Parameter estimation for Ornstein–Uhlenbeck processes driven by fractional Lévy process

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Abstract

We study the minimum Skorohod distance estimation \( \tilde{\theta}_\varepsilon \) and minimum \( L_1 \)-norm estimation \( \tilde{\theta}_\varepsilon \) of the drift parameter \( \theta \) of a stochastic differential equation

\[
dX_t = \theta X_t \, dt + \varepsilon \, dL_t, \quad X_0 = x_0,
\]

where \( \{ L_t, 0 \leq t \leq T \} \) is a fractional Lévy process, \( \varepsilon \in (0, 1] \).

We obtain their consistency and limit distribution for fixed \( T \), when \( \varepsilon \to 0 \). Moreover, we also study the asymptotic laws of their limit distributions for \( T \to \infty \).

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1 Introduction

Statistical inference for stochastic equations is a main research direction in probability theory and its applications. The asymptotic theory of parametric estimation for diffusion processes with small noise is well developed. Genon-Catalot [8] and Laredo [17] considered the efficient estimation for drift parameters of small diffusions from discrete observations as \( \varepsilon \to 0 \) and \( n \to \infty \). Using martingale estimating function, Sørensen [27] obtained consistency and asymptotic normality of the estimators of drift and diffusion coefficient parameters as \( \varepsilon \to 0 \) and \( n \) is fixed. Using a contrast function under suitable conditions on \( \varepsilon \) and \( n \), Sørensen and Uchida [28] and Gloter and Sørensen [9] considered the efficient estimation for unknown parameters in both drift and diffusion coefficient functions. Long [20], Ma [21] studied parameter estimation for Ornstein–Uhlenbeck processes driven by small Lévy noises for discrete observations when \( \varepsilon \to 0 \) and \( n \to \infty \) simultaneously. Shen and Yu [26] obtained consistency and the asymptotic distribution of the estimator for Ornstein–Uhlenbeck processes with small fractional Lévy noises.

Recently, Diop and Yode [4] obtained the minimum Skorohod distance estimate for the parameter \( \theta \) of a stochastic differential equation with a centered Lévy processes \( \{ Z_t, 0 \leq t \leq T \} \), \( \varepsilon \in (0, 1] \),

\[
dX_t = \theta X_t \, dt + \varepsilon \, dZ_t, \quad X_0 = x_0.
\]

When \( \{ Z_t, 0 \leq t \leq T \} \) is a Brownian motion, Millar [24] obtained the asymptotic behavior of the estimator of the parameter \( \theta \). The minimum uniform metric estimate of parameters
of diffusion-type processes was considered in Kutoyants and Pilibossian [14, 15]. Hénaff [10] considered the asymptotics of a minimum distance estimator of the parameter of the Ornstein–Uhlenbeck process. Prakasa Rao [25] studied the minimum $L_1$-norm estimates of the drift parameter of Ornstein–Uhlenbeck process driven by fractional Brownian motion and investigated the asymptotic properties following Kutoyants and Pilibossian [14, 15]. Some surveys on the parameter estimates of fractional Ornstein–Uhlenbeck process can be found in Hu and Nualart [11], El Onsy, Es-Sebaiy and Ndiaye [5], Xiao, Zhang and Xu [29], Jiang and Dong [12], Liu and Song [19].

Motivated by the above results, in this paper we consider the minimum Skorohod distance estimation $\tilde{\theta}^*_{\varepsilon}$ and minimum $L_1$-norm estimation $\tilde{\theta}_{\varepsilon}$ of the drift parameter $\theta$ for Ornstein–Uhlenbeck processes driven by the fractional Lévy process $\{L_t^d, 0 \leq t \leq T\}$ which satisfies the following stochastic differential equation:

$$dX_t = \theta X_t dt + \varepsilon dL_t^d, \quad X_0 = x_0,$$

where the shift parameter $\theta \in \Theta = (\theta_1, \theta_2) \subseteq \mathbb{R}$ is unknown, $\varepsilon \in (0, 1]$. Denote by $\theta_0$ the true value of the unknown parameter $\theta$. Note that

$$X_t(\theta) = x_t(\theta) + \varepsilon e^{\theta t} \int_0^t e^{-\theta s} dL_s^d,$$

where $x_t(\theta) = x_0 e^{\theta t}$ is a solution of (1) with $\varepsilon = 0$.

Recall that fractional Lévy processes is a natural generalization of the integral representation of fractional Brownian motion. Analogously to Mandelbrot and Van Ness [22] for fractional Brownian motion we introduce the following definition.

**Definition 1.1** (Marquardt [23]) Let $L = (L(t), t \in R)$ be a zero-mean two-sided Lévy process with $E[L(1)^2] < \infty$ and without a Brownian component. For $d \in (0, \frac{1}{2})$, a stochastic process

$$L_t^d := \frac{1}{\Gamma(d + 1)} \int_{-\infty}^{\infty} \left[(t-s)^d - (-s)^d\right] L(ds), \quad t \in R,$$

is called a fractional Lévy process (flp), where

$L(t) = L_1(t), \quad t \geq 0, \quad L(t) = -L_2(-t), \quad t < 0.$

\{L_1(t), t \geq 0\} and \{L_2(t), t \geq 0\} are two independent copies of a one-side Lévy process.

**Lemma 1.1** (Marquardt [23]) Let $g \in H, H$ is the completion of $L^1(R) \cap L^2(R)$ with respect to the norm $\|g\|_H^2 = E[L(1)^2] \int_R (L^d g)^2(u) du$, then

$$\int_R g(s) dL_t^d = \int_R (L^d g)(u) dL(u),$$

where the equality holds in the $L^2$ sense and $L^d g$ denotes the Riemann–Liouville fractional integral defined by

$$(L^d g)(x) = \frac{1}{\Gamma(d)} \int_x^\infty g(t)(t-x)^{d-1} dt.$$
Lemma 1.2 (Marquardt [23]) Let \( f, g \in H. \) Then
\[
E \left[ \int_R f(s) \, dL^d_s \right] \int_R g(s) \, dL^d_s = \frac{\Gamma(1-2d)E[L(1)^2]}{\Gamma(d)\Gamma(1-d)} \int_R \int_R f(t)g(s) |t-s|^{2d-1} \, ds \, dt.
\] 

Lemma 1.3 (Bender et al. [2]) Let \( L^d \) be a \( fLp \). Then for every \( p \geq 2 \) and \( \delta > 0 \) such that \( d + \delta < \frac{1}{2} \) there exists a constant \( C_{p,\delta,d} \) independent of the driving Lévy process \( L \) such that for every \( T \geq 1 \)
\[
E \left( \sup_{0 \leq t \leq T} |L^d_t|^p \right) \leq C_{p,\delta,d}E[(L(1)|^p) T^{p(d+1/2)+\delta}].
\]

For the study of \( fLp \) see Bender et al. [3], Fink and Klüppelberg [7], Lin and Cheng [18], Benassi et al. [1], Lacaux [16], Engelke [6] and the references therein.

The rest of this paper is organized as follows. In Sect. 2, we consider the minimum Skorohod distance estimation \( \theta^*_\varepsilon \) of the drift parameter \( \theta \), its consistency and limit distribution are studied for fixed \( T \), when \( \varepsilon \to 0 \). Moreover, the asymptotic law of its limit distribution are also studied for \( T \to \infty \). The similar problems for minimum \( L_1 \)-norm estimation \( \tilde{\theta}_\varepsilon \) of the drift parameter \( \theta \) were studied in Sect. 3.

## 2 Minimum Skorohod distance estimation

In this section, we consider the minimum Skorohod distance estimation which defined by
\[
\theta^*_\varepsilon = \arg \min_{\theta \in \Theta} \rho(X, x(\theta)),
\]
where
\[
\rho(x, y) = \inf_{\mu \in \Lambda([0, T])} \left( H(\mu) + \sup_{t \in [0, T]} |x(\mu(t)) - y(t)| \right)
\]
on the Skorohod space \( D([0, T], \mathbb{R}) \) consists of càdlàg functions on \([0, T]\), \( \Lambda([0, T]) \) is the set of functions \( \mu \) defined on \([0, T]\) with values in \([0, T]\), continuous, strictly increasing such that \( \mu(0) = 0 \) and \( \mu(T) = T \), and
\[
H(\mu) = \sup_{s,t \in [0, T], s \neq t} \left| \log \left( \frac{\mu(s) - \mu(t)}{s - t} \right) \right| < \infty.
\]

Let
\[
\eta_T = \arg \min_{u \in \mathbb{R}} \rho \left( Y(\theta_0), u \dot{x}(\theta_0) \right),
\]
where \( \dot{x}(\theta_0) = x_0 \dot{\theta} \) is the derivative of \( x_t(\theta_0) \) with respect to \( \theta_0 \) and
\[
Y_t(\theta_0) = e^{\theta_0 t} \int_0^t e^{\theta_0 s} \, dL^d_s.
\]

Let
\[
f(\kappa) = \inf_{|\theta - \theta_0| > \kappa} \| X_t - x(\theta_0) \| = \inf_{|\theta - \theta_0| > \kappa} \sup_{0 \leq t \leq T} \| X_t - x(\theta_0) \|, \quad \kappa > 0
\]
and \( P^{(\varepsilon)}_{\theta_0} \) denotes the probability measure induced by the process \( X_t \) for fixed \( \varepsilon \).
**Theorem 2.1** (Consistency) For every $p \geq 2$ and $\kappa > 0$ such that, for every $T \geq 1$, we have

\[
P_{00}^{(c)}(|\theta^*_\varepsilon - \theta_0| > \kappa) \leq C_{p,\kappa,d}E([L(1)]^p) T^{p(d + 1/2 + \kappa)} \left(\frac{2\varepsilon e^{(\theta_0 + T)}}{f(\kappa)}\right)^p = O((f(\kappa))^{-p}e_p),
\]

where constant $C_{p,\kappa,d}$ is only dependent on $p, \kappa, d$.

**Proof** Fixed $\kappa > 0$ and let

\[I_0 = \left\{ \omega : \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) > \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) \right\}.
\]

Then we can obtain $I_0 = \{|\theta^*_\varepsilon - \theta_0| > \kappa\}$. In fact, for $\omega \in I_0$, we have

\[
\inf_{|\theta - \theta_0| > \kappa} \rho (X(\omega), x(\theta)) \geq \inf_{\theta \in \Theta} \rho (X(\omega), x(\theta)) = \rho (X(\omega), x(\theta^*_\varepsilon)),
\]

thus, $|\theta^*_\varepsilon (\omega) - \theta_0| > \kappa$. On the other hand, assume that $|\theta^*_\varepsilon (\omega) - \theta_0| > \kappa$,

\[
\rho (X(\omega), x(\theta^*_\varepsilon)) = \inf_{|\theta - \theta_0| > \kappa} \rho (X(\omega), x(\theta)) < \inf_{|\theta - \theta_0| > \kappa} \rho (X(\omega), x(\theta)).
\]

For any $\kappa > 0$, we have

\[
P_{00}^{(c)} (I_0) = P_{00}^{(c)} \left( \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) > \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) \right)
\leq P_{00}^{(c)} \left( \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) > \inf_{|\theta - \theta_0| > \kappa} \left| \rho (X, x(\theta)) - \rho (x(\theta_0), x(\theta)) \right| \right)
\leq P_{00}^{(c)} \left( \inf_{|\theta - \theta_0| > \kappa} \rho (X, x(\theta)) > \inf_{|\theta - \theta_0| > \kappa} \rho (x(\theta_0), x(\theta)) - \rho (X, x(\theta)) \right)
\leq P_{00}^{(c)} \left( \inf_{|\theta - \theta_0| < \kappa} \rho (x(\theta), x(\theta_0)) + 2\rho (X, x(\theta_0)) > \inf_{|\theta - \theta_0| > \kappa} \rho (x(\theta), x(\theta)) \right)
\leq P_{00}^{(c)} \left( \|X - x(\theta_0)\|_\infty > \frac{f(\kappa)}{2} \right).
\]

Besides, since the process $X_t$ satisfies the stochastic differential Eqs. (1), it follows that

\[
X_t - x_t(\theta_0) = x_0 + \theta_0 \int_0^t X_s ds + \varepsilon L^d_t - x_t(\theta_0) = \theta_0 \int_0^t (X_s - x_t(\theta_0)) ds + \varepsilon L^d_t.
\]

Then

\[
|X_t - x_t(\theta_0)| = |\theta_0 \int_0^t (X_s - x_t(\theta_0)) ds + \varepsilon L^d_t| \leq |\theta_0| \int_0^t |X_s - x_t(\theta_0)| ds + \varepsilon |L^d_t|.
\]

Hence, we have

\[
\|X - x(\theta_0)\|_\infty = \sup_{0 \leq t \leq T} |X_t - x_t(\theta_0)| \leq \varepsilon e^{(\theta_0 + T)} \sup_{0 \leq t \leq T} |L^d_t|
\]

because of the Gronwall–Bellman lemma. Thus,

\[
P_{00}^{(c)} \left( \|X - x(\theta_0)\|_\infty > \frac{f(\kappa)}{2} \right) \leq P \left( \sup_{0 \leq t \leq T} |L^d_t| \geq \frac{f(\kappa)}{2\varepsilon e^{(\theta_0 + T)}} \right).
\]
According to Lemma 1.3 and Chebyshev’s inequality, for all $p \geq 2$, we get

$$P_{\theta_0}^{(\epsilon)}(|\theta^*_\epsilon - \theta_0| > \kappa) \leq E\left(\sup_{0 \leq t \leq T} |L_t^{(\epsilon)}|^p \right)^p \left(2\epsilon e^{\theta_0 T} \right)^p \leq \frac{C_{p,d} E\left(\sup_{0 \leq t \leq T} |L_t^{(\epsilon)}|^p \right)^p \epsilon^p}{f(\kappa)^p} = O\left(f(\kappa)^{-p} \epsilon^p\right).$$

(16)

This completes the proof. □

**Remark 2.1** As a consequence of the above theorem, we obtain the result that $\theta^*_\epsilon$ converges in probability to $\theta_0$ under $P_{\theta_0}^{(\epsilon)}$-measure as $\epsilon \to 0$. Furthermore, the rate of convergence is of order $O(\epsilon^p)$ for every $p \geq 2$.

**Theorem 2.2** (Limit distribution) For any $h \in D([0, T], \mathbb{R})$ satisfying $h(0) = 0$, $\phi^u_h = \rho(h, u \cdot \alpha)$, $\alpha(t) = t e^{\alpha t}$, $\alpha \in \mathbb{R}$, $u \in \mathbb{R}$ admits a unique minimum at $u$. Then we have, as $\epsilon \to 0$, $\epsilon^{-1}(\theta^*_\epsilon - \theta_0) \xrightarrow{d} \zeta_T$, where the notation ”$\xrightarrow{d}$” denotes ”convergence in distribution”.

**Remark 2.2** $\phi^u_h$ is a convex function and $\phi^u_h \to +\infty$ when $|u| \to +\infty$, so $\phi^u_h$ admits a minimum.

The following lemma due to Diop and Yode [4] which is vital for our proof of Theorem 2.2.

**Lemma 2.1** Let $\{K_{\epsilon}\}_{\epsilon > 0}$ be a sequence of continuous functions on $R$ and $K_0$ be a convex function which admits a unique minimum $\eta$ on $R$. Let $\{L_{\epsilon}\}_{\epsilon > 0}$ be a sequence of positive numbers such that $L_{\epsilon} \to +\infty$ as $\epsilon \to 0$. We suppose that

$$\lim_{\epsilon \to 0} \sup_{|u| \leq L_{\epsilon}} |K_{\epsilon}(u) - K_0(u)| = 0.$$

Then

$$\lim_{\epsilon \to 0} \arg \min_{|u| \leq L_{\epsilon}} K_{\epsilon}(u) = \eta,$$

where if there are several minima of $K_{\epsilon}$, we choose one of them arbitrarily.

**Proof of Theorem 2.2** We introduce the following notations:

$$K_{\epsilon}(u) = \rho\left(Y, \frac{1}{\epsilon} \left(x(\theta_0 + \epsilon u) - x(\theta_0)\right)\right),$$

$$K_0(u) = \rho\left(Y, u \hat{x}(\theta_0)\right).$$

Since

$$|K_{\epsilon}(u) - K_0(u)| = \left|\inf_{\mu \in A([0, T])} \left(H(\mu) + \left\|Y_{\mu} - \frac{1}{\epsilon} \left(x(\theta_0 + \epsilon u) - x(\theta_0)\right)\right\|_{\infty}\right) - \inf_{\mu \in A([0, T])} \left(H(\mu) + \left\|Y_{\mu} - u \hat{x}(\theta_0)\right\|_{\infty}\right)\right|$$

This completes the proof. □
\[
\begin{align*}
&= \left. \inf_{\mu \in A([0,T])} \left( H(\mu) + \left\| Y_{\mu} - \frac{1}{2} \varepsilon u^2 \bar{x}(\bar{\theta}) \right\|_{\infty} \right) \right| \\
&= \inf_{\mu \in A([0,T])} \left( H(\mu) + \left\| Y_{\mu} - \frac{1}{2} \varepsilon u^2 \bar{x}(\bar{\theta}) \right\|_{\infty} \right)
\end{align*}
\]

with \( \bar{\theta} = \bar{\theta}_{t,M} \in (\theta_0, \theta_0 + \varepsilon u) \), where the second equality is because of the Taylor expansion. If we take \( L_\varepsilon = \varepsilon^{d-1} \) with \( \delta \in (1/2, 1) \), we get

\[
\sup_{|u| \leq L_\varepsilon} \left| K_u(u) - K_0(u) \right| = \left. \inf_{\mu \in A([0,T])} \left( H(\mu) + \left\| Y_{\mu} - \frac{1}{2} \varepsilon u^2 \bar{x}(\bar{\theta}) \right\|_{\infty} \right) \right| \\
\leq \sup_{|u| \leq L_\varepsilon} \left[ \frac{1}{2} \varepsilon u^2 \sup_{0 \leq t \leq T} \bar{x}(\bar{\theta}) \right] \leq \frac{\varepsilon L_\varepsilon^2}{2} |x_0| T^2 e^{(\theta_0 + \varepsilon L_\varepsilon)T} \\
= \frac{\varepsilon^{2d-1}}{2} |x_0| T^2 e^{(\theta_0 + \varepsilon L_\varepsilon)T} \to 0 \quad (\varepsilon \to 0).
\]

Therefore, we get the desired results by Lemma 2.1. \( \square \)

In the following, we will consider the limiting behavior of \( \eta_T \) for \( T \to +\infty \). Let us introduce the following notations:

\[
A_t = \int_t^{+\infty} e^{-\theta_0 s} dL_s^d, \\
B_t = \int_0^t e^{-\theta_0 s} dL_s^d.
\]

From Theorem 3.6.6 of Jurek and Mason [13] and Lemma 4 of Diop and Yode [4], we can get the logarithmic moment condition is necessary and sufficient for the existence of the improper integral \( A_0 \).

**Lemma 2.2** Suppose that \( E(\log(1 + |L_1|)) < +\infty \). Then

\[
A_t^d = e^{-\theta_0 s} A_0
\]

where \( A_t^d \) denotes “identical distribution”.

**Proof** It is not hard to see,

\[
A_t = \int_t^{+\infty} e^{-\theta_0 s} dL_s^d = \int_t^{+\infty} (t^d e^{-\theta_0 u}) (s) dL(s) \\
= \int_t^{+\infty} \left( \frac{1}{\Gamma(d)} \right) \int_s^{+\infty} e^{-\theta_0 u} (u - s)^{d-1} du \right) dL(s) \\
= \int_t^{+\infty} \left( \frac{1}{\Gamma(d)} \right) \int_0^{+\infty} e^{-\theta_0 (s(x))} x^{d-1} dx \right) dL(s) \\
= \int_t^{+\infty} \left( \frac{1}{\Gamma(d)} \right) e^{-\theta_0 \theta_0^{-d} \int_s^{+\infty} e^{-\theta_0 x} x^{d-1} d(\theta_0 x)} \right) dL(s) \\
= \theta_0^{-d} \int_t^{+\infty} e^{-\theta_0 s} dL(s).
\]
In a similar way,

\[ A_0 = \theta_0^d \int_0^{+\infty} e^{-\theta_0 s} dL(s). \]

From Lemma 4 of Diop and Yode [4], we have immediately

\[ A_t \equiv e^{-\theta_0 s} A_0. \]

The next theorem gives the asymptotic behavior of the limit distribution \( \eta_T \) for large \( T \).

**Theorem 2.3** Suppose that \( \theta_0 > 0 \) and \( E(\log(1 + |L_1|)) < +\infty \). Then \( \xi_T = x_0 T \eta_T \) converges in distribution to \( A_0 \) as \( T \to +\infty \).

**Proof** Recall that

\[ \eta_T = \arg \min_{u \in R} \rho(Y(\theta_0), u \dot{x}(\theta_0)). \]

By changing variable, we have

\[ \xi_T = \arg \min_{\omega \in R} \rho(Y(\theta_0), M_{\xi}(\omega)) : = \arg \min_{\omega \in R} N(\omega), \quad (18) \]

where \( M_{\xi}(\omega) = \frac{\omega \theta_0 t}{T} \) and \( N(\cdot) = \rho(Y(\theta_0), M(\cdot)) \).

We want to show that, for every \( \Delta > 0 \),

\[ \lim_{T \to +\infty} P_{\theta_0} \{ |\xi_T - A_0| > \Delta \} = 0. \quad (19) \]

Therefore, let us consider the set

\[ V_{\Delta} = \{ \omega : |\omega - A_0| > \Delta \}, \]

where \( P_{\theta_0} \) is the probability measure induced by the process \( X_t \) when \( \theta_0 \) is the true parameter and \( \epsilon \to 0 \). We can get

\[
N(A_0) = \rho(Y(\theta_0), M(A_0)) \leq \left\| Y(\theta_0) - M(A_0) \right\|_{\infty}
\]

\[
= e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - \frac{A_0 t}{T} - A_0 + A_0 \right) \left\|_{\infty}
\]

\[
= e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - A_0 + \left( 1 - \frac{t}{T} \right) A_0 \right) \left\|_{\infty}
\]

\[
= e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - A_0 \right) \left\|_{\infty} + |A_0| t \left( 1 - \frac{t}{T} \right) e^{\theta_0 t} \right\|_{\infty}.
\]

On the other hand, for \( \omega \in V_{\Delta} \), we have

\[
N(\omega) = \rho(Y(\theta_0), M(\omega)) \geq \rho(M(A_0), M(\omega)) - \rho(Y(\theta_0), M(A_0))
\]
\[ M(\omega) - M(A_0) = |\omega - A_0| \frac{te^{\theta t}}{T} \rightarrow -N(A_0) \]

\[ \geq \Delta \frac{te^{\theta t}}{T} \rightarrow N(A_0). \]

Hence, we have

\[ \frac{N(\omega)}{N(A_0)} \geq \Delta \frac{\|te^{\theta t}\|_{\infty}}{N(A_0)} - 1, \]

\[ \inf_{\omega \in V_\Delta} \frac{N(\omega)}{N(A_0)} \geq \Delta \left[ \frac{\| e^{\theta t}(e^{s}A_{x} - A_0)\|_{\infty}}{\|te^{\theta t}\|_{\infty}} + \frac{|A_0| \| (T-t)e^{\theta t}\|_{\infty}}{\|te^{\theta t}\|_{\infty}} \right]^{-1} = 1 \]

\[ = \Delta \left[ \frac{e^{-\theta t} \| e^{\theta t} A_t \|_{\infty} + |A_0|}{T \theta e} \right]^{-1} - 1, \]

where we get the maximum value of the function \((T-t)e^{\theta t}\) by taking the derivative.

We obtain

\[ \frac{|A_0|}{T \theta e} \rightarrow 0 \quad \text{a.s. as } T \rightarrow +\infty. \] (20)

Using Lemma 2.2 we have

\[ P_{\theta_0} (e^{\theta_0 T} \| e^{\theta_0 T} A_t \|_{\infty} > \Delta) = P_{\theta_0} (|A_0| > e^{\theta_0 T} \Delta) \leq e^{-\theta_0 T} E_{\theta_0} (|A_0|) \Delta \rightarrow 0, \quad T \rightarrow +\infty. \] (21)

By (20) and (21), we obtain

\[ \inf_{\omega \in V_\Delta} \frac{N(\omega)}{N(A_0)} \rightarrow +\infty, \quad T \rightarrow +\infty. \] (22)

In addition, using (18), \(\xi_T \in V_\Delta\), we have

\[ N(\xi_T) = \inf_{\omega \in V_\Delta} N(\omega) \leq N(A_0). \] (23)

We can get the desired result (19) by Eqs. (22) and (23).

\section*{3 Minimum L_1-norm estimation}

In this section, we will study the minimum \(L_1\)-norm estimation \(\tilde{\theta_0}\) of the drift parameter \(\theta\). Let

\[ D_T(\theta) = \int_0^T |x_t - x_t(\theta)| \, dt. \] (24)

It is well known that \(\tilde{\theta_0}\) is the minimum \(L_1\)-norm estimator if there exists a measurable selection \(\tilde{\theta_0}\) such that

\[ D_T(\tilde{\theta_0}) = \inf_{\theta \in \Theta} D_T(\theta). \] (25)
Suppose that there exists a measurable selection \( \tilde{\theta} \) satisfying the above equation. We can also define the estimator \( \tilde{\theta}_e \) by the relation

\[
\tilde{\theta}_e = \arg \inf_{\theta \in \Theta} \mathbb{E} \left[ \int_0^T |X_t - x_t(\theta)| \, dt \right].
\] (26)

For any \( \kappa > 0 \), we define

\[
\tilde{f}(\kappa) = \inf_{|\theta - \theta_0| > \kappa} \mathbb{E} \left[ \int_0^T |X_t(\theta) - x_t(\theta_0)| \, dt \right], \quad \text{for any } \kappa > 0.
\] (27)

**Theorem 3.1 (Consistency)** For any \( p \geq 2 \), there exists a constant \( C_{p,K} \) (only depending the \( p, \kappa, \epsilon \)), such that for every \( \kappa > 0 \), we have

\[
P_{\theta_0}^{(\epsilon)} \left( |\tilde{\theta}_e - \theta_0| > \kappa \right) \leq C_{p,K} \mathbb{E} \left[ \left( \frac{\sup_{0 \leq t \leq T} |L^\epsilon_0|}{\tilde{f}(\kappa)} \right)^p \right] = O \left( \frac{\epsilon^p}{\tilde{f}(\kappa)} \right).
\] (28)

**Proof** Set \( \| \cdot \| \) denotes the \( L_1 \)-norm, then we have

\[
P_{\theta_0}^{(\epsilon)} \left( |\tilde{\theta}_e - \theta_0| > \kappa \right) = P_{\theta_0}^{(\epsilon)} \left\{ \inf_{|\theta - \theta_0| \leq \kappa} \mathbb{E} \left[ \int_0^T |X - x(\theta)| \, dt \right] \right\}
\]

\[
\leq P_{\theta_0}^{(\epsilon)} \left\{ \inf_{|\theta - \theta_0| \leq \kappa} \left( \mathbb{E} \left[ \int_0^T |X - x(\theta)| \, dt \right] + \mathbb{E} \left[ \int_0^T |x(\theta) - x(\theta_0)| \, dt \right] \right) \right\}
\]

\[
= P_{\theta_0}^{(\epsilon)} \left\{ \sup_{0 \leq t \leq T} \mathbb{E} \left[ |X - x(\theta)| \right] \geq \frac{1}{2} \tilde{f}(\kappa) \right\}.
\]

Since the process \( X_t \) satisfies the stochastic differential equation (1), it follows that

\[
X_t - x_t(\theta_0) = x_0 + \theta_0 \int_0^T X_s \, ds + \epsilon L^\epsilon_t - x_t(\theta_0) = \theta_0 \int_0^T (X_s - x_t(\theta_0)) \, ds + \epsilon L^\epsilon_t,
\] (29)

where \( x_t(\theta) = x_0 e^{\theta t} \).

Similar to the proof of Theorem 2.1, we have

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ |X_t - x_t(\theta_0)| \right] \leq \epsilon e^{\theta_0 T} \sup_{0 \leq t \leq T} \mathbb{E} \left[ L^\epsilon_t \right].
\] (30)

Thus,

\[
P_{\theta_0}^{(\epsilon)} \left\{ \mathbb{E} \left[ |X - x(\theta)| \right] \geq \frac{1}{2} \tilde{f}(\kappa) \right\} \leq P \left( \sup_{0 \leq t \leq T} \mathbb{E} \left[ L^\epsilon_t \right] \geq \frac{\tilde{f}(\kappa)}{2 \epsilon e^{\theta_0 T}} \right).
\] (31)

Applying Lemma 1.3 to the estimate obtained above, we have

\[
P_{\theta_0}^{(\epsilon)} \left( |\tilde{\theta}_e - \theta_0| > \kappa \right) \leq E \left( \sup_{0 \leq t \leq T} \mathbb{E} \left[ L^\epsilon_t \right] \right)^p \left( \frac{2 \epsilon e^{\theta_0 T}}{\tilde{f}(\kappa)} \right)^p
\]
\[
\leq C_{p,r,d}E[(L(1)^p) T^p(d \frac{1}{2} + r) 2^p e^{\theta_0 T^p} (\tilde{f}(\kappa))^{-p} \tilde{\epsilon}^p] = O((\tilde{f}(\kappa))^{-p} \tilde{\epsilon}^p).
\]

This completes the proof. □

**Remark 3.1** It follows from Theorem 3.1 that we have \( \tilde{\theta}_\epsilon \) converges in probability to \( \theta_0 \) under \( P_{\theta_0}^\epsilon \) measure as \( \epsilon \to 0 \). Furthermore, the rate of convergence is of order \( O(\epsilon^p) \) for every \( p \geq 2 \).

**Theorem 3.2** (Limit distribution) As \( \epsilon \to 0 \),
\[
\epsilon^{-1}(\tilde{\theta}_\epsilon - \theta_0) \xrightarrow{d} \xi,
\]
where \( \xi \) has the same probability distribution as \( \tilde{\eta} \) under \( P_{\theta_0}^\epsilon \).

**Proof** Let
\[
Z_\epsilon(u) = \| Y - \epsilon^{-1}(x(\theta_0 + \epsilon u) - x(\theta_0)) \|
\]
and
\[
Z_0(u) = \| Y - ux(\theta_0) \|.
\]
Furthermore, let
\[
A_\epsilon = \{ \omega : |\tilde{\theta}_\epsilon - \theta_0| < \delta_\epsilon \}, \quad \delta_\epsilon = \epsilon^\tau, \tau \in \left( \frac{1}{2}, 1 \right), \quad L_\epsilon = \epsilon^{-1}.
\]
It is easy to see that the random variable \( \tilde{\eta}_\epsilon = \epsilon^{-1}(\tilde{\theta}_\epsilon - \theta_0) \) satisfies the equation
\[
Z_\epsilon(\tilde{\eta}_\epsilon) = \inf_{|u| \leq L_\epsilon} Z_\epsilon(u), \quad \omega \in A_\epsilon.
\]
Define
\[
\tilde{\eta}_\epsilon = \arg \inf_{|u| \leq L_\epsilon} Z_0(u).
\]
Observe that, with probability one,
\[
\sup_{|u| \leq L_\epsilon} |Z_\epsilon(u) - Z_0(u)| = \| Y - \epsilon^{-1}(x(\theta_0) - 1/2\epsilon u^2 \tilde{\xi}(\tilde{\theta})) - \| Y - \epsilon^{-1}(x(\theta_0)) \|
\leq \frac{\epsilon}{2} L_\epsilon^2 \sup_{|\theta - \theta_0| \leq \delta_\epsilon} \int_0^T |\tilde{\xi}(\theta)| dt \leq C \epsilon^{2\tau - 1} \to 0, \quad \epsilon \to 0,
\]
where \( \tilde{\theta} = \theta_0 + \alpha(\theta - \theta_0) \) for some \( \alpha \in (0, 1) \). Note that the last term in the above inequality tends to zero as \( \epsilon \to 0 \). This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian [14, 15]. In addition, we can choose the interval \([-L, L] \) such that
\[
P_{\theta_0}^\epsilon \{ u_\epsilon^* \in (-L, L) \} \geq 1 - \frac{\epsilon}{\tilde{\epsilon}^p} (\tilde{f}(L)^{-p}
\]
and

\[ P\{u^* \in (-L, L)\} \geq 1 - \beta \tilde{f}(L)^p, \quad \beta > 0. \]  \hfill (40)

Note that \( \tilde{f}(L) \) increases as \( L \) increases. The process \( \{Z_t(u), u \in [-L, L]\} \) and \( \{Z_0(u), u \in [-L, L]\} \) satisfy the Lipschitz conditions and \( Z_t(u) \) converges uniformly to \( Z_0(u) \) over \( u \in [-L, L] \). Hence the minimizer of \( Z_t(\cdot) \) converges to the minimizer of \( Z_0(\cdot) \). This completes the proof. \( \square \)

Although the distribution of \( \tilde{\eta} \) is not clear, we can consider its limiting behaviors as \( T \to +\infty \).

**Theorem 3.3** (Asymptotic law) Suppose that \( \theta_0 > 0 \) and \( E(\log(1 + |L_1|)) < +\infty \). Then

\[ \tilde{\xi}_T = x_0 T \tilde{\theta}_T \overset{d}{\to} A_0, \quad T \to +\infty, \]

where \( L_1, A_0 \) and other notations in the following are the same as Theorem 2.3.

**Proof** Recall that

\[ \tilde{\eta}_T = \arg \inf_{u \in \mathbb{R}} \int_0^T \| Y_t(\theta_0) - utx_0 e^{\theta_0 t} \| dt. \]

Let \( \| \cdot \| \) denote the \( L_1 \)-norm. By changing variable, we have the following:

\[ \tilde{\xi}_T = \arg \inf_{\omega \in \mathbb{R}} \| Y - \tilde{M}(\omega) \| := \arg \inf_{\omega \in \mathbb{R}} \tilde{N}(\omega), \]  \hfill (41)

where \( \tilde{M}_t(\omega) = \frac{\omega e^{\theta_0 t}}{T} \) and \( \tilde{N}(\cdot) = \| Y - \tilde{M}(\cdot) \| \).

We want to show that, for every \( \Delta > 0 \),

\[ \lim_{T \to +\infty} P_{\theta_0} \{ |\tilde{\xi}_T - A_0| > \Delta \} = 0. \]  \hfill (42)

Therefore, we consider the set

\[ V_\Delta = \{ \omega : |\omega - A_0| > \Delta \}, \]

where \( P_{\theta_0} \) is the probability measure induced by the process \( X_t \) when \( \theta_0 \) is the true parameter and \( \varepsilon \to 0 \).

Besides, we have

\[ \tilde{N}(A_0) = \| Y - \tilde{N}(A_0) \| \]
\[ = \| e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - \frac{A_0 t}{T} + A_0 + A_0 \right) \| \]
\[ = \| e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - A_0 + \left( 1 - \frac{t}{T} \right) A_0 \right) \| \]
\[ = \| e^{\theta_0 t} \left( \int_0^t e^{-\theta_0 s} dL_s^d - A_0 \right) \| + |A_0 t| \left\| \left( 1 - \frac{t}{T} \right) e^{\theta_0 t} \right\|. \]
On the other hand, for $\omega \in V_\Delta$, we can get

$$\tilde{N}(\omega) = \|Y - \tilde{M}(\omega)\|$$

$$\geq \|\tilde{M}(A_0) - \tilde{M}(\omega)\| - \|Y - \tilde{M}(A_0)\|$$

$$= \|\tilde{M}(\omega) - \tilde{M}(A_0)\| - \tilde{N}(A_0)$$

$$= |\omega - R_0| \frac{\|te^{\theta_0t}\|}{T} - \tilde{N}(A_0)$$

$$\geq \Delta \frac{\|te^{\theta_0t}\|}{T} - \tilde{N}(A_0).$$

Obviously, we have

$$\frac{\tilde{N}(\omega)}{\tilde{N}(A_0)} \geq \Delta \frac{\|te^{\theta_0t}\|}{\tilde{N}(A_0)} - 1,$$

$$\inf_{\omega \in V_\Delta} \frac{\tilde{N}(\omega)}{\tilde{N}(A_0)} \geq \Delta \left[ \frac{T\|e^{\theta_0t}(\int_0^t e^{-\theta_0s} dM^d - A_0)\| + |A_0||T - te^{\theta_0t}\| \right]^{-1}$$

$$= \Delta \left[ T\frac{\|e^{\theta_0t}(B_t - A_0)\|}{\|te^{\theta_0t}\|} + |A_0||T - te^{\theta_0t}\| \right]^{-1} - 1$$

$$= \Delta (I_1 + I_2)^{-1} - 1,$$

with

$$I_1 = \frac{T\|e^{\theta_0t}(B_t - A_0)\|}{\|te^{\theta_0t}\|} = \frac{T\|e^{\theta_0t} A_t\|}{\int_0^T te^{\theta_0t} dt} = \frac{T\|e^{\theta_0t} A_t\|}{\theta_0^1 Te^{\theta_0T} - \theta_0^2 e^{\theta_0T} + \theta_0^2},$$

$$I_2 = \frac{|A_0||T - te^{\theta_0t}\|}{\|te^{\theta_0t}\|} = \frac{|A_0|\int_0^T e^{\theta_0t} dt}{\int_0^T te^{\theta_0t} dt} = \frac{|A_0|(\theta_0 - e^{\theta_0T} - \theta_0 T - \theta_0^2)}{\theta_0^1 Te^{\theta_0T} - \theta_0^2 e^{\theta_0T} + \theta_0^2.}$$

We obtain with probability one

$$\lim_{T \to +\infty} I_2 = \lim_{T \to +\infty} \frac{|A_0|(\theta_0 - e^{\theta_0T} - \theta_0 T - \theta_0^2)}{\theta_0^1 Te^{\theta_0T} - \theta_0^2 e^{\theta_0T} + \theta_0^2}$$

$$= \lim_{T \to +\infty} |A_0| \frac{\theta_0 - e^{\theta_0T}}{\theta_0^1 Te^{\theta_0T}} = \lim_{\theta_0 \to +\infty} \frac{|A_0|}{\theta_0 T} = 0. \quad (43)$$

Moreover, using Lemma 2.2 we obtain

$$\lim_{T \to +\infty} P_{\theta_0}(I_1 > \Delta) = \lim_{T \to +\infty} P_{\theta_0} \left( \frac{T\|e^{\theta_0t} R_t\|}{\theta_0^1 Te^{\theta_0T} - \theta_0^2 e^{\theta_0T} + \theta_0^2} > \Delta \right)$$

$$= \lim_{T \to +\infty} P_{\theta_0} \left( \frac{T\|e^{\theta_0t} R_t\|}{\theta_0^1 Te^{\theta_0T}} > \Delta \right)$$

$$= \lim_{T \to +\infty} P_{\theta_0} (|R_0| > \theta_0 e^{\theta_0T} \Delta) \leq \lim_{T \to +\infty} \theta_0^{-1} e^{-\theta_0T} E_{\theta_0}(|R_0|) \Delta = 0. \quad (44)$$
By (43) and (44), we obtain as $T \to +\infty$

$$\inf_{\omega \in V_\Delta} \widetilde{N}(\omega) \sim N(A_0) \quad p \to +\infty. \tag{45}$$

Using (41), $\widetilde{\xi}_T \in V_\Delta$ implies

$$\widetilde{N}(\xi_T) = \inf_{\omega \in V_\Delta} \widetilde{N}(\omega) \leq \widetilde{N}(A_0). \tag{46}$$

Therefore, from Eqs. (45) and (46), we have the result (42). $\square$

**Remark 3.2** If $L^d_t$ is a Brownian motion, then $\widetilde{\xi}_T$ is asymptotically Gaussian, this is treated by Kutoyants and Pilibossian [14, 15].

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**Competing interests**
The authors declare that they have no competing interests.

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