On bending angles by gravitational lenses in motion

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Accepted 2002 November 21. Received 2002 November 13; in original form 2002 May 13

ABSTRACT

The bending of light rays by the gravitational field of a ‘lens’ that is moving relative to the observer is calculated within the approximation of weak fields, small angles and thin lenses. Up to first order in $v/c$ (and assuming the acceleration to be much smaller than $v/c$), the bending angle, time delay and redshift of the images are found to be affected by the component of the speed of the deflector along the line of sight. The correction takes the form of an overall factor of $1 + v/c$ accompanying the mass of the deflector, leading to an indeterminacy of the order of $v/c$ in the mass of the lens inferred on the basis of the separation of multiple images. The consequent correction to the microlensing light curve is pointed out, as well as scenarios where the correction is potentially relevant.

Key words: gravitation – gravitational lensing – relativity.

1 INTRODUCTION

In order to derive a formula for the bending angle of light rays in the presence of a gravitational deflector, in gravitational lensing it is commonly assumed that the observer, deflector and source are at rest in some global coordinate system (see for instance Schneider, Ehlers & Falco 1992). However, most gravitational deflectors will not be at rest in the observer’s frame of reference. Typical gravitational lenses are of two kinds: galaxies and clusters of galaxies at cosmological distances; or massive compact objects within the Milky Way galaxy. Cosmological deflectors have peculiar velocities on top of the Hubble flow, and objects in the Milky Way usually move by the line of sight rather quickly. Still, the effect of the motion of a thin deflector to observable lensing quantities has traditionally been regarded as too small to merit discussion even in reference books in gravitational lensing. However, there have been indications that the effect is actually linear in $v/c$.

The earliest indication that the leading correction is probably linear in $v/c$ can be found in Schneider et al. (1992), where it is shown that the internal motion of a deflector at rest affects the effective index of refraction in first order (equation 4.16 of Schneider et al. 1992). Cosmological deflectors have peculiar velocities on top of the Hubble flow, and objects in the Milky Way usually move by the line of sight rather quickly. Still, the effect of the motion of a thin deflector to observable lensing quantities has traditionally been regarded as too small to merit discussion even in reference books in gravitational lensing. However, there have been indications that the effect is actually linear in $v/c$.

The next indication appears in Pyne & Birkinshaw (1993), where the authors develop a method for integrating the null geodesics of a space–time that is a small perturbation of flat space. As an application of the method, the authors calculate the bending angle due to a point mass $m$, obtaining

$$\alpha = \frac{4m}{r_0^2} \left(1 - v \cdot n_0\right),$$

where $r_0$ is the distance of closest approach of the light ray to the point mass $m$, and $n_0$ is the direction of the null ray at that point. The term $-v \cdot n_0$ is the Doppler shift of the deflector, $z_d$, so we can read this result as

$$\alpha = (1 + z_d) \alpha_s,$$

which means that the bending angle is corrected by a factor $(1 + z_d)$ with respect to the bending angle of the same deflector at rest in a Minkowski background. Here the quantity $z_d$ should not be confused with the cosmological redshift of the deflector.

The third time that the motion of the deflector was found interesting was in Capozziello et al. (1999), the first of a series of papers where the leading correction to the bending angle is calculated for different types of deflectors. By pursuing the integration of null geodesics in a weak perturbation of flat space–time by a slightly different method, for a point mass, (Capozziello et al. 1999, equation 33) find

$$\alpha = (1 - 2z_d) \alpha_s,$$

More recently, in Frittelli, Kling & Newman (2002), we use a completely different method based on envelopes of null surfaces (as opposed to null geodesics) to rederive the lens equation, and we find $\alpha = (1 + z_d) \alpha_s$ for the case of a lens moving along the line of sight.

Aside from a purely academic interest in lensing by moving deflectors, there has been a growing interest on the part of the microlensing sector of practising astrophysicists, stemming from improvements in observational accuracy. Microlensing results when a background star in the Magellanic Clouds (for instance) lines up with the line of sight between us and a compact object in our Galaxy. The line-up may happen due to the motion of the star, or to the motion of the compact object, but the same treatment is normally used in both instances. The motion creates a time-dependent brightness curve that peaks sharply when the star is closest to a caustic

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created by the compact object in our past light cone. The light curve is usually studied using the static lens equation, assuming the source moves with approximately uniform transverse velocity on the source plane. Inverting the lens equation, one obtains a curve for the motion of the images on the lens plane, which can be used to evaluate the magnification as a function of time – see for instance Paczyński (1986). The method is used indiscriminately for the case that the lens moves right through the line of sight to a static source. The point to be made is that there are cases of moving lenses of interest, and corrections to the lens equation of first order in $v/c$ (longitudinal speed) might become observable as the technology keeps improving. An accurate determination of the leading correction of order $v/c$ to the bending angle of light rays due to the relative motion between the lens and the observer may become necessary in the near future.

Here the calculation of the integration of the null geodesics in the case of a time-dependent perturbation of a certain kind is pursued once more with the methods and notation of Schneider et al. (1992). Our method uses elements of both of the calculations in Pyne & Birkinshaw (1993) and Capozziello et al. (1999), as well as new elements, and thus functions as an independent cross-check. We find full agreement with Pyne & Birkinshaw (1993) and Frittelli et al. (2002), and we back up our calculation with a check of consistency with the leading correction to the time delay, which, to our knowledge, has not appeared before except in Frittelli et al. (2002).

The geodesic equation for linearized perturbations corresponding to a non-stationary deflector of certain generality is written down in Section 2, where the notation and approximations are defined. The bending angle is evaluated in Section 3. The effects on a microlensing light curve are obtained in Section 4. Concluding remarks are offered in Section 5.

## 2 BENDING ANGLE BY A DEFLCTOR IN SLOW MOTION

Although we are not interested in the most general linearized localized perturbation off flat space, in this section we write the equation for the null geodesics in an asymptotically flat space–time representing a slowly moving lens of certain generality. This is because most steps in the derivation of the bending angle in the case of interest are more general and would perhaps be useful in a different context. Assuming that the metric deviates only slightly from a flat metric, we use Cartesian coordinates $x^a = \{t, x\}$ [with $a = 0, \ldots, 4$ and $x = (x, y, z)$]. We assume that the metric $d\mathbf{s}^2 = g_{ab} dx^a dx^b = (\eta_{ab} + \gamma_{ab}) dx^a dx^b$, where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is given in the following form

$$d\mathbf{s}^2 = (1 + 2\phi)c^2 dt^2 - 8c V \cdot dx \cdot dt - (1 - 2\phi) dx \cdot dx,$$

where a dot between two three-dimensional quantities represents the Euclidean scalar product $(u \cdot w \equiv \sum u^i w^i = \sum u_i w_i)$ and $\phi = \phi(x, t)$ and $V = V(x, t)$. The meaning of $\phi$ or $V$ is irrelevant to the calculations that follow; only their size matters. We assume that we have two independent smallness parameters $\epsilon$ and $\delta$ such that $\phi \sim \delta$ and $V \sim \epsilon \delta$. The parameter $\epsilon$ represents, in some sense, the ratio of the speed of motion of the deflector to the speed of light, and $\delta$ represents the strength of the Newtonian potential of the deflector’s mass density. All the calculations that follow assume that any integer powers of $\delta$ or $\epsilon$ greater than unity can be neglected, but the the product $\epsilon \delta$ will be kept in view of the independence of the two parameters. The reader should be aware that our particular choice of metric and sizes is motivated by the case of a perfect fluid in harmonic coordinates, in which the following expressions for the potentials,

$$\phi(x, t) \equiv -\frac{G}{c^2} \int \frac{\rho(x', t) \, d^3 x'}{|x - x'|},$$

$$V(x, t) \equiv -\frac{G}{c^2} \int \frac{\rho(x', t) \omega(x', t) \, d^3 x'}{|x - x'|},$$

hold in the near zone. However, we are not interested in these particular expressions for the potentials $\phi$ and $V$, and in fact we do not use them for any of the calculations in this work. Any expressions for the potentials are allowed as long as they lead to a metric that satisfies the Einstein equations up to terms of order $\delta, \epsilon$ or $\epsilon \delta$. This includes the potentials representing a lens that moves rigidly without changing shape, which will be used in the next section, and obviously the slowly moving fluid as well.

We are interested in light rays $x^a(s)$ with tangent vector $u^a = (u^0, u) = dx^a/ds$ travelling in this space–time. Because $u^0$ is null, namely $u^0 u_0 = 0$, it is defined only up to overall rescalings, and the scalefactor can be adjusted with an appropriate choice of parameter along the path. For convenience, we choose the scaling so that $u \cdot u = 1$, namely the space part of the tangent has unit length in the Euclidean background. This is equivalent to choosing to parametrize the light ray with its Euclidean length $t$, in contrast to the wider practice of using affine parametrizations for null paths. The Euclidean line element is precisely defined by

$$dt^2 = dx \cdot dx,$$

evaluated along the path. We thus have, by construction, a light ray given by $x^a(\ell) = (t(\ell), x(\ell))$ with tangent

$$u^0 = \frac{dt}{d\ell},$$

$$u = \frac{dx}{d\ell} = \hat{u},$$

where a hat $\hat{\cdot}$ is used to denote a three vector of unit length in the Euclidean space – a directional vector. With this choice of parametrization, the nullity condition $u^0 u_0 = 0$ reads explicitly

$$1 + 2\phi c^2 u^0 \omega^i - 8c V \cdot u = (1 - 2\phi) = 0.$$

Solving this equation for $u^0$ in a power series of $\epsilon$ and $\delta$ of the form

$$u^0 = \tilde{u}^0 + \delta \tilde{u}^0_\delta + \epsilon \tilde{u}^0_\epsilon + \epsilon \delta \tilde{u}^0_{\epsilon \delta}$$

we find

$$u^0 = 1 - 2(\phi - 2\tilde{u} \cdot V/c).$$

Thus the time component of the null tangent is found readily from the nullity condition. It is significant that the deviation of $c u^0$ from zeroth order is of order $\delta$ at least, but there is no deviation of order $\epsilon$. Next, we wish to obtain an equation for the spatial part of the null tangent, which, because of our choice of parametrization, is reduced to the vector $\hat{u}$, which gives the direction of the null ray. We obtain it from the geodesic equation in the following manner. The equation for geodesic curves with generic parametrization (not necessarily affine) reads

$$\frac{d\mathbf{u}'}{d\ell} + \Gamma_{ab}^c u^a u^b - \mu u^0 = 0,$$

where the factor $\mu$ is free, reflecting the freedom in the choice of parametrization [see, for instance, equation (3.3.2) of Wald (1984) and the subsequent discussion, or page 33 of Hawking & Ellis (1973) as well]. A vanishing $\mu$ gives rise to an affine parametrization by definition. In our case, however, $\mu$ is not zero, but is so far unspecified. We need only consider the space components, namely

$$\frac{d\mathbf{u}'}{d\ell} + \Gamma_{ab}^c u^a u^b + 2\Gamma_{ab}^c u^a u^b + \Gamma_{ab}^c u^a u^b - \mu u^0 = 0.$$
For the connection coefficients of the linearized metric $\Gamma^a_{bc} \equiv \frac{1}{2} \eta^{ad} (h_{bd,c} + h_{cd,b} - h_{bd,c})$, we have

$$\Gamma^0_{00} = 4cV^t + c^2 \phi,$$

$$\Gamma^0_{ij} = 2c(V_{i,j} - V_{j,i}) - \delta_{ij} \phi,$$

$$\Gamma^j_{ik} = \delta_{jk} \phi_{,i} - \delta_{ik} \phi_{,j} - \delta_{ij} \phi_{,k}$$

where $\phi_{,i} \equiv \partial \phi / \partial x^i$ and $\phi_{,ij} \equiv \partial^2 \phi / \partial x^i \partial x^j$. With these expressions for the connection coefficients, equation (13) reads

$$\frac{d}{dt}(4cV^t + c^2 \phi) + 2|2c(V_{i,j} - V_{j,i}) - \delta_{ij} \phi| \frac{u^i}{u^0} + \frac{\Gamma^j_{ik} u^k}{u^0} = 0. \quad (17)$$

Because $\Gamma^a_{0a}$ are all of order $\delta$ at least, only the zeroth order of $u^i$ is needed. Also notice that

$$u^i(V_{i,j} - V_{j,i}) = -[\hat{u} \wedge (\nabla \wedge V)].$$

Using these two facts, equation (17) becomes

$$\frac{d\hat{u}}{dt} + \frac{4}{c} \hat{V} + 2 \nabla \phi - 4 \hat{u} \wedge (\nabla \wedge V) - \left( \frac{2}{c} \phi + 2 \hat{u} \cdot \nabla \phi + \mu \right) \hat{u} = 0. \quad (18)$$

But since $\hat{u}$ has unit length, then $\hat{u} \cdot \frac{d\hat{u}}{dt} = 0$. Taking the scalar product of (18) with $\hat{u}$ we obtain

$$\frac{4}{c} \hat{u} \cdot \hat{V} - \frac{2}{c} \phi - \mu = 0. \quad (19)$$

which allows us to determine the unknown function $\mu$. Using (19) into (18) we finally have

$$\frac{d\hat{u}}{dt} = -2[V \phi - \hat{u} \cdot \nabla \phi - 2 \hat{u} \wedge (\nabla \wedge V)] - \frac{4}{c} (\hat{V} - \hat{u} \cdot \hat{V}).$$

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(20)

If for any vector $w$ we define the components that are perpendicular and parallel to the light ray $\hat{u}$ by

$$w_\perp \equiv w - \hat{u} (\hat{u} \cdot w),$$

and, in the case of the gradient,$$

\nabla_\perp \equiv \nabla - \hat{u} (\hat{u} \cdot \nabla).$$

this equation reads

$$\frac{d\hat{u}}{dt} = -2 \left[ \nabla \phi - \hat{u} \cdot \nabla \phi = 2 \hat{u} \wedge (\nabla \wedge V) + \frac{4}{c} \hat{V} \right].$$

(21)

Integrating $\frac{d\hat{u}}{dt}$ along the path of the light ray from the emission point to the observation point gives the change in the direction of the light ray’s path at the observation point relative to the emitted direction:

$$\hat{u}_{\text{out}} - \hat{u}_{\text{in}} = \int_{\text{source}}^{\text{observer}} \frac{d\hat{u}}{dt}(x^a(\ell)) \, d\ell. \quad (22)$$

We now assume that the function $\phi$ is vanishing outside of a region where it can accurately be described as the Newtonian potential of a mass density localized in a region that is small compared to the distance to the source and the observer. In such conditions, the light ray will suffer a total deviation that will be small compared with its original direction, and the change will take place only within a small range of values of $\ell$. In such conditions, the argument $x^a(\ell)$ of the integrand functions $\phi(x^a(\ell))$ and $V(x^a(\ell))$ can be taken as the original unperturbed path of the light ray, since it deviates from such an unperturbed path only in first-order quantities, but $\phi$ and $V$ are already small. In this context, then, the argument $x^a(\ell) = (t(\ell), x(\ell))$ can be taken as

$$x = t \hat{u}_{\text{in}} + x_{\text{in}},$$

(23)

$$t = \frac{\ell}{c} + t_{\text{in}},$$

(24)

where $\hat{u}_{\text{in}}$ is a constant unit vector, and $(t_{\text{in}}, x_{\text{in}})$ is an arbitrary reference point along the unperturbed path of the light ray. Typically, since the deflector has a small size and is moving slowly compared to the speed of light, it defines a localized neighbourhood where $(t_{\text{in}}, x_{\text{in}})$ may be thought to lie. In the case of static deflectors, $x_{\text{in}}$ would be taken as the impact parameter of the light ray with respect to a defined centre of the lens, whereas $t_{\text{in}}$ would denote the time of closest approach of the light ray to the deflector. In the general case, $(t_{\text{in}}, x_{\text{in}})$ should be thought of as an average space–time location of the interaction between the light ray and the deflector.

With these assumptions, we can define, as usual, the bending angle $\alpha$ as the difference between the initial and final directions of the light ray:

$$\alpha \equiv \hat{u}_{\text{in}} - \hat{u}_{\text{out}},$$

(25)

and we have an explicit formula

$$\alpha = 2 \int_{\text{source}}^{\text{observer}} \left[ \nabla \phi - 2 \hat{u}_{\text{in}} \wedge (\nabla \wedge V) + \frac{2}{c} \hat{V} \right] \, d\ell. \quad (26)$$

All ‘\hat{u}’s in the integrand, including those implicit in the label $\perp$, take the value $\hat{u}_{\text{in}}$, and the quantities $\phi$ and $V$ are evaluated at $x^a(\ell)$ as given by (23)–(24).

It is interesting to notice that the motion of the deflector may also affect the observed redshift of the source, as noted in Pyne & Birkinshaw (1993). The wavevector of a photon travelling along the null path is given by

$$k_a = (\omega, k) = \frac{dx^a}{ds}, \quad (27)$$

if $s$ is an affine parameter along the null geodesic. In our case, an affine parameter $s(\ell)$ must satisfy

$$\frac{ds^0}{d\ell} = \mu \frac{ds}{d\ell}, \quad (28)$$

where $\mu$ is small and is given by (19). Since

$$\frac{dx^0}{ds} = \frac{dx^0}{d\ell} / \frac{d\ell}{d\ell}$$

and since

$$\frac{dx^0}{d\ell} = u^0 = \frac{1}{c}$$

to zeroth order, then

$$\lambda \equiv \frac{2\pi}{\omega} = 2\pi c \frac{ds}{d\ell},$$

(29)

and equation (28) can be interpreted directly as an equation for the wavelength of the travelling photon:

$$\frac{d}{d\ell} \ln \lambda = \mu.$$ \hspace{1cm} (30)

Since $\mu$ is small, this integrates to

$$\frac{\lambda_0 - \lambda_x}{\lambda_x} \equiv \frac{\Delta \lambda}{\lambda} = -\frac{2}{c} \int_{\text{source}}^{\text{observer}} \phi - 2 \hat{u}_{\text{in}} \cdot \hat{V} \, d\ell. \quad (31)$$
3 APPLICATION TO A MOVING RIGID LENS

The situation that we wish to examine is that of a deflector that is moving without changing shape. We can obtain the metric of such a deflector by applying a coordinate transformation to the metric of the same deflector at rest. The metric of any deflector at rest in coordinates \((x', t')\) is entirely determined by its Newtonian potential \(\phi = \phi(x')\) in the form

\[
\text{d}x^2 = (1 + 2\phi) c^2 \text{d}t^2 - (1 - 2\phi) \text{d}x' \cdot \text{d}x'.
\]

(32)

To say that the deflector is moving rigidly along a path \(\gamma(t')\) is equivalent to making a change of coordinates \(x' \rightarrow x = x' + \gamma(t')\). Assuming that the acceleration of the path, \(\dot{\gamma}\), is of order quadratic in the velocity at least (namely, much smaller than the current order of approximation), we can accompany the shift in spatial positions with a change in the time coordinate \(t' \rightarrow t\) such that \(\text{d}t = \text{d}t' + \dot{\gamma}(t') \cdot \text{d}x' / c^2\). The metric takes the form (4) with

\[
\phi(x, t) = \phi_t(x - \gamma(t)),
\]

(33)

\[
V(x, t) = \frac{v}{c} \phi(x, t),
\]

(34)

where we are using the notation \(\gamma(t') = v\). [The reader should be aware that, if the acceleration is of linear order in \(v / c\), one can still ‘fix’ a transformation for the time coordinate, but in that case the metric will not take the form (4), and the calculation of the bending angle must be done afresh from the beginning.] Using these expressions we have

\[
\mathbf{u}_m \land (\nabla \land V) = \frac{\mathbf{\hat{u}}_m \cdot v}{c} \nabla \hat{\phi} - \mathbf{u}_m \cdot \nabla \frac{v}{c},
\]

(35)

and also

\[
\hat{V} = \frac{\mathbf{v}}{c},
\]

(36)

where contributions of the form \(\phi \dot{\gamma}(t)\) have been neglected. With (35) and (36), equation (21) becomes

\[
\frac{\text{d}\hat{u}}{\text{d}t} = -2 \left[ (1 - 2\mathbf{\hat{u}} \cdot v / c) \nabla \hat{\phi} + 2 \frac{v}{c} \frac{\text{d}\phi}{\text{d}t} \right],
\]

(37)

where \(\text{d}\phi / \text{d}t = \dot{\phi} / c + \mathbf{\hat{u}} \cdot \nabla \phi\). Integrating equation (37) yields the bending angle:

\[
\alpha = 2 \int_{\text{source}} \nabla \hat{\phi} \cdot \mathbf{x}' \, \text{d}t.
\]

(38)

In physically reasonable situations we expect the deflector to be both thin and weak, in the sense that the actual path of light \(\mathbf{\hat{u}}\) deviates at most in first order from \(\mathbf{\hat{u}}_m\) during the transit time through the deflector, which, under such conditions, is also small. During the transit time the velocities are approximately constant and can be pulled out of the integral sign. As a consequence, the second term on the right-hand side of equation (38) is proportional to the difference of the values of the potential \(\phi\) at the source and observer. Both values vanish, with the consequence that the second term does not contribute to the bending angle. The leading contribution to the bending angle (for small transverse accelerations, as assumed in the beginning) thus results from the first term on the right-hand side of equation (38):

\[
\alpha = 2(1 - 2\mathbf{\hat{u}}_m \cdot v / c) \int_{\text{source}} \nabla \hat{\phi} \cdot \mathbf{x}' \, \text{d}t.
\]

(39)

where \(\phi\) is a function of \(t\) through (33) where \(x\) and \(t\) are given by (23)–(24). For comparison, the bending angle of the same deflector at rest is

\[
\alpha_s = 2 \int_{\text{source}} \nabla \phi_s \cdot \mathbf{x}' \, \text{d}t.
\]

(40)

with \(x\) given by (23). The relationship between the two can be found, in this case, by defining a new integration variable for (39), which we here denote by \(q\):

\[
q = \ell - \mathbf{\hat{u}}_m \cdot \gamma(t' \ell).
\]

(41)

In terms of this variable we have

\[
x(t\ell) - \gamma(t' \ell) = q \mathbf{\hat{u}}_m + x_m - \gamma' \ell,
\]

(42)

where \(\gamma'_\ell\) is essentially a constant, as is \(x_m\). The differential form of equation (41) is

\[
\text{d}q = (1 - \mathbf{\hat{u}}_m \cdot v / c) \text{d}t,
\]

(43)

With this change of variable, equation (39) becomes

\[
\alpha = (1 - \mathbf{\hat{u}}_m \cdot v / c) \alpha_s.
\]

(44)

In terms of the Doppler shift of the deflector \(\Delta d\), since \(\mathbf{\hat{u}}_m\) points towards the observer, we have \(\mathbf{\hat{u}}_m \cdot v / c = -\Delta d\), so the bending angle by a moving deflector in terms of the same deflector at rest is

\[
\alpha = (1 + \Delta d) \alpha_s.
\]

(45)

This result is in agreement with Frittelli et al. (2002), where a significantly different method was used, and the deflector was assumed not to move across the line of sight. An important point to emphasize is the fact that the transverse motion of the deflector has no effect on the bending angle, as long as it is approximately uniform, as noted by Pyne & Birkinshaw (1993).

A consistency check for the validity of (45) has already been suggested by Pyne & Birkinshaw (1993): in the limit in which the thickness of the deflector vanishes, this equation verifies the aberration due to a Lorentz transformation between the frames in which the (absolutely thin) deflector is at rest or in motion with speed \(v\).

There is also a non-vanishing leading correction to the time delay that a light ray suffers in the presence of a deflector in motion. In order to calculate it, we set \(\Delta t^2 = 0\) and solve for \(\Delta t\) from equation (4). In the current order of approximation this yields

\[
\Delta t = \left[ \frac{1}{1 - 2 \phi + \frac{\mathbf{\hat{u}} \cdot V}{c}} \right] \text{d}t,
\]

(46)

which, in the particular case of the deflector in rigid motion, becomes

\[
\Delta t = \left[ \frac{1}{1 - 2 \phi - \frac{\mathbf{\hat{u}} \cdot v}{c}} \right] \text{d}t.
\]

(47)

Integrating along the path of the light ray, we have the total traveltime

\[
\Delta t = L - 2 \int_{\text{source}} \left[ \frac{1 - 2 \hat{u} \cdot v}{c} \right] \phi \text{d}t,
\]

(48)

where \(L\) is the Euclidean length of the path. The gravitational time delay is thus

\[
\Delta t^g = -2 \int_{\text{source}} \left[ 1 - 2 \mathbf{\hat{u}}_m \cdot v / c \right] \phi \text{d}t.
\]

(49)

We can evaluate the integral using the same methods as in the case of the bending angle, and the same approximations: the transit time is small, so the velocities can be pulled out of the integral sign, and the same change of variable helps put the result in terms of quantities associated with the same deflector at rest. We find

\[
\Delta t^g = (1 - \mathbf{\hat{u}}_m \cdot v / c) \int_{\text{source}} \text{d}q.
\]

(50)

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The quantity \( \int (-2\phi_t)\, dq \) is the gravitational time delay suffered by the light ray in the presence of the deflector at rest. So we have

\[
\Delta t^g = (1 - \hat{u}_m \cdot v/c) \Delta t^g.
\]

(51)

This formula is consistent with our bending angle because it implies the correct relationship between time delay and bending:

\[
\alpha = \nabla_\perp \Delta t^g.
\]

(52)

Finally, we can evaluate the redshift formula (31) in the case of interest. In the first place, notice that

\[
\phi = -v \cdot \nabla \phi = -v_\parallel \cdot \nabla \phi - \hat{u}_m \cdot v \hat{u}_m \cdot \nabla \phi.
\]

(53)

On the other hand,

\[
\phi = c(\partial \phi/\partial t - \hat{u}_m \cdot \nabla \phi_\parallel).
\]

From these two relationships, we obtain

\[
(1 - \hat{u}_m \cdot v/c)\hat{u}_m \cdot \nabla \phi = \frac{d\phi}{dt} + \frac{v_\parallel}{c} \cdot \nabla \phi_\parallel.
\]

(54)

and thus \( \hat{u}_m \cdot \nabla \phi_\parallel \) is equal to \( d\phi/dt \) up to zeroth order in \( v/c \). This fact can be used in (53) to obtain

\[
\phi = -v_\parallel \cdot \nabla \phi - \hat{u}_m \cdot \frac{d\phi}{dt}.
\]

(55)

Also, from (34) and (55) we have

\[
\ddot{V} = 0
\]

(56)

to leading order. Using (55) and (56) in equation (31) we have

\[
\frac{\Delta \lambda}{\lambda_0} = \frac{2}{c} \int_{\text{observer}}^{\text{source}} v_\perp \cdot \nabla \phi_\parallel + \hat{u}_m \cdot v \frac{d\phi}{dt} \, dt.
\]

(57)

The integrals can be evaluated using the same methods as in the case of the bending angle and time delay, and the same approximations. The reader will have no difficulty obtaining the following

\[
\frac{\Delta \lambda}{\lambda_0} = \frac{v_\perp}{c} \alpha_s.
\]

(58)

The source will thus appear redshifted due to the speed of the deflector across the line of sight. This agrees with Pyne & Birkinshaw (1993).

4 MICROLENSING EFFECTS

The observation of image separation by a lens that is moving along the line of sight will necessarily be affected by the motion of the lens. However, since the effect amounts to an overall factor, it may not be separable from the mass of the lens, usually not known with sufficient certainty. This entails an inherent error (which could be as large as 10 per cent) in the determination of the mass of the deflector on the basis of the separation of multiple images — in particular the size of an Einstein ring — independently of the accuracy of the observations.

Additionally, since the bending angle affects not only the location but also the brightness of the images, we can straightforwardly trace corrections to the magnification equation in the general case, and in particular to the light curves of microlensing events, in which case the images may not be observationally separated but their combined brightness is regularly observed as a function of time.

The theory of microlensing light curves as put forth by Paczyński (1986) calculates the combined magnification of the two images of a light source by a point mass:

\[
A = A_1 + A_2 = \frac{u^2 + 2}{u(a^2 + 4)^{1/2}}, \quad u \equiv \frac{r_0}{R_0}.
\]

(59)

where \( r_0 \) is the location of the source projected on the lens plane (namely, \( r_0 = \beta D_1 \) in the notation of Schneider et al. 1992), and

\[
R_0^2 \equiv \frac{4GM D_l D_s}{c^2 D_0} = r_0 \frac{D_l D_s}{D_0},
\]

(60)

where \( r \) is the location of the image on the lens plane \( (r = \theta D_1 \) in the notation of Schneider et al. 1992) and \( \alpha_s = 4GM/(c^2r) \) is the bending angle by a point mass at the origin of coordinates on the lens plane. The light curve \( \log \left( \frac{t}{t} \right) \) is then obtained by setting \( r_0 = (d^2 + v_\perp t^2)^{1/2} \) and letting the time flow, for a given impact parameter \( d \). This corresponds exactly to the case where the lens is at rest in the observer’s frame of reference and the source moves across the lens plane with a transverse velocity \( v_\perp \). The light curve is applied to the opposite case: that of a lens moving in the frame where both the source and observer are at rest, under the assumption that the motion of the lens does not affect the bending angle. What we have shown is that the application to this case is accurate if the lens is moving transversely, but there is a correction if the lens has also a component of the motion along the line of sight.

It is not complicated to calculate the correction to the light curve due to the longitudinal motion of the lens. This calculation amounts to the substitution \( R_0^2 \rightarrow (1 + v_\perp/c) R_0^2 \) in the light curve. A generic microlensing event will provide the time-scale \( t_0 = R_0/v_\parallel \) of the event, where \( R_0 \) will carry the factor \( 1 + v_\parallel/2c \). Consequently, unless the transverse velocity of the lens or its distance are known, its (corrected) mass cannot be determined from the time-scale \( t_0 \).

Statistical analysis cannot yield the deflector’s mass to a better certainty than a factor of 3 (page 454 of Paczyński 1996). Therefore, an entangled correction to the mass of the size of \( v_\parallel/c \) would not even seem relevant.

Nevertheless, following up on Paczyński (1996), there are at least two potential cases in which the masses of individual lenses can be determined unambiguously, because the distances and proper motions of the lenses can be measured independently of the microlensing event. The first case is that of stars in globular clusters acting as lenses for the stellar background of either the Galactic bulge, Large Magellanic Cloud (LMC) or Small Magellanic Cloud (SMC) (Paczyński 1994). The second case is that of nearby stars with high proper motions acting as lenses for the more distant stars in the Milky Way, LMC or SMC (Paczyński 1995). Our analysis in this section shows that, in both cases, in addition to the proper motion and distance of the lensing star, an unambiguous determination of the mass requires that the redshift of the lensing star be measured, in order for the correction factor of \( (1 + v_\parallel/c) \) to be determined. The mass could finally be obtained from the time-scale \( t_0 \) of the microlensing event. The relevance of the correction of order \( v_\parallel/c \) to the mass would depend on the accuracy of the measurements of \( t_0, v_\perp, v_\parallel, v_\parallel \) and the distance to the lensing star, all of which can be expected to improve steadily in the future.

5 CONCLUDING REMARKS

Our leading correction to the bending angle by a point mass differs from that obtained by Capozziello et al. (1999) by a factor of \( -2 \). Both calculations use very much the same method for the propagation of light and a metric perturbation of the same form (borrowed directly from Schneider et al. 1992). The calculations depart from each other in the parametrization of the null geodesics. We use the Euclidean length of the path instead of the affine length, which simplifies our calculations in great measure. Our equation for the propagation of the light ray in the general case, equation (21), differs from equation (26) in Capozziello et al. (1999) by the presence of
the time derivative of the potential $V$. However, because the time derivatives of the potentials do not contribute to the bending angle of a point mass if the acceleration is small, the extra term is negligible in the application of an almost uniformly moving point mass. The major departure appears to occur in the integration of this equation to yield the bending angle of the point mass.

On the other hand, our result for the point mass agrees with Pyne & Birkinshaw (1993), a work that seems to have been overlooked, perhaps because their method lies quite outside of the standard. In fact, Pyne & Birkinshaw use a metric perturbation of flat space based on a linearization of the Schwarzschild metric in Schwarzschild coordinates, instead of harmonic coordinates. Their metric perturbation representing a point mass contains off-diagonal terms, which are normally absent in the standard representation of the metric of a stationary mass distribution in the Newtonian limit, equation (32). Additionally, their calculation hinges strongly on the assumption that the Euclidean length is an affine parameter of the null geodesics of the perturbed space–time, an assumption that is unwarranted in the intended order of approximation since, as implied by equations (28) and (19), the affine length differs from the Euclidean length in terms that are linear in the strength of the Newtonian potential. Thus some question as to the validity of the Pyne & Birkinshaw calculation of the leading order correction in $v/c$ may have lingered, which we hope to have dispelled in our current work.

Our work, in fact, extends the Pyne & Birkinshaw (1993) result to extended lenses of any given deflection potential, being, thus, independent of the model. This agrees with the intuitive notion that the deflection caused by a distribution of point masses moving with the same speed should be equal to the sum of the deflections caused by the individual point masses in the distribution. However, our result in this extended case departs quite sharply from that derived by Capozziello & Re (2001), where the bending angle due to the moving lens is not proportional to the bending angle due to the same deflector at rest, but, on the contrary, it is quite complicated and model-dependent.

ACKNOWLEDGMENTS

I wish to thank Ted Newman for extremely valuable insights relevant to this calculation, and for the pleasure of many enlightening conversations. I am indebted to Olaf Wucknitz for pointing out an oversight in an earlier version of this work. This work was supported by NSF under grant No. PHY-0070624 to Duquesne University.

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