Comparison principle for stochastic heat equations driven by $\alpha$-stable white noises

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Abstract

For a class of non-linear stochastic heat equations driven by $\alpha$-stable white noises for $\alpha \in (1, 2)$ with Lipschitz coefficients, we first show the existence and pathwise uniqueness of $L^p$-valued càdlàg solutions to such an equation for $p \in (\alpha, 2]$ by considering a sequence of approximating stochastic heat equations driven by truncated $\alpha$-stable white noises obtained by removing the big jumps from the original $\alpha$-stable white noises. If the $\alpha$-stable white noise is spectrally one-sided, under additional monotonicity assumption on noise coefficients, we prove a comparison theorem on the $L^2$-valued càdlàg solutions of such an equation. As a consequence, the non-negativity of the $L^2$-valued càdlàg solution is established for the above stochastic heat equation with non-negative initial function.

Keywords: Stochastic heat equations; $\alpha$-stable white noises; truncated $\alpha$-stable white noises; comparison principle; non-negative solutions.

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1 Introduction

In this paper we study the comparison principle for a class of stochastic heat equations. More precisely, we want to show that if both the initial functions and the drift coefficients are ordered, then the solutions of the following non-linear stochastic heat equation

$$
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} &= \frac{1}{2} \frac{\partial^2 u(t, x)}{\partial x^2} + f(t, x, u(t, x)) \\
&\quad + \varphi(t-, x, u(t-, x)) \dot{L}_\alpha(t, x), \\
u(0, x) &= u_0(x), \\
u(t, 0) &= u(t, L) = 0,
\end{aligned}
$$

(1.1)

$\dot{L}_\alpha(t, x)$ is the $\alpha$-stable white noise, $\varphi(t-, x, u(t-, x))$ is an additional right-monotonicity assumption on noise coefficients. The symbol $\varphi(t-, x, u(t-, x))$ indicates that the drift coefficient $f(t, x, u(t, x))$ is right-monotonic.

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are also ordered. In equation (1.1), $T > 0, L > 0$ are arbitrary fixed constants, $\hat{L}_\alpha \equiv \{\hat{L}_\alpha(t, x) : (t, x) \in [0, T] \times [0, L]\}$ denotes an $\alpha$-stable white noise on $[0, T] \times [0, L]$ with $\alpha \in (1, 2)$, the initial function $u_0$ can be random, functions $f : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R}$ and $\varphi : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R}$ are the drift coefficient and the noise coefficient, respectively.

For the main results in this paper we need the following hypothesis:

**Hypothesis 1.1.** (i) Functions $f(t, x, \hat{u})$ and $\varphi(t, x, \hat{u})$ in equation (1.1) are globally Lipschitz continuous in $\hat{u}$, that is, there exists a constant $C$ such that given any $(t, x) \in [0, \infty) \times [0, L]$,

$$|f(t, x, \hat{u}) - f(t, x, \hat{v})| + |\varphi(t, x, \hat{u}) - \varphi(t, x, \hat{v})| \leq C|\hat{u} - \hat{v}|$$

for all $\hat{u}, \hat{v} \in \mathbb{R}$;

(ii) $\varphi : \hat{u} \in \mathbb{R} \mapsto \varphi(t, x, \hat{u}) \in \mathbb{R}$ is non-decreasing for all $(t, x) \in [0, \infty) \times [0, L]$.

The comparison principle for stochastic partial differential equations (SPDEs for short) driven by Gaussian (continuous) type noises has been extensively studied, see, for example, Chen and Kim [5], Donati-Martin and Pardoux [9] and Moreno Flores [13] for Gaussian space-time white noises; Chen and Huang [6] and Xiong and Yang [24] for Gaussian colored noises that is white in time and colored in space; Denis et al. [8] and Kotelenez [12] for cylindrical Brownian motions and references therein.

Although the existence and uniqueness of solutions to SPDEs driven by Lévy (discontinuous) type noises have been well studied, see, for example, Albeverio et al. [1] for Poisson white noise; Bo and Wang [4], Peszat and Zabczyk [18] and Truman and Wu [20] for Lévy space-time white noise; Peszat and Zabczyk [19] for infinite dimensional Lévy processes; Wang et al. [22] for truncated $\alpha$-stable white noises; Balan [3], Mytnik [16] and Yang and Zhou [25] for $\alpha$-stable white noises; Xiong and Yang [23] for $\alpha$-stable colored noises that is white in time and colored in space and references therein, there are few investigations on comparison principle for SPDEs driven by Lévy noises.

To the best of our knowledge, Niu and Xie [17] first prove a comparison principle for random field solutions of stochastic heat equations driven by Lévy space-time white noises, and show the non-negativity of the solution for non-negative initial function. The approach for showing the comparison principle in [17] is first introducing a sequence of symmetric mollifiers and a sequence of bounded linear operators to smooth the Lévy space-time white noise and to approximate the Laplace operator, respectively, so that one can construct a sequence of approximating stochastic differential equations with semi-martingale solutions, driven by the mollified Lévy processes, and then show that the random field solutions of these approximating stochastic differential equations indeed converge to the random field solution of the original stochastic heat equation. Consequently, the comparison principle for the original stochastic heat equation follows by the comparison principle for these approximating stochastic differential equations applying Itô’s formula.

For a comparison principle on function-valued solutions to stochastic integro-differential equations driven by Lévy processes, we refer to Dareiotis and Gyöngy [7] and references therein. Note that the above two studies are both based on a crucial hypothesis that the noises are square integrable that excludes the case of $\alpha$-stable noises or general heavy-tailed noises, which motivates our consideration in the present paper.

The main contribution of this paper is to study the existence and pathwise uniqueness of function-valued solutions to equation (1.1) and the associated comparison principle. We first show that there exists a pathwise unique strong $L^p([0, L])$-valued càdlàg solution to
equation (1.1) for \( p \in (\alpha, 2] \) by constructing a sequence of approximating stochastic heat equations driven by truncated \( \alpha \)-stable white noises. More precisely, the approach is to first solve an equation of the form (1.1) driven by truncated \( \alpha \)-stable white noises obtained by removing all the big jumps of size exceeding a fixed value \( K \) from \( \dot{L}_\alpha \), which results in a pathwise unique strong \( L^p([0, L]) \)-valued càdlàg solution \( u^K_t \) for \( p \in (\alpha, 2] \), and then show that any two solutions \( u^M_t \) and \( u^K_t \) are consistent. Such a localization method is similar to that in Balan [3] for showing the existence of random field solutions and to that in Peszat and Zabczyk [18] for showing the existence of weak Hilbert-space-valued solutions to SPDEs driven by \( \alpha \)-stable white noises.

We then show the comparison principle for \( L^2([0, L]) \)-valued (Hilbert-space-valued) càdlàg solutions to equation (1.1) with ordered initial functions and drift coefficients using the method of convolution approximation. The key of this approach is to take convolution of the equations driven by the truncated \( \alpha \)-stable white noise with the fundamental solution of heat equation with homogeneous Dirichlet boundary condition to obtain a series of real value semi-martingale so that one can use the classical Itô formula to obtain the associated comparison principle. As a consequence, the comparison principle for equation (1.1) then follows from the localization method mentioned earlier. The idea of convolution approximation is also applied in [24] in showing a comparison principle for stochastic heat equations with non-homogeneous boundary conditions driven by Gaussian colored noises that are white in time and colored in space. Compared to [24], an additional monotonicity assumption on the noise coefficient \( \varphi \) and more technical issues arise in our setting because of the discontinuous \( \alpha \)-stable white noises with \( \alpha \in (1, 2) \) in equation (1.1).

Applying the comparison principle, we also show that for non-negative initial function there exists a non-negative strong \( L^2([0, L]) \)-valued càdlàg solution to equation (1.1) under the hypothesis.

Our comparison principle respectively generalizes those in [9] and [17] in some sense. However, our comparison principle is on function-valued càdlàg solutions, while the random field solutions are considered in [9] and [17].

The rest of this paper is organized as follows. In the next section, some notation and main results of the existence and pathwise uniqueness of solutions, the comparison principle, and the non-negativity of the solution to equation (1.1) are stated. Section 3 contains proofs of the existence and pathwise uniqueness of strong \( L^p([0, L]) \)-valued càdlàg solutions to equation (1.1) for \( p \in (\alpha, 2] \). Section 4 contains proofs of the comparison principle on the strong \( L^2([0, L]) \)-valued càdlàg solutions to equation (1.1). In addition, a method based on the Itô formula for infinite dimensional semi-martingales for proving the comparison principle is briefly summarized at the end of this section.

2 Notation and main results

In this section we review some basic facts about \( \alpha \)-stable white noises and stochastic heat equations mainly to state the notation, and then list the main contributions of this paper.

2.1 Notation

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. Let \((U_n)\) be a disjoint partition of \( \mathbb{R} \setminus \{0\} \) such that \( \nu_\alpha(U_n) < \infty \) for
each \( n \), where \( \nu, \alpha \in (1,2) \) is the so-called Lévy measure given by

\[
\nu_n(dz) = (c_+ z^{-\alpha-1} \mathbf{1}_{(0,\infty)}(z) + c_- (z)^{-\alpha-1} \mathbf{1}_{(-\infty,0)}(z))dz
\]

with \( c_+ + c_- = 1 \). Given \( T, L > 0 \), let \((\xi^n_j), (x^n_j), (z^n_j)\) be independent random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) taking values in \([0,T], [0,L], \) and \( \mathbb{R} \setminus \{0\} \), respectively, with their distributions specified by

\[
\mathbb{P}[\xi^n_j > t] = \exp(-Lt\nu_n(U_n)), \quad t \in [0,T],
\]

\[
\mathbb{P}[x^n_j \in A] = \frac{|A \cap [0,L]|}{L}, \quad A \in \mathcal{B}([0,L]),
\]

and

\[
\mathbb{P}[z^n_j \in B] = \frac{\nu_n(B \cap U_n)}{\nu_n(U_n)}, \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}),
\]

respectively, where \( |\cdot| \) denotes the Lebesgue measure and \( \mathcal{B}(\cdot) \) denotes the Borel \( \sigma \)-field.

For any \( n,j \geq 1 \) set \( \tau^n_j = \xi^n_1 + \ldots + \xi^n_j \), then

\[
N(dt,dx,dz) := \sum_{n,j \geq 1} \delta_{(\tau^n_j,x^n_j,z^n_j)}(dt,dx,dz)
\]

(2.2)
is a Poisson random measure with the intensity measure \( dtdx\nu_n(dz) \), where \( \delta \) denotes the Dirac delta distribution. Given \( T > 0 \), let \( \{L_\alpha(t,dx), t \in [0,T]\} \) be a sign-measure valued process formally defined by

\[
L_\alpha(t,dx) := \sum_{\tau^n_\leq t} z^n_\delta_{\tau^n_j}(dx) - tdx \int_{\mathbb{R} \setminus \{0\}} z\nu_n(dz).
\]

By Peszat and Zabczyk [19, Example 7.26], \( L_\alpha(t,dx) \) is an \( \alpha \)-stable martingale measure and the corresponding distribution-valued derivative \( \{L_\alpha(t,x) : t \in [0,T], x \in [0,L]\} \) is an \( \alpha \)-stable white noise. Moreover, by (2.2),

\[
L_\alpha(dt,dx) = \int_{\mathbb{R} \setminus \{0\}} z(N(dt,dx,dz) - dtdx\nu_n(dz))
\]

\[
= \int_{\mathbb{R} \setminus \{0\}} z\tilde{N}(dt,dx,dz),
\]

(2.3)
where \( \tilde{N}(dt,dx,dz) = N(dt,dx,dz) - dtdx\nu_n(dz) \) is the so-called compensated Poisson random measure.

Let \( \mathcal{G}^\alpha \) be the class of almost surely \( \alpha \)-integrable functions defined by

\[
\mathcal{G}^\alpha := \left\{ g \in \mathcal{B} : \int_0^t ds \int_0^L |g(s,x)|^\alpha dx < \infty, \mathbb{P}\text{-a.s. for all } t \in [0,T] \right\},
\]

where \( \mathcal{B} \) is the space of progressively measurable functions on \([0,T] \times [0,L] \times \Omega \). By [16, Section 5], the stochastic integral with respect to \( \{L_\alpha(ds,dx)\} \) is well defined for all \( g \in \mathcal{G}^\alpha \).

Given \( T > 0 \) and \( p \geq 1 \), we denote by \( h_t \equiv \{h(t,\cdot), t \in [0,T]\} \) the \( L^p([0,L]) \)-valued process equipped with norm

\[
||h_t||_p := \left( \int_0^L |h(t,x)|^p dx \right)^{\frac{1}{p}}.
\]
In particular, in the case of \( p = 2 \), we write \( H = L^2([0, L]) \) for the Hilbert space with norm \( ||·||_H = ||·||_2 \).

Let \( B([0, L]) \) be the space of all Borel functions on \([0, L]\). For any \( b, c \in B([0, L]) \) define

\[
\langle b, c \rangle := \int_0^L b(x)c(x)dx
\]

if it exists. Let \( C([0, L]) \) be the space of all continuous functions on \([0, L]\), and let \( C^n([0, L]) \) be the subset of \( C([0, L]) \) with the bounded continuous derivatives up to the order \( n \geq 1 \).

Throughout this paper, \( C \) denotes an arbitrary positive constant whose value might vary from line to line. If \( C \) depends on some parameters such as \( p \) and \( T \), we denote it by \( C_{p,T} \).

Let \( G_t(x, y) \) be the fundamental solution of the following heat equation with homogeneous Dirichlet boundary conditions:

\[
\begin{aligned}
\frac{\partial G_t(x, y)}{\partial t} &= \frac{1}{2} \frac{\partial^2 G_t(x, y)}{\partial x^2}, \quad t \in (0, T], \ x, y \in (0, L), \\
\lim_{t\downarrow 0} G_t(x, y) &= \delta_y(x), \quad x, y \in [0, L], \\
G_t(x, 0) &= G_t(x, L) = 0, \quad t \in [0, T], \ x \in [0, L].
\end{aligned}
\]  

(2.4)

Its explicit formula (see, e.g., Feller [10, Page 341]) is specified by

\[
G_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \exp\left(-\frac{(2kL + x - y)^2}{2t}\right) = \exp\left(-\frac{(2kL - x - y)^2}{2t}\right)
\]

for \( t \in (0, T], x, y \in [0, L] \). Moreover, it holds that for any \( s, t \in [0, T] \) and \( x, y, z \in [0, L] \),

\[
\int_0^L G_t(x, y)dy \leq C_T,
\]  

(2.5)

\[
\int_0^L G_s(x, y)G_t(y, z)dy = G_{t+s}(x, z),
\]  

(2.6)

\[
\int_0^L |G_t(x, y)|^pdy \leq Ct^{-\frac{p}{p-1}}, \ p \geq 1,
\]  

(2.7)

and by setting \( G_t^x = G_t(x, \cdot) \) and \( G_t^y = G_t(\cdot, y) \), it holds that for any function \( h \in L^p([0, L]), p \geq 1 \),

\[
\lim_{t \downarrow 0} ||\langle h, G_t^x \rangle - h||_p^p = 0, \ \lim_{t \downarrow 0} ||\langle h, G_t^y \rangle - h||_p^p = 0, \ x, y \in [0, L].
\]  

(2.8)

### 2.2 Main results

Stochastic heat equation (1.1) is a formal SPDE. Given \( T, L > 0 \), by a solution to equation (1.1) we mean a process \( u_t \equiv \{u(t, \cdot), t \in [0, T]\} \) taking values from measurable functions on \([0, L]\), adapting to the filtration generated by \( L_\alpha \) and satisfying the following weak (variational) form equation:

\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \langle u_s, \phi'' \rangle ds + \int_0^t \langle f(s, \cdot, u_s), \phi \rangle ds \\
+ \int_0^{t+} \int_0^L \varphi(s-, y, u(s-, x))\phi(x)L_\alpha(ds, dx)
\]
The proof of Theorem 2.2 is deferred to Section 3 below.

For drift coefficients, we consider

\[ E \] 

strong\(H\) to the filtrations generated by the truncated \(\alpha\)

and exists a sequence of stopping times \(K\) that for any

\(E\) the following mild form equation:

\[ \text{for all } t \in [0,T] \text{ and for any } \phi \in C^2([0,L]) \text{ with } \phi(0) = \phi(L) = \phi'(0) = \phi'(L) = 0 \] 

or equivalently satisfying the following mild form equation:

\[ u(t, x) = \int_0^L G_t(x, y)u_0(y)dy + \int_0^t \int_0^L G_{t-s}(x, y)f(s, y, u(s, y))dsdy \]

\[ + \int_0^t \int_0^L G_{t-s}(x, y)\varphi(s-, y, u(s-, y))L_\alpha(ds, dy) \]

\[ \text{for all } t \in [0,T] \text{ and for a.e. } x \in [0,L], \text{ where the second equality in (2.9) or (2.10) follows} \]

from (2.3). For the equivalence between the weak form (2.9) and mild form (2.10), we refer to Walsh \([21]\) and references therein.

For the equivalence between the weak form (2.9) and mild form (2.10), we refer to Walsh \([21]\) and references therein.

The definition of a strong \(L^p([0,L])\)-valued solution to equation (1.1) for some \(p \geq 1\) is given as follows.

**Definition 2.1.** Stochastic heat equation (1.1) has a strong \(L^p([0,L])\)-valued càdlàg solution with initial function \(u_0\) for some \(p \geq 1\) if for a given \(\alpha\)-stable martingale measure \(L_\alpha\) there exists an \(L^p([0,L])\)-valued càdlàg process \(u_t \equiv \{u(t, \cdot), t \in [0,T]\}\) adapted to the filtration generated by \(L_\alpha\) such that either equation (2.9) or equation (2.10) holds.

We now state the main results in this paper. The existence and pathwise uniqueness of strong \(L^p([0,L])\)-valued solutions to equation (1.1) for \(p \in (\alpha, 2]\) is first given by the following theorem.

**Theorem 2.2.** (Existence and uniqueness) Given \(T > 0\), if the initial function \(u_0\) satisfies \(\mathbb{E}[||u_0||_{p}] < \infty\) for some \(p \in (\alpha, 2]\), then under Hypothesis 1.1 (i) there exists a pathwise unique strong \(L^p([0,L])\)-valued càdlàg solution \(u_t \equiv \{u(t, \cdot), t \in [0,T]\}\) to equation (1.1) and exists a sequence of stopping times \((R_K)_{K \geq 1}\) with \(R_K \uparrow +\infty\) \(\mathbb{P}\)-a.s. as \(K \uparrow +\infty\) such that for any \(K \geq 1\)

\[ \sup_{0 \leq t \leq T} \mathbb{E} [||u_t 1_{t < R_K}||_{p}] < \infty. \]  

(2.11)

The sequence of stopping times \((R_K)_{K \geq 1}\) in Theorem 2.2 is defined in (3.1) with respect to the filtrations generated by the truncated \(\alpha\)-stable white noise \((L^K_\alpha)_{K \geq 1}\) defined in (3.3). The proof of Theorem 2.2 is deferred to Section 3 below.

To show the comparison theorem for equation (1.1) with different initial functions and drift coefficients, we consider \(H\)-valued solutions to equation (1.1). For this, we further assume that \(\mathbb{E}[||u_0||^2_H] < \infty\). By Theorem 2.2 with \(p = 2\), there exists a pathwise unique strong \(H\)-valued càdlàg solution \(u_t \equiv \{u(t, \cdot), t \in [0,T]\}\) satisfying

\[ \sup_{0 \leq t \leq T} \mathbb{E} [||u_t 1_{t < R_K}||^2_H] < \infty. \]
To present the comparison principle, we further consider the following non-linear stochastic heat equation driven by the same \( \alpha \)-stable white noise \( L_\alpha \) with the same noise coefficient \( \varphi \) as in equation (1.1), that is,

\[
\begin{aligned}
\frac{\partial v(t, x)}{\partial t} &= \frac{1}{2} \frac{\partial^2 v(t, x)}{\partial x^2} + g(t, x, v(t, x)) \\
&\quad + \varphi(t-, x, v(t-, x)) \dot{L}_\alpha(t, x), \\
v(0, x) &= v_0(x), \\
v(t, 0) &= v(t, L) = 0,
\end{aligned}
\tag{2.12}
\]

where the initial function \( v_0 \) is different from \( u_0 \) in equation (1.1) and satisfies \( \mathbb{E}[|v_0|^2_H] < \infty \), the drift coefficient \( g : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R} \) and noise coefficient \( \varphi : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R} \) also satisfy Hypothesis 1.1. Then by Theorem 2.2 with \( p = 2 \), there exists a pathwise unique strong \( H \)-valued càdlàg solution \( v_t \equiv \{v(t, \cdot), t \in [0, T]\} \) to equation (2.12) such that for any \( K \geq 1 \)

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \|v_t 1_{\{t < R_K\}}\|_{H}^2 \right] < \infty
\]

for the same stopping times \( (R_K)_{K \geq 1} \) as in Theorem 2.2.

We next state the comparison theorem on the strong \( H \)-valued càdlàg solutions \( u_t \) and \( v_t \) to equations (1.1) and (2.12), respectively.

**Theorem 2.3.** (Comparison principle) Suppose that Hypothesis 1.1 holds and \( c_\zeta = 0 \) in (2.1). Let \( u_t \equiv \{u(t, \cdot), t \in [0, T]\} \) and \( v_t \equiv \{v(t, \cdot), t \in [0, T]\} \) be the strong \( H \)-valued càdlàg solutions to equations (1.1) and (2.12), respectively. If \( u_0(x) \leq v_0(x) \) for a.e. \( x \in [0, L] \) and \( f(t, x, y) \leq g(t, x, y) \) for all \( t, x, y \in [0, T] \times [0, L] \times \mathbb{R} \), we have

\[
P[u_t \leq v_t \text{ for all } t \in [0, T]] = 1.
\]

The precise proof of Theorem 2.3 is given in Section 4 below. Based on Theorem 2.3, one can easily obtain that with non-negative initial function in \( H \), there is a non-negative \( H \)-valued càdlàg solution to equation (1.1). More precisely, the following theorem holds.

**Theorem 2.4.** (Non-negative solution) Suppose that Hypothesis 1.1 holds and \( c_\zeta = 0 \) in (2.1). Let \( u_t \equiv \{u(t, \cdot), t \in [0, T]\} \) be the strong \( H \)-valued càdlàg solution to equation (1.1). If \( u_0(x) \geq 0 \) for a.e. \( x \in [0, L] \) and \( f(t, x, 0) = \varphi(t, x, 0) = 0 \) for all \( t, x \in [0, T] \times [0, L] \), we have

\[
P[u_t \geq 0 \text{ for all } t \in [0, T]] = 1.
\]

**Proof.** Let \( u_0(x) \equiv 0 \) for a.e. \( x \in [0, L] \). Then by the assumptions of Theorem 2.4 we have that \( u_t \equiv 0 \) is the unique strong \( H \)-valued càdlàg solution to equation (1.1). Therefore, the non-negativity of the solution \( u_t \) to equation (1.1) for general non-negative initial function \( u_0 \) follows from Theorem 2.3. \( \square \)

**Remark 2.5.** Note that Theorems 2.3 and 2.4 can also be established under Hypothesis 1.1 for which \( \varphi : \tilde{u} \in \mathbb{R} \mapsto \varphi(t, x, \tilde{u}) \in \mathbb{R} \) is non-increasing for all \( t, x \in [0, \infty) \times [0, L] \) and assumption \( c_\zeta = 0 \) in (2.1); see the crucial estimate (4.9) in proof to Theorem 2.3 in Section 4 for more details.
3 Proof of Theorem 2.2

The proof of Theorem 2.2 proceeds in the following three steps. Given $T > 0$, we first construct a sequence of truncated $\alpha$-stable white noises $(\hat{L}^K_\alpha)_{K \geq 1}$ in (3.3) and stopping times $(R_K)_{K \geq 1}$ increasing to infinity in (3.1) with respect to the filtrations generated by $(\hat{L}^K_\alpha)_{K \geq 1}$; see Lemma 3.1. We then construct a sequence of stochastic heat equations driven by the truncated $\alpha$-stable white noises $(\hat{L}^K_\alpha)_{K \geq 1}$ and prove that for any fixed $K \geq 1$ there exists a pathwise unique strong $L^p([0, L])$-valued càdlàg solution $u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\}$ to the stochastic heat equation for $p \in (\alpha, 2]$ by using the Banach fixed point principle; see Proposition 3.2. Finally, we proceed to show the consistency of solutions to equations (3.8) for different $K$, that is, it holds for any $M \geq K \geq 1$ that $u^K_t = u^K_M$ for all $t \in [0, R_K)$; see Lemma 3.4 and prove that $u_t := u^K_t$ is indeed a pathwise unique solution of equation (1.1) for $t \in [0, R_K)$. Therefore, we obtain a pathwise unique solution $u_t \equiv \{u(t, \cdot), t \in [0, T]\}$ of equation (1.1) by letting $K \uparrow \infty$.

We now construct a sequence of truncated $\alpha$-stable white noise $(\hat{L}^K_\alpha)_{K \geq 1}$ and stopping times $(R_K)_{K \geq 1}$. For each $K \geq 1$ and $T > 0$, recall Section 2.1 and define

$$Y(t) := \sum_{\tau^n_j \leq t} z^n_j, \quad t \geq 0,$$

$$R_K := \inf\{t : |Y(t) - Y(t^-)| > K\}, \quad (3.1)$$

$$\nu^K_\alpha(dz) := (c_+ z^{-\alpha-1}1_{[0, K]}(z) + c_-(-z)^{-\alpha-1}1_{[-K, 0]}(z))dz, \quad 1 < \alpha < 2, \quad c_+ + c_- = 1, \quad (3.2)$$

$$L^K_\alpha(t, dx) := \sum_{\tau^n_j \leq t, z^n_j \leq K} z^n_j \delta_{z^n_j}(dx) - t dx \int_{\mathbb{R}\setminus\{0\}} z^n_\alpha(dz), \quad (3.3)$$

and

$$\Gamma_{K,T} := \{\omega \in \Omega : z^n_j \leq K \text{ for all } n, j \text{ for which } \tau^n_j \leq T, x^n_j \in [0, L]\}. \quad (3.4)$$

From the above definitions, it easy to see that $(R_K)_{K \geq 1}$ is a sequence of stopping times with respect to the filtration generated by $L^K_\alpha$ and $R_K \leq R_M$ for any $K \leq M$, and that for each $K \geq 1$ and $T > 0$

$$\{\omega : R_K > T\} = \Gamma_{K,T}, \quad (3.5)$$

and that

$$L^K_\alpha(t, dx)(\omega) = L_\alpha(t, dx)(\omega), \quad \text{on } (t, \omega) \in [0, T] \times \Gamma_{K,T}. \quad (3.6)$$

Similar to [18, Section 19.5.1], one can show that the stopping times sequence $(R_K)_{K \geq 1}$ converges to infinity with probability one as $K$ tends to infinity in the following lemma.

**Lemma 3.1.** For each $K \geq 1$ and $T > 0$, we have

$$\mathbb{P}[R_K > T] = \exp \left(-\frac{TLK^{-\alpha}}{\alpha}\right),$$

and

$$\lim_{K \to +\infty} \mathbb{P}[R_K > T] = 1 \quad \text{and} \quad \lim_{K \to +\infty} R_K = \infty \mathbb{P} - \text{a.s.} \quad (3.7)$$
With the truncated \( \alpha \)-stable white noises \((\hat{L}_K^\alpha)_{K \geq 1}\) in hand, for each fixed \( K \geq 1 \), we construct the approximating stochastic heat equation for equation (1.1) as follows:

\[
\begin{aligned}
\left\{ \frac{\partial u^K(t, x)}{\partial t} = & \frac{1}{2} \frac{\partial^2 u^K(t, x)}{\partial x^2} + f(t, x, u^K(t, x)) \\
& + \varphi(t, -x, u^K(t, -x)) \hat{L}_K^\alpha(t, x), \quad (t, x) \in (0, T) \times (0, L), \\
u^K(0, x) = & u_0(x), \quad x \in [0, L], \\
u^K(t, 0) = & u^K(t, L) = 0, \quad t \in [0, T],
\end{aligned}
\]

(3.8)

where the drift coefficient \( f \), the noise coefficient \( \varphi \) and the initial function \( u_0 \) are the same as in equation (1.1).

Similar to (2.2), for each \( K \geq 1 \), define

\[
N^K(dt, dx, dz) := \sum_{z^n \leq K} \delta_{(\nu^n, x^n, z^n)}(dt, dx, dz).
\]

Then \( N^K(dt, dx, dz) \) is a Poisson random measure with intensity measure \( dt dx \nu^K_\alpha(dz) \), where \( \nu^K_\alpha \) is given by (3.2). By (3.3), one can show that

\[
L^K_\alpha(dt, dx) = \int_{\mathbb{R} \setminus \{0\}} z (N^K(dt, dx, dz) - dt dx \nu^K_\alpha(dz))
\]

\[
= \int_{\mathbb{R} \setminus \{0\}} z \hat{N}^K(dt, dx, dz),
\]

where \( \hat{N}^K(dt, dx, dz) = N^K(dt, dx, dz) - dt dx \nu^K_\alpha(dz) \) is the compensated Poisson random measure corresponding to the truncated \( \alpha \)-stable measure \( L^K_\alpha \).

Given \( K \geq 1 \), by a solution to equation (3.8) we mean a process \( u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\} \) taking values from measurable functions on \([0, L]\], adapting to the filtration generated by \( L^K_\alpha \) and satisfying the following weak form equation:

\[
\langle u^K_t, \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \langle u^K_s, \phi'' \rangle ds + \int_0^t \langle f(s, \cdot, u^K_s), \phi \rangle ds
\]

\[
+ \int_0^{t^+} \int_0^L \int_{\mathbb{R} \setminus \{0\}} \varphi(s, -x, u^K(s, -x)) \phi(x) z \hat{N}^K(ds, dx, dz)
\]

(3.9)

for all \( t \in [0, T] \) and for any \( \phi \in C^2([0, L]) \) with \( \phi(0) = \phi(L) = \phi'(0) = \phi'(L) = 0 \) or equivalently satisfying the following mild form equation:

\[
u^K(t, x) = \int_0^L G_t(x, y) u_0(y) dy + \int_0^t \int_0^L G_{t-s}(x, y) f(s, y, u^K(s, y)) ds dy
\]

\[
+ \int_0^{t^+} \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y) \varphi(s, -y, u^K(s, -y)) z \hat{N}^K(ds, dy, dz)
\]

(3.10)

for all \( t \in [0, T] \) and for a.e. \( x \in [0, L] \).

According to Definition 2.1, stochastic heat equation (3.8) has a strong \( L^p([0, L]) \)-valued càdlàg solution with initial function \( u_0 \) for some \( p \geq 1 \), if for a given truncated \( \alpha \)-stable martingale measure \( L^K_\alpha \) defined by (3.3) there exists a \( L^p([0, L]) \)-valued càdlàg process \( u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\} \) such that either equation (3.9) or equation (3.10) holds.
We now show that there exists a pathwise unique strong $L^p([0, L])$-valued càdlàg solution $u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\}$ to equation (3.8) for $p \in (\alpha, 2]$ by using Banach fixed point principle in the following proposition. The same method was also applied in [20] for showing the existence and uniqueness of solution to stochastic Burgers equation driven by Lévy space-time white noise.

**Proposition 3.2.** Given $K \geq 1$ and $T > 0$, If the initial function $u_0$ satisfies $E[\|u_0\|_p^p] < \infty$ for some $p \in (\alpha, 2]$, then under Hypothesis 1.1 (i) there exists a pathwise unique strong $L^p([0, L])$-valued càdlàg solution $u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\}$ to equation (3.8) and

$$
\sup_{0 \leq t \leq T} E[\|u^K_t\|_p^p] < \infty. \quad (3.11)
$$

**Proof.** Let $\mathcal{H}_T$ be the space of all $L^p([0, L])$-valued and $\mathcal{F}_t$-adapted càdlàg processes $h_t \equiv \{h(t, \cdot), t \in [0, T]\}$. The norm $\| \cdot \|_{\mathcal{H}_T}$ of the space $\mathcal{H}_T$ is defined by

$$
\|h\|_{\mathcal{H}_T} := \left( \sup_{0 \leq t \leq T} E[\|h_t\|_p^p] \right)^{\frac{1}{p}}, \quad p \in (\alpha, 2]. \quad (3.12)
$$

Then under the above norm, $\mathcal{H}_T$ is a Banach space; see, e.g., [4] Page 236].

Let $(J^K)_{K \geq 0}$ be the operators defined by

$$
J^0(h)(t, x) := \int_0^L G_t(x, y) h_0(y) dy,
$$

$$
J^K(h)(t, x) := \int_0^t \int_0^L G_{t-s}(x, y) f(s, y, h(s, y)) ds dy
$$
$$
+ \int_0^{t+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y) \varphi(s-, y, h(s-, y)) z \tilde{N}(ds, dy, dz), \quad K \geq 1,
$$

for all $h \in \mathcal{H}_T$, where $(t, x) \in [0, T] \times [0, L]$ and $h_0$ satisfies that $E[\|h_0\|_p^p] < \infty$.

We first consider the case of $t \in [0, t_1]$, where $0 < t_1 \leq T$ is sufficiently small.

**Step (i):** Prove that for each $h \in \mathcal{H}_{t_1}$, $J^K(h) \in \mathcal{H}_{t_1}, K \geq 0$.

By Jensen’s inequality and (2.5), we have

$$
E[\|J^0(h)(t, \cdot)\|_p^p] = E \left[ \int_0^L \left( \int_0^L G_t(x, y) h_0(y) dy \right)^p dx \right]
$$
$$
\leq E \left[ \int_0^L \left( \int_0^L G_t(x, y) dx \right)^p |h_0(y)|^p dy \right]
$$
$$
\leq C_T E[\|h_0\|_p^p].
$$

Since $E[\|h_0\|_p^p] < \infty$, then $J^0(h) \in \mathcal{H}_{t_1}$.

For $J^K(h)$, $K \geq 1$, the Hölder inequality and the Burkholder-Davis-Gundy inequality imply that

$$
E[\|J^K(h)(t, \cdot)\|_p^p]
$$
$$
\leq C_p \int_0^L E \left[ \int_0^L G_{t-s}(x, y) f(s, y, h(s, y)) ds dy \right]^p dx.
$$
By Hypothesis (3.14), where the third inequality follows from the fact that

\[
\sum_{i=1}^{k} a_i^q \leq \sum_{i=1}^{k} |a_i|^q
\]

for \( a_i \in \mathbb{R}, k \geq 1, \text{ and } q \in (0, 2). \)

By (3.12), it holds for \( p \in (\alpha, 2) \) that

\[
\int_{\mathbb{R} \setminus \{0\}} |z|^p \nu^K_{\alpha}(dz) = c_+ \int_0^K z^{p-\alpha-1} dz + c_- \int_{-K}^0 (-z)^{p-\alpha-1} dz = \frac{K^{p-\alpha}}{p-\alpha}.
\]

By Hypothesis (1.1) (i) and (2.7), there exists a constant \( C_{p,t,a,K} \) such that

\[
\mathbb{E}[||J^K(h)(t, \cdot)||^p_p] \\
\leq C_{p,t,a,K} \int_0^L \mathbb{E} \left[ \left( \int_0^t \int_0^L |G_{t-s}(x,y)|^p (|f(s,y,h(s,y))|^p + |\varphi(s,y,h(s,y))|^p) dy ds \right) \right] dt
\]

for any \( K \geq 1. \) Let \( (\mathcal{J}^K)_{K \geq 1} \) be the operators defined by

\[
\mathcal{J}^K(h)(t,x) := J^K(h)(t,x) + J^K(h)(t,x), \quad K \geq 1,
\]
for all $h \in \mathcal{H}_t$, where $(t, x) \in [0, T] \times [0, L]$.

**Step (ii):** Prove that for each $K \geq 1$ the operator $J^K$ is a contraction on $\mathcal{H}_t$, i.e., for any $h, \tilde{h} \in \mathcal{H}_t$ with $h_0 = \tilde{h}_0$, there exists a constant $\kappa \in (0, 1)$ such that

$$||J^K(h) - J^K(\tilde{h})||_{\mathcal{H}_t} \leq \kappa ||h - \tilde{h}||_{\mathcal{H}_t}.$$ 

Clearly $J^K : \mathcal{H}_t \to \mathcal{H}_t$ for each $K \geq 1$. For any $h, \tilde{h} \in \mathcal{H}_t$ with $h_0 = \tilde{h}_0$, we have

$$J^K(h) - J^K(\tilde{h}) = J^K(h) - J^K(\tilde{h}), ~ K \geq 1.$$

The Hölder inequality, the Burkholder-Davis-Gundy inequality, (3.13) - (3.14), Hypothesis 1.1 (i) and (2.7) together imply that

$$||J^K(h) - J^K(\tilde{h})||_{\mathcal{H}_t}^p \leq C_p \sup_{0 \leq t \leq t_1} \int_0^L \mathbb{E}\left[ \int_0^t \int_0^L G_{t-s}(x, y) [f(s, y, h(s, y)) - f(s, y, \tilde{h}(s, y))] ds dy \right]^{p/2} \right]^{1/2} dx$$

$$+ C_p \sup_{0 \leq t \leq t_1} \int_0^L \mathbb{E}\left[ \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y) \times [\varphi(s-, y, h(s-, y)) - \varphi(s-, y, \tilde{h}(s-, y))] z N(ds, dy, dz) \right]^{p/2} \right]^{1/2} dx$$

$$\leq C_{p, t_1} \sup_{0 \leq t \leq t_1} \int_0^L \mathbb{E}\left[ \int_0^t \int_0^L |G_{t-s}(x, y)| f(s, y, h(s, y)) - f(s, y, \tilde{h}(s, y)) \right]^{p/2} ds dy \right]^{1/2} dx$$

$$+ C_{p, t_1, \alpha, K} \sup_{0 \leq t \leq t_1} \int_0^L \mathbb{E}\left[ \int_0^t \int_0^L |G_{t-s}(x, y)| \varphi(s, y, h(s, y)) - \varphi(s, y, \tilde{h}(s, y)) \right]^{p/2} ds dy \right]^{1/2} dx$$

$$\leq C_{p, t_1, \alpha, K} \sup_{0 \leq t \leq t_1} \int_0^L \mathbb{E}\left[ |G_{t-s}(x, y)| \mathbb{E}[|h(s, y) - \tilde{h}(s, y)|^p] dy ds \right]^{1/2} dx$$

$$\leq C_{p, t_1, \alpha, K} \sup_{0 \leq t \leq t_1} \int_0^L \left[ t - s \right]^{-\alpha} \mathbb{E}[|h(s, y) - \tilde{h}(s, y)|^p] dy ds$$

$$\leq C_{p, t_1, \alpha, K} \left( \sup_{0 \leq t \leq t_1} \int_0^L \left[ t - s \right]^{-\alpha} ds \right) \||h - \tilde{h}||_{\mathcal{H}_t}^p$$

$$= \frac{2C_{p, t_1, \alpha, K} l_1^{\frac{3-p}{2}}}{3 - p} \||h - \tilde{h}||_{\mathcal{H}_t}^p.$$ 

Since $t_1$ is sufficiently small and $p \in (\alpha, 2]$, then for each fixed $K \geq 1$ we can choose

$$\kappa = \frac{2C_{p, t_1, \alpha, K} l_1^{\frac{3-p}{2}}}{3 - p} \in (0, 1).$$

Based on the results of **Step (i)** and **Step (ii)**, by using the Banach fixed point principle on the set $\{h \in \mathcal{H}_t : h(0, x) = u_0(x), x \in [0, L]\}$ and the right continuity of sample paths, one can conclude that for each $K \geq 1$ the equation (3.8) has a pathwise unique strong $L^p([0, L])$-valued càdlàg solution $u^K_t \in \mathcal{H}_t$ up to time $t_1$ for $p \in (\alpha, 2]$. For $t > t_1$, one can start with the time $t_1$ and find a sufficiently small time interval $\Delta t_1$ so that $t_2 = t_1 + \Delta t_1$. Then similar to the previous proof, one can obtain the solution, still denoted by $u^K_t$, in $\mathcal{H}_t$ with $t \in [0, t_2]$. Therefore, the proof of existence and pathwise uniqueness of solutions is completed via the above successive procedure, and the moment estimate (3.11) follows from (3.12). \qed
Lemma 3.4. Under the assumptions in Proposition 3.2, for any $1 \leq K \leq M$ we have
\[ u_t^K = u_t^M, \quad \mathbb{P}\text{-a.s. on } t \in [0, R_K). \]

Proof. It suffices to consider the case of $R_K < T$. By (3.10),
\[
(u^K(t, x) - u^M(t, x))1_{\{t<R_K\}}
= 1_{\{t<R_K\}} \int_0^t \int_0^L G_{t-s}(x, y)[f(s, y, u^K(s, y)) - f(s, y, u^M(s, y))]1_{\{s<R_K\}} ds dy
+ 1_{\{t<R_K\}} \int_0^{t+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)
\times [\varphi(s-, y, u^K(s-, y)) - \varphi(s-, y, u^M(s-, y))] z 1_{\{s<R_K\}} \tilde{N}^K(ds, dy, dz).
\]

Let us define
\[
U_t := \mathbb{E}\left[\left\|(u^K_t - u^M_t)1_{\{t<R_K\}}\right\|_p^p\right], \quad t \in [0, T].
\]

Then by the Hölder inequality, the Burkholder-Davis-Gundy inequality, (3.13)-(3.14) and Hypothesis 1.1 (i), we have
\[
U(t) \leq C_{p,T,\alpha,K} \int_0^t \int_0^L \int_0^L |G_{t-s}(x, y)|^p \mathbb{E}\left[\left\|(u^K(s, y) - u^M(s, y))1_{\{s<R_K\}}\right\|_p^p\right] dy ds dx
\leq C_{p,T,\alpha,K} \int_0^t |t-s|^{\frac{p-1}{2}} \mathbb{E}\left[\left\|(u^K_t - u^M_t)1_{\{t<R_K\}}\right\|_p^p\right] ds
= C_{p,T,\alpha,K} \int_0^t |t-s|^{\frac{p-1}{2}} U(s) ds.
\]

Therefore, the generalized Gronwall Lemma (see, e.g., Lin [13, Theorem 1.2]) implies that $U(t) \equiv 0$, which completes the proof.

Remark 3.3. By Hypothesis 1.1 (i) and estimate (3.14), the (stochastic) integrals on the right-hand side of (3.10) are well defined.

From Proposition 3.2, we know that, for each $K \geq 1$, there exists a pathwise unique strong $L^p([0, L])$-valued càdlàg solution $u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\}$ to equation (3.8) for $p \in (\alpha, 2]$. To find an $L^p([0, L])$-valued càdlàg solution of equation (1.1) using solutions $(u^K_t)_{K \geq 1}$, we first prove the consistency of the solutions $(u^K_t)_{K \geq 1}$ in the following lemma.

Lemma 3.4. Under the assumptions in Proposition 3.2, for any $1 \leq K \leq M$ we have
\[ u_t^K = u_t^M, \quad \mathbb{P}\text{-a.s. on } t \in [0, R_K). \]

Proof of Theorem 2.2. We first prove that the process $u_t \equiv \{u(t, \cdot), t \in [0, T]\}$ defined by
\[ u(t, \cdot, \omega) := u^K(t, \cdot, \omega), \quad \omega \in \Omega^*_t, t < R_K, \]
is a strong $L^p([0, L])$-valued solution of equation (1.1) for $p \in (\alpha, 2]$. By Lemma 3.4, (3.15) is well-defined. From Proposition 3.2, we know that for a given $K \geq 1$ there exists a unique $L^p([0, L])$-valued process $u^K_t$ satisfies equation (3.10) for $p \in (\alpha, 2]$. Therefore, for each $K \geq 1$, it holds by (3.15) that

$$u(t, x)1_{\{t < R_K\}} = u^K(t, x)1_{\{t < R_K\}}$$

$$= 1_{\{t < R_K\}} \int_0^L G_t(x, y)u_0(y)dy$$

$$+ 1_{\{t < R_K\}} \int_0^t \int_0^L G_{t-s}(x, y)f(s, y, u^K(s, y))dsdy$$

$$+ 1_{\{t < R_K\}} \int_0^{t+} \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K(s-, y))z\tilde{N}^K(ds, dy, dz)$$

for all $t \in [0, T]$ and for a.e. $x \in [0, L]$.

By [3.4]-[3.6] and [18] Remark 19.1 or [3] Proposition C1, we have

$$u(t, x)1_{\{t < R_K\}} = 1_{\{t < R_K\}} \int_0^L G_t(x, y)u_0(y)dy$$

$$+ 1_{\{t < R_K\}} \int_0^t \int_0^L G_{t-s}(x, y)f(s, y, u^K(s, y))1_{\{s < R_K\}}dsdy$$

$$+ 1_{\{t < R_K\}} \int_0^{t+} \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K(s-, y))z1_{\{s < R_K\}}\tilde{N}(ds, dy, dz)$$

for all $t \in [0, T]$ and for a.e. $x \in [0, L]$, and the desired existence of solution follows from (3.7) by letting $K \uparrow \infty$.

For the pathwise uniqueness of solutions to equation (1.1), without loss of generality, for each fixed $K \geq 1$ we consider the case of $t < R_K < T$. Suppose that $u_t \equiv \{u(t, \cdot), t \in [0, T]\}$ and $\bar{u}_t \equiv \{\bar{u}(t, \cdot), t \in [0, T]\}$ are two solutions of equation (1.1) with initial functions $u_0$ and $\bar{u}_0$, respectively. If $u_0(x) = \bar{u}_0(x)$ for a.e. $x \in [0, L]$, by (2.10) and (3.4)-(3.6) one can show that

$$1_{\{t < R_K\}}(u(t, x) - \bar{u}(t, x))$$

$$= 1_{\{t < R_K\}} \int_0^t \int_0^L G_{t-s}(x, y)[f(s, y, u(s, y)) - f(s, y, \bar{u}(s, y))1_{\{s < R_K\}}dsdy$$

$$+ 1_{\{t < R_K\}} \int_0^{t+} \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y)$$

$$\times [\varphi(s-, y, u(s-, y)) - \varphi(s-, y, \bar{u}(s-, y))]1_{\{s < R_K\}}\tilde{N}^K(ds, dy, dz).$$

By the Burkholder-Davis-Gundy inequality, (3.13)-(3.14), Hypothesis 1.1 (i) and (2.7), we have

$$\mathbb{E} \left[ \| (u_t - \bar{u}_t)1_{\{t < R_K\}} \|^p \right]$$

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can be regarded as the limit of solutions \((u_t) \geq t\) the pathwise uniqueness holds for all \(t \in [0, R_K]\). The generalized Gronwall Lemma (see, e.g., \[13, \text{Theorem 1.2}\]) implies that

\[
\mathbb{E} \left[ \left| \int_0^t u(s, y) \, ds dydx \right|^p \right] \leq C_p \mathbb{E} \left[ \int_0^t \int_0^t |G_t-s(x, y)|^p u(s, y) - \bar{u}(s, y) |p1_{s<R_K} ds dy dx \right]
+ C_{p,a,K} \mathbb{E} \left[ \int_0^t \int_0^t \int_0^t \int_0^t |G_t-s(x, y)|^p u(s, y) - \bar{u}(s, y) |p1_{s<R_K} ds dy dx \right]
\leq C_{p,a,K} \mathbb{E} \left[ \int_0^t |t-s|^{-\frac{p}{2}} \mathbb{E} \left[ \left| \int_0^t (u_s - \bar{u}_s) 1_{s<R_K} \right|^p \right] ds \right].
\]

The generalized Gronwall Lemma (see, e.g., \[13, \text{Theorem 1.2}\]) implies that

\[
\mathbb{E} \left[ \left| \int_0^t u(s, y) \, ds dydx \right|^p \right] \equiv 0.
\]

Therefore, by the fact that the sample paths (on variable \(t\)) of \(u_t, \bar{u}_t\) are right continuous, the pathwise uniqueness holds for all \(t \in [0, R_K]\). Combining Lemmas \[3.1\] and \[3.4\] the desired result follows by letting \(K \uparrow \infty\). Recalling \(3.15\), it is easy to see that the moment estimate \((2.1)\) follows from \((3.11)\), which completes the proof.

**Remark 3.5.** By Remark \[2.3\], \[3.15\] and Lemma \[3.1\] the (stochastic) integrals on the right-hand side of \((2.1)\) are well defined.

### 4 Proof of Theorem 2.3

Recalling \(3.15\) and the proof of Theorem \[2.2\], we know that the solution \(u_t\) of equation \((1.1)\) can be regarded as the limit of solutions \((u^K_t)_{K \geq 1}\) to equation \((3.8)\). Therefore, to prove the comparison theorem on solutions of equation \((1.1)\), it suffices to prove the comparison theorem on solutions \((u^K_t)_{K \geq 1}\) of equation \((3.8)\). To this end, for any given \(K \geq 1\) and \(T > 0\), similar to equation \((3.8)\), we consider the stochastic heat equation

\[
\begin{cases}
\frac{\partial v^K(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 v^K(t, x)}{\partial x^2} + g(t, x, v^K(t, x)) \\
+ \varphi(t-, x, v^K(t-, x)) \hat{L}^K_{\alpha}(t, x), \quad (t, x) \in (0, T) \times (0, L), \\
v^K(0, x) = v_0(x), \quad x \in [0, L], \\
v^K(t, 0) = v^K(t, L) = 0, \quad t \in [0, T].
\end{cases}
\]

where the initial function \(v_0\) satisfies \(\mathbb{E}[|v_0|^p] < \infty\), and the drift coefficient \(g : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R}\) and noise coefficient \(\varphi : [0, T] \times [0, L] \times \mathbb{R} \to \mathbb{R}\) also satisfy Hypothesis \[1.1\].

Proposition \[3.2\] implies that there exists a pathwise unique strong \(H\)-valued càdlàg solution \(v^K_t \equiv \{v^K(t, \cdot), t \in [0, T]\}\) to equation \((1.1)\) and

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \left| v^K t \right|^p \right] < \infty.
\]

Given \(K \geq 1\), we now present the comparison theorem on \(H\)-valued càdlàg solutions \(u^K_t, v^K_t\) to equations \((3.8)\) and \((4.1)\) in the following proposition.

**Proposition 4.1.** Suppose that Hypothesis \[1.1\] holds and \(c_- = 0\) in \((2.1)\). Given \(K \geq 1\), let \(u^K_t \equiv \{u^K(t, \cdot), t \in [0, T]\}\) and \(v^K_t \equiv \{v^K(t, \cdot), t \in [0, T]\}\) are the strong \(H\)-valued càdlàg solutions to equations \((3.8)\) and \((4.1)\), respectively. If \(u_0(x) \leq v_0(x)\) for a.e. \(x \in [0, L]\) and \(f(t, x, y) \leq g(t, x, y)\) for all \((t, x, y) \in [0, T] \times [0, L] \times \mathbb{R}\), we have

\[
\mathbb{P} \left[ u^K_t \leq v^K_t \text{ for all } t \in [0, T] \right] = 1.
\]
Proof. For any $\epsilon > 0$ and $x, y \in [0, L]$, we know that $G_\epsilon(x, y)$ is the fundamental solution of the heat equation (2.4) satisfying (2.8). Since $u^K_t, v^K_t$ are two strong $H$-valued càdlàg solutions of equations (3.8) and (4.1), respectively, it holds by setting $G^K_\epsilon = G_\epsilon(x, \cdot)$ that

$$
\langle u^K_t, G^K_\epsilon \rangle = \langle u_0, G^K_\epsilon \rangle + \frac{1}{2} \int_0^t \langle \partial^2 \nabla G^K_\epsilon, u^K_s \rangle ds + \int_0^t \langle f(s, \cdot, u^K_s), G^K_\epsilon \rangle ds
$$

$$
+ \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} \varphi(s, y, u^K(s, y)) G_\epsilon(x, y) z \tilde{N}^K(ds, dy, dz)
$$

where the second equality follows from the symmetry on the variables $x, y$ of the function $G_\epsilon(x, y)$. Similarly,

$$
\langle v^K_t, G^K_\epsilon \rangle = \langle v_0, G^K_\epsilon \rangle + \frac{1}{2} \int_0^t \langle \partial^2 \nabla G^K_\epsilon, v^K_s \rangle ds + \int_0^t \langle g(s, \cdot, v^K_s), G^K_\epsilon \rangle ds
$$

$$
+ \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} \varphi(s, y, v^K(s, y)) G_\epsilon(x, y) z \tilde{N}^K(ds, dy, dz).
$$

(4.3)

Setting $w^K_t = u^K_t - v^K_t$ and $w^K_{t, \epsilon}(x) = \langle w^K_t, G^K_\epsilon \rangle$, then it follows from (1.2)-(4.3) that

$$
w^K_{t, \epsilon}(x) = w^K_{0, \epsilon}(x) + \frac{1}{2} \int_0^t \langle \partial^2 \nabla w^K_{t, \epsilon}(x), t \rangle ds + \int_0^t \langle F^K_{t, \epsilon 1}(x) + F^K_{t, \epsilon 2}(x), t \rangle ds
$$

$$
+ \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} H^K_{x, \epsilon}(x, y, z) \tilde{N}^K(ds, dy, dz),
$$

(4.4)

where

$$
F^K_{t, \epsilon 1}(x) := \langle f(s, \cdot, u^K_s) - f(s, \cdot, v^K_s), G^K_\epsilon \rangle,
$$

$$
F^K_{t, \epsilon 2}(x) := \langle f(s, \cdot, v^K_s) - g(s, \cdot, u^K_s), G^K_\epsilon \rangle,
$$

$$
H^K_{x, \epsilon}(x, y, z) := [\varphi(s, y, u^K(s, y)) - \varphi(s, y, v^K(s, y))] G_\epsilon(x, y) z.
$$

For any $\eta > 0$, let $\Psi_\eta : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi_\eta : \mathbb{R} \rightarrow \mathbb{R}$ be two functions defined by

$$
\Psi_\eta(x) := \begin{cases} 0, & x \leq 0, \\ 2\eta x, & x \in [0, \frac{1}{\eta}], \\ 2, & x \geq \frac{1}{\eta}, \end{cases}
$$

and

$$
\psi_\eta(x) := 1_{\{x \geq 0\}} \int_0^y dy \int_0^z dz \Psi_\eta(z),
$$

respectively. Then it follows that
(a) $\psi_\eta$ is a $C^2$-function on $\mathbb{R}$;
(b) $0 \leq \psi_\eta'(x) \leq 2x^+ = 2\max\{x, 0\}$, $0 \leq \psi_\eta''(x) \leq 2$, for all $x \in \mathbb{R}$;
(c) $\lim_{\eta \to \infty} \psi_\eta(x) = (x^+)^2$, $\lim_{\eta \to \infty} \psi_\eta'(x) = 2x^+$, $0 \leq \lim_{\eta \to \infty} \psi_\eta''(x) \leq 2$, for all $x \in \mathbb{R}$.

Since $w_t^{K,\epsilon}(x)$ given by (4.4) is a real valued semi-martingale for each fixed $x \in [0, L]$, then by (3) in the above and Itô’s formula (see, e.g., Applebaum [2, Theorem 4.4.7]),

$$
\psi_\eta(w_t^{K,\epsilon}(x)) = \psi_\eta(w_0^{K,\epsilon}(x)) + \frac{1}{2} \int_0^t \psi_\eta'(w_s^{K,\epsilon}(x)) \frac{\partial^2}{\partial x^2} w_s^{K,\epsilon}(x) ds
+ \int_0^t \psi_\eta'(w_s^{K,\epsilon}(x)) [F_s^{K,\epsilon,1}(x) + F_s^{K,\epsilon,2}(x)] ds
+ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \{\psi_\eta(w_s^{K,\epsilon}(x) + H_s^{K,\epsilon}(x, y, z)) - \psi_\eta(w_s^{K,\epsilon}(x))\} \tilde{N}_s^{K}(ds, dy, dz)
+ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \{\psi_\eta(w_s^{K,\epsilon}(x) + H_s^{K,\epsilon}(x, y, z)) - \psi_\eta(w_s^{K,\epsilon}(x))
- \psi_\eta'(w_s^{K,\epsilon}(x)) H_s^{K,\epsilon}(x, y, z)\} ds dy \nu^K_\alpha(dz).
$$

By stochastic Fubini’s theorem, we get

$$
E \left[ \int_0^L \psi_\eta(w_t^{K,\epsilon}(x)) dx \right]
= E \left[ \int_0^L \psi_\eta(w_0^{K,\epsilon}(x)) dx \right] + \frac{1}{2} E \left[ \int_0^t \int_0^L \psi_\eta'(w_s^{K,\epsilon}(x)) \frac{\partial^2}{\partial x^2} w_s^{K,\epsilon}(x) ds dx \right]
+ E \left[ \int_0^t \int_0^L \psi_\eta'(w_s^{K,\epsilon}(x)) [F_s^{K,\epsilon,1}(x) + F_s^{K,\epsilon,2}(x)] ds dx \right]
+ E \left[ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \{ \int_0^L [\psi_\eta(w_s^{K,\epsilon}(x) + H_s^{K,\epsilon}(x, y, z)) - \psi_\eta(w_s^{K,\epsilon}(x))
- \psi_\eta'(w_s^{K,\epsilon}(x)) H_s^{K,\epsilon}(x, y, z)\} ds dy \nu^K_\alpha(dz) \right].
$$

(4.5)

The assumption $u_0(x) \leq v_0(x)$ for a.e. $x \in [0, L]$ implies that

$$
\int_0^L \psi_\eta(w_0^{K,\epsilon}(x)) dx = 0.
$$

(4.6)

Since $\psi_\eta'(w_s^{K,\epsilon}(0)) = \psi_\eta'(w_s^{K,\epsilon}(L)) = 0$, it holds by the integration-by-parts formula and (3) that

$$
\int_0^L \psi_\eta'(w_s^{K,\epsilon}(x)) \frac{\partial^2}{\partial x^2} w_s^{K,\epsilon}(x) dx = - \int_0^L \psi_\eta''(w_s^{K,\epsilon}(x)) \left| \frac{\partial}{\partial x} w_s^{K,\epsilon}(x) \right|^2 dx \leq 0.
$$

(4.7)

The assumption on $f(t, x, w) \leq g(t, x, w)$ for all $(t, x, w) \in [0, T] \times [0, L] \times \mathbb{R}$ and (3) imply that

$$
\int_0^L \psi_\eta'(w_s^{K,\epsilon}(x)) F_s^{K,\epsilon,2}(x) dx \leq 0.
$$

(4.8)
Therefore, by substituting (4.6)-(4.8) into (1.5), we obtain

\[
E \left[ \int_0^L \psi_\eta(w_t^{K_\epsilon}(x))dx \right]
\leq E \left[ \int_0^t \int_0^L \psi_\eta(w_s^{K_\epsilon}(x))F_s^{K_\epsilon}(x)dxds \right]
+ E \left[ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_0^L [\psi_\eta(w_s^{K_\epsilon}(x) + H_s^{K_\epsilon}(x, y, z)) - \psi_\eta(w_s^{K_\epsilon}(x))
- \psi_\eta(w_s^{K_\epsilon}(x))H_s^{K_\epsilon}(x, y, z)]dy \right\} dsdx\nu_\alpha^K(dz) \right].
\]

Letting \( \eta \uparrow \infty \), it holds by (i) and Hypothesis 1.1 (\( f \) is Lipschitz continuous) that

\[
E \left[ \int_0^L |(w_t^{K_\epsilon}(x))^+|^2dx \right]
\leq CE \left[ \int_0^t \int_0^L (w_s^{K_\epsilon}(x))^+|w_s^{K_\epsilon}(x)|dxds \right]
+ E \left[ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_0^L [(w_s^{K_\epsilon}(x) + H_s^{K_\epsilon}(x, y, z))^+|^2 - |(w_s^{K_\epsilon}(x))^+|^2
- 2(w_s^{K_\epsilon}(x))^+H_s^{K_\epsilon}(x, y, z)]dy \right\} dsdx\nu_\alpha^K(dz) \right].
\]

By Hypothesis 1.1 (\( \varphi \) is non-decreasing) and assumption \( c_- = 0 \) in (2.1),

\[
|(w_s^{K_\epsilon}(x) + H_s^{K_\epsilon}(x, y, z))^+|^2 - |(w_s^{K_\epsilon}(x))^+|^2 - 2(w_s^{K_\epsilon}(x))^+H_s^{K_\epsilon}(x, y, z)
= 1_{\{w_s^{K_\epsilon}(x) \geq 0\}}|H_s^{K_\epsilon}(x, y, z)|^2.
\]

(4.9)

(Note that the above equality also holds under Hypothesis 1.1 for which \( \varphi \) is non-increasing and assumption \( c_+ = 0 \) in (2.1).)

Recalling (2.8) and by letting \( \epsilon \downarrow 0 \) and using Lebesgue’s dominated convergence theorem, we obtain

\[
E \left[ \int_0^L |(w_t^{K}(x))|^2dx \right] \leq CE \left[ \int_0^t \int_0^L (w_s^{K}(x))^+|w_s^{K}(x)|dxds \right]
+ E \left[ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} 1_{\{w_s^{K}(x) \geq 0\}}|H_s^{K}(x, z)|^2dsdx\nu_\alpha^K(dz) \right].
\]

where

\[
w_t^{K}(x) = u^K(t, x) - v^K(t, x),
F_s^{K}(x) = f(s, x, u^K(s, x)) - f(s, x, v^K(s, x)),
H_s^{K}(x, z) = |\varphi(s, x, u^K(s, x)) - \varphi(s, x, v^K(s, x))|z.
\]

By Hypothesis 1.1 (\( \varphi \) is Lipschitz continuous) and (3.14) with \( p = 2 \),

\[
E \left[ \int_0^L |(w_t^{K}(x))|^2dx \right] \leq CE \left[ \int_0^t \int_0^L (w_s^{K}(x))^+|w_s^{K}(x)|dxds \right]
\]

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that is, 
\[ E \left[ \| (w_s^K)^+ \|_H^2 \right] \leq C_{\alpha,K} E \left[ \int_0^t \| (w_s^K)^+ \|_H^2 ds \right]. \]

Applying the Gronwall lemma, one can conclude that 
\[ \sup_{0 \leq t \leq T} E[\| (u_t^K)^+ \|_H^2] = 0. \]

By the right continuity of sample paths (on variable \( t \)) of \( u^K_t \) and \( v^K_t \), we obtain 
\[ P \left[ u^K_t \leq 0 \text{ for all } t \in [0,T] \right] = 1, \]
that is, 
\[ P \left[ u^K_t \leq v^K_t \text{ for all } t \in [0,T] \right] = 1, \]
which completes the proof. \( \square \)

**Proof of Theorem 2.3** Recalling Lemma 3.1 and (3.15), the proof is finished by Proposition 4.1 and by letting \( K \uparrow \infty \). \( \square \)

At the end of this section we outline another approach based on infinite dimensional stochastic differential equations theory to prove Theorem 2.3. This kind of method first appears in [9] in showing a comparison principle for stochastic heat equations driven by Gaussian space-time white noises. The key of this approach is to project the truncated \( \alpha \)-stable white noise \( \tilde{\mathcal{N}}_K \) given in equation (3.8) onto a finite dimensional spatial space to construct a sequence of evolution equations having semi-martingale form so that the Itô formula for infinite dimensional semi-martingales can be applied in showing the comparison principle for the evolution equations.

We now construct a sequence of evolution equations. Let \( \{h_n\}_{n \geq 1} \) be an orthonormal basis of the Hilbert space \( H \) such that \( h_n \in L^\infty([0,L]) \) for all \( n \geq 1 \), that is, 
\[ \sup_{x \in [0,L]} |h_n(x)| \leq C, \text{ for all } n \geq 1. \] (4.10)

Given \( n, K \geq 1 \) and \( T > 0 \), let \( L_n^K \equiv \{L_n^K(t), t \in [0,T]\} \) be a real-valued process defined by 
\[ L_n^K(t) := \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} h_n(x) z \tilde{\mathcal{N}}_K(ds,dx,dz), \text{ } t \in [0,T]. \] (4.11)

Then \( (L_n^K)_{n \geq 1} \) is a family of mutually uncorrelated real-valued truncated \( \alpha \)-stable processes. Given \( m, K \geq 1 \) and \( T > 0 \), let \( Z_m^K \equiv \{Z_m^K(t), t \in [0,T]\} \) be a \( H \)-valued truncated \( \alpha \)-stable process defined by 
\[ Z_m^K(t) := \sum_{n=1}^m L_n^K(t) h_n, \text{ } t \in [0,T]. \] (4.12)
Then we consider a evolution equation of the form

\[
\begin{aligned}
&\left\{ \begin{array}{l}
du^K_m(t) = Au^K_m(t)dt + f(t, \cdot, u^K_m(t))dt + \Sigma(t, \cdot, u^K_m(t))dz^K_m(t), \quad t \in (0, T], \\
u^K_m(0, x) = u_0(x), \\
u^K_m(t, 0) = u^K_m(t, L) = 0,
\end{array} \right.
\end{aligned}
\]

where \(u^K_m(t) \equiv \{u^K_m(t, \cdot), t \in [0, T]\}\), \(A = \frac{1}{2}\partial^2_x\), and \(\Sigma\) is the operator of multiplication by \(\varphi\), that is, for any \(t \in [0, T]\) and \(w \in H\), \(\Sigma(t, \cdot, w) : L^\infty([0, L]) \to H\) is defined by \(\Sigma(t, \cdot, w)(h) := \varphi(t, \cdot, w)h, h \in L^\infty([0, L])\).

By Gyöngy [11, Theorems 2.9-2.10], if the initial function \(u_0\) satisfies \(E[||u_0||_H^2] < \infty\), then under Hypothesis [11] (i) there exists a pathwise unique strong \(H^1_0\)-valued càdlàg solution \(u^K_m(t) \equiv \{u^K_m(t, \cdot), t \in [0, T]\}\) to equation (4.13) satisfying

\[
u^K_m(t, \cdot) = u_0(\cdot) + \int_0^t A\nu^K_m(s, \cdot)ds + \int_0^t f(s, \cdot, \nu^K_m(s, \cdot))ds + \int_0^{t+} \Sigma(s, \cdot, \nu^K_m(s, \cdot))dz^K_m(s),
\]

where \(H^1_0\) is the closure of \(C^\infty([0, L])\) in \(H^1\), and \(H^1\) is the usual Sobolev space consists of all functions in \(H\) for which the first order weak (generalized) derivative belongs to \(H\).

Replacing \(u_0\) and \(f\) in equation (4.13) by \(v_0\) and \(g\), respectively, then there exists a pathwise unique strong \(H^1_0\)-valued càdlàg solution \(v^K_m(t, \cdot) \equiv \{v^K_m(t, \cdot), t \in [0, T]\}\) satisfying

\[
v^K_m(t, \cdot) = v_0(\cdot) + \int_0^t A\nu^K_m(s, \cdot)ds + \int_0^t g(s, \cdot, v^K_m(s, \cdot))ds + \int_0^{t+} \Sigma(s, \cdot, v^K_m(s, \cdot))dz^K_m(s).
\]

Let \(\Phi_\eta : H \to \mathbb{R}\) be an operator defined by \(\Phi_\eta(h) := \int_0^L \psi_\eta(h(x))dx, h \in H\). Then, by [11], \(\Phi_\eta\) is twice Fréchet differentiable. Set \(w^K_m(t, \cdot) := v^K_m(t, \cdot) - v^K_m(t, \cdot)\). By Itô’s formula for infinite dimensional semi-martingale (see, e.g., [14, Theorem 3.7.2]), one can show that

\[
\Phi_\eta(w^K_m(t, \cdot)) = \Phi_\eta(w^K_m(0, \cdot)) + \int_0^t \langle \psi'_\eta(w^K_m(s, \cdot)), A\nu^K_m(s, \cdot) \rangle ds \\
+ \int_0^t \langle \psi'_\eta(w^K_m(s, \cdot)), f(s, \cdot, \nu^K_m(s, \cdot)) - g(s, \cdot, v^K_m(s, \cdot)) \rangle ds \\
+ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \Phi_\eta(\nu^K_m(s, \cdot) + G^K_m(s, \cdot, y, z)) \\
- \Phi_\eta(\nu^K_m(s, \cdot)) \{N^K_m(ds, dy, dz) \\
+ \int_0^t \int_0^L \int_{\mathbb{R}\setminus\{0\}} \Phi_\eta(w^K_m(s, \cdot) + G^K_m(s, \cdot, y, z)) - \Phi_\eta(w^K_m(s, \cdot)) \\
- \langle \psi'_\eta(w^K_m(s, \cdot)), G^K_m(s, \cdot, y, z) \rangle \} ds dy dv^K_m(\cdot)dz.
\]

Here \((\cdot, \cdot)\) is the inner product of \(H\), \((\cdot, \cdot)\) is the duality product between \(H^{-1}\) (dual space of \(H^1_0\)) and \(H^1_0\), and

\[
G^K_m(s, \cdot, y, z) := \sum_{n=1}^m [\varphi(s, \cdot, u^K_m(s, \cdot)) - \varphi(s, \cdot, v^K_m(s, \cdot))]h_n(\cdot)h_n(y)z.
\]

Similar to the estimates in the proof of Theorem [23], one can get

\[
E[||w^K_m(t, \cdot)||_H^2] \leq C_{m, a, k} \int_0^t E[||w^K_m(s, \cdot)||_H^2] ds.
\]

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The Gronwall lemma and the right continuity of the sample path of $w^K_m$ imply that
\[ P\left[u^K_m(t, \cdot) \leq v^K_m(t, \cdot) \text{ for all } t \in [0, T]\right] = 1. \tag{4.15} \]

We now show that for any given $T > 0$
\[ \lim_{m \to \infty} \sup_{0 \leq t \leq T} E[\|u^K(t, \cdot) - u^K_m(t, \cdot)\|_H^2] = 0, \tag{4.16} \]
\[ \lim_{m \to \infty} \sup_{0 \leq t \leq T} E[\|v^K(t, \cdot) - v^K_m(t, \cdot)\|_H^2] = 0, \tag{4.17} \]
where $u^K(t, \cdot)$ and $v^K(t, \cdot)$ are the strong $H$-valued càdlàg solutions of equations (3.8) and (4.11). We just prove (4.16) since (4.17) is similar. By (4.11) and (4.12), the mild form of equation (4.13) is

\[
\begin{align*}
    u^K_m(t, x) &= \int_0^L G_t(x, y)u_0(y)dy + \int_0^t \int_0^L G_{t-s}(x, y)f(s, y, u^K_m(s, y))dsdy \\
    &+ \sum_{n=1}^m \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K_m(s-, y))h_n(y)dyh_n(w)z\tilde{N}^K(ds, dw, dz)
\end{align*}
\]

for all $t \in [0, \infty)$ and a.e. $x \in [0, L]$. By rewriting (3.10) with the form as

\[
\begin{align*}
    u^K(t, x) &= \int_0^L G_t(x, y)u_0(y)dy + \int_0^t \int_0^L G_{t-s}(x, y)f(s, y, u^K(s, y))dsdy \\
    &+ \sum_{n=1}^m \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K(s-, y))h_n(y)dyh_n(w)z\tilde{N}^K(ds, dw, dz) \\
    &+ \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K(s-, y))z\tilde{N}^K(ds, dy, dz) \\
    &- \sum_{n=1}^m \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)\varphi(s-, y, u^K(s-, y))h_n(y)dyh_n(w)z\tilde{N}^K(ds, dw, dz),
\end{align*}
\]

we have
\[
    u^K(t, x) - u^K_m(t, x) = A^K_m(t, x) + B^K_m(t, x) + C^K_m(t, x),
\]
where
\[
\begin{align*}
    A^K_m(t, x) &= \int_0^t \int_0^L G_{t-s}(x, y)(f(s, y, u^K(s, y)) - f(s, y, u^K_m(s, y)))dsdy, \\
    B^K_m(t, x) &= \sum_{n=1}^m \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} \int_0^L G_{t-s}(x, y) \\
    &\times [\varphi(s-, y, u^K(s-, y)) - \varphi(s-, y, u^K_m(s-, y))]h_n(y)dyh_n(w)z\tilde{N}^K(ds, dw, dz), \\
    C^K_m(t, x) &= \int_0^{t^+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} (\xi^{K}_L(s-, y) - \xi^{m}_L(s-, y))z\tilde{N}^K(ds, dy, dz),
\end{align*}
\]
in which
\[
    \xi^{K}_L(s, y) = G_{t-s}(x, y)\varphi(s, y, u^K(s, y)), \tag{4.18}
\]
\[
\xi_{t,x}^{m,K}(s, y) = \sum_{n=0}^{m} \left( \int_{0}^{L} G_{t-s}(x, r) \varphi(s, r, u^{K}(s, r)) h_{n}(r) dr \right) h_{n}(y). \tag{4.19}
\]

By (4.13) - (4.19),
\[
\xi_{t,x}^{m,K}(s, y) = \sum_{n=1}^{m} (\xi_{t,x}^{K}(s, \cdot), h_{n}(\cdot)) h_{n}(y).
\]

For each fixed \( K \geq 1, s \leq t \in [0, T] \) and \( x \in [0, L] \), it holds by Hypothesis (4.11 (i) and (3.11) for \( p = 2 \) that
\[
||\xi_{t,x}^{K}(s, \cdot)||_{H}^{2} \leq C \int_{0}^{L} |G_{t-s}(x, y)|^{2} (1 + |u^{K}(s, y)|^{2}) dy
\]
\[
\leq C \sup_{y \in [0, L]} |G_{t-s}(x, y)|^{2} (L + \sup_{s \in [0, T]} ||u^{K}(s, \cdot)||_{H}^{2}) < \infty.
\]

Therefore, for each fixed \( K \geq 1, s \leq t \in [0, T] \) and \( x \in [0, L] \)
\[
||\xi_{t,x}^{m,K}(s, \cdot)||_{H} \leq ||\xi_{t,x}^{K}(s, \cdot)||_{H} \text{ and } \lim_{m \to \infty} ||\xi_{t,x}^{K}(s, \cdot) - \xi_{t,x}^{m,K}(s, \cdot)||_{H} = 0. \tag{4.20}
\]

We now respectively estimate \( A_{m}^{K}(t, x), B_{m}^{K}(t, x) \) and \( C_{m}^{K}(t, x) \). For \( A_{m}^{K}(t, x) \), it follows from the Hölder inequality and (2.7) that
\[
E[||A_{m}^{K}(t, \cdot)||_{H}^{2}] \leq C E \left[ \int_{0}^{L} \int_{0}^{L} |t-s|^{-\frac{3}{2}} |f(s, y, u^{K}(s, y) - f(s, y, u^{m}(s, y))|^{2} ds dy \right]. \tag{4.21}
\]

For \( B_{m}^{K}(t, x) \), it follows from the uncorrelatedness of \( (L^{K}_{n})_{n \geq 1} \), the Burkholder-Davis-Gundy inequality and (3.14) with \( p = 2 \) that
\[
E[||B_{m}^{K}(t, \cdot)||_{H}^{2}] \leq C_{\alpha,K} \int_{0}^{L} E \left[ \int_{0}^{L+} \int_{0}^{L} \sum_{n=0}^{m} \left( \int_{0}^{L} G_{t-s}(x, y)[\varphi(s-, y, u^{K}(s-, y)) - \varphi(s-, y, u^{m}(s-, y))] h_{n}(y) dy \right)^{2} h_{n}(w)^{2} dw \right] dx.
\]

Note that
\[
\sum_{n=0}^{m} \left( \int_{0}^{L} G_{t-s}(x, y)[\varphi(s-, y, u^{K}(s-, y)) - \varphi(s-, y, u^{m}(s-, y))] h_{n}(y) dy \right)^{2}
\]
\[
\leq ||G_{t-s}(x, \cdot)[\varphi(s-, \cdot, u^{K}(s-, \cdot)) - \varphi(s-, \cdot, u^{m}(s-, \cdot))]| |_{H}^{2}.
\]

Then by (4.10) and (2.7), one can show that
\[
E[||B_{m}^{K}(t, \cdot)||_{H}^{2}] \leq C_{\alpha,K} \int_{0}^{L} E \left[ \int_{0}^{L} ||G_{t-s}(x, \cdot)[\varphi(s-, \cdot, u^{K}(s-, \cdot)) - \varphi(s-, \cdot, u^{m}(s-, \cdot))]| |_{H}^{2} ds \right] dx
\]
\[
\leq C_{\alpha,K} E \left[ \int_{0}^{L} \int_{0}^{L} |t-s|^{-\frac{3}{2}} |\varphi(s, y, u^{K}(s, y)) - \varphi(s, y, u^{m}(s, y))|^{2} dy ds \right]. \tag{4.22}
\]
Given \( m \geq 1 \) and \( K \geq 1 \), let us define \( F^K_m(t) := \mathbb{E} \left[ ||u^K(t, \cdot) - u^K_m(t, \cdot)||^2_H \right], \ t \in [0, T]. \) By (4.21)-(4.22) and Hypothesis 1.1 (i), we have

\[
\mathbb{E}[||A^K_m(t, \cdot) + B^K_m(t, \cdot)||^2_H] \leq C_{p, T, \alpha, K} \int_0^t |t - s|^{-\frac{1}{2}} F^K_m(s) ds. \tag{4.23}
\]

For \( C^K_m(t, x) \), the Burkholder-Davis-Gundy inequality and (3.13)-(3.14) together imply that

\[
\mathbb{E}[||C^K_m(t, \cdot)||^2_H] \leq C_{\alpha, K} \int_0^T \mathbb{E} \left[ \int_0^t ||\xi^K_{t,x}(s, \cdot) - \xi^K_{m,t,x}(s, \cdot)||^2_H ds \right] dx.
\]

By (4.20), Hypothesis 1.1 (i) and (3.11) for \( p = 2 \), we obtain

\[
\mathbb{E}[||C^K_m(t, \cdot)||^2_H] \leq C_{\alpha, K} \int_0^T \mathbb{E} \left[ \int_0^t ||u^K(t, \cdot)||^2_H ds \right] dx \leq C_{\alpha, K} T \int_0^T |T - s|^{-\frac{1}{2}} ds \left( 1 + \sup_{0 \leq t \leq T} \mathbb{E}[||u^K(t, \cdot)||^2_H] \right) < \infty.
\]

Therefore, Lebesgue’s dominated convergence theorem implies that \( \lim_{m \to \infty} \mathbb{E}[||C^K_m(t, \cdot)||^2_H] = 0 \). Combine the estimate (4.23), for arbitrary \( \epsilon > 0 \), there exists a constant \( M \) such that for all \( m \geq M \) we have

\[
F^K_m(t) \leq C_{T, \alpha, K} \int_0^t |t - s|^{-\frac{1}{2}} F^K_m(s) ds + C \epsilon, \ t \in [0, T].
\]

By the generalized Gronwall lemma (see, e.g., [13, Theorem 1.2]) and by letting \( m \uparrow \infty, \epsilon \downarrow 0 \), we obtain

\[
\lim_{m \to \infty} \sup_{0 \leq t \leq T} \mathbb{E}[||u^K(t, \cdot) - u^K_m(t, \cdot)||^2_H] = \lim_{m \to \infty} \sup_{0 \leq t \leq T} F^K_m(t) = 0.
\]

Therefore, for each \( K \geq 1 \) and \( t \in [0, \infty) \), there exists a sub-sequence \( (m_k)_{k \geq 1} \) such that

\[
P \left[ u^K(t, \cdot) = \lim_{k \to \infty} u^K_{m_k}(t, \cdot) \right] = P \left[ v^K(t, \cdot) = \lim_{k \to \infty} v^K_{m_k}(t, \cdot) \right] = 1.
\]

Hence, by (4.15)-(4.17) and the right continuity of sample paths of \( u^K \) and \( v^K \), we obtain

\[
P \left[ u^K(t, \cdot) \leq v^K(t, \cdot) \text{ for all } t \geq 0 \right] = 1.
\]

Recalling (3.15), the proof is finished by letting \( K \uparrow +\infty \).

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