Dark Matter Spin Precession

Alexander I Nesterov
Departamento de Física, CUCEI, Universidad de Guadalajara, CP 44420, Jalisco, México

Gennady P Berman
Los Alamos National Laboratory, T-4, Los Alamos, NM 87545, USA

Vladimir Tsifrinovich
Department of Applied Physics, NYU Tandon School of Engineering, NY 11201, USA

Xidi Wang
Department of Chemistry & Biochemistry, University of California San Diego, La Jolla CA 92093, USA

Marco Merkli
Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, NL, Canada A1C 5S7

(Dated: July 4, 2019)

We suggest that the pseudo-scalar vacuum field (PSV) in the dark matter (DM) sector of the Universe may be as important as the electromagnetic vacuum field in the baryonic sector. In particular, the spin-spin interaction between the DM fermions, mediated by PSV, may represent the strongest interaction between the DM fermions due to the absence of the electric charge and the magnetic dipole moment. Based on this assumption, we consider the influence of the spin-spin interaction, mediated by PSV, on the spin precession of the DM fermions. In the secular approximation, we obtain the exact expression describing the frequency of the precession and estimate the decoherence time.

Keywords: dark matter, spin-spin interaction

I. INTRODUCTION

It is well known that the pseudo-scalar vacuum (PSV) field (e.g. axionic field) interacts with the spins of the baryonic matter (see, for example, [1]). As a result, PSV mediates the spin-spin interactions in the baryonic matter (see, for example, [2]). Certainly, interactions between the PSV and baryonic matter represent a tiny correction to the electromagnetic interactions. We suggest that in the dark sector of the Universe the situation may be opposite: due to the absence of the electric charge and the magnetic moment, the interaction between the DM fermions and PSV may manifest the leading interaction in DM. In particular, the DM fermionic spins could produce permanent fields similar to the magnetic fields in the baryonic sector. The spin-spin interaction, mediated by PSV, may represent the main interaction between DM fermions.

Based on this assumption we analyze precession of the two DM fermionic spins 1/2, coupled through PSV, in a non-uniform external field produced by the other DM fermions. We assume that the two DM spins are located at the points with the position vectors, \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). The frequencies of the spin precession in the external field are given by \( \varepsilon_1 \) and \( \varepsilon_2 \). In order to obtain the analytical expressions describing the spin precession, we assume that \( \varepsilon_1, \varepsilon_2, |\varepsilon_1 - \varepsilon_2| \gg 1 \). This allows us to use the secular approximation well-known in magnetic resonance (see, for example, [3]).

With no external field, the interaction between the two spins and PSV can be written as (see, for example, [1]),

\[
H_{\text{int}} = \lambda \sum_{\alpha=1,2} \nabla \varphi \cdot \mathbf{\sigma}^\alpha.
\]

(1)

Here \( \varphi \) is the pseudo-scalar vacuum field, \( \lambda \) is the coupling constant, and \( \mathbf{\sigma}^\alpha \equiv (\sigma^\alpha_x, \sigma^\alpha_y, \sigma^\alpha_z) \) is the vector built by Pauli matrices of spin \( \alpha \), located at \( \mathbf{r}_\alpha \). Through this paper, we will use the natural convention, \( \hbar = c = 1 \).

II. PRECESSION OF THE SPINS

Our total Hamiltonian, \( H \), describes, in the secular approximation, the two fermionic spins interacting with the external field, oriented in the \( z \)-direction, and the PSV:

\[
H = \frac{1}{2} \sum_{\alpha=1,2} \varepsilon_\alpha \sigma^\alpha_z + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \\
+ \frac{i\lambda}{\sqrt{V}} \sum_{\alpha=1,2} \sigma^\alpha_z \sum_k \frac{k_z}{\sqrt{2\omega_k}} \left( \hat{a}_k e^{ik\cdot\mathbf{r}_\alpha} - \hat{a}_k^\dagger e^{-ik\cdot\mathbf{r}_\alpha} \right).
\]

(2)

Here \( \varepsilon_\alpha \) is the transition frequency of spin \( \alpha \) in the external permanent field, \( \omega_k = \sqrt{k^2 + m^2} \) is the frequency...
of the field mode with wave number $k$, where $m$ is the mass of a DM boson, and $V$ is the quantization volume.

We will consider the initial state, $|\psi\rangle$, of the spin-PSV system as the tensor product of the spin $|\psi_s\rangle$ and the PSV $|\psi_V\rangle$ states: $|\psi\rangle = |\psi_s\rangle \otimes |\psi_V\rangle$. Below, we use the $\sigma_z$-representation for the single spin states: vectors $|0\rangle$ and $|1\rangle$ $(\alpha = 1, 2)$ denote the spin pointing in the negative and positive $z$-direction, respectively. For two spins, we use the basis: $|00\rangle \equiv |0_01\rangle$, $|1\rangle \equiv |0_21\rangle$, $|2\rangle \equiv |1_201\rangle$, $|3\rangle \equiv |1_211\rangle$. Then, the initial spin wave function is, $|\psi_s\rangle = \sum_{i=0}^{3} C_i |i\rangle$, $(\sum_{i=0}^{3} |C_i|^2 = 1)$. The corresponding initial spin density matrix is,

$$\rho_s(0) = \sum_{i,j=0}^{3} \rho_{ij}(0) |i\rangle \langle j|, \quad (\rho_{ij}(0) = C_i C_j^*) .$$  (3)

The concrete values of the amplitudes, $C_i$, are not important for us. In the interaction representation, the evolution operator of the spin-PSV system can be written as ${\tilde U}$,

$$U(t) = \tilde{U} \exp \left(- \frac{i}{\hbar} \int_0^t dt' H_{int}(t') \right) \exp \left( \frac{i}{\hbar} \sum_{\alpha,\beta=1,2} \sigma_z^\alpha \sigma_z^\beta \nu_{\alpha\beta}(t) \right) \exp \left( \sum_{\alpha=1,2} \sigma_z^\alpha \sum_k \left( e^{-ik \cdot r_\alpha \xi_k(t)} \hat{a}_k^\dagger - e^{ik \cdot r_\alpha \xi_k(t)} \hat{a}_k \right) \right),$$  (4)

where

$$\xi_k(t) = \frac{\lambda k_z (e^{i\omega_k t} - 1)}{\sqrt{2V} \omega_k ^{3/2}}, \quad (5)$$

and

$$\nu_{\alpha\beta} = i \sum_{k} \cos(k \cdot r_{\alpha\beta}) \int_0^t dt' (\xi_k(t') \hat{\xi}_k(t') - \xi_k^*(t') \hat{\xi}_k(t')). \quad (6)$$

Here $r_{\alpha\beta} = r_\beta - r_\alpha$.

Using the evolution operator $U(t)$ of Eq. (4), one can write the total density matrix as,

$$\rho_{tot}(t) = U(t) \rho_{tot}(0) U^{-1}(t), \quad (7)$$

where $\rho_{tot}(0) = |\psi\rangle \langle \psi|$. Next, we have obtained the $4 \times 4$ reduced spin density matrix, $\rho_s(t)$, by tracing out the PSV degrees of freedom. (The details of computations are given in Appendix A.)

Using Eqs. (3) and (7), we calculate the $x$-component of the total spin:

$$\tilde{S}_x(t) = \sum_{\alpha=1,2} Tr(\rho_s(t) \sigma_x^\alpha) = \sum_{\alpha=1,2} \sum_{i=0}^{3} \langle i | \rho_s(t) \sigma_x^\alpha |i \rangle = \rho_{01}(t) + \rho_{02}(t) + \rho_{13}(t) + \rho_{23}(t) + c.c., \quad (8)$$

where $\rho_{ij}(t)$ are the time-dependent components of the reduced spin density matrix. One can see that $\tilde{S}_x(t)$ is associated with the precession of a single spin while the other spin remains in a basis state.

For definiteness, consider the element $\rho_{01}(t)$ of the reduced spin density matrix. We have derived the following expression:

$$\rho_{01}(t) = \rho_{01}(0) e^{i\delta(t)} - \gamma(t), \quad (9)$$

where,

$$\delta(t) = -2(\nu_{12}(t) - \sum_k \sin(k \cdot r_{12}) |\xi_k(t)|^2), \quad (10)$$

$$\gamma(t) = 2 \sum_k |\xi_k(t)|^2. \quad (11)$$

The functions, $\delta(t)$ and $\gamma(t)$, describe the frequency shift and decoherence of the spin precession due to the indirect interaction between the fermionic spins through the PSV. Substituting the expression [3] for $\xi_k(t)$ and changing sums to integrals, $(1/V) \sum_k \rightarrow (1/(2\pi)^3) \int d^3 k$, we obtain:

$$\delta(t) = \frac{\lambda^2}{4\pi^3} \int d^3 k \left( \frac{k^2 \cos(k \cdot r_{12}) (\omega_k t - \sin(\omega_k t))}{\omega_k^3} - \frac{2k_z^2 \sin(k \cdot r_{12}) \sin^2(\omega_k t/2)}{\omega_k^3} \right), \quad (12)$$

$$\gamma(t) = \frac{\lambda^2}{4\pi^3} \int d^3 k \frac{k_z^2 \sin^2(\omega_k t/2)}{\omega_k^3}. \quad (13)$$

The frequency shift. – We computed the 3D integral in Eq. (12), assuming that $mL \ll 1$, where $L = |r_{12}|$, (see Appendix B for details), and obtained the following expression for the phase shift:

$$\delta(t) = \frac{\lambda^2}{4\pi} \cdot \frac{3 \cos^2 \theta - 1}{r_{12}^3} \cdot t. \quad (14)$$

Here $\theta$ is the polar angle of the vector $r_{12}$ connecting the two spins. Formula (14) is valid for $t > r_{12}$; for $t < r_{12}$ we have $\delta(t) = 0$.

One can see that the phase shift is proportional to time. The coefficient at $t$ is the frequency shift which is proportional to $(3 \cos^2 \theta - 1)/r_{12}^3$. This factor is similar to that obtained for the magnetic dipole-dipole interaction in the baryonic matter [2]. In both cases the frequency shift changes its sign at the magic angle, $\cos^2 \theta = 1/3$. Thus, we come to the conclusion that...
the spin-spin fermionic interaction, mediated by PSV, is similar to the magnetic dipole-dipole interaction mediated by the vacuum electromagnetic field.

Decoherence. – The integral in Eq. 13, describing the decoherence, is diverging. In Appendix A we compute this integral using the dimensional regularization.

Generation of entanglement entropy. – By tracing out the pseudo-scalar degrees of freedom, we generate the entanglement entropy (EE), $S(t) = -\text{Tr}(\rho_s(t) \ln \rho_s(t))$, in the spin sub-system. The EE is created independently of the initial spin wave function (with $S(0) = 0$) disentangled or entangled. Namely, in both cases, the trace over the pseudo-scalar vacuum field creates entanglement for spin sub-system, at $t > 0 \ (S(t) > 0)$.

III. CONCLUSION

We suggested that in the dark sector of the Universe the interaction between the PSV of the DM bosons (e.g., axions) and DM fermions (e.g., neutralinos) may be as important as the electromagnetic interaction in the baryonic sector. In particular, the spin-spin interaction between the DM fermions mediated by PSV may represent the leading interaction in DM.

Based on this assumption, we studied the influence of the spin-spin interaction, mediated by PSV, on precession of the two DM fermionic spins in the external field produced by the other DM fermions. In the secular approximation, we have obtained an exact expression for the reduced spin density matrix describing precession of the spins. We have found an analytical expression for the shift of the precession frequency caused by the DM spin-spin interaction mediated by PSV and estimated the decoherence time. We have also noted generation of the entanglement entropy is the sub-spin system.

ACKNOWLEDGMENTS

The work by G.P.B. was done at Los Alamos National Laboratory managed by Triad National Security, LLC, for the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. 89233218CNA000001. A.I.N. acknowledges the support by CONACyT (Network Project No. 294625, “Agujeros Negros y Ondas Gravitatorias”). M.M. was supported by an National Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant.

Appendix A: Supplemental Material

1. Evolution operator

In this section, we derive the elements of the reduced spin density matrix. First, consider the evolution opera-
where \( \nu = (1/2)(\nu_{11} + \nu_{22}) \).

The matrix elements of the reduced density matrix are given by,
\[
\rho_{ij}(t) = \langle i | \text{Tr}_R U(t) \rho(0) U^{-1}(t) | j \rangle, \quad i,j = 0,1,2,3, \tag{A13}
\]
where \( \rho(0) = \rho_s(0) \otimes |\psi_V\rangle \langle \psi_V| \). We start with calculation of the matrix element \( \rho_{01}(t) \). Using (A9) – (A12) in (A13), we obtain,
\[
\rho_{01}(t) = e^{-2i\nu_{12}(t)} e^{-\mu(t)} \rho_{01}(0), \tag{A14}
\]
where
\[
\mu = \frac{1}{2} \sum_{\alpha,\beta} (1 + (-1)^{\alpha-\beta}) \mu_{\alpha\beta} - \xi_k^* \xi_k. \tag{A15}
\]
Substituting \( \xi_k^2 = \kappa_k \xi_k \), into expression for \( \mu \), after some algebra we obtain,
\[
\mu = 2 \sum_k |\xi_k(t)|^2 + 2i \sum_k |\xi_k(t)|^2 \sin(k \cdot r_{12}). \tag{A16}
\]
Now, taking into account all these relations, we get,
\[
\rho_{01}(t) = e^{i\delta(t)} e^{-\gamma(t)} \rho_{01}(0), \tag{A17}
\]
where,
\[
\delta(t) = -2(\nu_{12}(t) + \sum_k \sin(k \cdot r_{12}) |\xi_k(t)|^2), \tag{A18}
\]
\[
\gamma(t) = 2 \sum_k |\xi_k(t)|^2. \tag{A19}
\]
Similar computation of the other elements of the reduced density matrix yields our final result:
\[
\rho_{ii}(t) = \rho_{ii}(0), \quad i = 0,1,2,3, \tag{A20}
\]
\[
\rho_{01}(t) = e^{i\delta(t)} e^{-\gamma(t)} \rho_{01}(0), \tag{A21}
\]
\[
\rho_{02}(t) = e^{i\delta(t)} e^{-\gamma(t)} \rho_{02}(0), \tag{A22}
\]
\[
\rho_{03}(t) = e^{-\gamma(t)} \rho_{03}(0), \tag{A23}
\]
\[
\rho_{12}(t) = e^{-\gamma(t)} \rho_{12}(0), \tag{A24}
\]
\[
\rho_{13}(t) = e^{-i\delta(t)} e^{-\gamma(t)} \rho_{13}(0), \tag{A25}
\]
\[
\rho_{23}(t) = e^{-i\delta(t)} e^{-\gamma(t)} \rho_{23}(0). \tag{A26}
\]
Here,
\[
\delta(t) = -2\nu_{12}(t) + 2 \sum_k \sin(k \cdot r_{12}) |\xi_k(t)|^2, \tag{A27}
\]
\[
\gamma(t) = 2 \sum_k |\xi_k(t)|^2, \tag{A28}
\]
\[
\gamma_1(t) = 8 \sum_k \cos^2 \left( \frac{k \cdot r_{12}}{2} \right) |\xi_k(t)|^2, \tag{A29}
\]
\[
\gamma_2(t) = 8 \sum_k \sin^2 \left( \frac{k \cdot r_{12}}{2} \right) |\xi_k(t)|^2. \tag{A30}
\]
2. The frequency shift

Using Eqs. [(A3)] and [(A4)], one can recast (A18) as,
\[
\delta(t) = -\frac{2\lambda^2}{V} \sum_k \left( k^2 \cos(k \cdot r_{12}) (\omega_k t - \sin(\omega_k t)) \right) \tag{A31}
\]

To proceed further, we replace a sum by an integral: \((1/V) \sum_k \rightarrow (1/(2\pi)^3) \int d^3k\), and write the total phase as \( \delta = \delta_1 + \delta_2 \), where,
\[
\delta_1 = -\frac{\lambda^2}{2\pi^3} \int d^3k \frac{k^2 \cos(k \cdot r_{12}) (\omega_k t - \sin(\omega_k t))}{\omega_k^3}, \tag{A32}
\]
\[
\delta_2 = -\frac{\lambda^2}{2\pi^3} \int d^3k \frac{k^2 \sin(k \cdot r_{12}) \sin^2(\omega_k t/2)}{\omega_k^3}. \tag{A33}
\]

We assume that spins are located at the points with the position vectors, \( r_1 \) and \( r_2 \). Without loss of generality, one can choose the orientation of the coordinate system in such a way that \( r_2 = L \sin(\theta, 0, \cos \theta) \), where \( L = |r_{12}| \equiv r_{12} \). Using the spherical coordinates \((k, \theta, \varphi)\), one can recast Eqs. [(A32), (A33)] as,
\[
\delta_1 = -\frac{\lambda^2}{2\pi^3} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cos^2 \theta \sin \theta \cos(k \cdot r_{12}), \tag{A34}
\]
\[
\delta_2 = -\frac{\lambda^2}{\pi^2} \int_0^\infty d\omega k \frac{d^3k I_2(kL)(\sin(\omega_k t/2))}{\omega_k^3}. \tag{A35}
\]

where,
\[
I_1(kL) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cos^2 \theta \sin \theta \cos(k \cdot r_{12}), \tag{A36}
\]
\[
I_2(kL) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cos^2 \theta \sin \theta \sin(k \cdot r_{12}). \tag{A36}
\]

The computation of the integrals \( I_{1,2} \) yields (see Appendix B),
\[
I_1(x) = \frac{2}{x^3} \left( x^2 \sin x + 3x \cos x - 3 \sin x \right) \cos^2 \theta_0 \tag{A37}
\]
\[
+ \frac{2}{x} \left( \sin x - x \cos x \right), \quad \tag{A37}
\]
\[
I_2(x) = 0, \tag{A38}
\]
where a new dimensionless variable, \( x = kL \), is introduced. Since \( I_2 = 0 \), we conclude that \( \delta_2 = 0 \).

Then, Eq. [(A31)], determining the total phase, can be rewritten as,
\[
\delta = -\frac{\lambda^2}{2\pi^2 L^3} \int_0^\infty dx x^2 I_1(x)(\omega t - L \sin(\omega t/L)), \tag{A39}
\]
where \( \omega = \sqrt{x^2 + (xL)^2} \).
In what follows, we assume that $mL \ll 1$. Then, one can recast (A39) as,

$$\delta = \frac{\lambda^2 S_0(t)}{2\pi^2 L^3} + \mathcal{O}((mL)^2). \quad (A40)$$

where

$$S_0(t) = \int_0^\infty dx I_1(x) (L \sin(\omega t/L) - x t). \quad (A41)$$

Performing the integration, we obtain (for detail see Appendix B),

$$\delta(t) = \begin{cases} 0, & \text{if } t < L, \\ \frac{\lambda^2 t}{\pi} \cdot \frac{(3 \cos^2 \theta - 1)}{L^3}, & \text{if } t > L. \end{cases} \quad (A42)$$

### 3. Decoherence

Here we limit ourselves by considering only the spin decoherence produced by the vacuum field. By replacing a sum by an integral, we have,

$$\gamma(t) = \frac{\lambda^2}{4\pi^3} \int \frac{d^3 k k^2 \sin^2(\omega_k t/2)}{\omega_k^3}, \quad (A43)$$

$$\gamma_1(t) = \frac{\lambda^2}{\pi^3} \int \frac{d^3 k k^2 \cos^2(\frac{k r_0}{\omega_k}) \sin^2(\omega_k t/2)}{\omega_k^3}, \quad (A44)$$

$$\gamma_2(t) = \frac{\lambda^2}{\pi^3} \int \frac{d^3 k k^2 \sin^2(\frac{k r_0}{\omega_k}) \sin^2(\omega_k t/2)}{\omega_k^3}. \quad (A45)$$

In what follows, we restrict ourselves by consideration only $\gamma(t)$. Performing integration over angle variables we obtain,

$$\gamma(t) = \frac{\lambda^2}{3\pi^2} \int_0^\infty \frac{d k k^4 \sin^2(\omega_k t/2)}{\omega_k^3}. \quad (A46)$$

The asymptotic value of $\gamma(t)$, while $t \to \infty$, is given by,

$$\gamma(t) \to \gamma_0 = \frac{\lambda^2}{6\pi^3} \int_0^\infty \frac{d k k^4 \sin^2(\omega_k t/2)}{\omega_k^3}. \quad (A47)$$

This integral is diverging as $k \to \infty$. There are different approaches to deal with this ultra-violet (UV) catastrophe, which occurs in many domains of the field theory. One of them is to introduce in (A47) a cutoff, $k_c$. In this approach, the question on the value of $k_c$ usually arises.

To regularize the divergent integral (A47) we use the dimensional regularization, following the procedure described in [a]. For this purpose, consider the auxiliary integral,

$$\Sigma_\epsilon = \frac{2\pi^{5-\epsilon}}{\Gamma(\frac{3}{2} - \epsilon)} \int_0^\infty \frac{x^4 - 2x dx}{(x^2 + 1)^{3/2}}. \quad (A48)$$

In order to calculate this integral we will use the formula:

$$\int_0^\infty \frac{r^\alpha dr}{(a + br^\beta)^\gamma} = \frac{\Gamma(\gamma - \frac{\alpha + 1}{\beta})}{\Gamma(\frac{\alpha + 1}{\beta})} \frac{\Gamma(\frac{3}{2} - \epsilon)}{\Gamma(\frac{3}{2})} (A49)$$

The computation yields,

$$\Sigma_\epsilon = \frac{\pi^{\frac{3}{2} - \epsilon} \Gamma(\frac{3}{2} - \epsilon)}{\Gamma(\frac{3}{2}) \Gamma(\epsilon - 1)}. \quad (A50)$$

Using the identities $\Gamma(z + 1) = z \Gamma(z)$ and $\pi^{-\epsilon} = e^{-\epsilon \ln \pi}$, we get

$$\Sigma_\epsilon = 2\pi e^{-\epsilon \ln \pi} \frac{\Gamma(\frac{3}{2} - \epsilon)}{\Gamma(\epsilon - 1)} \quad (A51)$$

For $\epsilon \ll 1$, we have $e^{-\epsilon \ln \pi} = 1 - \epsilon \ln \pi + \mathcal{O}(\epsilon^2)$. To proceed further, we use the Laurent series expansion in a neighborhood of the pole $z = -1$:

$$\Gamma(z) = \Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon), \quad (A52)$$

where $\gamma = 0.57722$ is the Euler constant. After some algebra we obtain,

$$\Sigma_\epsilon = 3\pi \left(-\frac{1}{\epsilon} + \gamma - 1 + \frac{2}{3} \ln \pi + \mathcal{O}(\epsilon)\right). \quad (A53)$$

We truncate the singular term $1/\epsilon$ of the function $\Sigma_\epsilon$ at the point $\epsilon = 0$ and obtain the regularized expression,

$$[\Sigma]_{reg} = \lim_{\epsilon \to 0} [\Sigma_\epsilon]_{reg} = \pi(3\gamma - 1 + 2 \ln \pi). \quad (A54)$$

Returning to Eq. (A22), we rewrite it as,

$$\gamma_0 = \frac{\lambda^2 m^2}{6\pi^3} \int_0^\infty \frac{x^4 dx}{(x^2 + 1)^{3/2}}. \quad (A55)$$

where $x = k/m$. This can be recast as follows:

$$\gamma_0 = \frac{\lambda^2 m^2}{24\pi^3} \lim_{\epsilon \to 0} \Sigma_\epsilon. \quad (A56)$$

Employing (A54), we obtain,

$$\gamma_0 \to \gamma_0^{reg} = \frac{\lambda^2 m^2}{24\pi^3}(3\gamma - 1 + 2 \ln \pi). \quad (A57)$$

Since for DM the product $\lambda m \ll 1$, we obtain $\gamma_0^{reg} \ll 1$. As a result, we have $e^{-\gamma_0^{reg}} \approx 1$. Thus, the partial phase decoherence takes place for $\rho_{01}(t)$. Similar conclusion can be made for other off-diagonal elements of the reduced spin density matrix. Note that the “dimensional regularization procedure”, used here for decoherence rate, requires additional justification, which is not a subject of this paper.
Appendix B: Appendix B: Calculation of some useful integrals

Here we calculate the typical integrals emergent in the main text of the paper. We start with the integrals:

\[ I_1 = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \cos^2 \vartheta \sin \vartheta \cos(k \cdot r_{12}), \]  
\[ I_2 = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \cos^2 \vartheta \sin \vartheta \sin(k \cdot r_{12}). \]  
(B1)  
(B2)

Here \( k = k(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \). Without loss of generality, we can choose the orientation of the coordinate system in such a way that \( r_{12} = L(\sin \theta, 0, \cos \theta) \), where, as before, \( L = |r_{12}| \equiv r_{12} \). Before proceeding further, it is convenient to introduce new variables: \( p = kL \sin \theta \) and \( q = kL \cos \theta \). We obtain,

\[ k \cdot r_{12} = p \sin \vartheta \cos \varphi + q \cos \vartheta. \]  
(B3)

First, consider the integral \( I_1 \) written as,

\[ I_1 = \int_0^\pi f_1(\vartheta) \cos^2 \vartheta \sin \vartheta d\vartheta, \]  
(B4)

where

\[ f_1(\vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \cos(p \sin \vartheta \cos \varphi + pq \cos \vartheta) d\varphi = \cos(q \cos \vartheta) J_0(p \sin \vartheta). \]  
(B5)

Here \( J_0(x) \) denotes the Bessel function. Introducing a new variable, \( z = \sin \varphi \), one can recast (B4) as,

\[ I_1 = 2R \int_0^1 z \sqrt{1 - z^2} e^{i\varphi \sqrt{1 - z^2}} J_0(pz) dz. \]  
(B6)

Performing the integration, we obtain \[ 1^{11} \]

\[ I_1(kL) = -2 \frac{\partial^2}{\partial q^2} \left( \frac{\sin \sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2}} \right). \]  
(B7)

This yields

\[ I_1(x) = \frac{2}{x^2} \left( x^2 \sin x + 3x \cos x - 3 \sin x \right) \cos^2 \theta \]
\[ + \frac{2}{x} \left( \sin x - x \cos x \right). \]  
(B8)

Next, consider the integral \( I_2 \). Using \( \text{B2} \), one can recast \( \text{B2} \) as,

\[ I_2 = 2 \int_0^{\pi/2} f_2(\vartheta) \cos^2 \vartheta \sin \vartheta \cos(q \cos \vartheta)) d\vartheta, \]  
(B9)

where

\[ f_2(\vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \sin(p \sin \vartheta \cos \varphi) d\varphi. \]  
(B10)

As one can easily see, \( f_2(\vartheta) = 0 \), and thus the integral \( I_2 = 0 \).

Now, consider the integral,

\[ S_0(t) = \int_0^\infty I_1(x)(L \sin(xt/L) - xt) dx, \]  
(B11)

where \( L > 0 \). To evaluate this integral, we use the identities \( (n \geq 0) \):

\[ \int_0^\infty \tau^n \cos(k \tau) d\tau = \begin{cases} (-1)^{n/2} \pi \delta^{(n)}(k), & n \text{ even} \\ (-1)^{(n+1)/2} \frac{n!}{\kappa^{n+1}}, & n \text{ odd} \end{cases} \]  
(B12)

\[ \int_0^\infty \tau^n \sin(k \tau) d\tau = \begin{cases} (-1)^{n/2} \frac{n!}{\kappa^{n+1}}, & n \text{ even} \\ (-1)^{(n+1)/2} \pi \delta^{(n)}(k), & n \text{ odd} \end{cases} \]  
(B13)

where \( \delta^{(n)} \) denotes the \( n \)th derivative of the Dirac delta function.

Taking into account that,

\[ \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}, \]  
(B14)

after some algebra we obtain,

\[ S_0(t) = \pi t (3 \cos^2 \theta - 1) + I_0, \]  
(B15)

where

\[ I_0 = L \int_0^\infty I_1(x) \sin(xt/L) dx. \]  
(B16)

The computation yields,

\[ I_0 = L \int_0^\infty I_1(x) \sin(xt/L) dx = \pi L \delta(t/L - 1) \]
\[ - \pi L \delta(t/L + 1) + \frac{\pi t}{2} (3 \cos^2 \theta - 1)(H(t/L - 1) - 1), \]  
(B17)

where \( H(x) \) denotes the Heaviside function. Finally, we obtain

\[ S_0(t) = \begin{cases} 0, & t < L \\ 2\pi t (3 \cos^2 \theta - 1), & t > L \end{cases} \]  
(B18)
[1] Y.V. Stadnik and V.V. Flambaum, “Axion-induced effects in atoms, molecules, and nuclei: Parity nonconservation, anapole moments, electric dipole moments, and spin-gravity and spin-axion momentum couplings”, Phys. Rev. D, 89, 043522 (2014).

[2] R. Daido and F. Takahashi, “The sign of the dipole-dipole potential by axion exchange”, Phys. Lett. B, 772, 127 (2017).

[3] R.T. Schumacher, Introduction to Magnetic Resonance (W.A. Benjamin, Inc, New York, 1970).

[4] G. Bertone and D. Hooper, “History of dark matter,” Rev. Mod. Phys. 90, 045002 (2018).

[5] G.P. Berman, D.I. Kamenev, and V.I. Tsifrinovich, Perturbation Theory for Solid-state Quantum Computation With Many Quantum Bits (Rinton Pr Inc., 2005).

[6] E. Zeidler, Quantum Field Theory II: Quantum Electrodynamics. A Bridge between Mathematicians and Physicists (Springer-Verlag, Berlin Heidelberg, 2009).

[7] A.P. Prudnikov, Yu.A. Brychkov, and O.J. Marichev, Integrals and Series: Volume 1: Elementary Functions; Volume 2: Special Functions (CRC Press, 1986).