AN INFINITE-DIMENSIONAL VERSION OF GOWERS’ FIN±k THEOREM

JAMAL K. KAWACH

Abstract. We prove an infinite-dimensional version of an approximate Ramsey theorem of Gowers, initially used to show that every Lipschitz function on the unit sphere of $c_0$ is oscillation stable. To do so, we use the theory of ultra-Ramsey spaces developed by Todorcevic in order to obtain an Ellentuck-type theorem for the space of all infinite block sequences in FIN±k.

1. Introduction

Let $X$ be a Banach space and let $S_X$ be its unit sphere. A function $f : S_X \to \mathbb{R}$ is oscillation stable if for every $\varepsilon > 0$ and every closed infinite-dimensional subspace $Y$ of $X$ there is a closed infinite-dimensional subspace $Z$ of $Y$ such that

$$\text{osc}(f, S_Z) := \sup\{|f(x) - f(y)| : x, y \in S_Z\} < \varepsilon.$$ 

Gowers’ $c_0$ theorem, originally proved in [5], states that every Lipschitz (or, more generally, uniformly continuous) function $f : S_{c_0} \to \mathbb{R}$ is oscillation stable. The proof of this theorem relies on a Ramsey-type result about the space of all finitely-supported functions $p : \omega \to \{0, \pm 1, \ldots, \pm k\}$ which take at least one of the values $\pm k$. The main goal of this note is to extend this latter result to its natural infinite-dimensional analogue (Theorem 1.2 below).

Before we can state these results, we fix some notation. Let $\omega$ denote the set of all non-negative integers, and $\mathbb{N}$ the set of all positive integers. We will often identify each ordinal $m < \omega$ with the set $\{0, \ldots, m - 1\}$ of its predecessors. Given $k \in \mathbb{N}$, let $\text{FIN}_{\pm k}$ denote the set of all functions $p : \omega \to \{0, \pm 1, \ldots, \pm k\}$ such that

$$\text{supp } p := \{n < \omega : p(n) \neq 0\}$$

is finite and such that $p$ achieves at least one of the values $\pm k$. Given $p, q \in \text{FIN}_{\pm k}$ we write $p < q$ whenever $\max \text{supp } p < \min \text{supp } q$. Whenever $p < q$ and $p, q \in \text{FIN}_{\pm k}$, $p + q$ will denote the element of $\text{FIN}_{\pm k}$ given by the coordinate-wise sum of $p$ and $q$. This operation gives $\text{FIN}_{\pm k}$ the structure of a partial semigroup.

We also have an operation between various FIN spaces: The tetris operation $T : \text{FIN}_{\pm k} \to \text{FIN}_{\pm (k-1)}$ is defined by

$$T(p)(n) = \begin{cases} 
    p(n) - 1 & \text{if } p(n) > 0, \\
    0 & \text{if } p(n) = 0, \\
    p(n) + 1 & \text{if } p(n) < 0.
\end{cases}$$

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We can now state the following theorem of Gowers, originally proved in [5] using the framework of topological Ramsey spaces as in [10] to prove an infinite-dimensional analogue using the theory of topological Ramsey spaces developed in [10]. Our goal is to show that such an analogue can still be obtained even though there is no pigeonhole principle for FIN\(^k\) \pm. Our main result is the following.

For each \(k \in \mathbb{N}\) and every \(c : \text{FIN}_k \rightarrow k + 1\) there is \(i < r\) such that \(\{c^{-1}(i)\}\) contains a partial subsemigroup of \(\text{FIN}_k\) generated by an infinite block sequence.

It is worth mentioning here that, although Gowers’ theorem is an approximate Ramsey-theoretic result, there is an exact version (also proved in [4]) for the spaces FIN\(^k\) consisting of all finitely-supported functions \(p : \omega \rightarrow k + 1\) which achieve the value \(k\). This latter result acts as a pigeonhole principle and can be used via the theory of idempotent ultrafilters in order to show that every real-valued Lipschitz function on \(S_\alpha\) is oscillation stable (see [1, 6, 10] for other proofs).

**Theorem 1.1** (Gowers). For every \(k, r \in \mathbb{N}\) and every \(c : \text{FIN}_k \rightarrow r\) there is \(i < r\) such that \(\{c^{-1}(i)\}\) contains a partial subsemigroup of \(\text{FIN}_k\) generated by an infinite block sequence.

We will work exclusively with the \(\ell_\infty\) norm given by

\[
\|p\| := \sup_{n \in \omega} |p(n)|
\]

where \(p \in \text{FIN}_k\) and \(k \in \mathbb{N}\). For a subset \(A \subseteq \text{FIN}_k\) and \(\varepsilon > 0\), define

\[
(A)_\varepsilon := \{p \in \text{FIN}_k : (\exists q \in A)\|p - q\| \leq \varepsilon\}.
\]

We can now state the following theorem of Gowers, originally proved in [5] using the theory of idempotent ultrafilters in order to show that every real-valued Lipschitz function on \(S_\alpha\) is oscillation stable (see [11, 6, 10] for other proofs).

**Theorem 1.1** (Gowers). For every \(k, r \in \mathbb{N}\) and every \(c : \text{FIN}_k \rightarrow r\) there is \(i < r\) such that \(\{c^{-1}(i)\}\) contains a partial subsemigroup of \(\text{FIN}_k\) generated by an infinite block sequence.

For \(X \subseteq \text{FIN}_k\) and \(\varepsilon > 0\), define

\[
(X)_\varepsilon := \{p \in \text{FIN}_k : (\exists Q \in X)\|P - Q\| \leq \varepsilon\}.
\]
It is well-known that infinite-dimensional Ramsey-theoretic results do not hold in general for all colourings. To obtain positive results, a topological restriction on the permitted colourings is needed. In our case we work with the *metrizable topology* on $\FIN_{\pm k}[\infty]$ which is generated by basic open sets of the form

$$[(q_0, \ldots, q_{m-1})] := \{ (p_n)_{n<\omega} \in \FIN_{\pm k}[\infty] : q_i = p_i \text{ for all } i < m \}$$

where $m < \omega$ and $(q_0, \ldots, q_{m-1}) \in \FIN_{\pm k}[m]$. This is the topology inherited by $\FIN_{\pm k}[\infty]$ when viewed as a subspace of the Tychonoff product $\left(\FIN_{\pm k}[\infty] \right)^{\omega}$ via the natural mapping

$$P = (p_n)_{n<\omega} \mapsto (r_n(P))_{n<\omega}$$

where $r_n(P) := (p_i)_{i<n}$, and where $\FIN_{\pm k}[\infty]$ is given the discrete topology.

We now describe the topological restriction mentioned above. First recall that a *Souslin scheme* is a family of sets $(X_s)_{s \in \omega < \omega}$ indexed by finite sequences of non-negative integers. The *Souslin operation* turns a Souslin scheme $(X_s)_{s \in \omega < \omega}$ into the set

$$\bigcup_{x \in \omega^\omega} \bigcap_{n < \omega} X_{s|n}$$

where $\omega^\omega$ denotes the set of all infinite sequences in $\omega$. Given a topological space $X$, the field of Souslin measurable sets is the smallest field of subsets of $X$ which contains all open subsets of $X$ and is closed under the Souslin operation. In particular, every analytic (and hence Borel) subset of $X$ is Souslin measurable (see, e.g., [7, Section 25.C]). Finally, a colouring $c: X \to r$ is Souslin measurable if $c^{-1}\{i\}$ is Souslin measurable for each $i < r$.

Let $\langle P \rangle_{\pm k}^{\infty}$ denote the set of all $Q \in \FIN_{\pm k}[\infty]$ such that $Q \leq P$. The purpose of this note is to extend Gowers’ $\FIN_{\pm k}$ theorem to the following analogue for $\FIN_{\pm k}[\infty]$. The proof will involve a synthesis of techniques introduced by Todorcevic in [10] and Kanellopoulos in [6].

**Theorem 1.2.** For every $k, r \in \mathbb{N}$ and every Souslin measurable $c : \FIN_{\pm k}[\infty] \to r$ there are $i < r$ and an infinite block sequence $P \in \FIN_{\pm k}[\infty]$ such that

$$\langle P \rangle_{\pm k}^{\infty} \subseteq (c^{-1}\{i\})^1.$$
2. An ultra-Ramsey space of infinite block sequences in $\mathsf{FIN}_{\pm k}$

In the setting of ultra-Ramsey theory, we work with a special class of trees of countably infinite height which branch according to a given ultrafilter. Recall that an ultrafilter on a set $X$ is a collection $\mathcal{U}$ of subsets of $X$ satisfying the following four properties:

1. $\emptyset \notin \mathcal{U}$.
2. $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$.
3. $A \in \mathcal{U}$, $B \supseteq A$ implies $B \in \mathcal{U}$.
4. For every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Let $\beta X$ denote the set of all ultrafilters on $X$; then $\beta X$ is a compact Hausdorff space under the topology generated by basic open sets of the form $A := \{ U \in \beta X : A \in U \}$ where $A$ is a non-empty subset of $X$. It is useful to view ultrafilters as quantifiers (e.g. as in Blass [3]) in the following way. Let $U$ be an ultrafilter on a set $X$. Given a first-order formula $\varphi(x)$ with a free variable $x$ ranging over elements of $X$, we write

$$\langle Ux \rangle \varphi(x) \iff \{ x \in X : \varphi(x) \} \in U.$$ 

Using the ultrafilter properties above it is easy to check that ultrafilter quantifiers commute with conjunction and negation of first-order formulas, i.e. we have

$$\langle Ux \rangle \varphi(x) \land \langle Ux \rangle \psi(x) \iff \langle Ux \rangle (\varphi(x) \land \psi(x)),$$

$$\neg \langle Ux \rangle \varphi(x) \iff \langle Ux \rangle (\neg \varphi(x))$$

for any first-order formulas $\varphi(x)$ and $\psi(x)$.

We will primarily be concerned with ultrafilters on $\mathsf{FIN}_{\pm k}$. Given two ultrafilters $\mathcal{U}, \mathcal{V} \in \beta \mathsf{FIN}_{\pm k}$, define the sum of $\mathcal{U}$ and $\mathcal{V}$ by declaring

$$A \in \mathcal{U} + \mathcal{V} \iff \langle Up \rangle \langle Vq \rangle \ (p + q \in A)$$

for $A \subseteq \mathsf{FIN}_{\pm k}$. To ensure that this operation is always defined we restrict our attention to the set of all cofinite ultrafilters on $\mathsf{FIN}_{\pm k}$, i.e. ultrafilters $U \in \beta \mathsf{FIN}_{\pm k}$ which satisfy

$$X_m := \{ p \in \mathsf{FIN}_{\pm k} : p(n) = 0 \text{ for all } n < m \} \in U$$

for all $m < \omega$. Let $\gamma \mathsf{FIN}_{\pm k}$ denote the set of all cofinite $U \in \beta \mathsf{FIN}_{\pm k}$. Then $(\gamma \mathsf{FIN}_{\pm k}, +)$ is a compact semigroup. (We refer the reader to [10, Chapter 2] for details.) We also extend the tetris operation $T : \mathsf{FIN}_{\pm k} \to \mathsf{FIN}_{\pm (k-1)}$ to a map $T : \gamma \mathsf{FIN}_{\pm k} \to \gamma \mathsf{FIN}_{\pm (k-1)}$ by setting

$$A \in T(\mathcal{U}) \iff T^{-1}(A) \in \mathcal{U}$$

for each $A \subseteq \mathsf{FIN}_{\pm (k-1)}$. This extension is a continuous surjective homomorphism. Below we will consider the sign-flipped version of the tetris operation given by

$$-T : \mathsf{FIN}_{\pm k} \to \mathsf{FIN}_{\pm (k-1)} : p \mapsto -T(p)$$

together with its extension to $\gamma \mathsf{FIN}_{\pm k}$ (the definition of which is analogous to the extension of $T$ to $\gamma \mathsf{FIN}_{\pm k}$ above).

Given $A \subseteq \mathsf{FIN}_{\pm k}$ let $-A := \{-x : x \in A\}$. We will need the following result, the proof which of uses the general theory of idempotents in compact semigroups (see [9] or [6, Lemma 4] for details).
Lemma 2.1. There exists a cofinite ultrafilter $\mathcal{U}$ on $\text{FIN}_{\pm k}$ such that $\mathcal{U} + (−T)^j\mathcal{U} = (−T)^j\mathcal{U} + \mathcal{U} = \mathcal{U}$ for all $j \in \{0, \ldots, k\}$. Furthermore, $\mathcal{U}$ is subsymmetric: For every $A \in \mathcal{U}$ we have $−(A)_1 \in \mathcal{U}$.

The first part of the above result can be found in [6]. The second part follows from the first (see [6] Lemma 11) but we point out here that the theory of subsymmetric ultrafilters was first developed in [10, Chapter 2] (and in the earlier manuscript [9]) and is used there to give an ultrafilter proof of Gowers’ Theorem. Note that the ultrafilters was first developed in [10, Chapter 2] (and in the earlier manuscript [9]) and is used there to give an ultrafilter proof of Gowers’ Theorem. Note that the ultrafilter $\mathcal{U}$ given by Lemma 2.1 has the property that, for any $A \in \mathcal{U}$ and $j < k$, 

$$(\mathcal{U}f)(\mathcal{U}g) \left( \{ f, g + (−T)^j(g), (−T)^j(f) + g \} \subseteq A \right).$$

Since ultrafilter quantifiers commute with finite conjunctions it follows that 

$$(\mathcal{U}f)(\mathcal{U}g) \left( \{ f, g + (−T)^j(g), (−T)^j(f) + g : j < k \} \subseteq A \right)$$

for any $A \in \mathcal{U}$.

We now proceed to describe a class of trees which form the basis for the required ultra-Ramsey theory. To this end, for each $k \in \mathbb{N}$ we view the space $\text{FIN}_{\pm k}^{[<\infty]}$ as a tree ordered by end-extension $\sqsubseteq$ and with stem $\emptyset$. Unless otherwise specified, for the rest of this paper we fix $k \in \mathbb{N}$ together with an ultrafilter $\mathcal{U}$ on $\text{FIN}_{\pm k}$ given by Lemma 2.1.

Definition 2.2. A $\mathcal{U}$-tree is a downward closed subtree $U \subseteq \text{FIN}_{\pm k}^{[<\infty]}$ such that 

$$U_t := \{ p \in \text{FIN}_{\pm k} : (t, p) \in U \} \in \mathcal{U}$$

for all $t \in U$. The stem of $U$, denoted stem$(U)$, is the $\sqsubseteq$-maximal element of $U$ which is comparable to every other node of the tree.

Given a $\mathcal{U}$-tree $U$, the set of infinite branches of $U$ is denoted by 

$$[U] := \{(p_m)_{m<\omega} \in \text{FIN}_{\pm k}^{[\infty]} : (p_0, \ldots, p_m) \in U \text{ for all } m < \omega\}.$$ 

For $t \in U$ let $|t|$ denote the length of $t$, which is just the domain of $t$ when viewed as a finite sequence in $\text{FIN}_{\pm k}^{[<\infty]}$. For $m < \omega$, the $m$th level $U(m)$ of $U$ is the set of all $t \in U$ of length $m$.

In order to prove an infinite-dimensional version of Theorem 1.1 we work with a topology defined using $\mathcal{U}$-trees and which extends the usual metrizable topology on $\text{FIN}_{\pm k}^{[<\infty]}$. Working in this topology allows us to remedy the fact that the space $\text{FIN}_{\pm k}$ lacks an exact pigeonhole principle.

Definition 2.3. Let $\mathcal{X} \subseteq \text{FIN}_{\pm k}^{[\infty]}$. $\mathcal{X}$ is $\mathcal{U}$-open if for every $A \in \mathcal{X}$ there is a $\mathcal{U}$-tree $U$ such that $A \in [U] \subseteq \mathcal{X}$. $\mathcal{X}$ is $\mathcal{U}$-Ramsey if for every $\mathcal{U}$-tree $U$ there is a $\mathcal{U}$-subtree $U' \subseteq U$ with stem$(U) = \text{stem}(U')$ such that $[U'] \subseteq \mathcal{X}$ or $[U'] \subseteq \mathcal{X}^c$. If the second alternative always holds then we say $\mathcal{X}$ is $\mathcal{U}$-Ramsey null.

The collection of all $\mathcal{U}$-open subsets of $\text{FIN}_{\pm k}^{[\infty]}$ forms a topology, called the $\mathcal{U}$-topology, which refines the metrizable topology of $\text{FIN}_{\pm k}^{[\infty]}$. The next two results are adapted from [10] Chapter 7.2 by replacing the tree $\mathbb{N}^{[<\infty]}$ of finite subsets of $\mathbb{N}$ ordered by end-extension with the tree $\text{FIN}_{\pm k}^{[<\infty]}$. We state them in our context without proof. First, recall that a subset $A$ of a topological space $X$ has the property of Baire if there is an open set $U \subseteq X$ such that the symmetric difference of $A$
and $U$ is meager in $X$. We then have the following version of Todorcevic’s Ultra-Ellentuck Theorem, which builds on a theorem of Ellentuck \cite{4} relating the notions of Baire and Ramsey in the setting of $\mathbb{N}^{[\infty]}$, the set of all infinite subsets of $\mathbb{N}$.

**Theorem 2.4.** Let $\mathcal{X} \subseteq \text{FIN}_k^{[\infty]}$. Then $\mathcal{X}$ has the property of Baire relative to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey. Furthermore, $\mathcal{X}$ is meager with respect to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey null.

The next result uses a classical fact of Nikodym (see, e.g., \cite[Chapter 4.1]{10}) which says that, in any topological space, the property of Baire is preserved under the Souslin operation.

**Corollary 2.5.** For every $r \in \mathbb{N}$ and every Souslin measurable $c : \text{FIN}_k^{[\infty]} \to r$ there are $i < r$ and a $\mathcal{U}$-tree $U$ with stem $\emptyset$ such that $[U] \subseteq c^{-1}\{i\}$.

3. $S$-CLOSED $\mathcal{U}$-TREES

In this brief section we define a class of subtrees which will allow us to inductively construct certain block sequences during the proof of Theorem \cite{12}. First, notice that if $p, q \in \text{FIN}_k$ satisfy $||p - q|| \leq 1$, then

$$n \in (\text{supp } p \setminus \text{supp } q) \cup (\text{supp } q \setminus \text{supp } p) \implies |p(n)|, |q(n)| \leq 1.$$ 

This motivates the following version of the tetris operation: Given $p \in \text{FIN}_k$ define $S(p) \in \text{FIN}_k$ by

$$S(p)(n) := \begin{cases} p(n) & \text{if } |p(n)| \neq 1 \\ 0 & \text{if } |p(n)| = 1. \end{cases}$$

We will repeatedly use the fact that $|p - S(p)| \leq 1$ for all $p \in \text{FIN}_k$. In particular, notice that $||p - q|| \leq 1$ implies $\text{supp } S(p) \subseteq \text{supp } q$ and $||S(p) - q|| \leq 2$. This will allow us to control the supports of elements which are close to a fixed $q \in \text{FIN}_k$. Also note that $S$ is idempotent, i.e. $S \circ S = S$. The following lemma allows us to replace a given $\mathcal{U}$-tree with one which behaves well with respect to $S$, at the cost of adding an approximate constant.

**Lemma 3.1.** Suppose $V$ is a $\mathcal{U}$-tree with stem$(V) = \emptyset$. Then there is a $\mathcal{U}$-tree $U$ with stem$(U) = \emptyset$ such that $[U] \subseteq ([V])_1$ and such that $U$ is $S$-closed: For every $t \in U$ and every $p \in \text{FIN}_k$, we have

$$(t, p) \in U \implies (t, S(p)) \in U.$$ 

**Proof.** Fix a well-ordering $<$ of $\text{FIN}_k^{[<\infty]}$. We construct, by induction on $n \geq 1$, each level $U(n)$ of $U$ above $\emptyset$ together with projections $\pi_n : U(n) \to V(n)$ satisfying $||t - \pi_n(t)|| \leq 1$ for all $t \in U(n)$. To begin, take $U_0 := V_0 \cup S''V_0$ and hence

$$U(1) := \{(\emptyset, p) : p \in U_0\}.$$ 

The projection $\pi_1 : U(1) \to V(1)$ is defined by setting, for $t = (\emptyset, p) \in U(1)$,

$$\pi_1(t) := \begin{cases} (\emptyset, p) & \text{if } p \in V_0 \\ (\emptyset, \min (V_0 \cap S^{-1}(p))) & \text{otherwise} \end{cases},$$

where the minimum is taken with respect to $<$. Note that such a minimum exists, since if $p \in U_0 \setminus V_0$ then we must have $p \in S''V_0$ and so there is $q \in V_0$ such that $S(q) = p$. Furthermore, since $||p - S(p)|| \leq 1$ we have $||t - \pi_1(t)|| \leq 1$ for all $t \in U(1)$.
Now suppose we have constructed the first $m > 1$ levels $U(1), \ldots, U(m)$ of $U$ with their corresponding projections $\pi_1, \ldots, \pi_m$. For each $t \in U(m)$, set $U_t := V_{\pi_m(t)} \cup S^0 V_{\pi_m(t)}$. We then define

$$U(m + 1) := \bigcup \{(s, p) : s \in U(m), p \in U_s\}.$$ 

The projection $\pi_{m+1} : U(m + 1) \to V(m + 1)$ is defined by setting, for $t = (s, p) \in U(m + 1)$ with $s \in U(m)$ and $p \in U_s$,

$$\pi_{m+1}(t) := \begin{cases} (\pi_m(s), p) & \text{if } p \in V_{\pi_m(s)} \\ (\pi_m(s), \min (V_{\pi_m(s)} \cap S^{-1}(p))) & \text{otherwise} \end{cases}$$

where the minimum is taken with respect to $\prec$. Inductively we have $||s - \pi_m(s)|| \leq 1$ and so by definition of $S$ we have $||t - \pi_{m+1}(t)|| \leq 1$. This completes the inductive construction of $U$.

The fact that $U$ is $S$-closed follows easily from the above construction. To finish, we check that $[U] \subseteq ([V])_1$. Let $P = (p_n)_{n \in \mathbb{N}}$ be an infinite block sequence corresponding to a branch of $U$. We define a projection $\pi_\infty : [U] \to [V]$ by setting

$$\pi_\infty(P) := (\pi_n \circ r_n(P))_{n \in \mathbb{N}}$$

where $r_n : [U] \to U(n)$ is the $n$th restriction mapping given by

$$r_n(P) := (\emptyset, p_1, \ldots, p_n).$$

Note that $\pi_\infty(P)$ is indeed a branch in $V$ since $s \subseteq t$ implies $\pi_s(t) \subseteq \pi_{t_1}(t)$ for any $s, t \in U$. Since for every $P \in [U]$ we have $||P - \pi_\infty(P)|| \leq 1$ and $\pi_\infty(P) \in [V]$, we obtain that $[U] \subseteq ([V])_1$.

4. The Proof of Theorem 1.2

In this section we give a proof of the main theorem of this note. To do so, we first need to consider the following modification of the usual notion of block subsequence. Given a block sequence $P = (p_n)_{n < \omega} \in \FIN_{\pm k}^{[\omega]}$, let $\langle P \rangle_{(-T)}$ be the partial subsemigroup consisting of all vectors of the form

$$(-T)^{j_0}(p_{n_0}) + \cdots + (-T)^{j_m}(p_{n_m})$$

where $m < \omega, n_0 < \cdots < n_m < \omega$ and $j_0, \ldots, j_m < k$ are such that $\text{min } j_i = 0$. If $Q = (q_n)_{n < \omega}$ is another block sequence, write $Q \leq_{(-T)} P$ to denote that $q_n \in \langle P \rangle_{(-T)}$ for every $n < \omega$. We define $\langle P \rangle_{(-T)}$ for finite block sequences $P = (p_n)_{n < m}$ similarly; in this case we write $\langle p_0, \ldots, p_{m-1} \rangle_{(-T)}$ for the corresponding (finite) partial subsemigroup.

Lemma 4.1. Let $U$ be a $U$-tree with stem $\emptyset$. There is $P = (p_n)_{n < \omega} \in \FIN_{\pm k}^{[\omega]}$ such that $Q \leq_{(-T)} P$ implies $Q \in [U]$.

Proof. By induction on $n < \omega$ we define two sequences $A_0 \supseteq A_1 \supseteq \ldots$ and $p_0 < p_1 < \ldots$ such that, for all $n < \omega$,

1. $p_n \in A_n \in U$,
2. $A_{n+1} \subseteq \{q \in \FIN_{\pm k} : \langle p_n, q \rangle_{(-T)} \subseteq A_n\}$, and
3. $A_n \subseteq U_t \cap -(U_t)_1$ for every $t \in U$ such that

$$\supp U_t \subseteq \bigcup_{i < n} \supp p_i$$

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where, for a node $t = (t_0, \ldots, t_{m-1}) \in U$, $\bigcup t$ is the element $\sum_{i \leq m} t_i \in \text{FIN}_{\pm k}$. To start, take $A_0 := U_\emptyset \cap - (U_\emptyset)_1$ and note that $A_0 \in \mathcal{U}$ since $\mathcal{U}$ is subsymmetric and $U_\emptyset \in \mathcal{U}$. By definition of $\mathcal{U}$ we have

$$\langle \mathcal{U}p, \mathcal{U}q \rangle_{(-T)} \subseteq A_0$$

and so we take any $p_0 \in \text{FIN}_{\pm k}$ such that $\langle \mathcal{U}q \rangle_{(-T)} \subseteq A_0$; in particular $p_0 \in A_0$ by definition of $\langle \mathcal{U}p, \mathcal{U}q \rangle_{(-T)}$. We then take $A_1$ to be the intersection of the set $\{q \in \text{FIN}_{\pm k} : \langle p_0, q \rangle_{(-T)} \subseteq A_0\}$ with

$$\bigcap \left\{ U_t \cap -(U_t)_1 : t \in U \text{ and } \supp t \subseteq \supp p_0 \right\}.$$ 

Note that $A_0 \supseteq A_1$ and $A_1 \in \mathcal{U}$ since there are only finitely many $t \in U$ satisfying $\supp t \subseteq \supp p_0$, and since each $U_t \cap -(U_t)_1 \in \mathcal{U}$ using the fact that $\mathcal{U}$ is subsymmetric.

Now suppose $A_0, \ldots, A_n$ and $p_0, \ldots, p_{n-1}$ have been constructed. Since $\mathcal{U}$ is cofinite, pick any $p_n \in \text{FIN}_{\pm k}$ such that $p_n > p_{n-1}$ and $\langle \mathcal{U}q \rangle_{(-T)} \subseteq A_n$; in particular $p_n \in A_n$. Then take $A_{n+1}$ to be the intersection of the set $\{q \in \text{FIN}_{\pm k} : \langle p_n, q \rangle_{(-T)} \subseteq A_n\}$ with

$$\bigcap \left\{ U_t \cap -(U_t)_1 : t \in U \text{ and } \supp t \subseteq \bigcup_{i < n+1} \supp p_i \right\}.$$ 

As before, we have $A_{n+1} \subseteq U$ and $A_n \supseteq A_{n+1}$. This completes the induction.

To check that $P$ is the desired block sequence, we prove the following properties:

1. $\langle p_m, \ldots, p_n \rangle_{(-T)} \subseteq A_m$ for all $m \leq n < \omega$.
2. If $Q = \langle q_n \rangle_{n<\omega} \subseteq (-T) P$, then $\langle q_0, \ldots, q_m \rangle \in U$ for all $m < \omega$.

We check (4) by downward induction on $m \leq n$ for $n < \omega$ fixed. The case $m = n$ follows from (1), while the case $m = n - 1$ follows using (1) and (2) to obtain $\langle p_{n-1}, p_n \rangle_{(-T)} \subseteq A_{n-1}$. Now suppose inductively that (4) holds for some $m \leq n$; we aim to show $\langle p_{m-1}, p_m, \ldots, p_n \rangle_{(-T)} \subseteq A_{m-1}$. Take any

$$q = \sum_{i=m}^n (-T)^{j_i}(p_i)$$

with $j_{m-1}, \ldots, j_n \in \{0, \ldots, k\}$ and $\min j_i = 0$. We consider two cases: Suppose first that there is $i > m - 1$ such that $j_i = 0$. Then

$$q' := \sum_{i=m}^n (-T)^{j_i}(p_i) \in \langle p_m, \ldots, p_n \rangle_{(-T)} \subseteq A_m$$

where the inclusion comes from the inductive hypothesis. Then $q' \in A_m$ and so

$$q \in \langle p_{m-1}, q \rangle_{(-T)} \subseteq A_{m-1}$$

by (2). Now suppose $j_i > 0$ for each $i > m - 1$ (so that, in particular, $j_{m-1} = 0$). Let $l := \min\{j_m, \ldots, j_n\} > 0$ and write

$$q = p_{m-1} + (-T)^{l} \left( \sum_{i=m}^n (-T)^{j_i-l}(p_i) \right).$$

By the inductive hypothesis we have

$$q'' := \sum_{i=m}^n (-T)^{j_i-l}(p_i) \in \langle p_m, \ldots, p_n \rangle_{(-T)} \subseteq A_m,$$
and so \( q \in \langle p_{m-1}, q'' \rangle (\neg T) \subseteq A_{m-1} \) by (2). This completes the proof of (4).

Let \( Q \) be as in the statement of (5) and fix \( q = (q_0, \ldots, q_{m-1}) \). We prove (5) by induction on \( m < \omega \). If \( m = 0 \) then \( q = \emptyset \), which belongs to \( U \) since \( \text{stem}(U) = \emptyset \).

So suppose \( m > 0 \) and \( q = (t, q_{m-1}) \) where \( t \in U, |t| = m - 1 \) and \( q_{m-1} \in \text{FIN}_{\pm k} \).

Write
\[
q_{m-1} = \sum_{i<l} (-T)^{j_i}(p_{n_i})
\]
for some \( l < \omega, n_0 < \cdots < n_{l-1} < \omega \) and \( j_i \in \{0, \ldots, k\} \) with \( \min j_i = 0 \). Then \( q_{m-1} \in \langle p_{n_0}, \ldots, p_{n_{l-1}} \rangle (\neg T) \) and so by (4) we have \( q_{m-1} \in A_{n_0} \).

If \( n_0 = 0 \) then since \( q \) is a block sequence we must have \( t = \emptyset \) and \( m = 1 \), and so \( A_{n_0} = A_0 \subseteq U \emptyset \) which yields \( q_{m-1} = (\emptyset, p) \in U \).

Now assume \( n_0 > 0 \). Since \( q_{m-2} < q_{m-1} \) it must be the case that
\[
\text{supp } \bigcup_{i<n_0} t \subseteq \bigcup_{i<n_0} \text{supp } p_i.
\]
Then by (3) we obtain \( q_{m-1} \in U_1 \) and so \( q = (t, q_{m-1}) \in U \). This finishes the inductive proof of (5) and hence the proof of the lemma is complete. \( \square \)

In what follows, we will only need the following corollary of the above proof.

**Corollary 4.2.** For every \( U \)-tree \( U \) with stem \( \emptyset \) there is \( P = (p_n)_{n<\omega} \in \text{FIN}_{\pm k}^{[\infty]} \)

together with a sequence \( A_0 \supseteq A_1 \supseteq \ldots \) of subsets of \( \text{FIN}_{\pm k} \) such that:

1. \( A_n \subseteq U_t \cap -\langle U \rangle_1 \) for every \( t \in U \) such that \( \text{supp } \bigcup_{i<n} \text{supp } p_i \).
2. \( \langle p_m, \ldots, p_n \rangle (\neg T) \subseteq A_m \) for all \( m \leq n < \omega \).

Recall that for a block sequence \( P = (p_n)_{n<\omega} \) in \( \text{FIN}_{\pm k} \), \( \langle P \rangle^{[\infty]}_{\pm k} \) denotes the set of all infinite block subsequences of \( P \) in \( \text{FIN}_{\pm k} \). We then have the following key lemma which makes use of the \( S \)-closed \( U \)-trees defined in the previous section.

**Lemma 4.3.** Let \( U \) be an \( S \)-closed \( U \)-tree with stem \( U = \emptyset \). Then there is an infinite block sequence \( P = (p_n)_{n<\omega} \) in \( \text{FIN}_{\pm k} \) such that \( \langle P \rangle^{[\infty]}_{\pm k} \subseteq \langle [U] \rangle_3 \).

**Proof.** Find an infinite block sequence \( P \) as in Corollary 4.2. We claim that \( P \) satisfies the conclusion of the lemma. To see this, fix an infinite block subsequence \( Q = (q_n)_{n<\omega} \) of \( P \). For convenience, we fix some notation: For each \( n < \omega \) let \( I_n \) be the smallest set of non-negative integers such that
\[
q_n \in \langle p_i : i \in I_n \rangle_{\pm k}.
\]
Notice that since \( Q \) is a block subsequence of \( P \) we have \( \max I_n < \min I_m \) whenever \( n < m \).

We will find a block sequence \( Q' = (q'_n)_{n<\omega} \in [U] \) such that \( ||q_n - q'_n|| \leq 3 \) and \( \text{supp } q'_n \subseteq \text{supp } q_n \) for all \( n < \omega \). We define \( Q' \) recursively as follows. For \( n = 0 \), write
\[
q_0 = \sum_{i \in I_0} \varepsilon_i T^{j_i}(p_i)
\]
for some (necessarily unique) \( \varepsilon_i \in \{\pm 1\} \) and \( j_i < k \) such that \( \min j_i = 0 \). We consider the following two cases:

**Case 1.** There is \( i \in I_0 \) such that \( \varepsilon_i = +1 \) and \( j_i = 0 \).

For each \( i \in I_0 \), set \( r_i := \varepsilon_i T^{j_i}(p_i) \) for convenience. We consider the following two subcases:
In Case 2, for every $i \in I_0$, if $j_i = 0$ then $\varepsilon_i = -1$. Apply Case 1 to $-q_0$ to obtain $r \in \langle p_i : i \in I_0 \rangle_{(-T)}$ such that $\|q_0 - r\| \leq 1$ and $\text{supp}(r) \subseteq \text{supp}(-q_0)$. By Corollary [1.2] we have
\[
\langle p_i : i \in I_0 \rangle_{(-T)} \subseteq A_{\min I_0}
\]
(using the notation of Corollary [1.2] and so $q'_0 \in U_t$ for every $t \in U$ such that
\[
\text{supp} \bigcup_{i < \min I_0} t \subseteq \bigcup_{i < \min I_0} \text{supp} p_i.
\]
In particular, $q'_0 \in U_\varnothing$ and so $(\varnothing, q'_0) \in U$.

In particular, $-r \in (U_\varnothing)_1$ and so there is $r' \in U_\varnothing$ such that $\|r - r'\| \leq 1$. Since $U$ is $S$-closed, we have $(\varnothing, S(r')) \in U$ and so we set $q'_0 := S(r')$. Note that by definition of $S$ we have
\[
\text{supp} q'_0 \subseteq \text{supp}(-r) = \text{supp} r \subseteq \text{supp} q_0.
\]
Furthermore, using the fact that $\|r' - S(r')\| \leq 1$ we have
\[
\|q_0 - q'_0\| \leq \|q_0 - (-r)\| + \|(-r) - r'\| + \|r' - S(r')\| \leq 3
\]
and so $q'_0$ satisfies our requirements.

Now suppose for $m > 0$ we have defined $q'_0, \ldots, q'_{m-1}$ such that
\[
s := (q'_0, \ldots, q'_{m-1}) \in U,
\]
\[
\|q_i - q'_i\| \leq 3 \text{ and } \text{supp} q'_i \subseteq \text{supp} q_i \text{ for all } i < m.
\]
Write
\[
q_m = \sum_{i \in I_m} \varepsilon_i T^{j_i}(p_i)
\]
for some $\varepsilon_i \in \{\pm 1\}$ and $j_i < k$ such that $\min j_i = 0$. Note that since
\[
\text{supp} q'_i \subseteq \text{supp} q_i \subseteq \bigcup_{j \in I_i} \text{supp} p_i,
\]
we must have
\[
\text{supp} \bigcup s \subseteq \bigcup_{i < \min I_m} \text{supp} p_i.
\]
As in the base case of the induction, we consider the following two cases:
Case 1. There is $i \in I_m$ such that $\varepsilon_i = +1$ and $j_i = 0$.

For each $i \in I_m$, set $r_i := \varepsilon_i T^i(p_i)$ for convenience. We consider the following two subcases:

(a) $\varepsilon_i = +1$ and $j_i$ is even, or $\varepsilon_i = -1$ and $j_i$ is odd. In either case, set $r'_i := r_i$ and note that $r'_i = (-T)^{j_i}(p_i)$.

(b) $\varepsilon_i = +1$ and $j_i$ is odd, or $\varepsilon_i = -1$ and $j_i$ is even. In either case, set $r'_i := T(r_i)$ and note that $r'_i = (-T)^{j_i+1}(p_i)$.

We then set

$$q'_m := \sum_{i \in I_m} r'_i.$$

As before, we have $\text{supp} q'_m \subseteq \text{supp} q_m$ and $||q_m - q'_m|| \leq 1$. Furthermore, we have

$$\langle p_i : i \in I_m \rangle (-T) \subseteq A_{\min I_m}$$

and so $q'_m \in U_t$ for every $t \in U$ such that

$$\text{supp} \bigcup_{i < \min I_m} t \subseteq \bigcup_{i < \min I_m} \text{supp} p_i.$$

In particular, $q'_m \in U_s$ and so $(s, q'_m) \in U$.

Case 2. For every $i \in I_m$, if $j_i = 0$ then $\varepsilon_i = -1$.

Apply Case 1 to $-q_m$ to obtain $r \in \langle p_i : i \in I_m \rangle (-T)$ such that $||(-q_m) - r|| \leq 1$ and $\text{supp} r \subseteq \text{supp}(-q_m)$. As before, $r \in U_t \cap -(U_t)_1$ for every $t \in U$ such that

$$\text{supp} \bigcup_{i < \min I_m} t \subseteq \bigcup_{i < \min I_m} \text{supp} p_i.$$

In particular, $-r \in (U_s)_1$ and so there is $r' \in U_s$ such that $||(-r) - r'|| \leq 1$. Since $U$ is $S$-closed, we have $(s, S(r')) \in U$ and so we set $q'_m := S(r')$. As before, we check that $q'_m$ satisfies our requirements. This completes the inductive construction of $Q'$. It is clear from the above construction that $Q' \in [U]$ and $||q_n - q'_m|| \leq 3$ for all $n < \omega$ and so $Q \in ([U])_3$. □

To finish the proof of Theorem 11, we need the following mapping which was originally used in [6] to give an alternate proof of Gowers’ Theorem. Given $m \in \mathbb{N}$, let $\Phi_m : \text{FIN}_{\pm 2m} \to \text{FIN}_{\pm m}$ be defined by setting, for $p \in \text{FIN}_{\pm 2m}$ and $n < \omega$,

$$\Phi_m(p)(n) := \begin{cases} \frac{2}{p(n)} & \text{if } p(n) \text{ is even,} \\ \frac{p(n)-1}{2} & \text{if } p(n) > 0 \text{ and } p(n) \text{ is odd,} \\ \frac{p(n)+1}{2} & \text{if } p(n) < 0 \text{ and } p(n) \text{ is odd.} \end{cases}$$

The following lemma is easy to check.

Lemma 4.4. For each $m \in \mathbb{N}$, the mapping $\Phi_m$ has the following properties:

(i) $\Phi_m$ is a surjective homomorphism of partial semigroups which, in addition, satisfies $\Phi_m(-p) = -\Phi_m(p)$ for every $p \in \text{FIN}_{\pm 2m}$.

(ii) For every $p_0 < p_1 \in \text{FIN}_{\pm 2m}$ and every $j_0, j_1 < k + 1$ with $\min\{j_0, j_1\} = 0$, we have

$$\Phi_m(T^{2j_0}(p_0) + T^{2j_1}(p_1)) = T^{j_0}(\Phi_m(p_0)) + T^{j_1}(\Phi_m(p_1)).$$

(iii) For every $p_0, p_1 \in \text{FIN}_{\pm 2m}$ and every $l < \omega$, we have

$$||p_0 - p_1|| \leq 2l \implies ||\Phi_m(p_0) - \Phi_m(p_1)|| \leq l.$$
Now, for $k \in \mathbb{N}$ fixed as in the previous sections, let $\Psi : \text{FIN}_{\pm 4k} \to \text{FIN}_{\pm k}$ be given by $\Psi := \Phi_4 \circ \psi_2 k$. Using the properties listed in Lemma 4.4, it is easy to verify that $\Psi$ is a surjective homomorphism which satisfies:

(a) For every $p_0 < p_1 \in \text{FIN}_{\pm 4k}$ and every $j_0, j_1 < k + 1$ with $\min \{ j_0, j_1 \} = 0$, we have

$$\Psi (T^{j_0}(p_0) + T^{j_1}(p_1)) = T^{j_0}(\Psi(p_0)) + T^{j_1}(\Psi(p_1)).$$

(b) For every $p_0, p_1 \in \text{FIN}_{\pm 4k}$, if $||p_0 - p_1|| \leq 4$ then $||\Psi(p_0) - \Psi(p_1)|| \leq 1.$

We extend $\Psi$ to $\text{FIN}_{\pm 4k}$ by setting

$$\Psi((p_n)_{n<\omega}) := (\Psi(p_n))_{n<\omega}.$$ 

It is straightforward to check that $\Psi$ is continuous with respect to the usual metrizable topologies.

Recall that $||p_n||$ is defined to be the supremum of $||p_n||$ for $n < \omega$. Note that if $P$ and $P'$ are two block sequences in $\text{FIN}_{\pm 4k}$ which satisfy $||P - P'|| \leq 4$, then $||\Psi(P) - \Psi(P')|| \leq 1$. We are now ready to finish the proof of the main theorem.

**Proof of Theorem 1.2.** Let $c : \text{FIN}_{\pm 4k}^[\infty] \to r$ be Souslin measurable. We define a colouring $\tilde{c} : \text{FIN}_{\pm 4k}^[\infty] \to r$ by setting $\tilde{c} := c \circ \Psi$. Then $\tilde{c}$ is Souslin measurable and so by Corollary 2.3 there are $i < r$ and a $\mathcal{U}$-tree $V$ with stem $\emptyset$ such that $[V] \subseteq \tilde{c}^{-1}\{i\}$. Applying Lemma 2.2, we find an $S$-closed $\mathcal{U}$-tree $U$ such that $[U] \subseteq ([V])_1$; in particular we get

$$[U] \subseteq (\tilde{c}^{-1}\{i\})_1.$$ 

Since $U$ is $S$-closed, by Lemma 4.3 we can find an infinite block sequence $\tilde{P} = (\tilde{p}_n)_{n<\omega}$ in $\text{FIN}_{\pm 4k}$ such that $\langle \tilde{P} \rangle_{\pm 4k} \subseteq ([U])_3$ and hence

$$\langle \tilde{P} \rangle_{\pm 4k} \subseteq (\tilde{c}^{-1}\{i\})_4.$$ 

Let $P := \Psi(\tilde{P}) \in \text{FIN}_{\pm 4k}^[\infty]$ and set $p_n := \Psi(\tilde{p}_n)$ for each $n < \omega$. We claim that $P$ satisfies

$$\langle P \rangle_{\pm 4k} \subseteq (c^{-1}\{i\})_1.$$ 

Indeed, if $Q = (q_n)_{n<\omega} \in \text{FIN}_{\pm 4k}^[\infty]$ is an infinite block subsequence of $P$, then for each $n < \omega$ we have

$$q_n = \sum_{i<m} \varepsilon_i T^{j_i}(p_n)$$

for some $\varepsilon_i \in \{ \pm 1 \}, n_0 < \cdots < n_{m-1}$ and $j_i < k$ such that $\min j_i = 0$. Then using property (a) of $\Psi$ listed above we see that $q_n = \Psi(\tilde{q}_n)$, where

$$\tilde{q}_n := \sum_{i<m} \varepsilon_i T^{j_i}(\tilde{p}_n) \in \langle \tilde{P} \rangle_{\pm k}$$

and so, setting $\tilde{Q} := (\tilde{q}_n)_{n<\omega}$, we see that $Q = \Psi(\tilde{Q})$. Since $\tilde{Q}$ is a block subsequence of $\tilde{P}$, by our choice of $\tilde{P}$ we can find $Q' \in \tilde{c}^{-1}\{i\}$ such that $||\tilde{Q} - Q'|| \leq 4$. Then, as observed above, property (b) of $\Psi$ implies $||\Psi(\tilde{Q}) - \Psi(Q)|| \leq 1$. Since

$$i = \tilde{c}(Q') = c(\Psi(Q'))$$

we obtain $\Psi(Q') \in c^{-1}\{i\}$ and so $Q \in (c^{-1}\{i\})_1$ as required. □
In fact, we can do a bit better: Given an infinite block sequence \( P \) in \( \text{FIN}_{\pm k} \), the proof of Lemma 2.1 from \([6]\) can be adapted to show the existence of an ultrafilter \( \mathcal{U} \) on the partial semigroup \( \langle P \rangle_{\pm k} \) which has the properties listed in Lemma 2.1. One can then develop the theory of \( \mathcal{U} \)-trees on \( \langle P \rangle_{\pm k}^{<\infty} \) and prove a corresponding analogue of Corollary 2.5. By equipping \( \langle P \rangle_{\pm k}^{[\infty]} \) with its natural analogue of the metrizable topology and replacing \( \text{FIN}^{[\infty]} \) with \( \langle P \rangle_{\pm k}^{[\infty]} \) in the proof of the main result, we obtain the following relativized version of Theorem 1.2.

**Theorem 4.5.** For every \( k, r \in \mathbb{N} \), every infinite block sequence \( P \) in \( \text{FIN}_{\pm k} \) and every Souslin measurable \( c : \text{FIN}^{[\infty]}_{\pm k} \to r \) there are \( i < r \) and an infinite block sequence \( Q \leq P \) such that

\[
(Q)_{\pm k}^{[\infty]} \subseteq (c^{-1}\{i\})_1.
\]

The previous result can be used to “diagonalize” Theorem 1.2 as follows. First note that, for each \( j < k \in \mathbb{N} \), the \( j \)th iterate of the tetris operation \( \mathcal{T}^{(j)} : \text{FIN}_{\pm k} \to \text{FIN}_{\pm (k-j)} \) can be extended to \( \mathcal{T}^{(j)} : \text{FIN}^{[\infty]}_{\pm k} \to \text{FIN}^{[\infty]}_{\pm (k-j)} \) by setting

\[
\mathcal{T}^{(j)}((p_n)_{n<\omega}) := (\mathcal{T}^{(j)}(p_n))_{n<\omega}.
\]

We then have the following:

**Corollary 4.6.** For every \( k, r \in \mathbb{N} \) and every Souslin measurable colouring

\[
c : \bigcup_{j=1}^{k} \text{FIN}^{[\infty]}_{\pm j} \to r
\]

there are \( i_1, \ldots, i_k < r \) and \( P \in \text{FIN}^{[\infty]}_{\pm k} \) such that

\[
(T^{(k-j)}(P))_{\pm j}^{[\infty]} \subseteq (c^{-1}\{i_j\})_1
\]

for each \( j = 1, \ldots, k \).

**Proof.** By Theorem 1.2 we can find \( P_1 \in \text{FIN}^{[\infty]}_{\pm 1} \) and \( i_1 < r \) such that \( \langle P_1 \rangle_{\pm 1}^{[\infty]} \subseteq (c^{-1}\{i_1\})_1 \). Take any \( Q_2 \in \text{FIN}^{[\infty]}_{\pm 2} \) such that \( T(Q_2) = P_1 \) and apply Theorem 4.5 to \( Q_2 \) to obtain \( P_2 \leq Q_2 \) and \( i_2 < r \) such that \( \langle P_2 \rangle_{\pm 2}^{[\infty]} \subseteq (c^{-1}\{i_2\})_1 \). Continue inductively to obtain \( P_j \leq Q_j \in \text{FIN}^{[\infty]}_{\pm j} \) and \( i_j < r \), for \( j = 2, \ldots, k \), such that \( T(Q_j) = P_{j-1} \) and \( \langle P_j \rangle_{\pm j}^{[\infty]} \subseteq (c^{-1}\{i_j\})_1 \).

We claim that setting \( P := P_k \) works. Indeed, for a fixed \( j = 1, \ldots, k \) we have \( T^{(k-j)}(P) \leq P_j \) by construction (and using the general fact that \( T(P) \leq T(Q) \) whenever \( P \leq Q \)) and so the desired conclusion follows from the choice of \( P_j \). □

We conclude with a proof of the multi-dimensional version of Theorem 1.2. Recall that, for \( d \in \mathbb{N} \), \( \text{FIN}^{[d]}_{\pm k} \) denotes the set of all block sequences in \( \text{FIN}_{\pm k} \) of length \( d \). Given an infinite block sequence \( P \) let \( \langle P \rangle_{\pm k}^{[d]} \) be the set of all \( Q = (q_n)_{n<d} \in \text{FIN}^{[d]}_{\pm k} \) such that \( q_n \in \langle P \rangle_{\pm k} \) for each \( n < d \).

**Corollary 4.7.** For every \( k, d, r \in \mathbb{N} \) and every colouring \( c : \text{FIN}^{[d]}_{\pm k} \to r \) there are \( i < r \) and an infinite block sequence \( P \in \text{FIN}^{[\infty]}_{\pm k} \) such that

\[
\langle P \rangle_{\pm k}^{[d]} \subseteq (c^{-1}\{i\})_1.
\]
Proof. Given a colouring $c$ as above, let $\bar{c}: \text{FIN}^{[\omega]}_{\pm k} \to r$ be given by
$$\bar{c}((p_n)_{n<\omega}) := c((p_n)_{n<d}).$$
Then $\bar{c}$ is continuous and hence Souslin measurable since for each $i \in r$ we have
$$\bar{c}^{-1}\{i\} = \bigcup \left\{ \{Q\} : Q \in \text{FIN}^{[d]}_{\pm k} \cap c^{-1}\{i\} \right\},$$
(recall that $[Q]$ denotes the basic open set consisting of all infinite block sequences which begin with $Q$). By Theorem 1.2 there are $i \in r$ and $P \in \text{FIN}^{[\omega]}_{\pm k}$ such that
$$\langle P \rangle_{\pm k} \subseteq (\bar{c}^{-1}\{i\})_1.$$ Given $Q = (q_n)_{n<d} \in \langle P \rangle_{\pm k}$ extend $Q$ arbitrarily to any $\tilde{Q} \in \langle P \rangle^{[\omega]}_{\pm k} \cap [Q]$. By choice of $P$ there is $Q' = (q'_n)_{n<\omega} \in \bar{c}^{-1}\{i\}$ such that $||q_n - q'_n|| \leq 1$ for all $n < \omega$. Then $c((q'_n)_{n<d}) = i$ and so $Q \in (c^{-1}\{i\})_1$. \hfill $\square$

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Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada.

E-mail address: jamal.kawach@mail.utoronto.ca