CUBULATING SMALL CANCELLATION FREE PRODUCTS

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Abstract. We give a simplified approach to the cubulation of small-cancellation quotients of free products of cubulated groups. We construct fundamental groups of compact nonpositively curved cube complexes that do not virtually split.

1. Introduction

Martin and Steenbock recently showed that a small-cancellation quotient of a free product of cubulated groups is cubulated [MS17]. In this paper we revisit their theorem in a slightly weaker form, and reprove it in a manner that capitalizes on the available technology. Combined with an idea of Pride’s about small-cancellation groups that do not split, we answer a question posed to us by Indira Chatterji by constructing an example of a compact nonpositively curved cube complex $X$ such that $\pi_1 X$ is nontrivial but does not virtually split.

Section 2 recalls the definitions and theorems that we will use from cubical small-cancellation theory. Section 3 recalls properties of the dual cube complex in the relatively hyperbolic setting. Section 4 recalls the definition of small-cancellation over free products, and describe associated cubical presentations. Section 5 reproves Pride’s result about small-cancellation groups that do not split. Section 6 relates small-cancellation over free products to cubical small-cancellation theory, and proves our main result which is Theorem 6.2. Finally, Section 7 combines Pride’s method with Theorem 6.2 to provide cubulated groups that do not virtually split in Example 7.1.

2. Background on Cubical Small Cancellation

2.1. Nonpositively curved cube complexes. We shall assume that the reader is familiar with $CAT(0)$ cube complexes which are $CAT(0)$ spaces having cell structures, where each cell is isometric to a cube. We refer the reader to [BH99, Sag95, Lea, Wis21]. A nonpositively curved cube complex is a cell-complex $X$ whose universal cover $\tilde{X}$ is a $CAT(0)$ cube complex. A hyperplane $\tilde{U}$ in $\tilde{X}$ is a subspace whose intersection with each $n$-cube $[0,1]^n$ is either empty or consists of the subspace where exactly one coordinate is restricted to $\frac{1}{2}$. For a hyperplane $\tilde{U}$ of $\tilde{X}$, we let $N(\tilde{U})$ denote its carrier, which is the union of all closed cubes intersecting $\tilde{U}$. The hyperplanes $\tilde{U}$ and $\tilde{V}$ osculate.
if $N(\tilde{U}) \cap N(\tilde{V}) \neq \emptyset$ but $\tilde{U} \cap \tilde{V} = \emptyset$. We will use the combinatorial metric on a nonpositively curved cube complex $X$, so the distance between two points is the length of the shortest combinatorial path connecting them. The systole $\|X\|$ is the infimal length of an essential combinatorial closed path in $X$. A map $\phi : Y \to X$ between nonpositively curved cube complexes is a local isometry if $\phi$ is locally injective, $\phi$ maps open cubes homeomorphically to open cubes, and whenever $a, b$ are concatenatable edges of $Y$, if $\phi(a)\phi(b)$ is a subpath of the attaching map of a 2-cube of $X$, then $ab$ is a subpath of a 2-cube in $Y$.

2.2. Cubical presentations and Pieces.

Definition 2.1. A cubical presentation $\langle X \mid Y_1, \ldots, Y_m \rangle$ consists of a nonpositively curved cube complex $X$, and a set of local isometries $Y_i \hookrightarrow X$ of nonpositively curved cube complexes. We use the notation $X^*$ for the cubical presentation above. As a topological space, $X^*$ consists of $X$ with a cone on $Y_i$ attached to $X$ for each $i$.

We often consider the universal cover $\tilde{X}^*$, whose cubical part is the preimage of $X$ under the covering map. The cubical part serves as a “Cayley graph”, whose “relators” are the cones.

Definition 2.2. A cone-piece of $X^*$ in $Y_i$ is a component of $\tilde{Y}_i \cap \tilde{Y}_j$, where $\tilde{Y}_i$ is a lift of $Y_i$ to the universal cover $\tilde{X}^*$, excluding the case where $i = j$. A wall-piece of $X^*$ in $Y_i$ is a component of $\tilde{Y}_i \cap N(\tilde{U})$, where $\tilde{U}$ is a hyperplane that is disjoint from $\tilde{Y}_i$. For a constant $\alpha > 0$, we say $X^*$ satisfies the $C'(\alpha)$ small-cancellation condition if $\text{diam}(P) < \alpha \|Y_i\|$ for every cone-piece or wall-piece $P$ involving $Y_i$.

When $\alpha$ is small, the quotient $\pi_1 X^*$ has good behavior. For instance, when $X^*$ is $C'(\frac{1}{12})$ then each immersion $Y_i \hookrightarrow X$ lifts to an embedding $Y_i \hookrightarrow \tilde{X}^*$. This is proven in [Wis21, Thm 4.1], and we also refer to [Jan17] for analogous results at $\alpha = \frac{1}{9}$.

2.3. The $B(8)$ condition. We now describe a special case of the $B(8)$ condition within the context of $C'(\alpha)$ metric small-cancellation. A piece-path in $Y$ is a path in a piece of $Y$.

Definition 2.3. A cubical presentation $X^*$ satisfies the $B(8)$ condition if there is a wallspace structure on each $Y_i$ as follows:

(1) The collection of hyperplanes of each $Y_i$ are partitioned into classes such that no two hyperplanes in the same class cross or osculate, and the union $U = \bigcup U_k$ of the hyperplanes in a class forms a wall in the sense that $Y_i - U$ is the disjoint union of a left and right halfspace.

(2) If $P$ is a path that is the concatenation of at most 8 piece-paths and $P$ starts and ends on the carrier $N(U)$ of a wall then $P$ is path-homotopic into $N(U)$.

(3) The wallspace structure is preserved by the group $\text{Aut}(Y_i \to X)$ which consists of automorphisms $\phi : Y_i \to Y_i$ such that $\phi \downarrow_X$ commutes.
2.4. Properness Criterion. A closed-geodesic $w \rightarrow Y$ in a nonpositively curved cube complex, is a combinatorial immersion of a circle whose universal cover $\tilde{w}$ lifts to a combinatorial geodesic $\tilde{w} \rightarrow \tilde{Y}$ in the universal cover of $\tilde{Y}$.

We quote the following criterion from [FW21, Thm 3.5].

**Theorem 2.4.** Let $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$ be a cubical presentation. Suppose $X$ is compact, and each $Y_i$ is compact and deformation retracts to a closed combinatorial geodesic $w_i$. Additionally, suppose that for every hyperplane $U$ of $Y_i$ the complement $Y_i \setminus U$ is contractible, and $U$ has an embedded carrier with diameter $N(U) < \frac{1}{20} \|Y_i\|$. If $X^*$ is $C'(\frac{1}{20})$ then $X^*$ is $B(8)$ and $\pi_1 X^*$ acts properly and cocompactly on the CAT(0) cube complex dual to the wallspace on $\tilde{X}^*$.

Moreover, if each $\pi_1 Y_i \subset \pi_1 X$ is a maximal cyclic subgroup, then $\pi_1 X^*$ acts freely and cocompactly on the associated dual CAT(0) cube complex.

The wallspace that is assigned to each $Y_i$ in the above theorem has a wall for hyperplanes dual to pairs of antipodal edges in $w_i$. (The complex $X$ is subdivided to ensure that each $|w_i|$ is even.)

2.5. The wallspace structure.

**Definition 2.5 (The walls).** When $X^*$ satisfies the $B(8)$ condition, $\tilde{X}^*$ has a wallspace structure which we now briefly describe: Two hyperplanes $H_1, H_2$ of $\tilde{X}^*$ are cone-equivalent if $H_1 \cap Y_i$ and $H_2 \cap Y_i$ lie in the same wall of $Y_i$ for some lift $Y_i \rightarrow \tilde{X}^*$. Cone-equivalence generates an equivalence relation on the collection of hyperplanes of $\tilde{X}^*$. A wall of $\tilde{X}^*$ is the union of all hyperplanes in an equivalence class. When $X^*$ is $B(8)$, the hyperplanes in an equivalence class are disjoint, and a wall $w$ can be regarded as a wall in the sense that $\tilde{X}^*$ is the union of two halfspaces meeting along $w$.

**Lemma 2.6.** Let $W$ be a wall of $\tilde{X}^*$. Let $Y \subset \tilde{X}^*$ be a lift of some cone $Y_i$ of $X^*$. Then either $W \cap Y = \emptyset$ or $W \cap Y$ consists of a single wall of $Y$.

The carrier $N(W)$ of a wall $W$ of $\tilde{X}^*$ consists of the union of all carriers of hyperplanes of $W$ together with all cones intersected by hyperplanes of $W$. The following appears as [Wis21, Cor 5.30]:

**Lemma 2.7 (Walls quasi-isometrically embed).** Let $X^*$ be $B(8)$. Suppose that pieces have uniformly bounded diameter. Then for each wall $W$, the map $N(W) \rightarrow \tilde{X}^*$ is a quasi-isometric embedding with uniform quasi-isometry constants.

We will need the following result of Hruska which is proven in [Hru10, Thm 1.5]:

**Theorem 2.8.** Let $G$ be a f.g. group that is hyperbolic relative to $\{G_i\}$. Let $H \subset G$ be a f.g. subgroup that is quasi-isometrically embedded. Then $H \subset G$ is relatively quasiconvex.

3. Relative Cocompactness

The following is a simplified restatement of [HW14, Thm 7.12] in the case $\heartsuit = \star$. We focus it on our application where the wallspace arises from a cubical presentation. We use the notation $N_d(S)$ for the closed $d$-neighborhood of $S$.\footnote{There is a small misstatement in [HW14 Thm 7.12], as it requires that $r \geq r_0$ for some constant $r_0$.}
Theorem 3.1. Consider the wallspace $((\tilde{X}^*, W))$. Suppose $G$ acts properly and cocompactly on the cubical part of $\tilde{X}^*$ preserving both its metric and wallspace structures, and the action on $W$ has only finitely many $G$–orbits of walls. Suppose $\text{Stabilizer}(W)$ is relatively quasiconvex and acts cocompactly on $W$ for each wall $W \in W$. Suppose $G$ is hyperbolic relative to $\{G_1, \ldots, G_r\}$. For each $G_i$ let $\tilde{X}_i \subset \tilde{X}^*$ be a nonempty $G_i$–invariant $G_i$–cocompact subspace. Let $C(\tilde{X}^*)$ be the cube complex dual to $(\tilde{X}^*, W)$ and for each $i$ let $C(\tilde{X}_i)$ be the cube complex dual to $(\tilde{X}^*, W_i)$ where $W_i$ consists of all walls $W$ with the property that $\text{diam}(W \cap N_d(\tilde{X}_i)) = \infty$ for some $d = d(W)$.

Then there exists a compact subcomplex $K$ such that $C(\tilde{X}^*) = GK \cup \bigcup_i GC(\tilde{X}_i)$. Hence $G$ acts cocompactly on $C(\tilde{X}^*)$ provided that each $C(\tilde{X}_i)$ is $G_i$–cocompact.

In our application of Theorem 3.1, $X$ is a “long” wedge of cube complexes $X_1, \ldots, X_r$ (see Construction 4.3 for the definition) and $\tilde{X}_i$ is a lift of the universal cover of $X_i$ to $\tilde{X}^*$. The wallspace structure of $X^*$ is described in Section 2.5 (see also Lemma 4.4). We will be able to apply Theorem 3.1 because the cube complex $C_{\ast}(\tilde{X}_i)$ will be $G_i$–cocompact for the following reason:

Lemma 3.2. Let $G$, $(X^*, W)$ be as in Theorem 3.1 and suppose that $X$ satisfies $C'(\frac{1}{10})$. Additionally assume that each $\tilde{X}_i$ has the property that if $s$ is a square with an edge in $\tilde{X}_i$ then $s \subset \tilde{X}_i$. Let $W$ be a wall of $\tilde{X}^*$. Suppose $\text{diam}(W \cap N_d(\tilde{X}_i)) = \infty$ for some $i, d$. Then $W$ contains a hyperplane of $\tilde{X}_i$. Hence $C_{\ast}(\tilde{X}_i) = \tilde{X}_i$ for each $i$.

Proof. Suppose $\text{diam}(N(W) \cap N_d(\tilde{X}_i)) = \infty$. By cocompactness of the action $\text{Stabilizer}(W)$ on $N(W)$ and $G_i$ on $\tilde{X}_i$ there is an infinite order element $g$ stabilizing both $W$ and $\tilde{X}_i$.

Each $\tilde{X}_i \subset \tilde{X}^*$ is convex by [Wis21] Lem 3.74, and we may therefore choose a geodesic $\tilde{\gamma}$ in $\tilde{X}_i$ that is stabilized by $g$, and let $\tilde{\lambda}$ be a path in $N(W)$ that is stabilized by $g$. We thus obtain an annular diagram $A$ between closed paths $\gamma$ and $\lambda$ which are the quotients of $\tilde{\gamma}$ and $\tilde{\lambda}$ by $(g)$. Suppose moreover that $A$ has minimal complexity among all such choices $(A, \gamma, \lambda)$ where $\gamma \rightarrow X_i$ has the property that $\tilde{\gamma}$ is a geodesic, and $\lambda \rightarrow N(W)$ is a closed path. By [Wis21] Thm 5.61], $A$ is a square annular diagram, and we may assume it is has no spur. Note that [Wis21] Thm 5.61] requires “tight innerpaths” which holds at $C'(\frac{1}{10})$ by [Wis21] Lem 3.70].

Observe that if $s$ is a square with an edge in $\tilde{X}_i$, then $s \subset \tilde{X}_i$. Consequently, the minimality of $A$ ensures that $A$ has no square, and so $\gamma = A = \lambda$.

There are now two cases to consider: Either $\tilde{\lambda} \subset N(U)$ for some hyperplane $U$ of $W$, or $\tilde{\lambda}$ has a subpath $u_1yju_2$ traveling along $N(U_1), Y_j, N(U_2)$, where $U_1, U_2$ are distinct hyperplanes of $W$, and $U_1, U_2$ intersect the cone $Y_j$ in antipodal hyperplanes.

In the latter possibility the $B(8)$ condition is contradicted for $Y_j$, since $\tilde{X}_i \cap Y_j$ contains the single piece-path $y_j$ which starts and ends on carriers of distinct hyperplanes of the same wall of $Y_j$.

In the former possibility, $N(U) \cap \tilde{X}_i \neq \emptyset$, and so the above square observation ensures that $N(U) \subset \tilde{X}_i$. Hence $W$ intersects $\tilde{X}_i$ as claimed.
Example 3.3. Consider the quotient: \( G = \mathbb{Z}^2 \ast \mathbb{Z}^2/\langle \langle w_1, w_2 \rangle \rangle \), with the following presentation for some number \( m > 0 \):

\[
\left\langle a, b \mid aba^{-1}b^{-1} \right\rangle \ast \left\langle c, d \mid cdc^{-1}d^{-1} \right\rangle \mid a^1c^1a^2c^2 \cdots a^mc^m, b^1d^1b^2d^2 \cdots b^md^m
\]

Note that each piece consists of at most 2 syllables, whereas the syllable length (see Definition 4.1) of each relator is \( 2m \). Hence the \( C_*(\frac{1}{m-1}) \) small-cancellation condition over free products is satisfied. See Definition 4.1.

The associated space \( X \) is the long wedge (see Construction 4.3) of two tori \( X_1, X_2 \) corresponding to \( \langle a, b \rangle \) and \( \langle c, d \rangle \). For \( i \in \{1, 2\} \), let \( Y_i \) be a square complex built out from an alternating sequence of rectangles and arcs as in Figure 2.

The cube complex dual to \( \tilde{X} \) has \( m(m+1) \) -dimensional cubes arising from the cone-cells \( Y_1 \) and \( Y_2 \). More interestingly, the cube complex dual to \( (\tilde{X}, W_1) \) where \( W_1 \) consists of the walls intersecting a copy of \( \tilde{X}_1 \), has dimension \( 2m \). This is because all hyperplanes dual to the path \( a^m \) cross each other because of \( Y_1 \) and likewise all hyperplanes dual to the path \( b^m \) cross each other because of \( Y_2 \), and every hyperplane dual to the path \( a^m \) crosses every hyperplane dual to the path \( b^m \) because \( \tilde{X}_1 \) is a 2-flat.

4. Small cancellation over free products

Definition 4.1 (The \( C_*(\frac{1}{n}) \) small cancellation over a free product). Every element \( R \) in the free product \( G_1 \ast \cdots \ast G_r \) has a unique normal form which is a word \( h_1 \cdots h_n \) where each \( h_i \) lies in a factor of the free product and \( h_i \) and \( h_{i+1} \) lie in different factors for \( i = 1, \ldots, n - 1 \). The number \( n \), which we denote by \( |R|_s \), is the syllable length of \( R \). We say \( R \) is cyclically reduced if \( h_1 \) and \( h_n \) also lie in different factors. We say that \( R \) is weakly cyclically reduced if \( h_n^{-1} \neq h_1 \) or if \( |R|_s \leq 1 \). We refer to each \( h_i \) as a syllable.

There is a cancellation in the concatenation \( P \cdot U \) of two normal forms if the last syllable of \( P \) is the inverse of the first syllable of \( U \).

Consider a presentation over a free product \( \langle G_1 \ast \cdots \ast G_r \mid R_1, \ldots, R_s \rangle \) where each \( R_i \) is a cyclically reduced word in the free product. A word \( P \) is a piece in \( R_i, R_j \) if \( R_i, R_j \) have weakly cyclically reduced conjugates \( R_i', R_j' \) that can be written as concatenations \( P \cdot U_i \) and \( P \cdot U_j \) respectively with no cancellations. The presentation is \( C_*(\frac{1}{n}) \) small cancellation if \( |P|_s < \frac{1}{n} |R_i|_s \) whenever \( P \) is a piece.
If $G$ is a $C_4' \left( \frac{1}{6} \right)$ small-cancellation quotient of a free product $G_1 \ast \cdots \ast G_r$, then each factor $G_i$ embeds in $G$. In particular, $G$ is nontrivial if some $G_i$ is nontrivial. We quote the following result from [Osi06]:

**Lemma 4.2.** Let $G$ be a quotient of $G_1 \ast \cdots \ast G_r$ arising as a $C_4' \left( \frac{1}{6} \right)$ small-cancellation presentation over a free product. Then $G$ is hyperbolic relative to $\{G_1, \ldots, G_r\}$.

### 4.1. Cubical presentation associated to a presentation over a free product.

**Construction 4.3.** Let $T_r$ be the union of directed edges $e_1, \ldots, e_r$ identified at their initial vertices. The long wedge of a collection of spaces $X_1, \ldots, X_r$ is obtained from $T_r$ by gluing the basepoint of each $X_j$ to the terminal vertex of $e_j$. We will later subdivide the edges of $T_r$. Given groups $G_1, \ldots, G_r$ such that for each $1 \leq j \leq r$, let $G_j = \pi_1 X_j$ where $X_j$ is a nonpositively curved cube complex, the long wedge $X$ of the collection $X_1, \ldots, X_r$ is a cube complex with $\pi_1 X = G_1 \ast \cdots \ast G_r$.

Given an element $R \in G_1 \ast \cdots \ast G_r$ with $|R|_* > 1$, there exists a local isometry $Y \to X$ where $Y$ is a compact nonpositively curved cube complex with $\pi_1 Y = \langle R \rangle$. Indeed, let $R = h_1 h_2 \cdots h_t$ where each $h_k$ is an element of some $G_{m(k)}$. For each $k$ let $V_k$ be the compact cube complex that is the combinatorial convex hull of the basepoint $p$ and its translate $h_k p$ in the universal cover $\tilde{X}_{m(k)}$. We call $p$ the initial vertex of $V_k$ and $h_k p$ the terminal vertex of $V_k$. For each $1 \leq k \leq t$ let $\sigma_k$ be a copy of $e_{m(k)}^{t} e_{m(k+1)}^{t}$ where $m(t + 1) = m(1)$. Finally we form $Y$ from $\bigsqcup_{k=1}^{t} V_k$ and $\bigsqcup_{k=1}^{t} \sigma_k$ by gluing the terminal vertex of $V_k$ to the initial vertex of $\sigma_k$ and the terminal vertex of $\sigma_k$ to the initial vertex of $V_{k+1}$. Note that there is an induced map $Y \to X$ which is a local isometry. See Figure 2

Given a presentation $\langle G_1, \ldots, G_r \mid R_1, \ldots, R_s \rangle$ over a free product there is an associated cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_s \rangle$ where each $Y_i \to X$ is a local isometry associated to $R_i$ as above. Finally, any subdivision of the edges $e_1, \ldots, e_r$ induces a subdivision of $X$, and accordingly a subdivision of each $Y_i$. We thus obtain a new cubical presentation that we continue to denote by $X^*$. 
Lemma 4.4. Suppose \( \langle X \mid Y_1, \ldots, Y_s \rangle \) is \( B(8) \) (after subdividing). And let \( \tilde{X}_k \) be the universal cover of \( X_k \) with the wallspace structure such that each hyperplane is a wall. Then \( \langle X \mid Y_1, \ldots, Y_s, \tilde{X}_1, \ldots, \tilde{X}_r \rangle \), where the maps \( \tilde{X}_j \to X \) are the local isometries factoring as \( \tilde{X}_j \to X_j \to X \), is \( B(8) \). Moreover, the two wallspace structures can be chosen so that the walls of \( \tilde{X}^* \) induced by the two structures are identical.

Proof. We choose the same wallspace structure on each \( Y_i \) as before, and the natural wallspace structure given by the hyperplanes on each \( \tilde{X}_j \). The cone-pieces between \( \tilde{X}_j \) and \( Y_i \) are copies of the \( V_k \) associated to \( X_j \) that appear in \( Y_i \), and hence Condition 2.3[2] holds for each \( Y_i \) as before. For each \( \tilde{X}_j \), Conditions 2.3[1] and 2.3[3] hold automatically by our choice of wallspace structure, and Condition 2.3[2] holds since \( \tilde{X}_j \) is contractible. \( \square \)

Corollary 4.5. For each wall \( W \) of \( \tilde{X}^* \), the intersection of \( W \cap \tilde{X}_j \) is either empty or consists of a single hyperplane.

Proof. This follows by combining Lemma 4.4 and Lemma 2.6 \( \square \)

5. CONSTRUCTION OF PRIDE

The following result was proven by Pride in [Pri83]. We give a slightly more geometric version of his proof, which was originally proven only for a \( C(n) \) presentation instead of a \( C'(\frac{1}{n}) \) presentation, which we can obtain as in Remark 5.2.

Lemma 5.1. Let \( G = \langle x, y \mid R_1, R_2, R_3, R_4, R_5, R_6 \rangle \) where the relators \( R_i \) are specified below for associated positive integers \( \alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i \) for each \( 1 \leq i \leq k \), and \( k \geq 1 \). Then \( G \) does not split as an amalgamated product or HNN extension.

\[
R_1(x, y) = xy^{\alpha_1}xy^{\alpha_2} \cdots xy^{\alpha_k}
R_2(x, y) = yx^{\beta_1}yx^{\beta_2} \cdots yx^{\beta_k}
R_3(x, y) = x^{\gamma_1}yx^{-\delta_1}x^{\gamma_2}yx^{-\delta_2} \cdots x^{\gamma_k}yx^{-\delta_k}
R_4(x, y) = xy^{\rho_1}xy^{-\rho_1}xy^{\rho_2}xy^{-\rho_2} \cdots xy^{\rho_k}xy^{-\rho_k}
R_5(x, y) = yx^{\sigma_1}yx^{-\rho_1}yx^{\sigma_2}yx^{-\rho_2} \cdots yx^{\sigma_k}yx^{-\rho_k}
R_6(x, y) = (xy)^{\tau_1}(x^{-1}y^{-1})^{\theta_1}(xy)^{\tau_2}(x^{-1}y^{-1})^{\theta_2} \cdots (xy)^{\tau_k}(x^{-1}y^{-1})^{\theta_k}
\]

Proof. Suppose \( G = A \ast_C B \) or \( G = A \ast_C \) and let \( T \) be the associated Bass-Serre tree. Without loss of generality, assume that the translation length of \( y \) is at least as large as the translation length of \( x \). Choose a vertex \( v \in \text{Min}(x) \) for which \( d_T(y,v,v) \) is minimal.

For use in the argument below, given a decomposition of \( w \in G \) as a product \( w = w_1w_2 \cdots w_t \), the path \( [v, w_1 \cdot v][w_1 \cdot v, w_1w_2 \cdot v] \cdots [w_1w_2 \cdots w_{t-1} \cdot v, w_1w_2 \cdots w_t \cdot v] \) is said to read \( w \).

We now show that \( v \in \text{Min}(y) \). First suppose that \( x, \) and hence \( y, \) is a hyperbolic isometry. If \( v \notin \text{Min}(y) \), i.e. \( \text{Min}(x) \cap \text{Min}(y) = \emptyset \), then the axes of \( x \) and \( y \) in \( T \) are disjoint, and \( v \) is a vertex in the axis of \( x \) minimizing the distance between the two axes. In particular, the concatenation of two nontrivial geodesics \( [x^{-1}v,v][v,yv] \) would be a geodesic. See Figure 3 \( \square \) Similarly \( [x^{-1}v,v][v,yv] \), \( [x^{-1}v,v][y^{-1}v,v] \) and \( [x^{-1}v,v][y^{-1}v,v] \) would be geodesics. Consequently, regarding \( R_6 \) as a product of elements \( \{x^{\pm 1}, y^{\pm 1}\} \), we
see that the path reading $R_6$ would be a geodesic, which contradicts that $R_6 = G$. Now, suppose that $x$ is elliptic and so $x \cdot v = v$. Let $e$ denote the initial edge of $[v, y \cdot v]$ and note that $e$ is also the initial edge of $[v, y^{-1} \cdot v]$ since $v \notin \text{Min}(y)$. The choice of $v$ implies $x \cdot e \neq e$, as otherwise the other endpoint $v'$ of $e$ would satisfy $d_T(y \cdot v', v') < d_T(y \cdot v, v)$. Thus the concatenation of the nontrivial geodesics $[y^{-1} \cdot v, v, x \cdot y \cdot v]$ is a geodesic, and similarly for $[y^{-1} \cdot v, v, x^{-1} \cdot y^{-1} \cdot v], [y \cdot v, v, x \cdot y \cdot v]$ and $[y \cdot v, v, x^{-1} \cdot y^{-1} \cdot v]$. It follows that regarding $R_6$ as a product of elements $\{x, x^{-1} y^{-1}\}$, the path reading $R_6$ is a geodesic, which contradicts that $R_6 = G$. Therefore $v \in \text{Min}(y)$.

Since $v \in \text{Min}(x) \cap \text{Min}(y)$, the element $y$ is a hyperbolic isometry, because otherwise $x, y$ are elliptic and so $v$ is a global fixed point. Suppose $x$ is also a hyperbolic isometry. At least one of $[y^{-1} \cdot v, v, x \cdot v]$ or $[x^{-1} \cdot v, v, y \cdot v]$ is not a geodesic, because otherwise the path reading $R_1$ regarded as a product of $\{x^\pm 1, y^\pm 1\}$ would be a geodesic. Consequently, both $[x \cdot v, v, y \cdot v]$ and $[x^{-1} \cdot v, v, y \cdot v]$ are geodesics, and hence regarding $R_3$ as a product of elements $\{x^\pm 1, y^\pm 1\}$, the path reading $R_3$ must be a geodesic, which is a contradiction. Thus, $x$ is an elliptic isometry.

Let $e_+$ and $e_-$ denote the initial edges of $[v, y \cdot v]$ and $[v, y^{-1} \cdot v]$ respectively. See Figure 4. Let us explain why $x \cdot e_+ = e_-$. Otherwise $[y^{-1} \cdot v, v, x \cdot y \cdot v]$ would be a geodesic since the last edge of $[y^{-1} \cdot v, v, x \cdot y \cdot v]$ is $e_-$ and the first edge of $[v, y \cdot v]$ is $x \cdot e_+$. Likewise, for $n, m > 0$ the path $[y^n \cdot v, v, x \cdot y^m \cdot v]$ would be a geodesic, and so too would be its translate $[v, x \cdot y^n \cdot v, x \cdot y^m \cdot v]$. Regarding $R_1$ as a product $(xy^a)(xy^b) \cdots (xy^k)$, the path reading $R_1$ would be a geodesic, contradicting $R_1 = G$.

Since $x \cdot e_+ = e_-$, neither $e_-$ nor $e_+$ is fixed by $x$. For any $n, m > 0$ the last edge of $[y^n \cdot v, v, x \cdot y^m \cdot v]$ is $e_+$ and the first edge of $[v, x \cdot y^m \cdot v]$ is $x \cdot e_+ = e_- \neq e_+$, and so the path $[y^n \cdot v, v, x \cdot y^m \cdot v]$ is a geodesic, and so is $[v, y^{-n} \cdot v, y^{-n} \cdot x \cdot y^m \cdot v]$. Similarly, the last edge of $[y^n \cdot v, v, x \cdot y^m \cdot v]$ is $e_-$ and the first edge of $[v, x \cdot y^m \cdot v]$ is $x \cdot e_- \neq e_-$, and so the path $[y^n \cdot v, v, x \cdot y^m \cdot v]$ is a geodesic as is $[v, x \cdot y^n \cdot v, x \cdot y^m \cdot v]$. Regarding $R_4$ as a product $(xy^a)(xy^b) \cdots (xy^k)(xy^l)$, we see that the path reading $R_4$ is a geodesic, contradicting $R_4 = G$. This completes the proof. \hfill \square

**Remark 5.2.** In the context of Lemma 5.1, for each $n$ there are choices of $k$ and $\{\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i : 1 \leq i \leq k\}$, such that the presentation is $C^{(\frac{1}{k})}$.

Given $n > 1$, let $k = 3n$ and choose $8k$ numbers $\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i$ that are all different and lie between $50n$ and $75n$. Then any piece $P$ in $R_i$ where $i \neq 6$ is of the form $x^i y x^m$ or $y^i x y^m$ for some $i, m$ (possibly 0). Thus $|P| \leq l + m + 1 \leq 150n + 1$. We also have $|R_i| \geq (k + 1)50n = (3n + 1)50n$ and so $|P| \leq \frac{1}{n}(150n + 1) \leq \frac{1}{n}|R_i|$. If $P$ is a piece in $R_6$, then $P$ is of the form $(xy)^l(x^{-1} y^{-1})^m$ and so $|P| \leq 2(l + m) \leq 300n$. We also have $|R_6| = 2(\tau_1 + \theta_1 + \tau_2 + \cdots + \theta_k) \geq 2(2k)50n = 600n^2$. Hence $|P| \leq \frac{1}{n}|R_6|$.
Corollary 5.3. Let $G_1, \ldots, G_r$ be nontrivial groups generated by finite sets of infinite order elements, and suppose $r > 1$. For each $n > 0$ there is a finitely related $C_n^r(\frac{1}{n})$ quotient $G$ of $G_1 \ast \cdots \ast G_r$ that does not split.

Proof. Let $S_p$ be the given generating set of $G_p$ for each $p$, and assume no proper subset of $S_p$ generates $G_p$. The desired quotient $G$ arises from a presentation $\langle G_1 \ast \cdots \ast G_r \mid R \rangle$, where following Lemma 5.1 the set of relators is:

$$\mathcal{R} = \{ R_\ell(x, y) : 1 \leq \ell \leq 6, \ (x, y) \in S_p \times S_q, \text{ where } 1 \leq p < q \leq r \}$$

where $k(x, y) = 3n$ for each $(x, y) \in S_p \times S_q$, and where the constants $\alpha_i(x, y), \beta_i(x, y), \gamma_i(x, y), \delta_i(x, y), \rho_i(x, y), \sigma_i(x, y), \tau_i(x, y), \theta_i(x, y)$ will be described below. For each $(x, y) \in S_p \times S_q$, let $\alpha_i(x, y), \delta_i(x, y), \rho_i(x, y)$ be distinct integers $> 1$ such that $y^m \notin \langle z \rangle$ for $m \in \{ \alpha_i(x, y), \delta_i(x, y), \rho_i(x, y) \}$ and $z \in S_q - \{ y \}$. This is possible because $y$ has infinite order and $y \notin \langle z \rangle$. Similarly, let $\beta_i(x, y), \gamma_i(x, y)$ and $\sigma_i(x, y)$ be distinct integers $> 1$ such that $x^m \notin \langle z \rangle$ for $m \in \{ \beta_i(x, y), \gamma_i(x, y), \sigma_i(x, y) \}$ and $z \in S_p - \{ x \}$.

Finally, let $\tau_i(x, y)$ and $\theta_i(x, y)$ be distinct integers between $10n$ and $20n$.

Having chosen the above constants for each $(x, y) \in S_p \times S_q$, we now show that the presentation for $G$ is $C_n^r(\frac{1}{n})$. We begin by observing that each $|R_\ell(x, y)|_s \geq 6n$. Let $P$ be a piece in $R_1 = R_{\ell_1}(x_1, y_1)$ and $R_2 = R_{\ell_2}(x_2, y_2)$ where $x_1 \in S_{p_1}, y_1 \in S_{q_1}, x_2 \in S_{p_2}, y_2 \in S_{q_2}$. If $\{p_1, q_1\} \neq \{p_2, q_2\}$ then $|P|_s \leq 1$. Assume that $\{p_1, q_1\} = \{p_2, q_2\}$. First suppose that $\ell_1 \neq 6$, then $|P|_s \leq 3$. Indeed, if $|P|_s \geq 4$ then two consecutive syllables would appear in distinct cyclically reduced forms of relators, which contradicts our choice of the constants. If $\ell_1 = 6$, then $|P|_s \leq \max\{ \tau_i(x, y) \} + \max\{ \theta_i(x, y) \} \leq 80n$. We also have $|R_6(x, y)|_s = 2(\tau_1(x, y) + \theta_1(x, y) + \cdots + \tau_k(x, y) + \theta_k(x, y)) \geq 2(2k)10n = 120n^2$, so $|P|_s \leq \frac{1}{n}|R_6(x, y)|_s$.

We now show that $G$ does not split as an amalgamated product. For each $x \in S_p, y \in S_q$ with $p \leq q$ we let $H(x, y) = (x, y \mid R_\ell(x, y) : 1 \leq \ell \leq 6)$. By Lemma 5.1 we see that $H(x, y)$ does not split. As there is a homomorphism $H(x, y) \to G$, we deduce that for any splitting of $G$ as an amalgamated free product $G = A \ast_C B$, the elements $x, y$ are either both in $A$ or both in $B$. Otherwise, the action of $H(x, y)$ on the Bass-Serre tree of $G = A \ast_C B$ induces a non-trivial splitting. Considering all such pairs $(x, y)$, we find that the generators of $G$ are either all in $A$ or all in $B$. Moreover $G$ cannot split as an HNN extension, since the relators $R_4(x, y)$ and $R_5(x, y)$ show that all generators have finite order in the abelianization of $G$. \hfill $\square$
6. Main theorem

The small cancellation over a free product condition $C'_s(\frac{1}{n})$ was defined in Definition 4.1. We start with the following Lemma.

Lemma 6.1. If $(G_1, \ldots, G_r | R_1, \ldots, R_s)$ is $C'_s(\frac{1}{n})$ then for a sufficient subdivision of $e_1, \ldots, e_r$ the cubical presentation $X^*$ is $C'(\frac{1}{n})$.

Proof. Let $X'$ be a subdivision of $X$ induced by a $q$-fold subdivision of each $e_j$. We accordingly let $Y'_i$ be the induced subdivision of $Y_i$, so $Y'_i = \bigsqcup V_k \cup \bigsqcup \sigma_k$ as in Construction 4.3 and with each $\sigma$-edge subdivided. We thus obtain a new cubical presentation $(X' \mid Y'_1, \ldots, Y'_s)$. Since $Y_i$ has $|R_i|_s$ $\sigma$-edges, the systole $\|Y'_i\| = \|Y_i\| + 2|R_i|_s(q - 1)$.

Note that $\|Y'_i\| > \sum_{i=1}^{\max(R_i|s)} |\sigma_i| = 2q|R_i|_s$ and so $\|Y'_i\| > 2(1 + \epsilon)q|R_i|_s$ for sufficiently small $\epsilon > 0$. Let $M_i = \max_k \{\text{diam}(V_k)\}$. For a wall-piece $P$ we have $\text{diam}(P) < M_i$. Consider a maximal cone-piece $P$ in $Y'_i$, and suppose it intersects $\ell$ different $V_k$’s and contains $\ell'$ different $e_k$ edges. Note that $2\ell \geq \ell'$ since if $P$ starts or ends with an entire $\sigma_k$ arc, then it intersects an additional $V_k$ (possibly trivially). We have $\text{diam}(P) \leq \ell M_i + q\ell'$. When $\ell' > 0$, for any $\epsilon > 0$ we can choose $q > 0$ so that $\text{diam}(P) < (1 + \epsilon)q\ell'$. Since $P$ corresponds to a length $\ell$ syllable piece, the $C'_s(\frac{1}{n})$ hypothesis implies that $\ell < \frac{1}{n}|R_i|_s$, and so $\text{diam}(P) < (1 + \epsilon)q\ell' < (2(1 + \epsilon)q|R_i|_s) < \frac{1}{n}\|Y'_i\|$. When $\ell' = 0$, then assuming $q > nM_i$ we have $\text{diam}(P) \leq M_i < 2\frac{q}{n}|R_i|_s < \frac{1}{n}\|Y'_i\|$. \hfill \qed

Theorem 6.2. Suppose $G = \langle G_1, \ldots, G_r | R_1, \ldots, R_s \rangle$ satisfies $C'_s(\frac{1}{27})$. If each $G_i$ is the fundamental group of a [compact] nonpositively curved cube complex, then $G$ acts properly and compactly on a CAT(0) cube complex.

Moreover, $G$ acts freely if each $(R_i)$ is a maximal cyclic subgroup.

Proof. Let $X^*$ be the associated cubical presentation. Lemma 6.1 asserts that $X^*$ is $C'(\frac{1}{27})$ after a sufficient subdivision. For each hyperplane $U$ in $Y_i$ we have $\text{diam}(N(U)) < \frac{1}{27}\|Y_i\|$ if the subdivision is sufficient. Theorem 2.4 asserts that $\pi_1 X^*$ acts freely (or with finite stabilizers if relators are proper powers) on a CAT(0) cube complex $C$ dual to $X^*$.

Let $X^{*\flat}$ be the cubical presentation $\langle X \mid \{Y_i\}, \{\tilde{X}_j\} \rangle$. By Lemma 4.4, $X^{*\flat}$ satisfies $B(8)$ with our previously chosen wallspace structure on each $Y_i$ and the hyperplane wallspace structure on each $\tilde{X}_j$. Thus by Lemma 2.6 each $\tilde{X}_j$ in $\tilde{X}^* = \tilde{X}^{*\flat}$ intersects the walls of $\tilde{X}^*$ in hyperplanes of $\tilde{X}_j$.

Lemma 4.2 asserts that $\pi_1 X^*$ is hyperbolic relative to $\{G_1, \ldots, G_r\}$.

The pieces in $X^* = \langle X \mid \{Y_i\} \rangle$ are uniformly bounded since $\text{diam}(Y_i)$ is uniformly bounded. Thus $N(W) \to X^*$ is quasi-isometrically embedded by Lemma 2.7. Hence $\text{Stabilizer}(N(W))$ is relatively quasiconvex with respect to $\{\pi_1 X_j\}$ by Theorem 2.8.

Theorem 3.1 asserts that $\pi_1 X^*$ acts relatively cocompactly on $C$. Lemma 3.2 asserts that each $C_*(\tilde{X}_i) = \tilde{X}_i$. Hence if each $X_i$ is compact, we see that $C$ is compact. \hfill \qed

7. A cubulated group that does not virtually split

Examples were given in [Wis21] of a compact nonpositively curved cube complex $X$ such that $X$ has no finite cover with an embedded hyperplane. It is conceivable that those groups have no (virtual) splitting, but this was not confirmed there.
Example 7.1. There exists a nontrivial group $G$ with the following two properties:

1. $G = \pi_1 X$ where $X$ is a compact nonpositively curved cube complex.
2. $G$ does not have a finite index subgroup that splits as an amalgamated product or HNN extension.

Let $G_1$ be the fundamental group of $X_1$ which is a compact nonpositively curved cube complex with a nontrivial fundamental group but no nontrivial finite cover. For instance, such complexes were constructed in [Wis96] or [BM97]. By Corollary 5.4, there exists a $C'_{\frac{1}{20}}$ quotient $G$ of the free product $G_1 * \cdots * G_1$ of $r$ copies of $G_1$, such that $G$ does not split. The group $G$ has no finite index subgroups since $G_1 * \cdots * G_1$ has none. Since $G_1 = \pi_1 X_1$, by Theorem 6.2, $G$ is the fundamental group of a compact nonpositively curved cube complex.

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