Toda equations associated with loop groups of complex classical Lie groups

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Abstract
A Toda equation is specified by a choice of a Lie group and a \( \mathbb{Z} \)-gradation of its Lie algebra. The Toda equations associated with loop groups of complex classical Lie groups, whose Lie algebras are endowed with integrable \( \mathbb{Z} \)-gradations with finite dimensional grading subspaces, are described in an explicit form.

1 Introduction
The Toda equations constitute a wide class of nonlinear integrable equations arising in many mathematical and physical problems having as fundamental as application significance. Recall that according to the general scheme, a concrete Toda equation arises when one chooses a Lie group and specifies a \( \mathbb{Z} \)-gradation of its Lie algebra, see, for example, the books [1, 2]. Hence, to describe one or another class of Toda equations one should start with some class of \( \mathbb{Z} \)-gradations of Lie algebras. The Toda equations associated with the complex classical Lie groups were classified and described in an explicit form in the papers [3, 4, 5]. In the present paper we explicitly describe a wide class of Toda equations associated with loop groups\(^1\) of classical Lie groups.

There are two main definitions of the loop Lie algebras. In accordance with the first definition a loop Lie algebra is the set of finite Laurent polynomials with coefficients in a finite dimensional Lie algebra, see, for example, the book [6]. The main disadvantage of this definition is that it is impossible to associate a Lie group with such a Lie algebra. However, using the results on \( \mathbb{Z} \)-gradations of the affine Kac–Moody algebras [6], in the case when the underlying Lie algebra is complex and simple, one can classify all \( \mathbb{Z} \)-gradations of the loop Lie algebras with finite dimensional grading subspaces.

In accordance with the second definition a loop Lie algebra is the set of smooth mappings from the circle \( S^1 \) to a finite dimensional Lie algebra \( \mathfrak{g} \), see, for example, the book [7]. This definition is more convenient for applications to the theory of integrable systems, because in this case we always have an appropriate Lie group formed

\(^1\)In the present paper by a loop group we mean either a usual loop group or a twisted loop group. Similarly, by a loop Lie algebra we mean either a usual loop Lie algebra or a twisted loop Lie algebra.
by smooth mappings from the circle $S^1$ to a Lie group $G$ whose Lie algebra is $\mathfrak{g}$. Therefore, it would be interesting and useful to obtain a classification of $\mathbb{Z}$-gradations for loop Lie algebras defined as in the book [7]. This problem was partially solved in the paper [8]. In that paper a concept of an integrable $\mathbb{Z}$-gradation was introduced and all integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces of loop Lie algebras of finite dimensional complex simple Lie algebras were classified. Any loop Lie algebra defined in accordance with the book [6] is a subalgebra of the corresponding loop Lie algebra defined in accordance with the book [7], and in the case when the underlying Lie algebra is complex and simple any $\mathbb{Z}$-gradation with finite dimensional grading subspaces of the former is the restriction of an integrable $\mathbb{Z}$-gradation with finite dimensional grading subspaces of the latter.

In the present paper we use the results of the paper [8] to describe in an explicit form the Toda equations associated with loop groups of complex classical Lie groups based on integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces. Here we use the convenient block matrix representations for the elements of the corresponding classical Lie algebras. This representation appeared to be effective to analyze the Toda equations associated with finite dimensional Lie groups [3, 4, 5, 9, 10].

There is a large number of works devoted to Toda equations associated with loop groups. From our point of view, an incomplete list of references which are the most relevant to the contents of the present investigation consists of the papers [11, 12, 13, 14, 15, 16, 17, 18].

There exist so-called higher grading [19, 20, 21, 22] and multi-dimensional [23, 24] generalizations of the Toda equations. The approach of the present paper can be generalized and used for classification of those equations.

Concluding this introductory section let us describe the notation used in the paper. We denote by $I_n$ the unit diagonal $n \times n$ matrix and by $J_n$ the unit skew diagonal $n \times n$ matrix. For an even $n$ we also define

$$K_n = \begin{pmatrix} 0 & J_{n/2} \\ -J_{n/2} & 0 \end{pmatrix}.$$ 

When it does not lead to a misunderstanding, we write instead of $I_n$, $J_n$, and $K_n$ just $I$, $J$, and $K$ respectively.

If $m$ is an $n_1 \times n_2$ matrix, $A$ is an $n_2 \times n_2$ nonsingular matrix, $B$ is an $n_1 \times n_1$ matrix, we denote

$$A^B m = A^{-1} t_m B,$$

where $t_m$ is the transpose of the matrix $m$. We also write $A^m$ instead of $A^A m$. Note that $t_m$ is actually the transpose of $m$ with respect to the skew diagonal.

It is useful to have in mind that $A(t_m) = B^{-1} t_m B$, where $B = t(A^{-1}) A$. In particular, one can show that $t(t_m) = m$ and $k(k_m) = m$.

### 2 Toda equations associated with loop groups

#### 2.1 Loop Lie algebras and loop groups

We start this section with an explanation of what we mean by loop Lie algebras and loop groups. Here we mainly follow the book [7] and the papers [25, 26, 8].
Let $\mathfrak{g}$ be a real or complex Lie algebra. Define the loop Lie algebra $\mathcal{L}(\mathfrak{g})$ as the linear space $C^\infty(S^1, \mathfrak{g})$ of smooth mappings from the circle $S^1$ to $\mathfrak{g}$ with the Lie algebra operation defined pointwise. We think of the circle $S^1$ as consisting of complex numbers of modulus one. Since $\mathcal{L}(\mathfrak{g})$ is infinite dimensional, one has to deal with infinite sums of its elements. Therefore, it is necessary to have some topology on $\mathcal{L}(\mathfrak{g})$ consistent with the Lie algebra operation. We assume that the required topology is introduced by supplying $\mathcal{L}(\mathfrak{g})$ in an appropriate way with the structure of a Fréchet space, see, for example, [25, 26, 8].

Now, let $G$ be a Lie group with the Lie algebra $\mathfrak{g}$. The loop group $\mathcal{L}(G)$ is the set $C^\infty(S^1, G)$ with the group law defined pointwise. We endow $\mathcal{L}(G)$ with the structure of a Fréchet manifold modeled on $\mathcal{L}(\mathfrak{g})$ in such a way that it becomes a Lie group, see, for example, [25, 26, 8]. Here the Lie algebra of the Lie group $\mathcal{L}(G)$ is naturally identified with the loop Lie algebra $\mathcal{L}(\mathfrak{g})$.

Let $A$ be an automorphism of a Lie algebra $\mathfrak{g}$ satisfying the relation $A^M = \text{id}_\mathfrak{g}$ for some positive integer $M$. The twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is a subalgebra of the loop Lie algebra $\mathcal{L}(\mathfrak{g})$ formed by the elements $\xi$ which satisfy the equality
\[
\xi(\varepsilon_M p) = A(\xi(p)),
\]
where $\varepsilon_M = \exp(2\pi i/M)$ is the $M$th principal root of unity. Similarly, given an automorphism $a$ of a Lie group $G$ which satisfies the relation $a^M = \text{id}_G$, we define the twisted loop group $\mathcal{L}_{a,M}(G)$ as the subgroup of the loop group $\mathcal{L}(G)$ formed by the elements $\gamma$ satisfying the equality
\[
\gamma(\varepsilon_M p) = a(\gamma(p)).
\]

The Lie algebra of a twisted loop group $\mathcal{L}_{a,M}(G)$ is naturally identified with the twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$, where we denote the automorphism of the Lie algebra $\mathfrak{g}$ corresponding to the automorphism $a$ of the Lie group $G$ by $A$.

It is clear that a loop Lie algebra $\mathcal{L}(\mathfrak{g})$ can be treated as a twisted loop Lie algebra $\mathcal{L}_{\text{id}_\mathfrak{g},M}(\mathfrak{g})$, where $M$ is an arbitrary positive integer. In its turn, a loop group $\mathcal{L}(G)$ can be treated as a twisted loop group $\mathcal{L}_{\text{id}_G,M}(G)$, where $M$ is again an arbitrary positive integer. In the present paper by loop Lie algebras and loop groups we mean twisted loop Lie algebras and twisted loop groups.

To construct a Toda equation associated with a Lie group one should first endow its Lie algebra with a $\mathbb{Z}$-gradation. We say that the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is endowed with a $\mathbb{Z}$-gradation if for any $k \in \mathbb{Z}$ there is given a closed subspace $\mathcal{L}_{A,M}(\mathfrak{g})_k$ of $\mathcal{L}_{A,M}(\mathfrak{g})$ such that

(a) for any $k, l \in \mathbb{Z}$ one has $[\mathcal{L}_{A,M}(\mathfrak{g})_k, \mathcal{L}_{A,M}(\mathfrak{g})_l] \subset \mathcal{L}_{A,M}(\mathfrak{g})_{k+l}$,

(b) any element $\xi$ of $\mathcal{L}_{A,M}(\mathfrak{g})$ can be uniquely represented as an absolutely convergent series
\[
\xi = \sum_{k \in \mathbb{Z}} \xi_k,
\]
where $\xi_k \in \mathcal{L}_{A,M}(\mathfrak{g})_k$. The subspaces $\mathcal{L}_{A,M}(\mathfrak{g})_k$ are called the grading subspaces of $\mathcal{L}_{A,M}(\mathfrak{g})$ and the elements $\xi_k$ the grading components of $\xi$. Note that $\mathcal{L}_{A,M}(\mathfrak{g})_0$ is a subalgebra of $\mathcal{L}_{A,M}(\mathfrak{g})$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and $F$ be an isomorphism from a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ to a loop Lie algebra $\mathcal{L}_{B,N}(\mathfrak{h})$. Taking the subspaces $\mathcal{L}_{B,N}(\mathfrak{h})_k = F(\mathcal{L}_{A,M}(\mathfrak{g})_k)$ as grading subspaces we endow $\mathcal{L}_{B,N}(\mathfrak{h})$ with a $\mathbb{Z}$-gradation.
In general, if the grading subspaces of two loop Lie algebras \( \mathcal{L}_{A,M}(\mathfrak{g}) \) and \( \mathcal{L}_{B,N}(\mathfrak{h}) \) are connected by the relation \( \mathcal{L}_{B,N}(\mathfrak{h})_k = F(\mathcal{L}_{A,M}(\mathfrak{g})_k) \), where \( F \) is an isomorphism from \( \mathcal{L}_{A,M}(\mathfrak{g}) \) to \( \mathcal{L}_{B,N}(\mathfrak{h}) \), we say that the corresponding \( \mathbb{Z} \)-gradations are \textit{conjugated} by the isomorphism \( F \). It is clear that if the grading components of an element \( \xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \) are \( \xi_k \), then the grading components of the element \( F(\xi) \in \mathcal{L}_{B,N}(\mathfrak{h}) \) are \( F(\xi_k) \).

### 2.2 Toda equations

Let \( \mathcal{M} \) be either the real manifold \( \mathbb{R}^2 \) or the complex manifold \( \mathbb{C} \). Denote the standard coordinates on \( \mathbb{R}^2 \) by \( z^- \) and \( z^+ \). In the case of the manifold \( \mathbb{C} \) we denote by \( z^- \) the standard complex coordinate \( z \) and by \( z^+ \) its complex conjugate \( \bar{z} \). We use the usual notation

\[
\partial_- = \partial / \partial z^-, \quad \partial_+ = \partial / \partial z^+.
\]

Let \( G \) be a complex simple Lie group. The Lie algebra \( \mathfrak{g} \) of \( G \) is certainly simple. Actually without any loss of generality one can assume that \( G \) is a matrix Lie group.

Consider a loop group \( \mathcal{L}_{a,M}(G) \) and the corresponding loop Lie algebra \( \mathcal{L}_{A,M}(\mathfrak{g}) \). Suppose that \( \mathcal{L}_{A,M}(\mathfrak{g}) \) is endowed with a \( \mathbb{Z} \)-gradation. Let for some positive integer \( L \) the subspaces \( \mathcal{L}_{A,M}(\mathfrak{g})_{-k} \) and \( \mathcal{L}_{A,M}(\mathfrak{g})_{+k} \) for \( 0 < k < L \) be trivial. Denote by \( \mathcal{L}_{a,M}(G)_0 \) the connected Lie subgroup of \( \mathcal{L}_{a,M}(G) \) corresponding to the subalgebra \( \mathcal{L}_{A,M}(\mathfrak{g})_0 \).

The \textit{Toda equation} associated with the loop group \( \mathcal{L}_{a,M}(G) \) is an equation for a smooth mapping \( \Xi \) from \( \mathcal{M} \) to \( \mathcal{L}_{a,M}(G)_0 \) which has the following form

\[
\partial_+(\Xi^{-1}\partial_-\Xi) = [\mathcal{F}_-, \Xi^{-1}\mathcal{F}_+\Xi],
\]

(1)

see, for example, the books [1] [2]. Here \( \mathcal{F}_- \) and \( \mathcal{F}_+ \) are some fixed smooth mappings from \( \mathcal{M} \) to \( \mathcal{L}_{A,M}(\mathfrak{g})_{-L} \) and \( \mathcal{L}_{A,M}(\mathfrak{g})_{+L} \), respectively, satisfying the conditions

\[
\partial_+\mathcal{F}_- = 0, \quad \partial_-\mathcal{F}_+ = 0.
\]

(2)

When the Lie group \( \mathcal{L}_{a,M}(G)_0 \) is Abelian, the corresponding Toda equation is said to be \textit{Abelian}, otherwise we deal with a \textit{non-Abelian Toda equation}.

The authors of the paper [15] consider equations of the form (1) for the case when \( L \) does not satisfy the condition that the subspaces \( \mathcal{L}_{A,M}(\mathfrak{g})_{-k} \) and \( \mathcal{L}_{A,M}(\mathfrak{g})_{+k} \) for \( 0 < k < L \) are trivial. Actually it is convenient to consider such equations as a reduction of higher grading Toda equations [19, 20, 21].

It is clear that to classify Toda equations associated with the loop groups \( \mathcal{L}_{a,M}(G) \) one has to classify \( \mathbb{Z} \)-gradations of the loop Lie algebras \( \mathcal{L}_{A,M}(\mathfrak{g}) \). Here, if we endow the Lie algebras of two loop groups connected by an isomorphism by conjugated \( \mathbb{Z} \)-gradations, we will obtain equivalent Toda equations.

In the present paper we restrict ourselves to the case of integrable \( \mathbb{Z} \)-gradations with finite dimensional grading subspaces. In accordance with the definition given in the paper [8] a \( \mathbb{Z} \)-gradation of \( \mathcal{L}_{A,M}(\mathfrak{g}) \) is \textit{integrable} if the mapping

\[
(\tau, \xi) \in \mathbb{R} \times \mathcal{L}_{A,M}(\mathfrak{g}) \mapsto \sum_{k \in \mathbb{Z}} e^{-i\tau k} \xi_k \in \mathcal{L}_{A,M}(\mathfrak{g})
\]

\[\text{2Actually one can assume that } \mathcal{M} \text{ is an arbitrary two-dimensional real manifold, or one-dimensional complex manifold, see, for example, the paper [27] and the book [2]. Here } z^- \text{ and } z^+ \text{ are some local coordinates.}\]
is smooth. As usually, we denote by $\xi_k$ the grading components of the element $\xi$ with respect to the $\mathbb{Z}$-gradation under consideration.

An important example of an integrable $\mathbb{Z}$-gradation is the standard gradation of a loop Lie algebra $\mathcal{L}_{A,M}(g)$. Here the grading subspaces are

$$\mathcal{L}_{A,M}(g)_k = \{\xi = \lambda^k x \in \mathcal{L}_{A,M}(g) \mid x \in g, A(x) = \epsilon^k_M x\},$$

where by $\lambda$ we denote the restriction of the standard coordinate on $\mathbb{C}$ to $S^1$.

Suppose that the loop Lie algebra $\mathcal{L}_{A,M}(g)$ is endowed with an integrable $\mathbb{Z}$-gradation with finite dimensional grading subspaces. As was proved in the paper [8], this $\mathbb{Z}$-gradation is conjugated by an isomorphism to the standard $\mathbb{Z}$-gradation of some other loop Lie algebra $\mathbb{L}_{B,N}(g)$. Here the automorphisms $A$ and $B$ differ by an inner automorphism of $g$. For the case of Toda equations the automorphism $A$ of the Lie algebra $g$ should be generated by an automorphism $a$ of the Lie group $G$. It is clear that in this case the automorphism $B$ can be lifted up to an automorphism $b$ of $G$. Moreover, one can show that the isomorphism that conjugates the $\mathbb{Z}$-gradation of the loop Lie algebra $\mathcal{L}_{A,M}(g)$ with the standard $\mathbb{Z}$-gradation of the loop Lie algebra $\mathcal{L}_{B,N}(g)$ can be lifted to an isomorphism connecting the loop groups $\mathcal{L}_{A,M}(G)$ and $\mathcal{L}_{B,N}(G)$. Thus, the corresponding Toda equations associated with the loop groups $\mathcal{L}_{A,M}(G)$ and $\mathcal{L}_{B,N}(G)$ are equivalent. Therefore, to describe the Toda equations belonging to the class under consideration it suffices to classify the Lie groups $\mathcal{L}_{A,M}(G)$. This problem is evidently equivalent to classification of finite order automorphisms of $G$.

Let $c$ be an arbitrary automorphism of $G$. Note that the mapping $\gamma \mapsto c \circ \gamma$ is an isomorphism from a loop group $\mathcal{L}_{A,M}(G)$ to the loop group $\mathcal{L}_{B,M}(G)$, where $b = c \circ a \circ c^{-1}$. This isomorphism generates the isomorphism from the loop Lie algebra $\mathcal{L}_{A,M}(g)$ to the loop Lie algebra $\mathcal{L}_{B,M}(g)$, where $B$ is the automorphism of $g$ corresponding to the automorphism $b$ of $G$. It is clear that the standard gradations of $\mathcal{L}_{A,M}(g)$ and $\mathcal{L}_{B,M}(g)$ are conjugated by the isomorphism under consideration, and the corresponding Toda equations are equivalent. Therefore, to classify Toda equations it suffices to classify finite order automorphisms of $G$ up to conjugation.

Note that any automorphism $A$ of $g$ satisfying the relation $A^M = id_g$ determines a $\mathbb{Z}^M$-gradation of $g$. Indeed, since the automorphism $A$ is of finite order, it is semisimple. The eigenvalues of $A$ are of the form $\epsilon^k_M$, where $0 \leq k \leq M - 1$. Denote by $[k]_M$ the element of the ring $\mathbb{Z}_M$ corresponding to the integer $k$, and define the subspaces $g_{[k]_M}$ of $g$ as

$$g_{[k]_M} = \{x \in g \mid A(x) = \epsilon^k_M x\}.$$

It is evident that

$$g = \bigoplus_{k=0}^{M-1} g_{[k]_M} = \bigoplus_{s \in \mathbb{Z}_M} g_s$$

and this decomposition is a $\mathbb{Z}^M$-gradation of $g$. Vice versa, any $\mathbb{Z}^M$-gradation of $g$ determines in an evident way an automorphism $A$ of $g$ which satisfies the relation $A^M = id_g$. If $A$ is an inner automorphism of $g$ we say that the corresponding $\mathbb{Z}^M$-gradation is of inner type, otherwise we say that it is of outer type. In terms of the $\mathbb{Z}^M$-gradation corresponding to the automorphism $A$, the grading subspaces of the standard $\mathbb{Z}$-gradation of the loop Lie algebra $\mathcal{L}_{A,M}(g)$ can be described as

$$\mathcal{L}_{A,M}(g)_k = \{\xi \in \mathcal{L}_{A,M}(g) \mid \xi = \lambda^k x, \ x \in g_{[k]_M}\}.$$
Above we denoted by $L$ a positive integer such that the grading subspaces with the grading index $k$ satisfying the condition $0 < |k| < L$ are trivial. It is not difficult to understand that for the standard $\mathbb{Z}$-gradation $L \leq M$. The case $L = M$ arises if and only if $A = \text{id}_g$ and $M$ is an arbitrary positive integer. In this case the nontrivial grading subspaces are $\mathcal{L}_{\text{id}_g, M}(g)_{kM}$, $k \in \mathbb{Z}$, and it is clear that

$$\mathcal{L}_{\text{id}_g, M}(g)_{kM} = \{ \xi \in \mathcal{L}_{\text{id}_g, M}(g) \mid \xi = \lambda^{kM}x, \ x \in g \}.$$ 

It is clear that for the standard $\mathbb{Z}$-gradation the subalgebra $\mathcal{L}_{A, M}(g)_0$ is isomorphic to $g|_{[0]_M}$ and the Lie group $\mathcal{L}_{a, M}(G)_0$ is isomorphic to the connected Lie subgroup $G_0$ of $G$ corresponding to the Lie algebra $g|_{[0]_M}$. Hence, the mapping $\mathcal{Z}$ is actually a mapping from $M$ to $G_0$, for consistency with the notation used earlier we will denote it by $\gamma$. The mappings $\mathcal{F}_-$ and $\mathcal{F}_+$ are given by the relation

$$\mathcal{F}_- (p) = \lambda^{-L}c_- (p), \quad \mathcal{F}_+ (p) = \lambda^L c_+ (p), \quad p \in M,$$

where $c_-$ and $c_+$ are mappings from $M$ to $g_{-[L]_M}$ and $g_{+[L]_M}$ respectively. Thus, the Toda equation (1) can be written as

$$\partial_+ (\gamma^{-1} \partial_- \gamma) = [c_-, \gamma^{-1}c_+ \gamma], \quad (3)$$

where $\gamma$ is a smooth mapping from $M$ to the connected Lie subgroup $G_0$ of $G$ corresponding to the Lie algebra $g|_{[0]_M}$, and the mappings $c_-$ and $c_+$ are fixed smooth mappings from $M$ to $g_{-[L]_M}$ and $g_{+[L]_M}$ respectively. The conditions (2) imply that

$$\partial_+ c_- = 0, \quad \partial_- c_+ = 0. \quad (4)$$

Summarizing one can say that a Toda equation associated with a loop group of a simple complex Lie group whose Lie algebra is endowed with an integrable $\mathbb{Z}$-gradation with finite dimensional grading subspaces is equivalent to the equation of the form (3).

The simplest case is when $a = \text{id}_G$, $A = \text{id}_g$ and $M$ is an arbitrary positive number. Remind that in this case $L = M$. The mapping $\gamma$ is a mapping from $M$ to $G$, and the mappings $c_+$ and $c_-$ are mappings from $M$ to $g$. For consistency with the notation used below we reddenote $\gamma$ by $\Gamma$, $c_+$ by $C_+$, and $c_-$ by $C_-$. Here the Toda equation (3) becomes

$$\partial_+ (\Gamma^{-1} \partial_- \Gamma) = [C_-, \Gamma^{-1}C_+ \Gamma], \quad (5)$$

and the conditions (4) read

$$\partial_+ C_- = 0, \quad \partial_- C_+ = 0. \quad (6)$$

It is natural to consider an equation of the type (3) in a more general setting. Namely, let $G$ be an arbitrary finite dimensional Lie group and $a$ be an arbitrary finite order automorphism of $G$. The corresponding automorphism $A$ of the Lie algebra $g$ of the Lie group $G$ generates a $\mathbb{Z}_M$-gradation of $g$. Assume that for some positive integer $L \leq M$ the grading subspaces $g_{+[k]_M}$ and $g_{-[k]_M}$ for $0 < k < L$ are trivial. Choose some fixed mappings $c_+$ and $c_-$ from $M$ to $g_{+[L]_M}$ and $g_{-[L]_M}$ respectively, satisfying the relations (4). Now Eq. (3) is equivalent to the Toda equation associated with the loop group $\mathcal{L}_{a, M}(G)$ whose Lie algebra $\mathcal{L}_{A, M}(g)$ is endowed with the standard $\mathbb{Z}$-gradation.

Note that the authors of the paper [17] also suggest Eq. (3) as a convenient form of a Toda equation associated with a loop group. They also explicitly show how to
go back from Eq. (3) to a Toda equation associated with a loop group. Repeat that we proved that in the case when $G$ is a simple complex Lie group such a procedure gives all nonequivalent Toda equations for $\mathcal{L}_{a,M}(G)$ based on integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces.

Below we describe a concrete form which Eq. (3) takes in the case when $G$ is a complex classical Lie group. To this end we classify up to conjugation the finite order automorphisms of the corresponding Lie algebras. Such a classification was performed earlier using root technique, see, for example, [6, 28]. Although this approach gives an answer, it appears to be inconvenient for description of the structure of grading subspaces that is important for us. Therefore, we develop and use another classification based on the appropriate block matrix representations of the Lie algebras under consideration.

3 Toda equations associated with loop groups of complex general linear groups. Gradations of inner type

Let $a$ be an inner automorphism of the Lie group $\mathrm{GL}_n(\mathbb{C})$ satisfying the relation $a^M = \mathrm{id}_{\mathrm{GL}_n(\mathbb{C})}$. Denote the corresponding inner automorphism of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ by $A$. This automorphism satisfies the relation $A^M = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$. In other words, $A$ is a finite order automorphism of $\mathfrak{gl}_n(\mathbb{C})$. Since we are interested in the automorphisms of $\mathfrak{gl}_n(\mathbb{C})$ up to conjugations, we assume that the automorphism $A$ under consideration is given by the relation

$$A(x) = h x h^{-1},$$

where $h$ is an element of the subgroup $D_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ formed by all complex non-singular diagonal matrices, see Appendix C.1. It is clear that multiplying $h$ by an arbitrary nonzero complex number we obtain an element of $D_n(\mathbb{C})$ which generates the same automorphism of $\mathfrak{gl}_n(\mathbb{C})$ as the initial element.

The equality $A^M = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$ gives $h^M x h^{-M} = x$ for any $x \in \mathfrak{gl}_n(\mathbb{C})$. Therefore, $h^M = \nu I$, where $\nu$ is a nonzero complex number.

Using inner automorphisms of $\mathfrak{gl}_n(\mathbb{C})$ which permute the rows and columns of the matrix $h$ synchronously, we collect coinciding diagonal matrix elements together, and come to the following block diagonal form of the element $h$:

$$h = \begin{pmatrix} \mu_1 I_{n_1} & & \\ & \mu_2 I_{n_2} & \\ & & \ddots \\ & & & \mu_p I_{n_p} \end{pmatrix}. \tag{7}$$

Here $n_\alpha, \alpha = 1, \ldots, p$, are positive integers, such that $\sum_{\alpha=1}^p n_\alpha = n$, and $\mu_\alpha$ are complex numbers, such that $\mu_\alpha^M = \nu$ for each $\alpha = 1, \ldots, p$. We assume that $p > 1$. The case $p = 1$ corresponds to $A = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$. Here we come to Eq. (5), where $\Gamma$ is a mapping from $\mathcal{M}$ to $\mathrm{GL}_n(\mathbb{C})$, $C_+$ and $C_-$ are mappings from $\mathcal{M}$ to $\mathfrak{gl}_n(\mathbb{C})$ satisfying the conditions (6).

Let $\rho$ be any complex number such that $\rho^M = \nu$. The number $\rho$ is defined up to multiplication by an $M$th root of unity. Represent the numbers $\mu_\alpha$ in the form

$$\mu_\alpha = \rho \epsilon_{\mathcal{M}}^{n_\alpha}, \tag{8}$$

where $\epsilon_{\mathcal{M}}$ is a $M$th root of unity.
where \( m_\alpha \) are integers satisfying the condition \( 0 < m_\alpha \leq M \). It is clear that \( p \leq M \). Without loss of generality, we assume that \( m_1 > m_2 > \ldots > m_p \). This can be provided by an appropriate inner automorphism of \( \mathfrak{gl}_n(\mathbb{C}) \). Thus, we assume that \( h \) has the form

\[
h = \rho \begin{pmatrix} e_M^{m_1} I_{n_1} & & & \\
& e_M^{m_2} I_{n_2} & & \\
& & \ddots & \\
& & & e_M^{m_p} I_{n_p}
\end{pmatrix},
\]

where \( M \geq m_1 > m_2 > \ldots > m_p > 0 \).

Now consider the corresponding \( \mathbb{Z} \)-gradation. Represent a general element \( x \) of \( \mathfrak{gl}_n(\mathbb{C}) \) in the block matrix form

\[
x = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\
x_{21} & \cdots & x_{2p} \\
\vdots & \ddots & \vdots \\
x_{p1} & \cdots & x_{pp}
\end{pmatrix},
\]

where \( x_{\alpha \beta} \) is an \( n_\alpha \times n_\beta \) matrix. It is clear that

\[
(h x h^{-1})_{\alpha \beta} = \mu_{\alpha} \mu_{\beta}^{-1} x_{\alpha \beta} = e_{M}^{m_{\alpha} - m_{\beta}} x_{\alpha \beta}.
\]

Hence, if for fixed \( \alpha \) and \( \beta \) only the block \( x_{\alpha \beta} \) of the element \( x \) is different from zero, then \( x \) belongs to the grading subspace \([m_\alpha - m_\beta]_M\). It is convenient to introduce integers \( k_\alpha, \alpha = 1, \ldots, p-1 \), defined as \( k_\alpha = m_\alpha - m_{\alpha+1} \). By definition, for each \( \alpha \) the integer \( k_\alpha \) is positive and \( \sum_{\alpha=1}^{\beta-1} k_\alpha = m_1 - m_p < M \). It is clear that for \( \alpha < \beta \) one has

\[
[m_\alpha - m_\beta]_M = [\sum_{\gamma=\alpha}^{\beta-1} k_\gamma]_M,
\]

and for \( \alpha > \beta \)

\[
[m_\alpha - m_\beta]_M = -[m_\beta - m_\alpha]_M = -[\sum_{\gamma=\beta}^{\alpha-1} k_\gamma]_M = -[M - \sum_{\gamma=\beta}^{\alpha-1} k_\gamma]_M.
\]

Now one can easily understand that the grading structure of the \( \mathbb{Z}_M \)-gradation generated by the automorphism \( A \) under consideration can be depicted by the scheme given in Figure 1. Here the elements of the ring \( \mathbb{Z}_M \) are the grading indices of the corresponding blocks in the block matrix representation (10) of a general element of \( \mathfrak{gl}_n(\mathbb{C}) \). Note, in particular, that the subalgebra \( \mathfrak{g}_{[0]} \) is formed by all block diagonal matrices and is isomorphic to the Lie algebra \( \mathfrak{gl}_{n_1}(\mathbb{C}) \times \cdots \times \mathfrak{gl}_{n_p}(\mathbb{C}) \). The group \( G_0 \) is also formed by block diagonal matrices and is isomorphic to \( \text{GL}_{n_1}(\mathbb{C}) \times \cdots \times \text{GL}_{n_p}(\mathbb{C}) \).

Assume now that there are given \( p \) positive integers \( n_\alpha \) such that \( \sum_{\alpha=1}^{p} n_\alpha = n \) and \( p - 1 \) positive integers \( k_\alpha \) such that \( \sum_{\alpha=1}^{p-1} k_\alpha < M \). Let \( m_p \) be a positive integer such that \( \sum_{\alpha=1}^{p-1} k_\alpha + m_p \leq M \). Using the relation

\[
m_\alpha = \sum_{\beta=\alpha}^{p-1} k_\beta + m_p, \quad \alpha = 1, \ldots, p-1,
\]
one defines the remaining $p - 1$ numbers $m_\alpha$. Now using (8) with an arbitrary nonzero $\rho$, we obtain the numbers $\mu_\alpha$ and construct an element $h \in \text{GL}_n(\mathbb{C})$ in accordance with (7). This element determines an inner automorphism of $gl_n(\mathbb{C})$ of order $M$ which in its turn generates a $\mathbb{Z}_M$-gradation of $gl_n(\mathbb{C})$ corresponding to our choice of numbers $n_\alpha$ and $k_\alpha$. Note that using the arbitrariness of the number $\rho$ one can make $h$ meet some additional requirements. In particular, one can always choose $h$ so that $\text{det} h = 1$.

Thus, up to a conjugation, a $\mathbb{Z}_M$-gradation of $gl_n(\mathbb{C})$ of inner type can be specified by a choice of $p \leq n$ positive integers $n_\alpha$, satisfying the equality $\sum_{\alpha=1}^{p} n_\alpha = n$, and $p - 1$ positive integers $k_\alpha$, satisfying the inequality $\sum_{\alpha=1}^{p-1} k_\alpha < M$.

Below we show that a $\mathbb{Z}_M$-gradation of an arbitrary complex classical Lie algebra is characterized by $p$ positive integers $n_\alpha$ and $p - 1$ positive integers $k_\alpha$, and up to the conjugation by an isomorphism the structure of the $\mathbb{Z}_M$-gradation is described by Figure 1 except for the case considered in Section 6. Therefore, we call this structure canonical. The only difference from the case of complex general linear groups is that for other groups the numbers $n_\alpha$, $k_\alpha$ and the blocks of the block matrix representation (10) satisfy some additional conditions.

Consider now the corresponding Toda equations. Choose some $\mathbb{Z}_M$-gradation of inner type of the Lie algebra $\mathfrak{g} = gl_n(\mathbb{C})$. Let $L$ be a positive integer such that the grading subspaces $\mathfrak{g}_{+}[k]M$ and $\mathfrak{g}_{-}[k]M$ for $0 < k < L$ are trivial. One can get convinced that if $x \in \mathfrak{g}_{+}[L]M'$ then only the blocks $x_{\alpha,\alpha+1}$, $\alpha = 1, \ldots, p - 1$, and $x_{p,1}$ in the block matrix representation (10) can be different from zero. Thus, the mapping $c_+$ has the structure given in Figure 2 where for each $\alpha = 1, \ldots, p - 1$ the mapping $C_{+\alpha}$ is a

\[
\begin{pmatrix}
0 & C_{+1} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
& 0 & \ddots & \ddots & 0 \\
C_{+p-1} & & \ddots & \ddots & 0 \\
0 & & & 0 & 0 \\
\end{pmatrix}
\]

Figure 2: The canonical structure of the mapping $c_+$ mapping from $\mathcal{M}$ to the space of $n_\alpha \times n_{\alpha+1}$ complex matrices, and $C_{+0}$ is a mapping.
from $\mathcal{M}$ to the space of $n_p \times n_1$ complex matrices. Here we assume that if some blocks among $x_{a,\alpha+1}$, $\alpha = 1, \ldots, p - 1$, and $x_{p1}$ in the general block matrix representation (10) have the grading index different from $+ [L]_{M}$, then the corresponding blocks in the block representation of $c_+$ are zero matrices.

Similarly, we see that the mapping $c_-$ has the structure given in Figure 3 where for each $\alpha = 1, \ldots, p - 1$ the mapping $C_{-\alpha}$ is a mapping from $\mathcal{M}$ to the space of $n_{\alpha+1} \times n_{\alpha}$ complex matrices, and $C_{-0}$ is a mapping from $\mathcal{M}$ to the space of $n_1 \times n_p$ complex matrices. Here we assume that if some blocks among $x_{a,\alpha+1}$, $\alpha = 1, \ldots, p - 1$, and $x_{1p}$ in the general block matrix representation (10) have the grading index different from $- [L]_{M}$, then the corresponding blocks in the block representation of $c_-$ are zero matrices. It is clear that to provide the validity of the conditions (4) the mappings $C_{-\alpha}$ and $C_{+\alpha}$, $\alpha = 0, \ldots, p - 1$, must satisfy the relations

$$\partial_+ C_{-\alpha} = 0, \quad \partial_- C_{+\alpha} = 0. \quad (11)$$

In the case of other complex classical Lie group using conjugations we always bring a $\mathbb{Z}_M$-gradation under consideration to the form described by Figure 1. Here the mappings $c_+$ and $c_-$ take the form described by Figures 2 and 3 respectively, where the mappings $C_{+\alpha}$ and $C_{-\alpha}$ satisfy the relations (11). Therefore, we call the structure of the mappings $c_+$ and $c_-$ described by Figures 2 and 3 canonical.

Figure 3: The canonical structure of the mapping $c_-$

\[ \begin{pmatrix}
0 & C_{-0} \\
C_{-1} & 0 \\
\vdots & \vdots & \ddots & \vdots \\
& & & 0 \\
& & & C_{-(p-1)} & 0
\end{pmatrix} \]

Return to the case of the complex general linear groups and parameterize the mapping $\gamma$ as

$$\gamma = \begin{pmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_p
\end{pmatrix}, \quad (12)$$

where for each $\alpha = 1, \ldots, p$ the mapping $\Gamma_\alpha$ is a mapping from $\mathcal{M}$ to the Lie group $\text{GL}_{n_\alpha}(\mathbb{C})$. In this parametrization the Toda equation (3) for the mapping $\gamma$ is equivalent to the following system of equations for the mappings $\Gamma_\alpha$:

$$\begin{align*}
\partial_+ \left( \Gamma_1^{-1} \partial_- \Gamma_1 \right) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + C_{-0} \Gamma_p^{-1} C_{+0} \Gamma_1, \\
\partial_+ \left( \Gamma_2^{-1} \partial_- \Gamma_2 \right) &= -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2, \\
\vdots \\
\partial_+ \left( \Gamma_{p-1}^{-1} \partial_- \Gamma_{p-1} \right) &= -\Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p C_{-(p-1)} + C_{-(p-2)} \Gamma_{p-2}^{-1} C_{+(p-2)} \Gamma_{p-1}, \\
\partial_+ \left( \Gamma_p^{-1} \partial_- \Gamma_p \right) &= -\Gamma_p^{-1} C_{+p} \Gamma_1 C_{-0} + C_{-(p-1)} \Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p.
\end{align*} \quad (13)$$
If for some $a$ one has $C_{+a} = 0$ or $C_{-a} = 0$, then the system of equations (13) is actually the system of equations which arises when we consider a Toda equation associated with the Lie group $GL_n(\mathbb{C})$, see, for example, the papers [4, 5]. Hence, to have equations which are really associated with a loop group of $GL_n(\mathbb{C})$ one must assume that all mappings $C_{+a}$ and $C_{-a}$ are nontrivial. This is possible only when $k_a = L$ and $M = pL$. Actually without any loss of generality one can assume that $L = 1$.

It is often convenient to suppose that the mappings $\Gamma_\alpha$, $C_{+\alpha}$ and $C_{-\alpha}$ are defined for all integer values of the index $\alpha$ and satisfy the periodicity condition

$$\Gamma_{\alpha+p} = \Gamma_\alpha, \quad C_{-(\alpha+p)} = C_{-\alpha}, \quad C_{+(\alpha+p)} = C_{+\alpha}.$$ 

Now the system (13) can be treated as the infinite periodic system

$$\partial_+ (\Gamma_\alpha^{-1}\partial_- \Gamma_\alpha) = -\Gamma_\alpha^{-1}C_{+\alpha}\Gamma_{\alpha+1}C_{-\alpha} + C_{-(\alpha+1)}\Gamma_{\alpha-1}^{-1}C_{+(\alpha-1)}\Gamma_\alpha.$$

In particular, when $n = rp$ and $n_a = r$, $C_{-\alpha} = C_{+\alpha} = I_r$ one comes to the system

$$\partial_+ (\Gamma_\alpha^{-1}\partial_- \Gamma_\alpha) = -\Gamma_\alpha^{-1}\Gamma_{\alpha+1} + \Gamma_{\alpha-1}^-\Gamma_\alpha.$$

It is not difficult to understand that the Toda equations associated with loop groups of $SL_n(\mathbb{C})$ in the case of $\mathbb{Z}_M$-gradations of inner type have the same form (13) as the Toda equations associated with loop groups of $GL_n(\mathbb{C})$. Here the mappings $\Gamma_\alpha$ should satisfy the condition $\prod_{\alpha=1}^p \det \Gamma_\alpha = 1$. Actually, if there is a solution of a Toda equation associated with a loop group of $GL_n(\mathbb{C})$ one can easily obtain a solution of the corresponding Toda equation associated with the corresponding loop group of $SL_n(\mathbb{C})$. Indeed, suppose that the mappings $\Gamma_\alpha, \alpha = 1, \ldots, p$, satisfy the system (13). Let us denote

$$\Delta = \prod_{\alpha=1}^p \det \Gamma_\alpha.$$

It is evident that

$$\partial_+ (\Delta^{-1}\partial_- \Delta) = \sum_{\alpha=1}^p \partial_+ [(\det \Gamma_\alpha)^{-1}\partial_- \det \Gamma_\alpha].$$

The well known formula for the differentiation of a determinant gives

$$\partial_+ [(\det \Gamma_\alpha)^{-1}\partial_- \det \Gamma_\alpha] = \text{tr} [\partial_+ (\Gamma_\alpha^{-1}\partial_- \Gamma_\alpha)].$$

Taking into account the system (13), we obtain

$$\partial_+ (\Delta^{-1}\partial_- \Delta) = \sum_{\alpha=1}^p \text{tr} [\partial_+ (\Gamma_\alpha^{-1}\partial_- \Gamma_\alpha)] = 0.$$ 

Now one can show that the mappings

$$\bar{\Gamma}_\alpha = \Delta^{-1/n} \Gamma_\alpha, \quad \alpha = 1, \ldots, p,$$

satisfy the condition $\prod_{\alpha=1}^p \det \bar{\Gamma}_\alpha = 1$ and the system (13). Note that every solution of a Toda equation associated with a loop group of $SL_n(\mathbb{C})$ can be obtained from a solution of the corresponding Toda equation associated with the corresponding loop group of $GL_n(\mathbb{C})$.

The Toda equations arising when we use $\mathbb{Z}_M$-gradations of $\mathfrak{gl}_n(\mathbb{C})$ of outer type, are considered in Section 6.
4 Toda equations associated with loop groups of complex orthogonal groups. Gradients of inner type

Let $a$ be an inner automorphism of the Lie group $SO_n(C)$ satisfying the relation $a^M = \text{id}_{SO_n(C)}$. The corresponding inner automorphism $A$ of the Lie algebra $so_n(C)$ satisfies the relation $A^M = \text{id}_{so_n(C)}$. In the simplest case $A = \text{id}_{so_n(C)}$ and $M$ is an arbitrary positive integer. In this case we come to Eq. (5), where $\Gamma$ is a mapping from $M$ to $SO_n(C)$, $C_+$ and $C_-$ are mappings from $\mathcal{M}$ to $so_n(C)$ satisfying the conditions (6).

In a general case without any loss of generality we can assume that the automorphism $A$ is a conjugation by an element $h$ which belongs to the Lie group $SO_n(C) \cap D_n(C)$, see Appendix C.2.

For any element $x \in so_n(C)$ one has $h^M x h^{-M} = x$, therefore,

$$h^M = \nu I$$

for some complex number \(\nu\). This equality implies that $(h^M)^M = \nu I$, therefore, $(h^M)^M h^M = \nu^2 I$. From the other hand, using the equality $hh = I$, we obtain $(h^M)^M h^M = (h^M)^M = I$. Thus, we see that $\nu^2 = 1$. In other words, either $\nu = 1$, or $\nu = -1$.

4.1 $\nu = 1$

In this case $h$ is a diagonal matrix with the diagonal matrix elements of the form $\epsilon^{m}_{M}$, where we assume that $m$ is a nonnegative integer satisfying the conditions $0 < m \leq M$. Let $m_{\alpha}$, $\alpha = 1, \ldots, p$, be the different values of $m$ taken in the decreasing order, $M \geq m_{1} > m_{2} > \ldots > m_{p} > 0$, and $n_{\alpha}$, $\alpha = 1, \ldots, p$, be their multiplicities. In other words, for each $\alpha$ the element $h$ has $n_{\alpha}$ diagonal elements equal to $\epsilon^{m_{\alpha}}_{M}$.

As follows from the equality $hh = I$, if the diagonal element of $h$ at position $i$ is $\epsilon^{m}_{M}$, then the diagonal element of $h$ at position $n - i + 1$ is equal to $\epsilon^{M-m}_{M}$. Note that if $n$ is odd, then the equality $\det h = 1$ implies that the central diagonal matrix element of $h$ is equal to 1.

Below, to bring the element $h$ into the desirable form, we use two special types of automorphisms of the Lie group $SO_n(C)$. The automorphisms of the first type are conjugations by the matrices of the following block matrix form

$$
\begin{pmatrix}
\begin{array}{ccc}
I_{i-1} & 0 & 0 \\
0 & I_{n-2i} & 0 \\
1 & 0 & 0 \\
\end{array}
\end{pmatrix},
$$

(14)

where $i \leq n/2$. The corresponding automorphism permutes the diagonal elements of a diagonal matrix at positions $i$ and $n - i + 1$. The automorphisms of the second type
are conjugations by the matrices of the form

\[
\begin{pmatrix}
I_{i-1} & 0 & 1 \\
0 & 1 & 0 \\
I_{n-2i-2} & 0 & 1 \\
1 & 0 & I_{i-1}
\end{pmatrix},
\]

(15)

where \( i \leq n/2 - 1 \). The corresponding automorphism of \( \text{SO}_n(\mathbb{C}) \) permutes the diagonal elements of a diagonal matrix at positions \( i \) and \( i + 1 \) and the diagonal elements at positions \( n - i + 1 \) and \( n - i \).

### 4.1.1 \( m_1 = M \)

Assume now that \( m_1 = M \). It is not difficult to see that \( n - n_1 \) is an even integer. Using automorphisms generated by elements of the type (14) and of the type (15), we bring the element \( h \) to the form

\[
h = \begin{pmatrix} h' & \\ I_{n_1} & (h')^{-1} \end{pmatrix},
\]

where \( h' \) is a square diagonal matrix of order \((n-n_1)/2\) which satisfies the relation \( h'^M = I_{(n-n_1)/2} \) and the condition that the diagonal matrix elements of \( h' \) have the form \( e_M^m \) where \( M > m \geq M/2 \), and \( m \) does not increase when we go along the diagonal from the upper left corner to the bottom right one.

The conjugation by the matrix

\[
\begin{pmatrix}
0 & I_{n_1} \\
I_{(n-n_1)/2} & 0 \\
I_{(n-n_1)/2} & I_{(n-n_1)/2}
\end{pmatrix}
\]

(16)

maps \( \text{so}_n(\mathbb{C}) \) isomorphically to the Lie algebra \( \mathfrak{gl}_n^B(\mathbb{C}) \) with

\[
B = \begin{pmatrix} J_{n_1} & 0 \\
0 & J_{n-n_1} \end{pmatrix}.
\]

(17)

The same conjugation maps the Lie group \( \text{O}_n(\mathbb{C}) \) isomorphically to the Lie group \( \text{GL}_n^B(\mathbb{C}) \). Here the Lie group \( \text{SO}_n(\mathbb{C}) \) is mapped isomorphically to the Lie subgroup of \( \text{GL}_n^B(\mathbb{C}) \) formed by the elements \( g \) of \( \text{GL}_n^B(\mathbb{C}) \) with \( \det g = 1 \).

Now the element \( h \) takes the form

\[
h = \begin{pmatrix} I_{n_1} & h'' \\
0 & h'' \end{pmatrix},
\]

where

\[
h'' = \begin{pmatrix} h' & (h')^{-1} \end{pmatrix}.
\]

13
Note that $J = I_{n_1}$. Using automorphisms of $GL_n^B(C)$ similar to automorphisms of $SO_n(C)$ generated by the elements of the form (14) or (15), we reduce $h$ to the form (7), where the numbers $n_\alpha$ satisfy the relations

$$n_\alpha = n_{p-\alpha+2}, \quad \alpha = 2, \ldots, p.$$  \hfill (18)

It is clear that $\sum_{\alpha=1}^p n_\alpha = n$. The numbers $\mu_\alpha$ satisfy the equalities $\mu_\alpha^M = 1, \alpha = 1, \ldots, p$, and

$$\mu_\alpha = \mu_{p-\alpha+2}, \quad \alpha = 2, \ldots, p.$$  \hfill (19)

The integers $m_\alpha$ form a decreasing sequence and satisfy the relations

$$m_\alpha = M - m_{p-\alpha+2}, \quad \alpha = 2, \ldots, p.$$  \hfill (20)

The positive integers $k_\alpha, \alpha = 1, \ldots, p-1$, defined as $k_\alpha = m_\alpha - m_{\alpha+1}$, satisfy the relations

$$k_\alpha = k_{p-\alpha+1}, \quad \alpha = 2, \ldots, p-1.$$  \hfill (21)

One, certainly, has

$$m_\alpha = \sum_{\beta=\alpha}^{p-1} k_\beta + m_p.$$  \hfill (22)

In particular, remembering that $m_1 = M$, we obtain

$$\sum_{\beta=1}^{p-1} k_\beta + k_1 = M.$$  \hfill (23)

Assume now that there are given $p$ positive integers $n_\alpha$ such that $\sum_{\alpha=1}^p n_\alpha = n$, the integer $n - n_1$ is even, and the relations (18) are satisfied. Assume also that there are given $p - 1$ positive integers $k_\alpha$ satisfying the relations (21) and the equality (23). Using the relation (22), we define $p$ positive integers $m_\alpha$. Here the relation (23) guarantees that $m_1 = M$. It follows from (21) and (23) that the integers $m_\alpha$ satisfy the relations (20). Hence, the numbers $\mu_\alpha = e_{\alpha}^{m_\alpha}$ satisfy the relations (19). The matrix $h$ defined by (7) belongs to the Lie group $GL_n^B(C)$ with $B$ defined by (17). The element $h$ satisfies the equality $h^M = I$.

Thus, a $Z_M$-gradation of $so_n(C)$ for which $\nu = 1$ and $m_1 = M$ is specified by a choice of $p$ positive integers $n_\alpha$ such that $\sum_{\alpha=1}^p n_\alpha = n$, the integer $n - n_1$ is even, and the relations (18) are satisfied, and by a choice of $p - 1$ positive integers $k_\alpha$ satisfying the relations (21) and the equality (23).

The structure of the $Z_M$-gradation generated by the automorphism $A$ under consideration is again described by Figure 1. Here an element $x$ of the Lie algebra $g_\alpha^{[0]}(C)$ is treated as a matrix having the block matrix structure (10). Now the blocks $x_{\alpha\beta}$ are not arbitrary. They satisfy the restrictions which follow from the equality $b^B x = -x$. The Lie subalgebra $g_{[0]}^M$ is formed by block diagonal matrices. It is clear that the Lie
group $G_0$ is also formed by block diagonal matrices. It is the desire to have such a simple structure of $g_{[0]}M$ and $G_0$ that is the reason to use the Lie algebra $gl_n^B(C)$ and a Lie subgroup of the Lie group $GL_n^B(C)$ with $B$ defined by the relation (17) instead of the Lie algebra $so_n(C)$ and the Lie group $SO_n(C)$.

Below, using conjugations, we always bring $g_{[0]}M$ and $G_0$ to block diagonal form. It allows one to use for the mapping $\gamma$, entering the Toda equation (3), the parametrization (12). Here, in general, the mappings $\Gamma_\alpha, \alpha = 1, \ldots, p$, are not independent. We denote the number of independent mappings $\Gamma_\alpha$ uniquely determining the mapping $\gamma$ by $s$. In the cases considered below it is always possible to choose the first $s$ mappings $\Gamma_\alpha$ as a complete set of independent mappings.

Describe now the Toda equations arising in the case where $\nu = 1$ and $m_1 = M$. Choose some $\mathbb{Z}_M$-gradation of the Lie algebra $so_n(C)$ of the type under consideration. Then map $so_n(C)$ isomorphically onto the Lie algebra $g = gl_n^B(C)$ with $B$ given by (17), and consider the corresponding conjugated $\mathbb{Z}_M$-gradation of $gl_n^B(C)$. Let $L$ be a positive integer such that the grading subspace $g_{[k]}M$ is trivial if $0 < |k| < L$. One can get convinced that if $x \in g_{+[L]}M$, then only the blocks $x_{\alpha+1,1}, \alpha = 1, \ldots, p-1$, and $x_{1p}$ in the block matrix representation (10) can be different from zero. Similarly, if $x \in g_{-[L]}M$, then only the blocks $x_{\alpha+1,\alpha}, \alpha = 1, \ldots, p-1$, and $x_{1p}$ can be different from zero. It is convenient to consider the cases of an even and odd $p$ separately.

First let $p$ be an even integer equal to $2s - 2, s \geq 2$. It follows from (23) and (21) that this is possible only if $M$ is even. Note that the integer $n_s$ is also even in this case. Having in mind the restrictions which follow from the equality $b_\gamma = -x$, we see that the mapping $c_\alpha$ may be parameterized as it is given in Figure 2, where in our case $C_{\alpha} = -t_1C_{\alpha+1}$, and $C_{\alpha} = -t_1C_{(p-\alpha+1)}, \alpha = 2, \ldots, p-1$. Here we again assume that if some blocks among $x_{\alpha, \alpha+1}, \alpha = 1, \ldots, p-1$, and $x_{1p}$ in the block matrix representation (10) have the grading index different from $[L]M$, then the corresponding blocks in the block representation of $c_\alpha$ are zero matrices. A similar convention is used for the block representation of the mapping $c_\alpha$, which we choose in the form given in Figure (3) where in our case $C_{-0} = -t_1C_{-1}$, and $C_{-\alpha} = -t_1C_{-(p-\alpha+1)}, \alpha = 2, \ldots, p-1$.

Parameterize the mapping $\gamma$ as is described by the equality (12). Here $\Gamma_\alpha = (l_\Gamma_{\alpha+1})^{-1}$ and $\Gamma_\alpha = (l_\Gamma_{p-\alpha+2})^{-1}$ for each $\alpha = 2, \ldots, p$. It follows from the last relation that $\Gamma_s = (l_\Gamma_s)^{-1}$. Therefore, among the mappings $\Gamma_\alpha$ there are only $s$ independent ones, and the Lie group $G_0$ is isomorphic to the Lie group $SO_{n_1}(C) \times GL_{n_2}(C) \times \cdots \times GL_{n_{s-1}}(C) \times SO_{n_s}(C)$. Recall that in the case under consideration $n_s$ is even. One can get convinced that the Toda equation (3) is equivalent to the following system of equations

\[
\begin{align*}
\partial_+ (\Gamma_1^{-1} \partial_+ \Gamma_1) &= -\Gamma_1^{-1}C_{+1} \Gamma_2 C_{-1} + l(\Gamma_1^{-1}C_{+1} \Gamma_2 C_{-1}), \\
\partial_+ (\Gamma_2^{-1} \partial_+ \Gamma_2) &= -\Gamma_2^{-1}C_{+1} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1}C_{+1} \Gamma_2, \\
\vdots \\
\partial_+ (\Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1}) &= -\Gamma_{s-1}^{-1}C_{+(s-1)} \Gamma_s C_{-(s-1)} + C_{-(s-2)} \Gamma_{s-2}^{-1}C_{+(s-2)} \Gamma_{s-1}, \\
\partial_+ (\Gamma_s^{-1} \partial_+ \Gamma_s) &= -l(C_{-(s-1)} \Gamma_{s-1}^{-1}C_{+(s-1)} \Gamma_s) + C_{-(s-1)} \Gamma_{s-1}^{-1}C_{+(s-1)} \Gamma_s.
\end{align*}
\]

Let now $p$ be an odd integer equal to $2s - 1, s \geq 2$. Parameterize the mapping $c_\alpha$ as is given in Figure 2, where $C_{\alpha} = -t_1C_{\alpha+1}$, and $C_{\alpha} = -t_1C_{(p-\alpha+1)}, \alpha = 2, \ldots, p-1$. For the mapping $c_\alpha$ we use the representation given in Figure 3, where $C_{-0} = -t_1C_{-1}$,
and \( C_{-\alpha} = -C_{-(p-\alpha+1)}, \alpha = 2, \ldots, p-1 \).

Parameterize again the mapping \( \gamma \) as is described by the equality (12). Here \( \Gamma_1 = (I \Gamma_1)^{-1} \) and \( \Gamma_\alpha = (I \Gamma_{p-\alpha+2})^{-1} \) for each \( \alpha = 2, \ldots, p \). It is clear that there are only \( s \) independent mappings \( \Gamma_\alpha \) and the Lie group \( G_0 \) is isomorphic to the Lie group \( SO_n(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \times \cdots \times GL_{n_k}(\mathbb{C}) \). Now we see that in the case under consideration the following system of equations

\[
\begin{align*}
\partial_+ \left( \Gamma_1^{-1} \partial_+ \Gamma_1 \right) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + l(\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}), \\
\partial_+ \left( \Gamma_2^{-1} \partial_+ \Gamma_2 \right) &= -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2, \\
&\vdots \\
\partial_+ \left( \Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1} \right) &= -\Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s C_{-(s-1)} + C_{-(s-2)} \Gamma_{s-2}^{-1} C_{+(s-2)} \Gamma_{s-1}, \\
\partial_+ \left( \Gamma_s^{-1} \partial_+ \Gamma_s \right) &= -\Gamma_s^{-1} C_{+s} l(\Gamma_1^{-1} C_{+s}) C_{s-1} + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} C_{s-1},
\end{align*}
\]

is equivalent to the Toda equation (3). Note that \( C_{+s} = -lC_{+s} \) and \( C_{-s} = -lC_{-s} \).

As well as for the case of Toda equations associated with loop groups of complex general linear groups, to have equations which cannot be reduced to Toda equations associated with the corresponding finite dimensional groups, we should require all mappings \( C_{+\alpha} \) and \( C_{-\alpha} \) entering the systems (25) and (24) to be nontrivial. This is again possible only if we choose \( k_\alpha = L \) for each \( \alpha \) and \( M = pL \).

4.1.2 \( m_1 < M \)

As we noted above, if \( n \) is odd, then the equality \( \det h = 1 \) implies that the central diagonal matrix element of \( h \) is 1. When \( m_1 < M \) it is impossible. Therefore, in the case under consideration \( n \) is even. Using automorphisms of \( so_n(\mathbb{C}) \), generated by elements of the form (14) and of the form (15), we bring the element \( h \) to the form (7), where \( n_\alpha, \alpha = 1, \ldots, p \), are positive integers, such that \( \Sigma_{\alpha=1}^p n_\alpha = n \), and \( \mu_\alpha \) are complex numbers, such that \( \mu_\alpha^M = 1 \) for each \( \alpha = 1, \ldots, p \).

It follows from the equality \( l^0 h h = I_n \) that the integers \( n_\alpha \) satisfy the relations

\[
n_\alpha = n_{p-\alpha+1}, \quad \alpha = 1, \ldots, p,
\]

and the numbers \( \mu_\alpha \) satisfy the relations

\[
\mu_\alpha = \mu_{p-\alpha+1}^{-1}, \quad \alpha = 1, \ldots, p.
\]

For the corresponding integers \( m_\alpha \) one obtains

\[
m_\alpha = M - m_{p-\alpha+1}, \quad \alpha = 1, \ldots, p,
\]

that implies that the integers \( k_\alpha, \alpha = 1, \ldots, p-1 \), defined as \( k_\alpha = m_\alpha - m_{\alpha+1} \), satisfy the equalities

\[
k_\alpha = k_{p-\alpha}, \quad \alpha = 1, \ldots, p-1.
\]

Using the relations

\[
m_\alpha = \sum_{\beta=\alpha}^{p-1} k_\beta + m_p, \quad \alpha = 1, \ldots, p-1,
\]

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and (28), one obtains
\[ m_p = M - m_1 = M - \sum_{\beta=1}^{p-1} k_\beta - m_p. \]

Hence, we come to the equality
\[ 2m_p = M - \sum_{\beta=1}^{p-1} k_\beta, \]

which implies that \( M - \sum_{\beta=1}^{p-1} k_\beta \) is an even positive integer.

Let now \( n \) be an even positive integer, and \( M \) be a positive integer. Assume that there are given \( p \) positive integers \( n_\alpha \) satisfying the equality \( \sum_{\alpha=1}^{p} n_\alpha = n \) and the relations (26). Assume also that there are given \( p - 1 \) positive integers \( k_\alpha \) such that \( M - \sum_{\beta=1}^{p-1} k_\beta \) is an even positive integer, and which satisfy the relations (29). Putting
\[ m_p = \frac{1}{2} \left( M - \sum_{\beta=1}^{p-1} k_\beta \right), \]

and using the relation (30), we obtain \( p \) integers \( m_\alpha \) which satisfy the relations (28). In this case the numbers \( \mu_\alpha = \epsilon_{M_\alpha}^{m_\alpha} \) satisfy the relations (27) and the matrix \( h_\alpha \), defined by the equality (7), belongs to \( \text{SO}_n(\mathbb{C}) \cap D_n(\mathbb{C}) \). Here we have \( h^M = I \) and \( m_1 < M \).

Thus, for an even \( n \) a \( \mathbb{Z}_M \)-gradation of inner type of the Lie algebra \( \mathfrak{so}_n(\mathbb{C}) \) for which \( \nu = 1 \) and \( m_1 < M \) is specified by a choice of positive integers \( n_\alpha \) satisfying the equality \( \sum_{\alpha=1}^{p} n_\alpha = n \) and the relations (26), and positive integers \( k_\alpha \) such that \( M - \sum_{\beta=1}^{p-1} k_\beta \) is an even positive integer and which satisfy the relations (29).

Proceed now to the corresponding Toda equations. Choose some \( \mathbb{Z}_M \)-gradation of the Lie algebra \( \mathfrak{so}_n(\mathbb{C}) \) of the type under consideration. Let \( L \) be a positive integer such that the grading subspace \( g[k]_M \) is trivial if \( 0 < |k| < L \). As above, if \( x \in g_\alpha[L]_M \), then only the blocks \( x_{\alpha,\alpha+1}, \alpha = 1, \ldots, p - 1, \) and \( x_{p1} \) in the block matrix representation (10) can be different from zero, and if \( x \in g_-[L]_M \), then only the blocks \( x_{\alpha+1,\alpha}, \alpha = 1, \ldots, p - 1, \) and \( x_{1p} \) can be different from zero.

Let \( p \) be an odd integer equal to \( 2s - 1 \), \( s \geq 2 \). Having in mind the restrictions which follow from the equality \( lx = -x \), parameterize the mapping \( c_+ \) as is given in Figure 2, where \( C_{\alpha} = -l\tilde{C}_{\alpha}(p-a), \alpha = 1, \ldots, p - 1, \) and \( C_{+0} = -l\tilde{C}_{+0} \). For the mapping \( c_- \) we use the parametrization given in Figure 3, where \( C_{-\alpha} = -l\tilde{C}_{-(p-a)}, \alpha = 1, \ldots, p - 1, \) and \( C_{-0} = -l\tilde{C}_{-0} \).

Parameterize the mapping \( \gamma \) in accordance with the relation (12). Here for each \( \alpha \) we have \( \Gamma_{p-a+1} = (\Gamma_\alpha)^{-1} \). It is clear that \( \Gamma_s = (\Gamma_s)^{-1} \), and that the Lie group \( G_0 \) is isomorphic to the Lie group \( \text{GL}_{n_1}(\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}}(\mathbb{C}) \times \text{SO}_{n_s}(\mathbb{C}) \). We come to the
following system of equations

\[
\begin{align*}
\partial_+ (Γ_1^{-1} ∂_Γ Γ_1) &= −Γ_1^{-1} C_{+1} Γ_2 C_{−1} + C_{−0} l Γ_1 C_{+0} Γ_1, \\
\partial_+ (Γ_2^{-1} ∂_Γ Γ_2) &= −Γ_2^{-1} C_{+2} Γ_3 C_{−2} + C_{−1} Γ_1^{-1} C_{+1} Γ_2, \\
&\vdots \\
\partial_+ (Γ_{s−1}^{-1} ∂_Γ Γ_{s−1}) &= −Γ_{s−1}^{-1} C_{+(s−1)} Γ_s C_{−(s−1)} + C_{−(s−2)} Γ_{s−2}^{-1} C_{+(s−2)} Γ_{s−1}, \\
\partial_+ (Γ_s^{-1} ∂_Γ Γ_s) &= −l (C_{−(s−1)} Γ_{s−1} C_{+(s−1)} Γ_s) + C_{−(s−1)} Γ_{s−1}^{-1} C_{+(s−1)} Γ_s.
\end{align*}
\]

which is equivalent to the Toda equation (3). Actually the substitution \( Γ_α \rightarrow l (Γ_{s−α+1}^{-1}) \) and \( C_{±α} \rightarrow −l C_{±(s−α)} \) transforms the system (31) into the system (25). Hence in the case under consideration we do not obtain new Toda equations.

Let now \( p \) be an even integer equal to \( 2s, s ≥ 1 \). Parameterize the mapping \( c_+ \) as is given in Figure 2 where \( C_{+α} = −l C_{+(−p−α)}, α = 1, \ldots , p − 1, \) and \( C_{+0} = −l C_{+0} \). For the mapping \( c_- \) we use the parametrization given in Figure 3 where \( C_{−α} = −l C_{−(p−α)}, α = 1, \ldots , p − 1, \) and \( C_{−0} = −l C_{−0} \). An appropriate parametrization of the mapping \( γ \) is given by the equality (12), where for each \( α = 1, \ldots , p \) we have \( Γ_{p−α+1} = (l Γ_α)^{−1} \). The Lie group \( G_0 \) is isomorphic to the Lie group \( GL_n (C) × \cdots × GL_n (C) \), and the Toda equation (3) is equivalent in the case under consideration to the system

\[
\begin{align*}
\partial_+ (Γ_1^{-1} ∂_Γ Γ_1) &= −Γ_1^{-1} C_{+1} Γ_2 C_{−1} + C_{−0} l Γ_1 C_{+0} Γ_1, \\
\partial_+ (Γ_2^{-1} ∂_Γ Γ_2) &= −Γ_2^{-1} C_{+2} Γ_3 C_{−2} + C_{−1} Γ_1^{-1} C_{+1} Γ_2, \\
&\vdots \\
\partial_+ (Γ_{s−1}^{-1} ∂_Γ Γ_{s−1}) &= −Γ_{s−1}^{-1} C_{+(s−1)} Γ_s C_{−(s−1)} + C_{−(s−2)} Γ_{s−2}^{-1} C_{+(s−2)} Γ_{s−1}, \\
\partial_+ (Γ_s^{-1} ∂_Γ Γ_s) &= −Γ_s^{-1} C_{+s} l (Γ^{-1}) C_{−s} + C_{−(s−1)} Γ_{s−1}^{-1} C_{+(s−1)} Γ_s.
\end{align*}
\]

Only in the case \( k_α = L \) and \( M = pL \) the systems (31) and (32) cannot be reduced to Toda equations associated with the finite dimensional Lie groups \( SO_n (C) \). Here the condition that \( M − \sum_{β=1}^{p−1} k_β \) is an even positive integer implies that \( L \) should be even.

4.2 \( ν = −1 \)

Since in the case under consideration \( h^M = −I \) and \( det h = 1 \), the integer \( n \) must be even. The matrix \( h \) is a diagonal matrix with the diagonal matrix elements of the form \( e_μ^m / e_{2m} \), where we assume that \( 0 < m ≤ M \). As before, we denote by \( m_α, α = 1, \ldots , p \), the different values of \( m \) taken in decreasing order, and by \( n_α, α = 1, \ldots , p \), their multiplicities. Using automorphisms of \( so_n (C) \), generated by elements of the type (14) and of the type (15), we bring the element \( h \) to the form (7).

It follows from the equality \( h^M h = I \) that the integers \( n_α \) satisfy the relations (26), and the numbers \( μ_α \) satisfy the relations (27). For the corresponding integers \( m_α \) instead of (28) one obtains the equalities

\[
m_α = M − m_{p−α+1} + 1, \quad α = 1, \ldots , p.
\]

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One can show that now we have the equality

\[ 2m_p = M - \sum_{\beta=1}^{p-1} k_\beta + 1, \]

where the integers \( k_\alpha \) are defined as \( k_\alpha = m_\alpha - m_{\alpha+1} \) as before. This equality implies that \( M - \sum_{\beta=1}^{p-1} k_\beta + 1 \) is an even positive integer.

Let now \( n \) be an even positive integer, and \( M \) be a positive integer. Assume that there are given \( p \) positive integers \( n_\alpha \) satisfying the equality \( \sum_{\alpha=1}^{p} n_\alpha = n \) and the relations (29). Assume also that there are given \( p - 1 \) positive integers \( k_\alpha \) such that \( M - \sum_{\beta=1}^{p-1} k_\beta + 1 \) is an even positive integer, and which satisfy the relations (26). Using the relation (30) and putting

\[ m_p = \frac{1}{2} \left( M - \sum_{\beta=1}^{p-1} k_\beta + 1 \right), \]

we obtain \( p \) integers \( m_\alpha \) which satisfy the relations (26). In this case the numbers \( \mu_\alpha = \epsilon_{M}^{m_\alpha} / \epsilon_{2M} \) satisfy the relations (27), and the matrix \( h \) defined by (7) satisfies the equality \( h h = 1 \). Moreover, \( h \) satisfies the equalities \( \det h = 1 \) and \( h^M = -I \).

Thus, for an even \( n \) a \( \mathbb{Z}_M \)-gradation of the Lie algebra \( \mathfrak{so}_n(\mathbb{C}) \) of inner type for which \( v = -1 \) is specified by a choice of positive integers \( n_\alpha \) satisfying the equality \( \sum_{\alpha=1}^{p} n_\alpha = n \) and the relations (26), and positive integers \( k_\alpha \) such that \( M - \sum_{\beta=1}^{p-1} k_\beta + 1 \) is an even positive integer, and which satisfy the relations (29).

It is not difficult to see that using \( \mathbb{Z}_M \)-gradations of the type under consideration we again come to the systems (31) and (32). The only difference is that to have Toda equations which cannot be reduced to Toda equations associated with the finite dimensional Lie groups \( \text{SO}_n(\mathbb{C}) \) one uses \( \mathbb{Z}_M \)-gradations for which the integer \( L \) is odd.

5 Toda equations associated with loop groups of complex symplectic groups

There are only inner automorphisms of the Lie group \( \text{Sp}_n(\mathbb{C}) \) and the Lie algebra \( \mathfrak{sp}_n(\mathbb{C}) \). Let \( a \) be an automorphism of \( \text{Sp}_n(\mathbb{C}) \) satisfying the relation \( a^M = \text{id}_{\text{Sp}_n(\mathbb{C})} \). The corresponding automorphism \( A \) of the Lie algebra \( \mathfrak{sp}_n(\mathbb{C}) \) satisfies the relation \( A^M = \text{id}_{\mathfrak{sp}_n(\mathbb{C})} \). In the case when \( A = \text{id}_{\mathfrak{sp}_n(\mathbb{C})} \) and \( M \) is an arbitrary positive integer we come to Eq. (5), where \( \Gamma \) is a mapping from \( \mathcal{M} \) to \( \text{Sp}_n(\mathbb{C}) \), \( C_+ \) and \( C_- \) are mappings from \( \mathcal{M} \) to \( \mathfrak{sp}_n(\mathbb{C}) \) satisfying the conditions (6).

In a general case without any loss of generality we can assume that the automorphism \( A \) is given by the relation

\[ A(x) = h x h^{-1}, \]

where \( h \in \text{Sp}_n(\mathbb{C}) \cap D_n(\mathbb{C}) \), see Appendix C.3. By definition we have \( k^h = h^{-1} \). It is easy to see that since \( h \in \text{Sp}_n(\mathbb{C}) \cap D_n(\mathbb{C}) \), one has \( k^h = h \), therefore, \( h \in \text{SO}_n(\mathbb{C}) \cap D_n(\mathbb{C}) \). Furthermore, one has \( h^M = v I \), where \( v \) is either 1, or -1. Thus, the matrix \( h \) satisfies the same relations as for the case of complex orthogonal groups. Actually, to describe Toda equations associated with loop groups of complex symplectic groups, one can almost literally follow the lines of Section 4.
5.1 \( \nu = 1 \)

Assume that \( \nu = 1 \). Here, as for the case of complex orthogonal groups, \( h \) is a diagonal matrix with the diagonal matrix elements of the form \( e_{m}^{m} \), where we assume that \( m \) is a nonnegative integer satisfying the conditions \( 0 < m \leq M \). Let \( m_{\alpha}, \alpha = 1, \ldots, p \), be the different values of \( m \) taken in the decreasing order, \( M \geq m_{1} > m_{2} > \ldots > m_{p} > 0 \), and \( n_{\alpha}, \alpha = 1, \ldots, p \), be their multiplicities.

5.1.1 \( m_{1} = M \)

Suppose that \( m_{1} = M \). To bring the element \( h \) to a form convenient for our purposes we permute its diagonal matrix elements by two types of automorphisms of \( \mathfrak{sp}_{n}(\mathbb{C}) \). The automorphisms of the first type are conjugations by the matrices of the block matrix form

\[
\begin{pmatrix}
I_{i-1} & 0 & 0 & i \\
0 & I_{n-2i} & 0 & \\
i & 0 & 0 & \\
0 & 0 & I_{i-1}
\end{pmatrix},
\]

where \( i \leq n/2 \). The corresponding automorphism permutes the diagonal elements of a diagonal matrix at positions \( i \) and \( n - i + 1 \). The automorphisms of the second type are conjugations by the matrices of the form described by (15), where \( i \leq n/2 - 1 \). The corresponding automorphism of \( \mathfrak{sp}_{n}(\mathbb{C}) \) permutes the diagonal elements of a diagonal matrix at positions \( i \) and \( i + 1 \) and the diagonal elements at positions \( n - i + 1 \) and \( n - i \). Using automorphisms of these two types and the conjugation by the matrix of the form (16), we bring the element \( h \) to the same form as in the case of complex orthogonal groups, see Section 4.1.1. Here the conjugation by the matrix of the form (16) maps the Lie algebra \( \mathfrak{sp}_{n}(\mathbb{C}) \) isomorphically to the Lie algebra \( \mathfrak{gl}_{n}^{R}(\mathbb{C}) \) with

\[
B = \begin{pmatrix}
K_{n_{1}} & 0 \\
0 & K_{n-n_{1}}
\end{pmatrix}.
\]

The same conjugation maps the Lie group \( \text{Sp}_{n}(\mathbb{C}) \) isomorphically to the Lie group \( \text{GL}_{n}^{R}(\mathbb{C}) \).

Repeating the discussion given in Section 4.1.1, we conclude that in the case under consideration a \( \mathbb{Z}_{M} \)-gradation of the Lie algebra \( \mathfrak{sp}_{n}(\mathbb{C}) \) is specified by a choice of \( p \) positive integers \( n_{\alpha} \) such that \( \sum_{\alpha=1}^{p} n_{\alpha} = n \), satisfying the relations (15), and a choice of \( p - 1 \) positive integers \( k_{\alpha} \), satisfying the relations (21) and the equality (23). The structure of the \( \mathbb{Z}_{M} \)-gradation is described by Figure 1.

Choose a \( \mathbb{Z}_{M} \)-gradation of the type under consideration and describe the corresponding Toda equation. Consider first the case of an even \( p = 2s - 2, s \geq 2 \). Using the restrictions which follow from the equality \( Bx = -x \), parameterize the mapping \( c_{+} \) as is given in Figure 2 where \( C_{+0} = jkC_{+1}, \ C_{+s} = -jkC_{+(s-1)}, \) and \( C_{+\alpha} = -jC_{+(p-\alpha+1)} \) for \( \alpha = 2, \ldots, s-2 \) and \( \alpha = s+1, \ldots, 2s-3 \). An appropriate parametrization of the mapping \( c_{-} \) is given in Figure 3 where \( C_{-0} = jkC_{-1}, \ C_{-s} = -jkC_{-(s-1)}, \) and \( C_{-\alpha} = -jC_{-(p-\alpha+1)} \) for \( \alpha = 2, \ldots, s-2 \) and \( \alpha = s+1, \ldots, 2s-3 \).

Parameterize the mapping \( \gamma \) in accordance with (12), where \( k_{1} = \Gamma_{1}^{-1}, k_{s} = \Gamma_{s}^{-1}, \) and \( \Gamma_{\alpha} = (j\Gamma_{p-\alpha+2})^{-1} \) for each \( \alpha = 2, \ldots, s-1 \) and \( \alpha = s+1, \ldots, 2s-2 \). The Lie group
$G_0$ is isomorphic to the Lie group $\text{Sp}_{n_1} (\mathbb{C}) \times \text{GL}_{n_2} (\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}} (\mathbb{C}) \times \text{Sp}_{n_s} (\mathbb{C})$, and the Toda equation (3) is equivalent to the system of equations

$$\partial_+ \left( \Gamma_1^{-1} \partial_1 \right) = -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + K(\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}),$$
$$\partial_+ \left( \Gamma_2^{-1} \partial_1 \right) = -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2,$$
$$\vdots$$
$$\partial_+ \left( \Gamma_{s-1}^{-1} \partial_1 \right) = -\Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_{s} C_{-(s-1)} + C_{-(s-2)} \Gamma_{s-2}^{-1} C_{+(s-2)} \Gamma_{s-1},$$
$$\partial_+ \left( \Gamma_s^{-1} \partial_1 \right) = -K(\Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_{s}) + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_{s}.$$

Consider now the case of an odd $p = 2s - 1$, $s \geq 2$. Parameterize the mapping $c_+$ as is given in Figure 2, where $C_{+0} = lC_{+0}$, $C_{+s} = lC_{+s}$, and $C_{+a} = -lC_{+(p-a+1)}$ for $a = 2, \ldots, s - 1$ and $\alpha = s + 1, \ldots, 2s - 2$. An appropriate parametrization of the mapping $c_-$ is given in Figure 3, where $C_{-0} = kC_{-1}$, $C_{-s} = kC_{-s}$, and $C_{-a} = -kC_{-(p-a+1)}$ for $a = 2, \ldots, s - 1$ and $\alpha = s + 1, \ldots, 2s - 2$. The mapping $\gamma$ is parameterized in accordance with (12), where $K \Gamma_1 = \Gamma_1^{-1}$, and $\Gamma_\alpha = (\Gamma_{p-\alpha+2})^{-1}$ for each $\alpha = 2, \ldots, 2s - 1$. The Lie group $G_0$ is isomorphic to the Lie group $\text{Sp}_{n_1} (\mathbb{C}) \times \text{GL}_{n_2} (\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}} (\mathbb{C}) \times \text{Sp}_{n_s} (\mathbb{C})$ and the Toda equation (3) is equivalent to the system of equations

$$\partial_+ \left( \Gamma_1^{-1} \partial_1 \right) = -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + K(\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}),$$
$$\partial_+ \left( \Gamma_2^{-1} \partial_1 \right) = -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2,$$
$$\vdots$$
$$\partial_+ \left( \Gamma_{s-1}^{-1} \partial_1 \right) = -\Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_{s} C_{-(s-1)} + C_{-(s-2)} \Gamma_{s-2}^{-1} C_{+(s-2)} \Gamma_{s-1},$$
$$\partial_+ \left( \Gamma_s^{-1} \partial_1 \right) = -\Gamma_s^{-1} C_{+(s-1)} (\Gamma_s^{-1}) C_{-s} + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_{s}.$$

Remind that in the case under consideration $lC_{+s} = C_{+s}$ and $lC_{-s} = C_{-s}$.

### 5.1.2 $m_1 < M$

In accordance with the results of Section 4.1.2, a $\mathbb{Z}_M$-gradation of the Lie algebra $\mathfrak{sp}_n (\mathbb{C})$ of inner type for which $\nu = 1$ and $m_1 < M$ is specified by a choice of positive integers $n_\alpha$ satisfying the equality $\sum_{\alpha=1}^p n_\alpha = n$ and the relations (26), and positive integers $k_\alpha$ such that $M - \sum_{\beta=1}^{p-1} k_\beta$ is an even positive integer and which satisfy the relations (29).

In the case of an odd $p = 2s - 1$, $s \geq 2$, we parameterize the mapping $c_+$ as is given in Figure 2, where $C_{+0} = lC_{+0}$, $C_{+s} = lC_{+s}$ and $C_{+a} = -lC_{+(p-a)}$ for $a = 1, \ldots, s - 2$ and $\alpha = s + 1, \ldots, 2s - 2$. An appropriate parametrization of the mapping $c_-$ is described by Figure 3, where $C_{-0} = lC_{-0}$, $C_{-s} = -lC_{-(s-1)}$ and $C_{-a} = -lC_{-(p-a)}$ for $a = 1, \ldots, s - 2$ and $\alpha = s + 1, \ldots, 2s - 2$. The mapping $\gamma$ can be parameterized in accordance with (12), where $K \Gamma_1 = \Gamma_1^{-1}$, and $\Gamma_\alpha = (\Gamma_{p-\alpha+1})^{-1}$ for $\alpha = 1, \ldots, s - 1$ and $\alpha = s + 1, \ldots, 2s - 1$. The Lie group $G_0$ is isomorphic to $\text{GL}_{n_1} (\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}} (\mathbb{C}) \times \text{Sp}_{n_s} (\mathbb{C})$. One can show that in the case under consideration the
Let a Toda equation associated with loop groups of complex general linear groups. Without any loss of generality we assume that the automorphism \( A \) of the Lie group \( GL_n(\mathbb{C}) \) of order \( M \) and the corresponding outer automorphism of the Lie algebra \( gl_n(\mathbb{C}) \) respectively. Consider the \( \mathbb{Z}_M \)-gradation of the Lie algebra \( gl_n(\mathbb{C}) \) generated by the automorphism \( A \). Without any loss of generality we assume that the automorphism \( A \) is given by the relation

\[
A(x) = -h^\dagger x h^{-1},
\]

where \( h \) is an element of \( GL_n(\mathbb{C}) \cap D_n(\mathbb{C}) \), such that

\[
\dagger h h = I,
\]

see Appendix [C.1]. From the equality (36) it follows that either \( \det h = 1 \), or \( \det h = -1 \). Since \( h \) is a diagonal matrix and satisfies the equality (36), we conclude that the
equality \( \det h = -1 \) could be valid only if \( n \) is odd. In this case, multiplying \( h \) by \( -1 \), we obtain a matrix with unit determinant, which satisfies (36) and generates the same automorphism of \( \mathfrak{gl}_n(\mathbb{C}) \) as the initial matrix \( h \). Thus, we can assume that the automorphism \( A \) generating the \( \mathbb{Z}_M \)-gradation under consideration is given by the equality (35), where \( h \) is an element of the group \( \text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C}) \).

Suppose now that \( M \) is odd. The equality \( A^M = \text{id}_{\mathfrak{gl}_n(\mathbb{C})} \) implies that \( -h^M J x h^{-M} = x \) for any \( x \in \mathfrak{gl}_n(\mathbb{C}) \). It is not difficult to show that it is impossible. Hence, \( M \) cannot be odd, and we assume that \( M \) is even and equals \( 2N \). In this case we obtain \( h^M J x h^{-M} = x \) for any \( x \in \mathfrak{gl}_n(\mathbb{C}) \). Hence, the matrix \( h \) satisfies the equality

\[
h^M = v I
\]

for some complex number \( v \). It follows from this equality that

\[
J(h^M) h^M = v^2 I.
\]

From the other hand, using (36), we obtain

\[
J(h^M) h^M = (J h h)^M = I.
\]

Thus, the number \( v \) should satisfy the equality \( v^2 = 1 \). This means that either \( v = 1 \), or \( v = -1 \).

In the simplest case \( h = I \). In this case there are only two nontrivial grading subspaces,

\[
\mathfrak{g}[0]_{2N} = \{ x \in \mathfrak{gl}_n(\mathbb{C}) \mid J x = -x \}
\]

and

\[
\mathfrak{g}[N]_{2N} = \{ x \in \mathfrak{gl}_n(\mathbb{C}) \mid J x = x \}.
\]

The Lie group \( G_0 \) coincides with \( \text{SO}_n(\mathbb{C}) \) and we come to Eq. (5), where \( \Gamma \) is a mapping from \( \mathcal{M} \) to \( \text{SO}_n(\mathbb{C}) \), the mappings \( C_+ \) and \( C_- \) are mappings from \( \mathcal{M} \) to the space of \( n \times n \) complex matrices \( x \) satisfying the equality \( J x = x \).

Before proceeding to the consideration of a general case, it is useful to make a few remarks.

Let \( B \) be an arbitrary nonsingular matrix. For any \( h \in \text{GL}_n(\mathbb{C}) \) the mapping \( A : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}) \) defined by the equality

\[
A(x) = -h B x h^{-1}
\]

is an automorphism of \( \mathfrak{gl}_n(\mathbb{C}) \). By an inner automorphism of \( \mathfrak{gl}_n(\mathbb{C}) \) generated by an element \( k \in \text{GL}_n(\mathbb{C}) \) the automorphism \( A \) is conjugated to an automorphism \( A' \) of \( \mathfrak{gl}_n(\mathbb{C}) \) defined by the equality

\[
A'(x) = -h' B' x h'^{-1},
\]

where

\[
h' = k h k^{-1}, \quad B' = k^{-1} B k^{-1}.
\]

Here if \( B \) is a skew-diagonal symmetric matrix, then using an inner automorphism generated by a diagonal matrix \( k \) we can make \( B \) coincide with the matrix \( J \). If \( B \) is a skew-diagonal skew-symmetric matrix, then using an inner automorphism also generated by a diagonal matrix \( k \) we can make \( B \) coincide with the matrix \( K \).
Note also that for any \( k \in \text{GL}_n(\mathbb{C}) \) we can represent the action of the automorphism \( A \) given by (37) on an element \( x \in \mathfrak{gl}_n(\mathbb{C}) \) as

\[
A(x) = -h'B_x h'^{-1}
\]

where

\[
h' = h k, \quad B' = B k.
\]

6.1 \( \nu = 1 \)

Represent each diagonal matrix element of the diagonal matrix \( h \) either as \( \varepsilon^m \) or as \(-\varepsilon^m\), where \( m \) is an integer satisfying the condition \( 0 < m \leq M/2 = N \). Let \( m_\alpha \), \( \alpha = 1, \ldots, p \), be the different values of \( m \) needed for such representation taken in the decreasing order, \( N \geq m_1 > m_2 > \ldots > m_p > 0 \). Denote by \( n'_\alpha \) and \( n''_\alpha \) the numbers of matrix elements equal to \( \varepsilon^m \) and \(-\varepsilon^m\), respectively, and by \( n_\alpha \) the sum of \( n'_\alpha \) and \( n''_\alpha \). Note that the integer \( n - n_1 \) is always even.

6.1.1 \( m_1 = N \)

The conditions \( \det h = 1 \) and \( h h = I \) imply that the integer \( n'_1 \) is even. Moreover, one can show that

\[
m_\alpha = N - m_{p-\alpha+2}, \quad \alpha = 2, \ldots, p,
\]

and

\[
n'_\alpha = n''_{p-\alpha+2}, \quad \alpha = 2, \ldots, p.
\]

The last equality, in particular, gives

\[
n_\alpha = n_{p-\alpha+2}, \quad \alpha = 2, \ldots, p.
\]

Using automorphisms of \( \mathfrak{gl}_n(\mathbb{C}) \) interchanging simultaneously rows and the corresponding columns of the matrices, one can bring the element \( h \) to the form

\[
h = \begin{pmatrix}
-I_{n'_1}/2 & & & \\
 & I_{n''_1} & & \\
 & -I_{n'_1}/2 & & \\
 & & \varepsilon^m_{M_1} I_{n'_2} & \\
 & & -\varepsilon^m_{M_1} I_{n''_2} & \\
 & & & \ddots \\
 & & & & \varepsilon^m_{M_p} I_{n'_p} \\
 & & & & -\varepsilon^m_{M_p} I_{n''_p}
\end{pmatrix}.
\]

Here the matrix \( J \) goes to the matrix \( B \) defined as

\[
B = \begin{pmatrix}
J_{n_1} & 0 \\
0 & J_{n-n_1}
\end{pmatrix}.
\]
Now multiplying $h$ and $B$ from the right by the appropriate diagonal matrix we bring the matrix $h$ to the form

$$h = \begin{pmatrix} \epsilon_{M}^{m_1} I_{n_1} \\ \epsilon_{M}^{m_2} I_{n_2} \\ \vdots \\ \epsilon_{M}^{m_p} I_{n_p} \end{pmatrix}. \quad (38)$$

Here the matrix $B$ takes the form

$$B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix},$$

where $B'$ is an $n_1 \times n_1$ skew-diagonal symmetric matrix, and $B''$ is an $(n - n_1) \times (n - n_1)$ skew-diagonal skew-symmetric matrix. Using an inner automorphism of $\mathfrak{gl}_n(\mathbb{C})$ generated by a diagonal matrix we bring the matrix $B$ to the form

$$B = \begin{pmatrix} J_{n_1} & 0 \\ 0 & K_{n-n_1} \end{pmatrix}. \quad (39)$$

Here the matrix $h$ remains unchanged.

Thus, in the case under consideration the outer automorphism $A$ of $\mathfrak{gl}_n(\mathbb{C})$ defined by the relation (35) is conjugated to the automorphism $A$ defined by the relation (37) with $h$ and $B$ given by the equalities (38) and (39) respectively. We assume that the automorphism $A$ under consideration has this form.

It can be easily verified that

$$B(Bx) = \tilde{I}^{-1}x\tilde{I},$$

where

$$\tilde{I} = \begin{pmatrix} I_{n_1} \\ -I_{n-n_1} \end{pmatrix}.$$

Now using the equality

$$B_h h = \tilde{I},$$

one obtains

$$A^2(x) = h^2 x h^{-2}.$$

Let us denote

$$\tilde{h} = h^2 = \begin{pmatrix} \epsilon_{M}^{m_1} I_{n_1} \\ \epsilon_{M}^{m_2} I_{n_2} \\ \vdots \\ \epsilon_{M}^{m_p} I_{n_p} \end{pmatrix}, \quad (40)$$

and rewrite the above relation as

$$A^2(x) = \tilde{h} x \tilde{h}^{-1}. \quad (41)$$

Let $k$ be an integer such that $0 \leq k < 2N$. Assume that an element $x$ belongs to the grading subspace with the grading index $[k]_{2N}$ of $\mathbb{Z}_{2N}$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ defined by the automorphism $A$. By definition we have

$$A(x) = \epsilon_{2N}^{k} x, \quad (42)$$
therefore,
\[ A^2(x) = e^k_N x. \]  

Hence, the element \( x \) belongs to the grading subspace with the grading index \([k]_N\) of inner type \( \mathbb{Z}_N \)-gradation of \( \mathfrak{gl}_n(\mathbb{C}) \) defined by the automorphism \( A^2 \). Using the block matrix representation (10) and the relations (41), (40), we see that the block \( x_{\alpha\beta} \) of \( x \) is different from zero only if
\[ \epsilon_{N}^{m_{\alpha}-m_{\beta}} = e^k_N. \]

There are four variants for the restrictions on possible values of \( m_{\alpha} \) and \( m_{\beta} \) arising from this equality. They are described by Table 1.

| \( \alpha \leq \beta \) | \( N \leq k < 2N \) |
|---|---|
| \( m_{\alpha} - m_{\beta} = k \) | \( m_{\alpha} - m_{\beta} = k - N \) |

| \( \alpha > \beta \) | \( N \leq k < 2N \) |
|---|---|
| \( m_{\alpha} - m_{\beta} = k - N \) | \( m_{\alpha} - m_{\beta} = k - 2N \) |

Table 1: The restrictions on possible values of \( m_{\alpha} \) and \( m_{\beta} \) for a fixed value of \( k \)

Introduce positive integers \( k_\alpha = m_{\alpha} - m_{\alpha+1}, \alpha = 1, \ldots, p - 1 \). They satisfy the relation
\[ k_\alpha = k_{p-\alpha+1}, \quad \alpha = 2, \ldots, p - 1. \]

Similarly as in section 4.1.1 one can show that the numbers \( k_\alpha \) satisfy the condition
\[ p-1 \sum_{\beta=1}^{\alpha} k_\beta + k_1 = N, \]  

and that the integers \( m_{\alpha} \) are connected with the numbers \( k_\alpha \) by the relation
\[ m_{\alpha} = \sum_{\beta=\alpha}^{p-1} k_\beta + k_1. \]

Now one can get convinced that the grading structure of the \( \mathbb{Z}_{2N} \)-gradation under consideration can be depicted by the scheme given in Figure 4. Here the pairs of elements of the ring \( \mathbb{Z}_{2N} \) are the possible grading indices of the corresponding blocks in the block matrix representation (10) of a general element of \( \mathfrak{gl}_n(\mathbb{C}) \). Note that this structure is obtained by using only the equality (43) which follows from the equality (42). The latter equality imposes additional restrictions.

Using again the block matrix representation (10), we see that the equality (42) gives
\[ \epsilon_{2N}^{m_{\alpha}-m_{\beta}} (B x)_{\alpha\beta} = -\epsilon_{2N}^k x_{\alpha\beta}. \]

Having in mind Table 1, we come to the restrictions on the block structure of an element \( x \) belonging to a grading subspace of the \( \mathbb{Z}_{2N} \)-gradation under consideration collected in Table 2.

Consider now the corresponding Toda equations. As in the cases considered before, to obtain equations which can not be reduced to Toda equations associated with finite dimensional group we should put \( k_\alpha = L, \alpha = 1, \ldots, p - 1 \). It follows from (44) that in this case we should assume that \( N = pL \).
For the case of an even $p = 2s - 2$, $s \geq 2$, the group $G_0$ is isomorphic to $SO_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \times \cdots \times GL_{n_{s-1}}(\mathbb{C}) \times Sp_{n_s}(\mathbb{C})$. The Toda equation (3) is equivalent to the system

$$
\begin{align*}
\partial_+ (\Gamma_1^{-1} \partial_+ \Gamma_1) &= -\Gamma_1^{-1}C_1 \Gamma_2 C_{-1} + \Gamma_2^{-1}(\Gamma_1^{-1}C_1 \Gamma_2 C_{-1}), \\
\partial_+ (\Gamma_2^{-1} \partial_+ \Gamma_2) &= -\Gamma_2^{-1}C_2 \Gamma_3 C_{-2} + \Gamma_1^{-1}(\Gamma_1^{-1}C_2 \Gamma_3 C_{-2}), \\
&\vdots \\
\partial_+ (\Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1}) &= -\Gamma_{s-1}^{-1}C_{s(s-1)} \Gamma_s C_{-(s-1)} + \Gamma_s^{-1}(\Gamma_{s-1}^{-1}C_{s(s-1)} \Gamma_s C_{-(s-1)}), \\
\partial_+ (\Gamma_s^{-1} \partial_+ \Gamma_s) &= -K(\Gamma_{s-1}^{-1}C_{s(s-1)} \Gamma_s C_{-(s-1)} + \Gamma_s^{-1}(\Gamma_{s-1}^{-1}C_{s(s-1)} \Gamma_s C_{-(s-1)}),
\end{align*}
$$

where $I \Gamma_1 = \Gamma_1^{-1}$ and $K \Gamma_s = \Gamma_s^{-1}$. One can get convinced that using appropriate substitutions it is possible to reduce the above system to the system of the same form but with $f$ and $K$ interchanged.
For the case of an odd \( p = 2s - 1, \ s \geq 2 \), the Lie group \( G_0 \) is isomorphic to the
group \( \text{SO}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \times \cdots \times \text{GL}_{n_s}(\mathbb{C}) \). The Toda equation (3) is equivalent to the
system of equations (25), where \( J_1 = \Gamma_1^{-1}, \ j_{C+s} = C_{+s}, \) and \( j_{C-s} = C_{-s} \).

6.1.2 \( m_1 < N \)

For the case when \( \nu = 1 \) and \( m_1 < N \) the automorphism \( A \) under consideration
is conjugated to the automorphism \( A \) given by the equality (37), where \( B = K \), the
matrix \( h \) has the form (38) and satisfies the equality \( h'h = I \). The integers \( k_\alpha, \ \alpha = 1, \ldots, p - 1 \), satisfy the relation (29). The structure of the \( \mathbb{Z}_{2N} \)-gradation generated by
the automorphism \( A \) is described by Figure 4, and the nontrivial blocks of an element
\( x \) belonging to the grading subspace with the grading index \([k]_{2N}\) are subjected to the
conditions given in Table 2.

For the case of an odd \( p = 2s - 1 \) the Lie group \( G_0 \) is isomorphic to the Lie group
\( \text{GL}_{n_1}(\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}}(\mathbb{C}) \times \text{Sp}_{n_s}(\mathbb{C}) \) and the Toda equation (3) is equivalent to a
system of equations which can be reduced to the system (34), where \( \Gamma_1 = \Gamma_{1,s}^{-1} \), and
\( j_{C+0} = -C_{+0}, j_{C-0} = -C_{-0} \).

For the case of an even \( p = 2s \) the Lie group \( G_0 \) is isomorphic to the Lie group
\( \text{GL}_{n_1}(\mathbb{C}) \times \cdots \times \text{GL}_{n_s}(\mathbb{C}) \) and the Toda equation (3) is equivalent to the system (32),
where \( C_{+0} = -C_{+0}, \ j_{C+s} = C_{+s}, \ j_{C-0} = -C_{-0}, \) and \( j_{C-s} = C_{-s} \).

6.2 \( \nu = -1 \)

In the case when \( \nu = -1 \), as for the inner \( \mathbb{Z}_M \)-gradations of the complex orthogonal
algebras and the complex symplectic algebras, we do not obtain new equations.

7 Toda equations associated with loop groups of com-
plex orthogonal groups. Gradations of outer type

Recall that the Lie group \( \text{SO}_n(\mathbb{C}) \) has outer automorphisms only if \( n \) is even. There-
fore, we assume that \( n \) is even and discard the trivial case \( n = 2 \). Let \( a \) be an outer
automorphism of \( \text{SO}_n(\mathbb{C}) \) of order \( M \) and \( A \) be the corresponding automorphism of
\( \mathfrak{so}_n(\mathbb{C}) \). Without any loss of generality we assume that the automorphism \( A \) is given
by the equality

\[
A(x) = (h'u)x(h'u)^{-1},
\]

where the matrix \( u \) is given by the equality (C.6) and \( h' \) is an element of \( \text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C}) \) satisfying the relation \( u'hu^{-1} = h' \), see Appendix C.2.

It is not difficult to show that the two central diagonal matrix elements of the matrix
\( h' \) are equal either to 1 or to \(-1\). We exclude the latter case multiplying the matrix \( h' \)
by \(-1\). In other words, we assume that the matrix \( h = h'u \) has the form

\[
h = \begin{pmatrix}
    h'' & 0 & 1 \\
    0 & 1 & 0 \\
    0 & 0 & j(h''^{-1})
\end{pmatrix},
\]

28
where $h''$ is an $(n - 2)/2 \times (n - 2)/2$ nonsingular diagonal matrix. From the equality $A^M = \text{id}_{\mathfrak{so}_n(\mathbb{C})}$ it follows that

$$h^M = \nu I$$

for some complex number $\nu$. Using the above explicit form of the matrix $h$, we conclude that it is possible only for an even $M = 2N$ and $\nu = 1$.

The conjugation by the matrix

$$k = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} I_{(n-2)/2} & 1 & 1 \\ -i & 1 \\ \sqrt{2} I_{(n-2)/2} \end{pmatrix}$$

brings the matrix $h$ to the form

$$h = \begin{pmatrix} h'' & 1 \\ -1 & I(h'' - 1) \end{pmatrix},$$

and maps $\mathfrak{so}_n(\mathbb{C})$ isomorphically onto the Lie algebra $\mathfrak{gl}_n^B(\mathbb{C})$, where

$$B = \begin{pmatrix} 1 & I_{(n-2)/2} \\ I_{(n-2)/2} & I \end{pmatrix}.$$  

The same conjugation maps $\text{SO}_n(\mathbb{C})$ isomorphically onto the Lie group $\text{GL}_n^B(\mathbb{C})$.

Note that $h$ is a diagonal matrix with the diagonal matrix elements of the form $e_{2N}^m$, where we assume that $m$ is a nonnegative integer satisfying the condition $0 < m \leq 2N$. Let $m_\alpha, \alpha = 1, \ldots, p$, be the different values of $m$ taken in the decreasing order, $2N \geq m_1 > m_2 > \ldots > m_p > 0$, and $n_\alpha, \alpha = 1, \ldots, p$, be their multiplicities. It is clear that $m_1 = 2N$. One can get convinced that $p$ is an even number, let it be equal to $2s - 2$. The integers $m_\alpha$ satisfy the relations

$$m_\alpha = 2N - m_{p-\alpha+2}.$$  

In particular, one has $m_s = N$. It is also almost evident that $n_1$ and $n_s$ are odd.

Simultaneously permuting the rows and columns of the matrices, we bring $h$ to the form (9) with $\rho = 1$ and $M = 2N$. Here the Lie algebra under consideration is mapped isomorphically onto the Lie algebra $\mathfrak{gl}_n^B(\mathbb{C})$ with $B$ given by the equality (17) and the Lie group under consideration is mapped onto Lie group $\text{GL}_n^B(\mathbb{C})$ with the same $B$.

It is clear that the Lie group $G_0$ is isomorphic to the Lie group $\text{SO}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \times \cdots \times \text{GL}_{n_{s-1}}(\mathbb{C}) \times \text{SO}_{n_s}(\mathbb{C})$. The Toda equation (3) is equivalent now to the system (24), where $I_{\Gamma_1} = \Gamma_1^{-1}$ and $I_{\Gamma_s} = \Gamma_s^{-1}$. Recall that in the case under consideration the integers $n_1$ and $n_s$ are odd.
8 Conclusions

Thus, there are four types of Toda equations associated with loop groups of complex classical Lie groups.

Let \( n_\alpha, \alpha = 1, \ldots, p \), be a set of positive integers. A Toda equation of the first type is equivalent to the system of equations

\[
\partial_+ \left( \Gamma_1^{-1} \partial_- \Gamma_1 \right) = -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + C_{-0} \Gamma_p^{-1} C_{+0} \Gamma_1,
\]

\[
\partial_+ \left( \Gamma_2^{-1} \partial_- \Gamma_2 \right) = -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2,
\]

\[
\vdots
\]

\[
\partial_+ \left( \Gamma_p^{-1} \partial_- \Gamma_p \right) = -\Gamma_p^{-1} C_{+(p-1)} \Gamma_p C_{-(p-1)} + C_{-(p-2)} \Gamma_p^{-1} C_{+(p-2)} \Gamma_p^{-1} \Gamma_{p-1},
\]

\[
\partial_+ \left( \Gamma_p^{-1} \partial_- \Gamma_p \right) = -\Gamma_p^{-1} C_{+1} \Gamma_1 C_{-1} + C_{-0} \Gamma_p^{-1} C_{+0} \Gamma_p.
\]

Here each of \( \Gamma_\alpha \) for \( \alpha = 1, \ldots, p \) is a mapping from the manifold of independent variables \( \mathcal{M} \) to the Lie group \( \text{GL}_{n_\alpha}(\mathbb{C}) \). \( C_{+\alpha} \) and \( C_{-\alpha} \) for each \( \alpha = 1, \ldots, p-1 \) are fixed mappings from \( \mathcal{M} \) to the space of \( n_\alpha \times n_{\alpha+1} \) and \( n_{\alpha+1} \times n_\alpha \) complex matrices respectively. \( C_{+0} \) and \( C_{-0} \) are mappings from \( \mathcal{M} \) to the space of \( n_p \times n_1 \) and \( n_1 \times n_p \) complex matrices respectively. The mappings \( C_{-\alpha} \) and \( C_{+\alpha} \) satisfy the conditions

\[
\partial_+ C_{-\alpha} = 0, \quad \partial_- C_{+\alpha} = 0.
\]

The mappings \( C_{-\alpha} \) and \( C_{+\alpha} \) for the other types of Toda equations should also satisfy these conditions.

To describe the Toda equations of the other types we introduce a set of positive integers \( n_\alpha, \alpha = 1, \ldots, s \). A Toda equation of the second type is equivalent to the system of equations

\[
\partial_+ \left( \Gamma_1^{-1} \partial_- \Gamma_1 \right) = -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + f_1(\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}),
\]

\[
\partial_+ \left( \Gamma_2^{-1} \partial_- \Gamma_2 \right) = -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2,
\]

\[
\vdots
\]

\[
\partial_+ \left( \Gamma_s^{-1} \partial_- \Gamma_s \right) = -\Gamma_s^{-1} C_{+(s-1)} \Gamma_{s-1} C_{-(s-1)} + C_{-(s-2)} \Gamma_{s-2}^{-1} C_{+(s-2)} \Gamma_{s-1},
\]

\[
\partial_+ \left( \Gamma_s^{-1} \partial_- \Gamma_s \right) = -f_2(C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s) + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s.
\]

Here each of \( F_1 \) and \( F_2 \) can be either \( J \) or \( K \). \( \Gamma_{1} \) is a mapping from \( \mathcal{M} \) either to the Lie group \( \text{SO}_{n_1}(\mathbb{C}) \) or to the Lie group \( \text{Sp}_{n_1}(\mathbb{C}) \), subject to the choice of \( F_1 \). Similarly, \( \Gamma_s \) is a mapping from \( \mathcal{M} \) either to the Lie group \( \text{SO}_{n_s}(\mathbb{C}) \) or to the Lie group \( \text{Sp}_{n_s}(\mathbb{C}) \), subject to the choice of \( F_2 \). For each \( \alpha = 2, \ldots, s-1 \) the mapping \( \Gamma_\alpha \) is a mapping from \( \mathcal{M} \) to the Lie group \( \text{GL}_{n_\alpha}(\mathbb{C}) \). \( C_{+\alpha} \) and \( C_{-\alpha} \) for each \( \alpha = 1, \ldots, s-1 \) are fixed mappings from \( \mathcal{M} \) to the space of \( n_\alpha \times n_{\alpha+1} \) and \( n_{\alpha+1} \times n_\alpha \) complex matrices respectively.
A Toda equation of the third type is equivalent to the system of equations

\[
\partial_+ \left( \Gamma_1^{-1} \partial_+ \Gamma_1 \right) = -\Gamma_1^{-1} C_{1+} F_1 C_- + \Gamma_1^{-1} C_{1+} F_1 C_- , \\
\partial_+ \left( \Gamma_2^{-1} \partial_+ \Gamma_2 \right) = -\Gamma_2^{-1} C_{2+} F_3 C_- + C_- \Gamma_2^{-1} C_{2+} F_3 , \\
\vdots \\
\partial_+ \left( \Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1} \right) = -\Gamma_{s-1}^{-1} C_{s+} (s-1) F_s C_- + C_- \Gamma_{s-1}^{-1} C_{s+} (s-1) F_s , \\
\partial_+ \left( \Gamma_s^{-1} \partial_+ \Gamma_s \right) = -\Gamma_s^{-1} C_{s+} (s-1) F_s C_- + C_- \Gamma_s^{-1} C_{s+} (s-1) F_s .
\]

Here \( F \) is either \( J \) or \( K. \) \( \Gamma_1 \) is a mapping from \( \mathcal{M} \) either to the Lie group \( \text{SO}_{n_1}(\mathbb{C}) \) or to the Lie group \( \text{Sp}_{n_1}(\mathbb{C}) \), subject to the choice of \( F. \) For each \( \alpha = 2, \ldots, s \) the mapping \( \Gamma_\alpha \) is a mapping from \( \mathcal{M} \) to the Lie group \( \text{GL}_{n_\alpha}(\mathbb{C}) \). \( C_{\alpha+} \) and \( C_{\alpha-} \) for each \( \alpha = 1, \ldots, s-1 \) are fixed mappings from \( \mathcal{M} \) to the space of \( n_\alpha \times n_{\alpha+1} \) and \( n_{\alpha+1} \times n_\alpha \) complex matrices respectively. The mappings \( C_{s+} \) and \( C_{s-} \) satisfy either the relations \( lC_{s+} = -C_{s+} \) and \( lC_{s-} = -C_{s-} \) or the relations \( lC_{s+} = C_{s+} \) and \( lC_{s-} = C_{s-} \).

A Toda equation of the fourth type is equivalent to the system of equations

\[
\partial_+ \left( \Gamma_1^{-1} \partial_+ \Gamma_1 \right) = -\Gamma_1^{-1} C_{1+} F_1 C_- + C_- \Gamma_1^{-1} C_{1+} F_1 , \\
\partial_+ \left( \Gamma_2^{-1} \partial_+ \Gamma_2 \right) = -\Gamma_2^{-1} C_{2+} F_3 C_- + C_- \Gamma_2^{-1} C_{2+} F_3 , \\
\vdots \\
\partial_+ \left( \Gamma_{s-1}^{-1} \partial_+ \Gamma_{s-1} \right) = -\Gamma_{s-1}^{-1} C_{s+} (s-1) F_s C_- + C_- \Gamma_{s-1}^{-1} C_{s+} (s-1) F_s , \\
\partial_+ \left( \Gamma_s^{-1} \partial_+ \Gamma_s \right) = -\Gamma_s^{-1} C_{s+} (s-1) F_s C_- + C_- \Gamma_s^{-1} C_{s+} (s-1) F_s .
\]

Here for each \( \alpha = 1, \ldots, s \) the mapping \( \Gamma_\alpha \) is a mapping from \( \mathcal{M} \) to the Lie group \( \text{GL}_{n_\alpha}(\mathbb{C}) \). \( C_{\alpha+} \) and \( C_{\alpha-} \) for each \( \alpha = 1, \ldots, s-1 \) are fixed mappings from \( \mathcal{M} \) to the space of \( n_\alpha \times n_{\alpha+1} \) and \( n_{\alpha+1} \times n_\alpha \) complex matrices. \( C_{s+} \) and \( C_{s-} \) are fixed mappings from \( \mathcal{M} \) to the space of \( n_1 \times n_1 \) complex matrices. The mappings \( C_{s+} \) and \( C_{s-} \) satisfy either the relations \( lC_{s+} = -C_{s+} \) and \( lC_{s-} = -C_{s-} \) or the relations \( lC_{s+} = C_{s+} \) and \( lC_{s-} = C_{s-} \). The mappings \( C_{s+} \) and \( C_{s-} \) satisfy either the relations \( lC_{s+} = C_{s+} \) and \( lC_{s-} = C_{s-} \) or the relations \( lC_{s+} = C_{s+} \) and \( lC_{s-} = C_{s-} \).

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A Automorphisms of Lie groups and Lie algebras

Let $G$ be a group. A bijective mapping $a: G \to G$ is called an automorphism of $G$ if

$$a(g_1g_2) = a(g_1)a(g_2)$$

for all $g_1, g_2 \in G$. With respect to the composition operation the set $Aut\ G$ of all automorphisms of the group $G$ is a group called the automorphism group of $G$. When $G$ is a real (complex) Lie group, an automorphism $a$ of $G$ is called a Lie group automorphism if $a$ is a diffeomorphism (biholomorphic mapping). Usually for the case of a Lie group, saying ‘automorphism’ one means ‘Lie group automorphism’, and saying ‘automorphism group’ one means the group formed by all Lie group automorphisms, if the opposite is not stated explicitly.

Let $\mathfrak{g}$ be a Lie algebra. A nonsingular linear mapping $A: \mathfrak{g} \to \mathfrak{g}$ is called an automorphism of $\mathfrak{g}$ if

$$A([x_1, x_2]) = [A(x_1), A(x_2)]$$

for all $x_1, x_2 \in \mathfrak{g}$. With respect to the composition operation the set $Aut\ \mathfrak{g}$ of all automorphisms of $\mathfrak{g}$ is a group called the automorphism group of $\mathfrak{g}$. The group $Aut\ \mathfrak{g}$ is a Lie subgroup of the Lie group $GL(\mathfrak{g})$.

Let $G$ be a Lie group. We identify the Lie algebra $\mathfrak{g}$ of $G$ with the tangent space $T_eG$, where $e$ is the identity element of $G$. Let $a$ be an automorphism of $G$, the differential of the mapping $a$ at $e$ is an automorphism of $\mathfrak{g}$. The mapping $a \in Aut\ G \mapsto a_{e\mathfrak{g}} \in Aut\ \mathfrak{g}$ is a homomorphism. In the case when $G$ is connected this mapping is an injective homomorphism, and when $G$ is simply connected it is an isomorphism.

For any real (complex) connected Lie group $G$ the group $Aut\ G$ can be supplied in a natural way with the structure of a real (complex) Lie group so that the action of $Aut\ G$ on $G$ be smooth (holomorphic).

For an arbitrary element $h$ of a group $G$ the conjugation $C(g): h \in G \mapsto ghg^{-1} \in G$ is an automorphism of $G$. Such an automorphism is said to be an inner automorphism. The corresponding automorphism $C(g)_{e\mathfrak{g}}$ of $\mathfrak{g}$ is denoted $Ad(g)$. Any automorphism of $G$ which is not inner is called an outer automorphism of $G$. The inner automorphisms of $G$ form a subgroup $Inn\ G$ of $Aut\ G$ called the inner automorphism group of $G$. When $G$ is a connected Lie group, the group $Inn\ G$ is a Lie subgroup of $Aut\ G$.

For a general Lie group $G$ the group $Inn\ G$ is a normal subgroup of $Aut\ G$ and one can consider the quotient group $Out\ G = Aut\ G / Inn\ G$. It is customary, slightly abusing language, to call this group the outer automorphism group of $G$. When $G$ is a connected real (complex) Lie group and the group $Inn\ G$ is a closed subgroup of $Aut\ G$, the group $Out\ G$ has the natural structure of a real (complex) Lie group.

A linear mapping $D$ from a Lie algebra $\mathfrak{g}$ to itself is called a derivation of $\mathfrak{g}$ if

$$D([x_1, x_2]) = [D(x_1), x_2] + [x_1, D(x_2)]$$

for all $x_1, x_2 \in \mathfrak{g}$. A linear combination of two derivations of $\mathfrak{g}$ and their commutator are again derivations of $\mathfrak{g}$. Hence, the set of all derivations of $\mathfrak{g}$ is a Lie algebra denoted $Der\ \mathfrak{g}$. The Lie algebra of the Lie group $Aut\ \mathfrak{g}$ is isomorphic to $Der\ \mathfrak{g}$ and usually identified with it.

For any $x \in \mathfrak{g}$ the linear mapping $ad(x): y \in \mathfrak{g} \mapsto [x, y] \in \mathfrak{g}$ is a derivation of $\mathfrak{g}$ called an inner derivation. The set of all inner derivations of $\mathfrak{g}$ is an ideal of $Der\ \mathfrak{g}$. The corresponding connected Lie subgroup of the Lie group $Aut\ \mathfrak{g}$ is called the inner
automorphism group of $g$ and denoted $\text{Inn} g$. An element of $\text{Inn} g$ is called an inner automorphism of $g$. An element of $\text{Aut} g$ which does not belong to $\text{Inn} g$ is said to be an outer automorphism of $g$. The quotient group $\text{Aut} g / \text{Inn} g$ is called the outer automorphism group of $g$.

Let $G$ be a connected Lie group. In this case the restriction of the mapping $a \in \text{Aut} G \mapsto a \ast e \in \text{Aut} g$ to $\text{Inn} G$ is an isomorphism of the Lie groups $\text{Inn} G$ and $\text{Inn} g$. If $G$ is simply connected, the groups $\text{Out} G$ and $\text{Out} g$ are isomorphic.

B Automorphisms of complex simple Lie algebras

Let $g$ be a complex simple Lie algebra of rank $r$. It is well known that any derivation of $g$ is an inner derivation. Therefore, the connected component of the identity of $\text{Aut} g$ coincides with the group $\text{Inn} g$ which in this case is a closed Lie subgroup of $\text{Aut} g$. Here the group $\text{Out} g$ is discrete.

Let $h$ be a Cartan subalgebra of $g$, $\Delta$ be the root system of $g$ with respect to $h$, and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base of $\Delta$. Denote by $(\cdot, \cdot)$ a bilinear form on $h^*$ induced by the restriction of the Killing form of $g$ to $h$. A permutation $\sigma \in S_r$ is called an automorphism of the base $\Pi$ if

$$
\frac{(\alpha_{\sigma(i)}, \alpha_{\sigma(j)})}{(\alpha_{\sigma(i)}, \alpha_{\sigma(j)})} = \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}
$$

for all $i, j = 1, \ldots, r$. All automorphisms of $\Pi$ form a group called the automorphism group of the base $\Pi$ and denoted $\text{Aut} \Pi$. Visually an automorphism of $\Pi$ can be represented as a permutation of the vertices of the corresponding Dynkin diagram which leaves it invariant. Therefore, it is customary to call an automorphism of $\Pi$ an automorphism of the Dynkin diagram of $\Pi$ and the group $\text{Aut} \Pi$ the automorphism group of the Dynkin diagram.

Let $h_i, i = 1, \ldots, r$ be Cartan generators of $h$, $x_{+i}$ and $x_{-i}, i = 1, \ldots, r$ be Chevalley generators of $g$. For any $\sigma \in \text{Aut} \Pi$ there exists a unique automorphism $\Sigma$ of $g$ such that $\Sigma(h_i) = h_{\sigma(i)}$, $\Sigma(x_{+i}) = x_{+\sigma(i)}$ and $\Sigma(x_{-i}) = x_{-\sigma(i)}$ for each $i = 1, \ldots, r$. Hence, one can identify the group $\text{Aut} \Pi$ with a subgroup of $\text{Aut} g$. It can be also shown that

$$
\text{Aut} g = \text{Inn} g \rtimes \text{Aut} \Pi,
$$

see, for example, the monograph [29].

In this paper we are interested in finite order automorphisms of Lie algebras up to conjugations, which can be lifted up to automorphisms of the corresponding Lie groups. Here the following statement appears to be very useful. Any finite order automorphism of $g$ is conjugated by an inner automorphism of $g$ to an automorphism $A$ of the form

$$
A = \exp \text{ad}(x) \circ \Sigma,
$$

where $\Sigma$ is the automorphism of $g$ corresponding to some automorphism $\sigma$ of $\Pi$, and $x$ is an element of $h$ such that $\Sigma(x) = x$, see, for example, [6, Proposition 8.1].
C Complex classical Lie groups and their finite order automorphisms

By complex classical Lie groups we mean the Lie groups \( GL_n(\mathbb{C}) \), \( O_n(\mathbb{C}) \) and \( Sp_{2n}(\mathbb{C}) \). Let us recall the definition of these Lie groups and describe their automorphisms.

C.1 Complex general linear groups

The complex general linear group \( GL_n(\mathbb{C}) \) is formed by all nonsingular complex \( n \times n \) matrices with matrix multiplication as the group law. It is a complex connected Lie group. Identifying a complex \( n \times n \) matrix \( g \) with a linear endomorphism of the vector space \( \mathbb{C}^n \), whose matrix with respect to the standard basis of \( \mathbb{C}^n \) coincides with \( g \), we identify \( GL_n(\mathbb{C}) \) with the group of linear automorphisms of \( \mathbb{C}^n \). The Lie algebra \( gl_n(\mathbb{C}) \) of the Lie group \( GL_n(\mathbb{C}) \) is formed by all complex \( n \times n \) matrices with matrix commutator as the Lie algebra law.

The Lie group \( GL_n(\mathbb{C}) \) and the Lie algebra \( gl_n(\mathbb{C}) \) are not simple. It has a normal subgroup formed by the matrices of the form \( \kappa I \), where \( \kappa \in \mathbb{C} \times \). Actually \( GL_n(\mathbb{C}) \) is isomorphic to the Lie group \( (\mathbb{C}^x \times SL_n(\mathbb{C}))/\mathbb{Z}_n \), where \( SL_n(\mathbb{C}) \) is the complex special linear group formed by the elements of \( GL_n(\mathbb{C}) \) with unit determinant. The Lie algebra \( gl_n(\mathbb{C}) \) is the direct sum of the Lie subalgebra formed by the complex \( n \times n \) matrices proportional to the unit \( n \times n \) matrix and the complex special linear Lie algebra \( sl_n(\mathbb{C}) \) formed by the complex traceless \( n \times n \) matrices. The Lie group \( SL_n(\mathbb{C}) \) is connected and simple. The Lie algebra \( sl_n(\mathbb{C}) \) is also simple.

The rank \( r \) of the Lie algebra \( sl_n(\mathbb{C}) \) is equal to \( n - 1 \). The corresponding Dynkin diagram is of the form

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & r - 1 & r
\end{array}
\]

It is clear that there is only one nontrivial automorphism of the Dynkin diagram of \( sl_n(\mathbb{C}) \) given by the equality

\[
\sigma(i) = r - i + 1.
\] (C.1)

Denote by \( D_n(\mathbb{C}) \) the Lie group formed by all nonsingular complex diagonal \( n \times n \) matrices, and by \( d_n(\mathbb{C}) \) its Lie algebra. Choose the Lie algebra \( sl_n(\mathbb{C}) \cap d_n(\mathbb{C}) \) as a Cartan subalgebra of \( sl_n(\mathbb{C}) \). In this case the Cartan and Chevalley generators can be chosen as

\[
h_i = e_{ii} - e_{i+1,j+1}, \quad x_{+i} = e_{i,j+1}, \quad x_{-i} = e_{i+1,j}.
\]

Here and below \( e_{ij} \) is the matrix with the matrix elements

\[
(e_{ij})_{kl} = \delta_{ik}\delta_{jl}.
\]

One can verify that with this choice of Cartan and Chevalley generators the automorphism \( \Sigma \) defined as

\[
\Sigma(x) = -d^t x d^{-1},
\]

where \( d \) is an element of \( SL_n(\mathbb{C}) \cap D_n(\mathbb{C}) \) such that

\[
d x_{+i} d^{-1} = -x_{+i}, \quad d x_{-i} d^{-1} = -x_{-i}
\]

for each \( i = 1, \ldots, r \), is an automorphism of \( sl_n(\mathbb{C}) \) induced by the automorphism \( \sigma \) of the Dynkin diagram of \( sl_n(\mathbb{C}) \) defined by the equality (C.1). It is convenient to assume that \( d d^t = I \).
Thus, a finite order inner automorphism of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{sl}_n(\mathbb{C})$ to an automorphism $A$ given by the equality

$$A(x) = h x h^{-1},$$

(C.2)

where $h$ is an element of $\text{SL}_n(\mathbb{C}) \cap D_n(\mathbb{C})$, and a finite order outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{sl}_n(\mathbb{C})$ to an automorphism $A$ given by the equality

$$A(x) = -h' d x d^{-1} h'^{-1},$$

where $h'$ is an element of $\text{SL}_n(\mathbb{C}) \cap D_n(\mathbb{C})$ satisfying the relation

$$d ^{h'} d^{-1} = h'^{-1}.$$

Denoting $h = h'd$, we see that a finite order outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{sl}_n(\mathbb{C})$ to an automorphism $A$ given by the equality

$$A(x) = -h ^{d} x h^{-1},$$

where $h$ is an element of $\text{SL}_n(\mathbb{C}) \cap D_n(\mathbb{C})$ satisfying the relation $h h = h^{-1}$.

Now it is not difficult to understand that a finite order inner automorphism of the Lie group $\text{SL}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{SL}_n(\mathbb{C})$ to an automorphism $a$ given by the equality

$$a(g) = h g h^{-1},$$

(C.3)

where $h$ is an element of $\text{SL}_n(\mathbb{C}) \cap D_n(\mathbb{C})$, and a finite order outer automorphism of $\text{SL}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{SL}_n(\mathbb{C})$ to an automorphism $a$ given by the equality

$$a(g) = h ^{1} (g ^{-1}) h^{-1},$$

(C.4)

where $h$ is an element of $\text{SL}_n(\mathbb{C}) \cap D_n(\mathbb{C})$ satisfying the relation $h h = h^{-1}$.

One can easily extend the automorphisms of $\text{SL}_n(\mathbb{C})$ described by the relations (C.3) and (C.4) to automorphisms of $\text{GL}_n(\mathbb{C})$ respectively. It can be shown that any finite order inner automorphism of $\text{GL}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{GL}_n(\mathbb{C})$ to an automorphism defined by the relation (C.3), and any finite order outer automorphism of $\text{GL}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{GL}_n(\mathbb{C})$ to an automorphism defined by the relation (C.4).

The other complex classical Lie groups can be defined as Lie subgroups of the Lie group $\text{GL}_n(\mathbb{C})$ via the following general procedure. Let $B$ be a complex nonsingular $n \times n$ matrix. It is not difficult to get convinced that the elements $g$ of $\text{GL}_n(\mathbb{C})$ satisfying the condition

$$B g = g ^{-1}$$

form a Lie subgroup of $\text{GL}_n(\mathbb{C})$ which we denote by $\text{GL}_n^B(\mathbb{C})$. One can get convinced that the Lie algebra $\mathfrak{gl}_n^B(\mathbb{C})$ of $\text{GL}_n^B(\mathbb{C})$ is a subalgebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ formed by the complex $n \times n$ matrices $x$ satisfying the condition

$$B x = -x.$$
C.2 Complex orthogonal groups

For any symmetric nonsingular $n \times n$ matrix $B$ the Lie group $GL_B^\mathbb{C}(\mathbb{C})$ is isomorphic to the Lie group $GL_n^\mathbb{C}(\mathbb{C})$. This group is called the complex orthogonal group and denoted $O_n(\mathbb{C})$. For an element $g \in O_n(\mathbb{C})$ from the equality $^t g = g^{-1}$ we obtain that $\det g$ is equal either to 1, or to $-1$. The elements of $O_n(\mathbb{C})$ with unit determinant form a connected Lie subgroup of $O_n(\mathbb{C})$ called the complex special orthogonal group and denoted $SO_n(\mathbb{C})$. This subgroup is the connected component of the identity of $O_n(\mathbb{C})$. The Lie algebra of $SO_n(\mathbb{C})$ is denoted $so_n(\mathbb{C})$. It is clear that the Lie algebra of $O_n(\mathbb{C})$ coincides with the Lie algebra of $SO_n(\mathbb{C})$. The Lie group $SO_n(\mathbb{C})$ and the Lie algebra $so_n(\mathbb{C})$ are simple.

For an odd $n = 2r + 1$ the rank of the Lie algebra $so_n(\mathbb{C})$ is equal to $r$ and the corresponding Dynkin diagram has the form

$$
\begin{array}{cccccccc}
1 & 2 & \cdots & r-1 & r
\end{array}
$$

Its automorphism group is trivial, and any automorphism of $so_{2r+1}(\mathbb{C})$ is an inner automorphism. One can choose the Lie algebra $so_{2r+1}(\mathbb{C}) \cap d_{2r+1}(\mathbb{C})$ as a Cartan subalgebra of $so_{2r+1}(\mathbb{C})$. Therefore, for an odd $n$ any finite order automorphism of $so_n(\mathbb{C})$ is conjugated by an inner automorphism of $so_n(\mathbb{C})$ to an automorphism given by the relation (C.2) with $h \in SO_n(\mathbb{C}) \cap D_n(\mathbb{C})$. Any finite order automorphism of $SO_n(\mathbb{C})$ for an odd $n$ is conjugated by an inner automorphism of $SO_n(\mathbb{C})$ to an automorphism given by the relation (C.3) with $h \in SO_n(\mathbb{C}) \cap D_n(\mathbb{C})$.

For an even $n = 2r$ the rank of the Lie algebra $so_n(\mathbb{C})$ is equal to $r$ and the corresponding Dynkin diagram is

$$
\begin{array}{cccccccc}
1 & 2 & \cdots & r-1 & r
\end{array}
$$

For $r \neq 4$ the only nontrivial automorphism of this Dynkin diagram is given by the equalities

$$
\sigma(i) = i, \quad i = 1, \ldots, r-2, \quad \sigma(r-1) = r, \quad \sigma(r) = r - 1. \quad \text{(C.5)}
$$

When $r = 4$ the group of automorphisms of the Dynkin diagram consists of all permutations of the vertices with the numbers 1, 3 and 4. However, the automorphisms of the Lie algebra $so_8(\mathbb{C})$ corresponding to these additional automorphisms of the Dynkin diagram cannot be lifted up to automorphisms of the Lie group $SO_8(\mathbb{C})$ and we do not consider them here.

Choose the Lie algebra $so_{2r}(\mathbb{C}) \cap d_{2r}(\mathbb{C})$ as a Cartan subalgebra of $so_{2r}(\mathbb{C})$. In this case the Cartan and Chevalley generators can be chosen as

$$
\begin{align*}
h_i &= e_{ii} - e_{i+1,i+1} + e_{2r-i,2r-i} - e_{2r-i+1,2r-i+1}, & i = 1, \ldots, r - 1, \\
h_r &= e_{r-1,r-1} + e_{r,r} - e_{r+1,r+1} - e_{r+2,r+2}, \\
x_{+i} &= e_{i,i+1} - e_{2r-i,2r-i+1}, & i = 1, \ldots, r - 1, \\
x_{+r} &= e_{r-1,r-1} - e_{r,r+2}, \\
x_{-i} &= e_{i+1,i} - e_{2r-i+1,2r-i}, & i = 1, \ldots, r - 1, \\
x_{-r} &= e_{r+1,r-1} - e_{r+2,r}.
\end{align*}
$$

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Now one can get convinced that the linear mapping $\Sigma$ defined by the relation

$$\Sigma(x) = u x u^{-1},$$

where

$$u = \begin{pmatrix} I_{r-1} & 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ I_{r-1} \end{pmatrix},$$

is an automorphism of $\mathfrak{so}_{2r}(\mathbb{C})$ induced by the automorphism $\sigma$ of the Dynkin diagram of $\mathfrak{so}_{2r}(\mathbb{C})$ defined by the equalities (C.5).

Thus, for an even $n$ not equal to 8 a finite order inner automorphism of the Lie algebra $\mathfrak{so}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{so}_n(\mathbb{C})$ to an automorphism $A$ given by the equality (C.2), where $h$ is an element of $\text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$, and a finite order outer automorphism of $\mathfrak{so}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{so}_n(\mathbb{C})$ to an automorphism $A$ given by the equality

$$A(x) = (hu) x (hu)^{-1},$$

where $h$ is an element of $\text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$ satisfying the relation $u h u^{-1} = h$. In the case of $n = 8$ there are finite order outer automorphisms of $\mathfrak{so}_8(\mathbb{C})$ different from the automorphisms defined by the relation (C.7) but they cannot be lifted to automorphisms of $\text{SO}_8(\mathbb{C})$.

It is clear now that for an even $n$ a finite order inner automorphism of the Lie group $\text{SO}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{SO}_n(\mathbb{C})$ to an automorphism $a$ given by the equality (C.3), where $h$ is an element of $\text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$, and a finite order outer automorphism of $\text{SO}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{SO}_n(\mathbb{C})$ to an automorphism $a$ given by the equality

$$a(g) = (hu) g (hu)^{-1},$$

where $u$ is given by (C.6) and $h$ is an element of $\text{SO}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$ satisfying the relation $u h u^{-1} = h$.

### C.3 Complex symplectic groups

Let for an even $n$ the $n \times n$ matrix $B$ be skew-symmetric and nonsingular. It can be shown that in this case the Lie group $\text{GL}^B_n(\mathbb{C})$ is isomorphic to the Lie group $\text{GL}^K_n(\mathbb{C})$. This group is called the complex symplectic group and denoted $\text{Sp}_n(\mathbb{C})$. The Lie group $\text{Sp}_n(\mathbb{C})$ is connected and simple. The corresponding Lie algebra $\mathfrak{sp}_n(\mathbb{C})$ is also simple.

The rank of the Lie algebra $\mathfrak{sp}_n(\mathbb{C})$ is equal to $r = n/2$ and the corresponding Dynkin diagram is of the form

$$\begin{align*}
1 & \quad 2 \quad \cdots \quad r-1 \quad r
\end{align*}$$

The automorphism group of this diagram is trivial, and any automorphism of $\mathfrak{sp}_n(\mathbb{C})$ is an inner automorphism. One can choose the Lie algebra $\mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{d}_n(\mathbb{C})$ as a Cartan subalgebra of $\mathfrak{sp}_n(\mathbb{C})$. Therefore, any finite order automorphism of $\mathfrak{sp}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\mathfrak{sp}_n(\mathbb{C})$ to an automorphism given by the relation (C.2) with $h \in \text{Sp}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$. Any finite order automorphism of $\text{Sp}_n(\mathbb{C})$ is conjugated by an inner automorphism of $\text{Sp}_n(\mathbb{C})$ to an automorphism given by the relation (C.3) with $h \in \text{Sp}_n(\mathbb{C}) \cap \text{D}_n(\mathbb{C})$. 

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