EXTREMAL RAYS IN
THE HERMITIAN EIGENVALUE PROBLEM FOR
ARBITRARY TYPES

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Abstract. The Hermitian eigenvalue problem asks for the possible eigenvalues of a
sum of Hermitian matrices given the eigenvalues of the summands. This is a problem
about the Lie algebra of the maximal compact subgroup of $G = \text{SL}(n)$. There is a
polyhedral cone (the “eigencone”) determining the possible answers to the problem.
These eigencones can be defined for arbitrary semisimple groups $G$, and also control
the (suitably stabilized) problem of existence of non-zero invariants in tensor products of
irreducible representations of $G$.

We give a description of the extremal rays of the eigencones for arbitrary semisimple
groups $G$ by first observing that extremal rays lie on regular facets, and then classifying
extremal rays on an arbitrary regular face. Explicit formulas are given for some extremal
rays, which have an explicit geometric meaning as cycle classes of interesting loci, on
an arbitrary regular face. The remaining extremal rays on that face are understood by
a geometric process we introduce, and explicite numerically, called induction from Levi
subgroups. Several numerical examples are given. The main results, and methods, of this
paper generalize [B3] which handled the case of $G = \text{SL}(n)$.

1. Introduction

The Hermitian eigenvalue problem asks for the possible eigenvalues of a sum of
Hermitian matrices given the eigenvalue of the summands (see, e.g., [Br], [Ku] for
recent surveys). In its Lie-theoretic formulation, this is a problem about the Lie
algebra of the maximal compact subgroup $K = \text{SU}(n)$ of $G = \text{SL}(n)$. There is a
polyhedral cone, $\Gamma(s, G)$, controlling the possible eigenvalues, and the problem can
be generalised to an arbitrary semisimple group $G$. The corresponding polyhedral
cones are called eigencones, and are important objects in representation theory
which also control saturated versions of the tensor decomposition problem. In this
paper we give an inductive determination of extremal rays of these eigencones.

Fix a Cartan decomposition of $G$. It is known that $\Gamma(s, G) \subseteq \mathfrak{h}_+^*$ is a polyhedral
cone (here $\mathfrak{h}_+$ is the positive Weyl chamber of $G$, see below). It is known that
$\Gamma(s, G)$ is cut out inside $\mathfrak{h}_+^*$ by a system of inequalities controlled by the Schubert

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calculus of homogenous spaces $G/P$ where $P$ runs through all standard maximal parabolics of $G$. The regular faces (see Definition 2) of the polyhedral cone $\Gamma(s,G)$ have also been determined, see [KL], [B1], [KTW], [BS], [KLM], [BK1], [Re1], [Re2], and the survey [Ku]. The results cited above on regular faces do not include explicit (i.e., with formulas) procedures of manufacturing elements in $\Gamma(s,G)$ on faces, and do not give information about the extremal rays of $\Gamma(s,G)$, or the structure of regular faces beyond their dimension.

The central results of this paper are:

1. The (elementary) observation that all extremal rays lie on regular facets (Definition 2, and Lemma 37), to which we then focus our attention.

2. A construction of some of the extremal rays $\vec{r}_i$ on an arbitrary regular face $\mathcal{F}$ (Theorems 7 and 8). These rays have an explicit geometric meaning as cycle classes of interesting loci. They form a linearly independent set inside $\mathcal{F}$ and, furthermore, lend a decomposition $\mathcal{F} = \mathcal{F}_{II} \times \prod \mathbb{R}_{\geq 0} \vec{r}_i$, from which we deduce that the remaining extremal rays of $\mathcal{F}$ are just those of $\mathcal{F}_{II}$.

3. A description of $\mathcal{F}_{II}$ and its rays via a geometric process we introduce, called induction from Levi subgroups (Theorem 12). Specifically, we realize $\mathcal{F}_{II}$ as the image of a cone $\Gamma(s,G')$ for a group $G'$ of smaller rank (the semi-simplification of a Levi subgroup) under a linear mapping of the ambient vector spaces. We have a good understanding of the linear mapping (see Remark 14): There is an exact formula for the induction map (Theorem 12), and the kernel is the span of a geometrically significant linearly independent subset of the set of extremal rays of $\Gamma(s,G')$ (Lemmas 64 and 62). The extremal rays of $\mathcal{F}_{II}$ are therefore images of the extremal rays of $\Gamma(s,G')$, although not all extremal rays of $\Gamma(s,G')$ map to extremal rays of $\mathcal{F}_{II}$ (indeed, in examples they can map to zero, or non-extremal non-zero rays).

Theorems 8 and 12 produce explicit formulas, featuring intersection numbers, which generate the regular faces of $\Gamma(s,G)$. These formulas were used crucially to prove that the tensor cone of type $D_6$ is saturated in [K]. While cones can be described by listing either their extremal rays or their faces, the former was the computationally suitable method in order to find the Hilbert basis (of the lattice points in the cone) needed in [K].

1.1. The methods and relations to prior work

In many ways, this work is a synthesis or culmination of the mathematical ideas of [B3], [BKR], [BK1], [Re1]. The main results and methods of this paper generalize [B3] in two ways: from the special case $G = SL(n)$ to arbitrary simple algebraic groups, and from a description of regular facets to arbitrary regular faces. While the results of [B3] used the classical geometry of flag varieties in type $A$, we rely here on more general Lie-theoretic methods. The codimension one ramification behavior is very simple in the type $A$ maximal parabolic case, but not so in general type. Accordingly, we use the ramification results of [BKR] (see Section 2.1) crucially in the proof of Theorem 7 and in the construction of the induction operation.

- The explicit geometry of type $A$ is not available in the general group setting, and the key formula for induction (Theorem 12) is proved in an entirely different
way (see Section 9.3) than in [B3, Sect. 8.3] (the relevant line bundles were expressed there in terms of tautological bundles and maps on flag varieties).

- Several statements were formulated in [B3] for type $A$ using the combinatorics of partitions. Their proofs also used the geometry and combinatorics of type $A$. The formulations in our paper are in the language of general type Weyl groups and are more subtle than their type $A$ analogues because of differences in codimension one behavior of Schubert cells and classes. The proofs of these statements are similar but differ in significant geometric details (e.g., the proof of Theorem 8 in Section 4, and the proof of Proposition 35).

- A range of new phenomena occur in our generalizations, and we have provided many such examples (Corollary 65, Example 1, Section 1.6). In Section 11 we have tested our examples against known lists of extremal rays from [KKM]. We build upon earlier work on the minimal set of inequalities defining $(s;G) [B1], [BK1], [Re1], and the attendant ramification theory in enumerative problems [B2], [BK1], [BKR]. The induction operation is inspired by [Re1], and the basic divisors $D(j,v)$ ((6) below), which are shown here to give extremal rays, appear implicitly in [BKR]. These basic divisors can also be seen to correspond to $E_j$ considered in [Re1, §4.1] for an optimal choice of $X_o$ (as in loc. cit.), given the results of [BKR].

1.2. The eigencones

Let $G$ be a semisimple, connected complex algebraic group, with a Borel subgroup $B$ and a maximal torus $T \subset B$. Let $W = W_G = N_G(T)/T$ be the associated Weyl group, where $N_G(T)$ is the normalizer of $T$ in $G$. Our choice of $B$ and $T$ fixes a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Let $\mathfrak{h}$ be the Lie algebra of $T$ and $\mathfrak{h}_\mathbb{R}$ the real vector space spanned by the co-roots of $G$. Let $K$ be a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$, chosen such that $i\mathfrak{h}_\mathbb{R}$ is the Lie algebra of a maximal torus of $K$.

Let $\mathfrak{h}_+$ be the positive Weyl chamber in $\mathfrak{h}_\mathbb{R}$. There is a bijection $C : \mathfrak{h}_+ \rightleftharpoons \mathfrak{k}/K$ where $K$ acts on $\mathfrak{k}$ by the natural action of a Lie group on its Lie algebra. Let $\Gamma(s,G) \subseteq \mathfrak{h}_+$ be the “eigencone”, with $s \geq 3$:

$$\Gamma(s,G) = \left\{ (h_1, \ldots, h_s) \left| \exists k_1, \ldots, k_s \in \mathfrak{k}, \ C(h_j) = k_j \in \mathfrak{k}/K \ \forall j, \ \sum_{j=1}^s k_j = 0 \right. \right\} \quad (1)$$

To state the minimal set of inequalities determining $\Gamma(s,G)$, we introduce some notation first.

1.3. Notation

Our choice of $T$ and $B$ fixes a Cartan decomposition of $\mathfrak{g}$. Let $R \subseteq \mathfrak{h}^*$ (resp. $R^+, R^-$) be the set of roots (resp. positive roots, negative roots) of $G$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset R^+$ be the set of simple roots. Let $\{x_1, \ldots, x_r\}$ be the basis of $\mathfrak{h}$ dual to $\Delta$; i.e., $\alpha_i(x_j) = \delta_{i,j}$. Let $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ be the set of simple coroots. Let $\varpi_1, \ldots, \varpi_r \in \mathfrak{h}^+$ (dominant fundamental weights) be the basis dual to the simple coroots, so that $\varpi_i(\alpha_j^\vee) = \delta_{ij}$.

Let $\mathfrak{h}_\mathbb{Q}$ denote the $\mathbb{Q}$-span of the simple coroots, and set $\mathfrak{h}_{+,\mathbb{Q}} = \mathfrak{h}_+ \cap \mathfrak{h}_\mathbb{Q}$. We have an isomorphism $\kappa : \mathfrak{h}_{+,\mathbb{Q}}^* \rightarrow \mathfrak{h}_\mathbb{Q}$ induced by the Killing form, where, with $\mathfrak{h}_\mathbb{Z}$ denoting the integer span of the fundamental weights, $\mathfrak{h}_\mathbb{Z}^* = \mathfrak{h}_\mathbb{Z} \otimes \mathbb{Q}$. The mapping...
takes the rational cone generated by dominant fundamental weights \( h_{+,\mathbb{Q}} \) to the rational Weyl chamber \( h_{+,\mathbb{Q}} \). \( \Gamma(s, G) \subseteq h_{+,\mathbb{Q}} \) is a rational polyhedral cone, and we denote by \( \Gamma_0(s, G) \subseteq h_{+,\mathbb{Q}} \) the corresponding rational cone.

Let \( P \) be any standard parabolic of \( G \) (not necessarily maximal) and \( U = U_P \) be its unipotent radical. Let \( L = L_P \) be the Levi subgroup of \( P \), which has a Borel subgroup \( B_L = B \cap L \). The Lie algebras of \( G, B, T, P, U, L, B_L \) and \( K \) are denoted by \( \mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L \) and \( \mathfrak{k} \) respectively. Let \( X(T) \) be the group of multiplicative characters \( T \to \mathbb{C}^* \). Let \( R_L \) be the subset of roots (resp. positive roots, negative roots) of \( L \) and \( \Pi \) the set of simple roots in \( R_L \).

The Weyl group of \( P \), \( \mathcal{W}_P \), is by definition the Weyl group of \( L \). In any coset of \( W/W_P \), there is a unique element \( w \) of minimal length, and it satisfies \( wB_Lw^{-1} \subseteq B \). Let \( W_P \subseteq W \) be the set of minimal length representatives in the cosets of \( W/W_P \).

**Definition 1.** Let \( w \in W^P \subseteq W \). Define the Schubert cell \( C_w \subseteq G/P \) by \( C_w = BwP/P \). Let \( \left[ X_w \right] \in H^{2 \dim G/P - 2l(w)}(G/P, \mathbb{Z}) \) be the cycle class of \( X_w \).

**1.4. The system of inequalities determining \( \Gamma(s, G) \)**

It is known (see [BK1]) that \( (h_1, \ldots, h_s) \in \Gamma(s, G) \) if and only if, for every standard parabolic \( P \subseteq G \) and \( w_1, \ldots, w_s \in W^P \) such that

\[
\left[ X_{w_1} \right] \odot_0 \left[ X_{w_2} \right] \odot_0 \cdots \odot_0 \left[ X_{w_s} \right] = \left[ X_e \right] \in H^*(G/P), \tag{2}
\]

the inequalities

\[
\sum_{j=1}^{s} \omega_k (w_j^{-1}h_j) \leq 0 \tag{3}
\]

hold, as \( k \) ranges over all values such that \( \alpha_k \in \Delta \setminus \Delta(P) \). Here \( \odot_0 \) is the deformation of the cup product on \( H^*(G/P, \mathbb{C}) \) introduced in [BK1].

**Definition 2.** A face of \( \Gamma(s, G) \subseteq h^*_{+} \) is said to be regular if it is not contained in one of the Weyl chamber walls on any of the \( s \) factors, i.e., the face is not contained in \( \{(h_1, \ldots, h_s) \mid \alpha_j(h_i) = 0\} \) for some \( (i, j) \).

In fact, the inequalities arising only from maximal parabolics \( P \) suffice to determine \( \Gamma(s, G) \), and these are irredundant: each inequality (3) corresponding to the above data with \( P \) maximal determines a regular facet of \( \Gamma(s, G) \) and all regular facets arise this way [Re1]. It will be shown that any extremal ray of \( \Gamma(s, G) \) lies on some such regular facet (Lemma 37). Therefore it suffices to determine the extremal rays of all regular facets.

Fix a (possibly non-maximal) parabolic \( P \) in \( G \), and \( w_1, \ldots, w_s \in W^P \) such that (2) holds. Define the face \( \mathcal{F}(\bar{w}, P) \) of \( \Gamma(s, G) \subseteq h^*_{+} \) by

\[
\mathcal{F}(\bar{w}, P) = \left\{ (h_1, \ldots, h_s) \in \Gamma(s, G) \mid \sum_{j=1}^{s} \omega_k (w_j^{-1}h_j) = 0, \ \alpha_k \notin \Delta(P) \right\}. \tag{4}
\]
We will often simply write $\mathcal{F}$ when the context is clear. We consider the more general problem of determining all extremal rays of $\mathcal{F}_Q = \mathcal{F} \cap \Gamma_Q(s, G) \subseteq \Gamma_Q(s, G)$. This problem can be refined as follows: Let $L$ be the Levi subgroup of $P$, and $L^{ss} = [L, L] \subset L$. Note that $L^{ss}$ is semisimple and simply connected. We pose ourselves the following more general, related problems:

1. Describe all extremal rays of $\mathcal{F}_Q$.
2. Describe $\mathcal{F}_Q$ in terms of $\Gamma_Q(s, L^{ss})$.

1.5. Tensor cones and eigencones

For any $h \in \mathfrak{h}_Z^*$ one can associate a line bundle $L$ on $G/B$ (briefly: $\mathcal{L}_\lambda = G \times_B \mathbb{C}$ as a line bundle on $G/B$ where $\mathbb{C}$ is the $B$ representation given by $\lambda^{-1}$). Via the Borel–Weil theorem, this sets up a bijection between $\mathfrak{h}_+^*$ and $\mathfrak{h}_+^*$, the semigroup generated by the dominant fundamental weights and Pic$(G/B)$, the semigroup of line bundles with non-zero global sections: $H^0(G/B, L_\lambda)$ is the dual of the irreducible representation $V_\lambda$ with highest weight $\lambda$.

**Definition 3.** We have a cone

$$\text{Tens}_{s, G, Q} \subseteq \text{Pic}_Q^+(G/B)^s = (\mathfrak{h}_+^*)^s$$

formed by tuples $(\lambda_1, \ldots, \lambda_s)$ such that for some $N > 0$, $H^0((G/B)^s, L_\lambda)^G \neq 0$ where

$$\mathcal{L} = L_{\lambda_1} \boxtimes L_{\lambda_2} \boxtimes \cdots \boxtimes L_{\lambda_s} \in \text{Pic}(G/B)^s,$$

equivalently, for some $N > 0$, $(V_{N\lambda_1} \otimes V_{N\lambda_2} \otimes \cdots \otimes V_{N\lambda_s})^G \neq 0$.

**Proposition 4.** The (Killing form) bijection $\kappa^s : (\mathfrak{h}_Q^*)^s \to (\mathfrak{h}_Q^*)^s$ restricts to a bijection between $\text{Tens}_{s, G, Q}$ and $\Gamma_Q(s, G)$.

**Proof.** See, for example, Mumford’s appendix in [N], or [S, Thm. 7.6], or [Br, Thm. 1.3]. □

1.6. Basic extremal rays

Fix a face $\mathcal{F} = \mathcal{F}(\bar{w}, P)$ as in Section 1.4 (see Equation (4)), with $P$ an arbitrary standard parabolic subgroup of $G$.

**Definition 5** ([BGG]). Let $v, w \in W$ (not necessarily in $W_P$) and $\beta \in R^+$. The notation $v \xrightarrow{\beta} w$ stands for the following two (simultaneous) conditions: $w = s_\beta v$ and $\ell(w) = \ell(v) + 1$. Note that if $v \xrightarrow{\beta} w$, then $w^{-1} \beta \in R^-$ and $v^{-1} \beta \in R^+$.

Codimension one Schubert cells $C_v \subseteq X_w$ correspond to $v \xrightarrow{\beta} w$ with $v, w \in W_P$ and $\beta$ a positive root (not necessarily simple). As was observed in [BKR], one should divide the set of such $v$ into two types; this division influences ramification behaviour in intersection-theoretic problems:

**Definition 6.** Let $w \in W_P$. A codimension one Schubert cell $C_v \subseteq X_w$, $v \in W_P$, is said to be simple if $v \xrightarrow{\beta} w$ with $\beta$ a simple root.

Definition 5 (resp. Definition 6) corresponds to the strong (resp. weak) Bruhat ordering of elements of the Weyl group.
Divisors in \((G/B)^s\). In order to construct extremal rays on a face \(F(w_1, \ldots, w_s, P)\) given by (4) (i.e., line bundles on \((G/B)^s\), see Proposition 4), we will identify a series of \(G\)-invariant divisors on \((G/B)^s\), with \(G\) acting diagonally on \((G/B)^s\). For a pair \(j, v\) such that \(v \rightarrow^\beta w_j\) with \(\beta\) simple (i.e., \(C_v \subset X_w\) is simple, see Definition 6; in this case \(v \in W^P\) automatically), first set

\[
u_i = \begin{cases} w_i, & i \neq j, \\ v, & i = j. \end{cases}
\]

Then define

\[
D(j, v) = \left\{ (g_1, \ldots, g_s) \in (G/B)^s \left| \bigcap_{i=1}^s g_i X_{u_i} \neq \emptyset \right. \right\} \subseteq (G/B)^s,
\]

which is given the reduced scheme structure, making it a subvariety of \((G/B)^s\). In Theorem 7, we will show that \(D(j, v)\) is irreducible and codimension one in \((G/B)^s\). In particular, we can express

\[
\mathcal{O}(D(j, v)) = \mathcal{L}_{\lambda_1} \boxtimes \mathcal{L}_{\lambda_2} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda_s} \in \text{Pic}(G/B)^s,
\]

for some \(\lambda_i \in \mathfrak{h}^*_+ \subseteq \mathfrak{h}^*\). Since \(D(j, v)\) is diagonal-\(G\)-invariant, \(H^0((G/B)^s, \mathcal{O}(D(j, v)))^G \neq 0\), and under the bijection of Proposition 4, we get

\[
[D(j, v)] = (\kappa(\lambda_1), \ldots, \kappa(\lambda_s)) \in \Gamma_Q(s, G).
\]

The main properties of \(D(j, v)\) are laid out in the following crucial result.

**Theorem 7.**

(a) \(D(j, v)\) is an irreducible codimension one\(^1\) subvariety of \((G/B)^s\);
(b) \(\dim H^0((G/B)^s, \mathcal{O}(mD(j, v)))^G = 1\) for all \(m \geq 0\);
(c) \(\mathbb{Q}_{\geq 0}[D(j, v)]\) is an extremal ray of \(\Gamma_Q(s, G)\);
(d) \([D(j, v)] \in \mathcal{F}\).

**Remark 1.** Fulton conjectured the following: Suppose \(\lambda, \mu, \nu\) are dominant integral weights for \(\text{SL}(n)\) such that \(\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\text{SL}(n)} = 1\), then the same remains true after scaling each of \(\lambda, \mu, \nu\) by an arbitrary positive integer \(m\). This type A conjecture was proved by Knutson, Tao and Woodward [KTW], but its direct generalization fails in other types. Therefore, by (b), the \(D(j, v)\) above are rather special. An arbitrary type generalization of Fulton’s conjecture (specializing to the original type A conjecture) was proved in [BKR].

The following gives formulas for \(\lambda_1, \ldots, \lambda_s\), and hence for \([D(j, v)]\):

**Theorem 8.** Write \(\lambda_k = \sum c_{k, \ell} \omega_\ell\). The coefficient \(c_{k, \ell} = \lambda_k(\alpha_\ell^+)\) is computed as follows. Fix \(k\) and \(\ell\) and let \(\widehat{u}_k = s_{\alpha_\ell} u_k\).

\(^1\)The codimension condition fails without the simpleness condition on \(C_v \subset X_{w_j}\), see Remarks 6, 7 and Example 1.
If $\hat{u}_k \in W^P$ and $u_k \overset{\alpha}{\to} \hat{u}_k$, set $\hat{u}_i = u_i$ for $i \neq k$. Then $c_{k,\ell}$ is the (possibly zero) intersection number $c$ in

$$
\prod_{i=1}^s [X_{\hat{u}_i}] = c[pt] \in H^*(G/P)
$$

(9)

where the product is in the usual cohomology (and not in the deformed product $\mathcal{C}_0$, see the example in Section 1.6).

If $\hat{u}_k \not\in W^P$, or $u_k \overset{\alpha}{\to} \hat{u}_k$ is false, then $c_{k,\ell} = 0$.

Remark 2. We could also have expressed the coefficient $c_{k,\ell}$ as follows: If $\hat{u}_k \in W^P$ and $u_k^{-1} \alpha_{\ell} \in R^+$ (which is equivalent to $\ell(u_k) < \ell(\hat{u}_k)$), then the coefficient $c_{k,\ell}$ is the number $c$ in (9) and zero otherwise.

Remark 3. The pairs $(j, v)$ are in one-one correspondence with simple roots $\beta = \alpha_{\ell}$ and $j$ such that $w_j^{-1} \alpha_{\ell} \in R^-$. This is because if we set $v = s_{\alpha_i} w$ then $\ell(w) > \ell(v)$ but $\ell(w) = \ell(v) + 1$ since $\alpha_{\ell}$ is a simple root, and hence $\ell(w) = \ell(v) + 1$. We have assumed that $w \in W^P$; this implies $v \in W^P$.

An example. Let $G$ be of type $D_4$, with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (using standard notation [Bou] here and elsewhere), and corresponding simple reflections $s_i$. Let $P = P_2$ be the standard maximal parabolic for which $\Delta(P) = \Delta \setminus \{\alpha_2\}$. Let $u = s_4 s_3 s_1 s_2$, $v = s_3 s_1 s_2 s_4 s_3 s_1 s_2$, and $w = s_1 s_2 s_4 s_2 s_3 s_1 s_2$; one verifies that $u, v, w \in W^P$, and that

$$
[X_u] \circ_0 [X_v] \circ_0 [X_w] = [X_e] \in H^*(G/P).
$$

This can be calculated using the multiplication table for $(G/P_2, \mathcal{C}_0)$ found in [KKM] (in their notation, $[X_u] = e_{\theta^u} b_2^2$, $[X_v] = e_{\theta^v} b_2^2$, $[X_w] = e_{\theta^w} b_2^2$, and $[X_e] = b_0$) and was also verified by computer. Therefore $u, v, w, P$ give rise to a regular facet $\mathcal{F}$.

Observe that, for instance, $s_3 v \overset{\alpha_3}{\to} v$. According to Theorem 7, $D(2, s_3 v)$ is a divisor on $(G/B)^s$, and we now compute the $\lambda_1$ appearing in $\mathcal{O}(D(2, s_3 v)) = L_{\lambda_1} \otimes L_{\lambda_2} \otimes L_{\lambda_3}$ using Theorem 8. First, $\lambda_1$: testing each of $s_1 u, s_2 u, s_3 u, s_4 u$, we see that only $s_2 u$ satisfies $\ell(s_2 u) = \ell(u) - 1$ and $s_2 u \in W^P$. As $[X_{s_2 u}] \cdot [X_{s_3 v}] \cdot [X_w] = [X_e]$, $\lambda_1 = \varpi_2$. On may check that this product would equal 0 if $\varpi_3$ were used instead.

For $\lambda_2$: $s_3 (s_3 v)$ and $s_4 (s_3 v)$ satisfy the two required conditions (the first is obvious). As $[X_u] \cdot [X_{s_4 s_3 v}] \cdot [X_w] = 0$, $\lambda_2 = \varpi_3$. Finally $\lambda_3$: only $s_3 v$ satisfies the requirements, and $[X_u] \cdot [X_{s_3 v}] \cdot [X_{s_1 w}] = [X_e]$, so $\lambda_3 = \varpi_3$. Indeed, $(\varpi_2, \varpi_3, \varpi_3)$ is an extremal ray of the tensor cone for $D_4$, cf. [KKM], and lies on $\mathcal{F}$. All standard cup product calculations here were done by computer.

1.7 Other extremal rays

Definition 9. A ray $Q_{\geq 0}(h_1, \ldots, h_s)$ is a type I ray of $\mathcal{F}_Q$ if there is a pair $(j, v)$ such that $v \overset{\beta}{\to} w_j$ with $\beta$ simple, $v \in W^P$ (i.e., $C_v \subset X_{w_j}$ is simple, see Definition 6) such that $\beta(h_j) > 0$.

Rays of $\mathcal{F}_Q$ which are not type I are called type II rays of $\mathcal{F}_Q$; they span a face $\mathcal{F}_{II, Q}$ of $\mathcal{F}_Q$: This face is defined inside $\mathcal{F}_Q$ by the system of equalities $\beta(h_j) = 0$
whenever \((j, v)\) is a pair such that \(v \xrightarrow{\beta} w_j\) with \(\beta\) simple, \(v \in WP\) (note that \(\beta(h_j) \geq 0\) on \(F\)).

It is an easy consequence of Theorem 8 that the extremal rays \(D = D(j, v)\) of Theorem 7 are type I (see Corollary 34 (1)).

Let \(Q_{\geq 0} \delta_1, \ldots, Q_{\geq 0} \delta_q\) be the type I extremal rays of \(F\) produced by Theorem 7, with \(q\) the number of possible \((j, v)\) with \(v \xrightarrow{\beta} w_j\) and \(\beta\) simple. We have a natural cone map

\[
\prod_{b=1}^{q} Q_{\geq 0} \delta_b \times F_{II, Q} \to F_Q. \tag{10}
\]

**Theorem 10.** The mapping (10) is an isomorphism of pointed rational cones.

Therefore, the general problems enumerated above reduce to the problem of describing \(F_{II, Q}\):

1. Describe all extremal rays of \(F_{II, Q}\).
2. Describe \(F_{II, Q}\) in terms of \(\Gamma_Q(s, L^{ss})\). We have defined eigencones for semi-simple groups, and therefore seek an answer in terms of \(L^{ss}\) instead of \(L\). In the setting of the (saturated) tensor invariant problems we can replace \(L^{ss}\) by \(L\), see Lemma 53.

We will do this by the process of induction.

### 1.8. Induction

We briefly indicate the basic geometry behind the induction operation in the special case of \(G = SL(n)\) and \(P\) maximal parabolic such that \(G/P\) is the Grassmannian \(Gr(r, n)\). Here \((w_1, \ldots, w_s)\) are as before: Given a point \((g_1, \ldots, g_s) \in (G/B)^s\), i.e., a tuple of complete flags on \(\mathbb{C}^n\), we locate (for generic \(g_i\), a unique point of intersection of open Schubert cells

\[
V \in \bigcap_{i=1}^{s} g_i C_{w_i} \subseteq Gr(r, n).
\]

We also obtain induced complete flags on \(V\) and on \(Q = \mathbb{C}^n/V\), and hence a birational map from \((G/B)^s\) to the moduli stack that parametrizes vector spaces \(V\) and \(Q\) of dimensions \(r\) and \(n-r\) respectively, each equipped with \(s\) complete flags. This moduli stack can be thought of as the stack quotient \([(L/B_L)^s/L]\), where \(B_L\) is the Borel subgroup of \(L = \{(A, B) \in GL(r) \times GL(n-r) \mid \det A \det B = 1\}\), and \(h\) in \(L\) acts on \((L/B_L)^s\) via left multiplication by \(h^{-1}\). The map \((G/B)^s \to [(L/B_L)^s/L]\) is not defined in codimension 1 at the divisors \(D(j, v)\), but one can extend it across generic points of such divisors by factoring it through an intermediate space of partial flags (denoted \(Fl'_G\)). Given this, for codimension reasons, we can transport \(L\)-equivariant line bundles on \((L/B_L)^s\) to line bundles on \((G/B)^s\). We use ideas from [BKR] to generalize this construction to arbitrary groups and hence obtain the induction map.
**Definition 11.** Given \((h_1, \ldots, h_s) \in \mathfrak{h}_{L,s,L}^{s} \subseteq \mathfrak{h}_{Q_s}^{s}\), set \(y_j = w_j h_j\) for \(j = 1, \ldots, s\). Define \(\text{Ind}_L^G : \mathfrak{h}_{L,s,L}^{s} \rightarrow \mathfrak{h}_{Q_s}^{s}\) by the following formula

\[
\text{Ind}_L^G(h_1, \ldots, h_s) = (y_1, \ldots, y_s) - \sum_{\ell} \frac{2}{\alpha_{\ell}} \alpha_{\ell}(y_j)[D(j, v)] \in \mathfrak{h}_{Q_s}^{s}
\]

where the sum is over triples \((j, v)\) such that \(v \alpha_{j} \sim w_j\) with \(v \in W_P\) \((|D(j, v)|\) is defined in (8)).

The induction operation \(\text{Ind}_L^G\) is constructed geometrically, and in a more general setting: Using the equivalences of Definition 3 and Proposition 4, we construct an injective morphism from \(L\)-equivariant line bundles on \((L = L)\) to \(G\)-equivariant line bundles on \((G = B)\). The formulas have a simpler form in the setting of the equivalent (saturated) tensor invariant problem (as in Proposition 4), see Theorem 55.

**Theorem 12.** The restriction of \(\text{Ind}_L^G\) to \(\Gamma_q(s, L^{ss})\) defines a surjective mapping of cones

\[
\text{Ind}_L^G : \Gamma_q(s, L^{ss}) \rightarrow \mathcal{F}_{II,Q} \subseteq \mathcal{F}_Q \subseteq \Gamma_q(s, G)
\]

Therefore all extremal rays of \(\mathcal{F}_{II,Q}\) can be obtained by induction (11) from extremal rays of \(\Gamma_q(s, L^{ss})\). Not all extremal rays of \(\Gamma_q(s, L^{ss})\) induct to extremal rays; some may even map to zero. For examples of these phenomena, see Section 11.

The surjection (11) is of a special type, and the kernel of the associated mapping of vector spaces is controlled by ramification divisors in the associated enumerative problem, see Remark 14.

**Remark 4.** One could ask whether \(\mathcal{F}_{II,Q}\) cannot be decomposed further. In Section 11.4, we give an example in which it is simplicial (hence can be decomposed further as a product of smaller cones). We also give an example when it is not simplicial, and cannot be written as a product of a ray and a cone.

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### 2. Basic extremal rays

Throughout this paper, we choose a set-theoretic lifting \(W \rightarrow N(T)\) of \(W \rightarrow N(T)/T\). Let \(W\) be an arbitrary standard parabolic of \(G\). For \(w \in W_P\), let \(Z_w\) denote the smooth locus of \(X_w\). There is a largest group \(Q_w \subseteq G\), a standard parabolic, that acts on the closed Schubert variety \(X_w\). Let \(Y_w \subseteq X_w\) be the open \(Q_w\)-orbit in \(X_w\). Then \(C_w \subseteq Y_w \subseteq Z_w \subseteq X_w\). By [BP] (also see [BKR, Lem. 7.1]),

\[
\Delta(Q_w) = \Delta_w = \Delta \cap w(R_1^+ \cup R^-).
\]

Note that if \(\alpha \in \Delta(Q_w)\), then \(s_\alpha \in Q_w\).

By [BP, §2.6], (also see [BKR, Prop. 7.2]):
Lemma 13. Let \( v \overset{\beta}{\rightarrow} w \) with \( v, w \in WP \). The following are equivalent:

1. \( C_v \subseteq Y_w \).
2. \( C_v \subseteq X_w \) is simple (i.e., \( \beta \) is a simple root).

Proof. This is just a restatement of the formulation in [BKR]. If \( C_v \subset Y_w \), [BKR, Prop. 7.2] shows that \( \beta \) is a simple root.

From \( v \overset{\beta}{\rightarrow} w \), we know \( w^{-1} \beta \in R^- \). If \( \beta \) is a simple root then \( \beta \in \Delta(Q_w) \) (by Equation 12), and \( C_v \subseteq Y_w \) by [BKR, Prop. 7.2]. \( \square \)

2.1. The geometry

Here and throughout the rest of the paper, we fix an \( s \geq 3 \), a standard parabolic \( P \), and a collection \( w_1, \ldots, w_s \in WP \) satisfying (2).

Define the universal intersection locus

\[ \mathcal{X} = \left\{ (g_1, \ldots, g_s, \bar{z}) \in (G/B)^s \times G/P \mid \bar{z} \in g_i X_{w_i} \ \forall i \right\}, \]

and similarly define subloci \( \mathcal{Z} \supseteq \mathcal{Y} \supseteq \mathcal{C} \) using the \( Z_{w_i}, Y_{w_i}, C_{w_i} \), respectively, in place of the \( X_{w_i} \). (For the scheme structures, see [BKR, §5].) As shown in [BKR, Lem. 5.2], \( \mathcal{X} \) is irreducible. Furthermore, the projection map \( \pi : \mathcal{X} \rightarrow (G/B)^s \) is birational, for the following reason. By [BKR, Lem. 5.4], the degree of \( \pi \) is equal to the intersection number \( \bigcap_{i=1}^{s} [X_{w_i}] \). When the deformed product is in top degree, it is known to agree with the intersection number. Thus (2) implies that the degree of \( \pi \) is one, and hence that \( \pi \) is birational. Moreover, because the \( X_{w_i} \) are all normal, each \( X_{w_i} \setminus Z_{w_i} \) has codimension \( \geq 2 \) in \( X_{w_i} \); accordingly, \( \mathcal{X} \setminus \mathcal{Z} \) has codimension \( \geq 2 \) in \( \mathcal{X} \).

Let \( \mathcal{R} \subseteq \mathcal{Z} \) be the ramification divisor of the map \( \pi \). It follows from the birationality of \( \pi \) that \( \overline{\pi(\mathcal{R})} \subseteq (G/B)^s \) is codimension \( \geq 2 \) in \( (G/B)^s \). Furthermore, it was demonstrated in [BKR]*Proposition 8.1 that \( \mathcal{Z} \setminus \mathcal{Y} \subset \mathcal{R} \cup \mathcal{A} \) for some \( \mathcal{A} \subset \mathcal{Z} \) of codimension at least 2. This result is of crucial importance in our work. Note that \( \mathcal{Z} = \mathcal{Y} \) in type \( A \) and maximal parabolic \( P \).

Definition 14. With notation as in (5), let

\[ \tilde{D}(j, v) = \left\{ (\tilde{g}_1, \ldots, \tilde{g}_s, \bar{z}) \in (G/B)^s \times G/P \mid \bar{z} \in \bigcap_{i=1}^{s} \tilde{g}_i X_{w_i} \right\} \subseteq \mathcal{X} \subseteq (G/B)^s \times G/P. \]

Note that \( \tilde{D}(j, v) \) has a natural scheme structure as a fibre-product of schemes, just like \( \mathcal{X} \) as described in [BKR, §5]. Clearly, as sets, \( D(j, v) = \pi(\tilde{D}(j, v)) \). We will see in Corollary 15 that this is an equality of cycles.

2.2. Proof of Part (a) of Theorem 7

By symmetry, we may assume \( j = 1 \). Then we are in the situation \( w_1 = s_\beta v \) with \( \beta \) simple and \( \ell(v) = \ell(w_1) - 1 \). Set \( \tilde{D} = \tilde{D}(1, v) \).

First, we observe that \( \tilde{D} \) has dimension \( s \cdot \dim(G/B) - 1 \); this follows from counting dimensions in the fibre-product construction of \( \tilde{D} \) à la [BKR, §5]. It is
also easy to see that $\tilde{D}$, hence its image $D$ is irreducible by the same argument as for the irreducibility of $\mathcal{X}$ (see, e.g., [BKR, Lem. 5.2]).

Second, we claim that it suffices to show $\tilde{D} \not\subseteq \mathcal{R}$. Indeed, $\pi$ is an isomorphism away from the closed subset $\mathcal{R}$; therefore if the claim holds, an open subset (namely $\tilde{D} \setminus \mathcal{R}$) of $\tilde{D}$ is isomorphic to an open subset of $D$. Since $\tilde{D}$ and $D$ are both irreducible, dimensions can be counted on open subsets. Therefore we conclude

$$\dim D = \dim \pi(\tilde{D} \setminus \mathcal{R}) = \dim \tilde{D} \setminus \mathcal{R} = \dim \tilde{D} = s \cdot \dim(G/B) - 1,$$

as desired.

Lastly, we now show $\tilde{D} \not\subseteq \mathcal{R}$. Starting with an arbitrary point $(g_1, \ldots, g_s, \tilde{z}) \in \mathcal{C} \setminus \mathcal{R}$, we can produce a point in $\tilde{D} \setminus \mathcal{R}$. By the definitions,

$$\{zP\} \subseteq \bigcap_{i=1}^{s} g_i C_{w_i} \subseteq \bigcap_{i=1}^{s} g_i X_{w_i};$$

furthermore, we must have equalities above since $\pi$ is an isomorphism away from $\mathcal{R}$. We can write $z = g_1 b w_1 p$ for suitable $b, p$. Then translating by $b^{-1} g_1^{-1}$, we have produced the intermediate point $(\tilde{e}, \tilde{g}_2', \ldots, \tilde{g}_s', \tilde{w}_1) \in \mathcal{C} \setminus \mathcal{R}$. Just as above,

$$\{w_1 P\} = X_{w_1} \cap \bigcap_{i=2}^{s} g_i' X_{w_1}.$$

We claim that $a = (\tilde{s}_\beta, \tilde{g}_2', \ldots, \tilde{g}_s', \tilde{w}_1)$ is in $\tilde{D} \setminus \mathcal{R}$. Indeed, $s_\beta \in Q_{w_1}$ and therefore $s_\beta X_{w_1} = X_{w_1}$, so

$$s_\beta X_{w_1} \cap \bigcap_{i=2}^{s} g_i' X_{w_1} = \{w_1 P\},$$

which puts $a \in \mathcal{X} \setminus \mathcal{R}$ (actually in $\mathcal{Y} \setminus \mathcal{R}$). Furthermore, $w_1 P = s_\beta v P \in s_\beta C_v \subseteq s_\beta X_v$, so

$$w_1 P \in s_\beta X_v \cap \bigcap_{i=2}^{s} g_i' X_{w_1},$$

which puts $a \in \tilde{D}$ as claimed.

This finishes the proof of (a). The above argument has the following corollary:

**Corollary 15.** As Weil divisors, $D(j, v) = \pi_* (\tilde{D}(j, v))$. Here $\pi_*$ is the pushforward operation on cycles under proper morphisms [F, §1.4].

**Proof.** Since $\pi$ is an isomorphism on $Z \setminus \mathcal{R}$, we only need to observe that $\tilde{D}(j, v)$ is irreducible and generically smooth. Both are standard (and follow by studying the fiber bundle $\tilde{D}(j, v) \rightarrow G/P$ as in [BKR, Lem. 5.2]).

---

### 2.3. Proof of parts (b) and (c) of Theorem 7

For what follows, we will need to use the following theorem on functions on the universal intersection, which will be proved in Section 3.10:
Theorem 16.

\[ H^0(\mathcal{C} \setminus \mathcal{R}, \mathcal{O})^G = \mathbb{C}. \]

Part (c) follows from part (b) as in [B3, Lem. 2.1] (briefly): Suppose some multiple \( \mathcal{O}(mD) \) is a tensor product of two line bundles \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) with non-zero invariants \( s_1 \) and \( s_2 \). Clearly the zero sets of \( s_1 \) and \( s_2 \) should be supported on \( D \), since, by (b), \( \mathcal{O}(mD) \) has only one non-zero invariant section up to scaling. Since \( D \) is reduced and irreducible, and \( (G/B)^s \) is smooth, \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are both multiples of \( \mathcal{O}(D) \), as desired.

We deduce part (b) from Theorem 16. For this we use:

Lemma 17. \( \pi(\mathcal{C} \setminus \mathcal{R}) \subseteq (G/B)^s \setminus D(j, v) \).

Proof. From the definitions it follows that \( \widetilde{D}(j, v) \cap \mathcal{C} = \emptyset \); therefore

\[ \mathcal{C} \setminus \mathcal{R} \subseteq (\mathcal{X} \setminus \mathcal{R}) \setminus (\widetilde{D}(j, v) \setminus \mathcal{R}). \]

Setting \( \mathcal{B} = \pi(\mathcal{R}) \), we have that \( \pi \) restricted to \( \mathcal{X} \setminus \mathcal{R} \) is an isomorphism onto \( (G/B)^s \setminus \mathcal{B} \). The map \( \pi \) further restricts to isomorphisms \( \mathcal{C} \setminus \mathcal{R} \simeq \pi(\mathcal{C} \setminus \mathcal{R}) \) and \( \widetilde{D}(j, v) \setminus \mathcal{R} \simeq D(j, v) \setminus \mathcal{B} \). By the above inclusion and these isomorphisms,

\[ \pi(\mathcal{C} \setminus \mathcal{R}) \subseteq ((G/B)^s \setminus \mathcal{B}) \setminus (D(j, v) \setminus \mathcal{R}) \]

and, in particular,

\[ \pi(\mathcal{C} \setminus \mathcal{R}) \subseteq (G/B)^s \setminus D(j, v). \]

We now prove part (b) of Theorem 7: a section \( \sigma \in H^0((G/B)^s, \mathcal{O}(mD))^G \) gives a \( G \)-invariant function, \( f \), on \( (G/B)^s \setminus D \), and hence one on \( \mathcal{C} \setminus \mathcal{R} \) (using Lemma 17): namely, \( f \circ \pi \). By Theorem 16, \( f \circ \pi \) is constant on \( \mathcal{C} \setminus \mathcal{R} \); thus \( f \) is constant on \( \pi(\mathcal{C} \setminus \mathcal{R}) \), a dense open subset of \( (G/B)^s \setminus D \). Therefore \( f \) must be constant on all of \( (G/B)^s \setminus D \), which proves the claim.

Theorem 7, part (d) is proved in Section 3.11.

3. Some parameter spaces

3.1. Principal \( G \)-spaces

We first point out why a stack-theoretic approach is convenient. Suppose \( G = \text{SL}(n) \) and \( P \) a maximal parabolic with \( G/P = \text{Gr}(r, n) \), the Grassmannian of \( r \)-dimensional subspaces of \( \mathbb{C}^n \). Then \( \mathcal{C} \) parameterizes the set of \( r \)-dimensional subspaces \( V \) of \( \mathbb{C}^n \) and \( s \) full flags of subspaces such that \( V \) is in prescribed Schubert cells with respect to these flags. Now one can consider induced flags on such a \( V \), and on \( Q = \mathbb{C}^n/V \). But \( V \) and \( \mathbb{C}^n/V \) are not \( \mathbb{C}^r \) and \( \mathbb{C}^{n-r} \) canonically; therefore we do not get a map to a product of flag varieties. Stacks provide a convenient setting to still make this work to pull back objects defined invariantly. We may just study pairs \( V \) and \( Q \) with \( s \) full flags on each. This is a stack and we do get a map from \( \mathcal{C} \) to this stack. But we will have to work with objects without trivializations, hence with principal \( G \)-spaces (principal bundles over a point).
3.2. Principal spaces and relative positions

A principal $G$-space is a variety $E$ with a right $G$-action that is principal homogenous for the action of $G$ (i.e., for any $x \in E$ the map $G \to E$ given by $g \mapsto xg$ is an isomorphism). If $\phi : G \to G'$ is a map of affine algebraic groups and $E$ a principal $G$-space then $E \times_G G' = E \times_{\phi} G'$ is a principal $G'$-space.

Suppose $\bar{g} \in E/B$ and $\bar{z} \in E/P$. We define the relative position $[\bar{g}, \bar{z}] \in W^P$ as follows. It is the element $w \in W^P$ such that

\[ z = gbwp^{-1} \text{ for some } b \in B, p \in P. \quad (13) \]

Here $g, z \in E$ are coset representatives of $\bar{g} \in E/B$ and $\bar{z} \in E/P$. It is easy to see that $w$ is independent of choices. If we choose a trivialization $e \in E$, we get corresponding elements $\bar{g} \in G/B$ and $\bar{z} \in G/P$. Equation (13) indicates that $\bar{z} \in G/P$ is in the Schubert cell $gBwP/P$.

3.3. Good representatives

Suppose $\bar{g} \in E/B$ and $\bar{z} \in E/P$ with $w = [\bar{g}, \bar{z}] \in W^P$. Consider $p$ as in (13). Write $z = gbwp^{-1}$ as $zp = gbw$, and change $z$ to $zp$ and $g$ to $gb$. The equation simplifies to $z = gw$. Therefore we may choose a (“good”) representative $(g, z)$ of $(\bar{g}, \bar{z})$ so that $z = gw$. The choice of “good representative” is unique up to the action of $(w^{-1}Bw \cap P)$: If $(zp, gb)$ is another choice of a good representative then $z = gw$ and $zp = gbw$ and hence $gwp = gbw$ and hence $p = w^{-1}bw \in (w^{-1}Bw \cap P)$.

Alternative formulation of relative position. One of the reviewers has formulated another, equivalent explanation of relative position and good representatives, which we quote here.

Given $\bar{x} \in E/B$ and $\bar{y} \in E/P$, lift them to $x, y \in E$. Since $E$ is a principal $G$-bundle there is a unique $g \in G$ such that $y = x \cdot g$. Running over all lifts of $x$ and $y$, the corresponding elements of $G$ give the double coset $BgP$. From the decomposition $G = \bigsqcup_{w \in W^P} BwP$, this double coset $BgP$ is equal to $BwP$ for a unique $w \in W^P$. We say that $\bar{x}$ and $\bar{y}$ have relative position $w$, and set $[\bar{x}, \bar{y}] = w$ (this agrees with above).

Choosing an equivariant isomorphism $E \simeq G$, and hence isomorphisms $E/P \simeq G/P$ and $E/B \simeq G/B$, the condition $[\bar{x}, \bar{y}] = w$ is equivalent to the condition that $\bar{y} = \bar{x} \cdot C_w$, and is independent of the isomorphism to $G$ chosen.

Finally, we say that $x$ and $y$ are good representatives for $\bar{x}$ and $\bar{y}$ if $y = x \cdot w$.

**Definition 18.** Let $E$ be a principal $G$-space. An element $\bar{z} \in E/P$ defines a principal $P$-space $E_P(\bar{z})$ (the coset in $E/P$), and hence a principal $L$-space $E_L(\bar{z}) = E_P(\bar{z}) \times_P L$, using the quotient map $P \to L = P/U$.

**Lemma 19.** Under the map $P \to L$, the subgroup $w^{-1}Bw \cap P$ maps to $B_L \subseteq L$, in fact onto it.

**Proof.** First we show that $w^{-1}Bw \cap P$ is connected: $B \cap wPw^{-1} = T \cdot (U \cap wPw^{-1})$ since both $B$ and $wPw^{-1}$ contain $T$. Now $T$ acts on both $U$ and $wPw^{-1}$ by conjugation. By [Bo, §14.4, Prop. (2)] applied to $U \cap wPw^{-1} \subset U$, we see that $U \cap wPw^{-1}$, and hence $B \cap wPw^{-1}$ is connected.
Clearly $wB_L w^{-1} \subseteq B$. Therefore $B_L \subseteq w^{-1} B w$ and $B_L \subseteq P$, therefore $B_L \subseteq w^{-1} B w \cap P$. Since $w^{-1} B w \cap P$ is connected we may prove the mapping property at the level of Lie algebras, which is easy because $w \alpha < 0$ for any negative root $\alpha$ in $R_L$. \hfill \Box

### 3.4. Universal Schubert intersection stacks

We introduce the following stacks:

1. Let $\text{Fl}_G$ be the stack parametrizing principal $G$-spaces $E$ together with elements $g_i \in E/B, i = 1, \ldots, s$ (i.e., to consider families of such objects over a scheme $X$, we consider principal $G$-bundles $E$ on $X$ locally trivial in the fppf topology, and sections $g_i$ of $E/B$ over $X$). One can see that $\text{Fl}_G = [(G/B)^s/G]$ (the stack quotient, where $g \in G$ acts via diagonal left multiplication by $g^{-1}$ to ensure a right action. All other quotient stacks in this paper also have the diagonal right action on the left by the group in question.) Let $\text{Fl}_L = [(L/B_L)^s/L]$, which parameterizes principal $L$-spaces $F$ together with elements $g_i \in F/B_L, i = 1, \ldots, s$.

2. Let $\tilde{\mathcal{C}}$ be the moduli stack parametrizing principal $G$-spaces $E$ together with elements $g_i \in E/B$ and a single element $\bar{z} \in E/P$ so that $[g_i, \bar{z}] = w_i$ for all $i$. One can check that $\tilde{\mathcal{C}} = [C/G]$. There is a natural map $\pi : \tilde{\mathcal{C}} \to \text{Fl}_G$.

**Lemma 20.** Let $E$ be a principal $G$-space, and $g \in E/B$ and $\bar{z} \in E/P$ with $w = [g, \bar{z}] \in W^P$. Consider $p$ as in (13). The element $z p \in E_P(\bar{z})/(w^{-1} B w \cap P)$ is well defined. As a result, the corresponding element in $E_L(\bar{z})/B_L$ is well defined (see Lemma 19).

**Lemma 21.** The stack $\tilde{\mathcal{C}}$ parameterizes principal $P$-spaces $E'$ together with $\bar{z}_i \in E'/(w_i^{-1} B w_i \cap P), i = 1, \ldots, s$.

**Proof.** This follows from Lemma 20: From $(E', \bar{z}_1, \ldots, \bar{z}_s)$, we let $E = E' \times_P G$ and $g_i = z_i \times w_i^{-1}$. The point $\bar{z} \in E/P$ is the tautological point. Lemma 20 gives the reverse correspondence, where $E' := z P$ and $\bar{z}_i := \varpi_i$.

### 3.5. A diagram of spaces

Lemmas 21 and 19 and the map $P \to L$ result in a map $\tau : \tilde{\mathcal{C}} \to \text{Fl}_L$. The inclusion $L \to P$ gives rise to $i : \text{Fl}_L \to \tilde{\mathcal{C}}$, so that $\tau \circ i$ is the identity on $\text{Fl}_L$. The map $\tilde{i}$ is $\pi \circ i$.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\pi} & \tilde{\mathcal{C}} \\
\downarrow \pi & & \downarrow \pi \\
(G/B)^s & \xrightarrow{\tau} & \text{Fl}_G \xleftarrow{i} \text{Fl}_L
\end{array}
$$

(14)

Note that $\tilde{i}$ sends a tuple $(F, \bar{l}_1, \ldots, \bar{l}_s)$ to $(E, \bar{g}_1, \ldots, \bar{g}_s)$ where $E = F \times_L G$, and $g_i = l_i \times w_i^{-1}$

### 3.6. Levification

**Definition 22.** Let $Z^0(L) \subseteq Z(L)$ be the connected component of the identity of the center $Z(L)$ of $L$. 

The space \( \hat{C} \) retracts to \( \text{Fl}_L \) by a process called Levification (cf. [BK2, §3.8], also [R, Prop 3.5]), which we recall now. Consider an element \( x_L = \sum_k N_k x_k \) where the sum is over \( k \) such that \( \alpha_k \not\in \Delta(P) \), with \( N_k \) such that \( N_k x_k \) is in the co-root lattice. The centralizer of the one-parameter subgroup \( t^{x_L} = \exp((\ln t)x_L), t \in \mathbb{C}^* \) equals \( L \).

For \( t \in \mathbb{C}^* \) we have an automorphism \( \phi_t : P \to P \) given by (cf. [BK2, §3.8] and [R, Lem. 3.1.12]) \( \phi_t(p) = t^{x_L} pt^{-x_L} \), with \( \phi_1 \) the identity on \( P \). This extends to a group homomorphism \( \phi_0 : P \to L \) (which coincides with the standard projection of \( P \) to \( L \)) giving rise to a morphism \( \hat{\phi} : P \times \mathbb{A}^1 \to P \). Clearly, \( \phi_t : L \to L \) is the identity on \( L \) for all \( t \).

**Definition 23.** Let \( (E', \tilde{z}_1, \ldots, \tilde{z}_s) \) be a point of \( \hat{C} \) where \( E' \) is a principal \( P \)-space and \( \tilde{z}_k \in E'/\langle w_k^{-1} Bw_k \rangle \cap P \). Define the Levification family \( (E'_t, \tilde{z}_1(t), \ldots, \tilde{z}_s(t)), t \in \mathbb{A}^1 \), by \( E'_t := E' \times_{\phi_t} P \) and \( \tilde{z}_k(t) := \tilde{z}_k \times_{\phi_t} e \). Clearly, at \( t = 0 \), \( (E'_0, \tilde{z}_1(0), \ldots, \tilde{z}_s(0)) \) is in the image of \( i : \text{Fl}_L \to \hat{C} \), and equals \( i \circ \tau((E', \tilde{z}_1, \ldots, \tilde{z}_s)) \).

Consider a point \( (\tilde{g}_1, \ldots, \tilde{g}_s, \tilde{z}) \in C \), which gives a point of \( \hat{C} \). Write equations \( g_k b_k w p_k^{-1} = z \), or \( g_k b_k w = zp_k \). We get the principal \( P \)-space \( E' = zP \) (independent of the lift of \( \tilde{z} \)) and well-defined points \( z_k = zp_k \in E'/\langle w_k^{-1} Bw_k \cap P \rangle \).

All the spaces \( E'_t = E' \times_{\phi_t} P \) are trivialized by \( z \). Under this trivialization \( (E'_t, \tilde{z}_1(t), \ldots, \tilde{z}_s(t)) \) is the point

\[
(P, \phi_t(\tilde{z}_1), \phi_t(\tilde{z}_2), \ldots, \phi_t(\tilde{z}_s)) = (P, t^{x_L} p_1 t^{-x_L}, \ldots, t^{x_L} p_s t^{-x_L}).
\]

The corresponding \( E \) spaces are also trivial, and hence we obtain a lifting of this part in \( \mathcal{C} \): the points \( (t^{x_L} p_1 w_1^{-1}, \ldots, t^{x_L} p_s w_s^{-1}) \): at \( t = 0 \) we get

\[
(p_1 w_1^{-1}, \ldots, p_s w_s^{-1}) = z^{-1}(g_1 b_1, \ldots, g_s b_s).
\]

### 3.7. Comparison of line bundles and sections

**Definition 24.** Let \( \mathcal{M} \) be a line bundle on \( \text{Fl}_L \), \( Z(L) \) acts on fibers of \( \mathcal{M} \) and gives rise to a (multiplicative) character \( \gamma_{\mathcal{M}} : Z^0(L) \to \mathbb{C}^* \) (note that the group of characters of \( Z^0(L) \) is discrete). More generally, this map can be defined if \( \mathcal{M} \) is defined over an open substack of \( \text{Fl}_L \) since \( \gamma_{\mathcal{M}} \) is constant over connected families.

**Proposition 25.** Let \( U \) be a non-empty open substack of \( \text{Fl}_L \), \( \mathcal{L} \) be a line bundle on \( \tau^{-1}(U) \) and \( \mathcal{M} = i^* \mathcal{L} \), a line bundle on \( U \), where \( i : \text{Fl}_L \to \hat{C} \) and \( \tau : \hat{C} \to \text{Fl}_L \). Then

1. \( \mathcal{L} = \tau^* \mathcal{M} \). Therefore \( \tau^* \) and \( i^* \) set up isomorphisms \( \text{Pic}(U) \cong \text{Pic}(\tau^{-1}(U)) \).
2. If \( \gamma_{\mathcal{M}} \) is trivial then \( H^0(\tau^{-1}(U), \mathcal{L}) \to H^0(U, \mathcal{M}) \) is an isomorphism.

**Proof.** The second part is essentially [BK1, Thm. 15 and Rem. 31(a)], and the main point is that if \( E_t \) is a Levification family then a section of \( \mathcal{L} \) (under the assumption of (2)) at \( E_1 \) can be propagated in a unique way to all \( E_t, t \neq 0 \) (since \( E_1 \) is isomorphic to \( E_t \) for \( t \neq 0 \)), and there are no poles or zeroes of this extended section at \( t = 0 \). For the surjectivity we can extend any section of \( \mathcal{L} \) at \( E_0 \) to all of \( E_t \), since the corresponding \( \mathbb{C}^* \)-equivariant line bundle on \( \mathbb{A}^1 \) is trivial.

For the first part consider \( \mathcal{L}' = \mathcal{L} \otimes \tau^* \mathcal{M}^{-1} \). Note that \( \mathcal{M}' = i^* \mathcal{L}' \) is trivial, and \( \gamma_{\mathcal{M}'} \) is trivial. We can apply (2) to \( (\mathcal{L}', \mathcal{M}') \). The nowhere vanishing global
section of $H^0(\mathcal{F}_L, M')$ gives a global section of $H^0(\mathcal{E}, \mathcal{L})$. It can be seen that this is nowhere vanishing as well (see [BK2, Lem. 3.17]: Consider the corresponding Levification family (Definition 23), if a global section vanishes at $E_t$ then it will also vanish at $E_0$.)

3.8. Notation for Picard groups of stacks

Let $\mathcal{A}$ be a stack (for us $\mathcal{A}$ will be a quotient stack):

- $\text{Pic}(\mathcal{A})$ will denote the Picard group of line bundles on $\mathcal{A}$.
- $\text{Pic}_\mathbb{Q}(\mathcal{A}) = \text{Pic}(\mathcal{A}) \otimes \mathbb{Q}$.
- $\text{Pic}^+ (\mathcal{A}) \subseteq \text{Pic}(\mathcal{A})$ is the monoid of line bundles with non-zero global sections.
- $\text{Pic}^+_\mathbb{Q}(\mathcal{A}) \subseteq \text{Pic}_\mathbb{Q}(\mathcal{A})$ is the subset of elements such that some multiple has a non-zero global section.

3.9. Ramification divisors

Definition 26 (cf. [BKR, §4]). Consider a linear map $p : V \to W$ between vector spaces of the same dimension $m$. Let

$$\mathcal{D}(p) := (\wedge^m V)^* \otimes (\wedge^m W) = \text{Hom}(\wedge^m V, \wedge^m W).$$

Denote by $\theta(p)$ ("the theta section") the canonical element of $\mathcal{D}(p)$ induced by the top exterior power of $p$. We call $\mathcal{D}(p)$ the "determinant line," and $\theta(p)$ is, of course, the usual $\text{det}(p)$ if $p$ happens to be an endomorphism.

At a point $a = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_s, \tilde{z})$ of $\mathcal{C}$, we may consider two maps

$$TC_a \to T(G/B)^s_{(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_s)}$$

and

$$T(G/P)_{\tilde{z}} \to \bigoplus T(G/P)_{\tilde{z}} \frac{T(G/P)_{\tilde{z}}}{T(g_i C_{w_i})_{\tilde{z}}}. $$

The theta sections and determinant lines of the two maps above are isomorphic [BKR, Lems. 4.1 and 4.2], and describe the ramification divisor and the associated line bundle of the birational map $\pi : \mathcal{C} \to (G/B)^s$. They give rise to the line bundle $\mathcal{O}(\mathcal{R})$ and the divisor $\mathcal{R}$ on $\mathcal{C}$.

The line bundle $\mathcal{O}(\mathcal{R})$ is the pullback of a natural line bundle on $\mathcal{C}$: Consider a $P$-bundle $E'$ together with $\tilde{z}_i \in E'/(w_i^{-1} B w_i \cap P)$. This gives a point of $\mathcal{C}$ as in Lemma 21. Consider the map of vector spaces

$$E' \times_p T(G/P)_{\tilde{e}} \to \bigoplus \frac{E' \times_p T(G/P)_{\tilde{e}}}{z_i \times T(w_i^{-1} C_{w_i})_{\tilde{e}}}. $$

(15)

The determinant line and theta section for this map give a line bundle and a section on $\mathcal{C}$, denoted by $\mathcal{O}(\mathcal{R})$ and $\mathcal{R}$. Recall that if $E$ is a principal $P$-space and $P \to \text{GL}(V)$ a homomorphism ($V$ a vector space), then $E \times_p V$ is associated vector space (a quotient of $E \times P$ by an appropriate action of $P$).
We claim that $\mathcal{O}(\widehat{\mathcal{R}})$ and $\widehat{\mathcal{R}}$ pull back to $\mathcal{O}(\mathcal{R})$ and the divisor $\mathcal{R}$ on $\mathcal{C}$. To see this consider a $(\tilde{g}_1, \ldots, \tilde{g}_s, \tilde{z}) \in \mathcal{C}$. Let $E' = zP$ and find good pairs $(g_i, z_i)$ lifting $(\tilde{g}_i, \tilde{z})$. There is a natural map $xP \times_P T(G/P)_{\tilde{e}} \to T(G/P)_{\tilde{z}}$ and similarly maps $\mathcal{E}_0 = zP$ and find good pairs $(g_i, z_i)$ lifting $(\tilde{g}_i, \tilde{z})$. There is a natural map $xP\mathcal{T}(G/P)_{\tilde{e}} \to \mathcal{T}(G/P)_{\tilde{z}}$.

The claim now follows from the functoriality of the determinant line and its theta section [BKR, Lems. 4.1 and 4.2].

Pulling back $\mathcal{O}(\widehat{\mathcal{R}})$ and $\widehat{\mathcal{R}}$ via $i$ we get a divisor $\mathcal{R}$ and line bundle $\mathcal{O}(\mathcal{R})$ on $\mathcal{F}_L$. For the assumption (2), for $M = \mathcal{O}(\mathcal{R})$, $M$ is trivial (see Definition 24): This follows from the definition of the deformed product given in [BK1]. Start with $(l_1, \ldots, l_s) \in (L/B_L)^s$; one gets a point of $\mathcal{F}_L$. Then the fiber of $\mathcal{O}(\mathcal{R})$ is identified (using (15), $E'$ trivial) with the determinant line of $\mathcal{E}_0$.

The center of $L$ acts on the trivial $E'$ by automorphisms, and therefore we have an action of $Z^0(L)$ on the determinant line of this morphism. The triviality of this action is therefore implied by the assumption (2).

Since we have assumed non-zeroness in the deformed product, by Proposition 25,

**Corollary 27.**

1. $\mathcal{O}(\widehat{\mathcal{R}}) = \tau^* \mathcal{O}(\mathcal{R})$.
2. $\widehat{\mathcal{R}} = \tau^{-1} \mathcal{R}$ so that $i : \mathcal{F}_L \setminus \mathcal{R} \to \hat{\mathcal{C}} \setminus \hat{\mathcal{R}}$.

Finally we recall the generalization of Fulton’s conjecture proved in [BKR, Thm. 8.2] in two equivalent forms:

**Proposition 28.**

1. $\dim H^0(\mathcal{F}_L, \mathcal{O}(m\mathcal{R})) = 1$ for all $m \geq 0$.
2. $H^0(\mathcal{F}_L \setminus \mathcal{R}, \mathcal{O}) = \mathbb{C}$.

**3.10. Proof of Theorem 16**

Since the pullback of $\widehat{\mathcal{R}}$ to $\mathcal{C}$ is $\mathcal{R}$,

$$H^0(\mathcal{C} \setminus \mathcal{R}, \mathcal{O})^G = H^0(\hat{\mathcal{C}} \setminus \hat{\mathcal{R}}, \mathcal{O}),$$

and the theorem reduces to showing $H^0(\hat{\mathcal{C}} \setminus \hat{\mathcal{R}}, \mathcal{O}) = \mathbb{C}$.

Then note that from Corollary 27, $H^0(\hat{\mathcal{C}} \setminus \hat{\mathcal{R}}, \mathcal{O}) = H^0(\hat{\mathcal{C}} \setminus \tau^{-1} \mathcal{R}, \mathcal{O})$. Applying Proposition 25 (2) on $\mathcal{L} = \mathcal{O}$ ($\gamma_{\mathcal{O}}$ is trivial) and $U = \mathcal{F}_L \setminus \mathcal{R}$, we see that $H^0(\hat{\mathcal{C}} \setminus \tau^{-1} \mathcal{R}, \mathcal{O}) = H^0(\mathcal{F}_L \setminus \mathcal{R}, \mathcal{O})$. The last space is $\mathbb{C}$ by Proposition 28.
3.11. Proof of Theorem 7(d)
For (d), we follow arguments of Ressayre [Re1]: First it is enough to show (see Lemma 29 below) that the pullback of $O(D)$ along $Fl_L \to Fl_G$ has a non-zero global section on $Fl_L$. But this is clear since the pullback of the section of $O(D)$ vanishing only along $D$ does not vanish along $Fl_L \setminus R_L$ because $Fl_L \setminus R_L$ does not meet $D$ by Proposition 17.

Lemma 29. Suppose $L = L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_s} \in \text{Pic}(G/B)^s = \text{Pic}(Fl_G)$. Let $M$ be the pullback to $Fl_L$ of $L$ (via $\tilde{i}$). Then the following are equivalent:

1. $x$ satisfies the linear equalities defining the face $F$; i.e., setting $x = (\kappa(\lambda_1), \ldots, \kappa(\lambda_s)) = (h_1, \ldots, h_s) \in \mathfrak{h}^*$, we have
   $$\sum_{j=1}^s \omega_k(w_j^{-1}h_j) = 0, \ \forall \alpha_k \notin \Delta(P).$$

2. $\gamma_M : Z^0(L) \to \mathbb{C}^*$ is trivial.

Furthermore, if $H^0(Fl_L, M) \neq 0$ then the equivalent conditions above hold.

Proof. Since $Z^0(L)$ is connected (2) is equivalent to $\sum \lambda_j(w_jx_k) = 0$ for all $\alpha_k \in \Delta \setminus \Delta(P)$ which is equivalent to (1).

If $\sigma \neq 0 \in H^0(Fl_L, M)$, then the action of the center of $L$ on $\sigma$ is trivial on $M$ at any point where $\sigma$ is not zero, and hence $\gamma_M : Z^0(L) \to \mathbb{C}^*$ is trivial.

4. Divisor classes

4.1. Schubert cells are affine spaces
Let $v \in W_P$, let $U_v = \{u \in U | v^{-1}uv \in U^-\} = (vUv^{-1}) \cap U$. Then the map $U_v \to C_v$ which takes $u \to uv\dot{e} \in G/P$ is an isomorphism (see [BGG, Prop. 5.1] and the references therein).

Let $\Phi_v$ be the set of positive roots $\alpha$ such $v^{-1}\alpha \in R^-$. Then the product mapping (in any order, different orders give different mappings)
$$\prod_{\alpha \in \Phi_v} U_\alpha \sim \to U_v \sim \to C_v$$
is an isomorphism of varieties [Bo, §14.4]. Here $U_\alpha \sim \to \mathbb{G}_\alpha \sim \to \mathbb{A}^1$ is the subgroup corresponding to the positive root $\alpha$.

Assume in the lemmas below that $v, w \in W_P$ and $v \overset{\beta}{\to} w$, with $\beta = \alpha_\ell$ a simple root.

Lemma 30.

1. $s_\beta \Phi_v \subseteq \Phi_w$.
2. $\Phi_v \setminus s_\beta(\Phi_v) = \{\beta\}$.

Proof. For (1): If $\alpha \in \Phi_v$, then we need to show that $\gamma = s_\beta \alpha \in \Phi_w$. We know $\alpha \neq \beta$ since $v^{-1}\beta \in R^+$. Therefore $\gamma = s_\beta \alpha \in R^+$ (see [FH, Lem. D.25]), and $w^{-1} \gamma = v^{-1} s_\beta \alpha = v^{-1} \alpha \in R^-$ since $\alpha \in \Phi_v$.

For (2): From $v \overset{\beta}{\to} w$, we find that $w^{-1} \beta \in R^-$, and hence $\beta \in \Phi_w$. We claim $\beta \notin s_\beta \Phi_v$. If $\beta = s_\beta \alpha$, then $\alpha = s_\beta \beta = -\beta \in R^-$, and therefore we are done using $|\Phi_v| = \ell(v)$ (similarly for $w$) and $\ell(w) = \ell(v) + 1$. \qed
Lemma 31.

1. \( s_\beta C_v \subseteq C_w \).
2. The map \( \mathbb{A}^1 \times s_\beta C_v \to G/P \) which sends \((t, s_\beta x)\) to \(\exp(tE_\beta)s_\beta x\) sets up an isomorphism \(\mathbb{A}^1 \times s_\beta C_v \to C_w \).

Proof. The first statement follows from the first part of Lemma 30. The second part follows from [Bo, §14.4]. 

Recall that \( Q_w \) is the largest subgroup of \( G \) that preserves \( X_w \). Since \( w^{-1} \beta \in R^- \) and \( \beta \) is simple, \( s_\beta \) in \( Q_w \) and \( s_\beta C_w \subseteq X_w \). The inclusion \( s_\beta C_v \subseteq C_w \) therefore yields a factorization of the canonical inclusion: \( C_v \subseteq s_\beta C_w \subseteq X_w \).

4.2. Universal Schubert varieties and their cycle classes

Let \( u \in W^P \) and consider the universal Schubert variety

\[
S_u = \{ (g, z) \mid z \in gX_u \} \subseteq G/B \times G/P.
\]

Recall that \( X_u \subseteq G/P \) is the closure of the Schubert cell \( C_u \). Let \( m = \dim G/P - \ell(u) \), the codimension of \( X_u \) in \( G/P \).

We want to determine the first two terms \((j = 0, 1 \text{ below})\) of the cycle class \([S_u] \in A^e(G/B \times G/P)\) of \( S_u \) in the decomposition

\[
A^m(G/B \times G/P) = \bigoplus_{j=0}^{m} A^j(G/B) \otimes A^{m-j}(G/P).
\]

We may intersect with \([\mathcal{E}] \times g[X_w]\), with \( g \) general, and \( w \in W^P \) arbitrary such that \( \ell(w) = \dim G/P - \ell(u) \) and see that the \( j = 0 \) term is \( 1 \otimes [X_u] \).

Write the \( j = 1 \) term as \( \sum \beta \mathcal{L} \otimes \beta \).

Proposition 32. Let \( \hat{u}_\ell = s_{\alpha_\ell} u \).

1. If \( \hat{u}_\ell \not\in W^P \) or \( u \not\in W^P \) is false then \( \beta \ell = 0 \).
2. If \( \hat{u}_\ell \in W^P \) and \( u \not\in W^P \) then \( \beta \ell = [X_u] \).

4.3. Proof of Proposition 32

For every simple root \( \beta = \alpha_\ell \) there is an associated \( \mathbb{P}^1 \to G/B \) which sends \( t \in \mathbb{A}^1 \) to \( \exp(tE_\beta)s_\beta \in G/B \) and \( t = \infty \) to \( \hat{e} \). In fact the entire Schubert cell \( Bs_\beta B/B \) is the image of \( \mathbb{A}^1 \), and the degree of the line bundle \( \mathcal{L}_{\alpha_\ell} \) along this curve is \( \delta_{k, \ell} \).

Therefore \( \mathcal{L}_{\alpha_\ell} \in \text{Pic}(G/B) = A^1(G/B) \), and \( [Bs_{\alpha_\ell}B/B] \in A_1(G/B) \) give dual bases under the intersection pairing \( A^1(G/B) \times A_1(G/B) \to \mathbb{Z} \).

To prove Proposition 32, we first intersect \( S_u \) with \( [Bs_{\alpha_\ell}B/B] \times g[X_w] \) for a general \( g \in G \), and \( w \in W^P \) arbitrary such that \( \ell(w) + 1 = \dim G/P - \ell(u) \). This shows that

- The intersection number \( I \) of \( \beta_\ell \) and \( [X_w] \) will equal the number of points of the form \((t, \hat{z})\) where (note that \( G/B \times G/P \) has a transitive action of a group, and we can use Kleiman transversality):
  1. \( \hat{z} \in tX_u \in G/P \) (since \((t, \hat{z}) \in S_u)\),
  2. \( t \in Bs_{\alpha_\ell}B/B = A^1 \), and
  3. \( \hat{z} \in X_w \).
If $\alpha_\ell \in \Delta(Q_u)$, then $tX_u$ does not vary with $t$ (since $s_\alpha X_u \subseteq X_u$), and by Kleiman transversality, the intersection number $I$ is zero. Therefore if $u^{-1}\alpha_\ell \in R_i^+$ or $u^{-1}\alpha_\ell \in R^-$, then $tX_u$ does not vary with $t$. Therefore unless $u^{-1}\alpha_\ell \in R^+ \setminus R_i^+$, the intersection number $I$ is zero (independently of $w$). The following Lemma therefore shows the first part of Proposition 32.

**Lemma 33.** The following are equivalent:

1. $u^{-1}\alpha_\ell \in R^+ \setminus R_i^+$;
2. $\widehat{u}_\ell \in W^P$ and $u^\alpha \to \widehat{u}_\ell$.

**Proof.** Assume (1). We first show that $\widehat{u}_\ell \in W^P$. We need to show that $\widehat{u}_\ell R_i^+ \subseteq R_i^+$. Assume the contrary. Now $\widehat{u}_\ell R_i^+ = s_{\alpha_\ell} u R_i^+ \subseteq s_{\alpha_\ell} R_i^+$. The only positive root which $s_{\alpha_\ell}$ takes to a negative root is $\alpha_\ell$, so we will have $\alpha_\ell \in u R_i^+$ which contradicts our assumptions. Therefore we have shown that $\widehat{u}_\ell \in W^P$. From $u^{-1}\alpha_\ell \in R^+$, we get $\ell(\widehat{u}_\ell) \geq \ell(u) + 1$, which should be an equality since $\widehat{u}_\ell = s_{\alpha_\ell} u$. Therefore (2) holds.

Now assume (2). The length condition in $u^\alpha \to \widehat{u}_\ell$ implies that $u^{-1}\alpha_\ell \in R^+$. If $u^{-1}\alpha_\ell \in R_i^+$, then $\alpha_\ell \in u R_i^+$, then $\widehat{u}_\ell R_i^+$ contains $-\alpha_\ell$ which is a negative root. This contradicts $\widehat{u}_\ell \in W^P$. \(\square\)

Now assume $\widehat{u}_\ell \in W^P$ and $u^\alpha \to \widehat{u}_\ell$. We need to show that $\beta_\ell = [X_{\widehat{u}_\ell}]$. We will show that the intersection number $I$ of $\beta$ and $[X_w]$ is the same as the intersection number of $[X_{\widehat{u}_\ell}]$ and $[X_u]$. This will finish the proof of Proposition 32, since $w$ was arbitrary.

The intersection number $I$ is the count of pairs $(z, t)$ satisfying conditions (a), (b), (c) above. By Lemma 31, the sets $tC_u$ are distinct and have $C_{\widehat{u}_\ell}$ for their union. Therefore $I$ equals the intersection number of $X_{\widehat{u}_\ell}$ and $gX_w$, as desired (we can assume that the intersection takes place in the open Schubert cells in each by dimension counting).

**4.4. Proof of Theorem 8**

Let $(u_1, \ldots, u_s)$ be as in Proposition 1.7. Recall that

$$\sum_{i=1}^s (\dim G/P - \ell(u_i)) = \dim G/P + 1.$$

Let $\widetilde{D}(j, v) \subseteq (G/B)^s \times G/P$ be as defined in Definition 14. Note that $D(j, v) = \pi_*(\widetilde{D}(j, v))$ (use the fact that $\widetilde{D}(j, v)$ is not contained in $R$ as proved in Section 2.2).

We have $s$ morphisms $p_i : (G/B)^s \times G/P \to (G/B) \times (G/P)$. The scheme-theoretic intersection of $p_i^{-1}S_{u_i}$ equals $\widetilde{D}(j, v)$. This intersection is proper because the codimension of $\widetilde{D}(j, v)$ in $(G/B)^s \times G/P$ is the sum of codimensions of $X_{u_i}$. The cycle class of $\widetilde{D}(j, v)$ is therefore the cup product of the pullbacks of cycle classes of $S_{u_i}$. Theorem 8 now follows from $D(j, v) = \pi_*(\widetilde{D}(j, v))$.

**Remark 6.** Suppose we consider a codimension one Schubert cell $C_v \subseteq X_{w_j}$ with $v \mapsto w_j$, and $\beta$ not simple, $v \in W^P$. Define $u_1, \ldots, u_s$ as in Equation (5), and let
$D$ be the right-hand side of (6). Let $D = G = P$, satisfying $b_s$ are $C$ and has been reduced in length by 1 from $b^*$.

Then by [BKR, Prop. 8.1], $D$ lies in the closure of $R$, and hence $D$, the image of $D$, is of codimension $\geq 2$ in $(G/B)^s$. The element $\pi_*(\tilde{D}) \in A^1((G/B)^s)$ is zero, and the formulas of Theorem 8 apply also in this case. Therefore one gets vanishing of several intersection numbers.

**Remark 7.** The proof of Theorem 8 shows that one obtains formulas for a divisor class supported on the locus (with possible multiplicities) given by the right side of (6) (as a suitable pushforward) for arbitrary $u_1, \ldots, u_s$ satisfying $\sum (\dim(G/P) - \ell(u_i)) = \dim G/P$ and $u_i \in W^i$. This divisor class is zero if and only if the right side of (6) is not codimension one in $(G/B)^s$ (it is always irreducible). The following is an example of a divisor class which is 0.

**Example 1.** Let $F$ be the face for $\Gamma(3, \text{Spin}(8))$ (type $D_4$) given by maximal parabolic $P_2$ (removal of $a_2$ from the simple roots) and

$$ u = s_2s_4s_3s_2, \quad v = s_1s_2s_4s_3s_2, \quad w = s_2s_3s_1s_2s_4s_3s_1s_2. $$

Then consider the reflection $t = s_1s_2s_1$ associated to the non-simple root $\alpha_1 + \alpha_2$. Observing that $tv = s_1s_4s_3s_2$ is in $W^P$, and has been reduced in length by 1 from $v$, we have $tv \xrightarrow{\alpha_1 + \alpha_2} v$ and therefore $C_{tv} \subseteq X_v \setminus Y_v$. As indicated by the preceding remarks, the class of the “divisor” $D = D(2, tv)$ should be 0. This can be explicitly verified using the formulas of Theorem 8 as follows: the only allowable increments of $u, tv, or w$ (still landing in $W^P$) are

$$ u \xrightarrow{\alpha_1} s_1u = s_1s_2s_4s_3s_2 \quad \text{and} \quad tv \xrightarrow{\alpha_2} s_2tv = s_2s_1s_4s_3s_2, $$

and one can calculate that both

$$ [X_{s_1s_2s_4s_3s_2}] \cdot [X_{s_1s_4s_3s_2}] \cdot [X_{s_2s_3s_1s_2s_4s_3s_1s_2}] = 0 $$

and

$$ [X_{s_2s_4s_3s_2}] \cdot [X_{s_2s_1s_4s_3s_2}] \cdot [X_{s_2s_3s_1s_2s_4s_2s_3s_1s_2}] = 0. $$

(In the notation of [KKM], those calculations are $b_1^4 \cdot b_3^5 \cdot 1 = 0$ and $b_5^4 \cdot b_2^4 \cdot 1 = 0$ and follow from the duality rule $b_i^4 \cdot b_j^4 = \delta_{i,j} b_9$ or from [KKM]*Corollary 4.1 and the multiplication table for $G/P$ under $\circ a_0$.) Therefore $[D] = 0$.

5. Faces of the eigencone

5.1. The face $F$ as a product

Our aim in this section is to prove Theorem 10. Consider the map (10). We first show that it is an injection. Suppose

$$ \sum a_b \delta_b + f = \sum a'_b \delta_b + f', \quad a_b, a'_b \in \mathbb{Q}. \quad (17) $$

It suffices to show $a_b = a'_b$ for all $b$. Fix $b$ and suppose $\delta_b = [D(j, v)]$ where $v \xrightarrow{\alpha_b} w_j$. Then we may apply $\alpha_{b'}$ to the $j$th coordinate of (17). This gives $a_b = a'_b$ by using Corollary 34 below.

The following is an easy corollary of Theorem 8.
Corollary 34. Consider a pair $(j, v)$ with $v \underset{\alpha}{\rightarrow} w_j$, and set

$$O(D(j, v)) = \mathcal{L}_{\lambda_1} \boxtimes \mathcal{L}_{\lambda_2} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda_s} \in \text{Pic}(G/B)^s.$$ 

Then

1. $\lambda_j(\alpha_\vee) = 1$, and hence $\alpha_\ell(\kappa(\lambda_j)) > 0$.
2. Suppose $(j', v') \neq (j, v)$ with $v' \underset{\alpha}{\rightarrow} w_{j'}$. Then, $\lambda_{j'}(\alpha_\vee) = 0$, and hence $\alpha_{\ell'}(\kappa(\lambda_{j'})) = 0$.

Proof. Set

$$u_i = \begin{cases} w_i, & i \neq j, \\ v, & i = j. \end{cases}$$

Using Theorem 8, for (1), the coefficient $c_{j, \ell} = \lambda_j(\alpha_\vee)$ is just the multiplicity in the intersection product (2) in ordinary cohomology product which is one, since it is one in the deformed product $\otimes_0$ by assumption.

For (2), consider the case $j = j'$ first: We start by showing that $v^{-1} \alpha_{\ell'}$ is not a positive root. Observe that $v^{-1} \alpha_{\ell'} = w^{-1}(s_{\alpha_\ell'} \alpha_{\ell'}) = w^{-1}(\alpha_{\ell'} + m \alpha_\ell)$ with $m \geq 0$ since $\ell \neq \ell'$. Now both $w^{-1}(\alpha_\ell)$ and $w^{-1}(\alpha_{\ell'})$ are negative roots by assumption and hence $v^{-1} \alpha_{\ell'}$ is not in $R^+$, and hence $\lambda_j(\alpha_\vee) = 0$ using Theorem 8.

If $j \neq j'$ then we need to show that $w_{j'}^{-1} \alpha_{\ell'}$ is not a positive root, which follows from $v' \underset{\alpha}{\rightarrow} w_{j'}$.  

The surjection part of Theorem 10 follows from Corollary 34, and the following:

Proposition 35. Suppose

$$\mathcal{L} = \mathcal{L}_{\mu_1} \boxtimes \mathcal{L}_{\mu_2} \boxtimes \cdots \boxtimes \mathcal{L}_{\mu_s} \in \text{Pic}(G/B)^s$$

and $x = (\kappa(\mu_1), \ldots, \kappa(\mu_s))$. Let $(j, v)$ with $v \underset{\alpha}{\rightarrow} w_j$ and $v \in W_P$. Assume

1. $H^0((G/B)^s, \mathcal{L})^G \neq 0$. Assume also that $x \in \mathcal{F}$.
2. $\alpha_\ell(\kappa(\mu_j)) > 0$, i.e., $\mu_j(\alpha_\vee) > 0$.

Let $m = \mu_j(\alpha_\vee) \in \mathbb{Z}_{>0}$, and

$$\mathcal{L}' = \mathcal{L}(-mD(j, v)) = \mathcal{L}_{\mu_1'} \boxtimes \mathcal{L}_{\mu_2'} \boxtimes \cdots \boxtimes \mathcal{L}_{\mu_s'} \in \text{Pic}(G/B)^s$$

and $x' = (\kappa(\mu_1'), \ldots, \kappa(\mu_s'))$. Then

1. $H^0((G/B)^s, \mathcal{L})^G \neq 0$. Thus all $\mu_i'$ are dominant and $x' \in \mathcal{F}$.
2. $\mu_j'(\alpha_\vee) = 0$.

Proof. Start with a non-zero invariant section $\sigma \in H^0((G/B)^s, \mathcal{L})^G$. We will show that $\sigma$ vanishes on $D(j, v)$: This will show that $\mathcal{L}(-D(j, v))$ has invariant sections and lies on $\mathcal{F}$ (also use Theorem 7 (d)). Writing

$$\mathcal{L}(-D(j, v)) = \mathcal{L}_{\nu_1} \boxtimes \mathcal{L}_{\nu_2} \boxtimes \cdots \boxtimes \mathcal{L}_{\nu_s} \in \text{Pic}(G/B)^s$$

we see using Corollary 34 that $\nu_j(\alpha_\vee) = \mu_j(\alpha_\vee) - 1$, and we can iterate this procedure to get the desired result.
For the vanishing of $\sigma$ on $D(j, v)$, start with a general point $x = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_s) \in D(j, v)$. Applying the considerations of Section 5.2 below, set

$$u_i = \begin{cases} w_i, & i \neq j, \\ v, & i = j. \end{cases}$$

We will show that inequality (21) below fails: i.e., show that for a suitable $\alpha_k \notin \Delta(P)$,

$$\sum_{i=1}^{s} u_i^{-1} \mu_i(x_k) > 0. \tag{18}$$

However, we know that the point $x$ is on the face $\mathcal{F}$, and hence

$$\sum_{i=1}^{s} w_i^{-1} \mu_i(x_k) = 0. \tag{19}$$

Therefore it suffices to show that $(w_j^{-1} \mu_j - v^{-1} \mu_j)(x_k) \leq 0$ for some $\alpha_k \notin \Delta(P)$, with a strict inequality for at least one $\alpha_k \notin \Delta(P)$. Now

$$w_j^{-1} \mu_j - v^{-1} \mu_j = v^{-1}(s_{\alpha_\ell} \mu_j - \mu_j) = -\mu_j(\alpha_\ell \vee)v^{-1}\alpha_\ell. \tag{20}$$

By assumption $\mu_j(\alpha_\ell \vee) > 0$. Also we know $\beta = v^{-1}\alpha_\ell \in R^+$. Therefore the inequality holds. We now show that at least one inequality holds strictly.

We claim that $\beta = v^{-1}\alpha_\ell \notin R^+_1$ because if $\beta \in R^+_1$, then $\alpha_\ell = v\beta$, and $-\alpha_\ell = s_{\alpha_\ell}v\beta = w_j\beta$, but $w_j\beta$ is a positive root since $w \in W_P$.

Therefore in the expression of the positive root $\beta$ as a sum of simple roots, at least one root $\alpha_\ell \in \Delta \setminus \Delta(P)$ appears with a non-zero coefficient, and $v^{-1}\alpha_\ell(x_k) > 0$. For this $k$, by (20), $(w_j^{-1} \mu_j - v^{-1} \mu_j)(x_k) < 0$, as desired. □

### 5.2. Necessary inequalities

Suppose

1. $x = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_s)$ is an arbitrary point of $(G/B)^s$.
2. $\sigma \in H^0((G/B)^s, \mathcal{L}_{\mu_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\mu_s})^G$ with $\sigma(x) \neq 0$. Assume further that
3. $\bigcap g_i C_{u_i} \neq \emptyset \subseteq G/P$. Here $P$ is a standard parabolic of $G$.

We want to recall the (standard) proof of

$$\sum_{i=1}^{s} u_i^{-1} \mu_i(x_k) \leq 0 \tag{21}$$

whenever $\alpha_k \in \Delta \setminus \Delta(P)$, under these conditions.

First assume that $e \in \cap g_i C_{u_i}$ by translations in $G$. Next write down equations $g_i u_i = p_i$ or $g_i = p_i u_i^{-1}$. Consider the (rational) one parameter subgroup $t^{x_k}$ and the limit point

$$\lim_{t \to 0} t^{x_k}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_s) = (\bar{h}_1, \ldots, \bar{h}_s) \in (G/B)^s.$$

The action of $t^{x_k}$ on the fiber of $\mathcal{L}_{\mu_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\mu_s}$ over the limit point (since this measures the order of vanishing of $\sigma$ as $t \to 0$) should be $\leq 0$.

The desired inequality (21) follows from
Lemma 36. The action of the (rational) one parameter subgroup $t^{x_k}$ on the fiber of $\mathcal{L}_{\mu_i}$ at $h_i$ is given by the exponent $-\mu_i(u_ix_k)$.

Proof. This is because $h_i$ is of the form $l_iv_iu_i^{-1}$ where $l_i$ is in the Levi subgroup $L$ and $v_i$ is in the unipotent radical and commutes with $t^{x_k}$ (start with $p_i = l_iv_i'$ and write $v_i'$ as a product of one parameter subgroups). Therefore we need to compute the action of $t^{x_k}$ on the fiber of $\mathcal{L}_{\mu_i}$ at $u_i^{-1}e$, which is a standard computation. □

5.3. Extremal rays lie on regular facets of the eigencone

Lemma 37. Suppose that $\hat{\rho} = \mathbb{Q}_{\geq 0}(h_1, \ldots, h_s)$ is an extremal ray of $\Gamma_{\mathbb{Q}}(s, G)$. Then $(h_1, \ldots, h_s)$ satisfies some inequality (3) with equality (and maximal parabolic $P$).

Proof. Assume not. By symmetry, we may assume $h_1, \ldots, h_m$ are all nonzero for some $1 \leq m \leq s$, and that $h_{m+1}, \ldots, h_s$ are all 0. By definition there exist $k_1, \ldots, k_s \in \mathfrak{k}$ such that $C(h_j) = k_j \in \mathfrak{k}/K$ and $\sum k_j = 0$. If $m = 1$, then $k_1 \neq 0$ but $k_j = 0$ for each $j > 1$, contradicting the sum condition. So $m$ is at least 2.

The condition $h_1 \in \mathfrak{h}_+$ is invariant under multiplication by $\mathbb{R}_{> 0}$; furthermore, we have assumed each inequality (3) holds strictly, so for arbitrarily small values of $\epsilon > 0$,

$$((1 + \epsilon)h_1, h_2, \ldots, h_s) \text{ and } ((1 - \epsilon)h_1, h_2, \ldots, h_s)$$

are elements of $\Gamma_{\mathbb{Q}}(s, G)$ which are not proportional (since $h_2 \neq 0$), but their sum gives the same ray $\hat{\rho}$. This contradicts the extremality of $\hat{\rho}$. □

6. Building blocks for induction

We need to find the remaining extremal rays on $F_{\mathbb{Q}}$, i.e., the extremal rays of $F_{\mathbb{II}, \mathbb{Q}}$ under the Killing form bijection of Proposition 4; we are interested in the extremal rays of the $\mathbb{Q}$ cone generated by line bundles

$$\mathcal{L} = \mathcal{L}_{\lambda_1} \boxtimes \mathcal{L}_{\lambda_2} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda_s} \in \text{Pic}(G/B)^s$$

such that

1. $H^0((G/B)^s, \mathcal{L}_N)^G \neq 0$ for some $N > 0$, and
2. for each of the $q$ pairs $(j, v)$ with $v \overset{\lambda_j}{\sim} w_j$ and $v \in W^P$, we have $\lambda_j(\alpha_j^\vee) = 0$.

We want to replace $(G/B)^s$ by a product of partial flag varieties $\prod_{i=1}^s G/Q'_{w_i}$ so that line bundles on the latter pull back to line bundles

$$\mathcal{L} = \mathcal{L}_{\lambda_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda_s}$$

on $(G/B)^s$ which satisfy all the linear equalities required in (2) above.

Definition 38. For $w \in W^P$, define

$$\Delta'_w = \{ \alpha \in \Delta \mid s_\alpha w < w \} = \Delta \cap wR^- \subseteq \Delta_w$$

and let $Q'_w \subseteq Q_w$ be the corresponding standard parabolic subgroup.
Let $\mathcal{L}_\lambda$ be the pullback to $G/B$ of a line bundle on $G/Q_w$, and $v \overset{\alpha_\ell}{\rightarrow} w$, $v, w \in W^P$. Then, $\lambda(\alpha_\ell \gamma) = 0$.

**Proof.** This follows from $\alpha_\ell \in \Delta'_w$ since $w^{-1} \alpha_\ell \in R^-$. □

The group $w^{-1} Bw \cap P$ played a key role in various constructions in the previous sections. It was important in those arguments that it mapped onto $B_L$ under the projection to $L$. The group $Q'_w$ has the same property (but not $Q_w$). The following proof was communicated to us by S. Kumar:

**Lemma 40.** $w^{-1} Q'_w w \cap L = B_L$.

**Proof.** The inclusion $w^{-1} Q'_w w \cap L \supset B_L$ is easy because $wB_L w^{-1} \subset B \subseteq Q'_w$. For the other direction, we are reduced to proving that (also see Lemma 19),

$$w^{-1} R(Q'_w) \cap R^-(L) = \emptyset. \quad (22)$$

Pick a $w^{-1} \gamma = \beta$ in the intersection. Clearly $\gamma$ is a negative root, since $w\beta$ is a negative root (use $wR^-(L) \subset R^-$). Write $\gamma = -\sum \gamma_i$ with $\gamma_i \in \Delta'_w$ and simple. Now $w^{-1} \gamma_i$ are negative, and hence $w^{-1} \gamma$ is a positive root, a contradiction. □

### 6.1. Enlargement of Schubert cells

Let $C'_w \subseteq Y_w \subseteq X_w$ be the open $Q'_w$-orbit in $X_w$. The following lemma relates $C'_w$ to simple codimension one Schubert cells in $X_w$ (see Lemma 13).

**Lemma 41.** Let $v \overset{\beta}{\rightarrow} w$ with $v, w \in W^P$. The following are equivalent:

1. $C_v \subseteq Y_w$.
2. $\beta \in \Delta'_w$ (in particular, $\beta$ is a simple root).
3. $C_v \subseteq C'_w$.

Therefore $Y_w \setminus C'_w$ is codimension $\geq 2$ in $Y_w$.

**Proof.** If $C_v \subseteq Y_w$ then $\beta \in \Delta = \Delta \cap w(R^+_1 \sqcup R^-)$, and hence $\beta \in \Delta$. We will also have $w^{-1} \beta \in R^-$, and hence $\beta \in \Delta'_w = \Delta \cap wR^-$. Therefore (1) implies (2).

Under the assumption (2), $s_\beta \in Q'_w$, which implies $s_\beta w \in C'_w$, and hence $C_v \subseteq C'_w$. Therefore (2) implies (3). Since $C'_w \subseteq Y_w$, it follows that (3) implies (1). □

### 6.2. Universal families over $\prod_{i=1}^s (G/Q'_{w_i})$

Define the universal intersection locus

$$\mathcal{X}' = \left\{ (g_1, \ldots, g_s, z) \in \prod_{i=1}^s \frac{G/Q'_{w_i}}{G/P} \bigg| z \in g_i X_{w_i} \; \forall i \right\},$$

and similarly define subloci $Z' \supseteq C'$ using the $Z_{w_i}, C'_{w_i}$, respectively, in place of the $X_{w_i}$. (For the scheme structures, see [BKR].) As shown in [BKR], the projection map $\pi' : \mathcal{X}' \to \prod_{i=1}^s (G/Q'_{w_i})$ is birational. Moreover, because the $X_w$ are all normal, each $X_{w_i} \setminus Z_{w_i}$ has codimension $\geq 2$ in $X_{w_i}$; accordingly, $\mathcal{X}' \setminus Z'$ has codimension $\geq 2$ in $\mathcal{X}'$. Let $\pi' : \mathcal{X}' \to \prod_{i=1}^s (G/Q'_{w_i})$.

**Lemma 42.** $\overline{\pi'(\mathcal{X}' \setminus Z')} \subseteq \prod_{i=1}^s (G/Q'_{w_i})$ is of codimension $\geq 2$. 

Proof. It suffices to show that $\pi'(\mathcal{X} \setminus \mathcal{C}) \subseteq \prod_{i=1}^{s}(G/Q'_{w_i})$ is of codimension $\geq 2$. Consider the fiber product diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
(G/B)^s & \xrightarrow{\phi} & \prod_{i=1}^{s}(G/Q'_{w_i})
\end{array}
$$

(23)

Now $\phi$ is a smooth fiber bundle over a smooth base, and it suffices to show that $\phi^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{C}'))$ is of codimension $\geq 2$ in $(G/B)^s$. We have (see the remark below)

$$
\phi^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{C}')) = \pi(\tilde{\phi}^{-1}(\mathcal{X}' \setminus \mathcal{C}'))
$$

(24)

and $\tilde{\phi}^{-1}(\mathcal{X}' \setminus \mathcal{C}') \supseteq \mathcal{X} \setminus \mathcal{Y}$ with complement of codimension $\geq 2$ in $\mathcal{X}$ (by Lemma 41, note also that $\mathcal{X} \setminus \mathcal{Z}$ is codimension $\geq 2$ in $\mathcal{X}$). The desired statement follows from [BKR, Prop. 8.1], which shows that up to codimension two, $\mathcal{Z} \setminus \mathcal{Y}$ lies in the ramification divisor of $\pi$. \qed

Remark 8. The equality (24) can be verified (analytic) locally on $(G/B)^s$. Let $U$ be an open subset of $\prod_{i=1}^{s}(G/Q'_{w_i})$, such that $U = \phi^{-1}(U) = U \times \Lambda$ for a suitable $\Lambda$. Then if $\Gamma = (\mathcal{X}' \setminus \mathcal{C}') \cap \pi'^{-1}U$ then the left-hand side looks (over $U'$) like $\pi'((\Gamma) \times \Lambda$ and the right-hand side like $\pi((\Gamma \times \Lambda)$.

7. Parameter spaces for induction

Remark 9. The constructions of Section 3.4 can be generalized: let $\psi : P \to L = P/U$ be the quotient map, and let $M_1, \ldots, M_s$ be standard parabolic subgroups satisfying

$$
\psi(w_i^{-1}M_iw_i) = B_L
$$

for each $i = 1, \ldots, s$. For brevity, write $\tilde{M} = (M_1, \ldots, M_s)$. Then one can define $\text{Fl}_G(\tilde{M}) = [\prod_{i=1}^{s}G/M_i]/G$, which parameterizes $G$-spaces $E$ together with elements $\tilde{g}_i \in E/M_i, i = 1, \ldots, s$, and $\mathcal{C}(\tilde{M})$, which parameterizes principal $G$-spaces $E$ together with elements $\tilde{g}_i \in E/M_i$ and a single element $\tilde{z} \in E/P$ so that $\tilde{z} \in g_iM_iw_iP$ for all $i$.

Analogues of Lemma 21, the diagram in Section 3.5, the Levification process, Proposition 25, the ramification loci of Section 3.9, and Corollary 27 all exist/hold in this general setting. The particular case in which we are now interested is where $M_i = Q'_{w_i}$ for each $i$, as we now convey in detail.

In the induction operation, we will continue to use $\text{Fl}_L$; however, we also introduce the following new stacks:

Definition 43. Let $\text{Fl}'_G$ be the stack parameterizing principal $G$-spaces $E$ together with elements $\tilde{g}_i \in E/Q'_{w_i}$; i.e.,

$$
\text{Fl}'_G = \left[\prod_{i=1}^{s}G/Q'_{w_i}\right]/G.
$$
**Definition 44.** Let \( \widehat{\mathcal{C}}' \) be the moduli stack parameterizing principal \( G \)-spaces \( E \) together with elements \( \bar{g}_i \in E/Q'_{w_i} \) and a single element \( \bar{z} \in E/P \) so that \( \bar{z} \in g_i C_{w_i} \) for \( i = 1, \ldots, s \). There is a natural map \( \pi' : \widehat{\mathcal{C}}' \to \text{Fl}_{L} \), and as before one can check that \( \widehat{\mathcal{C}}' = [C'/G] \) (\( C' \) was defined in Section 6.2).

Similar to Lemma 21, we have the following:

**Lemma 45.** The stack \( \widehat{\mathcal{C}}' \) parameterizes principal \( P \)-bundles \( E' \) together with \( \bar{z}_i \in E'/ (w_i^{-1}Q'_{w_i} w_i \cap P), i = 1, \ldots, s \).

This lemma, Lemma 40 and the map \( P \to L \) give rise to \( \tau' : \widehat{\mathcal{C}}' \to \text{Fl}_{L} \). The map \( L \to P \) gives a map \( i' : \text{Fl}_{L} \to \widehat{\mathcal{C}}' \), and the map \( \widetilde{i}' = \pi' \circ i' \). Similar to (14) we have the following diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i'} & \mathcal{C}' \\
\downarrow \pi' & & \downarrow \pi' \\
(G/B)^s & \xrightarrow{\prod_{i=1}^s (G/Q'_{w_i})} & \text{Fl}_{L} \\
\end{array}
\]

(25)

7.1. Levification in the new setting

\( \widehat{\mathcal{C}}' \) retracts to \( \text{Fl}_{L} \) by Levification, generalizing the constructions in Section 3.6: Let \( \phi_t : P \to P \) be as in Section 3.6.

**Definition 46.** Let \( (E', \bar{z}_1, \ldots, \bar{z}_s) \) be a point of \( \widehat{\mathcal{C}}' \) where \( E' \) is a principal \( P \)-space and \( \bar{z}_k \in E'/ (w_k^{-1}Q'_{w_k} w_k \cap P) \). Define the Levification family \( (E'_t, \bar{z}_1(t), \ldots, \bar{z}_s(t)), t \in \mathbb{A}^1 \), by \( E'_t := E' \times_{\phi_t} P \) and \( \bar{z}_k(t) := \bar{z}_k \times_{\phi_t} e \). Clearly at \( t = 0 \), \( (E_t, \bar{z}_1(t), \ldots, \bar{z}_s(t)) \) is in the image of \( i : \text{Fl}_{L} \to \widehat{\mathcal{C}} \).

Proposition 25 generalizes (with the same proof) to this new setting with \( \tau' : \widehat{\mathcal{C}}' \to \text{Fl}_{L} \).

**Proposition 47.** Let \( U \) be a non-empty open substack of \( \text{Fl}_{L} \), \( \mathcal{L} \) be a line bundle on \( \tau^{-1}(U) \) and \( \mathcal{M} = i^* \mathcal{L} \), a line bundle on \( U \), where \( i^* : \text{Fl}_{L} \to \widehat{\mathcal{C}} \). Then

1. \( \mathcal{L} = \tau^* \mathcal{M} \). Therefore \( \tau^* \) and \( i^* \) set up isomorphisms \( \text{Pic}(U) \cong \text{Pic}(\tau^{-1}(U)) \).
2. If \( \gamma_{\mathcal{M}} : Z^0(L) \to \mathbb{C}^* \) is trivial then \( i^* : H^0(\tau^{-1}(U), \mathcal{L}) \to H^0(U, \mathcal{M}) \) is an isomorphism.

7.2. Ramification divisors in the new setting

Let \( \mathcal{R}' \) be the ramification divisor of the map \( \pi' : \mathcal{C}' \to \prod_{i=1}^s (G/Q'_{w_i}) \). Similarly as in Section 3.9, the line bundle \( \mathcal{O}(\mathcal{R}') \) is the pullback of a natural line bundle on \( \widehat{\mathcal{C}}' \). Consider a \( P \)-bundle \( E' \) together with \( \bar{z}_i \in E'/ (w_i^{-1}Q'_{w_i} w_i \cap P) \). Consider the map of vector spaces

\[
E' \times_P T(G/P)_{\bar{e}} \to \bigoplus_{\bar{z}_i} E' \times_P T(G/P)_{\bar{e}} / z_i \times_P T(w_i^{-1}C'_{w_i} \bar{e}) \tag{26}
\]

The determinant line and theta section for this map give a line bundle and a section on \( \widehat{\mathcal{C}}' \), denoted by \( \mathcal{O}(\mathcal{R}') \) and \( \mathcal{R}' \).
As in Section 3.9, \( \mathcal{O}(\hat{\mathcal{R}}') \) and \( \hat{\mathcal{R}}' \) pull back to \( \mathcal{O}(\mathcal{R}') \) and the divisor \( \mathcal{R}' \) on \( \mathcal{C}' \). Pulling back \( \mathcal{O}(\hat{\mathcal{R}}') \) and \( \hat{\mathcal{R}}' \) via \( i' \) we get the same divisor \( \mathcal{R}_L \) and line bundle \( \mathcal{O}(\mathcal{R}_L) \) on \( \text{Fl}_L \) as in Section 3.9 (by looking at the complex computing the pullback determinant line, for example).

Similar to Corollary 27, we have

**Corollary 48.**

1. \( \mathcal{O}(\hat{\mathcal{R}}') = \tau'^* \mathcal{O}(\mathcal{R}_L) \)
2. \( \hat{\mathcal{R}}' = \tau'^{-1} \mathcal{R}_L \) so that \( i' : \text{Fl}_L \setminus \mathcal{R}_L \to \hat{\mathcal{C}}' \setminus \hat{\mathcal{R}}' \).

The restricted flag setting has one new feature, which follows from Lemma 42 and Zariski’s main theorem:

**Lemma 49.** \( \pi' : \mathcal{C}' \setminus \mathcal{R}' \to \prod_{i=1}^{s}(G/Q_{w_i}) \) is an open immersion whose complement has codimension \( \geq 2 \).

**8. Picard groups**

**Corollary 50.** \( \pi'^*: \text{Pic}(\text{Fl}_G) \to \text{Pic}(\hat{\mathcal{C}}' \setminus \hat{\mathcal{R}}') \) is an isomorphism.

**Proof.** We need to compare the set of \( G \)-equivariant line bundles on \( \prod_{i=1}^{s}(G/Q'_{w_i}) \) and on the open subset \( U = \mathcal{C}' \setminus \mathcal{R}' \). If a \( G \) equivariant line bundle on \( \prod_{i=1}^{s}(G/Q'_{w_i}) \) becomes trivial on \( U \), then it is trivial as a line bundle on \( \prod_{i=1}^{s}(G/Q_{w_i}) \) (by Lemma 49) and hence trivial as a \( G \)-bundle.

To show surjectivity, extend a \( G \)-equivariant line bundle \( \mathcal{L} \) on \( U \) first as a line bundle to all of \( \prod_{i=1}^{s}(G/Q_{w_i}) \). We have isomorphisms \( \mathcal{L} \to \phi_Y^* \mathcal{L} \) on \( U \) (here \( \phi_Y \) is the action of \( G \) on \( \prod_{i=1}^{s}(G/Q_{w_i}) \)). These actions extend to all of \( \prod_{i=1}^{s}(G/Q_{w_i}) \), since a section of the line bundle \( \text{Hom}(\mathcal{L}, \phi_Y^* \mathcal{L}) \) on \( U \) will extend to the whole space by codimension considerations. \( \square \)

**Definition 51.** Let \( U \) be any non-empty open substack of \( \text{Fl}_L \). Let \( \text{Pic}^{\text{deg}=0}(U) \subset \text{Pic}(U) \) denote the subgroup of line bundles \( \mathcal{M} \) with \( \gamma_\mathcal{M} \) trivial (i.e., \( Z^0(L) \) acts trivially).

1. Let \( \text{Pic}^{\text{deg}=0}(\text{Fl}_G) \) denote the subgroup of line bundles whose pullback under \( \pi \) is in \( \text{Pic}^{\text{deg}=0}(\text{Fl}_L) \).
2. Let \( \text{Pic}^{\text{deg}=0}(\text{Fl}_G') \) denote the subgroup of line bundles whose pullbacks under the natural map \( \text{Fl}_G \to \text{Fl}_G' \) are in \( \text{Pic}^{\text{deg}=0}(\text{Fl}_G) \).
3. Finally, \( \text{Fl}_{Lss} \) denotes the stack parametrizing principal \( L^{ss} \)-spaces \( F \) together with elements \( \tilde{q}_i \in F/B_{L^{ss}} \); i.e., \( \text{Fl}_{Lss} = [(L^{ss}/B_{L^{ss}})^s/L^{ss}] \).

**8.1. Picard groups for the Levi subgroup**

**Lemma 52.**

1. \( \text{Pic}^L(L/B_L) = \text{Hom}(T, \mathbb{C}^*) = \text{Pic}(G/B) = \text{Pic}^G(G/B) \).
2. There is a surjection \( (\text{Pic}^L(L/B_L))^s \to \text{Pic}(\text{Fl}_L) \) whose kernel is the set of tuples \( (\mu_1, \ldots, \mu_s) \) (using (1)) such that \( \mu_i \) are trivial on \( T(L^{ss}) \), the maximal torus of \( L^{ss} \), and \( \sum \mu_i = 0 \) (i.e., given the triviality of \( \mu_i \) on \( T(L^{ss}) \), equivalent to \( \sum \mu_i(x_k) = 0 \) for \( \alpha_k \notin \Delta(P) \)).
Proof. For (1), note that $L$ equivariant line bundles on $L/B_L$ are in one-one correspondence with characters of $B_L$, which coincide with characters of $T$.

For (2), Pic$(Fl_L)$ is the set of (diagonal-) $L$-equivariant bundles on $(L/B_L)^s$. Therefore there is a map $(Pic^L(L/B_L))^s \to Pic(Fl_L)$. This is surjective because an $L$-equivariant line bundle on $(L/B_L)^s$, as a line bundle on $(L^{ss}/B_{L^{ss}})^s$, is of the form $L = L_{\mu_1} \boxtimes \cdots \boxtimes L_{\mu_s}$, where $\mu_i$ are characters of $T'$. These can be extended to characters of $T$ since $T = T' \times (\mathbb{C}^*)^a$, where $a$ corresponds to the number of simple coroots $\alpha_k^\vee$, $\alpha_k \notin \Delta(P)$, and we can view $L_{\mu_1} \boxtimes \cdots \boxtimes L_{\mu_s}$ as an element of $(Pic^L(L/B_L))^s$. This gives rise to a diagonal-$L$-equivariant line bundle on $(L/B_L)^s$, call it $L'$. As line bundles on $(L/B_L)^s$, $L$ and $L'$ coincide, and therefore as equivariant line bundles, differ by a character $\lambda$ of $L$. We replace $\mu_1$ by $\mu_1 + \lambda$ (and leave other $\mu_i$ unchanged) and then have $L = L' \in Pic(Fl_L)$.

The kernel of the map in (2) is identified similarly: A tuple $(L_{\mu_1}, \ldots, L_{\mu_s})$ which maps to zero gives the trivial line bundle on $(L^{ss}/B_{L^{ss}})^s$, hence $\mu_i$ are trivial restricted to $T(L^{ss})$. The center of $L$ should also under the diagonal action act trivially, so $\sum \mu_i$ is trivial on $Z^0(L)$. Now $L^{ss}$ and $Z^0(L)$ generate $L$, and we get (2). \qed

8.2. Comparison of Picard groups of flag varieties for $L$ and for $L^{ss}$

Lemma 53.

(1) The natural mapping (see Definition 51) $Pic^{deg=0}(Fl_L) \to Pic(Fl_{L^{ss}})$ is an injection of semigroups, which is an isomorphism $\otimes \mathbb{Q}$:

$$Pic^{deg=0}(Fl_L) \xrightarrow{\sim} Pic_{\mathbb{Q}}(Fl_{L^{ss}}).$$

(2) $Pic_{\mathbb{Q}}^+(Fl_L) \subseteq Pic_{\mathbb{Q}}^{deg=0}(Fl_L)$ and (27) gives a cone bijection

$$Pic_{\mathbb{Q}}^{+, deg=0}(Fl_L) \xrightarrow{\sim} Pic_{\mathbb{Q}}^+(Fl_{L^{ss}}).$$

Proof. The injection statement follows from Lemma 52. For the surjection, given a line bundle $L'$ on $Fl_{L^{ss}}$ we can find a line bundle $L$ on $Fl_L$ which maps to $L'$ under the natural restriction map. The action of $Z^0(L)$ may not be trivial, but we can tensor $L$ by a line bundle of the form $L_0 \boxtimes \mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O}$ where $\lambda = \sum a_k \varpi_k$, the sum taken over $\alpha_k \notin \Delta(P)$ but possibly with $\alpha_k \in \mathbb{Q}$, to make the action of $Z^0(L)$ trivial (note that $Z^0(L) \times L^{ss} \to L$ is an isogeny, so that any $\gamma$ in the dual of the Lie algebra of $Z^0(L)$ is the restriction of some element of $\mathfrak{h}_Q^*$ that vanishes on the Lie algebra of $T(L^{ss})$). This proves (1).

It is also easy to check that the map $Pic^{deg=0}(Fl_L) \to Pic(Fl_{L^{ss}})$ preserves global sections and (2) follows. quad \qed

9. The induction operation

Define

$$Ind_L^G : Pic(Fl_L \setminus \mathcal{R}_L) \xrightarrow{\sim} Pic(Fl_G)$$

as a composite of the isomorphism from Proposition 47(1) applied to $U = Fl_L \setminus \mathcal{R}_L$ (and Corollary 48):

$$Pic(Fl_L \setminus \mathcal{R}_L) \to Pic(\tilde{C}' \setminus \tilde{\mathcal{R}}')$$

and the inverse of the isomorphism of Corollary 50. Note that $(\tilde{i})^* Ind_L^G(\mathcal{M})$ is isomorphic to $\mathcal{M}$ (here $\mathcal{M} \in Pic(Fl_L \setminus \mathcal{R}_L)$).
Lemma 54. Using notation defined in Definition 51,

(a) The restriction mapping $\text{Pic}(\text{Fl}_L) \to \text{Pic}(\text{Fl}_L \setminus \mathcal{R}_L)$ is surjective.
(b) The restriction mapping $\text{Pic}^{\deg=0}(\text{Fl}_L) \to \text{Pic}^{\deg=0}(\text{Fl}_L \setminus \mathcal{R}_L)$ is surjective.
(c) The isomorphism $\text{Ind} : \text{Pic}(\text{Fl}_L \setminus \mathcal{R}_L) \xrightarrow{\sim} \text{Pic}(\text{Fl}_G)$ restricts to an isomorphism $\text{Ind} : \text{Pic}^{\deg=0}(\text{Fl}_L \setminus \mathcal{R}_L) \xrightarrow{\sim} \text{Pic}^{\deg=0}(\text{Fl}_G)$.

Proof. Part (c) follows from the definition and part (a) implies (b). For (a), we define a linear section as follows: a line bundle $\mathcal{M}$ on $\text{Fl}_L \setminus \mathcal{R}_L$ inducts to a line bundle $\mathcal{L}$ on $\text{Fl}_G$ which can then be restricted to $\text{Fl}_L$ via $\tilde{i}$; this is the desired lift: via $\pi'$, $\tilde{C}' \setminus \tilde{\mathcal{R}}'$ is an open substack of $\text{Fl}_G$. Therefore the pullback of $\mathcal{L}$ to $\tilde{C}' \setminus \tilde{\mathcal{R}}'$ is isomorphic to $\tau'^* \mathcal{L}$, and since $\tilde{i}' = \pi \circ i'$, and $\tau' \circ i'$ is the identity on $\text{Fl}_L$, the result follows. 

Recall that Lemma 52 gives a surjection from $(\text{Pic}^L(\text{L}/B_L))^s = \text{Pic}(G/B)^s$ to $\text{Pic}(\text{Fl}_L)$. Therefore a tuple $(\mu_1, \ldots, \mu_s)$ of weights for $G$ gives rise to an element of $\text{Pic}(\text{Fl}_L)$. The following theorem gives a formula for its image, under induction, in $\text{Pic}(\text{Fl}_G)$.

Theorem 55. The composite induction map

$$\text{Pic}(\text{Fl}_L) \to \text{Pic}(\text{Fl}_L \setminus \mathcal{R}_L) \xrightarrow{\sim} \text{Pic}(\text{Fl}_G) \subseteq \text{Pic}(\text{Fl}_G)$$

(30)

takes $(\mu_1, \ldots, \mu_s)$ to $(\lambda_1, \ldots, \lambda_s)$ where

$$(\lambda_1, \ldots, \lambda_s) = (w_1 \mu_1, w_2 \mu_2, \ldots, w_s \mu_s) - \sum_{j=1}^{s} \sum_{v} w_j \mu_j (\alpha_v^\vee) \mathcal{O}(D(j, v))$$

(31)

Here the sum is over $v \in W^P$, $j = 1, \ldots, s$ and simple roots $\alpha_v$ such that $v \frac{\alpha_v^\vee}{w_j}$.

9.1. The induction operation and global sections

Lemma 56. If $\mathcal{M} \in \text{Pic}^{\deg=0}(\text{Fl}_L \setminus \mathcal{R}_L)$, then

$$H^0(\text{Fl}'_G, \text{Ind}(\mathcal{M})) \xrightarrow{(\tilde{i}')^*} H^0(\text{Fl}_L \setminus \mathcal{R}_L, \mathcal{M})$$

(32)

is an isomorphism. Furthermore,

$$H^0(\text{Fl}'_G, \text{Ind}(\mathcal{M})) \xrightarrow{(\tilde{i})^*} H^0(\text{Fl}_L, \tilde{i}'^* \text{Ind}(\mathcal{M}))$$

(33)

is also an isomorphism.

We note that (33) also follows from results of Roth [Ro], and is therefore not new.

Proof. The first isomorphism (32) follows from Lemma 49 and Proposition 47:

$$H^0(\text{Fl}'_G, \text{Ind}(\mathcal{M})) = H^0(\tilde{C}' \setminus \tilde{\mathcal{R}}', \pi'^* \text{Ind}(\mathcal{M})) = H^0(\tilde{C}' \setminus \tilde{\mathcal{R}}', \tau'^* \mathcal{M}) = H^0(\text{Fl}_L \setminus \mathcal{R}_L, \mathcal{M})$$

(32) factors through the inclusion $H^0(\text{Fl}_L, \tilde{i}'^* \text{Ind}(\mathcal{M})) \subseteq H^0(\text{Fl}_L \setminus \mathcal{R}_L, \mathcal{M})$ which gives (33). ⊐
Remark 10. Suppose $\mathcal{M} \in \text{Pic}^{\deg=0}(\text{Fl}_L)$, then in general $\tilde{i}^* \text{Ind}(\mathcal{M})$ and $\mathcal{M}$ may be different. They are identified on $\text{Fl}_L \setminus \mathcal{R}_L$ (even without the condition of action on center on $\mathcal{M}$).

Remark 11. Taking $\mathcal{M} = \mathcal{O}$ in (32), we see that $h^0(\text{Fl}_L \setminus \mathcal{R}_L, \mathcal{O}) = h^0(\text{Fl}'_G, \mathcal{O}) = 1$, and hence we recover the generalization of Fulton’s conjecture proved in [BKR] (this proof is not really a different proof).

Now note that $\text{Pic}^+(\text{Fl}_L) \subseteq \text{Pic}^{\deg=0}(\text{Fl}_L)$.

Lemma 57.

(1) The restriction mapping $\text{Pic}^+(\text{Fl}_L) \to \text{Pic}^+(\text{Fl}_L \setminus \mathcal{R}_L)$ is surjective, with a linear section.

(2) The isomorphism (29) restricts to an isomorphism $\text{Ind} : \text{Pic}^+(\text{Fl}_L \setminus \mathcal{R}_L) \simeq \text{Pic}^{+,\deg=0}(\text{Fl}'_G)$.

Proof. For (1), the lift is just $e_i^* \text{Ind}(\mathcal{M})$ as in Lemma 56, and (1) follows from (32) and (33). (2) follows from (47), applied to $U = \text{Fl}_L \setminus \mathcal{R}_L$, and Lemma 49.

Theorem 55 has the following corollary:

Corollary 58. The induction map (30) restricts to a surjection

$$\text{Pic}^+(\text{Fl}_L) \to \text{Pic}^+(\text{Fl}_L \setminus \mathcal{R}_L) \simeq \text{Pic}^{\deg=0}(\text{Fl}'_G).$$

9.2. Proof of Theorem 12

Under the identification of $\text{Pic}_Q(\text{Fl}_G)$ with $h^*_Q$, it is easy to see that $\text{Pic}_Q(\text{Fl}'_G)$ corresponds to tuples $(h_1, \ldots, h_s)$ such that $\alpha_m(h_j) = 0$ whenever $(j, v)$ is such that $v \alpha_j \rightarrow w_j$, and $\text{Pic}_Q^{+,\deg=0}(\text{Fl}'_G)$ to the face $\mathcal{F}_{II,Q}$. Similarly, $\text{Pic}_Q^+(\text{Fl}_{L^ss}) = \Gamma(s, \mathcal{L}_{L^ss})$ under the Killing form isomorphism $h^*_{L^ss} \rightarrow h_{L^ss}$ induced from $G$.

Theorem 12 follows immediately from Corollary 58 and the following lemma,

Lemma 59. The Killing form isomorphism takes $h^*_{L^ss} \subseteq h$ to $(h^*)^{\deg=0}$, the set of $\lambda$ such that $\lambda(x_i) = 0$ for all $\alpha_i \notin \Delta(P)$.

Proof. Let $I$ be the set of indices $i$ such that $\alpha_i \notin \Delta(P)$. Observe that

$$\lambda \in (h^*)^{\deg=0} \iff \lambda(x_i) = 0 \forall i \in I \iff \omega_i(\kappa(\lambda)) = 0 \forall i \in I \iff \kappa(\lambda) = \sum_{k \notin I} c_k \alpha_k^\vee \text{ for suitable } c_k \iff \kappa(\lambda) \in h^*_{L^ss};$$

the lemma is therefore proved. □

9.3. Proof of Theorem 55

Denote the right-hand side of (31) by $(\nu_1, \ldots, \nu_s)$. We divide the proof into steps:

(1) We first verify that $(\nu_1, \ldots, \nu_s)$ is indeed in $\text{Pic}(\text{Fl}'_G)$, since a priori it is only in $\text{Pic}(\text{Fl}_G)$. Consider a pair $(j, v)$ with $v \alpha_j \rightarrow w_j$. We want $\nu_j(\alpha_i^\vee) = 0$. This vanishing follows immediately from Corollary 34.
(2) Next, we verify that the line bundle $N$ given by $(\nu_1, \ldots, \nu_s)$ agrees with $\text{Ind}(L)$, on $C \setminus R$, considered an open subset of $F\ell_G$ where $L$ is the line bundle on $F\ell_u$ given by $(\mu_1, \ldots, \mu_s)$. To do this we only need to show, by Lemma 25, that the pullbacks to $F\ell_L \setminus R_L$ via $i$ agree. By Lemma 17, the line bundles $O(D(j, v))$ pull back to trivial line bundles on $F\ell_L \setminus R_L$. It is now easy to verify the desired agreement.

(3) Therefore one has a relation $N = \text{Ind}(L)(D)$ on $F\ell_G$ with $D$ supported on the complement of the image of $C \setminus R$, i.e., a sum of divisors $D(j, v)$.

(4) But such a sum of divisors $D$ needs to be zero because both sides are in $\text{Pic}(F\ell_G)$, and Corollary 34. This concludes the proof of Theorem 55.

We note that steps (2) and (3) are very similar to Ressayre’s proof [Re1] of the irredundancy of inequalities (2) for maximal parabolics $P$. The divisors analogous to $D$ in op. cit. are not determined. We are able to determine it (as zero) because of the enumerative computations of Theorem (8) (as in Corollary 34).

**Corollary 60.** If $\alpha_k \notin \Delta(P)$, then induction of the various $(0, \ldots, 0, \omega_k, 0, \ldots, 0)$ in the $j$th place coincide, i.e., the following elements of $\text{Pic}(F\ell_G) \subseteq \text{Pic}(F\ell_L)$ are the same:

$$(0, \ldots, 0, w_j \omega_k, 0, \ldots, 0) - \sum_v w_j \omega_k(\alpha^\vee_k)O(D(j, v)).$$

Moreover, this element fails the inequality defining $F$ and is thus not a member of $F(s, G)$.

**Proof.** This last claim is seen from the following: each $O(D(j, v))$ appearing in the sum vanishes on the inequality for $F$ by Theorem 7(d), and

$$w_j^{-1}(w_j \omega_k)(x_k) = \omega_k(x_k) > 0,$$

since $\omega_k(x_k) = c(\omega_k, \omega_k)$ for some $c > 0$. □

**Remark 12.** The quantity $w_j \mu_j(\alpha^\vee_\ell)$ in (31) is $\leq 0$ if $\mu_j$ is dominant:

$$w_j \mu_j(\alpha^\vee_\ell) = \frac{2}{(\alpha_\ell, \alpha_\ell)}(w_j \mu_j, \nu_\ell) = \frac{2}{(\alpha_\ell, \alpha_\ell)}(\mu_j, w_j^{-1} \alpha_\ell)$$

but $w_j^{-1} \alpha_\ell \in R^-$.  

If $(\mu_1, \ldots, \mu_s)$ is a tuple of dominant weights for $G$ then the $\lambda_j$ in (31) are also dominant: We compute $\lambda_j(\alpha^\vee_\ell)$: If $w_j \mu_j(\alpha^\vee_\ell) \geq 0$, then there is nothing to show, given the non-negativity from the previous paragraph. If $w_j \mu_j(\alpha^\vee_\ell) < 0$ then clearly $w^{-1} \alpha_\ell \in R^-$, and the divisor $D(j, v)$ with $v = s_\alpha w_j$ appears in the sum, and hence $\lambda_j(\alpha^\vee_\ell) = 0$.

We note however that an element in $\text{Pic}^+(F\ell_L)$ may not necessarily be representable by a tuple $(\mu_1, \ldots, \mu_s)$ of dominant weights for $G$, and it is therefore a nontrivial consequence of our results that the formulas for induction of elements in $\text{Pic}^+(F\ell_L)$ produce tuples of dominant weights $(\lambda_1, \ldots, \lambda_s)$ for $G$ in (31).  

**Remark 13.** By [BK1, §3], $O(R_L)$ is the line bundle on $F\ell_L$ given by (see Lemma 52(2)) the $s$-tuple $(\chi_{w_1} - \chi_{e}, \chi_{w_2}, \ldots, \chi_{w_s})$ where $\chi_w = \rho - 2 \rho^L + w^{-1} \rho$. Here $\rho$ (resp. $\rho^L$) is the half sum of roots in $R^+$ (resp. in $R^+_1$). Since the induction of $O(R_L)$ is zero, we get an interesting relation from (31).
10. Related results

Let $P$ and the $w_j$ be as earlier, and let

$$\mathcal{H} = \left\{ (h_1, \ldots, h_s) \in \mathfrak{h}_Q^s \mid \sum_{j=1}^s w_k(w_j^{-1}h_j) = 0, \quad \alpha_k \not\in \Delta(P) \right\} \subseteq \mathfrak{h}_Q^s. \tag{34}$$

Lemma 61. $\mathcal{H} \subseteq \mathfrak{h}_Q^s$ is of codimension $|\Delta \setminus \Delta(P)|$, and not contained in any root hyperplane $\alpha_m(h_j) = 0$.

Proof. We need to show that these equations are linearly independent: If fewer equations cut out the same set, it will also be the case if we restrict to $h_1 = \cdots = h_s = 0$. Therefore the equations $w_k(h) = 0$ where $\alpha_k \not\in \Delta(P)$ are linearly dependent which is clearly false. Finally, $\mathcal{H}$ is not contained in any root hyperplane because it contains points of the form $(w_1h, -w_2h, \ldots, 0)$, $h$ arbitrary. \hfill $\square$

Clearly $\mathcal{F}_Q \subseteq \mathcal{H}$, and we will show that it generates $\mathcal{H}$ as a vector space. This will show that $\mathcal{F}_Q$ is a regular face of codimension $|\Delta \setminus \Delta(P)|$, a result first proved by Ressayre [Re2].

To do this we define

$$\mathcal{H}[2] = \{(h_1, \ldots, h_s) \in \mathcal{H} \mid \beta(h_j) = 0, \forall (j, v), v \mapsto w_j, \beta \in \Delta, v \in W^P \}. \tag{35}$$

Clearly $\mathcal{F}_{11, Q} = \mathcal{H}[2] \cap \mathcal{F}_Q = \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_G')$, and parallel to $\mathcal{F}_Q = Q_2^{\geq 0} \times \mathcal{F}_{11, Q}$ we have a decomposition $\mathcal{H} = Q_2^{\geq 0} \times \mathcal{H}[2]$, therefore it suffices to show that $\mathcal{F}_{11, Q}$ generates $\mathcal{H}[2]$. Therefore we need to show that $\text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_G')$ is generated by $\text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_G')$.

There are surjections (Lemmas 54 and 57),

$$\text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L) \twoheadrightarrow \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_G'), \quad \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L) \twoheadrightarrow \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_G'),$$

and identifications (Lemma 53),

$$\text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L) = \text{Pic}_{\mathfrak{h}_Q^s}^{+}(\text{Fl}_L^\text{ss}), \quad \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L) = \text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L^\text{ss}).$$

The desired statement that $\text{Pic}_{\mathfrak{h}_Q^s}^{+}(\text{Fl}_L^\text{ss})$ generates $\text{Pic}_{\mathfrak{h}_Q^s}^{\text{deg}=0}(\text{Fl}_L^\text{ss})$, now follows by reducing to simple factors, and the known fact that $\Gamma_Q(s, G)$ generates $\mathfrak{h}_Q^s$ ($s \geq 3$) for simple, simply connected $G$.

10.1. Irreducible components

Consider the inverse image $\mathcal{R}_{(L/B_L)^s} \subseteq (L/B_L)^s$ of the divisor $\mathcal{R}_L \subseteq \text{Fl}_L$. Let $c$ be the number of irreducible components of $\mathcal{R}_{(L/B_L)^s}$. Let $\mathcal{R}_1, \ldots, \mathcal{R}_c$ be the irreducible (reduced) components of this divisor. It is easy to see that each is left invariant by the connected group $L$, and hence they give line bundles $\mathcal{O}(\mathcal{R}_1), \ldots, \mathcal{O}(\mathcal{R}_c)$ on $\text{Fl}_L$. Since the space $H^0(\text{Fl}_L \setminus \mathcal{R}_L, \mathcal{O})$ is one-dimensional, we have that $H^0(\text{Fl}_L, \bigotimes_{i=1}^c \mathcal{O}(\mathcal{N}_i \mathcal{R}_i))$ is one-dimensional for $N_i > 0$, $i = 1, \ldots, c$.

Lemma 62. $\mathcal{O}(\mathcal{R}_1), \ldots, \mathcal{O}(\mathcal{R}_c)$ give a $\mathbb{Z}$-basis for ker $(\text{Pic}(\text{Fl}_L) \rightarrow \text{Pic}(\text{Fl}_L \setminus \mathcal{R}_L))$. 

Proof. It is easy to see that the line bundles $\mathcal{O}(\mathcal{R}_1), \ldots, \mathcal{O}(\mathcal{R}_c)$ are in the kernel. These line bundles are linearly independent because any isomorphism of line bundles $\mathcal{O}(\sum_{i \in I} a_i \mathcal{R}_i) = \mathcal{O}(\sum_{j \in J} b_j \mathcal{R}_j)$, with $I, J$ disjoint and $a_i > 0$ and $b_j > 0$, produces two linearly independent sections in the isomorphic line bundles.

They span, because if $\mathcal{L} \in \text{Pic}(\text{Fl}_L)$ maps to zero, then first the action of $Z^0(L)$ is trivial, and hence we have to show that $\mathcal{L}$ is isomorphic to a linear combination of the pullbacks of the line bundles $\mathcal{O}(\mathcal{R}_i)$, when pulled back to $(L/B_L)^s$, without any equivariance conditions. Let $\sigma$ be a non-zero section of $\mathcal{L}$ on $\text{Fl} \setminus \mathcal{R}_L$; clearly the pullback of $\sigma$ to $(L/B_L)^s$ has an associated divisor supported on the union of $\mathcal{R}_i \subseteq (L/B_L)^s$ which completes the argument. \hfill \box

Proposition 63. $c = q - (s - 1)|\Delta \setminus \Delta(P)|$. Recall that $q$ is the number of divisors $D(j, v)$.

Proof. We count dimensions in the isomorphism

$$\text{Pic}^\text{deg}=0_Q(\text{Fl}_L \setminus \mathcal{R}_L) = \text{Pic}^\text{deg}=0_Q(\text{Fl}_G').$$

The right-hand side has dimension $\dim \mathcal{H} - q = s \cdot |\Delta| - |\Delta \setminus \Delta(P)| - q$.

Using the surjection $\text{Pic}^\text{deg}=0_Q(\text{Fl}_L) \to \text{Pic}^\text{deg}=0_Q(\text{Fl}_L \setminus \mathcal{R}_L)$, we see that the left-hand side has dimension equal to $\dim \text{Pic}^\text{deg}=0_Q(\text{Fl}_L) - c = s \cdot |\Delta| - s |\Delta \setminus \Delta(P)| - c$ (see Lemma 57(2)). The result follows. \hfill \box

Lemma 64. $\mathcal{O}(\mathcal{R}_1), \ldots, \mathcal{O}(\mathcal{R}_c)$ give (some) extremal rays of $\text{Pic}^+_Q,\text{deg}=0(\text{Fl}_L)$.

Proof. This follows the same method of proof as of Theorem 7, (b) implies (c), using the fact noted above that $H^0(\text{Fl}_L, \mathcal{O}(N\mathcal{R}_i))$ is one-dimensional if $N \geq 0$ for any $i$. \hfill \box

10.2. The face $\mathcal{F} = \mathcal{F}(\mathbf{w}, P)$ when $P = B$

Clearly $L^{ss} = \{e\}$ when $P = B$ and hence $\mathcal{F} \cap \mathcal{Q}_0 = 0$, and $\mathcal{F}_Q$ is the cone spanned by the linearly independent $\delta_1, \ldots, \delta_q$. Therefore the dimension of $\mathcal{F}_Q$ is $q$, while at the same time it is $sr - r = (s - 1)r$, $r = |\Delta|$, and $c = 0$.

Corollary 65. The regular faces of $\Gamma(s, G)$ of codimension $|\Delta|$ (the maximum possible) are simplicial cones.

Remark 14. The following types of surjection of cones $\mathcal{C} \to \overline{\mathcal{C}}$ can be considered special: Let $V = \mathbb{Q}^n$, and $\mathcal{C} \subseteq V$ a (spanning) cone that has the basis vectors $e_1, \ldots, e_c$ among its extremal rays. Let $V \to \mathbb{Q}^{n-c}$ be the projection to the remaining $n - c$ coordinates. Let $\overline{\mathcal{C}} \subseteq \mathbb{Q}^{n-c}$ be the image of $\mathcal{C}$.

The surjection of cones (11) is of the above special type (take $V = \text{Pic}^\text{deg}=0_Q(\text{Fl}_L)$, and $e_1, \ldots, e_c$ the elements $\mathcal{O}(\mathcal{R}_1), \ldots, \mathcal{O}(\mathcal{R}_c)$, using the bijection of Proposition 4, and Lemma 53).

Under the bijection of Proposition 4, the surjection of cones (11) becomes $\text{Pic}^+_Q,\text{deg}=0(\text{Fl}_L) \to \text{Pic}^+_Q,\text{deg}=0(\text{Fl}_G')$. We note that this has a section arising from Lemma 57.
11. Examples

In the following, we first examine several facets of the $D_4$ tensor cone ($s = 3$), producing type I and type II rays according to the formulas given earlier. All rays produced here can also be found in the (complete up to symmetrization) list of 81 extremal rays for $D_4$ in [KKM]. In fact, all 81 extremal rays are type I on some face. Type I extremal rays, under the bijection of Proposition 4, have the property that any multiple has an exactly one-dimensional space of invariant global sections, see Theorem 7, (b). There are examples in type A in [B3], due to Derksen–Weyman [DW, Example 7.13] and Ressayre, of extremal rays for $SL(8)$ and $SL(9)$ respectively, which do not have this property, and give examples of extremal rays which are not type I on any face. There are similar examples which do not have this property for $D_5$ in [K]. We also include an exploration of the simpliciality of $F_{\Pi, \mathbb{Q}}$ for a couple facets of the $A_3$ tensor cone with $s = 3$.

Some rudimentary computer code, written using the free math software Sage [Sag], was used to find tuples $(u, v, w, P)$ giving rise to facets and to implement the formulas found in Theorem 8 and Theorem 55. See [K] for further details of these computer algorithms.

11.1. A face coming from $P_2$

Let $G$ be of type $D_4$, with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and corresponding simple reflections $s_i$. Let $P = P_2$ and $u, v, w$ be specified as in the example in 1.6. On the corresponding face $F$, there are 7 type I extremal rays, generated by:

- $(\varpi_1, \varpi_4, \varpi_3)$,
- $(\varpi_3, 0, \varpi_3)$ and $(\varpi_4, \varpi_4, 0)$,
- $(\varpi_2, \varpi_3, \varpi_3)$ and $(\varpi_2, \varpi_4, \varpi_4)$,
- $(\varpi_2, \varpi_1 + \varpi_4, \varpi_3)$ and $(\varpi_2, \varpi_4, \varpi_1 + \varpi_3)$.

One may note that under the operation of switching entries 2 and 3 (i.e., $(u, v, w)$ becomes $(u, w, v)$, $(\lambda, \mu, \nu)$ becomes $(\lambda, \nu, \mu)$) while simultaneously switching indices 3 and 4 (on all simple roots, fundamental dominant weights, simple reflections; this is a Dynkin diagram automorphism), the specific $(u, v, w, P)$ of the example remains unchanged. Therefore the face $F$ is also invariant under the induced cone automorphism; the above type I rays are listed in pairs according to this (order 2) automorphism (the first is fixed).

The induction map gives the following four type II rays:

- $(\varpi_2, \varpi_2, 0)$ and $(\varpi_2, 0, \varpi_2)$,
- $(\varpi_2, \varpi_2, 2\varpi_3)$ and $(\varpi_2, 2\varpi_4, \varpi_2)$,

again given in pairs. The Levi associated to $P$ is of type $A_1 \times A_1 \times A_1$. The tensor cone for type $A_1$ is generated (over $\mathbb{Z}$ as well as over $\mathbb{Q}$) by three extremal rays: $(\varpi, \varpi)$ and its two permutations, where $\varpi$ is the single dominant fundamental weight. The dominant fundamental weights for $L$ are $\varpi_1$, $\varpi_3$, and $\varpi_4$, each representing a copy of $A_1$. Extremal rays for the type $A_1 \times A_1 \times A_1$ subcone are therefore given by the permutations of $(\varpi_i, \varpi_i, 0)$, where $i$ runs through 1, 3, 4, yielding a total of 9.
These 9 rays are shifted by a multiple of $\varpi_2$ in each entry so that the result evaluates to 0 against $x_2$ (in each entry); i.e., each ray is shifted to become degree 0 (see Lemma 59). The formula for induction (31) is then applied, with the following results:

\[
\begin{align*}
(\varpi_1, \varpi_1, 0) &\mapsto \vec{0}, & (\varpi_3, \varpi_3, 0) &\mapsto \vec{0}, \\
(\varpi_4, \varpi_4, 0)(\varpi_2, \varpi_2, 0), & & (0, \varpi_1, \varpi_1) &\mapsto \vec{0}, \\
(0, \varpi_3, \varpi_3) &\mapsto (\varpi_2, 2\varpi_4, \varpi_2), & (0, \varpi_4, \varpi_4) &\mapsto (\varpi_2, \varpi_2, 2\varpi_3), \\
(\varpi_1, 0, \varpi_1) &\mapsto \vec{0}, & (\varpi_3, 0, \varpi_3) &\mapsto (\varpi_2, 0, \varpi_2), \\
(\varpi_4, 0, \varpi_4) &\mapsto \vec{0}.
\end{align*}
\]

These 11 rays are indeed all of the extremal rays on $\mathcal{F}$. Notice that $c = \#$ irreducible components of $\mathcal{R}_L = 5$, the number of extremal rays going to 0 under induction. Here $q = 7, s = 3$, and $|\Delta \setminus \Delta(P)| = 1$, so $5 = 7 - (3 - 1)(1)$ illustrates Proposition 63.

Finally, in this example, any extremal ray for $L$ which does not go to $\vec{0}$ is induced to a type II ray. This is not always the case:

11.2. Illustration of Corollary 60

Maintaining $P = P_2$ and $u, v, w$ as above, we examine the induction operation (without any shifting) applied to $(u \cdot \varpi_2, 0, 0), (0, v \cdot \varpi_2, 0),$ and $(0, 0, w \cdot \varpi_2)$.

First $(u \cdot \varpi_2, 0, 0)$: one may check that $u \cdot \varpi_2 = 2\varpi_2 - \varpi_1 - \varpi_3 - \varpi_4$,

\[
s_4s_3s_1s_2(\epsilon_1 + \epsilon_2) = s_4s_3s_1(\epsilon_1 + \epsilon_3) = s_4s_3(\epsilon_2 + \epsilon_3) = s_4(\epsilon_2 + \epsilon_4) = \epsilon_2 - \epsilon_3,
\]

and indeed

\[
2\varpi_2 - \varpi_1 - \varpi_3 - \varpi_4 = 2(\epsilon_1 + \epsilon_2) - \epsilon_1 - \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) - \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)
\]

\[
= \epsilon_2 - \epsilon_3.
\]

The type I rays and coefficients coming from divisors $D(j, v)$ for $j = 1$ are:

| $\ell$ | $\mathcal{O}(D(j, v))$ | $u \cdot \varpi_2(\alpha_\ell^v)$ |
|-------|----------------------|-----------------|
| 1     | $(\varpi_1, \varpi_4, \varpi_3)$ | $-1$ |
| 3     | $(\varpi_3, 0, \varpi_3)$ | $-1$ |
| 4     | $(\varpi_4, \varpi_4, 0)$ | $-1$ |

Therefore $(u \cdot \varpi_2, 0, 0)$ is mapped to

\[
(2\varpi_2 - \varpi_1 - \varpi_3 - \varpi_4, 0, 0) + (\varpi_1, \varpi_4, \varpi_3) + (\varpi_3, 0, \varpi_3) + (\varpi_4, \varpi_4, 0)
\]

\[
= (2\varpi_2, 2\varpi_4, 2\varpi_3),
\]

and one may check that $(u^{-1} \cdot 2\varpi_2 + v^{-1} \cdot 2\varpi_4 + w^{-1} \cdot 2\varpi_3)(x_2) = 2 \leq 0$, so this induced triple is not in the cone.

Second $(0, v \cdot \varpi_2, 0): v \cdot \varpi_2 = -\varpi_1 - \varpi_3 + \varpi_4$. The type I rays and coefficients coming from divisors $D(j, v)$ with $j = 2$ are

| $\ell$ | $\mathcal{O}(D(j, v))$ | $v \cdot \varpi_2(\alpha_\ell^v)$ |
|-------|----------------------|----------------|
| 1     | $(\varpi_2, \varpi_1 + \varpi_4, \varpi_3)$ | $-1$ |
| 3     | $(\varpi_2, \varpi_3, \varpi_3)$ | $-1$ |
Therefore \((0, v \cdot \varpi_2, 0)\) is mapped to

\[
(0, -\varpi_1 - \varpi_3 + \varpi_4, 0) + (\varpi_2, \varpi_1 + \varpi_4, \varpi_3) + (\varpi_2, \varpi_3, \varpi_3) = (2\varpi_2, 2\varpi_4, 2\varpi_3)
\]
as well.

Finally \(w \cdot \varpi_2 = -\varpi_1 + \varpi_3 - \varpi_4\). The type I rays and coefficients coming from divisors \(D(j, v)\) with \(j = 3\) are

\[
\begin{array}{ccc}
\ell & O(D(j, v)) & w \cdot \varpi_2(\alpha_i^\vee) \\
1 & (\varpi_2, \varpi_4, \varpi_1 + \varpi_3) & -1 \\
4 & (\varpi_2, \varpi_4, \varpi_4) & -1 \\
\end{array}
\]

Therefore \((0, 0, w \cdot \varpi_2)\) is mapped to

\[
(0, 0, -\varpi_1 + \varpi_3 - \varpi_4) + (\varpi_2, \varpi_4, \varpi_1 + \varpi_3) + (\varpi_2, \varpi_4, \varpi_4) = (2\varpi_2, 2\varpi_4, 2\varpi_3)
\]
yet again.

11.3. The faces coming from \(P_4\)

Take \(P = P_4\). The Levi associated to \(P\) is of type \(A_3\), whose tensor cone is generated by 18 extremal rays: \((\varpi_1, \varpi_3, 0), (\varpi_2, \varpi_2, 0), (\varpi_2, \varpi_3, \varpi_3), (\varpi_2, \varpi_2, \varpi_1 + \varpi_3), (\varpi_2, \varpi_1, \varpi_1),\) and permutations: To apply induction, we want to get them in deg = 0 part of \(\text{Pic}(\mathcal{F}L)\) (as in Section 11.1). We shift each entry by a multiple of \(\varpi_4\) so that the result evaluates to 0 against \(x_4\) (in each entry), see Lemma 59.

It is possible for an induced ray to be non-zero and non-extremal (call such a ray “exotic”); this happens on several faces arising from \(P_4\). For instance, on the face \(\mathcal{F}(s_2s_4, s_3s_1s_2s_4, s_4s_2s_3s_1s_2s_4, P_4)\), the extremal ray \((\varpi_2, \varpi_2, \varpi_1 + \varpi_3)\) for \(A_3\) is induced to \((\varpi_1 + \varpi_3 + \varpi_4, \varpi_2 + \varpi_4, \varpi_1 + \varpi_3)\), which is not an extremal ray for \(\mathcal{F}\) because it can be expressed as the sum of two distinct extremal rays of \(\mathcal{F}\):

\[
(\varpi_1 + \varpi_3 + \varpi_4, \varpi_2 + \varpi_4, \varpi_1 + \varpi_3) = (\varpi_1 + \varpi_4, \varpi_2, \varpi_3) + (\varpi_3, \varpi_4, \varpi_1).
\]

The following table summarizes some characteristics of the 7 faces (up to symmetrization) coming from \(P_4\):

| Weyl triple                      | \(q\) | \(c\) | exotic | total rays \((q + 18 - c - e)\) |
|----------------------------------|-------|-------|--------|-------------------------------|
| \((1, s_4s_2s_3s_1s_2s_4, s_4s_2s_3s_1s_2s_4)\) | 2     | 0     | none   | 20                            |
| \((s_4, s_2s_3s_1s_2s_4, s_4s_2s_3s_1s_2s_4)\) | 3     | 1     | none   | 20                            |
| \((s_2s_4, s_3s_1s_2s_4, s_4s_2s_3s_1s_2s_4)\) | 4     | 2     | 1      | 19                            |
| \((s_2s_4, s_2s_3s_1s_2s_4, s_2s_3s_1s_2s_4)\) | 3     | 1     | 6      | 14                            |
| \((s_3s_2s_4, s_1s_2s_4, s_4s_2s_3s_1s_2s_4)\) | 3     | 1     | 1      | 19                            |
| \((s_3s_2s_4, s_3s_1s_2s_4, s_2s_3s_1s_2s_4)\) | 4     | 2     | 3      | 17                            |
| \((s_1s_2s_4, s_3s_1s_2s_4, s_2s_3s_1s_2s_4)\) | 4     | 2     | 3      | 17                            |
11.4. Examples where $\mathcal{F}_{II,Q}$ is simplicial, or non-simplicial of positive dimension

It follows from simple computations that $\Gamma_Q(3, SL(2))$ is a 3-dimensional cone with 3 extremal rays $(\varpi_1, \varpi_1, 0)$ and permutations. Similarly, $\Gamma_Q(3, SL(3))$ is a 6-dimensional cone with 8 extremal rays $(\varpi_1, \varpi_2, 0)$ and permutations, $(\varpi_1, \varpi_1, \varpi_1)$ and $(\varpi_2, \varpi_2, \varpi_2)$.

Let $G = SL(4)$ and $P$ the parabolic corresponding to the Grassmannian $Gr(2, 4)$. Let $w_1, w_2, w_3$ correspond to the classes of a divisor, divisor and of subspaces contained in a fixed 3-dimensional subspace respectively. There are 5 type I extremal rays and so $\mathcal{F}_{II,Q}$ is $3(3) - 1 - 5 = 3$-dimensional. By Proposition 63, the ramification divisor has $5 - 2 = 3$ connected components. Since $L^{ss} = SL(2) \times SL(2)$, $\Gamma_Q(s, L^{ss})$ is simplicial of dimension 6. By Lemmas 62, 64, and Theorem 12, $\mathcal{F}_{II,Q}$ is a cone of dimension 3 generated by 6 extremal rays, and is hence simplicial. The reason why this example worked out was because $L^{ss} = SL(2) \times SL(2)$, and hence $\Gamma_Q(3, L^{ss})$ is simplicial.

Now let $G = SL(4)$ again and $P$ be a maximal parabolic so that $G/P \simeq \mathbb{P}^3$. If we consider the case of $(w_1, w_2, w_3)$ corresponding to classes ([pt], $[\mathbb{P}^3]$, $[\mathbb{P}^3]$), we see that there are two type I extremal rays, and hence the ramification divisor is empty (Proposition 63, or by a direct inspection). The face $\mathcal{F}_{II,Q}$ is therefore isomorphic to $\Gamma_Q(s, L^{ss})$. Now $L^{ss} = SL(3)$, and $\Gamma_Q(s, SL(3))$ has 8 extremal rays, and the cone is 6-dimensional. Therefore $\mathcal{F}_{II,Q}$ is not simplicial. In fact any 7 of the generators generate a vector space of dimension 6, and hence $\mathcal{F}_{II,Q}$ cannot be decomposed as a product of a ray and a cone.

As pointed out by one of the reviewers, it is an interesting problem to determine conditions under which $\mathcal{F}_{II,Q}$ can be decomposed further. The answer will be subtle, as the above examples show.

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