Contact forces in regular 3D granular pile

G. Oron and H.J. Herrmann
Laboratoire de Physique et Mécanique des Milieux Hétérogènes,
UMR CNRS 7636, École de Physique et de Chimie Industrielles
de la ville de Paris,
10 rue Vauquelin, 75231 Paris Cedex 05, FRANCE
email : oron@pmmh.espci.fr

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Abstract

We present exact results for the contact forces in a three dimensional static piling of identical, stiff and frictionless spheres. The pile studied is a pyramid of equilateral triangular base (“stack of cannonballs”) with a FCC (face centered cubic) structure. We show in particular that, as for the two dimensional case, the pressure on the base of such a pile is uniform.

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The force propagation through a heap of granular materials is a controversial issue (see refs. [1, 2] and references herein) and still attracts much attention. Experiments in conical sand piles show the existence of a depression in the normal force on the base (a dip) under the pile’s apex rather than the intuitively expected maximum [3, 4]. Much work has been done in this subject in order to explain this effect numerically and theoretically. Most of the results obtained in this domain are for models considering two dimensional (2D) piles of discs or rods [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. It is unclear however, whether in general, these results can be directly generalized to three dimensions (3D). In the case of the continuum approach proposed in ref. [3] it was noticed that the choice of the second closure equation needed for the 3D case did not have an important impact on the results obtained, so that the 2D results survive the passage from 2 to 3 dimensions. This case is a priori unlikely when discrete models are considered. One difficulty might be that a simple 3D extension of a 2D pile, which is obtained by a revolution around the symmetry axis of a 2D pile is not any regular 3D piling of spheres but rather a 3D piling of tori. Another possible
problem is that in 3D there are more packing possibilities than in 2D. Even if one accepts that each layer of spheres is arranged on a triangular lattice, there are different ways to pile layers one on top of the other, giving different arrangements; FCC, HCP, DHCP etc.

Another issue is the shape of the base. A circular base, as for the real conical sandpile, is “unnatural” for spheres in the 3D context, while in 2D it is the simplest one since for a 2D pile, a circular base is simply a segment.

Here we study a 3D case that might be considered as an extension of the one proposed in refs. [14, 12] for 2D. The configuration we study is presented in fig. 1. It is a regular 3D pyramid with triangular base and an FCC-like structure, composed of identical, spherical, stiff and frictionless particles of mass \( m \) and radii \( R = 1 \) (this piling is also known as the “spanish stack of cannonballs”) [15]. In each horizontal layer the distance between neighboring spheres is \( 2a \), where \( 1 < a < 3/2 \) so that there is no overlap between spheres, horizontally or vertically. Our aim is to calculate the contact forces everywhere in the pile, especially the normal and shear forces applied by the pile on the supporting surface. For the reduced 2D case it was shown [14] that the pressure profile on the supporting surface is uniform. This feature was also proved in ref. [16] for a similar pile with periodic vacancies.

In the following, we will index the spheres using a triplet of integers \((i, j, k)\) where \( k \) is the horizontal layer index counted from the top of the pile down (the topmost layer has \( k = 0 \)) and the couple \((i, j)\) indices the spheres in the \( k \)-th layer as shown in fig. 4. If the pile has a total of \( L \) layers then, \( 0 \leq k \leq L - 1 \), \( 0 \leq j \leq k \) and \( 0 \leq i \leq k - j \). Components of the force vectors will be given in the vectorial base shown in fig. 1.

Let us take a closer look at one of the spheres, indexed: \((i, j, k)\). We denote, as shown in fig. 3, by \( a, b, c \) the forces applied by the downwards neighboring spheres, and by \( a', b', c' \) the forces applied by the upward spheres. The weight of the particle is \( w \).

One can easily conclude from fig. 3 that

\[
\begin{align*}
a &= \alpha \hat{e}_a, & a' &= -\alpha' \hat{e}_a & \text{with} & \hat{e}_a &= (\sin \theta \sin \phi, \sin \theta \cos \phi, -\cos \theta), \\
b &= \beta \hat{e}_b, & b' &= -\beta' \hat{e}_b & \text{with} & \hat{e}_b &= (\sin \theta \sin \phi, -\sin \theta \cos \phi, -\cos \theta), \\
c &= \gamma \hat{e}_c, & c' &= -\gamma' \hat{e}_c & \text{with} & \hat{e}_c &= (-\sin \theta, 0, -\cos \theta),
\end{align*}
\]

where \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) are the norms of the vectors \( a, b, c, a', b', c' \) respectively, and \( \hat{e}_a, \hat{e}_b, \hat{e}_c \) are the unit vectors along the the vectors \( a, b, c \) respectively.

Since the pile is on top of an horizontal surface the weight of each sphere is give by

\[
w = mg(0, 0, 1).
\]

The equilibrium of each sphere is given by

\[
\sum F = a + b + c + w = 0.
\]
Combining eqs. 1, 2 and 3 and simplifying we get the following linear system:

\[ \alpha \sin \phi - \alpha' \sin \phi + \beta \sin \phi - \beta' \sin \phi - \gamma + \gamma' = 0 \]  
(4a)

\[ \alpha - \alpha' - \beta + \beta' = 0 \]  
(4b)

\[ \alpha \cos \theta - \alpha' \cos \theta + \beta \cos \theta - \beta' \cos \theta + \gamma \cos \theta - \gamma' \cos \theta = mg \]  
(4c)

Giving finally

\[ \Delta \alpha + \Delta \beta - \frac{\Delta \gamma}{\sin \phi} = 0 \]  
(5a)

\[ \Delta \alpha - \Delta \beta = 0 \]  
(5b)

\[ \Delta \alpha + \Delta \beta + \Delta \gamma = \frac{mg}{\cos \theta}, \]  
(5c)

where \( \Delta \alpha = \alpha - \alpha' \), \( \Delta \beta = \beta - \beta' \) and \( \Delta \gamma = \gamma - \gamma' \). This system is easily solved yielding

\[ \Delta \alpha = \Delta \beta = \frac{mg}{2(1 + \sin \phi) \cos \theta} \quad \text{and} \quad \Delta \gamma = \frac{\sin \phi \cdot mg}{1 + \sin \phi \cos \theta}. \]  
(6)

In the case of an equilateral triangular base we have \( \phi = \pi/6 \), so that

\[ \Delta \alpha = \Delta \beta = \Delta \gamma = \frac{mg}{3 \cos \theta}, \]  
(7)

In order to get the force acting on a specific contact one should count the number of spheres up to the free surface in the corresponding direction, each one of these spheres contributing \( mg/(3 \cos \theta) \). We denote \( N_\alpha(i, j, k) \), \( N_\beta(i, j, k) \) and \( N_\gamma(i, j, k) \) the number of particles on top of the \((i, j, k)\) sphere in the corresponding direction, so that

\[ \alpha'(i, j, k) = mg/(3 \cos \theta) \cdot N_\alpha(i, j, k), \]  
(8a)

\[ \beta'(i, j, k) = mg/(3 \cos \theta) \cdot N_\beta(i, j, k), \]  
(8b)

\[ \gamma'(i, j, k) = mg/(3 \cos \theta) \cdot N_\gamma(i, j, k). \]  
(8c)

One can easily show that (see appendix for details)

\[ N_\alpha(i, j, k) = k - i - j, \quad N_\beta(i, j, k) = i \quad \text{and} \quad N_\gamma(i, j, k) = j, \]  
(9)

yielding, when introduced into eq. 8c

\[ \alpha'(i, j, k) = (k - i - j)mg/(3 \cos \theta) \]  
(10a)

\[ \beta'(i, j, k) = img/(3 \cos \theta) \]  
(10b)

\[ \gamma'(i, j, k) = jmg/(3 \cos \theta), \]  
(10c)
which gives us the amplitude of the contact forces anywhere in the pile. If we suppose that the pile is \( L \) layers high and that the bottom layer spheres are “glued” to a flat surface, the force applied by the bottom layer spheres on the surface; \( \mathbf{F}_s(i, j) \) can easily be calculated, since from the equilibrium of a bottom layer particle we get

\[
\mathbf{F}_s(i, j) = a'(i, j, L - 1) + b'(i, j, L - 1) + c'(i, j, L - 1) + \mathbf{w},
\]

and using eqs. [4], [2] and [10] in this last equation gives

\[
\mathbf{F}_s(i, j) = \frac{mg}{3 \cos \theta} \left( -\frac{1}{2} (L - 1 - 3j) \sin \theta, -\sqrt{3}/2 (L - 1 - 2i - j) \sin \theta, (L + 2) \cos \theta \right).
\]

Thus, the normal force on the base of the pile applied by each one of the spheres is the same and has an amplitude of

\[
mg(L + 2)/3.
\]

This force is independent of \( \theta \) and hence, of \( a \). One can easily verify that the sum of the normal forces on the surface is equal to the total weight of the pile \( mg L(L + 1)(L + 2)/6 \).

The shear force applied on the surface is given by

\[
\frac{-mg \tan \theta}{3} \left( (L - 1 - 3j)/2, \sqrt{3}/2 (L - 1 - 2i - j) \right).
\]

An example of this vector field is given in fig. 4. Notice that the shear force vanishes under the apex of the pile as required from symmetry.

In conclusion, in the case of a FCC-like piling with equilateral, triangular base of stiff, identical, spherical particles we were able to calculate the contact forces between any couple of neighboring particles (eq. [10]). In particular we have shown that normal forces on the base are constant and are given by eq. [13] while the shear force is given by eq. [14]. These results are similar to those obtained in ref. [14] for the 2D case and might suggest some applicability of 2D model results to 3D or, at least, that the starting point in 3D is the same as for 2D. It is clear, however, that horizontal contacts should be considered in order to get more physical behavior like arching. But in this case, the system is hyperstatic and more complicated techniques are to be used, exactly [4, 5] or numerically [7], in order to get the correct contact network. Further investigations should be carried out in order to obtain similar results for other base shapes like hexagons and other structures, especially in the RHCP (Random Hexagonal Compact Packing) case that can be a convenient way to introduce disorder in the system so to approach a modelization of real granular piles.
Figure 1: A pile of four layers with $a = 1.2 \, R$. 
Figure 2: Indexing convention of the spheres for a layer in the pile. The layer shown is the one for \( k = 3 \) (forth from the top). Remark that force components are given in the vectorial base shown in fig. 1 and not in the one used for indexing.
Figure 3: A schematic drawing of the forces acting on a given sphere centered at $O$ (not shown). The neighboring spheres are the ones centered at $A, B$ and $C$ (layer beneath) which apply the contact forces $a, b$ and $c$ respectively, and $A', B'$ and $C'$ (layer above) which apply the contact forces $a', b'$ and $c'$ respectively. $w$ is the weight of the particle.
Figure 4: A plot of the shear force applied by the pile on the supporting surface for a 10 layer pile. Notice that the shear force vanishes under the apex of the pile as required by the symmetry of the pile.
1 Appendix

Let us start by showing by induction over \( k \) that: 
\[
N_\gamma(i, j, k) = j \quad \forall \{0 \leq j \leq k | 0 \leq i \leq k - j\}. 
\]
When \( k = 0 \) it is clear that \( N_\gamma(0, 0, 0) = 0 \) since the top particle does not have any neighbor on it. Now let us suppose that 
\[
N_\gamma(i, j, k) = j \quad \forall \{0 \leq j \leq k | 0 \leq i \leq k - j\} 
\]
for a certain \( k \) and prove it for \( k + 1 \). If \( j = 0 \) the particle is on the surface of the pile and \( N_\gamma(i, 0, k + 1) = 0 = j \). When \( j > 0 \) the \((i, j, k + 1)\) sphere is in contact with the \((i, j - 1, k)\) particle, so if \((i, j, k + 1)\) particle is centered at point \( O \) in fig. 3, the \((i, j - 1, k)\) particle is centered at \( C' \) and the contact point is at the middle of the segment \( OC' \). Thus, we conclude, using the hypothesis, that 
\[
N_\gamma(i, j, k + 1) = N_\gamma(i, j - 1, k) + 1 = j, \quad \text{Q.E.D.} 
\]

\( N_\alpha(i, j, k) \) and \( N_\beta(i, j, k) \) are obtained by a rotation of the pile around the \( z \) axis of \( 2\pi/3 \) and \( 4\pi/3 \) respectively, which correspond to applying the transformation \( \{ j \rightarrow k - i - j, i \rightarrow j \} \) once and twice respectively.

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