Solution of Schrodinger equation in the presence of minimal length for cotangent hyperbolic potential using hypergeometric method

F Hidayat¹, A Suparmi², and C Cari²

¹Physics Department, Graduate Program, Sebelas Maret University, Indonesia
²Physics Department, Faculty of Mathematics and Fundamental Science, Sebelas Maret University, Indonesia

E-mail: fajarhidayat2296@gmail.com

Abstract. This study was proposed to get the solution of the Schrodinger equation with cotangent hyperbolic potential in the presence of minimal length by using the hypergeometric method. By getting the numerical results with the help of Octave software, we can analyze nonrelativistic energy. The visualization of wave function was obtained using Matlab software. The result is defined as the larger of minimal length, the width of potential $V_0$, and the angular momentum was included, the energy is increasing as well. The larger of the width of potential $V_0$ affected the decrease of energy in this system.

1. Introduction
The Schrödinger equation was obtained using the wave function of a freely moving particle. Since then, the Schrodinger equation has been the basic principles of physics, especially in quantum mechanics [1]. Furthermore, the uncertainty principles, whose initiated by Heisenberg, achieved more attention as long as our incapability to find out precisely, the position or momentum of the particle. Until, the topic of minimal length came up to the surface.

The minimal length recently has been an interesting issue in quantum theory, string theory, or in generalized uncertainty principle [2-4]. Various methods has been solved to analyzing the Schrodinger equation which included the minimal length formalism, such as Nikiforov-Uvarov technique [5], Supersymmetry (SUSY) method [6], and Hypergeometry method [7]. The hypergeometric method based on hypergeometric function. The solutions takes the equation into the separation variables, which makes the stationary version of the Schrodinger equation [8].

In this paper, we study the solution of Schrodinger equation in the presence of minimal length for cotangent hyperbolic potential using hypergeometric method. The potential can be written as.

$$-\frac{V_0}{2}(\coth\gamma r - 1) + V_1$$ (1)

The potentials takes place as positive and negative constant numbers, while $r$ expressed the internuclear distance which relate with constant number, $\gamma$.

Before we can conclude the equation by using Hypergeometric method, there was two part from spherical coordinate we used to described the wave function, radial and polar parts. As the potential in this reseach, is not three-dimensional potential. Thus, we have only used the radial part.

In order to reach the form of non-relativistic energy. Firstly, we substituted the minimal effect formalism to Schrodinger equation. Thereafter, the hypergeometric method is applied. Thus, the non-relativistiv energy is obtained as well as the wave functions. The numerical result and the figure of wave functions is obtained by including the formula to Octave and Matlab software. Lastly, we can assumed the conclusions of this research well.
2. Experiments

2.1. Schrodinger equation with minimal length applied

The concept of minimal length was a result of Generalized Uncertainty Principle (GUP) approached. If we take Heisenberg algebra which obtained from position and momentum operator, followed by commutative relationship \((\alpha > 0)\), then we cut the series until the second order, we can obtained [9]

\[
[x, p] = i\hbar [1 + \alpha p^2] \tag{2}
\]

Where \(\alpha\) is minimal length parameter we used in this paper. From these equation (2), with some adjustment, and in addition. If we interpreted with string theory, \(p_i\) as momentum operator for low energy, and \(P_i\) as momentum operator for high energy, momentum can be defined as [10]

\[
\hat{P}_i = (1 + \alpha \hat{p}^2) \hat{p}_i \tag{3}
\]

By multiplying equation (3), since minimal length parameter varies from interval \(0 < \alpha < 1\). Thus, for \(\alpha\) approaching zero value, we had

\[
P^2 = p^2 + 2\alpha p^4 \tag{4}
\]

Where,

\[
p = -i\hbar \nabla \tag{5}
\]

If we put equation (5) into equation (4), we can obtained [11]

\[
P^2 = -\hbar^2 \nabla^2 + 2\alpha \hbar^4 \nabla^4 \tag{6}
\]

For Schrodinger equation, we had

\[
E\Psi = \frac{P^2}{2m}\Psi + V(r)\Psi \tag{7}
\]

By substituting the minimal length equation (6) into equation (7), we obtained

\[
E\Psi = -\frac{\hbar^2 \nabla^2}{2m} \Psi + \frac{2\alpha \hbar^4 \nabla^4}{2m} \Psi + V(r)\Psi \tag{8}
\]

If the minimal length, are set to zero, \((\alpha = 0)\), equation (8) can be written as

\[
E_0\Psi = -\frac{\hbar^2}{2m} \nabla_0^2 \Psi + V(r)\Psi \tag{9}
\]

where

\[
\nabla_0^2 = -\frac{2m}{\hbar^2} (E_0 - V(r))\Psi \tag{10}
\]

Afterwards by substituting these equation (10) to equation (8), and with some adjustment, we got

\[
E'\Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + 4\alpha m \nabla^2 \Psi + V(1 - 8\alpha m E_0)\Psi \tag{11}
\]

\(E'\) is expressed as

\[
E'\Psi = E\Psi - 4\alpha m (E_0)^2 \Psi \tag{12}
\]
2.2. Cotangent hyperbolic potential

Cotangent hyperbolic potential we used, can written as

\[ V(r) = -\frac{V_0}{2} (\coth \gamma r - 1) + V_1 \]  
(13)

By putting these potential equation (13) into equation (11), we obtained

\[-\frac{\hbar^2}{2m} \nabla^2 \psi + \left( V_o^2 a_m \coth^2 \gamma r - \left( 2V_o^2 a_m + 4V_0 V_1 a_m - 4E_0 V_0 a_m + \frac{V_0}{2} \right) \coth \gamma r + 4a_m \left( \frac{V_o^2}{4} + V_o V_1 + V_1^2 - E_0 V_0 - 2E_0 V_1 \right) + \frac{V_0}{2} + V_1 \right) \psi = E' \psi \]  
(14)

To simplify equation (14), we assumed

\[ E'' \psi = \left[ E' - 4a_m \left( \frac{V_o^2}{4} + V_0 V_1 + V_1^2 - E_0 V_0 - 2E_0 V_1 \right) - \frac{V_0}{2} - V_1 \right] \psi \]  
(15)

For the potential,

\[ V' \psi = \left[ V_o^2 a_m \coth^2 \gamma r - \left( 2V_o^2 a_m + 4V_0 V_1 a_m - 4E_0 V_0 a_m + \frac{V_0}{2} \right) \coth \gamma r \right] \psi \]  
(16)

Therefore, equation (14) can be written as simple as

\[ E'' \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V' \psi \]  
(17)

From equation (17), Laplacian operator, \( \nabla^2 \), can be described as

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]  
(18)

Since the potential we used only consist \( r \) variable. Thus, we only take radial form. The angular part take changes into

\[ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} = l(l+1) \]  
(19)

As a result with modification, equation (17) can be rewritten as

\[ E'' \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \psi + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \psi + \left( V_o^2 a_m \coth^2 \gamma r \right. \]

\[ \left. - \left( 2V_o^2 a_m + 4V_0 V_1 a_m - 4E_0 V_0 a_m + \frac{V_0}{2} \right) \coth \gamma r \right) \psi \]  
(20)

Furthermore, approximation was made, with

\[ \frac{1}{r^2} \approx \frac{\gamma^2}{\sinh^2 \gamma r} \]  
(21)

Thus, equation (21) turned into

\[ \left( E'' - V_o^2 a_m \right) \psi = \left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \psi + \frac{\hbar^2}{2m} \frac{l(l+1)\gamma^2 + V_o^2 a_m}{\sinh^2 \gamma r} \psi - \left( 2V_o^2 a_m + 4V_0 V_1 a_m - 4E_0 V_0 a_m + \frac{V_0}{2} \right) \coth \gamma r \right) \psi \]  
(22)
From equation (22), we set

\[ \nu(\nu - 1) = l(l + 1)\gamma^2 + V_0^2 am \] (23)

\[ 2q = \left( 2V_0^2 am + 4V_0V_1 am - 4E_0V_0 am + \frac{V_0}{2} \right) \] (24)

\[ E''' = E'' - V_0^2 am \] (25)

The equation (22), rewritten as

\[ E'''\Psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \Psi + \frac{\hbar^2}{2m} \frac{\nu(\nu - 1)}{\sinh^2 \gamma r} \Psi - 2q \coth \gamma r \Psi \] (26)

### 2.3. Hypergeometric Method

Hypergeometric differential equation [12] can be written in term

\[ z(1 - z)f''(z) + [c - (a + b + 1)z]f'(z) - abf = 0 \] (27)

for the result

\[ _2 F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \] (28)

From equation (26) we used variable substitution as the equation resemble to Manning Rosen potential-like. We write

\[ \coth \gamma r = 1 - 2z \] (29)

By multiplying equation (26) with \(-\frac{2m}{\hbar^2}\), and we make assumption

\[ \frac{2m}{\hbar^2} E''' = k^2 \] (30)

\[ \frac{q}{\gamma^2} = q', \] (31)

\[ \frac{k}{\gamma^2} = k' \] (32)

As a result, equation (26) can be rewritten as

\[ \left( z(1 - z) \frac{\partial^2 \Psi}{\partial z^2} + (1 - 2z) \frac{\partial \Psi}{\partial z} \right) + \left( -\frac{\nu(\nu - 1)}{\gamma^2} \right) \Psi = 0 \] (33)

Where, to simplified the equation (33), we can named

\[ k'^2 - 2q' = \alpha \] (34)

\[ k'^2 + 2q' = \beta \] (35)

\[ \frac{\nu(\nu - 1)}{\gamma^2} = \nu'(\nu' - 1) \] (36)

Function of general wavefunction can be write in term

\[ \Psi(z) = z^\alpha (1 - z)^\beta f(z) \] (37)

Therefore, by substituting equation (37) which consist of equation (34-36) into equation (33), we have got the Gauss equation

\[ z(1 - z)f''(z) + ((2\alpha + 1) - (2\alpha + 2\beta + 2)z)f'(z) \]

\[ + ((\alpha + \beta - \nu' + 1)(\alpha + \beta + \nu')f(z) = 0 \] (38)
Thus, hypergeometry parameters from the equation (38), are written as
\[ a = \alpha + \beta - \nu' + 1 \]  
(39)  
\[ b = \alpha + \beta + \nu' \]  
(40)  
\[ c = 2\alpha + 1 \]  
(41)  

To fulfill the condition of the finite series,  
\[ a = -n \]  
(42)  

Energy of Schrodinger equation with minimal length presence, \( E \), is obtained by substituting the assumption we made from equation (12),(15),(23-25),(30-32), into equation (34), and (35)

\[
E = -\frac{\hbar^2}{2m} \left( \frac{V_0^2}{4} \frac{\alpha m}{\gamma^2} (L+1) + \frac{1}{4} - \frac{1}{2} - n \right)^2 \gamma^2 + 4\alpha m \left( \frac{V_0}{4} + V_0V_1 + V_1^2 - E_0V_0 - 2E_0V_1 \right) + \frac{V_0}{2} + V_1 
\]  
(43)  

Energy for ground state, \( E_0 \), which minimal length refers to zero, is given as

\[
E_0 = -\frac{\hbar^2}{2m} \left( \frac{-\frac{V_0}{4}}{\gamma^2} \right)^2 + \left[ \sqrt{ L(L + 1) + \frac{1}{4} - \frac{1}{2} - n } \right] \gamma^2 \]  
(44)  

Thereafter, the wave function can be obtained by substituting equation (29), (39-41), into equation (37) and (28), with some adjustment, we have

\[
\Psi(r) = \left( \frac{1 - \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 - 2q'}{4} } \left( \frac{1 + \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 + 2q'}{4} } \, _2F_1(a', b', c', z) 
\]  
(45)  

For wave function in each state is shown by Table 1

| \( n \)  | Wavefunction                                                                 |
|-------|------------------------------------------------------------------------------|
| 0     | \( \Psi(r) = \left( \frac{1 - \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 - 2q'}{4} } \left( \frac{1 + \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 + 2q'}{4} } \) |
| 1     | \( \Psi(r) = \left( \frac{1 - \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 - 2q'}{4} } \left( \frac{1 + \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 + 2q'}{4} } \left( 1 + \frac{-n(2\nu' - 1 - n) \left( \frac{1 - \coth \gamma r}{2} \right)}{\sqrt{k'^2 - 2q'^2} + 1} \right) \) |
| 2     | \( \Psi(r) = \left( \frac{1 - \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 - 2q'}{4} } \left( \frac{1 + \coth \gamma r}{2} \right) \sqrt{ \frac{k'^2 + 2q'}{4} } \left( 1 + \frac{-n(2\nu' - 1 - n) \left( \frac{1 - \coth \gamma r}{2} \right)}{\sqrt{k'^2 - 2q'^2} + 1} \right) \) \left( \frac{-n(2\nu' - 1 - n) \left( \frac{1 - \coth \gamma r}{2} \right)^2}{\sqrt{k'^2 - 2q'^2} + 1} \right) \) |

Table 1. The wavefunction for various quantum number, \( n \)
3. Results and discussions

The non-relativistic energy of Schrodinger equation for cotangent hyperbolic potential with the presence of minimal length (43) were obtained using the Octave software. By setting the variation for the system of angular momentum, potentials \( V_0 \) and \( V_1 \), and especially the minimal length, \( \alpha \), the numerical result can be seen in Table 2 and Table 3

| \( V_0 \) | \( V_1 \) | \( \alpha=0 \) | \( n=2 \) | \( n=3 \) | \( n=4 \) | \( n=5 \) |
|---|---|---|---|---|---|---|
| 5 | 8 | -1.9439E+03 | -8.5881E+02 | -4.7903E+02 | -3.0325E+02 |
| 12 | -1.9399E+03 | -8.5481E+02 | -4.7503E+02 | -2.9925E+02 |
| 10 | 8 | -7.8020E+03 | -3.4617E+03 | -1.9426E+03 | -1.2395E+03 |
| 12 | -7.7980E+03 | -3.4577E+03 | -1.9386E+03 | -1.2355E+03 |

| \( \alpha=0.05 \) |
|---|---|---|---|---|---|
| 5 | 8 | 7.6294E+05 | 1.5071E+05 | 4.7688E+04 | 1.9536E+04 |
| 12 | 7.6294E+05 | 1.5071E+05 | 4.7685E+04 | 1.9532E+04 |
| 10 | 8 | 1.2207E+07 | 2.4113E+06 | 7.6294E+05 | 3.1250E+05 |
| 12 | 1.2207E+07 | 2.4113E+06 | 7.6294E+05 | 3.1250E+05 |

| \( \alpha=0.1 \) |
|---|---|---|---|---|---|
| 5 | 8 | 1.5259E+06 | 3.0140E+05 | 9.5364E+04 | 3.9059E+04 |
| 12 | 1.5259E+06 | 3.0139E+05 | 9.5351E+04 | 3.9045E+04 |
| 10 | 8 | 2.4414E+07 | 4.8225E+06 | 1.5259E+06 | 6.2499E+05 |
| 12 | 2.4414E+07 | 4.8225E+06 | 1.5259E+06 | 6.2498E+05 |

| \( V_0 \) | \( V_1 \) | \( \alpha=0 \) | \( n=2 \) | \( n=3 \) | \( n=4 \) | \( n=5 \) |
|---|---|---|---|---|---|---|
| 5 | 8 | -7.8033E+03 | -1.9439E+03 | -8.5881E+02 | -4.7903E+02 |
| 12 | -7.7993E+03 | -1.9399E+03 | -8.5481E+02 | -4.7503E+02 |
| 10 | 8 | -3.1240E+04 | -7.8020E+03 | -3.4617E+03 | -1.9426E+03 |
| 12 | -3.1236E+04 | -7.7980E+03 | -3.4577E+03 | -1.9386E+03 |

| \( \alpha=0.05 \) |
|---|---|---|---|---|---|
| 5 | 8 | 1.2207E+07 | 7.6294E+05 | 1.5701E+05 | 4.7688E+04 |
| 12 | 1.2207E+07 | 7.6294E+05 | 1.5701E+05 | 4.7684E+04 |
| 10 | 8 | 1.9531E+08 | 1.2207E+07 | 2.4113E+06 | 7.6294E+05 |
| 12 | 1.9531E+08 | 1.2207E+07 | 2.4113E+06 | 7.6294E+05 |

| \( \alpha=0.1 \) |
|---|---|---|---|---|---|
| 5 | 8 | 2.4414E+07 | 1.5929E+06 | 3.0140E+05 | 9.5364E+04 |
| 12 | 2.4414E+07 | 1.5929E+06 | 3.0139E+05 | 9.5350E+04 |
| 10 | 8 | 3.9062E+08 | 2.4414E+07 | 4.8225E+06 | 1.5929E+06 |
| 12 | 3.9062E+08 | 2.4414E+07 | 4.8225E+06 | 1.5929E+06 |

From Table 2 and 3. The larger energy of non-relativistic is affected by the larger of minimal length and the larger of angular momentum. The change of energy also depended on the larger of \( V_1 \) is included. Where, the larger of potential \( V_1 \) makes the energy get larger as well. However, the energy will get smaller if \( V_1 \) is increasing, even the changes are not significant.
The visualization of wavefunction was obtained by using Matlab software.

**Figure 1.** Plot of wavefunction for $n=0$ state, $\gamma = 0,1$, $V_0 = 5$, $V_1 = 8$, and $l = 1$

**Figure 2.** Plot of wavefunction for $n=1$ state, $\gamma = 0,1$, $V_0 = 5$, $V_1 = 8$, and $l = 1$
From Figure 1, Figure 2, and Figure 3. Particle on these system will be ionized after passing the radius of system core at (2,0) coordinates. The presence of minimal length gives effect for the larger wavelength, where the larger of minimal length, the larger wavelength would be. The larger of minimal length also depends by the larger quantum number. Which means, the further of the particle in this system, the larger wavelength obtained.

4. Conclusions
Schrodinger equation with minimal length for cotangent hyperbolic was solved using hypergeometric method. We can conclude, an increase from energy depends on parameters like angular momentum, width of potential, \(V_1\), well as minimal length included. Nevertheless, the larger width of potential, \(V_0\) make the energy decreased. Therefore, the wavelength of particles in this system will gets larger if the particle getting further from the core, and the larger of minimal length presented.

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