Two- and three dimensional Brownian motion under non-instantaneous resetting

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We consider Brownian motion under resetting in two and three dimensions for the case when the return of the particle to the origin occurs at a constant speed. We investigate the behavior of the probability density function (PDF) and of the mean-squared displacement (MSD) in this process under Poissonian and deterministic resetting. We moreover discuss a general problem of the invariance of the PDF with respect to the return speed, as observed in the one-dimensional system for Poissonian resetting, and show, that this one dimensional situation is the only one in which such an invariance can be observed.

I. INTRODUCTION

The investigation of different aspects of the behavior of random processes under resetting has attracted much attention in the last years, see \cite{1} for a recent review. In such processes the stochastic motion is interrupted from time to time and restarts anew. These processes may be applicable in order to describe the computer search algorithms, motion of animals returning home to rest and of the machines returning in order to recharge.

The very first model studied, see \cite{2}, corresponded to a one-dimensional Brownian motion interrupted by the resetting events following a Poisson process. While the probability density function (PDF) to the unrestricted Brownian motion spreads with time, the PDF of diffusion with resetting reaches, after a short relaxation period, a stationary state, which corresponds to a Laplace distribution of the coordinate \cite{2}. Later, Brownian motion in two and higher dimensions has been studied \cite{3, 1}. Other types of processes under resetting, such as Lévy flights \cite{4, 6}, continuous-time random walks with or without drift \cite{7, 10}, and scaled Brownian motion \cite{11, 12} have been also considered. Also a variety of waiting time distributions, such as δ-distribution (resetting at a fixed time after starting the stochastic motion) \cite{13, 14}, power-law \cite{11, 12, 15}, and other types \cite{16, 17} have been investigated.

The return to the initial position (assumed at the origin of the coordinate system in what follows) is treated as instantaneous in the classical approach \cite{1}. It means that at the resetting event the particle teleports to the initial position. While this model is immediately applicable to the computer algorithms \cite{18}, it is unrealistic with respect to the motion of foraging animals, robots and other objects. Recently, the diffusion under resetting with non-instantaneous return has been considered, but for one-dimensional systems only \cite{19, 22}. Surprisingly, it has been revealed, that the PDF and MSD of normal diffusion under Poisson resetting do not depend on the return velocity, if it remains constant during the whole return path \cite{23, 24}. However, this invariance does not hold for non-Poissonian resetting, such as resetting at fixed time \cite{22} and for resetting with Pareto distribution of the waiting times \cite{23}.

The motion of real objects, e.g. of a robotic vacuum cleaner performing stochastic motion during the cleaning phase and returning to its base in order to recharge, as well as of foraging mammals, is usually two-dimensional (2d); the motion of birds, fish and drones is quasi three-dimensional (3d). This gave us motivation to consider two-and three dimensional Brownian motion under resetting with non-instantaneous return.

In the current study we generalize the approach, developed in \cite{22} and consider MSD and PDF of the exponential resetting and the resetting at fixed time in higher dimensions. We investigate if the invariance of the PDF and MSD with respect to the return velocity can be observed also in higher dimensions. We proceed as follows. In Section II we describe our model, give the general expressions for MSD and for PDF and provide the details of computer simulations. In Section III we give both the numerical and analytical results for PDF and MSD of normal diffusion under Poisson resetting with return at constant velocity. Section IV is devoted to the resetting at fixed time. In Section V we discuss the invariance of the PDF and MSD with respect to the return velocity. Finally, concluding remarks are given in Section VI.

II. MODEL

We consider two- and three-dimensional motion under non-instantaneous resetting by generalizing the approach used in Ref.\cite{22} for the one-dimensional case. We assume that the object (particle) start its motion at the origin $R = 0$. The motion of the particle is a renewal process consisting of repeated runs. Each run is a sequence of two processes (phases of motion): The stochastic displacement process which ends at the resetting event, and the deterministic return process, which ends when the particle returns to the origin $R = 0$ We assume that the motion is statistically isotropic (i.e. mirror, rotationally or spherically symmetric in dimensions $1$, $2$, and $3$ correspondingly), so that the probability density function (PDF) of displacements is determined only by the distance $R$ from the origin. The resetting occurs with waiting time density $\psi(t)$, which in our explicit calculations is taken to be either exponential, $\psi(t) = re^{-rt}$, or
deterministic with fixed time interval, \( \psi(t) = \delta(t - t_r) \), and concentrate on the properties of the stationary distribution of the particle’s positions.

A. General expressions

The stationary PDF follows by noting that a time at which the particle’s position is observed may fall into the displacement phase of the motion or into the return phase, and these events are mutually exclusive. The probability of the first, \( P_1 \), and of the second, \( P_2 \), are given by

\[
P_1 = \frac{\langle t_{\text{res}} \rangle}{\langle t_{\text{res}} \rangle + \langle t_{\text{ret}} \rangle}, \quad P_2 = \frac{\langle t_{\text{ret}} \rangle}{\langle t_{\text{res}} \rangle + \langle t_{\text{ret}} \rangle},
\]

where \( \langle t_{\text{res}} \rangle \) is the mean duration of the displacement phase, \( \langle t_{\text{ret}} \rangle \) is the mean duration of the return phase, and

\[
\langle t_{\text{res}} \rangle + \langle t_{\text{ret}} \rangle = \langle t_{\text{run}} \rangle
\]

represents the mean duration of a run. The PDF of the particle’s positions can thus be represented as

\[
p(R) = p_1(R)P_1 + p_2(R)P_2,
\]

where \( p_1(R) \) and \( p_2(R) \) are the PDFs of the particle’s displacements from the origin conditioned on the fact that the time at which the position was measured belongs to the displacement phase and to the return phase of the motion correspondingly. Let us now introduce the rescaled PDF in the displacement phase \( \rho_1(R) = p_1(R)P_1 \langle t_{\text{run}} \rangle \) and the rescaled PDF of the return phase \( \rho_2(R) = p_2(R)P_2 \langle t_{\text{run}} \rangle \). According to Ref. [22] these are given by

\[
\rho_1(R) = \int_0^\infty dt \, \rho(t) \int_0^{t_{\text{res}}} P(R, t) dt
\]

under general conditions, and

\[
\rho_2(R) = \frac{1}{v} \int_R^\infty dR_0 \int_0^\infty dt \, P(R_0, t) \psi(t)
\]

for the return at a constant speed \( v \). Accordingly, the mean duration of the displacement phase is

\[
\langle t_{\text{res}} \rangle = \int_0^\infty \rho_1(R) dR = \int_0^\infty dt \psi(t) t
\]

and the mean duration of the return phase is

\[
\langle t_{\text{ret}} \rangle = \int_0^\infty \rho_2(R) dR = \frac{1}{v} \int_0^\infty dt \psi(t) \int_0^\infty dR' R' p(R'|t)
\]

The Eq. [3] then takes the form

\[
p(R) = \frac{\rho_1(R) + \rho_2(R)}{\langle t_{\text{res}} \rangle + \langle t_{\text{ret}} \rangle}
\]

B. Brownian motion in displacement phase

The propagator of free Brownian particle in \( d \) dimensions reads

\[
P(R|t) = \frac{dR^{d-1}}{(4Dt)^{d/2}} \frac{e^{-\frac{d^2}{4Dt}}}{\Gamma \left( 1 + \frac{d}{2} \right)},
\]

where \( D \) is the diffusion coefficient. Using Eq. [7] we get for the ballistic return time

\[
\langle t_{\text{ret}} \rangle = \lambda(d) \langle t_{\text{res}} \rangle^{1/2}
\]

with the coefficients

\[
\lambda(d) = 2\sqrt{D} \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)}
\]

and

\[
\langle t_{\text{res}} \rangle = \int_0^\infty dt \, t^{1/2} \psi(t).
\]

C. Mean squared displacement

The MSD

\[
\langle R^2 \rangle = \int_0^\infty R^2 p(R') dR'
\]

may be obtained by integration of Eq. [8]

\[
\langle R^2 \rangle = \frac{R^2_1 + R^2_2}{\langle t_{\text{res}} \rangle + \langle t_{\text{ret}} \rangle}
\]

where

\[
R^2_{1,2} = \int_0^\infty R^2 \rho_1,2(R) dR.
\]

The integration in Eq. [15] gives for normal diffusion:

\[
R^2 = d D \langle t_{\text{res}} \rangle
\]

and integration of Eq. [5] yields

\[
R^2 = \frac{1}{3v} \int_0^\infty dt \psi(t) \int_0^\infty dR' R^2 p(R'|t) = \frac{\kappa(d) D^{3/2} \langle t_{\text{res}} \rangle}{v}
\]

with coefficients \( \kappa(d) \) given by

\[
\kappa(d) = \frac{8\Gamma \left( \frac{3+d}{2} \right)}{3D \left( \frac{d}{2} \right)}.
\]

Hence we get for the MSD of free Brownian motion under non-instantaneous return at constant velocity in the steady state

\[
\langle R^2 \rangle = d D \langle t_{\text{res}} \rangle + \frac{1}{\lambda(d) D^{1/2} \langle t_{\text{res}} \rangle} \langle t_{\text{res}} \rangle^{1/2} \langle t_{\text{res}} \rangle.
\]

Below we calculate PDF and MSD of free Brownian motion under exponential and deterministic resetting with non-instantaneous return.
FIG. 1: Probability density function of the distance of the particle to the origin $p(R)$ for the exponential resetting (a) in 2d (b) in 3d with $D = 1$, $r = 1$. The colored solid lines show the results of numerical simulations, the thin black dashed and dotted lines show PDF given by Eqs. (28) and (33) for 2d and 3d, correspondingly. For comparison we give also the results for the instantaneous resetting ($v = \infty$).

D. Computer simulations

The analytical predictions for MSD and PDF will be compared with numerical simulations directly following from the discretization of Langevin equations in the motion phase and of the equation of motion for the return at a constant speed. The time axis is discretized with the step $dt = t_{i+1} - t_i$, and the time of the first resetting event is generated according to its probability density $\psi(t)$. For the deterministic resetting the resetting time is fixed. During the displacement phase the particle performs stochastic motion according to a finite-difference analogue of the Langevin equation in 3d

\begin{align*}
x_{i+1} &= x_i + \xi_{ix}\sqrt{2Ddt} \\
y_{i+1} &= y_i + \xi_{iy}\sqrt{2Ddt} \\
z_{i+1} &= z_i + \xi_{iz}\sqrt{2Ddt}
\end{align*}

Here $x_i = x(t_i)$, $y_i$ and $z_i$ are the coordinates of the particle at time $t_i$, $R^2 = x_i^2 + y_i^2 + z_i^2$ and $\xi_{ix,y,z}$ are the random numbers undergoing a standard normal distribution and generated using the Box-Muller transform. In 2d $z_i = 0$ and is not updated during the simulation. In 1d both $z_i = 0$ and $y_i = 0$. When the resetting event occurs, the particle starts moving to the origin at a constant speed: $R_{i+1} = R_i - vdt$. When the particle reaches the origin, the time of the next resetting event is generated, and the particle starts performing Brownian motion again until the new resetting event. All simulations are performed with $N = 10^5$ particles.

III. EXPONENTIAL RESETTING

Let us first consider the Poissonian resetting process. In this case the waiting time distribution for the resetting event is exponential:

$$\psi(t) = r \exp(-rt)$$

with $r$ being the rate of the resetting events. The survival probability

$$\Psi(t) = 1 - \int_0^t \psi(t')dt'$$

is the probability that no resetting event occurred up to time $t$. For Poisson resetting it is equal to

$$\Psi(t) = e^{-rt}.$$ 

The mean duration of the displacement phase is (according to Eq. (6)) $\langle t_{res} \rangle = \frac{1}{r}$.

A. One dimension

Here we briefly review the results obtained for the Brownian motion with Poisson resetting in 1d. In this case the remarkable fact of the invariance of the PDF and MSD with respect to the return speed has been observed [20, 23]. The PDF is given by the Laplace distribution and has the same form as for the Brownian motion under instantaneous resetting [2]:

$$p(R) = \frac{1}{\sqrt{D/r}} \exp \left( -\sqrt{\frac{2r}{D}}R \right).$$

is the same as the 1d distribution obtained by Evans and Majumdar (Eq. 26) [2]. In the case of the finite return velocity the evolution of the coordinates is coupled, because the duration of the return along the x-coordinate depends on the y-coordinate as well: the larger y, the longer the return. The PDF $p_x(x)$ may be derived from $p(R)$ given by Eq. (28) in the following way:

$$p_x(x) = \int_0^\infty dR \int_0^{2\pi} d\phi \delta (x - R \cos \phi) p(R) p(\phi). \quad (30)$$

Due to the isotropy of our system $p(\phi) = \frac{1}{2\pi}$, and the integration over $\phi$ yields

$$p_x(x) = \frac{1}{\pi} \int_{|x|}^\infty dR \frac{p(R)}{\sqrt{R^2 - x^2}}. \quad (31)$$

Substituting Eq. (28) into Eq. (31) gives

$$p_x(x) = \frac{1}{2\sqrt{\pi D} e^{-\sqrt{\pi D}|x|}} + \frac{\sqrt{\pi x}}{4\sqrt{\pi D^3}} G_{1,3}^{3,0}\left(\frac{x^2}{4D}, \frac{0}{4D} \left| -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right. \right)$$

$$\frac{1}{\pi} + \frac{\pi}{4\sqrt{\pi D}} \right)$$

with $G_{1,3}^{3,0}$ being the Meijer G function. This probability density function $p(x)$ is shown in Fig. 2. The distributions of position projections for non-instantaneous return processes are stronger peaked at the origin $x = 0$ compared to such for instantaneous return. This is due to the fact that a considerable portion of return processes starts around $x = 0$ but with large value of y, so that the particles stays in the vicinity of $x = 0$ during the whole return path.
C. Three dimensions

The probability density function $p(R)$ in 3d may be obtained similarly to the two-dimensional case:

$$p(R) = \frac{R e^{-\sqrt{\frac{D}{r}} R} + \frac{1}{2} (1 + R \sqrt{\frac{D}{r}}) e^{-\sqrt{\frac{D}{r}} R}}{\frac{1}{r} + \frac{2}{v} \sqrt{\frac{D}{r}}}.$$  \hfill (33)

The PDF given by Eq. (33) is in a nice agreement with numerical simulations (see Fig. 1b). Again, three-dimensional Brownian motion with Poisson resetting is not invariant with respect to the return velocity. The MSD in three dimensions is given by

$$\langle R^2 \rangle = 4 D/v + 8 \sqrt{D r} \sqrt{rac{D}{v}}.$$  \hfill (34)

Numerical simulations reveal that the MSD rapidly reaches its steady state values given by Eqs. (29) and (34) in 2d and 3d, correspondingly (see Fig. 3). The MSD grows with increasing of the return velocity $v$ and tends to values $\langle R^2 \rangle = 4D/r$ and $\langle R^2 \rangle = 6D/r$ for large velocities in 2d and 3d, respectively. In 1d it is invariant with respect to the return velocity.

For the distribution of coordinate $x$ we get in three-dimensional systems:

$$p_x(x) = \frac{1}{2} \int_{|x|}^{\infty} dR p(R) \frac{R}{R}.$$  \hfill (35)

The integration yields

$$p_x(x) = \frac{1}{\sqrt{D r} e^{-\sqrt{\frac{D}{r}} x} + \frac{1}{2} e^{-\sqrt{\frac{D}{r}} x} + \frac{1}{2} \Gamma (0, \sqrt{\frac{D}{r}} x)}{\frac{1}{r} + \frac{2}{v} \sqrt{\frac{D}{r}}}.$$  \hfill (36)

The analytical prediction for $p_x(x)$ is compared with computer simulations at Fig. 2b and nice agreement is observed.

IV. DETERMINISTIC ResetsING

Let us now consider normal diffusion under resetting at a constant pace:

$$\psi(t) = \delta(t - t_r).$$  \hfill (37)

The survival probability for resetting events is now $\Psi(t) = \Theta(t_r - t)$. The mean duration of the displacement phase is $\langle t_{res} \rangle = t_r$.

A. One dimension

The invariance of the PDF, observed for the normal diffusion with exponential resetting in 1d, does not hold any more for diffusion with deterministic resetting. The PDF may be calculated using Eq. (8):

$$p(R) = \frac{\sqrt{\frac{4 t_r^2}{\pi}} e^{-R^2/(4 t_r^2)} - R \text{erfc} \left(\frac{R}{\sqrt{4 D t_r}}\right)}{t_r + \frac{4}{v} \sqrt{\frac{4 D t_r}{\pi}}}.$$  \hfill (38)

For the MSD we have

$$\langle R^2 \rangle = D t_r \left(1 + \frac{1}{t_r v \sqrt{\pi} + 2 \sqrt{D t_r}}\right).$$  \hfill (39)
motion with deterministic resetting at constant speed: Eqs. (8-5), we get PDF for two-dimensional Brownian motion Eq. (29) into  and 3d, respectively. The case \( v = \infty \) ever.

FIG. 4: Probability density function for the deterministic resetting in (a) 2d and (b) 3d with \( t_r = 10 \). The colored solid lines show the results of numerical simulations, the thin black dashed and dotted lines show PDF given by Eqs. \((10)\) and \((13)\) for 2d and 3d, respectively. The case \( v = \infty \) corresponds to the case of the instantaneous resetting.

B. Two dimensions

Inserting the waiting time density Eq. \((37)\), and the PDF of two-dimensional Brownian motion Eq. \((29)\) into Eqs. \((8)\), we get PDF for two-dimensional Brownian motion with deterministic resetting at constant speed:

\[
p(R) = \frac{R}{2D} \Gamma \left(0; \frac{R^2}{4Dt_r} \right) + \frac{1}{v} \exp \left(-\frac{R^2}{4Dt_r} \right) \frac{1}{t_r + \sqrt{\pi Dt_r} / v} .
\]

(40)

Here \( \Gamma(0, R) \) is the incomplete Gamma-function. The PDF is shown in Fig. 4(a). The MSD is obtained by inserting Eq. \((10)\) into Eq. \((13)\):

\[
\langle R^2 \rangle = 2Dt_r .
\]

(41)

While the PDF (Eq. \((10)\) explicitly depends on the return velocity, the MSD in the steady state, surprisingly, does not. In the simulations the MSD oscillates before it reaches the speed-independent stationary value (Fig. 5(a)). The higher the velocity, the longer the oscillations persist. In the case of the instantaneous resetting \( (v = \infty) \) the oscillations are not damped and persist forever.

The PDF \( p_x(x) \) of the position’s projection can be derived according to Eq. \((31)\) in full analogy to the case of the exponential resetting:

\[
p_x(x) = \left[ \frac{1}{2D} \sqrt{\frac{4Dt_r}{\pi}} e^{- \frac{x^2}{8Dt_r}} - x \text{erfc} \left( \frac{x}{\sqrt{4Dt_r}} \right) \right] + \frac{1}{2\pi v} \exp \left(-\frac{x^2}{8Dt_r} \right) K_0 \left( \frac{x^2}{8Dt_r} \right) / \left( t_r + \sqrt{\pi Dt_r} / v \right) .
\]

(42)

Here \( \text{erfc}(R) \) is the complementary error function. \( p_x(x) \) is shown at Fig. 4(a) and is in a nice agreement with the simulation results.

C. Three dimensions

In 3d the PDF can be expressed as

\[
p(R) = \frac{R}{D} \text{erfc} \left( \frac{R}{\sqrt{4Dt_r}} \right) + \frac{2}{v \sqrt{\pi}} \Gamma \left( \frac{3}{2}; \frac{R^2}{4Dt_r} \right) \frac{1}{t_r + 4\sqrt{D Dt_r} / v}\]

(43)

and is shown in Fig. 4(b). The invariance of the steady-state MSD with respect to the return velocity is not observed:

\[
\langle R^2 \rangle = Dt_r + \frac{32}{3\nu} \sqrt{\frac{Dt_r}{\pi}} : (44)
\]

the dependence of MSD in the steady state on the return velocity \( v \) for the deterministic resetting is shown in Fig. 5(b). For comparison, also the MSD for one-dimensional system is given. Note that the asymptotic values of the MSD for \( v \rightarrow \infty \) are in different dimensions are achieved in different ways: The MSD is a growing function of \( v \) in three-dimensional systems, a decreasing function of \( v \) in one-dimensional systems, and does not depend on \( v \) in two dimensions.

Now let us calculate the PDF of the projections of the locations of particle \( p_x(x) \) on x-axis. Using Eq. \((35)\), we have

\[
p_x(x) = \left[ \frac{1}{2D} \sqrt{\frac{4Dt_r}{\pi}} e^{- \frac{x^2}{8Dt_r}} - x \left( \text{erf} \left( \frac{x}{\sqrt{4Dt_r}} \right) - 1 \right) \right]
\]

\[+ \frac{2}{9\nu \sqrt{\pi} (4Dt_r)^3} x^2 \left( \frac{3}{2} \frac{3}{2} \frac{3}{2} \frac{1}{2} \right) F_2 \left( \frac{3}{2} \frac{3}{2} \frac{1}{2} 2 ; \frac{2}{4Dt_r} - x^2 / 4Dt_r \right)
\]

\[\left. - \frac{1}{4\nu} \left(-2 + \gamma + \log \left( \frac{x^2}{4Dt_r} \right) \right) \right] / \left( t_r + \frac{4\sqrt{D Dt_r}}{v \sqrt{\pi}} \right) .
\]

(45)
Here \( \text{erf}(x) \) is the error function and \( _2F_1 \) is the generalized hypergeometric function. This function \( p_2(x) \) is shown in Fig. 2(b) and fits very well the simulation data.

V. THE INVARIANCE OF PDF AND MSD WITH RESPECT TO THE RETURN VELOCITY

A. The invariance of the PDF

As our calculations for the specific examples discussed above show, the PDF of particle’s displacements is invariant with respect to the return speed in the one-dimensional case for exponential waiting time density, and not invariant with respect to this speed in two and in three dimensions both for exponential and for the deterministic resetting. This motivates a general discussion of circumstances under which the PDF becomes invariant with respect to the return speed.

Let us return to our Eq. (8) and note that according to Eqs. (5) and (7) the rescaled PDFs in the displacement phase and in the return phase, \( \rho_1(R) \) and \( \rho_2(R,v) \), Eqs. (8) and (9), can be expressed in terms of the function

\[
p(R) = \frac{\rho_1(R) + v^{-1} \rho_2(R)}{\langle t_{\text{res}} \rangle + v^{-1} \langle |R_0| \rangle}.
\]

with \( \Psi(t) \) being the survival probability, Eq. (24). Then the condition given by Eq. (45) becomes equivalent to the following integral relation [22]:

\[
\int_0^\infty \left[ C \frac{\partial}{\partial R} F(R,t) + \frac{\partial}{\partial t} F(R,t) \right] \Psi(t) dt = 0. \tag{52}
\]

1. One dimension

Let us first show that in 1d the exponential waiting time density, Eq. (24), is the only solution of integral equation Eq. (52). The integration of the Gaussian PDF, Eq. (9), of a Brownian particle in 1d yields

\[
F(R,t) = \frac{1}{2} \text{erf} \left( \frac{R}{2\sqrt{Dt}} \right). \tag{53}
\]

Substituting the partial derivatives of \( F(R,t) \) into (Eq. 52), we get

\[
\int_0^\infty \sqrt{\frac{1}{t}} \exp \left( -\frac{R^2}{2Dt} \right) (C - \frac{R}{2t}) \Psi(t) dt = 0. \tag{54}
\]

Introducing new variables \( z = 1/t \) and \( u = R^2/4Dt \) we get

\[
\int_0^\infty e^{-uz} \left( \frac{C}{z} - \sqrt{Du} \right) \Psi \left( \frac{1}{z} \right) \frac{dz}{\sqrt{z}} = 0. \tag{55}
\]

Introducing a new function

\[
Q(z) = \frac{1}{\sqrt{z}} \Psi \left( \frac{1}{z} \right)
\]

we rewrite this equation as

\[
C \int_0^\infty \frac{1}{z} e^{-uz} Q(z) dz = \sqrt{Du} \int_0^\infty e^{-uz} Q(z) dz. \tag{57}
\]

On the r.h.s. we immediately recognize the Laplace transform \( \tilde{Q}(u) \) of \( Q(z) \). Now we differentiate both parts of this equation with respect to \( u \) and rearrange the terms. Thus, in the Laplace domain our integral relation turns into the differential equation

\[
\frac{d}{du} \tilde{Q}(u) = - \left( \frac{1}{2u} + \frac{C}{\sqrt{Du}} \right) \tilde{Q}(u). \tag{58}
\]

This differential equation can be easily integrated

\[
\tilde{Q}(u) = \frac{C_1}{\sqrt{u}} \exp \left( -\frac{2C}{\sqrt{D}} \sqrt{u} \right), \tag{59}
\]

with \( C_1 \) being the constant of integration, and the inverse Laplace transform gives

\[
Q(z) = \frac{C_1}{\sqrt{\pi z}} \exp \left( -\frac{C^2}{Dz} \right). \tag{60}
\]
with $C_2$ deriving from $C_1$. Returning back to $\Psi(t)$ and using the initial condition on the survival probability $\Psi(0) = 1$ one gets
\[ \Psi(t) = \exp \left( -\frac{C^2}{D}t \right). \]
Introducing the resetting rate $r = \frac{C^2}{D}$ we turn to Eq. (25) and therefore obtain exponential resetting as a unique solution for the invariance problem in one dimension.

2. Two dimensions

The integration of the diffusion propagator in 2d, Eq. (9), yields
\[ F(R,t) = 1 - \exp \left( -\frac{R^2}{4Dt} \right). \]
The integral relation, Eq. (52), becomes
\[ \int_0^\infty \frac{1}{t} \exp \left( -\frac{R^2}{4Dt} \right) \left( C - \frac{R}{2t} \right) \Psi(t) dt = 0. \]
Now we introduce the same variables $z = 1/t$ and $u = R^2/4D$ as in 1d, which gives us
\[ \int_0^\infty e^{-uz} \left( \frac{C_2}{z} - \sqrt{Du} \right) \Psi \left( \frac{1}{z} \right) dz = 0. \]
Denoting
\[ Q(z) = \Psi \left( \frac{1}{z} \right), \]
we return to the same Eq. (52), with the solution given by Eq. (60). In such a way we get
\[ \Psi(t) = \tilde{C}_2 \sqrt{t} \exp \left( -\frac{C^2}{D}t \right). \]
This function, however, vanishes at $t = 0$ for any integration constant $\tilde{C}_2$, therefore the condition $\Psi(0) = 1$ cannot be satisfied, and $\Psi(t)$ can not be a survival probability. Thus, we have shown, that no physical solution of Eq. (52) exists in two dimensions, and the PDF depends on the return velocity $v$ for all possible resetting time distributions.

3. Three dimensions

In 3d we integrate Eq. (9) and obtain
\[ F(R,t) = \text{erfc} \left( \frac{R}{\sqrt{4Dt}} \right) + \frac{R}{\sqrt{\pi Dt}} e^{-\frac{R^2}{4Dt}}. \]
As in previous cases, we obtain Eq. (57), with the solution given by Eq. (60), but now with
\[ Q(z) = \sqrt{z} \Psi \left( \frac{1}{z} \right). \]
The function
\[ \Psi(t) = \tilde{C}_3 t \exp \left( -\frac{C^2}{D}t \right) \]
again can not be considered as a survival probability of the resetting process with any waiting time distribution.

B. Invariance of the MSD

Let us consider the question of the invariance of $\langle R^2 \rangle$ w.r.t. the return velocity. The MSD is given by Eq. (19) and depends only on the dimension of space and on four (fractional) moments of the waiting time distribution. Since such a distribution is not defined by these four parameters uniquely, there are many waiting time densities,
for which an invariance of the MSD with respect to the return speed could take place.

1. One dimension

Using Eq. [19], we get in 1d
\[
\langle R^2 \rangle = D \frac{\langle t_{res}^2 \rangle + \frac{8}{\pi} \sqrt{\frac{2}{\pi}} \langle t_{res}^{3/2} \rangle}{\langle t_{res} \rangle}.
\] (70)

The MSD remains invariant if
\[
\frac{\langle t_{res}^2 \rangle}{\langle t_{res} \rangle} = \frac{4 \langle t_{res}^{3/2} \rangle}{3 \langle t_{res}^{1/2} \rangle},
\] (71)

which is fulfilled for the exponential resetting, Eq. (23), as expected.

2. Two dimensions

In two dimensions using Eq. [19], we get
\[
\langle R^2 \rangle = 2D \frac{\langle t_{res}^2 \rangle + \frac{8}{\pi} \sqrt{\frac{2}{\pi}} \langle t_{res}^{3/2} \rangle}{\langle t_{res} \rangle} + \frac{\Delta E}{v} \frac{\langle t_{res}^{1/2} \rangle}{\langle t_{res} \rangle}.
\] (72)

The MSD remains invariant if
\[
\frac{\langle t_{res}^2 \rangle}{\langle t_{res} \rangle} = \frac{\langle t_{res}^{3/2} \rangle}{\langle t_{res}^{1/2} \rangle},
\] (73)

which holds true for the resetting at fixed time.

3. Three dimensions

For the invariance of MSD we in three dimensions get from Eq. [19] the following condition:
\[
\langle R^2 \rangle = 3D \frac{\langle t_{res}^2 \rangle + \frac{32}{9\pi} \sqrt{\frac{2}{\pi}} \langle t_{res}^{3/2} \rangle}{\langle t_{res} \rangle} + \frac{4}{v} \sqrt{\frac{D}{\pi}} \langle t_{res}^{1/2} \rangle.
\] (74)

The MSD remains invariant if
\[
\frac{\langle t_{res}^2 \rangle}{\langle t_{res} \rangle} = \frac{8 \langle t_{res}^{3/2} \rangle}{9 \langle t_{res}^{1/2} \rangle}.
\] (75)

To fit a single equation one needs only a one-parametric family of waiting time distributions with non-negative support. For example, the MSD remains invariant under resetting with the following waiting time distribution:
\[
\psi(t) = \frac{1}{2} (\delta(t - t_0) + \delta(t - a t_0))
\] (76)

where the parameter \(a\) can be found from the equation
\[
\frac{8}{9} a^{5/2} - a^2 - \frac{1}{9} a^{3/2} + \frac{8}{9} - a^{1/2} - \frac{1}{9} = 0
\] (77)

and is equal either to \(a = 1.95242\) or to \(a = 0.442352\).

VI. CONCLUSIONS

We have investigated the stationary probability density function (PDF) and the mean-squared displacement (MSD) of two- and three-dimensional Brownian motion under resetting with return at constant velocity, concentrating on particular cases of Poissonian resetting and of resetting at a constant pace. While the stationary PDF (and thus the MSD) in one dimension does not depend on the return velocity under exponential resetting, no such invariance holds in two or three dimensions. In these cases the probability to find the particle closer to the origin becomes higher and the tails of the pdf become lighter for smaller return velocities. Surprisingly, the steady-state MSD of Brownian motion with deterministic resetting in 2d remains invariant with respect to the return velocity.

The problem of the possible invariance of the stationary PDF with respect to return velocity was investigated also in a general setting. We have shown that the exponential waiting time distribution (characteristic for Poissonian resetting) is the only one for which such invariance holds in one dimension. We have moreover shown that in two and three dimensions such an invariance is impossible under any waiting time distribution.

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