Iterating Random Functions on a Finite Set

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Abstract

Choose random functions $f_1, f_2, f_3, \ldots$ independently and uniformly from among the $n^n$ functions from $[n]$ into $[n]$. For $t > 1$, let $g_t = f_t \circ f_{t-1} \circ \cdots \circ f_1$ be the composition of the first $t$ functions, and let $T$ be the smallest $t$ for which $g_t$ is constant (i.e. $g_t(i) = g_t(j)$ for all $i, j$). We prove that, for any positive real number $x$,

$$
\lim_{n \to \infty} \Pr\left( \frac{T}{n} \leq x \right) = \frac{x}{0} \int f(y) dy,
$$

where

$$
f(y) = \sum_{k \geq 2} (-1)^{k} e^{-y(k)} (2k - 1) \binom{k}{2}.
$$

We make our proof available here, but acknowledge that the result is already “well known.”
1 Introduction

Let \( f_1, f_2, f_3, \ldots \) be a sequence of functions chosen independently and uniformly randomly from the \( n^n \) functions on \([n]\). Let \( g_1 = f_1 \), and for \( t > 1 \) let \( g_t = f_t \circ g_{t-1} \) be the composition of the first \( t \) random functions. Define \( T(\langle f_i \rangle_{i=1}^{\infty}) \) to be the smallest \( t \) for which \( g_t \) is a constant function. (i.e. \( g_t(i) = g_t(j) \) for all \( i \neq j \).) This manuscript contains a simple derivation of the asymptotic distribution of \( T \). We had originally intended to publish it in a journal, but we recently learned that the asymptotic distribution of \( T \) is “well known”. It was apparently known to Kingman twenty years ago \([7],[8],[9] \), and stronger results are fully proved in Donnelly\([3] \). There is a lot of related work by M˝ohle and others, e.g. \([11],[12],[5] \).

For \( m > 1 \), let \( \tau_m = |\{ t : |\text{Range}(g_t)| = m \}| \) be the the number of iterates for which the range has exactly \( m \) elements. Thus \( T = \sum_{m=2}^{\infty} \tau_m \). The random variables \( \{\tau_m\}_{m=2}^{\infty} \) are not independent. They are however conditionally independent once we specify the set of visited states. Fortunately this set is well behaved, has some convenient properties that enable us to do computations.

Let \( \xi = \lfloor \log \log n \rfloor \), and decompose \( T \) as \( T = T_1 + T_2 \), where \( T_1 = \sum_{m=2}^{\xi} \tau_m \) and \( T_2 = \sum_{m=\xi+1}^{n} \tau_m \). Let \( \mathcal{A} = \bigcap_{m=1}^{\xi} [\tau_m > 0] \). The following facts from \([2] \) will be needed (See also Theorem 5 of \([3] \)):

**Theorem 1** \( \Pr(\mathcal{A}) = 1 - o(1) \), and \( E(T_2) = o(n) \).

2 Characteristic Function

Let \( \lambda_k = \prod_{j=1}^{k-1} (1 - \frac{j}{n}) \). Then we have

**Theorem 2** \( E(e^{itT_1}|\mathcal{A}) = e^{it(\xi-1)} \prod_{k=2}^{\xi} \frac{(1-\lambda_k)}{-\lambda_k e^{it}} \).

**Proof:**

Suppose \( g_{t-1} \) has an \( m \) element range \( R = \{r_1, r_2, \ldots, r_m\} \). What is the chance that the next function \( g_t \) still has an \( m \) element range? On \( R \) we have \( n \) choices for \( f_t(r_1) \), then \( n-1 \) choices for \( f_t(r_2) \) etc. For \( x \notin R \), \( f_t(x) \) can be chosen arbitrarily. Hence the number of functions \( f_t \) for which \( g_t = f_t \circ g_{t-1} \) has an an \( m \) element range is

2
\[ n^{n-m} \prod_{j=0}^{m-1} (n-j). \text{ Hence} \]

\[ \Pr(\tau_m = k|\tau_m > 0) = \lambda_m^{k-1}(1 - \lambda_m), \quad (1) \]

and consequently

\[ E(e^{i\tau_m}|\tau_m > 0) = \sum_{k=1}^{\infty} \lambda_m^{k-1}(1 - \lambda_m) e^{ikt} = \frac{(1 - \lambda_m)e^{it}}{1 - \lambda_m e^{it}}. \]

Now let \( \phi_n(t) = E(e^{iT_1/n}|A) \) be the characteristic function of the normalized random variable \( T_1/n \) on \( A \). Then the following corollary follows immediately from Theorem 2.

**Corollary 3** \( \phi_n(t) = \prod_{m=2}^{\xi} \frac{(1 - \lambda_m)e^{it/n}}{1 - \lambda_m e^{it/n}} = e^{it(\xi-1)/n} \prod_{m=2}^{\xi} \frac{(1 - \lambda_m)}{1 - \lambda_m e^{it/n}} \)

**Lemma 4** \( \phi_n(t) = \sum_{m=2}^{\infty} \frac{(\xi)}{(\xi)} - it + o(1). \)

**Proof:** Note that, for \( m \leq \xi, \)

\[ 1 - \lambda_m = \frac{1}{n} \left( \begin{array}{c} m \\ 2 \end{array} \right) + O(\xi^4/n^2) \quad (2) \]

and

\[ 1 - \lambda_m e^{it/n} = \frac{1}{n} \left( \begin{array}{c} m \\ 2 \end{array} \right) - it + O(\xi^4/n^2) \quad (3) \]

Therefore

\[ \frac{(1 - \lambda_m)e^{it/n}}{1 - \lambda_m e^{it/n}} = \frac{\left( \begin{array}{c} m \\ 2 \end{array} \right) + O(\xi^4/n^2)}{(\xi)} - it + O(\xi^4/n^2) = \frac{\left( \begin{array}{c} m \\ 2 \end{array} \right) - it}{(\xi)}(1 + O(\xi^4/n^2)). \quad (4) \]

Therefore

\[ \phi_n(t) = (1 + O(\xi^4/n)) \prod_{m=2}^{\xi} \frac{\left( \begin{array}{c} m \\ 2 \end{array} \right)}{(\xi) - it} \]

\[ = (1 + o(1)) \prod_{m=2}^{\xi} \frac{\left( \begin{array}{c} m \\ 2 \end{array} \right)}{(\xi) - it}. \]

Finally, note that the infinite product \( \prod_{m=2}^{\infty} \frac{\left( \begin{array}{c} m \\ 2 \end{array} \right)}{(\xi) - it} = \prod_{m=2}^{\infty} 1 - \frac{1}{(\xi)} \) converges since \( \sum (\xi) \) is convergent.
3 Simplification

To facilitate inversion, we reexpress the characteristic function $\phi$ in a more convenient form. Working with the reciprocal, we have

$$1 \phi_n(t) + o(1) = \prod_{k \geq 1} (1 - \frac{2it}{k(k+1)}) = \prod_{k \geq 1} \frac{(k - \alpha)(k - \beta)}{k(k+1)},$$

where $\alpha = \frac{-1 - \sqrt{1+8it}}{2}$, $\beta = \frac{-1 + \sqrt{1+8it}}{2}$. It is well known [4] that

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n \geq 1} (1 + \frac{z}{n} e^{-z/n}).$$

Since $\alpha + \beta = -1$, the right side of equation (5) becomes

$$\prod_{k \geq 1} (1 - \frac{\alpha}{k}) e^{\frac{\alpha}{k}(1 - \frac{\beta}{k})} = \frac{1}{\alpha \beta \Gamma(-\alpha) \Gamma(-\beta)} = \frac{\cos(\frac{\pi}{2} \sqrt{1 + 8it})}{-2\pi it}.$$ 

Hence

$$\phi_n(t) = \frac{-2\pi it}{\cos(\frac{\pi}{2} \sqrt{1 + 8it})} + o(1).$$

4 Fourier Inversion

Inverting, we get the conditional density function $f_n$ for $T_1/n$:

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{-2\pi it}{\cos(\frac{\pi}{2} \sqrt{1 + 8it})} + o(1) dt.$$ 

Note that $\frac{-2\pi it}{\cos(\frac{\pi}{2} \sqrt{1 + 8it})}$ has simple poles at $t = -i\frac{k}{2}$, for $k = 2, 3, 4, \ldots$

Since the residue at $t = -i\frac{k}{2}$ is $i(-1)^k (2k - 1) \frac{(k)}{2} e^{-k^2 x}$, contour integration yields, for $x > 0$, $f_n(x) = f(x) + o(1)$ where

$$f(x) = \sum_{k \geq 2} (-1)^k e^{-\frac{k}{2}} \binom{k}{2} (2k - 1).$$

5 Main Result

For $x > 0$, let $F(x) = \int_0^x f(t) dt$. Our main result is
**Theorem 5** For any $x > 0$, $\lim_{n \to \infty} \Pr(T/n \leq x) = F(x)$.

**Proof:** For any $x$,
\[
\Pr(T/n \leq x) \leq \Pr(T_1/n \leq x)
= \Pr(T_1/n \leq x | A) \Pr(A) + \Pr(T_1/n \leq x | A^c) \Pr(A^c)
= \Pr(T_1/n \leq x | A)(1 + o(1)) + o(1)
= F(x)(1 + o(1)) + o(1).
\]

In the other direction, let $\epsilon$ be a fixed but arbitrarily small positive number. Then
\[
\Pr(T/n \leq x) \geq \Pr(T_1/n \leq x - \epsilon \text{ and } T_2/n \leq \epsilon)
\geq \Pr(T_1/n \leq x - \epsilon) - \Pr(T_2/n > \epsilon)
\geq \Pr(T_1/n \leq x - \epsilon) - \frac{E(T_2/n)}{\epsilon}
= F(x - \epsilon) + o(1).
\]

The theorem follows from this and the fact that $F$ is continuous. ■

6 Discussion

Although the ultimate behaviour of our chain is like Kingman’s coalescent [7], there are differences. In that process every state is visited, whereas in our process few of the high numbered states are visited.

Let $N = \sum_{m=2}^{n} I_{[\tau_m > 0]}$, the number of states visited. In an earlier version of this manuscript, we conjectured that $E(N) \sim \sqrt{2\pi n}$. Robin Pemantle recently proved our conjecture and the corresponding central limit theorem. He may also be able to prove stronger and more general versions of this result, e.g. a functional limit theorem. [14].
Acknowledgement There is a large literature in applied probability that can be traced back to Kingman’s work, and it was not immediately obvious to us what is relevant. We are grateful to Simon Tavare for pointing our way to the work of Donnelly and Möhle.

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