STABLE ULRICH BUNDLES

MARTA CASANELLAS AND ROBIN HARTSHORNE

With an Appendix by Florian Geiß and Frank-Olaf Schreyer

Abstract. The existence of stable ACM vector bundles of high rank on algebraic varieties is a challenging problem. In this paper, we study stable Ulrich bundles (that is, stable ACM bundles whose corresponding module has the maximum number of generators) on nonsingular cubic surfaces $X \subset \mathbb{P}^3$. We give necessary and sufficient conditions on the first Chern class $D$ for the existence of stable Ulrich bundles on $X$ of rank $r$ and $c_1 = D$. When such bundles exist, we prove that the corresponding moduli space of stable bundles is smooth and irreducible of dimension $D^2 - 2r^2 + 1$ and consists entirely of stable Ulrich bundles (see Theorem 1.1). As a consequence, we are also able to prove the existence of stable Ulrich bundles of any rank on nonsingular cubic threefolds in $\mathbb{P}^4$, and we show that the restriction map from bundles on the threefold to bundles on the surface is generically étale and dominant.

1. Introduction

The study of moduli spaces of stable vector bundles of given rank and Chern classes on algebraic varieties is a very active topic in algebraic geometry. See for example the book [20]. In recent years attention has focused on ACM bundles, that is vector bundles without intermediate cohomology. Recently ACM bundles on hypersurfaces have been used to provide counterexamples to a conjecture of Griffiths and Harris about whether subvarieties of codimension two of a hypersurface can be obtained by intersecting with a subvariety of codimension two of the ambient space [22]. There have been numerous studies of rank 2 ACM bundles on surfaces and threefolds (see [3], [10], [23], [9], [6], and the references in those papers), and a few studies of ACM bundles of higher rank [2], [3], [24]. There have also been a few examples of indecomposable ACM bundles of arbitrarily high rank [26], [27]. But as far as we can tell, examples of stable ACM bundles of higher ranks are essentially unknown.

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In our earlier paper [8] we began such a study by constructing stable ACM bundles of all ranks \( r \) on a nonsingular cubic surface, with first Chern class \( rH \). The bundles we constructed are of a particular kind, the so-called Ulrich bundles: that is ACM bundles whose corresponding module has the maximum number of generators. We refer to the introduction of our earlier paper [8] for the history and motivation for considering these concepts and the corresponding notions of maximal Cohen-Macaulay modules and Ulrich modules in local algebra.

In this paper we continue that study by determining for which divisor classes on the cubic surface there are Ulrich bundles or stable Ulrich bundles. The following theorem summarizes our main results on cubic surfaces.

**Theorem 1.1.** (Theorem 4.3) Let \( D \) be a divisor on a nonsingular cubic surface \( X \subset \mathbb{P}^3 \), and let \( r \geq 2 \) be an integer. Then there exist stable Ulrich bundles \( \mathcal{E} \) of rank \( r \) with \( c_1(\mathcal{E}) = D \) if and only if \( 0 \leq D.L \leq 2r \) for all lines \( L \) on \( X \), and \( D.T \geq 2r \) for all twisted cubic curves \( T \) on \( X \), with one exception.

Moreover, if \( D \) satisfies the conditions above, the moduli space \( \mathcal{M}^s_X(r; c_1, c_2) \) of stable vector bundles on \( X \) of rank \( r \), \( c_1 = D \) and \( c_2 = \frac{D^2 - r^2}{2} \), is smooth and irreducible of dimension \( D^2 - 2r^2 + 1 \) and consists entirely of stable Ulrich vector bundles.

In particular, if \( D = rH \), where \( H \) is the hyperplane class, this gives a new proof of the main theorem (5.7) of [8].

Using this result we are also able to prove the existence of stable Ulrich bundles of any rank on any nonsingular cubic threefold \( Y \subset \mathbb{P}^4 \).

**Theorem 1.2.** (see Theorem 5.7 and Corollary 5.12) For any \( r \geq 2 \), the moduli space of stable rank \( r \) Ulrich bundles on a general cubic threefold \( Y \) in \( \mathbb{P}^4 \) is non-empty and smooth of dimension \( r^2 + 1 \). Furthermore, it has an open subset for which restriction to a hyperplane section gives an étale dominant map to the moduli of stable bundles on the cubic surface.

The motivation for writing this paper was to clarify the proof of stability for orientable (i.e. \( c_1 = rH \)) Ulrich bundles given in our previous paper [8, 5.3]. We thank R. M. Miró-Roig for pointing out that the proof of [8, 5.3] was not clear enough.

In section 2 we prove some generalities about Ulrich bundles. The most important result in this section shows that Ulrich bundles of any rank on nonsingular projective varieties of any dimension are semistable (cf. Theorem 2.9). In this section we also discuss modular families of simple Ulrich bundles, which will be crucial to show the existence of stable bundles. Indeed, our technique for
proving Theorem 1.1 is first to show the existence of simple Ulrich bundles. For these, we can compute the dimension of modular families by deformation theory. Then we show that the non-stable simple bundles form a family of smaller dimension, so the general bundle of that modular family must be stable.

In section 3 we study Ulrich bundles on a nonsingular cubic surface $X$. We first prove that, if they exist, they form an irreducible family. Then in Theorem 3.9 we give necessary and sufficient conditions on the first Chern class $D$ for the existence of some Ulrich bundle of rank $r$ with this first Chern class. This requires a careful analysis of the intersection properties of $D$ with lines and twisted cubic curves on the surface.

In section 4 we prove our main theorem giving necessary and sufficient conditions on $c_1(E)$ for the existence of a stable Ulrich bundle $E$ (see Theorem 4.3). As a consequence we recover Faenzi’s results [12] on stable rank 2 Ulrich bundles. As an illustration of our results we give the classification of Ulrich bundles of rank 3 on $X$.

In the last section we consider nonsingular cubic 3-folds and prove the existence of stable Ulrich bundles of all ranks (Theorem 5.7). We would like to thank Florian Geiß and Frank-Olaf Schreyer for computations in Macaulay2 that provide essential ingredients for extending our results from the cubic surface to cubic threefolds (see Appendix).

We expect that our results will generalize naturally using the same ideas to other del Pezzo surfaces and Fano threefolds. We have restricted our attention to the cubic surface and the cubic threefold for simplicity. What might be more interesting would be to explore the existence of higher rank stable Ulrich bundles on surfaces in $\mathbb{P}^3$ and threefolds in $\mathbb{P}^4$ of higher degree. Here is what is known about the existence of Ulrich bundles of rank $r \geq 2$ on a general hypersurface $X$ of degree $d \geq 3$ and dimension $N \geq 2$ in $\mathbb{P}^{N+1}$. It follows from the work of Beauville [4] that rank 2 Ulrich bundles exist for $N = 2$ if and only if $d \leq 15$; and for $N = 3$ if and only if $d \leq 5$. For $N \geq 4$ and $d \geq 3$ there are no rank 2 Ulrich bundles [10]. On the other hand, a theorem of Herzog, Ulrich and Backelin [19] shows that every hypersurface admits an Ulrich bundle of some high rank. A simple calculation with Chern classes (see Remark 2.5) shows that $r(d - 1)$ must be even for the existence of an Ulrich bundle, so the first open cases would seem to be $N = 2$, $r = 3$, $d = 5$; and $N = 3$, $r = 4$, $d = 6$. See also [21], [23], [24], [3], [6], [9] for further details on Ulrich bundles and more general ACM bundles.

Throughout this paper we work over an algebraically closed field $k$ of arbitrary characteristic.
2. Generalities on Ulrich sheaves on projective varieties

In this section we review the definition and cohomological properties of Ulrich bundles. We show that Ulrich bundles on any nonsingular projective variety are semistable, and if stable they are also \(\mu\)-stable. We discuss the two kinds of moduli spaces we will use in the paper. And for future reference we compute \(\chi(\mathcal{E} \otimes \mathcal{F}^\vee)\) for Ulrich bundles \(\mathcal{E}, \mathcal{F}\) on an algebraic surface.

Let \(X\) be an integral projective variety with a fixed very ample invertible sheaf \(\mathcal{O}_X(1)\), over an algebraically closed field \(k\) of arbitrary characteristic. Let \(d\) be the degree of \(X\) in the embedding defined by \(\mathcal{O}_X(1)\). A coherent sheaf \(\mathcal{E}\) on \(X\) is arithmetically Cohen-Macaulay (briefly ACM) if \(\mathcal{E}\) is locally Cohen-Macaulay, and \(H^i_*(\mathcal{E}) := \oplus_{t \in \mathbb{Z}} H^i(\mathcal{E}(t)) = 0\) for \(0 < i < \dim X\). If \(\mathcal{E}\) is an ACM sheaf of rank \(r\), one knows that the number \(m(\mathcal{E})\) of generators of the graded module \(H^0_*(\mathcal{E})\) is \(\leq dr\) [8, 3.1]. In our previous paper [8] we defined \(\mathcal{E}\) to be an Ulrich sheaf if the number of generators of \(H^0_*(\mathcal{E})\) achieved this maximum. To simplify terminology, in this paper we give a more restrictive definition:

**Definition 2.1.** Let \(X, \mathcal{O}_X(1)\) be an algebraic variety of degree \(d\). We say an ACM sheaf \(\mathcal{E}\) on \(X\) of rank \(r\) is an Ulrich sheaf if the module \(H^0_*(\mathcal{E})\) has the maximum number of generators \(dr\), and the generators are all in degree zero. (Thus an Ulrich sheaf is what we called a “normalized Ulrich sheaf generated by global sections” in [8].) If \(X\) is a nonsingular variety then an Ulrich sheaf is always locally free, so we may call it a vector bundle.

**Lemma 2.2.** If \(\mathcal{E}\) is an ACM sheaf on an integral projective variety \(X\) of degree \(d\), with \(h^0_*(\mathcal{E})(-1) = 0\) and \(h^0(\mathcal{E}) = dr\), then \(\mathcal{E}\) is Ulrich.

*Proof.* \(h^0_*(\mathcal{E})(-1) = 0\) implies that \(h^0(\mathcal{E}) \leq m(\mathcal{E})\) which is \(\leq dr\) by [8, 3.1]. Then equality \(h^0(\mathcal{E}) = dr\) implies \(m(\mathcal{E}) = dr\) and the generators are all in degree zero, so \(\mathcal{E}\) is Ulrich. \(\square\)

**Proposition 2.3.** Let \(X, \mathcal{O}_X(1)\) be a nonsingular projective curve of degree \(d\) and genus \(g\), and let \(\mathcal{E}\) be a rank \(r\) locally free sheaf on \(X\). Assume \(h^0(\mathcal{E}(-1)) = 0\). Then

\[
\begin{align*}
(a) & \quad h^0(\mathcal{E}) \leq dr \\
(b) & \quad \deg \mathcal{E} \leq r(d + g - 1) \\
(c) & \quad \chi(\mathcal{E}(n)) \leq dr(n + 1) \text{ for all } n \in \mathbb{Z}.
\end{align*}
\]

Furthermore, equality in any one of (a), (b), (c) implies equality in the other two, and is equivalent to \(\mathcal{E}\) being an Ulrich sheaf.
Proof. This result follows from \cite[section 4]{11} but for convenience we include a self-contained proof. First of all (a) follows from \cite[Theorem 3.1]{8}, since any locally free sheaf on a curve is ACM. Applying the Riemann-Roch theorem to $E(-1)$ and using the hypothesis $h^0(E(-1)) = 0$, we find
\[
\chi(E(-1)) = \deg E - rd + r(1 - g) \leq 0,
\]
which gives (b). Then Riemann-Roch for $E(n)$ says
\[
\chi(E(n)) = \deg E + nrd + r(1 - g),
\]
and substituting (b) gives (c).

Note that equality in (b) is equivalent to equality in (c). Equality in (b) implies $\chi(E(-1)) = 0$, so $H^1(E(-1)) = 0$. Hence from the sequence $0 \to E(-1) \to E \to E_H \to 0$ where $H$ is a general hyperplane section, consisting of $d$ points, we see that $h^0(E) = h^0(E_H) = dr$, which gives equality in (a).

Conversely, equality in (a), using the same exact sequence, shows that $\alpha : H^1(E(-1)) \to H^1(E)$ is injective. Since the map $H^0(E(n)) \to H^0(E_H(n))$ will also be surjective for $n \geq 0$, we find $\alpha(n) : H^1(E(n - 1)) \to H^1(E(n))$ is also injective for $n \geq 0$. So by Serre’s vanishing theorem, $H^1(E(n)) = 0$ for $n \gg 0$ and hence for all $n \geq -1$. Therefore $\chi(E(-1)) = 0$, and we get equality in (b), equivalent to equality in (c).

Moreover, equality in (a) is equivalent to $E$ being Ulrich. Indeed, if $E$ is Ulrich, then it has all its generators in degree 0 and therefore $h^0(E) = dr$. Conversely, if we have equality in (a), since we have assumed $h^0(E(-1))$, $E$ is Ulrich by Lemma 2.2.

The following lemma implies that the definition of Ulrich sheaf given in this paper and the one given in \cite{11} coincide.

Lemma 2.4. Let $E$ be an Ulrich bundle of rank $r$ on a nonsingular projective variety $X$ of dimension $N$ and $\deg X = d$. Then,

(i) The restriction $E_H$ of $E$ to a general hyperplane section is also an Ulrich bundle.
(ii) $h^0(E) = dr$ and $H^N(E(i)) = 0$ for any $i \geq -N$.
(iii) $\deg E = r(d + g - 1) g$ is the genus of $X \cap H^{N-1}$ for a general hyperplane section $H$.
(iv) If furthermore $X$ is subcanonical with $\omega_X = O_X(m)$ for some $m \in \mathbb{Z}$, then the twisted dual sheaf $E^*(N + m + 1)$ is also an Ulrich bundle.

Proof. If $N = 1$, then (i), (ii), and (iii) follow from Proposition 2.3. For (iv) note that $h^0(E^*(m + 2))$ is equal to $h^1(E(-2))$ by duality. But $h^0(E(-2)) = 0$ and
χ(E(−2)) = −dr by Proposition 2.3, so h⁰(E∨(m + 2)) = dr, which shows that E∨(m + 2) is Ulrich.

For N ≥ 2 we use induction on N. By Bertini’s theorem, we may assume that a general hyperplane section H is also nonsingular.

(i) Consider the exact sequence

\[ 0 \rightarrow E(-1) \rightarrow E \rightarrow E_H \rightarrow 0 \]

for a general hyperplane section H. It is easy to see that E_H is an ACM sheaf on H. Furthermore, from the exact sequence we see that H⁰(E_H(−1)) = 0 and h⁰(E_H) = dr, where d = deg(X) and r = rk E, so that E_H is an Ulrich sheaf on H by Lemma 2.2.

(ii) Since E_H is Ulrich by (i), the induction hypothesis applies to E_H so H^{N−1}(E_H(t)) = 0 for t ≥ −(N − 1) and h⁰(E_H) = dr. Therefore h⁰(E) = dr and H⁰(E(t − 1)) ∼= H⁰(E(t)) for all t ≥ −N + 1. By Serre’s vanishing, all these higher cohomologies are 0 and we are done.

(iii) Since deg E coincides with deg E_H for a general hyperplane section H of X and E_H is Ulrich by (i), we can apply the induction hypothesis to it. Therefore we obtain deg E = r(d + g − 1).

(iv) To show that E∨(N + m + 1) is Ulrich, we verify the conditions of Lemma 2.2. First, h⁰(E∨(N + m)) is dual to h⁰(E(−N)), which is zero by (ii) above. Next, for 0 < i < N, h^i(E∨(n)) is dual to h^{N−i}(E(m − n)) = 0, so we see that E∨ is an ACM sheaf. Now from h⁰(E∨(N + m)) = 0 it follows that h^0(E∨(N + m + 1)) = h^0(E_H∨(N + m + 1)). Now H has dimension N − 1 and ω_H = O_H(m + 1), so by induction E_H∨(N − 1 + m + 1 + 1) = E_H∨(N + m + 1) is Ulrich, and so its h^0 is dr. Hence also h^0(E∨(N + m + 1)) = dr and E∨(N + m + 1) is Ulrich.

Remark 2.5. If E is an Ulrich bundle of rank r on a general hypersurface X in P^{N+1} of dimension N and degree d ≥ 3, then r(d − 1) must be even. So for example there is no Ulrich bundle of rank 3 of a general quartic hypersurface. The reason for this is that under these hypotheses, because of the theorem of Noether-Lefschetz, Pic X = Z, so c₁(E) = mH for some integer m. Then deg E = md. But also deg E = r(d + g − 1) by Lemma 2.4 and X being a hypersurface, its linear curve section is a plane curve of genus g = 1/2(d − 1)(d − 2). Equating these two expressions for deg E, we find m = 1/2r(d − 1), so r(d − 1) must be even.

For future reference we compute the Hilbert polynomial of an Ulrich bundle on a nonsingular variety.
Lemma 2.6. If $\mathcal{E}$ is an Ulrich bundle of rank $r$ on a nonsingular projective variety $X$ of degree $d$ and dimension $N$, then its Hilbert polynomial is

$$P_{\mathcal{E}}(n) = rd\left(\frac{n + N}{N}\right).$$

Proof. (See also [11, 2.2]). By induction on $N$, the case $N = 1$ being Proposition 2.3 above. For $N \geq 2$, using the sequence $0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}_H \to 0$ we find $P_{\mathcal{E}}(n) - P_{\mathcal{E}}(n - 1) = P_{\mathcal{E}_H}(n)$ which by induction is $rd\left(\frac{n + N - 1}{N - 1}\right)$.

Since $\mathcal{E}$ is Ulrich, $H^i(\mathcal{E}(n)) = 0$ for all $i > 0$ and all $n \geq 0$. Hence $P_{\mathcal{E}}(n) = h^0(\mathcal{E}(n))$ for $n \geq 0$, and similarly for $\mathcal{E}_H$. But $h^0(\mathcal{E}(n)) = \sum_{j=0}^{n} h^0(\mathcal{E}_H(n)) = \sum_{j=0}^{n} rd\left(\frac{n + N - 1}{N - 1}\right)$. We conclude by summing the binomial coefficients. □

We now turn our attention to the stability and semistability property of these bundles.

Definition 2.7. Let $X$ be a nonsingular projective variety and let $\mathcal{E}$ be a vector bundle on it. Then $\mathcal{E}$ is said to be semistable if for every nonzero coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$P_{\mathcal{F}}/\rk(\mathcal{F}) \leq P_{\mathcal{E}}/\rk(\mathcal{E}),$$

where $P_{\mathcal{F}}$ and $P_{\mathcal{E}}$ are the Hilbert polynomials of the sheaves. It is stable if one always has strict inequality above.

The slope $\mu(\mathcal{E})$ of $\mathcal{E}$ is defined as $\deg(c_1(\mathcal{E}))/\rk(\mathcal{E})$. We say that $\mathcal{E}$ is $\mu$-semistable if for every subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \rk(\mathcal{F}) < \rk(\mathcal{E})$, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. We say $\mathcal{E}$ is $\mu$-stable if strict inequality $<$ always holds. The two definitions are related as follows

$$\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu - \text{semistable}$$

(see [20, 1.2.13]). Note that on a nonsingular curve stable is equivalent to $\mu$-stable and semistable is equivalent to $\mu$-semistable.

Proposition 2.8. Let $X, \mathcal{O}_X(1)$ be a nonsingular projective curve. Any Ulrich bundle $\mathcal{E}$ on $X$ is semistable. Furthermore, any coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with the same slope is also an Ulrich bundle.

Proof. From Proposition 2.3 (b) we have $\mu(\mathcal{E}) = d + g - 1$ where $d$ and $g$ are the degree and genus of $X$ respectively. If $\mathcal{F}$ is any coherent subsheaf of $\mathcal{E}$, then $\mathcal{F}$ is locally free and $h^0(\mathcal{F}(-1)) = 0$, so by Proposition 2.3 (b), $\mu(\mathcal{F}) \leq d + g - 1$. Hence $\mathcal{E}$ is semistable. If $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then we have equality in Proposition 2.3 (b) for $\mathcal{F}$, hence $\mathcal{F}$ is also Ulrich. □
For varieties of higher dimension we have the following result on Ulrich bundles (if they exist).

**Theorem 2.9.** Let $X, \mathcal{O}_X(1)$ be a nonsingular projective variety, and let $\mathcal{E}$ be an Ulrich bundle on $X$. Then

(a) $\mathcal{E}$ is semistable and $\mu$-semistable
(b) If $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is an exact sequence of coherent sheaves with $\mathcal{G}$ torsion-free, and $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then $\mathcal{F}$ and $\mathcal{G}$ are both Ulrich bundles.
(c) If $\mathcal{E}$ is stable, then it is also $\mu$-stable.

**Proof.** We will use induction on the dimension of $X$.

First suppose $\dim X = 1$. Part (a) follows by 2.8 since on a curve semistable is equivalent to $\mu$-semistable. For part (b) we have shown in Proposition 2.8 that $\mathcal{F}$ is Ulrich. Since $\mathcal{G}$ is assumed to be torsion-free, it is locally free on the curve $X$. By reason of degrees in the exact sequence, the slope of $\mathcal{G}$ will also be $d + g - 1$, so by Proposition 2.3(b) equality, $\mathcal{G}$ will be Ulrich also. Part (c) follows directly from the equivalence of stability and $\mu$-stability on curves.

Now let $\dim X \geq 2$. Given $\mathcal{E}$, we take a generic hyperplane section $H$ of $X$, which will also be a nonsingular variety of the same degree, and consider the restriction $\mathcal{E}_H$ of $\mathcal{E}$ to $H$. By Lemma 2.4 $\mathcal{E}_H$ is an Ulrich bundle on $H$. Thus we can apply the induction hypothesis to $\mathcal{E}_H$.

First we show that $\mathcal{E}$ is $\mu$-semistable. Indeed, if $\mathcal{F} \subset \mathcal{E}$ is any coherent subsheaf, then $\mathcal{F}_H$ is a subsheaf of $\mathcal{E}_H$ for a general hyperplane section $H$, so $\mu(\mathcal{F}_H) \leq \mu(\mathcal{E}_H)$ since $\mathcal{E}_H$ is $\mu$-semistable by induction. But the slope is preserved by passing to a hyperplane section, so $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ and $\mathcal{E}$ is $\mu$-semistable.

Next we prove part (b). Let

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

be an exact sequence with $\mathcal{G}$ torsion-free and $\mu(\mathcal{F}) = \mu(\mathcal{E})$. Choosing $H$ general, we have an exact sequence

$$0 \to \mathcal{F}_H \to \mathcal{E}_H \to \mathcal{G}_H \to 0$$

with $\mathcal{G}_H$ also torsion-free, so by the induction hypothesis, $\mathcal{F}_H$ and $\mathcal{G}_H$ are both Ulrich sheaves on $H$. Using the sequence $0 \to \mathcal{F}(n) \to \mathcal{F} \to \mathcal{F}_H \to 0$ and ditto for $\mathcal{G}$ we conclude that $h^0(\mathcal{F}(n)) = h^0(\mathcal{G}(n)) = 0$ for $n < 0$. Hence $h^0(\mathcal{F}) \leq ds$, where $s = \text{rk}(\mathcal{F})$, and $h^0(\mathcal{G}) \leq dt$, where $t = \text{rk}(\mathcal{G})$. But the rank of $\mathcal{E}$ is $r = s + t$, and $h^0(\mathcal{E}) \leq h^0(\mathcal{F}) + h^0(\mathcal{G})$, so we must have equality in both cases. From this we conclude that $H^0(\mathcal{F}) \to H^0(\mathcal{F}_H)$ is surjective and hence...
$H^0(\mathcal{F}(n)) \rightarrow H^0(\mathcal{F}_H(n))$ is also surjective for all $n \in \mathbb{Z}$, since $h^0(\mathcal{F}_H(-1)) = 0$ and $\mathcal{F}_H$ is generated by global sections. The same holds also for $\mathcal{G}$. Therefore $H^1(\mathcal{F}(n-1)) \rightarrow H^1(\mathcal{F}(n))$ is injective for all $n \in \mathbb{Z}$, and ditto for $\mathcal{G}$. But these groups are zero for $n >> 0$ by Serre vanishing, so they are all zero.

For $2 \leq i < \dim X$, since $\mathcal{F}_H$ is ACM, we have $H^{i-1}(\mathcal{F}_H(n)) = 0$ for all $n$. Hence $H^i(\mathcal{F}(n-1)) \rightarrow H^i(\mathcal{F}(n))$ is injective for all $n$, and again by Serre vanishing we conclude these groups are all zero.

To conclude that $\mathcal{F}$ and $\mathcal{G}$ are both Ulrich sheaves, it remains to show that they are locally Cohen-Macaulay sheaves, which amounts on the nonsingular variety $X$, to showing that they are locally free. The argument is the same for $\mathcal{G}$ and for $\mathcal{F}$, so we write $\mathcal{F}$ only. To say $\mathcal{F}$ is locally free is equivalent to saying depth $\mathcal{F} = \dim X$, and this is equivalent to the vanishing of the sheaf $\mathcal{E}.\text{ext}^i(\mathcal{F}, \omega_X)$ for all $i > 0$, since $\omega_X$ is locally free on $X$. Using Serre’s vanishing and the spectral sequence of local and global Ext, this is equivalent to saying $\text{Ext}^i(\mathcal{F}, \omega_X(n)) = 0$ for $i > 0$ and for all $n >> 0$. By Serre duality on $X$, this is equivalent to saying $H^i(X, \mathcal{F}(-n)) = 0$ for $i < \dim X$ and all $n >> 0$. But this we have already established, so $\mathcal{F}$ is locally free, and hence is an Ulrich sheaf. This completes the proof of part (b) of the Theorem.

Next, to show that $\mathcal{E}$ is also semistable, as in part (a), let $\mathcal{F}$ be any coherent subsheaf of $\mathcal{E}$. If $\mu(\mathcal{F}) < \mu(\mathcal{E})$ then clearly $\frac{P_{\mathcal{F}}}{rk_{\mathcal{F}}} < \frac{P_{\mathcal{E}}}{rk_{\mathcal{E}}}$, because the slope dominates the Hilbert polynomial. On the other hand, if $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then $P_{\mathcal{F}} \leq P_{\mathcal{F}'}$ where $\mathcal{F}'$ is a slightly larger sheaf, obtained by pulling back torsion from $\mathcal{E}/\mathcal{F}$, and then by part (b), $\mathcal{F}'$ is also Ulrich of the same slope. In this case $\frac{P_{\mathcal{F}'}(n)}{rk_{\mathcal{F}'}} \leq \frac{P_{\mathcal{F}}(n)}{rk_{\mathcal{F}}} = \frac{P_{\mathcal{E}}(n)}{rk_{\mathcal{E}}} = d(\frac{n+N}{N})$ where $N = \dim X$ by Lemma 2.6. Hence $\mathcal{E}$ is also semistable.

Finally, to prove part (c), suppose that $\mathcal{E}$ is stable. We wish to show that $\mathcal{E}$ is also $\mu$-stable, i.e. for any $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$. We know in any case $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. If equality holds, then by (b), replacing $\mathcal{F}$ by $\mathcal{F}'$ as above if necessary, $\mathcal{F}$ will also be Ulrich, and $\frac{P_{\mathcal{F}}}{rk_{\mathcal{F}}} = \frac{P_{\mathcal{E}}}{rk_{\mathcal{E}}}$, contradicting the hypothesis $\mathcal{E}$ stable.

In this paper we will use two kinds of moduli spaces. One is the usual moduli of semistable sheaves, as explained in [20, Chapter 4]. For fixed rank and Chern classes, we obtain a quasiprojective variety $M^s_X(r; c_1, \ldots, c_r)$ containing an open subset $M^s_X(r; c_1, \ldots, c_r)$ of stable bundles. The points of $M^s$ correspond to isomorphism classes of stable bundles, while the points of $M^s \setminus M^s$ correspond to $S$-equivalence classes of semistable bundles (see [20, 4.3.4]). Here two semistable bundles are called $S$-equivalent if the associated direct sum of stable bundles that occur as factors in a Jordan-Hölder sequence are the same. Thus we may
regard $M^{ss} \setminus M^s$ as parametrizing polystable bundles, i.e. direct sums of two or more stable bundles. The space $M^{ss}$ also corepresents the functor of families of semistable bundles in the sense that, for any flat family $E$ on $X \times T/T$, there is a corresponding morphism of $T \to M^{ss}$, and $M^{ss}$ is universal with this property [20, 2.2.1].

The property of being Ulrich in a family of vector bundles is an open condition. Indeed, the property of being locally free is open, the property of being an ACM sheaf is open, and the property $H^0(E(-1)) = 0$ is open. Since $H^i(E) = 0$ for $i > 0$ because $E$ is ACM and by Lemma [2.4] we see that $h^0(E)$ is of constant dimension in the family. Thus as soon as there is an Ulrich bundle, $h^0(E) = \chi(E) = rd$ is constant and thus the subset corresponding to Ulrich bundles is open. Therefore we have an open subset $M^{ss,U}$ in $M^{ss}$ and a corresponding open subset $M^{s,U}$ of $M^s$ corresponding to semistable and stable Ulrich bundles, respectively.

For a point $m \in M^s$ corresponding to a stable vector bundle $E$, the usual obstruction theory [20, 4.5.2] allows us to compute the Zariski tangent space to $M^s$ as $\text{Ext}^1(E, E)$, and obstructions as $\text{Ext}^2(E, E)$. Thus in our case, if we know there exist stable Ulrich bundles, we can estimate the dimension of the moduli space in this way. However, this method does not help us prove the existence of stable vector bundles.

To prove the existence of stable bundles, we introduce another space, the modular family of simple vector bundles. Recall [18, §26] that a modular family for a class of objects, say vector bundles on $X$, is a flat family $E$ on $X \times S/S$, with $S$ a scheme of finite type such that

a) each isomorphism class of bundles occurs at least once, and at most finitely many times in the family
b) For each $s \in S$, the local ring $\hat{O}_{S,s}$ together with the induced family, pro-represents the local deformation functor
c) for any other flat family $E'$ on $X \times S'/S'$ of such bundles, there exists a surjective étale map $S'' \to S'$ for some scheme $S''$, and a morphism $S'' \to S$ such that $E' \times_{S'} S''/X \times S'' \cong E \times_S S''/X \times S''$.

Recall that a vector bundle $E$ on $X$ is simple if $\text{End}(E) = k$. For simple vector bundles, the local deformation functor is pro-representable [18, 19.2]. Then, just as in the case of vector bundles on curves [18, 28.4], we obtain a modular family. Surely such a modular family exists quite generally, but we give the proof only in a restricted case since it is easier.

**Proposition 2.10.** On a nonsingular projective variety $X$, any bounded family of simple bundles $E$ with given rank and Chern classes satisfying $H^2(E \otimes E^\vee) = 0$ has a smooth modular family.
Proof. We follow the proof of [18, 28.4]. To eliminate the automorphisms induced by scalar multiplication, we consider pairs \((E, \theta)\) as in [18, 28.4]. We pick \(m > 0\) so that the sheaves \(E^m\) are generated by global sections, and consider the Quot scheme of quotients \(\mathcal{O}^N \rightarrow E(m) \rightarrow 0\). Then the main point is to observe from the sequence

\[
0 \rightarrow Q \rightarrow \mathcal{O}^N \rightarrow E(m) \rightarrow 0
\]

taking its dual and tensoring with \(\mathcal{E}(m)\), we have

\[
0 \rightarrow \mathcal{E} \otimes \mathcal{E}^\vee \rightarrow \mathcal{E}(m)^N \rightarrow Q^\vee \otimes \mathcal{E}(m) \rightarrow 0.
\]

Since \(H^1(\mathcal{E}(m)) = 0\) for \(m > 0\), and \(H^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 0\), we find that \(H^1(Q^\vee \otimes \mathcal{E}(m)) = 0\), so the Quot scheme of quotients \(\mathcal{O}^N \rightarrow E(m) \rightarrow 0\) is smooth, and also \(H^0(Q^\vee \otimes \mathcal{E}(m)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{E})\) is surjective. This enables us to show, as in the proof of [18, 27.2], that \(\widehat{\mathcal{O}}_{S,s}\) pro-represents the local deformation functor. The rest of the proof proceeds as in [18, 28.4]. \(\square\)

Remark 2.11. In particular, there is a smooth modular family for simple Ulrich bundles on the cubic surface of given rank and first Chern class. Indeed, they form a bounded family by Proposition 3.1 below, \(c_2\) is determined from \(c_1\) and \(r\) (because the Hilbert polynomial of an Ulrich bundle is determined by its rank by 2.6) and \(H^2(\mathcal{E} \otimes \mathcal{E}^\vee) \perp H^0((\mathcal{E} \otimes \mathcal{E}^\vee)(-1))\) which is 0 because \(\mathcal{E}\) is simple.

To aid in the computation of dimensions of families of extensions and moduli spaces, we compute \(\chi(\mathcal{E} \otimes \mathcal{F}^\vee)\) for Ulrich bundles \(\mathcal{E}, \mathcal{F}\) on any algebraic surface.

Proposition 2.12. Let \(X\) be a nonsingular projective surface of degree \(d\). Let \(\mathcal{E}, \mathcal{F}\) be Ulrich bundles of ranks \(r, s\), respectively, and with first Chern classes \(C, D\), respectively. Then

\[
\chi(\mathcal{E} \otimes \mathcal{F}^\vee) = rD.K - C.D + rs(2d - 1 - p_a),
\]

\[
\chi(\mathcal{E} \otimes \mathcal{E}^\vee) = rC.K + C^2 + r^2(2d - 1 - p_a),
\]

where \(K\) is the canonical divisor on \(X\) and \(p_a\) is the arithmetic genus of \(X\).

Proof. Let \(c_2\) be the second Chern class of \(\mathcal{E}\). Then Riemann-Roch theorem says

\(\chi(\mathcal{E}) = \frac{1}{2}(C^2 - 2c_2) - \frac{1}{2}C.K + r(1 + p_a).\)

Now call \(d_2\) the second Chern class of \(\mathcal{F}\). A straightforward computation of Chern classes shows that \(c_1(\mathcal{E} \otimes \mathcal{F}^\vee) = sC - rD\) and

\[
c_2(\mathcal{E} \otimes \mathcal{F}^\vee) = \frac{1}{2}s(s - 1)C^2 + sc_2 + \frac{1}{2}r(r - 1)D^2 + rd_2 - (rs - 1)C.D.
\]

Applying Riemann-Roch to the bundle \(\mathcal{E} \otimes \mathcal{F}^\vee\) of rank \(rs\), we find

\[
\chi(\mathcal{E} \otimes \mathcal{F}^\vee) = \frac{1}{2}(sC^2 + rD^2) - sc_2 - rd_2 - C.D - \frac{1}{2}(sC - rD).K + rs(1 + p_a).
\]
Now as $E$ is an Ulrich bundle of rank $r$ on the surface $X$ of degree $d$ we know that $\chi(E) = rd$ (see Lemma 2.6). But by Riemann-Roch this is also equal to $1$, so we can solve for $c_2$ and substitute in the above formula for $\chi(E \otimes F^\vee)$. Doing the same for the second Chern class $d_2$ of $F$ we obtain the desired result.

Corollary 2.13. Let $X$ be a del Pezzo surface of degree $d$. Let $E$, $F$ be as in Proposition 2.12. Then $\chi(E \otimes F^\vee) = (d-1)rs - C.D$ and $\chi(E \otimes E^\vee) = (d-1)r^2 - C^2$.

Proof. This immediately follows from Proposition 2.12 because in this case $p_a = 0$, $K = -H$, $\deg C = rd$ and $\deg D = sd$ by Lemma 2.4.

Corollary 2.14. If $E$ is a simple or stable Ulrich bundle of rank $r$ and $c_1(E) = D$ on the cubic surface $X$, then the modular family or the coarse moduli space at that point is smooth of dimension $D^2 - 2r^2 + 1$.

Proof. The modular family or the moduli space is smooth because $H^2(E \otimes E^\vee) = 0$ (cf. Remark 2.11). Then, as $h^0(E \otimes E^\vee) = 1$ for simple and stable bundles, the dimension of this space is $h^1(E \otimes E^\vee) = 1 - \chi(E \otimes E^\vee)$, which by Corollary 2.13 is equal to $D^2 - 2r^2 + 1$.

3. Existence of Ulrich sheaves on cubic surfaces

In this section for any Ulrich bundle $E$ on the nonsingular cubic surface $X$, we denote by $D \in \text{Pic} X$ the divisor class of its first Chern class $c_1(E)$. We express the Ulrich bundle as an extension of the twisted ideal sheaf $J_Z(D)$ for a certain zero-scheme $Z$ by a trivial sheaf $O_X^{-1}$. This allows us to show that the Ulrich bundles of given rank and first Chern class $D$ form an irreducible family in the moduli space. For ranks 1 and 2 we list the possible divisors $D$ that correspond to Ulrich bundles, using ad hoc arguments. The technical heart of this paper is Proposition 3.8 that shows for a divisor $D$ satisfying certain incidence conditions, how to choose a twisted cubic curve $T$, so that the new divisor $D' = D - T$ satisfies the same conditions. To prove this we express $D$ in a “standard form” in Pic $X$, and do a case-by-case analysis. This proposition is used in the induction needed to characterize those divisors $D$ corresponding to Ulrich bundles of any rank, in terms of their intersection numbers with lines. The same proposition is used again in the next section to characterize those $D$ that correspond to stable Ulrich bundles.

Proposition 3.1. (a) Let $E$ be an Ulrich bundle of rank $r \geq 2$ on the cubic surface $X$. Then there is an exact sequence

$$0 \to O_X^{-1} \to E \to J_Z(D) \to 0$$
where $D$ is a divisor of positive degree, $Z$ is a zero-scheme in $X$, and furthermore $H^0(\mathcal{J}_Z(D-H)) = 0$.

(b) Conversely, given an integer $r \geq 2$, a divisor $D$ of positive degree and a zero-scheme $Z$ of length $n$ such that $h^0(\mathcal{J}_Z(D-H)) = 0$, the collection of coherent sheaves $\mathcal{E}$ obtained as extensions as above forms an irreducible family (for $r, D, n$ fixed and $Z$ and choice of extension variable).

Proof. (a) Since $\mathcal{E}$ is generated by global sections by definition, a well-known lemma states that dividing by $r-1$ general sections will leave a quotient of rank 1 that is torsion-free. Any such torsion-free sheaf can be written as $\mathcal{J}_Z(D)$ for some divisor $D$ and some zero-scheme $Z$. Since $D$ represents $c_1(\mathcal{E})$, we have $\deg D = 3r > 0$.

Since $\mathcal{E}$ is Ulrich, we have $H^0(\mathcal{E}(-1)) = 0$. Since also $H^1(\mathcal{O}_X(-1)) = 0$, we find $H^0(\mathcal{J}_Z(D-H)) = 0$.

(b) We first note that the Hilbert scheme of zero-schemes in $X$ of length $n$ is irreducible [13]. The condition $H^0(\mathcal{J}_Z(D-H)) = 0$ is an open condition on $Z$, so the family of $Z'$s used here is irreducible. Next we observe that such an extension is defined by a choice of $r-1$ elements of $\text{Ext}^1(\mathcal{J}_Z(D), \mathcal{O}_X)$ which is dual to $H^1(\mathcal{J}_Z(D-H))$. Because of the hypothesis $H^0(\mathcal{J}_Z(D-H)) = 0$ there is an exact sequence

$$0 \to H^0(\mathcal{O}_X(D-H)) \to H^0(\mathcal{O}_Z) \to H^1(\mathcal{J}_Z(D-H)) \to H^1(\mathcal{O}_X(D-H)) \to 0.$$ 

Now $H^2(\mathcal{O}_X(D-H))$ is dual to $H^0(\mathcal{O}_X(-D))$, which is zero because $D$ has positive degree. Therefore

$$h^1(\mathcal{J}_Z(D-H)) = \text{length } Z - \chi(\mathcal{O}_X(D-H))$$

depends only on $D$ and $n$. Thus the dimension of $\text{Ext}^1(\mathcal{J}_Z(D), \mathcal{O}_X)$ is constant as $Z$ varies, so the family of all these sheaves is parametrized by a Grassmannian of a vector bundle over an open subset of the Hilbert scheme, and hence is an irreducible family. □

Corollary 3.2. The Ulrich bundles $\mathcal{E}$ on the cubic surface $X$ of given rank and first Chern class $c_1(\mathcal{E}) \in \text{Pic}(X)$ (if they exist) form an irreducible family. More precisely, there is a smooth irreducible variety $T$ and a vector bundle $\mathcal{E}$ on $X \times T$ such that each fiber for $t \in T$ is an Ulrich bundle on $X$ with given rank and $c_1(\mathcal{E})$, and all such Ulrich bundles appear in the family $\mathcal{E}$ (in general many times).

Proof. Since $\mathcal{E}$ is an Ulrich bundle, its Hilbert polynomial is already determined by its rank (see Lemma 2.6), so in particular the length $n$ of the zero-scheme $Z$ of part (a) of the previous proposition, which represents the second Chern class of $\mathcal{E}$, is determined by $r$. Then using the given $r, D, n$, we consider the irreducible
family of coherent sheaves described in 3.1(b). The property of being Ulrich is open (see discussion on moduli spaces in section 2). This open subset gives the parameter space $T$ for the required irreducible family.

**Corollary 3.3.** The moduli space $\mathcal{M}_X^{ss,U}(r; D, n)$ of semistable Ulrich bundles is irreducible.

**Proof.** As all Ulrich bundles are semistable by Theorem 2.9, there is a map from the irreducible family in Corollary 3.2 to the coarse moduli space $\mathcal{M}_X^{ss}(r; D, n)$. The image of this map is $\mathcal{M}_X^{ss,U}(r; D, n)$ which is therefore irreducible.

**Proposition 3.4.** If $\mathcal{E}$ is an Ulrich bundle of rank $r \geq 1$ on a cubic surface $X$ with first Chern class $D$, then

a) $\deg D = 3r$

b) $D^2 = 2c_2(\mathcal{E}) + r > 0$

c) $0 \leq D.L \leq 2r$ for all lines $L$ on $X$

d) $D$ can be represented by an irreducible nonsingular curve.

**Proof.** a) $\deg D = \deg \mathcal{E} = 3r$ by Lemma 2.4

b) The Riemann-Roch theorem applied to $\mathcal{E}$ says

$$\chi(\mathcal{E}) = r + \frac{1}{2} c_1(\mathcal{E}).H + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})).$$

On the other hand, by Lemma 2.6 $\chi(\mathcal{E}) = 3r$. Since $c_1(\mathcal{E}) = D$ and $D.H = 3r$ we can solve for $c_1(\mathcal{E})^2 = D^2$ to get b). Note that by Proposition 3.1, the second Chern class $c_2(\mathcal{E})$ is represented by an effective zero-scheme, so $c_2(\mathcal{E}) \geq 0$.

c) If $\mathcal{E}$ is an Ulrich bundle of rank $r$ with $c_1(\mathcal{E}) = D$, then it has a resolution over $\mathbb{P}^3$ of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{3r} \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{3r} \longrightarrow \mathcal{E} \longrightarrow 0$$

by [8, 3.7]. If $L$ is a line in $X$, restricting to $L$ we get

$$\ldots \longrightarrow \mathcal{O}_L(-1)^{3r} \overset{\alpha}{\longrightarrow} \mathcal{O}_L^{3r} \longrightarrow \mathcal{E}_L \longrightarrow 0.$$

The image of $\alpha$ must be a subbundle of $\mathcal{O}_L^{3r}$ of rank $2r$, which is also a quotient of $\mathcal{O}_L(-1)^{3r}$. On $L$, vector bundles are completely decomposable, so it must be of type $\mathcal{O}_L(-1)^s \oplus \mathcal{O}_L^t$ for some $s, t \geq 0$, $s + t = 2r$. The degree of this bundle is $-s$, so the degree of the quotient $\mathcal{E}_L = \mathcal{O}_L^{3r}/\operatorname{Im} \alpha$ (which is equal to $D.L$) is $s$ and $0 \leq s \leq 2r$.

d) This follows from b) and c) [15, V, Exercise 4.8].
Example 3.5. Ulrich bundles of rank 1. These will be line bundles $L$, with
$\deg L = 3, h^0(L) = 3, h^0(L(-1)) = 0$, and $L$ generated by global sections. Thus
$L$ corresponds to a curve of degree 3 irreducible and nonsingular by Proposition
3.4. The only such curves on the cubic surface are $H$, (a hyperplane section, in
which case $h^0(L(-1)) \neq 0$) and twisted cubic curves. Thus $L \cong O(T)$ where $T$
is a twisted cubic curve. There are 72 classes of twisted cubic curves in Pic $X$,
represented by (using notation of [15, V, §4])

$T_A = (1; 0, 0, 0, 0, 0, 0)$
$T_B = (2; 1, 1, 1, 0, 0, 0)$
$T_C = (3; 2, 1, 1, 1, 1, 0)$
$T_D = (4; 2, 2, 2, 1, 1, 1)$
$T_E = (5; 2, 2, 2, 2, 2, 2)$

and their permutations. The permutations give 1 of type $T_A$, 20 of $T_B$, 30 of $T_C$,
20 of $T_D$ and 1 of $T_E$, making 72 in all. For future reference we record that
$T_A^2 = 1$ for any twisted cubic curve $T$, and that the intersection of two distinct classes
can be 2, 3, 4 or 5 (for example $T_A \cdot T_B = 2, T_A \cdot T_C = 3, T_A \cdot T_D = 4, T_A \cdot T_E = 5$).

Example 3.6. Rank 2 Ulrich bundles on the cubic surface. We start
by listing the possible irreducible nonsingular curves $D$ (see Proposition 3.4) of
degree 6 on the cubic surface, given as divisors $(a; b_1, \ldots, b_6)$ in standard form
with $a \geq b_1 + b_2 + b_3$, and $b_1 \geq \cdots \geq b_6$ (see Proposition 3.7).

| $D$ | $D^2$ | $\sum T_i$ | $\exists$ Ulrich | $\dim \{\text{ss simple}\}$ | $\dim \{\text{stable}\}$ |
|-----|-------|-------------|-----------------|-----------------|-----------------|
| $a$ | (2;0,0,0,0,0,0) | 4 | $2A$ | ✓ | - | - |
| $b$ | (3;1,1,1,0,0,0) | 6 | $A + B$ | ✓ | - | - |
| $c$ | (3;2,1,0,0,0,0) | 4 | - | - | - |
| $d$ | (4;2,1,1,1,1,0) | 8 | $A + C$ | ✓ | 0 | 1 |
| $e$ | (4;1,1,1,1,1,1) | 10 | $B + B'$ | ✓ | 1 | 3 |
| $f$ | (6;2,2,2,2,2,2) | 12 | $A + E$ | ✓ | 2 | 5 |

We have written the various divisors $D$ as sums of twisted cubic curves (labeled
as in Example 3.5) in the third column, by inspection. The sign $'$ on a letter,
such as $B'$, means a permutation of the form listed for that letter. In this case,
$B' = (2; 0, 0, 0, 1, 1, 1)$. The column $\dim \{\text{ss simple}\}$ refers to the dimension of
properly semistable simple Ulrich bundles and $\dim \{\text{stable}\}$ to the dimension of
stable Ulrich bundles. We will justify these two last columns later in section 4.

To determine the possible existence of an Ulrich bundle with $c_1 = D$, first
note that if there are stable bundles corresponding to $D$, then the dimension of
the moduli space is $D^2 - 2r^2 + 1 = D^2 - 7$ (see Corollary 2.14), and this number
must be nonnegative, so $D^2 \geq 7$. Thus for the first three cases $a, b, c$ above there
can be no stable Ulrich bundles. If there is any Ulrich bundle $E$ at all, then it must
be semistable by Theorem 2.9, hence an extension of two rank 1 Ulrich bundles: take any rank 1 subsheaf \( F \) of the same slope, pull back torsion of \( G = E/F \) if necessary, and apply Theorem 2.9. Each of these is a twisted cubic curve, so \( D = T_1 + T_2 \) is a sum of two twisted cubic curves. Since the divisor in case \( c \) above cannot be written in this way, there is no Ulrich bundle corresponding to that \( D \). On the other hand, in the other five cases, just taking a direct sum \( \mathcal{O}(T_1) \oplus \mathcal{O}(T_2) \) shows the existence of Ulrich bundles for those values of \( D \).

**Proposition 3.7.** (Standard form) Let \( D \) be a divisor on a nonsingular cubic surface \( X \).

(a) There is a choice of 6 points in \( \mathbb{P}^2 \) such that when \( X \) is represented as the blow-up of these 6 points, \( D \) can be represented as \( (a; b_1, \ldots, b_6) \) with \( b_1 \geq \cdots \geq b_6 \) and \( a \geq b_1 + b_2 + b_3 \) in the notation of [15, V, §4].

(b) The integers \( a, b_1, \ldots, b_6 \) in a representation satisfying the conditions of (a) are uniquely determined. In particular, \( b_3, b_4, b_5, b_6 \) are the four smallest numbers of the set \( \{ D.L \mid L \text{ is a line on } X \} \).

(c) In that representation \( b_6 = \min \{ D.L \mid L \text{ is a line on } X \} \) and \( D.G_1 = 2a - \sum_{i=2}^{6} b_i = \max \{ D.L \mid L \text{ is a line on } X \} \) (\( G_1 \) refers to the notation of [15, V, 4.9]). One can also write \( D.G_1 = d - a + b_1 \) where \( d = \deg D \).

(d) The integer \( a \) can be characterized as

\[
  a = \min \{ D.T \mid T \text{ is a twisted cubic on } X \}.
\]

**Proof.** (a) (cf. [15, proof of V, 4.11]) Given \( D \) choose six mutually skew lines \( E_1', \ldots, E_6' \) as follows. Take \( E_6' \) so that \( D.E_6' = \min \{ D.L \mid L \text{ line on } X \} \). Choose \( E_5' \) such that \( D.E_5' \) is minimum among lines that do not meet \( E_6' \). Choose \( E_4' \) and \( E_3' \) similarly such that \( D - E_i' \) is minimum among those lines not meeting any of the \( E_j' \) already chosen. There remain three lines not meeting any of \( E_3', E_4', E_5', E_6' \), one of which meets the other two. Take the two that do not meet each other to be \( E_1', E_2' \) with \( D.E_1' \geq D.E_2' \). Then \( E_1', \ldots, E_6' \) being six mutually skew lines, we can represent them as \( E_1, \ldots, E_6 \) for a suitable projection \( X \to \mathbb{P}^2 \) [15, V, 4.10]. We write \( D = a.l - \sum b_i e_i \) in this basis for statement (a). As \( b_i = D.E_i \), by the choices we have made we have \( b_1 \geq b_2 \geq \cdots \geq b_6 \). Note that \( F_{12} \) was available (notation of [15, V, 4.9]), not meeting \( E_4, E_5, E_6 \) at the time we chose \( E_3 \). Therefore \( D.F_{12} \geq D.E_3 \), i.e \( a - b_1 - b_2 \geq b_3 \). This says \( a \geq b_1 + b_2 + b_3 \).

(b) Suppose a divisor \( D \) is represented as \( (a; b_1, \ldots, b_6) \) as in (a) with \( b_1 \geq \cdots \geq b_6 \) and \( a \geq b_1 + b_2 + b_3 \). Then (using the notation \( E_i, F_{ij}, G_i \) of [15, V 4.9]) we have

\[
\begin{align*}
  D.E_i &= b_i \\
  D.F_{ij} &= a - b_i - b_j \\
  D.G_j &= 2a - \sum_{i \neq j} b_i
\end{align*}
\]
From the condition \( a \geq b_1 + b_2 + b_3 \) it follows that the minimum of \( D.F_{ij} \) is \( D.F_{12} \geq b_3 \). The maximum of the \( D.F_{ij} \) is \( D.F_{56} = a - b_5 - b_6 \). This is \( \leq D.G_4 = 2a - \sum_{i \neq 4} b_i \) since \( a \geq b_1 + b_2 + b_3 \). Clearly \( D.G_4 \leq D.G_3 \leq D.G_2 \leq D.G_1 \). Moreover \( D.G_3 \geq D.G_6 \) which is \( \geq b_3 \) because \( a \geq b_1 + b_2 + b_3 \geq b_3 + b_4 + b_5 \). Thus the recipe of (a) will chose the same \( E_i \), and we have the same representation. Note that even though we chose \( E'_5 \) minimum among the lines not meeting \( E'_6 \), and similarly \( E'_4, E'_3 \), it follows from the proof of (b) that \( b_3, b_4, b_5, b_6 \) are the four smallest values of \( \{ D.L \} \).

(c) From the analysis just given, \( b_6 = \min \{ D.L \} \) and \( D.G_1 = \max \{ D.L \} \). Also \( D.G_1 = 2a - \sum_{i \neq 1} b_i = d - a + b_1 \) since \( d = 3a - \sum_{i=1}^{6} b_i \).

(d) To compute \( D.T \) for various twisted cubic curves \( T \), we use the notation in Example 3.5 above and their permutations. Note that the basis for \( \text{Pic} \ X \) was chosen to make \( D \) have standard form, so the \( T \)'s need not be in standard form. Then

- \( D.T_A = a \)
- \( D.T_B = 2a - b_1 - b_2 - b_3 \geq a \). If we use a permutation of \( B \) we get \( 2a - b_1 - b_j - b_k \geq a \) because \( b_1 \geq b_2 \geq \cdots \geq b_6 \).
- \( D.T_C = 3a - 2b_1 - b_2 - b_3 - b_4 - b_5 \). Again, since \( a \geq b_1 + b_2 + b_3 \) and \( a \geq b_1 + b_4 + b_5 \) this is \( \geq a \). Moreover, this is the minimum of \( D.T_C \) for any permutation of \( C \).
- \( D.T_D = 4a - 2(b_1 + b_2 + b_3) - (b_4 + b_5 + b_6) \). Again, even after permutation of \( T_D \) this is \( \geq a \).
- \( D.T_E = 5a - 2 \sum b_i \geq a \).

Thus \( a = D.T_A = \min \{ D.T \mid T \text{ twisted cubic curve} \} \), which proves (d).

Next we describe an algorithm that will be used later in some inductive proofs. Given a divisor \( D \) of degree \( 3r \) we subtract a suitable twisted cubic curve to get a new divisor \( D' \) of degree \( 3r - 3 \).

**Proposition 3.8.** Let \( D \) be a divisor on \( X \) of degree \( 3r \), with \( r \geq 2 \), satisfying \( 0 \leq D.L \leq 2r \) for all lines \( L \). Write \( D \) as \( (a; b_1, \ldots, b_6) \) in standard form (see Proposition [3.7]). Then

(a) There exists a twisted cubic curve \( T \) with \( D.T = a \) so that the new divisor \( D' = D - T \) verifies \( 0 \leq D'.L \leq 2r - 2 \) for all lines \( L \).

(b) Furthermore, assuming \( r \geq 3 \), if \( a \geq 2r \) and \( D \) is not a multiple of \( D_0 = (4; 2, 1^4, 0) \), then \( D' \) is not a multiple \( mD_0 \) for any \( m \geq 2 \), and also \( a' \geq 2r - 2 \) where \( D' = (a'; b_1', \ldots, b_6') \) is in standard form.
Proof. We distinguish the following cases.

Case 1: \(a > b_1 + b_2 + b_3\). Take \(T = T_A = (1; 0^6)\). Then \(D.T = a\) and \(D' = D - T = (a - 1; b_1, \ldots, b_6)\). By hypothesis \(a - 1 \geq b_1 + b_2 + b_3\), so it is still standard form. Since \(D.L \geq 0\) for all \(L\), we have \(b_6 \geq 0\), so \(D'.L \geq 0\) for all \(L\) also. To find the maximum of \(D'.L\), we use Proposition 3.7(c) to write it as \(d' - a' + b'_1\) where \(d' = \deg D'\). Now \(D.L \leq 2r\) for all \(L\), so \(d - a + b_1 \leq 2r\). But \(d' = 3r - 3, a' = a - 1\) and \(b'_1 = b_1\), so \(d' - a' + b'_1 \leq 2r - 2\).

If \(a \geq 2r\), then \(a' = a - 1 \geq 2r - 1 \geq 2r - 2\) as required. Moreover if \(a \geq 2r\) then \(D'\) cannot be a multiple of \(D_0\). Indeed, if it were, say \(D' = mD_0\) with \(m \geq 2\), then \(a - 1 = 4m\), and \(\deg D' = 3r - 3 = m \deg D_0 = 6m\), so \(r = 2m + 1\). But \(a = 4m + 1 \geq 2r\), a contradiction.

Case 2: \(a = b_1 + b_2 + b_3\) and either \(b_2 > b_4\) or \(b_3 > b_5\). We take \(T = T_B\), so \(D.T = 2a - \sum b_i\). Since \(a = b_1 + b_2 + b_3\) this gives \(D.T = a\). Write \(D' = D - T_B = (a - 2; b_1 - 1, b_2 - 1, b_3 - 1, b_4, b_5, b_6)\). Then we reorder the six latter numbers so that they are in descending order.

Case 2a. If \(b_2 > b_4\), then \(b_4\) may move up to the third place (in case \(b_3 = b_4\), but no further. So the first three \(b'_i\) include \(b_1 - 1, b_2 - 1\) and either \(b_3 - 1\) or \(b_4\). In either case \(a - 2\) is \(\geq\) their sum.

Case 2b. If \(b_3 > b_5\), then again \(b_4\) may move up to the first place (if \(b_1 = b_2 = b_3 = b_4\), but we still have \(b_1 - 1, b_2 - 1\) in the first three positions so again \(a - 2\) \(\geq\) their sum.

Hence \(D'\) is in standard form and since \(b_6 \geq 0\), \(D'.L \geq 0\) for all \(L\). Moreover, \(d - a + b_1 \leq 2r\) implies \(d' - a' - b'_1 \leq 2r - 2\) unless \(3r - a + b_1 = 2r\) and \(b_1 = b_2 = b_3 = b_4 =: b\). In this case \(a = r + b\). But \(a = b_1 + b_2 + b_3 = 3b\), so \(r = 2b\). However this contradicts the fact that degree of \(D\) is equal to \(3r = 6b\) as the degree is given by \(3a - \sum b_i = 5b - b_5 - b_6\).

Moreover, assume now that \(a \geq 2r\). Then \(a' = a - 2 \geq 2r - 2\). Furthermore \(D'\) cannot be equal to \(mD_0\) for any \(m \geq 2\), because then \(D = (4m + 2; 2m + 1, (m + 1)^2, m^2, 0)\), which is not in standard form.

Case 3: \(a = b_1 + b_2 + b_3, b_2 = b_3 = b_4 = b_5 > b_6\). Take \(T = T_C\) so that \(D.T_C = 3a - 2b_1 - \sum b_i = a\). Then \(D' = D - T_C = (a - 3; b_1 - 2, b_2 - 1, b_3 - 1, b_4 - 1, b_5 - 1, b_6)\). We first verify that \(b_1 \geq 2\). If not, then \(D = (a; 1^5, 0)\) and \(\deg D = 3a - 5\) which is not a multiple of \(3\). Hence \(b_1 \geq 2\) so all the new \(b'_i\) are \(\geq 0\). Rearrange \(b_i\) in descending order: since \(b_5 > b_6\) we still have \(b_6\) at the last place. Hence the first three places will contain \(b_2 - 1, b_3 - 1, b_4 - 1\) or possibly \(b_1 - 2\) in place of one of these if \(b_1 > b_2 + 1\), so \(a - 3 \geq b'_1 + b'_2 + b'_3\). Thus \(D'\) is in standard form, so \(b'_6 = b_6 \geq 0\) and hence \(D'.L \geq 0\) for any line \(L\).
If $b_1 > b_2$ then $d' - a' + b'_1 = 3r - 3 - (a - 3) + b_1 - 2$ so the condition $D.L \leq 2r$ for all lines implies $D'.L \leq 2r - 2$ for all lines. If $b_1 = b_2$ then $d' - a' + b'_1 = 3r - 3 - (a - 3) + b_1 - 1$ so $D'.L \leq 2r - 2$ also for all lines unless max $D.L = 2r$. In that case $b_1 = b_2$, $d - a + b_1 = 2r$, and $D = (3b; b, b, b, b, b)$. Then $d = 2r + 2b = 3r$, so $r = 2b$ and $d = 6b$. This implies $b_0 = -2b$, a contradiction.

Now if $a > 2r$, then $a' = a - 3 \geq 2r - 2$ and we conclude. If $a = 2r$, then $a' = 2r - 3$ so we have to prove that this case does not happen. In this case we obtain $b_1 = 2r - 2b$ for $b = b_2 = \cdots = b_5$ and $D$ has degree $6r - (2r - 2b) - 4b - b_6$ which must be equal to $3r$ so $r = 2b + b_6$. But $d - a + b_1 \leq 2r$, which holds only if $r \leq 2b$. Therefore $b_6 = 0$, $r = 2b$ and $D = (4b; 2b, b^4, 0) = bD_0$, contrary to hypothesis.

Finally, if $D' = mD_0$ with $m \geq 2$, then $D = (4m + 3; 2m + 2, (m + 1)^4, 0)$ which cannot hold as $D$ is in standard form.

**Case 4**: $a = b_1 + b_2 + b_3$, $b_1 > b_2 = b_3 = b_4 = b_5 = b_6 = b$. Notice that in this case we have $b \neq 0$ because a divisor of type $(b_1; b_1, 0^5)$ has degree $2b_1 = 3r$ and $d - a + b_1 = 2b_1$ which is not $\leq 2r$.

As in case 3 we take $T = T_C$, thus $D.T_C = a$ as before. Then $D' = D - T_C = (a - 3; b_1 - 2, b - 1, b - 1, b - 1, b - 1, b)$. Now $b_6$ may move up, even to the first place, but we still have $b_1 - 2, b - 1$ among the first three, so $a - 3 \geq$ the sum of the first three positions, so $D'$ is in standard form. Since $b_6 > 0$, $D'.L \geq 0$ for all $L$. Also $D'.L \leq 2r - 2$ as in case 3.

If $D$ satisfies $a > 2r$ then $a' = a - 3$ is $\geq 2r - 2$. The case $a = 2r$ cannot occur. Indeed, it implies $b_1 = 2r - 2b$ and degree of $D$ equal to $4r - 3b$ so $r = 3b$. But then $D$ does not satisfy the hypothesis $d - a + b_1 \leq 2r$ because $d - a + b_1 = 7b$.

Finally note that in this case $D$ and $D'$ are never multiples of $D_0$ because $b_6 = b'_6 > 0$.

**Case 5**: $a = 3b$ and all $b_i$ are equal to $b \geq 0$. In this case deg $D = 3 \cdot 3b - 6b = 3b$ so $r = b$ and $D = rH$ where $H = (3; 1^b)$ is the hyperplane class.

We take $T = T_E$ so $D.T_E = 5 \cdot 3b - 2 \cdot 6 \cdot b = 3b = a$. Then $D' = D - T_E = (3b - 5; (b - 2)^6)$. This is in standard form and it still satisfies $D'.L \geq 0$ for all $L$. Also $\max \{D'.L\} = d' - a' + b'_1 = b \leq 2r$. Note that $D$ and $D'$ are never multiples of $D_0$ in this case. Moreover, assuming $r \geq 3$, as $a = 3r$ we have $a \geq 2r + 3$ and so $a' = a - 5 \geq 2r - 2$.

**Theorem 3.9.** Let $D$ be a divisor on $X$ and let $r \geq 1$ be an integer. Then the following are equivalent:

(i) $D$ is linearly equivalent to a sum of twisted cubic curves $\sum_{i=1}^{r} T_i$. 
(ii) There exists an Ulrich bundle of rank r with first Chern class equal to D.

Moreover if $r \geq 2$ these two conditions are equivalent to

(iii) $\deg D = 3r$ and $0 \leq D.L \leq 2r$ for all lines $L$ in $X$.

Proof. For $r = 1$ we refer to Example 3.5. For $r \geq 2$ we will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii.) Take the bundle $\mathcal{E} = \bigoplus_i \mathcal{O}(T_i)$.

(ii) $\Rightarrow$ (iii). Follows from Proposition 3.4 a) and c).

(iii) $\Rightarrow$ (i). We proceed by induction on $r$. If $r = 2$, then $D$ has degree 6 and we just need to look at the table in Example 3.6 to note that any divisor $D$ satisfying $0 \leq D.L \leq 4$ is linearly equivalent to a sum of two twisted cubic curves (use $\max\{D.L\} = 6 - a + b_1$). If $r > 2$, by Proposition 3.8 (a) we choose a twisted cubic curve $T$ so that $D' = D - T$ satisfies the same condition (iii). Therefore $D'$ is itself linearly equivalent to a sum of $r - 1$ twisted cubic curves, so $D = T + D'$ is a sum of $r$ twisted cubic curves. □

Remark 3.10. a) In particular, this theorem implies the existence of orientable Ulrich bundles of rank $r \geq 2$ on the cubic surface, that is, Ulrich bundles with $c_1 = rH$. Indeed, $rH$ satisfies condition (iii) of Theorem 3.9. Note that in this proof we did not need the minimal resolution conjecture for points on $X$ [8, 4.3], which was an essential ingredient of the existence proof in our earlier paper [8].

b) Actually, the existence of orientable Ulrich bundles of rank $r$ on the cubic surface $X$ is equivalent to the existence of sets of $\frac{1}{2}(3r^2 - r)$ points on $X$ satisfying the Minimal Resolution Conjecture on $X$.

Indeed, it was proven in [8, 4.4(b)] that if $Z$ is a set of $\frac{1}{2}(3r^2 - r)$ points on $X$ satisfying the Minimal Resolution Conjecture, then there exists an extension

\begin{equation}
0 \longrightarrow \mathcal{O}^{-1}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_Z(r) \longrightarrow 0
\end{equation}

where $\mathcal{E}$ is an orientable Ulrich bundle of rank $r$.

Conversely, given an orientable Ulrich bundle $\mathcal{E}$ of rank $r$, by Proposition 3.1 (a), there is an exact sequence like (2). As $\mathcal{E}$ is generated by its global sections, $\mathcal{J}_Z(r)$ is also generated by global sections and $h^0(\mathcal{J}_Z(r)) = 2r + 1$. On the other hand, $\mathcal{E}$ has the following minimal free resolution in $\mathbb{P}^3$ (see [8, 3.7(b)])

\begin{equation}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{3r} \longrightarrow \mathcal{O}^{3r}_{\mathbb{P}^3} \longrightarrow \mathcal{E} \longrightarrow 0.
\end{equation}
Using the mapping cone procedure with the free resolutions of $\mathcal{O}_X$ and $\mathcal{E}$ in sequence (2) we get the following free resolution of $J_Z(r)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{r-1} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{3r} \oplus \mathcal{O}_{\mathbb{P}^3}^{3r-1} \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{3r} \longrightarrow J_Z(r) \longrightarrow 0.$$

The terms $\mathcal{O}_{\mathbb{P}^3}^{r-1}$ can be split off because $J_Z(r)$ is generated by global sections and $h^0(J_Z(r)) = 2r + 1$. Therefore $J_Z$ has the minimal free resolution predicted by the Minimal Resolution Conjecture.

c) Using this, we obtain a new proof for the Minimal resolution conjecture for general sets of $\frac{1}{2}(3r^2 - r)$ points on $X$, which was firstly proven in [7].

4. Stable Ulrich bundles on the cubic surface

In this section we prove our main theorem, characterizing those divisors $D$ on the cubic surface $X$ associated to a stable Ulrich bundle of rank $r$. From the previous section we already know the existence of some Ulrich bundles, those of rank 1 being trivially stable. To show the existence of stable bundles of higher ranks, we proceed inductively. Using extensions of stable bundles of lower rank we can construct simple Ulrich bundles. These belong to a modular family whose dimension we can compute. Then we show that the possible non-stable simple bundles form families of lower dimension, so that the general bundle of the modular family must be stable.

We start with rank 2, which is elementary (see Example 4.1 below). For rank $r \geq 3$ we use the inductive procedures of the previous section, expressing the divisor $D$ as $D' + T$ for a suitable twisted cubic curve $T$, and using induction on $D'$. The computation of dimensions is achieved using the computation of Euler characteristics done earlier (Corollary 2.13). We also give a second proof, modeled on the ideas of the original proof in our earlier paper [8, 5.7], that does not use modular families.

As an example, we list all the divisors that correspond to rank 3 Ulrich bundles, showing which ones are stable and the dimensions of the families. A curious byproduct of our investigation is that when there are no stable Ulrich bundles, it is not because there are too few semistable bundles: rather there are too many! In these cases there are “oversize” families of polystable bundles (meaning direct sums of two or more stable bundles) of dimension bigger than the expected dimension of the moduli of stable bundles.

Example 4.1. Rank 2 Ulrich bundles on the cubic surface (cont.). Here we recover the results of Faenzi [12] on stable Ulrich bundles of rank 2. We have seen in Example 3.6 that the only possibilities for stable Ulrich bundles of rank
2 are cases d, e, f. To show the existence of stable bundles in these cases, we consider extensions of line bundles corresponding to twisted cubic curves

$$0 \longrightarrow \mathcal{O}(T_1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(T_2) \longrightarrow 0$$

where \((T_1, T_2) = (T_A, T_C), (T_B, T_{B'}), (T_A, T_E)\) for cases d, e, f respectively. Since \(\mathcal{O}(T_1) \not\cong \mathcal{O}(T_2), h^0(\mathcal{O}(T_1 - T_2)) = 0, h^2(\mathcal{O}(T_1 - T_2)) \perp h^0(\mathcal{O}(T_2 - T_1 - H)) = 0,\) so \(h^1(\mathcal{O}(T_1 - T_2)) = \dim \text{Ext}^1(\mathcal{O}(T_2), \mathcal{O}(T_1)) = -\chi(\mathcal{O}(T_1) \otimes \mathcal{O}(T_2)^\vee),\) which is equal to \(2 \cdot 1 \cdot 1 - T_1, T_2\) according to Corollary 2.13. Now \(T_A, T_C = 3, T_B, T_{B'} = 4, T_A, T_E = 5,\) so \(\dim \text{Ext}^1(\mathcal{O}(T_2), \mathcal{O}(T_1)) = 1, 2, 3\) respectively. Therefore there exist non-split extensions, and these bundles are necessarily Ulrich, being extensions of Ulrich bundles. Furthermore they are simple because of Lemma 4.2 below.

Thus in cases d, e, f there are simple bundles, and we can look at a modular family of simple bundles. Its dimension is computed as \(\dim \text{Ext}^1(\mathcal{E}, \mathcal{E}) = -\chi(\mathcal{E} \otimes \mathcal{E}) + 1 = D^2 - 7,\) which gives dimension 1, 3, 5 respectively in these cases. On the other hand, the dimension of the family of simple extensions of \(\mathcal{O}(T_2)\) by \(\mathcal{O}(T_1)\) constructed above is \(\dim \text{Ext}^1(\mathcal{O}(T_2), \mathcal{O}(T_1)) - 1\) which we found above to be 0, 1, 2 in these cases. Since any non-stable simple Ulrich bundle \(\mathcal{E}\) must be an extension of rank 1 Ulrich bundles, it will be an extension as above for some \(T_1\) and \(T_2.\) Now \(c_1(\mathcal{E}) = D = T_1 + T_2,\) and \(D^2 = T_1^2 + 2T_1T_2 + T_2^2 = 8, 10, 12\) in cases d, e, f, respectively, so \(T_1, T_2 = 3, 4, 5\) respectively, and from the previous dimension count we conclude that the remaining bundles in the larger modular families must be stable. This explains the last two columns of Example 3.6.

**Lemma 4.2.** On a nonsingular projective variety \(X,\) let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

be a non-split extension of non-isomorphic \(\mu\)-stable vector bundles \(\mathcal{F}, \mathcal{G}\) of the same slope. Then \(\mathcal{E}\) is a simple vector bundle.

**Proof.** If \(\mathcal{E}\) is not simple, then there is a map \(\mathcal{E} \xrightarrow{\alpha} \mathcal{E}\) of rank less than \(\text{rk}\mathcal{E}.\) If \(\alpha(\mathcal{F}) = 0,\) then \(\alpha\) factors through \(\mathcal{G},\) and the map \(\mathcal{E} \longrightarrow \mathcal{G}\) splits. Therefore \(\alpha(\mathcal{F}) \neq 0.\) The composed map \(\mathcal{F} \longrightarrow \mathcal{E} \xrightarrow{-\alpha} \mathcal{E} \longrightarrow \mathcal{G}\) must be zero since \(\mathcal{F}, \mathcal{G}\) are stable, not isomorphic, and of the same slope. Therefore \(\alpha(\mathcal{F}) \subset \mathcal{F}\) and so \(\alpha|_{\mathcal{F}}\) must be an isomorphism, since \(\mathcal{F}\) is stable. Therefore \(\alpha\) induces a map from \(\mathcal{G}\) to \(\mathcal{G}.\) If this is 0 then \(\alpha\) maps \(\mathcal{E}\) to \(\mathcal{F}\) and the sequence splits again. Hence the map on \(\mathcal{G}\) is an isomorphism and so \(\alpha\) is an isomorphism, contradicting \(\text{rk}\alpha < \text{rk}\mathcal{E}.\) \(\square\)

**Theorem 4.3.** Let \(D\) satisfy the equivalent conditions of Theorem 3.7 with \(r \geq 2.\) Then the following are equivalent.

(i) \(D.T \geq 2r\) for all twisted cubic curves \(T,\) but \(D \neq mD_0\) for any \(m \geq 2,\) where \(D_0 = (4; 2, 1^4, 0).\)
(ii) There exist stable Ulrich bundles corresponding to $D$.

Moreover, if $D$ satisfies these conditions, then the moduli space $M_X^r(c_1, c_2)$ of stable vector bundles of rank $r$ on $X$ with $c_1 = D$ and $c_2 = \frac{1}{2}(D^2 - r)$ is smooth and irreducible of dimension $D^2 - 2r^2 + 1$ and consists entirely of stable Ulrich bundles.

Proof. (i) $\Rightarrow$ (ii). By induction on $r$. For $r = 2$, this follows by inspection of the table of degree 6 curves in Example 3.6.

For $r \geq 3$, we use Proposition 3.8 to choose a twisted cubic curve $T$ such that $D' = D - T$ satisfies the same conditions as $D$, namely $a' = \min\{D'.T\} \geq 2r - 2$, and $D' \neq m'D_0$ for any $m' \geq 2$. Then by induction, there exist stable Ulrich bundles $F$ corresponding to $D'$. We consider extensions

$$0 \to \mathcal{O}(T) \to \mathcal{E} \to F \to 0.$$ 

First we use Corollary 2.13 to compute $\chi(F^\vee(T)) = 2(r - 1) - D'.T$. Since $D'.T = a - 1$ (by Proposition 3.8 we have $D.T = a$, so $D'.T = a - T^2 = a - 1$) and $a \geq 2r$, this is strictly negative, and so there are non-split extensions in $\text{Ext}^1(F, \mathcal{O}(T))$. These are simple by Lemma 1.2. Moreover $H^0(F^\vee(T)) = 0$ and $H^2(F^\vee(T)) = 0$ because $F$ is stable, and thus the dimension of $\text{Ext}^1(F, \mathcal{O}(T))$ is equal to $a + 1 - 2r > 0$.

Consider the modular family of simple bundles $\mathcal{E}$ (see Remark 2.11). We can compute its dimension as $\dim H^1(\mathcal{E} \otimes \mathcal{E}^\vee) = D^2 - 2r^2 + 1$ (Corollary 2.14). If the general member of this modular family is not stable, then it must be of the same type as $\mathcal{E}$ above, namely an extension of a stable rank $r - 1$ bundle by a stable line bundle $\mathcal{O}(T)$, because no other semistable splitting type can specialize to this one (see [20, 2.3.1]).

On the other hand, the dimension of the family of simple bundles obtained by this construction is

$$\dim\{\mathcal{F}\} + \dim\text{Ext}^1(F, \mathcal{O}(T)) - 1 = (D')^2 - 2(r - 1)^2 + 1 + a - 2r.$$

Since $D' = D - T$ and $D.T = a$ by Proposition 3.8, this number is

$$D^2 - 2a + 1 - 2r^2 + 4r - 2 + 1 + a - 2r = D^2 - 2r^2 - a + 2r.$$

Since $a \geq 2r$, this number is strictly less than $D^2 - 2r^2 + 1$. Thus the general bundle in the modular family of simple bundles must be stable.

(ii) $\Rightarrow$ (i). For this implication we will show that if $a < 2r$, or if $D = mD_0$ for some $m \geq 2$, then there exists a family of properly semistable polystable bundles of dimension $\geq D^2 - 2r^2 + 1$. In this case there cannot be stable bundles, because the moduli space $M^{ss,U}$ is irreducible (see Corollary 3.2), and the stable
bundles, if they exist, fix its dimension as \( D^2 - 2r^2 + 1 \). In that case the polystable bundles would have dimension \( \leq D^2 - 2r^2 \) in the moduli space.

We assume \( a < 2r \), or \( D = mD_0 \) for some \( m \geq 2 \) and proceed again by induction on \( r \). For \( r = 2 \), looking at the table of rank 2 bundles in Example 3.6, we have \( D = 2T_A \) or \( D = T_A + T_B \). These give polystable bundles, forming families of dimension 0, while \( D^2 - 2r^2 + 1 \) is \( D^2 - 8 + 1 \) which is either -3 or -1.

For \( r \geq 3 \) and \( a < 2r \), we choose \( T \) as in Proposition 3.3 and let \( D' = D - T \). If \( a' \geq 2r - 2 \) then by the above implication (i) \( \Rightarrow \) (ii) there is a family of Ulrich stable bundles \( \mathcal{F} \) of rank \( r - 1 \) and dimension \( D^2 - 2(r - 1)^2 + 1 \). We take \( \mathcal{O}(T) \oplus \mathcal{F} \) as polystable bundles and thus we obtain a family of polystable bundles of this same dimension which is \( D^2 - 2r^2 + 4r - 2a \). On the other hand, if \( a' < 2r - 2 \) or if \( D' = m'D_0 \) for some \( m' \geq 2 \), then by induction, there exists a family of polystable bundles for \( D' \) of dimension at least of this same dimension \( D^2 - 2r^2 + 4r - 2a \). Taking the direct sum with \( \mathcal{O}(T) \) gives polystable bundles for \( D \) of at least the same dimension. Now since \( a < 2r \), \( a \leq 2r - 1 \), and this family has dimension \( \geq D^2 - 2r^2 + 2 \).

Finally, consider the case \( D = mD_0 \) with \( m \geq 2 \). Since \( D_0 \) corresponds to a 1-dimensional family of rank 2 bundles (see case \( d \) in the table in Example 3.6), by taking direct sums of \( m \) of them we obtain an \( m \)-dimensional family of polystable bundles. On the other hand, since \( D_0^2 = 8 \) and \( r = 2m \), in this case \( D^2 - 2r^2 + 1 = 8m^2 - 8m^2 + 1 = 1 \), so again there are no stable bundles.

For the last statement, we recall that the Ulrich bundles form an irreducible family (see Corollary 3.2) and the corresponding moduli space is smooth if dimension \( D^2 - 2r^2 + 1 \) (see Corollary 2.11). Thus we have only to show that any other stable bundle of the same rank with the same Chern classes is an Ulrich bundle. Since the Hilbert polynomial depends only on the Chern classes, this will be a consequence of the following lemma, using Lemma 2.6. \( \square \)

**Lemma 4.4.** On a nonsingular cubic surface \( X \), any stable vector bundle \( \mathcal{E} \) of rank \( r \) with Hilbert polynomial \( P_{\mathcal{E}}(n) = 3r(\binom{n+2}{2}) \) is an Ulrich bundle, and therefore also \( \mu \)-stable.

**Proof.** Since \( \frac{1}{r}P_{\mathcal{E}(-1)}(n) = 3\binom{n+1}{2} \) and \( P_{\mathcal{O}_X}(n) = 3\binom{n+1}{2} + 1 \), it follows from the stability of \( \mathcal{E} \) that \( h^0(\mathcal{E}(-1)) = 0 \). Hence \( h^0(\mathcal{E}(n)) = 0 \) for all \( n < 0 \).

Next, we note that \( \chi(\mathcal{E}^{\vee}(n+2)) = \chi(\mathcal{E}(-n-3)) = 3r(-n-1)^2 = 3r^{n+2} \), so \( \mathcal{E}^{\vee}(2) \) is also stable satisfying the same hypothesis as \( \mathcal{E} \). It follows that \( h^0(\mathcal{E}^{\vee}(n)) = 0 \) for all \( n \leq -1 \). By duality, this shows \( h^2(\mathcal{E}(n)) = 0 \) for all \( n \geq -2 \).
Now $\chi(\mathcal{E}(n)) = 0$ for $n = -1, -2$, so it follows that $h^1(\mathcal{E}(n)) = 0$ for $n = -1, -2$. Then by Castelnuovo-Mumford regularity, $\mathcal{E}$ is regular, so $h^1(\mathcal{E}(n)) = 0$ for all $n \geq -2$. The same argument applies to $\mathcal{E}^\vee(2)$ so $h^1(\mathcal{E}^\vee(n)) = 0$ for all $n \geq 0$. By duality this implies $h^1(\mathcal{E}(n)) = 0$ for all $n \leq -1$. Thus $h^1(\mathcal{E}(n)) = 0$ for all $n$, and $\mathcal{E}$ is an ACM sheaf.

Finally, since $h^i(\mathcal{E}) = 0$ for $i = 1, 2$ and $\chi(\mathcal{E}) = 3r$, we find $h^0(\mathcal{E}) = 3r$, so $\mathcal{E}$ is Ulrich by Lemma 2.2, hence also $\mu$-stable by Theorem 2.9.

We thus obtain a new proof for the following statement of our previous paper [8, 5.7]:

**Corollary 4.5.** There exist stable orientable Ulrich bundles of every rank $r \geq 2$ on the cubic surface $X$ and their moduli space is smooth and irreducible of dimension $r^2 + 1$.

**Proof.** Here orientable means $D = rH$, and in this case $D.L = r$ for all $L$, $D.T = 3r$ for all $T$, and $D \neq mD_0$ for any $m$, so Theorem 4.3 applies. □

**Remark 4.6.** We give here another proof of Theorem 4.3 (i) $\Rightarrow$ (ii), without using modular families or counting dimension of families. This proof is an adaptation of the proof of Theorem 5.3 given in [8] and thus validates the idea of that proof if not all of its details.

As in the previous proof, we use induction on $r$, the case $r = 1$ being trivial (the only Ulrich bundles are $O(T)$, $T$ a twisted cubic, and these are stable). For $r = 2$ we use Example 4.1 or [12]. For $r \geq 3$, given a divisor $D$ satisfying the conditions of Theorem 4.3(i), we choose a $T$ such that $D' = D - T$ satisfies the same conditions by 3.8(b), so by induction there exists a stable bundle $\mathcal{F}$ of rank $r - 1$ corresponding to $D'$. Then as in the proof of Theorem 4.3 we consider extensions

$$0 \rightarrow O(T) \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$ 

Since there is an irreducible family containing all Ulrich bundles corresponding to $r$ and $D$, if there are no stable ones, then the general one would have to be an extension of some stable bundle $\mathcal{F}$ of rank $r - 1$ by $O(T)$: for no other splitting type could specialize to this one.

By Proposition 3.11, any one of these Ulrich bundles can be represented as an extension

$$0 \rightarrow O_X^{-1} \rightarrow \mathcal{E} \rightarrow \mathcal{J}_Z(D) \rightarrow 0,$$

for some subscheme $Z$ with $h^0(\mathcal{J}_Z(D - H)) = 0$. This last condition is an open condition on $Z$, so there will exist extensions $\mathcal{E}$ corresponding to a general set of points $Z$, and these will also be Ulrich, since the Ulrich condition is open (see...
section 2 above). For such an \( \mathcal{E} \) we will show there are no non-zero morphisms from \( \mathcal{O}(T) \) to \( \mathcal{E} \). We want to compute \( \text{Hom}(\mathcal{O}(T), \mathcal{E}) = H^0(\mathcal{E}(-T)) \). So we consider the sequence

\[
0 \longrightarrow \mathcal{O}(-T)^{r-1} \longrightarrow \mathcal{E}(-T) \longrightarrow \mathcal{J}_Z(D-T) \longrightarrow 0.
\]

Now \( h^0(\mathcal{O}(-T)) = 0 \). So to show \( h^0(\mathcal{E}(-T)) = 0 \) it is sufficient to show \( h^0(\mathcal{J}_Z(D-T)) = 0 \). We compute both numbers. From Proposition 3.4 we find \( h^0(\mathcal{O}(D)) = \frac{1}{2}(D^2 - 2r) \). On the other hand, since \( D' = D - T \) can be represented by an irreducible nonsingular curve (see Proposition 3.4), we have \( h^i(\mathcal{O}(D')) = 0 \) for \( i = 1, 2 \), and so \( h^0(\mathcal{O}(D')) \) can be computed by Riemann-Roch:

\[
h^0(\mathcal{O}(D')) = \frac{1}{2}(D' + H)D' + 1 = \frac{1}{2}(D^2 - 2D.T + T^2 + 3(r - 1)) + 1 = \frac{1}{2}(D^2 - 2D.T + 3r).
\]

Our hypothesis on \( D \) says \( D.T \geq 2r \) for all twisted cubic curves \( T \), and so \( h^0(\mathcal{O}(D')) \leq \frac{1}{2}(D^2 - r) = n \). Hence \( n \) general points will make \( h^0(\mathcal{J}_Z(D-T)) = 0 \) so \( h^0(\mathcal{E}(-T)) = 0 \), and the general \( \mathcal{E} \) must be stable.

Example 4.7. Rank 3 Ulrich bundles on the cubic surface. To illustrate our main theorem, we classify Ulrich bundles of rank 3. To do this, in the following table we list all divisor classes representing an irreducible nonsingular curve of degree \( 3r = 9 \) (see Proposition 3.4), in standard form and then retain only those that satisfy \( 0 \leq D.L \leq 2r = 6 \) for all lines \( L \). These are exactly the divisors \( D \) that can correspond to Ulrich bundles.

| \( D \) | \( D^2 \) | \( \sum T_i \) | \( D^2 - 2r^2 + 1 \) | comments |
| --- | --- | --- | --- | --- |
| (3;0,0,0,0,0,0) | 9 | \( 3A \) | -8 | \( \oplus \mathcal{O}(T_i) \) |
| (4;1,1,0,0,0,0) | 13 | \( 2A + B \) | -4 | \( \oplus \mathcal{O}(T_i) \) |
| (5;1,1,1,1,1,1) | 19 | \( A + B + B' \) | 2 | 3-dim. polystable |
| (5;2,1,1,1,1,0) | 17 | \( 2A + C \) | 0 | 1-dim. polystable |
| (5;2,2,1,1,0,0) | 15 | \( A + B + B'' \) | -2 | \( \oplus \mathcal{O}(T_i) \) |
| (6;2,2,2,1,1,1) | 21 | \( 2B + B' \) | 4 | \( \equiv \) stable |
| (6;2,2,2,2,1,0) | 19 | \( A + B' + C \) | 2 | \( \equiv \) stable |
| (6;3,2,1,1,1,1) | 19 | \( A + B' + C \) | 2 | \( \equiv \) stable |
| (7;2,2,2,2,2) | 25 | \( 2A + E \) | 8 | \( \equiv \) stable |
| (7;3,2,2,2,2,1) | 23 | \( A + B + D \) | 6 | \( \equiv \) stable |
| (9;3,3,3,3,3,3) | 27 | \( B + B' + E(= 3H) \) | 10 | \( \equiv \) stable |
Note that by our main Theorem 4.3 there exist stable bundles in the last six cases, and the corresponding moduli spaces have dimension $D^2 - 2r^2 + 1$. In the third and fourth row of the table if there were stable bundles, the moduli space would have dimension 2, 0 respectively. In these cases there are no stable bundles, but we have “oversize” families of polystable bundles of dimensions 3, 1 respectively. In each case they are direct sums of a rank 1 bundle corresponding to a twisted cubic curve and a stable rank 2 bundle corresponding to $D = (4; 1^6)$. In these cases there are no stable bundles. In the third and fourth row of the table if there were stable bundles, the moduli spaces have dimension 2, 0 respectively. In these cases there are no stable bundles, but we have “oversize” families of polystable bundles of dimensions 3, 1 respectively. In each case they are direct sums of a rank 1 bundle corresponding to a twisted cubic curve and a stable rank 2 bundle corresponding to $D' = (4; 1^6)$ and $(4; 2, 1^4, 0)$ respectively. In particular, the necessary condition $D^2 - 2r^2 + 1 \geq 0$ is not sufficient for the existence of stable bundles.

Remark 4.8. In proving the nonexistence of stable bundles in case $a < 2r$, we constructed oversize families of polystable bundles. One can see from the proof, inductively, that the polystable bundles we construct there are all direct sums of stable bundles of ranks 1 or 2. One might ask if there are other types of polystable bundles that are not specializations of stable bundles. That this does not happen is a consequence of the following corollary.

Corollary 4.9. Let $\mathcal{F}$ and $\mathcal{G}$ be stable Ulrich bundles on the cubic surface $X$, of ranks $s, t \geq 2$. Then the polystable bundle $\mathcal{F} \oplus \mathcal{G}$ is a specialization of a stable bundle unless $s = t = 2$ and both $\mathcal{F}$ and $\mathcal{G}$ are associated to the same divisor class $D_0 = (4; 2, 1^4, 0)$.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are both stable, corresponding to divisors $C, D$, say, by Theorem 4.3 we know that $C.T \geq 2s$ and $D.T \geq 2t$ for all twisted cubic curves $T$. Therefore $(C + D).T \geq 2(s + t)$ for all $T$, and we conclude by Theorem 4.3 that $C + D$ corresponds to a family of stable bundles that will specialize to $\mathcal{F} \oplus \mathcal{G}$, unless $C + D = mD_0$ for some $m \geq 2$.

In this case, we write $D_0$ in standard form as $(4; 2, 1^4, 0)$, so that $mD_0$ is $(4m; 2m, m^4, 0)$, and the rank $r = s + t$ of the corresponding Ulrich bundle is $2m$. Having chosen a basis for Pic $X$ so that $D_0$ is in standard form, we cannot assume that $C, D$ are in standard form. So let $C = (a; b_1, \ldots, b_6)$, $D = (a'; b'_1, \ldots, b'_6)$. Then $a + a' = 4m$, $b_1 + b'_1 = 2m$, $b_i + b'_i = m$ for $i = 2, 3, 4, 5$, and $b_6 = b'_6 = 0$ since both are $\geq 0$ (because $C.L \geq 0$, $D.L \geq 0$ by Theorem 3.9) and their sum is 0.

Next, since $C.T \geq 2s$ for all $T$, taking $T = T_A = (1; 0^6)$, we find that $a \geq 2s$. Similarly, $a' \geq 2t$. But $a + a' = 4m = 2r$, so we must have equality in both cases. On the other hand, taking $T = T_B = (2; 1^3, 0^3)$ or one of its permutations, we find that $2a - b_i - b_j - b_k \geq 2s$ for any three $i, j, k \in \{1, \ldots, 6\}$. Since $a = 2s$, we find $a \geq b_i + b_j + b_k$. This shows that $C$ is in standard form except for the ordering of the $b_i$, which we do not know yet.
Consider the line \( L = G_1 \) (notation of [15, V.49]). We know \( C.L \leq 2s \) and \( D.L \leq 2t \) by Theorem 3.9 (iii). On the other hand \( mD_0.L = 4m = 2r \). Therefore again we have equality in both cases. Thus \( C.L = 2s \) and \( D.L = 2t \). Therefore we have equality in both cases. Thus \( C.L = 2s \) and \( D.L = 2t \).

Finally, since we have assumed \( C, D \) correspond to stable bundles, by Theorem 4.3 we must have \( s = t = 2 \) and \( C = D = D_0 \).

5. Stable Ulrich bundles on the cubic threefold

In this section we construct stable Ulrich bundles of all ranks \( r \geq 2 \) on a general cubic threefold \( Y \) in \( \mathbb{P}^4 \), and we show that the corresponding moduli space is smooth of the expected dimension \( r^2 + 1 \). Stable bundles of rank 2 are well known [4], [5], [1], [25]. We construct stable rank 3 Ulrich bundles using curves whose existence is proven by a Macaulay2 computation due to Geiβ and Schreyer (see Appendix). Then we use a method analogous to the case of surfaces, creating simple bundles as extensions of stable bundles of lower rank, and counting dimensions to show that the general bundles in a modular family must be stable. Finally, for each \( r \), we show that there is at least one component of the moduli space of stable Ulrich bundles of rank \( r \) on \( Y \) for which the restriction map to the moduli of stable rank \( r \) bundles on the hyperplane section, a cubic surface, is generically étale and dominant.

For completeness, we include a proof of existence of stable rank 2 Ulrich bundles on \( Y \).

**Proposition 5.1.** On a nonsingular cubic threefold \( Y \), there exist stable rank 2 Ulrich bundles on \( Y \) with first Chern class \( c_1 = 2H \), where \( H \) is the hyperplane class, and \( c_2 = 5 \). The moduli space of these bundles is smooth of dimension 5.

**Proof.** If \( E \) is such a bundle, a general section will vanish along a quintic elliptic curve \( C \) in \( Y \), not contained in any hyperplane. Thus, to construct such bundles using the Serre correspondence, we need to show the existence in \( Y \) of a nonsingular elliptic curve \( C \) of degree 5 that is not contained in any hyperplane section. Let \( H \) be a general hyperplane section of \( Y \). This is a nonsingular cubic surface, and on it we can find a quintic elliptic curve \( C_0 \), for example in the divisor class \((3, 1^4, 0^2)\). This curve has self-intersection \( C_0^2 = 5 \). There is an exact sequence for
the normal bundle of $C_0$ in $Y$

$$0 \longrightarrow \mathcal{N}_{C_0/H} \longrightarrow \mathcal{N}_{C_0/Y} \longrightarrow \mathcal{N}_{H/Y}|_{C_0} \longrightarrow 0.$$ 

The first of these is $\mathcal{O}_{C_0}(0)$, the third is $\mathcal{O}_{C_0}(1)$. Both have degree 5, $h^0 = 5$ and $h^1 = 0$. Hence $h^0(\mathcal{N}_{C_0/Y}) = 10$ and $h^1(\mathcal{N}_{C_0/Y}) = 0$. Thus the Hilbert scheme of quintic elliptic curves in $Y$ is smooth of dimension 10 at the point corresponding to $C_0$.

Let us count how many quintic elliptic curves there are contained in hyperplane sections $H$ of $Y$. The choice of $H$ is four parameters, and the dimension of the linear system $|C_0|$ on $H$ is five, so there is a 9-dimensional family of quintic elliptic curves in hyperplanes of $Y$. We conclude that a general quintic elliptic curve on $Y$ is not contained in any hyperplane section. Let $C$ be one of these.

We apply the Serre construction \[16, 1.1\] to obtain a bundle of rank 2 as an extension

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_C(2) \longrightarrow 0.$$ 

The extension is determined by an element of $\text{Ext}^1(\mathcal{J}_C(2), \mathcal{O}_Y) \cong \text{Ext}^2(\mathcal{O}_C(2), \mathcal{O}_Y) \cong H^0(\omega_C)$. Since $C$ is an elliptic curve, $\omega_C \cong \mathcal{O}_C$ and there is just one choice of section that is nowhere vanishing. Hence $\mathcal{E}$ is locally free of rank 2. Since $C$ is an ACM curve in $Y$ (cf. \[17, 3.4\]), it follows that $\mathcal{E}$ is an ACM bundle on $Y$. Indeed, $C$ being ACM means $H^1(\mathcal{J}_C) = 0$, and so $H^1(\mathcal{E}) = 0$. Since $\mathcal{E}$ has rank 2, its dual $\mathcal{E}^\vee$ is isomorphic to $\mathcal{E}(-c_1)$, and then by duality $H^2(\mathcal{E})$, dual to $H^2(\mathcal{E}^\vee(-1))$, is also zero. Since $C$ is not contained in a hyperplane, $H^0(\mathcal{E}(-1)) = 0$. We see that $h^0(\mathcal{E}) = 6$, so $\mathcal{E}$ is an Ulrich bundle. Since there are no Ulrich bundles of rank 1 on $Y$, by Theorem \[2.9\] it cannot be properly semistable, so it is stable. To show that the moduli space is smooth of dimension 5, we just compute $H^1(\mathcal{E} \otimes \mathcal{E}^\vee) = 5$ and $H^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 0$. This is elementary (left to the reader). \[\square\]

**Remark 5.2.** In fact, Beauville shows that this moduli space is isomorphic to an open subset of the intermediate Jacobian of the cubic threefold \[5, 5.2\] and therefore is irreducible, but we will not make use of this fact.

**Definition 5.3.** For the rest of this section, a general cubic threefold in $\mathbb{P}^4$ will denote a cubic hypersurface in a suitable Zariski open subset of the Hilbert scheme of cubic hypersurfaces in $\mathbb{P}^4$ over an algebraically closed field either of characteristic zero, or of prime characteristic $p$, except possibly for finitely many values of $p$.

**Proposition 5.4.** On a general cubic threefold $Y$ in $\mathbb{P}^4$, there exist stable Ulrich bundles of rank 3.

**Proof.** In the Appendix it is shown that $Y$ contains smooth ACM curves $C$ of degree 12 and genus 10 with the additional property that $\omega_C(-1)$ has two global...
sections that generate the graded module $H^0_*(\omega_C)$ over the homogeneous coordinate ring $S$ of $\mathbb{P}^4$.

We use the Serre construction to create an exact sequence

$$0 \rightarrow O^2_Y \rightarrow \mathcal{E} \rightarrow J_{C(3)} \rightarrow 0$$

by using two generators of $H^0(\omega_C(-1)) \cong \text{Ext}^1(J_{C(3)}, O_Y)$ to define the extension. Since $C$ is an ACM curve, it follows that $H^1_*\mathcal{E} = 0$. The dual sequence is

$$0 \rightarrow O_Y(-3) \rightarrow \mathcal{E}^\vee \rightarrow O^2_Y \rightarrow \omega_C(-1) \rightarrow 0,$$

and now the fact that the two sections of $\omega_C(-1)$ generate the module $H^0_*\mathcal{E}$ shows that $H^1_*\mathcal{E}^\vee \cong H^2_*\mathcal{E}(3)$. Thus $\mathcal{E}$ is an ACM bundle. Again using the fact that $C$ is an ACM curve, we compute $h^0(O_C(2)) = 15$, so $h^0(J_C(2)) = 0$, and $h^0(\mathcal{E}(1)) = 0$; furthermore $h^0(\mathcal{O}_C(3)) = 27$, so $h^0(J_C(3)) = 0$ and $h^0(\mathcal{E}) = 9 = 3r$. Hence by Lemma 2.2, $\mathcal{E}$ is a rank 3 Ulrich bundle on $Y$. It is necessarily stable, because there are no rank 1 Ulrich bundles on $Y$. □

**Remark 5.5.** The existence of rank 3 Ulrich bundles on $Y$ was announced earlier [1, Example 4.4], but the proof given there is incorrect because the ACM curve of degree 12, genus 10 that they used, Gorenstein linked to a conic, does not satisfy the additional condition that the sections of $\omega_C(-1)$ should generate $H^0_*\mathcal{E}$.

**Proposition 5.6.** Let $\mathcal{E}$, $\mathcal{F}$, be Ulrich bundles on the cubic threefold $Y$. Then

a) $\chi(\mathcal{E} \otimes \mathcal{F}^\vee(-1)) = 0$

b) $\chi(\mathcal{E} \otimes \mathcal{F}^\vee) = \chi(\mathcal{E}_H \otimes \mathcal{F}_H^\vee) = 0$, where $H$ is a general hyperplane section

c) $H^i(\mathcal{E} \otimes \mathcal{F}^\vee) = 0$ for $i = 2, 3$.

**Proof.** a) By Serre duality, $\chi(\mathcal{E} \otimes \mathcal{F}^\vee(-1)) = -\chi(\mathcal{E}^\vee \otimes \mathcal{F}(-1))$. Let $\mathcal{E}' = \mathcal{E}^\vee(2)$ and $\mathcal{F}' = \mathcal{F}^\vee(2)$. These are Ulrich bundles, by Lemma 2.4 (iv). We can rewrite our equation as $\chi(\mathcal{E} \otimes \mathcal{F}'(-3)) = -\chi(\mathcal{E}' \otimes \mathcal{F}(-3))$. Note that $\chi$ depends only on the Chern classes of the bundles in question; the Chern classes of a tensor product are determined by those of the two factors; and the Chern classes of an Ulrich bundle are determined by its rank (Lemma 2.5). Since $\mathcal{E}$ and $\mathcal{E}'$ have the same rank, and $\mathcal{F}$ and $\mathcal{F}'$ have the same rank, $\mathcal{E} \otimes \mathcal{F}'$ and $\mathcal{E}' \otimes \mathcal{F}$ have the same Chern classes, so $\chi(\mathcal{E} \otimes \mathcal{F}'(-3)) = \chi(\mathcal{E}' \otimes \mathcal{F}(-3))$. Combining with the above, both must be zero.

b) This is a direct consequence of a), tensoring $\mathcal{E} \otimes \mathcal{F}^\vee$ with the exact sequence

$$0 \rightarrow O_Y(-1) \rightarrow O_Y \rightarrow O_H \rightarrow 0.$$

c) Since $\mathcal{E}$ is Ulrich, it has a linear resolution over $\mathbb{P}^4$ [5, 3.7],

$$0 \rightarrow O_{\mathbb{P}^4}(-1)^{3r} \rightarrow O_{\mathbb{P}^4}^{3r} \rightarrow \mathcal{E} \rightarrow 0.$$
Tensoring with $F^\vee$, we get a right exact sequence on $Y$,
\[ F^\vee(-1)^{3r} \longrightarrow F^{\vee3r} \longrightarrow E \otimes F^\vee \longrightarrow 0. \]
Now $F^\vee(2)$ is an Ulrich bundle by Lemma 2.4(iv), so $F^\vee$ and $F^\vee(-1)$ have no cohomology. It follows from cohomology sequences on $Y$, that $H^2(E \otimes F^\vee) = H^3(E \otimes F^\vee) = 0$.

Theorem 5.7. For any $r \geq 2$, the moduli space of stable rank $r$ Ulrich bundles on a general cubic threefold $Y$ in $\mathbb{P}^4$ is non-empty and smooth of dimension $r^2 + 1$.

Proof. For any Ulrich bundle $E$, we have $H^2(E \otimes E^\vee) = 0$ by Proposition 5.6(c), and this implies the smoothness of the moduli space. Furthermore, by the same Proposition, $\chi(E \otimes E^\vee) = \chi(E_H \otimes E_H^\vee) = -r^2$ by Corollary 2.13 since $c_1(E) = rH$. For $E$ stable or simple we have $h^0(E \otimes E^\vee) = 1$, and $h^2(E \otimes E^\vee) = h^3(E \otimes E^\vee) = 0$ by Proposition 5.6(c), so $h^1(E \otimes E^\vee) = r^2 + 1$ is the dimension of the moduli space.

It remains to show the existence. We proceed by induction on $r$, the cases $r = 2, 3$ having been done above. So let $r \geq 4$, and choose $E$ stable of rank 2, and $F$ stable of rank $r - 2$, different from $E$. Then $h^i(E \otimes F^\vee) = 0$ for $i = 0, 2, 3$, so $h^1(E \otimes F^\vee) = -\chi(E \otimes F^\vee) = -\chi(E_H \otimes F_H^\vee) = 2(r - 2)$ by Corollary 2.13. In particular, this number is positive, so there exist nonsplit extensions
\[ 0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0, \]
and the new bundle $G$ will be a simple Ulrich bundle of rank $r$ (see Lemma 4.2). We consider the modular family of these simple bundles, which will be smooth of dimension $r^2 + 1$ by the above observations.

If the general simple bundle in this family is not stable, it must have the same splitting type as the ones just constructed. However, the dimension of the family of extensions above is
\[
\dim\{E\} + \dim\{F\} + \dim(\text{Ext}^1(F, E)) - 1 \\
= 2^2 + 1 + (r - 2)^2 + 1 + 2(r - 2) - 1 \\
= r^2 - 2r + 5.
\]
Since $r \geq 4$, this number is strictly less than $r^2 + 1$. We conclude that the general simple bundle of rank $r$ is stable, so stable bundles exist. □

Our next goal is to study the restriction map from Ulrich bundles on $Y$ to bundles on a hyperplane section $H$. We will show in many cases that there is an open set of stable bundles on $Y$ that restricts by an étale dominant map to stable bundles on $H$.  

Proposition 5.8. Suppose that $E$ is a stable Ulrich bundle of rank $r$ on $Y$ with the property that $H^i(E \otimes E^\vee(-1)) = 0$ for all $i$ (in which case we say $E \otimes E^\vee(-1)$ has no cohomology). Then the restriction map from bundles on $Y$ to bundles on the general hyperplane section $H$ induces an étale dominant map from an open subset of a modular family of stable rank $r$ Ulrich bundles on $Y$ to a modular family of stable rank $r$ Ulrich bundles on $H$.

Proof. First we recall that the restriction of an Ulrich bundle $E$ on $Y$ to $H$ is also an Ulrich bundle $E_H$ (Lemma 2.4). The condition that $E \otimes E^\vee(-1)$ has no cohomology implies that $H^i(E \otimes E^\vee) \to H^i(E_H \otimes E_H^\vee)$ is an isomorphism for all $i$. If $E$ is stable, then $h^0(E \otimes E^\vee) = 1$ and $h^1(E \otimes E^\vee) = r^2 + 1$. Therefore the same is true for $E_H$, hence $E_H$ is simple. The condition that $E \otimes E^\vee(-1)$ has no cohomology is an open condition, so we obtain a morphism from an open subset of a modular family of stable bundles on $Y$ to a modular family of simple bundles on $H$. This map induces an isomorphism on the Zariski tangent spaces at the point $E$, hence is étale and dominant in a neighborhood of $E$. The modular family on $H$ contains a nonempty open subset of stable bundles (Corollary 4.5), and the inverse image of this open set gives an open set of stable bundles on $Y$ which restricts by an étale dominant map to stable bundles on $H$, as required. □

Lemma 5.9. Let $E$ be a rank $r$ Ulrich bundle on $Y$ corresponding to a nonsingular curve $C$ via the exact sequence

$$0 \to O_Y^{r-1} \to E \to J_C(r) \to 0.$$ 

Then

a) $H^i(E \otimes E^\vee(-1)) = 0$ for $i = 0, 3$

b) $H^i(E \otimes E^\vee(-1)) \cong H^{i-1}(N_{C/Y}(-1))$ for $i = 1, 2$ where $N_{C/Y}$ is the normal bundle of $C$ in $Y$.

Proof. a) Since $E$ is a quotient of $O_Y^{3r}$, it follows that $E \otimes E^\vee(-1)$ is a quotient of $E^\vee(-1)^{3r}$. But since $E^\vee(2)$ is an Ulrich sheaf, this sheaf has no $H^3$, and it follows that $H^3(E \otimes E^\vee(-1)) = 0$. By duality also $H^0(E \otimes E^\vee(-1)) = 0$.

b) Tensoring the exact sequence above with $E^\vee(-1)$ we get

$$0 \to E^\vee(-1)^{r-1} \to E \otimes E^\vee(-1) \to E^\vee \otimes J_C(r - 1) \to 0.$$ 

Now $E^\vee(2)$ is Ulrich, so $E^\vee(-1)$ has no cohomology, and $H^i(E \otimes E^\vee(-1)) \cong H^i(E^\vee \otimes J_C(r - 1))$ for all $i$.

The dual of the sequence for $E$, twisted by $r - 1$ is

$$0 \to O_Y(-1) \to E^\vee(r - 1) \to O_Y(r - 1)^{r-1} \to \omega_C(1) \to 0.$$
Tensoring with $O_C$ this gives

\[(3) \quad E_C(r - 1) \rightarrow O_C(r - 1)^{r-1} \rightarrow \omega_C(1) \rightarrow 0.\]

On the other hand, tensoring the original sequence with $O_C$ gives

\[
O_C^{-1} \rightarrow E_C \rightarrow J_C/J_C^2(r) \rightarrow 0,
\]

and since these are locally free sheaves on $C$, we can dualize and twist by $r - 1$ to get

\[(4) \quad 0 \rightarrow \mathcal{N}_{C/Y}(-1) \rightarrow E_C^\vee(r - 1) \rightarrow O_C(r - 1)^{r-1} \rightarrow \omega_C(1) \rightarrow 0.\]

The map in the middle of sequences (3) and (4) is the same, so we can combine to get

\[
0 \rightarrow \mathcal{N}_{C/Y}(-1) \rightarrow E_C^\vee(r - 1) \rightarrow O_C(r - 1)^{r-1} \rightarrow \omega_C(1) \rightarrow 0.
\]

Finally, we tensor the sequence $0 \rightarrow J_C \rightarrow O_Y \rightarrow O_C \rightarrow 0$ with $E^\vee(r - 1)$ and put our sequences together in the following diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
O_Y \\
\downarrow \\
\mathcal{N}_{C/Y}(-1)
\end{array}
\begin{array}{c}
0 \\
\downarrow \\
E_Y^\vee(J_C(r - 1)) \\
\downarrow \alpha \\
E_Y^\vee(r - 1) \\
\downarrow \\
E_C^\vee(r - 1) \\
\downarrow \gamma \\
E_C(r - 1)^{r-1} \\
\downarrow \\
\omega_C(1) \\
\downarrow \\
0
\end{array}
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{N}_{C/Y}(-1) \\
\downarrow \\
E_C(r - 1)^{r-1} \\
\downarrow \\
O_Y(r - 1)^{r-1} \\
\downarrow \\
O_C(r - 1)^{r-1} \\
\downarrow \\
\omega_C(1) \\
\downarrow \\
0
\end{array}
\]

Since $E$ is Ulrich, it follows that $h^i(J_C(r - 1)) = 0$ for all $i$. Hence the map $\gamma$ induces an isomorphism on cohomology, and therefore an isomorphism on cohomology of $\text{Im} \alpha$ to $\text{Im} \beta$. Since $O_Y(-1)$ has no cohomology, $\alpha$ also induces an isomorphism of cohomology from $E^\vee(r - 1)$ to $\text{Im} \alpha$. It follows that $H^0(E^\vee \otimes J_C(r - 1)) = 0$ and $H^1(E^\vee \otimes J_C(r - 1)) \cong H^0(N_{C/Y}(-1))$. Furthermore, $\beta$ induces a surjective map on $H^0$, and $H^1(\text{Im} \beta) = 0$, so $H^1(N_{C/Y}(-1)) \rightarrow H^1(E_C^\vee(r - 1)) \rightarrow H^2(E^\vee \otimes J_C(r - 1))$ are all isomorphisms. Combining with the isomorphisms already proved above gives the desired conclusion b).

\[\square\]

**Corollary 5.10.** There exist rank 2 and rank 3 stable Ulrich bundles $E$ on a general cubic threefold $Y$ such that $E \otimes E^\vee(-1)$ has no cohomology.

**Proof.** Indeed, this follows from Lemma 5.9 using the computations of Geiß and Schreyer in the Appendix, since they constructed ACM curves $C$ of degree 5 and genus 1 and of degree 12 and genus 10 having $H^i(N_{C/Y}(-1)) = 0$ for $i = 0, 1,$
Proposition 5.11. For each \( r \geq 2 \), there is a stable rank \( r \) Ulrich bundle \( E \) on the general cubic threefold \( Y \) such that \( E \otimes E^\vee(-1) \) has no cohomology.

Proof. Let \( E_0 \) be a stable rank 2 Ulrich bundle such that \( E_0 \otimes E_0^\vee(-1) \) has no cohomology (Corollary 5.10). We will prove by induction the following statement

(*) For each \( r \geq 2 \) there is a stable rank \( r \) Ulrich bundle \( F \) on \( Y \), \( F \not\cong E_0 \), such that \( F \otimes F^\vee(-1) \) and \( F \otimes E_0^\vee(-1) \) have no cohomology.

The condition of having no cohomology is an open condition, so for \( r = 2 \) we can take \( F \) to be a deformation of \( E_0 \). Then by semicontinuity, both \( F \otimes F^\vee(-1) \) and \( F \otimes E_0^\vee(-1) \) will have no cohomology.

For \( r = 3 \), we make use of the third section of the Appendix, which shows that on a general cubic threefold \( Y \), there are curves \( E \) and \( C \) as in the earlier theorems [A.1] and [A.3] respectively, such that if \( E_0 \) (changing notation) is the rank 2 bundle corresponding to \( E_0 \):

\[
0 \to \mathcal{O}_Y \to E_0 \to J_{E/Y}(2) \to 0,
\]

then we have also the additional property that \( H^i(E_0 \otimes J_{C/Y}) = 0 \) for \( i = 1, 2 \).

Let \( F \) be a stable rank 3 bundle corresponding to \( C \) (as in Proposition [5.4]):

\[
0 \to \mathcal{O}_Y^2 \to F \to J_{C/Y}(3) \to 0.
\]

Tensoring with \( E_0^\vee(-1) \) we have

\[
0 \to E_0^\vee(-1)^2 \to F \otimes E_0^\vee(-1) \to E_0^\vee(-1) \otimes J_{C/Y}(3) \to 0.
\]

Now \( E_0 \) has rank 2, so \( E_0^\vee \cong E_0(-2) \). Thus \( E_0^\vee(-1) \cong E_0(-3) \), which has no cohomology. Furthermore, since \( F \) and \( E_0 \) are distinct stable bundles, already \( H^0(F \otimes E_0^\vee) = 0 \), so also \( H^0(F \otimes E_0^\vee(-1)) = 0 \), and by duality also \( H^3(F \otimes E_0^\vee(-1)) = 0 \). To show therefore that \( F \otimes E_0^\vee \) has no cohomology, we have only to check the vanishing of \( H^i \) for \( i = 1, 2 \). Since \( E_0^\vee(-1) \) has no cohomology, the groups on question are isomorphic to \( H^i(E_0^\vee(-1) \otimes J_{C/Y}(3)) = H^i(E_0 \otimes J_{C/Y}) \), and these are zero by Proposition [A.6] of the Appendix. We have shown that \( F \otimes F^\vee(-1) \) has no cohomology earlier (Corollary 5.10). (Note that at this step we have redefined the rank 2 bundle \( E_0 \) chosen before, but we can just as well use this one from the beginning.)

For \( r \geq 4 \), choose by the induction hypothesis a stable bundle \( F_0 \) of rank \( r - 2 \), different from \( E_0 \), such that \( F_0 \otimes F_0^\vee(-1) \) and \( F_0 \otimes E_0^\vee(-1) \) have no cohomology. As in the proof of existence of stable bundles (Theorem 5.4), consider an extension

\[
0 \to E_0 \to G \to F_0 \to 0.
\]
Then $G$ will be simple of rank $r$. Tensoring with $E_0^\vee(-1)$ and using our hypotheses on $E_0$ and $F_0$, we see that $G \otimes E_0^\vee(-1)$ has no cohomology. Similarly tensoring with $F_0^\vee(-1)$ we find that $G \otimes F_0^\vee(-1)$ has no cohomology. (Note that $E_0 \otimes F_0^\vee(-1) = (F_0 \otimes E_0^\vee(-1))^\vee \odot \omega_Y$ so by Serre duality it has also no cohomology.) Now tensor $G(-1)$ with the dual sequence

$$0 \rightarrow F_0^\vee \rightarrow G^\vee \rightarrow E_0^\vee \rightarrow 0$$

to see that $G \otimes G^\vee(-1)$ has no cohomology. Finally, as in Theorem 5.7 we can deform $G$ into a stable bundle, call it $F$, and by semicontinuity it will satisfy $F \otimes F^\vee(-1)$ and $F \otimes E_0^\vee(-1)$ have no cohomology.

□

**Corollary 5.12.** For each $r \geq 2$, there is a nonempty open set of a modular family of stable rank $r$ Ulrich bundles on the general cubic threefold $Y$ restricting by an étale dominant map to a modular family of stable rank $r$ bundles on a hyperplane section $H$.

**Proof.** Follows from Propositions 5.11 and 5.8 □

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**Appendix A. ACM Curves of Small Degree on Cubic Threefolds**

by Florian Geiß\(^1\) and Frank-Olaf Schreyer\(^2\)

\(^1\)Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, 66123 Saarbrücken, Germany. email: fg@math.uni-sb.de. The first author is supported by DFG grant Schr 307/5-1 of the second author within the priority program SSP 1409.

\(^2\)Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, 66123 Saarbrücken, Germany. email: schreyer@math.uni-sb.de
Abstract. In this note we prove the following: A general elliptic normal curve $E$ of degree 5 on a general cubic threefold $X \subset \mathbb{P}^4$ over an algebraically closed field of characteristic 0 has a twisted normal bundle which splits as $\mathcal{N}_{E/X}(-1) \cong L \oplus L^{-1}$ with $L \in \text{Pic}^0(E)$, $L \not\cong \mathcal{O}_E$. In particular, we have $H^1(E, \mathcal{N}_{E/X}(-1)) = 0$. Similarly, we prove the vanishing $H^1(C, \mathcal{N}_{C/X}(-1)) = 0$ for a general arithmetically Cohen-Macaulay curve $C$ of genus 10 and degree 12 on a general cubic threefold $X \subset \mathbb{P}^4$. Finally, we prove the existence of a nonempty open subset of triples $C, E, X$ as above which satisfy in addition $E \cap C = \emptyset$, $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{E/X}(2), \mathcal{O}_X)$ is 1-dimensional and for the nontrivial extension $\mathcal{F}$ the vanishing $H^1(\mathcal{F} \otimes \mathcal{I}_{C/X}) = H^2(\mathcal{F} \otimes \mathcal{I}_{C/X}) = 0$ holds. The proofs are based on a computation in Macaulay2 over a finite field and semi-continuity.

A.1. Quintic elliptic curves.

Theorem A.1. Let $E \subset X \subset \mathbb{P}^4$ be a general pair of an elliptic normal curve on a general cubic threefold over an algebraically closed field of characteristic 0. Then the twisted normal bundle of $E$ in $X$ splits as $\mathcal{N}_{E/X}(-1) = L \oplus L^{-1}$ with $L \in \text{Pic}^0(E)$, $L \not\cong \mathcal{O}_E$. In particular, $H^1(\mathcal{N}_{E/X}(-1)) = 0$.

Proof. First, we check the corresponding statement for a general pair $E \subset X \subset \mathbb{P}^4$ defined over a finite field $\mathbb{F}_p$ by computation in Macaulay2. Initialization:

```plaintext
i1: p=101 -- a fairly small prime number
Fp=ZZ/p -- a finite ground field
R=Fp[x_0..x_4] -- coordinate ring of P^4
setRandomSeed("beta")
```

We start by randomly choosing a smooth cubic threefold $X$ and a smooth quintic elliptic curve $E$ on it.

```plaintext
i2 : m1=random(R^6,R^{6:-1});
m=m1-transpose m1;
   -- a random skew symmetric 6x6 matrix of linear forms
I=pfaffians(4,m_{0..4}^{0..4});
   -- the ideal an elliptic normal curve E
singE=minors(codim I,jacobian I)+I;
   (codim I==3, degree I==5, genus I==1, codim singE==5)
```

o2 = (true, true, true, true)
\[ i3 : f=pfaffians(6,m) \quad \text{-- ideal of } X \]
\[ \text{singf=ideal jacobian } f; \]
\[ \text{(codim } f==1, \text{ degree } f==3, \text{ codim singf == 5)} \]

\[ o3 = (\text{true, true, true}) \]

Next, we compute the normal bundle and the first values of its Hilbert function:

\[ i4 : I2=saturate(I^2+f); \]
\[ \text{coN=prune (image( gens I)/ image gens I2);} \]
\[ \quad \text{-- a module whose sheafication is the conormal sheaf} \]
\[ \quad \text{-- of } E \text{ in } X \]
\[ \quad N=\text{Hom(coN,R^1/I);} \quad \text{-- the module of global sections} \]
\[ \quad \text{-- of the normal bundle} \]
\[ \quad \text{apply(toList(-1..2),i->hilbertFunction(i,N))} \]

\[ o4 = \{0, 10, 20, 30\} \]

Hence, \( N_{E/X}(-1) \) has no sections, and since \( \det N_{E/X}(-1) \cong O_E \) has degree 0, we have \( H^1(N_{E/X}(-1)) = 0 \) as well. There are two possibilities for the rank 2 vector bundle \( N_{E/X}(-1) \) according to the Atiyah classification \cite{2}. Either

\[ N_{E/X}(-1) \cong L_1 \oplus L_2 \]

with \( L_2 \cong L_1^{-1} \in \text{Pic}^0(E) \) or \( N_{E/X}(-1) \) is an extension

\[ 0 \to L \to N_{E/X}(-1) \to L \to 0 \]

with \( L \in \text{Pic}^0(E) \) is 2-torsion. We check that we are in the first case:

\[ i5 : Nminus1 = N**R^{-1}; \]
\[ \text{time betti(EndN=Hom(Nminus1,Nminus1))} \]

\[ o5 = \text{total: 12 40} \]
\[ \text{0: 2} \]
\[ \text{1: 10 40} \]
Thus, $H^0(\mathcal{E}nd(\mathcal{N}_{E/X}(-1)))$ is two-dimensional. We compute the characteristic polynomial and the eigenvalues of this pencil of endomorphisms. The command `SetRandomSeed("beta")` above was chosen such that the characteristic polynomial decomposes completely over $\mathbb{F}_p$ in this step of the computation.

```
i6 : h0=homomorphism EndN_{0};
h0a=map(R^10,R^10,h0)
h1=homomorphism EndN_{1};
h1a=map(R^10,R^10,h1) -- the corresponding matrices

i7 : T=Fp[t] -- an extra ring
chiA=det(sub(h0a,T)-t*sub(h1a,T));
   -- the characteristic polynomial
chiAFactors = factor chiA

o7 = (t - 47) (t - 14)

i8 : -- We compute the eigenvalues and eigenspaces
eigenValues=apply(2,c-> -((chiAFactors#c)#0)%ideal t)
betti (N1=syz(h0a-eigenValues_0*h1a)
betti (N2=syz(h0a-eigenValues_1*h1a)) -- the eigenspaces
betti N
L1=prune coker(presentation N|N1)**R^{-1};
L2=prune coker(presentation N|N2)**R^{-1};
   -- the corresponding line bundles
betti res L1 -- L1 (and L2) has a linear resolution

0 1 2 3
o8 = total: 5 15 15 5
   1: 5 15 15 5

Finally, we check that $L1 \oplus L2 \cong \mathcal{N}_{E/X}(-1)$.

i9 : time betti (homL1L2=Hom(L1**L2,R^1/I)) -- used 32.25 seconds
   -- => L1 tensor L2 = 0_E
   annihilator homL1L2==I -- check

o9 = true
```
\texttt{i10 : time betti(iso=Hom(L1++L2,N)) -- used 9.22 seconds}
\begin{verbatim}
iso0=homomorphism iso_{0}
iso1=homomorphism iso_{1}
\end{verbatim}
\begin{verbatim}
o10 =
\begin{array}{ccccccccccc}
  | 0 0 0 0 0 10 42 31 7 -9 |
  | 0 0 0 0 0 16 -27 -30 -21 -35 |
  | 0 0 0 0 0 6 -13 -19 -5 -29 |
  | 0 0 0 0 0 38 9 41 22 -30 |
  | 0 0 0 0 0 -3 -9 34 -31 1 |
  | 0 0 0 0 0 20 -4 -19 -5 6 |
  | 0 0 0 0 0 17 -2 -37 -6 -19 |
  | 0 0 0 0 0 -46 -18 -31 -26 -20 |
  | 0 0 0 0 0 43 -23 -47 -33 -43 |
  | 0 0 0 0 0 34 41 -35 -13 1 |
\end{array}
\end{verbatim}

\texttt{i11 : det map(R^10,R^10,iso0+iso1)!=0 -- N(-1) is isomorphic to L1++L2}
\begin{verbatim}
o11 = true
\end{verbatim}

\texttt{i12 : prune ker(iso0+iso1)==0 and prune coker(iso0+iso1)==0}
\begin{verbatim}
o12 = true
\end{verbatim}

Since \( L_1 \in \text{Pic}^0(E)(\mathbb{F}_p) \) it has finite order. We compute the order, just for fun, in the most naive way. If the prime \( p \) is larger a better method is necessary.

\texttt{i13 : time betti(twoL1=prune Hom(L2,L1))}
\begin{verbatim}
k=2;
L1=twoL1;
time while (rank target gens kL1!=1) do (k=k+1;
kL1=prune Hom(L2,kL1)); -- used 29 seconds
\end{verbatim}
\begin{verbatim}
-- in a case where the order \( k=52 \)
k -- the order of \( L_1 \) in \( \text{Pic} E \).
\end{verbatim}
\begin{verbatim}
o13 = 52
\end{verbatim}

\texttt{i15 : betti kL1;}
\begin{verbatim}
kL1==R^1/I
\end{verbatim}
\begin{verbatim}
o15 = true
\end{verbatim}
To conclude from these computations the desired result in characteristic zero, we argue that the computation above can be seen as the reduction mod $p$ of computation over $\mathbb{Z}$. By semi-continuity the vanishing

$$H^0(\mathcal{N}_{E_\mathbb{Q}/X_\mathbb{Q}}(-1)) = H^1(\mathcal{N}_{E_\mathbb{Q}/X_\mathbb{Q}}(-1)) = 0$$

holds for the corresponding pair $(E_\mathbb{Q}, X_\mathbb{Q})$ defined over $\mathbb{Q}$ as well. The splitting into line bundles will be defined over a quadratic extension field $K$ of $\mathbb{Q}$ and the line bundle most likely will have infinite order in $\text{Pic}^0(E_\mathbb{Q})(K)$. □

Remark A.2. By the computation above we know from the example $E \subset X$ defined over an open part $\text{Spec} \mathbb{Z}$, that the same result holds for algebraically closed fields of positive characteristic except for possible finitely many primes $p$. In principle, one could try to compute these exceptional primes by computing an example over $\mathbb{Q}$, and then try to verify the result one by one for the remaining primes. We believe that this is currently computationally out of reach.

A.2. ACM curves of genus 10 and degree 12. In this section we prove the following

**Theorem A.3.** The space of pairs $C \subset X \subset \mathbb{P}^4$ of smooth arithmetically Cohen-Macaulay curves $C$ of degree 12 and genus 10 on a cubic threefold $X$ is unirational and dominates the moduli space $\mathcal{M}_{10}$ of curves of genus 10 and the Hilbert scheme of cubic threefolds in $\mathbb{P}^4$ with the maps defined over $\mathbb{Q}$. Moreover, for a general pair $C \subset X$ the following holds:

(i) The line bundle $\mathcal{O}_C(1)$ is a smooth isolated point of the Brill-Noether space $W^1_{12}(C) \subset \text{Pic}^{14}(C)$.

(ii) The module of global sections $\sum_{n \in \mathbb{Z}} H^0(\omega_C(n))$ of the dualizing sheaf $\omega_C$ is generated by its two sections in degree $-1$ as an $S = \sum_{n \in \mathbb{Z}} H^0(\mathbb{P}^4, \mathcal{O}(n))$-module.

(iii) The twisted normal bundle of $C$ in $X$ satisfies $h^1(\mathcal{N}_{C/X}(-1)) = 0$.

As in section 1, we will prove the result by a computation over a finite field and semi-continuity. The key ingredient is the following unirational construction of the desired curves. Suppose $C$ is a smooth projective curve of genus 10 defined over a field $k$ together with line bundles $L_1, L_2$ with $|L_1| a g_6$ and $|L_2|$ a $g_5$. Let $C'$ denote the image under the map

$$C \xrightarrow{|L_1|, |L_2|} \mathbb{P} H^0(C, L_1) \times \mathbb{P} H^0(C, L_2) = \mathbb{P}^1 \times \mathbb{P}^2.$$ 

We say that $C$ is of maximal rank if the map $H^0(\mathcal{O}_\mathbb{P}^2(n, m) \to H^0(L_1^{\otimes n} \otimes L_2^{\otimes m})$ is of maximal rank for all $n, m \geq 1$. Under the assumption of maximal rank of $C$
the image \( C' \) is isomorphic to \( C \) and the Hilbert series of the truncated vanishing ideal

\[
I_{\text{trunc}} = \bigoplus_{n \geq 3, m \geq 3} H^0(I_{C'}(n, m))
\]

in the Cox-Ring \( S = k[x_0, x_1, y_0, y_1, y_2] \) of \( \mathbb{P}^1 \times \mathbb{P}^2 \) is

\[
H_{I_{\text{trunc}}}(s, t) = \frac{3s^4t^5 - 6s^4t^4 - 3s^3t^5 + 3s^3t^4 + 4s^3t^3}{(1 - s)^2(1 - t)^3}.
\]

In other words, we expect a bigraded free resolution of type

\[
0 \to F_2 \to F_1 \to F_0 \to I_{\text{trunc}} \to 0
\]

with modules \( F_0 = S(-3, -3)^4 \oplus S(-3, -4)^3, F_1 = S(-3, -5)^3 \oplus S(-4, -4)^6 \) and \( F_2 = S(-4, -5)^3 \).

Turning things around, we find the following unirational construction for such curves: For a general map \( M : F_2 \to F_1 \) let \( K \) be the cokernel of the dual map \( M^* : F_1^* \to F_2^* \). For the first terms of a minimal free resolution of \( K \) we expect

\[
\ldots \to G \xrightarrow{N'} F_1^* \to F_2^* \to K \to 0
\]

with \( G = S(2, 4)^3 \oplus S(3, 3)^9 \oplus S(3, 4)^3 \oplus S(4, 2)^6 \). Composing \( N' \) with a general map \( F_0^* \to G \) and dualizing again yields a map \( N : F_1 \to F_0 \). Finally, \( \text{Ker}(F_0^* \xrightarrow{N^*} F_1^*) \cong S \) and the entries of the matrix \( S \to F_0^* \) generate \( I_{\text{trunc}} \). The following Code for Macaulay2 realizes this construction over an arbitrary field, here in particular for random choices over a finite field \( \mathbb{F}_p \):

```plaintext
i1 : setRandomSeed"I am feeling lucky"; -- initiate random generator p=32009; -- a prime number Fp=ZZ/p; -- a prime field S=Fp[x_0,x_1,y_0..y_2, Degrees=>{2:{1,0},3:{0,1}}]; -- Cox ring of P^1 x P^2 m=ideal basis({1,1},S); -- irrelevant ideal

i2 : randomCurveGenus10Withg16=(S)->( M:=random(S^{6:{-4,-4},3:{-3,-5}}),S^{3:{-4,-5}}); -- a random map F1 <---- F2
N':=syz transpose M; -- syzygy-matrix of the dual of M N:=transpose(N'*random(source N',S^{3:{3,4},4:{3,3}})); ideal syz transpose N) -- the vanishing ideal of the curve

i3 : IC'=saturate(randomCurveGenus10Withg16(S),m);
```
As being of maximal rank is an open condition this computation proves the existence of a nonempty unirational component $H$ in the Hilbert scheme $\text{Hilb}_{(6,9),10}(\mathbb{P}^1 \times \mathbb{P}^2)$ of curves of bidegree $(6,9)$ and genus 10.

By semi-continuity we get the first half of the following Theorem.

**Theorem A.4.** The Hilbert scheme $\text{Hilb}_{(6,9),10}(\mathbb{P}^1 \times \mathbb{P}^2)$ has a unirational component $H$ over $\mathbb{Q}$ that dominates the moduli space $\mathcal{M}_{10}$.

**Proof.** The main missing ingredient is to prove that in our example above the line bundles $L_1$ and $L_2$ will behave like general line bundles in $W^1_6(C)$ and $W^2_9(C)$ for a general curve $C$. Recall the following facts from Brill-Noether theory \cite{N}: For a general smooth curve $C$ of genus $g$ the Brill-Noether loci $W^r_d(C) = \{ L \in \text{Pic}^d(C) \mid h^0(L) \geq r + 1 \}$ are non-empty and smooth away from $W^{r+1}_d(C)$ of dimension $\rho$ if and only if the Brill-Noether number

$$\rho = \rho(g, r, d) = g - (r + 1)(g - d + r) \geq 0.$$ 

Moreover, $W^r_d(C)$ is connected if $\rho > 0$ and the tangent space at a linear series $L \in W^r_d(C) \setminus W^{r+1}_d(C)$ is the dual of the cokernel of the Petri-map

$$H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C).$$

Now let $\eta : C \to C'$ be a normalization of our given point $C' \in H$. $\eta$ will be an isomorphism, but we do not know this yet. We can check computationally that the linear systems $L_1 = \eta^*O_{\mathbb{P}^1}(1)$ and $L_2 = \eta^*O_{\mathbb{P}^2}(1)$ are smooth points in the respective $W^{r+1}_{d_i}(C)$ for $i = 1, 2$:

In order to check $L_2$, we start by computing the plane model $\Gamma \subset \mathbb{P}^2$ of $C'$:

```plaintext
i4 : Sel=Fp[x_0,x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
-- elimination order
R=Fp[y_0..y_2]; -- coordinate ring of P^2
IGammaC=sub(ideal selectInSubring(1,gens gb sub(IC',Sel)),R);
-- ideal of the plane model
```

We check that $\Gamma$ is a curve of desired degree and genus and its singular locus $\Delta$ consists only of ordinary double points:
i5 : distinctPoints=(J)->(
  singJ:=minors(2,jacobian J)+J;
  codim singJ==3)

i6 : IDelta=ideal jacobian IGammaC + IGammaC; -- singular locus
  distinctPoints(IDelta)
o6 = true

i7 : delta=degree IDelta;
  d=degree IGammaC;
  g=binomial(d-1,2)-delta;
  (d,g,delta)==(9,10,18)
o7 = true

We compute the free resolution of $I_{\Delta}$:

i8 : IDelta=saturate IDelta;
  betti res IDelta

| 0 | 1 | 2 |
|---|---|---|
| 0: | 1 | . |
| 1: | . | . |
| 2: | . | . |
| 3: | . | . |
| 4: | . | 3 |
| 5: | . | 1 |
o8 = total: 1 4 3

(We can deduce that $\Gamma$ is irreducible from this information: Suppose $\Gamma$ decomposes in two parts of degree $a$ and $b$ with $a + b = 9$ and, say, $a < b$ then the intersection points of two components would be among the points of $\Delta$. The cases $(a,b) = (1,8)$ and $(2,7)$ are excluded because $I_{\Delta}$ is generated by sextics, $(4,5)$ is excluded because $20 > 18$ and $(3,6)$ is excluded because $\Delta$ is not a complete intersection. Thus $C$ the normalization of $\Gamma$ is isomorphic to a smooth irreducible curve of genus $g = 10$, and $C'$ is smooth because $10 = g \leq p_a C' \leq 10$.)

From Riemann–Roch we deduce $h^0(C, L_2) = 3$ since $h^1(C, L_2) = h^0(C, \omega_C \otimes L_2^{-1}) = h^0(\mathbb{P}^2, \mathcal{I}_\Delta(5)) = 3$. The Petri map for $L_2$ can be identified with

$$H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d - 4)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d - 3)).$$
This map is injective since there are no linear relations among the three quintic generators of $I_{\Delta}$. So $L_2 \in W^3_9(C)$ is a smooth point of dimension $p_2 = 1$.

Turning to $L_1$, we compute the embedding $C \to \mathbb{P}H^0(C, \omega_C \otimes L_1^{-1}) = \mathbb{P}^4$ as follows

```plaintext
i9 : LK=(mingens IDelta)*random(source mingens IDelta, R^10:{-6}));
   -- compute a basis of the Riemann-Roch space L(Omega_C)
Pt=random(Fp^1,Fp^2); -- random point in P^1
L1=substitute(IC',Pt|vars R); -- L1 is the fiber over Pt
KD=LK*(syz(LK % gens L1))_{0..4};
   -- compute a basis of those elements in L(Omega_C) that
   -- vanish in L1
T=Fp[z_0..z_4]; -- coordinate ring of P^4
phiKD=map(R,T,KD); -- embedding
IC=preimage_phiKD(IGammaC);
degree IC==12 and genus IC==10
o9 = true

i10 : betti(FC=res IC)

0 1 2 3
0: 1 . . .
1: . . . .
2: . 8 9 .
3: . . . 2
```

From the length of the resolution $F_C$ we see that the image of $C$ in $\mathbb{P}^4$ is arithmetically Cohen-Macaulay. The dual complex $\text{Hom}_S(F_C, S(-5))$ is a resolution of $\bigoplus_{n \in \mathbb{Z}} H^0(\omega_C(n))$. Thus this module is generated by its two sections in degree $-1$ and $h^0(L_1) = h^0(C, \omega_C(-1)) = 2$. The Petri map can be identified with

$$H^0(C, \omega_C(-1)) \otimes H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(1)) \to H^0(C, \omega_C).$$

Here, this map is an isomorphism, because there is no linear relation among the two generators, and $L_1$ is a smooth isolated point in $W^4_6(C)$. Thus our random example over the finite field is as expected, and semi-continuity proves that the same is true for the triple $(C, L_1, L_2)$ defined over an open part of Spec $\mathbb{Z}$ whose reduction mod $p$ is the given randomly selected curve.
The map $H \rightarrow \mathcal{M}_{10}$ factors over $Z = \mathcal{W}_6^1 \times_{\mathcal{M}_{10}} \mathcal{W}_9^2$ and the fiber of $H \rightarrow Z$ for a triple $(C, L_1, L_2)$ (without automorphisms) with $h^0(C, L_1) = 2$ and $h^0(C, L_2) = 3$ is $\text{PGL}(2) \times \text{PGL}(3)$. The fiber dimension of $Z \rightarrow \mathcal{M}_g$ is $\rho_1 + \rho_2 = 0 + 1 = 1$, as expected.

\textbf{Remark A.5.} Note that $H$ dominates the Severi variety $V_{9,10}$ of reduced and irreducible plane curves of degree 9 and genus 10 as well as the Hurwitz scheme $H_{6,10}$ of 6-gonal curves of genus 10.

In fact, the outlined approach allows to prove the existence of unirational components of $\text{Hilb}_{d_1,d_2,g}(\mathbb{P}^1 \times \mathbb{P}^2)$ for several values $(d_1, d_2, g)$. Particularly, we find that the Severi variety $V_{10,11}$ and the Hurwitz schemes $H_{6,g}$ of 6-gonal covers for $g \leq 40$ are unirational. The last statement is proved using liaison in $\mathbb{P}^1 \times \mathbb{P}^2$; see [5].

\textbf{Proof of Theorem A.3.} We are nearly done. The embedding of

$C \hookrightarrow \mathbb{P}H^0(C, \omega_C \otimes L_1^{-1}) \cong \mathbb{P}^4$

is a curve which satisfies (1) and (2). Since $L_1$ and equivalently $\mathcal{O}_C(1) \in \mathcal{W}_{12}^4(C)$ is Petri general this proves the existence of a unirational component

$H' \subset \text{Hilb}_{12t+1-10}(\mathbb{P}^4)$.

Since the Hurwitz scheme $H_{6,10}$ is irreducible, we can conclude that the induced rational map $H'//\text{PGL}(5) \rightarrow \mathcal{M}_{10}$ is generically finite of degree

$g! \prod_{i=0}^r \frac{i!}{(g - d + r + i)!} = 42 = \deg \mathcal{W}_6^1(C)$

as is well-known ([1], Ch.V). Choosing a cubic threefold containing $C$ is the same as choosing a point in the projective space $\mathbb{P}H^0(\mathbb{P}^4, \mathcal{I}_C(3))$. Hence,

$V = \{(C, X) \mid C \in H' \text{ ACM and } X \in \mathbb{P}H^0(\mathbb{P}^4, \mathcal{I}_C; 3) \text{ smooth}\}$

is unirational as well. For a random pair $(C, X) \in V$ we compute the normal sheaf $\mathcal{N}_{C/X}$ of $C$ in $X$ and check that $h^i(\mathcal{N}_{C/X}(-1)) = 0$ for $i = 0, 1$:

```
46

The map $H \rightarrow \mathcal{M}_{10}$ factors over $Z = \mathcal{W}_6^1 \times_{\mathcal{M}_{10}} \mathcal{W}_9^2$ and the fiber of $H \rightarrow Z$ for a triple $(C, L_1, L_2)$ (without automorphisms) with $h^0(C, L_1) = 2$ and $h^0(C, L_2) = 3$ is $\text{PGL}(2) \times \text{PGL}(3)$. The fiber dimension of $Z \rightarrow \mathcal{M}_g$ is $\rho_1 + \rho_2 = 0 + 1 = 1$, as expected.

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$V = \{(C, X) \mid C \in H' \text{ ACM and } X \in \mathbb{P}H^0(\mathbb{P}^4, \mathcal{I}_C; 3) \text{ smooth}\}$

is unirational as well. For a random pair $(C, X) \in V$ we compute the normal sheaf $\mathcal{N}_{C/X}$ of $C$ in $X$ and check that $h^i(\mathcal{N}_{C/X}(-1)) = 0$ for $i = 0, 1$:

```
i11 : IX=ideal((mingens IC)*random(source mingens IC,T^1:-3));
   IC2=saturate(IC^2+X);
   cNCX=image gens IC/image gens IC2; -- the conormal sheaf in X
   NCX=sheaf Hom(cNCX,T^1/IC); -- the normal sheaf in X

i12 : HH^0 NCX(-1)==0 and HH^1 NCX(-1)==0

o12 = true
```
With a similar computation for $N_{C/P^4}$ we check that $H'$ is a generically smooth component of the Hilbert scheme $\text{Hilb}_{12t+1-10}(P^4)$ of expected dimension 51 and $C$ is a smooth point in $H'$.

Consider the maps

$$
\begin{array}{cccc}
V & \pi_1 \rightarrow & H' & \pi_2 \rightarrow \\
\downarrow & & \downarrow & \\
\mathbb{P}^0(P^4, O_{P^4}(3)) \cong \mathbb{P}^7 & & & \end{array}
$$

The fibre of $\pi_1$ over a point $C$ is exactly $\mathbb{P}^0(P^4, I_C(3)) \cong \mathbb{P}^7$, hence $V$ is irreducible of dimension 58. The map $\pi_2$ is smooth of dimension $h^0(C, N_{C/X}) = 24$ at $(C, X)$. Thus $\pi_2$ is surjective. By semi-continuity the desired vanishing holds for the general curve on a general cubic. □
A.3. Cohomology of Extensions. In order to prove Corollary 5.12 for arbitrary rank the following statement is needed:

**Proposition A.6.** Let $k$ be an algebraically closed field of characteristic 0. There is an open subset $U$ of the space of triples $C, E \subset X$ with $C$ and ACM curve of genus 10 and degree 12, $E$ an elliptic normal curve of degree 5 not meeting $C$ and $X$ a smooth cubic threefold over $k$ with the following properties:

(i) $U$ dominates the space $\mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$ of cubic threefolds and the spaces of pairs $E \subset X$ and $C \subset X$. In particular the pair $E \subset X$ and the pair $C \subset X$ satisfy all assertions of Theorem A.1 and A.3 respectively.

(ii) For every triple $C, E \subset X$ in $U$ the extension group $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{E/X}(2), \mathcal{O}_X))$ is 1-dimensional and for the non-trivial extension

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_{E/X}(2) \to 0$$

we have the vanishing $H^1(\mathcal{F} \otimes \mathcal{I}_{C/X}) = H^2(\mathcal{F} \otimes \mathcal{I}_{C/X}) = 0$.

**Proof.** Again, our strategy is to construct a triple $C, E \subset X$ over a finite field with the help of Macaulay2 and then establishing the theorem in characteristic 0 with semi-continuity.

The bottleneck of this approach is to construct $E$ and $C$ such that there is a cubic threefold which contains both curves. Since $H^0(\mathcal{O}_{\mathbb{P}^4}(3))$ is 35-dimensional, for a general pair $(E, C)$ the 20-dimensional subspace $W_E = H^0(\mathcal{I}_{E/\mathbb{P}^4}(3))$ and the 8-dimensional subspace $W_C = H^0(\mathcal{I}_{C/\mathbb{P}^4}(3))$ will have a trivial intersection.

More precisely, the locus $M$ of pairs $(E, C)$ with $W_E \cap W_C \neq 0$ has expected codimension 8 in $H = H_1 \times H_2 \subset \text{Hilb}_{56}(\mathbb{P}^4) \times \text{Hilb}_{12-9}(\mathbb{P}^4)$ where $H_1$ is the subscheme whose points correspond to smooth elliptic normal curves and $H_2$ the subscheme whose points correspond to smooth ACM curves.

One way to find points in $M$ is by searching over a small finite field. Heuristically, the probability for a random point $(E, C) \in H(\mathbb{F}_p)$ to lie in $M(\mathbb{F}_p)$ is

$$\frac{\#M(\mathbb{F}_p)}{\#H(\mathbb{F}_p)} \approx \frac{1}{p^8}.$$  

From the Weil formula we see that this approximation is asymptotically correct as $p \to \infty$. Practice shows that it is a reasonable heuristic even for small $p$ in many cases. However, over very small fields ($p = 2, 3$) most hypersurfaces are singular, see [3]. Hence we must not choose $p$ too small in order to minimize the total runtime. Empirically, $p = 5$ is a good choice. Turning to the construction, we start with a random smooth arithmetically Cohen-Macaulay curve $C$ of genus 10 and degree 12 in $\mathbb{P}^4$. To keep things clear we capsulated the construction of the preceding section in a function that returns the vanishing ideal of such a curve:
such an

To do this is in a space-saving way, we write down the $9 \times 8$ matrix $m_C$ with linear entries in the free resolution of $I_C$. From $m_C$ the curve can easily be regained:

In our example, we have

$$m_C = \begin{pmatrix}
-2z_3 & 2z_1 - 2z_3 - z_4 & -2z_0 - z_1 - 2z_3 + 2z_4 & 20 + z_3 - 2z_4 \\
2z_1 - z_2 + z_3 - 2z_4 & -2z_0 + 2z_1 + 2z_4 & -2z_0 + 2z_4 & z_3 - 4z_3 \\
-2z_1 + z_3 & 2z_0 - z_1 - 2z_3 - 2z_4 & 20 + 2z_3 + 4z_4 & 2z_3 - 4z_4 \\
-2z_1 - z_3 - 4z_2 & 2z_0 - z_1 - 2z_3 - 2z_4 & z_0 - z_3 - 4z_3 & -2z_3 + 2z_4 \\
2z_3 & 2z_1 + z_3 + z_4 & -2z_0 - z_2 - 2z_4 & z_3 - 4z_3 \\
-2z_1 - z_3 + 2z_4 & z_0 + z_3 + 4z_4 & z_0 - z_2 - 2z_4 & -2z_3 - 4z_4 \\
z_3 + 2z_3 & -2z_0 - 2z_1 + 2z_3 + z_4 & 2z_0 - 2z_3 - 2z_4 & -z_2 \\
z_2 + 2z_3 - 4z_4 & -2z_0 - 2z_1 - 2z_4 & z_2 + 2z_3 - 4z_4 & -z_2 \\
z_1 - 2z_4 & -2z_0 - z_1 + z_3 - 4z_4 & z_0 - z_3 - 2z_4 & -z_3 + 2z_4 \\
\end{pmatrix}$$

In the next step we search for an elliptic normal curve $E$ such that $C$ and $E$ lie on a common cubic threefold $X$. Picking $E$ at random and checking whether there is a relation between the generators of $H^0(I_C/P^4(3))$ and $H^0(I_E/P^3(3))$ takes about 0.01 seconds a time on a 2.4 GHz processor. Hence we expect to find a such an $E$ within a span of about one hour.
i5 : getEllipticWithCommonThreefold=(IC)->(
   max3:=ideal basis(3,T);
   -- third power of the maximal ideal
   for attemptsHS from 1 do (mEtmp:=random(T^5,T^{5:-1});
      mE:=mEtmp-transpose mEtmp;
      -- the 5x5 skew-symmetric matrix
      IE:=pfaffians(4,mE);
      -- the elliptic curve E
      if rank source gens intersect(IE+IC,max3)<28 then (
            rltn:=(syz(gens IC|gens intersect(IE,max3)))_{0};
            -- the relation between the generators
            X:=ideal (gens IC*rltn^{0..7});
            -- the cubic threefold
            "attempts hypersurface = "<<attemptsHS;
            -- print number of attempts
            return(mE,X)));

We also have to check that the cubic hypersurface $X$ is smooth and that the twisted normal bundles $N_{E/X}(-1)$ has no global sections, as expected.

i6 : normalSheaf=(I,X)->(
   I2:=saturate(I^2+X);
   cNIX:=image gens I/ image gens I2;
   sheaf Hom(cNIX,(ring I)^1/I))

i7 : sectionsTwistedNormalBundle=(mE,X)->(
   IE:=pfaffians(4,mE);
   NEX:=normalSheaf(IE,X);
   HH^0(NEX(-1)))

Recall from [4], that $O_E$ has an eventually 3-periodic free resolution as an $O_X$-module

$$
\ldots \rightarrow O_X(-6)^6 \rightarrow O_X(-5)^6 \rightarrow O_X(-3)^6 \rightarrow O_X(-2)^5 \rightarrow O_X \rightarrow O_E \rightarrow 0
$$

whose higher syzygy modules are independent of choice of the section $s \in H^0(F)$ defining $E$. Thus the number of sections

$$
N_{E/X}(-1) \cong I_{E/X}/I_{E/X}^2(1) \cong F \otimes O_E(-1)
$$

depends only on $F$ but not on $s \in H^0(F)$: Tensoring the periodic resolution with $F(-1)$ and the fact that $F$ has no intermediate cohomology yields

$$
H^0(N_{E/X}(-1)) \cong \text{Ker}(H^3(F(-1)) \otimes K_3) \rightarrow H^3(F^6(-4)),
$$
and $K_3 = \text{Ker}(\mathcal{O}_X(-3)^6 \to \mathcal{O}_X(-2)^5) \cong \text{Im}(\mathcal{O}_X(-4)^6 \to \mathcal{O}_X(-3)^6)$ is independent of $E$. So if $H^0(N_{E/X}(-1)) = 0$ then for any other elliptic curve $E'$ corresponding to a global section of $F$ the cohomology $H^0(N_{E'/X}(-1))$ is also vanishing. Putting everything together, we have the following search routine:

```plaintext
i8 : time for attemptsN from 1 do (time for attemptsS from 1 do (
    time (mE,X)=getEllipticWithCommonThreefold(IC);
    if isSmooth X then (<<"attempts smooth = "<<attemptsS;
        break));
    if sectionsTwistedNormalBundle(mE,X)==0 then (<<"attempts normalbundle = "<<attemptsN;
        break));

-- the output:
attempts hypersurface = 25831 -- used 221.619 seconds
attempts hypersurface = 206719 -- used 1825.24 seconds
attempts hypersurface = 132506 -- used 1154.79 seconds
attempts smooth = 3 -- used 3201.66 seconds
attempts normalbundle = 1 -- used 3202.02 seconds
```

The extension is given as the cokernel of $m$ which is accessible through the resolution of $\mathcal{O}_E$:

```plaintext
i9 : m0=sub((res sub(pfaffians(4,mE),T/X)).dd_4,T);
    -- the matrix in the resolution
    baseChange=Hom(coker m0,coker transpose m0);
    b=map(T^6,T^6,homomorphism baseChange_{0});
    -- we compute a skewsymmetrization of m0
    m=b*m0;
    pfaffians(6,m)==X
```

In our example, we have

$$ m = \begin{pmatrix}
0 & -2z_1 - z_2 + 2z_4 & -2z_3 - z_4 \\
2z_1 + z_2 - 2z_4 & 0 & z_0 + 2z_2 - 2z_3 + z_4 \\
2z_3 + z_4 & -2z_0 - 2z_2 + 2z_3 - z_4 & 0 \\
2z_2 - 2z_3 - 2z_4 & -2z_0 + 2z_1 - z_2 + 2z_3 - 2z_4 & -2z_2 + 2z_3 + z_4 \\
-z_2 + z_3 + z_4 & 2z_0 - z_2 - z_3 - 2z_4 & z_0 + 2z_2 - 2z_3 \\
-2z_0 + z_2 - z_4 & z_0 + 2z_1 + z_2 + 2z_3 + 2z_4 & -z_1 - z_2 + 2z_3 + 2z_4
\end{pmatrix} $$

The output:

```
o9 = true
```

In our example, we have
A smooth random section $E'$ of the bundle $F$ can also be obtained very easily:

```plaintext
i10 : IE'=for i from 1 do (
    b:=random(T^6,T^6);
    m':=b*m*transpose b;
    IE':=pfaffians(4,m'_{0..4}^{0..4});
    if isSmooth IE' and dim(IC+IE')==0 then break(IE'));
```

In order to check that $C$ and $E'$ are smooth points in $\text{Hilb}_{12t-9}(X)$ and $\text{Hilb}_{5t}(X)$, respectively, we compute the cohomology groups of the normal sheaves:

```plaintext
i11 : NE'X=normalSheaf(IE',X);
HH^0(NE'X)==Fp^10 and HH^1(NE'X)==0

o11 = true
```

```plaintext
i12 : NCX=normalSheaf(IC,X);
HH^0(NCX)==Fp^24 and HH^1(NCX)==0

o12 = true
```

Finally, we compute the cohomology groups of $F \otimes I_C/X$:

```plaintext
i13 : M=coker sub(m,T/X);
    -- this is a module whose sheafification is an extension
    sheafMIC=sheaf(M)**sheaf(module sub(IC,T/X));
    HH^1 sheafMIC==0 and HH^2 sheafMIC==0

o13 = true
```

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Department de Matemàtica Aplicada I. ETSEIB. Universitat Politècnica de Catalunya. Avinguda Diagonal 647. 08028 Barcelona. Spain.

E-mail address: marta.casanellas@upc.edu

Department of Mathematics. Evans Hall. University of California. Berkeley, CA, 94720-3840. USA

E-mail address: robin@math.berkeley.edu