ON HIGHER DIMENSIONAL EXTREMAL VARIETIES OF GENERAL TYPE

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Abstract. Relations among fundamental invariants play an important role in algebraic geometry. It is known that an $n$-dimensional variety of general type with nef canonical divisor and canonical singularities, whose image $Y$ under the canonical map is of maximal dimension, satisfies $K^n_X \geq 2(p_g - n)$. We investigate the very interesting extremal situation $K^n_X = 2(p_g - n)$, which appears in a number of geometric situations. Since these extremal varieties are natural higher dimensional analogues of Horikawa surfaces, we name them Horikawa varieties. These varieties have been previously dealt with in the works of Fujita [Fuj83] and Kobayashi [Kob92]. We carry out further studies of Horikawa varieties, proving new results on various geometric and topological issues concerning them. In particular, we prove that the geometric genus of those Horikawa varieties whose image under the canonical map is singular is bounded. We give an analogous result for polarized hyperelliptic subcanonical varieties, in particular, for polarized Calabi-Yau and Fano varieties. The pleasing numerology that emerges puts Horikawa’s result on surfaces in a broader perspective. We obtain a structure theorem for Horikawa varieties and explore their pluriregularity. We use this to prove optimal results on projective normality of pluricanonical linear systems. We study the fundamental groups of Horikawa varieties, showing that they are simply connected, even if $Y$ is singular. We also prove results on deformations of Horikawa varieties, whose implications on the moduli space make them the higher dimensional analogue of curves of genus 2.

1. Introduction

For a minimal surface of general type it is a well known result of Noether that its canonical divisor $K_S$ satisfies $K_S^2 \geq 2p_g - 4$. The surfaces for which $K_S^2 = 2p_g - 4$ have been dealt by Horikawa in his well known work [Hor76]. Horikawa shows that, for surfaces of general type on the Noether line, $K_S$ is indeed base

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point free and the complete linear system $|K_S|$ maps $S$ as a canonical double cover of a surface of minimal degree. These and, more generally, canonical double covers of rational surfaces have a ubiquitous presence in the geometry of algebraic surfaces.

In this article, we prove higher dimensional analogues of the results of Horikawa. Some very interesting work has been previously done by Fujita and Kobayashi. Fujita and Kobayashi proved (see [Fuj75] and [Kob92, Theorem 2.4]) that, if $X$ is a variety of general type with nef canonical divisor and at worst canonical singularities, whose image under the canonical map has maximum dimension, then $K^n_X \geq 2(p_g(X) - n)$. In this article we study, Horikawa varieties, i.e., those varieties for which the equality holds:

**Definition 1.1.** Let $X$ be a variety of general type of dimension $n$, $n \geq 2$ with at worst canonical singularities and whose canonical divisor $K_X$ is nef. We say that $X$ is a Horikawa variety if

1. the image of the canonical map of $X$ has dimension $n$; and
2. $K^n_X = 2(p_g(X) - n)$.

If $X$ is a Horikawa variety of arbitrary dimension, then $K_X$ is a Cartier divisor (and, hence, it has Gorenstein singularities) and the linear system $|K_X|$ is base–point–free (see [Kob92, Propositions 2.2, 2.5]). In addition, the canonical morphism of $X$ is a generically finite morphism of degree 2 onto a variety of minimal degree (see [Kob92, Proposition 2.5]). In fact this property characterizes those varieties of general type with nef canonical divisor and canonical singularities which are Horikawa varieties.

In Section 2 we study various geometric and cohomological aspects of these extremal varieties. We show that Horikawa varieties are regular (see Theorem 2.6). We also study the deformations of the canonical morphism of Horikawa varieties. We prove that if $X$ is a Horikawa variety of any dimension, then the deformation of its, degree 2, canonical morphism are again canonical of degree 2 whose image is a variety of minimal degree. Some of the cases have an overlap with results in [Fuj83], but, even in those instances, the methods of this article are very different and more transparent. There are implications of our results to the moduli space of varieties of general type. In this article, we also prove a structure theorem for Horikawa varieties (see Theorem 2.12): If $X$ is a Horikawa variety of dimension $n$ and $Y$ is smooth, then it is pluriregular and, in most cases, it possesses a fibration, with fibers $F$ of general type and $p_g(F) = n$.

In Section 2, we prove an interesting result (see Theorem 2.7) which shows that geometric genus of a Horikawa variety of dimension $n$ is bounded if the image under the canonical morphism is a singular variety $Y$. Precisely, we show that \( p_g(X) \leq n + 4 \). This gives raise to a beautiful numerology; indeed, this bound generalizes Horikawa’s bound in the case of surfaces, which is \( p_g \leq
6. In our proof, we consider the pullback $X_2$ of the intersection $Y_2$ of $n - 2$ hyperplane sections passing through a given point of $Y$. The surface $X_2$ maps onto $Y_2$ by a subcanonical linear series. Even though $X$ has at worst canonical singularities, the singularities might a priori get worse and worse, since we are taking special hyperplane sections to obtain $X_2$. We are able to show that $X_2$ has, nevertheless, only canonical singularities. This allows us to use in a crucial way the subtle arguments developed in the proof of Theorem 2.6 to bound the dimension of the projective space containing $Y_2$.

In Section 3 we study topological aspects of Horikawa varieties. Precisely, we show that Horikawa varieties are simply connected (see Theorem 3.3). Although Fujita proved previously in [Fuj83] the simple connectedness of some families of Horikawa varieties, our proof is very transparent and covers all the cases. Our methods are quite different from the methods in [Fuj83]. We crucially use results of Nori in [Nor83]. Even though simple connectedness implies regularity, the proof of Theorem 2.6 is independent, and necessary because of the reasons explained in the paragraph above.

The classification of Horikawa varieties involves canonical covers of varieties of minimal degree. The canonical covers of varieties of minimal degree have a significant presence in the geometry of algebraic surfaces and higher dimensional varieties of general type and they occur in a variety of contexts such as determination of very ampleness of linear series, ring generation, deformation theory and construction of varieties with given invariants (see [GP03], [GP98], [GGP10], [GGP16a], [GGP16b], [Hor76]). One of the most natural contexts where the canonical covers of minimal degree varieties occur is at the boundary of the geography of surfaces of general type. The results in this article and those in [Kob92] and [Fuj83], show that this is true for all higher dimensional varieties of general type as well on what can be called the “Noether faces”.

It is indeed compelling to note the beauty of the analogy with lower dimensional results, even though methods are different. The analogies are striking, despite the existence of much worse singularities. In view of Theorem 2.9 smooth Horikawa varieties can be considered as analogues of curves of genus 2, from the point of view of deformations and moduli. Moreover, the implications of our results on classification, regularity and deformations of Horikawa varieties yield interesting consequences for the moduli space of varieties of general type (see Corollary 2.10).

The question of projective normality of pluricanonical systems of a surface of general type has a long history dating back to Kodaira and Bombieri. In this article we also show that if $X$ is a Horikawa variety of dimension $n$ with an optimal condition on its geometric genus, then $|mK_X|$ embeds $X$ as a projectively normal variety if and only if $m \geq n + 1$. The standard methods involving Castelnuovo-Mumford regularity and vanishing theorems do not yield this result. We do in fact show a general statement that follows from these standard
methods but, to get the optimal statements, one has to use in an essential way the structure of Horikawa varieties proved in this article and find different methods, as we do in Section 4.

Finally, in Section 5 we show that the main results of Sections 2 and 3 (precisely, Theorems 2.6, 2.7, 2.12 (1) and 3.3), when considered for strong Horikawa varieties, fit in a broader setting, that is, they hold for hyperelliptic subcanonical polarized varieties. For instance, if \((X, A)\) is an \(n\)-dimensional hyperelliptic, subcanonical polarized variety with canonical singularities, then \(X\) is simply connected and, if the image of the morphism induced by \(|A|\) is singular, then \(h^0(A) \leq n + 4\). In particular, this holds if \((X, A)\) is a Calabi-Yau or Fano hyperelliptic polarized variety with canonical singularities.

2. COHOMOLOGICAL PROPERTIES OF HORIKAWA VARIETIES

We will use the following conventions and notations throughout the article.

**Convention 2.1.** We will work over \(\mathbb{C}\). By a variety we will mean an irreducible variety.

**Notation 2.2.** Let \(X\) be an algebraic projective normal variety.

1. If \(D\) is a Cartier divisor on \(X\), we will use indistinctly the notations \(H^0(\mathcal{O}_X(D))\) and \(H^0(X, D)\) (and \(h^0(\mathcal{O}_X(D))\) and \(h^0(X, D)\)) and \(|D|\) will denote the complete linear series of \(D\).
2. We will denote by \(K_X\) the canonical divisor of \(X\).
3. If \(X\) is a Horikawa variety as in Definition 1.1, then \(\varphi\) will denote the canonical morphism of \(X\), \(Y\) will be the image of \(\varphi\) (which is a variety of minimal degree) and \(\overline{X}\) will denote the canonical model of \(X\).

We now explore the topological and cohomological properties of Horikawa varieties and its applications. It is quite interesting that just one numerical equality gives raise to so much geometry. We start with a general study of the singularities that might occur in the context of this article.

**Proposition 2.3.** If \(X\) be a Horikawa variety, then the Stein factorization of the canonical morphism

\[\varphi : X \rightarrow Y\]

of \(X\) is

\[X \rightarrow \overline{X} \stackrel{\phi}{\rightarrow} Y,\]

where \(\phi\) is the canonical map of \(\overline{X}\), which is therefore, a morphism. In particular, the Stein factorization of \(X\) has at worst canonical singularities and \(Y\) has at worst log terminal singularities.

**Proof.** Since \(|K_X|\) is base-point-free by [Kob92, Propositions 2.2, 2.5], it is known (see [Laz04, Theorem 2.1.15]) that \(|K_X|\) gives the Stein factorization \(\tilde{X}\)
of the canonical morphism of $X$, for any sufficiently large integer $l$. Then, for a suitable $l' \in \mathbb{N}$
\[ \text{Proj} \left( \bigoplus_{m \geq 0} H^0(X, mK_X) \right) = \text{Proj} \left( \bigoplus_{m \geq 0} H^0(X, l'mK_X) \right) = \hat{X}, \]
so the canonical model $\overline{X}$ of $X$ is nothing but $\hat{X}$ and $\phi$ is induced by $|K_X|$. Therefore, $\hat{X}$ has at worst canonical singularities. It is a well-known fact that Kawamata log terminal singularities are preserved under finite maps (cf. [KM98, Proposition 5.20]), hence it follows that $Y$ has at worst log terminal singularities.

\[ \square \]

**Notation 2.4.** Given a Horikawa variety, we will keep the name $\phi$ used in Proposition 2.3 for the canonical morphism of $X$.

Recall (see [Kob92, Propositions 2.5]) that the canonical morphism of a Horikawa variety is generically of degree 2 onto its image. We then say that a Horikawa variety is a strong Horikawa variety if its canonical morphism is finite. Proposition 2.3 implies the following corollary for strong Horikawa varieties:

**Corollary 2.5.** A Horikawa variety $X$ is a strong Horikawa variety if and only if its canonical model is $X$. In particular, the canonical model of a Horikawa variety is a strong Horikawa variety.

The irregularity is an important topological invariant of any variety, so we study it in Theorem 2.6 for any Horikawa variety. In our proof we make a reduction to the case of algebraic surfaces. However, the linear series involved is not canonical, unlike in the study of algebraic surfaces made by Horikawa. Moreover, the singular case has to be handled carefully using a resolution of singularities. As a byproduct of our proof, we do obtain a more transparent proof of Horikawa’s results on surfaces as well.

Although Theorem 2.6 also follows from Theorem 3.3, we give an independent proof below. Among other things, the interest of this proof lies on the arguments employed. These arguments are used in a crucial way in the proof of Theorem 2.7. For example, we build on them to show that, for the intersection of $n-2$ hyperplane section of $Y$, which is a Hirzebruch surface $F_e$, $e$ is bounded. This is a key part in proving the boundedness of $p_g$ for singular $X$.

**Theorem 2.6.** A Horikawa variety $X$ is regular.

**Proof.** If we show that $\overline{X}$ is regular, then so is $X$. From Corollary 2.5 and [Kob92, Proposition 2.5], we know that the base point free complete canonical linear series $|K_X|$ induces a finite morphism $\phi: \overline{X} \to Y \subset \mathbb{P}^N$, of degree 2 onto a variety of minimal degree $Y$. By taking $(n-2)$ general hyperplane sections of $Y$ and then taking their respective pullbacks to $\overline{X}$ we end up with a finite morphism $\phi_2: \overline{X}_2 \to Y_2$, of degree 2, where, by [CKM88, Lemma 6.6], $\overline{X}_2$ has canonical singularities and $Y_2$ is a surface of minimal degree. We divide the situation into two cases: when $Y_2$ is smooth and when $Y_2$ is singular.
Case 1. $Y_2$ is a smooth surface of minimal degree.

Then $Y_2$ is either $\mathbb{P}^2$, the Veronese surface in $\mathbb{P}^5$ or a Hirzebruch surface embedded as a scroll. Since $φ_2$ is flat, $φ_2^* \mathcal{O}_{Y_2} = \mathcal{O}_{Y_2} \oplus \mathcal{L}^{-1}$, where $\mathcal{L}^{\otimes 2} = \mathcal{O}_{Y_2}(B)$, with $B$ being the branch divisor of the double cover.

First let $Y_2$ be a Hirzebruch surface embedded as scroll. Then $Y_2$ is embedded by a very ample linear series $|C_0 + mf|$, where $C_0$ is a minimal section of $Y_2$. Then $m > e$, where $C_0^2 = -e$ and $e$ is the invariant of the Hirzebruch surface. In this case note that, by the ramification formula for double covers, we have

$$O_{X_2}(K_{X_2}) = φ_2^*(O_{Y_2}(K_{Y_2}) \otimes \mathcal{L}).$$

By adjunction we also have

$$O_{X_2}(K_{X_2}) = φ_2^*(O_{Y_2}(n - 1)).$$

Since we are working on a rational surface,

$$H^1(O_{X_2}) = H^1(O_{Y_2}) \oplus H^1(\mathcal{L}^{-1}) = 0,$$

because $Y_2$ is regular and $H^1(Y_2, O_{Y_2}(K_{Y_2}) \otimes \mathcal{L})$ vanishes (the latter follows from \[2.6.1\] and from the Leray spectral sequence applied to the projection from the ruled surface $Y_2$ to $\mathbb{P}^1$). Therefore $X_2$ is regular.

The case that $Y_2$ is $\mathbb{P}^2$ or the Veronese surface in $\mathbb{P}^5$ can be treated similarly.

Case 2. $Y_2$ is a singular surface of minimal degree.

In this case $Y_2$ is a cone over a rational curve of degree $e \geq 2$. Let $q: W_2 \to Y_2$ be the minimal desingularization of $Y_2$. There exists the following commutative diagram:

(2.6.2)

where $Z_2$ is the normalization of the reduced part of the fiber product $W_2 \times_{Y_2} X_2$, which is irreducible. The map $p_2: Z_2 \to W_2$, which is a finite map of degree 2, and $\overline{\pi}: Z_2 \to X_2$, which is a birational map, are induced by the projections from the fiber product onto each factor. Since $W_2$ is smooth and $Z_2$ is normal, $p_2$ is a flat morphism of degree 2, and $Z_2$ is Gorenstein. Moreover, $p_2^*(O_{Z_2}) = O_{W_2} \oplus \mathcal{L}^{-1}$ where now $\mathcal{L}^{\otimes 2} = O_{W_2}(B)$ and $B$ is the branch divisor of the double cover. By adjunction

$$φ_2^*(O_{Y_2}(n - 1)) = φ_2^*(O_{Y_2}(K_{Y_2})).$$

Let $y \in Y_2$ be the vertex of the cone. The surface $W_2$ is a Hirzebruch surface with the minimal section $C_0^2 = -e$, where $C_0 = q^{-1}(y)$. Let $F := p_2^{-1}(C_0)$.
There are two possible cases for $\phi_2^{-1}(y)$, either it consists of one point or it consists of two points.

**Case 2.1.** $\phi_2^{-1}(y)$ is one point.

First we show $C_0$ is in the branch locus of $p_2$. Suppose the contrary. If $C_0$ is not contained in the branch locus of $p_2$, then $p_2^*C_0 = F$ and $F^2 = -2e$. There exist canonical divisors $K_{Z_2}$ and $K_{\mathbb{X}_2}$ and a nonnegative integer $a$ such that $K_{Z_2} = q^*K_{\mathbb{X}_2} + aF$. Applying adjunction we have

$$(K_{Z_2} + F) \cdot F = (q^*K_{\mathbb{X}_2} + (a + 1)F) \cdot F = -2e(a + 1) \leq -4,$$

and this is impossible because $F$ is a reduced and connected curve.

Therefore the minimal section $C_0$ is in the branch locus of $p_2$. Since $C_0$ is in the branch locus of $p_2$, we have that $F$ is isomorphic to $\mathbb{P}^1$, that $p_2^*(C_0) = 2F$ and that $F^2 = -\frac{2}{e}$. We also have $K_{Z_2} = q^*(K_{\mathbb{X}_2}) + aF$, with $a$ nonnegative because $\mathbb{X}_2$ has canonical singularities. By (2.6.2) and (2.6.3) we obtain,

$$(2.6.4) \quad K_{Z_2} = q^*(K_{\mathbb{X}_2}) + aF = q^*\phi_2^*O_{Y_2}(n - 1) + aF = p_2^*q^*O_{Y_2}(n - 1) + aF.$$

Comparing (2.6.4) with

$$O_{Z_2}(K_{Z_2}) = p_2^*(O_{W_2}(K_{W_2}) \otimes L),$$

one sees that $O_{Z_2}(aF) = p^*N$ for some line bundle $N$ on $W_2$. Therefore, $p_2^*N^\otimes 2 = O_{Z_2}(2aF) = p_2^*O_{W_2}(aC_0)$. This implies in particular that $N^\otimes 2$ and $aC_0$ are numerically equivalent. Since $W_2$ is a rational ruled surface, $N^\otimes 2$ and $aC_0$ are linearly equivalent. Therefore $N \sim O_{W_2}(a'C_0)$ with $2a' = a$, where $a'$ is an integer. Thus $a$ is a nonnegative even integer. On the other hand, we have

$$(2.6.5) \quad K_{Z_2} = p_2^*K_{W_2} + R \sim p_2^*(-2C_0 - (e + 2)e) + R,$$

where $R$ is the ramification divisor of $p_2$. From (2.6.4) and (2.6.3) we get

$$(2.6.6) \quad R \sim p_2^*((n + 1)C_0 + (ne + 2)f) + aF \sim p_2^*((ne + 2)f) + (2n + 2 + a)F.$$

Since $Z_2$ is normal, $R$ can be written as $R = R_1 + F$, where $R_1$ is a divisor that does not contains $F$ in its support.

$$(2.6.7) \quad (a + 1)e \leq 4.$$

Since $a$ is even and $e \geq 2$, we have

$$(2.6.8) \quad a = 0 \text{ and } e = 2, 3 \text{ or } 4.$$

In particular, $q^*$ is crepant. Then we have (see (2.6.6))

$$R \sim p_2^*((n + 1)C_0 + (ne + 2)f).$$

Now, the ramification formula for $p_2$ reads

$$O_{Z_2}(K_{Z_2}) = p_2^*(O_{W_2}(K_{W_2}) \otimes L),$$

with

$$(2.6.9) \quad L = O_{W_2}((n + 1)C_0 + (ne + 2)f).$$
Since \( p_{2*}(\mathcal{O}_{Z_2}) = \mathcal{O}_{W_2} \oplus \mathcal{L}^{-1} \), it follows from projection formula and duality that
\[
h^1(\mathcal{O}_{Z_2}) = h^1(\mathcal{L}^{-1}) = h^1(\mathcal{O}_{W_2}((n-1)(C_0 + e f))) = 0.
\]
Thus \( Z_2 \) is regular and hence so is \( \overline{X}_2 \) in this case.

**Case 2.2.** \( \phi_2^{-1}(y) = \{x_1, x_2\} \) are two distinct points.

This implies that \( \phi_2 \) is étale in the analytic neighborhood of \( x_1 \) and \( x_2 \). Also, \( p_2 \) is étale on an analytic neighborhood of \( C_0 \). Let \( E_1 \) and \( E_2 \) be exceptional divisors of \( \overline{y} \). We have \( p_2^* (C_0) = E_1 + E_2 \), \( E_1 \cdot E_2 = 0 \) and \( E_i^n = -e = C_0^n \), with \( E_i \cong \mathbb{P}^1 \). Recall \( Z_2 \) is Gorenstein as explained above. We then have
\[
K_{Z_2} = \overline{y}^* K_{\overline{X}_2} + a(E_1 + E_2),
\]
with a nonnegative. Applying the adjunction formula we obtain:
\[
(2.6.10) \quad 2 = (K_{Z_2} + E_i) \cdot E_i = (\overline{y}^* K_{\overline{X}_2} + a(E_1 + E_2) + E_i) \cdot E_i = -e(a+1).
\]
Since \( e \geq 2 \), then
\[
(2.6.11) \quad a = 0 \text{ and } e = 2.
\]
This means that \( \overline{y} \) is crepant. By a similar computation as in Case 2.1, we have \( H^1(\mathcal{O}_{\overline{X}_2}) = 0 \).

We thus conclude that \( \overline{X}_2 \) is regular in all cases.

Since \( \overline{X}_2 \) is regular, so is \( X_2 \). Now we prove the Horikawa variety \( X \) is regular. Recall that \( X_2 \) is obtained as a complete intersection of members of the linear system \( |K_X| \). Let \( L_i = \mathcal{O}_{X_i}(K_X|_{X_i}) \), where \( X_i \) is the variety obtained from the intersection of \( n-i \) general members of \( |K_X| \). Then \( X_i \) has canonical singularities by [CKM88, Lemma 6.6]. Consider the following short exact sequence:
\[
(2.6.12) \quad 0 \rightarrow L_i^{-1} \rightarrow \mathcal{O}_{X_i} \rightarrow \mathcal{O}_{X_{i-1}} \rightarrow 0.
\]
We note that \( H^1(L_i^{-1}) = 0 \) by Serre duality and the Kawamata-Viehweg vanishing theorem so, if \( X_{i-1} \) is a regular variety, then so is \( X_i \). We have shown that \( X_2 \) is regular, so we have that \( X_i \) is regular for all \( 2 \leq i \leq n \) by induction. \( \square \)

**Theorem 2.7.** Let \( X \) be a Horikawa variety of dimension \( n \). If the image of \( X \) by its canonical morphism is singular, then \( p_9(X) \leq n + 4 \).

**Proof.** By [Kob92, Proposition 2.5] we know that the base point free complete canonical linear series \( |K_X| \) induces a finite morphism \( \phi: \overline{X} \rightarrow Y \subset \mathbb{P}^N \), of degree 2 onto \( Y \), where \( Y \) is a singular variety of minimal degree in \( \mathbb{P}^N \). Then \( Y \) is a cone over a smooth variety of minimal degree. Let \( y \) be a point of the vertex of \( Y \). If we choose \( n - 2 \) general hyperplanes through \( y \), by considering their intersection and its pullback by \( \phi \), we get a finite, degree 2 morphism \( \phi_2: \overline{X}_2 \rightarrow Y_2 \), where \( Y_2 \) is an irreducible, singular surface of minimal degree (i.e., a cone over a (smooth) rational normal curve) and \( \overline{X}_2 \) is also irreducible and locally Gorenstein. Since \( Y_2 \) has log terminal singularities, it follows from
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[KM98, Proposition 5.20] that $\overline{X}_2$ also has log terminal singularities. Then, since $\overline{X}_2$ is locally Gorenstein, $\overline{X}_2$ has canonical singularities. Therefore $\overline{X}_2$ and $\phi_2$ are like $X_2$ and $\phi_2$ in the proof of Theorem 2.6, and $Y_2$ is like $Y_2$ of Case 2 of the proof of Theorem 2.6, so from (2.6.8) and (2.6.11) it follows that $Y_2$ is a nondegenerate surface in $\mathbb{P}^3$, $\mathbb{P}^4$ or $\mathbb{P}^5$. Thus $Y$ is a nondegenerate variety in $\mathbb{P}^{n+1}$, $\mathbb{P}^{n+2}$ or $\mathbb{P}^{n+3}$, so $p_g(X) = p_g(\overline{X}) \leq n + 4$.

Now we present a very interesting result, Proposition 2.9, on the deformations of canonical morphisms of Horikawa variety. This crucially depends on Theorem 2.6 and the base point freeness of $K_X$. First we make clear what we mean by a deformation of a morphism:

**Definition 2.8.** Let $X$ be an algebraic projective normal variety and let $\psi : X \rightarrow \mathbb{P}^N$ be a morphism. Let $T$ be a smooth disc. A deformation of $\psi$ is a $T$–morphism $\Psi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ such that

1. the variety $\mathcal{X}$ is irreducible and reduced;
2. the morphism $\mathcal{X} \rightarrow T$ is proper and surjective;
3. $\mathcal{X}_0 = X$; and
4. $\Psi_0 = \psi$.

**Proposition 2.9.** Let $X$ be a Horikawa variety of dimension $n$. Let $\overline{X}$ be its canonical model. Then the general deformation of the canonical morphism of $X$ is again a generically finite canonical morphism of degree 2 onto a variety of minimal degree. Also, the general deformation of $\overline{X}$ is again a canonical model of a Horikawa variety.

**Proof.** We will use [GGP10, Lemma 2.4]. Note that, although this statement requires smoothness, in fact, it holds for varieties with canonical singularities. Thus, since by Theorem 2.6 $X$ and $\overline{X}$ are regular, [GGP10, Lemma 2.4] applies to both. Therefore, for any small deformation (by a small deformation we mean that we shrink $T$ if needed)

$$\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N,$$

of the canonical morphism $\varphi$ of $X$ (respectively, any small deformation

$$\Phi_t : \mathcal{X}_t \rightarrow \mathbb{P}_T^N,$$

of the canonical morphism $\phi$ of $\overline{X}$, $\Phi_t$ (respectively, $\overline{\Phi}_t$) is a canonical morphism, for all $t \in T$. In addition, the image of $\mathcal{X}_t$ (respectively, of $\overline{\mathcal{X}}_t$) under its canonical morphism $\Phi_t$ (respectively, its canonical morphism $\overline{\Phi}_t$) is of maximum dimension $n$. By the main theorem of [Kaw99], the deformation of canonical singularities is again canonical and, by [Kaw99, Theorem 6], $K_{\mathcal{X}_t}^n$ and $p_g(\mathcal{X}_t)$
(respectively, $K^n_{X_t}$ and $p_g(\overline{X}_t)$) are invariant under deformations. Thus, $X_t$ (respectively, $K^n_{X_t}$) is a Horikawa variety and, by [Kob92, Proposition 2.5], $\Phi_t$ is a generically finite morphism of degree 2 (respectively, $\overline{\Phi}_t$ is a finite morphism of degree 2) onto a variety of minimal degree. In particular $\overline{X}_t$ is a strong Horikawa variety so, by Corollary 2.5, is its own canonical model.

Proposition 2.9 has this obvious implication for the components of the moduli space of varieties of general type:

**Corollary 2.10.** The general points of the components of the moduli of varieties of general type that contain a canonical model of a Horikawa variety are canonical models of Horikawa varieties.

We now recall the generalized notion of regularity for an algebraic variety, for varieties of arbitrary dimension.

**Definition 2.11.** A variety $X$ of dimension $n$ is said to be pluriregular if $H^1(O_X) = \cdots = H^{n-1}(O_X) = 0$.

**Theorem 2.12.** Let $X$ be a Horikawa variety of dimension $n$.

1. If the image $Y$ of its canonical morphism $\varphi$ is smooth, then $X$ is pluriregular.

2. If, in addition, $Y$ is a rational normal scroll then the general fiber of $X$ over $\mathbb{P}^1$ is a Horikawa variety with geometric genus $n$.

**Proof.** For the proof of (1) we may assume $n > 2$, for the result for $n = 2$ has been proved in Theorem 2.6. In view of Proposition 2.8 we will work with the canonical model $\overline{X}$ of $X$, since $X$ is pluriregular if and only if $\overline{X}$ is pluriregular. The canonical morphism $\phi$ of $\overline{X}$ is finite of degree 2. Since $\overline{X}$ is locally Cohen-Macaulay and $Y$ is smooth, the morphism $\phi$ is flat. Then

$$\phi_* O_{\overline{X}} = O_Y \oplus L^{-1},$$

where $L$ is line bundle. Let $R$ be the ramification divisor and let $B$ be the branch divisor of $\phi$. Then $O_Y(B) = L^{\otimes 2}$ and

$$(2.12.1) \quad O_{\overline{X}}(K_{\overline{X}}) = \phi^*(O_Y(K_Y) \otimes L) = \phi^* O_Y(1).$$

Showing that $\overline{X}$ is pluriregular is equivalent to showing the vanishing of the intermediate cohomology of $O_Y$ and $L^{-1}$, because $R^i \phi_* O_{\overline{X}} = 0$ for all $i > 0$ ($\phi$ is finite). By the classification of varieties of minimal degree, we have three possible cases for $Y$:

1. $Y$ is $\mathbb{P}^n$, $n \geq 3$.
2. $Y$ is a smooth quadric hypersurface in $\mathbb{P}^{n+1}$, $n \geq 3$.
3. $Y$ is a smooth rational normal scroll of dimension $n \geq 3$. In this case $Y$ is fibered over $\mathbb{P}^1$, hence $\overline{X}$ is also fibered over $\mathbb{P}^1$.\]
If $Y$ is $\mathbb{P}^n$ or a smooth quadric in $\mathbb{P}^{n+1}$, then the vanishing of the intermediate cohomology of $\mathcal{O}_Y$ and $\mathcal{L}^{-1}$ follows from the vanishing of the intermediate cohomology of line bundles in $\mathbb{P}^n$ and $\mathbb{P}^{n+1}$ respectively.

It remains to consider the case that $Y$ is a smooth rational normal scroll, which is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$. We may write

$$p: Y = \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^1,$$

where

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $0 < a_1 \leq \ldots \leq a_n$. We have

$$h^i(\mathcal{O}_{\overline{X}}) = h^i(\phi_* \mathcal{O}_X) = h^i(\mathcal{O}_Y) + h^i(\mathcal{L}^{-1}) = h^i(\mathcal{O}_Y) + h^{n-i}(\mathcal{O}_Y(1)),$$

because $\mathcal{O}_Y(K_Y) \otimes \mathcal{L} = \mathcal{O}_Y(1)$ by (2.12.1). For any $0 < i < n$

$$h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_{\mathbb{P}^1}) = h^i(\mathcal{O}_{\mathbb{P}^1}) = 0$$

and

$$h^i(\mathcal{O}_Y(1)) = h^i(p_* \mathcal{O}_Y(1)) = h^i(\mathcal{E}) = 0.$$  

The pluriregularity of $\overline{X}$ now follows.

Let us now prove (2). Let $F$ be a general fiber of $p$ and let $G' = \phi^{-1}(F)$. We have the following long exact sequence of cohomology

$$0 \rightarrow H^0(\mathcal{O}_{\overline{X}}(K_{\overline{X}} - G')) \rightarrow H^0(\mathcal{O}_{\overline{X}}(K_{\overline{X}})) \rightarrow H^0(\mathcal{O}_{\overline{X}}(K_{\overline{X}}|_{G'})) \rightarrow H^1(\mathcal{O}_{\overline{X}}(K_{\overline{X}} - G')).$$

Because of $R^i \phi_* \mathcal{O}_X = 0$, $\mathcal{O}_{\overline{X}}(K_{\overline{X}} - G') = \phi^*(\mathcal{O}_Y(1) \otimes \mathcal{O}_Y(-F))$ and (2.12.1), we have

$$h^1(\mathcal{O}_{\overline{X}}(K_{\overline{X}} - G')) = h^1(\mathcal{O}_Y(1) \otimes \mathcal{O}_Y(-F)) = h^1(\mathcal{O}_Y(2) \otimes \mathcal{O}_Y(\mathcal{L}^{-1}) \otimes \mathcal{O}_Y(\mathcal{L}) \otimes \mathcal{O}_Y(-F)).$$

Pushing forward to $\mathbb{P}^1$ via $p$, it is easy to see that both $h^1(\mathcal{O}_Y(1) \otimes \mathcal{O}_Y(-F))$ and $h^{n-1}(\mathcal{O}_Y(2) \otimes \mathcal{O}_Y(-F))$ vanish. This implies that the complete linear series $|K_{\overline{X}}|$ restricts to the complete linear series $|K_{\overline{X}}|_{G'}$, which is $|K_{G'}|$. Hence $|K_{G'}|$ maps $G'$ onto $G = \mathbb{P}^{n-1}$ as a finite double cover. By [KM98 Theorem 5.20] and since $K_{G'}$ is Cartier, we have that $G'$ has canonical singularities. Therefore $G'$ is a strong Horikawa variety and $p_\#(G') = n$. Let $G$ be the fiber of $X$ that corresponds to $G'$. Then $p_\#(G) = p_\#(G')$. Then the restriction of $\varphi$ to $G$ is the canonical morphism of $G$, which is generically of degree 2 onto $\mathbb{P}^{n-1}$. Since $G'$ has canonical singularities, so does $G$, therefore $G$ is a Horikawa variety. □

**Remark 2.13.** It might be illuminating to observe how beautifully the situation fits with the classical case of surfaces. In Horikawa’s work in [Hor70], it turns out that smooth Horikawa surfaces are indeed regular genus two fibrations over $\mathbb{P}^1$ (in this case, $G'$ is a curve so $p_\#(G')$ is the genus of $G'$). In the higher
dimensional case Theorem 2.12 shows the exact analogy. That the numerical situation dictates this in all dimensions is indeed compelling.

3. Simple connectedness of Horikawa varieties

We devote this section to proving the simple connectedness of Horikawa varieties. For this, first we need to state two results. The first one is follows from a well-known result on the fundamental group of the complement of smooth submanifold of (real) codimension more than 1. The second one is based in the ideas and results of M. Nori in [Nor83].

**Lemma 3.1.** Let \( V \) be a smooth complex variety and let \( W \) a complex, reduced subvariety of \( V \), not necessarily smooth, and let
\[
\pi_1(V \setminus W) \longrightarrow \pi_1(V)
\]
be the homomorphism induced by inclusion. This homomorphism is surjective if \( W \) has codimension 1 in \( V \) and an isomorphism if \( W \) has codimension more than 1 in \( V \).

**Lemma 3.2.** Let \( p : V' \longrightarrow V \) be a finite morphism of degree 2 among smooth, (quasiprojective) complex varieties \( V' \) and \( V \) of dimension \( n \geq 2 \), branched along a smooth, irreducible divisor \( B \). If \( V \) is simply connected, then so is \( V' \).

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccccccccc}
1 & \longrightarrow & K' & \longrightarrow & \pi_1(V' \setminus p^{-1}(B)) & \longrightarrow & \pi_1(V') & \longrightarrow & 1 \\
\downarrow & & \downarrow \iota' & & \downarrow \iota & & \downarrow j & & \\
1 & \longrightarrow & K & \longrightarrow & \pi_1(V \setminus B) & \longrightarrow & \pi_1(V) & \longrightarrow & 1,
\end{array}
\]
where the horizontal exact sequences are exact at the right hand side by Lemma 3.1. Since \( p \) restricted to \( V' \setminus p^{-1}(B) \) is a 2-sheet unramified cover, \( \iota' \) is injective (and so is \( \iota \)) and its cokernel is isomorphic to \( \mathbb{Z}_2 \). Since \( V \) is simply connected, chasing the diagram we get the following short exact sequence
\[
0 \longrightarrow L \longrightarrow K/K' \longrightarrow \mathbb{Z}_2 \longrightarrow 0,
\]
where \( L \) is the kernel of \( j \).

By [Nor83, 1.2, 1.4 B], \( K \) is a cyclic group, so is \( K' \) and the generator of \( K' \) maps to the square of the generator of \( K \). Therefore \( K/K' \) is also isomorphic to \( \mathbb{Z}_2 \), so \( L = 0 \).

We further explore the topological properties of the Horikawa varieties. We prove that any Horikawa variety is simply connected. The similarities with the surface case is striking. The fact that just two invariants \( p_g(X) \) and \( K_X^q \) determine the topology entirely is rather remarkable. Note that a strong Horikawa variety together with its canonical divisor is a hyperelliptic polarized variety as defined by Fujita in [Fuj83, Definition 1.1] (see also [BGG20, Definition 0.1]). Then Theorem 3.3 extends what [Fuj83, Corollary 5.17] says with respect to...
the simple connectedness of strong Horikawa varieties. Indeed, [Fuj83, Corollary 5.17] states the simply connectedness only of strong Horikawa varieties whose canonical morphism is a double cover of a (smooth) rational normal scroll with connected branch divisor, while Theorem [3.3] states that all Horikawa varieties are simply connected.

**Theorem 3.3.** Any Horikawa variety $X$ is simply connected.

**Proof.** We will work with the canonical model $\overline{X}$ of $X$ and, by taking the intersections of $n - 2$ general hyperplane sections, from the canonical morphism $\phi$ of $\overline{X}$ we obtain a finite morphism $\phi_2 : \overline{X}_2 \rightarrow Y_2$, of degree 2 onto a surface of minimal degree $Y_2$, as we did in the proof of Theorem 2.6. Our goal now is to prove that $\overline{X}_2$ is simply connected. We split the argument in several cases.

**Case 1:** $\overline{X}_2$ and $Y_2$ are smooth. Therefore $Y_2$ is either $\mathbb{P}^2$, the Veronese surface in $\mathbb{P}^5$ or a Hirzebruch surface embedded as a rational normal scroll. Let $B$ be the branch divisor of $p$. Since $\overline{X}_2$ is smooth, so is $B$. The case in which $B$ is ample, that includes the cases of $Y_2$ being $\mathbb{P}^2$ or the Veronese surface in $\mathbb{P}^5$, is straightforward. Indeed, by [Nor83, Corollary 2.7], $\overline{X}_2$ is simply connected.

Now we deal with the case of $Y_2$ being a rational normal scroll and $B$ not ample. Then $Y_2$ is a Hirzebruch surface embedded by $|C_0 + mf|$, where $C_0$ is its minimal section, $f$ is a fiber, $C_0^2 = -e$ and $m \geq e + 1$. is a Hirzebruch surface. By (2.6.1),

$$B \sim 2(n + 1)C_0 + 2((n - 1)m + e + 2)f).$$

Let $B \sim \alpha C_0 + \beta f$. Since $B$ is smooth, then either $\beta = \alpha e$ or $\beta = (\alpha - 1)e$. If $\beta = \alpha e$, then $B$ is big and base point free, so $B$ is irreducible. Since $Y_2$ is simply connected, by Lemma 3.2 so is $\overline{X}_2$.

If $\beta = (\alpha - 1)e$, then $B = C_0 + B_1$, with $B_1$ big and base point free (thus $B_1$ is smooth and irreducible) and $C_0$ and $B_1$ disjoint. Since $Y_2 \setminus C_0$ is an $\mathbb{A}^1$-fibration over $\mathbb{P}^1$, $Y_2 \setminus C_0$ is simply connected. By Lemma 3.2, $\overline{X}_2 \setminus p^{-1}(C_0)$ is also simply connected and, by Lemma 3.1 so is $\overline{X}_2$. Thus we have showed that, if $Y_2$ is a smooth rational normal scroll and $B$ is not ample, then $\overline{X}_2$ is also simply connected.

**Case 2:** $\overline{X}_2$ is singular and $Y_2$ is smooth.

Then, as in Case 1 of the proof of Theorem 2.6, $\phi_2$ is a flat double cover determined by a branch divisor $B$ (which is necessarily singular).

**Case 2.1.** $Y_2$ is $\mathbb{P}^2$ or a Veronese surface in $\mathbb{P}^5$.

Although $B$ is singular, since $B$ is base-point-free, a general member of $|B|$ is smooth and, in fact, we can consider a deformation $B$ of $B$ over a disc $T$, such that $B_0 = B$, $B$ is smooth and $B_t$ is smooth for all $t \neq 0$. Using $B$ as a relative
branch divisor, we can construct a deformation
\[ \mathcal{X} \to Y_2 \times T, \]
flat over \( T \) of \( \phi_2 \), such that \( \mathcal{X}' = \mathcal{X}_2, \mathcal{X} \) is smooth and \( \mathcal{X}' \) is smooth for all \( t \neq 0 \). Shrinking \( T \) if necessary, by [BHPV04, Theorem I.8.8] and [Nor83, Lemma 1.5.C], \( \pi_1(\mathcal{X}_t) \) surjects onto \( \pi_1(\mathcal{X}_0) \) for any general \( t \in T \). Since \( \mathcal{X}_t \) is a finite double cover of \( Y_2 \) and \( \mathcal{X}_t \) is smooth, by the arguments used in Case 1, \( \mathcal{X}_t \) is simply connected, and so is \( \mathcal{X}_2 \).

**Case 2.2.** \( Y_2 \) is a rational normal scroll.
Then \( Y_2 \) is a Hirzebruch surface. If \( B \) is big and base–point–free, then the general member of \( |B| \) is smooth. Therefore, arguing as in Case 2.1, we can deform \( \mathcal{X}_2 \) to a smooth, simply connected surface \( \mathcal{X}_1 \), so \( \mathcal{X}_2 \) is also simply connected. Now assume \( B \) is not big and base–point–free and, with the same notation of Case 1, let \( B \sim \alpha C_0 + \beta f \). Then, by [2.6.1], \( \beta < \alpha e \). Since \( \mathcal{X}_2 \) is normal, \( C_0 \) cannot occur in the branch locus with multiplicity more than 1, so \( B \sim B' + C_0, \beta \geq (\alpha - 1)e \), and \( B' \cdot C_0 \geq 0 \). Then, by [2.6.1], \( B' \) is big and base–point–free and any general member in \( |B'| \) is irreducible and nonsingular. In addition, since \( H^1(\mathcal{O}_Y(B' - C_0)) = 0 \), we may assume that any general member of \( |B'| \) intersects \( C_0 \) transversally. Thus there is a family of divisors \( \mathcal{B}_t \) and \( \mathcal{B}_t' \) of \( Y_2 \) over a disc \( T \) such that \( \mathcal{B}_0 = B, \mathcal{B}_t' = B' \) and, for all \( t \in T, t \neq 0 \), the divisor \( \mathcal{B}_t' \) is smooth and meets \( C_0 \) transversally. Let
\[ \Phi : \mathcal{X} \to Y_2 \times T, \]
be the double cover of \( Y_2 \times T \), branched along the total space of the family formed by the divisors \( \mathcal{B}_t \). As in Case 2.1, \( \Phi \) is a deformation, flat over \( T \), of \( \phi_2 \). Let
\[ \Psi : \hat{\mathcal{X}} \to \mathcal{X} \]
be the minimal desingularization of \( \mathcal{X} \). Since \( C_0 \) is the section of the rational ruled surface \( \mathcal{Y}_2 \), \( \mathcal{Y}_2 \times C_0 \) is an \( \mathbb{A}^1 \)-fibration over \( \mathbb{P}^1 \), hence \( \mathcal{Y}_2 \times C_0 \) is simply connected. Then, by Lemma 3.2, \( \hat{\mathcal{X}} \times \Psi^{-1}(C_0) \) is also simply connected. Since \( \hat{\mathcal{X}} \times \Psi^{-1}(\Phi_t^{-1}(C_0)) \) and \( \hat{\mathcal{X}} \times \Phi_t^{-1}(C_0) \) are isomorphic, we have \( \hat{\mathcal{X}} \times \psi^{-1}(\Phi_t^{-1}(C_0)) \) is simply connected and, by Lemma 3.1, so is \( \hat{\mathcal{X}} \). Then, by [Nor83, Lemma 1.5.C], \( \pi_1(\hat{\mathcal{X}}) \) surjects onto \( \pi_1(\mathcal{X}) \), so \( \hat{\mathcal{X}} \) is also simply connected. Since \( \mathcal{X} \) has canonical singularities, by [Tak03, Theorem 1.1], \( \mathcal{X} \) is simply connected. Shrinking \( T \) if necessary, by [BHPV04, Theorem I.8.8], \( \pi_1(\mathcal{X'}) \) and \( \pi_1(\mathcal{X}_0) \) are isomorphic, so \( \mathcal{X}_0 = \overline{\mathcal{X}}_2 \) is simply connected.

**Case 3.** \( Y_2 \) is singular. In this case we consider the desingularization diagram (2.6.2). Since the singularities of \( \overline{\mathcal{X}}_2 \) and \( Z_2 \) are canonical, it follows from [Ko93, Theorem 7.8] or [Tak03, Theorem 1.1] that \( \pi_1(\overline{\mathcal{X}}_2) \) and \( \pi_1(Z_2) \) are isomorphic. We consider now the double cover
\[ p : Z_2 \to W_2 \]
in the diagram (2.6.2). As calculated in Theorem 2.6, the branch divisor $B$ of $p$ satisfies

$$B \sim 2(n + 1)C_0 + 2(ne + 2)f.$$ 

If $e = 2$, then $B$ is big and base–point–free, so the general member of $|B|$ is smooth. Arguing like in Case 2.1 we conclude that $Z_2$ is simply connected and, as observed above, so is $\overline{X}_2$. If $e > 2$, since $Z_2$ is normal, $C_0$ occurs with multiplicity 1 in $B$. This shows that $B = B' + C_0$, where $B'$ is linearly equivalent to $(2n + 1)C_0 + 2(ne + 2)f$. As we saw in (2.6.8) and (2.6.11), we have $e \leq 4$.

Thus we have seen that, in all three cases, $\overline{X}_2$ is simply connected. Then, by Lefschetz hyperplane section theorem, $\overline{X}$ is also simply connected and, by [Tak03, Theorem 1.1], so is $X$. \hfill $\square$

4. Birationality and projective normality

Recall that, if $X$ is a Horikawa variety, then $K_X$ is base point free and the canonical morphism of $X$ maps $X$ onto a variety of minimal degree. We now study the pluricanonical morphisms of Horikawa varieties. We want to know when a pluricanonical morphism of $X$ is birational and maps $X$ to a projectively normal variety. More precisely, we want to know when a pluricanonical morphism of the canonical model of $X$ is an embedding and its image is projectively normal.

We will use the following notation throughout this section:

**Notation 4.1.** Let $X$ be a Horikawa variety of dimension $n$.

According to the context, $X_n$ will be $X$ or $\overline{X}$. Let $X_1 \subset \cdots \subset X_{n'} \subset \cdots \subset X_n$ be irreducible subsequent $n'$-dimensional, complete intersections of general members of $|K_X|$ or of $|K_{\overline{X}}|$. Note that $X_1$ is a smooth and irreducible curve, let us call it $C$. According to the context, we denote the line bundle, on $X_{n'}$ or on $\overline{X}_{n'}$, associated to $K_X|_{X_{n'}}$ or $K_{\overline{X}}|_{\overline{X}_{n'}}$, by $L_{n'}$. In this notation, $L_n$ is $\mathcal{O}_X(K_X)$ or $\mathcal{O}_{\overline{X}}(K_{\overline{X}})$ accordingly. Finally, let

$$H^0(L_{n'}^s) \otimes H^0(L_{n'}^t) \xrightarrow{\alpha(s,t,n')} H^0(L_{n'}^{s+t})$$

be the usual multiplication map of global sections.

**Proposition 4.2.** Let $X$ be a Horikawa variety of dimension $n$. If $1 \leq s \leq n$ or if $s = n + 1$ and $p_g(X) = n + 1$, then $|sK_X|$ does not induce a birational morphism.
Proof. From (2.6.12) we obtain, for all \(2 \leq n' \leq n\),
\[
(4.2.1) \quad 0 \to \mathcal{O}_{X_{n'}} \to L_{n'} \to L_{n'-1} \to 0.
\]
From (2.6.12), in the proof of Theorem 2.6 we also saw that, for all \(2 \leq n' \leq n\),
\(X_{n'}\) is regular. This together with (4.2.1) implies
\[
h^0(L_1) = h^0(\mathcal{O}_X(K_X)) - n + 1.
\]
Since \(X\) is a Horikawa variety,
\[
\deg L_1 = 2h^0(L_1) - 2,
\]
so, by Clifford’s theorem, \(C\) is hyperelliptic and \(L_1\) is a multiple of the \(g^1_2\) of \(C\). By adjunction, \(L_1^\otimes n = \mathcal{O}_C(nK_X|_C) = \mathcal{O}_C(K_C)\) so, if \(1 \leq s \leq n\), then \(L_1^\otimes s\) is in the special range of \(C\) and \(|L_1^\otimes s|\) induces a morphism of degree \(2\). If \(p_g(X) = n + 1\), then \(\deg L_1 = K_X^n = 2\), so \(L_1\) is the \(g^1_2\) of \(C\). By adjunction we have \(L_1^\otimes n+1 = \mathcal{O}_C((n+1)K_X|_C) = \mathcal{O}_C(K_C) \otimes L_1\). Since \(C\) is hyperelliptic, \(|L_1^\otimes n+1|\) also induces a morphism of degree \(2\) in this case. Therefore, under the hypothesis of the statement, the restriction of the \(s\)-canonical morphism of \(X\) to \(C\) has degree \(2\). Since \(C\) is the intersection of \(n-1\) general divisors of \(|K_X|\), there is an open set of \(X\) where the \(s\)-canonical morphism of \(X\) has degree \(2\), hence it is not birational. \(\square\)

Before we deal with the birationality and projective normality of pluricanonical systems of a Horikawa variety, we will state a particular case of a very useful more general result (see [GP98, Observation 1.2]).

**Lemma 4.3.** Let \(X\) be a projective variety, \(E\) a coherent sheaf, \(L_1, L_2, \ldots, L_r\) be line bundles on \(X\). Let
\[
\gamma: H^0(E) \otimes H^0(L_1 \otimes L_2 \otimes \cdots \otimes L_r) \to H^0(E \otimes L_1 \otimes L_2 \otimes \cdots \otimes L_r)
\]
be the multiplication map of global sections. If the multiplication maps
\[
\alpha_j: H^0(E \otimes L_1 \otimes \cdots \otimes L_{j-1}) \otimes H^0(L_j) \to H^0(E \otimes L_1 \otimes L_2 \otimes \cdots \otimes L_j)
\]
are surjective for all \(1 \leq j \leq r\), then \(\gamma\) is surjective.

Lemma 4.3 will be crucial in proving Proposition 4.4 which is the following general result on projective normality of pluricanonical images of varieties of general type. Proposition 4.4 shows that our main result on projective normality of pluricanonical images of Horikawa varieties, Theorem 4.6, has the correct bounds as far as geometry of Horikawa varieties is concerned. Proposition 4.4 also shows that general standard methods using Castelnuovo-Mumford regularity (see Remark 4.5) yield limited results:

**Proposition 4.4.** Let \(X\) be a variety of general type of dimension \(n\) with Gorenstein canonical singularities and base-point-free canonical bundle. Then the image of \(X\) by the morphism induced by \(|sK_X|\) is a projectively normal variety for all \(s \geq n + 2\).
Proof. Let $L := \mathcal{O}_X(sK_X)$. It is enough to show that

$$H^0(L^\otimes r + 1) \xrightarrow{\gamma_r} H^0(L^\otimes (r+1))$$

is surjective for all $r \geq 1$. By Lemma 4.3, it is enough to show that the maps

$$\alpha_j : H^0(X,jK_X) \otimes H^0(X,K_X) \rightarrow H^0(X,(j+1)K_X)$$

are surjective for all $j \geq s$. Note that since $s \geq n+2$ then $s-i \geq 2$ for all $1 \leq i \leq n$. It follows from the Kawamata-Viehweg vanishing that $H^i(X,(s-i)K_X) = 0$ for all $1 \leq i \leq n$, so $\mathcal{O}_X(K_X)$ is s-regular. By Castelnuovo-Mumford Lemma (see [Mum70, p. 41, Theorem 2]), the surjection of $\alpha_j$ for all $j \geq s$ now follows. Hence by Lemma 4.3, the multiplication map $\gamma_r$ surjects for all $r \geq 1$. □

Remark 4.5. What is at the heart of the proof of the above Proposition 4.4 is the fact that the multiplication map

$$\alpha_s : H^0(X,sK_X) \otimes H^0(X,K_X) \rightarrow H^0(X,(s+1)K_X)$$

is surjective, for all $s \geq n+2$, for a variety of general type of dimension $n$. This follows by the Castelnuovo-Mumford Lemma and the Kawamata-Viehweg vanishing as noted in the proof of the Proposition 4.4. Note that, if $\alpha_{n+1}$ were surjective, Castelnuovo-Mumford Lemma would be of no help whatsoever to prove it.

In view of Remark 4.5, the following result on projective normality needs non standard methods of proof. The proof of Theorem 4.6 crucially uses the structure of Horikawa varieties, namely, that its canonical map is a morphism which is a generically double cover a variety of minimal degree and Theorem 2.6.

**Theorem 4.6.** Let $X$ be a Horikawa variety of dimension $n$. Then $K^\otimes s_X$ embeds the canonical model $\overline{X}$ of $X$ as a projectively normal variety if and only if $s \geq n+2$ or $s = n+1$ and $p_g(X) \neq n+1$.

**Proof.** The “only if” part of the statement follows from Proposition 4.2.

We will prove the much more subtle “if” part of the statement in several steps. We define $X_{n'}$ and $L_{n'}$ starting from $\overline{X}$, as explained in Notation 4.1. We set $L = K^\otimes s_X$.

**Step 1:** Since $\overline{X}$ is regular by Theorem 2.6, the varieties $X_{n'}$ are all regular (in fact, this was proved at the end of the proof of Theorem 2.6). It follows by using adjunction inductively that, in addition, the varieties $X_i$ are of general type.

**Step 2:** To prove the theorem, it would be enough to show that the multiplication maps

$$H^0(L^\otimes t) \otimes H^0(L) \rightarrow H^0(L^\otimes (t+1))$$

which, according to Notation 4.1 are

$$H^0(L_{n'}^\otimes s) \otimes H^0(L_{n'}^\otimes (s)) \xrightarrow{\alpha_{ts,s;n}} H^0(L_{n'}^\otimes (t+1)s),$$
surject for all $t \geq 1$. For this, by Lemma 4.3 it is enough to prove that the multiplication maps

$$H^0(L_n^{\otimes n}) \otimes H^0(L_{n'}) \xrightarrow{\alpha(t',1;n)} H^0(L_{n'}^{\otimes (t'+1)})$$

surject for all $t' \geq n + 1$. Recall that $L_n = K_X$. As observed in Remark 4.5 if $t' \geq n + 2$, then $\alpha(t',1;n)$ surjects by [Mum70, p. 41, Theorem 2] and the Kawamata-Viehweg vanishing theorem. However, Remark 4.5 also says that these standard methods do not yield the surjectivity of $\alpha(n + 1,1;n)$. Therefore the proof of the surjectivity of the map $\alpha(n + 1,1;n)$ will be handled in a completely a different way. We point out (it will be crucial in Step 5) that, because of our hypothesis, we may assume $p_g(X) > n + 1$ when studying $\alpha(n + 1,1;n)$.

**Step 3:** In order to prove the surjectivity of $\alpha(n + 1,1;n)$ we consider, for any $2 \leq n' \leq n$, the commutative diagram

$$\begin{array}{ccccccc}
0 & \to & H^0(L_n^{\otimes n+1}) \otimes H^0(\mathcal{O}_{X_{n'}}) & \to & H^0(L_n^{\otimes n+1}) \otimes H^0(L_{n'}) & \to & H^0(L_n^{\otimes n+1}) \otimes H^0(L_{n'-1}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(L_{n'}^{\otimes n+1}) & \to & H^0(L_{n'}^{\otimes n+2}) & \to & H^0(L_{n'-1}^{\otimes n+2}) & \to & 0 \\
\end{array}$$

Both horizontal sequences in commutative diagram (4.6.1) are exact. Indeed, the top horizontal row is exact on the right because $X_{n'}$ is regular, as shown in Step 1, and the bottom horizontal row is exact on the right because $H^1(L_{n'}^{\otimes n+1}) = 0$ by the Kawamata-Viehweg vanishing theorem. Note that $\alpha(n + 1,1;n')$ is the middle vertical map of (4.6.1), so $\alpha(n + 1,1;n)$ is the middle vertical map of (4.6.1) when $n' = n$. Then we prove the surjectivity of $\alpha(n + 1,1;n)$ by induction on $n'$. Thus assume $\alpha(n + 1,1;n'-1)$ surjects for $2 \leq n' \leq n$. The left-hand-side vertical map of (4.6.1) is an isomorphism. The right-hand-side vertical map of (4.6.1) is the composition of the map

$$H^0(L_{n'}^{\otimes n+1}) \otimes H^0(L_{n'-1}) \to H^0(L_{n'}^{\otimes n+1}) \otimes H^0(L_{n'-1})$$

and $\alpha(n + 1,1;n'-1)$. The map in (4.6.2) surjects because $H^1(L_{n'}^{\otimes n}) = 0$, by adjunction and the Kawamata-Viehweg vanishing theorem, since $n' \geq 2$. The map $\alpha(n + 1,1;n'-1)$ surjects by induction hypothesis, so the right-hand-side vertical map of (4.6.1) surjects. Thus the middle vertical map $\alpha(n + 1,1;n')$ surjects. Therefore the only thing left to check is the first step of the induction, namely, the surjectivity of the multiplication map

$$H^0(L_1^{\otimes n+1}) \otimes H^0(L_1) \xrightarrow{\alpha(n+1,1;1)} H^0(L_1^{\otimes n+2})$$

of global sections on $C$. For brevity we denote $\beta = \alpha(n + 1,1;1)$ and $\theta = L_1$. By adjunction we have $\theta^{\otimes n} = K_C$. No conventional results on curves will suffice to show the surjectivity of $\beta$ due to lack of sufficient positivity of the line bundles.
\(\theta^{n+1}\) and \(\theta\). So we have to handle this in a different way. This leads us to the two, final steps of this proof.

**Step 4:** In the previous steps we have just used that \(X\) is a regular variety of general type. The final steps require the full force of the fact that \(X\) is a Horikawa variety. By [Kob92, Propositions 2.2, 2.5], we know that the canonical map of \(X\) is a morphism, which is finite of degree 2 onto a variety of minimal degree. Thus the linear system \(|\theta|\) induces the morphism \(\pi : C \to D\), where \(\pi\) is finite of degree 2. Since \(Y\) is a variety of minimal degree, \(D\) is a rational normal curve of degree \(r\). In addition, since \(D\) is cut-out by general hyperplane sections, \(D\) is smooth. Then we have

\[
\pi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a).
\]

By relative duality we have \(a = nr + 2\), so

\[
\pi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-nr - 2).
\]

**Step 5:** Recall that to finish the proof we have to show that the multiplication map of global sections on \(C\)

\[
H^0(\theta^{n+1}) \otimes H^0(\theta) \xrightarrow{\beta} H^0(C, \theta^{n+1})
\]

is surjective. Due to Step 4 and since \(\theta = \pi^*(\mathcal{O}_{\mathbb{P}^1}(r))\), we have

\[
H^0(\theta) = H^0(\pi_* \theta) = H^0(\mathcal{O}_{\mathbb{P}^1}(r)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}((1-n)r - 2)),
\]

hence \(H^0(\theta) = H^0(\mathcal{O}_{\mathbb{P}^1}(r))\). We also have

\[
H^0(\theta^{n+1}) = H^0(\pi_*(\theta^{n+1})) = H^0(\mathcal{O}_{\mathbb{P}^1}((n + 1)r)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(r - 2)).
\]

Note that

\[
\pi_*(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-nr - 2)
\]

is a sheaf of \(\mathcal{O}_{\mathbb{P}^1}\)-algebras whose ring multiplication decomposes in the following way: The map

\[
\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}
\]

is the ring multiplication in \(\mathcal{O}_{\mathbb{P}^1}\). The maps

\[
\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-nr - 2) \to \mathcal{O}_{\mathbb{P}^1}(-nr - 2)
\]

are the left and right module multiplication of \(\mathcal{O}_{\mathbb{P}^1}(-nr - 2)\). The map

\[
\mathcal{O}_{\mathbb{P}^1}(-nr - 2) \otimes \mathcal{O}_{\mathbb{P}^1}(-nr - 2) \to \mathcal{O}_{\mathbb{P}^1}
\]

is given by structure of double cover of \(\pi\). Denote

\[
A(m) = H^0(\mathcal{O}_{\mathbb{P}^1}(mr)),
\]

\[
A'(m) = H^0(\mathcal{O}_{\mathbb{P}^1}((m - n)r - 2)).
\]
In this notation
\[ H^0(\theta^{\otimes n+1}) = A(n+1) \oplus A'(n+1), \]
\[ H^0(\theta) = A(1) \oplus A'(1) = A(1), \]
since \( A'(1) = 0 \), and
\[ H^0(\theta^{\otimes n+2}) = A(n+2) \oplus A'(n+2). \]
Therefore the map \( \beta \) splits as direct sum of the maps
\[ A(n+1) \otimes A(1) \overset{\beta_1}{\rightarrow} A(n+2) \]
\[ A'(n+1) \otimes A(1) \overset{\beta_2}{\rightarrow} A'(n+2). \]
The map \( \beta \) surjects if and only if the maps \( \beta_1 \) and \( \beta_2 \) both surject. On \( \mathbb{P}^1 \) the
multiplication map of global sections
\[ H^0(\mathcal{O}_{\mathbb{P}^1}(c_1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(c_2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(c_1 + c_2)) \]
surjects if and only if \( c_1 \) and \( c_2 \) are both nonnegative. Recall that \( A(1) = H^0(\mathcal{O}_{\mathbb{P}^1}(r)) \) and \( A(n+1) = H^0(\mathcal{O}_{\mathbb{P}^1}((n+1)r)) \) and \( r, (n+1)r > 0 \). On the other
hand, \( A'(n+1) = H^0(\mathcal{O}_{\mathbb{P}^1}(r-2)) \). Under our hypothesis, \( p_g > n+1 \), so the
degree of \( D \) is \( r = p_g(X) - n \geq 2 \) and \( r - 2 \geq 0 \). Then \( \beta \) surjects as wished. \( \Box \)

5. HYPERELLIPTIC SUBCANONICAL POLARIZED VARIETIES

Even though this article focuses on Horikawa varieties, the main results of
Sections 2 and 3, when considered for strong Horikawa varieties, hold more
generally. Indeed, if \( X \) is a strong Horikawa variety, the polarized variety \((X, \omega_X)\)
is hyperelliptic according to the following definition (compare with \cite{Fuj83},
Definition 1.1):

**Definition 5.1.** Let \((X, A)\) be a polarized variety. If \( A \) is base-point-free, the
morphism induced by \(|A|\) has degree 2 and its image is a variety of minimal
degree, then \((X, A)\) is a hyperelliptic polarized variety.

We also recall the definition of subcanonical variety (compare with \cite{BGMR20},
Definition 2.11):

**Definition 5.2.** Let \((X, A)\) be a polarized variety, let \( A \) be base-point-free. If
\( \omega_X = A^{\otimes s} \), then we say \((X, A)\) is \( s \)-subcanonical.

**Proposition 5.3.** Let \((X, A)\) be an \( s \)-subcanonical polarized variety of dimen-
sion \( n \). If \((X, A)\) is hyperelliptic, then \( s \geq -n + 1 \).

**Proof.** We call \( \varphi \) to the morphism induced by \(|A|\) and argue as in the proof of
Theorem 2.6 and construct \( \overline{X}_2, \phi_2 \) and \( Y_2 \) in the same fashion. If \( Y_2 \) is \( \mathbb{P}^2 \), then
the branch divisor of \( \phi_2 \) is \( 2(n + s + 1) \) times a line in \( \mathbb{P}^2 \). Since the branch
divisor is effective and \( \overline{X}_2 \) is connected, then \( n + s \geq 0 \). If \( n + s = 0 \), then \( \varphi \) is
not induced by the complete linear series \(|A|\) of \( A \), hence \( n + s \geq 1 \) in this case.
If $Y_2$ is the Veronese surface, then the branch divisor of $\phi_2$ is $2(2n + 2s - 1)$ times a line in $\mathbb{P}^2$. Since the branch divisor is effective and $X_2$ is connected, $n + s \geq 1$ in this case.

If $Y_2$ is smooth rational normal scroll or a cone over a rational normal curve (in the latter case, we carry out a construction analogous to (2.6.2)), then we get results analogous to (2.6.1), (2.6.8), (2.6.9) and (2.6.11). Thus the branch divisor of $\phi_2$ is linearly equivalent to

$$2(n + s)C_0 + 2((n + s - 2)m + e + 2)f,$$

with $m \geq e + 1$ and the branch divisor of $p_2$ is linearly equivalent to

$$2(n + s)C_0 + 2(n + s - e + 2)f.$$

Since these branch divisors are effective, we get $n + s \geq 0$. If $n + s = 0$ and $Y_2$ is smooth, since $X_2$ is connected, $2m \leq e + 1$, but this contradicts $m \geq e + 1$. If $n + s = 0$ and $Y_2$ is singular, since $X_2$ is connected, $e \leq 1$, which again is a contradiction. Thus $n + s \geq 1$ in both cases.

Now we are ready to state the analogues of the main results of Sections 2 and 3. The analogue of Theorem 3.3 generalizes what [Fuj83, Corollary 5.17] says with respect to the simple connectedness of hyperelliptic, subcanonical polarized varieties (see the comment about this just before Theorem 3.3). The analogues of Theorems 2.7 and 2.12 (1) are novel though.

**Theorem 5.4.** If in the statements of Theorems 2.6, 2.7, 2.12 (1) and 3.3 we replace “$X$ a Horikawa variety” by “$(X, A)$ a hyperelliptic, subcanonical polarized variety with canonical singularities”, “the canonical morphism of $X$” by “the morphism induced by $|A|$”, and “$p_2(X)$” by “$h^0(A)$”, then the same conclusions of those theorems hold in this new setting.

In particular, the above statements hold for $(X, A)$ polarized, hyperelliptic Calabi–Yau varieties with canonical singularities and for $(X, A)$ polarized, hyperelliptic Fano varieties of index $i \leq n - 3$ (in the sense of Fujita, see [Fuj83 Definition 1.5]), with canonical singularities.

**Proof.** In the proofs of Theorems 2.6, 2.7 and 3.3 we showed that $X_2$ is regular, that $p_2(X_2) \leq 6$ if $Y_2$ and that $X_2$ is simply connected. If we set $s' = n - 1$, in all those cases, we in fact proved those statements for any hyperelliptic, $s'$–subcanonical polarized normal surface $(X_2, \phi_2^*(O_Y(1)))$ with canonical singularities and $s' \geq 1$. Actually, it is easy to check that the same arguments go through and the same statements are true for $s' = 0$ and for $s' = -1$, with the possible exception of Theorem 5.3 in the latter case. Thus, if $(X, A)$ is a hyperelliptic, $s$–subcanonical polarized with canonical singularities and $s \geq -n + 1$, then we define $X_2$ from $(X, A)$ in a way analogous to the proofs of Theorems 2.6, 2.7 and 3.3 so we get analogous conclusions except maybe for the above mentioned exception.
We deal now with the exception. In that case, the arguments of the proof of Theorem 3.3 do work except maybe if $Y_2 = F_0$. Then (see (5.3.1))

$$B \sim 2C_0 + (4 - 2m)f.$$}

Since $B$ is effective, then $m = 1$, in which case $B$ is ample, or $m = 2$. Since $\overline{X}_2$ is normal, in the latter case $B$ is the union of two lines and $\overline{X}_2 = F_0$, so $\overline{X}_2$ is simply connected in both cases and, arguing as in the proof of Theorem 3.3, $X$ is also simply connected.

We deal now with the analogue of Theorem 2.12 (1). If $Y$ is $\mathbb{P}^n$ or a smooth hyperquadric in $\mathbb{P}^{n+1}$, the same arguments of the proof of Theorem 2.12 work. Now let $Y$ be a smooth rational normal scrol. Using the same notation as there, the analogue of (2.12.1) yields

$$O_Y(K_Y) \otimes L = O_Y(s),$$

so

$$h^i(L^{-1}) = h^{n-i}(O_Y(s)).$$

If $s < 0$, then $h^{n-i}(O_Y(s)) = 0$ by Serre duality and the Kodaira vanishing theorem. If $s \geq 0$, then

$$h^{n-i}(O_Y(s)) = h^{n-i}(S^s(E)) = 0.$$

Remark 5.5. Because of the analogue of Theorem 2.6, hyperelliptic, 0–subcanonical polarized varieties with canonical singularities are Calabi–Yau.

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