Accelerated over relaxation iterative method using triangle element approximation for solving 2D Helmholtz Equations

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Abstract. Weighted iterative methods particularly Accelerated Over Relaxation (AOR) method are used to solve linear system generated from triangle finite element approximation equation in solving 2D Helmholtz equation. The development of the AOR iterative method were also presented. Numerical experiments have been carried out and the results obtained confirm the superiority of the proposed iterative method.

1. Introduction
The two dimensional elliptic equation particularly the 2D Helmholtz equation can be represented mathematically as

\[ \Delta u - \alpha u = f(x, y), \quad (x, y) \in \Omega^h = [a, b] \times [a, b] \]  \hspace{1cm} (1)

in a unit square $\Omega^h$ with Dirichlet boundary conditions using finite element methods [1], resulted a large system of equations which is usually solved iteratively. Here $f(x, y)$ and $\alpha$ is a given function and known parameter respectively. Assume (1) as our model problem defined in a unit square $\Omega^h$ with spacing $\Delta x = \Delta y = \frac{1}{n} = h$ in both $x$ and $y$ directions. The most commonly used approximations is the standard triangle element approximation equation given by [6, 7]

\[ \int \int_{\Omega} \nabla u. \nabla v \, dx\,dy = \int \int_{\Omega} fv \, dx\,dy \]  \hspace{1cm} (2)

With the interpolation functions $\Omega^h$ defined on a triangulation. Essentially the linear system in Eq. (2) can also simplify as follows [4, 10, 11]

\[ \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_2 \end{bmatrix} u_{i,j} = \Psi_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} f_{i,j}, \]  \hspace{1cm} (3)

where,

\[ \Psi_0 = h^2/12, \quad \Psi_1 = -(4 + \alpha 6 \Psi_1), \quad \Psi_2 = 1 - r \Psi_1, \quad \Psi_3 = r \Psi_1. \]
In the next section, we will discuss the simplified triangle element properties of the above Eq. (2).

2. Simplification of The Triangle Element for Helmholtz Equation
In this section we will apply the simplification of the triangle element for problem (1). By considering triangle element interpolation function [4], Eq. (2) can be reduce to a simple form of Eq. (3) as follows

**Theorem 1** (see[12]). The general triangle element can be determined by \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\). Let

\[
a_i = x_i y_m - x_m y_j
\]

\[
b_i = y_j - y_m
\]

\[
c_i = x_m - x_j
\]

where \(i, j, m\) is a positive permutation of 1, 2, 3, e.g., \(i = 1, j = 2\) and \(m = 3\); \(i = 2, j = 3\) and \(m = 1\); and \(i = 3, j = 1\) and \(m = 2\). Now the element of three nonzero interpolation functions are

\[
\zeta_i (x, y) = \frac{a_i + b_i x + c_i y}{2\Delta}, \quad i, j = 1, 2, 3
\]

where \(\zeta_i (x_i, y_i) = 1, \quad \zeta_i (x_i, y_i) = 0 \quad \text{if} \quad i \neq j\), and

\[
\Delta = \frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \pm \text{area of the triangle}
\]

Proof: From Theorem 1, \(\zeta_1 (x, y) = 1\). Thus

\[
\zeta_1 (x, y) = \frac{a_1 + b_1 x + c_1 y}{2\Delta},
\]

\[
= \frac{(x_2 y_3 - x_3 y_2) + (y_3 - y_2) x + (x_3 - x_2) y}{2\Delta}
\]

so

\[
\zeta_1 (x_2, y_2) = \frac{(x_2 y_3 - x_3 y_2) + (y_2 - y_3) x + (x_3 - x_2) y}{2\Delta} = 0
\]

\[
\zeta_1 (x_3, y_3) = \frac{(x_2 y_3 - x_3 y_2) + (y_3 - y_2) x + (x_3 - x_2) y}{2\Delta} = 0
\]

\[
\zeta_1 (x_1, y_1) = \frac{(x_2 y_3 - x_3 y_2) + (y_3 - y_2) x + (x_3 - x_2) y}{2\Delta} = \frac{2\Delta}{2\Delta} = 1
\]

Following the same step afore, we can show the same feature for \(\zeta_2\) and \(\zeta_3\). Next, the following theorem, is compulsory for the simplification triangle element approach.
Theorem 2 (see[12]). By consider the same approach as in Theorem 1, we have

\[ \int \int_{\Omega} (\zeta_1)^m (\zeta_2)^n (\zeta_3)^i \, dxdy = \frac{m!n!i!}{(m+n+i+2)^2} \Delta \]  

(9)

\[ \int \int_{\Omega} \nabla \zeta_i \cdot \nabla \zeta_j \, dxdy = \frac{b_i b_j + c_i c_j}{4\Delta}, \]

\[ F_1^e = \int \int_{\Omega} \zeta_1 f(x, y) \simeq f_1 \frac{\Delta}{6} + f_2 \frac{\Delta}{12} + f_3 \frac{\Delta}{12}, \]

\[ F_2^e = \int \int_{\Omega} \zeta_2 f(x, y) \simeq f_1 \frac{\Delta}{12} + f_2 \frac{\Delta}{6} + f_3 \frac{\Delta}{12}, \]

\[ F_3^e = \int \int_{\Omega} \zeta_3 f(x, y) \simeq f_1 \frac{\Delta}{12} + f_2 \frac{\Delta}{12} + f_3 \frac{\Delta}{6}, \]

where \( f_i = f(x_i, y_j) \). Since we get the analytic form for \( \zeta_i \), the approximate interpolation function \( f(x, y) \) can be obtained by using

\[ f(x, y) \simeq f_1 \zeta_1 + f_2 \zeta_2 + f_3 \zeta_3, \]

(10)

where

\[ F_2^e \simeq \int \int_{\Omega} \zeta_1 f(x, y) \]

\[ = f_1 \int \int_{\Omega} \zeta_1^2 \, dxdy + f_2 \int \int_{\Omega} \zeta_1 \zeta_2 \, dxdy + f_3 \int \int_{\Omega} \zeta_1 \zeta_3 \, dxdy. \]

(11)

Therefore, the last part terms can be obtained from the formula (11). Similarly we can get approximation \( F_2^e \) and \( F_3^e \).

3. Accelerated Over Relaxation Iterative Method

Weighted iterative methods such as AOR iterative method has been invented by [8] via two weighted parameter \( \omega \) dan \( r \). Motivated by these concept, this paper will only focus on combination AOR iterative method and triangle element approximation equations (3) in order to get an efficient solver.

The general scheme for the AOR iterative method with two weighted \( (r \) dan \( \omega \)) parameters can be stated as [1, 2]

\[ v^{(k+1)} = T_{r,\omega} v^{(k)} + \omega(D - rL)^{-1}b \]

(12)

and \( T_{r,\omega} = (D - rL)^{-1} [(1 - \omega)D + (\omega - r)D^{-1}L + \omega D^{-1}V] \) is AOR iteration matrix. Some special well-known weighted and non-weighted iterative methods can be derived from the AOR iterative method by assigning special values to the parameters \( \omega \) and \( r \). Gauss-Seidel (GS) and Successive Overrelaxation (SOR) are special cases of the AOR iterative method can be listed as follows.

- \( r = 1 \), \( \omega = 1 \): GS
- \( r = \omega \): SOR
4. Numerical Results

In order to clarify the performance triangle element approximation equation and AOR iterative method, Eq. (12) were applied to the following problem [4]

\[ \Delta u - \alpha u = -(\cos(x + y) + \cos(x - y)) - \alpha \cos(x) \cos(y) \quad (x, y) \in \Omega = [0, 1] \times [0, 1] \quad (13) \]

subject to the Dirichlet conditions and satisfying the exact solution \( u(x, y) = \cos(x) \cos(y) \) for \( (x, y) \in \Omega^b \). Throughout the experiments, the local convergence test was the maximum absolute error with the error tolerance \( \varepsilon = 10^{-10} \). Criteria that were considered for these the proposed iterative method are number of iterations (k), execution time (t) (in seconds) and maximum errors (\( e_{\lambda} \)). Results of numerical experiments obtained have been illustrated in Table 1.

5. Conclusions

In this work, we introduced the application of a specific simplification of the triangle element approximation equation with AOR iterative method for solving Helmholtz equation. Through numerical results observed in Table 1, the findings show that number of iterations for each mesh sizes, have declined approximately 96.14 – 97.47% and 95.22 – 99.05% correspond to the SOR and AOR iterative methods compared to the GS iterative method. Indeed, the execution time of these mesh sizes for both SOR and AOR iterative methods have demonstrated faster by 93.10 – 95.15% and 93.30 – 96.18% respectively than the iterative GS method. This is due to the AOR iterative method have two weighted (\( r \) and \( \omega \)) parameters, while SOR iterative method have one weighted (\( \omega \)) parameter.

It can be concluded that the AOR method is far better than among existing SOR and GS iterative methods. Overall it seems that the whole results for the AOR iterative method is obviously parallel in good agreement with the results compared to the SOR and GS iterative methods. Finally, we have also point out that the AOR iterative method leads to a great convergence improvement. For our future works, further extension can be proceeded to improve simplification triangle element approach by using half-sweep approach [3, 4, 5] as a smoother for the AOR iterative method.

References

[1] Akhir M K M and Sulaiman J 2005. Investigation of Triangle Element Analysis for the Solutions of 2D Poisson Equations via AOR method J. of Adv. in. Math. 11 4033-4040.
[2] Akhir M K M and Sulaiman J 2005. The 4-EGAOR method for Solving Triangle Element Approximation of 2D Poisson equations App. Math. Sci. 9 5561-5571.
[3] Akhir M K M and Sulaiman J 2005. HSAOR Iterative Method for the Finite Element Solution of 2D Poisson Equations Int. J. of Math. and Comp. 2 5561-5571.
[4] Akhir M K M and Sulaiman J 2005. HSGS Method for the Finite Element Solution of Two-Dimensional Helmholtz Equations Glo. J. of. Math. 4 367-373.
[5] Akhir M K M and Sulaiman J 2005. Numerical Performance of Triangle Element Approximation Equations using 4-EDGAOR Method Int. J. of. Math. Any. 4 367-373.
[6] Fletcher C A J 1978. The Galerkin method: An introduction. In. Noye, J. (nyt.). Numerical Simulation of Fluid Motion, North-Holland Publishing Company, Amsterdam. 113-170.
[7] Fletcher C A J 1984. Computational Galerkin method. Springer Series in Computational Physics. Springer-Verlag, New York.
[8] Hajidimos A 1978. Accelerated Over Relaxation method Math. of. Comp. 149-157.
[9] Vichnevetsky R 1981. Computer Methods for Partial Differential Equations, Vol I. New Jersey, Prentice-Hall.
[10] White R E 1985. An introduction to the FE method with applications to non-linear problems, Wiley-Interscience.
[11] Zienkiewicz O C 1975. Why finite elements?. In. Gallagher R H, Oden J T C.Taylor C Zienkiewicz O C (Eds). Finite Elements In Fluids Volume. London, John Wiley and Sons.
Table 1. Comparison of $k$, $t$ and $e_N$ for the iterative methods with $\omega$ and $r$ optimum.

| $\alpha$ | Methods | $k$   |     |     |
|----------|---------|-------|-----|-----|
|          |         | 284   | 308 | 332 | 356 |
| 0        | GS      | 115954| 134823| 154979| 176405|
|          | SOR     | 3428  | 4288 | 5960 | 6815 |
|          | AOR     | 1890  | 1939 | 2668 | 3032 |
| 5        | GS      | 93990 | 109305| 125668| 148976|
|          | SOR     | 2376  | 3100 | 4154 | 5254 |
|          | AOR     | 1180  | 1784 | 2073 | 2788 |

| $e_N$ |
|-------|
| $\alpha$ | GS      | 1.0913e-6| 1.1936e-6| 1.3165e-6| 1.4576e-6|
| SOR     | 3.0217e-7| 2.6481e-7| 2.4124e-7| 2.1950e-7|
| AOR     | 2.8268e-7| 2.4128e-7| 2.0723e-7| 1.8023e-7|
| $\alpha$ | GS      | 1.6410e-6| 1.6046e-6| 1.6095e-6| 1.6476e-6|
| SOR     | 4.1942e-7| 4.2610e-7| 4.4910e-7| 4.2509e-7|
| AOR     | 1.1380e-7| 1.3179e-7| 1.3199e-7| 1.5167e-7|