Reply to the Comment on “Negative Landau damping in bilayer graphene”

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In the preceding Comment [1] Svintsov and Ryzhii (SR) criticize the conductivity derived in our Letter using the self-consistent field (SCF) approach [2, 3]

\[ \sigma_{\text{drift}}(\omega, q_x) = (\omega/\bar{\omega})\sigma_{\text{g}}(\bar{\omega}, q_x) . \] (1)

Here, \( \sigma_{\text{g}}(\omega, q_x) \) is the nonlocal conductivity with no drift and \( \bar{\omega} = \omega - q_x v_0 \). The drift effect was modeled by the interaction Hamiltonian \( \hat{H}_{\text{int,drift}} = v_0 \cdot \hat{p} \). \( v_0 = v_0 \hat{x} \) being the drift velocity and \( \hat{p} = -i\hbar \nabla \) [2]. The use of this interaction Hamiltonian was motivated by an analogy with moving media [4]. Equation (1) extends to graphene the well-known result \( \varepsilon_{\text{drift}}(\omega, q_x) = \varepsilon(\omega - q_x v_0, q_x) \) for a drift-biased plasma [5-7] (for 3D materials \( \varepsilon(\omega) = 1 + \sigma / (-i\omega \varepsilon_0) \)).

SR argue that because the electrons in graphene are massless the Galilean Doppler shift cannot be used. They rely on the distribution \( f_{\text{drift}}(k) = f^0(\varepsilon_k^0 - \hbar k \cdot v_0) \), which is applicable when the electron-electron (e-e) scattering predominates [8]. Here, \( \varepsilon_k^0 \) is the energy dispersion of the relevant electronic band and \( f^0(\varepsilon) \) is the Fermi-Dirac distribution. SR use \( f_{\text{drift}}(k) \) in the Lindhard formula. However, they miss a subtle point. In the shifted Fermi distribution \( \hbar k \) is a kinetic momentum rather than a canonical momentum (see Ref. [9, App. H]). The canonical momentum is \( p = \hbar k - eA \).
with $A$ the vector potential due to the static electric field $E_0 = E_0 \hat{x}$. The vector potential is $A(t) = -(t - t_0)E_0$ in the intervals between e-e collisions ($t = t_0$ is the time instant of a collision).

The Lindhard formalism relies on the time evolution of Bloch states ($\psi_{n\kappa}$). The Bloch wave vector $\kappa$ determines the canonical momentum. This means that the relevant distribution for the Lindhard formula is a canonical momentum distribution [10]. It is roughly $\tilde{f}_{\text{drift}}(\kappa) \approx f_{\text{drift}}(\kappa + \hbar^{-1}eA) \approx f^0(\varepsilon_\kappa^0 + e\langle A \rangle \cdot v_\kappa^0 - \hbar \kappa \cdot v_0)$ where $v_0^0 = \hbar^{-1}\partial_\kappa \varepsilon_\kappa^0$. In the second identity, we used a Taylor expansion, replaced $A(t)$ by its time average $\langle A \rangle$, and dropped the term $eA \cdot v_0$ because it is of second order ($-E_0^2$). Moreover, since $E_0 = E_0 \hat{x}$ is space independent the canonical momentum of an electron must be preserved by the static field (it is also preserved by the e-e collisions as on average they are independent of the space coordinates). This implies that $\tilde{f}_{\text{drift}}(\kappa) = f^0(\varepsilon_\kappa^0)$. Thus, when the e-e collisions predominate one must have $e\langle A \rangle \cdot v_\kappa^0 = \hbar \kappa \cdot v_0$.

Substitution of $\tilde{f}_{\text{drift}}(\kappa) = f^0(\varepsilon_\kappa^0)$ in the Lindhard formula yields (taking the band overlap integral $F_{\kappa,k'\kappa} \approx 1$):

$$\sigma_{\text{ne},q}^{\text{drift}} = \frac{i\omega e^2}{q^2 (2\pi)^2} \int d^2 \kappa \frac{f^0(\varepsilon_\kappa^0) - f^0(\varepsilon_{\kappa'k}^0)}{\hbar \omega + \varepsilon_\kappa^0 - \varepsilon_{\kappa'k}^0}.$$  \hfill (2)

Here, $\varepsilon_\kappa^* \approx \langle \psi_{\kappa} | \hat{H}_0(\hat{p} + e\langle A \rangle) | \psi_{\kappa} \rangle \approx \varepsilon_\kappa^0 + (e\langle A \rangle \cdot \hat{\kappa}) \langle \psi_{\kappa} | \hat{\partial}_p \hat{H}_0(\hat{p}) | \psi_{\kappa} \rangle$ is the average electron energy during the interaction with the static field. Combining $\langle \psi_{\kappa} | \hat{\partial}_p \hat{H}_0(\hat{p}) | \psi_{\kappa} \rangle = v_\kappa^0$ and $e\langle A \rangle \cdot v_\kappa^0 = \hbar \kappa \cdot v_0$, it is found that $\varepsilon_\kappa^* \approx \varepsilon_\kappa^0 + \hbar \kappa \cdot v_0$. Note that $\varepsilon_\kappa^* \neq \varepsilon_\kappa^0$ because the electron is accelerated by $E_0$. Substituting $\varepsilon_\kappa^* \approx \varepsilon_\kappa^0 + \hbar \kappa \cdot v_0$ in Eq. (2), we recover Eq. (1) and the Galilean Doppler-shift (see [10] for additional discussion and a derivation with the Boltzmann equation).
Fig. 1. (a) Re\{\sigma^\text{drift}_g\} in the UHP as a function of \(\hbar\omega'^2/E_F\) for \(q_x=1.6k_F\), \(E_F=0.1\text{ eV}\), \(v_0=0.5v_F\), and \(\omega^*=0^+\). The NLD region is shaded in gray. (b) A drift-current biased graphene sheet and a metal half-space are separated by the distance \(d\) with a dielectric; (c) \(\sqrt{\omega^2+\omega^2}=(f'+if'')\) for the unstable mode as a function of \(q_x\).

Regarding the second point raised by SR about the long-wavelength approximation, we underline that the nonlocality neither precludes the negative Landau damping (NLD) nor the emergence of instabilities in graphene platforms. Indeed, the square root singularity of \(\sigma_g\) is compatible with gain regimes because when \(\omega_0 = \omega - q_xv_0\) is negative the pre-factor \(\omega/\omega_0\) of (1) is also negative. Thus, the NLD (Re\{\sigma^\text{drift}_g(\omega,q_x)\} < 0) can occur in the real-frequency axis (Fig. 1a), or in the upper-half frequency plane (UHP), \(\omega^* = \text{Im}\{\omega\} \geq 0\) with \(\omega = \omega^'+i\omega^*\) [10, 11]. The same result is predicted by the collisionless SR, Levitov’s [12] and Polini’s [13] models (see Fig. 1a). Thus, similar to Ref. [2], by coupling the drift-current biased graphene to a resonant system (here a metal half-space) (Fig. 1b) it is possible to spontaneously generate THZ and IR radiation (Fig. 1c). The collisionless models of SR and Levitov predict quantitatively similar unstable regimes; all the models predict solutions with \(f'^* = \omega^*/(2\pi) > 0\) that grow exponentially with time.

In summary, the drift-biased conductivity is ruled by a Galilean transformation when the e-e collisions force the electron gas to move with a constant velocity. The
instabilities predicted in our Letter may be observed in properly designed drift-current biased graphene platforms.

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References

[1] Svintsov, V. Ryzhii, Comment on “Negative Landau damping in bilayer graphene,” arXiv:1812.03764.

[2] T. A. Morgado, M. G. Silveirinha, “Negative Landau damping in bilayer graphene,” Phys. Rev. Lett. 119, 133901 (2017).

[3] T. A. Morgado, M. G. Silveirinha, “Drift-Induced Unidirectional Graphene Plasmons,” ACS Photonics 5(11), pp. 4253-4258 (2018).

[4] M. G. Silveirinha, “Optical instabilities and spontaneous light emission by polarizable moving matter,” Phys. Rev. X 4, 031013 (2014).

[5] A. B. Mikhailovskii, Theory of Plasma Instabilities, vol.1: Instabilities of a Homogeneous Plasma, Springer, 1974.

[6] E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics, Butterworth-Heinemann, 1981.

[7] K. Y. Bliokh, F. J. R. Fortuño, A. Y. Bekshaev, Y. S. Kivshar, and F. Nori, “Electric current induced unidirectional propagation of surface plasmon-polaritons,” Opt. Lett. 43, 963-966 (2018).

[8] V. F. Gantmakher and Y. B. Levinson, Carrier Scattering in Metals and Semiconductors, North-Holland, Amsterdam, 1987, p. 176.

[9] N. W. Ashcroft and N. D. Mermin, Solid State Physics (Brooks/Cole, Thomson Learning, Cornell, 1976).

[10] Attached supplementary information.

[11] Note that a passive medium has \( \text{Re}\{\sigma(\omega, q_x)\} \geq 0 \) in the UHP for any real-valued \( q_x \) [10]. A response with \( \text{Re}\{\sigma(\omega, q_x)\} < 0 \) may lead to power flows emerging from the graphene sheet [10].
[12] D. S. Borgnia, T. V. Phan, L. S. Levitov, “Quasi-Relativistic Doppler Effect and Non-Reciprocal Plasmons in Graphene”, arXiv: 1512.09044

[13] B. V. Duppen, A. Tomadin, A. N. Grigorenko, and M. Polini, “Current-induced birefringent absorption and non-reciprocal plasmons in graphene,” 2D Mater. 3, 015011 (2016).
Supplementary Information

A. Further discussion on the distribution function and Lindhard formula

The Lindhard formalism relies on the time evolution (described with perturbation theory) of the crystal Bloch states ($\psi_{\alpha \kappa}$) under the influence of an electric field. The time evolution of the wave function is ruled by:

$$i\hbar \hat{\psi} = \left[ \hat{H}_0 \left(-i\hbar \nabla + e\mathbf{A}\right) - e\phi \right] \psi.$$  \hspace{1cm} (S1)

Here, $\phi, \mathbf{A}$ are the scalar and the vector potentials. As explained in the main text, the static electric field is modeled by a space independent vector potential. The dynamic (and longitudinal) electric field (with space and time variation of the form $e^{i\epsilon t}e^{-i\omega t}$) is described by $\phi$.

Let us first ignore the dynamic field ($\phi \approx 0$) which is assumed weak compared to the static bias. Then, for a initial state of the form $\psi(\mathbf{r}, t=0) = \psi_{\alpha \kappa}$ (where $\psi_{\alpha \kappa}$ is a Bloch eigenstate of $\hat{H}_0(-i\hbar \nabla, \mathbf{r})$) the time evolution determined by Eq. (S1) preserves the Bloch wave vector (this is so because $\mathbf{A}$ depends exclusively on time). In other words, for any time $t$ the function $\psi(\mathbf{r}, t)e^{-i\mathbf{k} \cdot \mathbf{r}}$ is periodic in space. Thus, the crystal momentum ($\mathbf{k}$) is preserved by the interaction with the static bias. Due to this reason the canonical momentum distribution is simply $\tilde{f}_{\text{drift}}(\mathbf{k}) = f^{0}(\epsilon_{\mathbf{k}}^{0})$, as it is unaffected by the static bias. In contrast, the kinetic momentum distribution is affected by the static bias as $\hbar \mathbf{k} = \hbar \mathbf{k} + e\langle \mathbf{A} \rangle_{\mathbf{k}}$. This leads to a shifted (in the direction of the drift) (kinetic) Fermi distribution $f_{\text{drift}}$. Notice that the time averaged vector potential $\langle \mathbf{A} \rangle_{\mathbf{k}}$ in general may depend on the canonical momentum.

Since the Lindhard formula gives the conductivity in terms of a sum (integral) of the contributions of the electronic states, and since each state is parameterized by the crystal
(canonical) momentum, it is evident that the integral must be over $\hbar \kappa$ rather than over $\kappa$ (kinetic momentum). Thus, the Lindhard formula must be written as in Eq. (2) of the main text, repeated here for convenience:

$$
\sigma_{\omega,q}^{\text{drift}} = \frac{i\omega e^2}{q^2} \left[ \frac{g_z g_v}{(2\pi)^2} \right] \int d^2 \kappa \frac{f_{\text{drift}}(\kappa) - f_{\text{drift}}(\kappa + q)}{\hbar \omega + \mathcal{E}_k - \mathcal{E}_{k+q}},
$$

(S2)

where $\mathcal{E}_k \approx \mathcal{E}_k^0 + \hbar \kappa \cdot v_0$ and $f_{\text{drift}}(\kappa) = f^0(\mathcal{E}_k^0)$. In contrast the formula used in the derivation of SR is:

$$
\sigma_{\omega,q}^{\text{SR}} = \frac{i\omega e^2}{q^2} \left[ \frac{g_z g_v}{(2\pi)^2} \right] \int d^2 \kappa \frac{f_{\text{drift}}(\kappa) - f_{\text{drift}}(\kappa + q)}{\hbar \omega + \mathcal{E}_k^0 - \mathcal{E}_{k+q}},
$$

(S3)

It is relevant to note that since $e\langle A \rangle \cdot v_k^0 = \hbar \kappa \cdot v_0$ and $\hbar \kappa = \hbar \kappa + e\langle A \rangle$, the mean electron energy ($\mathcal{E}_k$) can be expressed in terms of the kinetic momentum as $\mathcal{E}_k \approx \mathcal{E}_k^0$ (notice the different indices). Moreover, it is possible to write $f_{\text{drift}}(\kappa) = f^0(\mathcal{E}_k^0 - \hbar \kappa \cdot v_0) \approx f^0(\mathcal{E}_k^0 + e\langle A \rangle \cdot v_k^0 - \hbar \kappa \cdot v_0) \approx f^0(\mathcal{E}_k^0) = f_{\text{drift}}(\kappa)$, where we used again $e\langle A \rangle \cdot v_k^0 = \hbar \kappa \cdot v_0$ and $\hbar \kappa = \hbar \kappa + e\langle A \rangle$, and ignored terms that are of second order in the field interaction.

The main reason why Eqs. (S2)-(S3) predict very different results is that the identities $\mathcal{E}_k \approx \mathcal{E}_k^0$ and $f_{\text{drift}}(\kappa) \approx f_{\text{drift}}(\kappa)$ discussed in the previous paragraph do not imply that $\mathcal{E}_{k+q} \approx \mathcal{E}_{k+q}^0$ and $f_{\text{drift}}(\kappa + q) \approx f_{\text{drift}}(\kappa + q)$ because the relation between the kinetic and canonical momentum is nonlinear ($\hbar \kappa = \hbar \kappa + e\langle A \rangle_k$; again we underline that $\langle A \rangle_k$ generally depends on the canonical momentum). This is where Eq. (S3) goes wrong.

Note that the effect of the dynamical field $\phi \sim e^{i q \cdot x} e^{-i \omega t}$ in Eq. (S1) is to create a weak perturbation of the wave function that has canonical momentum $\kappa + q$. Thus $\sigma_{\omega,q}^{\text{drift}}$ must unquestionably depend on the interactions between states with canonical momentum
\( \hbar \mathbf{k} \) and \( \hbar (\mathbf{k} + \mathbf{q}) \), consistent with the term \( \tilde{f}_{\text{drift}}(\mathbf{k}) - \tilde{f}_{\text{drift}}(\mathbf{k} + \mathbf{q}) \) in the integrand of Eq. (S1). In contrast, the formula used by SR predicts the interactions between states with kinetic momentum \( \hbar \mathbf{k} \) and \( \hbar (\mathbf{k} + \mathbf{q}) \) (because of the term \( f_{\text{drift}}(\mathbf{k}) - f_{\text{drift}}(\mathbf{k} + \mathbf{q}) \)), which has no physical basis.

**B. Derivation of the graphene conductivity with the Boltzmann theory**

The Boltzmann equation predicts that the electron distribution function \( f = f(\mathbf{r}, \mathbf{p}, t) \) satisfies (we use a distribution function \( f \) such that \( \mathbf{p} \) represents the canonical momentum rather than the kinetic momentum, \( \mathbf{p} + eA \)):

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \left( \frac{\partial f}{\partial \mathbf{r}} \right) + \mathbf{p} \cdot \left( \frac{\partial f}{\partial \mathbf{p}} \right) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.
\]

(S4)

The semiclassical Hamiltonian is determined by \( \mathcal{H}_{\text{sc}}(\mathbf{r}, \mathbf{p}) = \mathcal{E}_{\mathbf{k}}^{\text{sc}} - e\phi \) with \( \phi, A \) the scalar and the vector potentials [R1, Appendix H] \( (E = -\nabla \phi - \partial_t A) \). By definition \( \dot{r} = \partial \mathcal{H}_{\text{sc}} / \partial \mathbf{p} \) and \( \dot{p} = -\partial \mathcal{H}_{\text{sc}} / \partial \mathbf{r} \). Here, \( \mathcal{E}_{\mathbf{k}}^{\text{sc}} \) determines the energy dispersion of the relevant electronic band.

The collisions with the ionic lattice are modeled with the relaxation time approximation \( \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = \left( f^{\text{eq}}(\mathcal{E}_{\mathbf{k}}^{\text{sc}}) - f \right) / \tau_{\text{ion}} \) where \( f^{\text{eq}}(\mathcal{E}) \) is the Fermi-Dirac distribution. On the other hand, the electron-electron (e-e) collisions are modeled by a periodic in time (e.g., with a saw-tooth type profile) vector potential, such that \( A(t) = -\left( t - t_i \right) E_0 \) in the generic interval \( t_i < t < t_{i+1} \) with \( t = t_i \) the time instants of the e-e collisions which may depend on \( \mathbf{p} \) (thereby \( A = A(\mathbf{p}, t) \)). Here, \( E_0 = E_0 \hat{x} \) is the static electric field. The time-averaged \( \mathbf{A} \) is \( \langle \mathbf{A} \rangle = -\tau_{\mathbf{k}} E_0 \) where \( \tau_{\mathbf{k}} \) determines the e-e scattering time and \( \mathbf{k} = \hbar^{-1} \mathbf{p} \). It is assumed that \( \tau_{\mathbf{k}} \ll \tau_{\text{ion}} \) so that the e-e collisions predominate. The scalar potential determines the dynamic (longitudinal) electric field \( -\nabla \phi = E_t e^{i\mathbf{k} \cdot \mathbf{x}} e^{-i\omega t} \).
The solution of the Boltzmann equation without the dynamic field \((\phi = 0)\) is exactly
\[ f = f^0\left(\mathcal{E}_k^0\right). \]
With the dynamic field, the solution can be written as \(f = f^0 + f^1\) with \(f^1\) satisfying:
\[
\partial_t f^1 + \partial_p \left[\mathcal{E}_k^{00} + e\mathbf{A} \cdot \mathbf{v}_k^0\right] \cdot \left(\partial f^0 / \partial \mathbf{r}\right) + \left(-eE_k e^{\phi_k} e^{-\text{int}}\right) \cdot \left(\partial f^0 / \partial \mathbf{p} + \partial f^1 / \partial \mathbf{p}\right) = -f^1 / \tau_{\text{ion}}. \tag{S5}
\]
In a linear response approximation, the term \(\partial f^1 / \partial \mathbf{p}\) must be dropped. Using \(\mathcal{E}_k^{00} \approx \mathcal{E}_k^{00} + e\mathbf{A} \cdot \mathbf{v}_k^0\), one can write \(\mathbf{r} = \partial_p \left[\mathcal{E}_k^{00} + e\mathbf{A} \cdot \mathbf{v}_k^0\right] \approx \partial_p \left[\mathcal{E}_k^{00} + e\mathbf{A} \cdot \mathbf{v}_k^0\right]\) where \(\mathbf{v}_k^0 = \hbar^{-1} \partial_k \mathcal{E}_k^0\). The function \(\partial_p \left[\mathcal{E}_k^{00} + e\mathbf{A} \cdot \mathbf{v}_k^0\right]\) varies periodically in time (because the vector potential also does) and its time-average is \(\langle \mathbf{r} \rangle \approx \partial_p \left[\mathcal{E}_k^0 - e\mathbf{E}_0 \cdot \mathbf{v}_k^0\right] = \mathbf{v}_k^0 + \delta \mathbf{v}_k\) with \(\delta \mathbf{v}_k = -e\hbar^{-1} \partial_k \left(\mathbf{E}_0 \cdot \mathbf{v}_k^0\right)\). Replacing \(\mathbf{r}\) by \(\langle \mathbf{r} \rangle\) in Eq. (S5) we get:
\[
\left(\tau_{\text{ion}}^{-1} + \partial_t\right) f^1 + \left(\mathbf{v}_k^0 + \delta \mathbf{v}_k\right) \cdot \left(\partial f^1 / \partial \mathbf{r}\right) = eE_k e^{\phi_k^e} e^{-\text{int}} \cdot \partial f^0 / \partial \mathbf{p}. \tag{S6}
\]
The solution of this equation is
\[
f^1 = eE_k e^{\phi_k^e} e^{-\text{int}} \frac{\partial f^0}{\tau_{\text{ion}}^{-1} - i\left(\omega - \mathbf{v}_k^0 \mathbf{q}_k - \delta \mathbf{v}_{s,k} \mathbf{q}_s\right) \partial \mathbf{p}} = eE_k e^{\phi_k^e} e^{-\text{int}} \frac{\partial f^0 \left(\mathcal{E}_k^0\right)}{\tau_{\text{ion}}^{-1} - i\left(\omega - \mathbf{v}_k^0 \mathbf{q}_k - \delta \mathbf{v}_{s,k} \mathbf{q}_s\right) \partial \mathcal{E}} V_{s,k}. \tag{S7}
\]
The impact of the static electric field on the dynamic response depends critically on \(\tau_k\), i.e., on how the scattering time varies with the canonical momentum. If the e-e collisions act to create a “drift”, i.e., a coherent motion of the whole electron gas with a constant velocity, then the velocity \(\delta \mathbf{v}_k\) must be independent of \(\mathbf{k}\). In the graphene case, this situation can occur when \(\tau_k \sim |\mathbf{k}| \sim \mathcal{E}_k^0\), i.e., when the collision time is proportional to the energy for states near the Fermi level. In such a situation, \(\mathbf{v}_k^0 \tau_k = v_0 \mathbf{k} \tau_k \sim \mathbf{k}\) and therefore \(\delta \mathbf{v}_{s,k}\) is of the form \(\delta \mathbf{v}_{s,k} = \mathbf{v}_0\), with \(\mathbf{v}_0\) the drift velocity. Under these conditions, \(\langle \mathbf{r} \rangle = \mathbf{v}_k^0 + \mathbf{v}_0\) and
The dynamic conductivity can now be evaluated noting that the current density is given by:

\[ j_x = \frac{eg_x g_v}{(2\pi)^2} \int d^2 \kappa \langle \hat{x} \rangle f^1, \]  

(S9)

with \( \langle \hat{x} \rangle = v_{i,k} + v_0 \) the mean electron velocity and \( g_x g_v = 2 \times 2 \) the valley and spin factors. The longitudinal conductivity (defined such that \( j_x = \sigma_{q,\omega} E q^{i \omega} e^{-i\omega t} \)) is evidently (for simplicity we take now the collisionless limit \( \tau_{ion}^{-1} \rightarrow 0^+ \)):

\[ \sigma_{q,\omega} = \frac{-ie^2}{\pi^2} \int d^2 \kappa \frac{\partial f^0 (E_{\kappa}^0)}{\partial \mathcal{E}} v_{i,k} (v_{i,k} + v_0) \frac{1}{\tilde{\omega} - v_{i,k} q_s}, \]  

(S10)

with \( \tilde{\omega} = \omega - v_0 q_s \) the Doppler shifted frequency. In the zero-temperature limit it is possible to write \( \frac{\partial f^0 (E_{\kappa}^0)}{\partial \mathcal{E}} v_{i,k} = \frac{1}{\hbar} \frac{\partial}{\partial k_x} \left[ f^0 (E_{\kappa}^0) \right] \approx \frac{1}{\hbar} \frac{\partial}{\partial k_x} [u(k_x - \kappa)] \) where \( k_F \) is the Fermi wave number and \( u \) is the Heaviside step function. We used the dispersion \( E_{\kappa}^0 = \hbar v_F \kappa \) of the conduction band of graphene, with \( v_F \) the Fermi velocity. This approximation yields:

\[ \sigma_{q,\omega} = \frac{ie^2}{\pi^2 \hbar} \int d^2 \kappa \delta (k_x - k_F) \frac{v_0}{v_F} \left( v_{i,k} + v_0 \right) \frac{1}{\tilde{\omega} - v_{i,k} q_s}, \]  

(S11)

Using polar coordinates \( \kappa, \varphi \) one obtains:

\[ \sigma_{q,\omega} = \frac{ie^2 E_F}{\pi^2 \hbar^2} \int_0^{2\pi} d\varphi \frac{\cos^2 \varphi + \frac{v_F}{v_0} \cos \varphi}{\tilde{\omega} - q_x v_F \cos \varphi}, \]  

(S12)

where \( E_F \) is the Fermi level and we took into account the linear dispersion of graphene \( (E_{\kappa}^0 = \hbar v_F \kappa) \) near the Dirac cones. The integral can be evaluated analytically as
\[ \sigma_{q,\omega} = \frac{i\omega e^2}{h^2} \frac{2E_{\nu}}{\pi} \frac{1}{\omega^2} \frac{1}{\sqrt{1-A^2}} \left( \frac{1}{1+\frac{\nu_0\omega}{\nu_\nu}} \right) \]  

(S13)

with \( A = \frac{q_\omega v_\nu}{\omega} \). Using \( \frac{1}{A^2} \left( \frac{1}{\sqrt{1-A^2}} - 1 \right) = \frac{1}{\left( 1+\frac{\nu_0\omega}{\nu_\nu} \right)\sqrt{1-A^2}} \) and \( 1 + \frac{\nu_0\omega}{\nu_\nu} = \frac{\omega}{\omega} \)

we obtain the final and exact result for the drift-current biased graphene:

\[ \sigma_{q,\omega} = \frac{i\omega e^2}{h^2} \frac{2E_{\nu}}{\pi} \frac{1}{\omega^2} \frac{1}{\left( \omega_\omega - \left( q_\omega v_\nu \right)^2 \right) + \omega^2 - \left( q_\omega v_\nu \right)^2} \]  

(S14)

This result is fully consistent with the moving medium analogy of our earlier article [R2] as \( \sigma_{q,\omega}^{\text{drift}} = \sigma_{q,\omega}^{\text{no-drift}} \frac{\nu_0}{\omega} \). The no-drift conductivity obtained from Eq. (S14) agrees with the result of SR and Levitov [R3, R4].

The previous analysis confirms that the Galilean Doppler shift theory is the correct answer when the e-e collisions force the electron gas to move as a whole with some drift velocity \( \nu_0 \). In such a situation, the drift effectively shifts the electrons’ velocity by \( \nu_0 \), consistent with the assumption of our earlier work [R2]. In general, if the scattering is not dominated by the e-e collisions, the conductivity of the drift-biased graphene is not determined by a Galilean transformation.

We note in passing that the static current density obtained with our theory is determined from \( j = \frac{-e g_\nu g_\nu}{(2\pi)^2} \int d^2k \langle \hat{r} \rangle f^0 \) with \( \langle \hat{r} \rangle = \nu_\nu - \hbar^{-1}e\partial_{\nu} (E_0 \cdot \nu_\nu \tau_\nu) \). After integration by parts it can be written as \( j = \frac{-e^2 g_\nu g_\nu}{(2\pi)^2} \int d^2k \frac{\partial f^0}{\partial \nu} (\nu_\nu \nu_\nu \tau_\nu \cdot E_0) \), which agrees with the standard theory of conduction in solids [R1].

\section*{C. Overview of the different graphene conductivity models in the collisionless regime}
Here we present a brief overview of the nonlocal conductivity of the drift-current biased graphene predicted by different models [R2-R5].

In the Svinitsov and Ryzhii (SR) theory the graphene conductivity for the waves propagating parallel to the drift current is given by (restoring explicitly all the units) [R3],

\[ \sigma_{SR}^{\omega,q_x} = -ie^2 \frac{\omega}{q_x^2} \left( \frac{1}{\hbar v_f^2} \right) \Pi(\omega,q_x), \]  

with \( E_f \) the Fermi energy, \( s = \omega / |v_F q_x| \), \( \beta = v_0 / v_e \), and \( \omega \) in the upper-half frequency plane (UHP).

The conductivity obtained from Levitov’s collisionless theory is [R4]:

\[ \sigma_{Levitov}^{\omega,q_x} = \frac{i\omega e^2}{h^2} \frac{2E_f}{\pi} \frac{1}{\gamma(\omega - v_0 q_x)\sqrt{\omega^2 - v_e^2 q_x^2 + \omega^2 - v_e^2 q_x^2}}, \]  

for \( \omega \) in the UHP and \( \gamma = 1 / \sqrt{1 - v_0^2 / v_e^2} \).

The real-part of the conductivity obtained from Polini’s theory [R5] for the waves propagating parallel to the drift current at zero-temperature (considering only the intraband term and \( \omega \) real-valued) is:

\[ \text{Re}\left\{ \sigma_{intra}^{\omega,q_x} \right\} = \frac{i\omega e^2}{q_x^2} \left\{ \begin{array}{ll} 2f(\omega_n,k_n)(H_{intra}(A_s,B) - H_{intra}(A_s,B)) & \text{if } k_n > \omega_n \\ 0 & \text{if } k_n < \omega_n \end{array} \right., \]  

for 1

1 Levitov and co-authors also introduce an alternative hydrodynamic model applicable when the electron-electron scattering rate is larger than \( \omega \).
where \( \omega_n = \hbar \omega / E_F \), \( k_n = \hbar v_F q_x / E_F \), \( f(\omega_n, k_n) = \frac{|E_F|}{8\pi\hbar^2 v_F} \frac{k_n^2}{\sqrt{k_n^2 - \omega_n^2}} \),

\[ A_x = 2 + \alpha(\omega_n - \beta k_n), \quad B = \beta \omega_n - k_n \quad \text{and} \quad \beta = v_0 / v_F. \]

The function \( H_{\text{intra}}(A_x, B) \) is defined by:

\[
H_{\text{intra}}(A_x, B) = \begin{cases} 
G_{\text{intra}}(x_\alpha), & A_x + B > 0 \quad \text{and} \quad B < 0 \\
-G_{\text{intra}}(x_\alpha) \text{sgn}(B), & A_x + B < 0 \quad \text{and} \quad A_x - |B| > 0, \\
0, & \text{otherwise}
\end{cases}
\] (S18)

where \( x_\alpha = -A_x / B \) and \( G_{\text{intra}}(x) = x\sqrt{x^2 - 1} - \log\left(x + \sqrt{x^2 - 1}\right) \).

The SR and Levitov (collisionless) theories agree exactly when the drift velocity vanishes, such that:

\[
\sigma_{g}^{\text{no,drift}}(\omega, q_x) \approx \frac{i\omega e^2}{\hbar^2} \frac{2E_F}{\pi} \frac{1}{\omega^2 - \omega^2 q_x^2 + \omega^2 - v_F^2 q_x^2}. \] (S19)

Finally, the model introduced in our article [R2] predicts (considering only the intraband effects):

\[
\sigma_{g}^{\text{Doppler}}(\omega, q_x) = (\omega / \bar{\omega}) \sigma_{g}^{\text{no,drift}}(\bar{\omega}, q_x), \quad \bar{\omega} = \omega - q_x v_0. \] (S20)

The simulations of the main text are based on the above formulas.

In our model [Eq. (S20)] the “polarizability” \( \sigma_{g} / (i\omega) \) is transformed by the drift-current bias using the Galilean formulas \( \omega \to \bar{\omega} = \omega - q_x v_0 \) and \( q_x \to \bar{q}_x = q_x \). In contrast a relativistic-type Doppler shift transformation would give instead \( \omega \to \bar{\omega} = \gamma (\omega - q_x v_0) \) and \( q_x \to \bar{q}_x = \gamma \left(q_x - \omega v_0 / v_F^2\right) \) with \( \gamma = 1 / \sqrt{1 - v_0^2 / v_F^2} \) the graphene Lorentz factor. Importantly, the no-drift polarizability \( \sigma_{g}^{\text{no,drift}} / (i\omega) \) is transformed by the relativistic Doppler shift as:

\[
\frac{\sigma_{g}^{\text{no,drift}}}{i\omega} \to \frac{e^2}{\hbar^2} \frac{2E_F}{\pi} \frac{1}{\gamma (\omega - q_x v_0) \sqrt{\omega^2 - v_F^2 q_x^2 + \omega^2 - v_F^2 q_x^2}} = \frac{\sigma_{g}^{\text{Levitov}}}{i\omega}. \] (S21)
because the terms $\omega^2 - v_F^2 q_x^2$ are “Lorentz invariant”. In other words, the collisionless
model by Levitov [R4] is “fully” relativistic and is essentially the result of applying a
relativistic Doppler transformation to the no-drift polarizability.
From the previous discussion, it follows that Eq. (S20) and the model of Levitov will
give similar results provided the Lorentz factor $\gamma$ is near unit and if in addition $\omega v_0 / v_F^2$
is negligible as compared to $q_x$. In the negative Landau damping region one has
$|q_x| \sim 2\omega / v_F$, and hence provided $v_0 / v_F$ is not too large (let us say $v_0 / v_F < 0.6$) it
follows that $\tilde{q}_x \approx q_x$ is typically a good approximation for $\tilde{q}_x = \gamma (q_x - \omega v_0 / v_F^2)$. Thus,
Eq. (S20) typically agrees very well with the SR and Levitov models in the negative
Landau damping region.

D. Dispersion equation for the natural modes of oscillation in the
graphene-dielectric-metal cavity

The dispersion equation for a system formed by a graphene sheet with drifting electrons
and a plasmonic slab separated by a dielectric gap can be written as
$1 - R_1 R_2 e^{-2\gamma d} = 0$
(here $R_1$ and $R_2$ are the magnetic-field reflection coefficients at the graphene-dielectric
and dielectric-metal interfaces, respectively, $\gamma_d = \sqrt{q_x^2 - \varepsilon_d \omega^2 / c^2}$, and $d$ is the gap
distance between the graphene sheet and the plasmonic slab). In the quasi-static limit
$\gamma_d \approx |q_x| = q_\parallel$, and hence the characteristic equation becomes [R2]

$$1 - R_1 R_2 e^{-2\gamma d} = 0 .$$

(S22)

The reflection coefficient at the graphene-dielectric interface is given by [R2]

$$R_1 (q_x, \omega) = \frac{q_\parallel}{q_\parallel - 2\varepsilon_d \kappa_y} ,$$

(S23)
where $\kappa_g(\omega) = \frac{i\omega\varepsilon_0}{\sigma_g^{\text{drift}}(\omega, q_s)}$ and $\sigma_g^{\text{drift}}(\omega, q_s)$ is the nonlocal graphene conductivity with drifting electrons. In our model, $\sigma_g^{\text{drift}}(\omega, q_s)$ is calculated with Eq. (S20) and in the SR model with Eq. (S15).

In the quasi-static approximation, the magnetic field reflection coefficient at the dielectric-metal interface is given by:

$$R_z(\omega) = \frac{\varepsilon_m(\omega) - \varepsilon_d}{\varepsilon_m(\omega) + \varepsilon_d}. \quad (S24)$$

**E. Incidence of an evanescent wave on a drift-current biased graphene sheet**

To illustrate the consequences of the negative Landau damping, we consider a plane wave incidence problem with the graphene sheet biased with a drift current. The incident wave is TM-polarized [see Fig. S1(a)] and is characterized by the wave number $q_s$. We focus in the case in which $|q_s| > \sqrt{\varepsilon_d \omega / c}$, which corresponds to an evanescent incident plane wave. The wave reflected by the interface is also an evanescent wave, and the superposition of the two waves generally generates a power flux towards the graphene sheet due to the material absorption. Using standard methods and the nonlocal graphene conductivity formula [Eq. (1) of the main text with the bare graphene nonlocal conductivity evaluated as in Ref. [R6], we numerically computed the $z$-component of the Poynting vector at the interface [see Fig. S1(b)].

As expected, without the drift current [blue line in Fig. S1(b)] the $z$-component of the Poynting vector is negative ($S_z < 0$) due to the material absorption by the graphene. In contrast, with the drift-current (green and purple lines in Fig. S1(b)) the $z$-component of the power flux direction may be flipped so that the energy flows *away* from the graphene sheet [R7]. This happens due to the negative Landau damping effect predicted
in [R2], which enables the transfer of kinetic energy from the drifting electrons to the radiation field.

Fig. S1. (a) Sketch of a graphene sheet (with a drift-current bias) surrounded by a dielectric with relative permittivity \( \varepsilon_r = 4 \). (b) Poynting vector component perpendicular to the interface (in arbitrary units) as a function of \( q_x \) for different drift velocities and \( f = 15 \text{ THz} \). The remaining parameters are \( E_F = 0.1 \text{ eV} \) and \( \Gamma_{\text{ins}} = 1/(0.17 \text{ ps}) \).

**F. Conductivity of a passive material in the upper-half frequency plane**

Here, we show explicitly that for a passive material (with no gain) it is necessary that

\[ \text{Re} \{ \sigma(\omega,q_x) \} > 0 \text{ for } \omega = \omega' + i\omega'' \text{ in the upper-half frequency plane (UHP) (} \omega'' > 0 \text{)} \]

and \( q_x \) real-valued. For simplicity, in the following we omit the dependence of \( \sigma \) on \( q_x \).

Assuming that the conductivity is an analytic (scalar) function in the UHP it is possible to write (from Cauchy theorem):

\[
\sigma(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(\xi)}{\xi - \omega} d\xi
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma(\xi)}{\xi - \omega'} \left(\omega'' - i(\xi - \omega')\right) d\xi\]

\[\text{for } \omega'' > 0. \quad \text{(S25)}\]

Thus, the real-part of the conductivity in the UHP satisfies (with \( \sigma = \sigma' + i\sigma'' \)):
\[ \sigma'(\omega) = \frac{1}{2\pi} \int_{\omega}^{\infty} \frac{\sigma'(\xi)\omega'^*}{(\xi - \omega')^2 + \omega'^2} + \frac{\sigma'^*(\xi)}{(\xi - \omega')^2 + \omega^2} d\xi. \]  

(S26)

On the other hand, the Kramers-Kronig relations imply that in the real-frequency axis [R8]:

\[ \sigma'^*(\xi) = -\frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\xi} \frac{\sigma'(x)}{x - \xi} dx. \]  

(S27)

This allows us to write:

\[
\int_{-\infty}^{\infty} \frac{\sigma'^*(\xi)}{(\xi - \omega')^2 + \omega'^2} (\xi - \omega') d\xi \\
= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \sigma'(x) \text{P.V.} \int_{-\infty}^{\infty} \frac{\xi - \omega'}{\xi - x} \frac{1}{(\xi - \omega')^2 + \omega'^2} d\xi \\
= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \sigma'(x) \frac{\pi \omega'^*}{(x - \omega')^2 + \omega'^2}.
\]  

(S28)

Hence, we get simply:

\[ \sigma'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma'(\xi)\omega'^*}{(\xi - \omega')^2 + \omega'^2} d\xi, \quad \text{for} \quad \omega'^* > 0. \]  

(S29)

It is well-known that a passive material has \( \text{Re}\{\sigma(\omega, q_z)\} > 0 \) for \( \omega \) and \( q_z \) real-valued [R8]. Hence, the above formula implies that the same relation holds for \( \omega \) in the UHP and \( q_z \) real-valued.

**Supplementary information references:**

[R1] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Brooks/Cole, Thomson Learning, Cornell, 1976).

[R2] T. A. Morgado, M. G. Silveirinha, “Negative Landau damping in bilayer graphene,” *Phys. Rev. Lett.* **119**, 133901 (2017).

[R3] D. Svintssov, V. Ryzhii, Comment on “Negative Landau damping in bilayer graphene,” arXiv:1812.03764.
[R4] D. S. Borgnia, T. V. Phan, L. S. Levitov, “Quasi-Relativistic Doppler Effect and Non-Reciprocal Plasmons in Graphene”, arXiv: 1512.09044.

[R5] B. V. Duppen, A. Tomadin, A. N. Grigorenko, and M. Polini, “Current-induced birefringent absorption and non-reciprocal plasmons in graphene,” 2D Mater. 3, 015011 (2016).

[R6] X. Lin et al., “All-angle negative refraction of highly squeezed plasmon and phonon polaritons in graphene-boron nitride heterostructures,” Proc. Natl. Acad. Sci. USA 114, 6717–6721 (2017).

[R7] M. G. Silveirinha, “Optical Instabilities and Spontaneous Light Emission by Polarizable Moving Matter,” Phys. Rev. X 4, 031013 (2014).

[R8] V. Agranovich and V. Ginzburg, Spatial Dispersion in Crystal Optics and the Theory of Excitons. New York: Wiley-Interscience, 1966.