Strengthening of weak convergence for Radon measures in separable Banach spaces.

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Abstract.

We prove in this short report that for arbitrary weak converging sequence of sigma-finite Borelian measures in the separable Banach space there is a compact embedded separable subspace such that this measures not only are concentrated in this subspace but weak converge therein.

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1 Introduction. Notations. Statement of problem. Previous works.

Let $(X = \{x\}, || \cdot ||_X)$ be a separable Banach space relative the norm function $|| \cdot ||_X$; $(\Omega, B, P)$ be a non-trivial probability space, $\xi$, $\{\xi_n\}$, $n = 1, 2, \ldots$ be a sequence of random variables with values in the space $X$ having Borel distributions

\[ \mu_n(A) = P(\xi_n \in A), \quad \mu(A) = P(\xi \in A). \]  

(1.0)

Recall that the Banach subspace $(Y = \{y\}, || \cdot ||_Y)$ of the space $X$ is named compact embedded into the space $X$, write

\[ Y \subset \subset X, \]

iff the space $Y$ is linear subspace of the space $X$ : $Y \subset X$ and the closed unit ball $B_Y = \{y : ||y||_Y \leq 1\}$ of the space $Y$ is pre-compact set in the space $X$, i.e. the
closure $[B_Y]X$ is compact set in the space $X$ relative the source topology generated by the norm $\| \cdot \|_X$.

It is known, see [13], [4], [14], that for one Borelian distribution, say $\mu = \mu_\xi$, on the space $X$, probabilistic or at last sigma finite, there exists a separable compact embedded Banach subspace $(Y = \{y\}, \| \cdot \|_Y)$, such that

$$\mu(X \setminus Y) = 0.$$  

Note that this proposition is false in the Linear Topological Spaces instead the Banach space $X$, see [14].

Obviously, for the enumerable, or more generally dominated family of sigma-
finite measures $\mu_n$ there exists a single Banach separable compact embedded into
the space $X$ subspace $(Y = \{y\}, \| \cdot \|_Y)$, $Y \subset X$ such that

$$\forall n \Rightarrow \mu_n(X \setminus Y) = 0. \quad (1.1)$$

Suppose now in addition that the sequence $\{\mu_n\}$ converges weakly as $n \to \infty$ to the measure $\mu$ in the classical Prokhorov-Skorokhod sense, i.e. such that for arbitrary continuous bounded functional $F : X \to R$

$$\lim_{n \to \infty} \int_X F(x) \, \mu_n(dx) = \int_X F(x) \, \mu(dx). \quad (1.2)$$

Write $\mu_n \xrightarrow{X,w} \mu$.

**Question 1.1.** One can choose either the compact embedded subspace $(Y = \{y\}, \| \cdot \|_Y)$ in (1.1) such that the sequence $\{\mu_n\}$ is not only concentrated in the space $Y$, but convergent also in the space $Y$?

Our aim this short report is to ground the positive answer on this question.

## 2 Main result.

**Theorem 2.1.** Let $(X, \| \cdot \|_X)$ be a separable Banach space and $\mu, \mu_n$ be an enumeratable set of Borelian probability measures (distributions) on $X$ converging weakly to the measure $\mu$. There exists a separable compact embedded into $X$ Banach subspace $(Y, \| \cdot \|_Y)$ of the space $X : Y \subset X$, such that all the measures $\mu, \mu_n$ are concentrated on the $Y$ and moreover weak converge also in the space $Y$:

$$\mu_n \xrightarrow{Y,w} \mu. \quad (2.1)$$

**Proof.**

1. It is sufficient to consider by virtue of universality only the case when $X = C[0,1]$, see [13], [4], [14]. On the other words, we can and will suppose $\xi =$
ξ(t), ξ_n = ξ_n(t), t ∈ [0, 1] are continuous a.e. numerical values random processes. As before,

\[ \mu_n \xrightarrow{C[0,1], w} \mu. \]  \hfill (2.2)

2. Further, we intend to use the famous Skorokhod’s representation theorem, [20], see also [3]. Indeed, there exists a sufficiently rich new probability space \((\Omega_1, B_1, P_1)\) and identically with \(ξ(t), \xi_n(t)\) distributed separable r.p. \(η(t), \eta_n(t)\):

\[ ξ(t) \overset{\text{dis}}{=} η(t), \quad ξ_n(t) \overset{\text{dis}}{=} η_n(t), \]  \hfill (2.3)

such that with the \(P_1\) probability one the sequence \(\{η_n(·)\}\) converges uniformly to the random process \(η(·)\).

Here the symbol \(\overset{\text{dis}}{=}\) denotes the coincidence of distribution. Evidently, the r.p. \(η, η_n\) are continuous a.e.

The corresponding "accompanying" sequence \(\{η(t), η_n(t)\}\) is said to be Strengthened Converging Copy of the sequence for the initial one \(\{ξ(t), ξ_n(t)\}\), not necessary to be unique, write

\[ \{η(t), η_n(t)\} \overset{\text{SCC}}{=} \{ξ(t), ξ_n(t)\}. \]

One can assume

\[ \lim_{n \to \infty} ζ_n = 0, \quad ζ_n := \sup_{t \in [0,1]} |η_n(t) - η(t)| = 0. \]  \hfill (2.4)

3. It follows from (2.4) that there exists a deterministic sequence \(ε_n\) tending to zero and a random variable \(τ\) defined on the new probability space such that

\[ ζ_n ≤ τ \cdot ε_n, \]  \hfill (2.5)

see e.g. [9], chapter 2, section 3.

Moreover, since the sequence of continuous functions \(η(t), η_n(t)\) converges uniformly (a.e.), therefore it is compact set in the space \(C[0,1]\). It follows from the Arzela-Ascoli theorem that they are equicontinuous, i.e. there exists the (random) non-negative continuous increasing function \(δ \to h(ω, δ), δ \in [0,1]\):

\[ \lim_{δ \to 0^+} h(ω, δ) = 0, \]  \hfill (2.6)

such that

\[ Δ(η_n - η, δ) \leq h(ω, δ) \to 0, \quad δ \to 0^+. \]  \hfill (2.7)

In what follows \(Δ(f, δ)\) will be denote the ordinary module (modulus) of continuity of an uniform continuous function \(f \in C[0,1]\).

4. It follows from the main result of the preprint [13] that there exists a random variable \(θ\) defined on the new probability space and deterministic non-negative
continuous function $\delta \to g(\delta)$, which takes zero value at the origin: $g(0) = g(0+) = 0$ such that

$$h(\omega, \delta) \leq \theta \cdot g(\delta).$$

Then

$$\Delta(\eta_n - \eta, \delta) \leq \theta \cdot g(\delta).$$

5. Let us introduce the following modification of the classical Hölder’s spaces $H^\alpha(\sqrt{g})$. By definition, $H^\alpha(\sqrt{g})$ consists on all the (continuous) functions $f : [0, 1] \to R$ for which

$$\lim_{\delta \to 0^+} \frac{\Delta(f, \delta)}{\sqrt{g(\delta)}} = 0,$$

with (finite) norm

$$\|f\|_{H^\alpha(\sqrt{g})} \overset{def}{=} \max_{t \in [0,1]} |f(t)| + \sup_{\delta \in (0,1)} \left[ \frac{\Delta(f, \delta)}{\sqrt{g(\delta)}} \right].$$

These Banach spaces are separable and compact embedded into the space $C[0,1]$, see, e.g. the monograph [7], chapter 1, where these spaces are used in particular in the theory of non-linear singular integral equation; some another applications, for example, in the theory of CLT in Banach spaces, may be found in [15].

6. We can suppose without loss of generality that the r.f. $\eta(\cdot), \eta_n(\cdot)$ belong to the introduced above space $H^\alpha(\sqrt{g})$.

Indeed, as long as the sequence $\eta_n(\cdot)$ converges uniformly to $\eta(\cdot)$, it is also compact. It remains to repeat the considerations of fourth item (2.9):

$$\Delta(\eta_n, \delta) \leq \tilde{\theta} \cdot \tilde{g}(\delta),$$

$$\Delta(\eta, \delta) \leq \tilde{\theta} \cdot \tilde{g}(\delta),$$

and choose $\tilde{g}(\delta) := \max(\tilde{g}(\delta), g(\delta))$.

7. It follows immediately from the equality (2.9) that the sequence of the r.p. \{\eta_n(\cdot)\} converge as $n \to \infty$ to the r.p. $\eta(\cdot)$ also in the norm $H^\alpha(\sqrt{g})$ with $P_1$ probability one:

$$\|\eta_n(\cdot) - \eta(\cdot)\|_{H^\alpha(\sqrt{d})} \to 0.$$

The last equality implies for the source sequence of r.p. $\xi(\cdot), \xi_n(\cdot)$ weak its distribution convergence in the space $H^\alpha(\sqrt{d})$.

This completes the proof of theorem 2.1.
**Example 2.1.** Suppose that the sequence of centered continuous random fields $\xi_n(s), s \in S$, somehow dependent, where $(S = \{s\}, r)$ is compact relative certain distance $r(\cdot, \cdot)$ metric space, converges weakly to the continuous *Gaussian* random field $\xi = \xi(s)$ in the ordinary space of all continuous functions $C(S)$; on the other words, CLT in $C(S)$, [11], chapter 9, [12], chapter 4; uniform CLT [5].

We deduce based on the theorem 2.1 that there exists some modified Hölder space $H^\alpha(g)$ over $(S, r)$ such that the sequence $\xi_n(\cdot)$ convergent weakly in the space $H^\alpha(g)$, i.e. is subgaussian, as well.

### 3 Orlicz norm estimates for the tail of random coefficient.

**The case of the space of continuous functions.**

In this subsection $X = C(S)$, where the set $S = \{s\}$ is compact set relative certain distance $\rho = \rho(s_1, s_2), s_1, s_2 \in S$.

It is interest by our opinion for the practical using, for instance, in the Monte Carlo method, to estimate the tails of distribution for the r.v. $\theta$ in the estimation 2.9. For this purpose assume that the r.v.

$$\nu \overset{\text{def}}{=} \sup_n ||\xi_n||_{C(S)} = \sup_n \sup_{s \in S} |\xi_n(s)|$$

belongs to certain Orlicz space $L(\Phi)$ with Luxemburg norm $|| \cdot ||_{L(\Phi)} = || \cdot ||_{\Phi}$ constructed over source probability space; here $\Phi(\cdot)$ is some Young-Orlicz function.

The detail investigation of Orlicz’s spaces may be found in the classical books [10], [18], [19].

We can and will suppose without loss of generality

$$\mathbb{E}\Phi(\nu) = 1. \quad (3.2)$$

**Proposition 3.1.**

It follows in particular from one of results of the recent preprint [16] that if the function $\Phi$ satisfies the so-called $\Delta_2$ condition: $\Phi \in \Delta_2$, then the scaling function $g = g(\delta)$ in (2.9) may be picked such that also $\theta \in L(\Phi)$.

The converse predicate is trivially true.

In the opposite case the situation is more complicated. Recall that the other Orlicz function $\Psi(\cdot)$ is called weaker than the function $\Phi$, notation $\Psi << \Phi$, if for all positive constant $v; v = \text{const} > 0$

$$\lim_{u \to \infty} \frac{\Psi(uv)}{\Phi(u)} = 0. \quad (3.3)$$

It is alleged that if the function $\Psi, \Psi << \Phi$ is given, then the scaling function $g = g(\delta) = g_{\Phi, \Psi}(\delta)$ may be picked such that also $\theta \in L(\Psi)$. 


Example 3.1. For instance, if

$$\sup_n \mathbb{E}|\|\xi_n\|C(S)|^p < \infty, \exists p = \text{const} \geq 1,$$

then the scaling function $g = g(\delta)$ may be picked such that also $\mathbb{E}|\theta|^p = 1$.

Example 3.2.

Let us consider now the so-called Gaussian centered case, i.e. when the common distribution of the infinite-dimensional vector $\mathbf{\xi} = \{\xi, \xi_1, \xi_2, \xi_3, \ldots\}$ has a mean zero Gaussian distribution. We have to take

$$\Phi(u) = \Phi_G(u) := e^{u^2/2} - 1.$$

It follows from one of the main results of an article X.Fernique [6], see also [4], [16] that for any choice of the function $g(\delta)$ satisfying the relation (2.9) the r.v. $\theta$ belongs also to the Orlicz’s space $L(\Phi_G)$, i.e. is subgaussian.

The last example show us that the second assertion of proposition 3.1 is in general case improvable.

4 Bernstein’s moment convergence for compact embedded subspace.

The general case of the arbitrary separable Banach space.

The Bernstein’s moment convergence for weakly convergent sequence of measures $\{\mu_n\}$ imply by definition the following integral convergence

$$\int_X V(x) \mu_n(dx) \to \int_X V(x) \mu(dx)$$

for certain continuous unbounded functional $V: X \to \mathbb{R}$. This problem goes back to S.N.Bernstein [2]; see also [8], [17].

As a rule, in the aforementioned articles the functional $V(x)$ has a form $V(x) = \|x\|^p, x \in X, p = \text{const} \geq 2$.

Let again $(X, \| \cdot \| X)$ be separable Banach space and let $\bar{\mu} = (\mu, \{\mu_n\})$ be the family of Borelian probability measures defined on all the Borelian subsets $X$. Let also $V : X \to \mathbb{R}$ be continuous functional acting from $X$ to the real axis $R : V = V(x), x \in X$.

**Definition 4.1.** The functional $V(\cdot)$ is named uniform integrable relative the family $\bar{\mu}$, if

$$\lim_{N \to \infty} \sup_n \int_{x : |V(x)| > N} |V(x)| \mu_n(dx) = 0. \quad (4.1)$$
Lemma 4.1. Suppose that $\mu_n \overset{w,X}{\rightarrow} \mu$ and that the functional $V(\cdot)$ is uniform integrable relative the family $\bar{\mu}, \mu$. We propose

$$\lim_{n \to \infty} \int_X V(x) \, \mu_n(dx) = \int_X V(x) \, \mu(dx). \quad (4.2)$$

Proof is elementary. Let the positive number $\epsilon > 0$ be a given. We introduce the truncated functional, also continuous,

$$V_N(x) = V(x), \quad |V(x)| \leq N; \quad V_N(x) = -N, \; V(x) < -N;$$

$$V_N(x) = +N, \; V(x) > N; \quad N = 1, 2, \ldots ,$$

and denote

$$\kappa := \bigg| \int_X V(x) \, \mu_n(dx) - \int_X V(x) \, \mu(dx) \bigg| .$$

We get using the triangle inequality $\kappa \leq \kappa_1 + \kappa_2 + \kappa_3$, where

$$\kappa_1 = \bigg| \int_X V(x) \, d\mu_n - \int_X V_N(x) \, d\mu_n \bigg| ,$$

$$\kappa_2 = \bigg| \int_X V_N(x) \, d\mu_n - \int_X V_N(x) \, d\mu \bigg| ,$$

$$\kappa_3 = \bigg| \int_X V(x) \, d\mu - \int_X V_N(x) \, d\mu \bigg| .$$

Since the functional $V = V(x)$ is uniform integrable relative the family $\bar{\mu}$, there exists the value $N_0 = N_0(\epsilon)$ such that for all the values $N > N_0(\epsilon)$

$$\kappa_1 < \epsilon/3, \quad \kappa_3 \leq \epsilon/3.$$

Further, as long as $\mu_n \overset{X,w}{\rightarrow} \mu$, there is a value $n_0 = n_0(\epsilon, N_0(\epsilon))$ so that for all the values $n > n_0$

$$\kappa_2 < \epsilon/3.$$

Totally, $n > n_0 \Rightarrow \kappa < \epsilon$, Q.E.D.

We deduce applying this assertion and the last section the following proposition.

Theorem 4.1. Let all the conditions of Lemma 4.1 be satisfied. There exists common support for the all the measures $\mu, \mu_n$ compact embedded Banach subspace $(Y, ||\cdot||Y)$ such that the functional $V(\cdot)$ is uniform integrable also for these measures inside the new space $Y$ and hence

$$\lim_{n \to \infty} \int_Y V(y) \, \mu_n(dy) = \int_Y V(y)\mu(dy). \quad (4.3)$$
5 Concluding remarks.

A. Generalization on the non-normed measures.

All the references about the random variables $\xi, \xi_n$ generating the correspondent distributions $\mu, \mu_n$ may be eliminated; it is sufficient to consider only the sequence of Borelian (Radon) sigma-finite measures $\{\mu_n\}$ converging weakly in at the same separable Banach space $X$ to the measure $\mu$, which also also Borelian.

Wherein theorem 2.1 remains true, as long as each Borelian sigma-finite measure in equivalent in the Radon-Nikodym sense to the probability distribution.

B. Generalization on the arbitrary family of measures.

At the same result (theorem 2.1) remains true for an arbitrary dominated and convergent net $\mu_\alpha$, $\alpha \in A$, $\alpha \to \alpha_0 \in A$ of sigma-finite Borelian measures in the separable Banach space $X$.

C. Open question.

V.V.Buldygin in [4] proved that in the probabilistic case $\mu(X) = \mu_n(X) = 1$ the single subspace $(Y, \| \cdot \|)$ may be constructed to be reflexive and with continuous differentiable on the unit sphere in the Freshet sense norm.

We do not know either or not possible to choose this ”good” subspace $(Y, \| \cdot \|)$ such that all the r.v. are concentrate in this space and in addition weakly converges therein.

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