Symmetry of solutions of a mean field equation on flat tori

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Abstract

We study symmetry of solutions of the mean field equation

\[ \Delta u + \rho \left( \frac{Ke^u}{\int_{T_\epsilon} Ke^u} - \frac{1}{|T_\epsilon|} \right) = 0 \]

on the flat torus \( T_\epsilon = [-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}] \times [-\frac{1}{2}, \frac{1}{2}] \) with \( 0 < \epsilon \leq 1 \), where \( K \in C^2(T_\epsilon) \) is a positive function with \( -\Delta \ln K \leq \frac{\rho}{|T_\epsilon|} \) and \( \rho \leq 8\pi \). We prove that if \( (x_0, y_0) \) is a critical point of the function \( u + \ln(K) \), then \( u \) is evenly symmetric about the lines \( x = x_0 \) and \( y = y_0 \), provided \( K \) is evenly symmetric about these lines. In particular we show that all solutions are one-dimensional if \( K \equiv 1 \) and \( \rho \leq 8\pi \). The results are sharp and answer a conjecture of Lin and Lucia affirmatively. We also prove some symmetry results for mean field equations on annulus.

1 Introduction

Consider the mean field equation

\[ \Delta u + \rho \left( \frac{e^u}{\int_{T_\epsilon} e^u} - \frac{1}{|T_\epsilon|} \right) = 0, \quad (x, y) \in T_\epsilon, \] (1)

on the flat torus with fundamental domain

\[ T_\epsilon = [-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}] \times [-\frac{1}{2}, \frac{1}{2}], \quad 0 < \epsilon \leq 1. \] (2)

Define

\[ \mathcal{H}(T_\epsilon) = \{ u \in H^1(T_\epsilon) : \int_{T_\epsilon} u = 0 \}. \]
Solutions of (1) are critical points of the functional $F_{\rho, \epsilon} : \mathcal{H}(T_\epsilon) \to \mathbb{R}$ defined by

$$F_{\rho, \epsilon} := \frac{1}{2} \int_{T_\epsilon} |\nabla u|^2 - \rho \ln(\frac{1}{|T_\epsilon|} \int_{T_\epsilon} e^u).$$

Notice that, since both equation (1) and functional $F_{\rho, \epsilon}$ are invariant under dilations, we do not lose generality by taking the vertical size of $T_\epsilon$ to be 1.

Equations of type (1) arise in Onsager’s vortex theory for one specie (see, [4, 5, 6, 16]). It is also obtained in the context of Chern-Simons gauge theory (see [3, 11, 14, 15, 25], etc). Tarantello [25] showed that when the Chern-Simons coupling constant tends to zero, the asymptotic behavior of a class of solutions is described by

$$\Delta u + \rho (\frac{K e^u}{\int_{T_\epsilon} K e^u} - \frac{1}{|T_\epsilon|}) = 0, \quad (x, y) \in T_\epsilon,$$  (3)

where $\rho = 4\pi N$, $N$ is an integer called the vortex number, and $K$ is a prescribed non-negative function.

Various results have been obtained regarding the existence and qualitative properties of mean field equations on flat tori (see, e.g., [2, 7, 8, 9, 10, 16, 17, 18, 19, 20, 22, 23, 24]). If $\rho \leq 0$, it is easy to see that the functional $F_{\rho, \epsilon}$ is strictly convex and consequently zero is the only solution of (1) which is also a minimizer of $F_{\rho, \epsilon}$. For $\rho > 0$ it follows from the Moser-Trudinger inequality [21, 26] that the functional $F_{\rho, \epsilon}$ is bounded from below if and only if $\rho \leq 8\pi$. Moreover if $\rho < 8\pi$, then $F_{\rho, \epsilon}$ is coercive and therefore it admits a minimizer. For $\rho = 8\pi$ existence of a minimizer is discussed in [8] and [22].

For $\rho > 8\pi$, the functional $F_{\rho, \epsilon}$ is unbounded below. It is shown in [10] by a minmax method that (3) has a solution for $\rho \in (8\pi, 16\pi)$. In [24] it is proved that if $\epsilon = 1$ and $8\pi < \rho < 4\pi^2$, then the equation (1) has two-dimensional solutions and $u \equiv 0$ is only a local minimizer of $F_{\rho, 1}$. On the other hand it is proved in [23] that $\rho > 4\pi^2\epsilon$ is a necessary and sufficient condition for the existence of at least one non-zero one-dimensional solution. Moreover if $4\pi^2\epsilon < 8\pi$ and $\rho \in (4\pi^2\epsilon, 8\pi]$, then any minimizer is nonzero [23].

As stated above, there exist two-dimensional solutions for $\rho \in (8\pi, 4\pi^2)$ and $\epsilon = 1$. So it is a natural question to ask whether all solutions of (1) are one-dimensional for $\rho \leq 8\pi$. In [2], Cabré, Lucia, and Sanchón proved that if

$$\rho \leq \rho^* := \frac{16\pi^3}{\pi^2 + \frac{2}{R_\epsilon} + \sqrt{(\pi^2 + \frac{2}{R_\epsilon})^2 - \frac{8\pi^2}{|T_\epsilon|}}} \leq 0.879 \times 8\pi,$$

then every solution $u$ of (1) depends only on $x$-variable. Here $R_\epsilon$ is the maximum conformal radius of the rectangle $T_\epsilon$.

In [19], Lin and Lucia proved that constants are the only solutions of (1) whenever

$$\rho \leq \begin{cases} 8\pi & \text{if } \epsilon \geq \frac{4}{3}, \\ \frac{32\epsilon}{4} & \text{if } \epsilon \leq \frac{4}{3}, \end{cases}$$

for $\epsilon$ sufficiently small.
Later in [20] Lin and Lucia obtained optimal symmetry results for minimizers of the functional \( F_{\rho, \epsilon} \). Indeed they proved the following theorem.

**Theorem A.** (Theorem 1.2 in [20]) Let \( T_\epsilon \) be the flat torus defined in (2) and suppose \( \rho \leq 8\pi \). Then any global minimizer of \( F_{\rho, \epsilon} \) is one-dimensional. In addition for \( \rho \leq \min\{8\pi, 4\pi^2\epsilon\} \), \( u \equiv 0 \) is the unique global minimizer of the functional \( F_{\rho, \epsilon} \).

However, one-dimensional symmetry of solutions of (1) still remained open. In particular, Lin and Lucia [20] conjectured that \( u \equiv 0 \) is the unique solution of (1) whenever \( \rho \leq \min\{8\pi, 4\pi^2\epsilon\} \). In this paper, among other results, we prove this conjecture. Indeed we prove the following optimal result which improves the results in [2], [19], and [20], discussed above.

**Theorem 1.1** Suppose \( \rho \leq 8\pi \) and let \( u \) be a solution of (1). Then \( u \) must be one-dimensional. In particular, \( u \) is constant if \( \rho \leq \min\{8\pi, 4\pi^2\epsilon\} \).

We also prove the following theorem for the general mean field equation (3).

**Theorem 1.2** Let \( \rho \leq 8\pi \), \( T_\epsilon \) be a flat torus defined in (2), and \( K \in C^2(T_\epsilon) \) be a positive function with \(-\Delta \ln K \leq \frac{\rho}{|T_\epsilon|} \) in \( T_\epsilon \). Suppose that \( K \) is evenly symmetric in \( x, y \), and \( u \) is a solution of (3) with the origin being a critical point of \( u \). Then \( u \) is evenly symmetric in \( x \) and \( y \), i.e.,

\[ u(x, y) = u(-x, y) = u(x, -y), \quad \forall (x, y) \in T_\epsilon. \]

Letting \( v = \ln K + u \), we obtain the following corollary of Theorem 1.2.

**Corollary 1.3** Let \( \rho \leq 8\pi \), \( T_\epsilon \) be a flat torus defined in (2), and \( K \in C^2(T_\epsilon) \) be a positive function with \(-\Delta \ln K = C \leq \frac{\rho}{|T_\epsilon|} \) in \( T_\epsilon \), where \( C \) is a constant. Suppose \((x_0, y_0)\) is a critical point of \( \ln K + u \). Then \( \ln K + u \) is symmetric with respect to the lines \( x = x_0 \) and \( y = y_0 \).

We also present symmetry results for mean field equations on annulus in Section 3, improving corresponding results in [7].

## 2 Proof of the symmetry results on tori

In this section we present the proofs of our main results, Theorems 1.1 and 1.2. The proofs are based on the Sphere Covering Inequality recently proved by the authors in [13].
Theorem B. (Theorem 3.1 in [13]) Let \( \Omega \) be a simply-connected subset of \( \mathbb{R}^2 \) and assume 
\[ w_i \in C^2(\Omega), \quad i = 1, 2 \] 
satisfy
\[ \Delta w_i + e^{w_i} = f_i, \]
where \( f_2 \geq 0 \) and \( f_2 \geq f_1 \) in \( \Omega \). If \( w_2 \neq w_1 \) in \( \Omega \) and \( w_2 = w_1 \) on \( \partial \Omega \), then
\[ \int_\Omega (e^{w_1} + e^{w_2}) \geq 8\pi. \]
Moreover, the equality only holds when \( f_2 \equiv f_1 \equiv 0 \) and \((\Omega, e^{w_i} dy), \quad i = 1, 2\) are isometric to two complimenting spherical caps on the standard sphere with radius \( \sqrt{2} \).

Proof of Theorem 1.2. Let \( v := u + \ln(K) + \ln \rho - \ln(\int_{T_\epsilon} Ke^u) \). Then \( v \) satisfies
\[ \Delta v + e^v = \frac{\rho}{|T_\epsilon|} + \Delta \ln K \geq 0 \quad (x, y) \in T_\epsilon. \]
We claim that \( u \) is even in \( x \) and \( y \), i.e.,
\[ v(x, y) = v(-x, y) = v(x, -y), \quad \forall (x, y) \in T_\epsilon. \]
To prove the claim, define
\[ w(x, y) = v(x, y) - v(x, -y), \quad (x, y) \in T_\epsilon. \]
Suppose that \( w \neq 0 \), and set
\[ \Omega^+ := \{ x \in [-1/2\epsilon, 1/2\epsilon] \times [0, 1/2] : \quad w(x) > 0 \} \]
and
\[ \Omega^- := \{ x \in [-1/2\epsilon, 1/2\epsilon] \times [0, 1/2] : \quad w(x) < 0 \}. \]
Note that \( w = 0 \) on \( \Gamma_0 \cup \Gamma_1 \), where
\[ \Gamma_0 = \{(x, 0) : -\frac{1}{2\epsilon} \leq x \leq \frac{1}{2\epsilon}\} \quad \text{and} \quad \Gamma_1 = \{(x, 1) : -\frac{1}{2\epsilon} \leq x \leq \frac{1}{2\epsilon}\}. \]
Since \((0,0)\) is a critical point of \( u \) and \( K \) is evenly symmetric in \( y \),
\[ \frac{\partial}{\partial y} w(0,0) = 0. \]
Therefore, by Hopf’s lemma, the nodal set of \( w \) must contain a curve \( \Gamma \) originating from the origin and having a transversal intersection with \( \Gamma_0 \). We now discuss two cases:

(a) \( \Gamma \) reaches the boundary curve \( \Gamma_1 \).
(b) \( \Gamma \) does not reach the boundary curve \( \Gamma_1 \).

In case (a), there must be another nodal curve of \( w \) connecting \( \Gamma_1 \) and \( \Gamma_2 \), and hence there are at least two simply connected nonempty regions \( \Omega_1, \Omega_2 \subset [-1/2\epsilon, 1/2\epsilon] \times [0, 1/2] \) such
that $\Omega_1 \cap \Omega_2 = \emptyset$ and $w = 0$ on $\partial \Omega_1 \cup \partial \Omega_2$. Hence on each $\Omega_i$, $i = 1, 2$, equation (6) has two distinct solutions $v(x, y)$ and $v(x, -y)$ with $v(x, y) = v(x, -y) = 0$ on $\partial \Omega_i$. Therefore, by the Sphere Covering Inequality (Theorem B) we conclude that

$$\int_{\Omega_i} (e^{v(x,y)} + e^{v(x,-y)}) > 8\pi, \quad i = 1, 2.$$ 

Hence

$$\rho = \int_{T_c} e^v \geq \sum_{i=1}^{2} \int_{\Omega_i} (e^{v(x,y)} + e^{v(x,-y)}) > 16\pi,$$

which is contradiction.

In Case (b), there exists at least one simply-connected region $\Omega_1 \subset \subset [-\frac{1}{2\pi}, \frac{1}{2\pi}] \times [0, \frac{1}{2}]$ such that $w = 0$ on $\partial \Omega_1$. Thus the equation (6) has two distinct solutions $v(x, y)$ and $v(x, -y)$ on $\Omega_1$ with $v(x, y) = v(x, -y) = 0$ on $\partial \Omega_1$. Therefore, by the Sphere Covering Inequality (Theorem B) again we conclude that

$$\int_{\Omega_1} (e^{v(x,y)} + e^{v(x,-y)}) > 8\pi.$$ 

Consequently

$$\rho = \int_{T_c} e^v \geq \int_{\omega_1} (e^{v(x,y)} + e^{v(x,-y)}) > 8\pi,$$ 

which is a contradiction. In both cases, we conclude $\rho > 8\pi$ which contradicts the assumption $\rho \leq 8\pi$. Hence $u(x, -y) \equiv u(x, y)$ in $T_c$. The proof of $u(x, y) \equiv u(-x, y)$ in $T_c$ is similar. □

**Remark 2.1** Notice that if $u$ has a critical point at the origin and another critical point $X^* = (x^*, 1)$ with $-\frac{1}{2\pi} < x^* < \frac{1}{2\pi}$, then case (b) in the proof of Theorem 1.2 can not happen and therefore $u$ must be symmetric with respect to $x$-axis if $\rho \leq 16\pi$. Similarly if $u$ has another critical point $Y^* = (\frac{1}{2\pi}, y^*)$ with $-\frac{1}{2} < y^* < \frac{1}{2}$, then $u$ will be symmetric with respect to $y$-axis for $\rho \leq 16\pi$. In particular, if $u$ has a critical point at the origin and another critical point at the corner of the torus, then $u$ must be symmetric with about $x$ and $y$-axis for $\rho \leq 16\pi$.

From the above proof, we can easily see the following.

**Remark 2.2** Let $\rho \leq 8\pi$, $T_c$ be a flat torus defined in (2), and $K \in C^2(T_c)$ be a positive function with $-\Delta \ln K \leq \frac{\rho}{|T_c|}$ in $T_c$. If $K$ is only evenly symmetric in $y$ and $u_y(x^*, 0) = 0$ for some $x^* \in [-\frac{1}{2\pi}, \frac{1}{2\pi}]$, then $u$ is evenly symmetric in $y$. Similarly, if $K$ is only evenly symmetric in $x$ and $u_x(0, y^*) = 0$ for some $y^* \in [-\frac{1}{2}, \frac{1}{2}]$, then $u$ is evenly symmetric in $x$.

Next we prove Theorem 1.1. Following [20] let us first define Steiner symmetric solutions.
**Definition 1** Let \( T = (-a, a) \times (-b, b) \) be a flat torus. A function \( u \in H^1(T) \) is said to be Steiner symmetric on \( T \) if

\[
\begin{cases}
  u(x, y) = u(-x, y) = u(x, -y) & \forall (x, y) \in T, \\
  \frac{\partial u}{\partial x}(x) \leq 0 & \forall (x, y) \in (0, a) \times (-b, b), \\
  \frac{\partial u}{\partial y}(x) \leq 0 & \forall (x, y) \in (-a, a) \times (0, b).
\end{cases}
\]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Without loss of generality we may assume that \( u \) has a maximum point at \((0, 0)\). Thus it follows from Theorem 1.2 that \( u \) is symmetric about \( x \) and \( y \)-axis.

Assume now that \( u_x(x^*, y^*) = 0 \) for some \( x^* \in (0, \frac{1}{2l}) \). Reflecting \( u \) about \( x = x^* \) and applying an argument similar to the one in the proof of Theorem 1.2 we can conclude that \( u \) is evenly symmetric about \( x = x^* \). Similarly if \( u_y(x^*, y^*) = 0 \), then we conclude that \( u \) is symmetric about \( y = y^* \). Therefore, there exist positive integers \( l, m \) such that \( u \) is a periodic function with periods \( \frac{1}{l \epsilon}, \frac{1}{m \epsilon} \) in \( x, y \) variables, respectively. Moreover

\[
u_x(x, y) \leq 0, \quad u_y(x, y) \leq 0 \quad \text{for} \quad (x, y) \in [0, \frac{1}{2l \epsilon}] \times [0, \frac{1}{2m \epsilon}].\]

In other words, \( u \) can be regarded as a Steiner symmetric solution on a flat torus with fundamental domain as \( [-\frac{1}{2l \epsilon}, \frac{1}{2l \epsilon}] \times [-\frac{1}{2m \epsilon}, \frac{1}{2m \epsilon}] \). Since \( \rho \leq 8 \pi \), it follows from Theorem 1.2 b) in [20] that \( u \) is one-dimensional in \( [-\frac{1}{2l \epsilon}, \frac{1}{2l \epsilon}] \times [-\frac{1}{2m \epsilon}, \frac{1}{2m \epsilon}] \), and consequently \( u \) must be one-dimensional in \( T_\epsilon \).

Riccardi and Tarantello in [23] showed that \( \rho > 4 \pi^2 \epsilon \) is a necessary and sufficient condition for the existence of a non constant one dimensional solution. Hence \( u \) must be constant if \( \rho \leq \min\{8 \pi, 4 \pi^2 \epsilon\} \). The proof is now complete. \( \square \)

### 3 Mean field equations on annulus

In this section we prove symmetry results for mean field equations on an annulus. Let \( \mathcal{A} \) be an arbitrary annulus in \( \mathbb{R}^2 \), i.e.

\[
\mathcal{A} := \{(x, y) \in \mathbb{R}^2 : a < |(x, y)| < b\} \quad \text{for some} \quad a < b.
\]

We consider the following mean field equation

\[
\begin{cases}
  \Delta u + \rho \int_{\mathcal{A}} \frac{K(x, y) u}{K(x, y) + u} \, dx = f(x, y) \geq 0 \quad \text{in} \quad \mathcal{A} \\
  u(x, y) = \alpha & \text{if} \quad |(x, y)| = a \\
  u(x, y) = \beta & \text{if} \quad |(x, y)| = b.
\end{cases}
\]

(9)
where $K$ is a positive radial function with $\Delta \ln K \geq 0$, $f$ is a non-negative radial function, and $\alpha, \beta \in \mathbb{R}$.

For $\rho \in (8\pi, 16\pi)$, it is shown in [10] that there exists a solution to (9) when $\alpha = \beta = 0$. In [7], it is proved that if $\rho_i \to 8\pi$ from left and $\rho_i \to 16\pi$ from right, then for large $i$ the solutions have even symmetry about a line passing through the origin, when blow-up of the solution happens. Below (see Theorem 3.2) we shall show that the solutions always have even symmetry.

We will need the following lemma (see, e.g. [1, 12, 20]).

**Lemma 3.1** Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and assume that $w \in C^2(\overline{\Omega})$ satisfies $\Delta w + e^w \geq 0$ in $\overline{\Omega}$ and $\int_{\Omega} e^w \leq 8\pi$. Consider an open set $\omega \subset \Omega$ and define the first eigenvalue of the operator $\Delta + e^w$ in $H^1_0(\omega)$ by

$$
\lambda_{1,w}(\omega) := \inf_{\phi \in H^1_0(\omega)} \left( \int_{\omega} |\nabla \phi|^2 e^w - \int_{\omega} \phi^2 e^w \right) \leq 0.
$$

Then $\int_{\Omega} e^w \geq 4\pi$ if $\lambda_{1,w}(\Omega) \leq 0$.

**Theorem 3.2** Assume $\rho < 16\pi$. Then, after a proper rotation, every solution of (9) must be evenly symmetric about $x$-axis. Furthermore, either $u$ is radially symmetric or the angular derivative $u_\theta$ of $u$ doesn’t change sign in

$$
\mathcal{A}^+ := \{(x, y) \in \mathbb{R}^2 : a < |(x, y)| < b, \ y > 0\}.
$$

**Proof.** Define

$$
v = u + \ln(K) + \ln(\rho) - \ln(\int_{\mathcal{A}} K e^u dx).
$$

Then $v$ satisfies

$$
\Delta v + e^v = f + \Delta \ln K \geq 0.
$$

(10)

Fix $x_0 \in (a, b)$. Without loss of generality, we may assume that at $u$ attains its minimum on the circle $|(x, y)| = x_0$ at the point $(x_0, 0)$. Note that $\frac{\partial v}{\partial \theta}(x_0, 0) = 0$. Define

$$
w(x, y) := v(x, y) - v(x, -y).
$$

Since

$$
\frac{\partial v}{\partial \theta}(x_0, 0) = \frac{\partial v}{\partial y}(x_0, 0) = 0,
$$

it follows from the Hopf’s lemma that the nodal line of $w$ divided a neighborhood of $x_0$ into at least four regions. Hence there exists two simply connected regions

$$
\Omega_1, \Omega_2 \subset \{(x, y) \in \mathcal{A} : y > 0\}
$$

such that $w = 0$ on $\partial \Omega_1 \cup \partial \Omega_2$. Therefore on each $\Omega_i$, $i = 1, 2$, the equation (10) has two solutions $v(x, y)$ and $v(x, -y)$ with $v(x, y) = v(x, -y)$ on $\partial \Omega_i$. Thus it follows from the Sphere Covering Inequality (Theorem B) that

$$
\rho = \int_{\mathcal{A}} e^v \geq 2 \sum_{i=1}^2 \int_{\Omega_i} (e^{v(x,y)} + e^{v(x,-y)}) \geq 16\pi.
$$
This is a contradiction to the assumption $\rho < 16\pi$. Therefore, we conclude that $w \equiv 0$ in $\mathcal{A}$, and $u$ is evenly symmetric about $x$-axis. Furthermore, $\phi := v_\theta = u_\theta$ satisfies the linearized equation
\[
\begin{aligned}
\Delta \phi + e^v \phi &= 0 \quad \text{in } \mathcal{A}, \\
\phi(x, y) &= 0 \quad \text{if } |(x, y)| = a, \text{ or } |(x, y)| = b, \text{ or } y = 0.
\end{aligned}
\] (11)
Assume $u$ is not radially symmetric. If $\phi$ changes sign in $\mathcal{A}^+$, then there are at least two regions $\Omega_1, \Omega_2 \subset \mathcal{A}^+$ with $\phi = 0$ on $\partial \Omega_1 \cup \partial \Omega_2$. Hence it follows from Lemma 3.3 that
\[
\rho = 2 \int_{\mathcal{A}^+} e^v \geq 2 \sum_{i=1}^2 \int_{\Omega_i} e^v \geq 16\pi.
\]
This is a contradiction. Hence $u_\theta$ does not change sign in $\mathcal{A}^+$. The proof is now complete. □

Remark 3.3 If $f + \Delta \ln K \neq 0$ in $\mathcal{A}$, then the condition $\rho < 16\pi$ in the statement of Theorem 3.2 can be replaced by $\rho \leq 16\pi$.

Remark 3.4 Assume that $K, f$ are only evenly symmetric about $x$-axis in Theorem 3.2. The above proof also indicates that $u$ must be evenly symmetric about $x$-axis if there is a point $(x_0, 0)$ on $x$-axis such that $u_y(x_0, 0) = 0$.

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