The classical master equation

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Abstract. We formalize the construction by Batalin and Vilkovisky of a solution of the classical master equation associated with a regular function on a nonsingular affine variety (the classical action). We introduce the notion of stable equivalence of solutions and prove that a solution exists and is unique up to stable equivalence. A consequence is that the associated BRST cohomology, with its structure of Poisson$_0$-algebra, is independent of choices and is uniquely determined up to unique isomorphism by the classical action. We give a geometric interpretation of the BRST cohomology sheaf in degree 0 and 1 as the cohomology of a Lie–Rinehart algebra associated with the critical locus of the classical action. Finally we consider the case of a quasi-projective varieties and show that the BRST sheaves defined on an open affine cover can be glued to a sheaf of differential Poisson$_0$-algebras.

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1. Introduction

Batalin and Vilkovisky [3] [4], in their study of generalized gauge symmetries in quantum field theory, proposed to associate to a function $S_0$, called the classical action, on the space of fields $\mathcal{X}$, taken here to be an affine variety, a solution $S$ of

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We show that, given the critical points of the notion of stable equivalence and that automorphisms act trivially on cohomology.

isomorphism of Poisson algebras, by the elements of degree \( \geq 1 \) and a Poisson bracket of degree 1. The BRST cohomology in degree 0 consists of regular functions on the critical locus of \( S_0 \) that are annihilated by the vector fields that annihilate \( S_0 \). The ultimate aim is to study (or make sense of) the asymptotic expansion of oscillatory integrals \( \int \exp(iS_0(x)/\hbar)f(x)dx \) as \( \hbar \to 0 \) in cases where the critical points of \( S_0 \) are not isolated, particularly in the infinite dimensional case. The classical master equation, considered here, appears in the study of the critical locus, which is preliminary to the study of the oscillatory integrals where the quantum master equation arises. We plan to extend our approach to the quantum case in a future publication.

The starting point of this paper is the remark that the Batalin–Vilkovisky construction depends on several choices, so that the invariant meaning of the BRST cohomology remained unclear. Our aim is to formalize the construction by introducing the notion of \( BV \) variety associated with a regular function \( S_0 \) on a nonsingular affine variety, comprising a solution \( S \) of the classical master equation. We show that, given \( S_0 \), such a \( BV \) variety exists and is unique up to a natural notion of stable equivalence and that automorphisms act trivially on cohomology. A consequence is that the BRST cohomology is uniquely determined, up to unique isomorphism of Poisson algebras, by \( S_0 \).

Let us describe the result in more detail. Let \( k \) be a field of characteristic 0 and \( X \) a nonsingular affine variety. We use the language of \( \mathbb{Z} \)-graded varieties. A \( \mathbb{Z} \)-graded variety with support \( X \) is a \( \mathbb{Z} \)-graded commutative ringed space \( V = (X, \mathcal{O}_V) \) with structure sheaf \( \mathcal{O}_V = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_V^i \) locally isomorphic to the completed symmetric algebra of a free graded \( \mathcal{O}_X \)-module with homogeneous components of finite rank. The completion is defined by the filtration \( F^p \mathcal{O}_V \) of ideals generated by the elements of degree \( \geq p \). The \((-1)\)-shifted cotangent bundle of a \( \mathbb{Z} \)-graded variety \( V \) such that \( \mathcal{O}_V^i = 0 \) for \( i < 0 \) is \( M = T^*[1]V = (X, \mathcal{O}_M) \) with \( \mathcal{O}_M = S\text{ym}_{\mathcal{O}_X} T_V[1] \), the completed graded symmetric algebra of the tangent sheaf of \( V \), with degree shift \( T_V[1]^i = T_V^{i+1} \). Then \( \mathcal{O}_M \) is, in the terminology of [14], a sheaf of Poisson or \( P_0 \)-algebras; namely, it comes with a graded commutative product of degree zero and a Poisson bracket of degree 1. The Poisson bracket comes from the canonical symplectic structure of degree \(-1\) on \( T^*[1]V \). More generally we define a \((-1)\)-symplectic variety with support \( X \) to be a graded manifold \( (X, \mathcal{O}_M) \) such that \( \mathcal{O}_M \) is locally isomorphic as a \( P_0 \)-algebra to a shifted cotangent bundle. The classical master equation for a function \( S \in \Gamma(X, \mathcal{O}_M) \) of degree 0 on a \((-1)\)-symplectic manifold \( M \) is \( [S, S] = 0 \). A solution \( S \) defines a differential \( d_S = [S, \_] \) on the sheaf of \( P_0 \)-algebras \( \mathcal{O}_M \). Let \( I_M = F^1 \mathcal{O}_M \) be the ideal of \( \mathcal{O}_M \) generated by elements of positive degree. Then \( d_S \) preserves \( I_M \) and induces a differential on the non-positively graded complex of sheaves \( \mathcal{O}_M/I_M \).

\footnote{In general a \( P_j \)-algebra is a graded commutative algebra with a Poisson bracket of degree \( 1 - j \).}
Definition 1.1. Let \( S_0 \in \Gamma(X, \mathcal{O}_X) \) be a regular function on \( X \in \mathbb{C} \). A BV variety with support \((X, S_0)\) is a pair \((M, S)\) consisting of a \((-1)\)-symplectic variety \( M \) with support \( X \) and a function \( S \in \Gamma(X, \mathcal{O}_M^0) \) such that

(i) \( S|_X = S_0 \).
(ii) \( S \) is a solution of the classical master equation \([S, S] = 0\).
(iii) The cohomology sheaf of the complex \((\mathcal{O}_M/I_M, d_S)\) vanishes in non-zero degree.

The complex of sheaves \((\mathcal{O}_M, d_S)\) is called BRST complex and its cohomology is called BRST cohomology (after Becchi, Rouet, Stora and Tyutin, who introduced it in the case of ordinary gauge theory [5]). The BRST complex is a sheaf of differential \( P_0 \)-algebras, namely a sheaf of \( P_0 \)-algebras with a differential that is a derivation for both the product of and the bracket, so that the BRST cohomology is a \( P_0 \)-algebra.

Before stating our results we now add a few comments on the origin and meaning of our axiomatic setting. We refer to [18] for the physical background of this construction and to [31] for its mathematical context. The origin of the story is in the Faddeev–Popov description [16] of path integrals over the quotient of the space of fields by the action of the gauge group. With the work of Becchi, Rouet, Stora and Tyutin, see [5], who identified gauge invariant observables as cocycles in a differential graded algebra, the BRST complex, and of Zinn-Justin [36] who introduced a version of the master equation, it became clear that \(-1\)-symplectic manifolds and the master equation are the structure underlying perturbative gauge theory and renormalization, see [13] for a mathematical approach to this subject.

In our finite-dimensional classical setting the solutions of the master equation covered by the approach of Faddeev and Popov arise in the case of a classical action \( S_0 \) invariant under the action of a connected Lie group, see Example 6.3 in Section 6. To these data one associates a solution (6.1) of the master equation, the Faddeev–Popov action. This solution obeys (i) and (ii) but in general not (iii). Condition (iii) is satisfied under additional conditions of freeness of the action, see Section 6, Example 6.3. In the physics literature it was noticed that the Faddeev–Popov construction had to be extended if the gauge group does not act freely, or in more general situations in which one would like to quotient by symmetries of the classical actions that are not described by a group action. For example, in certain quantum field theories, such as the Poisson sigma model underlying Kontsevich’s deformation quantization [21, 11], the classical action \( S_0 \) is invariant under (i.e., annihilated by) a distribution of tangent planes which is integrable only when restricted to the critical locus of \( S_0 \). A finite-dimensional model for this phenomenon is given by Example 6.7 in Section 6, where \( S_0 \) is the square of the norm on a Euclidean vector bundle with orthogonal connection. The contribution of Batalin and Vilkovisky was to introduce a general construction of solutions of the master equation replacing previous ad hoc attempts to generalize the Faddeev–Popov solution. Their idea was to start from a classical action without assuming a priori the existence of a group of symmetries. Our observation is that the Batalin–Vilkovisky approach amounts to add axiom (iii) to the wish list for solutions of the master equation. From the point of view of this paper, axiom (iii) is important as it implies existence and uniqueness results: the existence and uniqueness up to stable equivalence of a solution.
obeying (i)–(iii) given $S_0$ and the existence and uniqueness of the corresponding differential $P_0$-algebra of observables (the BRST complex) up to a contractible space of isomorphisms, see Prop. 8.1 in Section 8.

Finally let us remark that there are other interesting solutions of the master equation, that do not necessarily obey property (iii) of the definition of BV varieties. They are effective actions obtained by reduction from solutions in infinite dimensional spaces of fields of a quantum field theory, see [23, 5, 7, 12].

We now turn to the description of our results.

The main local result of this paper is that a BV variety $(M, S)$ with given support $(X, S_0)$ such that $X$ is affine exists and is essentially unique. Two BV varieties $(M_1, S_1)$, $(M_2, S_2)$ with support $(X, S_0)$ are called equivalent if there is a Poisson isomorphism $M_1 \to M_2$ inducing the identity on $X$ whose pull-back sends $S_2$ is $S_1$. They are stably equivalent if they become equivalent after taking the product with solutions of BV varieties with support $(X = \{ pt \}, S_0 = 0)$, see Section 3.6 for the precise definition. The main property is that stably equivalent solutions give rise to BRST complexes that are quasi-isomorphic as sheaves of differential $P_0$-algebras.

**Theorem 1.2.** Let $S_0$ be a regular function on a nonsingular affine variety $X$ over a field $k$ of characteristic zero.

1. There exists a BV variety $(M, S)$ with support $(X, S_0)$ such that $M \cong T^*[−1]V$ for some non-negatively graded variety $V$. It is unique up to stable equivalence.

2. Poisson automorphisms of $(M, S)$ act as the identity on the cohomology of the BRST complex. Thus the BRST cohomology $\mathcal{H}^*(O_M, d_S)$ is determined by $(X, S_0)$ up to unique isomorphism.

The existence proof is based on the construction described in [3, 4] and is in two steps. In the first step one extends the map $dS_0: T_X \to O_X$, sending a vector field $\xi$ to $\xi(S_0)$, to a semi-free resolution of the Jacobian ring, namely a quasi-isomorphism $(R, \delta) \to (J(S_0), 0)$ of differential graded commutative $O_X$-algebras, where $R$ is the symmetric algebra of a negatively graded locally free $O_X$-module with homogeneous components of finite rank. The existence of such resolutions is due to Tate [33] and $R$ is called Tate (or Koszul–Tate) resolution. Geometrically $\delta$ is a cohomological vector field on the coisotropic subvariety of a shifted cotangent bundle $M = T^*[−1]V$ determined by the ideal $I_M$. In the second step one extends this vector field to a Hamiltonian cohomological vector field $[S, ]$ on $T^*[−1]V$. This existence proof is basically adapted from [18], Chapter 17, but we avoid using the “regularity condition” on the smoothness of the critical locus assumed there.

To show uniqueness up to stable equivalence we remark that all BV varieties with given support are isomorphic to BV varieties obtained from some Tate resolution and the question reduces to comparing different Tate resolutions. It is a standard result that different Tate resolutions of the same algebra are related by a quasi-isomorphism that is unique up to homotopy. We prove in the Appendix the stronger result that any two such resolutions become isomorphic as differential graded commutative algebras after taking the tensor product with the symmetric algebra of an acyclic complex.

The existence part of Theorem 1.2 (i) is proved in Section 4.2 (Theorem 4.5); the uniqueness up to stable equivalence is Theorem 4.10 in Section 4.4. Part (ii) is proved in Section 4.6 (Theorem 4.13 and Corollary 4.15).
The next result is a partial description of the cohomology of the BRST complex. The cokernel of the map $dS_0: T_X \to \mathcal{O}_X$ is the Jacobian ring, the quotient of $\mathcal{O}_X$ by the ideal generated by partial derivatives of $S_0$. Vector fields in the kernel $L(S_0)$ of $dS_0$ are infinitesimal symmetries of $S_0$. They form a sheaf of Lie subalgebras of $T_X$.

**Theorem 1.3.**

(i) The cohomology sheaf of the BRST complex is supported on the critical locus of $S_0$ and vanishes in negative degree.

(ii) The zeroth BRST cohomology algebra is isomorphic to the algebra of invariants

$$J(S_0)^{L(S_0)} = \{ f \in J(S_0) | \xi(f) = 0, \forall \xi \in L(S_0) \},$$

of the Jacobian ring for the Lie algebra of infinitesimal symmetries of $S_0$.

The computation of the BRST cohomology is based on the spectral sequence associated with the filtration $F^\bullet \mathcal{O}_M$ and is presented in Section 5. Theorem 1.3 follows from the description of the $E_2$-term in Theorem 5.1, see Corollary 5.4 and Proposition 5.5.

Can one describe the BRST cohomology in terms of the geometry of the critical locus? We give a conjectural description of this kind, which we prove in degree 0 and 1: the BRST cohomology for an affine variety is isomorphic to the cohomology of a Lie–Rinehart algebra naturally associated to the critical locus, see Section 7. One encouraging fact is that the bracket $H^0 \otimes H^0 \to H^1$ induced by the Poisson bracket has a very natural geometric description in terms of this Lie–Rinehart algebra.

Theorem 1.3 refers to affine varieties and it is natural to ask whether affine BV varieties glue well to build global objects defined on general nonsingular varieties. We have a partial existence result in this direction. We show in Corollary 8.3 that if $S_0$ is a (possibly multivalued) function on a quasi-projective variety, then there is a sheaf of differential $P_0$-algebras which is locally quasi-isomorphic to the BRST complex of a BV variety associated to $S_0$. The necessary homotopy gluing technique is explained in Appendix B by Tomer Schlank.

Apart from the extension of our results to the quantum case, namely the theory of the quantum master equation and Batalin–Vilkovisky integration, see [3, 18, 28, 20, 2, 29, 1], it is important to study the higher dimensional case of local functionals in field theory, see [3, 18, 14, 25]. It would also be interesting to compare our approach to the derived geometry approach of [14], developed in [35, 24], and consider, as these authors do, the more general situation of an intersection of Lagrangian submanifolds in a symplectic manifold (the case studied in this paper is the intersection of the zero section with the graph of $dS_0$ in the cotangent bundle).

In most of the paper we formulate our results for a nonsingular affine variety $X$ over a field $k$ of characteristic zero for consistency of language, but our results hold also, with the same proofs, for smooth manifolds (with $k = \mathbb{R}$) or complex Stein manifolds (with $k = \mathbb{C}$).

Another straightforward generalization to which our results apply with the same proofs is the case where $S_0$ is a multivalued function defined modulo constants (alias a closed one-form). By this we mean a formal indefinite integral $S_0 = \int \lambda$, where $\lambda$ is a closed 1-form. The point is that it is not $S_0$ that matters but the differential $[S_0, ]$, which depends on $S_0$ through $dS_0$, see [28] for a more formal treatment.
The paper is organized as follows. In Section 2 we introduce a notion of graded variety suitable for our problem. It is patterned on Manin’s definition of supermanifolds and supervarieties [22]. Shifted cotangent bundles are also introduced there. BV varieties and their cohomology are introduced in Section 3. In Section 4 we prove the existence and uniqueness result for BV varieties, and the computation of the BRST cohomology is contained in Section 5. We then discuss several examples in Section 6 and in Section 7 we give a geometric description of the cohomology in degree 0 and 1 and of the induced bracket $H^0 \otimes H^0 \to H^1$. We conclude our paper with Section 8 where we extend our existence result to the case of quasi-projective varieties.

Conventions. We work over a field $k$ of characteristic zero. The homogeneous component of degree $i$ of a graded object $E$ is denoted by $E^i$ and $E[i]$ is $E$ with degrees shifted by $i$: $E[i]^j = E^{i+j}$. Differentials have degree 1. To avoid conflicts of notation we denote by $I^{(j)}(j)$ the $j$-th power $I \cdots I$ of an ideal $I$.

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2. Graded varieties

2.1. Symmetric algebras of graded modules. Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a $\mathbb{Z}$-graded module over a commutative unital ring $B$ with free homogeneous components $V^i$ of finite rank such that $V^0 = 0$. If $a \in V^i$ is homogeneous, we set $\deg a = i$. The symmetric algebra $\text{Sym}(V) = \text{Sym}_B(V)$ is the quotient of the tensor algebra of $V$ by the relations $ab = (-1)^{\deg a \deg b}ba$. It is a graded commutative algebra with grading induced by the grading in $V$. Let $F^p \text{Sym}(V)$ be the ideal generated by elements of degree $\geq p$. These ideal form a descending filtration $\text{Sym}(V) \supset F^1 \text{Sym}(V) \supset F^2 \text{Sym}(V) \supset \cdots$.

**Definition 2.1.** The completion $\widehat{\text{Sym}}(V)$ of the graded algebra $\text{Sym}(V)$ is the inverse limit of $\text{Sym}(V)/F^p \text{Sym}(V)$ in the category of graded modules. Namely, $\widehat{\text{Sym}}(V) = \bigoplus_{i \in \mathbb{Z}} \text{Sym}(V)^i$ with $\widehat{\text{Sym}}(V)^i = \varprojlim_{\leftarrow} \text{Sym}(V)^i/(F^p \text{Sym}(V) \cap \text{Sym}(V)^i)$.

Then $\widehat{\text{Sym}}(V)$ is a graded commutative algebra and comes with the induced filtration $F^p \widehat{\text{Sym}}(V)$. Note that the completion has no effect if $V$ is $\mathbb{Z}_{\geq 0}$-graded or $\mathbb{Z}_{\leq 0}$-graded, namely if $V^i = 0$ for all $i < 0$ or for all $i > 0$.

**Remark 2.2.** The assumption that $V^0 = 0$ is not essential. It can be achieved by replacing $B$ by $\text{Sym}_B(V^0)$.
2.2. Graded manifolds and graded algebraic varieties. We adapt the constructions and definitions of Manin [22], who introduced a general notion of \( \mathbb{Z}/2\mathbb{Z}\)-graded spaces (or superspaces), to the \( \mathbb{Z}\)-graded case.

**Definition 2.3.** Let \( M_0 \) be a topological space. A **graded space** with support\(^2\) \( M_0 \) is a ringed space \( M = (M_0, \mathcal{O}_M) \) where \( \mathcal{O}_M \) (the **structure sheaf** of \( M \)) is a sheaf of \( \mathbb{Z}\)-graded commutative rings on \( M_0 \) such that the stalk \( \mathcal{O}_{M,x} \) at every \( x \in M_0 \) is a local graded ring (namely it has a unique maximal proper graded ideal). Morphisms are morphisms of locally ringed spaces: a morphism \( M \to N \) is a pair \((f, f^*)\) where \( f : M_0 \to N_0 \) is a homeomorphism and \( f^* : \mathcal{O}_N \to f_* \mathcal{O}_M \) is a grading preserving morphism of sheaves of rings, such that, for all \( x \in M_0 \), \( f^* \) maps the maximal ideal of \( \mathcal{O}_{N,f(x)} \) to the maximal ideal of \( \mathcal{O}_{M,x} \).

**Definition 2.4.** An **open subspace** of a graded space \( M = (M_0, \mathcal{O}_M) \) is a graded space of the form \((U_0, \mathcal{O}_{M|U_0})\) for some open subset \( U_0 \subset M_0 \). A **closed subspace** is a graded space of the form \((N_0, (\mathcal{O}_M/J)|_{N_0})\) for some sheaf of ideals \( J \subset \mathcal{O}_M \) such that \( \mathcal{O}_M/J \) has support on \( N_0 \).

Both open and closed subspaces come with inclusion morphisms to \( M \).

**Definition 2.5.** Let \( M \) be a graded variety. Let \( J_M \) be the ideal sheaf of \( \mathcal{O}_M \) generated by sections of non-zero degree. The **reduced space** is the locally ringed space \( M_{rd} = (M_0, \mathcal{O}_M/J_M) \).

Thus \( M_{rd} \) is a closed subspace of \( M \) and it is a locally ringed space in the classical sense, with strictly commutative structure sheaf sitting in degree 0.

An important class of graded spaces is obtained from graded locally free sheaves. Let \((X, \mathcal{O}_X)\) be a commutative locally ringed space (no grading). If \( \mathcal{E} \) is a graded module with homogeneous components \( \mathcal{E}^i \) of finite rank and \( \mathcal{E}^0 = 0 \), then \( U \mapsto \bigoplus \mathcal{O}_X(U)(\mathcal{E}(U)) \) is a sheaf of rings on \( X \) whose stalks are local rings. Thus \( M = (X, \bigoplus \mathcal{O}_X(\mathcal{E})) \) is a graded space with \( M_{rd} = (X, \mathcal{O}_X) \).

**Definition 2.6.** Let \( \mathcal{C} \) be a subcategory of the category of locally ringed space, such as algebraic varieties, smooth manifolds or complex manifolds. A graded \( \mathcal{C}\)-variety with support \( M_0 \in \mathcal{C} \) is a graded space \( M = (M_0, \mathcal{O}_M) \) such that every point \( x \in M_0 \) has an open neighborhood \( U \) such that \( (U, \mathcal{O}_M|_U) \) is isomorphic to \((U, \bigoplus \mathcal{O}_X(\mathcal{E}))\) for some free graded \( \mathcal{O}_X\)-module \( \mathcal{E} \) with homogeneous components of finite rank and \( \mathcal{E}^0 = 0 \). Morphisms are morphisms of locally ringed spaces restricting to morphisms in \( \mathcal{C} \) on their supports, namely such that there is a commutative diagram

\[
\begin{array}{ccc}
M & \to & N \\
\uparrow & & \uparrow \\
M_0 & \to & N_0
\end{array}
\]

with lower arrow in \( \mathcal{C} \).

Depending on \( \mathcal{C} \), we call graded \( \mathcal{C}\)-varieties graded smooth manifolds, graded algebraic varieties, graded affine varieties, and so on.

**Definition 2.7.** A graded \( \mathcal{C}\)-variety \( M \) is called \( \mathbb{Z}_{\geq 0}\)-graded (\( \mathbb{Z}_{\leq 0}\)-graded) if \( \mathcal{O}_M^j = 0 \) for \( j < 0 \) (\( j > 0 \)).

\(^2\)or body; we use the terminology of [8]
Examples of graded $C$-varieties are $(X, \widehat{\text{Sym}}_{\mathcal{O}_X}(E))$ for some locally free graded \(\mathcal{O}_X\)-module $\mathcal{E}$ with finite rank homogeneous components such that $\mathcal{E}^0 = 0$. Conversely, if $(X, \mathcal{O}_M)$ is a graded $C$-variety with support $(X, \mathcal{O}_X = \mathcal{O}_M/J_M) \in \mathcal{C}$, then $\mathcal{E} = J_M/J^2_M$ is a locally free $\mathcal{O}_X$ module and $(X, \mathcal{O}_M)$ is locally isomorphic to $(X, \widehat{\text{Sym}}_{\mathcal{O}_X}(E))$. The obstructions to patch local isomorphisms to a global isomorphism lie in $H^1$ of a certain vector bundle on $X$. Thus if $X$ is a smooth manifold or an affine algebraic variety, then the obstruction vanish and Batchelor’s Theorem holds: every graded variety with support $X$ is isomorphic to $(X, \widehat{\text{Sym}}_{\mathcal{O}_X}(E))$ for some locally free $\mathcal{E}$ with homogeneous components of finite rank.

### 2.3. $P_0$-algebras

A $P_0$-algebra over a field $k$ is a graded commutative algebra $A = \oplus_{d \in \mathbb{Z}} A^d$ over $k$ with a Poisson bracket $[\ , ] : A \otimes A \to A$ of degree 1. A differential $P_0$-algebra is a $P_0$-algebra together with a differential of degree 1 which is a derivation for both the product and the bracket. For completeness (and to fix sign conventions) let us write out the axioms for the bracket $[\ , ]$ and the differential $d$. For any homogeneous elements $a, b, c$,

(i) The bracket is a bilinear map sending $A^r \otimes A^s$ to $A^{r+s+1}$

(ii) $[a, b] = -(-1)^{(\deg a - 1)(\deg b - 1)}[b, a]$,

(iii) $[ab, c] = a[b, c] + (-1)^{\deg a \deg b}[a, c]$,

(iv) $d(ab) = (da)b + (-1)^{\deg a}a \, db$,

(v) $d(a, b) = [da, b] + (-1)^{\deg a - 1}[a, db]$,

(vi) $(-1)^{(\deg a - 1)(\deg c - 1)}[a, b, c] + \text{cyclic permutations} = 0$.

If $S \in A^0$ obeys the classical master equation $[S, S] = 0$ then the map $d_S : b \to [S, b]$ is a differential. We call $d_S$ a *hamiltonian differential* with *hamiltonian* $S$.

### 2.4. Shifted cotangent bundles and $(−1)$-symplectic varieties

We consider $C$-varieties, where $\mathcal{C}$ is the category of nonsingular algebraic varieties over a field $k$ of characteristic zero or of smooth manifolds (with $k = \mathbb{R}$) or of complex manifolds (with $k = \mathbb{C}$). Let $V$ be such a graded variety with support $X$ and suppose that $O_X^{i+1} = 0$ for all $i < 0 \ (V \text{ is } \mathbb{Z}_{\geq 0}\text{-graded})$. To $V$ we associate its shifted cotangent bundle $M = T^*[−1]V$ by the following construction. A (left) derivation of $\mathcal{O}_V$ of degree $d$ is a section $\xi$ of the sheaf $\Pi_{j=0}^{\infty} \text{Hom}(\mathcal{O}_V, \mathcal{O}_V^{j+d})$ of degree $d$ endomorphisms of $\mathcal{O}_V$ such that $\xi(ab) = \xi(a)b + (-1)^{d \deg a}a \xi(b)$. Derivations of $\mathcal{O}_V$ of degree $d$ form a sheaf $T^d_V$ and $T^d_V = \oplus_{d} T^d$ is a sheaf of graded Lie algebras acting on $\mathcal{O}_V$ by derivation. Then the bracket extends to a Poisson bracket of degree 1 on

$$\mathcal{O}_M = \text{Sym}_{\mathcal{O}_V} T^*_V[1],$$

which thus becomes a sheaf of $P_0$-algebras. We will need a completion $\mathcal{O}_M$ of $\mathcal{O}_M$. Let $F^p \mathcal{O}_M$ be the ideal in $\text{Sym}_{\mathcal{O}_V}(T^*_V[1])$ generated by elements of degree at least $p$. These sheaves of ideals form a descending filtration of $\mathcal{O}_M = F^0 \mathcal{O}_M$.

**Definition 2.8.** The $(-1)$-shifted cotangent bundle of $V$ is the graded variety $M = T^*[−1]V = (X, \mathcal{O}_M)$, where

$$\mathcal{O}_M = \lim_{\longleftarrow} F^p \mathcal{O}_M / F^{p+1} \mathcal{O}_M.$$

The inverse limit is taken in the category of $\mathbb{Z}$-graded sheaves, i.e., degree by degree.

We denote by $F^p \mathcal{O}_M = \lim_{\longleftarrow} F^p \mathcal{O}_M / F^{p+q} \mathcal{O}_M$ the filtration by ideals topologically generated by elements of degree $\geq p$ in $\mathcal{O}_M$. 


Let $V$ be a $\mathbb{Z}_{>0}$-graded variety and $M = T^*[-1]V$. Then the Poisson bracket on $\tilde{\mathcal{O}}_M$ extends to the completion making $\mathcal{O}_M$ a sheaf of $P_0$-algebras over $k$.

Let $\tilde{\mathcal{O}}_M = \text{Sym}_{\mathcal{O}_X} T_V[1]$. The product on $\tilde{\mathcal{O}}_M$ is compatible with the filtration in the sense that $F^p\tilde{\mathcal{O}}_M \cdot F^q\tilde{\mathcal{O}}_M \subset F^{p+q}\tilde{\mathcal{O}}_M$, and thus passes to the completion $\mathcal{O}_M$ but this is not true for the bracket, making the statement not completely trivial. However, we have the following observation, which suffices to show that the bracket is defined on the completion.

**Lemma 2.10.** Let $p \geq 0$.

(i) If $d \geq -1$ then $[\tilde{\mathcal{O}}^d_M, F^p\tilde{\mathcal{O}}_M] \subset F^p\tilde{\mathcal{O}}_M$.

(ii) If $d < -1$ and $p + d \geq 0$ then $[\tilde{\mathcal{O}}^d_M, F^p\tilde{\mathcal{O}}_M] \subset F^{p+d+1}\tilde{\mathcal{O}}_M$.

Thus the bracket passes to the completion $\mathcal{O}_M = \oplus_i \lim_{\to} \tilde{\mathcal{O}}^i_M/F^i\tilde{\mathcal{O}}_M$ and (i), (ii) hold for $\tilde{\mathcal{O}}_M$ replaced by $\mathcal{O}_M$.

**Proof.** Let us adopt the convention that $a_j, b_j, \ldots$ denote local sections of $\tilde{\mathcal{O}}_M$ of degree $j$ and $a, b, \ldots$ sections of unspecified degree. Then, if $d + p + 1 \geq 0$ (which is trivially true in (i)), and $p' \geq p$,

\[ [a_d, b_{p'}] = [a_d, b]c_{p'} \pm [a_d, c_{p'}]b \in F^p\tilde{\mathcal{O}}_M + F^{d+p+1}\tilde{\mathcal{O}}_M \subset F^{\min(p,d+p+1)}\tilde{\mathcal{O}}_M, \]

since $c_{p'}$ has degree $\geq p$ and $[a_d, c_{p'}]$ has degree $d + p' + 1 \geq d + p + 1$. \hfill \Box

We note that with the same construction we can define $T^*[-1]N$ for a $\mathbb{Z}_{\leq 0}$-graded variety $N$.

**Definition 2.11.** A Poisson morphism $T^*[-1]V \to T^*[-1]W$ is a map of graded varieties respecting the Poisson bracket.

An étale morphism $\varphi: V \to W$ of graded varieties (namely one which is étale on supports and for which $\varphi^*$ is locally invertible) induces a morphism $T_V \to T_W$ of sheaves of graded Lie algebras, defined by $\theta \mapsto (\varphi^*)^{-1} \circ \theta \circ \varphi^*$, and thus induces a Poisson isomorphism $\Phi: T^*[-1]V \to T^*[-1]W$, called the symplectic lift of $\varphi$.

**Definition 2.12.** A $(1)$-symplectic variety is a graded variety $M = (X, \mathcal{O}_M)$ locally Poisson isomorphic to a $(1)$-shifted cotangent bundle: every point $x \in X$ has an open neighborhood $U$ so that $\mathcal{O}_M|_U$ is Poisson isomorphic to $\mathcal{O}_{T^*[-1]V}$ for some non-negatively graded variety $V = (U, \mathcal{O}_V)$.

The structure sheaf $\mathcal{O}_M$ of a $(1)$-symplectic variety comes with a filtration $\mathcal{O}_M = F^0\mathcal{O}_M \supset \cdots \supset F^p\mathcal{O}_M \supset F^{p+1}\mathcal{O}_M \supset \cdots$. This filtration can be used to uniquely reconstruct $V$ from $M$ up to isomorphism:

**Proposition 2.13.** If $M = (X, \mathcal{O}_M)$ is Poisson isomorphic to $T^*[-1]V$ for a $\mathbb{Z}_{\geq 0}$-graded variety $V$ then $V$ is isomorphic to $(X, \text{Sym}_{\mathcal{O}_X} \mathcal{E})$, where the graded $\mathcal{O}_X$-module $\mathcal{E}$ has homogeneous components

\[ \mathcal{E}^p = \mathcal{O}_M^p/F^{p+1}\mathcal{O}_M^p \oplus I_M \cdot I_M^\perp \oplus \mathcal{O}_M^p, \quad I_M = F^1\mathcal{O}_M, \quad p \geq 1. \]

**Proof.** Suppose $\mathcal{O}_V = (X, \text{Sym}_{\mathcal{O}_X} \hat{\mathcal{E}})$. Then we have a monomorphism $\hat{\mathcal{E}} \hookrightarrow \mathcal{O}_V \hookrightarrow \mathcal{O}_M \rightarrow \mathcal{E}$.
But $\mathcal{O}_M^p$ is spanned over $\mathcal{O}_X$ by the image of $\mathcal{E}^p$ and products of sections of non-zero degree. Products with at least two factors of positive degree are in $I_M \cdot I_M$ and products with at least a factor of negative degree have a factor of degree $\geq p$ and lie therefore in $F^{p+1} \mathcal{O}_M$. 

2.5. Local description. Let $X$ be an $n$-dimensional nonsingular algebraic variety over a field $k$ of characteristic zero. Then every point $p \in X$ has an affine open neighborhood $U$ with an étale map $U \rightarrow \mathbb{A}^n$ to the affine $n$-space. Thus there are functions $x^1, \ldots, x^n \in \mathcal{O}_X(U)$ generating the maximal ideal at $p$ and commuting vector fields $\partial_1, \ldots, \partial_n \in T_X(U)$ with $\partial_i x^j = \delta_{ij}$. Similarly if $V$ is a graded variety with support $i$, then $U$ can be chosen so that $\mathcal{O}_V|_U \cong \text{Sym}_{\mathcal{O}_X} \mathcal{E}(U)$ where $\mathcal{E}$ is a free $\mathcal{O}_X$-module with homogeneous components $\mathcal{E}^i$ of finite rank and $\mathcal{E}^0 = 0$. Let us assume that $\mathcal{O}_V|_U = 0$ for $i < 0$ (the case $i > 0$ is treated similarly). Then there are sections $\beta^i, \beta^j, \ldots$ of $\mathcal{E}(U)$, such that, for all $i > 0$, those of degree $i$ are a basis of the free $\mathcal{O}_X(U)$-module $\mathcal{E}^i(U)$. Let $\beta^+_i$ be the dual basis of $\mathcal{E}^*[1]$, so that

$$\deg \beta^+_i = -\deg \beta^i - 1 \leq -2.$$ 

Then $\mathcal{O}_V^{[-1]}(U) \cong \text{Sym}(T_X[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1])(U)$. Its homogeneous component of degree $j$ consists of formal power series with coefficients in $\mathcal{O}_X(U)$ whose terms are monomials of degree $j$ in generators $x^*_i = -\partial_i \in T_X[1]$, $i = 1, \ldots, n$ of degree -1 and $\beta^i, \beta^+_i$, $j = 1, 2, \ldots$. The Poisson bracket is

$$[f, x^*_i] = \partial_i f, \quad f \in \mathcal{O}_X, \quad [\beta^i, \beta^+_j] = \delta_{ij},$$

and vanishes on other pairs of generators.

2.6. Associated graded. It will be useful to have a description of the associated graded of the structure sheaf $\mathcal{O}_M$ of the cotangent bundle $M = T^*[1]V$ of a $\mathbb{Z}_{\geq 0}$-graded variety $V$. Let as above $F^p \mathcal{O}_M$ be the ideal generated by sections of degree $\geq p$ and $\text{gr} \mathcal{O}_M = \oplus_{p \geq 0} F^p \mathcal{O}_M / F^{p+1} \mathcal{O}_M$. Then $R_M = \mathcal{O}_M / F^1 \mathcal{O}_M$ is an algebra over $\mathcal{O}_X = \mathcal{O}_V / F^1 \mathcal{O}_V$ and for each $p$, $\text{gr}^p \mathcal{O}_M$ is naturally an $R_M$-module. Also there is a natural $\mathcal{O}_X$-linear map $\iota: \mathcal{O}_V \rightarrow \text{gr} \mathcal{O}_M$ induced from the inclusion $\mathcal{O}_V = F^0 \mathcal{O}_M$.

**Lemma 2.14.** The composition of $\iota$ with the structure map of the $R_M$-module $R_M \otimes \mathcal{O}_V \rightarrow R_M \otimes \text{gr} \mathcal{O}_M \rightarrow \text{gr} \mathcal{O}_M$

factors through $R_M \otimes_{\mathcal{O}_X} \mathcal{O}_V$ and induces an isomorphism of graded algebras

$$R_M \otimes_{\mathcal{O}_X} \mathcal{O}_V \cong \text{gr} \mathcal{O}_M.$$

**Proof.** Both the map $\iota$ and the action of $R_M$ are $\mathcal{O}_X$-linear so the map factors through the tensor product over $\mathcal{O}_X$. In the local description of the preceding section $\mathcal{O}_V = \text{Sym}_{\mathcal{O}_X} \mathcal{E}$ for some locally free sheaf $\mathcal{E}$. On the other hand, $F^p \mathcal{O}_M / F^{p+1} \mathcal{O}_M$ is a free $R_M$-module spanned by monomials in $\mathcal{O}_V$ of total degree $p$. Thus $\text{gr} \mathcal{O}_M$ is a free module over $R_M$ generated by $\mathcal{O}_V$. 

2.7. Poisson center and Poisson derivations. Let $M = T^*[1]V$ be the shifted cotangent bundle of a $\mathbb{Z}_{\geq 0}$-graded variety $V$ with support $X$. The Poisson center $Z_M$ is the subalgebra of $\mathcal{O}_M$ of sections $z$ such that $[z, g] = 0$ for all sections $g \in \mathcal{O}_M$. Let $k_X$ be the locally constant sheaf with fiber $k$.

**Proposition 2.15.** $Z_M = k_X$
Proof. Using the local description we see that every point of \( X \) has an open neighborhood \( U \) such that \( \mathcal{O}_M(U) \) is a completion of a free \( \mathcal{O}_X(U) \)-module generated by monomials in \( x_i^1, \beta_j^1, \beta_j^* \). The condition that the bracket of \( z \in Z_M(U) \) with \( \beta_i \) vanishes implies that \( \beta_i \) cannot appear in the monomials contributing to \( z \) with nontrivial coefficients. Similarly \( \beta_j^1 \) cannot appear and the vanishing of bracket with \( \mathcal{O}_X(U) \) and with \( x_i^1 \) implies that \( z \) is locally constant. □

A Poisson derivation of degree zero of \( \mathcal{O}_M \) is a vector field \( D \in \Gamma(X, T^0_M) \) of degree zero such that \( D[a, b] = [D(a), b] + [a, D(b)] \) for all local sections \( a, b \in \mathcal{O}_M \). If \( h \in \mathcal{O}_M(X)^{-1} \) is a global section of degree \(-1\) then \( a \mapsto [h, a] \) is a Poisson derivation by the Jacobi identity. Derivations of this form are called Hamiltonian and \( h \) is called a Hamiltonian of the derivation.

Proposition 2.16. All Poisson derivations of degree zero of \( \mathcal{O}_M \) are Hamiltonian with unique Hamiltonian.

Proof. The basic case is the one dimensional case: let \( B \) be a graded commutative ring and \( A = B[z, z^*] = B[z] + B[z]z^* \) where \( z \) is an even variable of degree \( j \in 2\mathbb{Z} \) and \( z^* \) has degree \(-j - 1\). Let \( [\ , ] \) be a \( P_0 \)-bracket such that \( [z, z^*] = 1 \) and \( [z, B] = 0 = [z^*, B] \). Then the action of a Poisson derivation on generators has the form \( D(z) = f_0 + z^* f_1, D(z^*) = g_0 + z g_1 \) with \( f_i, g_i \in B[z] \). The conditions for \( D(z) \) and \( D(z^*) \) to define a derivation are

\[
[D(z), z] + [z, D(z)] = 2[D(z), z] = 0,
\quad [D(z), z^*] + [z, D(z^*)] = 0.
\]

They imply that \( f_1 = 0 \) and \( g_1 = -[f_0, z^*] \) and it follows that

\[
(2.1) \quad h = G - z^* f_0,
\]

where \( G = \int g_0 \, dz \in B[z] \) with \( \int b z^j \, dz = b z^{j+1} / (j + 1) \), defines a Hamiltonian derivation with the same action as \( D \) on \( z \) and \( z^* \). Thus after subtracting a Hamiltonian derivation we get a derivation vanishing on \( z \) and \( z^* \). Moreover, a Poisson derivations obeys \( D(b), z] = 0 = [D(b), z^*] \) for all \( b \in B \). It follows that \( D(B) \subset B \).

Thus adding to any Poisson derivation of \( B[z, z^*] \) a suitable Hamiltonian derivation we obtain the \( k[z, z^*] \)-linear extension of a derivation of \( B \).

Now let \( D \) be a Poisson derivation of \( \mathcal{O}_M \) and use the local description to study the action of \( D \) on \( \mathcal{O}_M(U) \) for some open neighborhood \( U \) of a point of \( X \). By the repeating the above reasoning for each pair \( \beta_j^1, \beta_j^* \) of variables, we may subtract a Hamiltonian derivation to get a derivation vanishing on \( \beta_j^1, \beta_j^* \). Note that this works even if \( V \) is infinite dimensional as the Hamiltonian for each pair lies in \( F^p \mathcal{O}_M \) with \( p \) increasing as the degree of \( \beta_j^1 \) increases (in Eq. 2.1) \( h \) is of degree \(-1\) and \( G \) is divisible by \( z \) of degree \( j \) so both terms are a product of an element of degree \( j \) and one of degree \(-j - 1\). We are left with a derivation of \( \mathcal{O}_{T^*[-1]X}(U) = \text{Sym}_{\mathcal{O}_X} T_X[1](U) \) extended to \( \mathcal{O}_M(U) \) by linearity over \( k[\beta_j^1, \beta_j^*] \). The restriction to \( \mathcal{O}_X(U) \) is a vector field \( \xi \) on \( U \). By subtracting a Hamiltonian derivation with Hamiltonian \( \xi \in T_X[1] \) we may assume that \( D \) vanishes on \( \mathcal{O}_X \). We claim that \( D \) also vanishes on \( T_X[1] \) and is thus zero. Indeed \( D \) maps \( T_X[1] \) to itself and being a derivation obeys \( [D(\xi), f] + [\xi, D(f)] = 0 \) for all \( \xi \in T_X[1], f \in \mathcal{O}_X \); since \( D(f) = 0 \) and \( [D(\xi), f] \) is the action of the vector field \( D(\xi) \) on the function \( f \), we see that \( D(\xi) = 0 \). Then \( D \) vanishes on generators. Since it preserves the filtration, it is well-defined and vanishes on each \( \mathcal{O}_M / F^p \mathcal{O}_M \), and thus vanishes on the inverse limit \( \mathcal{O}_M \).
Now let $D \in \Gamma(X, T_V)$ be a global derivation of degree zero. Then we have an open cover $(U_i)$ of $X$ such that $D|_{U_i} = [h_i]$ for some Hamiltonian $h_i \in \mathcal{O}_M(U_i)$. On intersections $U_i \cap U_j$, $h_i - h_j$ is a Poisson central element of degree $-1$ and thus vanishes by Prop. 2.15. Hence the Hamiltonians $h_i$ agree on intersections and are restrictions of a globally defined Hamiltonian $h$ of degree $-1$ with $D = [h]$. Since the Poisson center is trivial in degree $-1$, $h$ is unique.

2.8. Duality. The following is an extension of a result of Roytenberg [27], who considered the case of smooth graded manifolds.

**Proposition 2.17.** Let $V = (X, \text{Sym}_{\mathcal{O}_X} \mathcal{E})$ for some positively graded locally free $\mathcal{O}_X$-module $\mathcal{E}$ with homogeneous components of finite rank on a nonsingular algebraic variety. Let $V^\vee = (X, \text{Sym}_{\mathcal{O}_X} \mathcal{E}^*[1])$. Then $T^*[−1]V$ is Poisson isomorphic to $T^*[−1]V^\vee$.

**Proof.** Let us first assume that $X$ is affine. Then $\mathcal{E}$ admits an algebraic connection $\nabla : T_X \to \text{End}(\mathcal{E})$ (the obstruction to the existence of connection lies in $H^1(\Omega(X) \otimes \text{End}(\mathcal{E}))$ and thus vanishes for affine varieties). Such a connection extends to a map of $\mathcal{O}_X$-modules $T_X \to \text{Der}(\text{Sym}_{\mathcal{O}_X} \mathcal{E})$. Also the pairing between $\mathcal{E}$ and $\mathcal{E}^*$ defines an inner multiplication $\iota : \mathcal{E}^* \to \text{Der}(\text{Sym}_{\mathcal{O}_X} \mathcal{E})$. Then the sheaf of derivations of $\text{Sym}_{\mathcal{O}_X} \mathcal{E}$ is isomorphic to $T_X \oplus \mathcal{E}^*$, where $T_X$ acts via $\nabla$ and $\mathcal{E}^*$ via $\iota$. Thus

\begin{equation}
\mathcal{O}_{T^*[−1]V} \cong \text{Sym}_{\mathcal{O}_X} (TX[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]).
\end{equation}

By using the dual connection on $\mathcal{E}^*$, the same result is obtained for $V^\vee$ as $\mathcal{E}$ and $\mathcal{E}^*[1]$ are interchanged. It remains to show that the resulting isomorphism respects the bracket and that it is independent of the choice of connection, implying that the isomorphisms on affine subsets glue to a global isomorphism. For this we work locally on an open subset $U$ and use the local description of the previous section. Then $\mathcal{O}_U(U) = \mathcal{O}_X(U)[x^i, j \in I]$, $\mathcal{O}_{V^\vee}(U) = \mathcal{O}_X(U)[\beta^i, j \in I]$ and the right-hand side of (2.2) is $A(U) = \mathcal{O}_X(U)[x^i, \beta^i, j \in I]$.

The degree $d$ component of $A(U)$ consists of formal power series with coefficients in $\mathcal{O}_X(U)$ such that all terms have degree $d$. Let $\nabla \partial_i = \sum_j a_{ij} \beta^j$ for some $a_{ij} \in \Omega^1(X)$. Then the isomorphism $\phi : A(U) \to \mathcal{O}_{T^*[−1]V}(U)$ is

\[ x^i \mapsto -\partial_i - \sum_{j,k} \beta^j a_{kj}^i \frac{\partial}{\partial \beta^k}, \quad \beta^i \mapsto \beta^i, \quad \beta^j \mapsto -\frac{\partial}{\partial \beta^j}, \]

and the isomorphism $\phi' : A(U) \to \mathcal{O}_{T^*[−1]V^\vee}(U)$ is

\[ x^i \mapsto -\partial_i + \sum_{k,j} \beta^k a_{ij}^k \frac{\partial}{\partial \beta^j}, \quad \beta^i \mapsto \frac{\partial}{\partial \beta^i}, \quad \beta^j \mapsto \beta^j. \]

The composition $\phi' \circ \phi^{-1}$ sends $\partial_i$ to $\partial_i$, $\beta^j$ to $\partial/\partial \beta^j$ and $\beta^j/\partial \beta^i$ to $-\beta^i$, therefore it is independent of the choice of connection and it is easy to check that it is an isomorphism of Poisson algebras.

2.9. The group of gauge equivalences. Let $M = T^*[−1]V$ then $\mathcal{O}_M^{-1}$, the homogeneous component of $\mathcal{O}_M$ of degree $−1$, is a sheaf of Lie algebras acting on the sheaf $\mathcal{O}_M$ by derivations (for both the product and the bracket) of degree $0$. Thus the action preserves the filtration $\mathcal{F}^*\mathcal{O}_M$. Let $I_M = \mathcal{F}^1\mathcal{O}_M$ be the graded
ideal generated by elements of positive degree and \( I_M^{(j)} \) be the \( j \)-th power of \( I_M \):

\[
I_M^{(0)} = \mathcal{O}_M; \quad I_M^{(j+1)} = I_M \cdot I_M^{(j)}.
\]

Clearly \( I_M^{(j)} \subset F^{p}O_M \).

**Lemma 2.18.**

(i) \([I_M^{(2)}, \mathcal{O}_M] \subset I_M \).

(ii) If \( d \geq -1 \), and \( j \geq 0 \), \([I_M^{(j)}, \mathcal{O}_M^d] \subset I_M^{(j)} \).

(iii) Let \( p \geq 1 \), \( d \in \mathbb{Z} \). Then \([\mathcal{O}_M^{-1} \cap I_M^{(2)}, F^{p}\mathcal{O}_M^d] \subset I_M^{(2)} \cap F^{p}\mathcal{O}_M^d \).

**Proof.** (i) This follows from the Leibniz rule: \([I_M^{(2)}, \mathcal{O}_M] \subset I_M[I_M, \mathcal{O}_M] \subset I_M \). The same argument proves (ii) by induction, by taking into account that \([I_M, \mathcal{O}_M^d] \subset I_M \) for \( d \geq -1 \), as the bracket of an element of positive degree with one of degree \(-1\) has positive degree. (iii) We have \([\mathcal{O}_M^{-1}, F^{p}\mathcal{O}_M^d] \subset F^{p}\mathcal{O}_M^d \) for degree reasons as the bracket has degree 1. Similarly, \( I_M \) is closed under the Poisson bracket and therefore \([I_M^{(2)}, I_M] \subset I_M[I_M, I_M] \subset I_M^{(2)} \).

**Corollary 2.19.** Let \( j \geq 0 \). Then \([I_M^{(2)} \cap \mathcal{O}_M^{-1}, I_M^{(j)}] \subset I_M^{(j+1)} \). In particular \( \mathcal{O}_M^{-1} \) is a promilpotent Lie algebra acting nilpotently on \( \mathcal{O}_M/F^{p}\mathcal{O}_M \).

**Proof.** By Lemma 2.18 (iii) for \( p = 1 \), we have \([I_M^{(2)} \cap \mathcal{O}_M^{-1}, I_M] \subset I_M^{(2)} \) and the claim follows with the Leibniz rule by induction on \( j \).

Thus the adjoint action of the Lie algebra \( g_M = \mathcal{O}_M^{-1} \cap I_M^{(2)} \) exponentiates to a sheaf of groups \( G_M = \exp(\text{ad} g_M) \).

**Definition 2.20.** Let \( g(M) = \Gamma(X, g_M) \). The group of Poisson automorphisms \( G(M) = \exp(\text{ad} g(M)) \) is called group of **gauge equivalences**.

### 3. Solutions of the classical master equation

Let \( \mathcal{C} \) be as in Section 2.4. We formulate most of the statements for \( \mathcal{C} \) consisting of smooth algebraic varieties, but they are valid with slight change of vocabulary to the other cases.

#### 3.1. The classical master equation

Let \( M \) be a \((-1)\)-symplectic variety with support \( X \in \mathcal{C} \). The classical master equation is the equation \([S, S] = 0 \) for a function \( S \in \Gamma(X, \mathcal{O}_M) \) of degree 0 on \( M \). If \( S \) is a solution of the master equation then the operator \( dS = [S, ] \) is a differential on the sheaf of Poisson algebras \( \mathcal{O}_M \). Moreover, being a derivation of degree 1, it preserves \( I_M = F^1\mathcal{O}_M \) and thus defines a differential on the sheaf of \( \mathbb{Z}_{\leq 0} \)-graded algebras \( \mathcal{O}_M/I_M \).

#### 3.2. BV varieties

**Definition 3.1.** Let \( S_0 \) be a regular function on \( X \in \mathcal{C} \). A **BV variety** with support \((X, S_0)\) is a pair \((M, S)\) consisting of a \((-1)\)-symplectic variety \( M \) with support \( X \) and a function \( S \in \Gamma(X, \mathcal{O}_M) \) such that

(i) \( S|_X = S_0 \).

(ii) \( S \) is a solution of the classical master equation \([S, S] = 0 \).

(iii) The cohomology sheaf of the complex \((\mathcal{O}_M/I_M, dS)\) vanishes in non-zero degree.

**Remark 3.2.** The inclusion \( X \) in \( M \) is described by the map \( \mathcal{O}_M^0 \to \mathcal{O}_X \) which has kernel \( \mathcal{I}_M^0 = I_M \cap \mathcal{O}_M^0 \). Thus we can write (i) as \( S_0 \equiv S \mod \mathcal{I}_M^0 \).
3.3. Multivalued BV varieties. As the bracket of two functions depends only on their differential it is natural to consider a slight generalization where we allow $S$ to be multivalued. Let $\hat{O}_X$ be the $O_X$ module $\Omega^1_X$ of closed differentials. We think of it as the space of “regular multivalued functions modulo constants”, namely as formal expressions $S_0 = \int \lambda$ where $\lambda \in \Omega^1_X$.

**Definition 3.3.** Let $S_0 \in \Gamma(X, \hat{O}_X)$ be a multivalued function on $X \in C$. A BV variety with support $(X, S_0)$ is a pair $(M, S_0 + S')$ consisting of a $\mathbb{Z}_{\geq 0}$-graded variety $V$ with support $X$ and a function $S' \in \Gamma(X, O^0_M)$ such that

(i) $S'|_X = 0$.
(ii) $S = S_0 + S'$ is a solution of the classical master equation $[S, S] = 0$.
(iii) The cohomology sheaf of the complex $(O_M/I_M, d_S)$ vanishes in non-zero degree.

We notice that $d_{S_0} = [S_0, \cdot]$ is well defined on $O_M$ as the bracket involves only the derivative of $S_0$ and it is easy to check that it is a differential. Then condition (ii) means $d_{S_0}S' + \frac{1}{2}[S', S'] = 0$.

Our results hold in the more general setting of multivalued BV varieties with the same proofs, except for notational adjustments.

3.4. The resolution of the Jacobian ring associated with a BV variety. Let $(M, S)$ be a BV variety with support $(X, S_0)$. Then $S_0 = S|_X$ is a regular function on $X$. The complex $R_M = (O_M/F^1O_M, d_S)$ looks like

$$\cdots \to T_X \to O_X,$$

and the last map is $\xi \mapsto [S_0, \xi] = \xi(S_0)$, so the cohomology of $R_M$ is the Jacobian ring $J(S_0)$, the quotient of $O_X$ by the ideal generated by partial derivatives of $S_0$. Thus $R_M$ is a resolution of $J(S_0)$.

3.5. The BRST complex of a BV variety.

**Definition 3.4.** The BRST complex of a BV variety $(M, S)$ is the sheaf of differential $P_0$-algebras $(O_M, d_S)$.

**Proposition 3.5.** Let $(M, S)$ be a BV variety. The subgroup $G(M, S)$ of gauge equivalences of $M$ fixing $S$ acts as the identity on the cohomology sheaf of the BRST complex.

**Proof.** Let $ad_a$ be the operator $b \mapsto [a, b]$. If $\exp(ad_a)S = S$ and $a \in g(M)$ then

$$\left(\frac{\exp(ad_a) - \text{id}}{ad_a}\right)[a, S] = 0.$$

The expression in parentheses is a power series $\text{id} + \frac{1}{2}ad_a + \cdots$ starting with the identity and is thus an invertible operator acting on $\hat{O}_M(X)$. Thus $[a, S] = 0$, i.e., $a$ is a cocycle of degree $-1$. Since the cohomology vanishes in this degree, there is a $b \in O_M(X)^{-2}$ such that $a = [S, b]$. It follows that

$$ad_a = d_S \circ ad_b + ad_b \circ d_S,$$

and thus $ad_a$ is homotopic to the zero map. $\square$
3.6. Products, trivial BV varieties and stable equivalence. If \((M', S')\) and \((M'', S'')\) are BV varieties with supports \((X', S'_0)\), \((X'', S''_0)\) then \(M' \times M''\) with \(S = S' \otimes 1 + 1 \otimes S'' \in \Gamma(X' \times X'', \mathcal{O}_{M' \times M''}) = \Gamma(X' \times X'', \mathcal{O}_M \otimes \mathcal{O}_M)\) is also a BV variety, called the product of the BV varieties \((M', S')\), \((M'', S'')\). It is denoted by slight abuse of notation \((M' \times M'', S' + S'')\).

Let \(W = \bigoplus_{i<0} W^i\) be a negatively graded vector space over \(k\) with finite dimensional homogeneous components \(W^i\) and set \(W^* = \bigoplus_{i>0} (W^*)^i\) with \((W^*)^i = (W^{-i})^*\). Then \(W\) may be considered as a graded variety \(W = (\{\text{pt}\}, \text{Sym} W^*)\) supported at a point. Its shifted cotangent bundle is \(T^*[-1]W = (\{\text{pt}\}, \text{Sym}(W^* \oplus W[1]))\). Let \(d_W\) be a differential on \(W^*\) with trivial cohomology and set \(S_W\) be the element of the completion of \(W^* \otimes W[1]\) in \(\mathcal{O}_{T^*[-1]W} = \widehat{\text{Sym}}(W^* \oplus W[1])\) corresponding to \(d_W^*:\) for any homogeneous basis \((\beta^j)\) of \(W^*\) and dual basis \((\beta^*_j)\) of \(W[1]\),

\[
S_W = \sum_j d_W^*(\beta^*_j) \beta^j \in \mathcal{O}_{T^*[-1]W}.
\]

Then \((T^*[-1]W, S_W)\) is a BV variety with support \((\{\text{pt}\}, S_0 = 0)\). Its BRST cohomology is \(k\) and the BRST complex is \(\text{Sym}(W^* \oplus W[1])\) with differential induced from \(d_W^*\) on \(W^*\) and its dual on \(W\).

**Definition 3.6.** A BV variety of the form \((T^*[-1]W, S_W)\) for a negatively graded acyclic complex of vector spaces \(W\) is called trivial.

The Künneth formula implies:

**Lemma 3.7.** If \((M, S)\) is a BV variety with support \((X, S_0)\) and \((T^*[-1]W, S_W)\) is a trivial BV variety then their product is a BV variety with support \((X, S_0)\) and the canonical map

\[
\mathcal{O}_M \to \mathcal{O}_{M \times T^*[-1]W}
\]

is a quasi-isomorphism of sheaves of \(P_0\)-algebras between the corresponding BRST complexes.

**Definition 3.8.** Two BV varieties \((M, S)\), \((M', S')\) with support \((X, S_0)\) are called equivalent if there is a Poisson isomorphism \(\Phi: M \to M'\) such that \(\Phi^* S' = S\). They are called stably equivalent if they become equivalent after taking products with trivial BV varieties.

**Remark 3.9.** If \(R_M = (\mathcal{O}_M/I_M, \delta)\) is the resolution of the Jacobian ring associated with \((M, S)\), see \[3.4\] then the resolution associated with the product with the trivial BV variety \((W, S_W)\) is \(R_M \otimes \text{Sym}(W[1])\) with differential \(\delta \otimes \text{id} + \text{id} \otimes d_W\), where \(d_W\) is induced from the differential dual to \(d_W^*\).

4. Existence and uniqueness for affine varieties

Let \(S_0\) be a regular function on a nonsingular affine algebraic variety over a field \(k\) of characteristic zero. In this section we prove the existence and uniqueness up to stable equivalence of a BV variety with support \((X, S_0)\) and thus prove Theorem 1.2 (i). The existence result occupies Sections 4.1–4.3 and is contained in Theorem 4.5. The uniqueness up to stable equivalence is Theorem 4.10 and is discussed in Section 4.4. A variant of the argument, discussed in Section 4.5, shows that adding a square of a linear function to \(S_0\) does not influence the BRST cohomology. Finally,
in [10] we show that Poisson automorphism act trivially on cohomology thereby proving Theorem 1.2 (ii).

The existence proof is an adaptation of the construction proposed in [33, 34, 18] to our context. The idea is to start from a resolution of the Jacobian ring of $S_0$ and show that there exists a BV variety $(M, S)$ with $M$ a shifted cotangent bundle $M = T^*[-1]V$ such that $\mathcal{O}_M/I_M$ with the differential induced from $d_S = [S, ]$ is the given resolution.

4.1. Tate resolutions. Let $S_0 \in \mathcal{O}_X(X)$. The Jacobian ring $J(S_0)$ of $S_0$ is the cokernel of the map $\delta: T_X \rightarrow \mathcal{O}_X$ sending $\xi$ to $\xi(S_0)$. The first step of the existence proof is the extension of $\delta$ to a Tate resolution $R$ of the Jacobian ring, namely a quasi-isomorphism of differential graded commutative $\mathcal{O}_X$-algebras $(R, \delta) \rightarrow (J(S_0), 0)$ such that $R = \text{Sym}_{\mathcal{O}_X}(\mathcal{W})$ for some graded $\mathcal{O}_X$-module $\mathcal{W} = \oplus_{j \leq -1} \mathcal{W}^j$ with locally free $\mathcal{O}_X$-modules $\mathcal{W}^j$ such that $\mathcal{W}^{-1} = T_X$. See the Appendix for more properties of Tate resolutions.

By construction, $\mathcal{W} = T_X[1] \oplus \mathcal{E}^*[1]$, where $\mathcal{E}$ is concentrated in positive degree. Tate resolutions exist by the classical recursive construction of Tate [33]. Assume that $\mathcal{W}$ is constructed down to degree $-d$ in such a way that the negative degree cohomology of $\text{Sym}_{\mathcal{O}_X}(\mathcal{W})$ is zero down to degree $-d$. Then the cohomology of degree $-d$ is a finitely generated $\mathcal{O}_X$-module so that it can be killed by adding to $\mathcal{W}$ a direct summand of degree $-d - 1$.

Given a Tate resolution, let $V = (X, \text{Sym}_{\mathcal{O}_X}(\mathcal{E}))$ the corresponding graded variety and $M = T^*[-1]V$ its (-1)-shifted cotangent bundle. We have

$$\mathcal{O}_M = \text{Sym}_{\mathcal{O}_X}(T_X[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1]),$$

and $R$ is identified with the quotient $R_M = \mathcal{O}_M/F^1\mathcal{O}_M$ by the ideal generated by elements of positive degree.

The first approximation to a solution of the master equation is $S_0 + \delta|_{\mathcal{E}^*[1]}$, where the restriction $\delta|_{\mathcal{E}^*[1]} \in \text{Hom}(\mathcal{E}^*[1], (\text{Sym}(T_X[1] \oplus \mathcal{E}^*[1]))[1])$ of $\delta$ to $\mathcal{E}^*$ is viewed as an element of $\mathcal{E} \otimes \text{Sym}(T^*[1] \oplus \mathcal{E}^*[1]) \subset \mathcal{O}_M(X)$. More explicitly,

$$S_{\text{lin}} = S_0 + \sum_i \delta_{\beta_i^*} \beta^i,$$

for any basis $\beta^i$ of $\mathcal{E}$ and dual basis $\beta_i^*$ of $\mathcal{E}^*$. Although $S_{\text{lin}}$ does not obey the master equation, its Hamiltonian vector field $[S_{\text{lin}}, ]$ does define a differential on the associated graded of $\mathcal{O}_M$.

**Proposition 4.1.** The operator $[S_{\text{lin}}, ]$ preserves the filtration $F^*\mathcal{O}_M$ and thus induces a differential on the graded $\mathcal{O}_X$-algebra $\text{gr} \mathcal{O}_M$. The canonical isomorphism $\text{gr} \mathcal{O}_M \cong R_M \otimes_{\mathcal{O}_X} \mathcal{O}_V$ (see Lemma 2.17) identifies the differential with $\delta \otimes \text{id}$.

**Proof.** The operator preserves the filtration because it is of degree one. Let us compute the action of $[S_{\text{lin}}, ]$ on the associated graded starting with $R_M = \text{gr}^0\mathcal{O}_M$: $R_M$ is a locally free graded $\mathcal{O}_X$-module whose generators are the classes of $x_i^*$, a local basis of $T_X[-1]$ and of $\beta_i^*$, a local basis of $\mathcal{E}^*[-1]$. The non-zero brackets are $[f, x_i^*] = \delta_i f$, $f \in \mathcal{O}_X$ and $[\beta^j, \beta_i^*] = \delta_{ji}$. Thus $[S_{\text{lin}}, x_i^*] \equiv \delta_i S_0 \mod F^1\mathcal{O}_M$ and $[S_{\text{lin}}, \beta_i^*] \equiv \delta_{\beta_i} \mod F^1\mathcal{O}_M$. Also $[S_{\text{lin}}, \mathcal{O}_X] \equiv 0 \mod F^1\mathcal{O}_M$. Thus $[S_{\text{lin}}, ]$ coincides with $\delta$ on $R_M = \text{gr}^0\mathcal{O}_M$. The claim in the general case follows from the fact that if $a \in \mathcal{O}_V = \text{Sym}_{\mathcal{O}_X}(\mathcal{E})$ has positive degree $p$ then $[S_{\text{lin}}, a]$ has degree $p + 1$ and thus vanishes modulo $F^{p+1}\mathcal{O}_M$. \[\square\]
Remark 4.2. Note that $\text{gr} \mathcal{O}_M$ is bigraded, with
\[ \text{gr}^{p,q} \mathcal{O}_M = F^p \mathcal{O}_M^p / F^{p+1} \mathcal{O}_M^p, \]
and nonzero components for $p \geq \max(q,0)$.

4.2. BV varieties associated to a Tate resolution. Let $I_M = F^1 \mathcal{O}_M$
be the ideal generated by elements of positive degrees and $I_M^{(j)}$ its $j$-th power, cf. Section 2.9.

Lemma 4.3. $[I_M^{(2)} \cap \mathcal{O}_M^0, F^p \mathcal{O}_M] \subset F^{p+1} \mathcal{O}_M$

Proof. Let us again adopt the convention that $a_j, b_j, \ldots$ denote arbitrary local sections of degree $j$ and $a, b, \ldots$ sections of unspecified degree. Then an element of $F^p \mathcal{O}_M$ is a sum of terms of the form $ab'_p$ with $p' \geq p$ and if $c_0 \in I_M^{(2)} \cap \mathcal{O}_M^0$, $[c_0, ab'_p] = [c_0, a]b'_p \pm [c_0, b_p]a$. The bracket in the second term has degree $p' + 1$ so the second term is in $F^{p+1} \mathcal{O}_M$. By Lemma 2.18, the first term belongs to $[I_M^{(2)}, \mathcal{O}_M] F^p \mathcal{O}_M \subset I_M F^p \mathcal{O}_M \subset F^{p+1} \mathcal{O}_M$. \hfill $\square$

Thus $I_M^{(2)} \cap \mathcal{O}_M$ acts trivially on the associated graded. In particular any $S \in \mathcal{O}_M^0$ such that $S \equiv S_{\text{lin}} \mod I_M^{(2)}$ will induce the same differential as $S_{\text{lin}}$ on $\text{gr} \mathcal{O}_M$. The idea is to construct a solution of $S \in \Gamma(X, \mathcal{O}_M)$ of the classical master equation such that $S \equiv S_{\text{lin}} \mod I_M^{(2)}$. This is done recursively:

Definition 4.4. Let $R = (\mathcal{O}_M/I_M, \delta)$ be a Tate resolution of the Jacobian ring $J(S_0)$ with $M = T^* [-1] V$ and $S_{\text{lin}}$ be the corresponding hamiltonian function. We say that a solution $S \in \Gamma(X, \mathcal{O}_M)$ of the master equation is associated with $R$ if $S \equiv S_{\text{lin}} \mod I_M^{(2)}$.

Theorem 4.5. Let $S_0 \in \mathcal{O}(X)$. Let $R = (\mathcal{O}_M/I_M, \delta)$ be a Tate resolution of the Jacobian ring $J(S_0)$. Then there exists a solution $S \in \mathcal{O}_M(X)$ of the classical master equation
\[ [S, S] = 0 \]
asociated with $R$. If $S'$ is another solution with this property then $S' = g \cdot S$ for some gauge equivalence $g \in G(M)$.

4.3. Proof of Theorem 4.5.

(a) Filtration and bracket. The following Lemma gives a compatibility condition between bracket and filtration needed for the recursive construction of the solution of the classical master equation.

Lemma 4.6. Let $p \geq 0$.

(i) $[F^p \mathcal{O}_M^0, \mathcal{O}_M^0] \subset F^p \mathcal{O}_M^1$.

(ii) $[F^p \mathcal{O}_M^1, F^p \mathcal{O}_M^1] \subset F^{p+1} \mathcal{O}_M^1$.

Proof. Clearly $[\mathcal{O}_M^0, \mathcal{O}_M^0] \subset \mathcal{O}_M^1$ since the bracket has degree 1. Also $F^1 \mathcal{O}_M^1 = \mathcal{O}_M^1$ so if $p = 0$ there is nothing to prove. So let us assume that $p \geq 0$. Let $a_i, b_i, \ldots$ denote general local sections of $\mathcal{O}_M$ of degree $i$. Then, for $j \geq p$, the bracket
\[ [a_{-j} b_j, c_0] = a_{-j} [b_j, c_0] \pm b_j [a_{-j}, c_0] \]
lies in $F^p \mathcal{O}_M$ since $\deg [b_j, c_0] = j + 1 \geq p + 1 \geq p$ and $\deg b_j = j \geq p$. This proves (i). Now let us assume that $c_0 \in F^p \mathcal{O}_M^0$. If $j > p$, or if $c_0 \in F^{p+1} \mathcal{O}_M^0$ (ii) follows
from (i), so let \( j = p \) and \( c_0 = d_{-p} e_p \). Then the first term in \((4.2)\) is in \( F^{p+1}O_M \) and the second term is
\[
b_p[a_{-p}, c_0] = b_p([a_{-p}, d_{-p}] e_p \pm [a_{-p}, e_p] d_{-p}) \in F^{p+1}O_M,
\]
and the second term is
\[
\text{deg } b_p e_p = 2p > p \text{ and } \text{deg}[a_{-p}, e_p] = 1.
\]

The construction of \( S \) and the construction of the gauge equivalence in the uniqueness proof both rely on the vanishing of the cohomology of a complex of sheaves, that we now introduce. Let \( 0 \leq q \leq p \). Let
\[
\pi_p : F^p O_M \to \text{gr}^p O_M,
\]
be the canonical projection and consider the subcomplex of \( \text{gr}^p O_M \)
\[
\mathcal{G}^*_{p,q} = \pi_p(\text{gr}^p O_M \cap I^{(q)}_M),
\]

**Lemma 4.7.**
\[
H^j(\Gamma(X, \mathcal{G}^*_{p,q})) = 0, \quad \text{if } j < p.
\]

**Proof.** The canonical isomorphism \( \text{gr} O_M \cong (R_M \otimes_{\mathcal{O}_X} \mathcal{O}_V, \delta \otimes \text{id}) \) of Lemma 2.14 identifies \( \mathcal{G}^*_{p,q} \) with with \( R_M \otimes_{\mathcal{O}_X} (I^{(q)}_V \cap \mathcal{O}_V) \). The cohomology sheaf \( H^j(X, \mathcal{G}^*_{p,q}) \) is zero in degree \( j < p \) because the cohomology groups of \( R_M \) are trivial in negative degree and \( I^{(q)}_V \) is a locally free \( \mathcal{O}_X \)-module. Since \( X \) is affine and \( \mathcal{G}^*_{p,q} \) is quasi-coherent the same holds for the complex of global sections. \( \square \)

(b) **Existence proof.** We prove by induction that for each \( p \geq 1 \) there is an \( S_{\leq p} \in \Gamma(X, \mathcal{O}^{0}_{M}) \) such that
\[
(i) \quad S_{\leq p} \equiv S_{\text{lin}} \mod I^{(2)}_{M},
(ii) \quad [S_{\leq p}, S_{\leq p}] \in I^{(2)}_{M} \cap F^{p+1}O_M,
(iii) \quad S_{\leq p+1} = S_{\leq p} \mod F^{p+1}O_M.
\]
We set \( S_{\leq 1} = S_{\text{lin}} \) Then obviously (i) holds for \( p = 1 \) and (iii) does not apply. To prove (ii) for \( p = 1 \) we use the local description and choose a local basis of sections \( \beta^{i} \) of generators of \( \mathcal{O}_V \) with dual sections \( \beta^{*} \), see \[2.3\]. Then \( S_{\text{lin}} = S_0 + \sum_{i} \delta \beta^{*}_{i} \beta^{i} \).
Using the fact that \( [S_0, S_0] = [S_0, \beta^{i}] = 0 \), we obtain
\[
[S_{\leq 1}, S_{\leq 1}] = 2 \sum_{j} [S_0, \delta \beta^{*}_{j}] \beta^{j} + \sum_{i,j} [\delta \beta^{*}_{i} \beta^{i}, \delta \beta^{j}_{j} \beta^{j}].
\]
The summand in the second term modulo \( I^{(2)}_{M} \) can be written as
\[
[\delta \beta^{*}_{i} \beta^{i}, \delta \beta^{j}_{j} \beta^{j}] = \delta \beta^{*}_{i}[\beta^{i}, \delta \beta^{j}_{j} \beta^{j}] + \delta \beta^{*}_{j}[\beta^{j}, \delta \beta^{*}_{i} \beta^{i}] \quad \text{mod } I^{(2)}_{M},
\]
\[
= [\delta \beta^{*}_{j} \beta^{j}, \delta \beta^{*}_{i} \beta^{i}] + [\delta \beta^{*}_{i} \beta^{i}, \delta \beta^{*}_{j} \beta^{j}] \quad \text{mod } I^{(2)}_{M}.
\]
Summing over \( i,j \) and inserting in \[4.3\] yields
\[
[S_{\leq 1}, S_{\leq 1}] = 2 \sum_{j} [S_{\leq 1}, \delta \beta^{*}_{j}] \beta^{j} \quad \text{mod } I^{(2)}_{M},
\]
\[
= 2 \sum_{j} \delta^{2}(\beta^{*}_{j}) \beta^{j} \quad \text{mod } I^{(2)}_{M},
\]
\[
= 0 \quad \text{mod } I^{(2)}_{M}.
\]
For the induction step we write
\[
S_{\leq p+1} = S_{\leq p} + \nu,
\]
with \( v \in \Gamma(X, I_M^{(2)} \cap F^{p+1}O_{M}) \) to be determined. Then \( S_{\leq p+1} \) obeys (i) and (iii). As for (ii) we notice that by Proposition 4.1, \([S_{\leq p}, v] \equiv [S_{\text{lin}}, v] \equiv \delta v \mod F^{p+2}O_{M}\) where \( \delta \) is the differential of \( \text{gr} O_{M} \), and, by Lemma 4.6(ii), \([v, v] \equiv 0 \mod F^{p+2}O_{M} \). Thus

\[
[S_{\leq p+1}, S_{\leq p+1}] \equiv [S_{\leq p}, S_{\leq p}] + 2\delta v \mod F^{p+2}O_{M}.
\]

On the other hand, by the Jacobi identity, Lemma 4.3 and Proposition 4.1,

\[
\text{Then } \Gamma(p, q) \text{ also globally. Then}
\]

\[
0 = [S_{\leq p}, [S_{\leq p}, S_{\leq p}]] \equiv \delta[S_{\leq p}, S_{\leq p}] \mod F^{p+2}O_{M}.
\]

Then \([S_{\leq p}, S_{\leq p}] \mod F^{p+2}O_{M} \in \Gamma(X, G_{p+1,2}^{0})\) is a cocycle of degree 1. Since by Lemma 4.7 the cohomology vanishes in degree \( p + 1 \geq 2 \) there exists a \( \bar{v} \in \Gamma(X, G_{p+1,2}^{0}) \) with \( 2\delta \bar{v} + [S_{\leq p}, S_{\leq p}] \equiv 0 \mod F^{p+2}O_{M} \). Let \( v \in \Gamma(X, F^{p+1}O_{M}^{0} \cap I_{M}^{(2)}) \) such that \( \pi_{p}v = \bar{v} \). Such a lift certainly exists locally and, because \( X \) is affine, also globally. Then \( S_{\leq p+1} = S_{\leq p} + v \) is a solution of the master equation modulo \( F^{p+2}O_{M} \).

It remains to show that \([S_{\leq p+1}, S_{\leq p+1}] \in I_{M}^{(2)} \). It is clear that \( I_{M} \) is a Lie subalgebra. Thus \([v, v] \in I_{M}^{(2)} \) and \([S_{\leq p}, v] = [S_{\text{lin}}, v] \equiv 0 \mod I_{M}^{(2)} \) since \( S_{\leq p} \equiv S_{\text{lin}} \mod I_{M} \). But clearly \([S_{\text{lin}}, I_{M}] \subset I_{M} \) and therefore \([S_{\text{lin}}, v] \in I_{M}^{(2)} \) for \( v \in I_{M}^{(2)} \).

This completes the induction step.

(c) Uniqueness up to gauge equivalence. Next we prove the transitivity of the action of the group of gauge equivalences on the space of solutions \( S \) of the master equation associated with a given Tate resolution \((R_{M}, \delta)\). Assume that \( S, S' \) are two such solutions. Since both \( S \) and \( S' \) are congruent to \( S_{\text{lin}} \) modulo \( I_{M}^{(2)} \), we know that

\[
S - S' \equiv 0 \mod I_{M}^{(2)} \subset F^{2}O_{M}.
\]

The proof is by induction: we show that if \( S - S' \in F^{p}O_{M}(X) \), with \( p \geq 2 \), we can find a gauge equivalence \( g \) such that \( g \cdot S - S' \in \Gamma(X, F^{p+1}O_{M}) \). This induction step is done by a second induction: for fixed \( p \) let us suppose inductively over \( q \) that the difference between the two solutions is a section of \( I_{M}^{(q)} \cap F^{q}O_{M} + F^{p+1}O_{M} \) with \( p \geq q \geq 2 \) and show that we can find a gauge equivalence \( g \) so that

\[
g \cdot S - S' \in \Gamma(X, I_{M}^{(q+1)} \cap F^{p}O_{M} + F^{p+1}O_{M}).
\]

Since \( I_{M}^{(p+1)} \subset F^{p+1}O_{M} \) this shows by induction that we may achieve that \( S - S' \equiv 0 \mod F^{p+1}O_{M} \) and in fact, because of (4.4), in \( I_{M}^{(2)} \cap F^{p+1}O_{M} \), completing the induction step in \( p \). So let us assume that \( v = S - S' \) is a section of \( I_{M}^{(q)} \cap F^{p}O_{M} + F^{p+1}O_{M} \) with \( p \geq q \geq 2 \) so that it defines a section

\[
\bar{v} \in \Gamma(X, G_{p,q}^{0}).
\]

Since \( S \) and \( S' \) both obey the master equation, we see that \( 0 = [S + S', S - S'] = [S + S', v] \equiv 2\delta \bar{v} \mod F^{p+1}O_{M} \) and thus \( \bar{v} \) is a cocycle. As the cohomology vanishes in degree 0 (it starts in degree \( p \geq 2 \)), \( \bar{v} \) is exact and there exists a \( \bar{u} \in \Gamma(X, G_{p,q}^{-1}) \) such that \( \delta \bar{u} = \bar{v} \). Let \( u \in \Gamma(X, I_{M}^{(q)} \cap F^{p}O_{M}^{-1}) \) be a lift of \( u \), so that \( v \equiv [S, u] \mod F^{p+1}O_{M} \). As in the existence proof, it is clear that such a lift exists locally and we use the fact that \( X \) is affine to show that it exists globally on \( X \). Since \([u, S] = -[S, u] \), we have

\[
v + [u, S] \in \Gamma(X, F^{p+1}O_{M}).
\]
Let $g = \exp(\text{ad}_u)$. Then
\[
    g \cdot S - S' = g \cdot S - S + v = v + [u, S] + \frac{1}{2} [u, [u, S]] + \cdots = \frac{1}{2} [u, [u, S]] + \cdots \mod F^{p+1}O_M.
\]
By Lemma 2.10 (i) and Lemma 2.18 (ii), $[u, S] \in F^pO_M \cap I_M^{(q)}$; also $u \in I_M^{(q)} \cap O_M^{-1}$ so by Lemma 2.18 (i) and the fact that $[I_M^{(q)}, I_M^{(q)}] \subseteq I_M^{(q+1)}$, we conclude that $[u, [u, S]]$, and by the same argument any of the higher brackets in the sum, is a section of $I_M^{(q+1)} \cap F^pO_M$. Therefore $g \cdot S - S' \in \Gamma(X, I_M^{(q+1)} \cap F^pO_M + F^{p+1}O_M)$, as required.

The proof of Theorem 4.5 is complete.

**4.4. Relating Tate resolutions.** Let $S_0$ be a regular function on a non-singular affine variety $X$ and suppose $(M = T^*[−1]V, S), (M' = T^*[−1]V', S')$ are two BV varieties with support $(X, S_0)$. Then the quotients $R_M = O_M / I_M$, $R_{M'} = O_{M'}/I_{M'}$ by the ideals generated by positive elements are both resolutions of the Jacobian sheaf of rings $J(S_0)$. Let us first consider the case where $R_M$ is isomorphic to $R_{M'}$ as a differential graded algebra by an isomorphism that is the identity in degrees $−1$ and $0$ (so that in particular $\varphi$ is a morphism of $O_X$-modules).

**Proposition 4.8.** Suppose $\varphi : R_M \to R_{M'}$ is an isomorphism of sheaves of differential graded algebras which is the identity in degree $0$ and $−1$. Then $\varphi$ is induced by a Poisson isomorphism $M' \to M$ sending $S$ to $S'$.

**Proof.** We use the duality, see 2.8, to represent $R_M$ as $O_{M'} / I_{M'}$ with $M' = T^*[−1]V'$, $M' = T^*[−1]V'$, $S'$ are two BV varieties with support $(X, S_0)$. Then the quotients $R_M = O_M / I_M$, $R_{M'} = O_{M'}/I_{M'}$ by the ideals generated by positive elements are both resolutions of the Jacobian sheaf of rings $J(S_0)$. Let us first consider the case where $R_M$ is isomorphic to $R_{M'}$ as a differential graded algebra by an isomorphism that is the identity in degrees $−1$ and $0$ (so that in particular $\varphi$ is a morphism of $O_X$-modules).

Let $R \to J(S_0)$ be a Tate resolution of the Jacobian ring and let $(W, \delta_W)$ be an acyclic negatively graded complex of $k$-vector spaces with finite dimensional homogeneous components. Then the differential $\delta_W^{[1]}$ extends uniquely as a derivation of $\text{Sym}(W^{[1]})$ and $(\text{Sym}(W^{[1]}), \delta_W^{[1]})$ is a differential graded algebra with cohomology $\cong k$. The tensor product $R \otimes \text{Sym}(W^{[1]})$ is another resolution of $J(S_0)$. 


Proposition 4.9. Let $R_j \to J(S_0)$, $j = 1, 2$ be Tate resolutions extending $\delta: T_X \to \mathcal{O}_X$. Then there exist acyclic negatively graded complexes $W_1, W_2$ of $k$-vector spaces and an isomorphism of differential graded commutative algebras

$$R_1 \otimes \text{Sym}(W_1[1]) \cong R_2 \otimes \text{Sym}(W_2[1]),$$

that is the identity in degree $-1$ and $0$.

We prove this Proposition in a slightly more general context in Appendix A.

Theorem 4.10. Let $X$ be affine. Any two BV varieties $(M_1, S_1), (M_2, S_2)$ with support $(X, S_0)$ are BV varieties with the same support $(X, S_0)$ and corresponding Tate resolutions $(R_1, \delta_1), (R_2, \delta_2)$. By Prop. 4.12 there are acyclic negatively graded complexes $W_1, W_2$ and an isomorphism of differential graded commutative algebras $R_1 \otimes \text{Sym}(W_1[1]) \cong R_2 \otimes \text{Sym}(W_2[1])$, which is the identity in degree $0$ and $-1$. Then the product of $(M_1, S_1)$ with the trivial BV varieties $(T^*[1]V_1, S_1)$, see Section 5.0, give stably equivalent BV varieties whose Tate resolutions are isomorphic by Remark 5.9. By Prop. 4.8, the BV varieties $V_1 \times W_1, V_2 \times W_2$ are equivalent.

Together with Prop. 4.7, this implies:

Corollary 4.11. If $(T^*[1]V_1, S_1)$ and $(T^*[1]V_2, S_2)$ are BV varieties with support $(X, S_0)$, then there exists a BV variety $(T^*[1]V, S)$ with support $(X, S_0)$ and morphisms of sheaves of differential $P_0$-algebras

$$(\mathcal{O}_{T^*[1]V_1}, d_{S_1}) \to (\mathcal{O}_{T^*[1]V}, d_S) \leftarrow (\mathcal{O}_{T^*[1]V_2}, d_{S_2}),$$

between the corresponding BRST complexes, inducing isomorphisms on the cohomology.

4.5. Adding a square.

Proposition 4.12. Let $(M, S)$ be a BV variety with support $(X, S_0)$. Let $X' = X \times \mathbb{A}^1$ and $S'_0 = S_0 + at^2 \in \mathcal{O}(X')$, where $t$ is a coordinate on $\mathbb{A}^1$ and $a \in k^*$. Then $(M' = M \times T^*[1]A^1, S' = S + at^2)$ is a BV variety with support $(X', S'_0)$ and the corresponding BRST complexes are quasi-isomorphic differential $P_0$-algebras.

Proof. The argument is similar to the one for trivial solutions: we are taking the product with the BV variety $(T^*[1]A^1, at^2)$. The shifted cotangent bundle $T^*[1]A^1$ has coordinates $t, t^*$ and differential such that $t \mapsto 0, t^* \mapsto 2at$ which is clearly acyclic. Thus the natural map $\mathcal{O}_M \to \mathcal{O}_{M'}$ is a quasi-isomorphism of sheaves of differential $P_0$-algebras by the K"unneth formula.

4.6. Automorphisms of a BV variety.

Theorem 4.13. Let $(M, S)$ be a BV variety and $\phi: M \to M$ a Poisson automorphism preserving $X$ such that $\phi^*S = S$. Then $\phi^*$ induces the identity on the cohomology sheaf of the BRST complex.

Proof. Since the statement is local we can assume that $M = T^*[1]V$ is a cotangent bundle. Let $F = \phi^*: \mathcal{O}_M \to \mathcal{O}_M$. The automorphism $F$ induces an automorphism $f$ of the sheaf of differential graded algebras $\mathcal{O}_M/I_M$. Then $f$ and $\text{id}$ are both automorphisms of the Tate resolution $R_M = \mathcal{O}_M/I_M$ of the Jacobian.
ring that are the identity in degree 0 and −1 and are thus related by a homotopy $H$, namely a morphism of differential graded algebras $R_M \to R_M[t, dt]$ such that $ev_0 \circ H = id$ and $ev_1 \circ H = f$, see Lemma A.1 in the Appendix. In more detail,

$$H = f_t + dt \, h_t,$$

and $f_t$ is a morphism of differential graded algebras $R_M \to R_M[t] = R_M \otimes_k k[t]$ which is the identity in degree 0 and −1. Its $k[t]$-linear extension $R_M[t] \to R_M[t]$ is not invertible in general, but since it is at $t = 0$ and $t = 1$, we can invert it at the generic point. More precisely, in each degree $i$ we have a rational map $(f_t^{-1})^i \in \text{End}(R^i_M) \otimes_k k(t)$ regular at $t = 0$ and $t = 1$, and such that $(f_0^{-1})^i = id$ and $(f_1^{-1})^i = f^{-1} |_{R^i_M}$. This map is the inverse of $f_t |_{R^i_M}$ for $t$ in a Zariski open subset $U_i$ of $\mathbb{A}^1$ containing 0, 1. We next use Prop. 4.8 to lift $f_t$ to a family of Poisson automorphisms of $\mathcal{O}_M$. Recall that the lift is constructed using the duality isomorphism $T^*[-1]V = T^*[-1]V^\vee$. Under this identification a morphism of Tate resolutions is the same as an automorphism of $\mathcal{O}_{V^\vee}$ and the lift is the canonical symplectic lift of an automorphism of the base of a cotangent bundle. The latter is given in terms of $f_t$ and $f_t^{-1}$ and thus the lift is defined for all $t$ for which $f_t^{-1}$ is defined. Moreover the action of the lift on generators of degree bounded by $n$ is defined by the restriction of $f_t, f_t^{-1}$ to $R^i_M$ for finitely many $i$ depending on $n$. The result is that there is a family $F_t$ of automorphisms of $\mathcal{O}_M$ given by a sequence of compatible maps

$$\mathcal{O}_M / F^p \mathcal{O}_M \to \mathcal{O}_M / F^p \mathcal{O}_M,$$

parametrized by $t$ in a Zariski open ($p$-dependent) subset $V_p$ (an intersection of finitely many $U_i$) of $\mathbb{A}^1$ containing 0 and 1. By construction $F_0$ is the identity and we may assume (by possibly composing $F_t$ with a gauge equivalence of the form $\exp(ta)$, $a \in \mathfrak{g}(M)$) that $F_1 = F$. Thus $S_t := F_t(S) \in \text{lim}_\rightarrow \mathcal{O}(M) \otimes k(t) / F^p \mathcal{O}(M) \otimes k(t)$ is a family of solutions of the master equation for the Tate resolution $R_M$. It is given by a compatible sequence $S^{(p)}_t = S_t \mod F^p \mathcal{O}_M$ that is defined for $t$ in the open set $V_p \subset \mathbb{A}^1$ containing 0 and 1 such that $S_0 = S_1 = S$.

The next step is to replace $F_t$ by $G_t \circ F_t$ for some gauge equivalence $G_t$ such that $G_t(S_t) = S$ and such that $G_0 = G_1 = id$. We need to check that $G_t$ can be chosen this way and that it is defined when $F_t$ is. The construction of a gauge equivalence relating two solutions of the master equation associated with the same Tate resolution is done recursively in the filtration degrees, see part (c) of the previous section, and the induction step relies on the vanishing of the cohomology of the complexes $\mathcal{G}_{p,q}$ of locally free $\mathcal{O}_X$-modules of finite rank. Now $S_t - S$ mod $F^p \mathcal{O}(M)$ vanishes at $t = 0$ and $t = 1$ and is defined for $t \in V_p$. In other words $S_t - S \in \mathcal{O}(M) / F^p \mathcal{O}(M) \otimes t(1-t)k[V_p]$, where $k[V_p] \subset k(t)$ is the space of rational functions that are regular on $V_p$. Thus the construction of the previous section applies to the complex $\mathcal{G}_{p,q} \otimes_k t(1-t)k[V_p]$ (the tensor product with the free and thus flat $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_k t(1-t)k[V_p]$) gives recursively a gauge equivalence $G_t$ such that $G_0 = G_1 = id$ and $G_t(S_t) = S$, as required.

We thus have a compatible family of morphisms $F^{(p)}_t \in \text{End}(\mathcal{O}_M / F^p \mathcal{O}_M) \otimes k[V_p]$ whose value at every $t \in V_p$ is an automorphism such that $F^{(p)}_t(S) = S \mod F^p \mathcal{O}(M)$, so that $F^{(p)}_t$ commutes with $d_S$. The inverse limit $F_t$ is defined for $t$ in a countable intersection of Zariski open subsets and is given by a sequence $F^{(p)}_t$ each parametrized by $t$ in a Zariski open subset of $\mathbb{A}^1$. 
Claim: $F_t^{(p)}$ acts trivially on the cohomology $H^j(O_M/F^pO_M, d_S)$ for $p$ large enough depending on $j$. To prove this claim we use the following result.

**Lemma 4.14.** For any $j$ there exist a $p_0(j)$ such that $H^j(O_M/F^pO_M, d_S) \simeq H^j(O_M, d_S)$ for all $p \geq p_0(j)$.

This Lemma is proved in Section 5; see Theorem 5.1. Given the claim, it also implies that $F = F_1$ acts trivially on the cohomology. To prove the claim, we take the derivative $F_t$ of $F_1$ with respect to $t$.

The endomorphism $F_t^{-1} \circ F_t$ is a Poisson derivation of degree 0 of $O_T[-1]V$. By Prop. 2.16 all Poisson derivations of degree zero are uniquely Hamiltonian. Thus there exists an element $K_t$ of degree $-1$ such that

$$F_t^{-1} \circ F_t = [K_t, \cdot].$$

As above, these expression have to be understood as sequences of families of endomorphisms of $O_M/F^pO_M$ parametrized by $t \in V_p$. By the uniqueness of the Hamiltonian, $K_t$ is defined whenever $F_t$ is. As $F_1(S) = S$, we see that $K_t$ is a cocycle: $[S, K_t] = 0$. But by Theorem 4.13 (proved in Section 5) the cohomology in degree -1 is trivial, so there exists a family of elements $L_t^{(p)} \in \Gamma(X, O_M/F^pO_M) \otimes k[V_p]$ for $p$ large of degree $-2$ such that $K_t = [S, L_t^{(p)}] \mod F^pO_M$. With the Jacobi identity we obtain the homotopy formula

$$F_t^{-1} \circ F_t(a) = [S, [L_t^{(p)}, a]] + [L_t^{(p)}, [S, a]], a \in O_M/F^pO_M,$$

from which it follows that the action on the cohomology is trivial. □

**Corollary 4.15.** Let $S_0$ be a function on a nonsingular affine variety $X$ over $k$. Then the BRST cohomology sheaf $\mathcal{H}^*(M, S)$ is determined by $(X, S_0)$ up to unique isomorphism.

**Proof.** By Lemma 3.7 the BRST complexes of BV varieties differing by taking products with trivial BV varieties are canonically quasi-isomorphic. By Theorem 4.10 any two BV varieties with support $(X, S_0)$ become equivalent after taking such products. The equivalence induces a quasi-isomorphism of the corresponding BRST-complexes as differential $P_0$-algebras. By Theorem 4.12 any two equivalences differ by an automorphism, so they induce the same map on BRST cohomology. □

**5. Computing the BRST cohomology**

The BRST complex of a BV variety $(M, S)$ with support $(X, S_0)$ is a sheaf of differential graded $P_0$-algebras over $X$. We consider here the cohomology sheaf $\mathcal{H}(M, S)$ of the BRST complex for $X$ affine and $M = T^*[-1]V$. We use the local description of $O_M$ of 2.4.

**5.1. The spectral sequence.** The main tool for the computation of the BRST cohomology is the spectral sequence of the filtered complex $(O_M, d_S)$. Recall that the sheaf of Jacobian rings $J(S_0)$ is by definition the cokernel of the map $dS_0: T_X \to O_X$.

**Theorem 5.1.** Let $(M, S)$ be a BV variety with support $(X, S_0)$ such that $X$ is affine.
(i) There is a fourth quadrant spectral sequence degenerating at $E_2$ such that

$$E_2^{p,q} = \begin{cases} \mathcal{H}^p(J(S_0) \otimes_{\mathcal{O}_X} \mathcal{O}_V, d_1), & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases}$$

The differential $d_1$ is described as follows. Let $S^{(1)}$ the component in $\Gamma(X, T_V[1])$ of $S \in \Gamma(X, \Sym^2_{\mathcal{O}_V} T_V[1])$. Then the derivation $S^{(1)}$ vanishes on the kernel of the canonical projection $\mathcal{O}_V \to J(S_0) \otimes_{\mathcal{O}_X} \mathcal{O}_V$. The induced differential on the image is $d_1$.

(ii) This spectral sequence converges to the cohomology $\mathcal{H}(M, S)$. More precisely, the edge homomorphism

$$\mathcal{H}^*(M, S) \to E_2^{*, 0}$$

is an isomorphism of graded commutative algebras.

(iii) The natural map $\mathcal{H}(\mathcal{O}_M, d_S) \to \mathcal{H}(\mathcal{O}_M / F^{p+1} \mathcal{O}_M)$ is an isomorphism for $j < p$ and a monomorphism for $j = p$.

PROOF. The first term in the spectral sequence of the filtered complex is (see, e.g., [9], Chap. XV, §4)

$$E_0^{p,q} = F^p \mathcal{O}_M^{p+q} / F^{p+1} \mathcal{O}_M^{p+q}.$$ 

By Remark 4.2, $E_0$ lives in the fourth quadrant $p \geq 0, q \leq 0$ and, as a sheaf of $k$-algebras,

$$E_0^{p,q} \cong R^p_M \otimes_{\mathcal{O}_X} \mathcal{O}_V,$$

where $(R_M = \mathcal{O}_M / I_M, \delta)$ is the resolution of $J(S_0)$ associated with $S$. By Prop. 4.1 the differential is $\delta \otimes \id$. It follows that

$$E_1^{p,q} \cong \begin{cases} J(S_0) \otimes_{\mathcal{O}_X} \mathcal{O}_V, & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases}$$

Let us compute the differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ by decomposing $S$ according to the power in the symmetric algebra:

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \cdots, \quad \text{with } S^{(j)} \in \Gamma(X, \Sym^j_{\mathcal{O}_V} T_V[1]).$$

Since the natural map $\mathcal{O}_V \to F^p \mathcal{O}_M / F^{p+1} \mathcal{O}_M = E^{p,0}$ is an isomorphism, we may compute $d_1$ by acting with $d_S$ on representatives in $\mathcal{O}_V$. We have $S^{(0)} = S_0$ and thus $[S^{(0)}, \mathcal{O}_V] = 0$. Let $I_M$ be the ideal of $\mathcal{O}_M$ generated by elements of negative degree. Then, if $j \geq 2, S^{(j)} \in I_M \cdot I_M$ and thus $[S^{(j)}, \mathcal{O}_V] \subseteq I_M \cap \mathcal{O}_M^{p+1} \subset F^{p+2} \mathcal{O}_M$. Hence for $a \in \mathcal{O}_V$ and $j \geq 2$ the class of $[S^{(j)}, a]$ in $E^{p+1,0} = F^{p+1} \mathcal{O}_M^{p+1} / F^{p+2} \mathcal{O}_M^{p+1}$ vanishes. By definition of the Poisson structure on $\mathcal{O}_M$, the bracket of $S^{(1)}$ with $\mathcal{O}_V$ is the action of $S^{(1)}$ viewed as a derivation.

Since $E_1^{p,q} = 0$ for $q \neq 0$, all higher differentials vanish for degree reasons and the spectral sequence degenerates.

It remains to show that the spectral sequence converges to the cohomology $\mathcal{H}(M, S)$. Recall from [9], p. 324 that a descending filtration $\cdots \supset F^p A \supset F^{p+1} A \supset \cdots$ of a cochain complex $A$ is called regular if for each $m$ there exists a $p_0 = p_0(m)$ such that the cohomology $H^m(F^p A)$ vanishes for $p \geq p_0$. By [9], Chap. XV, Prop. 4.1 the spectral sequence of a regular filtration converges to the cohomology.
LEMMA 5.2. Suppose $\cdots \supset F^p A \supset F^{p+1} A \supset \cdots$ is a filtration of a cochain complex $(A, d)$ such that, for each $j$, the natural map

\begin{equation}
A^j \rightarrow \lim_{\to} A^j / F^p A^j
\end{equation}

is an isomorphism and assume that $E_1^{p,m-p} = H^m(F^p A / F^{p+1} A) = 0$ for $p \geq p_0(m)$ sufficiently large. Then $H^m(F^p A) = 0$ for $p \geq p_0(m)$ and thus the filtration is regular.

PROOF. Let $p \geq p_0$ and $z \in F^p A^m$ be a cocycle representing a class in $H^m(F^p A)$. By the assumption on $E_1$, $z \equiv dy_0 \mod F^{p+1} A$ for some $y_0 \in F^p A^{m-1}$. Thus $z - dy_0 \in F^{p+1} A$ and by the same argument we find $y_1 \in F^{p+1} A$ such that $z - dy_0 - dy_1 \in F^{p+2} A$. Iterating we conclude that $z \equiv d(y_0 + \cdots + y_r) \mod F^{p+r+1} A^{m-1}$, for some $y_j \in F^{p+j} A^{m-1}$. The sequence $(y_0 + \cdots + y_r)_r \geq 0$ defines an element of $\lim_{\to} F^p A^{m-1} / F^{p+r} A^{m-1}$. Let $y \in F^p A^{m-1}$ be its inverse image by the isomorphism (5.1). Then $z = dy$ and it follows that $H^m(F^p A) = 0$. \qed

Since the completed complex $O_M$ obeys (5.1) and the assumption on $E_1$ holds with $p_0(m) = m + 1$, we have

\begin{equation}
\mathcal{H}^m(F^p O_M, d_S) = 0, \quad \text{for } p > m
\end{equation}

and the proof of convergence is complete. For the statement about the map $\mathcal{H}^p(M, S) \rightarrow E_2^{p,0}$ see [9], Chap. XV, Theorem 5.12. Since the product is compatible with the filtration, the edge homomorphism is an algebra homomorphism.

The statement (iii) follows from the long exact sequence associated with the short exact sequence

$0 \rightarrow F^{p+1} O_M \rightarrow O_M \rightarrow O_M / F^{p+1} O_M \rightarrow 0,
$}

and (5.2).

\begin{remark}
5.3. Theorem 5.1 shows that although in general Tate resolutions require in general an infinite dimensional $V$, computing the BRST cohomology in a given degree is a finite process: computing $\mathcal{H}^j(M, S)$ requires knowing $S$ modulo $F^{j+1}$, which in turn can be computed from the Tate resolution down to degree $-j - 1$.

COROLLARY 5.4. The BRST cohomology sheaf vanishes in negative degree.

PROOF. This is obvious as $E_2 \cong \mathcal{H}(M, S)$ is the cohomology of a complex $E_1$ concentrated in non negative degree. \qed

5.2. BRST cohomology in degree 0. Let us compute $\mathcal{H}^0(M, S)$ from the spectral sequence:

$\mathcal{H}^0(M, S) \cong \text{Ker}(d_1 : j(S_0) \rightarrow \oplus_{i=1}^r J(S_0) \beta^i),$

where $\beta^1, \ldots, \beta^r$ is a local basis of the locally free $O_X$-module $O^1_X$. The differential $d_1$ is induced from the bracket with the terms in $S$ linear in the dual variables $x_i^* \in O^1_{M_i}$. To compute it we need to construct a Tate resolution down to degree $-2$.

The Lie algebra $L(S_0) = \{ \xi \in T_X \mid \xi(S_0) = 0 \}$ acts on the algebra $J(S_0)$ by derivations. The vector fields $\xi(S_0) \eta - \eta(S_0) \xi$, for $\xi, \eta \in T$, act by zero and span a Lie ideal $L_0(S_0)$. Let $L^\text{eff}(S_0) = L(S_0) / L_0(S_0)$. It is a Lie algebra and an $O_X$-module and comes with an $O_X$-linear Lie algebra homomorphism $L^\text{eff}(S_0) \rightarrow$
Der(\(J(S_0)\)). As \(L_0(S_0)\) acts trivially on \(J(S_0)\), the action of \(L^{\text{eff}}(S_0)\) on \(J(S_0)\) is defined and we have

\[
J(S_0)^{L(S_0)} = J(S_0)^{L^{\text{eff}}(S_0)}.
\]

**Proposition 5.5.** \(H^q(M, S) \cong J(S_0)^{L(S_0)}\)

**Proof.** The Tate resolution down to degree \(-2\) looks like

\[
\cdots \rightarrow \bigwedge^2 T_X \oplus \bigoplus_{i=1}^r \mathcal{O}_X \beta_i^* \rightarrow T_X \rightarrow \mathcal{O}_X.
\]

The map \(\bigwedge^2 T_X \rightarrow T_X\) sends \(\xi \wedge \eta\) for vector fields \(\xi, \eta \in T_X\) to \(\xi(S_0)\eta - \eta(S_0)\xi\). Its image is \(L_0(S_0)\). Since the cohomology must vanish in degree \(-1\), \(\delta\) must map generators \(\beta_i^*\) of degree \(-2\) to vector fields \(\xi_i = \delta(\beta_i^*)\) which together with \(L_0(S_0)\) span the kernel \(L(S_0)\) of \(dS_0\): \(T_X \rightarrow \mathcal{O}_X\). In other words the classes of \(\xi_i\) generate the \(\mathcal{O}_X\)-module \(L^{\text{eff}}(S_0)\). Next we use the fact that \(d_1\) is given by the induced action of the component in \(\Gamma(X, T_V[1])\) of \(S\). Now \(S = S_{\text{lin}} \mod I_{(2)}\)

The only term contributing to \(d_1: J(S_0) \rightarrow \bigoplus_i J(S_0)\beta_i^*\) is then \(\sum \delta(\beta_i^*)\beta_i^* = \sum \xi_i \beta_i\) appearing in \(S_{\text{lin}}\): indeed, by degree reasons, the terms linear in \(T_V[1]\) in \(I_{(2)}\) cannot have a component in \(T_X[1]\) and thus vanish when acting on \(J(S_0)\). Therefore \(d_1 f = \sum [\xi_i \beta_i, f] = -\xi_i(f)\beta_i^*\). Thus the kernel consists of elements of \(J(S_0)\) annihilated by vector fields spanning \(L(S_0)\). By (5.3) this proves the claim. 

**Corollary 5.6.** Let \((M, S)\) be a BV variety with support \((X, S_0)\) and suppose that \(S_0\) has no critical points. Then \(H(M, S) = 0\).

**Proof.** In this case \(J(S_0) = 0\) and thus \(E_2 = 0\). 

Thus the sheaf \(H(M, S)\) has support on the critical locus of \(S_0\).

### 5.3. Hypercohomology.**

The **BRST cohomology** \(H(M, S)\) of a BV variety with support \((X, S_0)\) is the hypercohomology of the BRST complex of sheaves. Then there is a hypercohomology spectral sequence converging to \(H^{p+q}(M, S)\) and whose \(E_2\)-term is

\[
E_2^{p,q} = H^p(X, H^q(M, S)).
\]

The results on \(\mathcal{H}\) in non-positive degree imply:

**Corollary 5.7.** The **BRST cohomology** of a BV variety \((M, S)\) with support \((X, S_0)\) vanishes in negative degree and

\[
H^0(M, S) = \Gamma(X, J(S_0)^{L(S_0)}).
\]

Also, there is a second hypercohomology spectral sequence whose \(E_2^{p,q}\) term is the \(p\)-th cohomology of \([S, ]\) on \(H^q(X, \mathcal{O}_M)\). If \(X\) is affine, the hypercohomology coincides with the cohomology of global sections and we obtain:

**Corollary 5.8.** Let \(X\) be an affine variety. Then

\[
H^\bullet(M, S) = H^\bullet(\Gamma(X, \mathcal{O}_M), d_S).
\]

If \(X\) is affine, the BRST cohomology is determined up to unique isomorphism by \((X, S_0)\), see Corollary 4.15. It then makes sense to define the BRST cohomology of \((X, S_0)\) as

\[
H^\bullet(X, S_0) = H^\bullet(M, S),
\]
for any choice of \((M, S)\) with support \((X, S_0)\) as in Theorem 1.2. From the Mayer–Vietoris sequence we then obtain:

**Corollary 5.9.** Let \(X\) be an affine variety. Suppose that the critical locus of \(S_0 \in \Gamma(X, \mathcal{O}_X)\) has two disjoint components \(C_1 \subset U_1, C_2 \subset U_2\) contained in open sets \(U_1, U_2\) such that \(C_1 \cap U_2 = \emptyset = C_2 \cap U_1\). Then

\[
H^\bullet(X, S_0) = H^\bullet(U_1, S_0|_{U_1}) \oplus H^\bullet(U_2, S_0|_{U_2}).
\]

6. Examples

In this section we discuss some examples of functions \(S_0\) and corresponding BV varieties \((M, S)\), see Definition 3.1.

The simplest non-trivial examples are the quadratic forms.

**Example 6.1.** Suppose that \(S_0 = a_1(x^1)^2 + \cdots + a_n(x^n)^2 \in k[x^1, \ldots, x^n]\) for some \(a_i \in k^\times\) with \(0 \leq j \leq n\). Then we can choose the Tate resolution

\[
R = k[x_1^*, \ldots, x_n^*, \beta_{j+1}^*, \ldots, \beta_n^*], \quad \deg(x_i^*) = -1, \quad \deg(\beta_i^*) = -2,
\]

and \(\delta(x_i^*) = 2a_i x^i, \delta(\beta_i^*) = x_i^* (i > j)\). Then

\[
S = S_0 + \sum_{i=j+1}^n x_i^* \beta_i^*
\]

is a solution of the classical master equation. The BRST cohomology is one-dimensional concentrated in degree 0. Indeed this example is obtained from the zero example \((X = \{\text{pt}\}, S_0 = 0)\) by adding squares (Prop. 4.12) and taking the product with trivial BV varieties, see 3.6.

**Example 6.2.** Let \(S_0\) be a regular function on a nonsingular affine variety \(X\). Suppose that \(\Gamma(X, T_X)\) is spanned by vector fields \(\xi_1, \ldots, \xi_n\) with the property that \(\xi_1(S_0), \ldots, \xi_n(S_0)\) form a regular sequence. Then we can choose \(\text{Sym}_{\mathcal{O}_X}(T_X[1]) = \wedge T_X\), the exterior algebra of the \(\mathcal{O}_X\)-module \(T_X\), as a Tate resolution: it is the Koszul resolution associated with the regular sequence. Then \(S = S_0\) and the BRST complex is concentrated in non-positive degree and has cohomology \(H^0(M, S) \cong J(S_0), H^j(M, S) = 0, j \neq 0\). This class includes the case of isolated critical points.

**Example 6.3.** (Faddeev–Popov action) Let \(\mathfrak{g}\) be a finite dimensional Lie algebra over \(k\) acting on a nonsingular algebraic variety \(X\). We assume that the action of \(\mathfrak{g}\) is **infinitesimally free and transitive** on the fibers of a flat morphism \(\pi: X \to Y\) with \(Y\) smooth and irreducible smooth fibers, namely that it is given by a Lie algebra homomorphism \(\theta: \mathfrak{g} \to \Gamma(X, T_X)\) such that (a) \(\theta(\mathfrak{g}) \subset \Gamma(X, T_X|_Y)\) where \(T_X|_Y\) is the sheaf of vector fields tangent to the fibers and (b) the action \(\theta: \mathcal{O}_X \otimes \mathfrak{g} \to T_X|_Y\) is an isomorphism of \(\mathcal{O}_X\)-modules. For example \(\pi: X \to Y\) could be a principal G-bundle where \(G\) is a connected algebraic group whose Lie algebra is \(\mathfrak{g}\).

Suppose that \(S_0 \in \Gamma(X, \mathcal{O}_X)^g\) is the pull-back of a function \(S_0|_Y\) on \(Y\) with isolated critical points. Then a BV variety with support \((X, S_0)\) is given by the classical construction of Faddeev and Popov in the context of gauge theory [16]. The graded variety \(V\) is \((X, \mathcal{O}_V)\) where \(\mathcal{O}_V = \mathcal{O}_X \otimes \wedge^\bullet \mathfrak{g}^*\) with \(\mathfrak{g}^*\) in degree 1. Thus

\[
\mathcal{O}_{T^*|-1|V} = \wedge \mathcal{O}_X T_X \otimes \text{Sym} \mathfrak{g} \otimes \wedge \mathfrak{g}^*.
\]
with \( g \) in degree \(-2 \) and \( T_X \) in degree \(-1 \). The solution of the master equation is 
\[ S = S_0 + a + b, \]
where \( a \in T_X \otimes g^* \cong \text{Hom}_X(g, T_X) \) is the infinitesimal action and \( b \in \wedge^2 g^* \otimes g \cong \text{Hom}_X(\wedge^2 g, g) \) is \((-1/2)\) times the bracket. More explicitly, let \( \beta_1, \ldots, \beta_m \) (the “Faddeev–Popov ghosts”) be a basis of \( g^* \) with dual basis \( \beta^*_1, \ldots, \beta^*_m \) of \( g \) (the “antighosts”), with commutation relations 
\[ [\beta^*_i, \beta^*_j] = \sum \ell c_{ij}^\ell \beta^*_\ell \] 
and fundamental vector fields \( \theta_i = \theta(\beta_i) \). Then 
\[ S = S_0 + \sum_{i=1}^m \theta_i \beta_i - \frac{1}{2} \sum_{i,j=1}^m c_{ij}^\ell \beta^*_\ell \beta^*_i \beta^*_j. \]

It is a well-known exercise to check that \([S, S] = 0\) and it follows from the more general result of the next example. Since \( S|_X = S_0 \) axioms (i), (ii) of BV varieties, see Definition 3.1, are fulfilled.

Let us check that \((M, S)\) obeys axiom (iii). We have \( R_M = \mathcal{O}_M/I_M = \text{Sym} g \otimes \wedge T_X \) with \( g \) in degree \(-2 \) and \( T_X \) in degree \(-1 \). The induced differential is the \( \mathcal{O}_X \)-linear derivation sending \( a \in g \) to \( \theta a \in T_X \) and \( \xi \in T_X \) to \( \xi(S_0) \in \mathcal{O}_X \). Consider the ascending filtration \( 0 \subset F_0 R_M \subset F_1 R_M \subset F_2 R_M \subset \cdots \) where \( F_p R_M \) is spanned by \( a \otimes \xi_1 \wedge \cdots \wedge \xi_m \in \text{Sym} g \otimes \wedge^m X \), where at most \( p \) among \( \xi_1, \ldots, \xi_m \) do not belong to \( T_X \). The associated graded is then 
\[ E^0_{p-q} = F_p R_M^{p-q}/F_{p-1} R_M^{p-q} = \bigoplus_{2j+\ell=q} \text{Sym}^j g \otimes \wedge^j T_X \otimes \mathcal{O}_X \wedge^\ell (T_X/T_X|_Y), \quad p, q \geq 0. \]

Here we switched to chain complex conventions to get a first quadrant homology spectral sequence instead of the third quadrant cohomology spectral sequence \( E^{-p,q} = E^r_{p-q} \) corresponding to the descending filtration \( F^{-p} = F_p \). The differential \( d_0 \) is the derivation vanishing on \( T_X|_Y \) and on \( T_X/T_X|_Y \), and coinciding with \( \theta \): \( g \to T_X|_Y \otimes \mathcal{O}_X \) on \( g \). Since \( d_0: \mathcal{O}_X \otimes g \to T_X|_Y \otimes \mathcal{O}_X \) is an isomorphism, the cohomology of \( (E^0_{p,*}, d_0) \) is concentrated in degree 0 and equal to \( E^1_{p,0} = \wedge^p T_X/T_X|_Y = \pi^* \wedge^p T_Y \). The next differential is \( d_1 = \pi^* \delta \), where \( \delta(\xi) = \xi(S_0|_Y) \). For isolated singularities \( \wedge^p T_Y, \delta \) is a Koszul resolution and thus has cohomology concentrated in degree 0. Since \( \pi \) is flat, \( \pi^* \) is exact and \( E^2 \) is concentrated in bidegree \((0,0)\). The spectral sequence degenerates and the cohomology of \( \mathcal{O}_M/I_M \) is trivial in nonnegative degrees. Thus \((M, S)\) obeys (iii) and is indeed a BV variety.

The BRST complex is quasi-isomorphic to \((J(S_0) \otimes \wedge g^*, d_S)\). The induced differential \( d_S \) is the Chevalley–Eilenberg differential of the \( g \)-module \( J(S_0) \).

**Example 6.4.** (Lie algebroids) The previous example is a special case of a more general construction: let \( X \) be a nonsingular variety, \( \mathcal{L} \) a Lie algebroid, i.e. a locally free sheaf \( \mathcal{L} \) of \( \mathcal{O}_X \)-modules with a Lie bracket \([ \cdot, \cdot ]: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \) and a Lie algebra homomorphism \( \rho: \mathcal{L} \to T_X \), called the anchor, such that \([v, f w] \mathcal{L} = f[v, w] \mathcal{L} + \rho_v(f) w \) for all \( v, w \in \mathcal{L}, f \in \mathcal{O}_X \). As shown by Vaintrob [34], Lie algebroid structures on a locally free \( \mathcal{L} \) are in one-to-one correspondence with homological vector fields on the graded variety \( V = (X, \text{Sym}_{\mathcal{O}_X} \mathcal{L}^*[\cdot]) \), namely vector fields \( Q \in \Gamma(X, T_V) \) of degree 1 such that \([Q, Q] = 0\). The vector field \( Q \) corresponding to a Lie algebroid structure is the Chevalley–Eilenberg differential on \( \mathcal{O}_V = \wedge \mathcal{O}_X \mathcal{L}^* \) namely the derivation sending \( f \in \mathcal{O}_X \) to \( Q(f) \in \mathcal{L}^* \) with \( Q(f)(v) = \rho_v(f) \) and \( \beta \in \mathcal{L}^* \) to \( Q(\beta) \in \wedge^2 \mathcal{L} \cong \text{Hom}_{\mathcal{O}_X}(\wedge^2 \mathcal{L}, \mathcal{O}_X) \) given by 
\[ u \wedge v \mapsto \rho_u \beta(v) - \rho_v \beta(u) - \beta([u, v] \mathcal{L}), \quad u, v \in \mathcal{L}. \]
The corresponding Hamiltonian function $S_Q$ on $M = T^*[-1]V$ of degree zero is then a solution of the master equation.

If $S_0 \in \Gamma(X, \mathcal{O}_X)$ is a regular function invariant under the Lie algebroid, in the sense that $\rho_u(S_0) = 0$ for all local sections $u \in \mathcal{L}$, then $S = S_0 + S_Q$ is a solution of the master equation. It has the same local form as \((6.1)\) with $\xi_i = \rho_{\beta_i^*}$ for some local basis $\beta_i^*$ of $\mathcal{L}$. The main difference is that the structure constants $c_{ij}^k$ are not constants but functions in $\mathcal{O}_X$.

The pair $(M, S)$ is a BV variety if the cohomology of $\mathcal{O}_M/I_M$ is trivial in nonzero degree. As in the case of the previous example, this follows if there is a flat morphism $p: X \to Y$, such that $\rho: \mathcal{L} \to T_{X/Y}$ is an isomorphism and $S_0$ is the pull-back of a function on $Y$ with isolated critical points.

In the next example we consider the case of the function $0$ on a (not necessarily affine) variety $X$. We show that the $(-1)$-shifted cotangent bundle of the 1-shifted tangent bundle of any variety $X$ has a canonical structure of BV variety with support $(X, 0)$. Its BRST cohomology is the de Rham cohomology with trivial bracket.

**Example 6.5.** Let $\Omega^*_X = \text{Sym}_{\mathcal{O}_X} T^*_X[-1]$ be the sheaf of differential forms on a nonsingular variety $X$ and let $T[1]X$ (the 1-shifted tangent bundle to $X$) denote the $\mathbb{Z}_{\geq 0}$-graded variety $T[1]X = (X, \Omega^*_X)$. The de Rham differential $d$ is a derivation of $\Omega^*_X$ of degree 1 obeying $[d, d] = 0$. Let $M = T^*[-1](T[1]X)$. Let $S$ be $d$, viewed as a section of degree 0 of $\text{Der}(\Omega^*_X)[1] \subset \mathcal{O}_M$. The function $S \in \Gamma(X, \mathcal{O}_M)$ is a hamiltonian function of $d$ and is a solution of the classical master equation.

**Proposition 6.6.** The pair $(M = T^*[-1](T[1]X), S)$ is a BV variety with support $(X, 0)$. The canonical inclusion of scalars

$$i: \Omega^*_X \to \mathcal{O}_M = \text{Sym}_{\mathcal{O}_X}(\text{Der}(\Omega^*_X)[1])$$

is a quasi-isomorphism of sheaves of differential $P_0$-algebras from the de Rham algebra $(\Omega^*_X, d)$, viewed as a $P_0$-algebra with trivial bracket to the BRST complex $(\mathcal{O}_M, d_S)$.

The first statement follows from the fact that this example is a special case of the previous one, as the tangent bundle with the Lie bracket of sections and identity anchor is a Lie algebroid. To prove the second statement, notice that the morphism $i$ is the inclusion of the first summand in

$$\mathcal{O}_M = \Omega^*_X \oplus \text{Der}(\Omega^*_X)[1] \oplus \text{Sym}^2_{\mathcal{O}_X}(\text{Der}(\Omega^*_X)[1]) \oplus \cdots$$

and thus comes with a projection $p: \mathcal{O}_M \to \Omega^*_X$ such that $p \circ i = \text{id}$. The left inverse $p$ is a morphism of differential graded commutative algebras. It is sufficient to show that the chain map $i \circ p$ is locally homotopic to the identity. To show this notice that every point has an open neighborhood $U$ such that $\text{Der}(\Omega^*_X)|_U$ is a free $\Omega^*_U$-module generated by interior multiplication $\iota_j$ and Lie derivative $L_j = d_S(\iota_j) = [d, \iota_j]$ by vector fields $\partial_j$, $j = 1, \ldots, \dim X$ trivializing $T_X$. Then the $\Omega^*_X$-linear derivation $h$ of $\mathcal{O}_M|_U$ sending $L_j$ to $\iota_j$ and $\iota_j$ to zero obeys

$$d_S \circ h + h \circ d_S = m \text{id}$$

on $\text{Sym}^m_{\mathcal{O}_X}(\text{Der}(\Omega^*_X)[1])|_U$, and $H = 0 \oplus \bigoplus_{m > 0} \frac{1}{m}h$ is a homotopy between $i \circ p$ and the identity of $\mathcal{O}_M$.

\[\text{There is no completion here as the degrees are bounded above}\]
Example 6.7. Let $\pi : X \to Y$ be a vector bundle with a nondegenerate symmetric bilinear form $(\ , \ )$ on the fibers. The associated quadratic form

$$S_0(v) = \frac{1}{2}(v, v)$$

is a regular function on $X$ whose critical locus is the zero section. BV varieties with support $(X, S_0)$ can be obtained from orthogonal connections $\nabla$ on $X$. Let $E$ be the locally free $\mathcal{O}_Y$ module of sections of $\pi$ so that $\mathcal{O}_X = \text{Sym}_{\mathcal{O}_Y} E^*$. Suppose that $\nabla \in \text{Hom}(E, E \otimes \mathcal{O}_Y, \Omega^1_Y)$ is a connection such that $d(u, v) = \langle \nabla u, v \rangle + \langle u, \nabla v \rangle$ for all $u, v \in E$. Let $V$ be the graded variety $(X, \pi^* \Omega^*_Y)$. The solution $S$ of the classical master equation is a function on $T^*[-1]V$. Notice first that for an open set $U \subset Y$,

$$(6.2) \quad \mathcal{O}_V(U) = \pi^* \Omega^*_Y(\pi^{-1}(U)) = \Gamma(U, \text{Sym}_{\mathcal{O}_Y} E^* \otimes \mathcal{O}_V \Omega^*_Y).$$

The connection $\nabla$ induces a connection $\nabla^*$ on $E^*$ and therefore a derivation of degree 1, also denoted $\nabla^*$ of $\text{Sym}_{\mathcal{O}_Y} E^* \otimes \mathcal{O}_Y \Omega^*_Y$ and via (6.2) a vector field of degree 1 on $V$. Let $\nabla^*$ be the corresponding Hamiltonian: it is just $\nabla^*$ viewed as a function on $T^*[-1]V$. Let $F^* \in \Gamma(Y, \Omega^2_Y \otimes \mathcal{O}_Y \text{End}(E^*))$ be the curvature of $\nabla^*$. Since the bilinear form is non-degenerate, it defines an isomorphism $b : E^* \to E$; therefore $b \circ F^*$ may be viewed as bilinear form on $E^*$ which, by the orthogonality of the connection, is skew-symmetric and thus defines an element $4S_F$ of $\Omega^2_Y \otimes \Lambda^2 E \subset \text{Sym}^2_{\mathcal{O}_Y}(\mathcal{T}_V[1])$. The claim is that

$$S = S_0 + S_V + S_F$$

is a solution of the master equation. To check it, let us realize $\mathcal{O}_M$ as the symmetric algebra of $T_V[1]$ and notice that $S_0$ is a function in $\mathcal{O}_V$, so $[S_0, S_0] = 0$. The fact that the connection is orthogonal is equivalent to $[S_V, S_0] = \nabla^* S_0 = 0$. Then $[S_V, S_V]$ is the derivation $[\nabla^*, \nabla^*]$ which is $2F^*$ on generators in $E^*$ and is extended as a derivation of $\mathcal{O}_V$. We claim that

$$[S_V, S_V] + 2[S_0, S_F] = 0.$$ 

Indeed, if $S_0 = \frac{1}{2} \sum g_{ij} v^i v^j$ for some local basis $v^i$ of $E^*$ and $F^*(v^i) = \sum F^i_j v^j$ for local sections $F^i_j$ of $\Omega^2_Y$, then $S_F = \frac{1}{4} \sum F^i_j g_{ij} \mu \nu \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu}$ where $\sum F^i_j g_{ij} = F_0$. Thus $[S_0, S_F] = \sum F^i_j g_{ij} \mu \nu \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu}$, which is the derivation sending $v^i$ to $\sum F^i_j v^j$ and is thus equal to $-F^*$. Finally $[S_V, S_F] = 0$ follows from the Bianchi identity $\nabla F = 0$ and $[S_F, S_F]$ clearly vanishes.

Alternatively, one can do an explicit calculation in the local description using local functions $y^1, \ldots, y^m \in \mathcal{O}_Y(U)$ such that $dy^1, \ldots, dy^m$ are a basis of $\mathcal{O}_Y(U)$ on some neighborhood $U$ of a point of $Y$ and local sections $v^i$ as above. Then one obtains a slightly more general statement: let $A$ be the graded polynomial algebra over $\mathcal{O}_Y$ in variables $y^i, v^i, v^*_i, \beta^{ij}, \beta^*_i, \mu = 1, \ldots, m, i = 1, \ldots, r$ of degrees $-1, 0, -1, 1, -2$ and nontrivial Poisson brackets among generators $[f, y^i] = \partial_i f$, $f \in \mathcal{O}_Y$, $[\beta^{ij}, \beta^*_i] = \delta^{ij}$, $[v^i, v^*_j] = \delta_{ij}$. Suppose $(g_{ij})$ is a (not necessarily invertible) matrix with entries in $\mathcal{O}_Y$, $A^i_j = A^i_{\mu j} dy^{\mu}$ are one-forms on $Y$, $F^{ij} = \frac{1}{2} F^{ij}_{\mu \nu} dy^\mu \wedge dy^\nu$ are two-forms with $F^{ij} = -F^{ji}$, such that (a) $\partial_\mu g_{ij} + 2A^i_{\mu j} g_{ij} = 0$ (orthogonality), (b) $dA^i_j + A^i_j \wedge F^{ij} = F^{ik} g_{kj}$ (structure equation) and (c) $dF^{ij} + A^i_j \wedge F^{ej} - A^j_i \wedge F^{dj} = 0$ (Bianchi identity). Then

$$S = \frac{1}{2} g_{ij} v^i v^j + \beta^\mu (y^\mu - A^i_{\mu j} v^i v^*_j) + \frac{1}{4} F^{ij}_{\mu \nu} \beta^\mu \beta^\nu v^i v^*_j,$$
The induced differential $\delta$ on

$$R_M(\bar{U}) = \mathcal{O}_Y(U)[v^i, v^*_i, y^*_\mu, \beta^*_\mu]$$

is the $\mathcal{O}_Y$-linear derivation such that

$$\delta y^*_\mu = A^i_\mu v^i v^*_i, \quad \delta v^*_i = g_{ij} v^j, \quad \delta v^i = 0, \quad \delta y^*_\mu = \frac{1}{2} \partial_\mu g_{ij} v^i v^j.$$ 

Let $(g^{ij}) \in \mathcal{O}_Y(U)$ the matrix inverse to $(g_{ij})$. Let $h$ be the $\mathcal{O}_Y$-linear derivation of degree -1 such that

$$hv^i = g^{ij} v^*_j, \quad hy^*_\mu = \beta^*_\mu + A^i_\mu g^{jk} v^*_k v^*_i, \quad h\beta^*_\mu = 0 = hv^i.$$ 

Then it is easy to check that for $x = v^i, v^*_i, y^*_\mu, \beta^*_\mu$, one has $(h \circ \delta + \delta \circ h)(x) = x$. It follows that the cohomology is spanned by polynomials of degree 0 in these variables, namely those in $\mathcal{O}_Y(U)$. Thus $(R_M, \delta)$ is indeed a resolution of the Jacobian ring $J(S_0)(\bar{U}) \cong \mathcal{O}_Y(U)$.

The next example illustrates the fact that polynomial functions $S_0$ on affine spaces with critical locus of positive dimension usually lead to complicated solutions of the master equation. It is also an example with nontrivial BRST cohomology in positive degree.

**Example 6.8.** Let $X = \mathbb{A}^2$ be the affine plane with coordinates $x, y$, and set $S_0 = (x^2 + y^2 - 1)^2/4$.

The Jacobian ring is $J(S_0) = k[x, y]/(xh, yh)$, where $h = x^2 + y^2 - 1$. The critical locus is $h^{-1}(0) \cup \{0\}$. We have $L(S_0) = k[x, y]_\tau$, $\tau = y \partial_x - x \partial_y$, $L_0(S_0) = k[x, y]h\tau$. $L^\text{eff}(S_0)$ is the $\mathcal{O}$-module generated by $\tau$ with relation $h\tau = 0$. There does not appear to be a finitely generated Tate resolution in this case. Here is a BV variety $(M, S)$ up to filtration degree 4: $V$ has coordinates $x, y$ of degree 0, $\beta$ of degree 1, $\gamma$ of degree 2, $\xi, \eta$ of degree 3, $\rho, \mu, \nu, \phi$ of degree 4. The solution of the master equation up to filtration degree four, computed with the help of Macaulay2 is

$$S \equiv S_0 + (xy^* - yx^*)\beta + (h\beta^* - x^* y^*)\gamma + (x\gamma^* - x^* \beta^*)\xi + (y\gamma^* - y^* \beta^*)\eta + (xy^* \gamma^* - yx^* \gamma^* + x^* y^* \beta^*)\rho$$

$$(h\xi^* - x^* \gamma^*)\mu + (h\eta^* - y^* \gamma^*)\nu + (\beta^*)^2 \gamma^2 - \eta^* \beta^2 + \xi^* \eta^* + (\beta^*)^2/2)\phi \mod F^5.$$ 

It is probably hopeless to compute the BRST cohomology using this formula. Instead we can use Corollary 5.3 to write $X = U_0 \cup U_1$ with $U_0 = \{(x, y) \in \mathbb{A}^2 | x^2 + y^2 \neq c\}$. Then $S_0|_{U_1}$ has an isolated critical point with one dimensional Jacobian ring, so that $H^*(U_1, S_0|_{U_1}) = k$, concentrated in degree 0. The map $h: U_0 \to \mathbb{A}^1$ is a principal $SO(2)$-bundle and $S_0|_{U_0}$ is the pull-back of $h^2 \in \mathcal{O}(\mathbb{A}^1) = k[h]$, so we are in the setting of Faddeev–Popov as in Example 6.3.

The BRST complex is quasi-isomorphic to the Chevalley–Eilenberg complex of the one-dimensional Lie algebra $so(2) = k$ of the rotation group with values in the algebra $A = \Gamma(U_0, J(S_0))$ of functions on the critical set. By definition
A = k[x,y,(x^2 + y^2)^{-1}]/I where I is the ideal generated by xh,yh. But I is also generated by h since h = (x^2h + y^2h)(x^2 + y^2)^{-1}. So over an algebraic closure \( \overline{k} \), \( A \otimes \overline{k} = \overline{k}[t, t^{-1}] \), \( t = x + \sqrt{-1}y \). The Chevalley–Eilenberg complex is concentrated in degree 0 and 1 with differential \( \tau : A \to A \). Since \( \tau(t^j) = j\sqrt{-1}t^j \) we see that \( H^0(\mathfrak{so}(2), A) = k1 \) and \( H^1(\mathfrak{so}(2), A) = ka \), where \( a \) is the class of 1. Thus

\[
H^*(U_0, S_0[\nu_0]) = k[\alpha], \quad \deg \alpha = 1, \quad \alpha^2 = [\alpha, \alpha] = 0.
\]

By Corollary 5.9 the BRST cohomology is three-dimensional and is the direct sum of algebras

\[
H^*(X, S_0) = k \oplus k[\alpha], \quad \deg \alpha = 1,
\]

with trivial bracket.

The next is an example, due to Pavel Etingof, with infinite dimensional BRST cohomology.

**Example 6.9.** Let \( X = \mathbb{A}^4 \) with coordinates \( x, y, z, w \) and

\[
S_0 = x^3 + y^3 + z^3 - 3wxyz.
\]

This function is of weight 3 if we assign weight 1 to \( x, y, z \) and 0 to \( w \). We claim that the zeroth BRST cohomology \( J(S_0) / L(S_0) \) is infinite dimensional. The Jacobian ring \( J(S_0) = k[x, y, z, w] / (x^2 - wyz, y^2 - wxz, z^2 - wxy, xyz) \) is the quotient by an ideal generated by homogeneous elements and is thus graded with respect to the weight. In weights \( 0, 1, 2 \) it is a free \( k[w] \)-module generated by \( 1, x, y, z, xy, xz, yz \). In weight 3 we have a free \( k[w] / (w^3 - 1)k[w] \)-module generated by \( x^2y, xy^2 \). It follows that every \( f \in J(S_0) \) of weight \( \geq 3 \) obeys \( (w^3 - 1)f = 0 \). Every vector field in \( L(S_0) \) is a sum of weight homogeneous vector fields. It is clear that there are no vector fields of weight \(-1 \) in \( L(S_0) \). A vector field of weight 0 in \( L(S_0) \) has the form \( \xi = a\partial_x + b\partial_y + c\partial_z + d\partial_w \) with \( a, b, c \) of weight 1 and \( d \in k[w] \). Since \( \xi(S_0) \equiv 3(ax^2 + by^2 + cz^2 - dxyz) \mod w \) it follows that \( a, b, c, d \) must all be divisible by \( w \). But if \( \xi \in L(S_0) \) is divisible by \( w \) then also \( w^{-1}\xi \in L(S_0) \). It follows that \( \xi = 0 \). Thus all homogeneous vector fields in \( L(S_0) \) have weight at least 1. Let \( g \in J(S_0) \) be of weight 2. Then \( h = (w^3 - 1)^2g \) is annihilated by \( L(S_0) \) since \( \xi(h) \) is of weight at least three and is divisible by \( w^3 - 1 \). But the space of such \( h \) is infinite dimensional.

**7. BRST cohomology in degree 0 and 1 and Lie–Rinehart cohomology**

In this Section we show that, up to degree 1, the BRST cohomology of a BV variety with support \((X, S_0)\), with \( X \) an affine variety, coincides with the de Rham cohomology of a Lie–Rinehart algebra associated with the critical locus of \( S_0 \). We also describe the induced bracket \( H^0(M, S) \otimes H^0(M, S) \to H^1(M, S) \) geometrically. The degree 1 cohomology appears as an obstruction to extend a solution of the classical master equation to a solution of the quantum master equation. The bracket \( H^0 \otimes H^0 \to H^1 \) appears in deformation theory as the obstruction to extend an infinitesimal deformation of a solution of the classical master equation to a solution in formal power series.
7.1. Lie–Rinehart algebras. Recall that a Lie–Rinehart algebra over \( k \) is a pair \((A, \mathfrak{g})\) where \( A \) is a commutative algebra over \( k \) and \( \mathfrak{g} \) is both a Lie algebra over \( k \) acting on \( A \) by derivations and an \( A \)-module. The required compatibility conditions are \([\tau, f \cdot \sigma] = \tau(f) \cdot \sigma + f \cdot [\tau, \sigma]\) and \((f \cdot \tau)(g) = f \cdot (\tau(g))\) for all \( f, g \in A \) and \( \tau, \sigma \in \mathfrak{g} \). Here \( f \otimes \tau \mapsto f \cdot \tau \) denotes the module structure and \( \tau \otimes f \mapsto \tau(f) \) the action of \( \mathfrak{g} \) on \( A \). The main example is \((A, \text{Der}(A))\) for an algebra \( A \).

Lie–Rinehart algebras come with a differential graded algebra, the de Rham (or Chevalley–Eilenberg) complex \( \Omega^*_\text{dR}(A, \mathfrak{g}) = \text{Hom}_A(\wedge^*_A \mathfrak{g}, A) \) introduced by Rinehart [26]. We only consider this complex for \( p \leq 2 \). For \( g \in \Omega^p_\text{dR}(A, \mathfrak{g}) = A \) and \( \alpha \in \Omega^1_\text{dR}(A, \mathfrak{g}) \) we have

\[
dg(\tau) = \tau(g), \quad d\alpha(\tau, \sigma) = \tau(\alpha(\sigma)) - \sigma(\alpha(\tau)) - \alpha([\tau, \sigma]), \quad \tau, \sigma \in \mathfrak{g}.
\]

An ideal of a Lie–Rinehart algebra \((A, \mathfrak{g})\) is a pair \((I, \mathfrak{h})\) such that

(i) \( I \) is an ideal of \( A \).
(ii) \( \mathfrak{h} \) is a Lie ideal of \( \mathfrak{g} \).
(iii) \( i \cdot \tau \in \mathfrak{h} \) and \( \tau(i) \in I \) for all \( i \in I, \tau \in \mathfrak{g} \).
(iv) \( f \cdot \sigma \in \mathfrak{h} \) and \( \sigma(f) \in I \) for all \( f \in A, \sigma \in \mathfrak{h} \).

**Lemma 7.1.** Let \((I, \mathfrak{h})\) be an ideal of the Lie–Rinehart algebra \((A, \mathfrak{g})\). Then \((A/I, \mathfrak{g}/\mathfrak{h})\) is naturally a Lie–Rinehart algebra.

**Proof.** The conditions (i)–(iv) guarantee that the structure maps defined on representatives are well-defined on the quotient. \( \square \)

7.2. Cohomology in degree \( \leq 1 \). Let \( S_0 \in \mathcal{O}(X) \) be a regular function and denote as before \( J = J(S_0) = \mathcal{O}_X/\text{Im}(dS_0; T_X \to \mathcal{O}_X) \) the Jacobian ring, \( L = L(S_0) \) the Lie algebra of vector fields annihilating \( S_0 \), \( L_0 = L_0(S_0) \) the \( \mathcal{O}_X \)-submodule of \( L \) spanned by \( \xi(S_0) - \eta(S_0)\xi, \xi, \eta \in T \).

**Lemma 7.2.** \((\text{Im}(dS_0; T_X \to \mathcal{O}_X), L_0)\) is an ideal of the Lie–Rinehart algebra \((\mathcal{O}_X, L)\).

**Proof.** (i) The subspace \( \text{Im}(dS_0) \) consists of functions of the form \( \xi(S_0), \xi \in T_X \). It is clearly an ideal of \( \mathcal{O}_X \). (ii) Let \( \tau \in L \). By exploiting the fact that \( \tau(S_0) = 0 \), we have

\[
[\tau, \xi(S_0)\eta - \eta(S_0)\xi] = [\tau, \xi([\eta(S_0)\eta - \eta(S_0)]\xi + \xi(S_0)[\tau, \eta] - \tau, \eta(S_0)\xi],
\]

which lies in \( L_0 \). (iii) Let \( \tau \in L, \xi(S_0) \in I \). Then, again because \( \tau(S_0) = 0 \), \( \xi(S_0)\tau = \xi(S_0)\tau - \tau(S_0)\xi \in L_0 \) and \( \tau(S_0) = [\tau, \xi(S_0)] \in I \). (iv) Let \( f \in \mathcal{O}_X, \nu = \xi(S_0)\eta - \eta(S_0)\xi \in L_0 \). Then clearly \( f\nu \in L_0 \) and \( \nu(f) = (\eta(f)\xi - \xi(f)\eta)(S_0) \in I \). \( \square \)

It follows that \((J, \mathfrak{g} = L/L_0)\) is a Lie–Rinehart algebra. We use the notation \( \mathfrak{g} = g(S_0) \) to denote the \( J \)-module \( L/L_0 \). It is convenient to distinguish it from \( L^\mathfrak{h} \) which is \( L/L_0 \) considered as an \( \mathcal{O}_X \)-module.

**Theorem 7.3.** The BRST cohomology is isomorphic to the de Rham cohomology up to degree 1:

\[
H^p(M, S) \cong H^p_\text{dR}(J(S_0), g(S_0)) \quad \text{for } p = 0, 1.
\]

**Remark 7.4.** The de Rham cohomology of a Lie–Rinehart algebra \((A, \mathfrak{g})\) is the appropriate cohomology for Lie–Rinehart algebras only if the Lie algebra \( \mathfrak{g} \) is a projective module over the commutative algebra \( A \). If it is not one should replace...
By a quasi-isomorphic differential graded Lie algebra which is projective over the commutative algebra, see [10]. The resulting cohomology groups \( H^p(A, g) \) coincide with the de Rham groups for \( p = 0,1 \) but are in general different. We conjecture that \( H^p(M, S) \cong H^p(J(S_0), g(S_0)) \) for all \( p \geq 0 \).

We have seen that \( H^0(M, S) \cong J^L = J^g \) since \( L_0 \) acts trivially. This is the claim for \( p = 0 \). To prove this theorem for \( p = 1 \) we need to construct a solution of the master equation modulo \( F^3 \).

Let \( \tau_1, \ldots, \tau_r \) be a system of generators of \( L/L_0 \) as an \( \mathcal{O}_X \)-module. Then

\[
\tau_i, \tau_j = \sum_{k=1}^r f_{ij}^k \tau_k,
\]

for some (non-unique) \( f_{ij}^k \in \mathcal{O}_X \).

Let \( r_{aj} \in \mathcal{O}_X \), \( a = 1, \ldots, s, j = 1, \ldots, r \), a generating system of relations, namely a set of elements of \( \mathcal{O}_X \) such that

(i) \( \sum_{j=1}^r r_{aj} \tau_j = 0 \), \( a = 1, \ldots s \)

(ii) if \( \sum_{j=1}^r r_j \tau_j = 0 \) with \( r_j \in \mathcal{O}_X \) then \( r_j = \sum_{a=1}^s f_{a} r_{aj} \) for some \( f_a \in \mathcal{O}_X \).

In other words we choose a presentation of the \( \mathcal{O}_X \)-module \( L_{\text{eff}}(S_0) \):

\[
\mathcal{O}_X^r \to \mathcal{O}_X \to L_{\text{eff}}(S_0) \to 0.
\]

Using this presentation we construct a Tate resolution \( R = (\text{Sym}_{\mathcal{O}_X} \mathcal{W}, \delta) \) down to degree \(-3\), with \( \mathcal{W} \) of the form \( \mathcal{W} = T_X[1] \oplus \mathcal{E}^*[1] \), see [4.1] and a solution \( S \in \Gamma(X, \Omega_{T^* X}[-1]^\ast) \) of the classical master equation modulo \( F^3 \) on the shifted cotangent bundle of \( V = (X, \text{Sym}_{\mathcal{O}_X} \mathcal{E}) \). We set \( \mathcal{W}^{-1} = T_X, \mathcal{W}^{-2} = \mathcal{O}_X^r \) with basis \( \beta_1^1, \ldots, \beta_r^s \), \( \mathcal{W}^{-3} = \mathcal{O}_X^r \oplus E \), where \( \mathcal{O}_X^r \) has basis \( \gamma_1^1, \ldots, \gamma_s^s \) and \( E \) is a free \( \mathcal{O}_X \)-module to be determined and whose precise form will not be important for the computation of the cohomology. The beginning of the Tate resolution looks like

\[
R^{-3} \xrightarrow{\delta} R^{-2} \xrightarrow{\delta} R^{-1} \xrightarrow{\delta} R^0,
\]

with

\[
R^{-3} = \wedge^3 T_X \oplus (T_X \otimes \mathcal{O}_X^r) \oplus \mathcal{O}_X^r \oplus E, \quad R^{-2} = \wedge^2 T_X \oplus \mathcal{O}_X^r, \quad R^{-1} = T_X, \quad R^0 = \mathcal{O}_X.
\]

Here \( \wedge^\bullet T_X = \wedge^\bullet \mathcal{O}_X T_X \) is the exterior algebra of the \( \mathcal{O}_X \)-module \( T_X \). The differential \( \delta: R^{-1} \to R^0 \) is \( \delta(\xi) = \xi(S_0) \). The differential on the other components depends on a choice of lifts of \( \tau_1, \ldots, \tau_r \in L/L_0 \) to vector fields \( \hat{\tau}_1, \ldots, \hat{\tau}_r \in L \). We set

\[
\delta(\beta_j^a) = \hat{\tau}_j,
\]

so that \( \delta^2 |_{\mathcal{O}_X} = 0 \). The relations for the lifts hold modulo \( L_0 \):

\[
\sum_j r_{aj} \hat{\tau}_j + dS_0 \cdot v_a = 0,
\]

for some \( v_a \in \wedge^2 T_X \). Here \( \cdot \) is the contraction operator: \( dS_0 \cdot \xi \land \eta = \xi(S_0) \eta - \eta(S_0) \xi \in L_0 \). By the Leibniz rule we have \( \delta(v) = dS_0 \cdot v \) for \( v \in \wedge^2 T_X \). Thus Eq. (7.2) implies that \( v_a + \sum_j r_{aj} \beta_j^a \) is a cocycle for \( a = 1, \ldots, s \). We set

\[
\delta(\gamma_a^s) = v_a + \sum_j r_{aj} \beta_j^s \in \wedge^2 T_X \oplus \mathcal{O}_X^r.
\]

Thus \( \delta^2 |_{\mathcal{O}_X} = 0 \).
Let $R_0^{-2}$ be the $O_X$-submodule spanned by $\xi(S_0)r$ with $\xi \in T_X$, $r \in R^{-2}$. The differential on $E$ will be chosen so that $\delta(E) \subset R_0^{-2}$.

Let us check that we indeed get a resolution down to degree $-3$. The kernel of $\delta: T_X \to O_X$ is $L$. Since $\delta(\wedge^2 T_X) = L_0$ and $r_1, \ldots, r_r$ span $L/L_0$, the complex is exact at $R^{-1}$. The kernel of the next differential consists of pairs $v \in \wedge^2 T_X$, $\sum r_i \beta_i^* \in O_X^*$ such that $\sum r_i \tilde{\tau}_i + dS_0.w = 0$. We need to show that these cocycles are equivalent to cocycles lying in $R_0^{-2}$, so that they can be killed by a judicious choice of $E$. Reducing modulo $L_0$, we see that $\sum r_i \tilde{\tau}_i = 0$ so that $r_i = \sum f_{ai} r_{ai}$, see (ii). Possibly subtracting an element of $\delta(O_X^*)$, we may assume that $r_i = 0$, $i = 1, \ldots, r$. Thus $dS_0.w = 0$ and therefore $v$ is a sum of bivector fields of the form $\xi \wedge \eta$ where $\eta \in L$, i.e., $\eta$ is a linear combination of $\tilde{\tau}_i$ and $dS_0.w$, $w \in \wedge^2 T_X$, the latter being in $R_0^{-2}$. We can get rid of terms $\xi \wedge \tilde{\tau}_i$ modulo $R_0^{-2}$ by subtracting a coboundary from $\delta(T_X \otimes O_X^*)$: indeed we have

$$\delta(\xi \wedge \tilde{\tau}_i) = \xi(S_0)\beta_i^* - \xi \wedge \tilde{\tau}_i \equiv -\xi \wedge \tilde{\tau}_i \mod R_0^{-2}.$$ 

Thus any $(-2)$-cocycle is equivalent modulo coboundaries to a cocycle lying in $R_0^{-2}$. Exactness at $R^{-2}$ is achieved by defining $\delta$ to map $E$ onto the space of cocycles in $R_0^{-2}$.

With this information on the Tate resolution we can construct a solution of the master equation modulo $F^3$ along the line of the existence proof in Section 4.3 (b). The first approximation is

$$S_{\leq 1} \equiv S_0 + \sum \delta(\beta_i^*)\beta_i + \sum \delta(\gamma_a^*)\gamma^a + \sum \delta(\rho_b^*)\rho^b \mod F^3,$$

where $(\rho_b^*)$ is a basis of $E$. By construction, $[S_{\leq 1}, S_{\leq 1}]$ is a section of $I^{(2)}$, see (b) (ii). So the first violation of the master equation appears from the bracket of the second term on the right-hand side with itself: $[S_{\leq 1}, S_{\leq 1}] \equiv \sum[\tilde{\tau}_i, \tilde{\tau}_j]\beta_i^*\beta_j^* \mod F^3$. The relation lifts to

$$[\tilde{\tau}_i, \tilde{\tau}_j] = \sum f_{ij}^k \tilde{\tau}_k + dS_0.w_{ij},$$

for some bivector field $g_{ij} \in \wedge^2 T_X$. This dictates the form of the corrections to $S$. We obtain

$$S \equiv S_0 + \sum_i \tilde{\tau}_i \beta_i + \sum_{a,j} r_{ai} \beta_j^* \gamma^a + \frac{1}{2} \sum a \gamma^a$$

$$+ \sum_b \delta(\rho_b^*)\rho^b + \frac{1}{2} \sum_{i,j} g_{ij} \beta_i^* \beta_j^* - \frac{1}{2} \sum_{i,j,k} f_{ijk}^k \beta_i^* \beta_j^* \beta_k^* \mod F^3.$$

The BRST cohomology is isomorphic to the cohomology of the first term $E_1^{\bullet,0} = J[\beta^i, \gamma^a, \rho^b, \ldots]$ of the spectral sequence. The beginning of the complex $E_1^{\bullet,0}$ is

$$J \to \oplus_i J\beta^i \to (\oplus_{i<j} J\beta^i \beta^j) \oplus (\oplus_a J\gamma^a) \oplus (\oplus_b J\rho^b) \to \cdots.$$

The differential is obtained from the terms in $S$ linear in the generators, underlined in the formula for $S$ above. We have

$$d_1(f) = -\sum_i \tau_i(f)\beta_i^*, \quad d_1(\beta_i^*) = -\sum_a r_{ai} \gamma^a + \frac{1}{2} f_{ij}^k \beta_i^* \beta_j^* \beta_k^*.$$ 

In $\delta(\rho_b^*)\rho^b$ there could be a linear term in $\beta_i^*$ that would contribute to $d_1(\beta_i^*)$. But since $\delta(\rho_b^*)$ lies in $R_0^{-2}$ its image in $E_1$ vanishes.
The Leibniz rule gives the differential of a general 1-cochain
\[ d_1(\sum_j g_j \beta^j) = -\sum_{i,j} \tau_i(g_j) \beta^i \beta^j + \frac{1}{2} \sum_{i,j,k} g_i f_{jk} \beta^j \beta^k - \sum_{a,i} r_{ai} g_i \gamma^a. \]

Thus \( H^1(M, S) \cong Z^1/B^1 \) with
\[ Z^1 = \left\{ \sum_i g_i \beta^i \in \oplus_i J \beta^i \text{ such that} \right. \\
(a) \ \tau_i(g_j) - \tau_j(g_i) - \sum_k f_{jk} g_k = 0, \forall i, j, \ (b) \ \sum_k r_{ak} g_k = 0, \forall a \right\}, \]

\[ B^1 = \left\{ \sum_i \tau_i(f) \beta^i \mid f \in J \right\}. \]

Cochains \( \sum_i g_i \beta^i \) obeying (b) are in one-to-one correspondence with linear functions \( \alpha \in \text{Hom}_{\mathcal{O}_X}(L/\mathfrak{L}_0, J) \) via \( \alpha(\tau_i) = g_i \). This space is canonically isomorphic to \( \text{Hom}_J(J \otimes_{\mathcal{O}_X} L/\mathfrak{L}_0, J) = \Omega^{1}_{\text{dR}}(J, \mathfrak{g}) \) and (a) translates to the cocycle condition for the de Rham differential. Thus
\[ Z^1 \cong \text{Ker}(d: \Omega^0_{\text{dR}}(J, \mathfrak{g}) \to \Omega^1_{\text{dR}}(J, \mathfrak{g})). \]

Similarly \( B^1 \cong d(\Omega^0_{\text{dR}}(J, \mathfrak{g})) \). This concludes the proof of Theorem 7.3.

7.3. The \( P_0 \)-algebra structure. As the filtration is compatible with the product, it follows that the isomorphism of Theorem 7.3 respects the product \( \Lambda^p \otimes \Lambda^q \to \Lambda^{p+q} \), \( p + q \leq 1 \). The situation with the bracket is more tricky and potentially more interesting.

Let, as above, \( \mathfrak{g} = L(S_0)/L_0(S_0) \) and \( f \in H^0_{\text{dR}}(J, \mathfrak{g}) = J(S_0)^{L(S_0)} \) be an invariant function on the critical locus. Let \( \tilde{f} \in \mathcal{O}_X \) be a representative of \( f \) in \( \mathcal{O}_X \). Then for each \( \tau \in L(S_0) \)
\[ \tau(\tilde{f}) = \xi(S_0), \]
for some \( \xi = \xi(\tau, \tilde{f}) \in T \) defined modulo \( L(S_0) \) depending \( \mathcal{O}_X \)-linearly on \( \tau \) and \( k \)-linearly on \( \tilde{f} \). This defines a homomorphism of \( \mathcal{O}_X \)-modules
\[ L(S_0) \to T/L(S_0), \quad \tau \mapsto \xi(\tau, \tilde{f}) \text{ mod } L(S_0). \]
Recall that, by Lemma 7.2, \( L_0(S_0) \) acts trivially on \( J \), so that the map descends to a map \( \mathfrak{g} \to T/L(S_0) \). We define a symmetric \( k \)-bilinear map \( H^0(J, \mathfrak{g}) \times H^0(J, \mathfrak{g}) \to H^1(J, \mathfrak{g}) \)
\[ B(f, g) = \text{class of } (\tau \mapsto \xi(\tau, \tilde{f})(\tilde{g}) + \xi(\tau, \tilde{g})(\tilde{f})) \in \text{Hom}_J(\mathfrak{g}, \mathfrak{g}), \]
for any choice of lifts \( \tilde{f}, \tilde{g} \in \mathcal{O}_X \) of \( f, g \in J \).

**Lemma 7.5.** The bilinear map \( B \) is well-defined.

**Proof.** Any two lifts \( \tilde{f} \) of \( f \in J \) differ by \( \xi(S_0) \) for some vector field \( \zeta \in T \).
We need to show that \( B(\xi(S_0), \tilde{g}) = 0 \). We have \( \tau(\xi(S_0)) = [\tau, \zeta](S_0) \) for \( \tau \in L \).
Thus
\[ \xi(\tau, \xi(S_0)) = [\tau, \zeta], \]
and therefore
\[ B(\xi(S_0), \tilde{g})(\tau) = [\tau, \zeta](\tilde{g}) + \xi(\tilde{g}, \tau)(\xi(S_0)). \]
The second term is equal to \( \zeta(\tilde{g}, \tau) = \zeta(\tilde{g} \circ \tau) \) by definition of \( \zeta \). The bracket evaluated on \( S_0 \) vanishes in \( J \) and we get

\[
B(\zeta(S_0), \tilde{g})(\tau) = \tau(\zeta(\tilde{g})),
\]

which belongs to the zero cohomology class in \( H^1(J, g) \).

\[ \square \]

**Theorem 7.6.** The bilinear form \( B \) coincides with the induced bracket

\[
H^0(M, S) \times H^0(M, S) \rightarrow H^1(M, S)
\]

under the identification \( H^p(M, S) \cong H^p(J, g) \) of Theorem 7.3.

**Proof.** To compute the induced bracket

\[
[ , ] : H^0(M, S) \otimes H^0(M, S) \rightarrow H^1(M, S),
\]

using the description of Theorem 7.3, we need to lift the cocycles of \( E_1 \) to cocycles of the BRST complex modulo \( F_2 \). So let \( \tilde{f} \in \mathcal{O}_X \) represent a cocycle \( f \) in \( E_1^{0,0} = J \).

Then \( \tilde{\tau}_i(\tilde{f}) = \xi_i(S_0) \) for some vector fields \( \xi_i \).

\[
[S, f] = [\sum_i \tilde{\tau}_i \beta^i, f] \mod F^2
\]

\[
= -\sum_i \tilde{\tau}_i(f) \beta^i \mod F^2
\]

\[
= -\sum_i \xi_i(S_0) \beta^i \mod F^2.
\]

\[
= [S, \sum_i \xi_i \beta^i] \mod F^2.
\]

It follows that a cocycle in the BRST complex corresponding to \( f \) is

\[
F \equiv f - \sum_i \xi_i \beta^i \mod F^2.
\]

Let us do the same for a second function \( g \in J^1 \) : \( G \equiv g + \sum_i \eta_i \beta^i \).

Then

\[
[F, G] = [f - \sum_i \xi_i \beta^i, g - \sum_i \eta_i \beta^i] \mod F^2
\]

\[
= (\xi_i(g) + \eta_i(f)) \beta^i \mod F^2.
\]

The right-hand side is the image in \( E_1^{1,0} \) of \( [F, G] \) and coincides with the claimed formula. \( \square \)

8. The case of quasi-projective varieties

Let \( X \) be a nonsingular algebraic variety over a field \( k \) of characteristic 0, \( S_0 \in \mathcal{O}(X)/k_X \) a function defined modulo constants or more generally, in the setting of Sect. 3.3, a multivalued function \( S_0 = \int \lambda \) for a closed 1-form \( \lambda \in \Gamma(X, \Omega^1_X) \).

Then on an open affine subset \( U \subset X \), the restriction of \( S_0 \) to \( U \) gives rise to a BV variety \( (M = T^*[-1]V, S) \) with support \( (U, S_0|_U) \) which is unique up to stable equivalence. In particular the BRST complex \( (\mathcal{O}_{T^*[-1]V}, d_S) \) is uniquely determined, up to quasi-isomorphism of sheaves of differential \( P_0 \)-algebras on \( U \), by \( S_0 \). Is it possible to glue these local data to form a sheaf of differential \( P_0 \)-algebras on \( X \), whose restriction to any open affine subset is quasi-isomorphism to the local BRST complex? How unique is the construction?
We cannot answer these questions in general, but we show existence in the case of quasi-projective varieties \( X \). In this case, every coherent sheaf admits a resolution by locally free sheaves and, as a consequence, a global Tate resolution of the sheaf \( J(S_0) \) of Jacobian rings exists. Let us fix such a Tate resolution. Then for any covering \((U_i)\) by affine open sets, the Tate resolution restricts on every non-empty intersection \( U = U_{i_1} \cap \cdots \cap U_{i_k} \) to a Tate resolution of \( J(S_0)|_U \) and we can consider the class of BV varieties on \( U \) associated with this Tate resolution (see Sect. 4.2).

As the Tate resolution is defined globally, the corresponding solutions of the master equation are sections \( S \in \Gamma(U, \mathcal{O}_U[-1]) \) for some globally defined graded variety \( V \) with support \( X \). We know by Theorem 4.3 that the group of gauge equivalences acts transitively on the BV varieties with support on an affine set and associated with a fixed Tate resolutions. To glue, we need a stronger form of this result and show that the corresponding action groupoid extends to a simplicial groupoid whose sets of morphisms are contractible Kan complexes. We first introduce the necessary notions to formulate the result.

Let \( U \) be nonsingular affine, \( S_0 \in \mathcal{O}(U) \) and \((R, \partial)\) be a Tate resolution of the Jacobian ring of \( S_0 \). We realize \( R \) as in Section 4.1 as \( \mathcal{O}_M/I_M \) where \( M = T^*[1]-V \) is the cotangent bundle of a \( \mathbb{Z}_{>0} \)-graded variety \( V \). Let \( \mathcal{B} = \mathcal{B}(M, \partial) \) be the set of BV varieties \((T^*[1]-V, S)\) with support \((U, S_0)\) associated with the Tate resolution \((\mathcal{O}_M/I_M, \partial)\), see Def. 4.1. Then by Theorem 4.3 the group of gauge equivalences \( G(M) = \exp \text{ad}(g(M)) \subset \text{Aut}(\mathcal{O}_M) \), see Sect. 2.9, fixes the Tate resolution and acts transitively on \( \mathcal{B} \). The Lie algebra \( \mathfrak{g}(M) \) is the degree 0 component of the graded Lie algebra \( \mathfrak{g}^n(M) = \Gamma(U, \mathcal{O}_U)[-1] \). We will need the other components as well.

Let \( \mathcal{G}_0 \) denote the action groupoid of the action of \( G(M) \) on \( \mathcal{B} \): its object set is \( \mathcal{B} \) and the set of morphisms \( S_1 \to S_2 \) is

\[
\mathcal{G}_0(S_1, S_2) = \{ g \in G(M) \mid g \cdot S_1 = S_2 \}.
\]

We will show that \( \mathcal{G}_0(S_1, S_2) \) is the set of 0-simplices of a contractible Kan complex, namely a simplicial set \( \mathcal{G}(S_1, S_2) \) such that every simplicial sphere \( \partial \Delta^n \to \mathcal{G}(S_1, S_2) \) can be filled to a map of simplicial sets \( \Delta^n \to \mathcal{G}(S_1, S_2) \). Here \( \Delta^n \) is the standard combinatorial \( n \)-simplex. It is the simplicial set \( \text{Hom}_\Delta([n]) \) represented by the object \([n] = \{0, \ldots, n\}\) of the simplicial category \( \Delta \) of finite ordered sets and non-decreasing maps.

Let \( A \) be the \( P_0 \)-algebra \( A = \Gamma(U, \mathcal{O}_M) \). Let \( \Omega^\bullet(|\Delta^n|) \) be the de Rham algebra of polynomial differential forms on the geometric \( n \)-simplex \(|\Delta^n|\):

\[
\Omega^\bullet(|\Delta^n|) = k[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(\sum_{i=0}^n t_i - 1, \sum_{i=0}^n dt_i), \quad \text{deg}(t_i) = 0, \quad \text{deg}(dt_i) = 1.
\]

Then the sequence \( g^n_\bullet = \mathfrak{g}^n(M) \otimes \Omega^\bullet(|\Delta^n|) \) is a simplicial differential graded Lie algebra. The Lie bracket is \([a \otimes \omega, b \otimes \eta] = (-1)^{\text{deg} a \cdot \text{deg} \omega} [a, b] \otimes \omega \eta \). The simplicial algebra \( \Omega(|\Delta^n|) \) is the functor from \( \Delta \) to differential graded algebras sending \([n] \) to \( \Omega(|\Delta^n|) \) and \( f: [n] \to [m] \) to the map of differential graded algebras such that \( t_i \mapsto \sum_{j \in f^{-1}(i)} t_j \).

Let \( G_n = \exp(\text{ad}(g^n_\bullet)) \). It is a subgroup of the group \( \text{Aut}_{\Omega^\bullet(|\Delta^n|)}(A \otimes \Omega^\bullet(|\Delta^n|)) \) of \( \Omega^\bullet(|\Delta^n|) \)-linear automorphism of the graded \( P_0 \)-algebra \( A \otimes \Omega^\bullet(|\Delta^n|) \). A solution \( S \in \mathcal{B} \) of the classical master equation defines a differential \( d_S = [S, _\cdot] \) on \( A \). Let us use the same notation for the differential on the product of complexes \( A \otimes \Omega^\bullet(|\Delta^n|) \)
given by

\[ d_S(a \otimes \omega) = [S, a] \otimes \omega + (-1)^{\deg a} a \otimes d\omega. \]

We set \( \mathcal{G}_n(S_1, S_2) = \{ g \in G_n \mid g \circ d_{S_1} = d_{S_2} \circ g \} \).

For \( n = 0 \) this condition means that \([g \cdot S_1, a] = [S_2, a]\) for all \( a \in A \). Since the Poisson center consists of constants and \( S_1 \) and \( S_2 \) are both congruent to \( S_0 \) modulo \( I_M \), the definition coincides with the previous definition of \( \mathcal{G}_0 \). The collection \( \mathcal{G}(S_1, S_2) = (\mathcal{G}_n(S_1, S_2))_{n=0,1,...} \) is then naturally a simplicial set.

**Proposition 8.1.** Let \( S_1, S_2 \in B \). The restriction map

\[ \text{Hom}_{\mathcal{SSet}}(\Delta^n, \mathcal{G}(S_1, S_2)) \to \text{Hom}_{\mathcal{SSet}}(\partial \Delta^n, \mathcal{G}(S_1, S_2)) \]

is surjective.

**Proof.** Since \( G_0 \) acts transitively on \( B \), post-composing \( g \) with a map in \( G_0 \) mapping \( S_2 \) to \( S_1 \) reduces the problem to the case where \( S_1 = S_2 = S \).

By the Yoneda lemma \( \text{Hom}(\Delta^n, \mathcal{G}(S, S)) = \mathcal{G}_n(S, S) \).

**Lemma 8.2.** \( \mathcal{G}_n(S, S) = \exp(\text{ad}(g^0_n(S, S))) \) where \( g^0_n(S, S) = \{ \omega \in g_n \mid d\omega + [S, \omega] = 0 \} \).

**Proof.** This is a special case of a general result: let \((L, d_L)\) be a differential nilpotent graded Lie algebra and \((M, d_M)\) be a differential graded faithful \( L \)-module on which \( L \) acts locally nilpotently. This means that \( M \) is an \( L \)-module such that

1. For all \( a \in L, m \in M, d_M(a \cdot m) = d_L(a) \cdot m + a \cdot d_L(m) \)
2. If \( a \in L \) and \( a \cdot m = 0 \) for all \( m \in M \) then \( a = 0 \).
3. For all \( a \in L, m \in M \), there exists an \( n \) such that \( a^n \cdot m = 0 \).

The claim is then:

\[ a \in L^n, \exp(a) \circ d_M = d_M \circ \exp(a) \iff d_L(a) = 0. \]

The proof relies on the identity

\[ (\exp(a) \circ d_M \circ \exp(-a) - d_M)m = b \cdot m, \quad m \in M, a \in A, \]

\[ b = \frac{\text{id} - \exp(\text{ad}(a))}{\text{ad}(a)}(d_L(a)) \in L. \]

Since \( b \) is obtained by the action of an invertible operator on \( d_L(a) \) it follows that \( b = 0 \) iff \( d_L(a) = 0 \). To check the identity, introduce

\[ b_t = \exp(t a) \circ d_M \circ \exp(-t a) - d_M \in \text{End}(M)[t]. \]

Differentiating with respect to \( t \) yields

\[ \dot{b}_t = \exp(t a) \circ (a \cdot d_M - d_M \circ a) \circ \exp(-t a) = -\exp(t a) \circ d_L(a) \circ \exp(-t a). \]

Thus \( \dot{b}_t \) is the action of \(-\exp(t \text{ad}(a))d_L(a) \in L \) and we conclude that

\[ b_{t=1} = -\int_0^1 \exp(t \text{ad}(a))d_L(a)dt, \]

implying the identity after integration.

In our case we have \( L = g^\bullet_n \) and \( M = A \otimes \Omega^\bullet(|\Delta^n|) \) with differential \( d + d_S \).

The action is faithful because the part of the Poisson center in \( g^\bullet(M) \) is trivial: \( Z_M \cap g^\bullet_M = 0 \), see Prop. 2.15. The Lie algebra and the action are pronilpotent: they
become nilpotent modulo \( I_M^{(k)} \) for all \( k \). The statement of the Lemma is equivalent to the statement modulo \( I_M^{(k)} \) for all \( k \).

Let us decompose \( \omega \) with respect to the de Rham degree

\[
\omega = \omega_0 + \omega_1 + \cdots + \omega_n, \quad \omega_j \in A^{-n-j} \otimes \Omega^j(\mid \Delta^n \mid).
\]

Then the condition for \( \exp(\text{ad}(\omega)) \in G_n \) to lie in \( \mathcal{G}_n(S, S) \) is

\[
[S, \omega_0] = 0,
\]

\[
d\omega_0 + [S, \omega_1] = 0,
\]

\[
\vdots
\]

\[
d\omega_{n-1} + [S, \omega_n] = 0.
\]

We need to show that \( \omega \in g^* \otimes \Omega^*([\partial \Delta^n]) \) of degree 0 obeying \( S.1 \) extends to a differential form on the boundary of a simplex obeying \( S.1 \).

The basic fact about polynomial differential forms is that every differential form on the boundary of a simplex extends to the simplex (see \( S.2 \)). Thus the restriction homomorphism

\[
\Omega^*([\Delta^n]) \rightarrow \Omega^*([\partial \Delta^n])
\]

is surjective. Let us choose a right inverse \( s : \omega \mapsto \tilde{\omega} \) and use the same notation to denote the right inverse \( \text{id} \otimes s \) of the \( A \)-linear extension

\[
A \otimes \Omega^*([\Delta^n]) \rightarrow A \otimes \Omega^*([\partial \Delta^n]).
\]

Let now \( \omega = \omega_0 + \cdots + \omega_{n-1} \in A \otimes \Omega^*([\partial \Delta^n]) \) obey \( S.1 \) and \( \tilde{\omega}_0, \ldots, \tilde{\omega}_{n-1} A \)-linear extensions to \( [\Delta^n] \). Clearly \( [S, \cdot] \) commutes with \( \text{id} \otimes s \) and thus

\[
[S, \tilde{\omega}_0] = 0.
\]

Let us assume inductively that, possibly after modifying the choice of extensions \( \tilde{\omega}_i \),

\[
d\tilde{\omega}_{j-1} + [S, \tilde{\omega}_j] = 0, \text{ for } j \leq k.
\]

In particular, for \( j = k \) this implies

\[
0 = d[S, \tilde{\omega}_k] = [S, d\tilde{\omega}_k].
\]

Thus \( \alpha = d\tilde{\omega}_k + [S, \tilde{\omega}_{k+1}] \) obeys \( [S, \alpha] = 0, \alpha|_{\partial \Delta^n} = 0 \), where we set \( \tilde{\omega}_{k+1} = 0 \) for \( k = n - 1 \). Thus \( \alpha \) is a cocycle for \( [S, \cdot] \) in \( A^{-k-1} \otimes J^k \) where \( J \) is the ideal of differential forms on \( [\Delta^n] \) whose restriction to the boundary vanishes. Since the BRST cohomology vanishes in negative degree, there exists a \( \beta \) vanishing on the boundary such that \( \alpha = [S, \beta] \). Replacing \( \tilde{\omega}_k \) by \( \tilde{\omega}_k - \beta \) yields

\[
d\tilde{\omega}_k + [S, \tilde{\omega}_{k+1}] = 0,
\]

which proves the induction step.

Thus \( \mathcal{G}_n(S, S') \) is the set of \( n \)-simplices of a contractible Kan complex. Moreover, we have composition maps

\[
\mathcal{G}_n(S, S') \times \mathcal{G}_n(S', S'') \rightarrow \mathcal{G}_n(S, S''),
\]

induced by the product in \( A \) and the wedge product of differential forms. The compositions are maps of simplicial sets if we define the Cartesian product \( C \times D \) of simplicial sets \( C, D \) as the sequence of sets \( C_n \times D_n \) with face maps \( \partial_i(x, y) = (\partial_i x, \partial_i y) \) and degeneracy maps \( s_i(x, y) = (s_i x, s_i y) \).
Thus there is a simiplicial category $\mathcal{G}$, namely a category enriched over simplicial sets, with objects $B$ and morphisms $S \to S'$ given by the contractible Kan complex $\mathcal{G}(S, S')$.

**Corollary 8.3.** Let $X$ be a nonsingular quasi-projective variety, $\lambda \in \Omega^1(X)$ a closed 1-form, $S_0 = \int \lambda$. Then there exists a sheaf $\mathcal{P}$ of differential $P_0$-algebras on $X$ which is locally the BRST complex of a local BV variety with support $(X, S_0)$. More precisely, every point of $X$ has an open affine neighborhood $U$ such that $\mathcal{P}|_U$ is quasi-isomorphic, as a sheaf of differential $P_0$-algebras, to $(O_M, dS)$ for some BV variety $(M, S)$ with support $(U, S_0|_U)$.

The sheaf $\mathcal{P}$ is constructed by gluing the BV varieties associated with the restriction of a Tate resolution to the affine open subsets of a covering by using the contractibility result of Prop. [8.1]. This construction is described in Appendix B.

### Appendix A. Stable isomorphism of Tate resolutions

Let $A$ be a Noetherian unital commutative ring containing $\mathbb{Q}$ and $C_A$ be the monoidal category of non-positively graded differential graded unital commutative algebras $C = \bigoplus_{i \leq 0} C^i$ over $A$ whose homogeneous components $C^i$ are finitely generated $A$-modules. Differential have degree 1. An object of $C_A$ is called semi-free if it is isomorphic, as a graded commutative algebra, to the symmetric algebra $\text{Sym}_A(V)$ of some projective negatively graded $A$-module $V = \bigoplus_{i < 0} V^i$.

Let $C$ be an object of $C_A$ concentrated in degree 0. A Tate resolution of $C$ is a quasi-isomorphism $E \to C$ in $C_A$ where $E = \text{Sym}_A(V)$ is semi-free. Tate resolutions exist by Tate’s recursive construction: the subalgebra $E_{\geq -d}$ of $E$ generated by $\bigoplus_{i = -d}^0 V^i$ is constructed out of $E_{\geq -(d-1)}$ as

\[(A.1) \quad E_{\geq -d} = E_{\geq -(d-1)}[T_1, \ldots, T_n], \quad \deg(T_i) = -d,
\]

with differential extending the differential on $E_{\geq -(d-1)}$ and such that

\[d T_i = c_i,
\]

for some set of cocycles $c_i$ spanning the cohomology of $E_{\geq -d}$ in degree $-d$ (the Noetherian hypothesis ensures that the cohomology of fixed degree is a finitely generated $A$-module). One then takes $V^{-d}$ at the free module with basis $T_1, \ldots, T_n$.

A morphism of Tate resolutions $p_E : E \to C, p_F : F \to C$ is a morphism of differential graded algebras $f : E \to F$ such that $p_F \circ f = p_E$.

Any two Tate resolutions are related by a quasi-isomorphism which is unique up to homotopy. More precisely:

**Lemma A.1.** Let $p_E : E \to C, p_F : F \to C$ be two Tate resolutions.

(i) Any morphism $f : E_{\geq -d} \to F_{\geq -d}$ of differential graded Lie algebras such that $p_F \circ f = p_E$ extends to a morphism $E \to F$ of Tate resolutions.

(ii) Any two morphisms $f_0, f_1 : E \to F$ of Tate resolutions are homotopic, namely, there is a morphism $h : E \to F \otimes_A A[t, dt]$ with $\text{id} \otimes \epsilon_j \circ h = f_j, \quad j \in \{0, 1\}$. Here $\epsilon_j : A[t, dt] \to A$ is the map $t \mapsto j, \ dt \mapsto 0$.

(iii) Every morphism $f : E \to F$ of Tate resolution admits an inverse up to homotopy: there is a morphism $g : F \to E$ of Tate resolutions so that $f \circ g$ and $g \circ f$ are homotopic to the identity. In particular all morphisms of Tate resolutions of $C$ are quasi-isomorphisms.
The proof is standard: the required maps (extension of \( f \), homotopy, inverse up to homotopy) on free algebras are uniquely determined by the images of generators, that are to obey cohomological conditions. The existence of images obeying the conditions follows recursively by degree by the acyclicity of the resolution in the previous degrees.

The next result is a stronger statement relating Tate resolutions of an object by an isomorphism (rather than a quasi-isomorphism) after tensoring with a semi-free acyclic algebra.

Let \( \text{Sym}_A(V \oplus V[1]) \) the acyclic symmetric algebra on a free graded \( A \)-module \( V \) with differential defined on generators \( V \oplus V[1] = \oplus (V^i \oplus V^{i+1}) \) as \( d(v \oplus w) = w \oplus 0 \). If \( E \to C \) is a Tate resolution, then also \( E \otimes_A \text{Sym}_A(V \oplus V[1]) \to C \), sending \( V \oplus V[1] \) to zero, is a Tate resolution and the obvious map \( E \to E \otimes_A \text{Sym}_A(V \oplus V[1]) \) is a quasi-isomorphism of Tate resolutions.

**Theorem A.2.** Let \( d \geq 0 \), \( C \in C_A \). Let \( f : E \to F \) be a morphism of Tate resolutions of \( C \) which is an isomorphism in degrees \( \geq -d + 1 \). Then there is an isomorphism in \( C_A \)

\[
E \otimes_A \text{Sym}_A(V \oplus V[1]) \cong F \otimes_A \text{Sym}_A(W \oplus W[1]),
\]

restricting to \( f \) in degrees \( \geq -d + 1 \), for some free graded \( A \)-modules \( V, W \) concentrated in degree \( \leq -d \).

**Proof.** We prove inductively the following statement.

**Lemma A.3.** Let \( f_{d-1} : E \to F \) be a morphism of Tate resolutions of \( C \), which restricts to an isomorphism \( E_{\geq -(d-1)} \to F_{\geq -(d-1)} \) for some \( d > 0 \). Then there are graded free \( A \)-modules \( V_d, W_d \) of finite rank concentrated in degree \( -d \) and a morphism of Tate resolutions

\[
f_d : E \otimes_A \text{Sym}_A(V_d \oplus V_d[1]) \to F \otimes_A \text{Sym}_A(W_d \oplus W_d[1]),
\]

restricting to an isomorphism in degree \( \geq -d \) and coinciding with \( f_{d-1} \) on \( E_{\geq -(d-1)} \).

By using this Lemma and starting from any morphism \( f_0 : E \to F \) of Tate resolutions, we obtain a morphism \( f \) as in the claim of the theorem with \( V = \oplus V_d \) and \( W = \oplus W_d \).

To prove the Lemma, we first of all notice that if \( E = \text{Sym}_A(W) \) we can assume without loss of generality that the projective \( A \)-module \( W^{-d} \) is free: if it is not, we replace \( W \) by \( W \oplus U[d] \oplus U[d + 1] \), where \( W^{-d} \oplus U \) is free and the differential is extended so that \( \delta : U[d + 1] \to U[d][1] \) is the identity. Similarly we can assume that the generators of degree \( -d \) of \( F \) form a free \( A \)-module.

Let \( T_1, \ldots, T_n \) be a basis of the space of generators of degree \( -d \) of \( E \) as in (A.1) and \( S_1, \ldots, S_m \) be a basis of the space generators of degree \( -d \) of \( F \). We then take \( V_d \) to be a vector space of dimension \( m \) with basis \( S_1', \ldots, S_m' \). Then

\[
\text{Sym}_A(V_d \oplus V_d[1]) = A[S_1', \ldots, S_m', S_1'', \ldots, S_m''], \quad \deg(S_j'') = -d - 1, \quad d S_j'' = S_j'.
\]

Similarly,

\[
\text{Sym}_A(W_d \oplus W_d[1]) = A[T_1', \ldots, T_n', T_1'', \ldots, T_n''], \quad \deg(T_i'') = -d - 1, \quad d T_i'' = T_i'.
\]

Let \( g_{d-1} : F \to E \) be a morphism extending the inverse of the restriction of \( f_{d-1} \) to \( E_{\geq -(d-1)} \). Such an extension exists by Lemma [A.1](i). Let \( f_d \) coincide with \( f_{d-1} \) on \( E_{\geq -(d-1)} \) and similarly for \( g_{d} \). We set

\[
(A.2) \quad f_d(T_i) = f_{d-1}(T_i) + T_i', \quad g_d(S_j) = g_{d-1}(S_j) + S_j'.
\]
This defines \( f_d \) as a morphism of graded algebras \( E_{>-d} \to F_{>-d} \). Since \( dT'_i = 0, \) \( f_d \) commutes with differentials on \( E_{>-d} \). The same holds for \( g_d \). The next definition is devised to make \( g_d \) the inverse of \( f_d \) in degree \( -d \):
\[
\begin{align*}
  f_d(S'_j) &= S_j - f_d(g_{d-1}(S_j)), \\
  g_d(T'_i) &= T_i - g_d(f_{d-1}(T_i)).
\end{align*}
\]
The right-hand sides are already defined since \( g_{d-1}(S_j) \in E_{>-d} \) and \( f_{d-1}(T_i) \in F_{>-d} \). Also, since \( f_d \) coincides with \( f_{d-1} = g_{d-1}^{-1} \) in degree \( \geq -d + 1 \), \( f_d \), and similarly \( g_d \), commutes with the differential:
\[
d(f_d(S'_j)) = dS_j - f_d(g_{d-1}(dS_j)) = dS_j - f_{d-1}(g_{d-1}(dS_j)) = 0 = f_d(dS'_j).
\]
The inversion property is first checked on \( T_i, S_j \).
\[
\begin{align*}
  g_d \circ f_d(T_i) &= g_d(f_{d-1}(T_i)) + g_d(T'_i) \\
  &= g_d(f_{d-1}(T_i)) + T_i - g_d(f_{d-1}(T_i)) \\
  &= T_i.
\end{align*}
\]
This and the induction hypothesis proves that \( g_d \circ f_d \) is the identity on \( E_{>-d} \).
Similarly, \( f_d \circ g_d \) is the identity on \( F_{>-d} \). We use these facts for the next check:
\[
\begin{align*}
  g_d \circ f_d(S'_j) &= g_d(S_j) - g_d \circ f_d(g_{d-1}(S_j)) \\
  &= S'_j + g_{d-1}(S_j) - g_{d-1}(S_j) \\
  &= S'_j.
\end{align*}
\]
Similarly, \( f_d(g_d(T'_i)) = T'_i \).

We have constructed an isomorphism of differential graded algebras
\[
f_d : (E \otimes_{A} \mathcal{Sym}_A(V_d \oplus V_d[1]))_{>-d} \to (F \otimes_{A} \mathcal{Sym}_A(W_d \oplus W_d[1]))_{>-d},
\]
with inverse \( g_d \). By Lemma \ref{lem:extension}, \( f_d \) admits an extension to a morphism of Tate resolution \( E \otimes_{A} \mathcal{Sym}_A(V_d \oplus V_d[1]) \to F \otimes_{A} \mathcal{Sym}_A(W_d \oplus W_d[1]) \). Any such extension (in particular defining \( f_d(T'_i) \)) has the properties required in the claim of the Lemma.  

\section*{Appendix B. Gluing sheaves of differential graded algebra}

by Tomer M. Schlank

The aim of this Appendix is to describe how to glue sheaves of differential \( P_0 \)-algebras.

More precisely, let \( X \) be a non-singular quasi-projective variety. For an open subvariety \( U \subset X \) we denote by \( \mathfrak{A}(U) \) the category of sheaves of differential \( P_0 \)-algebras on \( U \). Let \( U_0, \ldots, U_n \) be a finite affine cover of \( X \). Further let \( A_i \in \mathfrak{A}(U_i) \) be a sheaf of differential \( P_0 \)-algebras on \( U_i \), our goal is to glue all the \( A_i \)'s to a sheaf of differential \( P_0 \)-algebras \( A \in \mathfrak{A}(X) \) such that \( A|_{U_i} \) is quasi-isomorphic \( A_i \). To better understand the issues at hand consider first the case where we are trying to glue sheaves given on two open subset \( U_0, U_1 \). In this case our gluing data consist of object \( A_i \in \mathfrak{A}(U_i), 0 \leq i \leq 1, \) and a quasi-isomorphism \( K_{0,1} : A_0|_{U_0 \cap U_1} \to A_1|_{U_0 \cap U_1} \). Here we already see the first difference between classical gluing of sheaves and gluing of sheaves in our case, since we only assume that \( K_{0,1} \) is a quasi-isomorphism and not necessarily an isomorphism as in the classical case. The difference from the classical case becomes even more apparent when we consider 3 open sets \( U_0, U_1, U_2 \). In this case in addition to \( A_i \in \mathfrak{A}(U_i), 0 \leq i \leq 2 \) and the quasi-isomorphism \( K_{0,1}, K_{0,2}, K_{1,2} \) our gluing data will consist of a homotopy \( K_{0,1,2} \) between \( K_{1,2} \circ K_{0,1} \).
and $K_{0,2}$. Note that when gluing classical sheaves one will assume that the two maps $K_{1,2} \circ K_{0,1}$ and $K_{0,2}$ are the same.

To conclude we have several steps to go through so we can glue our sheaves of differential $P_0$-algebras. The first step will be to define gluing data for sheaves of differential $P_0$-algebras. The second step will be to show that given such gluing data we indeed can construct the desired $\mathcal{A}$. The last step will be to show that in the situation described in this paper we will indeed have such gluing data.

The datum and argument below are very categorical in nature. So we first have to take good care of the notations.

**B.1. Notation.** Throughout this appendix we fix a non-singular quasi-projective variety $X$. Let $\mathcal{S}$ be the category of simplicial sets.

For an open subvariety $U \subset X$ we denote by $\mathfrak{A}(U)$ the category of sheaves of differential $P_0$-algebras on $U$. The de Rham algebras $\Omega^*_n = \Omega^*_n(\Delta^n)$ of polynomial differential forms on the geometric $n$-simplices form a simplicial differential graded commutative algebra (DGCA) $\Omega^*$. Let us denote by the same letter the functor $\Omega^*: \mathcal{S} \to \text{DCGA}$, $K \mapsto \mathfrak{A}(K, \Omega^*)$ from the category of simplicial sets to the category of DCGA (so that $\Omega^*(\Delta^n) = \Omega^*_n$). Let us denote by the same letter the functor

$$A^\Delta := A \otimes \Omega^*(\Delta^n) \in \mathfrak{A}(U).$$

Similarly for every simplicial set $K \in \mathcal{S}$ we can define

$$A^K := A \otimes \Omega^*(K) \in \mathfrak{A}(U).$$

For every $A_0, A_1 \in \mathfrak{A}(U)$ we define the simplicial set

$$\mathfrak{M}_U(A_0, A_1) \in \mathcal{S},$$

$$\mathfrak{M}_U(A_0, A_1)_n := \text{Hom}_{\mathfrak{A}(U)}(A_0, A_1^\Delta^n).$$

Note that we can view this also as $\text{Hom}_{\mathfrak{A}(U) \otimes \Omega^*(\Delta^n)}(A_0^\Delta^n, A_1^\Delta^n)$, so that compositions of maps in $\mathfrak{M}_U$ are defined. Given two open subsets $U \subset V \subset X$, We denote by

$$\downarrow_U^V: \mathfrak{A}(V) \to \mathfrak{A}(U)$$

$$A \mapsto A \downarrow_U^V,$$

the restriction functor, $\downarrow_U^V$ has a right adjoint which is the pushforward functor, we denote this functor by

$$\uparrow_U^V: \mathfrak{A}(U) \to \mathfrak{A}(V)$$

$$A \mapsto A \uparrow_U^V.$$

**Lemma B.1.** The functors $\downarrow_U^V$ and $\uparrow_U^V$, have the following properties:

1. $\downarrow_U^V$ is the left adjoint of $\uparrow_U^V$.
2. Let $U \subset V \subset W \subset X$; we have

$$\downarrow_V^W \downarrow_U^V = \downarrow_U^W,$$

$$\uparrow_U^W \uparrow_V^W = \uparrow_U^W.$$

3. Let $U, V \subset W$; we have

$$\uparrow_U^W \uparrow_V^W = \uparrow_{U \cup W}^W \uparrow_{U \cup V}^W.$$
For $n \in \mathbb{N}$ we denote by $[n]$ the ordered set $0 < 1 < \cdots < n$. For any $I \subset [n]$ we denote $P_I$ to be the partially ordered set of subsets of $I$ containing the first and last element. Now let 
\[ \{a_0, \ldots, a_d\} = I \subset [n], \]
and let $0 < i < d$. We denote:
\[
I_{\leq i} := \{a_0, \ldots, a_i\}, \quad I_{\geq i} := \{a_i, \ldots, a_d\}, \quad I_i := \{a_0, \ldots, a_i, \ldots, a_d\},
\]
\[ a_0 = n(I), \quad a_d = x(I). \]

We introduce the following maps:
\[
T_{I,i} : P_{I_{\leq i}} \times P_{I_{\geq i}} \to P_I, \quad T_{I,i}(K, L) = K \cup L,
\]
\[
S_{I,i} : P_{I_i} \to P_I, \quad S_{I,i}(K) = K,
\]
\[
\hat{S} : P_{I_i} \to P_I, \quad \hat{S}_{I,i}(K) = K \cup \{i\}.
\]

Let $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ be a cover of $X$. For every non-empty $I \subset [n]$ we denote
\[
U_I := \bigcap_{i \in I} U_i.
\]

For any partially ordered set $P$ we denote by $N(P) \in \mathcal{S}$ the nerve of $P$. Note that for $I = \{a_0, \ldots, a_d\} \subset [n]$, $d \geq 1$, $N(P_I)$ is naturally isomorphic as a simplicial set to the $d - 1$-dimensional cube $(\Delta^d)^{d-1}$. We denote by $\partial N(P_I) \subset N(P_I)$ the boundary of the cube. Note that $\partial N(P_I)$ is the union of the $2(d-1)$ faces of $N(P_I)$, each of which is a $d - 2$-dimensional cube. It easy to see that each of these faces is one of the images of the maps
\[
N(S_{I,i}) : N(P_{I_i}) \to N(P_I)
\]
\[
N(\hat{S}_{I,i}) : N(P_{I_i}) \to N(P_I),
\]
for $0 < i < d$.

**B.2. Gluing data.** In this section we shall define the gluing data required in order to glue sheaves of differential $P_0$-algebras. Let $\mathcal{U} := \{U_0, U_1, \ldots, U_n\}$ be a finite open cover of $X$. Let $(A_0, \ldots, A_n; \{K_I\})$ consist of:

1. A sheaf of differential $P_0$-algebras, $A_i \in \mathfrak{A}(U_i)$ for every $i = 0, \ldots, n$.
2. For every $I \subset [n]$, $|I| \geq 2$ a map
\[
K_I \in \mathcal{S}(N(P_I), \mathfrak{M}_{U_I}(A_{n(I)} \downarrow_{U_I} U_{n(I)}, A_{x(I)} \downarrow_{U_I} U_{x(I)})).
\]

Then $(A_0, \ldots, A_n; \{K_I\})$ are called **gluing data on $\mathcal{U}$** if they satisfy the following conditions

1. For every $I \subset [n]$, $|I| = 2$, $K_I$ is a quasi-isomorphism.
2. For every $I = \{a_0, \ldots, a_d\}$ and $0 < i < d$ the diagram:
\[
\begin{array}{ccc}
N(P_{I_{\leq i}}) \times N(P_{I_{\geq i}}) & \xrightarrow{N(T_{I,i})} & N(P_I) \\
\downarrow K_{I_{\leq i}} \times K_{I_{\geq i}} & & \downarrow K_I \\
\mathfrak{M}_{U_{I_{\leq i}}} \times \mathfrak{M}_{U_{I_{\geq i}}} & \xrightarrow{C_{I,i}} & \mathfrak{M}_{U_I}(A_{a_0} \downarrow_{U_I} U_{a_0}, A_{a_d} \downarrow_{U_I} U_{a_d})
\end{array}
\]

with
\[
\mathfrak{M}_{U_{I_{\leq i}}} = \mathfrak{M}_{U_{I_{\leq i}}}(A_{a_0} \downarrow_{U_{I_{\leq i}}} U_{a_0}, A_{a_i} \downarrow_{U_{I_{\leq i}}} U_{a_i}).
\]
\( \mathfrak{M}_{U_{I,i}} = \mathfrak{M}_{U_{I,i}}(A_{n_i} \downarrow_{U_{I,i}}, A_{n_d} \downarrow_{U_{I,i}}), \)

commutes. Here \( C_{I,i} = \bigcirc \circ (\downarrow_{U_{I,i}} \times \downarrow_{U_{I,i}}) \) and \( \bigcirc \) is the composition morphism.

(3) For every \( I = \{a_0, \ldots, a_d\} \) and \( 0 < i < d \) the diagram:

\[
\begin{array}{ccc}
N(P_I) & \xrightarrow{N(S_{i,i})} & N(P_I) \\
\downarrow K_I & & \downarrow K_I \\
\mathfrak{M}_{U_{I,i}}(A_{n_0} \downarrow_{U_{I,i}}, A_{n_d} \downarrow_{U_{I,i}}) & \xrightarrow{U_{I_i}} & \mathfrak{M}_{U_{I,i}}(A_{n_0} \downarrow_{U_{I,i}}, A_{n_d} \downarrow_{U_{I,i}})
\end{array}
\]

commutes.

**Remark B.2.** For \( I = \{i\} \subset [n] \) it will be convenient to denote:

\[ K_I = \text{Id}_{A_i} \in \mathfrak{M}_{U_i}(A_i, A_i). \]

**Remark B.3.** If the \( K_I \) are given only for \( I \subset [n], \ 2 \leq |I| \leq d \) for some integer \( d \), we shall call \((A_0, \ldots, A_n; \{K_I\})\) \( d \)-partial gluing data. Note that this definition makes sense since the compatibility conditions for \( K_I \) involve only \( K_J \) with \(|J| < |I|\).

We shall define a morphism between two gluing data on \( \mathcal{U}, (A_0, \ldots, A_n; K_I), (B_0, \cdots, B_n; L_I) \) as a collection of morphism

\[ f_i \in \mathfrak{M}_{U_i}(A_i, B_i), \]

such that the following squares commute.

\[
\begin{array}{ccc}
A_{n(I)} & \xrightarrow{U_{n(I)}} & A_{n(I)} \\
\downarrow f_{n(I)} & & \downarrow f_{n(I)} \\\nB_{n(I)} & \xrightarrow{L_{n(I)}} & B_{n(I)}
\end{array}
\]

\[
\begin{array}{ccc}
A_{n(I)} & \xrightarrow{U_{n(I)}} & A_{n(I)} \\
\downarrow f_{n(I)} & & \downarrow f_{n(I)} \\\nB_{n(I)} & \xrightarrow{L_{n(I)}} & B_{n(I)}
\end{array}
\]

For \((A_0, \ldots, A_n)\) given sheaves of differential \( P_0\)-algebras, \( A_i \in \mathfrak{A}(U_i) \) we say that \( \{K_I\} \) are gluing data for \((A_0, \ldots, A_n)\) iff \((A_0, \ldots, A_n; \{K_I\})\) are gluing data on \( \mathcal{U} := \{U_0, U_1, \ldots, U_n\} \).

**B.3. The space of gluing data.** Let \((A_0, \ldots, A_n)\) be sheaves of differential \( P_0\)-algebras, \( A_i \in \mathfrak{A}(U_i) \). The set of all possible gluing data \( \{K_I\} \) for \((A_0, \ldots, A_n)\) can be considered as the zero simplices of a naturally defined simplicial set.

**Definition B.4.** Let \((A_0, \ldots, A_n)\) be sheaves of differential \( P_0\)-algebras, \( A_i \in \mathfrak{A}(U_i) \). We define the space of gluing data for \((A_0, \ldots, A_n)\) to be the simplicial set \( \mathbb{G}(A_0, \ldots, A_n) \) such that \( \mathbb{G}(A_0, \ldots, A_n)_m \) consists of all collections of maps

\[ K_I \in \mathcal{S}(\Delta^m \times N(P_I), \mathfrak{M}_{U_I}(A_{n(I)} \downarrow_{U_I}, A_{n(I)} \downarrow_{U_I})), \]

satisfying compatibly conditions as above.

Given a finite cover of \( X, \mathcal{U} := \{U_0, U_1, \ldots, U_n\} \) we can take the disjoint union of \( \mathbb{G}(A_0, \ldots, A_n) \) over all the possible \((A_0, \ldots, A_n)\), \( A_i \in \mathfrak{A}(U_i) \). We denote the resulting space by \( \mathbb{G}(\mathcal{U}) \).
Similarly the collection of all morphism between gluing data on a cover \( \mathcal{U} \) can be considered as the zero simplices of a naturally defined simplicial set.

Namely let \((A_0, \ldots, A_n), (B_0, \ldots, B_n)\) be sheaves of differential \( P_0 \)-algebras, \( A_i, B_i \in \mathfrak{A}(U_i) \). We shall denote by \( \mathbb{H}((A_0, \ldots, A_n), (B_0, \ldots, B_n)) \) the simplicial set of morphisms between gluing data for \((A_0, \ldots, A_n), (B_0, \ldots, B_n)\). Namely the \( m \)-simplices of \( \mathbb{H}((A_0, \ldots, A_n), (B_0, \ldots, B_n)) \) are collections of maps:

\[
K_I \in \mathcal{S}(\Delta^m \times \mathcal{N}(P)), \mathfrak{M}_{U_I}(A_{\alpha(I)} \downarrow_{U_I} \cup_{U_I} B_{\alpha(I)} \downarrow_{U_I}),
\]

\[
L_I \in \mathcal{S}(\Delta^m \times \mathcal{N}(P)), \mathfrak{M}_{U_I}(A_{\alpha(I)} \downarrow_{U_I} \cup_{U_I} B_{\alpha(I)} \downarrow_{U_I}),
\]

\[
f_i \in \mathcal{S}(\Delta^m, \mathfrak{M}_{U_i}(A_i, B_i)),
\]

satisfying the natural computability conditions analogous to those specified in the previous subsection. Again one takes the disjoint union of all possible such \((A_0, \ldots, A_n)\) and \((B_0, \ldots, B_n)\) and get a simplicial set \( \mathbb{H}(\mathcal{U}) \). Thus we get that the collection of gluing data on \( \mathcal{U} \) can be best described as a category object \((\mathcal{G}(\mathcal{U}), \mathbb{H}(\mathcal{U}))\) in the category of simplicial sets. A category object in a category is by definition a pair of objects \( O \) and \( M \) with two morphisms \( t, d: M \rightarrow O \) (target, source), a morphism \( O \rightarrow M \) (identity) and a composition morphism \( c: M \times_O M \rightarrow M \) satisfying compatibility conditions of a category. We take \( \mathbb{N}(\mathcal{U}) \) to be the nerve bi-simplicial set corresponding to this category object.

\[
\mathbb{N}(\mathcal{U})_m := \mathbb{H}(\mathcal{U}) \times_{\mathcal{G}(\mathcal{U})} \cdots \times_{\mathcal{G}(\mathcal{U})} \mathbb{H}(\mathcal{U}),
\]

where the product is taken over \( m \) copies of \( \mathbb{H}(\mathcal{U}) \) in an analogous fashion to the classical nerve construction. The diagonal of the bi-simplicial set \( \mathbb{N}(\mathcal{U}) \) is a simplicial set. We denote this simplicial set by \( \mathbb{N}^\Delta(\mathcal{U}) \)

### B.4. Gluing

We shall prove the following statement.

**Theorem B.5.** Let \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \) be a finite open cover of \( X \) and let \((A_0, \ldots, A_n; \{K_I\})\) be gluing data on \( \mathcal{U} \). Then there exist a sheaf of differential \( P_0 \)-algebra \( \mathbf{A} \in \mathfrak{A}(X) \), such that for all \( 0 \leq i \leq n \), \( \mathbf{A} \downarrow_{U_i} \) is quasi-isomorphic to \( A_i \).

**Proof.** The proof will be done by induction on the number of open sets in the cover, i.e., we shall first construct the gluing for \( n = 1 \), then we shall show that given gluing data on \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \), one can define gluing data on \( \mathcal{V} = \{U_0, U_1, \ldots, U_{n-2}, U_{n-1} \cup U_n\} \), by gluing \( A_{n-1} \) and \( A_n \).

**Definition B.6.** Let \( \mathcal{U} = \{U_0, U_1\} \) be a cover of \( X \) (i.e. \( X = U_0 \cup U_1 \)) and let

\[
A_0 \in \mathfrak{A}(U_0),
\]

\[
A_1 \in \mathfrak{A}(U_1),
\]

\[
K_{0,1} \in \text{Hom}_{\mathfrak{A}(U_{0,1})}(A_0 \downarrow_{U_{0,1}} A_1, U_{0,1})
\]
be gluing data on \( \mathfrak{U} \). We denote by \( A_0 \coprod_{K_{0,1}} A_1 \) the limit of the following diagram in \( \mathfrak{U}(X) \):

\[
\begin{array}{ccc}
A_0 \xrightarrow{u_0} & A_1 \xrightarrow{u_1} & A_1 \\
\downarrow A_0 \times_{U_0,1} \downarrow & A_1 \times_{U_1,1} \downarrow & A_1 \times_{U_1,1} \\
A_1 \xleftarrow{u_1} & A_1 \xleftarrow{u_1} & A_1 \\
\end{array}
\]

Where \( \beta_0, \beta_1 \) are the maps corresponding to the two maps \( \Delta^0 \to \Delta^1 \)

**Lemma B.7.** Given data as above, \( (A_0 \coprod_{K_{0,1}} A_1) \xrightarrow{i_{U_0}} \) is naturally quasi-isomorphic to \( A_i \).

**Proof.** First we prove this for \( i = 0 \): since the restriction functor commutes with limits, we have that \( (A_0 \coprod_{K_{0,1}} A_1) \xrightarrow{i_{U_0}} \) is the limit of the diagram:

\[
\begin{array}{ccc}
A_0 & A_1 \times_{U_0,1} \downarrow & A_1 \times_{U_0,1} \\
\downarrow & \downarrow & \downarrow \\
A_1 & A_1 \times_{U_1,1} & A_1 \times_{U_1,1} \\
\end{array}
\]

Thus we have a pullback diagram

\[
(A_0 \coprod_{K_{0,1}} A_1) \xrightarrow{i_{U_0}} \xrightarrow{i_{U_0}} A_1 \times_{U_0,1} \xrightarrow{i_{U_0}} \]

Now we need to show that left vertical map is a quasi-isomorphism. Note that this can be checked stalk-wise. Now the right vertical map is stalk-wise surjective and quasi-isomorphism (i.e., with acyclic kernel) and thus so is the left vertical map.

For \( i = 1 \): since the restriction functor commutes with limits we have that \( (A_0 \coprod_{K_{0,1}} A_1) \xrightarrow{i_{U_1}} \) is the limit of the diagram:

\[
\begin{array}{ccc}
A_0 \xrightarrow{u_0} & A_1 \times_{U_1,1} \downarrow & A_1 \\
\downarrow K_{0,1} \downarrow & \downarrow & \downarrow \\
A_1 \times_{U_0,1} & A_1 \times_{U_0,1} & A_1 \\
\end{array}
\]

Let \( P \) be the pullback of left side of the diagram. Since \( K_{0,1} \), \( \beta_0 \) and \( \beta_1 \) are quasi-isomorphisms and \( \beta_0 \) is surjective on stalks, we get that the map induced
by $\beta_1, b_1 : P \to A_1 \downarrow_{U_0}^V$ is a quasi-isomorphism. Further, it is easy to see that $b_1$ is also surjective on stalks. Thus as in the case $i = 0$ we get that the map 
\((A_0 \coprod_{K_{0,1}} A_1) \downarrow_{U_i}^X = P \times A_1 \downarrow_{U_0}^V \to A_1 \downarrow_{U_0}^V\) is a quasi-isomorphism. \(\square\)

We shall now take care of the induction step in the proof

**Proposition B.8.** Let $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ be a finite open cover of $X$. Let 
\((A_0, \ldots, A_n; \{K_I\})\) be gluing data on $\mathcal{U}$. Consider the finite open cover of $X$, 
\[\mathcal{V} = \{V_0 := U_0, V_1 := U_1, \ldots, V_{n-2} := U_{n-2}, V_{n-1} := U_{n-1} \cup U_n\}.\]

There exist gluing data $(B_0, \ldots, B_{n-1}; \{L_I\})$ on $\mathcal{V}$, such that $B_{n-1} = A_0 \coprod_{K_{0,1}} A_1$ and $B_i = A_i$ for $0 \leq i \leq n - 2$.

**Proof.** The proof will be done by an explicit construction. A complete description of the construction will take some effort and place. In order to simplify the exposition and improve readability we will follow the following notations and guidelines.

1. We shall construct the required data, but leave checking the compatibility conditions to the reader.

2. We shall use repeatedly properties (1), (2) and (3) from Lemma B.1. When we show that two objects are isomorphic by using one or more of these properties will use subscript next to the equal sign: e.g. if $O \subset U \subset W \subset X$ we write 
\[\mathfrak{M}_U(A \downarrow_U^W, B \downarrow_O^U) = \mathfrak{M}_O(A \downarrow_O^W, B \downarrow_O^W).\]

3. When two mapping spaces are isomorphic by applying natural adjunctions we will abuse notation and treat them as equal.

Since 
\[B_i := A_i, \quad 0 \leq i \leq n - 2, \quad B_{n-1} = A_0 \coprod_{K_{0,1}} A_1,\]

we are left with defining $L_I$ for every $\emptyset \neq I \subset \{0, \ldots, n - 1\}, |I| > 1$. If $n - 1 \not\in I$ we will just take $L_I := K_I$. Now assume $n - 1 \in I$; we denote $I_- := I \cup \{n\}$, $I_- = I_+ \setminus \{n - 1\}$. Note that we then have $V_I = U_I \cup U_{I_-}, U_I \cap U_{I_-} = U_{I_+}$. For every $I$ such that $n - 1 \in I$ we require a map 
\[L_I \in \mathcal{S}(N(P_I), \mathfrak{M}_V(A_n \downarrow_{V_I}^{U_n(I)}, B_{n-1} \downarrow_{V_I}^{V_{n-1}(I)})).\]

Note that $|I| > 1$ so $n(I) < n - 1$ and $B_{n(I)} = A_{n(I)}$. Further, since $B_{n-1} = A_0 \coprod_{K_{0,1}} A_1$ is a limit, $L_I$ can be described as three maps 
\[L_I^0 \in \mathcal{S}(N(P_I), \mathfrak{M}_V(A_n(I) \downarrow_{V_I}^{U_n(I)}), (A_{n-1} \downarrow_{U_n}^{V_{n-1}} \downarrow_{V_I}^{V_{n-1}})),\]
\[L_I^+ \in \mathcal{S}(N(P_I), \mathfrak{M}_V(A_n(I) \downarrow_{V_I}^{U_n(I)}), (A_n \downarrow_{U_n}^{V_{n-1}} \downarrow_{V_I}^{V_{n-1}}), (A_{n-1} \downarrow_{U_n}^{V_{n-1}} \downarrow_{V_I}^{V_{n-1}})),\]
\[L_I^- \in \mathcal{S}(N(P_I), \mathfrak{M}_V(A_n(I) \downarrow_{V_I}^{U_n(I)}), (A_n \downarrow_{U_n}^{V_{n-1}} \downarrow_{V_I}^{V_{n-1}}),\]

satisfying certain compatibility conditions. Again we will construct $L_I^0, L_I^+$ and $L_I^-$ and leave checking the compatibility to the reader. Let us start with defining $L_I^-$. Note that we have 
\[(A_n \downarrow_{U_n}^{V_{n-1}} \downarrow_{U_n}^{V_{n-1}} \downarrow_{V_I}^{V_{n-1}}) = 2,3\]
Thus
\[ A_n^I \downarrow_{U_I}^V L_I^+ \uparrow_{U_I}^L . \]

Thus
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) = 2.3 \]
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) = 1 \]
\[ \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) . \]

Now we have:
\[ S(N(P_I)), \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) = \]
\[ S(N(P_I)), \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) = \]
\[ S(N(P_I) \times I, \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) = \]
\[ S(N(P_I)), \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) . \]

Thus we can take \( L_I^+ = K_I^+ . \)

We shall now define
\[ L_I^0 \in S(N(P_I)), \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) . \]

Note that we have
\[ (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V = 3 A_n^I \downarrow_{U_I}^V L_I^+ \uparrow_{U_I}^L . \]

Thus
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) = 3 \]
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) = 1,2 \]
\[ \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^+ ) . \]

Thus we can take \( L_I^0 = K_I^+ . \)

Finally we define
\[ L_I^- \in S(N(P_I)), \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) . \]

Note that
\[ (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V = 3 A_n^I \downarrow_{U_I}^V L_I^- \uparrow_{U_I}^L \uparrow_{U_I}^L . \]

Thus
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, (A_n^I \uparrow_{U_I}^V n_{-1} \downarrow_{U_{-1}I}) \downarrow_{V_I}^V) = 3 \]
\[ \mathcal{M}_{V_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^- ) = 1 \]
\[ \mathcal{M}_{U_I} (A_n(I) \downarrow_{V_I}^U, A_n^I \uparrow_{U_I}^V L_I^- ) . \]

So we can take \( L_I^- = K_I^- . \)

\[ \square \]

\[ \square \]

Above we discussed the process of gluing inductively, now we can also describe the final result:
Lemma B.9. Let \((A_0, \ldots, A_n; \{K_I\}_I)\) be gluing data on \(\mathfrak{U} = \{U_0, U_1, \ldots, U_n\}\). Then the glued sheaf of differential \(P_0\)-algebra \(A \in \mathfrak{A}(X)\) constructed in the previous section in the equalizer of a diagram

\[
\prod_{I \subseteq [n], 1 \leq |I|} A^{(\Delta^1)^{[I]-1}}_{\partial X(I)} \downarrow_{U_I} \uparrow_{U_I} \prod_{I \subseteq [n], 1 \leq |I|} A^{|\Delta^1|^{[I]-1}}_{\partial I} \downarrow_{U_I} \uparrow_{U_I}
\]

one of the map is induced by the inclusions \(\partial((\Delta^1)^{[I]-1}) \subseteq (\Delta^1)^{[I]-1}\) and the second one is defined by \((A_0, \ldots, A_n; \{K_I\}_I)\).

Proof. By induction on \(n\). \(\square\)

B.5. Gluing as a functor. Given a finite cover \(\mathfrak{U} = \{U_0, U_1, \ldots, U_n\}\), gluing is a way to construct from gluing data on \(\mathfrak{U} = \{U_0, U_1, \ldots, U_n\}\) new gluing data on \(\mathfrak{V} = \{U_0, U_1, \ldots, U_{n-2}, U_{n-1} \cup U_n\}\). It is clear from the construction that it is functorial with respect to morphisms of gluing data. However, the collection of gluing data has an additional structure as a category object in simplicial sets and to understand the behavior of gluing with respect to this structure one needs additional care. Let us e.g. assume that we have \(\mathfrak{U} = \{U_0, U_1\}\). A path in \((G(\mathfrak{U}))_1\) is given by sheaves of \(P_0\)-algebras \(A_i \in \mathfrak{A}(U_i)\) and a map

\[
K_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1^\Delta \downarrow_{U_{0,1}},
\]

by restricting this path to its two endpoints we get two different gluing data,

\[
K^0_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1 \downarrow_{U_{0,1}},
\]

\[
K^1_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1 \downarrow_{U_{0,1}}.
\]

We shall denote the result of gluing according to \(K^0_{0,1}\) by \(B^0 \in \mathfrak{A}(U_0 \cup U_1)\). There is no simplicial path connecting \(B^0\) and \(B^1\), but it is possible to connect them by a zigzag of maps. Namely the map

\[
K_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1^\Delta \downarrow_{U_{0,1}}
\]

naturally defines a map

\[
K^{0,1}_{0,1} : A_0^\Delta \downarrow_{U_{0,1}} \rightarrow A_1^\Delta \downarrow_{U_{0,1}}.
\]

We denote the resulting gluing by \(B^{0,1} \in \mathfrak{A}(U_0 \cup U_1)\). Note that restricting to the endpoints results in a zigzag:

\[
B^0 \leftarrow B^{0,1} \rightarrow B^1.
\]

Similarly a triangle

\[
K_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1^\Delta \downarrow_{U_{0,1}}
\]
gives rise to a commutative diagram of the form:

![Diagram]

To conclude, in general we have a natural map from
\[ N^\Delta(\mathfrak{U})_m \to S(\Delta_m \times sd\Delta^m, N^\Delta(\mathfrak{V})) \]
where \( sd \) denote the barycentric subdivision. Thus we get a map
\[ N^\Delta(\mathfrak{U}) \to N^\Delta(\mathfrak{V}) \]
defined up to homotopy, “realizing” the gluing construction.

**B.6. Gluing as homotopy limit.** In this section we shall describe the process of gluing as a certain homotopy limit in the category \( \mathfrak{A}(X) \). Usually when discussing homotopy limits one uses the language of model categories. However, the definition of a homotopy limit actually depends only on the weak equivalences, and can be defined using only them (see [15]). Although we have not presented a model category structure on \( \mathfrak{A}(X) \) we do have a natural notion of weak equivalences as the quasi-isomorphisms in \( \mathfrak{A}(X) \).

Further complication results from the fact that we take a homotopy limit of a simplicial functor
\[ F : D_n \to \mathfrak{A}(X), \]
where \( D_n \) is a simplicial diagram. The notion of a limit in the enriched situation is discussed in [19] and some general results about homotopy limits in this realm are discussed in [30]. In this section we shall use the following definitions.

**Definition B.10** ([15], §33). A homotopical category is a category \( \mathfrak{M} \) equipped with a class of morphisms called weak equivalences that contains all the identities and satisfies the 2-out-of-6 property i.e.: if \( hg \) and \( gf \) are weak equivalences, then so are \( f \), \( g \), \( h \), and \( hgf \).

The category one gets from inverting all the weak equivalences is denoted by \( \text{Ho}(\mathfrak{M}) \). Note that there is a natural map:
\[ \delta_{\mathfrak{M}} : \mathfrak{M} \to \text{Ho}(\mathfrak{M}) \]

Given a functor
\[ G : \mathfrak{M} \to \mathfrak{N} \]
between two homotopical categories we say that \( G \) is homotopical if \( G \) takes weak equivalences to weak equivalences. Similarly given a functor
\[ G : \mathfrak{M} \to \text{Ho}(\mathfrak{N}) \]
we say that \( G \) is homotopical if \( G \) takes weak equivalences to isomorphisms.
Definition B.11. Let \( \mathcal{M} \) and \( \mathcal{N} \) be two homotopical category and let
\[ G : \mathcal{M} \to \mathcal{N} \]
be a functor. A right derived functor of \( G \) is a functor
\[ RG : \mathcal{M} \to \mathcal{N} \]
equipped with a comparison map
\[ \delta_{\mathcal{M}} G \to RG \]
such that \( RG \) is homotopical and initial among homotopical functors equipped with maps from \( \delta_{\mathcal{M}} G \).

Definition B.12. Let \( \mathcal{M} \) and \( \mathcal{N} \) be two homotopical category and let
\[ G : \mathcal{M} \to \mathcal{N} \]
be a functor. A point-set right derived functor of \( G \) is a homotopical functor
\[ RG : \mathcal{M} \to \mathcal{N} \]
equipped with a comparison map \( G \to RG \) such that the induced map \( \delta_{\mathcal{M}} G \to \delta_{\mathcal{N}} RG \) makes \( \delta_{\mathcal{N}} RG \) into a right derived functor of \( G \).

To present gluing as a homotopy limit we shall first explain how given a gluing data \((A_0, \ldots, A_n; K_I)\) one can construct a diagram \( F : D_n \to \mathfrak{A}(X) \) were \( D_n \) depends on \( n \) alone (but \( F \) depends on \((A_0, \ldots, A_n; K_I)\)). Then, the result of gluing \((A_0, \ldots, A_n; K_I)\) would turn out to be the homotopy limit of this diagram i.e the result of applying a point-set right derived functor of the limit functor on \( F \).

Now let \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \) be a finite cover of \( X \). Let \((A_0, \ldots, A_n; \{K_I\}_I)\) be gluing data on \( \mathcal{U} \). We shall construct a finite simplicial category \( D_n \) and a simplicial functor \( F : D_n \to \mathfrak{A}(X) \) where the result of gluing along \((A_0, \ldots, A_n; \{K_I\}_I)\) is a homotopy limit of \( F \).

First we shall describe \( D_n \).

Definition B.13. Let \( \emptyset \neq I \subset J \subset [n] \) be two non-empty subsets we denote
\[ H(I, J) := \{t \in J | x(I) \leq t \} \subset J \]
We also denote
\[ \mathbb{P}_{I,J} := P_{H(I,J)} \]
For convenience for \( \emptyset \neq I, J \subset [n] \) such that \( I \not\subset J \) we shall take \( P_{I,J} = \emptyset \) to be the empty poset.

Definition B.14. for \( n \geq 0 \) We denote by \( D_n \) the simplicial category such that
\begin{enumerate}
  \item The objects of \( D_n \) are non-empty subsets, \( I \subset [n] \),
  \item Given \( I, J \in \text{Ob}D_n \), we take
    \[ \text{Map}_{D_n}(I, J) := N(\mathbb{P}_{I,J}) \].
  \item For \( I \subset J \subset K \), we take the composition
    \[ N(\mathbb{P}_{I,J}) \times N(\mathbb{P}_{J,K}) \xrightarrow{\mathbb{P}_{I,J,K}} N(\mathbb{P}_{I,K}) \]
to be the nerve of the map induced by taking union of sets. (note that in all other cases there is only one possible map)
\end{enumerate}
We are now ready to define our simplicial functor $F$.

$$F : D_n \to \mathcal{A}(X)$$

First, on objects we define:

$$F(I) := A_{x(I)} \downarrow_{U_I} \upto_{U_J} \in \mathcal{A}(X)$$

now it is left to define for every $I, J$,

$$F : N(\mathcal{P}_{I,J}) = \text{Map}_{D_n}(I, J) \to \mathcal{M}_X(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J})$$

Clearly it is enough to consider the case $I \subset J$. Now since $U_J \subset U_I$, we have

$$\upto_I \upto_J = \upto_U = \upto_{U_J} \upto_{U_I} \upto_{U_J}$$

Thus

$$\mathcal{M}_X(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J}) =$$

$$\mathcal{M}_{U_J}(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J}) =$$

$$\mathcal{M}_{U_J}(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J}) =$$

$$\mathcal{M}_{U_J}(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J})$$

We have to define:

$$F_{I,J} : N(\mathcal{P}_{I,J}) \to \mathcal{M}_{U_J}(A_{x(I)} \downarrow_{U_I} \upto_{U_J}, A_{x(J)} \downarrow_{U_J} \upto_{U_J})$$

Now recall that for $H = H(I, J)$ we have map

$$K_H : N(\mathcal{P}_{I,J}) = N(\mathcal{P}_H) \to \mathcal{M}_{U_H}(A_{n(H)} \downarrow_{U_H} \upto_{U_H}, A_{x(H)} \downarrow_{U_H} \upto_{U_H})$$

since $H \subset J$ we have also a map

$$K_H : N(\mathcal{P}_H) \to \mathcal{M}_{U_J}(A_{n(H)} \downarrow_{U_J} \upto_{U_J}, A_{x(H)} \downarrow_{U_J} \upto_{U_J}).$$

Now since $n(H) = x(I), x(H) = x(J)$, we can just take $F_{I,J} = K_H \downarrow_{U_H}$. We leave it to the reader to verify that the compatibility conditions on $(A_0, \ldots, A_n; \{K_I\}_I)$ insures that $F$ respects composition.

Now, along the lines of 30 we have a functor

$$\lim \mathcal{A}(X)^{D_n} \to \mathcal{A}(X).$$

We shall construct $\mathbb{R}\lim$ as a special form of weighted limit, i.e., we shall define a “coefficients functor” $C : D_n \to \mathcal{S}$ such that for any $F' : D_n \to \mathcal{A}(X)$ we have $\mathbb{R}\lim(F') = \lim^C F'$. We will get the required natural transformation $\lim \to \mathbb{R}\lim$ by taking the unique natural transformation $C \to \ast$, where $\ast$ is the constant functor on the terminal object and by using the identification $\lim^C F = \lim F$.

We define $C : D_n \to \mathcal{S}$ on objects to be $C(I) := N(Q_I)$ where $Q_I$ is the poset of subsets of $I$ that contain the last element $x(I)$ . note that $N(Q_I) \cong (\Delta^1)^{|I|-1}$, Now we need to define a map

$$N(\mathcal{P}_{I,J}) \times N(Q_I) \to N(Q_J).$$

Thus it is enough to define a map

$$C_{I,J} : \mathcal{P}_{I,J} \times Q_I \to Q_J,$$

for every $\emptyset \neq I \subset J \subset [n]$. We shall take:

$$C_{I,J}(K, I') = K \cup I'.$$
We leave it to the reader to check compatibly.

**Lemma B.15.** Let \((A_0, \ldots, A_n; K_I)\) be gluing data and let
\[
F : D_n \to \mathfrak{A}(X)
\]
be the corresponding functor, then the weighted limit \(\lim^C F \in \mathfrak{A}(X)\) is isomorphic to result of gluing of according to \((A_0, \ldots, A_n; K_I)\).

**Proof.** The weighed limit \(\lim^C F\) can be computed as an equalizer of the form
\[
\prod_{\emptyset \neq I \subseteq [n]} A^{N(I)}_{x(I)} \xrightarrow{U_{x(I)}} \prod_{\emptyset \neq I \subseteq [n]} A^{W_I}_{x(I)} \xrightarrow{U_{x(I)}}
\]
Where \(W_I\) is some simplicial set. We leave it to reader the completely formal but tedious check that this equalizer is isomorphic to the one described in Lemma 3.39.

To complete the proof we need to show that \(\lim^C\) is indeed a point set right derived functor of \(\lim\). The proof relies on two essential facts:

1. \(D_n\) is a Reedy category and \(C : D_n \to \mathcal{S}\) is Reedy cofibrant — this can be verified directly by computing latching objects.
2. The category \(\mathfrak{A}(X)\) can be given the structure of a category of fibrant objects in a way that is compatible with the simplicial enrichment.

However giving a complete account of this proof will exceed the scope of this appendix, thus it is omitted.

**B.6.1. A different indexing category.** It is convenient to consider a alternative simplicial category \(E_n\) which is more symmetric and comes naturally with a simplicial functor
\[
W : E_n \to D_n
\]
which is a categorical equivalence. Thus we get that for every functor
\[
F : D_n \to \mathfrak{A}(X)
\]
we have
\[
\mathbb{R} \lim(F \circ W) \approx \mathbb{R} \lim(F)
\]
where the homotopy limit is taken over \(E_n\) in the left hand side and on \(D_n\) on the right-hand side. As a conclusion we will have that gluing can also be considered as homotopy limit over \(E_n\).

To define \(E_n\) we take:

1. The objects of \(E_n\) are all the non-empty subsets \(\emptyset \neq I \subseteq [n]\).
2. For \(\emptyset \neq I, J \subseteq [n]\) we take define the poset \(E_{I,J}\) to be the poset of all chains
\[
I = I_0 \subset I_1 \subset \cdots \subset I_n = J
\]
ordered by inclusion. We define
\[
\text{Map}_{E_n}(I, J) := N(E_{I,J})
\]
(3) We take the composition to be the nerve of the map induced by taking the union of chains.
We are now left with defining the functor

\[ W : E_n \to D_n \]

On objects we shall just take \( W(I) = I \). To define \( W \) on morphisms it is enough to give for every \( \emptyset \neq I \subset J \subset [n] \) a map

\[ W_{I,J} : E_{I,J} \to P_{I,J} \]

we take the map:

\[ W_{I,J}(I = I_0 \subset I_1 \subset \cdots \subset I_n = J) = x(I) = x(I_0) < x(I_1) < \cdots < x(I_n) = x(J) \]

note that \( W \) is an isomorphism on objects and a weak equivalence on morphism spaces and thus a categorical equivalence.

**B.7. The existence of gluing data.** In this section we shall demonstrate the existence and suitable uniqueness of gluing data in our case.

**Theorem B.16.** Let \( X \) be a non-singular quasi-projective variety, \( \lambda \in \Omega^1(X) \) a closed 1-form, \( S_0 = \int \lambda \). Let \( \mathcal{U} = \{ U_0, \ldots, U_n \} \) be a finite affine cover of \( X \). Let \( (M_i, T_i) \) be a BV variety with support \( (U_i, S_0 |_{U_i}) \). Denote \( A_i := (O_{T_i^*}[-1]|_{V_i}, d_{T_i}) \), and consider the sub simplicial sets

\[ \mathcal{G}(U_i)(T_{n(i)}, T_{x(i)}) \subset \mathfrak{M}_{U_i}(A_{n(i)} \downarrow_{U_i}^{U_{n(i)}}, A_{x(i)} \downarrow_{U_i}^{U_{x(i)}}). \]

Then the subspace of \( \mathcal{G}(A_0, \ldots, A_n) \) consisting of gluing data \( \{ K_I \}_I \), such that the image of

\[ K_I : N(P_I) \to \mathfrak{M}_{U_i}(A_{n(i)} \downarrow_{U_i}^{U_{n(i)}}, A_{x(i)} \downarrow_{U_i}^{U_{x(i)}}) \]

lands in

\[ \mathcal{G}(U_i)(T_{n(i)}, T_{x(i)}) \subset \mathfrak{M}_{U_i}(A_{n(i)} \downarrow_{U_i}^{U_{n(i)}}, A_{x(i)} \downarrow_{U_i}^{U_{x(i)}}), \]

is contractible. In particular it is non-empty.

**Proof.** First let us denote by \( \mathcal{G}_d \) the space of all \( d \)-partial gluing data \( \{ K_I \} \) for \( (A_0, \ldots, A_n) \) such that the image of

\[ K_I : N(P_I) \to \mathfrak{M}_{U_i}(A_{n(i)} \downarrow_{U_i}^{U_{n(i)}}, A_{x(i)} \downarrow_{U_i}^{U_{x(i)}}) \]

will land

\[ \mathcal{G}(U_i)(T_{n(i)}, T_{x(i)}) \subset \mathfrak{M}_{U_i}(A_{n(i)} \downarrow_{U_i}^{U_{n(i)}}, A_{x(i)} \downarrow_{U_i}^{U_{x(i)}}). \]

note that \( \mathcal{G}_1 = * \) and our goal is to prove that \( \mathcal{G}_{n+1} \to \mathcal{G}_{d} \) is a trivial Kan fibration. For this it is enough to show that the projection map \( \mathcal{G}_{d+1} \to \mathcal{G}_{d} \) is a trivial Kan fibration. Thus we need to exhibit a lift for diagrams of the sort:

\[
\begin{array}{ccc}
\Delta^n & \to & \mathcal{G}_{d+1} \\
\downarrow & & \downarrow \\
\Delta^m & \to & \mathcal{G}_{d} 
\end{array}
\]
unraveling the definitions and compatibility conditions and using standard adjunctions, specifying such a lift is the same as specifying a collection of lifts:

\[ \partial N(P_I) \times \Delta^n \prod_{\partial N(P_I) \times \Lambda^m_n} N(P_I) \times \Lambda^m_n \rightarrow G(U_I)(T_{n(I)}, T_{x(I)}) \]

for every \( I \subset [n] \) with \( |I| = d + 1 \). But this all exist since \( G(U_I)(T_{n(I)}, T_{x(I)}) \) is Kan contractible.

\[ \Box \]

B.8. Permuting the ordering of the cover. In this appendix we took a minimalistic approach to describe gluing data, i.e., we took the minimal amount of data required to perform the gluing. This minimality came with a price of some symmetry breaking (as often happens). Specifically, note that gluing data \( (A_0, \ldots, A_n; K_I) \) is presented in a way which is not symmetric with respect to the ordering of the open subsets \( U_0, \ldots, U_n \). In this section we shall explain why the resulting glued object will be the same (up to weak equivalence) regardless of the chosen order.

The essential point lies in the Kan contractible groupoid that we discussed above. Note that the lifting conditions we have are not sensitive to ordering. Thus one could get gluing data for any possible ordering. Further the contractibility allows us to get homotopies that relate the different ordering. To show how this works we give a partial account of the case of two open sets, and a very rough sketch for the general case. Let \( U_0, U_1 \) be a cover of \( X \). By considering the groupoid above we get gluing data \( (A_0, A_1; K_{0,1}) \) as well as \( (A_1, A_0; L_{1,0}) \) when \( K_{0,1} \) and \( L_{1,0} \) are maps

\[ K_{0,1} : A_0 \downarrow_{U_{0,1}} \rightarrow A_1 \downarrow_{U_{1,1}} \]
\[ L_{1,0} : A_1 \downarrow_{U_{1,1}} \rightarrow A_0 \downarrow_{U_{0,1}} \]

Further the contractibility supply us with a homotopy:

\[ H : A_0 \downarrow_{U_{0,1}} \rightarrow A_1 \downarrow_{U_{1,1}} \]

between the identity map and \( L_{1,0} \circ K_{0,1} \). Thus we get a commutative diagram

\[ \begin{array}{ccc}
A_0 & \xrightarrow{K_{0,1}} & A_1 \\
\downarrow H & & \downarrow L_{1,0} \\
A_0^{\Delta^{1}} & \xrightarrow{L_{1,0}} & A_1^{\Delta^{1}}
\end{array} \]

where all the vertical maps are weak equivalences.

Since the gluing along \( (A_1, A_0; L_{1,0}) \) is the homotopy limit of the bottom row and the gluing along \( (A_0, A_1; K_{0,1}) \) is the homotopy limit of the top row, we get that the two results are weakly equivalent. In general in order to prove the contractibly
of space of results for arbitrary \( n \) one uses the higher homotopies supplied by the contractible groupoid to show equivalences (or equivalence between equivalence etc.) of corresponding diagrams \( E_n \rightarrow \mathfrak{A}(X) \). This is here were the symmetries of \( E_n \) becomes useful.

Now let \( \mathfrak{U} := \{ U_0, U_1, \ldots, U_n \}, \mathfrak{V} := \{ V_0, V_1, \ldots, V_m \} \) be two affine coverings of \( X \). We would like to show that the result of our construction is independent of the choice of a covering. To see this consider the covering

\[
\mathfrak{W} := \{ U_0 \cap V_0, U_1 \cap V_0, \ldots, U_n \cap V_0, U_0 \cap V_1, U_1 \cap V_1, \ldots, U_n \cap V_1, \ldots, U_0 \cap V_m, U_1 \cap V_m, \ldots, U_n \cap V_m \}.
\]

Indeed we get that the gluing with respect either \( \mathfrak{U} \) or \( \mathfrak{V} \) can be described by gluing along \( \mathfrak{W} \) according to different orderings.

References

1. C. Albert, B. Bleile, and J. Fröhlich, Batalin-Vilkovisky integrals in finite dimensions, J. Math. Phys. 51 (2010), no. 1, 015213, 31. MR 2605846 (2011g:81299)
2. M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, The geometry of the master equation and topological quantum field theory, Internat. J. Modern Phys. A 12 (1997), no. 7, 1405–1429. MR 1432574 (98a:81235)
3. I. A. Batalin and G. A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1981), no. 1, 27–31. MR 616572 (82f:81079)
4. , Quantization of gauge theories with linearly dependent generators, Phys. Rev. D (3) 28 (1983), no. 10, 2567–2582. MR 726170 (85i:81068a)
5. Carlo Becchi, Alain Rouet, and Raymond Stora, Renormalization of gauge theories, Ann. Phys. 98 (1976), 287–321.
6. Felix Alexandrovich Berezin, Introduction to superanalysis, Mathematical Physics and Applied Mathematics, vol. 9, D. Reidel Publishing Co., Dordrecht, 1987, Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký, Translation edited by Dimitri Leites. MR 914369 (89b:58006)
7. Francesco Bonechi, Alberto S. Cattaneo, and Pavel Mnëv, The Poisson sigma model on closed surfaces, J. High Energy Phys. (2012), no. 1, 099, 26. MR 2949286
8. Francesco Bonechi, Pavel Mnëv, and Maxim Zabzine, Finite-dimensional AKSZ-BV theories, Lett. Math. Phys. 94 (2010), no. 2, 197–228. MR 2733561 (2012b:81220)
9. Henri Cartan and Samuel Eilenberg, Homological algebra, Princeton University Press, Princeton, N. J., 1956. MR 0077480 (17,1040c)
10. J. M. Casas, M. Ladra, and T. Pirashvili, Triple cohomology of Lie-Rinehart algebras and the canonical class of associative algebras, J. Algebra 291 (2005), no. 1, 144–163. MR 2158515 (2006f:17020)
11. Alberto S. Cattaneo and Giovanni Felder, A path integral approach to the Kontsevich quantization formula, Comm. Math. Phys. 212 (2000), no. 3, 591–611. MR 1779159 (2002b:53141)
12. Alberto S. Cattaneo and Pavel Mnëv, Remarks on Chern-Simons invariants, Comm. Math. Phys. 293 (2010), no. 3, 803–836. MR 2566163 (2011e:58038)
13. Kevin Costello, Renormalization and effective field theory, Mathematical Surveys and Monographs, vol. 170, American Mathematical Society, Providence, RI, 2011. MR 2778558 (2012f:81177)
14. Kevin Costello and Owen Gwilliam, Factorization algebras in perturbative quantum field theory, http://math.northwestern.edu/~costello/factorization_public.html.
15. William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, Homotopy limit functors on model categories and homotopical categories, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR 2102294 (2005k:18027)
16. Ludvig D. Faddeev and Victor N. Popov, Feynman diagrams for the Yang-Mills field, Phys. Lett. B 25 (1967), 29–30.
17. Daniel R. Grayson and Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, http://www.math.uiuc.edu/Macaulay2/.
18. Marc Henneaux and Claudio Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton, NJ, 1992. MR 1191617 (94h:81003)
19. G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301
20. Hovhannes M. Khudaverdian, *Laplacians in odd symplectic geometry*, Quantization, Poisson brackets and beyond (Manchester, 2001), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 199–212. MR 1958837 (2003k:58005)
21. Maxim Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), no. 3, 157–216. MR 2062626 (2005j:53122)
22. Yuri I. Manin, *Gauge field theory and complex geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 1988, Translated from the Russian by N. Koblitz and J. R. King. MR 954833 (89d:32001)
23. P. Mnëv, *Notes on simplicial BF theory*, Mosc. Math. J. 9 (2009), no. 2, 371–410, back matter (English, with English and Russian summaries). MR 2568441 (2010j:57045)
24. T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Quantization and derived moduli spaces I: shifted symplectic structures*, arxiv:1111.3209 [math.AG].
25. Frédéric Paugam, *Homotopical reduction of gauge theories*, arxiv:1106.4955 [math-ph].
26. George S. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. 108 (1963), 195–222. MR 0154906 (27 #4850)
27. Dmitri Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, Quantization, Poisson brackets and beyond (Manchester, 2001), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 169–185. MR 1958835 (2004i:53116)
28. Albert Schwarz, *Semiclassical approximation in Batalin-Vilkovisky formalism*, Comm. Math. Phys. 158 (1993), no. 2, 373–396. MR 1249600 (95c:58083)
29. Pavol Ševera, *On the origin of the BV operator on odd symplectic supermanifolds*, Lett. Math. Phys. 78 (2006), no. 1, 55–59. MR 2271128 (2007h:58010)
30. Michael Shulman, *Homotopy limits and colimits and enriched homotopy theory*, arxiv:math/0610194 [math.AT].
31. Jim Stasheff, *The (secret?) homological algebra of the Batalin-Vilkovisky approach*, Secondary calculus and cohomological physics (Moscow, 1997), Contemp. Math., vol. 219, Amer. Math. Soc., Providence, RI, 1998, pp. 195–210. MR 1640453 (2000f:18011)
32. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR 0646078 (58 #31119)
33. John Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. 1 (1957), 14–27. MR 0086072 (19,119b)
34. A. Yu. Vaintrob, *Lie algebroids and homological vector fields*, Uspekhi Mat. Nauk 52 (1997), no. 2(314), 161–162 (Russian). MR 1480150
35. Gabriele Vezzosi, *Derived critical loci I - Basics*, arxiv:1109.5213 [math.AG].
36. Jean Zinn-Justin, *Renormalization of gauge theories*, Trends in elementary particle theory (H. Rollnik and K. Dietz, eds.), Lecture Notes in Physics, vol. 37, Springer, Berlin, 1975, pp. 2–39.

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