ON IMPROVING A SCHUR-TYPE THEOREM IN SHIFTED PRIMES

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Abstract. We show that if $N \geq \exp(\exp(\exp(kO(1))))$, then any $k$-colouring of the primes that are less than $N$ contains a monochromatic solution to $p_1 - p_2 = p_3 - 1$.

1. Introduction

Schur [12] proved that the equation $x + y = z$ is partition regular over the set of the natural numbers in his celebrated paper published over a hundred years ago. More specifically, let $r(k)$ be the smallest positive integer such that every $k$-colouring of the set $[r(k)]$ contains a monochromatic solution to $x + y = z$ where $x, y, z \in [r(k)]$. It follows from Schur’s paper that one has

$$
\exp(\Omega(k)) \leq r(k) \leq \exp(O(k \log k)).
$$

There is an extensive collection of works on improving bounds for $r(k)$, see Irving [6] for bounding it from above, see Heule [5] and its reference concerning the lower bound.

Li and Pan [9] showed that the equation $x + y = z$ is also partition regular over the sparse set $\mathbb{P} - 1$, where $\mathbb{P}$ denotes the set of primes. Let $r_p(k)$ be the smallest integer such that for any $N \geq r_p(k)$, any $k$-colouring of $\mathbb{P} \cap [N]$ contains a monochromatic solution to

$$p_1 - p_2 = p_3 - 1.$$

Although not explicitly stated, the following bound follows from the proof of Li and Pan.
Theorem 1.1 Li and Pan [9]. We have
\[ r_p(k) \leq \exp(\exp(\exp(\exp(k^{O(1)})))) \]

We prove the following quantitative strengthening.

Theorem 1.2. We have
\[ r_p(k) \leq \exp(\exp(\exp(k^{O(1)}))) \]

Li and Pan used a somewhat similar strategy as Green’s [3] proof of Roth’s theorem in the primes which can be summarised as follows. Firstly, one needs the counting result in the setting of the integers, which are essentially quantitative versions of Schur’s theorem and Roth’s theorem, respectively. Secondly, one aims to prove the corresponding counting result in the primes. This is done by the transference principle, which typically involves two parts — an analytic framework which transfers the result in the setting of the integers to suitably weighted functions, and finding an appropriate way to weight the integers in a way that resembles the primes\(^1\).

We shall take a rather different approach by using the method from [11] and applying existing proofs [10,13] concerning the upper bound on the cardinality of subsets of \([N]\) whose difference sets avoid \(P - 1\). These proofs [10, 13] follow from a Fourier concentration argument which differs from the \(L^\infty\)-transference principle used in the proof of Li and Pan. There are three major steps involved in the proof of Theorem 1.2. Firstly, we derive an iteration lemma which helps us to find many shifted primes in the difference set of any set \(A\) with large density. By the pigeonhole principle, there exists a large monochromatic set amongst those shifted primes, and it turns out that a translate of this monochromatic set has a large intersection with the original set \(A\). These considerations ultimately yield a colouring bootstrapping lemma. The final part of the proof is to bootstrap this lemma. We eventually conclude that there is always a monochromatic solution to \(p_1 - p_2 = p_3 - p\), where \(p_1, p_2, p_3 \in [N] \cap \mathbb{P}\), unless \(N\) is small in terms of \(k\), which completes the proof.

Since the major arc estimate used to prove Lemma 4.2 holds only for \(\mathbb{P} \pm 1\), our method cannot be used to study the partition regularity of \(p_1 - p_2 = p_3 - t\), where \(p_1, p_2, p_3 \in \mathbb{P}\), when \(t \neq \pm 1\). The easiest example to compare the problems is when \(t = p\) where \(p\) is a prime number. In this case, the equation \(p_1 - p_2 = p_3 - p\) always has the trivial monochromatic solution \(p_1 = p_2 = p_3 = p\). Nevertheless, as there always exists an arithmetic

\(^1\)Precisely speaking, the weight used in the work of Green and Li–Pan resembles a subset of primes which is contained in an arithmetic progression. One needs to choose the common difference of the progression carefully to avoid the Fourier bias caused by small primes.
progression whose difference set avoids \( \mathbb{P} - p \), we obviously cannot prove a corresponding difference set result in this case.

There has been a noticeable interest in understanding partition regularity problems over sparse sets. Lê [8] proved a more general partition regularity result for linear equations over shifted primes, showing that any system of linear equations which is partition regular over \( \mathbb{N} \) is also partition regular over \( \mathbb{P} - 1 \). The author uses ideas from the paper of Green and Tao [4] concerning more general linear equations in the primes instead of Green’s work on 3-term arithmetic progressions, together with Deuber’s work [1] on partition regularity. On the other hand, a famous problem of Erdős and Graham [2] asking whether the equation \( x + y = z \) is partition regular over perfect squares is still open.

The note is organised as follows. Section 2 is a list of notation. We summarise necessary estimates for the Fourier transform in Section 3, which are from the existing literature. In Section 4, we modify an iteration lemma of Ruzsa and Sanders and apply the lemma iteratively to locate shifted primes. The key bootstrapping lemma is shown in Section 5. In the final section, we apply the bootstrapping lemma from Section 5 to prove Theorem 1.2.

2. Notation

The set of primes is denoted by \( \mathbb{P} \), and \( \Lambda \) denotes the von Mangoldt function. For any positive integers \( N, d \), let

\[
F_{N,d}(n) = \Lambda(dn + 1)1_{[N]}(n).
\]

Let \( \chi \) be a Dirichlet character of modulus \( q \). We use \( L(s, \chi) \) to denote the associated Dirichlet \( L \)-function. Let

\[
\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).
\]

Let \( f \in \ell^1(\mathbb{Z}) \). The Fourier transform of \( f \) is defined as the function \( \hat{f}: \mathbb{T} \to \mathbb{C} \) given by

\[
\hat{f}(\theta) = \sum_{x \in \mathbb{Z}} f(x) e(-x\theta),
\]

where \( e(\theta) = e^{2\pi i \theta} \). The convolution of two functions \( f, g \in \ell^1(\mathbb{Z}) \) is defined by

\[
f \ast g(x) = \sum_{y \in \mathbb{Z}} f(x - y)g(y).
\]
For any functions \( f, g \in \ell^1(\mathbb{Z}) \) with finite support, we define their inner product by
\[
\langle f, g \rangle = \sum_{x \in \mathbb{Z}} f(x)g(x).
\]

Throughout the article, we use capital letter \( C \) with subscripts to denote absolute constants which tend to be large, and \( c \) with subscripts to denote absolute constants which are small (and at least less than 1).

3. Fourier transform of \( F_{N,d} \)

In this section, we summarise and simplify several existing results from the literature [10,13].

3.1. Preliminaries. Using the classical zero-free region [7, Theorem 5.26] and a result of Landau [7, Theorem 5.28], one can show the following lemma.

**Lemma 3.1.** There exists a positive constant \( c \) such that the following holds for any \( D \geq 2 \). If there exists a primitive character \( \chi_D \) such that \( \chi_D \) has modulus \( d_D \leq D \) and \( L(s, \chi_D) \) has a zero \( \beta_D \) in the region
\[
\Re(s) \geq 1 - \frac{c}{\log(D(|\Im(s)| + 3))},
\]
then
(i) the zero \( \beta_D \) is real and simple, and it is the only zero of \( L(s, \chi_D) \) in the region (3);
(ii) there does not exist any other primitive character \( \chi \) of modulus \( q \leq D^{10} \) such that \( L(s, \chi) \) has a zero in the region (3).

As a consequence of the lemma above, the following definition is exhaustive.

**Definition 3.2.** Let \( c \) be as in Lemma 3.1 and \( D \geq 2 \). We say that \( D \) is exceptional if there exists a unique primitive character \( \chi_D \) such that \( \chi_D \) has modulus \( d_D \leq D \), and \( L(s, \chi_D) \) has a zero \( \beta_D \) which is real and simple and satisfies \( \beta_D \geq 1 - c/\log(3D) \). We call \( \chi_D \) the exceptional character and \( \beta_D \) the exceptional zero. Otherwise, we say that \( D \) is unexceptional.

If \( \chi \pmod{q} \) has an exceptional zero \( \beta \), then it follows from [7, Theorem 5.28] that
\[
1 - \beta \gg q^{-1/2},
\]

\(^2\)The absolute constant 10 is chosen for the purpose of proving Lemma 4.2.
where the implicit constant is effective. We can use (4) and a prime number theorem for arithmetic progressions to prove the following estimate for the Fourier transform of $F_{N,d}$ at 0.

**Proposition 3.3.** Let $D \geq 2$ and let $N, d$ be positive integers. Suppose that either

1. $D$ is unexceptional, $\bar{d} = 1$ and $d \leq D$,
2. or $D$ is exceptional, $\bar{d} = d_D$ and $d \leq D^9$.

Then

$$|\hat{F}_{N,\bar{d}d}(0)| \gg \frac{dN}{\phi(\bar{d}d)} + O\left(\frac{\log(\bar{d}d)}{\sqrt{\log(\bar{d}d) + \log(\bar{d})}}(\log(\bar{d}d))^4\right).$$

Here $c$ is a positive absolute constant which does not depend on $N, D, d$.

**Sketch of the proof.** The unexceptional case is a direct consequence of the definition of Fourier transform and a well-known prime number theorem for arithmetic progressions [7, Theorem 5.27]. When $D$ is exceptional, we have

$$\hat{F}_{N,\bar{d}d}(0) = \psi(\bar{d}dN + 1; \bar{d}, 1)$$

$$= \frac{\bar{d}dN}{\phi(\bar{d}d)} - \frac{(\bar{d}dN)^{\beta_D}}{\phi(\bar{d}d)\beta_D} + O\left(\frac{\log(\bar{d}d)}{\sqrt{\log(\bar{d}d) + \log(\bar{d})}}(\log(\bar{d}d))^4\right).$$

Using the inequality $1 - (\bar{d}dN)^{\beta_D - 1}/\beta_D \geq 1 - \beta_D$ which holds\(^3\) for $N \geq 100$ and $\beta_D \geq 1/2$, we conclude that

$$\hat{F}_{N,\bar{d}d}(0) \gg \frac{\bar{d}dN}{\phi(\bar{d}d)}(1 - \beta_D)$$

$$+ O\left(\frac{\log(\bar{d}d)}{\sqrt{\log(\bar{d}d) + \log(\bar{d})}}(\log(\bar{d}d))^4\right).$$

Combining the inequality above with (4) yields the proposition. \qed

**3.2. The major arcs.** The following major arc estimate is used in the proof of the iteration lemma.

**Proposition 3.4.** Let $D \geq 2$. Let $N, d, a, q$ be positive integers such that $(a, q) = 1$. Suppose that either

1. $D$ is unexceptional, $\bar{d} = 1$ and $dq \leq D$,
2. or $D$ is exceptional, $\bar{d} = d_D$ and $dq \leq D^9$.

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\(^3\)This can be verified by taking derivatives of both sides and noticing the upper bound (4) for $\beta_D$.  

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Then for any $\delta \in [-1/2, 1/2]$, we have

$$\left| \widehat{F_{N,\bar{d}d}} \left( \frac{a}{q} + \delta \right) \right| \ll \frac{|\widehat{F_{N,\bar{d}d}}(0)|}{\phi(q)} + O\left( (1 + |\delta|N)\bar{d}dqN \exp\left( -c \frac{\log(\bar{d}dq^2N)}{\sqrt{\log N + \log D}} \right) \log^4(\bar{d}dq) \right).$$

Here $c$ is a positive absolute constant which does not depend on $N, D, d, a, q, \delta$.

SKETCH OF THE PROOF. We sketch a proof for the exceptional case. The unexceptional case can be shown in a similar manner. By a change of variables, one can show that

$$\widehat{F_{N,\bar{d}d}} \left( \frac{a}{q} + \delta \right) = \sum_{m=0}^{q-1} e\left( -\frac{am}{q} \right) \sum_{\substack{1 \leq l \leq \bar{d}dN+1 \\lfloor l \equiv 1 (mod(\bar{d}d)) \\lfloor l \equiv m (mod(q))}} \Lambda(l) e^{-2\pi i l(1-\delta)/(\bar{d}d)}.$$

It then follows from Lemma 3.1, a prime number theorem for arithmetic progressions [7, Theorem 5.27] and standard analytic number theory techniques (such as integration by parts and partial summation) that

$$\widehat{F_{N,\bar{d}d}} \left( \frac{a}{q} + \delta \right) = \frac{1}{\phi(\bar{d}dq)} \sum_{m=0}^{q-1} e\left( -\frac{am}{q} \right) \times \int_{1}^{\bar{d}dN+1} e^{-2\pi i \delta t/(\bar{d}d)} \left( \chi_0(\bar{d}dm+1) - \chi_0\chi_D(\bar{d}dm+1)t^{\beta_D-1} \right) dt + O\left( (1 + |\delta|N)\bar{d}dq^2N \exp\left( -c \frac{\log(\bar{d}dqN)}{\sqrt{\log N + \log D}} \right) \log^4(\bar{d}dq) \right),$$

where $\chi_0$ is the principal character of modulus $\bar{d}dq$. Since $\chi_0\chi_D(\bar{d}k+1) = \chi_D(1) = 1$ for every nonnegative integer $k$ such that $(\bar{d}k+1, \bar{d}dq) = 1$, and $\chi_0\chi_D(\bar{d}k+1) = 0$ otherwise, it follows that

$$\sum_{m=0}^{q-1} e\left( -\frac{am}{q} \right) \int_{1}^{\bar{d}dN+1} e^{-2\pi i \delta t/(\bar{d}d)} \left( \chi_0(\bar{d}dm+1) - \chi_0\chi_D(\bar{d}dm+1)t^{\beta_D-1} \right) dt = \int_{1}^{\bar{d}dN+1} e^{-2\pi i \delta t/(\bar{d}d)} \left( 1 - t^{\beta_D-1} \right) dt \sum_{m=0}^{q-1} e\left( -\frac{am}{q} \right).$$

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Notice that the inner sum over \( m \) is 0 if \((dd, q) > 1\), and it follows from the well-known bound for Ramanujan sum (see Iwaniec–Kowalski [7, (3.3)]) that its absolute value is bounded by 1 if \((dd, q) = 1\).

Since \( 1 - t^{\beta_D - 1} \geq 1 - \beta_D \) for all \( t \gg 1 \), it follows from Hölder’s inequality that
\[
\left| \widehat{F_{N, dd}}\left( \frac{a}{q} + \delta \right) \right| \leq \frac{1}{\phi(dd)\phi(q)} \cdot (1 - \beta_D)\dd N + O\left((1 + |\delta|N)\dd q^2 N \exp\left(-c \frac{\log(\dd q N)}{\sqrt{\log N + \log D}}\right) \log^4(\dd q) \right).
\]

Comparing the inequality above with (5) yields the result. \( \square \)

### 3.3. The minor arcs.

The minor arc estimate we need is a rather well-known consequence of the minor arc estimate of Vinogradov (see, for instance, Iwaniec and Kowalski [7, Theorem 13.6]).

**Proposition 3.5.** Let \( N, Q, d \) be positive integers such that \( d \leq N \). For any positive integers \( q, a \) such that \( q \leq Q \) and \((a, q) = 1\) and any \(|\theta - a/q| \leq 1/(qQ)\), we have
\[
\left| \widehat{F_{N, d}}(\theta) \right| \ll d(\log N)^4 \left( \frac{N}{\sqrt{q}} + N^{4/5} + \sqrt{NQ} \right).
\]

### 4. Locating shifted primes in difference sets

This section consists of two lemmas. The first one is an iteration lemma used in the difference set problem. Then we iterate this result to deduce the second one.

The set up of the lemma differs depending on whether a possible exceptional zero occurs or not. In the exceptional case, the iteration starts from an arithmetic progression whose common difference is a multiple of the exceptional modulus. This is motivated by considering the model situation where the Fourier transform is concentrated near the arcs whose denominators are multiples of the exceptional modulus.

We need some notation to describe the circle method. Let \( Q \) be a positive integer. For any integers \( q \leq Q \) and \( 1 \leq a \leq q \), we define
\[
\mathcal{M}_{a, q} = \left\{ \theta \in \mathbb{T} : \left| \theta - \frac{a}{q} \right| \leq \frac{1}{qQ} \right\} \quad \text{and} \quad \mathcal{M}_q^* = \bigcup_{(a,q) = 1, 1 \leq a \leq q} \mathcal{M}_{a, q}.
\]

By Dirichlet’s pigeonhole principle, we have \( \mathbb{T} = \bigcup_{q \leq Q} \mathcal{M}_q^* \).

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The following result of Ruzsa and Sanders [10, Corollary 7.3] is used to locate density increment. It is purposely designed according to the major arc estimate for $\hat{F}_{N,d}$.

**Proposition 4.1.** Let $N$ be a positive integer and $A \subseteq [N]$ have density $\alpha > 0$. Let $Q \geq Q_1 \geq 1$. Suppose

$$\alpha^{-1}|A|^{-1}\sum_{q=1}^{Q_1} \frac{1}{\phi(q)} \int_{M_q^*} |(1_A - \alpha 1_{[N]})^\wedge(\theta)|^2 \, d\theta \geq c$$

for some $c > 0$, then there exists an arithmetic progression $P$ with common difference $q \leq Q_1$ and $|P| \gg Q_1^{-1} \min\{Q, \alpha N\}$ such that

$$|A \cap P| \geq \alpha(1 + 2^{-5}c)|P|.$$ 

The proof of the lemma below is essentially the same as the proof of Ruzsa and Sanders [10]. Instead of looking at whether the difference set contains any shifted primes, we quantify the number of shifted primes that can be located using the same idea.

**Lemma 4.2.** There exist positive absolute constants $C_1, c_1$ such that we can obtain the following result.

Let $D \geq 2$ and let $N$ be a positive integer such that $N \leq \exp(D^{1/10})$. Let $d$ be a positive integer. Let $\bar{d} = 1$ if $D$ is unexceptional, or $\bar{d} = d_D$ if $D$ is exceptional.

Suppose $A \subseteq [N]$ has density $\alpha > 0$. Suppose also that

\begin{equation}
C_1 (\log D)^2 \leq \log N \quad \text{and} \quad \log d + \log \alpha^{-1} \leq C_1^{-1} \log D.
\end{equation}

Then one of the following assertions must be true.

(i) There exists an arithmetic progression $P'$ with common difference $\ll \alpha^{-3}$ and length $\gg (\alpha/\bar{d}d \log N)^8 N$ such that $|A \cap P'| \geq \alpha(1 + c_1)|P'|$.

(ii) There exists $N' \gg \alpha N$ such that

$$\left|(A - A) \cap \left(\frac{P - 1}{dd} \right)\right| \geq \frac{c_1 \alpha N'}{d \log N'}.$$

**Proof.** Let $c > 0$ be an absolute constant to be optimised later. Then, either $N \ll c^{-1} \alpha^{-1}$ which is impossible for $C_1 \gg c$ due to (6), or $N' := \lfloor c \alpha N \rfloor \geq 1$. By Proposition 3.3, we have

\begin{equation}
|\hat{F}_{N',\bar{d}d}(0)| \gg \frac{N'}{d}.
\end{equation}

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\footnote{We need to introduce an upper bound on $N$ due to the factor $(\log N)^4$ in the minor arc estimate.}
Suppose first that

\[(8) \quad \langle 1_A * 1 - A, F_{N', \bar{d}d} \rangle \geq \frac{\alpha^2 N |\hat{F}_{N', \bar{d}d}(0)|}{2}.\]

Notice that there are at most \(O((\bar{d}dN')^{1/2})\) composite prime powers which are less than \(\bar{d}dN' + 1\). Combine this observation with the trivial upper bounds \(1_A * 1 - A(n) \leq \alpha N\) and \(F_{N', \bar{d}d}(n) \ll \log(\bar{d}dN')\), we have

\[
\sum_{n: \bar{d}dn + 1 = p^k} 1_A * 1 - A(n) F_{N', \bar{d}d}(n) \ll \alpha N \sqrt{\bar{d}dN' \log(\bar{d}dN')}.
\]

Since \(C_1\) is large, it follows from (6) that \(|\hat{F}_{N', \bar{d}d}(0)|^{2/3} \gg \alpha \sqrt{\bar{d}dN' \log(\bar{d}dN')}\), and so

\[
\sum_{n: \bar{d}dn + 1 = p^k} 1_A * 1 - A(n) F_{N', \bar{d}d}(n) \ll \alpha^2 N |\hat{F}_{N', \bar{d}d}(0)|^{2/3}.
\]

By (7) and the definition of \(N'\), if \(|\hat{F}_{N', \bar{d}d}(0)| = O(1)\) then (6) can no longer hold. Thus, subtracting the inequality above from (8) yields

\[
\sum_{n: \bar{d}dn + 1 \in \mathbb{P}} 1_A * 1 - A(n) F_{N', \bar{d}d}(n) \geq \frac{\alpha^2 N |\hat{F}_{N', \bar{d}d}(0)|}{4}.
\]

Using the bounds \(1_A * 1 - A(n) \leq \alpha N\) and \(F_{N', \bar{d}d}(n) \leq \log(\bar{d}dN' + 1)\) again, together with Hölder’s inequality, (6) and (7), we can conclude that

\[(9) \quad |(A - A) \cap \{n : \bar{d}dn + 1 \in \mathbb{P}\}| \gg \frac{\alpha N'}{d \log N'}.
\]

Otherwise, we have

\[(10) \quad \langle 1_A * 1 - A, F_{N', \bar{d}d} \rangle \leq \frac{\alpha^2 N |\hat{F}_{N', \bar{d}d}(0)|}{2}.
\]

Let \(I\) denote the interval \([N]\) and consider the inner product

\[
\langle (1_A - \alpha 1_I) * (1 - A - \alpha 1_{-I}), F_{N', \bar{d}d} \rangle.
\]

By (10) and direct computations, we have

\[
\langle (1_A - \alpha 1_I) * (1 - A - \alpha 1_{-I}), F_{N', \bar{d}d} \rangle \leq \alpha^2 N |\hat{F}_{N', \bar{d}d}(0)| (-1/2 + O(c)).
\]
Therefore, we can choose a small constant \( c \gg 1 \) which guarantees
\[
|\langle (1_A - \alpha 1_I) \ast (1_A - \alpha 1_I), F_{N', \dd} \rangle| \gg \alpha^2 N |\hat{F}_{N', \dd}(0)|.
\]

It follows from Plancherel’s theorem that
\[
\int_{T} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 \left| \hat{F}_{N', \dd}(\theta) \right| d\theta \gg \alpha^2 N |\hat{F}_{N', \dd}(0)|.
\]

Let \( c', c'' \) be two positive absolute constants to be chosen later. Let
\[
Q' = \frac{(\dd)^4 \log^8 N'}{c'^2 \alpha^2}, \quad Q = \frac{N'}{Q'} \quad \text{and} \quad Q'' = c'' \alpha^{-3}.
\]

If \( Q' \geq Q'' \) then let
\[
\mathcal{M}' := \bigcup_{Q'<q \leq Q} \mathcal{M}^*_q, \quad \mathcal{M}'' := \bigcup_{Q'<q \leq Q'} \mathcal{M}^*_q \quad \text{and} \quad \mathcal{M}''' := \bigcup_{q \leq Q''} \mathcal{M}^*_q,
\]

which are defined at the beginning of this section. If \( Q' < Q'' \) we define \( \mathcal{M}'' = \emptyset \) and \( \mathcal{M}', \mathcal{M}''' \) as above. Notice that it follows from Dirichlet’s pigeonhole principle that \( T = \mathcal{M}' \cup \mathcal{M}'' \cup \mathcal{M}''' \), and so
\[
\int_{\theta \in T} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 \left| \hat{F}_{N', \dd}(\theta) \right| d\theta
\]
\[
\leq \int_{\theta \in \mathcal{M}'} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 \left| \hat{F}_{N', \dd}(\theta) \right| d\theta
\]
\[
+ \int_{\theta \in \mathcal{M}''} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 \left| \hat{F}_{N', \dd}(\theta) \right| d\theta
\]
\[
+ \int_{\theta \in \mathcal{M}'''} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 \left| \hat{F}_{N', \dd}(\theta) \right| d\theta.
\]

By (6), we can apply the minor arc estimate given in Theorem 3.5 with our choices\(^5\) of \( N', Q' \) and \( Q \) to see that
\[
|\hat{F}_{N', \dd}(\theta)| \ll \frac{N'}{\dd} (c' \alpha + \dd N'^{-1/10}) \ll \frac{c' \alpha N'}{\dd} \ll c' \alpha |\hat{F}_{N', \dd}(0)|
\]
for all \( \theta \in \mathcal{M}' \). By Plancherel’s theorem,
\[
\int_{T} \left| \left( \hat{1}_A - \alpha \hat{1}_I \right)(\theta) \right|^2 d\theta \leq \alpha N.
\]

\(^5\) We also used \( N \leq \exp(D^{1/10}) \) in the inequality below.
Hence, it follows from Hölder’s inequality and (13) that

\begin{equation}
\int_{\theta \in \mathcal{M}'} \left| (\hat{1}_A - \alpha \hat{1}_I)(\theta) \right|^2 \left| \hat{F}_{N', \bar{d}d}(\theta) \right| \, d\theta \ll c' \alpha^2 N \left| \hat{F}_{N', \bar{d}d}(0) \right|.
\end{equation}

We employ our major arc estimate on \( \mathcal{M}'' \) and \( \mathcal{M}''' \). Since for all \( C_1 \gg c' 1 \), we have \( dQ' \leq D^9 \) if \( D \) is exceptional or \( dQ' \leq D \) if \( D \) is unexceptional, it follows from Proposition 3.4 that for \( a/q + \delta \in \mathcal{M}_{a,q} \) we have

\begin{equation}
\left| \hat{F}_{N', \bar{d}d}(a/q + \delta) \right| \ll \frac{|\hat{F}_{N', \bar{d}d}(0)|}{\phi(q)}
\end{equation}

\begin{equation}
+ O \left( (1 + |\delta| N') \bar{d}N' \exp \left( -\Omega \left( \frac{\log(\bar{d}qN')}{\log N' + \log D} \right) \right) \log^4(\bar{d}q) \right).
\end{equation}

Notice that for any \( C_1 \) which is sufficiently large in terms of \( c' \) and the implicit constants in (7) and (15), it follows from our choices of \( Q, Q' \) and (6) that for each \( q \leq Q' \), the second term in the major arc estimate (15) is at most \( |\hat{F}_{N', \bar{d}d}(0)| / \phi(q) \). Therefore, uniformly for all \( q \leq Q' \) and \( a/q + \delta \in \mathcal{M}_{a,q} \) we have

\begin{equation}
\left| \hat{F}_{N', \bar{d}d}(a/q + \delta) \right| \ll \frac{|\hat{F}_{N', \bar{d}d}(0)|}{\phi(q)}
\end{equation}

It follows from the choice of \( Q'' \) and the definition of \( \mathcal{M}'' \) that

\begin{equation}
\sup_{\theta \in \mathcal{M}''} \left| \hat{F}_{N', \bar{d}d}(\theta) \right| \ll c'' \alpha \left| \hat{F}_{N', \bar{d}d}(0) \right|.
\end{equation}

By (17) and Plancherel’s theorem, we have

\begin{equation}
\int_{\theta \in \mathcal{M}''} \left| (\hat{1}_A - \alpha \hat{1}_I)(\theta) \right|^2 \left| \hat{F}_{N', \bar{d}d}(\theta) \right| \, d\theta \ll c'' \alpha^2 N \left| \hat{F}_{N', \bar{d}d}(0) \right|.
\end{equation}

Therefore, combining (11), (14), (18) and (12), we can choose \( c' \gg 1 \) and \( c'' \gg 1 \) so that

\begin{equation}
\int_{\theta \in \mathcal{M}''} \left| (\hat{1}_A - \alpha \hat{1}_I)(\theta) \right|^2 \left| \hat{F}_{N', \bar{d}d}(\theta) \right| \, d\theta \gg \alpha^2 N \left| \hat{F}_{N', \bar{d}d}(0) \right|.
\end{equation}

By the definition of \( \mathcal{M}''' \) and the triangle inequality, we have

\[ \sum_{q \leq Q''} \int_{\theta \in \mathcal{M}_{q}} \left| (\hat{1}_A - \alpha \hat{1}_I)(\theta) \right|^2 \left| \hat{F}_{N', \bar{d}d}(\theta) \right| \, d\theta \gg \alpha^2 N \left| \hat{F}_{N', \bar{d}d}(0) \right|. \]
Combine this with (16), we can deduce that

\[ \sum_{q \leq Q''} \frac{|\overline{F_{N', \overline{d}d}(0)}|}{\phi(q)} \int_{\theta \in \mathbb{R}} |(1_A - \alpha \hat{1}_I)(\theta)|^2 \, d\theta \gg \alpha^2 N |\overline{F_{N', \overline{d}d}(0)}|. \]

Since $|\overline{F_{N', \overline{d}d}(0)}| > 0$, we obtain the density increment outcome (i) by applying Proposition 4.1. □

Now we apply iteratively the lemma above to find shifted primes in the difference set. Assuming the input parameters satisfy certain conditions, which roughly say that $A$ has large density and the input common difference is not too large, we rule out density increment and locate many shifted primes in an arithmetic progression. Compared to the work of Wang [13], we follow the same strategy and quantify the number of shifted primes we can find from the difference set after iterations.

**Lemma 4.3.** Let $N$ and $d$ be positive integers. Let $D \geq 2$ and $\alpha > 0$. Suppose that $N < \exp(D^{1/10})$ and $\alpha = O((\log N)^{-1})$. Suppose also that

\[ C_2 (\log D)^2 \leq \log N \quad \text{and} \quad \log d + (\log(\alpha^{-1}))^2 \leq C_2^{-1} \log D \]

for some sufficiently large constant $C_2$. Let $\overline{d} = d_D$ be the modulus of the exceptional zero if $D$ is exceptional, or $\overline{d} = 1$ if $D$ is unexceptional.

Suppose that $A$ is contained in an arithmetic progression of length $N$ with common difference $\overline{d}d$, and has $|A| = \alpha N$. Then, there exists $A' \subseteq A - A$ which is contained in an arithmetic progression of length

\[ N' \geq \alpha^{C_3 (\log \alpha^{-1})^2 (\overline{d}d)^{-C_3 \log \alpha^{-1} (\log N)^{-C_3 \log \alpha^{-1}}} N \]

with common difference $\overline{d}d'$, where $d' \leq \alpha^{-C_3 \log \alpha^{-1}} d$ and $d \mid d'$, such that

\[ |A' \cap (\mathbb{P} - 1)| \geq \frac{c_1 \alpha N'}{d \log N'}. \]

Here, $C_3$ is an absolute constant which does not depend on $N, \alpha, d, \overline{d}$ or $D$, and $c_1$ is the same one as in Lemma 4.2.

**Proof.** Let $A_0$ be the affine transformation of $A$ so that $A_0 \subseteq [N]$. We choose

\[ A_1 = A_0, \quad N_1 = N, \quad \alpha_1 = \alpha \quad \text{and} \quad d_1 = d. \]

Since $C_2$ is large, $D$, $\overline{d}$, $A_1$, $N_1$, $\alpha_1$, $d_1$ satisfy the hypotheses of Lemma 4.2.

Suppose for some positive integer $k$, we have obtained $A_k, N_k, \alpha_k$ and $d_k$ such that $A_k$ is a subset of $[N_k]$, $|A_k| = \alpha_k N_k$, $d \mid d_k$, $A_k$ is dilated by a
factor $\tilde{d}^{-1}d_{k}^{-1}$ times a shifted subset of $A$, the inputs $D$, $\tilde{d}$, $A_{k}$, $N_{k}$, $\alpha_{k}$, $d_{k}$ satisfy the hypotheses of Lemma 4.2, and $N_{k}$, $\alpha_{k}$, $d_{k}$ satisfy the bounds

\begin{equation}
\alpha_{k} \geq (1 + c_{1})^{k-1} \alpha,
\end{equation}

\begin{equation}
d_{k} \leq \alpha^{-C(k-1)}d,
\end{equation}

and

\begin{equation}
N_{k} \geq \frac{\alpha^{Ck^{2}N}}{(dd)^{C(k-1)}(\log N)^{C(k-1)}},
\end{equation}

where $C$ is a large absolute constant which is independent of any parameters. Let us apply Lemma 4.2 to $D$, $\tilde{d}$, $A_{k}$, $N_{k}$, $\alpha_{k}$ and $d_{k}$. Either we are in outcome (ii), or it follows from outcome (i) that there exist $A_{k+1}$, $N_{k+1}$, $\alpha_{k+1}$ and $d_{k+1}$ such that $A_{k+1}$ is a subset of $[N_{k+1}]$, $|A_{k+1}| = \alpha_{k+1}N_{k+1}$, $d \mid d_{k+1}$, $A_{k+1}$ is dilated by a factor $\tilde{d}^{-1}d_{k}^{-1}$ times a shifted subset of $A$, the inputs $N_{k+1}$, $\alpha_{k+1}$, $d_{k+1}$ satisfy the bounds $\alpha_{k+1} \geq (1 + c_{1})^{k}\alpha$, $d_{k+1} \leq \alpha^{-Ck}d$, and $N_{k+1} \geq \alpha^{C(k+1)^{2}}N(\tilde{d}d)^{-Ck}(\log N)^{-Ck}$. Moreover, for sufficiently large $C_{2}$, condition (6) from Lemma 4.2 holds for all $k \leq 2(\log \alpha^{-1})/\log(1 + c_{1})$, where $c_{1}$ is the constant from Lemma 4.2.

Since outcome (i) at every step of the iteration would produce a set with density bigger than 1 when $k > (\log \alpha^{-1})/\log(1 + c_{1})$, it follows that we must be in outcome (ii) for some

\begin{equation}
k_{0} \leq \frac{\log \alpha^{-1}}{\log(1 + c_{1})}.
\end{equation}

Using the size bounds (20), (21) and (22) and the bound given in outcome (ii), we have

\begin{equation}
\left| (A_{k_{0}} - A_{k_{0}}) \cap \left( \frac{p - 1}{dd_{k_{0}}} \right) \right| \geq \frac{c_{1}\alpha_{k_{0}}N'_{k_{0}}}{d\log N'_{k_{0}}},
\end{equation}

where

\begin{equation}
N'_{k_{0}} \gg \alpha_{k_{0}}N_{k_{0}} \gg \frac{\alpha^{Ck_{0}^{2}+2N}}{(dd)^{Ck_{0}}(\log N)^{Ck_{0}}}.
\end{equation}

Let

\[ A'_{k_{0}} = \{ n \in A_{k_{0}} - A_{k_{0}} : n \leq N'_{k_{0}}, \tilde{d}d_{k_{0}}n + 1 \text{ is a prime} \}. \]
It follows from (24) that

\begin{equation}
|A'_{k_0}| \geq \frac{c_1 \alpha k_0 N'_{k_0}}{d \log N'_{k_0}}.
\end{equation}

Let $A'$ be the pre-image of $A'_{k_0}$ under the affine transformations carried out in the proof above. Since $A_{k_0}$ is dilated by a factor $\bar{d} d_{k_0}^{-1}$ times a shifted subset of $A$, it follows that $A'$ is contained in an arithmetic progression of length $N'_{k_0}$ with common difference $\bar{d} d_{k_0}$, and $A'$ consists of numbers which are one less than a prime. Therefore, one can conclude that the assertion of the lemma holds for sufficiently large $C_3$ by (21), (23), (25) and (26).

\[ \square \]

5. The bootstrapping lemma

The aim of this section is to establish the bootstrapping lemma which is the key tool used in the proof of Theorem 1.2.

We need the following preliminary lemma which follows from averaging.

**Lemma 5.1.** Let $X \subseteq \mathbb{Z}$ be a set contained in an arithmetic progression of length $N$ with common difference $d$. Suppose $Y \subseteq \mathbb{Z}$ is contained in an arithmetic progression of length $N'$ with common difference $d'$ where $d'$ is a positive integer, then there exists $n \in \mathbb{Z}$ such that

\[ |(n + Y) \cap X| \geq \frac{|X||Y|}{N + d'N'}. \]

**Proof.** We assume $d = 1$ in the proof below, since the general result follows from applying the $d = 1$ result to affine transformations of the sets $X$ and $Y$.

Notice that the set

\[ \{ x \in \mathbb{Z} : (x + Y) \cap X \neq \emptyset \} \]

is contained in an interval of length at most $N + d'N'$. Also, by a direct counting argument, we have

\[ \sum_{x \in \mathbb{Z}} |(x + Y) \cap X| = |X||Y|. \]

Therefore, it follows from the pigeonhole principle that there exists some $n \in \mathbb{Z}$ such that

\[ |(n + Y) \cap X| \geq \frac{|X||Y|}{N + d'N'}, \]

which completes the proof. \[ \square \]
We combine Lemma 4.3 and the averaging lemma above to deduce the bootstrapping lemma. There are two steps involved in the proof of the bootstrapping lemma below. The first one is to apply the iteration lemma to an initial set and the pigeonhole principle to find either a desired monochromatic solution, or a large monochromatic subset of shifted primes in its difference set. In the latter case, we then use the averaging lemma above to locate one translate whose intersection with the initial set is large. Since difference sets are translation invariant, the difference set of this particular translate lies in the difference set of the initial set, and so it does not contain the colour of the initial set.

**Lemma 5.2.** Let \( N_0 \) and \( k \) be positive integers. Let \( N \leq N_0 \) and \( d \) be positive integers. Let \( D \geq 2 \) be a positive number such that \( N < \exp(D^{1/10}) \) and let \( \alpha = O((\log N)^{-1}) \). Let \( d = D \) be the modulus of the exceptional zero if \( D \) is exceptional, or \( d = 1 \) if \( D \) is unexceptional.

Suppose \( N, d, \alpha, D \) satisfy (19), then for any \( k \) colouring of \((\mathbb{P} - 1) \cap [N_0] \) and \( j \leq k \), we have the following.

Suppose \( A \subseteq [N_0] \) is a monochromatic subset of \( \mathbb{P} - 1 \) contained in an arithmetic progression of length \( N \) with common difference \( d \), and has \( |A| = \alpha N \). Suppose also that \( A \cup ((A - A) \cap (\mathbb{P} - 1)) \) is \( j \)-coloured, then

1. either there exists a monochromatic solution to \( x - y = z \) where \( x, y, z \in (\mathbb{P} - 1) \cap [N_0] \);
2. or there exists a monochromatic set \( A'' \subseteq \mathbb{P} - 1 \) satisfying the following:

   a. \( A'' \) is contained in an arithmetic progression of length \( N'' \) with common difference \( d'' \), where \( N'' \geq \alpha^C (\log \alpha^{-1})^2 (dD)^{-C \log \alpha^{-1}} (\log N)^{-C \log \alpha^{-1}} N \), \( d \mid d'' \) and \( d'' \leq \alpha^{-C \log \alpha^{-1}} d ");

   b. \( |A''| \geq \alpha^C ((j - 1)^d \log N)^{-1} N'' \);

   c. \( A'' \cup ((A'' - A'') \cap (\mathbb{P} - 1)) \) is \((j - 1)\)-coloured.

**Proof.** Suppose there exists \( z \in (A - A) \cap (\mathbb{P} - 1) \) which shares the same colour as \( A \), then we are in the first outcome. Hence, from now on, we assume that \( (A - A) \cap (\mathbb{P} - 1) \) is at most \((j - 1)\)-coloured, and every element of \((A - A) \cap (\mathbb{P} - 1) \) has a different colour from \( A \).

By applying Lemma 4.3 with \( A, \alpha, d, D, N \), we can find a set \( A' \subseteq A - A \) such that \( A' \) is contained in an arithmetic progression of length

\[
N' \geq \frac{\alpha^{C_3 (\log \alpha^{-1})^2} N}{(dD)^{C_4 \log \alpha^{-1}} (\log N)^{C_4 \log \alpha^{-1}}}
\]

with common difference \( d' \) where \( d' \leq \alpha^{-C_3 \log \alpha^{-1}} d \), \( d \mid d' \), and

\[
|A' \cap (\mathbb{P} - 1)| \geq \frac{c_1 \alpha}{d' \log N'} N'.
\]

Here \( c_1 \) and \( C_3 \) are the same constants as the ones in Lemma 4.3.
Let $B$ be the largest colour class in $A' \cap (\mathbb{P} - 1)$. It follows that $B$ is contained in an arithmetic progression of length $N'$ with common difference $d'd$, and

$$|B| \geq \frac{c_1 \alpha}{(j - 1)d \log N'} N'.$$

By Lemma 5.1, there exists some $n$ such that

$$|(n + B) \cap A| \geq \frac{|A||B|}{N + (d'/d)N'}.$$

Since $A' \subseteq A - A$, we have $d'N' \leq 2dN$, and so

$$|(n + B) \cap A| \geq \frac{|A||B|}{3N} \geq \frac{c_1 \alpha^2}{3(j - 1)d \log N} N'.$$

Let

$$A'' := B \cap (A - n).$$

Since $A'' \subseteq B$, it follows that $A''$ is monochromatic and satisfies condition (a), with $N'' = N'$ and $d'' = d'$. We have

$$|A''| = |(n + B) \cap A| \geq \frac{c_1 \alpha^2}{3(j - 1)d \log N} N''$$

which completes the proof of assertion (b). By our assumption which rules out outcome (i), we know that $A'' \subseteq B$ is monochromatic and has a different colour from $A$. Since $A'' \subseteq A - n$, we have $A'' - A'' \subseteq A - A$, and so $(A'' - A'') \cap (\mathbb{P} - 1)$ contains at most $j - 1$ colour classes and they are different from the colour of $A$. By combining these observations, we can conclude that assertion (c) is true. □

6. Proof of Theorem 1.2

The following lemma is used to modify the initial set if an exceptional zero occurs.

**Lemma 6.1.** Let $A$ be a subset of an arithmetic progression of length $N$ with common difference $d$ and have $|A| = \alpha N$ for some $\alpha > 0$. For every positive integer $\bar{d}$, there exists an arithmetic progression $P$ with common difference $\bar{d}d$ and length at least $\alpha N/\bar{d}$ such that $|A \cap P| \geq \alpha |P|/2$.

**Proof.** The lemma follows from applying Lemma 5.1 with $d' = \bar{d}$ and $Y = \{dd'n : 0 \leq n \leq \alpha N/\bar{d}, n \in \mathbb{Z}\}$. □
The proof of the main theorem essentially follows from bootstrapping Lemma 5.2 from the previous section. The initial monochromatic set follows from the prime number theorem and a basic application of the pigeonhole principle. Afterwards, we shall repeatedly apply Lemma 5.2 until we find a monochromatic solution to \( x + y = z \) in \( \mathbb{P} - 1 \), or we shall run out of the colour classes which is impossible. In the proof below, we shall clarify our choices of sets at each step of the bootstrapping, and keep a detailed record of the quantities involved there.

**Proof of Theorem 1.2.** Let \( N_0 = N \). By the prime number theorem and the pigeonhole principle, the largest colour class in \( (\mathbb{P} - 1) \cap [N_0] \) must have cardinality at least \( (1 + o(1))N_0(k \log N_0)^{-1} \). Thus, we can always find a monochromatic subset \( A_0 \subseteq [N_0] \) such that \( |A_0| = \alpha_0N_0 \) with \( \alpha_0 = (2k \log N_0)^{-1} \).

We shall proceed by contradiction. Suppose the theorem is false, then we claim that for each \( 1 \leq i \leq k \), we can find \( D_i, A_i, N_i, \alpha_i, d_i, A''_i, N''_i, d''_i, \alpha''_i \) which satisfy the following conditions:

(I) \( 2^{-k-4+i}(\log(2\alpha_i)^{-1})^3 \leq \log D_i \leq 2^{-k-4+i}(\log\alpha_i^{-1})^3 \);

(II) \( 2^{-k-4+i}(\log(2\alpha_i^{-1})^{-1})^3 \leq \log\alpha_i^{-1} \leq 2^{-k-3+i}(\log\alpha_i^{-1})^3 \);

(III) \( \log d_i \leq 2^{-k-4+i}(\log\alpha_i^{-1})^3 \);

(IV) \( \log N_i \geq \log N_{i-1} - C'(\log\alpha_{i-1}^{-1})^9 \) where \( C' \) is a large absolute constant;

(V) \( A''_i \) is contained in an arithmetic progression of length \( N''_i \) with common difference \( d''_i/d_i \), where

\[
N''_i \geq \alpha_i^{C(\log\alpha_i^{-1})^2}(d_iD_i)^{-C\log\alpha_i^{-1}}(\log N_i)^{-C\log\alpha_i^{-1}}N_i,
\]

\( d_i \mid d''_i \) and \( d''_i \leq \alpha_i^{-C\log\alpha_i^{-1}}d_i \);

(VI) \( |A''_i| = \alpha''_iN''_i \) where \( \alpha''_i/2 \leq \alpha_i^{C\log\alpha_i^{-1}}((k-i)d_iD_i\log N_i)^{-1} \leq \alpha''_i \); and

(VII) \( A''_i \cup ((A''_i - A''_i) \cap (\mathbb{P} - 1)) \) is \( (k-i) \)-coloured.

In particular, since condition (VI) and condition (VII) are impossible for \( i = k \), we come to a contradiction at the \( k \)th step and so the theorem must be true.

Let \( D_1 = \exp((\log\alpha_0^{-1})^3/2^{k+3}) \) and \( d_1 = 1 \).

- if \( D_1 \) is unexceptional, we let \( N_1 = N_0, \overline{d_1} = 1 \), and \( A_1 \) be a subset of \( A_0 \) such that \( |A_1| = \alpha_1N_1 \) where\(^6\) \( 2^{-k-3}(\log(2\alpha_0)^{-1})^3 \leq \log\alpha_1^{-1} \leq 2^{-k-2}(\log\alpha_0^{-1})^3 \);

- if \( D_1 \) is exceptional, then let \( \overline{d_1} \) be the exceptional modulus. We choose \( A_1 \) to be a subset of the set obtained from Lemma 6.1, so that \( A_1 \)

---

\(^6\)This is always possible since \( \alpha_1 \) stated here is way smaller than \( \alpha_0 \) since \( k \leq (\log\log N_0)^c \) for some \( c < 1 \). Similarly, we can make such a choice in the exceptional case.

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is contained in an arithmetic progression with common difference \( \overline{d_1} \) and length \( N_1 \gg \alpha_0 N_0/\overline{d_1} \), and \( A_1 \) has density \( \alpha_1 \) in this arithmetic progression where \( 2^{-k-3}(\log(2\alpha_0)^{-1})^3 \leq \log \alpha_1^{-1} \leq 2^{-k-2}(\log \alpha_0^{-1})^3 \).

By our choice of \( D_1 \) and the bound \( k \leq (\log \log \log N_0)^c \) for some small constant \( c \), the parameters satisfy the conditions listed in (19), and so we can apply Lemma 5.2 to conclude that either the theorem is true (when the first outcome of Lemma 5.2 occurs) which contradicts our assumption, or we can find a monochromatic set \( A''_1 \subseteq \mathbb{P} - 1 \) such that

- \( A''_1 \) is contained in an arithmetic progression of length \( N''_1 \) and common difference \( \overline{d''_1} \), where \( N''_1 \geq \alpha_1^{(\log \alpha_1^{-1})^2} (d_1 D_1)^{-C \log \alpha_1^{-1}} (\log N)^{-C \log \alpha_1^{-1}} N_1 \), \( d \mid d'' \) and \( d''_1 \leq \alpha_1^{-C \log \alpha_1^{-1}} d_1 \);

- \( |A''_1| = \alpha''_1 N''_1 \) where \( \alpha''_1 \leq \alpha_1^{\log \alpha_1^{-1}} ((k - 1) d_1 D_1 \log N_1)^{-1} \leq \alpha''_1 \);

- \( A''_1 \cup ((A''_1 - A''_1) \cap (\mathbb{P} - 1)) \) is \( (k - 1) \)-coloured.

In particular, it follows from the properties above that conditions (I)-(VII) hold for \( i = 1 \).

Suppose for some \( 1 \leq j \leq k - 1 \), we have defined \( D_i, A_i, N_i, \alpha_i, \overline{d_i}, A''_i, N''_i, d''_i, \alpha''_i \) which satisfy conditions (I)-(VII) for all \( 1 \leq i \leq j \). We choose \( D_{j+1}, A_{j+1}, N_{j+1}, \alpha_{j+1}, d_{j+1}, A''_{j+1}, N''_{j+1}, d''_{j+1}, \alpha''_{j+1} \) as follows. Fix

\[
D_{j+1} = \exp \left( (\log \alpha''_{j+1}^{-1})^3 / 2^{k-j+3} \right) \quad \text{and} \quad d_{j+1} = \overline{d_{j+1}} \cdot d_j.
\]

Depending on whether \( D_{j+1} \) is exceptional, we define \( A_{j+1}, N_{j+1}, \overline{d_{j+1}}, \alpha_{j+1} \) as follows:

(A.I) if \( D_{j+1} \) is unexceptional, we take \( A_{j+1} = A''_{j+1}, N_{j+1} = N''_{j+1}, d_{j+1} = 1 \) and \( \alpha_{j+1} = \alpha''_{j+1} \);

(A.II) if \( D_{j+1} \) is exceptional, then let \( \overline{d_{j+1}} \) be the exceptional modulus. Let \( A_{j+1} \) be the set obtained from Lemma 6.1, so that \( A_{j+1} \) is contained in an arithmetic progression with common difference \( d_{j+1} \overline{d_{j+1}} \) and length \( N_{j+1} \gg \alpha''_{j+1} N''_{j+1} / \overline{d_{j+1}} \), and \( A_{j+1} \) has density \( \alpha_{j+1} \geq \alpha''_{j+1} / 2 \) in this arithmetic progression.

We shall first show that conditions (I)-(IV) hold for \( j + 1 \).

Condition (I) is a direct consequence of the definitions of \( D_{j+1} \) and \( \alpha_{j+1} \).

By (A.I), (A.II) and (VI), we have

\[
\log \alpha_{j+1} \geq \log \alpha''_{j+1} - \log 2 \\
\geq -C(\log \alpha_{j+1}^{-1})^2 - \log(k - j) - \log d_j - \log D_j - \log \log N_j - \log 2.
\]

Since \( k = (\log \log \log N_0)^c \) for some small constant \( c < 1 \), and \( \alpha_i \leq (k \log N_0)^{-1} \) for all \( i \leq j \), we have

\[
C(\log \alpha_{j+1}^{-1})^2 + \log(k - j) + \log \log N_j + \log 2 \leq 2^{-k-5+j}(\log \alpha_{j+1}^{-1})^3.
\]
By (II), (III) and $\alpha_j \leq (k \log N_0)^{-1}$, we also have

$$\log d_j \leq 2^{-k-4+j}(\log \alpha_j^{-1})^3 \leq 2^{-k-1+j} \log \alpha_j^{-1} \leq 2^{-k-5+j}(\log \alpha_j^{-1})^3.$$ 

Combining the bounds above with (I), we conclude that

$$\log \alpha_j^{-1} \leq 2^{-k-4+j}(\log \alpha_j^{-1})^3 + \log D_j \leq 2^{-k-3+j}(\log \alpha_j^{-1})^3.$$ 

On the other hand, since $\log \alpha_j^{-1} \geq 2^{-k-4+j}(\log(2\alpha_j)^{-1})^3$, it follows from (I) and (VI) that

$$\log \alpha_j^{-1} \geq 2^{-k-4+j}(\log(2\alpha_j)^{-1})^3.$$ 

Thus, we conclude that condition (II) holds for $j + 1$.

To see that condition (III) is true, notice that combining our definitions of $d_{j+1}$ and $\overline{d}_j$, (I), (III) and (V) yields

$$\log d_{j+1} = \log d''_j + \log \overline{d}_j \leq C(\log \alpha_j^{-1})^2 + \log d_j + \log D_j \leq C(\log \alpha_j^{-1})^2 + 2^{-k-4+j}(\log \alpha_j^{-1})^3 + 2^{-k-4+j}(\log \alpha_j^{-1})^3.$$ 

Since $k \leq (\log \log \log N_0)^c$ and $\alpha_j \leq (k \log N_0)^{-1}$, it follows that one has $C(\log \alpha_j^{-1})^2 \leq 2^{-k-5+j}(\log \alpha_j^{-1})^3$. On the other hand, by condition (II), $k \leq (\log \log \log N_0)^c$ and $\alpha_j \leq (k \log N_0)^{-1}$, we have

$$2^{-k-4+j}(\log \alpha_j^{-1})^3 \leq \log 2^{-k-1+j}\alpha_j^{-1} \leq 2^{-k-5+j}(\log \alpha_j^{-1})^3.$$ 

Therefore,

$$\log d_{j+1} \leq 2^{-k-3+j}(\log \alpha_j^{-1})^3.$$ 

By (A.I), (A.II) and (V), one obtains

$$\log N_{j+1} \geq \log N''_j - \log D_{j+1} + \log \alpha''_j + O(1) \geq -C(\log \alpha_j^{-1})^3 - C(\log d_j + \log D_j) \log \alpha_j^{-1} - C \log \alpha_j^{-1} \log \log N_j + \log N_j - \log D_{j+1} - C(\log \alpha_j^{-1})^2 - \log(k - j) - \log d_j - \log D_j - \log \log N_j + O(1).$$ 

Notice that

$$(\log d_j)(\log \alpha_j^{-1}) + \log \alpha_j^{-1} \log \log N_j \ll (\log \alpha_j^{-1})^2$$

which follows from (II), (III) and the inequality $\alpha_j \leq \alpha_0 \leq (k \log N)^{-1}$. Hence, by (I), (II), (A.I) and (A.II), one has

$$\log N_{j+1} \geq -C(\log \alpha_j^{-1})^3 - 2C(\log D_j) \log \alpha_j^{-1} + \log N_j - \log D_{j+1}$$

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and so (IV) holds for \( j+1 \).

In fact, it follows from the verifications above that the estimates from (19) hold for \( A_{j+1}, D_{j+1}, \alpha_{j+1}, d_{j+1} \). The first inequality there follows from iterating (II) and (IV) and using (I). More specifically, one has

\[
\log \alpha_{j+1}^{-1} \leq (\log \alpha_0^{-1})^{3^{j+1}}
\]

and

\[
\log N_{j+1} \geq \log N_0 - (j+1)\, C'(\log \alpha_j^{-1})^9.
\]

Therefore, the first inequality from (19) holds if

\[
2^{-k-2+j} (\log \alpha_0^{-1})^{3^{j+3}} \ll \log N_0 - (j+1)\, C'(\log \alpha_j^{-1})^{3^{j+3}}
\]

for some sufficiently large implicit constant, which is true since \( j \leq (\log \log N_0)^c \) where \( c < 1 \) and \( \alpha_0 \gg (k \log N_0)^{-1} \).

By (III), we have

\[
\log d_{j+1} \leq 2^{-k-3+j} (\log \alpha_j^{-1})^3.
\]

By (VI), we have \( \log \alpha_j^{-1} + \log 2 \geq C(\log \alpha_j^{-1})^2 \), and so

\[
\log \alpha_j^{-1} \leq (2(\log \alpha_j^{-1})/C)^{1/2}.
\]

Thus, it follows that

\[
\log d_{j+1} \leq 2^{-k-3+j} (2/C)^{3/2} (\log \alpha_j^{-1})^{3/2}.
\]

Combining this inequality with the definition of \( D_{j+1} \), \( k \leq (\log \log N_0)^c \) and \( \alpha_j \leq (k \log N_0)^{-1} \) yields

\[
\log d_{j+1} \leq C_2^{-1} (\log D_{j+1})/2.
\]

On the other hand, it follows from condition (I), \( k \leq (\log \log N_0)^c \) and \( \log \alpha_{j+1}^{-1} \gg \log (k \log N_0) \) that

\[
(\log \alpha_{j+1}^{-1})^2 \leq C_2^{-1} (\log D_{j+1})/2.
\]

Thus, the second inequality of (19) also holds.

Therefore, by Lemma 5.2, either the theorem holds (if the first outcome of Lemma 5.2 occurs) which contradicts our assumption, or we can find a monochromatic set \( A''_{j+1} \subseteq \mathbb{P} - 1 \) such that

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(B.I) $A''_{j+1}$ is contained in an arithmetic progression of length $N''_{j+1}$ and common difference $d''_{j+1} \overline{d}_{j+1}$, where $d_{j+1} \mid d''_{j+1}$, $d''_{j+1} \leq \alpha_{j+1}^{-C \log \alpha_{j+1}^{-1}} d_{j+1}$, and
\[
N''_{j+1} \geq \alpha_{j+1}^{C \log \alpha_{j+1}^{-1}} (d_{j+1}D_{j+1})^{-C \log \alpha_{j+1}^{-1}} (\log N_{j+1})^{-C \log \alpha_{j+1}^{-1}} N_{j+1};
\]
(B.II) $|A''_{j+1}| = \alpha_{j+1}'' N''_{j+1}$ where
\[
\alpha_{j+1}''/2 \leq \alpha_{j+1}^{C \log \alpha_{j+1}^{-1}} ((k - j - 1)d_{j+1}D_{j+1} \log N_{j+1})^{-1} \leq \alpha_{j+1}'';
\]
(B.III) $A''_{j+1} \cup ((A''_{j+1} - A''_{j+1}) \cap (\mathbb{P} - 1))$ is $(k - j - 1)$-coloured.
In particular, this implies conditions (V)–(VII) hold for $i = j + 1$. \hfill \Box

7. Concluding remarks

It is interesting to understand the true size of $r_p(k)$. The possible exceptional zero has an impact on the size of $|\hat{F}_{N,d}(0)|$, which ultimately leads to the bound in Theorem 1.2. It seems that we cannot rule out the possibility that almost all energy is concentrated on $\mathcal{M}_{a,q}$ where $q$ is a multiple of the modulus of the exceptional zero $d_D$. A model case is to consider a set $A$ which has large density on $\{d_Dn + b_i : n \leq N, 1 \leq i \leq j\}$, where $b_i$ are integers such that $\psi(d_DN; d_D, b_i) \geq d_DN/\phi(d_D)$ and $d_D$ is a product of small primes.

Due to the analytic nature of the method presented in this article, it seems unlikely that any nontrivial lower bound for $r_p(k)$ could be produced easily, either assuming the Generalised Riemann Hypothesis or assuming the existence of an exceptional zero very close to 1.

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