1. Introduction and statement of results

In this paper, we prove the following result:

**Theorem 1.**

$$
\sum\sum_{\ell^2 + \ell m + m^2 \leq x} A(2\ell - m)A(\ell^2 - \ell m + m^2) \sim \sigma x
$$

for some $\sigma > 0$.

We shall prove Theorem 1 by following along the lines of the proof of Theorem 20.3 in [FI2], by using $Q(\omega)$ rather than $Q(i)$ when working with the bilinear forms that arise in Section 20.4 of [FI2]. A related result was proved by Fouvry and Iwaniec in [FoI] where it is shown that there are infinitely many primes of the form $\ell^2 + m^2$ such that $\ell$ is prime.

2. Preliminaries

Let $\gamma_\ell = \log \ell$ when $\ell$ is a prime greater than 2 and 0 otherwise. Then, let

$$
a_n = \sum_{\ell^2 - \ell m + m^2 = n} \gamma_2 \ell - m = \sum_{r^2 + 3s^2 = 4n} \gamma_r.
$$

Let

$$
A(x) = \sum_{n \leq x} a_n
$$

and let

$$
A_d(x) = \sum_{n \leq x, n \equiv 0 (\mod d)} a_n
$$

Let $\rho(d) = |\{v \in \mathbb{Z}/(d) : v^2 + 3 \equiv 0 (\mod d)\}|$.

We expect that $A_d(x)$ is well approximated by

$$
M_d(x) = \frac{\rho(4d)}{4d} \sum_{r \leq \sqrt{4x}} \frac{1}{2} \gamma_r \sqrt{\frac{4x - r^2}{3}}
$$

so we let the remainder terms $r_d(x)$ be such that

$$
A_d(x) = M_d(x) + r_d(x)
$$

For $d$ even, this is clearly equal to 0, while for $d$ odd, since $\rho(d)$ is multiplicative, this is equal to

$$
\frac{\rho(d)}{4d} \sum_{r \leq \sqrt{4x}} \gamma_r \sqrt{\frac{4x - r^2}{3}}
$$

We then have the following:
Proposition 1. Suppose that for some \( \sqrt{x} < D \leq x(\log x)^{-20} \),
\[
(2.1) \quad R(x; D) = \sup_{y \leq x, d \leq D} \sum_{y \leq n \leq \sqrt{x}} |r_d(n)| \ll A(x) \log^{-2} x
\]
and let
\[
(2.2) \quad T(x; D) = \sum_{\ell \leq D} \left| \sum_{\ell \leq x} a_{\ell m}(m) \right|
\]
Then, we have that
\[
(2.3) \quad \sum_{n \leq x} a_n A(n) = HA(x) \left\{ 1 + O((\log x)^{-1}) \right\} + O(T(x, D) \log x)
\]
where \( A(x) = A_1(x), g(d) = M_d(x)/A(x) \), and
\[
H = \prod_p \left( 1 - g(p) \right) \left( 1 - \frac{1}{p} \right)^{-1}
\]
Proof. This is Theorem 18.6 in [FI2] for our particular sequence. \qed

3. THE REMAINDER TERM

In this section, we verify that (2.1) holds. From this point on, \( e(\alpha) = e^{2\pi i \alpha} \). First, we study the distribution of the roots of the congruence \( v^2 + 3 \equiv 0 \mod d \) by studying Weyl sums related to these quadratic roots.

In order to do so, we will establish a well-spacing of the points \( v/d \mod 1 \). It is easy to show that for odd \( d \), the roots to \( v^2 + 3 \equiv 0 \mod d \) each correspond to a representation
\[
d = r^2 + rs + s^2 = \frac{(r-s)^2 + 3(r+s)^2}{4}
\]
such that \( (r,s) = 1, -r-s < r-s \leq r+s \) where \( v(r-s) \equiv (r+s) \mod d \).

It then follows that
\[
\frac{v}{d} \equiv -\frac{4(r-s)}{r+s} + \frac{r-s}{d(r+s)} \mod 1
\]
where \( r-s \) is such that \( (r-s)(r-s) \equiv 1 \mod r+s \).

Note that we then have that
\[
\frac{|r-s|}{d(r+s)} < \frac{1}{2(r+s)^2}
\]
Now, restrict \( d \) to the range \( 4D < d \leq 9D \). It then follows that \( 2D^{1/2} < r+s < 3D^{1/2} \), so for any two points \( v_1/d_1, v_2/d_2 \),
\[
\left\| \frac{v_1}{d_1} - \frac{v_2}{d_2} \right\| > \frac{4}{(r_1+s_1)(r_2+s_2)} - \max \left\{ \frac{1}{(r_1+s_1)^2}, \frac{1}{(r_2+s_2)^2} \right\} \gg \frac{1}{D}
\]
Then by the large sieve inequality of Davenport and Halberstam, we have the following

Lemma 2. For all \( \alpha_1, \alpha_2, \ldots \in \mathbb{C} \), we have that
\[
\sum_{D < d \leq 2D} \sum_{d \equiv 1 \mod 2} \left| \sum_{n \leq N} \alpha_n e\left( \frac{vn}{d} \right) \right|^2 \ll (D + N) \left( \sum_n \alpha_n^2 \right)
\]
Applying Cauchy’s inequality yields

**Proposition 2.** For all \( \alpha_1, \alpha_2, \cdots \in \mathbb{C} \), we have that

\[
\sum_{D<d\leq 2D} \sum_{d \equiv 1 \pmod{2}} \left| \sum_{n \leq N} \alpha_n e\left(\frac{vn}{d}\right) \right| \ll D^{1/2} (D + N)^{1/2} \left( \sum_n \alpha_n^2 \right)^{1/2}.
\]

Now, let

\[
\rho_h(d) = \sum_{v^2 + 3w \equiv 0 \pmod{d}} e\left(\frac{vh}{d}\right).
\]

Then, the following holds:

**Proposition 3.**

\[
\sum_{d \leq D} \left| \sum_{h \leq N} \alpha_h \rho_h d \right| \ll D^{1/2} (D + N)^{1/2} \left( \sum_n \alpha_n^2 \right)^{1/2}.
\]

Now, we prove that (2.1) holds by proving the following:

**Proposition 4.** For all \( D \leq x \)

\[
\sum_{d \leq D} \left| r_d(x) \right| \ll D^{1/4} x^{3/4+\varepsilon}.
\]

**Proof.** Note that

\[
A_d(x) = \sum_{\frac{r^2 + 3s^2}{4} \leq x} \gamma_r.
\]

It is more convenient for now to consider only the contribution of the terms with \((r, d) = 1\). To that end, note that it is possible to replace \(A_d(x)\) with

\[
A_d^*(x) = \sum_{\frac{r^2 + 3s^2}{4} \leq x} \gamma_r
\]

since

\[
\sum_{d \leq D} \left| A_d(x) - A_d^*(x) \right| \leq \sum_{d \leq D} \sum_{\ell \mid d} \left| \gamma_\ell \right| \sum_{r^2 + 3s^2 \leq 4x} \tau(r^2 + 3) \ll x^{1/2+\varepsilon}.
\]

Now, rather than approximating \(A_d^*(x)\), we shall approximate

\[
A_d^*(f) = \sum_{r^2 + 3s^2 \equiv 0 \pmod{4d}} \gamma_r f\left(\frac{r^2 + 3s^2}{4}\right)
\]

for some smooth \(f\) supported on \([1, x]\) satisfying

\[
f(u) = 1, \text{ for } y \leq u \leq x - y.
\]
\[
  f^{(j)}(x) \ll x^{-j}
\]
where \(y = \min\{x^{3/4}D^{1/4}, \frac{1}{x}\}\). Note that bounding this is sufficient, since
\[
  \sum_{d \leq D} |A_d^*(f) - A_d^*(x)| \leq \sum_{\ell^2 - \ell m + m^2 \in I} \tau(\ell^2 - \ell m + m^2) \ll yx^\varepsilon
\]
where \(I = \mathbb{Z} \cap ([1, y] \cup [x - y, x])\). Note that since \(\gamma_r\) is supported on odd primes, we have that
\[
  A_d^*(f) = \sum_{v^2 + 3 \equiv 0 \pmod{4d}} \sum_{(r, d) = 1} \gamma_r \sum_{s \equiv r \pmod{4d}} f\left(\frac{r^2 + 3s^2}{4}\right).
\]
Now, let
\[
  A_d(f) = \sum_{v^2 + 3 \equiv 0 \pmod{4d}} \sum_{r} \gamma_r \sum_{s \equiv r \pmod{4d}} f\left(\frac{r^2 + 3s^2}{4}\right).
\]
We can replace \(A_d^*(f)\) with \(A_d(f)\) with an error of \(O(y \log x)\), which is small enough. We then have that by Poisson’s formula
\[
  A_d(f) = \frac{1}{4d} \sum_{r} \gamma_r \sum_{k \in \mathbb{Z}} \rho_{kr}(4d) F_r\left(\frac{k}{4d}\right)
\]
where
\[
  F_r(v) = \int_{\mathbb{R}} f\left(\frac{v^2 + 3r^2}{4}\right) e(-vt) dt = 2 \int_{0}^{\infty} f\left(\frac{v^2 + 3r^2}{4}\right) \cos(2\pi vt) dt.
\]
Note that the contribution from when \(k = 0\) is equal to \(M_d(x) + O(y)\), so it is necessary and sufficient to bound the contribution from \(k \neq 0\). To that end, note that by the change of variable \(t = w\sqrt{x}/k\),
\[
  F_r\left(\frac{k}{4d}\right) = \frac{2\sqrt{x}}{k} \int_{0}^{\infty} f\left(\frac{v^2 + 3w^2}{4}\right) \cos\left(\frac{2\pi \sqrt{x}w}{4d}\right) dw.
\]
Integrating by parts twice yields that this equals
\[
  \frac{16\sqrt{x}d^2}{\pi^2 k^3} \int_{0}^{\infty} \left( f' + \frac{2w^2 x}{k^2} f'' \right) \left(\frac{v^2 + 3w^2}{4}\right) \cos\left(\frac{\pi \sqrt{x}w}{2d}\right) dw.
\]
Now, let
\[
  R(f, D) = \sum_{D < d \leq 2D} \left| \frac{1}{4d} \sum_{r} \gamma_r \sum_{k \in \mathbb{Z}} \rho_{kr}(4d) F_r\left(\frac{k}{4d}\right) \right|.
\]
We then have that
\[
  R(f, D) \ll \frac{1}{D} \sum_{D < d \leq 2D} \sum_{kr \neq 0} \gamma_r F_r\left(\frac{k}{4d}\right).
\]
To estimate this, we split this into sums with \(|k|\) restricted to certain ranges. In particular, we write
\[
  R_k(f, D) = \frac{1}{D} \sum_{D < d \leq 2D} \sum_{2^k \leq |k| < 2^{k+1}} \sum_{r} \gamma_r F_r\left(\frac{k}{4d}\right).
\]
Then, we have that by (3.4) and Proposition 3, \(R_n(f, D)\) is
\[
  \frac{1}{D} \sum_{D < d \leq 2D} \left| \sum_{2^k \leq |k| < 2^{k+1}} \sum_{r} \gamma_r \rho_{kr}(d) \frac{2\sqrt{x}}{k} \int_{0}^{\infty} f\left(\frac{r^2 + 3w^2}{4}\right) \cos\left(\frac{\pi \sqrt{x}w}{2d}\right) dw \right| \ll \frac{\sqrt{x}}{D} \int_{0}^{2^{k+1}} \sum_{D < d \leq 2D} \sum_{2^k \leq |k| < 2^{k+1}} 4 \sum_{r} \gamma_r \rho_{kr}(d) f\left(\frac{r^2 + 3w^2}{4}\right) dw.
\]
\[
\ll \frac{x^{1/2+\epsilon}}{D} D^{1/2} (D + 2^n \sqrt{x})^{1/2} (2^n \sqrt{x})^{1/2}.
\]

Similarly, we also have that by (3.5) and Proposition 3 \( R_n(f, D) \) is

\[
\ll \frac{D^{3/2}}{y^2 2^n} (D + 2^n \sqrt{x})^{1/2} (2^n \sqrt{x})^{1/2}
\]

by Proposition 2.

Proposition 4 then follows from summing over all \( n \).

\[\square\]

4. The bilinear form

Now, we shall bound the bilinear form in (2.2) by estimating the following sum:

\[(4.1) \quad B_1(M, N) = \sum_{N < n \leq N'} \left| \sum_{M < m \leq M'} a_{mn} \mu(m) \right| \]

for some unspecified \( M < M' \leq 2M, N < N' \leq 2N \) by showing the following:

**Proposition 5.** For \( \delta \) a sufficiently small positive number, we have that

\[(4.2) \quad B(M, N) \ll MN (\log MN)^{-A} \]

for all \( A > 0 \), where \( M = N^\delta \).

**Proof.** First, note that it is sufficient to estimate

\[(4.3) \quad B_1(M, N) = \sum_{N < n \leq N'} \left| \sum_{M < m \leq M'} a_{mn} \mu(m) \right| \]

since if \( (m, n) = d \), if \( d < M^{1/2} \), we can just transfer the factor of \( d \) to \( n \), and otherwise use the trivial bound.

Write \( \gamma(a) \) to denote \( \gamma_{2 \text{Re} a} \).

Note that we have that

\[ a_n = \sum_{Na=n} \gamma(a) \]

so by unique factorization in \( \mathbb{Q}(\omega) \), we have that for relatively prime \( m, n \), we have that

\[ a_{mn} = \frac{1}{6} \sum_{Nm=m} \sum_{Nn=n} \gamma(mn) \]

where the factor of 1/6 accounts for the six units \( \pm 1, \pm \omega, \pm \omega^2 \) in \( \mathbb{Z}[\omega] \). It follows that

\[ B_1(M, N) = \frac{1}{6} \sum_{N < N(n) \leq N'} \left| \sum_{M < N(m) \leq M'} (m, n) \mu(m) \right| . \]
The coprimality condition can easily be dropped by a similar argument by which it was added, so it follows
that it is sufficient to show that
\[ B_2(M, N) = \sum_{N < (n) \leq N'} \sum_{M < (m) \leq M'} \gamma(nm) \mu(m) \ll MN(\log MN)^{-A} \]

By Cauchy, we have that it is sufficient to show that
\[ B_3(M, N) = \sum_{N < (n) \leq N'} \sum_{M < (m) \leq M'} \gamma(nm) \mu(m) \ll M^2N(\log MN)^{-A}. \]

We then have that
\[ B_3(M, N) = \sum_{M < (m_1, N(m_2) \leq M')} \mu(m_1) \mu(m_2) S(m_1, m_2) \]
where
\[ S(m_1, m_2) = \sum_{N < (n) \leq N'} \gamma(nm_1) \gamma(nm_2). \]

Now, let \( \ell_1, \ell_2 \) be such that
\[ nm_1 + \overline{\imath}m_1 = \ell_1 \]
\[ nm_2 + \overline{\imath}m_2 = \ell_2 \]
and let \( \Delta(m_1, m_2) = \Delta = i(m_1 \overline{m_2} - m_1 m_2) \). Note that \( \ell_1, \ell_2 \leq 4\sqrt{MN} \). When \( \Delta = 0 \), note that the
contribution \( B_0(M, N) \) satisfies
\[ B_0(M, N) \ll N(\log N)^2 \sum \sum 1 \]
which is clearly \( \ll NM^2(\log MN)^{-A} \).

Otherwise, we have that
\[ \overline{\imath} = \frac{i(\ell_1 m_2 - \ell_2 m_1)}{\Delta} \]
so it follows that
\[ \ell_1 m_2 \equiv \ell_2 m_1 \pmod{\Delta} \]
and that
\[ \Delta^2 N < N(\ell_1 m_2 - \ell_2 m_1) \leq \Delta^2 N' \]

It then follows that
\[ S(m_1, m_2) = \sum_{\ell_1, \ell_2 \pmod{\Delta}} \gamma_{\ell_1} \gamma_{\ell_2} \]
where \( \Delta^2 N < N(\ell_1 m_2 - \ell_2 m_1) \leq \Delta^2 N' \)

Now, we state Proposition 20.9 in [FTI], which is used below:

Proposition 6.

\[ \sum_{q \leq Q} \max_{a \in \mathbb{Z}, (a, q) = 1} \sum_{\substack{\ell_1, \ell_2 \leq x \\ell_1 \equiv \ell_2 \pmod{\overline{\imath} q} \\ell_1 \ell_2 \leq y}} \gamma_{\ell_1} \gamma_{\ell_2} - \phi(q)^{-1} \sum_{\substack{\ell_1, \ell_2 \leq x \\ell_1 \ell_2 \leq y}} \sum_{\ell_1 \equiv \ell_2 \pmod{\overline{\imath} q}} \gamma_{\ell_1} \gamma_{\ell_2} \ll x^2(\log x)^{-A} \]
where $Q = x(\log x)^{-B}$ for some $B > 0$ that depends on $A$.

We can split up $S(m_1, m_2)$ into classes restricted to

$$
\ell_1 \equiv a \ell_2 \pmod{\Delta}
$$

for $a \in (\mathbb{Z}/(\Delta))^*$ such that $am_2 \equiv m_1 \pmod{\Delta}$ and apply Proposition 6. It then follows that

$$
B_0(M, N) \ll B_4(M, N) + O(NM^2(\log MN)^{-A})
$$

where

$$
B_4(M, N) = \sum_{M < N(m_1), N(m_2) \leq M'} \mu(m_1)\mu(m_2) \frac{\eta(\Delta)}{\phi(\Delta)} \sum_{\ell_1, \ell_2 \leq x} \gamma_{\ell_1} \gamma_{\ell_2}
$$

where $\eta(\Delta)$ is the total number of $a \in (\mathbb{Z}/(\Delta))^*$ such that $am_2 \equiv m_1 \pmod{\Delta}$.

By the prime number theorem, we have that the inner sum satisfies

$$
\sum_{\ell_1, \ell_2 \leq x} \gamma_{\ell_1} \gamma_{\ell_2} = X + O(MN(\log MN)^{-A})
$$

where

$$
X = \int \int_{\Delta \sqrt{N} < |\ell_1 m_2 - \ell_2 m_1| \leq \Delta \sqrt{N'}} d\ell_1 d\ell_2 = |\Delta| \int \int_{N < |u+\omega| \leq N'} dudv = \frac{1}{2} \pi \sqrt{3}|\Delta|(N' - N).
$$

It therefore now remains to estimate

$$
S_1 = \sum_{M < N(m_1), N(m_2) \leq M'} \mu(m_1)\mu(m_2) \frac{\eta(\Delta)|\Delta|}{\phi(\Delta)}
$$

Splitting this up for all $(m_1, m_2) = \delta$, we then have that

$$
S_1 = \sum_{\delta} \mu^2(\delta) \sum_{M < N(m_1, \delta), N(m_2, \delta) \leq M'} \frac{\eta(\Delta N(\delta))|\Delta|N(\delta)}{\phi(\Delta N(\delta))}
$$

$$
= \sum_{\delta} \mu^2(\delta) \sum_{M < N(m_1, \delta), N(m_2, \delta) \leq M'} \frac{\mu(m_1)\mu(m_2) \eta(\Delta N(\delta))|\Delta|N(\delta)}{\phi(\Delta N(\delta))}
$$

Note that we have that

$$
\eta(\Delta N(\delta)) = \sum_{\substack{a \in (\mathbb{Z}/(\Delta N(\delta)))^* \\delta \equiv m_2^{-1} \pmod{7\Delta}}} 1 = N(\delta) \prod_{p | N(\delta), p \nmid \Delta} \left(1 - \frac{1}{p}\right).
$$

It then follows that

$$
S_1 = \sum_{\delta} \mu^2(\delta)N(\delta) \sum_{M < N(m_1, \delta), N(m_2, \delta) \leq M'} \frac{\mu(m_1)\mu(m_2) |\Delta|}{\phi(\Delta)}
$$

By multiplicativity, we have that

$$
\frac{|\Delta|}{\phi(\Delta)} = \sum_{d|\Delta} \mu^2(d)\phi(d)^{-1}.
$$

Using this and reversing the order of summation, we have that

$$
S_1 = \sum_{\delta} \mu^2(\delta)N(\delta) \sum_{M < N(m_1, \delta), N(m_2, \delta) \leq M'} \mu(m_1)\mu(m_2) \sum_{d|\Delta} \mu^2(d)\phi(d)^{-1}
$$
\[
= \sum_{\mathfrak{d}} \mu^2(d) N(\mathfrak{d}) \sum_{d \leq 2M} \phi(d)^{-1} \sum_{M < N(m_1, d), N(m_2, d) \leq M'} \mu(m_1) \mu(m_2) \\
= \sum_{\mathfrak{d}} \mu^2(d) N(\mathfrak{d}) \sum_{d \leq 2M} \phi(d)^{-1} \sum_{\chi} \sum_{M < N(m_1, d), N(m_2, d) \leq M'} \mu(m_1) \mu(m_2) \psi(m_1) \overline{\psi}(m_2).
\]
by orthogonality where \( \chi \) runs over the characters of \( \mathbb{Z}[\omega]/(d) \) and \( \psi(m) = \chi(m) \overline{\psi(m)} \).

To estimate this, we use the following version of the Siegel-Walfisz Theorem that follows from the main result in [G]:

**Proposition 7.** For any character \( \psi \) on ideals

\[
\sum_{N(m) \leq x} \mu(m) \psi(m) \ll_A x (\log x)^{-A}
\]
for all \( A > 0 \).

Now, let

\[
S_{\mathfrak{d}, d, \psi}(M) = \sum_{M < N(m_1, d), N(m_2, d) \leq M'} \mu(m_1) \mu(m_2) \psi(m_1) \overline{\psi}(m_2).
\]

Then, it is easy to see that

\[
S_{\mathfrak{d}, d, \psi}^*(M) = S_{\mathfrak{d}, d, \psi}(M) + O(M^{1+\epsilon})
\]
where

\[
S_{\mathfrak{d}, d, \psi}(M) = \sum_{M < N(m_1, d), N(m_2, d) \leq M'} \mu(m_1) \mu(m_2) \psi(m_1) \overline{\psi}(m_2).
\]

We then have that

\[
\sum_{\mathfrak{d}_1 \in \mathbb{Z}[\omega] \setminus \{0\}} \mu^2(\mathfrak{d}_1) S_{\mathfrak{d}_1, d, \psi}(M/N(\mathfrak{d}_1))
\]

\[
= \left( \sum_{M < N(m_1, \mathfrak{d}) \leq M'} \mu(m_1) \psi(m_1) \right) \left( \sum_{M < N(m_2, \mathfrak{d}) \leq M'} \mu(m_2) \overline{\psi}(m_2) \right)
\]
so by a variant of Möbius inversion, we have that

\[
S_{\mathfrak{d}, d, \psi}(M) \ll (M/N(\mathfrak{d}))^2 (\log M/N(\mathfrak{d}))^{-A}.
\]

The desired result then follows. \( \square \)

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