Exponential equations for the quantum “$az + b$” group.

Małgorzata Rowicka - Kudlicka*
Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warszawa, Poland
e-mail: rowicka@fuw.edu.pl

November 2, 2018

Abstract

We consider quantum group theory on the Hilbert space level. We find all solutions for scalar and general exponential equations for the quantum “$az + b$” group. It turns out that there is a simple formula for all of them involving the quantum exponential function $F_N$. The very interesting theorem we prove by the way is the one on the existence of normal extension of certain sum of normal operators.

To put it differently, we find all unitary representations of the braided quantum group related to the quantum “$az + b$” group. This is the most difficult result needed to classify all unitary representations of the quantum “$az + b$” group. Eventually this enables us to give a formula for all unitary representations of the quantum “$ax + b$” group in our next paper [17].

key words: unbounded operators – Hilbert space
MSC-class: 20G42 (Primary), 47B25 (Secondary).

1 Introduction

To explain what is going on in this paper let us use an analogy with the classical case. One of the goals of the classical group theory is to find all unitary representations of the group considered. For example, by SNAG theorem, we know that $U$ is a (strongly continuous) unitary representation of the group $\mathbb{R}^2$ acting on Hilbert space $\mathcal{H}$ iff $C_\infty(\mathbb{R}^2)$ there exists a pair of strongly commuting selfadjoint operators $(a, b)$ acting on $\mathcal{H}$ such that for any $(x, y) \in \mathbb{R}^2$ we have

$$U(x, y) = e^{ixa+iyb}.$$ 

It means that all unitary representations of $\mathbb{R}^2$ are “numbered” by elements from the set of all pairs of strongly commuting selfadjoint operators.

On the other hand, every such pair gives rise, through the functional calculus of normal operators, to a representation of the algebra of all continuous vanishing at infinity functions on the group dual to $\mathbb{R}^2$, i.e. in this case to representations of $C_\infty(\mathbb{R}^2) = C_\infty(\mathbb{R}^2)$. This phenomenon, i.e. correspondence between unitary representations of the locally compact group $G$ and representations of the algebra $C_\infty(G)$, is known as the Pontryagin duality. On the other hand, the exponential function is a solution of the exponential equation

$$F(x + y) = F(x)F(y),$$ \hspace{1cm} (1)
where $x,y \in \mathbb{R}^2$. Moreover, if we look for a solution such that $F$ is measurable and $|F(x)|$ has a modulus 1 for every $x$, all the solutions of (1) is given by the formula

$$F(x,y) = e^{ixa+iyb},$$

where $(a,b) \in \mathbb{R}^2$.

The solution of the general exponential equation, i.e. with the one unitary-operator-valued function $F$, is a direct integral of the solutions of the scalar case. So once one knows solutions of the general exponential equation, one knows also all unitary representations of the group involved, in this case $\mathbb{R}^2$.

One can also consider equation (1) in more general setting. One can allow $x$ and $y$ to be “coordinates” on the two copies of a space $G$, classical or quantum. They are usually sets of operators acting on Hilbert space and satisfying certain conditions. 8888

This paper is very similar in spirit to the our forthcoming paper [15], where braided quantum groups related to the quantum “$ax+b$” group were studied and their unitary representations classified.

In this paper we consider the quantum group theory on the Hilbert space level. The quantum “$az+b$” group was constructed recently by S.L. Woronowicz in [28]. It is the natural deformation of the group of affine transformations of the complex plane, with the deformation parameter $q$ being an even primitive root of unity (for details see citeaz+b). However, what we are mainly interested in in this paper is a braided quantum group related to the quantum “$az+b$” group.

Let us begin with an explanation, what we mean by a quantum group there.

In fact, the definition of locally compact quantum group, and such is “$az+b$”, is still under construction [7], however one knows approximately what a quantum group should be.

We like most the approach using operator domain and operator function (described in [27, 6, 15]). We believe it is a very insightful one. What we say below is not really necessary to understand the paper, but we hope it will be useful to understand idea that lie behind and the connections with the quantum “$az+b$” group.

We only outline general ideas here, for more detailed treatment we refer the Reader to [27, 6, 15].

The description of an operator domain is similar to the global one of a manifold, where coordinates and relation satisfied by them are given. In our noncommutative case, the coordinates are closed (so in general unbounded) operators and the relations are arbitrary commutation rules that are invariant with respect to unitary transformations and direct sum decomposition. For this invariance S.L. Woronowicz, to whom this idea is due, coined a term “respecting symmetry of the Hilbert space”.

Easy example of an operator domain is the one crucial in this paper: an operator domain $D$ related to the quantum “$az+b$” group. The relations in this case are

$$RR^* = R^*R \quad \text{and} \quad \text{Sp}R \subset \Gamma$$

where $\Gamma$ is the multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ given by (1). The unbounded operator $R$ entering descriptions of the operator domain $D$ can be thus thought of as a “coordinate on a quantum space”. Observe, that this space is entirely classical, it can be identified with $\overline{\Gamma}$.

The operator functions can be thought of as a recipe what to do with a $N$-tuple of closed operators $(a_1,a_2,...,a_N)$ to obtain another closed operator $F(a_1,a_2,...,a_N)$. An operator map are similar to operator functions, the only difference is they may transform
$N$-tuple of closed operators $(a_1, a_2, ..., a_N)$ to obtain another $k$-tuple of closed operators $(F_1(a_1, a_2, ..., a_N), F_1(a_1, a_2, ..., a_N),)$

Let $G$ be an operator domain and let $G \times G$ denote an operator domain

$$G \times G := \{(x, y) \mid x, y \in G \text{ and } xy = yx\}$$

Let $\cdot$ be an operator map

$$\cdot : G \times G \ni (x, y) \mapsto xy \in G$$

Loosely speaking, a quantum group $G$ is such an operator domain $G$ equipped with an associative operator map $\cdot$.

**Example 1.1 (Quantum “az + b” at roots of unity)**

$$G = \left\{ (a, b) : \begin{array}{l}
aa^* = a \ast a, \quad b^*b = b^*b, \\
a \text{ invertible} \\
ab = qba \\
\text{Sp}a, \text{Sp}b \subset \Gamma, \\
\end{array} \right\}$$

where $q$ is an even primitive root of unity and $\Gamma$ is given by (4).

Group operation in $G$ is given by

$$\cdot : G \times G \ni ((a_1, b_1), (a_2, b_2)) \mapsto (a, b) \in G$$

where

$$a = a_1 \otimes a_2 \quad \text{and} \quad b = a_1 \otimes b_2 + b_1 \otimes I,$$

where $\dot{+}$ denotes closure. The spectral condition is there to make sure that $G$ is closed under the operation $\cdot$.

The main difference between a braided quantum group and a quantum group is that a group operation on a braided quantum group $G$ is defined on a smaller operator domain

$$G^2 := \{(x, y) \mid x, y \in G \text{ and } x, y \text{ satisfy certain relations} \}.$$ 

Usually we do not assume that operators from both copies of $G$ commute, so in general a braided quantum group is not a quantum group. A group operation on a braided quantum group $G$ should be the operator map

$$\boxdot : G^2 \ni (x, y) \mapsto x \boxdot y \in G$$

which is associative.

**Example 1.2 (Braided quantum group $D$) Let us define operator domains**

$$D = \{ R \mid RR^* = R^*R \text{ and } \text{Sp}R \subset \Gamma \}$$

and

$$D^2 = \{ (R, S) \mid R, S \in D \quad \text{Phase} \ S R = q R \text{Phase} \ S \}$$

and on $(\ker S) \perp$ we have $|S|^it = e^{-\frac{i\pi}{2}t}R|S|^it$ for any $t \in \mathbb{R}$.
We define an operation
\[ \bigcirc_D : D^2 \to D \]
by
\[ R \bigcirc_D S = R \dot{+} S. \]
Thus defined operation \( \bigcirc_D \) is associative and \( D \) with this operation forms a braided quantum group.

Moreover, let us observe that the braided quantum group \( D \) is related to the quantum “az+b” group, denoted here by \( G \), in the following way
\[(a, b) \in G \iff \left( (b, a) \in D^2 \quad \ker a = \{0\} \right).\]

In this paper we find all solutions of the general exponential equation for the quantum “az+b” group. In fact, we find all the unitary representations of the braided quantum group \( D \). The main result is Theorem 7.1, which gives formula for all such representations. This result is essential for classification of all unitary representations of the quantum group ”az+b”, which is achieved in our forthcoming paper [17].

The second important result in this paper is Proposition 5.1, which solves the problem of the existence of a normal extension of a sum \( \mu R + S \), where \((R, S) \in D^2 \) and \( \mu \in \Gamma \). We hope that this result will be useful in the construction of the quantum \( GL(2, \mathbb{C}) \) group [11].

In the remaining part of this section we introduce some non-standard notation and notions used in this paper.

In Section 2 we introduce commutation rules related to the quantum “az+b” group. In the next section we discuss properties of pair of operators \((R, S)\) satisfying these commutation rules. In Section 4 we repeat the definition of the quantum exponential function for the “az+b” group after [28]. Then we investigate in Section 5 the existence of normal extensions of \( \mu R + S \), where \((R, S) \in D^2 \) and \( \mu \in \Gamma \). In Section 6 we give all solutions of the scalar exponential equation for the quantum “az+b” group and finally in Section 7 we do the same for the general exponential equation.

In Appendix A we prove the formula we use in Section 7.

1.1 Notation

We denote Hilbert spaces by \( \mathcal{H} \) and \( K \), the set of all closed operators acting on \( \mathcal{H} \) by \( \mathcal{C}(\mathcal{H}) \), the set of bounded operators by \( B(\mathcal{H}) \) and the sets of compact and unitary ones by \( CB(\mathcal{H}) \) and \( \text{Unit}(\mathcal{H}) \), respectively. The set of all continuous vanishing at infinity functions on a space \( X \) will be denoted by \( C_\infty(X) \). We consider only separable Hilbert spaces, usually infinite dimensional. We denote scalar product by \( (\cdot | \cdot) \) and it is antilinear in the first variable. We consider mainly unbounded linear operators. All operators considered are densely defined. We use functional calculus of normal operators [12, 13, 18]. We also use the symbol Phase \( T \) for partial isometry obtained from polar decomposition of a normal operator \( T \).

We use a non-standard, but very useful notation for orthogonal projections and their images [23], as explained below. Let \( a \) and \( b \) be strongly commuting selfadjoint operators acting on a Hilbert space \( \mathcal{H} \). Then by spectral theorem there exists a common spectral measure \( dE(\lambda) \) such that
\[ a = \int_{\mathbb{R}^2} \lambda dE(\lambda, \mu), \quad b = \int_{\mathbb{R}^2} \mu dE(\lambda, \mu). \]
For every complex measurable function \( f \) of two variables

\[
f(a, b) = \int_{\mathbb{R}^2} f(\lambda, \lambda') \, dE(\lambda, \lambda').
\]

Let \( f \) be a logical sentence and let \( \chi(f) \) be 0 if is false, and 1 otherwise. If \( \mathcal{R} \) is a binary relation on \( \mathbb{R} \) then \( f(\lambda, \lambda') = \chi(\mathcal{R}(\lambda, \lambda')) \) is a characteristic function of a set

\[
\Delta = \{(\lambda, \lambda') \in \mathbb{R}^2 : \mathcal{R}(\lambda, \lambda')\}
\]

and assuming that \( \Delta \) is measurable \( f(a, b) = E(\Delta) \). From now on we will write \( \chi(\mathcal{R}(a, b)) \) instead of \( f(a, b) \):

\[
\chi(\mathcal{R}(a, b)) = \int_{\mathbb{R}^2} \chi(\mathcal{R}(\lambda, \lambda')) \, dE(\lambda, \lambda') = E(\Delta).
\]

Image of this projector will be denoted by \( \mathcal{H}(\mathcal{R}(a, b)) \), where ‘\( \mathcal{H} \)’ is a Hilbert space, on which operators \( a, b \) act.

Thus we defined symbols \( \chi(a > b) \), \( \chi(a^2 + b^2 = 1) \), \( \chi(a = 1) \), \( \chi(b < 0) \), \( \chi(a \neq 0) \) etc. They are orthogonal projections on appropriate spectral subspaces. For example \( \mathcal{H}(a = 1) \) is is an eigenspace of operator \( a \) for eigenvalue 1 and \( \chi(a = 1) \) is orthogonal projector on this eigenspace.

Generally, whenever \( \Delta \) is a measurable subset of \( \mathbb{R} \), then \( \mathcal{H}(a \in \Delta) \) is spectral subspace of an operator \( a \) corresponding to \( \Delta \) and \( \chi(a \in \Delta) \) is its spectral projection.

Let \( \hat{q} \) and \( \hat{p} \) denote the position and momentum operators in Schrödinger representation, i.e. we set \( \mathcal{H} = L^2(\mathbb{R}) \). Then the domain of \( \hat{q} \)

\[
D(\hat{q}) = \{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |\psi(x)|^2 dx < \infty \}
\]

and \( \hat{q} \) is multiplication by coordinate operator on that domain

\[
(\hat{q} \psi)(x) = x \psi(x).
\]

The domain of \( \hat{p} \) consists of all distributions from \( L^2(\mathbb{R}) \) such that

\[
D(\hat{p}) = \{ \psi \in L^2(\mathbb{R}) : \psi' \in L^2(\mathbb{R}) \}
\]

and for any \( \psi \in D(\hat{p}) \)

\[
(\hat{p} \psi)(x) = \frac{\hbar}{i} \frac{d\psi(x)}{dx},
\]

where \(-\pi < \hbar < \pi\).

2 Commutation rules related to the quantum ”az+b” group

Let

\[
q = e^{\frac{2\pi i}{N}},
\]

where \( N \) is an even number and \( N \geq 6 \), i.e. \( q \) is a primitive root of unity: \( q^N = 1 \). Let us introduce notation

\[
\hbar = \frac{2\pi}{N}.
\]
Note that $\hbar < \pi$ and $q = e^{i\hbar}$.

The assumption that $N$ is an even number was made by S.L. Woronowicz in [28], where the quantum "az+b" was constructed. We need the assumption that $N \geq 6$ to use formulas (1.31), (1.32) and (1.34) from [23] (or equivalently (1.10), (1.11) and (1.12) from [28]). We will use these formula to derive relations we need later on.

Let

$$\Gamma = \bigcup_{k=0}^{N-1} q^k \mathbb{R}_+ .$$

(4)

It means that $\Gamma$ is a multiplicative group.

Let $\overline{\Gamma}$ denote a closure of $\Gamma$ in $\mathbb{C}$, i.e.

$$\overline{\Gamma} = \Gamma \cup \{0\} .$$

(5)

Our goal is to find an exponential equation for a pair of operators $(R, S)$ acting on Hilbert space $\mathcal{H}$ and satisfying commutation relations described in [28] and denoted by $(R, S) \in D_{\mathcal{H}}$ there. We recall below definition of these relations.

**Definition 2.1** Let $\mathcal{H}$ be a separable Hilbert space. We say that closed operator $(R, S) \in D_{\mathcal{H}}$ if

1. $R, S$ are normal
2. $\ker R = \ker S = \{0\}$
3. $\text{Sp} R, \text{Sp} S \subset \overline{\Gamma}$
4. $$(\text{Phase} S) R = q R (\text{Phase} S)$$
   and $$|S|^it R = e^{-\frac{2\pi}{N}it} R |S|^it ,$$
   for any $t \in \mathbb{R}$.

**Remark 2.2** Condition 4. can be written in equivalent form using polar decomposition of operators $R$ and $S$

$$ (\text{Phase} S) |R| = |R| (\text{Phase} S)$$
$$ |S| (\text{Phase} R) = (\text{Phase} R) |S|$$
$$ (\text{Phase} S)(\text{Phase} R) = q (\text{Phase} R) (\text{Phase} S)$$

and

$$ |S|^it |R| |S|^{-it} = e^{-\frac{2\pi}{N}it} |R| ,$$

(6)

for any $t \in \mathbb{R}$.

Observe also that if one performs analytical continuation and substitutes $t = i$ in (6), one gets $|S| |R| = q |R| |S|$ . Hence, using other equalities introduced in Remark 2.2 we get

$$SR = q^2 RS \quad \text{and} \quad SR^* = R^* S .$$

More precise derivation of these formulas can be found in proof of Proposition 2.1 [28].
3 Properties of operators \((R, S) \in D_\mathcal{H}\)

We prove now analogues of Proposition 3.1, Proposition 3.2 and Theorem 3.3 [23]. The proofs below are modifications of those presented in the cited above paper.

Let us consider Hilbert space \(L^2(\Gamma, d\gamma)\), where \(d\gamma\) is Haar measure of the group \(\Gamma\), i.e.

\[
\int_{\Gamma} f(\gamma)d\gamma = \sum_{k=1}^{N} \int_{0}^{\infty} f(q^k r) \frac{dr}{r},
\]

for any \(f \in L^2(\Gamma, d\gamma)\) and \(r \in \mathbb{R}_+\) and \(k = 0, 1, ..., N - 1\).

Let

\[
\Lambda_k = \left\{ z : \frac{2k\pi}{N} < \arg z < \frac{2(k+1)\pi}{N} \right\}
\]

and

\[
\overline{\Lambda}_k = \left\{ z : \frac{2k\pi}{N} \leq \arg z \leq \frac{2(k+1)\pi}{N} \text{ or } z = 0 \right\} \quad (7)
\]

Then

\[
\mathbb{C} = \Gamma \cup \bigcup_{k=0}^{N-1} \Lambda_k.
\]

Let

\[
\Gamma_k = e^{\frac{2k\pi i}{N}} \mathbb{R}_+.
\]

Then

\[
\Gamma = \bigcup_{k=0}^{N-1} \Gamma_k.
\]

**Definition 3.1** Let \(H_{\text{bounded}}\) denote the set of all functions \(f \in C(\Gamma)\), such that there exists a continuous and bounded function \(\tilde{f}\) defined on \(\overline{\Lambda}_0 \times q\mathbb{Z}\) and such that

1. for any \(k \in \mathbb{Z}\) function

\[
\Lambda_0 \ni z \to \tilde{f}(z, q^k)
\]

is holomorphic

2. for any \(r \in \mathbb{R}_+\) and \(k \in \mathbb{Z}\)

\[
f(q^k r) = \tilde{f}(r, q^k) \quad (8)
\]

If \(f \in H_{\text{bounded}}\), it follows that there exist the described above function \(\tilde{f}\). We will use notation

\[
f(q \cdot q^k r) := \tilde{f}(qr, q^k).
\]

**Proposition 3.2** Let \((R, S) \in D_\mathcal{H}\) and \(f \in H_{\text{bounded}}\). Then

\[
f(R)D(S) \subset D(S)
\]

and for any \(x \in D(S)\) we have

\[
f(q \cdot qR)Sx = Sf(R)x. \quad (9)
\]
Proof: Let $\Sigma$ be the stripe $\{\tau \in \mathbb{C} : 0 < \Re \tau < 1\}$ and let $H(\Sigma)$ be the space of all functions continuous on $\Sigma$ and holomorphic in the interior of $\Sigma$. Observe that space $H(\Sigma)$ equipped with sup norm is a Banach space.

For any $\lambda \in \Gamma$ and $\tau \in \Sigma$ let us introduce notation

$$\varphi_\lambda(\tau) = \hat{f}\left(e^{i\lambda \tau}, q^k\right),$$

where $k = 0, 1, \ldots, N-1$, and the relation between $\hat{f}$ and $f$ is given by (8). Then $\varphi_\lambda \in H(\Sigma)$ and $\|\varphi_\lambda\| \leq C$, where $C = \sup\{|f(r, q^k)| : r \in \Lambda_0 \text{ and } k \in \{0, 1, \ldots, N-1\}\}$. Therefore

$$\int_{\Gamma} \varphi_\lambda(\tau)d\mu(\lambda) \in H(\Sigma) \quad (10)$$

for any (complex-valued) finite measure $d\mu(\lambda)$ on $\Gamma$.

Let $dE_R(\lambda)$ be a spectral measure of a normal operator $R$, $x, y \in D(S)$ and $d\mu(\lambda) = (y|dE_R(\lambda)Sx)$. Then

$$\int_{\Gamma} \varphi_\lambda(\tau)d\mu(\lambda) = (y|\hat{f}(e^{i\lambda \tau}|R|, \text{Phase } R)Sx).$$

Moreover, by (10) the map

$$\Sigma \ni \tau \longrightarrow (y|\hat{f}(e^{i\lambda \tau}|R|, \text{Phase } R)Sx) \in \mathbb{C}$$

is a function continuous on $\overline{\Sigma}$ and holomorphic inside $\Sigma$.

Since $x, y \in D(S)$, it follows that

$$\Sigma \ni \tau \longrightarrow |S|^{1+i\tau}x \in \mathcal{H},$$

$$\Sigma \ni \tau \longrightarrow |S|^{-i\tau}y \in \mathcal{H}$$

are continuous. The former function is holomorphic, whereas the latter is antiholomorphic on $\Sigma$. Therefore the function

$$\Sigma \ni \tau \longrightarrow ((\text{Phase } S)^*|S|^{-i\tau}y|\hat{f}(|R|, \text{Phase } R)|S|^{1+i\tau}x) \in \mathbb{C}$$

is continuous and holomorphic on $\Sigma$.

Consider for a while case $\tau \in \mathbb{R}$. Since $(R, S) \in D_H$, it follows that

$$(\text{Phase } S)^*|S|^{-i\tau}y|\hat{f}(|R|, \text{Phase } R)|S|^{1+i\tau}x) = ((\text{Phase } S)^*y|S^{-i\tau}\hat{f}(|R|, \text{Phase } R)|S|^{1+i\tau}y) =$$

$$(y|\hat{f}(e^{i\lambda \tau}|R|, e^{i\lambda \tau}\text{Phase } R)Sx).$$

The above equality remains true after analytic continuation to the $\tau \in \overline{\Sigma}$. In particular for $\tau = i$ we get

$$(S^*y|f(R)x) = (y|\hat{f}(i|R|, \text{Phase } R)Sx) = (y|f(q \cdot qR)x).$$

The above formula holds for all $y \in D(S^*)$. Moreover, $f(R)x \in D(S)$ and $Sf(R)x = f(q \cdot qR)x$.

For any $z \in \Gamma$ we define

$$\ell(z) = \log |z|. \quad (11)$$
Let us define a new space of holomorphic functions

\[ H = \left\{ f \in C(\Gamma) : \text{for any } \lambda > 0 \text{ function } e^{-\mathcal{M}(r)^2} f(r) \in H_{\text{bounded}} \right\} . \] (12)

Note that \( H \) is a vector space. Moreover, if we define for any \( f \in H \) and \( z \in \Gamma \)
\[ f^*(z) = f(q \cdot q^z) , \]
then \( f^* \in H \) and \((f^*)^* = f\). By the above consideration and functional calculus for normal operators, for any operator \( R \) with spectrum contained in \( \Gamma \), we get
\[ f^*(R) = f(q \cdot q R^*)^* , \] (13)
for any function \( f \in H \).

Observe that for any \( \lambda > 0 \), function \( e^{-\mathcal{M}(\cdot)^2} \in H_{\text{bounded}} \).

Let \((R, S) \in D_H \). Then for any \( \lambda > 0 \), operator \( e^{-\mathcal{M}(R)^2} \) is bounded and converges strongly to \( I \) when \( \lambda \to +0 \). So the set
\[ D_0 = \bigcup_{\lambda > 0} e^{-\mathcal{M}(R)^2} D(S) , \] (14)
is dense in \( \mathcal{H} \), because \( D(S) \) is dense in \( \mathcal{H} \). Moreover, we have

**Proposition 3.3** Let \( f \in H \). Then

0. \( D_0 \subset D(f(R)) \),
1. \( f(R)D_0 \subset D_0 \),
2. \( D_0 \subset D(S) \),
3. \( SD_0 \subset D(f(q \cdot q R)) \).

**Proof:** Let \( \lambda > 0 \). One can easily check that function \( g(r) = e^{-\mathcal{M}(r)^2} \) satisfies assumptions of Proposition 3.2. Therefore \( e^{-\mathcal{M}(R)^2} D(S) \subset D(S) \) and we thus proved point 2.

Let now \( f \in H \) and \( \lambda > 0 \). Then the function
\[ g(r) = f(r) e^{-\mathcal{M}(r)^2} \] (15)
is bounded. Therefore \( e^{-\mathcal{M}(R)^2} D(S) \subset D(f(R)) \) and point 0. follows.

One can easily check that function \( g(r) \) defined by (15) satisfies assumptions of Proposition 3.2. Therefore
\[ f(R) e^{-\mathcal{M}(R)^2} D(S) \subset D(S) . \]
Moreover
\[ f(R) e^{-2\mathcal{M}(R)^2} D(S) \subset e^{-\mathcal{M}(R)^2} D(S) \subset D_0 \]
and point 1. follows.

Let \( x \in D_0 \). Then
\[ x = e^{-\mathcal{M}(R)^2} x' , \] (16)
where \( x' \in D(S) \) and \( \lambda > 0 \). By (15)
\[ e^{-\mathcal{M}(q q R)^2} S x' = S x . \]
If \( f \in H \) then function \( f(r)e^{-\lambda r^2} \) is bounded, \( f(q \cdot qR)e^{-\lambda(qqR)} \in B(H) \) and \( Sx = e^{-\lambda(qqR)}Sx' \in D(f(q \cdot qR)) \). This completes the proof of point 3. \( \square \)

Let \((R, S) \in D_H \) and let \( f \in H \). The just proved Proposition shows that operators \( S \circ f(R) \) and \( f(q \cdot qR) \circ S \) are densely defined, in particular their domains contain \( D_0 \). We will prove that these operators are closable. To this end we show that their adjoints are densely defined. It means that adjoints of these adjoints are well defined - and they are exactly closures of the considered operators.

Using formula \((13)\) we deduce that

\[
(S \circ f(R))^* \supset f^*(q \cdot qR^*) \circ S^* \quad \text{and} \quad (f(q \cdot qR) \circ S)^* \supset S^* \circ f^*(R^*).
\]

Moreover, Proposition \(3.7\) yields that if \((R, S) \in D_H\), then also \((R^*, S^*) \in D_H\). We know also that if \( f \in H \), then \( f^* \in H \), too. Therefore, by the Proposition we just proved, operators \((S \circ f(R))^*\) and \((f(q \cdot qR) \circ S)^*\) are densely defined. Hence \( S \circ f(R) \) and \( f(q \cdot qR) \circ S \) are closable operators. We will denote their closures by \( Sf(R) \) and \( f(q \cdot qR)S \), respectively.

**Theorem 3.4** Let \((R, S) \in D_H \) and let \( D_0 \) be defined by \((14)\). Then for any function \( f \in H \) we have

1. \( D_0 \) is a core for \( f(q \cdot qR)S \),
2. \( (f(q \cdot qR)S)^* = S^* f^*(R^*) \),
3. \( f(q \cdot qR)S \subset Sf(R) \),

**Proof:** Ad 1. We already know that \( (f(q \cdot qR)S)^* \supset S^* f^*(R^*) \), it is enough to prove that the opposite inclusion holds. Let \( y \in D((f(q \cdot qR)S)^*) \) and \( z = (f(q \cdot qR)S)^* y \). Then

\[
(y|f(q \cdot qR)Sx) = (z|x)
\]

for any \( x \in D(f(q \cdot qR)S) \). In particular (see points 2. and 3. in Proposition \(3.3\)) the above relation holds for all \( x \in D_0 \). Therefore

\[
(y|f(q \cdot qR)Se^{-\lambda(R)^2}x') = (z|e^{-\lambda(R)^2}x')
\]

for any \( \lambda > 0 \) and \( x' \in D(S) \). Using \((3)\) we get

\[
(y|f(q \cdot qR)e^{-\lambda(qqR)}Sx') = (z|e^{-\lambda(R)^2}x').
\]

By \((13)\)

\[
(f^*(R^*)e^{-\lambda(R^*)^2}y|x) = (e^{-\lambda(R^*)^2}z|x').
\]

This relation holds for for any \( x' \in D(S) \). Hence we obtain

\[
f^*(R^*)e^{-\lambda(R^*)^2}y \in D(S^*) \quad \text{and} \quad e^{-\lambda(R^*)^2}y \in D(S^* f^*(R^*)).
\]

Moreover

\[
S^* f^*(R^*)e^{-\lambda(R^*)^2}y = e^{-\lambda(R^*)^2}z.
\]

The above formula holds for any \( \lambda > 0 \) and the operator \( S^* f^*(R^*) \) is closed. When \( \lambda \to +0 \) we obtain \( y \in D(S^* f^*(R^*)) \) and \( S^* f^*(R^*)y = z \). Thus we proved that \((f(q \cdot qR)S)^* \subset S^* f^*(R^*) \) and hence point 1. follows.
Proposition 3.7 (Proposition 2.2 [28])

If \((R, S)\), precisely, by Stone-von Neumann Theorem, every pair \((R, S)\) satisfies Definition 2.1, one can easily show that if \(\gamma_1 R, \gamma_2 S \in D(H)\), then also \((\gamma_1 R, \gamma_2 S) \in D(H)\). In particular for \(f = 1\) we obtain \(S x = e^{-\lambda(qqR)^2} S x'\). Comparing these two formulas we get

\[ f(q \cdot q R) S x = S f(R) x \]

This formula holds for any \(x \in D_o\). Remembering that \(D_o\) is a core for \((q \cdot q R) S\), we obtain \(f(q \cdot q R) S \subset S f(R)\).

Before we proceed to discuss properties of pairs \((R, S) \in D(H)\), we give an example of such operators.

Example 3.5 (The most important one: Schrödinger’s pair) Let \(H = L^2(\Gamma, d\gamma)\). Then for any \(z \in \Gamma\), \(z = q^k r\),

\[ (R f)(z) = z f(z), \]

and

\[ (\text{Phase } R f)(q^k r) = q^k f(q^k r) \]

and \( (|R| f)(q^k r) = r f(q^k r) \),

where

\[ D(R) = \{ f \in L^2(\Gamma) : \sum_{k=1}^{N} \int_{0}^{\infty} |f(q^k r)|^2 dr < \infty \}. \]

Moreover

\[ (S f)(z) = f(q^{-1} \cdot q^{-1} z), \]

and

\[ (\text{Phase } S f)(q^k r) = f(q^{-1} r) \]

and \( (|S| f)(q^k r) = f(q^{-1} \cdot q^k r) \).

The domain of \(D(|S|)\) consists of all functions \(f \in L^2(\Gamma)\), such that there exists a function \(g \in L^2(\Gamma)\) and a function \(\tilde{f}\) holomorphic in \(\Lambda_o \times q^{-1} \), and such that for any \(k = 0, 1, ..., N - 1\) we have

\[ \lim_{\varphi \to 0^-} \tilde{f}(e^{i\varphi} r, q^k) = f(q^k r) \]

and

\[ \lim_{\varphi \to \frac{2\pi}{N}^+} \tilde{f}(e^{i\varphi} r, q^k) = g(q^{-1} r) \]

where limits are taken in \(L^2\)-norm. Moreover, for fixed \(\varphi\) not equal multiply of \(\frac{2\pi}{N}\) and for any \(\lambda > 0\), function \(e^{-\lambda(r^{1/2} q^2)} \tilde{f}(r e^{i\varphi})\) should be bounded. From now on we will use notation

\[ f(q^{-1} \cdot q^k r) = g(q^{-1} r) \]

One can prove that thus defined operators \(R\) and \(S\) satisfy Definition 2.1, i.e. \((R, S) \in D(L^2(\Gamma))\).

The example above is crucial and all other examples are built up from this one. More precisely, by Stone-von Neumann Theorem, every pair \((R, S) \in D(H)\) is a direct sum of a certain number of copies of pairs, which are all unitarily equivalent to the Schrödinger’s pair.

We will use often the following remark and proposition.

Remark 3.6 Checking one by one conditions in Definition 2.1 one can easily show that if \((R, S) \in D(H)\) \(\gamma_1, \gamma_2 \in \Gamma\), then also \((\gamma_1 R, \gamma_2 S) \in D(H)\).

Proposition 3.7 (Proposition 2.2 [28]) If \((R, S) \in D(H)\), then also \((R^*, S^*) \in D(H)\) and \((S^{-1}, R) \in D(H)\) and \((S, R^{-1}) \in D(H)\) and \((R, SR) \in D(H)\).
4 The quantum exponential function for the ”az+b” group

As we explained in Introduction, a sum of closed operators may not be closed. Therefore, we will not consider \(R+S\) itself but its appropriate closure. Contrary to the case of selfadjoint \(R\) and \(S\) considered in \([23, 15]\), closure of the sum \(R+S\) is the desired operator, i.e. it is normal and its spectrum is contained in \(\Gamma\) (Theorem 2.4 w \([28]\)). Let \(\tilde{R+S}\) denote closure of the sum \(R+S\). We give now a useful formula for \(\tilde{R+S}\), where \((R,S)\in \mathcal{D}_H\), involving the quantum exponential function \(F_N\). The definition of \(F_N\) is given below.

**Proposition 4.1** \([28]\) Let \((R,S)\in \mathcal{D}_H\). Then

\[
\tilde{R+S} = F_N(S^{-1}R)^*SF_N(S^{-1}R) = F_N(R^{-1}S)RF_N(R^{-1}S)
\]

The special function \(F_N: \Gamma \rightarrow \mathbb{C}\) is given by \([28]\)

\[
F_N(q^{k}r) = \begin{cases} 
\prod_{s=1}^{k} \left( \frac{1+q^{2s}r}{1+q^{-2s}r} \right) \frac{f_o(qr)}{1+r} & \text{for } 2|k \\
\prod_{s=0}^{k-1} \left( \frac{1+q^{2s+1}r}{1+q^{-2s-1}r} \right) f_o(r) & \text{for } 2 \nmid k
\end{cases}, \tag{18}
\]

where

\[
f_o(z) = \exp \left\{ \frac{1}{\pi i} \int_0^{\infty} \log(1 + a^{-\frac{N}{2}}) \frac{da}{a + z^{-1}} \right\}, \tag{19}\]

for any \(z \in \mathbb{C} \setminus \{\mathbb{R} \cup \{0\}\}\).

Note that if \(N\) would be an odd number, this definition would not be good - values of function \(F_N\) at the point \(q^{k}r = q^{k+N}r\) could be calculated in two different ways giving different results. Whereas for \(N\) even we get, regardless of the way of calculating it, all the time the same value at the same point, i.e. \(F_N\) is well defined for an even \(N\).

By \([19]\) it follows that relation between \(f_o\) and the special function \(V_\theta\) used in \([23, 15]\) is given by

\[
f_o(q^{k}r) = V_{\frac{N}{2}}(\log r + i\hbar k)^2, \tag{20}\]

where \(r \in \mathbb{R}_+\) and \(k = 0, 1, \ldots, N-1\).

From \([24]\) we know that function \(V_{\frac{N}{2}}\) is holomorphic in the stripe \(\Im x < \pi\), so in particular it is continuously differentiated along lines \(x = \log r\) and \(x = \log r + i\hbar\), for any \(r \in \mathbb{R}_+\).

Moreover, by (1.37) and (1.38) \([23]\) it follows that

\[
V_{\frac{N}{2}}(\log r + i\hbar l) = 1 + \frac{q^{l}r}{2i\sin \hbar} + \mathcal{R}_o(q^{l}r), \tag{21}\]

where

\[
\lim_{r \to 0} \frac{\mathcal{R}_o(q^{l}r)}{r} = 0
\]

and \(l = 0\) or \(l = 1\).

Hence

\[
\lim_{r \to 0} V_{\frac{N}{2}}(\log r)^2 = 1 = \lim_{r \to 0} V_{\frac{N}{2}}(\log r + i\hbar)^2.
\]

Therefore setting \(F_N(0) = 1\) will make \(F_N\) continuous on \(\overline{\Gamma}\).

Moreover, by (1.7) \([28]\)

\[
|F_N(\gamma)| = 1 \quad \text{for} \quad \gamma \in \overline{\Gamma}.
\]
In order to prove the theorem of this Section we will need a formula for an expansion of $F_N$ around 0. Let us derive a formula for derivative at the point zero for $k$ even

$$F_N'(q^k r) = \left( \frac{1 + q^2 r}{1 + q^{-2} r} \right)' \left( \frac{1 + q^4 r}{1 + q^{-4} r} \right) \cdots \left( \frac{1 + q^{2^k} r}{1 + q^{-2^k} r} \right) f_o(q r) + \cdots$$

$$+ \left( \frac{1 + q^2 r}{1 + q^{-2} r} \right)' \left( \frac{1 + q^4 r}{1 + q^{-4} r} \right) \cdots \left( \frac{1 + q^{2^k} r}{1 + q^{-2^k} r} \right) f_o(q r)$$

Moreover

$$\left( \frac{1 + q^{2s} r}{1 + q^{-2s} r} \right)' = \frac{2i\Im q^{2s}}{(1 + q^{-2s} r)^2}$$

and

$$\left( \frac{f_o(q r)}{1 + r} \right)' = \frac{(1 + r) f_o'(q r) - f_o(q r)}{(1 + r)^2}.$$

By \(20\) and \(21\) one can calculate right derivatives of function $f_o$ at the point zero

$$f_o'(r)|_{r=0} = \frac{1}{i \sin h}$$

and

$$f_o'(q r)|_{r=0} = \frac{q}{i \sin h}.$$

Hence

$$F_N'(q^k r)|_{r=0} = -\frac{iq}{\sin h} - 1 + 2i \sum_{s=0}^{k-1} 3q^{2s+1} = \frac{\cos h}{i \sin h} + 2i \sum_{s=1}^{k} 3q^{2s}$$

Let us compute

$$\sum_{s=1}^{k} 3q^{2s} = 3q^2 \sum_{s=0}^{k-1} q^{2s} = 3 \left( q^2 1 - q^k \right) = 3 \left( q 1 - q^k \right) =$$

$$= \left( \frac{3(i(q - q^{k+1}))}{2 \sin h} \right) = \frac{\cos h - \cos(k + 1)h}{2 \sin h}$$

Hence

$$F_N'(q^k r)|_{r=0} = \frac{\cos h}{i \sin h} + \frac{-\cos h + \cos(k + 1)h}{i \sin h} = \frac{\cos(k + 1)h}{i \sin h}$$

Finely

$$F_N'(q^k r)|_{r=0} = \frac{q^{k+1} + q^{-k-1}}{2i \sin h}.$$

Similar calculations show that the above formula remains true also for $k$ odd. From the Taylor formula we obtain an expansion of $F_N$ around 0 for $\lambda \in \mathbb{R}_+$ and $t \in \Gamma$

$$F_N(\lambda t) = 1 + \lambda \frac{\lambda}{2i \sin h} (qt + \overline{q}t) + r(\lambda t)\lambda|t|,$$

where

$$\lim_{\lambda \to 0} r(\lambda t) = 0.$$

Moreover, using \(22\) one can deduce that

$$\lim_{\lambda \to +\infty} r(\lambda t) = -F_N'(\lambda t)|_{t=0}.$$

Obviously, function $r$ is continuous on $\gamma$. Hence, there exist a constant $M$, independent of $\lambda$, and such that

$$|r(\lambda t)| < M, \text{ for any } t \in \Gamma.$$
By (22) we have
\[ \frac{F_N(\lambda t) - 1}{\lambda} = \frac{qt + \overline{q}t}{2i \sin \hbar} + r(\lambda t)|t|, \] (25)
in particular
\[ \lim_{\lambda \to 0} \frac{F_N(\lambda t) - 1}{\lambda} = \frac{qt + \overline{q}t}{2i \sin \hbar}. \] (26)

Our next objective is to derive (4), which we will use to transform (48). Let \( dE(t) \) be a spectral measure \([12, \text{Chapter 8}]\) of normal operator \( T \). According to rules of functional calculus \([12, \text{Theorem VIII.6}]\), a continuous, bounded function of an operator \( T \) has form
\[ \left( u \begin{array}{c} \frac{F_N(\lambda T)}{\lambda} - I \\ v \end{array} \right) = \int_{\Gamma} \frac{F_N(\lambda t) - 1}{\lambda} (u \mid dE(t) \mid v), \]
for any \( u \in \mathcal{H} \) and \( v \in D(T) \).

Moreover, by (23)
\[ \int_{\Gamma} \frac{F_N(\lambda t) - 1}{\lambda} (u \mid dE(t) \mid v) = \int_{\Gamma} \frac{qt + \overline{q}t}{2i \sin \hbar} (u \mid dE(t) \mid v) + \int_{\Gamma} r(\lambda t) (u \mid t \mid dE(t) \mid v). \] (27)

Observe that \( (u \mid t \mid dE(t) \mid v) = (u \mid dE(t)|T| \mid v) \) and the measure \( (u \mid dE(t)|T| \mid v) \) is finite.

Constant function \( M \), which majorises function \( r \) (see (24)), is integrable with respect to the measure \( (u \mid t \mid dE(t) \mid v) \) and is the function majorising \( r(\lambda t) \).

Therefore, by the Lebesgue dominated convergence theorem \([12, \text{Theorem I.16}]\) and by (23)
\[ \lim_{\lambda \to 0} \int_{\Gamma} r(\lambda t) (u \mid t \mid dE(t) \mid v) = 0. \]

Hence by (27)
\[ \lim_{\lambda \to 0} \int_{\Gamma} \frac{F_N(\lambda t) - 1}{\lambda} (u \mid dE(t) \mid v) = \frac{1}{2i \sin \hbar} \int_{\Gamma} (qt + \overline{q}t) (u \mid dE(t) \mid v), \]
so
\[ \lim_{\lambda \to 0} \left( u \begin{array}{c} \frac{F_N(\lambda T)}{\lambda} - I \\ v \end{array} \right) = \frac{1}{2i \sin \hbar} (u \mid (qT + \overline{q}T^*) \mid v), \]
for any \( u \in \mathcal{H} \) and \( v \in D(T) \). This formula will be used in proof of the main theorem of this Section.

The proposition below explains why \( F_N \) is called the quantum exponential function

**Proposition 4.2 (Theorem 2.6, [28])** Let \( (R, S) \in D_H \). Then
\[ F_N(R)F_N(S) = F_N(R + S) \]
Later on in this Section we will prove that the quantum exponential function is unique (up to a parameter) solution of (4.2).

\section{Normal extensions of \( \mu R + RS \)}

**Proposition 5.1** Let \((R, S) \in D_H \) and let \( \mu \in \mathbb{C} \). Operator \( \mu R + RS \) has a normal extension if and only if, when \( \mu \in \Gamma \).

14
Proof: \[\Leftarrow\] Obvious, because according to Proposition (3.7) we have \((RS, S) \in D_{\mathcal{H}}\), and on account of Remark 3.6 it follows that \((RS, \mu S) \in D_{\mathcal{H}}\), so \(\mu S + RS\) is by Theorem 2.4 \([28]\) a normal extension of \(\mu S + RS\).

\[\Rightarrow\] We show that for any \(\mu \in \mathbb{C} \setminus \Gamma\) operator \(\mu S + RS\) has not a normal extension. Let us introduce notation

\[Q = \mu S + RS.\]

We first prove that operator \(Q\) is closed.

Let us define

\[m := \sup_{z \in \Gamma} \frac{1}{|\mu + z|} = \frac{1}{d(-\mu, \Gamma)},\]

where \(d(-\mu, \Gamma)\) denotes Euclidean distance of the point \(-\mu\) from the set \(\Gamma\). Note that \(m < \infty\), since \(\Gamma\) is closed and \(-\mu \notin \Gamma\). Hence for any \(z \in \Gamma\) we have

\[c|\mu + z| \geq 1.\]

We know that spectrum of operator \(R\) is contained in \(\Gamma\) and that \(R\) is normal operator, i.e. there exists a representation in which it is a multiplication by a \(\Gamma\)-valued function operator. Hence for any \(\psi \in D(\mu i + R)\)

\[c||\mu i + R\psi|| \geq ||\psi||,\]

so for any \(f \in D(Q)\) we have

\[c||Qf|| \geq ||Sf||.\] (28)

Consider a sequence \(\phi_n\) of elements of \(D(Q)\) norm-converging to \(\phi\) and such that \(Q\phi_n\) is norm-converging to certain \(y\). Since \(D(Q) \subset D(S)\) and by inequality (28), we conclude that also \(\phi \in D(S)\) and \(S\phi_n\) is norm-converging to \(S\phi\), as operator \(S\) is normal, so it is closed. Since \(\phi_n\) belongs to the domain of operator \(Q\), hence it belongs also to the domain of operator \(RS\). Moreover, \(RS\phi_n\) is converging to \(y - \mu S\phi\). As we know that operator \(RS\) is closed we see that \(\phi\) belongs also to the domain of \(Q\) and \(y = RS\phi + \mu S\phi = Q\phi\). It means that operator \(Q\) is closed.

By definition, the domain of operator \(Q^*\) consists of all \(x \in \mathcal{H}\) such that there exists \(w \in \mathcal{H}\) such that for any \(y \in D(Q)\) we have

\[\langle x, Qy \rangle = \langle w, y \rangle.\] (29)

Set

\[f(r) = \mu + q^{-2}r.\]

Observe that function \(f\) belongs to space \(H\). With this notation

\[Q = f(q \cdot qR) \circ S.\]

Since \(D_o\) is a core for \(Q\), one can assume in formula (24) that \(y \in D_o\). Every element \(y \in D_o\) has form \(y = e^{-\lambda(R)^2}z\) for certain \(\lambda > 0\) and \(z \in D(S)\). Hence

\[\langle x, Qy \rangle = \langle x|f(q \cdot qR)Se^{-\lambda(R)^2}z \rangle\]

Since function \(e^{-\lambda(R)^2} \in H_{\text{bounded}}\), (3) shows that

\[\langle x, Qy \rangle = \langle x|f(q \cdot qR)e^{-\lambda(q^{-2}R)^2}Sz \rangle.\]
Moreover, since operator $f(q \cdot q R)e^{-\lambda q^2}$ is bounded, we see that the domain of its adjoint operator is the whole space $H$. Hence by (32)

$$\langle x, Qy \rangle = \langle f^*(R^*) e^{-\lambda (R^*)^2} x | Sz \rangle = \langle f^*(R^*) x | e^{-\lambda q^2} S z \rangle.$$  

Consequently, if vector $x \in H$ belongs to the domain $Q^*$, then

$$\langle \pi i + R^* x | e^{-\lambda (R^*)^2} x \rangle \in D(S^*) = D(S).$$  

(30)

Moreover

$$S^*(\pi i + R^*) e^{-\lambda (R^*)^2} x = e^{-\lambda (R^*)^2} Q^* x.$$  

(31)

Let

$$Q' = (Q^*|_{D(Q)})^*.$$  

On account of Proposition (3.4) point 2

$$Q^* = (f(q \cdot q R) \circ S)^* = S^* f^*(R^*).$$

We will prove that

$$Q^*|_{D(Q)} = f(q \cdot q R)^* \circ S^*.$$  

(32)

Vector $x \in D(Q)$ belongs to the domain of operator $Q^*|_{D(Q)}$, if for any $y \in D(Q)$ there exists $w \in H$ such that

$$\langle x | Qy \rangle = \langle x | y \rangle.$$  

We know, that $D_o$ is a core for $Q$. Therefore we can set $y \in D_o$ in the above formula. Every element $y \in D_o$ has form

$$y = e^{-\lambda(R)^2} z,$$

for any $\lambda > 0$ and $z \in D(S)$. Hence

$$\langle x, Qy \rangle = \langle x, Qf(q \cdot q R)S e^{-\lambda(R)^2} z \rangle.$$  

Since $z \in D(S)$ and $e^{-\lambda(R)^2} \in H_{bounded}$, formula (23) implies

$$\langle x, f(q \cdot q R) e^{-\lambda(q \cdot q R)^2} S z \rangle.$$  

Moreover function $f(\cdot)e^{-\lambda(\cdot)^2} \in H_{bounded}$, so using once more (32) we obtain

$$\langle x, S f(R) e^{-\lambda(R)^2} z \rangle.$$  

Since $x \in D(Q)$, it follows that $x \in D(S) = D(S^*)$. Hence

$$\langle x, S f(R) e^{-\lambda(R)^2} z \rangle = \langle S^* x | f(R) e^{-\lambda(R)^2} z \rangle.$$  

Moreover $Sx \in D(f(q \cdot q R)) = D(f(R)^*)$. It is easily seen that the domain of $f(R)^*$ is invariant with respect to the action of operator Phase $S$. Hence also $S^* x \in D(f(R)^*)$. Consequently

$$\langle S^* x | f(R) e^{-\lambda(R)^2} z \rangle = \langle f(R)^* S^* x | e^{-\lambda(R)^2} z \rangle = \langle f(R)^* S^* x | y \rangle.$$  

Thus we proved (32).
We proceed to derive condition which \( x \in H \) has to satisfy to belong to the domain of \( Q' \). By definition, \( x \in D(Q') \) if for any \( y \in D(Q) \)

\[
\langle x|Q^*y \rangle = \langle Q'x|y \rangle .
\]

By (32) and (13)

\[
\langle x|Q^*y \rangle = \langle x|f^*(R^*) \circ S^*y \rangle .
\]

On account of (3.4) point 1, \( D_o \) is a core of operator \( Q^*|_{D(Q)} \). Hence we may assume that \( y \in D_o \). Every element \( y \in D_o \) has form

\[
y = e^{-\lambda (R^*)^2}z,
\]

for any \( \lambda > 0 \) and \( z \in D(S^*) \). Hence

\[
\langle x, Q^*y \rangle = \langle x|f^*(R^*)S^*e^{-\lambda (R^*)^2}z \rangle .
\]

By (9), as previously, we obtain

\[
\langle x, Q^*y \rangle = \langle e^{-\lambda (R^*)^2}f(q \cdot qR)S^*x \rangle .
\]

Moreover, since operator \( f^*(R^*)e^{-\lambda (qR^*)^2} \) is bounded, the domain of its adjoint is the whole Hilbert space \( H \).

Hence by (9)

\[
\langle x, Q^*y \rangle = \langle e^{-\lambda (R^*)^2}f(q \cdot qR)x|S^*z \rangle .
\]

From this we conclude that if \( x \in D(Q') \), then

\[
(\mu I + R)e^{-\lambda (R^*)^2}x \in D(S) .
\]

(33)

Moreover

\[
S(\mu I + R)e^{-\lambda (R^*)^2}x = e^{-\lambda (R^*)^2}Q'x .
\]

(34)

Observe also that by (11) for any \( z \in \Gamma \)

\[
\ell^*(z) = \ell^*(z) .
\]

It implies that

\[
e^{-\lambda (R^*)^2} = e^{-\lambda (R^*)^2},
\]

for any \( \lambda > 0 \). This result allows one to replace (30) with a more convenient, as we will soon see, condition

\[
(\pi I + R^*)e^{-\lambda (R^*)^2}x \in D(S) .
\]

(35)

Assume that \( x \in D(Q^*) \cap D(Q') \). It means that \( x \) satisfies simultaneously (35) and (33), i.e.

\[
(\pi I + R^*)e^{-\lambda (R^*)^2}x \in D(S) \text{ and } (\mu I + R)e^{-\lambda (R^*)^2}x \in D(S) .
\]

Because the domain of operator \( S \) is invariant with respect to the action of the phase of \( R \), the second condition yields that

\[
(\pi(\text{Phase } R)^2 + R^*)e^{-\lambda (R^*)^2}x \in D(S) .
\]

(36)
Since a domain of any linear operator is a linear subspace of $\mathcal{H}$, hence by (35) and (36) it follows that

$$
(\pi(\text{Phase } R)^2 - \mu I)e^{-\lambda \ell^* (R)^2} x \in D(S).
$$

(37)

Let $\rho$ be the $N$th primitive root of unity, i.e. let $\rho \in \mathbb{C}$ and $\rho^N = 1$. Let function $h$ be defined on the set of $N$th roots of unity and be given by

$$
h(\rho) = \mu \rho^2 - \mu.
$$

Note that $h(\rho) \neq 0$, since in order to have equality $\pi q^{2k} = \mu$, the phase of $\mu$ would have to be equal to the multiply of $\frac{2\pi}{N}$. But this contradicts our assumption that $\mu \notin \Gamma$.

We know that Phase $R$ is a unitary operator which eigenvalues are all $N$th roots of unity. Consequently the operator $h(\text{Phase } R) = \mu \text{Phase } R$ is invertible. Moreover it is easily seen that the operator $h^{-1}(\text{Phase } R)$ is given by

$$
h^{-1}(\text{Phase } R) = \sum_{k=0}^{N-1} a_k \text{Phase } R^k,
$$

where $a_0, a_1, \ldots, a_{N-1} \in \mathbb{C}$.

Therefore, because $h(\text{Phase } R) e^{-\lambda \ell^* (R)^2} x \in D(S)$ and the domain of the operator $S$ is invariant with respect to the action of the phase of operator $R$, it follows that

$$
h^{-1}(\text{Phase } R) h(\text{Phase } R) e^{-\lambda \ell^* (R)^2} x \in D(S),
$$

and finally

$$
e^{-\lambda \ell^* (R)^2} x \in D(S) \tag{38}
$$

for any $\lambda > 0$.

Let $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}_+$. By (34)

$$
S e^{-\lambda \ell^* (R)^2} x = e^{-\lambda_1 \ell^* (q \circ q R)^2} S e^{-\lambda_2 \ell^* (R)^2} x.
$$

Moreover the operator $Re^{-\lambda \ell^* (q \circ q R)^2}$ is bounded, hence for any $y \in \mathcal{H}$

$$
e^{-\lambda_1 \ell^* (q \circ q R)^2} y \in D(R).
$$

Moreover

$$(\mu + \cdot) e^{-\lambda_1 \ell^*(\cdot)^2} \in H_{\text{bounded}}.
$$

Hence by (33) for any $x \in \mathcal{H}$ we have

$$
S(\mu I + R) e^{-\lambda \ell^* (R)^2} x = (\mu I + q^2 R) e^{-\lambda_1 \ell^* (q \circ q R)^2} S e^{-\lambda_2 \ell^* (R)^2} x = (\mu I + q^2 R) S e^{-\lambda \ell^* (R)^2} x. \tag{39}
$$

On the other hand by (32)

$$
S(\mu I + R) e^{-\lambda \ell^* (R)^2} x = e^{-\lambda (\ell^*(R)^*^2)} Q' x.
$$

(40)

Combining (33) and (30) we obtain

$$
(\mu I + q^2 R) S e^{-\lambda \ell^* (R)^2} x = e^{-\lambda (\ell^*(R)^*^2)} Q' x. \tag{41}
$$
It has been already proved that the operator $Q = (\mu I + R) \circ S$ is closed. Therefore on account of remark 3.6 the operator $(\mu I + q^2 R) \circ S$ is also closed. Hence setting $\lambda \to 0^+$ into formula 41 yields

$$x \in D((\mu I + q^2 R) \circ S) = D((\mu I + R) \circ S) = D(Q).$$

We have proved that if $x \in D(Q^*) \cap D(Q')$, then $x \in D(Q)$. Rephrasing, we have proved that

$$D(Q) \supset D(Q^*) \cap D(Q').$$

On the other hand, from the definitions of operators $Q^*$ and $Q'$ it is obvious that

$$D(Q) \subset D(Q^*) \cap D(Q').$$

Consequently, we have proved that

$$D(Q) = D(Q^*) \cap D(Q'). \quad (42)$$

Note that the domain of $Q^*$ is strictly greater than this of $Q$.

In order to give an example of a function $\psi \in L^2(\Gamma)$, such that $\psi \in D(Q^*) \setminus D(Q)$, let us assume that $(R^*, S^*)$ is the Schrödinger pair (see Example 3.5). Then operator $Q^*$ acts on functions $\psi$ from its domain as follows

$$(Q^* \psi)(z) = (S^*([\pi + z] \psi))(z).$$

Let function $\psi$ be for any $z \in \Gamma$ given by

$$\psi(z) = e^{-\ell(z)^2} \frac{1}{z + \bar{\pi}}.$$

Observe that $\psi \in L^2(\Gamma)$, since $\mu \notin \Gamma$. Moreover the function $\tilde{(\pi + \cdot)} \psi = e^{-\ell(\cdot)^2}$ corresponding to the function $(\pi + \cdot) \psi$ in the sense described in Example 3.5, is holomorphic and bounded inside $\Lambda_0 \times q^2$. Moreover, one can easily check that this function satisfies also other conditions of belonging to the domain of $|S|$ given in Example 3.5. Moreover, $Q^* \psi \in L^2(\Gamma)$. Consequently, $\psi$ belongs to the domain of $Q^*$.

However, $\psi$ does not belong to the domain of operator $Q$, since it does not belong to the domain of $|S|$. Function $\psi$ does not belong to the domain of $|S|$, since its corresponding function $\psi$ is meromorphic (but it is not holomorphic). Namely, function $\tilde{\psi}$ has a simple pole at the point $z = -\bar{\pi}$. It shows that $D(Q) \subseteq D(Q^*)$ and $Q$ is closed, so $Q$ is not normal.

We repeat now S.L. Woronowicz’s reasoning from the proof of Theorem 2.2 [25]. Assume, that $\tilde{Q}$ is a normal extension of $Q$. Then

$$D(Q) \subset D(\tilde{Q}) = D(\tilde{Q}^*) \subset D(Q^*). \quad (43)$$

On the other hand

$$Q^*|_{D(Q)} \subset \tilde{Q}^* \subset Q^*,$$

hence

$$D(\tilde{Q}) \subset D(Q'), \quad (44)$$
since \( Q' = (Q^*|_{D(Q)})^* \).

Using (42), (43) and (44) we obtain that
\[
D(\tilde{Q}) \subset D(Q).
\]

But by assumption \( \tilde{Q} \) is a normal extension of \( Q \), so \( \tilde{Q} = Q \).

This last result implies that if \( Q \) is not normal, it has not a normal extension. We recall that we have already proved that \( Q \) is normal.

We have thus proved that if \( \mu \notin \Gamma \) then the operator \( \mu S + RS \) has no normal extension, which completes the proof. \( \square \)

6 Solutions of the scalar exponential equation

We proceed to the proof of the main theorem of this Section.

**Theorem 6.1** Let \((R, S) \in DH\) and let \( f : \Gamma \to S^1 \) be a Borel function. The following conditions are equivalent

1). \( f(R)f(S) = f(R S) \) (45)

2). \( f(z) = F_N(\gamma z) \) (46)

for any \( \gamma \in \Gamma \) and almost all \( z \in \Gamma \).

**Proof:** 1). \( \Leftarrow \) 2). By Remark 3.6 we know \((\gamma R, \gamma S) \in DH\) for any \( \gamma \in \Gamma \). Moreover \( \gamma R \gamma S = \gamma (R S) \). Hence by Proposition 4.2
\[
F_N(\gamma R)F_N(\gamma S) = F_N(\gamma (R S)),
\]
which shows that function (46) satisfies 1..\( \)

2). \( \Rightarrow \) 1). Applying Proposition 4.4 to the right-hand-side of (45) we obtain
\[
f(R)f(S) = F_N(S^{-1}R)^*f(S)F_N(S^{-1}R).
\]

Let \( \lambda > 0 \) be an arbitrary real positive number. Then by Remark 3.6 it follows that \((\lambda R, \lambda S) \in DH\). Substituting \( \lambda R \) instead of \( R \) into equation (47) and setting \( T = S^{-1}R \) we obtain
\[
f(\lambda R)f(S) = F_N(\lambda T)^*f(S)F_N(\lambda T).
\]

Equivalently
\[
f(\lambda R)f(S) - f(S) = F_N(\lambda T)^*f(S)F_N(\lambda T) - F_N(\lambda T)^*f(S) + F_N(\lambda T)^*f(S) - f(S).
\]

(Note the all the operators above are bounded, so adding and subtraction them do not change theirs domains. However, if the operators involved would be unbounded, one could get a false inequality, because such a procedure could change domains of operators.)

Dividing both sides by \( \lambda > 0 \) we get
\[
\frac{f(\lambda R) - I}{\lambda}f(S) = F_N(\lambda T)^*f(S)F_N(\lambda T) - I + \frac{F_N(\lambda T)^* - I}{\lambda}f(S) \quad \text{(48)}
\]
Let \( RHS \) and \( LHS \) denote right and and left hand side of (48) and let \( x, z \in D(T) \). Then
\[
\lim_{\lambda \to 0} \left( z \left| RHS \right| x \right) = \frac{-i}{2\sin h} (z \left| f(S)(qT + \overline{q}T^*) \right| x) + \frac{i}{2\sin h} ( (qT + \overline{q}T^*) z \left| f(S) \right| x). \tag{49}
\]

By the reasoning above

**Observation 6.2** *For any \( x, z \in D(T) \) there exists the limit*
\[
\lim_{\lambda \to +0} \left( z \left| f(\lambda R) - I \right| \lambda \right) f(S)x .
\]

We will use the Lemma below to write this limit in the form convenient for future computation

**Lemma 6.3 (Lemma 7.2 z [23])** Let \( f \) be a bounded Borel function on \( \mathbb{R}_+ \) and let \( Y \) be a selfadjoint positive operator acting on Hilbert space \( \mathcal{H} \). Let \( \Xi \) be the set of pairs \((z, x) \in D(Y) \times \mathcal{H}\) such that there exists the limit
\[
\lim_{\lambda \to +0} \left( z \left| f(\lambda Y) - I \right| \lambda \right) x .
\]

If there is a pair \((y, u) \in \Xi \) such that \((Yy|u) \neq 0\), then there is also a constant \( \mu_f \) such that
\[
\lim_{\lambda \to +0} \left( z \left| f(\lambda Y) - I \right| \lambda \right) x = \mu_f(Yz|x)
\]
for any \((z, x) \in \Xi\), and \( \mu_f \) depends only on \( f \).

Note the the Lemma above can be applied to the function \( f \) mentioned in Theorem 6.1 and satisfying condition \([\mathbb{R}]\) (because this function is by assumption Borel and bounded) and to the operator \( Y = |R| \), which is clearly selfadjoint and positive.

Let \( \mathcal{H}_k \) denote an eigenspace of Phase \( R \) corresponding to the eigenvalue \( q^k \). We have the following decomposition of the Hilbert space \( \mathcal{H} \):
\[
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_N .
\]

The restriction of Operator \( R \) to the space \( \mathcal{H}_k \) is \( q^k|R| \). (Matrix representation of phases of operators \((R, S) \in D_\mathcal{H} \) is discussed in Section 7 and given by \((60) \) and \((61) \).) By Lemma 6.3 applied independently in every space \( \mathcal{H}_k \) to the function \( f(q^k) \) and the operator \( |R| \) and by Observation 6.2 it follows that
\[
\lim_{\lambda \to 0} \left( z \left| f(\lambda k |R|) - I \right| \lambda \right) y = \frac{-i}{2\sin h} \left( \mu(q^k) |R| z \right| y) , \tag{50}
\]
where \( z \in D(R) \cap D(T) \) and \( y \in f(S)D(T) \) and \( \mu \) is a complex-valued function defined on the set of all \( N \)th roots of unity. Observe that the function \( \mu \) is determined uniquely by complex numbers \( b_1, b_2, ..., b_N \) such that
\[
\mu(\rho) = b_1 + b_2\rho + b_3\rho^2 + ... + b_N\rho^{N-1} , \tag{51}
\]
where \( \rho \in \mathbb{C} \) and \( \rho^N = 1 \). Note that \( \rho \in \Gamma \).

To make further computations easier we adopt different \( \mu \) in order to have \( \frac{-i}{2\sin h} \) before parentheses in (50).
Substituting \( y = f(S)x \), where \( x \in D(T) \), to (50)

\[
\lim_{\lambda \to 0} \left( z \left| \frac{f(\lambda R) - I}{\lambda} f(S)x \right| \right) = \lim_{\lambda \to 0} \left( z \left| LHS \right| x \right) = \frac{-i}{2\sin h} (\mu(\text{Phase} R)^*|R|z | f(S)x) .
\]  

Comparing (52) and (49) we conclude that

\[
(\mu(\text{Phase} R)^*|R|z | f(S)x) = (z | f(S)(qT + \overline{T}^*) x) - ((qT + \overline{T}^*)z | f(S)x) ,
\]

for any \( z \in D(R) \cap D(T) \) and \( x \in D(T) \).

From Remark 3.3 it follows that if operators \((R, S) \in D_H \) satisfy the above equation they satisfy also \( (\rho R, S) \in D_H \). Let us substitute \( \rho R \) instead of \( R \) to the equation \((53)\) and write function \( \mu \) in the form \( \gamma R \) is normal. It means that the operator \( \gamma R \) has a normal extension. Proposition 5.1 implies that in such a case \( \gamma \in \Gamma \). Consequently, by Remark 3.6 we have \((T, \gamma R) \in D_H \). By Theorem 2.4 \( \overline{T} \) the closure of \( \gamma R + T \), denoted by \( \gamma R + T \), is normal for any \( \gamma \in \Gamma \). Moreover, Proposition 4.2 yields

\[
\gamma R + T \subset (f(S)T^* f(S))^* = f(S)T^* f(S) .
\]

The operator on the right-hand-side is unitarily equivalent to a normal operator, so it is normal. It means that the operator \( \gamma R + T \) has a normal extension. Proposition 5.1 implies that in such a case \( \gamma \in \Gamma \). Consequently, by Remark 3.6 we have \((T, \gamma R) \in D_H \). By Theorem 2.4 \( \overline{T} \) the closure of \( \gamma R + T \), denoted by \( \gamma R + T \), is normal for any \( \gamma \in \Gamma \). Moreover, Proposition 4.2 yields

\[
\gamma R + T = F_N(\gamma S)T F_N(\gamma S)^* .
\]

Because by Proposition 5.1 \( \gamma R + T \) has a normal extension and by Theorem 2.4 \( \overline{T} \) this normal extension of \( \gamma R + T \) is \( \gamma R + T \), by (53) and since normal operators do not have normal extensions, one may conclude that

\[
\gamma R + T = f(S)T^* f(S)^* .
\]
Comparing (57) and (58) we obtain

\[ F_N(\gamma S)TF_N(\gamma S)^* = f(S)Tf(S)^* . \]

Hence

\[ TF_N(\gamma S)^* f(S) = F_N(\gamma S)^* f(S)T , \]

so \( T \) commutes with a bounded operator \( F_N(\gamma S)^* f(S) \). By spectral theorem for normal operators [18, Theorem 13.33] it follows that \( F_N(\gamma S)^* f(S) \) commutes also with functions \(|T|^i t\), where \( t \in \mathbb{R} \). Hence

\[ |T|^i t F_N(\gamma S)^* f(S)|T|^{-i t} = F_N(\gamma S)^* f(S) , \]

so

\[ F_N(\lambda \gamma S)^* f(\lambda S) = F_N(\gamma S)^* f(S) , \quad (58) \]

where \( \lambda = e^{-ht} \). Considerations preceding Lemma 13 imply that for any \((T,R) \in D_H\)

\[ \lim_{\lambda \to 0} (z | f(\lambda R) \xi) = (z | \xi) , \]

for any \( z \in D(T) \) and \( \xi \in f(S)D(T) \). Since \((T,S) \in D_H\), hence we obtain

\[ \lim_{\lambda \to 0} (\zeta | F_N(\lambda \gamma S)^* f(\lambda S) \xi) = (\zeta | \xi) , \]

where \( \zeta \in F_N(\gamma S)^* D(T) \) and \( \xi \in f(S^{-1})D(T) \). Note that \( D(T) \) is linearly dense in \( \mathcal{H} \), as the operator \( T \) is densely defined. Similarly, we see that \( f(S^{-1})D(T) \) and \( F_N(\gamma S)^* D(T) \) are densely defined, since \( f(S^{-1}) \) and \( F_N(\gamma S) \) are unitary. Hence \( F_N(\lambda \gamma S)^* f(\lambda S) \) converges weakly to \( I \), when \( \lambda \) goes to 0. Since the right-hand-side of (58) is independent on \( \lambda \), it follows that

\[ F_N(\lambda \gamma S)^* f(\lambda S) = I , \]

so

\[ f(S) = F_N(\gamma S) , \]

where \( \gamma \in \Gamma \), which completes the proof.

\[ \square \]

7 Solutions of the general exponential equation for the quantum "az+b" group

We prove now the generalization of the Theorem 6.1 to the case of an operator-valued function \( f \). Our earlier results from [15, 14] will make this proof much simpler.

Let \( f \) be a function defined on \( \Gamma \) and such, that for any \( z \in \Gamma \), \( f(z) \) is a unitary operator acting on a Hilbert space \( \mathcal{K} \), i.e. \( f(z) \in \text{Unit} (\mathcal{K}) \). We will call \( f \) a Borel function iff for any \( \varphi, \psi \in \mathcal{K} \) the function

\[ z \to (\varphi | f(z) \psi) \]

is Borel. Then for any normal operator \( R \) with spectrum contained in \( \Gamma \) we define function \( f(R) \) by

\[ f(R) = \int_{\Gamma} f(z) \otimes dE_R(z) , \]

where \( dE_R \) is the spectral measure of the operator \( R \) and \( f(z) \) is a unitary operator acting on a Hilbert space \( \mathcal{K} \).
Theorem 7.1 Let \( f \) be a Borel function defined on \( \Gamma \) with values in unitary operators acting on \( \mathcal{K} \) and let \((R, S) \in \mathcal{D}_H\). Then

\[
\left( f(R)f(S) = f(R\hat{+}S) \right) \iff \begin{cases} 
\text{there exists an invertible normal operator } M, \\
\text{such that } \text{Sp}M \subset \Gamma \\
\text{and } f(z) = F_N(Mz) \\
\text{for a. a. } z \in \Gamma.
\end{cases}
\]

Proof: \( \Leftarrow \) If \( f(z) = F_N(Mz) \), then \( f(R) = F_N(M \otimes R) \), where

\[
F_N(M \otimes R) = \int_{\Gamma} F_N(Mz) \otimes dE_R(z),
\]

where \( dE_R \) is the spectral measure of \( R \). By assumption \((R, S) \in \mathcal{D}_H\) and \( M \) is an invertible normal operator such that \( \text{Sp}M \subset \Gamma \). Let us set \( R' = M \otimes R \) and \( S' = M \otimes S \). Then \((R', S') \in \mathcal{D}_{H \otimes K}\). Therefore by Theorem 5.1 we get

\[
F_N(M \otimes R)F_N(M \otimes S) = F_N(M \otimes (R\hat{+}S)).
\]

\( \Rightarrow \) We will use the same method as in the proof of Theorem 2.6 \([5]\), i.e. we show that

\[
f(z)f(x) = f(x)f(z),
\]

for any \( z, x \in \Gamma \). For the Reader’s convenience we recall why it is enough to prove this. Observe that if \( \dim \mathcal{K} = k < \infty \), then from commutation of unitary operators \( f(z) \) and \( f(x) \) follows that there exists orthonormal basis in which these operators are represented by diagonal matrices for any \( x, y \in \Gamma \). Thus the problem reduces to finding solutions of \( k \) scalar equations

\[
f_o(R)f_o(S) = f_o(R\hat{+}S),
\]

where \( f_o \) is a complex-valued function defined on \( \Gamma \). The above reasoning can be generalized to the case of arbitrary many dimensional separable Hilbert space \( \mathcal{K} \). This is so because operators \( f(q^k r) \) and \( f(q^l s) \) belong to a commutative \(*\)-subalgebra of \( B(\mathcal{K}) \). Therefore, by spectral theorem and its consequences \([4\text{, Chapter X}]\), operators \( f(q^k r) \) and \( f(q^l s) \) have the same spectral measure

\[
f(q^k r) = \int_0^{2\pi} f_o(q^k r, t) dE_K(t) \quad \text{and} \quad f(q^l s) = \int_0^{2\pi} f_o(q^l s, t) dE_K(t).
\]

Hence

\[
f(R) = \int_{\Gamma} \int_0^{2\pi} f_o(q^k r, t) dE_K(t) \otimes dE_R(z)
\]

and

\[
f(S) = \int_{\Gamma} \int_0^{2\pi} f_o(q^l s, t) dE_K(t) \otimes dE_S(z),
\]

where the function \( f_o \) is complex-valued. Thus Theorem 7.1 reduces to the already proved "scalar" Theorem 5.1.

We proceed to prove that really for any \( r, s \in \mathbb{R}_+ \) and \( k, l = 0, 1, \ldots, N - 1 \) we have

\[
f(q^k r)f(q^l s) = f(q^l s)f(q^k r). \tag{59}
\]
Observe that operators Phase $R$ and Phase $S$ have in a certain basis the following matrix representation

$$
\text{Phase } R = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & q & 0 & \ldots & 0 \\
0 & 0 & q^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & q^{N-1}
\end{bmatrix}
$$

(60)

and

$$
\text{Phase } S = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}.
$$

(61)

Let $T = S^{-1}R$. Then

$$
\text{Phase } T = q^{-\frac{1}{2}}(\text{Phase } S)^*\text{Phase } R,
$$

so

$$
\text{Phase } T = q^{-\frac{1}{2}}\begin{bmatrix}
0 & 0 & \ldots & 0 & q^{N-1} \\
1 & 0 & \ldots & 0 & 0 \\
0 & q & \ldots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \ldots & q^{N-2} & 0
\end{bmatrix}.
$$

(62)

We use the notion of generalized eigenvectors. It is well known that a selfadjoint operator with continuous spectrum acting on $\mathcal{H}$ does not have eigenvectors. Still one can show that in the general case the generalized eigenvectors are continuous linear functionals on a certain dense locally convex subspace $\Phi \subset \mathcal{H}$, provided with a much stronger topology than $\mathcal{H}$. Then we get the same formulas as for discreet spectrum provided we replace scalar product by the duality relation between $\Phi$ and $\Phi'$. This will be explained by the example below, for general considerations see [10].

**Example 7.2** Let $\mathcal{H} = L^2(\mathbb{R})$ and

$$
|R| = e^\beta \quad \text{and} \quad |S| = e^{\tilde{\beta}} \quad \text{and} \quad |T| = e^{\frac{i\hbar}{2}}|S|^{-1}|R| = e^{\tilde{\beta}-\tilde{\gamma}}.
$$

(63)

These operators have continuous spectra, so they do not have eigenvectors. There are however tempered distributions on $\mathbb{R}$ such that for every function $f$ from the Schwartz space of smooth functions on $\mathbb{R}$ decreasing rapidly at infinity $S(\mathbb{R})$ we have

$$
\langle f | R | \Omega_r \rangle = r \langle f | \Omega_r \rangle \quad \text{and} \quad \langle f | S | \Phi_s \rangle = s \langle f | \Phi_s \rangle \quad \text{and} \quad \langle f | T | \Psi_t \rangle = t \langle f | \Psi_t \rangle.
$$

(64)

Such $|\Omega_r\rangle$, $|\Phi_s\rangle$ and $|\Psi_t\rangle$ are called generalized eigenvectors of operators $|R|$, $|S|$ and $|T|$ with eigenvalues respectively $r$, $s$ and $t$.

An example of generalized eigenvectors of operators (63) is

$$
|\Omega_r\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i\pi}{2}x \log r} \quad \text{and} \quad |\Phi_s\rangle = \delta(\log s - x) \quad \text{and} \quad |\Psi_t\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i\pi}{2}x \log t}.
$$

Moreover, we will use notation of a type $\langle \Omega_r | \Phi_s \rangle$. 

25
It should be understood in the following way: for any \( f \in \mathcal{S}(\mathbb{R}) \) we have
\[
\langle \Omega_r | f \rangle = \int_{\mathbb{R}} \langle \Omega_r | \Phi_s \rangle \langle \Phi_s | f \rangle ds .
\] (65)

To shorten notation from now on we skip the integration symbol, i.e. we write
\[
\langle \Omega_r | f \rangle = \langle \Omega_r | \Phi_s \rangle \langle \Phi_s | f \rangle .
\] instead of (65)

The generalized eigenvectors \( \Omega \) are said to have the Dirac \( \delta \) normalization if
\[
\langle \Omega_r | \Omega_s \rangle = \delta(r - s) ,
\]
where \( \delta \) is the Dirac \( \delta \) distribution. Note that generalized eigenvectors \( \Omega, \Phi \) and \( \Psi \) given above have the Dirac \( \delta \) normalization.

Let \( | \Omega_r \rangle \) be a generalized eigenvector of \( R \) with real eigenvalue \( r \) and with Dirac delta normalization. Analogously, let \( | \Phi_s \rangle \) and \( | \Psi_t \rangle \) denote generalized eigenvectors of operators \( S \) and \( T \) with real eigenvalues respectively \( s \) and \( t \) and with the Dirac delta normalization.

Let us define \( | \Omega_{k,r} \rangle \) using the vector \( | \Omega_r \rangle \) introduced above
\[
| \Omega_{k,r} \rangle = e_k \otimes | \Omega_r \rangle
\]
where and \( k = 0,1,\ldots,N - 1 \) and
\[
e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ,
\]
\[
e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ,
\]
\[
\ldots,
\]
\[
e_{N-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .
\]

Similarly, we define the vector \( | \Phi_{l,s} \rangle \) using the vector \( | \Phi_s \rangle \) introduced above
\[
| \Phi_{l,s} \rangle = f_l \otimes | \Phi_s \rangle
\]
where
\[
f_l = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ q_l \\ q^{2l} \\ \vdots \\ q^{(N-1)l} \end{bmatrix}
\]
and \( l = 0,1,\ldots,N - 1 \).

Analogously, the vector \( | \Psi_{m,t} \rangle \) is given by
\[
| \Psi_{m,t} \rangle = g_m \otimes | \Psi_t \rangle
\]
where
\[
(g_m)_p = \frac{1}{\sqrt{N}} q^{\frac{1}{2}(p^2 - 2p(m + 1))}
\]
and \( m, p = 0,1,\ldots,N - 1 \).
Note that $|\Omega_{k,r}|$, $|\Phi_{l,s}|$ and $|\Psi_{m,t}|$ are generalized eigenvectors of $R$, $S$ and $T$, respectively, corresponding to generalized eigenvalues $q^k r, q^l s$ and $q^m t$.

In order to prove (59) we compute matrix elements
\[
(\Omega_{k,r}|f(R)f(S)|\Phi_{l,s}) .
\]

Since the function $f$ satisfies exponential equation, is Borel and by (4.1)
\[
f(R)f(S) = f(R+s) = F_N(T)^* f(S)F_N(T) .
\]

Therefore
\[
(\Omega_{k,r}|f(R)f(S)|\Phi_{l,s}) = (\Omega_{k,r}|F_N(T)^* |\Psi_{m,t}|f(S)|\Phi_{n,s}|\Phi_{n,s}|F_N(T)|\Psi_{o,t}|\Phi_{p,t}) .
\]

Hence
\[
f(q^k r)f(q^l s) = (\Omega_{k,r}|\Phi_{l,s})^{-1}(\Omega_{k,r}|F_N(T)^* |\Psi_{m,t}|f(S)|\Phi_{n,s}|\Phi_{n,s}|F_N(T)|\Psi_{o,t}|\Phi_{p,t}) .
\]

It is easily checked that
\[
(\Omega_{k,r}|\Phi_{l,s}) = q^{kl} (\Omega_{r}|\Phi_{s})
\]
\[
(\Omega_{k,r}|\Psi_{m,t}) = \frac{1}{N} q^{(k^2 - 2k(m+1))} (\Omega_{r}|\Psi_{t})
\]
\[
(\Psi_{m,t}|\Phi_{n,s}) = (\Psi_{t}|\Phi_{s}) \frac{1}{N} \left( \sum_{p=0}^{N-1} q^{p(m+n+1-\frac{p}{2})} \right)
\]

As we see from the last formula, one has to compute for any integer $\alpha$ a sum
\[
\sum_{p=0}^{N-1} e^{\frac{2\pi i}{N} p(\alpha - \frac{p}{2})} = e^{\frac{N\pi i}{2}} \sum_{p=0}^{N-1} e^{-\frac{N\pi i}{N} (p-\alpha)^2} = e^{\frac{N\pi i}{2}} \sum_{p=0}^{N-1} e^{-\frac{N\pi i}{N} p^2} .
\]

In the appendix we derive the formula
\[\sum_{p=0}^{N-1} e^{-\frac{N\pi i}{N} p^2} = \sqrt{N} e^{-\frac{\pi i}{4}} \tag{66} \]

Hence
\[
\sum_{p=0}^{N-1} e^{\frac{2\pi i}{N} p (\alpha - \frac{p}{2})} = \sqrt{N} e^{\frac{N\pi i}{2} \alpha^2} e^{-\frac{\pi i}{4}}
\]

so
\[
(\Psi_{m,t}|\Phi_{n,s}) = \frac{1}{\sqrt{N}} e^{-\frac{\pi i}{4}} e^{\frac{N\pi i}{2} (m+n+1)^2} (\Psi_{t}|\Phi_{s})
\]

Using Proposition 1.1 [28] one can easily prove that

**Proposition 7.3** For any $m = 0, 1, \ldots, N - 1$ and $t \in R_+$ we have
\[
F_N(q^m t) = e^{\frac{\pi i}{N} (\frac{N}{2} + \frac{m}{N})} e^{-\frac{\pi i}{N} (m+1)^2} e^{\frac{4\pi}{N} \log t} F_N(q^{-m-2} t^{-1}) , \tag{67}
\]
By Proposition above
\[
\langle \Omega_{k,r}|F_N(T)^*|\Phi_{n,s} \rangle = e^{\frac{N}{2}(\frac{m}{2} + \frac{N}{2})} e^{-\frac{N}{2}(m+1)^2} e^{\frac{N}{2} \log^2 t} F_N(q^{-m-2}t^{-1}) \langle \Omega_{k,r}|\Psi_{m,t}|\Psi_{m,t}|\Phi_{n,s} \rangle = \\
= e^{\frac{N}{2}(\frac{m}{2} + \frac{N}{2})} e^{-\frac{N}{2}(m+1)^2} e^{\frac{N}{2} \log^2 t} F_N(q^{-m-2}t^{-1}) e^{-\frac{N}{4}(k^2-2k-2km+m^2+n^2+2mn+2n+2m+1)} \langle \Omega_r|\Psi_t|\Psi_t|\Phi_z \rangle = \\
= \frac{e^{-\frac{N}{4}}}{N} \langle \frac{m}{2} + \frac{N}{2} \rangle e^{\frac{N}{2} \log^2 t} e^{\frac{N}{2}(k^2+2k+2km+n^2+2mn+2n)} F_N(q^{-m-2}t^{-1}) \langle \Omega_r|\Psi_t|\Psi_t|\Phi_z \rangle.
\]
Inserting \(-m - 2\) in the place of \(m\), we get
\[
\langle \Omega_{k,r}|F_N(T)^*|\Phi_{n,s} \rangle = \frac{e^{-\frac{N}{4}}}{N} \langle \frac{m}{2} + \frac{N}{2} \rangle e^{\frac{N}{2} \log^2 t} e^{\frac{N}{2}(k^2+2k+2km+n^2+2mn+2n)} F_N(q^{-m}t) \langle \Omega_r|\Phi_z \rangle.
\]
Observed that by formula 1.36 [23], for any \(t \in \mathbb{R}\)
\[
V_\theta(\log t) = e^{-i\frac{N}{4}} e^{i\frac{N}{2} \log^2 t} V_\theta(-\log t),
\]
where \(c_\theta = e^{\frac{N}{4}}(\frac{m}{2} + \frac{N}{2})\).
Moreover, we have prove in [15] that
\[
\langle \Omega_r|V_\theta(\log T)^*|\Phi_s \rangle = c_\theta^{-1} e^{\frac{iN}{2} \log^2 T} \langle \Phi_s|V_\theta(\log T)|\Phi_r \rangle.
\]
By (28) and (29)
\[
e^{-\frac{N}{4}} e^{\frac{N}{2} \log^2 t} F_N(q^{-m}t^{-1}) \langle \Omega_r|\Psi_t|\Psi_t|\Phi_z \rangle = e^{-\frac{iN}{4}} e^{\frac{N}{2} \log^2 t} F_N(q^{-m}t) \langle \Phi_z|\Psi_t|\Psi_t|\Phi_r \rangle.
\]
Moreover
\[
\frac{1}{N} F_N(q^{-m}t) e^{\frac{N}{2}(k^2+2km+2k+n^2+2n)} = e^{i\frac{N}{2} \log^2 t} (f_{m}g_{m}) F_N(q^{-m}t) (g_{m}f_{k})
\]
Hence
\[
\langle \Omega_{k,r}|F_N(T)^*|\Phi_{n,s} \rangle = e^{\frac{N}{4}(\frac{m}{2} + \frac{N}{2})} e^{i\frac{N}{2} \log^2 T} e^{-\frac{iN}{4}} \langle \Phi_{n,s}|F_N(T)|\Phi_{k,r} \rangle.
\]
Therefore
\[
f(q^{k}r)f(q^{l}s) = q^{-kl} f(q^{n}s)^{-1} \langle \Omega_{k,r}|F_N(T)^*|\Phi_{n,s} \rangle \langle \Phi_{n,s}|F_N(T)|\Phi_{k,r} \rangle = \\
= e^{\frac{N}{4}(\frac{m}{2} + \frac{N}{2})} e^{i\frac{N}{2} \log^2 t} f(q^{n}s)^{-1} \langle \Phi_{n,s}|F_N(T)|\Phi_{k,r} \rangle \langle \Phi_{n,s}|F_N(T)|\Phi_{l,s} \rangle.
\]
The expression below is clearly symmetric with respect to swapping \(k \leftrightarrow l\) with \(r \leftrightarrow s\), which completes the proof. \(\square\)

A Deriving formula (69)

We will derive the formula
\[
\sum_{p=0}^{N-1} e^{\frac{N}{2} p^2},
\]
where \(N\) is a non-zero even number.
Let us introduce the notation
\[ a_p = e^{\frac{\pi i}{N} p^2} \quad \text{and} \quad S_N = \sum_{p=0}^{N-1} a_p . \]

In order to compute (71) we integrate the function
\[ f(z) = \frac{e^{\frac{\pi i}{N} z^2}}{e^{2\pi iz} - 1} \quad (72) \]
over the contour \( \Gamma \) as on the Figure 1.

Since \( N \) is an even number, it follows that for any \( z \in \mathbb{C} \) from the domain of \( f \) we have
\[ f(z + N) - f(z) = e^{\frac{\pi i}{N} z^2} . \quad (73) \]

Let us introduce the notation
\[ I_1 = \int_{-R}^{R} f(-\frac{1}{2} + iy)dy \quad \text{and} \quad I_2 = \int_{-\frac{1}{2}}^{N-\frac{1}{2}} f(x - iR)dx . \]
\[ I_3 = \int_{-R}^{R} f(N - \frac{1}{2} + iy)dy \quad \text{and} \quad I_4 = \int_{N-\frac{1}{2}}^{-\frac{1}{2}} f(x + iR)dx . \]

Observe that
\[ \int_{\Gamma} f(z)dz = I_1 + I_2 + I_3 + I_4 \quad (74) \]

Function \( f \) in the interior of \( \Gamma \) has simple poles at the points \( z = 1, 2, \ldots, \frac{N}{2} \), and residues at these points are
\[ \text{Res}_{z=p} f(z) = \frac{a_p}{2\pi i} . \]

Therefore by the Residue Theorem
\[ \int_{\Gamma} f(z)dz = 2\pi i \sum_{p=0}^{N-1} \text{Res}_{z=p} f(z) = a_0 + a_2 + \ldots + a_{N-1} = S_N . \quad (75) \]

Comparing (74) and (75) we get
\[ S_N = I_1 + I_2 + I_3 + I_4 . \quad (76) \]

We proceed now to calculate integrals \( I_2 \) and \( I_4 \).
\[ I_2 = \int_{-\frac{1}{2}}^{N-\frac{1}{2}} \frac{e^{\frac{\pi i}{N} (x-iR)^2}}{e^{2\pi i(x-iR)} - 1} \frac{2\pi R}{e^{2\pi i(x+iR)} - 1} \frac{dx}{e^{2\pi i(x+iR)} - 1} \]

Since
\[ |e^{\frac{\pi i}{N} (x^2-R^2)} e^{\frac{2\pi R}{N} Rx}| \leq e^{\frac{2\pi R}{N} Rx} \quad \text{and} \quad |e^{2\pi i(x+iR)} - 1| \geq e^{2\pi R} - 1 , \]

it follows that
\[ I_2 < \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{\frac{\pi i}{N} (x-iR)^2}}{e^{2\pi i(x+iR)} - 1} \frac{dx}{e^{2\pi i(x+iR)} - 1} \right| \leq \frac{1}{e^{2\pi R} - 1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{\frac{2\pi R}{N} Rx}}{e^{2\pi i(x+R/2)}} \frac{dx}{e^{2\pi R} - 1} = \frac{N e^{-\pi R}}{2\pi R} \frac{e^{2\pi R} - 1}{e^{2\pi R} - 1} = \frac{N e^{-\pi R}}{2\pi R} . \]
Hence $I_2$ converges to 0, when $R$ goes to $+\infty$. Using similar estimation one can prove that $I_4$ converges also to 0, when $R$ goes to $+\infty$. Moreover, using formula (73) we get

$$I_1 + I_3 = 2 \int_{-R}^{R} \left( f(N - \frac{1}{2} + it) - f(-\frac{1}{2} + it) \right) dy = 2 \int_{-R}^{R} e^{\frac{\pi i (t - \frac{1}{2})^2}{N}} dy = 2i \int_{-R}^{R} e^{-\frac{\pi i}{N} (t + \frac{1}{2})^2} dy .$$

When $R \to +\infty$, then $I_1 + I_3$ takes form

$$2i \int_{-\infty}^{+\infty} e^{-\frac{\pi i}{N} (t + \frac{1}{2})^2} dy = 2i \frac{\sqrt{N}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-i(t + \frac{1}{2})^2} dy$$

After passing to the limit $R \to +\infty$ into the formula (76) we get

$$S_N = 2i \frac{\sqrt{N}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-i(t + \frac{1}{2})^2} dy$$

It is easily seen that

$$S_2 = 1 + i .$$

Hence

$$\int_{-\infty}^{+\infty} e^{-iy^2} dy = \frac{\sqrt{\pi}}{2\sqrt{2i}} (i + 1) = -i \frac{\sqrt{\pi}}{2} e^{\frac{\pi i}{4}} .$$

Thus for any even $N \geq 2$ we have

$$\sum_{p=0}^{N-1} e^{\frac{\pi i p^2}{N}} = \sqrt{Ne^{\frac{\pi i}{4}}}$$
Acknowledgments

This is part of the author’s Ph.D. thesis, written under the supervision of Professor Stanislaw L. Woronowicz at the Department of Mathematical Methods in Physics, Warsaw University. The author is greatly indebted to Professor Stanislaw L. Woronowicz for stimulating discussions and important hints and comments. The author also wishes to thank Professor Wiesław Pusz and Professor Marek Bożejko for several helpful suggestions and Piotr Soltan for reading carefully the manuscript.

References

[1] Bohm, A.: Quantum mechanics: Foundations and Applications. Springer-Verlag, New York, 1986.

[2] Connes, A.: Noncommutative Geometry. Academic Press, 1995.

[3] Dixmier, J.: Les algèbres d’opérateurs dans l’espace Hilbertien, Gauthier-Villars, Paris, 1969.

[4] Dunford N., Schwartz, J.T.: Linear operators, Part II: Spectral Theory. Interscience Publishers, New York, London 1963.

[5] Klimyk A., Schmüdgen, K.: Quantum Groups and Their Representations. Springer-Verlag Berlin – Heidelberg 1997.

[6] Kruszyński P., Woronowicz, S.L.: A Non-commutative Gelfand-Naimark Theorem- J. Operator Theory 8, 361-389 (1982).

[7] Kustermans J., Vaes S.: A simple definition for locally compact quantum group C.R. Acad. Sci. Paris, Sér. and 328 (10), 871-876 (1999).

[8] Majid, S.: Foundations of Quantum Group Theory, Cambridge University Press, Cambridge 1995.

[9] Maurin, K.: Methods of Hilbert spaces, Warszawa 1967.

[10] Maurin, K.: General eigenfunction expansions and unitary representations of topological groups, Warszawa 1968.

[11] Pusz, W.: Quantum GL(2, C) group - in preparation.

[12] Reed M., Simon, B.: Methods of Modern Mathematical Physics, Part I. Academic Press, New York, San Francisco, London 1975.

[13] Reed M., Simon, B.: Methods of Modern Mathematical Physics, Part II. Academic Press, New York, San Francisco, London 1975.

[14] Rowicka - Kudlicka, M.: PhD Thesis, Warsaw University, Warsaw 2000.

[15] Rowicka - Kudlicka, M.: Braided quantum groups related to the quantum ’ax+b’ group -math.QA/0101003.

[16] Rowicka - Kudlicka, M.: Unitary representations of the quantum ’ax+b’ group -math.QA/0102151.
[17] Rowicka - Kudlicka, M.: Unitary representations of the quantum 'az+b' group at roots of unity - in preparation.

[18] Rudin, W.: Functional analysis, McGraw-Hill, Inc., 1991.

[19] van Daele, A.: The Haar measure on some locally compact quantum groups -in preparation.

[20] Woronowicz, S.L.: Pseudospaces, pseudogroups, and Pontryagin duality. Proceedings of the International Conference on Mathematical Physics, Lausanne 1979.

[21] Woronowicz, S.L.: Operator systems and their application to the Tomita-Takesaki theory, J. Operator Theory, 2(1979), 169-209.

[22] Woronowicz, S.L.: Duality in the C*-algebra a Theory. Proceedings of the International Congress of Mathematicians, Warszawa 1983.

[23] Woronowicz, S.L.: Quantum exponential function - Rev. Math. Phys. Vol. 12, No. 6 (2000) 873-920.

[24] Woronowicz S.L., Zakrzewski, S.: Quantum 'ax+b' group - submitted to Comm. Math. Phys.

[25] Woronowicz, S.L.: Operator Equalities Related to the Quantum E (2) Group - Comm. Math. Phys. 144, 417-428 (1992).

[26] Woronowicz, S.L.: Quantum E (2) Group and its Pontryagin Dual- Lett. Math. Phys. 23: 251-263, 1991.

[27] Woronowicz, S.L.: C*-algebras generated by unbounded elements. Rev. Math. Phys. Vol. 7 No. 3 (1995) 481-521.

[28] Woronowicz, S.L.: Quantum 'az+b' group on complex plane - KMMF Preprint 1999.