

Spontaneous polarization of spin 1 analogue of the eight-vertex model

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Abstract

The spin 1 analogue of the eight-vertex model is considered on the basis of free field representations of vertex operators in the $2 \times 2$-fold fusion SOS model and vertex-face transformation. The spontaneous polarization of the model is obtained in terms of one-fold integral formula. Some limiting cases are discussed in order to examine the validity of the formula. Furthermore, we also present the integral formulae of the one-point function for the inhomogeneous twenty-one-vertex model.

1 Introduction

In this paper we consider the spin 1 analogue of Baxter’s eight-vertex model [1], on the basis of vertex operator approach [2]. The model is often called twenty-one-vertex model since the $R$ matrix has twenty one non-zero elements. The eight-vertex model is related to spin $\frac{1}{2}$ anisotropic Heisenberg spin chain. It was found by Lashkevich and Pugai [3] that the correlation functions of the eight-vertex model can be obtained by using the free field realization of the vertex operators in the eight-vertex SOS model [4], with insertion of the nonlocal operator $\Lambda$, called ‘the tail operator’. In [5] Lashkevich obtained integral formulae for form factors of the eight-vertex model.

There are some researches which generalize the study of [3, 5]. The vertex operator approach for higher spin generalization of the eight-vertex model was presented in [6]. As for higher rank generalization, the integral formulae for correlation functions of Belavin’s $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [7] were presented in [8], and those form factor formulae were presented in [9].

We are interested in the spontaneous polarization, a kind of one-point function of vertex models. Baxter and Kelland [10] derived the expression of the spontaneous polarization of the eight-vertex model. In [11] the same result was reproduced by solving a set of difference equations. Baxter-Kelland formula was also reproduced in [12] on the basis of vertex operator approach. The explicit expression for the

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spontaneous polarization of \((\mathbb{Z}/n\mathbb{Z})\)-symmetric model was found in [12], which was reproduced in [8], on the basis of vertex operator approach.

Let us mention on the trigonometric limit cases of elliptic vertex model. In [13] the spontaneous polarization formulae of the higher spin analogue of the six vertex model, the trigonometric limit of the eight-vertex model, were obtained by using Bethe Ansatz method. Idzumi [14] reproduced those formulae for spin 1 case in terms of vertex operator formalism. In the critical limit, the spin \(\frac{k}{2}\) (isotropic) Heisenberg spin chain is described by level \(k\) Wess-Zumino-Witten model [15], whose central charge is given by \(c = \frac{3k}{k+2}\). Since \(c = 1\) for the spin \(\frac{1}{2}\) case, the eight-vertex model can be described in terms of one boson. Spin 1 analogue of the eight-vertex model (twenty-one-vertex model) can be described in terms of one boson and one fermion, because \(c = \frac{3}{2} = 1 + \frac{1}{2}\) for \(k = 2\). Actually, Idzumi [14], Bougourzi and Weston [16] constructed level 2 irreducible highest weight representations of the quantum affine Lie algebra \(U_q(\hat{\mathfrak{sl}}_2)\) in terms of one boson and one fermion.

Let us turn to the elliptic case. Baxter [17, 18, 19] found the vertex-face transformation which relates the eight-vertex model and the SOS model. Boos et al. [20] proposed a conjectural formula for multipoint correlation functions of the \(Z\)-invariant (inhomogeneous) eight-vertex model. The restricted SOS (RSOS) model was constructed in [21]. The higher spin generalization of RSOS model was introduced in [22, 23] on the basis of the fusion procedure. Kojima, Konno and Weston [6] constructed vertex operator formalism for the higher spin analogue of the eight-vertex model, by using vertex-face transformation onto \(k \times k\) fusion SOS model.

The present paper is organized as follows. In section 2 we review the basic objects of the twenty-one-vertex model, the corresponding fusion face model [22, 23], the vertex-face correspondence of these two model, and the tail operators which translate correlation functions of fusion SOS model into those of the twenty-one-vertex model. Some detail definitions of the models concerned are listed in Appendix A. In section 3 we introduce a field representation for \(2 \times 2\) fusion SOS model. The type I vertex operators, the tail operators and the CTM Hamiltonian can be realized in terms of bosons and fermions. Correlation functions of the twenty-one-vertex model can be obtained by these objects, in principle. Section 4 is devoted to derivation of the spontaneous polarization of the twenty-one-vertex model. Useful operator product expansion (OPE) formulae and commutation relations for basic operators are given in Appendix B. Some limiting cases are considered in order to examine the validity of the formula. In particular, the result is compared with that of the trigonometric model obtained by Idzumi [14]. Furthermore, we present the corresponding formula for the inhomogeneous case. In section 5 we give some concluding remarks.

2 Basic objects

The present section aims to formulate the problem, thereby fixing the notation.
2.1 Theta functions

Jacobi theta function with two pseudo-periods 1 and \(\tau (\text{Im}\,\tau > 0)\) are defined as follows:

\[
\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (u;\tau) := \sum_{m \in \mathbb{Z}} \exp \left\{ \pi \sqrt{-1}(m+a) \left[ (m+a)\tau + 2(u+b) \right] \right\}, \tag{2.1}
\]

for \(a, b \in \mathbb{R}\). In what follows we use the functions \(\theta_1(u;\tau), \theta_2(u;\tau), \theta_3(u;\tau)\), \(\theta_4(u;\tau)\) when \((a,b) = (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, 0), (0, 0), (0, \frac{1}{2})\) on (2.1), respectively. Let \(\tau > 2\) and \(\epsilon > 0\) be fixed, and let

\[
h_j^{(r)}(u) := \theta_j \left( \frac{u; z^+\epsilon}{\tau}; \frac{z^+\epsilon}{\tau} \right), \quad (j = 1, 2, 3, 4)
\]

for \(r > 0\). We put \(h_j^{(r)}(u) = h_j(u)\) for simplicity. We will use the abbreviations,

\[
[u] := x^\frac{\pi}{2} - u \Theta_{2\tau}(x^{2u}), \quad \{u\} := x^\pi 2 - u \Theta_{2\tau}(-x^{2u}), \quad [u] := x^\frac{\pi}{2} \Theta_{2\tau}(x^{2u + r}), \quad \{u\} := x^\pi \Theta_{2\tau}(-x^{2u + r}), \tag{2.2}
\]

where

\[
\Theta_q(z) = (z; q)\infty (q^{-1}; q)\infty (q; q)\infty = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2} (-z)^m,
\]

\[
(z; q_1, \ldots, q_m)\infty = \prod_{i_1, \ldots, i_m \geq 0} (1 - z q_1^{i_1} \cdots q_m^{i_m}).
\]

Note that

\[
h_1(u) = \sqrt{\frac{\pi}{\tau}} \exp \left( -\frac{\pi\epsilon}{2} \right) [u], \quad h_2(u) = \sqrt{\frac{\pi}{\tau}} \exp \left( -\frac{\pi\epsilon}{2} \right) \{u\}, \quad h_3(u) = \sqrt{\frac{\pi}{\tau}} \{u\},
\]

where \(x = e^{-\epsilon}\).

In the present paper we often use the following abbreviations:

\[r' = r - 1, \quad r'' = r - 2, \quad [u]' = [u]_{r\rightarrow r-1}, \quad [u]'' = [u]_{r\rightarrow r-2},\]

and so on.

2.2 Spin 1 analogue of the eight-vertex model

The twenty-one-vertex model is constructed from the original spin \(\frac{1}{2}\) eight-vertex model by fusion procedure. Let

\[
R(u)v_{e_1} \otimes v_{e_2} = \sum_{e_1', e_2'} v_{e_1'} \otimes v_{e_2'} R(u)_{e_1 e_2}^{e_1' e_2'} \tag{2.3}
\]

be the \(R\)-matrix of the eight-vertex model. Non-zero elements of the \(R\)-matrix are given as follows:

\[
R(u)_{\epsilon \epsilon}^{\epsilon \epsilon} = \frac{1}{\kappa(u)} \frac{h_2(2r)(1)h_2(2r)(u)}{h_2(2r)(0)h_2(2r)(1 - u)}, \quad R(u)_{\epsilon \epsilon}^{-\epsilon} = \frac{1}{\kappa(u)} \frac{h_2(2r)(1)h_1(2r)(u)}{h_2(2r)(0)h_1(2r)(1 - u)}, \quad R(u)_{\epsilon \epsilon}^{-\epsilon} = \frac{1}{\kappa(u)} \frac{h_2(2r)(1)h_2(2r)(u)}{h_2(2r)(0)h_2(2r)(1 - u)} \tag{2.4}
\]
where,
\[
\kappa(u) = \zeta^{-\frac{1}{2}} \frac{\rho(z)}{\rho(z-1)} , \quad (z = \zeta^2 = x^{2u}, x = e^{-t})
\]
\[
\rho(z) = \frac{(2z^2;x^2)_\infty(x^{2r+2};x^4)_\infty(x^{2r};x^4)_\infty}{(x^4;x^2)_\infty(x^{2r};x^4)_\infty}.
\]

Let
\[
R^{(1,1)}(u)v_{j_1} \otimes v_{j_2} = \sum_{j'_1, j'_2=-1}^{1} v_{j'_1} \otimes v_{j'_2} R^{(1,1)}(u)_{j_1,j_2}^{j'_1,j'_2}
\]
be the twenty-one-vertex model. This $R^{(1,1)}(u)$ can be obtained from $R(u)$ in terms of fusion procedure. The following property
\[
PR(1) = -R(1), \quad P(x \otimes y) = y \otimes x,
\]
is important in the fusion procedure. The explicit expressions of the matrix elements of $R$-matrix of the twenty-one-vertex model are given in Appendix A.

We assume that the parameters $u$, $\epsilon$ and $r$ on (2.4) and (A.2) lie in the so-called principal regime:
\[
\epsilon > 0, \quad r > 2, \quad 0 < u < 1.
\]

This is the antiferroelectric region of the parameters. The twenty-one-vertex model has three kinds of ground states labeled by $i$ for $i = 0, 1, 2$. Accordingly, there are three spaces of physical states $H^{(i)}$ ($i = 0, 1, 2$). Here, the space $H^{(i)}$ is the $\mathbb{C}$-vector space spanned by the half-infinite pure tensor vectors of the forms
\[
v_{s_1} \otimes v_{s_2} \otimes v_{s_3} \otimes \cdots \quad \text{with } s_j \in \{-1, 0, 1\}, \quad \text{for } j = 1, 2, 3, \cdots
\]
and
\[
s_j = \begin{cases} 
1 - i & (j \equiv 0 \text{ mod } 2) \\
i - 1 & (j \equiv 1 \text{ mod } 2)
\end{cases} \quad \text{for } j \gg 0.
\]

Note that $H^{(i)}$ is isomorphic to the level 2 highest weight module of affine Lie algebra $A^{(1)}_1$, with the highest wight
\[
\lambda_i := (2 - i)\lambda_0 + i\lambda_1 \quad (i = 0, 1, 2),
\]
respectively. Here, $\lambda_i$'s ($i = 0, 1$) denote the fundamental weights of $A^{(1)}_1$.

Let $H^{(i)}$ be the dual of $H^{(i)}$ spanned by the half-infinite pure tensor vectors of the forms
\[
\cdots \otimes v_{s_{-2}} \otimes v_{s_{-1}} \otimes v_{s_0} \quad \text{with } s_j \in \{-1, 0, 1\}, \quad \text{for } j = 1, 2, 3, \cdots
\]
and
\[
s_j = \begin{cases} 
1 - i & (j \equiv 0 \text{ mod } 2) \\
i - 1 & (j \equiv 1 \text{ mod } 2)
\end{cases} \quad \text{for } j \ll 0.
\]

Let us consider the so-called low temperature limit $x \to 0$ of (A.2) with $\zeta = x^u$ be fixed. Then the $R^{(1,1)}(u)$ behaves as
\[
R^{(1,1)}(u)_{s_1,s_2}^{s'_1,s'_2} \sim \zeta^{H(s_1,s_2)} \delta_{s_1}^{s'_1} \delta_{s_2}^{s'_2} \quad (x \to 0)
\]
where

\[ H(s, s') = |s + s'| = \begin{cases} 
0 & \text{if } (s, s') = (±1, ±1), (0, 0) \\
1 & \text{if } (s, s') = (±1, 0), (0, ±1) \\
2 & \text{if } (s, s') = (±1, ±1) 
\end{cases} \]  

(2.14)

Thus, the South-East corner transfer matrix behaves

\[ A_{SE}^{(i)}(u)^{s_1s_2\ldots} \sim \chi_{\text{CTM}}^{(i)}(s_1, s_2, \ldots) \delta_{s_1}^{s_1} \delta_{s_2}^{s_2} \ldots ; H^{(i)} \rightarrow H^{(i)}, \]  

(2.15)
in the low temperature limit \( x \rightarrow 0 \), where

\[ H_{\text{CTM}}^{(i)}(s_1, s_2, \ldots) = \sum_{j=1}^{\infty} jH(s_j, s_{j+1}). \]  

(2.16)

We assume that (2.15) is valid not only for low temperature limit \( x \rightarrow 0 \) but also for finite \( 0 < x < 1 \).

Likewise other three types of the corner transfer matrices are introduced as follows:

\[ A_{NE}^{(i)}(u) : H^{(i)} \rightarrow H^{*(i)}, \]

\[ A_{NW}^{(i)}(u) : H^{*(i)} \rightarrow H^{*(i)}, \]

\[ A_{SW}^{(i)}(u) : H^{*(i)} \rightarrow H^{(i)}, \]  

(2.17)

where NE, NW and SW stand for the corners North-East, North-West and South-West. It seems to be rather general that the product of four CTMs in the infinite lattice limit is independent of \( u \):

\[ \rho^{(i)} = A_{SE}^{(i)}(u)A_{SW}^{(i)}(u)A_{NW}^{(i)}(u)A_{NE}^{(i)}(u) = x^{2H_{\text{CTM}}^{(i)}}. \]  

(2.18)

The trace of \( \rho^{(i)} \) coincides with the principally specialized character of \( \lambda_i \), up to some factors:

\[ \chi^{(i)} := \text{tr}_{H^{(i)}}(x^{2H_{\text{CTM}}^{(i)}}) = x^i \chi_{\lambda_i}(x) = \begin{cases} 
-x^2; x^2 \in (-x^2; x^2) & (i = 0, 2) \\
-x^2; x^2 \in (-x^2; x^2) & (i = 1)
\end{cases} \]  

(2.19)

Introduce the type I vertex operator by the following half-infinite transfer matrix

\[ \Phi^j(u_1 - u_2) = \begin{array}{cccc}
\vdots \\
u_2 & u_2 & u_2 & u_2 \\
u_1 & \vdots \\
u_2 & u_2 & u_2 & u_2 \\
u_2 & u_2 & u_2 & u_2 \\
u_1 & \vdots \\
\end{array} \]  

(2.20)

Then the operator (2.20) is an intertwiner from \( H^{(i)} \) to \( H^{(2-i)} \). The type I vertex operators satisfy the following commutation relation:

\[ \Phi^{j_1}(u_1)\Phi^{j_2}(u_2) = \sum_{j_1' j_2'} R^{(1,1)}(u_1 - u_2)^{j_1' j_2'} \Phi^{j_1'}(u_2)\Phi^{j_2'}(u_1). \]  

(2.21)

Furthermore, the type I vertex operator \( \Phi^j(u) \) and \( \rho^{(i)} \) introduced on (2.18) satisfy the homogeneity relation

\[ \Phi^j(u)\rho^{(i)} = \rho^{(2-i)}\Phi^j(u - 2). \]  

(2.22)

Note that the \( u \)-dependence of \( R^{(1,1)}(u) \) is actually \( \zeta \)-dependence, where \( \zeta = x^u \). Since the eigenvalues \( \lambda_s \) of \( A_{NE}^{(i)}(u) \) should be invariant under the shift \( u \rightarrow u + 2\pi \sqrt{-1}/\log x \), we have \( \lambda_s = \zeta^{n_s} \) \( (n_s \in \mathbb{Z}) \). Owing to the discreteness property of eigenvalues, (2.18) should be valid even for finite \( 0 < x < 1 \), in the sense of similarity transformation.
2.3 $2 \times 2$ fusion SOS model

The SOS model was introduced by Baxter [17, 18, 19] in order to solve the eight-vertex model. The state variables of the SOS model take integer values. A pair $(a, b) \in \mathbb{Z}^2$ is called admissible if $b = a \pm 1$. Let $(a, b)$ be the state variables at adjacent sites. Then the pair $(a, b)$ is admissible. For $(a, b, c, d) \in \mathbb{Z}^4$, let $W \begin{bmatrix} c & d \\ b & a \end{bmatrix} u$ be the Boltzmann weight of the SOS model for the state configuration $\begin{bmatrix} c & d \\ b & a \end{bmatrix}$ round a face. Here the four states $a, b, c$ and $d$ are ordered clockwise from the SE corner. In this model $W \begin{bmatrix} c & d \\ b & a \end{bmatrix} u = 0$ unless the four pairs $(a, b), (a, d), (b, c)$ and $(d, c)$ are admissible. Non-zero Boltzmann weights are given as follows:

$$
W \begin{bmatrix} k \pm 2 & k \pm 1 \\ k + 1 & k \end{bmatrix} u = \frac{1}{\kappa(u)}, \\
W \begin{bmatrix} k & k \pm 1 \\ k + 1 & k \end{bmatrix} u = \frac{1}{\kappa(u)} \frac{[1][k \pm u]}{[1 - u][k]}, \\
W \begin{bmatrix} k \pm 1 & k \pm 1 \\ k \mp 1 & k \end{bmatrix} u = -\frac{1}{\kappa(u)} \frac{[u][k \pm 1]}{[1 - u][k]).}
$$ (2.23)

The twenty-one-vertex model can be transformed into $2 \times 2$ fusion SOS model in terms of vertex-face correspondence. Let $(a, b)$ be the state variables of $2 \times 2$ fusion SOS model at adjacent sites. Then $b = a \pm 2$, or $b = a$. In what follows we denote $b \sim a$ when $b - a \in \{-2, 0, 2\}$. Non-zero Boltzmann weights $W_{22}(u)$ are given in Appendix A.

Here we again assume that the parameters $u$, $\epsilon$ and $r$ on (2.23) and (A.3) lie in (2.8). This region of the parameters is called regime III in the SOS-type model. For $k, l \in \mathbb{Z}$ and $i = 0, 1, 2$, let $\mathcal{H}^{(i)}_{l,k}$ be the space of admissible paths $(k_0, k_1, k_2, \cdots)$ such that

$$k_0 = k, \quad k_{j+1} \sim k_j \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, \quad (2.24)$$

and

$$k_j = \begin{cases} l + i & (j \equiv 0 \mod 2) \\
     l + 2 - i & (j \equiv 1 \mod 2) \end{cases} \quad \text{for} \quad j \gg 0. \quad (2.25)$$

Also, let $\mathcal{H}^{(i)}_{l,k}$ be the space of admissible paths $(\cdots, k_{-2}, k_{-1}, k_0)$ such that

$$k_0 = k, \quad k_{j-1} \sim k_j \quad \text{for} \quad j = 0, -1, -2, 3, \cdots, \quad (2.26)$$

and

$$k_j = \begin{cases} l + i & (j \equiv 0 \mod 2) \\
     l + 2 - i & (j \equiv 1 \mod 2) \end{cases} \quad \text{for} \quad j \ll 0. \quad (2.27)$$

After gauge transformation [22, 23], the Boltzmann weights $W_{22}(u)$ in the so-called low temperature limit $x \to 0$ behave as

$$W_{22} \begin{bmatrix} c & d \\ b & a \end{bmatrix} u \sim \delta_{bd} \zeta^{\frac{1}{2}|c-a|}. \quad (2.28)$$
Here we take the limit \( x \to 0 \) with \( \zeta = x^{u} \) be fixed. Let \( A_{l,k}^{(i)} \), \( B_{l,k}^{(i)} \), \( C_{l,k}^{(i)} \) and \( D_{l,k}^{(i)} \) be the SE, SW, NW, NE corner transfer matrix. Then the SE corner transfer matrix behaves as follows:

\[
A_{l,k}^{(i)}(u)^{k_0,k_1,k_2,\cdots} \sim \zeta H_{l,k}^{(i)}(u)^{k_0,k_1,k_2,\cdots}, H_{l,k}^{(i)} \to H_{l,k}^{(i)},
\]

in the low temperature limit \( x \to 0 \), where

\[
H_{l,k}^{(i)}(k_0,k_1,k_2,\cdots) = \frac{1}{2} \sum_{j=1}^{\infty} j|k_{j+1} - k_{j-1}|.
\]

Likewise other three types of the corner transfer matrices are introduced as follows:

\[
B_{l,k}^{(i)}(u) : \quad H^{(i)} \to H^{(i)},
\]

\[
C_{l,k}^{(i)}(u) : \quad H^{(i)} \to H^{(i)},
\]

\[
D_{l,k}^{(i)}(u) : \quad H^{(i)} \to H^{(i)}.
\]

It seems to be rather general \[1\] that the product of four CTMs in the infinite lattice limit is independent of \( u \):

\[
\rho_{l,k}^{(i)} = A_{l,k}^{(i)}(u) B_{l,k}^{(i)}(u) C_{l,k}^{(i)}(u) D_{l,k}^{(i)}(u) = [k] x^{2H_{l,k}^{(i)}}.
\]

Let \( k \equiv l + i + 2j \) (mod 4), where \( i = 0, 1, 2 \) and \( j = 0, 1 \). Then the trace of \( \rho_{l,k}^{(i)} \) can be given as follows \[25\]:

\[
\chi_{l,k}^{(i)} := \text{tr}_{H^{(i)}}(\rho_{l,k}^{(i)}) = [k] c_{\lambda_{i+2j}}(x^{4}) x^{\frac{2r(r^{2} - 1)}{2} - ik + \frac{r^{2}k}{2}}.
\]

Here \( c_{\lambda_{i}}(x^{4}) \) is the string function \[26\], up to some factors. We change the definitions for the present purpose as follows:

\[
c_{\lambda_{i}}(x^{4}) = \left( \frac{+x^{2}; x^{4})}{(x^{4}; x^{4})} \right)_{\infty}^{(x^{2}; x^{4})},
\]

\[
c_{\lambda_{i}}(x^{4}) = \left( \frac{x^{2}(-x^{4}; x^{4})}{(x^{4}; x^{4})} \right)_{\infty},
\]

\[
c_{\lambda_{i}}(x^{4}) = 0 \quad \text{(for } j \neq i \text{ mod 2)}.
\]

Besides \[23\] we have the following symmetry

\[
c_{\lambda_{i}}(x^{4}) = c_{\lambda_{i+4}}(x^{4}) = c_{\lambda_{2i-1}}(x^{4}).
\]

From \[23\], \[24\], \[25\] and \[26\], we obtain the following relation \[6\]:

\[
\sum_{k \in l+i+2\mathbb{Z}} \chi_{l,k}^{(i)} = [l]^{(i)} \chi^{(i)}.
\]

Introduce the type I vertex operator by the following half-infinite transfer matrix

\[
\Phi(u_{1} - u_{2})^{k}_{k} = \left[
\begin{array}{ccc}
\cdots & 0 & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{array}
\right].
\]

Then the operator \[27\] is an intertwiner from \( H_{l,k}^{(i)} \) to \( H_{l,k'}^{(2-i)} \). The type I vertex operators satisfy the following commutation relation:

\[
\Phi(u_{1})^{a}_{b} \Phi(u_{2})^{k}_{d} = \sum_{d} W_{22}^{cd} \left[
\begin{array}{cc}
c & d \\
b & a
\end{array}
\right] u_{1} - u_{2} \Phi(u_{2})^{a}_{b} \Phi(u_{1})^{d}.\]

\[\text{(2.38)}\]
The free field realization of $\Phi(u)_b^k$ was constructed in [6]. See section 3.2.

Furthermore, the type I vertex operator $\Phi(u)_b^k$ and $\rho^{(i)}_{1,k}$ introduced on (2.32) satisfy the homogeneity relation
\[
\Phi(u)_b^k \rho^{(i)}_{1,k} [k] = \rho^{(2-i)}_{1,k'} [k'] \Phi(u - 2)_b^{k'}.
\tag{2.39}
\]

### 2.4 Vertex-face correspondence

Baxter [17, 18, 19] introduced the intertwining vectors which relate the eight-vertex model and the SOS model. Let
\[
t(u)_b^k = \sum_{\varepsilon = \pm} v_\varepsilon t^\varepsilon(u)_b^k = \frac{f(u)}{\sqrt{2}} \left[ h_3^{(2\varepsilon)}(k \mp u) \right],
\tag{2.40}
\]
where the scalar function $f(u)$ satisfies
\[
h_1(u)f(u)f(u - 1) = 1.
\]

Then the following relation holds: (cf. Figure 1)
\[
R(u_1 - u_2)t(u_1)^d_a \otimes t(u_2)^e_b = \sum_b t(u_1)_b^c \otimes t(u_2)_a^d W \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] \left[ u_1 - u_2 \right].
\tag{2.41}
\]

![Figure 1. Picture representation of vertex-face correspondence.](image)

Note that the present intertwining vectors are different from the ones used in [17, 18, 19], which relate the $R$-matrix of eight-vertex model in the disordered phase and Boltzmann weights $W$ of $A_{n-1}^{(1)}$-model in the regime III.

Let us introduce the dual intertwining vectors (see Figure 2) satisfying
\[
\sum_{\varepsilon = \pm} t^\varepsilon_*(u)_b^k t^{\varepsilon'}_*(u)_b^k = \delta^{k'}_{k\varepsilon'} , \quad \sum_{k' = k, k \pm 1} t^\varepsilon(u)_b^k t^{\varepsilon'}(u)_b^k = \delta_\varepsilon^{k'}.
\tag{2.42}
\]

![Figure 2. Picture representation of the dual intertwining vectors.](image)
From (2.41) and (2.42), we have (cf. Figure 3)

\[ t^*(u_1)_c^b \otimes t^*(u_2)_b^a R(u_1 - u_2) = \sum_d W \begin{bmatrix} c & d \\ b & a \end{bmatrix} t^*(u_1)_d^a \otimes t^*(u_2)_c^d. \]  

(2.43)

\[ L \begin{bmatrix} a_0' & a_1' \\ a_0 & a_1 \end{bmatrix} : u_0 := \sum_j t^*_j (-u_0)^a_{a_0} t^j (-u_0)^a_{a_1}. \]  

(2.44)

\[ L \begin{bmatrix} a_0 & a_1' \\ a_0 & a_1 \end{bmatrix} = \delta^a_{a_1}. \]  

(2.45)

Figure 3. Vertex-face correspondence by dual intertwining vectors.

Intertwining vectors which relate the twenty-one-vertex model and the $2 \times 2$ fusion SOS model can be constructed by fusion procedure. The explicit expressions of the fused intertwining vectors are given in Appendix A.

Let

\[ L \begin{bmatrix} a_0' & a_1' \\ a_0 & a_1 \end{bmatrix} = \sum_j t^*_j (-u_0)^a_{a_0} t^j (-u_0)^a_{a_1}. \]  

(2.44)

Then form (A.7)

\[ L \begin{bmatrix} a_0 & a_1' \\ a_0 & a_1 \end{bmatrix} = \delta^a_{a_1}. \]  

(2.45)

The explicit expressions of $L$ are given in Appendix A.

Assume that $0 < R(u_0) < 2$. Then it follows from (A.4) and (A.6) that for $i = 0, 1, 2$,

\[ |t^*_{i-1}(-u_0)^{i+2-i} t^{i-1}(-u_0)|_{i+2-i} \sim 1 \quad (x \to 0) \]  

(2.46)

is much greater than other products $t^*_j (-u_0)^{j+2-i} t^{j+1} (-u_0)$ for $j \neq i$, in the low temperature limit. Thus, the boundary condition $H^{(i)}$ of the twenty-one-vertex model (2.10) corresponds to that of $H^{(i)}_{l,k}$ of the $2 \times 2$ fusion SOS model (2.25).

2.5 Tail operators and commutation relations

Tail operators were originally introduced in [3, 5], in order to translate correlation functions of the eight-vertex model into those of SOS model. Tail operators for higher spin case were constructed in [6], and those for higher rank case were constructed in [8, 9].

Let us introduce the intertwining operators between $H^{(i)}$ and $H^{(i)}_{l,k}$:

\[ T(u_0)^{lk} = \prod_{j=0}^{\infty} t^{j}_{i+kj} (-u_0)^{i+kj}_{i+kj+1} : H^{(i)} \to H^{(i)}_{l,k}, \]  

(2.47)

\[ T(u_0)^{lk} = \prod_{j=0}^{\infty} t^{j}_{i+kj} (-u_0)^{i+kj}_{i+kj+1} : H^{(i)}_{l,k} \to H^{(i)}. \]
From (A.3) and (A.8) the following intertwining relations hold:

\[ T(u_0)^{k'k} \Phi^j(u) = \sum_k t^j(u - u_0)^{k'} \Phi(u)^{k} T(u)^{jk}, \]

(2.48)

\[ T(u_0)^{k'k} = \frac{1}{t^{-1}} T(u - u_0)_{k'} \Phi^j(u) T(u_0)^{ij}. \]

(2.49)

Tail operator is defined by the product of these two objects (see Figure 4):

\[ \lambda(u_0)^{l''k'} = T(u_0)^{l''k'} T(u_0)^{lk} : H_{l''k'}^{(i)} \rightarrow H_{l,k}^{(i)}. \]

(2.50)

From (2.45), (2.48) and (2.49), we have

\[ \lambda(u_0)_{lk}^{l''k'} = \sum_{l''} L \left[ \begin{array}{cc} c & d \\ a & b \end{array} \right] (u_0 - u) \Phi(u)^{l''} \lambda(u_0)_{lk}^{l''k'}. \]

(2.51)

In this paper we only need \( \lambda(u_0)_{lk}^{l''k'} \), which is diagonal with respect to the boundary conditions. In what follows we suppress \( l \)-dependence to denote \( \lambda(u_0)_{lk}^{l''k'} \) by \( \lambda(u_0)_{k}^{l''k'} \). From (2.47), (2.50) and (2.44) we have

\[ \lambda(u_0)^{k'k} = \prod_{j=0}^{\infty} L \left[ \begin{array}{cc} k'_{j} & k_{j+1} \\ k_{j} & k_{j+1} \end{array} \right] u_0. \]

(2.52)

It is obvious from (2.45), we have

\[ \lambda(u_0)_{k}^{k} = 1. \]

(2.53)

The relation (2.36) implies that

\[ \text{tr}_{H_{l,k}^{(i)}} (\rho^{(i)}) = \frac{1}{|T|^L} \sum_{k' \in \ell + i + 2Z} \text{tr}_{H_{l,k}^{(i)}} (\rho_{k,k'}^{(i)}). \]

(2.54)

Insert unity (2.53) into the RHS of (2.54). Then we have

\[ \text{tr}_{H_{l,k}^{(i)}} (\rho^{(i)}) = \sum_{k' \in \ell + i + 2Z} \text{tr}_{H_{l,k}^{(i)}} \left( \frac{\rho_{k,k'}^{(i)} T(u_0)^{ik} T(u)^{lk}}{|T|^L} \right) \]

\[ = \sum_{k' \in \ell + i + 2Z} \text{tr}_{H_{l,k}^{(i)}} \left( T(u)^{lk} \frac{\rho_{k,k'}^{(i)} T(u_0)^{ik}}{|T|^L} \right). \]

(2.55)
Thus in what follows we assume that
\[
\rho^{(i)} = \sum_{k \in \ell + i + 2 \mathbb{Z}} T(u)_k \rho^{(i)}_{l,k} T(u)^k.
\] (2.56)

3 Free field realization

One of the most standard ways to calculate correlation functions and form factors is the vertex operator approach [2] on the basis of free field representation. The face type elliptic quantum group $B_{q,\lambda}(\hat{sl}_2)$ was introduced in [27]. The elliptic algebra $U_q,p(\hat{sl}_2)$ associated with fusion SOS models was defined in [28], and its free field representations were constructed in [28, 29]. Using these representations we derive the free field representation of the tail operator in this section.

3.1 Bosons and fermions

Let us consider the bosons $\beta_m (m \in \mathbb{Z} \setminus \{0\})$ with the commutation relations
\[
[\beta_m, \beta_{m'}] = m \frac{[x^m][x^{m'}]}{[rm][x^r]} \delta_{m + m', 0}.
\] (3.1)

Here the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. The relation between the present $\beta_m$ and the bosons $a_m$ in [6] is as follows:
\[
\beta_m = \begin{cases} 
\frac{m[x^m]}{[2m][x^r]} a_m & (m > 0) \\
\frac{mx^{2m}}{[2m]} a_m & (m < 0)
\end{cases}
\] (3.2)

We will deal with the bosonic Fock spaces $\mathcal{F}^{(i)}_{l,k}, (l, k \in \mathbb{Z})$ generated by $\beta_{-m} (m > 0)$ and $e^\alpha$ over the vacuum vectors $|l, k\rangle$:
\[
\mathcal{F}^{(i)}_{l,k} = \mathbb{C}[\beta_{-1}, \beta_{-2}, \cdots] \otimes (\otimes_{n \in \mathbb{Z}} \mathbb{C} e^{\lambda_i + n\alpha}) |l, k\rangle,
\]
where
\[
\beta_m |l, k\rangle = 0 \ (m > 0),
\]
\[
e^{\pm \alpha} |l, k\rangle = |l, k \pm 2\rangle.
\]

Let $K$ and $L$ be the operators which act diagonally on $\mathcal{F}^{(i)}_{l,k}$:
\[
K |l, k\rangle = k |l, k\rangle, \quad L |l, k\rangle = l |l, k\rangle.
\]

Furthermore, let us consider the fermions
\[
\phi(w) = \sum_m \phi_m w^{-m}
\] (3.3)
with the anticommutation relations
\[
[\phi_m, \phi_{m'}]_+ = \delta_{m + m', 0} \frac{x^{2m} + x^{-2m}}{x + x^{-1}}.
\] (3.4)
We refer to $\phi_m$’s for $m \in \mathbb{Z} + \frac{1}{2}$ as Neveu-Schwarz fermions, and $\phi_m$’s for $m \in \mathbb{Z}$ as Ramond fermions.

Let

$$\mathcal{F}^\phi = \begin{cases} \mathbb{C}[\phi_{-\frac{1}{2}}, \phi_{-\frac{3}{2}}, \cdots] & \text{(for } i = 0, 2) \\ \mathbb{C}[\phi_{-1}, \phi_{-2}, \cdots] & \text{(for } i = 1) \end{cases}$$

be the fermionic Fock space.

Note that the following anticommutation relation holds:

$$[\phi(w_1), \phi(w_2)]_+ = \frac{1}{x + x^{-1}} \left( \delta \left( \frac{x^2 w_2}{w_1} \right) + \delta \left( \frac{x^2 w_1}{w_2} \right) \right).$$

(3.5)

Here we use $\phi_0^2 = 1/(x + x^{-1})$ for Ramond fermion sector.

The total space of states $\mathcal{H}_{l,k}^{(i)}$ is isomorphic to

$$\mathcal{H}_{l,k}^{(i)} = \mathcal{F}_{l,k}^{(i)} \otimes \mathcal{F}^\phi$$

(3.6)

### 3.2 Free field realization of type I vertex operators

Let us introduce the following basic operators

$$\Phi_1(u) = z^\frac{\alpha'}{2} : \exp \left( -\sum_{m \neq 0} \frac{\beta_m}{m} z^{-m} : e^{\alpha_2 - \frac{1}{2} L + \frac{\alpha'}{2} K} (\sqrt{-1})^{K-L}, \right)$$

$$A(v) = w^\alpha : \exp \left( \sum_{m \neq 0} \frac{\beta_m}{m} w^{-m} : e^{-\alpha_2 w^{\frac{1}{2} L - \frac{\alpha'}{2} K}} \phi(w), \right)$$

(3.7)

where $z = x^{2u}$, $w = x^{2v}$. As for some useful OPE formulae and commutation relations, see Appendix B.

Then the type I vertex operators (half transfer matrices) on $\mathcal{H}_{l,k}^{(i)}$ can be realized in terms of bosons and fermions:

$$\Phi(u)^{k+2} = \frac{[1]}{[k][k+1]} \Phi_1(u),$$

$$\Phi(u)^k = \frac{[2]}{[k-1][k+1]} \Phi_1(u) X(u),$$

$$\Phi(u)^{k-2} = \frac{[1]}{[k-1]} \Phi_1(u) X(u)^2,$$

(3.8)

where $w_j = x^{2v_j}$ and

$$X(u) = \oint_C \frac{dw}{2\pi \sqrt{-1} w} A(v) \phi(v) \frac{[v-u-K]}{[v-u-1]}$$

(3.9)

Considering the denominators $[v_j - u - 1]$’s together with the OPE formulae [12,2], the expressions [3.8] has poles at $w_j = x^{k(2+2nr)}z$ ($n \in \mathbb{Z}_{\geq 0}$). The integral contour $C$ for $w_j$-integration is the anti-clockwise circle such that all integral variables lie in the common convergence domain; i.e., the contour $C$ encircles the poles at $w_j = x^{2+2nr}z$ ($n \in \mathbb{Z}_{\geq 0}$), but not the poles at $w_j = x^{2-2nr}z$ ($n \in \mathbb{Z}_{\geq 0}$).

Note that

$$\Phi(u)_k^{k'} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}_{l,k'}^{(2-i)}.$$  (3.10)
These type I vertex operators satisfy the following commutation relations on $H_{i,k}^{(i)}$:
\[
\Phi(u_1)^b_1 \Phi(u_2)^b_2 = \sum_d W \begin{bmatrix} c & d \\ b & a \end{bmatrix} u_1 - u_2 \Phi(u_2)^c_2 \Phi(u_1)^d_a.
\] (3.11)

Dual vertex operators are likewise defined as follows:
\[
\Phi^*(u)^k_{k-2} = \frac{(-1)^{k-l+1}}{\lambda} \Phi_1(u - 1),
\]
\[
\Phi^*(u)^k_k = \frac{(-1)^{k-l+1}}{\lambda} \Phi_1(u - 1) X(u - 1),
\] (3.12)
\[
\Phi^*(u)^k_{k+2} = \frac{(-1)^{k-l+1}}{\lambda} \Phi_1(u - 1) X(u - 1)^2.
\]

Here the normalization factor can be determined as
\[
\lambda = \frac{(x^{2r}; x^{2r})_\infty}{(x + x^{-1})(x^{2r}; x^{2r})_\infty^2},
\]
such that $\Phi(u)^k_k$ and $\Phi^*(u)^k_k$ satisfy the inversion relation:
\[
\sum_{k' \sim k} \Phi^*(u)^k_{k'} \Phi(u)^{k'}_k = 1.
\] (3.13)

As explained below (3.9), the integral contour $C = C_u$ actually depends on $u$. On eqs. (3.12) the $w_j$-integration contour $C_{u-1}$ of $X(u - 1)$ encircles the poles at $z_j = x^{2nr} z (n \in \mathbb{Z}_{\geq 0})$, but not the poles at $w_j = x^{-4 - 2nr} z (n \in \mathbb{Z}_{\geq 0})$. Note that
\[
\Phi^*(u)^k_{k': H_{i,k}^{(i)}} \rightarrow H_{i,k}^{(2-i)}.
\] (3.14)

A level 2 representation of the elliptic algebra $U_{x,0}(\hat{sl}_2)$ was obtained in terms of one free boson and one free fermion in [30].

### 3.3 Free field realization of tail operators

Another ingredient of the present scheme is the tail operators $\Lambda(u_0)^{k'}_k$. In this paper we use a different normalization from the one used in [6]. Thus we briefly explain how to derive free field representations of $\Lambda(u_0)^{k'}_k$. When $k' \leq k - 2$, let us consider (2.51) for $(a, b, c) = (k, k + 2, k')$:
\[
\Lambda(u_0)^{k'}_{k+2} \Phi(u)^{k+2}_k = \sum_{k'' \sim k'} L \begin{bmatrix} k' & k'' \\ k & k+2 \end{bmatrix} u_0 - u \Phi(u_0)^{k''}_{k''} \Lambda(u_0)^{k'}_k.
\] (3.15)

It follows from (A.9) that $L(u_0 - u)$ has simple poles at $u_0 - u = \pm \frac{1}{2}$. Note that
\[
[u_0 - u + \frac{1}{2}]L \begin{bmatrix} k' & k'' \\ k & k - 2 \end{bmatrix} u_0 - u \bigg|_{u_0=u+\frac{1}{2}}
\]
for $k'' = k', k' \pm 2$ are all equal. Thus if we assume that the LHS of (3.15) has no pole at $u_0 = u + \frac{1}{2}$, we have the following necessary conditions:
\[
\sum_{k'' \sim k'} \Phi(u)^{k''}_k \Lambda(u - \frac{1}{2})^{k''}_k = 0,
\] (3.16)
We can realize the CTM Hamiltonian of $2 \times 3$. Free field realization of CTM Hamiltonian

and we use (2.48), (2.49), (2.56) and (2.50).

Furthermore, we can check that (3.18) for generic operators of fusion SOS model and tail operators as follows:

Let $k' = k-2$. Then the LHS of (3.17) contains $\Lambda(u-\frac{1}{2})^k = 1$. By changing $k' = k-2, k-4, k-6, \ldots$, we can solve (3.17) iteratively as follows:

$$\Lambda(u_0)^{k-2s} = (-X(u_0 + \frac{1}{2}))^s \left[\frac{s + 1}{1} \right] \frac{s + 1}{k-s + 1} \frac{1}{[k][k+1]}.$$  

Here we use the identity:

$$\left[\frac{s + 1}{1} \right] \frac{s + 1}{k-s + 1} \frac{2}{[k-2s-1][k-2s-3]} = 0.$$  

Furthermore, we can check that (3.18) for generic $u_0$ satisfies (3.15).

Eq. (3.18) is expressions of $\Lambda(u_0)^{k'}$ for $k' \leq k$. When $k' > k$, we should realize another free field representation of $H_{t,l,k}^{(i)}$ on the Fock space $F_{l-1-k}^{(i)} \otimes \Phi_0$. Then $\Lambda(u_0)^{k+2s}$ can be identified with $\Lambda(u_0)^{k-2s}$, in addition to the identification $\Phi(a_0)^{k'}$ and $\Phi^*(a_0)^{k'}$ with $\Phi(a_0)^{-k'}$ and $\Phi^*(a_0)^{-k'}$, respectively.

Correlation functions in the twenty-one-vertex model can be constructed in terms of type I vertex operators of fusion SOS model and tail operators as follows:

$$\frac{1}{\chi^{(i)}} \text{tr}_{\mathcal{H}^{(i)}}(\Phi_1^*(u_1) \cdots \Phi_{j_n}^*(u_n)) \Phi^{j_n}(u_n) \cdots \Phi^{j_1}(u_1) \rho^{(i)}$$

$$= \frac{1}{\chi^{(i)}} \sum_{k \in 1+i+2k} \text{tr}_{\mathcal{H}^{(i),1,k}} \left( T(u_0)^{k_1} \Phi_1^*(u_1) \cdots \Phi_{j_n}^*(u_n) \Phi^{j_n}(u_n) \cdots \Phi^{j_1}(u_1) T(u_0) k \rho^{(i,k)} \right)$$

$$= \frac{1}{\chi^{(i)}} \sum_{k, k_1, \ldots, k_2n} \phi_1(u_1)^{k_1} \cdots \phi_{j_n}(u_n)^{k_{j_n}} \phi^{j_n}(u_n)^{k_{j_n}} \phi(u_1)^{k_1} \Lambda(u_0)^{k_1} \rho^{(i,k)} \frac{1}{\rho^{(i,k)}}$$

Here, the sum on the third line is taken over

$$\{k, k_2n, \ldots, k_{j_n} \mid k_1 \sim k_2, \ldots, k_2n \sim k; k \in l + i + 2k\},$$

and we use (2.48), (2.49), (2.56) and (2.59).

### 3.4 Free field realization of CTM Hamiltonian

We can realize the CTM Hamiltonian of $2 \times 2$ fusion SOS model in terms free fields as follows:

$$H_{t,l,k}^{(i)} = H_a^{(i,k)} + H_{\phi}^{(i)},$$

where

$$\frac{1}{2} H_a^{(i,k)} = \sum_{m=1}^{\infty} \frac{r_m}{r_m^2} \beta_m \beta_m + \frac{1}{4} \left( \frac{r'}{2r'} L^2 - KL + \frac{r''}{2r} K^2 \right),$$

$$\frac{1}{2} H_{\phi}^{(i)} = \sum_{n>0} \frac{x'}{x'^2} \phi_n + \frac{i(2-i)}{8}.$$
Let us examine the validity of these expressions. First of all, (3.20) satisfies the homogeneity relation

\[ \Phi(u)_{k'}^{x H_{l,k}^{(i)}} = x^{2 H_{l,k}^{(i)} - 1} \Phi(u - 2)^{k'}. \]  
(3.22)

Secondly, the traces on the bosonic/fermionic Fock space are given as follows:

\[ \text{tr}_{\mathcal{F}_{l,k}} \left( x^{2 H_{l,k}^{(i)}} \right) \text{tr}_{\mathcal{F}_{\phi}} \left( x^{2 H_{l,k}^{(i)}} \right) = x^{\sum_{r=-l}^{r=+l} (r^2 + 1) k^2} \left\{ \begin{array}{ll} c_{\lambda_i} + c_{\lambda_{i+1}} & (i = 0, 2) \\ c_{\lambda_i} & (i = 1) \end{array} \right. \]  
(3.23)

which implies (2.51). From these checks we conclude that \( \mathcal{H}_{l,k}^{(i)} = \mathcal{F}_{l,k}^{(i)} \otimes \mathcal{F}_{\phi} \) and \( \rho_{l,k}^{(i)} = [k] x^{2 H_{l,k}^{(i)}} \).

4 Spontaneous polarization

4.1 Integral formulae

We are now in a position to calculate the spontaneous polarization, i.e., the vacuum expectation value of \( S_z \) at the center site of the lattice:

\[ S_z^i = \sum_{j=-1}^{1} j \Phi_j^*(u) \Phi^j(u). \]  
(4.1)

From (3.19), we have

\[ \langle S_z^i \rangle = \frac{1}{\chi^{(i)}} \sum_{j=-1}^{1} \sum_{k_{l,k}} \sum_{k_{l,k}} \sum_{k_{l,k}} \sum_{k_{l,k}} t_j^*(u - u_0) k_j^2 t_j^*(u - u_0) k_j^2 \times \text{tr}_{\mathcal{H}_{l,k}^{(i)}} \left( \Phi^*(u) k_j^2 \Phi(u) k_j^2 \Lambda(u_0) k_j^2 [k] x^{2 H_{l,k}^{(i)}} \right). \]  
(4.2)

for \( i = 0, 1, 2 \). It is difficult to simplify the sum of (4.2). We thus use another representation of (4.1):

\[ S_z^i = \sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \frac{j_1 + j_2}{2} \Phi^j(u) \Phi^j(u). \]  
(4.3)

Using this trick, we have

\[ \langle S_z^i \rangle = \langle S_z^i \rangle_1 + \langle S_z^i \rangle_2, \]  
(4.4)

where

\[ \langle S_z^i \rangle_1(a) = \frac{1}{2\chi^{(i)}} \sum_{k_{l,k}} \sum_{k_{l,k}} \sum_{k_{l,k}} \sum_{k_{l,k}} j_1^a t_j^*(u - u_0) k_j^2 \sum_{j_2 = \pm 1} j_2^a t_2^*(u - u_0) k_j^2 \times \text{tr}_{\mathcal{H}_{l,k}^{(i)}} \left( \Phi^*(u) k_j^2 \Phi(u) k_j^2 \Lambda(u_0) k_j^2 [k] x^{2 H_{l,k}^{(i)}} \right). \]  
(4.5)

for \( a = 1, 2 \).

First consider \( \langle S_z^i \rangle_1(1) \). Since the twenty-one-vertex model does not contain the parameter \( u_0 \), a correlation function such as (4.5) should be \( u_0 \)-independent. For simplicity of calculation, let \( u_0 = u - \frac{r+1}{2} \). Then from

\[ \sum_{j_1 = \pm 1} j_1^a t_j^*(u - u_0) k_j^2 = 0, \]  
15
we can express (4.5) as follows:

$$\langle S_{ij}^{(1)} \rangle = \frac{1}{2^{|l|^\prime}} (H_l^{(i)} + H_l^{(j)}), \quad (4.6)$$

where

$$H_{k^l}^{(i)} = \sum_{k \equiv l+1 (2)} \sum_{k_1 \sim k_2} \sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \frac{j_1 t_j (r_{r_1}^t \cdot 1)^k}{h_1} \sum_{j_2 = \pm 1} \left( \sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \frac{j_1 t_j (r_{r_1}^t \cdot 1)^k}{h_1} \right) \times \text{tr}_{H_{k^l}^{(i)}} \left( \Phi^*(u)_{k^l} \Phi(u)_{k^l} \Lambda(u - \frac{r_{r_1}^t \cdot 1}{2}, k) \right), \quad (4.7)$$

with $k_2 = k + 2$. Here, $H_{k^l}^{(i)}$ is a contribution from $k_2 = k + 2$.

By using

$$\sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \frac{j_1 t_j (r_{r_1}^t \cdot 1)^k}{h_1} = \frac{1}{h_1 (k-2)},$$

and

$$\sum_{j_2 = \pm 1} \frac{j_1 t_j (r_{r_1}^t \cdot 1)^k}{h_1} = \frac{h_3 (k-2)}{h_1 (k-2)},$$

the $H_l^{(i)}$ after taking the sum over $k_1$ reduces to

$$H_l^{(i)} = \sum_{k \equiv l+1 (2)} \frac{1}{\lambda} \int_C \frac{dw_1}{2\pi\sqrt{-1}w_1} \int_C \frac{dw_2}{2\pi\sqrt{-1}w_2} \left( g(v_1, v_2; u, k) + h(v_1, v_2; u, k) \right) \times \text{tr}_{H_{k^l}^{(i)}} \left( \Phi_1(u-1)\Phi_1(u)A(v_1)A(v_2)\frac{2H_{(k)}^{(k)}_{\text{CTM}}}{[l]^\prime} \right), \quad (4.8)$$

where $w_j = x^{2v_j}$, and

$$g(v_1, v_2; u, k) = \frac{h_1(v_1 - u + 1)h_4(k-2)h_2(v_1 - v_2)h_4(v_1 + v_2 - 2u + 1 - k)}{h_1(v_1 - u - 1)h_3(v_1 - v_2)h_2(v_1 + v_2 - 2u + 1 - k) h_3(h_1(v_1 - u)h_2(v_1 - u)h_1(v_2 - u)h_2(v_2 - u))},$$

$$h(v_1, v_2; u, k) = \frac{h_1(v_2 - v_1 - 1)h_2(v_1 + v_2 - 2u + 2 - k)h_3(k - 1)h_4(0)}{h_1(v_1 - u)h_2(v_1 - u)h_1(v_2 - u)h_2(v_2 - u)}.$$

Note that $h(v_1, v_2; u, k)/h_1(v_2 - v_1 - 1)$ is symmetric with respect to $v_1$ and $v_2$, and also note that the integral contours for $w_1$ and $w_2$ are the same. Owing to the commutation relation (3.3), the term proportional to $h(v_1, v_2; u, k)$ on (4.8) vanishes after the integrals. Thus we have

$$H_l^{(i)} = \sum_{k \equiv l+1 (2)} \frac{1}{\lambda} \int_C \frac{dw_1}{2\pi\sqrt{-1}w_1} \int_C \frac{dw_2}{2\pi\sqrt{-1}w_2} \tilde{g}(v_1, v_2; u, k) \times \text{tr}_{H_{k^l}^{(i)}} \left( \Phi_1(u-1)\Phi_1(u)A(v_1)A(v_2)\frac{2H_{(k)}^{(k)}_{\text{CTM}}}{[l]^\prime} \right), \quad (4.9)$$

where

$$\tilde{g}(v_1, v_2; u, k) = \frac{h_1(1)h_1(v_1 - u + 1)h_2(v_1 - v_2)h_4(v_1 + v_2 - 2u + 1 - k)}{h_1(v_1 - u - 1)h_1(v_1 - u)h_2(v_1 - u)h_1(v_2 - u)h_2(v_2 - u)}.$$ 

---

2The expression (4.4) is due to the symmetry of the space of states $H_{k^l}^{(i)} = H_{l-k}^{(i)}$, and the so-called $\sigma$-invariance of the Boltzmann weights with respect to the simultaneous transformation $\sigma : k_j \rightarrow -k_j$ for each site $j$. The free field representations (3.8), (3.12) and (3.18) are not invariant under the transformation $\sigma$. Thus in order to calculate $H_{k^l}^{(i)}$, we should substitute $\Phi(u)_{k^l}, \Phi^*(u)_{k^l}$ and $A(0_k)^{1-k^2}$ into $\Phi(u)_{-k^l}, \Phi^*(u)_{-k^l}$ and $A(0_k)^{-1-k^2}$ on (3.8), (3.12) and (3.18), respectively.
Let us recall (3.10), and first perform the trace on the fermionic Fock space $\mathcal{F}^\phi$. Simple calculation shows

$$\text{tr}_{\mathcal{F}^\phi} \left( \phi(w_1)\phi(w_2)x^2H_{\phi}^{(i)} \right) = \begin{cases} \left( -x^2; x^4 \right)_\infty D(w_1, w_2) & (i = 0, 2) \\ \left( -x^4; x^4 \right)_\infty (D(w_1, w_2) - 1) & (i = 1) \end{cases} \quad (4.10)$$

where

$$D(w_1, w_2) = \frac{1}{x + x^{-1}} \left( \delta \left( \frac{x^2 w_2}{w_1} \right) + \delta \left( \frac{x^{-2} w_2}{w_1} \right) \right). \quad (4.11)$$

Secondly, let us perform the trace on the bosonic Fock space $\mathcal{F}^{(i)}$. By using OPE formulae (3.4), we have

$$\Phi_1(u - 1)\Phi_1(u) \tilde{A}(v_1) \tilde{A}(v_2) = \left( x^2 z^2 w_1 w_2 \right)^{\frac{m}{2}} \frac{\left( x^4 w_1 w_2 \right)_\infty}{\left( x^2 w_1 w_2 \right)_\infty} \times \prod_{j=1}^{2} (x^{-2} z^2)^{-\frac{m}{2}} \frac{\left( x^{2r_2 - 2} w_1 z \right)_\infty}{\left( x^{2r_2} w_1 z \right)_\infty} \times \frac{\left( z^2 \right)_\infty}{\left( x^2 w_1 w_2 \right)_\infty} \times \exp \left( \sum_{m \neq 0} \beta_m \frac{B_m}{m} \right) \quad (4.12)$$

where $B_m = -(x^2 + 1)z^m + w_1^m + w_2^m$, and $\tilde{A}(v)$ denotes the fermion contraction

$$\tilde{A}(v) = w \frac{\omega}{x^2} : \exp \left( \sum_{m \neq 0} \frac{\beta_m}{m} w^{-m} \right) : e^{-\frac{\omega}{x^2} L - \frac{\omega}{x^2} K}.$$

What we want to calculate is the following trace:

$$\text{tr}_{\mathcal{F}^{(i)}} \left( \Phi_1(u - 1)\Phi_1(u) \tilde{A}(v_1) \tilde{A}(v_2) x^{2H^{(i)}_{\phi}} \right) = \left( x^2 z^2 w_1 w_2 \right)^{\frac{m}{2}} \frac{\left( x^4 w_1 w_2 \right)_\infty}{\left( x^2 w_1 w_2 \right)_\infty} \times \prod_{j=1}^{2} (x^{-2} z^2)^{-\frac{m}{2}} \frac{\left( x^{2r_2 - 2} w_1 z \right)_\infty}{\left( x^{2r_2} w_1 z \right)_\infty} \times \frac{\left( z^2 \right)_\infty}{\left( x^2 w_1 w_2 \right)_\infty} \times \exp \left( \sum_{m \neq 0} \beta_m \frac{B_m}{m} \right) \quad (4.13)$$

Following [2], we explain how to calculate the trace (4.13). Note that

$$e^{-\frac{\beta_m}{m} B_m} e^{\frac{\beta_m}{m} B_m} = e^{-\frac{\beta_m}{m} B_m}$$

Multiply $x^{4mn}$ by the coefficient of $\beta_m^{n} |l,k\rangle$ on (4.14), and take the sum with respect to $n$. Then we obtain

$$\sum_{n=0}^{\infty} x^{4mn} \sum_{r=0}^{n} \frac{n!}{s!} \left( \frac{[r^m]_m B_m}{m} \right)^s = \sum_{s=0}^{\infty} \frac{1}{s!} \left( \frac{[r^m]_m B_m}{m} \right)^s \sum_{n=s}^{\infty} C_n x^{4mn}$$

$$= \frac{1}{1 - x^{4m}} \exp \left( -\frac{x^{4m} [r^m]_m B_m}{m} \right)$$

(4.15)
Thus we have
\[
\prod_{m=1}^{\infty} \sum_{n=0}^{\infty} \langle l, k | \beta_n^m e^{-\frac{x}{m} B_m - \frac{1}{m} B_m \beta_n^m} | l, k \rangle x^{4mn} = \frac{1}{(x^4; x^4)_\infty} \exp \left( - \sum_{m=1}^{\infty} \frac{x^{4m} \langle \rho^m | B_m B_m^* \rangle}{m} \right) = 1 \left( \frac{(x^4; x^4)_\infty}{(x^4; x^4)^2_\infty} \right)^2 \times \prod_{j=1}^{2} (x^2 w_j/z; x^4)^\infty \times (x^2 z/w; x^4)^\infty (x^4 w_2/w_1; x^4)^\infty (x^2 w_2/w_1; x^4)^\infty \quad (4.16)
\]

Hence, by performing the trace on the $F^{(i)}_{l,k}$ we have
\[
H^{(i)}_l = C \sum_{k=1}^{l+i+1(2)} \int_C \frac{dw_1}{2\pi \sqrt{-1w_1}} \int_C \frac{dw_2}{2\pi \sqrt{-1w_2}} [v_1 - v_2 + 1]_{p=2} h_1(v_1 - u + 1) \times [1] h_2(0) h_2(v_1 - v_2) \left( v_1 + v_2 - 2u + 1 - k \right) \prod_{j=1}^{2} \left[ v_j - u \right]_{m=1}^{[v_j - u - 1]} \times x^{(l-\frac{m}{2})} (v_1 + v_2 - 2u + 1 + \frac{m}{2}) \left[ x^{\rho^2-\frac{l+k}{2}} + x^{\frac{l+k}{2}} \right] \frac{1}{[l]_{\rho}''} F^{(i)}(w_1, w_2),
\]

where
\[
C = \frac{x^{\frac{m}{2}} (x + x^{-1}) (x^2; x^4)_{\infty} (x^2; x^2)^2_\infty (x^4; x^4)^2_\infty (x^2; x^2)^2_\infty (x^4; x^4)^2_\infty (x^4; x^4)^2_\infty},
\]

and $F^{(i)}(w_1, w_2)$ denotes the Fermionic trace derived on (4.10).

Next, let us take the sum over $k$ on (4.17). Let $s = v_1 + v_2 - 2u + 1$. From
\[
\{ s - k \} = x^{\frac{i-k^2}{2}} (s-k) \sum_{n=0}^{\infty} x^{r(n-1)} (x^{2(s-k)})^n,
\]

we have
\[
\sum_{k=1}^{l+i+1(2)} \{ s - k \} c_{\lambda_{k-1}}^{l}(x) x^{\frac{i-k^2}{2}} s + \frac{m}{2} s^2 x^{\rho^2-\frac{l+k}{2}} + \frac{m}{2} s^2 \frac{1}{[l]''}.
\]

(4.18)

Here, the order of the sum over $k$ and $n$ can be exchanged because the double sums absolutely converge.

When $i = 0, 2$, the explicit form of (4.18) is as follows:
\[
\left\{ \begin{array}{l}
\sum_{n=0}^{\infty} x^{r(2n)} x^{4n} \left( c_{\lambda_0}^{l}(x) \sum_{k=0}^{\infty} x^{4k} (2k+1) + c_{\lambda_0}^{l}(x) \sum_{k=0}^{\infty} x^{4k} (4k+1) \right) \\
+ \sum_{n=0}^{\infty} x^{r(2n+1)} x^{2(2n+1)} \left( c_{\lambda_2}^{l}(x) \sum_{k=0}^{\infty} x^{4k} (2k+1) + c_{\lambda_0}^{l}(x) \sum_{k=0}^{\infty} x^{4k} (4k+1) \right)
\end{array} \right.
\]

(4.19)

Here, add the corresponding term on $H^{(i)}_{l-1}$ to (4.19), by noting that
\[
\frac{x^{\frac{m}{2}} + x^{\frac{m}{2}}}{[l]''} \sum_{n=0}^{\infty} x^{r(2n-1)} x^{4n} + \frac{x^{\frac{m}{2}} - x^{\frac{m}{2}}}{[-l]''} \sum_{n=0}^{\infty} x^{r(2n-1)} x^{4n} = -1.
\]

(4.20)
Then we obtain the following expression for $i = 0, 2$:

$$
\langle S_i \rangle_i^{(1)} = \frac{C}{2^{2i}} \int_C \frac{dw_1}{2\pi \sqrt{-1}w_1} \int_C \frac{dw_2}{2\pi \sqrt{-1}w_2} (c_{\lambda_i}^\lambda(x) - c_{\lambda_0}^\lambda(x))F^{(i)}(w_1, w_2)
\times [1][v_1 + v_2 - 2u + 1]_{r_{i+2}} [v_1 - u + 1][v_2 - u - 1]
\times \frac{h_2(0)h_3(v_1 - v_2)}{h_3(v_1 - u)h_3(v_2 - u)}.
$$

Next let us calculate $\langle S_i \rangle_i^{(2)}$ on (4.21). Put $u_0 = u - \frac{r_0+r_{i+1}}{2}$ and perform the integral once on (4.22). Then we have

$$
\langle S_i \rangle_i^{(2)} = \frac{1}{2}\pi \sqrt{-1}w_1 \int_C \frac{dw_2}{2\pi \sqrt{-1}w_2} (c_{\lambda_i}^\lambda(x) - c_{\lambda_0}^\lambda(x))F^{(i)}(w_1, w_2)
\times [1][v_1 + v_2 - 2u + 1]_{r_{i+2}} [v_1 - u + 1][v_2 - u - 1]
\times \frac{h_2(0)h_3(v_1 - v_2)}{h_3(v_1 - u)h_3(v_2 - u)} + \frac{h_3(0)h_3(v_1 - v_2)}{h_3(v_1 - u)h_3(v_2 - u)}.
$$

Thus we obtain the following integral formulae of $\langle S_i \rangle_i$ for $i = 0, 2$:

$$
\langle S_i \rangle_i^{(1)} = \frac{C}{2^{2i}} \int_C \frac{dw_1}{2\pi \sqrt{-1}w_1} \int_C \frac{dw_2}{2\pi \sqrt{-1}w_2} (c_{\lambda_i}^\lambda(x) - c_{\lambda_0}^\lambda(x))F^{(i)}(w_1, w_2)
\times [1][v_1 + v_2 - 2u + 1]_{r_{i+2}} [v_1 - u + 1][v_2 - u - 1]
\times \frac{h_2(0)h_3(v_1 - v_2)}{h_3(v_1 - u)h_3(v_2 - u)} + \frac{h_3(0)h_3(v_1 - v_2)}{h_3(v_1 - u)h_3(v_2 - u)}.
$$

Here, the integral contour $C$ is the anti-clockwise circle defined by $|z| < |w_j| < x^{-2}|z| (j = 1, 2)$.

From (4.22) with (2.31) and (2.35) we can show

$$
\langle S_i \rangle_i^{(1)} = -\langle S_i \rangle_i^{(2)}.
$$

When $i = 1$, we obtain an analogous expression to (4.22), by replacing

$$
(c_{\lambda_2}^\lambda(x) - c_{\lambda_0}^\lambda(x))h_1^{(2)}(v_1 + v_2 - 2u + 1) \quad \text{by} \quad (c_{\lambda_2}^\lambda(x) - c_{\lambda_0}^\lambda(x))h_1^{(2)}(v_1 + v_2 - 2u),
$$

and $F^{(i)}(w_1, w_2)$ for $i = 0, 2$ by the one for $i = 1$. Since $c_{\lambda_i}^\lambda(x) = c_{\lambda_0}^\lambda(x)$, we have

$$
\langle S_i \rangle_i^{(1)} = 0.
$$

### 4.2 Some limiting cases

Let $i = 2$ and perform the integral once on (4.22). Then we have

$$
\langle S_i \rangle_i^{(2)} = \frac{x^2}{2\pi \sqrt{-1}w_1} \int_C \frac{dw_2}{2\pi \sqrt{-1}w_2} (c_{\lambda_i}^\lambda(x) - c_{\lambda_0}^\lambda(x))F^{(i)}(w_1, w_2)
\times [1][v_1 + v_2 - 2u + 1]_{r_{i+2}} [v_1 - u + 1][v_2 - u - 1]
\times \left( \begin{array}{c} [0] \{1\} \\ [v_1 - u][v_1 - u + 1] \end{array} + \begin{array}{c} [1] \{0\} \\ [v_1 - u][v_1 - u + 1] \end{array} \right).
$$

A very analogous result to (4.25) was reported by H. Konno in [31]. Konno’s formula is given in terms of a contour integral of a monomial of a product of elliptic functions, whereas our formula (4.25) is that of a binomial of the products of elliptic functions.
In order to examine the validity of the formulae, let us consider some limiting cases. We derived (4.25) under the assumption of \( r > 2 \). However, this expression is well defined even for \( r = 2 \). Let \( r = 2 \). Then the model describes the spin 1 analogue of the Ising model, and the expression (4.26) can be simplified as follows:

\[
\langle S_1^z \rangle_2 = \frac{\langle x^2; x^2 \rangle_\infty}{\langle -x^2; x^2 \rangle_\infty (x^4; x^4)_\infty} \int_C \frac{dw_1}{2\pi\sqrt{-1}w_1} \frac{h_1^{(1)}(v_1 - u)}{h_1^{(1)}(v_1 - u)}
\]  

(4.26)

Note that by transforming \( w_1 = x^{-2}w'_1 \), the integral (4.26) reduces to the one along the contour \( x^2C \) with changing the sign of the integrand. Thus (4.26) can be evaluated by the half of the residue at \( w_1 = z \):

\[
\langle S_1^z \rangle_2^{(1)} = -\frac{\langle x^2; x^2 \rangle_\infty}{\langle -x^2; x^2 \rangle_\infty (x^4; x^4)_\infty} \text{Res}_{w_1} \frac{dw_1}{w_1} \frac{h_1^{(1)}(v_1 - u)}{h_1^{(1)}(v_1 - u)} = \frac{\langle -x^4; x^4 \rangle_\infty}{\langle -x^2; x^2 \rangle_\infty (x^4; x^4)_\infty}.
\]  

(4.27)

A one-point function of the twenty-one-vertex model is like a nearest neighbor two-point function of the eight-vertex model. The integral formula for the nearest neighbor two-point correlation function of the inhomogeneous (Z-invariant) eight-vertex model was obtained in [32]. Their formula is given in terms of a two-fold integral. The structure of their formula is similar to ours (4.22). In [32] Lashkevich and Pugai derived an infinite product formula for the nearest neighbor diagonal correlation functions of the inhomogeneous Ising model in the ferromagnetic regime, by performing the integrals twice of their integral formula at \( r = 2 \).

The expression (4.25) can be also simplified when \( r \) is an odd integer. In this case the sign of the integrand changes under the transformation \( v_1 \mapsto v_1 + r \). Thus the (4.25) can be evaluated by the half of the sum of residues on the annulus between \( C \) and \( x^{2r}C \). On this annulus, there exist simple poles at \( x^{2n}z \) (\( 1 \leq n \leq r - 2 \)), \( \pm x^r z \) and \( \pm x^{r+2} z \). Thus we obtain ‘the sum of products formulae’ for odd integers \( r \)’s.

Next let us consider the limit \( \epsilon \rightarrow +0 \), called the critical limit. This limit describes the massless regime of the twenty-one-vertex model. Let \( \nu = \frac{1}{\pi}, \beta = \pi\sqrt{-1}u \) and \( \alpha = \pi\sqrt{-1}v_1 \). Then the interval of integration becomes \([ -\frac{\pi^2}{\epsilon}, \frac{\pi^2}{\epsilon} ] \rightarrow (-\infty, \infty) \). Note that

\[
\Theta_{x^1}(x^2) = \sqrt{\frac{\pi}{2\epsilon}} h^{(2)}_1(1) \sim \sqrt{\frac{2\pi}{\epsilon}} \exp \left( -\frac{x^2}{2\epsilon}\right),
\]

in this limit. Thus the formula (4.25) in the limit \( \epsilon \rightarrow +0 \) reduced to

\[
\langle S_1^z \rangle_2 = \mu \int_{-\infty}^{+\infty} \frac{d\alpha}{\pi^2\sqrt{1}\text{sh}(\alpha - \beta)\text{sh}(\alpha - \beta + \pi\sqrt{-1})} \text{sh}(\alpha - \beta + \pi\sqrt{-1}),
\]  

(4.28)

where

\[
\mu = -\frac{\sqrt{2\pi\epsilon}}{2\nu\sin(\nu\pi)(-x^4; x^4)_\infty} \exp \left( -\frac{\nu^2\pi^2}{2\epsilon}\right).
\]

Note that the integrand of (4.28) has no poles on the real axis.

When \( r \rightarrow 2 + 0 \) (\( \nu \rightarrow \frac{1}{2} \)), we should let \( \beta \) be located above the real axis. The expression (4.25) in the limit \( r \rightarrow 2 + 0 \) reduces to

\[
\langle S_1^z \rangle_2 = \sqrt{\frac{2\pi\epsilon}{(-x^4; x^4)_\infty}} \int_{-\infty}^{+\infty} \frac{d\alpha}{\pi^2\sqrt{-1}\text{sh}(\alpha - \beta)} \text{sh}(\alpha - \beta + \pi\sqrt{-1}),
\]  

(4.29)
which is consistent with (4.27) in the limit $\epsilon \to +0$. This is due to the following approximation property
\[
\Theta_x^2(-x^2) = \sqrt{\frac{1}{2 \pi}} h_4^{(2)}(1) - \sqrt{\frac{1}{2 \pi}} (\epsilon \to +0).
\]
Thus, the order of the double limits $r \to 2 + 0$ and $\epsilon \to +0$ are commutative.

When $r$ is an odd integer, the integral (4.28) can be evaluated by the half of the sum of residues on the strip between the real axis $\mathbb{R}$ and $\mathbb{R} + r\pi \sqrt{-1}$. We obtain
\[
\langle S_z^1 \rangle_2 = \mu' \left( \frac{-1}{\nu} \cot(\nu \pi) + \sum_{a=1}^{r-2} (-1)^a \tan(a \nu \pi) \tan(a + 1) \nu \pi \right),
\]
with
\[
\mu' = \sqrt{\frac{\epsilon}{2 \pi \nu \sin(\nu \pi)}} \frac{1}{(-x^4; x^4)_\infty},
\]
for $r = 2n + 1 \ (n \geq 2)$. Anyway, the expressions (4.29–4.30) vanish when $\epsilon \to +0$. It is very likely for generic $r > 2$ that the spontaneous polarization (4.28) vanishes since the overall factor $\mu \to 0$ when $\epsilon \to +0$.

Finally, let us consider the trigonometric limit $r \to \infty$. This limit describes the nineteen-vertex model in the antiferroelectric regime, whose integral formulae for correlation functions were obtained by Idzumi [14], Bougourzi and Weston [16]. The spontaneous polarization of the twenty-one-vertex model (4.25) in the limit $r \to \infty$ reduces to:
\[
\langle S_z^1 \rangle_2 = \left( 1 - x^2 \right) \left( -x^4; x^4 \right)_\infty \left( x^8; x^8 \right)_\infty,
\]
where
\[
G(v_1; u) := \left( \frac{z}{w_1} - 1 \right) \left( 1 - \frac{x^2 w_1}{z} \right) = - (1 + x^2) + \frac{x^2 w_1}{w_1} + \frac{x}{w_1}.
\]
This expression should give another integral formula for the spontaneous polarization of the nineteen-vertex model. The terms proportional to $- (1 + x^2)$ on (4.31) can be evaluated by the half of the residue at $w_1 = z$. Furthermore, the term proportional to $\frac{x^2 w_1}{w_1}$ on (4.31) can be formally evaluated by the sum of the residues at $w_1 = x^{2n} z$ for $n \geq 0$. Concerning the term proportional to $\frac{x}{w_1}$, the sequence of the residues inside the contour $C$ diverge so that we can evaluate the integral formally by the sum of the residues outside $C$, i.e., those at $w_1 = x^{-2n} z$ for $n \geq 1$. Thus we have
\[
\langle S_z^1 \rangle_2 = \frac{1 - x^2 \left( x^2; x^2 \right)_\infty}{1 + x^2 \left( x^8; x^8 \right)_\infty}.
\]

Idzumi performed his integral formula up to $x^{150}$ to obtain the following infinite product representation [14]:
\[
\langle S_z^1 \rangle_2 = \frac{(x^2; x^2)_\infty}{(-x^4; x^4)_\infty}.
\]
Unfortunately, (4.32) do not coincide with (4.33). We have no resolution of this mystery at present.
4.3 Inhomogeneous model

In this subsection let us consider the inhomogeneous model, and let us calculate the following quantity:

\[
\langle S_i^1 \rangle_i (u_1, u_2) = \frac{1}{\chi^{(i)}} \sum_{j=-1}^{1} \sum_{k=1+i+1(2)}^{k+2} \sum_{k_2=k-2}^{k_2+2} \sum_{k_1=k-1(2)}^{k_1+1} t_j^* (u_2 - u_0)_{k_2}^k t_j (u_1 - u_0)_{k_1}^k \times \text{tr}_{H_{l,k}} \left( \Phi^* (u_2)_{k_2}^k \Phi (u_1)_{k_1}^k \Lambda (u_0)_{k}^k \right)_{(l,k)} \times e^{2H_{cT}(l,k)} .
\] (4.34)

As was done in section 4.1, we divide \( \langle S_i^1 \rangle_i (u_1, u_2) \) into two parts, \( \langle S_i^1 \rangle_i^{(1)} (u_1, u_2) \) and \( \langle S_i^1 \rangle_i^{(2)} (u_1, u_2) \). For the former part we put \( u_0 = u_2 - \frac{t_2 + t_3}{2} \), and for the latter part we put \( u_0 = u_2 - \frac{t_2 - t_3}{2} \), where \( \tau = \frac{x}{t \tau} \). By repeating the same procedure we obtain

\[
\langle S_i^1 \rangle_i^{(1)} (u_1, u_2) = \frac{(x^2; x^2)^2_\infty}{2(x^2; x^2)_\infty} \frac{h_{l_1}^{(1)} (1)^2 h_{l_2}^{(1)} (u_2)}{h_1 (1) h_1 (u_1)} \int \frac{dw_1}{2\pi \sqrt{-1}w_1} h_1 (v_1 - u_2) h_1 (v_1 - u_2 + 1)
\times \frac{h_{l_1}^{(2)} (v_1 - u_1) h_{l_2}^{(2)} (v_1 - u_2 + 1)}{h_2 (v_1 - u_2) h_2 (v_1 - u_2 + 1)} .
\] (4.35)

where \( u_{12} = u_1 - u_2 \). By transforming \( v_1 \mapsto v_1 + u_2 \), (4.35) actually depends only on \( u_{12} \), i.e., \( \langle S_i^1 \rangle_i^{(2)} (u_1, u_2) = \langle S_i^1 \rangle_i^{(2)} (u_1, u_2) \). Furthermore, \( \langle S_i^1 \rangle_i^{(1)} (0) \) coincides the expression for spontaneous polarization (4.28), as expected.

4.4 Calculation of \( \langle S_0^x \rangle_i \)

In this paper we presented vertex operator approach for correlation functions of spin 1 analogue of the eight-vertex model, the twenty-one-vertex model. Using the present formalism, we can obtain the integral formulae for any correlation functions of the twenty-one-vertex model, in principle. However, it is very difficult to perform the sum over \( k \)'s, the local states of the fusion SOS model, for general \( n \)-point functions with \( n \geq 2 \).

In order to show the relevance of our formalism, let us calculate \( \langle S_0^x \rangle_i \) for \( i = 0, 1, 2 \), the vacuum expectation value of \( S_0^x \) at the center site of the lattice:

\[
S_0^x = \frac{1}{\sqrt{2}} \sum_{j=-1}^{1} \Phi^{(i)} (u) (\Phi^{(i+1)} (u) + \Phi^{(i-1)} (u)) \equiv \frac{1}{\sqrt{2}} \left[ \Phi^0 (u) (\Phi^1 (u) + \Phi^{-1} (u)) + \Phi^{(i)} (u) (\Phi^{(i+1)} (u)) + \Phi^{(i)} (u) (\Phi^{(i-1)} (u)) \right] .
\] (4.36)

Let us briefly explain the procedures of obtaining \( \langle S_0^x \rangle_i \). From (3.19) and (4.36), we have

\[
\langle S_0^x \rangle_i = \langle S_0^x \rangle_i^{(1)} + \langle S_0^x \rangle_i^{(2)} ,
\] (4.37)

where

\[
\langle S_0^x \rangle_i^{(1)} = \frac{1}{\sqrt{2} \chi^{(i)}} \sum_{k=1+i+1(2)}^{k+2} \sum_{k_2}^{k_2+2} \sum_{k_1}^{k_1+1} t_j^* (u_2 - u_0)_{k_2}^k t_j (u_1 - u_0)_{k_1}^k T ,
\]

\[
\langle S_0^x \rangle_i^{(2)} = \frac{1}{\sqrt{2} \chi^{(i)}} \sum_{k=1+i+1(2)}^{k+2} \sum_{k_2}^{k_2+2} \sum_{k_1}^{k_1+1} t_j^* (u_2 - u_0)_{k_2}^k t_j (u_1 - u_0)_{k_1}^k T ,
\] (4.38)
and
\[ T = \text{tr}_{A_{ik}} \left( \Phi^* (u)_{k_1} \Phi (u)_{k_2} \Lambda (u_0)_{l_k} \frac{[k]}{[l_k]} x^{2H_{23}^{(1,2)}} \right). \]

We repeat the similar calculation as was done for \( \langle S_{1}^{x} \rangle_i \). Let \( u_0 = u - \frac{1}{2} + \frac{\pi}{2g} \) for the calculation of \( \langle S_{2}^{x} \rangle_{i}^{(1)} \). In this case we should replace \( \tilde{g} (v_1, v_2; u, k) \) on (4.9) by
\[
\tilde{g}_1 (v_1, v_2; u, k) = \frac{h_1 (v_1 - u + 1)}{h_1 (v_1 - u - 1)} \frac{h_2 (v_1 + v_2 - 2u + 1 - k)h_3 (v_1 - v_2)h_4 (0)}{h_1 (v_1 - u) h_2 (v_1 - u) h_3 (v_1 - u) h_4 (v_1 - v_2)}. 
\]
Let \( u_0 = u - \frac{r + 1}{2} - \frac{\pi}{2g} \) for the calculation of \( \langle S_{2}^{x} \rangle_{i}^{(1)} \). In this case we should replace \( g (v_1, v_2; u, k) \) on (4.9) by
\[
\tilde{g}_2 (v_1, v_2; u, k) = \frac{h_1 (v_1 - u + 1)}{h_1 (v_1 - u - 1)} \frac{h_2 (v_1 + v_2 - 2u + 1 - k)h_3 (0)h_4 (v_1 - v_2)}{h_1 (v_1 - u) h_2 (v_1 - u) h_3 (v_1 - u) h_3 (v_2 - u)}. 
\]
Thus, the sum over \( k \) on (4.38) can be reduced to (4.18) with \( s - k \) replaced by
\[
[s - k] := x^{-\frac{k^2}{2}} \sum_{n \in \mathbb{Z}} x^{rn^2} (-x^2 (s - k))^n. 
\]

Here note that \( s = v_1 + v_2 - 2u + 1 \). Accordingly, the corresponding equality (4.20) for the calculation of \( \langle S_{1}^{x} \rangle_i \) should be given by
\[
\sum_{n \in \mathbb{Z}} x^{rn^2} \sum_{n \in \mathbb{Z}} x^{r(n+2p)^2} x^{-2l(2n+p)} + \sum_{n \in \mathbb{Z}} x^{rn^2} \sum_{n \in \mathbb{Z}} x^{r(n+2p)^2} x^{2l(2n+p)} = 0, 
\]
for \( p = 0, 1 \). Therefore we obtain the expected result for \( \langle S_{1}^{x} \rangle_i \):
\[
\langle S_{1}^{x} \rangle_i = 0. \quad (4.39) 
\]

5 Concluding remarks

In this paper we have derived integral formulæ for the spontaneous polarization of the twenty-one-vertex model. For that purpose we constructed the free field representations of type I vertex operators \( \Phi (u)_{k}^{I} \) in \( 2 \times 2 \) fusion SOS model, the tail operators \( \Lambda (u_0)_{k}^{I} \) and the corner transfer Hamiltonian \( H_{23}^{(1,2)} \).

Our integral formulæ are given by (4.22). The formula (4.22) is given in terms of the two-fold integral. By performing the integral once, we further obtain the one-fold integral formula (4.25). We examined the validity of our results by considering some limiting case. When \( r \to 2 + 0 \) we obtain an infinite product representation by performing the remained one-fold integral. When \( r \) is an odd integer our formula (4.25) reduces to the sum of infinite product formulæ. The critical limit \( \epsilon \to +0 \) was considered to obtain the spontaneous polarization of the twenty-one-vertex model in the massless regime. We take the trigonometric limit \( r \to \infty \) to compare with the spontaneous polarization of the nineteen-vertex model obtained by Date et al. [13] and Idzumi [13]. It is a future issue to show that our formulæ reproduce their results in the trigonometric limit. Furthermore, we presented the corresponding formula for the inhomogeneous twenty-one-vertex model.

Our approach is based on some assumptions. We assumed that the vertex operator algebra (2.48) and (2.50) correctly describes the intertwining relation between the twenty-one-vertex model and
Let $R$.

**A Appendix A Definitions of the models concerned**

I would like to thank H. Konno, A. Nakayashiki and M. Okado for discussion and their interests in the state sectors. We wish to address this issue in a separate paper.

In order to derive integral formulae for form factors of the twenty-one-vertex model, we need a free field representation of $\Lambda(u_0)^k_k$, non-diagonal components of the tail operator with respect to the ground state sectors. We wish to address this issue in a separate paper.

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#### A.1 $R$-matrix of the spin 1 analogue of the eight-vertex model

Let $R^{(s,s')}(u)$ ($s' = 1, \frac{1}{2}, \cdots$) be the $R$-matrix of vertically $2s$-fold and horizontally $2s'$-fold fusion of $R^{(1,1)}(u)$, the $R$-matrix of the eight-vertex model. Then non-zero elements of $R^{(1,1)}(u)$ are given as follows:

\[
R^{(1,1)}_{\mp s, s}(u) = \begin{cases} 
\frac{1}{\kappa_{1,2}(u)} \theta_2^s \left( \frac{1}{2\sqrt{2}} \right) \theta_2^s \left( \frac{1}{2} \sqrt{2} \right), & \text{if } s = \pm 1, \\
\frac{1}{\kappa_{1,2}(u)} \theta_2^{-s} \left( \frac{1}{2\sqrt{2}} \right) \theta_2^{-s} \left( \frac{1}{2} \sqrt{2} \right), & \text{if } s = 0.
\end{cases}
\]

where $\theta_i \left( \frac{u}{\sqrt{2}} \right) = \theta_i \left( \frac{u}{\sqrt{2}}; \frac{x^2 r^2}{2 \sqrt{2}} \right)$, and

\[
\kappa_{1,2}(u) = (x^{-1} z)^{-1} \left( \frac{e^{x^2 z^{-1} r^2}}{e^{x^2 z^{-1} r^2}} \right) \left( e^{x^2 z^{-1} r^2} \right) \left( e^{x^2 z^{-1} r^2} \right).
\]

The case $(s, s') = (1, 1)$ is of interest in the present study. There are twenty one non-zero elements of $R^{(1,1)}(u)$ so that the spin 1 analogue of the eight-vertex model is also called twenty-one-vertex model.
The explicit expressions of non-zero elements of $R^{(1,1)}(u)$ are given as follows:

\[
R^{(1,1)}(u)_{\pm 1 \pm 1}^{\pm 1 \pm 1} = \frac{1}{\kappa_{2,2}(u)} \begin{pmatrix}
\theta_2^z \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{2N} \right) & -\theta_1^z \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{2N} \right) \\
\theta_2^z \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{2N} \right) & \theta_1^z \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{2N} \right)
\end{pmatrix},
\]

\[
R^{(1,1)}(u)_{00}^{00} = \frac{1}{\kappa_{2,2}(u)} \begin{pmatrix}
\theta_2^z \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{2N} \right) & -\theta_1^z \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{2N} \right) \\
\theta_2^z \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{N} \right) \theta_2 \left( \frac{1}{2N} \right) & \theta_1^z \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{N} \right) \theta_1 \left( \frac{1}{2N} \right)
\end{pmatrix} = R^{(1,1)}(u)_{00}^{00},
\]

(A.2)

Here,

\[
\kappa_{2,2}(u) = z^{-\frac{a}{2}} \left( x^{2r-1}; x^{2r} \right)_\infty \left( x^{2r-2} z^{-1}; x^{2r} \right)_\infty.
\]

Note that some of components are modified by symmetrization of the $R$-matrix.

In this article we assume that the parameters $v$, $\epsilon$ and $r$ lie in the so-called principal regime (2.8).

### A.2 Boltzmann weights of $2 \times 2$ fusion SOS model

In what follows we use the following symbols:

\[
\begin{bmatrix}
u \\
m
\end{bmatrix} = \begin{bmatrix}
[u]_m \\
m \cdot m_m
\end{bmatrix}, \quad [u]_m = [u][u-1] \cdots [u-m+1].
\]

Let $W_{22}$ be the Boltzmann weights of $2 \times 2$ fusion SOS model, and let

\[
W_{22} \begin{bmatrix}
c & d \\
b & a
\end{bmatrix} = \bar{R}^{(2,2)}(u) \begin{pmatrix}
2 - u \\
2
\end{pmatrix} W_{22} \begin{bmatrix}
c & d \\
b & a
\end{bmatrix} u
\]

\[
= \bar{R}^{(2,2)}(u) \begin{pmatrix}
2 - u \\
2
\end{pmatrix} W_{22} \begin{bmatrix}
c & d \\
b & a
\end{bmatrix} u
\]

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be unnormalized weights. Then the non-zero \( W_{22} \) are given as follows:

\[
\begin{align*}
W_{22} & \begin{bmatrix} k \pm 4 & k \pm 2 & u \end{bmatrix} = \begin{bmatrix} 2 - u \\ 2 \end{bmatrix}, \\
W_{22} & \begin{bmatrix} k \pm 2 & k \pm 2 & u \end{bmatrix} = \frac{1 - u |k \pm 1 \pm u|}{|k| |k^2|}, \\
W_{22} & \begin{bmatrix} k \pm 2 & k \pm 2 & u \end{bmatrix} = \frac{k + 1}{|k|} \begin{bmatrix} 1 - u \\ 2 \end{bmatrix}, \\
W_{22} & \begin{bmatrix} k \pm 2 & k \pm 2 & u \end{bmatrix} = \frac{2 |k + 2| |u|}{|k| ^2 |k^2 - 1| |k + 1|}, \\
W_{22} & \begin{bmatrix} k \pm 2 & k \pm 2 & u \end{bmatrix} = \frac{k - 1 + u |k - u|}{|k| |k^2 - 1|} + \frac{k - 1 |k + 2|}{|k| |k + 1|} \begin{bmatrix} 1 - u \\ 2 \end{bmatrix}.
\end{align*}
\tag{A.3}
\]

Note that some of weights are modified by symmetrization of the Boltzmann weights. In this paper we consider so-called Regime III in the model, i.e., \( 0 < u < 1 \).

### A.3 Fused intertwining vectors

For \( k' = k, k \pm 2 \), let

\[
\begin{align*}
t(u)_{k'}^k & = \sum_{j=-1}^{1} v_j t^j(u)_{k'}, \\
t(u)_{k+2}^k & = \frac{1}{2h_1(u + \frac{1}{2})} \begin{bmatrix} h_3^{(2r)}(k \mp u \pm \frac{1}{2}) h_3^{(2r)}(k \mp u \pm \frac{1}{2}) \\ 2h_4(1) h_4(k \mp u \pm \frac{1}{2}) h_4^{(2r)}(k \mp u \pm \frac{1}{2}) \\ h_4^{(2r)}(k \mp u \pm \frac{1}{2}) h_4^{(2r)}(k \mp u \pm \frac{1}{2}) \end{bmatrix}, \\
t(u)_{k}^k & = \frac{1}{2h_1(u + \frac{1}{2})} \begin{bmatrix} h_3^{(2r)}(k - u \mp \frac{1}{2}) h_3^{(2r)}(k + u \mp \frac{1}{2}) \\ 2h_4(k) h_4(u \mp \frac{1}{2}) h_4^{(2r)}(k + u \mp \frac{1}{2}) \\ h_4^{(2r)}(k - u \mp \frac{1}{2}) h_4^{(2r)}(k + u \mp \frac{1}{2}) \end{bmatrix}.
\end{align*}
\tag{A.4}
\]

Then the following relation holds:

\[
R^{(1,1)}(u_1 - u_2) t(u_1)_{a}^d \otimes t(u_2)_{b}^d = \sum_{b} \sum_{k} t(u_1)_{k}^d \otimes t(u_2)_{b}^d W_{22} \begin{bmatrix} c \\ d \\ b \\ a \end{bmatrix} W_{22} \begin{bmatrix} c \\ d \end{bmatrix} W_{22} \begin{bmatrix} a \\ b \end{bmatrix}.
\tag{A.5}
\]

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The dual intertwining vectors are given as follows:

\[ t^*(u)^{k'}_k = \sum_{j=-1}^{1} v^{*j} t_j^*(u)^{k'}_k \]

\[ t^*(u)^{k+2}_k = \begin{cases} \frac{h_4^{(2r)}(k \pm u \pm \frac{1}{2})}{2h_1(k)h_1(k \pm 1)} & \text{for } k \neq 0, \\ \frac{h_4^{(2r)}(k \pm u)}{2h_1(k)h_1(k \pm 1)} & \text{for } k = 0 \end{cases} \]

\[ t^*_1(u)^{k'}_k = -\frac{h_4^{(2r)}(k + u + \frac{1}{2})}{2h_1(k)h_1(k + 1)}h_4(k + 1) + h_4(k - 1) \]

\[ t^*_0(u)^{k'}_k = -\frac{h_4^{(2r)}(k + u + \frac{1}{2})}{2h_1(k)h_1(k + 1)}h_4(k - 1) \]

The intertwining vectors and their dual vectors satisfy the following inversion relations:

\[ \sum_{j=-1}^{1} t_j^*(u)^{k'}_k t^j(u)^{k''}_{k'} = \delta^{k''}_{k''}, \quad \sum_{j=-1}^{1} t^j(u)^{k'}_k t^*_j(u)^{k''}_{k'} = \delta^{k''}_j. \] (A.7)

Then the following relation holds:

\[ t^*(u_1)^b_1 \otimes t^*(u_2)^b_2 R_{11}(u_1 - u_2) = \sum_d W_{22} \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] u_1 - u_2 t^*(u_1)^d_1 \otimes t^*(u_2)^d_2. \] (A.8)

The explicit expressions of the \( L \)-operators defined by (2.44) are given as follows:

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]

\[
L \begin{bmatrix} k' & k' + 2 \\ k & k + 2 \end{bmatrix} u_0 = \begin{cases} \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \\ \frac{k + k'}{2} & u_0 = \frac{k - k' + 1}{2} \end{cases}
\]
Appendix B  OPE formulae and commutation relations

In this Appendix we list some useful formulae for the basic operators. In what follows we denote \( z = x^{2u}, \ w = x^{2v} \).

First, useful OPE formulae are:

\[
\begin{align*}
\Phi_1(u)\Phi_1(v) &= z^{\alpha}(x^{2w}/z;x^{2w})_\infty : \Phi_1(u)\Phi_1(v) : , \\
\Phi_1(u)A(v) &= -z^{\alpha}(x^{2w}/z;x^{2w})_\infty : \Phi_1(u)A(v) : , \\
A(v)\Phi_1(u) &= w^{\beta}(x^{2z}/w;x^{2z})_\infty : A(v)\Phi_1(u) : , \\
\hat{A}(u)\hat{A}(v) &= z^{\alpha}(x^{2w}/z;x^{2w})_\infty : \hat{A}(u)\hat{A}(v) : , \\
A(u)A(v) &= z^{\alpha}(x^{2w}/z;x^{2w})_\infty : A(u)A(v) : + f(z, w) : \hat{A}(u)\hat{A}(v) : .
\end{align*}
\]

Here \( \hat{A}(v) \) denotes the fermion contraction

\[
\hat{A}(v) = w^{\alpha} \exp \left( \sum_{m \neq 0} \frac{\beta_m}{m} w^{-m} \right) : e^{-\alpha w^2 L - \frac{\alpha}{2} \kappa},
\]

and

\[
f(z, w) = \frac{1}{x + x^{-1}} \sum_{m > 0} \left( \left( \frac{x^{2w}}{z} \right)^m + \left( \frac{x^{-2w}}{z} \right)^m \right).
\]

From these, we obtain the following commutation relations:

\[
\begin{align*}
\Phi_1(u)\Phi_1(v) &= \frac{[v - u + 1]}{[u - v + 1]} \Phi_1(v)\Phi_1(u), \\
A(v)\Phi_1(u) &= \frac{[v - u + 1]}{[v - u - 1]} \Phi_1(u)A(v), \\
[u - v + 1] A(u)A(v) &= [u - v - 1] A(v)A(u).
\end{align*}
\]

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