Consistent Deformed Bosonic Algebra in Noncommutative Quantum Mechanics

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Abstract

In two-dimensional noncommutative space for the case of both position - position and momentum - momentum noncommuting, the consistent deformed bosonic algebra at the non-perturbation level described by the deformed annihilation and creation operators is investigated. A general relation between noncommutative parameters is fixed from the consistency of the deformed Heisenberg - Weyl algebra with the deformed bosonic algebra. A Fock space is found, in which all calculations can be similarly developed as if in commutative space and all effects of spatial noncommutativity are simply represented by parameters.
1 Introduction

Physics in noncommutative space \([1-3]\) has been extensively investigated in literature. In the low energy sector one expects that quantum mechanics in noncommutative space (NCQM) may clarify some low energy phenomenological consequences, lead to qualitative understanding of effects of spatial noncommutativity and shed some light on the problem at the level of noncommutative quantum field theory. In literature the perturbation aspects of NCQM and its applications \([4-13]\) have been studied in detail. The perturbation approach is based on the Weyl - Moyal correspondence \([14-16]\), according to which the usual product of functions should be replaced by the star-product. Because of the exponential differential factor in the Weyl - Moyal product the non-perturbation treatment is difficult. But non-perturbation investigations may explore some essentially new features of NCQM. A suitable example for non-perturbation investigations is a two dimensional isotropic harmonic oscillator which is exactly soluble, and fully explored in literature. In the first paper of Ref. \([9]\) through the non-perturbation investigation of this example it was clarified that the consistent ansatz of commutation relations of phase space variables should simultaneously include space-space noncommutativity and momentum-momentum noncommutativity; The consistent deformed bosonic algebra at the non-perturbation level was obtained and a relation between noncommutative parameters was fixed. But this example is special. In many systems, the potential can be modelled by a harmonic oscillator through an expansion about its minimum. But there are some potentials which are not the case. Since NCQM is peculiar in many ways, it is necessary to clarify the situation for general cases in detail.

In this paper we elucidate this topic for general cases in the context of non-relativistic quantum mechanics. The point is how to maintain Bose-Einstein statistics at the non-perturbation level described by the deformed annihilation and creation operators in non-commutative space when the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics. For this purpose, first we need to find the general structure of the deformed annihilation and creation operators which satisfy a complete and closed deformed bosonic algebra at the non-perturbation level. We obtain the consistent
deformed bosonic algebra at the non-perturbation level. A relation between noncommu-
tative parameters is fixed from the consistency of the deformed Heisenberg - Weyl algebra
with the deformed bosonic algebra. A construction of Fock space in general cases at the
non-perturbation level is complicated. We found a Fock space in which all calculations can
be similarly developed as if in commutative space and all effects of spatial noncommutativ-
ity are simply represented by parameters, not represented by noncommutative operators.

In the following, in section 2 the background of the deformed Heisenberg - Weyl algebra
is reviewed. In section 3 the consistent deformed bosonic algebra is investigated. In
section 4 the two-dimensional isotropic harmonic oscillator is revisited. Its exact (non-
perturbative) eigenvalues are obtained in a simple way in commutative Fock space.

2 The Deformed Heisenberg - Weyl Algebra

In this paragraph we review the necessary background first. The starting point is the
deformed Heisenberg - Weyl algebra. We consider the case of both position - position
noncommutativity (space-time noncommutativity is not considered) and momentum - mo-
momentum noncommutativity. In this case the consistent deformed Heisenberg - Weyl algebra
is \cite{9}:

\[
\begin{align*}
[\hat{x}_i, \hat{x}_j] &= i\xi^2 \theta \epsilon_{ij}, \\
[\hat{p}_i, \hat{p}_j] &= i\xi^2 \eta \epsilon_{ij}, \\
[\hat{x}_i, \hat{p}_j] &= i\hbar \delta_{ij}, \quad (i, j = 1, 2),
\end{align*}
\]

(2.1)

where $\theta$ and $\eta$ are constant parameters, independent of the position and mo-
momentum. Here we consider the noncommutativity of the intrinsic canonical momentum. \footnote{The intrinsic noncommutativity of the canonical momenta discussed here is essentially different from the noncommutativity of the mechanical momenta of a particle in an external magnetic field with a vector potential $A_i(x_j)$ in commutative space. In the former case the difference between $\hat{p}_i$ and $p_i$ must be extremely small, see footnote 3. In the later case the mechanical momentum is $p_{\text{mech},i} = \mu \dot{x}_i = p_i - \frac{q}{c} A_i$, where $p_i = -i\hbar \partial_i$ is the canonical momentum in commutative space, satisfying $[p_i, p_j] = 0$. The commutator between $p_{\text{mech},i}$ and $p_{\text{mech},j}$ is $[p_{\text{mech},i}, p_{\text{mech},j}] = -\frac{q}{c} ([p_i, A_j] + [A_i, p_j]) = \frac{\hbar q}{c} (\partial_i A_j - \partial_j A_i) = \frac{\hbar q}{c} \epsilon_{ij3} B_3$. Such a noncommutativity is determined by the external magnetic field $\vec{B}$ which, unlike the noncommutative parameter $\eta$, may be strong so that the difference between $p_{\text{mech},i}$ and $p_i$ may not be extremely small.}

It means that
the parameter \( \eta \), like the parameter \( \theta \), should be extremely small. This is guaranteed by a direct proportionality provided by a constraint between them (See Eq. (3.4) below). The \( \epsilon_{ij} \) is a two-dimensional antisymmetric unit tensor, \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0 \). In Eq. (2.1) the scaling factor \( \xi = (1 + \theta \eta / 4\hbar^2)^{-1/2} \) is a dimensionless constant.

The deformed Heisenberg - Weyl algebra (2.1) can be realized by undeformed phase space variables \( x_i \) and \( p_i \) as follows (henceforth summation convention is used)

\[
\hat{x}_i = \xi (x_i - \frac{1}{2\hbar} \theta \epsilon_{ij} p_j), \quad \hat{p}_i = \xi (p_i + \frac{1}{2\hbar} \eta \epsilon_{ij} x_j),
\]

(2.2)

where \( x_i \) and \( p_i \) satisfy the undeformed Heisenberg - Weyl algebra \([x_i, x_j] = [p_i, p_j] = 0, [x_i, p_j] = i\hbar \delta_{ij}\).

It should be emphasized that for the case of both position - position and momentum - momentum noncommuting the scaling factor \( \xi \) in Eqs. (2.1) and (2.2) guarantees consistency of the framework, and plays an essential role in dynamics. One may argues that only three parameters \( \hbar, \theta \) and \( \eta \) can appear in three commutators (2.1), thus \( \xi \) is an additional spurious parameter and can be set to 1. If one re-scales \( \hat{x}_i \) and \( \hat{p}_i \) so that \( \xi = 1 \) in Eqs. (2.1) and (2.2), it is easy to check that Eq. (2.2) leads to \([\hat{x}_i, \hat{p}_j] = i\hbar (1 + \theta \eta / 4\hbar^2) \delta_{ij}\), thus the Heisenberg commutation relation cannot be maintained.

3 The Consistent Deformed Bosonic Algebra

The investigation of the deformed bosonic algebra at the non-perturbation level includes two aspects. The first aspect is to find the general structures of the deformed annihilation and creation operators which satisfy the complete and closed deformed bosonic algebra at the non-perturbation level. Because there is a new type of the deformed bosonic commutation relation which correlates different degrees of freedom at the level of the deformed annihilation and creation operators, the second aspect is, by generalizing one - particle quantum mechanics to many particle system, how to establish the Fock space of identical bosons, see footnote 5 below.

We review the first respect [9]. In the context of quantum mechanics the general representations of the deformed annihilation and creation operators \( \hat{a}_i \) and \( \hat{a}^\dagger_i \) at the non-perturbation level are represented by the deformed phase space variables \( \hat{x}_i \) and \( \hat{p}_i \) as
follows:

\[
\hat{a}_i = c_1(\hat{x}_i + ic_2\hat{p}_i), \quad \hat{a}^\dagger_i = c_1(\hat{x}_i - ic_2\hat{p}_i),
\]  

(3.1)

where \(c_1\) and \(c_2\) are real constants which can be fixed as follows. Operators \(\hat{a}_i\) and \(\hat{a}^\dagger_i\) should satisfy the bosonic commutation relations \([\hat{a}_1, \hat{a}^\dagger_1] = [\hat{a}_2, \hat{a}^\dagger_2] = 1\) (to keep the physical meaning of \(\hat{a}_i\) and \(\hat{a}^\dagger_i\) at the non-perturbation level). From this requirement and the deformed Heisenberg-Weyl algebra (2.1) it follows that

\[
c_1 = \sqrt{1/2}\hbar c_2. \tag{3.2}
\]

Following the standard procedure in quantum mechanics, starting from a system with one particle, at the level of the annihilation and creation operators the state vector space of a many-particle system can be constructed by generalizing one-particle formalism. Then Bose-Einstein statistics for an identical-boson system can be developed in the standard way. Bose-Einstein statistics should be maintained at the non-perturbation level described by \(\hat{a}_i\), thus operators \(\hat{a}_i\) and \(\hat{a}_j\) should be commuting: \([\hat{a}_i, \hat{a}_j] = 0\). From this equation and the deformed Heisenberg-Weyl algebra (2.1) it follows that

\[
ic_1^2\xi^2\epsilon_{ij}(\theta - c_2^2\eta) = 0. \tag{3.3}
\]

This requirement leads to the following condition between \(\eta\) and \(\theta\)

\[
\eta = c_2^{-2}\theta. \tag{3.4}
\]

From Eqs. (3.1), (3.2) and (3.4) we obtain the following deformed annihilation and creation operators \(\hat{a}_i\) and \(\hat{a}^\dagger_i\):

\[
\hat{a}_i = \sqrt{\frac{1}{2\hbar}}\sqrt{\frac{\eta}{\theta}} \left(\hat{x}_i + i\sqrt{\frac{\theta}{\eta}}\hat{p}_i\right), \quad \hat{a}^\dagger_i = \sqrt{\frac{1}{2\hbar}}\sqrt{\frac{\eta}{\theta}} \left(\hat{x}_i - i\sqrt{\frac{\theta}{\eta}}\hat{p}_i\right), \tag{3.5}
\]

From Eqs. (2.1) and (3.5) it follows that the deformed bosonic algebra of \(\hat{a}_i\) and \(\hat{a}^\dagger_j\) reads

\[
[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij} + \frac{i}{\hbar}\xi^2\sqrt{\theta\eta}\epsilon_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0, \quad (i, j = 1, 2). \tag{3.6}
\]

In Eqs. (3.6) the three equations \([\hat{a}_1, \hat{a}^\dagger_1] = [\hat{a}_2, \hat{a}^\dagger_2] = 1, \quad [\hat{a}_1, \hat{a}_2] = 0\) are the same as the undeformed bosonic algebra in commutative space. They constitute a closed algebra. The
equation

\[ [\hat{a}_1, \hat{a}^\dagger_2] = \frac{i}{\hbar} \xi^2 \sqrt{\bar{\theta} \eta} \]  

(3.7)
is a new type. Eqs. (3.6), including Eq. (3.7), constitute not only a closed but also a complete deformed bosonic algebra. Because of noncommutativity of space, different degrees of freedom are correlated at the level of the deformed Heisenberg-Weyl algebra (2.1); Eq. (3.7) represents such correlations at the level of the deformed annihilation and creation operators.

The deformed annihilation and creation operators \( \hat{a}_i \) and \( \hat{a}^\dagger_i \) can be represented by the undeformed ones \( a_i \) and \( a^\dagger_i \). The general structure of the undeformed annihilation operator \( a_i \) represented by the undeformed phase space variables \( x_i \) and \( p_i \) is \( a_i = c'_i (x_i + ic'_2 p_i) \), where the constants \( c'_i \) can be fixed as follows. Operators \( a_i \) and \( a^\dagger_i \) should satisfy bosonic commutation relations \( [a_1, a^\dagger_2] = [a_2, a^\dagger_1] = 1 \). From this requirement the undeformed Heisenberg-Weyl algebra leads to \( c'_2 = \sqrt{1/2\hbar c'_2} \). The undeformed bosonic commutation relation \( [a_i, a_j] = 0 \) is automatically satisfied, so \( c'_2 \) is a free parameter. Thus the general structure of the undeformed annihilation and creation operators reads

\[
a_i = \frac{1}{\sqrt{2\hbar c'_2}} (x_i + ic'_2 p_i), \quad a^\dagger_i = \frac{1}{\sqrt{2\hbar c'_2}} (x_i - ic'_2 p_i). \]  

(3.8)
The operators \( a_i \) and \( a^\dagger_i \) satisfy the undeformed bosonic algebra \( [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \), \( [a_i, a^\dagger_j] = i\delta_{ij} \) which constitute a closed and complete algebra in commutative space.

In the limit \( \theta, \eta \to 0 \), the deformed operators \( \hat{x}_i, \hat{p}_i, \hat{a}_i, \hat{a}^\dagger_i \) reduce to the undeformed ones \( x_i, p_i, a_i, a^\dagger_i \). Eq. (3.4) indicates that in this limit \( \theta/\eta \) and \( \eta/\theta \) should keep finite. It follows that \( c_2 = c'_2 \). From this result and Eqs. (2.1), (2.2), (3.1) and (4.2) it follows that \( \hat{a}_i \) and \( \hat{a}^\dagger_i \) can be represented by \( a_i \) and \( a^\dagger_i \) as follows:

\[
\hat{a}_i = \xi (a_i + \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a_j), \quad \hat{a}^\dagger_i = \xi (a^\dagger_i - \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a^\dagger_j). \]  

(3.9)

Similar to Eqs. (2.1) and (2.2), it should be emphasized that for the case of both position - position and momentum - momentum noncommuting the scaling factor \( \xi \) in Eq. (3.9) guarantees consistency of the framework. Specially, it maintains the bosonic commutation relation. If one sets \( \xi = 1 \) in Eq. (3.9), it follows that \( [\hat{a}_1, \hat{a}^\dagger_1] = [\hat{a}_2, \hat{a}^\dagger_2] = (1 + \theta \eta/4\hbar^2) \), the bosonic commutation relation \( [\hat{a}_1, \hat{a}^\dagger_1] = [\hat{a}_2, \hat{a}^\dagger_2] = 1 \) cannot be maintained.
If momentum-momentum is commuting, η = 0, Eq. (3.3) shows that the second equation in (3.6) cannot be obtained. It is clear that in order to maintain Bose-Einstein statistics for identical bosons at the non-perturbation level described by $\hat{a}_i$ and $\hat{a}_i^\dagger$ we should consider both space-space noncommutativity and momentum-momentum noncommutativity. Eq. (3.5) is the most general representation of the physical annihilation and creation operators in noncommutative space.  

Eq. (3.4) shows that for any system the general feature of a relation between noncommutative parameters is a direct proportionality. It is fixed by

\[ \frac{\nu}{\sqrt{2\hbar c^2}} = \frac{\xi(1 - \theta \eta/4\hbar^2)}{\sqrt{2\hbar c^2}}. \]

where ν = ξ(1 − θη/4ℏ^2). These operators automatically satisfy the bosonic commutation relations $[\hat{a}_i', \hat{a}_j'] = \delta_{ij}$, $[\hat{a}_i', \hat{a}_j^\dagger] = [\hat{a}_i^\dagger, \hat{a}_j'] = 0$. Moreover no constraint on the parameters θ and η is required apart from the obvious one $\eta \theta \neq 4\hbar^2$.

Then it follows a related tacit understanding that:

"it is possible to construct an infinity of the creation and annihilation operators which satisfy exactly the bosonic commutation relations, but do not require any constraint on the parameters such as Eq. (3.4)."

For example, similar to the Landau creation and annihilation operators (acting within or across Landau levels) involve mixing of spatial directions in an external magnetic field, we may define the following annihilation operator

\[ \hat{a}_i' = \frac{\nu^{-1}}{\sqrt{2\hbar c^2}} \left[ \left( \delta_{ij} - \frac{i c^2 \eta \epsilon_{ij}}{2\hbar} \right) \hat{x}_j + i \left( c^2 \delta_{ij} - \frac{i \theta \epsilon_{ij}}{2\hbar} \right) \hat{p}_j \right], \]

where $\nu = \xi(1 - \theta \eta/4\hbar^2)$. These operators automatically satisfy the bosonic commutation relations $[\hat{a}_i', \hat{a}_j'] = \delta_{ij}$, $[\hat{a}_i', \hat{a}_j^\dagger] = [\hat{a}_i^\dagger, \hat{a}_j'] = 0$. Moreover no constraint on the parameters θ and η is required.

Then it follows a related tacit understanding that:

"the previous construction also indicates that it is not compulsory to consider both position and momentum noncommutativity."

Indeed, if we take η = 0, ν = ξ = 1 in the previous expression for the creation and annihilation operators, we get:

$\hat{a}_i'' = \frac{1}{\sqrt{2\hbar c^2}} \left[ \hat{x}_i + i \left( c^2 \delta_{ij} - \frac{i \theta \epsilon_{ij}}{2\hbar} \right) \hat{p}_j \right],$

This is also perfectly consistent.

In order to clarify the meaning of $\hat{a}_i'$ we insert Eqs. (2.2) into it. It follows that $[(\delta_{ij} - ic^2 \eta \epsilon_{ij}/2\hbar)\hat{x}_j + i(c^2 \delta_{ij} - i\theta \epsilon_{ij}/2\hbar)\hat{p}_j] = \xi(1 - \theta \eta/4\hbar^2) (x_i + ic^2 p_i)$, thus

$\hat{a}_i' = \frac{1}{\sqrt{2\hbar c^2}} (x_i + ic^2 p_i),$

which elucidates that $\hat{a}_i'$ is just the undeformed annihilation operator $a_i$ in Eq. (4.2), not the annihilation operator in noncommutative space. This explains that $\hat{a}_i'$ and $\hat{a}_i'^\dagger$ automatically satisfy the undeformed bosonic commutation relations, and no constraint on the parameters θ and η is required.

For the case η = 0, ν = ξ = 1, inserting Eqs. (2.2) into $a_i''$, we obtain $a_i'' = (x_i + ic^2 p_i)/\sqrt{2\hbar c^2}$ which is the annihilation operator in commutative space again.
the consistency of the deformed Heisenberg-Weyl algebra (2.1) with the deformed bosonic algebra (3.6).

Normal quantum mechanics in commutative space is a most successful theory, fully confirmed by experiments. It is correct from the atomic scale $10^{-10} m$ down to at least the scale $10^{-18} m$. It means that any corrections originated from spatial noncommutativity must be extremely small, thus both noncommutative parameters $\eta$ and $\theta$ must be extremely small. This is guaranteed by Eq. (3.4).  

In the context of quantum mechanics how to fix the proportional coefficient $K$ in Eq. (3.4) from a first principle is open.

Now we consider the second aspect. Following the standard procedure of constructing the Fock space of many-particle systems at the level of annihilation and creation operators in commutative space, in the following we take Eqs. (3.6) as the definition relations for the complete and closed deformed bosonic algebra without making further reference to its \( \hat{x}_i \), \( \hat{p}_i \) representations, generalize it to many-particle systems and find a basis of the Fock space.

We introduce the following auxiliary operators, the tilde annihilation and creation operators

\[
\tilde{a}_1 = \frac{1}{\sqrt{2\alpha_1}}(\hat{a}_1 + i\hat{a}_2), \quad \tilde{a}_2 = \frac{1}{\sqrt{2\alpha_2}}(\hat{a}_1 - i\hat{a}_2),
\]

where \( \alpha_{1,2} = 1 \pm \xi_2 \sqrt{\theta \eta / \bar{h}} \). From Eqs. (3.6) it follows that the commutation relations of \( \tilde{a}_i \) and \( \tilde{a}_j \) read

\[
[\tilde{a}_i, \tilde{a}_j] = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = \left[ \tilde{a}_i^\dagger, \tilde{a}_j^\dagger \right] = 0, \quad (i, j = 1, 2). \tag{3.11}
\]

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3 There are different bounds on the parameter $\theta$ set by experiments. The space-space noncommutative theory from string theory violates Lorentz symmetry and therefore strong bounds can be placed on the parameter $\theta$, the existing experiments [17] give $\theta/(\hbar c)^2 \leq (10 \text{TeV})^{-2}$. Comparing with the above estimation, other bounds on $\theta$ exist: measurements of the Lamb shift [4] give a weaker bound; clock-comparison experiments [18] claim a stronger bound. The magnitude of $\theta$ is surely extremely small.

4 In literature different relations between $\eta$ and $\theta$ were consideration. For example, Ref. [19] considered a general D-dimensional case, according their theoretical framework, the realization of the deformed Heisenberg-Weyl algebra $\hat{x}^{\mu}, \hat{x}^{\nu} = i \theta^{\mu\nu}$, $[\hat{x}^{\mu}, \hat{p}_\nu] = i \delta^{\mu\nu}$, and $[\hat{p}_\mu, \hat{p}_\nu] = i \eta^{\mu\nu}$ with a special relation $\eta^{\mu\nu} = -(\theta^{-1})_{\mu\nu}$ takes the form: $\hat{x}^{\mu} = \frac{1}{2} x^{\mu} - \theta^{\mu\nu} p_\nu$ and $\hat{p}_\mu = p_\mu - \frac{1}{2}(\theta^{-1})_{\mu\nu} x^{\nu}$, where Greek indices $\mu, \nu$ run over $0, 1, \ldots, D - 1$. An extremely small $\theta$ guarantees that corrections originated from spatial noncommutativity for $\hat{x}^{\mu}$ is extremely small (In a correct theoretical framework $\frac{1}{2} x^{\mu}$ should be $x^{\mu}$ so that $\hat{x}^{\mu}$ reduces to $x^{\mu}$ in the limit $\theta \to 0$). But an extremely small $\theta$ corresponds to an extremely large $\theta^{-1}$. It leads to that corrections from spatial noncommutativity for $\hat{p}_\mu$ is extremely large. The case $\eta^{\mu\nu} = -(\theta^{-1})_{\mu\nu}$ does not correspond to real physics.
Thus \( \tilde{a}_i \) and \( \tilde{a}_i^\dagger \) are explained as the deformed annihilation and creation operators in the tilde system. The tilde number operators \( \tilde{N}_1 = \tilde{a}_i^\dagger \tilde{a}_i \) and \( \tilde{N}_2 = \tilde{a}_2^\dagger \tilde{a}_2 \) commute each other, \([\tilde{N}_1, \tilde{N}_2] = 0\). A general tilde state

\[
|m, n\rangle \equiv (m!n!)^{-1/2}(\tilde{a}_1^\dagger)^m(\tilde{a}_2^\dagger)^n|\tilde{0}, \tilde{0}\rangle,
\]

where the vacuum state \( |\tilde{0}, \tilde{0}\rangle \) in the tilde system is defined as \( \tilde{a}_i|\tilde{0}, \tilde{0}\rangle = 0 \) \((i = 1, 2)\), is the common eigenstate of \( \tilde{N}_1 \) and \( \tilde{N}_2 \): \( \tilde{N}_1|m, n\rangle = m|m, n\rangle, \tilde{N}_2|m, n\rangle = n|m, n\rangle \), \((m, n = 0, 1, 2, \cdots)\), and satisfies \( \langle m', n'|m, n\rangle = \delta_{m'm}\delta_{n'n} \). Thus \( \{m, n\}\) constitute an orthogonal normalized complete basis of the tilde Fock space. In the tilde Fock space all calculations are the same as the case in commutative Fock space, thus the concept of identical particles is maintained and the formalism of the deformed Bosonic symmetry which restricts the states under permutations of identical particles in multi-boson systems can be similarly developed.

Ref. [10] also investigated the structure of a noncommutative Fock space and obtained eigenvectors of several pairs of commuting hermitian operators which can serve as basis vectors in the noncommutative Fock space. Calculations in such a noncommutative Fock space are much complex than the above (commutative) tilde Fock space.

4 Example

In literature the noncommutative-commutative correspondence has been investigated and noncanonical changes of variables in search of new characteristics have been undertaken [20–23]. In the tilde system constructed above calculations are easy for systems whose dynamical behavior can be treated at the level of annihilation-creation operators, where spatial noncommutativity are simply represented by parameters \( \alpha_i \), not represented by noncommutative operators.

As usual, harmonic oscillators serve as typical examples. In the hat system the Hamiltonian of the two-dimensional isotropic harmonic oscillator reads

\[
\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2\mu}\hat{p}_i\hat{p}_i + \frac{1}{2}\mu\omega^2\hat{x}_i\hat{x}_i.
\]

where \( \mu \) and \( \omega \) are the mass and frequency. In order to maintain the physical meaning of deformed annihilation-creation operators \( \hat{a}_i, \hat{a}_i^\dagger \) \((i = 1, 2)\) the relations among \((\hat{a}_i, \hat{a}_i^\dagger)\) and
(\hat{x}_i, \hat{p}_i) should keep the same formulation as the ones in commutative space. Thus they are
defined by
\[ \hat{a}_i = \sqrt{\frac{\mu \omega}{2 \hbar}} (\hat{x}_i + \frac{i}{\mu \omega} \hat{p}_i), \quad \hat{a}_i^\dagger = \sqrt{\frac{\mu \omega}{2 \hbar}} (\hat{x}_i - \frac{i}{\mu \omega} \hat{p}_i). \] (4.2)

From the condition \([\hat{a}_i, \hat{a}_j] = 0\), it follows that
\[ \eta = \mu^2 \omega^2 \theta. \] (4.3)

Using Eqs. (4.2), the Hamiltonian \(\hat{H}\) is rewritten by \(\hat{a}_i\) and \(\hat{a}_i^\dagger\) as
\[ \hat{H} = \hbar \omega \left( \hat{N}_1 + \hat{N}_2 + 1 \right), \] (4.4)

where \(\hat{N}_1 = \hat{a}_i^\dagger \hat{a}_1\) and \(\hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2\) are the number operators in the hat system. Because
the bosonic commutation relation (3.7) correlates different degrees of freedom, \(\hat{N}_1\) and \(\hat{N}_2\)
do not commute, \([\hat{N}_1, \hat{N}_2] \neq 0\). They have not common eigenstates. Though \([\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0\), the deformed Bosonic symmetry is not guaranteed in the hat system.\(^5\) This
difficulty can be simply solved in the tilde system. Using Eqs. (3.10) the Hamiltonian \(\tilde{H}\) can be represented by \(\tilde{a}_i\) and \(\tilde{a}_i^\dagger\) as
\[ \tilde{H} = \tilde{H} = \hbar \omega \left( \alpha_1 \tilde{N}_1 + \alpha_2 \tilde{N}_2 + 1 \right). \] (4.5)

It is worthy noting that all effects of spatial noncommutativity are included in the parameters \(\alpha_i\), not represented by noncommutative operators. Because the commutation relations

\(^5\) In the hat system the vacuum state is defined as \(\hat{a}_i |0,0\rangle = 0, (i = 1, 2)\). A general hat state \(|\tilde{m}, n\rangle\) is defined as
\[ |\tilde{m}, n\rangle \equiv c(\hat{a}_1^\dagger)^m (\hat{a}_2^\dagger)^n |0,0\rangle \]
where \(c\) is the normalization constant, these hat states \(|\tilde{m}, n\rangle\) are not the eigenstate of \(\hat{N}_1\) and \(\hat{N}_2\):
\[ \hat{N}_1 |\tilde{m}, n\rangle = m |\tilde{m}, n\rangle + \frac{i}{\hbar} m \xi^2 \sqrt{\theta \eta} |\tilde{m} + 1, n - 1\rangle, \]
\[ \hat{N}_2 |\tilde{m}, n\rangle = n |\tilde{m}, n\rangle + \frac{i}{\hbar} n \xi^2 \sqrt{\theta \eta} |\tilde{m} - 1, n + 1\rangle. \]

Because of Eq. (3.7), in calculations of the above equations we should take care of the ordering of \(\hat{a}_i\) and \(\hat{a}_j^\dagger\) for even \(i \neq j\) in the state \(|\tilde{m}, n\rangle\). The states \(|\tilde{m}, n\rangle\) are not orthogonal each other. For example, the
inner product between \(|1, 0\rangle\) and \(|0, 1\rangle\) is
\[ (1, 0 | 1, 0) = -\frac{i}{\hbar} \xi^2 \sqrt{\theta \eta}. \]
Thus \(|\tilde{m}, n\rangle\) do not constitute an orthogonal complete basis of the Fock space of a identical - boson system.
among \( \tilde{a}_i, \tilde{a}_i^\dagger \) and \( \tilde{N}_i \) are the same as ones in commutative space, eigenvalues of \( \tilde{H} \) can be directly read out from Eq. (4.5),

\[
\tilde{E}_{n_1,n_2} = \hbar \omega (\alpha_1 n_1 + \alpha_2 n_2 + 1) = \hbar \omega (n_1 + n_2 + 1) + \xi^2 \mu \omega^2 \theta (n_1 - n_2), \quad (n_1, n_2 = 0, 1, 2, \cdots).
\]

The last term in the above second equation is \( \theta \) dependent which represents the corrections of the energy level originated from the deformed bosonic algebra (3.6). There is no correction for the zero-point energy \( \hbar \omega \). It is worth noting that Eq. (4.6) gives the exact (non-perturbative) eigenvalues.

In order to appreciate noncommutative corrections of the deformed bosonic algebra (3.6) to the physical observables, we compare the above results with ones obtained from the case of only position - position noncommuting. For the later case Ref. [23] obtained the energy spectrum of noncommutative oscillators with mass \( \mu \) and frequency \( \omega \),

\[
E'_{n_1,n_2} = \hbar \Omega (n_1 + n_2 + 1) - M \Omega^2 \theta (n_1 - n_2)/2, \quad \text{where} \quad 1/M \equiv 1/\mu + \mu \theta^2 \omega^2/4\hbar^2 \quad \text{and} \quad M \Omega^2 \equiv \mu \omega^2,
\]

thus \( \Omega = \left(1 + \mu^2 \theta^2 \omega^2 / 4\hbar^2 \right)^{1/2} \approx \omega + \mu^2 \theta^2 \omega^3 / 8\hbar^2 \). The energy spectrum \( E'_{n_1,n_2} \) can be approximately represented as

\[
E'_{n_1,n_2} \approx \hbar \omega (n_1 + n_2 + 1) + \frac{\mu^2 \theta^2 \omega^3}{8\hbar} (n_1 + n_2 + 1) - \frac{1}{2} \mu \omega^2 \theta (n_1 - n_2) .
\]

In the above the \( \theta \) dependent terms appreciate the noncommutative corrections of energy spectrum of noncommutative oscillators to the commutative ones. It shows that their behavior of noncommutative corrections is different from ones in Eq. (4.6) originated from the deformed bosonic algebra (3.6). Specially, there is a shift \( \mu^2 \theta^2 \omega^3 / 8\hbar \) of the zero-point energy in \( E'_{n_1,n_2} \).

5 Summary and Discussions

(i) In the tilde Fock space the deformed Bosonic symmetry is maintained, the investigation of the consistent deformed bosonic algebra is completed, all calculations can be similarly developed as if in commutative space and all effects of spatial noncommutativity are simply represented by parameters \( \alpha_i \). Such a noncommutative-commutative correspondence in the
tilde system works for general systems whose dynamical behavior can be investigated at the level of annihilation-creation operators.

(ii) On the fundamental level of quantum field theory the annihilation and creation operators appear in the expansion of the (free) field operator \( \Psi(x) = \int d^3 k a_k(t) \Phi_k(x) + H.c. \)

The consistent multi-particle interpretation requires the usual (anti)commutation relations among \( a_k \) and \( a_k^\dagger \). The noncommutative extension of quantum field theory was obtained by deforming the ordinary product between quantum fields into the Moyal "star" product. For the case of both position - position and momentum - momentum noncommutativity, however, the corresponding investigation on the fundamental level of noncommutative quantum field theory is involved. Noncommutative quantum mechanics, as the one-particle sector of noncommutative quantum field theory, can be treated in a more or less self-contained way so that a more detailed study of quantum systems at the level of noncommutative quantum mechanics should be useful. It is expected that some qualitative features obtained at the level of noncommutative quantum mechanics may survive at the level of noncommutative quantum field theory. Therefore investigations of the deformed bosonic algebra at the level of noncommutative quantum mechanics may give some clue for further development in noncommutative quantum field theory. Studies on noncommutative corrections of the deformed bosonic algebra (3.6) on the fundamental level of quantum field theory will be the next step.

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