Synchronization analysis between exchange rates based on purchasing power parity using the Hilbert transform

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Abstract
Synchronization is a phenomenon when a pair of fluctuations adjust their rhythms when they interact with each other. We measure the degree of synchronization between the exchange rates of the U.S. dollar (USD) and the euro, and between those of the USD and the Japanese yen based on purchasing power parity (PPP) over time. We employ a method of synchronization analysis using the Hilbert transform which is common in the field of nonlinear science. We find that the synchronization degree is high most of the time, suggesting a PPP establishment. The synchronization degree does not remain high across periods containing economic events with asymmetric effects, such as the U.S. real estate bubble.

Keywords: Synchronization, Hilbert transform, Band-pass filter, Exchange rate, U.S. dollar, Euro, Japanese yen, Purchasing power parity

JEL Classification: C02, C14, C65, F31
1 Introduction

Exchange rates have various theories, such as purchasing power parity (PPP) (Cassel 1918) and interest rate parity (IRP) (Keynes 1923). These theories hold for different time scales; PPP is a long-term theory, whereas IRP is a short-term theory. If PPP holds, then the exchange rate adjusts a deviation from PPP level within several years. (See Rogoff 1996). Ito (1997) and Taylor (2002) apply a unit root test to real exchange rates. Hasan (2006) applies a unit root test and a cointegration test to real exchange rate of Australia and Canada. Each study shows that deviation tends to take several years to adjust to PPP level depending on data. Enders and Hurn (1994), Sarno (1997), Ogawa and Kawasaki (2008), and Mishra and Sharma (2010) analyze long-term equilibrium values of PPP in multiple real exchange rates.

However, the degree to which the exchange rate theory holds may differ between periods. In other words, the exchange rate adjusts to the equilibrium value indicated by the theory in a certain period, whereas in other periods, it does not. Thus, this study quantifies the degree to which the exchange rate theory holds at each period. We focus on exchange rate synchronization based on PPP. Synchronization is a phenomenon when a pair of fluctuations adjust their rhythms when they interact with each other, that is, the phase difference between the two fluctuations maintains a constant level during a certain time interval. Approximately, phase denotes a specific position $(-\pi, \pi]$ in one amplitude of a big fluctuation. The synchronization analysis quantifies the degree of rhythm adjustment between two time series. The following studies apply the synchronization concept to economic analysis. Flood and Rose (2010), Ikeda et al. (2013), and Esashi et al. (2018) apply synchronization to study business cycles. (See Onozaki 2018). Vodenska et al. (2016) perform synchronization analysis to examine the interaction and lead–lag relationship between the stock market and foreign exchange market. Wälti (2011) investigates the relationship between stock market co-movements and monetary integration.

When two numeraire-denominated exchange rates synchronize around the PPP level, we consider that the ratio of the two data adjusts to the original exchange rate’s PPP level. Therefore, synchronization degree measures the impact of PPP on the original exchange rate. PPP impact denotes the strength to which the exchange rate adjusts to the PPP level. If factors other than PPP significantly impact exchange rates, then PPP impact can be small.

We analyze the exchange rates between the U.S. dollar (USD) and euro (EUR) and between the USD and Japanese yen (JPY). A monetary authority’s foreign exchange intervention hinders theory establishment, and mutual influence is necessary in this analysis. The USD, EUR, and JPY are the three most traded currencies and have floating exchange rate systems.

From our analysis, we find that synchronization degree is stably high between the USD and

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3 In addition to PPP and IRP, the flexible-price monetary model (Frenkel 1976) and sticky-price monetary model (Dornbusch 1976) are also used.

4 IRP analysis should require high-frequency tick data. IRP analysis will be a future task due to data constraints.
EUR and between the USD and JPY during certain periods. This result suggests that the USD/EUR and the USD/JPY exchange rates adjust to PPP level during the period. In other periods, the synchronization degree does not maintain a high level either for the USD/EUR or USD/JPY. This result suggests that certain factors other than PPP affect the exchange rate during a period.

The remainder of this paper is structured as follows. Section 2 defines synchronization, and Section 3 explains the data used in this analysis. Section 4 provides calculations of exchange rate deviation from PPP level and introduces a frequency-based filter. Section 5 conducts a synchronization analysis using the Hilbert transform. Section 6 shows our synchronization analysis results. Section 7 discusses the difference between synchronization analysis and correlation coefficient. Finally, Section 8 concludes our paper.

2 Synchronization

Synchronization is a phenomenon when a pair of fluctuations adjust their rhythms through mutual interaction. Rhythm adjustment indicates that the phase difference between two time series adjusts to remain constant for a certain time interval. We use the notion of phase to capture the synchronization degree. The synchronization concept is explained the simple vibration \( f(t) = \sin(2\pi t) + 5 \) and \( g(t) = 2\sin(2\pi(t - 0.25)) + 5 \) [Figure 1(a)]. Phase represents a specific position \((-\pi, \pi]\) in 1 amplitude of oscillation data. The phases of \( f(t) \) and \( g(t) \) are \( 2\pi t \) and \( 2\pi(t - 0.25) \), respectively, where each phase is converted into \((-\pi, \pi]\) value. Two time series, namely, \( f(t) \) and \( g(t) \), are synchronized if the phase difference is constant in time. In the case above, the phase difference is \( 0.5\pi \) for all \( t \), which are synchronized. This synchronization is called phase synchronization.

Figure 1(b) shows the ratio of two synchronized time series, whose ratio \( \frac{2\sin(2\pi(t - 0.25)) + 5}{\sin(2\pi t) + 5} \) fluctuates around a reference value of approximately 1. In other words, this ratio has a property of returning to the reference value. We use this property to quantify the strength of the exchange rate returning to the PPP level.

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5 More precisely, it is synchronization between the deviations of the exchange rate from PPP level of numeraire-denominated USD and EUR, or that of the denominated USD and JPY. See section 4.3
6 For details, see Section 5.1 for the definition of the phase.
3 Data

We use the monthly average nominal exchange rate data of the USD/EUR, USD/JPY, Australian dollar (AUD)/USD, AUD/EUR, AUD/JPY, the New Zealand dollar (NZD)/USD, NZD/EUR, NZD/JPY during 1999:01–2017:12, 1987:01–2017:12, 1999:01–2017:12, 1999:01–2017:12, 1999:01–2017:12, 1999:01–2017:12, 1999:01–2017:12, and 1987:01–2017:12, respectively. These data are obtained from Datastream. We use a monthly producer price index data for price level data of the United States, the euro area, and Japan, where the 2010 data are normalized as 100. They are obtained from the International Financial Statistics of the International Monetary Fund (IMF) website. Current account balances (percent of GDP) of the United States, the euro area, and Japan are yearly data obtained from the World Economic Outlook of the IMF website.

4 PPP and frequency band

4.1 PPP

PPP level is calculated as

$$PPP_t^{USDj} = S_{base}^{USDj} \frac{P_t^{j}}{P_{base}^{j}} \frac{P_t^{USD}}{P_{base}^{USD}}$$

(1)

where $S_{t}^{USDj}$ denotes the USD/currency $j$ ($j = EUR, JPY$) exchange rate at time $t$, $S_{base}^{USDj}$ denotes the USD/currency $j$ ($j = EUR, JPY$) exchange rate at a base time, $P_t^{j}$ denotes the country of currency $j$ price index at time $t$, and $P_{base}^{j}$ denotes the country of currency $j$ price index at a base time. To calculate PPP a base time is selected when the current account balances are close to zero, $PPP_t^{USDEUR}$ is in 2010, and $PPP_t^{USDJPY}$ is in 1991. Exchange rate fluctuation around PPP ($FPPP$) is calculated as
\[ FPPP_{i}^{USD} = \frac{S_{i}^{(SD)}}{PFP_{i}^{(SD)}}. \]  

Figure 2(a) represents the USD/EUR exchange rate and PPP level. When the USD/EUR exchange rate deviates from the PPP level, it returns to its level in approximately 1–5 years. The exchange rate has deviated from the PPP level since 2015. Figure 2(b) represents the divergence of the USD/EUR exchange rate from PPP, which fluctuates around 1. Similarly, Figure 2(c) represents the USD/JPY exchange rate and PPP level. The USD/JPY exchange rate returns to the PPP level approximately every 2–5 years. In addition, the return time is longer than that of the USD/EUR exchange rate. The exchange rate has deviated from the PPP level since 2013. Figure 2(d) represents the divergence of the USD/JPY exchange rate from the PPP level, which fluctuates around 1.

Figure 2. (a) USD/EUR exchange rate and \( PPF_{i}^{USD} \). Note: “●” (blue) represents the USD/EUR exchange rate, and “×” (orange) represents the PPP level. (b) \( FPPP_{i}^{USD} \). Note: “●” (purple) represents the divergence of the USD/EUR exchange rate from PPP. (c) USD/JPY exchange rate and \( PPF_{i}^{USD} \). Note: “●” (blue) represents the USD/JPY exchange rate, and “×” (orange) represents the PPP level. The USD/JPY exchange rate has returned to the PPP level over a long period. (d) \( FPPP_{i}^{USD} \). Note: “●” (purple) represents the divergence of the USD/JPY exchange rate from the PPP level.
4.2 Power spectrum

Exchange rates have many determinants other than PPP. If data contain various fluctuation sizes, then the appropriate phase cannot be clearly defined. Therefore, we need to identify the frequency band that holds PPP and generates data with the frequency. We use the power spectrum to identify the frequency band of the exchange rate fluctuation around PPP. Figure 3(a) shows the log power of $F_{t}^{\text{USDEUR}}$, where the horizontal axis represents the monthly frequency. For example, the leftmost part of the figure suggests that a frequency component with amplitude of 228 months has the largest power. The arrow represents the band used in this analysis. Figure 3(b) shows the log power of $F_{t}^{\text{USDJPY}}$.

![Log power of $F_{t}^{\text{USDEUR}}$](image1)

![Log power of $F_{t}^{\text{USDJPY}}$](image2)

Figure 3. (a) Log power of $F_{t}^{\text{USDEUR}}$  (b) Log power of $F_{t}^{\text{USDJPY}}$

The power spectrum indicates frequency power included in the time series. We focus on low-frequency components. From the power spectrum, we can expect that the USD/EUR exchange rate fluctuates around PPP in frequencies ranging approximately 32.6–228.0 and 38.0–228.0 months. Then, we extract the time series with the frequency band of that range using the lower cutoff frequency of $k_0 = 1$ and upper cutoff frequency of $k_1 = 7$, which correspond to 228.0 ($\approx 228/k_0$) and 32.6 ($\approx 228/k_1$) months, respectively. (See the Appendix For $k_0$ and $k_1$.) Thus, if PPP holds, then deviations from PPP of the USD/EUR exchange rate have vanished from 16.3 (32.6/2) to 114.0 (228.0/2) months. Therefore, we focus on the frequency band ranging 32.6–228.0 and 38.0–228.0 months in the USD/EUR analysis.

Similarly, for the USD/JPY exchange rate, the corresponding frequency bands of 37.2–186.0 or 41.3–186.0 months are expected. Thus, if PPP holds, then the USD/JPY exchange rate deviations from PPP vanish from 18.6 (37.2/2) to 93.0 (186.0/2) months. We focus on 37.2–186.0 and 41.3–86.0 months the frequency band in the USD and JPY analysis. We consider that the band movements represent price adjustments through trade and productivity adjustments.

4.3 Numeraire

We analyze the synchronization between the fluctuations of the USD and currency $j$ of

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7 See Appendix A for the power spectrum details.
Using numeraire to divide $FPPP_t^{USD}$ into the USD and currency $j$ parts. Then, we study the synchronization between these two time series. Frankel and Wei (1994) and McKinnon and Schnabl (2004) use this method to analyze the linkage between multiple exchange rates using numeraire-denominated exchange rates. $FPPP_t^{USD}$ can be written using the AUD numeraire as\(^8\)

$$FPPP_t^{USD} \approx \frac{FPPP_t^{AUD}}{FPPP_t^{USD}} = \frac{S_t^{AUD}}{S_t^{USD}} \left( \frac{P_t^j}{P_t^i} \right) = \frac{FPPP_t^{AUD}}{FPPP_t^{USD}}, \quad (3)$$

where $FPPP_t^{USD} = \frac{S_t^{AUD}}{S_t^{USD}} \left( \frac{P_t^j}{P_t^i} \right)$ and $FPPP_t^{AUD} = \frac{S_t^{AUD}}{S_t^{USD}} \left( \frac{P_t^j}{P_t^i} \right)$. When $FPPP_t^{AUD}$ and $FPPP_t^{USD}$ exclude the AUD inflation rate from $FPPP_t^{USD}$ and $FPPP_t^{AUD}$. When $FPPP_t^{AUD}$ and $FPPP_t^{USD}$ have the same fluctuation, $FPPP_t^{USD}$ fluctuates around 1 (see Figure 1) therefore, suggesting PPP establishment.

### 4.4 Band-pass filter

We extract a frequency band of numeraire-denominated exchange rate based on PPP as described above. We generate time series data with frequency estimated from each $FPPP_t^{USD}$ and $FPPP_t^{AUD}$ using a band-pass filter.\(^10\) Figure 4(a) shows $BFPPP_t^{USD}$ and $BFPPP_t^{AUD}$ with a 32.6–228.0-month band-pass filter applied to $FPPP_t^{USD}$ and $FPPP_t^{AUD}$. The two time series fluctuation rhythms do not adjust around year 2015 but in other periods. Figure 4(b) shows $BFPPP_t^{USD}$ and $BFPPP_t^{AUD}$ with a 38.0–228.0-month band-pass filter applied to $FPPP_t^{USD}$ and $FPPP_t^{AUD}$. The two time series fluctuation rhythms do not adjust around 2006 but in other periods. Figure 4(c) shows $BFPPP_t^{USD}$ and $BFPPP_t^{AUD}$ with a 37.2–186.0-month band-pass filter applied to $FPPP_t^{USD}$ and $FPPP_t^{AUD}$. The two time series fluctuation rhythms adjust around 2001 but not in other periods. Figure 4(d) shows $BFPPP_t^{USD}$ and $BFPPP_t^{AUD}$ with a 41.3–186.0-month band-pass filter applied to $FPPP_t^{USD}$ and $FPPP_t^{AUD}$. The two time series fluctuation rhythms adjust around 2001 but not in other periods.

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\(^8\) Numeraire often uses the currency of the floating exchange rate system. The Swiss franc (CHF) and NZD are also often used for numeraire. We do not use the CHF because Swiss National Bank sets minimum exchange rate at CHF 1.20 per EUR from 2011 to 2015. Changing the NZD to numeraire does not affect the result. See Appendix B for details.

\(^9\) $FPPP_t^j$ is not PPP in a strictly sense because it excludes the inflation rate of numeraire. However, we use $FPPP_t^j$ because of the small sample size of the PPI data of Australia. As shown in Eq. (3), this condition does not affect the analysis because the $FPPP_t^j$ ratio is the same as that of $FPPP_t$.

\(^10\) See Appendix A for the band-pass filter details. Rodriguez et al. (1999) and Varela et al. (2001) perform a synchronization analysis using band-pass-filtered data for brain science research. Business cycle studies often use band-pass-filtered economic time series data (Baxter and King 1999; Calderón, Chong, and Stein 2007).
Figure 4. (a) $BFPPP_{t}^{AUDUSD}$ and $BFPPP_{t}^{AUDEUR}$ with a 32.6–228.0-month band-pass filter applied to $FPPP_{t}^{AUDUSD}$ and $FPPP_{t}^{AUDEUR}$. (b) $BFPPP_{t}^{AUDUSD}$ and $BFPPP_{t}^{AUDEUR}$ with a 38.0–228.0-month band-pass filter applied to $FPPP_{t}^{AUDUSD}$ and $FPPP_{t}^{AUDEUR}$. (c) $BFPPP_{t}^{AUDUSD}$ and $BFPPP_{t}^{AUDJPY}$ with a 37.2–186.0-month band-pass filter applied to $FPPP_{t}^{AUDUSD}$ and $FPPP_{t}^{AUDJPY}$. (d) $BFPPP_{t}^{AUDUSD}$ and $BFPPP_{t}^{AUDJPY}$ with a 41.3–186.0-month band-pass filter applied to $FPPP_{t}^{AUDUSD}$ and $FPPP_{t}^{AUDJPY}$.

5 Synchronization analysis using the Hilbert transform

5.1 Hilbert transform and instantaneous phase

We define a phase to measure the phase difference. We assume that $BFPPP_{t}^{AUDUSD}$ and $BFPPP_{t}^{AUDEUR}$ are obtained from the real part of the complex variable data. We can generate the imaginary part of the complex data using the Hilbert transform value of the real part. The Hilbert transform can be realized by an ideal filter, for which the amplitude and phase responses are unity and a constant $\pi/2$ lag at all Fourier frequencies (Pikovsky et al. 2003, pp. 362-363). The Hilbert transform is expressed as

$$s_{t}^{H} = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{s_{\tau}}{t - \tau} d\tau,$$

(4)

Ikeda et al. (2013) employs the Hilbert transform for synchronization analysis of the business cycle, and Vodenska et al. (2016) employs the Hilbert transform for synchronization analysis between the stock market and the foreign exchange market.
where \( s_t \) denotes time series data at time \( t \) and \( P.V. \) denotes the Cauchy principal value integrals. Phase is defined by the angle \( \phi_t \) formed by the horizontal axis and complex variable data

\[
\phi_t = \begin{cases} 
\tan^{-1}\left( \frac{s_t''}{s_t} \right) & (s_t > 0) \\
\tan^{-1}\left( \frac{s_t''}{s_t} \right) + \pi & (s_t > 0) 
\end{cases}
\] (5)

Phase at a certain time is called an instantaneous phase. The value can be discontinuous over time because it ranges from \( -\pi \) to \( \pi \). Figure 5(a) shows \( s_t = \sin(2\pi t) \) and its Hilbert transform value \( s_t'' = \sin(2\pi(t-0.25)) \), which implies \( s_t'' = s_t - \sqrt{2} \). Figure 5(b) shows a behavior of \((s_t, s_t'')\) in a complex plane.\(^{12}\)

Phase is identified by the angle \( \phi_t \) formed by the real axis and complex variable data.

![Figure 5](image_url)

Figure 5. (a) \( s_t = \sin(2\pi t) \) and its Hilbert transform value \( s_t'' = \sin(2\pi(t-0.25)) \). Note: “\( \times \)” (blue) represents \( s_t'' = \sin(2\pi(t-0.25)) \) delayed by \( \pi/2 \) from \( s_t = \sin(2\pi t) \) and is created by the Hilbert transform. (b) Behavior of \((s_t, s_t'')\) on a complex plane.

Figures 6(a) and (b) show a behavior of \((s_t, s_t'')\), where \( s_t = BFPP_{t, AUDEUR} \) and \( BFPP_{t, AUDUSD} \) with 32.6–228.0 months, respectively. Figures 6(c) and (d) show a behavior of \((s_t, s_t'')\), where \( s_t = BFPP_{t, AUDUSD} \) and \( BFPP_{t, AUDJPY} \) with 37.2–186.0 months, respectively.

\(^{12}\) An outlier occurs at both ends of the Hilbert transform values when we perform numerical computations. Thus, certain complex variable data near \((0, -1)\) in the complex plane are slightly deviated from the unit circle. Therefore, we exclude 10 data points from both ends in the following analysis.
Figure 6. (a) Behavior of \((s_t, s_t^H)\), where \(s_t = BFPPP_t^{AUDUSD}\) with 32.6–228.0 months. (b) Behavior of \((s_t, s_t^H)\), where \(s_t = BFPPP_t^{AUDEUR}\) with 32.6–228.0 months. (c) Behavior of \((s_t, s_t^H)\), where \(s_t = BFPPP_t^{AUDUSD}\) with 37.2–186.0 months. (d) Behavior of \((s_t, s_t^H)\), where \(s_t = BFPPP_t^{AUDJPY}\) with 37.2–186.0 months.

Figures 7(a)–(d) show instantaneous phase of 32.6–228.0, 38.0–228.0, 37.2–186.0, and 41.3–186.0 months \(BFPPP_t^{AUDUSD}\) and \(BFPPP_t^{AUDEUR}\), \(BFPPP_t^{AUDUSD}\) and \(BFPPP_t^{AUDEUR}\), \(BFPPP_t^{AUDUSD}\) and \(BFPPP_t^{AUDJPY}\), and \(BFPPP_t^{AUDUSD}\) and \(BFPPP_t^{AUDJPY}\), respectively. The phase differences in Figures 7(a)–(d) are not constant around 2006, 2006, 2005 and 2012, and 2005, respectively, but all are almost constant in other periods.
Phase difference discontinuity affects its analysis. Therefore, we use an unwrapped instantaneous phase, which is defined, to allow continuous change in the time development of instantaneous phase.

The phase difference is expressed as

\[ \psi_t = \hat{\phi}_t^{AUDUSD} - \hat{\phi}_t^{AUDJ}, \]  

where \( \hat{\phi}_t^{AUDUSD} \) and \( \hat{\phi}_t^{AUDJ} \) denote unwrapped instantaneous phases of \( BFPPP_t^{AUDUSD} \) and \( BFPPP_t^{AUDJ} \) both at time \( t \), respectively. We say that \( BFPPP_t^{AUDUSD} \) and \( BFPPP_t^{AUDJ} \) synchronize in the time interval \( [t_0, t_1] \) if there exists a constant \( d \) and a sufficiently small positive constant \( \varepsilon \), such that:

\[ |\psi_t - d| < \varepsilon. \]
for $t_0 \leq \forall t \leq t_i$.

5.2 Synchronization index

We employ the synchronization index $\gamma^2$ (Rosenblum et al. 2001) for the time interval $1 \leq i \leq W$ using phase difference $\psi_i$ at time $i$ to measure the synchronization degree between two time series,

$$\gamma^2 = \left(\frac{1}{W} \sum_{n=1}^{W} \cos \psi_i\right)^2 + \left(\frac{1}{W} \sum_{n=1}^{W} \sin \psi_i\right)^2. \quad (8)$$

Index $\gamma^2$ ranges from 0 to 1. When phase difference between two time series is constant in time, $\psi_i$ takes a constant value. Thus, $\gamma^2$ takes a value close to 1 because $(\cos \psi_i, \sin \psi_i)$ moves in the vicinity of one point on the unit circle. Therefore, if $\gamma^2$ is close to 1, then the two time series has high synchronization degree. In contrast, if $\gamma^2$ is close to 0, then the synchronization degree is low.

Figure 8 shows the expected value of synchronization index $\gamma^2$ relevant to $W$. When we choose $W = 13$, the expected value of $\gamma^2$ is 0.077 and the standard deviation of $\gamma^2$ is 0.074. Therefore, if the value of $\psi_i$ is given at random, then the synchronization index takes a value close to 0.077, which will be a reference value for judging the randomness of $\psi_i$.

![Figure 8](image)

We measure the synchronization index for the time interval $W$ at each time $t$,

$$\gamma_t^2 = \left(\frac{1}{W} \sum_{n=p}^{W} \cos \psi_i\right)^2 + \left(\frac{1}{W} \sum_{n=p}^{W} \sin \psi_i\right)^2, \quad (9)$$

where $p = (W-1)/2$ and $0 < p < t$. In the following analysis, we set the window size of 13 (approximately 1-year period).\textsuperscript{13}

\textsuperscript{13} Changing the window size does not affect the result. We perform the synchronization analysis using four window sizes: 7 (approximately 0.5 year), 13 (approximately 1 year), 19 (approximately 1.5 years), and 25 (approximately 2 years). Although the amplitude magnitude is different, periods for which the synchronization index peaks do not change. In addition, periods for which the synchronization index maintains a high value do not change.
6 Results and interpretation

6.1 Result summary

Figure 9 shows the synchronization index. If it is stably high, the synchronization degree between $BFPP_{t}^{\text{AUD/USD}}$ (the AUD/USD exchange rate band-pass-filtered fluctuation around PPP, excluding the AUD inflation rate) and $BFPP_{t}^{\text{AUD}}$ is high. We call the synchronization index stably high when it maintains a high for a certain period. It suggests that the USD/currency $j$ exchange rate fluctuates aroundPPP level, which confirms the PPP establishment (see Figure 1). Conversely, if the synchronization index does not maintain a high level, the synchronization degree between $BFPP_{t}^{\text{AUD/USD}}$ and $BFPP_{t}^{\text{AUD}}$ is low. Although the synchronization index is temporarily high, the two instantaneous phase rhythms may match by chance. Therefore, the synchronization degree in such cases is not necessarily high. The low synchronization degree suggests that the USD/currency $j$ exchange rate fluctuated during this period due to factors other than PPP. For example, interest rate difference affects exchange rates. If the exchange rate fluctuates due to interest rate differences, then PPP may not hold. Therefore, the two time series has low synchronization degree in such periods.

Figure 9(a) shows the time development of the synchronization index $\hat{\gamma}_{p}^{2}$ between $BFPP_{t}^{\text{AUD/USD}}$ and $BFPP_{t}^{\text{AUD/EUR}}$. The synchronization degree is stably high around 2000:05–2005:08 and 2007:09–2013:04. This finding suggests that the USD/EUR exchange rate fluctuates around the PPP level this period. From 2002:04 to 2002:09, the synchronization degree decreases slightly in the short term. Therefore, we consider that this period belongs to a stably high period. Similarly, from 2013:05 to 2014:07, the synchronization degree decreases slightly. However, when we replace numeraire, the synchronization degree is greatly reduced (see Figure 10 in Appendix B). Therefore, this period is not stably high.

Figure 9(b) shows the time development of the synchronization index $\hat{\gamma}_{p}^{2}$ between $BFPP_{t}^{\text{AUD/USD}}$ and $BFPP_{t}^{\text{AUD/JPY}}$. The synchronization degree is stably high during 1991:08–1997:12, 1999:04–2003:10, and 2007:09–2010:10. This suggests that the USD/JPY exchange rate fluctuates around the PPP level during this period. From these results, the synchronization degree between $BFPP_{t}^{\text{AUD/USD}}$ and $BFPP_{t}^{\text{AUD/EUR}}$ and between $BFPP_{t}^{\text{AUD/USD}}$ and $BFPP_{t}^{\text{AUD/JPY}}$ is high for the most parts. Therefore, PPP holds in the frequency band used in this analysis. The synchronization degree is low in certain periods due to two reasons. First, an economic event may have an asymmetric effect on each $BFPP_{t}$. Second, factors other than PPP affect exchange rates. For example, exchange rates change when interest rate difference between two countries increases.
Figure 9. (a) Synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDEUR}$. Note: The gray-colored interval indicates that the synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDEUR}$ from 32.6 to 228.0 months is the synchronization index $\gamma^2 \geq \gamma^2 (\approx 0.92)$, where $\gamma^2$ denotes the average value $\pm 0.25$ (standard deviation).

(b) Synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDJPY}$. Note: The gray-colored interval indicates that the synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDJPY}$ from 37.2 to 186.0 months is $\gamma^2 \geq \gamma^2 (\approx 0.85)$.

6.2 Results interpretation

This section discusses the relationship between fluctuations in the synchronization indices in the previous section and economic events. First, for the synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDEUR}$, the synchronization index is stably high in many periods, suggesting the PPP establishment. However, the synchronization degree does not remain at a high level around 2005:09–2007:08 and after 2013:05. Around 2005–2006, a housing bubble and subsequent housing market collapse occurred in the United States. House prices fluctuated frequently in this period. Therefore, PPP cannot explain the price index fluctuation of the United States during this period. Since the asymmetric event occurs, the synchronization degree does not maintain a high level after 2013:05.

Second, for the synchronization index between $BFPPP^t_{AUDUSD}$ and $BFPPP^t_{AUDJPY}$, the Lehman Brothers’ bankruptcy in 2008:09 marked the beginning of a worldwide recession. During the economic crisis, countries’ economic variables tend to move in the same direction. Therefore, the synchronization degree is high around 2008:9 and may not be affected by PPP.
synchronization index is stably high in several periods, suggesting the PPP establishment. However, the synchronization degree does not remain at a high level from 1988:05 to 1991:07, around 1999, from 2003:11 to 2007:08, and after 2010:11. From 1986 to 1989, Japan experienced an economic bubble, and stocks and real estate prices soared. In addition, a technology bubble occurred around 1999. These events may have hindered the PPP establishment from 1988:06 to 1991:07 and around 1999. Around 2005–2006, a housing bubble and subsequent housing market collapse occurred in the United States. The period of the event overlaps with the period when the synchronization degree does not maintain a high level between 2003:11 and 2007:08. The Great East Japan Earthquake in 2011:03 caused violent JPY fluctuations. During this period, the JPY appreciated due to the JPY purchase for insurance claim payments following the earthquake. In addition, investors may have bought the JPY in anticipation of a strong JPY. The synchronization degree is low around 2011, which may be due to JPY fluctuations. Moreover, the JPY has depreciated against the USD since 2013, when the exchange rate has remained deviated from the PPP level (see Figure 2[c]). The JPY has depreciated since 2013 due to quantitative easing by the Bank of Japan and the rebound to the highest JPY value around 2011. PPP may not explain this exchange rate fluctuation. Therefore, the synchronization degree does not remain high after 2010:11. The Lehman Brothers’ collapse in 2008:09 initiated a worldwide recession. Countries’ economic variables during the economic crisis tend to move in the same direction. Therefore, the synchronization degree is high around 2008:09 and may not be affected by PPP.

7 Comparison with the correlation coefficient

A correlation coefficient is often used in economic studies to measure the strength of the relationship between the movements of two time series. However, time difference in phases between two synchronized time series affects correlation coefficients. In contrast, the synchronization index is not affected by time difference in phase. Figure 10(a) shows the synchronization index and correlation coefficient between \( \sin(2\pi t) \) and \( \sin(2\pi(t + \Delta t)) \) relevant to \( \Delta t \). The synchronization index is 1, regardless of the existence of time difference in phase. Depending on \( \Delta t \) size, the correlation coefficient can take any value between -1 and 1. The synchronization index is useful for measuring the synchronization of two time series with time difference in phase.

If the time difference in phase of two time series does not significantly change over time or that in each period is clearly known, then the correlation coefficient can be used by shifting time series with the time difference in phase. However, an appropriate time difference is difficult to find from our data. We compare the synchronization index and correlation coefficient with time difference in phase between \( BFPP_{AUDUSD} \) and \( BFPP_{AUDY} \). Figures 10(b) and (c) show time series of the synchronization index and the absolute value of a
correlation coefficient and that of a correlation coefficient with time difference in phase. The correlation coefficient is calculated using the same moving window (window size $W = 13$) as that of the synchronization index. In addition, the time difference correlation coefficient considers time difference in phase between two time series. Figure 10(b) shows the synchronization index and correlation coefficient between $BFPPP_{AUDUSD}$ and $BFPPP_{AUDEUR}$ from 32.6 to 228.0 months. Figure 10(c) shows the synchronization index and correlation coefficient between $BFPPP_{AUDUSD}$ and $BFPPP_{AUDJPY}$ from 37.2 to 186.0 months. Based on these figures, when the synchronization index greatly decreases during a period, the correlation coefficient also decreases. Time series of synchronization index and correlation coefficient sometimes behave differently, when the synchronization index is stably high. An appropriate time difference in phase is difficult to find from our data in which lead–lag relationship varies over the short term. Thus, we employ the synchronization index.

\[ \text{(a)} \]

\[ \text{(b)} \]

14 See Appendix C for the calculation method of the correlation coefficient with time difference in phase.
Figure 10. (a) Synchronization index and correlation coefficient between two time series \( \sin(2\pi t) \) and \( \sin(2\pi(t+\Delta t)) \). (b) Synchronization index and correlation coefficient between \( BFPPP^\text{AUDUSD} \) and \( BFPPP^\text{AUDEUR} \) from 32.6 to 228.0 months. Note: “●” (orange), “×” (blue), and “▲” (green) represent the synchronization index, absolute value of the correlation coefficient, and absolute value of the correlation coefficient with the time difference, respectively, between \( BFPPP^\text{AUDUSD} \) and \( BFPPP^\text{AUDEUR} \) from 32.6 to 228.0 months. (c) Synchronization index and correlation coefficient between \( BFPPP^\text{AUDUSD} \) and \( BFPPP^\text{AUDJPY} \) from 37.2 to 186.0 months. Note: “●” (orange), “×” (blue), and “▲” (green) represent the synchronization index, absolute value of the correlation coefficient, and absolute value of the correlation coefficient with the time difference, respectively, between \( BFPPP^\text{AUDUSD} \) and \( BFPPP^\text{AUDJPY} \) from 37.2 to 186.0 months.

8 Conclusions

We determine the synchronization degree between \( BFPPP^\text{AUDUSD} \) (band-pass-filtered fluctuation around the PPP of the AUD/USD” exchange rate, excluding the AUD inflation rate) and \( BFPPP^\text{AUDEUR} \) and between \( BFPPP^\text{AUDUSD} \) and \( BFPPP^\text{AUDJPY} \) is high for most of the times. This result suggests that PPP holds in the long term and that the PPP level is the long-term equilibrium value of the USD/EUR and USD/JPY exchange rates. However, the high-level synchronization degree is not maintained for several periods. This property can be attributed to the occurrence of asymmetrical economic events and factors other than PPP that affect the exchange rate, for example, during the U.S. real estate bubble and the Great East Japan Earthquake.

Correlation coefficients are inappropriate in this study due to frequent time difference in phase between two time series. In contrast, we use synchronization index in our analysis to measure the synchronization degree without identifying the time difference value in phase at each time. Therefore, synchronization analysis is suitable for our study.
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Appendix A. Fourier band-pass filter and power spectrum

Fourier band-pass filter. We focus on recurrent patterns in some specific time scale to measure the synchronization degree between two time series data. We employ a band-pass filter using a Fourier series representation and briefly review the Fourier series of a function $f$. For simplicity, let $f$ be a real-valued continuous periodic function on $[0, L)$. The function $f$ can be represented as a Fourier series in Eq. (10) as

$$f(x) = \frac{a_0}{2} + \sum_{k \neq 0} \left( a_k \cos \left( \frac{2\pi kx}{L} \right) + b_k \sin \left( \frac{2\pi kx}{L} \right) \right), \quad (10)$$

where

$$a_k = \frac{1}{L} \int_0^L f(x) \cos \left( \frac{2\pi kx}{L} \right) \, dx \quad (k = 0, 1, 2, 3, \ldots), \quad (11)$$

$$b_k = \frac{1}{L} \int_0^L f(x) \sin \left( \frac{2\pi kx}{L} \right) \, dx \quad (k = 1, 2, 3, \ldots). \quad (12)$$

We can consider the Fourier series for more general functions (e.g., Korner 2008). By taking a partial sum in Eq. (10), we can create a band-pass-filtered periodic function $\tilde{f}$ using bands $k$ for $1 \leq k_0 \leq k \leq k_i$, from a given function $f$

$$\tilde{f}(x) = \sum_{k=k_0}^{k_i} \left( a_k \cos \left( \frac{2\pi kx}{L} \right) + b_k \sin \left( \frac{2\pi kx}{L} \right) \right). \quad (13)$$

The transformation procedure from a discrete non-periodic time series data $g_n = g(x_0 + n\Delta x) \ (n = 0, \ldots, N-1)$ to the band-pass-filtered discrete periodic time series data $\tilde{f}_n = \tilde{f}(x_0 + n\Delta x) \ (n = 0, \ldots, N-1)$ is as follows:

1. Using the linear transformation determined by $g_0$ at $x_0$ and $g_{N-1}$ at $x_{N-1}$, convert a given set of uniformly discretized $N + 1$ time series data $(g_n)_{n=0, \ldots, N}$ into a periodic data $(f_n)_{n=0, \ldots, N}$ such that $f_0 = f_{N-1} = g_0$.
2. Compute Fourier coefficients $a_k, b_k$ for $(f_n)_{n=0, \ldots, N}$.
3. Construct a band-pass-filtered periodic time series data $(\tilde{f}_n)_{n=0, \ldots, N}$ using $a_k, b_k$ for $k_0, k_i (1 \leq k_0 \leq k \leq k_i)$.

Step 1

$$f_n = g_n - sx_n \Delta x, \quad n = 0, \ldots, N-1, \quad (14)$$
where \( s = (g_{N-t} - g_0)/(x_{N-t} - x_0) \), \( x_n = x_0 + n\Delta x \) and \( \Delta x = (x_{N-t} - x_0)/(N - 1) \).

Step 2
Compute

\[
a_k = \frac{1}{L} \sum_{n=0}^{L-1} f_n \cos \left( \frac{2\pi k x_n}{L} \right) \Delta x \quad (k = 0, \ldots, K),
\]

\[
b_k = \frac{1}{L} \sum_{n=0}^{L-1} f_n \sin \left( \frac{2\pi k x_n}{L} \right) \Delta x \quad (k = 1, \ldots, K),
\]

for \( k_0, k_1 (k_0 < k < k_1) \).

Step 3

\[
\tilde{f}_n = \sum_{k=1}^{k_1} \left( a_k \cos \left( \frac{2\pi k x_n}{L} \right) + b_k \sin \left( \frac{2\pi k x_n}{L} \right) \right).
\]

Power spectrum. We compute power spectrum \( E(k) \) for each \( k \),

\[
E(k) = \frac{a_k^2 + b_k^2}{2}.
\]

If \( f \) is \( C^1 \) function, then \( k^1|a_k|, k^1|b_k| < \infty \), implying that the Fourier coefficients decrease exponentially as \( k \) increases. Notably, we plot \( \sqrt{E(k)} \), not \( E(k) \) (Figure 3).

Appendix B. Robustness check of numeraire

The third country’s currency is introduced as numeraire to analyze the synchronization between two currencies (e.g., USD and EUR). We use the AUD as numeraire in the main body. We conform the same results by using NZD as numeraire in this appendix.

Figure 11(a) shows the time development of the synchronization index between \( BFPPP_{t}^{AUDUSD} \) and \( BFPPP_{t}^{AUDEUR} \) and between \( BFPPP_{t}^{NZDUSD} \) and \( BFPPP_{t}^{NZDEUR} \) from 32.6 to 228.0 months. The time series of the two synchronization indices behave similarly, except around 2014. Figure 11(b) shows the time development of the synchronization index between \( BFPPP_{t}^{AUDUSD} \) and \( BFPPP_{t}^{AUDJPY} \) and between \( BFPPP_{t}^{NZDUSD} \) and \( BFPPP_{t}^{NZDJPY} \) from 37.2 to 186.0 months. The time series of the two synchronization indices behave similarly except around 1989:05.
Figure 11. (a) Synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDEUR}$ and between $BFPPP_t^{NZDUSD}$ and $BFPPP_t^{NZDEUR}$. Note: “●” (orange) and “×” (blue) represent the synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDEUR}$ and between $BFPPP_t^{NZDUSD}$ and $BFPPP_t^{NZDEUR}$ from 32.6 to 228.0 months, respectively. The gray-colored interval indicates that the synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDEUR}$ from 32.6 to 228.0 months is $\gamma^2 \geq \gamma^2 (\approx 0.92)$, where $\gamma^2$ denotes average value $+0.25$(standard deviation). (b) Synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDJPY}$ and between $BFPPP_t^{NZDUSD}$ and $BFPPP_t^{NZDJPY}$. Note: “●” (orange) and “×” (blue) represent the synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDJPY}$ and between $BFPPP_t^{NZDUSD}$ and $BFPPP_t^{NZDJPY}$ from 37.2 to 186.0 months, respectively. The gray-colored interval indicates that the synchronization index between $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUDJPY}$ from 37.2 to 186.0 months is $\gamma^2 \geq \gamma^2 (\approx 0.85)$.

Appendix C. Calculation method for a correlation coefficient with time difference in phase

Time difference in phase between two time series affects a correlation coefficient. Thus, the correlation coefficient is not useful when the time difference is not sufficiently small. However, the correlation coefficient may show a result similar to that by the synchronization index when time difference in phase is considered. The correlation coefficient with time difference in phase is calculated as follows:

1. Identify the leading time series, and calculate phase difference at each time. Find $m_t (= \hat{m}_t)$ and $q_t (= \hat{q}_t)$ that satisfy $\min_{m \in \mathbb{Z}} (|m_t|)$, where $m_t = \hat{\phi}_t^{AUDUSD} - \hat{\phi}_t^{AUD} + 2q_t \pi$. The variables $\hat{\phi}_t^{AUDUSD}$ and $\hat{\phi}_t^{AUD}$ denote unwrapped instantaneous phases of $BFPPP_t^{AUDUSD}$ and $BFPPP_t^{AUD}$, respectively, both at time $t$.

2. Calculate time difference in phase at each time and $\hat{t}_i = l - 1$ that satisfies
\[
\begin{align*}
\max_{i \neq j, W} \left( \phi_{i}^{\text{AUDUSD}} - \phi_{i}^{\text{AUDj}} + 2\hat{q}_{i} \pi \right) & < 0 \quad (\hat{m}_{i} > 0) \\
\min_{i \neq j, W} \left( \phi_{i}^{\text{AUDUSD}} - \phi_{i}^{\text{AUDj}} + 2\hat{q}_{i} \pi \right) & > 0 \quad (\hat{m}_{i} < 0)
\end{align*}
\]

where \( L \) is chosen depending on the data. The time difference in phase at each time is expressed as

\[
lag_{i} = \begin{cases} \\
\hat{l}_{i} + \frac{\phi_{i}^{\text{AUDUSD}}}{\phi_{i+1}^{\text{AUDUSD}}} - \frac{\phi_{i}^{\text{AUDj}}}{\phi_{i+1}^{\text{AUDj}}} \quad (\hat{m}_{i} > 0) \\
\hat{l}_{i} + \frac{\phi_{i}^{\text{AUDUSD}}}{\phi_{i+1}^{\text{AUDUSD}}} - \frac{\phi_{i}^{\text{AUDj}}}{\phi_{i+1}^{\text{AUDj}}} \quad (\hat{m}_{i} < 0)
\end{cases}
\]

3. Compute the absolute value of the correlation coefficient with time difference in phase

\[
\hat{z}_{p}^{2} = \begin{cases} \\
\frac{1}{W} \left[ \sum_{t=1}^{1+W} (BFPPP_{t}^{\text{AUDUSD}} - BFPPP_{t}^{\text{AUDj}}) \left( BFPPP_{t+1+W}^{\text{AUDUSD}} - BFPPP_{t+1+W}^{\text{AUDj}} \right) \right] \quad (\hat{m}_{i} > 0) \\
\frac{1}{W} \left[ \sum_{t=1}^{1+W} (BFPPP_{t}^{\text{AUDUSD}} - BFPPP_{t}^{\text{AUDj}}) \left( BFPPP_{t+1+W}^{\text{AUDUSD}} - BFPPP_{t+1+W}^{\text{AUDj}} \right) \right] \quad (\hat{m}_{i} < 0)
\end{cases}
\]

where \( p = t + \frac{W+1}{2} \) and \( \overline{\text{lag}}_{i} = \frac{1}{W} \sum_{j=1}^{i-W} \text{lag}_{j} \). \( W \) denotes the window size.

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