Viscosity and Principal-Agent Problem∗

Ruoting Gong† Christian Houdré‡

November 5, 2009

We develop a stochastic control system from a continuous-time Principal-Agent model in which both the principal and the agent have imperfect information and different beliefs about the project. We consider the agent’s problem in this stochastic control system, i.e., we attempt to optimize the agent’s utility function under the agent’s belief. Via the corresponding Hamilton-Jacobi-Bellman equation the value function is shown to be jointly continuous and to satisfy the Dynamic Programming Principle. These properties directly lead to the conclusion that the value function is a viscosity solution of the HJB equation. Uniqueness is then also established.

Key Words: Stochastic Control Problem, Principal-Agent Problem, Second-order Hamilton-Jacobi-Bellman Equation, Viscosity Solution, Dynamic Programming Principle.

1 Introduction

Real-world principal-agent relationship such as the venture capitalist-entrepreneur relationship or the shareholder-manager relationship are usually characterized by at least two components: (i) the presence of uncertainty and heterogeneous beliefs about the intrinsic qualities of the project in which they are involved; (ii) Different attitudes towards this project’s risks. To investigate how heterogeneous beliefs and agency conflicts affect principal-agent relationship, Giat, Hackman and Subramanian [13] have developed a dynamic structural model. In their dynamic framework, both the principal and the agent take action that affect the project’s output, and the dynamic contracts between the principal and the agent are characterized as the principal’s investment policy, the agent’s effort policy and the payoffs

∗It is a pleasure to thank A. Subramanian who introduced us to this problem and A. Swiech for discussions, bibliographical help, and setting us straight.
†School of Mathematics, Georgia Institute of Technology, rgong@math.gatech.edu
‡School of Mathematics, Georgia Institute of Technology, houdre@math.gatech.edu.
Research supported in part by the NSA Grant H98230-09-1-0017
of the agent. The principal and the agent both have imperfect information about the project’s intrinsic quality which was the main output growth period of the project’s value process under a suitable probability measure. That is, they had different mean assessments of the normally distributed project quality while they agree on the variance of their respective assessments. In that framework the principal is risk-neutral while the agent is risk-averse (so that his/her utility function is a negative exponential) with inter-temporal constant absolute risk aversion (CARA). Under the optimal dynamic contracts between the principal and the agent (optimal in the sense of optimizing the agent’s conditional expected utility at each date), they showed that the agent’s pay-off process evolved as a diffusion process whose drift and volatility both depend on the principal’s optimal investment and the agent’s optimal effort.

In order to provide a rigorous approach of the optimal dynamic contract (optimal in the sense of optimizing the agent’s absolute expected utility, see Williams [35]), we only consider the agent’s problem, in which all the random processes are distributed under the agent’s belief. Moreover, motivated by the results in [13], we impose a diffusion pay-off process into the principal-agent model with its drift and volatility depending on the principal’s investment policy and the agent’s effort policy. Then a stochastic control system is built in which the state variable is a three dimensional process formed by the pay-off of the agent, the Girsanov density (which is used for the change of measure when characterizing the value process) and the project’s intrinsic quality, whereas the control domain is a two dimensional process consisting of the principal’s investment and the agent’s effort. We assume that the CARA constant of the agent is relatively small. As a result, in order to maximize the agent’s negative exponential expected utility (the cost functional of the control system), we can approximately maximize the expectation of the exponent. To solve this control system, we refer to the weak formulation and the corresponding second-order Hamilton-Jacobi-Bellman equation. Since uniform parabolicity conditions do not hold for the state equations, there is no hope to find a classical (differentiable) solution for the HJB equation. Nevertheless, we verify that the value function is a viscosity solution of the HJB equation by verifying its joint continuity and establishing the Dynamic Programming Principle. Moreover, we prove that the value function is the unique viscosity solution with polynomial growth.

Let us describe the content of this paper: Section 2 is mainly a review of the literature on Principal-Agent models and stochastic control problems. In Section 3, we state a principal-agent model following Giat, Hackman and Subramanian [13]. In Section 4, we formulate the stochastic control system and give some necessary conditions for later proofs. In Section 5 we first prove the joint continuity of the value function and establish the Dynamic Programming Principle. We then verify the existence and uniqueness of the viscosity solution of the HJB equation.
2 A Brief Literature Survey

In this first part, we briefly review the literature on dynamic principal-agent models. In a seminal study, Spear and Srivastava [31] developed a significant approach that applied “dynamic contracting” models to a number of problems in corporate finance such as CEO compensation (Wang [33], Spear and Wang [32]) and financial contracting (Quadrini [29], Clementi and Hopenhayn [4], DeMarzo and Sannikov [9]). Holmstrom and Milgrom [16] presented a continuous-time principal-agent framework in which the principal and the agent had CARA (exponential) preferences while the payoffs were normally distributed. They verified that the principal and the agent produced and saved via linear technologies so that the optimal contract for the agent is affine in the project’s performance. Schattler and Sung [30] studied incentive provision in a continuous time model and provided a rigorous development of the first-order approach to the analysis of continuous-time principal-agent problems with exponential utility using martingale methods. Building on the studies of Holmstrom and Milgrom [16] and Holmstrom [15], Gibbons and Murphy [14] developed a dynamic framework with imperfect public information to study how reputation concerns affect incentive contracts for workers. Giat, Hackman and Subramanian [13] extended those frameworks in several ways. First, they analyzed a continuous-time framework in which both the principal and the agent took productive actions with continuous support. Second, they considered the general scenario in which there is imperfect information about the project’s intrinsic quality and, moreover, the principal and the agent have heterogeneous belief. Under those assumptions, they derived the optimal contract explicitly which maximizes the agent’s continuation utility ratio. They also verified that under the optimal contract the payoff process satisfied a certain diffusion equation. We follow the economic model of Giat, Hackman and Subramanian [13] but further assume that the payoff process is a diffusion, and search for the optimal contract which maximizes the agent’s expected utility via the corresponding HJB equation. For a general approach of the optimal contract on Dynamic Principal-Agent Problems, we refer the reader to the paper of Williams [35].

Our study is also related to the literature that analyzes the stochastic control problem and viscosity solutions. Many authors have introduced different notions of generalized solutions in order to prove the value function to be the solution of the HJB equation. Krúzkov [22]-[26] built a systematic theory for first-order Hamilton-Jacobi (HJ) equations with smooth and convex Hamiltonians, Fleming [10]-[11] independently introduced the vanishing of viscosity, combining with the differential games technique, to study the HJ equations. On the other hand, Clarke and Vinter [4] used Clarke’s notion of generalized gradients to introduce generalized solutions of the HJB equations. In that framework, the HJB equation can have more
than one solution, and that value function is one of them. However, generalized gradients may not be readily used to solve second-order HJB equations that correspond to stochastic problems. A survey of many studies of HJB equations is given in Bardi-Capuzzo-Dolcetta [1].

In the early 1980s, Crandall and Lions [8] made a breakthrough in controlled systems by introducing the notion of a viscosity solution for first-order Hamilton-Jacobi-Bellman equations. The first treatment of viscosity solutions of second-order dynamic programming equations was given by Lions [28]. Lions investigated the degenerate second-order HJB equation using a Feynman-Kac-type technique, representing the solutions of the second-order PDEs by the value functions of some stochastic optimal control problems. For general second-order equations which are not necessarily dynamic programming equations, this technique is clearly not appropriate. Jenson [20] first proved a uniqueness result for a general second-order equation. In Jenson [20] semiconvex and concave approximations of a function were given by using the distance to the graph of this function. Another important step in the development of the second-order problems is Ishii’s Lemma [17]. Since then the proofs and the statements of the results have been greatly improved. In particular, the analysis results in Crandall and Ishii [6] have been used in almost all comparison result. We refer to the survey article of Crandall, Ishii and Lions [7] for more information. Fleming and Soner [12] provides a rigorous approach to the control of Markov diffusion processes in $\mathbb{R}^n$. When the uniform parabolicity condition is satisfied, they show that the value function is a classical solution of the corresponding second-order HJB equation. In case that the assumption of uniform parabolicity is abandoned, they provide a systematic analysis of the value function and establish a strong version of the dynamic programming principle by making approximations which reduce the result to the uniformly parabolic case. Similar results were summarized and developed in Yong and Zhou [36] via independent approaches.

3 The Principal-Agent Model

Our time horizon is $[0, T]$ for some fixed $T > 0$. At date zero, a cash-constrained agent with a project approaches a principal for funding. The project can potentially generate value through capital investments from the principal and human effort investments from the agent. Both the principal and the agent have imperfect information about the project and different assessments of the project’s intrinsic quality. The key variable in the model is the project’s termination value process $\{V(t), t \in [0, T]\}$, which is the total value of the project if the principal-agent relationship is terminated at date $t$. The incremental termination value $dV(t)$, that is, the change in the termination value over the infinitesimal period $[t, t + dt]$, is the sum of
a base output, i.e., a Gaussian process that is unaffected by the actions of
the principal and the agent, and a discretionary output, i.e., a deterministic
component that depends on the physical capital investments by the principal
and the human effort by the agent (See (3.4) below).

More rigorously, consider an underlying measurable space \((\Omega, \mathcal{F})\) with
probability measure \(\mathbb{P}^l, l \in \{Pr, Ag\}\), representing the different belief of
the principal and the agent. Let \(w\) be a standard Brownian motion and
\((\mathcal{F}_t)_{t \geq 0}\) be the complete and augmented filtration generated by \(w\). Let
\(\eta := \{\eta(t), t \in [0, T]\}, c := \{c(t), t \in [0, T]\}\) be \((\mathcal{F}_t)_{t \geq 0}\)-progressively measurable processes on \([0, T]\), describing respectively the agent’s choice of efforts
and the principal’s choice of investments over time. We assume that \((\eta, c)\)
takes value in \(U = [0, N] \times [0, C]\), where \(N\) represents the maximal effort the
agent can make per-period and \(C\) represents the maximal capital investment
the principal can afford per-period. Let \(\Theta := \{\Theta(t), t \in [0, T]\}\) be normally
distributed with mean \(\theta_l\) and variance \(\sigma^2_l\) under the measure \(\mathbb{P}^l\), which repre-
sents the intrinsic quality of the underlying project. Let \(\ell := \{\ell(t), t \in [0, T]\}\)
be a deterministic continuous process which describes the operating costs of
the project. Consider the value process \(V = \varrho w\), where \(\varrho^2\) is the intrinsic
risk of the project, \(\varrho > 0\). Define the Girsanov Density as

\[
\xi(t) = \exp \left\{ \int_0^t (\Theta(r) + Ac(r)^\alpha \eta(r)^\beta - \ell(r))\varrho^{-1}dr(r) - \frac{1}{2} \int_0^t (\Theta(r) + Ac(r)^\alpha \eta(r)^\beta - \ell(r))^2 \varrho^{-2}dr \right\}; \tag{3.1}
\]

\[
\hat{B}(t) = w(t) - \int_0^t (\Theta(r) + Ac(r)^\alpha \eta(r)^\beta - \ell(r))\varrho^{-1}dr, \tag{3.2}
\]

where \(\Phi(\eta, c) = Ac^\alpha \eta^\beta\) is the Cobb-Douglas production function (see [5])
with \(\alpha > 0, \beta > 0\). By Girsanov’s Theorem (see [21], we temporarily
assume the Novikov condition whose validity will be checked in Remark 4.1
of Section 4), under the new probability measure \(\Pi^l\) such that

\[
\frac{d\Pi^l}{d\mathbb{P}^l} = \xi(T), \tag{3.3}
\]

the process \(\hat{B}\) is a Brownian motion. Moreover, under the same measure,
the value process \(V\) evolves as:

\[
dV(t) = \left(\Theta(t) + Ac(t)^\alpha \eta(t)^\beta - \ell(t)\right)dt + \varrho d\hat{B}(t)
= \left(\Theta(t) - \ell(t)\right)dt + \varrho d\hat{B}(t) + \Phi(\eta(t), c(t))dt. \tag{3.4}
\]

Let \(P := \{P(t), t \in [0, T]\}\) be the pay-off process to the agent. We
assume that the agent has a constant certainty equivalent reservation utility
$R > 0$, that is, $R$ is the constant payoff that the agent would accept for the chance at a higher, but uncertain, amount. Next, a contract $(P, \eta, c)$ is said to be feasible at date $t$ if $P(t) \geq R$. As in the introduction, we assume that the agent is risk-averse with inter-temporal constant absolute risk aversion (CARA) preferences described by a negative exponential utility function. Moreover, the agent incurs a disutility of exerting effort in each period; that is, for any given investment policy $\eta$ and effort policy $c$, if the project is terminated at date $\tau$ ($\tau$ is a $(\mathcal{F}_t)_{t \geq 0}$-stopping time), then the agent’s expected utility is given by

$$-\mathbb{E}^{Ag}_\Pi \left[ \exp \left\{ -\lambda \left( P(\tau) - \int_0^\tau k\eta(t)\gamma dt \right) \right\} \right],$$

where the parameter $\lambda \geq 0$ characterizes the agent’s risk aversion. The agent’s rate of disutility from effort in the period $[t, t + dt]$ is given by $k\eta(t)\gamma$ with $k > 0, \gamma > 0$. Our goal is, to find an optimal pair $(\eta, c)$ which maximizes the agent’s expected utility (3.5).

Let us now assume that the parameter $\lambda > 0$ is small enough, then maximizing the utility function (3.5) over all $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable processes $(\eta, c)$ is approximately equivalent to minimizing the exponent

$$-\mathbb{E}^{Ag}_\Pi \left[ \int_0^\tau k\eta(t)\gamma dt - P(\tau) \right] = \mathbb{E}^{Ag}_\mathbb{P} \left[ \int_0^\tau k\eta(t)\gamma \xi(t) dt - P(\tau)\xi(\tau) \right]. \quad (3.5)$$

This can be seen from the first order expansion. For more information about this type of approximation, see Whittle [34] and Yong [36]. Therefore, in the next section and throughout, we will formulate the stochastic control problem using (3.5) as the cost functional.

### 4 The Stochastic Control Model

Let us first reiterate our settings. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a given filtered probability space satisfying the usual conditions, on which a standard Brownian motion $w := \{w(t), t \geq 0\}$ (with $w(0) = 0$) is given. Here $(\mathcal{F}_t)_{t \geq 0}$ is the augmented filtration generated by the Brownian motion. Let $Q = (R, +\infty) \times (0, +\infty) \times (-H, H)$ be the state space where $R > 0$ and $H > 0$. Let $\mathcal{U} = [0, N] \times [0, C]$ be the control domain, where the positive constants $N > 0$ and $C > 0$ represent respectively the maximal effort that the agent can make per-period and the maximal amount of investment that the principal can afford per-period. Define

$$U[0, T] = \left\{ u : [0, T] \times \Omega \rightarrow \mathcal{U} : u \text{ is } (\mathcal{F}_t)_{t \geq 0}\text{-progressively measurable} \right\}. $$
The state equations under the control $u(t) = (\eta(t), c(t)), t \in [0, T]$ for $x(t) = (P(t), \xi(t), \Theta(t)), t \in [0, T]$ are then given by

\begin{align*}
    dP(t) &= b(t, P(t), \eta(t), c(t))dt + \sigma(t, P(t), \eta(t), c(t))dw(t) \quad (4.1) \\
    d\xi(t) &= -\xi(t)\varphi^{-1}(\Theta(t) + A\eta(t)^\alpha c(t)^\beta - \ell(t))dw(t) \quad (4.2) \\
    d\Theta(t) &= \theta(t)\zeta_H(\Theta(t))dt + \sigma(t)\zeta_H(\Theta(t))dw(t), \quad (4.3)
\end{align*}

with initial conditions

$$P(0) = P_0 \geq R, \quad \xi(0) = 1, \quad \Theta(0) = \theta_0 \in [-H, H], \quad (4.4)$$

with $H > 0$ large enough. Above, $b, \sigma : [0, T] \times [R, +\infty) \times \mathcal{U} \to \mathbb{R}$ are measurable functions, while $\theta, \sigma, l$ are continuously differentiable and deterministic functions. Moreover, $\zeta_H : \mathbb{R} \to \mathbb{R}$ belongs to $C^\infty(\mathbb{R})$ and satisfies $\zeta_H(\Theta) = 1$ for $|\Theta| \leq H - 1$ and $\zeta_H(\Theta) = 0$ for $|\Theta| > H$. Finally, $\alpha$ and $\beta$ are positive constants.

**Remark 4.1** In (4.3), we assume that $\Theta$ is a Brownian motion with drift which is moreover absorbed at $-H$ and $H$ for some large $H > 0$. This assumption ensures the validity of the Novikov condition for $\xi$ which in turn guarantees the validity of the Girsanov transformation in Section 3. Assuming that $\Theta$ is absorbed at $\pm H$ is indicating that the absolute value of the project’s intrinsic value can be large but is bounded. Moreover, using the $C^\infty$ function $\zeta_H$ instead of the indicator function $I_{|\Theta|\leq H}$ ensures the smoothness of the drift and the volatility, which will be used in later proofs.

**Remark 4.2** Giat, Hackman and Subramanian [13] have shown that under the optimal investment policy $c^*$ of the principal and the optimal human effort $\eta^*$ of the agent, the pay-off to the agent is a diffusion process whose drift and volatility are functions of the optimal pair $(\eta^*, c^*)$. So we impose a diffusion process on the pay-off as in (4.1) and try to solve the stochastic control problem (SC) described below.

The cost function is given by

$$J(u) = \mathbb{E}\left\{\int_0^\tau k\eta(t)^\gamma \xi(t)dt - P(\tau)\xi(\tau)\right\}, \quad (4.5)$$

where $k > 0, \gamma > 0$ and

$$\tau = \inf\{t \geq 0 : x(t) \notin \mathcal{Q}\} \wedge T. \quad (4.6)$$

**Problem (SC).** Minimize (4.5) over $U[0, T]$, i.e. find $u^* \in U[0, T]$ satisfying

$$J(u^*) = \inf_{u \in U[0, T]} J(u).$$

with the corresponding state equations and initial conditions (4.1)-(4.4).
Remark 4.3 To search for the optimal control, we need to consider a weak formulation. The idea of studying first a weak formulation actually comes from deterministic optimal problems in which one needs to consider a family of optimal problems with different initial times and states. But in the stochastic case the states along a trajectory become random variables in the original probability space. In order to get deterministic initial condition under different initial time, naturally we need to consider the conditional probability space.

For any fixed \( s \in [0, T] \), let \( \Lambda[s, T] \) denote the set of all five-tuples \((\Omega, \mathcal{F}, \{F_t\}_{t \geq s}, \mathbb{P}, \{w(t)\}_{s \leq t \leq T})\) satisfying the following two conditions:

(i). \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space.

(ii). \(\{w(t)\}_{s \leq t \leq T}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), and \(\{F_t\}_{s \leq t \leq T} = \{\sigma\{w(r)\mid s \leq r \leq t\}\}_{s \leq t \leq T}\) is the augmented filtration generated by the Brownian motion.

For any \( \nu \in \Lambda[s, T] \), let \( U_{\nu}[s, T] = \left\{ u : [s, T] \times \Omega \to U : u \text{ is } (F_t)_{t \geq s}\text{-progressively measurable} \right\} \).

For any fixed \( (s, y) \in [0, T] \times \overline{Q} \), where \( \overline{Q} = [R, +\infty) \times [0, +\infty) \times [-H, H] \), consider the state equations \((4.1), (4.2), (4.3)\) with initial conditions

\[
\begin{align*}
 x(s) = (P(s), \xi(s), \Theta(s)) = (P_s, \xi_s, \Theta_s) = y,
\end{align*}
\]
(4.7)

then the cost function and the value function respectively become

\[
\begin{align*}
 J_{\nu}(s, y, u) &= \mathbb{E}_{(s,y)} \left\{ \int_{s}^{T} k\eta(t)\xi(t)dt - P(\tau)\xi(\tau) \right\}, \quad (4.8) \\
 V_{\nu}(s, y) &= \inf_{u \in U_{\nu}[s, T]} J_{\nu}(s, y, u), \quad (4.9) \\
 V(s, y) &= \inf_{\nu \in \Lambda[s, T]} V_{\nu}(s, y), \quad (4.10) \\
 \tau = \tau(s, y) &= \inf\{t \geq s : x(t) \notin \overline{Q}, x(s) = y\} \land T. \quad (4.11)
\end{align*}
\]

Motivated by Remark 4.3, we now introduce:

**Problem (SC’).** Given any \((s, y) \in [0, T] \times \overline{Q}\), minimize \((4.8)\) over all \( u \in U_{\nu}[s, T], \nu \in \Lambda[s, T]\), i.e., find a five-tuples \( \nu^* \in \Lambda[s, T] \) and \( u^* \in U_{\nu^*}[s, T]\) such that

\[
J_{\nu}(s, y, u^*) = V(s, y),
\]
with the corresponding state equations and initial conditions given by \((4.1), (4.2), (4.3)\) and \((4.7)\).

Before moving forward, let us make some further assumptions which will be useful in the sequel. These assumptions correspond to a Lipschitz
condition in the pay-off variable.

(C1) There exists $L_1 > 0$, such that for $\phi = b, \sigma$,

$$|\phi(t, P, \eta, c) - \phi(t, \hat{P}, \eta, c)| \leq L_1 |P - \hat{P}|, \forall t \in [0, T], P, \hat{P} \in [R, +\infty), (\eta, c) \in U.$$

(C2) The functions $b$ and $\sigma$ satisfy a polynomial growth condition in the variable $P$, i.e., there exists $L_2 > 0$, such that

$$|b(t, P, \eta, c)|^2 + |\sigma(t, P, \eta, c)|^2 \leq L_2^2 (1 + P^2), \forall t \in [0, T], P \in [R, +\infty), (\eta, c) \in U.$$

(C3) The functions $b(\cdot, \cdot, \eta, c), \sigma(\cdot, \cdot, \eta, c) \in C^{1,2}([0, \infty) \times [R, \infty))$ for any $(\eta, c) \in U$, i.e., $b_t, b_P, b_{PP}, \sigma_t, \sigma_P, \sigma_{PP}$ exist and are continuous.

We now verify that the expectation in (4.8) is finite, making our problem well defined.

**Lemma 4.1** Let the conditions (C1) and (C2) be satisfied. Then for any $(s, y) \in [0, T] \times \overline{Q}, \nu \in \Lambda[s, T], u \in U_{\nu}[s, T]$,

$$|J_\nu(s, y, u)| \leq \xi_y T[kN^7 + K(1 + P_s^2)e^{KT} + \exp\{H + AC^\alpha N^3 + \ell^2g^{-2}T\}].$$

(4.12)

where $K > 0$ depends on $L_1, L_2$ and $T$, $\ell^* = \max_{t \in [0, T]} |\ell(t)|, y = (P_y, \xi_y, \Theta_y)$.

In particular, $V := \{V(s, y), (s, y) \in \overline{Q}\}$ is well defined.

**Proof:** Given $(s, y) \in [0, T] \times \overline{Q}, \nu \in \Lambda[s, T], u \in U_{\nu}[s, T]$, let $\xi_{(1)}$ be given via

$$\xi(t) = \xi_y \exp\int_s^t (\Theta(r) + Ac(r)\eta(r)^3 - \ell(r))g^{-1}dw(r)$$

$$- \frac{1}{2} \int_s^t (\Theta(r) + Ac(r)\eta(r)^3 - \ell(r))^2g^{-2}dr$$

$$= \xi_y \xi_{(1)}(t).$$

Then the cost functional is estimated via

$$|J_\nu(s, y, u)| \leq E_{(s, y)}\left\{\int_s^T k\eta(t)^7\xi(t)dt\right\} + E_{(s, y)}\left\{\xi_y P(t)\xi_{(1)}(t)\right\}$$

$$\leq E_{(s, y)}\left\{\int_s^T k\eta(t)^7\xi(t)dt\right\} + E_{(s, y)}\left\{\xi_y P_2(t) + \xi_y \xi_{(1)}^2(t)\right\}$$

$$\leq kN^7 \int_s^T E_{(s, y)}(\xi(t))dt + \xi_y \int_s^T (\xi_{(1)}^2(t) + P^2(t))dt.$$  (4.13)

With the help of the conditions (C1) and (C2), the second moment of $P(t)$ can be estimated as (e.g. see Theorem 5.2.9 of Karatzas and Shreve [21])

$$E_{(s, y)}|P(t)|^2 \leq K(1 + P_s^2)\exp(KT), \ s \leq t \leq T;$$  (4.14)
where $K > 0$ is a constant depending on $L_1, L_2$ and $T$. Also, we note that $\xi_{(1)}$ satisfies the following stochastic differential equation

$$d\xi_{(1)}(t) = -\xi_{(1)}(t)(\Theta(t) + Ac(t)^{\alpha} \eta(t)^{\beta} - \ell(t))\varrho^{-1}dw(t).$$

By Itô’s isometry and Gronwall’s inequality, we also have, for all $(s, y) \in [0, T] \times \overline{Q}$, $s \leq t \leq T$,

$$E_{(s,y)}(\xi_{(1)}^{2}(t)) = \int_s^t E_{(s,y)}[\xi_{(1)}^{2}(r)(\Theta(r) + Ac(r)^{\alpha} \eta(r)^{\beta} - \ell(r))^2\varrho^{-2}]dr \leq H + AC^{\alpha}N^{\beta} + \ell^* \varrho^{-2}T.$$  (4.15)

Combining (4.13), (4.14) and (4.15), we obtain (4.12). The last statement in the theorem then follows since whenever $\nu, \hat{\nu} \in \Lambda[s, T]$, for all $u \in U_{\nu}[s, T]$, there exists $\hat{u} \in U_{\hat{\nu}}[s, T]$ having the same distribution as $u$. □

5 The HJB Equation and Viscosity Solutions

Let $S^n$ be the set of all $n \times n$ symmetric matrices, and $S^n_+$ be the set of symmetric, nonnegative-definite $n \times n$ matrices. For $(t, x) \in [0, T] \times \overline{Q}$, $M \in S^n_+$, $z \in \mathbb{R}^3$, let

$$\mathcal{H}(t, x, z, M) = \sup_{u \in \mathcal{U}} \left\{ -\tilde{f}(t, x, u) \cdot z - \frac{1}{2}tr(a(t, x, u)M) - L(t, u) \right\}.  \quad (5.1)$$

where (Recalling that $x = (P, \xi, \Theta) \in \overline{Q}$ and $u = (\eta, c) \in \mathcal{U}$)

$$\tilde{f}(t, x, u) = (b(t, P, u), 0, \theta(t)\zeta_H(\Theta)),$$
$$\tilde{\sigma}(t, x, u) = (\sigma(t, P, u), -\varrho^{-1}\xi(\Theta + Ac^{\alpha}\eta^{\beta} - \ell(t)), \sigma(t)\zeta_H(\Theta),$$
$$a(t, x, u) = \tilde{\sigma}(t, x, u)\tilde{\sigma}^T(t, x, u),$$
$$L(x, u) = k\eta^2\xi.$$

Then the Hamilton-Jacobi-Bellman (HJB) partial differential equation associated with the stochastic control problem (4.8), (4.9) and (4.10) can be written as:

$$-\frac{\partial V}{\partial t} + \mathcal{H}(t, x, D_xV, D_{xx}^2V) = 0, \quad (t, x) \in O := [0, T) \times Q,  \quad (5.2)$$

with the boundary condition

$$V(t, x) = -P\xi \quad \text{for} \quad (t, x) \in \partial^*O := ([0, T] \times \partial Q) \cup \{T\} \times Q.  \quad (5.3)$$
By standard stochastic control theory, the value function \((4.10)\) can be expected to be a classical solution of the HJB system \((5.2)\) and \((5.3)\) provided the uniform parabolicity condition holds, i.e., provided

\[
\exists h > 0 : \forall (t,x,u) \in \mathcal{O} \times \mathcal{U} \text{ and } \zeta \in \mathbb{R}^3, \sum_{i,j=1}^{3} a_{ij}(t,x,u)\zeta_i\zeta_j \geq h|\zeta|^2. \quad (5.4)
\]

Unfortunately, the stochastic control system does not satisfy the condition \((5.4)\); in particular, the matrix \(a(t,x,u) = \vec{\sigma}(t,x,u)\vec{\sigma}^T(t,x,u)\) is not even positive definite. Instead, we can search for a so-called viscosity solution of the HJB equations, and try to connect it with our value function.

Let \(C(0,T] \times \overline{Q})\) be the set of continuous functions \(f(t,x) : [0,T] \times \overline{Q} \rightarrow \mathbb{R}\) and \(C^{1,2}(0,T] \times \overline{Q})\) be the set of all \(f\) whose partial derivatives \((\partial f/\partial t), (\partial f/\partial x_i), (\partial f/\partial x_i \partial x_j)\) exist and are continuous on \([0,T] \times \overline{Q}\), \(1 \leq i,j \leq 3\). Next, recall:

**Definition 5.1** A function \(v \in C([0,T] \times \overline{Q})\) is called a viscosity subsolution of \((5.2)\) and \((5.3)\), if

\[
v(t,x) \leq -P\xi \quad \forall (t,x) \in \partial^*O, \quad (5.5)
\]

and for any \(\varphi \in C^{1,2}([0,T] \times \overline{Q})\), whenever \(v - \varphi\) attains a local maximum at some \((\bar{t},\bar{x}) \in O\), we have

\[
-\varphi_t(\bar{t},\bar{x}) + \mathcal{H}(\bar{t},\bar{x},-\varphi_t(\bar{t},\bar{x}),-\varphi_{xx}(\bar{t},\bar{x})) \leq 0. \quad (5.6)
\]

A function \(v \in C([0,T] \times \overline{Q})\) is called a viscosity supersolution of \((5.2)\) and \((5.3)\) if in \((5.5)\) and \((5.6)\) the inequalities “\(\leq\)” are changed to “\(\geq\)” and “local maximum” is changed to “local minimum”. When \(v \in C([0,T] \times \overline{Q})\) is both a viscosity subsolution and a viscosity supersolution of \((5.2)\) and \((5.3)\), then it is called a viscosity solution of \((5.2)\) and \((5.3)\).

In studying viscosity solutions of a second-order, nonlinear parabolic HJB equation, an equivalent definition in terms of second-order subdifferentials and superdifferentials is also useful.

**Definition 5.2** Let \(v \in C([0,T] \times \overline{Q})\).

(i) The set of second (parabolic) superdifferentials of \(v\) at \((t,x) \in [0,T] \times Q\) is

\[
D^{(1,2),+}v(t,x) = \left\{ (q,p,A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathcal{S}^3 : v(t+h,x+y) - v(t,x) \leq qh + p \cdot y + \frac{1}{2} Ay \cdot y + o(|h| + |y|^2) \right\}.
\]
(ii) The set of second (parabolic) subdifferentials of \( v \) at \((t, x) \in [0, T) \times Q \) is
\[
D^{(1,2),-} v(t, x) = -D^{(1,2),+} (-v)(t, x) = \left\{ (q, p, A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathcal{S}^3 : v(t+h, x+y) - v(t, x) \geq qh + p \cdot y + \frac{1}{2} Ay \cdot y + o(|h| + ||y||^2) \right\}.
\]

**Definition 5.3** For \( v \in C([0, T] \times \overline{Q}) \), \((t, x) \in [0, T) \times Q \), the closure of the set of sub and superdifferentials are
\[
\overline{D}^{(1,2),\pm} v(t, x) = \left\{ (q, p, A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathcal{S}^3 : \exists (t_n, x_n) \in [0, T) \times Q \to (t, x), (q_n, p_n, A_n) \in D^{(1,2),\pm} v(t_n, x_n) \to (q, p, A) \right\}.
\]

From Definition 5.2, it follows that if \( \varphi \in C^{1,2}([0, T) \times Q) \), then
\[
D^{(1,2),+} (v - \varphi)(t, x) = \{(q - \varphi_t(t, x), p - D_x \varphi(t, x), A - D^2 \varphi(t, x)) : (q, p, A) \in D^{(1,2),+} v(t, x) \}.
\]
As a consequence, the same statement holds if \( D^{(1,2),-} \) is replaced everywhere by \( D^{(1,2),-} \) or \( \overline{D}^{(1,2),+} \).

It is not hard to see that the above characterization of the second-order sub and superdifferentials yields
\[
-q + \mathcal{H}(t, x, p, A) \leq 0, \ \forall (q, p, A) \in \overline{D}^{(1,2),+} v(t, x), \quad (5.7)
\]
\[
-q + \mathcal{H}(t, x, p, A) \geq 0, \ \forall (q, p, A) \in \overline{D}^{(1,2),-} v(t, x). \quad (5.8)
\]

The above inequalities form an equivalent requirement to viscosity solutions. We start with stating two results towards obtaining this equivalence. The proof is given in Fleming and Soner [12] (Lemma 5.4.1).

**Lemma 5.1** Let \((t, x) \in [0, T) \times Q\) be given. Then \((q, p, A) \in D^{(1,2),+} v(t, x)\) if and only if there exists \( \tilde{v} \in C^{1,2}([0, T] \times \overline{Q}) \) satisfying
\[
\left( \frac{\partial}{\partial t} \tilde{v}(t, x), D_x \tilde{v}(t, x), D^2_x \tilde{v}(t, x) \right) = (q, p, A), \quad (5.9)
\]
such that \( v - \tilde{v} \) achieves its maximum at \((t, x) \in [0, T) \times Q\). Similarly, \((q, p, A) \in D^{(1,2),-} v(t, x)\) if and only if there exists \( \tilde{v} \in C([0, T] \times \overline{Q}) \) satisfying \((5.9)\) such that \( v - \tilde{v} \) achieves its minimum at \((t, x) \in [0, T) \times Q\).

An immediate corollary of the above result is this.

**Proposition 5.1** \( v \in C([0, T] \times \overline{Q}) \) is a viscosity subsolution of \((5.4)\) and \((5.3)\) if and only if \((5.7)\) holds for all \((t, x) \in [0, T) \times Q\) and \((5.3)\) holds for all \((t, x) \in \partial^* O\). Similarly, \( v \in C([0, T] \times \overline{Q}) \) is a viscosity supersolution of \((5.2)\) and \((5.3)\) if and only if \((5.5)\) holds for all \((t, x) \in [0, T) \times Q\) and \((5.3)\) holds for all \((t, x) \in [0, T) \times Q\) with “\( \leq \)” changed to “\( \geq \)”. 

12
5.1 Continuity of the Value Function

As seen in Definition 5.1, in order to prove that the value function \( (4.10) \) is a viscosity solution of \((5.2)\) and \((5.3)\), the first step is to prove the joint continuity of \( V \), in the variables \((t, x) \in [0, T] \times \mathcal{Q} \). The main difficulty in trying to do so is the fact that, for different starting time, the underlying probability spaces vary (indeed the starting points of the Brownian motions are different). So one cannot prove this joint continuity directly by estimating the difference of two expectations. In order to get the joint continuity, we first give some more conditions on the drift \( b \) and on the volatility \( \sigma \) of the pay-off process.

\[(C4)\] The functions \( b \) and \( \sigma \) satisfy a linear growth in the \( t \) variable, i.e., there exists \( L_5 > 0 \), such that
\[
|b(t, P, \eta, c)| + |\sigma(t, P, \eta, c)| \leq L_5 t
\]
for all \( t \in [0, T] \), \( P \in [0, +\infty) \), \((\eta, c) \in \mathcal{U} \).

To begin with, here is an elementary tail estimate.

**Lemma 5.2** Let \( X = \{X_t : t \in [0, T]\} \) be a semimartingale, i.e., \( X_t = x + M_t + C_t \) with \( x \in \mathbb{R} \), \( M \) a continuous local martingale, \( C \) a continuous process of bounded variation. Assume that there exists a constant \( \kappa > 0 \) such that
\[
|C_t| + \langle M \rangle_t \leq \kappa t
\]
for all \( t \geq 0 \) is valid almost surely. Then for any fixed \( T > 0 \) and sufficiently large \( n \geq 1 \), we have
\[
\mathbb{P}\left\{ \max_{0 \leq t \leq T} |X_t| \geq n \right\} \leq \frac{12}{n} \sqrt{\frac{\kappa T}{2\pi}} \exp \left\{ -\frac{n^2}{18\kappa T} \right\}. \tag{5.10}
\]

**Proof:** Taking \( n > 3 \max \{|x|, \kappa T\} \) and letting \( R_n := \inf\{t \geq 0 : |B_t| \geq n/3\} \), where \( B \) is a suitable Brownian representation of the continuous local martingale \( M \) (e.g. Theorem 4.6 of Karatzas and Shreve [21]), we have
\[
\left\{ \max_{0 \leq t \leq T} |X_t| \geq n \right\} \subseteq \left\{ \max_{0 \leq t \leq T} |M_t| \geq \frac{n}{3} \right\} \subseteq \left\{ \max_{0 \leq t \leq T} |B_{\langle M \rangle_t}| \geq \frac{n}{3} \right\} = \left\{ \langle M \rangle_T \geq R_n \right\} \subseteq \{\kappa T \geq R_n\},
\]
which lead to \( (T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}) \)
\[
\mathbb{P}\left\{ \max_{0 \leq t \leq T} |X_t| \geq n \right\} \leq \mathbb{P}\{R_n \leq \kappa T\} \leq 2\mathbb{P}\{T_{n/3} \leq \kappa T\} = 4\mathbb{P}\{B_{\kappa T} \geq \frac{n}{3}\}
\]
\[
\leq \frac{4}{\sqrt{2\pi}} \int_{\frac{n}{3\sqrt{2\pi}}}^{\infty} \exp \left\{ -\frac{z^2}{2} \right\} dz \leq \frac{12}{n} \sqrt{\frac{\kappa T}{2\pi}} \exp \left\{ -\frac{n^2}{18\kappa T} \right\}.
\]

The proof is complete. \( \square \)

It is now time to state and prove the main theorem of this section which shows the joint continuity of the value function \( (4.10) \). The idea of the proof
is similar to Fleming and Soner \[12\] Lemma 4.7.1 and Theorem 4.7.2. We first restrict the state space to some compact area with smooth boundary and add some small perturbation on the volatility of the state equation to make it satisfy the uniform parabolicity condition. Then we show that the value function corresponding to the new state equation, with compact state space, is continuous. Moreover, we verify that this value function does not depend on the choice of the reference probability space, i.e., \( V^{(n)}_\rho = V^{(n)}_{\rho,\mu} \) in the notations of (5.13) and (5.16) below. Finally, we show some uniform convergence as the perturbation goes to zero and the bounded state space goes to the unbounded state space. This implies that the same statement holds for the value functions as in (4.9) and (4.10).

**Theorem 5.1** Assume that the conditions (C1)-(C4) hold. Then the value function \( V \) as given in (4.10) is continuous on \([0, T] \times \overline{Q} \). Moreover, \( V^{(n)}_\rho(s,y) = V^{(n)}_{\rho,\mu}(s,y) \), for all \( \mu \in \Lambda(s,T) \).

**Proof:** Step 1. Fix any \( \rho > R \), and consider first the following region:

\[
Q_\rho = \left\{ x = (P, \xi, \Theta) \in Q : P \in (R, \rho), \xi \in \left( \frac{1}{\rho}, 1 \right), \Theta \in (-H, H) \right\}.
\]

We then smooth \( \partial Q_\rho \) so that \( \partial Q_\rho \in C^3 \). Consider the function \( \alpha_\rho : \overline{Q} \to [0,1] \) such that \( \alpha_\rho \in C^\infty(Q) \), \( \alpha_\rho(x) = 1 \) for \( x \in Q_\rho \) and \( \alpha_\rho(x) = 0 \) for \( \overline{Q} \setminus Q_{\rho+1} \). Next, let

\[
\begin{align*}
\tilde{f}_\rho(t, x, u) &= f(t, x, u) \cdot \alpha_\rho(x), \\
\tilde{\sigma}_\rho(t, x, u) &= \sigma(t, x, u) \cdot \alpha_\rho(x), \\
L_\rho(x,u) &= k\eta^\gamma \xi \cdot \alpha_\rho(x).
\end{align*}
\]

Now fix \( s \in [0,T] \), \( y \in \overline{Q}_\rho \) and \( 0 < \epsilon < 1 \), given any reference probability space \( \mu = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq s}, \mathbb{P}, w, w_1) \) where \( w_1 \) is another Brownian motion independent of \( w \) with starting points \( w(s) = w_1(s) = 0 \), a.e., for each \( n \in \mathbb{N} \), consider the following stochastic control system

\[
\begin{align*}
dx^{(n)}_{\rho}(t) &= \tilde{f}_\rho(t, x^{(n)}_{\rho}(t), u(t))dt + \tilde{\sigma}_\rho(t, x^{(n)}_{\rho}(t), u(t))dw(t) \\
&\quad + \epsilon^3 I_3 dw_1(t), \\
x^{(n)}_{\rho}(s) &= y \in \overline{Q}_\rho, \quad (5.11)
\end{align*}
\]

\[
J^{(n)}_{\rho,\mu}(s,y,u) = \mathbb{E}_{(s,y)} \left[ \int_s^{\tau^{(n)}_{\rho}} L_\rho(x^{(n)}_{\rho}(t), u(t))dt - P^{(n)}_{\rho}(\tau^{(n)}_{\rho}) \xi^{(n)}_{\rho}(\tau^{(n)}_{\rho}) \right] \quad (5.13)
\]

\[
\tau^{(n)}_{\rho} = \inf\{t \geq s : x^{(n)}_{\rho}(t) \notin \overline{Q}_\rho, x^{(n)}_{\rho}(s) = y, x^{(n)}_{\rho}(t) \notin \overline{Q}_\rho, x^{(n)}_{\rho}(s) = y \} \wedge T, \quad (5.14)
\]

\[
I^{(3)} = (1, 1, 1)^T.
\]
stead of a five-tuples Cing and Soner [12] and noting that the drift and volatility vectors satisfy equation (5.4). By classical stochastic control theory (e.g., Theorem 4.1 of Flem-

Note that the equations (5.11)-(5.12) satisfy the uniform parabolicity condi-

Note that the other hand, given \( \nu \) depend on \( \omega \), \( F \) placed by \( \Omega \). For \( (\omega, \omega') \in (\Omega, \Omega') \), \( t \geq s \), consider the Brownian motions \( \hat{w}(t, \omega, \omega') = w(t, \omega) \) and \( \hat{w}_1(t, \omega, \omega') = w'(t, \omega') \). Thus, setting \( \mu = (\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \{\mathcal{F}_t\}_{t \geq s}, \mathbb{P} \times \mathbb{P}', \hat{w}, \hat{w}_1) \) and if \( u \in U_{\mu}[s, T] \), we can regard \( u \) as an element of \( U_{\mu}[s, T] \) which does not depend on \( \omega' \).

Now for fixed \( n \in \mathbb{N} \), the HJB equation corresponding to the stochastic control system (5.11)-(5.16) is given by:

\[
- \frac{\partial V^{(n)}_{\rho}}{\partial t} + \mathcal{H}^{(n)}_{\rho}(t, x, D_x V^{(n)}_{\rho}, D^2_x V^{(n)}_{\rho}) = 0 \quad (t, x) \in [0, T) \times Q_{\rho},
\]

with the boundary condition (setting \( O_{\rho} := [0, T) \times Q_{\rho} \)),

\[
V^{(n)}_{\rho}(t, x) = -P \xi \quad (t, x) \in \partial^* O_{\rho},
\]

where

\[
\mathcal{H}^{(n)}_{\rho}(t, x, z, M) = \sup_{a \in \mathcal{A}} \left[ -\bar{f}_{\rho}(t, x, u) \cdot z - \frac{1}{2} tr(a^{(n)}_{\rho}(t, x, u)M) - L_{\rho}(x, u) \right],
\]

\[
a^{(n)}_{\rho} = (\bar{\sigma}_{\rho}, \epsilon^m I_3)(\bar{\sigma}_{\rho}, \epsilon^m I_3)^T.
\]

Note that the equations (5.11)-(5.12) satisfy the uniform parabolicity condition (5.3). By classical stochastic control theory (e.g., Theorem 4.1 of Fleming and Soner [12]) and noting that the drift and volatility vectors satisfy conditions (C2) and (C3), (5.17)-(5.18) has a unique solution \( W^{(n)}(t, x) \in C^{1,2}(O_{\rho}) \cap C(\overline{O_{\rho}}) \). For any \( \delta > 0 \) small enough such that \( \rho - \delta > R \), the partial derivatives \( W_t^{(n)}, W_x^{(n)}, W_{xx}^{(n)} \) are uniformly continuous on \( \overline{O_{\rho-\delta}} = [0, T - \delta] \times \overline{Q_{\rho-\delta}} \), where \( Q_{\rho-\delta} \) is defined in the same way as \( Q_{\rho} \) with \( \rho \) replaced by \( \rho - \delta \). Hence for any \( \epsilon > 0 \), there exists \( \lambda > 0 \), such that whenever \( (t, x), (t', x') \in \overline{O_{\rho-\delta}} \) with \( |t - t'| < \lambda, \|x - x'\| < \lambda \), then

\[
|A^n W^{(n)}(t, x) + L_{\rho}(t, x, u) - A^n W^{(n)}(t', x') - L_{\rho}(t', x', u)| < \epsilon / 4T,
\]
for all \( u \in \mathcal{U} \). Here \( A^u W^{(n)} \) is given by

\[
A^u W^{(n)} = W_t^{(n)} + \frac{1}{2} \sum_{i,j=1}^{3} a^{(n)}_{\rho,ij}(t,x,u) W_{x_i x_j}^{(n)} + \sum_{i=1}^{3} f_{\rho,i}(t,x,u) W^{(n)}_{x_i},
\]

with \( a^{(n)}_{\rho}(t,x,u) = \tilde{\sigma}_{\rho}^{(n)} \cdot (\tilde{\sigma}_{\rho}^{(n)})^T, \sigma_{\rho}^{(n)} = (\sigma_{\rho}, e^n I^{(3)})^T \), and the HJB equation (5.17) can be written as

\[
\min_{u \in \mathcal{U}} [A^u W^{(n)} + L_{\rho}] = 0.
\]

Now we choose \( M \) large enough so that \( M^{-1}(T-s) < \min(\lambda,1) \), and partition \([s,T-\delta]\) into \( M \) subintervals \( I_i = [t_i, t_{i+1}), i = 1, \ldots, M \). Also, we choose \( K_0 > 0 \) large enough and partition \( Q_{\rho-\delta} = B_1 \cup B_2 \cup \ldots \cup B_{K_0} \), where the \( B_j, j = 1, \ldots, K_0 \) are disjoint Borel sets of diameter less than \( \lambda/2 \). Pick \( x_j \in B_j \). For each \( i = 1, \ldots, M, j = 1, \ldots, K_0 \), by (5.20), there exists \( u_{ij} \in \mathcal{U} \), such that

\[
A^{u_{ij}} W^{(n)}(t_i, x_j) + L_{\rho}(t_i, x_j, u_{ij}) < \frac{\epsilon}{4T}.
\]

Combining (5.19) and (5.21), for \( t \in I_i, |x - x_j| < \lambda \), we have

\[
A^{u_{ij}} W^{(n)}(t,x) + L_{\rho}(t,x, u_{ij}) < \frac{\epsilon}{2T}.
\]

Pick an arbitrary \( u_0 \in \mathcal{U} \) and define the discrete Markov control policy \( \underline{u} = (\underline{u}_1, \ldots, \underline{u}_M) \), corresponding to the partition \( I_i = [t_i, t_{i+1}), i = 1, \ldots, M \), by

\[
\underline{u}_i(x) = \begin{cases} u_{ij} & x \in B_j, j = 1, \ldots, K_0, \\ u_0 & (t, x) \in \overline{O} \setminus \overline{O}_{\rho-\delta}. \end{cases}
\]

Define, by induction on \( i, u \in U_{\mu}[s,T] \) and solution \( x_{\rho}^{(n)}(t) \) to (5.11) and (5.12) with control \( u \) such that

\[
u(t) = \underline{u}(x(t_i)), \quad t \in I_i, \quad i = 1, \ldots, M,
\]

(This is done by induction on \( i \) since for \( t \in I_i, x(t) \) is the solution to (5.11) with \( F_{t_i} \)-measurable initial data \( x(t_i) \), and for \( t \in [T-\delta, T], x(t) \) is the solution to (5.11) with \( F_{t_M} \)-measurable initial data \( x(t_{M+1}) \)). Thus

\[
u(t) = u_{ij}, \quad t \in I_i, \quad x(t_i) \in B_j.
\]

By Dynkin’s formula, for any \( \{F_t\}_{t \geq s} \)-stopping time \( \theta, (s,y) \in \overline{O}_{\rho-\delta}, \tau_{\rho}^{(n,\delta)} = \)

16
\[
\inf\{ t \geq s : x^{(n)}_{\rho}(t) \not\in \overline{Q_{\rho,\delta}} \} \land (T - \delta), \theta^{(n,\delta)}_{\rho} = \tau^{(n,\delta)}_{\rho} \land \theta,
\]

\[
W^{(n)}(s, y) = \mathbb{E}_{(s, y)}\left(-\int_{s}^{\theta^{(n,\delta)}_{\rho}} \mathcal{A}^{u}W^{(n)}(t, x^{(n)}_{\rho}(t))dt + W^{(n)}(\theta^{(n,\delta)}_{\rho}, x^{(n)}_{\rho}(\theta^{(n,\delta)}_{\rho}))\right)
\]

\[
= \mathbb{E}_{(s, y)}\left(\int_{s}^{\theta^{(n,\delta)}_{\rho}} L_{\rho}(x^{(n)}_{\rho}(t), u(t))dt + W^{(n)}(\theta^{(n,\delta)}_{\rho}, x^{(n)}_{\rho}(\theta^{(n,\delta)}_{\rho}))\right)
\]

\[
- \mathbb{E}_{(s, y)}\left(\int_{s}^{\theta^{(n,\delta)}_{\rho}} [\mathcal{A}^{u}W^{(n)}(t, x^{(n)}_{\rho}(t)) + L_{\rho}(x^{(n)}_{\rho}(t), u(t))]dt\right)\quad(5.23)
\]

(since \(W^{(n)} \in C^{1,2}(\overline{Q_{\rho,\delta}})\)). We need to estimate the second term in (5.23).

To that effect, define

\[
\Gamma = \left\{ \omega : x^{(n)}_{\rho}(t) \in \overline{Q_{\rho,\delta}}, \|x^{(n)}_{\rho}(t) - x^{(n)}_{\rho}(t_{i})\| < \frac{\lambda}{2}, t \in I_{i}, 1 \leq i \leq M \right\},
\]

then for \(t \in [s, T - \delta)\), by (5.22),

\[
\mathcal{A}^{u}W^{(n)}(t, x^{(n)}_{\rho}(t)) + L_{\rho}(x^{(n)}_{\rho}(t), u(t)) < \frac{\varepsilon}{2T} \quad \text{on} \quad \Gamma, \quad (5.24)
\]

By the very definition of \(\tilde{f}_{\rho}, \tilde{\sigma}_{\rho}\), the drift and volatility vector are both bounded. Hence for all \(i = 1, \ldots, M\),

\[
\mathbb{P}_{(s, y)}\left(\max_{t \in I_{i}}\|x^{(n)}_{\rho}(t) - x^{(n)}_{\rho}(t_{i})\|_{\infty} \geq \frac{\lambda}{2}\right) \leq \lambda^{-4}D_{2}(t_{i+1} - t_{i})^{2},
\]

from which it follows that

\[
\mathbb{P}_{(s, y)}\left(\max_{i, t \in I_{i}}\|x^{(n)}_{\rho}(t) - x^{(n)}_{\rho}(t_{i})\|_{\infty} \geq \frac{\lambda}{2}\right) \leq M^{-1}\lambda^{-4}D_{2}T^{2}, \quad (5.25)
\]

where, above, we used the following result (e.g. (D.12) on P.406 in Fleming and Soner [12]): for each \(k \in \mathbb{N}, \delta' > 0\),

\[
\mathbb{P}\left\{ \max_{t \in [t_{1}, t_{2}]} \|x(t) - x(t_{1})\| \geq \delta' \right\} \leq \delta'^{-2k}D_{k}(t_{2} - t_{1})^{k}, \quad (5.26)
\]

where \(D_{k} > 0\) is a universal constant depending only on \(\rho, T, L_{1}\) and \(L_{2}\) (as in (C1) and (C2)). Since

\[
\mathbb{P}_{(s, y)}(\Gamma^{c}) \leq \mathbb{P}_{(s, y)}\left(\max_{i, t \in I_{i}}|x^{(n)}_{\rho}(t) - x^{(n)}_{\rho}(t_{i})| \geq \frac{\lambda}{2}\right),
\]

we obtain that

\[
\mathbb{E}_{(s, y)}\left(\int_{s}^{\theta^{(n,\delta)}_{\rho}} [\mathcal{A}^{u}W^{(n)} + L_{\rho}]dt\right) \leq \frac{\varepsilon}{2} + \max_{(t, x) \in [s, T - \delta] \times \overline{Q_{\rho,\delta}}} \|\mathcal{A}^{u}W^{(n)} + L_{\rho}\|_{\mathbb{P}_{(s, y)}(\Gamma^{c})}
\]

\[
\leq \frac{\varepsilon}{2} + \|\mathcal{A}^{u}W^{(n)} + L_{\rho}\|_{(t, x) \in [s, T - \delta] \times \overline{Q_{\rho,\delta}}}M^{-1}\lambda^{-4}DT^{2}.
\]
Therefore, for fixed $\rho$, $\delta > 0$, for any $(s, y) \in [0, T - \delta] \times \overline{Q_{\rho - \delta}}$, if we pick $M > 0$ large enough, then for any reference probability system $\mu \in \Lambda[s, T]$, there exists $u \in U_{\mu}[s, T]$, such that for any stopping time $\theta$,

$$W^{(n)}(s, y) + \varepsilon \geq \mathbb{E}_{(s, y)} \left[ \int_s^{\theta_{\rho, \delta}^{(n)}} L_{\rho}(x_{\rho, \delta}^{(n)}(t), u(t)) dt + W^{(n)}(\theta_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)})) \right]$$

(5.27)

Also, by (5.26), for any $s \in [0, T]$, $\mu \in \Lambda[s, T]$, $u \in U_{\mu}[s, T]$, $0 \leq \mathcal{A}^{n} W^{(n)} + L_{\rho}$, so (5.29) implies

$$W^{(n)}(s, y) \leq \mathbb{E}_{(s, y)} \left[ \int_s^{\theta_{\rho, \delta}^{(n)}} L_{\rho}(x_{\rho, \delta}^{(n)}(t), u(t)) dt + W^{(n)}(\theta_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)})) \right]$$

(5.28)

We need to take $\delta \to 0$ in both (5.27) and (5.28). Note that $W^{(n)}$ is continuous and thus uniformly continuous on $\overline{Q_{\rho}}$, and so there exists $0 < \kappa_{1} < \min\{1, 2\varepsilon\}$, such that, whenever $|t - s| < \kappa_{1}/2$, $\|x - x'\| < \kappa_{1}/2$, $(t, x)$, $(t', x') \in \overline{Q_{\rho}}$,

$$|W^{(n)}(t, x) - W^{(n)}(t', x')| < \varepsilon.$$  

(5.29)

From (5.26) and (5.29), we obtain that,

$$\mathbb{E}_{(s, y)} |W^{(n)}(\theta_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)}))| - W^{(n)}(\theta_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)}))|$$

$$\leq 2\|W^{(n)}\|_{(s, y) \in \overline{Q_{\rho}}}(\varepsilon + D_{\delta} \varepsilon) + \varepsilon.$$  

(5.30)

Moreover, given any $\mu \in \Lambda[s, T]$, $u \in U_{\mu}[s, T]$, note that clearly there exists $\delta_{0} > 0$, such that for any $0 < \delta < \delta_{0}$,

$$P_{(s, y)} \{ \tau_{\rho, \delta}^{(n)} - \tau_{\rho, \delta}^{(n, \delta)} > \kappa_{1}/8 \} \leq \varepsilon,$$

(since the events $\{ \tau_{\rho, \delta}^{(n)} - \tau_{\rho, \delta}^{(n, \delta)} > \kappa_{1}/8 \}$ are monotonically decreasing to the empty set, as $\delta \to 0$). So

$$\mathbb{E}_{(s, y)} \left[ \int_{\theta_{\rho, \delta}^{(n)}}^{\tau_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)})} L_{\rho}(x_{\rho, \delta}^{(n)}(t), u(t)) dt \right] \leq \|L_{\rho}\|_{\varepsilon} + \|L_{\rho}\|_{TP_{(s, y)}} \left[ \tau_{\rho, \delta}^{(n)} - \tau_{\rho, \delta}^{(n, \delta)} > \kappa_{1}/8 \right]$$

$$\leq \|L_{\rho}\|_{(T + 1)\varepsilon}.$$  

(5.31)

Combining (5.30) and (5.31), we obtain that for any $s \in [0, T]$, $y \in Q_{\rho}$, $\mu \in \Lambda[s, T]$, $u \in U_{\mu}[s, T]$, any stopping time $\theta$,

$$\lim_{\delta \to 0} \sup_{\theta} \mathbb{E}_{(s, y)} |W^{(n)}(\theta_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta_{\rho, \delta}^{(n)})) - W^{(n)}(\theta \wedge \tau_{\rho, \delta}^{(n)}, x_{\rho, \delta}^{(n)}(\theta \wedge \tau_{\rho, \delta}^{(n)}))|$$

$$+ \mathbb{E}_{(s, y)} \left[ \int_{\theta_{\rho, \delta}^{(n)}}^{\tau_{\rho, \delta}^{(n)}(\theta)} L_{\rho}(x_{\rho, \delta}^{(n)}(t), u(t)) dt \right] = 0.$$  

(5.32)
Hence in (5.28), taking $\delta \to 0$, we obtain that for any $s \in [0, T)$, $y \in Q_\rho$, $\mu \in \Lambda[s, T]$, $u \in U_\mu[s, T]$, any stopping time $\theta$,

$$W^{(n)}(s, y) \leq \mathbb{E}_{(s, y)}\left(\int_s^{\theta \wedge \tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt + W^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}))\right).$$

(5.33)

Moreover, by (5.32), for any $\Delta > 0$, any reference probability system $\mu \in \Lambda[s, T]$, $u \in U[s, T]$, there exists $\delta_1 > 0$, such that whenever $\delta < \delta_1$, for any stopping time $\theta$,

$$\mathbb{E}_{(s, y)}|W^{(n)}(\theta^{(n, \delta_0)}_\rho, x_\rho^{(n)}(\theta^{(n, \delta_0)}_\rho)) - W^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}))| + \mathbb{E}_{(s, y)}\left|\int_{\theta^{(n, \delta_0)}_\rho}^{\tau_\rho^{(n)} \wedge \theta} L_\rho(x_\rho^{(n)}(t), u(t)) dt\right| \leq \frac{\Delta}{2}. \quad (5.34)$$

Also, for any $(s, y) \in [0, T) \times Q_\rho$, pick $\delta_2 > 0$ small enough so that $(s, y) \in O_{\rho - \delta_2}$, then by (5.27), for $\delta_0 = \min\{\delta_1, \delta_2\}$, for any reference probability system $\mu \in \Lambda[s, T]$, there exists $u \in U_\mu[s, T]$, such that for any stopping time $\theta$,

$$W^{(n)}(s, y) + \frac{\Delta}{2} \geq \mathbb{E}_{(s, y)}\left(\int_s^{\theta^{(n, \delta_0)}_\rho} L_\rho(x_\rho^{(n)}(t), u(t)) dt + W^{(n)}(\theta^{(n, \delta_0)}_\rho, x_\rho^{(n)}(\theta^{(n, \delta_0)}_\rho))\right).$$

(5.35)

Hence, combining (5.34) and (5.35), we obtain that

$$W^{(n)}(s, y) + \frac{\Delta}{2} \geq \mathbb{E}_{(s, y)}\left(\int_s^{\theta \wedge \tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt + W^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}))\right).$$

(5.36)

Note that (5.33) and (5.36) are trivially true on $\partial^* O_\rho$, so we can extend those two inequalities to all $(s, y) \in [0, T] \times \overline{Q_\rho}$. In particular, by picking the stopping time $\theta \equiv T$, then for any reference probability system $\mu$, we have by (5.33) and (5.36),

$$W^{(n)}(s, y) = \inf_{u \in \mathcal{U}_\mu} \mathbb{E}_{(s, y)}\left(\int_s^{\tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt - P_\rho^{(n)}(\tau_\rho^{(n)}) \xi_\rho^{(n)}(\tau_\rho^{(n)})\right),$$

$$V^{(n)}_{\rho, \mu}(s, y), \quad (5.37)$$

which implies that for $(s, y) \in [0, T] \times \overline{Q_\rho}$,

$$W^{(n)}(s, y) = \inf_{\mu \in \Lambda[s, T]} V^{(n)}_{\rho, \mu}(s, y) = V^{(n)}_{\rho}(s, y).$$

Therefore, for any $\rho > 0$, $s \in [0, T]$, $y \in \overline{Q_\rho}$, $\mu \in \Lambda[s, T]$,

$$V^{(n)}_{\rho}(s, y) = V^{(n)}_{\rho, \mu}(s, y) \in C(O_\rho). \quad (5.38)$$
which achieves the proof of Step 1.

**Step 2** For the region \( O_\rho \) considered in Step 1, let the stochastic control system be described via:

\[
\begin{align*}
    dx_\rho(t) &= f_\rho(t, x_\rho(t), u(t))dt + \sigma_\rho(t, x_\rho(t), u(t))dw(t), \\
    x_\rho(s) &= y \in \overline{Q}_\rho, \\
    J_{\rho,\mu}(s, y, u) &= E_{(s,y)} \left[ \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t))dt - P_\rho(\tau_\rho)\xi_\rho(\tau_\rho) \right],
\end{align*}
\]

\( \tau_\rho = \tau_\rho(s, y) = \inf\{ t \geq s : x_\rho(t) \notin \overline{Q}_\rho, x_\rho(s) = y \} \wedge T, \)

\( V_{\rho,\mu}(s, y) = \inf_{u \in U_\mu[s, T]} J_{\rho,\mu}(s, y, u), \)

\( V_\rho(s, y) = \inf_{\mu \in \Lambda[s, T]} V_{\rho,\mu}(s, y). \)

By Remark 5.1, we can build a one-to-one correspondence between all five-tuplet \( \nu = (\Omega, F, \{ F_t \}_{t \geq s}, \mathbb{P}, w, w_1) \) and six-tuplet \( \mu = (\Omega, F, \{ F_t \}_{t \geq s}, \mathbb{P}, w, w_1) \), and so we can regard those two expectations in \( J_{\rho,\nu}(s, y, u) \) and \( J_{\rho,\mu}^{(n)}(s, y, u) \) as defined on the same probability space. Then for \( (s, y) \in [0, T] \times \overline{Q}_\rho, \)

\[
|J_{\rho,\mu} - J_{\rho,\mu}^{(n)}| \leq \mathbb{E}_{(s,y)} \left| \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t))dt - \int_s^{\tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t))dt \right| \\
+ \mathbb{E}_{(s,y)}|P_\rho(\tau_\rho)\xi_\rho(\tau_\rho) - P_\rho^{(n)}(\tau_\rho^{(n)})\xi_\rho^{(n)}(\tau_\rho^{(n)})| \\
\leq (kN^\gamma + \max_{x \in \overline{Q}_\rho} \| x \|_\infty) \mathbb{E}_{(s,y)} \int_s^T \| x_\rho(t) - x_\rho^{(n)}(t) \|_\infty dt \\
\leq (kN^\gamma + \max_{x \in \overline{Q}_\rho} \| x \|_\infty) B_0 T \| \sigma_\rho - \sigma_\rho^{(n)} \|_\infty \\
= (kN^\gamma + \max_{x \in \overline{Q}_\rho} \| x \|_\infty) B_0 T \omega^n,
\]

where we note that the drift function for \( x_\rho \) and for \( x_\rho^{(n)} \) are the same and the estimation

\( E \| x_\rho(\cdot) - x_\rho^{(n)}(\cdot) \|_\infty \leq B_0 \| \sigma_\rho - \sigma_\rho^{(n)} \|_\infty, \)

for some constant \( B_0 \) depending only on \( \rho, T, L_1 \) and \( L_2 \) (e.g. (D.9) on P.405 Fleming and Soner [12]).

Therefore as \( n \to \infty, J_{\rho,\mu}^{(n)}(s, y, u) \) converges to \( J_{\rho,\nu}(s, y, u) \) uniformly for all \( (s, y) \in \overline{Q}_\rho \), \( \mu \in \Lambda[s, T], u \in U_\mu[s, T] \). It follows that uniformly in all \( (s, y) \in \overline{Q}_\rho, \)

\( V_\rho^{(n)}(s, y) \to V_\rho(s, y), \quad n \to \infty, \)

20
which implies that $V_\rho(s, y) \in C(\overline{\rho})$. Moreover, since for any $(s, y) \in \overline{\rho}$, $\mu \in \Lambda[s, T]$, $V_{\rho, \mu}^{(n)}(s, y)$ converges to $V_{\rho, \mu}(s, y)$, as $n \to \infty$, and in a pointwise manner, $V_{\rho, \mu}^{(n)}(s, y) = V_{\rho}^{(n)}(s, y)$, we have

$$V_\rho(s, y) = V_{\rho, \mu}(s, y),$$

for all $\mu \in \Lambda[s, T]$, $(s, y) \in [0, T] \times \overline{\rho}$. This finishes Step 2.

**Step 3.** We finally consider the stochastic control system (4.1)-(4.3). For any compact subset $C \subseteq \overline{Q} \setminus \{\xi = 0\}$, we can find $\rho > 0$, such that $C \subseteq \overline{\rho}$. Note that when $s \leq \tau_\rho$, the trajectories of $x_\rho(t)$ and $x(t)$ are exactly the same. Also, at time $\tau_\rho$, $x(t)$ passes through the surface $\{x = (P, \xi, \Theta) \in \overline{Q} : P = R\}$, then $\tau_\rho = \tau$. We first show that for $\rho$ large enough, with large probability $x(t)$ will first penetrate $\{x = (P, \xi, \Theta) \in \overline{Q} : P = R\}$ before it passes through the other surfaces of $\overline{\rho}$. In fact, since $\Theta$ is bounded, and since the condition (C4) guarantees that $P$ satisfies Lemma 5.2, for any $\varepsilon > 0$, there exists $\rho_0 > 0$ independent of the starting points, the reference system and the control, with $C \subseteq \overline{\rho_0}$, such that for any $\rho > \rho_0$,

$$\mathbb{P}_{(s, y)} \left\{ \max_{s \leq t \leq T} P(t, u) \geq \rho \right\} \leq \frac{\varepsilon}{2},$$

for any $u \in U_\mu[s, T]$, $\mu \in \Lambda[s, T]$, $(s, y) \in [0, T] \times C$.

Note that $\xi$ is a positive martingale. So by the (sub)martingale maximum inequality, for $\rho > 4 \max_{x \in C} |\xi| \varepsilon^{-1}$, we obtain that for any $u \in U_\mu[s, T]$, $\mu \in \Lambda[s, T]$, $(s, y) \in [0, T] \times C$,

$$\mathbb{P}_{(s, y)} \left\{ \max_{s \leq t \leq T} \xi(t) \geq \rho \right\} \leq \rho^{-1} \mathbb{E}_{(s, y)}(\xi(T)) \leq \rho^{-1} \max_{x \in C} |\xi| \leq \frac{\varepsilon}{4}. \quad (5.49)$$

Also, let $\zeta(t) := \{\xi^{-1}(t), \, t \in [s, T]\}$, note that

$$\phi(t) := \zeta(t) \exp \left\{ - \int_s^t \left( \Theta(r) + Ac(r)^\alpha \eta(r)^\beta - \ell(r) \right) \bar{q}^{-2} dr \right\},$$

is also a positive martingale (since it is also a Girsanov density). So if we pick $\rho > 4T \max_{x \in C} |\xi^{-1}| \exp \{(H + AC^\alpha N^\beta + \ell^\alpha)q^{-2}T\} \varepsilon^{-1} := L(T, \mathcal{C}) \varepsilon^{-1}$, it follows from Markov’s inequality that

$$\mathbb{P}_{(s, y)} \left\{ \min_{s \leq t \leq T} \zeta(t) \leq \rho^{-1} \right\} = \mathbb{P}_{(s, y)} \left\{ \max_{s \leq t \leq T} \zeta(t) \geq \rho \right\} \leq \rho^{-1} \mathbb{E}_{(s, y)}(\max_{s \leq t \leq T} \zeta(t))$$

$$\leq \rho^{-1} \exp \left\{ (H + AC^\alpha N^\beta + \ell^\alpha)q^{-2}T \right\} \mathbb{E}_{(s, y)}(\max_{s \leq t \leq T} \phi(t))$$

$$\leq \rho^{-1} \exp \left\{ (H + AC^\alpha N^\beta + \ell^\alpha)q^{-2}T \right\} \mathbb{E}_{(s, y)} \left( \int_s^T \phi(t) dt \right)$$

$$\leq \rho^{-1} T \max_{x \in C} |\xi^{-1}| \exp \{ (H + AC^\alpha N^\beta + \ell^\alpha)q^{-2}T \} \leq \frac{\varepsilon}{4}. \quad (5.50)$$
for any \( u \in U_\mu[s,T], \mu \in \Lambda[s,T], (s,y) \in [0,T] \times \mathcal{C} \). Combining (5.48) - (5.50), we have for \( \rho > \max\{\rho_0, 4\max_{x \in \mathcal{C}}|\xi|\epsilon^{-1}, L(T,\mathcal{C})\epsilon^{-1}\}, \)

\[
\mathbb{P}_{(s,y)}\{\gamma_\rho < \tau\} < \epsilon, \tag{5.51}
\]

for all \( u \in U_\mu[s,T], \mu \in \Lambda[s,T], (s,y) \in [0,T] \times \mathcal{C} \).

Now using conditions (C1) and (C2), from (D.7) of [12] P.405, we obtain

\[
\mathbb{E}_{(s,y)}|P(t)|^m \leq B_m(1 + |P_m|^m), \quad m \geq 1, \tag{5.52}
\]

where \( B_m \) is a constant depends on \( \rho, L_1, L_2 \) and \( T \). Also, note that

\[
\xi^t(t) = \xi^s \exp \left\{ 4 \int_s^t (\Theta(r) + Ac(r)\eta(r)\beta - \ell(r))\theta^{-1}dw(r) \right. \\
- 2 \int_s^t (\Theta(r) + Ac(r)\eta(r)\beta - \ell(r))^2\theta^{-2}dr \left. \right\}
\]

\[
= \psi(t)\xi^s \exp \left\{ 6 \int_s^t (\Theta(r) + Ac(r)\eta(r)\beta - \ell(r))^2\theta^{-2}dr \right\},
\]

where

\[
\psi(t) := \exp \left\{ 4 \int_s^t (\Theta(r) + Ac(r)\eta(r)\beta - \ell(r))\theta^{-1}dw(r) \right. \\
- \frac{1}{2} \int_s^t 16(\Theta(r) + Ac(r)\eta(r)\beta - \ell(r))^2\theta^{-2}dr \left. \right\},
\]

is a Girsanov density satisfying the Novikov condition and is thus a martingale. Hence, we can estimate the moment of \( \xi^t(t) \) as:

\[
\mathbb{E}_{(s,y)}\xi^t(t) \leq \exp \left\{ 6T(H + AC^\alpha N^\beta + \ell^\star)^2\theta^{-2}\right\} \xi^s \mathbb{E}_{(s,y)}\psi(t)
\]

\[
= \exp \left\{ 6T(H + AC^\alpha N^\beta + \ell^\star)^2\theta^{-2}\right\} \xi^s, \tag{5.53}
\]

for any \( (s,y) \in [0,T] \times \mathcal{C}, \mu \in \Lambda[s,T], u \in U_\mu[s,T] \). Therefore, by (4.15), (5.52) and (5.53),

\[
\mathbb{E}_{s,y}\left\{ \int_s^{\tau(s,y)} k\eta(t)\gamma(t)dt - P(\tau(s,y))\xi(\tau(s,y)) \right\}^2 \\
\leq \mathbb{E}_{s,y}\left( \int_s^{\tau(s,y)} k\eta(t)\gamma(t)dt \right)^2 + \mathbb{E}_{s,y}\left[ P^2(\tau(s,y))\xi^2(\tau(s,y)) \right]
\]

\[
\leq kN^2\mathbb{E}_{s,y}\left( \int_s^T \xi(t)^2dt \right) + \frac{1}{2}\mathbb{E}_{s,y}\left[ \int_s^T (P^4(t) + \xi^4(t))dt \right]
\]

\[
\leq kN^2T \max_{x \in \mathcal{C}}|\xi|^2 \exp \left\{ (H + AC^\alpha N^\beta + \ell^\star)^2\theta^{-2}T \right\} + \frac{1}{2}B_4(1 + \max_{x \in \mathcal{C}}|P|^4)
\]

\[
+ \frac{1}{2}T \max_{x \in \mathcal{C}}|\xi|^4 \exp \left\{ 6T(H + AC^\alpha N^\beta + \ell^\star)^2\theta^{-2} \right\}
\]

\[
:= K(T,\mathcal{C}) < \infty. \tag{5.54}
\]
Further, by (5.51) and (5.54), for any \( \varepsilon > 0 \), \((s,y) \in [0,T] \times \mathcal{C} \), \( u \in U_\mu[s,T] \), \( \mu \in \Lambda[s,T] \), when \( \rho > \max\{\rho_0, 6\max_{x \in \mathcal{C}} |\xi|^{-1}, L(T,\mathcal{C})\varepsilon^{-1}\} \), we have by the Cauchy-Schwarz Inequality that

\[
|J_\mu - J_{\rho,\mu}| \leq \mathbb{E}_{(s,y)} \{ I_{(\tau_\rho < \tau)} \left[ \int_0^{\tau(s,y)} k\eta(t)^\gamma \xi(t) dt - P(\tau(s,y))\xi(\tau(s,y)) \right] \} \\
\leq \left( \mathbb{E}_{(s,y)} \left\{ \int_0^T k\eta(t)^\gamma \xi(t) dt - P(\tau)\xi(\tau) \right\}^2 \right)^{1/2} (\mathbb{P}_{(s,y)}\{\tau_\rho < \tau\})^{1/2} \\
\leq K(T,C)^{1/2}\varepsilon^{1/2},
\]

which implies that uniformly in all \((s,y) \in [0,T] \times \mathcal{C} \), \( V_\rho(s,y) \rightarrow V(s,y) \), as \( \rho \rightarrow \infty \), and hence \( V(s,y) \in C([0,T] \times \mathcal{C}) \). Moreover for all \( \mu \in \Lambda[s,T] \), we have \( V(s,y) = V_\mu(s,y) \). Since \( \mathcal{C} \) is an arbitrary compact set, we obtain that \( V(s,y) \in C([0,T] \times \{\mathcal{Q} \setminus \{x \in \mathcal{Q} : \xi = 0\}\}) \), and \( V_{\rho,\mu}(s,y) = V_\mu(s,y) \) on \([0,T] \times \{\mathcal{Q} \setminus \{x \in \mathcal{Q} : \xi = 0\}\} \) (and thus on \([0,T] \times \mathcal{Q} \) since on \( \{x \in \mathcal{Q} : \xi = 0\} \) these two value functions are both zero). The continuity on the boundary \([0,T] \times \{x \in \mathcal{Q} : \xi = 0\}\) follows from (4.12) of Lemma (4.1) since \( V(s,y) \equiv 0 \) on \([0,T] \times \{x \in \mathcal{Q} : \xi = 0\}\). Therefore, we finally have

\[
V(s,y) \in C([0,T] \times \mathcal{Q}), \quad V(s,y) = V_\mu(s,y), \forall \mu \in \Lambda[s,T], \forall s \in [0,T] \quad (5.56)
\]

The proof is complete. \( \square \)

### 5.2 The Dynamic Programming Principle

In order to prove that the value function is a viscosity solution of the corresponding HJB equation, and besides multivariate continuity, we also need to show that the Dynamic Programming Principle is satisfied. In fact, a result which is a stronger version of the traditional dynamic programming principle is verified in this section.

**Definition 5.4** The value function \( V \) is said to satisfy the Dynamic Programming Principle if, for any \((s,y) \in [0,T] \times \mathcal{Q} \) and \((\mathcal{F}_t)_{t \geq s}\)-stopping time \( \theta \),

\[
V(s,y) = \inf_{u \in U_\mu[s,T], \mu \in \Lambda[s,T]} \mathbb{E}_{(s,y)} \left\{ \int_s^{\tau \land \theta} L(x(t),u(t)) dt + V(\tau \land \theta, x(\tau \land \theta)) \right\} \quad (5.57)
\]

**Definition 5.5** The value function \( V \) is said to satisfy the property \((DP)\) if, for any \((s,y) \in [0,T] \times \mathcal{Q} \), the following two conditions hold:

(i) For any \( \mu \in \Lambda[s,T], u \in U_\mu[s,T] \) and \((\mathcal{F}_t)_{t \geq s}\)-stopping time \( \theta \),

\[
V(s,y) \leq \mathbb{E}_{(s,y)} \left\{ \int_s^{\theta \land \tau} L(x(t),u(t)) dt + V(\theta \land \tau, x(\theta \land \tau)) \right\}. \quad (5.58)
\]

23
(ii) For every $\delta > 0$, there exists $\mu \in \Lambda[s,T]$ and $u \in U_{\mu}[s,T]$ such that for every $(\mathcal{F}_t)_{t \geq s}$-stopping time $\theta$

$$V(s, y) + \delta \geq E_{(s, y)} \left\{ \int_{s}^{\theta \wedge \tau} L(x(t), u(t)) dt + V(\theta \wedge \tau, x(\theta \wedge \tau)) \right\}. \quad (5.59)$$

Clearly the property (DP) immediately implies the validity of the Dynamic Programming Principle. In the next theorem we establish the validity of the property (DP) for our value function (4.10), using an approach similar to that of Theorem 5.1. But we first need the following lemma.

**Lemma 5.3** Given $(s, y) \in [0, T] \times \mathcal{Q}_{\rho}, \mu \in \Lambda[s,T], u \in U_{\mu}[s,T]$, then as $n \to \infty$,

$$\tau^{(n)}_{\rho} \to \tau_{\rho} \quad a.e. - P_{(s, y)},$$

where $\tau^{(n)}_{\rho}$ and $\tau_{\rho}$ are respectively defined in (5.14) and (5.42).

**Proof:** By (5.45), for any $\varepsilon > 0$,

$$P_{(s, y)} \left\{ \| x^{(n)}_{\rho}(\cdot) - x_{\rho}(\cdot) \| > \varepsilon \right\} \leq \varepsilon^{-1} E_{(s, y)} \| x^{(n)}_{\rho}(\cdot) - x_{\rho}(\cdot) \| \leq \varepsilon^{-1} B_0 \| \bar{\sigma}_{\rho} - \bar{\sigma}^{(n)}_{\rho} \| \leq \varepsilon^{-1} B_0 \| \sigma^{n} \| \leq \varepsilon^{-1} B_0 \| \rho \| ^{\alpha} \| n \| ^{\beta},$$

which immediately implies that

$$\sum_{n=1}^{\infty} P_{(s, y)} \left\{ \| x^{(n)}_{\rho}(\cdot) - x_{\rho}(\cdot) \| > \varepsilon \right\} \leq \varepsilon^{-1} B_0 \| \rho \| ^{\alpha} \frac{\varepsilon^{\beta}}{1 - \varepsilon^{\beta}} < \infty.$$ 

Hence by the Borel-Cantelli Lemma, there exists $\Omega_0 \subseteq \Omega$ with $P_{(s, y)}(\Omega_0) = 1$, such that

$$x^{(n)}_{\rho}(t) \to x_{\rho}(t) \quad uniformly \ in \ t \in [s, T], \ n \to \infty, \ \forall \omega \in \Omega_0.$$ 

Therefore, for any $\omega \in \Omega_0$, $\tau^{(n)}_{\rho}(\omega) \to \tau_{\rho}(\omega)$ as $n \to \infty$, which completes the proof of lemma.

**Theorem 5.2** Assume that the conditions (C1)-(C3) are satisfied. Then the property (DP) holds for the value function $V$. 

**Proof:** Step 1. The property (DP) of $V_{\rho}^{(n)}$ has been established in the proof of the previous theorem as (5.33) and (5.36).
Since $V_I$, the value function $V_\rho$ as in (5.44). For any $(s,y) \in [0,T] \times Q_\rho$, $\mu \in A[s,T]$, $u \in U_\mu[s,T]$ and $(\mathcal{F}_t)_{t \geq s}$-stopping time $\theta$,

$$
\begin{align*}
&\mathbb{E}_{(s,y)} \left[ V_\rho(\theta \wedge T, x_\rho(\theta \wedge T)) - V_\rho^{(n)}(\theta \wedge T, x_\rho^{(n)}(\theta \wedge T)) \right] \\
\leq & \mathbb{E}_{(s,y)} \left[ I_{(\tau_\rho^{(n)})^c < \tau_\rho} \left[ V_\rho(\theta \wedge \tau_\rho, x_\rho(\theta \wedge \tau_\rho)) - V_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)})) \right] \\
& + \mathbb{E}_{(s,y)} \left[ I_{(\tau_\rho^{(n)})^c \geq \tau_\rho} \left[ V_\rho(\theta \wedge \tau_\rho, x_\rho(\theta \wedge \tau_\rho)) - V_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)})) \right] \right) \\
\leq & \mathbb{E}_{(s,y)} \left[ I_{(\tau_\rho^{(n)})^c < \tau_\rho} \left[ V_\rho(\theta \wedge \tau_\rho, x_\rho(\theta \wedge \tau_\rho)) - V_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)})) \right] \\
& + \mathbb{E}_{(s,y)} \left[ I_{(\tau_\rho^{(n)})^c \geq \tau_\rho} \left[ V_\rho(\theta \wedge \tau_\rho, x_\rho(\theta \wedge \tau_\rho)) - V_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)}, x_\rho^{(n)}(\theta \wedge \tau_\rho^{(n)})) \right] \right) \\
& + 2\| V_\rho - V_\rho^{(n)} \| \\
= & I_1 + I_2 + I_3.
\end{align*}
$$

By (5.46), for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that whenever $n \geq N_1$,

$$
I_3 = 2\| V_\rho - V_\rho^{(n)} \| \leq \varepsilon.
$$

Next, we estimate $I_1$ as follows:

$$
\begin{align*}
I_1 &\leq \mathbb{E}_{(s,y)} \left[ I_{(\theta < \tau_\rho^{(n)})^c < \tau_\rho} \left[ V_\rho(\theta, x_\rho(\theta)) - V_\rho^{(n)}(\theta, x_\rho^{(n)}(\theta)) \right] \right] \\
& + \mathbb{E}_{(s,y)} \left[ I_{(\theta \leq \tau_\rho^{(n)})^c < \tau_\rho} \left[ V_\rho(\theta, x_\rho(\theta)) - V_\rho^{(n)}(\tau_\rho^{(n)}, x_\rho^{(n)}(\tau_\rho^{(n)})) \right] \right] \\
& + \mathbb{E}_{(s,y)} \left[ I_{(\tau_\rho^{(n)})^c < \tau_\rho < \theta} \left[ V_\rho(\tau_\rho, x_\rho(\tau_\rho)) - V_\rho^{(n)}(\tau_\rho^{(n)}, x_\rho^{(n)}(\tau_\rho^{(n)})) \right] \right] \\
= & J_1 + J_2 + J_3.
\end{align*}
$$

Since $V_\rho$ is uniformly continuous in $[0, T] \times Q_\rho$, for any $\varepsilon > 0$, there exists $0 < \delta_0 < \min\{1, 2\varepsilon\}$, such that whenever $(t, x)$, $(t', x') \in [0,T] \times Q_\rho$ satisfy $|t - t'| + \| x - x' \| < \delta_0$, we have

$$
| V_\rho(t, x) - V_\rho(t', x') | < \varepsilon.
$$

Also, using (5.45) and Lemma 5.3, there exists $N_2 \in \mathbb{N}$, such that whenever $n \geq N_2$,

$$
\mathbb{P}_{(s,y)} \left\{ \| x_\rho(\cdot) - x_\rho^{(n)}(\cdot) \| = \frac{\delta_0}{2} \leq \varepsilon, \right. \tag{5.63}
\mathbb{P}_{(s,y)} \left\{ | \tau_\rho^{(n)} - \tau_\rho^{(n)} | > \frac{\delta_0}{8} \leq \varepsilon, \right. \tag{5.64}
$$

25
which implies that
\[
J_1 \leq \mathbb{E}_{(s,y)} \left[ I_{\{\phi < \tau_{\rho}^{(n)}} < \tau_{\rho}, \|x_{\rho}(\cdot) - x_{\rho}^{(n)}(\cdot)\|_{\infty} > \frac{\delta_0}{4} \} \left( V_{\rho}(\theta, x_{\rho}(\theta)) - V_{\rho}(\theta, x_{\rho}^{(n)}(\theta)) \right) \right]
+ \mathbb{E}_{(s,y)} \left[ I_{\{\phi < \tau_{\rho}^{(n)}} < \tau_{\rho}, \|x_{\rho}(\cdot) - x_{\rho}^{(n)}(\cdot)\|_{\infty} \leq \frac{\delta_0}{4} \} \left( V_{\rho}(\theta, x_{\rho}(\theta)) - V_{\rho}(\theta, x_{\rho}^{(n)}(\theta)) \right) \right]
\leq (2 \max_{(t,x) \in O_{\rho}} |V_{\rho}(t, x)| + 1)\varepsilon. \tag{5.65}
\]

For \(J_2\) and \(J_3\), set \(A_1 := \{\|x_{\rho}(\tau_{\rho}) - x_{\rho}^{(n)}(\tau_{\rho}^{(n)})\| > \delta_0/2\}\), \(A_2 := \{\|x_{\rho}(\tau_{\rho}^{(n)}) - \tau_{\rho}\| > \delta_0/8\}\) and \(A_3 := \{\|x_{\rho}(\cdot) - x_{\rho}^{(n)}(\cdot)\|_{\infty} > \delta_0/4\}\), then by \((5.63), (5.64)\) and \((5.26)\), we have
\[
\mathbb{P}_{(s,y)}(A_1) \leq \mathbb{P}_{(s,y)}(A_2) + \mathbb{P}_{(s,y)}(A_1 \cap A_2^c)
\leq \mathbb{P}_{(s,y)}(A_2) + \mathbb{P}_{(s,y)}(A_3) + \mathbb{P}_{(s,y)}(A_2^c \cap \{\|x_{\rho}(\tau_{\rho}) - x_{\rho}^{(n)}(\tau_{\rho}^{(n)})\| > \delta_0/4\})
\leq 2\varepsilon + D_1 \frac{\delta_0}{2} \leq (2 + D_1)\varepsilon. \tag{5.66}
\]

Hence by \((5.63), (5.64)\) and \((5.66)\), for \(n \geq N_2\),
\[
J_3 = \mathbb{E}_{(s,y)} \left[ I_{\{\tau_{\rho}^{(n)} < \tau_{\rho} < \theta, \tau_{\rho} \in (A_1 \cup A_2)\}} \left| V_{\rho}(\tau_{\rho}, x_{\rho}(\tau_{\rho})) - V_{\rho}(\tau_{\rho}^{(n)}, x_{\rho}^{(n)}(\tau_{\rho}^{(n)})) \right| \right]
+ \mathbb{E}_{(s,y)} \left[ I_{\{\tau_{\rho}^{(n)} < \tau_{\rho} < \theta, \tau_{\rho} \in (A_1 \cup A_2)^c\}} \left| V_{\rho}(\tau_{\rho}, x_{\rho}(\tau_{\rho})) - V_{\rho}(\tau_{\rho}^{(n)}, x_{\rho}^{(n)}(\tau_{\rho}^{(n)})) \right| \right]
\leq (4 \max_{(t,x) \in O_{\rho}} |V_{\rho}(t, x)| + 1)\varepsilon, \tag{5.67}
\]
\[
J_2 \leq \mathbb{E}_{(s,y)} \left[ I_{\{\tau_{\rho}^{(n)} < \tau_{\rho} < \theta, \tau_{\rho} \in (A_1 \cup A_2)\}} \left| V_{\rho}(\theta, x_{\rho}(\theta)) - V_{\rho}(\tau_{\rho}^{(n)}, x_{\rho}^{(n)}(\tau_{\rho}^{(n)})) \right| \right]
+ \mathbb{E}_{(s,y)} \left[ I_{\{\tau_{\rho}^{(n)} < \tau_{\rho} < \theta, \tau_{\rho} \in (A_1 \cup A_2)^c\}} \left| V_{\rho}(\theta, x_{\rho}(\theta)) - V_{\rho}(\tau_{\rho}^{(n)}, x_{\rho}^{(n)}(\tau_{\rho}^{(n)})) \right| \right]
\leq (4 \max_{(t,x) \in O_{\rho}} |V_{\rho}(t, x)| + 1)\varepsilon. \tag{5.68}
\]

Combining \((5.61), (5.63), (5.67)\) and \((5.68)\), for \(n \geq N_1\),
\[
I_1 \leq (10 \max_{(t,x) \in O_{\rho}} |V_{\rho}(t, x)| + 3)\varepsilon.
\]

Similarly, we fix \(N := \max\{N_1, N_2\} \in I_2\) and then by \((5.60)\),
\[
I_2 \leq (10 \max_{(t,x) \in O_{\rho}} |V_{\rho}^{(N)}(t, x)| + 3)\varepsilon \leq (10 \max_{(t,x) \in O_{\rho}} |V_{\rho}(t, x)| + 8)|\varepsilon.
\]

Hence, whenever \(n \geq N\), for any \(\{\mathcal{F}_t\}_{t \geq s}\)-stopping time \(\tau_{\rho,\theta}^{(n)} := \theta \wedge \tau_{\rho}^{(n)}, \tau_{\rho,\theta} \in \theta \wedge \tau_{\rho}\),
\[
\mathbb{E}_{(s,y)} \left| V_{\rho}(\tau_{\rho,\theta}, x_{\rho}(\tau_{\rho,\theta})) - V_{\rho}(\tau_{\rho,\theta}^{(n)}, x_{\rho}^{(n)}(\tau_{\rho,\theta}^{(n)})) \right| \leq (20\|V_{\rho}\|_{\infty} + 12)\varepsilon,
\]
which directly implies that,

$$
\lim_{n \to \infty} \sup_{\theta} \mathbb{E}_{(s,y)} \left[ V_\rho(\tau_\rho, x_\rho(\tau_\rho, \theta)) - V_\rho^{(n)}(\tau_\rho, x_\rho^{(n)}(\tau_\rho, \theta)) \right] = 0. \quad (5.69)
$$

Moreover, by (5.64),

$$
\mathbb{E}_{(s,y)} \left| \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t)) dt - \int_s^{\tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt \right| \leq \| L_\rho \| T \mathbb{P}_{(s,y)} \left\{ |\tau_\rho - \tau_\rho^{(n)}| > \frac{\delta_0^3}{\delta} \right\} + \| L_\rho \| \varepsilon \leq (T + 1) \| L_\rho \| \varepsilon. \quad (5.70)
$$

Therefore, combining (5.69) and (5.70), we obtain (uniformly in all \((s,y) \in O_\rho, \mu \in \Lambda[s,T], u \in U_\mu[s,T]\),

$$
\lim_{n \to \infty} \sup_{\theta} \mathbb{E}_{(s,y)} \left[ \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t)) dt + V_\rho(\tau_\rho, x_\rho(\tau_\rho, \theta)) - \int_s^{\tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt - V_\rho^{(n)}(\tau_\rho, x_\rho^{(n)}(\tau_\rho, \theta)) \right] = 0. \quad (5.71)
$$

Together with (5.33) in Step 1 of the proof of Theorem 5.1, for any \((s,y) \in \overline{O_\rho}, \mu \in \Lambda[s,T], u \in U_\mu[s,T]\), for any \((\mathcal{F}_t)_{t \geq s}\)-stopping time \(\theta\),

$$
V_\rho(s,y) \leq \mathbb{E}_{(s,y)} \left\{ \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t)) dt + V_\rho(\tau_\rho, x_\rho(\tau_\rho, \theta)) \right\}, \quad (5.72)
$$

which is the inequality (5.58) for the value function \(V_\rho\).

To get the inequality (5.59), for any \(\delta > 0\), pick \(n \geq 1\) large enough in (5.71), so that

$$
\sup_{\theta} \mathbb{E}_{(s,y)} \left[ \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t)) dt + V_\rho(\tau_\rho, x_\rho(\tau_\rho, \theta)) - \int_s^{\tau_\rho^{(n)}} L_\rho(x_\rho^{(n)}(t), u(t)) dt - V_\rho^{(n)}(\tau_\rho, x_\rho^{(n)}(\tau_\rho, \theta)) \right] \leq \frac{\delta}{2}.
$$

For this \(n\), for any \((s,y) \in \overline{O_\rho}, \mu \in \Lambda[s,T]\), it follows from (5.36) that we can pick a six-tuple \(\mu \in \Lambda[s,T]\) and \(u \in U_\mu[s,T]\), such that

$$
V_\rho^{(n)}(s,y) + \frac{\delta}{2} \geq \mathbb{E}_{(s,y)} \left\{ \int_s^{\tau_\rho} L_\rho(x_\rho^{(n)}(t), u(t)) dt + V_\rho^{(n)}(\tau_\rho, x_\rho^{(n)}(\tau_\rho, \theta)) \right\}. \quad (5.73)
$$

Note that by Remark 5.1, \(u \in U_\nu[s,T]\), where \(\nu\) is obtained by omitting the last component \(W_t\) of the six-tuple \(\mu\). Therefore we have

$$
V_\rho(s,y) + \delta \geq \mathbb{E}_{(s,y)} \left\{ \int_s^{\tau_\rho} L_\rho(x_\rho(t), u(t)) dt + V_\rho(\tau_\rho, x_\rho(\tau_\rho, \theta)) \right\}, \quad (5.74)
$$

27
for the choice of \( u \) as in (5.73). Therefore, the property (DP) holds for the value function \( V_\rho \).

**Step 3.** We finally establish the validity of the property (DP) for \( V \). Without loss of generality, for any \((s,y) \in [0,T] \times \overline{Q} \setminus \{ \xi = 0 \} \) (on the boundary \( \{ \xi = 0 \} \) the property (DP) is trivially verified), \( \mu \in \Lambda[s,T] \), \( u \in U_\mu[s,T] \), pick \( \rho_0 > 0 \) large enough so that \( y \in \overline{Q}_{\rho_0} \). Note that whenever \( \rho > \rho_0 \), if \( P_\rho(\tau_{\rho_0}) = R \), then \( \tau = \tau_\rho = \tau_{\rho_0} \) and \( x(t) = x_\rho(t) = x_{\rho_0}(t) \), for \( s \leq t \leq \tau_{\rho_0} \). In particular, \( x_\rho(\theta \land \tau_\rho) = x(\theta \land \tau) \), for any \( \{ F_t \}_{t \geq s} \) stopping time \( \theta \), where \( \tau \) and \( \tau_\rho \) are defined as in (4.6) and (5.42) respectively. Hence

\[
\begin{align*}
&\mathbb{E}_{(s,y)}[V_\rho(\theta \land \tau_\rho, x_\rho(\theta \land \tau_\rho)) - V(\theta \land \tau, x(\theta \land \tau))] \\
\leq &\mathbb{E}_{(s,y)}[I_{(P_\rho(\tau_{\rho_0})=R)}] V_\rho(\theta \land \tau_\rho, x_\rho(\theta \land \tau_\rho)) - V(\theta \land \tau, x(\theta \land \tau))] \\
&+ \mathbb{E}_{(s,y)}[I_{(P_\rho(\tau_{\rho_0})\neq R)}] (|V_\rho(\theta \land \tau_\rho, x_\rho(\theta \land \tau_\rho))| + |V(\theta \land \tau, x(\theta \land \tau))|) \\
\leq &\sup_{(s,y) \in U_{\rho_0}} \|V_\rho - V\|.
\end{align*}
\]

By Step 3 and in particular (5.55) in Theorem 5.1, \( V_\rho(s,y) \) converges to \( V(s,y) \) uniformly for \((s,y) \in \overline{Q}_{\rho_0} \), as \( \rho \to \infty \). Hence, we have

\[
\sup_{(s,y) \in \overline{Q}_{\rho_0}} \|V_\rho - V\| \to 0, \quad \rho \to \infty. \tag{5.75}
\]

Also, by (5.54), for any \((s,y) \in [0,T] \times \overline{Q} \) with \( y = (P_s, \xi_s, \Theta_s) \), for any \( \mu \in \Lambda[s,T], \ u \in U_\mu[s,T] \),

\[
\begin{align*}
|V(s,y)|^2 &\leq kN^\gamma T^2 \xi_s^2 \exp \left\{ (H + AC^\alpha N^\beta + \ell^\epsilon)^2 g^{-2} T \right\} + \frac{1}{2} B_4 (1 + |P_s|^4) \\
&+ \frac{1}{2} T \xi_s^4 \exp \left\{ 6T(H + AC^\alpha N^\beta + \ell^\epsilon)^2 g^{-2} \right\}.
\end{align*}
\]

Thus setting \( L(T,C) = \exp((H + AC^\alpha N^\beta + \ell^\epsilon)^2 g^{-2} T) \), it follows that

\[
\begin{align*}
&\mathbb{E}_{(s,y)}[V(\theta \land \tau, x(\theta \land \tau))]^2 \\
\leq &kN^\gamma T L(T,C) \mathbb{E}_{(s,y)}[\xi_s^2(\theta \land \tau)] + \frac{1}{2} B_4 (1 + \mathbb{E}_{(s,y)}(P^4(\theta \land \tau))) \\
&+ \frac{1}{2} T L^6(T,C) \mathbb{E}_{(s,y)}[\xi_s(\theta \land \tau)] \\
\leq &kN^\gamma T L(T,C) \int_s^T \mathbb{E}_{(s,y)}[\xi_s^2(t)] dt + \frac{1}{2} B_4 \left( 1 + \int_s^T \mathbb{E}_{(s,y)}(P^4(t)) dt \right) \\
&+ \frac{1}{2} T L^6(T,C) \int_s^T \mathbb{E}_{(s,y)}[\xi_s(t)] dt. \tag{5.76}
\end{align*}
\]
Using arguments as in (4.15), the following estimation for the expectation of $\xi^2$ holds true:

$$
E_{(s,y)}(\xi^2(t)) \leq \exp \{ (H + AC^\alpha N^\beta + \ell^\gamma)g^{-2}T \} := L(T, C). \quad (5.77)
$$

Together with (5.52) and (5.53), we obtain from (5.76) that

$$
E_{(s,y)}|V(\theta \wedge \tau, x(\theta \wedge \tau))|^2 \leq kN^\gamma T^2 L^2(T, C) + \frac{1}{2}B_d(1 + B_4 T(1 + P_s^4)) + \frac{1}{2}T^2 L^2(T, C)\xi^4. \quad (5.78)
$$

Using exactly the same type of arguments, the same bound can be obtained for $E_{(s,y)}|V_\rho(\theta \wedge \tau_\rho, x(\theta \wedge \tau_\rho))|^2$.

Note that by (5.51), for any $\varepsilon > 0$, we can pick $\rho_0 > 0$ large enough, such that

$$
P_{(s,y)}\{P_{\rho_0}(\tau_{\rho_0}) \neq R\} = P_{(s,y)}\{\tau_{\rho_0} < \tau\} < \varepsilon \quad (5.79)
$$

Combining (5.75) - (5.79) and using the Cauchy-Schwarz Inequality, we first pick $\rho_0$ satisfying (5.79) then letting $\rho \to \infty$ in (5.75), we conclude that

$$
limit_{\rho \to \infty, \theta} E_{(s,y)}|V_\rho(\theta \wedge \tau_\rho, x(\theta \wedge \tau_\rho)) - V(\theta \wedge \tau, x(\theta \wedge \tau))| = 0. \quad (5.80)
$$

Moreover, for fixed $(s, y) \in [0, T] \times Q/\{\xi = 0\}$, $\mu \in \Lambda[s, T]$, $u \in U_\mu[s, T]$, by (5.51) again, we pick $\rho_1$ large enough so that (5.51) holds and $y \in \overline{Q}_{\rho_1}$, then

$$
E_{(s,y)}\left| \int_s^{\theta \wedge \tau} L(x(t), u(t))dt - \int_s^{\theta \wedge \tau_\rho} L_{\rho_1}(x_{\rho_1}(t), u(t))dt \right|
\leq E_{(s,y)}\left[ I(P_{\rho_1}(\tau_{\rho_1}) \neq R) \int_{\tau_{\rho_1}}^{\tau} L(x(t), u(t))dt \right]
\leq \left( \frac{1}{2} \right)^{1/2} \left( \int_s^T E_{(s,y)} L^2(x(t), u(t))dt \right)^{1/2}
\leq kN^\gamma T^{1/2} L^{1/2}(T, C) \varepsilon^{1/2},
$$

which is independent of the choice of the $\{F_t\}_{t \geq s}$-stopping time $\theta$. Therefore,

$$
limit_{\rho \to \infty, \theta} E_{(s,y)}\left| \int_s^{\theta \wedge \tau} L(x(t), u(t))dt - \int_s^{\theta \wedge \tau_\rho} L_{\rho}(x_{\rho}(t), u(t))dt \right| = 0. \quad (5.81)
$$

Combining (5.80) and (5.81), we finally obtain that

$$
limit_{\rho \to \infty, \theta} \left| E_{(s,y)}\left[ \int_s^{\theta \wedge \tau} L(x(t), u(t))dt + V(\theta \wedge \tau, x(\theta \wedge \tau)) \right]
- E_{(s,y)}\left[ \int_s^{\theta \wedge \tau_\rho} L_{\rho}(x_{\rho}(t), u(t))dt + V_{\rho}(\theta \wedge \tau_\rho, x(\theta \wedge \tau_\rho)) \right] \right| = 0.
$$
As in the last part of Step 2 of the proof of Theorem 5.1, we conclude that the property (DP) holds for the value function \( V(s,y) \). The proof is complete.

\[ \square \]

5.3 Existence of Viscosity Solutions

Now that we have established that the value function is jointly continuous and satisfies the Dynamic Programming Principle, it is time to show that it is indeed a viscosity solution of the HJB equation (5.2) and (5.3). Recall the notations used in Section 4:

\[
\begin{align*}
\tilde{f}(t,x,u) &= (b(t,P,u),0,\theta(t)\zeta_H(\Theta(t))), \\
\tilde{\sigma}(t,x,u) &= (\sigma(t,P,u),-\eta^{-1}\xi(\Theta + Ac^\alpha \eta^\beta - \ell(t)),\sigma(t)\zeta_H(\Theta(t))), \\
a(t,x,u) &= \tilde{\sigma}(t,x,u)\tilde{\sigma}^T(t,x,u), \\
L(x,u) &= k\eta^7\xi.
\end{align*}
\]

**Theorem 5.3** Under the conditions of Theorem 5.1 and 5.2, the value function defined in (4.10) is a viscosity solution of the HJB equation (5.2) with the boundary condition (5.3).

**Proof:** The boundary condition is clearly satisfied. For any \( \varphi \in C_{1,2}(O) \), let \( V - \varphi \) attain a local maximum at \((s,y)\) \( \in O \). Fix any \( u \in \mathcal{U} \), consider the constant control \( u(t) \equiv u \), \( s \leq t \leq T \), and let

\[
G(t,x,u,z,B) := -\tilde{f}(t,x,u) \cdot z - \frac{1}{2} \text{tr}(a(t,x,u)B) - L(x,u).
\]

By the Dynamic Programming Principle, for any reference probability space \( \mu \in \Lambda[s,T] \), any \( s' > s \) with \( s' - s > 0 \), small enough, \( \tau = \tau(s,y) \),

\[
0 \leq \frac{\mathbb{E}(s,y)[V(s,y) - \varphi(s,y) - V(s' \land \tau, x(s' \land \tau)) + \varphi(s' \land \tau, x(s' \land \tau))]}{s' - s} \\
\leq \frac{1}{s' - s} \mathbb{E}(s,y) \left\{ \int_s^{s' \land \tau} L(t,x(t),u)dt - \varphi(s,y) + \varphi(s' \land \tau, x(s' \land \tau)) \right\},
\]

Note that from Lemma 4.1, the value function satisfies a polynomial growth property. Without loss of generality, we may assume that the test function \( \varphi \) also satisfies certain polynomial growth property, namely,

\[
|\varphi(s,y)| \leq C_0(1 + \|y\|^m), \quad \text{for some } m > 1, \quad (5.82)
\]

where \( C_0 > 0 \) is some constant which is independent of both the time and the state variable. Using (5.32) and the facts that \( \Theta \) is a bounded process.
and \( \psi \) is a Girsanov density for which any polynomial moments are finite, we see that for any \( p > 1 \),
\[
E_{(s,y)}\|x(t)\|^p \leq C_p(1 + \|y\|^p), \tag{5.83}
\]
where \( C_p > 0 \) is some universal constant depending only on \( p \). Then,
\[
\frac{1}{s' - s} \mathbb{E}_{(s,y)} \left\{ \int_s^{s' \wedge \tau} L(t, x(t), u) dt \right\} \leq \frac{1}{s' - s} kN^{+} \mathbb{E}_{(s,y)} \left\{ \int_s^{s' \wedge \tau} \xi(t) dt \right\} < \infty,
\]
and thus the Dominated Convergence Theorem ensures that
\[
\frac{1}{s' - s} \mathbb{E}_{(s,y)} \left\{ \int_s^{s' \wedge \tau} L(t, x(t), u) dt \right\} \rightarrow L(s, y, u), \text{ as } s' \rightarrow s.
\]
Also, by Itô’s formula and the polynomial growth assumption of \( \phi \), we have
\[
\frac{1}{s' - s} \mathbb{E}_{(s,y)} [\phi(s' \wedge \tau, x(s' \wedge \tau)) - \phi(s, y)] = \frac{1}{s' - s} \mathbb{E}_{(s,y)} \left\{ \int_s^{s' \wedge \tau} \varphi_1(t, x(t)) dt + \int_s^{s' \wedge \tau} D_x \varphi(t, x(t)) \cdot \tilde{f}(t, x(t), u) dt + \frac{1}{2} \sum_{i,j=1}^3 \int_s^{s' \wedge \tau} D_{x_ix_j} \varphi(t, x(t)) a_{ij}(t, x(t), u) dt \right\}
\]
From (5.82) and (5.83), we first pick \( p > m \), then for any \( \varepsilon > 0 \), there exists \( R > 0 \) large enough, such that
\[
\mathbb{P}_{(s,y)} \{ \max_{t \in [s, s']} \|x(t)\| > R \} \leq R^{-p} C_p(1 + \|y\|^p)(s' - s). \tag{5.84}
\]
Thus,
\[
\frac{1}{s' - s} \int_s^{s' \wedge \tau} |\varphi_1(t, x(t))| dt = \frac{1}{s' - s} \int_{(\max_{t \in [s, s']} \|x(t)\| > R)}^{s' \wedge \tau} |\varphi_1| dt + \frac{1}{s' - s} \int_{(\max_{t \in [s, s']} \|x(t)\| \leq R)}^{s' \wedge \tau} |\varphi_1| dt \leq \frac{1}{s' - s} \int_{(\max_{t \in [s, s']} \|x(t)\| > R)}^{s' \wedge \tau} C_0 (1 + \|x(t)\|^m) dt
\]
\[
+ \frac{1}{s' - s} \int_{(\max_{t \in [s, s']} \|x(t)\| \leq R)}^{s' \wedge \tau} \max_{(t,x) \in [s, s'] \times \{\|x\| \leq R\}} \|\phi\| (s' - s)
\]
which has finite moment by (5.83) and (5.84), and the Dominated convergence Theorem ensures that
\[
\frac{1}{s' - s} \mathbb{E}_{(s,y)} \int_s^{s' \wedge \tau} \varphi_1(t, x(t)) dt \rightarrow \varphi_1(s, y), \text{ as } s' \rightarrow s.
\]
Similarly, we can show that
\[ s' - s \quad D_x \varphi(t, x(t)) \cdot \vec{f}(t, x(t), u)dt \quad \longrightarrow D_x \varphi(s, y) \cdot \vec{f}(s, y, u), \]
\[ \frac{1}{s' - s} \mathbb{E}_{(s, y)} \left\{ \int_s^{s' \wedge \tau} L(t, x(t), u)dt \right\} \quad \longrightarrow L(s, y) + A^\mu \varphi(s, y), \]
\[ \frac{1}{s' - s} \mathbb{E}_{(s, y)} \left\{ \frac{1}{2} \sum_{i,j=1}^3 \int_s^{s' \wedge \tau} D_{x_{ij}} \varphi(t, x(t)) a_{ij}(t, x(t), u)dt \right\} \quad \longrightarrow \frac{1}{2} \sum_{i,j=1}^3 D_{x_{ij}} \varphi(s, y) a_{ij}(s, y, u). \]

This lead to
\[ 0 \leq \frac{1}{s' - s} \mathbb{E}_{(s, y)} \left\{ \int_s^{s' \wedge \tau} L(t, x(t), u)dt - \varphi(s, y) + \varphi(s' \wedge \tau, x(s' \wedge \tau)) \right\} \quad \longrightarrow L(s, y) + A^\mu \varphi(s, y), \]
and therefore
\[ - \varphi_t(s, y) + \sup_{u \in U} G(s, y, u, -\varphi_x(s, y), -\varphi_{xx}(s, y)) \leq 0. \quad (5.85) \]

On the other hand, assume that \( V - \varphi \) attains a local minimum at \((s, y) \in O\).

By (5.59), for any \( \varepsilon > 0 \) and \( s' > s \) with \( s' - s > 0 \) small enough, there exists \( \mu \in \Lambda[s, T] \) and \( u \in U^\mu[s, T] \) such that
\[ 0 \geq \mathbb{E}_{(s, y)} \{ V(s', x(s')) - V(s, x(s')) + \varphi(s', x(s')) \} \quad \geq -\varepsilon (s' - s) + \mathbb{E}_{(s, y)} \left\{ \int_s^{s' \wedge \tau} L(t, x(t), u(t))dt + \varphi(s' \wedge \tau, x(s' \wedge \tau)) - \varphi(s, y) \right\}. \]

Hence, by arguments as above, we have
\[ -\varepsilon \leq \frac{1}{s' - s} \mathbb{E}_{(s, y)} \int_s^{s' \wedge \tau} \left\{ -\varphi_t(t, x(t)) + G(t, x(t), u, -\varphi_x(t, x(t)), -\varphi_{xx}(t, x(t))) \right\} dt \]
\[ \leq \frac{1}{s' - s} \mathbb{E}_{(s, y)} \int_s^{s' \wedge \tau} \left\{ -\varphi_t(t, x(t)) + \sup_{u \in U} G(t, x(t), u, -\varphi_x(t, x(t)), -\varphi_{xx}(t, x(t))) \right\} dt \quad \longrightarrow -\varphi(s, y) + \sup_{u \in U} G(s, y, u, -\varphi_x(s, y), -\varphi_{xx}(s, y)). \quad (5.86) \]

Combining (5.85) and (5.86), we conclude that \( V(s, y) \) is a viscosity solution of the HJB equation (5.2) with boundary condition (5.3). \( \square \)

### 5.4 Uniqueness of the Viscosity Solution

In this section, we establish the uniqueness of the viscosity solution of (5.2) and (5.3). By Lemma 4.1, the value function satisfies a polynomial
growth condition in the state variables. Hence, we will prove a uniqueness theorem for those viscosity solutions with polynomial growth. We start with introducing the second order sub and superdifferentials with a complete second order expansion in all variables, as given in the User’s guide of Crandall, Ishii and Lions [7] (see Remark 5.1(ii)). Let $\mathcal{O}$ be a locally compact subset in $\mathbb{R}^4$. Let

$$USC(\mathcal{O}) = \{\text{upper semicontinuous functions } v: \mathcal{O} \subset \mathbb{R}^4 \rightarrow \mathbb{R}\}$$

$$LSC(\mathcal{O}) = \{\text{lower semicontinuous functions } v: \mathcal{O} \subset \mathbb{R}^4 \rightarrow \mathbb{R}\}$$

For $x \in \mathcal{O}$, and for $v \in USC(\mathcal{O})$, we set

$$J^{2,+}v(x) = \{ (p, X) \in \mathbb{R}^4 \times \mathcal{S}^4 : v(x+h) - v(x) \leq p \cdot h + \frac{1}{2}Xh \cdot h + o(\|h\|^2) \}$$

$$= \{ (D\phi(x), D^2\phi(x)) : \phi \in C^2, v - \phi \text{ has a local maximum at } x \},$$

$$J^{2,-}v(x) = \{ (p, X) \in \mathbb{R}^4 \times \mathcal{S}^4 : \exists x_n \in \mathcal{O} \rightarrow x, (p_n, X_n) \in J^{2,+} v(x_n) \rightarrow (p, X) \},$$

and similarly for $v \in LSC(\mathcal{O})$,

$$J^{2,-}v(x) = \{ (p, X) \in \mathbb{R}^4 \times \mathcal{S}^4 : v(x+h) - v(x) \geq p \cdot h + \frac{1}{2}Xh \cdot h + o(\|h\|^2) \}$$

$$= \{ (D\phi(x), D^2\phi(x)) : \phi \in C^2, v - \phi \text{ has a local minimum at } x \},$$

$$J^{2,-}v(x) = \{ (p, X) \in \mathbb{R}^4 \times \mathcal{S}^4 : \exists x_n \in \mathcal{O} \rightarrow x, (p_n, X_n) \in J^{2,-} v(x_n) \rightarrow (p, X) \},$$

Using the above notations, we state the following Maximum Principle for semicontinuous functions. For a proof, see Theorem 3.2 in Crandall, Ishii and Lions [7].

**Theorem 5.4** Let $\mathcal{O}_i$ be a locally compact subset of $\mathbb{R}^{N_i}$ for $i = 1, \ldots, k$,

$$\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_k,$$

$v_i \in USC(\mathcal{O}_i)$, and $\varphi \in C^2(\mathcal{O})$. For $x = (x_1, \ldots, x_k) \in \mathcal{O}$, set also

$$v(x) = v_1(x_1) + \cdots + v_k(x_k).$$

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k) \in \mathcal{O}$ be a local maximum of $v - \varphi$ in $\mathcal{O}$, then for any $\varepsilon > 0$, there exists $X_i \in \mathcal{S}^{N_i}$ such that

$$(D_{x_i} \varphi(\bar{x}), X_i) \in J^{2,+} v_i(\bar{x}_i), \text{ for } i = 1, \ldots, k,$$
and the block diagonal matrix with entries $X_i$ satisfies

$$
\begin{pmatrix}
X_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X_k
\end{pmatrix} \leq A + \varepsilon A^2,
$$

where $A = D^2 \varphi(\bar{x}) \in S^N_+$, $N = \sum_{j=1}^k N_j$.

The next theorem is the main result of this subsection. It proves the uniqueness of viscosity solutions satisfying a polynomial growth condition, of the boundary problem of (5.2) and (5.3).

**Theorem 5.5** Assume that the conditions (C1) and (C2) are valid. Let $W$ and $V$ be respectively any subsolution and supersolution of (5.2) with boundary and terminal condition (5.3), such that for any $(t, x), (s, y) \in [0, T] \times \overline{Q}$

$$
|W(t, x)| \leq L_0(\|x\|^m + 1), \quad |V(s, y)| \leq L_0(\|y\|^m + 1),
$$

(5.87)

for some $L_0 > 0$ and $m > 3$. Moreover, for $(t, x), (s, y) \in \partial^* O$, let

$$
W(t, x) = -P_1 \xi_1, \quad V(s, y) = -P_2 \xi_2, \quad x = (P_1, \xi_1, \Theta_1), \quad y = (P_2, \xi_2, \Theta_2). \tag{5.88}
$$

Then $W(t, x) \leq V(t, x)$ for any $(t, x) \in [0, T] \times \overline{Q}$. In particular, (5.2) and (5.3) have a unique viscosity solution under a polynomial growth condition.

**Remark 5.2** We assume in the above theorem that both the subsolution and the supersolution satisfy the boundary and terminal condition with "". This avoids appealing to some extra conditions such as uniform continuity on boundary and terminal values of the solutions.

**Proof:** To prove Theorem 5.4, we argue by contradiction. Suppose there exists $(t_0, x_0) \in [0, T] \times \overline{Q}$ and $\gamma_0 > 0$ so that

$$
W(t_0, x_0) - V(t_0, x_0) > \gamma_0 > 0.
$$

Choose $r > 0$ and $\delta > 0$ small enough so that

$$
W(t_0, x_0) - V(t_0, x_0) - \frac{2r}{t_0} - 2\delta e^{-T}(\|x_0\|^{m+1} + 1) > \frac{\gamma_0}{2}. \tag{5.89}
$$

(Here without loss of generality, we assume $t_0 > 0$. Otherwise we can replace $r/t_0$ by $r/(T - t_0)$ and the argument is similar.)

**Step 1** For $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $b > 1$, set

$$
\Phi(t, x, y) = W(t, x) - V(s, y) - \delta e^{-bt}(\|x\|^{m+1} + 1) - \delta e^{-bs}(\|y\|^{m+1} + 1) - \frac{\|x - y\|^2}{2\varepsilon_1} - \frac{|t - s|^2}{2\varepsilon_2} - \frac{r - r}{t - s}.
$$
We first show that \( \Phi \) achieves its maximum in the interior of the region \(([0, T] \times Q)^2\). Define

\[
m_0 = \lim_{\eta \to 0} \lim_{\epsilon \to 0} \sup \left\{ W(t, x) - V(s, y) - \frac{r}{t} - \frac{r}{s} - \delta e^{-bs}(\|x\|^{m+1} + 1) - \delta e^{-bs}(\|y\|^{m+1} + 1) : |t - s| < \epsilon, \|x - y\| < \eta \right\},
\]

\[
m_1(\epsilon_1, \epsilon_2) = \sup \{ \Phi(t, x, s, y) : (t, x, s, y) \in [0, T] \times Q \},
\]

\[
m_2(\epsilon_2) = \sup_{\eta \to 0} \left\{ W(t, x) - V(s, y) - \frac{|t - s|^2}{2 \epsilon_2} - \delta e^{-bs}(\|x\|^{m+1} + 1), - \delta e^{-bs}(\|y\|^{m+1} + 1) - \frac{r}{t} - \frac{r}{s} : \|x - y\| < \eta \right\}.
\]

It is easy to see that

\[
\lim_{\epsilon_1 \to 0} m_1(\epsilon_1, \epsilon_2) = m_2(\epsilon_2), \quad \lim_{\epsilon_2 \to 0} m_2(\epsilon_2) = m_0. \tag{5.90}
\]

Note that for \( x, y \in \overline{Q} \) with \( \|x\|, \|y\| \) large enough, \( \Phi(t, x, s, y) \) becomes negative. On the other hand, (5.89) guarantees that \( m_1 > \gamma_0/2 \). Therefore \( \Phi \) achieves its maximum value in some bounded region. Suppose the maximum of \( \Phi \) is attained at \((\bar{t}, \bar{x}, \bar{s}, \bar{y})\), and so

\[
m_1(\epsilon_1, \epsilon_2) = \Phi(\bar{t}, \bar{x}, \bar{s}, \bar{y}) - \delta e^{-bs}(\|\bar{x}\|^{m+1} + 1) - \delta e^{-bs}(\|\bar{y}\|^{m+1} + 1)
\]

\[
- \frac{\|\bar{x} - \bar{y}\|^2}{2 \epsilon_1} - \frac{|\bar{t} - \bar{s}|^2}{2 \epsilon_2} - \frac{r}{\bar{t}} - \frac{r}{\bar{s}} \leq m_1(2\epsilon_1, 2\epsilon_2) - \frac{\|\bar{x} - \bar{y}\|^2}{4 \epsilon_1} - \frac{|\bar{t} - \bar{s}|^2}{4 \epsilon_2}.
\]

Hence, by (5.90), we have

\[
\lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \left( \frac{\|\bar{x} - \bar{y}\|^2}{4 \epsilon_1} + \frac{|\bar{t} - \bar{s}|^2}{4 \epsilon_2} \right) = 0. \tag{5.91}
\]

We now claim that \( 0 < \bar{t}, \bar{s} < T \) and \( \bar{x}, \bar{y} \in Q \). From the expression of \( \Phi \), it is easy to see that \( \bar{t} > 0, \bar{s} > 0 \). Assume that \( \bar{t} = T \). By (5.87), for fixed \( r > 0, \delta > 0 \) and \( b > 0 \) satisfying (5.89), we can pick \( R > 0 \) such that for any \( 0 \leq t, s \leq T, x, y \in \overline{Q} \) with \( \|x\| > R, \|y\| > R, \|

\[
|W(t, x)| < \delta e^{-bT} \|x\|^{m+1}, |V(s, y)| < \delta e^{-bT} \|y\|^{m+1}.
\]

Hence \( \|\bar{x}\| \leq R \) or \( \|\bar{y}\| \leq R \), since otherwise \( \Phi(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \) would achieve a negative value for \( \|\bar{x}\| > R \) and \( \|\bar{y}\| > R \), contradicting (5.89). Hence without loss of generality, we set \( \|\bar{x}\| \leq R \).

Now since \( W \) and \( V \) are uniformly continuous on \([0, T] \times (\overline{Q} \cap (z \in \mathbb{R}^3 : \|z\| \leq R + 1))\), for any \( 0 < \lambda < r/(2(R + 1)T) \), there exists \( \Delta_0 > 0 \), such

35
that whenever \((t,x),(s,y)\) \(\in\) \([0,T] \times (\overline{Q} \cap (z \in \mathbb{R}^3 : ||x|| \leq R + 1))\), with 
\(|t-s| < \Delta_0||x-y|| < \Delta_0,\)
\(|W(t,x) - W(s,y)| < \lambda, |V(t,x) - V(s,y)| < \lambda\).

If we pick \(\varepsilon_1, \varepsilon_2\) small enough so that \(|t-\bar{s}| < \Delta_0\) and \(||\bar{x} - \bar{y}|| < \min\{\Delta_0, r/2(R+1)\}\), then \(\bar{x}, \bar{y} \in (\overline{Q} \cap (z \in \mathbb{R}^3 : ||x|| < R+1))\), and so \(\bar{x} = (\overline{P}_1, \xi_1, \Theta_1), \bar{y} = (\overline{P}_2, \xi_2, \Theta_2)\),
\(|W(\bar{t}, \bar{x}) - V(\bar{s}, \bar{y})| \leq |W(T, \bar{x}) - V(T, \bar{y})| + |V(T, \bar{y}) - V(\bar{s}, \bar{y})|
\leq \left| \frac{P_1 \xi_1 - P_2 \xi_2}{r/s} + \lambda \right|
\leq \left( ||\bar{x}|| + ||\bar{y}|| \right)||\bar{x} - \bar{y}|| + \lambda
\leq \frac{2r}{T},\)

Therefore,
\[W(\bar{t}, \bar{x}) - V(\bar{s}, \bar{y}) - \frac{r}{t} - \frac{r}{s} \leq 0,\]
which again contradicts (5.89). Therefore \(\bar{t} < T\). Similarly we can show that \(\bar{s} < T\) and \(\bar{x}, \bar{y} \in Q\).

**Step 2** We now apply Theorem 5.4 to obtain some contradiction. Set \(\mathcal{O}_1 = \mathcal{O}_2 = (0,T) \times (Q \cap \{||z|| < R + 1\})\), \(\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2, z_1 = (t,x), z_2 = (s,y)\), and define
\[
\tilde{W}(z_1) = \tilde{W}(t,x) = W(t,x) - de^{-bt}(||x||^m + 1) - \frac{r}{t},
\tilde{V}(z_2) = \tilde{V}(s,y) = V(s,y) + de^{-bs}(||y||^m + 1) + \frac{r}{s},
\varphi(z_1, z_2) = \varphi(t,x,s,y) = \frac{||x-y||^2}{2\varepsilon_1} + \frac{||t-s||^2}{2\varepsilon_2}.
\]

The arguments in Step 1 above show that \(\tilde{W} - \tilde{V} - \varphi\) achieves a local maximum at \((\tilde{z}_1, \tilde{z}_2) = (\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in \mathcal{O}\). By Theorem 5.4, taking \(k = 2\), \(v_1 = \tilde{W}, v_2 = -\tilde{V}\), and since \(\mathcal{J}^2,\tilde{V} = -\mathcal{J}^{2,+}(-\tilde{V})\),
\[
D_{z_1} \varphi(\tilde{z}_1, \tilde{z}_2) = D_{z_2} \varphi(\tilde{z}_1, \tilde{z}_2) = \left(\begin{array}{c} \frac{\bar{x} - \bar{y}}{\varepsilon_1} - \frac{\bar{t} - \bar{s}}{\varepsilon_2} \end{array}\right)^T,
A = D_{\varepsilon}^2 \varphi(\tilde{z}_1, \tilde{z}_2) = \left(\begin{array}{ccc} \varepsilon_2^{-1} & 0 & -\varepsilon_2^{-1} I_3 \\ 0 & \varepsilon_1^{-1} I_3 & 0 & -\varepsilon_1^{-1} I_3 \\ -\varepsilon_1^{-1} I_3 & 0 & \varepsilon_2^{-1} & 0 \\ 0 & -\varepsilon_1^{-1} I_3 & 0 & \varepsilon_1^{-1} I_3 \end{array}\right),
\]
and we conclude that for \(\varepsilon = \varepsilon_1\), there exist \(X, Y \in S^4\) such that
\[
\left(\begin{array}{c} \frac{\bar{x} - \bar{y}}{\varepsilon_1} - \frac{\bar{t} - \bar{s}}{\varepsilon_2} \end{array}\right)^T \in \mathcal{J}^{2,+}(-\tilde{W}(\tilde{z}_1)), \left(\begin{array}{c} \frac{\bar{x} - \bar{y}}{\varepsilon_1} - \frac{\bar{t} - \bar{s}}{\varepsilon_2} \end{array}\right)^T \in \mathcal{J}^{2,-}(\tilde{V}(\tilde{z}_2)),
\]

36
and

\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix} \leq A + \varepsilon \epsilon A^2.
\]

Taking submatrices by omitting the elements of the first and fourth rows and columns of the matrices on both sides of the above inequality leads to

\[
\begin{pmatrix}
\tilde{X} & 0 \\
0 & -\tilde{Y}
\end{pmatrix} \leq \frac{1}{\varepsilon} \begin{pmatrix}
I_3 & -I_3 \\
-I_3 & I_3
\end{pmatrix} + 2\varepsilon \begin{pmatrix}
\varepsilon I_3 & -2I_3 \\
-2I_3 & \varepsilon I_3
\end{pmatrix}
= \frac{3}{\varepsilon} \begin{pmatrix}
I_3 & -I_3 \\
-I_3 & I_3
\end{pmatrix},
\]

where we set

\[
X = \begin{pmatrix}
X_{11} & X_1^T \\
X_1 & X
\end{pmatrix},
Y = \begin{pmatrix}
Y_{11} & Y_1^T \\
Y_1 & Y
\end{pmatrix}
\]

with \(\tilde{X}, \tilde{Y} \in S^3\).

Moreover, we claim that

\[
\left( \frac{\bar{x} - \bar{s}}{\varepsilon_2}, \frac{\bar{y} - \bar{\bar{s}}}{\varepsilon_1}, \tilde{X} \right) \in \mathcal{D}^{(1,2), +} \tilde{W}(\bar{t}, \bar{x}), \quad \left( \frac{\bar{x} - \bar{s}}{\varepsilon_2}, \frac{\bar{y} - \bar{\bar{s}}}{\varepsilon_1}, \tilde{Y} \right) \in \mathcal{D}^{(1,2), -} \tilde{V}(\bar{s}, \bar{y}).
\]

In fact, since \(\left( \frac{x - \bar{s}}{\varepsilon_1}, \frac{\bar{x} - \bar{s}}{\varepsilon_2} \right)^T, X \right) \in \mathcal{J}^{2, +} \tilde{W}(\bar{x}_1), \) there exists \((t_n, x_n) \in \mathcal{O}, ((q_n, p_n), X_n) \in \mathcal{D}^{2, +} \tilde{W}(t_n, x_n), \) such that as \(n \to \infty, \)

\[
(t_n, x_n) \to \left( \frac{\bar{x} - \bar{s}}{\varepsilon_2}, \frac{\bar{y} - \bar{\bar{s}}}{\varepsilon_1}, X \right),
\]

where \(q_n \in \mathbb{R}, p_n \in \mathbb{R}^3, n \in \mathbb{N}. \) Hence for any \(h \in \mathbb{R}, y \in \mathbb{R}^3, \)

\[
\tilde{W}(t_n + h, x_n + y) - \tilde{W}(t_n, x_n) \leq q_n h + p_n y + \frac{1}{2} X_n (h, y)^T \cdot (h, y)^T + o(|h|^2 + |y|^2).
\]

By setting \(X_n = \begin{pmatrix}
X_{11}^{(n)} & X_{1}^{(n), T} \\
X_1^{(n)} & X_n
\end{pmatrix}, \) we have

\[
\tilde{W}(t_n + h, x_n + y) - \tilde{W}(t_n, x_n) \leq q_n h + p_n y + \frac{1}{2} \tilde{X}_n y \cdot y + o(|h|^2 + |y|^2).
\]

which implies that \((q(n), p(n), \tilde{X}_n) \in \mathcal{D}^{(1,2), +} \tilde{W}(t_n, x_n). \) Therefore, taking \(n \to \infty, \) we obtain

\[
\left( \frac{\bar{t} - \bar{s}}{\varepsilon_2}, \frac{\bar{x} - \bar{s}}{\varepsilon_1}, \tilde{X} \right) \in \mathcal{D}^{(1,2), +} \tilde{W}(\bar{t}, \bar{x}).
\]
Similarly, we can also prove that \( \left( \frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}, \tilde{\nu} \right) \in \mathcal{D}^{(1,2),-}\tilde{V}(s, \tilde{y}) \).

Now by Remark 5.1(i) and the very definitions of \( \tilde{W} \) and \( \tilde{V} \),
\[
\begin{align*}
\left( \frac{t-s}{\varepsilon} + \frac{\partial}{\partial t} \varphi_1(\tilde{t}, \tilde{x}), \frac{x-y}{\varepsilon} + D_x \varphi_1(\tilde{t}, \tilde{x}), \tilde{X} + D_x^2 \varphi_1(\tilde{t}, \tilde{x}) \right) &\in \mathcal{D}^{(1,2),+}W(\tilde{t}, \tilde{x}), \\
\left( \frac{t-s}{\varepsilon} - \frac{\partial}{\partial t} \varphi_2(\bar{s}, \bar{y}), \frac{x-y}{\varepsilon} - D_y \varphi_2(\bar{s}, \bar{y}), \bar{Y} - D_y^2 \varphi_2(\bar{s}, \bar{y}) \right) &\in \mathcal{D}^{(1,2),-}V(\bar{s}, \bar{y}),
\end{align*}
\]
where \( \varphi_1(t, x) = \delta e^{-bt}(\|x\|^{m+1} + 1) + r/t \) and \( \varphi_2(s, y) = \delta e^{-bs}(\|y\|^{m+1} + 1) + r/s \). Hence, (5.7) and (5.8) yield
\[
\begin{align*}
\frac{t-s}{\varepsilon} + \frac{\partial}{\partial t} \varphi_1(\tilde{t}, \tilde{x}) + \mathcal{H} \left( \tilde{t}, \tilde{x}, \frac{x-y}{\varepsilon}, D_x \varphi_1(\tilde{t}, \tilde{x}), \tilde{X} + D_x^2 \varphi_1(\tilde{t}, \tilde{x}) \right) &\leq 0 \text{ (5.93)}, \\
\frac{t-s}{\varepsilon} - \frac{\partial}{\partial t} \varphi_2(\bar{s}, \bar{y}) + \mathcal{H} \left( \bar{s}, \bar{y}, \frac{x-y}{\varepsilon}, -D_y \varphi_2(\bar{s}, \bar{y}), \bar{Y} - D_y^2 \varphi_2(\bar{s}, \bar{y}) \right) &\geq 0 \text{ (5.94)}.
\end{align*}
\]
Subtracting (5.94) from (5.93) gives
\[
\begin{align*}
\mathcal{H}(\tilde{t}, \tilde{x}, p_1(\tilde{t}, \tilde{x}), B_1(\tilde{t}, \tilde{x})) - \mathcal{H}(\bar{s}, \bar{y}, p_2(\bar{s}, \bar{y}), B_2(\bar{s}, \bar{y})) &+ b \delta e^{-bs}(\|y\|^{m+1} + 1) \\
&\leq \frac{r}{t^2} - \frac{r}{s^2} \leq -\frac{2r}{T^2} \tag{5.95}
\end{align*}
\]
where
\[
\begin{align*}
p_1(\tilde{t}, \tilde{x}) &= \frac{x-y}{\varepsilon} + D_x \varphi_1(\tilde{t}, \tilde{x}), \quad p_2(\bar{s}, \bar{y}) = \frac{x-y}{\varepsilon} - D_y \varphi_2(\bar{s}, \bar{y}), \\
B_1(\tilde{t}, \tilde{x}) &= \tilde{X} + D_x^2 \varphi_1(\tilde{t}, \tilde{x}), \quad B_2(\bar{s}, \bar{y}) = \bar{Y} - D_y^2 \varphi_2(\bar{s}, \bar{y}).
\end{align*}
\]
Now for fixed \( u \in U \), we obtain
\[
\begin{align*}
G(\tilde{t}, \tilde{x}, u, p_1(\tilde{t}, \tilde{x}), B_1(\tilde{t}, \tilde{x})) - G(\bar{s}, \bar{y}, u, p_2(\bar{s}, \bar{y}), B_2(\bar{s}, \bar{y}))
&= \left[ -\tilde{f}(\tilde{t}, \tilde{x}, u) \cdot p_1(\tilde{t}, \tilde{x}) - \frac{1}{2} tr(a(\tilde{t}, \tilde{x}, u)B_1(\tilde{t}, \tilde{x})) - L(\tilde{x}, u) \right] \\
&\quad - \left[ -\bar{f}(\bar{s}, \bar{y}, u) \cdot p_2(\bar{s}, \bar{y}) - \frac{1}{2} tr(a(\bar{s}, \bar{y}, u)B_2(\bar{s}, \bar{y})) - L(\bar{y}, u) \right] \\
&\geq -\frac{1}{2} \left[ tr(a(\tilde{t}, \tilde{x}, u)B_1(\tilde{t}, \tilde{x}) - a(\bar{s}, \bar{y}, u)B_2(\bar{s}, \bar{y})) \right] \\
&= -I_1 - I_2 - I_3.
\end{align*}
\]
By the conditions (C1) and (C2),
\[
\begin{align*}
I_1 &\leq \sup_{u \in U} \left( |\tilde{f}(\tilde{t}, \tilde{x}, u) - \bar{f}(\bar{s}, \bar{y}, u)| \cdot \frac{x-y}{\varepsilon} + \delta(m+1)\|x\|^{m-1} \sup_{u \in U} |\tilde{f}(\tilde{t}, \tilde{x}, u) \cdot \tilde{x}| \right) \\
&\quad + \delta(m+1)\|y\|^{m-1} \sup_{u \in U} |\bar{f}(\bar{s}, \bar{y}, u) \cdot \bar{y}| \\
&= O \left( \frac{|x-y|^2}{\varepsilon^2} \right) + \delta \tilde{L}(1 + \|\tilde{x}\|^{m+1} + \|\bar{y}\|^{m+1}),
\end{align*}
\]

where $I_2$ is a universal constant independent of $b$ and $\delta$, and

$$I_2 = kN^2 \|\bar{x} - \bar{y}\|.$$  

For $I_3$, using (5.92) (with the control variable $u$ is omitted for simplicity),

$$tr\left( \begin{pmatrix} \sigma(\bar{t}, \bar{x}) \sigma^T(\bar{t}, \bar{x}) & \sigma(\bar{t}, \bar{x}) \sigma^T(\bar{s}, \bar{y}) \\ \sigma(\bar{s}, \bar{y}) \sigma^T(\bar{t}, \bar{x}) & \sigma(\bar{s}, \bar{y}) \sigma^T(\bar{s}, \bar{y}) \end{pmatrix} \begin{pmatrix} \tilde{X} & 0 \\ 0 & -\tilde{Y} \end{pmatrix} \right) \leq \frac{3}{\varepsilon_1} tr\left( \begin{pmatrix} (\sigma(\bar{t}, \bar{x}) - \sigma(\bar{s}, \bar{y})) (\sigma^T(\bar{t}, \bar{x}) - \sigma^T(\bar{s}, \bar{y})) \end{pmatrix} \right),$$

which implies that

$$tr(a(\bar{t}, \bar{x}) \tilde{X} - a(\bar{s}, \bar{y}) \tilde{Y}) \leq \frac{3}{\varepsilon_1} tr\left( (\sigma(\bar{t}, \bar{x}) - \sigma(\bar{s}, \bar{y})) (\sigma^T(\bar{t}, \bar{x}) - \sigma^T(\bar{s}, \bar{y})) \right).$$

Also, for $r > 0$, $\delta > 0$ and $b > 0$ satisfying (5.89), using arguments as at the end of Step 1 above, we can pick $R > 0$ so that $\Phi(t, x, s, y)$ is negative when both $\|x\| > R$ and $\|y\| > R$. Then we can choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough so that $\|\bar{x}\| \leq R + 1$ and $\|\bar{y}\| \leq R + 1$. Together with the conditions (C1)-(C3) and the uniform continuity of $\sigma$ in $[0, T] \times (\overline{\mathcal{O}} \cap \{\|x\| \leq R + 1\})$, we have

$$I_3 = \frac{1}{2} \left[ tr\left( (\sigma(\bar{t}, \bar{x}, u) - \sigma(\bar{s}, \bar{y}, u)) (\sigma^T(\bar{t}, \bar{x}, u) - \sigma^T(\bar{s}, \bar{y}, u)) \right) \right]$$

$$+ \frac{1}{2} \left[ tr\left( a(\bar{t}, \bar{x}, u) D^2_x \varphi(\bar{t}, \bar{x}) \right) \right]$$

$$\leq \frac{1}{2} \left[ tr\left( (\sigma(\bar{t}, \bar{x}, u) - \sigma(\bar{s}, \bar{y}, u)) (\sigma^T(\bar{t}, \bar{x}, u) - \sigma^T(\bar{s}, \bar{y}, u)) \right) \right]$$

$$+ \frac{1}{2} \left[ tr\left( a(\bar{t}, \bar{x}, u) D^2_x \varphi(\bar{t}, \bar{x}) \right) \right]$$

$$\leq O(\|\bar{x} - \bar{y}\|^2) + O(\|\bar{t} - \bar{s}\|^2) + \delta \tilde{K}(1 + \|\bar{x}\|^{m+1} + \|\bar{y}\|^2),$$

where $\tilde{K} > 0$ is independent of $b$ and $\delta$. Hence

$$G(\bar{t}, \bar{x}, p_1(\bar{t}, \bar{x}), B_1(\bar{t}, \bar{x}) - G(\bar{s}, \bar{y}, p_2(\bar{s}, \bar{y}), B_2(\bar{s}, \bar{y}))$$

$$\geq -\delta(\tilde{L} + \tilde{K})(1 + \|\bar{x}\|^{m+1} + \|\bar{y}\|^m) + O(\|\bar{t} - \bar{s}\|^2) + O(\|\bar{x} - \bar{y}\|^2) + O\left( \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon_1} \right),$$

and thus

$$H(\bar{t}, \bar{x}, p_1(\bar{t}, \bar{x}), B_1(\bar{t}, \bar{x}) - H(\bar{s}, \bar{y}, p_2(\bar{s}, \bar{y}), B_2(\bar{s}, \bar{y}))$$

$$\geq -\delta(\tilde{L} + \tilde{K})(1 + \|\bar{x}\|^{m+1} + \|\bar{y}\|^m) + O(\|\bar{t} - \bar{s}\|^2) + O(\|\bar{x} - \bar{y}\|^2) + O\left( \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon_1} \right).$$
Therefore, (5.95) now becomes,

\[ -\delta (\tilde{L} + \tilde{K})(1 + \|\bar{x}\|^{m+1} + \|\bar{y}\|^{m+1}) + O(\|\bar{x} - \bar{y}\|^2) + O\left(\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon_1}\right) + O(|\bar{t} - \bar{s}|^2) \]

\[ + b\delta e^{-b\delta}(\|\bar{x}\|^{m+1} + 1) + b\delta e^{-b\delta}(\|\bar{y}\|^{m+1} + 1) \leq -\frac{2r}{T^2}. \]  

(5.96)

Note that in (5.96), if we choose \(b > 1\) large enough so that

\[ b\delta e^{-b\delta}(\|\bar{x}\|^{m+1} + 1) + b\delta e^{-b\delta}(\|\bar{y}\|^{m+1} + 1) - \delta (\tilde{L} + \tilde{K})(1 + \|\bar{x}\|^{m+1} + \|\bar{y}\|^{m+1}) > 0, \]

then (5.96) implies that

\[ O(\|\bar{x} - \bar{y}\|^2) + O\left(\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon_1}\right) + O(|\bar{t} - \bar{s}|^2) \leq -\frac{2r}{T^2}. \]  

(5.97)

Therefore, by taking \(\varepsilon_2 \to 0\) and then \(\varepsilon_1 \to 0\), we finally obtain

\[ 0 = \limsup_{\varepsilon_1 \to 0} \left( \limsup_{\varepsilon_2 \to 0} \left( O(\|\bar{x} - \bar{y}\|^2) + O\left(\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon_1}\right) \right) \right) \leq \frac{2r}{T^2}, \]

which is clearly a contradiction. The proof is now complete.
References

[1] Bardi, M. and Capuzzo-Dolcetta, I. (1997), *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston.

[2] Borkar, V.S. (1989), *Optimal Control of Diffusion Processes*, Longman Group UK Limited.

[3] Clarke, F.H. and Vinter, R.B. (1983), Local Optimality Conditions and Lipschitzian Solutions to the Hamilton-Jacobi Equations, *SIAM J. Control and Optim.*, 21, 856-870.

[4] Clementi, G. and Hopenhayn, H. (2006), A Theory of Financing Constraints and Firm Dynamics, *Quarterly Journal of Economics* 121(1), 229-265.

[5] Cobb, C. W. and Douglas, P. H. (1928), A Theory of Production, *American Economic Review* 18, 139-165.

[6] Crandall, M.G. and Ishii, H. (1990) The Maximum Principle for Semicontinuous functions, *Differential Integral Equations*, 3, 1001-1014.

[7] Crandall, M.G., Ishii, H. and Lions, P.L. (1992), A User’s guideto viscosity solutions, *Bulletin A. M. S.*, 27, 1-67.

[8] Crandall, M.G. and Lions, P.L. (1983), Viscosity Solutions of Hamilton-Jacobi Equations, *Trans. Amer. Math. Soc.* 277, 487-502.

[9] DeMarzo, P. and Sannikov, Y. (2006), Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model, *Journal of Finance* 61, 2681-2724.

[10] Fleming, W.H. (1964), The Cauchy Problem for Degenerate Parabolic Equations, *J. Math. Mech.* 13, 987-1008.

[11] Fleming, W.H. (1969), The Cauchy Problem for a Nonlinear First-order Differential Equation, *J. Diff. Eqs.*, 5, 530-555.

[12] Fleming, W.H. and Soner, H.M. (2006), *Controlled Markov Processes and Viscosity Solutions*, Second Edition, Springer Science and Business Media, Inc. U.S.A.

[13] Giat, Y., Hackman, S.T. and Subramanian, A. (2007), Dynamic Contracting under Imperfect Information and Heterogeneous Beliefs: A Continuous-Time Principal-Agent Model, Preprint.
[14] Gibbons, R. and Murphy, K.J. (1992), Optimal Incentive Contracts in the Presence of Career Concerns: Theory and Evidence, *Journal of Political Economy* 100(3), 468-505.

[15] Holmstrom, B. (1999), Managerial Incentive Problems: A Dynamic Perspective, *Review of Economic Studies* 66, 169-182.

[16] Holmstrom, B. and Milgrom, P. (1987), Aggregation and Linearity in the Provision of Intertemporal Incentive, *Econometrica* 55(2), 303-328.

[17] Ishii, H. (1989), On Uniqueness and Existence of Viscosity Solutions of fully Nonlinear Second order Elliptic PDE’s, *Commun. Pure Appl. Math.*, 42, 15-45.

[18] Ishii, H. (1991), Viscosity Solutions of a Class of Hamilton-Jacobi Equations in Hilbert Spaces, *Annals of Functional Analysis* 105, 301-341.

[19] Ishii, H. (1993), Viscosity Solutions of Nonlinear Second-order Partial Differential Equations in Hilbert Spaces, *Commun. in Partial Differential Equations* 18(3,4), 601-650.

[20] Jenson, R (1988), The Maximum Principle for Viscosity Solutions of second order fully Nonlinear Partial Differential Equations, *Arch. Rat. Mech. Anal.*, 101, 1-27.

[21] Karatzas, L. and Shreve, S.E. (1998), *Brownian Motion and Stochastic Calculus, Second Edition*, Springer Science and Business Media, LLC.

[22] Krulkov, S.N. (1960), The Cauchy Problem in the large for certain Nonlinear First-order Differential Equations, *Soviet math. Dokl.*, 1, 474-477.

[23] Krulkov, S.N. (1964), The Cauchy Problem in the large for Nonlinear Equations and for certain Quasilinear Systems of the First-order with several variables, *Soviet Math. Dokl.*, 5, 493-496.

[24] Krulkov, S.N. (1966), On Solutions of First-order Nonlinear Equations, *Soviet Math. Dokl.*, 7, 376-379.

[25] Krulkov, S.N. (1967), Generalized Solutions of Nonlinear First-order Equations with several independent variables II, *Math. USSR Sbornik*, 1, 93-116.

[26] Krulkov, S.N. (1970), First-order Quasilinear Equations in several independent variables, *Math. USSR Sbornik*, 10, 217-243.

[27] Krylov, N.V. (1980), *Controlled Diffusion Processes*, Springer-Verlag, New York, U.S.A.
[28] Lions, P.L. (1983), Optimal Control of Diffusion Processes and Hamilton-Jacobi-Bellman Equation, Part I: The dynamic programming principle and applications; Part II: Viscosity solution and uniqueness; Part III: Regularity of the optimal cost function, *Comm. PDEs*, Vol. 8, 1101-1174; 1229-1276; *Nonlinear PDEs and Appl., College de France Seminar* Vol. V, Pitman Boston.

[29] Quadrini, V. (2004) Investment and Liquidation in Renegotiation-Proof Contracts with Moral Hazard, *Journal of Monetary Economics* 51(4), 713-751.

[30] Schattler, H. and Sung, J. (1993), The First Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility, *Journal of Economic Theory* 61, 331-371.

[31] Spear, S. and Srivastava, S. (1987), On Repeated Moral Hazard with Discounting, *Review of Economic Studies* 54(4), 599-617.

[32] Spear, S. and Wang, C. (2005), When to Fire a CEO: Optimal Termination in Dynamic Contracts, *Journal of Economic Theory* 120(2), 239-256.

[33] Wang, C. (1997), Incentives, CEO Compensation and Shareholder Wealth in a Dynamic Agency Model, *Journal of Economic Theory* 76(1), 72-105.

[34] Whittle, P. (1990), *Risk-sensitive Optimal Control*, John Wiley and Sons Ltd. Baffins Lane, Chichester, England.

[35] Williams, N. (2006), On Dynamic Principal-Agent Problems in Continuous Time, Preprint.

[36] Yong, J. and Zhou, X.Y. (1999), *Stochastic Control: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, U.S.A.