CORONA PROBLEM WITH DATA
IN IDEAL SPACES OF SEQUENCES

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Abstract. Let $E$ be a Banach lattice on $\mathbb{Z}$ with order continuous norm. We show that for any function $f = \{f_j\}_{j \in \mathbb{Z}}$ from the Hardy space $H_\infty(E)$ such that $\delta \leq \|f(z)\|_E \leq 1$ for all $z$ from the unit disk $\mathbb{D}$ there exists some solution $g = \{g_j\}_{j \in \mathbb{Z}} \in H_\infty(E')$, $\|g\|_{H_\infty(E')} \leq C\delta$ of the Bézout equation $\sum_j f_j g_j = 1$, also known as the vector-valued corona problem with data in $H_\infty(E)$.

The (classical) Corona Problem (see, e.g., [8, Appendix 3]) has the following equivalent formulation: given a finite number of bounded analytic functions $f = \{f_j\}_{j=1}^N \subset H_\infty$ on the unit disk $\mathbb{D}$, is the condition $\inf_{z \in \mathbb{D}} \max_j |f_j(z)| > 0$ sufficient as well as necessary for the existence of some solutions $g = \{g_j\}_{j=1}^N \subset H_\infty$ of the Bézout equation $\sum_j f_j g_j = 1$? A positive answer to this problem was established for the first time by L. Carleson in [1], and later a relatively simple proof was found by T. Wolff (see, e.g., [8, Appendix 3]); we also mention another approach to the proof based on the theory of analytic multifunctions (see, e.g., [12]).

These important results set ground for many subsequent developments. One question to ask is what estimates are possible for the solutions $g$ in terms of the estimates on $f$. In particular, estimates such as the $H_2(l^2)$ norm of $g$ make it possible to extend these results to infinite sequences $f$. We formalize this problem somewhat vaguely for now; see Section 1 below for exact definitions. Notation $\langle f, g \rangle = \sum_j f_j g_j$ will be used with suitable sequences $f = \{f_j\}$ and $g = \{g_j\}$. Let $E$ be a normed space of sequences, and let

$$E' = \{g \mid |\langle f, g \rangle| < \infty \text{ for all } f \in E\}$$

be the space of sequences dual to $E$ with respect to the pairing $\langle \cdot, \cdot \rangle$. Suppose that $f \in H_\infty(E)$ satisfies $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$ with some $\delta > 0$. We are interested in the existence of a function $g \in H_\infty(E')$ satisfying $\langle f, g \rangle = 1$, and in the possible norm estimates of $g$ in terms of $\delta$. If this is the case for all such $f$ then we say that $E$ has the corona property.

T. Wolff’s argument allowed M. Rosenblum, V. A. Tolokonnikov and A. Uchiyama to obtain (independently from one another) a positive answer.
to this question in the Hilbert space case, showing that $l^2$ has the corona property. The corresponding estimates were later improved many times; apparently, the best one at the time of this writing can be found in [10]. A. Uchiyama also obtained in [11] a different estimate using a rather involved argument based on the original proof of L. Carleson, thus establishing that $l^\infty$ also has the corona property. The intermediate spaces $E = l^p$, $2 < p < \infty$ between these two cases can be reduced to the case $p = 2$ (see [6, §3]), following Tolokonnikov’s (unpublished) remark that Wolff’s method can be directly extended to this case at least for even values of $p > 2$, but very little was known about the corona property for other spaces $E$.

Recently, in [5] S. V. Kislyakov extended the corona property to a large class of Banach ideal sequence spaces $E$. Specifically, in this result $E$ is assumed to be $q$-concave with some $q < \infty$ and satisfy the Fatou property, and space $L^\infty(E)$ is assumed to be BMO-regular. These conditions are satisfied by all UMD lattices with the Fatou property, and in particular by spaces $l^p$, $1 \leq p < \infty$. A novel and somewhat counterintuitive idea leading to this result is that for suitable spaces $E_0$ and $E_1$ the corona property of $E_0$ implies the corona property of their pointwise product $E_0E_1$. The proof uses the theory of interpolation for Hardy-type spaces to reduce the result to the well-known case $E = l^2$.

In the present work we show how the approach of [5] can be modified to obtain a complete answer: it turns out that all ideal sequence spaces with order continuous norm have the corona property. Note that this, in particular, includes all finite-dimensional ideal sequence spaces. These conditions are more general than the result [5]; see Proposition 9 at the end of Section 2. It remains unclear if the assumption of the order continuity of the norm can be weakened.

Compared to [5], our proof relies on somewhat less elementary means, namely we use a fixed point theorem and a selection theorem to reduce the problem to Uchiyama’s difficult case $E = l^\infty$, but otherwise the reduction appears to be rather simple and straightforward. Moreover, for $q$-concave lattices $E$ with $q < \infty$ the problem is still reduced in this manner to the relatively easy standard case $E = l^2$ using the same argument as in [5].

We also mention that very recently in [13] the method described in the present work was also applied to the problem of characterizing the ideals $I(f) = \{\langle f, g \rangle \mid g \in H^\infty(E')\}$. Certain classical results concerning the case $E = l^2$ were extended to the case $E = l^1$. The approach [5] based on
interpolation of Hardy-type spaces does not seem to lend itself to such an extension.

We briefly outline the implications for the estimates $C_{E,\delta}$ of the norms of the solutions $g$ in terms of $\delta$. Theorem 2 below can be stated quantitatively:

\[ C_{E_0,\delta} \leq C_{E,\delta}^0 \leq c_1 \delta^{-c_2} \]

with some constants $c_1, c_2 > 0$ independent of $E$. Furthermore, if a Banach lattice $E$ is $q$-concave with some $1 < q < \infty$ then $E = l^q E_1$ with a Banach lattice $E_1$ (see the proof of Proposition 3 below), and we get an estimate $C_{E,\delta} \leq C_{l^q,\delta}^0$ that may be sharper than (1) for some values of $q$. Indeed, we also have an estimate $C_{l^p,\delta} \leq C_{l^q,\delta}^0$ for $p \geq 2$ (see, e.g., [6, §3]) and $C_{l^2,\delta} \leq \frac{1}{\delta} + c \log \frac{1}{\delta}$ by [10] with an explicit constant $c \approx 8.4$. The latter estimate is known to be close to optimal in terms of the rate of growth as $\delta \to 0$. Thus, for $q$-concave lattices $E$ with some $2 \leq q < \infty$ we also have an estimate $C_{E,\delta} \leq C_{l^p,\delta}^0$ for small enough $\delta$. Our knowledge about sharp estimates for the value of $C_{l^p,\delta}$ with $p \neq 2$ seems to be lacking.

1. Statements of the results

A quasi-normed lattice $X$ of measurable functions on a measurable space $\Omega$, also called an ideal lattice, is a quasi-normed space of measurable functions such that $f \in X$ and $|g| \leq |f|$ implies $g \in X$ and $\|g\|_X \leq \|f\|_X$. Ideal spaces of sequences $E$ are lattices on $\Omega = \mathbb{Z}$. A lattice $X$ is said to have order continuous quasi-norm if for any sequence $f_n \in X$ such that $\sup_n |f_n| \in X$ and $f_n \to 0$ almost everywhere one also has $\|f_n\|_X \to 0$. Lattices $l^p$ have order continuous quasi-norm if and only if $p < \infty$. For a lattice $X$ of measurable functions the order dual $X'$, also called the associate space, is the lattice of all measurable functions $g$ such that the norm

\[ \|g\|_{X'} = \sup_{f \in X, \|f\|_X \leq 1} \int |fg| \]

is finite. For example, the order dual of $l^p$ is $l^{p'}$ for all $1 \leq p \leq \infty$. If $X$ is a Banach lattice, the order dual is contained in the topological dual space $X^*$ of all continuous linear functionals on $X$, and $X' = X^*$ if and only if $X$ has order continuous norm. For more on lattices see, e.g., [4].

**Definition 1.** Suppose that $E$ is a normed lattice on $\mathbb{Z}$. We say that $E$ has the corona property with constant $C_{\delta}$, $0 < \delta < 1$, if for any $f \in H_\infty (E)$ such
that \( \delta \leq \|f(z)\|_E \leq 1 \) for all \( z \in \mathbb{D} \) there exists some \( g \in H_\infty(E') \) such that \( \|g\|_{H_\infty(E')} \leq C_\delta \) and \( \langle f(z), g(z) \rangle = 1 \) for all \( z \in \mathbb{D} \). Such a function \( f \) is called the data for the corona problem with lower bound \( \delta \), and such a function \( g \) is called the solution for the corona problem with data \( f \).

For any two quasi-normed lattices \( E_0 \) and \( E_1 \) on the same measurable space the set of pointwise products

\[ E_0 E_1 = \{ h_0 h_1 \mid h_0 \in E_0, h_1 \in E_1 \} \]

is a quasi-normed lattice with the quasi-norm defined by

\[ \|h\|_{E_0 E_1} = \inf_{h = h_0 h_1} \|h_0\|_{E_0} \|h_1\|_{E_1}. \]

For example, the Hölder inequality shows that \( l^p l^q = l^r \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). It is easy to see that \( X L_\infty = X \) for any lattice \( X \).

**Theorem 2.** Suppose that \( E_0, E_1 \) are Banach lattices on \( \mathbb{Z} \) such that \( E = E_0 E_1 \) is also a Banach lattice having order continuous norm. If \( E_0 \) has the corona property with constants \( C_\delta, 0 < \delta < 1 \), then \( E \) also has it with constants \( C_{\delta/2} \).

The proof of Theorem 2 is given in Section 2 below. Since \( l^\infty \) has the corona property by [11], applying Theorem 2 with \( E_0 = l^\infty \) and \( E_1 = E \) yields the main result, stated as follows.

**Theorem 3.** Every Banach lattice on \( \mathbb{Z} \) with order continous norm has the corona property.

2. **Proof of Theorem 2**

We begin with some preparations. The following result shows that finite-dimensional approximation of the corona property is possible under the assumption that \( E \) has order continous norm.

**Proposition 4.** Suppose that \( E \) is a Banach lattice on \( \mathbb{Z} \) with order continuous norm such that for any \( \varepsilon > 0 \) and a finite \( I \subset \mathbb{Z} \) the restriction of \( E \) onto \( I \) has the corona property with constants \( (1 + \varepsilon)C_\delta, 0 < \delta < 1 \), where \( C_\delta \) are independent of \( I \) and \( \varepsilon \). Then \( E \) has the corona property with constants \( C_\delta, 0 < \delta < 1 \).

\(^1\) Replacing this condition with the nonstrict inequality \( \delta \leq \|f(z)\|_E \leq 1 \) would allow to simplify somewhat the arguments in Section 3 below; however, the closed form looks nicer.
The proof of Proposition 4 is given in Section 3 below.

A set-valued map $\Phi : X \to 2^Y$ between normed spaces is called lower semicontinuous if for any $x_n \in X$, $x_n \to x$ in $X$ and $y \in \Phi(x)$ there exists a subsequence $n'$ and some $y_{n'} \in \Phi(x_{n'})$ such that $y_{n'} \to y$ in $Y$. We need the following well-known result.

**Michael Selection Theorem** ([7]). Let $Y$ be a Banach space, $X$ a paracompact space and $\varphi : X \to 2^Y$ a lower semicontinuous multivalued map taking values that are nonempty, convex and closed. Then there exists a continuous selection $f : X \to Y$ of $\varphi$, i.e. $f(x) \in \varphi(x)$ for all $x \in X$.

This allows us to conclude that the factorization corresponding to the product of finite-dimensional Banach lattices can be made continuous (in the more general infinite-dimensional cases this seems to be unclear).

**Proposition 5.** Suppose that $F_0$ and $F_1$ are finite-dimensional Banach lattices of functions on the same measurable space. Then for every $\varepsilon$ there exists a continuous map $\Delta : F_0F_1 \setminus \{0\} \to F_1$ taking nonnegative values such that $\|\Delta f\|_{F_1} \leq 1$ and $\|f(\Delta f)^{-1}\|_{F_0} \leq (1 + \varepsilon)\|f\|_{F_0F_1}$.

Indeed, we first consider a set-valued map $\Delta_0 : F_0F_1 \setminus \{0\} \to 2^{F_1}$ defined by

$$\Delta_0(f) = \left\{ g \in F_1 \mid g \geq 0 \text{ everywhere, } \|g\|_{F_1} < 1, \|fg^{-1}\|_{F_0} < (1 + \varepsilon)\|f\|_{F_0F_1} \right\}$$

for $f \in F_0F_1 \setminus \{0\}$. By the definition of the space $F_0F_1$, the map $\Delta_0$ takes nonempty values. It is easy to see that $\Delta_0$ has open graph, and hence $\Delta_0$ is a lower semicontinuous map. The graph of a map $\overline{\Delta}_0 : F_0F_1 \setminus \{0\} \to 2^{F_1}$ defined by

$$\overline{\Delta}_0(f) = \left\{ g \in F_1 \mid g \geq 0, \|g\|_{F_1} \leq 1, \|fg^{-1}\|_{F_0} \leq (1 + \varepsilon)\|f\|_{F_0F_1} \right\}$$

(with the conventions $0 \cdot 0^{-1} = 0$ and $a \cdot 0^{-1} = \infty$ for $a \neq 0$) is easily seen to be the closure of the graph of the map of $\Delta_0$, therefore $\overline{\Delta}_0$ is also a lower semicontinuous map. The values of $\Delta_0$ are convex and closed, so by the Michael selection theorem $\overline{\Delta}_0$ admits a continuous selection $\Delta$, that is, $\Delta(f) \in \overline{\Delta}_0(f)$ for all $f \in F_0F_1$. This selection satisfies the conclusion of Proposition 5.

**Fan–Kakutani Fixed Point Theorem** ([2]). Suppose that $K$ is a compact set in a locally convex linear topological space. Let $\Phi$ be a mapping from $K$ to

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2 For the sake of simplicity we omitted the converse part of this famous theorem.
the set of nonempty subsets of $K$ that are convex and compact, and assume that the graph of $\Phi$ is closed. Then $\Phi$ has a fixed point, i.e., $x \in \Phi(x)$ for some $x \in K$.

A quasi-normed lattice $X$ of measurable functions is said to have the Fatou property if for any $f_n, f \in X$ such that $\|f_n\|_X \leq 1$ and the sequence $f_n$ converges to $f$ almost everywhere it is also true that $f \in X$ and $\|f\|_X \leq 1$.

The following formula (also appearing in [5, Lemma 1]) seems to be rather well known; see, e.g., [9, Theorem 3.7].

**Proposition 6.** Suppose that $X$ and $Y$ are Banach lattices of measurable functions on the same measurable space having the Fatou property such that $XY$ is also a Banach lattice. Then $X' = (XY)'Y$.

In order to achieve the best estimate possible with the method used without assuming that $C_\delta$ is continuous in $\delta$, we take advantage of the fact that the decomposition in the definition of the pointwise product of Banach lattices can be made exact if both lattices satisfy the Fatou property.

**Proposition 7.** Let $X$ and $Y$ be Banach lattices of measurable functions on the same measurable space having the Fatou property. Then for every function $f \in XY$ there exist some $g \in X$ and $h \in Y$ such that $f = gh$ and $\|g\|_X \|h\|_Y \leq \|f\|_{XY}$.

This appears to be rather well known but hard to find in the literature, so we give a proof. We may assume that $\|f\|_{XY} = 1$. Let $\varepsilon_n \to 0$ be a decreasing sequence. Sets

$$F_n = \{ g \mid g \geq 0, \text{ supp } g = \text{ supp } f \text{ up to a set of measure } 0, \|g\|_X \leq 1, \|fg^{-1}\|_Y \leq 1 + \varepsilon_n \} \subset X$$

are nonempty and form a nonincreasing sequence. It is easy to see that $F_n$ are convex (one uses the convexity of the map $t \mapsto t^{-1}$, $t > 0$). By the Fatou property of $X$ and $Y$ sets $F_n$ are closed with respect to the convergence in measure, and they are bounded in $X$. The intersection of such a sequence of sets is nonempty (see [41, Chapter 10, §5, Theorem 3]), so there exists some $g \in \bigcap_n F_n$, which together with $h = fg^{-1}$ yields the required decomposition.

Now we are ready to prove Theorem 2. Suppose that under its assumptions $E = E_0E_1$, lattice $E_0$ has the corona property with constant $C_\delta$ for some $0 < \delta \leq 1$ and we are given some $f \in H_\infty(E)$ such that $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$; we need to find a suitable $g \in H_\infty(E')$ solving $\langle f, g \rangle = 1$. 

Proposition 4 allows us to assume that the lattices have finite support
$I \subset \mathbb{Z}$, and moreover, we may relax the claimed estimate for the norm of
a solution to $(1 + \varepsilon)C_{\frac{1}{2}}$ for arbitrary $\varepsilon > 0$. It is easy to see that finite-
dimensional lattices always have the Fatou property, so we may assume
that both $E_0$ and $E_1$, and thus both $L_\infty(E_0)$ and $L_\infty(E_1)$ have the Fatou
property. By Proposition 7 there exist some $u \in L_\infty(E_0)$ and $v \in L_\infty(E_1)$
such that $|f| = uw$ and $\|u\|_{L_\infty(E_0)} \|v\|_{L_\infty(E_1)} \leq \|f\|_{L_\infty(E)} \leq 1$. We may further
assume that $\|u\|_{L_\infty(E_0)} \leq 1$ and $\|v\|_{L_\infty(E_1)} \leq 1$.

Since $f = \{f_j\}_{j \in I}$ is analytic and bounded, if we restrict $I$ so that $\mathbb{T} \times I$
becomes the support of $f$, we may assume that $\log |f_j| \in L_1$ for all $j \in I$.
Boundedness of $u$ and $v$ further implies that $\log |v_j| \in L_1$ for all $j \in I$.

Let us fix some $\varepsilon > 0$ and a sequence $0 < r_j < 1$ such that $r_j \to 1$. We
denote by $P_r$ the operator of convolution with the Poisson kernel for radius
$0 < r < 1$, that is, $P_r a(z) = a(rz)$ for any harmonic function $a$ on $\mathbb{D}$ and
any $z \in \mathbb{D}$.

Let

$$B = \{ \log w \mid w \in L_\infty(E_1), \|w\|_{L_\infty(E_1)} \leq 2, w \geq v \} \subset L_1. \tag{2}$$

This set is convex, which follows from the well-known logarithmic convexity
of the norm of a Banach lattice. We endow $B$ with the weak topology
of $L_1$. By the Fatou property of $L_\infty(E_1)$ it is easy to see that $B$ is closed
with respect to the convergence in measure, so $B$ is also closed in $L_1$ and
thus weakly closed. The Dunford–Pettis theorem easily shows that $B$ is a
compact set, since the functions from $B$ are uniformly bounded from above
and below by some summable functions.

For convenience, we denote by $B_Z$ the closed unit ball of a Banach
space $Z$. We endow $H_\infty(E_0')$ with the topology of uniform convergence on
compact sets in $\mathbb{D} \times I$, and define a (single-valued) map $\Phi_0^{(j)} : C_{\frac{1}{2}}B_{H_\infty(E_0')} \to B$
by

$$\Phi_0^{(j)}(h) = \log \left( \Delta(|P_r h|) + v \right), \quad h \in C_{\frac{1}{2}}B_{H_\infty(E_0')}$$

with a map $\Delta$ from Proposition 5 applied to $F_0 = E'$ and $F_1 = E_1$ (observe
that by Proposition 8 we have $E_0' = E'E_1$) and the chosen value of $\varepsilon$. It is
ey easy to see that $\Phi_0^{(j)}$ is continuous.

We endow $H_\infty(E_1)$ with the topology of uniform convergence on compact
sets in $\mathbb{D} \times I$ and define a (single-valued) map $\Phi_1 : B \to 2B_{H_\infty(E_1)}$ by

$$\Phi_1(\log w)(z, \omega) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log w(e^{i\theta}, \omega) \, d\theta \right) \tag{3}$$
for all \( \log w \in B, \ z \in \mathbb{D} \) and \( \omega \in I \). This map is easily seen to be continuous and (since the integral under the exponent is the convolution with the Schwarz kernel) \(|\Phi_1(\log w)| = w\) almost everywhere.

Observe that if \( \psi = \Phi_1(\log w) \) for some \( \log w \in B \) and \( \varphi = \frac{f}{\psi} \), then
\[
|\varphi| = \frac{|f|}{\psi} \leq \frac{|f|}{|\psi|} = u \quad \text{and we have} \quad \varphi \in H_\infty(E_0) \quad \text{with} \quad \|\varphi\|_{H_\infty(E_0)} \leq 1.
\]
On the other hand,
\[
\delta \leq \|f(z)\|_E = \|\varphi(z)\psi(z)\|_{E_0E_1} \leq 2\|\varphi(z)\psi(z)\|_{E_1} \leq 2\|\varphi(z)\|_{E_0},
\]
so \( \frac{\delta}{2} \leq \|\varphi(z)\|_{E_0} \leq 1 \) for all \( z \in \mathbb{D} \). This means that \( \varphi \) belongs to the set
\[
D = \left\{ \varphi \in H_\infty(E_0) \mid \frac{\delta}{2} \leq \|\varphi(z)\|_{E_0} \leq 1 \quad \text{for all} \quad z \in \mathbb{D} \right\}
\]
of corona data functions corresponding to the assumed corona property of \( E_0 \). Thus we may define a (single-valued) map
\[
\Phi_2 : \Phi_1(B) \to D
\]
by \( \Phi_2(\psi) = \frac{f}{\psi} \) for \( \psi \in \Phi_1(B) \). We endow \( D \) with the topology of uniform convergence on compact sets in \( \mathbb{D} \times I \). The continuity of \( \Phi_2 \) is evident.

We define a set-valued map \( \Phi_3 : D \to 2^{C_{\frac{1}{2}}B_{H_\infty(E_0')}} \) by
\[
\Phi_3(\varphi) = \left\{ h \in H_\infty(E_0') \mid \langle \varphi, h \rangle = 1, \|h\|_{H_\infty(E_0')} \leq C_{\frac{1}{2}} \right\}
\]
for \( \varphi \in D \). By the assumed corona property of \( E_0 \) map \( \Phi_3 \) takes nonempty values. Since the condition \( \langle \varphi, h \rangle = 1 \) is equivalent to \( \langle \varphi(z), h(z) \rangle = 1 \) for all \( z \in \mathbb{D} \), it is easy to see that the values of \( \Phi_3 \) are convex and closed, and thus they are compact. Similarly, the closedness of the graph of \( \Phi_3 \) is easily verified.

Now we define a set-valued map \( \Phi^{(j)} : C_{\frac{1}{2}}B_{H_\infty(E_0')} \to 2^{C_{\frac{1}{2}}B_{H_\infty(E_0')}} \) by \( \Phi^{(j)} = \Phi_3 \circ \Phi_2 \circ \Phi_1 \circ \Phi^{(j)}_0 \). The graph of \( \Phi^{(j)} \) is closed since all individual maps are continuous in the appropriate sense (specifically, as a composition of upper semicontinuous maps, but it is easy to establish the continuity in this case directly using compactness). The domain \( C_{\frac{1}{2}}B_{H_\infty(E_0')} \) with the introduced topology is a compact set in a locally convex linear topological space. Thus \( \Phi^{(j)} \) satisfies the assumptions of the Fan–Kakutani fixed point theorem, which implies that the maps \( \Phi^{(j)} \) admit some fixed points \( h_j \in C_{\frac{1}{2}}B_{H_\infty(E_0')} \), that is, \( h_j \in \Phi^{(j)}(h_j) \) for all \( j \). This means that with \( \log w_j = \Phi^{(j)}_0(h_j), \ \psi_j = \Phi_1(\log w_j) \) and \( \varphi_j = \Phi_2(\psi_j) \) we have \( h_j \in \Phi_3(\varphi_j) \). The first two conditions imply that \( |\psi_j| = \Delta(|P_r h_j|) + v \geq \Delta(|P_r h_j|) \), so
\[
\left\| (P_r h_j)(\psi_j)^{-1} \right\|_{L_\infty(E')} \leq \left\| \frac{|P_r h_j|}{\Delta(|P_r h_j|)} \right\|_{L_\infty(E')} \leq (1 + \varepsilon) C_{\frac{1}{2}}
\]
by Proposition 3. Thus
\begin{equation}
\| h_j(r_j z) \|_{E'} \leq (1 + \varepsilon) C_\delta^j
\end{equation}
for all \( z \in \mathbb{D} \), and condition \( h_j \in \Phi_3(\varphi_j) \) implies that
\begin{equation}
1 = \left\langle \frac{f(z)}{\psi_j(z)}, h_j(z) \right\rangle = \left\langle f(z), \frac{h_j(z)}{\psi_j(z)} \right\rangle
\end{equation}
for all \( z \in \mathbb{D} \). Since sequences \( \psi_j, h_j \) are uniformly bounded on compact sets in \( \mathbb{D} \times I \), by passing to a subsequence we may assume that \( \psi_j \to \psi \) with some \( \psi \in \Phi_1(B) \) and \( h_j \to h \) with some \( h \in C_\delta B_{H_\infty(E'_0)} \) uniformly on compact sets in \( \mathbb{D} \times I \). Thus we may pass to the limits in (4) and (5) to see that \( h \psi \) is a suitable solution for the corona problem with data \( f \), which concludes the proof of Theorem 2.

We remark that this construction can be modified to use the Tychonoff fixed point theorem, which is the particular case of single-valued maps in the setting of the Fan–Kakutani theorem. It suffices to find a continuous selection for the slightly enlarged map \( \Phi_3 \), which is the purpose of the next result; the arbitrarily small increase in the estimate is inconsequential for the scheme of the proof.

**Proposition 8.** Suppose that a finite-dimensional lattice \( E \) has the corona property with constant \( C_\delta \) for some \( 0 < \delta < 1 \). Let
\[ D_E = \{ f \in H_\infty(E) \mid \delta \leq \| f(z) \|_E \leq 1 \text{ for all } z \in \mathbb{D} \} . \]
Then for any \( \varepsilon > 0 \) there exists a continuous map
\[ K : D_E \to (1 + \varepsilon) C_\delta B_{H_\infty(E')} \]
such that \( \langle f, K(f) \rangle = 1 \) for any \( f \in D_E \).

Indeed, let \( 0 < \alpha < 1 \). We define a set-valued map
\[ K_0 : D_E \to (1 + \alpha) C_\delta B_{H_\infty(E')} \]
by
\[ K_0(f) = \{ g \in H_\infty(E') \mid \| \langle f, g \rangle - 1 \|_{H_\infty} < \alpha, \| g \|_{H_\infty(E')} < (1 + \alpha) C_\delta \} \]
for \( f \in D_E \). By the corona property assumption on \( E \) map \( K_0 \) takes nonempty values. It is easy to see that \( K_0 \) has open graph and thus \( K_0 \) is a lower semicontinuous map. The graph of a map \( \overline{K}_0 : D_E \to (1 + \alpha) C_\delta B_{H_\infty(E')} \)
defined by
\[ \overline{K}_0(f) = \{ g \in H_\infty(E') \mid \| \langle f, g \rangle - 1 \|_{H_\infty} \leq \alpha, \| g \|_{H_\infty(E')} \leq (1 + \alpha) C_\delta \} \]
is easily seen to be the closure of the graph of the map of $K_0$, and hence $\overline{K}_0$ is also a lower semicontinuous map. The values of $K_0$ are convex and closed. By the Michael selection theorem there exists a continuous selection $K_1$ of the map $\overline{K}_0$, that is, $K_1(f) \in \overline{K}_0(f)$ for all $f \in D_E$. Now observe that $|\langle f(z), K_1(f)(z) \rangle - 1| \leq \alpha$ implies $|\langle f(z), K_1(f)(z) \rangle| \geq 1 - \alpha$ for all $z \in \mathbb{D}$ and $f \in D_E$, so we may set $K(f) = \frac{K_1(f)}{\langle f, K_1(f) \rangle}$ and have $\langle f, K(f) \rangle = 1$ with $\|K(f)\|_{H_\infty(E')} \leq \frac{1+\alpha}{1-\alpha}C_\delta$. Choosing $\alpha$ small enough yields the claimed range of $K$.

Finally, we mention that Theorem 3 includes the result [5, Corollary 2]. This is implied by the following known observation; we give a proof for convenience.

**Proposition 9.** Suppose that $X$ is a Banach lattice of measurable functions having the Fatou property and $X$ is $q$-concave with some $1 < q < \infty$. Then $X$ has order continuous norm.

Lattice $Z^\delta$ is defined by the norm $\|f\|_{Z^\delta} = \left\| \frac{|f|}{Z} \right\|_Z^\delta$ for a quasi-normed lattice $Z$ of measurable functions and $\delta > 0$. Lattice $X'$ is $q'$-convex, and hence $Y = (X')^{q'}$ is a Banach lattice with the Fatou property. Then $X' = Y^{q'}$. The Fatou property is equivalent to the order reflexivity $X = X''$, and using the well-known formula for the duals of the Calderón-Lozanovsky products (see, e. g., [9, Theorem 2.10]) we may write

$$X = (X')' = (Y^{q'}L_{\infty}^{q'})' = Y'^{q'}L_1^{q'} = Y'^{q'}L_q.$$

Since lattice $L_q$ has order continuous norm, it suffices to establish the following.

**Proposition 10.** Suppose that $X$ and $Y$ are quasi-normed lattices of measurable functions and $Y$ has order continuous quasi-norm. Then $XY$ also has order continuous quasi-norm.

Let $f_n \in XY$ be a sequence with $f = \sup_n |f_n| \in XY$ such that $f_n \to 0$ almost everywhere. Then $f = gh$ with some $g \in X$ and $h \in Y$. We may assume that $g,h \geq 0$. Sequence $h_n = \frac{f_n}{g}$ also converges to 0 almost everywhere, and $|h_n| \leq \frac{f}{g} = h$, so $\sup_n |h_n| \in Y$. By the order continuity of the quasi-norm of $Y$ we have $\|h_n\|_Y \to 0$, hence $\|f_n\|_{XY} \leq \|g\|_X \|h_n\|_Y \to 0$.

3. Proof of Proposition 4

First, observe that if a lattice $E$ on $\mathbb{Z}$ has order continuous norm then $L_1(E)$ also has order continuous norm, we have $L_\infty(E') = [L_1(E)]^* = [L_1(E)]^*$, and $H_\infty(E')$ is easily seen to be $w^*$-closed in $L_\infty(E')$ (see, e. g., [6].
\[ \text{§1.2.1}. \] The \( w^* \)-convergence of a sequence \( h_k \in H_\infty(E') \) to some \( h \) implies that \( h_k(z) \to h(z) \) in the \( * \)-weak topology of \( E' = E^* \) for all \( z \in \mathbb{D} \).

Now suppose that under the assumptions of Proposition \[ 4 \] \( 0 < \delta < 1 \) and \( f \in H_\infty(E) \) satisfies \( \delta \leq \|f(z)\|_E \leq 1 \) for all \( z \in \mathbb{D} \). Let \( I_k \subset \mathbb{Z} \) be a nondecreasing sequence such that \( \bigcup_k I_k = \mathbb{Z} \), and fix a sequence \( \varepsilon_j > 0 \), \( \varepsilon_j \to 0 \). We consider the natural approximations \( f_{A,r,k}(z) = Af(rz)\chi_{I_k}, \) \( z \in \mathbb{D} \), for \( 0 < r \leq 1 \) and \( A \geq 1 \). If there exists some sequence of parameters \( A_j \to 1, r_j \to 1 \) and \( k_j \to \infty \) such that \( f_j = f_{A_j,r_j,k_j} \) is a data for the corona problem with lower bound \( \delta \) then by the assumptions there exist some \( g_j \) such that \( \langle f_j, g_j \rangle = 1 \) and \( \|g_j\|_{H_\infty(E')} \leq (1 + \varepsilon_j)C_\delta \). By passing to a subsequence we may assume that \( A_j \) is nonincreasing, \( r_j \) and \( k_j \) are nondecreasing, and \( g_j \to g \) in the \( * \)-weak topology of \( L_\infty(E') \) for some \( g \in H_\infty(E') \).

Observe that

\[ (6) \quad 1 = \langle f_j(z), g_j(z) \rangle = \langle f(z), g_j(z) \rangle + \langle f(r_jz) - f(z), g_j(z) \rangle + \langle f_j(z) - f(r_jz), g_j(z) \rangle \]

for all \( z \in \mathbb{D} \). The first term in (6) converges to \( \langle f(z), g(z) \rangle \). Since the \( E \)-valued analytic function \( f \) is strongly continuous at every \( z \in \mathbb{D} \), the second term in (6) converges to 0. By the assumptions

\[ (7) \quad |f_j(z) - f(r_jz)| = |A_jf(r_jz)\chi_{I_{k_j}} - f(r_jz)| \leq \chi_{I_{k_j}}|A_jf(r_jz)\chi_{I_{k_j}} - f(r_jz)| + \chi_{\setminus I_{k_j}}|A_jf(r_jz)\chi_{I_{k_j}} - f(r_jz)| = \chi_{I_{k_j}}|A_jf(r_jz) - f(r_jz)| + \chi_{\setminus I_{k_j}}|f(r_jz)| \leq (A_j - 1)|f(r_jz)| + \chi_{\setminus I_{k_j}}|f(z)| + |f(r_jz) - f(z)|. \]

The norm in \( E \) of the first term in (7) is estimated by \( (A_j - 1) \), and thus it converges to 0. The second term in (7) converges to 0 in \( E \) by the assumption that \( E \) has order continuous norm. The third term in (7) converges to 0 in \( E \) by the strong continuity of \( f \) in \( \mathbb{D} \). It follows that the third term in (6) is dominated by \( \|f_j(z) - f(r_jz)\|_E\|g_j(z)\|_{E'} \leq \|f_j(z) - f(r_jz)\|_E(1 + \varepsilon_j)C_\delta \), and so it also converges to 0. Therefore passing to the limit in (6) yields \( \langle f(z), g(z) \rangle = 1 \) for all \( z \in \mathbb{D} \). We also have \( \|g\|_{H_\infty(E')} \leq \limsup_j \|g_j\|_{H_\infty(E')} \leq C_\delta \), so \( g \) is a solution for the corona problem with data \( f \) having the claimed constant \( C_\delta \).

Thus it suffices to find a suitable sequence of parameters. We consider two cases. In the first case \( \|f(z)\|_E = 1 \) for all \( z \in \mathbb{D} \). We take \( A_j = 1 \) and any increasing sequence \( r_j \to 1 \). By the order continuity of norm we have \( \|f(z)\chi_{I_k}\|_E \to \|f(z)\|_E = 1 \) for every \( z \in \mathbb{D} \), so by the compactness of
closed sets in $\mathbb{D}$ and the assumption that $\delta < 1$ we have $\|f(r_j z)\chi_{I_k}\|_E \geq \delta$
for large enough $k_j$. Thus $f_j$ is a suitable corona data in this case.

In the second case $\|f(z_0)\|_E < 1$ for some $z_0 \in \mathbb{D}$. With the help of an automorphism we may assume for convenience that $z_0 = 0$. We also fix any increasing sequence $r_j \to 1$. A simple consequence of the Schwarz lemma (see, e. g., [3, Chapter 1, Corollary 1.3]) shows that

$$\langle f(z), e' \rangle \leq \frac{|\langle f(0), e' \rangle| + |z|}{1 + |\langle f(0), e' \rangle| |z|}
$$

for all $z \in \mathbb{D}$ and $e' \in E' = E^*$ with $\|e'\|_{E'} \leq 1$. Since function $(x, y) \mapsto \frac{x + y}{1 + xy}$ is increasing in both $x \in [0, 1]$ and $y \in [0, 1]$, taking the supremum in (8) over all such $e'$ yields $\|f(z)\|_E \leq \frac{\|f(0)\|_E + |z|}{1 + \|f(0)\|_E |z|}$, thus $\alpha_j = \sup_{z \in \mathbb{D}} \|f(r_j z)\|_E < 1.$

Setting $A_j = \frac{1}{\alpha_j}$ yields $\|A_j f(r_j z)\|_E \leq 1$ for all $z \in \mathbb{D}$. Again, since $E$ has order continuous norm we have $\|A_j f(z)\chi_{I_k}\|_E \to \|A_j f(z)\|_E \geq A_j \delta > \delta$
for every $z \in \mathbb{D}$, and we also have $\|A_j f(r_j z)\chi_{I_{kj}}\|_E \geq \delta$ for large enough $k_j$, so $f_j$ is a suitable corona data in this case as well. The proof of Proposition 4 is complete.

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