FLUCTUATION THEORY FOR LEVEL-DEPENDENT LÉVY RISK PROCESSES

IRMINA CZARNA, JOSÉ-LUIS PÉREZ, TOMASZ ROLSKI, AND KAZUTOSHI YAMAZAKI

Abstract. A level-dependent Lévy process solves the stochastic differential equation 
\[ dU(t) = dX(t) - \phi(U(t)) \, dt, \]
where \( X \) is a spectrally negative Lévy process. A special case is a multi-refracted Lévy process with 
\[ \phi_k(x) = \sum_{j=1}^{k} \delta_j 1_{\{x \geq b_j\}}. \]
A general rate function \( \phi \) that is non-decreasing and continuously differentiable is also considered. We discuss solutions of the above stochastic differential equation and investigate the so-called scale functions, which are counterparts of the scale functions from the theory of Lévy processes. We show how fluctuation identities for \( U \) can be expressed via these scale functions. We demonstrate that the derivatives of the scale functions are solutions of Volterra integral equations.

Keywords. Refracted Lévy process, multi-refracted Lévy process, level-dependent Lévy process, Lévy process, Volterra equation, fluctuation theory.

Date: December 4, 2017.

2000 Mathematics Subject Classification. 60J99, 91B30, 60G40.

I. Czarna is partially supported by the National Science Centre Grant No. 2015/19/D/ST1/01182. T. Rolski is partially supported by the National Science Centre Grant No. 2015/17/B/ST1/01102. J. L. Pérez is supported by CONACYT, project no. 241195. K. Yamazaki is supported by MEXT KAKENHI grant no. 17K05377.
1. Introduction

In this paper, we consider a level-dependent Lévy process \( U(t) \), which solves the following stochastic differential equation (SDE):

\[
dU(t) = dX(t) - \phi(U(t)) \, dt,
\]

where \( X(t) \) is a spectrally negative Lévy process. In Chapter VII of the book of Asmussen and Albrecher [3], the following alternative form of (1) is analysed:

\[
dU(t) = -dS(t) + p(U(t)) \, dt,
\]

where \( S(t) \) is a compound Poisson process with non-negative summands, and \( p(x) > 0 \) for all \( x \in \mathbb{R} \). The function \( p(x) \) is a level-dependent premium rate. Notice that in (1), if \( X(t) \) has paths of bounded variation, then we can write \( X(t) = -A(t) + ct \), where \( A(t) \) is a pure jump subordinator, and (1) can be rewritten as

\[
dU(t) = -dA(t) - (\phi(U(t) - c)) \, dt = -dA(t) + p(U(t)) \, dt,
\]

by setting \( p(x) = c - \phi(x) \). Such a level-dependent risk process is dual to a storage process \( V \) with a general release rate, which solves

\[
dV(t) = -dX(t) - p(V(t)) \, dt,
\]

with the additional condition that \( p(0) = 0 \) (see, e.g., Chapter XIV of Asmussen [4]).

The main contribution of this paper is to develop a theory of scale functions for level-dependent Lévy risk processes. If \( X \) is a spectrally negative Lévy process with Lévy exponent \( \psi(\lambda) \), then for all \( q \geq 0 \) the unique solution \( W(q) \) of the equation \( (\psi(\lambda) - q)L[W(q)](\lambda) = 1 \) is said to be a scale function, where by \( L[f](\lambda) = \int_{0}^{\infty} e^{-\lambda x} f(x) \, dx \) we denote the Laplace transform of the function \( f \). Moreover, \( W(q)(x) = 0 \) for all \( x < 0 \). Required notations, definitions, and facts regarding Lévy processes are recalled in Section 1.1.

It turns out that some exit probabilities, and moreover the solutions of interest in risk theory for Lévy processes, can be expressed by scale functions. The name scale function comes from the formula \( W(0)(x - d)/W(0)(a - d) \) for the probability of exiting the interval \((d, a)\) via the point \( a \) for the process \( X \) such that \( X(0) = x \). A useful survey paper regarding the theory of scale functions of Lévy processes and their applications in risk theory was provided by Kuznetsov et al. [9]. We also refer the reader to the books of Kyprianou [10, 11].

Kyprianou and Loeffen [12] developed a parallel theory of processes fulfilling equation (1) with \( \phi(x) = \delta \mathbf{1}_{\{x>b\}} \), and called the solution \( U \) a refracted Lévy process. In their theory, it is essential that \( \delta > 0 \). In this paper, we extend the theory first to solutions \( U \) of (1) with some non-decreasing function

\[
\phi_k(x) = \sum_{j=1}^{k} \delta_j \mathbf{1}_{\{x \geq b_j\}},
\]

called a multi-refracted Lévy process, and then to the case of a general non-decreasing continuously differentiable \( \phi \). We prove the existence and uniqueness of a solution to (1) for rate functions \( \phi \) fulfilling the conditions mentioned above. Notice that without these assumptions the uniqueness problem for SDE (1) does not have an obvious solution. Most known results in the literature require either some Lipschitzian properties for \( \phi \) or a Brownian component of \( X \). See, for example, [14].
The paper consists of two parts. In the first part, we develop some fluctuation formulas for multi-refracted Lévy processes. This theory represents a direct, but non-trivial, extension of the paper of Kyprianou and Loeffen [12]. We derive formulas for the two-sided exit problem, one-sided exit problem, resolvents, and ruin probability. We express the solutions to these problems using the scale functions \( w_k^{(q)} \) and \( z_k^{(q)} \). We also demonstrate that the derivatives of \( w_k^{(q)} \) and \( z_k^{(q)} \) fulfill Volterra integral equations of the second kind. In the second part, we analyze processes with general \( \phi \) (or \( p \)). Here, we assume that \( \phi \) is non-decreasing and continuously differentiable. Formulas are developed by approximating \( \phi \) by the so-called approximating sequence \( \phi_n \), where \( \phi_n \) are rate functions of some multi-refracted processes. In the limit, we obtain a level-dependent Lévy risk process \( U \) and the scale functions \( w^{(q)} \) and \( z^{(q)} \). In this case, the uniqueness of the solution to (1) is clear. The derivatives of these scale functions are the solutions of Volterra integral equations of the second kind, as in the multi-refracted case. As corollaries, we derive ruin probabilities.

The theory of level-dependent Lévy risk processes or storage processes has a long history. It was mainly developed for compound Poisson processes with the ruin function being the main interest (or the stationary distribution in the context of storage processes). There has been little work regarding scale functions for such processes. Of particular interest is the paper by Brockwell et al. [6], where in the setting of storage processes, the process \( X \) is a bounded variation Lévy process. The existence of a solution (3) was studied there, and stationary distributions of \( V \) were characterised as solutions of Volterra equations. For further references and historical comments, we refer the reader to Asmussen [4] and Asmussen and Albrecher [3].

1.1. Basic concepts and notations for Lévy processes. Here we present basic concepts and notations from the theory of Lévy processes (which can be found in the books of Kyprianou [10], [11]). In this paper, \( X = \{X(t), t \geq 0\} \) is a spectrally negative Lévy process on the filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P}) \). To avoid trivialities, we exclude the case where \( X \) has monotone paths. As the Lévy process \( X \) has no positive jumps, its moment generating function exists for all \( \lambda \geq 0 \):

\[
\mathbb{E} \left[ e^{\lambda X(t)} \right] = e^{\psi(\lambda)},
\]

where

\[
\psi(\lambda) := \gamma \lambda + \frac{1}{2} \sigma^2 \lambda^2 + \int_0^\infty \left( e^{-\lambda z} - 1 + \lambda z 1_{(0,1]}(z) \right) \Pi(dz),
\]

for \( \gamma \in \mathbb{R} \) and \( \sigma \geq 0 \), and where \( \Pi \) is a \( \sigma \)-finite measure on \( (0, \infty) \) such that

\[
\int_0^\infty (1 \wedge z^2) \Pi(dz) < \infty.
\]

The measure \( \Pi \) is called the Lévy measure of \( X \). Finally, note that \( \mathbb{E} [X(1)] = \psi'(0+) \).

We will use the standard Markovian notation: the law of \( X \) when starting from \( X_0 = x \) is denoted by \( \mathbb{P}_x \), and the corresponding expectation by \( \mathbb{E}_x \). We simply write \( \mathbb{P} \) and \( \mathbb{E} \) when \( x = 0 \).

When a surplus process \( X \) has paths of bounded variation, that is, when \( \int_0^1 z \Pi(dz) < \infty \) and \( \sigma = 0 \), we can write

\[
X(t) = ct - A(t),
\]

where \( c := \gamma + \int_0^1 z \Pi(dz) \) is the drift of \( X \) and \( A = \{A(t), t \geq 0\} \) is a pure jump subordinator.
2. Multi-refracted Lévy processes

For \( k \geq 1, 0 < \delta_1, \ldots, \delta_k, \) and \( -\infty < b_1 < \cdots < b_k < \infty, \) we consider the function

\[
\phi(x) = \phi_k(x) = \sum_{i=1}^{k} \delta_i 1_{\{x > b_i\}}.
\]

The corresponding SDE (1) is given by

\[
dU_k(t) = dX(t) - \sum_{i=1}^{k} \delta_i 1_{\{U_k(t) > b_i\}} dt.
\]

In this section, we show that (5) admits a unique solution in the strong sense for the case that \( X \) is a spectrally negative Lévy process (not the negative of a subordinator). Therefore, when \( X \) has paths of bounded variation we assume that

\[
0 < \delta_1 + \cdots + \delta_k < c := \gamma + \int_{\langle 0,1 \rangle} z \Pi(dz).
\]

Note that the special case with \( k = 1 \) was already studied in Kyprianou and Loeffen [12]. Furthermore, we study the dynamics of multi-refracted Lévy processes by establishing a suite of identities, written in terms of scale functions, related to the one- and two-sided exit problems and resolvents. We also present the formula for the ruin probability. Finally, we show that the scale functions for the multi-refracted processes satisfy a Volterra-type integral equation, which we use in Section 3 to define the scale functions for general level-dependent processes.

Observe that from the SDE (5), for any \( 0 \leq j \leq k \) in each level interval \((b_j, b_{j+1}] \) (where \( b_0 := -\infty \) and \( b_{k+1} := \infty \)) the process \( U_k \) evolves as \( X_j := \{X(t) - \sum_{i=1}^{j} \delta_i t : t \geq 0\} \), which is a spectrally negative Lévy process that is not the negative of a subordinator, because we assume (6). The Laplace exponent of \( X_j \) on \([0, \infty)\) is given by

\[
\lambda \mapsto \psi_j(\lambda) := \psi(\lambda) - (\delta_j + \cdots + \delta_j) \lambda,
\]

with right-inverse \( \varphi_j(q) = \sup\{\lambda \geq 0 \mid \psi_j(\lambda) = q\} \). We use the notation \( U_0 = X_0 := X \).

Moreover, for all \( 0 \leq j \leq k, \) \( X_j \) has the same Lévy measure \( \Pi \) and diffusion coefficient \( \sigma \) as \( X \). Furthermore, it is easy to notice the recursive relationship between the processes \( X_j \) and \( X_{j+1} \), i.e., \( X_{j+1} = \{X_j(t) - \delta_{j+1} t : t \geq 0\} \).

**Theorem 1 (Existence).** For \( k \geq 1, 0 < \delta_1, \ldots, \delta_k, \) and \( -\infty < b_1 < \cdots < b_k < \infty, \) there exists a strong solution \( U_k \) to the SDE (5).

**Proof.** As in Section 3 of [12], we provide a pathwise solution to (5) when the driving Lévy process \( X \) is of bounded variation. We defer the proof for the unbounded variation case to Appendix B. We prove the result by induction. First, the base case \( k = 1 \) holds by [12]. Next, we assume that there exists a strong solution \( U_{k-1} \) to the SDE

\[
dU_{k-1}(t) = dX(t) - \sum_{i=1}^{k-1} \delta_i 1_{\{U_{k-1}(t) > b_i\}} dt
\]

with the initial condition \( U_{k-1}(0) = x \) for any arbitrary \( x \), in order to show that we can construct a process \( U_k \) that solves the SDE (5).

To this end, we define a sequence of times \((S_n)_{n \geq 0}\) and processes \((\overline{U}_{k-1}(t), t \geq S_n)_{n \geq 0}\) recursively as follows. First, we set \( \overline{U}_{k-1}(t) := U_{k-1}(t), t \geq S_0 := 0, \) which exists and
solves (7) by the inductive hypothesis. For \( n \geq 1 \), we recursively set
\[
T_n := \inf \left\{ t > S_{n-1} : U_{k-1}^{n-1}(t) \geq b_k \right\},
\]
\[
S_n := \inf \left\{ t > T_n : U_{k-1}^{n-1}(t) - \delta_k(t - T_n) < b_k \right\},
\]
and \( \{U_{k-1}^{n}, t \geq S_n\} \), starting at \( U_{k-1}^{n}(S_n) = U_{k-1}^{n-1}(S_n) - \delta_k(S_n - T_n) \) and solving
\[
dU_{k-1}^{n}(t) = dX(t) - \sum_{i=1}^{k-1} \delta_i 1_{U_{k-1}^{n}(t) > b_i} dt,
\]which again exists by the inductive hypothesis.

Here, one can observe that the difference between any two consecutive times \( S_n \) and \( T_n \) is strictly positive. This is because of the fact that on \([T_n, S_n]\), \( dU_{k-1}^{n} = dX_{k-1} - \delta_k \), and \( X_{k-1} \) is of bounded variation with drift \( c - \sum_{j=1}^{k-1} \delta_j > 0 \). Thus \( b_k \) is irregular for \((-\infty, b_k)\), and hence \( T_n < S_n \). On the other hand, \( U_{k-1} \) always jumps at \( S_n \), while it is continuous (creeps upwards) at \( T_n \), and so \( T_n < S_n < T_{n+1} \).

Now, proceeding as in [12], we construct a solution \( \{U_k(t) : t \geq 0\} \) to (5) as follows:
\[
U_k(t) = \begin{cases} 
U_{k-1}^{n}(t) & \text{for } t \in [S_n, T_{n+1}) \text{ and } n = 0, 1, 2, \ldots, \\
U_{k-1}^{n-1}(t) - \delta_k(t - T_n) & \text{for } t \in [T_n, S_n) \text{ and } n = 1, 2, \ldots.
\end{cases}
\]Our final step is to prove that the above pathwise-constructed process \( U_k \) is a strong solution to the equation (5).

First, for \( t \in [S_0, T_1] \) we note that \( U_k(t) = U_{k-1}^{0}(t) = U_{k-1}(t) \) and \( 1_{\{U_k(t) > b_k\}} = 0 \), and therefore
\[
U_k(t) = X(t) - \sum_{i=1}^{k-1} \delta_i \int_0^t 1_{\{U_{k-1}(s) > b_i\}} ds = X(t) - \sum_{i=1}^{k} \delta_i \int_0^t 1_{\{U_k(s) > b_i\}} ds,
\]which solves (5). Now, let \( t \in [T_1, S_1] \). Then, \( U_k(t) = U_{k-1}(t) - \delta_k(t - T_1) \), and hence
\[
U_k(t) = U_{k-1}(t) - \delta_k(t - T_1) = X(t) - \sum_{i=1}^{k-1} \delta_i \int_0^t 1_{\{U_{k-1}(t) > b_i\}} dt - \delta_k(t - T_1)
\]
\[
= X(t) - \sum_{i=1}^{k-1} \delta_i \int_0^t 1_{\{U_{k-1}(t) > b_i\}} dt - \sum_{i=1}^{k-1} \delta_i \int_{T_1}^t 1_{\{U_{k-1}(t) > b_i\}} dt - \delta_k(t - T_1)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_0^t 1_{\{U_k(t) > b_i\}} dt - \sum_{i=1}^{k-1} \delta_i \int_{T_1}^t 1_{\{U_{k-1}(t) > b_i\}} dt
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_0^t 1_{\{U_k(t) > b_i\}} dt,
\]where the second to last equality holds because \( U_k = U_{k-1} \leq b_k \) on \([0, T_1]\), and the last equality holds because \( U_{k-1} \geq U_k \geq b_k \geq b_i \) for \( i \leq k \) on \([T_1, S_1]\). In particular, we have that
\[
U_k(S_1) = U_{k-1}(S_1) - \delta_k(S_1 - T_1) = X(S_1) - \sum_{i=1}^{k} \delta_i \int_0^{S_1} 1_{\{U_k(t) > b_i\}} dt.
\]
From (8) and (10), it holds that
\[
\overline{U}_{k-1}(t) = U_k(S_1) + (X(t) - X(S_1)) - \sum_{i=1}^{k-1} \delta_i \int_{S_1}^t 1_{\{U_{k-1}(t) > b_i\}} \, dt
\]
\[
= X(S_1) - \sum_{i=1}^{k} \delta_i \int_{0}^{S_1} 1_{\{U_k(t) > b_i\}} \, dt + (X(t) - X(S_1)) - \sum_{i=1}^{k-1} \delta_i \int_{S_1}^t 1_{\{U_{k-1}(t) > b_i\}} \, dt
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_{0}^{S_1} 1_{\{U_k(t) > b_i\}} \, dt - \sum_{i=1}^{k-1} \delta_i \int_{S_1}^t 1_{\{U_{k-1}(t) > b_i\}} \, dt.
\]
\[
(11)
\]

Then, for \( t \in [S_1, T_2) \), by (8), (9), and the fact that \( \overline{U}_{k-1} = U_k \leq b_k \) on \([S_1, T_2)\), we have
\[
U_k(t) = \overline{U}_{k-1}(t) - \delta_k(t - T_2)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_{0}^{S_1} 1_{\{U_k(t) > b_i\}} \, dt - \sum_{i=1}^{k-1} \delta_i \int_{S_1}^{T_2} 1_{\{U_{k-1}(t) > b_i\}} \, dt
\]
\[
- \sum_{i=1}^{k-1} \delta_i \int_{T_2}^{t} 1_{\{U_{k-1}(t) > b_i\}} \, dt - \delta_k(t - T_2)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_{0}^{t} 1_{\{U_k(t) > b_i\}} \, dt.
\]

On \([T_2, S_2)\), it follows from (8) and (9) that
\[
U_k(t) = \overline{U}_{k-1}(t) - \delta_k(t - T_2)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_{0}^{S_1} 1_{\{U_k(t) > b_i\}} \, dt - \sum_{i=1}^{k-1} \delta_i \int_{S_1}^{T_2} 1_{\{U_{k-1}(t) > b_i\}} \, dt
\]
\[
- \sum_{i=1}^{k-1} \delta_i \int_{T_2}^{S_2} 1_{\{U_{k-1}(t) > b_i\}} \, dt - \delta_k(t - T_2)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_{0}^{t} 1_{\{U_k(t) > b_i\}} \, dt.
\]

The previous identity follows from the fact that \( U_{k-1}^1 = U_k \leq b_k \) on \([S_1, T_2)\) and \( \overline{U}_{k-1}^1 \geq U_k \geq b_k \geq b_i \) for all \( 1 \leq i \leq k \) on \([T_2, S_2)\).

Hence, by proceeding by induction on the time intervals \([S_n, T_{n+1})\) and \([T_{n+1}, S_{n+1})\) for \( n = 2, 3, \ldots \) we obtain that, for any \( t > 0 \), the process defined in (9) fulfills (5).

This completes the proof of the existence of a strong solution for the bounded variation case.

The proof of the uniqueness follows verbatim from Proposition 15 of [12] (see also Example 2.4 on p. 286 of [8],) using the fact that the function \( \phi_k \) is non-decreasing for \( k = 1, 2, \ldots \). Hence, we have the following result.

**Lemma 2. (Uniqueness)** Assume that \( \delta_j > 0 \) for all \( j = 1, 2, \ldots, k \). Then, there exists a unique solution to (7).

Using the argument given in Remark 3 of [12], we can obtain the following.

**Lemma 3. (Strong Markov property)** For each \( k \geq 1 \), the process \( U_k \), which is the unique solution to (5), is a strong Markov process.
2.1. **Scale functions.** In this section, we present a few facts concerning scale functions that are important for writing many fluctuation identities for spectrally negative Lévy processes.

First, for any $k \geq 0$ the scale functions $W_k^{(q)}$ and $Z_k^{(q)}$ of $X_k$ are defined as follows. For $q \geq 0$, the $q$-scale function $W_k^{(q)}$ of the process $X_k$ is defined as the continuous function on $[0, \infty)$ whose Laplace transform satisfies

$$
\int_0^\infty e^{-\lambda y} W_k^{(q)}(y) dy = \frac{1}{\psi_k(\lambda) - q}, \quad \text{for } \lambda > \varphi_k(q).
$$

The scale function $W_k^{(q)}$ is positive, strictly increasing and continuous for $x \geq 0$. We extend $W_k^{(q)}$ to the whole real line by setting $W_k^{(q)}(x) = 0$ for $x < 0$. In particular, we write $W_k = W_k^{(0)}$ when $q = 0$. We also define

$$
Z_k^{(q)}(x) = 1 + q \int_0^x W_k^{(q)}(y) dy, \quad x \in \mathbb{R}.
$$

Note that $Z_k = Z_k^{(0)} = 1$ when $q = 0$. In particular, for $k = 0$, we set $W^{(q)} = W_0^{(q)}$ and $Z^{(q)} = Z_0^{(q)}$ for the scale functions of the spectrally negative Lévy process $X$.

For any $k \geq 0$, the initial value of $W_k^{(q)}$ is given by

$$
W_k^{(q)}(0) = \begin{cases} 
\frac{1}{e^{-\sum_{j=1}^k \delta_j}}, & \text{when } \sigma = 0 \text{ and } \int_0^1 z \Pi(dz) < \infty, \\
0, & \text{otherwise}.
\end{cases}
$$

In [12], many fluctuation identities, including the probability of ruin for $U_1$, have been derived using the scale functions for $U_1$. For $q \geq 0$, $x, d \in \mathbb{R}$ and $b_1 > d$, define

$$
w_1^{(q)}(x; d) := W^{(q)}(x - d) + \delta_1 \int_{b_1}^x W_1^{(q)}(x - y) W^{(q)}(y - d) dy.
$$

In the remainder of this paper, we will use the convention that $\int_a^b = 0$ if $b < a$. Hence, note that when $x \leq b_1$ we have that

$$
w_1^{(q)}(x; d) = W^{(q)}(x - d).
$$

The following definition will be useful for a compact presentation of the main results of Section 2.2

**Definition 4.** For any $0 \leq k$, $-\infty =: b_0 < b_1 < \cdots < b_k < b_{k+1} := \infty$, and $y \in \mathbb{R}$, define

$$
\Xi_{\phi_k}(y) := 1 - W^{(q)}(0) \phi_k(y).
$$

For the unbounded variation case, we note that the fact that $W^{(q)}(0) = 0$ implies that $\Xi_{\phi_k}(y) = 1$ for all $y \in \mathbb{R}$. On the other hand, in the bounded variation case with $y \in (b_i, b_{i+1}]$ and $i \leq k - 1$, we have that

$$
\Xi_{\phi_k}(y) = 1 - W^{(q)}(0) \sum_{j=1}^i \delta_j = \prod_{j=1}^i \left(1 - \delta_j W_j^{(q)}(0)\right),
$$

and similarly for $y > b_k$, we obtain $\Xi_{\phi_k}(y) = \prod_{j=1}^k \left(1 - \delta_j W_j^{(q)}(0)\right)$. 

2.2. Exit problems and resolvents. For $a \in \mathbb{R}$ and $k \geq 1$, define the following first-passage stopping times:
\[
\tau_k^{a,-} := \inf\{t > 0 : X_k(t) < a\} \quad \text{and} \quad \tau_k^{a,+} := \inf\{t > 0 : X_k(t) \geq a\},
\]
\[
\kappa_k^{a,-} := \inf\{t > 0 : U_k(t) < a\} \quad \text{and} \quad \kappa_k^{a,+} := \inf\{t > 0 : U_k(t) \geq a\},
\]
with the convention that $\inf \emptyset = \infty$.

First, we state the result for the two-sided exit problem for $k$-multi-refracted Lévy processes. The special case with $k = 1$ was already derived in Theorem 4 of Kyprianou and Loeffen [12]. We remark that for $X_k$ and $U_1$ we have, for example for $a > d$ and $x \leq a$, that
\[
E_x \left[ e^{-q \tau_k^{a,+}} 1_{\{\tau_k^{a,+} < \tau_k^{d,-}\}} \right] = \frac{W_k^{(q)}(x-d)}{W_k^{(q)}(a-d)}, \quad k \geq 0,
\]
\[
E_x \left[ e^{-q \kappa_k^{a,+}} 1_{\{\kappa_k^{a,+} < \kappa_k^{d,-}\}} \right] = \frac{w_1^{(q)}(x;d)}{w_1^{(q)}(a;d)},
\]
where the latter holds by [12].

**Theorem 5. (Two-sided exit problem)**

Fix $k \geq 1$ and $q \geq 0$.

(i) For $d < b_1 < \cdots < b_k \leq a$ and $d \leq x \leq a$, we have
\[
E_x \left[ e^{-q \kappa_k^{a,+}} 1_{\{\kappa_k^{a,+} < \kappa_k^{d,-}\}} \right] = \frac{w_k^{(q)}(x;d)}{w_k^{(q)}(a;d)},
\]
where $w_k^{(q)}$ is defined by the recursion
\[
w_k^{(q)}(x;d) := w_{k-1}^{(q)}(x;d) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) w_{k-1}^{(q)}(y;d)dy.
\]

The function $w_k^{(q)}(x;d)$ is the scale function associated with the process $U_{k-1}$, and the initial function $w_1^{(q)}(x;d)$ is defined in [14].

(ii) For $0 < b_1 < \cdots < b_k \leq a$ and $0 \leq x \leq a$,
\[
E_x \left[ e^{-q \kappa_k^{a,-}} 1_{\{\kappa_k^{a,-} < \kappa_k^{d,+}\}} \right] = z_k^{(q)}(x) - \frac{z_k^{(q)}(a)}{w_k^{(q)}(a)} w_k^{(q)}(x),
\]
where $z_k^{(q)}$ is defined by the recursion
\[
z_k^{(q)}(x) := z_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) z_{k-1}^{(q)}(y)dy.
\]

The scale function $z_k^{(q)}(x)$ is associated with the process $U_{k-1}$, and the initial function $z_1^{(q)}(x)$ given by
\[
z_1^{(q)}(x) = Z^{(q)}(x) + \delta_1 q \int_{b_1}^{x} W_1^{(q)}(x-y) W^{(q)}(y)dy.
\]

For $q = 0$, we write $w_k^{(0)}(x;d) = w_k(x;d)$ with $k \geq 0$. Moreover, for $d = 0$ we denote $w_k^{(q)}(x;0) = w_k^{(q)}(x)$. Furthermore, we also use the same convention for the functions $z_k^{(q)}$ and $u_k^{(q)}$. 
Corollary 6. (One-sided exit problem)

Fix $k \geq 1$.

(i) For $x \geq 0$, $b_1 > 0$ and $q > 0$, we have

$$
\mathbb{E}_x \left[ e^{-q \kappa_{k}^{-} \cdot} 1_{\{ \kappa_{k}^{-} < \infty \}} \right] = z_k^{(q)}(x) - \int_{b_k}^{\infty} e^{-q \varphi_k(z)} z_k^{(q)'}(z) dz \frac{w_k^{(q)}(x)}{\int_{b_k}^{\infty} e^{-q \varphi_k(z)} w_k^{(q)'}(z) dz} w_k^{(q)}(x). 
$$

(ii) For $b_1 < \cdots < b_k \leq a$, $x \leq a$ and $q \geq 0$,

$$
\mathbb{E}_x \left[ e^{-q \kappa_{k}^{+} \cdot} 1_{\{ \kappa_{k}^{+} < \infty \}} \right] = \frac{u_k^{(q)}(x)}{u_k^{(q)}(a)}.
$$

Here, $u_k^{(q)}$ is defined by the recursion

$$
 u_k^{(q)}(x) := u_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) u_{k-1}^{(q)'}(y) dy, 
$$

where the function $u_{k-1}^{(q)}(x)$ is associated with the process $U_{k-1}$, and the initial function $u_1^{(q)}(x)$ is given by

$$
 u_1^{(q)}(x) = e^{\Phi(q)x} + \delta_1 \Phi(q) \int_{b_1}^{x} e^{\Phi(q)y} W_1^{(q)}(x-y) dy. 
$$

Theorem 7. (Resolvents)

Fix a Borel set $\mathcal{B} \subseteq \mathbb{R}$, $k \geq 1$.

(i) For $d < b_1 < \cdots < b_k \leq a$, $d \leq x \leq a$ and $q \geq 0$,

$$
\mathbb{E}_x \left[ \int_{0}^{\kappa_{k}^{+} \wedge \kappa_{k}^{-}} e^{-qt} 1_{\{ U_k(t) \in \mathcal{B} \}} dt \right] = \int_{\mathcal{B} \cap (d,a)} \frac{w_k^{(q)}(x;d)}{w_k^{(q)}(a;d)} w_k^{(q)}(a;y) - w_k^{(q)}(x;y) \Xi \phi_k(y) dy,
$$

where the scale function $w_k^{(q)}(x;z)$ is defined in Theorem 5 (i).

(ii) For $x \geq 0$, $b_1 > 0$ and $q > 0$,

$$
\mathbb{E}_x \left[ \int_{0}^{\kappa_{k}^{-}} e^{-qt} 1_{\{ U_k(t) \in \mathcal{B} \}} dt \right] = \int_{\mathcal{B} \cap (0,\infty)} \frac{v_k^{(q)}(y)}{v_k^{(q)}(0)} v_k^{(q)}(y) - w_k^{(q)}(x;y) \Xi \phi_k(y) dy,
$$

where $v_k^{(q)}(y) := \delta_k \int_{b_k}^{\infty} e^{-q \varphi_k(z)} w_{k-1}^{(q)'}(z;y) dz$, and the scale function $w_k^{(q)}(x;z)$ is defined in Theorem 5 (i).

(iii) For $x, b_k \leq a$ and $q \geq 0$,

$$
\mathbb{E}_x \left[ \int_{0}^{\kappa_{k}^{+}} e^{-qt} 1_{\{ U_k(t) \in \mathcal{B} \}} dt \right] = \int_{\mathcal{B} \cap (-\infty,a)} \frac{u_k^{(q)}(x)}{u_k^{(q)}(a)} w_k^{(q)}(a;y) - w_k^{(q)}(x;y) \Xi \phi_k(y) dy,
$$

where the functions $w_k^{(q)}(x;y)$ and $u_k^{(q)}(x)$ are defined in Theorems 5 and 6 respectively.

(iv) For $x \in \mathbb{R}$ and $q > 0$,

$$
\mathbb{E}_x \left[ \int_{0}^{\infty} e^{-qt} 1_{\{ U_k(t) \in \mathcal{B} \}} dt \right] = \int_{\mathcal{B}} \frac{u_k^{(q)}(x) \int_{b_k}^{\infty} e^{-q \varphi_k(z)} w_{k-1}^{(q)'}(z;y) dz}{\int_{b_k}^{\infty} e^{-q \varphi_k(z)} u_{k-1}^{(q)'}(z) dz} - w_k^{(q)}(x;y) \Xi \phi_k(y) dy,
$$

where $\Xi \phi_k(y)$ is associated with the process $U_k$.
where the functions $w_k^{(q)}(x;y)$ and $u_k^{(q)}(x)$ are defined in Theorems 2 and 4 respectively.

Note that in the above expressions, the derivatives of the scale functions $w_k^{(q)}$, $z_k^{(q)}$, and $u_k^{(q)}$ appear. We refer to Lemma 9 for a further explanation regarding why the integrals given in the above identities are well defined.

**Corollary 8. (Ruin probability)** For any $k \geq 1$, $b_1 > 0$, $x \geq 0$, and $0 < \sum_{j=1}^{k} \delta_j < \mathbb{E}[X_1]$, we have

\[
\mathbb{P}_x \left( k_{-}^{0,-} < \infty \right) = 1 - \frac{\mathbb{E}[X_1] - \sum_{j=1}^{k} \delta_j}{1 - \sum_{j=1}^{k} \delta_j b_{j-1}(b_j)} w_k(x). \tag{29}
\]

The proofs of the above theorems and corollaries are given in the Appendix, because the arguments tend to be technical, and the results intuitively hold in a similar manner to the case presented in [12]. In Appendix A, we begin by providing the proofs of the identities [13] and [24] under the assumption that $X$ is a Lévy process that has paths of bounded variation. The proofs for the case of unbounded variation are presented in Appendix B. We remark here that our further reasoning is common for a general Lévy process $X$, and hence using formulas [13] and [25] we obtain the remainder of the identities for a general class of Lévy processes. For details, we refer the reader to Appendix C.

2.3. **Analytical properties of multi-refracted scale functions.** In this section, we summarize the properties of the multi-refracted scale functions $w_k^{(q)}$ and $z_k^{(q)}$, which will be crucial for further proofs in this paper. Moreover, the following results are important for many applied probability models, such as optimal dividend problems.

**Lemma 9. (Smoothness)** In general, $w_k^{(q)}(\cdot;d)$ is a.e. continuously differentiable. In particular, if $X$ is of unbounded variation, then $w_k^{(q)}(\cdot;d)$ is $C^1(d,\infty)$. On the other hand, if we assume that $W_k^{(q)}(\cdot-d) \in C^1(d,\infty)$ for all $k \geq 0$ and $X$ is of bounded variation, then $w_k^{(q)}(\cdot;d)$ is also $C^1((d,\infty)\setminus\{b_1,\ldots,b_k\})$, where in particular $w_0^{(q)}(\cdot;d)$ is $C^1(d,\infty)$.

**Proof.** First take $k = 0$. Because $w_0^{(q)}(\cdot,d) = W^{(q)}(\cdot-d)$, the statement of the Lemma follows verbatim from [3].

Next, for induction, we assume that for the bounded variation case $w_k^{(q)}(x;d) \in C^1(d,\infty)$ except at $b_1,\ldots,b_{k-1}$. From [19], we compute

\[
w_k^{(q)'}(x;d) = \begin{cases} 
  w_k^{(q)'}(x;d) & \text{for } x < b_k \\
  w_{k-1}^{(q)'}(x;d)(1 + \delta_k W_k^{(q)}(0)) + \delta_1 \int_{b_k}^{x} W_k^{(q)'}(x-z)w_{k-1}^{(q)'}(z;d)dz & \text{for } x > b_k.
\end{cases}
\]

The above equation, together with the inductive assumption and the fact that function $W_k^{(q)}$ is in general a.e. continuously differentiable, gives that $w_k^{(q)}(x;d)$ is also a.e. continuously differentiable. Moreover, for the unbounded variation case, because $W_k^{(q)}(0) = 0$ we have that $\lim_{x \to b_k^-} w_k^{(q)'}(x;d) = w_k^{(q)'}(b_k;d) = \lim_{x \to b_k^+} w_k^{(q)'}(x;d)$. Then, because $W_k^{(q)}(\cdot-d)$, $w_{k-1}^{(q)}(\cdot;d) \in C^1(d,\infty)$ we obtain that $w_k^{(q)}(\cdot;d)$ is $C^1(d,\infty)$. Finally, for the bounded variation case we have that $W_k^{(q)}(0) > 0$, and then

\[
\lim_{x \to b_k^-} w_k^{(q)'}(x;d) = w_{k-1}^{(q)'}(b_k;d) \neq w_{k-1}^{(q)'}(b_k;d)(1 + \delta_k W_k^{(q)}(0)) = \lim_{x \to b_k^+} w_k^{(q)'}(x;d),
\]
so that the derivative does not exist at $b_k$. Furthermore, for the bounded variation case it follows from the inductive assumption and the assumption that $W_k^{(q)}(\cdot - d) \in C^1((d, \infty))$ that $w_k^{(q)}$ belongs to $C^1((d, \infty) \setminus \{b_1, \ldots, b_k\})$.

Lemma 10. (Monotonicity) The scale function $w_k^{(q)}$ defined in (19) is an increasing function for $k \geq 0$ and $x \geq d$.

Proof. Because we have for $k = 0$ that $w_k^{(q)}(\cdot; d) = W^{(q)}(\cdot - d)$, the monotonicity follows directly from the definition.

Now, assume for induction that $w_{k-1}^{(q)}$ is an increasing function for $x \geq d$. Then, again from (19), for any $x > y$ we obtain that

$$w_k^{(q)}(x; d) - w_k^{(q)}(y; d) = w_{k-1}^{(q)}(x; d) + \delta_k \int_{b_k}^{x} W_k^{(q)} (x - z) w_{k-1}^{(q)}(z; d) dz - w_{k-1}^{(q)}(y; d) - \delta_k \int_{b_k}^{y} W_k^{(q)} (y - z) w_{k-1}^{(q)}(z; d) dz$$

As a consequence of the previous lemma, we have the following result.

Corollary 11. For any $d \in \mathbb{R}$ and $k \geq 0$, $w_k^{(q)}(x; d) > 0$ for a.e. $x > d$.

Lemma 12. (Behavior at 0) For any $d \in \mathbb{R}$, $b_1 > d$ and $k \geq 1$, $w_k^{(q)}(d; d) = W^{(q)}(0)$ and $w_k^{(q)}(0) = Z^{(q)}(0) = 1$. In particular, $w_k^{(q)}(0) = W^{(q)}(0)$.

Proof. From (14), it is easy to check that $w_1^{(q)}(d; d) = W^{(q)}(0)$. Hence from (19) it follows that $w_k^{(q)}(d; d) = w_{k-1}^{(q)}(d; d)$, and by induction we obtain that $w_k^{(q)}(d; d) = W^{(q)}(0)$ for any $k \geq 1$. A similar argument yields that $z_k^{(q)}(0) = Z^{(q)}(0) = 1$ for any $k \geq 1$.

By solving the recursion given in (19) with the initial condition (14), we get the following result.

Proposition 13. For $k \geq 1$, we have

$$w_k^{(q)}(x; d) = W^{(q)}(x - d) + \sum_{i=1}^{k} \delta_i \prod_{0 < l < i} \left(1 + \delta_l W_i^{(q)}(0)\right) \int_{b_i}^{x} W_i^{(q)}(x - y) W^{(q)}(y - d) dy$$

$$+ \sum_{1 \leq i_1 < i_2 \leq k} \delta_{i_1} \delta_{i_2} \prod_{0 < l \neq i_1 < i_2} \left(1 + \delta_l W_i^{(q)}(0)\right) \int_{b_{i_1}}^{y_{i_1}} W_{i_1}^{(q)}(y_1 - y_2) W^{(q)}(y_2 - d) dy_2 dy_1$$

$$\times \int_{b_{i_2}}^{y_{i_2}} W_{i_2}^{(q)}(y_1 - y_2) W^{(q)}(y_2 - y_1) dy_2 dy_1$$

$$+ \sum_{1 \leq i_1 < i_2 < i_3 \leq k} \delta_{i_1} \delta_{i_2} \delta_{i_3} \prod_{0 < l \neq i_1 < i_2 < i_3} \left(1 + \delta_l W_i^{(q)}(0)\right) \int_{b_{i_1}}^{y_{i_1}} W_{i_1}^{(q)}(y_1 - y_2) \int_{b_{i_2}}^{y_{i_2}} W_{i_2}^{(q)}(y_1 - y_2) \int_{b_{i_3}}^{y_{i_3}} W_{i_3}^{(q)}(y_1 - y_2) \times$$
The above equation together with (32) implies that, for Lemma 15. Suppose that 
\[\Phi_1 \leq 0, b_0 = -\infty, \text{ and } b_k = \max_{1 \leq j \leq k} b_j < y \text{ for } k \geq 0,\]

\[w_k^{(q)}(x; y) = \Xi_\Phi_k(y)W_k^{(q)}(x - y).\]  

**Proof.** We prove the result by induction. The base case \((k = 0)\) is clear, because this holds with \(\Xi_\Phi_0(y) = 1\) and \(w_0^{(q)}(x; y) = W_0^{(q)}(x - y).\)

Assume for induction that \(w_{k-1}^{(q)}(x; y) = \Xi_\Phi_{k-1}(y)W_{k-1}^{(q)}(x - y)\) for \(b_{k-1} < y\). Then, by (19), for \(b_{k-1} < b_k < y\)

\[w_k^{(q)}(x; y) = w_{k-1}^{(q)}(x; y) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z)w_{k-1}^{(q)}(z; y)dz\]

\[= w_{k-1}^{(q)}(x; y) + \Xi_\Phi_{k-1}(y)\delta_k \int_{b_k}^{x-y} W_k^{(q)}(x - y - z)W_{k-1}^{(q)}(z)dz\]

\[= \Xi_\Phi_{k-1}(y) \left( W_{k-1}^{(q)}(x - y) + \delta_k \int_{0}^{x-y} W_k^{(q)}(x - z - y)W_{k-1}^{(q)}(z)dz \right).\]  

(32)

According to the proofs of Theorems 4 and 5 of [12], we obtain that for \(z \geq 0\) the scale functions \(W_k^{(q)}\) and \(W_{k-1}^{(q)}\) are related by the following formula:

\[\delta_k \int_{0}^{z} W_k^{(q)}(z - y)W_{k-1}^{(q)}(y)dy = \int_{0}^{z} W_k^{(q)}(y)dy - \int_{0}^{z} W_{k-1}^{(q)}(y)dy.\]  

(33)

Differentiating the above formula in \(z\) gives us

\[\delta_k \int_{0}^{z} W_k^{(q)}(y)W_{k-1}^{(q)}(z - y)dy + \delta_k W_k^{(q)}(z)W_{k-1}^{(q)}(0) = W_k^{(q)}(z) - W_{k-1}^{(q)}(z).\]

The above equation together with (32) implies that, for \(b_k < y\),

\[w_k^{(q)}(x; y) = \Xi_\Phi_k(y)W_k^{(q)}(x - y).\]

**Lemma 15.** Suppose that \(q > 0\). For \(\delta > \varphi_k(q)\), we have that \(w_k^{(q)}(x; d)e^{-\delta x} \xrightarrow{x \to \infty} 0\) for \(k \geq 0\).

**Proof.** We prove the result by induction. The base case with \(k = 0\) is clear, as we know that \(W^{(q)}(x)e^{-\varphi(q)x}\) converges to a finite value as \(x \to \infty\). Now, suppose that the claim holds for \(k - 1\) (i.e., \(w_{k-1}^{(q)}(x; d)e^{-\delta x} \xrightarrow{x \to \infty} 0\) for \(\delta > \varphi_{k-1}(q)\)). Then we will show that it also holds for \(k\). We have

\[e^{-\delta x}w_k^{(q)}(x; d) = e^{-\delta x}w_{k-1}^{(q)}(x; d) + \delta_x e^{-\delta x} \int_{b_k}^{x} W_k^{(q)}(x - y)w_{k-1}^{(q)}(y; d)dy\]
Remark 16. It is easy to check that similar results as those in Lemma 9 and Lemma 15 hold for the functions $z^{(q)}$ and $w^{(q)}$.

2.4. Integral equations for the multi-refracted scale functions. In this section, we will show that the scale functions $w^{(q)}_k$ and $z^{(q)}_k$ are the solutions to some integral equation. This representation will be important for defining the scale functions associated with the level-dependent Lévy process that we study in the next section.

Proposition 17. Suppose that $\phi_k(x) = \sum_{i=1}^{k} \delta_i 1_{\{x > b_i\}}$ and $k, q \geq 0$. Then, for $d, x \in \mathbb{R}$, $d < b_1$ the functions $w^{(q)}_k(x; d)$ and $z^{(q)}_k(x)$ are unique solutions to the following equations:

$$w^{(q)}_k(x; d) = W^{(q)}(x - d) + \int_d^x W^{(q)}(x - y) \phi_k(y) w^{(q)\prime}_k(y; d) \, dy$$

(34)

and

$$z^{(q)}_k(x) = Z^{(q)}(x) + \int_0^x W^{(q)}(x - y) \phi_k(y) z^{(q)\prime}_k(y) \, dy.$$  

(35)

Proof. In the proof, we use the partial Laplace transform

$$\mathcal{L}_b[f(x; d)](\lambda) = \int_b^{\infty} e^{-\lambda x} f(x; d) \, dx$$

and the following property that we obtain using Fubini’s theorem:

$$\mathcal{L}_b \left[ \int_b^x g(x - y) f(y; d) \, dy \right](\lambda) = \mathcal{L}_b[f(x; d)](\lambda) \mathcal{L}[g(x)](\lambda).$$

Moreover, integration by parts gives us that

$$\mathcal{L}_b[f'(x; d)](\lambda) = \lambda \mathcal{L}_b[f(x; d)](\lambda) - e^{-\lambda b} f(b; d).$$

Using the principle of induction, we will demonstrate that for $k = 0, 1, 2, \ldots$ it holds that

$$w^{(q)}_k(x; d) = W^{(q)}(x - d) + \sum_{i=1}^{k} \delta_i \int_{b_i}^x W^{(q)}(x - y) w^{(q)\prime}_k(y; d) \, dy,$$

(36)

which is equivalent to (34).

First, the identity (36) for the base case ($k = 0$) holds because $w^{(q)}_0(x; d) = W^{(q)}(x - d)$, and this is the unique solution.

Next, under the inductive hypothesis that $w^{(q)}_{k-1}(x; d)$ is a unique solution to (36) with $k$ replaced with $k - 1$, we show that $w^{(q)}_k(x; d)$ must be a unique solution to (36). To
this end, given a function \( u \) solving
\[
    u(x; d) = W^q(x) - d + \sum_{i=1}^{k} \delta_i \int_{b_i}^{x} W^q(x - y) u'(y; d) \, dy,
\]
we show it necessarily holds that \( u = w^q_k(x; d) \). The proof is given separately for \( x \leq b_k \) and \( x > b_k \).

Suppose that \( x \leq b_k \). Then, (37) reduces to
\[
    u(x; d) = W^q(x) - d + \sum_{i=1}^{k} \delta_i \int_{b_i}^{x} W^q(x - y) u'(y; d) \, dy,
\]
which is precisely \( w^q_{k-1}(x; d) \) by the inductive hypothesis. Because \( w^q_{k-1}(x; d) \) and \( w^q_k(x; d) \) coincide for \( x \leq b_k \) (in view of (19)), we must have that \( u = w^q_k(x; d) \).

To prove the identity for \( x > b_k \), we show that \( \mathcal{L}_{b_k}[u(x; d)](\lambda) = \mathcal{L}_{b_k}[w^q_k(x; d)](\lambda) \) for all \( \lambda > 0 \).

Consequently,
\[
    u(x; d) = W^q(x) - d + \sum_{i=1}^{k} \delta_i \int_{b_i}^{x} W^q(x - y) u'(y; d) \, dy
\]
\[
    = W^q(x) - d + \sum_{i=1}^{k-1} \delta_i \int_{b_i}^{b_k} W^q(x - y) u'(y; d) \, dy
\]
\[
    + \sum_{i=1}^{k} \delta_i \int_{b_k}^{x} W^q(x - y) u'(y; d) \, dy
\]
\[
    = W^q(x) - d + \sum_{i=1}^{k-1} \delta_i \int_{b_i}^{b_k} W^q(x - y) u'(y; d) \, dy
\]
\[
    + \sum_{i=1}^{k} \delta_i \int_{b_k}^{x} W^q(x - y) u'(y; d) \, dy
\]
\[
    = w^q_{k-1}(x; d) + \sum_{i=1}^{k} \delta_i \int_{b_i}^{x} W^q(x - y) u'(y; d) \, dy
\]
\[
    - \sum_{i=1}^{k-1} \delta_i \int_{b_k}^{x} W^q(x - y) w^q_{k-1}(y; d) \, dy.
\]

Applying the operator \( \mathcal{L}_{b_k} \) to both sides of the previous equation, we obtain that
\[
    \mathcal{L}_{b_k}[u(x; d)](\lambda) = \mathcal{L}_{b_k}[w^q_{k-1}(x; d)](\lambda) + \sum_{i=1}^{k} \delta_i \left( \lambda \mathcal{L}_{b_k}[u(x; d)](\lambda) - e^{-\lambda b_k} u(b_k; d) \right)
\]
\[
    - \sum_{i=1}^{k-1} \delta_i \left( \lambda \mathcal{L}_{b_k}[w^q_{k-1}(x; d)](\lambda) - e^{-\lambda b_k} w^q_{k-1}(b_k; d) \right)
\]
\[
    = \frac{\psi_{k-1}(\lambda)}{\psi(\lambda) - q} \mathcal{L}_{b_k}[w^q_{k-1}(x; d)](\lambda) + \frac{\sum_{i=1}^{k} \delta_i}{\psi(\lambda) - q} \lambda \mathcal{L}_{b_k}[u(x; d)](\lambda)
\]
\[
    - \frac{\delta_k}{\psi(\lambda) - q} e^{-\lambda b_k} w^q_{k-1}(b_k; d).
\]
This implies that
\[
\mathcal{L}_{b_k}[u(x;d)](\lambda) = \frac{\psi_{k-1}(\lambda) - q}{\psi_k(\lambda) - q} \mathcal{L}_{b_k}[w_{k-1}^{(q)}(x;d)](\lambda) - \frac{\delta_k}{\psi_{k-1}(\lambda) - q} e^{-\lambda b_k} w_{k-1}^{(q)}(b_k; d). \tag{38}
\]
Applying the operator \( \mathcal{L}_{b_k} \) on both sides of (19) gives
\[
\mathcal{L}_{b_k}[w_k^{(q)}(x;d)](\lambda) = \mathcal{L}_{b_k}[w_{k-1}^{(q)}(x;d)](\lambda) + \frac{\delta_k}{\psi_k(\lambda) - q} \mathcal{L}_{b_k}[w_{k-1}^{(q)}(x;d)](\lambda)
+ \frac{\delta_k}{\psi_k(\lambda) - q} \left( \lambda \mathcal{L}_{b_k}[w_{k-1}^{(q)}(x;d)](\lambda) - e^{-\lambda b_k} w_{k-1}^{(q)}(b_k; d) \right)
= \frac{\psi_{k-1}(\lambda) - q}{\psi_k(\lambda) - q} \mathcal{L}_{b_k}[w_{k-1}^{(q)}(x;d)](\lambda) - \frac{\delta_k}{\psi_k(\lambda) - q} e^{-\lambda b_k} w_{k-1}^{(q)}(b_k; d). \tag{39}
\]
Hence, by (38) and (39) we obtain that \( \mathcal{L}_{b_k}[u(x;d)](\lambda) = \mathcal{L}_{b_k}[w_k^{(q)}(x;d)](\lambda) \) for all \( \lambda > 0 \), which in turn implies that \( u \) is the unique solution of (37) and that \( u(x;d) = w_k^{(q)}(x;d) \) for \( x \geq 0 \). In a similar manner as in the proof of Proposition 17 we can show that the scale function \( z_k^{(q)} \) can be characterised as the solution to the following integral equation:
\[
z_k^{(q)}(x) = Z^{(q)}(x) + \sum_{i=1}^{k} \delta_k \int_{b_i}^{x} W^{(q)}(x-y) z_k^{(q)}(y) dy.
\]

3. Theory for a General Premium Rate Function

In this section, we extend the theory of multi-refracted processes to the solutions of (1) with a general premium rate function \( \phi \). We prove the existence and uniqueness of these solutions using the theory already developed for the multi-refracted case. To this end, we approximate a general rate function \( \phi \) by a sequence of rate functions \( (\phi_n)_{n \geq 0} \) of the form given in (1). This will define a sequence of multi-refracted Lévy processes, which will allow us to prove the existence and uniqueness of solutions to (1) by a limiting argument.

Because most of the results presented in this section are based on the previous results for multi-refracted processes, we adopt the following assumptions:

[A] The function \( \phi \) is non-decreasing, non-negative, and continuously differentiable, and \( \phi(x) = 0 \) for \( x \leq 0 \). In the bounded variation case, we assume that \( \phi(x) < c \) for all \( x \in \mathbb{R} \).

Remark 18. The assumption that \( \phi(0) = 0 \) is not very restrictive. Suppose that \( \phi(0) > 0 \), then
\[
dU(t) = dX(t) - \phi(U(t)) dt = dX(t) - \phi(0) dt - (\phi(U(t)) - \phi(0)) dt = d\tilde{X}(t) - \tilde{\phi}(U(t)) dt,
\]
where \( \tilde{X} \) is a Lévy process defined by \( (\gamma - \phi(0), \sigma, \Pi) \) and \( \tilde{\phi}(t) = \phi(t) - \phi(0) \).
Now, consider $U$, which is the solution of (2). Between jumps, the evolution of $U$ is deterministic, according to $\dot{x} = p(x)$ with $x(0) = x_0$. To solve this equation, we introduce

$$\Omega_x(y) = \int_x^y \frac{1}{p(v)} dv.$$ 

Notice that $\frac{d}{dt} \Omega_{x_0}(x_t) = \dot{x}_t(p(x_t))^{-1} = 1$, and so $\Omega_{x_0}(x_t) = t$. Hence, $x_t = \Omega_{x_0}^{-1}(t)$. A problem appears if $\Omega_0(y) < \infty$ for all $0 < y \leq \infty$. In this case, $\Omega_x^{-1}(t)$ is defined for all $x \geq 0$, and $0 < t \leq \Omega_x(\infty) < \infty$. In the paper by Albrecher et al. [1], the authors considered the case in which the process $U$ is driven by a compound Poisson process and a rate function $p(x) = C + x^2$. In view of the previous remark, we can see that the process $U$ cannot be constructed for all $t \geq 0$.

3.1. **Existence, uniqueness, and monotonicity.** In this section we will prove the existence and uniqueness of a solution to (1) with a rate function $\phi$ that satisfies Assumption [A]. As we already mentioned, we will use an approximating argument, in which a monotonicity property of the solutions based on their driving rate functions will be crucial.

We now consider a sequence of functions $(\phi_n)_{n \geq 1}$ that satisfy the following conditions:

(a) $\lim_{n \to \infty} \phi_n = \phi$ uniformly on compact sets.

(b) For $x \in \mathbb{R}$,

$$\phi_1(x) \leq \phi_2(x) \leq \ldots \leq \phi(x).$$

(c) For each $n \geq 1$ and $x \in \mathbb{R}$, we have that $\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{(x,b_j^n)}$, for some $m_n \in \mathbb{N}$, $0 < b_1^n < \ldots < b_{m_n}^n$, and $\delta_j^n > 0$ for each $j = 1, \ldots, m_n$.

For each $n \geq 1$ we denote the solution of (1) with the rate function $\phi_n$ by $U_n$.

**Remark 19.** According to convention from Section 2, $\phi_n$ is an $m_n$-multi-refracted rate function. Notice that from now on $\phi_n$ is not exactly the rate function from Section 2 because now we are not indexing the sequence $(\phi_n)_{n \geq 1}$ only by the number of barriers. We decided to use the above notation for clarity of Section 3.

We now show how to construct a specific sequence $(\phi_n)_{n \geq 1}$ that satisfies the conditions mentioned above. For each $n \geq 1$ we choose a grid $\Pi^n = \{b_l^n = l2^{-n} : l = 1, \ldots, m_n = n2^n\}$ and set $\delta_j^n = \phi(b_j^n) - \phi(b_{j-1}^n)$, with $b_0^n = 0$. Furthermore we define the approximating sequence of the rate function $\phi$ as follows:

$$\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{(x,b_j^n)} \quad \text{for } n \geq 1 \text{ and } x \in \mathbb{R}.$$ 

For any $n \geq 1$, we have from Lemma 11 that there exists a unique solution $U_n$ to (1) with the rate function $\phi_n$. Moreover, the following lemma implies that the sequence $(U_n(t))_{n \geq 1}$ is non-decreasing for any $t \geq 0$.

**Lemma 20.** Suppose that for each $n \geq 1$, $\phi_n(x) \leq \phi_{n+1}(x)$ for all $x \in \mathbb{R}$. Then $U_{n+1}(t) \leq U_n(t)$ for all $t \geq 0$.

**Proof.** Consider $\varepsilon > 0$, and define the function $\phi_{n+1}^\varepsilon(x) := \phi_{n+1}(x) + \varepsilon$. Then, $\phi_n(x) < \phi_{n+1}^\varepsilon(x)$ for all $x \in \mathbb{R}$. Consider the process $U_{n+1}^\varepsilon$, which is a solution to the following SDE:

$$U_{n+1}^\varepsilon(t) = X(t) - \int_0^t \phi_{n+1}^\varepsilon(U_{n+1}^\varepsilon(s)) \, ds, \quad t \geq 0.$$
Moreover, define
\[ \varsigma := \inf\{ t > 0 : U_n(t) < U_{n+1}^\varepsilon(t) \}, \]
and assume that \( \varsigma < \infty \). We remark that because \( U_n \) and \( U_{n+1}^\varepsilon \) have the same jumps, the crossing cannot occur at a jump instant. Hence, \( U_n(t) - U_{n+1}^\varepsilon(t) \) is non-increasing in some \([\varsigma - \varepsilon, \varsigma)\) for small enough \( \varepsilon \), and
\[
0 \geq \frac{d}{dt} \left( U_n(t) - U_{n+1}^\varepsilon(t) \right) \bigg|_{t=\varsigma-} = \phi_{n+1}^\varepsilon(U_{n+1}^\varepsilon(\varsigma)) - \phi_n(U(\varsigma)) > 0,
\]
which yields that \( \varsigma = \infty \), implying that \( U_{n+1}^\varepsilon \leq U_n \).

Now, because \( \phi_{n+1}(x) < \phi_{n+1}^\varepsilon(x) \) for all \( x \in \mathbb{R} \), using the same argument as above we obtain that \( U_{n+1}^\varepsilon \leq U_{n+1} \).

On the other hand, let
\[
\Delta^\varepsilon(t) := U_{n+1}(t) - U_{n+1}^\varepsilon(t) = \int_0^t \phi_{n+1}^\varepsilon(U_{n+1}^\varepsilon(s)) - \phi_{n+1}(U_{n+1}(s)) \, ds.
\]
Then, from classical calculus
\[
(\Delta^\varepsilon(t))^2 = 2 \int_0^t \Delta^\varepsilon(s)(\phi_{n+1}^\varepsilon(U_{n+1}^\varepsilon(s)) - \phi_{n+1}(U_{n+1}(s))) \, ds.
\]
\[
= 2 \int_0^t \Delta^\varepsilon(s)(\phi_{n+1}(U_{n+1}(s)) - \phi_{n+1}(U_{n+1}(s))) \, ds + 2\varepsilon \int_0^t \Delta^\varepsilon(s) \, ds.
\]
Since we have that \( U_{n+1}^\varepsilon(t) \leq U_{n+1}(t) \) for all \( t \geq 0 \) and \( \phi_{n+1} \) is a non-decreasing function, we obtain that
\[
\int_0^t \Delta^\varepsilon(s)(\phi_{n+1}(U_{n+1}^\varepsilon(s)) - \phi_{n+1}(U_{n+1}(s))) \, ds \leq 0,
\]
and finally
\[
(\Delta^\varepsilon(t))^2 \leq 2\varepsilon \int_0^t \Delta^\varepsilon(s) \, ds \xrightarrow{\varepsilon \downarrow 0} 0.
\]
From this we conclude that
\[
U_{n+1}(t) = \lim_{\varepsilon \downarrow 0} U_{n+1}^\varepsilon(t) \leq U_n(t).
\]

The following result establishes the existence and uniqueness of a solution to the SDE \((\mathbb{I})\) with a general premium rate function \( \phi \), which satisfies condition \([\mathbb{A}]\). In the literature, there exist general results regarding the existence of solutions of SDEs driven by Lévy processes. See, for example, the book of Applebaum \([2]\). In particular, we wish to quote Brockwell et al. \([9]\) and references therein, where the existence and uniqueness of a solution to \((\mathbb{I})\) were considered for storage processes in the case that the driving Lévy process is of bounded variation. However, in this paper we will proceed in a natural manner, by demonstrating the uniform convergence on compact sets of a sequence of \( m_n \)-multi-refracted Lévy processes \((U_n)_{n \geq 0}\), defined by the approximating sequence of rate functions \((\phi_n)_{n \geq 0}\) to the solution \( U \) of \((\mathbb{I})\) with the corresponding rate function \( \phi \).

For the proof of the following proposition we introduce the following notation, for any \( t > 0 \),
\[
\overline{X}(t) := \sup_{0 \leq s \leq t} X(s) \quad \text{and} \quad \underline{X}(t) := \inf_{0 \leq s \leq t} X(s).
\]
Proposition 21. Suppose that the rate function $\phi$ satisfies condition [A]. Then, there exists a unique solution $U$ to the SDE (1) with rate function $\phi$. Furthermore, the sequence $(U_n)_{n \geq 1}$ converges uniformly to $U$ a.s. on compact time intervals.

Proof. We will first show the existence, by proving the uniform convergence of the sequence $(U_n)_{n \geq 1}$ on compact sets to a solution of (1) with rate function $\phi$. To this end, let $(\phi_n)_{n \geq 1}$ be the approximating sequence for $\phi$. For each $n \geq 1$, we consider $U_n$ as a unique solution to

$$U_n(t) = X(t) - \int_0^t \phi_n(U_n(s)) \, ds.$$ 

Since $\phi_1(x) \leq \phi_2(x) \leq \ldots \leq \phi(x)$, we have by Lemma 20 that $U_1(t) \geq U_2(t) \geq \ldots$. Now, fix $T > 0$. It follows that

$$U_n(t) \leq X(t) \leq |X(T)|.$$ 

On the other hand, using the fact that $\phi(x) \geq 0$ for $x \in \mathbb{R}$, we have that

$$U_n(t) = X(t) - \int_0^t \phi_n(U_n(s)) \, ds \geq X(t) - \phi(|X(T)|)T.$$ 

Hence,

$$|U_n(t)| \leq (|X(T)| \vee |X(T)|) + \phi(|X(T)|)T := K_T, \quad 0 \leq t \leq T, \, n \geq 1.$$ 

Because the sequence $U_n$ is non-increasing and bounded below, we can define $U(t) = \lim_{n \to \infty} U_n(t)$. Clearly $U(t) \leq U_n(t) \leq U_1(t)$ and again using Gronwall’s inequality we can show that

$$|U_n(t) - U(t)| \leq \int_0^t |\phi(U(s)) - \phi(U_n(s))| \, ds + \int_0^t |\phi(U_n(s)) - \phi_n(U_n(s))| \, ds$$

$$\leq T \sup_{s \in I} |\phi(s) - \phi_n(s)| + \sup_{s \in I} |\phi'(s)| \int_0^t |U(s) - U_n(s)| \, ds,$$

where $I = [-K_T, K_T]$. Hence,

$$\sup_{0 \leq t \leq T} |U(t) - U_n(t)| \leq T \sup_{s \in I} |\phi(s) - \phi_n(s)| e^{\sup_{s \in I} |\phi'(s)| T}.$$ 

Because $\sup_{s \in I} |\phi(s) - \phi_n(s)| \to 0$ the uniform convergence of $U_n$ to $U$ follows. Hence, by the uniform convergence of $\phi_n$ to $\phi$ and $U_n$ to $U$ on compact sets, together with the continuity of $\phi$, we obtain that

$$U(t) = \lim_{n \to \infty} U_n(t) = X(t) - \lim_{n \to \infty} \int_0^t \phi_n(U_n(s)) \, ds = X(t) - \int_0^t \phi(U(s)) \, ds.$$ 

Now, in order to show the uniqueness, consider two solutions of (1), say $U$ and $\tilde{U}$. Then,

$$|U(t) - \tilde{U}(t)| \leq \int_0^T |\phi(U(s)) - \phi(\tilde{U}(s))| \, ds \leq \sup_{x \in I} |\phi'(x)| \int_0^T |U(s) - \tilde{U}(s)| \, ds.$$ 

This implies that

$$\sup_{t \in [0, T]} |U(t) - \tilde{U}(t)| \leq \sup_{x \in I} |\phi'(x)| \int_0^T \sup_{s \in [0, T]} |U(s) - \tilde{U}(s)| \, ds.$$
Then, Gronwall’s Lemma implies that
\[ \sup_{t \in [0, T]} |U(t) - \tilde{U}(t)| = 0, \]
and therefore there exists a unique solution to (11).

3.2. Theory of scale functions for level-dependent Lévy processes. In this section, we introduce the scale function \( w^{(q)} \) for the level-dependent Lévy process \( U \) with rate function \( \phi \), as a unique solution to some integral equation. Recall that we denote the scale function of the driving Lévy process \( X \) as \( W^{(q)} \). Here, we define \( w^{(q)} \) as the solution to the following integral equation:
\[
w^{(q)}(x; d) = W^{(q)}(x - d) + \int_{d}^{x} W^{(q)}(x - y)\phi(y)w^{(q)\prime}(y; d) \, dy, \quad x \geq d, \tag{41}\]
provided that \( w^{(q)} \) is a.e. differentiable. For any \( x \geq 0 \), we denote \( w^{(q)}(x) := w^{(q)}(x; 0) \).

Now, for any \( x \geq 0 \) define \( \Xi_{\phi}(x) := 1 - W^{(q)}(0)\phi(x) \), which is strictly positive by Assumption [A]. However, a more useful form for our subsequent analysis is obtained by differentiating equation (41). Thus, for \( x \geq d \), we obtain
\[
w^{(q)\prime}(x; d) = \frac{1}{1 - \phi(x)W^{(q)}(0)}W^{(q)\prime}((x - d) +) + \int_{d}^{x} \frac{\phi(y)}{1 - \phi(x)W^{(q)}(0)}W^{(q)\prime}(x - y)w^{(q)\prime}(y; d) \, dy
\]
\[= \Xi_{\phi}(x)^{-1}W^{(q)\prime}((x - d) +) + \int_{d}^{x} \Xi_{\phi}(x)^{-1}\phi(y)W^{(q)\prime}(x - y)w^{(q)\prime}(y; d) \, dy, \tag{42}\]
with the boundary condition \( w^{(q)}(d; d) = W^{(q)}(0) \).

Remark 22. Because in general the derivative of the function \( W^{(q)} \) is not defined for all \( x \in \mathbb{R} \), we take the right first derivatives of \( W^{(q)} \), which always exist (see, for example, the proof of Lemma 2.3 in Kuznetsov et al. [9]). Then, \( w^{(q)\prime} \) is the solution of the Volterra equation (12), which together with the initial condition defines the scale function \( w^{(q)} \) uniquely.

Lemma 23. Assume that \( u \) is differentiable a.e. Then, \( u \) is the solution of (41) if and only if \( \prime \) is the solution of (12), with the boundary condition \( u(d; d) = W^{(q)}(0) \).

Proof. Suppose that \( u \) has a derivative \( \prime \) and fulfills (41). Then, differentiation of (41) yields (42). Conversely, suppose that \( \prime \) fulfills (42). Then, defining \( u(x; d) = W^{(q)}(x - d) + \int_{d}^{x} \prime(t; d) \, dt \), we can verify by inspection that \( u \) fulfills (41).

In a similar manner we define the scale function \( z^{(q)} \) as the solution to
\[
z^{(q)\prime}(x) = \frac{q}{1 - \phi(x)W^{(q)}(0)}W^{(q)}(x) + \int_{0}^{x} \frac{\phi(y)}{1 - \phi(x)W^{(q)}(0)}W^{(q)\prime}(x - y)z^{(q)\prime}(y) \, dy
\]
\[= \Xi_{\phi}(x)^{-1}qW^{(q)}(x) + \int_{0}^{x} \Xi_{\phi}(x)^{-1}\phi(y)W^{(q)\prime}(x - y)z^{(q)\prime}(y) \, dy, \tag{43}\]
with the boundary condition \( z^{(q)}(0) = 1 \).

In the remainder of this section we will prove the existence and uniqueness of solutions to equations (12) and (13), which belong to the family of Volterra equations. To this
end, we now present an outline of the theory of such equations. For \( x \geq d \), a Volterra equation is given by
\[
    u(x; d) = g(x; d) + \int_d^x K(x, y)u(y; d) \, dy,
\]
where \( K, g, \) and \( u \) are measurable, and the integrals are well defined. We set
\[
    g(x; d) = W^{(q)}((x - d)+)/\Xi_\phi(x)
\]
and
\[
    g(x) = g(x; 0) = qW^{(q)}(x)/\Xi_\phi(x)
\]
to obtain \( w^{(q)}(x; d) \) and \( z^{(q)}(x) \) respectively.

The idea of how to solve Volterra equations (44) is as follows. For \( T > d \), we consider a kernel \( K : D = \{(x, y) : d < y < x < T\} \to \mathbb{R}_+ \), which is assumed to be measurable, and define the following operator for a nonnegative measurable function \( f \):
\[
    K \odot f(x; d) = \int_d^x K(x, y)f(y; d) \, dy,
\]
for \( d < x < T \).

Now, consider the following Pickard iteration: We set \( u_1 = g \) and \( u_{n+1} = g + K \odot u_n \) for \( n \geq 1 \). By induction, one can prove that
\[
    u_{n+1}(x; d) = g(x; d) + \int_d^x \sum_{l=1}^{n} K^{(l)}(x, y)g(y; d) \, dy,
\]
for \( n \geq 1 \), and \( x \in [d, T] \), where \( K^{(1)} = K \) and \( K^{(l+1)}(x, y) = \int_y^x K^{(l)}(x, w)K(w, y) \, dw \).

In order to find the solution to (44), we need to show the convergence of the sequence \( \{u_n\}_{n \geq 1} \) to the function \( u \) defined by
\[
    u(x; d) = g(x; d) + \int_d^x K^*(x, y)g(y; d) \, dy,
\]
for \( x \in [d, T] \) (47) where \( K^*(x, y) = \sum_{l=1}^{\infty} K^{(l)}(x, y) \). Notice that the sequence \( \left( \sum_{l=1}^{j} K^{(l)}(x, y) \right)_{j \geq 1} \) is non-decreasing, and it is convergent if there exists a majorant function \( \zeta(x, y) < \infty \) such that
\[
    \sum_{l=1}^{j} K^{(l)}(x, y) \leq \zeta(x, y) \quad \text{for all } j \geq 1 \text{ and } (x, y) \in D.
\]
Furthermore, a unique solution exists and is given by (17) if \( \int_d^x \zeta(x, y)g(y; d) \, dy < \infty \) for all \( x \in [d, T] \). We remark that in the theory of Volterra equations, the continuity of \( g \) and \( K \) is not required. The presented theory is in the spirit of \( L^1 \) kernels as in Chapter 9.2 of [7].

For a given scale function \( W^{(q)} \) and rate function \( \phi \), we now consider equations (42) and (43), which are Volterra equations of the type (44). For both equations, the kernel is given by
\[
    K(x, y) = \Xi_\phi(x)^{-1}\phi(y)W^{(q)}((x - y)+)
\]
On the other hand for the function \( g \) in (44) we use (45) and (46) to obtain the functions \( w^{(q)} \) and \( z^{(q)} \) respectively.

Before we determine a majorant \( \zeta(x, y) \) for the Neymann series \( K^* \) we require two preparatory lemmas. The proof of the first is obvious. For the proof of the second we refer the reader to Appendix D.
Lemma 24. Assume that $f$ is a non-negative measurable function. Suppose that for some $s_0 > 0$, we have that $\int_0^\infty e^{-s_0 x} f(x) \, dx < \infty$. Then, $f$ is finite a.e., and $\int_0^\infty e^{-sx} f(x) \, dx < 1$ for sufficiently large $s$.

Lemma 25. Let $a > 0$. The function
\[ \zeta(x) = \sum_{l=1}^\infty a^l (W^{(q)})^l(x+) \] is finite for all $x > 0$.

Because $\phi(x) < 1/W^{(q)}(0)$ for all $x > 0$ in the bounded variation case, and $W^{(q)}(0) = 0$ in the unbounded variation case, for any $T > 0$ we have that
\[ a_T := \sup_{d \leq y \leq T} \left| \Xi_T^{-1} \phi(y) \right| < \infty. \] For clarity, we defer the proof of the following lemma and proposition to Appendix D, because the arguments are of a technical nature.

Lemma 26. For any $T > d$, we have
\[ K^*(x, y) \leq \zeta(x - y), \quad \text{for any } d \leq y \leq x \leq T, \]
where
\[ \zeta(x) = \sum_{l=1}^\infty a^l_T (W^{(q)})^l(x+) < \infty, \quad x > 0. \]

The next result proves the existence and uniqueness of solutions to (42) and (43), and also provides expressions for the functions $w^{(q)}(\cdot; d)$ and $z^{(q)}$ in terms of the scale function $W^{(q)}$ of the driving Lévy process $X$. Because the scale functions are solutions of the Volterra equation one might try to solve them numerically (see, e.g., Remark VIII.1.10 of [3]).

Proposition 27. For all $T > 0$, the following hold.

(i) For all $d \leq x \leq T$, we have that
\[ \int_d^x K^*(x, y) \Xi_T(x)^{-1} W^{(q)}(y - d) \, dy < \infty, \]
and hence
\[ w^{(q)}(x; d) = \Xi_T(x)^{-1} W^{(q)}((x - d)+) + \int_d^x K^*(x, y) \Xi_T(x)^{-1} W^{(q)}(y - d) \, dy, \] is the unique solution to (42).

(ii) For $0 \leq x \leq T$,
\[ \int_0^x K^*(x, y) \Xi_T(x)^{-1} qW^{(q)}(y) \, dy < \infty, \]
and hence
\[ z^{(q)}(x) = \Xi_T(x)^{-1} qW^{(q)}(x) + \int_0^x K^*(x, y) \Xi_T(x)^{-1} qW^{(q)}(y) \, dy \] is the unique solution to (43).
Consider now an approximating sequence \((\phi_n)_{n \geq 1}\) for \(\phi\). Then, by definition we have that for any \(x \in \mathbb{R}\) the sequence \((\phi_n(x))_{n \geq 1}\) is non-decreasing, which implies that the sequence \((\Xi_{\phi_n}(x)^{-1})_{n \geq 1}\) is also non-decreasing. Therefore, if we define
\[
K_n(x, y) := \Xi_{\phi_n}(x)^{-1} \phi_n(y) W^{(q)\prime}(x - y) + , \quad \text{for } n \geq 1 \text{ and } x \geq y \geq d,
\]
then the sequence \((K_n(x, y))_{n \geq 1}\) is non-decreasing for any \(d \leq x \leq y\). From Proposition\(^27\) we have that, for each \(n \geq 1\), the scale functions \(w_n^{(q)}(\cdot; d)\) and \(z_n^{(q)}\) associated with the level-dependent Lévy process \(U_n\) with rate function \(\phi_n\) satisfy the following equations:
\[
w_n^{(q)\prime}(x; d) = \Xi_{\phi_n}(x)^{-1} W^{(q)\prime}((x - d) +) + \int_d^x K_n^*(x, y) \Xi_{\phi_n}(x)^{-1} W^{(q)\prime}(y - d) \, dy \tag{51}
\]
and
\[
z_n^{(q)\prime}(x) = \Xi_{\phi_n}(x)^{-1} q W^{(q)\prime}(x) + \int_0^x K_n^*(x, y) \Xi_{\phi_n}(x)^{-1} q W^{(q)\prime}(y) \, dy,
\]
where, for each \(d \leq y \leq x\),
\[
K_n^*(x, y) := \sum_{l=1}^\infty K_n^l(x, y).
\]

**Theorem 28.** For any \(x \geq d\), we have
\[
\lim_{n \to \infty} w_n^{(q)\prime}(x; d) = w^{(q)\prime}(x; d) \quad \text{and} \quad \lim_{n \to \infty} z_n^{(q)\prime}(x) = z^{(q)\prime}(x),
\]
where the functions \(w^{(q)\prime}(\cdot; d)\) and \(z^{(q)\prime}\) are the unique solutions to equations\(^42\) and \((43)\), respectively.

**Proof.** We prove the result for the function \(w^{(q)}\). The case for the function \(z^{(q)}\) can be treated similarly. We now show that
\[
\lim_{n \to \infty} K_n^{(l)}(x, y) = K^{(l)}(x, y), \quad \text{for } d < y < x. \tag{52}
\]
Using the fact that \((\phi_n)_{n \geq 1}\) is an approximating sequence for \(\phi\), we obtain for any \(x \geq y \geq d\) that
\[
\lim_{n \to \infty} K_n(x, y) = \lim_{n \to \infty} \Xi_{\phi_n}(x)^{-1} \phi_n(y) W^{(q)\prime}(x - y) = K(x, y).
\]
Hence, the result follows for the case that \(l = 1\). Assuming that the result holds true for \(l \geq 1\), we note that
\[
K_n^{(l+1)}(x, y) = \int_y^x K_n^{(l)}(x, z) \Xi_{\phi_n}(z)^{-1} \phi_n(y) W^{(q)\prime}(z - y) \, dz, \quad \text{for } d < y < x.
\]
By assumption, we have that
\[
\lim_{n \to \infty} K_n^{(l)}(x, z) \Xi_{\phi_n}(z)^{-1} \phi_n(y) = K^{(l)}(x, z) \Xi_{\phi}(z)^{-1} \phi(y), \quad \text{for } d < y < z < x.
\]
On the other hand, by the monotonicity of the sequence \((\phi_n)_{n \geq 1}\) we obtain for all \(n \geq 1\)
\[
a_n := \sup_{d \leq y \leq x} \Xi_{\phi_n}(x)^{-1} \phi_n(y) \leq \sup_{d \leq y \leq x} \Xi_{\phi}(x)^{-1} \phi(y) =: a.
\]
Hence, by Lemma\(^26\) we have that
\[
K_n^{(l)}(x, z) \leq a^l (W^{(q)\prime})^l((x - z) +), \quad \text{for } 0 < z < x, \quad n \geq 1. \tag{53}
\]
Therefore, we can deduce by dominated convergence that
\[
\lim_{n \to \infty} K_n^{(l+1)}(x, z) = K^{(l+1)}(x, z), \quad \text{for } d < z < x.
\]
Therefore, proceeding by induction we obtain\(^52\).
Hence, using Exercise 8.5 in [12], the fact that
\[ \lim_{n \to \infty} K^*_n(x, y) = K^*(x, y). \]

We now observe that for \( x > d \)

\[ \int_d^x K^*_n(x, y) \Xi_\phi(x)^{-1} W^{(q)}(y-d) \, dy \leq a \int_d^x \zeta(x-y) W^{(q)}(y-d) \, dy < \infty, \quad \text{for every } n \geq 1. \]

Hence, by the dominated convergence theorem we obtain
\[
\lim_{n \to \infty} \int_d^x K^*_n(x, y) \Xi_\phi(x)^{-1} W^{(q)}(y-d) \, dy = \int_d^x K^*(x, y) \Xi_\phi(x)^{-1} W^{(q)}(y-d) \, dy, \quad \text{for } x \geq d.
\]

Finally, using (51) we obtain for \( x > d \) that
\[
\lim_{n \to \infty} u^{(q)}_n(x; d) = \lim_{n \to \infty} \left[ \Xi_\phi(x)^{-1} W^{(q)}((x-d)+) + \int_d^x K^*(x, y) \Xi_\phi(x)^{-1} W^{(q)}(y-d) \, dy \right] = \Xi_\phi(x)^{-1} W^{(q)}((x-d)+) + \int_d^x K^*(x, y) \Xi_\phi(x)^{-1} W^{(q)}(y-d) \, dy = u^{(q)}(x; d).
\]

The next result will introduce new functions that will be used in the next section. Consider now the scale function \( w^{(q)} \) associated with the level-dependent Lévy process \( U \) with rate function \( \phi \), which is given by (50). Then, we have the following result.

**Lemma 29.** (i) For any \( x \in \mathbb{R} \),

\[
u^{(q)}(x) := \lim_{d \to -\infty} \frac{w^{(q)}(x; d)}{W^{(q)}(-d)} = e^{\Phi(q)x}
\]

\[ + \int_0^x W^{(q)}(x-y) \phi(y) \left( \Xi_\phi(y)^{-1} \Phi(q)e^{\Phi(q)y} + \int_0^y \Phi(q) \Xi_\phi(y)^{-1} K^*(y, z)e^{\Phi(q)y} \, dz \right) \, dy. \tag{54}\]

(ii) For any \( d \in \mathbb{R} \),

\[
u^{(q)}(d) := \lim_{a \to -\infty} \frac{w^{(q)}(a; d)}{W^{(q)}(a)} = e^{-\Phi(q)d} + \int_d^x e^{-\Phi(q)y} \phi(y) w^{(q)}(y; d) \, dy. \tag{55}\]

**Proof.** (i) First, we note by (50) that

\[
\frac{w^{(q)}(x; d)}{W^{(q)}(-d)} = \Xi_\phi(x)^{-1} W^{(q)}((x-d)+) + \int_d^x K^*(x, y) \Xi_\phi(y)^{-1} \frac{W^{(q)}(y-d)}{W^{(q)}(-d)} \, dy.
\]

Hence, using Exercise 8.5 in [12], the fact that \( K^*(x, y) = 0 \) for \( y < 0 \), and dominated convergence, we have that

\[
\lim_{d \to -\infty} \frac{w^{(q)}(x; d)}{W^{(q)}(-d)} = \Phi(q) \Xi_\phi(x)^{-1} e^{\Phi(q)x} + \int_0^x \Phi(q) \Xi_\phi(x)^{-1} K^*(x, y)e^{\Phi(q)y} \, dy.
\]

Therefore, using (11) and dominated convergence we obtain for \( x \in \mathbb{R} \)

\[
\lim_{d \to -\infty} \frac{w^{(q)}(x; d)}{W^{(q)}(-d)} = \lim_{d \to -\infty} \left[ \frac{W^{(q)}(x-d)}{W^{(q)}(-d)} + \int_d^x W^{(q)}(x-y) \phi(y) \frac{w^{(q)}(y; d)}{W^{(q)}(-d)} \, dy \right],
\]

\[ = e^{\Phi(q)x} + \int_0^x W^{(q)}(x-y) \phi(y) \left( \Xi_\phi(y)^{-1} \Phi(q)e^{\Phi(q)y} + \int_0^y \Phi(q) \Xi_\phi(y)^{-1} K^*(y, z)e^{\Phi(q)y} \, dz \right) \, dy. \]
Lemma 30. We note that using (11), and dominated convergence

\[ v^{(q)}(d) = \lim_{a \to \infty} \frac{w^{(q)}(a; d)}{W^{(q)}(a)} = \lim_{a \to \infty} \left[ \frac{W^{(q)}(a - d)}{W^{(q)}(a)} + \int_{d}^{\infty} \frac{W^{(q)}(a - y)}{W^{(q)}(a)} \phi(y) w^{(q)'}(y; d) \, dy \right], \]

\[ = e^{-\Phi(q)d} + \int_{d}^{\infty} e^{-\Phi(q)y} \phi(y) w^{(q)'}(y; d) \, dy, \quad d \in \mathbb{R}. \]

Hence, we have the result. \[ \square \]

3.3. Fluctuation identities for level-dependent Lévy processes. We recall that \( w^{(q)} \) is a scale function if it fulfills equation (11). That is, if it satisfies

\[ w^{(q)}(x; d) = W^{(q)}(x - d) + \int_{d}^{\infty} W^{(q)}(x - y) \phi(y) w^{(q)'}(y; d) \, dy. \]  

(56)

A similar definition is given for the scale function \( z^{(q)} \), which is the solution of

\[ z^{(q)}(x) = Z^{(q)}(x) + \int_{0}^{x} W^{(q)}(x - y) \phi(y) z^{(q)'}(y) \, dy. \]  

(57)

This definition can be justified as follows. Consider now a level-dependent Lévy process \( U \) with rate function \( \phi \), and an approximating sequence \( (\phi_n)_{n \geq 1} \) for \( \phi \). For each \( n \geq 1 \), we consider the associated multi-refracted Lévy process \( U_n \) with rate function \( \phi_n \). By Proposition 21, we have the convergence of the sequence \( (U_n)_{n \geq 1} \) to the process \( U \) uniformly on compact sets.

On the other hand, we showed in Section 2 that with the use of the \( w^{(q)}_n \) and \( z^{(q)}_n \) scale functions we can compute important fluctuation identities for the process \( U_n \) for each \( n \geq 1 \). Finally, by Theorem 28, we found that the sequences of scale functions \( (w^{(q)}_n)_{n \geq 0} \) and \( (z^{(q)}_n)_{n \geq 0} \) converge to the corresponding scale functions \( w^{(q)} \) and \( z^{(q)} \) of the process \( U \), respectively.

These facts imply that we can obtain fluctuation identities for the process \( U \) as the limits of the respective identities for the sequence of multi-refracted Lévy processes \( (U_n)_{n \geq 1} \), and hence these will be given in terms of the scale functions \( w^{(q)} \) and \( z^{(q)} \). We will first prove a preliminary result, which follows verbatim from [12].

Lemma 30. Let \( \overline{U}(t) := \sup_{0 \leq s \leq t} U(s) \). For each given \( x, a \in \mathbb{R} \), the level-dependent Lévy process \( U \) with rate function \( \phi \) satisfies \( \mathbb{P}_x(\overline{U}(t) = a) = 0 \) for Lebesgue almost every \( t > 0 \).

Let \( a \in \mathbb{R} \) and define the following first-passage stopping times for the level-dependent process:

\[ \kappa^{a,-} := \inf\{ t > 0 : U(t) < a \} \quad \text{and} \quad \kappa^{a,+} := \inf\{ t > 0 : U(t) \geq a \}. \]

Theorem 31. (Resolvents)
Fix a Borel set \( \mathcal{B} \subseteq \mathbb{R} \), then

(i) For \( q \geq 0 \) and \( d \leq x \leq a \),

\[ \mathbb{E}_x \left[ \int_{0}^{\kappa^{a,+} \wedge \kappa^{a,-}} e^{-qt} 1_{\{U(t) \in \mathcal{B}\}} \, dt \right] = \int_{\mathcal{B} \cap (d,a)} \Xi \phi(y)^{-1} \left( \frac{w^{(q)}(x; d)}{w^{(q)}(a; d)} w^{(q)}(a; y) - w^{(q)}(x; y) \right) \, dy. \]  

(58)
(ii) For \( q > 0 \) and \( x \geq 0 \),
\[
\mathbb{E}_x \left[ \int_0^{s_n^-} e^{-qt} 1_{\{U(t) \in B\}} dt \right] = \int_{B \cap (0,\infty)} \Xi_\phi(y)^{-1} \left( \frac{w^{(q)}(x)}{v^{(q)}(0)} v^{(q)}(y) - w^{(q)}(x;y) \right) dy,
\]
where \( v^{(q)} \) is given by (55).

(iii) For \( q \geq 0 \) and \( x \leq a \),
\[
\mathbb{E}_x \left[ \int_0^{s_n^+} e^{-qt} 1_{\{U(t) \in B\}} dt \right] = \int_{B \cap (-\infty,a)} \Xi_\phi(y)^{-1} \left( \frac{w^{(q)}(x)}{w^{(q)}(a)} w^{(q)}(a;y) - w^{(q)}(x;y) \right) dy,
\]
where \( w^{(q)} \) is given by (54).

(iv) For \( q > 0 \) and \( x \in \mathbb{R} \),
\[
\mathbb{E}_x \left[ \int_0^{\infty} e^{-qt} 1_{\{U(t) \in B\}} dt \right] = \int_{B} \Xi_\phi(y)^{-1} \left( \frac{u^{(q)}(x)v^{(q)}(y)}{A(q)} - w^{(q)}(x;y) \right) dy,
\]
where \( A(q) := \psi'(\Phi(q)) + \int_0^\infty e^{-q(t)v^{(q)}(y)} (\Phi(q)e^{\Phi(q)}y + \int_y^\infty \Phi(q)K^*(y,z)e^{\Phi(q)z} dz) dy. \)

Proof. (i) Consider an approximating sequence \((\phi_n)_{n \geq 1}\) for the rate function \( \phi \) of the level-dependent Lévy process \( U \). If we denote by \((U_n)_{n \geq 0}\) the sequence of non-increasing multi-refracted Lévy processes, then by Proposition 21 we know that the sequence converges uniformly on compact sets to \( U \). Because
\[
|\overline{U}_n(t) - \overline{U}(t)| \vee |\underline{U}_n(t) - \underline{U}(t)| \leq \sup_{s \in [0,t]} |U_n(s) - U(s)|,
\]
we have for \( t > 0 \) that
\[
\lim_{n \to \infty} (U_n(t), \overline{U}_n(t), \underline{U}_n(t)) = (U(t), \overline{U}(t), \underline{U}(t)),
\]
where \( \overline{U}(t) = \inf_{0 \leq s \leq t} U(s) \). Now, using the fact that for each \( t \geq 0 \) the sequence \((U_n(t))_{n \geq 1}\) is non-increasing, we have for \( a, y \geq 0 \)
\[
\{ U(t) \geq 0 \} = \bigcap_{n \geq 1} \{ U_n(t) \geq 0 \},
\]
\[
\{ U(t) \geq a \} = \bigcap_{n \geq 1} \{ U_n(t) \geq a \},
\]
\[
\{ U(t) \geq y \} = \bigcap_{n \geq 1} \{ U_n(t) \geq y \}.
\]
This means that for any \( x \in \mathbb{R} \) and \( t > 0 \), we have
\[
\mathbb{P}_x \left( U(t) \geq y, \overline{U}(t) \geq a, \underline{U}(t) \geq 0 \right) = \mathbb{P}_x \left( \bigcap_{n \geq 1} \{ U_n(t) \geq y, \overline{U}_n(t) \geq a, \underline{U}_n(t) \geq 0 \} \right)
\]
\[
= \lim_{n \to \infty} \mathbb{P}_x \left( U_n(t) \geq y, \overline{U}_n(t) \geq a, \underline{U}_n(t) \geq 0 \right).
\]
By Lemma 30, we have that \( \mathbb{P}_x(\overline{U}(t) = a) = 0 \) for any \( x \in \mathbb{R} \) and for Lebesgue almost every \( t > 0 \). This in turn implies that
\[
\mathbb{P}_x \left( U(t) \geq y, \overline{U}(t) \leq a, \underline{U}(t) \geq 0 \right) = \mathbb{P}_x \left( U(t) \geq y, \overline{U}(t) \geq 0 \right)
\]
\[
- \mathbb{P}_x \left( U(t) \geq y, \overline{U}(t) \geq a, \underline{U}(t) \geq 0 \right)
\]
Thus, we obtain by bounded convergence that

\[
\lim_{n \to \infty} E_x \left[ \int_0^{a_n+\Lambda n_{\alpha-}} e^{-qt} 1_{\{U(t) \geq y\}} dy \right] = \lim_{n \to \infty} \int_0^\infty e^{-qt} P_x \left( U(t) \geq y, U_n(t) \leq a, U_n(t) \geq 0 \right) dt
\]

\[
= \int_0^\infty e^{-qt} P_x \left( U(t) \geq y, U(t) \leq a, U(t) \geq 0 \right) dt
\]

\[
= E_x \left[ \int_0^{\kappa \Lambda n_{\alpha-}} e^{-qt} 1_{\{U(t) \geq y\}} dy \right].
\]

We recall that

\[
\Xi_{\phi_n}(y) = 1 - W(q)(0)\phi_n(y).
\]

Hence,

\[
E_x \left[ \int_0^{\kappa \Lambda n_{\alpha-}} e^{-qt} 1_{\{U(t) \geq y\}} dt \right] = \lim_{n \to \infty} E_x \left[ \int_0^{\kappa \Lambda n_{\alpha-}} e^{-qt} 1_{\{U(t) \geq y\}} dt \right]
\]

\[
= \lim_{n \to \infty} \int_y^\infty \Xi_{\phi_n}(s)^{-1} \left\{ \frac{w_n \nu(q;x;d)}{w_n \nu(q;a;d)} w_n \nu(q;a; s) - w_n \nu(q;x; s) \right\} ds
\]

\[
= \int_y^\infty \Xi_{\phi}(s)^{-1} \left\{ \frac{w(q;x;d)}{w(q;a;d)} w(q;a; s) - w(q;x; s) \right\} ds.
\]

(ii) Identity \((59)\) follows by taking \(a \uparrow \infty\) in \((58)\) and applying Lemma \(29\) (ii).

(iii) The result follows by taking the limit \(d \downarrow -\infty\) in \((58)\) and applying Lemma \(29\) (i).

(iv) We note that using \((54)\), Exercise 8.5 in \([10]\), and dominated convergence, it follows that

\[
\lim_{a \to \infty} \frac{u(q)(a)}{W(q)(a)} = \lim_{a \to \infty} \left[ \frac{e^{\Phi(q)}a}{W(q)(a)} + \int_0^a \frac{W(q)(a-y)}{W(q)(a)} \Phi(y) \left( \frac{\Phi(q)e^{\Phi(q)y}}{1 - \phi(y)W(q)(0)} + \int_0^y \Phi(q) e^{\Phi(q)z} dz \right)dy \right]
\]

\[
+ \int_0^y \Phi(q) \frac{K^*(y,z)}{1 - \phi(y)W(q)(0)} e^{\Phi(q)z} dz dy
\]

\[
= \psi'(\Phi(q))
\]

\[
+ \int_y^\infty \Xi_{\phi}(y)^{-1} e^{-\Phi(q)y} \phi(y) \left( \Phi(q)e^{\Phi(q)y} + \int_0^y \Phi(q) K^*(y,z) e^{\Phi(q)z} dz \right) dy = A(q).
\]

The result follows by taking the limit \(a \uparrow \infty\) in \((60)\) and applying \((62)\).  

Now, we will prove the identities related to the two-sided exit problem for the level-dependent Lévy process \(U\).

**Theorem 32. (Two-sided exit problem)**
(i) For \( d \leq x \leq a \) and \( q \geq 0 \),
\[
\mathbb{E}_x \left[ e^{-q\kappa^{+}_n} 1_{\{\kappa^{+}_n, < \kappa^{d,-}_n \}} \right] = \frac{w^{(q)}(x; d)}{w^{(q)}(a; d)}.
\]

(ii) For \( 0 \leq x \leq a \) and \( q \geq 0 \),
\[
\mathbb{E}_x \left[ e^{-q\kappa^{-}_n} 1_{\{\kappa^{-}_n, < \kappa^{d,+}_n \}} \right] = \frac{z^{(q)}(a) - z^{(q)}(a)}{w^{(q)}(a)} w^{(q)}(x).
\]

**Proof.** (i) As for the previous result, we can take a non-increasing sequence of multi-refracted Lévy process \((U^n)_n \geq 0\) that converges uniformly on compact sets to the level-dependent Lévy process \(U\) with the rate function \(\phi\). Then, by the proof of Theorems 4 and 5 in [12],
\[
\mathbb{E}_x \left[ e^{-q\kappa^{+}_n} 1_{\{\kappa^{+}_n, < \kappa^{d,-}_n \}} \right] = q \int_0^\infty e^{-qt} \mathbb{P}_x (U(t) \in \mathbb{R}, U(t) \geq 0) dt - \int_0^\infty e^{-qt} \mathbb{P}_x (U(t) \in [0, a], U(t) \geq 0, U(t) \leq a) dt
\]

On the other hand, by the proof of Theorem 31, it follows that
\[
\lim_{n \to \infty} q \int_0^\infty e^{-qt} \mathbb{P}_x (U^n(t) \in \mathbb{R}, U^n(t) \geq 0) dt - \int_0^\infty e^{-qt} \mathbb{P}_x (U^n(t) \in [0, a], U^n(t) \geq 0, U^n(t) \leq a) dt
\]

Hence, by Theorem 33 (i) and Theorem 28, we obtain
\[
\mathbb{E}_x \left[ e^{-q\kappa^{+}_n} 1_{\{\kappa^{+}_n, < \kappa^{d,-}_n \}} \right] = \lim_{n \to \infty} q \int_0^\infty e^{-qt} \mathbb{P}_x (U^n(t) \in \mathbb{R}, U^n(t) \geq 0) dt - \int_0^\infty e^{-qt} \mathbb{P}_x (U^n(t) \in [0, a], U^n(t) \geq 0, U^n(t) \leq a) dt
\]

\[
= \lim_{n \to \infty} \frac{w^{(q)}(x; d)}{w^{(q)}(a; d)} = \frac{w^{(q)}(x; d)}{w^{(q)}(a; d)}.
\]

(ii) By (i) and the proof of Theorems 4 and 5 in [12], we have
\[
\mathbb{E}_x \left[ e^{-q\kappa^{-}_n} 1_{\{\kappa^{-}_n, < \kappa^{d,+}_n \}} \right] = 1 - q \int_0^\infty \mathbb{P}_x (U(t) \in \mathbb{R}, U(t) \geq 0) dt - \frac{w^{(q)}(x)}{w^{(q)}(a)} \left( 1 - q \int_0^\infty \mathbb{P}_a (U(t) \in \mathbb{R}, U(t) \geq 0) dt \right).
\]

Then, by the proof Theorem 31 for any \( x \geq 0 \) it holds that
\[
\lim_{n \to \infty} \left( 1 - q \int_0^\infty \mathbb{P}_x (U^n(t) \in \mathbb{R}, U^n(t) \geq 0) dt \right) = 1 - q \int_0^\infty \mathbb{P}_x (U(t) \in \mathbb{R}, U(t) \geq 0) dt.
\]

Therefore, using Theorems 33 (i) and 28, we obtain that
\[
\mathbb{E}_x \left[ e^{-q\kappa^{-}_n} 1_{\{\kappa^{-}_n, < \kappa^{d,+}_n \}} \right] = \lim_{n \to \infty} \left( 1 - q \int_0^\infty \mathbb{P}_x (U^n(t) \in \mathbb{R}, U^n(t) \geq 0) dt \right)
\]
Proposition 34. Hence, we have the following result.

Theorem 33. (One-sided exit problem)

(i) For \( x \geq 0 \) and \( q > 0 \),
\[
\mathbb{E}_x \left[ e^{-q\kappa_0^- \cdot 1_{\{\kappa_0^- < \infty\}}} \right] = z^{(q)}(x) - \frac{\phi(y) z^{(q)\mu}(y) - w^{(q)}(x)}{1 + \int_0^\infty e^{-\Phi(q)y} \phi(y) w^{(q)\mu}(y) dy}.
\] (65)

(ii) For \( x \leq a \) and \( q \geq 0 \),
\[
\mathbb{E}_x \left[ e^{-q\kappa_a^+ \cdot 1_{\{\kappa_a^+ < \infty\}}} \right] = \frac{u^{(q)}(x)}{\Phi(q)}. \] (66)

Proof. (i) First, we note that
\[
\lim_{a \to \infty} \frac{z^{(q)}(a)}{W^{(q)}(a)} = \lim_{a \to \infty} \left( \frac{Z^{(q)}(a)}{W^{(q)}(a)} + \int_0^a \frac{W^{(q)}(a - y)}{W^{(q)}(a)} \phi(y) z^{(q)\mu}(y) dy \right)
\]
\[
= \frac{q}{\Phi(q)} + \int_0^\infty e^{-\Phi(q)y} \phi(y) z^{(q)\mu}(y) dy. \] (67)

Therefore by taking \( a \uparrow \infty \) in (64) and using (55) and (67), we obtain the result.

(ii) The result follows as in the proof of Theorem 32 (i). 

Now, we will compute the ruin probability for the level-dependent Lévy process \( U \) with rate function \( \phi \), under the assumption that \( \mathbb{E}[X(1)] > 0 \). Following the considerations from Corollary 8, the ruin function is given by
\[
\Psi(x) := \mathbb{P}_x (\kappa_0^- < \infty) = 1 - \lim_{a \to \infty} \frac{w(x)}{w(a)}.
\]
Hence, we have the following result.

Proposition 34. Assume that \( x \geq 0 \).

(i) If \( \mathbb{E}[X(1)] \leq 0 \), then \( \Psi(x) = 1 \) for all \( x \geq 0 \).

(ii) If \( \mathbb{E}[X(1)] > 0 \) and \( \int_0^\infty \phi(x) w'(x) dx \) exists, then the ruin function
\[
\Psi(x) = 1 - A^{-1}w(x),
\]
where
\[
A = \frac{1 + \int_0^\infty \phi(x) w'(x) dx}{\mathbb{E}[X(1)]}
\]
and \( \Psi \) satisfies the following Volterra equation:
\[
\Psi(x) = 1 - A^{-1}W(x) + \int_0^x W(x - y) \phi(y) \Psi'(y) dy.
\]
Moreover, when \( \int_0^\infty \phi(x)w'(x) \, dx = \infty \) it follows that \( A = \infty \), and hence \( \Psi = 1 \).

**Proof.** We recall that \( w \) satisfies the following integral equation:

\[
w(x) = W(x) + \int_0^x W(x - y)\phi(y)w'(y) \, dy.
\]

Hence, if \( E[X(1)] \leq 0 \) then \( \lim_{a \to \infty} W(a) = \infty \), which using the fact that \( w(a) \geq W(a) \), for every \( a \geq 0 \) implies that

\[
\lim_{x \to \infty} w(x) = \infty.
\]

On the other hand, if \( E[X(1)] > 0 \) then \( \lim_{a \to \infty} W(a) = 1/E[X(1)] \), and hence

\[
\lim_{a \to \infty} w(a) = \frac{1}{E[X(1)]} \left( 1 + \int_0^\infty \phi(y)w'(y) \, dy \right).
\] (68)

Finally, we note that

\[
\Psi(x) = 1 - A^{-1}w(x) = 1 - A^{-1} \left( W(x) + \int_0^x W(x - y)\phi(y)w'(y) \, dy \right)
= 1 - A^{-1}W(x) + \int_0^x W(x - y)\phi(y)\Psi(y) \, dy.
\]

\[\blacksquare\]

**APPENDIX A. PROOFS FOR THE BOUNDED VARIATION CASE**

In this appendix, we prove Theorems 5(i) and 7(i) for the case that \( X \) has paths of bounded variation. The proofs for the case of unbounded variation are deferred to Appendix 13.

**A.1. Proof of Theorem 5(i).** The proof is inductive. First, it is easy to check that for \( k = 1 \), formula (18) agrees with (17).

Now, we shall assume that equation (18) holds true for \( k - 1 \), and show that it also holds for \( k \). We let \( p(x; \delta_1, ..., \delta_k) := \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\varphi_k < \kappa_k \}} \right] \). We follow the main idea given in Theorem 16 of [12]. For \( x \leq b_k \), it follows from the strong Markov property and the assumption that the identity holds for \( k - 1 \) that

\[
p(x; \delta_1, ..., \delta_k) = \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\varphi_k < \kappa_k \}} \right] p(b_k; \delta_1, ..., \delta_k).
\] (69)

For \( b_k < x \leq a \), by applying the strong Markov property once more and considering equation (69) and the expectation given in Theorem 23 (iii) of [12], we obtain that

\[
p(x; \delta_1, ..., \delta_k) = \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\varphi_k < \kappa_k \}} \right] + \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\varphi_k < \kappa_k \}} \mathbf{1}_{\{\tau_k > \tau_k \}} \right]
= \frac{W_k(x - b_k)}{W_k(a - b_k)} + \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\varphi_k < \kappa_k \}} p(U_k(\tau_k); \delta_1, ..., \delta_k) \right]
= \frac{W_k(x - b_k)}{W_k(a - b_k)} + \frac{p(b_k; \delta_1, ..., \delta_k)}{w_k(b_k; d)} \mathbb{E}_x \left[ e^{-\kappa_k a^+} \mathbf{1}_{\{\tau_k < \varphi_k \}} w_k(b_k; d) \right]
= \frac{W_k(x - b_k)}{W_k(a - b_k)} + \frac{p(b_k; \delta_1, ..., \delta_k)}{w_k(b_k; d)} \int_{\varphi_k}^{\infty} w_k(b_k + y - \theta; d)
\].
The transform of the left-hand side of (73) becomes

\[ \times \left[ \frac{W_k^{(q)}(x - b_k)W_k^{(q)}(a - b_k - y)}{W_k^{(q)}(a - b_k)} - W_k^{(q)}(x - b_k - y) \right] \Pi(d\theta)dy. \]  

(70)

Now, by setting \( x = b_k \) in (70) and using the fact that \( W_k^{(q)}(0) = \frac{1}{e^{-(\delta_1 + \ldots + \delta_k)}} \), we obtain

\[ p(b_k; \delta_1, \ldots, \delta_{k-1}, \delta_k) = w_{k-1}^{(q)}(b_k; d) \left\{ (c - \delta_1 - \ldots - \delta_k)W_k^{(q)}(a - b_k)w_{k-1}^{(q)}(b_k; d) \right. \]

\[ \left. - \int_0^{a-b_k} \int_{(y,\infty)} w_{k-1}^{(q)}(b_k + y - \theta; d)W_k^{(q)}(a - b_k - y)\Pi(d\theta)dy \right\}^{-1}. \]  

(71)

We shall now simplify this expression.

First, if we take \( \delta_k = 0 \), then by the inductive hypothesis we obtain that

\[ p(b_k; \delta_1, \ldots, \delta_{k-1}, 0) = E_{\delta_k} \left[ e^{-qs_{k-1}^{\alpha_k^+}1_{\left\{ \kappa_k^+ < \kappa_{k-1}^+ \right\}}} \right] = \frac{w_{k-1}^{(q)}(b_k; d)}{w_k^{(q)}(a; d)}. \]  

(72)

Then, it follows from (71) and (72) and the fact that \( W_k = W_{k-1} \) when \( \delta_k = 0 \) that

\[ \int_0^{a-b_k} \int_{(y,\infty)} w_{k-1}^{(q)}(b_k + y - \theta; d)W_k^{(q)}(a - b_k - y)\Pi(d\theta)dy = \left( c - \sum_{i=1}^{k-1} \delta_i \right) w_{k-1}^{(q)}(a - b_k)w_{k-1}^{(q)}(b_k; d) - w_{k-1}^{(q)}(a; d). \]  

(73)

Let \( \lambda > \varphi_k(q) \). As \( a \geq b_k \) is taken arbitrarily, we set \( a = u \) and take the Laplace transforms from \( b_k \) to \( \infty \) of both sides of (73). By Fubini’s theorem, the Laplace transform of the left-hand side of (73) becomes

\[ \int_b^\infty \int_0^\infty w_{k-1}^{(q)}(b_k + y - \theta; d)W_k^{(q)}(u - b_k - y)\Pi(d\theta)dydu = \frac{e^{-\lambda b_k}}{\psi_{k-1}(\lambda) - q} \int_0^\infty e^{-\lambda y} \int_y^\infty w_{k-1}^{(q)}(b_k + y - \theta; d)\Pi(d\theta)dydu. \]  

(74)

On the other hand, the Laplace transform of the right-hand side of (73) becomes

\[ \int_b^\infty e^{-\lambda u} \left( c - \delta_1 - \ldots - \delta_{k-1} \right)W_k^{(q)}(u - b_k)w_{k-1}^{(q)}(b_k; d) - w_{k-1}^{(q)}(u; d) \right)du = \frac{e^{-\lambda b_k} \left( c - \delta_1 - \ldots - \delta_{k-1} \right)}{\psi_{k-1}(\lambda) - q} w_{k-1}^{(q)}(b_k; d) - \int_b^\infty e^{-\lambda u}w_{k-1}^{(q)}(u; d)du. \]

Hence, by matching these, we obtain

\[ \int_0^\infty \int_{(y,\infty)} e^{-\lambda y}w_{k-1}^{(q)}(b_k + y - \theta; d)\Pi(d\theta)dydu = \left( c - \sum_{i=1}^{k-1} \delta_i \right) w_{k-1}^{(q)}(b_k; d) - \left( \psi_{k-1}(\lambda) - q \right) e^{\lambda b_k} \int_b^\infty e^{-\lambda u}w_{k-1}^{(q)}(u; d)du. \]  

(75)

Now using (74) and (75) and Fubini’s theorem we obtain that
Remark 35. It is easy to see that from equation (77) we obtain the following identity, which will be crucial for the remainder of the paper:

\[
\int_{0}^{\infty} \int_{(y, \infty)} w_{k-1}^{(q)}(b_k + y - \theta; d) W_k^{(q)}(u - b_k - y) \Pi(d\theta) \, dy \, du \\
\quad = \frac{e^{-\lambda b_k}}{\psi_k(\lambda) - q} \left( (c - \delta_1 - \ldots - \delta_{k-1}) w_{k-1}^{(q)}(b_k; d) \\
\quad - (\psi_{k-1}(\lambda) - q) e^{\lambda b_k} \int_{b_k}^{\infty} e^{-\lambda u} w_{k-1}^{(q)}(u; d) \, du \right). \tag{76}
\]

Finally, inversion of the Laplace transform (75) with respect to \( \lambda \) (for details we refer the reader to p. 34 of [12]) gives that, for all \( u \geq b_k \),

\[
\int_{0}^{\infty} \int_{(y, \infty)} w_{k-1}^{(q)}(b_k + y - \theta; d) W_k^{(q)}(u - b_k - y) \Pi(d\theta) \, dy \\
\quad = \left( c - \sum_{i=1}^{k-1} \delta_i \right) w_{k-1}^{(q)}(b_k; d) W_k^{(q)}(u - b_k) - W_k^{(q)}(u - b_k) - \delta_k \int_{b_k}^{x} W_k^{(q)}(u - y) w_{k-1}^{(q)}(y; d) \, dy. \tag{77}
\]

Then by using the above formula together with (71), we compute that

\[
p(b_k; \delta_1, \ldots, \delta_k) = w_{k-1}^{(q)}(b_k; d) \left\{ w_{k-1}^{(q)}(a; d) - \delta_k \int_{b_k}^{a} W_k^{(q)}(a - y) w_{k-1}^{(q)}(y; d) \, dy \right\}^{-1}. \tag{78}
\]

Finally, putting (78) into (39) and (70) gives the equation (18). 

\[\blacksquare\]

**A.2. Proof of Theorem 7(i).** The proof is inductive. Using equation (31) it is easy to check for \( k = 1 \) that the formula (25) agrees with [12]Theorem 6(i). Now, we assume that equation (25) holds true for \( k - 1 \). By the strong Markov property, we have for \( x \leq b_k \) that

\[
V^{(q)}(x, dy) := \mathbb{E}_x \left[ \int_{0}^{\kappa_k^{+} \wedge \kappa_k^{-}} e^{-qt} 1_{\{U_k(t) \in dy\}} \, dt \right] \\
\quad = \mathbb{E}_x \left[ \int_{0}^{b_k^{+} \wedge \kappa_k^{-}} e^{-qt} 1_{\{U_k(t) \in dy\}} \, dt \right] + \mathbb{E}_x \left[ e^{-q_1^{b_k^{+}}} 1_{\{U_k^{b_k^{+}} < \kappa_k^{-}\}} \int_{0}^{b_k^{+}} e^{-qt} 1_{\{U_k(t) \in dy\}} \, dt \right] \\
\quad = \sum_{i=0}^{k-1} \frac{w_{k-1}^{(q)}(b_k; y) - w_{k-1}^{(q)}(b_k; y)}{\Xi_{\phi_i}(y)} 1_{\{y \in (b_i, b_{i+1}]\}} dy + \frac{w_{k-1}^{(q)}(x; d)}{w_{k-1}^{(q)}(b_k; d)} V^{(q)}(b_k, dy). \tag{79}
\]
where the last equality holds by the inductive hypothesis and Theorem 5(i).

For $b_k \leq x \leq a$, we have that

$$V^{(q)}(x, dy) = \mathbb{E}_x \left[ \int_0^{t_{k-1}} e^{-qt_k} 1_{(X_k(t) \in \mathbb{D}_q)} dt \right] + \mathbb{E}_x \left[ e^{-qt_k} 1_{(t_{k-1} \leq t \leq t_k)} V^{(q)}(X_k(t_k), dy) \right].$$

The former expectation on the right-hand side is equal to

$$\int_0^\infty e^{-qt_k} \mathbb{P}_x \left( X_k(t) \in dy, y \in [b_k, a], t < t_{k-1} \land t_k \right) dt$$

while by (80), the latter becomes

$$\int_0^\infty \int_0^\infty V^{(q)}(z - \theta, dy) \left[ \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} W_k^{(q)}(a - b_k - z) - W_k^{(q)}(x - b_k - z) \right] \Pi(d\theta) \, dz$$

$$= \left\{ \int_0^\infty \int_0^\infty \sum_{i=0}^{k-1} \frac{1}{\Xi(x)} \left( \frac{w_{k-1}^{(q)}(z - \theta + b_k; d)}{w_k^{(q)}(b_k; d)} w_{k-1}^{(q)}(b_k; y) - w_{k-1}^{(q)}(z - \theta + b_k; y) \right) 1_{\{y \in [b_k, b_{k+1}]\}} \times \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} W_k^{(q)}(a - b_k - z) - W_k^{(q)}(x - b_k - z) \right\} \Pi(d\theta) \, dz \right\} dy$$

$$+ \int_0^\infty \int_0^\infty \sum_{i=0}^{k-1} \frac{w_{k-1}^{(q)}(z - \theta + b_k; d)}{w_k^{(q)}(b_k; d)} V^{(q)}(b_k, dy)$$

$$\times \left[ \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} W_k^{(q)}(a - b_k - z) - W_k^{(q)}(x - b_k - z) \right] \Pi(d\theta) \, dz.$$

Next, we use Remark 33 to simplify the expression for $V^{(q)}(x, dy)$ for $b_k \leq x \leq a$ as follows:

$$V^{(q)}(x, dy) = \left\{ \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} W_k^{(q)}(a - y) - W_k^{(q)}(x - y) \right\} 1_{\{y \in [b_k, a]\}} \cdot dy$$

$$+ \left\{ \sum_{i=0}^{k-1} \left[ \frac{w_{k-1}^{(q)}(b_k; y)}{w_k^{(q)}(b_k; d)} \left( \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} w_k^{(q)}(a; d) + w_k^{(q)}(x; d) \right) \right. \right.$$

$$+ \frac{W_k^{(q)}(x - b_k)}{w_k^{(q)}(b_k)} w_k^{(q)}(a; y) - w_k^{(q)}(x; y) \right\} \left. \frac{1\{y \in [b_k, b_{k+1}]\}}{\Xi(x)} \right] \cdot dy$$

$$+ \frac{V^{(q)}(b_k, dy)}{w_k^{(q)}(b_k)} \left( - \frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} w_k^{(q)}(a; d) + w_k^{(q)}(x; d) \right).$$

Finally, setting $x = b_k$ in (81) and using the fact that $w_k^{(q)}(b_k) = w_{k-1}^{(q)}(b_k)$ leads us to an explicit formula for $V^{(q)}(b_k, dy)$. Then, putting the expression for $V^{(q)}(b_k, dy)$ into
\((81)\) gives us that
\[
\mathbb{E}_x \left[ \int_0^{a_k^+ \wedge d_k^-} e^{-q(x)} 1_{\{U_k(t) \in dy\}} dt \right] = \sum_{i=0}^k \frac{w_k^{(q)}(x;d) w_k^{(q)}(x,y) - w_k^{(q)}(x,y)}{\Xi \phi_k(y)} 1_{\{y \in (b_i, b_{i+1}] \cap (-\infty, a)\}} dy
\]
\[
= \Xi \phi_k(y)^{-1} \left\{ \frac{w_k^{(q)}(x;d) w_k^{(q)}(a,y) - w_k^{(q)}(x,y)}{w_k^{(q)}(a,d)} \right\} 1_{\{y \in (-\infty, a)\}} dy.
\]

\[\Box\]

**Appendix B. Proofs for the unbounded variation case**

In this appendix, we first show the existence of the multi-refracted Lévy process \(U_k\) (as in Theorem \(1\) for the case that \(X\) is of unbounded variation. Together with Lemma \(2\), this implies that \(U_k\) is the unique strong solution of the SDE \((5)\). We then prove Theorems \(5(i)\) and \(7(i)\) for the case of unbounded variation.

**B.1. Existence.** Now, we will prove that there exists a process \(U_k\) that is a solution to \((5)\) under the assumption that \(X\) has unbounded variation paths. To this end, we will use an approximation method as in \([12]\). First, recall that it is known (see, e.g., Bertoin \([5]\) or \([12]\), Lemma 12) that for any spectrally negative Lévy process \(X\) of unbounded variation we can find a sequence of bounded variation Lévy processes \(X^{(n)}\) such that, for each \(t > 0\),
\[
\lim_{n \to \infty} \sup_{s \in [0,t]} |X^{(n)}(s) - X(s)| = 0 \quad a.s.
\]
Hence, let the constants \(t > 0\) and \(\eta > 0\) be fixed, and let \(N \in \mathbb{N}\) be sufficiently large that for all \(n,m \geq N\), \(\sup_{s \in [0,t]} |X^{(n)}(s) - X^{(m)}(s)| \leq \eta\).

Now, by \(U_k^{(n)}\) denote the sequence of \(k\)-multi-refracted pathwise solutions associated with \(X^{(n)}\) for \(n \geq 1\). That is the solutions to the following SDE:
\[
dU_k^{(n)}(t) = dX^{(n)}(t) - \sum_{i=1}^k \delta_i 1_{\{U_k^{(n)}(t) > b_i\}} dt.
\]
The next step is to prove that
\[
\lim_{n,m \to \infty} \sup_{s \in [0,t]} |U_k^{(n)}(s) - U_k^{(m)}(s)| = 0 \quad a.s.,
\]
from which we deduce that \(\{U_k^{(n)}(s)\}_{s \in [0,t]}\) is a Cauchy sequence in the Banach space consisting of càdlàg mappings equipped with the supremum norm. To this end, for any \(k \geq 1\), we adopt the reasoning given in the proof of Lemma 12 of \([12]\), because in our case we have that
\[
A_k^{(n,m)}(s) := (U_k^{(n)}(s) - U_k^{(m)}(s)) - (X^{(n)}(s) - X^{(m)}(s))
\]
\[
= \sum_{i=1}^k \delta_i \int_0^s \left( 1_{\{U_k^{(m)}(u) > b_i, U_k^{(n)}(u) \leq b_i\}} - 1_{\{U_k^{(m)}(u) \leq b_i, U_k^{(n)}(u) > b_i\}} \right) du.
\]
Then, taking the advantage of the fact that \(0 < \delta_1, ..., \delta_k\), we show as in \([12]\) that \(\sup_{s \in [0,t]} |A_k^{(n,m)}(s)| \leq \eta\). Thus, we have shown that
\[
\lim_{n \to \infty} U_k^{(n)}(t) = U_k^{(\infty)}(t) \quad a.s.
\]
for a stochastic process \(U_k^{(\infty)} = \{U_k^{(\infty)}(t) : t \geq 0\}\). Now, repeating the reasoning presented in Lemma 21 of \([12]\) (using the fact that the resolvent obtained in Theorem
for the bounded variation case has a density), we show that for all driving Lévy processes \(X\) with paths of unbounded variation when \(x\) is fixed we have for \(1 \leq j \leq k\) that
\[
\mathbb{P}_x(U_k^{(x)}(t) = b_j) = 0 \text{ for almost every } t \geq 0.
\] (83)

Finally, using (83) we obtain that
\[
\lim_{n \to \infty} U_k^{(n)}(t) = \lim_{n \to \infty} \left(X^{(n)}(t) - \sum_{i=1}^{k} \delta_i \int_0^t 1_{\{U_k^{(n)}(s) > b_i\}} ds\right)
\]
\[
= X(t) - \sum_{i=1}^{k} \delta_i \int_0^t 1_{\{U_k^{(x)}(s) > b_i\}} ds \text{ a.s.,}
\]
which gives us that \(U_k := U_k^{(x)}\) solves (5), as desired.

B.2. Proof of Theorem (7(i)). Following the argument given in the proof of Lemma (4), we conclude that for any \(k \in \mathbb{N}\) the sequence \((U_k^{(n)})_{n \geq 1}\) converges uniformly on compact sets to the process \(U_k\).

Consider now for any \(k \geq 1\) the sequence of scale functions \((w_k^{(q), (n)})_{n \geq 1}\) associated with the sequence of \(k\)-multi-refracted Lévy processes \(U_k^{(n)}\). In the next step, we will prove the convergence of \((w_k^{(q), (n)})_{n \geq 1}\) to the scale function \(w_k^{(q)}\) defined in (19).

Therefore, for the case that \(k = 1\) we have by the proof of Lemma 20 in [12] that, for any \(x \geq d\),
\[
\lim_{n \to \infty} w_1^{(q), (n)}(x; d) = \lim_{n \to \infty} W^{(q), (n)}(x - d) + \delta_1 \int_{b_1}^{x} W_1^{(q), (n)}(x - y)W^{(q), (n)}(y - d) dy
\]
\[
= W^{(q)}(x - d) + \delta_1 \int_{b_1}^{x} W_1^{(q)}(x - y)W^{(q)}(y - d) dy = w_1^{(q)}(x; d),
\]
(84)

where \(W^{(q), (n)}(W^{(q)})\) and \(W_1^{(q), (n)}(W_1^{(q)})\) are the scale functions associated with the processes \((X^{(n)}(t))_{t \geq 0}\) \(((X(t))_{t \geq 0}\)) and \((X_1^{(n)}(t) := X^{(n)}(t) - \delta_1 t)_{t \geq 0}\) \(((X_1(t))_{t \geq 0}\)), respectively.

For each \(k \geq 1\) and \(x \geq d\), we define the function \(w_k^{(q), (n)}\) by the following recursion:
\[
w_k^{(q), (n)}(x; d) := w_{k-1}^{(q), (n)}(x; d) + \delta_k \int_{b_k}^{x} W_k^{(q), (n)}(x - y)w_{k-1}^{(q), (n)}(y; d) dy
\]
\[
= w_{k-1}^{(q), (n)}(x; d) + \delta_k W_k^{(q), (n)}(0)w_k^{(q), (n)}(x; d) - \delta_k w_{k-1}^{(q), (n)}(b_k; d)W_k^{(q)}(x - b_k)
\]
\[+ \delta_k \int_{b_k}^{x} W_k^{(q), (n)}(x - y)w_{k-1}^{(q), (n)}(y; d) dy,
\]
(85)

where the last equality follows from integration by parts. Assume now for the inductive step that for any \(x \geq d\) it holds that
\[
\lim_{n \to \infty} w_{k-1}^{(q), (n)}(x; d) = w_{k-1}^{(q)}(x; d).
\]

We will prove that the above limit holds for \(k\). To this end, we recall that the scale function \(w_k^{(q), (n)}\) for the process \(U_k^{(n)}\) satisfies the recurrence relation (85), where \(W_k^{(q), (n)}\) is the scale function of the process \((X_k^{(n)}(t) := X^{(n)}(t) - \sum_{j=1}^{k} \delta_j t)_{t \geq 0}\). Now, by the proof of Lemma 20 in [12] (see also Remark 3.2 of [13]) we have that \(W_k^{(q), (n)}\) and \(W_k^{(q), (n)}\)
converge to $W_k^{(q)}$ and $W_k^{(q)'}$, respectively. Hence by the dominated convergence theorem we have for any $x \in \mathbb{R}$ and $x \geq d$ that

$$
\lim_{n \to \infty} w_k^{(q), (n)}(x; d) = \lim_{n \to \infty} \left( w_{k-1}^{(q), (n)}(x; d) + \delta_k W_k^{(q), (n)}(0) w_k^{(q), (n)}(x; d) \right) \nonumber \\
- \delta_k w_{k-1}^{(q), (n)}(b_k; d) W_k^{(q), (n)}(x - b_k) + \delta_k \int_{b_k}^x W_k^{(q), (n)'}(x - y) w_{k-1}^{(q), (n)}(y; d) \, dy 
$$

$$
= w_k^{(q), (n)}(x; d) - \delta_k w_{k-1}^{(q), (n)}(b_k; d) W_k^{(q)}(x - b_k) + \delta_k \int_{b_k}^x W_k^{(q)'}(x - y) w_{k-1}^{(q)}(y; d) \, dy 
$$

$$
= w_k^{(q), (n)}(x; d) + \delta_k \int_{b_k}^x W_k^{(q)}(x - y) w_{k-1}^{(q)'}(y; d) \, dy 
$$

$$
= w_k^{(q)}(x; d). 
$$

Therefore, we have that for any $k \geq 1$ the sequence $(w_k^{(q), (n)})_{n \geq 1}$ converges pointwise to $w_k^{(q)}$.

If we denote by $V_k^{(q), (n)}$ the potential measure of the process $U_k^{(q), (n)}$ killed when exiting the interval $[0, a]$, then for each $n \geq 1$ and any open interval $B \subset [0, \infty]$ we have

$$
V^{(q), (n)}(x, B) := \int_0^\infty e^{-qt} \mathbb{P}_x \left( U_k^{(n)}(t) \in B, \overline{U}_k^{(n)}(t) \leq a, U_k^{(n)}(t) \geq 0 \right) \, dt, 
$$

where $\overline{U}_k^{(n)}(t) := \sup_{0 \leq s \leq t} U_k^{(n)}(s)$ and $U_k^{(n)}(t) := \inf_{0 \leq s \leq t} U_k^{(n)}(s)$.

As in the proof of Theorem 6 in [12], the uniform convergence on compact sets of the sequence $(U_k^{(n)})_{n \geq 1}$ to $U_k$ implies that for each $t > 0$, it holds a.s. that

$$
\lim_{n \to \infty} ((U_k^{(n)}(t), \overline{U}_k^{(n)}(t), U_k^{(n)}(t)) = ((U_k(t), \overline{U}_k(t), U_k(t)). 
$$

Using the ideas in the proof of Theorem 6 in [12], it is not difficult to see that $\mathbb{P}_x(U_k(t) \in \partial B) = 0$.

Hence by using the convergence of the sequence $(w_k^{(q), (n)})_{n \geq 1}$ we obtain that

$$
V^{(q)}(x, B) := \int_0^\infty e^{-qt} \mathbb{P}_x \left( U_k(t) \in B, \overline{U}_k(t) \leq a, U_k(t) \geq 0 \right) \, dt 
$$

$$
= \lim_{n \to \infty} \int_0^\infty e^{-qt} \mathbb{P}_x \left( U_k^{(n)}(t) \in B, \overline{U}_k^{(n)}(t) \leq a, U_k^{(n)}(t) \geq 0 \right) \, dt 
$$

$$
= \lim_{n \to \infty} \int_B \left( \sum_{i=0}^k \frac{w_k^{(q), (n)}(x)}{w_k^{(q), (n)}(a)} w_k^{(q), (n)}(a; y) - w_k^{(q), (n)}(x; y) \right) \frac{1_{\{y \in [b_i, b_{i+1})\}}}{\prod_{j=1}^i \left( 1 - \delta_j W_j^{(q), (n)}(0) \right)} \, dy 
$$

$$
= \int_B \left( \sum_{i=0}^k \frac{w_k^{(q)}(x)}{w_k^{(q)}(a)} w_k^{(q)}(a; y) - w_k^{(q)}(x; y) \right) \frac{1_{\{y \in [b_i, b_{i+1})\}}}{\prod_{j=1}^i \left( 1 - \delta_j W_j^{(q)}(0) \right)} \, dy 
$$

$$
\Xi_{\phi_k}^{-1}(y) \left( \frac{w_k^{(q)}(x)}{w_k^{(q)}(a)} w_k^{(q)}(a; y) - w_k^{(q)}(x; y) \right) \, dy, 
$$

which proves the identity (23) for the unbounded variation case.
B.3. Proof of Theorem 5(i). To obtain the identity (18) for a general Lévy process \( X \), it suffices to note that for \( q > 0 \), by applying the strong Markov property, we have that

\[
\mathbb{E}_x \left[ e^{-q \kappa_k^+} 1_{\left( \kappa_k^+ < \kappa_k^{-} \right)} \right] \cdot q \int_0^\infty e^{-qt} \mathbb{P}_a (U_k(t) \in \mathbb{R}_+, t < \kappa_k^{-}) \, dt
\]

\[
= q \int_0^\infty e^{-qt} \mathbb{P}_x (U_k(t) \in \mathbb{R}_+, t < \kappa_k^{-}) \, dt
\]

\[
- q \int_0^\infty e^{-qt} \mathbb{P}_x (U_k(t) \in [0, a], t < \kappa_k^{-} \wedge \kappa_k^+) \, dt.
\]

The above probabilities can be obtained directly from the potential measures given in Theorem 7.

\[\blacksquare\]

Appendix C. Proofs for General Case

C.1. Proof of Theorem 7(ii)-(iv). We start by providing the proof of identity (26). To compute the resolvent of the first passage time below 0, we take \( d = 0 \) and \( a \to \infty \) in (25). To this end, we note that from Lemma 15 we get the following:

\[
\lim_{y \to \infty} e^{-q \kappa_k^+} 1_{\left( \kappa_k^+ < \kappa_k^{-} \right)} = q \int_0^\infty e^{-qt} \mathbb{P}_a (U_k(t) \in \mathbb{R}_+, t < \kappa_k^{-}) \, dt
\]

\[
- q \int_0^\infty e^{-qt} \mathbb{P}_x (U_k(t) \in [0, a], t < \kappa_k^{-} \wedge \kappa_k^+) \, dt.
\]

The above probabilities can be obtained directly from the potential measures given in Theorem 7.

\[\blacksquare\]
Using (26) we note that
\[ a = \Phi(q) e^{\Phi(q)x} \left( 1 + \delta_1 W_1^{(q)}(0) \right) + \delta_1 \Phi(q) \int_{b_1}^{x} e^{\Phi(q)z} W_1^{(q)}(x - z) dz = u_1^{(q)}(x). \]

Thus, proceeding by induction we assume that
\[ \lim_{d \to -\infty} \frac{w_k^{(q)}(x; d)}{W(q)(-d)} = u_k^{(q)}(x) \quad \text{and} \quad \lim_{d \to -\infty} \frac{w_{k-1}^{(q)}(x; d)}{W(q)(-d)} = u_{k-1}^{(q)}(x). \] (87)

Therefore
\[
\lim_{d \to -\infty} \frac{w_k^{(q)}(x; d)}{W(q)(-d)} = \lim_{d \to -\infty} \frac{w_k^{(q)}(x; d) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) u_{k-1}^{(q)}(z; d) dz}{W(q)(-d)}
\]
\[ = u_k^{(q)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) u_{k-1}^{(q)}(z) dz = u_k^{(q)}(x). \]

We obtain the equality in (28) by taking the limit \( a \to \infty \) in (27) and applying Lemma 15. Moreover, using arguments given in Lemma 15 we obtain that \( \lim_{a \to \infty} \frac{u_k^{(q)}(a)}{W_k^{(q)}(a)} = 0 \), and finally
\[ \lim_{a \to \infty} \frac{u_k^{(q)}(a)}{W_k^{(q)}(a)} = \delta_k \int_{b_k}^{\infty} e^{-\phi_k(q)z} u_{k-1}^{(q)}(z) dz. \]

C.2. Proof of Theorem 5(ii). The proof follows by noting that, for \( x \leq a \),
\[ \mathbb{E}_x \left[ e^{-q_k^{a-1} \{1_{\kappa_k^{a-1} - \kappa_k^{a-1}} \}} \right] = \mathbb{E}_x \left[ e^{-q_k^{a-1} 1_{\{\kappa_k^{a-1} - \kappa_k^{a-1}\}}} \right] - \mathbb{E}_x \left[ e^{-q_k^{a-1} 1_{\{\kappa_k^{a-1} < \kappa_k^{a-1}\}}} \right]. \]

Now, we compute the first term of the right-hand side of the above equality as
\[ \mathbb{E}_x \left[ e^{-q_k^{a-1} 1_{\{\kappa_k^{a-1} < \kappa_k^{a-1}\}}} \right] = 1 - q \int_{0}^{\infty} e^{-\theta_t} \mathbb{P}_x(U_k(t) \in \mathbb{R}_+, t < \kappa_k^{a-1}) dt. \]

Using (28) we note that
\[
\int_{0}^{\infty} e^{-\theta_t} \mathbb{P}_x(U_k(t) \in \mathbb{R}_+, t < \kappa_k^{a-1}) dt = \int_{0}^{\infty} \frac{w_k^{(q)}(x)}{w_k^{(q)}(y)} W_k^{(q)}(y) - W_k^{(q)}(x; y) dy
\]
\[ = \sum_{i=0}^{k} \int_{b_i}^{b_{i+1}} \frac{w_i^{(q)}(x)}{\Xi_{\phi_i}(y)} W_i^{(q)}(y) - W_i^{(q)}(x; y) dy \cdot 1_{\{y \in [b_i, b_{i+1}]\}} dy. \]

Using equalities (14), (31), and (33) we obtain that
\[
\sum_{i=0}^{1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_i^{(q)}(x; y) dy = \int_{0}^{b_1} w_1^{(q)}(x; y) dy + \int_{b_1}^{x} w_1^{(q)}(x; y) dy
\]
\[ = \int_{0}^{x} W^{(q)}(x - y) dy + \delta_1 \int_{0}^{b_1} W_1^{(q)}(x - z) W^{(q)}(z - y) dz dy
\]
\[ + \int_{b_1}^{x} W_1^{(q)}(x - y) dy - \int_{b_1}^{x} W^{(q)}(x - y) dy
\]
\[ = \int_{0}^{x} W^{(q)}(y) dy + \delta_1 \int_{b_1}^{x} W_1^{(q)}(x - y) \left( W^{(q)}(y) - W^{(q)}(y - b_1) \right) dy
\]
Hence,

\[ z_1^{(q)}(x) := 1 + q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_k^{(q)}(x; y) \, dy = Z^{(q)}(x) + q \delta_1 \int_{b_1}^{x} W_1^{(q)}(y) \, dy. \]

Now, assume by induction that

\[ z_{k-1}^{(q)}(x) := 1 + q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_k^{(q)}(x; y) \, dy, \quad \text{where} \quad b_0 = -\infty, b_k = x. \]

Then, by using identities (19), (31), and (33) we have that

\[ z_k^{(q)}(x) := 1 + q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_k^{(q)}(x; y) \, dy \]

\[ = 1 + q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_k^{(q)}(x; y) \, dy + q \int_{b_k}^{x} W_k^{(q)}(x - y) \, dy \]

\[ = 1 + q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_k^{(q)}(x; y) \, dy + q \int_{b_k}^{x} W_k^{(q)}(x - y) \, dy \]

\[ = z_{k-1}^{(q)}(x) + q \delta_k \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} W_k^{(q)}(x - z) w_{k-1}^{(q)}(z; y) \, dz \, dy \]

\[ = z_{k-1}^{(q)}(x) + q \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) W_{k-1}^{(q)}(z - b_k) \, dz \]

\[ = z_{k-1}^{(q)}(x) + q \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} w_{k-1}^{(q)}(z; y) \, dy \, dz \]

\[ = z_{k-1}^{(q)}(x) + q \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) \frac{d}{dz} \left( \int_{b_k}^{z} W_{k-1}^{(q)}(z - y) \, dy \right) \, dz \]

\[ = z_{k-1}^{(q)}(x) + q \delta_k \int_{b_k}^{x} W_k^{(q)}(x - z) z_{k-1}^{(q)}(y) \, dy. \]

Hence, we can write

\[ \mathbb{E}_x \left[ e^{-\alpha_k^{(q)}} 1_{\{\alpha_k^{(q)} < \infty\}} \right] = 1 - q \sum_{i=0}^{k-1} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} \left\{ \frac{w_k^{(q)}(x)}{v_k^{(q)}(0)} v_k^{(q)}(y) \, dy - w_k^{(q)}(x; y) \right\} \, dy. \]
Finally, by putting the pieces together we obtain that
\[
\mathbb{E}_x \left[ e^{-q \kappa_k^0} \mathbf{1}_{\{\kappa_k^0 < \kappa_k^{a^+}\}} \right] = 1 - q \sum_{i=0}^{k} \frac{1}{\Xi_{\phi_i}} \int_{b_i}^{b_{i+1}} \left\{ \frac{w_k^{(q)}(x)}{v_k^{(q)}(0)} v_k^{(q)}(y) dy - \frac{w_k^{(q)}(y)}{v_k^{(q)}(0)} w_k^{(q)}(a; y) \right\} dy \\
- \frac{w_k^{(q)}(x)}{w_k^{(q)}(a)} \left( 1 - q \sum_{i=0}^{k} \frac{1}{\Xi_{\phi_i}(y)} \int_{b_i}^{b_{i+1}} \left\{ \frac{w_k^{(q)}(a)}{v_k^{(q)}(0)} v_k^{(q)}(y) dy - \frac{w_k^{(q)}(y)}{v_k^{(q)}(0)} w_k^{(q)}(a; y) \right\} dy \right) \\
= z_k^{(q)}(x) - \frac{w_k^{(q)}(x)}{w_k^{(q)}(a)} z_k^{(q)}(a).
\]

C.3. Proof of Corollary 6(i)-(ii). To obtain (22), we take the limit \( a \to \infty \) in (20). The result follows because from (80) we have
\[
v_k^{(q)}(0) = \lim_{a \to \infty} \frac{w_k^{(q)}(a)}{W_k^{(q)}(a)} = \delta_k \int_{b_k}^{\infty} e^{-\varphi_k(q)z} w_k^{(q)}(z) dz
\]
Then, using Lemma 15 and Remark 16 we get
\[
\lim_{a \to \infty} z_k^{(q)}(a) = \lim_{a \to \infty} \left( \frac{z_{k-1}^{(q)}(a) W_{k-1}^{(q)}(a)}{W_k^{(q)}(a) W_{k+1}^{(q)}(a)} + \delta_k \int_{b_k}^{a} \frac{W_{k-1}^{(q)}(a-z)}{W_{k-1}^{(q)}(z)} dz \right) = 0.
\]
Therefore,
\[
\lim_{a \to \infty} \frac{z_k^{(q)}(a)}{W_k^{(q)}(a)} = \lim_{a \to \infty} \left( \frac{z_{k-1}^{(q)}(a)}{W_{k-1}^{(q)}(a)} + \delta_k \int_{b_k}^{a} \frac{W_{k-1}^{(q)}(a-z)}{W_{k-1}^{(q)}(z)} dz \right) = \delta_k \int_{b_k}^{\infty} e^{-\varphi_k(q)z} z_{k-1}^{(q)}(z) dz.
\]
To obtain (23), we take the limit \( d \to -\infty \) in (18) and use equation (87).

C.4. Proof of Corollary 8. To obtain the ruin probability, we first take \( q = 0 \) in (20) giving that
\[
\mathbb{P}_x(\kappa_k^{0^+} < \kappa_k^{a^+}) = z_k(x) - \frac{w_k(x)}{w_k(a)}.
\]
From the definition of \( z_k \) given in Theorem 5(ii), and because \( Z = 1 \), it follows immediately by induction that \( z_k = 1 \). Next, we will prove by induction that
\[
\lim_{a \to \infty} w_k(a) = \frac{1 - \sum_{j=1}^{k} \delta_j w_{j-1}(b_j)}{\mathbb{E}[X_1] - \sum_{j=1}^{k} \delta_j},
\]
where \( w_0 = W \). Now, for \( k = 1 \), using the fact that \( W(\infty) = 1/\mathbb{E}[X_1] \) (which is well defined because \( \mathbb{E}[X_1] > 0 \) by assumption), we obtain the following:
\[
w_1(\infty) = \lim_{a \to \infty} \left( W(a) + \delta_1 \int_{b_1}^{a} W_1(a-y)W'(y)dy \right) = \frac{1}{\mathbb{E}[X_1]} + \frac{\delta_1}{\mathbb{E}[X_1] - \delta_1} \int_{b_1}^{\infty} W'(y)dy = \frac{1}{\mathbb{E}[X_1]} + \frac{\delta_1}{\mathbb{E}[X_1] - \delta_1} \left( \frac{1}{\mathbb{E}[X_1]} - W(b_1) \right) = 1 - \frac{1}{\mathbb{E}[X_1] - \delta_1}.
\]
Hence,
\[
\mathbb{P}_x(\kappa_1^{0^+} < \kappa_1^{a^+}) = 1 - \left[ \frac{\mathbb{E}[X_1] - \delta_1}{1 - \delta_1 W(b_1)} \right] w_1(x),
\]
which agrees with the result in Theorem 5 (ii) in [12].
Now, for the induction step assume that
\[
\lim_{a \to \infty} w_{k-1}(a) = \frac{1 - \sum_{j=1}^{k-1} \delta_j w_{j-1}(b_j)}{\mathbb{E}[X_1] - \sum_{j=1}^{k-1} \delta_j}.
\] (90)

Then, we compute using (90)
\[
w_k(\infty) = \lim_{a \to \infty} \left( w_{k-1}(a) + \delta_k \int_{b_k}^a W_k(a-y)w_{k-1}'(y)dy \right)
= w_{k-1}(\infty) + \frac{\delta_k (w_{k-1}(\infty) - w_{k-1}(b_k))}{\mathbb{E}[X_1] - \sum_{j=1}^{k-1} \delta_j} = \frac{1 - \sum_{j=1}^{k-1} \delta_j w_{j-1}(b_j)}{\mathbb{E}[X_1] - \sum_{j=1}^{k-1} \delta_j}.
\]

Finally, we obtain (29). Then, applying (89) in (88) gives the result.

\[\blacksquare\]

**Appendix D. Proofs for Section 3.2**

**D.1. Proof of Lemma 25.** To show that \( \zeta(x) \) is well defined, we consider its Laplace transform. Thus, we have that
\[
\int_0^\infty e^{-sx} \sum_{l=1}^\infty a^l (W(q)^{\prime})^l(x) dx = \sum_{l=1}^\infty \left( a \int_0^\infty e^{-sx} W(q)^{\prime}(x) dx \right)^l.
\]

For the convergence, we have to verify that for sufficiently large \( s > 0 \), it holds that
\[a \int_0^\infty e^{-sx} W(q)^{\prime}(x) dx < 1.\]

For \( s > \Phi(q) \)
\[
\int_0^\infty e^{-sx} W(q)^{\prime}(x) dx = \frac{1}{s} W(q)(0) + \frac{1}{s} \int_0^\infty e^{-sx} W(q)(x) dx
= \frac{1}{s} W(q)(0) + \frac{1}{s(\psi(s) - q)},
\]
which is less than 1 for for sufficiently large \( s \). The proof is completed by applying Lemma 24.

\[\blacksquare\]

**D.2. Proof of Lemma 26.** We start by noting that \( K(x, y) \leq a_T W(q)^{\prime}((x - y)+) \).

Now, assume that
\[
K^{(l)}(x, y) \leq a_T^l (W(q)^{\prime})^l((x - y)+) \] (91)
for \( l \in \mathbb{N} \). Then, for \( x \geq y \geq 0 \), we have that
\[
K^{(l+1)}(x, y) = \int_y^x K^{(l)}(x, z) K(z, y) dz
\leq a_T^{l+1} \int_y^x (W(q)^{\prime})^l(x - z) W(q)^{\prime}(z - y) dw
= a_T^{l+1} \int_0^{x-y} (W(q)^{\prime})^l(x - y - z) W(q)^{\prime}(z) dz
= a_T^{l+1} (W(q)^{\prime})^{l+1}((x - y)+).
\]
Hence, by induction we can conclude that (91) holds for every \( l \in \mathbb{N} \). Therefore,

\[
K^*(x, y) \leq \sum_{l=1}^{\infty} a_l^l W^{(q)^l}(x-y) = \zeta(x-y).
\]

\[\square\]

D.3. **Proof of Proposition 27**. \( (i) \) Considering the discussion in Section 3.1, we only need to show that

\[
\int_d^x K^*(x, y) \Xi(x)^{-1} W^{(q)}(y-d) \, dy < \infty.
\]

Because \( \phi(x) < 1/W^{(q)}(0) \) for all \( x \geq 0 \) in the bounded variation case and \( W^{(q)}(0) = 0 \) in the unbounded variation case, we have that \( b := \Xi(x)^{-1} < \infty \). Therefore,

\[
\int_d^x \zeta(x-y) g(y) \, dy \leq b \int_d^x \zeta(x-y) W^{(q)}(y-d) \, dy = b \int_0^{x-d} \zeta(x-y-d) W^{(q)}(y) \, dy
\]

\[
= b \sum_{l=2}^{\infty} a^l (W^{(q)^l})^l((x-d)+),
\]

which is convergent by Lemma 26 for any \( x \in [d, T] \).

\( (ii) \) In this case, we need only to check that

\[
\int_0^x K^*(x, y) \Xi(x)^{-1} qW^{(q)}(y) \, dy < \infty.
\]

Then,

\[
\int_0^x \zeta(x-y) g(y) \, dy = \int_0^x g(x-y) \zeta(y) \, dy
\]

\[
= \sum_{l=1}^{\infty} a^l \int_0^x qW^{(q)}(x-y)(W^{(q)^l}(y) \, dy
\]

\[
\leq qW^{(q)}(x) \sum_{l=1}^{\infty} a^l \int_0^x (W^{(q)^l}(y) \, dy = qW^{(q)}(x) \int_0^x \zeta(y) \, dy.
\]

Furthermore, we have that

\[
\int_d^\infty e^{-sx} \int_0^x (W^{(q)^l}(y) \, dydx = \frac{1}{s} \int_0^\infty e^{-sy} (W^{(q)^l}(y) \, dy
\]

\[
= \frac{1}{s} \left( \int_0^\infty e^{-sx} W^{(q)}(x) \, dx \right)^l
\]

\[
= \frac{1}{s} \left( \frac{1}{s} W^{(q)}(0) + \frac{1}{s(\psi(s) - q)} \right)^l.
\]

Hence,

\[
\int_0^\infty e^{-sx} \int_0^x \zeta(y) \, dy \, dx < \infty
\]

for sufficiently large \( s > 0 \), which yields the finiteness of \( \int_0^x \zeta(y) \, dy \) for any \( x \in [0, T] \). \[\square\]
References

[1] H. Albrecher, C. Constantinescu, Z. Palmowski, G. Regensburger, and M. Rosnekranz, Exact and asymptotic results for insurance risk models with surplus-dependent premiums, SIAM Journal of Applied Mathematics 73 (2013), 47–66.

[2] D. Applebaum, Lévy processes and stochastic calculus, Cambridge University Press, 2004.

[3] S. Asmussen and H. Albrecher, Ruin probabilities, Second, World Scientific, 2010.

[4] S. Assmusen, Applied probability and queues. 2nd ed., Springer, New York, 2003.

[5] J. Bertoin, Lévy processes, Cambridge University Press, 1996.

[6] E. Brockwell, S.I. Resnick, and R.L. Tweedie, Storage processes with general release rule and additive inputs, Adv. in Appl. Probab. 14 (1982), 392–433.

[7] G. Gripenberg, S. Londen, and O. Staffans, Volterra integral and functional equations, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 1990.

[8] I. Karatzas and S. Shreve, Brownian motion and stochastic calculus, 2nd edition, Graduate Texts in Mathematics 113, Springer, New York, 1991.

[9] A. Kuznetsov, A. E. Kyprianou, and V. Rivero, The theory of scale functions for spectrally negative Lévy processes, Lévy Matters - Springer Lecture Notes in Mathematics, 2012.

[10] A. E. Kyprianou, Introductory lectures on fluctuations of Lévy processes with applications, Universitext, Springer-Verlag, 2006.

[11] ______, Fluctuations of Lévy processes with applications - Introductory lectures, Second, Universitext, Springer, Heidelberg, 2014.

[12] A. E. Kyprianou and R. L. Loeffen, Refracted Lévy processes, Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 24–44.

[13] J.-L. Pérez and K. Yamazaki, Refraction-reflection strategies in the dual model, Astin Bulletin 47 (2017), no. 1, 199–238.

[14] R. Situ, Theory of stochastic differential equations with jumps and applications, Springer US, 2005.

Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: irmina.czarna@gmail.com

Department of Probability and Statistics, Centro de Investigación en Matemáticas A.C. Calle Jalisco s/n. C.P. 36240, Guanajuato, Mexico.
E-mail address: jluis.garmendia@cimat.mx

Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: tomasz.rolski@gmail.com

Department of Mathematics, Faculty of Engineering Science, Kansai University, 3-3-35 Yamatecho, Suita-shi, Osaka 564-8680, Japan
E-mail address: kyamazak@kansai-u.ac.jp