SUBFACTORS AND HECKE GROUPS

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ABSTRACT. We study a relation between the Hecke groups and the index of subfactors in a von Neumann algebra. Such a problem was raised by V. F. R. Jones. We solve the problem using the notion of a cluster $C^*$-algebra.

1. Introduction

The following problem can be found in [Jones 1991] [5, p.24]:

“Consider the subgroup $G_{\lambda}$ of $SL_2(\mathbb{R})$ generated by $(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$.

For what values of $\lambda > 0$ is it discrete? Answer: $\lambda = 2 \cos \left( \frac{\pi}{n} \right)$, $n = 3, 4, \ldots$ or $\lambda \geq 2$. (…) We have been unable to find any direct connection between this result and Theorem 3.1 (Jones Index Theorem). It is a tantalizing situation.”

The aim of our note is to solve the problem in terms of the cluster $C^*$-algebras [6, Section 4.4.3]. To give an idea, let $\mathcal{D} = \{ z = x + iy \in \mathbb{C} | r \leq |z| \leq R \}$ be an annulus in the complex plane. Consider the Schottky uniformization of $\mathcal{D}$, i.e.

$$\mathcal{D} \cong \mathbb{C}P^1 / \mathbb{Z},$$

(1.1)

where $\mathbb{C}P^1 := \mathbb{C} \cup \{ \infty \}$ is the Riemann sphere and $A \in SL_2(\mathbb{C})/ \pm I$ is a matrix acting on the $\mathbb{C}P^1$ by the M"obius transformation. It follows from [Glubokov & Nikolaev 2018] [3] and Section 2.2, that the index of subfactors in a von Neumann algebra coincides with the square of trace of matrix $A$, i.e.

$$\text{tr}^2 (A) \in [4, \infty) \bigcup \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) | n \geq 3 \right\}.$$  

(1.2)

To solve the Jones Problem, we prove in Section 3 that $\mathcal{D}$ is a ramified double cover of the orbifold $\mathbb{H}/G_{\lambda}$, where $\mathbb{H} := \{ x + iy \in \mathbb{C} | y > 0 \}$ is the Lobachevsky half-plane and the group $G_{\lambda}$ acts on $\mathbb{H}$ by the linear fractional trasformations. Since such a cover takes the square root of the moduli parameter $\text{tr}^2 (A)$ of $\mathcal{D}$, we conclude that $\lambda = \text{tr} (A)$. In other words, the Jones Index Theorem (1.2) is equivalent to the following well-known result:

**Theorem 1.1.** ([Hecke 1936] [4, Satz 1,2 & 6]) The $G_{\lambda}$ is a discrete subgroup of $SL_2(\mathbb{R})$ if and only if

$$\lambda \in [2, \infty) \bigcup \left\{ 2 \cos \left( \frac{\pi}{n} \right) | n \geq 3 \right\}.$$  

(1.3)

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Remark 1.2. The group $G_\lambda$ appears in the study of the Riemann zeta function [Hecke 1936] [4]. In particular, Hecke’s Theorem says that the space of automorphic functions corresponding to $G_\lambda$ is (i) infinite-dimensional, if $\lambda > 2$ or (ii) finite-dimensional, if $\lambda \in \{2 \cos \left(\frac{\pi}{n}\right) \mid n \geq 3\}$. The proof of this fact is purely analytic [Hecke 1936] [4]. On the other hand, cases (i) and (ii) follow from the Sherman-Zelevinsky Theorem for the cluster $C^*$-algebras of rank 2, see [Glubokov & Nikolaev 2018] [3].

The article is organized as follows. Section 2 contains a brief review of preliminary results. Theorem 1.1 is proved in Section 3.

2. Preliminaries

The cluster $C^*$-algebras and their $K$-theory are covered in [6, Section 4.4.3]. A correspondence between the cluster $C^*$-algebra of an annulus $\mathcal{D}$ and the Jones Index Theorem was established in [Glubokov & Nikolaev 2018] [3]. The Hecke groups were introduced in [Hecke 1936] [4].

2.1. Cluster $C^*$-algebras. A cluster algebra $\mathcal{A}(x, B)$ of rank $n$ is a subring of the field of rational functions in $n$ variables depending on a cluster of variables $x = (x_1, \ldots, x_n)$ and a skew-symmetric matrix $B = (b_{ij}) \in M_n(\mathbb{Z})$; the pair $(x, B)$ is called a seed. A new cluster $x' = (x_1, \ldots, x'_k, \ldots, x_n)$ and a new skew-symmetric matrix $B' = (b'_{ij})$ is obtained from $(x, B)$ by the exchange relations:

$$x_k x'_k = \prod_{i=1}^{n} x_i^{\max(b_{ik},0)} + \prod_{i=1}^{n} x_i^{\max(-b_{ik},0)},$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2} & \text{otherwise.} \end{cases} \quad (2.1)$$

The seed $(x', B')$ is said to be a mutation of $(x, B)$ in direction $k$, where $1 \leq k \leq n$; the algebra $\mathcal{A}(x, B)$ is generated by cluster variables $\{x_i\}_{i=1}^{\infty}$ obtained from the initial seed $(x, B)$ by the iteration of mutations in all possible directions $k$. The Laurent phenomenon says that $\mathcal{A}(x, B)$ is generated by cluster variables $x = (x_1, \ldots, x_n)$ depending on an initial seed $(x, B)$ by the iteration of mutations in all possible directions $k$. The Laurent phenomenon in variables $x = (x_1, \ldots, x_n)$ depending on an initial seed $(x, B)$, The $\mathcal{A}(x, B)$ is a commutative algebra with an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. Thus the algebra $\mathcal{A}(x, B)$ is a countable abelian group with an order satisfying the Riesz interpolation property, i.e. a dimension group. A cluster $C^*$-algebra $\mathcal{A}(x, B)$ is an AF-algebra, such that

$$K_0(\mathcal{A}(x, B)) \cong \mathcal{A}(x, B), \quad (2.2)$$

where $\cong$ is an isomorphism of the dimension groups [6, Section 4.4.3].

2.2. Schottky uniformization of $\mathcal{D}$. Consider the Riemann surface $\mathcal{D}$ (an annulus) defined by the formula (1.1). We shall use the Schottky uniformization of $\mathcal{D}$ by the loxodromic transformations. Namely, let $\mathbb{C}P^1 := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and consider the Möbius transformation of $\mathbb{C}P^1$ given by the formula:

$$z \mapsto k z, \quad \text{where } z \in \mathbb{C}P^1 \text{ and } |k| \neq 1. \quad (2.3)$$
It is easy to see, that (2.3) can be written in the matrix form:

\[
A = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} \in PSL_2(\mathbb{C}) := SL_2(\mathbb{C})/\pm I. \tag{2.4}
\]

It is well known, that (2.3) is a loxodromic transformation if and only if

\[
tr^2 (A) = \frac{(k + 1)^2}{k} \in \mathbb{C} \setminus [0, 4]. \tag{2.5}
\]

The Schottky uniformization of \(\mathcal{D}\) is given by the formula:

\[
\mathcal{D} \cong \mathbb{C}P^1 / A\mathbb{Z}. \tag{2.6}
\]

2.3. Admissible values of \(tr^2 (A)\). Recall that the moduli space of the annulus \(\mathcal{D}\) is given by the formula:

\[
T_{\mathcal{D}} = \left\{ t = \frac{R}{r} \mid t > 1 \right\}. \tag{2.7}
\]

We consider a cluster \(\mathcal{C}^*\)-algebra \(\mathcal{A}(\mathcal{D})\) associated to a canonical triangulation of \(\mathcal{D}\) [Fomin, Shapiro & Thurston 2008] [1, Example 4.4]. It follows from the Sherman-Zelevinsky Theorem for the algebra \(\mathcal{A}(\mathcal{D})\), that the admissible values of the “index” \(\frac{(t+1)^2}{t}\) must belong to the set:

\[
[4, \infty) \bigcup \{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \}, \tag{2.8}
\]

see [Glubokov & Nikolaev 2018] [3] for the proof.

We set \(k = t\) in the formulas (2.3) - (2.5). Comparing (2.5) and (2.8), one gets

\[
tr^2 (A) \in [4, \infty) \bigcup \{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \}. \tag{2.9}
\]

Remark 2.1. It follows from (2.5) that \(A\) is a loxodromic transformation if and only if \(tr^2 (A) \in (4, \infty)\). The case \(tr^2 (A) \in \{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \}\) corresponds to an elliptic transformation \(A\) of order \(n\). Finally, that case \(tr^2 (A) = 4\) gives a parabolic transformation \(A\). Note that for the elliptic and parabolic transformations, the values of parameter \(t\) in (2.7) are the \(n\)-th roots of unity.

3. Proof of theorem 1.1

We split the proof in a series of lemmas.

Lemma 3.1. Let \(p_1, p_2 \in \mathbb{C}P^1\) be two points on the Riemann sphere, such that \(p_1 \neq p_2\). Then there exists a double covering map

\[
p : \mathbb{C}P^1 \to \mathbb{C}P^1 \tag{3.1}
\]

ramified over the points \(p_1\) and \(p_2\).

Proof. (i) Recall that the necessary condition for the existence of \(p\) is given by the Riemann-Hurwitz formula:

\[
\chi(\mathbb{C}P^1) = 2\chi(\mathbb{C}P^1) - \sum_{i=1}^{2} (e_i - 1), \tag{3.2}
\]
where \( \chi(\mathbb{C}P^1) \) is the Euler characteristic and \( e_i \) is the degree of the map \( z \mapsto z^{e_i} \) in the ramification point \( p_i \). Since \( \chi(\mathbb{C}P^1) = e_1 = e_2 = 2 \), we conclude that the condition (3.2) is satisfied.

(ii) The sufficient condition for the existence of \( p \) can be verified directly using [Gersten 1987] [2, Theorem 1.5]. We leave it as an exercise to the reader.

Lemma 3.1 is proved. \( \square \)

**Remark 3.2.** We assume further that in lemma 3.1 we have \( p_1 = 0 \) and \( p_2 = \infty \). Our assumption is not restrictive, since any two points \( p_1, p_2 \in \mathbb{C}P^1 \) can be put into such a position by a Möbius transformation.

\[
\rho \quad \text{D(\text{tr}^2(A))} \quad \longrightarrow \quad \text{D(\lambda)}
\]

**Figure 1.** Covering map \( \rho : \mathcal{D} \to \mathbb{H}/G_\lambda \).

**Lemma 3.3.** Consider a Riemann surface:

\[
\mathcal{H} := \mathbb{C}P^1 \setminus \{ D_0, \infty \},
\]

where \( D_0 \) is a disk containing point \( 0 \in \mathbb{C}P^1 \). Then there exists a double covering map

\[
\rho : \mathcal{D} \to \mathcal{H}.
\]

**Proof.** Let \( \sigma : \mathbb{C}P^1 \to \mathbb{C}P^1 \) be the double covering map ramified at 0 and \( \infty \), see lemma 3.1 and remark 3.2. Recall that

\[
\mathcal{D} \cong \mathbb{C}P^1 \setminus \{ D_0, D_\infty \},
\]

where \( D_\infty \) is a disk containing point \( \infty \in \mathbb{C}P^1 \). We use a homotopy to contract \( D_\infty \) to the point \( \infty \), and set the map \( \rho \equiv \sigma \). Comparing (3.3) and (3.5), we conclude that \( \rho \) is the required double covering map. Lemma 3.3 is proved. \( \square \)

**Lemma 3.4.** \( \mathcal{H} \cong \mathbb{H}/G_\lambda \), where \( \lambda \in [2, \infty) \cup \{ 2 \cos \left( \frac{\pi}{n} \right) \mid n \geq 3 \} \).
Proof. It is well known, that the Hecke orbifold \( \{ \mathbb{H}/G_{\lambda} \mid \lambda > 2 \} \) is a topological sphere \( S^2 \) with a hole \( D(\lambda) \) of radius \( \lambda/2 \), one elliptic fixed point \( e_2 \) of order 2 and one puncture \( c \), see e.g. [Schmidt & Sheingorn 1995] [7, p. 255]. The elliptic point \( e_2 = i \) is a fixed point of the matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) having order 2 in the group \( SL_2(\mathbb{Z})/\pm I \).

Since \( S^2 \cong \mathbb{CP}^1 \), we use (3.3) to identify \( D(\lambda) \equiv D_0 \) and \( e_2 \equiv \infty \). Thus one gets \( \mathcal{H} \cong \mathbb{H}/G_{\lambda} \). Using lemma 3.3, we obtain a double covering map \( \rho \) ramified in the points \( e_2 \) and \( 0 \in D(\lambda) \), i.e.

\[
\rho : \mathcal{D} \rightarrow \mathbb{H}/G_{\lambda}. \tag{3.6}
\]

Remark 3.5. Notice that \( \rho^{-1}(D(\lambda)) \) is a disk and \( \rho^{-1}(e_2) \) is a regular point of \( \mathcal{D} \). The corresponding ramification points are shown in Figure 1.

To determine admissible values of the moduli parameter \( \lambda \), observe that \( \lambda = \frac{1}{\pi} |\partial D(\lambda)| \), where \( \partial D(\lambda) \) is the boundary of the disk \( D(\lambda) \). Observe that the local map at the point \( 0 \in D(\lambda) \) is given by the formula \( z \mapsto z^2 \). Therefore using the polar coordinates, we conclude that:

\[
\lambda^2 = \frac{1}{\pi} |\rho^{-1}(\partial D(\lambda))| \tag{3.7}
\]
is a moduli parameter of the Riemann surface \( \mathcal{D} \). But according to (1.2) any such a parameter must coincide with the \( tr^2 \) \((A)\), where \( A \) is the matrix in the Schottky uniformization (1.1). Taking positive values of the square root, one gets from (1.2)

\[
\lambda = tr (A) \in [2, \infty) \bigcup \{2 \cos \left( \frac{\pi}{n} \right) \mid n \geq 3 \}. \tag{3.8}
\]

Lemma 3.4 is proved. \( \square \)

Remark 3.6. For the sake of brevity, lemma 3.4 is proved for the continuous moduli \( \lambda \in (2, \infty) \). The case of the discrete moduli is treated likewise, see remark 2.1.

Theorem 1.1 follows from lemma 3.4.

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