BOUNDDEDNESS OF WEIGHTED ITERATED HARDY-TYPE OPERATORS INVOLVING SUPREMA FROM WEIGHTED LEBESGUE SPACES INTO WEIGHTED CESÀRO FUNCTION SPACES

R.CH. MUSTAFAYEV AND N. BİLGİÇLİ

Abstract. In this paper the boundedness of the weighted iterated Hardy-type operators \( T_{a,b} \) and \( T_{a,b}^* \) involving suprema from weighted Lebesgue space \( L^p(v) \) into weighted Cesàro function spaces \( \text{Ces}_q(w,a) \) are characterized. These results allow us to obtain the characterization of the boundedness of the supremal operator \( R_{a,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions. For the convenience of the reader, we formulate the statement on the boundedness of the weighted Hardy operator \( P_{a,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions. Under additional condition on \( a \) and \( b \), we are able to characterize the boundedness of weighted iterated Hardy-type operator \( T_{a,b} \) involving suprema from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions. At the end of the paper, as an application of obtained results, we calculate the norm of the fractional maximal function \( M_f \) from \( L^p(v) \) into \( \Gamma^q(w) \).

1. Introduction

Many Banach spaces which play an important role in functional analysis and its applications are obtained in a special way: the norms of these spaces are generated by positive sublinear operators and by \( L_p \)-norms.

In connection with Hardy and Copson operators

\[
(Pf)(x) := \frac{1}{x} \int_0^x f(t) \, dt \quad \text{and} \quad (Qf)(x) := \int_x^{\infty} \frac{f(t)}{t} \, dt, \quad (x > 0),
\]

the classical Cesàro function space

\[
\text{Ces}(p) := \left\{ f : \|f\|_{\text{Ces}(p)} := \left( \int_0^\infty \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \right)^{1/p} \right\} < \infty,
\]

and the classical Copson function space

\[
\text{Cop}(p) := \left\{ f : \|f\|_{\text{Cop}(p)} := \left( \int_0^\infty \left( \int_x^{\infty} \frac{|f(t)|}{t} \, dt \right)^p \right)^{1/p} \right\} < \infty,
\]

where \( 1 < p \leq \infty \), with the usual modifications if \( p = \infty \), are of interest.

The classical Cesàro function spaces \( \text{Ces}(p) \) have been introduced in 1970 by Shiue [48]. These spaces have been defined analogously to the Cesàro sequence spaces that appeared two years earlier in [40] when the Dutch Mathematical Society posted a problem to find a representation of their dual spaces. In 1971 Leibowitz proved that \( \text{ces}_1 = \{0\} \) and for \( 1 < q < p \leq \infty \), \( \ell_p \) and \( \text{ces}_q \) sequence spaces are proper subspaces of \( \text{ces}_p \) [32]. The problem posted [40] was resolved by Jagers [28] in 1974 who gave an explicit isometric description of the dual of Cesàro sequence space. In [51], Sy, Zhang and Lee gave a description of dual spaces of \( \text{Ces}(p) \) spaces based on Jagers’ result. In 1996 different, isomorphic description due to Bennett appeared in [4]. In [4, Theorem 21.1] Bennett observes that the classical Cesàro function space and the classical Copson function space coincide for \( p > 1 \). He also derives estimates for the norms of the corresponding inclusion operators. The same result, with different estimates, is due to Boas [7], who in fact obtained the integral analogue of the Askey-Boas Theorem [6, Lemma 6.18] and [1].

These results generalized in [27] using the blocking technique. In [2] they investigated dual spaces for \( \text{Ces}(p) \) for \( 1 < p < \infty \). Their description can be viewed as being analogous to one given for sequence spaces in [4]. For a long time, Cesàro function spaces have not attracted a lot of attention contrary to their sequence counterparts. In fact there is quite rich literature concerning different topics studied in Cesàro sequence spaces as for instance in [11–15]. However, recently in a series of papers, Astashkin and Maligranda started to study the structure of Cesàro...
function spaces. Among others, in [2] they investigated dual spaces for $\text{Ces}(p)$ for $1 < p < \infty$. Their description can be viewed as being analogous to one given for sequence spaces in [4] (For more detailed information about history of classical Cesàro spaces see recent survey paper [3]).

Throughout the paper we assume that $I := (a, b) \subseteq (0, \infty)$. By $\mathcal{M}(I)$ we denote the set of all measurable functions on $I$. The symbol $\mathcal{M}^+(I)$ stands for the collection of all $f \in \mathcal{M}(I)$ which are non-negative on $I$, while $\mathcal{M}^{+1}(I)$ is used to denote the subset of those functions which are non-increasing on $I$, respectively. A weight is a function $v \in \mathcal{M}^+(0, \infty)$ such that $0 < V(x) < \infty$ for all $x \in (0, \infty)$, where

$$V(x) := \int_0^x v(t) dt.$$  

The family of all weight functions (also called just weights) on $(0, \infty)$ is given by $\mathcal{W}(0, \infty)$.

For $p \in (0, \infty]$ and $w \in \mathcal{M}^+(I)$, we define the functional $\|\cdot\|_{p, w, I}$ on $\mathcal{M}(I)$ by

$$\|f\|_{p, w, I} := \left\{ \begin{array}{ll} \left( \int |f(x)|^p w(x) \, dx \right)^{1/p} & \text{if } p < \infty, \\ \text{ess sup} |f(x)|v(x) & \text{if } p = \infty. \end{array} \right.$$  

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w, I)$ is given by

$$L^p(w, I) = \{ f \in \mathcal{M}(I) : \|f\|_{p, w, I} < \infty \}$$  

and it is equipped with the quasi-norm $\|\cdot\|_{p, w, I}$.

When $I = (0, \infty)$, we write $L^p(w)$ instead of $L^p(w, (0, \infty))$.

We adopt the following usual conventions.

Convention 1.1. We adopt the following conventions:

- Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.
- If $p \in [1, +\infty]$, we define $p'$ by $1/p + 1/p' = 1$.
- If $0 < q < p < \infty$, we define $r$ by $1/r = 1/q - 1/p$.
- If $I = (a, b) \subseteq \mathbb{R}$ and $g$ is monotone function on $I$, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x \to a^+} g(x)$ and $\lim_{x \to b^-} g(x)$, respectively.

Throughout the paper, we always denote by $c$ and $C$ a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript or superscript such as $c_1$ does not change in different occurrences. By $a \leq b$, $(b \geq a)$ we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \leq b$ and $b \leq a$, we write $a \approx b$ and say that $a$ and $b$ are equivalent.

Unless a special remark is made, the differential element $dx$ is omitted when the integrals under consideration are the Lebesgue integrals.

The weighted Cesàro and Copson function spaces are defined as follows:

Definition 1.2. Let $0 < p \leq \infty$, $u \in \mathcal{M}^+(0, \infty)$ and $v \in \mathcal{W}(0, \infty)$. The weighted Cesàro and Copson spaces are defined by

$$\text{Ces}_p(u, v) := \left\{ f \in \mathcal{M}^+(0, \infty) : \|f\|_{\text{Ces}_p(u, v)} := \|\|f\|_{1, v, (0, \infty)}\|_{p, u, (0, \infty)} \right\} < \infty,$$

and

$$\text{Cop}_p(u, v) := \left\{ f \in \mathcal{M}^+(0, \infty) : \|f\|_{\text{Cop}_p(u, v)} := \|\|f\|_{1, v, (\infty, \infty)}\|_{p, u, (0, \infty)} \right\} < \infty,$$

respectively.

When $v \equiv 1$ on $(0, \infty)$, we simply write $\text{Ces}_p(u)$ and $\text{Cop}_p(u)$ instead of $\text{Ces}_p(u, v)$ and $\text{Cop}_p(u, v)$, respectively.

Recall that $\text{Ces}_p(u, v)$ and $\text{Cop}_p(u, v)$ are contained in the scale of weighted Cesàro and Copson function spaces $\text{Ces}_{p, q}(u, v)$ and $\text{Cop}_{p, q}(u, v)$ defined in [22]. Obviously, $\text{Ces}(p) = \text{Ces}_p(x^{-1})$ and $\text{Cop}(p) = \text{Cop}_p(x^{-1})$. In [29], Kamińska and Kubiak computed the dual norm of the Cesàro function space $\text{Ces}_p(u)$, generated by $1 < p < \infty$ and an arbitrary positive weight $u$. A description presented in [29] resembles the approach of Jagers [28] for sequence spaces.
Let \( u \in W(0, \infty) \cap C(0, \infty) \), \( b \in W(0, \infty) \) and \( B(t) := \int_0^t b(s) \, ds \). Assume that \( b \) is a weight such that \( b(t) > 0 \) for a.e. \( t \in (0, \infty) \). The weighted iterated Hardy-type operators involving suprema \( T_{u,b} \) and \( T^*_{u,b} \) are defined at \( g \in \mathcal{W}(0, \infty) \) by

\[
(T_{u,b}g)(t) := \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y)b(y) \, dy, \quad t \in (0, \infty),
\]

\[
(T^*_{u,b}g)(t) := \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \int_\tau^\infty g(y)b(y) \, dy, \quad t \in (0, \infty).
\]

Such operators have been found indispensible in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [30]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [38], [39]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [18], [17], [16], [45]. Recall that \( T_{u,b} \) successfully controls non-increasing rearrangements of wide range of maximal functions (see, for instance, [34] and references therein).

It was shown in [23] that for every \( h \in \mathcal{W}(0, \infty) \) and \( t \in (0, \infty) \)

\[
(T_{u,b}h)(t) = (T_{\bar{u},b}h)(t),
\]

where

\[
\bar{u}(t) := B(t) \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)}, \quad t \in (0, \infty).
\]

Moreover, if the condition

\[
\sup_{0 < t < \infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau) \, d\tau}{u(\tau)} < \infty.
\]

holds, then for all \( f \in \mathcal{W}^{1,1}(0, \infty) \),

\[
(T_{u,b}f)(t) \approx (R_u f)(t) + (P_{u,b} f)(t), \quad t \in (0, \infty),
\]

where the supremal operator \( R_u \) and the weighted Hardy operator \( P_{u,b} \) are defined for \( h \in \mathcal{W}^{1,1}(0, \infty) \) and \( t \in (0, \infty) \) by

\[
(R_u h)(t) = \sup_{t \leq \tau} u(\tau) h(\tau),
\]

\[
(P_{u,b}h)(t) = \frac{u(t)}{B(t)} \int_0^t h(\tau) b(\tau) \, d\tau,
\]

respectively.

Recall that the boundedness of \( R_u \) from \( L^p(v) \) into \( L^q(w) \) on the cone of monotone non-increasing functions, that is, the validity of the inequality

\[
\|R_u f\|_{L^q(w)} \leq C \|f\|_{L^p(v)}, \quad f \in \mathcal{W}^{1,1}(0, \infty)
\]

was completely characterized in [23] in the case \( 0 < p \leq q < \infty \). In the case \( 0 < q < p < \infty \), [23] provides solution when \( u \) is equivalent to a non-decreasing function on \( (0, \infty) \). The complete solution of inequality (1.3) using a certain reduction method was presented in [21]. Another solution of (1.3) was obtained in [31].

Note that inequality

\[
\|P_{u,b} f\|_{L^q(w,0,\infty)} \leq c \|f\|_{L^p(v,0,\infty)}, \quad f \in \mathcal{W}^{1,1}(0, \infty)
\]

was considered by many authors and there exist several characterizations of this inequality (see, papers [5, 8, 9, 19, 20, 26]).

The complete characterizations of inequality

\[
\|T_{u,b} f\|_{L^q(w,0,\infty)} \leq C \|f\|_{L^p(v,0,\infty)}, \quad f \in \mathcal{W}^{1,1}(0, \infty)
\]

for \( 0 < q \leq \infty \), \( 0 < p \leq \infty \) were given in [21] and [34]. Inequality (1.5) was characterized in [23, Theorem 3.5] under condition (1.1). Note that the case when \( 0 < p \leq 1 < q < \infty \) was not considered in [23]. It is also worth to mention that in the case when \( 1 < p < \infty \), \( 0 < q < p < \infty \), \( q \neq 1 \) [23, Theorem 3.5] contains only discrete condition. In [25] the new reduction theorem was obtained when \( 0 < p \leq 1 \), and this technique allowed to characterize inequality (1.5)
when \( b = 1 \), and in the case when \( 0 < q < p \leq 1 \), [25] contains only discrete condition. Using the results in [41–44], another characterization of (1.5) was obtained in [50] and [46].

In this paper we investigate the boundedness of \( T_{u,b} \) and \( T_{u,b}^* \) from the weighted Lebesgue spaces \( L^p(v) \) into the weighted Cesàro spaces \( \text{Ces}_q(w,a) \), when \( 1 < p, q < \infty \) (see, Theorems 3.1 and 3.3). These results allow us to obtain the characterization of the boundedness of \( R_u \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions (see, Theorem 4.1). For the convenience of the reader, we formulate the statement on the boundedness of \( P_{u,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions (see, Theorem 5.1). In view of (1.2), we are able to characterize the boundedness of \( T_{u,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions (see, Theorem 6.1). At the end of the paper, as an application of obtained results, we calculate the norm of the fractional maximal function \( M_f \) from \( L^p(v) \) into \( \Gamma_q(w) \).

The paper is organized as follows. We start with formulations of ”an integration by parts” formula in Section 2. The boundedness results for \( T_{u,b} \) and \( T_{u,b}^* \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) are presented in Section 3. The characterizations of the boundedness of \( R_u, P_{u,b} \) and \( T_{u,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions are given in Sections 4, 5 and 6, respectively. Finally, the obtained in previous sections results are applied to calculate the norm of the operator \( M_f : \Lambda^p(v) \rightarrow \Gamma_q(w) \) in Section 7.

2. ”AN INTEGRATION BY PARTS” FORMULA

We recall the following ”an integration by parts” formula. For the convenience of the reader we give the proof here (cf. [49, Lemma, p. 176]).

**Theorem 2.1.** Let \( \alpha > 0 \). Let \( g \) be a non-negative function on \((0,\infty)\) such that \( 0 < \int_0^\infty g < \infty \), \( t > 0 \) and let \( f \) be a non-negative non-increasing right-continuous function on \((0,\infty)\). Then

\[
A_1 := \int_0^\infty \left( \int_0^t g \right)^\alpha g(t)[f(t) - \lim_{t \to +\infty} f(t)] dt < \infty \quad \iff \quad A_2 := \int_{(0,\infty)} \left( \int_0^t g \right)^{\alpha+1} d[-f(t)] < \infty.
\]

Moreover, \( A_1 \approx A_2 \).

**Proof.** Assume at first that \( \lim_{t \to +\infty} f(t) = 0 \). Let

\[
A_1 = \int_0^\infty \left( \int_0^t g \right)^\alpha g(t)f(t) dt < \infty.
\]

Then

\[
\int_0^x \left( \int_0^t g \right)^\alpha g(t)f(t) dt \to 0, \quad \text{as} \quad x \to 0+.
\]

Since

\[
\int_0^x \left( \int_0^t g \right)^\alpha g(t)f(t) dt \geq f(x) \int_0^x \left( \int_0^t g \right)^\alpha g(t) dt \approx f(x) \left( \int_0^x g \right)^{\alpha+1}, \quad x > 0,
\]

we have that

\[
f(x) \left( \int_0^x g \right)^{\alpha+1} \to 0, \quad \text{as} \quad x \to 0+.
\]

Integrating by parts, we get that

\[
A_2 = \int_{(0,\infty)} \left( \int_0^t g \right)^{\alpha+1} d[-f(t)] = -f(t) \left( \int_0^t g \right)^{\alpha+1} \bigg|_0^\infty + \int_{(0,\infty)} f(t) d\left( \int_0^t g \right)^{\alpha+1}
\]

\[
= \lim_{t \to 0+} f(t) \left( \int_0^t g \right)^{\alpha+1} - \lim_{t \to +\infty} f(t) \left( \int_0^t g \right)^{\alpha+1} + (\alpha + 1) \int_0^\infty \left( \int_0^t g \right)^\alpha g(t)f(t) dt \n\]

\[
\leq (\alpha + 1) \int_0^\infty \left( \int_0^t g \right)^\alpha g(t)f(t) dt = (\alpha + 1)A_1.
\]

Thus

\[
A_2 \leq A_1.
\]
Now assume that
\[ A_2 := \int_{(0, \infty)} \left( \int_0^x \right)^{\alpha+1} g(t) dt \left[-f(t)\right] < \infty. \]
Then
\[ \int_{[1, \infty)} \left( \int_0^x \right)^{\alpha+1} g(t) dt \left[-f(t)\right] \to 0, \quad \text{as } x \to +\infty. \]
Since
\[ \int_{[1, \infty)} \left( \int_0^x \right)^{\alpha+1} g(t) dt \left[-f(t)\right] \geq \int_0^x \left( \int_0^x \right)^{\alpha+1} g(t) dt \left[-f(t)\right] \]
\[ = \left( \int_0^x \right)^{\alpha+1} \left[ f(x) - \lim_{x \to +\infty} f(x) \right] = f(x) \left( \int_0^x \right)^{\alpha+1}, \quad x > 0, \]
we obtain that
\[ f(x) \left( \int_0^x \right)^{\alpha+1} \to 0, \quad \text{as } x \to +\infty. \]
Thus, integrating by parts, we get that
\[ A_1 = \int_0^\infty \left( \int_0^x \right)^{\alpha+1} g(t) f(t) dt \approx \int_0^\infty f(t) d\left( \int_0^x \right)^{\alpha+1} \]
\[ = f(t) \left( \int_0^x \right)^{\alpha+1} \bigg|_0^\infty + \int_0^\infty f(t) \left( \int_0^x \right)^{\alpha+1} dt \left[-f(t)\right] \]
\[ = \lim_{t \to \infty} f(t) \left( \int_0^x \right)^{\alpha+1} - \lim_{t \to 0+} f(t) \left( \int_0^x \right)^{\alpha+1} + \int_0^\infty \left( \int_0^x \right)^{\alpha+1} dt \left[-f(t)\right] \]
\[ \leq \int_0^\infty \left( \int_0^x \right)^{\alpha+1} dt \left[-f(t)\right] = A_2. \]
Hence
\[ A_1 \leq A_2. \]
We have shown that if \( \lim_{x \to +\infty} f(x) = 0 \), then
\[ A_1 < \infty \iff A_2 < \infty, \]
and
\[ A_1 \approx A_2. \]
Now assume that \( \lim_{x \to +\infty} f(x) > 0 \). Then, applying previous statement to the function \( f(x) - \lim_{x \to +\infty} f(x) \), we arrive at
\[ \int_0^\infty \left( \int_0^x \right)^{\alpha+1} g(t) \left[ f(t) - \lim_{x \to +\infty} f(x) \right] dt \approx \int_{(0, \infty)} \left( \int_0^x \right)^{\alpha+1} g(t) dt \left[-f(t)\right]. \]
The proof is completed. \( \square \)

**Remark 2.2.** Note that if \( f \in \mathcal{M}^{\alpha+1}(0, \infty) \) is such that \( \lim_{x \to +\infty} f(x) > 0 \), then
\[ \int_0^\infty \left( \int_0^x \right)^\alpha g(t) f(t) dt < \infty \quad \implies \quad \int_0^\infty g(x) dx < \infty. \]
Indeed: for each \( x \in (0, \infty) \)
\[ \int_0^\infty \left( \int_0^x \right)^\alpha g(t) f(t) dt \leq \int_0^x \left( \int_0^x \right)^\alpha g(t) f(t) dt \]
\[ \geq f(x) \int_0^x \left( \int_0^x \right)^\alpha g(t) dt \approx f(x) \left( \int_0^x \right)^{\alpha+1} \]
holds. Thus
\[ \lim_{x \to +\infty} f(x) \cdot \left( \int_0^x \right)^{\alpha+1} \leq f(x) \left( \int_0^x \right)^{\alpha+1} \leq \int_0^\infty \left( \int_0^x \right)^\alpha g(t) f(t) dt < \infty. \]
Hence
\[
\lim_{x \to +\infty} f(x) \cdot \left( \int_0^\infty g \right)^{\alpha+1} < \infty.
\]
Therefore
\[
\int_0^\infty g < \infty.
\]

**Corollary 2.3.** Let \( \alpha > 0 \). Let \( g \) be a non-negative function on \((0, \infty)\) such that \( 0 < \int_1^\infty g < \infty \), \( t > 0 \) and let \( f \) be a non-negative non-decreasing right-continuous function on \((0, \infty)\). Then
\[
\int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt \approx \int_0^\infty \left( \int_0^\infty g \right)^{\alpha+1} d[-f(t)] + \lim_{x \to +\infty} f(x) \cdot \left( \int_0^\infty g \right)^{\alpha+1}.
\]

**Proof.** If \( \lim_{x \to +\infty} f(x) = 0 \), then the statement follows by Theorem 2.1. If \( \lim_{x \to +\infty} f(x) > 0 \), then by Remark 2.2, we know that
\[
\int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt < \infty \quad \Rightarrow \quad \int_0^\infty g(x) dx < \infty.
\]
Therefore, by Theorem 2.1, we get that
\[
\int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt = \int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt + \lim_{x \to +\infty} f(x) \cdot \int_0^\infty \left( \int_0^\infty g \right)^{\alpha+1} d[-f(t)] + \lim_{x \to +\infty} f(x) \cdot \left( \int_0^\infty g \right)^{\alpha+1}.
\]
The proof is completed. \( \square \)

**Theorem 2.4.** Let \( \alpha > 0 \). Let \( g \) be a non-negative function on \((0, \infty)\) such that \( 0 < \int_1^\infty g < \infty \), \( t > 0 \) and let \( f \) be a non-negative non-decreasing left-continuous function on \((0, \infty)\). Then
\[
B_1 := \int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt < \infty \quad \Leftrightarrow \quad B_2 := \int_0^\infty \left( \int_0^\infty g \right)^{\alpha+1} d[f(t)] < \infty.
\]
Moreover, \( B_1 \approx B_2 \).

**Proof.** Assume at first that \( f(0+) = 0 \). Let
\[
B_1 := \int_0^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt < \infty.
\]
Then
\[
\int_x^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) g(t) dt \to 0, \quad \text{as} \quad x \to \infty.
\]
Since
\[
\int_x^\infty \left( \int_0^\infty g \right)^{\alpha} f(t) dt \geq f(x) \int_x^\infty \left( \int_0^\infty g \right)^{\alpha} g(t) dt \approx f(x) \left( \int_0^\infty g \right)^{\alpha+1}, \quad x > 0,
\]
we have that
\[
f(x) \left( \int_0^\infty g \right)^{\alpha+1} \to 0, \quad \text{as} \quad x \to \infty.
\]
Hence, integrating by parts, we get that
\[
B_2 = \int_0^\infty \left( \int_0^\infty g \right)^{\alpha+1} d[f(t)] = f(t) \left( \int_0^\infty g \right)^{\alpha+1} \left|_0^\infty \right. - \int_0^\infty f(t) \left( \int_0^\infty g \right)^{\alpha+1} dt
\]
\[
= \lim_{t \to \infty} f(t) \left( \int_0^\infty g \right)^{\alpha+1} - \lim_{t \to 0^+} f(t) \left( \int_0^\infty g \right)^{\alpha+1} + (\alpha+1) \int_0^\infty \left( \int_0^\infty g \right)^{\alpha} g(t) f(t) dt
\]
\[
\leq (\alpha+1) \int_0^\infty \left( \int_0^\infty g \right)^{\alpha} g(t) f(t) dt = (\alpha+1)B_1.
\]
Now assume that
\[
B_2 := \int_0^\infty \left( \int_0^\infty g \right)^{\alpha+1} d[f(t)] < \infty.
\]
Then
\[
\int_{[0,x]} \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)] \to 0, \quad \text{as } x \to 0^+.
\]

Since
\[
\int_{[0,x]} \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)] \geq \left( \int_x^\infty g \right)^{\alpha+1} \int_{[0,x]} d[f(t)]
\]
\[
= \left( \int_x^\infty g \right)^{\alpha+1} [f(x) - f(0^+)] = f(x) \left( \int_x^\infty g \right)^{\alpha+1}, \quad x > 0,
\]
we obtain that
\[
f(x) \left( \int_x^\infty g \right)^{\alpha+1} \to 0, \quad \text{as } x \to 0^+.
\]

Thus, integrating by parts, we get that
\[
B_1 = \int_0^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) f(t) dt \approx \int_0^\infty f(t) d \left[ - \left( \int_t^\infty g \right)^{\alpha+1} \right]
\]
\[
= -f(t) \left( \int_t^\infty g \right)^{\alpha+1} \Big|_0^\infty + \int_0^\infty \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)]
\]
\[
= \lim_{t \to 0^+} f(t) \left( \int_t^\infty g \right)^{\alpha+1} - \lim_{t \to 0^+} f(t) \left( \int_t^\infty g \right)^{\alpha+1} + \int_0^\infty \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)]
\]
\[
\leq \int_0^\infty \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)] = B_2.
\]

We have shown that if \( f(0^+) = 0 \), then
\[
B_1 < \infty \quad \iff \quad B_2 < \infty,
\]
and
\[
B_1 \approx B_2.
\]

Now assume that \( f(0^+) > 0 \). Then, applying previous statement to the function \( f(x) - f(0^+) \), we arrive at
\[
\int_0^\infty \left( \int_t^\infty g \right)^{\alpha} [f(t) - f(0^+)] dt \approx \int_{(0,\infty)} \left( \int_t^\infty g \right)^{\alpha+1} d[f(t)].
\]

The proof is completed. \( \square \)

**Remark 2.5.** Note that if \( f \) is a non-negative non-decreasing function on \((0,\infty)\) such that \( f(0^+) > 0 \), then
\[
\int_0^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) f(t) dt < \infty \quad \implies \quad \int_0^\infty g(t) dt < \infty.
\]

Indeed: for each \( x \in (0,\infty) \)
\[
\infty > \int_0^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) f(t) dt \geq \int_x^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) f(t) dt
\]
\[
\geq f(x) \int_x^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) dt = f(x) \left( \int_x^\infty g \right)^{\alpha+1}
\]
holds. Thus
\[
f(0^+) \left( \int_x^\infty g \right)^{\alpha+1} \leq f(x) \left( \int_x^\infty g \right)^{\alpha+1} \leq \int_0^\infty \left( \int_t^\infty g \right)^{\alpha} g(t) f(t) dt < \infty.
\]

Hence
\[
f(0^+) \left( \int_0^\infty g \right)^{\alpha+1} < \infty.
\]

Therefore
\[
\int_0^\infty g < \infty.
\]
Corollary 2.6. Let \( \alpha > 0 \). Let \( g \) be a non-negative function on \((0, \infty)\) such that \( 0 < \int_0^\infty g < \infty \), \( t > 0 \) and let \( f \) be a non-negative non-decreasing left-continuous function on \((0, \infty)\). Then

\[
\int_0^\infty \left( \int_0^\infty g(t) \, dt \right)^\alpha g(t) \, dt \approx \int_0^{\infty} \left( \int_0^\infty g(t) \, dt \right)^{\alpha + 1} \, f(t) + f(0+) \left( \int_0^\infty g \, dt \right)^{\alpha + 1}.
\]

Proof. If \( f(0+) = 0 \), then the statement follows by Theorem 2.4. If \( f(0+) > 0 \), then by Remark 2.5, we know that

\[
\int_0^\infty \left( \int_0^\infty g(t) \, dt \right)^\alpha g(t) \, dt < \infty \quad \Rightarrow \quad \int_0^\infty g(x) \, dx < \infty.
\]

Therefore, by Theorem 2.4, we get that

\[
\int_0^\infty \left( \int_0^\infty g(t) \, dt \right)^\alpha g(t) \, dt = \int_0^\infty \left( \int_0^\infty g(t) \, dt \right)^\alpha g(t) \, dt \approx \int_0^{\infty} \left( \int_0^\infty g(t) \, dt \right)^{\alpha + 1} \, f(t) + f(0+) \left( \int_0^\infty g \, dt \right)^{\alpha + 1}.
\]

The proof is completed.

3. The boundedness of \( T_{u,b} \) and \( T_{u,b}^* \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \)

In this section we give solutions of the following two inequalities

\[
\left( \int_0^\infty \left( \int_0^\infty \frac{u(t)}{B(\tau)} \, dt \right)^\alpha \int_0^\tau h(y)b(y) \, dy \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+(0, \infty) \tag{3.1}
\]

and

\[
\left( \int_0^\infty \left( \int_0^\infty \frac{u(t)}{B(\tau)} \, dt \right)^\alpha \int_0^\tau h(y)b(y) \, dy \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+(0, \infty) \tag{3.2}
\]

where \( 1 < p \leq q < \infty \) and \( a, u, v, w \in \mathcal{W}(0, \infty) \). Using the duality argument, we reduce the problem to the boundedness for the dual of integral Volterra operator with a kernel satisfying Oinarovs condition and weighted Stieltjes operator.

Note that the characterization of inequalities

\[
\left( \int_0^\infty \left( \int_0^\infty \frac{u(t)}{B(\tau)} \, dt \right)^\alpha \int_0^\tau h(y)b(y) \, dy \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+(0, \infty) \tag{3.3}
\]

and

\[
\left( \int_0^\infty \left( \int_0^\infty \frac{u(t)}{B(\tau)} \, dt \right)^\alpha \int_0^\tau h(y)b(y) \, dy \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+(0, \infty) \tag{3.4}
\]

can be reduced to the solutions of (3.1) and (3.2).

Recall that, if \( F \) is a non-negative non-decreasing function on \((0, \infty)\), then

\[
\text{ess sup}_{t \in (0, \infty)} F(t)G(t) = \text{ess sup}_{t \in (0, \infty)} F(t) \text{ess sup}_{\tau \in (t, \infty)} G(\tau),
\]

likewise, when \( F \) is a non-negative non-increasing function on \((0, \infty)\), then

\[
\text{ess sup}_{t \in (0, \infty)} F(t)G(t) = \text{ess sup}_{t \in (0, \infty)} F(t) \text{ess sup}_{\tau \in (0,t)} G(\tau)
\]

(see, for instance, [24, p. 85]).

We need the following notations:

\[
A(t) := \int_0^t a(s) \, ds, \quad U(t) := \int_0^t u(s) \, ds, \quad W(t) := \int_0^t w(s) \, ds.
\]
Theorem 3.1. Let $1 < p, q < \infty$. Assume that $u \in W(0, \infty) \cap C(0, \infty)$ and $a, v, w \in W(0, \infty)$. Moreover, assume that

$$0 < \int_0^x t^{1-p'} \, dt < \infty \quad \text{for all} \quad x > 0.$$

(i) If $p \leq q$, then

$$\sup_{h \geq 0} \left( \int_0^\infty \left( \frac{\sup_{s \leq t} u(\tau) \int_0^\tau h(y) \, dy}{w(x)} \right)^q \right) \frac{1}{p} \left( \int_0^\infty h(s) v(s) \, ds \right)^{\frac{1}{p}}$$

$$\approx \sup_{t \in (0, \infty)} \left( \int_0^t v(x)^{1-p'} \left( \int_0^\infty \left( \sup_{s \leq t} u(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_t^\infty \frac{w(y) \, dy}{y^q} \right)^{\frac{1}{q}}$$

$$+ \sup_{t \in (0, \infty)} \left( \int_0^t v(x)^{1-p'} \, dx \right)^{\frac{1}{p'}} \left( \int_t^\infty \left( \sup_{s \leq t} u(\tau) \right) a(s) \, ds \right)^q \frac{1}{q} \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

$$+ \sup_{x \in (0, \infty)} \left( \int_{[x, \infty)} A(t) v(t) \frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

(ii) If $q < p$, then

$$\sup_{h \geq 0} \left( \int_0^\infty \left( \frac{\sup_{\tau \leq t} u(\tau) \int_0^\tau h(y) \, dy}{w(x)} \right)^q \right) \frac{1}{p} \left( \int_0^\infty h(s) v(s) \, ds \right)^{\frac{1}{p}}$$

$$\approx \int_0^\infty \left( \int_0^\infty \frac{v(x)^{1-p'} \, dx}{x^q} \right)^{\frac{1}{p'}} \frac{1}{p} \left( \int_0^\infty \left( \sup_{s \leq t} u(\tau) \right) a(s) \, ds \right)^q \frac{1}{q} \left( \int_0^\infty w(z) \, dz \right)^{\frac{1}{q}} \frac{1}{x^p} \, dx$$

$$+ \left( \int_0^\infty \left( \int_0^\infty \frac{v(x)^{1-p'} \, dx}{x^q} \right)^{\frac{1}{p'}} \frac{1}{p} \left( \int_0^\infty \left( \sup_{s \leq t} u(\tau) \right) a(s) \, ds \right)^q \frac{1}{q} \left( \int_0^\infty w(z) \, dz \right)^{\frac{1}{q}} \frac{1}{x^p} \, dx$$

$$+ \left( \int_0^\infty \left( \int_{[x, \infty)} A(t) v(t) \frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_{[0, x]} A(t) v(t) \frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_{[0, x]} A(t) v(t) \frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_{[0, x]} A(t) v(t) \frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

$$\frac{1}{p} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_0^\tau \int_0^\tau h(y) \, dy \right) \frac{1}{p'} \right) \right)A(\tau)^q w(\tau) \, d\tau \left( \int_0^\infty w(y) \, dy \right)^{\frac{1}{q}}$$

Proof. Assume that $1 < p \leq q < \infty$. By duality, using Fubini’s Theorem, and interchanging the suprema, we get that

$$\sup_{h \geq 0} \frac{1}{p} \left( \int_0^\infty h(s) v(s) \, ds \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{w(x) \, dx}{x^q} \right)^{\frac{1}{q}}$$

$$= \sup_{h \geq 0} \frac{1}{p} \left( \int_0^\infty h(s) v(s) \, ds \right)^{\frac{1}{p}} \sup_{g \geq 0} \frac{1}{g} \left( \int_0^\infty g(x) v(x)^{1-p'} \, dx \right)^{\frac{1}{p'}}$$
Similarly, integrating by parts (applying Corollary 4.4), on using (3.5), we arrive at

\[
\frac{\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq \tau} u(\tau) \int_{0}^{t} h(y) dy \right) \left( \int_{0}^{\infty} g(x) dx \right) \left( a(t) dt \right)^{1-p} \right)}{\left( \int_{0}^{\infty} h(s)^p v(s) ds \right)^{1/p}} \approx D + E,
\]

where

\[
D := \left( \int_{0}^{\infty} \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} v(s)^{1-p'} ds \right) \right) \left( \int_{t}^{\infty} g(x) dx \right) a(t) dt \right)^{1/p},
\]

\[
E := \left( \int_{0}^{\infty} \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} v(s)^{1-p'} ds \right) \right) \left( \int_{t}^{\infty} g(x) dx \right) a(t) dt \right)^{1/p}.
\]

Integrating by parts (applying Corollary 2.6), on using Fubini’s Theorem, we arrive at

\[
D \approx \left( \int_{0}^{\infty} \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} v(s)^{1-p'} ds \right) a(t)^{1-p'} dt \right)^{1/p}
\]

\[
= \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} v(s)^{1-p'} ds \right) a(t)^{1-p'} dt \right)^{1/p}.
\]

Similarly, integrating by parts (applying Corollary 2.3), on using Fubini’s Theorem, we get at

\[
E \approx \left( \int_{0}^{\infty} \left( \sup_{t \leq \tau} u(\tau) \left( \int_{0}^{t} v(s)^{1-p'} ds \right) \right) \left( \int_{t}^{\infty} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{1-p'} dt \right)^{1/p}
\]

\[
= \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{p'} \left( \int_{t}^{\infty} \left( \sup_{s \leq \tau} u(\tau) \int_{0}^{s} g(x) dx \right) a(s) ds \right)^{1-p'} dt \right)^{1/p}.
\]

(i) Let \( p \leq q \). By [35, Theorem 1.1], we obtain that

\[
\sup_{g \geq 0} \left( \int_{0}^{\infty} g^{q'} w^{1-q'} \right)^{1/q}
\]

\[
= \sup_{g \geq 0} \left( \int_{0}^{\infty} g^{q'} w^{1-q'} \right)^{1/q} \left( \int_{0}^{\infty} \left( \sup_{y \leq \tau} u(y) \int_{0}^{y} a(s) ds \right)^{p'} \left( \int_{0}^{\infty} v(t)^{1-p'} dt \right)^{1/p} \right)
\]

\[
= \sup_{r \in (0, \infty)} \left( \int_{0}^{\infty} v(t)^{1-p'} dt \right)^{1/p} \left( \int_{0}^{\infty} \left( \sup_{y \leq \tau} u(y) \int_{0}^{y} a(s) ds \right)^{q} w(y) dy \right)^{1/q}
\]

\[
+ \sup_{r \in (0, \infty)} \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \sup_{y \leq \tau} u(y) \int_{0}^{y} a(s) ds \right)^{q} w(z) dz \right)^{1/q}
\]

\[
= \sup_{r \in (0, \infty)} \left( \int_{0}^{\infty} \left( \sup_{y \leq \tau} u(y) \int_{0}^{y} a(s) ds \right)^{q} \right)^{1/q} \left( \int_{0}^{\infty} w(z) dz \right)^{1/q}.
\]
By [33, Theorem 1, p. 40 and Theorem 3, p. 44], respectively, we have that

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q \, E_1 \right) \frac{1}{x} = \sup_{g \geq 0} \left( \int_0^\infty g^q \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}}
\]

\[
= \sup_{g \geq 0} \left( \int_0^\infty g^q \, A(x) \, dx \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}}
\]

\[
\approx \sup_{x \in (0, \infty)} \left( \int_0^x A(t) \, dt \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}} \left( \int_0^\infty A(y)^p w(y) \, dy \right)^\frac{1}{x}
\]

and

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q \, E_2 \right) \frac{1}{x} = \sup_{g \geq 0} \left( \int_0^\infty g^q \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}}
\]

\[
= \sup_{g \geq 0} \left( \int_0^\infty g^q \, A(x) \, dx \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}}
\]

\[
\approx \sup_{x \in (0, \infty)} \left( \int_0^x A(t) \, dt \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}} \left( \int_0^\infty w(y) \, dy \right)^\frac{1}{x}.
\]

By duality, we have that

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q \, E_3 \right) \frac{1}{x} = \sup_{g \geq 0} \left( \int_0^\infty g^q \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}}
\]

\[
= \sup_{g \geq 0} \left( \int_0^\infty g^q \, A(x) \, dx \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}} \lim_{t \to \infty} \left( \sup_{r \leq t} \left( \int_0^r v(s)^{1-p'} \, ds \right)^\frac{1}{p} \left( \int_0^\infty A(y)^p w(y) \, dy \right)^\frac{1}{x} \right)
\]

\[
= \left( \int_0^\infty A(y)^p w(y) \, dy \right)^\frac{1}{x} \lim_{t \to \infty} \left( \sup_{r \leq t} \left( \int_0^r v(s)^{1-p'} \, ds \right)^\frac{1}{p} \right).
\]

Thus, we get that

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q \, E \right) \frac{1}{x} \approx \sup_{x \in (0, \infty)} \left( \int_0^x A(t) \, dt \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}} \left( \int_0^\infty A(y)^p w(y) \, dy \right)^\frac{1}{x}
\]

\[
+ \sup_{x \in (0, \infty)} \left( \int_0^x A(t) \, dt \right) \frac{1}{x} \frac{1}{(\int_0^\infty g^q \, w_1 \, w_1')^\frac{1}{x}} \left( \int_0^\infty w(y) \, dy \right)^\frac{1}{x}
\]

\[
+ \left( \int_0^\infty A(y)^p w(y) \, dy \right)^\frac{1}{x} \lim_{t \to \infty} \left( \sup_{r \leq t} \left( \int_0^r v(s)^{1-p'} \, ds \right)^\frac{1}{p} \right).
\]

(3.8)
Combining (3.7) and (3.8), we arrive at

\[
\left( \sup_{h \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \int_0^\tau h(y)dy \right)^q w(x)d\tau \right)^\frac{1}{q} \right)^{\frac{1}{p}} \left( \int_0^\infty h(s)^p v(s)ds \right)^{\frac{1}{p}} \approx \sup_{t \in (0, \infty)} \left( \int_0^\infty v(x)^{1-p'} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \right)^{p'} v(s)ds \right)^{q/p'} dx \right)^{\frac{1}{p'}} \left( \int_0^\infty w(y)dy \right)^{\frac{1}{q'}}
\]

\[
+ \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \right)^{p'} v(s)ds \right)^q w(y)dy \right)^{\frac{1}{q'}}
\]

\[
+ \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \right) v(s)ds \right)^{q/p'} dx \right)^{\frac{1}{p'}} \left( \int_0^\infty A(y)^q w(y)dy \right)^{\frac{1}{q'}}
\]

\[
+ \left( \int_0^\infty A(y)^q w(y)dy \right)^{\frac{1}{q'}} \lim_{t \to \infty} \left( \int_0^\infty v(s)^{1-p'} ds \right)^{\frac{1}{q'}}.
\]

(ii) Let now \( q > p \). By [35, Theorem 1.2], we obtain that

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q w^{1-q'} \right)^{\frac{1}{p'}} = \sup_{g \geq 0} \left( \int_0^\infty g^{q} \int_0^x \left( \sup_{s \leq y} u(y) \right) a(s)ds dx \right)^{\frac{1}{p'}} \left( \int_0^\infty w^{1-q'} \right)^{\frac{1}{p'}}
\]

\[
\approx \left( \int_0^\infty \left( \int_0^\infty g^q \int_0^x \left( \sup_{s \leq y} u(y) \right) a(s)ds dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{q'}} \left( \int_0^\infty w^{1-q'} \right)^{\frac{1}{p'}} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \right)^{p'} v(s)ds \right)^q w(z)dz \right)^{\frac{1}{q'}} dt
\]

\[
+ \left( \int_0^\infty \left( \int_0^\infty g^q \int_0^x \left( \sup_{s \leq y} u(y) \right) a(s)ds dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{q'}} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{t \leq \tau} u(t) \right)^{p'} v(s)ds \right)^q w(z)dz \right)^{\frac{1}{q'}} dt.
\]

By [33, Theorem 2, p. 48], we have that

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q w^{1-q'} \right)^{\frac{1}{p'}} = \sup_{g \geq 0} \left( \int_0^\infty g^{q} \left( \int_0^x \left( \sup_{s \leq y} A(x) \right) dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{q'}} \left( \int_0^\infty w^{1-q'} \right)^{\frac{1}{p'}}
\]

\[
= \sup_{g \geq 0} \left( \int_0^\infty g^{q} \left( \int_0^x \left( \sup_{s \leq y} A(x) \right) dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{q'}} \left( \int_0^\infty w^{1-q'} \right)^{\frac{1}{p'}}
\]

\[
\approx \left( \int_0^\infty \left( \int_0^x \left( \sup_{s \leq y} A(x) \right) dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{q'}} \left( \int_0^\infty A(y)^q w(y)dy \right)^{\frac{1}{q'}} \left( \int_0^\infty A(x)^q w(x)dx \right)^{\frac{1}{q'}}
\]

and

\[
\sup_{g \geq 0} \left( \int_0^\infty g^q w^{1-q'} \right)^{\frac{1}{p'}}
\]
\[
\text{Let Theorem 3.2. (i)}
\]

Consequently, we arrive at

\[
\sup_{g \geq 0} \left( \int_0^\infty g' w^{1-q} \right)^{\frac{1}{q'}} \\
\approx \left( \int_0^\infty \left( \int_{[0,1]} A(t)^{\nu'} d\left( -\left( \sup_{t \leq \tau} u(\tau)^{\nu'} \left( \int_0^\tau v(s)^{1-\nu'} ds \right) \right) \right) \right)^{\frac{1}{q'}} \left( \int_0^\infty w(y) dy \right)^{\frac{1}{q'}} \left( \int_0^\infty w(x) dx \right)^{\frac{1}{q'}}.
\]

Combining (3.9) and (3.10), we arrive at

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)^{\nu} \int_0^\tau h(y) dy \right) a(t) dt \right)^q w(x) dx \\
= \sup_{g \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)^{\nu} \left( \int_0^\tau v(s)^{1-\nu'} ds \right) \right)^{\frac{1}{q'}} \left( \int_0^\infty g' w^{1-q} \right)^{\frac{1}{q'}} \\
\approx \left( \int_0^\infty \left( \int_{[0,1]} A(t)^{\nu'} d\left( -\left( \sup_{t \leq \tau} u(\tau)^{\nu'} \left( \int_0^\tau v(s)^{1-\nu'} ds \right) \right) \right) \right)^{\frac{1}{q'}} \left( \int_0^\infty w(y) dy \right)^{\frac{1}{q'}} \left( \int_0^\infty w(x) dx \right)^{\frac{1}{q'}}.
\]

The proof is completed. \qed

**Theorem 3.2.** Let \(1 < p, q < \infty\) and \(b \in W(0, \infty)\) be such that \(b(t) > 0\) for a.e. \(t \in (0, \infty)\). Assume that \(u \in W(0, \infty) \cap C(0, \infty)\) and \(a, v, w \in W(0, \infty)\). Moreover, assume that

\[
0 < \int_0^x v(t)^{1-\nu'} dt < \infty \quad \text{for all} \quad x > 0.
\]

(i) If \(p \leq q\), then

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \int_0^\tau h(y) dy \right) a(t) dt \right)^q w(x) dx \\
\approx \sup_{r \in (0, \infty)} \left( \int_0^r b(x)^{\nu'} v(x)^{1-\nu'} \left( \int_x^r \sup_{s \leq \tau} \frac{u(\tau)}{B(\tau)} a(s) ds \right)^{\nu'} dx \right)^{\frac{1}{q'}} \left( \int_0^\infty w(y) dy \right)^{\frac{1}{q'}}.
\]
Theorem 3.3.

(i) If \( q \geq 0 \), then

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \int_0^x b(x)^p v(x)^{1-p'} \, dx \right)^{p'} \left( \int_t^\infty \left( \int_s^{s+y} \frac{u(t)}{B(s)} a(s) \, ds \right)^q w(y) \, dy \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} = \sup_{h \geq 0} \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p'}}.
\]

(ii) If \( q < p \), then

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \int_0^x b(x)^p v(x)^{1-p'} \, dx \right)^{p'} \left( \int_t^\infty \left( \int_s^{s+y} \frac{u(t)}{B(s)} a(s) \, ds \right)^q w(y) \, dy \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} = \sup_{h \geq 0} \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p'}}.
\]

Proof. The statement follows by Theorem 3.1 at once if we note that

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \int_0^x b(x)^p v(x)^{1-p'} \, dx \right)^{p'} \left( \int_t^\infty \left( \int_s^{s+y} \frac{u(t)}{B(s)} a(s) \, ds \right)^q w(y) \, dy \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} = \sup_{h \geq 0} \left( \int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p'}}.
\]

\[\square\]

Theorem 3.3. Let \( 1 < p, q < \infty \) and \( b \in \mathcal{W}(0, \infty) \) be such that \( b(t) > 0 \) for a.e. \( t \in (0, \infty) \). Assume that \( u \in \mathcal{W}(0, \infty) \cap C(0, \infty) \) and \( a, v, w \in \mathcal{W}(0, \infty) \). Moreover, assume that \( 0 < \int_x^\infty v(t)^{1-p'} \, dt < \infty \) for all \( x > 0 \).

Denote by

\[
\psi(x) := \left( \int_x^\infty b(t)^p v^{1-p'}(t) \, dt \right)^{-\frac{p'}{p-1}} b(x)^p v^{1-p'}(x).
\]
Proof. If $q \leq p$, then

\[
\Psi(x) := \left( \int_{x}^{\infty} b(t)^{q} \nu_{1}^{-q}(t) dt \right)^{\frac{1}{q}}.
\]

(i) If $p \leq q$, then

\[
\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{q} w(x) dx \right)^{\frac{1}{q}}
\]

\[
\approx \sup_{\tau \in (0,\infty)} \left( \int_{0}^{\tau} \Psi(x)^{-p} \psi(x) \left( \int_{x}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{p} \left( \int_{x}^{\infty} w(y)dy \right)^{\frac{1}{p}} \left( \int_{x}^{\infty} (A(y)^{p}w(y)dy) \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{q} \right) \right) w(x) dx \right)^{\frac{1}{q}}.
\]

(ii) If $q < p$, then

\[
\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{q} w(x) dx \right)^{\frac{1}{q}}
\]

\[
\approx \left( \int_{0}^{\infty} \left( \int_{0}^{t} \Psi(x)^{-p} \psi(x) dx \right)^{\frac{1}{q}} \psi(t)^{-p} \psi(t) \left( \int_{0}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{q} w(z) dz \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}.
\]

Proof. By [20, Corollary 3.5], we have that

\[
\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq x \leq \frac{\tau(\psi)}{B(\tau)}} s \frac{(\int_{0}^{\infty} h(y)b(y)dy)a(t)dt}{w(x)} \right)^{q} w(x) dx \right)^{\frac{1}{q}}
\]

\[
\left( \int_{0}^{\infty} h(s)^{p} \nu(s) ds \right)^{\frac{1}{q}}
\]
\[
\begin{align*}
&= \sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \int_{t}^{\infty} h(y) dy \right) a(t) dt \right)^{q} w(x) dx \right) \frac{1}{q} \\
&\approx \sup_{h \geq 0} \left( \int_{0}^{\infty} h(s)^p b(s)^{-p} v(s) ds \right) \frac{1}{p} \\
&+ \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t) \left( \int_{0}^{\infty} h(y) dy \right)^{q} a(t) dt \right)^{\frac{1}{q}} w(x) dx \right) \frac{1}{q} \\
&+ \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t) \left( \int_{0}^{\infty} h(y) dy \right)^{q} \right)^{\frac{1}{q}} \psi(s) ds \right) \frac{1}{q}.
\end{align*}
\]

(i) Let \( p \leq q \). By Theorem 3.1, (i), we get that
\[
\begin{align*}
&\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \int_{t}^{\infty} h(y) b(y) dy \right) a(t) dt \right)^{q} w(x) dx \right) \frac{1}{q} \\
&\approx \sup_{s \in (0, \infty)} \left( \int_{0}^{\infty} \Psi(x)^{-p} \psi(x) \left( \int_{s}^{\infty} \left( \sup_{y \leq T} \frac{u(y)}{B(y)} \Psi(y)^2 a(s) ds \right)^{p} \right)^{\frac{1}{p}} \left( \int_{s}^{\infty} w(y) dy \right)^{\frac{1}{q}} \\
&+ \sup_{s \in (0, \infty)} \left( \int_{0}^{\infty} A(t)^p d \left( - \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \right) \right) \left( \int_{s}^{\infty} w(y) dy \right)^{\frac{1}{q}} \\
&+ \sup_{x \in (0, \infty)} \left( \int_{0}^{\infty} A(x)^p \right) \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \right) \right) \left( \int_{0}^{\infty} w(y) dy \right)^{\frac{1}{q}} \\
&+ \left( \int_{0}^{\infty} A(x)^p w(y) dy \right)^{\frac{1}{q}} \lim_{t \to 0} \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \left( \int_{0}^{\infty} w(y) dy \right)^{\frac{1}{q}} \\
&+ \left( \int_{0}^{\infty} \psi(s) ds \right)^{-\frac{1}{q}} \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \right) a(t) dt \right)^{q} w(x) dx \right)^{\frac{1}{q}}.
\end{align*}
\]

(ii) Let \( q < p \). By Theorem 3.1, (ii), we obtain that
\[
\begin{align*}
&\sup_{h \geq 0} \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \int_{t}^{\infty} h(y) b(y) dy \right) a(t) dt \right)^{q} w(x) dx \right) \frac{1}{q} \\
&\approx \left( \int_{0}^{\infty} \Psi(x)^{-p} \psi(x) \left( \int_{s}^{\infty} \left( \sup_{y \leq T} \frac{u(y)}{B(y)} \Psi(y)^2 a(s) ds \right)^{q} w(z) dz \right)^{\frac{1}{q}} dt \right) \frac{1}{q} \\
&+ \left( \int_{0}^{\infty} \Psi(x)^{-p} \psi(x) \left( \int_{s}^{\infty} \left( \sup_{y \leq T} \frac{u(y)}{B(y)} \Psi(y)^2 a(s) ds \right)^{p} \right)^{\frac{1}{p}} \left( \int_{s}^{\infty} w(z) dz \right)^{\frac{1}{q}} dt \right) \frac{1}{q} \\
&+ \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} d \left( - \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \right) \right) \left( \int_{0}^{\infty} w(y) dy \right)^{\frac{1}{q}} A(x)^p \right) \left( \int_{0}^{\infty} \left( \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \right) \right) \left( \int_{0}^{\infty} w(y) dy \right)^{\frac{1}{q}} w(x) dx \right)^{\frac{1}{q}} \\
&+ \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} d \left( - \sup_{t \leq T} \frac{u(t)}{B(t)} \Psi(t)^{2p} \left( \int_{0}^{\infty} \Psi(s)^{-p} \psi(s) ds \right) \right) \right) \left( \int_{0}^{\infty} w(y) dy \right)^{\frac{1}{q}} A(x)^p w(x) dx \right)^{\frac{1}{q}}.
\end{align*}
\]
\[
+ \left( \int_0^\infty A(y)^q w(y) dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \Psi(\tau)^2 \left( \int_0^\tau \Psi(s)^{-p'} \psi(s) ds \right)^{\frac{1}{p'}} \right) + \left( \int_0^\infty \psi(s) ds \right)^{\frac{1}{p'}} \left( \int_0^\infty \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \Psi(\tau)^2 \right) a(t) dt \right)^q w(x) dx \right)^{\frac{1}{q}}.
\]

The proof is completed. \qed

4. The boundedness of \( R_u \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions

In this section we characterize the boundedness of \( R_u \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions.

**Theorem 4.1.** Let \( 1 < p, q < \infty \). Assume that \( u \in \mathcal{W}(0, \infty) \cap C(0, \infty) \) and \( a, v, w \in \mathcal{W}(0, \infty) \).

(i) If \( p \leq q \), then
\[
\sup_{f \in \mathcal{W}^+_{q-1}(0, \infty)} \frac{\left( \int_0^\infty \left( \int_0^\infty (R_u f)(t) a(t) dt \right)^q w(x) dx \right)^{\frac{1}{q}}}{\left( \int_0^\infty f(s)^p v(s) ds \right)^{\frac{1}{p'}}} \\
\approx \sup_{r \in (0, \infty)} \left( \int_0^r V(x)^{p'} v(x) \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) + \sup_{r \in (0, \infty)} \left( \int_0^r V(x)^{p'} v(x) \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) + \sup_{r \in (0, \infty)} \left( \int_0^r A(t)^{p'} a \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) \int_t^\infty (w(y) dy)^{\frac{1}{q}} \right) + \left( \int_0^\infty A(y)^q w(y) dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \sup_{t \leq \tau} \left( u(\tau) V(\tau)^{-2} \left( \int_0^\tau V(s)^{p'} v(s) ds \right)^{\frac{1}{p'}} \right) \right).
\]

(ii) If \( q < p \), then
\[
\sup_{f \in \mathcal{W}^+_{q-1}(0, \infty)} \frac{\left( \int_0^\infty \left( \int_0^\infty (R_u f)(t) a(t) dt \right)^q w(x) dx \right)^{\frac{1}{q}}}{\left( \int_0^\infty f(s)^p v(s) ds \right)^{\frac{1}{p'}}} \\
\approx \left( \int_0^\infty \left( \int_0^\infty V(x)^{p'} v(x) \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(z) dz)^{\frac{1}{q}} \right) dt \right)^{\frac{1}{q}} \right) + \left( \int_0^\infty \left( \int_0^\infty V(x)^{p'} v(x) \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(z) dz)^{\frac{1}{q}} \right) dt \right)^{\frac{1}{q}} \right) + \left( \int_0^\infty \left( \int_0^\infty A(t)^{p'} a \left( \int_{x(t)}^{x(\tau)} \left( \sup \left( u(\tau) V(\tau)^{-2} \right) a(s) ds \right)^{p'} dx \int_t^\infty (w(z) dz)^{\frac{1}{q}} \right) dt \right)^{\frac{1}{q}} \right) + \left( \int_0^\infty A(y)^q w(y) dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \sup_{t \leq \tau} \left( u(\tau) V(\tau)^{-2} \left( \int_0^\tau V(s)^{p'} v(s) ds \right)^{\frac{1}{p'}} \right) \right).
\]
Proof. By [26, Theorem 3.2] (cf. [20, Theorem 2.3]), we get that

\[
\sup_{f \in \mathbb{R}^{p+1}(0,\infty)} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)f(\tau)a(t)dt \right)^q \frac{w(x)dx}{h(x)} \right)^{\frac{1}{q}}
\]

\[
\leq \sup_{h \geq 0} \left( \int_0^\infty h(x)^{p}V(x)^{-p}V(s)^{-1-p} ds \right)^{\frac{1}{p}}
\]

By Theorem 3.1, we have that

(i) if \( p \leq q \), then

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2} \int_0^\tau h(y)dy a(t)dt \right)^q \frac{w(x)dx}{h(x)} \right)^{\frac{1}{q}}
\]

\[
\leq \sup_{t \in (0,\infty)} \left( \int_0^t V(x)^p v(x) \left( \int_x^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2}a(s)ds \right)^q dx \right)^{\frac{1}{q}} \left( \int_0^\infty w(y)dy \right)^{\frac{1}{q}}
\]

\[
+ \sup_{x \in (0,\infty)} \left( \int_0^\tau V(x)^p v(x) dx \right)^{\frac{1}{p}} \left( \int_t^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2}a(s)ds \right)^q w(y)dy \right)^{\frac{1}{q}}
\]

\[
+ \sup_{x \in (0,\infty)} \left( \int_0^\tau A(\tau)^{q} w(y)dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \int_0^\tau V(s)^p v(s)ds \right)^{\frac{1}{p}}
\]

(ii) if \( q < p \), then

\[
\sup_{h \geq 0} \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2} \int_0^\tau h(y)dy a(t)dt \right)^q \frac{w(x)dx}{h(x)} \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_0^\infty \left( \int_0^\tau V(x)^p v(x)dx \right)^{\frac{1}{p}} V(t)^p v(t) \left( \int_0^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2}a(s)ds \right)^q w(z)dz \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_0^\infty \left( \int_0^\tau V(x)^p v(x) \left( \int_x^\infty \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2}a(s)ds \right)^q dx \right)^{\frac{1}{q}} \left( \int_0^\infty w(s)ds \right)^{\frac{1}{p}} \left( \int_t^\infty w(t)dt \right) \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_0^\infty \left( \int_{[0,\infty)} A(\tau)^{q} w(y)dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \int_0^\tau V(s)^p v(s)ds \right)^{\frac{1}{p}} A(x)^q w(x)dx \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_0^\infty \left( \int_{[0,\infty)} A(\tau)^{q} w(y)dy \right)^{\frac{1}{q}} \lim_{t \to \infty} \left( \sup_{t \leq \tau} u(\tau)V(\tau)^{-2} \int_0^\tau V(s)^p v(s)ds \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}
\]
5. The boundedness of $P_{a,b}$ from $L^p(v)$ into $\text{Ces}_q(w,a)$ on the cone of monotone non-increasing functions

In this section we characterize the boundedness of weighted Hardy operator $P_{a,b}$ from $L^p(v)$ into $\text{Ces}_q(w,a)$ on the cone of monotone non-increasing functions.

Theorem 5.1. Let $1 < p, q < \infty$ and $b \in \mathcal{W}(0,\infty)$ be such that $b(t) > 0$ for a.e. $t \in (0,\infty)$. Assume that $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ and $a, v \in \mathcal{W}(0,\infty)$.

(i) If $p \leq q$, then

$$\sup_{f \in \mathcal{B}^{1-/(0,\infty)}} \frac{\left( \int_0^\infty \left( \int_0^t (P_{a,b,f}(t)a(t)dt)^q w(x)dx \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}}{\left( \int_0^\infty f(s)^p v(s)ds \right)^{\frac{1}{p}}}$$

$$\approx \sup_{x \in (0,\infty)} \frac{\left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty V(s)^{-\beta} v(s)ds \right)^{\frac{1}{p}}}$$

$$+ \sup_{x \in (0,\infty)} \frac{\left( \int_0^\infty w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty V(s)^{-\beta} v(s)ds \right)^{\frac{1}{p}}}$$

$$+ \sup_{x \in (0,\infty)} \frac{\left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty \frac{B(s)}{V(s)} \right)^{\frac{1}{p}}}$$

$$+ \sup_{x \in (0,\infty)} \frac{\left( \int_0^\infty w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty \frac{B(s)}{V(s)} \right)^{\frac{1}{p}}}$$

$$+ \left( \int_0^\infty \left( \int_0^x w(t)dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}$$

(ii) If $q < p$, then

$$\sup_{f \in \mathcal{B}^{1-/(0,\infty)}} \frac{\left( \int_0^\infty \left( \int_0^t (P_{a,b,f}(t)a(t)dt)^q w(x)dx \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}}{\left( \int_0^\infty f(s)^p v(s)ds \right)^{\frac{1}{p}}}$$

$$\approx \left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^\infty \left( \int_0^x a(y)u(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}}$$

Proof. By [26, Theorem 3.1], using Fubini’s Theorem, we get that
(i) Let \( p \leq q \). Using the characterizations of weighted Hardy-type inequalities (see, for instance, [37, Section 1]), by [35, Theorem 1.1], we obtain that

\[
\sup_{f \in \mathcal{H}_{p}^{q}(0, \infty)} \left( \frac{\int_{0}^{\infty} \left( \int_{0}^{x} (P_{u,b,f}(t) a(t) dt) \right)^{q} w(x) dx}{\left( \int_{0}^{\infty} f(s) v(s) ds \right)^{\frac{q}{p}}} \right)
\approx \sup_{h \geq 0} \left( \frac{\int_{0}^{\infty} h(s)^{p} V(s)^{-p} v(s)^{-p} ds}{\left( \int_{0}^{\infty} h(s)^{p} V(s)^{p} v(s)^{1-p} ds \right)^{\frac{1}{p}}} \right)
+ \sup_{h \geq 0} \left( \frac{\int_{0}^{\infty} h(s)^{p} B(s)^{-p} V(s)^{p} v(s)^{1-p} ds}{\left( \int_{0}^{\infty} h(s)^{p} V(s)^{p} v(s)^{1-p} ds \right)^{\frac{1}{p}}} \right). 
\]
(ii) Let now \( q < p \). Using the characterizations of weighted Hardy-type inequalities (see, for instance, [37, Section 1]), by [35, Theorem 1.2], we obtain that

\[
\sup_{f \in \mathcal{B}^{1,1}(0,\infty)} \left( \int_0^\infty \left( \int_0^x \left( \int_0^t (P_{u,b,f}(t)a(t)dt)^q \right)^{\frac{1}{q}} \right) \right)^{\frac{1}{p}} \left( \int_0^\infty f(s)^p \nu(s)ds \right)^{\frac{1}{p}} \approx \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \left( \int_0^x a(y)u(y)dy \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

The proof is completed. \( \square \)

6. The boundedness of \( T_{u,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions

In this section we combine the results from previous two sections to present the characterization of the boundedness of \( T_{u,b} \) from \( L^p(v) \) into \( \text{Ces}_q(w,a) \) on the cone of monotone non-increasing functions.

**Theorem 6.1.** Let \( 1 < p, q < \infty \) and \( b \in \mathcal{W}(0,\infty) \) be such that \( b(t) > 0 \) for a.e. \( t \in (0,\infty) \). Assume that \( u \in \mathcal{W}(0,\infty) \cap C(0,\infty) \) and \( u, v, w \in \mathcal{W}(0,\infty) \). Moreover, assume that condition (1.1) holds.

(i) If \( p \leq q \), then

\[
\sup_{f \in \mathcal{B}^{1,1}(0,\infty)} \left( \int_0^\infty \left( \int_0^x \left( \int_0^t (T_{u,b,f}(t)a(t)dt)^q \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \left( \int_0^\infty f(s)^p \nu(s)ds \right)^{\frac{1}{p}} \right) \approx \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \left( \int_0^x a(y)u(y)dy \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty \left( \int_0^x \left( \int_0^t a(y)u(y)dy \right)^q \right) \left( \int_y^\infty V(z)^{-1} \nu(z)dz \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty a(y)u(y)dy \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty a(y)u(y)dy \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty a(y)u(y)dy \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]

\[+ \left( \int_0^\infty a(y)u(y)dy \right)^{\frac{1}{q}} \left( \int_0^\infty V(x)^{-1} \nu(x)dx \right)^{\frac{1}{q}}
\]
\[ \begin{align*}
&+ \sup_{x \in (0,\infty)} \left( \int_0^\infty w(t) dt \right)^\frac{1}{p'} \left( \int_0^\infty \left( \int_0^\infty a(y)\bar{u}(y) dy \right)^{p'} V(s)^{-p'} v(s) ds \right)^\frac{1}{p} \\
&+ \sup_{x \in (0,\infty)} \left( \int_0^\infty \left( \int_x^\infty \frac{a(\tau)}{B(\tau)} \bar{u}(\tau) d\tau \right)^q w(t) dt \right)^\frac{1}{q} \left( \int_0^\infty \left( \frac{B(s)}{V(s)} \right)^{p'} v(s) ds \right)^\frac{1}{p} \\
&+ \sup_{x \in (0,\infty)} \left( \int_0^\infty w(t) dt \right)^\frac{1}{q} \left( \int_0^\infty \left( \int_x^\infty a(\tau)\bar{u}(\tau) d\tau \right)^p \left( \frac{B(s)}{V(s)} \right)^{p'} v(s) ds \right)^\frac{1}{p'} \\
&+ \left( \int_0^\infty v(s) ds \right)^{-\frac{1}{p}} \left( \int_0^\infty \left( \int_x^\infty a(\bar{u}(\bar{t})) dt \right)^q w(x) dx \right)^\frac{1}{q};
\end{align*} \]

(ii) If \( q < p \), then

\[
\sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (T_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q} \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \\
\approx \left( \int_0^\infty \left( \int_0^\infty V(x)^{p'} v(x) dx \right)^\frac{1}{q'} V(t)^{p'} v(t) \left( \int_t^\infty \left( \int_0^\infty \left( \sup_{x \in [y]} V(y)^{-2} \right) a(s) ds \right)^q w(z) dz \right) dt \right)^\frac{1}{q'} \\
+ \left( \int_0^\infty \left( \int_0^\infty V(x)^{p'} v(x) \left( \int_0^\infty \left( \sup_{x \in [y]} V(y)^{-2} \right) a(s) ds \right)^p w(t) dt \right)^\frac{1}{q'} \right)^{\frac{q}{p}} \left( \int_0^\infty w(s) ds \right)^\frac{1}{p} \\
+ \left( \int_0^\infty \left( \int_0^\infty A(\tau)^p \left( \sup_{t \leq \tau} V(\tau)^{-2p} \left( \int_0^\infty V(s)^{p'} v(s) ds \right) \right)^q \left( \int_0^\infty w(y) dy \right)^\frac{1}{q'} \right)^{\frac{q}{p}} \left( \int_0^\infty w(x) dx \right)^\frac{1}{q'} \\
+ \left( \int_0^\infty \left( \int_0^\infty (\tau)^{q(\tau)^{-2p}} v(\tau) w(y) dy \right) \lim_{\tau \to \infty} \left( \sup_{t \leq \tau} V(\tau)^{-2p} \left( \int_0^\infty V(s)^{p'} v(s) ds \right) \right)^\frac{q}{p} \\
+ \left( \int_0^\infty \left( \int_0^\infty q(\tau)^{q(\tau)^{-2p}} v(\tau) w(y) dy \right) \lim_{\tau \to \infty} \left( \sup_{t \leq \tau} V(\tau)^{-2p} \left( \int_0^\infty V(s)^{p'} v(s) ds \right) \right)^\frac{q}{p} \\
+ \left( \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty a(\bar{y}(\bar{y})) dy \right)^q w(t) dt \right)^\frac{1}{q'} \left( \int_0^\infty V(z)^{-p'} v(z) dz \right)^\frac{1}{q'} \left( \int_0^\infty a(y)u(y) dy \right)^q \left( \int_0^\infty w(x) dx \right)^\frac{1}{q'} \\
+ \left( \int_0^\infty \left( \int_0^\infty w(t) dt \right)^\frac{1}{q'} \left( \int_0^\infty \left( \int_0^\infty a(y)\bar{u}(y) dy \right)^p V(z)^{-p'} v(z) dz \right)^\frac{1}{q'} \left( \int_0^\infty w(x) dx \right)^\frac{1}{q'} \\
+ \left( \int_0^\infty \left( \int_0^\infty V(t)^{p'} v(t) \left( \int_0^\infty \left( \sup_{x \in [y]} V(y)^{-2} \right) a(s) ds \right)^q w(z) dz \right) dt \right)^\frac{1}{q'} \\
+ \left( \int_0^\infty \left( \int_0^\infty a(\tau) B(\tau) \bar{u}(\tau) d\tau \right)^q \left( \int_0^\infty \left( \frac{B(s)}{V(s)} \right)^{p'} v(s) ds \right)^\frac{1}{p'} \left( \int_0^\infty \left( \frac{B(s)}{V(s)} \right)^{p'} v(s) ds \right)^\frac{1}{p'} \\
+ \left( \int_0^\infty \left( \int_0^\infty \left( \int_x^\infty a(\tau) B(\tau) \bar{u}(\tau) d\tau \right)^q v(s) ds \right) \left( \frac{B(s)}{V(s)} \right)^{p'} v(s) ds \right)^\frac{1}{p'} \\
+ \left( \int_0^\infty \left( \int_0^\infty a(y)\bar{u}(y) dy \right)^q \left( \int_0^\infty \left( \int_x^\infty a(\bar{u}(\bar{t})) dt \right)^q w(x) dx \right)^\frac{1}{q'} \right)^{\frac{q}{p}} \left( \int_0^\infty \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \\
+ \sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (P_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q'} \right)^{\frac{q}{p}} \left( \int_0^\infty \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \\
+ \sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (R_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q'} \right)^{\frac{q}{p}} \left( \int_0^\infty \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \right) \right) \right).
\]

Proof. By (1.2), we have that

\[
\sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (T_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q} \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \approx \sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (R_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q} \left( \int_0^\infty \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \right) \right) + \sup_{f \in \mathfrak{M}^{-1}(0,\infty)} \left( \int_0^\infty \left( \int_0^\infty (P_{u,b}f)(t) a(t) dt \right)^q w(x) dx \right)^\frac{1}{q} \left( \int_0^\infty \left( \int_0^\infty f(s)^p v(s) ds \right)^\frac{1}{p} \right) \right).
\]
Let $f$ be a measurable a.e. finite function on $\mathbb{R}^n$. Then its non-increasing rearrangement $f^*$ is given by
$$f^*(t) = \inf\{ \lambda > 0 : |\{ x \in \mathbb{R}^n : |f(x)| > \lambda \}| \leq t \}, \quad t \in (0, \infty),$$
and let $f^{**}$ denotes the Hardy-Littlewood maximal function of $f^*$, i.e.
$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau) \, d\tau, \quad t > 0.$$
Quite many familiar function spaces can be defined by using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0, \infty)$ and $w \in \mathcal{W}(0, \infty)$. Then the classical Lorentz spaces $L^p(w)$ and $\Gamma^p(w)$ consist of all measurable functions $f$ on $\mathbb{R}^n$ for which $\|f\|_{L^p(w)} := \|f^*\|_{p,w,(0,\infty)} < \infty$ and $\|f\|_{\Gamma^p(w)} := \|f^{**}\|_{p,w,(0,\infty)} < \infty$, respectively. For more information about the Lorentz $\Lambda$ and $\Gamma$ spaces see e.g. [9] and the references therein.

The fractional maximal operator, $M_{\gamma}$, $\gamma \in (0, n)$, is defined at a locally integrable function $f$ on $\mathbb{R}^n$ by
$$(M_{\gamma}f)(x) := \sup_{Q,x} Q^{-\gamma/(n-1)} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$
It was shown in [10, Theorem 1.1] that
$$(M_{\gamma}f)^*(t) \leq \sup_{\gamma > 1} \int_0^t f^*(y) \, dy \leq (M_{\gamma}\bar{f})^*(t)$$
(7.1)
for every locally integrable function $f$ on $\mathbb{R}^n$ and $t \in (0, \infty)$, where $\bar{f}(x) := f^*(\omega_n|x|^p)$ and $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The characterization of the boundedness of $M_{\gamma}$ between classical Lorentz spaces $L^p(v)$ and $L^q(w)$ was obtained in [10] for the particular case when $1 < p \leq q < \infty$ and in [36, Theorem 2.10] in the case of more general operators and for extended range of $p$ and $q$ (For the characterization of the boundedness of more general fractional maximal functions between $L^p(v)$ and $L^q(w)$, see [34], and the references therein).

As an application of obtained results, we calculate the norm of the fractional maximal function $M_{\gamma}$ from $L^p(v)$ into $\Gamma^q(w)$.

**Theorem 7.1.** Let $1 < p, q < \infty$ and $0 < \gamma < n$. Assume that $v, w \in \mathcal{W}(0, \infty)$.

(i) If $p \leq q$, then
$$\|M_{\gamma}\| \|L^p(v) \rightarrow \Gamma^q(w)\|$$
$$\approx \sup_{t \in (0, \infty)} \left( \int_0^t V(x)^{p'} v(x) \left( \int_x^t \left( \sup_{r \leq \tau} \frac{1}{t^{\gamma/(n-1)}} \left( \int_r^\infty \frac{1}{y^{1/m}} \, dy \right)^{\gamma/m} \right)^{1/p'} \left( \int_0^\infty w(y) \, dy \right)^{1/q'} \right) \right)^{1/p'}$$
$$+ \sup_{t \in (0, \infty)} \left( \int_0^t V(x)^{p'} v(x) \, dx \right)^{1/p'} \left( \int_t^\infty \left( \int_s^\infty \frac{1}{y^{1/m}} \, dy \right)^{\gamma/m} \left( \int_0^\infty w(y) \, dy \right)^{1/q'} \right)^{1/p'}$$
$$+ \sup_{x \in (0, \infty)} \left( \int_0^t \frac{1}{x} \left( \sup_{r \leq \tau} \frac{1}{t^{\gamma/(n-1)}} \left( \int_r^\infty \frac{1}{y^{1/m}} \, dy \right)^{\gamma/m} \right)^{1/p'} \left( \int_0^\infty w(y) \, dy \right)^{1/q'} \right) \left( \int_0^\infty \frac{1}{x} \, dx \right)^{1/2}$$
$$+ \left( \int_0^\infty w(y) \, dy \right)^{1/2} \lim_{t \to \infty} \sup_{r \leq \tau} \frac{1}{t^{\gamma/(n-1)}} \left( \int_r^\infty \frac{1}{y^{1/m}} \, dy \right)^{\gamma/m} \left( \int_0^\infty v(s)^{-p'} v(s) \, ds \right)^{1/p'}$$
$$+ \sup_{x \in (0, \infty)} \left( \int_0^\infty w(y) \, dy \right)^{1/2} \left( \int_0^\infty \frac{1}{x} \left( \sup_{r \leq \tau} \frac{1}{t^{\gamma/(n-1)}} \left( \int_r^\infty \frac{1}{y^{1/m}} \, dy \right)^{\gamma/m} \right)^{1/p'} \left( \int_0^\infty w(y) \, dy \right)^{1/q'} \right)^{1/p'}$$
+ \sup_{x \in (0, \infty)} \left( \int_x^\infty y^{-q} w(y) \, dy \right)^\frac{1}{\gamma} \left( \int_x^\infty \left( \int_y^\infty \sup_{s \leq y} \frac{z^{-q} w(z)}{z^\gamma} \, dz \right) \frac{q}{\gamma} \, ds \right)^\frac{1}{\gamma} \frac{d \gamma}{\gamma} + \left( \int_x^\infty s^{-q} w(s) \, ds \right)^\frac{1}{\gamma} \left( \int_x^\infty \left( \int_0^s y^{-q} w(y) \, dy \right)^\frac{1}{\gamma} \right)^\frac{1}{\gamma} ;

(ii) If q < p, then
\begin{align*}
\|M_f\|_{\Lambda^p(v) \to \Gamma^q(w)} &\approx \left( \int_0^\infty \left( \int_0^s \left( \int_v^x V(x)^{p'} V(x) \, dx \right)^\frac{1}{p'} \right) V(t)^{p'} V(t) \left( \int_t^\infty \left( \int_t^\infty \left( \sup_{s \leq y} \frac{z^{-q} w(z)}{z^\gamma} \, dz \right) \frac{q}{\gamma} \, ds \right)^\frac{1}{\gamma} \right)^\frac{1}{p'} \frac{d \gamma}{\gamma} + \left( \int_x^\infty s^{-q} w(s) \, ds \right)^\frac{1}{\gamma} \left( \int_x^\infty \left( \int_0^s \left( \int_v^x V(x)^{p'} V(x) \, dx \right)^\frac{1}{p'} \right)^\frac{1}{p'} \right)^\frac{1}{p'} ;
\end{align*}

Proof. From inequalities (7.1), we have that
\begin{align*}
\|M_f\|_{\Lambda^p(v) \to \Gamma^q(w)} &\approx \sup_{f \in \mathcal{H}} \left( \int_0^\infty \left( \int_0^s \left( \int_v^x \frac{T_{a,b,f}(t)}{B(t)} \, dt \right)^q x^{-q} w(x) \, dx \right)^\frac{1}{q} \right)^\frac{1}{q} ;
\end{align*}

with u(\tau) = \tau^{p/n} and b \equiv 1. Note that
\begin{align*}
\sup_{0 < t < \infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)} \, d\tau < \infty
\end{align*}
in this case. So, it remains to apply Theorem 6.1.

References

[1] R. Askey and R. P. Boas Jr., Some integrability theorems for power series with positive coefficients, Mathematical Essays Dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, pp. 23–32. MR0277956 (43 #3689)
[35] R. Oinarov, Two-sided estimates for the norm of some classes of integral operators, Trudy Mat. Inst. Steklov. 204 (1993), no. Issled. po Teor. Differ. Funktii Mnozikh Peremen. i ee Prilozh. 16, 240–250 (Russian); English transl., Proc. Steklov Inst. Math. 3 (204) (1994), 205–214. MR1320028

[36] B. Opic, On boundedness of fractional maximal operators between classical Lorentz spaces, Function spaces, differential operators and nonlinear analysis (Pudlásjarvi, 1999), Acad. Sci. Czech Repub., Prague, 2000, pp. 187–196. MR1755309 (2001g:42043)

[37] B. Opic and A. Kufner, Hardy-type inequalities, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990. MR1069756 (92b:26028)

[38] L. Pick, Supremum operators and optimal Sobolev inequalities, Function spaces, differential operators and nonlinear analysis (Pudasjarvi, 1999), Acad. Sci. Czech Repub., Prague, 2000, pp. 207–219. MR1755311 (2000m:46075)

[39] , Optimal Sobolev embeddings—old and new, Function spaces, interpolation theory and related topics (Lund, 2000), de Gruyter, Berlin, 2002, pp. 403–411. MR1943297 (2003j:46054)

[40] Programma van Jaarlijkse Prijsvragen (Annual Problem Section), Nieuw Arch. Wiskd. 16 (1968), 47–51.

[41] D. V. Prokhorov and V. D. Stepanov, On weighted Hardy inequalities in mixed norms, Proc. Steklov Inst. Math. 283 (2013), 149–164.

[42] , Weighted estimates for a class of sublinear operators, Dokl. Akad. Nauk 453 (2013), no. 5, 486–488 (Russian); English transl., Dokl. Math. 88 (2013), no. 3, 721–723. MR3203323

[43] , Estimates for a class of sublinear integral operators, Dokl. Akad. Nauk 456 (2014), no. 6, 645–649 (Russian); English transl., Dokl. Math. 89 (2014), no. 3, 372–377. MR3287911

[44] D. V. Prokhorov, On the boundedness of a class of sublinear integral operators, Dokl. Akad. Nauk 92 (2015), no. 2, 602–605 (Russian).

[45] E. Pustylnik, Optimal interpolation in spaces of Lorentz-Zygmund type, J. Anal. Math. 79 (1999), 113–157, DOI 10.1007/BF02788238. MR1749309 (2001a:46028)

[46] G. È. Shambilova, Weighted inequalities for a class of quasilinear integral operators on the cone of monotone functions, Sibirsk. Mat. Zh. 55 (2014), no. 4, 912–936 (Russian, with Russian summary); English transl., Sib. Math. J. 55 (2014), no. 4, 745–767. MR3242605

[47] J.-S. Shiue, On the Cesáro sequence spaces, Tamkang J. Math. 1 (1970), no. 1, 19–25.

[48] , A note on Cesáro function space, Tamkang J. Math. 1 (1970), no. 2, 91–95. MR0276751 (44 #2491)

[49] V. D. Stepanov, The weighted Hardy’s inequality for nonincreasing functions, Trans. Amer. Math. Soc. 338 (1993), no. 1, 173–186, DOI 10.2307/2154450. MR1097171

[50] V. D. Stepanov and G. È. Shambilova, Weight boundedness of a class of quasilinear operators on the cone of monotone functions, Dokl. Math. 90 (2014), no. 2, 569–572.

[51] P. W. Sy, W. Y. Zhang, and P. Y. Lee, The dual of Cesáro function spaces, Glas. Mat. Ser. III 22(42) (1987), no. 1, 103–112 (English, with Serbo-Croatian summary). MR940098 (89g:46059)

RZA MUSTAFAYEV, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KARAMANOGLU MEHMETBEC UNIVERSITY, KARAMAN, 70100, TURKEY
E-mail address: rzamustafayev@gmail.com

NEVIN BILGİÇLİ, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, KIRIKKALE UNIVERSITY, 71450 YAHSİHAN, KIRIKKALE, TURKEY
E-mail address: nevinbilgicli@gmail.com