eventually disappear. Suppose a black hole has zero angular momentum and zero electric charge, the Hawking temperature is then $T_H = 1/(8\pi M)$. As it produces Hawking radiation, it will lose mass and therefore get hotter. It will radiate away all its mass on a timescale $\tau \sim M^3$. Eventually, the black hole will disappear leaving nothing but Hawking radiation.

Another view of black hole entropy is provided by a calculation of Gibbons and Hawking [17]. They calculated the partition function of stationary black holes using Euclidean field theory techniques. Euclidean spaces that are the analytic continuation of stationary black holes exhibit a periodicity in imaginary time corresponding to the inverse temperature. Under Euclideanization, the horizon becomes a conical singularity unless this periodicity is imposed. There is no analogue of the interior of the black hole in this picture so in effect one has traced over the interior states of the black hole, [18, 19]. Starting from the path integral for gravity using the Einstein action, Gibbons and Hawking found that the thermodynamic entropy of the black hole was again given by $A/4$.

Now consider black hole formation in classical physics. Some matter undergoes gravitational collapse and once it has become sufficiently compressed, an event horizon is expected to form and then a spacetime singularity will develop. An explicit example of how this comes about was provided by Oppenheimer and Snyder [20] for the case of dust undergoing spherically symmetric collapse. The interior of the dust cloud, mass $M$, is described by a contracting Friedmann-Robertson-Walker universe, whilst exterior to the body, the spacetime is that given by the Schwarzschild metric. Once the outer surface of the collapsing

\footnote{Recently, the data from the black hole merger GW150914 was examined with the aim of testing the area theorem. Agreement with the area theorem was found with 97% probability [16].}
cloud passes through the Schwarzschild radius, \( r = 2M \), an event horizon forms. The end point of collapse is a spacelike singularity.

Gravitational collapse is not usually spherically symmetric. Nevertheless, this general picture is expected to still hold. The hoop conjecture \([21]\) suggests that once matter is sufficiently condensed, an horizon will form, \([22–24]\). There is at present no complete proof of the hoop conjecture, but it is widely believed and is supported by numerical evidence \([25]\). Penrose’s singularity theorem \([26]\) guarantees that a singularity will form once an horizon has appeared. The weak cosmic censorship conjecture suggests that the singularity will be hidden from observers in the asymptotic region. Stated rather more precisely, it requires that in an asymptotically flat spacetime, future null infinity should be geodesically complete, \([27, 28]\). Again, there is no complete proof of the weak cosmic censorship conjecture but again it is widely believed and there is numerical evidence in its favour \([25]\).

If a black hole has rotation or electric charge and is stationary, its geometry is that of the Kerr-Newman solution and has an inner Cauchy horizon. The inner horizon has been shown to be unstable perturbatively \([29]\) and it is believed that in the exact theory, it will morph into a singularity. Unambiguous evidence for this has been provided by Dafermos and Luk \([30]\) who showed that in certain circumstances, a singularity forms in place of the Cauchy horizon. In general, singularities that replace a null Cauchy horizon are expected to be spacelike although it is possible that there are null segments. What seems to be ruled out in realistic situations, is the possibility of the timelike singularities of the type that are found in the maximal analytic extension of the stationary solutions in the Kerr-Newman family. As a consequence of all of this, we expect the Penrose diagram of an evaporating black hole to be as shown in Figure 1.

Now let us ask what happens quantum mechanically. Our only option is to try to perform a semi-classical treatment since, as yet, there is no satisfactory theory of quantum gravity. In Figure 1, \( \Sigma_i \) is a surface on which an initial state can be specified, \( \Sigma_f \) is a final surface outside the black hole after it has completely evaporated and where a final state can be described. Lastly, \( \Sigma_s \) is a surface close to the singularity and anything crossing this surface would seem to impact the singularity and be lost to spacetime. Quantum mechanics tells us that, in the Heisenberg picture, one expects there to be a unitary operator \( U(t) = e^{iHt} \) with \( H \) being the Hamiltonian such that any operator \( \mathcal{O} \) evolves with time by

\[
\mathcal{O}(t') = U(t' - t) \mathcal{O}(t) U(t - t'),
\]

whereas states remain constant. The density matrix \( \rho \) therefore remains constant and hence the von Neumann, or entanglement, entropy \([31]\) \( S \), given by

\[
S = -tr \rho \ln \rho
\]

also remains constant. In particular, the entropy on the surface \( \Sigma_i \) should be the same as the entropy on \( \Sigma_f \). The initial state could have been pure with zero von Neumann entropy. However, the semi-classical picture appears to tell us that after the black hole has disappeared, there is only thermal Hawking radiation and that has non-vanishing entropy. Such evolution cannot be described by quantum mechanics as the entropy is not constant.
This is the information paradox. It has come about because the matter falling into the black hole encounters a boundary to spacetime at the singularity. At this boundary, at least classically, one presumes the matter has left the spacetime.

Initially, Hawking suggested that quantum mechanics breaks down [32]. That is certainly a logical possibility. However, these ideas were criticised by Gross [33], Banks, Susskind and Banks [34] and Lee [35]. Despite such criticism, a recent review by Unruh and Wald [36] suggests that we somehow have to learn to live with information loss. There have been various other suggestions. There might be black hole remnants that have large entropy and whose states provide for the purification of the Hawking radiation. However, any such remnant would have to be small as there is no reason for the Hawking radiation to turn off until the black hole is of roughly the Planck scale where non-perturbative quantum gravity effects might take place and allow this to happen. However, for something small to contain a large amount of information appears unnatural. It might be that instead of a singularity, there is a baby universe that splits off and forms a disjoint part of spacetime from the original one. In either of these cases information will be lost as far as observers in the asymptotic region are concerned. For a review of this type of possibility, see [37]. Another possibility is that black holes never form, but instead one forms a fuzzball, a region of spacetime that at large distances looks like a black hole but does not contain either an horizon or a singularity [38]. These would be quantum gravity configurations that have no classical analogue as classical general relativity does not admit such solutions [39]. Furthermore, for very large black holes as found at the centers of galaxies, one expects fields outside the horizon to be very weak. One would not expect quantum gravity effects to be important at the horizon and so it is hard to see how in the fuzzball picture an horizon would fail to form.

The AdS-CFT correspondence [39] seems to indicate that quantum mechanics should work as expected for a description of black hole evaporation. That is because one can have evaporating black holes in a spacetime that is asymptotic to anti-de Sitter space. The boundary theory that is dual to the bulk theory in anti-de Sitter space is perfectly unitary and defined in a way that is non-perturbative, [40]. The puzzle then is to determine where the semi-classical picture is wrong. Whatever the problem is, one should be able to rely on semi-classical arguments as long as fields do not approach Planckian scales. That seems to indicate that quantum gravity effects are important but only as one gets close to the singularity.

This paper is an attempt to understand what properties of quantum gravity allow a resolution of the information paradox. We require there to be the standard type of quantum mechanical evolution for observers exterior to the black hole. What then are the implications of requiring the evolution from $\Sigma_i$ to $\Sigma_f$ to be unitary? For some relatively recent reviews of the information paradox see Mathur [41] and Harlow [42]. See also lecture six in [43] for an information theoretic view of the problem.

The primary obstacle is the lack of a microscopic, or fine-grained, theory of quantum gravity. Such a theory would allow a calculation of the complete details of how a black

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3At least not in dimension four.
hole evaporates. We do have a coarse-grained description based on the geometrical picture provided by general relativity but this a classical theory and does not contain a description of the quantum phenomenon of Hawking radiation. We do have a semi-classical picture based on the path integral quantisation of general relativity. The failure of renormalizability limits its usefulness [44]. This too is a geometrical picture but its rules seem to imply that a strict interpretation based on smooth, real, Lorentzian metrics needs to be abandoned. At the very least, it requires us to veer off into the complex and it does not really allow us to develop a coherent approach to singularities. The path integral is however equivalent to the canonical approach [45–47, 51, 52] 4. In the canonical approach, the Wheeler-DeWitt equation [53, 54] defines the wavefunction of the universe, \( \Psi[\gamma] \) as a functional on superspace. Superspace is the space of all 3-metrics modulo diffeomorphisms. Although \( \Psi[\gamma] \) has as its argument the 3-metric \( \gamma_{ij} \), it depends only on the geometry of the spacelike surface on which it is defined and not on the coordinate system chosen to define \( \gamma_{ij} \). Superspace includes the possibility that the 3-metric is singular and therefore there is the possibility that the wavefunction of the universe has something to say about singularities. In classical physics, it appears that the approach to singularities is chaotic as was first discussed by Belinsky, Khalatnikov and I. M. Lifshitz (BKL), [48–50]. Their method was based on the canonical approach to general relativity. As such, it lends itself to an extension to a quantum version by an appropriate adaptation of the Wheeler-DeWitt equation. A recent discussion of such methods as applied to the origin of the universe can be found in [55]. Although \( \Psi[\gamma] \) was originally defined by the Wheeler-DeWitt equation, it can also be found by path integral methods [51].

There are two principal difficulties posed by the black hole information paradox that require resolution by a fine-grained theory of quantum gravity. The first is that the interior of the black hole is causally disconnected from any external observer. There seems to be no way to recover material that falls into the black hole. The second is that the singularity simply swallows everything up, taking it beyond the boundary of spacetime.

In his work on cosmology, Hawking asserts that “There ought to be something very special about the boundary conditions of the universe and what can be more special than the condition that there is no boundary,” [56]. The laws of physics are CPT-invariant. So perhaps this provides us with a clue about how to proceed. We simply need to find what boundary conditions apply to the singularity inside the black hole. That one might need to consider boundary conditions in the future in quantum mechanics was first suggested by Einstein, Tolman and Podolsky [57]. Aharonov, Bergmann and Lebowitz [58] presented a formulation of quantum theory that incorporated the possibility of fixing boundary conditions in the future. Subsequently, their scheme was elaborated on by Gell-Mann and Hartle, [59].

Horowitz and Maldacena [60] made the suggestion that such methods may be used to explore and perhaps resolve the black hole information paradox. However, difficulties with these proposals were pointed out by Gottesmann and Preskill [61] and by Bousso

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4These references show the equivalence in a rather formal way and the treatment ignores issues of the spaces on which the path integral and the Wheeler-DeWitt equation are defined, operator ordering and ultraviolet divergences and therefore can be relied on only in the WKB approximation.
and Stanford [62]. Subsequently, Lloyd [63] and by Lloyd and Preskill [64] discussed these difficulties and showed how they could be resolved. But, in the absence of any concrete proposals about how to use such techniques, interest appears to have waned.

In this paper, we present a calculation that shows that there is a boundary condition for the singularity that emerges in a natural way from the coarse-grained semi-classical picture. By examining solutions of the Wheeler-DeWitt equation, we are able to show that singularities in gravitational collapse simply do not form. The probability amplitude for geometries on $\Sigma_s$, as shown in Figure 1, vanishes in the limit that $\Sigma_s$ tends towards the classical singularity. We conclude that there is no singularity. Nevertheless, in the above limit, $\Sigma_s$ presumably still represents a future boundary to the spacetime. One can impose a boundary condition on $\Sigma_s$ so that information does not escape from the spacetime. We have provided half of a solution to the information paradox. It is still necessary to address the issue of how the information emerges from the black hole. That problem will be addressed elsewhere.

The plan of the remainder of the paper is to explain how one necessarily arrives at our conclusion. We describe the results that motivate our treatment and briefly outline the tools required to see how everything fits together. Our aim being to make the work self-contained. In section two, we describe the Page curve and how it arises. In section three we discuss the issues raised by firewalls and the possibility of their avoidance due to appearance of extra gravitational contributions to the fine-grained black hole entropy. In section four, we describe the consistent histories approach to quantum mechanics and how it can implement a coarse-graining on a fundamental quantum theory. In section five, we discuss the time symmetric version of quantum theory and post-selection. In section six, we examine the formation of singularities in gravitational collapse and the BKL phenomenon. In section seven we apply the ideas of quantum gravity to see what the boundary condition must be. In the last section, we first briefly examine the effect of matter contributions to our picture, and the possible extension to dimensions higher than four. We conclude by looking at some of the unconventional phenomena that one might expect to find inside the horizon and then move on to discuss possible observable consequences. For strange behaviour not to be observable outside the horizon, we propose a quantum cosmic censorship conjecture. A brief outline of the work presented in this paper is to be found in [65].

## 2 Black Hole Evaporation and the Page Curve

In Hawking’s original picture, a Schwarzschild black hole will produce black body radiation and thereby lose mass at a rate

$$\dot{M} = -\sigma A T_H^4$$

(2.1)

where $\sigma$ is a number of order unity that depends on the particle species being radiated, the Stefan-Boltzmann constant and the grey-body factors of the black hole. Its precise value does not concern us here. Integrating (2.1) we find the mass of the black hole as a function of time, $M(t)$. Taking the mass to be $M_0$ at time $t = 0$ and neglecting any time dependence
\( M(t) \) is given by

\[
M(t) = \left( M_0^3 - \frac{3\sigma t}{256\pi^3} \right)^{1/3}
\]  

(2.2)

The lifetime of the black hole \( t_l \) is found from (2.2) giving

\[
t_l = \frac{256\pi^3 M_0^3}{3\sigma}.
\]  

(2.3)

Assuming the Hawking radiation to be thermal, the entropy of the radiation \( s_r(t) \),\(^5\) is generated at a rate

\[
\dot{s}_r = \frac{4}{3} \sigma A T_H^3.
\]  

(2.4)

After a time \( t \), the entropy is

\[
s_r(t) = \frac{16\pi}{3} \left[ M_0^2 - \left( M_0^3 - \frac{3\sigma t}{256\pi^3} \right)^{2/3} \right].
\]  

(2.5)

We note that when the black hole has completely disappeared at time \( t_l \), the entropy in the radiation is

\[
s_r(t_l) = \frac{16\pi M_0^2}{3}.
\]  

(2.6)

\( s_r(t_l) \) is greater than the Bekenstein-Hawking entropy \( 4\pi M_0^2 \) of the black hole when it is first formed. The generalised second law of thermodynamics supposes that the total thermodynamic entropy of matter together with that of the black hole to be an increasing function of time, [11, 13] and is supported by (2.6).

Page has presented some more refined versions of this calculation in [66–68]. He included the black hole grey-body factors, and the temperature dependence in \( \sigma \) coming from different particle species. His conclusions are in accord with our crude estimate presented above.

This result is at the heart of the information paradox. If quantum mechanics applies to an evaporating black hole and the black hole was formed from the collapse of some matter in a pure quantum state, the final von Neumann entropy should be zero. The conclusion is therefore that our treatment so far has included some kind of hidden coarse-graining. A true microscopic treatment would reveal where and how this has taken place. The difficulty is to see how this could have happened. The black hole uniqueness theorems appear to indicate that information about the collapse that formed the black hole cannot reside in the geometry outside the black hole. Soft hair provides for the possibility that some extra information beyond the mass, angular momentum and electric charge could be detectable outside the black hole [69], but this does not appear to be sufficient to resolve the issue. Then again, communication from the interior of the black hole to the exterior would violate our ideas of causality. So on the face of it, one appears stuck.

\(^5\)We reserve lower case \( s \) for the thermodynamic, or coarse-grained entropy and upper case \( S \) for the von Neumann or entanglement entropy.
It is generally presumed that conventional quantum mechanics works to describe physics outside a black hole. We now make some proposals as to how this can come about. Before we proceed, we need to be clear about what assumptions we are making. Firstly, we take it that the black hole behaves as a conventional quantum mechanical system as seen by asymptotic observers. The black hole density of states is given by \( e^{s_H} \) with \( s_H \) being the Bekenstein-Hawking entropy. Secondly, interaction of the black hole with its environment is determined by unitary time evolution. These assumptions have been termed the “central dogma” by Almheiri, Hartmann, Maldacena, Shaghoulian and Tajdini [70].

We now take these assumptions as controlling the physics of black hole evaporation. Working in the Heisenberg picture, we can start with the collapsing matter being described by a density matrix \( \rho \) with von Neumann, or entanglement, entropy \( S \). Once a black hole has formed, we can divide the system into two parts, \( r \) being the radiation with density matrix \( \rho_r \) and the black hole \( h \) with density matrix \( \rho_h \). Eventually, we will be left just with radiation which is again described by the density matrix \( \rho \) assuming that evolution really is unitary. Page examined such a system [71–73] and we very briefly review his result here.

The reduced density matrices of the radiation \( \rho_r \) and the black hole \( \rho_h \) are given by

\[
\rho_r = tr_h \rho \quad \rho_h = tr_r \rho
\]

(2.7)

where \( tr_r \) and \( tr_h \) are traces taken over the degrees of freedom of the radiation and black hole sectors respectively. These density matrices can be used to compute the von Neumann entropy of the radiation \( S_r \) or the black hole \( S_h \),

\[
S_r = -tr_r (\rho_r \ln \rho_r) \quad S_h = -tr_h (\rho_h \ln \rho_h).
\]

(2.8)

Let \( S \) be the entropy of the combined system \( r \cup h \). Then \( S, S_r \) and \( S_h \) obey the subadditivity and Araki-Lieb inequalities for a composite system with two components [74],

\[
S_r + S_h \geq S \geq |S_r - S_h|.
\]

(2.9)

This pair of inequalities are often referred to as the triangle inequalities.

Suppose that initially the collapsing matter is in a pure state so that \( S = 0 \). Then \( S_r = S_h \). Contrast \( S_r \) and \( S_h \) with the thermodynamic, or coarse-grained, entropy \( s_r \) and \( s_h \). They display marked differences to each other as \( s_r \neq s_h \). The Bekenstein-Hawking coarse-grained entropy of the black hole is initially extremely large. Similarly, the coarse-grained entropy of the radiation is low simply because the temperature is low. At late times, the Bekenstein-Hawking entropy is low because the black hole has become small. But the total entropy in the radiation is large as can be seen from (2.6).

Page’s insight follows from the “central dogma.” The black hole is treated as conventional quantum system described by a Hilbert space of dimension \( n \). The radiation is a conventional quantum system described by a Hilbert space of dimension \( m \). The dimensionality of the combined Hilbert space is thus \( nm \). Suppose the system is initially in a pure state \( |x\rangle \) so that its density matrix is \( \rho_x = |x\rangle \langle x| \). At the beginning of the evaporation process, it is assumed that the state of the radiation is described by picking states at random out of the \( nm \) possible states of the combined system. This amounts to selecting \( m \)
random Hermitian matrices $M_\alpha$, $\alpha = 1, \ldots, m$. In a diagonal basis this results in a density matrix for the radiation

$$\rho_r = \sum_{\alpha=1}^{m} p_{\alpha i} |i\rangle\langle i|, \quad i = 1, \ldots, mn$$  \hspace{1cm} (2.10)

where $p_{\alpha i}$ are the eigenvalues of $M_\alpha$. The probability $P$ of finding the density matrix $\rho_r$ given the initial density matrix $\rho_x$ is given by

$$P = \text{Tr} (\rho_r \rho_x) = \sum_{\alpha} p_{\alpha i} \langle i|x \rangle \langle x|i \rangle = \sum_{\alpha} p_{\alpha x}.$$ \hspace{1cm} (2.11)

The von Neumann entropy of $\rho_r$ is

$$S_r = -\sum_{\alpha} \langle p_{\alpha x} \ln p_{\alpha x} \rangle.$$ \hspace{1cm} (2.12)

The expectation value can be computed using random matrix techniques $[71–73, 75–79, 81]$. For the case $m \ll n$ corresponding to early times, one finds

$$S_r \sim \ln m - \frac{m}{2n}.$$ \hspace{1cm} (2.13)

The entropy in the radiation is close to its maximal value $\ln m$, since $m \ll n$. The quantum information in the radiation therefore starts off very low. $S_r$ is an increasing function of $m$ until $m \sim n$.

At late times, most of the system is Hawking radiation and so $m \gg n$. Now we can regard the state of the black hole as being modelled by choosing $n$ random matrices. Then we can find $S_h$ by the same method but since $m \gg n$ all we need to do to find $S_h$ is to interchange $n$ and $m$ in (2.13). Since $S_r = S_h$ and we get

$$S_r \sim \ln n - \frac{n}{2m}.$$ \hspace{1cm} (2.14)

Now $n$ is a decreasing function of time, so $S_r$ is decreasing and therefore contains a lot of quantum information.

The quantum information content of the radiation starts off low but starts to increase dramatically from a time where $m \sim n$, the Page time, $t_p$. Page showed that

$$t_p \sim \left(1 - \frac{8}{5^{3/2}}\right) t_0 \sim 0.28 t_0$$ \hspace{1cm} (2.15)

The Page time $t_0 \sim M_0^3$. The black hole does not start to allow information to appear in the radiation at some specific mass but rather at around a quarter of the lifetime of the black hole. The indications therefore are not that Planck scale quantum gravity effects change the evaporation process once the black hole has reached some mass threshold but rather information starts to leak out in a continuous fashion right from the outset being at first very gradual but later much more rapid. A black hole is said to be young before the Page time and old after it. For young black holes the entropy in the radiation is increasing but for old black holes the entropy in the radiation is decreasing. In addition, it means that the radiation from old black holes must be maximally entangled with that from young black holes.
3 Firewalls

The conventional picture of Hawking radiation is that it is the result of the production of particle anti-particle pairs close to the horizon. A particle escapes to infinity and its anti-particle partner falls into the black hole. However, there is a serious difficulty with this picture if unitary evolution of the black hole is to be retained, [41, 82].

Consider an old black hole; the existing Hawking radiation is in a state $r$ with density matrix $\rho_r$ and the black hole is in a state $h$ with density matrix $\rho_h$. The black hole then evolves for a short interval of time, creating more Hawking radiation in a state $y$. This radiation escapes to infinity and has density matrix $\rho_y$. The new composite system of radiation is $r'$ with density matrix $\rho_{r'}$. The anti-particles falling into the black hole are in a state $\bar{y}$, density matrix $\rho_{\bar{y}}$ and, being pair-produced with $y$, are maximally entangled with the particles that were radiated off to infinity. After this has happened, the black hole is in a new state $h'$ with density matrix $\rho_{h'}$. To each of these states, one can assign a von Neumann entropy. Since $y$ and $\bar{y}$ are maximally entangled, $S_y = S_{\bar{y}}$. $r'$ is a composite state formed from $r$ and $y$, so that $r' = r \otimes y$ and $\rho_{r'} = \rho_{ry}$. In the same way, the composite state $y \otimes \bar{y}$ has density matrix $\rho_{y\bar{y}}$. However, since $y$ and $\bar{y}$ are maximally entangled, $S_{y\bar{y}} = 0$. Similarly $h' = h \otimes \bar{y}$, has density matrix $\rho_{h\bar{y}}$ and von Neumann entropy $S_{h\bar{y}}$.

Strong subadditivity of entropy is an inequality that applies to a combination of three quantum systems, [74]. Provided conventional quantum mechanical properties hold for the states in question, then if one takes any three quantum systems $a, b,$ and $c$, the entanglement entropies of the composite states $ab$ and $ac$ obey $S_{ab} + S_{bc} \geq S_a + S_c$. Now putting $a = r$, $b = y$ and $c = \bar{y}$ results in $S_{ry} + S_{y\bar{y}} \geq S_r + S_{\bar{y}}$ which can be re-arranged to give $S_{ry} - S_r - S_{\bar{y}} \geq 0$ since $S_{y\bar{y}} = 0$. But for old black holes, the von Neumann entropy of the Hawking radiation is decreasing so that $S_{ry} \leq S_r$ and we reach an inconsistency. Essentially, the strong subadditivity inequality says that if there is some unitary process in which a particle escapes observation, in this case by disappearing into the black hole, then the entropy of the remainder of the system must increase. However, the total von Neumann entropy in radiation from old black holes is decreasing as is required by unitary time evolution. We conclude that the overall picture is inconsistent. The use of strong subadditivity, or perhaps the assumptions under which strong subadditivity was derived, are suspect.

There is another objection to this line of thought. Suppose the physics of the interior of the black hole is controlled by same ideas of causality and local quantum field theory as the exterior. Braunstein then showed that “energetic curtains” [83–86] form on, or just inside, the horizon. Subsequently, Almheiri, Marolf, Polchinski and Sully (AMPS) [87] independently discovered these ideas and gave the name “firewall” to the same phenomenon. For

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6One is reminded of the aphorism due to Arthur Eddington. “If someone points out to you that your pet theory of the universe is in disagreement with Maxwell’s equations — then so much the worse for Maxwell’s equations. If it is found to be contradicted by observation — well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.” [80]. Quantum entanglement entropy cannot increase.
some later reviews of firewalls, see Harlow and Hayden [89], Almheiri, Marolf, Polchinski, Stanford and Sully [88] and Bousso [90].

Firewalls make the quandary just presented somewhat worse. Consider the totality of the Hawking radiation that is described by some pure state. Divide the radiation up into its late and early components. At late times the radiation consists of wavepackets with energy \( \sim T_H \). Since asymptotic observers are believed to be able to describe the evaporation process in terms of semi-classical physics, they can observe these radiated particles. At late times, the number of available states is small compared to early times as the average energy of the emitted particles is increasing whereas the energy available in the evaporating black hole is decreasing. The number of particles emitted is decreasing and we therefore expect the density matrix of the outgoing radiation to be getting closer and closer to describing a pure state. As a result, at late times one can construct creation and annihilation operators \( b_i^\dagger \) and \( b_i \) for these excitations. Furthermore, the number operator for the \( i \)-th state, \( N_i = b_i^\dagger b_i \) (no sum) will be (reasonably) well-defined because the states we are describing are very close to being pure. That means the entirety of the outgoing radiation will be in an eigenstate of \( N_i \). Consequently, we can restrict our attention to the subspace of the Hawking radiation states for which this holds. These late outgoing states must be entangled with the early outgoing states which themselves are the result of pair production outside the horizon. The ingoing early Hawking modes can be described by creation and annihilation operators \( a_j^\dagger \) and \( a_j \). The \( a_j \) and \( a_j^\dagger \) are related to the \( b_i \) and \( b_i^\dagger \) by Bogoliubov coefficients \( A_{ij} \) and \( B_{ij} \) in the way that was originally described in this context by Hawking [1, 2, 32]. Thus

\[
b_i = \sum_j (A_{ij} a_j + B_{ij} a_j^\dagger) \tag{3.1}
\]

and therefore

\[
N_i = \sum_{jk} A_{ij}^* (A_{ik} a_j^\dagger a_k + B_{ik} a_j^\dagger a_k^\dagger) + B_{ij}^* (A_{ik} a_j a_k + B_{ik} a_j a_k^\dagger) \tag{3.2}
\]

Now suppose we are in the state \( |S\rangle \) and look at the expectation value of \( N_i \). We immediately see that if \( \langle N_i \rangle \neq 0 \), then \( a_i |S\rangle \neq 0 \). Thus, we cannot be in the \( a \)-vacuum. The firewall comes about because these modes will be Tolman blue-shifted relative to the asymptotic states. We conclude that the energy density blows up close to the horizon.\(^7\) Whilst this treatment is approximate, it nevertheless poses an immediate problem for our picture of black holes.

A freely falling observer should be able to fall through the horizon without experiencing anything unusual. The principle of equivalence would be called into question were this not the case. If the firewall really exists, one of the central tenets of relativity would be violated. Bousso has succinctly summarised the general reaction to firewalls, “The firewall shakes the foundations of what most of us believed about black holes. It essentially pits quantum mechanics against general relativity, without giving us any clues as to which direction to go next,” [91].

\(^7\)In fact, this is almost apparent in Hawking’s original treatment.
If unitarity of black hole evaporation is to be preserved, there must be something wrong with our assumptions. The obvious place to look is to ask what happens in the interior of the black hole. We assumed that the fine-grained entropy was given entirely by the von Neumann entropy. However, the AdS-CFT correspondence has led to the conjecture that there is an extra contribution to the fine-grained entropy. This idea originates with the observations of Ryu and Takayanagi [92–95]. These ideas in some way parallel the Bekenstein-Hawking entropy of the horizon. The additional contribution to the fine-grained entropy $S_{RT}(X)$ in the context of black hole physics comes from a spacelike 2-surface $X$ and is given by $S_{RT}(X) = A(X)/4$ where $A(X)$ is the proper area of $X$. The definition of $X$, sometimes called the quantum extremal surface, is a little complicated, [96–98]. In the black hole context, take a partial Cauchy surface $\Sigma$ that has a single boundary anchored somewhere outside the black hole on an $S^2$ that completely encloses the horizon. $\Sigma$ extends into the interior of the black hole without either hitting the singularity or having a further boundary. Now, if possible, find a minimal surface $Y$ on $\Sigma$. There may be no minimal surface on $\Sigma$ in which case $S_{RT} = 0$ and indeed such a minimal surface will not generally exist outside the black hole. To find $X$, take $S(Y)$ with

$$S(Y) = \frac{A(Y)}{4} + S_{vN}(\Sigma_Y)$$

(3.3)

with $S_{vN}(\Sigma_Y)$ being the von Neumann entropy of the quantum state on $\Sigma$ exterior to $Y$. Now maximize $S(Y)$ by varying $\Sigma$ with its anchor fixed. We might find multiple extrema by this process in which case it is the global minimum of (3.3) that defines $X$. [70]. The new definition of the fine-grained entropy of a gravitating system is then

$$S = S_{RT}(X) + S_{vN}(\Sigma_X).$$

(3.4)

Ryu and Takayanagi interpreted $S_{RT}$ as being due to a CFT on the holographic screen provided by $X$. It is not clear what, if any, fundamental degrees of freedom describe $S_{RT}$ in the bulk.

Because $X$ is determined by finding the global minimum of $S(Y)$, it seems that the definition of $X$ should be extended by allowing for $\Sigma_X$ to contain disconnected components, [99, 100]. In this case there could be “islands” of $\Sigma_X$ in the interior of $X$ as well as the exterior.

These ideas have been used to calculate the entropy of the Hawking radiation and the entropy of the black hole interior and they reinforce the correctness of the Page curve [97]. They do not give a way of reconstructing the quantum state of what gives rise to the black hole from the Hawking radiation. Nevertheless, they strongly suggest that our attention should be directed to what happens in the black hole interior.

For a slightly different viewpoint that also suggests that physics in the black hole interior needs modification, see Akhoury, [101].

4 Consistent Histories

The idea of an observer causing wavefunction collapse seems rather alien to our present understanding of quantum mechanics. An observer is usually thought to be exterior to
the system under observation. In the context of gravitational physics this is not really possible except maybe for asymptotic observers in an asymptotically flat spacetimes. More realistically, one should follow Everett [102] and Wheeler [103] and think about participants and treat them as quantum mechanical too. Such a reformulation of quantum mechanics has been achieved (but perhaps not yet in its final form) by Griffiths [104, 105] and subsequently refined and expanded on by Omnes [106] and by Hartle and Gell-Mann [107]. There are excellent reviews by Hartle, [108, 109].

We assume that below the Planck scale we can neglect the quantum fluctuations in spacetime geometry and take spacetime to be a differentiable manifold that has the usual causal structure and is time orientable. Unless we are within a few Planck lengths of the singularity this seems to be a reasonable assumption. We can take it that inside a black hole, but not too close to the singularity, conventional Hamiltonian evolution will occur and the usual rules of local quantum field theory apply. The singularity is a place where the idea of spacetime as a smooth differentiable manifold does not hold. We need to keep an open mind about what takes place as one gets close to the singularity as it is there that the established laws of physics must break down.

In the consistent histories approach to quantum mechanics, the basic concepts are a state and an event. An event $E_i$ takes place at some specified time $t_i$ and its possible outcomes are labelled by $j$ taking the values 1 or 0 depending on whether the result of the event $E_i$ is true or false. $E_i$ is represented by a projection operator on the Hilbert space of states. If the result of the event $E_i$ is $j$, then the projection operator is $P_i^{(j)}(t_i)$. An event could have a discrete outcome, such as asking if a particle goes through a particular slit in a diffraction grating. The outcome is either true or false depending on whether the particle went through the slit in question or not. On the other hand the event might probe part of a continuum, for example asking if a particle has an energy $E$ less than $E_0$. Again there are only two possible outcomes for such an event. If the particle has energy less than $E_0$, the outcome is true, otherwise it is false. In every case, each of the projection operators can only have eigenvalue 0 if false and 1 if true. The projection operators have properties that reflect the propositional calculus of the $E_i$. Summing over the possibilities $j$ for fixed $i$ must give unity

$$\sum_j P_i^{(j)} = 1, \quad (4.1)$$

and the projections must be orthogonal

$$P_i^{(j)} P_i^{(k)} = \delta_{jk} P_i^{(j)}. \quad (4.2)$$

These are the quantum mechanical versions of completeness and exclusivity in quantum logic [110, 111]. Quantum logic is a set of rules that define the possible collection of events $E_i$. It is not the same as classical logic as it must prevent the occurrence of non-commuting projections at null separations.

An event can be many different things. It could be an observation, the construction of a telescope in some time interval, asking if the radioactive decay of a particular nucleus has taken place or even if a coarse-graining of a more fundamental description has taken
place. In all cases it is a restriction applied to the Hilbert space. However, the point of the consistent histories approach is to include everything in a quantum mechanical setting by choosing a complete set of relevant projections. In doing so, we need to include the physics of the system being examined together with the apparatus doing the examining and the observers. There is no longer any distinction between an quantum mechanical system and a classical observer as is required in the Copenhagen interpretation of quantum mechanics.

A history is a time ordered series of events. It is represented in the Hilbert space by a class operator $C_\alpha$. $C_\alpha$ is a time-ordered series of projections, thus

$$C_\alpha = P_n(t_n)P_{n-1}(t_{n-1}) \cdots P_1(t_1)$$

with $t_n > t_{n-1} \cdots > t_1$.

Suppose one starts with a pure state in the Heisenberg picture $|\Psi\rangle$, then $C_\alpha|\Psi\rangle$ is the branch of $|\Psi\rangle$ that corresponds to a history $C_\alpha$. An alternative history might be $C_{\alpha'}$. If the state $C_{\alpha'}|\Psi\rangle$ is orthogonal to $C_\alpha|\Psi\rangle$ then the two states are said to be consistent. One can think of this as saying that such pairs of states are independent of each other. Suppose there is a collection of histories $\{C_\alpha\}$, one can then form the decoherence functional $D(\alpha, \alpha')$ for all elements $\alpha, \alpha' \in \{\alpha\}$.

$$D(\alpha, \alpha') = \langle \Psi|C_{\alpha'}^\dagger C_\alpha|\Psi\rangle.$$  

If $D(\alpha, \alpha')$ is diagonal, then whole set of histories $\{C_\alpha\}$ is said to decohere or be consistent. Under these circumstances, the probability $p_\alpha$ of the history $C_\alpha$ is given by

$$p_\alpha = \langle \Psi|C_{\alpha'}^\dagger \rho_i C_\alpha|\Psi\rangle.$$  

It might be that $D(\alpha, \alpha') = 0$ with $\alpha \neq \alpha'$ does not hold exactly, but is subject to small violations. Under these circumstances it is still possible to find probabilities but they will be subject to some kind of approximation. This type of behaviour could occur if some coarse-graining were not sufficiently precise.

It is straightforward to extend the computation of probabilities from pure states $|\Psi\rangle$ to states described by a density matrix $\rho_i$,

$$p_\alpha = \text{tr}(C_{\alpha'}^\dagger \rho_i C_\alpha).$$

Now we need to see if we can rewrite this expression as a path integral. Suppose that we choose a series of spacelike surfaces labelled by a time co-ordinate and projectors, or a coarse-graining, that selects out the metric and other quantum fields everywhere on these surfaces. Then we can construct a path integral, as explained in [108]. By doing this, we are ignoring the fundamental nature of quantum gravity as we do not have much of an idea of what it is. This is a particular kind of coarse-graining. One can then hope that the decoherence functional is well-behaved. The result is a path integral version of $D(\alpha', \alpha)$ that is essentially the same as the Schwinger-Keldysh [112–114] path integral and in which

$$D(\phi_i, \phi'_i) = \int D[\phi]D[\phi']\delta(\phi_f - \phi'_f)\rho_i(\phi_i, \phi'_i)e^{i[I[\phi] - I[\phi']]}.$$  

(4.7)
\( \phi \) represents all of the fields in the problem and \( I[\phi] \) is the action. The path integral in the forward direction is taken over all fields that start with \( \phi_i \) at time \( t_i \) and end with \( \phi_f \) at time \( t_f \). The path integral over the backward direction is taken over fields that end at time \( t_f \) with the field being \( \phi'_f \) and start at time \( t_i \) with the field being \( \phi'_i \) in a way that is consistent with the density matrix describing the initial state being \( \rho_i(\phi_i, \phi'_i) \). Since it is the initial state being specified by some density matrix, there is no requirement that \( \phi_i = \phi'_i \) For the final state, the delta function enforces the matching condition that \( \phi \) and \( \phi' \) be the same.

5 Time Symmetric Quantum Theory

We are familiar with the idea that the future is different to the past. There are many arrows of time that show us in what way the future differs from the past [115]. Despite this, the fundamental laws of physics are time reversal invariant.

Perhaps the most familiar is psychological time, in that we remember our own past whilst the future remains a mystery until we experience it. Each of us is what Gell-Mann calls an IGUS (Information Gathering and Utilizing System) [116], that is an entity capable of making observations and drawing conclusions from them. However, there is also some ambiguity about psychological time because of relativistic effects. Different observers may see the same events but in various orders and at different times. That is because each IGUS is an independent entity. That makes psychological time a concept that has a strong subjective component. Nevertheless, an observation can correspond to a quantum mechanical event and result in the insertion of a projection operator into the decoherence functional.

There are two arrows of time that appear unambiguously in classical physics. The first is the observation in electrodynamics that the fields due to some distribution of currents and charges is governed by the retarded solution to the inhomogeneous wave equation. In effect this is the observation that cause precedes effect and of course applies not only to electrodynamics but much more widely. Sound waves travel from the source to a receiver; not the other way around. Waves spread out from where a stone is thrown into a pond. The second is the observation that the Universe is expanding. That is inferred from the fact the galaxies exhibit a red-shift that grows with distance and also from the existence of the cosmic microwave background that shows that the universe has evolved from a very dense early state. It appears that the Universe originated in some kind of initial singularity as predicted by the singularity theorems of Hawking and Penrose [118]. As an aside, it presently appears that the Universe will never recollapse to a big crunch singularity but will expand forever powered by a small positive cosmological constant.

The second law of thermodynamics, that the entropy of an isolated system cannot decrease, defines a thermodynamic arrow of time. However, this arrow of time is really about complicated systems where there has been some kind of coarse-graining. A system that has some degree of order will evolve in a way given by the microscopic laws specifying the evolution of its most fundamental components. However, if you only look at the bulk properties of the system, most of the time it will appear to have become more disordered.
reflecting the randomizing nature of the coarse-graining process. That the second law is of limited applicability is demonstrated by the phenomenon of Poincaré recurrence.

In quantum mechanics, at first sight, there does seem to be a time direction. In the Copenhagen interpretation, where an observation is taken to result in the collapse of the wavefunction, there is clearly a time asymmetry. However, in post-Everett quantum mechanics as exemplified by decoherent histories, there is no collapse of the wavefunction, merely an application of a projection onto the state of the system. Of course, as we saw in section 4, projections are required to be time ordered, but appear in the decoherence functional in a time symmetric way. That is to say the time-ordered history $C_\alpha$ appears together with the anti-time-ordered history $C_\alpha^\dagger$. Additionally, there is no analogue of the second law of thermodynamics, since in a closed system the von Neumann entropy is constant.

There is one remaining type of time asymmetry and that is provided by the CP-violation in $K_0$ decay. It is strongly believed that CPT is an exact symmetry of nature and so CP-violation translates into a T-violation. However, this is a very weak phenomenon and can be accounted for in the standard model. It appears unrelated to the other arrows of time.

As just observed, CPT is believed to be an exact symmetry of nature. Nevertheless, arrows of time are real and must originate from somewhere. The action of CPT on a particle is to transform the particle into its antiparticle, perform a total spatial reflection and reverse its momentum. Once this has been done, CPT symmetry says that the same laws of physics apply now as to the original system. How then can there be any arrow of time? The answer can only come from the boundary conditions. Long ago, Einstein, Tolman and Podolsky [57] showed that in quantum mechanics once a measurement is made, not only does this result in a probabilistic view of the future but also it means you have a similar probabilistic view of the past. Thus quantum mechanics does not have an inbuilt arrow of time.

It therefore seems strange that the decoherence functional contains only a contribution from the initial density matrix but makes no reference to the future. Were one trying to retrodict the past rather than predict the future, the decoherence functional would contain a density matrix at its future endpoint instead. This observation stimulated Aharonov, Bergmann and Lebowitz [58] to suggest that under certain circumstances one should include in the decoherence functional both an initial state density matrix and a final state density matrix. The decoherence functional is then replaced by

$$D(\alpha, \alpha') = \frac{\text{tr} (\rho_f C_\alpha \rho_i C_{\alpha'})}{\text{tr} (\rho_f \rho_i)}$$

There is the trivial possibility that $\rho_f$ is the identity. This is what Gell-Mann and Hartle [116] call the principle of indifference and under these circumstances the conventional presentation of consistent histories is regained. For a more detailed discussion of the decoherence functional in time symmetric quantum mechanics, see Hartle, [117].

What we would like to investigate now is the possibility that the principle of indifference holds outside the black hole, but as one approaches the singularity, or more precisely what classically is called the singularity, one needs to specify some particular $\rho_f$. 
One might feel sceptical that a boundary condition in the future makes sense. However, in certain circumstances open to experimental verification, [119–121] these ideas have proved to be correct.

6 Singularities

Our ideas about black hole information loss are conditioned by the classical picture of black holes in general relativity. The singularity theorems show that once there is an horizon, the spacetime is singular. Some intuition can be derived from spherical collapse where a spacelike singularity forms which is the future boundary of all causal lines that get trapped inside the horizon. Classically, the generic situation appears to be that collapse results in a spacelike, or possibly a null, singularity in the future of all worldlines that pass through the horizon. Such worldlines terminate at the singularity. The singularity is the future boundary of the part of spacetime inside the horizon and so anything reaching it can no longer be thought of as being in the spacetime.

We need to study the nature of this singularity. Let us first examine what happens in classical physics. The most convenient approach for present purposes is to use the canonical formalism, [54, 122–124]. Spacetime is taken to be foliated by spacelike surfaces \( \Sigma(t) \) with spatial coordinates \( x^i \), a time coordinate \( t \). The induced metric on \( \Sigma(t) \) is \( \gamma_{ij} \). The line element for the spacetime can then be written as

\[
ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad (6.1)
\]

where \( N \) is the lapse and \( N^i \) the shift. \( N, N^i \) and \( \gamma_{ij} \) are in general functions of both \( x^i \) and \( t \). In what follows we will deal with pure Einstein gravity where the action is, including the boundary term [17, 125],

\[
I = \int_M R (-g)^{1/2} d^4x \pm 2 \int_{\partial M} \nabla_a n^a (\pm \sigma)^{1/2} d^3x \quad (6.2)
\]

with \( M \) being the spacetime manifold, \( \partial M \) its boundary, \( R \) the Ricci scalar of the four-dimensional spacetime and \( n^a \) the unit normal to the boundary where the induced metric is \( \sigma \) and the signs are determined by whether the boundary is spacelike or timelike. Using the 3 + 1 decomposition of the metric, the action becomes

\[
I = \int_M (K_{ij} K^{ij} - K^2 + (3)R(\gamma)) N \gamma^{1/2} d^3x \, dt \quad (6.3)
\]

where \((3)R(\gamma)\) is the Ricci scalar of \( \gamma_{ij} \), \( K_{ij} \) is the second fundamental form describing the embedding of \( \Sigma(t) \) in \( M \) and \( K = \gamma^{ij} K_{ij} \). Explicitly

\[
K_{ij} = \frac{1}{2N} \left( D_i N_j + D_j N_i - \frac{\partial \gamma_{ij}}{\partial t} \right) \quad (6.4)
\]

with \( D_i \) being the covariant derivative with respect to the three-metric \( \gamma_{ij} \).

The momentum conjugate to \( \gamma_{ij} \) is \( \pi^{ij} \) given by

\[
\pi^{ij} = -\gamma^{1/2} \left( K^{ij} - \gamma^{ij} K \right). \quad (6.5)
\]
The momenta conjugate to both $N$ and $N^i$ vanish and this results in a set of constraints. The diffeomorphism constraint coming from the vanishing of the momentum conjugate to $N^i$ is

$$\chi^i \equiv D^j \pi^{ij} = 0,$$

(6.6)

If it is satisfied at one instant of time, since it does not evolve, it will always be satisfied and so is a constraint on the data on $\Sigma$ at any moment. The other constraint is the Hamiltonian constraint

$$\mathcal{H} \equiv \left( G_{ijkl} \pi^{ij} \pi^{kl} - \gamma^{1/2} (3) R \right) = 0.$$

(6.7)

with

$$G_{ijkl} = \frac{1}{2} \gamma^{-1/2} (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - 2 \gamma_{ij} \gamma_{kl}).$$

(6.8)

The Hamiltonian constraint bears a superficial resemblance to the mass-shell condition for a massless particle, which suggests that one should interpret $G_{ijkl}$ as the inverse metric on the space $\mathbf{M}$ of all metrics $\gamma_{ij}$ at each point in space. The metric $G^{ijkl}$ on $\mathbf{M}$ can then calculated from

$$G^{ijkl} G_{klnm} = \frac{1}{2} (\delta^i_m \delta^j_n + \delta^i_n \delta^j_m)$$

(6.9)

and is

$$G^{ijkl} = \frac{1}{2} \gamma^{1/2} (\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2 \gamma^{ij} \gamma^{kl}).$$

(6.10)

$G^{ijkl}$ has signature $(-+++)$ with the negative direction being associated with conformal transformations. It should be carefully noted that the negative direction has nothing whatsoever to with the time direction in the original spacetime. In fact, the negative direction associated with conformal transformations is the same as that making the Euclidean formulation of gravity somewhat sick \cite{126}. The conformal factor can vanish and when this happens $\mathbf{M}$ is singular. DeWitt \cite{54} refers to the singularity in $\mathbf{M}$ as the “frontier” and it is an avatar of the spacetime singularities we are ultimately interested in. Quotienting out the conformal degree of freedom results in $G^{ijkl}$ being the metric on the symmetric space $SL(3, \mathbb{R})/SO(3)$. Finally, in order to eliminate gauge degrees of freedom conjugate to the constraints, one must fix both the lapse and shift and eliminate the diffeomorphisms from the metric $\gamma_{ij}$.

Although we expect to encounter a singularity inside a black hole, the picture presented by Oppenheimer and Snyder \cite{20} is somewhat misleading. The reason is that one does not expect exact spherical symmetry. The general approach to a singularity appears to be chaotic as was first shown by BKL, \cite{48–50}. Their results indicate that as one gets close to the singularity, each point of the spacetime behaves more or less independently of neighbouring points. That is not to say that neighbouring points do not interact, but rather that as one gets close to the singularity, their influence can be summarised in a way that results in a considerable simplification of the Einstein equations. Such a behaviour is termed ultralocal. The results of BKL show that the time evolution at each point can be described in terms of ordinary differential equations for metric components, with the

\footnote{Historically, the negative direction has been termed “extrinsic time” which is grossly misleading.}
independent variable being time. In the BKL idealisation, each point is independent but in reality it seems that by point, one really means an appropriately small spatial region. Scheel and Thorne [127] suggested that perhaps the BKL picture of singularity formation was relevant to black hole physics. Recent numerical work [128] is consistent with the BKL picture of the approach to singularities.

The starting point for an exploration of the BKL approximation is a minisuperspace [129] description of the Kasner universe. One firstly partially fixes the gauge by choosing pseudo-Gaussian normal coordinates for a Bianchi I spacetime. The metric is thus

\[ ds^2 = -N^2 dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2 \]  

(6.11)

with the lapse being a function only of \( t \). Substituting the above metric form into the Einstein action gives

\[ I = \int dt \left[ -2N \left( \dot{a}\dot{b}c + \dot{b}\dot{c}a + \dot{a}\dot{c}b \right) \right] \]  

(6.12)

where a dot denotes the derivative with respect to \( t \). The solutions are given by

\[ a = a_0(t_0 - t)^{p_1}, \quad b = b_0(t_0 - t)^{p_2}, \quad c = c_0(t_0 - t)^{p_3} \]  

(6.13)

with the Kasner exponents \( p_i \) being given by

\[ p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1 \]  

(6.14)

and \( a_0, b_0, c_0 \) and \( t_0 \) being constants. Thus, as is well-known, two of the \( p_i \) are positive and one negative. Hence as \( t \) approaches \( t_0 \) from below, two directions are contracting and one is expanding \(^9\). At \( t_0 \) there is a curvature singularity although the Ricci scalar remains zero. The volume of space, which is proportional to the product \( abc \sim (t - t_0) \), tends to zero as \( t \rightarrow t_0 \) from below. It should be noted that when the action is varied with respect to \( N \), it generates the Hamiltonian constraint.

A more convenient set of variables are

\[ a = e^{-\beta_1}, \quad b = e^{-\beta_2}, \quad c = e^{-\beta_3} \]  

(6.15)

and

\[ \tilde{N} = \frac{N}{abc} \]  

(6.16)

in terms of which the action becomes

\[ I = \int dt \left[ -\frac{2}{\tilde{N}} \left( \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 \right) \right]. \]  

(6.17)

The momenta \( \pi_i \) conjugate to \( \beta^i \) are

\[ \pi_i = 2\tilde{N} g_{ij} \dot{\beta}^j \]  

(6.18)

---

\(^9\)The Kasner universe is usually presented as starting at a singularity and then two directions are expanding. Here we are looking at a collapsing version of the Kasner universe so two directions are contracting and one is expanding.
where the inverse minisuperspace metric is

\[
G_{ij} = \begin{pmatrix}
0 & -1 & -1 \\
-1 & 0 & -1 \\
-1 & -1 & 0
\end{pmatrix}.
\]  

(6.19)

The Hamiltonian constraint can now be written as

\[
H = \frac{1}{4N} G^{ij} \pi_i \pi_j
\]  

(6.20)

with the minisuperspace metric being

\[
G^{ij} = \frac{1}{2} \begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}.
\]  

(6.21)

\(G_{ij}\) has signature \((-++\)), is flat and the negative direction is the overall scale. Solutions to the Einstein equations can be seen to be null geodesics in this space. The affine parameter \(\tau\) on these geodesics is a measure of \(t\) in which \(d\tau = \tilde{N} dt\). These geodesics are of the form

\[
\beta^i = w^i \tau + \beta^i_0
\]  

(6.22)

with both \(w^i\) and \(\beta^i_0\) being constants. \(w^i\) is null in the sense that \(G_{ij} w^i w^j = 0\). The \(w^i\) are related to the Kasner exponents \(p^i\) defined in (6.13) by

\[
p_i = \frac{w^i}{w^1 + w^2 + w^3}.
\]  

(6.23)

Defining \(\rho\) by

\[
\rho^2 = -\beta^1 \beta^2 G_{ij}
\]  

(6.24)

one observes that the singularity where \(t = t_0\) is approached as \(\rho \to \infty\) where the proper volume of space tends to zero.

A more geometrically pleasing understanding of the minisuperspace picture introduced above comes from exhibiting its relationship to the Lobachevskii plane, \([50, 130]\) the two-manifold of constant negative curvature, with polar coordinates \(0 \leq r < \infty, \phi \leq 0 \leq 2\pi\). Let \(\beta^i = \rho \gamma^i\) then

\[
\gamma^1 = \frac{1}{\sqrt{6}} \cosh r - \sqrt{\frac{2}{3}} \sin \left(\phi + \frac{\pi}{3}\right) \sinh r
\]  

(6.25)

\[
\gamma^2 = \frac{1}{\sqrt{6}} \cosh r - \sqrt{\frac{2}{3}} \sin \left(\phi - \frac{\pi}{3}\right) \sinh r
\]  

(6.26)

\[
\gamma^3 = \frac{1}{\sqrt{6}} \cosh r + \sqrt{\frac{2}{3}} \sin \phi \sinh r
\]  

(6.27)

The minisuperspace line element is now

\[
d\Sigma^2 = -d\rho^2 + \rho^2 (dr^2 + \sinh^2 r d\phi^2)
\]  

(6.28)
The BKL approximation is a relatively simple modification of this picture. The influence of neighbouring cells can be described by the introduction of perfectly reflecting walls constraining the null geodesic motion to the spatial region
\[ \gamma^1 - \gamma^2 \geq 0, \quad \gamma^3 - \gamma^2 \geq 0, \quad \gamma^1 \geq 0 \] (6.29)
or equivalently in the Lobachevskii plane
\[ \frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}, \quad \coth r \geq 2 \sin(\phi + \frac{\pi}{3}). \] (6.30)
At these walls, the geodesics are specularly reflected. Back in the Kasner picture, the effect of a reflection is to change the Kasner exponents and rotate the principal axes of the Kasner expansion or contraction.

Remarkably, mapping the spatial sections into the upper-half Poincaré plane reveals some hidden structure. Let
\[ x^0 = \cosh r, \quad x^1 = \sinh r \sin \phi, \quad x^2 = \sinh r \cos \phi. \] (6.31)
Now set
\[ \hat{x}^0 = \frac{2}{\sqrt{3}} x^0 - \frac{1}{\sqrt{12}} x^1 - \frac{1}{2} x^2, \] (6.32)
\[ \hat{x}^1 = \frac{1}{\sqrt{3}} x^0 - \frac{1}{\sqrt{3}} x^1 - x^2, \] (6.33)
\[ \hat{x}^2 = \frac{\sqrt{3}}{2} x^1 - \frac{1}{2} x^2, \] (6.34)
and then transform the “spatial” part into the upper-half complex plane using
\[ z = u + iv = \frac{\hat{x}^1 + i(\hat{x}^0 + \hat{x}^2 + 1)}{\hat{x}^0 - \hat{x}^2 + 1 + i\hat{x}^1}. \] (6.35)
The classical behavior of this system is that of null geodesics in the metric
\[ ds^2 = -d\rho^2 + \rho^2 \left( \frac{du^2 + dv^2}{v^2} \right). \] (6.36)
The action in terms of \( \rho, u \) and \( v \) can now be taken to be
\[ I = \int dt \left[ -\dot{\rho}^2 + \frac{\rho^2}{v^2} (u^2 + \dot{v}^2) \right]. \] (6.37)
The momenta are defined by
\[ \pi_\rho = -\dot{\rho} \] (6.38)
\[ \pi_u = \frac{\rho^2}{v^2} \dot{u} \] (6.39)
\[ \pi_v = \frac{\rho^2}{v^2} \dot{v} \] (6.40)
The Hamiltonian is now
\[
H = -\frac{1}{2} p_p^2 + \frac{v^2}{2\rho^2} (\pi_u^2 + \pi_v^2)
\] (6.41)
which must be supplemented by the constraint that \( H \) must vanish. The walls are now located at

- Wall 1: \( u = 0, \ v \geq 1 \) (6.42)
- Wall 2: \( u = \frac{1}{2}, \ v \geq \frac{\sqrt{3}}{2} \) (6.43)
- Wall 3: \( z = e^{i\theta}, \ \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \). (6.44)

This is a fundamental region \( F \) of \( PGL(2, \mathbb{Z}) \) and is precisely half of the more familiar fundamental region of \( PSL(2, \mathbb{Z}) \), see Figure 2. Again, there is specular reflection from the walls and this has the effect of transforming the trajectory, \( z(\rho) \) close to the wall into \( R_i(z) \) as

- Wall 1: \( R_1(z) = -z^* \) (6.45)
- Wall 2: \( R_2(z) = 1 - z^* \) (6.46)
- Wall 3: \( R_3(z) = \frac{1}{z^*} \). (6.47)

A series of reflections then is described by a word in this group. The Kasner evolution classically will typically involve an infinite number of reflections before reaching the singularity at \( \rho \to \infty \). It should be noted that the generators of \( PSL(2, \mathbb{Z}) \) are words in \( PGL(2, \mathbb{Z}) \). The generator of translations in \( PSL(2, \mathbb{Z}) \) is \( T : z \to z + 1 \) and therefore \( T = R_2 R_1 \). Similarly \( S : z \to -\frac{1}{z} \) is \( S = R_1 R_3 \). Although it is not of obvious direct relevance here, \( PGL(2, \mathbb{Z}) \) is the Weyl group of the Kac-Moody algebra \( A_1^{++} \). The Weyl group describes how the root vectors can be chosen and this in turn determines where we decide to place the fundamental region.

To summarise: classically, there are domains that for some period look like the Kasner universe in the sense that two directions are contracting and one is expanding. At the beginning and end of each period there is a discontinuity and the exponents and the directions of expansion and contraction change as a result of bouncing off the walls. To reach the singularity, there are generically an infinite number of bounces. However, the volume of any cell goes to zero as the singularity is approached since the product \( abc \) is proportional to \( (t_0 - t) \). This type of behaviour is what is expected classically as one approaches the singularity in any realistic collapse.

7 Quantum Gravity

How is the situation different in a quantum universe as compared to a classical universe? For quantum field theory in curved spacetimes, the situation seems to be essentially similar to the classical one.

Consider the Schwarzschild spacetime and massless scalar fields propagating in it, [131, 132]. The radial part of the Klein-Gordon equation has a resonant singularity at \( r = 0 \) and
the solutions of the Klein-Gordon equation have logarithmic singularities there. As a result, the probability flux vector for excitations crossing the horizon is future-pointing there and divergent and there appears to be an inevitable loss of information. Similar results hold for fields with spin or mass. The Kerr spacetime is just as bad, since the inner horizon is unstable for essentially the same reason, \cite{29}. Solutions of the radial part of the Teukolsky equation blow up on the inner horizon and it is for this reason that one believes a singularity to be formed there.

To make any progress, one should be looking at quantum gravity. There is no fine-grained theory of quantum gravity known. The best we can do is to look at a coarse-grained approach based on general relativity. The classical theory is the most coarse-grained picture. Semi-classical quantum gravity based on general relativity is less coarse-grained and allows for some exploration of the quantum world. One can explore semi-classical gravity either by the use of path integrals or via the canonical formulation. In what follows, we will motivate the use of the Wheeler-DeWitt equation by observing that the wavefunction of the universe derived from the path integral is a solution of the Wheeler-DeWitt equation.

Path integral expressions for a wavefunction in gravitation are of the form

$$\Psi[\gamma] = \int D[g] e^{iI[g]}$$

(7.1)

where $I[g]$ is the classical action and the integral is taken over all distinct metrics subject to the metric on the boundary being $\gamma$. $\Psi$ will obey the Wheeler-DeWitt equation. The Hamiltonian constraint $H$ is in general a functional of components of the metric $\gamma_{ij}$ and the canonical momenta $\pi^{ij}$. To quantize, we replace the momenta $\pi^{ij}$ by $-i\frac{\delta}{\delta \gamma_{ij}}$. Then the Wheeler-DeWitt equation is

$$H\Psi = 0$$

(7.2)

and in some sense replaces the Schrödinger equation for gravity. One must also impose the diffeomorphism constraint so that in addition

$$\chi^i\Psi = 0.$$  

(7.3)

Of course, the prescription for determining the wavefunction needs to be supplemented by physically appropriate boundary conditions.

Hartle and Hawking \cite{51} defined the scalar product of two wavefunctions $\Psi_a$ and $\Psi_b$ to be

$$(\Psi_a, \Psi_b)_{HH} = \int D[\gamma] \Psi_a^*[\gamma] \Psi_b[\gamma].$$

(7.4)

The Hartle-Hawking inner product is the integral over all spatial metrics at each point in space. We immediately recognise (7.4) as being related to the decoherence functional for gravity, (4.4).

An alternative definition of the inner product is due to DeWitt,

$$(\Psi_a, \Psi_b)_{DW} = \int D[\gamma]_{ij} \text{Im} \left( \Psi_a[\gamma]^* \mathcal{G}^{ij}_{kl} \frac{\delta}{\delta \gamma_{kl}} \Psi_b[\gamma] \right).$$

(7.5)
In this expression, the integral is again evaluated at each point in space, but instead of integrating over all metrics $\gamma$, one integrates over all metrics $\bar{\gamma}$ that have the conformal factor quotiented out, [54]. At each point in space, one integrates over the five-dimensional collection of metrics $\bar{\gamma}_{ij}$ that parametrise the symmetric space $SL(3, \mathbb{R})/SO(3)$. The subscript on the measure $D[\bar{\gamma}]_{ij}$ indicates that this is a vector in the space of all metrics in the direction of conformal transformations. In many ways, this is a functional analog of the Klein-Gordon norm where one is integrating the divergence-free Klein-Gordon current over a spacelike surface. Here one is integrating at each point in space a vector, the DeWitt current, that is similarly divergence-free in a functional way as a result of obeying the Wheeler-DeWitt equation. The surface one is integrating over is “spacelike” as it is the conformal direction in the space of all metrics that has a “timelike” direction. Because of the divergence-free condition, (7.5) is independent of precisely how one chooses the hypersurface determining $\bar{\gamma} \subset \gamma$ as long as the hypersurface intersects each conformal equivalence class once only.

Consider now the Wheeler-DeWitt equation for our minisuperspace describing the approach to the singularity. It is just the vanishing of the Laplacian of the metric (6.36) acting on $\Psi$. However, we need to take account of the effect of the walls. The walls generate specular reflection classically and therefore mark the locations where the potential becomes infinite and thereby restricts classical motion to the fundamental region, $F$. For the Wheeler-DeWitt equation this is equivalent to introducing a potential that is infinite beyond the walls. Standard arguments about the solution to differential equations of this type tell us that $\Psi$ must vanish on the walls that make up the boundary of the fundamental region.

For our minisuperspace the Wheeler-DeWitt equation becomes

$$\rho^2 \frac{\partial^2 \Psi}{\partial \rho^2} + 2 \rho \frac{\partial \Psi}{\partial \rho} + \Delta_F \Psi = 0$$

(7.6)

where $\Delta_F$ is the Laplacian in the fundamental region. An explicit expression for $\Delta_F$ is

$$\Delta_F = -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right).$$

(7.7)

One could solve the Wheeler-DeWitt equation in terms of Maass functions and appropriate cusp forms for $F$, [133]. Fortunately, a detailed computation of $\Psi$ is unnecessary here. The eigenfunctions $f_n$ of $\Delta_F$ obey

$$\Delta_F f_n = s_n (1 - s_n) f_n.$$ 

(7.8)

The $f_n$ are the odd Maass waveforms of $SL(2, \mathbb{Z})$ with $s_n = \frac{1}{2} \pm it$, with $t_n$ real. For each $n$, there is a complex conjugate pair of such eigenfunctions. There are two families of such wavefunctions. The first is a discrete set of odd cusp forms of $SL(2, \mathbb{Z})$ that are square integrable in $F$ with the norm inherited from constant negative curvature space,

$$\int_F \frac{dudv}{v^2} f_n^* f_n.$$ 

(7.9)
The second is a continuum of odd non-holomorphic Eisenstein series that are not integrable in this norm. Despite the fact that the non-holomorphic Eisenstein series are not normalizable, normalizable functions obeying our boundary conditions are represented by linear combinations of members of both families.

Suppose now that solutions of the Wheeler-DeWitt equation are separable and can be written in the form $\sum_n P_n(\rho)f_n(u,v)$. The $P_n(\rho)$ obey

$$\frac{\partial^2 P_n}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial P_n}{\partial \rho} + \frac{\lambda_n P_n}{\rho^2} = 0. \quad (7.10)$$

Solutions to the “radial” part of the Wheeler-DeWitt equation are then just powers $P_n(\rho) = \rho^{p_n}$ with $p_n = \frac{1}{2} \pm i t_n$. Thus as $\rho \to \infty$, $\Psi \to 0$.

The two inner products $(\Psi_a, \Psi_b)_{HH}$ and $(\Psi_a, \Psi_b)_{DW}$ both descend to inner products on our minisuperspace. Explicitly,

$$(\Psi_a, \Psi_b)_{HH} = \int \Psi_a^* \Psi_b \frac{\rho^2}{v^2} \, du \, dv \quad (7.11)$$

and

$$(\Psi_a, \Psi_b)_{DW} = \int \text{Im} \left( \Psi_a^* \frac{\partial}{\partial \rho} \Psi_b \right) \frac{\rho^2}{v^2} \, d\rho \, du \, dv. \quad (7.12)$$

Assuming that $\Psi$ is derived from functions that are normalizable on $F$, they are normalizable in the DeWitt norm, but not in the Hartle-Hawking norm.

From the path integral form of $\Psi[\gamma]$, one notes that $\Psi[\gamma]$ is the probability amplitude for finding the geometry described by the spatial metric $\gamma$. In the BKL minisuperspace, the geometry is described by $\rho, u$ and $v$ or equivalently $a, b$ and $c$. So imagine asking for the probability amplitude of finding the geometries with $\rho \to \infty$ corresponding to surfaces getting closer and closer to what classically is singular. For each eigenfunction $f_n$ of $\Delta_F$,

$$\Psi \sim \rho^{-1/2} e^{\pm it_n \ln \rho} f_n. \quad (7.13)$$

The probability of getting close to the singular surface is decreasing to zero. Presumably there is no singularity. There is however an oscillatory part of this wavefunction that looks like waves travelling both forward and backwards in $\rho$ depending on the sign chosen. For a given $f_n$, the choice of sign is completely arbitrary so one can always choose $\Psi$ to contain arbitrary linear combinations of both possible signs. In many ways, the DeWitt current looks like a probability current so if information is not to be lost close to the singularity, one wants not just for $\Psi$ to vanish, but the DeWitt norm also to vanish. Since each eigenfunction is of order $\rho^{-1/2}$ one concludes that the DeWitt norm generically independent of $\rho$. But it is easy to make the DeWitt norm vanish by making $\Psi$ real as can be done by replacing $\Psi$ by $\Psi + \Psi^*$, as this will still obey the Wheeler-DeWitt equation. This last condition prevents information from escaping from what appears to be the future boundary of the spacetime.

We propose to take seriously this boundary condition on the surfaces close to the classical singularity. It amounts to setting a boundary condition in the future that enforces

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10DeWitt [54] speculated that $\Psi$ would have to be set to zero at the frontier of $M$. 

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a reflection in the time direction. Its physical meaning is that nothing will escape the
spacetime through what is classically the singularity. When evaluating the time-symmetric
path integral, it amounts to selecting \( \rho_f \) to be a projection into a pure state \(|N\rangle\). Thus,
\[ \rho_f = |N\rangle\langle N| \]. The state \(|N\rangle\) is that which essentially says there is no spacetime beyond
what classically would be the singularity and reflects everything. Of course, this boundary
condition only applies to the future inside the black hole. Exterior to the black hole, we
take the path integral to be controlled by the principal of indifference. Alternatively, one
can regard the selection of the future final state \(|N\rangle\) to be a boundary condition on solu-
tions of the Wheeler-DeWitt equation. Of course, it remains to be seen if the information
paradox can be resolved using the time symmetric version of the path integral. Were this
boundary condition to result in some strange behaviour exterior to the black hole, it might
either prove to be something that is detectable leading to a test of these ideas or maybe
sufficiently catastrophic to invalidate the programme. Further investigation of the details
of this scenario need to be explored.

One might ask what happens if one insists on having spherically symmetric gravitational
collapse as opposed to a collapse into a BKL regime. The fact that superspace is stratified
\[ [135, 136] \] as a result of making such symmetry assumptions means that looking here is not
going to be relevant to generic collapse. However, spherically symmetric collapse has been
investigated by Bouhmadi-Lopez \textit{et al} \[137\]. Their results are not inconsistent with those
found here.

8 Unitarity, Causality and the Information Paradox.

The quantum picture of a black hole involves the vanishing of the gravitational wavefunc-
tional at what would classically be the singularity. If, in addition, one imposes the condition
that the wavefunctional is real, we find an affirmation of the ideas of Horowitz and Mal-
dacena \[60\]. The picture we now have has been nicely summarised by Gottesman and
Preskill, \[61\]. Particles fall into the black hole as it forms, and then are reflected close to
the singularity by a process that is the time reverse of the Hawking particle-antiparticle
creation process. Outside the horizon, the usual Hawking pair production process can be
interpreted as the reflection of antiparticles coming backwards in time out of the black hole
and scattering into the outgoing Hawking radiation. Quantum information is expected to
follow this flow of particles and is illustrated in Figure 3. The setting of a final boundary
condition destroys the usual notion of causality and unitarity and it is this that allows
violations of the entropy triangle inequalities and strong subadditivity \[138, 139\].

Bousso and Stanford \[62\] noted that inside the black hole, it seems that the decoherence
functional did not decohere, leaving the probability interpretation somewhat ambiguous.
However we can see this as a failure of the coarse-graining used. Whilst such behaviour can
be tolerated inside the black hole where no observer is going to make observations that we
can perceive, it should not happen outside the black hole.

However, Lloyd \[63\] observed that as long as normal causality and unitarity hold right
up to the final surface, classical information is preserved and quantum information escapes
with fidelity \((8/3\pi)^2\). The classical information is comprised of charges that are embedded
in the classical geometry: the momentum, angular momentum, mass and electric charge of the black hole together with a collection of soft charges \(^{11}\). The practical outcome of Lloyd’s result on quantum information is that, on average, only half a bit of quantum information will be lost, independently of the number of bits that escape from the black hole. Lloyd and Preskill \cite{64} also showed that this kind of final state model is able to avoid the difficulties presented by firewalls.

Penrose \cite{27} proposed the cosmic censorship hypothesis in order to prevent classical singularities being seen by asymptotic observers. The physical reason for censorship was to prevent singularities influencing physics outside collapsed bodies in an unpredictable fashion. A slightly different way of expressing these ideas is to ask that the Cauchy problem be well-defined when gravitational collapse occurs. Accordingly, singularities should always be hidden inside the horizon. There is the possibility that setting future boundary conditions inside the horizon could cause violation of the “central dogma.” We suggest a quantum cosmic censorship hypothesis, a conjecture that no such trouble occurs. Alternatively, if quantum cosmic censorship is violated, a weaker version might well hold, in which such violations occur, but are in practice undetectable.

We looked at minisuperspace for pure gravity. One obvious question is to ask what happens if other fields are included. Something like BKL behaviour seems inevitable. In four spacetime dimensions, ultralocal BKL type behaviour is almost inevitable. In almost all cases, BKL behaviour and chaos occurs just as it does in pure gravity, the only difference is that the pattern of the walls changes, \cite{50, 130, 134}. The only exception appears to be if there is a massless scalar that does not couple to other fields. If such a scalar exists, then the classical approach to singularity is a little more complicated and requires separate treatment. It might also be that quantum gravity requires spacetime dimensions greater than four in order to control divergences, as is expected to be the case in string/M-theory. In spacetime dimensions \(d \leq 10\), the approach to singularity in pure gravity is chaotic. In \(d = 11\) supergravity the approach to singularity is also chaotic. In all of these cases, the wavefunction at the singularity vanishes \cite{134} and these wavefunctions can be made real on the approach to singularity. String/M-theory is a promising avenue leading to quantum gravity, but these theories (at least in their present form) are not quantum theories of spacetime. However, they seem to be consistent with the idea that \(A_1^{++}\) that controls the approach to singularity in dimension four is replaced by \(E_{10}\), \cite{140}. It has been speculated that a fundamental theory of quantum gravity should be based on \(E_{10}\) \cite{130, 141}. If this were the case, it would add support for our mechanism for avoiding information loss.

Our proposal outlines a plausible route for the complete resolution of the information paradox. Further investigation of these ideas is called for, since it seems that the vanishing of the wavefunction in the future inside a black hole is unavoidable. Precisely how the information is retrieved and the placing of these ideas into the broader picture of both gravitational and quantum physics is an on-going project.

\(^{11}\)Gauge charges other than just electromagnetism should presumably be included here too, even though traditionally they are not mentioned.
Acknowledgments

I would like to thank the STFC for financial support under grant ST/L000415/1.

I would like to thank Ratindranath Akhoury, David Berman, Sam Braunstein, Jeremy Butterfield, Mihalis Dafermos, Fay Dowker, David Garfinkle, Gary Gibbons, Hadi Godazgar, Mahdi Godazgar, David Gross, Stephen Hawking, Hermann Nicolai, Frans Pretorius, Kip Thorne, Edward Witten and Anna Żytkow for stimulating discussions and advice.
Figure 1: The Penrose diagram for a black hole that evaporates completely. $\Sigma_i$ is an initial Cauchy surface. $\Sigma_f$ is a similar surface after the black hole has completely evaporated. $\Sigma_s$ is a spacelike surface close to the singularity where future boundary conditions are applied.
Figure 2: The fundamental region for $PGL(2, \mathbb{Z})$ is shaded in grey and is bounded by the lines $\text{Re } z = 0$, $\text{Im } z \geq 1$; $\text{Re } z = 1/2$, $\text{Im } z \geq \sqrt{3}/2$ and a segment of the unit circle. It is precisely half of the more familiar fundamental for $PSL(2, \mathbb{Z})$. 
Figure 3: The Penrose diagram of a black hole that evaporates completely together with a river showing the expected flow of quantum information.
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