Wire Construction of Topological Crystalline Superconductors

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We present a unified framework to construct and classify topological crystalline superconductors (TCSCs). The building blocks are one-dimensional topological superconductors (TSCs) protected solely by onsite symmetries, which are arranged and glued by crystalline symmetries in real space. We call this real-space scheme “wire construction”, and we show its procedure can be formulated mathematically. For illustration, we treat TCSCs of Altland-Zirnbauer (AZ) class[1] DIII protected by wallpaper group as well as layer group symmetries, with the resulting states by wire construction being TCSCs in two-dimension. We also discuss how these real-space TCSCs by wire construction are characterized by anomalous boundary states. Our method provides a real-space picture for TCSCs versus momentum-space pictures.

I. INTRODUCTION

Topological quantum matters have received wide concerns among the condensed matter physics community in recent years[2–27]. Beyond Landau’s theory of symmetry breaking which classifies matters according to symmetries, the studies of topological quantum matters reveal many different phases of matter with the same symmetries, which entail explanations from the perspective of topology. One significant problem in the studies of topological matters is how many topologically inequivalent classes of states exist for a specific kind of physical system, i.e., topological classification. This issue has been resolved, completely or incompletely, for many physical systems, especially for weakly interacting systems, which have single-particle band descriptions. For free fermion systems protected by only onsite symmetries, the classification problems have been solved by K-theory-based method, which is also known as the “tenfold way” [28–30]. The classification of topological insulators protected by spacial symmetries, i.e., topological crystalline insulators (TCIs), has been completed by different methods[31–37]. By contrast, the study of classification problems of topological crystalline superconductors (TCSCs) are still ongoing with many efforts [38–58].

Historically, topological classifications always began with momentum-space frameworks, where one can mathematically define topological invariants in terms of Bloch wave functions. Besides, the diagnosis of topological states sometimes can be easily accessible in momentum space, where one can acquire (usually partial) information of the topological invariants by using only the symmetry representations of Bloch wave functions at high symmetry points in Brillouin zone (BZ). The diagnosis of band topology through symmetries traced back to the Fu-Kane formula[8] and was further promoted into the theory of symmetry-based indicators[59–69]. However, momentum-space schemes for topological classification often involve sophisticated mathematical tools, such as K-theory, and the boundary anomalies characterized by the topological states could be obscure. A different type of methods developed later, which provide another perspective to crystalline topological states, are more physically transparent than momentum-space-based methods. These methods can be summarised as real-space construction[34, 37, 70–78], where topological states protected adjointly by onsite symmetries and spacial symmetries are constructed by arranging topological states protected by onsite symmetries in the ways that maintain spacial symmetries. One of the simplest examples is that, 3D weak topological insulators (TIs) can be constructed by stacking an array of 2D TIs in a specific direction. From this example, it is not hard for one to see crystalline topological states could be understood intuitively from real-space perspective.

Despite success in the classification of topological crystalline insulators (TCIs)[33, 34, 36, 37], the extension of real-space construction to TCSCs is not straightforward, as many detailed techniques become more sophisticated. In real-space construction of TCIs, only 2D building blocks, which are actually (mirror) Chern insulators and 2D TIs, are used, while for TCSC, due to the existence of particle-hole symmetry, the building blocks can be diverse. In addition to 2D building blocks, 1D building blocks also have to be taken into consideration, for TCSCs of several AZ classes. Compared with 2D building blocks which either coincide with mirror planes or not, 1D building blocks can have more complicated symmetries such as $C_n$-rotations or $C_{nv}$, which provide more ingredients for topological classifications. Additionally, while classifying TCIs involve group representation theory, classifying TCSCs involve projective representations, due to nontrivial representations of crystalline symmetries acting on the superconducting gap function.

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In this letter, we focus on real-space construction by 1D building blocks, i.e., “wire construction”, which we claim can give complete classification of TCSCs in 2D, including weak TSCs (protected by translation symmetries) and second-order TSCs. We deal with 2D TCSCs in AZ class DIII, which have always received special concerns. It is worth noting that, 2D strong TSCs with helical Majorana edge states are beyond the scope of wire construction. Therefore we remark our method does not give the full classification of TSCs in 2D. In addition, one should be careful when discussing 2D spatial symmetries give the full classification of TSCs in 2D. In addition, one should be careful when discussing 2D spatial symmetries and their symmetry representations on spin-1/2 particles are different, i.e., C_2^2 = -1 while T^2 = 1. This difference can significantly influence the TCSC classification, as will be shown in the following context. For a comprehensive discussion, in this work, we treat both wallpaper groups and layer groups.

Before going further, we briefly outline the contents. Sec.II is devoted to the formulation of wire construction, where we begin with the geometric structure in real space to decorate with “wires”, i.e., 1D TSCs protected solely by onsite symmetries, and then show the condition for the wire-like building blocks to be glued together without “opened boundaries” (see Sec.II.D), as well as the condition for them to be stable under real-space trivialization process (Sec.II.E). In Sec.III, we give an example of wallpaper group p4 to show the above procedure, and more examples can be found in the appendix. In Sec.IV we compare wallpaper groups and layer groups in terms of TCSC classification. In Sec.V we summarise the results of TCSC classification by wire construction, with detailed tables in Appendix.G. In Sec.VI, we discuss the topological invariants and anomalous boundary states which characterize the TCSCs by wire construction. Lastly, in Sec.VII we summarise this work and discuss further directions.

II. FORMULATION

In this section, we explicitly present the complete procedure of wire construction.

A. Prerequisite: symmetries in superconductors

Here we formulate the symmetries of BdG Hamiltonian following Ref.[43, 65], which is the basis for further discussions.

Consider a generic superconducting Hamiltonian

$$\mathcal{H} = \epsilon(\hat{c}^\dagger \hat{c} + h.c.) + \Delta \hat{c}^\dagger \hat{c}^\dagger + \Delta \hat{c} \hat{c}$$  \hspace{1cm} (1)

where $\hat{c}$ is the shorthand for $(\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_n)$, $\epsilon$ is the normal state spectrum, and $\Delta$ is the superconducting gap function.

A crystalline symmetry $g$ of normal state Hamiltonian could no longer be a symmetry for the superconducting Hamiltonian, as it may be broken by the superconducting gap function $\Delta$, i.e.,

$$\hat{U}_g(\Delta \hat{c} \hat{c}) \hat{U}_g^{-1} = e^{i\theta} \Delta \hat{c} \hat{c}$$  \hspace{1cm} (2)

where $e^{i\theta}$ is a $U(1)$ phase factor. Nevertheless, we can redefine the crystalline symmetry as

$$\hat{U}_g = \hat{U}_g \hat{V}_g = \hat{U}_g e^{i\frac{\pi}{2} \hat{Q}}$$  \hspace{1cm} (3)

where $\hat{Q}$ is the charge operator. Then we have

$$\hat{U}_g'(\Delta \hat{c} \hat{c}) \hat{U}_g^{-1} = \Delta \hat{c} \hat{c}$$  \hspace{1cm} (4)

So we get to restore the crystalline symmetry of the Hamiltonian. Now for $\hat{U}_g$, we have

$$\{\hat{U}_g, \hat{\mathcal{T}}\} = 0, \{\hat{U}_g, \hat{\mathcal{P}}\} = 0$$  \hspace{1cm} (5)

where $\hat{\mathcal{T}}$ is time-reversal symmetry and $\hat{\mathcal{P}}$ is particle-hole symmetry, by noting

$$\{\hat{U}_g, \hat{\mathcal{T}}\} = 0, \{\hat{U}_g, \hat{\mathcal{P}}\} = 0$$  \hspace{1cm} (6)

and

$$\{\hat{V}_g, \hat{\mathcal{T}}\} = 0, \{\hat{V}_g, \hat{\mathcal{P}}\} = 0$$  \hspace{1cm} (7)

In the first-quantization language, we can write

$$\mathcal{U}_g = \begin{pmatrix} U_g e^{i\frac{\pi}{2}} & 0 \\ 0 & U_g^* e^{-i\frac{\pi}{2}} \end{pmatrix}$$  \hspace{1cm} (8)

where $U_g$ is the matrix representation of $g$ for normal state.

However, we move on for a new convention as we hope to make $\mathcal{U}_g$ commutes with $\hat{\mathcal{T}}$ as it does for the normal state. We do this by

$$\mathcal{U}_g \to \mathcal{U}_g e^{-i\frac{\pi}{2}} = \begin{pmatrix} U_g & 0 \\ 0 & U_g^* e^{-i\theta} \end{pmatrix}$$  \hspace{1cm} (9)

This actually amounts to changing $\hat{c}$ and $\hat{c}^\dagger$ as

$$\hat{c} \to \hat{c} e^{-i\frac{\pi}{2}}, \quad \hat{c}^\dagger \to \hat{c}^\dagger e^{i\frac{\pi}{2}}$$  \hspace{1cm} (10)

In such a convention, we have

$$\{\hat{U}_g, \hat{\mathcal{T}}\} = 0, \quad \hat{U}_g \hat{\mathcal{P}} = e^{i\theta} \hat{\mathcal{P}} \hat{U}_g$$  \hspace{1cm} (11)

In the first-quantization language, the particle-hole symmetry is given by

$$\mathcal{P} = \begin{pmatrix} 0 & \xi 1_n \\ 1_n & 0 \end{pmatrix}$$  \hspace{1cm} (12)
where $K$ is the complex conjugation operator, $\xi = \pm 1$ depends on physical realization. In presence of $T$, $\theta$ is restricted to be 0 or $\theta$, thus we have

$$U_g = \begin{pmatrix} U_g & 0 \\ 0 & \pm U_g^* \end{pmatrix}$$

which leads to

$$\mathcal{P}U_g = \pm U_g\mathcal{P}$$

and further

$$SU_g = \pm U_gS$$

where $S = T\mathcal{P}$ is the Chiral symmetry.

To sum up, due to the nontrivial representation of superconducting gap function under crystalline symmetry $G_c$, the redefined crystalline symmetry $U_g$, in presence of time-reversal, could either commute or anti-commute with particle-hole symmetry $\mathcal{P}$ (or Chiral symmetry $S$), i.e.,

$$\mathcal{P}U_g = \pm U_g\mathcal{P} \iff SU_g = \pm U_gS$$

As a result, the whole symmetry group generated by crystalline symmetry group $G_c$, time-reversal symmetry $T$ and particle-hole symmetry $\mathcal{P}$ (or Chiral symmetry $S$), is projectively represented, with the factor system given by

$$z^g_s \mathcal{P} s \mathcal{P} s = U_g s \mathcal{P} s$$

where $z^g_s = \chi_g = \pm 1$ depending on whether $U_g$ commutes or anti-commutes with $S$.

**B. Cell decomposition**

The first step of wire construction (as well as for other real-space construction schemes) is to decompose the nD space into small units by applying crystalline symmetries, which we call $d$-cells, where $d = 0, 1, 2, \ldots, n$. The resulting geometric structure composed of all $d$-cells is called a cell complex, which is a CW-complex in mathematics. Each $d$-cell has no two points inside it related by crystalline symmetry operations. In our case, we have $n = 2$. The 2-cells are small patches that cover the whole 2D space without overlaps, and a single 2-cell is also called an asymmetric unit (AU). 1-cells are edges of 2-cells, and 0-cells are ends of 1-cells. Each $(d-1)$-cell is shared by more than one $d$-cells.

It is worth noting that there could be different ways to construct the cell complex for a specific crystalline symmetry group. However, these seemingly different cell complexes are essentially equivalent and can be deformed into each other while preserving symmetries.

Here we give an example, i.e., the cell complex of 2D space group $p2$, which is shown in Figure 1.

**C. Building blocks**

Upon having the cell complex, one has to decide what topological states can be decorated on the 1-cells as building blocks for wire construction. On each 1-cell, no crystalline symmetry operation can relate two distinct points, therefore the topological states that can be placed on it are those protected solely by onsite symmetries. Note that apart from $T$, $\mathcal{P}$ and $S$, the onsite symmetries could include crystalline operations that keep every point on the 1-cell invariant, which form a subgroup $G^{1}_c$ of the entire onsite symmetry group $G^1$. We remark that, although $G^{1}_c$ generically could contain lattice translations, here we ignore them and always regard $G^{1}_c$ as a point group.

For wallpaper groups, $G^{1}_c$ is always simple, which depends on whether the 1-cell coincides with some mirror line or not and turns out to be either 1 or $M$, where 1 stands for the point group with identity operation only, and $M$ for the point group having mirror reflection besides identity. Different types of 1-cells host different topological states protected by onsite symmetries, depending on specific $G^{1}_c$, thus $G^1$. And it further depends on the projective representations of $G^1$, as in presence of nontrivial symmetry representations of the gap function, the representations of $G^1$ are projective.

Actually, the projective irreducible representations of $G^1$ can be obtained from the irreducible representations (irreps) of $G^{1}_c$. Given the irreps of $G^{1}_c$, the inclusion of $T$, $\mathcal{P}$ and $S$ can have effects on these irreps, making them either invariant or paired, and the resulting representations, together with those of $T$, $\mathcal{P}$ and $S$, form projective
irreps of $G^1$. For each sector labeled by an irrep of $G^1_c$, we determine its effective Altland-Zirnbauer (AZ) class under $T$, $P$, and $S$, so that the topological classification of this sector can be directly known from the “tenfold way” results[30]. Finally, the total classification on the 1-cell is obtained by combining the classification of each sector. As two different sectors could be paired under $T$, $P$ and $S$, the independent sectors which are labeled by projective irreps of $G^1$ could be less than the sectors labeled by irreps of $G^1_c$, as two paired sectors always have the same or opposite topological numbers. Therefore we can define an independent topological index for each new sector labeled by projective irrep of $G^1$, see following contents.

To figure out the effective AZ classes and the behaviors of the irreps of $G^1_c$ under $T$, $P$ and $S$, one can employ a powerful tool, the generalized Wigner test[79–82], which can handle not only time-reversal but also particle-hole and chiral symmetries. It states that for a group $G$ which can be decomposed into left cosets as

$$G = aG_0 \oplus bG_0 \oplus cG_0 \oplus abG_0$$

(18)

where $G_0$ is a unitary group, $a$ is a $T$-like operator, $b$ is a $P$-like operator and $ab$ is $S$-like operator, there are three indices to compute in order to determine the effects of $T$, $P$ and $S$ on a irrep $\alpha$ of $G_0$:

$$W^\alpha (T) = \frac{1}{|G_0|} \sum_{g \in G_0} z_{ag,ag} (\bar{\chi}_\alpha((ag)^2))$$

(19)

$$W^\alpha (P) = \frac{1}{|G_0|} \sum_{g \in G_0} z_{bg,bg} (\bar{\chi}_\alpha((bg)^2))$$

(20)

$$W^\alpha (S) = \frac{1}{|G_0|} \sum_{g \in G_0} \frac{z_{g,ab}}{z_{ab,ab} g(ab)^{-1}} (\bar{\chi}_\alpha(g)\bar{\chi}_\alpha^*((ab)(ab)^{-1}))$$

(21)

with $W^\alpha (T) \in \{\pm 1, 0\}$, $W^\alpha (P) \in \{\pm 1, 0\}$, and $W^\alpha (S) \in \{1,0\}$. Here $|G_0|$ is the number of elements in $G_0$, $z_{a_1,a_2} u_{g_1g_2} = u_{g_1} u_{g_2}$ specifies the factor system, and $\bar{\chi}_\alpha$ denotes the character of the irrep $\alpha$. Specifically,

$$W^\alpha (U = T, P, S) = 1$$

(22)

indicates $\alpha$ is invariant under $U$, $W^\alpha (U = T, P, S) = -1$

(23)

indicates $\alpha$ is paired with itself, and $W^\alpha (U = T, P, S) = 0$

(24)

indicates $\alpha$ is paired with another irrep $\alpha'$. The effective AZ class for $\alpha$ is determined as

$$[W^\alpha (T), W^\alpha (P), W^\alpha (S)] = [T^2, P^2, S^2]$$

(25)

For instance,

$$[W^\alpha (T), W^\alpha (P), W^\alpha (S)] = [-1, 0, 0]$$

(26)

dictates $\alpha$ has effective AZ class AII.

In fact, it is straightforward to figure out all the building blocks for wallpaper groups. As the irreps of point group $1$ and $M$ are all 1D, it is each to see the behaviors of each sector under $T$, $P$ and $S$, i.e., whether it is unchanged or get paired with either itself or another irrep. The results are illustrated in Fig.2. By contrast, 1-cells with higher symmetries exist in layer groups, the building blocks on which are more complicated and not straightforward to determine.

1. Building blocks for wallpaper groups

1. 1-cells with $C_2 = 1$

These 1-cells do not coincide with any mirror lines, and their $G_c$ is trivial. Therefore the onsite symmetry group $G^1$ is generated by $T$ and $P$ (or $S$). Apparently, the effective AZ class is DIII with $Z_2$ classification in 1D, and the corresponding topolog-
ical invariant $\nu^T$ is defined as\cite{27}

\begin{equation}
(−1)\nu^T = e^{i\gamma(M)} , \quad \gamma(M) = \int_\pi^{-\pi} dk A^1_{(.)}(k) \tag{27}
\end{equation}

where $A^1_{(.)}$ and $A^1_{(.)}$ represent the Berry phase of the Kramer’s pair formed by the occupied states. The boundary state of such a building block with $Z_2 = 1$ is a Kramer’s pair of Majorana zero modes. Actually such a pair of zero modes can be eigenstates of chiral symmetry $\mathcal{S}$, with opposite eigenvalues of $\mathcal{S}$, i.e., $\pm 1$. One can write out the symmetry representation taken by the zero modes as

$$
\mathcal{T} = i\sigma_2 K \\
\mathcal{S} = \sigma_3 
$$

which satisfies $\mathcal{T}^2 = −1$, $\mathcal{S}^2 = 1$ and $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$.

2. 1-cells with $G_c = M$

This type of 1-cells lie on mirror lines. The Hilbert space is divided into two sectors with mirror eigenvalues $\pm i$ since $M^2 = −1$. One has to discuss whether the gap function is even or odd under $M$, i.e., $\chi_M = 1$ or $\chi_M = −1$.

(a) $\chi_M = 1$

Since $[M, \mathcal{T}] = 0$ and $[M, \mathcal{S}] = 0$, it is easy to see each of the two sectors is closed under $\mathcal{S}$, and they are interchanged under $\mathcal{T}$ and $\mathcal{P}$. Therefore each sector belongs to AZ class AIII with $Z_2$ classification in 1D. The corresponding topological invariant is the 1D winding number

$$
w^\pm = \frac{i}{4\pi} \int_\pi^{-\pi} dk \text{tr} [\mathcal{S} \mathcal{H}^{-1}(k) \partial_k \mathcal{H}(k)] \tag{30}
$$

where $\mathcal{H}_k$ stands for the Hamiltonian of the $M = \pm i$ sector. Since the two sectors are not independent but related by $\mathcal{T}$ and $\mathcal{P}$, there is only one independent $Z$ index and actually $w^+ = −w^-$. We can define the independent index as $w_M = \frac{1}{2}(w^+ − w^-)$.

For a $|w_M| = n$ state, at its 0D boundary there are $n$ Majorana zero modes with $M = +i$ and $\mathcal{S} = ±1$, as well as $n$ Majorana zero modes with $M = −i$ and $\mathcal{S} = ±1$, forming $n$ Majorana Kramer’s pairs. In fact, one can write out the symmetry representation carried by the zero modes as

$$
\mathcal{T} = i\sigma_2 K \\
\mathcal{S} = \sigma_3 \\
M = i\sigma_3 
$$

which satisfies $\mathcal{T}^2 = −1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $[\mathcal{T}, M] = 0$, and $[\mathcal{S}, M] = 0$.

(b) $\chi_M = −1$

Like the above case, it is easy to find each of the two mirror sectors is closed under $\mathcal{P}$ but change into the other one under $\mathcal{T}$ and $\mathcal{S}$. Therefore each sector belongs to AZ class D with $Z_2$ classification in 1D. The topological index $\nu$ is given by\cite{27}

\begin{equation}
(−1)^\nu = e^{i\gamma}, \quad \gamma = \int_\pi^{-\pi} dk A^1_{(.)}(k) \tag{34}
\end{equation}

where $A^1_{(.)}$ stands for Berry connection of occupied states. Since each of the two sectors is not independent and they are related by $\mathcal{T}$ and $\mathcal{S}$, they have the same $\nu$ and there is only one independent $Z_2$ index. The Majorana zero modes at its 0D boundary are described by the symmetries

$$
\mathcal{T} = i\sigma_2 K \\
\mathcal{S} = \sigma_1 \\
M = i\sigma_3 
$$

which satisfy $\mathcal{T}^2 = −1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $[\mathcal{T}, M] = 0$, and $[\mathcal{S}, M] = 0$.

2. building blocks for layer groups

Firstly note that all the building blocks for wallpaper groups discussed above are also for layer groups, as wallpaper groups can be viewed as subset of layer groups. Besides, for layer groups there can be mirror reflection vertical to the lattice plane, which we denote as $M$. When combined with $M$ (or $M_y$), the generated point group is $C_{2v}$. As a consequence, the 1-cells coinciding with such $C_{2v}$ axis have $C^1 = C_{2v}$. The determination of building blocks on such 1-cells is not straightforward and one need turn to the Wigner test.

For $C_{2v}$, there are four representations of the gap function, $A_1$, $B_1$, $A_2$ and $B_2$. Meanwhile, there is only one double-valued irrep, $\Gamma_3$. By careful analysis, we find that for $A_1$ representation, the classification on the 1-cell is trivial, while for the remaining three representations, i.e., $A_2$, $B_1$ and $B_2$, the classifications are all $Z$, with the corresponding topological index being the 1D winding numbers defined regarding either $M$ or $C_2$ sectors. Here we summarise the three nontrivial building blocks in Fig.3.

D. Gluing condition

By decorating 1-cells with the above building blocks, we have a resulting state in 2D, which we call a decoration. However, not all those decorations are what we need. As the building blocks are 1D TSCs, they naturally have topologically protected Majorana zero modes
at their 0D boundaries, which coincide with 0-cells. To make a decoration a TCSC fully gapped in the bulk, one has to ensure all the 0D zero modes at the 0-cells are gapped out, i.e., there are not “opened boundaries” in the interior of the decoration, which we call gluing condition at 0-cells.

To determine whether 1-cells decorated with building blocks can be glued together, one has to carefully study the boundary theories of Majorana modes. The crucial issue for gluing condition is to address the problem of group induction, which is illustrated in Fig.4(a) and explained as follows. Generally, there could be multiple 1-cells meeting at a 0-cell, and the onsite symmetry group of the 0-cell is larger than or equivalent to those of the surrounding 1-cells. Consider a set of symmetry-related 1-cells, each one having onsite symmetry group $G^1$, and the 0-cell that they share as a common end has onsite symmetry group $G^0$. Then $G^1$ must be a subgroup of $G^0$, i.e., $G^1 \subseteq G^0$, and the symmetry-related 1-cells are in one-to-one correspondence to the representatives of the cosets $G^0/G^1$. Therefore, once having a building block, i.e., choosing a representative among the symmetry-related 1-cells and decorating it by 1D TCS with 0D zero modes taking projective irreps of $G^1$, we can derive an induced projective representation of $G^0$ spanned by the entire set of zero modes from all the symmetry-related 1-cells. In another word, the symmetry group that describes the zero modes at the 0-cell is enhanced from $G_0$ to $G_1$. Mathematically, there is a standard formalism to derive induced representations, see Appendix B. This group induction procedure is necessary since the boundary modes of 1-cells locate at 0-cells and therefore their corresponding onsite symmetry group is enhanced from $G^1$ to $G^0$. One can further imagine the 0D boundary modes are contributed by 1D TSCs with onsite symmetry group lifted from $G^1$ to $G^0$. 

![FIG. 3. Nontrivial Building blocks on $C_{2y}$-axis. The three types of building blocks are for $A_2$, $B_1$, and $B_2$, respectively. The trivial building block for $A_1$ is not plotted. (a) Building block for $A_2$. The topological index is mirror winding number $w_{M_x} \in \mathbb{Z}$. The generator with $w_{M_x} = 1$ has a pair of Majorana zero modes at its 0D boundary (blue and orange dots), with $M_x$ eigenvalues $\pm i$ and $S$ eigenvalues $\pm 1$. (b) Building block for $B_1$. The topological index is mirror winding number $w_{M_y} \in \mathbb{Z}$. The generator with $w_{M_y} = 1$ has a pair of Majorana zero modes at its boundary (blue and orange dots), with $M_y$ eigenvalues $\pm i$ and $S$ eigenvalues $\pm 1$. (c) Building block for $B_2$. The topological index is $C_2$ winding number $w_{C_2} \in \mathbb{Z}$. The generator with $w_{C_2} = 1$ has a pair of Majorana zero modes at its boundary (blue and orange dots), with $C_2$ eigenvalues $\pm i$ and $S$ eigenvalues $\pm 1$.](image)

![FIG. 4. (a) Illustration of group induction. The 1-cells and 0-cell have onsite symmetry group $G^1$ and $G^0$, respectively. The zero modes $\{\gamma_1\}$ belong to a projective representation $\Gamma^1$ of $G^1$, which locate at the 0D boundary of a single 1-cell. There will be an induced projective representation $\Gamma^0$ of $G^0$, carried by the zero modes $\{\gamma_0\}$ from all the symmetry-related 1-cells which meet at the 0-cell. (c) and (d) Intuitive comprehension of the gluing condition. Two boundary modes of the same symmetry representation can not be gapped if they have the same topological number (c) and can be gapped if they have opposite topological numbers (d).](image)
of a linear map from all 1-cells to all 0-cells. Suppose we have all the projective irreps of \( G^1 \) for all 1-cells, and we denote the \( m \)-th projective irrep for the \( i \)-th 1-cell as \( \Gamma^1_{i,m} \). Correspondingly, the generator of the states spanning \( \Gamma^1_{i,m} \) is denoted as \( \phi^i_{1,m} \), i.e., if the sector of \( \Gamma^1_{i,m} \) has \( \mathbb{Z} \) topological classification, then \( \phi^i_{1,m} \) has \( \mathbb{Z} = 1 \). Likewise, we denote the \( n \)-th projective irrep for the \( j \)-th 0-cell as \( \Gamma^0_{j,n} \), and the generator as \( \phi^0_{j,n} \). By group induction, we get to know the mappings from all \( \Gamma^1 \)'s to \( \Gamma^0 \)'s, and the matrix elements from the linear space spanned by \( \{ \phi^1 \} \) to the space spanned by \( \{ \phi^0 \} \) are given by

\[
\left( \begin{array}{c}
\vdots \\
c_{j,n} \\
\vdots
\end{array} \right)
\]

(42)

where \( c_{j,n} \) is determined by the weight of \( \Gamma^0_{j,n} \) in the decomposition of the induced projective representation from \( \Gamma^1_{i,m} \), i.e.,

\[
\Gamma^1_{i,m} \rightarrow \Gamma^0 = \bigoplus_{j,n} c_{j,n} \Gamma^0_{j,n}
\]

(43)

### E. Bubble equivalence

However, we have not arrived at our destination even we get all decorations allowed by gluing condition, because some of them could be unstable against a kind of real-space trivialization process called “bubble equivalence”\cite{34, 75}. In our case, “bubbles” are 2D objects inside 2-cells, created from a generic point and growing larger and larger, until their edges coincide with 1-cells. Despite that its edges can be decorated with 1D TSCs, a bubble itself is intrinsically trivial as its interior is empty and it can shrink back into a point then vanish. Because bubbles can create 1D TSCs on 1-cells by their edges, which could be equivalent to some decorations, one has to retract those unstable decorations. As a result, the final topological classification is obtained by computing a quotient space \( \mathcal{E}_1/\mathcal{E}_2 \), where \( \mathcal{E}_1 \) stands for the space spanned by decorations satisfying gluing condition, and \( \mathcal{E}_2 \) for the space spanned by decorations trivialized by bubbles equivalence.

Like gluing condition, the bubble equivalence process also involves group induction. Consider a set of symmetry-related 2-cells (in our 2D case, at most two such bubbles, related by mirror or \( C_5 \) with onsite symmetry group \( G^2 \) intersecting at a 1-cell with onsite symmetry group \( G^1 \), which they share as a common edge. Then we must have \( G^2 \subseteq G^1 \), as well as \( G^2 \subseteq G^1 \). Since a single bubble locates inside a 2-cell, it has the same onsite symmetry group as the 2-cell does, i.e., \( G^2 \), and the symmetry group of the entire set of symmetry-related
bubbles should be enhanced to $G^1$. That is to say, the resulting states from the edges of surrounding bubbles on a 1-cell span an induced projective representation of $G^1$.

However, to give a simple example of bubble equivalence to the readers, we turn to TCI classification in 3D, in the presence of mirror symmetry and in absence of time-reversal[37], see Fig.6. The real-space construction can be realized by decorating a 2D mirror Chern insulator with mirror Chern numbers $(C_1^+, C_1^-)$ at the mirror plane. Due to the absence of time-reversal, $C_1^+$ and $-C_1^-$ are not necessarily equal, i.e., there could be nonvanishing net Chern number. Then we show the decoration with $(C_1^+, C_1^-) = (1, 1)$ can be trivialized by the generator of bubble equivalence. Here the bubbles are 3D objects with their 2D surfaces carrying Chern numbers. On the two sides of the mirror plane, there are two bubbles related by mirror reflection represented by an off-diagonal matrix, $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, by noting the two bubbles (as well as their surfaces) are interchanged under mirror reflection. When the bubble surfaces coincide with the mirror plane, the resulting states become two mirror eigenstates with mirror eigenvalues $+i$ and $-i$, as can be seen by the diagonalization of $M$. Meanwhile, since the sign of Chern number is invariant under mirror reflection by noting Berry curvature is a pseudovector, the two bubble surfaces carry the same Chern number $\pm 1$. As a result, the two bubbles create a state of $(C_1^+, C_1^-) = (1, 1)$ on the mirror 2-cell, thus trivialize all the decorations with $(C_1^+, C_1^-) = (n, n)$ where $n \in \mathbb{Z}$.

Nevertheless, we find for 2D TCSs in class DIII protected by wallpaper groups, any bubble equivalence has trivial effects, i.e., does not change the decorations on 1-cells, thus we need not consider such a process. For a detailed analysis see AppendixC.

On the contrary, for layer groups, there can be nontrivial bubble equivalence due to additional symmetries. In fact, nontrivial bubble equivalence happens if there is a vertical mirror reflection $M_z$, which is absent in wallpaper groups.

Consider layer group $pm2m$ as an example. This group is generated by mirror reflection in $x$ direction, $M_x$, and vertical mirror reflection $M_z$. The combination of them is a $C_{2v}$ rotation. As a result, there are 1-cells that coincide with $C_{2v}$ axes at $x = 0$ and $x = \frac{1}{2}$. When the representation of gap function is chosen such that $\chi_{M_x} = -1$ and $\chi_{M_z} = 1$, these two 1-cells can be decorated with 1D $C_{2v}$-symmetric TSC with $\mathbb{Z}$ classification. The topological index is the mirror winding number regarding $M_z$, i.e., $\chi_{M_z} = \frac{1}{2}(w_{M_z} - w_{M_z})$, and the generator of decoration has $w_{M_z} = 1$. Next we analysis the effect of bubbles on the $C_{2v}$ axis, as shown in Fig.7. For brievity we only consider the $C_{2v}$ axis at $x = 0$, and the effect of bubble equivalence on the $C_{2v}$ axis at $x = \frac{1}{2}$ is the same. The bubbles have $C_e = M_z$, and their edges are 1D mirror-symmetric TSCs with boundary zero modes described by symmetries

$$T = i\sigma_3 k$$
$$S = \sigma_3$$
$$M_z = i\sigma_3$$

When the two bubbles related by $M_z$ have their edges coinciding with the $C_{2v}$ axis, the $G_e$ of the resulting state contributed by the two bubble edges is enhanced from $M_z$ to $C_{2v}$, and by group induction the symmetries are
The relevant double-valued irreps are presented in the following table.

| $\Gamma_3$ | $\Gamma_6$ | $\Gamma_7$ | $\Gamma_8$ | $\Gamma_3$ | $\Gamma_4$ |
|------------|------------|------------|------------|------------|------------|
| $C_4$      | $e^{i\frac{\pi}{4}}$ | $e^{-i\frac{\pi}{4}}$ | $e^{i\frac{3\pi}{4}}$ | $e^{i\frac{\pi}{4}}$ |

TABLE I. Double-valued irreducible representations of $C_4$ and $C_2$

Applying the Wigner test to the irreps of $C_4$, we get

$$[W^{\alpha}(T), W^{\beta}(P), W^{\gamma}(S)] = [0, 0, 1]$$

for $\alpha = \bar{\Gamma}_{i,i=1,2,3,4}$, which means each irrep of $C_4$ belongs to class AIII characterised by the 1D winding number $w \in \mathbb{Z}$, and each of them is closed under $S$ but paired with its conjugate irrep, $\bar{S}$, under $T$ and $P$. It is easy to see $\bar{\Gamma}_5$, $\bar{\Gamma}_7$ are paired, and $\bar{\Gamma}_6$, $\bar{\Gamma}_8$ are paired. We denote the resulting projective irreps as $\bar{\Gamma}_{58}$ and $\bar{\Gamma}_{68}$, and write out them explicitly as

$$T = i\sigma_2 K$$

$$S = \sigma_3$$

$$C_4 = \begin{pmatrix} e^{i\frac{3\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}$$

for $\bar{\Gamma}_{58}$, and

$$T = i\sigma_2 K$$

$$S = \sigma_3$$

$$C_4 = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}$$

for $\bar{\Gamma}_{68}$. It is obvious that the sectors belong to $\bar{\Gamma}_5$ and $\bar{\Gamma}_7$ have opposite winding numbers, as well as for the sectors belong to $\bar{\Gamma}_6$ and $\bar{\Gamma}_8$. We can define the independent topological index from the winding numbers regarding $C_4$ sectors, for $\bar{\Gamma}_{58}$

$$w_{C_4} = \frac{1}{2}(w_{e^{i\frac{\pi}{4}}}-w_{e^{-i\frac{\pi}{4}}})$$

and for $\bar{\Gamma}_{68}$

$$w_{C_4} = \frac{1}{2}(w_{e^{i\frac{\pi}{4}}}-w_{e^{-i\frac{\pi}{4}}})$$

Likewise, we apply the Wigner test to the irreps of $C_2$ and get

$$[W^{\alpha}(T), W^{\beta}(P), W^{\gamma}(S)] = [0, 0, 1]$$

for $\alpha = \bar{\Gamma}_{i,i=3,4}$, which means each irrep of $C_2$ also belongs to AZ class AIII characterised by the 1D winding number $w \in \mathbb{Z}$. We denote the resulting projective irrep as $\bar{\Gamma}_{34}$, and write it out explicitly as

$$T = i\sigma_2 K$$

$$S = \sigma_3$$

$$C_2 = i\sigma_3$$

III. EXAMPLE: p4, A$_2$ REPRESENTATION

Here we use wallpaper group p4 with the representation of gap function $A_2$, ($\chi_{C_4} = 1$) to show the complete procedure of wire construction. There is another example for wallpaper group in Appendix.D, as well as an example for layer group in Appendix.E.

The cell complex of p4 is shown in Fig.8. All the three 1-cells have $G_c = 1$ and can be decorated with 1D DIII TSC with $\mathbb{Z}_2$ classification, thus there are three independent decorations before considering gluing condition.

For convenience of analyzing gluing condition, we first look into the projective irreps of $C_4^0$, which in this case are actually equivalent to linear irreps since $\chi_{C_4} = 1$. The irreps of $G_c^0$ are given in the following table. Among the 0-cells, $e_1^0$ and $e_2^0$ have $G_c = C_4$, and $e_3^0$ has $G_c = C_2$. (a) We have the 1-cells colored.

![Fig. 8. The cell complex of wallpaper group p4. (a) There are two independent 1-cells, colored by blue and red, respectively, and denoted as $e_1$ and $e_2$. There are three independent 0-cells, denoted as $e_1^0$, $e_2^0$, and $e_3^0$, among them $e_1^0$ and $e_2^0$ have $G_c^0 = C_4$, while $e_3^0$ has $G_c^0 = C_2$. We select the original point, i.e., the $C_4$ center, at $e_1^0$. (b) We have all the 1-cells colored.](image-url)
It is easy to see the two sectors of $C_2 = \pm i$ have opposite winding numbers. We can define the independent topological index as

$$w_{C_2} = \frac{1}{2}(w_i - w_{-i})$$  \hspace{1cm} (65)

Now we are prepared to consider gluing condition for each 1-cell decoration.

1. $e_1^0$ decoration

   From the cell complex, it is easy to see that $e_1^0$ and its symmetry-related partners only intersect at the 0-cells $e_3^0$ and $e_2^0$, but not at $e_1^0$. Thus we only have to consider gluing condition at $e_3^0$ and $e_2^0$.

   (a) gluing condition at $e_3^0$

   We have symmetry representation for the zero modes at the boundary of $e_1^0$ as

   $$T = i\sigma_2\mathcal{K}$$ \hspace{1cm} (66)
   $$S = \sigma_3\mathcal{K}$$ \hspace{1cm} (67)

   By group induction, we have the induced representation at $e_1^0$:

   $$T = s_0\mu_0i\sigma_2\mathcal{K}$$ \hspace{1cm} (68)
   $$S = s_0\mu_0\sigma_3$$ \hspace{1cm} (69)

   $$C_4 = \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{array}\right) \otimes \sigma_0$$ \hspace{1cm} (70)

   As $[C_4, S] = 0$, we can simultaneously diagonalize them and get

   $$C_4 = \sigma_0 \otimes \left(\begin{array}{cccc} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{array}\right)$$ \hspace{1cm} (71)

   $$S = \sigma_3 \otimes \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ \hspace{1cm} (72)

   By decomposition, we can see there are two $\Gamma_{5\oplus 7}$’s and two $\Gamma_{6\oplus 8}$’s to describe the zero modes. Explicitly, the two $\Gamma_{5\oplus 7}$’s are given by

   $$C_4 = \left(\begin{array}{cc} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{array}\right), \quad S = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$ \hspace{1cm} (73)

   and

   $$C_4 = \left(\begin{array}{cc} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{array}\right), \quad S = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right)$$ \hspace{1cm} (74)

   The first one has $w_{C_4} = 1$ while the second one has $w_{C_4} = -1$, thus the net topological number by them is zero. Likewise, the two $\Gamma_{6\oplus 8}$’s are given by

   $$C_4 = \left(\begin{array}{cc} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{array}\right), \quad S = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$ \hspace{1cm} (75)

   and

   $$C_4 = \left(\begin{array}{cc} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{array}\right), \quad S = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right)$$ \hspace{1cm} (76)

   with the first one having $w_{C_4} = 1$ and the second one having $w_{C_4} = -1$, thus the net topological number is also zero.

   We see that both the sector of $\bar{\Gamma}_{5\oplus 7}$ and the sector of $\bar{\Gamma}_{6\oplus 8}$ have vanishing net topological numbers, thus the gluing condition at $e_3^0$ is met.

   (b) gluing condition at $e_2^0$

   As what we did above, we have the induced representation at $e_2^0$ by group induction, i.e.,

   $$T = \mu_0\mu_0i\sigma_2\mathcal{K}$$ \hspace{1cm} (77)
   $$S = \mu_0\sigma_3$$ \hspace{1cm} (78)
   $$C_2 = i\mu_2\sigma_0$$ \hspace{1cm} (79)

   After simultaneous diagonalization of $C_2$ and $S$, we have

   $$C_2 = \left(\begin{array}{cccc} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{array}\right)$$ \hspace{1cm} (80)

   $$S = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$ \hspace{1cm} (81)

   This representation is reducible and can be decomposed in two $\bar{\Gamma}_{3\oplus 4}$’s, with the first one given by

   $$C_2 = \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right), \quad S = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$ \hspace{1cm} (82)

   and the second one given by

   $$C_2 = \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right), \quad S = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right)$$ \hspace{1cm} (83)

   We can see the first one has $w_{C_2} = 1$ while the second one has $w_{C_2} = -1$, thus the net topological number is zero. The vanishing topological number indicates the zero modes at $e_2^0$ can be gapped, therefore the gluing condition is satisfied.
2. $e_1^1$ decoration

This decoration is essentially the same with $e_1^1$, and the gluing condition is also satisfied at $e_0^2$ and $e_1^1$.

Through above analysis, we see that the decorations of $e_1^1$ and $e_2^2$ are allowed, both being $Z_2$-type and serving as the two generators for all TCSCs by wire construction. As bubble equivalence is trivial for wallpaper groups, the final classification is $Z_2^2$.

IV. WALLPAPER GROUPS VS LAYER GROUPS

In the introduction, we mentioned that wallpaper groups (WGs) sometimes can not give a full description for $C$ condition is not satisfied. Below we show it is different $1$-cells are represented as $\bar{1}$ with $I$ and $\bar{I}$ are implemented differently on spinful electrons. In Sec.II.D, we showed that $\bar{I}$ with $\chi_{\bar{I}} = 1$ prohibits nontrivial decorations as gluing condition is not satisfied. Below we show it is different for $C_2$.

![FIG. 9. Two 1-cells related by $\bar{I}$ or $C_2$. Each 1-cell is decorated with 1D DIII TSC.](image)

For comparison, we consider $\chi_{C_2} = 1$. The symmetries for the 0D zero modes contributed by the two $C_2$-related 1-cells are represented as

\[
\begin{align*}
T &= \mu_0 i \sigma_2 K \\
S &= \mu_0 \sigma_3 \\
C_2 &= \mu_1 i \sigma_3
\end{align*}
\]

(84) (85) (86)

The mass term proportional to $\mu_2 \sigma_1$ commutes with $C_2$ and $T$ while anti-commutes with $S$, and therefore can be added to gap out these zero modes. As bubble equivalence is not needed for wallpaper groups, the success of gluing condition for $C_2$ case indicates nontrivial TCSC classification. By contrast, the failure of gluing condition for $I$ case indicates trivial topological classification for even parity superconductors, and only odd-parity superconductors can be topologically nontrivial\cite{8385}.

It could be helpful for one to be clear with the relationships between WGs and LGs. In fact, each LG has one or more corresponding WGs which share the same 2D lattice as this WG, and the number of LGs (80) is much larger than that of WGs (17). Furthermore, among those LGs corresponding to the same WG, one of them shares completely the same group elements with the WG, and the remaining ones either have the same number of group elements although some elements differ, or can be generated from this WG by adding inversion or reflection (glide) in the vertical direction to the 2D plane. For instance, for the WG $p4mm$ with all symmetry elements defined in $x$-$y$ plane, LG $p4mm$ is completely the same with it, LG $p422$ has the same elements except that mirrors are replaced by $C_2$ rotations, and LG $p4mm$ is generated from $p4mm$ by adding mirror reflection in $z$ direction.

It is worth noting that, $p4mm$ and $p422$ share the same results of TCSC classification by wire construction, while $p4mm$ is more complicated due to the inclusion of $M_2$ and its TCSC classification can not be inferred from that of $p4mm$.

V. CLASSIFICATION RESULTS

Using wire construction, we obtain the classification of TCSCs protected by spatial symmetries in 2D, and we present the results as tables in Appendix.G. Specifically, in TABLE.VI, we show TCSC classification by wire construction for all WGs with all representations of gap function considered. In TABLE.VII, we list the LGs which share the same wire constructions thus TCSC classifications as those of LGs. In TABLE.VIII, we show the TCSC classification for a portion of LGs that can not be inferred directly from the results of WGs.

VI. INVARIANTS AND BOUNDARY STATES

When employing real-space method, one usually has the benefits that topological invariants and boundary states of the output states, i.e., those by real-space construction, are relatively easy to see. One can usually define topological invariants in real-space, and for each decoration, identify these topological invariants. In real-space construction of TCIs\cite{34}, for each crystalline symmetry operator $g$, its invariant is defined in the following way. Choose an generic point $O$ in a 3-cell, act it with $g$ and one gets the image point $gO$. Then draw an arbitrary path connecting $O$ and $gO$, and the invariant is determined by the 2-cell decorations, which are actually 2D TIs or (mirror) Chern insulators, that the path crosses. Here in our wire construction for 2D TCSCs, one could define invariants in a similar way.

Apart from topological invariants, the TCSCs by wire construction can be characterized in terms of their anomalous boundary states, in one-to-one correspondence with their invariants. We deal with decorations protected by single representative 2D spatial symmetries,
including translation, $C_n$ rotation, mirror reflection, and glide. For each independent generator of decorations, we figure out what its boundary states can be. If there are more complicated symmetries, i.e., any kinds of combinations of them, the boundary states can be comprehended as a superimposition of those protected by single symmetry.

Through our careful analysis, we identify some boundary states which are not touched by previous works, as shown in the following two cases. For boundary states protected by other symmetries, see Appendix.F.

1. wallpaper group $p1m1$, $\chi_M = -1$

The generators of wire construction are shown in Fig.10(a)-(c). Here we focus on the boundary state of the decoration as a superposition of (b) and (c), which is kind of tricky.

Firstly one notes that since $\{M, S\} = 0$, both the mirror-invariant lines in BZ, i.e., $k_x = 0$ and $k_y = \pi$, have effective AZ class D, with $\mathbb{Z}_2$ classification in 1D. Both the two generators in Fig.10 (b) and (c), i.e., the one decorated with 1D mirror-symmetric TSC at $x = 0$ and one decorated at $x = \frac{1}{2}$, have the same invariant $\mathbb{Z}_2 = 1$ at $k_x = 0$ as well as at $k_x = \pi$. Therefore, their superposition, i.e., decorating at both $x = 0$ and $x = \frac{1}{2}$, has $\mathbb{Z}_2 = 0$ at $k_x = 0$ as well as $Z = 0$ at $k_x = \pi$. Does the failure of description by the $\mathbb{Z}_2$ invariants in momentum space means this decoration is trivial? Otherwise, there should be another nonzero invariant accounting for this decoration.

To answer this question we carefully analyze the boundary state of this decoration. Consider an edge respecting mirror symmetry as well as translation symmetry in $x$ direction, see Figure.10(d). We denote the Majorana zero modes as $\chi_{is}(n)$, where $i = a, b$ for the subsite in a unit cell at $x = N$ or $x = N + \frac{1}{2}$ where $N \in \mathbb{Z}$, $s = \uparrow, \downarrow$ for the “spin”, and $n$ for the index of unit cell. A generic tight-binding Hamiltonian can be written as

$$\mathcal{H} = \sum_{n,n'} \sum_{i,j} \sum_{s,t} iF_{isjt}(n-n')\gamma_{is}(n)\gamma_{jt}(n')$$

(87)

with $F$ being the hopping amplitude. Before proceeding, we specify the symmetries of the Majorana zero modes. We choose

$$\mathcal{T} = i\sigma_2\mathcal{K}$$

(88)

$$\mathcal{P} = \mathcal{K}$$

(89)

$$M = \sigma_3$$

(90)

Note here we take another convention with $M^2 = 1$ and $[M, \mathcal{T}] = 0$, which is equivalent to the convention with $M^2 = -1$ and $[M, \mathcal{T}] = 0$ that we used before. Time-reversal symmetry requires

$$F_{i\uparrow,j\uparrow}(n-n') = -F_{i\downarrow,j\downarrow}(n-n')$$

(91)

$$F_{i\uparrow,j\downarrow}(n-n') = F_{i\downarrow,j\uparrow}(n-n')$$

(92)

FIG. 10. (a) The decoration protected by $y$-translation. (b)-(c) The decorations on mirror lines at $x = 0$ and $x = \frac{1}{2}$, respectively. (d) The boundary Majorana zero modes of the decoration by the superimposition of (b) and (c). The two zero modes in the same ellipse are in the same unit cell. (e) Boundary spectrum along $k_x$ with the parameters set to be $d_1 = 0.3$, $e_1 = 0.3$, $f_1 = 1.3$ and $g_1 = 0.4$. There are two generic gapless points, which are symmetric about $k_x = \pi$.

Mirror symmetry requires

$$F_{isjt}(n-n') = stF_{isjt}(n'-n)$$

(93)

$$F_{asbt}(n-n') = stF_{asbt}(n'-n + 1)$$

(94)

Fermion statistics requires

$$F_{isjt}(n-n') = -F_{jitis}(n'-n)$$

(95)

As a result, the hopping amplitudes are constrained as

$$F_{iss1}(n) = 0$$

(96)

$$F_{a\uparrow a\downarrow}(n) = F_{a\downarrow a\uparrow}(-n) = F_{a\downarrow a\uparrow}(n)$$

(97)

$$= -F_{a\downarrow a\uparrow}(-n) = d_n$$

(98)

$$F_{b\uparrow b\downarrow}(n) = -F_{b\downarrow b\uparrow}(-n) = F_{b\downarrow b\uparrow}(n)$$

(99)

$$= -F_{b\downarrow b\uparrow}(-n) = e_n$$

(100)

$$F_{a\uparrow b\downarrow}(n) = F_{a\downarrow b\uparrow}(-n + 1) = -F_{a\downarrow b\uparrow}(n)$$

(101)

$$= -F_{a\downarrow b\uparrow}(-n + 1) = f_n$$

(102)

$$F_{a\uparrow b\downarrow}(n) = -F_{a\downarrow b\uparrow}(-n + 1) = F_{a\downarrow b\uparrow}(n)$$

(103)

$$= -F_{a\downarrow b\uparrow}(-n + 1) = g_n$$

(104)
Because we choose $\mathcal{P} = K$, the Majorana operators are real and $F$’s must be real as well. Keeping only the nearest hoppings, i.e., $n = 1$, the Hamiltonian in momentum space reads

$$\mathcal{H} = \sum_k \left( \begin{array}{cccc} \gamma_{a\uparrow} & \gamma_{a\downarrow} & \gamma_{b\uparrow} & \gamma_{b\downarrow} \end{array} \right) H(k) \left( \begin{array}{c} \gamma_{a\uparrow} \\ \gamma_{a\downarrow} \\ \gamma_{b\uparrow} \\ \gamma_{b\downarrow} \end{array} \right)$$

(105)

where $H(k) =$

$$\begin{pmatrix} 0 & -d_k & if_1(1 + e^{ik}) & ig_1(1 - e^{ik}) \\ -d_k & 0 & ig_1(1 - e^{ik}) & -if_1(1 + e^{ik}) \\ if_1(1 + e^{-ik}) & -ig_1(1 - e^{-ik}) & 0 & -e_k \\ ig_1(1 - e^{-ik}) & if_1(1 + e^{-ik}) & -e_k & 0 \end{pmatrix}$$

(106)

and

$$d_k = d_1 \sin k$$

(107)

$$e_k = e_1 \sin k$$

(108)

Below we show the spectrum is always gapless at some generic point between $k = 0$ and $k = \pi$ (as well as between $k = \pi$ and $k = 2\pi$, by symmetry). First we set $k = 0$ and we have

$$\det(H(k = 0)) = -16f_1^4 \leq 0$$

(109)

Then we set $k = \pi$ and we have

$$\det(H(k = \pi)) = 16g_1^4 \geq 0$$

(110)

Since the determinant of $H(k)$ is a continuous function of $k$, according to the mean value theorem, there must exist at least one point $k_0$ between 0 and $\pi$ satisfying $\det(H(k = k_0)) = 0$, which indicates the spectrum of $H(k_0)$ is gapless.

For illustration, we plot the boundary spectrum in Fig.10(e) where $0 < k < 2\pi$, with the parameters chosen as $d_1 = 0.3$, $e_1 = 0.3$, $f_1 = 1.3$ and $g_1 = 0.4$.

Through the above discussion, we showed the existence of gapless boundary modes, which confirms that the decoration by wires at both $x = 0$ and $x = \frac{1}{2}$ is topologically nontrivial. To further understand the nontrivial topology, we note a generic momentum at $k_x$ line has effective AZ class BDI, because one has $\mathcal{T}' = \mathcal{T} \cdot M_z$ and $\mathcal{P}' = \mathcal{P} \cdot M_z$, both of which leaving any $k_x$ unchanged and satisfying $\mathcal{T}'^2 = 1$, $\mathcal{P}'^2 = 1$. As class BDI has $\mathbb{Z}_2$ topological invariant in 0 D, the existence of gapless mode at $k_x = k_0$ indicates different such $\mathbb{Z}_2$ values on the two sides of $k_0$.

2. wallpaper group $p1g1$, $\chi_g = \pm 1$

There are two decorations as the generators of wire construction for WG $p1g1$ with both $\chi_g = +1$ and $\chi_g = -1$, as shown in Fig.11 (a)-(b). Here we choose the glide to be $g = \{M_y | \frac{1}{2}\}$, i.e., a combination of mirror reflection in $x$ direction and a half-integer translation in $y$ direction. The first decoration has 1D DIII TCSs aligned along $x$ direction and is the one protected by $x$-translation. Its boundary state is characterized by Majorana band along $k_x$ direction, as that for WG $p1$, see Appendix.F. Here we focus on the the second decoration, which is the one protected by glide, with 1D DIII TSCs aligned along $y$, decorated at both $y = n$ and $y = n + \frac{1}{2}$, where $n \in \mathbb{Z}$.

To maintain the glide symmetry, the boundary geometry can only be made into a strip finite in $x$ direction while infinite in $y$ direction, see Fig.11(c). Note the left side and the right side are not independent but related by glide symmetry. On each side, there are Kramer’s pairs of Majorana zero modes from the terminated ends of 1D DIII TCS at both $y = n$ and $y = n + \frac{1}{2}$. One can group two nearest Majorana Kramer’s pairs together, e.g., the pair at $y = 0$ and the pair at $y = \frac{1}{2}$, and gap them, as they are not related by any symmetries (two pairs of Majorana modes related by glide symmetry locate at different sides). As a result, there are not any gapless states at the boundary. To see why the anomaly of this decoration does not manifest at the boundary, one notes that there is not any single edge which respects glide symmetry in 2D. One can further imagine each side equivalently as a 1D DIII TCS, i.e., time-reversal symmetric (TRS) Kitaev chain, as suggested in Fig.11(d). In each unit cell, there are Majorana Kramer’s pair at both integer subsite and half-integer subsite. Since an integer subsite on one side has its glide-related partner a half-integer.
subsite on the other side, the inter-cell pairing of Majorana modes on one side becomes intra-cell pairing on the other side. Therefore, the two TRS Kitaev chains on left and right sides have topological invariant $Z_2 = 1$ and $Z_2 = 0$, respectively, or vice versa. Suppose we could make the two TRS Kitaev chains contact with each other, then at their intersection, a domain wall is formed where the mass changes sign and gapless modes are supported. However, they are related by glide symmetry and are always separated as long as the sample has finite width in $x$ direction.

VII. DISCUSSION

In this work, we present a unified real-space framework to construct and classify TCSCs using 1D building blocks which we call wire construction. We deal with 2D TCSCs of AZ class DIII, and our products are second-order TSCs as well as weak TSCs, i.e., those protected by translation symmetries. We show that our framework can be formulated mathematically with central issues as projective representations and group induction, and we use several examples to illustrate the detailed procedure of wire construction. For an inclusive description of the symmetries for spinful electrons in 2D, we deal with both pure 2D systems which are described by wallpaper groups, and 2D systems embedded into 3D, which are described by layer groups. Furthermore, we discuss what kinds of boundary states these TCSCs by wire construction are characterized by.

We comment again that our results are not complete classifications for TSCs in 2D, as the 2D strong TSC cannot be obtained by wire construction. To include the 2D strong phase for complete TSC classification, one should first figure out whether it is compatible with the crystalline symmetries, i.e., whether the $Z_2$ invariant for 2D strong TSC is restricted to be zero by crystalline symmetries. For an incompatible example see the discussion of layer group $p11m$ in [86]. If compatible, one can further consider the group extension problem, as what was previously done in the full classification of TIs where both 3D strong TI and TCIs are included[32, 34]. As the 2D strong TSC has $Z_2$ classification itself, one should consider whether its double is still a trivial state in the presence of crystalline symmetries, or it becomes a second-order TSC as an ingredient in the complete TSC classification. Such procedure is presented in [87] for wallpaper group $p2$, where the authors formulate the problem of complete classification as solving a short exact sequence.

For 3D TCSCs, wire construction can still be applied, with the products being third-order TCSs in 3D. However, there is another problem to handle that is not encountered in 2D. According to the discussions in [75], there could be “$(d + 2)$-bubbles” in real-space construction of TCSs, which suggests that for 1-cell decorations in 3D, not only 2-cell bubbles need to be considered, but also 3-cell bubbles should be involved. That is to say, some 1-cell decorations, although stable under 2-cell bubble equivalence, may be trivialized by 3-cell bubbles. Since this problem is beyond the scope of the present work, we leave it, together with the full TSC classification in 2D, to future works.

VIII. ACKNOWLEDGEMENTS

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Appendix A: Building blocks for layer groups

1. $A_1$ representation

For this representation, we have $\chi_{M_x} = 1$, $\chi_{M_y} = 1$, and $\chi_{C_2} = 1$. Applying Wigner test, we have

$$[W^T_5 (T), W^T_5 (P), W^T_5 (S)] = [1, -1, 1] \quad (A1)$$

which dictates $\Gamma_5$ belongs to class CI with trivial classification in 1D, and two $\bar{\Gamma}_5$’s are paired since $W^T_5 (P) = -1$. We can write the projective irrep as

$$\mathcal{T} = \mu_0 i\sigma_2 K \quad (A2)$$
$$\mathcal{S} = \mu_3 \sigma_3 \quad (A3)$$
$$M_x = \mu_1 i\sigma_1 \quad (A4)$$
$$M_y = \mu_1 i\sigma_2 \quad (A5)$$

which satisfies $\mathcal{T}^2 = -1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $M_x^2 = -1$, $M_y^2 = -1$, $[\mathcal{T}, M_x] = 0$, $[\mathcal{T}, M_y] = 0$, $[\mathcal{S}, M_x] = 0$, and $[\mathcal{S}, M_y] = 0$. One can find mass term proportional to $\mu_0 \sigma_3$ to gap out the 0D zero modes described by these symmetries, verifying this building block is topologically trivial.

2. $A_2$ representation

For this representation, we have $\chi_{M_x} = -1$, $\chi_{M_y} = -1$, and $\chi_{C_2} = 1$. Applying Wigner test, we have

$$[W^T_5 (T), W^T_5 (P), W^T_5 (S)] = [1, 1, 1] \quad (A6)$$

which dictates $\bar{\Gamma}_5$ belongs to class BDI with $Z$ classification in 1D, and it is not paired with itself. We can write the projective irrep as

$$\mathcal{T} = i\sigma_2 K \quad (A7)$$
$$\mathcal{S} = \sigma_3 \quad (A8)$$
$$M_x = i\sigma_1 \quad (A9)$$
$$M_y = i\sigma_2 \quad (A10)$$

which satisfies $\mathcal{T}^2 = -1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $M_x^2 = -1$, $M_y^2 = -1$, $[\mathcal{T}, M_x] = 0$, $[\mathcal{T}, M_y] = 0$, $[\mathcal{S}, M_x] = 0$, and $[\mathcal{S}, M_y] = 0$. One can find no mass term to gap out the 0D zero modes described by these symmetries, as well as for the doubled system, verifying the $Z$ classification for this building block. Since $[C_2, S] = 0$, one can define 1D winding number for $C_2$ axis. We can define the independent topological index as $w_{C_2} = \frac{1}{2}(w_{C_2}^+ - w_{C_2}^-)$.

3. $B_1$ representation

For this representation, we have $\chi_{M_x} = 1$, $\chi_{M_y} = -1$, and $\chi_{C_2} = -1$. Applying Wigner test, we have

$$[W^T_5 (T), W^T_5 (P), W^T_5 (S)] = [1, 1, 1] \quad (A11)$$

which dictates $\bar{\Gamma}_5$ still belongs to class BDI with $Z$ classification in 1D. We can write the projective irrep as

$$\mathcal{T} = i\sigma_2 K \quad (A12)$$
$$\mathcal{S} = \sigma_1 \quad (A13)$$
$$M_x = i\sigma_1 \quad (A14)$$
$$M_y = i\sigma_2 \quad (A15)$$

which satisfies $\mathcal{T}^2 = -1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $M_x^2 = -1$, $M_y^2 = -1$, $[\mathcal{T}, M_x] = 0$, $[\mathcal{T}, M_y] = 0$, $[\mathcal{S}, M_x] = 0$, and $[\mathcal{S}, M_y] = 0$. Since $[M_x, S] = 0$, the $M_x = \pm i$ sectors have 1D winding number $w_{M_x}^\pm$, and they satisfy $w^+_{M_x} = w^-_{M_x}$. We can define the independent topological index as $w_{M_x} = \frac{1}{2}(w^+_{M_x} - w^-_{M_x})$.

4. $B_2$

For this representation, we have $\chi_{M_x} = -1$, $\chi_{M_y} = 1$ and $\chi_{C_2} = -1$. Wigner test shows that

$$[W^T_5 (T), W^T_5 (P), W^T_5 (S)] = [1, 1, 1] \quad (A16)$$

which dictates $\bar{\Gamma}_5$ belongs to class BDI again. We can write the projective irrep as

$$\mathcal{T} = i\sigma_2 K \quad (A17)$$
$$\mathcal{S} = \sigma_2 \quad (A18)$$
$$M_x = i\sigma_1 \quad (A19)$$
$$M_y = i\sigma_2 \quad (A20)$$

which satisfies $\mathcal{T}^2 = -1$, $\mathcal{S}^2 = 1$, $\mathcal{P}^2 = (\mathcal{T}\mathcal{S})^2 = 1$, $M_x^2 = -1$, $M_y^2 = -1$, $[\mathcal{T}, M_x] = 0$, $[\mathcal{T}, M_y] = 0$, $[\mathcal{S}, M_x] = 0$, and $[\mathcal{S}, M_y] = 0$. Since $[M_y, \mathcal{S}] = 0$, we can define the independent topological index as $w_{M_y} = \frac{1}{2}(w_{M_y}^+ - w_{M_y}^-)$.

Appendix B: Group induction

According to group theory, the representations of a group can be derived based on the representations of its subgroups, which are called induced representations. One application of induced representations in solid-state physics is the theory of elementary band representations[60, 88–90], where one can derive representations of space groups from representations of site symmetry groups of the orbitals at Wyckoff positions. In real-space construction, all cells are equivalent to Wyckoff positions along $C_n$ axis. In our wire construction, for a cell-complex respecting a given crystalline symmetry group $G$, the whole 1-cells consist of different sets of symmetry-related ones, and each set forms induced representations of $G$ from the representations of the onsite-symmetry group $G_1$.
First we note for point group, (B3) reduces to
\[ h g_\alpha = g_\beta g \] (B5)
and (B1) reduces to
\[ \bar{\rho}(h) a_{\alpha\beta} = \sum_{i' = 1}^{n_q} \rho_{i'1}(g_\beta^{-1} h g_\alpha) a_{i'\beta} = \sum_{i' = 1}^{n_q} \rho_{i'1}(g) a_{i'\beta} \] (B6)
as translations are now absent. As shown in Figure 12, we label the four 1-cells as 1, 2, 3, 4 in the counterclockwise direction, and the \( C_4 \) center, which is also the original point, is denoted as \( O \). The site symmetry group of each 1-cell is trivial, and the onsite symmetry group is generated by \( T \) and \( S \). We can write the representation as
\[ T = i \sigma_2 K \] (B7)
\[ S = \sigma_3 K \] (B8)
Then we use the formula (B6) to derive the induced representation at the 0-cell \( O \). We first deal with \( C_4 \), under which the four 1-cells transform as
\[ 1 \to 2, 2 \to 3, 3 \to 4, 4 \to 1 \] (B9)
i.e., in a permutation manner. We can assign the coset representatives as \( g_i = C_4^{i-1} \), i.e.,
\[ g_1 = 1, g_2 = C_4, g_3 = C_4^2 = C_2, g_4 = C_4^3 \] (B10)
where 1 stands for the identity operation. It is easy to find for \( g_\alpha = g_i \), we have \( g_\beta = g_{i+1} \), where \( i \in \mathbb{Z}_4 \). Therefore, for \( h = C_4 \), (120) reads
\[ g = g_\beta^{-1} C_4 g_\alpha = g_{i+1}^{-1} C_4 g_i \] (B11)
which leads to \( g = 1 \) for \( i = 1, 2, 3 \) while \( g = -1 \) for \( i = 4 \), by noting \( C_4^4 = -1 \). We can see that the induced representation of \( C_4 \) at the 0-cell can be derived based on the representation of the identity operation, \( 1 \), at the 1-cell, perhaps up to a minus sign. As a result, we have
\[ C_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \sigma_0 \] (B12)
Then we deal with \( T \). Since \( T \) acts locally and always commutes with \( C_4 \), one can directly write the induced representation of \( T \) with a thought
\[ T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes i \sigma_2 K \] (B13)
Lastly for \( S \), one has to consider \( \chi_{C_4} = \pm 1 \), i.e., both \([C_4, S] = 0 \) and \([C_4, S] = 0 \).
1. $[C_4, S] = 0$

In this case, it is similar with $T$, and we have

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \otimes \sigma_3
\]  

\tag{B14}

2. $\{C_4, S\} = 0$

In this case, we should be a little careful, since $C_4$ reverses the eigenvalue of $S$. Note $S$ does not change the position of 1-cells, so we always have $g_3 = g_a$. Specifically, for $g_a = g_{h,i=1,2,3,4}$, we have

\[
g = g_i^{-1} S g_i = (C_4^{-1})^{-1} S C_4^{-1} = (C_4^{-1})^{-1} C_4^{-1} S (-1)^{-1} = S (-1)^{-1}
\]  

\tag{B15}

where the extra factor $(-1)^{-1}$ comes from $\{C_4, S\} = 0$. Thus we have the induced representation.

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \otimes \sigma_0
\]  

\tag{B17}

**Appendix C: Trivial bubble equivalence in wallpaper groups**

Below we show that for wallpaper groups, bubble equivalence always has trivial effects.

Firstly we note in 2D, any 1-cell is always shared by two 2-cells, which means when considering bubble equivalence, a 1-cell has always two bubbles on its two sides. Recall that there are two types of 1-cells for wallpaper groups, one with $G^1_c = 1$ and the other with $G^1_c = M$. A 1-cell of the first type has $\mathbb{Z}_2$ classification, and the resulting states by two bubbles can only have $\mathbb{Z}_2 = 0$. For the second type, the trivial effects of bubble equivalence are not obvious, and we make careful analysis as follows.

1. $\chi_M = 1$

According to Sec.II.C1, the classification on the 1-cell is $\mathbb{Z}$, and the corresponding topological index can be defined as mirror winding number $w_M = \frac{1}{2}(w^+ - w^-)$, which indicates the number of zero modes at the 0D boundary. On the two sides of the 1-cell, two bubbles are related by mirror $M$. Each bubble edge is a 1D DIII TCS with boundary zero modes described by symmetries

\[
T = i \sigma_3 \mathcal{K}
\]  

\tag{C1}

\[
S = \sigma_3
\]  

\tag{C2}

When coinciding with the 1-cell, by group induction, the resulting state from the two mirror-related bubble edges have zero modes described by

\[
T = \mu_0 \sigma_2 \mathcal{K}
\]  

\tag{C3}

\[
S = \mu_0 \sigma_3
\]  

\tag{C4}

\[
M = i \mu_2 \sigma_0
\]  

\tag{C5}

Since $[M, S] = 0$, we can diagonalize them simultaneously, leading to

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, M = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}
\]  

\tag{C6}

which can be decomposed into $M = +i$ sector with $w = 0$

\[
S = \begin{pmatrix}1 & 0 \end{pmatrix}, M = \begin{pmatrix}i & 0 \end{pmatrix}
\]  

\tag{C7}

and $M = -i$ sector with $w = 0$

\[
S = \begin{pmatrix}1 & 0 \end{pmatrix}, M = \begin{pmatrix}-i & 0 \end{pmatrix}
\]  

\tag{C8}

Since both $w^+$ and $w^-$ are zero, $w_M = \frac{1}{2}(w^+ - w^-)$ is also zero. The vanishing topological number means the zero modes can be gapped out and the resulting 1D state from the two bubble edges is trivial. Consequently, the classification on the 1-cell is unchanged under bubble equivalence.

2. $\chi_M = -1$

According to Sec.II.C1, the topological classification on the 1-cell is $\mathbb{Z}_2$. Similar to the 1-cells with $C^1_c = 1$, bubble equivalence can not change the topological index by 1 thus has trivial effects on the 1-cell classification.

![Diagram](image-url)
The projective irrep can be written as which means the Wigner test to \( G \) with \( \mathbb{Z} \) invariant if \( \chi_M = 1 \) or \( \mathbb{Z}_2 \) invariant if \( \chi_M = -1 \). Meanwhile, all 0-cells have \( G^0_c = C_{2v} \), with the only double-valued irrep being \( \Gamma_5 \).

We first analysis the projective irreps of \( G^0 \), which describe the symmetries of zero modes at \( e^0 \)'s. Apply Wigner test to \( \Gamma_5 \), we have

\[
[W^T_{\mathcal{S}}(T), W^T_{\mathcal{S}}(\mathcal{P}), W^T_{\mathcal{S}}(S)] = [1, 1, 1]
\]

which means \( \Gamma_5 \) belongs to class BDI with \( \mathbb{Z} \) classification in 1D. The projective irrep can be written as

\[
\begin{align*}
T &= i\sigma_2 K \\
S &= \sigma_1 \\
M_x &= i\sigma_1 \\
M_y &= i\sigma_2
\end{align*}
\]

which satisfies \( [M_x, S] = 0 \) and \( \{M_y, S\} = 0 \). Since \( [M_x, S] = 1 \), we can define the topological index as

\[
\tilde{w}_{M_x} = \frac{1}{2}(w_{M_x} - w_{M_y}).
\]

Then we can analyze the gluing condition.

1. \( e_1^1 \) (\( e_3^1 \)) decoration

The decorations on \( e_1^1 \) and \( e_3^1 \) are essentially the same, and we take the one on \( e_1^1 \). The 1D TSC with \( \mathbb{Z} \) classification decorated on \( e_1^1 \) (\( e_3^1 \)) has 0D zero modes described by

\[
\begin{align*}
T &= i\sigma_2 K \\
S &= \sigma_1 \\
M_x &= i\sigma_1 \\
M_y &= i\sigma_2
\end{align*}
\]

By group induction, the zero modes at \( e_1^0 \) (\( e_3^0 \)) are described by

\[
\begin{align*}
T &= \mu_3 i\sigma_2 K \\
S &= \mu_3 \sigma_1 \\
M_x &= \mu_3 \sigma_1 \\
M_y &= i\mu_0 \sigma_0
\end{align*}
\]

Diagonalizing \( M_x \) and \( S \) simultaneously, we have

\[
M_x = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}, \quad S = \begin{pmatrix}1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1\end{pmatrix}
\]

One can see the topological index is

\[
w_{M_x} = \frac{1}{2}(2 - (-2)) = 2
\]

which is nonzero and suggests the gluing condition is failed at \( e_1^0 \) and \( e_3^0 \). Therefore the decorations on \( e_1^1 \) and \( e_3^1 \) are not allowed.

2. \( e_2^1 \) (\( e_4^1 \)) decoration

Then we analysis the decoration on \( e_2^1 \) (\( e_4^1 \)). The gluing condition for these two decorations are essentially the same, and we take \( e_2^1 \).

Since \( e_2^1 \) has \( G^1_c = M_y \) with \( \chi_{M_y} = -1 \), the decoration on has \( \mathbb{Z}_2 \) classification, with 0D zero modes described by

\[
\begin{align*}
T &= i\sigma_2 K \\
S &= \sigma_3 \\
M_y &= i\sigma_2
\end{align*}
\]

By group induction, the zero modes at \( e_2^0 \) (\( e_4^0 \)) are described by

\[
\begin{align*}
T &= \mu_0 i\sigma_2 K \\
S &= \mu_0 \sigma_3 \\
M_x &= i\mu_0 \sigma_0 \\
M_y &= \mu_3 i\sigma_2
\end{align*}
\]

Diagonalizing \( M_x \) and \( S \) simultaneously, we have

\[
M_x = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}, \quad S = \begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1\end{pmatrix}
\]
One can see the topological number is
\[ w_{M_x} = \frac{1}{2} (1 - 1 + 1 - 1) = 0 \]  
(D23)

which means the gluing condition is met and the decoration of \( e_1 \) (\( e_2 \)) is allowed.

As a result, decorations on \( e_1 \) and \( e_2 \) are the generators for all decorations. As they are both \( \mathbb{Z}_2 \)-type, the classification is \( \mathbb{Z}_2 \).

**Appendix E: Example: layer group \( pm2m \) with \( A_1 \) and \( B_1 \) representations**

Here we present an example of layer group, \( pm2m \). There are four representations of gap function, and here we deal with two of them, \( A_1 \) and \( B_1 \).

This LG is generated by adding mirror reflection in \( z \) direction, \( M_z \), to the WG \( plm1 \). Since \( plm1 \) has \( M_x \), the product \( M_z \cdot M_x \) generates a \( C_2 \) rotation in \( y \) direction, \( C_{2y} \). The cell complex of \( pm2m \) is the same as that of WG \( plm1 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{cell_complex.png}
\caption{(a) Cell complex of \( pm2m \). (b) Bubbles of \( pm2m \).}
\end{figure}

Note all the \( d \)-cells (\( d = 0, 1, 2 \)) have \( M_z \) as onsite symmetry. Besides, \( e_0^{d=1} \) and \( e_1^{d=1} \) have \( M_x \) as onsite symmetry thus they have \( G^c = C_{2v} \). Therefore we have the particular type of building blocks for layer groups to be decorated on \( e_1^1 \) and \( e_3^1 \), which have \( G^c = C_{2v} \). Nevertheless, we need not explicitly analyze gluing condition involving them. Actually, at \( e_1^0 \) (\( e_2^0 \)), the two relevant \( e_1^1 \)’s (\( e_3^1 \)’s) are related by \( y \) translation and one has its tail connected by the other’s head, which makes the gluing condition trivially satisfied, and the decorations of \( e_1^1 \) and \( e_3^1 \) are always allowed. As a result, for gluing condition we only need to explicitly consider the two \( e_2^1 \)’s related by \( C_{2y} \) at \( e_1^0 \) (\( e_2^0 \)).

1. \( A_1 \) representation

For this representation of gap function, one has \( \chi_{M_z} = 1 \), \( \chi_{M_x} = 1 \), and \( \chi_{C_{2y}} = 1 \). The projective irreps for the 1-cells and 0-cells are shown in Table II and Table III, respectively.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{IR} & \text{EAZ classification} \\
\hline
\Gamma_5 & \text{CI} & \text{N/A} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{PIR name} & \text{PIR} & \text{TI} \\
\hline
\Gamma_5 & S = \mu_0 \sigma_3 & \text{N/A} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{PM} & \text{Irrep} & \text{EAZ} \\
\hline
\text{CI} & \text{N/A} & \text{CI} \\
\hline
\end{array}
\]

According to the above tables, the only double-valued irrep of \( C_{2v} \), i.e., \( \Gamma_5 \), belongs to class CI with trivial classification in 1D. This means the building blocks on \( e_1^1 \) and \( e_3^1 \) with \( G^c = C_{2v} \) are trivial. What’s more, the gluing condition is trivially satisfied at \( e_1^0 \) and \( e_2^0 \) with \( G^c = C_{2v} \). Therefore one has 1-cell decoration of \( e_2^1 \) allowed by gluing condition.

Then we consider bubble equivalence. However, the two bubbles on the two sides of \( e_1^1 \) are related by translation in \( y \) direction and they have trivial effects on \( e_2^1 \).

As a result, the final classification is \( \mathbb{Z} \), contributed by \( e_2^1 \) decoration.

2. \( B_1 \) representation

For this representation of gap function, one has \( \chi_{M_z} = -1 \), \( \chi_{M_x} = 1 \) and \( \chi_{C_{2y}} = -1 \).

The projective irreps for the 1-cells and 0-cells are shown in Table IV and Table V, respectively.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{IR} & \text{EAZ classification} \\
\hline
\Gamma_5 & \text{CI} & \text{N/A} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{PIR name} & \text{PIR} & \text{TI} \\
\hline
\Gamma_5 & S = \mu_0 \sigma_3 & \text{N/A} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{PM} & \text{Irrep} & \text{EAZ} \\
\hline
\text{CI} & \text{N/A} & \text{CI} \\
\hline
\end{array}
\]
**From the above tables, we know the building blocks on \( e_{1,1,1}^{1,1} = \ldots \) are all nontrivial.**

Then we consider the gluing condition at \( e_{1,1}^{1,1} \) (or \( e_{1,3}^{1,3} \)), which involves \( e_{1,2}^{1,2} \). The induced representation from \( \bar{\Gamma}_{3b4} \) at \( e_{1,2}^{1,2} \) is given by

\[
\begin{align*}
T &= \sigma_2 \mathcal{K} \\
S &= \sigma_3 \\
M_e &= i\sigma_1 w_{M_z} \\
M_z &= i\sigma_3 
\end{align*}
\]

By diagonalizing \( S \) and \( M_z \) together, we obtain

\[
M_z = \begin{pmatrix}
  i & 0 & 0 & 0 \\
  0 & -i & 0 & 0 \\
  0 & 0 & i & 0 \\
  0 & 0 & 0 & -i \\
\end{pmatrix},
\quad
S = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

which equals to two \( \bar{\Gamma}_5 \)'s with the same \( S \), i.e., the same topological number \( w_{M_z} = \frac{1}{2}(w_{+1} - w_{-1}) = 1 \). This leads to the non-vanishing total topological number \( w_{M_z} = 2 \), which protects the 0D modes from being gapped out. Therefore the gluing condition is failed, which means decoration on \( e_{1,2}^{1,2} \) is not allowed.

Then one has to analyze whether the decorations on \( e_{1,1}^{1,1} \) and \( e_{1,3}^{1,3} \), which are trivially allowed the gluing condition, are stable against bubble equivalence.

For a single bubble, its edge can be 1D mirror-symmetric TSC with 0D zero modes described by

\[
\begin{align*}
T &= i\sigma_2 \mathcal{K} \\
S &= \sigma_3 \\
M_z &= i\sigma_3 
\end{align*}
\]

By group induction, the two \( M_z \)-related bubble edges contribute 1D states on \( e_{1,1}^{1,1} \) (or \( e_{1,3}^{1,3} \)) with 0D zero modes described by

\[
\begin{align*}
T &= \mu_0 i\sigma_2 \mathcal{K} \\
S &= \mu_0 \sigma_3 \\
M_z &= \mu_0 i\sigma_0 \\
M_z &= \mu_0 i\sigma_3 
\end{align*}
\]

This induced projective representation can be decomposed into two \( \bar{\Gamma}_5 \)'s with the same \( S \), which is exactly double of the irrep carried by \( e_{1,1}^{1,1} \) decoration. Thus we can see that the bubble equivalence generates double \( e_{1,1}^{1,1} \) decoration and double \( e_{1,3}^{1,3} \) decoration simultaneously, and the final classification is given by \( \mathbb{Z}^2 / 2\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}_2 \).

**Appendix F: Boundary states protected by other symmetries**

In Sec.VI of the main text, we discussed some TSCS boundary states. Here we give detailed analyses for other ones.

1. **wallpaper group \( \text{p1} \)**

This wallpaper group has only translation symmetries. The two decorations are simply 1D DIII TSCs stacking along \( x/y \) directions. Here we choose the decoration in \( x \) direction for discussion, and the decoration in \( y \) direction is essentially the same.

![Fig. 16](attachment:figure.png)

**FIG. 16.** (a) Decorations of \( \text{p1} \), which are 1D DIII TSCs aligned along \( x \) or \( y \) direction. (b) Boundary spectrum along \( x \) or \( y \).
The real-space topological invariant, i.e., the $Z_2$ topological number of the 1D DIII TSC, can be transformed into momentum space, denoted as $\nu$. It can be evaluated at $k_x = 0$ or $k_x = \pi$ line[27]

$$(-1)^\nu = e^{i\gamma^{(I)}}$$

$$\gamma^{(I)} = \int_{-\pi}^{\pi} dk_y A^{(I)} k_x = 0/\pi, k_y$$

where $A^I$ and $A^{II}$ are the Berry connection of the Kramer’s pair formed by the occupied states. When a boundary termination is made preserving $x$ translation, the array of 0D Majorana modes form Majorana band in $x$ direction, as shown in Fig.16. At $k_x = 0$ and $k_x = \pi$, the effective symmetry class is DIII, since these two momenta are invariant under $T$, $P$ and $S$. The $\nu = 1$ invariant grants a Kramer’s pair of Majorana zero modes, while at generic momentum, the zero modes are absent and a gap opens.

2. **wallpaper group** $p2$, $\chi_{C_2} = \pm 1$

We can see the decorations of $p1$ are still compatible with $p2$, as shown in Fig.17(a)-(b), and the corresponding boundary states are same as those of $p1$, as shown in Fig.17(d). Besides, there is an additional decoration protected by $C_2$ rotation, see Fig.17(c). Since this decoration has two 1D DIII TSCs in a unit cell, the translation invariant is zero due to its $Z_2$ essence, and the corresponding boundary state, i.e., Majorana band formed in $x/y$ direction, is absent. Instead, in a $C_2$-symmetric boundary termination, there are two $C_2$-related Majorana Kramer’s pairs as corner modes, as shown in Fig.17(e).

We remark that the authors in Ref.[87] did more elaborate works for DIII TSCs protected by $p2$, with the overlapped part in agreement with our results.

3. **wallpaper group** $p1m1$, $\chi_M = 1$

In Sec.VI 1 of the main text, we discussed the boundary state protected by $p1m1$ with $\chi_M = -1$. Here we focus on the $\chi_M = 1$ case.

The first decoration, as shown in Fig.18 (a), is a stacking of 1D DIII TSCs along $y$ direction, which has the same boundary state as that of $p1$. The other two decorations are protected by mirror $M_x$ and $x$-translation, and here we focus on them.

These two decorations are built by mirror-symmetric 1D TSCs, whose topological invariant is the mirror winding number $w_M = \frac{1}{2}(w_+ - w_-)$. The momentum-space mirror winding numbers can be derived from the real-space ones. Consider assigning mirror eigenstates with $M = \pm i$ at $x = n$ where $n \in \mathbb{Z}$, i.e.,

$$M \phi_x = \pm i \phi_{-x}$$

Fig. 17. Decorations and boundary states of $p2$. (a)The decoration with $x$-translation invariant. (b)The decoration with $y$-translation invariant. (c)The decoration with zero translation invariants but nonzero $C_2$ rotation invariant. (d) The boundary spectrum of either (a) or (b), which is the same as that of $p1$. (e) Majorana corner modes of decoration (c) with boundary geometry respecting $C_2$ symmetry, where we use a single circle to represent a Kramer’s pair of Majorana modes.

Fig. 18. Decorations and boundary states of $p1m1$ with $\chi_M = 1$. (a) Decoration protected by $y$-translation. (b)-(c) Decorations protected by mirror together with $x$-translation, with 1D building blocks placed at mirror line $x = 0$ and $x = \frac{1}{2}$, respectively. (d) Boundary spectrum along $k_x$ of the decoration in (b). (e) Boundary spectrum along $k_x$ of the decoration in (c). Note we use blue and orange dots to represent zero modes protected by different mirror winding numbers. (f)-(g) Boundary spectrum along $k_x$ of the states from the combinations of (b) and (c).
Fourier transformation, the momentum state satisfies

\[ M \phi_k = M \left( \sum_n \phi_n e^{i k n} \right) = M \sum_n e^{i k n} (M \phi_n) \]  
\[ = \sum_n e^{i k n} (\pm i) \phi_{-n} = \pm i \sum_n e^{-i k n} \phi_n \]  
\[ = \pm i \sum_n e^{i k n} e^{-i 2k n} \phi_n \]  
\[ = \pm i \sum_n e^{i k (n + \frac{1}{2})} e^{-i 2k (n + \frac{1}{2})} \phi_{n + \frac{1}{2}} \]  
\[ (F4) \]

At both \( k = 0 \) and \( k = \pi \), we have \( M \phi_k = \pm i \phi_k \), i.e., both \( \phi_{k=0} \) and \( \phi_{k=\pi} \) are mirror eigenstates with \( M = \pm i \).

Similarly, if one puts mirror eigenstates at \( x = n + \frac{1}{2} \) with \( M = \pm i \), which is the case for the decoration in Fig.18(b), the momentum state \( \phi_k \) satisfies

\[ M \phi_k = M \left( \sum_n \phi_{n + \frac{1}{2}} e^{i k (n + \frac{1}{2})} \right) \]  
\[ = \sum_n e^{i k (n + \frac{1}{2})} (M \phi_{n + \frac{1}{2}}) \]  
\[ = \sum_n e^{i k (n + \frac{1}{2})} (\pm i) \phi_{-(n + \frac{1}{2})} \]  
\[ = \pm i \sum_n e^{-i k (n + \frac{1}{2})} \phi_{n + \frac{1}{2}} \]  
\[ = \pm i \sum_n e^{i k (n + \frac{1}{2})} e^{-i 2k (n + \frac{1}{2})} \phi_{n + \frac{1}{2}} \]  
\[ (F7) \]

At \( k = 0 \), we have \( M \phi_k = \pm i \phi_k \), while at \( k = \pi \), we have \( M \phi_k = \mp i \phi_k \), i.e., \( \phi_{k=0} \) is a mirror eigenstate with \( M = \pm i \) while \( \phi_{k=\pi} \) is a mirror eigenstate with \( M = \mp i \).

Then we apply the above results to analyze the boundary states. The decoration in Fig.18(b) has mirror line at \( x = 0 \) decorated by 1D mirror-symmetric TSC with \( w_M = 1 \), i.e., \( w_+ = +1 \) and \( w_- = -1 \). It is straightforward to figure out its boundary state, as plotted in Fig.18(d). At \( k_x = 0 \) and \( k_x = \pi \), the mirror winding numbers \( w_M \) are both 1, protecting a pair of zero modes. At generic momentum, there is a gap opened. Similarly, the decoration in Fig.18(c), which has mirror line at \( x = \frac{1}{2} \) decorated by 1D mirror-symmetric TSC with \( w_M = +1 \), has boundary state shown in Fig.18(e). At \( k_x = 0 \), the mirror winding number \( w_M \) is +1, while at \( k_x = \pi \), \( w_M \) becomes -1 since the two mirror sectors are exchanged. At both the two momenta, there are a pair of zero modes. Different with the boundary state in Fig.18, here the zero modes at \( k_x = 0 \) and \( k_x = \pi \) are protected by opposite \( w_M \).

We further consider combinations of the two decorations. First we put 1D mirror-symmetric TSC of \( w_M = +1 \) at both \( x = 0 \) and \( x = \frac{1}{2} \). It is easy to see the momentum mirror winding numbers are given as

\[ w_M(k_x = 0) = 1 + 1 = +2 \]  
\[ (F12) \]

and

\[ w_M(k_x = \pi) = 1 - 1 = 0 \]  
\[ (F13) \]

Consequently, at \( k_x = 0 \) there are two pairs of zero modes, while at \( k_x = \pi \), there is no zero mode but a gap opened, as shown in Fig.18(f). Next we put 1D mirror-symmetric TSC of \( w_M = +1 \) at \( x = 0 \), while 1D mirror-symmetric TSC of \( w_M = -1 \) at \( x = \frac{1}{2} \). The momentum mirror winding numbers are given as

\[ w_M(k_x = 0) = 1 - 1 = 0 \]  
\[ (F14) \]

and

\[ w_M(k_x = \pi) = 1 + 1 = +2 \]  
\[ (F15) \]

Contrary to the first combination, there are no zero modes at \( k_x = 0 \) but two pairs of zero modes at \( k_x = \pi \), as shown in Fig.18(g).

4. Layer group symmetries

As we have mentioned previously, the generating symmetry operations of LGs are those of WGs plus mirror or glide in the vertical direction. For brevity, we only consider the boundary states protected solely by vertical mirror and vertical glide.

For the LG generated by vertical mirror reflection, i.e. \( p1lm \), its decorations are generated by stacking 1D mirror-symmetric TSCs with \( \chi_M = \pm 1 \) along \( x \) or \( y \) directions, which are similar as those of WG \( p1 \), with the difference of building blocks. The only thing to note is that, for \( \chi_M = 1 \), the decorations are \( \mathbb{Z}_2 \)-type, while for \( \chi_M = -1 \), the decorations are \( \mathbb{Z}_2 \)-type. At \( k_x/y = 0 \) and \( k_x/y = \pi \), there are the same number of Majorana zero modes, as also discussed in Ref.[86].

The LG generated by vertical glide \( \{ \mathbb{Z}_2 | t_x t_y 0 \} \), where \( t_x, t_y = 0, \frac{1}{2}, 1 \), is \( p1g1 \). The TCSC boundary states are already studied in Ref.[42].
Appendix G: Tables for classification results

| WG | Rep classification | decorations |
|----|--------------------|-------------|
| p1 | A                  | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p2 | A                  | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p1m | A′                 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p1m | A′′               | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p1gl | A′                 | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p1gl | A′′               | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| c1m | A′                 | $\mathbb{Z} \times \mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| c1m | A′′               | $\mathbb{Z}_2$ | \{e\}_1, \{e\}_2 |
| p2mm | A1               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| p2mm | A2               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | A1               | $\mathbb{Z}_2^4 \times \mathbb{Z}_2$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | B1               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | B2               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| p2mg | A1               | $\mathbb{Z}_2^2 \times \mathbb{Z}_2$ | \{e\}_1, \{e\}_2, \{e\}_3 |
| p2mg | A2               | $\mathbb{Z}_2^2$ | \{e\}_1, \{e\}_2, \{e\}_3 |
| p2mg | B1               | $\mathbb{Z}_2^2$ | \{e\}_1, \{e\}_2, \{e\}_3 |
| p2mg | B2               | $\mathbb{Z}_2^2 \times \mathbb{Z}$ | \{e\}_1, \{e\}_2, \{e\}_3 |
| c2mm | A1               | $\mathbb{Z}_2^4 \times \mathbb{Z}_2$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | A2               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | B1               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |
| c2mm | B2               | $\mathbb{Z}_2^4$ | \{e\}_1, \{e\}_2, \{e\}_3, \{e\}_4 |

TABLE VI. TCSCs by wire construction for WGs. The first column are WG names. The second column are representations of the gap function. The third column are classifications. The fourth column are 1-cell decorations in correspondence with classifications. N/A means trivial classification and decorations. For instance, for p6mm with A1 representation, there are two types of independent decorations, both with $\mathbb{Z}$ classification. The generator for the first is constructed by decorating $e_1$ and $e_2$ simultaneously, while the generator for the latter is constructed by decorating $e_3$. Readers can refer to the cell-complex that we give in Appendix H to see what these 1-cells are.

| WG | LG |
|----|----|
| p1 | p1 |
| p2 | p112 |
| p1m1 | p211, pm11, |
| p1g1 | p211, p611 |
| c1m | c211, cm11 |
| p2mm | p222, pm2 |
| p2mg | p22, pm2 |
| p2gg | p22, p6 |
| c2mm | c222, cm2 |
| p4 | p4 |
| p4mm | p422, p-42m, p-4mm |
| p4gm | p42, p-42m, p-4bm |
| p3 | p3 |
| p3m1 | p312, p3m |
| p3m1 | p312, p3m |
| p6 | p6 |
| p6mm | p622, p6mm |

TABLE VII. LGs whose wire construction results are the same as corresponding WGs. The blue one is the LG that shares completely the same group elements with the corresponding WG.

| LG | Rep classification | decorations |
|----|--------------------|-------------|
| p1 | $A_x$              | N/A         |
| p1 | $A_y$              | N/A         |
| p1 | $A_{y'}$           | N/A         |
| p1 | $A_{y''}$          | N/A         |
| p3 | $A_x$              | N/A         |
| p3 | $A_{y'}$           | N/A         |
| p3 | $A_{y''}$          | N/A         |
| p111 | $A'$              | N/A         |
| p111 | $A''$             | N/A         |
| p111a | $A'$               | N/A         |
| p111a | $A''$            | N/A         |
| p112/m | $A_x$            | N/A         |
| p112/m | $A_{y'}$       | N/A         |
| p112/m | $A_{y''}$      | N/A         |
| pm2 | $A_x$              | N/A         |
| pm2 | $A_{y'}$           | N/A         |
| pm2 | $A_{y''}$          | N/A         |
| p3/m | $A'$              | N/A         |
| p3/m | $A''$             | N/A         |
| p4/m | $A_x$              | N/A         |
| p4/m | $A_{y'}$           | N/A         |
| p4/m | $A_{y''}$          | N/A         |
| p6/m | $A_x$              | N/A         |
| p6/m | $A_{y'}$           | N/A         |
| p6/m | $A_{y''}$          | N/A         |

TABLE VIII. Wire construction results for some LGs, which can not be directly inferred from those of corresponding WGs. To avoid confusion, we remark that for LG pm2m, whose group elements are $1$, $M_x$, $C_{2y}$ and $M_y$, the representations of gap function are given as follows. $A_1$: $\chi_{M_x} = 1, \chi_{M_y} = 1$, $A_2$: $\chi_{M_x} = 1, \chi_{M_y} = -1$, $B_1$: $\chi_{M_x} = -1, \chi_{M_y} = 1$, $B_2$: $\chi_{M_x} = -1, \chi_{M_y} = -1$. To see what these 1-cells are, readers can refer to Appendix I, where we plot the cell complex for each LG in this table.
Appendix H: Cell complex for wallpaper groups

FIG. 20. WG $p_1$

FIG. 21. WG $p_2$

FIG. 22. WG $p_{1m1}$

FIG. 23. WG $p_{1g1}$

FIG. 24. Cell complex of WG $c1m1$.  

FIG. 25. WG $p_{2mm}$

FIG. 26. Cell complex of WG $p_{2mg}$. 
FIG. 27. cell complex of WG $p2gg$.

FIG. 28. cell complex of WG $c2mm$.

FIG. 29. WG $p4$.

FIG. 30. WG $p4mm$.

FIG. 31. WG $p4gm$.

FIG. 32. WG $p3$.

FIG. 33. WG $p3m1$.

FIG. 34. WG $p31m$. 
Appendix I: Cell complex for some LGs

FIG. 35. WG $p6$

FIG. 36. WG $p6mm$.

FIG. 37. LG $p-1$

FIG. 38. LG $p-3$.

FIG. 39. LG $p11m$.

FIG. 40. LG $p11a$.

FIG. 41. LG $p112/m$.

FIG. 42. LG $pm2m$. 
FIG. 43. LG $p3/m$.

FIG. 44. LG $p4/m$.

FIG. 45. LG $p6/m$. 