THE DIRICHLET PRINCIPLE FOR THE COMPLEX k-HESSIAN FUNCTIONAL

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ABSTRACT. We study the variational structure of the complex k-Hessian equation on bounded domain $X \subset \mathbb{C}^n$ with boundary $M = \partial X$. We prove that the Dirichlet problem $\sigma_k(\partial \bar{\partial} u) = 0$ in $X$, and $u = f$ on $M$ is variational and we give an explicit construction of the associated functional $\mathcal{E}_k(u)$. Moreover we prove $\mathcal{E}_k(u)$ satisfies the Dirichlet principle. In a special case when $k = 2$, our constructed functional $\mathcal{E}_2(u)$ involves the Hermitian mean curvature of the boundary, the notion first introduced and studied by X. Wang [34]. Earlier work of J. Case and and the first author of this article [7] introduced a boundary operator for the (real) $k$-Hessian functional which satisfies the Dirichlet principle. The present paper shows that there is a parallel picture in the complex setting.

1. Introduction

Let $X \subset \mathbb{C}^n$ be a bounded smooth domain with boundary $M = \partial X$. The usual Dirichlet principle states that

\begin{equation}
-\int_X u \Delta u \, dx + \oint_M f u \nu \, d\mu \geq \oint_M f (u_f) \nu \, d\mu
\end{equation}

for all $f \in C^\infty(M)$ and all $u \in C^\infty(X)$ such that $u|_M = f$. Here $u_\nu$ denotes the derivative of $u$ with respect to the unit outward normal vector $\nu$ along $M$, $u_f$ is the harmonic function in $X$ such that $u_f|_M = f$, and $dx$, $d\mu$ are the volume forms on $X$ and $M$, respectively. A standard density argument implies that the trace $u \mapsto u|_M := \text{tr} u$ extends to a bounded linear operator $\text{tr} : W^{1,2}(\mathbb{X}) \rightarrow W^{1/2,2}(M)$, while the extension $f \mapsto u_f := E(f)$ extends to a bounded linear operator $E : W^{1/2,2}(M) \rightarrow W^{1,2}(\mathbb{X})$ such that $\text{tr} \circ E$ is the identity.

The Dirichlet principle is a useful tool in many analytic and geometric problems. The initial observation is in regard to the Dirichlet-to-Neumann map $f \mapsto (u_f)_\nu$, a pseudodifferential operator of principle symbol $(-\Delta)^{1/2}$. When $X = \mathbb{R}^n_+$ is the upper half-space, it is the operator $(-\Delta)^{1/2}$. Thus (1.1) relates the energy of the local operator $\Delta$ on $X$ to the energy of the nonlocal Dirichlet-to-Neumann operator, providing a useful tool for establishing estimates of PDEs stated in terms of the latter operator. This strategy is a key motivation for the approach of Caffarelli and Silvestre [3] to study fractional powers of the Laplacian. As another example, Escobar [16, 17] proved an analogue of (1.1) on compact manifolds with boundary and used it to recover a sharp Sobolev-trace inequality when $X = \mathbb{R}^n_+$. It thus leads to the embedding $W^{1,2}(\mathbb{R}^n_+) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^{n-1})$ when $n \geq 3$. This work is important to studying the boundary Yamabe problem [18]. By considering weights or higher-order operators, Caffarelli and Silvestre [3] and R. Yang [35] established analogues of (1.1) for the energy of fractional powers of the Laplacian. Later, the work of [4, 5, 9, 10] established analogues of (1.1) for the energy of the conformal fractional Laplacian (the GJMS operators).

In [7], Case and Wang established a Dirichlet principle for the fully nonlinear operator $\sigma_k(D^2 u)$, where $D^2 u$ denotes the Hessian of $u$ on $\mathbb{R}^n$. Later, they [6] also developed this idea to study
the $k$-curvature, the $k$-th elementary symmetric function of the Schouten tensor, on manifolds for $k = 1, 2$ or when $g$ is a locally conformally flat metric. The purpose of this article is to study if the complex $k$-Hessian energy $\sigma_k(D^{1,1}u)$ on $\mathbb{C}^n$ also satisfies the Dirichlet principle and what functional gives rise to it.

To present the result, we first introduce some notations. In this paper, $D^{1,1}u$ denotes the complex Hessian of $u$, and the $k$-th elementary symmetric function $\sigma_k(A)$ of a Hermitian matrix $A$ (i.e. $A^\dagger = A$) is defined by

$$\sigma_k(A) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

for $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $A$. The complex $k$-Hessian equation with Dirichlet boundary condition

$$\begin{cases} 
\sigma_k(D^{1,1}u) = F(x, u), & \text{in } X, \\
u = f(x), & \text{on } M
\end{cases}$$

has been well-studied for functions $u$ in the elliptic $k$-cone

$$\Gamma_k^+ := \{ u \in C^\infty(X) \mid \sigma_j(D^{1,1}u) > 0, 1 \leq j \leq k \}.$$  

Note that the existence of a solution to (1.2) requires that $M$ is $(k - 1)$-pseudoconvex; i.e. the Levi form $\mathcal{L}$ of $M$ must satisfy $\sigma_j(\mathcal{L}) > 0$ for $1 \leq j \leq k - 1$. Also, in the degenerate case where $F \geq 0$, one needs to consider solutions $u \in \Gamma_k^+$, where the closure of the elliptic $k$-cone (1.3) is with respect to the $C^{1,1}$-norm in $\mathbb{X}$. (1.2) is a fully nonlinear analogue of (1.1), and is a generalization of complex Monge-Ampère equations (where $k = n$).

For real Monge-Ampère equation as well as the $k$-Hessian equations, the existence of a unique classical solution for the Dirichlet problem on a domain in $\mathbb{R}^n$ was proved by Caffarelli, Nirenberg and Spruck in [2]. Many other results related to (real) geometric problems were studied by Urbas [33], Guan and Li [23], Guan and Guan [22], Chang, Gursky and Yang [8], Guan and Ma [25], Guan, Lin and Ma [24], Guan [21], and references therein. With regard to the complex Monge-Ampère equations, the existence of plurisubharmonic solution on strictly pseudoconvex domain was proved by Caffarelli, Kohn, Nirenberg and Spruck [2]. In the meanwhile, existence and regularity of solutions to the degenerate $(F(x, u) \geq 0)$ Monge-Ampère equation were due to the work of Krylov [27, 30, 28, 29]. And it is generally believed that Krylov’s method could also prove the same results for degenerate complex $k$-Hessian equations, though we could not find a good reference. (See also [36]). There are many important results on complex Hessian equations on $\mathbb{C}^n$ and Kähler manifolds, e.g. [11], [20], [26], [31]. Recently the results of [32, 12, 14, 13, 15] have made a lot of new developments on degenerate complex Hessian equations on compact Kähler manifolds as well.

To further understand equation (1.2), it is natural to study the variational structure of equation (1.2). Namely, we would like to establish a fully nonlinear analogue of (1.1). There are two questions that arise here. First, is there a functional whose critical points satisfy (1.2)? And second, if the answer to the first question is yes, does this functional satisfy the Dirichlet principle? The main result of this paper is that the answers to both of these questions are affirmative.

Let

$$\mathcal{C}_f := \{ u \in C^\infty(X) \mid u|_M = f \}$$

be the class of functions with fixed trace $f \in C^\infty(M)$. Our contribution is the following Dirichlet’s principle for such solutions:
Theorem 1.1. Fix $k \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be a bounded $(k-1)$-pseudoconvex domain with boundary $M = \partial X$. Given $f \in C^\infty(M)$, let

$$C_{f,k} := \{ u \in C_f \mid D^{1,1}u \in \Gamma^+_k \}.$$ 

There is a functional $\mathcal{E}_{k+1} : C^2(X) \cap C^2(M) \cap C^1(\overline{X}) \to \mathbb{R}$ such that every $u \in \overline{C_{f,k}}$ satisfies

$$(1.4) \quad \mathcal{E}_{k+1}(u) \geq \mathcal{E}_{k+1}(u_f)$$

where $u_f$ is the unique solution to the Dirichlet problem

$$(1.5) \quad \begin{cases} \sigma_k(D^{1,1}u) = 0, & \text{in } X, \\ u = f, & \text{on } M, \end{cases}$$

and $\overline{C_{f,k}}$ is the closure of $C_{f,k}$ with respect to the $C^{1,1}$-norm in $\overline{X}$.

The way we prove the existence of $\mathcal{E}_k$ is by explicitly constructing it using an induction argument.

For any $1 \leq k \leq n$, let $\sigma_k(\cdot, \ldots, \cdot)$ be the polarization of the $k$-linear map $u \to \sigma_k(D^{1,1}u)$ and let $L|_{\mathcal{H}} : T^{1,0}M \times T^{0,1}M \to \mathbb{C}$ be the restriction of the second fundamental form $L$ to $T^{1,0}M \times T^{0,1}M$.

Then we prove that the functional defined below satisfies the Dirichlet principle (1.4):

$$(1.6) \quad \mathcal{E}_{k+1}(u) := - \int_X u\sigma_k(D^{1,1}u)dx + \frac{1}{k^2(k+1)} \sum_{i=2}^{k+1} S_i(u),$$

where each $S_i$ is a functional on $C^2(X) \cap C^2(M) \cap C^1(\overline{X})$ defined as

$$(1.7) \quad S_i(u) := (-1)^i \frac{k(k+1)}{2} \binom{k}{i-1} \int_M w u^i \sigma_{k-1}(L|_{\mathcal{H}}, \ldots, L|_{\mathcal{H}}, D^{1,1}u|_{\mathcal{H}}, \ldots, D^{1,1}u|_{\mathcal{H}}).$$

In the above formula, $D^{1,1}u|_{\mathcal{H}} : T^{1,0}M \times T^{0,1}M \to \mathbb{C}$ is the restriction of the complex Hessian $D^{1,1}u$ to $T^{1,0}M \times T^{0,1}M$.

In particular, when $k = 1$, we obtain $\mathcal{E}_2(u) = \frac{1}{2} \int_M w u^2 d\mu$, which recovers (1.1). When $k = 2$, we can further simplify the functional $\mathcal{E}_3(u)$ into:

$$(1.8) \quad \mathcal{E}_3(u) = - \int_X u \sigma_2(D^{1,1}u) + \frac{1}{2} \int_M w u^2 \Delta_b u + \frac{1}{8} \int_M w^2 H_b - \frac{\sqrt{-1}}{2} \int_M w u L(\nabla^{1,0}u - \nabla^{0,1}u, T),$$

where $\Delta_b$ is the sub-Laplacian on $M$ and $H_b$ is the Hermitian mean curvature of $M$ introduced in [34]. Here $\nabla u$ denotes the gradient vector field of $u$ on $M$. $\nabla^{1,0}u$ and $\nabla^{0,1}u$ are the projection of $\nabla u$ onto $T^{1,0}M$ and $T^{0,1}M$ respectively. Let $\nu$ be the unit outward normal vector along $M$ and $T := J\nu$ where $J$ is the complex structure on $\mathbb{C}^n$.

The detailed argument to construct $\mathcal{E}_{k+1}$ can be found in Section 4. It is deduced from the following technical result.

Proposition 1.2. Fix $k \in \mathbb{N}$ and let $X \subset \mathbb{C}^n$ be a bounded smooth domain with boundary $M = \partial X$. Then there is a multilinear differential operator

$$(1.9) \quad B_k : C^\infty(\overline{X})^k \to C^\infty(M)$$

such that the multilinear form $L_{k+1} : C^\infty(\overline{X})^{k+1} \to \mathbb{R}$ defined by

$$(1.10) \quad L_{k+1}(u, w^1, \ldots, w^k) := - \int_X u \sigma_k(D^{1,1}w^1, \ldots, D^{1,1}w^k)dx + \int_M u B_k(w^1, \ldots, w^k) d\mu$$

is symmetric, where $\sigma_k(D^{1,1}w^1, \ldots, D^{1,1}w^k)$ is the polarization of the $k$-linear map $w \to \sigma_k(D^{1,1}w)$. 
Let us explain briefly why Theorem 1.1 follows from Proposition 1.2. The energy functional in Theorem 1.1 is actually defined as \( \mathcal{E}_{k+1}(u) := L_{k+1}(u, \ldots, u) \). The fact that (1.10) defines a symmetric \((k+1)\)-linear form implies that if \( v \in C^\infty(X) \) is such that \( v|_M = 0 \), then

\[
\left. \frac{d}{dt} \right|_{t=0} \mathcal{E}_k(u + tv) = -\frac{(k+1)!}{(k+1-j)!} \int_X v \sigma_k(D^{1,1}v, \ldots, D^{1,1}v, D^{1,1}u, \ldots, D^{1,1}u) dx,
\]

for all \( 1 \leq j \leq k+1 \). That is, within the class \( \mathcal{C}_f \), the derivatives of the energies \( \mathcal{E}_k \) depend only on the interior integrals. In particular, it is straightforward to identify the critical points of \( \mathcal{E}_k \) and deduce the convexity of \( \mathcal{E}_k \) within the positive cone \( \Gamma_+^k \), from which Theorem 1.1 follows readily.

Given \( u \in C^2(X) \cap C^2(M) \cap C^1(\overline{X}) \) and \( k \in \mathbb{N} \), we set

\[
Q_k(u) := \frac{1}{2k} \sum_{i=2}^{k+1} (-1)^i \binom{k}{i-1} u_{i-1} \sigma_{k-1}(L|_{\mathcal{H}}, \ldots, L|_{\mathcal{H}}, D^{1,1}u|_{\mathcal{H}}, \ldots, D^{1,1}u|_{\mathcal{H}}).
\]

(1.11) and (1.6) and (1.7) imply that

\[
\mathcal{E}_{k+1}(u) = -\int_X u \sigma_k(D^{1,1}u) dx + \oint_M u Q_k(u) d\mu.
\]

Given \( f \in C^\infty(M) \) and \( k \in \mathbb{N} \), define

\[
Q_k(f) := Q_k(u_f)
\]

for \( u_f \) the solution to (1.5). It follows from [27, 30, 28, 29, 36] that \( Q_k \) is well-defined; it should be regarded as a fully nonlinear analogue of the Dirichlet-to-Neumann map \( f \rightarrow (u_f)_\nu \). In terms of this operator, Theorem 1.1 states that

\[
\mathcal{E}_k(u) \geq \oint_M f Q_k(f) d\mu
\]

(1.12) for all \( u \in \overline{C_{f,k}} \), with equality if and only if \( u = u_f \). Equation (1.12) gives a trace inequality which can be regarded as a norm computation for part of the trace embedding \( W^{k+2,1+k}(X) \subset W^{k+2,1+k+1}(M) \).

**Remark 1.3.** We conclude this introduction with a few additional comments on the boundary operators \( B_k \) of Proposition 1.2. The conditions of Proposition 1.2 do not uniquely determine the boundary operators \( B_k \) of Proposition 1.2. So neither the conditions in Theorem 1.1 determine the functional \( \mathcal{E}_{k+1} \) uniquely. Indeed, the operators are not unique even if we require additionally that the operators \( B_k \) commute with diffeomorphisms, as do the operators constructed in the proof of Proposition 1.2. A trivial source of nonuniqueness comes from the freedom to add symmetric zeroth-order terms to \( B_k \). For example, if \( B_k \) satisfies the conclusions of Proposition 1.2, so does the operator

\[
(w^1, \ldots, w^k) \mapsto B_k(w^1, \ldots, w^k) + c H w^1 \cdots w^k
\]

for any \( c \in \mathbb{R} \). More generally, one may add to the boundary operators \( B_k \) any symmetric multilinear operator which is also symmetric upon pairing with integration. For example, consider the operator \( D: (C^2(X))^2 \rightarrow C^\infty(M) \) defined by

\[
D(v, w) = \overline{\delta} (L(\nabla(vw))) - L(\nabla v, \nabla w),
\]

where \( \nabla \) is the Levi-Civita connection on \( M \) and \( \overline{\delta} \) is the divergence operator. It is readily verified that \( (u, v, w) \mapsto \oint_M u D(v, w) d\mu \) is a symmetric trilinear form, and thus \( D \) can be added to the operator \( B_2 \) to yield another operator \( \tilde{B}_2 \) which satisfies the conclusions of Proposition 1.2.
This article is organized as follows. In Section 2 we collect some useful facts involving the complex \( k \)-Hessian and the CR structure on \( M \). In Section 3 we shall first prove Proposition 1.2 for \( k = 2 \), since things are much simpler in this case, yet it still provides the essential insights to this problem. In Section 4 we prove Proposition 1.2 for any \( k \) by explicitly constructing a suitable boundary operator. In Section 5 we prove Theorem 1.1.

2. Preliminaries

In this note, we will use the Greek letters \( \alpha, \beta, \gamma, \cdots \) to denote indices ranging between 1 and \( n - 1 \) and use the Roman letters \( i, j, k, \cdots \) to denote the indices ranging between 1 and \( n \). In order to avoid tedium, we will always adopt the Einstein summation convention. Let us review some background materials.

2.1. The CR structure and the Kohn Laplacian. In this section, we give a brief review of the CR structure and the Kohn Laplacian on the real hypersurface in \( \mathbb{C}^n \). For more details, we refer the readers to [1, 34].

Let \( X \subset \mathbb{C}^n \) be a bounded domain with smooth boundary \( M = \partial X \). The boundary \( M \) has the induced metric from \( \mathbb{C}^n \) and its unit outward normal vector is denoted by \( \nu \). Then \( T := J\nu \) is a unit tangent vector field along \( M \), where \( J \) is the complex structure on \( \mathbb{C}^n \).

We denote by \( \langle \cdot, \cdot \rangle \) the standard Euclidean metric on \( \mathbb{C}^n \). The distribution \( \mathcal{H} = \{Y \in TM : (Y, T) = 0\} \) is invariant under the complex structure \( J \). Therefore, we can decompose \( \mathcal{H} \otimes \mathbb{C} \) into the direct sum of the \( \sqrt{-1} \) and \( -\sqrt{-1} \) eigenspaces of \( J \), which are denoted by \( T^{1,0}M \) and \( T^{0,1}M \) respectively. We have

\[
T^{1,0}M = \{Y - \sqrt{-1}JY : Y \in \mathcal{H}\}, \quad T^{0,1}M = \overline{T^{1,0}M}.
\]

In doing computations, we will use a local unitary frame \( \{Z_i : 1 \leq i \leq n\} \) for \( T^{1,0}\mathbb{C}^n \) and its dual frame \( \{\theta^i : 1 \leq i \leq n\} \). Let \( \nabla \) and \( \overline{\nabla} \) be the Levi-Civita connection of \( \mathbb{C}^n \) and \( M \) respectively. For any \( u \in C^\infty(X) \), we write the covariant derivatives as

\[
u_i = Z_i u, \quad u_i = \overline{Z_i} u, \quad u_{ij} = \nabla^2 u (Z_j, Z_i) = Z_j Z_i u - \nabla_{Z_j} Z_i u, \quad \text{etc.}
\]

As \( \mathbb{C}^n \) is flat, it follows immediately that

\[
\nabla_{ij} u = u_{ij}, \quad u_{ijk} = u_{ikj}, \quad u_{ijk} = u_{ikj}.
\]

Set

\[
\nabla u := u_\alpha \overline{Z_\alpha} + u_\alpha Z_\alpha + (Tu) T,
\]

which is the gradient vector field of \( u \) on \( M \). We denote its holomorphic part and antiholomorphic part respectively by

\[
\nabla^{1,0} u := u_\alpha \overline{Z_\alpha}, \quad \nabla^{0,1} u := u_\alpha Z_\alpha.
\]

In other word, \( \nabla^{1,0} u \) and \( \nabla^{0,1} u \) are respectively the projection of \( \nabla u \) to \( T^{1,0}M \) and \( T^{0,1}M \). And we use \( \Delta \) to denote the complex Laplacian, i.e., \( \Delta u = u_{\overline{\alpha} \alpha} \), which is half of the real Laplacian.

Along \( M \), we can assume that \( Z_n = \frac{1}{\sqrt{2}} (\nu - \sqrt{-1} T) \) and \( \{Z_\alpha : 1 \leq \alpha \leq n - 1\} \) is a local unitary frame of \( T^{1,0}M \). If we denote \( X_i = \sqrt{2} \text{Re} Z_i \) and \( Y_i = -\sqrt{2} \text{Im} Z_i \), then \( \{X_i, Y_i : 1 \leq i, j \leq n\} \) is an orthonormal basis of \( T\mathbb{C}^n \). In particular, \( X_n = \nu \) and \( Y_n = T \). The shape operator \( A : TM \to TM \) and the second fundamental form are defined in the usual way: for \( X, Y \in TM \),

\[
AX = \nabla_X \nu, \quad L(X, Y) = (AX, Y).
\]
The Hermitian mean curvature $H_b$ introduced in [34] is defined by
\[ H_b = H - L(J\nu, J\nu), \]
where $H$ is the mean curvature.

Let $\overline{\partial}_b$ be the tangential Cauchy-Riemann operator on $M$. In terms of the local frame,
\[ \overline{\partial}_b u = u_\alpha \bar{\theta}^\alpha. \]
Let $\overline{\partial}_b^*$ be the adjoint of $\overline{\partial}_b$ and let $\Box_b = -\overline{\partial}_b^* \overline{\partial}_b$ be the Kohn Laplacian. The Kohn Laplacian on functions is in general not a real operator and its real part, denoted by $\Delta_b = \text{Re} \Box_b$, is usually called the sub-Laplacian.

2.2. The $\Gamma^+_k$-cone. In this subsection, we describe some properties of the elementary symmetric functions and their associated convex cones.

**Definition 2.1.** The $k$-th elementary symmetric function for $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ is
\[ \sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \]

The elementary symmetric functions are special cases of hyperbolic polynomials [19]. As such, they enjoy many nice properties in their associated positive cones.

**Definition 2.2.** The positive $k$-cone is the connected component of \( \{ \lambda : \sigma_k(\lambda) > 0 \} \) which contains $(1, \cdots, 1)$. Equivalently,
\[ \Gamma^+_k = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0 \}. \]

For example, the positive $n$-cone is
\[ \Gamma^+_n = \{ \lambda \in \mathbb{R}^n : \lambda_1, \cdots, \lambda_n > 0 \}, \]
and the positive 1-cone is the half-space
\[ \Gamma^+_1 = \{ \lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0 \}. \]

Note that $\Gamma^+_k$ is an open convex cone and satisfies that
\[ \Gamma^+_n \subset \Gamma^+_{n-1} \subset \cdots \subset \Gamma^+_1. \]

Applying Gårding’s theory of hyperbolic polynomials [19], one concludes that $\sigma^+_k$ is a concave function in $\Gamma^+_k$.

Let $A$ be a Hermitian matrix, i.e., $\overline{A}^T = A$. Then all its eigenvalues are real and we can similarly define a positive $k$-cone of the Hermitian matrices.

**Definition 2.3.** A Hermitian matrix $A$ is in the $\tilde{\Gamma}^+_k$ cone if its eigenvalues satisfy that
\[ \lambda(A) := (\lambda_1(A), \cdots, \lambda_n(A)) \in \tilde{\Gamma}^+_k. \]

Suppose $f$ is a function on $\Gamma^+_k$. Denote by $F = f(\lambda(A))$ the function on $\tilde{\Gamma}^+_k$ induced by $f$. It is known in [?] that if $f$ is concave in $\Gamma^+_k$, then the induced function $F$ is concave in $\tilde{\Gamma}^+_k$. For this reason, we shall denote $\tilde{\Gamma}^+_k$ by $\Gamma^+_k$ and $\sigma_k(\lambda(A))$ by $\sigma_k(A)$ when there is no possibility of confusion.
For an $n \times n$ Hermitian matrix $A$, let $A_{ij}$ be its $(i, j)$ entry. An equivalent definition of $\sigma_k(A)$ is
\begin{equation}
\sigma_k(A) := \frac{1}{k!} \delta^{i_1 \ldots i_k}_{j_1 \ldots j_k} A_{i_1 j_1} \cdots A_{i_k j_k},
\end{equation}
where $\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ is the generalized Kronecker delta, that is to say, it is zero if $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$ and equals 1 (resp. -1) if $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ differ by an even (resp. odd) permutation. In particular, when $k = n$,
\begin{equation}
\sigma_n(A) = \frac{1}{n!} \delta^{i_1 \ldots i_n}_{j_1 \ldots j_n} A_{i_1 j_1} \cdots A_{i_n j_n} = \det A.
\end{equation}

The Newton transformation tensor is defined as
\begin{equation}
T_k(A)_{\bar{i}j} := \frac{1}{k!} \delta^{i_1 \ldots i_k}_{j_1 \ldots j_k} A_{i_1 j_1} \cdots A_{i_k j_k}.
\end{equation}
In fact, it is the linearized operator of $\sigma_{k+1}$:
\begin{equation}
T_k(A)_{\bar{i}j} = \frac{\partial \sigma_{k+1}(A)}{\partial A_{ij}}.
\end{equation}

**Definition 2.4.** The polarization of $\sigma_k$ is defined by
\begin{equation}
\sigma_k(A_1, \ldots, A_k) := \frac{1}{k!} \delta^{i_1 \ldots i_k}_{j_1 \ldots j_k} (A_1)_{i_1 \bar{j}_1} \cdots (A_k)_{i_k \bar{j}_k}.
\end{equation}

It is called the polarization of $\sigma_k$ because $\sigma_k(A_1, \ldots, A_k)$ is the symmetric multilinear form such that $\sigma_k(A) = \sigma_k(A, \ldots, A)$.

**Definition 2.5.** The polarized Newton transformation tensor is
\begin{equation}
T_k(A_1, \ldots, A_k)_{\bar{i}j} := \frac{1}{k!} \delta^{i_1 \ldots i_k}_{j_1 \ldots j_k} (A_1)_{i_1 \bar{j}_1} \cdots (A_k)_{i_k \bar{j}_k}.
\end{equation}

When some components in the polarizations are the same, we adopt the notational conventions
\begin{align*}
\sigma_k(B, \ldots, B, C, \ldots, C) &:= \sigma_k(B, \ldots, B, C, \ldots, C), \\
T_k(B, \ldots, B, C, \ldots, C)_{\bar{i}j} &:= T_k(B, \ldots, B, C, \ldots, C)_{\bar{i}j}.
\end{align*}

Some useful relations between the Newton transformation tensor $T_k$ and $\sigma_k$ are as follows.
\begin{equation}
\sigma_k(A) = \frac{1}{n-k} T_k(A)_{\bar{i}i} = \frac{1}{k} A_{ij} T_k(A)_{\bar{j}i}.
\end{equation}

Many useful algebraic inequalities for elements of $\Gamma^+_k$ can be deduced from Gårding’s theory of hyperbolic polynomials [19]. For us, the important such inequality is the fact that if $(A_1)_{i\bar{j}}, \ldots, (A_k)_{i\bar{j}} \in \Gamma^+_{k+1}$, then $T_k(A_1, \ldots, A_k)_{\bar{i}j}$ is a nonnegative Hermitian matrix.

For any $u \in C^\infty(\overline{\mathcal{X}})$, let $D^{1,1}u$ be its complex Hessian matrix, i.e., $D^{1,1}u = (u_{\bar{i}j})_{1 \leq i, j \leq n}$. For simplicity, we denote by $T_k(u)_{\bar{i}j}$ the Newton transformation tensor $T_k(D^{1,1}u)_{\bar{i}j}$. We will use its divergence free property in later sections.

**Proposition 2.6.** Given $k \in \mathbb{N}$ and $u \in C^\infty(\overline{\mathcal{X}})$, the Newton transformation tensor $T_k(D^{1,1}u)_{\bar{i}j}$ satisfies
\begin{equation}
T_k(u)_{\bar{i}j} := \nabla_{\bar{i}} T_k(u)_{\bar{j}i} = 0, \quad \text{and} \quad T_k(u)_{ij} := \nabla_i T_k(u)_{\bar{j}j} = 0.
\end{equation}
We first recall the divergence theorem in terms of the unitary frame \(\{Z_i : 1 \leq i \leq n\}\) with \(Z_n = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}T)\).

**Lemma 3.1.** We have the following integral identities for tensors \(a_i\theta^i\) or \(a_i\overline{\theta}^i\) on \(\mathcal{X}\).

\[
\int_X \nabla_i a_i = \frac{1}{\sqrt{2}} \oint_M a_n, \quad \text{and} \quad \int_X \nabla_i a_i = \frac{1}{\sqrt{2}} \oint_M a_n.
\]

**Proof.** Note that

\[\text{div} \ (a_i\theta^i) = \nabla_i a_i.\]

By the divergence theorem, we obtain

\[\int_X \nabla_i a_i = \int_X \text{div} \ (a_i\theta^i) = \oint_M a_i\theta^i(\nu) = \frac{1}{\sqrt{2}}a_n.\]

The last equality follows from that \(Z_n = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}T)\).

Similarly, we also have

\[\int_X \nabla_i a_i = \int_X \text{div} \ (a_i\overline{\theta}^i) = \oint_M a_i\overline{\theta}^i(\nu) = \frac{1}{\sqrt{2}}a_n.\]

\[\blacksquare\]

**Proof of Proposition 1.2 for \(k = 2\).** Note that \(S_0\) is symmetric. Our goal is to rewrite (3.1) in the desired form (1.10). By applying integration by parts to \(S_0\) and using the divergence free property (2.10), we have

\[S_0 = \int_X 2uv_{ij} T_1(w)_{j\bar{i}} + 2vw_{ij} T_1(u)_{j\bar{i}} + 2uw_{ij} T_1(v)_{j\bar{i}} - \frac{1}{\sqrt{2}} \oint_M uv_{ij} T_1(w)_{n\bar{i}} + vw_{ij} T_1(u)_{n\bar{i}} + uw_{ij} T_1(v)_{n\bar{i}} + vw_{ij} T_1(u)_{j\bar{n}} + uw_{ij} T_1(v)_{j\bar{n}}.\]

Note that by (2.5),

\[w_{ij} T_1(u)_{j\bar{i}} = \delta^{i_1}_{j_1} w_{ij} u_{i_1 j_1} = \delta^{i_1}_{j_1} u_{ij} w_{i_1 j_1} = u_{ij} T_1(w)_{j\bar{i}}.\]
Thus,
\[
\int_X vw_{ij}T_1(u)_{ji} = \int_X vu_{ij}T_1(w)_{ji} \\
= -\int_X v_j u_i T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_i T_1(w)_{ni} \\
= \int_X u v_{ij} T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_i T_1(w)_{ni} - uv_j T_1(w)_{jn}.
\]

Similarly,
\[
\int_X vw_{ij}T_1(u)_{ji} = \int_X vu_{ij}T_1(w)_{ji} \\
= -\int_X v_i u_j T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_j T_1(w)_{jn} \\
= \int_X u v_{ij} T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_j T_1(w)_{jn} - uv_i T_1(w)_{ni}.
\]

Combining these two identities,
\[
2 \int_X vw_{ij}T_1(u)_{ji} = 2 \int_X uv_{ij} T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_i T_1(w)_{ni} - uv_j T_1(w)_{jn} + uv_j T_1(w)_{jn} - uv_i T_1(w)_{ni}.
\]

Plugging this back into \( S_0 \), we have
\[
S_0 = \int_X 6uv_{ij} T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M vu_i T_1(w)_{ni} - uv_j T_1(w)_{jn} + uv_j T_1(w)_{jn} - uv_i T_1(w)_{ni} \\
- \frac{1}{\sqrt{2}} \oint_M uv_i T_1(u)_{ni} + uv_j T_1(u)_{jn} + uv_j T_1(u)_{jn} + uv_j T_1(u)_{jn} + uv_j T_1(v)_{jn}.
\]

Since \( S_0(u,v,w) \) is symmetric in \( v \) and \( w \), we also have
\[
S_0 = \int_X 6uv_{ij} T_1(w)_{ji} + \frac{1}{\sqrt{2}} \oint_M uv_i T_1(v)_{ni} - uv_j T_1(v)_{jn} + uv_j T_1(v)_{jn} - uv_i T_1(v)_{ni} \\
- \frac{1}{\sqrt{2}} \oint_M uv_i T_1(v)_{ni} + uv_j T_1(v)_{jn} + uv_j T_1(v)_{jn} + uv_j T_1(v)_{jn} + uv_j T_1(w)_{jn}.
\]

For simplicity, we will use \( uv_i \) and \( uv_j \) to denote \( uv_i T_1(w)_{ni} \) and \( uv_j T_1(w)_{jn} \) respectively when there is no ambiguity. Combining these two expressions of \( S_0 \), we obtain
\[
S_0 = 6 \int_X uv_{ij} T_1(w)_{ji} - \frac{3}{2\sqrt{2}} \oint_M u(v_i + w_i + v_j + w_j) \\
- \frac{1}{2\sqrt{2}} \oint_M vu_i + vw_j + uv_i + uv_j + \frac{1}{2\sqrt{2}} \oint_M vu_i + vu_j + wu_i + wu_j.
\]

Denote the boundary integral by \( P \):
\[
P := -\frac{3}{2\sqrt{2}} \oint_M u(v_i + w_i + v_j + w_j) - \frac{1}{2\sqrt{2}} \oint_M vu_i + vw_j + uv_i + uv_j \\
+ \frac{1}{2\sqrt{2}} \oint_M vu_i + vu_j + wu_i + wu_j.
\]

Thus,
\[
S_0 = 6 \int_X uv_{ij} T_1(w)_{ji} + P.
\]
Our goal is to write \( P \) as the sum of a symmetric term and a boundary integral of the form 
\[ \int_M uB(v, w)du \] for some bilinear differential operator \( B : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \). To this purpose, we take the symmetrization with respect to \( u, v, w \) for the second integral in (3.3):

\[
(3.4) \quad S_1 := - \frac{1}{2\sqrt{2}} \int_M vv_i + vw_j + vv_i + vv_j + uw_i + uu_i + uu_j + uu_j .
\]

Then

\[
(3.5) \quad S_0 - S_1 = 6 \int_X w_i \partial_1 T_i w_j - \frac{1}{\sqrt{2}} \int_M u(v_i + w_i + v_j) + \frac{1}{\sqrt{2}} \int_M u_i + v_i + u_j & + \nu_i .
\]

We denote the second and the last integrals by \( U_1 \) and \( Q \) respectively. That is,

\[
(3.6) \quad U_1 := - \frac{1}{\sqrt{2}} \int_M u(v_i T_i w_i) + w_i T_i v_i + v_j T_i v_j + w_j T_i w_j ,
\]

\[
Q := \frac{1}{\sqrt{2}} \int_M vu_i T_i w_i + vu_j T_i v_i + uu_i T_i w_i + uu_j T_i v_i .
\]

Since \( U_1 \) is already in the form of \( \int_M uB(v, w) \), we will focus on the term \( Q \). In order to express \( T_1(w)_{n\overline{n}} \), we recall some earlier work of Wang [34] on the Hermitian mean curvature and sub-Laplacian operator. We re-organize and summarize some computations of Lemma 1 and Proposition 1 of [34] and present them in Lemma 3.2 and Lemma 3.4 for the purpose of our setting.

As before let \( \nu \) be the unit outward normal vector along \( M \) and \( T = J\nu \). In addition, \( \nabla \) denotes the Levi-Civita connection on \( M \), \( \Box_b \) denotes the Kohn Laplacian on \( M \) and \( H_b \) is the Hermitian mean curvature of \( M \).

**Lemma 3.2.** For any \( f \in C^\infty(M) \), we have

\[
(3.7) \quad \Box_b f = \nabla^2 f (Z_\alpha, Z_\alpha) - \frac{\sqrt{-1}}{2} H_b T f + \sqrt{-1} L (Z_\alpha, T) \overline{Z_\alpha f} .
\]

**Proof.** For any \( g \in C^\infty(M) \), we have

\[
(\Box_b f, \overline{g}) = \int_M f g \overline{\alpha} = \int_M Z_\alpha (f g) - (Z_\alpha f \overline{g}) .
\]

Let \( V = f g Z_\alpha := V^\alpha Z_\alpha \) and we compute its divergence on \( M \).

\[
\text{div}^M (V) = (\nabla_\beta V, \overline{Z_\beta}) + (\nabla_T V, T)
\]

\[
= Z_\beta (f g) - (V, \nabla_\beta \overline{Z_\beta}) - (V, \nabla_T T)
\]

\[
= Z_\beta (f g) - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) f g - (J V, \nabla_TJT)
\]

\[
= Z_\beta (f g) - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) f g + \sqrt{-1} (V, \nabla_T V)
\]

\[
= Z_\beta (f g) - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) f g + \sqrt{-1} L (V, T)
\]

\[
= Z_\beta (f g) - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) f g + \sqrt{-1} f g L (Z_\beta, T) .
\]

Therefore,

\[
-(\Box_b f, \overline{g}) = \int_M (Z_\alpha, \nabla_\beta \overline{Z_\beta}) f g - \sqrt{-1} f g L (Z_\alpha, T) - (Z_\alpha f \overline{g}) .
\]
Since this identity works for any \( g \in C^\infty(M) \), we have
\[
\square_b f = Z_\alpha \overline{Z_\alpha f} - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) \overline{Z_\alpha f} + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}
\]
\[
= \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - (\nabla Z_\alpha, \overline{Z_\alpha}) f - (Z_\alpha, \nabla_\beta \overline{Z_\beta}) \overline{Z_\alpha f} + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}
\]
\[
= \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - (T, \overline{\nabla_\beta Z_\beta}) T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}
\]
\[
= \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - \sqrt{-1} (\nabla_\beta \nu, \overline{Z_\beta}) T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}
\]
\[
= \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - \sqrt{-1} L(Z_\alpha, \overline{Z_\alpha}) T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}.
\]
Recall that \( X_i = \sqrt{2} \Re Z_i \) and \( Y_i = -\sqrt{2} \Im Z_i \) for \( 1 \leq i \leq n \). Thus,
\[
L(Z_\alpha, \overline{Z_\alpha}) = \frac{1}{2} L(X_\alpha, X_\alpha) + \frac{1}{2} L(Y_\alpha, Y_\alpha) = \frac{1}{2} H_b.
\]
Therefore,
\[
\square_b f = \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - \frac{1}{2} \frac{\sqrt{-1}}{2} H_b T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}.
\]
\[\square_b f = \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - \frac{1}{2} \frac{\sqrt{-1}}{2} H_b T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}.
\]
\[\square_b f = \nabla^2 f (\overline{Z_\alpha}, Z_\alpha) - \frac{1}{2} \frac{\sqrt{-1}}{2} H_b T f + \sqrt{-1} L(Z_\alpha, T) \overline{Z_\alpha f}.
\]

**Remark 3.3.** By performing the same computation as in Lemma 3.2, we can in fact prove the following integration by parts identity on \( M \). For any smooth differential forms \( a_\alpha \theta^\alpha \) or \( a_\alpha \overline{\theta^\alpha} \) on \( M \) and any \( f \in C^\infty(M) \), we have
\[
\oint_M f \, a_\alpha = \oint_M f \left( Z_\alpha, \nabla_\beta \overline{Z_\beta} \right) a_\alpha - \sqrt{-1} f L(Z_\alpha, T) a_\alpha - f Z_\alpha a_\alpha,
\]
\[
\oint_M f \, a_\alpha = \oint_M f \left( Z_\alpha, \nabla_\beta \overline{Z_\beta} \right) a_\alpha + \sqrt{-1} f L(Z_\alpha, T) a_\alpha - f Z_\alpha a_\alpha.
\]
These integration by parts formulas yield that for any boundary integral with the holomorphic or antiholomorphic derivatives on \( f \), we can always write it into a boundary integral whose integrand factors through \( f \).

We compare complex the Laplacian \( \Delta \) on \( C^n \) and the Kohn Laplacian \( \square_b \) on \( M \) for latter purposes.

**Lemma 3.4.** For any \( u \in C^\infty(\overline{X}) \), we have
\[
\Delta u - u_{\bar{n}n} = \square_b u - \sqrt{-1} u_\alpha L(Z_\alpha, T) + \frac{1}{\sqrt{2}} H_b u_{\bar{n}} \text{ on } M.
\]

**Proof.** Recall that \( X_i = \sqrt{2} \Re Z_i \) and \( Y_i = -\sqrt{2} \Im Z_i \) for \( 1 \leq i \leq n \). In particular, \( X_\alpha = \nu \) and \( Y_\alpha = T = J \nu \). By the definition of the Hessian matrix,
\[
\nabla^2 u(X_\alpha, X_\beta) = X_\alpha X_\beta u - (\nabla X_\alpha X_\beta) u,
\]
\[
\nabla^2 u(Y_\alpha, Y_\beta) = Y_\alpha Y_\beta u - (\nabla Y_\alpha Y_\beta) u.
\]
Therefore,
\[
\nabla^2 u(X_\alpha, X_\beta) - \nabla^2 u(X_\alpha, X_\beta) = - (\nabla X_\alpha X_\beta) u + (\nabla X_\alpha X_\beta) u = L(X_\alpha, X_\beta) \nu u.
\]
Similarly,
\[
\nabla^2 u(Y_\alpha, Y_\beta) - \nabla^2 u(Y_\alpha, Y_\beta) = - (\nabla Y_\alpha Y_\beta) u + (\nabla Y_\alpha Y_\beta) u = L(Y_\alpha, Y_\beta) \nu u.
\]
Combining these two identities and taking the trace, we have
\[ 2\Delta u - \nabla^2 u(X_n, X_n) - \nabla^2 u(Y_n, Y_n) = \nabla^2 u(X_\alpha, X_\alpha) + \nabla^2 u(Y_\alpha, Y_\alpha) + H_b u_\nu, \]
where \( H_b = H - L(Y_n, Y_n) \) is the Hermitian mean curvature. By (3.7), it follows that
\[ 2\Delta u - \nabla^2 u(X_n, X_n) - \nabla^2 u(Y_n, Y_n) = 2\Box u + \sqrt{-1}H_b T u - 2\sqrt{-1}u_\alpha L(Z_\alpha, T) + H_b u_\nu. \]
Note that
\[ \nabla^2 u(Z_n, \overline{Z_n}) = \frac{1}{2} \nabla^2 u(X_n, X_n) + \frac{1}{2} \nabla^2 u(Y_n, Y_n). \]
So
\[ 2\Delta u - 2u_{n\bar{n}} = 2\Box u - 2\sqrt{-1}u_\alpha L(Z_\alpha, T) + \sqrt{-1}H_b T u + H_b u_\nu, \]
\[ = 2\Box u - 2\sqrt{-1}u_\alpha L(Z_\alpha, T) + \sqrt{2}H_b u_\bar{n}. \]
\[ \square \]

**Remark 3.5.** By the definition of the Newton transformation tensor (2.5), we have
\[ T_1(u)_{n\bar{n}} = \delta_{n\bar{j}}^{m\bar{j}} u_{i\bar{i}n} = \Delta u - u_{n\bar{n}}. \]
Combining this with (3.8) yields
\[ T_1(u)_{n\bar{n}} = \Box u - \sqrt{-1}u_\alpha L(Z_\alpha, T) + \frac{1}{\sqrt{2}} H_b u_\bar{n}. \]
By taking the real parts of both sides, we have
\[ T_1(u)_{n\bar{n}} = \Delta_b u - \sqrt{-1}u_\alpha L(Z_\alpha, T) + \sqrt{-1}u_\alpha L(Z_{\alpha}, T) + \frac{1}{2} H_b u_\nu, \]
where \( \Delta_b \) is the sub-Lapacian on \( M \).

Now we are ready to proceed by (3.9). First, we decompose \( Q \) in (3.6) as
\[ Q = \frac{1}{\sqrt{2}} \oint_M vu_\alpha T_1(w)_{n\bar{\alpha}} + vu_\beta T_1(w)_{\bar{\beta}n} + uu_\alpha T_1(v)_{n\bar{\alpha}} + uu_\beta T_1(v)_{\bar{\beta}n} \]
\[ + \frac{1}{\sqrt{2}} \oint_M vu_n T_1(w)_{n\bar{n}} + vu_n T_1(w)_{\bar{n}n} + uu_n T_1(v)_{n\bar{n}} + uu_n T_1(v)_{\bar{n}n} \]
\[ =: U_2 + Q_1. \]

By Remark 3.3, \( U_2 \) can be written into the form of \( \oint_M u \cdot B(v, w) \) for some bilinear differential operator \( B \). Now we consider the integral \( Q_1 \). Since \( u_n + u_{\bar{n}} = \sqrt{2}u_\nu, \) \( Q_1 \) simplifies into
\[ Q_1 = \oint_M vu_\nu T_1(w)_{n\bar{n}} + vu_\nu T_1(v)_{n\bar{n}}. \]
Take the symmetrization of \( Q_1 \):
\[ S_2 := \oint_M vu_\nu T_1(w)_{n\bar{n}} + vu_\nu T_1(v)_{n\bar{n}} + uv_\nu T_1(w)_{n\bar{n}} + uv_\nu T_1(u)_{n\bar{n}} + uv_\nu T_1(u)_{n\bar{n}} + uv_\nu T_1(v)_{n\bar{n}}. \]
Thus,
\[ Q_1 - S_2 = -\oint_M uv_\nu T_1(w)_{n\bar{n}} + uv_\nu T_1(v)_{n\bar{n}} - \oint_M vu_\nu T_1(u)_{n\bar{n}} + vu_\nu T_1(u)_{n\bar{n}}. \]
We set

\[ U_3 := - \oint_M w \nu T_1(w) n \bar{n} + uw \nu T_1(v) n \bar{n}, \]

which is already in the form of \( \oint_M uB(v, w) \). By (3.9), we have

\[
Q_1 - S_2 - U_3 = - \oint_M w \nu \Delta_b u - \sqrt{-1} w \nu (u_\alpha L(Z_\alpha, T) - u_\alpha L(Z_\bar{\alpha}, T)) + \frac{1}{2} w \nu H_b u_\nu \\
- \oint_M v \nu \Delta_b u - \sqrt{-1} v \nu (u_\alpha L(Z_\alpha, T) - u_\alpha L(Z_\bar{\alpha}, T)) + \frac{1}{2} w \nu H_b u_\nu.
\]

Set

\[ U_4 := - \oint_M (w \nu + v \nu) \Delta_b u - \sqrt{-1}(w \nu + v \nu)(u_\alpha L(Z_\alpha, T) - u_\alpha L(Z_\bar{\alpha}, T)). \]

Then

\[
Q_1 - S_2 - U_3 - U_4 = - \frac{1}{2} \oint_M w \nu H_b u_\nu + v w \nu H_b u_\nu := Q_2
\]

Using the fact that \( \Delta_b \) is self-adjoint and the integration by parts identities in Remark 3.3, we notice that \( U_4 \) is actually in the form of \( \oint_M uB(v, w) \). It remains to consider \( Q_2 \). Take the symmetrization of \( Q_2 \):

\[
S_3 = - \frac{1}{2} \oint_M w \nu H_b u_\nu + v w \nu H_b u_\nu + w^2 H_b u_\nu.
\]

Thus,

\[
Q_2 - S_3 = \frac{1}{2} \oint_M u w \nu H_b v_\nu,
\]

which is again in the form of \( \oint_M uB(v, w) \). By combining (3.5), (3.10), (3.12) and (3.14), we obtain

\[
S_0 - S_1 - S_2 - S_3 = 12 \int X u \sigma_2(D^{1,1} v, D^{1,1} w) + \sum_{j=1}^{4} U_j + \frac{1}{2} \oint_M w \nu H_b v_\nu.
\]

Since each \( U_j \) for \( 1 \leq j \leq 4 \) is in the form of \( \oint_M uB(v, w) \), there exists some bilinear differential operator \( B_2(\cdot, \cdot) \) such that

\[
S_0 - S_1 - S_2 - S_3 = 12 \int X u \sigma_2(D^{1,1} v, D^{1,1} w) - \oint_M uB_2(v, w).
\]

The result follows immediately by setting

\[
-L_3(u, v, w) := \frac{1}{12}(S_0 - S_1 - S_2 - S_3).
\]

Now we set the functional \( \mathcal{E}_3 \) as

\[ \mathcal{E}_3(u) := L_3(u, u, u). \]

By combining (3.5), (3.11) and (3.13), \( \mathcal{E}_3 \) writes into

\[
\mathcal{E}_3(u) = - \int_X u \sigma_2(D^{1,1} u) + \frac{1}{2} \oint_M u v T_1(u) n \bar{n} + \frac{1}{8} \oint_M u^2 H_b.
\]
By using (3.9), we can simplify $\mathcal{E}_3$ into

\begin{equation}
3.15 \quad \mathcal{E}_3(u) = -\int_X u\sigma_2(D^{1,1}u) + \frac{1}{2} \int_M uu_\nu\Delta_b u + \frac{1}{8} \int_M uu_\nu^2 H_b - \frac{\sqrt{-1}}{2} \int_M uu_\nu L(\nabla^{1,0}u - \nabla^{0,1}u, T),
\end{equation}

where $\nabla^{1,0}u = u_\alpha Z_\alpha$ and $\nabla^{0,1}u = u_\alpha \overrightarrow{\nu}_\alpha$.

**Remark 3.6.** Clearly, the functional $\mathcal{E}_3$ depends on at most second order tangential derivatives on $M$ and at most first order transverse derivative along $\nu$. Therefore, $\mathcal{E}_3(u)$ is well-defined if $u \in C^2(X) \cap C^2(M) \cap C^1(\overline{X})$.

### 4. Proof of Proposition 1.2 for General $k$

In this section, we will construct $B_k$ for general $k$ by an inductive argument. We begin our construction with the following symmetric multilinear differential operator

\begin{equation}
4.1 \quad S_0(u, w^1, \cdots, w^k) := -\sum_p \left[ \int_X u_i w_j^p T_{k-1}(D^{1,1}w^p)_{ji}dx \right] + \int_X w_i^p u_j T_{k-1}(D^{1,1}w^p)_{ji}dx - \sum_{p \neq q} \int_X w_i^p w_j^q T_{k-1}(D^{1,1}u, D^{1,1}w^{p,q})_{ji}dx.
\end{equation}

where $D^{1,1}w^p$ denotes the $(k-1)$-tuple $(D^{1,1}w^1, \cdots, D^{1,1}w^{p-1}, D^{1,1}w^{p+1}, \cdots, D^{1,1}w^k)$, obtained from $(D^{1,1}w^1, \cdots, D^{1,1}w^k)$ by removing the $p$-th entry $D^{1,1}w^p$, and likewise $D^{1,1}w^{p,q}$ denotes the $(k-2)$-tuple obtained from $(D^{1,1}w^1, \cdots, D^{1,1}w^q)$ by removing the $p$-th entry $D^{1,1}w^p$ and the $q$-th entry $D^{1,1}w^q$. Similar notations will be used to remove more entries from the list. We shall perform integration by parts repeatedly to rewrite (4.1) as a sum of an interior and a boundary integral, both of which have integrands which factor through $u$.

In the following, we use $S_i$ to denote terms that are symmetric in $u, w^1, \cdots, w^k$, no matter they are of interior integral or boundary integral. We use $U_i$ to denote some boundary integral of the form $\int uB(w^1, \cdots, w^k)d\mu$ for some multi-linear operator $B$. And we use $Q_i$ to denote terms that will be repeatedly decomposed in the induction, and eventually disappear when the induction terminates.

**Proof of Proposition 1.2.** Note that $S_0$ is symmetric. Our goal is to rewrite (4.1) in the desired form (1.10). To that end, writing (4.1) as a sum over pairs $p \neq q$ and then integrating by parts in $X$ yields

\begin{align*}
S_0 &= -\sum_{p \neq q} \left[ \frac{1}{k-1} \int_X u_i w_j^p T_{k-1}(D^{1,1}w^p)_{ji}dx + \frac{1}{k-1} \int_X w_i^p u_j T_{k-1}(D^{1,1}w^p)_{ji}dx \\
&\quad + \int_X w_i^p w_j^q T_{k-1}(D^{1,1}u, D^{1,1}w^{p,q})_{ji}dx \right] \\
&= \sum_{p \neq q} \left[ \frac{2}{k-1} \int_X u_i w_j^p T_{k-1}(D^{1,1}w^p)_{ji}dx + \int_X w_i^p u_j T_{k-1}(D^{1,1}w^p)_{ji}dx \\
&\quad - \frac{1}{(k-1)\sqrt{2}} \int_M uu_i^p T_{k-1}(D^{1,1}w^p)_{ji}d\mu - \frac{1}{(k-1)\sqrt{2}} \int_M uu_i^p T_{k-1}(D^{1,1}w^p)_{ji}d\mu \\
&\quad - \frac{1}{2\sqrt{2}} \int_M w_i^p w_j^q T_{k-1}(D^{1,1}u, D^{1,1}w^{p,q})_{ji}d\mu - \frac{1}{2\sqrt{2}} \int_M w_i^p w_j^q T_{k-1}(D^{1,1}u, D^{1,1}w^{p,q})_{ji}d\mu \right].
\end{align*}
Performing integration by parts two more times on the second term $\int_X w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} dx$, we have
\[
S_0 = \sum_{p \neq q} \left[ \frac{k + 1}{k - 1} \int_X u w^p_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} dx - \frac{1}{2k - 1} \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu - \frac{1}{2k - 1} \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right].
\]

Denote the boundary integral by $P$:
\[
P = - \frac{k + 1}{2\sqrt{2}} \sum_p \left[ \int_M u w^p T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu + \int_M w^p T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right]
\]
\[
+ \frac{1}{2\sqrt{2}} \sum_{p \neq q} \left[ - \int_M w^p w^q u T_{k-1}(D^{1,1} u, D^{1,1} w^{\wedge p-q})_{j\bar{i}} d\mu - \int_M w^p w^q T_{k-1}(D^{1,1} u, D^{1,1} w^{\wedge p-q})_{j\bar{i}} d\mu \right].
\]

Thus
\[
S_0 = k^2 (k + 1) \int_X u \sigma_k (D^{1,1} w^1, \ldots, D^{1,1} w^k) dx + P.
\]

We aim to write $T$ as the sum of a symmetric term and a boundary integral of the form $\int uB(w^1, \ldots, w^k) d\mu$. To that end, consider the symmetrization of the second line in (4.2):
\[
(4.3) \quad S_0 = \frac{k^2 (k + 1)}{2} \int_X u \sigma_k (D^{1,1} w^1, \ldots, D^{1,1} w^k) dx + P.
\]

Note that $S_1$ is symmetric with respect to $u, w^1, \ldots, w^k$. Combining (4.2) and (4.4) yields
\[
P = S_1 - \frac{k}{2\sqrt{2}} \sum_p \left[ \int_M u w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu + \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right]
\]
\[
+ \frac{k}{2\sqrt{2}} \sum_p \left[ \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu + \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right].
\]

Define
\[
U_1 := - \frac{k}{2\sqrt{2}} \sum_p \left[ \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu + \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right],
\]
\[
Q := \frac{k}{2\sqrt{2}} \sum_p \left[ \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu + \int_M w^p u_{ij} T_{k-1}(D^{1,1} w^{\wedge p})_{j\bar{i}} d\mu \right].
\]
Thus,

\[ P = U_1 + S_1 + Q. \]

Note that \( U_1 \) is in the desired form \( \oint uB(w^1, \ldots, w^p)d\mu \). It remains to deal with the term \( Q \). We can decompose \( Q \) as

\[ Q = \frac{k}{2\sqrt{2}} \sum_p \left[ \oint_M w^p u_{\alpha} T_{k-1}(D^{1,1}w^\nu)_{n\alpha} d\mu + \oint_M w^p u_{\beta} T_{k-1}(D^{1,1}w^\nu)_{\beta n} d\mu \right] + \frac{k}{2\sqrt{2}} \sum_p \left[ \oint_M w^p u_n T_{k-1}(D^{1,1}w^\nu)_{n\nu} d\mu + \oint_M w^p u_n T_{k-1}(D^{1,1}w^\nu)_{n\beta} d\mu \right]. \]

Recall that the Greek indices \( \alpha, \beta \in \{1, \ldots, n-1\} \) denote the holomorphic tangential directions in \( T^{1,0}M \) and \( n \) denotes the direction \( Z_n = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}J\nu) \in T^{1,0}\mathbb{C}^n \) along \( M \). By the definition of the Newton transformation tensor (2.5), \( T_{k-1}(D^{1,1}w^\nu)_{n\nu} = \sigma_{k-1}(D^{1,1}w^\nu|_{\mathcal{H}}) \), where \( \mathcal{H} \) is the distribution \( \{ Y \in TM : (Y, T) = 0 \} \) and \( D^{1,1}w^\nu|_{\mathcal{H}} \) denotes the list of the restrictions \( D^{1,1}w^1|_{\mathcal{H}}, \ldots, D^{1,1}w^n|_{\mathcal{H}} \) with the \( p \)-th element removed. In terms of the local frame, \( D^{1,1}w|_{\mathcal{H}} \) is the Hermitian matrix \( (w_{\alpha\beta})_{1 \leq \alpha, \beta \leq n-1} \).

We define

\[ U_2 := \frac{k}{2\sqrt{2}} \sum_p \left[ \oint_M w^p u_{\alpha} T_{k-1}(D^{1,1}w^\nu)_{n\alpha} d\mu + \oint_M w^p u_{\beta} T_{k-1}(D^{1,1}w^\nu)_{\beta n} d\mu \right], \]

\[ Q_1 := \frac{k}{2} \sum_p \oint_M w^p u_{\alpha} \sigma_{k-1}(D^{1,1}w^\nu|_{\mathcal{H}}) d\mu. \]

Note that

\[ u_n + u_{\bar{n}} = Z_n u + \overline{Z_n u} = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}T)u + \frac{1}{\sqrt{2}}(\nu + \sqrt{-1}T)u = \sqrt{2}u_{\nu}, \]

and thus we have

(4.6) \[ Q = U_2 + Q_1 \]

Performing integration by parts along \( M \) as in Remark 3.3 with \( f = u \), \( U_2 \) writes into the form of \( \oint uB(w^1, \ldots, w^p)d\mu \). Therefore, we only need to consider \( Q_1 \).

Take the symmetrization of \( Q_1 \):

\[ S_2 := \sum_{p \neq q} \left[ \frac{k}{2(k-1)} \oint_M w^p w^q u_{\alpha} \sigma_{k-1}(D^{1,1}w^\nu|_{\mathcal{H}}) d\mu + \frac{k}{2(k-1)} \oint_M w^p w^q \sigma_{k-1}(D^{1,1}w^\nu|_{\mathcal{H}}) d\mu \right] + \frac{k}{2} \oint_M w^p w^q \sigma_{k-1}(D^{1,1}u|_{\mathcal{H}}, D^{1,1}w^{\nu|-\omega}|_{\mathcal{H}}) d\mu. \]

Thus, \( S_2 \) is symmetric with respect to \( u, w^1, \ldots, w^k \). And we have

(4.7) \[ Q_1 = S_2 - \frac{k}{2(k-1)} \sum_{p \neq q} \oint_M u w^p w^q \sigma_{k-1}(D^{1,1}w^{\nu|-\omega}|_{\mathcal{H}}) d\mu - \frac{k}{2} \sum_{p \neq q} \oint_M w^p w^q \sigma_{k-1}(D^{1,1}u|_{\mathcal{H}}, D^{1,1}w^{\nu|-\omega}|_{\mathcal{H}}) d\mu. \]

For any \( v \in C^\infty(\overline{X}) \), denote by \( \overline{D}^2 v \) the Hessian with respect to the Levi-Civita connection \( \overline{\nabla} \) on \( M \) and let \( \overline{D}^{1,1}v = \overline{D}^2 v|_{\mathcal{H}} : T^{1,0}M \times T^{0,1}M \to \mathbb{C} \) be the restriction to the distribution \( \mathcal{H} \). In terms
of the local frame, \( \overline{D}^{1,1} v \) is the Hermitian matrix \((\nabla^2_{Z,\alpha,\beta} u)_{1<\alpha,\beta\leq n-1} \). Moreover, we denote by \( L = L(\cdot, \cdot) \) the second fundamental form of \( M \). Given \( v \in C^\infty(X) \), it holds that
\[
D^{1,1} v(Z, \overline{W}) = \overline{D}^{1,1} v(Z, \overline{W}) + v_\nu L(Z, \overline{W}), \quad \text{for any } Z, W \in T^{1,0} M.
\]
For simplicity, we write it as
\[
(4.8) \quad D^{1,1} v|_H = \overline{D}^{1,1} v + v_\nu L|_H,
\]
where \( L|_H : T^{1,0} M \times T^{0,1} M \to \mathbb{C} \) is the restriction of \( L \). Define
\[
U_3 := -\frac{k}{2(k-1)} \sum_{\nu \neq q} \int_M u w_\nu^p \sigma_{k-1}(D^{1,1} w|_H^{\nu, q}) d\mu,
\]
\[
U_4 := -\frac{k}{2} \sum_{\nu \neq q} \int_M w_\nu^p w_q^p \sigma_{k-1}(D^{1,1} u, D^{1,1} w|_H^{\nu, q}) d\mu.
\]
\[
Q_2 := -\frac{k}{2} \sum_{\nu \neq q} \int_M w_\nu^p w_q^p u_\nu \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q}) d\mu.
\]
Thus,
\[
(4.9) \quad Q_1 = S_2 + U_3 + U_4 + Q_2.
\]
By using the fact
\[
(k-1)\sigma_{k-1}(D^{1,1} u, D^{1,1} w|_H^{\nu, q}) = \nabla_{\alpha \beta}^2 u \cdot T_{k-2}(D^{1,1} w|_H^{\nu, q}),
\]
we have
\[
U_4 = -\frac{k}{2(k-1)} \sum_{\nu \neq q} \int_M w_\nu^p w_q^p \nabla_{\alpha \beta} u T_{k-2}(D^{1,1} w|_H^{\nu, q}) d\mu.
\]
Integrating by parts along \( M \) twice, we can rewrite \( U_4 \) into the form of \( \int_M u B(w^1, \ldots, w^p) d\mu \). As \( U_3 \) and \( U_4 \) are in the correct form, now we only need to consider \( Q_2 \). To that end, take the symmetrization of \( Q_2 \):
\[
S_3 := -\frac{k}{2} \sum_{\nu \neq q \neq r} \left[ \frac{1}{k-2} \int_M w_\nu^p w_q^p u_\nu \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q}) d\mu \right.
\]
\[
+ \frac{1}{2! (k-2)} \int_M u w_\nu^p w_q^p \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q}) d\mu \right.
\]
\[
+ \frac{1}{2!} \int_M w_\nu^p w_q^p w_r^p \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q, r}) d\mu \right].
\]
Note that \( S_3 \) is symmetric with respect to \( u, w^1, \ldots, w^k \). By defining
\[
U_5 := \frac{k}{4(k-2)} \sum_{\nu \neq q \neq r} \int_M w_\nu^p w_q^p \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q, r}) d\mu,
\]
\[
U_6 := \frac{k}{4} \sum_{\nu \neq q \neq r} \int_M w_\nu^p w_q^p w_r^p \sigma_{k-1}(L|_H, D^{1,1} w|_H^{\nu, q, r}) d\mu,
\]
\[
Q_3 := \frac{k}{4} \sum_{\nu \neq q \neq r} \int_M w_\nu^p w_q^p w_r^p u_\nu \sigma_{k-1}(L|_H, L|_H, D^{1,1} w|_H^{\nu, q, r}) d\mu,
\]
we have
\[
(4.10) \quad Q_2 = S_3 + U_5 + U_6 + Q_3.
\]
As above, integration by parts along \( M \) implies that both \( U_5 \) and \( U_6 \) are of the form \( \int uB(w^1, \cdots, w^k) \mu \). Thus, we only need to consider \( Q_3 \).

Proceeding in this way, for all \( 2 \leq i \leq k \) we make the following definitions. First, define

\[
S_i := \frac{(-1)^i k}{2} \sum_{p_1 \neq \cdots \neq p_i} \left[ \frac{1}{(i-2)! (k+1-i)} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \mu \right. \\
\times \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_{i-1})}) d\mu \\
+ \frac{1}{(i-1)! (k+1-i)} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_{i-1})}) d\mu \\
+ \left. \frac{1}{(i-1)!} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_{i-1})}) d\mu \right]
\]

Note that \( S_i \) is symmetric with respect to \( u, w^1, \cdots, w^k \). Next, define

\[
U_{2i-1} := \frac{(-1)^{i+1} k}{2(i-1)! (k+1-i)} \sum_{p_1 \neq \cdots \neq p_i} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \mu \\
\times \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_{i-1})}) d\mu,
\]

\[
U_{2i} := \frac{(-1)^{i+1} k}{2(i-1)!} \sum_{p_1 \neq \cdots \neq p_i} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} u, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_i)}) d\mu,
\]

\[
Q_i := \frac{(-1)^{i+1} k}{2(i-1)!} \sum_{p_1 \neq \cdots \neq p_i} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{i-1}} \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}, D^{1,1} w|_{H_1}^{(p_1, \cdots, p_i)}) d\mu.
\]

Then we have

\[
Q_{i-1} = S_i + U_{2i-1} + U_{2i} + Q_i.
\]

As above, integration by parts along \( M \) implies that both \( U_{2i-1} \) and \( U_{2i} \) are of the form \( \int uB(w^1, \cdots, w^k) \mu \). We can inductively perform this argument until \( i = k-1 \) and get

\[
Q_{k-1} = S_k + U_{2k-1} + U_{2k} + Q_k,
\]

where

\[
Q_k := \frac{(-1)^{k+1} k}{2(k-1)!} \sum_{p_1 \neq \cdots \neq p_k} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{k-1}} \sigma_{k-1}(L|_{H_1}, \cdots, L|_{H_1}) d\mu.
\]

It remains to write \( Q_k \) into as the sum of a symmetric integral and a boundary integral whose integrand factors through \( u \). To that end, we define

\[
S_{k+1} := \frac{(-1)^{k+1} k}{2(k-1)!} \sum_{p_1 \neq \cdots \neq p_k} \left[ \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{k-1}} (L|_{H_1}) d\mu \\
+ \frac{1}{k} \int_M u^{p_1} w_{p_2} w_{p_3} \cdots u_{p_{k-1}} (L|_{H_1}) d\mu \right].
\]
If we define
\[ U_{2k+1} := \frac{(-1)^k}{2(k-1)!} \sum_{p_1 \neq \cdots \neq p_k} \iint_M uu_{p_1} \cdots w_{p_k}^k \sigma_{k-1}(L|\mathcal{H})d\mu, \]
then
\[ (4.14) \quad Q_k = S_{k+1} + U_{2k+1}. \]
Note that \( S_{k+1} \) is symmetric with respect to \( u, w^1, \cdots, w^k \) and \( U_{2k+1} \) is of the form \( \oint uB(w^1, \cdots, w^k)d\mu \). Therefore, \( Q_k \) is now in the desired form.

In summary, we have shown that
\[ (4.15) \quad S_0 - \sum_{i=1}^{k+1} S_i = k^2(k+1) \int_X u \sigma_k(D^{1,1}w^1, \cdots, D^{1,1}w^k)dx + \sum_{i=1}^{2k+1} U_i \]
and observed that the left-hand side is symmetric in \( u, w^1, \cdots, w^k \) while the right-hand side is of the form \( \oint uB(w^1, \cdots, w^k)d\mu \). The result therefore follows by defining
\[ (4.16) \quad L_{k+1}(u, w^1, w^2, \cdots, w^k) := -\frac{1}{k^2(k+1)} \left( S_0 - \sum_{i=1}^{k+1} S_i \right). \]
\[ \square \]

For any \( u \in C^\infty(X) \), we define
\[ \mathcal{E}_{k+1}(u) := L_{k+1}(u, \cdots, u). \]
We are going to show in the proof of Proposition 4.1 that this definition of \( \mathcal{E}_{k+1} \) coincides with the expression given in (1.6), and from there one can see \( \mathcal{E}_{k+1}(u) \) does not involve second order transverse derivatives.

**Proposition 4.1.** The functional \( \mathcal{E}_{k+1} \) depends on at most second order tangential derivatives on \( M \) and at most first order transverse derivatives. In particular, we only need \( u \in C^2(X) \cap C^2(M) \cap C^1(X) \) to define \( \mathcal{E}_{k+1}(u) \).

**Proof.** Given \( u \in C^\infty(X) \), for \( 0 \leq i \leq k+1 \) and \( 1 \leq j \leq 2k+1 \), we denote
\[ S_i(u) := S_i(u, \cdots, u), \quad U_j(u) := U_j(u, \cdots, u). \]
If we set \( w^1 = w^2 = \cdots = w^k = u \in C^\infty(X) \) in (4.5), then we have \( P = S_1(u) \). Thus, (4.3) implies that
\[ S_0(u) = k^2(k+1) \int_X u \sigma_k(D^{1,1}u)dx + S_1(u). \]
Combining this with (4.16), we obtain
\[ (4.17) \quad \mathcal{E}_{k+1}(u) = -\int_X u \sigma_k(D^{1,1}u)dx + \frac{1}{k^2(k+1)} \sum_{i=2}^{k+1} S_i(u). \]
Note that by (4.11) and (4.13), we have that for \( 2 \leq i \leq k+1 \),
\[ (4.18) \quad S_i(u) = (-1)^i \frac{k(k+1)}{2} \sum_{j=0}^{i-2} \left( \begin{array}{c} k \\ i-1 \end{array} \right) \iint_M uu_{i-1}^{i-1} \sigma_{k-1}(L|\mathcal{H}, \cdots, L|\mathcal{H}, D^{1,1}u|\mathcal{H}, \cdots, D^{1,1}u|\mathcal{H}). \]
The result therefore follows immediately by (4.8). \[ \square \]
5. Proof of Theorem 1.1

It is straightforward to compute the first and second variations of the energy functional \( \mathcal{E}_{k+1} \) associated to the symmetric multilinear form constructed by Proposition 1.2.

**Proposition 5.1.** Let \( X \subset \mathbb{C}^n \) be a bounded smooth domain with boundary \( M = \partial X \). Let \( u, v \in C^\infty(X) \) and suppose that \( v|_M = 0 \). Then

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) = -(k + 1) \int_X v \sigma_k(D^{1,1}u, \ldots, D^{1,1}u) dx.
\]

**Proof.** Since \( L_{k+1} \) is symmetric, we compute that

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) = (k + 1)L_{k+1}(v, u, \ldots, u).
\]

Since \( v|_M = 0 \), we see that the boundary integral in (1.10) vanishes. The result therefore follows. □

**Proposition 5.2.** Let \( X \subset \mathbb{C}^n \) be a bounded smooth domain with boundary \( M = \partial X \). Let \( u, v \in C^\infty(X) \) and suppose that \( v|_M = 0 \). Then

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) = (k + 1) \int_X v_i v_j T_{k-1}(D^{1,1}u)_{ij} dx.
\]

In particular, if \( u \in \Gamma^+_{\mathcal{L}^k} \), then

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) \geq 0
\]

for all \( v \in C^\infty(X) \) such that \( v|_M = 0 \).

**Proof.** Since \( L_{k+1} \) is symmetric, we compute that

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) = k(k + 1)L_{k+1}(v, v, u, \ldots, u).
\]

Since \( v|_M = 0 \), it follows that

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{k+1}(u + tv) = -k(k + 1) \int_X v \sigma_k(D^{1,1}v, D^{1,1}u, \ldots, D^{1,1}u) dx,
\]

\[
= -(k + 1) \int_X v T_{k-1}(D^{1,1}u)_{ij} v_{ij} dx
\]

\[
= (k + 1) \int_X v_i v_j T_{k-1}(D^{1,1}u)_{ij} dx.
\]

The last conclusion follows from the fact that if \( u \in \Gamma^+_{\mathcal{L}^k} \), then \( T_{k-1}(D^{1,1}u)_{ij} \) is nonnegative. □

**Remark 5.3.** By Remark 3.6 and Proposition 4.1, the functional \( \mathcal{E}_{k+1} \) is actually well-defined on \( C^2(X) \cap C^2(M) \cap C^1(\overline{X}) \). Given \( u \in \overline{C^2_{f,k}} \), \( u \) is contained in \( C^{1,1}(\overline{X}) \cap C^2(M) \). We can construct \( u_{\varepsilon} \in C^\infty(\overline{X}) \) such that as \( \varepsilon \to 0 \), \( \|u_{\varepsilon} - u\|_{C^1(\overline{X})} \to 0 \) and \( \|u_{\varepsilon} - u\|_{W^{2,p}(\overline{X})} \to 0 \) for any \( 1 \leq p < \infty \). By a density argument, the variation identities (5.1), (5.2) and the convexity property (5.3) actually hold for \( u \in \overline{C^2_{f,k}} \) and \( v \in C^{1,1}(\overline{X}) \) with \( v|_M = 0 \).

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1. By Proposition 5.1 and Remark 5.3, the solution $u_f$ to (1.5) is a critical point of the functional $\mathcal{E}_{k+1}: \overline{C}_{f,k} \to \mathbb{R}$. By Proposition 5.2 and Remark 5.3, the restriction $\mathcal{E}_{k+1}: \overline{C}_{f,k} \to \mathbb{R}$ is a convex functional. Since $\overline{C}_{f,k}$ is convex, $u_f$ realizes the infimum of $\mathcal{E}_{k+1}: \overline{C}_{f,k} \to \mathbb{R}$. Indeed, if not, then there is a $u \in \overline{C}_{f,k}$ such that $\mathcal{E}_{k+1}(u) < \mathcal{E}_{k+1}(u_f)$. Since $\overline{C}_{f,k}$ is convex, it follows that $tu + (1-t)u_f \in \overline{C}_{f,k}$ for all $t \in [0,1]$. Denote $\mathcal{E}_{k+1}(t) := \mathcal{E}_{k+1}(tu + (1-t)u_f)$. Since $\mathcal{E}_{k+1}(u) < \mathcal{E}_{k+1}(u_f)$, there exists a $t^* \in [0,1]$ such that $\mathcal{E}_{k+1}'(t^*) < 0$. This contradicts the facts that $\mathcal{E}_{k+1}'(0) = 0$ and $\mathcal{E}_{k+1}'' \geq 0$ for all $t \in [0,1]$.

\[ \square \]

ACKNOWLEDGMENTS

The first author would like to thank Pengfei Guan for suggestions and interest of this work. The second author is thankful to Song-Ying Li and Xiangwen Zhang for the friendly discussions about the complex $k$-Hessian equation.

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