Coherent description of electrical and thermal impurity–and–phonon limited transport in simple metals

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Abstract

The electrical resistivity, thermoelectric power and electronic thermal conductivity of simple (isotropic) metals are studied in a uniform way. Starting from results of a variational solution of the Boltzmann equation, a generalized Matthiessen rule is used in order to superpose the inelastic (or not) electron-phonon and elastic electron-impurity scattering cross sections ("matrix elements"). The temperature dependence relative to these processes is given through simple functions and physical parameters over the usually investigated range of temperature for each transport coefficient. The coherence of such results is emphasized.

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I. INTRODUCTION

To consider the most simple models is sometimes necessary for studying the charge carrier and thermal transport in solids. Indeed mathematical approximations used on more sophisticated models may make less evident, or even completely suppress, certain effects. For the transport coefficients the latter statement surely concerns the thermoelectric power and the thermal conductivity. For studying the electrical resistivity theoretical methods are surely simpler, better developed and more reliable.

When there is more than a single scattering source the electrical resistivity is sufficiently accurately treated in metals under the assumption of Matthiessen rule (see exceptions in Concluding Remarks). Further assumptions are necessary to find simple
and effective expressions for the thermoelectric power and the thermal conductivity. The Gorter–Nordheim rule, which is usually applied to treat the thermoelectric power, is based on the assumption of the Mott(-Jones) formula validity, however, restricted to the elastic scattering and the adiabatic approximation. For the thermal conductivity the Wiedemann–Franz law is a usual basis on which the separation of the electronic and lattice contribution to the experimental thermal conductivity is made, whence for comparing experimental and theoretical data. The validity of the Wiedemann–Franz law is also restricted to elastic collisions. Clearly to resolve the question of elasticity and inelasticity of the scattering is essential for comparing experimental data and theoretical laws in particular for the thermoelectric power and the thermal conductivity.

The most simple inelastic scattering model of the charge carriers by phonons describes most features of the thermoelectric power in normal metals. Consequently, an approach well accounting for the energy transfer between the charge carriers and the scattering quasiparticles should be applied beyond that based on the Mott formula.

It is known that the elastic approximation is sufficient for considering the scattering of the conduction electrons by static impurities. This leads to a constant, temperature independent residual resistivity, when the potential is appropriately chosen and the electron gas is degenerate. The simplest and well founded potential which leads to the residual resistivity in metals is the screened Coulomb potential (or Yukawa potential) provided that impurity concentration is sufficiently low.

Phenomenological considerations show that without assuming a behavior for the electrical residual resistivity one cannot explain the low temperature behavior of the electronic contribution to the thermal conductivity. At moderate temperatures, along with the scattering by phonons the impurity scattering explains the magnitude and the basic features of the temperature dependence of the thermal conductivity of normal nonmagnetic and even magnetic metals, provided that the magnetic moment in the latter is periodic and well localized. It was shown indeed in Ref. devoted solely to the thermal conductivity, that the magnetic scattering was of equal importance to the impurity and the phonon scattering. However, was neither compared to the electrical resistivity , nor to the thermoelectric power , for example calculated in the same temperature interval in Ref. Moreover the final results were not presented through standard functions such as the Bloch–Grüneisen ones. We shall do so in the present paper, but confining ourselves to non magnetic impurity and phonon scattering only.

As in our previous papers we shall apply the Ziman variational approach within the Boltzmann equation formalism, an approach in which the notion of ”relaxation time” is not used. Thus, the relationship to the Mott formula is not straightforward. However, terms responsible for elastic and inelastic scattering can be easily selected, at least when the conduction band is parabolic. Within this approach one can generalize the Matthiessen rule: we consider a superposition of the transition probabilities themselves within the Boltzmann equation formalism instead of the (integrated) relaxation time components. Therefore the electrical resistivity, the thermoelectric power and the thermal conductivity are considered here with respect to impurity and phonon scattering on the same ground, i.e. both model and approximations. For this purpose it is sufficient to
introduce a single parameter which accounts for the mutual contribution of the phonon and the impurity scattering to the transport coefficients.

A similar model accounting for the scattering by phonons and impurities was considered in the fifties within the Kohler variational formalism. However, the mathematical method was more complicated and less physical than that developed later by Ziman. It did not allow to get qualitatively correct results for the thermoelectric power in contrast to those for the electrical resistivity and the electronic thermal conductivity.

The finding of an accurate solution of the (Bloch–)Boltzmann equation is not a past problem of the fifties or the seventies but still a present-day problem. Numerical methods are used for this purpose. Recently the Allen method of solving the equation with a good numerical convergence has been used to study the influence of the electron–electron scattering on the electrical and thermal conductivity. A comparative discussion of the approximation made while using the Ziman variation method and the Allen one is given in Sec. V.

In the following Section II we shortly describe the Ziman variational method and the general results when applied to the model here defined. Certain general results and formulae are also given in Appendix A for completeness. In Section III the matrix elements describing the phonon and impurity scattering are calculated and final formulas for the electrical resistivity, electronic thermal conductivity and thermoelectric power are systematically presented. In Section IV the temperature dependence of the transport coefficients and that of the Lorenz number is numerically illustrated. The visual presentation is made by using dimensionless functions. An analysis of appropriate material constants allowing to estimate the magnitude of the final transport coefficients is found in Appendix B. Section V contains the concluding remarks.

II. MODEL, METHOD AND APPROXIMATIONS.

We assume that the current carriers are free electrons with a parabolic isotropic energy band $\varepsilon(k) = \hbar^2 k^2 / 2m$ spectrum. We also assume that the concentration of the electron gas (of both subbands with spin up or down) $n_e = (1/3\pi^2)(2m/\hbar^2)^{3/2}\varepsilon_F^{3/2}$ (or the Fermi energy $\varepsilon_F$) is sufficiently high to consider the gas to be degenerate. Then one can assume that the chemical potential is $\zeta(T) \cong \varepsilon_F$, and all relevant functions of $z = \zeta/k_BT$ can be considered as asymptotic expressions of the argument $\varepsilon_F/k_BT$. In particular the Fermi-Dirac integrals

$$ F_n(z) = \int_0^\infty dx \frac{x^n}{1 + \exp[x - z]}, \quad \text{with} \quad z = \frac{\zeta(T)}{k_BT} \tag{1} $$

can be represented as

$$ F_n(z) \cong z^{n+1}/(n+1), \quad \text{with} \quad z \cong \varepsilon_F/k_BT \tag{2} $$

for sufficiently large $z$. 
The carriers are assumed to be scattered by static, nonmagnetic impurities, which concentration is sufficiently low to have a linear influence on the electrical resistivity. The scattering of carriers by phonons will be considered on the same footing. However, the phonon (quasiparticle) scattering will be treated as inelastic while the impurity scattering is elastic in its nature, since the impurity is usually assumed to be more heavy than the electron effective mass. Even though some calculations can be performed without assuming the form of the potential, we assume the form of the impurity potential to be the screened (Yukawa) potential
\[ U(r) = -\frac{Z e^2 \exp(-\lambda r)}{r} \] (3)
with \( r = |r| \), \( Z = Z_{\text{impurity}} - Z_{\text{host ion}} \) the effective valence and the screening constant \( \lambda \). The Fourier transform \( U(q) = 4\pi Z e^2 / (\lambda^2 + q^2) \) of the above function is responsible for the magnitude of the carrier scattering; \( q \equiv |q| \) is the electron wave vector transferred during the scattering. The transport relaxation time following from \( U(q) \) — and consequently the residual resistivity — is temperature independent for a degenerate electron gas. Their magnitude is related to the Fermi wave vector \( k_F \) or the energy \( \varepsilon_F \) through the function
\[ 2 \int_{0}^{2k_F} \frac{dq q^3}{(\lambda^2 + q^2)^2} = \ln(1 + 4k_{F\lambda}^2) - 4k_{F\lambda}^2/(1 + 4k_{F\lambda}^2) \equiv \Phi(2k_{F\lambda}), \] (4)
where \( k_{F\lambda} = k_F / \lambda \).

Note that the standard considerations\[^3\] yield that \( \lambda^2 \sim n_e^{1/3} \) and thus one can estimate \( 2k_F = (3\pi)^{1/6}(a_H k_F)^{1/2} \approx 1 \) since the effective Bohr radius \( a_H = \hbar/(me^2) \) is of the order of \( k_F \).

The phonon system and the electron–phonon scattering are considered in the Bloch approximation, \textit{i.e.} phonons in the harmonic approximation and the electron–phonon interaction in the deformation potential approximation. We restrict the magnitude of the phonon wave vector, \( q \leq q_D \), to be within the Debye sphere. Then, for sufficiently high values of the Fermi energy, \( z = \rightarrow \infty \) (see, e.g., Appendix B in Ref. \[^8\]), the phonon limited transport coefficients are expressed in terms of the Bloch(–Grüneisen) functions
\[ \mathcal{F}_n(t) = \int_0^{1/t} \frac{dx x^n}{(e^x - 1)(1 - e^{-x})}. \] (5)

A relation between the deformation potential approximation and the method based on the dynamical structure factor introduced by van Hove\[^\text{16}\] is described in Sec. V.

As we mentioned in the Introduction we apply the Ziman variation method\[^\text{10}\] to consider the combined effect of the impurity and phonon scattering on the transport coefficients. The method consists in calculating the elements of a matrix describing the scattering and two vectors governing the flow of the electric charge and heat. The "exact" results correspond to an infinite order matrix and infinite dimension current vectors
(see, e.g. Appendix A and Sec. IV in Ref. [5]). Here, we truncate the matrix and vectors to four elements and two components, respectively. This is the lowest order approximation, which accounts well for both the electrical and thermal current if the variational trial functions are appropriately chosen [6]. This choice is known to be

\[ \Omega_i(k) = (\mathbf{v} \cdot \mathbf{a}) (\varepsilon - \zeta)^{i-1}, \]  

where \( i = 1 \) and \( 2 \), while \( \mathbf{a} \) denotes the unit vector in the direction of the applied temperature gradient or that of the external electric field causing the charge flow, and \( \mathbf{v} = (\hbar/m) \nabla_k \varepsilon \) is the electron velocity.

In this approximation the transport coefficients: the electrical resistivity \( \rho \), the thermoelectric power or the Seebeck coefficient \( S \) and the (electronic) thermal conductivity \( \kappa \) for cubic crystals, are given as such

\[ \rho = (P_{11}P_{22} - P_{12}^2) \Pi^{-1}, \]  

\[ S = [P_{22}J_1U_2 - P_{12}(J_1U_2 + J_2U_1) + P_{11}J_2U_2] (T \Pi)^{-1} \]  

\[ \kappa = (J_1^2U_2^2 + J_2^2U_1^2 - 2J_1J_2U_1U_2) (T \Pi)^{-1}. \]

with

\[ \Pi = P_{22}J_1^2 - 2P_{12}J_1J_2 + P_{11}J_2^2, \]

the elements of the scattering matrix being

\[ P_{ij} = \int \! dk \int \! dk' C(k, k') u_{ij}(k, k') , \]

\[ u_{ij}(k, k') = \left[ \Omega_i(k) - \Omega_i(k') \right] \left[ \Omega_j(k) - \Omega_j(k') \right] , \]

and the components of the trial currents

\[ J_i = -e \int \! dk \left( -\frac{df}{d\varepsilon} \right) (\mathbf{v} \cdot \mathbf{a}) \Omega_i(k) , \]

\[ U_i = \int \! dk \left( -\frac{df}{d\varepsilon} \right) (\mathbf{v} \cdot \mathbf{a})(\varepsilon - \zeta) \Omega_i(k) , \]

\(^1\)The factor \((\mathbf{v} \cdot \mathbf{a})\) in the expression for the trial currents in Ref. 5 was inadvertently omitted. In the trial functions (12)-(13) this factor is used as in Ref. 2, and in our previous papers where it was denoted \((\mathbf{k} \cdot \mathbf{a})\).
where $C(k, k')$ is the transition probability to be determined on the ground of an appropriate Hamiltonian describing the scattering. The function $f$ denotes the Fermi–Dirac one, $f = 1/[1 + \exp(\varepsilon - \zeta)/k_B T]$, and $-e(e > 0)$ is the electron charge.

The integrals in (11) and (12)-(13) should be taken over the Brillouin zone but in our free electron considerations we let

$$\int \! dk \ldots \equiv \frac{2}{(2\pi)^3} \int_0^\infty \! d\varepsilon \int \left( \frac{dS_\varepsilon}{|\nabla_\varepsilon|} \right) \ldots,$$

where $dS_\varepsilon/|\nabla_\varepsilon|$ is an element of the isoenergy surface. The integrals with respect to the electron energy in (10) for the degenerate electron gas are described in Appendix B of Ref. 8. In the case of elastic scattering by impurities, as here, the integrals can be treated as simply. Namely, we take advantage of the approximate equalities

$$\int_0^\infty \! d\varepsilon (-df/d\varepsilon)(\varepsilon - \zeta)g(\varepsilon) = (\pi^2/3)(k_B T)^2 [dg/d\varepsilon]_\zeta,$$

$$\int_0^\infty \! d\varepsilon (-df/d\varepsilon)(\varepsilon - \zeta)^2g(\varepsilon) = (\pi^2/3)(k_B T)^2 g(\zeta).$$

where $g(\varepsilon)$ is an arbitrary function.

The trial currents (12)-(13) can be represented as a combination of the Fermi–Dirac integrals (see Appendix B):

$$J_1 = -J_0(k_B T)^{3/2}L_0(z), \quad J_2 = -J_0(k_B T)^{5/2}L_1(z), \quad U_1 = -U_0(k_B T)^{5/2}L_1(z), \quad U_2 = -U_0(k_B T)^7L_2(z),$$

where $U_0$ and $J_0 = eU_0$ are material constants.

For the electron–phonon system the $P$–matrix elements can be derived (see Ref. 8) without any further approximation. They are expressed by double integrals with respect to the electron energy and the transferred electron momentum (wave vector). For a Debye cutoff and in the limit $z \to \infty$ they can be represented by $P_{ij} = P_0(k_B T)^{i+j-2}P_{ij}$, where $P_0$ is a material constant and $P_{ij} = P_{ij}(t)$ are dimensionless combinations of the Bloch–Grüneisen functions of the argument $t = T/T_D$ ($T_D$ is the Debye temperature).

For the impurity scattering an expression similar to the above is readily obtained. In this case the $P_{ij}$’s are simply related to $\Phi(2k_F/\lambda)$ defined in (4). The explicit form of the functions $P_{ij}$ will be given in the next Section. The magnitude of $P_0$ and that of other material constants are estimated in Appendix B.

In view of the above, by calculating the scattering matrix elements in the degenerate limit, retaining only $P_{11}(\gg P_{12}/P_{22})$ in case of the electrical resistivity and the lowest order terms in the $1/z$ in the trial currents (12)-(13), we obtain

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2 For further convenience, the factor $t^5$ used in Ref.8 is introduced here directly in the definition of $P_{ij}$. 

6
\begin{align}
\rho &= \frac{P_{11}}{J_1^2} = \frac{P_0 \mathcal{P}_{11}}{J_0^2 \varepsilon_F^3},
\end{align}

\begin{align}
S &= S_0(t) + S_1(t) \frac{k_B T}{\varepsilon_F},
\end{align}

\begin{align}
\frac{1}{\kappa} &= \frac{P_{22}}{(\pi^2/3) \mathcal{L}_0 T (k_B T)^3 J_1^2} = \frac{P_0}{U_0^2} \frac{T \mathcal{P}_{22}}{(\pi^2/3)^2 (k_B T)^2 \varepsilon_F^3},
\end{align}

where \(J_1\) in (17) and (19) is the trial current in its asymptotic form and \(\mathcal{L}_0 = (\pi^2/3)(k_B/e)^2\) is the Sommerfeld value of the Lorenz number. The explicit form of the functions \(S_0(t)\) and \(S_1(t)\) of (18) – having a more complicated dependence on the matrix elements \(\mathcal{P}_{ij}\) – are discussed in Section III B.

### III. SCATTERING MATRIX CONTRIBUTIONS.

As seen from (17)-(19) it remains to calculate the elements of the \(P\)-matrix in order to find the final expressions for the transport coefficients. In our case the crux is to take into account the combined effect of the scattering by impurities and phonons.

As we mentioned in the Introduction we apply the Matthiessen rule in a more generalized way [1, 11, 12], i.e. to the microscopic transition probabilities \(C(k, k')\) themselves appearing in Eq. (11) rather than to the final coefficients: the electrical and thermal resistivity. The variational method enables us to use the same approximation to both these coefficients and the thermoelectric power through the integrated probabilities, i.e, we assume

\begin{align}
C(k, k') &= C^{(imp)}(k, k') + C^{(ph)}(k, k').
\end{align}

This means that the resultant scattering probability for the phonon and impurity scattering can be simply considered as a sum of the scattering probabilities of the impurities (the phonon scattering being quenched) and by phonons (those of the pure crystal). This, of course, implies that the elements of the resultant scattering matrix, (11), are also the sums of the corresponding matrix elements for the phonon and impurity scattering. Therefore, we write

\begin{align}
P_{ij} = P_{ij}^{(imp)} + P_{ij}^{(ph)}.
\end{align}

in formulae (7) - (9).

The electron-phonon contribution \(P_{ij}^{(ph)}\)'s have been derived in previous papers, e.g., in Ref. [8] they are expressed in terms of the Bloch(-Grüneisen) functions [9]. The appropriate expressions for the electron-impurity contribution \(P_{ij}^{(imp)}\) following from the Yukawa potential [3] are given in this Section, but first we write down the final expression for the electronic thermal conductivity with the influence of the residual electrical resistivity as a parameter. For this purpose we have to separate elastic and inelastic contribution to \(P_{22}\)
and therefore we write explicitly the matrices $u_{ij}$ of the kernels in (10). For the parabolic band $\varepsilon(k) = \hbar^2 k^2 / 2m$ they can be represented as

$$u_{11} = (\hbar/m)^2 (q \cdot a)^2,$$

$$u_{12} = u_{21} = (\hbar/m)^2 [(\varepsilon - \zeta) + (\varepsilon' - \varepsilon)] (q \cdot a)^2,$$

$$u_{22} = (\hbar/m)^2 [(\varepsilon - \zeta)^2 + 2(\varepsilon' - \varepsilon)(\varepsilon - \zeta)] (q \cdot a)^2 + (\varepsilon' - \varepsilon)^2 [(q \cdot a)^2 + (k \cdot a)^2].$$

where $(\varepsilon - \varepsilon)$ is the transferred energy during the scattering, $q = k' - k$ is the transferred wave vector and terms linear in $k$ and $k'$, which do not contribute to the integrals, have been neglected. It is seen that the second component in $u_{12}$ and the second and third one in $u_{22}$ contribute only when a change of the electron energy in the scattering is taken into account. It means that the latter terms describe the effects of inelasticity in the scattering. With the definition

$$P^{(ph)}_{11}(t) = P^{(ph)}_0 P^{(ph)}_{11}(t), \quad \text{with } t = T/T_D$$

and where $P^{(ph)}_0$ is a material constant (see Appendix B), we get

$$P^{(ph)}_{22} = P^{(ph)}_0 (k_b T)^2 [P^{(1,ph)}_{22}(t) + P^{(2,ph)}_{22}(t)] \equiv P^{(1,ph)}_{22} + P^{(2,ph)}_{22},$$

where

$$P^{(1,ph)}_{22}(t) = \frac{\pi^2}{3} P^{(ph)}_{11}(t),$$

$$P^{(2,ph)}_{22}(t) = t^5 \left[ \frac{1}{3} F_7(t) + \frac{\varepsilon_s}{(k_b T)^2} F_5(t) - \frac{\varepsilon_s}{2k_b T} F_6(t) \right],$$

and $\varepsilon_s$ is a constant (see Appendix B).

The explicit form of $P^{(ph)}_{11}(t)$ describing the electrical resistivity is

$$P^{(ph)}_{11}(t) = t^5 F_5(t).$$

The two terms in $u_{12}$, accounting respectively for the elastic and inelastic contributions, after integration with the transition probability for the phonon scattering can be represented by a single term,

$$P^{(ph)}_{12} = P^{(ph)}_{21} = P^{(ph)}_0 (k_b T) P^{(ph)}_{12}(t)$$

3These terms describe the basic inelastic contribution. For the phonon scattering the single term of $u_{11}$ and the first terms in the case of $u_{12}$ and $u_{22}$ are expressed by the Debye functions in case of pure elastic scattering and by the Bloch(-Grüneisen) functions in case of inelastic scattering (see Ref.8). The terms proportional to $(\varepsilon' - \varepsilon)$ and $(\varepsilon' - \varepsilon)^2$ in $u_{12}$ and $u_{22}$ lead to an essential qualitative effect in the thermoelectric power and the thermal conductivity.
with
\[ P_{12}^{(ph)}(t) = P_{21}^{(ph)}(t) = \frac{1}{2} t^5 F_6(t) . \] (28)

Two material constants influence the magnitude of \( P_{22}^{(2,ph)}(t) \). These are the Fermi energy, \( \varepsilon_F \), and \( \varepsilon_s = 2mv_s^2 \), where \( v_s \) is the sound velocity (see the Table I in Appendix). The latter energy (or more precisely \( \varepsilon_s/4 \)) is a threshold energy for the phonon emission in the electron–phonon scattering process (see Appendix J in Ref. [18]).

Assume now that the charged (static) impurity concentration is so low that one can consider them to be statistically independent. Then the scattering of the conduction electrons by a single impurity contributes solely to the \( P_{ij} \) matrix elements and they are linear in their concentrations. In consequence the electrical resistivity is linear in the electron concentration and for the degenerate electron gas it is also temperature independent.

The latter statement can be easily confirmed by the calculation of \( P_{11}^{(imp)} \) in the metallic limit. For further convenience we assume \( P_{11}^{(imp)} / P_0^{(ph)} \equiv \Gamma \) and due to the Sommerfeld expansion of terms of (22) corresponding to an elastic scattering we have
\[ P_{22}^{(imp)} = P_{22}^{(imp,1)} = P_0^{(ph)} (\pi^2/3)(k_B T)^2 \Gamma . \] Therefore, the final expressions for \( P_{11} \) and \( P_{22} \) describing the impurity and phonon scattering are
\[ P_{11} = P_0^{(ph)} P_{11}(t), \quad \text{with} \quad P_{11}(t) = P_0^{(ph)} [\Gamma + P_{11}^{(ph)}(t)] \]
\[ P_{22} = P_0^{(ph)} (k_B T)^2 P_{22}(t), \quad \text{with} \quad P_{22}(t) = \frac{\pi^2}{3} [\Gamma + P_{11}^{(ph)}(t)] + P_{22}^{(2,ph)}(t) . \] (29)

\[ \text{A. Electrical resistivity and thermal conductivity in metals} \]

The above expressions for the matrix elements of the \( P \)–matrix yield the electrical resistivity
\[ \rho = \rho_{0,ph} [\Gamma + P_{11}^{(ph)}(t)] , \] (30)
where \( \rho_{0,ph} \) is a constant (see Appendix B), and the electronic thermal conductivity
\[ \kappa = \kappa_{0,ph} \frac{t}{P_{22}(t)} , \quad \kappa_{0,ph} = \frac{\pi^2 L_0 T_D}{3 \rho_{0,ph}} . \] (31)

Note that the Lorenz number reads
\[ \mathcal{L} = \frac{\rho \kappa}{T} = L_0 \frac{\pi^2 P_{11}(t)}{3 P_{22}(t)} \] (32)
and reduces to the Sommerfeld value \( L_0 \) for elastic scattering, when one neglects \( P_{22}^{(2,ph)} \) in (25).
The results (26) describing the electrical resistivity are obviously standard ones and do not require any comments. As concerns the electronic thermal conductivity and the Lorenz number similar expressions in terms of the Bloch–Grüneisen functions as those in the present paper were obtained by Wilson and were numerically examined in Ref. manifesting the influence of the impurity scattering. The Wilson expression for the thermal resistivity in our notation reads

\[
\frac{2\pi^2}{3} t^5 F_5(t) + \frac{8\pi^2 \varepsilon_F}{(k_B T)^2} t^5 F_5(t) - \frac{1}{3} t^5 F_7(t).
\]  

(33)

B. Seebeck coefficient of metals.

In order to discuss the thermoelectric power, \(S\), we introduce, as for the discussion of the thermal conductivity in Ref. 11, a relative order of magnitude parameter \(\varphi = P_{\text{imp}}(0)/P_{\text{ph}}(0)\), in terms of which for the Yukawa potential we have \(\Gamma(\varepsilon_F) = \varphi \Phi(2k_F/\lambda)\). It will be also convenient to introduce here the function \(\Psi(x) = x^4/(1 + x^2)^2\) in terms of which we express \([d\Phi/d\varepsilon]|_{\varepsilon_F} = (\varepsilon_F)^{-1}\Psi(2k_F/\lambda)\). In so doing the off-diagonal component of the \(P\)-matrix responsible for the impurity scattering is linear in \(k_B T/\varepsilon_F\) and reads

\[
P_{12}(T) = P_0(\text{imp}) (k_B T) P_{12}^{(\text{imp})}, \quad P_{12}^{(\text{imp})} = \frac{\pi^2}{3} \varphi \Psi(2k_F/\lambda) \frac{k_B T}{\varepsilon_F}.
\]  

(34)

When these expressions are inserted into (8) and linear terms in \(k_B T/\varepsilon_F\) are only accounted for, one obtains the following expressions for the coefficients in (18)

\[
S_0(t) = \frac{\pi^2}{3} k_B \frac{P_{12}^{(\text{ph})}(t)}{P_{22}(t)}, \quad S_1(t) = -\frac{k_B}{e} \left\{ \frac{\pi^2}{2} \left[ 1 + \frac{\pi^2}{3} \frac{P_{11}(t)}{P_{22}(t)} \right] - \left( \frac{\pi^2}{3} \right)^2 \frac{\varphi \Psi(2k_F/\lambda)}{P_{22}(t)} \right\}.
\]  

(35)

Note that the impurity scattering has an influence on the term \(S_0(t)\) only through the matrix element \(P_{22}(t)\) in the denominator. If \(\varphi = 0\), the above formula corresponds to that obtained in Ref. 8 for pure phonon limited thermoelectric power.

Notice that the pure phonon limited thermoelectric power obtained within the Wilson formalism do not represent any characteristic features of the experimentally observed \(S\) behavior in simple monovalent metals. Such results for \(S\) obtained in Ref. 8 were already discussed there in terms of various approximations.

IV. TEMPERATURE DEPENDENCE OF THE TRANSPORT COEFFICIENTS AND THE LORENZ NUMBER.

To compute the transport coefficients as a function of \(t = T/T_D\), we assume the value of \(2k_F/\lambda = 0.75\) and \(0 \leq \varphi \leq 1\). The temperature dependence of the reduced electrical resistivity following from (26), \(\rho/\rho_0\), which is shown in Fig. 1, and the Seebeck coefficient in Fig. 2 have been computed for \(\varphi = 0.0, 0.2, 0.5, 1.0, 2.0\) and the corresponding
values of $\Gamma = \varphi \Phi(0.75) = 0.0, 0.0173, 0.0259, 0.0431, 0.0862, 0.173$ respectively. As it is seen the impurity scattering slightly lowers the maximum of $S$ following from the transfer of the energy between the electron and phonon system. The values of $\varepsilon_s$ and $\varepsilon_F$ have been chosen to be the same as those in Ref. 8, namely $\varepsilon_s = 6.0K$ and $\varepsilon_F = 1.5eV$. The electronic thermal conductivity is considerably affected by the impurity scattering, what is seen from the data represented in Fig. 3 and Fig. 4. In Fig. 3 the reduced thermal conductivity ($\kappa/\kappa_{0,ph}$) is shown as a function of $t$ in the low temperature range such as in Fig. 4.16 of Ref. 9, where the results of Wilson's considerations were already illustrated. In Fig. 4 the results are shown in the same temperature range as the thermoelectric power in Fig. 2. We note that he dependence of the thermal conductivity as a function of temperature here obtained by the Ziman variational method is qualitatively the same as that following from Wilson’s considerations. Furthermore, at physically realistic temperatures the values of $\kappa$ obtained by the latter method approach a saturation, which is characteristic of monovalent metals. Under the assumption of the same values of the parameters describing the impurity scattering the saturation of $\kappa$ following from Wilson formula occurs at a so high temperature that it ceases to be meaningful. According to Ziman, who examined the Wilson formula, physically reasonable results for monovalent metals (saturation and lack of minimum at intermediate temperatures) are obtained when the Umklapp scattering is taken into account. Experimental results for rare earth intermetallic compounds exhibit the existence of a minimum at temperatures such as those seen in Figs. 3 and 4 (see e.g. Fig. 7 in Ref. 20 and references therein), in which the magnetic scattering are also taken into account.

The reduced Lorenz number, $L/L_0$, as a function of $t$ is shown in Fig. 5 for the same values of the parameters as $\kappa/\kappa_{0,ph}$ in Fig. 4. The behavior of this quantity as a function of $t$ is qualitatively the same as that following from the Wilson considerations. Note that for only phonon scattering $L(0) = 0$, whereas $\kappa(0)$ tends to infinity at zero temperature.

V. CONCLUDING REMARKS.

Using a simple physical electron model, applying the Ziman variational method within the Boltzmann equation approach and using the same level of approximations for each property we have calculated the phonon-limited electrical resistivity, thermoelectric power and the electronic thermal conductivity of a simple metal and studied how the presence of charged impurities influences these transport coefficients. The method allows us to determine the temperature dependence of the coefficients in a wide temperature interval and to express them in terms of simple and standard functions already useful for examining experimental data. However we are aware that to take into account a more complicated band structure is surely necessary in many cases. In this purpose the Allen method is surely more appropriate though the task needs to bear upon numerical technicalities from the very beginning. Note that the latter method is strictly applicable to metallic systems, when the electron gas is degenerate, while by using the method of this present paper one advantage is that one can easily consider semiconductor or semimetallic systems as well.
A comment is in order on the results of the present paper obtained in the limit of the degenerate electron gas. As concerns the electrical resistivity and the electronic thermal conductivity (or resistivity) the final results are qualitatively the same as those obtained previously in Refs. 1, 9, 10. In the case of the thermoelectric power however we can obtain a new evidence as a conclusion, i.e. the maximum at intermediate temperatures is caused by the energy transfer between the conduction electrons and phonons and is lowered as a result of the elastic scattering of the electrons on impurities.

The final formal expression for the electronic thermal resistivity reads as a simple expression: this quantity is proportional to the scattering matrix element $P_{22}$, much like the electrical resistivity is proportional to $P_{11}$. The result is in fact model independent and is a consequence of confining the scattering matrix to four elements (see Appendix A).

The range of applicability of the method we have used here above cannot be finally discussed without referring to papers using more general and sophisticated methods and/or models. In the first place this concerns the papers in which the electron transport was studied within the formalism of the Boltzmann equation. In the second place a relation to the linear response theory and the quantum transport equation will be given. However, by using these methods one cannot obtain simple, compact, and useful expressions for the three transport coefficients, moreover being valid in the wide temperature region experimentally studied. Further complications have also to be expected when taking into account the phonon drag, the virtual recoil or the electron–phonon renormalisation.

Let us start our discussion with a comparison of our model and results with the other ones obtained by the Ziman variational method within the formalism of the Boltzmann equation. As we stressed at the beginning we confined ourselves to a model of the parabolic electron band and the Debye model of the phonon system. This oversimplified model allowed us to gain a coherent and qualitatively correct description of the three transport coefficients: the electrical resistivity, thermoelectric power and the thermal conductivity for a simple and standard model of the scattering. Any generalization of these results to an arbitrary band structure requires numerical studies at an initial stage. A compact formula in this general case of the Ziman variational method has been obtained only for the electrical resistivity among the three coefficients, a formula which has been called "Ziman’s resistivity formula". It has been used to compute the electrical resistivity for a realistic Fermi surface.

Again a more exact method is that of Allen though a numerical one. The electrical resistivity is again simplest to compute but the results for the thermal conductivity and the thermoelectric power were also obtained by this method. The essence of the method is the separation of the $k$-variation during the scattering into two types: (i) an "angular variation" over the Fermi surface, and (ii) a "radial" variation with $\varepsilon(k)$ perpendicular to the Fermi surface describing the inelastic electron scattering. It is obvious from our results for the thermoelectric power that taking into account inelastic electron scattering, which is equivalent to considering a sufficient number of harmonics describing the radial variation within the Allen formalism, is of major importance for obtaining a correspondence
to experimental results if the scattering of the electrons by excitations such as phonons is to be accounted. In Ref. 32, in which the lowest order radial harmonic was considered and the analysis was in fact restricted to the Mott formula, a poor correspondence to the experimental results for the thermoelectric power was obtained. Inelastic scattering and the radial harmonics are also important to describe properly the thermal conductivity. In Ref. 14 the radial harmonics were applied to discuss the influence of the electron–electron scattering on the thermal conductivity. According to our knowledge no such considerations were made to analyze the influence of the electron–phonon scattering neither on the thermal conductivity nor the thermoelectric power.

The assumed statistical independence between the impurities as scattering centers restrict our considerations to low impurity concentration. This is consistent with the experimental fact that the Matthiessen rule is valid below a critical residual resistivity above which the $T^5$ low temperature dependence (following, for instance, from the Bloch–Grüneisen dependence of the electrical resistivity) ceases to be valid. One can hardly estimate such a critical concentration without analyzing a specific system and experimental results.

To estimate the magnitude of the transport coefficients out of the reduced values illustrated on each figure in Sec. IV we present the order of magnitude of the relevant material constants in Appendix B. The magnitude of $\varepsilon_s$ in Table I is the same as in Ref. 8 and corresponds to the magnitude of the sound velocity and the effective electron mass(es) of real materials.

We start our discussion of more sophisticated methods than the Boltzmann equation formalism by noting that in the linear response theory the validity of the Matthiessen rule, which was assumed by us in a generalized version, is related to the fact that the self-energy of the conduction electrons is the same as the sum of the phonon and impurity self-energy. This holds only in the lowest Born approximation. When higher order processes are taken into account, i.e., when interference between the electron-phonon and electron-impurity interaction is to be considered, the Keldysh quantum transport equation formalism is apparently better.

In Ref. 23 the same model as in the present paper was considered to study the influence of the electron-phonon-impurity interference on the thermoelectric power. A considerable nonlinearity of $S$ as a function of temperature was found, the stronger the higher impurity effective charge. The effect can be hardly compared with that studied in the present paper since it has been obtained under an assumption of elasticity for the electron-phonon scattering whereas the assumption of inelasticity of the electron-phonon scattering is essential for obtaining the maximum of $S$ found in the present paper and Ref. 8. Note that as a result of a weak elastic impurity scattering (considered within the Born approximation) the bump in $S$ is found to be lowered with the strength of the impurity interaction or the impurity concentration.

The electron-phonon renormalization leads also to a nonlinearity of $S$ as a function of $T$. The effect was studied under the assumption of elastic electron-phonon interaction or in the adiabatic approximation. Again there can be no discussion between our results and those in Ref. 7 and 30 except for the region of very low temperatures, where the effect
of elasticity and renormalization are not effective and where \( S \) is a linear function of \( T \) with the same slope as in the pure crystal.

Though the impurity scattering is known to play an important role in the thermal conductivity at least in the low temperature region there is a very limited number of papers devoted to this coefficient aspect. The basic difficulties and problems related to applying the methods of the linear response formalism to treating the electronic thermal conductivity problem in the pure electron-phonon system are described in Ref. 22. However, such considerations as those in Ref. 22 have not be generalized for a system with impurities. On the other hand the considerations of Refs. 7, 21, devoted to thermoelectric effects for a model of Debye phonons with fixed impurities, have not been used by these authors in an appropriate way to consider the electronic thermal conductivity. Therefore, a complete theory of the electronic thermal conductivity still requires a thorough consideration of the mutual role of the electron-phonon-impurity interference, phonon renormalization and elasticity of the scattering, much the same as on the thermoelectric power. For comparison to data, in not too good metals, the phonon heat contribution should be also taken into account for the thermal conductivity.

A question may arise if our results can be compared to experimental ones in a straightforward, quantitative way. The answer seems that these are the alkali metals, which exhibit essentially spherical Fermi surface to within a few per cent, that should be first of all considered as materials for the comparison. It occurs, however, that the phonon system and the electron–phonon interaction in these metals are too complicated to be well accounted by the applied approximations. It concerns in the first place, (i), the continuous medium approximation used to treat the phonon system and the deformation potential approximation applied to the electron–phonon interaction (see, e.g., Ref. 12). In the second place, (ii), the Debye approximation is insufficient for the alkali metals. Finally, in the description of the electron scattering we also neglect the Umklapp processes, which occur important even at very low temperatures in the alkali metals.

The approximations (i) mean that the longitudinal phonons of long wavelength are only considered, whereas the transverse phonons and those of short wavelength (contributing also to the Umklapp process) have to be taken into account. Due to a strong elastic anisotropy the Debye approximation (ii) is not appropriate even at low temperatures since a constant frequency surface is distorted from the Debye spherical form. Furthermore, the strong influence of the electronic system on the behavior of the phonons cause that the basic symmetry relations between the elastic constants (Cauchy relations) are no longer valid and, therefore, even the basic results following from the continuous medium approximation of Ref. 12 may be questioned. All these factors cause that the simple Bloch approach have to be used with a great caution in attempts of the quantitative description of the electrical and thermal transport in the alkali metals.

The most early investigations of the alkali metals indicated that one can fix such parameters as those of Appendix B to fit the experimental electrical resistivity to the Bloch–Grüneisen function, provided that the temperature is not too low. Thorough analysis based on the measured phonon spectra of the alkali metals showed later that the problem is more complicated and the effects mentioned by us above have to be
in fact taken into account. Therefore, even in case of these simplest metals the correct quantitative theory based on first principles is in fact impossible and a quantitative analysis have to be based on a cross section for the scattering of the conduction electrons written in terms of the phonon spectrum resulting from neutron scattering data. Precise experimental measurements of the transport coefficients of the alkali metals at low temperatures support this conjecture and reveals other physical effects which should be taken into account for a proper quantitative analysis in this temperature region.

As concerns the thermoelectric power our results indicate that other factors than those in case of the electrical resistivity are important. The impurity scattering occurred less essential than expected. Within the assumed model this is the parameter $\varepsilon_s$ which besides $\varepsilon_F$ is solely responsible for the strength of the inelastic scattering of electrons by phonons and controls the magnitude and sign of $S$. The obtained temperature dependence of $S$ is the same in character as that obtained experimentally for all the the alkali metals except Li (which exhibits a peculiar behavior but the Fermi surface of which comes closest to the Brillouin zone boundaries). In search for a quantitative theory of $S(T)$ of a specific metal one should also estimate the order of the contribution of the phonon drag, that following from the electron–phonon renormalization and the electron–phonon–impurity interference. The described above factors which influence the electrical resistivity in the alkali metals may occur of secondary importance.

The effects of the impurity scattering manifest in most pronounced way in the behavior of the thermal conductivity but due to the validity the Wiedemann–Franz law in the limit of low temperatures the effects of the impurity scattering can be easily extracted out of our results at low temperatures for a comparison with experimental ones. As concerns the comparison of the values of appropriate coefficients following from the theory and experimental data for the alkali metals the same problems as in case of the studies of the electrical resistivity may arise. The Umklapp process are expected to be less dominant at low temperatures than in case of the electrical resistivity but the electron–electron interactions may occur essential for the thermal conductivity since both the experimental and theoretical investigations indicate that electron–electron scattering in the alkali metals impedes heat flow much more effectively than charge flow.

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APPENDIX A

The transport coefficients considered in the paper follow from the Onsager expressions for the electrical $J$ and thermal $U$ currents:

$$
\begin{align*}
J &= \sigma E + L \nabla T, \\
U &= M E + N \nabla T,
\end{align*}
$$

(36)

which in the "experimental" form are following

$$
\begin{align*}
E &= \rho J + S \nabla T, \\
U &= \pi J - \kappa \nabla T
\end{align*}
$$

(37)

with $\pi = -ST$ and

$$
\begin{align*}
\rho &= 1/\sigma, \\
S &= -\rho L, \\
\kappa &= M \rho L - N.
\end{align*}
$$

(38)

Within the Ziman variational method the coefficients of (36) are

$$
\begin{align*}
\sigma &= \sum_{i,j=1} J_i Q_{ij} J_j , \\
L &= -\frac{1}{T} \sum_{i,j=1} J_i Q_{ij} U_j , \\
M &= \sum_{i,j=1} U_i Q_{ij} J_j , \\
N &= -\frac{1}{T} \sum_{i,j=1} U_i Q_{ij} U_j ,
\end{align*}
$$

(39)

where $J_i$ and $U_i$ ($i, j = 1, 2, \ldots$) are respectively the electron and thermal trial currents defined by (12)-(13) and $Q_{ij}$ is the inverse matrix to the scattering matrix $P_{ij}$, i.e. $Q_{ij} = (P^{-1})_{ij}$. In the approximation consisting in abbreviating the combinations in (39) to two components ($i, j = 1, 2$) we obtain (7)-(9) from (38). For a degenerate electron gas, i.e. for sufficiently large $z \cong \varepsilon_F / k_B T$, the final expressions for the transport coefficients are (14) – (16). Notice that the thermal resistivity in particular is proportional to $P_{22}$. 

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APPENDIX B

The functions describing the trial currents, as written below (15)-(16), are

\[ L_0(z) = (3/2) F_{1/2}(z), \]
\[ L_1(z) = -(3/2) z F_{1/2}(z) + (5/2) F_{3/2}(z), \]
\[ L_2(z) = (3/2) z^2 F_{1/2}(z) - 5 z F_{3/2}(z) + (7/2) F_{5/2}(z). \]  

(40)

For completeness we also write down their asymptotic forms

\[ L_0(z) \approx z^{3/2}, \quad L_1(z) \approx (\pi^2/2) z^{1/2}, \quad L_2(z) \approx (\pi^2/3) z^{3/2}. \]  

(41)

The coefficients of the trial currents are \( J_0 = e U_0 \) with

\[ U_0 = \begin{cases} 
\frac{1}{6\pi^2} \left( \frac{2m}{\hbar^2} \right)^{5/2} \left( \frac{\hbar}{m} \right)^2 \ (\mathbf{v} \cdot \mathbf{a}) \\
\frac{1}{6\pi^2} \left( \frac{2m}{\hbar^2} \right)^{5/2} \left( \frac{\hbar}{m} \right)^2 \ (\mathbf{k} \cdot \mathbf{a}) 
\end{cases} \]  

when the factor is \( (\mathbf{v} \cdot \mathbf{a}) \), and the lower one to \( (\mathbf{k} \cdot \mathbf{a}) \) in the trial function \( \Phi \).

The coefficient which determines the dimension and the order of the magnitude of the phonon scattering matrix depends on the effective electron mass, \( m \), the volume of the primitive cell, \( V \), its total mass, \( M \), the sound velocity \( v_s \), the phonon wave vector cutoff, \( q_D \), and the deformation potential constant, \( E_1 \). It reads

\[ P_0^{(ph)} = \begin{cases} 
\frac{VE_1^2}{48\pi^3 M v_s} \left( \frac{2m}{\hbar^2} \right)^2 \left( \frac{\hbar}{m} \right)^2 q_D^5 \ (\mathbf{v} \cdot \mathbf{a}) \\
\frac{VE_1^2}{48\pi^3 M v_s} \left( \frac{2m}{\hbar^2} \right)^2 q_D^5 \ (\mathbf{k} \cdot \mathbf{a}) 
\end{cases} \]  

when the factor is \( (\mathbf{v} \cdot \mathbf{a}) \), and the lower one to \( (\mathbf{k} \cdot \mathbf{a}) \).

The coefficient \( \rho_{0,ph} = P_0^{(ph)}/(e^2 U_0^2 \varepsilon_F^3) \) in (30) can be represented by

\[ \rho_{0,ph} = 48\pi (\mu)^2 \left( \frac{m_0}{\hbar^2 e^2} \right) \left( \frac{m_0}{M} \right) V E_1^2 \tilde{\rho}_{0,ph} \]  

(43)

where \( m_0/(\hbar^2 e^2) = 9.113 \cdot 10^3 [eV^{-2} \text{nm}^{-3}] \),

\[ \tilde{\rho}_{0,ph} = \left( \frac{q_D}{2k_F} \right)^6 (v_s q_D)^{-1} = \left( \frac{q_D}{2k_F} \right)^6 \hbar (k_B T_D)^{-1}, \]  

(44)

where \( m_0 \) is the free electron mass and \( \mu = m/m_0 \). For the value of \( T_D = 200 K \) assumed for our representative data we have \( \hbar (k_B T_D)^{-1} = 3.437 \cdot 10^{-2} \Omega \text{ cm} \). In Table I we show some representative data of \( \tilde{\rho}_{0,ph} \) for \( \varepsilon_F = 1.5 eV, T_D = 200 K, \mu = 1 \) and the values of \( \varepsilon_s \) given in the table. They correspond to those in Ref. 3, where numerical relations between \( \varepsilon_s \) and other constants describing the phonon system are presented.

The screened Coulomb potential scattering contribution to the transport is described in terms of the dimensionless functions \( \Phi \) and \( \Psi \) defined by (11) and (14). The coefficient \( P_0^{(imp)} \) of the scattering matrix elements is
\[ P_0^{(\text{imp})} = \begin{cases} \frac{1}{3\pi^2} n_{\text{imp}} Z^2 e^4 \left( \frac{2m}{\hbar^2} \right)^2 \left( \frac{\hbar}{m} \right)^2 (\mathbf{v} \cdot \mathbf{a}) & \text{when the factor is 1} \\ \frac{1}{3\pi^2} n_{\text{imp}} Z^2 e^4 \left( \frac{2m}{\hbar^2} \right)^2 & (\mathbf{k} \cdot \mathbf{a}). \end{cases} \] (46)

where \( n_{\text{imp}} \) is the impurity concentration (their number per unit volume). The residual resistivity \( \rho^{(\text{imp})} = P_{11}^{(\text{imp})} / (e^2 U_0^2 \varepsilon_F^3) \) can be represented by

\[ \rho^{(\text{imp})} = \frac{Z^2 n_{\text{imp}}}{n_e \mu} \tilde{\rho}_{\text{imp}} \Phi(2k_F/\lambda) \] (47)

where \( n_e \) is the number of the conduction electrons per unit volume, with

\[ \tilde{\rho}_{\text{imp}} = \frac{12\pi^2 e^2}{m_0 v_F^3} \] (48)

\( v_F \) is the Fermi velocity, and \( 2k_F/\lambda \approx 1 \) as we mentioned in Sec. II. For the data represented in Table I (free electron gas, \( \varepsilon_F = 1.5 \text{ eV} \)) we have \( \tilde{\rho}_{\text{imp}} = 0.075 \Omega \text{ cm} \).

In order to relate the magnitude of \( \Gamma \) in (25)–(26) with the number of impurities in the sample we represent it as \( \tilde{\varphi} \)

\[ \Gamma = \frac{\rho^{(\text{imp})}}{\rho_{0,\text{ph}}} = \frac{\tilde{\rho}_{\text{imp}}}{\rho_{0,\text{ph}}} \Phi(2k_F/\lambda) \frac{Z^2 n_{\text{imp}}}{n_e \mu} \equiv \tilde{\varphi} \Phi(2k_F/\lambda) \frac{Z^2 n_{\text{imp}}}{n_e \mu} \] (49)

and shall estimate \( \tilde{\varphi} \). It means that \( \varphi \) (31) is \( \varphi = (Z^2 n_{\text{imp}}/n_e \mu) \tilde{\varphi} \). For the mass of the unit cell \( M = 10^6 m_0 \), its volume \( V = 0.1 \text{ nm}^3 \) and \( E_1 = (2/3) \varepsilon_F = 1 \text{ eV} \) we have \( \rho_{0,\text{ph}} \approx 0.1 \tilde{\rho}_{0,\text{ph}} \). This is an underestimated value in many cases, thus we may assume \( \rho_{0,\text{ph}} = a \tilde{\rho}_{0,\text{ph}} \) with \( 0.1 < a < 1 \). For \( \varepsilon_F = 1.5 \text{ eV}, T_D = 200 \text{ K} \mu = 1 \) the value of \( \tilde{\rho}_{\text{imp}} \) is: \( \tilde{\rho}_{\text{imp}} \approx 0.1 \Omega \text{ cm} \). The latter value along with those ones for \( \rho_{0,\text{ph}} \) following from the values of \( \tilde{\rho}_{0,\text{ph}} \) presented the Table I yields an estimate of \( \tilde{\varphi} \).

Finally, for information as to the magnitude of the thermal conductivity represented in dimensionless units in the figures we give the magnitude of

\[ \tilde{\kappa}_{0,\text{ph}} = \pi^2 \frac{L_0 T_D}{\rho_{0,\text{ph}}} \] (50)

for the Debye temperature \( T_D = 200 \text{ K} \) in the last column of Table I.
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TABLE I. The values of the parameters allowing to estimate the magnitude of the electrical resistivity and electronic thermal conductivity for a free electron effective mass, when the Fermi energy is $\varepsilon_F = 1.5 \text{ eV}$ and the Debye temperature $T_D = 200 \text{ K}$.

| no. | $\varepsilon_s [\text{K}]$ | $\tilde{\rho}_{0,ph} [\Omega \text{ cm}]$ | $\tilde{\varphi}$ | $\tilde{\kappa}_{0,ph} [\text{W/mK}]$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| (i) | 1.50            | $1.93 \cdot 10^{-3}$ | $10^0 - 10^1$   | $2.54 \cdot 10^{-5}$ |
| (ii)| 3.00            | $2.39 \cdot 10^{-4}$ | $10^1 - 10^2$   | $2.04 \cdot 10^{-4}$ |
| (iii)| 6.00           | $2.99 \cdot 10^{-5}$ | $10^2 - 10^3$   | $1.64 \cdot 10^{-3}$ |
| (iv)| 12.00           | $3.79 \cdot 10^{-6}$ | $10^3 - 10^4$   | $1.23 \cdot 10^{-2}$ |
| (v) | 24.0            | $4.58 \cdot 10^{-7}$ | $10^4 - 10^5$   | $1.07 \cdot 10^{-1}$ |
| (vi)| 48.0            | $5.76 \cdot 10^{-8}$ | $10^5 - 10^6$   | $8.48 \cdot 10^{-1}$ |
Figure Captions

**Figure 1** — Temperature dependence of $\rho/\rho_{0, ph}$ resulting from Eq.(26) for values of the parameters $2k_F/\lambda = 0.75$ and the relative scattering strength $\varphi = 0.0, 0.2, 0.5, 1.0$. They correspond to the values of $\Gamma = \varphi\Phi(0.75)$ which are 0.0, 0.0173, 0.0431, 0.0862, respectively.

**Figure 2** — Temperature dependence of $S/S_{k,e}$ for values of the parameters as in Fig.1 with $\varepsilon_F = 1.5$ eV, $\varepsilon_s = 6$ K, and the effective electron mass equal to that of a free electron. $S_{k,e} = k_B/e = 86 \mu$ V/K.

**Figure 3** — Temperature dependence of the reduced thermal conductivity below $T_D$, for $2k_F/\lambda = 0.75$ and $\varphi = 0.0, 0.2, 0.5, 1.0, 2.0$; the corresponding values of $\Gamma$ are 0.0, 0.0173, 0.0431, 0.0862, 0.173, respectively. For $\Gamma = 0$, the value of $\kappa$ is infinite. Note that for the magnitude of $\Gamma$ as large as 2.0 there is a plateau of $\kappa$ instead of the maximum at $T \approx 0.1T_D$.

**Figure 4** — Temperature dependence of the reduced thermal conductivity in the same temperature range as that displayed for the thermoelectric power in Fig.2. The value of $2k_F/\lambda$ is the same as in Fig.3, i.e. 0.75, and $\varphi = 0.0, 0.2, 0.5, 1.0, 2.0$.

**Figure 5** — Temperature dependence of the reduced Lorenz number in the same temperature range as that of the thermoelectric power in Fig.2. The value of $2k_F/\lambda$ is 0.75 as in Fig.3, and $\varphi = 0.0, 0.2, 0.5, 1.0, 2.0$. 

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\[ \frac{\kappa}{\kappa_0, ph} \] vs. \( \frac{T}{T_D} \)
