Partial immunization of trees

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Abstract

For a graph \( G \) and an integer-valued function \( \tau \) on its vertex set, a dynamic monopoly is a set of vertices of \( G \) such that iteratively adding to it vertices \( u \) of \( G \) that have at least \( \tau(u) \) neighbors in it eventually yields the vertex set of \( G \). We study the problem of maximizing the minimum order of a dynamic monopoly by increasing the threshold values of individual vertices subject to vertex-dependent lower and upper bounds, and fixing the total increase. We solve this problem efficiently for trees, which extends a result of Khoshkhah and Zaker (On the largest dynamic monopolies of graphs with a given average threshold, Canadian Mathematical Bulletin 58 (2015) 306-316).

Keywords: Dynamic monopoly; vaccination

1 Introduction

As a simple model for an infection process within a network \cite{12,13,16} one can consider a graph \( G \) in which each vertex \( u \) is assigned a non-negative integral threshold value \( \tau(u) \) quantifying how many infected neighbors of \( u \) are required to spread the infection to \( u \). In this setting, a dynamic monopoly of \((G,\tau)\) is a set \( D \) of vertices such that an infection starting in \( D \) spreads to all of \( G \), and the smallest order \( \text{dyn}(G,\tau) \) of such a dynamic monopoly measures the vulnerability of \( G \) for the given threshold values.

Khoshkhah and Zaker \cite{17} consider the maximum of \( \text{dyn}(G,\tau) \) over all choices for the function \( \tau \) such that the average threshold is at most some positive real \( \bar{\tau} \). They show that this maximum equals

\[
\max \left\{ k : \sum_{i=1}^{k} (d_G(u_i) + 1) \leq n(G)\bar{\tau} \right\},
\]

where \( u_1, \ldots, u_{n(G)} \) is a linear ordering of the vertices of \( G \) with non-decreasing vertex degrees \( d_G(u_1) \leq \ldots \leq d_G(u_{n(G)}) \). To obtain this simple formula one has to allow \( d_G(u) + 1 \) as a threshold value for vertices \( u \), a value that makes these vertices completely immune to the infection, and forces every dynamic monopoly to contain them. Requiring \( \tau(u) \leq d_G(u) \) for every vertex \( u \) of \( G \) leads to a harder problem; Khoshkhah and Zaker \cite{17} show hardness for planar graphs.
and describe an efficient algorithm for trees. In the present paper we consider their problem with additional vertex-dependent lower and upper bounds on the threshold values. As our main result, we describe an efficient algorithm for trees based on a completely different approach than the one in [17].

In order to phrase the problem and our results exactly, and to discuss further related work, we introduce some terminology. Let $G$ be a finite, simple, and undirected graph. A threshold function for $G$ is a function from the vertex set $V(G)$ of $G$ to the set of integers. For notational simplicity, we allow negative threshold values. Let $\tau \in \mathbb{Z}^{V(G)}$ be a threshold function for $G$. For a set $D$ of vertices of $G$, the hull $H_{(G,\tau)}(D)$ of $D$ in $(G,\tau)$ is the smallest set $H$ of vertices of $G$ such that $D \subseteq H$, and $u \in H$ for every vertex $u$ of $G$ with $|H \cap N_G(u)| \geq \tau(u)$. Clearly, the set $H_{(G,\tau)}(D)$ is obtained by starting with $D$, and iteratively adding vertices $u$ that have at least $\tau(u)$ neighbors in the current set as long as possible. With this notation, the set $D$ is a dynamic monopoly of $(G,\tau)$ if $H_{(G,\tau)}(D)$ equals the vertex set of $G$, and $\text{dyn}(G,\tau)$ is the minimum order of such a set. A dynamic monopoly of $(G,\tau)$ of order $\text{dyn}(G,\tau)$ is minimum. The parameter $\text{dyn}(G,\tau)$ is computationally hard [5,10]; next to general bounds [1,9,15] efficient algorithms are only known for essentially tree-structured instances [2,5,7,8,10].

We can now phrase the problem we consider: For a given graph $G$, two functions $\tau,\iota_{\max} \in \mathbb{Z}^{V(G)}$, and a non-negative integer budget $b$, let $\text{vacc}(G,\tau,\iota_{\max}, b)$ be defined as

$$\max \left\{ \text{dyn}(G, \tau + \iota) : \iota \in \mathbb{Z}^{V(G)}, 0 \leq \iota \leq \iota_{\max}, \text{ and } \iota(V(G)) = b \right\},$$

(2)

where inequalities between functions are meant pointwise, and $\iota(V(G)) = \sum_{u \in V(G)} \iota(u)$. The function $\iota$ is the increment of the original threshold function $\tau$. The final threshold function $\tau + \iota$ must lie between $\tau$ and $\tau + \iota_{\max}$, which allows to incorporate vertex-dependent lower and upper bounds. Note that no such increment $\iota$ exists if $\iota_{\max}(V(G))$ is strictly less than $b$, in which case $\text{vacc}(G,\tau,\iota_{\max}, b)$ equals $\max \emptyset = -\infty$. Note that we require $\iota(V(G)) = b$ in (2), which determines the average final threshold as $(\tau(V(G)) + b)/n(G)$. Since $\text{dyn}(G,\rho) \leq \text{dyn}(G,\rho')$ for every two threshold functions $\rho$ and $\rho'$ for $G$ with $\rho \leq \rho'$, for $\iota_{\max}(V(G)) \geq b$, the value in (2) remains the same when replacing '$\iota(V(G)) = b$' with '$\iota(V(G)) \leq b$' provided that $b \leq \iota_{\max}(V(G))$.

The results of Khoshkhah and Zaker [17] mentioned above can be phrased by saying

(i) that $\text{vacc}(G,0,d_G+1,n(G)\bar{\tau})$ equals (1) whenever $n(G)\bar{\tau}$ is a non-negative integer at most

$$\sum_{u \in V(G)} (d_G(u) + 1) = 2m(G) + n(G),$$

where $m(G)$ is the size of $G$, and

(ii) that $\text{vacc}(T,0,d_T,b)$ can be determined efficiently whenever $T$ is a tree.

Our main result is the following.

**Theorem 1.1.** For a given tuple $(T,\tau,\iota_{\max}, b)$, where $T$ is a tree of order $n$, $\tau,\iota_{\max} \in \mathbb{Z}^{V(G)}$, and $b$ is an integer with $0 \leq b \leq \iota_{\max}(V(T))$, the value $\text{vacc}(T,\tau,\iota_{\max}, b)$ as well as an increment $\iota \in \mathbb{Z}^{V(G)}$ with $0 \leq \iota \leq \iota_{\max}$ and $\iota(V(G)) = b$ such that $\text{vacc}(T,\tau,\iota_{\max}, b) = \text{dyn}(T,\tau + \iota)$ can be determined in time $O\left(n^2(b+1)^2\right)$.

While our approach relies on dynamic programming, Khoshkhah and Zaker show (ii) using the following result in combination with a minimum cost flow algorithm.
Theorem 1.2 (Khoshkhah and Zaker [17]). For a given tree $T$, and a given integer $b$ with $0 \leq b \leq 2m(T)$, there is a matching $M$ of $T$ such that $\text{vacc}(T, 0, d_T, b) = \text{dyn}(G, \tau_M)$ and $\tau_M(V(T)) \leq b$, where

$$\tau_M : V(T) \rightarrow \mathbb{Z} : u \mapsto \begin{cases} d_T(u) & \text{if } u \text{ is incident with a vertex in } M, \\ 0 & \text{otherwise.} \end{cases}$$

We believe that the threshold function $\tau_M$ considered in Theorem 1.2 is a good choice in general, and pose the following.

Conjecture 1.3. For a given graph $G$, and a given integer $b$ with $0 \leq b \leq 2m(G)$, there is a matching $M$ of $G$ such that $\text{vacc}(G, 0, d_G, b) \leq 2\text{dyn}(G, \tau_M)$ and $\tau_M(V(G)) \leq b$, where $\tau_M$ is as in Theorem 1.2 (with $T$ replaced by $G$).

As a second result we show Conjecture 1.3 for some regular graphs.

Theorem 1.4. Conjecture 1.3 holds if $G$ is $r$-regular and $b \geq (2r - 1)(r + 1)$.

Before we proceed to the proofs of Theorems 1.1 and 1.4, we mention some further related work. Centeno and Rautenbach [6] establish bounds for the problems considered in [17]. In [14], Ehard and Rautenbach consider the following two variants of (2) for a given triple $(G, \tau, b)$, where $G$ is a graph, $\tau$ is a threshold function for $G$ and $b$ is a non-negative integer:

$$\max \left\{ \text{dyn}(G - X, \tau) : X \in \binom{V(G)}{b} \right\} \quad \text{and} \quad \max \left\{ \text{dyn}(G, \tau_X) : X \in \binom{V(G)}{b} \right\},$$

where

$$\tau_X(u) = \begin{cases} d_G(u) + 1 & \text{if } u \in X, \\ \tau(u) & \text{if } u \in V(G) \setminus X, \end{cases}$$

and $\binom{V(G)}{b}$ denotes the set of all $b$-element subsets of $V(G)$. For both variants, they describe efficient algorithms for trees. In [3] Bhawalkar et al. study so-called anchored $k$-cores. For a given graph $G$, and a positive integer $k$, the $k$-core of $G$ is the largest induced subgraph of $G$ of minimum degree at least $k$. It is easy to see that the vertex set of the $k$-core of $G$ equals $V(G) \setminus H_{(G, \tau)}(\emptyset)$ for the special threshold function $\tau = d_G - k + 1$. Now, the anchored $k$-core problem [3] is to determine

$$\max \left\{ |V(G) \setminus H_{(G, \tau_X)}(\emptyset)| : X \in \binom{V(G)}{b} \right\}, \quad (3)$$

for a given graph $G$ and non-negative integer $b$. Bhawalkar et al. show that [3] is hard to approximate in general, but can be determined efficiently for $k = 2$, and for graphs of bounded treewidth. Vaccination problems in random settings were studied in [4, 11, 16].

2 Proofs of Theorem 1.1 and Theorem 1.4

Throughout this section, let $T$ be a tree rooted in some vertex $r$, and let $\tau, \iota_{\max} \in \mathbb{Z}^{V(T)}$ be two functions. For a vertex $u$ of $T$, and a function $\rho \in \mathbb{Z}^{V(T)}$, let $V_u$ be the subset of $V(T)$ containing
implies $\iota$ of $(T_{\text{monopoly}})$, if $x$ equals $\iota$ where, if possible, let $\iota$ be such that then let $\iota$ is, the index 0 or 1 indicates the amount of help that $b$ be the set of ordered partitions of $u$ for the following two quantities: For a vertex $u, b$ write ' $\rho$ be the subtree of $\rho$ function with $\rho$ Below we consider threshold functions of the form $\rho|_{V_u} + \rho'|_{V_u}$ for the subtrees $T_u$, where $\rho$ and $\rho'$ are defined on sets containing $V_u$. For notational simplicity, we omit the restriction to $V_u$ and write ' $\rho + \rho'$' instead of ' $\rho|_{V_u} + \rho'|_{V_u}$' in these cases. For an integer $k$ and a non-negative integer $b$, let $[k]$ be the set of positive integers at most $k$, and let

$$P_k(b) = \{(b_1, \ldots, b_k) \in \mathbb{N}_0^k : b_1 + \cdots + b_k = b\}$$

be the set of ordered partitions of $b$ into $k$ non-negative integers.

Our approach to show Theorem 1.1 is similar as in [14] and relies on recursive expressions for the following two quantities: For a vertex $u$ of $T$ and a non-negative integer $b$, let

- $x_0(u, b)$ be the maximum of $\text{dyn}(T_u, \tau + \iota)$ over all $\iota \in \mathbb{Z}^{V_u}$ with $0 \leq \iota(v) \leq \iota_{\text{max}}(v)$ for every $v \in V_u$, and $\iota(V_u) = b$, and

- $x_1(u, b)$ be the maximum of $\text{dyn}(T_u, (\tau + \iota)^{-u})$ over all $\iota \in \mathbb{Z}^{V_u}$ with $0 \leq \iota(v) \leq \iota_{\text{max}}(v)$ for every $v \in V_u$, and $\iota(V_u) = b$.

The increment $\iota$ captures the local increases of the thresholds within $V_u$. The value $x_1(u, b)$ corresponds to a situation, where the infection reaches the parent of $u$ before it reaches $u$, that is, the index 0 or 1 indicates the amount of help that $u$ receives from outside of $V_u$.

Note that $x_j(u, b) = -\infty$ if and only if $b > \iota_{\text{max}}(V_u)$ for both $j$ in $\{0, 1\}$. If $b \leq \iota_{\text{max}}(V_u)$, then let $\iota_0(u, b), \iota_1(u, b) \in \mathbb{Z}^{V_u}$ with $0 \leq \iota_j(u, b) \leq \iota_{\text{max}}$, and $\iota_j(u, b)(V_u) = b$ for both $j \in \{0, 1\}$, be such that

$$x_0(u, b) = \text{dyn}(T_u, \tau + \iota_0(u, b))$$

$$x_1(u, b) = \text{dyn}(T_u, (\tau + \iota_1(u, b))^{-u})$$

where, if possible, let $\iota_0(u, b) = \iota_1(u, b)$. As we show in Corollary 2.3 below, $\iota_0(u, b)$ always equals $\iota_1(u, b)$, which is a key fact for our approach.

**Lemma 2.1.** $x_0(u, b) \geq x_1(u, b)$, and if $x_0(u, b) = x_1(u, b)$, then $\iota_0(u, b) = \iota_1(u, b)$.

**Proof.** If $x_1(u, b) = -\infty$, then the statement is trivial. Hence, we may assume that $x_1(u, b) > -\infty$, which implies that the function $\iota_1(u, b)$ is defined. Let $D$ be a minimum dynamic monopoly of $(T_u, \tau + \iota_1(u, b))$. By the definition of $x_0(u, b)$, we have $x_0(u, b) \geq |D|$. Since $D$ is a dynamic monopoly of $(T_u, (\tau + \iota_1(u, b))^{-u})$, we obtain $x_0(u, b) \geq |D| \geq \text{dyn}(T_u, (\tau + \iota_1(u, b))^{-u}) = x_1(u, b)$. Furthermore, if $x_0(u, b) = x_1(u, b)$, then $x_0(u, b) = |D| = \text{dyn}(T_u, \tau + \iota_1(u, b))$, which implies $\iota_0(u, b) = \iota_1(u, b)$. \qed
Lemma 2.2. If \( u \) is a leaf of \( T \), and \( b \) is an integer with \( 0 \leq b \leq \tau_{\text{max}}(u) \), then, for \( j \in \{0, 1\}, \)
\[
x_j(u, b) = \begin{cases} 0 & \text{if } \tau(u) + b - j \leq 0, \\ 1 & \text{otherwise, and} \end{cases}
\]
\[
\iota_j(u, b)(u) = b.
\]

Proof. These equalities follow immediately from the definitions. \(
\)

Lemma 2.3. Let \( u \) be a vertex of \( T \) that is not a leaf, and let \( b \) be a non-negative integer. If \( v_1, \ldots, v_k \) are the children of \( u \), and \( \iota_0(v_i, b_i) = \iota_1(v_i, b_i) \) for every \( i \in [k] \) and every integer \( b_i \) with \( 0 \leq b_i \leq \tau_{\text{max}}(V_{v_i}) \), then, for \( j \in \{0, 1\}, \)
\[
x_j(u, b) = z_j(u, b), \quad \iota_0(u, b) = \iota_1(u, b), \quad \text{if } b \leq \tau_{\text{max}}(V_u),
\]
where \( z_j(u, b) \) is defined as
\[
\max \left\{ \delta_j(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i) : (b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b) \text{ with } b_u \leq \tau_{\text{max}}(u) \right\},
\]
and, for \( (b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b) \) with \( b_u \leq \tau_{\text{max}}(u), \)
\[
\delta_j(b_u, b_1, \ldots, b_k) := \begin{cases} 0 & \text{if } \left\{ i \in [k] : x_0(v_i, b_i) = x_1(v_i, b_i) \right\} \geq \tau(u) + b_u - j, \text{ and} \\ 1 & \text{otherwise.} \end{cases}
\]

Proof. By symmetry, it suffices to consider the case \( j = 0 \).

First, suppose that \( b > \tau_{\text{max}}(V_u) \). If \( (b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b) \) with \( b_u \leq \tau_{\text{max}}(u) \), then \( b_i > \tau_{\text{max}}(V_{v_i}) \) for some \( i \in [k] \), which implies \( z_0(u, b) = -\infty = x_0(u, b). \)

Now, let \( b \leq n(T_u) \), which implies \( x_0(u, b) > -\infty \). The following two claims complete the proof of (4).

Claim 1. \( x_0(u, b) \geq z_0(u, b) \).

Proof of Claim [7]. It suffices to show that \( x_0(u, b) \geq \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i) \) for every choice of \( (b_u, b_1, \ldots, b_k) \) in \( \mathcal{P}_{k+1}(b) \) with \( b_u \leq \tau_{\text{max}}(u) \) and \( b_i \leq \tau_{\text{max}}(V_{v_i}) \) for every \( i \in [k] \). Let \( (b_u, b_1, \ldots, b_k) \) be one such an element. Let \( \iota_u \in ZV_u \) be defined as
\[
\iota_u(v) = \begin{cases} b_u & \text{if } v = u, \text{ and} \\ 0 & \text{otherwise,} \end{cases}
\]
and let \( \iota = \iota_u + \sum_{i=1}^{k} \iota_1(v_i, b_i) \), where \( \iota_1(v_i, b_i)(u) \) is set to 0 for every \( i \in [k] \). Since \( \iota(V_u) = b \) and \( 0 \leq \iota \leq \tau_{\text{max}} \), we have \( x_0(u, b) \geq \text{dyn}(T_u, \tau + \iota) \).

Let \( D \) be a minimum dynamic monopoly of \( (T_u, \tau + \iota) \), that is, \( |D| \leq x_0(u, b) \). For each \( i \in [k] \), it follows that the set \( D_i = D \cap V_{v_i} \) is a dynamic monopoly of \( (T_{v_i}, (\tau + \iota)^{v_i}) \). Since,
restricted to $V_{u,i}$, the two functions $(\tau + i)^{\rightarrow x_i}$ and $(\tau + \ell_1(v_i, b_i))^{\rightarrow x_i}$ coincide, we obtain
\[
|D_i| \geq \text{dyn}\left(T_{v_i}, \left(\tau + \ell_1(v_i, b_i)\right)^{\rightarrow x_i}\right) \geq x_1(v_i, b_i).
\]
If $\delta_0(b_u, b_1, \ldots, b_k) = 0$, then $|D| \geq \sum_{i=1}^{k} |D_i| \geq \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i)$. Similarly, if $u \in D$, then $|D| = 1 + \sum_{i=1}^{k} |D_i| \geq \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i)$. Therefore, we may assume that $\delta_0(b_u, b_1, \ldots, b_k) = 1$ and that $u \notin D$. This implies that there is some $\ell \in [k]$ with $x_0(v_{\ell}, b_{\ell}) > x_1(v_{\ell}, b_{\ell})$ such that $D_\ell = D \cap V_{v_\ell}$ is a dynamic monopoly of $(T_{v_\ell}, \tau + \ell)$. Since, by assumption, $\ell_0(v_{\ell}, b_{\ell}) = \ell_1(v_{\ell}, b_{\ell})$, we obtain that, restricted to $V_{v_\ell}$, the two functions $\tau + \ell$ and $\tau + \ell_0(v_{\ell}, b_{\ell})$ coincide, which implies $|D_\ell| \geq \text{dyn}(T_{v_\ell}, \tau + \ell_0(v_{\ell}, b_{\ell})) = x_0(v_{\ell}, b_{\ell}) \geq 1 + x_1(v_{\ell}, b_{\ell})$. Therefore, also in this case, $|D| = |D_\ell| + \sum_{i \in [k] \setminus \{\ell\}} |D_i| \geq \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i).$

Claim 2. $x_0(u, b) \leq z_0(u, b)$.

Proof of Claim 2. Let $i = \iota_0(u, b)$, that is, $x_0(u, b) = \text{dyn}(T_u, \tau + i)$. Let $b_i = \iota(V_{v_i})$ for every $i \in [k]$, and let $b_u = b - \sum b_i$. Clearly, $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$ and $b_u \leq \iota_{\text{max}}(u)$. Let $D_i$ be a minimum dynamic monopoly of $(T_{v_i}, (\tau + i)^{\rightarrow x_i})$ for every $i \in [k]$. By the definition of $x_1(v_i, b_i)$, we obtain $|D_i| \leq x_1(v_i, b_i)$. Let $D = \{u\} \cup \bigcup_{i=1}^{k} D_i$. The set $D$ is a dynamic monopoly of $(T_u, \tau + i)$, which implies $x_0(u, b) \leq |D|$. If $\delta_0(b_u, b_1, \ldots, b_k) = 1$, then
\[
x_0(u, b) \leq |D| = 1 + \sum_{i=1}^{k} |D_i| \leq \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^{k} x_1(v_i, b_i) \leq z_0(u, b).
\]
Therefore, we may assume that $\delta_0(b_u, b_1, \ldots, b_k) = 0$. By symmetry, we may assume that $x_0(v_i, b_i) = x_1(v_i, b_i)$ for every $i \in [\tau(u) + b_u]$. Let $D'_i$ be a minimum dynamic monopoly of $(T_{v_i}, \tau + i)$ for every $i \in [\tau(u) + b_u]$. By the definition of $x_0(v_i, b_i)$, we obtain $|D'_i| \leq x_0(v_i, b_i) = x_1(v_i, b_i)$. Let $D' = \bigcup_{i \in [\tau(u) + b_u]} D'_i \cup \bigcup_{i \in [k] \setminus [\tau(u) + b_u]} D_i$. The set $D'$ is a dynamic monopoly of $(T_u, \tau + i)$. This implies
\[
x_0(u, b) \leq |D'| = \sum_{i \in [\tau(u) + b_u]} |D'_i| + \sum_{i \in [k] \setminus [\tau(u) + b_u]} |D_i| \leq \sum_{i \in [k]} x_1(v_i, b_i) \leq z_0(u, b),
\]
which completes the proof of the claim.

It remains to show (5). If $x_0(u, b) = x_1(u, b)$, then (5) follows from Lemma 2.1. Hence, we may assume that $x_0(u, b) > x_1(u, b)$. Since, by definition,
\[
\delta_1(b_u, b_1, \ldots, b_k) \leq \delta_0(b_u, b_1, \ldots, b_k) \leq \delta_1(b_u, b_1, \ldots, b_k) + 1
\]
for every $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$ with $b_u \leq \iota_{\text{max}}(u)$, we obtain $z_1(u, b) \leq z_0(u, b) \leq z_1(u, b) + 1$. 

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Together with (4), the inequality \( x_0(u, b) > x_1(u, b) \) implies that

\[
x_0(u, b) = z_0(u, b) > z_1(u, b) = x_1(u, b) \quad \text{and} \quad z_1(u, b) = z_0(u, b) - 1.
\]

Let \((b_u, b_1, \ldots, b_k) \in P_{k+1}(b)\) with \(b_u \leq \tau_{\text{max}}(u)\) be such that

\[
z_0(u, b) = \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^k x_1(v_i, b_i).
\]

We obtain

\[
z_1(u, b) \geq \delta_1(b_u, b_1, \ldots, b_k) + \sum_{i=1}^k x_1(v_i, b_i)
\]

\[
\geq \delta_0(b_u, b_1, \ldots, b_k) - 1 + \sum_{i=1}^k x_1(v_i, b_i)
\]

\[
= z_0(u, b) - 1
\]

\[
= z_1(u, b),
\]

which implies \( z_1(u, b) = \delta_1(b_u, b_1, \ldots, b_k) + \sum_{i=1}^k x_1(v_i, b_i) \), that is, the same choice of \((b_u, b_1, \ldots, b_k)\) in \(P_{k+1}(b)\) with \(b_u \leq \tau_{\text{max}}(u)\) maximizes the terms defining \(z_0(u, b)\) and \(z_1(u, b)\).

Since \(z_0(u, b) > z_1(u, b)\), we obtain \(\delta_1(b_u, b_1, \ldots, b_k) = 0\) and \(\delta_0(b_u, b_1, \ldots, b_k) = 1\), which, by the definition of \(\delta_j\), implies that there are exactly \(\tau(u) + b_u - 1\) indices \(i\) in \([k]\) with \(x_0(v_i, b_i) = x_1(v_i, b_i)\). By symmetry, we may assume that \(x_0(v_i, b_i) = x_1(v_i, b_i)\) for \(i \in [\tau(u) + b_u - 1]\) and \(x_0(v_i, b_i) > x_1(v_i, b_i)\) for \(i \in [k] \setminus [\tau(u) + b_u - 1]\).

Let \(\iota = \iota_u + \sum_{i=1}^k \iota_0(v_i, b_i)\), where \(\iota_0(v_i, b_i)(u)\) is set to 0 for every \(i \in [k]\) and \(\iota_u\) is as in (6). Note that, by assumption, we have \(\iota = \iota_u + \sum_{i=1}^k \iota_1(v_i, b_i)\). Let \(D\) be a minimum dynamic monopoly of \((T_u, \tau + \iota)\). By the definition of \(x_0(u, b)\), we have \(|D| \leq x_0(u, b)\). Let \(D_i = D \cap V_{v_i}\) for every \(i \in [k]\). Since \(D_i\) is a dynamic monopoly of \((T_{v_i}, (\tau + \iota)^{+v_i})\) for every \(i \in [k]\), we obtain \(|D_i| \geq x_1(v_i, b_i)\). Note that

- either \(u \in D\),
- or \(u \not\in D\) and there is some index \(\ell \in [k] \setminus [\tau(u) + b_u - 1]\) such that \(D_{\ell} = D \cap V_{v_\ell}\) is a dynamic monopoly of \((T_{v_\ell}, \tau + \iota)\).

In the first case, we obtain

\[
z_0(u, b) = x_0(u, b) \geq |D| = 1 + \sum_{i=1}^k |D_i| \geq 1 + \sum_{i=1}^k x_1(v_i, b_i) = z_0(u, b),
\]
and, in the second case, we obtain $|D| \geq x_0(v_\ell, b_\ell) \geq x_1(v_\ell, b_\ell) + 1$, and, hence,

$$z_0(u, b) = x_0(u, b) \geq |D| = |D_\ell| + \sum_{i \in [k] \setminus \{\ell\}} |D_i| \geq 1 + \sum_{i=1}^k x_1(v_i, b_i) = z_0(u, b).$$

In both cases we obtain $|D| = x_0(u, b)$, which implies that $\mu_0(u, b)$ may be chosen equal to $\ell$.

Now, let $D^-$ be a minimum dynamic monopoly of $(T_u, (\tau + \ell)^{-\infty})$. By the definition of $x_1(u, b)$, we have $|D^-| \leq x_1(u, b)$. Let $D^- = D^- \cap V_{\ell}$ for every $i \in [k]$. Since $D^-_i$ is a dynamic monopoly of $(T_{\ell}, (\tau + \ell)^{-\infty})$ for every $i \in [k]$, we obtain $|D^-_i| \geq x_1(v_i, b_i)$. Now,

$$z_1(u, b) = x_1(u, b) \geq |D^-| \geq \sum_{i=1}^k x_1(v_i, b_i) = z_1(u, b),$$

which implies that $|D^-| = x_1(u, b)$, and that $\mu_1(u, b)$ may be chosen equal to $\ell$. Altogether, the two functions $\mu_0(u, b)$ and $\mu_1(u, b)$ may be chosen equal, which implies $\mathcal{M}$.

Applying induction using Lemma 2.2 and Lemma 2.3, we obtain the following.

**Corollary 2.4.** $\mu_0(u, b) = \mu_1(u, b)$ for every vertex $u$ of $T$, and every integer $b$ with $0 \leq b \leq \mu_{\text{max}}(V_u)$.

Apart from the specific values of $x_0(u, b)$ and $x_1(u, b)$, the arguments in the proof of Lemma 2.3 also yield feasible recursive choices for $\mu_0(u, b)$. In fact, if

$$x_0(u, b) = \delta_0(b_u, b_1, \ldots, b_k) + \sum_{i=1}^k x_1(v_i, b_i) > -\infty$$

for $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$ with $b_u \leq \mu_{\text{max}}(u)$, and $\mu_u$ is as in $\mathcal{M}$, then $\mu_u + \sum_{i=1}^k \mu_0(v_i, b_i)$ is a feasible choice for $\mu_0(u, b)$.

Our next lemma explains how to efficiently compute the expressions in Lemma 2.3.

**Lemma 2.5.** Let $u$ be a vertex of $T$ that is not a leaf, let $b$ be an integer with $0 \leq b \leq \mu_{\text{max}}(V_u)$, and let $v_1, \ldots, v_k$ be the children of $u$. If the values $x_1(v_i, b_i)$ are given for every $i \in [k]$ and every integer $b_i$ with $0 \leq b_i \leq \mu_{\text{max}}(V_v)$, then $x_0(u, b)$ and $x_1(u, b)$ can be computed in time $O(k^2(b + 1)^2)$.

**Proof.** By symmetry, it suffices to explain how to compute $z_0(u, b)$.

For $p \in \{0\} \cup [k]$, an integer $p_\ell$, an integer $b_\ell' \in \{0\} \cup [b]$, and $b_u \in \{0\} \cup [\min\{\mu_{\text{max}}(u), b_\ell'\}]$, let $M(p, p_\ell, b_\ell', b_u)$ be defined as the maximum of the expression $\sum_{i=1}^p x_1(v_i, b_i)$ over all $(b_1, \ldots, b_p) \in \mathcal{P}_p(b_\ell' - b_u)$ such that $p_\ell$ equals $\left|\{i \in [p] : x_0(v_i, b_i) = x_1(v_i, b_i)\}\right|$. Clearly, $M(p, p_\ell, b_\ell', b_u) = -\infty$ if $p < p_\ell$ or $p_\ell < 0$ or $b_\ell' - b_u > \sum_{i=1}^p \mu_{\text{max}}(V_{v_i})$, and

$$M(0, 0, b_\ell', b_u) = \begin{cases} 0, & \text{if } b_\ell' = b_u, \\ -\infty, & \text{otherwise.} \end{cases}$$
For $p \in [k]$, the value of $M(p, p_\equiv, b', b_u)$ is the maximum of the following two values:

- The maximum of $M(p - 1, p_\equiv - 1, b_{\leq p - 1}, b_u) + x_1(v_p, b_p)$ over all $(b_{\leq p - 1}, b_p) \in \mathcal{P}_2(b' - b_u)$ with $x_0(v_p, b_p) = x_1(v_p, b_p)$, and
- the maximum of $M(p - 1, p_\equiv, b_{\leq p - 1}, b_u) + x_1(v_p, b_p)$ over all $(b_{\leq p - 1}, b_p) \in \mathcal{P}_2(b' - b_u)$ with $x_0(v_p, b_p) > x_1(v_p, b_p),$

which implies that $M(p, p_\equiv, b', b_u)$ can be determined in $O(b' + 1)$ time given the values $M(p - 1, p_\equiv - 1, b_{\leq p - 1}, b_u), M(p - 1, p_\equiv, b_{\leq p - 1}, b_u), x_0(v_p, b_p)$, and $x_1(v_p, b_p)$.

Altogether, the values $M(k, p_\equiv, b, b_u)$ for all $p_\equiv \in \{0\} \cup [k]$ can be determined in time $O\left(k^2(b + 1)\right)$.

For $b_u \in \{0\} \cup [\min\{\ell_{\text{max}}(u, b)\}]$, let $m(b_u)$ be the maximum of the two expressions

$$1 + \max\left\{M(k, p_\equiv, b, b_u) : p_\equiv \in \{0\} \cup [\tau(u) - b_u - 1]\right\}$$

and

$$\max\left\{M(k, p_\equiv, b, b_u) : p_\equiv \in [k] \setminus [\tau(u) - b_u - 1]\right\}.$$

Now, by the definition of $\delta_0(b_u, b_1, \ldots, b_k)$, the value of $z_0(u, b)$ equals $\max\left\{m(b_u) : b_u \in \{0\} \cup [\min\{\ell_{\text{max}}(u, b)\}]\right\}$. Hence, $z_0(u, b)$ can be computed in time $O\left(k^2(b + 1)^2\right)$. 

We proceed to the proof of our first theorem.

**Proof of Theorem 1.1.** Given $(T, \tau, \ell_{\text{max}}, b)$, Lemma 2.2 to Lemma 2.3 imply that the values of $x_0(u, b')$ and of $x_1(u, b')$ for all $u \in V(T)$ and all $b' \in \{0\} \cup [b]$ can be determined in time

$$O\left(\sum_{u \in V(T)} d_T(u)^2(b + 1)^2\right).$$

It is a simple folklore exercise that $\sum_{u \in V(T)} d_T(u)^2 \leq n^2 - n$ for every tree $T$ of order $n$, which implies the statement about the running time. Since $\text{vacc}(T, \tau, \ell_{\text{max}}, b) = x_0(r, b)$, the statement about the value of $\text{vacc}(T, \tau, \ell_{\text{max}}, b)$ follows. The statement about the increment $\ell$ follows easily from the remark after Corollary 2.4 concerning the function $\ell_0(u, b)$, and the proof of Lemma 2.3, where, next to the values $M(p, p_\equiv, b', b_u)$, one may also memorize suitable increments.

We conclude with the proof of our second theorem.

**Proof of Theorem 1.4.** Let $G$ be an $r$-regular graph of order $n$, and let $b$ be an integer with $(2r - 1)(r + 1) \leq b \leq rn = 2m(G)$.

Let $\ell \in \ell(G)$ with $0 \leq \ell \leq d_G$ and $\ell(V(G)) = b$ be such that $\text{vacc}(G, 0, d_G, b) = \text{dyn}(G, \ell)$. By a result of Ackerman et al. [1],

$$\text{vacc}(G, 0, d_G, b) = \text{dyn}(G, \ell) \leq \sum_{u \in V(G)} \frac{\ell(u)}{d_G(u) + 1} = \frac{\ell(V(G))}{r + 1} = \frac{b}{r + 1}. $$
First, suppose that the matching number $\nu$ of $G$ satisfies $2r\nu > b$. In this case, $G$ has a matching $M$ with $\tau_M(V(G)) = 2r|M| \leq b$ and $2r(|M| + 1) \geq b + 1$, where $\tau_M$ is as in the statement. We obtain $2\text{dyn}(G, \tau_M) \geq 2|M| \geq 2 \left( \frac{b+1}{2r} - 1 \right) \geq \frac{b}{r+1} \geq \text{vacc}(G, 0, d_G, b)$. Next, suppose that $2r\nu \leq b$. If $M$ is a maximum matching and $D$ is a minimum vertex cover, then $|D| \leq 2|M|$. Since $D$ is a dynamic monopoly of $(G, d_G)$, we obtain $2\text{dyn}(G, \tau_M) \geq 2|M| \geq |D| \geq \text{dyn}(G, d_G) \geq \text{vacc}(G, 0, d_G, b)$, that is, $2\text{dyn}(G, \tau_M) \geq \text{vacc}(G, 0, d_G, b)$ holds in both cases. 

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