ABSTRACT. In this work, we address some optimal control problems related to the evolution of two isothermal, incompressible, immisible fluids in a two dimensional bounded domain. A distributed optimal control problem is formulated as the minimization of a suitable cost functional subject to the controlled nonlocal Cahn-Hilliard-Navier-Stokes equations. We prove the existence of an optimal control and then establish the Pontryagin’s maximum principle for optimal control of such systems, which gives the first-order necessary conditions of optimality. We characterize the optimal control using the adjoint variable.

Key words: optimal control, Nonlocal Cahn-Hilliard-Navier-Stokes systems, Pontryagin’s maximum principle, data assimilation problem.

Mathematics Subject Classification (2010): 49J20, 35Q35, 76D03.

1. Introduction

We consider the evolution of two isothermal, incompressible, immisible fluids in a bounded domain $\Omega$ subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. The average velocity of the fluid is denoted by $u(x,t)$ and the relative concentration of one fluid is denoted by $\phi(x,t)$, for $(x,t) \in \Omega$. 

Acknowledgments: Tania Biswas would like to thank the Indian Institute of Science Education and Research, Thiruvananthapuram, for providing financial support and stimulating environment for the research. M. T. Mohan would like to thank the Indian Statistical Institute (ISI) Bangalore Center, for providing stimulating scientific environment and resources.
A general model for such a system is known as nonlocal Cahn-Hilliard-Navier-Stokes system in \( \Omega \times (0, T) \) and is given by

\[
\begin{align*}
\varphi_t + \mathbf{u} \cdot \nabla \varphi &= \text{div}(m(\varphi)\nabla \mu), \\
\mu &= a\varphi - J * \varphi + F'(\varphi), \\
\mathbf{u}_t - 2\text{div}(\nu(\varphi)\mathbf{D}\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi &= \mu \nabla \varphi + \mathbf{h}, \\
\text{div} \mathbf{u} &= 0, \\
\frac{\partial \mu}{\partial n} &= 0, \mathbf{u} = 0 \text{ on } \partial\Omega \times (0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \text{ in } \Omega,
\end{align*}
\]

(1.1)

where \( m \) is the mobility parameter, \( \mu \) is the chemical potential, \( \pi \) is the pressure, \( J \) is the spatial-dependent internal kernel, \( J * \varphi \) denotes the spatial convolution over \( \Omega \), \( a \) is defined by \( a(x) := \int_\Omega J(x - y)dy \), \( F \) is a double well potential, \( \nu \) is the kinematic viscosity and \( \mathbf{h} \) is the external forcing term acting in the mixture. (see [17]). Also, \( \mathbf{D}\mathbf{u} \) is the symmetric part of the gradient of the flow velocity vector, i.e., \( \mathbf{D}\mathbf{u} \) is the strain tensor \( \frac{1}{2}((\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)) \).

The density is supposed to be constant and is equal to one (i.e., matched densities). The system (1.1) is called nonlocal because of the term \( J \), which is averaged over the spatial domain. Various simplified models of this system are studied by several mathematicians and physicists. The local version of the system is obtained by replacing \( \mu \) equation by \( \mu = \Delta \varphi + F'(\varphi) \). Another simplification appeared in the literature is to assume the constant mobility parameter and/or constant viscosity. From the mathematical point of view, the nonlocal version is physically more relevant and mathematically challenging too. This model is more difficult to handle because of the nonlinear terms like the capillarity term (i.e., Korteweg force) \( \mu \nabla \varphi \) acting on the fluid. Even in two dimensions, this term can be less regular than the convective term \( (\mathbf{u} \cdot \nabla)\mathbf{u} \) (see [5]).

We now discuss some of the works available in literature for the solvability of the system (1.1) and also the simplified Cahn-Hilliard-Navier-Stokes models. In [4] author studies local Cahn Hilliard Navier Stokes system and establishes existence and uniqueness in dimension 2 and 3. In [5], the authors proved the existence of a weak solution for nonlocal Cahn-Hilliard-Navier-Stokes system with mobility parameter equal to one and constant viscosity. The uniqueness of weak solution for such systems remained open until 2016 and the authors in [17] resolved it for dimension 2. The authors in [17] also considered the case of nonlocal systems with non constant \( m \) and non constant \( \nu \) under certain assumptions on the kernel \( J \). The existence of a unique strong solution in two dimensions for the nonlocal system with constant viscosity and mobility parameter equal to 1 is proved in [18] and the authors showed that any weak solution regularizes in finite time uniformly with respect to bounded sets of initial data. As in the case of 3D Navier-Stokes, in three dimensions, the existence of a weak solution is known (see [5]), but the uniqueness of the weak solution for the nonlocal Cahn-Hilliad-Navier-Stokes system remains open.

Optimal control theory of fluid dynamic models has been one of the far-reaching areas of applied mathematics with several engineering applications (see for example [25, 19, 20]). Controlling fluid flow and turbulence inside a flow in a given physical domain, with known initial data with various means, for example, body forces, boundary values, temperature (cf. [1] [24] etc.), is an interesting problem in fluid mechanics. An another
interesting control problem is to find an optimal controlled initial data with a given external forcing such that a suitable cost functional is minimized (see [25]). The mathematical developments in infinite dimensional nonlinear system theory and partial differential equations in the past several decades, opened up a new window for the optimal control theory of Navier-Stokes equations. Such problems are extensively addressed in [16, 19, 1, 20, 23, 25], etc., to name a few.

We are interested to study some optimal control problems related to the system (1.1). In this paper, we give a systematic approach to the mathematical formulation and resolve the problem of minimizing total energy. We consider two dimensional fluid flows, since the nonlocal Cahn-Hilliard-Navier-Stokes equations are not known to be well-posed in three dimensions. To the best of authors knowledge, the control problems for nonlocal Cahn-Hilliard-Navier-Stokes system is completely open. However the optimal control for dynamic boundary control [7, 8] and optimal control problem for the viscous Cahn-Hilliard system [27, 28] are available in the literature. In [6, 9, 10], authors discuss about the control problems related to phase field system of Cahn-Hilliard-Navier-Stokes type. An optimal distributed control of a diffuse interface model of tumor growth is considered in [11]. The model studied in [11] is a kind of local Cahn-Hilliard-Navier-Stokes type system with some additional conditions on F.

In this paper, we formulate a distributed optimal control problem and a data assimilation problem (optimization of the initial velocity filed), and establish the Pontryagin maximum principle for the non local Cahn-Hilliard-Navier-Stokes system. The unique global strong solution of the system (2.1a)-(2.1f) (see below) established in [18] helps us to achieve this goal. The coupling in the system (1.1) makes the problem mathematically challenging and harder to resolve than that of the corresponding problem for the Navier-Stokes system and Cahn Hilliard system.

The paper is organized as follows: In the next section, we discuss the functional setting for the unique solvability of the system (2.1a)-(2.1f) (see below). We also state the existence and uniqueness of a weak as well as strong solution of the system in the same section (see Theorems 2.7, 2.10 and 2.11). The unique solvability of the linearized system is established in section 3 using a standard Galerkin approximation technique (see Theorem 3.1). An optimal control problem is formulated in section 4 as the minimization of a suitable cost functional (the sum of total energy and total effort by controls). The distributed control acting on the system controls the average velocity of the fluids. We first prove the existence of an optimal control (see Theorem 4.5) and then establish the Pontryagin maximum principle (see Theorem 4.7), which gives the first-order necessary optimality conditions associated with the problem. We characterize the optimal control using the adjoint system.

2. Mathematical Formulation

In this section, we mathematically formulate the two dimensional Cahn-Hilliard-Navier-Stokes system and discuss the necessary function spaces required to obtain the global solvability results for such systems. We mainly follow the papers [5, 17] for the mathematical formulation and functional setting.
2.1. Governing equations. A well known model which describes the evolution of an incompressible isothermal mixture of two immiscible fluids is governed by Cahn-Hilliard-Navier-Stokes system (see [5]). We consider the following controlled Cahn-Hilliard-Navier-Stokes system:

\[
\begin{align*}
\varphi_t + \mathbf{u} \cdot \nabla \varphi &= \Delta \mu, \quad \text{in } \Omega \times (0, T), \\
\mu &= a \varphi - J^\ast \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, T), \\
\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mu \nabla \varphi + \mathbf{h} + \mathbf{U}, \quad \text{in } \Omega \times (0, T), \\
\text{div } \mathbf{u} &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \mu}{\partial n} &= 0, \quad \mathbf{u} = 0 \quad \text{on } \partial \Omega \times (0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^2\) is a bounded domain with sufficiently smooth boundary and \(n\) is the unit outward normal to the boundary \(\partial \Omega\). In the system \(2.1a)-(2.1f)\), \(U\) is the distributed control acting in the system.

2.2. Functional Setting. Let us introduce the following functional spaces required for getting the unique global solvability results of the system \(2.1a)-(2.1f)\).

\[
\begin{align*}
\mathcal{G}_{\text{div}} := \{ u \in L^2(\Omega; \mathbb{R}^2) : \text{div } u = 0 \}, \\
\mathcal{V}_{\text{div}} := \{ u \in H^1_0(\Omega; \mathbb{R}^2) : \text{div } u = 0 \}, \\
\mathcal{H} := L^2(\Omega; \mathbb{R}), \quad \mathcal{V} := H^1(\Omega; \mathbb{R}), \\
\mathcal{V}' := H^{-1}_0(\Omega; \mathbb{R}).
\end{align*}
\]

Let us denote \(\| \cdot \|\) and \(\langle \cdot, \cdot \rangle\) the norm and the scalar product, respectively, on both \(\mathcal{H}\) and \(\mathcal{G}_{\text{div}}\). The duality between any Hilbert space \(\mathcal{X}\) and its dual \(\mathcal{X}'\) will be denoted by \(\langle \cdot, \cdot \rangle\).

We know that \(\mathcal{V}_{\text{div}}\) is endowed with the scalar product

\[
(\mathbf{u}, \mathbf{v})_{\mathcal{V}_{\text{div}}} = (\nabla \mathbf{u}, \nabla \mathbf{v}) = 2(\mathbf{D} \mathbf{u}, \mathbf{D} \mathbf{v}), \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}_{\text{div}}.
\]

Let us define the Stokes operator \(A : D(A) \cap \mathcal{G}_{\text{div}} \to \mathcal{G}_{\text{div}}\) by

\[
A = -P \Delta, \quad D(A) = H^2(\Omega) \cap \mathcal{V}_{\text{div}},
\]

where \(P : L^2(\Omega) \to \mathcal{G}_{\text{div}}\) is the Helmholtz-Hodge orthogonal projection. Note also that, we have

\[
\langle A \mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}} = (\nabla \mathbf{u}, \nabla \mathbf{v}) = 2(\mathbf{D} \mathbf{u}, \mathbf{D} \mathbf{v}), \quad \text{for all } \mathbf{u} \in D(A), \mathbf{v} \in \mathcal{V}_{\text{div}}.
\]

It should also be noted that \(A^{-1} : \mathcal{G}_{\text{div}} \to \mathcal{G}_{\text{div}}\) is a self-adjoint compact operator in \(\mathcal{G}_{\text{div}}\) and by the classical spectral theorems there exists a sequence \(\lambda_j\) with \(0 < \lambda_1 \leq \lambda_2 \leq \lambda_j \leq \cdots \to +\infty\) and a family of \(\mathbf{w}_j \in D(A)\) which is orthonormal in \(\mathcal{G}_{\text{div}}\) and such that \(A \mathbf{w}_j = \lambda_j \mathbf{w}_j\). Hence, we have the following form of the Poincaré inequality:

\[
\lambda_1 \| \mathbf{u} \|^2 \leq \langle A \mathbf{u}, \mathbf{u} \rangle = \| \nabla \mathbf{u} \|^2.
\]
For $u, v, w \in V_{\text{div}}$ we define the trilinear operator $b(\cdot, \cdot, \cdot)$ as
\[
b(u, v, w) = \int_{\Omega} (u(x) \cdot \nabla)v(x) \cdot w(x)\,dx = \sum_{i,j=1}^{2} \int_{\Omega} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)\,dx,
\]
and the bilinear operator $B$ from $V_{\text{div}} \times V_{\text{div}}$ into $V_{\text{div}}'$ defined by,
\[
\langle B(u, v), w \rangle = b(u, v, w), \quad \text{for all } u, v, w \in V_{\text{div}}.
\]
An integration by parts yields,
\[
\begin{cases}
  b(u, v, v) = 0, \quad \text{for all } u, v \in V_{\text{div}}, \\
  b(u, v, w) = -b(u, w, v), \quad \text{for all } u, v, w \in V_{\text{div}}.
\end{cases}
\]

For more details about the linear and nonlinear operators, we refer the readers to [26].

**Lemma 2.1** (Gagliardo-Nirenberg inequality, Theorem 2.1, [14]). Let $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, $p \geq 1$. Then for any fixed number $q, r \geq 1$, there exists a constant $C > 0$ depending only on $d, p, q$ such that
\[
\|u\|_{L^r} \leq C \|\nabla u\|_p^\theta \|u\|_{L^q}^{1-\theta}, \quad \theta \in [0, 1],
\]
where the numbers $p, q, r$ and $\theta$ satisfy the relation
\[
\theta = \left(\frac{1}{q} - \frac{1}{r}\right) \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{q}\right)^{-1}.
\]

The particular cases of Lemma 2.1 are the well known inequalities, due to Ladyzhenskaya (see Lemma 1 and 2, Chapter 1, [21]), which are given below.

**Lemma 2.2** (Ladyzhenskaya inequality). For $u \in C_0^\infty(\Omega; \mathbb{R}^n), n = 2, 3$, there exists a constant $C$ such that
\[
\|u\|_{L^4} \leq C^{1/4} \|\nabla u\|^{\frac{1}{2}} \|u\|^{\frac{3}{2}}, \quad \text{for } n = 2, 3,
\]
where $C = 2, 4$ for $n = 2, 3$ respectively.

For every $u, v, w \in V_{\text{div}}$ the following estimates hold
\[
|b(u, v, w)| \leq \sqrt{2} \|u\|^{1/2} \|\nabla u\|^{1/2} \|v\|^{1/2} \|\nabla v\|^{1/2} \|w\|,
\]
so that for all $u \in V_{\text{div}}$, we have
\[
\|B(u, u)\|_{V_{\text{div}}'} \leq \sqrt{2} \|u\| \|\nabla u\| \leq \sqrt{\frac{2}{\lambda_1}} \|u\|^2_{V_{\text{div}}'},
\]
by using the Poincaré inequality.

For every $f \in V'$ we denote $\overline{f}$ the average of $f$ over $\Omega$, i.e., $\overline{f} := |\Omega|^{-1} \langle f, 1 \rangle$, where $|\Omega|$ is the Lebesgue measure of $\Omega$. Let us also introduce the spaces (see [17])
\[
V_0 = \{v \in V : \overline{v} = 0\},
\]
\[
V_0' = \{f \in V' : \overline{f} = 0\},
\]
and the operator $\mathcal{A} : V \to V'$ is defined by
\[
\langle \mathcal{A}u, v \rangle := \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx \text{ for all } u, v \in V.
\]
Clearly $\mathcal{A}$ is linear and it maps $V$ into $V'_0$ and its restriction $\mathcal{B}$ of $\mathcal{A}$ to $V_0$ onto $V'_0$ is an isomorphism. We know that for every $f \in V'_0$, $\mathcal{B}^{-1}f$ is the unique solution with zero mean value of the Neumann problem:
\[
\begin{cases}
-\Delta u = f, \text{ in } \Omega, \\
\frac{\partial u}{\partial n} = 0, \text{ on } \partial \Omega.
\end{cases}
\]
In addition, we have
\[
\langle \mathcal{A}u, \mathcal{B}^{-1}f \rangle = \langle f, u \rangle, \text{ for all } u \in V, f \in V'_0, \tag{2.6}
\]
\[
\langle f, \mathcal{B}^{-1}g \rangle = \langle g, \mathcal{B}^{-1}f \rangle = \int_{\Omega} \nabla (\mathcal{B}^{-1}f) \cdot \nabla (\mathcal{B}^{-1}g) \, dx, \text{ for all } f, g \in V'_0. \tag{2.7}
\]
Note that $\mathcal{B}$ can be also viewed as an unbounded linear operator on $H$ with domain $\text{D}(\mathcal{B}) = \{ v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega \}$.

### 2.3. Weak Solution of the Governing Equations

Now we state the existence theorem and uniqueness theorem for the nonlocal Cahn-Hilliard-Navier-Stokes system given by (2.1a)-(2.1f). For, let us make the following assumptions:

**Assumption 2.3.** Let $J$ and $F$ satisfy:

1. $J \in W^{1,1}(\mathbb{R}^2; \mathbb{R})$, $J(x) = J(-x)$ and $a(x) = \int_{\Omega} J(x-y) \, dy \geq 0$, a.e., in $\Omega$.
2. $F \in C^2(\mathbb{R})$ and there exists $C_0 > 0$ such that $F''(s) + a(x) \geq C_0$, for all $s \in \mathbb{R}$, a.e., $x \in \Omega$.
3. Moreover, there exist $C_1 > 0$, $C_2 > 0$ and $q > 0$ such that $F''(s) + a(x) \geq C_1 |s|^{2q} - C_2$, for all $s \in \mathbb{R}$, a.e., $x \in \Omega$.
4. There exist $C_3 > 0$, $C_4 \geq 0$ and $r \in (1, 2]$ such that $|F'(s)|^r \leq C_3 |F(s)| + C_4$, for all $s \in \mathbb{R}$.

**Remark 2.4.** Assumption $J \in W^{1,1}(\mathbb{R}^2; \mathbb{R})$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta; \mathbb{R})$, where $B_\delta := \{ z \in \mathbb{R}^2 : |z| < \delta \}$ with $\delta := \text{diam}(\Omega)$, or also by
\[
\sup_{x \in \Omega} \int_{\Omega} (|J(x-y)| + |\nabla J(x-y)|) \, dy < +\infty. \tag{2.8}
\]

**Remark 2.5.** Since $F(\cdot)$ is bounded from below, it is easy to see that Assumption 2.3(4) implies that $F(\cdot)$ has a polynomial growth of order $r'$, where $r' \in [2, \infty)$ is the conjugate index to $r$. Namely, there exist $C_5$ and $C_6 \geq 0$ such that
\[
|F(s)| \leq C_5 |s|^{r'} + C_6, \quad \forall s \in \mathbb{R}. \tag{2.9}
\]
Observe that Assumption 2.3(4) is fulfilled by a potential of arbitrary polynomial growth. For example, (2)-(4) are satisfied for the case of the well-known double-well potential $F = (s^2 - 1)^2$.

Let us now give the definition of a weak solution for the system (2.1a)-(2.1f).
**Definition 2.6** (weak solution). Let \( u_0 \in G_{\text{div}}, \varphi_0 \in H \) with \( F(\varphi_0) \in L^1(\Omega) \) and \( 0 < T < \infty \) be given. Then \((u, \varphi)\) is a weak solution to the uncontrolled system \((2.1a)-(2.1f)\) on \([0, T]\) corresponding to initial conditions \( u_0 \) and \( \varphi_0 \) if

(i) \( u, \varphi \) and \( \mu \) satisfy

\[
\begin{aligned}
  u &\in L^\infty(0, T; G_{\text{div}}) \cap L^2(0, T; V_{\text{div}}), \\
  u_t &\in L^{2-\gamma}(0, T; W^1_{\text{div}}), \text{ for all } \gamma \in (0, 1), \\
  \varphi &\in L^\infty(0, T; H) \cap L^2(0, T; V), \\
  \varphi_t &\in L^{2-\delta}(0, T; V'), \text{ for all } \delta \in (0, 1), \\
  \mu &\in L^2(0, T; V),
\end{aligned}
\]

(ii) For every \( \psi \in V \), every \( v \in V_{\text{div}} \) if we define \( \rho \) by

\[
\rho(x, \varphi) := a(x) \varphi + F'(\varphi),
\]

and for almost any \( t \in (0, T) \), we have

\[
\langle \varphi_t, \psi \rangle + (\nabla \rho, \nabla \psi) = \int_\Omega (u \cdot \nabla \psi) \varphi dx + \int_\Omega (\nabla \ast \varphi) \cdot \nabla \psi dx,
\]

(iii) Moreover, the following initial conditions hold in the weak sense

\[
u(0) = u_0, \quad \varphi(0) = \varphi_0,
\]

i.e., for every \( v \in V_{\text{div}} \), we have \((u(t), v) \to (u_0, v)\) as \( t \to 0 \), and for every \( \chi \in V \), we have \((\varphi(t), \chi) \to (\varphi_0, \chi)\) as \( t \to 0 \).

Next, we discuss the existence and uniqueness of weak solution results for the system \((2.1a)-(2.1f)\).

**Theorem 2.7** (Existence, Theorem 1, Corollaries 1 and 2, [5]). Let the Assumption 2.3 be satisfied. Let \( u_0 \in G_{\text{div}}, \varphi_0 \in H \) such that \( F(\varphi_0) \in L^1(\Omega) \) and \( h \in L^2_{\text{loc}}([0, \infty), V'_{\text{div}}) \). Then, for every given \( T > 0 \), there exists a weak solution \((u, \varphi)\) to the uncontrolled equation \((2.1a)-(2.1f)\) such that \((2.10)\) is satisfied. Furthermore setting

\[
\mathcal{E}(u(t), \varphi(t)) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\varphi(x, t) - \varphi(y, t))^2 dx dy + \int_\Omega F(\varphi(t)) dx,
\]

the following energy estimate holds for almost any \( t > 0 \):

\[
\mathcal{E}(u(t), \varphi(t)) + \int_0^t \left( v \|\nabla u(s)\|^2 + \|\nabla \mu(s)\|^2 \right) ds \leq \mathcal{E}(u_0, \varphi_0) + \int_0^t \langle h(s), u(s) \rangle ds,
\]

or the weak solution \((u, \varphi)\) satisfies the following energy identity,

\[
\frac{d}{dt} \mathcal{E}(u(t), \varphi(t)) + v \|\nabla u(t)\|^2 + \|\nabla \mu(t)\|^2 = \langle h(t), u(t) \rangle.
\]
Furthermore, if in addition \( h \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}') \), then the following dissipative estimates are satisfied,

\[
\mathcal{E}(u(t), \varphi(t)) \leq \mathcal{E}(u_0, \varphi_0) \exp(-kt) + F(m_0)|\Omega| + K,
\]

where \( m_0 = (\varphi_0, 1) \) and \( k, K \) are two positive constants which are independent of the initial data, with \( K \) depending on \( \Omega, \nu, J, F, \|h\|_{L^2_{\text{loc}}(0, \infty; V_{\text{div}}')} \).

**Remark 2.8.** It can be proved that \( u \in C([0, T]; G_{\text{div}}) \) and \( \varphi \in C([0, T]; H) \). Here \( C_w \) means continuity on the interval \((0, T^*)\) with values in the weak topology of \( G_{\text{div}} \) and \( H \) respectively, i.e., \( u \in C_w([0, T^*]; G_{\text{div}}) \) means that for any fixed \( v \in G_{\text{div}} \) \( \equiv G_{\text{div}}(u, v) \) is a continuous scalar function on \((0, T^*)\). The above theorem also implies \( u \in C([0, T]; G_{\text{div}}) \) and \( \varphi \in C([0, T]; H) \).

**Remark 2.9.** We denote by \( Q \) a continuous monotone increasing function with respect to each of its arguments. As a consequence of energy inequality (2.16), we have the following bound:

\[
\|u\|_{L^\infty(0, T; G_{\text{div}}) \cap L^2(0, T; V_{\text{div}}')} + \|\varphi\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} + \|F(\varphi)\|_{L^\infty(0, T; H)} \\
\leq Q(\mathcal{E}(u_0, \varphi_0), \|h\|_{L^2(0, T; V_{\text{div}}')}), \tag{2.17}
\]

where \( Q \) also depends on \( F, J, \nu \) and \( \Omega \).

**Theorem 2.10** (Uniqueness, Theorem 2, [17]). Suppose that Assumption 2.3 is satisfied. Let \( u_0 \in G_{\text{div}}, \varphi_0 \in H \) with \( F(\varphi_0) \in L^1(\Omega) \) and \( h \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}') \). Then, the weak solution \((u, \varphi)\) corresponding to \((u_0, \varphi_0)\) and given by Theorem 2.7 is unique. Furthermore, for \( i = 1, 2 \), let \( z_i := (u_i, \varphi_i) \) be two weak solutions corresponding to two initial data \( z_{0i} := (u_{0i}, \varphi_{0i}) \) and external forces \( h_i, \) with \( u_{0i} \in G_{\text{div}}, \varphi_{0i} \in H \) with \( F(\varphi_{0i}) \in L^1(\Omega) \) and \( h_i \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}') \). Then the following continuous dependence estimate holds:

\[
\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2 \\
+ \int_0^t \left( c_0 \|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{4} \|\nabla(u_2(\tau) - u_1(\tau))\|^2 \right) d\tau \\
\leq \left( \|u_2(0) - u_1(0)\|^2 + \|\varphi_2(0) - \varphi_1(0)\|^2 \right) \Lambda_0(t) \\
+ \|\varphi_2(0) - \varphi_1(0)\|Q(\mathcal{E}(z_{01}), \mathcal{E}(z_{02}), \|h_1\|_{L^2(0, t; V')}, \|h_2\|_{L^2(0, t; V')}) \Lambda_1(t) \\
+ \|h_2 - h_1\|_{L^2(0, t; V_{\text{div}}')}^2 \Lambda_2(t),
\]

for all \( t \in [0, T], \) where \( \Lambda_0(t), \Lambda_1(t) \) and \( \Lambda_2(t) \) are continuous functions which depend on the norms of the two solutions. The functions \( Q \) and \( \Lambda_i(t) \) also depend on \( F, J \) and \( \Omega \).

The following theorem gives the existence and uniqueness of strong solution for the system (2.1a) - (2.1f).

**Theorem 2.11** (Global Strong Solution, Theorem 2, [18]). Let \( h \in L^2_{\text{loc}}([0, \infty); G_{\text{div}}) \), \( u_0 \in V_{\text{div}}, \varphi_0 \in V \cap L^\infty(\Omega) \) be given and the Assumption 2.3 be satisfied. Then, for a given \( T > 0 \), there exists a unique weak solution \((u, \varphi)\) of (2.1a) - (2.1f) such that

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\{ u \in L^\infty(0, T; V_{\text{div}}) \cap L^2(0, T; H^2(\Omega)), & \varphi \in L^\infty((0, T) \times \Omega) \times L^\infty(0, T; V), \\
u.t. \in L^2(0, T; G_{\text{div}}), & \varphi_t \in L^2(0, T; H).
\end{array}
\right.
\end{align*}
\]
Furthermore, suppose in addition that \( F \in C^3(\mathbb{R}) \), \( a \in H^2(\Omega) \) and that \( \varphi_0 \in H^2(\Omega) \). Then, the system (2.1a) - (2.1c) admits a unique strong solution on \([0, T]\) satisfying (2.18) and also
\[
\left\{ \begin{array}{l}
\varphi \in L^\infty(0, T; W^{1,p}), \ 2 \leq p < \infty, \\
\varphi_t \in L^\infty(0, T; H) \cap L^2(0, T; V) .
\end{array} \right.
\tag{2.19}
\]
If \( J \in W^{2,1}(\mathbb{R}; \mathbb{R}) \), we have in addition
\[
\varphi \in L^\infty(0, T; H^2(\Omega)).
\tag{2.20}
\]

**Remark 2.12.** The regularity properties given in (2.18) - (2.20) imply that
\[
u \in C([0, T]; V_{div}), \ \varphi \in C([0, T]; V) \cap C_w([0, T]; H^2(\Omega)).
\tag{2.21}
\]
If the assumptions of Theorem 2.11 guarantee that \( \varphi \) satisfies (2.20), then we also have strong continuity in time, that is,
\[
\varphi \in C([0, T]; H^2(\Omega)).
\tag{2.22}
\]

### 3. Existence and Uniqueness of the Linearized system

From the well known theory for the optimal control problems governed by partial differential equations, we know that the optimal control is derived in terms of the adjoint variable which satisfies a linear system. As a first step to this goal, we linearize the nonlinear system and obtain the existence and uniqueness of weak solution using a standard Galerkin approximation technique. Let us linearize the equations (2.1a)-(2.1f) around \((\tilde{u}, \tilde{\varphi})\) which is the unique weak solution of system (2.1a)-(2.1f) with control term \( U = 0 \) (uncontrolled system) and external forcing \( \tilde{h} \) such that
\[
\tilde{h} \in L^2(0, T; C_{div}), \ \tilde{u}_0 \in V_{div}, \ \tilde{\varphi}_0 \in V \cap L^\infty(\Omega),
\]
so that \((\tilde{u}, \tilde{\varphi})\) has the regularity given in (2.18).

Let us now rewrite the equation (2.1c). We know that
\[
\mu \nabla \varphi = (a \varphi - J * \varphi + F'(\varphi)) \nabla \varphi = \nabla \left( F(\varphi) + a \frac{\varphi^2}{2} \right) - \nabla a \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi.
\]
Hence we can rewrite (2.1c) as
\[
u \Delta u + (u \cdot \nabla) u + \nabla \tilde{\pi}_u = -\nabla a \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi + h + U,
\tag{3.1}
\]
where \(\tilde{\pi}_u = \pi - \left( F(\varphi) + a \frac{\varphi^2}{2} \right)\). Since pressure is also an unknown quantity, in order to linearize, we substitute \( u = w + \tilde{u}, \ \pi = \tilde{\pi} + \tilde{\pi}_u \) and \( \varphi = \psi + \tilde{\varphi} \) in (3.1) and (2.1a) to get
\[
u \Delta w + (w \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) w + \nabla \tilde{\pi}_w = -\nabla a \psi \tilde{\varphi} - (J * \psi) \nabla \tilde{\varphi} - (J * \tilde{\varphi}) \nabla \psi + \tilde{h} + U,
\]
where \(\tilde{\pi}_w = \tilde{\pi} - (F'(\tilde{\varphi}) + a \tilde{\varphi}) \psi, \ \tilde{h} = h - \tilde{h}\). Also, we have
\[
\psi_t + w \cdot \nabla \tilde{\varphi} + \tilde{u} \cdot \nabla \psi = \Delta \tilde{\mu}
\]
where \(\tilde{\mu} = a \psi - J * \psi + F''(\tilde{\varphi}) \psi\). Hence, we consider the following linearized system:
\[
u \Delta w + (w \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) w + \nabla \tilde{\pi}_w = -\nabla a \psi \tilde{\varphi} - (J * \psi) \nabla \tilde{\varphi} - (J * \tilde{\varphi}) \nabla \psi + \tilde{h} + U,
\tag{3.2a}
\[ \psi_t + \mathbf{w} \cdot \nabla \hat{\varphi} + \hat{\mathbf{u}} \cdot \nabla \psi = \Delta \tilde{\mu}, \quad (3.2b) \]
\[ \tilde{\mu} = a\psi - J \ast \psi + F'(\hat{\varphi})\psi, \quad (3.2c) \]
\[ \text{div } \mathbf{w} = 0, \quad (3.2d) \]
\[ \frac{\partial \tilde{\mu}}{\partial n} = 0, \mathbf{w} = 0 \text{ on } \partial \Omega \times (0, T), \quad (3.2e) \]
\[ \mathbf{w}(0) = \mathbf{w}_0, \psi(0) = \psi_0 \text{ in } \Omega. \quad (3.2f) \]

Note that in (3.2c), we used the Taylor formula:
\[ F'(\psi + \hat{\varphi}) = F'(\hat{\varphi}) + F''(\hat{\varphi})\psi + F'''(\hat{\varphi})\frac{\psi^2}{2} + \cdots, \]
and ignored the second order terms in \( \psi \), since we are considering a linear system, \( \hat{\varphi} \in L^\infty((0, T) \times \Omega) \) and \( F(\cdot) \) has a polynomial growth as discussed in Remark 2.5. In order to establish the Pontryagin maximum principle in next section, we take \( \tilde{\mathbf{h}} = \mathbf{w}_0 = \psi_0 = 0. \)

Next, we discuss the unique global solvability results for the system (3.2a)-(3.2f).

**Theorem 3.1 (Existence and Uniqueness of Linearized System).** Suppose that the Assumption 2.3 is satisfied. Let us assume \((\hat{\mathbf{u}}, \hat{\varphi})\) be the unique weak solution of the system (2.1a)-(2.1f) with the regularity given in (2.18).

Let \( \mathbf{w}_0 \in \mathbf{G}_\text{div} \) and \( \psi_0 \in \mathbf{V}' \) with \( \tilde{\mathbf{h}} \in L^2(0, T; \mathbf{G}_\text{div}), U \in L^2(0, T; \mathbf{V}'_\text{div}) \). Then for a given \( T > 0 \), there exists a unique weak solution \((\mathbf{w}, \psi)\) to the system (3.2a)-(3.2f) such that \( \mathbf{w} \in L^\infty(0, T; \mathbf{G}_\text{div}) \cap L^2(0, T; \mathbf{V}_\text{div}) \) and \( \psi \in L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; H) \).

**Proof.** To prove the existence and uniqueness of the linearized system (3.2a)-(3.2f) we use a standard Galerkin approximation scheme and show that these approximate solutions converge in appropriate spaces to the solution of the linearized system. Before that we prove some a-priori energy estimates which help us in proving the convergence of Galerkin approximations.

Taking inner product of (3.2a) with \( \mathbf{w} \), we get
\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{w}(t) \|^2 + \nu \| \nabla \mathbf{w}(t) \|^2 + b(\mathbf{w}, \hat{\mathbf{u}}, \mathbf{w}) + (\nabla \tilde{\pi}_w, \mathbf{w})
= -(\nabla a\psi\hat{\varphi}, \mathbf{w}) - ((J \ast \psi) \nabla \hat{\varphi}, \mathbf{w}) - ((J \ast \hat{\varphi}) \nabla \psi, \mathbf{w}) + (\tilde{\mathbf{h}}, \mathbf{w}) + (U, \mathbf{w}). \quad (3.3)
\]

Now we estimate each term of the above equality. In order to estimate \( b(\mathbf{w}, \hat{\mathbf{u}}, \mathbf{w}) \), we use the Hölder, Ladyzhenskaya and Young’s inequalities to obtain
\[
|b(\mathbf{w}, \hat{\mathbf{u}}, \mathbf{w})| \leq \| \nabla \hat{\mathbf{u}} \| \| \mathbf{w} \| \leq \sqrt{2} \| \nabla \hat{\mathbf{u}} \| \| \mathbf{w} \| \leq \frac{\nu}{12} \| \nabla \mathbf{w} \|^2 + \frac{6}{\nu} \| \nabla \hat{\mathbf{u}} \|^2 \| \mathbf{w} \|^2. \quad (3.4)
\]

Using an integration by parts and the divergence free condition given in (3.2d), we have
\[
(\nabla \tilde{\pi}_w, \mathbf{w}) = (\tilde{\pi}_w, \text{div } \mathbf{w}) = 0.
\]

Once again using the Hölder, Ladyzhenskaya and Young’s inequalities, we obtain
\[
|a(\mathbf{w}, \mathbf{w})| \leq \| \nabla \mathbf{a} \|_1 \| \mathbf{w} \| \| \nabla \mathbf{w} \| \leq 2^{1/4} \| \nabla \mathbf{a} \|_1 \| \mathbf{w} \|^{1/2} \| \nabla \mathbf{w} \|^{1/2} \]

Similarly, we obtain
\[
\left| (J * \psi) \nabla \tilde{\varphi}, w \right| \leq \left| J * \psi \nabla \tilde{\varphi} \right| \leq \|J\|_{L^\infty} \|\psi\|_{L^4} \|w\|_{L^4} \\
\leq 2^{1/4} \|J\|_{L^1} \|\psi\|_{L^4} \|w\|_{L^4}^{1/2} \| \nabla w \|^{1/2} \\
\leq \frac{C_0}{10} \|\psi\|_2^2 + \frac{v}{12} \| \nabla w \|_2^2 + \frac{75}{2vC_0^2} \| \nabla J \|_{L^1}^4 \| \tilde{\varphi} \|_{L^4}^4 \|w\|_2^2.
\] (3.6)

Using the Cauchy-Schwarz inequality and Young’s inequality, we also get
\[
\left| \langle \tilde{h}, w \rangle \right| \leq \frac{3}{\nu} \| \tilde{h} \|_{L^2_{\text{div}}}^2 + \frac{v}{12} \| \nabla w \|_2^2 \leq \frac{3C_0}{10} \| \tilde{h} \|_2^2 + \frac{3}{v} \| \tilde{h} \|_{L^2_{\text{div}}}^2 + \frac{3}{\nu} \|U\|_{L^2_{\text{div}}}^2 + \frac{6}{v} \| \nabla \tilde{u} \| \|w\|_2^2,
\] (3.8)

Combining (3.4)-(3.8) and substituting in (3.3) to find
\[
\frac{1}{2} \frac{d}{dt} \|w\|_2^2 + \frac{v}{2} \| \nabla w \|_2^2 \leq \frac{3C_0}{10} \| \psi \|_2^2 + \frac{3C_0}{v} \| \tilde{h} \|_2^2 + \frac{3}{v} \|U\|_{L^2_{\text{div}}}^2 + \frac{6}{v} \| \nabla \tilde{u} \| \|w\|_2^2
\] 
+ \left[ \frac{75}{2vC_0^2} \| \tilde{\varphi} \|_{L^4}^4 \left( \| \nabla a \|_{L^\infty}^2 + 2 \| \nabla J \|_{L^1}^4 \right) \right] \|w\|_2^2.
\] (3.9)

Now taking inner product of (3.2b) with $B^{-1}(\psi - \overline{\psi})$, we get
\[
\langle \psi, B^{-1}(\psi - \overline{\psi}) \rangle + \langle w \cdot \nabla \tilde{\varphi}, B^{-1}(\psi - \overline{\psi}) \rangle + \langle \tilde{u} \cdot \nabla \psi, B^{-1}(\psi - \overline{\psi}) \rangle = \langle \Delta \tilde{\mu}, B^{-1}(\psi - \overline{\psi}) \rangle.
\] (3.10)

We can write $\langle \Delta \tilde{\mu}, B^{-1}(\psi - \overline{\psi}) \rangle$ using (2.6) as
\[
-\langle -\Delta \tilde{\mu}, B^{-1}(\psi - \overline{\psi}) \rangle = -\langle \tilde{\mu}, \psi - \overline{\psi} \rangle
= -\langle a \psi + F''(\tilde{\varphi}) \psi, \psi - \overline{\psi} \rangle + (J * \psi, \psi - \overline{\psi}).
\] (3.11)

Using the Assumption 2.3(2) and (3.11) in (3.10), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|B^{-1/2}(\psi - \overline{\psi})\|_2^2 + C_0 \|\psi\|_2^2.
\]
\[ \leq (a\psi + F''(\tilde{\varphi})\psi, \tilde{\varphi}) - (w \cdot \nabla \tilde{\varphi}, B^{-1}(\psi - \tilde{\varphi})) - (\tilde{u} \cdot \nabla \psi, B^{-1}(\psi - \tilde{\varphi})) + (I \ast \psi, \psi - \tilde{\varphi}). \] (3.12)

The difficult term to estimate with the weak solution regularity of \((\tilde{u}, \tilde{\varphi})\) is \((F''(\tilde{\varphi})\psi, \tilde{\varphi})\), and hence we need more regular weak solution. Using the Cauchy-Schwarz, Hölder and Young's inequalities to estimate the first term from the right hand side of the inequality (3.12) as

\[ |(a\psi + F''(\tilde{\varphi})\psi, \tilde{\varphi})| \leq |\varphi| (|a|_{L^\infty} + |F''(\tilde{\varphi})|_{L^\infty}) |\psi|_{L^1} \]

\[ \leq \frac{C_0}{10} |\psi|^2 + \frac{5}{C_0} |\varphi|^2 |\Omega| \left( |\psi|_{L^1}^2 + |F''(\tilde{\varphi})|_{L^\infty}^2 \right). \] (3.13)

To estimate \((w \cdot \nabla \tilde{\varphi}, B^{-1}(\psi - \tilde{\varphi}))\), we use an integration by parts, \(w|_{\partial \Omega} = 0\) and the divergence free condition of \(w\) to get

\[ (w \cdot \nabla \tilde{\varphi}, B^{-1}(\psi - \tilde{\varphi})) = -(w \cdot \nabla B^{-1}(\psi - \tilde{\varphi}), \tilde{\varphi}). \]

Now using the Hölder, Ladyzhenskaya, Poincaré and Young's inequalities, we estimate the above term as

\[ |(w \cdot \nabla B^{-1}(\psi - \tilde{\varphi}), \tilde{\varphi})| \leq |w|_{L^4} |\nabla B^{-1}(\psi - \tilde{\varphi})| |\tilde{\varphi}|_{L^4} \]

\[ \leq 2^{1/4} |w|_{L^4}^{1/2} |\nabla w|^{1/4} |\nabla B^{-1}(\psi - \tilde{\varphi})| |\tilde{\varphi}|_{L^4} \]

\[ \leq \left( \frac{2}{\lambda_1} \right)^{1/4} |\nabla w|^{1/4} |\nabla B^{-1}(\psi - \tilde{\varphi})| |\tilde{\varphi}|_{L^4} \]

\[ \leq \frac{\sqrt{2}}{4} |\nabla w|^2 + \frac{1}{\sqrt{2}} \left( \frac{2}{\lambda_1} \right)^{1/2} |\nabla B^{-1}(\psi - \tilde{\varphi})|^2 |\tilde{\varphi}|_{L^4}^2, \] (3.14)

where \(\lambda_1\) is the first eigenvalue of the Stokes operator \(A\). Moreover, we have

\[ |(\tilde{u} \cdot \nabla \psi, B^{-1}(\psi - \tilde{\varphi}))| \leq |(\tilde{u} \cdot \nabla B^{-1}(\psi - \tilde{\varphi}), \psi)| \]

\[ \leq |\tilde{u}|_{L^4} |\nabla B^{-1}(\psi - \tilde{\varphi})|_{L^4} |\psi| \]

\[ \leq \frac{C_0}{20} |\psi|^2 + \frac{5}{C_0} |\nabla B^{-1}(\psi - \tilde{\varphi})|_{L^4}^2 |\tilde{u}|_{L^4}^2 \]

\[ \leq \frac{C_0}{20} |\psi|^2 + \frac{5\sqrt{2}}{C_0} |\nabla B^{-1}(\psi - \tilde{\varphi})|_{H^1} |\nabla B^{-1}(\psi - \tilde{\varphi})|_{H^1} |\tilde{u}|_{L^4}^2, \] (3.15)

where in the final step we used the Gagliardo-Nirenberg inequality. One can see that the \(H^2\)-norm of \(\zeta\) in \(D(B)\) is equivalent to the \(L^2\)-norm of \((B + I)\zeta\), i.e.,

\[ |\zeta|_{H^2} \approx |(B + I)\zeta|. \]

Now since \(B^{-1}(\psi - \tilde{\varphi}) \in D(B)\) and \(B\) is a linear operator, we have

\[ |\nabla B^{-1}(\psi - \tilde{\varphi})|_{H^1} \leq \tilde{C} |B^{-1}(\psi - \tilde{\varphi})|_{H^2} \leq \tilde{C} |(B + I)B^{-1}(\psi - \tilde{\varphi})| \leq \tilde{C} |\psi - \tilde{\varphi}|. \]

Substituting in (3.15), we get

\[ |(\tilde{u} \cdot \nabla \psi, B^{-1}(\psi - \tilde{\varphi}))| \leq \frac{C_0}{20} |\psi|^2 + \frac{\tilde{C}}{C_0} |\nabla B^{-1}(\psi - \tilde{\varphi})| |\psi - \tilde{\varphi}| |\tilde{u}|_{L^4}^2 \]
\[
\frac{1}{2} \frac{d}{dt} \| B^{-1/2}(\psi - \overline{\psi}) \|^2 + C_0 \| \psi \|^2 \leq \frac{3C_0}{10} \| \psi \|^2 + \frac{v}{4} \| \nabla w \|^2 + \left[ \frac{5}{C_0} \left( \| a \|^2_{L^\infty} + \| F''(\overline{\phi}) \|^2_{L^\infty} \right) + \frac{C_0}{20} \right] \overline{\psi}^2 |\Omega| \\
+ \left( \frac{1}{v} \frac{2}{\lambda_1} \right)^{1/2} \| \nabla a \|^2_{L^4} + \frac{10C_0}{C_0} \| \nabla a \|^2_{L^4} + \frac{5}{2C_0} \| \nabla J \|^2_{L^4} \| \nabla B^{-1}(\psi - \overline{\psi}) \|^2. \tag{3.18}
\]

By adding (3.9) and (3.18) we get,

\[
\frac{1}{2} \frac{d}{dt} \| w(t) \|^2 + \| B^{-1/2}(\psi - \overline{\psi}) \|^2 + \frac{v}{2} \| \nabla w \|^2 + \frac{C_0}{2} \| \psi \|^2 \leq \frac{3C_0}{v} \| \nabla w \|^2 + \left[ \frac{3}{v} \| U \|^2_{H^1_{div}} + \frac{v}{4} \| \nabla w \|^2 + \left[ \frac{5}{C_0} \left( \| a \|^2_{L^\infty} + \| F''(\overline{\phi}) \|^2_{L^\infty} \right) + \frac{C_0}{20} \right] \overline{\psi}^2 |\Omega| \\
+ \left( \frac{1}{2} + \frac{1}{v\lambda_1} \right) \| \nabla a \|^2_{L^4} + \frac{10C_0^2}{C_0^3} \| \nabla a \|^2_{L^4} + \frac{5}{2C_0} \| \nabla J \|^2_{L^4} \| \nabla B^{-1}(\psi - \overline{\psi}) \|^2, \tag{3.19}
\]

where we used the fact that \( 2x \leq (1 + x^2) \). Integrating from 0 to \( t \), we get

\[
\| w(t) \|^2 + \| B^{-1/2}(\psi - \overline{\psi})(t) \|^2 + \frac{v}{2} \int_0^t \| \nabla w(s) \|^2 ds + C_0 \int_0^t \| \psi(s) \|^2 ds \leq \| w_0 \|^2 + \| B^{-1/2}(\psi_0 - \overline{\psi}_0) \|^2 + \left[ \frac{10}{C_0} \| a \|^2_{L^\infty} + \frac{C_0}{10} \right] \overline{\psi}^2 |\Omega| t \\
+ \frac{10}{C_0} \overline{\psi}^2 |\Omega| \int_0^T \| F''(\overline{\phi}(s)) \|^2_{L^\infty} ds + \frac{6C}{v} \int_0^t \| \nabla h(s) \|^2 ds + \frac{6}{v} \int_0^t \| U(s) \|^2_{H^1_{div}} ds.
\]
\[ + \int_0^t \left[ \frac{12}{\nu} \left\| \nabla \tilde{u}(s) \right\|^2 + \frac{75}{\nu C_0^2} \left( \| \nabla a \|_{L^\infty}^4 + 2 \| \nabla J \|_{L^1}^4 \right) \right] \left\| w(s) \right\|^2 ds \\
+ \int_0^t \left[ \left( 1 + \frac{2}{\nu^2 \lambda_1} \left\| \tilde{\varphi}(s) \right\|_{L^4}^4 \right) + \frac{20C^2}{C_0^3} \left\| \tilde{u}(s) \right\|_{L^4}^4 + \frac{5}{C_0} \left\| \nabla J \right\|_{L^1}^2 \right] \left\| \nabla B^{-1}(\psi - \tilde{\varphi}(s)) \right\|^2 ds. \] 

Using the Grönwall inequality, we obtain

\[ \left\| w(t) \right\|^2 + \left\| B^{-1/2}(\psi - \tilde{\varphi})(t) \right\|^2 \leq \left( \left\| w_0 \right\|^2 + \left\| B^{-1/2}(\psi - \tilde{\varphi})_0 \right\|^2 \right) + \left[ \frac{10}{C_0} \left( \left\| a \right\|_{L^\infty}^2 + \left\| F''(\tilde{\varphi}) \right\|_{L^\infty((0,T) \times \Omega)}^2 \right) + \frac{C_0}{10} \right] \tilde{\varphi}^2 |\Omega| T \]

\[ + \frac{6C}{\nu} \int_0^T \left\| \tilde{h}(t) \right\|^2 dt + \frac{6}{\nu} \int_0^T \left\| U(t) \right\|^2_{V'_\div} dt \]

\[ \times \exp \left( \int_0^T \left[ \frac{12}{\nu} \left\| \nabla \tilde{u}(t) \right\|^2 + \frac{75}{\nu C_0^2} \left( \| \nabla a \|_{L^\infty}^4 + 2 \| \nabla J \|_{L^1}^4 \right) \right] \left\| \tilde{\varphi}(t) \right\|_{L^4}^4 dt \right) \]

\[ \times \exp \left( \int_0^t \left( 1 + \frac{2}{\nu^2 \lambda_1} \left\| \tilde{\varphi}(t) \right\|_{L^4}^4 \right) + \frac{20C^2}{C_0^3} \left\| \tilde{u}(t) \right\|_{L^4}^4 + \frac{5}{C_0} \left\| \nabla J \right\|_{L^1}^2 \right] dt, \]

for all \( t \in (0, T] \). Therefore, we get

\[ \left\| w(t) \right\|^2 + \left\| B^{-1/2}(\psi - \tilde{\varphi})(t) \right\|^2 \leq C, \text{ for all } t \in [0, T], \]

since \( J \in W^{1,1}(\mathbb{R}^2; \mathbb{R}) \), the weak solution regularity of \((\tilde{u}, \tilde{\varphi})\),

\[ \tilde{\varphi} \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \tilde{u} \in L^\infty(0, T; G_{\div}) \cap L^2(0, T; V'_\div), \]

along with \( \tilde{\varphi} \in L^\infty((0, T) \times \Omega) \), the forcing term \( \tilde{h} \) and control \( U \) satisfying \( \tilde{h} \in L^2(0, T; G_{\div}), \)

\( U \in L^2(0, T; V'_\div), \) and the embeddings \( L^\infty(0, T; G_{\div}) \cap L^2(0, T; V'_\div) \hookrightarrow L^4(0, T; L^4(\Omega)) \)

and \( L^\infty(0, T; H) \cap L^2(0, T; V) \hookrightarrow L^4(0, T; L^4(\Omega)) \), where \( \hookrightarrow \) denotes the dense and continuous embedding. Note that \( F(\cdot) \) has a polynomial growth (see \( 2.9 \)) and hence using the regularity of \( \tilde{\varphi} \), we get

\[ \left\| F''(\tilde{\varphi}) \right\|_{L^\infty((0,T) \times \Omega)} \leq C(\left\| \tilde{\varphi} \right\|_{L^\infty((0,T) \times \Omega)}) \]

Hence, it is immediate that \( w \in L^\infty(0, T; G_{\div}) \) and \( \psi \in L^\infty(0, T; V') \). Le us now substitute \( 3.21 \) in \( 3.20 \) to obtain

\[ \left\| w(t) \right\|^2 + \left\| B^{-1/2}(\psi - \tilde{\varphi})(t) \right\|^2 + \nu \int_0^t \left\| \nabla w(s) \right\|^2 ds + C_0 \int_0^t \left\| \psi(s) \right\|^2 ds \leq C, \]

where \( C \) is a generic constant depending on initial data, external forcing, control and different norms described above, so that we also get \( w \in L^2(0, T; V'_\div) \) and \( \psi \in L^2(0, T; H) \).

Using a standard Galerkin approximation technique one can easily obtain the global existence of the linearized system \((3.2a)-(3.2f)\) by making use of the above energy estimates. Uniqueness also follows easily from the energy estimates, since the system is linear. The unique solvability results for the nonlinear system \((2.1a)-(2.1f)\) is thoroughly given in Theorem 1, \([5]\) and Theorem 2, \([17]\). Also, the pressure \( \tilde{\pi} \in L^2(0, T; L^2(\Omega)/\mathbb{R}), \)
where $L^2(\Omega)/\mathbb{R} := \{ \tilde{\pi} \in L^2(\Omega) : \int_\Omega \pi(x) \, dx = 0 \}$, which is unique up to an additive constant, can be obtained using the same technique as in the case of Navier-Stokes equations (see [26]). Furthermore, one can show that $w_t \in L^2(0, T; \mathbb{V}'_{\text{div}})$ and $\psi_t \in L^2(0, T; H^{-2}(\Omega))$, so that we get $w \in C([0, T]; G_{\text{div}})$ and $\psi \in C([0, T]; \mathbb{V}')$. 

Now if we assume that $F \in C^3(\mathbb{R})$, $a \in H^2(\Omega)$, then one can also have $\psi \in C([0, T]; H) \cap L^2(0, T; V)$. This regularity result of the linearized system we need in the next section.

**Theorem 3.2.** Suppose that the assumptions given in Theorem 3.1 holds and assume that $F \in C^3(\mathbb{R})$, $a \in H^2(\Omega)$. Let $w_0 \in G_{\text{div}}$ and $\psi_0 \in \mathbb{V}'$ with $\tilde{h} \in L^2(0, T; G_{\text{div}})$, $U \in L^2(0, T; \mathbb{V}'_{\text{div}})$. Then for a given $T > 0$, there exists a unique weak solution $(w, \psi)$ to the system (3.2a)-(3.2f) such that

$$w \in L^\infty(0, T; G_{\text{div}}) \cap L^2(0, T; \mathbb{V}_{\text{div}}) \, \text{and} \, \psi \in L^\infty(0, T; H) \cap L^2(0, T; V).$$

**Proof.** Let us take inner product of $\tilde{\mu} + J \ast \psi$ with the equation (3.2b) to obtain

$$\left( \frac{d\psi}{dt}, \tilde{\mu} + J \ast \psi \right) + \| \nabla \tilde{\mu} \|^2 = (\Delta H, J \ast \psi) - (w \cdot \nabla \tilde{\phi}, \tilde{\mu} + J \ast \psi) - (\tilde{u} \cdot \nabla \psi, \tilde{\mu} + J \ast \psi).$$

(3.23)

We know that

$$\left( \frac{d\psi}{dt}, \tilde{\mu} + J \ast \psi \right) = \left( \frac{d\psi}{dt}, (a + F''(\phi)) \psi \right).$$

(3.24)

Note that $0 < C_0 \leq a + F''(s)$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Thus, we get

$$\frac{d}{dt} \left\| \sqrt{a + F''(\phi)} \psi \right\|^2 = \frac{d}{dt} \left( (a + F''(\phi)) \psi, \psi \right)$$

$$= \left( \frac{d}{dt} \left( (a + F''(\phi)) \psi \right), \psi \right) + \left( (a + F''(\phi)) \frac{d\psi}{dt}, \psi \right)$$

$$= (F''(\phi) \tilde{\phi} T, \psi) + 2 \left( \frac{d\psi}{dt}, (a + F''(\phi)) \psi \right).$$

(3.25)

Substituting (3.25) in (3.23), we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \sqrt{a + F''(\phi)} \psi \right\|^2 + \| \nabla \tilde{\mu} \|^2$$

$$= \frac{1}{2} (F''(\phi) \tilde{\phi} T, \psi) + (\Delta \tilde{\mu}, J \ast \psi) - (w \cdot \nabla \tilde{\phi}, \tilde{\mu} + J \ast \psi) - (\tilde{u} \cdot \nabla \psi, \tilde{\mu} + J \ast \psi).$$

(3.26)

An integration by parts, Cauchy-Schwarz inequality, convolution inequality and Young’s inequality yields

$$\left| (\Delta \tilde{\mu}, J \ast \psi) \right| = \left| (\nabla \tilde{\mu}, \nabla J \ast \psi) \right| \leq \| \nabla \tilde{\mu} \| \| \nabla J \ast \psi \| \leq \| \nabla \tilde{\mu} \| \| \nabla J \|_1 \| \psi \|$$

$$\leq \frac{1}{6} \| \nabla \tilde{\mu} \|^2 + \frac{3}{2} \| \nabla J \|_1 \| \psi \|^2.$$
Using an integration by parts, using the divergence free condition of $w$, Hölder, Ladyzhenskaya inequality and Young’s inequalities, we estimate $|(w \cdot \nabla \phi, \tilde{\mu})|$ as

$$
|(w \cdot \nabla \phi, \tilde{\mu})| = |(w \cdot \nabla \mu, \phi)| \leq \|w\|_{L^1} \|\nabla \mu\|_1 \|\phi\|_{L^4} \leq \frac{1}{6} \|\nabla \mu\|^2 + \frac{3}{2} \|w\|_{L^1} \|\phi\|_{L^4}^2
$$

$$
\leq \frac{1}{6} \|\nabla \mu\|^2 + \frac{v}{8} \|\nabla w\|^2 + \frac{9}{v} \|\phi\|_{L^4}^4 \|w\|^2.
$$

(3.28)

Once again an integration by parts, Hölder and Young’s inequality yields

$$
|(\tilde{u} \cdot \nabla \psi, \tilde{\mu})| = |(\tilde{u} \cdot \nabla \mu, \psi)| \leq \|\tilde{u}\|_{L^1} \|\nabla \mu\| \|\psi\|_{L^4} \leq \frac{1}{6} \|\nabla \mu\|^2 + \frac{3}{2} \|\tilde{u}\|_{L^1} \|\psi\|_{L^4}^2.
$$

(3.29)

We use (3.28) and (3.29) in (3.26) to obtain

$$
\frac{1}{2} \frac{d}{dt} \left(\sqrt{A + F''(\tilde{\phi}) \psi}\right)^2 + \frac{1}{2} \|\nabla \mu\|^2
$$

$$
\leq \frac{1}{2} (F'''(\tilde{\phi}) \tilde{\phi}_t \psi) + \frac{v}{8} \|\nabla w\|^2 + \frac{3}{2} \|\nabla J\|_{L^1} \|\psi\|_{L^4}^2 + \frac{9}{v} \|\phi\|_{L^4}^4 \|w\|^2 + \frac{3}{2} \|\tilde{u}\|_{L^1} \|\psi\|_{L^4}^2
$$

$$
- (w \cdot \nabla \phi, J \psi) - (\tilde{u} \cdot \nabla \psi, J \psi).
$$

(3.30)

Note that $C_0 \leq a(x) + F''(s)$, for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Using the Hölder inequality, convolution inequality, Ladyzhenskaya inequality and Young’s inequality, we have

$$
(-\Delta \psi, \tilde{\mu}) = (\nabla \psi, \nabla \tilde{\mu}) = (\nabla \psi, \nabla (a \psi - J \psi + F''(\tilde{\phi}) \psi))
$$

$$
= (\nabla \psi, a \nabla \psi + \psi \nabla a - \nabla J \psi + F''(\tilde{\phi}) \nabla \psi + F'''(\tilde{\phi}) \nabla \psi)
$$

$$
\geq C_0 \|\nabla \psi\|^2 - (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\psi\|_4 \|\nabla \psi\|_4 - \|F'''(\tilde{\phi})\|_4 \|\nabla \phi\|_{L^4} \|\nabla \psi\|_4 \|\psi\|_{L^4}^4
$$

$$
\geq C_0 \|\nabla \psi\|^2 - C_0 \|\nabla \psi\|^2 - \frac{3}{2C_0} (\|\nabla a\|_{L^\infty}^2 + \|\nabla J\|_{L^1}^2) \|\psi\|_4^2
$$

$$
- \sqrt{2} \|F'''(\tilde{\phi})\|_4 \|\nabla \phi\|_{L^4} \|\nabla \psi\|_4 \|\psi\|_4 \|\psi\|_{L^4}^2
$$

$$
\geq \frac{C_0}{2} \|\nabla \psi\|^2 - \frac{3}{2C_0} (\|\nabla a\|_{L^\infty}^2 + \|\nabla J\|_{L^1}^2) \|\psi\|_4^2 - \frac{729}{8C_0^4} \|F'''(\tilde{\phi})\|_4^4 \|\nabla \phi\|_{L^4}^4 \|\psi\|_4^4 \|\psi\|_{L^4}^2.
$$

(3.31)

But, we also have

$$
(\nabla \psi, \nabla \mu) \leq \|\nabla \psi\|_4 \|\nabla \mu\| \leq \frac{C_0}{4} \|\nabla \psi\|^2 + \frac{1}{C_0} \|\nabla \mu\|^2.
$$

(3.32)

Combining (3.31) and (3.32), we find

$$
\frac{C_0}{4} \|\nabla \psi\|^2 \leq \frac{1}{C_0} \|\nabla \mu\|^2 + \frac{3}{2C_0} (\|\nabla a\|_{L^\infty}^2 + \|\nabla J\|_{L^1}^2) \|\psi\|_4^2 + \frac{729}{8C_0^4} \|F'''(\tilde{\phi})\|_4^4 \|\nabla \phi\|_{L^4}^4 \|\psi\|_4^4 \|\psi\|_{L^4}^2.
$$

(3.33)

Let us substitute (3.33) in (3.30) to get

$$
\frac{1}{2} \frac{d}{dt} \left(\sqrt{A + F''(\tilde{\phi}) \psi}\right)^2 + \frac{C_0}{8} \|\nabla \psi\|^2
$$

$$
\leq \frac{1}{2} (F'''(\tilde{\phi}) \tilde{\phi}_t \psi) + \frac{v}{8} \|\nabla w\|^2 + \frac{3}{2} \|\nabla J\|_{L^1} \|\psi\|_4^2 + \frac{9}{v} \|\phi\|_{L^4}^4 \|w\|^2 + \frac{3}{2} \|\tilde{u}\|_{L^1} \|\psi\|_{L^4}^2
$$

(3.34)
\[ + \frac{3}{4} (\| \nabla a \|^2_{L^\infty} + \| \nabla J \|^2_{L^1}) \| \psi \|^2 + \frac{729}{16C_0^2} \| F'''(\bar{\phi}) \|^4 \| \nabla \bar{\phi} \|^4 \| \psi \|^2 \\
- (w \cdot \nabla \bar{\phi}, J \ast \psi) - (\bar{u} \cdot \nabla \psi, J \ast \psi). \]  

(3.34)

We use the Hölder inequality, Ladyzhenskaya inequality, Poincaré inequality, convolution inequality and Young’s inequality to estimate the terms as

\[ \frac{3}{2} \| \bar{u} \|^2_{L^1} \| \psi \|^2_4 \leq \frac{3}{\sqrt{2}} \| \bar{u} \|^2_{L^1} \| \psi \| \| \nabla \psi \| \leq \frac{C_0^2}{48} \| \nabla \psi \|^2 + \frac{54}{C_0^2} \| \bar{u} \|^4_{L^4} \| \psi \|^2, \]  

(3.35)

\[ |(F'''(\bar{\phi}) \bar{\phi}, \psi, \psi)| \leq \| F'''(\bar{\phi}) \|_{L^\infty} \| \bar{\phi} \|_4 \| \psi \|^2 \leq \sqrt{2} \| F'''(\bar{\phi}) \|_{L^\infty} \| \bar{\phi} \|_4 \| \psi \| \| \nabla \psi \| \leq \frac{C_0^2}{48} \| \nabla \psi \|^2 + \frac{24}{C_0^2} \| F'''(\bar{\phi}) \|_{L^\infty} \| \bar{\phi} \|^2 \| \psi \|^2, \]  

(3.36)

\[ |(w \cdot \nabla \bar{\phi}, J \ast \psi)| \leq \| w \|_{L^4} \| \nabla \bar{\phi} \|_{L^4} \| J \ast \psi \| \leq 2^{1/4} \| w \|^{1/2} \| \nabla w \|^{1/2} \| \nabla \bar{\phi} \|_{L^4} \| J \|_{L^1} \| \psi \| \leq \frac{\nu}{8} \| \nabla w \|^2 + \frac{2}{\nu} \sqrt{\frac{2}{\lambda_1}} \| \nabla \bar{\phi} \|^2_{L^4} \| J \|^2_{L^1} \| \psi \|^2, \]  

(3.37)

\[ |(\bar{u} \cdot \nabla \psi, J \ast \psi)| \leq \| \bar{u} \|_{L^4} \| \nabla \psi \| \| J \|_{L^3} \| \psi \|_{L^4} \leq \sqrt{2} \| \bar{u} \|_{L^4} \| J \|^3_{L^1} \| \nabla \psi \|^2 \| \psi \|^{1/2} \leq \frac{C_0^2}{48} \| \nabla \psi \|^2 + \frac{1296}{C_0^2} \| \bar{u} \|^4_{L^4} \| J \|^4_{L^1} \| \psi \|^2. \]  

(3.38)

We substitute (3.35)–(3.38) in (3.34) to obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \sqrt{a + F''(\bar{\phi})} \psi \right)^2 + \frac{C_0^2}{16} \| \nabla \psi \|^2 \leq \frac{\nu}{4} \| \nabla w \|^2 + \frac{9}{\nu} \| \bar{\phi} \|^4_{L^4} \| w \|^2 + \left( \frac{3}{4} \| \nabla a \|^2_{L^\infty} + \frac{9}{4} \| \nabla J \|^2_{L^1} \right) \| \psi \|^2 \\
+ \left( \frac{54}{C_0^2} \| \bar{u} \|^4_{L^4} + \frac{729}{16C_0^2} \| F'''(\bar{\phi}) \|^4 \| \nabla \bar{\phi} \|^4_{L^4} + \frac{24}{C_0^2} \| F'''(\bar{\phi}) \|^2 \| J \|^2_{L^1} \| \bar{\phi} \|^2 \\
+ \frac{2}{\nu} \sqrt{\frac{2}{\lambda_1}} \| \nabla \bar{\phi} \|^2_{L^4} \| J \|^3_{L^1} + \frac{1296}{C_0^2} \| \bar{u} \|^4_{L^4} \| J \|^4_{L^1} \right) \| \psi \|^2. \]  

(3.39)

Let us now add the inequalities (3.9) and (3.39) to get

\[ \frac{1}{2} \frac{d}{dt} \| w \|^2 + \frac{\nu}{4} \| \nabla w \|^2 + \frac{1}{2} \frac{d}{dt} \left( \sqrt{a + F''(\bar{\phi})} \psi \right)^2 + \frac{C_0^2}{16} \| \nabla \psi \|^2 \leq \frac{3C}{\nu} \| \tilde{u} \|^2 + \frac{3}{\nu} \| U \|^2_{V_{div}'} + \left( \frac{3C_0}{10} + \frac{3}{4} \| \nabla a \|^2_{L^\infty} + \frac{9}{4} \| \nabla J \|^2_{L^1} \right) \| \psi \|^2 \\
+ \left( \frac{54}{C_0^2} \| \bar{u} \|^4_{L^4} + \frac{729}{16C_0^2} \| F'''(\bar{\phi}) \|^4 \| \nabla \bar{\phi} \|^4_{L^4} + \frac{24}{C_0^2} \| F'''(\bar{\phi}) \|^2 \| J \|^2_{L^1} \| \bar{\phi} \|^2 \\
+ \frac{2}{\nu} \sqrt{\frac{2}{\lambda_1}} \| \nabla \bar{\phi} \|^2_{L^4} \| J \|^3_{L^1} + \frac{1296}{C_0^2} \| \bar{u} \|^4_{L^4} \| J \|^4_{L^1} \right) \| \psi \|^2. \]
Let us integrate the inequality (3.40) from 0 to $T$. Biswas, S. Dharmatti and M. T. Mohan

\[ + \left[ \frac{6}{v} \| \nabla \hat{u} \|^2 + \left( \frac{9}{v} + \frac{75}{2vC_0} \left( \| \nabla a \|_{L^\infty} + 2 \| \nabla J \|_{L^1}^4 \right) \right) \| \phi \|^4_{L^4} \right] \| w \|^2. \quad (3.40) \]

Let us integrate the inequality (3.40) from 0 to $t$ to obtain

\[
\| w(t) \|^2 + \left\| \sqrt{a + F''(\phi(t))} \psi(t) \right\|^2 + \frac{v}{2} \int_0^t \| \nabla w(s) \|^2 \, ds + \frac{C_0^2}{8} \int_0^t \| \nabla \psi(s) \|^2 \, ds \\
\leq \| w_0 \|^2 + \left\| \sqrt{a + F''(\phi(0))} \psi(0) \right\|^2 + \frac{6C}{v} \int_0^t \| \tilde{h}(s) \|^2 \, ds + \frac{6}{v} \int_0^t \| U(s) \|^2_{V_{\text{div}}} \, ds \\
+ \int_0^t \left[ \frac{6}{v} \| \nabla \hat{u}(s) \|^2 + \left( \frac{9}{v} + \frac{75}{2vC_0} \left( \| \nabla a \|_{L^\infty} + 2 \| \nabla J \|_{L^1}^4 \right) \right) \| \phi(s) \|^4_{L^4} \right] \| w(s) \|^2 \, ds \\
+ \left( \frac{3C_0}{10} + \frac{3}{4} \| \nabla a \|^2_{L^\infty} + \frac{9}{4} \| \nabla J \|^2_{L^1} \right) \int_0^t \| \psi(s) \|^2 \, ds \\
+ \int_0^t \left( \frac{54}{C_0} \| \tilde{u}(s) \|^4_{L^1} + \frac{729}{16C_0} \| F'''(\phi(s)) \|^4 \| \nabla \phi(s) \|^4_{L^4} \| \phi(s) \|^4_{L^4} \right) \| \tilde{h}(s) \|^4_{L^1} \| \psi(s) \|^2 \, ds \\
+ \frac{2}{v} \sqrt{\frac{2}{\lambda_1}} \| \nabla \phi(s) \|^2_{L^4} \| J \|^2_{L^1} + \frac{1296}{C_0} \| \tilde{u}(s) \|^4_{L^1} \| J \|^4_{L^1} \right) \| \psi(s) \|^2 \, ds. \quad (3.41) \]

An application of the Grönwall inequality in (3.41) yields

\[
\| w(t) \|^2 + \left\| \sqrt{a + F''(\phi(t))} \psi(t) \right\|^2 \\
\leq \left( \| w_0 \|^2 + \left\| \sqrt{a + F''(\phi(0))} \psi(0) \right\|^2 + \frac{6C}{v} \int_0^t \| \tilde{h}(s) \|^2 \, ds + \frac{6}{v} \int_0^t \| U(s) \|^2_{V_{\text{div}}} \, ds \right) \\
\times \exp \left( \int_0^t \left[ \frac{6}{v} \| \nabla \hat{u}(s) \|^2 + \left( \frac{9}{v} + \frac{75}{2vC_0} \left( \| \nabla a \|_{L^\infty} + 2 \| \nabla J \|_{L^1}^4 \right) \right) \| \phi(s) \|^4_{L^4} \right] \, ds \right) \\
\times \exp \left( \left( \frac{3C_0}{10} + \frac{3}{4} \| \nabla a \|^2_{L^\infty} + \frac{9}{4} \| \nabla J \|^2_{L^1} \right) T \right) \\
\times \exp \left( \int_0^t \left( \frac{54}{C_0} \| \tilde{u}(s) \|^4_{L^1} + \frac{729}{16C_0} \| F'''(\phi(s)) \|^4 \| \nabla \phi(s) \|^4_{L^4} \| \phi(s) \|^4_{L^4} \right) \| \tilde{h}(s) \|^4_{L^1} \| \psi(s) \|^2 \, ds \right) \\
+ \frac{2}{v} \sqrt{\frac{2}{\lambda_1}} \| \nabla \phi(s) \|^2_{L^4} \| J \|^2_{L^1} + \frac{1296}{C_0} \| \tilde{u}(s) \|^4_{L^1} \| J \|^4_{L^1} \right) \, ds, \quad (3.42) \]

for all $t \in [0, T]$. Note that the right hand side of the inequality (3.42) is finite, since $(\phi, \hat{u})$ is a unique strong solution of the uncontrolled system (2.1a)-(2.11). Using the Assumption 2.3 (2), we get

\[
\| w(t) \|^2 + C_0 \| \psi(t) \|^2 \leq C, 
\]

since we have

\[
\left\| \sqrt{a + F''(\phi(0))} \psi(0) \right\|^2 \leq \left( \| a \|_{L^\infty} + \| F''(\phi) \|_{L^\infty} \right) \| \psi(0) \|^2 \leq C \left( \| a \|_{L^\infty}, \| \phi(0) \|_{L^\infty} \right) \| \psi(0) \|^2.
\]
Thus, we obtain the regularity \( (\mathbf{w}, \psi) \in L^\infty(0, T; G_{\text{div}}) \times L^\infty(0, T; H) \). Let us substitute (3.42) in (3.41) to get the regularity \( (\mathbf{w}, \psi) \in L^2(0, T; V_{\text{div}}) \times L^2(0, T; V) \), since \( \psi \in L^2(0, T; H) \) also. Since the pressure \( \pi \in L^2(0, T; L^2(\Omega)/\mathbb{R}) \), it is immediate that \( (\mathbf{w}, \psi_i) \in L^2(0, T; V_{\text{div}}) \times L^2(0, T; V') \) and also
\[
(\mathbf{w}, \psi) \in (C([0, T]; G_{\text{div}}) \cap L^2(0, T; V_{\text{div}})) \times (C([0, T]; H) \cap L^2(0, T; V)),
\]
which completes the proof. \( \square \)

4. Optimal Control Problem

In this section, we formulate a distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled nonlocal Cahn-Hilliard-Navier-Stokes equations. The main aim is to establish the existence of an optimal control that minimizes the cost functional given below subject to the constraint (2.1a)-(2.1f). The associated cost functional is defined by

\[
J(u, \varphi, U) := \frac{1}{2} \int_0^T \|u(t) - u_d(t)\|^2 \, dt + \frac{1}{2} \int_0^T \|\varphi(t) - \varphi_d(t)\|^2 \, dt + \frac{1}{2} \int_0^T \|U(t)\|^2 \, dt,
\]

(4.1)

where \( u_d(\cdot) \) and \( \varphi_d(\cdot) \) are the desired states. Note that the cost functional is the sum of total energy and total effort by control.

Let us assume that
\[
F \in C^3(\mathbb{R}), \ a \in H^2(\Omega),
\]

(4.2)

and the initial data
\[
u_0 \in V_{\text{div}} \text{ and } \varphi_0 \in H^2(\Omega).
\]

(4.3)

By the embedding of \( V \) and \( L^\infty(\Omega) \) in \( H^2(\Omega) \), the initial concentration \( \varphi_0 \in H^2(\Omega) \) implies \( \varphi_0 \in V \cap L^\infty(\Omega) \). From now onwards, we assume that along with the Assumption 2.3 condition (4.2) holds true.

Definition 4.1. Let \( \mathcal{U}_{\text{ad}} \) be a closed and convex subset consisting of controls \( U \in L^2(0, T; G_{\text{div}}) \).

Definition 4.2 (Admissible Class). The admissible class \( \mathcal{A}_{\text{ad}} \) of triples \( (u, \varphi, U) \) is defined as the set of states \( (u, \varphi) \) with initial data given in (4.3), solving the system (2.1a)-(2.1f) with control \( U \in \mathcal{U}_{\text{ad}} \), which is a subspace of \( L^2(0, T; G_{\text{div}}) \). That is,
\[
\mathcal{A}_{\text{ad}} := \left\{ (u, \varphi, U) : (u, \varphi) \text{ is a unique strong solution of (2.1a) - (2.1f) with control } U \right\}.
\]

In view of the above definition, the optimal control problem we are considering is formulated as
\[
\min_{(u, \varphi, U) \in \mathcal{A}_{\text{ad}}} J(u, \varphi, U).
\]

(4.3) (Optimal Solution). A solution to the Problem (OCP) is called an optimal solution and the optimal triplet is denoted by \( (u^*, \varphi^*, U^*) \). The control \( U^* \) is called an optimal control.
In the rest of this section, we find an optimal solution to the problem \( \text{OCP} \) via adjoint variables characterization.

4.1. The Adjoint System. In this subsection, we formally derive the adjoint system corresponding to the problem (2.1a) - (2.1c). Let us take \( h = 0 \) in (2.1a) - (2.1c) and define

\[
\mathcal{N}_1(u, \varphi, U) := \nu \Delta u - (u \cdot \nabla) u - \nabla \pi - (J * \varphi) \nabla \varphi - \nabla a \frac{\varphi^2}{2} + U, \\
\mathcal{N}_2(u, \varphi) := - u \cdot \nabla \varphi + \Delta(a \varphi - J * \varphi + F'(\varphi)),
\]

where \( \pi = \pi - (F(\varphi) + a \frac{\varphi^2}{2}) \). Then the system (2.1a) - (2.1c) can be written as

\[
(\partial_t u, \partial_t \varphi) = (\mathcal{N}_1(u, \varphi, U), \mathcal{N}_2(u, \varphi)).
\]

As is well known from the control theory literature, in order to get the necessary conditions for the existence of an optimal control to the Problem (OCP), we need the adjoint equations corresponding to the system (2.1a)-(2.1f). With this motivation, we define the augmented cost functional \( \tilde{J} \) by

\[
\tilde{J}(u, \varphi, U, p, \eta) = J(u, \varphi, U) + \int_0^T \langle p, \partial_t u - \mathcal{N}_1(u, \varphi, U) \rangle dt + \int_0^T \langle \eta, \partial_t \varphi - \mathcal{N}_2(u, \varphi) \rangle dt,
\]

where \( p \) and \( \eta \) denote the adjoint variables to \( u \) and \( \varphi \) respectively.

Before establishing the Pontryagin maximum principle, we derive the adjoint equations formally by differentiating the augmented cost functional \( \tilde{J} \) in the Gâteaux sense with respect to each variable.

\[
\begin{align*}
\mathcal{J}_u + \int_0^T \langle p, \partial_t u - [\partial_u \mathcal{N}_1] \rangle dt + \int_0^T \langle \eta, \partial_t \varphi - [\partial_u \mathcal{N}_2] \rangle dt &= 0, \\
\mathcal{J}_\varphi + \int_0^T \langle p, \partial_t \varphi - [\partial_\varphi \mathcal{N}_1] \rangle dt + \int_0^T \langle \eta, \partial_t u - [\partial_\varphi \mathcal{N}_2] \rangle dt &= 0, \\
\mathcal{J}_U + \int_0^T \langle p, \partial_t U - [\partial_U \mathcal{N}_1] \rangle dt + \int_0^T \langle \eta, \partial_t \varphi - [\partial_U \mathcal{N}_2] \rangle dt &= 0.
\end{align*}
\]

Then the adjoint variables \( p, \eta \) and \( U \) satisfy the following system

\[
\begin{align*}
-p_t + \mathcal{J}_u - [\partial_u \mathcal{N}_1]^* p - [\partial_u \mathcal{N}_2]^* \eta &= 0, \\
-\eta_t + \mathcal{J}_\varphi - [\partial_\varphi \mathcal{N}_1]^* p - [\partial_\varphi \mathcal{N}_2]^* \eta &= 0, \\
\mathcal{J}_U - [\partial_U \mathcal{N}_1]^* p - [\partial_U \mathcal{N}_2]^* \eta &= 0, \quad \text{div } p = 0, \\
\left. p \right|_{\partial \Omega} &= \left. \eta \right|_{\partial \Omega} = 0, \\
p(T, \cdot) &= \eta(T, \cdot) = 0.
\end{align*}
\]
Note that differentiating $\hat{J}$ with respect to the adjoint variables recovers the original linearized system. We compute $[\partial_u M_1]^* p$, $[\partial_u M_2]^* \eta$, $[\partial \varphi N_1]^* p$, $[\partial \varphi N_2]^* \eta$ as

\[
\begin{align*}
[\partial_u M_1]^* p &= \nu \Delta p - (p \cdot \nabla) u - (u \cdot \nabla) p - \nabla q, \\
[\partial_u M_2]^* \eta &= (\nabla \varphi)' \eta, \\
[\partial \varphi N_1]^* p &= -J * (p \cdot \nabla \varphi) + (\nabla J * \varphi) \cdot p - \nabla a \cdot p \varphi, \\
[\partial \varphi N_2]^* \eta &= u \nabla \eta + a \Delta \eta - J * \Delta \eta + F''(\varphi) \Delta \eta.
\end{align*}
\]

(4.7)

Also it should be noted that the third condition in (4.6) gives $U = p$. Thus from (4.6), it follows that the adjoint variables $(p, \eta)$ satisfy the following adjoint system:

\[
\begin{align*}
-p_t - \nu \Delta p + (p \cdot \nabla) u + (u \cdot \nabla) p - (\nabla \varphi)' \eta + \nabla q &= -(u - u_d), \\
-\eta_t + J * (p \cdot \nabla \varphi) - (\nabla J * \varphi) \cdot p + \nabla a \cdot p \varphi - u \cdot \nabla \eta - a \Delta \eta \\
&\quad + J * \Delta \eta - F''(\varphi) \Delta \eta = -(\varphi - \varphi_d), \\
\text{div } p &= 0, \\
\left. p \right|_{\partial \Omega} &= \eta \mid_{\partial \Omega} = 0, \\
\left. p \right|_{(T, \cdot)} &= \eta(T, \cdot) = 0.
\end{align*}
\]

(4.8)

Remember that $q : \Omega \to \mathbb{R}$ is also an unknown. The following theorem gives the unique solvability of the system (4.8).

**Theorem 4.4 (Existence and Uniqueness of Adjoint System).** Let $(u, \varphi)$ be a unique strong solution of the nonlinear system (2.1a) - (2.1c). Then, there exists a unique weak solution

\[(p, \eta) \in (L^\infty(0, T; \mathcal{C}_{\text{div}}) \cap L^2(0, T; \mathcal{W}_{\text{div}})) \times (L^\infty(0, T; H) \cap L^2(0, T; V))\]

(4.9)

to the system (4.8).

**Proof.** In order to prove the existence and uniqueness of the adjoint system (4.8), we use a standard Galerkin approximation scheme and show that these approximations converge in the space given in (4.9) to the solution of the system (4.8). First we find an a-priori energy estimates satisfied by $(p, \eta)$.

Let us take the inner product with $p(\cdot)$ of the first equation in (4.8) to obtain

\[
-\frac{1}{2} \frac{d}{dt} \|p\|^2 + \nu \|\nabla p\|^2 = -((p \cdot \nabla u, p) + (\nabla \varphi)' \eta, p) - (u - u_d, p) =: I_1 + I_2 + I_3,
\]

(4.10)

where we used $((u \cdot \nabla)p, p) = 0$, since $p$ is divergence free. Let us estimate $I_1$ using the Hölder, Ladyzhenskaya and Young’s inequalities as

\[
|I_1| \leq \|p\|^2 \|\nabla u\| \leq \sqrt{2} \|p\| \|\nabla p\| \|\nabla u\| \leq \frac{\nu}{12} \|\nabla p\|^2 + \frac{6}{\nu} \|\nabla u\|^2 \|p\|^2.
\]

(11.11)

Using an integration by parts, the divergence free condition, Hölder, Ladyzhenskaya and Young’s inequalities, we estimate $I_2$ as

\[
|I_2| = |(\varphi \nabla \eta, p)| \leq \|\nabla \eta\| \|\varphi\| L^1 \|p\| L^1 \leq \frac{C_0}{10} \|\nabla \eta\|^2 + \frac{5}{2C_0} \|\varphi\|^2 \|p\| L^1 \|\nabla p\| L^1.
\]
\[
I_3 \leq \frac{3}{v} \| u - u_d \|_{\mathcal{V}_{div}} \| p \|_{\mathcal{V}_{div}} + \frac{\nu}{12} \| \nabla p \|^2 + \frac{75}{2vC_0^2} \| \varphi \|_{L^4}^4 \| p \|^2.
\]

(4.13)

We now take the inner product with \( \eta \) to the second equation in (4.8) to get

\[
-\frac{1}{2} \frac{d}{dt} \| \eta \|^2 - ((a + F''(\varphi))\Delta \eta, \eta) = -(J \ast (p \cdot \nabla \varphi), \eta) + ((\nabla J \ast \varphi) \cdot p, \eta) - (\nabla a \cdot p \varphi, \eta) \\
- (J \ast \Delta \eta, \eta) + (\varphi - \varphi_d, \eta),
\]

(4.14)

where we used \((u \cdot \nabla \eta, \eta) = 0\), since \( u \) is divergence free. Since \( \eta \big|_{\Omega} = 0 \), an integration by parts yields

\[
- ((a + F''(\varphi))\Delta \eta, \eta) \\
= -\frac{2}{v} \int_{\mathcal{M}} (a(x) + F''(\varphi(x))) \frac{\partial^2 \eta(x)}{\partial x_i^2} \eta(x) dx \\
= \frac{2}{v} \int_{\mathcal{M}} (a(x) + F''(\varphi(x))) \left( \frac{\partial \eta(x)}{\partial x_i} \right)^2 dx + \frac{2}{v} \int_{\mathcal{M}} \frac{\partial}{\partial x_i} (a + F''(\varphi(x))) \frac{\partial \eta(x)}{\partial x_i} \eta(x) dx \\
\geq C_0 \sum_{i=1}^{2} \int_{\mathcal{M}} \left( \frac{\partial \eta(x)}{\partial x_i} \right)^2 dx + \frac{2}{v} \int_{\mathcal{M}} \frac{\partial}{\partial x_i} (a + F''(\varphi(x))) \frac{\partial \eta(x)}{\partial x_i} \eta(x) dx \\
= C_0 \| \nabla \eta \|^2 + (\nabla (a + F''(\varphi)) \cdot \nabla \eta, \eta) \\
\]

(4.15)

Thus, from (4.14), we obtain

\[
-\frac{1}{2} \frac{d}{dt} \| \eta \|^2 + C_0 \| \nabla \eta \|^2 = -(J \ast (p \cdot \nabla \varphi), \eta) + ((\nabla J \ast \varphi) \cdot p, \eta) - (\nabla a \cdot p \varphi, \eta) \\
- (J \ast \Delta \eta, \eta) + (\varphi - \varphi_d, \eta) - (\nabla a \cdot \nabla \eta, \eta) - (\nabla F''(\varphi) \cdot \nabla \eta, \eta) \\
= : \sum_{k=4}^{10} I_k.
\]

(4.16)

Note that the properties of \( J(\cdot) \), an integration by parts, Hölder, Ladyzhenskaya and Young’s inequalities yields

\[
|I_4| = |(J \ast \nabla \varphi, p)| = |(\nabla (J \ast \varphi), p)| \leq \| \nabla J \|_{L^1} \| \eta \|_{L^1} \| p \|_{L^4} \| \varphi \|_{L^4} \\
\leq \frac{1}{2} \| \nabla J \|_{L^1}^2 \| \eta \|^2 + \frac{1}{2} \| p \|_{L^4}^2 \| \varphi \|_{L^4}^2 \leq \frac{1}{2} \| \nabla J \|_{L^1}^2 \| \eta \|^2 + \frac{1}{\sqrt{2}} \| p \|_{L^4} \| \nabla p \| \| \varphi \|_{L^4}^2 \\
\leq \frac{1}{2} \| \nabla J \|_{L^1}^2 \| \eta \|^2 + \frac{\nu}{12} \| \nabla p \|^2 + \frac{3}{2\nu} \| p \|^2 \| \varphi \|_{L^4}^4.
\]

(4.17)

In a similar way, we estimate \( I_5 \) as

\[
|I_5| \leq \| \nabla J \|_{L^1} \| \varphi \|_{L^4} \| p \|_{L^4} \| \eta \| \leq \frac{1}{2} \| \nabla J \|_{L^1} \| \eta \|^2 + \frac{\nu}{12} \| \nabla p \|^2 + \frac{3}{2\nu} \| p \|^2 \| \varphi \|_{L^4}^4.
\]

(4.18)
Once again using the Hölder and Young’s inequalities to estimate $I_6$ as

$$|I_6| \leq \| \nabla a \|_{L^\infty} \| \varphi \|_{L^4} \| p \|_{L^4} \| \eta \| \leq \frac{1}{2} \| \nabla a \|_{L^\infty} \| \eta \|^2 + \frac{\nu}{12} \| \nabla p \|^2 + \frac{3}{2\nu} \| p \|^2 \| \varphi \|^4_{L^4}. \quad (4.19)$$

We use the properties of $J(\cdot)$, an integration by parts, Cauchy-Schwarz, Hölder and Young’s inequality to estimate $I_7$ as

$$|I_7| = |(J * \eta, \Delta \eta)| = |(\nabla (J * \eta), \nabla \eta)| \leq \| \nabla J \|_{L^1} \| \eta \| \| \nabla \eta \|
\leq \frac{C_0}{10} \| \nabla \eta \|^2 + \frac{5}{2C_0} \| \nabla J \|^2 \| \eta \|^2. \quad (4.20)$$

Using the Cauchy-Schwarz and Young’s inequality, we estimate $I_8$ as

$$|I_8| \leq \| \varphi - \varphi_d \| \| \nabla \eta \| \leq \frac{5C}{2C_0} \| \varphi - \varphi_d \|^2 + \frac{C_0}{10} \| \eta \|^2 + \frac{C_0}{10} \| \nabla \eta \|^2. \quad (4.21)$$

Similarly, we have

$$|I_9| \leq \| \nabla a \|_{L^\infty} \| \nabla \eta \| \| \eta \| \leq \frac{C_0}{10} \| \nabla \eta \|^2 + \frac{5}{2C_0} \| \nabla a \|_{L^\infty} \| \eta \|^2. \quad (4.22)$$

The most difficult term, which is not possible to estimate using the weak solution regularity of $(u, \varphi)$ is $(\nabla F''(\varphi) \cdot \nabla \eta, \eta)$. Thus we need the solution $(u, \varphi)$ to be a strong solution. Using the Hölder, Young’s and Ladyzhenskaya inequalities, it is immediate that

$$|I_{10}| = |(F''(\varphi) \nabla \varphi \cdot \nabla \eta, \eta)| \leq \sup_{x \in \Omega} |F''(\varphi)| \| \nabla \varphi \|_{L^4} \| \nabla \eta \| \| \eta \|_{L^4}
\leq 2^{1/4} \sup_{x \in \Omega} |F''(\varphi)| \| \nabla \varphi \|_{L^4} \| \nabla \eta \|^{3/2} \| \eta \|^1/2
\leq \frac{C_0}{10} \| \nabla \eta \|^2 + C(C_0) \| F''(\varphi) \|_{L^4} \| \nabla \varphi \|_{L^4} \| \eta \|^2. \quad (4.23)$$

Let us now combine (4.11)-(4.13) and substitute in (4.10) and also combine (4.17)-(4.23) and then substitute in (4.16), and then add together to find

$$- \frac{1}{2} \frac{d}{dt} \left( \| p \|^2 + \| \eta \|^2 \right) + \frac{\nu}{2} \| \nabla p \|^2 + \frac{C_0}{2} \| \nabla \eta \|^2
\leq \left[ \frac{6}{\nu} \| \nabla u \|^2 + \frac{3}{2\nu} \left( \frac{25}{C_0} + 3 \right) \| \varphi \|^4_{L^4} \right] \| p \|^2 + \frac{3C}{\nu} \| u - u_d \|^2 + \frac{5C}{2C_0} \| \varphi - \varphi_d \|^2
\quad + \left[ \frac{1}{2} \| \nabla J \|^2_{L^1} + \frac{1}{2} \left( 1 + \frac{5}{C_0} \right) \left( \| \nabla a \|^2_{L^\infty} + \| \nabla J \|^2_{L^1} \right) + \frac{C_0}{10} \right] \| \eta \|^2
\quad + C(C_0) \| F''(\varphi) \|_{L^4} \| \nabla \varphi \|^2_{L^4} \| \eta \|^2. \quad (4.24)$$

Integrating the above inequality from $t$ to $T$ and using the data $p(T) = \eta(T) = 0$, we get

$$\| p(t) \|^2 + \| \eta(t) \|^2 + \nu \int_t^T \| \nabla p(s) \|^2 ds + C_0 \int_t^T \| \nabla \eta(s) \|^2 ds
\leq \int_t^T \left[ \frac{12}{\nu} \| \nabla u(s) \|^2 + \frac{3}{\nu} \left( \frac{25}{C_0} + 3 \right) \| \varphi(s) \|^4_{L^4} \right] \| p(s) \|^2 ds
\quad + \left[ \| \nabla J \|^2_{L^1} + \left( 1 + \frac{5}{C_0} \right) \left( \| \nabla a \|^2_{L^\infty} + \| \nabla J \|^2_{L^1} + \frac{C_0}{5} \right) \right] \int_t^T \| \eta(s) \|^2 ds.$$
\[ + C(C_0) \int_T^t \|F''(\varphi(s))\|_{L^\infty}^4 \|\nabla \varphi(s)\|_{L^4}^4 \|\eta(s)\|^2 ds \]
\[ + \frac{6C}{\nu} \int_T^0 \|u(s) - u_d(s)\|^2 ds + \frac{5C}{C_0} \int_T^0 \|\varphi(s) - \varphi_d(s)\|^2 ds. \]  
(4.25)

An application of the Grönwall inequality in (4.25) yields
\[
\|p(t)\|^2 + \|\eta(t)\|^2 \\
\leq \left( \frac{6C}{\nu} \int_0^T \|u(s) - u_d(s)\|^2 ds + \frac{5C}{C_0} \int_0^T \|\varphi(s) - \varphi_d(s)\|^2 ds \right) \times \exp \left\{ \left[ \|\nabla J\|_{L^1}^2 + \left( 1 + \frac{5}{C_0} \right) \left( \|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1} \right) + \frac{C_0}{5} \right] T \right\} \\
\times \exp \left\{ \int_0^T \left[ \frac{12}{\nu} \|\nabla u(t)\|^2 + \frac{3}{\nu} \left( \frac{25}{C_0^2} + 3 \right) \|\varphi(t)\|_{L^4}^4 \right] dt \right\} \times \exp \left( C(C_0) \sup_{t \in [0,T]} \|F''(\varphi(t))\|_{L^\infty}^4 \int_t^T \|\nabla \varphi(t)\|_{L^4}^4 dt \right) < +\infty, \]  
(4.26)

for all \( t \in [0,T] \). Since \((u, \varphi)\) is the unique strong solution of the nonlinear system, the right hand side of the inequality (4.26) is finite. Note that \( \varphi \in L^\infty((0,T) \times \Omega) \cap L^\infty(0,T; W^{1,p} \cap L^2(0,T; H^2(\Omega))) \), for \( 2 \leq p < \infty \). Thus, we get \( p \in L^\infty(0,T; G_{\text{div}}) \) and \( \eta \in L^\infty(0,T; H) \). Let us substitute (4.26) in (4.25) to obtain the regularity \( p \in L^2(0,T; V_{\text{div}}) \) and \( \eta \in L^2(0,T; V) \). Since \( \eta \in L^\infty(0,T; H) \) implies \( \eta \in L^2(0,T; H) \), and also the regularity \( \eta \in L^2(0,T; V) \) easily gives \( \eta \in L^2(0,T; V') \).

By employing a standard Galerkin approximation technique (see Theorem 4.2, [11]), we obtain the existence of a weak solution \((p,u)\) to the system (4.8) with the regularity \( p \in L^\infty(0,T; G_{\text{div}}) \cap L^2(0,T; V_{\text{div}}) \) and \( \eta \in L^\infty(0,T; H) \cap L^2(0,T; V) \). It can also be shown that \( q \in L^2(0,T; L^2(\Omega)/R) \), so that we also have \( p, \eta \in L^2(0,T; V_{\text{div}}') \) and \( \eta \in L^2(0,T; V') \). Uniqueness follows easily from the energy estimate (4.26), since the system is linear. Furthermore, Thus \( p \) is almost everywhere equal to a continuous function from \([0,T]\) to \( G_{\text{div}} \) and \( \eta \) is almost everywhere equal to a continuous function from \([0,T]\) to \( H \), and hence we have \( p \in C([0,T]; G_{\text{div}}) \) and \( \eta \in C([0,T]; H) \).

\[ \square \]

4.2. Existence of an Optimal Control. Let us now show that an optimal triplet \((u^*, \varphi^*, U^*)\) exists for the problem (OCP).

**Theorem 4.5** (Existence of an Optimal Triplet). Let the Assumption 2.3 along with the condition (4.2) holds true and the initial data \((u_0, \varphi_0)\) satisfying (4.3) be given. Then there exists at least one triplet \((u^*, \varphi^*, U^*)\) \( \in D_{\text{ad}} \) such that the functional \( J(u, \varphi, U) \) attains its minimum at \((u^*, \varphi^*, U^*)\), where \((u^*, \varphi^*)\) is the unique strong solution of (2.1a)-(2.1f) with the control \( U^* \).

**Proof.** **Claim (1):** \( D_{\text{ad}} \) is nonempty. If \( U = 0 \), then by the existence and uniqueness theorem (see Theorems 2.7 and 2.10), a unique weak solution \((u, \varphi)\) exists. Since the condition (4.2) holds true and the initial data \((u_0, \varphi_0)\) satisfies (4.3), the unique weak solution we obtained is also a strong solution. Hence, \( J(u, \varphi, 0) \) exists and belongs to \( D_{\text{ad}} \). Therefore the set \( D_{\text{ad}} \) is nonempty. 

Claim (2): There exists an optimal triplet \((u^*, \varphi^*, U^*) \in \mathcal{A}_{ad}\). Let us define

\[
\mathcal{J} := \inf_{U \in \mathcal{A}_{ad}} J(u, \varphi, U).
\]

Since, \(0 \leq \mathcal{J} < +\infty\), there exists a minimizing sequence \(\{U_n\} \in \mathcal{U}_{ad}\) such that

\[
\lim_{n \to \infty} \mathcal{J}(u_n, \varphi_n, U_n) = \mathcal{J},
\]

where \((u_n, \varphi_n)\) is the unique strong solution of (2.1a)-(2.1f) with the control \(U_n\) and \(u_n(0) = u_0^I \in \mathcal{V}_{div}\) and \(\varphi_n(0) = \varphi_0^I \in H^2(\Omega)\). (4.27)

Without loss of generality, we assume that \(\mathcal{J}(u_n, \varphi_n, U_n) \leq \mathcal{J}(u, \varphi, 0)\), where \((u, \varphi, 0) \in \mathcal{A}_{ad}\). From the definition of \(\mathcal{J}(\cdot, \cdot, \cdot)\), this gives

\[
\frac{1}{2} \int_0^T \|u_n(t) - u_d(t)\|^2 \, dt + \frac{1}{2} \int_0^T \|\varphi(t) - \varphi_d(t)\|^2 \, dt \leq \frac{1}{2} \int_0^T \|u_0(t)\|^2 \, dt.
\]

(4.28)

Since \(u, u_d \in L^\infty(0, T; G_{div})\) and \(\varphi, \varphi_d \in L^\infty(0, T; H)\), from the above relation, it is clear that, there exist a \(K > 0\), large enough such that

\[
0 \leq \mathcal{J}(u_n, \varphi_n, U_n) \leq K < +\infty.
\]

In particular there exists a large \(C > 0\), such that

\[
\int_0^T \|U_n(t)\|^2 \, dt \leq C < +\infty.
\]

Therefore the sequence \(\{U_n\}\) is uniformly bounded in the space \(L^2(0, T; G_{div})\). Since \((u_n, \varphi_n, U_n)\) is a unique weak solution of the system (2.1a)-(2.1f), from the energy estimates, one can easily show that the sequence \(\{u_n\}\) is uniformly bounded in \(L^\infty(0, T; G_{div}) \cap L^2(0, T; \mathcal{V}_{div})\) and \(\{\varphi_n\}\) is uniformly bounded in \(L^\infty(0, T; H) \cap L^2(0, T; V)\). Hence, by using the Banach–Alaoglu theorem, we can extract a subsequence \(\{(u_n, \varphi_n, U_n)\}\) such that

\[
\begin{align*}
&u_n \rightharpoonup u^* \text{ in } L^\infty(0, T; G_{div}), \\
u_n \rightharpoonup u^* \text{ in } L^2(0, T; \mathcal{V}_{div}), \\
&\varphi_n \rightharpoonup \varphi^* \text{ in } L^\infty(0, T; H), \\
&\varphi_n \rightharpoonup \varphi^* \text{ in } L^2(0, T; V), \\
&U_n \rightharpoonup U^* \text{ in } L^2(0, T; G_{div}).
\end{align*}
\]

(4.29)

A calculation similar to proof of Theorem 2.7, [5], Theorem 2] and theorem, 2.10 [17] Theorem 2] and using the Aubin-Lion’s compactness arguments and the convergence in (4.29), we get

\[
\begin{align*}
u_n &\to u^* \text{ in } L^2(0, T; G_{div}), \text{ a. e. in } \Omega \times (0, T), \\
&\varphi_n \to \varphi^* \text{ in } L^2(0, T; H), \text{ a. e. in } \Omega \times (0, T).
\end{align*}
\]

(4.30)
Therefore, we get
\[ u_n \in L^\infty(0, T; \mathbb{V}_{\text{div}}) \cap L^2(0, T; H^2(\Omega)), \quad \varphi_n \in L^\infty((0, T) \times \Omega) \times L^\infty(0, T; \mathbb{V}). \]
and
\[ \varphi_n \in L^\infty(0, T; W^{1,p}), \quad 2 \leq p < \infty. \]
Thus, we have (see Theorem 2, [18] also)
\[
\begin{cases}
    u_n &\xrightarrow{w^*} u^* \text{ in } L^\infty(0, T; \mathbb{V}_{\text{div}}), \\
    u_n &\rightharpoonup u^* \text{ in } L^2(0, T; H^2(\Omega)), \\
    \varphi_n &\rightarrow \varphi^* \text{ a.e., } (t, x) \in (0, T) \times \Omega,
\end{cases}
\tag{4.31}
\]
and the other convergences can be obtained from the regularity results. Thus the above convergences and Remark 2.12 (see 2.21) imply that \((u^*, \varphi^*, U^*)\) has the regularity given in \((2.18)\) and \((2.19)\), and also
\[ u^*(0) = u^*_0 \in \mathbb{V}_{\text{div}} \text{ and } \varphi^*(0) = \varphi^*_0 \in H^2(\Omega). \tag{4.32} \]
Hence \((u^*, \varphi^*, U^*)\) is a unique strong solution of \((2.1a)-(2.1f)\) with control \(U^* \in \mathcal{U}_{\text{ad}}\). This easily implies \((u^*, \varphi^*, U^*) \in \mathcal{A}_{\text{ad}}\).

**Claim (3):** \(J = J(u^*, \varphi^*, U^*)\). Recall that \(\mathcal{U}_{\text{ad}}\) is a closed and convex subset of \(L^2(0, T; G_{\text{div}})\).
Since the cost functional \(J(\cdot, \cdot, \cdot)\) is continuous and convex on \(L^2(0, T; G_{\text{div}}) \times L^2(0, T; H) \times \mathcal{U}_{\text{ad}}\), it follows that \(J(\cdot, \cdot, \cdot)\) is weakly lower semi-continuous (see Proposition 1, Chapter 5, [3]). That is, for a sequence
\[
(u_n, \varphi_n, U_n) \xrightarrow{w^*} (u^*, \varphi^*, U^*) \text{ in } L^2(0, T; \mathbb{V}_{\text{div}}) \times L^2(0, T; \mathbb{V}) \times L^2(0, T; G_{\text{div}}),
\]
we have
\[
J(u^*, \varphi^*, U^*) \leq \liminf_{n \to \infty} J(u_n, \varphi_n, U_n).
\]
Therefore, we get
\[
J \leq J(u^*, \varphi^*, U^*) \leq \liminf_{n \to \infty} J(u_n, \varphi_n, U_n) = \lim_{n \to \infty} J(u_n, \varphi_n, U_n) = J,
\]
and hence \((u^*, \varphi^*, U^*)\) is a minimizer. \(\square\)

### 4.3. Pontryagin’s Maximum Principle

In this subsection, we prove the Pontryagin’s maximum principle for the optimal control problem defined in \((\text{OCP})\). Pontryagin Maximum principle gives a first order necessary condition for the optimal control problem \((\text{OCP})\). We also characterize the optimal control in terms of the adjoint variables. Even though we announced the subsection title as Pontryagin’s maximum principle, our problem is a minimization of the cost functional given in \((4.1)\) and hence we obtain a minimum principle.

The following minimum principle is satisfied by the optimal triplet \((u^*, \varphi^*, U^*) \in \mathcal{A}_{\text{ad}}:\n\]
\[
\frac{1}{2}\|U^*(t)\|^2 - \langle p(t), U^*(t) \rangle \leq \frac{1}{2}\|W(t)\|^2 - \langle p(t), W \rangle,
\tag{4.33}
\]
for all $U \in G_{div}$, and a.e. $t \in [0, T]$. Equivalently the above minimum principle may be written in terms of the Hamiltonian formulation. Let us first define the Lagrangian by

$$\mathcal{L}(u, \varphi, U) = \frac{1}{2} (\|u - u_d\|^2 + \|\varphi - \varphi_d\|^2 + \|U\|^2).$$

Then, we can define the corresponding Hamiltonian by

$$\mathcal{H}(u, \varphi, U, p, \eta) = \mathcal{L}(u, \varphi, U) - \langle p, \mathcal{N}_1(u, \varphi, U) \rangle - \langle \eta, \mathcal{N}_2(u, \varphi) \rangle,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are defined by (4.4). Hence, we get the minimum principle as

$$\mathcal{H}(u^*(t), \varphi^*(t), U^*(t), p(t), \eta(t)) \leq \mathcal{H}(u^*(t), \varphi^*(t), W, p(t), \eta(t)), \quad \text{for all } W \in G_{div}, \text{ a.e., } t \in [0, T].$$

**Definition 4.6 (Subgradient, Subdifferential).** Let $\mathbb{X}$ be a real Banach space and $f : \mathbb{X} \to (-\infty, \infty]$ a functional on $\mathbb{X}$. A linear functional $u' \in \mathbb{X}'$ is called subgradient of $f$ at $u$ if

$$f(v) \geq f(u) + \langle u', v - u \rangle_{\mathbb{X}' \times \mathbb{X}},$$

holds for all $v \in \mathbb{X}$. The set of all subgradients of $f$ at $u$ is called subdifferential $\partial f(u)$ of $f$ at $u$.

Now we state the main result of our paper. For similar results regrading the incompressible Navier-Stokes equations, see for example [11, 25] and for the linearized compressible Navier-Stokes equations, see [15].

**Theorem 4.7 (Pontryagin’s Minimum Principle).** Let $(u^*, \varphi^*, U^*) \in \mathcal{A}_{ad}$ be the optimal solution of the Problem [OCP] obtained in Theorem 4.3 Then there exists a unique weak solution $(p, \eta)$ of the adjoint system (4.8) and for almost every $t \in [0, T]$ and $W \in G_{div}$, we have

$$\frac{1}{2} \|U^*(t)\|^2 - \langle p(t), U^*(t) \rangle \leq \frac{1}{2} \|W\|^2 - \langle p(t), W \rangle. \quad (4.35)$$

Furthermore, we can write (4.35) as

$$\langle p(t), W - U^*(t) \rangle \leq \frac{1}{2} \|W\|^2 - \frac{1}{2} \|U^*(t)\|^2, \quad \text{a.e., } t \in [0, T]. \quad (4.36)$$

In the above form, we see that $p \in \partial \frac{1}{2} \|U^*(t)\|^2$, where $\partial$ denotes the subdifferential. Since, $\frac{1}{2} \|\cdot\|^2$ is Gâteaux differentiable, the subdifferential consists of a single point and it follows that

$$p(t) = U^*(t), \quad \text{a.e., } t \in [0, T].$$

**Proof of Theorem 4.7** Let $(u^*, \varphi^*, U^*) \in \mathcal{A}_{ad}$ be the optimal triplet for the control problem [OCP]. Let $\mathcal{F}(U) = \mathcal{J}(u_U, \varphi_U, U)$, where $(u_U, \varphi_U, U)$ is the solution of (2.1a)-(2.1f) with control $U$. Then, we have

$$\mathcal{F}(U^* + \lambda U) - \mathcal{F}(U^*)$$

$$= \mathcal{J}(u_{U^* + \lambda U}, \varphi_{U^* + \lambda U}, U^* + \lambda U) - \mathcal{J}(u_{U^*}, \varphi_{U^*}, U^*)$$

$$= \frac{1}{2} \int_0^T \|u_{U^* + \lambda U}(t) - u_d(t)\|^2 dt + \frac{1}{2} \int_0^T \|\varphi_{U^* + \lambda U}(t) - \varphi_d(t)\|^2 dt + \frac{1}{2} \int_0^T \|U^*(t) + \lambda U(t)\|^2 dt$$

$$- \frac{1}{2} \int_0^T \|u_{U^*}(t) - u_d(t)\|^2 dt - \frac{1}{2} \int_0^T \|\varphi_{U^*}(t) - \varphi_d(t)\|^2 dt - \frac{1}{2} \int_0^T \|U^*(t)\|^2 dt$$

$$+ \int_0^T \mathcal{F}(u_{U^* + \lambda U}(t), \varphi_{U^* + \lambda U}(t), U^* + \lambda U(t)) dt - \int_0^T \mathcal{F}(u_{U^*}(t), \varphi_{U^*}(t), U^*(t)) dt.$$
\[
\begin{align*}
&= \frac{1}{2} \int_0^T (\mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t), \mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t) - 2\mathbf{u}_d(t)) \, dt \\
&+ \frac{1}{2} \int_0^T (\varphi_{U^* + \lambda U}(t) - \varphi_{U^*}(t), \varphi_{U^* + \lambda U}(t) + \varphi_{U^*}(t) - 2\varphi_d(t)) \, dt \\
&+ \frac{1}{2} \int_0^T 2\lambda(U(t), U^*)(t) \, dt + \frac{1}{2} \int_0^T \lambda^2(U(t), U(t)) \, dt \\
&= \frac{1}{2} \int_0^T (\mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t), \mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t)) \, dt \\
&+ \frac{1}{2} \int_0^T (\varphi_{U^* + \lambda U}(t) - \varphi_{U^*}(t), \varphi_{U^* + \lambda U}(t) - \varphi_{U^*}(t)) \, dt \\
&+ \frac{1}{2} \int_0^T \lambda(U(t), U^*)(t) \, dt + \frac{1}{2} \int_0^T \lambda^2(U(t), U(t)) \, dt.
\end{align*}
\]

Therefore, we get
\[
\mathcal{F}(U^* + \lambda U) - \mathcal{F}(U^*)
\]
\[
= \frac{1}{2} \int_0^T \|\mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t)\|^2 \, dt + \int_0^T (\mathbf{u}_{U^* + \lambda U}(t) - \mathbf{u}_{U^*}(t), \mathbf{u}_{U^*}(t) - \mathbf{u}_d(t)) \, dt \\
+ \frac{1}{2} \int_0^T \|\varphi_{U^* + \lambda U}(t) - \varphi_{U^*}(t)\|^2 \, dt + \int_0^T (\varphi_{U^* + \lambda U}(t) - \varphi_{U^*}(t), \varphi_{U^*}(t) - \varphi_d(t)) \, dt \\
+ \frac{1}{2} \int_0^T \lambda(U(t), U(t)) \, dt + \int_0^T \lambda^2(U(t), U(t)) \, dt.
\]

(4.37)

Since \((\mathbf{u}_{U^* + \lambda U}, \varphi_{U^* + \lambda U})\) and \((\mathbf{u}_{U^*}, \varphi_{U^*})\) are the unique solution of the system (2.1a)-(2.1f) with controls \(U^* + \lambda U\) and \(U^*\) respectively, using the estimates given in the uniqueness theorem (see Theorem 2.10), \(\|\mathbf{u}_{U^* + \lambda U} - \mathbf{u}_{U^*}\|_{L^2(0,T;V'_{\text{div}})}\) can be estimated by \(\lambda^2 \|\mathbf{u}\|_{L^2(0,T;V'_{\text{div}})}\).

Thus dividing by \(\lambda\), and then sending \(\lambda \to 0\), we have \(\|\mathbf{u}_{U^* + \lambda U} - \mathbf{u}_{U^*}\|_{L^2(0,T;G_{\text{div}})} \to 0\).

Similarly \(\|\varphi_{U^* + \lambda U} - \varphi_{U^*}\|_{L^2(0,T;V'_{\text{div}})} \to 0\) as \(\lambda \to 0\). Let us denote the Gâteaux derivative of \(\mathcal{F}\) at \(U^*\) in the direction of \(U\) by \(\mathcal{F}'(U^*) \cdot U\). Let \((w, \psi)\) satisfies the linearized system (3.2a)-(3.2b) with control \(U\), and initial condition and forcing term to be equal to zero, that is, \(w(0) = \psi(0) = 0\). From Lemma 4.8 (see below), we have
\[
\lim_{\lambda \to 0} \frac{\|\mathbf{u}_{U^* + \lambda U} - \mathbf{u}_{U^*} - \lambda \mathbf{w}\|_{L^2(0,T;V_{\text{div}})}}{|\lambda|} = 0,
\]

(4.38)

and
\[
\lim_{\lambda \to 0} \frac{\|\varphi_{U^* + \lambda U} - \varphi_{U^*} - \lambda \psi\|_{L^2(0,T;V'_{\text{div}})}}{|\lambda|} = 0.
\]

(4.39)

Dividing by \(\lambda\) and then taking \(\lambda \to 0\) in (4.37), we obtain
\[
0 \leq \mathcal{F}'(U^*) \cdot U = \lim_{\lambda \to 0} \frac{\mathcal{F}(U^* + \lambda U) - \mathcal{F}(U^*)}{|\lambda|}.
\]
Similarly if we take the directional derivative of $F$ a.e.

\[ \frac{\partial F}{\partial t} \cdot \psi = \int_0^T \frac{\partial F}{\partial t} \psi dt \]

we have the following minimum principle

\[ \int_0^T (U(t), U^*(t)) dt, \]

where

\[
\begin{align*}
\mathbf{w} &= \lim_{\lambda \to 0} \frac{\mathbf{u}_{\lambda} - \mathbf{u}_0}{|\lambda|}, \\
\psi &= \lim_{\lambda \to 0} \frac{\mathbf{u}_{\lambda} - \mathbf{u}_0}{|\lambda|}.
\end{align*}
\]

Identifying the inner product in $G_{div}$ with the duality pairing between $V_{div}$ and $V_{div}'$, and the inner product in $H$ with the duality pairing between $V'$ and $V$, using (4.8), we obtain

\[
0 \leq \mathcal{F}'(U^*) \cdot U = \int_0^T \left( -\mathbf{w} \right)_t + \nu \nabla \mathbf{w} - (\mathbf{p} \cdot \nabla) \mathbf{u}^* - (\mathbf{u}^* \cdot \nabla) \mathbf{w} - \nabla a \mathbf{q}^* \psi, \mathbf{p} \right) dt
\]

\[
+ \int_0^T \left( \nabla \cdot (\mathbf{w} \cdot \nabla \mathbf{u}^*) - \nabla a \mathbf{q}^* \psi, \mathbf{p} \right) dt
\]

\[
+ \int_0^T \left( -\psi_t - \nabla \mathbf{u}^* \cdot \nabla \mathbf{w} - \mathbf{u}^* \cdot \nabla \psi + \nabla \varphi^* \mathbf{w} \right) dt
\]

\[
+ \int_0^T (U(t), U^*) dt
\]

Thus, we have

\[
0 \leq \mathcal{F}'(U^*(t)) \cdot U(t) = \int_0^T (-U(t), \mathbf{p}(t)) dt + \int_0^T (U(t), U^*(t)) dt.
\]

Similarly if we take the directional derivative of $F$ in the direction of $-U$, we get $\mathcal{F}'(U^*(t)) \cdot U(t) \leq 0$. Hence, we obtain $\mathcal{F}'(U^*(t)) \cdot U(t) = 0$, so that

\[ \int_0^T (-U(t), \mathbf{p}(t)) dt + \int_0^T (U(t), U^*(t)) dt = 0. \]

Since the above equality is true for all $U \in G_{div}$, we get

\[ U^*(t) = \mathbf{p}(t), \]

a.e. $t \in [0, T]$. Since, the optimum is $U^* = \mathbf{p}$, it minimizes the Hamiltonian, and hence we have the following minimum principle

\[
\frac{1}{2} \|U^*(t)\|^2 - \langle \mathbf{p}(t), U^*(t) \rangle \leq \frac{1}{2} \|W\|^2 - \langle \mathbf{p}(t), W \rangle,
\]

ea.e. $t \in [0, T]$ and $W \in G_{div}$. \qed
Lemma 4.8. Let \((u_0, \varphi_0)\) satisfies (4.3), the mapping \(U \mapsto (u_U, \varphi_U)\) from \(L^2(0, T; \mathcal{V}_{div})\) into 
\((L^\infty(0, T; G_{div}) \cap L^2(0, T; \mathcal{V}_{div})) \times (L^\infty(0, T; H) \cap L^2(0, T; V))\) is Gâteaux differentiable. Furthermore, we have
\[
\left( \lim_{\lambda \to 0} \frac{u_{U^*+\lambda U} - u_{U^*}}{\lambda}, \lim_{\lambda \to 0} \frac{\varphi_{U^*+\lambda U} - \varphi_{U^*}}{\lambda} \right) = (w, \psi), \tag{4.41}
\]
where \((w, \psi)\) is the unique weak solution of
\[
w_t - \nu \Delta w + (w \cdot \nabla)u_{U^*} + (u_{U^*} \cdot \nabla)w + \nabla \pi_w
\]
\[
= \nabla a \varphi \varphi_{U^*} - (J * \psi) \nabla \varphi_{U^*} - (J * \varphi_{U^*}) \nabla \psi + U, \quad \text{in } \Omega \times (0, T),
\]
\[
\psi_t + w \cdot \nabla \varphi_{U^*} + u_{U^*} \cdot \nabla \psi = \Delta \tilde{\mu}, \quad \text{in } \Omega \times (0, T),
\]
\[
div w = 0, \quad \text{in } \Omega \times (0, T),
\]
\[
\frac{\partial \tilde{\mu}}{\partial n} = 0, \quad w = 0 \text{ on } \partial \Omega \times (0, T),
\]
\[
w(0) = 0, \quad \psi(0) = 0 \quad \text{in } \Omega.
\]
The pair \((u_{U^*}, \varphi_{U^*})\) and \((u_{U^*+\lambda U}, \varphi_{U^*+\lambda U})\) are the unique strong solutions of (2.1a)-(2.1f) with controls \(U^*\) and \(U^* + \lambda U\), respectively.

Proof. In order to prove (4.41), we need to prove (4.38) and (4.39). Let us set
\[
(\tilde{u}, \tilde{\varphi}) = (u_{U^*+\lambda U} - u_{U^*}, \varphi_{U^*+\lambda U} - \varphi_{U^*})
\]
and
\[
(y, \varphi) = (\tilde{u} - \lambda w, \tilde{\varphi} - \lambda \psi).
\]
Observe that, \((\tilde{u}, \tilde{\varphi})\) satisfies the following system:
\[
\begin{cases}
\tilde{u}_t - \nu \Delta \tilde{u} + (\tilde{u} \cdot \nabla)u_{U^*} + (u_{U^*+\lambda U} \cdot \nabla)\tilde{u} + \nabla \pi_{\tilde{u}} = \\
\quad = -\frac{\nabla a}{2} \tilde{\varphi}(\varphi_{U^*+\lambda U} + \varphi_{U^*}) - (J * \tilde{\varphi}) \nabla \varphi_{U^*}, \quad \text{in } \Omega \times (0, T),
\end{cases}
\]
\[
\tilde{\varphi}_t + \tilde{u} \cdot \nabla \varphi_{U^*+\lambda U} + u_{U^*} \cdot \nabla \tilde{\varphi} = \Delta \tilde{\mu}, \quad \text{in } \Omega \times (0, T),
\]
\[
\tilde{\mu} = a \tilde{\varphi} - J * \tilde{\varphi} + F'(\varphi_{U^*}) \tilde{\varphi}, \quad \text{in } \Omega \times (0, T),
\]
\[
div \tilde{u} = 0, \quad \text{in } \Omega \times (0, T),
\]
\[
\frac{\partial \tilde{\mu}}{\partial n} = 0, \quad \tilde{u} = 0 \text{ on } \partial \Omega \times (0, T),
\]
\[
\tilde{u}(0) = 0, \quad \psi(0) = 0 \quad \text{in } \Omega,
\]
where
\[
\pi_{\tilde{u}} = \pi_{U^*+\lambda U} - \pi_{U^*} - \left[F'(\varphi_{U^*}) \tilde{\varphi} + \frac{a}{2} \left(\varphi_{U^*+\lambda U} + \varphi_{U^*}\right) \tilde{\varphi}\right]. \tag{4.44}
\]

While considering (4.43), we ignored the second order terms involving \(\nabla^2\) in the equation for \(\tilde{\mu}\), since \(\varphi_{U^*} \in L^\infty((0, T) \times \Omega)\) and \(F(\cdot)\) has an arbitrary polynomial growth in \(\varphi_{U^*}\) (see Remark 2.5). A same kind of approximation we did for \(\pi_{\tilde{u}}\) in (4.44) also.
Moreover, it can be shown that \((y, \xi)\) satisfies the following system:

\[
\begin{aligned}
y_t - \nu \Delta y + (y \cdot \nabla) u^* + (u^* \cdot \nabla) y + \nabla \pi y + (J * \xi) \nabla \varphi u^* + (J * \varphi u^*) \nabla \xi - \nabla \delta \varphi
\quad &= \nabla \left( \frac{\varphi}{2} \right) - (\tilde{u} \cdot \nabla) \tilde{u} - \nabla \tilde{\varphi} - (J * \tilde{\varphi}) \nabla \tilde{\varphi}, \quad \text{in } \Omega \times (0, T), \\
\xi_t + y \cdot \nabla \varphi u^* + u^* \cdot \nabla \xi - \Delta \tilde{\mu}_\xi &= -\tilde{u} \cdot \nabla \tilde{\varphi}, \quad \text{in } \Omega \times (0, T), \\
\tilde{\mu}_\xi &= \delta \xi - J * \xi + F'(\varphi u^*) \xi, \quad \text{in } \Omega \times (0, T), \\
\text{div } y &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \tilde{\mu}_\xi}{\partial n} &= 0, \quad y = 0 \text{ on } \partial \Omega \times (0, T), \\
y(0) &= 0, \quad \xi(0) = 0 \quad \text{in } \Omega,
\end{aligned}
\]

where

\[
\pi y = (\pi u^* + \lambda u - \pi u^*) - \lambda \tilde{\pi} - [(F'(\varphi u^*) + a \varphi u^*) \xi].
\]

Let us denote the right hand side of the equations for \(y\) and \(\xi\) by \(k(x, t)\) and \(l(x, t)\), respectively. That is,

\[
\begin{aligned}
k(x, t) &= \nabla \left( \frac{\varphi}{2} \right) - (\tilde{u} \cdot \nabla) \tilde{u} - \nabla \tilde{\varphi} - (J * \tilde{\varphi}) \nabla \tilde{\varphi}, \\
l(x, t) &= -\tilde{u} \cdot \nabla \tilde{\varphi}.
\end{aligned}
\]

Our next aim is to show that

\[
\begin{align}
\|k(x, t)\|_{L^2(0, T; \mathcal{V}'_{\text{div}})} &\leq C \left( \|\tilde{u}\|_{L^\infty(0, T; G_{\text{div}})} \|\tilde{u}\|_{L^2(0, T; \mathcal{V}_{\text{div}})} + \|\tilde{\varphi}\|_{L^\infty(0, T; \mathcal{H})} \|\tilde{\varphi}\|_{L^2(0, T; \mathcal{V})} \right), \\
\|l(x, t)\|_{L^2(0, T; \mathcal{V}')} &\leq C \left( \|\tilde{u}\|_{L^\infty(0, T; G_{\text{div}})} \|\tilde{u}\|_{L^2(0, T; \mathcal{V}_{\text{div}})} + \|\tilde{\varphi}\|_{L^\infty(0, T; \mathcal{H})} \|\tilde{\varphi}\|_{L^2(0, T; \mathcal{V})} \right),
\end{align}
\]

which implies

\[
\begin{align}
\|y\|_{L^2(0, T; \mathcal{V}_{\text{div}})} &\leq C \|k(x, t)\|_{L^2(0, T; \mathcal{V}'_{\text{div}})} \leq C \lambda^2, \\
\|\xi\|_{L^2(0, T; \mathcal{V}')} &\leq C \|l(x, t)\|_{L^2(0, T; \mathcal{V}')} \leq C \lambda^2,
\end{align}
\]

by using Theorem [2.10]. Using the H"older and Ladyzhenskaya inequalities, it is clear that

\[
\begin{align}
\int_0^T \| \nabla \left( \frac{\varphi}{2} \right) \|_{\mathcal{V}'_{\text{div}}}^2 \, dt &\leq C \|a\|_{L^\infty} \int_0^T \|\varphi(t)\|^2 \, dt \leq C \|a\|_{L^\infty} \int_0^T \|\varphi(t)\|_{L^4}^4 \, dt \\
&\leq C \|a\|_{L^\infty} \int_0^T \|\varphi(t)\| \|\nabla \varphi(t)\|^2 \, dt \\
&\leq C \|a\|_{L^\infty} \sup_{t \in [0, T]} \|\varphi(t)\| \int_0^T \|\nabla \varphi(t)\|^2 \, dt,
\end{align}
\]

so that we find

\[
\| \nabla \left( \frac{\varphi}{2} \right) \|_{L^2(0, T; \mathcal{V}'_{\text{div}})} \leq C \|a\|_{L^\infty} \|\varphi\|_{L^\infty(0, T; \mathcal{H})} \|\varphi\|_{L^2(0, T; \mathcal{V})}.
\]

(4.50)
We now take inner product of \( k(x,t) \) and \( l(x,t) \) with \( y \) and \( B^{-1}(\varrho - \overline{\varrho}) \), respectively to obtain

\[
(k(x,t), y) = -((\tilde{u} \cdot \nabla)\tilde{u}, y) - \left( \frac{\nabla a_{2,\varrho}}{2}, \Phi \right) - ((J \ast \tilde{\varphi}) \nabla \tilde{\varphi}, y),
\] (4.51)

\[
(l(x,t), B^{-1}(\varrho - \overline{\varrho})) = - \left( \tilde{u} \cdot \nabla \tilde{\varphi}, B^{-1}(\varrho - \overline{\varrho}) \right).
\] (4.52)

The terms in the right hand side of (4.51) and (4.52) can be estimated in a similar way as in the proof of Theorem 2, [17]. Using (2.5), we know that

\[
\|((\tilde{u} \cdot \nabla)\tilde{u}, y)\| \leq \|\tilde{u}\|_{L^4}^2 \|\nabla y\| \leq \|\tilde{u}\|_{L^4}^2 \|y\|_{W^{1,4}}
\]

so that by using the Ladyzhenskaya inequality, we get

\[
\int_0^T \|((\tilde{u}(t) \cdot \nabla)\tilde{u}(t))\|_{W^{1,4}}^2 dt \leq \int_0^T \|\tilde{u}(t)\|_{L^4}^4 dt \leq 2 \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{L^2}^2 \int_0^T \|\nabla \tilde{u}(t)\|_{L^2}^2 dt.
\]

Thus, we obtain

\[
\|((\tilde{u} \cdot \nabla)\tilde{u})\|_{L^2(0,T;W^{1,4})} \leq \sqrt{2}\|\tilde{u}\|_{L^\infty(0,T;G_{div})} \|\tilde{u}\|_{L^2(0,T;W_{div})}. \tag{4.53}
\]

Using H"older’s inequality and Poincaré inequality, we also have

\[
\left| \left( \frac{\nabla a_{2,\varrho}}{2}, \Phi \right) \right| \leq \frac{1}{2} \|\nabla a\|_{L^\infty} \|\tilde{\varphi}\|_{L^4}^2 \|y\| \leq C\|\nabla a\|_{L^\infty} \|\tilde{\varphi}\|_{L^4}^2 \|y\|_{W^{1,4}}.
\]

Hence, we get

\[
\int_0^T \left| \frac{\nabla a_{2,\varrho}}{2} \right| \|\Phi(t)\|_{L^4}^2 dt \leq C\int_0^T \|\tilde{\varphi}(t)\|_{L^4}^4 dt \leq C \sup_{t \in [0,T]} \|\tilde{\varphi}(t)\|_{L^4}^2 \int_0^T \|\nabla \tilde{\varphi}(t)\|_{L^2}^2 dt.
\]

Thus, it follows that

\[
\left\| \frac{\nabla a_{2,\varrho}}{2} \right\|_{L^2(0,T;W_{div}')} \leq C\|\tilde{\varphi}\|_{L^\infty(0,T;H)} \|\tilde{\varphi}\|_{L^2(0,T;V)} \tag{4.54}
\]

Once again using an integration by parts, H"older’s inequality and Poincaré inequality, we obtain

\[
\left| \left( J \ast \tilde{\varphi} \right) \nabla \tilde{\varphi}, y \right| \leq \|\nabla J\|_{L^1} \|\tilde{\varphi}\|_{L^4}^2 \|y\| \leq C\|\nabla J\|_{L^1} \|\tilde{\varphi}\|_{L^4}^2 \|y\|_{W^{1,4}}.
\]

An estimate similar to (4.54) and (2.8) yields

\[
\|\left( J \ast \tilde{\varphi} \right) \nabla \tilde{\varphi}\|_{L^2(0,T;W_{div}')} \leq C\|\tilde{\varphi}\|_{L^\infty(0,T;H)} \|\tilde{\varphi}\|_{L^2(0,T;V)} \tag{4.55}
\]

Combining (4.50), (4.53)—(4.55), we finally obtain (4.46).

Now we estimate the right hand side term in (4.52) using an integration by parts and H"older’s inequality as

\[
\left| \left( \tilde{u} \cdot \nabla \tilde{\varphi}, B^{-1}(\varrho - \overline{\varrho}) \right) \right| \leq \|\tilde{u}\|_{L^4} \|B^{-1/2}(\varrho - \overline{\varrho})\|_{L^1} \|\tilde{\varphi}\|_{L^4}.
\]

It should be noted that \( \tilde{u} \cdot \nabla \tilde{\varphi} = 0 \), since an integration by parts, \( \tilde{u}|_{\partial \Omega} = 0 \) and the divergence free condition of \( \tilde{u} \) yields

\[
\tilde{u} \cdot \nabla \tilde{\varphi} = \frac{1}{|\Omega|} \sum_{i=1}^2 \int_{\Omega} \tilde{u}_i(x) \frac{\partial \tilde{\varphi}(x)}{\partial x_i} dx = \frac{1}{|\Omega|} \sum_{i=1}^2 \left[ \tilde{\varphi}(x) \frac{\partial \tilde{u}_i(x)}{\partial x_i} n_i(x) \right]_{\partial \Omega} - \int_{\Omega} \frac{\partial \tilde{u}_i(x)}{\partial x_i} \tilde{\varphi}(x) dx = 0.
\]
Thus to estimate $\|\tilde{u} \cdot \nabla \tilde{\varphi}\|_{L^2(0,T;V^*)}$, we use the Ladyzhenskaya, Hölder and Young’s inequalities to get

\[
\int_0^T \|B^{-1/2}(\tilde{u} \cdot \nabla \tilde{\varphi}(t))\|^2 dt = \int_0^T \langle (\tilde{u} \cdot \nabla \tilde{\varphi})(t), B^{-1}(\tilde{u} \cdot \nabla \tilde{\varphi})(t) \rangle dt \\
\leq \int_0^T \|\tilde{u}(t)\|^2_L \|\tilde{\varphi}(t)\|^2_L dt \leq 2 \left( \int_0^T \|\tilde{u}(t)\|^2 \|\nabla \tilde{\varphi}(t)\|^2 dt \right)^{1/2} \left( \int_0^T \|\tilde{\varphi}(t)\|^2 \|\nabla \tilde{\varphi}(t)\|^2 dt \right)^{1/2} \\
\leq \sup_{t \in [0,T]} \|\tilde{u}(t)\|^2 \left( \int_0^T \|\nabla \tilde{u}(t)\|^2 dt \right) + \sup_{t \in [0,T]} \|\tilde{\varphi}(t)\|^2 \left( \int_0^T \|\nabla \tilde{\varphi}(t)\|^2 dt \right),
\]

so that

\[
\|\tilde{u} \cdot \nabla \tilde{\varphi}\|_{L^2(0,T;V^*)} \leq C \left( \|\tilde{u}\|_{L^\infty(0,T;G_{div})} \|\tilde{u}\|_{L^2(0,T;V_{div})} + \|\tilde{\varphi}\|_{L^\infty(0,T;H)} \|\tilde{\varphi}\|_{L^2(0,T;V)} \right).
\]

(4.56)

In the above calculation, we used the fact that $(a + b)^p \leq 2^p(a^p + b^p)$, for all $p > 0$. Thus (4.47) holds true. Also, the validity of (4.48) and (4.49) is immediate. Hence by taking $\lambda \to 0$, we finally arrive at (4.38) and (4.39). Also, we know that $(\pi_{U^* + \lambda U} - \pi_{U^*}) - \lambda \tilde{\pi}$ has a zero average, so that from the above calculations, it is immediate that

\[
\int_0^T \|\pi_{U^* + \lambda U} - \pi_{U^*}\|_{L^2}^2 dt \to 0,
\]

as $\lambda \to 0$. Hence, we also have

\[
\tilde{\pi} = \lim_{\lambda \to 0} \frac{\pi_{U^* + \lambda U} - \pi_{U^*}}{|\lambda|},
\]

which completes the proof. \hfill \Box

**Remark 4.9.** One can also get the above convergence directly in the $V$—space as

\[
\lim_{\lambda \to 0} \frac{\|\varphi\|_{L^2(0,T;V)}}{|\lambda|} = \lim_{\lambda \to 0} \frac{\|\varphi_{U^* + \lambda U} - \varphi_{U^*} - \lambda \psi\|_{L^2(0,T;V)}}{|\lambda|} = 0,
\]

(4.57)

where

\[
\|\varphi\|_{L^2(0,T;V)}^2 = \int_0^T \left( \|\varphi(t)\|^2 + \|\nabla \varphi(t)\|^2 \right) dt.
\]

First let us estimate $\|\nabla ((a + F''(\varphi_{U^*}))\varphi)\|$ using the Hölder, Ladyzhenskaya and Young’s inequalities as

\[
\|\nabla ((a + F''(\varphi_{U^*}))\varphi)\| \\
\leq \|\nabla (a + F''(\varphi_{U^*}))\varphi\| + \|(a + F''(\varphi_{U^*})) \nabla \varphi\| \\
\leq \|\nabla a\|_{L^\infty} \|\varphi\| + \|F''(\varphi_{U^*})\|_{L\infty} \|\nabla \varphi_{U^*}\|_{L^4} \|\varphi\|_{L^4} + \|\nabla \varphi\| + \|F''(\varphi_{U^*})\|_{L^\infty} \|\nabla \varphi\| \\
\leq \left( \|a\|_{L^\infty} + \|\nabla a\|_{L^\infty} \right)^{1/2} \left( \|\varphi\|_{L^2}^2 + \|\nabla \varphi\|_{L^2}^2 \right)^{1/2} + 2^{1/4} \|F''(\varphi_{U^*})\|_{L^\infty} \|\nabla \varphi_{U^*}\|_{L^4} \|\varphi\|_{L^4}^{1/2} \|\nabla \varphi\|^{1/2} + \|F''(\varphi_{U^*})\|_{L^\infty} \|\nabla \varphi\| \\
\leq \left( \|a\|_{L^\infty} + \|\nabla a\|_{L^\infty} \right)^{1/2} \left( \|\varphi\|_{L^2}^2 + \|\nabla \varphi\|_{L^2}^2 \right)^{1/2} \\
+ (C \|F''(\varphi_{U^*})\|_{L\infty} \|\nabla \varphi_{U^*}\|_{L^4} + \|F''(\varphi_{U^*})\|_{L^\infty}) \left( \|\varphi\| + \|\nabla \varphi\| \right).
Remark 4.10. and hence we get (4.56). Thus it is immediate that
\[
\left( \|u\|_2 + \|\nabla u\|_{L^2}^2 \right)^{1/2} + \left( C \|F''(u^*)\|_2 \|\nabla u^*\|_{L^4}^2 + \|F''(u^*)\|_2 \right)^{1/2} \|\varepsilon\|_V
\]
=: M\|\varepsilon\|_V,
\]
where we have also used \((ac + bd) \leq (a^2 + b^2)^{1/2}(c^2 + d^2)^{1/2}\) for all \(a, b, c, d \in \mathbb{R}\). An integration by parts, Hölder and Ladyzhenskaya inequalities yield
\[
\left| (\tilde{u} \cdot \nabla \tilde{\varphi}, (a + F''(u^*))\varepsilon) \right| = \left| (u \cdot \nabla ((a + F''(u^*))\varepsilon), \tilde{\varphi}) \right| \leq \|\tilde{u}\|_{L^4} \|\tilde{\varphi}\|_{L^4} \|\nabla ((a + F''(u^*))\varepsilon)\| \leq \sqrt{2} M \|\tilde{u}\| \|\nabla \tilde{\varphi}\| \|\varepsilon\|_V.
\]
Thus, we have
\[
\int_0^T \|\tilde{u}(t) \cdot \nabla \tilde{\varphi}(t)\|_{V'}^2 dt
\]
\[
\leq 2 \int_0^T M(t)^2 \|\tilde{u}(t)\|^2 \|\nabla \tilde{u}(t)\|^2 \|\tilde{\varphi}(t)\|^2 \|\nabla \tilde{\varphi}(t)\|^2 dt
\]
\[
\leq 2 \sup_{t \in [0,T]} M(t) \left[ \sup_{t \in [0,T]} \|\tilde{u}(t)\|^2 \left( \int_0^T \|\nabla \tilde{u}(t)\|^2 dt \right) + \sup_{t \in [0,T]} \|\tilde{\varphi}(t)\|^2 \left( \int_0^T \|\nabla \tilde{\varphi}(t)\|^2 dt \right) \right],
\]
and hence we get (4.56). Thus it is immediate that
\[
\|I(x, t)\|_{L^2(0,T;V')} \leq C \left( \|\tilde{u}\|_{L^\infty(0,T;G_{div})} \|\tilde{u}\|_{L^2(0,T;V_{div})} + \|\tilde{\varphi}\|_{L^\infty(0,T;H)} \|\tilde{\varphi}\|_{L^2(0,T;V')} \right),
\]
and
\[
\|\varepsilon\|_{L^2(0,T;V)} \leq C \|I(x, t)\|_{L^2(0,T;V')} \leq C \lambda^2,
\]
and (4.57) follows easily.

Remark 4.10. When we considered the equation for \(\tilde{u}\) in (4.43), we ignored the second and higher order terms involving \(\tilde{\varphi}\). Suppose, we consider the term \(\frac{1}{2} F'''(u^*) \tilde{\varphi}^2\) also in the equation for \(\tilde{u}\) in (4.43), then one can estimate it as
\[
\left| \left( \Delta (F''(u^*) \tilde{\varphi}^2), B^{-1}(q - \overline{\sigma}) \right) \right| = \left| (F'''(u^*) \tilde{\varphi}^2, q - \overline{\sigma}) \right| \leq \sup_{x \in \Omega} |F'''(u^*)| \|\tilde{\varphi}\|_{L^4}^2 \|q - \overline{\sigma}\|,
\]
so that we find
\[
\|F'''(u^*) \tilde{\varphi}^2\| \leq \|F'''(u^*)\|_{L^\infty} \|\tilde{\varphi}\| \|\nabla \tilde{\varphi}\| \leq C(\|\varphi^*\|_{L^\infty}) \left( \|\tilde{\varphi}\|^2 + \|\nabla \tilde{\varphi}\|^2 \right).
\]
We ignored higher order terms in (4.44) also. It can be easily seen that
\[
\|\nabla (F''(u^*) \tilde{\varphi})\|_{V'_{div}} \leq \|F''(u^*)\|_{L^\infty} \|\tilde{\varphi}\| \leq \|F''(u^*)\|_{L^\infty} \|\tilde{\varphi}\|_{L^4}^2 \leq C(\|\varphi^*\|_{L^\infty}) \left( \|\tilde{\varphi}\|^2 + \|\nabla \tilde{\varphi}\|^2 \right).
\]
Thus (4.38) and (4.39) still holds true.
References

[1] F. Abergel and R. Temam, On Some Control Problems in Fluid Mechanics, *Theoretical and Computational Fluid Dynamics*, 1 (1990), 303–325.

[2] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, AMS Chelsea Publishing, Providence, RI, 1965.

[3] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Dover Publications, New York, 1984.

[4] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, *Asymptotic analysis* 20(2) (1999), 175–212.

[5] P. Colli, S. Frigeri and M. Grasselli, Global existence of weak solutions to a nonlocal Cahn–Hilliard Navier–Stokes system, *Journal of Mathematical Analysis and Applications* 386(1) (2012), 428–444.

[6] P. Colli, G. Gilardi, and J. Sprekels, Analysis and optimal boundary control of a nonstandard system of phase field equations, *Milan Journal of Mathematics*, 80(1) (2012), 1–31.

[7] P. Colli, and J. Sprekels, Optimal boundary control of a nonstandard Cahn–Hilliard system with dynamic boundary condition and double obstacle inclusions, In: Colli P., Favini A., Rocca E., Schimperna G., Sprekels J. (eds) Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs. Springer INdAM Series, vol 22, 151–182, Springer, Cham, 2017.

[8] P. Colli and G. Gilardi, Optimal boundary control of a nonstandard viscous Cahn–Hilliard system with dynamic boundary condition, arXiv preprint [arXiv:1609.07046] 2016.

[9] P. Colli and G. Gilardi, Distributed optimal control of a nonstandard nonlocal phase field system with double obstacle potential, arXiv preprint [arXiv:1607.01991] 2016.

[10] P. Colli and G. Gilardi, Distributed optimal control of a nonstandard nonlocal phase field system, arXiv preprint [arXiv:1605.07801] 2016.

[11] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth, *Nonlinearity*, 30 (2017), 2518–2546.

[12] P. Colli, G. Gilardi, and J. Sprekels, Distributed optimal control of a nonstandard nonlocal phase field system, *AIMS Mathematics*, 1(3) (2016), 225–260.

[13] P. Colli, M. H. Farshbaf-Shaker, G. Gilardi and J. Sprekels, Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials, *SIAM J. Control Optim.*, 53(2) (2015), 696–721.

[14] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[15] S. Doboszczak, M. T. Mohan, and S. S. Sritharan, Necessary conditions for distributed optimal control of linearized compressible Navier-Stokes equations, *Submitted*.

[16] D. E. Edmunds, Optimal control of systems governed by partial differential equations, *Bulletin of the London Mathematical Society* 4(2) (1972), 236–237.

[17] S. Frigeri, C. G. Gal and M. Grasselli, On nonlocal Cahn–Hilliard–Navier–Stokes systems in two dimensions, *Journal of Nonlinear Science*, 26(4) (2016), 847–893.

[18] S. Frigeri, M. Grasselli, and P. Krejci, Strong solutions for two-dimensional nonlocal Cahn–Hilliard-Navier–Stokes systems, *Journal of Differential Equations* 255(9) (2013), 2587–2614.

[19] A. V. Fursikov, *Optimal control of distributed systems: Theory and applications*, American Mathematical Society, Rhode Island, 2000.

[20] M. D. Gunzburger, *Perspectives in Flow Control and Optimization*, SIAM’s Advances in Design and Control series, Philadelphia, Society for Industrial and Applied Mathematics, 2003.

[21] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[22] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* 3 13 (1959), 115–162.

[23] J. P. Raymond, Optimal control of partial differential equations, *Université Paul Sabatier*, Lecture Notes, 2013.

[24] J. P. Raymond, *Boundary feedback stabilization of the two dimensional Navier-Stokes equations*, SIAM J. Control and Optimization, Vol. 45 (2006), 790-828.

[25] S. S. Sritharan, *Optimal Control of Viscous Flow*, SIAM Frontiers in Applied Mathematics, Philadelphia, Society for Industrial and Applied Mathematics, 1998.

[26] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.

[27] X. Zhao and C. Liu, Optimal control problem for viscous Cahn–Hilliard equation, *Nonlinear Analysis: Theory, Methods and Applications* 74(17) (2011), 6348–6357.
[28] J. Zheng and Y. Wang, Optimal control problem for Cahn–Hilliard equations with state constraint, *Journal of Dynamical and Control Systems* 21(2) (2015), 257–272.