REPARAMETRIZATIONS OF VECTOR FIELDS AND THEIR SHIFT MAPS

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Abstract. Let $M$ be a smooth manifold, $F$ be a smooth vector field on $M$, and $(F_t)$ be the local flow of $F$. Denote by $Sh(F)$ the subset of $C^\infty(M,M)$ consisting of maps $h : M \to M$ of the following form:

$$h(x) = F_{\alpha(x)}(x),$$

where $\alpha$ runs over all smooth functions $M \to \mathbb{R}$ which can be substituted into $F$ instead of $t$. This space often contains the identity component of the group of diffeomorphisms preserving orbits of $F$. In this note it is shown that $Sh(F)$ is not changed under reparametrizations of $F$, that is for any smooth strictly positive function $\mu : M \to (0, +\infty)$ we have that $Sh(F) = Sh(\mu F)$. As an application it is proved that $F$ can be reparametrized to induce a circle action on $M$ if and only if there exists a smooth function $\mu : M \to (0, +\infty)$ such that $F(x, \mu(x)) \equiv x$.

1. Introduction

Let $M$ be a smooth manifold and $F$ be a smooth vector field on $M$ tangent to $\partial M$. For each $x \in M$ its integral trajectory with respect to $F$ is a unique mapping $o_x : \mathbb{R} \supset (a_x, b_x) \to M$ such that $o_x(0) = x$ and $\frac{d}{dt} o_x = F(o_x)$, where $(a_x, b_x) \subset \mathbb{R}$ is the maximal interval on which a map with the previous two properties can be defined. The image of $o_x$ will be denoted by the same symbol $o_x$ and also called the orbit of $x$. It follows that from standard theorems in ODE the following subset of $M \times \mathbb{R}$

$$\text{dom}(F) = \bigcup_{x \in M} x \times (a_x, b_x),$$

is an open, connected neighbourhood of $M \times 0$ in $M \times \mathbb{R}$. Then the local flow of $F$ is the following map

$$F : M \times \mathbb{R} \ni \text{dom}(F) \to M, \quad F(x, t) = F_x(t).$$

It is well known that if $M$ is compact, or $F$ has compact support, then $F$ is defined on all of $M$.

Denote by $\text{func}(F) \subset C^\infty(M, \mathbb{R})$ the subset consisting of functions $\alpha : M \to \mathbb{R}$ whose graph $\Gamma_\alpha = \{(x, \alpha(x)) : x \in M\}$ is contained in

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dom(\(F\)). Then we can define the following map
\[
\varphi : C^\infty(M, \mathbb{R}) \supset \text{func}(F) \longrightarrow C^\infty(M, M),
\]
\[
\varphi(\alpha)(x) = F(x, \alpha(x)).
\]
This map will be called the shift map along orbits of \(F\) and its image in \(C^\infty(M, M)\) will be denoted by \(\text{Sh}(F)\).

It is easy to see, \([1, \text{Lm. 2}]\), that \(\varphi\) is \(S^{r,s}\)-continuous for all \(r \geq 0\), that is continuous between the corresponding \(S^r\) Whitney topologies of \(\text{func}(F)\) and \(C^\infty(M, M)\).

Moreover, if the set \(\Sigma_F\) of singular points of \(F\) is nowhere dense, then \(\varphi\) is locally injective, \([1, \text{Pr. 14}]\). Therefore it is natural to know whether it is a homeomorphism with respect to some Whitney topologies, and, in particular, whether it is \(S^{r,s}\)-open, i.e. open as a map from \(S^r\) topology of \(\text{func}(F)\) into \(S^s\) topology of the image \(\text{Sh}(F)\), for some \(r, s \geq 0\). These problems and their applications were treated e.g. in \([1, 2, 3]\).

In this note we prove the following theorems describing the behaviour of the image of shift maps under reparametrizations and pushforwards.

**Theorem 1.1.** Let \(\mu : M \to \mathbb{R}\) be any smooth function and \(G = \mu F\) be the vector field obtained by the multiplication \(F\) by \(\mu\). Then
\[
\text{Sh}(G) \subset \text{Sh}(F).
\]
Suppose that \(\mu \neq 0\) on all of \(M\). Then
\[
\text{Sh}(\mu F) = \text{Sh}(F).
\]
In this case the shift mapping \(\varphi : \text{func}(F) \to \text{Sh}(F)\) of \(F\) is \(S^{r,s}\)-open for some \(r, s \geq 0\), if and only if so is the shift mapping \(\psi : \text{func}(G) \to \text{Sh}(G)\) of \(G\).

**Theorem 1.2.** Let \(z \in M\), \(\alpha : (M, z) \to \mathbb{R}\) be a germ of smooth function at \(z\), and \(f : M \to M\) be a germ of smooth map defined by \(f(x) = F(x, \alpha(x))\). Suppose that \(f\) is a germ of diffeomorphism at \(z\). Then
\[
f_*F = (1 + F(\alpha)) \cdot F,
\]
where \(f_*F = Tf \circ F \circ f^{-1}\) is the vector field induced by \(f\), and \(F(\alpha)\) is the derivative of \(\alpha\) along \(F\). Thus \(f_*F\) is just a reparametrization of \(F\).

If \(\alpha : M \to \mathbb{R}\) is defined on all of \(M\) and \(f = \varphi(\alpha)\) is a diffeomorphism of \(M\), then
\[
\text{Sh}(f_*F) = \text{Sh}(F).
\]

Further in \([3, 3]\) we will apply these results to circle actions. In particular, we prove that \(F\) can be reparametrized to induce a circle action on \(M\) if and only if there exists a smooth function \(\mu : M \to (0, +\infty)\) such that \(F(x, \mu(x)) \equiv x\), see Corollary \([3, 3]\).
2. Proofs of Theorems 1.1 and 1.2

These theorems are based on the following well-known statement, see e.g. [7, 5, 4] for its variants in the category of measurable maps.

Lemma 2.1. Let $G = \mu F$ and $G : \text{dom}(G) \to M$ be the local flow of $G$. Then there exists a smooth function $\alpha : \text{dom}(G) \to \mathbb{R}$ such that

$$G(x, s) = F(x, \alpha(x, s)).$$

In fact,

$$\alpha(x, s) = \int_0^s \mu(G(x, \tau))d\tau.$$

In particular, for each $\gamma \in \text{func}(G)$ we have that

$$G(x, \gamma(x)) = F(x, \alpha(x, \gamma(x))),$$

whence $\text{Sh}(G) \subset \text{Sh}(F)$.

Proof. Put $G(x, s) = F(x, \alpha(x, s))$, where $\alpha$ is defined by (3). We have to show that $G = G$.

Notice that a flow $G$ of a vector field $G$ is a unique mapping that satisfies the following ODE with initial condition:

$$\frac{\partial G(x, s)}{\partial s} \bigg|_{s=0} = G(x) = F(x)\mu(x), \quad G(x, 0) = x.$$

Notice that

$$\alpha(x, 0) = 0, \quad \alpha'(x, 0) = \mu(G(x, 0)) = \mu(x).$$

In particular, $G(x, 0) = F(x, \alpha(x, 0)) = x$. Therefore it remains to verify that

$$\frac{\partial G(x, s)}{\partial s} \bigg|_{s=0} = F(x) \cdot \mu(x).$$

We have:

$$\frac{\partial G}{\partial s}(x, s) = \frac{\partial F}{\partial s}(x, \alpha(x, s)) = \frac{\partial F(x, t)}{\partial t} \bigg|_{t=\alpha(x, s)} \cdot \alpha'(x, s).$$

Substituting $s = 0$ in (6) we get (5). \qed

Proof of Theorem 1.1. Eq. (1) is established in Lemma 2.1.

Suppose that $\mu \neq 0$ on all of $M$. Then $F = \frac{1}{\mu} G$, and $\frac{1}{\mu}$ is smooth on all of $M$. Hence again by Lemma 2.1 $\text{Sh}(F) \subset \text{Sh}(G)$, and thus $\text{Sh}(F) = \text{Sh}(G)$.

To prove the last statement define a map $\xi : \text{func}(G) \to \text{func}(F)$ by

$$\xi(\gamma)(x) = \alpha(x, \gamma(x)) = \int_0^s \mu(G(x, \tau))d\tau, \quad \gamma \in \text{func}(G).$$
Then (4) means that the following diagram is commutative:

\[
\begin{array}{c}
\text{func}(G) \xrightarrow{\xi} \text{func}(F) \\
\downarrow \psi \hspace{1cm} \downarrow \varphi \\
\text{Sh}(G) \xrightarrow{\psi} \text{Sh}(F)
\end{array}
\]

We claim that \(\xi\) is a homeomorphism with respect to \(S^r\) topologies for all \(r \geq 0\). Indeed, evidently \(\xi\) is \(S^r,r\)-continuous. Put

\[
(7) \quad \beta(x,s) = \int_0^s \frac{d\tau}{\mu(F(x,\tau))}.
\]

Then the inverse map \(\xi^{-1} : \text{func}(F) \rightarrow \text{func}(G)\) is given by

\[
(8) \quad \xi^{-1}(\delta)(x) = \beta(x,\delta(x)) = \int_0^{\delta(x)} \frac{d\tau}{\mu(F(x,\tau))}, \quad \delta \in \text{func}(F),
\]

and is also \(S^r,r\)-continuous. Hence \(\psi\) is \(S^r,s\)-open iff so is \(\varphi\). Theorem 1.1 is completed.

**Proof of Theorem 1.2.** First we reduce the situation to the case \(\alpha(z) = 0\). Suppose that \(a = \alpha(z) \neq 0\) and let \(\beta(x) = \alpha(x) - a\). Define the following germ of diffeomorphisms \(g = F_a \circ f\) at \(z\):

\[
g(x) = F(F(x,\alpha(x)), -a) = F(x,\alpha(x) - a) = F(x,\beta(x)).
\]

Then \(g(z) = z\), and \(\beta(z) = 0\).

Since \(F\) preserves \(F\), i.e. \((F_t)_*F = F\) for all \(t \in \mathbb{R}\), we obtain that

\[
f_*F = f_*\left(F_a\right)_*F = (f \circ F_a)_*F = g_*F.
\]

Moreover, \(F(\alpha) = F(\beta)\). Therefore it suffices to prove our statement for \(g\).

If \(z\) is a singular point of \(F\), i.e. \(F = 0\), then both parts of (2) vanish. Therefore we can assume that \(z\) is a regular point of \(F\). Then there are local coordinates \((x_1, \ldots, x_n)\) at \(z = 0 \in \mathbb{R}^n\) in which \(F(x) = \frac{\partial}{\partial x_1}\) and

\[
F(x_1, \ldots, x_n, t) = (x_1 + t, x_2, \ldots, x_n).
\]

Then \(g(x_1, \ldots, x_n) = (x_1 + \beta(x), x_2, \ldots, x_n)\), whence

\[
Tg \circ F \circ g^{-1} = \begin{pmatrix}
1 + \beta'_{x_1} & \beta'_{x_2} & \cdots & \beta'_{x_n} \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
0 \\
\vdots \\
0
\end{pmatrix} = (1 + \beta'_{x_1})F = (1 + F(\beta))F.
\]

Suppose now that \(\alpha\) is defined on all of \(M\) and \(f\) is a diffeomorphism of all of \(M\). Then by 11 the function \(\mu = 1 + F(\alpha) \neq 0\) on all of \(M\), whence by Theorem 1.1 \(\text{Sh}(\mu F) = \text{Sh}(F)\).
3. Periodic shift maps

Let \( F \) be a vector field, and \( \varphi \) be its shift map. The set
\[
\ker(\varphi) = \varphi^{-1}(\text{id}_M)
\]
will be called the kernel of \( \varphi \), thus \( F(x, \nu(x)) \equiv x \) for all \( \nu \in \ker(\varphi) \). Evidently, \( 0 \in \ker(\varphi) \). Moreover, it is shown in [1, Lm. 5] that \( \varphi(\alpha) = \varphi(\beta) \iff \alpha - \beta \in \text{func}(F) \).

Suppose that the set \( \Sigma_F \) of singular points of \( F \) is nowhere dense in \( M \). Then, [1, Th. 12 & Pr. 14], \( \varphi \) is a locally injective map with respect to any weak or strong topologies, and we have the following two possibilities for \( \ker(\varphi) \):

a) Non-periodic case: \( \ker(\varphi) = \{0\} \), so \( \varphi : \text{func}(F) \rightarrow \text{Sh}(F) \) is a bijection.

b) Periodic case: there exists a smooth strictly positive function \( \theta : M \rightarrow (0, +\infty) \) such that \( F(x, \theta(x)) \equiv x \) and \( \ker(\varphi) = \{n\theta\}_{n \in \mathbb{Z}} \).

It also follows that every non-singular point \( x \) of \( F \) is periodic of some period \( \text{Per}(x) \),
\[
\theta(x) = n_x \text{Per}(x)
\]
for some \( n_x \in \mathbb{N} \), and in particular, \( \theta \) is constant along orbits of \( F \). We will call \( \theta \) the period function for \( \varphi \).

Lemma 3.1. Suppose that the shift map \( \varphi \) of \( F \) is periodic and let \( \theta \) be its period function. Let also \( \mu : M \rightarrow (0, +\infty) \) be any smooth strictly positive function. Put \( G = \mu F \). Then the shift map \( \psi \) of \( G \) is also periodic, and its period function is
\[
\tilde{\theta}(x) \overset{\text{[S]}}{=} \xi^{-1}(\theta)(x) = \beta(x, \theta(x)) = \int_0^{\theta(x)} \frac{d\tau}{\mu(F(x, \tau))}.
\]

If \( \mu \) is constant along orbits of \( F \), then the last formula reduces to the following one:
\[
\tilde{\theta} = \frac{\theta}{\mu}.
\]
In particular, for the vector field \( G = \theta F \) its period function is equal to \( \tilde{\theta} \equiv 1 \).

Proof. Let \( G : M \times \mathbb{R} \rightarrow M \) be the flow of \( G \). We have to show that \( G(x, \tilde{\theta}(x)) \equiv x \) for all \( x \in M \):
\[
G(x, \tilde{\theta}(x)) = G(x, \beta(x, \theta(x))) = F(x, \theta(x)) \equiv x.
\]
Since $\theta$ is the minimal positive function for which $F(x, \theta(x)) \equiv x$ and $\mu > 0$, it follows from (9) that so is $\bar{\theta}$ is also the minimal positive function for which (11) holds true. Hence $\bar{\theta}$ is the period function for the shift map of $G$.

Let us prove (10). Since $\mu$ is constant along orbits of $F$, we have that $\mu(F(x, \tau)) = \mu(x)$, whence

$$\bar{\theta}(x) = \beta(x, \theta(x)) = \int_0^{\theta(x)} \frac{d\tau}{\mu(F(x, \tau))} = \int_0^{\theta(x)} \frac{d\tau}{\mu(x)} = \frac{\theta(x)}{\mu(x)}.$$

Lemma is proved. $\square$

3.2. Circle actions. Regard $S^1$ as the group $U(1)$ of complex numbers with norm 1, and let $\exp : \mathbb{R} \to S^1$ be the exponential map defined by $\exp(t) = e^{2\pi it}$.

Let $\Gamma : M \times S^1 \to M$ be a smooth action of $S^1$ on $M$. Then it yields a smooth $\mathbb{R}$-cation (or a flow) $G : M \times \mathbb{R} \to M$ given by

$$G(x, t) = \Gamma(x, \exp(t)).$$

Moreover $G$ is generated by the following vector field

$$G(x) = \frac{\partial G(x, t)}{\partial t} \bigg|_{t=0}.$$

Evidently, any of $\Gamma$, $G$, and $G$ determines two others. In particular, a flow $G$ on $M$ is of the form (12) for some smooth circle action $\Gamma$ on $M$ if and only if $G_1 = \text{id}_M$, i.e. $G(x, 1) \equiv x$ for all $x \in M$.

In other words, the shift map of $G$ is periodic and its period function is the constant function $\theta \equiv 1$.

As a consequence of Lemma 3.1 we get the following:

**Corollary 3.3.** Let $F$ be a smooth vector field on $M$ and

$$\theta : M \to (0, +\infty)$$

be a smooth strictly positive function. Then the following conditions are equivalent:

(a) the vector field $G = \theta F$ yields a smooth circle action, i.e. $G(x, 1) = x$ for all $x \in M$;

(b) the shift map $\varphi$ of $F$ is periodic and $\theta$ is its period function, i.e. $F(x, \theta(x)) \equiv x$ for all $x \in M$.

**Corollary 3.4.** Suppose that the shift map $\varphi$ of $F$ is periodic and let $z \in M$ be a singular point of $F$. Then there are $k, l \geq 0$ such that $2k + l = \dim M$, non-zero numbers $A_1, \ldots, A_k \in \mathbb{R} \setminus \{0\}$, local coordinates $(x_1, y_1, \ldots, x_k, y_k, t_1, \ldots, t_l)$ at $z = 0 \in \mathbb{R}^{2k+l}$, and in which
the linear part of $F$ at $0$ is given by

$$j_0^1 F(x_1, y_1, \ldots, x_k, y_k, t_1, \ldots, t_l) = -A_1 y_1 \frac{\partial}{\partial x_1} + A_1 x_1 \frac{\partial}{\partial y_1} + \cdots$$

$$-A_k y_k \frac{\partial}{\partial x_k} + A_k x_k \frac{\partial}{\partial y_k}.$$

**Proof.** Let $\theta$ be the period function for $F$ and $G = \theta F$. Since $\theta > 0$, it follows that $\Sigma_F = \Sigma_G$ and for every $z \in \Sigma_F$ we have that

$$j_1^1 G = \theta(z) \cdot j_1^1 F.$$

Therefore it suffices to prove our statement for $G$.

By Corollary 3.3, $G$ yields a circle action, i.e. $G_1 = \text{id}_M$, where $G$ is the flow of $G$. Then $G$ yields a linear flow $T_z G_1$ on the tangent space $T_z M$ such that $T_z G_1 = \text{id}$. In other words we obtain a linear action (i.e. representation) of the circle group $U(1)$ in the finite-dimensional vector space $T_z M$. Now the result follows from standard theorems about presentations of $U(1)$. \[\square\]

**Remark 3.5.** Suppose that in Corollary 3.4 dim $M = 2$. Then we can choose local coordinates $(x, y)$ at $z = 0 \in \mathbb{R}^2$ in which

$$j_0^1 F(x, y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$ 

For this case the normal forms of such vector fields are described in [6].

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