RINGS OVER WHICH EVERY MATRIX IS THE SUM OF TWO IDEMPOTENTS AND A NILPOTENT

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Abstract. A ring $R$ is (strongly) 2-nil-clean if every element in $R$ is the sum of two idempotents and a nilpotent (that commute). Fundamental properties of such rings are discussed. Let $R$ be a 2-primal ring. If $R$ is strongly 2-nil-clean, we show that $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$. We also prove that the matrix ring is 2-nil-clean for a strongly 2-nil-clean ring of bounded index. These provide many classes of rings over which every matrix is the sum of two idempotents and a nilpotent.

1. Introduction

Throughout, all rings are associative with an identity. A ring is called (strongly) nil-clean if every element can be written as the sum of an idempotent and a nilpotent (that commute). A ring $R$ is weakly nil-clean provided that every element in $R$ is the sum or difference of a nilpotent element and an idempotent. Such rings have been the object of much investigation over the last decade, as they are related to the well-studied clean rings of Nicholson. Though nil and weakly clean rings are popular, the conditions a bit restrictive (for example, there are even fields which are not weakly nil clean). The subjects of nil-clean and weakly nil-clean rings are interested for so many mathematicians, e.g., [1, 3, 4, 5, 9, 10, 12] and [13]. In the current paper, we seek to remedy this by looking at an interesting generalization of nil and weakly nil cleanness, which they call 2-nil-clean. That is, a ring $R$ is (strongly) 2-nil-clean
provided that every element in \( R \) is the sum of two idempotents and a nilpotent (that commute). This new class enjoys many interesting properties and examples (for example, all tripotent rings are 2-nil-clean). We shall investigate when a matrix ring is 2-nil-clean, i.e., when every matrix over a ring can be written as the sum of two idempotents and a nilpotent. A ring \( R \) is 2-primal if its prime radical coincides with the set of nilpotent elements of the ring. Examples of 2-primal rings include commutative rings and reduced rings. Let \( R \) be a 2-primal ring. If \( R \) is strongly 2-nil-clean, we show that \( M_n(R) \) is 2-nil-clean for all \( n \in \mathbb{N} \). A ring \( R \) is of bounded index if there is a positive integer \( n \) such that \( a^n = 0 \) for each nilpotent element \( a \) of \( R \). We also prove that the matrix ring is 2-nil-clean for a strongly 2-nil-clean ring of bounded index. These provide many classes of rings over which every matrix is the sum of two idempotents and a nilpotent.

We use \( N(R) \) to denote the set of all nilpotent elements in \( R \) and \( J(R) \) the Jacobson radical of \( R \). \( \mathbb{N} \) stands for the set of all natural numbers.

2. Examples and Subclasses

The aim of this section is to construct examples of 2-nil-clean rings and investigate certain subclass of such rings. We begin with

Example 2.1. The class of 2-nil-clean rings contains many familiar examples.

(1) Every weakly nil-clean ring is 2-nil-clean, e.g., strongly nil-clean rings, nil-clean rings, Boolean rings, weakly Boolean rings.
(2) \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) is 2-nil-clean, while it is not weakly nil-clean.
(3) A local ring \( R \) is 2-nil-clean if and only if \( R/J(R) \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \), and \( J(R) \) is nil.

We also provide some examples illustrating which ring-theoretic extensions of 2-nil-clean rings produce 2-nil-clean rings.

Example 2.2.

(1) Any quotient of a 2-nil-clean ring is 2-nil-clean.
(2) Any finite product of 2-nil-clean rings is 2-nil-clean. But 
\( R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \) is an infinite product of 2-nil-clean rings, which is not 2-nil-clean. Here, the element \((0, 2, 2, 2, \cdots) \in R\) cannot be written as the sum of two idempotents and a nilpotent element.

(3) The triangular matrix ring \( T_n(R) \) over a 2-nil-clean ring \( R \) is 2-nil-clean.

(4) The quotient ring \( R[[x]]/(x^n)(n \geq 1) \) of a 2-nil-clean ring \( R \) is 2-nil-clean.

**Theorem 2.3.** Let \( I \) be a nil ideal of the ring \( R \). Then \( R \) is 2-nil-clean if and only if the quotient ring \( R/I \) is 2-nil-clean.

*Proof.* \( \Rightarrow \) It is obtained from Example 2.2 (1).

\( \Leftarrow \) Let \( a \in R \), there exist two idempotents \( \overline{e}, \overline{f} \in R/I \) and a nilpotent \( \overline{w} \in R/I \) such that \( \overline{a} = \overline{e} + \overline{f} + \overline{w} \). As idempotents and nilpotents lift modulo nil ideal, we can assume that \( e, f \) are idempotents in \( R \) and \( w \) is a nilpotent in \( R \). Then \( a = e + f + w + r \) for some \( r \in I \). Since \( w \in N(R) \), we may assume that \( w^k = 0 \) for some \( k \in \mathbb{N} \), this implies that \( (w + r)^k \in I \) and so \( w + r \in N(R) \). This completes the proof. \( \square \)

We use \( P(R) \) to denote the prime radical of a ring \( R \). That is, \( P(R) = \bigcap\{P \mid P \text{ is a prime ideal of } R\} \). We have

**Corollary 2.4.** A ring \( R \) is 2-nil-clean if and only if the quotient ring \( R/P(R) \) is 2-nil-clean.

*Proof.* As \( P(R) \) is a nil ideal of \( R \), the result follows from Theorem 2.3. \( \square \)

**Corollary 2.5.** Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is 2-nil-clean.
2. \( T_n(R) \) is 2-nil-clean for all \( n \in \mathbb{N} \).
3. \( T_n(R) \) is 2-nil-clean for some \( n \in \mathbb{N} \).
Proof. (1) ⇒ (2) Let $I = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid a_{ij} \in R \right\}$. Then $I$ is an ideal of $T_n(R)$. Clearly, $T_n(R)/I \cong R \times R \times \cdots \times R$. In light of Example 2.2 and Theorem 2.3, we show that $T_n(R)$ is 2-nil-clean.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Straightforward. □

A ring $R$ is tripotent if $a^3 = a$ for all $a \in R$. We have

**Lemma 2.6.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is tripotent.
2. $R$ is a commutative ring in which every element is the sum of two idempotents.
3. $R$ is the product of fields isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

**Proof.** (1) ⇔ (2) This is obvious, by [7, Theorem 1].

(1) ⇒ (3) Birkhoff’s Theorem, $R$ is isomorphic to a subdirect product of subdirectly irreducible rings $R_i$. Thus, $R_i$ satisfies the identity $x^3 = x$. In view of [7, Theorem 1], $R_i$ is commutative. But $R_i$ has no central idempotents except for 0 and 1. Thus, $x^2 = 0$ or $x^2 = 1$. Hence, $x = x^3 = 0$ or $x = 1$. If $x \neq 0, 1$, then $(x - 1)^2 = 1$, and $x(x - 2) = 0$. This implies that $x = 2$. Thus, $R_i \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, as desired.

(3) ⇒ (1) $R$ is the product of fields isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. As $\mathbb{Z}_2$ and $\mathbb{Z}_3$ satisfy the identity $x^3 = x$. This completes the proof. □

Clearly, strongly 2-nil-clean rings form a subclass of 2-nil-clean rings. For further use, we now consider strongly 2-nil-clean rings. We record the following.

**Lemma 2.7.** A ring $R$ is strongly 2-nil-clean if and only if

1. $J(R)$ is nil;
2. $R/J(R)$ is tripotent.

**Proof.** $\Rightarrow$ Let $a \in R$. Then we can find two idempotents $e, f \in R$ and a nilpotent $w \in R$ such that $a + 1 = e + f + w$ where $e, f$ and $w$ commute. Hence, $a = e - (1 - f) + w$. Clearly, $(e - (1 - f))^3 = $
e − (1 − f), we see that \(a^3 - a \in N(R)\). It follows by [8, Theorem A.1] that \(N(R)\) forms an ideal of \(R\). Hence, \(N(R) \subseteq J(R)\). This shows that every element in \(R/J(R)\) is the sum of two idempotents that commute. In view of Lemma 2.6, \(R/J(R)\) is tripotent. Let \(x \in J(R)\). Then \(x^3 - x \in N(R)\) by the preceding discussion. Hence, \(J(R)\) is nil, as desired.

\[\begin{align*}
\iff & \text{By hypothesis, } \overline{2} = \overline{2^3} \text{ in } R/J(R). \text{ Hence, } 6 \in J(R) \text{ is nil. Let } a \in R. \text{ Since } R/J(R) \text{ is tripotent, we see that } (a^2 - a) - (a^2 - a)^3, a^3 - a \in N(R), \text{ and so } 3a^2 - 3a \in N(R). \text{ This shows that } (-2a^2)^2 - (-2a^2) = 4a^4 + 2a^2 = (6a^4 - 4a^2) + 2a^2 = 6a^4 + 2a(a - a^3) \in N(R). \text{ Moreover, } (a + 2a^2)^2 - (a + 2a^2) = a^2 + 4a^3 + 4a^4 - a - 2a^2 = (3a^2 - 3a) + 4(a^3 - a) + 6a + 6a(a^3 - a) \in N(R). \text{ In light of [12, Lemma 3.5], there exist } f(t), g(t) \in \mathbb{Z}[t] \text{ such that } (-2a^2) - f(a), (a + 2a^2) - g(a) \in N(R), f(a) = f^2(a) \text{ and } g(a) = g^2(a). \text{ Therefore } a - (f(a) + g(a)) = ((a + 2a^2) - g(a)) + ((-2a^2) - f(a)) \in N(R). \text{ Hence, } a = f(a) + g(a) + w \text{ with } w \in N(R). \text{ One easily checks that } af(a) = f(a)a \text{ and } ag(a) = g(a)a, \text{ and then } f(a), g(a) \text{ and } w \text{ commute. Therefore } R \text{ is strongly } 2\text{-nil-clean, as asserted.}
\end{align*}\]

A ring \(R\) a right (left) quasi-duo ring if every maximal right (left) ideal of \(R\) is an ideal. For instance, local rings, duo rings and weakly right (left) duo rings are all right (left) quasi-duo rings. Every abelian exchange ring is a right (left) duo ring (cf. [16]).

**Theorem 2.8.** A ring \(R\) is strongly 2-nil-clean if and only if

1. \(R\) is 2-nil-clean;
2. \(R\) is right (left) quasi-duo;
3. \(J(R)\) is nil.

**Proof.** \(\implies (1)\) is obvious. By Lemma 2.7, \(R/J(R)\) is tripotent and then it is commutative. Let \(M\) be a right (left) maximal ideal of \(R\). Then \(M/J(R)\) is an ideal of \(R/J(R)\). Let \(x \in M, r \in R\). Then \(\overline{rx} \in M/J(R)\), and then \(rx \in M + J(R) \subseteq M\). This shows that \(M\) is an ideal of \(R\). Thus \(R\) is right (left) quasi-duo. \(\implies (3)\) is follows from Lemma 2.7.
As $R$ is 2-nil-clean, $R/J(R)$ is 2-nil-clean. Since $R$ right (left) is quasi-duo, then by [16, Lemma 2.3], every nilpotent in $R$ contains in $J(R)$. Let $e \in R/J(R)$ be an idempotent. As $J(R)$ is nil, we can find an idempotent $f \in R$ such that $e = f + J(R)$. For any $r \in R$, $fr(1 - f) \in J(R)$, and then $efr = efre$. Likewise, $re = efr$. Thus, $efr = re$, i.e., $R/J(R)$ is abelian. Hence, $R/J(R)$ is tripotent, by Lemma 2.6. As $J(R)$ is nil, it follows by Lemma 2.7 that $R$ is strongly 2-nil-clean. □

A natural problem is if the matrix ring over a strongly 2-nil-clean ring is strongly 2-nil-clean. The answer is negative as the following shows.

**Example 2.9.** Let $n \geq 2$. then matrix ring $M_n(R)$ is not strongly 2-nil-clean for any ring $R$.

**Proof.** Let $R$ be a ring, and let $A = \begin{pmatrix} 1_R & 1_R \\ 1_R & 0 \end{pmatrix}$. Then $A^3 - A = \begin{pmatrix} 2 & 1_R \\ 1_R & 1_R \end{pmatrix}$. One checks that $\left( \begin{pmatrix} 2 & 1_R \\ 1_R & 1_R \end{pmatrix} \right)^{-1} = \begin{pmatrix} 1_R & -1_R \\ -1_R & 2 \end{pmatrix}$, and so $A^3 - A$ is not nilpotent. If $M_n(R)$ is strongly 2-nil-clean, as in the proof of Lemma 2.7, $A^3 - A$ is nilpotent, a contradiction, and we are done. □

### 3. 2-Nil-clean Matrix Rings

In [6, Corollary 1], Han and Nicholson proved that every matrix ring of a clean ring (i.e., every element is the sum of an idempotent and a unit) is clean. By using a similar route, we easily see that every matrix over a 2-nil-clean ring is the sum of two idempotent matrices and an invertible matrix. As seen in Example 2.9, there exist some matrices over an arbitrary strongly 2-nil-clean ring which is not strongly 2-nil-clean. The purpose of this section is to investigate certain strongly 2-nil-clean rings over which every matrix is 2-nil-clean. We have

**Lemma 3.1.** $M_n(\mathbb{Z}_3)$ is 2-nil-clean.

**Proof.** As every matrix over a field has a Frobenius normal form, and that 2-nil-clean matrix is invariant under the similarity, we may
assume that 
\[ A = \begin{pmatrix}
0 & c_0 \\
1 & c_1 \\
& \ddots & \ddots \\
& & 0 & c_{n-2} \\
& & & 1 & c_{n-1}
\end{pmatrix}. \]

Case I. \( c_{n-1} = 1 \). Choose
\[ W = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
& \ddots & \ddots \\
& & 0 & 0 \\
& & & 1 & 0
\end{pmatrix},
E = \begin{pmatrix}
0 & c_0 \\
0 & 0 \\
& \ddots & \ddots \\
& & 0 & c_{n-2} \\
& & & 0 & 1
\end{pmatrix}. \]
Then \( E^2 = E \), and so \( A = E + 0 + W \) is \( 2 \)-nil-clean.

Case II. \( c_{n-1} = -1 \). Choose
\[ W = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
& \ddots & \ddots \\
& & 0 & 0 \\
& & & 1 & 0
\end{pmatrix},
E = \begin{pmatrix}
0 & c_0 \\
0 & 0 \\
& \ddots & \ddots \\
& & 0 & c_{n-2} \\
& & & 0 & -1
\end{pmatrix}. \]
Then \( E^2 = -E \), and so \( A = (I_2 - E) + I_2 + W \) is \( 2 \)-nil-clean.

Case III. \( c_{n-1} = 0 \).
If \( n = 2 \), then
\[ \begin{pmatrix}
0 & c_0 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix} + \begin{pmatrix}
0 & c_0 - 1 \\
0 & 0
\end{pmatrix} \] is \( 2 \)-nil-clean.

If \( n = 3 \), then
\[ \begin{pmatrix}
0 & 0 & c_0 \\
1 & 0 & c_1 \\
0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
1 & -1 & 1 \\
-1 & 1 & -1
\end{pmatrix} + \begin{pmatrix}
0 & 0 & c_0 \\
0 & 0 & c_1 - 1 \\
0 & 0 & 0
\end{pmatrix} \]
is 2-nil-clean. If $n \geq 4$, we have

$$A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & c_0 \\
0 & 0 & \cdots & 0 & 0 & 0 & c_1 \\
0 & 0 & \cdots & 0 & 0 & 0 & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & c_{n-3} \\
0 & 0 & \cdots & 0 & 0 & 0 & c_{n-2} - 1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

is the sum of two idempotents and a nilpotent. This implies that $A \in M_n(\mathbb{Z}_3)$ is 2-nil-clean. Therefore $M_n(\mathbb{Z}_3)$ is 2-nil-clean. □

**Lemma 3.2.** Let $R$ be tripotent. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Let $A \in M_n(R)$, and let $S$ be the subring of $R$ generated by the entries of $A$. That is, $S$ is formed by finite sums of monomials of the form: $a_1a_2\cdots a_m$, where $a_1, \cdots, a_m$ are entries of $A$. Since $R$ is a commutative ring in which $6 = 0$, $S$ is a finite ring in which $x = x^3$ for all $x \in S$. By virtue of Lemma 2.6, $S$ is isomorphic to finite direct product of $\mathbb{Z}_2$ and/or $\mathbb{Z}_3$. In terms of Lemma 3.1 and Example 2.2 (2), $M_n(S)$ is 2-nil-clean. As $A \in M_n(S)$, $A$ is the sum of two idempotent matrices and a nilpotent matrix over $S$, as desired. □

**Theorem 3.3.** Let $R$ be 2-primal. If $R$ is strongly 2-nil-clean, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Since $R$ is strongly 2-nil-clean, it follows by Lemma 2.7 that $J(R)$ is nil and $R/J(R)$ is tripotent. In virtue of Lemma 3.2,
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$M_n(R/J(R))$ is 2-nil-clean. Furthermore, $J(R) \subset N(R) = P(R) \subset J(R)$, we get $J(R) = P(R)$. Hence, $M_n(J(R)) = M_n(P(R)) = P(M_n(R))$ is nil. Since $M_n(R/J(R)) \cong M_n(R)/M_n(J(R))$, it follows by Theorem 2.3 that $M_n(R)$ is 2-nil-clean. This completes the proof.

**Corollary 3.4.** Let $R$ be a commutative 2-nil-clean ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Corollary 3.5.** Let $R$ be a commutative weakly nil-clean ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** As every commutative weakly nil-clean ring is strongly 2-nil-clean 2-primal ring, we obtain the result, by Theorem 3.3. □

**Example 3.6.** Let $m = 2^k3^l (k, l \in \mathbb{N})$. Then $M_n(\mathbb{Z}_m)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** In light of [1, Example 9], $\mathbb{Z}_m$ is a commutative weakly nil-clean ring, hence the result by Corollary 3.5. □

**Lemma 3.7.** ( [10, Lemma 6.6]) Let $R$ be of bounded index. If $J(R)$ is nil, then $M_n(R)$ is nil for all $n \in \mathbb{N}$.

**Theorem 3.8.** Let $R$ be of bounded index. If $R$ is strongly 2-nil-clean, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** By virtue of Lemma 3.7, $M_n(J(R))$ is nil. In view of Lemma 2.7, $R/J(R)$ is tripotent. Thus, $M_n(R/J(R))$ is 2-nil-clean, in terms of Lemma 3.2. Since $M_n(R/J(R))/J(M_n(R)) \cong M_n(R/J(R))$, according to Theorem 2.3, $M_n(R)$ is 2-nil-clean. □

**Corollary 3.9.** Let $R$ be a ring, and let $m \in \mathbb{N}$. If $(a - a^3)^m = 0$ for all $a \in R$, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Let $x \in J(R)$. Then $(x - x^3)^m = 0$, and so $x^m = 0$. This implies that $J(R)$ is nil. In light of [8, Theorem A.1], $N(R)$ forms an ideal of $R$, and so $N(R) \subset J(R)$. Hence, $J(R) = N(R)$ is nil. Further, $R/J(R)$ is tripotent. In light of Lemma 2.7, $R$ is strongly 2-nil-clean. If $a^k = 0 (k \in \mathbb{N})$, then $1 - a, 1 + a \in U(R)$, and so $1 - a^2 = (1 - a)(1 + a) \in U(R)$. By hypothesis, $a^m(1 - a^2)^m = 0$. Hence, $a^m = 0$, and so $R$ is of bounded index. This complete the proof, by Theorem 3.8. □
A ring $R$ is a 2-Boolean ring provided that $a^2$ is an idempotent for all $a \in R$.

**Corollary 3.10.** Let $R$ be a 2-Boolean ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Let $a \in R$. Then $a^2 = a^4$. Hence, $a^2(1 - a^2) = 0$. This shows that $(1 - a^2)^2a^2(1 - a^2)a = 0$, i.e., $(a - a^3)^3 = 0$. In light of Corollary 3.9, the result follows. □

Let $n \geq 2$ be a fixed integer. Following Tominaga and Yaqub, a ring $R$ is said to be generalized $n$-like provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$ (\cite{14}).

**Corollary 3.11.** Let $R$ be a generalized 3-like ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Let $a \in R$. Then $(a - a^3)^2 = 0$, hence the result by Corollary 3.9. □

Recall that a ring $R$ is strongly SIT-ring if every element in $R$ is the sum of an idempotent and a tripotent that commute (cf. \cite{15}). We have

**Corollary 3.12.** Let $R$ be a strongly SIT-ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.

**Proof.** Let $R$ be a strongly SIT-ring, and let $a \in R$. In view of \cite[Theorem 3.10]{15}, we see that $a^6 = a^4$; hence, $a^4(1 - a^2) = 0$. This implies that $(a - a^3)^5 = 0$. In light of Corollary 3.9, we complete the proof. □

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