On the Strong Feller Property and Well-Posedness for SDEs with Functional, Locally Unbounded Drift

Stefan Bachmann

Email: bachmann@math.uni-leipzig.de
Institut für Mathematik, Universität Leipzig,
Augustusplatz 10, 04109 Leipzig, Germany

August 20, 2018

Abstract. We study functional stochastic differential equations with a locally unbounded, functional drift focusing on well-posedness, stability and the strong Feller property. Following the non-functional case, we only consider integrability conditions and avoid continuity assumptions as far as possible. Our approach is mainly based on Zvonkin’s transformation [18], an extended version of the probabilistic approach of Maslowski and Seidler [12] and the convergence concept for random variables in topological spaces in [2].

Keywords: stochastic delay differential equations, stochastic functional differential equations, strong Feller property, pathwise uniqueness, regularization by white noise, singular drift, unbounded drift

MSC 2010: primary 34K50; secondary 60B10, 60B12, 60H10.

1. Introduction

In this paper, we consider stochastic functional differential equations of the following form

\[ \begin{align*}
    dX^x(t) &= B(t, X^x(t)) \, dt + \sigma(t, X^x(t)) \, dW(t) \\
    X_0 &= x \in \mathcal{C}
\end{align*} \tag{1} \]

where \( W \) is a \( d \)-dimensional Brownian motion, \( B : \mathbb{R}_{\geq 0} \times C(\mathbb{R}_{\geq -r}, \mathbb{R}^d) \rightarrow \mathbb{R}^d \) is non-anticipating and \( \sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \) is measurable, bounded, non-degenerate and Lipschitz in space.
Non-functional stochastic differential equations (SDEs) with discontinuous drift has been extensively studied: Portenko [13], Veretennikov [15] and Zvonkin [18] considered - among other things - well-posedness for SDEs with bounded, discontinuous drift terms. Krylov and Röckner have shown existence and uniqueness for locally unbounded drifts and constant, non-degenerate diffusion coefficients in [10]. Singular SDEs with non-constant, non-degenerate diffusion matrices have been studied by Martínez, Gyöngy [7] and Zhang [17]. Additionally, there are numerous results for the strong Feller property for non-functional, singular SDEs with the euclidean state space $\mathbb{R}^d$ (i.e. [17]).

However, we are interested in the strong Feller property for functional SDEs with the state space of path segments $C([-r, 0], \mathbb{R}^d)$ for some $r > 0$. Es-Sarhir, von Renesse and Scheutzow established a Harnack-inequality under Lipschitz conditions and constant, non-degenerate diffusion matrices in [4], which implies the strong Feller property. Wang and Yuan proved a log-Harnack inequality for non-constant, non-degenerate diffusion coefficients. In [1] and [8] well-posedness has been considered for SDEs with a drift consisting of a functional part and a non-functional, locally unbounded part. The strong Feller property has been shown in [2].

To prove the strong Feller property for functional, locally unbounded drifts, we follow an extended version of the probabilistic approach of Maslowski and Seidler [12]. Analogously to the non-functional case, the proofs for well-posedness and stability are based on Zvonkin’s transformation [18]. In both cases, we extensively make use of Krylov’s estimate for semimartingales [9] and the convergence concept for random variables in topological spaces from [2].

**Notation 1.1.** If not stated otherwise, $W$ will be a $d$-dimensional Brownian motion on some arbitrary but fixed probability space $(\Omega, \mathcal{F}, P)$ and every strong solution shall be defined on this space.

However, weak solutions of equation (11) might be defined on different filtrated probability spaces. Therefore, we use the short hand notation $(X^x, \tilde{W}^x, Q^x)$ where $X^x$ is an adapted, continuous stochastic process, $\tilde{W}^x$ is an adapted Brownian motion, both with respect to some filtrated probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{Q}, (\tilde{\mathcal{F}}_t)_{t \geq 0})$, and $(X^x, \tilde{W}^x)$ solves equation (11) with initial value $x$.

**Condition C1.** For each $T > 0$ there exist an $F \in L^{d+1}([0, T] \times \mathbb{R}^d)$ and $C_1 = C_1(T), C_2 = C_2(T) \geq 0$ with

$$
\int_0^t |B(s, x)|^2 \, ds \leq \int_0^t |F(s, x(s))| \, ds + C_1 \sup_{s \in [-r, t]} |x(s)|^2 + C_2
$$

for all $t \in [0, T]$ and $x \in C(\mathbb{R}_{\geq -r}, \mathbb{R}^d)$.

**Condition C2.** Assume that for all $T > 0$ there exists some $C_\sigma = C_\sigma(T) > 0$ such that

1. $C_\sigma^{-1} I_{d \times d} \leq \sigma(t, x)\sigma(t, x)^\top \leq C_\sigma I_{d \times d} \forall t \in [0, T], x \in \mathbb{R}^d,$
2. $||\sigma(t, x) - \sigma(t, y)||_{HS} \leq C_\sigma |x - y| \forall t \in [0, T], x, y \in \mathbb{R}^d.$
**Condition C3.** Assume that there is an \( r_{\tilde{B}} \in (0, r) \) such that
\[
B(t, x) = \tilde{B}(t, x) + b(t, x(t))
\]
with \( b \in L^{2d+2} (\mathbb{R}_{\geq 0} \times \mathbb{R}^d; \mathbb{R}^d) \) and \( \tilde{B} : \mathbb{R}_{\geq 0} \times C (\mathbb{R}_{\geq -r} \times \mathbb{R}^d) \rightarrow \mathbb{R}^d \) measurable where, for fixed \( t \geq 0 \), \( \tilde{B}(t, x) \) depends only on \( x_{[-r, t-r_{\tilde{B}}]} \), i.e.
\[
\tilde{B}(t, x) = \tilde{B}(t, y) \text{ if } x(s) = y(s) \forall s \in [-r, t - r_{\tilde{B}}].
\]

**Condition C4.** For \( t \in [0, r) \) the function \( x \mapsto B(t, x) \) is continuous. Moreover, for each \( T > 0 \) there exist functions \( \tilde{F} \in L^{d+1}_{\text{loc}} ([0, T] \times \mathbb{R}^d) \) and \( G, H : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) with \( G \) monotone increasing and
\[
\lim_{R \rightarrow \infty} \frac{H(R)}{R} = \infty
\]
such that
\[
\int_0^t H \left( |B(s, x)|^2 \right) \, ds \leq \int_0^t \left| \tilde{F}(s, x(s)) \right| \, ds + G \left( \sup_{s \in [-r, t]} |x(s)| \right)
\]
for all \( t \in [0, T] \) and \( x \in C (\mathbb{R}_{\geq -r}, \mathbb{R}^d) \).

**Notation 1.2.** In the sequel, let \( r > 0 \) be an arbitrary but fixed number and define
\[
C := C \left( [-r, 0], \mathbb{R}^d \right)
\]
equipped with the supremum norm \( \| \cdot \|_\infty \). For a process \( X \) defined on \([t-r, t]\) with \( t \geq 0 \), we write
\[
X_t(s) := X(t + s), \quad s \in [-r, 0].
\]

**Condition C5.** The non-anticipating function \( B \) has bounded memory, i.e. it holds
\[
B(t, x) = B(t, y) \text{ if } x(s) = y(s) \forall s \in [t, t - r].
\]
Then we use the abuse of notation
\[
B(t, x_t) = B(t, x) \forall x \in C \left( \mathbb{R}_{\geq -r}, \mathbb{R}^d \right)
\]
and similarly for \( \tilde{B} \) if \( \text{(C3)} \) is satisfied.

The main results read as follows.

**Theorem 1.3 (Existence).** Assume (C1) and (C2). Then for each initial value \( x \in C \), equation (II) has a global weak solution \((X^x, \tilde{W}^x, Q^x)\), which is unique in distribution.

**Theorem 1.4 (Pathwise Uniqueness).** Assume the localized versions of (C1), (C2) and (C3). Then local pathwise uniqueness holds for equation (II), i.e. let \((X^x, W)\) and \((\tilde{X}^x, \tilde{W})\) be two weak solutions of equation (II) with initial value \( x \in C \) on some time interval \([0, \tau]\) for some common Brownian motion \( W \) and stopping time \( \tau \). Then it follows \( X^x = \tilde{X}^x \) on \([0, \tau]\) almost surely.
Theorem 1.5 (Strong Feller Property). Assume \((C1), (C2), (C4)\) and \((C5)\). Let \((X^x, W^x, Q^x)\) be weak solutions with initial value \(x \in C\). Then one has the strong Feller property for all \(t > r\), i.e.

\[
\lim_{y \to x} E_{Q^y} f(X^y_t) = E_{Q^x} f(X^x_t) \forall f \in B_b(C).
\]

Theorem 1.6 (Stability). Assume \((C1), (C2), (C3), (C4)\) and \((C5)\). Let \(X^x\) be the strong solutions with initial value \(x \in C\). Then one has

\[
\lim_{y \to x} E \|X^y_t - X^x_t\|_\gamma = 0
\]

for all \(0 < \gamma < 2\) and for \(t > r\)

\[
\lim_{y \to x} E |f(X^y_t) - f(X^x_t)| = 0 \forall f \in B_b(C).
\]

Remark 1.7.

1. Conditions \((C1)\) and \((C4)\) are closed under linear combinations.

2. Assume, one has

\[
B(t, x_t) = \int_{-r}^{0} k(t, x(t + s)) d\mu(s)
\]

for some Borel measure \(\mu\) on \([-r, 0]\). Then \((C1)\) is fulfilled if \(k\) is of at most linear growth on \([0, r)\) and

\[
k \in L^{2d+2} \left([0, T] \times \mathbb{R}^d\right) \forall T > 0.
\]

If \(x \mapsto k(t, x)\) is additionally continuous for \(t \in [0, r)\) then condition \((C4)\) will be satisfied. The assumption \(\text{supp}\ \mu \subset [-r, -r_B]\) for some \(r_B \in (0, r)\) implies \((C3)\).

3. The continuity assumption in \((C4)\) is not artificial. Consider the following equation

\[
dX^x(t) = \text{sgn}(X^x(t - 1)) dt + dW(t)
\]

with

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

Then we have for the strong solutions

\[
X^0(1) = W(1) + 1,
\]

\[
X^{-1/n}(1) = W(1) - 1 - \frac{1}{n}
\]

where we denoted constant paths by real numbers. All conditions are fulfilled but the continuity assumption on the interval \([0, r)\). However, neither the strong Feller property nor convergence in probability hold.
2. A-priori Estimates and Existence

In the sequel, denote by $M^x$, $x \in C$ the global, unique strong solution of
\[
\begin{align*}
\text{d}M^x(t) &= \sigma(t, M^x(t)) \, \text{d}W(t), \\
M^x_0 &= x.
\end{align*}
\]

**Notation 2.1.** We denote by $\|\cdot\|_{OP}$ and $\|\cdot\|_{HS}$ the operator norm and respectively the Hilbert-Schmidt norm for matrices $A \in \mathbb{R}^{d \times d}$, i.e.
\[
\|A\|_{op} = \sup_{v \in \mathbb{R}^d, |v|=1} |Av|, \quad \|A\|_{HS} = \sqrt{\sum_{i,j=1}^d |A_{i,j}|^2}.
\]
Additionally, we write for $a, b \in [-\infty, +\infty]$
\[
a \land b := \min\{a, b\}, \quad a \lor b := \max\{a, b\}.
\]

**Remark 2.2.** Condition (C2) implies the following inequalities
\[
\|\sigma\|_{op}, \|\sigma^{-1}\|_{op} \leq \sqrt{C_\sigma}.
\]

**Lemma 2.3.** Assume (C2) and let $T > 0$, $p > \frac{d+2}{2}$ be given. Then one has for all $0 \leq S < T$ and $f \in L_p([S, T] \times \mathbb{R}^d)$ the estimate
\[
\mathbb{E} \left( \int_S^T f(t, M^x(t)) \, \text{d}t \right) \leq C \|f\|_{L_p([S, T] \times \mathbb{R}^d)}
\]
for some constant $C = C(d, p, T, C_\sigma)$. In particular, the constant $C$ is independent of the initial value $x \in C$.

**Proof.** This follows directly from Theorem 2.1 in [17]. \qed

**Lemma 2.4.** Assume (C2). Then for any $R, T > 0$ and $p > \frac{d+2}{2}$ there exists a constant $C_R = C_R(d, p, T, C_\sigma)$ such that
\[
\mathbb{E} \exp \left( \int_0^T f(t, M^x(t)) \, \text{d}t \right) \leq C_R
\]
for all $f \in L_p([0, T] \times \mathbb{R}^d)$ with $\|f\|_{L_p([0, T] \times \mathbb{R}^d)} \leq R$.

**Proof.** See Lemma 2.1 in [17]. \qed

**Lemma 2.5.** Assume (C2). Then for any $T > 0$ and $0 \leq \alpha < (2dC_\sigma T)^{-1}$, it holds
\[
\mathbb{E} \exp \left( \alpha \sup_{0 \leq t \leq T} |M^x(t)|^2 \right) \leq \frac{4}{\sqrt{1-2\alpha dC_\sigma T}} \exp \left( \frac{\alpha}{1-2\alpha dC_\sigma T} |x(0)|^2 \right).
\]
Proof. See Lemma 2.4 in [1]. □

Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\) be a filtrated probability space, \(S\) be a continuous, adapted, \(\mathbb{R}^d\)-valued process and \(A\) be a continuous, adapted, increasing process, which both have finite variation. Furthermore, let \(m\) be a local martingale with \(S(0) = A(0) = m(0) = 0\) and \(d(m)(t) \ll dA(t)\). Let \(r\) and \(c\) be nonnegative, progressively measurable stochastic processes such that

\[
y(t) := \int_0^t r(s) \, ds, \quad \varphi(t) := \int_0^t c(s) \, dA(s)
\]

are finite almost surely for all \(t \geq 0\). Set

\[
a^{ij}(t) := \frac{1}{2} \frac{d(m^1, m^j(t))}{dA(t)}, \quad X(t) := m(t) + S(t)
\]

and let \(\tau_R\) be the first exit time of \(X(t)\) from the ball \(B_R\).

**Lemma 2.6** (Krylov’s Estimate). For every \(p \geq d\), stopping time \(\gamma\) and nonnegative Borel function \(f : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}\) one has

\[
E \int_0^{\gamma \wedge \tau_R} e^{-r(t)} \left( c(t)^{p-1} r(t) \det a(t) \right)^\frac{1}{p+1} f(y(t), X(t)) \, dA(t)
\]

\[
\leq N(d) \left( S^2 + A \right)^{\frac{d}{2(p+1)}} \left( \int_0^\infty \int_{|x| \leq R} f^{p+1}(t, x) \, dx \, dt \right)^\frac{1}{p+1}
\]

where

\[
A := E \int_0^{\gamma \wedge \tau_R} e^{-r(t)} \, tr \, a(t) \, dA(t), \quad S := E \int_0^{\gamma \wedge \tau_R} e^{-r(t)} \, |dS(t)|
\]

and \(N(d)\) is a constant depending only on the dimension \(d\) (with the convention \(c(t)^0 = 1\)).

**Corollary 2.7.** Assume (C1), (C2) and let \(T > 0\). Furthermore, let \((X^x, \tilde{W}^x, Q^x)\) be a solution of equation (1) on some time interval \([-r, T]\) where \(\tau\) is some stopping time with \(0 \leq \tau \leq T\). Then one has

\[
E \int_0^\tau |B(t, X^x)|^2 \, dt \leq 2C_1 E \left[ \sup_{-r \leq t \leq \tau} |X^x(t)|^2 \right] + C
\]

where \(C = C \left( d, T, C_2, C_\sigma, \|F\|_{L^{d+1}([0,T] \times \mathbb{R}^d)} \right)\) is some constant.

Proof. The proof is similar to the one of Corollary 3.2. in [1]. By Krylov’s estimate and Young’s inequality, one has

\[
E \int_0^\tau |B(t, X^x)|^2 \, dt \leq E \int_0^\tau |F(t, x(t))| \, dt + C_1 E \left[ \sup_{-r \leq t \leq \tau} |X^x(t)|^2 \right] + C_2
\]

\[
\leq \frac{1}{2} E \int_0^\tau |B(t, X^x)|^2 \, dt + C_1 E \left[ \sup_{-r \leq t \leq \tau} |X^x(t)|^2 \right] + C
\]

where \(C = C \left( d, T, C_2, C_\sigma, \|F\|_{L^{d+1}([0,T] \times \mathbb{R}^d)} \right)\). □
Corollary 2.8. Assume \((C1), (C2)\) and let \(T > 0\). Furthermore, let \((X^x, \tilde{W}^x, Q^x)\) be a solution of equation (1) on some time interval \([-r, \tau]\) where \(\tau\) is some stopping time with \(0 \leq \tau \leq T\). Then one has
\[
E \left[ \sup_{-r \leq t \leq \tau \wedge T} |X^x(t)|^2 \right] \leq C \left( 1 + \|x\|_\infty^2 \right)
\]
where \(C = C \left( d, T, C_1, C_2, C_\sigma, \|F\|_{L^{d+1}(\mathbb{R}^d)} \right)\) is some constant.

Proof. Applying Gronwall’s lemma and Doob’s maximal inequality.

Corollary 2.9. Assume \((C1), (C2)\) and let \(T > 0\). Moreover, let \((X^x, \tilde{W}^x, Q^x)\) be a weak solution of equation (1) on \([-r, \tau]\) for some stopping time \(0 \leq \tau \leq T\). Then for any Borel function \(f : \mathbb{R}^{d+1} \to \mathbb{R} \geq 0\) and \(q \geq d + 1\), one has
\[
E \int_0^T f(t, X^x(t)) \, dt \leq N \|f\|_{L^q(T)}
\]
where \(N = N \left( d, T, C_1, C_2, C_\sigma, \|F\|_{L^{d+1}(\mathbb{R}^d)} , \|x\|_\infty \right)\) is a constant.

Proof. This follows directly from Krylov’s estimate and the Corollaries before.

Theorem 2.10. Assume \((C1)\) and \((C2)\). Then for every initial values \(x \in C\), equation (1) has a global weak solution. Moreover, for each weak solution \((X^x, \tilde{W}^x, Q^x)\) of equation (1) on some time interval \([-r, T]\), \(T > 0\), one has
\[
Q^x_{X^x}(A) = E_P \left[ 1_A(M^x) \exp \left( \int_0^T a^x(t)^\top dW(t) - \frac{1}{2} \int_0^T |a^x(t)|^2 \, dt \right) \right],
\]
where \(a^x(t) := \sigma(t, M^x(t))^{-1}B(t, M^x), t \in [0, T]\) for all measurable \(A \subset C([-r, T], \mathbb{R}^d)\).

Proof. At first, we show the existence of a weak solution. The strong solution \(M^x\) is by definition \((\mathcal{F}_t)_{t \geq 0}\)-adapted where \((\mathcal{F}_t)_{t \geq 0}\) is the augmented filtration generated by \(W\). Next, we construct a probability measure on
\[
\mathcal{F}_\infty := \sigma(\mathcal{F}_t : t \geq 0)
\]
such that \(M^x\) is a global weak solution for equation (1). By Lemma 2.4 Lemma 2.5 conditions \((C1)\) and \((C2)\), there exist for each \(T > 0\) a partition \(0 = T_0 \leq T_1, \cdots \leq T_{n-1} \leq T_n = T, n \in \mathbb{N}\) with
\[
E_P \exp \left( \frac{1}{2} \int_{T_{i-1}}^{T_i} |\sigma(t, M^x(t))^{-1}B(t, M^x)|^2 \, dt \right) < \infty, \quad i = 1, \ldots, n.
\]
Therefore, Novikov’s condition is fulfilled for each subinterval, which gives that
\[ t \mapsto \exp \left( \int_{t \wedge T_i}^{t \wedge T_{i-1}} (\sigma(s, M^x(s))^{-1} B(s, M^x))^\top dW(s) \right. \]
\[ \left. - \frac{1}{2} \int_{t \wedge T_i}^{t \wedge T_{i-1}} |\sigma(s, M^x(s))B(s, M^x)|^2 ds \right) \]
is a martingale for \( i = 1, \ldots, n \). Consequently,
\[ t \mapsto \exp \left( \int_0^t (\sigma(s, M^x(s))^{-1} B(s, M^x))^\top dW(s) \right. \]
\[ \left. - \frac{1}{2} \int_0^t |\sigma(s, M^x(s))B(t, M^x)|^2 ds \right) \]
is a martingale and by Girsanov’s theorem,
\[ \tilde{W}(t) := W(t) - \int_0^t \sigma(s, M^x(s))^{-1} B(s, M^x) ds, \quad t \geq 0 \]
is a Brownian motion on \([0, T]\) under the probability measure
\[ d\tilde{\mathbb{P}}_T := \exp \left( \int_0^T (\sigma(t, M^x(t))^{-1} B(t, M^x))^\top dW(t) \right. \]
\[ \left. - \frac{1}{2} \int_0^T |\sigma(t, M^x(t))B(t, M^x)|^2 dt \right) d\mathbb{P} \]
and \((M^x, \tilde{W}, \tilde{\mathbb{P}}_T)\) is a weak solution of \(\Pi\) on \([-r, T]\) for each \( T > 0 \). Additionally, one has for \(0 < T_1 < T_2\)
\[ \mathbb{P}_{T_1}(A) = \mathbb{P}_{T_2}(A) \quad \forall A \in \mathcal{F}_{T_1}, \]
so the probability measure on \(\mathcal{F}_\infty\) uniquely defined by
\[ \tilde{\mathbb{P}}(A) := \mathbb{P}_T(A) \quad \forall T > 0, A \in \mathcal{F}_T \]
is indeed well-defined and \((M^x, \tilde{W}, \tilde{\mathbb{P}})\) is a global weak solution.

Now, let \((X^x, \tilde{W}^x, Q^x)\) be a weak solution on some time interval \([0, T]\), \( T > 0 \). The following approach is inspired by the techniques used in \(\Pi\). Define
\[ \tau^n(\omega) := \inf \left\{ t \geq 0 : \int_0^t |B(s, \omega)|^2 ds \geq n \right\} \land T, \quad \omega \in C([-r, T], \mathbb{R}^d), \quad n \in \mathbb{N}. \]
Then the stopped process \(X^{x,n}(t) := X^x(t \wedge \tau^n(X^x)), t \in [-r, T]\) fulfills the equation
\[ dX^{x,n}(t) = \mathbf{1}_{\tau^n(X^x) \leq t} B(t, X^{x,n}) dt + \mathbf{1}_{\tau^n(X^x) \leq t} \sigma(t, X^{x,n}) d\tilde{W}^x \]
By construction, Novikov’s condition is fulfilled. Consequently, Girsanov’s theorem is applicable and
\[ \tilde{W}^{x,n}(t) := \int_0^{t \wedge \tau^n(X^{x,n})} \sigma(s, X^{x,n}(s))^{-1} B(s, X^{x,n}) ds + \tilde{W}^x(t), \quad t \geq 0 \]
is a Brownian motion with respect to the probability measure
\[
\mathbb{Q}^x_{n,t} := \exp\left( - \int_{0}^{t} \sigma(t, X^{x,n}(t))^{-1} B(t, X^{x,n}(t))^{\top} d\tilde{W}^x(t) \right.
\]
\[
\left. - \frac{1}{2} \int_{0}^{t} |\sigma(t, X^{x,n}(t)) B(t, X^{x,n}(t))|^{2} dt \right) d\mathbb{Q}^x.
\]

The process \( X^{x,n} \) solves the equation
\[
dX^{x,n}(t) = \sigma(t, X^{x,n}(t)) d\tilde{W}^{x,n}(t), \ t \in [0, \tau^{n}(X^{x,n})],
\]
\[X^{x,n}_0 = x.\]

Such a solution is (locally) pathwise unique, i.e.
\[
X^{x,n}(t) = M^{x,n}(t), \ t \in [-r, \tau^{n}(X^{x,n})]
\]
where \( M^{x,n} \) is the unique strong solution of
\[
dM^{x,n}(t) = \sigma(t, M^{x,n}(t)) d\tilde{W}^{x,n}(t),
\]
\[M^{x,n}_0 = x.\]

and it holds
\[
\tau^{n}(X^{x,n}) = \tau^{n}(M^{x,n}) \text{ a.s.}
\]

Moreover, \( \mathbb{Q}^x \) and \( \mathbb{Q}^{x,n} \) are equivalent. Thus,
\[
\mathbb{Q}^x(X^x \in A) = \lim_{n \to \infty} \mathbb{Q}^x(\tau^n(X^x) = T, X^x \in A)
\]
\[
= \lim_{n \to \infty} \mathbb{E}^{\mathbb{Q}^x} \left[ \mathbb{1}_{\tau^n(X^x,n)=T} \mathbb{1}_{A}(X^{x,n}) \exp\left( \int_{0}^{T} \sigma(t, X^{x,n}(t))^{-1} B(t, X^{x,n}(t))^{\top} d\tilde{W}^{x,n}(t) \right. \right.
\]
\[
\left. - \frac{1}{2} \int_{0}^{T} |\sigma(t, X^{x,n}(t))^{-1} B(t, X^{x,n}(t))|^{2} dt \right) \right]
\]
\[
= \lim_{n \to \infty} \mathbb{E}^{\mathbb{Q}^x} \left[ \mathbb{1}_{\tau^n(M^{x,n})=T} \mathbb{1}_{A}(M^{x}) \exp\left( \int_{0}^{T} \sigma(t, M^{x,n}(t))^{-1} B(t, M^{x,n}(t))^{\top} d\tilde{W}^{x,n}(t) \right. \right.
\]
\[
\left. - \frac{1}{2} \int_{0}^{T} |\sigma(t, M^{x,n}(t))^{-1} B(t, M^{x,n}(t))|^{2} dt \right) \right]
\]
\[
= \lim_{n \to \infty} \mathbb{E}^{P} \left[ \mathbb{1}_{\tau^n(M^x)=T} \mathbb{1}_{A}(M^{x}) \exp\left( \int_{0}^{T} \sigma(t, M^x(t))^{-1} B(t, M^{x}(t))^{\top} dW^{x}(t) \right. \right.
\]
\[
\left. - \frac{1}{2} \int_{0}^{T} |\sigma(t, M^x(t))^{-1} B(t, M^{x}(t))|^{2} dt \right) \right]
\]
\[
= \mathbb{E}^{P} \left[ \mathbb{1}_{A}(M^{x}) \exp\left( \int_{0}^{T} \sigma(t, M^x(t))^{-1} B(t, M^{x}(t))^{\top} dW^{x}(t) \right. \right.
\]
\[
\left. - \frac{1}{2} \int_{0}^{T} |\sigma(t, M^x(t))^{-1} B(t, M^{x}(t))|^{2} dt \right) \right]
\]
for all measurable $A \subset C([-r, T], \mathbb{R}^d)$. \hfill \square

**Lemma 2.11.** Assume $\mathbf{(C1)}$ with $C_1 = 0$, $\mathbf{(C2)}$ and let $T > 0$, $q \geq d + 1$ be given. Moreover, let $(X^x, \tilde{W}^x, Q^x)$ be a weak solution of equation (1) on $[-r, \tau]$ for some stopping time $0 \leq \tau \leq T$. Then one has

$$
\sup_{f \in L^q([0, T] \times \mathbb{R}^d): \|f\|_q \leq R} \mathbb{E}_{Q^x} \exp \left( \int_0^T f(t, X^x(t)) \, dt \right) < \infty
$$

for all $R > 0$.

**Proof.** Let

$$a^x(t) := \sigma(t, M^x(t))^{-1} B(t, M^x), \quad t \in [0, T].$$

Analogous proceeding as in proof of Theorem 2.10 gives

$$
\mathbb{E}_{Q^x} \exp \left( \int_0^T f(t, X^x(t)) \, dt \right) \leq \mathbb{E}_P \exp \left( \int_0^T f(t, M^x(t)) \, dt + \int_0^T a^x(t)^\top dW(t) - \frac{1}{2} \int_0^T |a^x(t)|^2 \, dt \right)
$$

$$
\leq \left[ \mathbb{E}_P \exp \left( \int_0^T 2f(t, M^x(t)) \, dt \right) \right]^{\frac{1}{2}} \left[ \mathbb{E}_P \exp \left( 2 \int_0^T a^x(t)^\top dW(t) - \int_0^T |a^x(t)|^2 \, dt \right) \right]^{\frac{1}{2}}
$$

$$
\leq \left[ \mathbb{E}_P \exp \left( \int_0^T 2f(t, M^x(t)) \, dt \right) \right]^{\frac{1}{2}} \cdot \left[ \mathbb{E}_P \exp \left( 6 \int_0^T |a^x(t)|^2 \, dt \right) \right]^{\frac{1}{2}}.
$$

The uniform bound follows from condition $\mathbf{(C1)}$ and Lemma 2.4. \hfill \square

**Lemma 2.12.** Assume $\mathbf{(C1)}$ with $C_1 = 0$, $\mathbf{(C2)}$ and let $T > 0$ be given. Let $(X^x, \tilde{W}^x, Q^x)$ be a weak solution of equation (1) on $[-r, \tau]$ for some stopping time $0 \leq \tau \leq T$. Then the following inequality holds.

$$
\mathbb{E}_{Q^x} \exp \left( \alpha \sup_{-r \leq t \leq \tau} |X^x(t)|^2 \right) \leq \frac{C}{\sqrt{1 - 4\alpha dC \sigma T}} \exp \left( \frac{\alpha}{1 - 4\alpha dC \sigma T} \|x\|_\infty^2 \right)
$$

for all $0 \leq \alpha < (4dC \sigma T)^{-1}$ and a constant $C = C \left( d, T, C_2, C \sigma, \|F\|_{L^{d+1}([0, T] \times \mathbb{R}^d)} \right)$.

**Proof.** As before, let

$$a^x(t) := \sigma(t, M^x(t))^{-1} B(t, M^x), \quad t \in [0, T].$$
By the assumed conditions and Lemma 2.4 one has
\[
\mathbb{E}_p \exp \left( 6 \int_0^T |a^x(t)|^2 \, dt \right) \\
\leq K_1 \mathbb{E}_p \exp \left( 6C_\sigma \int_0^T |B(t, M^x)|^2 \, dt \right) \\
\leq K_2 \mathbb{E}_p \exp \left( 6C_\sigma \int_0^T |F(t, M^x(t))| \, dt \right) \\
\leq K_3
\]
for constants \( K_1, K_2 \) and \( K_3 \) that only depend on \( d, T, C_2, C_\sigma, \|F\|_{L^{d+1}([0,T] \times \mathbb{R}^d)} \). By Theorem 2.11 one obtains
\[
\mathbb{E}_q^x \left( \alpha \sup_{-r \leq t \leq \tau} |X^x(t)|^2 \right) \\
\leq \mathbb{E}_p \exp \left( \alpha \sup_{-r \leq t \leq T} |M^x(t)|^2 + \int_0^T a^x(t)^\top dW(t) - \frac{1}{2} \int_0^T |a^x(t)|^2 \, dt \right) \\
\leq \left[ \mathbb{E}_p \exp \left( 2\alpha \sup_{-r \leq t \leq T} |M^x(t)|^2 \right) \right]^{\frac{1}{2}} \cdot \left[ \mathbb{E}_p \exp \left( 2 \int_0^T a^x(t)^\top dW(t) - \int_0^T |a^x(t)|^2 \, dt \right) \right]^{\frac{1}{2}} \\
\leq \left[ \mathbb{E}_p \exp \left( 2\alpha \sup_{-r \leq t \leq T} |M^x(t)|^2 \right) \right]^{\frac{1}{2}} \cdot \left[ \mathbb{E}_p \exp \left( 6 \int_0^T |a^x(t)|^2 \, dt \right) \right]^{\frac{1}{4}} \\
\leq \frac{C}{\sqrt{1 - 4\alpha dC_\sigma T}} \exp \left( \frac{\alpha}{1 - 4\alpha dC_\sigma T} \|x\|_\infty^2 \right)
\]
for a constant \( C = C \left( d, T, C_2, C_\sigma, \|F\|_{L^{d+1}([0,T] \times \mathbb{R}^d)} \right) \).

### 3. Strong Feller Property

The following theorem is a consequence of a log-Harnack inequality that has been shown in \[16\] and requires the Lipschitz-continuity of \( \sigma \) in space.

**Theorem 3.1.** Assume \( (C2) \). Then one has for all \( t > r \)
\[
\lim_{y \to x} \mathbb{E}_f(M^y_t) = \mathbb{E}_f(M^x_t) \forall f \in B_b(C).
\]

**Lemma 3.2.** Assume \( (C1), (C2), (C4) \) and \( (C5) \). Then one has
\[
\lim_{y \to x} \mathbb{P} \left( \int_0^T |B(t, M^y_t) - B(t, M^y_t)|^2 \, dt > \varepsilon \right) = 0 \forall \varepsilon > 0.
\]
Proof. By Theorems 3.1 and A.1 one has for all $t>r$
\[
\lim_{y \to x} E \left| f(M_t^x) - f(M_t^y) \right| = 0 \quad \forall f \in B_b(C).
\]
Therefore, one has for all $f \in B_b([0,T] \times C)$
\[
\lim_{y \to x} E \int_r^T \left| f(t, M_t^x) - f(t, M_t^y) \right| \, dt = 0.
\]
Consequently,
\[
\lim_{y \to x} P \otimes \lambda_{|[r,T]} \left( |B(\cdot, M^y) - B(\cdot, M^x)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]
By condition (C4), one has also
\[
\lim_{y \to x} P \otimes \lambda_{|[0,r]} \left( |B(\cdot, M^y) - B(\cdot, M^x)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]
Therefore, it holds
\[
\lim_{y \to x} P \otimes \lambda_{|[0,T]} \left( |B(\cdot, M^y) - B(\cdot, M^x)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]
Now, define for $R > 0$
\[
B^R(t, x) := 1_{\sup_{-r \leq s \leq t} |x(s)|^2 < R} B(t, x), \ x \in C \left( \mathbb{R}^{d+1}, \mathbb{R}^d \right)
\]
Each $B^R$, $R > 0$ fulfills condition (C4) and the suitably modified version of (C4) with bounded $G$. It follows
\[
\sup_{y \in C} E \int_0^T H \left( |B^R(t, M^y)|^2 \right) \, dt < \infty, \ R > 0
\]
by Lemma 2.3, 2.5 and condition (C4). Hence, $\left\{ |B^R(\cdot, M^y)|^2 : y \in C \right\}$ is uniformly integrable for each $R > 0$. Since
\[
\lim_{y \to x} E \|M_t^y - M_t^x\|_\infty \quad \forall t \geq 0,
\]
it holds
\[
\lim_{y \to x} P \otimes \lambda_{|[0,T]} \left( |B^R(\cdot, M^y) - B^R(\cdot, M^x)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]
Furthermore,
\[
\lim_{R \to \infty} \sup_{x \in C, \|x\| \leq \tilde{R}} P \left( \sup_{t \in [-r,T]} |M^x(t)| \geq R \right) = 0 \quad \forall \tilde{R} > 0
\]
holds by condition (C2). Thus,

\[
\limsup_{y \to x} P \left( \int_0^T |B(t, M^x(t)) - B(t, M^y(t))|^2 \, dt > \varepsilon \right) \\
\leq \limsup_{y \to x} P \left( \int_0^T |B^R(t, M^x(t)) - B^R(t, M^y(t))|^2 \, dt > \varepsilon \right) \\
+ 2 \sup_{z \in C, \|z\| \leq 2} \|z\|_{\infty} P \left( \sup_{t \in [-r, T]} |M^z(t)| \geq R \right)
\]

\[
= 2 \sup_{z \in C, \|z\| \leq 2} \|z\|_{\infty} P \left( \sup_{t \in [-r, T]} |M^z(t)| \geq R \right)
\]

Now, one can let \( R \to \infty \), which proofs the claim. \( \square \)

**Proof of Theorem 1.5.** Let \( t > r \) and \( f \in B_b(C) \), then one has by Theorem 2.10

\[
E_{Q^x} f(X^x_T) - E_{Q^y} f(X^y_T) \\
= EP(D^x(t)f(M^x_t)) - EP(D^y(t)f(M^y_t)) \\
= EP[D^x(t)(f(M^x_t) - f(M^x_t))] + EP[|D^x(t) - D^y(t)|f(M^y_t)] \\
\leq EP[|D^x(t) - f(M^y_t)|] + \|f\|_{\infty} EP[|D^x(t) - D^y(t)|]
\]

where we define for every \( z \in C \)

\[
a^z(t) := \sigma(t, M^z(t))^{-1} B(t, M^z_t), \\
D^z(t) := \exp \left( \int_0^t a^z(s)^\top dW(s) - \frac{1}{2} \int_0^t |a^z(s)|^2 ds \right).
\]

By condition (C2), Itô’s formula and the stochastic Gronwall Lemma A.6 it holds

\[
\lim_{y \to x} P \left( |M^y_t - M^x_t| > \varepsilon \right) = 0 \ \forall \varepsilon > 0. \tag{2}
\]

Applying Theorems A.1 and 3.1 gives

\[
\lim_{y \to x} EP[|f(M^y_t) - f(M^x_t)|] = 0
\]

and in particular,

\[
\lim_{y \to x} P (|D^x(t)f(M^x_t) - D^y(t)f(M^y_t)| > \varepsilon) = 0 \ \forall \varepsilon > 0.
\]

By the dominated convergence theorem, it follows

\[
\lim_{y \to x} EP[D^x(t)(f(M^y_t) - f(M^x_t))] = 0.
\]

Consequently, it remains to show that

\[
\lim_{y \to x} EP[|D^y(t) - D^x(t)|] = 0.
\]
Since one has $E_P D^z(t) = 1$ for all $z \in C$, it suffices to show
\[
\lim_{y \to x} P \left( |D^y(t) - D^x(t)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0
\]
by standard measure theoretic arguments. Therefore, it is sufficient to show
\[
\lim_{y \to x} P \left( \int_0^t |a^y(s) - a^x(s)|^2 \, ds > \varepsilon \right) = 0 \quad \forall \varepsilon > 0
\]
by the martingale isometry. One has
\[
\int_0^t |a^y(s) - a^x(s)|^2 \, ds 
\leq 2 \int_0^t \left\| \sigma(s, M^y(s))^{-1} - \sigma(s, M^x(s))^{-1} \right\|_{op}^2 |B(s, M^x_s)|^2 \, ds 
+ 2C_\sigma \int_0^t |B(s, M^y_s) - B(s, M^x_s)|^2 \, dt.
\]
The second term converges to zero by the assumed conditions and Lemma 3.2. Moreover,
\[
\lim_{y \to x} P \otimes \lambda_{[0,t]} \left( \left\| \sigma(\cdot, M^y(\cdot))^{-1} - \sigma(\cdot, M^x(\cdot))^{-1} \right\|_{op} > \varepsilon \right) = 0 \quad \forall \varepsilon > 0
\]
holds by (2), the continuity of $\sigma$ in space and the continuity of the inverting map $A \mapsto A^{-1}$ on the space of invertible matrices. Additionally, one can bound the first integrand
by
\[
2C_\sigma |B(\cdot, M^x)|^2,
\]
which is $P \otimes \lambda_{[0,t]}$-integrable by Lemma 2.3. Consequently, one can apply the dominated convergence theorem and the proof is complete.

4. Pathwise Uniqueness and Stability

Notation 4.1. We introduce - as in [17] - the following function space. For $p, \in (1, \infty)$ and $0 \leq S < T$, denote by $W^{1,2}_{p}([S,T] \times \mathbb{R}^d)$ the closure of compactly supported, smooth functions on $[S,T] \times \mathbb{R}^d$ with respect to the norm
\[
\|u\|_{W^{1,2}_{p}([S,T] \times \mathbb{R}^d)} := \|\partial_t u\|_{L^{p}([S,T])} + \| u \|_{L^{p}([S,T]; W^{2,p}(\mathbb{R}^d))}, \ u \in C_c([S,T] \times \mathbb{R}^d).
\]
Let $p := 2d + 2$. By Theorem A.2 for every $0 < T \leq T_0$, there exists a solution
\[
\tilde{u}(\cdot; T) \in \left( W^{1,2}_{p}([0,T_0] \times \mathbb{R}^d) \right)^d
\]
of the coordinatewise PDE system
\[
\partial_t \tilde{u}(t, x; T) + L_t \tilde{u}(t, x; T) + b(t, x) = 0, \\
\tilde{u}(T, x; T) = 0
\]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \) where

\[
L_t v(t, x) := \frac{1}{2} \sum_{i,j,k=1}^d \sigma^{i,k}(t, x)\sigma^{j,k}(t, x)\partial_i\partial_j v(t, x) + b(t, x) \cdot \nabla v(t, x), \quad v \in W^{1,2}_p([0, T_0] \times \mathbb{R}^d).
\]

Additionally, it holds

\[
\sup_{T \in [0, T_0]} \|\tilde{u}(\cdot; T)\|_{W^{1,2}_p([0, T] \times \mathbb{R}^d)} < \infty, \quad i = 1, \ldots, d
\]

and by the embedding Theorem \([A,B]r\) there exists a uniform \( \delta \) such that for all \( 0 \leq S \leq T \) with \( T - S \leq \delta \)

\[
|\tilde{u}(t, x; T) - \tilde{u}(t, y; T)| \leq \frac{1}{2} |x - y|
\]

for all \( t \in [S, T] \) and \( x, y \in \mathbb{R}^d \). Furthermore, the function

\[
u(t, x; T) := \tilde{u}(t, x; T) + x
\]

satisfies coordinatewise the equation

\[
\partial_t \nu(t, x; T) + L_t \nu(t, x; T) = 0,
\]

\[
u(T, x; T) = x.
\]

**Proof of Theorem \([L,4]\).** Let \((X^x, W)\) and \((\hat{X}^x, W)\) be two weak solutions of equation \((1)\) with initial value \( x \in C \) for some common Brownian motion \( W \) on the time interval \([0, \tau]\) for some stopping time \( \tau \). By localization, we can assume that condition \((C1)\) is fulfilled with \( C_1 = 0 \) and that \( \tau \) is bounded by some \( T_0 > 0 \). Choose \( \delta > 0 \) like above with the additional restraint \( \delta < r_B \). By induction, it suffices to prove for every \( 0 \leq S \leq T \leq T_0 \) with \( T - S \leq \delta \)

\[
X^x_{[-r, S \wedge \tau]} = \hat{X}^x_{[-r, S \wedge \tau]},
\]

\[
\Rightarrow X^x_{T \wedge \tau} = \hat{X}^x_{T \wedge \tau}.
\]

For the sake of simplicity, we write \( u(\cdot) := u(\cdot; T) \). Furthermore, define

\[
Y(t) := u(t, X(t)), \quad S \wedge \tau \leq t \leq T \wedge \tau,
\]

\[
\hat{Y}(t) := u(t, \hat{X}^x(t)), \quad S \wedge \tau \leq t \leq T \wedge \tau.
\]

By the choice of \( \delta \), one has for the difference processes \( Z(t) := X^x(t) - \hat{X}^x(t) \) and \( \hat{Z}(t) := Y(t) - \hat{Y}(t) \)

\[
\frac{1}{2} |\hat{Z}(t)| \leq |Z(t)| \leq \frac{3}{2} |\hat{Z}(t)|, \quad S \wedge \tau \leq t \leq T \wedge \tau.
\]

15
Due to Lemma 2.11, Lemma A.4 is applicable, which gives for $S \land \tau \leq t \leq T \land \tau$

$$\tilde{Z}(t) = \int_t^T \left( Du(s, X^x(s))\bar{B}(s, X^x) - Du(s, \hat{X}^x(s))\bar{B}(s, \hat{X}^x) \right) ds$$

$$+ \int_t^T \left( Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, \hat{X}^x(s))\sigma(s, \hat{X}^x(s)) \right) dW(s)$$

$$= \int_t^T \left( Du(s, X^x(s))\bar{B}(s, X^x) - Du(s, \hat{X}^x(s))\bar{B}(s, \hat{X}^x) \right) ds$$

$$+ \int_t^T \left( Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, \hat{X}^x(s))\sigma(s, \hat{X}^x(s)) \right) dW(s)$$

and consequently

$$d\left|\tilde{Z}(t)\right|^2 = -2\tilde{Z}(t)^\top \left( Du(t, X^x(t))\bar{B}(t, X^x) - Du(t, \hat{X}^x(t))\bar{B}(t, \hat{X}^x) \right) dt$$

$$+ 2\tilde{Z}(t)^\top (Du(t, X^x(t))\sigma(t, X^x(t)) - Du(t, X^{x_n}(t))\sigma(t, X^{x_n}(t))) dW(t)$$

$$+ \left|Du(t, X^x(t))\sigma(t, X^x(t)) - Du(t, \hat{X}^x(t))\sigma(t, \hat{X}^x(t))\right|_{HS}^2 dt$$

Using the boundedness of $Du$, condition [C1] and Young’s inequality gives for $S \leq t_1 \leq t_2 \leq T$

$$\left|\tilde{Z}(t_2)\right|^2 - \left|\tilde{Z}(t_1)\right|^2 \leq \int_{t_1}^{t_2} \left|\tilde{Z}(s)\right| \left\|Du(s, X^x(s)) - Du(s, X^{x_n}(s))\right\|_{op} \left|\bar{B}(s, X^x)\right| ds$$

$$+ c \int_{t_1}^{t_2} \tilde{Z}(s)^\top (Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^{x_n}(s))\sigma(s, X^{x_n}(s))) dW(s)$$

$$+ c \int_{t_1}^{t_2} \left|Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^{x_n}(s))\sigma(s, X^{x_n}(s))\right|_{HS}^2 ds$$

where $c > 0$ is a constant. As in [6], one can use a suitable multiplier of the form

$$\exp(-A(t))$$

where $A$ is an adapted, continuous process. Here, we choose

$$A(t) := c \int_S^t \left|\bar{B}(s, X^x)\right| \left\|Du(s, X^x(s)) - Du(s, \hat{X}^x(s))\right\|_{op} 1_{\tilde{Z}(s) \neq 0} ds$$

$$+ c \int_S^t \left|Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, \hat{X}^x(s))\sigma(s, \hat{X}^x(s))\right|^2_{HS} \left|\tilde{Z}(s)\right|^2 1_{\tilde{Z}(s) \neq 0} ds$$

for $S \leq t \leq T$. To show that $A$ is indeed well defined - namely finite - it suffices to show

$$\mathbb{E} \exp\left(\frac{1}{2} A(T)\right) < \infty.$$
Since $u$ belongs coordinatewise to $W_{p}^{1,2}\left([0,T_{0}] \times \mathbb{R}^{d}\right)$ and by conditions (C2), it holds
\[(Du \cdot \sigma)^{i,j} \in L^{p}\left(T_{0}; W_{1,p}^{1}(\mathbb{R}^{d})\right), \ i,j = 1, \ldots, d.\]

Additionally, $C_{c}^{\infty}(\mathbb{R}^{d+1})$ is dense in $L^{p}\left(T_{0}; W_{1,p}^{1}(\mathbb{R}^{d})\right)$. Hence, by Young's inequality, Lemma 2.11 and Lemma 2.12, it suffices to show for all $R > 0$ the existence of a constant $C_{R} > 0$ such that
\[
E \exp\left( \int_{S}^{T} \frac{\left| f(s, X^{x}(s)) - f(s, \hat{X}^{x}(s)) \right|^{2}}{\left| \tilde{Z}(s) \right|^{2}} 1_{\tilde{Z}(s) \neq 0} \, ds \right) \leq C_{R}
\]
for all $f \in C^{\infty}(\mathbb{R}^{d+1})$ with $\|f\|_{L^{p}(T_{0}; W_{1,p}^{1}(\mathbb{R}^{d}))} \leq R$. By Lemmas 2.11 and A.5 one obtains
\[
E \exp\left( \int_{S}^{T} \frac{\left| f(s, X^{x}(s)) - f(s, \hat{X}^{x}(s)) \right|^{2}}{\left| \tilde{Z}(s) \right|^{2}} 1_{\tilde{Z}(s) \neq 0} \, ds \right) \leq E \exp\left( C_{d}^{2} \int_{S}^{T} \left( M |\nabla f| (X^{x}(s)) + M |\nabla f| (\hat{X}^{x}(s)) \right)^{2} \, ds \right) < \infty.
\]
Now, it holds for $S \leq t \leq T$
\[
eq -A(t) \left| \tilde{Z}(t) \right|^{2} \leq c \int_{S}^{t} e^{-A(s)} \left| \tilde{Z}(s) \right|^{2} \, ds + \text{local martingale}
\]
by the Itô formula. Applying the stochastic Gronwall Lemma A.6 gives
\[
E \left[ \sup_{t \in [S,T]} e^{-\frac{1}{2} A(t)} \left| \tilde{Z}(t) \right| \right] = 0,
\]
which finishes the proof. \qed

The following result is a rather technical one, which will be used to proof Theorem 1.6.

**Proposition 4.2.** Assume (C1), (C2), (C3), (C4) and (C5). Furthermore, let $X^{x}$, $x \in \mathcal{C}$ be the strong solutions to equation (1) with initial value $x$ and assume that
\[
\lim_{y \to x} P \left( \|X_{S}^{y} - X_{S}^{x}\|_{\infty} > \varepsilon \right) = 0 \ \forall \varepsilon > 0, \forall x \in \mathcal{C}
\]
for some $S \geq 0$. Then one has for each $R > 0$
\[
\lim_{n \to \infty} E \int_{S + \tau_{R}^{x,y}}^{S + r_{R}^{x,y}} \left| \tilde{B}(s, X^{y}_{s}) - \tilde{B}(s, X^{x}_{s}) \right|^{2} \, ds = 0
\]
where
\[
\tau_{R}^{x,y} := \sup \left\{ t \geq 0 : \sup_{-r \leq s \leq t} |X^{x}(s)|^{2} < R, \sup_{-r \leq s \leq t} |X^{y}(s)|^{2} < R \right\}.
\]
Proof. By condition (C3), one can write
\begin{equation}
\tilde{B}(t, X^x_t) = g(t, X^x_t) \quad t \in [S, S + r_{\tilde{B}}], x \in C.
\end{equation}

If \( S > r \), Theorem 1.5 gives
\[
\lim_{y \to x} E f(X^x_y) = E f(X^x_x) \quad \forall f \in B_b(\sigma).
\]

Consequently, combining it with (3), (4) and Theorem A.1 gives
\[
\lim_{y \to x} \mathbb{P} \left( \left| g(t, X^x_y) - g(t, X^x_x) \right| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0, t \in [S, S + r_{\tilde{B}}].
\]

If \( S \leq r \), one can use the continuity assumption (C4) and (3) to deduce (5), too.

Therefore, combining it with (3), (4) and Theorem A.1 gives
\[
\lim_{y \to x} \mathbb{P} \otimes \lambda_{|[S,S+r_{\tilde{B}}]} \left( \left| B(\cdot, X^y_s) - B(\cdot, X^x_s) \right|^2 > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]

It follows
\[
\lim_{y \to x} \mathbb{P} \otimes \lambda_{|[S,S+r_{\tilde{B}}]} \left( \mathbf{1}_{t \in [S,T]} \left| B(\cdot, X^y_t) - B(\cdot, X^x_t) \right|^2 > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]

Now, one can use Lemma 2.3 and condition (C4) to obtain
\[
\sup_{y \in C} \mathbb{E} \int_{[S,T]} H \left( |B(t, X^y_t)|^2 \right) \ dt < \infty,
\]
which guarantees the uniform integrability of \( \left\{ \mathbf{1}_{t \in [S,T]} |B(\cdot, M^y_t)|^2 : y \in C \right\} \) with respect to the measure \( \mathbb{P} \otimes \lambda_{|[S,S+r_{\tilde{B}}]} \).

Proof of Theorem 1.6. Choose \( \delta > 0 \) like before with the additional restraint \( \delta < r_{\tilde{B}} \). By induction and Lemma (2.12), it suffices to prove for every \( 0 \leq S \leq T \leq T_0 \) with \( T - S \leq \delta \) the implication
\[
\lim_{y \to x} \mathbb{E} \left| X^y_S - X^x_S \right|_\gamma^\gamma = 0 \quad \forall x \in C, \quad 0 < \gamma < 2
\]

\[
\implies \lim_{y \to x} \mathbb{P} \left( \sup_{s \in [S,T]} |X^y_s - X^x(s)| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0, x \in C.
\]

For the sake of simplicity, we write \( u(\cdot) := u(\cdot; T) \). Furthermore, define
\[
Y^x(t) := u(t, X(t)), \quad S \leq t \leq T,
\]
\[
Y^y(t) := u(t, X^y(t)), \quad S \leq t \leq T.
\]

By the choice of \( \delta \), one has for the difference processes \( Z(t) := X^x(t) - X^y(t) \) and \( \tilde{Z}(t) := Y^x(t) - Y^y(t) \)
\[
\frac{1}{2} \left| \tilde{Z}(t) \right| \leq |Z(t)| \leq \frac{3}{2} \left| \tilde{Z}(t) \right|, \quad S \leq t \leq T.
\]
Due to Lemma 2.11, Lemma A.4 is applicable, which gives

\[
\tilde{Z}(t) = \int_S^t \left( Du(s, X^x(s)) \tilde{B}(s, X^x_s) - Du(s, X^y(s)) \tilde{B}(s, X^y_s) \right) \, ds \\
+ \int_S^t (Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^y(s))\sigma(s, X^y(s))) \, dW(s)
\]

and consequently

\[
d \left| \tilde{Z} \right|^2(t) \\
= 2\tilde{Z}(t)^T \left( Du(t, X^x(t)) \tilde{B}(t, X^x_t) - Du(t, X^y(t)) \tilde{B}(t, X^y_t) \right) \, dt \\
+ 2\tilde{Z}(t)^T \left( Du(t, X^x(t))\sigma(t, X^x(t)) - Du(t, X^y(t))\sigma(t, X^y(t)) \right) \, dW(t) \\
+ \| Du(t, X^x(t))\sigma(t, X^x(t)) - Du(t, X^y(t))\sigma(t, X^y(t)) \|^2_{HS} \, dt
\]

Using the boundedness of \( Du \) and Young’s inequality gives for \( S \leq t_1 \leq t_2 \leq T \)

\[
\left| \tilde{Z}(t_2) \right|^2 - \left| \tilde{Z}(t_1) \right|^2 \\
\leq c \int_{t_1}^{t_2} \left| \tilde{Z}(s) \right|^2 \, ds \\
+ c \int_{t_1}^{t_2} \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds \\
+ c \int_{t_1}^{t_2} \left| \tilde{Z}(s) \right| \left\| Du(s, X^x(s)) - Du(s, X^y(s)) \right\|_{op} \left| \tilde{B}(s, X^x_s) \right| \, ds \\
+ c \int_{t_1}^{t_2} \tilde{Z}(s)^T \left( Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^y(s))\sigma(s, X^y(s)) \right) \, dW(s) \\
+ c \int_{t_1}^{t_2} \left\| Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^y(s))\sigma(s, X^y(s)) \right\|_{HS}^2 \, ds
\]

where \( c > 0 \) is a constant. Like before, one can use the multiplier \( \exp( -A(t) ) \) where

\[
A(t) := c \int_S^t \left[ B(s, X^x_s) \left\| Du(s, X^x(s)) - Du(s, X^y(s)) \right\|_{op} \left| \tilde{Z}(s) \right| \mathbf{1}_{\tilde{Z}(s) \neq 0} \right] \, ds \\
+ c \int_S^t \left\| Du(s, X^x(s))\sigma(s, X^x(s)) - Du(s, X^y(s))\sigma(s, X^y(s)) \right\|_{HS}^2 \mathbf{1}_{\tilde{Z}(s) \neq 0} \, ds
\]

for \( S \leq t \leq T \). Again, one has

\[
E \exp \left( \frac{1}{2} A(T) \right) \leq \hat{C}
\]
where $\hat{C}$ is some constant not depending on $x, y \in C$. By the Itô formula, it holds for $S \leq t \leq T$

\[
e^{-A(t)} \left| \tilde{Z}(t) \right|^2 \leq \left| \tilde{Z}(S) \right|^2 + c \int_S^t \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds + c \int_S^t e^{-A(s)} \left| \tilde{Z}(s) \right|^2 \, ds + \text{local martingale.}
\]

Applying the stochastic Gronwall Lemma A.6 gives

\[
E \left[ \sup_{t \in [S,T]} e^{-\frac{1}{2} A(t)} \left| \tilde{Z}(t) \right| \right] \leq \hat{C} E \left| \tilde{Z}(S) \right| + \hat{C} E \left( \int_S^T \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds \right)^{\frac{1}{2}}
\]

for a constant $\hat{C}$ which does not depend on $x, y \in C$. By Lemma 2.8, $\lim_{R \to \infty} \sup_{z \in C, \|z\|_{\infty} \leq \|x\|_{\infty}} P \left( \sup_{-r \leq t \leq T} |X^z(t)|^2 > R \right) = 0$ holds. Thus, applying Lemma 1.2 and the induction hypothesis gives

\[
\lim_{y \to x} P \left( \int_S^T \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.
\]

By Corollary 2.8, one has

\[
\sup_{z \in C, \|z\|_{\infty} \leq 2\|x\|_{\infty}} E \int_S^T \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds < \infty
\]

and consequently,

\[
\lim_{y \to x} E \left( \int_S^T \left| \tilde{B}(s, X^x_s) - \tilde{B}(s, X^y_s) \right|^2 \, ds \right)^{\frac{1}{2}} = 0.
\]

\[\Box\]

\section{A. Appendix}

\textbf{Theorem A.1.} Let $(\Omega, \mathcal{F}, P)$ be some probability space and $(E,d)$ be a metric space. Furthermore, let $X, X_n : \Omega \to E$, $n \in \mathbb{N}$ be measurable maps. Then the statement

1. a) $\lim_{n \to \infty} P^* (d(X, X_n) \geq \varepsilon) = 0 \quad \forall \varepsilon > 0,$

b) $\lim_{n \to \infty} P_{X_n}(O) = P_X(O)$ for all open $O \subset E$

implies

2. $\lim_{n \to \infty} E |f(X) - f(X_n)| = 0 \quad \forall f \in B_b(C).$
Additionally, if there exists some null set $N \subset \Omega$ such that $X(\Omega \setminus N)$ is separable, then the converse implication is also true.

Proof. See Theorem 1.7 in [2]. □

Theorem A.2. Assume (C2) and $b \in L^p([0, T] \times \mathbb{R}^d)$ with $p > d + 2$. Then for any $T > 0$ and $f \in L^p([0, T] \times \mathbb{R}^d)$, there exists a unique solution $u \in W^{1,2}_p([0, T] \times \mathbb{R}^d)$ of the following PDE

$$
\partial_t u(t, x) + L_t u(t, x) + f(t, x) = 0,
$$

$$
u(T, x) = 0
$$

with the bound

$$
\|u\|_{W^{1,2}_p([S, T] \times \mathbb{R}^d)} \leq C \|f\|_{L^p([S, T] \times \mathbb{R}^d)}
$$

for any $S \in [0, T]$ and some constant $C = C(T, C_\sigma, p, \|b\|_{L^p([0,T] \times \mathbb{R}^d)}) > 0$.

Proof. See Theorem 10.3 in [10]. □

Theorem A.3. Let $p \in (1, \infty)$, $T > 0$ and $u \in W^{1,2}_p([0, T] \times \mathbb{R}^d)$.

1. If $p > \frac{d+2}{2}$, then $u$ is a bounded Hölder continuous function on $[0, T] \times \mathbb{R}^d$ and for any $0 < \varepsilon, \delta \leq 1$ satisfying

$$
\varepsilon + \frac{d + 2}{p} < 2, \quad 2\delta + \frac{d + 2}{p} < 2,
$$

there exists a constant $N = N(p, \varepsilon, \delta)$ such that

$$
|u(t, x) - u(s, x)| \leq N |t - s|^\varepsilon \|u\|_{L^p(T; W^{2,p} (\mathbb{R}^d))}^{1 - \frac{1}{p} - \delta} \|\partial_t u\|_{L^p([0,T] \times \mathbb{R}^d)}^{\frac{1}{p} + \delta},
$$

$$
|u(t, x)| + \frac{|u(t, x) - u(t, y)|}{|x - y|^\delta} \leq NT^{-\frac{1}{p}} \left( \|u\|_{L^p(T; W^{2,p} (\mathbb{R}^d))} + T \|\partial_t u\|_{L^p([0,T] \times \mathbb{R}^d)} \right)
$$

for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d, x \neq y$.

2. If $p > d + 2$, then $\nabla u$ is a bounded Hölder continuous function on $[0, T] \times \mathbb{R}^d$ and for any $\varepsilon \in (0, 1)$ satisfying

$$
\varepsilon + \frac{d + 2}{p} < 1,
$$

there exists a constant $N = N(p, \varepsilon)$ such that

$$
|\nabla u(t, x) - \nabla u(s, x)| \leq N |t - s|^\varepsilon \|u\|_{L^p(T; W^{2,p} (\mathbb{R}^d))}^{1 - \frac{1}{p} - \frac{\varepsilon}{2}} \|\partial_t u\|_{L^p([0,T] \times \mathbb{R}^d)}^{\frac{1}{p} + \frac{\varepsilon}{2}},
$$

$$
|\nabla u(t, x)| + \frac{|\nabla u(t, x) - \nabla u(t, y)|}{|x - y|^\delta} \leq NT^{-\frac{1}{p}} \left( \|u\|_{L^p(T; W^{2,p} (\mathbb{R}^d))} + T \|\partial_t u\|_{L^p([0,T] \times \mathbb{R}^d)} \right)
$$

for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d, x \neq y$. 21
Proof. See [5, p. 22, 23, 36]. □

In the next lemma we identify every $u \in W_{1,2}^{p}$ with its regular version.

**Lemma A.4** (Itô formula for $W_{1,2}^{p}$-functions). Let $T > 0$, $p > d+2$. Let $X : \Omega \times [0, T] \to \mathbb{R}^{d}$ be a semimartingale on some filtrated probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ of the form
\[
dX(t) = b(t) \, dt + \sigma(t) \, dW(t)
\]
where $W$ is a $d$-dimensional Brownian motion, $b : \Omega \times [0, T] \to \mathbb{R}^{d}$ and $\sigma : \Omega \times [0, T] \to \mathbb{R}^{d \times d}$ are progressively measurable with
\[
\mathbb{P} \left( \|b\|_{L^1[0,T]} + \|a^{i,j}\|_{L^\delta[0,T]} < \infty \right) = 1, \ i,j = 1, \ldots, d
\]
for some $1 < \delta \leq \infty$ where $a := \sigma \sigma^\top$. Furthermore, assume that there exists a constant $C > 0$ with
\[
\mathbb{E} \int_{0}^{T} f(t, X(t)) \, dt \leq C \|f\|_{L^{p/\delta^\star}([0,T] \times \mathbb{R}^d)}
\]
for all $f \in L^{p/\delta^\star}([0,T] \times \mathbb{R}^d)$ where $\delta^\star$ denotes the conjugate exponent of $\delta$. Then for any $u \in W_{1,2}^{p}([0,T] \times \mathbb{R}^d)$, the Itô formula holds, i.e.
\[
u(t, X(t)) - u(0, X(0)) = \int_{0}^{t} \partial_t u(s, X(s)) \, ds + \int_{0}^{t} \nabla u(s, X(s))^\top b(s) \, ds
\]
\[
+ \int_{0}^{t} \nabla u(s, X(s))^\top \sigma(s) \, dW(s)
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} \partial_i \partial_j u(s, X(s)) a^{i,j}(s) \, ds.
\]

Proof. See [1]. □

Let $\phi$ be a locally integrable function on $\mathbb{R}^{d}$. The Hardy-Littlewood maximal function is defined by
\[
\mathcal{M} \phi(x) := \sup_{0 < r < \infty} \frac{1}{|B_r|} \int_{B_r} \phi(x + y) \, dy
\]
where $B_r$ is the Euclidean ball of radius $r$. The following result is cited from Appendix A in [3].

**Lemma A.5.**

1. There exists a constant $C_d > 0$ such that for all $\phi \in C^\infty(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,
\[
|\phi(x) - \phi(y)| \leq C_d |x - y| \left( \mathcal{M} |\nabla \phi| (x) + \mathcal{M} |\nabla \phi| (y) \right).
\]

2. For any $p > 1$, there exists a constant $C_{d,p}$ such that for all $\phi \in L^p(\mathbb{R}^d)$,
\[
\|\mathcal{M} \phi\|_{L^p} \leq C_{d,p} \|\phi\|_{L^p}.
\]
For a real-valued process denote $Y^*(t) := \sup_{0 \leq s \leq t} Y(s)$.

Lemma A.6. Let $Z$ and $H$ be nonnegative, adapted processes with continuous paths and assume that $\psi$ is nonnegative and progressively measurable. Let $M$ be a continuous local martingale starting at 0. If

$$Z(t) \leq \int_0^t \psi(s) Z(s) \, ds + M(t) + H(t)$$

holds for all $t \geq 0$, then for $p \in (0, 1)$ and $\mu, \nu > 1$ such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $p \nu < 1$, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^p \leq (c_p \nu + 1)^{1/\nu} \left( \mathbb{E} \exp \left\{ p \mu \int_0^t \psi(s) \, ds \right\} \right)^{1/\mu} \left( \mathbb{E} (H^*(t))^{p \nu} \right)^{1/\nu}$$

where

$$c_p := \left( 4 \wedge \frac{1}{p} \right) \frac{\pi p}{\sin(\pi p)}.$$

If $\psi$ is deterministic, then

$$\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^p \leq (1 + c_p) \exp \left\{ p \int_0^t \psi(s) \, ds \right\} \left( \mathbb{E} (H^*(t))^{p} \right)$$

and

$$\mathbb{E} Z(t) \leq \exp \left\{ \int_0^t \psi(s) \, ds \right\} \mathbb{E} H^*(t).$$

Proof. See [14].

References

[1] S. Bachmann. Well-posedness and stability for a class of stochastic delay differential equations with singular drift. *Stochastics and Dynamics*, 0(0):1850019, 0.

[2] S. Bachmann. On the Strong Feller Property of Stochastic Delay Differential Equations with Singular Drift. *ArXiv e-prints*, September 2017.

[3] Gianluca Crippa and Camillo De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.*, 616:15–46, 2008.

[4] A. Es-Sarhir, M.-K. von Renesse, and M. Scheutzow. Harnack inequality for functional sdes with bounded memory. *Electron. Commun. Probab.*, 14:560–565, 2009.

[5] E. Fedrizzi. Uniqueness and flow theorems for solutions of SDEs with low regularity of the drift. *tesi di Laurea in Matematica, Università di Pisa*, 2009.

[6] E. Fedrizzi and F. Flandoli. Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. *Stochastics*, 83(3):241–257, 2011.
[7] I. Gyöngy and T. Martínez. On stochastic differential equations with locally unbounded drift. *Czechoslovak Math. J.*, 51(126)(4):763–783, 2001.

[8] Xing Huang. Strong solutions for functional sdes with singular drift. *Stochastics and Dynamics*, 0(0):1850015, 0.

[9] N. V. Krylov. Estimates of the maximum of the solution of a parabolic equation and estimates of the distribution of a semimartingale. *Mat. Sb. (N.S.)*, 130(172)(2):207–221, 284, 1986.

[10] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.

[11] R. S. Liptser and A. N. Shiryaev. *Statistics of Random Processes: I. General Theory*, pages 286–297. Springer, 2001.

[12] B. Maslowski and J. Seidler. Probabilistic approach to the strong feller property. *Probability Theory and Related Fields*, 118(2):187–210, Oct 2000.

[13] N. I. Portenko. *Generalized diffusion processes*, volume 83 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990. Translated from the Russian by H. H. McFaden.

[14] M. Scheutzow. A stochastic gronwall lemma. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 16(02):1350019, 2013.

[15] A. Ju. Veretennikov. Strong solutions of stochastic differential equations. *Teor. Veroyatnost. i Primenen.*, 24(2):348–360, 1979.

[16] F.-Y. Wang and C. Yuan. Harnack inequalities for functional SDEs with multiplicative noise and applications. *Stochastic Process. Appl.*, 121(11):2692–2710, 2011.

[17] X. Zhang. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.*, 16:no. 38, 1096–1116, 2011.

[18] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.