The Level Two and Three Modular Invariants of SU(n)

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Abstract

In this paper we explicitly classify all modular invariant partition functions for $A_r(1)$ at level 2 and 3. Previously, these were known only for level 1. The level 2 exceptionals exist at $r = 9, 15, \text{ and } 27$; the level 3 exceptionals exist at $r = 4, 8, \text{ and } 20$. One of these is new, but the others were all anticipated by the “rank-level duality” relating $A_r(1)$ level $k$ and $A_{k-1}^{(1)}$ level $r+1$. The main recent result which this paper rests on is the classification of “ADE-type invariants”.

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1. Introduction

The problem of classifying all modular invariant partition functions corresponding to a given affine algebra $X_r^{(1)}$ and level $k$, has been around for about a decade. The problem is easy to state. The level $k$ highest weights $\lambda$ of $X_r^{(1)}$ form a finite set $P^k(X_r^{(1)})$; the problem is to find all sesquilinear combinations

$$Z = \sum_{\lambda, \mu \in P^k(X_r^{(1)})} M_{\lambda, \mu} \chi_\lambda \chi_\mu^* \quad (1)$$

of the characters $\chi_\lambda$, which satisfy three properties:

- $Z$ is modular invariant (i.e. its coefficient matrix $M$ commutes with the matrices $S$ and $T$ defined in eqs.(3) below);
- $M_{\lambda, \mu} \in \mathbb{Z}_{\geq} \{0, 1, 2, \ldots \}$ for all $\lambda, \mu \in P^k(X_r^{(1)})$;
- $M_{k\Lambda_0, k\Lambda_0} = 1$.

These functions $Z$ or matrices $M$ are called physical invariants.

In spite of considerable effort, for few $X_r^{(1)}$ and $k$ do we have a complete classification. The original result is the A-D-E classification [1] for $A_1^{(1)}$, $\forall k$. Also, $A_2^{(1)}$ for all $k$ is known [2], as is $k = 1$ for all (simple) $X_r^{(1)}$ [3,4]. In this paper we add two more results: $k = 2, 3$ for all $A_r^{(1)}$.

Nevertheless, there has been considerable progress, on a more abstract level, towards solving this problem. In particular, in [5] we find for $A_r^{(1)}$ at all levels $k$, all physical invariants satisfying in addition the condition

$$M_{\lambda, k\Lambda_0} \neq 0 \text{ or } M_{k\Lambda_0, \lambda} \neq 0 \implies \lambda = J'(k\Lambda_0) \quad (2)$$

for some simple current $J'$ (the simple currents for $A_r^{(1)}$ are simply the rotation symmetries of its extended Coxeter-Dynkin diagram). These physical invariants are called $\mathcal{ADE}_7$-type invariants. Almost every physical invariant is expected to satisfy eq.(2).

This paper is essentially two corollaries to [5]. The main purpose here is simply to illustrate the value of the $\mathcal{ADE}_7$ classification, and to help clarify the final step in the physical invariant classification: the determination of anomalous “$\rho$-couplings”. Another immediate consequence of [5] would be physical invariant classifications for $A_r^{(1)}$ for several “small” pairs $(r, k)$ (though admittedly, the value of those results is not so clear).

In the following section we review the basic tools we will use, and list all of the level $k \leq 3$ physical invariants for $A_r^{(1)}$. In section 3 we give the completeness proof for $k = 2$, and in section 4 we give it for $k = 3$.

2. Review

The level $k$ weights $\lambda$ form the set

$$P^r_{+} = P^k_{+}(A_r^{(1)}) = \{ (\lambda_0, \lambda_1, \ldots, \lambda_r) | \lambda_i \in \mathbb{Z}_{\geq}, \sum_i \lambda_i = k \}.$$
We will also write these as $\lambda = \sum \lambda_i \Lambda_i$. For convenience put $\Lambda^i = (k - 1)\Lambda_0 + \Lambda_i$. Define $\rho = (1, 1, \ldots, 1)$ and put $\bar{r} = r + 1$, $\bar{k} = k + \bar{r}$. The affine characters $\chi_\lambda$ at fixed level $k$ define a natural unitary representation of $SL_2(\mathbb{Z})$ [6]:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \chi_\lambda = \sum_{\mu \in P_{r,k}^+} S_{\lambda,\mu} \chi_\mu \quad (3a)
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \chi_\lambda = \sum_{\mu \in P_{r,k}^+} T_{\lambda,\mu} \chi_\mu \quad . \quad (3b)
\]

Then $Z$ in (1) is modular invariant iff

\[
MT = TM \quad (4a)
\]
\[
MS = SM \quad . \quad (4b)
\]

(4a) is equivalent to the selection rule

\[
M_{\lambda,\mu} \neq 0 \implies (\lambda + \rho | \lambda + \rho) \equiv (\mu + \rho | \mu + \rho) \pmod{2\bar{k}} , \quad (4c)
\]

where $(-|-)$ denotes the familiar invariant form of $A_r$ (normalized so that roots have norm 2), and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_r)$. Eq.(4b) is more difficult to interpret, though explicit expressions for $S_{\lambda,\mu}$ exist [6]. $S$ is unitary and symmetric.

Let $J$ and $C$ denote the following permutations of $P_{r,k}^+$:

\[
J(\lambda_0, \lambda_1, \ldots, \lambda_r) = (\lambda_r, \lambda_0, \lambda_1, \ldots, \lambda_{r-1})
\]
\[
C(\lambda_0, \lambda_1, \ldots, \lambda_r) = (\lambda_0, \lambda_r, \lambda_{r-1}, \ldots, \lambda_1) .
\]

They are called simple currents and conjugations, respectively. Let $J_d$ denote the group generated by $J^d$. Write $[\lambda]$ for the orbit of $\lambda$ with respect to both $C$ and $J$, and write $[\lambda]_d$ for the orbit with respect to $J_d$. $C$ defines a physical invariant in the obvious way, also denoted by $C$. In a more subtle way [7], so does $J_d$. First define

\[
t(\lambda) = \sum_{j=1}^{r} j\lambda_j ,
\]

and put $k' = k$ unless both $\bar{r}$ and $k$ are odd, in which case put $k' = \bar{k}$. Then for any divisor $d$ of $\bar{r}$ for which $k'd$ is even, we get a physical invariant $I[\mathcal{J}_d]$ given by

\[
I[\mathcal{J}_d]|_{\lambda,\mu} = \sum_{j=1}^{\bar{r}/d} \delta^{\bar{r}/d}(t(\lambda) + djk'/2) \delta_{\mu,J^d\lambda} \quad (5)
\]

where $\delta^y(x) = 1$ or 0 depending respectively on whether or not $x/y \in \mathbb{Z}$. Any physical invariant which cannot be expressed as the matrix product $C^a \cdot I(\mathcal{J}_d)$ for some $a, d$, is called exceptional.
The level 2 exceptionals $\mathcal{E}^{(r,2)}$ are

$$\mathcal{E}^{(9,2)} = \sum_{i=0}^{9} |\chi J^i \Lambda^0 + \chi J^i (\Lambda_3 + \Lambda_7)|^2 + \sum_{i=0}^{4} |\chi J^i \Lambda^3 + \chi J^i (\Lambda_5 + \Lambda_8)|^2$$

$$\mathcal{E}^{(15,2)} = \sum_{i=0}^{7} (|\langle \chi J^i \Lambda^0 \rangle^2| + |\langle \chi J^i \Lambda^4 \rangle^2| + |\langle \chi J^i \Lambda^6 \rangle^2| + |\langle \chi J^i \Lambda^8 \rangle^2| + |\langle \chi J^i (\Lambda_3 + \Lambda_5) \rangle^2| \chi J^i \Lambda^8 + \chi J^i \Lambda^8 \langle \chi J^i (\Lambda_3 + \Lambda_5) \rangle^2|)$$

$$\mathcal{E}^{(27,2)} = \sum_{i=0}^{13} (|\langle \chi J^i \Lambda^0 \rangle_{14} + \langle \chi J^i (\Lambda_5 + \Lambda_{23}) \rangle_{14}|^2 + |\langle \chi J^i (\Lambda_3 + \Lambda_{25}) \rangle_{14} + \langle \chi J^i (\Lambda_6 + \Lambda_{22}) \rangle_{14}|^2)$$

together with the matrix products $C \cdot \mathcal{E}^{(9,2)}$, $C \cdot \mathcal{E}^{(15,2)}$, $\frac{1}{2} I[J_4] \cdot \mathcal{E}^{(15,2)}$, and $C \cdot \mathcal{E}^{(29,2)}$. In these equations we use the short-hand

$$\langle \chi \lambda \rangle_d \overset{\text{def}}{=} \sum_{j=1}^{\ell/d} \chi J^j \delta \lambda$$

$\mathcal{E}^{(9,2)}$ first appeared in [8], which also anticipated the other two – although to this author’s knowledge neither $\mathcal{E}^{(15,2)}$ nor $\mathcal{E}^{(29,2)}$ have appeared explicitly in the literature before (however [9] found the projection $\frac{1}{2} I[J_4] \cdot \mathcal{E}^{(15,2)}$). The level 3 exceptionals are

$$\mathcal{E}^{(4,3)} = \sum_{i=0}^{4} (|\langle \chi J^i \Lambda^0 \rangle + \langle \chi J^i (\Lambda_0 + \Lambda_2 + \Lambda_3) \rangle|^2 + |\langle \chi J^i (2\Lambda_1 + \Lambda_3) \rangle + \langle \chi J^i (\Lambda_2 + 2\Lambda_4) \rangle|^2)$$

$$\mathcal{E}^{(8,3)} = \sum_{i=0}^{2} (|\langle \chi J^i (\Lambda_0 + \Lambda_2 + \Lambda_3) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_2 + \Lambda_4) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_3 + \Lambda_6) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_3 + \Lambda_8) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_3 + \Lambda_{10}) \rangle|^2)$$

$$\mathcal{E}^{(8,3)}' = \sum_{i=0}^{2} (|\langle \chi J^i (\Lambda_0 + \Lambda_2 + \Lambda_4 + \Lambda_5) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_2 + \Lambda_7) \rangle|^2)$$

$$\mathcal{E}^{(8,3)}'' = |\langle \chi \Lambda^0 \rangle|^2 + |\langle \chi \Lambda_0 + \Lambda_4 + \Lambda_5 \rangle|^2 + |\langle \chi \Lambda_0 + \Lambda_2 + \Lambda_7 \rangle|^2 + \langle \chi \Lambda_0 + \Lambda_2 + \Lambda_4 + \Lambda_5 \rangle|^2 + |\langle \chi \Lambda_1 + \Lambda_2 + \Lambda_5 \rangle|^2 + \langle \chi \Lambda_1 + \Lambda_2 + \Lambda_4 + \Lambda_5 \rangle|^2$$

$$\mathcal{E}^{(20,3)} = \sum_{i=0}^{6} (|\langle \chi J^i \Lambda^0 \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_4 + \Lambda_{17}) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_6 + \Lambda_{15}) \rangle|^2 + |\langle \chi J^i (\Lambda_0 + \Lambda_{10} + \Lambda_{11}) \rangle|^2 + |\langle \chi J^i (\Lambda_1 + \Lambda_8 + \Lambda_{12}) \rangle|^2 + |\langle \chi J^i (\Lambda_9 + \Lambda_{13} + \Lambda_{20}) \rangle|^2 + |\langle \chi J^i (2\Lambda_2 + \Lambda_{17}) \rangle|^2 + |\langle \chi J^i (\Lambda_4 + 2\Lambda_{19}) \rangle|^2)
we will generally put tilde’s over the quantities of a diagram of some weight in \( \lambda \). Note the strong resemblance of the exceptionals in eqs. (6), (7) (except \( \tilde{S} \)) of this section, and reflect it through the diagonal (i.e. take its transpose). Deleting all columns (if any) of length \( k \), this will be the Young diagram of some weight in \( P_{+}^{k-1,r+1} \) which we will denote by \( T(\lambda) \). To avoid confusion we will generally put tilde’s over the quantities of \( A_{r}^{(1)} \) level \( k \), and \( A_{k-1}^{(1)} \) level \( r + 1 \). In particular, choose any \( \lambda \in P_{+}^{r,k} \). Construct its Young diagram, and reflect it through the diagonal (i.e. take its transpose).

The main result of this paper is that these exhaust all physical invariants for \( A_{r}^{(1)} \) at levels 2,3. At level 1, there are no exceptionals: all physical invariants are given by eq. (5). Note the strong resemblance of the exceptionals in eqs. (6), (7) (except \( \tilde{S} \)) of this section, and reflect it through the diagonal (i.e. take its transpose). Deleting all columns (if any) of length \( k \), this will be the Young diagram of some weight in \( P_{+}^{k-1,r+1} \) which we will denote by \( T(\lambda) \). To avoid confusion we will generally put tilde’s over the quantities of \( A_{r}^{(1)} \) level \( k \), and \( A_{k-1}^{(1)} \) level \( r + 1 \). In particular, choose any \( \lambda \in P_{+}^{r,k} \). Construct its Young diagram, and reflect it through the diagonal (i.e. take its transpose).

Then

\[
S_{\lambda,\mu} = \sqrt{\frac{k}{\bar{r}}} \exp \left[ \frac{2\pi i}{\bar{r}k} \{ t(\lambda) \} \{ t(\mu) \} \right] \tilde{S}_{T(\lambda),T(\mu)}
\]

with a similar expression relating \( T_{\lambda,\mu} \) and \( \tilde{T}_{T(\lambda),T(\mu)} \). From (8a) we find that \( T'(J_\lambda) \in \tilde{T}_1 T'(\lambda) \). With the exception of \( \tilde{S} \), the map \( T' \) connects the exceptionals of \( A_{1}^{(1)} \) and \( A_{2}^{(1)} \) with those in eqs. (6), (7) above.

We conclude this section by reviewing the basic lemmas. A well-known result is [6]

\[
S(0,\lambda) \geq S(0,\lambda^0) > 0 ,
\]

with equality in (9a) iff \( \lambda \in [\Lambda^0] \). Two useful identities are

\[
\frac{S(\lambda,\rho) \exp(2\pi i (bt(\lambda) + at(\mu) + kab) / \bar{r})}{2k} \equiv \frac{-2at(\lambda) + ka(\bar{r} - a)}{2\bar{r}} + \frac{(\lambda + \rho, [\lambda + \rho])}{2k} .
\]

For the remainder of this section let \( M \) be any physical invariant. Useful definitions are

\[
\mathcal{P}_L = \{ \lambda \in P_{+}^{r,k} | \exists \mu \in P_{+}^{r,k} \text{ such that } M_{\lambda,\mu} \neq 0 \}
\]

\[
s_L(\lambda) = \sum_{\mu \in P_{+}^{r,k}} S_{\lambda,\mu} M_{\mu,\lambda^0}
\]

and define \( \mathcal{P}_R \) and \( s_R(\mu) \) analogously. Our first result comes from [5,11].

**Cyclotomy Lemma**

(a) For each \( \lambda \in P_{+}^{r,k} \), \( s_L(\lambda) \geq 0 \). Also, \( s_L(\lambda) > 0 \) iff \( \lambda \in \mathcal{P}_L \).

(b) If \( M_{J_\rho,\lambda,\mu} \neq 0 \), then \( M_{J_\mu,\lambda,J_\rho} = M_{\lambda,\mu} \) for all \( \lambda, \mu \in P_{+}^{r,k} \), and \( M_{\lambda,\mu} \neq 0 \) only if \( at(\lambda) \equiv bt(\mu) \) (mod \( \bar{r} \)).
We may decompose $M$ into a direct sum of submatrices $M^{(i)}$, and apply Perron-Frobenius theory \[12\] to each submatrix. Let $M^{(0)}$ be the submatrix containing the index $\Lambda^0$, and let $e(M^{(i)})$ denote the largest real eigenvalue of $M^{(i)}$.

**Perron-Frobenius Lemma** \[11\] $e(M^{(i)}) \leq e(M^{(0)})$ for all $i$. If $(M^{(0)})^2 = e M^{(0)}$ for some number $e$, then $e(M^{(i)}) = e$ for each nonzero $M^{(i)}$.

A final very important result is the Galois symmetry \[13\] obeyed by $S$. Choose any $\lambda \in P^{r,k}_+$, and any integer $\ell$ coprime to $\bar{r} \bar{k}$. Then there will exist a Weyl group element $w$ of $A_r$, a root lattice element $\alpha$ of $A_r$, and a weight in $P^{r,k}_+$ which we will denote by $(\ell \lambda)_+$, for which

$$
\ell \lambda + \rho = w((\ell \lambda)_+) + \bar{k} \alpha
$$

$$
\epsilon_\ell(\lambda) S_{(\ell \lambda)_+, \mu} = \epsilon_\ell(\mu) S_{\lambda, (\ell \mu)_+} \quad \forall \lambda, \mu \in P^{r,k}_+,
$$

where $\epsilon_\ell(\lambda) = \det w \in \{\pm 1\}$. Eq.(10b) also equals the value of the Galois automorphism corresponding to $\ell$, applied to $S_{\lambda, \mu}$ (upto irrelevant sign independent of $\lambda$ and $\mu$). This is important because rank-level duality and (9a) then imply that

$$
\epsilon_\ell(\lambda) \epsilon_\ell(\Lambda^0) = \tilde{\epsilon}_\ell(T'(\lambda)) \tilde{\epsilon}_\ell(\tilde{\Lambda}^0)
$$

$$
T'((\ell \lambda)_+) \in \tilde{J}_1(\ell T'(\lambda))_+.
$$

In particular, (10c) follows by applying the Galois automorphism to $S_{\lambda, \Lambda^0}/S_{\Lambda^0, \Lambda^0} = \tilde{S}_{T'(\lambda), \tilde{\Lambda}^0}/\tilde{S}_{\tilde{\Lambda}^0, \Lambda^0}$. Eq.(10d) follows because rank-level duality $T'$ is an exact (ignoring the irrelevant $\sqrt{k/r}$ factor) bijection for weights $\mu \in P^{r,k}_+$ with $t(\mu) \equiv 0 \pmod{\bar{r}}$: that implies

$$
\tilde{S}_{T'((\ell \lambda)_+), \tilde{\mu}} = \tilde{S}_{(\ell T'(\lambda))_+, \bar{\mu}}
$$

for all $\tilde{\mu} \in P^{-1,r+1}_+$ with $\tilde{t}(\mu) \equiv 0 \pmod{\bar{k}}$. From this equation, (10d) immediately follows.

**Galois Lemma** \[4,14\]

(a) $M_{\lambda, \mu} \neq 0$ only if $\epsilon_\ell(\lambda) = \epsilon_\ell(\mu)$ for all $\ell$ coprime to $\bar{r} \bar{k}$.

(b) $M_{\lambda, \mu} = M_{(\ell \lambda)_+, (\ell \mu)_+}$ for all $\lambda, \mu \in P^{r,k}_+$, and all $\ell$ coprime to $\bar{r} \bar{k}$.

We are most interested in applying Galois (a), together with Cyclotomy (a) and eq.(4c), to find the possibilities $\lambda$ with either $M_{\lambda, \Lambda^0}$ or $M_{\Lambda^0, \lambda}$ nonzero. What we will find is that, for all but finitely many pairs $(r, 2), (r, 3)$, eq.(2) will necessarily be satisfied. In [5], all such physical invariants were classified; for $k \leq 3$ we found that they are either of the form $C^a \cdot I[J_d]$, or equal $C^a \cdot E(15,2)$, $C^a \cdot E(8,3)$, or $\frac{1}{2} I[J_4] \cdot E(15,2)$.

### 3. The level 2 physical invariants of $A_r^{(1)}$

Throughout this section let $M$ denote any physical invariant of $A_r^{(1)}$ level 2, not satisfying eq.(2). The condition $\epsilon_\ell(\lambda) = \epsilon_\ell(\Lambda^0)$ in Galois (a) was explicitly solved for $A_1^{(1)}$ in [11]. The result is that it forces $\lambda \in [\Lambda^0]_1$, unless $\bar{k} = 6, 10, 12$ or 30. Thus by (10c), it suffices to consider $r = 3, 7, 9$ or 27. $r = 3$ and $r = 7$ are handled directly by (4c).
A_9(1) level 2: By Galois (a), the only possibilities \( \lambda \) with \( M_{\lambda, A^0} \) or \( M_{A^0, \lambda} \) nonzero are \( \lambda \in [A^0] \cup [A^4] \). Eq.(4c) now forces \( \lambda = A^0 \) or \( \lambda = A_3 + A_7^{\text{def}} = \lambda' \). Now compute \( s_L(A^1) \) – this is trivial using (8). Then Cyclotomy (a) tells us

\[
\sin(\pi/6) - M_{\lambda', A^0} \sin(\pi/6) \geq 0.
\]

(11a)

Thus \( M_{\lambda', A^0} = 0 \) or 1. Similarly for \( M_{A^0, \lambda'} \). But (4b) evaluated at \( (A^0, A^0) \) forces \( M_{\lambda', A^0} = M_{A^0, \lambda'} = 1 \).

Next, consider the possible \( \lambda \) with \( M_{J A^0, \lambda} \neq 0 \). Again from Galois (a) and (4c), we find that the only possibilities are \( \lambda \in \{ J^{\pm 1}A^0, J^{\pm 1}\lambda' \} \). Now if \( M_{J A^0, J^i A^0} = 0 \) for all \( i \), then (4b) evaluated at \( (J^i A^0, A^0) \) would give

\[
(M_{J A^0, J^i A^0} + M_{J A^0, J^{-i} A^0} - 1) S_{\lambda', A^0} = S_{A^0, A^0},
\]

(11b)

which contradicts (9a). Thus, multiplying \( M \) if necessary by \( C \), we may assume \( M_{J A^0, J A^0} \neq 0 \).

Note that \( M_{\lambda', \lambda'} = 1 \) by Galois (b) with \( \ell = 7 \) (again this is easiest to see using rank-level duality). Finally, consider the possible \( \lambda \) with \( M_{A^3, \lambda} \neq 0 \). By Galois (a), \( \lambda \in [A^3] \), so by Cyclotomy (b), \( \lambda \in \{ A^3, J^5 A^3 \} \). Putting \( (A^3, A^1) \) in (4b) forces \( M_{A^3, J^5 A^3} = M_{A^3, A^3} \), and now Perron-Frobenius forces them to equal 1.

All other entries of \( M \) are fixed by Cyclotomy (b), and we find \( M = \mathcal{E}^{(9,2)} \).

A_{27}^{(1)} level 2: As before, Galois (a) and (4c) tell us \( M_{A^0, \lambda} \neq 0 \) requires \( \lambda \in \{ [A^0]_{14}, [\lambda_5 + \lambda_2]_{14} \} \). Put \( \lambda' = \lambda_5 + \lambda_2 \). Assume first that \( M_{A^0, J^1 A^0} = 0 \). Write \( m = M_{A^0, \lambda'} \), \( m' = M_{A^0, J^1 A^0} \). Then \( s_L(A^{\ell-1}) \geq 0 \) requires

\[
\sin(\pi j/30) + m \sin(11\pi j/30) + m' \sin(19\pi j/30) \geq 0.
\]

(12)

We get \( m = m' = 0 \) by taking \( j = 2, 4, 15 \) in (12). Similarly for \( M_{\lambda, A^0} \neq 0 \). Hence \( M \) will obey eq.(2), contrary to hypothesis.

Thus \( M_{A^0, J^1 A^0} = M_{J^1 A^0, A^0} = 1 \). Hence by Cyclotomy (b), \( M_{A^0, \lambda'} = M_{A^0, J^1 A^0} = m \).

s_L(\lambda^{15}) \geq 0 \) forces \( m = 1 \).

As before (by conjugating \( M \) if necessary), we may assume \( M_{J A^0, J A^0} \neq 0 \). \( M_{\lambda', \lambda'} = 1 \) is forced by Galois (b) with \( \ell = 11 \). All other entries of \( M \) are now found by Cyclotomy (b) and Galois (b) (\( \ell = 13 \)). We obtain \( \mathcal{E}^{(27,2)} \).

4. The level 3 physical invariants of \( A_{1}^{(1)} \)

Throughout this section let \( M \) denote any physical invariant of \( A_{1}^{(1)} \) level 3 which does not satisfy eq.(2). The condition \( \epsilon_\ell(\lambda) = \epsilon_\ell(A^0) \) was also solved for \( A_{1}^{(1)} \) in [11], though in places the proof used (4c). Fortunately we are saved the hassle of verifying that the argument in [11] also works if one replaces (4c) with its rank-level dual, by a remarkable coincidence discovered (I believe) by Ph. Ruelle: The Galois condition \( \epsilon_\ell(\lambda) = \epsilon_\ell(\mu) \) for \( A_{1}^{(1)} \) also appears naturally in the analysis of Jacobians of Fermat curves! In particular, Thm. 0.3 of [15] solves this condition for all but 31 levels \( k \) (the highest exception being \( k = 177 \)).
For now let us avoid these 31 anomalous $k$. Then we learn from [15] that for $\bar{k}$ odd, only $\lambda \in [\Lambda^0]$ satisfies Galois (a) with $\mu = \Lambda^0$. When $\bar{k} \equiv 0 \pmod{4}$, we get

$$\lambda \in [\Lambda^0] \cup [\Lambda_0 + 2\Lambda_s] \cup [\Lambda_0 + \Lambda_s + 2\Lambda_s],$$

and when $\bar{k} \equiv 2 \pmod{4}$, we get

$$\lambda \in [\Lambda^0] \cup [\Lambda_0 + 2\Lambda_s] \cup [\Lambda_0 + 2\Lambda_{(r-2)/4}],$$

where we put $s = r/2$.

Now apply (4c) to $\lambda \in [\Lambda_0 + 2\Lambda_s]$ and $\mu = \Lambda^0$. Multiplying (4c) by $2\bar{r}$ and using (9c), eq.(4c) implies $(\bar{r}^2 - \bar{r} + 1)/2 - 2/\bar{k} \in \mathbb{Z}$, which can never hold. The other possibilities for $\lambda$ can be analysed similarly, as can the 31 anomalous levels. What we find is that (4c) and Galois (a) require any $\lambda$ satisfying $M_{\Lambda^0, \lambda} \neq 0$ or $M_{\lambda, \bar{\Lambda}^0} \neq 0$ to be:

(i) for $\bar{k} \not\equiv 0 \pmod{4}$: $\lambda \in [\Lambda^0]$;

(ii) for $\bar{k} \equiv 0 \pmod{4}$, $\bar{k} \neq 24, 60$: $\lambda \in [\Lambda^0]_d \cup [\lambda^*_d]_d$, where $d$ is the smallest positive solution to $k_1^2d^2 \equiv 0 \pmod{2\bar{r}}$;

(iii) $\bar{k} = 24$: $\lambda \in [\Lambda^0]_7 \cup [\lambda^*_7] \cup [\Lambda^0]_7$;

(iv) $\bar{k} = 60$: $\lambda \in [\Lambda^0]_19 \cup [\lambda^*]_19 \cup [\Lambda^0]_19 \cup [\Lambda^0]_19$,

where we put $\lambda^i = \Lambda_0 + \Lambda_i + \Lambda_{r-1}$ in (ii)-(iv), and where $k'$ in (ii) is defined near eq.(5).

The arguments applying eq.(4b) and Cyclotomy (a) reduce by eq.(8) to the $A_2^{(1)}$ calculations explicitly given in [11]. We will give here one example. Suppose $M_{\Lambda^0, \lambda} = 0$ unless $\lambda \in [\Lambda^0]_d \cup [\lambda^*_d]$, as in (ii). Define $m = \sum_{\lambda \in [\Lambda^0]} M_{\Lambda^0, \lambda}$, and $m' = \sum_{\lambda \in [\lambda^*]} M_{\Lambda^0, \lambda}$. Then by Cyclotomy (b), $m' \geq m$. Now $s_R(2\Lambda_1 + \Lambda_{r-1}) \geq 0$ then reduces to [11]

$$0 \leq (m + m') \sin\left[\frac{2\pi}{k} \left( m + m' \right) \sin\left[\frac{8\pi}{k} \right] - \sin\left[\frac{10\pi}{k} \right] \right] \leq (m + m') \left\{ \sin\left[\frac{2\pi}{k} \right] - \sin\left[\frac{10\pi}{k} \right] \right\}.$$

This forces $\bar{k} = 8$ or $\bar{k} = 12$.

$A_4^{(1)}$ level 3: Here $d = 5, m = 1$. $s_R(\Lambda^1) \geq 0$ forces $m' = 1$, and (4b) evaluated at $(\Lambda^0, \Lambda^0)$ forces $M_{\Lambda^0, \lambda^*} = M_{\lambda^*, \Lambda^0}$. Conjugating if necessary, the usual argument forces $M_{\Lambda^0, \lambda^*} = 1$. $M_{\lambda^*, \lambda^*} = 1$ follows from Galois (b) ($\ell = s + 1$). We are done if we know the values of $M_{2\Lambda_1 + \Lambda_3, \lambda}$. For this to be nonzero, Cyclotomy (b) and Galois (a) says $\lambda$ must equal either $\mu^1 \overset{\text{def}}{=} 2\Lambda_1 + \Lambda_3$ or $C\mu^1$. Now (4b) evaluated at $(\Lambda^1, \mu^1)$ gives us $M_{\mu^1, \mu^1} = M_{\mu^1, C\mu^1}$, and Perron-Frobenius forces $M_{\mu^1, \mu^1} = 1$.

$A_8^{(1)}$ level 3: The argument here is also analogous to the calculations given in [11]. The only difference is that we try to prove $M_{\Lambda^0, \lambda^*} \neq 0$. Galois (a) and eq.(4c) permit an unexpected possibility for $M_{\Lambda^0, \lambda} \neq 0$: $\lambda \in [\mu^3]_3$ where $\mu^3 = \Lambda_1 + \Lambda_3 + \Lambda_5$. If $M_{\Lambda^0, \mu^3} = 0$, then all proceeds as before, but if that entry is nonzero we get $M = E^{(8,3)}m$ by using the familiar arguments.

$A_{20}^{(1)}$ level 3: Again the only difference here arises in the proof that $M_{\Lambda^0, \lambda} \neq 0$. If $M_{\Lambda^0, \lambda_i} = 0$ for all $i$, then we get from (4b) evaluated at $(\Lambda^0, \Lambda^0)$ the equality

$$\frac{m_4 - 3}{3} S_{\Lambda^0, \lambda^4} + \frac{m_6 - 3}{3} S_{\Lambda^0, \lambda^6} + \frac{m_{10} - 3}{3} S_{\Lambda^0, \lambda^{10}} = S_{\Lambda^0, \lambda^0},$$

where $m_i$ are non-negative integer multiples of 3 defined in the obvious way. Since $S_{\Lambda^0, \lambda^i}/S_{\Lambda^0, \lambda^0} \approx 81.2, 137.9, 57.7$ for $i = 4, 6, 10$, respectively, we see this requires $m_4 = m_{10} = 6$ and $m_6 = 0$. This however is readily seen to violate Galois (b) ($\ell = 31$).
$A^{(1)}_{56}$ level 3: The identical contradiction used in [11] works here.

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