CANONICAL BUNDLE FORMULA AND DEGENERATING FAMILIES OF VOLUME FORMS

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Abstract. General canonical bundle formula due to Kawamata and others has played fundamental roles in algebraic geometry. We show that the canonical bundle formula has analytic characterization in terms of fiberwise integration, which confirms a folklore conjecture. As an application, we identify the singularity of the Ohsawa measure from an $L^2$ extension theorem of Demailly for log canonical pairs. In particular, this has the effect of weakening a strictly positive curvature condition to a semipositive one from a previous $L^2$ extension theorem of the author. As another application, we give a partial answer to a question of Berndtsson on semipositivity theorems.

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1. Introduction

General canonical bundle formula [Ka98] (cf. [FM00], [Am04], [Ko07]) has played fundamental roles in higher dimensional algebraic geometry. The first instance of such formula goes back to the classical work of Kodaira on elliptic fibrations [K64 Thm. 12]. The setting of the canonical bundle formula is as follows.

Let $f : X \to Y$ be a surjective projective morphism with connected fibers between complex manifolds. We will say that $f : (X, R) \to (Y, B)$ satisfies Kawamata’s
condition ([Ka98], see Definition 4.1) when \( R \) is a simple normal crossing (or snc) divisor on \( X \) and \( B \) is a reduced snc divisor on \( Y \) such that \( f \) is smooth over \( Y \setminus B \), \( \text{red}(R) + f^*B \) is an snc divisor on \( X \) and the vertical parts of \( R \) maps into \( B \). Moreover, the horizontal parts of \( R \) satisfy the conditions in Definition 4.1.

1.1. Statements of the main results.

Now let \( L := \mathcal{O}(R) \) be the \( \mathbb{Q} \)-line bundle associated to \( R \). Let \( M \) be the \( \mathbb{Q} \)-line bundle on \( Y \) defined by the relation \( K_X + L = f^*(K_Y + M) \). Our first main result is the following metric version of the canonical bundle formula [Ka98].

**Theorem 1.1.** Let \( f : (X, R) \to (Y, B) \) be as above (cf. Definition 4.1). Assume that the horizontal divisor \( R_h \) is effective, so that its coefficients are in the interval \([0, 1)\).

Let \( \lambda \) be a singular hermitian metric of \( L \) given by (a defining meromorphic section of) the divisor \( R \). Then the \( L^2 \) metric \( \mu \) for the \( \mathbb{Q} \)-line bundle \( M \) is decomposed as the product of two singular hermitian metrics \( (\mathcal{O}(B_R), \eta) \) and \( (J := M - \mathcal{O}(B_R), \psi) \), i.e.

\[
\mu = \eta + \psi \quad (e^{-\mu} = e^{-\eta}e^{-\psi}),
\]

where \( \eta \) is a singular hermitian metric given by the discriminant divisor \( B_R \) and \( \psi \) is a singular hermitian metric of \( J \) with semipositive curvature current and with zero Lelong numbers at every point.

The \( L^2 \) metric \( \mu \) is defined in Proposition 3.6, Theorem 3.7. Note that \( R \) and \( B_R \) are not necessarily effective divisors. Thus \( \lambda, \mu \) and \( \eta \) are not necessarily with semipositive curvature currents. Also unlike \( \mu \), the metrics \( \psi \) and \( \eta \) are not uniquely determined by \( \lambda \): they can be added constants \( c \) and \(-c\), for example.

The discriminant divisor \( B_R \) due to [Ka98] is a particular linear combination (see (7)) of the components of \( B \), which is supposed to capture the singularities of both \( R \) and the singular fibers of \( f \), generalizing the case of elliptic fibrations [K63]. The definition of \( B_R \) was originally motivated by algebraic geometry: according to the author of [Ka98], the coefficients of the discriminant divisor in [Ka98] were defined so that they behave well under semi-stable reduction.

Our next main result shows that the discriminant divisor has analytic characterization in terms of fiberwise integration along \( f \) of singular volume forms. This proves a folklore conjecture, cf. [Ka96, p.81], [Ka98], [Ka99, p.10], [T07]. Together with Theorem 1.1, these also answer a question of [EFM18] which asked if there is a metric version of Kawamata’s canonical bundle formula [Ka98].

\footnote{See conventions for our notation of hermitian line bundles.}
Theorem 1.2. Let \( f : (X, R) \rightarrow (Y, B) \) be as in Theorem 1.1 except that we do not require \( R_h \) to be effective. Let \( u \) be a singular volume form on \( X \) with poles along the snc divisor \( R \) (as in (2), §2.3). Then the fiberwise integration of \( u \) along the smooth fibers \( f_0 : X_0 \rightarrow Y_0 \) is a singular volume form \( v \) on \( Y \) with poles along \( B_R \), the discriminant divisor of \( R \), up to a plurisubharmonic weight with vanishing Lelong numbers, i.e. it is locally written as

\[
v(w) = g(w)(\prod_{i=1}^{m} |w_i|^{-2a_i})e^{-\psi(w)} |dw_1 \wedge \ldots \wedge dw_m|^2
\]

whenever local coordinates \( w = (w_1, \ldots, w_m) \) are adapted to \( B \) on an open subset \( U \subset Y \) (of dimension \( m \)), where \( \psi \) is psh with vanishing Lelong numbers, \( \sum a_i \text{div}(w_i) = B_R \) on \( U \) and \( g : U \rightarrow \mathbb{R} \) is a positive and bounded function.

As in Theorem 1.1 this is also a metric version in disguise of the canonical bundle formula in terms of the \( L^2 \) metric, which is a metric determined by the fiberwise integration (cf. [PT, (3.2.2)], [F78]). Since \( R_h \) is not necessarily effective, our metric is for \( M \otimes f_*A \) in the sense of Definition 3.5 where \( A = \mathcal{O}(\lceil -R_h \rceil) \), i.e. it is nothing but a usual psh metric for a \( \mathbb{Q} \)-line bundle outside a subset of codimension \( \geq 2 \) where \( f_*A \) is locally free of rank 1. The definition of the \( L^2 \) metric in our \( \mathbb{Q} \)-line bundle setting (see §3.2) is obtained by a simple way of Proposition 3.6. After all, singular volume forms and fiberwise integration are obviously well defined in the \( \mathbb{Q} \)-line bundle setting as well.

Once the \( L^2 \) metric on \( Y \) is prepared as a psh metric by Theorem 3.7, our methods of proof of the main theorems in Section 4 are via identifying the singularities of the \( L^2 \) metric in terms of all divisorial valuations over \( Y \). Since a general psh metric does not have a log resolution, it is necessary to consider all possible divisors in proper modifications over \( Y \). This is done in the key technical result Theorem 4.5 by carefully applying the defining condition of the discriminant divisor to show that, in fact, the \( L^2 \) metric is \textit{valuatively equivalent} to the psh metric given by the discriminant divisor.

This is indeed a powerful viewpoint for our purpose: the psh metric \( \psi \) in Theorem 1.1 is obtained immediately from comparing the Siu decomposition of the curvature currents of the two metrics in comparison, cf. Lemma 4.9.

Remark. Theorem 1.1 in particular recovers nefness of \( J \) in [Ka98] by Proposition 2.1. Such refinement of nefness was also given in a recent work [T22] by a different method.\(^2\)

\(^2\)The current paper first appeared on arXiv and [T22] was received by a journal, both on October 2019 whereas a manuscript of the current paper was sent to the author of [T22] on March 2019. We note that the approach of the current paper is not ‘similar’, as opposed to a mention in [T22].
1.2. Applications to $L^2$ extension theorems.

As a consequence of Theorem 1.2, we obtain algebro-geometric identification of the $L^2$ norm from an $L^2$ extension theorem of Demailly for log canonical pairs. This will have the effect of weakening a strictly positive condition to a semipositive one from a previous $L^2$ extension theorem of the author [K07].

Let $Y \subset X$ be a subvariety of a complex manifold. Let $L$ be a holomorphic line bundle on $X$ and $K_X$ the canonical line bundle of $X$. An $L^2$ extension theorem is a type of a statement that (under suitable conditions on $X, Y, L, \ldots$) if a certain $L^2$ norm $\|s\|_Y$ is finite for a holomorphic section $s$ on $Y$ of $(K_X \otimes L)|_Y$, then there exists $\tilde{s} \in H^0(X, K_X \otimes L)$ such that $\tilde{s}|_Y = s$ and $\|\tilde{s}\|_X \leq c \|s\|_Y$ for some constant $c > 0$.

(See [OT87], [O01], [D15], [K07] and many others: see also the introduction and references of [K21].) Here the $L^2$ norm $\|s\|_Y$ plays the crucial role of determining whether $s$ can be extended or not: we will call it as the input norm of the statement.

In [D15, Thm. 2.8], Demailly gave an influential $L^2$ extension theorem when $Y$ is taken to be the non-klt locus of a log canonical pair. The input norm $\|s\|_Y$ is taken with respect to the Ohsawa measure (cf. [O01]) which is defined in terms of a certain limit which is taken over some tubular neighborhoods shrinking to the pole set of $\Psi$, see Definition 6.1. Although this definition is natural from the analytic point of view, its singularities remained to be understood.

In our recent work [K21], we studied the Ohsawa measure by examining its restriction to each irreducible component (say $Z$) of the non-klt locus. We showed that if $Z$ has at least two log canonical places, then the Ohsawa measure is the infinity measure, which makes the $L^2$ extension statement virtually unusable on such $Z$ since only the constant zero function will satisfy the input norm condition.

In this paper, we apply Theorem 1.2 to the case when $Z$ has a unique log canonical place to complete the understanding of the Ohsawa measure. We show that the Ohsawa measure $dV[\Psi]_Z$ has the singularities described by Kawamata’s subadjunction: see Theorem 6.2 for the precise statement.

Using Theorem 6.2, we can then compare the input norms of two $L^2$ extension theorems: [K07, Thm. 4.2] and [D15, Thm. 2.8] both for log canonical pairs (on a smooth projective variety). It turns out that the input norm of the latter is the same as the one of the former, only modulo a psh weight of vanishing Lelong numbers which does not affect the $L^2$ conditions.

For this comparison, we first derive [K21, Thm. 3.9], a version of $L^2$ extension for a maximal log canonical center, from [D15, Thm. 2.8] (which was extension from the entire non-klt locus). See Theorem 6.3.

Let $\|s\|_1$ be the input norm of [K07, Thm. 4.2] which was taken with respect to a ‘Kawamata metric’, a singularity described by Kawamata’s subadjunction (see
Theorem 6.4). Let $\|s\|_2$ be the input norm in [K21, Thm. 3.9] with respect to the Ohsawa measure $dV[\psi]_Y$. From Theorem 6.2 we have the following corollary (see Theorem 6.4 for the full statement).

**Corollary 1.3** (= Theorem 6.4). For a holomorphic function $s$ on $Y$, we have $\|s\|_1$ locally finite if and only if $\|s\|_2$ locally finite. In other words, the two input norms give the same criterion for extension of sections.

Moreover this comparison in Corollary 1.3 means that, for all possible purposes, [K21, Thm. 3.9] (derived from [D15, Thm. 2.8]) can be regarded as giving a generalization of [K07, Thm. 4.2] (i.e. once [K21, Thm. 3.9] is equipped with the additional information on the Ohsawa measure provided by Theorem 6.2) in that a strictly positive curvature condition (“small ample” $A$) in [K07, Thm. 4.2] is replaced with a semipositive one in [D15] (see (28) in Theorem 6.3).

**1.3. Applications to semipositivity theorems.**

Using Theorem 1.1, we give a partial answer to a question of Berndtsson (2005) who asked whether it would be possible to use the theory of positivity of direct images (along the line of [B09], [BP08], [PT], cf. Section 3) in the proof of the celebrated semipositivity theorems in [Ka81], [Ka00] and others, which used Hodge theory. By Theorem 7.1 we answer this question when the direct image of the relative canonical line bundle is locally free of rank 1.

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**2. Singular hermitian metrics and plurisubharmonic functions**

We refer to [D11], [B20], [BBJ21, Appendix B] for introduction to singular hermitian metrics of a line bundle and plurisubharmonic functions. We have some conventions used in this paper.

- We will often write holomorphic ($\mathbb{Q}$-) line bundles additively as in $L_1 + L_2 := L_1 \otimes L_2$. 
• We will often denote a singular hermitian metric \( e^{-\varphi} \) of a line bundle simply by \( \varphi \) so that we can write additively both line bundles and metrics as in \((L_1 + L_2, \varphi_1 + \varphi_2)\).

2.1. Singular hermitian metrics on line bundles.

A psh metric of a holomorphic \( \mathbb{Q} \)-line bundle \( L \) on a complex manifold \( X \) is a singular hermitian metric of \( L \) with semipositive curvature current (so that its local weight functions can be taken as psh functions), cf. [D11], [HPS]. When a \( \mathbb{Z} \)-line bundle \( L \) is given transition functions \( \{g_{ij}\} \) on a locally trivializing open cover \( \{U_i\}_{i \in I} \), a (smooth or singular) hermitian metric of \( L \) can be identified with a collection of functions \( e^{-\varphi_i} = |g_{ij}|^{-2} e^{-\varphi_j} \). (A similar description holds when \( L \) is a \( \mathbb{Q} \)-line bundle by taking some multiple \( mL \) \((m \geq 1)\) that is a genuine holomorphic line bundle.)

A holomorphic section \( s \in H^0(X, L) \) defines a psh metric \( \varphi \) of \( L \) by taking \( \varphi_i = \log |s_i|^2 \) (from \( s_i = s_j g_{ij} \)), which can be also denoted by \( e^{-\varphi} = \frac{1}{|s|^2} \).

The following fact is due to [D92] (cf. [FF17, (3.5)]). Note that the converse does not hold, see [D11] Example after (6.11).

**Proposition 2.1.** Let \( X \) be a compact complex manifold. If \( L \) admits a psh metric with vanishing Lelong numbers, then it is nef.

2.2. Valuative equivalence of psh singularities.

If two psh functions (or two psh metrics) \( \varphi, \psi \) satisfy that \( \varphi - \psi \) is locally bounded, we say that \( \varphi \) and \( \psi \) have equivalent singularities following [D11] (6.3)]. We introduce the following notion.

**Definition 2.2.** We say that two psh functions \( \varphi \) and \( \psi \) on a complex manifold \( X \) are **valuatively equivalent** (and write \( \varphi \sim_v \psi \)) if the following two equivalent conditions (due to [BFJ08] when combined with the strong openness [GZ15]) hold:

1. For all real \( m > 0 \), the multiplier ideals are equal: \( \mathcal{J}(m\varphi) = \mathcal{J}(m\psi) \).
2. At every point of a proper modification \( \pi : X' \to X \), the Lelong numbers of \( \pi^*\varphi \) and \( \pi^*\psi \) coincide. In other words, for every divisorial valuation \( v \) centered on \( X \), we have \( v(\varphi) = v(\psi) \).

When two psh functions \( \varphi \) and \( \psi \) on a complex manifold are valuatively equivalent, we have the following:

1. If, moreover, \( \varphi - \psi \) (or \( \psi - \varphi \)) happens to be psh, it should have vanishing Lelong numbers. Of course, in general, none of \( \varphi - \psi \) and \( \psi - \varphi \) needs to be psh or quasi-psh (even if one of \( \varphi, \psi \) has analytic singularities): see, for example, [KS19] (2.3), (2.9)].
2. If \( \psi \) has analytic singularities, then \( \varphi \leq \psi + O(1) \), cf. [K15] Thm. 4.3].
See [KR22], [KS19] for more examples and results on valuative equivalence of plurisubharmonic singularities.

2.3. Singular volume forms.

First recall that $|K_X|^2 = K_X \otimes K_X$ is the real line bundle of volume forms (i.e. real $(n,n)$ forms, cf. [Ko95, Chap.7]) on a complex manifold $X$. A singular volume form $u$ on $X$ can be defined, in the greatest generality, as a general measurable section of $|K_X|^2$ such that when locally written as

$$u(w) = f(w) |dw_1 \wedge \ldots \wedge dw_n|^2$$

in local analytic coordinates $w = (w_1, \ldots, w_n)$, $f \geq 0$ is a local measurable function with values in $\mathbb{R} \cup \{+\infty\}$. We will mainly use the following more concrete cases.

We will say that a singular volume form $u$ on $X$ has poles along an snc divisor (not necessarily effective) $R = \sum_{i=1}^m a_i R_i$ on $X$ if it can be written in local coordinates on $V \subset X$ adapted to $R$ as

$$(2) \quad u(w) = g(w) \left( \prod_{i=1}^k |w_i|^{-2a_i} \right) |dw_1 \wedge \ldots \wedge dw_n|^2$$

where $R|_V = \sum_{i=1}^k a_i R_i$, $R_i = \text{div}(w_i)$ and $g$ is a $C^\infty$ locally bounded positive function (i.e. $g$ has no ‘poles and zeros’).

Similarly, we say that a real-valued function $t$ has poles along such $R = \sum a_i R_i$ if it can be written locally

$$(3) \quad t(w) = g(w) \left( \prod_{i=1}^k |w_i|^{-2a_i} \right)$$

as in (2).

3. $L^2$ metrics

In [PT] (cf. [BP08], [HPS]), the authors studied semipositivity of direct images of the form $M := f_*(K_{X/Y} + L)$ for a surjective projective morphism $f : X \to Y$ in terms of certain naturally defined singular hermitian metrics on $M$. We will use the special case of [PT] (3.2.2) when $M$ is torsion-free of rank 1.

\footnote{Of course, the smoothness of $g$ is not particularly relevant and it can be weakened to continuity or mere boundedness, but in practice it is enough to work with these. We should regard $a_i R_i$ as having zero of order $-a_i$ along $R_i$ if $a_i < 0$.}
3.1. Direct images of adjoint line bundles.

In this paper, we will use the following special case of [PT, Thm. 3.3.5] (cf. [HPS, (21.2)]).

**Theorem 3.1** (Positivity of direct images, a special case). [PT, Thm. 3.3.5], cf. [PT, Set-up 3.2.1]

Let \( f : X \to Y \) be a surjective projective morphism with connected fibers between two connected complex manifolds. Let \( Y_0 \) be the set of regular values of \( f \).

- Let \( L \) and \( M \) be \( \mathbb{Z} \)-line bundles on \( X \) and \( Y \) such that \( K_X + L = f^*(K_Y + M) \) holds.
- Let \( A \) be a \( \mathbb{Z} \)-line bundle on \( X \) such that \( f_* A \) is torsion-free of rank 1. Let \( Y_1, \text{free} \) be the Zariski open subset of \( Y \) where \( f_* A \) is locally free of rank 1.
- Let \((L \otimes A, \lambda)\) be a psh metric such that the inclusion

\[ f_*(K_X/Y \otimes L \otimes A \otimes J(\lambda)) \to f_*(K_X/Y \otimes L \otimes A) \]

is generically an isomorphism.

Then the \( L^2 \) metric \( \mu \) for \( M \otimes f_* A = f_*(K_X/Y \otimes L \otimes A) \) on \( Y_0 \cap Y_{1,\text{free}} \) is a psh metric and it extends to a psh metric for \( M \otimes f_* A \) on \( Y_{1,\text{free}} \).

Let us recall the definition of this \( L^2 \) metric \( \mu \) in Theorem 3.1 from [PT, (3.2.2)]. We define the \( L^2 \) metric on \( Y_0 \cap Y_{1,\text{free}} \). For the simplicity of notation, we denote \( Y := Y_0 \cap Y_{1,\text{free}} \) in the following paragraph (before Definition 3.2).

Let \( u \in H^0(Y, M \otimes f_* A) \) be a section. Since \( M \otimes f_* A = f_*(K_X + L + A + f^*K_Y^{-1}) \) (where the connected fibers assumption \( f_* \mathcal{O}_X = \mathcal{O}_Y \) is used), we have

\[ H^0(Y, M \otimes f_* A) = H^0(Y, \mathcal{H}om(K_Y, f_*(K_X + L + A))). \]

Hence viewing \( u \) as a sheaf morphism \( u : K_Y \to f_*(K_X + L + A) \), for a nowhere vanishing local section \( \eta \) of \( K_Y \) on an open subset \( V \subset Y \), we have \( u(\eta) \in H^0(f^{-1}(V), K_X + L + A) \). From [PT, (3.2.2)], we have locally

\[ u(\eta) = \sigma_i \wedge f^* \eta \]

for some \( \sigma_i \), an \((L + A)\)-valued holomorphic \( n \)-form \((n = \dim X - \dim Y)\) defined on \( U_i \) from an open cover \( \{U_i\}_{i \in I} \) of \( f^{-1}(V) \). The existence of such \( \sigma_i \) follows from computation of elementary nature in terms of local coordinates (‘admissible coordinates’ [MT08 §2.2]) which locally make \( f \) a projection. The restrictions \( \sigma_i|_{X_y} \) glue together to define \( \sigma|_{X_y} \) in the integral below which defines the \( L^2 \) metric \( \mu \).
**Definition 3.2** \((L^2 \text{ metric on } Y_0 \cap Y_{1, \text{free}}). \) \([\text{PTI} \ (3.2.2)]\)

In the setting of Theorem 3.1, let \((L + A, \lambda)\) be the given hermitian line bundle. We define the \(L^2\) metric \(\mu\) (induced from \(\lambda\)) for the direct image \(M \otimes f_*A\) on \(Y_0 \cap Y_{1, \text{free}}\) by

\[
(|u|^2 e^{-\mu})(y) := \int_X c_n \sigma|_{X_y} \wedge \overline{\sigma}|_{X_y} e^{-\lambda} = \int_X |\sigma_y|^2 e^{-\lambda}
\]

where \(c_n = i^n\). Here we write \(\sigma_y := \sigma|_{X_y}\) noting that \(\sigma\) itself may not be defined. Also note that the family \(\sigma_y\) is determined by \(u\) only.

We recall the following fiberwise integration property of the \(L^2\) metric from [PTI], cf. [HPS].

**Proposition 3.3.** Let \(V \subset Y_0\) and \(u(\eta) \in H^0(f^{-1}(V), K_X + L + A)\) be as above. The volume form defined by \(|u(\eta)|^2 e^{-\lambda}\) on \(f^{-1}(V)\) has its fiberwise integration along \(f\) equal to \(|u \cdot \eta|^2 e^{-\mu}\) where \(u \cdot \eta \in H^0(V, K_Y + M)\).

**Proof.** As in (5), we have \(u(\eta) = \sigma \wedge f^*\eta\). Note that this last wedge product makes sense even though \(\sigma\) alone may not be defined globally. For simplicity of notation, let us write \(f : X \to Y\) in the place of \(f : f^{-1}(V) \to V\). Since \(|u(\eta)|^2\) is a \(|L + A|^2\)-valued volume form on \(X\), its multiplication with a metric \(e^{-\lambda}\) for \(L + A\),

\[|u(\eta)|^2 e^{-\lambda} = |\sigma \wedge f^*\eta|^2 e^{-\lambda} = |\sigma|^2 e^{-\lambda} \cdot |f^*\eta|^2\]

is a volume form, i.e. an \((m + n, m + n)\) form on \(X\). By the projection formula of fiberwise integration [DX] and using (6), the fiberwise integration of this along \(f\) is equal to \(|u|^2 e^{-\mu}, |\eta|^2\).

The following example illustrates Theorem 3.1.

**Example 3.4.** (1) Let \(f : X \to Y\) be a ruled surface where \(X = \mathbb{P}(E)\) for a holomorphic vector bundle \(E\) of rank 2 on a smooth projective curve \(Y\), cf. \([H] \S V.2\). When \(L = O_{\mathbb{P}(E)}(2)\) and \(M = \det E\), we have \(K_X + L = f^*(K_Y + M)\). When \(L\) is ample and provided with a smooth psh metric, Theorem 3.1 says that \(M\) is semipositive, which is consistent with \([H] \S V, (2.20), (2.21)\], characterization of ample line bundles on \(X\).

(2) Now let \(X = \mathbb{P}(E)\) be as in [DPS94] (1.7) with \(L = O_{\mathbb{P}(E)}(2)\) : it is shown there that the only possible psh metrics \(\lambda\) for \(L\) are ones whose curvature current is equal to \(2[C]\) where \(C \subset X\) is a section of \(f\) (in particular, \(L\) is not semipositive). It is clear that this \(\lambda\) does not satisfy the condition (1) of Theorem 3.1.
3.2. Generalization to Q-line bundles.

In this subsection, we will generalize Theorem 3.1 to Theorem 3.7. Let $L$ and $M$ be Q-line bundles satisfying the relation $K_X + L = f^*(K_Y + M)$. Note that even when $K_X, L$ and $K_Y$ are Z-line bundles, it is possible that $M$ is only a Q-line bundle (see e.g. (22)). The only case of the direct images in [PT, Thm. 3.3.5] is of the form $M \otimes f_*A$ as in Theorem 3.1. To generalize this for a Q-line bundle $M$, it is sufficient for our purpose to regard $M \otimes f_*A$ as a formal object and adopt the following definition.

**Definition 3.5.** Let $M$ be a Q-line bundle. Let $A$ be a Z-line bundle on $X$ such that $f^*A$ is torsion-free of rank 1. We define a psh metric for $M \otimes f_*A$ to be a psh metric in the usual sense for the Q-line bundle on the Zariski open subset $Y_{1,free}$ where $f_*A$ is locally free of rank 1.

In order to define the $L^2$ metric for $M$ over $Y_0 \cap Y_{1,free}$, let $N = (k-1)M$ where $k \geq 1$ is the smallest integer such that $kM$ is a Z-line bundle. Then we have $K_X + L + f^*N = f^*(K_Y + M + N)$. Let $(N, \psi)$ be a smooth hermitian metric.

Let $L + A + f^*N$ be given the metric $\lambda + f^*\psi$. Apply Definition 3.2 of the $L^2$ metric in the Z-line bundle case, (6). Note that, since $M + N$ is a Z-line bundle, so are $f^*(K_Y + M + N)$ and then $L + f^*N$. Hence we get an induced $L^2$ metric $\mu_\psi$ for $M + N$. Take $\mu_\psi - \psi$ for $M$. This is the definition of the $L^2$ metric.

**Proposition 3.6.** The $L^2$ metric $\mu := \mu_\psi - \psi$ for $M \otimes f_*A$ over $Y_0 \cap Y_{1,free}$ is well-defined.

**Proof.** We need to check that $\mu_\psi - \psi$ is independent of the choice of the smooth metric $\psi$. It suffices to see this at each point $y$, hence we may fix a local trivialization of $kM$ in a neighborhood $V$ of $y$. Then $\mu_\psi$ and $\psi$ are functions on $V$.

Now let $u$ be a holomorphic section of $M + N = kM$. From (6), we have

$$|u|^2 e^{-\mu_\psi(y)} = \int_{X_y} |\sigma_y|^2 e^{-\lambda} e^{-f^*\psi} = e^{-\psi(y)} \int_{X_y} |\sigma_y|^2 e^{-\lambda}$$

where we get the second equality since $f^*\psi$ is constant on $X_y$. □

Also using this definition involving $M + N = kM$ and $\psi$, we have the fiberwise integration property for the $L^2$ metric inherited from Proposition 3.3.

After these preparations, we now derive the following version of Theorem 3.1 when $L$ and $M$ are Q-line bundles.

**Theorem 3.7.** Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two connected complex manifolds.
Let $L$ and $M$ be $\mathbb{Q}$-line bundles on $X$ and on $Y$ such that $K_X + L = f^*(K_Y + M)$ holds as an equality of $\mathbb{Q}$-line bundles.

Let $A$ be a $\mathbb{Z}$-line bundle on $X$ such that $f_*A$ is torsion-free of rank 1. Let $Y_{1,\text{free}}$ be the Zariski open subset of $Y$ where $f_*A$ is locally free of rank 1.

Let $(L + A, \lambda)$ be a psh metric such that $J(\lambda|_F) = 0$ for a general fiber $F$ of $f$.

Then the $L^2$ metric $\mu$ for $M \otimes f_*A$ on $Y_0 \cap Y_{1,\text{free}}$ is a psh metric and extends to $Y$.

The case when $A = \mathcal{O}_X$ will be used in the proof of Theorem 1.1 while the general case will be used in the proof of Theorem 1.2.

Proof. Consider the restriction of $f$ to $X_0 := f^{-1}(Y_0) \to Y_0$ of smooth fibers where $Y_0 \subset Y$ is the subset of regular values of $f$. From its definition, the $L^2$ metric we want is already defined on the subset $Y_0$ as a metric of the $\mathbb{Q}$-line bundle $M|_{Y_0}$. We need to show that it is a psh metric on $Y_0$ and extends to $Y$ as a psh metric.

Note that $M$ is a line bundle on $Y$. Considering local weight functions on a locally trivializing open cover for $M$, this is a local problem on $Y$: it suffices to work on a neighborhood $U \subset Y$ of an arbitrary point $p \in Y \setminus Y_0$. This is simply a local problem of extending psh functions now: take $\psi = 0$ and we are done by Theorem 3.1.

□

4. Proofs of the main results

In this section, we will give the proof of the main theorems.

4.1. The setting of Kawamata’s canonical bundle formula.

We will give the setting of the main theorems, which is the same as in [Ka98] (cf. [Ko07]) except that we are in the generality of complex manifolds (where the necessary Hironaka resolution theorems also hold, cf. [H64], [AHV77], [W09]).

Let $(X, R)$ and $(Y, B)$ be two pairs of complex manifolds and snc $\mathbb{Q}$-divisors. For a divisor $R = \sum a_iR_i$, define $\text{red}(R) := \sum R_i$.

Let $f : X \to Y$ be a surjective projective morphism with connected fibers. An irreducible component $R_i$ of $R$ is called horizontal if $f(R_i) = Y$. Otherwise it is called vertical. We write $R = R_h + R_v$ where $R_h$ is the horizontal part and $R_v$ is the vertical part.

Definition 4.1. [Ko07, Def. 8.3.6], [Ka98, Thm. 2] We will say that $f : (X, R) \to (Y, B)$ satisfies Kawamata’s condition if the following hold:

1. $X, Y$ are complex manifolds.
2. $B$ is a reduced snc divisor on $Y$, i.e. $B = \text{red}(B)$.
3. $f(\text{Supp}(R_v)) \subset B$. 


(4) \( \text{red}(R) + f^*B \) is an snc divisor on \( X \).
(5) \( f \) is a submersion over \( Y \setminus B \).
(6) \( R_h \) is relative snc over \( Y \setminus B \).
(7) Coefficients of \( R_h \) are in the interval \( (-\infty, 1) \).
(8) \( \text{rank} \ f_*\mathcal{O}_X([-R_h]) = 1 \).
(9) \( K_X + R \) is \( \mathbb{Q} \)-linearly equivalent to the pullback of some \( \mathbb{Q} \)-Cartier divisor on \( Y \).

Let \( B_R \) be the discriminant divisor induced by \( R \) which is defined as (when \( B = \sum B_i \)) \( B_R = \sum c_i B_i \) where

\[
\tag{7} c_i := 1 - \sup \{ c : (X, R + cf^*B_i) \text{ is lc over the general point of } B_i \} \]

([Am99, Def. 3.1]). The condition in the sup is that, in other words, the non-lc locus of the snc divisor \( R + cf^*B_i \) is contained in the inverse image of the union of some Zariski closed proper subsets of \( B_i \)'s. Since \( \text{red}(R) + f^*B \) is snc, the lc (resp. klt) conditions are determined by coefficients of all the components being \( \leq 1 \) (resp. \( < 1 \)), cf. [Ko97, Cor. 3.12]. Now write \( R_v = R_{v1} + R_{v2} \) where the sum \( R_{v1} \) (resp. \( R_{v2} \)) consists of components whose images under \( f \) are of codimension 1 (resp. of codimension at least 2) in \( Y \). Since the coefficients of \( R_h \) are less than 1, the condition in (7) is then equivalent to \( (X, R_{v1} + cf^*B_i) \) being lc over the general point of \( B_i \). Hence \( B_R \) is also characterized as the unique smallest \( \mathbb{Q} \)-divisor supported on \( B \) such that

\[
\tag{8} R_{v1} + f^*(B - B_R) \leq \text{red}(f^*B) \]

cf. [Ko07] (8.3.7) (2)]. \(^4\) (Also see [Ka98, p.895] for the equivalent original definition of \( B_R \).)

Remark 4.2. It can be easily seen (from local equations (18)) that the components of \( R_{v1} \) are contained in the components of \( f^*B \).

Later we will need the following lemma.

**Lemma 4.3.** In the setting of (4.1), let \( S \) be a \( \mathbb{Q} \)-divisor supported on \( B \). The discriminant divisor \( B_{R + f^*S} \) associated to \( R + f^*S \) is equal to \( B_R + S \).

**Proof.** It follows from the definition putting \( R' = R + f^*S \) and \( R'_v = R_v + f^*S \). \( \square \)

In the setting of Definition 4.1 (simply using the condition (8)), a \( \mathbb{Q} \)-line bundle \( J(X/Y, R) \) (called the moduli part) is defined on \( Y \) by the following relation (cf. [Ka98])

\(^4\)We thank Hyunsuk Kim for pointing out to replace \( R_v \) by \( R_{v1} \) in (8).
(9) \[ K_X + R \sim_Q f^*(K_Y + B_R + J(X/Y, R)) \]

For the purpose of this paper involving metrics, we also view (9) as the corresponding equality of \( \mathbb{Q} \)-line bundles

(10) \[ K_X + L = f^*(K_Y + M) = f^*(K_Y + J + H) \]

where we define the corresponding \( \mathbb{Q} \) line bundles \( L := \mathcal{O}(R), H := \mathcal{O}(B_R), J := J(X/Y, R), M := J + H \).

Also we prepare the following obvious statement.

**Lemma 4.4.** Let \( R_h = \sum a_i D_i \) be the snc divisor in Definition 4.1 with the coefficients \( a_i \) in the interval \((-\infty, 1)\). Then the divisor \( R_h + \lceil -R_h \rceil \) has all the coefficients in the interval \([0, 1)\).

Let \( A = \mathcal{O}([-R_h]) \) be the corresponding \( \mathbb{Z} \)-line bundle on \( X \). Note that \( A = \mathcal{O}_X \) in the setting of Theorem 1.1.

4.2. Valuative equivalence result.

In the setting of the previous subsection, the following is the key technical result in the proof of the main results.

**Theorem 4.5.** Let \( f : X \to Y, K_X + L = f^*(K_Y + M) \) and \( A \) be as in the previous subsection. Assume that the snc divisor \( R_{v1} \) (i.e. those vertical components whose images are of codimension 1 in \( Y \)) is effective.

Let \( \lambda \) be a singular hermitian metric of \( L + A \) given by the divisor \( R + \lceil -R_h \rceil \) (so that \( \mathcal{J}(\lambda|_F) = \mathcal{O}_F \) holds for a general fiber \( F \) of \( f \) due to Definition 4.1 (6), (7)).

(1) The \( L^2 \) metric \( \mu \) (induced from \( \lambda \)) for \( M \otimes f_*A \) (defined in Proposition 3.6) is a psh metric (in the sense of Definition 3.5).

(2) The \( L^2 \) metric \( \mu \) is valuatively equivalent to the psh metric \( \varphi_{B_R} \) (for the \( \mathbb{Q} \)-line bundle \( \mathcal{O}(B_R) \)) given by the divisor \( B_R \geq 0 \), in the Zariski open subset where \( f_*A \) is locally free (of rank 1).

**Proof.** (1) Let \( Y' = Y \setminus Z \) where \( Z \) is the image of the components in \( R_{v2} \) (hence \( Z \) is of codimension \( \geq 2 \)). Apply Theorem 3.7 to the restriction \( f^{-1}(Y') \to Y' \).

Note that \( \lambda \) is a psh metric on \( f^{-1}(Y') \) since it is taking the complement of the possibly noneffective divisor \( R_{v2} \). Hence the \( L^2 \) metric \( \mu \) is a psh metric on \( Y \setminus Z \) by Theorem 3.7. Since \( Z \) is of codimension \( \geq 2 \), \( \mu \) extends to \( Y \).

(2) We need to show that \( v(\mu) = v(B_R) \) for every divisorial valuation \( v = \text{ord}_G \) where \( G \) is a divisor lying over \( Y \) with nonempty center in \( Y \). Consider a higher model
\( \pi : Y' \to Y \) such that \( G \) appears as a prime divisor on \( Y' \). Consider the following diagram (11) where \( \rho : X' \to X \) is bimeromorphic and \( f' \) also satisfies Kawamata’s condition (11). (The morphism \( f' : X' \to Y' \) in this diagram is obtained by first taking the fibration \( X'' \to Y' \) induced by the bimeromorphic base change \( Y' \to Y \) and then taking further blow-ups over \( X'' \) : cf. [Ko07, 8.4.8].)

We take a Zariski open subset \( Y_0 \subset Y \) and let \( X_0 := f^{-1}(Y_0) \) so that \( f : X_0 \to Y_0 \) is submersion. Also determine \( Y_0' \) and \( X_0' := f'^{-1}(Y_0') \) in the following diagram so that they are isomorphic to \( Y_0 \) under \( \pi \) and to \( X_0 \) under \( \rho \) respectively.

\[
\begin{array}{ccc}
X'_0 & \subset & X' \\
\downarrow^{f'} & & \downarrow^f \\
Y'_0 & \subset & Y'
\end{array}
\] (11)

The restrictions \( f : X_0 \to Y_0 \) and \( f' : X'_0 \to Y'_0 \) can be identified with each other and so are the fiberwise integrations taken along them. This can be expressed as

\[
\pi^*(f_\ast u) = f'_\ast(\rho^*u)
\]
where the domains are all restricted to \( X_0, Y_0, X'_0, Y'_0 \). Now let \( R' \) be the divisor on \( X' \) defined by \( K_{X'} + R' = \rho^*(K_X + R) \). Note that while \( u \) has poles along \( R \), the pullback \( \rho^*u \) has poles along \( R' = \rho^*R - K_{X'/X} \).

Consider the following projection formula of fiberwise integration combined with (12):

\[
f'_\ast(\rho^*u \wedge f'^t) = f'_\ast(\rho^*u) \wedge t = \pi^*(f_\ast u) \wedge t.
\]

In order to show \( v(\mu) = v(B_R) \), we will apply (13) for the fiberwise integration taken along the restriction of \( f' \) over a neighborhood \( U \subset Y' \) of a general point of \( G \) taking \( t \) with poles \( \delta B' - (v(\mu) - \alpha)G \) first and then with poles \( \delta B' - (v(B_R) - \alpha)G \) secondly. Here \( \alpha := \text{ord}_G(K_{Y'/Y}) \), i.e. the coefficient of the prime divisor \( G \) in the relative canonical divisor \( K_{Y'/Y} \).

For arbitrary \( \delta < 1 \), take a function \( t \) with poles \( \delta B' - (v(\mu) - \alpha)G \). We claim that

\[
R'_{\omega_1} + f'^\ast(\delta B' - (v(\mu) - \alpha)G) < \text{red}(f'^\ast B').
\]

To verify the claim (14), first note that since \( f_\ast u \) has the poles given by the psh weight \( e^{-\mu} \), its pullback \( \pi^*(f_\ast u) \) on \( Y' \) has poles given by the pullback of \( e^{-\mu} \) divided by the contribution of the jacobian of the morphism \( Y' \to Y \). When \( g = 0 \) is a local equation of the divisor \( G \), this pole is expressed as \( e^{-\pi^\ast\mu} |g|^{2\alpha} \).
Since we are looking at a neighborhood $U$ of a general point of $G$ and considering the restriction of the divisor $\delta B' - (v(\mu) - \alpha)G$ to $U$, which is nothing but $(\delta - v(\mu) + \alpha)G$, the local expression of the pole of $\pi^*(f_*u) \wedge t$ in (13) is now given by

$$e^{-\pi^*\mu} |g|^{2\alpha} |g|^{-2(\delta - v(\mu) + \alpha)} = e^{-\pi^*\mu} |g|^{2(v(\mu) - \delta)}.$$

By Lemma 4.7 applied for $W = Y'$, $\varphi = \pi^*\mu$, $\psi = 2(v(\mu) - \delta) \log |g|$, this is locally integrable. Thus from the LHS of (13), the singular volume form inside $f'_* \pi'$ is also locally integrable, which gives (14).

Now from (14), it follows that $R_{\mathfrak{v}_1} + f'^*(B' - (v(\mu) - \alpha)G) \leq \text{red}(f'^*B')$. From the definition of the discriminant divisor $B'_{R'}$ associated to $R'$ (see (8)), we see that $v(B'_{R'})$ is the smallest possible coefficient for $G$ (in the place of $(v(\mu) - \alpha)$) to make this inequality hold. Hence we have

$$v(\mu) - \alpha \geq v(B'_{R'}) = v(B_R) - \alpha \tag{15}$$

where the equality is from Lemma 4.6. Thus we have $v(\mu) \geq v(B_R)$.

Now suppose that $v(\mu) - v(B_R) > 0$. This time, take a function $t$ to be with poles $\delta B' - (v(B_R) - \alpha)G$ for $\delta < 1$. From (13), we will have contradiction since the LHS is locally integrable while the RHS is not. Indeed, the RHS of (13) has poles along $$(v(\mu) - \alpha)G + \delta B' - (v(B_R) - \alpha)G = (v(\mu) - v(B_R))G + \delta B'$$
and it is not locally integrable when $v(\mu) - v(B_R) + \delta > 1$ (note that $G$ appears in $B'$ as we may assume so).

On the other hand, the LHS of (13) is locally integrable since $\rho^*u \wedge f'^*w$ is locally integrable before taking the fiberwise integration: it has poles along $R' + f'^*(\delta B' - (v(B'_{R'})G)$ which is klt (since $\delta < 1$), from the definition of the discriminant divisor $B'_{R'}$. This is contradiction. Hence we have $v(\mu) = v(B_R)$, which completes the proof of Theorem 4.5. $\square$

**Lemma 4.6.** For $f$ and $f'$ in (11) satisfying Kawamata’s conditions, we have $v(B'_{R'}) = v(B_R) - \alpha$.

**Proof.** This follows from the equality of divisors $K_{Y'} + B'_{R'} = \pi^*(K_Y + B_R)$ which in turn follows immediately from the pull back property of the moduli part line bundles $J(X'/Y', R') = \pi^*J(X/Y, R)$ [Ko07, 8.4.9 (3)] (originally due to [Ka98]). The pull back property results from the Hodge theoretic characterization of the moduli part line bundle and the fact that the canonical extension in Hodge theory commutes with pull backs. $\square$

Also the following lemma was used, a weaker variant (using only one divisorial valuation) of valuative characterization of multiplier ideals (cf. [BFJ08], [BBJ21]).
Lemma 4.7. Let \( W \) be a complex manifold and \( H \) a prime divisor on \( W \). Let \( \varphi \) and \( \psi \) be psh functions on \( W \). Assume that \( \psi \) has analytic singularities. If \( \text{ord}_H(\psi) > \text{ord}_H(\varphi) - 1 \), then \( e^{\psi - \varphi} \) is locally integrable at a general point of \( H \).

Note the minor point that here \( e^{\psi - \varphi} \) is locally \( L^1 \) instead of \( L^2 \) as in [B20] Thm. 10.11 since our convention from §2.1 is taking, for example, \( \varphi = \log |f|^2 \) instead of \( \varphi = \log |f| \) throughout this paper.

\textbf{Proof.} (1) First assume that \( \varphi \) also has analytic singularities. Let \( \mu : W' \rightarrow W \) be a log resolution of both \( \varphi \) and \( \psi \). We may choose a general point \( p \) of \( H \) and shrink \( W \) to some \( U \), a neighborhood of \( p \), in order to avoid prime divisors on \( W \) along which \( \varphi \) or \( \psi \) has positive generic Lelong numbers and also to avoid the images of exceptional divisors of \( \mu \) along which \( \mu^* \varphi \) or \( \mu^* \psi \) has positive generic Lelong numbers. Then it is easy to see from \( \mu : \mu^{-1}(U) \rightarrow U \) that the condition \( v(\psi - \varphi) > -1 \) for \( v = \text{ord}_H \) alone determines local integrability of \( e^{\psi - \varphi} \).

(2) Now let \( \varphi \) be a general psh function. Let \((\varphi_m)_{m \geq 1}\) be a Demailly approximation sequence of \( \varphi \). (cf. [D13] Thm. 7 for the case of a bounded pseudoconvex domain in \( \mathbb{C}^n \). In general, one can use partitions of unity as in [D11] to choose such a sequence.) Since \( v(\psi) > v(\varphi) - 1 \), we can fix \( \lambda > 1 \) such that \( v(\psi) > \lambda v(\varphi) - 1 \). Then for every \( m \geq 1 \), we have \( v(\psi) > \lambda v(\varphi_m) - 1 \) since \( \varphi_m \) is less singular than \( \varphi \) in the sense that \( \varphi_m \geq \varphi + O(1) \). Applying (1) to \( \varphi_m \) (having analytic singularities), we then have \( e^{\psi - \lambda \varphi_m} \in L^1_{\text{loc},x} \) at a general point \( x \) of \( H \), i.e. for \( x \in H \setminus V_m \) where \( V_m \subset H \) is a Zariski closed subset (which may depend on \( m \)). By Lemma 4.8, we have \( e^{\psi - \varphi} \in L^1_{\text{loc},x} \) for the same \( x \).

\textbf{Lemma 4.8.} [D13] Let \( \psi, \varphi \) be psh functions on a complex manifold \( X \). Let \((\varphi_m)_{m \geq 1}\) be a Demailly approximation sequence of \( \varphi \). Let \( \lambda > 1 \) be a real number. For every \( x \in X \) and every integer \( m \geq \left\lceil \frac{\lambda}{2(\lambda - 1)} \right\rceil \), we have the implication

\[ e^{\psi - \lambda \varphi_m} \in L^1_{\text{loc},x} \implies e^{\psi - \varphi} \in L^1_{\text{loc},x}. \]

\textbf{Proof.} When \( \psi = \log |f|^2 \) for a holomorphic function \( f \), this is [D13] Cor. 4 which states the inclusion of multiplier ideals \( \mathcal{J}(\lambda \varphi_m) \subset \mathcal{J}(\varphi) \). The same argument (which uses [D13] Lem. 2) works when we put \( e^{\psi} \) in the place of \( |f|^2 \). \qed

4.3. Proofs of the main theorems.

Now using Theorem 4.3, we complete the proof of Theorem 1.1.

\textbf{Proof of Theorem 1.1.} First note that \( R_v \) is not necessarily effective. By Remark 4.2 there exists an effective divisor \( S \) supported on \( B \) such that \( R_{v1} + f^*S \geq 0 \) and \( B_R + S \geq 0 \). Let \( N = \mathcal{O}(S) \) be the associated \( \mathbb{Q} \)-line bundle. Consider the equality of \( \mathbb{Q} \)-line bundles.

\[ e^{\psi - \lambda \varphi_m} \in L^1_{\text{loc},x} \implies e^{\psi - \varphi} \in L^1_{\text{loc},x}. \]
\[ K_X + L + f^* N = f^* (K_Y + J + H + N). \]

Equip \( L + f^* N \) with a psh metric \( \lambda \) given by the effective divisor \( R + f^* S \). Then its \( L^2 \) metric \( \mu \) for \( J + H + N \) is a psh metric by Theorem 3.7. Here note that the condition \( J(\lambda|_F) = \mathcal{O}_F \) in (3.7) is satisfied since the coefficients of the horizontal snc divisor \( R_h \) are assumed to be less than 1. Now consider the Siu decomposition [D11, (2.18)], [B04, 2.2.1] of the curvature current \( \Theta_\mu \) of \( \mu \):

\[
T := \Theta_\mu = \sum \nu(T, Y_k)[Y_k] + R_T.
\]

Here \( \nu(T, Y_k) \) is the generic Lelong number of \( T \) along the codimension 1 irreducible subvariety \( Y_k \). Recall that by Theorem 4.5, the current \( \Theta_\mu \) (or its psh potential) is valuatively equivalent to the current given by the effective divisor \( B_R + f^* S = B_R + f^* S \) (by Lemma 4.3). Thus the divisor part \( \sum \nu(T, Y_k)[Y_k] \) is a finite sum which is precisely given by the discriminant divisor \( B_R + f^* S \) of the snc divisor \( R + f^* S \).

We apply Lemma 4.9 to the curvature current \( T \) of \( (J + H + N, \mu) \) and the curvature current \( Q \) of \( (H + N, \varphi_{B_R+S}) \) where \( \varphi_{B_R+S} \) is a psh metric given by the effective divisor \( B_R + S \). Since the closed positive \((1,1)\) current \( R_Q = 0 \), we see that \( R_T \) has zero Lelong numbers at every point by Lemma 4.9.

Since the closed positive \((1,1)\) current \( R_T \) belongs to the first Chern class of the \( \mathbb{Q} \)-line bundle \( J + H + N - (H + N) = J \), there exists a singular hermitian metric \( \psi \) of \( J \) whose curvature current is equal to \( R_T \) (as is well-known, see e.g. [B04, p.50]). This \( \psi \) is the one we wanted in the statement of (1) of the theorem: it has vanishing Lelong numbers.

Choose a singular hermitian metric \( \varphi_S \) given by the divisor \( S \) such that \( \varphi_{B_R+S} = \varphi_{B_R} + \varphi_S \). From \( (J + H + N, \mu) \), we subtract \( (J, \psi) \) and get a psh metric \( (H + N, \mu - \psi) \) given by the effective divisor \( B_R + S \). Now subtracting again \( (N, \varphi_S) \), we get \( (H, \eta := \mu - \psi - \varphi_S) \) which is a singular hermitian metric given by the original discriminant divisor \( B_R \). Since \( B_R \) may not be effective, \( \eta \) may not be a psh metric. This \( \eta \) is the one we were looking for, which completes the proof of Theorem 1.1.

Lemma 4.9. Let \( X \) be a complex manifold. Let \( T \) and \( Q \) be closed semipositive \((1,1)\) currents on \( X \). Suppose that \( T \) and \( Q \) are valuatively equivalent, i.e. they have the same Lelong numbers at every point in \( X \) and at every point in all proper modifications \( \tilde{X} \to X \). Then in the Siu decomposition of the closed positive \((1,1)\) currents \( T \) and \( Q \),

\[
T = \sum \nu(T, Y_k)[Y_k] + R_T
\]
\[
Q = \sum \nu(Q, Y_k)[Y_k] + R_Q.
\]
the closed positive \((1, 1)\) currents \(R_T\) and \(R_Q\) are valuatively equivalent.

**Proof.** This is immediate from the construction of the Siu decomposition as in \[B04, 2.2.1\]. Note that \(v(R_T) = v(R_Q)\) is nonzero only for \(v = \ord_G\) where \(G\) is a divisor lying over \(X\) and its center (the image) on \(X\) is of codimension \(\geq 2\) in \(X\).

Now we turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Note that it is a local problem on \(Y\). We apply Theorem 4.5 to the given \(f : (X, R) \to (Y, B)\) by taking \(A = \mathcal{O}([-R_h])\). Since \(u\) is continuous, its fiberwise integration \(v\) has continuity. Hence we may restrict to the Zariski open subset of \(Y\) where \(f_*A\) is locally free of rank 1.

In the notation of Proposition 3.3 (cf. the remark after Proposition 3.6), the fiberwise integration of the singular volume form \(|u(\eta)|^2 e^{-\lambda} \) is equal to \(|u \cdot \eta|^2 e^{-\mu}\) where \(\lambda\) is a psh metric for \(L + A\) given by the divisor \(R + [-R_h]\). By Lemma 4.10 and Lemma 4.4, this fiberwise integration has the same singularities as the one we are considering, namely \(v\) in Theorem 1.2.

Theorem 4.5 says that the singular volume form \(|u \cdot \eta|^2 e^{-\mu}\) is locally of the form as in (1) in Theorem 1.2 (by the same reason as in the proof of Theorem 1.1), hence so is \(v\).

**Lemma 4.10.** Let \(f : (X, R) \to (Y, B)\) be a surjective projective morphism with connected fibers satisfying Kawamata’s condition in Definition 4.1. Let \(u\) and \(v\) be singular volume forms on \(X\) with poles along the snc divisor \(R = R_h + R_v\) (as in (2)) and along the snc divisor \(R + T\) respectively, where \(T\) is a divisor having the same support as \(R_h\). Suppose that both \(R_h\) and \(R_h + T\) have all coefficients less than 1. Then the fiberwise integrations (along \(f\)) \(f_*u\) and \(f_*v\) have the same singularities, i.e. their quotient is locally bounded.

**Proof.** It is shown by computation in the next section: see Remark 5.1.

This lemma can be viewed as the fiberwise integration analogue of the algebro-geometric fact that \(B_R\) depends only on \(R_v\) (not on \(R_h\)) as in \[Ko07, (8.3.7.2)\].

5. **On the fiberwise integration**

In this section, we have two independent subsections which supplement Section 4.

5.1. **Computation of the fiberwise integration.**

In this subsection, we explain the computation of the fiberwise integration, to complete the proof of Lemma 4.10. We first remark that when \(\dim Y = 1\), such computation has been dealt with by various authors, especially recently by \[BJ17, EFM18 (2.1)\] (see also related works by \[Y10, GTZ16 (2.1), GTZ19, Be16 (3.8)\] and...
others). Such computations for \( \int_X |\sigma_t|^2 \) can be roughly (and typically) summarized as (for a local holomorphic coordinate \( t \) on \( Y \) with \( \dim Y = 1 \))

\[
(16) \quad |t|^{-2\alpha} |\log |t|^2|^{\beta}
\]

where \( \alpha \geq 0 \) can be interpreted in terms of log-canonical thresholds.

Now in our setting of general dimension of \( Y \), let \( p \in X_0 = f^{-1}(Y_0) \). We take local coordinates \( w = (w_1, \ldots, w_{n+m}) \) in a neighborhood \( U_1 \) of \( p \in X \) and \( z = (z_1, \ldots, z_m) \) in a neighborhood \( V \) of \( f(p) \in Y \). We assume that these coordinates are adapted to the given snc divisors \( R + f^*B \) and \( B \), respectively. Eventually we need different coordinate neighborhoods \( U_2, \ldots, U_k \) in addition to \( U_1 \) to take care of the entire \( R \).

We can first restrict \( \alpha \) to \( U_1 \) and compute its fiberwise integration, i.e. its contribution to \( f_*\alpha \).

For \( f \) satisfying the condition of [Ka98] (see also [Ka14, Thm. 3.7.5]), we may assume that \( f \) is locally given as follows: \( f(w) = z \) and (for \( i = 1, \ldots, m \))

\[
(17) \quad z_i = g_i(w_1, \ldots, w_{n+m}) w_1^{a_{1i}} \cdots w_n^{a_{ni}} w_{n+m}^{a_{ni+m}}
\]

where each \( a_{ji} \geq 0 \) is an integer \( (1 \leq j \leq n + m) \). Since the divisor of each \( f^*z_i \) is supported on the divisor \( w_1 \cdots w_{n+m} = 0 \), we may assume \( g_i \) is nowhere zero in the domain of the coordinates and thus may assume that they are constantly 1 for the purpose of the computation that follows:

\[
(18) \quad z_i = w_1^{a_{1i}} \cdots w_{n+m}^{a_{ni+m}}.
\]

The matrix \((a_{ji})\) has rank \( m \) since otherwise there would be a multiplicative relation among \( z_i \)'s contradicting to surjectivity of \( f \). Now in these coordinates, the singular volume form \( \alpha \) with poles along \( R \) is given by

\[
(19) \quad \alpha = \frac{1}{\prod |w_i|^{2r_i}} |dw_1 \wedge \cdots \wedge dw_{n+m}|^2
\]

where \( R = \sum r_i R_i = \sum r_i(w_i = 0) \) is the given divisor. We will compute \( f_*\alpha \) by integrating \( \alpha \) with respect to \( n \) fiber coordinates (to be determined soon) among \( w_1, \ldots, w_{n+m} \) on each smooth fiber of \( f \). On the other hand, from (18), we get

\[
dz_i = \sum_{j=1}^{n+m} z_i a_{ji} \frac{w_j}{w_i} dw_j.
\]
Since the $m \times (m + n)$ matrix $(a_{ij})$ has rank $m$, we may assume that the $m \times m$ matrix $(a_{ij})$ is invertible by renaming coordinates. Solving (18) for $w_i$’s, we have the relation (for $i = 1, \ldots, m$)

$$w_i = \prod_{l=1}^{m} z_i^{b_l} \prod_{k=1}^{m} w_k^{b_k}. \tag{20}$$

where $b_l, b_{k+m} \in \mathbb{Q}$. Then we can replace $dw_1, \ldots, dw_m$ in (19) by taking $d$ of both sides of (20) (or (18)). Note that by the implicit function theorem, $w_{m+1}, \ldots, w_{m+n}$ are fiber coordinates i.e. local coordinates on a smooth fiber. We can rewrite (19) as

$$\alpha = \frac{1}{\prod_{a_i \neq 0} \prod_i |a_i/w_i|^{2r_i}} \left| \prod_i |dz_i|^2 \right|^2 \left| \prod_i |dw_i|^2 \right|^2.$$

Now we take fiberwise integration on a smooth fiber using $2n$ real fiber coordinates associated to $w_{1+m}, \ldots, w_{m+n}$. We divide coordinates into the following three groups:

$A = \{ w_1, \ldots, w_m \}$

$B = \{ w_{m+1}, \ldots, w_{m+v} \}$: fiber coordinates corresponding to $\text{red}(R_v) + f^*B$. These can appear in (18).

$C = \{ w_{m+v+1}, \ldots, w_{m+n} \}$: fiber coordinates corresponding either to $\text{red}(R_v)$ or to none of $\text{red}(R) + f^*B$. These do not appear in (18) (i.e. the corresponding exponent is zero). Consider the following factor in (19)

$$\prod_i |w_i|^{2r_i} = \prod_i |w_i|^{2r_i} \prod_i |w_i|^{2r_i} \prod_i |w_i|^{2r_i}.$$ 

The third $C$ group factor on the right hand side is locally integrable on each smooth fiber since $(X, R_v)$ is klt. Using (20), we change the first $A$ group factor into two factors: one involving $z$ coordinates and the other involving $B$ and $C$ group coordinates. For the factor consisting of $z$ coordinates, we have nothing to do: they stay the same when we do the fiberwise integration.

Now the remaining argument is to do the fiberwise integration with respect to $B$ group coordinates. That is, we apply Fubini theorem with $B$ and $C$ group coordinates, but with $C$ group coordinates, the result of integration will be just bounded by the above klt reason.

Remark 5.1. At this point, Lemma 4.10 is confirmed since local coordinates corresponding to $R_h$ are contained in $C$ group coordinates.

5.2. Elliptic fibrations.
The classical elliptic fibrations studied by Kodaira \cite{K63}, \cite{K64} provide the important initial case of the canonical bundle formula. Even in this case, our main result seems new, cf. Theorem 5.2.

Let $f : X \to Y$ be a relatively minimal elliptic fibration with $\dim Y = 1$ (see e.g. \cite[Thm. V.12.1]{BPV}). When $f$ has multiple fibers $m_1F_1, \ldots, m_kF_k$, we have the canonical bundle formula

\[(21) \quad K_X = f^*(K_Y + G) + \sum (m_i - 1)F_i\]

where $G$ is the line bundle equal to $f^*(K_{X/Y})$. We turn this into the following equality of $\mathbb{Q}$-line bundles:

\[(22) \quad K_X = f^* \left( K_Y + G + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}Q_i \right)\]

where $Q_i \in Y$ is a point viewed as a divisor such that $f^*Q_i = m_i F_i$ as Cartier divisors. This is a situation where we can apply Theorem 3.7 viewing \[(22) \quad K_X + L = f^*(K_Y + M) \] with $L$ trivial and $M := G + \mathcal{O}(\sum_{i=1}^{k} \frac{m_i - 1}{m_i}Q_i)$. Note that in general, the equality $K_{X/Y} = f^*M$ holds as $\mathbb{Q}$-line bundles only: the direct image $f_* (K_{X/Y})$ is a $\mathbb{Z}$-line bundle which is not necessarily equal to $M$.

Furthermore, by \cite[(2.9)]{F86}, \cite[(8.2.1)]{Ko07} (also see e.g. the introduction to \cite{FM00}), we have equality of $\mathbb{Q}$-line bundles

\[(23) \quad G = \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + \mathcal{O}(\sum_{k \in K} \sigma_k P_k)\]

where $\sigma_k$ is the well-known coefficients (see e.g. \cite[(2.6)]{F86}) from the list of singular fibers \cite{K63} and $j : Y \to \mathbb{P}^1$ is the map into the moduli.

The trivial metric 1 is a psh metric for the trivial line bundle $L$, thus we get the corresponding $L^2$ metric $\mu$ for the $\mathbb{Q}$-line bundle $M$ in a canonical way.

**Theorem 5.2.** In this case of an elliptic fibration $f : X \to Y$, the $L^2$ metric $\mu$ is the product of a singular hermitian metric given by the divisor $\sum_{k \in K} \sigma_k P_k + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}Q_i$, and a singular hermitian metric with vanishing Lelong numbers for $\frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1)$.

**Proof.** Let $g : X' \to X$ be a proper birational morphism so that Kawamata’s condition \[(1.1) \] is satisfied for $X' \to Y$: we will apply Theorem 1.1. The $L^2$ metric for $X' \to Y$ coincides with the $L^2$ metric $\mu$.

Define the divisor $R$ (in Theorem 1.1) by $K_{X'} + R = g^* K_X$. For dimension reasons, $R_h = 0$. The discriminant divisor $B_R$ is equal to $\sum_{k \in K} \sigma_k P_k + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}Q_i$. Thus as $\mathbb{Q}$-line bundles, $J$ (in Theorem 1.1) is equal to $\frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1)$. This concludes the proof.
Thus Theorem 1.1 recovers the slightly weaker metric version of the highly non-trivial fact [F86] that the $\mathbb{Q}$-line bundle $J$ is semiample with the morphism $j$.

We see that even in this simplest case, it was essential to formulate our results (1.1), (3.7) in the generality of $\mathbb{Q}$-line bundles to be able to equally deal with multiple and non-multiple singular fibers in the canonical bundle formula. One can compare, for example, with [EFM18, Prop. 2.1] where the $L^2$ metric is for the line bundle $f^* (K_{X/Y})$ which will only count non-multiple singular fibers as in (23).

6. Applications to $L^2$ extension theorems

Let $(X, \psi)$ be a pair where $X$ is a complex manifold and $\psi$ is a psh metric with neat analytic singularities for a $\mathbb{Q}$-line bundle $L$ on $X$. A log canonical center (or an lc center) of $(X, \psi)$ is an irreducible subvariety $Y \subset X$ that is the image of a prime divisor $E$ in a log resolution $X' \rightarrow X$ of the pair, with its discrepancy $a(E, X, \psi)$ equal to $-1$ (i.e. the lowest possible value for an lc pair), cf. [Ko97, Def. 3.3], [K21, Def. 2.3]. We will call such $E$ an lc place of $Y$. When $Y$ is an lc center and there is no other lc center $Y_1$ such that $Y_1 \supset Y$, we call $Y$ a maximal lc center. The maximal lc centers are precisely the irreducible components of the non-klt locus of $(X, \psi)$. (We refer to [K21, Section 2] for more details on this setting.)

6.1. Ohsawa measure.

Now let $Y$ be a maximal lc center of an lc pair $(X, \psi)$ as above. Let $\Psi$ be a quasi-psh function on $X$ with analytic singularities determined by the relation $e^{-\Psi} h = e^{-\psi}$ where $h$ is a smooth hermitian metric of $L$. In this setting, the Ohsawa measure $dV[\Psi]|_Y$ is defined following the original definition of [O01], cf. [D15].

Definition 6.1. [K21, Def. 3.1, Prop. 3.2], cf. [D15, Section 2] Let $dV_X$ be a smooth volume form on $X$. The Ohsawa measure $dV[\Psi]|_Y$ of $\Psi$ on $Y$ (with respect to $dV_X$) is a positive measure $d\mu$ on $Y_{\text{reg}}$ satisfying the following condition: for every $g$, a real-valued compactly supported continuous function on $Y_{\text{reg}}$ and for every $\tilde{g}$, a compactly supported extension of $g$ to $X$, we have the relation

$$(24) \quad \int_{Y_{\text{reg}}} g \, d\mu = \lim_{t \to -\infty} \int_{\{x \in X, t < \Psi(x) < t+1\}} \tilde{g} e^{-\Psi} dV_X.$$ 

Now assume that $Y$ has a unique lc place $E \rightarrow Y$ where $E \subset X'$ is on a log resolution $\mu : X' \rightarrow X$ of $(X, \psi)$. Then the morphism $E \rightarrow Y$ has connected fibers (cf. [K21, Cor. 2.12]). Writing the discrepancy relation as in [K21, (5) in Proof of Prop. 3.2], we have
By [K21, Prop. 3.2], the Ohsawa measure $dV[\Psi]_Y$ is equal to the direct image $(\mu|_E)_*d\nu$ where $d\nu$ is a measure corresponding to a singular volume form with poles along the snc divisor $R := F|_E$.

Following [Ka98], we may assume (by choosing $\mu : X' \to X$) that the restriction $\mu|_E : E \to Y$ factors through a proper modification $\pi : Y' \to Y$ and $f : (E,R) \to (Y',B)$ satisfies Kawamata’s conditions of Definition 4.1 for a suitable reduced snc divisor $B$ on $Y'$. We have the following diagram.

Now applying Theorem 1.2 to $E \to Y'$, we have the following description of the Ohsawa measure in terms of the discriminant divisor $B_R$ supported on $B$ as a consequence of Theorem 1.2.

**Theorem 6.2.** Let $(X, \Psi)$ and $Y$ be as above. The Ohsawa measure $dV[\Psi]_Y$ is equal to the direct image under $\pi : Y' \to Y$ of a measure $v$ on $Y'$ with poles along Kawamata’s discriminant divisor $B_R$ on $Y'$ multiplied by a local psh weight with vanishing Lelong numbers. In other words, $v$ can be locally written as (up to a bounded positive factor)

$$v(w) = \left(\prod_{i=1}^{m} |w_i|^{-2a_i}\right)e^{-\psi(w)}|dw_1 \wedge \ldots \wedge dw_m|^2$$

in local coordinates $w = (w_1, \ldots, w_m)$ adapted to $B$ on an open subset $U \subset Y'$ (with $\dim Y = m$) where $\psi$ is a psh function on $U$ with vanishing Lelong numbers and $\sum a_i \text{div}(w_i) = (B_R)|_U$.

**6.2. Comparison of two $L^2$ extension theorems.**

Using Theorem 6.2, we can compare two $L^2$ extension theorems of [K07, Thm. 4.2] and [D15, Thm. 2.8] (for a smooth complex projective variety $X$). For this comparison, we first derive [K21, Thm. 3.9], a version of $L^2$ extension for a maximal
log canonical center, from [D15, Thm. 2.8]. For the convenience of readers, we recall the statement.

**Theorem 6.3.** [K21, Thm. 3.9] (cf. [D15, Thm. 2.8 and Remark 2.9 (b)])

Let \((X, \omega)\) be a weakly pseudoconvex \(\mathbb{K}\)ähler manifold. Let an lc pair \((X, \psi)\), a \(\mathbb{Q}\)-line bundle \(L\), a psh metric \(\psi\) for \(L\) and a quasi-psh function \(\Psi\) be as in Definition 6.1, so that \(e^{-\psi} = he^{-\Psi}\). Assume that, for some \(\delta > 0\),

\[
(28) \quad \frac{i}{\alpha} \Theta(L, h) + \sqrt{-1} \partial \bar{\partial} \Psi = \frac{i}{\alpha - 1} \Theta(L, \psi) + \sqrt{-1} \partial \bar{\partial} \Psi \geq 0
\]

for all \(\alpha \in [1, 1 + \delta]\). Let \(B\) be a \(\mathbb{Q}\)-line bundle on \(X\) such that \(K_X + L + B\) is a \(\mathbb{Z}\)-line bundle. Let \(b\) be a psh metric of \(B\).

Let \(Y\) be a maximal lc center of \((X, \psi)\) with a unique lc place. If a holomorphic section \(s \in H^0(Y, (K_X + L + B)|_Y)\) satisfies

\[
(29) \quad \int_{Y_{\text{reg}}} |s|_{\omega, h, b}^2 dV[\Psi]_Y < \infty,
\]

then there exists a holomorphic section \(\tilde{s} \in H^0(X, K_X + L + B)\) such that we have \(\tilde{s}|_Y = s\) and moreover

\[
(30) \quad \int_X |\tilde{s}|_{\omega, h, b}^2 \gamma(\delta \Psi)e^{-\Psi} dV_{\omega} \leq \frac{34}{\delta} \int_{Y_{\text{reg}}} |s|_{\omega, h, b}^2 dV[\Psi]_Y.
\]

Here the Ohsawa measure \(dV[\Psi]_Y\) is taken with respect to the smooth volume form \(dV_{\omega}\) and \(\gamma\) is as in [D15, (2.7)].

Let \(\|s\|_1\) be the input norm of [K07, Thm. 4.2] which was given as the \(L^2\) norm for an adjoint line bundle with respect to a Kawamata metric [K07, Def. 3.1]. Let \(\|s\|_2\) be the input norm in [K21, Thm. 3.9] with respect to the Ohsawa measure \(dV[\Psi]_Y\).

From Theorem 6.2, we have the following comparison of two \(L^2\) extension theorems from [K07] and [D15].

**Theorem 6.4** (= Corollary 1.3). Let \(Y\) be a maximal lc center of \((X, \psi)\) with a unique lc place. Let \(B\) be a \(\mathbb{Q}\)-line bundle on \(X\) such that \(K_X + L + B\) is a \(\mathbb{Z}\)-line bundle. Let \(b\) be a psh metric of \(B\). Let \(s\) be a holomorphic section of \(K_Y + M + B|_Y = (K_X + L + B)|_Y\) on \(Y\) (where \(K_Y + M\) is defined as in [K07, Def. 3.1]).

Let \(p \in Y\) be a point. Then we have the following equivalence: the \(L^2\) norm of \(s\) with respect to the Ohsawa measure is locally finite at \(p\), i.e. there exists a neighborhood \(U \subset Y\) of \(p\) such that

\[
\|s\|_2 := \int_U |s|_{\omega, h, b}^2 dV[\Psi]_Y < \infty
\]
if and only if the $L^2$ norm of $s$ with respect to a Kawamata metric $e^{-\kappa}$ is locally finite at $p$, i.e. there exists a neighborhood $V \subset Y$ of $p$ such that

$$\|s\|_1 := \int_V |s|^2 \cdot e^{-\kappa} \cdot b|_Y < \infty.$$  

**Proof.** We will use the notation as in the setting of (26). In particular, we have the relation $K_Y' + M' = \pi^*(K_Y + M)$ where $\pi : Y' \to Y$ is the modification as before.

From the definition of the Kawamata metric $e^{-\kappa}$ [K07, Def. 3.1], locally we can write $e^{-\kappa} = e^{-\varphi}e^{-\eta}$ as a metric for $M'$ on $Y'$ where $e^{-\varphi}$ is a singular hermitian metric given by the discriminant divisor and $e^{-\eta}$ is a choice of a smooth metric. On the other hand, by Theorem 6.2, the Ohsawa measure has density locally equal to the product $e^{-\varphi}e^{-\chi}$ where $\chi$ is a local psh weight with vanishing Lelong numbers. Let $\beta$ be a psh function such that $b = e^{-\beta}$ near $p$. Then by Lemma 6.5 (2), we have the equivalence:

$$\int_{\pi^{-1}(V)} |\pi^* s|^2 e^{-\varphi} e^{-\eta} e^{-\beta} < \infty \text{ if and only if } \int_{\pi^{-1}(U)} |\pi^* s|^2 e^{-\varphi} e^{-\chi} e^{-\beta} < \infty$$

where $U$ and $V$ are as in the statement. This completes the proof. Note that $\varphi$ is locally the difference of two psh functions with analytic singularities, which is why we need (2) of Lemma 6.5.

**Lemma 6.5.** Let $u$ and $v$ be two psh functions on a complex manifold $X$. Suppose that $v$ has zero Lelong numbers at every point of $X$. Then the following hold:

(1) For the multiplier ideals, we have $J(u + v) = J(u)$.

(2) More generally, let $\psi$ be a psh function with analytic singularities. We have $e^{\psi - u} \in L^2_{\text{loc}}$ if and only if $e^{\psi - u - v} \in L^2_{\text{loc}}$.

**Proof.** It is clear that (2) implies (1). (1) is from [K15, Prop. 2.3] and (2) from [B20, Cor. 10.15]. Both of them use the strong openness theorem of [GZ15].

**7. Applications to semipositivity theorems**

In this section, we discuss applications of the main results to semipositivity theorems, Theorem 7.1. We first recall the following series of semipositivity theorems for an algebraic fiber space $f : X \to Y$ (i.e. a surjective morphism of smooth projective varieties with connected fibers) under some general conditions.

(1) [Ka81, Thm. 5] : Nefness of the locally free sheaf $E := f_* K_{X/Y}$. (Cf. [F78, FFS].)

(2) [Ka98, Thm. 2] : Log version of (1) for log Calabi-Yau fibrations $f$. 

Refinement of (1) replacing nefness by the existence of a singular hermitian metric with vanishing Lelong numbers for \( \mathcal{O}_{\mathcal{P}(E)}(1) \).

We showed in Theorem 1.1 that the moduli part line bundle in the canonical bundle formula [Ka98, Thm. 2] for \( f : X \to Y \) admits a psh metric with vanishing Lelong numbers when \( R_h \geq 0 \). In some cases, the moduli part can coincide with the direct image \( f_* (K_X/Y) \) of the relative canonical line bundle of \( f \).

In this way, we obtain as a corollary of Theorem 1.1, an alternative proof of the following Kawamata semipositivity theorem [Ka00, Thm. 1.1 (3)], [Ka81] when \( f_* (K_X/Y) \) is of rank 1, which does not use some heavy machineries from the theory of variation of Hodge structure, cf. [CKS86], [S73].

**Theorem 7.1.** [Ka00, Thm. 1.1 (3)], cf. [FF17, Cor. 1.2] Let \( f : X \to Y \) be a surjective proper morphism with connected fibers between connected complex manifolds. Let \( B \) be an snc divisor on \( Y \) such that \( f \) restricted over \( Y \setminus B \) is a holomorphic submersion. Let \( X_0 := f^{-1}(Y \setminus B) \). Let \( n := \dim X - \dim Y \).

1. Suppose that a general smooth fiber \( F \) satisfies \( K_F \sim 0 \).
2. Suppose that \( R^n f_* C_{X_0} \) has unipotent monodromies around the components of \( B \).

Then the line bundle \( f_* (K_X/Y) \) admits a singular hermitian metric with vanishing Lelong numbers.

The fact that \( f_* (K_X/Y) \) is a line bundle in this setting is due to [Ka81] §4 (see also e.g. [Ko07, (8.4.4)] for related exposition). This uses some basic results from Hodge theory, for which we follow the exposition in [Ko07].

In [Ko07], they are used to define the moduli part line bundle \( J(X/Y, R) \) and show the equality in (8.5.1) of \( \mathbb{Q} \)-line bundles \( K_X + R = f^*(K_Y + J(X/Y, R) + B_R) \), which is nontrivial this time unlike (9) since here \( J(X/Y, R) \) has its own Hodge-theoretic definition unlike (9).

With this Hodge-theoretic characterization of the moduli part line bundle, one knows that it is equal to \( f_* K_X/Y \) as in [Ko07, (8.4.4)] which is used in the following proof.

**Proof.** First observe that, by [Ko07, Lem. 8.3.4], there exists a vertical divisor \( R \) on \( X \) such that \( K_X + R \) is \( \mathbb{Q} \)-linearly equivalent to the pullback by \( f \) of a \( \mathbb{Q} \)-line bundle on \( Y \). Let \( \mu : X' \to X \) be a proper modification (given by composition of blow-ups) such that \( f' : X' \to X \to Y \) satisfies Kawamata’s condition (1.1). Then one can write (for a divisor \( R' \)) that \( K_{X'} + R' = \mu^*(K_X + R) = f'^*(K_Y + J(X'/Y, R') + B_{R'}) \) from [Ko07, Thm. 8.3.7].

As [Ko07] Thm. 8.3.7 (1) states, the moduli part depends only on the generic fibers \( F \) of \( f \) and the pairs on them \((F, R_h|_F)\). In the present case, \( R \) and \( R' \) are vertical
divisors, thus we have $R_h = 0, R'_h = 0$. For a generic fiber $F'$ of $X' \to Y$, we have isomorphism $F' \to F$ and in view of the pairs $(F, R_h|_F) = (F, 0), (F', R'_h|_{F'}) = (F', 0)$, we have

\begin{equation}
J(X'/Y, R') = J(X/Y, R)
\end{equation}

on $Y$ where both sides are defined in [Ko07, (8.4.6)].

Now considering the open set $X'_0 \subset X'$ that is isomorphic to $X_0$ under the proper modification $X' \to X$ (indeed $X' \to X$ can be taken to satisfy this), we have $R^nf_*C_{X_0} = R^nf'_*C_{X'_0}$. Since the unipotent monodromies condition is satisfied for both of them, from [Ko07] (8.4.6)], we have $J(X'/Y, R') = f'_*(K_{X'/Y})$ and $J(X/Y, R) = f_*(K_{X/Y})$ respectively. Combining with (31), we have $f_*(K_{X/Y}) \cong J(X'/Y, R')$. Since $J(X'/Y, R')$ admits a psh metric with vanishing Lelong numbers $\lambda$ by Theorem 1.1, so does $f_*(K_{X/Y})$ by taking the image (call it $\lambda'$) of $\lambda$ under the above isomorphism. Both $\lambda'$ and $\lambda$ are the $L^2$ metrics. ∎

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