ANNIHILATORS FOR CUSP FORMS OF WEIGHT 2
AND LEVEL $4p^m$

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Abstract. We obtain operators, given essentially by formal sums
of Hecke operators, that annihilate spaces of cusp forms of weight
2 for $\Gamma_1(p^m) \cap \Gamma(4)$, whose dimensions will be specified. Moreover,
we obtain the principal part (mod $p$), over the cusps, of certain
meromorphic modular functions of level $4p^m$.

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References

1. Introduction

Previous notation: In this work $p$ is a prime integer $\neq 2, 3$. We fix $\zeta_{4p^m}$, a primitive $4p^m$-root of unity. Let $p$ be a place over $p$ in $\mathbb{Z}[\zeta_{4p^m}]$ and $R_p$ is the local ring $\mathbb{Z}[\zeta_{4p^m}]_{(p)}$; $k$ and $K$ are its residual field and fraction field, respectively. The cardinality of $k$ is $p^\delta$, where $\delta = 1$ when $4| (p - 1)$, and $\delta = 2$ in other cases. If $X$ is a curve over $\text{Spec}(R_p)$, then $X_k$ denotes $X \otimes_{R_p} k$. Let us denote by $\text{Fr}$ the $p^\delta$-Frobenius morphism. We denote by $A^0_m$ the $p^m$-cyclic subgroup of $(\mathbb{Z}/p^m)^\oplus 2$ generated by $(1, 0)$ and $A^0_i \subset A^0_m$ its $p^i$-cyclic subgroup. Let $H/L$ be a Galois extension; the group $\text{Aut}_L(H)$ is called the Galois group of $H/L$.

Part of this section is extracted from [DR], [L] and [M]. Let us denote by $\mathcal{H}$ the upper plane of complex numbers with $\text{Im} z > 0$. Let us denote by $\mathcal{H}^*$ the union of the upper plane with $\infty$ and $\mathbb{Q}$. The modular congruence subgroups $\Gamma(n)$ and $\Gamma_1(n)$ consist of the elements $\gamma \in \text{Sl}_2(\mathbb{Z})$ such that

$$\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n, \quad \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod n,$$

(where $b$ is arbitrary), respectively. The quotients $\mathcal{H}^*/\Gamma(n)$ and $\mathcal{H}^*/\Gamma_1(n)$ give smooth, compact modular curves over $\mathbb{C}$, usually denoted by $X(n)$ and $X_1(n)$, respectively. In the same way, if one considers commensurable subgroups $\Gamma \subset \text{Sl}_2(\mathbb{Z})$, one obtains modular curves $X_\Gamma$. 
A modular form of weight 2 for \( \Gamma(n) \), (respectively \( \Gamma_1(n) \)), is a holomorphic function \( g(z) : \mathcal{H} \to \mathbb{C} \) such that \( g(\frac{az+b}{cz+d}) = (cz+d)^2 g(z) \) for each \( \gamma \in \Gamma(n) \), (respectively \( \Gamma_1(n) \)), with

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Therefore, if \( g(z) \) is a modular form for \( \Gamma(n) \), (respectively \( \Gamma_1(n) \)) then \( g(z+n) = g(z) \), (respectively, \( g(z+1) = g(z) \)). If we denote \( q := e^{2\pi i z} \), then \( g(z) \) has a \( q \)-expansion:

\[
g(z) = \sum_{i=r}^{\infty} a_i q^{i/n}, \quad \text{(respectively, } g(z) = \sum_{i=r}^{\infty} a_i q^i),\]

for some \( r \in \mathbb{Z} \).

The open subset \( X^0(n) := \mathcal{H}/\Gamma(n) \) of \( X(n) \) has the following modular interpretation: its points are identified with isomorphism classes of elliptic curves \( E \) over \( \mathbb{C} \), together with an isomorphism of the \( n \)-torsion group of \( E \) with \( (\mathbb{Z}/n)^{\oplus 2} \). The points of the closed subset \( X(n) \setminus X^0(n) \), (respectively \( X_1(n) \setminus X^0_1(n) \)) are called the cusps of \( X(n) \), (respectively \( X_1(n) \)).

A cusp form of weight 2 for \( \Gamma_1(n) \) is a modular form \( g(z) \) for \( \Gamma_1(n) \) of weight 2 that has a zero at each cusp point. Via the identification \( g(z) \to g(z)dz \), the space of cusp forms for \( \Gamma_1(n) \) is isomorphic to the space of 1-holomorphic differentials on \( X_1(n) \).

The meromorphic functions on \( X(n) \) are called meromorphic modular functions of level \( n \).

Let us set \( R_i \in \Gamma(1)/\Gamma_1(p^m) \) as

\[
R_i \equiv \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix} \mod \Gamma_1(p^m) \quad \text{and} \quad R_i \equiv \text{Id} \mod \Gamma(4)
\]

with \( 1 \leq i < p^m \) and \((i,p) = 1\). We use the symbol \( T_p \) for the Hecke operator \( T_p(\sum_{i=r}^{\infty} a_i q^i) = \sum_{i=r}^{\infty} a_{ip} q^i \) and \( T_{ps} = T_p^s \). Moreover, \( R_i \) and \( T_p \) operate over the cusp forms of level \( \Gamma_1(p^m) \cap \Gamma(4) \).
The Deligne-Rapoport model $M(4p^m)$, (c.f. [DR] IV, Corollary 3.9.2.) provides us with proper models of $X(4p^m)$ over $\mathbb{Z}[1/4, \zeta_{p^m}]$. There exists an open subscheme $M^0(4p^m) \subset M(4p^m)$ with

$$M^0(4p^m) \otimes_{\mathbb{Z}[1/4, \zeta_{p^m}]} \mathbb{C} = X^0(4p^m),$$

Moreover, there exists a moduli scheme $M_1(4p^m)$ which is proper and flat over $\mathbb{Z}[1/4]$ and is a model for $X(4p^m)/\Gamma_1(p^m)$, (c.f. [DR] IV, Proposition 3.10).

In this work, we consider the moduli $M(4p^m)$ and $M_1(4p^m)$ defined over the category of schemes over $R_p$. We consider the open subscheme $M^0_1(4p^m) \subset M_1(4p^m)$ such that $M^0_1(4p^m) \otimes_{R_p} \mathbb{C} = X^0_1(4p^m)$.

In this article for each $0 \leq i \leq d - 1$ we shall define a set $H_i$ and an operator $R_h$ for each $h \in H_i$ such that:

The operator $\sum_{i=0}^{d-1} (\sum_{h \in H_i} R_h T_{p^i})$ annihilates a space of cusp forms, of weight 2 for the modular group $\Gamma_1(p^m) \cap \Gamma(4)$, of dimension $N$, where $N = \dim(\text{Pic}^0_{M_1(4p^m)_{k/k}})^{ab}$. Let $G$ be an algebraic group, smooth and connected over $k$. There exists a smallest linear algebraic subgroup $L \subset G$ with $G/L$ an abelian variety. $G^{ab} =: G/L$ is called the abelian part of $G$, c.f. [Ro].

In the second result of this article, we obtain the principal parts of certain meromorphic modular functions of level $4p^m \pmod{p}$ over $M(4p^m)$. Let $\text{Spec}(B) = M^0(4p^m)$ and let $\bar{p}^{1/4p^m}$ be the reduction mod $\mathfrak{p}$ of the local parameter $q$ for the $\infty$. We prove in Theorem 2 that:

For each $m \in \mathbb{N}$, there exists an element of $B_k[p(\sigma)^{-1}]$ such that its principal is $\sum_{i=0}^{d-1} (\sum_{h \in H_i} \frac{1}{(\sigma_{p^{i+1}})^{R_{-i}}})$.

The function field of $M_1[1/4] \otimes_{\mathbb{Z}[\mathfrak{d}]} k$ is $k(\sigma)$, (c.f [H] 4.3), and $p(\sigma) \in k[\sigma]$ is defined in section 4.1.
To prove the above two results, we use the correspondences found by Anderson and Coleman, (c.f. [An], [C]), which are trivial (up to vertical and horizontal ones) and that are defined over abelian extensions of rational function fields. These correspondences are given by Stickelberger’s elements for function fields and they provide proof of the Brumer-Stark theorem in the function field case. To prove the first result of this article, we need the Eichler-Shimura Theorem for $T_p$ over $M_1(4p^m)$.

The results of the first part of this work is useful for obtaining, in an algorithmic way, the $p$-term in the Euler product of the Hasse-Weil $L$-function of $M_1(4p^m)$ (section 4.3), while the second result could be considered as an additive version of the arithmetic problem of explicitly obtaining generators for principal ideals of global fields.

2. Preliminaries

2.1. Models for modular curves. We consider $M(4p^m)$ and $M_1(4p^m)$ defined over $R_p$. Note that $M(n)$ for small $n$ is not a scheme. For the modular interpretation of these models we use the Drinfeld level structures for elliptic curves, c.f. [KM].

**Definition 2.1.** (c.f. [KM], chapter 3) Let $E$ be an elliptic curve over a scheme $S$. A $\Gamma(p^m)$ (respectively $\Gamma_1(p^m)$)-Drinfeld level structure is an homomorphism of groups $\phi_{p^m} : (\mathbb{Z}/p^m)^{\oplus 2} \to E_{p^m}(S)$, (respectively $\phi_{p^m}^1 : \mathbb{Z}/p^m \to E_{p^m}(S)$), such that the divisor $\sum_{a \in (\mathbb{Z}/p^m)^{\oplus 2}} \phi_{p^m}(a)$, (respectively $\sum_{a \in \mathbb{Z}/p^m} \phi_{p^m}^1(a)$) gives a direct sum of two cyclic subgroup subschemes of $E$ of order $p^m$ (respectively a cyclic subgroup subscheme of $E$ of order $p^m$). Note that in the case of $S = \text{Spec}(k)$ with $\text{char}(k) = p$, a $p^m$-cyclic subgroup subscheme of $E_{p^m}$ could be $\text{Spec}(k[t]/(t^{p^m}))$ and hence the divisor given by $\phi_{p^m}^1$ could be $p^m \cdot 0$. When
Definition 2.2. (c.f. [KM], chapter 3) A $\Gamma_1(p^m)$-level structure over an elliptic curve $E$ over $S$ is an object $(E, C_{p^m}, \phi_{p^m})$, with $\phi_{p^m}$ a $\Gamma_1(p^m)$-Drinfeld level structure and $C_{p^m} = \phi_{p^m}(\mathbb{Z}/p^m)$. The objects $(E, \phi_{p^m})$ are said to be $\Gamma(p^m)$-level structures.

Bearing in mind that $M(4)$ is a fine moduli over $R_p$ for the 4-level structures (c.f. [DR] IV, Corollaire 2.9), by [KM] 5.5.1, and [DR] IV, we have that $M_0^0(4p^m)$ (respectively $M_1^0(4p^m)$) are schemes and they have the following modular interpretation: Let $S$ be a scheme over $R_p$, $M^0(4p^m)(S)$ (respectively $M_1^0(4p^m)(S)$) are the sets

\[ \{(E, \iota_4, \phi_{p^m})\} \text{ (respectively } \{(E, \iota_4, C_{p^m}, \phi_{p^m})\}\),

where $E$ is an elliptic curve over $S$ and $\iota_4$ is a usual 4-level structure.

Let $A_m$ be a cyclic subgroup of $(\mathbb{Z}/p^m)^{\oplus 2}$ of cardinality $p^m$. We define the functor $C_{A_m}^0(S)$ as the set of objects $(E, \iota_4, C_{p^m}, \phi_{p^m})$ such that $\phi_{p^m}(A_m) = C_{p^m}$ is an etale $p^m$-cyclic subgroup of $E$ over the points of $S$ where $E$ is not supersingular. By [DR] V, 4.8, $C_{A_m}^0 \subset M^0(4p^m)$ gives a curve over $R_p$.

Let $\{A_{m}^{i}\}_{0 \leq i \leq l-1}$ be the set of $p^m$-cyclic subgroups of $(\mathbb{Z}/p^m)^{\oplus 2}$ and $l$ the cardinality of this set. According to [DR] V, Theorem 4.12, the decomposition into the reduced and irreducible components of the scheme $M(4p^m)_k$ takes the form $\bigcup_{i=0}^{l-1}(C_{A_m^i})_k$. The curves $(C_{A_m^i})_k$ are smooth over $k$. We denote $(C_{A_m^i})_k = (C_{A_m})_k \cap M^0(4p^m)_k$.

Remark 2.1. Let $A_m^0$ and $A_i^0 \subset A_m^0$ be as in the Introduction. We have an injective morphism of functors $\Psi_i : (C_{A_i^0})_k \hookrightarrow M_1^0(4p^m)$. Let us consider an object $(E, \iota_4, \phi_{p^m}) \in (C_{A_i^0})_k$ and the morphism of elliptic curves

\[ h_i : E \to E/\phi_{p^i}(A_i^0) \overset{Fr_m-i}{\longrightarrow} (Fr_m-i)^*(E/\phi_{p^i}(A_i^0)), \]
where, \( \text{Fr} \) denotes the \( p \)-Frobenius morphism. Thus, we define
\[
\Psi_i(E, \iota_4, \phi_{p^m}) = (E, \iota_4, \text{Ker}(h_i), \phi_{p^m}^1),
\]
where \( \phi_{p^m}^1 : \mathbb{Z}/p^m \cong A^0_m \to E_{p^m}(S) \) with \( \sum_{a=0}^{p^m-1} \phi_{p^m}^1(a) = p^m - 1 \) and \( \phi_{p^m}^1|_{A^0_m} = \phi_{p^m}|_{A^0_m} \). We have that \( \text{Ker}(h_i) = \phi_{p^m}^1(\mathbb{Z}/p^m) \).

Let \( A^0_m, \ldots, A^m_m \) be elements for the equivalence classes of the quotient set for the action of \( \Gamma_1(p^m) \) on the set of \( p^m \)-cyclic subgroups \( A^i_m \) of \( (\mathbb{Z}/p^m)^{\oplus 2} \), where \( A^0_i = A^0_m \cap A^m_m \). The decomposition into reduced and irreducible components of \( M^0_{11}(4p^m)_k \) is \( \bigcup_{i=0}^{m} \Psi_i((C^0_{A^0_i})_k) \). This result is deduced from both [DR] V, 4.8, 4.11 and 4.12, and [KM] 13.5.6.

2.2. Néron models. Let \( R \) be a discrete valuation ring, \( K \) its fraction field, and \( k \) the residual field, which we assume to be perfect. The Néron model for an abelian variety \( J_K \) defined over \( K \) is a group scheme \( J \) smooth over \( R \), such that \( J_K \cong J_K \) and for each scheme \( \mathcal{H} \) over \( R \) the natural morphism
\[
(*) \text{Hom}_R(\mathcal{H}, J) \to \text{Hom}_K(\mathcal{H}_K, J_K)
\]
is bijective.

Let \( X \) be a flat, proper curve over \( R \) with \( X_K \) smooth and geometrically irreducible over \( K \). We assume that \( X_k \) has an irreducible component of geometric multiplicity 1. In the above conditions, the identity component for the Néron model \( J \) for \( \text{Pic}^0_{X_K/K} \) is \( \text{Pic}^0_{X/R} \), where \( \overline{X} \to X \) is a desingularization for \( X \). (C.f. [BLR], 9.5 Theorem 4 and 9.7 Theorem 1).

**Definition 2.3.** Let \( G, H \) be two abelian group schemes over \( R \). \( f : G \to H \) is said to be a quasi-isogeny if there exists a homomorphism \( g : H \to G \), with \( g \cdot f = [l] \), for some \( l \in \mathbb{N} \).

**Lemma 2.4.** Let \( J, L \) be the Néron models for the abelian varieties \( J_K \) and \( L_K \), respectively. Let \( f : J_K \to L_K \) be an isogeny and \( \tilde{f} : J \to L \)
the morphism associated with $f$ by the bijection $(*)$. We have that $f_k : J_k \to L_k$ is a quasi-isogeny and $f_{ab}^k : J_{ab}^k \to L_{ab}^k$ is an isogeny. We denote by $f_{ab}^k$ the homomorphism given by $f_k$ between the abelian parts of $J_k$ and $L_k$.

Proof. Since $f$ is an isogeny $f$ is a quasi-isogeny and therefore $f_k$ is also a quasi-isogeny. The remaining assertion is proved because over abelian varieties quasi-isogenies are isogenies. □

Lemma 2.5. Let $g$ be an endomorphism of an abelian variety $J_K$. Let $g_{ab}^k : J_{ab}^k \to J_{ab}^k$ be the endomorphism given by $g_k$. If $\dim \ker(g_{ab}^k) \geq r$ then $\dim \ker(g) \geq r$. Here, we assume that $\text{char}(K) = 0$.

Proof. By [BLR] 7.1, Corollary 6, $\ker(g)$ and $\text{im}(g)$ admit Néron models $N$ and $I$, respectively. Because $N$ is flat over $R$ we have $\dim \ker(g) = \dim N_k$. Thus, to prove the Lemma it suffices to prove that $N_{ab}^k$ is isogenous to $\ker(g_{ab}^k)$.

We have the complex $N \to J \overset{g}{\to} I$. Moreover, $J_K$ is isogenous to $\ker(g) \times \text{im}(g)$ so by the above Lemma $J$ and $J_k$ are quasi-isogenous to $N \times I$ and $N_k \times I_k$, respectively and $J_{ab}^k$ is isogenous to $N_{ab}^k \times I_{ab}^k$. Thus, we deduce an exact sequence, up to isogenies

$$0 \to N_{ab}^k \to J_{ab}^k \overset{g_{ab}^k}{\to} I_{ab}^k \to 0$$

and therefore $\ker(g_{ab}^k)$ is isogenous to $N_{ab}^k$. □

3. The Eichler-Shimura Theorem

3.1. Convention for correspondences. Let us consider a proper smooth and geometrically irreducible curve $Y$ over a field $k$. A correspondence $C \subset Y \times Y$, given by a Weil divisor, defines an endomorphism, denoted by $\hat{C}$, over the Jacobian $J_Y$ in the following way and convention: Let $R$ be an $k$-algebra and $L(D)$ a line bundle associated with a locally principal ideal $I_D \subseteq \mathcal{O}_{Y_R}$ (i.e: $I_D$ is given by an effective
Cartier divisor $D$ on $Y_R$), then, $\tilde{C}(L(D))$ is the line bundle on $Y_R$ with the Cartier divisor given (locally over $Y_R$) by the line bundle obtained from the kernel ideal of the ring morphism:

$$\phi : \mathcal{O}_{Y_R} \to \frac{\mathcal{O}_{Y_R} \otimes_R \mathcal{O}_{Y_R}}{I_D \otimes_R \mathcal{O}_{Y_R} + I_C},$$

with $\phi(a) := 1 \otimes a$, and $I_C$ is the ideal associated with $C$. We denote by $C(D)$ the Cartier divisor given by Ker($\phi$). By linear extension, one can obtain $\tilde{C}(L(D))$ and $C(D)$, with $D$ not effective.

Let $\tau : Y \to Y$ be a scheme morphism. We denote $\Gamma(\tau) = \{(x, \tau(x)) : x \in Y\}$ and its transpose $\Gamma(\tau) = \{(\tau(x), x) : x \in Y\}$.

**Lemma 3.1.** Let $k$ be a finite field of $q$ elements. If we denote by $D$ a Cartier divisor over $Y_R$, then $\gamma^*D = (\text{Fr} \times \text{Id})^*D$ and $\Gamma(\text{Fr})(D) = (\text{Id} \times \text{Fr})^*D$. Here, $\text{Fr}$ denotes the $q$-Frobenius morphism and $\gamma^*D$ denotes the pullback of $D$ by a morphism $\gamma$.

**Proof.** Let $I \subset \mathcal{O}_{Y_R}$ be an ideal and $I_\Delta \subset \mathcal{O}_{Y_R} \otimes_R \mathcal{O}_{Y_R}$ the diagonal ideal. The ideal $I_\Gamma$ given by $\Gamma(\text{Fr})$ is $(\text{Id}_Y \otimes \text{Id}_R) \otimes (\text{Fr} \otimes \text{Id}_R))(I_\Delta)$. Thus, $I \otimes_R \mathcal{O}_{Y_R} + I_\Gamma = (\text{Fr} \otimes \text{Id}_R)I \otimes_R \mathcal{O}_{Y_R} + I_\Gamma$. Thus, $\Gamma(\text{Fr})(D) = (\text{Fr} \times \text{Id}_R)^*D$ for any Cartier divisor $D$.

To prove the remaining equality we consider the composition of the correspondences $\Gamma(\text{Fr}) * \Gamma(\text{Fr}) = q \Delta$, where $\Delta \subset Y \times Y$ is the diagonal subscheme. Moreover, $(\text{Fr} \times \text{Fr})^*D = D^q$ for each Cartier divisor $D$ on $Y_R$ and $\Gamma(\text{Fr})[\Gamma(\text{Fr})(D) - (\text{Id} \times \text{Fr})^*D] = 0$ and we conclude, since $\Gamma(\text{Fr})$ is injective acting on the Cartier group of divisors of $Y_R$. \hfill $\square$

**3.2. The Eichler-Shimura Theorem for $T_p$.** We fix an isomorphism of groups $p/p^{m+1} \simeq \mathbb{Z}/p^m$ and we follow the notation of section 2.1

First, we shall define the Hecke correspondence $T_p$ on $M_1^0(4p^m)$.

Following [R], we consider the morphisms,

$$\beta, \alpha : M_1^0(4p^{m+1}) \to M_1^0(4p^m)$$
\[ \beta(E, \iota_4, C_{p^{m+1}}, \phi^{1}_{p^{m+1}}) = (E/C_p, \iota'_4, C_{p^{m+1}}/C_p, \phi^{-1}_{p^{m+1}}), \]

\[ \alpha(E, \iota_4, C_{p^{m+1}}, \phi^{1}_{p^{m+1}}) = (E, \iota_4, C_{p^{m}}, \phi^{1}_{p^{m+1}}|_{p/p^{m+1}}), \]

\[ \iota'_4 \text{ is deduced from the isomorphism } E_4 \cong E_4/C_p, \phi^{-1}_{p^{m+1}} \text{ is } \phi^{1}_{p^{m+1}} \mod p^{m} \text{ and } C_{p^i} = \phi^{1}_{p^{m+1}}(p^{m+1-i}/p^{m+1}). \]

Because \( M_1^0(4p^m) \) is an affine scheme it is a separated scheme over \( R_p \). Moreover, \( \alpha \) is a finite morphism, and therefore proper, because the finite morphism of forgetting the \( p^m \)-level structures, \( M^0(4p^m) \to M^0(4) \), factorizes through \( \alpha \). Thus, we have the closed subscheme \( \Gamma^0_p = \{ (\beta(x), \alpha(x)) \text{ with } x \in M^0_1(4p^{m+1}) \} \subset M^0_1(4p^m) \times_{R_p} M^0_1(4p^m). \)

Let \( S \) be an \( R_p \)-scheme. Then, \( \Gamma^0_p \) has the following modular interpretation

\[ \Gamma^0_p(S) = \{ (E, \iota_4, C_{p^m}, \phi^1_{p^m}), (E, \iota_4, C_{p^m}, \phi^{-1}_{p^m}) \} \]

such that there exists an isogeny \( \psi : E \to \overline{E} \) of degree \( p \) with \( \iota_4 = \psi \cdot \iota_4, \phi^{-1}_{p^m} = \psi \cdot \phi^1_{p^m} \) and \( \psi(C_{p^m}) = \overline{C}_{p^m}. \)

According to [DR] IV, Proposition 3.10, we consider \( M_1(4p^m) := (M(4p^m)/G)_{R_p}, \) which is a proper and flat scheme over \( R_p \), where

\[ G := \{ \gamma \in \text{Gl}_2(\mathbb{Z}) \text{ such that } \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \mod p^m \}. \]

This scheme is a compactification of \( M^0_1(4p^m) \).

Now, let \( f : \overline{M} \to M_1(4p^m) \) be a desingularization of \( M_1(4p^m) \). By [KM], Theorem 10.12.2, \( \overline{M} \) can be obtained from the \( \Gamma_1(p^m) \)-balanced moduli problem over \( M(4) \). Moreover, for each automorphism \( R_j \), there exists an automorphism \( R_j : M \to \overline{M} \), (which we also denote by \( R_j \)), such that \( R_j \cdot f = f \cdot R_j. \)

The schematic closure of \( \Gamma^0_p \subset M_1(4p^m) \times_{R_p} M_1(4p^m) \) will be denoted by \( \Gamma_p \) and \( \Gamma_p := (f \times f)^{-1}(\Gamma_p). \)

**Lemma 3.2.** \( \Gamma_p \) provides an endomorphism, which we denote by \( T_p \), of the scheme \( \text{Pic}^0_{M/R_p}. \)
Proof. According to [KM], 13.5.6, we are in the conditions of [BLR], Theorem 1, 9.7, and hence $\text{Pic}^0_{\mathcal{M}/R_p}$ is a smooth scheme over $\text{Spec}(R_p)$. It is the identity component of the Néron model for $\text{Pic}^0_{\mathcal{M}_1(4p^m)K/K} = \text{Pic}^0_{\mathcal{M}/K}$.

To find the endomorphism $T_p$ it suffices to give for each smooth morphism $W \rightarrow \text{Spec}(R_p)$ an endomorphism

$$(T_p)_W : \text{Pic}^0_{\mathcal{M}/R_p}(W) \rightarrow \text{Pic}^0_{\mathcal{M}/R_p}(W).$$

Let $\pi_i, i = 1, 2$, be the natural projections $\pi_i : \overline{\mathcal{M}}_W \times_W \overline{\mathcal{M}}_W \rightarrow \overline{\mathcal{M}}_W$, and $\overline{\pi}_i$ its restrictions to $(\overline{\Gamma}_p)_W$. Following the convention fixed in the previous section, let $L$ be a line bundle over $\overline{\mathcal{M}}_W$, $\overline{\pi}_1(L)$ is a line bundle on $(\overline{\Gamma}_p)_W$ with Weil divisor $Z$. Therefore, the pushforward $(\overline{\pi}_2)_*(Z)$ is a Weil divisor on the regular scheme $\overline{\mathcal{M}}_W$. We define $(T_p)_W(L)$ as the line bundle on $\overline{\mathcal{M}}_W$ associated with this Weil divisor. Note that $\overline{\mathcal{M}}_W$ is a regular scheme because $\overline{\mathcal{M}}_W \rightarrow \mathcal{M}$ is smooth and $\mathcal{M}$ is regular. □

For a morphism $R_p \rightarrow \mathbb{C}$, one can write down $T_p$ explicitly in terms of the modular parameter $\tau$. Let $E_\tau$ be the elliptic curve given by the torus obtained from the lattice $1 \cdot \mathbb{Z} + \tau \cdot \mathbb{Z}$, $\tau \in \mathbb{C}$. Let $(E, C_{p^m}, \phi_{p^m})$ be given by the pair $(E_\tau, E_{p^{-m}\mathbb{Z}}, \phi_{p^m}^1)$. Here, $\phi_{p^m}^1$ is given by the natural isomorphism $\mathbb{Z}/p^m\mathbb{Z} \simeq p^{-m}\mathbb{Z}/\mathbb{Z}$. We have:

$$T_p(E, p^{-m}\mathbb{Z}/\mathbb{Z}, \phi_{p^m}^1) = \sum_{i=0}^{p-1} (E_{i+\tau} \cdot \frac{p^{-m}\mathbb{Z}}{\mathbb{Z}}, \phi_{p^m}^1).$$

If $f(\tau)$ is a cusp form of weight 2 and level $p^m$, with $q$-expansion $\sum_{i=0}^{\infty} a_i q^i$, then $T_p(f(z)) = \sum_{i=0}^{\infty} a_ip^iq^i$.

**Notation** We denote $M := M_1(4p^m)$. We consider the desingularization morphisms $h : (\mathcal{M}_k)_{\text{red}} \rightarrow M_k$ and $\overline{h} : (\overline{\mathcal{M}}_k)_{\text{red}} \rightarrow \overline{M}_k$.

Let us consider $\tilde{f}_k : (\overline{\mathcal{M}}_k)_{\text{red}} \rightarrow (\overline{\mathcal{M}}_k)_{\text{red}}$, the morphism induced by $f_k : \overline{\mathcal{M}}_k \rightarrow M_k$. The morphism $\tilde{f}_k$ is surjective because $f$ is surjective. Moreover, we have $h \cdot \tilde{f}_k = f_k \cdot \overline{h}$.
We denote $\widetilde{(\Gamma_p)_k} := (h \times \bar{h})^{-1}(\Gamma_p)_k$ and $\widetilde{\Gamma_p}_k := (\bar{h} \times h)^{-1}(\Gamma_p)_k$.

Note that $(\Gamma_p)_k = (\bar{\Gamma}_p)_k \cap (h \times \bar{h})^{-1}(M_k^0 \times M_k^0)$.

We have that $(M_k)_\text{red} = \bigcup_{i=0}^m \Psi_i((\mathcal{C}_{A^0})_k)$ and therefore $\text{Pic}^0_{(M_k)_\text{red}/k} = \prod_{i=0}^m \text{Pic}^0_{(\mathcal{C}_{A^0})_k/k}$. Moreover, $\text{Pic}^0_{(\mathcal{C}_{A^0})_k/k} = \{0\}$, c.f \cite{KM} 13.5.6. Now, we prove the Eichler-Shimura Theorem for $T_p$ and $M_1(4p^m)$.

**Lemma 3.3.** We have that $\widetilde{(\Gamma_p)_k}(x) = \Gamma(\text{Fr})(x)$, where $x$ is a geometric point of $\Psi_i((\mathcal{C}_{A^0})_k)$, $(i > 0)$, given by $\Psi_i(E, \iota_4, \phi_{p^i})$ with $E$ an ordinary elliptic curve. Moreover, $(\Gamma_p)_k((\tilde{f}_k)^{-1}(x)) = (\tilde{f}_k)^{-1}(\Gamma(\text{Fr})(x)))$.

**Proof.** Let us consider $\Psi_i(E, \iota_4, \phi_{p^i}) = (E, \iota_4, C_{p^m}, \phi_{p^m}^1) = x$. Since $i > 0$ the subgroup $C_p \subset C_{p^m}$ is etale because $\phi_{p^i}(A^0)$ is an etale subgroup of $C_{p^m}$ of cardinality $p^i$, (see Remark 2.1). Let us denote $(\widetilde{\Gamma_p})_k(x) = \bar{x}$ with $\bar{x} = (\bar{E}, \bar{\iota}_4, \bar{C}_{p^m}, \bar{\phi}_{p^m}^1)$ and we have an isogeny of degree $p$, $\psi : E \to \bar{E}$, such that $\bar{\iota}_4 = \psi \cdot \iota_4$, $\bar{\phi}_{p^m}^1 = \psi \cdot \phi_{p^m}^1$ and $\psi(C_{p^m}) = \bar{C}_{p^m}$. Since $\psi(C_{p^m}) = \bar{C}_{p^m}$ we have $\text{Ker}(\psi) \cap C_p = \{0\}$ and thus we have that $\psi$ is the $p$-Frobenius morphism $E \to \text{Fr}^*E = \bar{E}$, (recall that $C_p$ is etale), and therefore $\bar{x} = \text{Fr}(x)$. The remaining assertion is deduced bearing in mind the above result and $(\Gamma_p)_k := (h \cdot \tilde{f}_k \times h \cdot \tilde{f}_k)^{-1}(\Gamma_p)_k$. Recall that, $h \cdot \tilde{f}_k = f_k \cdot \bar{h}$. 

$\square$

**Lemma 3.4.** Let $(T_p)_K : \text{Pic}^0_{M_K/K} \to \text{Pic}^0_{M_K/K}$ be the endomorphism given by $T_p$ and $T_p : \mathcal{J} \to \mathcal{J}$ the endomorphism given by $(T_p)_K$ for the Néron model $\mathcal{J}$ of $\text{Pic}^0_{M_K/K}$. We have that the endomorphism $T^0_p : \mathcal{J}^0 \to \mathcal{J}^0$ between the identity component of $\mathcal{J}$ is $T_p$. Recall that $M := M_1(4p^m)$ and that $k$ is the function field of $R_p$.

**Proof.** This is deduced from the equalities

$$\text{Hom}_R(\text{Pic}^0_{M/K}, \mathcal{J}) = \text{Hom}_K(\text{Pic}^0_{M_K/K}, \text{Pic}^0_{M_K/K}) = \text{Hom}_R(\mathcal{J}, \mathcal{J}).$$
These equalities are obtained because \( J \) is the Néron model for \( \text{Pic}^0_{M_K/K} \).

\[ \square \]

4. ANNIHILATORS FOR CUSP FORMS AND A CALCULATION

4.1. Annihilators for cusp forms. An elliptic curve over \( \mathbb{F}_p \) in the Legendre form \( y^2 = x(x - 1)(x - \lambda) \) is supersingular if and only if \( H(\lambda) = 0 \), with

\[
H(\lambda) = (-1)^m \sum_{i=0}^{m} \binom{m}{i}^2 \lambda^i
\]

the Deuring polynomial \( (m = (p - 1)/2) \). For more details see [H] 13.3.

The function field of \( M_4 \otimes_{\mathbb{Z}[i]} k \) is \( k(\sigma) \), with \( \lambda = (\sigma + 1/2\sigma)^2 \), C.f [H] 4.3. We denote \( p(\sigma) := \sigma^{p-1}H((\sigma + 1/2\sigma)^2) \). Here, \( k \) denotes the residual field of \( R_p \).

Let us consider \( q(\sigma) \in k[\sigma] \). We denote by \( K_{q(\sigma)} \) the \( q(\sigma)\mathfrak{m}_\infty \)-ray class field for \( k(\sigma) \) that is the maximal abelian extension, totally ramified, of \( k(\sigma) \) and with ideal of ramification given by \( q(\sigma)\mathfrak{m}_\infty \). Here, \( \mathfrak{m}_\infty \) is the maximal ideal associated with \( \infty, (1/\sigma = 0) \).

**Lemma 4.1.** We have that the Galois group of \( K_{q(\sigma)}/k(\sigma) \) is isomorphic to \( (k[\sigma]/q(\sigma))^\times \).

**Proof.** From class field theory, the abelian extension \( K_{q(\sigma)}/k(\sigma) \) has as Galois group \( I/t^Z_\infty \cdot k(\sigma)^\times \cdot U(q(\sigma)) \), where \( I \) is the idele group of \( k(\sigma) \), \( U(q(\sigma)) \) denotes the idele subgroup \( \{ \mu; \mu \equiv 1 \text{ mod } q(\sigma)\mathfrak{m}_\infty \} \) and \( t^Z_\infty \) is the idele whose entries are \( \sigma^{-1} \) at \( \infty \) and 1 elsewhere.

Let \( T \) be the zero locus of \( q(\sigma)\mathfrak{m}_\infty \). Let \( I^T \) and \( U^T \) the ideles of \( I \) and \( U \) outside \( T \), respectively, and \( K^\times(q(\sigma)) := k(\sigma)^\times \cap U(q(\sigma)) \). We have an isomorphism of groups

\[
\alpha : I^T/K^\times(q(\sigma)) \cdot U^T \rightarrow I/t^Z_\infty \cdot k(\sigma)^\times \cdot U(q(\sigma)),
\]

where \( \alpha(\mu_T) \) is the class of the idele in \( I \), whose entries are given by \( \mu_T \) outside \( T \) and 1 over the places of \( T \). Moreover, we have an
isomorphism
\[ \delta : \mathcal{I}^T / K^\times(q(\sigma)) \cdot U^T \to (k[\sigma]/p(\sigma))^\times \]
defined as follows. Let \( \mathfrak{h} \) be a place on \( k(\sigma) \) with \( \mathfrak{h} \notin T \) and \( h(\sigma) \) the irreducible polynomial associated with \( \mathfrak{h} \). Let \( t_\mathfrak{h} \in \mathcal{I}^T \) be the idele whose entries are \( h(\sigma) \) at \( \mathfrak{h} \) and 1 elsewhere. We define \( \delta(t_\mathfrak{h}) \) as the class of \( h(\sigma)^{-1} \) within \( (k[\sigma]/p(\sigma))^\times \).

We have that \( \Sigma \) is a totally ramified abelian extension of \( k(\sigma) \) whose ideal of ramification divides \( q(\sigma)\mathfrak{m}_\infty \) if and only if \( \Sigma \) is a subextension of \( K_{q(\sigma)} \) (see [Ha] Chapter 9).

Let \( Y_{q(\sigma)} \) be the smooth, proper and geometrically irreducible curve over \( k \) with function field \( K_{q(\sigma)} \). We have the following Anderson-Coleman result, (c.f. [An], [C]): there exists a trivial correspondence, up to vertical and horizontal correspondences on \( Y_{q(\sigma)} \times Y_{q(\sigma)} \)

\[ D_k := \sum_{i=0}^{d-1} \left( \sum_{\substack{r(\sigma)(\text{monic}) \\
 \deg(r(\sigma)) = d-1-i}} \Gamma(\beta_{r(\sigma)})(\Gamma(\text{Fr}^i)) \right), \]

with \( \deg(q(\sigma)) = d \), \( \beta_{r(\sigma)} := (\delta^{-1} \cdot \alpha)(r(\sigma)) \in \text{Aut}_{k(\sigma)}(K_{q(\sigma)}) \) and \( r(\sigma) \) are monic polynomials in \( \sigma \). We denote the composition of correspondences by * and \( \text{Fr} \) denotes the \( p^d \)-Frobenius morphism.

**Lemma 4.2.** The extension \( \Sigma_{A_{1p}} / k(\sigma) \), given by \( (C_{A_{1p}})_k \to M(4)_k \), is an abelian extension of group \( (\mathbb{Z}/p^m)^\times \). Moreover, \( \Sigma_{A_{1p}} \) is a subextension of \( K_{q(\sigma)} \), where \( q(\sigma) = p(\sigma)^r k[\sigma] \) for certain \( r \in \mathbb{N} \).

**Proof.** By [DR] V, Lemma 4.16, the morphism \( (C_{A_{1p}})_k \to M(4)_k \) is a ramified abelian covering of group \( (\mathbb{Z}/p^m)^\times \) and it is totally ramified over the pairs \( (E, \iota_4) \), with \( E \) a supersingular elliptic curve. This means that it is totally ramified for the values of \( \sigma \) where \( E \) is supersingular. These values are given by the roots of the Deuring polynomial \( \sigma^{p-1}H(\lambda), \lambda = (\sigma + 1/2\sigma)^2 \), (see for example, [H] Chapter 13, §3). \( \square \)
We have that $\Gamma(1)$ operates transitively over the irreducible components of $M(4p^m)_k = \cup_{i=0}^{d-1}(C_{A_{m}}^0)_k$. By fixing $A_{m}^0$ as in the introduction, there exists $g_i \in \Gamma(1)$, with $g_i((C_{A_{m}}^0)_k) = (C_{A_{m}}^0)_k$. The action of the elements $R_i$ on $g_i((C_{A_{m}}^0)_k)$ is given by $g_i \cdot R_i \cdot g_i^{-1}$.

The Galois group of $\Sigma_{A_{m}^0}/k(\sigma)$ is identified with the subgroup of $\Gamma(1)/\Gamma(p^m)$ formed by the classes of $\{R_i\}_{1 \leq i < p^m} \in \Gamma(1)$, where $R_i \equiv (i_0 \ 0 \ i-1) \mod \Gamma(p^m)$, $R_i \equiv \text{Id} \mod \Gamma(4)$, with $1 \leq i < p^m$ and $(i, p) = 1$.

**Remark 4.1.** Let $\pi : Y_{q(\sigma)} \to (C_{A_{m}^0})_k$ be the morphism induced by the inclusion $\Sigma_{A_{m}^0} \subset K_{q(\sigma)}$. Let us denote $\rho : \text{Gal}(K_{q(\sigma)}/k(\sigma)) \to (\mathbb{Z}/p^m)^\times$, the surjective morphism between the Galois groups of the extensions $K_{q(\sigma)}/k(\sigma)$ and $\Sigma_{A_{m}^0}/k(\sigma)$. The pushforward to $(C_{A_{m}^0})_k \times (C_{A_{m}^0})_k$ of the correspondence $D_k$,

$$(\pi \times \pi)_*(D_k) = \deg(\pi) \cdot \sum_{i=0}^{d-1} \left( \sum_{r(\sigma) \text{monic}} \Gamma(R_i(\sigma)) \ast \Gamma(Fr^i) \right)$$

is again a trivial correspondence, up to vertical and horizontal correspondences, where $R_i(\sigma) = \rho(\beta_i(\sigma))$. Thus, as the ring of classes of correspondences is $\mathbb{Z}$-free-torsion, $\deg(\pi)^{-1} \cdot (\pi \times \pi)_*(D_k)$ is also trivial.

Note that given a proper morphism $f : Z \to \bar{Z}$, between varieties, if $U$ is a $k$-cycle of $Z$, with $\text{dim} U = \text{dim} f(U)$, then the pushforward $f_*(U)$ is defined by

$$f_*(U) = [\Sigma_U : \Sigma_{f(U)}] \cdot f(U).$$

Where, $\Sigma_U$ and $\Sigma_{f(U)}$ are the function fields associated with $U$ and $f(U)$, respectively.

**Remark 4.2.** Let $Y$ be a proper curve over a field $k$ and $\tilde{Y}_{\text{red}}$ the normalization of the largest reduced subscheme $Y_{\text{red}}$ of $Y$. Then, by [BLR], 9.3 Corollary 11, $(\text{Pic}^0_Y/k)^{ab} \simeq \text{Pic}^0_{\tilde{Y}_{\text{red}}/k}$. We set $N = \dim(\text{Pic}^0_{\tilde{M}_1(4p^m)_k/k}^{ab})$.
**Theorem 1.** The operator

\[ D_K = \sum_{i=0}^{d-1} \left( \sum_{r(\sigma) \text{ (monic)}} \sum_{\deg(r(\sigma)) = d-1-i} R_{r(\sigma)} \cdot T_{p^i} \right) \]

annihilates a space of dimension \( N \) of cusp forms of level \( \Gamma(4) \cap \Gamma_1(p^m) \) and weight 2.

**Proof.** We have \((M^0_1(4p^m)_{k})_{\text{red}} = \bigcup_{i=0}^{m} \psi_i((\mathcal{C}_A^0)_k)\) (see Remark 2.1) and by the last Remark we have a surjective morphism

\[ \text{Pic}^0_{\text{M}_1(4p^m)_{k}/k} \to \prod_{i=1}^{m} \text{Pic}^0_{(\mathcal{C}_A^0)_{k}/k}. \]

Since \( A_0^0 \subset A_0^m \), we have morphisms of curves \( \pi_i : (\mathcal{C}_A^0)_k \to (\mathcal{C}_A^0)_k \) defined by \( \pi_i(E, \ell_4, \phi_{p^m}) = (E, \ell_4, \overline{\phi}_{p^m}) \), with \( \overline{\phi}_{p^m} = \phi_{p^m} \mid p^m - (Z/p^m)^{2} \). Moreover, because \( \pi_i \cdot \overline{\text{Fr}} = \overline{\text{Fr}} \cdot \pi_i \) we have that \((\pi_i \times \pi_i)_*, \Gamma(\text{Fr}) = \deg(\pi_i) \cdot \Gamma(\text{Fr})\). Here, we have denoted the Frobenius morphism over \( \mathcal{C}_A^0 \) and \( \mathcal{C}_A^0 \) in the same way. This latter statement also is true for the correspondences associated with the graphics of the automorphisms \( R_j \).

By Remark 4.1 since \( D_k \) annihilates \( \text{Pic}^0_{(\mathcal{C}_A^0)_{k}/k} \), the correspondence \( \deg(\pi_i)^{-1} \cdot (\pi_i \times \pi_i)_*(D_k) \) also annihilates \( \text{Pic}^0_{(\mathcal{C}_A^0)_{k}/k} \), with \( 0 \leq i \leq m \).

The Theorem is deduced from the Eichler-Shimura Theorem (c.f: Lemma 3.3), from Lemma 2.5 applied to \( \text{Pic}^0_{\text{M}_1(4p^m)_{k}/k} \) and to the endomorphism \( D_K \), knowing that \( D_k \) annihilates \( \prod_{i=1}^{m} \text{Pic}^0_{(\mathcal{C}_A^0)_{k}/k} \).

We bear in mind that the homomorphism

\[ (\hat{f}_k)^* : (\text{Pic}^0_{\text{M}_1(4p^m)_{k}/k})^{ab} \to (\text{Pic}^0_{M_{k}/k})^{ab} \]

has a finite kernel because \( \hat{f}_k : (\overline{\text{M}_k})_{\text{red}} \to (\overline{\text{M}_1(4p^m)})_{\text{red}} \) is surjective. \( \square \)
4.2. An explicit calculation. The morphisms \((C_{A_0^1})_k \to (C_{A_0^0})_k\) and \((C_{A_0^m})_k \to (C_{A_0^0})_k\) give us extensions \(k(\sigma) \subset \Sigma_{A_0^1}\) and \(\Sigma_{A_0^0} \subset \Sigma_{A_0^m}\), respectively, which are abelian extensions of groups \((\mathbb{Z}/p)^\times\) and \(\mathbb{Z}/p^{m-1}\), respectively. One can obtain the generator of the extension \(k(\sigma) \subset \Sigma_{A_0^1}\), because it is a totally ramified extension and its ramification is given by the polynomial \(p(\sigma)\), (c.f.\cite{I}). Thus, \(\Sigma_{A_0^0} = k(\sigma, p(\sigma)^{1/p-1})\).

According to \cite{K}, Lemma 2.5 and (5.5), together with the following two Lemmas, allow us to make explicit calculations for the Artin-Schreier generators for the extensions \(\Sigma_{A_0^0} \subset \Sigma_{A_0^m}\). We denote \(P(a) = a^p - a\).

**Lemma 4.3.** The extension \(k((t))(\mathcal{P}^{-1}(1/t^n))/k((t))\) is totally ramified over \(t = 0\).

*Proof.* An element \(1/x \in \mathcal{P}^{-1}(1/t)\) satisfies \(x^p + x^{p-1} \cdot t = t - t = 0\). This equation is a non-singular curve at the point \((0,0)\). The Lemma is proved for \(n = 1\) by considering the support of the differential \(k[[t]][x]\)-module

\[\Omega_{k[[t]][x]/x^p+x^{p-1},t-t}/k[[t]].\]

Bearing in mind the totally ramified extension \(k((t^n)) \to k((t))\) and \(k((t))(\mathcal{P}^{-1}(1/t^n)) = k((t^n))(\mathcal{P}^{-1}(1/t^n)) \otimes_{k((t^n))} k((t))\), one proves the Lemma for any \(n\). \(\square\)

**Lemma 4.4.** If \(k(t, \mathcal{P}^{-1}(r(t)/s(t))))/k(t)\) is an extension such that its ideal of ramification divides the polynomial \(p(t)\), then there exists \(l \in \mathbb{N}\) with \(r(t)/s(t) = h(t)/p(t)^l + \mathcal{P}(u(t)/v(t))\) and \(\deg(h(t)) \leq \deg(p(t)^l)\).

*Proof.* This is deduced from the above Lemma bearing in mind the decomposition in simple fractions of \(r(t)/s(t)\) and the fact that \(\mathcal{P}\) is additive. \(\square\)
By the previous Lemma to make explicit calculations of the Artin-Schreier generators of $\Sigma_{A_1} \subset \Sigma_{A_0}$ it suffices to calculate $l$ and $h(t)$ for these extensions; $p(t)$ is the polynomial $p(\sigma)$ defined in section 4.1.

As example we shall make this calculation precise for $m = 2$. Following the notation of [K] §2 and [Se] 2.2, the Artin-Shreier generator for $\Sigma_{A_2} \subset \Sigma_{A_1}$ is the reduction mod $p$ of $\psi := \beta(1)(\frac{a_p - 1}{p}) = \frac{a_p - 1}{p(E_{p-1})} \in \mathbb{F}_p((q))$, where $a_p$ is a modular form of weight $p - 1$. By [Se], 2.2, we have that

$$(\psi)^p - \psi = -\frac{1}{24} \cdot \beta(1)(\theta^{p-2}(E_{p+1})) = \frac{\theta^{p-2}(E_{p+1})}{E_{3}^{p-1}} \pmod{p},$$

with $\theta = a\frac{\lambda}{\lambda^3}$ and $\theta^{p-2}(E_{p+1})$ a modular form of weight $3(p - 1)$.

Here, $E_{p+1}$ and $E_{p-1}$ denote the normalization, (constant term = 1), of the Eisenstein series of weights $p + 1$ and $p - 1$, respectively.

The algebra of modular functions is given by the graded ring of polynomials $\mathbb{Z}[g_2, g_3]$, where $g_2$ and $g_3$ have grades 4 and 6, respectively. If $A(g_2, g_3)(\frac{dx}{2y})^{(p-1)}$, $(A(g_2, g_3) \in \mathbb{F}_p[g_2, g_3])$, is the Hasse invariant of the universal elliptic curve in the Weierstrass form $y^2 = 4x^3 - g_2x - g_3$ then the class of $E_{p-1}$ in $\mathbb{F}_p[g_2, g_3]$ is $A(g_2, g_3)$.

Since the Deuring polynomial gives the Hasse invariant for the universal elliptic curve in the Legendre form $y^2 = x(x - 1)(x - \lambda)$, we have that $\Sigma_{A_2} = \Sigma_{A_1}[\psi]$ with $\psi^p - \psi - \frac{h(\sigma)}{p(\sigma)}$. Here, $p(\sigma) = \sigma^{p-1} \cdot H(\lambda)$ with $\lambda = (\sigma + 1/2\sigma)^2$.

Moreover, because $\frac{\theta^{p-2}(E_{p+1})}{E_{3}^{p-1}}$ only has poles over the zeroes of $E_{p-1}(z)dz$ we have that $\deg(h(\sigma)) \leq \deg(p(\sigma)^3)(= 3(p - 1))$. To calculate $h(\sigma)$ it suffices to bear in mind the expansions on the parameter $q^{1/4}$ of $\sigma$, $\frac{\theta^{p-2}(E_{p+1})}{E_{3}^{p-1}}$ and the congruence

$$\frac{\theta^{p-2}(E_{p+1})}{E_{3}^{p-1}} \equiv \frac{h(\sigma)}{p(\sigma)^3} \pmod{p}.$$
Bearing in mind this latter calculation and the action of \((k[\sigma]/p(\sigma)^3)^{\times}\) on \(\psi\), given by the class field theory, (c.f. [Ha] Chapter 9), one can explicitly obtain the group homomorphism

\[\rho : (k[\sigma]/p(\sigma)^3)^{\times} \rightarrow (\mathbb{Z}/p^m)^{\times}\].

4.3. A way to calculate the \(p\)-term of the Hasse-Weil \(L\)-function of \(M_1(4p^m)\). To obtain the \(p\)-term in the Euler product of the Hasse-Weil \(L\)-function of \(M_1(4p^m)\) it suffices to obtain the Euler zeta function of the curve \((\mathcal{C}_{A_0^m})_k\).

We, now, consider the incomplete \(L\)-function associated with the Galois extension \(K_{q(\sigma)}/k(\sigma)\):

\[\prod_{x \in [\mathbb{P}^1(k)] \setminus T} (1 - F_x \cdot t^{\deg(x)})^{-1} = \frac{U(t)}{1 - p^4 \cdot t},\]

(C.f [An] 1.3), where \(U(t)\) is an explicit polynomial in \(\mathbb{Z}[G][t]\); \(G = (k[\sigma]/p(\sigma))^{\times}\); \(F_x\) is the Frobenius element for \(x \in [\mathbb{P}^1(k)] \setminus T\), \(T\) being the places in \([\mathbb{P}^1(k)]\) given by the divisors of \(q(\sigma)\mathfrak{m}_\infty\).

We define \(\text{Norm}_{\mathbb{Z}[G]/\mathbb{Z}[H]}(U(t))\) as the determinant of the morphism of multiplication by \(U(t)\) in the \(\mathbb{Z}[t][H]\)-module \(\mathbb{Z}[t][G]\). \(H\) being the kernel of the group morphism

\[\rho : (k[\sigma]/q(\sigma))^{\times} \rightarrow (\mathbb{Z}/p^m)^{\times}\]

given by the Galois extensions \(k(\sigma) \subset \Sigma_{A_0^m} \subset K_{q(\sigma)}\).

Let \(\pi : (\mathcal{C}_{A_0^m})_k \rightarrow \mathbb{P}^1(k)\) be the morphism of smooth curves given by \(k(\sigma) \subset \Sigma_{A_0^m}\). Bearing in mind that

\[\text{Norm}_{\mathbb{Z}[G]/\mathbb{Z}[H]}(1 - F_x \cdot t^{\deg(x)}) = \prod_{z \in \pi^{-1}(x)} (1 - F_z \cdot t^{\deg(z)})^{-1},\]

we can calculate the incomplete \(L\)-function

\[\prod_{z \in (\mathcal{C}_{A_0^m})_k \setminus \pi^{-1}(T)} (1 - F_z \cdot t^{\deg(z)})^{-1}\]
as a faction of polynomials $\frac{V(t)}{W(t)}$, where
\[
\frac{V(t)}{W(t)} = \frac{\text{Norm}_{\mathbb{Z}[G]/\mathbb{Z}[H]}(U(t))}{\text{Norm}_{\mathbb{Z}[G]/\mathbb{Z}[H]}(1 - p^g \cdot t)}.
\]

By making $h = 1$ for each $h \in H$ in the quotient $\frac{V(t)}{W(t)}$, we obtain a fraction of polynomials $\frac{v(t)}{w(t)}$. The Euler zeta function of $(C_{\alpha_1^2})_k$ is
\[
\frac{v(t)}{w(t)} \cdot \prod_{z \in \pi^{-1}(T)} (1 - t^{\deg(z)})^{-1}.
\]

By taking account the above section we can obtain $H$ in an explicit way and we can calculate $\frac{v(t)}{w(t)}$. Moreover, we obtain the finite product $\prod_{z \in \pi^{-1}(T)} (1 - t^{\deg(z)})^{-1}$ by considering $\pi^{-1}(\infty)$ and the roots of $q(\sigma)$, which are given by the Deuring polynomial $H(\lambda).

5. The cuspidal principal part (mod $p$) of certain meromorphic modular functions

5.1. Line bundles from an adelic point of view. Let $Y$ be a smooth, projective and geometrically irreducible curve over a field $k$ of $q = p^g$ elements; $\Sigma$ denotes its function field. Let $q$ be a place of $\Sigma$, $t_q$ a local parameter for $q$ and $k(q)$ its residual field. For each $k$-algebra $R$, we consider the group $(R \otimes k(q))( (t_q) ) := (R \otimes k(q))[t_q][t_q^{-1}]$ and the adele group
\[
A_{\Sigma}(R) := \prod_{q \in |Y|}^\prime (R \otimes k(q))( (t_q) ),
\]
where $\prod^\prime$ refers to the adeles with a non-trivial principal part only on a finite number of places of $\Sigma$. The idele group $I_{\Sigma}(R)$ is the group of units of $A_{\Sigma}(R)$. Thus,
\[
I_{\Sigma}(R) = ( \prod_{q \in |Y|}^\prime (R \otimes k(q))( (t_q) ))^\times.
\]

Let $U_{\Sigma}(R)$ be the subgroup $\prod_{q \in |Y|} (R \otimes k(q))[t_q]^\times$. Let $\Sigma \otimes_k R$ be denoted by $\Sigma_R$. One can embed the group $(\Sigma_R)^\times$ diagonally into $I_{\Sigma}(R)$. 

The quotient group
\[
\frac{I_\Sigma(R)}{\Sigma_R^\times \cdot U_\Sigma(R)}
\]
is isomorphic to the class group of line bundles \( L \) on \( Y \otimes R \), such that if \( j : \text{Spec}(\Sigma_R) \hookrightarrow Y_R \) is the natural inclusion then the pullback \( j^*L \) is isomorphic to the trivial line bundle. See, for example, [BL], Section 2, and Lemma 3.4.

If we denote by \( J_Y \) the Jacobian variety of \( Y \) and \( \mu \) is an idele of degree 1, we have a monomorphism of groups
\[
\frac{I_\Sigma(R)}{\mu^\Sigma \cdot \Sigma_R^\times \cdot U_\Sigma(R)} \hookrightarrow J_Y(R).
\]

This monomorphism is an isomorphism if and only if \( R \) is a finite \( k \)-algebra. Recall that \( J_Y(R) \) is the group of isomorphism classes of line bundles over \( Y_R \) of degree 0.

In the following Lemma we recall that the tangent space of the Jacobian on the zero element can be identified with the cohomology group \( H^1(Y, \mathcal{O}_Y) \) and by duality with \( H^0(Y, \omega)^\vee \) (\( \omega \) is the dualizing sheaf of \( Y \)):

**Lemma 5.1.** We have an exact sequence of groups
\[
0 \to H^1(Y, \mathcal{O}_Y) \to J_Y(k[\epsilon]) \xrightarrow{\epsilon = 0} J_Y(k) \to 0, \quad (\epsilon^2 = 0).
\]

*Proof.* It suffices to take into account that \( k[\epsilon] \) is a finite \( k \)-algebra and that
\[
J_Y(k[\epsilon]) \simeq \frac{I_\Sigma(k[\epsilon])}{\mu^\Sigma \cdot \Sigma^\times_{k[\epsilon]} \cdot U_\Sigma(\mathbb{F}_q[\epsilon])} = \frac{I_\Sigma}{\mu^\Sigma \cdot \Sigma^\times \cdot U_\Sigma} + \epsilon \cdot \frac{\mathbb{A}}{\Sigma + O_\Sigma},
\]
where \( O_\Sigma \) denotes the integer adeles. \( \square \)

**Remark 5.1.** Let \( D \) be a Cartier divisor given by the class of an idele
\[
\mu_0 + \mu_1 \epsilon \in \frac{I_\Sigma(k[\epsilon])}{U_\Sigma(k[\epsilon])} = \frac{I_\Sigma}{U_\Sigma} + \epsilon \cdot \frac{\mathbb{A}}{O_\Sigma}.
\]
According to Lemma 3.1, \( \Gamma(\text{Fr})(D) \) and \( \Gamma(\text{Fr})(D) \) are given by \( \mu_0 + \mu_1 \epsilon^q \) and \( \mu_0^q + \mu_1^q \epsilon \), respectively. Note that \( \epsilon^q = 0 \).
5.2. Correspondences in an additive setting. We now prove a Lemma that will be used to make explicit calculations in the next section. Let \( \text{Spec}(B) \subset Y \) be an open subscheme and let \( C \) be a correspondence on \( Y \) that is trivial over \( \text{Spec}(B) \otimes \text{Spec}(B) \). We set that \( C \) over \( \text{Spec}(B \otimes B) \) will be given by the zero locus of a regular function \( H(b, \bar{b}) \in B \otimes B \). Let \( q \in \text{Spec}(B) \) be a geometric point and \( t_q \) a local parameter for \( q \). For easy notation, we can assume \( q \) to be rational. We denote by \( D^{-r \cdot q}, D^{r \cdot q} \) the Cartier divisors on \( Y \otimes k[\epsilon] \) given by the line bundles associated with the ideles of \( I_\Sigma(k[\epsilon]) \), whose entries are \( 1 - \epsilon t_q^{-r} \) and \( t_q^r - \epsilon \) at \( q \) and 1 elsewhere, respectively.

**Lemma 5.2.** With the notations and conventions of section 3.1, \( C(D^{-r \cdot q}) \) is a Cartier divisor such that over the open subset \( \text{Spec}(B[\epsilon]) \subset Y[\epsilon] \) is given by \( 1 + \epsilon \text{Res}_q(t_q^{-r} \text{d log } H(t_q, \bar{b})) \in 1 + \epsilon \cdot \Sigma \).

To calculate this residue, one considers \( H(b, \bar{b}) = H(t_q, \bar{b}) \in B((t_q)) \) and \( \text{d log } H(t_q, \bar{b}) = \frac{\text{d log } H(t_q, \bar{b})}{\text{d } t_q} \text{d } t_q \).

**Proof.** We have that \( C(D_{r \cdot q}) \) over \( \text{Spec}(B) \) is given by the kernel ideal of the \( B \)-algebra morphism:

\[
\phi : B[\epsilon] \to B[\epsilon, t_q]/(t_q^r - \epsilon, H(t_q, \bar{b})), \quad \phi(\epsilon) = \epsilon.
\]

Since \( t_q^{2r} = 0 \) as an element of

\[
B[\epsilon, t_q]/(t_q^r - \epsilon, H(t_q, \bar{b})),
\]

one can assume that \( H(t_q, \bar{b}) = u(t_q) = b_1 t_q^{2r-1} + \cdots + b_{2r-1} t_q + b_{2r}, \)

with \( b_i \in B \).

The ideal \( \text{Ker}(\phi) \) is generated by \( \prod_i (\alpha_i^r - \epsilon) \); this product is taken over the roots of \( u(t_q) \).

Because \( C(D^{-r \cdot q}) = C(-r \cdot q)C(D_{r \cdot q}) \) and because the Cartier divisor \( C(-r \cdot q) \) is given over \( \text{Spec}(B) \) by the element \( \prod_i \alpha_i^{-r} \), we have that the Cartier divisor defined over \( \text{Spec}(B[\epsilon]) \), \( C(D^{-r \cdot q}) = C(t_q^{-r})C(t_q^r - \epsilon) \) is given by the element \( \prod_i (1 - \epsilon \alpha_i^{-r}) = 1 - \epsilon(\sum_i \alpha_i^{-r}) \).
By using the residues theorem for 1-differential forms of $\Sigma(t_q)$, where the field of constants is $\Sigma$, we conclude because

$$\text{Res}_q(t_q^{-r}d \log H(t_q, b)) = - \sum_y \text{Res}_y(t_q^{-r}d \log H(t_q, \bar{b})) = \sum_i \alpha_i^{-r}.$$ 

The second sum is taken over the maximal ideals $y$ of $\Sigma[t_q^{-1}]$. □

Remark 5.2. It is not hard to prove that if $a \in B((t_q))$ with either $a \in B$ or $a \in k((t_q))$ then $\text{Res}_q(t_q^{-r}d \log (aH(t_q, \bar{b}))) = \text{Res}_q(t_q^{-r}d \log H(t_q, \bar{b})) + \alpha$, for some $\alpha \in k$. Here, $aH(t_q, \bar{b}) \in B((t_q))$.

From Lemma 5.2 and the above Remark, to calculate $C(D^{-r\cdot q})$ as a Cartier divisor on $Y_R$, one can forget the vertical and horizontal components of $C$. Thus, this Cartier divisor is given, up to units of $k[\epsilon]$, by $1 + \epsilon \cdot \text{Res}_q(t_q^{-r}d \log H(t_q, \bar{b}))$.

5.3. The principal part (mod $p$) for certain meromorphic modular functions of level $4p^m$ with poles over the cusps. By [DR] VI, 2.3.1, we have a section $s_n : \text{Spec}(\mathbb{Z}[\zeta_n]) \to M(n)$. The completion of $M(n)$ along $s_n$ is identified with $\text{Spec}(\mathbb{Z}[\zeta_n][[q^{1/n}]]) \to M(n)$. This morphism can be obtained from the Tate elliptic curve with an $n$-level structure. The completion along the cusps of $M(n)$ is a finite number of copies of $\text{Spec}(\mathbb{Z}[\zeta_n][[q^{1/n}]]).$ From this, one deduces that the abelian ramified covering $(C_{A_0^m})_k \to M(4)_k$ splits completely over the cusps $M(4)_k \setminus M^0(4)_k$.

By tensoring $s_{4p^m}$ by $\otimes_{R_q} k$, one obtains a section $(s_{4p^m})_k : \text{Spec}(k) \to M(4p^m)_k$. We denote by $\infty$ the geometric point of $M(4p^m)_k$ given by $(s_{4p^m})_k$ and we denote by $q$ the local parameter $q^{1/4p^m} \pmod{p}$.

Recall that $M(4p^m)_k = \bigcup_{k=0}^{l-1}(C_{A^m_{l-1}})_k$. We denote by $B$ the ring such that $\text{Spec}(B) = M^0(4p^m)$.

Bearing in mind [DR] V, 4.19 and 4.20, we have an isomorphism

$$\phi : B_k[p(\sigma)^{-1}] \simeq (H_{A^m_0})_{S_0} \times \cdots \times (H_{A^m_{l-1}})_{S_{l-1}},$$
where we denote Spec$(H_{A_m}^i) = (\mathcal{O}_{A_m}^0)_{k}$ and $S_0, \cdots, S_{l-1}$ are multiplicative systems given by the powers of the image of the polynomial $p(\sigma)$ in $H_{A_m}^0, \cdots, H_{A_m}^{l-1}$, respectively.

**Theorem 2.** For each $m \in \mathbb{N}$, there exists an element of $B_k[p(\sigma)^{-1}]$ such that its principal part is

$$\sum_{i=0}^{d-1} \left( \sum_{r(\sigma) \text{monic}} \frac{1}{(\mu_{mpi\delta})^{R_{r(\sigma)}}} \right).$$

**Proof.** We assume that $\infty \in (\mathcal{C}_{A_m}^i)_{k}$ and we denote by $\Sigma_{A_m}^i$ its fraction field and by $\mu_m$ the idele whose entries are $1 + \epsilon q^{-m}$ at $\infty$ and $1$ elsewhere. Following the notation of section 5.2, let $D^{-m\infty}$ be the Cartier divisor associated with $\mu_m$. Now, if we consider the transpose of the trivial correspondence $\deg(\pi)^{-1}(D_k)$ of the Remark 4.1 then $\deg(\pi)^{-1}(t D_k)(D^{-m\infty})$ is the idele on $k[\epsilon]$

$$\prod_{i=0}^{d-1} \left( \prod_{r(\sigma) \text{monic}} \frac{1}{(\mu_{mpi\delta})^{R_{r(\sigma)}}} \right) \in 1 + \epsilon \frac{A_k}{O_{\Sigma_{A_m}^i}}.$$

Bearing in mind Lemmas 5.1 and 5.2 we obtain an element $b_m \in (H_{A_m}^i)_{S_j}$ such that $1 + \epsilon b_m \in 1 + \epsilon \frac{A_k}{O_{\Sigma_{A_m}^i}}$ is the above idele. We finish by considering the element $\phi^{-1}(0, \cdots, b_m, \cdots, 0) \in B_k[p(\sigma)^{-1}]$. □

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