Formulas for central critical values of twisted L-functions attached to paramodular forms

Nathan C. Ryan
Department of Mathematics, Bucknell University
nathan.ryan@bucknell.edu

Gonzalo Tornaría
Centro de Matemática, Universidad de la República
tornaria@cmat.edu.uy

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Abstract

In the 1980s Böcherer formulated a conjecture relating the central values of the imaginary quadratic twists of the spin L-function attached to a Siegel modular form $F$ to the Fourier coefficients of $F$. This conjecture has been proved when $F$ is a lift. More recently, we formulated in [RT11] an analogous conjecture for paramodular forms $F$ of prime level, even weight and in the plus-space.

In this paper, we examine generalizations of this conjecture. In particular, our new formulations relax the conditions on $F$ and allow for twists by real characters. Moreover, these formulation are more explicit than the earlier version. We prove the conjecture in the case of lifts and provide numerical evidence in the case of nonlifts.

1 Introduction

Many problems in number theory are related to central values of L-functions associated to modular forms, and central values of twisted L-functions are tools used to make progress on these problems. In this paper we focus our attention on paramodular forms of level $N$ and the spin L-function associated to them.

In the 1980s a conjecture was formulated by Böcherer [Böc86] that relates the coefficients of a Siegel modular form $F$ of degree 2 and the central value of the spin L-function associated to $F$. One fixes a discriminant $D$ and, roughly speaking, adds up all the coefficients of $F$ indexed by quadratic forms of discriminant $D$. One also computes the central value of the spin L-function twisted by the quadratic character $\chi_D$. The conjecture asserts that the central value, up to a constant that depends only on $F$ (and not on $D$) and a suitable normalization, is the square of the sum of coefficients. In Böcherer’s original paper
it was proved for $F$ that are Saito-Kurokawa lifts and later Böcherer and Schulze-Pillot [BSP92] proved the conjecture in the case when $F$ is a Yoshida lift. Kohnen and Kuss [KK02] gave numerical evidence in the case when $F$ is of level one, degree 2 and is not a Saito-Kurokawa lift (these computations have recently been extended by Raum [Rau10]). A much more general approach to the conjecture has been pursued by Furusawa, Martin and Shalika [Fur93, FM11, FS99, FS00, FS03].

In [RT11] we investigated an analogous conjecture in the setting of paramodular forms and our goal here is to state some generalizations of the conjecture and to point out some subtleties in the statement of the original conjecture. Paramodular forms have been gaining a great deal of attention because, for example, the most explicit analogue of Taniyama-Shimura for abelian surfaces has been formulated by Brumer and Kramer [BK10] and been verified computationally by Poor and Yuen [PY09].

We summarize the main results in [RT11]. Fix a paramodular eigenform $F$ of level $N$ with Fourier coefficient $a(T; F)$ for each positive semidefinite quadratic form $T$. One defines

$$A(D) = A_F(D) := \frac{1}{2} \sum_{\{T > 0 : \text{disc } T = D\}} \frac{a(T; F)}{\epsilon(T)}$$

where $\epsilon(T) := \#\{U \in \Gamma_0(N) : T[U] = T\}$. We provided evidence for a conjecture, which can be considered a generalization of Waldspurger’s theorem [Wal81]. We state a different version of that conjecture now:

**Conjecture A** (Paramodular Böcherer’s Conjecture). Let $p$ be prime and let $F \in S^k(\Gamma_{\text{para}}[p])^+$ be a paramodular Hecke eigenform of even weight $k$. Then, for fundamental discriminants $D < 0$ we have

$$A_F(D)^2 = \alpha_D C_F L(F, 1/2, \chi_D) |D|^{k-1}$$

where $C_F$ is a nonnegative constant that depends only on $F$, and where $\alpha_D = 1 + \left(\frac{D}{p}\right)$. Moreover, when $F$ is a Gritsenko lift we have $C_F > 0$, and when $F$ is not a lift, we have $C_F = 0$ if and only if $L(F, 1/2) = 0$.

This conjecture corrects a defect of the corresponding conjecture in [RT11], the defect being that it might be wrong in the case of nonlifts since the conjecture in [RT11] requires the constant be positive. It is a theorem in [RT11] that $C_F > 0$ when $F$ is a Gritsenko lift and we know that $C_F > 0$ in all the examples of nonlifts that we computed. A minor difference is that the formula here uses the factor $\alpha_D$ instead of the factor $\star$ in the previous version of the conjecture. Though $\alpha_D$ here can vanish (while $\star$ could not), this does not affect the conjecture since $\left(\frac{D}{p}\right) = -1$ implies $A(D)$ is an empty sum and $L(F, 1/2, \chi_D) = 0$ because in this case the functional equation has sign $-1$. However, this factor $\alpha_D$ will be essential in the case of composite levels.

**Remark 1.** We make the simple observation that if Conjecture A is true and if $F$ is a nonlift for which $L(F, 1/2) = 0$, then the average $A(D)$ of its Fourier coefficients is zero.
coefficients would be zero for all $D$. This is a step in characterizing the kinds of forms that might violate the conjecture as stated in \cite{RT11}.

**Remark 2.** Our Conjecture A has a noteworthy difference from the original version of the conjecture first stated by Böcherer in \cite{Boc86}. Namely, in his conjecture the constant $C_F$ is required to be positive while we only require that ours be nonnegative, though we do characterize exactly when the constant is zero. This was subsequently addressed in a limited way in \cite{BSP92}, where the conjecture for Siegel modular forms of level $N$ was considered.

**Remark 3.** In order to verify Conjecture A in a case where $F$ is a nonlift and $L(F, 1/2) = 0$, one would have to compute the Fourier coefficients of a paramodular form whose L-function vanishes to even order greater that zero. Here we note that if the Paramodular Conjecture holds, there should be such a paramodular of, for example, levels 3319, 3391, 3571, 4021, 4673, 5113, 5209, 5449, 5501, 5599 since there is an hyperelliptic curve for each of these conductors whose Hasse-Weil L-function vanishes to even order at least two \cite{Sto08}. This last assertion about the order of vanishing was verified by directly computing the central value of these L-functions using lcalc \cite{Rub08}.

### 1.1 Two surprises

After carrying out the computations used to verify the conjecture in \cite{RT11}, we made two observations that we describe now. We asked about what happens if you do not restrict the computations to forms in the plus space. To do this, we first noticed that the two sides of (1) do indeed make sense. We also carried out a simple computation \cite[Section 4]{RT11} that shows the averages $A(D)$ add to zero when a form is in the minus space. Undaunted, we carried out the computations and tabulated the following data for $F_{-587}$ (see Table I for a list of all the examples considered in this paper).

| $D$ | -4 | -7 | -31 | -40 | -43 | -47 |
|-----|----|----|-----|-----|-----|-----|
| $L_D/L_{-3}$ | 1.0 | 1.0 | 4.0 | 9.0 | 144.0 | 1.0 |

Here $L_D := L(F_{587}, 1/2, \chi_D)\cdot |D|$ and the table shows fundamental discriminants for which $(D_{587}) = -1$. The obvious thing to notice is that the numbers in the table appear to be squares and so the natural question to ask is: squares of what?

This first experiment was a natural extension of our previous work in \cite{RT11} as we had a paramodular form to compute the right-hand side of (1) and both sides of the equation make sense. Emboldened by the results of the first experiment, we decided to change another hypothesis in the conjecture: we decided to look at the case when $D > 0$. This is somewhat unnatural as the sum $A(D)$ is an empty sum in this case. Nevertheless we get the following data for $F_{277}$.

| $D$ | 12 | 13 | 21 | 28 | 29 | 40 |
|-----|----|----|----|----|----|----|
| $L_D/L_1$ | 225.0 | 225.0 | 225.0 | 225.0 | 2025.0 | 900.0 |
Here \( L_D := L(F_{277}, 1/2, \chi_D) \cdot |D| \) and \( (\frac{D}{277}) = +1 \). Again, these seem to be squares, but squares of what? (Also, the observant reader may have noticed that all these squares are divisible by \( 15^2 \). See Section 5 for more about this.)

In Section 2 we will show how to define, for an auxiliary discriminant \( \ell \), a twisted average \( B_{\ell}(D) \). When \( \ell \) is properly chosen, the squares of \( B_{\ell}(D) \) are exactly the squares we see in the previous two tables.

Given a discriminant \( \Delta \), we put

\[
\alpha_\Delta := \prod_{p|\Delta} \left( 1 + \left( \frac{\Delta_0}{p} \right) \right) ,
\]

where \( \Delta_0 \) is the fundamental discriminant associated to \( \Delta \).

**Conjecture B.** Let \( N \) be squarefree. Suppose \( F \in S^k(\Gamma_{\text{para}}[N]) \) is a Hecke eigenform and not a Gritsenko lift. Let \( \ell \) and \( D \) be fundamental discriminants such that \( \ell D < 0 \). Then

\[
B_{\ell,F}(D)^2 = \alpha_{\ell D} C_{\ell,F} L(F, 1/2, \chi_D) |D|^{k-1} ,
\]

where \( C_{\ell,F} \) is a constant independent of \( D \). Moreover, \( C_{\ell,F} = 0 \) if and only if \( L(F, 1/2, \ell) = 0 \).

The notation in this conjecture is further explained in Section 2 but the analogy with Conjecture A will be made clear now. First, note that \( B_{\ell}(D) \) is a twisted average of the Fourier coefficients of \( F \) indexed by quadratic forms of discriminant \( D \). Essentially, if \( k \) is even, \( N \) is prime, and \( \ell = 1 \) then \( B_{\ell}(D) = |A(D)| \) so we recover Conjecture A.

In a later section, we are interested in verifying Conjecture B in the case of nonlifts. To do this, we attempt to understand the constant \( C_{\ell,F} \) a little better. In Conjecture B, one can think of the discriminant \( \ell \) as being fixed. In this next conjecture, we think of it as a parameter.

**Conjecture C.** Let \( N \) be squarefree. Suppose \( F \in S^k(\Gamma_{\text{para}}[N]) \) is a Hecke eigenform and not a Gritsenko lift. Let \( \ell \) and \( D \) be fundamental discriminants such that \( \ell D < 0 \). Then

\[
B_{\ell,F}(D)^2 = \alpha_{\ell D} k_F L(F, 1/2, \chi_{\ell}) L(F, 1/2, \chi_D) |D\ell|^{k-1} ,
\]

for some positive constant \( k_F \) independent of \( \ell \) and \( D \).

This gives us a very explicit statement of a conjecture for forms that are not Gritsenko lifts. It is this formula that we verify in Section 3. We observe that Conjecture C implies Conjecture B by letting \( C_{\ell,F} = k_F L(F, 1/2, \chi_{\ell}) |\ell|^{k-1} \).

We note that when \( F \) is a Gritsenko lift the formula of Conjecture B is valid in the case \( \ell = 1 \) with \( C_{\ell,F} > 0 \), as shown in Theorem 4.1 below; the formula of Conjecture C is valid provided \( \ell \neq 1 \) and \( D \neq 1 \), but uninteresting with both sides being zero for trivial reasons (see Proposition 4.2 and Proposition 4.3).
1.2 Notation

The main objects of study in this paper are paramodular forms of level $N$ and their L-functions.

Suppose $R$ is a commutative ring with identity. The symplectic group is $Sp(4, R) := \{ x \in GL(4, R) : x^t J_2 x = J_2 \}$, where the transpose of matrix $x$ is denoted $x^t$ and for the $n \times n$ identity matrix $I_n$ we set $J_n = (\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix})$. When $R \subset \mathbb{R}$, the group of symplectic similitudes is $GSp^+(4, R) := \{ x \in GL(4, R) : \exists \mu \in \mathbb{R}_{>0} : x^t J_2 x = \mu J_2 \}$.

The paramodular group of level $N$ is

$$\Gamma_{\text{para}}[N] := Sp(4, \mathbb{Q}) \cap \left\{ \begin{pmatrix} * & * & \ast/N & * \\ N* & * & * & * \\ N* & N* & * & N* \\ N* & * & * & * \end{pmatrix} \right\}, \text{ where } \ast \in \mathbb{Z}. $$

Paramodular forms of degree 2, level $N$ and weight $k$ are modular forms with respect to the group $\Gamma_{\text{para}}[N]$. We denote the space of such modular forms by $M^k(\Gamma_{\text{para}}[N])$ and the space of cuspforms by $S^k(\Gamma_{\text{para}}[N])$. The space $S^k(\Gamma_{\text{para}}[N])$ can be split into a plus space and a minus space according to the action of the Atkin-Lehner operator $\mu_N$: in particular, $S^k(\Gamma_{\text{para}}[N])^\pm = \{ f \in S^k(\Gamma_{\text{para}}[N]) : f | \mu_N = \pm f \}$.

Every $F \in M^k(\Gamma_{\text{para}}[N])$ has a Fourier expansion of the form

$$F(Z) = \sum_{T = [N a, b, c] \in \mathcal{Q}_N} a(T; F) q^N \zeta^b q^c$$

where $q := e^{2\pi i z}$, $q' := e^{2\pi i z'}$ ($z, z' \in \mathcal{H}_1$), $\zeta := e^{2\pi i \tau}$ ($\tau \in \mathbb{C}$) and

$$\mathcal{Q}_N := \{ [N a, b, c] \geq 0 : a, b, c \in \mathbb{Z} \};$$

here we use Gauss’s notation for binary quadratic forms.

We will want to decompose $\mathcal{Q}_N$ by discriminant $D < 0$ so we also define

$$\mathcal{Q}_{N, D} = \{ T \in \mathcal{Q}_N : \text{disc } T = D \}.$$

This is useful, for example, so that we can write

$$A_F(D) := \frac{1}{2} \sum_{T \in \mathcal{Q}_{N, D}/\Gamma_0(N)} a(T; F) \frac{\varepsilon(T)}{\varepsilon(T)}.$$

For $F \in S^k(\Gamma_{\text{para}}[N])$, we have $a(T[U]; F) = a(T; F)$ for every $U \in \Gamma_0(N)$, where $\Gamma_0(N)$ is the congruence subgroup of $SL(2,\mathbb{Z})$ with lower lefthand entry congruent to $0$ mod $N$, and $a(T[1 0 0 1]; F) = (-1)^k a(T; F)$. Moreover, cusp forms are supported on the positive definite matrices in $\mathcal{Q}_N$.

Suppose we are given a paramodular form $F \in S^k(\Gamma_{\text{para}}[N])$ so that for all Hecke operators $T(n)$, $F[T(n) F] = \lambda_n F$. Then we can define the spin L-series by the Euler product

$$L(F, s) := \prod_{q \text{ prime}} L_q(q^{-s-k+3/2})^{-1},$$
where the local Euler factors are given by
\[
L_q(X) := 1 - \lambda_q X + (\lambda_q^2 - \lambda_q^2 - q^{2k-4})X^2 - \lambda_q q^{2k-3}X^3 + q^{4k-6}X^4
\]
for \( q \nmid N \), and has a similar formula but of lower degree for \( q \mid N \).

The Paramodular Conjecture [BK10] asserts that the L-function of a paramodular form is the same as the L-function of an associated abelian surface. In all the examples we consider, the paramodular forms have corresponding abelian surfaces isogenous to Jacobians of hyperelliptic curves that can be found in tables of Stoll [Sto08]; thus we compute hyperelliptic curve L-functions when we carry out our computations. A table in [Dok04] summarizes the data that we use to write down the functional equation of the L-function of an hyperelliptic curve:
\[
L^*(F, s) = \left(\frac{\sqrt{N}}{4\pi^2}\right)^s \Gamma(s + 1/2)\Gamma(s + 1/2)L(F, s).
\]
so that conjecturally
\[
L^*(F, s) = \epsilon L^*(F, 1 - s),
\]
when \( F \in S^2(\Gamma_{\text{para}}[N])^\epsilon \).

Let \( D \) be a fundamental discriminant, and denote by \( \chi_D \) the unique quadratic character of conductor \( D \). For the spin L-series \( L(F, s) = \sum_{n \geq 1} a(n) n^{-s} \) of a paramodular form \( F \), we define the quadratic twist
\[
L(F, s, \chi_D) := \sum_{n \geq 1} \chi_D(n) a(n) n^{-s},
\]
which is conjectured to have an analytic continuation and satisfy a functional equation. Suppose \( N \) is squarefree, let \( N' = ND^4/\gcd(N, D) \), and define
\[
L^*(F, s, \chi_D) := \left(\frac{\sqrt{N'}}{4\pi^2}\right)^s \Gamma(s + 1/2)\Gamma(s + 1/2)L(F, s, \chi_D)
\]
so that assuming standard conjectures,
\[
L^*(F, s, \chi_D) = \epsilon' L^*(F, s, \chi_D).
\]
The global root number \( \epsilon' \) of the functional equation for \( L(F, s, \chi_D) \) is given in terms of the local root numbers \( \epsilon_p \) of \( L(F, s) \) by the following lemma [Sch].

**Lemma 1.1.** Let \( F \in S^k(\Gamma_{\text{para}}[N]) \) be a Hecke eigenform, with \( N \) squarefree. Denote the local root numbers of, respectively, \( L(F, s, \chi_D) \) and \( L(F, s) \), as follows: \( \epsilon' = \prod_{p \leq \infty} \epsilon'_p \) and \( \epsilon = \prod_{p \leq \infty} \epsilon_p \). Then

1. At the infinite place, \( \epsilon'_\infty = \epsilon_\infty = (-1)^k \).
2. Assume \( p \nmid N \), then \( \epsilon'_p = \epsilon_p = +1 \).
3. Assume \( p \mid N \) and \( p \mid D \), then \( \epsilon'_p = +1 \).
4. Assume \( p \mid N \) and \( p \nmid D \), then \( e'_p = \chi_D(p)e_p \).

In particular, if \( N_0 = N/\gcd(N, D) \),
\[
e' = e \cdot \chi_D(N_0) \cdot \prod_{p \mid \gcd(D, N)} e_p
\]

## 2 Generalizations of the Paramodular Böcherer’s Conjecture

In this section, we motivate Conjectures B and C. We do it by describing what happens for particular \( F \) that are not Gritsenko lifts. We place particular emphasis on the transition from the hypotheses in Conjecture A (namely, \( F \) in the plus-space, of prime level and even weight) to Conjectures B and C which have no such hypotheses.

### 2.1 A simple case

For \( F = F_{249} \), the unique Hecke eigenform of weight 2 and level \( 249 = 3 \cdot 83 \), we have \( A(D) = 0 \) for all \( D \) (see Lemma 2.1, below) since its eigenvalues under the Atkin-Lehner operators \( \mu_3 \) and \( \mu_{83} \) are \( e_3 = e_{83} = -1 \). However, we have

\[
L_D/L_{-4} = \begin{pmatrix} 1.0 & 1.0 & 4.0 & 1.0 & 16.0 & 4.0 & 1.0 & 4.0 & 0.0 \\
-7 & -8 & -20 & -31 & -35 & -40 & -47 & -56 & -71
\end{pmatrix}
\]

where \( L_D := L(F_{249}, 1/2, \chi_D) \cdot |D| \) and \( (D_{249}) = +1 \).

In this and the next section we will show where the squares in this table come from. Before we do that, we show that for \( F_{249} \) (as well as \( F_{587} \) and \( F_{713} \)), we really do get \( A(D) = 0 \) for all \( D \).

**Lemma 2.1.** Let \( F \) be a paramodular form of weight \( k \) and level \( N \). Assume \( F \) is an eigenform under the Atkin-Lehner operators \( \mu_p \) for every \( p \mid N \), so that \( F \mid \mu_p = e_p F \). If \( e_p = -1 \) for any \( p \mid N \) or if \( k \) is odd, then \( A_F(D) = 0 \) for all \( D \).

**Proof.** For \( N_0 \mid N \), one can define an involution \( W_{N_0} \) over the set \( Q_N/\Gamma_0(N) \) (see [GKZ87, p. 507]). This involution is related to the Atkin-Lehner operators in the following way:

\[
a(W_p(T); F) = a(T; F | \mu_p) = e_p a(T; F)
\]

where \( p^i \) is the largest power of \( p \) dividing \( N \). Taking the sum over all classes \( T \in Q_{N,D}/\Gamma_0(N) \) shows that \( A(D) = e_p A(D) \), and it follows that \( A(D) = 0 \) if \( e_p = -1 \). The case of odd \( k \) is similar using \( a(T^{[1 \ 0 \ -1]}; F) = (-1)^k a(T; F) \).
We will show now how to define a more refined average \( B(D) \) on the coefficients of \( F \) for which Lemma 2.1 does not apply. In order to do that, we further decompose \( Q_{N,D} \) as follows. Note that for any \( T = [Na, b, c] \in Q_{N,D} \) we have \( b^2 \equiv D \pmod{4N} \) and we can thus define
\[
R_D := \{ \rho \mod 2N : \rho^2 \equiv D \pmod{4N} \}.
\]
For each \( \rho \in R_D \) we set
\[
Q_{N,D,\rho} := \{ T = [Na, b, c] \in Q_{N,D} : b \equiv \rho \pmod{2N} \}.
\]
We observe that \( Q_{N,D} \) is the disjoint union of \( Q_{N,D,\rho} \) for \( \rho \in R_D \). Now for each \( \rho \in R_D \) we put
\[
B(D, \rho) = B_F(D, \rho) := \sum_{T \in Q_{N,D,\rho}/\Gamma_0(N)} a(T; F) \varepsilon(T).
\]

**Lemma 2.2.** We note the following:
1. \( A_F(D) = \frac{1}{2} \sum_{\rho \in R_D} B_F(D, \rho) \),
2. \( B_F(D, -\rho) = (-1)^k B_F(D, \rho) \), and
3. \( |B_F(D, \rho)| \) is independent of \( \rho \).

**Proof.** The first statement is obvious, and the second follows from the fact that \( a(T\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; F) = (-1)^k a(T; F) \). The last statement follows by noting that the Atkin-Lehner involutions \( W_N \), mentioned in the proof of Lemma 2.1 transitively permute the sets \( Q_{N,D,\rho} \).

Now we can finally define the new average:
\[
B(D) = B_F(D) := \frac{1}{2} \sum_{\rho \in R_D} |B(D, \rho)|.
\]
We observe by Lemma 2.2 that \( B(D) = \frac{1}{7} |R_D| |B(D, \rho)| \). We also note that when \( k \) is even and \( N \) is prime, we have \( B(D) = |A(D)| \) since \( R_D = \{ \pm \rho \} \) and \( B(D, -\rho) = B(D, \rho) \) in this case.

We return to the example \( F_{249} \). We get the following table:

| \( D \) | \(-7\) | \(-8\) | \(-20\) | \(-31\) | \(-35\) | \(-40\) | \(-47\) | \(-56\) | \(-71\) |
|---|---|---|---|---|---|---|---|---|---|
| \( L_D/L_{-4} \) | 1.0 | 1.0 | 4.0 | 1.0 | 16.0 | 4.0 | 1.0 | 4.0 | 0.0 |
| \( B(D)/B(-8) \) | 1 | 2 | 4 | 1 | 2 | 0 |

where again \( L_D := L(F_{249}, 1/2, \chi_D) \cdot |D| \) and \( (\frac{D}{249}) = +1 \). When \( (\frac{D}{249}) = (\frac{D}{249}) = -1 \) the definition of \( B(D) \) gives an empty sum; this is indicated in the table above with an empty space. In the next section we describe where the remaining squares come from.

**Remark 4.** Note that the value of \( B(-71) = 0 \) is a non-trivial zero average, predicting the vanishing of the twisted \( L \)-function at the center. We will investigate such phenomena in a future paper.
2.2 A general case

In the previous section we looked at a form $F_{249}$ of composite level for which the averages $A(D)$ vanish trivially. We introduced a refined average $B(D)$ that explained some of the data in the tables, but in the case $\left( \frac{D}{83} \right) = -1$ the sum $B(D)$ is empty, although the central values $L_D/L_{-4}$ are (nonzero) squares.

Consider the form $F_{587}$ as described in the Introduction. It was shown in [RT11, Section 4] that for the discriminants $D$ so that $\left( \frac{D}{587} \right) = -1$ the sum $A(D)$ was empty and so, in particular, our new sum $B(D)$ is also empty, and cannot explain the fact that its normalized twisted central values are (nonzero) squares.

Also, in the definition of $A(D)$ and of $B(D)$ we require that $D$ be negative, so neither average can make sense of the data in the Introduction related to real quadratic twists of the $L$-functions of $F_{277}$. In this section, using the genus theory for $\Gamma_0(N)$-classes of quadratic forms, we fully explain these examples by defining another new average $B_\ell(D)$ weighted by a genus character. We define this now.

Fix a fundamental discriminant $\ell$. Then we define a genus character $\chi_\ell$ similar to the generalized genus character defined in [GKZ87]. Let $T = [Na, b, c] \in \mathbb{Q}_{N,\ell D}$ so that $\gcd(a, b, c, \ell) = 1$ and let $g = \gcd(N, b, c, \ell)$. Define $T = [Na/g, b, cg]$ and note that it represents an integer $n$ relatively prime to $\ell$. Now

$$\chi_\ell(T) := \left( \frac{\ell}{n} \right) \prod_{p \mid g} s_p$$

where

$$s_p = \begin{cases} \left( \frac{-\ell/p}{p} \right) & p \text{ odd} \\ \left( \frac{2}{p} \right) & p = 2, t \text{ the odd part of } \ell. \end{cases}$$

We note that $\chi_\ell$ has the following properties. First, it is completely multiplicative: if $\ell = \ell_1\ell_2$, then $\chi_{\ell_1}\chi_{\ell_2}$. Second, it behaves predictably with respect to $W_p$. Namely, if $p \nmid \ell$, then $\chi_{\ell}(W_p T) = \left( \frac{\ell}{p} \right)\chi_{\ell}(T)$ and otherwise $\chi_{p^*}(W_p T) = \chi_{p^*}(T)$ where $p^* = \left( \frac{-1}{p} \right)p$ for odd $p$, and $p^* = -4, 8$ or $-8$ for $p = 2$.

Then we define, for $D$ a fundamental discriminant such that $\ell D < 0$,

$$B_\ell(D, \rho) = B_{\ell,F}(D, \rho) := \sum_{T \in \mathbb{Q}_{N,\ell D,\rho} / \Gamma_0(N)} \chi_\ell(T) \frac{\alpha(T; F)}{\varepsilon(T)}$$

and

$$B_\ell(D) = B_{\ell,F}(D) := \frac{1}{2} \sum_{\rho \in \mathbb{R}_{\ell D}} B_{\ell}(D, \rho).$$

We note that $B_1(D) = B(D)$ as defined in the previous section. One can also prove, using quadratic reciprocity, that $B_\ell(D) = B_D(\ell)$. 

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Remark 5. If \( \left( \frac{D}{p} \right) = -1 \) for some \( p \mid N \), then \( B_\ell(D) = 0 \), because \( \mathcal{Q}_{N,\ell D} \) is empty in this case. Nevertheless, for any fundamental discriminant \( D \), there exists some \( \ell \) for which \( B_\ell(D) \) is not an empty sum.

We will now complete the explanation of the examples we have discussed so far. We start with the form \( F_{249} \). In the previous section we were able to explain the case \( (D_{3249}, D_{83}) = +1 \) with auxiliary discriminant \( \ell = 1 \) (implicitly). We can explain the other case, \( (D_{3249}, D_{83}) = -1 \), by choosing \( \ell = 5 \).

\[
\begin{array}{c|ccccccc}
D & -7 & -8 & -20 & -31 & -35 & -40 & -47 \\
L_D/L_{-4} & 1.0 & 1.0 & 4.0 & 1.0 & 16.0 & 4.0 & 1.0 \\
B_1(D)/B_1(-8) & 1 & 2 & 4 & 1 & 2 & 0 \\
B_5(D)/B_5(-4) & 1 & 1 & 2 & 3 & 12 & 1 \\
\end{array}
\]

where \( L_D := L(F_{249}, 1/2, \chi_D) \cdot |D| \) and \( \left( \frac{D}{249} \right) = +1 \). The empty entries correspond to empty sums as noted in the remark above.

In order to explain the case of the form \( F_{587} \), in the minus space, we need to use an auxiliary discriminant \( \ell > 0 \) such that \( (\ell_{587}) = -1 \). Using \( \ell = 5 \):

\[
\begin{array}{c|ccccccc}
D & -4 & -7 & -31 & -40 & -43 & -47 \\
L_D/L_{-5} & 1.0 & 1.0 & 4.0 & 9.0 & 144.0 & 1.0 \\
B_5(D)/B_5(-3) & 1 & 1 & 2 & 3 & 12 & 1 \\
\end{array}
\]

where \( L_D := L(F_{587}, 1/2, \chi_D) \cdot |D| \) and the table shows fundamental discriminants for which \( (\frac{D}{587}) = -1 \).

Finally, in order to handle positive discriminants \( D \), we can choose a negative discriminant \( \ell \). In the example of \( F_{277} \) we choose \( \ell = -3 \):

\[
\begin{array}{c|ccccccc}
D & 12 & 13 & 21 & 28 & 29 & 40 \\
L_D/L_{-1} & 225.0 & 225.0 & 225.0 & 225.0 & 225.0 & 900.0 \\
B_{-3}(D)/B_{-3}(1) & 15 & 15 & 15 & 15 & 45 & 30 \\
\end{array}
\]

where \( L_D := L(F_{277}, 1/2, \chi_D) \cdot |D| \) and \( (\frac{D}{277}) = +1 \).

3 The Case of Nonlifts

In the previous section, we highlighted some tables that give evidence for our Conjectures B and C. Now we describe how those tables were computed and how the tables in Section 6 were computed.

The Paramodular Conjecture asserts that for each rational Hecke eigenform \( F \) that is not a Gritsenko lift, there is an abelian surface \( A \) so that the Hasse-Weil L-function of \( A \) and the spin L-function of \( F \) are the same. Suppose we have such an \( F \) and such an \( A \). In all our examples, \( A \) is isogenous to the Jacobian of a hyperelliptic curve \( C \). We list a sampling of such hyperelliptic curves in Table 1, and more examples can be found in [RT12].

Consider such a \( C \). Then the Euler product of \( C \) can be found as in [RT14, Section 3] and the functional equation can be found, for example, in [Dok04].
| $F$   | $N$ | $C$                                                                 | $T$ |
|-------|-----|----------------------------------------------------------------------|-----|
| $F_{249}$ | 249 | $y^2 + (x^3 + 1)y = x^2 + x$                                        | 14  |
| $F_{277}$ | 277 | $y^2 + y = x^3 - 2x^3 + 2x^2 - x$                                    | 15  |
| $F_{295}$ | 295 | $y^2 + (x^3 + 1)y = -x^4 - x^3$                                      | 14  |
| $F_{587}$ | 587 | $y^2 + (x^3 + x + 1)y = -x^3 - x^2$                                  | 1   |
| $F_{713}$ | 713 | $y^2 + (x^3 + x + 1)y = -x^4$                                        | 9   |
| $F_{713}$ | 713 | $y^2 + (x^3 + x + 1)y = x^5 - x^3$                                  | 1   |

Table 1: Hyperelliptic curves $C$ used to compute the L-series of the paramodular form $F$ associated to $C$ via the Paramodular Conjecture. Here $T$ denotes the torsion of the abelian surface $\text{Jac}(C)$.

though we give it the analytic normalization. The central values were then computed using Michael Rubinstein’s \texttt{lcalc} \cite{Rub08}.

Now we describe how the averages are computed. The Fourier coefficients of the 6 paramodular forms whose L-functions correspond to the L-functions of the curves listed in Table 1 were computed by Cris Poor and David Yuen. The paramodular forms of level 277 and 587 are publicly available and computed via the methods of [PY09]. The other four paramodular forms were computed by Poor and Yuen for us, using an as of yet unpublished method [PY].

The sum $B_\ell(D)$ is computed using these Fourier coefficients using a combination of Sage \cite{Ste11} code and custom-written Python code. In particular, we implemented a class that represent binary quadratic forms modulo $\Gamma_0(N)$ one of whose methods computes the generalized genus character $\chi_\ell$. In Table 2 we summarize the forms and discriminants for which we have computed both twisted averages and twisted central values.

The following theorem summarizes the cases in which Conjecture C has been verified.

**Theorem 3.1.** Let $F$ be one of the paramodular forms listed in Table 2. Let $\ell$ and $D$ be fundamental discriminants such that $\ell D < 0$ satisfying the constraints described in the same table. Then

$$B_\ell,F(D)^2 \approx \alpha_{\ell D} k_F L(F,1/2,\chi_\ell) L(F,1/2,\chi_D) |D\ell|^{k-1}$$

numerically, with $k_F$ a positive constant listed in the table.

In addition to the cases listed in Table 2 we point out that more cases and more tables can be found at \cite{RT12}, providing evidence for Conjecture C using forms and curves that are not in this table.

### 4 The Case of Lifts

A Gritsenko lift $F$ \cite{Gri95} is a paramodular form that comes from a Jacobi form $\phi$ which in turn corresponds to an elliptic modular form $f$. The standard refer-
Table 2: Summary of forms and discriminants for which we have computed both twisted averages $B_\ell(D)$ and the corresponding twisted central values, and for which Conjecture C has been numerically verified. The discriminants for which we computed satisfy $0 > \Delta \geq \Delta_{\text{min}}$, where $\Delta = \ell D$, with the following exceptions:

- If a discriminant $\Delta$ is in the last column, it means that we did not have all the Fourier coefficients necessary to compute the averages.
- In the case of $F_{277}$, we have the further restriction $|\ell| \leq 500$ and $|D| \leq 500$ due to loss of precision in computing $L_\ell$ and $L_D$.

We will now state and prove a theorem that gives evidence for Conjecture B in the case of lifts.

**Theorem 4.1.** Let $N$ be squarefree. Suppose $F \in S^k(\Gamma_{\text{para}}[N])$ is a Hecke eigenform and a Gritsenko lift. Let $D < 0$ be a fundamental discriminant. Then

$$B_F(D)^2 = \alpha_D C_F L(F, 1/2, \chi_D) |D|^{k-1}$$

where $C_F$ is a positive constant independent of $D$.

Let $F = \text{Grit}(\phi)$ where

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{r^2 \leq 4nN} c(n, r) q^n \zeta^r$$

is a Jacobi form of weight $k$ and index $N$. We note [EZ85, Theorem 2.2, p. 23] that $c(n, r)$ depends only on $D = r^2 - 4nN$ and $r \mod 2N$; for each $\rho \in R_D$ we let

$$c_\rho(D) := c \left( \frac{\rho^2 - D}{4N}, \rho \right)$$

and

$$c^*(D) := \frac{1}{2} \sum_{\rho \in R_D} |c_\rho(D)|.$$

We remark that $|c_\rho(D)|$ is independent of $\rho$ and that $c^*(D)$ is, up to sign, the coefficient of a weight $k - 1/2$ modular form [EZ85, Theorem 5.6, p. 69].

**Proposition 4.2.** If $D < 0$ is a fundamental discriminant, then

$$B_F(D) = c^*(D) \frac{h(D)}{w_D}.$$
If \( \ell \neq 1 \) and \( D \neq 1 \) are fundamental discriminants, we have \( B_{\ell,F}(D) = 0 \).

**Proof.** By the definition of the Gritsenko lift, we know that \( a(T; F) = c_b(\text{disc } T) \) for \( T = [Na,b,c] \in \mathbb{Q}_N \), provided \( T \) is primitive, which is always the case for \( \text{disc } T \) fundamental. Thus

\[
|B_{\ell}(D, \rho)| = \left| \sum_{T \in \mathbb{Q}_N, \ell D, \rho, N(\Gamma_0(N)} \chi_\ell(T) \frac{a(T; F)}{\varepsilon(T)} \right| = |c_\rho(\ell D)| \sum_{T \in \mathbb{Q}_N, \ell D, \rho, N(\Gamma_0(N)} |\chi_\ell(T) - 1| \frac{\varepsilon(T)}{\varepsilon(T)}.
\]

When \( \ell = 1 \) the sum in the last term is \( \sum \frac{1}{\varepsilon(T)} = \frac{h(D)}{w_D} \), since \( |\mathbb{Q}_{N,D,\rho,\Gamma_0(\mathbb{N})}| = h(D) \) for fundamental \( D \) and \( \varepsilon(T) = w_D \). On the other hand if \( \ell \neq 1 \) and \( D \neq 1 \) then \( \chi_\ell \) is a nontrivial character in \( \mathbb{Q}_{N,D,\rho,\Gamma_0(\mathbb{N})} \), hence the sum vanishes. \( \square \)

Let \( f \) be the elliptic modular form corresponding to the Jacobi form \( \phi \) as in [SZ88 Theorem 5]. It is a standard fact that \( L(F,s) = \zeta(s + 1/2) \zeta(s - 1/2) L(f, s) \) (using the analytic normalization, so that the center is at \( s = 1/2 \)). Twisting by \( \chi_D \) we obtain

\[
L(F,s,\chi_D) = L(s + 1/2, \chi_D) L(s - 1/2, \chi_D) L(f, s, \chi_D)
\]

valid on the region of convergence. It follows from this that \( L(F,s,\chi_D) \) has an analytic continuation (with a pole at \( s = 3/2 \) for \( D = 1 \)) and, using Dirichlet’s class number formula for the special values \( L(0,\chi_D) \) and \( L(1,\chi_D) \), we have

**Proposition 4.3.**

\[
L(F,1/2,\chi_D) = \begin{cases} 
\frac{4\pi^2}{w_D} \frac{h(D)^2}{\sqrt{|D|}} L(f,1/2,\chi_D) & \text{if } D < 0, \\
0 & \text{if } D > 1, \\
-\frac{1}{2} L'(f,1/2) & \text{if } D = 1.
\end{cases}
\]

**Proof of Theorem 4.1.** By Waldspurger’s formula [Wal81, Koh85], we have

\[
c^*(D)^2 = \alpha_D k_f L(f,1/2,\chi_D) |D|^{k-3/2}
\]

with \( k_f > 0 \). The theorem thus follows from Proposition 4.2 and Proposition 4.3 with \( k_F = k_f/4\pi^2 \). \( \square \)

5 Torsion

In the Introduction, we observed that for \( L(F_{277},1/2,\chi_D) \cdot |D|/L(F_{277},1/2) \) is divisible by \( 15^2 \) when \( D > 1 \):

**Proposition 5.1.** Let \( D > 1 \) and assume Conjecture B. Then, the ratio of special values \( L(F_{277},1/2,\chi_D) \cdot |D|/L(F_{277},1/2) \) is divisible by \( 15^2 \).
Proof. We recall [PY09, Theorem 7.3] which asserts: suppose \( \phi \) is the first Fourier-Jacobi coefficient of \( F_{277} \) and let \( G = \text{Grit}(\phi) \). Then, for all \( T \in \mathbb{Q}_N \),

\[
a(T; F) \equiv a(T; G) \pmod{15}.
\]

In Proposition 4.2 we observed that \( B_{1, G}(D) = 0 \) when \( \ell, D \neq 1 \); hence, it follows that \( B_{1, F}(D) \equiv 0 \pmod{15} \). Finally, Conjecture B implies that

\[
L(F, 1/2, \chi_D) \cdot |D|/L(F, 1/2) = * \frac{B_{-3}(D)}{B_{-3}(1)}^2
\]

where \( B_{-3}(1) = 1 \) (see Table 4) and where \(* = 1 \) if \( p \mid D \) and \(* = 2 \) if \( p \nmid D \).

We recall (see Table 1) that the Jacobian of the abelian surface associated to \( F_{277} \) by the Paramodular Conjecture has torsion of size 15. In [PY09], it is suggested that this phenomenon holds in generality. Observe in Tables 3–8 each entry (both the integers \( B_{\ell}(D) \) and the normalized central values) is divisible by the corresponding curve’s torsion unless \( \ell = 1 \) or unless \( D = 1 \). This provides further (indirect) evidence for Poor and Yuen’s observation holding in general.

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6 Tables
Table 3: Data for the modular form $F_{249}$ based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + (x^3 + 1) y = x^2 + x$, whose Jacobian has 14-torsion. The constant $C_\ell := k_{249} L_\ell$ with $k_{249} = 0.831968$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT12]. The values $L_\ell$ and $L_D$ are $L(F_{249}, 1/2, \chi_\ell) \cdot |\ell|$ and $L(F_{249}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$   | -8  | -20 | -35 | -47 | -56 | -71 |
|-------|-----|-----|-----|-----|-----|-----|
| $\alpha_D C_1 L_D$ | 4.0 | 16.0 | 64.0 | 4.0 | 16.0 | 0.0 |
| $B_1(D)$ | 2   | 4   | 8   | 2   | 4   | 0   |
| $D$   | -3  | -4  | -7  | -31 | -40 | -51 |
| $\alpha_{5D} C_5 L_D$ | 196.0 | 784.0 | 784.0 | 784.0 | 3136.0 | 19600.0 |
| $B_5(D)$ | 14 | 28 | 28 | 28 | 56 | 140 |
| $D$   | -3  | -4  | -7  | -31 | -40 | -51 |
| $\alpha_{8D} C_8 L_D$ | 196.0 | 784.0 | 784.0 | 784.0 | 3136.0 | 19600.0 |
| $B_8(D)$ | 14 | 28 | 28 | 28 | 56 | 140 |
| $D$   | 5   | 8   | 24  | 53  | 56  | 60  |
| $\alpha_{-3D} C_{-3} L_D$ | 196.0 | 196.0 | 784.0 | 3136.0 | 3136.0 | 3136.0 |
| $B_{-3}(D)$ | 14 | 14 | 28 | 56 | 56 | 56 |
| $D$   | 5   | 8   | 24  | 53  | 56  | 57  |
| $\alpha_{-4D} C_{-4} L_D$ | 784.0 | 784.0 | 784.0 | 12544.0 | 12544.0 | 0.0 |
| $B_{-4}(D)$ | 28 | 28 | 28 | 112 | 112 | 0 |
| $D$   | 5   | 8   | 24  | 53  | 56  | 57  |
| $\alpha_{-7D} C_{-7} L_D$ | 784.0 | 784.0 | 784.0 | 12544.0 | 12544.0 | 0.0 |
| $B_{-7}(D)$ | 28 | 28 | 28 | – | – | – |
| $D$   | 1   | 28  | 37  | 40  | 61  | 109 |
| $\alpha_{-8D} C_{-8} L_D$ | 4.0 | 0.0 | 3136.0 | 3136.0 | 3136.0 | 28224.0 |
| $B_{-8}(D)$ | 2 | 0 | – | – | – | – |
Table 4: Data for the modular form $F_{277}$ based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + y = x^5 - 2x^3 + 2x^2 - x$, whose Jacobian has 15-torsion. The constant $C_\ell := k_{277} L_\ell$ with $k_{277} = 0.537716$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT12]. The values $L_\ell$ and $L_D$ are $L(F_{277}, 1/2, \chi_\ell) \cdot |\ell|$ and $L(F_{277}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$   | -3  | -4  | -7  | -19 | -23 | -39 |
|-------|-----|-----|-----|-----|-----|-----|
| $\alpha D C_1 L_D$ | 1.0 | 1.0 | 1.0 | 4.0 | -0.0 | 1.0 |
| $B_1(D)$ | 1   | 1   | 1   | 2   | 0   | 1   |
| $D$   | -3  | -4  | -7  | -19 | -23 | -39 |
| $\alpha_{12D} C_{12} L_D$ | 225.0 | 225.0 | 225.0 | 900.0 | -0.0 | 225.0 |
| $B_{12}(D)$ | 15  | 15  | 15  | 30  | 0   | 15  |
| $D$   | -3  | -4  | -7  | -19 | -23 | -39 |
| $\alpha_{13D} C_{13} L_D$ | 225.0 | 225.0 | 225.0 | 900.0 | -0.0 | 225.0 |
| $B_{13}(D)$ | 15  | 15  | 15  | 30  | 0   | 15  |
| $D$   | 1   | 12  | 13  | 21  | 28  | 29  |
| $\alpha_{-3D} C_{-3} L_D$ | 1.0 | 225.0 | 225.0 | 225.0 | 225.0 | 2025.0 |
| $B_{-3}(D)$ | 1   | 15  | 15  | 15  | 15  | 45  |
| $D$   | 1   | 12  | 13  | 21  | 28  | 29  |
| $\alpha_{-4D} C_{-4} L_D$ | 1.0 | 225.0 | 225.0 | 225.0 | 225.0 | 2025.0 |
| $B_{-4}(D)$ | 1   | 15  | 15  | 15  | 15  | 45  |
| $D$   | 1   | 12  | 13  | 21  | 28  | 29  |
| $\alpha_{-7D} C_{-7} L_D$ | 1.0 | 225.0 | 225.0 | 225.0 | 225.0 | 2025.0 |
| $B_{-7}(D)$ | 1   | 15  | 15  | 15  | 15  | 45  |
Table 5: Data for the modular form $F_{295}$ based on the Hasse-Weil L-series for the
hyperelliptic curve $y^2 + (x^3 + 1) y = -x^4 - x^3$, whose Jacobian has 14-torsion.
The constant $C_\ell := k_{295} L_\ell$ with $k_{295} = 0.224745$. The table displays the first
few twists by real and by imaginary characters. More comprehensive data for
this curve can be found at [RT12]. The values $L_\ell$ and $L_D$ are $L(F_{295}, 1/2, \chi_\ell) \cdot |\ell|
and $L(F_{295}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$               | $-11$ | $-24$ | $-31$ | $-39$ | $-40$ | $-55$ |
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_D C_1 L_D$| 4.0   | 4.0   | 4.0   | 4.0   | 16.0  | 4.0   |
| $B_1(D)$          | 2     | 2     | 2     | 2     | 4     | 2     |

| $D$               | $-11$ | $-24$ | $-31$ | $-39$ | $-55$ | $-56$ |
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_5 D C_5 L_D$ | 196.0 | 196.0 | 196.0 | 196.0 | 784.0 | 196.0 |
| $B_5(D)$          | 14    | 14    | 14    | 14    | 28    |       |

| $D$               | $-3$  | $-7$  | $-68$ | $-87$ | $-88$ | $-107$|
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_8 D C_8 L_D$ | 196.0 | 196.0 | 3136.0| 784.0 | 3136.0| 15876.0|
| $B_8(D)$          | 14    | 14    |       |       |       |       |

| $D$               | $8$   | $13$  | $33$  | $37$  | $73$  | $77$  |
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_{-3} D C_{-3} L_D$ | 196.0 | 196.0 | -0.0  | 1764.0| -0.0  | 784.0 |
| $B_{-3}(D)$       | 14    | 14    |       | 42    |       |       |

| $D$               | $8$   | $13$  | $33$  | $37$  | $73$  | $77$  |
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_{-7} D C_{-7} L_D$ | 196.0 | 196.0 | -0.0  | 1764.0| -0.0  | 784.0 |
| $B_{-7}(D)$       | 14    | 14    |       |       |       |       |

| $D$               | $1$   | $5$   | $21$  | $29$  | $41$  | $60$  |
|-------------------|-------|-------|-------|-------|-------|-------|
| $\alpha_{-11} D C_{-11} L_D$ | 4.0   | 196.0 | 784.0 | 3136.0| 784.0 | 3136.0|
| $B_{-11}(D)$      | 2     | 14    |       |       |       |       |
Table 6: Data for the modular form $F_{-587}$ based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + (x^3 + x + 1) y = -x^3 - x^2$, whose Jacobian has 1-torsion. The constant $C_ℓ := k_{-587} L_ℓ$ with $k_{-587} = 0.002681$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT12]. The values $L_ℓ$ and $L_D$ are $L(F_{-587}, 1/2, \chi_ℓ) \cdot |ℓ|$ and $L(F_{-587}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$ | -3 | -4 | -7 | -31 | -40 | -43 |
|-----|----|----|----|-----|-----|-----|
| $α_{5D} C_5 L_D$ | 4.0 | 4.0 | 4.0 | 16.0 | 36.0 | 576.0 |
| $B_5(D)$ | 2 | 2 | 2 | 4 | 6 | 24 |
| $D$ | -3 | -4 | -7 | -31 | -40 | -43 |
| $α_{8D} C_8 L_D$ | 4.0 | 4.0 | 4.0 | 16.0 | 36.0 | 576.0 |
| $B_8(D)$ | 2 | 2 | 2 | 4 | 6 | 24 |
| $D$ | -3 | -4 | -7 | -31 | -40 | -43 |
| $α_{13D} C_{13} L_D$ | 4.0 | 4.0 | 4.0 | 16.0 | 36.0 | 576.0 |
| $B_{13}(D)$ | 2 | 2 | 2 | 4 | 6 | 24 |
| $D$ | 5 | 8 | 13 | 24 | 33 | 37 |
| $α_{-3D} C_{-3} L_D$ | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 16.0 |
| $B_{-3}(D)$ | 2 | 2 | 2 | 2 | 2 | 4 |
| $D$ | 5 | 8 | 13 | 24 | 33 | 37 |
| $α_{-4D} C_{-4} L_D$ | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 16.0 |
| $B_{-4}(D)$ | 2 | 2 | 2 | 2 | 2 | 4 |
| $D$ | 5 | 8 | 13 | 24 | 33 | 37 |
| $α_{-7D} C_{-7} L_D$ | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 16.0 |
| $B_{-7}(D)$ | 2 | 2 | 2 | 2 | 2 | 4 |
Table 7: Data for the modular form $F^+_{713}$ based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + (x^3 + x + 1) y = -x^4$, whose Jacobian has 9-torsion. The constant $C_\ell := k^+_{713} L_\ell$ with $k^+_{713} = 0.422122$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT12]. The values $L_\ell$ and $L_D$ are $L(F^+_{713}, 1/2, \chi_\ell) \cdot |\ell|$ and $L(F^+_{713}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$ | $-11$ | $-15$ | $-23$ | $-43$ | $-68$ | $-79$ |
|-----|-------|-------|-------|-------|-------|-------|
| $C_D L_D$ | $16.0$ | $16.0$ | $-0.0$ | $144.0$ | $64.0$ | $64.0$ |
| $B_1(D)$ | $4$ | $4$ | $0$ | $12$ | $8$ | $8$ |

| $D$ | $-11$ | $-15$ | $-23$ | $-43$ | $-68$ | $-79$ |
|-----|-------|-------|-------|-------|-------|-------|
| $C_8 L_D$ | $1296.0$ | $1296.0$ | $-0.0$ | $11664.0$ | $5184.0$ | $5184.0$ |
| $B_8(D)$ | $36$ | $36$ | $0$ | $-$ | $-$ | $-$ |

| $D$ | $-4$ | $-8$ | $-35$ | $-39$ | $-47$ | $-59$ |
|-----|-------|-------|-------|-------|-------|-------|
| $C_{17} L_D$ | $0.0$ | $0.0$ | $0.0$ | $-0.0$ | $0.0$ | $-0.0$ |
| $B_{17}(D)$ | $0$ | $0$ | $-$ | $-$ | $-$ | $-$ |

| $D$ | $17$ | $21$ | $37$ | $44$ | $53$ | $57$ |
|-----|-----|-----|-----|-----|-----|-----|
| $C_{-4} L_D$ | $0.0$ | $1296.0$ | $-0.0$ | $1296.0$ | $1296.0$ | $-0.0$ |
| $B_{-4}(D)$ | $0$ | $36$ | $0$ | $36$ | $36$ | $0$ |

| $D$ | $17$ | $21$ | $37$ | $44$ | $53$ | $57$ |
|-----|-----|-----|-----|-----|-----|-----|
| $C_{-8} L_D$ | $0.0$ | $1296.0$ | $-0.0$ | $1296.0$ | $1296.0$ | $-0.0$ |
| $B_{-8}(D)$ | $0$ | $36$ | $-$ | $-$ | $-$ | $-$ |

| $D$ | $1$ | $8$ | $41$ | $69$ | $93$ | $101$ |
|-----|-----|-----|-----|-----|-----|-----|
| $C_{-11} L_D$ | $16.0$ | $1296.0$ | $1296.0$ | $20736.0$ | $20736.0$ | $20736.0$ |
| $B_{-11}(D)$ | $4$ | $36$ | $-$ | $-$ | $-$ | $-$ |
Table 8: Data for the modular form $F_{713}$ based on the Hasse-Weil $L$-series for the hyperelliptic curve $y^2 + (x^3 + x + 1) y = x^5 - x^3$, whose Jacobian has 1-torsion. The constant $C_\ell := k_{713} L_\ell$ with $k_{713} = 0.005249$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT12]. The values $L_\ell$ and $L_D$ are $L(F_{713}, 1/2, \chi_\ell) \cdot |\ell|$ and $L(F_{713}, 1/2, \chi_D) \cdot |D|$, respectively.

| $D$ | $-3$ | $-24$ | $-52$ | $-55$ | $-104$ | $-116$ |
|-----|------|-------|-------|-------|--------|--------|
| $\alpha_{5D} C_5 L_D$ | 16.0 | 16.0 | 400.0 | 64.0 | 16.0 | 144.0 |
| $B_5(D)$ | 4 | 4 | 20 | – | – | – |
| $D$ | $-7$ | $-19$ | $-20$ | $-40$ | $-51$ | $-56$ |
| $\alpha_{12D} C_{12} L_D$ | 16.0 | 256.0 | 16.0 | 400.0 | 576.0 | 16.0 |
| $B_{12}(D)$ | 4 | 16 | 4 | – | – | – |
| $D$ | $-7$ | $-19$ | $-20$ | $-40$ | $-51$ | $-56$ |
| $\alpha_{13D} C_{13} L_D$ | 16.0 | 256.0 | 16.0 | 400.0 | 576.0 | 16.0 |
| $B_{13}(D)$ | 4 | 16 | 4 | – | – | – |
| $D$ | 5 | 28 | 33 | 40 | 56 | 76 |
| $\alpha_{-3D} C_{-3} L_D$ | 16.0 | 16.0 | 0.0 | 16.0 | 16.0 | -0.0 |
| $B_{-3}(D)$ | 4 | 4 | 0 | 4 | 4 | 0 |
| $D$ | 12 | 13 | 24 | 29 | 73 | 77 |
| $\alpha_{-7D} C_{-7} L_D$ | 16.0 | 16.0 | 16.0 | 144.0 | 0.0 | 576.0 |
| $B_{-7}(D)$ | 4 | 4 | 4 | 12 | – | – |
| $D$ | 12 | 13 | 24 | 29 | 73 | 77 |
| $\alpha_{-19D} C_{-19} L_D$ | 256.0 | 256.0 | 256.0 | 2304.0 | 0.0 | 9216.0 |
| $B_{-19}(D)$ | 16 | 16 | – | – | – | – |