I. INTRODUCTION

The Einstein-Hilbert action is the simplest possible gravitational action. It can be easily generalized to the higher dimensional models of gravity

\[ W = \frac{1}{16\pi} \int d^Dx \sqrt{-g} R. \]  

Black hole solutions with the spherical topology of the horizon in this theory are well known now. The most general solution (in the presence of a cosmological constant) is a so-called Kerr-NUT-(A)dS spacetime. In the absence of rotation and for vanishing NUT parameters and the cosmological constant, this solution reduces to the Tangherlini metrics.

\[ ds^2 = -Adt^2 + \frac{dr^2}{A} + r^2 d\omega_n^2, \quad A = 1 - \left(\frac{r_0}{r}\right)^{n-1}. \]

For a $D$-dimensional spacetime, $n = D - 2$, and $d\omega_n^2$ is the metric of a unit round sphere $S^n$.

Modifications of the Einstein-Hilbert action are commonly considered in the modern literature. Besides the needs of a purely phenomenological description of gravity in different models, there exists a more fundamental reason for considering a generalized Einstein gravity.

It is well known that quantum corrections in a gravitational field can be described by using the DeWitt’s effective action formalism. In a general case, such an effective action contains higher curvature corrections to the Einstein-Hilbert Lagrangian, as well as non-local contributions (see, e.g., review and references therein). Higher in curvature terms naturally arise in the Starobinsky model of inflation. The higher curvature terms naturally arise in the string theory (see e.g. and references therein). One of the most interesting aspects connected to the introduction of the higher in curvature effective actions is a possibility to resolve a curvature singularity problem. Such singularities, according to the Penrose and Hawking theorems, always exist in cosmology (initial singularity) and inside black holes (final singularity). It is well known that the interior of (non-rotating) black hole in the vicinity of the singularity is similar to a contracting anisotropic universe and it exhibits the Kasner type behavior with infinitely growing curvature invariants. In the models where the curvature is limited by some (say Planckian) values, one may expect newly born universes instead of the singularity formation (see, e.g., and references therein). For more general discussion of the quantum effects in black holes see e.g. the book and references therein.

Let us notice that the consistent consideration of quan-
tum effects in gravity requires an introduction of the higher derivative terms, even at the semiclassical level (see [15, 16] for the introduction and [14] for the recent review of semiclassical approach). One of the first papers where the quantum curvature corrections were considered in the gravitational collapse model was [15]. In particular, it was shown that a collapse of a null shell with the mass smaller than the Planckian mass does not create a black hole. In other words, in this problem there exists a mass gap for the black hole formation. In the modern language it means that the black hole formation in the gravitational collapse of null shells is a first order phase transition. Later, black holes in the theories with higher curvature terms were considered, e. g. in [16].

In the four-dimensional (4D) case the lowest in curvature quadratic corrections to the Einstein gravity do not modify the Schwarzschild solution. The reason is the following. In 4D case there exists the Gauss-Bonnet relation

$$C_{abcd}C^{abcd} - 2\left( R_{ab}R^{ab} - \frac{1}{3} R^2 \right) = \text{topological term.} \quad (3)$$

As a consequence, the general 4D action with quadratic in curvature corrections can be always written as

$$W = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R + a R_{ab}R^{ab} + b R^2 \right) . \quad (4)$$

It is clear that any vacuum solution of the Einstein equations, $R_{ab} = 0$ is, at the same time, a solution of the theory (3). For this reason, in order to study the quantum gravity corrections we need to use higher in curvature Lagrangians. One of the interesting option is to consider 4D theories with the corrections of the form $f(C^2)$, which in many respects are similar to the $f(R)$ theories in the cosmological models.

In this paper we use another approach. We consider higher dimensional spherically symmetric black holes with quadratic in the curvature corrections. Since the Gauss-Bonnet argument does not work directly in the higher dimensions, the quadratic in the Weyl tensor corrections necessarily modify the background Tangherlini solution. This results in the creation of what is called secondary or induced hair [17, 18]. In the present paper we focus our attention on the study of these ‘hair’. Namely, we consider static spherically symmetric vacuum solutions of the gravitational theory with $C^2$ corrections. Using the reduced action approach we derive equations for such spacetimes (sections II and III). In section IV we demonstrate that in the special (seminull) coordinated these equations can be reduced to a single third order ordinary differential equation (ODE) for a quantity, connected to the Weyl curvature. Linearized version of the field equations and their static solutions are discussed in section V. In order to obtain corrections to the Tangherlini metric describing the induced hair generated by the $C^2$ correction, we use the iteration procedure developed in section VI. This section also contains the numerical results. Specific properties of the 4D case are discussed in section VII. Section VIII contains discussion and lists some of the perspective problems.

Throughout the paper we use the sign conventions adopted in [19].

## II. REDUCED ACTION

In a $D$-dimensional spacetime with metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (5)$$

we consider a gravitational action of the form

$$W = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left( R - \frac{a}{2} C^2 \right) , \quad (6)$$

where $C^2 = C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$.

Here $R$ and $C_{\alpha\beta\gamma\delta}$ are the Ricci scalar and the Weyl tensor, respectively. The constant $a$ has the dimensionality of $[\text{length}]^2$, and, for physically realistic case, it is positive. The reason is that this sign corresponds to the positively defined theory, e.g., to the positive energy of the massless mode (graviton) in the flat space limit. This requirement is also relevant because it provides the correct low-energy limit of the theory [14, 20]. The constant $a$ can be interpreted as a square of the characteristic fundamental scale $l$ for an appropriate quantum gravity theory. In a ‘standard quantum gravity’ $l$ is of the order of the Planckian length. In the popular recently models with large extra dimensions the corresponding to $l$ fundamental energy scale is of order of TeV. We shall study spherically symmetric black hole solutions. The characteristic scale of such a solution is its gravitational radius $r_0$. This dimensional parameter can be chosen as a general metric scale. After this in the theory (5) there remains only one essential dimensionless parameter $l/r_0$ (or, equivalently, $a/r_0^2$). One can expect that when $r_0 \gg l$, the Weyl scalar (Weyl curvature-squared) term plays the role of a small correction. We shall focus on this case later on, but at the initial part of the work there is no need to make additional assumptions about this parameter.

Our purpose is to explore static spherically symmetric solutions of the theory formulated above. The corresponding metric can be written in the form

$$ds^2 = r^2 ds_0^2 = r^2 (d\gamma^2 + d\omega_n^2) , \quad (8)$$

where $n = D - 2$ and $d\omega_n^2$ is the metric on a unit sphere $S^n$

$$d\omega_n^2 = \omega_{ij} d\theta^i d\theta^j , \quad i, j, \ldots = 2, \ldots, n + 1 . \quad (9)$$
Quantity $r$ is a scalar function on a 2D manifold with the metric
\[ d\gamma^2 = \gamma_{ab} dx^a dx^b, \quad a, b, \ldots = 0, 1. \]
(10)
For this class of the metrics the action (10) can be reduced to the 2D action. Because of the spherical symmetry neither $R$ nor $C^2$ depend on the angle variables. Using the relation
\[ \sqrt{-g} = r^{n+2} \sqrt{-\gamma \omega}, \]
and integrating over the unit sphere one obtains
\[ W = \frac{A_n}{10 \pi} \mathcal{W}, \quad \mathcal{W} = \int d^2x \sqrt{-\hat{\gamma}} \mathcal{L}, \]
(12)
\[ \mathcal{L} = r^{n+2} \left( R - \frac{a}{2} C^2 \right). \]
(13)
Here
\[ A_n = \frac{2 \pi^{(n+1)/2}}{\Gamma(n/2+1)} \]
is a surface area of the $n$-dimensional unit sphere $S^n$.
We denote by symbols with bars the objects constructed for the metric $d\hat{\gamma}^2$. Since this metric is a direct sum of the two metrics, $d\gamma^2$ and $d\omega^2$, the curvature for it is a direct sum of two curvatures and it has the following non-vanishing components
\[ \hat{R}_{abcd} = \frac{1}{2} \hat{R} (\gamma_{ac} \gamma_{bd} - \gamma_{ad} \gamma_{bc}), \]
\[ \hat{R}_{ijkl} = \omega_{ik} \omega_{jl} - \omega_{il} \omega_{jk}. \]
(14)
We also have
\[ \hat{R}_{ac} = \frac{1}{2} \hat{R} \gamma_{ac}, \quad \hat{R}_{ik} = (n-1) \omega_{ik}, \]
\[ \hat{R} = \hat{R} + n(n-1). \]
(15)
We denote by symbols with hats the objects constructed for the 2D metric $d\hat{\gamma}^2$. In particular, $\hat{R}$ and $\hat{C}^2$ denote the Ricci scalar and the Weyl-square invariant, calculated for the metric $\gamma_{ab}$. It is easy to show that (see, e.g., [21])
\[ R = r^{-2} \left[ \hat{R} + n(n+1) - 2(n+1) \hat{\nabla}^2 r + \frac{1}{r} \hat{\nabla}^4 r \right]. \]
\[ (n+1)(n-2) \left[ \hat{\nabla}^2 r \right]^2. \]
(16)
The conformal invariance of the Weyl tensor implies
\[ \hat{C}^2 = r^{-4} \hat{C}^2. \]
(17)
To calculate $\hat{C}^2$ we use a relation
\[ \hat{C}^2 = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{4}{n} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{n(n+1)} \hat{R}^2. \]
(18)
By using (14–15) one can obtain
\[ \bar{R}_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} + 2 n(n-1), \]
\[ \bar{R}_{\alpha\beta} = \frac{1}{2} \hat{R}_{\alpha\beta} + 2 n(n-1), \]
\[ \hat{R}^2 = \left[ \hat{R} + n(n-1) \right]^2. \]
(19)
(20)
(21)
Then we find
\[ \bar{C}^2 = \frac{n-1}{n+1} (\hat{R} + 2)^2. \]
(22)
Thus the dimensionally reduced action on the 2D manifold with the metric $\gamma_{ab}$ is
\[ \mathcal{W} = \mathcal{W}_{(c)} + \mathcal{W}_{(q)} = \int d^2x \sqrt{-\hat{\gamma}} \left[ \mathcal{L}_{(c)} + \mathcal{L}_{(q)} \right], \]
\[ \mathcal{L}_{(c)} = r^a \hat{R} + n(n-1) - 2(n-2) \left( \hat{\nabla} r \right)^2, \]
\[ \mathcal{L}_{(q)} = - \frac{a(n-1) r^{n-2}}{2(n+1)} (\hat{R} + 2)^2. \]
(23)
(24)
**III. FIELD EQUATIONS**

The reduced action $\mathcal{W}$ is a functional of the 2D metric $\gamma_{ab}$ and of a scalar function $r$. Its variation has the form
\[ \delta \mathcal{W} = \int d^2x \sqrt{-\hat{\gamma}} \left[ \mathcal{P} \delta r + \mathcal{G}^{ab} \delta \gamma_{ab} \right]. \]
(25)
Consider a coordinate transformation generated by a vector field $\xi^a$
\[ x^a \rightarrow x^a + \xi^a. \]
(26)
Under this transformation $r$ and $\gamma_{ab}$ transform as follows
\[ \delta_x r = - \xi^a r_a, \quad \delta_x \gamma_{ab} = - 2 \hat{\nabla} (\hat{\nabla} \xi_b). \]
(27)
Since $\mathcal{W}$ is invariant under this transformation one has $\delta \mathcal{W} = 0$. Using (25) and (27) one obtains the following *Noether identity*
\[ 2 \hat{\nabla} \mathcal{G}^b - \mathcal{P} \hat{\nabla}_a r = 0. \]
(28)
To obtain the field equations for the reduced action we use the following relations
\[ \delta \sqrt{-\hat{\gamma}} = \frac{1}{2} \sqrt{-\gamma} \gamma_{ab} \delta \gamma_{ab}, \]
\[ \int d^2x \sqrt{-\hat{\gamma}} (\delta \hat{R}) S = \int d^2x \sqrt{-\gamma} \delta \gamma_{ab} \left[ \hat{\nabla} a \hat{\nabla} b S - \gamma_{ab} \hat{\nabla} S - \hat{\nabla}^2 S \right]. \]
(29)
The variation of (22) with respect to \( r \) gives

\[
\mathcal{P} = \mathcal{P}_c + \mathcal{P}_q ,
\]

\[
\mathcal{P}_c = nrK - n(n+1)r^{n-3}[(n-2)(\hat{\nabla}r)^2 \\
+ 2r\hat{\Box}r - (n-2)r^2] ,
\]

\[
\mathcal{P}_q = -\frac{a(n-1)(n-2)}{2(n+1)r^{n-1}}K^2 .
\]

Here and later we use subscripts \( c \) and \( q \) for quantities connected with ‘classical’ and ‘quantum’ parts of the action, respectively. Similarly, the variation of (22) with respect to \( \gamma_{ab} \) gives

\[
\mathcal{G}^{ab} = \mathcal{G}^{ab}_c + \mathcal{G}^{ab}_q ,
\]

where

\[
\mathcal{G}^{ab}_c = nr^{n-2}\left\{\hat{\nabla}^a\hat{\nabla}^b r - 2\hat{\nabla}^a r\hat{\nabla}^b r \\
+ \frac{1}{2}\gamma^{ab}[(n-1)r^2 - (n-3)(\hat{\nabla}r)^2 - 2r\hat{\Box}r]\right\}
\]

and

\[
\mathcal{G}^{ab}_q = -\frac{a(n-1)}{(n+1)}\left[\hat{\nabla}^a\hat{\nabla}^b K + \gamma^{ab}(K - \hat{\Box}K \\
- \frac{1}{4}r^{2-n}K^2)\right].
\]

In the above relations we use, instead of \( \hat{R} \), the related quantity,

\[
K = r^{n-2}(\hat{R} + 2) ,
\]

which is directly connected to the Weyl scalar in the ‘physical’ spacetime

\[
|K| = \sqrt{\frac{n+1}{n-1}}r^n(C^2)^{1/2} .
\]

Furthermore, it is easy to show that

\[
\hat{\nabla}_b\mathcal{G}^{b}_{(q)a} = -\frac{a(n-2)(n-1)}{4(n+1)r^{n-1}}K^2\hat{\nabla}_a r .
\]

This relation allows one to check the validity of the Noether identity (28) for the ‘quantum’ part of the action. This serves as a good test of the rather long calculations required to derive relations (30)-(34).

To summarize, the vacuum field equations for the reduced action (22) are

\[
\mathcal{P} = 0 , \quad \mathcal{G}_{ab} = 0 .
\]

IV. EXPLICIT FORM OF THE FIELD EQUATIONS

In the absence of the Weyl term in the action one can show that any spherically symmetric vacuum solution of the higher dimensional Einstein equations is static (a generalized Birkhoff’s theorem). In the presence of the Weyl term this is not true anymore. In the present work we restrict ourselves by considering static solutions of the theory (4), that is a special subclass of its solutions.

Let us describe the form of the action which is well adapted to our problem. We use the coordinate freedom \( r \to \tilde{r}(r) \) in order to put \( g_{rr} = 0 \), and write the metric \( \gamma_{ab} \) in the form

\[
d\gamma^2 = -\frac{A}{\tilde{r}^2}e^{2C}dv^2 + 2\frac{a}{\tilde{r}^2}e^Cdvdr ,
\]

where \( A = A(r) \) and \( C = C(r) \) are some unknown functions. The corresponding ‘physical’ metric is

\[
ds^2 = -Ae^{2C}dv^2 + 2e^Cdvdr + \tilde{r}^2d\omega^2 .
\]

We assume that the spacetime is asymptotically flat. This implies that \( A(r = \infty) = 1 \). The function \( C(r) \) takes a finite value \( \tilde{C} \) at infinity. This value depends on the normalization of the advanced time coordinate \( v \). We choose its normalization so that \( \tilde{C} = 0 \). For this choice \( \xi = \partial_v \), which is a Killing vector, has a canonical normalization at the infinity: \( \tilde{\xi}^2 = -1 \).

The event horizon, if it exists, is located at the surface where \( A = 0 \). The metric (41) in the coordinates \((v, r)\) is regular at this surface, provided \( C \) is regular there. In particular, this property enables one easily to consider modifications of the Schwarzschild black hole solution due to the quantum corrections, in a case when the latter are small and can be treated as perturbations (see section 4). In the presence of a black hole, its event horizon is located at the point \( r_H \) where the function \( A \) vanishes. The surface gravity \( \kappa \) of a static black hole is defined as

\[
\kappa = \left(\frac{1}{2}\tilde{\xi}_{ab}\tilde{\xi}^{ab}\right)^{1/2} ,
\]

where \( \tilde{\xi} = \partial_v \) is the Killing vector. For the metric (41) this expression can be rewritten in a more simple form,

\[
\kappa = \frac{1}{2}(e^C A')_{|r = r_H} .
\]

The Hawking temperature of such a black hole is \( T_H = 2\pi/\kappa \). For the unperturbed metric, when \( a = 0 \), one has

\[
\kappa_0 = \frac{n-1}{2r_0} .
\]
Calculations using GRTensor program allows one to obtain the following relations. The non-vanishing components of the Christoffel symbols are

$$\Gamma_{vv}^v = -\Gamma_{rv}^r = V, \quad \Gamma_{vv}^r = AVe^C,$$

$$\Gamma_{rr}^r = C' - \frac{2}{r}.$$  \hspace{1cm} (45)

where

$$V = \frac{1}{2r}(A' + 2AC'r - 2A)e^C.$$  \hspace{1cm} (46)

One also has $G_{rv} = G_{vr}$, due to algebraic properties of the metric (49) and corresponding components of the Ricci tensor. Finally, it proves useful to introduce a parameter $p$, related to $a$ as

$$p = -\frac{a}{2(n+1)}.$$  

Notice that according to our assumption this parameter is negative. The non-vanishing field equations can be presented in the form

$$(n-1)p\left[4r^2AK'' + 2r(A' + 2A)K' - 4K + \frac{K^2}{r^n-2}\right]$$

$$+ nr^n[rA' + (n-1)(A - 1)] = 0,$$  \hspace{1cm} (47)

$$nr^nC' - 2(n-1)p[rK'' - rC'K' + 2K'] = 0,$$  \hspace{1cm} (48)

$$(n-1)(n-2)pK^2 + 2(n+1)rAC' + A') - K r^{2-n} = 0,$$  \hspace{1cm} (49)

$$K = -r^{n-2}\left[3r^2A'C'' - 2 - 2r(A' + AC') + 2A + r^2A'' + 2ArC'' + C'^2\right].$$  \hspace{1cm} (50)

The equations (47) and (48) follow from $(vr)$ and $(rr)$ components of the equation $G_{ab} = 0$, while (49) follows from the equation $P = 0$. The last equation is the definition of $K$.

A remarkable property of the system of equations (47)–(49) is that they can be reduced to a single third order ODE. We notice first that this system does not contain the function $C(r)$, but only its derivatives. The (48) enables one to express $C'$ as a function of $K$ and its derivatives

$$C' = \frac{2(n-1)p(rK'' + 2K')}{nr^n + 2(n-1)prK}.$$  \hspace{1cm} (51)

After substituting this expression into (47) and (49), one obtains two linear equations for $A$ and $A'$. Solving these equation one determines $A$ and $A'$ as functions of $K$ and its derivatives up to the second order. One can present the expression for $A$ in the form

$$A = \frac{S_1K' + S_0}{n(T_2K'' + T_1K' + T_0)}.$$  \hspace{1cm} (52)

where

$$S_1 = -2p(n-1)\times$$

$$\times \left[\frac{n(n-2)(n+1)r - p(n^3 - 3n + 2)}{r^{2-n}}\right],$$

$$S_0 = n\left[2(n^2 + 1)pK - n^2K - \frac{p(n-1)(n+1)K^2}{r^n-2}\right],$$

$$T_2 = 4(n^2 - 1)p^2, \quad T_1 = -2p(n^2 - 2)\frac{n(n-2)(n-1)r}{n^2(n+1)r^n}.$$  

A similar expression is valid for $A'$. Taking a derivative of $A$ given by (52) and putting it equal to $A'$, one arrives at a single ordinary differential equation of the third order for $K$. We call it master equation. We have obtained this equation using the MAPLE program but do not reproduce it here since it is too bulky.

V. LINEARIZED EQUATIONS AND THEIR SOLUTIONS

Since the equations of our interest are rather complicated, it is worthwhile to consider first their linearized form. This exercise is especially useful, because it helps to control the correct classical $(p \to 0)$ limit in the region far from the center $r = 0$. Let us denote

$$A = 1 + \alpha(r), \quad C = \beta(r), \quad K = \nu(r).$$  \hspace{1cm} (53)

We regard the functions $\alpha$, $\beta$, and $\nu$ as small perturbations and hence, in the course of our calculations, we shall keep only those terms which are linear in these functions. The linearization of equations (47)–(49) gives

$$n(n-1)\alpha + n\nu = 0,$$  \hspace{1cm} (54)

$$-4(n-1)p r^{-n}(\nu - r\nu' - r^2\nu'') = 0,$$  \hspace{1cm} (55)

$$2(n-1)p(\nu'' + 2\nu') - nr^n\beta' = 0,$$  \hspace{1cm} (56)

$$[(n-2)\alpha + 2r(\alpha' + \beta')] - \frac{\nu r^{2-n}}{n+1} = 0,$$  \hspace{1cm} (57)

$$\nu r^{2-n} + r^2(\alpha'' + 2\beta'') - 2r(\alpha' + \beta') + 2\alpha = 0.$$  

It is easy to check that the last equation follows from the first 3 equations. This system of equations can be reduced to the single third order ODE (master equation) for the function $\nu$. This can be done by adopting the procedure described in the previous section to linearized case. Namely, let us find $\beta'$ from (55) and substitute it into (54) and (56). Use these two equations to find $\alpha$ and $\alpha'$. Substitute the obtained expression for $\alpha'$ into (55) and differentiate the obtained equation with respect to $r$. After this, again substitute $\alpha'$ in the equation.
These operations produce the following master equation which contains only \( \nu \) and its derivatives \[ \begin{align*}
\nu &\left( \nu + \nu' \right) + 4(n^2 - 1) (n + 2) \nu (\nu - \nu') \\
+ 4p(n^2 - 1)^2 \nu'' - (n - 1) \nu''' = 0 .
\end{align*} \] (58)

Before we solve this equation, it proves useful to perform a linear change of variables. Let us define the coefficient \( p \), with the dimensionality \([\text{length}]^{-1}\), according to

\[ p = -\frac{n}{(n^2 - 1) \lambda^2} , \] (59)

and introduce a new dimensionless quantity \( \xi = \lambda r \) and a new dimensionless function, \( Q(\xi) \), according to

\[ \nu(\xi) = \lambda^2 \xi \xi^2 - nQ(\xi) . \] (60)

Notice that the dimensionality of \( \nu \) is \([\text{length}]^{n-2}\), while \( Q \) is dimensionless. The master equation (65) can be cast into the form

\[ 4\xi^3 \frac{d^3Q}{d\xi^3} - 4\xi^2 (n-1) \frac{d^2Q}{d\xi^2} - \xi \left( \xi^2 + 4n + 8 \right) \frac{dQ}{d\xi} + (4n + 8 - \xi^2) Q = 0 . \] (61)

The last equation can be rewritten as

\[ 4\xi^2 \frac{d^2T}{d\xi^2} - 4\xi (n+2) \frac{dT}{d\xi} + (4n + 8 - \xi^2) T = 0 , \] (62)

where

\[ Q(\xi) = \frac{S(\xi)}{\xi} \quad \text{and} \quad dS = T(\xi) . \]

The equation (62) can be easily solved in terms of the modified Bessel functions \( K_\nu(z) \) and \( I_\nu(z) \). The final solution for \( Q(\xi) \) has the form

\[ Q(\xi) = \frac{C_0}{\xi} + \xi^{(n+1)/2} \left[ C_1 K_{\frac{n+1}{2}}(\xi/2) + \tilde{C}_2 I_{\frac{n+1}{2}}(\xi/2) \right] . \] (63)

Since we assume that at infinity the spacetime is flat we have to put one of the integration constants zero, \( \tilde{C}_2 = 0 \). In this way we arrive at the final form of the physically interesting solution

\[ Q(\xi) = \frac{C_0}{\xi} + \xi^{(n+1)/2} \left[ C_1 K_{\frac{n+1}{2}}(\xi/2) \right] . \] (64)

For completeness, we present also the same solution in terms of original variables

\[ \nu = \frac{\tilde{C}_0}{r} + \tilde{C}_1 r^{(n+1)/2} K_{\frac{n+1}{2}}(\lambda r/2) . \] (65)

Let us now find the expressions for \( \alpha \) and \( \beta \). For this end we can use the results of the previously considered procedure, which involved the equations (55), (55), (56) and led to the master equation (65). Solving equations (55), (55), (60) we find an algebraic solution for \( \alpha \). After using (64), we arrive at the following result:

\[ \alpha(\xi) = \frac{1}{\xi^n} \left[ \frac{4\xi^2 \frac{dQ}{d\xi} - (\xi^2 + 8) Q}{\xi^{n(n+1)}/2} \right] \] (66)

\[ = - \frac{C_0}{n(n+1)} \xi^{1-n} - \frac{2C_1}{n+1} \xi^{(3-n)/2} K_{\frac{n+1}{2}}(\xi/2) . \]

Equation (55) leads to the equation for \( \beta \),

\[ \frac{d\beta}{d\xi} = - \frac{2}{\xi^n} \left[ \xi \frac{dQ}{d\xi} + \frac{dQ}{d\xi} \right] \]

\[ = \frac{C_1}{2(n+1)} \xi^{1-n} \left[ 2 K_{\frac{n+1}{2}}(\xi/2) - \xi K_{\frac{n+1}{2}}(\xi/2) \right] . \]

Integration of this relation gives

\[ \beta(\xi) = C_2 + \frac{C_1}{n+1} \xi^{1-n} \left[ \xi K_{\frac{n+1}{2}}(\xi/2) - 2n K_{\frac{n+1}{2}}(\xi/2) \right] . \] (67)

Substitution of the obtained solutions for \( \nu(\xi) \), \( \alpha(\xi) \), and \( \beta(\xi) \) into the original set of the equations (55)–(56) enables one to make an additional check of the correctness of the obtained solutions.

The modified Bessel functions \( K_\nu(z) \), which enter the expressions for \( \nu \), \( \alpha \) and \( \beta \), have the following asymptotic behavior at the infinity

\[ K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + \frac{4\mu^2 - 1}{8z} + \ldots \right) . \] (68)

Notice, that at the leading order these asymptotics are universal, that is they do not depend on the index \( \mu \). Using (65) we obtain the following asymptotic expressions at large \( \xi \) for the functions \( \nu \), \( \alpha \) and \( \beta \)

\[ \nu \sim \lambda^{2-n} Q , \quad Q \sim \frac{C_0}{\xi} + C_1 \sqrt{\pi} \xi^{n/2} e^{-\xi^2/2} , \] (69)

\[ \alpha \sim - \frac{C_0}{n(n+1)} \xi^{1-n} - \frac{2\sqrt{\pi} C_1}{n+1} \xi^{1-n/2} e^{-\xi^2/2} , \] (70)

\[ \beta \sim C_2 + \frac{\sqrt{\pi}}{n+1} \xi^{1-n/2} e^{-\xi^2/2} . \] (71)

VI. FINDING A SOLUTION BY ITERATIONS

Now we study solutions of the system (64)–(66) without making an assumption that the gravitation field is weak. As we already mentioned in Section 2 the corresponding black hole solutions has a characteristic scale.
connected with its gravitational radius \( r_0 \). For example, in the absence of quantum corrections the corresponding (Tangherlini) metric is

\[
ds^2 = -[1 - (r_0/r)^{n-1}] dv^2 + 2 dv dr + r^2 d\Omega^2_n. \tag{72}\]

One can always rescale this metric as follows

\[
ds^2 = r_0^2 ds^2, \tag{73}\]

where \( ds^2 \) is the dimensionless form of the metric. Making the calculations in this dimensionless form is equivalent putting \( r_0 = 1 \) in the ‘physical’ metric \( ds^2 \). In what follows we always assume \( r_0 = 1 \). It is convenient to rewrite the main system of equations (77)–(79) in terms of the new coordinate \( x = 1/r \). We also introduce the new functions \( M(x) \) and \( \Psi(x) \) according to

\[
A = 1 - x^{n-1} M(x), \quad \Psi = \dot{C}. \tag{74}\]

We denote by a ‘dot’ a derivative with respect to the coordinate \( x \). Our normalization means that the asymptotic values of \( M \) at \( x \to 0 \) is always equal to 1.

In these coordinates and notations the system (77)–(79) takes the form

\[
\dot{M} = \frac{p(1-n)}{n} \left[ -2x^n \{[(n+1)\dot{K} + 2x \dot{K}]M + x \ddot{K}M \right] + 4(x^2 \ddot{K} + x \dot{K} - K) + x^{n-2} K^2 \right], \tag{75}\]

\[
\Psi = -\frac{2p(n-1)}{n} x^{n-1} [\ddot{K} - K \dot{\Psi}], \tag{76}\]

\[
K = (n+1)[nxM + 2x \dot{M} + 2x^2 \dot{\Psi}M - 2x^{3-n} \dot{\Psi} - \frac{p(n-1)(n-2)}{n} x^n K^2]. \tag{77}\]

In order to solve this system for small \( p \) we use the following procedure. We start with a ‘classical’ solution. Thus we put \( p = 0 \) and obtain

\[
M_0 = 1, \quad \Psi_0 = 0, \quad K_0 = n(n+1)x. \tag{78}\]

One can replace these expressions into (75) and (76) to obtain \( M_1 \) and \( \Psi_1 \). After this, using (77) we obtain \( K_1 \) in which all the terms higher that the first order in \( p \) are omitted. It is possible to repeat this procedure to obtain an approximate solution up to any order in powers of \( p \), by means of iterations.

Denote by \( \{M_k, \Psi_k, K_k\} \) the result of the \( k \)-th iteration. Then one has

\[
M_{k+1} = 1 + p \int_0^x dx F_M(M_k, K_k), \tag{79}\]

\[
\Psi_{k+1} = p F_\Psi(\Psi_k, K_k), \tag{80}\]

\[
K_{k+1} = F_K^{(0)}(M_{k+1}, \Psi_{k+1}) + p F_K^{(1)}(K_k). \tag{81}\]

Here

\[
F_M(M_k, K_k) = \frac{(n-1)}{n} \left[ 2x^n \{(n+1)\dot{K}M - 2x\dot{K}M - xK\dot{M} \right] - 4(x^2 \ddot{K} + x K - K) - x^{n-2} k^2 \right], \tag{82}\]

\[
F_\Psi(\Psi_k, K_k) = -\frac{2(n-1)}{n} x^{n+1} [\ddot{K} - K \dot{\Psi}], \tag{83}\]

\[
F_K^{(0)}(M_k, \Psi_k) = (n+1) [nxM + 2x^2 \dot{M} + 2x^2 \dot{\Psi}M - 2x^{3-n} \dot{\Psi} - \frac{p(n-1)(n-2)}{n} x^n K^2], \tag{84}\]

\[
F_K^{(1)}(K_k) = \frac{(n-1)(n-2)}{n} x^n K^2. \tag{85}\]

In the formulas above the subscript symbol \( [\leq k] \) means the expansion into power series up to the order \( k \) in the parameter \( p \). This prescription provides the correct run of the iteration procedure, such that the \( M_k, \Psi_k \) and \( K_k \) include powers of \( p \) up to the order \( k \).

The results of the analytical calculations of the first two iterations for \( M \), \( \Psi \) and \( K \) obtained by using the Maple program and checked by using Mathematica are given below:

\[
M_2 = 1 - (n-2)(n-1)(n+1)p x^{n+1} \tag{86}\]

\[
- \frac{2}{n} (n-2)(n^2-1)^2 p^2 \left[ (4n^2+15n+6)x^{n+2} - 4(n+1)(2n+1)x^{n+3} \right], \tag{87}\]

\[
\Psi_2 = \frac{4p^2 x^{n+1}}{n} (n+1)^2(n+1)^3(n^2-4)(2n+1), \tag{88}\]

\[
K_2 = n(n+1)x \tag{89}\]

\[
- 2(n+1)(n-2)(n-1)(2n+1)x^{n+2} p \tag{90}\]

\[
- \frac{2}{n} (n-2)(n^2-1)^2(n+1)^3 p^2 \times \tag{91}\]

\[
\times \left[ (8n^3+69n^2+66n+16)x^{2n+3} - 8(n+1)(n+2)(2n+1)x^{n+4} \right]. \tag{92}\]

It is easy to see that the solution presented above include zero order terms (which are of course the same as in (73)), first order and second order terms, all of them have corresponding powers of the expansion parameter \( p \). It is not difficult to perform further iterations, but starting from the third order the formulas become very bulky and hence we do not show them. (The third order iteration results for \( M_3, \Psi_3 \), and \( K_3 \) can be found in the Appendix.)
FIG. 1: Iterations of the mass function $M$ up to the second order. The lines 0, 1, and 2 show the functions $M_0(x)$, $M_1(x)$ and $M_2(x)$, respectively. The plot is constructed for $n = 3$ and $a = 0.03$.

The plot in Figure 1 shows that the mass function $M(x)$ grows from its value 1 at infinity to a larger value at the horizon. A natural interpretation of this result is that introducing the Weyl term to the action is qualitatively equivalent to the addition of a negative mass density distribution into the black hole exterior. Let us remember that we assume that the Weyl term in the action produces the change of the black hole solution which is close to the one of the vacuum polarization effect.

We use equation

$$M(x_H) = x_H^{1-n}, \quad (87)$$

to obtain a position of the horizon. We also use the expression (43) for the surface gravity

$$\kappa = -\frac{1}{2} [x^2 e^C A]_{x=x_H}. \quad (88)$$

The results of the calculations of $x_H$ and $\kappa$ are given in the Table.

| $a$ | $x_H$ | $\kappa/\kappa_0$ |
|-----|-------|-----------------|
| 0.01 | 0.994 | 1.014 |
| 0.02 | 0.987 | 1.028 |
| 0.03 | 0.979 | 1.041 |

TABLE I: Position of the horizon, $x_H$, and the surface gravity $\kappa$ of distorted black holes.

This table shows that, as expected, for smaller values of $a$, the solution are closer to their unperturbed ones. The size of the event horizon, $r_0/x_H$, is always greater than its unperturbed value $r_0$. As far as the value of $a$ increase, the gravitational radius $r_H = x_H^{1-n}$ grows up and the surface gravity grows up as well.

VII. 4D CASE

As we already mentioned in the Introduction, the $4D$-case is a special one. Let us consider it in more details, using the results of the iteration procedure presented in the previous section.

The iteration formulas (83), (84) and (85) imply that in the 4-dimensional case, when $n = D - 2 = 2$, the corrections to the ‘classical’ solution (78) vanish. One can check this directly by substituting (78) into the equations (75)-(77). In principle, there might exist non-perturbative in $p$ solutions close to the Schwarzschild metric. Let us demonstrate that at least for small perturbations which can be treated in the linearized approach, such regular both at the infinity and at the horizon solutions do not...
exist. For this purpose let us consider small perturbations near the ‘classical’ solution

\[ M = 1 + \epsilon m(x), \quad \Psi = \epsilon \psi(x), \quad K = 6x[1 + \epsilon k(x)] \]  \hspace{1cm} (89)

Here \( \epsilon \) is a small parameter which should be set to unity after performing the series expansion. Substituting these relations into the basic equations (75)-(77) one obtains that the terms of the order of \( \epsilon^3 \) vanish. Keeping leading linear in \( \epsilon \) terms one obtains a following set of equations

\[
6px^2[2x(1-x)\ddot{k} + (6 - 7x)\dot{k} + 3k] \\
+ (1 - 6px^3)\dot{m} - 18px^2m = 0,  \hspace{1cm} (90)
\]

\[
v + 6px^3[x\ddot{k} + 2\dot{k} - \dot{v}] = 0, \hspace{1cm} (91)
\]

\[
m + x\ddot{m} - k - (1 - x)v = 0. \hspace{1cm} (92)
\]

We obtain now a master equation for this system. For this purpose we use (91) to find \( v(x) \)

\[
v = \frac{6px^3}{6px^3 - 1} (x\ddot{k} + 2\dot{k}). \hspace{1cm} (93)
\]

Let us substitute this expression into (92). Using the obtained equation and (90) one can solve these linear with respect to \( m \) and \( \dot{m} \) equations to obtain the both quantities. One gets

\[
m = k + \frac{6px^3}{1 + 12px^3} \left[ x(1-x)\ddot{k} + (4 - 5x)\dot{k} \right], \hspace{1cm} (94)
\]

\[
\dot{m} = \frac{6px^2}{(1 - 6px^3)(1 + 12px^3)} \left[ 2x(x-1)(3px^3 + 1)\ddot{k} \right. \\
+ \left. (7x - 6px^3 - 6)k \right]. \hspace{1cm} (95)
\]

Differentiating (93) and putting it equal to \( \dot{m} \) defined by (95) one obtains a master equation for \( k \). From the structure of the expressions for \( m \) and \( \dot{m} \) it is evident that the master equation does not contain \( k \), but only its derivatives. This means that \( k = \text{const} \) is a solution. This solution results in a simple change of the gravitational radius \( r_0 \) which we originally put equal to 1. We do not consider this renormalization ambiguity and in what follows will keep \( r_0 = 1 \).

Let us now study possible non-trivial perturbed solutions. By denoting \( k = Y \) we write the master equation in the form

\[
F_2 \dddot{Y} + F_1 \ddot{Y} + F_0 Y = 0, \hspace{1cm} (96)
\]

\[
F_0 = 1 + 108px^2 - 144px^3 - 108p^2x^6 + 1728p^3x^9, \\
F_1 = 12px^3(5 - 6x + 21px^3 - 27px^4 - 144p^2x^6 + 216p^3x^7), \\
F_2 = 6px^4(1 - x)(1 + 12px^3)(1 - 6px^3). \hspace{1cm} (97)
\]

As any second order ODE, (96) can be written in a self-adjoint form

\[
\frac{d}{dx} \left( f_1 \frac{dY}{dx} \right) + f_0 Y = 0. \hspace{1cm} (98)
\]

By comparing (96) and (98) we have

\[
\frac{\dot{f}_1}{f_1} = \frac{F_1}{F_2}, \quad \frac{f_0}{\dot{f}_1} = \frac{F_0}{F_2}. \hspace{1cm} (99)
\]

Integrating the first of these equations we obtain

\[
f_1 = \frac{(1 - x)^2x^{10}}{(1 - 6px^3)(1 + 12px^3)^2}. \hspace{1cm} (100)
\]

A general solution contains as a common factor an arbitrary integration constant \( C \). We put this constant equal to 1. With this choice and \( p > -1/12 \), \( f_1(x) \) is positive in the black hole exterior. Using the second equation of (99) we get

\[
f_0 = \frac{(1 - x)x^6Q}{6p(1 - 6px^3)^2(1 + 12px^3)^2}, \hspace{1cm} (101)
\]

\[
Q = 1 + 108px^2 - 144px^3 - 108p^2x^6 + 1728p^3x^9. \hspace{1cm} (102)
\]

The horizon, \( x = 1 \), and the infinity, \( x = 0 \), are singular points of the equation (98). Let us consider first this equation in the vicinity of the horizon. Denote \( x = 1 - y \), then in the region \( y \approx 0 \) one has

\[
y \frac{d^2Y}{dy^2} + 2 \frac{dY}{dy} + cY = 0, \quad c = 1 - 30p - 288p^2. \hspace{1cm} (103)
\]

Putting \( Y \sim y^n \) one finds that \( \alpha(1 + \alpha) = 0 \). It means that one of the two linearly independent solutions is singular, \( Y \sim y^{-1} \), at this point. Hence only one of these two solutions is regular at this point, and it is uniquely determined by the boundary condition \( Y(y = 0) = Y_0 \).

Let us analyze solutions of (98) near the infinity, that is near \( x = 0 \). In the close vicinity of this point one has

\[
F_2 \sim 6px^4, \quad F_1 \sim 60px^3, \quad F_0 \sim 1, \hspace{1cm} (104)
\]

and the equation (96) takes the form

\[
6px^4\dddot{Y} + 60px^3\ddot{Y} + Y = 0. \hspace{1cm} (105)
\]

Changing the coordinates \( r = 1/x \) and keeping leading at \( r \to \infty \) terms we obtain the following asymptotic form of the equation

\[
6p \frac{d^2Y}{dr^2} - 48p \frac{dY}{dr} + rY = 0. \hspace{1cm} (106)
\]

The asymptotic form of the solutions at infinity is

\[
Y \sim \exp(\pm r/2\lambda), \quad \lambda = \sqrt{-6p}. \hspace{1cm} (107)
\]
A solution of (106) which decreases at infinity is

\[
Y = C(r^4 + 10\lambda r^3 + 45\lambda^2 r^2 + 105\lambda^3 r + 105\lambda^4)e^{-r/\Lambda}.
\] (108)

Since a regular solution is uniquely fixed by its value at the horizon, in a general case such a solution would become increasing at infinity. Thus in a general case \( Y = 0 \) is the only solution which is regular both, at the horizon and infinity. Based on this analysis, one however cannot exclude the existence of regular solutions for some discrete values of \( p \). We demonstrate now that it never happens when \( p > -1/12 \).

For this purpose let us multiply (106) by \( Y \) and integrate the obtained expression in the interval \( x \in (0,1) \). After integration by parts one has

\[
\int_0^1 dx [f_1 Y^2 - f_0 Y^2] = [f_1 \dot{Y} Y]_0^1.
\] (109)

For a solution which decreases at infinity and is bounded at the horizon \( x = 1 \), where \( f_1(1) = 0 \), the expression in the right-hand side vanishes. For \( p < -1/12 \) the function \( f_1(x) \) is positive in the interval \( x \in (0,1) \). Numerical analysis shows (see Figure 4) that for \( p < -1/12 \) the function \( f_0(x) \) is negative in the interval \( (0,1) \). Hence both \( Y \) and \( \dot{Y} \) must vanish identically. This proves that for \( p < -1/12 \) the equation has only one bounded solution, \( Y(x) = 0 \). It means that for small \( p \) the Schwarzschild metric is the only one solution of the modified by \( C^2 \) correction Einstein gravitational equations (at least in the close vicinity of this solution).

VIII. DISCUSSIONS

We study black hole solutions in the theory with the Weyl action correction to the Einstein gravity in higher dimensional spacetimes. We demonstrated that in the higher dimensional case with \( D \geq 5 \) this correction results in the modification of the Tangherlini metric, which is uniquely determined the one parameter, the value of the gravitational radius. We developed the iteration procedure which for a small value of \( \alpha = a/r_0^2 \) allows one to obtain a solution outside the horizon. This iteration procedure does not work uniformly. In particular for a given value of \( \alpha \) there always exist such a value of \( r \) where the iteration is not convergent. For small enough value of \( \alpha \) this value of \( r \) is inside the gravitational radius. In this domain terms with higher in curvature corrections must play an important role. It is an interesting problem to develop a method of solving the equations in this domain either by obtaining a reliable analytic approximation or by developing stable numerical schemes. Another interesting open question is to study uniqueness of the exterior solution obtained by the iteration method. We analyze a similar problem in the 4D-case where the situation is much simpler. We first demonstrate that the classical Schwarzschild metric is a fixed point of the iteration procedure. After this we studied metrics in the vicinity of the Schwarzschild metric and demonstrated that at least for the value of \( \alpha < 1/2 \) \((p > -1/12)\) such regular perturbative solutions are absent. We also showed that in a general case additional to the Schwarzschild solutions do not exist. It is interesting to generalize these results to the higher dimensional case. It should be emphasized, that the action with the \( C^2 \) term does not modify the 4D black hole solution. However a similar to the higher dimensional case happens, for example, if one considers a correction to the Einstein action of the form \( f(C^2) \), or even more general functions of the curvature invariants.

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Appendix. Results of iterations in the third order

Here we present the third order results for the iteration procedure developed in the section 6.

\[ M_3 = 1 - (n-2)(n-1)(n+1)px^{n+1} - \frac{2}{n} (n-2)(n^2-1)^2 p^2 \left[ (4n^2+15n+6)x^{2n+2} - 4(n+1)(2n+1)x^{n+3} \right] \]

\[ - \frac{4(n-2)(n^2-1)^3}{3n^2} p^3 \left[ (68n^4 + 745n^3 + 1699n^2 + 1166n + 240)x^{3n+3} \right. \]

\[ -12(n+1)^2(12n^2 + 97n + 44)x^{2n+4} + 48(n+1)(n+2)(n+3)(2n+1)x^{n+5} \]. \hspace{1cm} (A1)

\[ \Psi_3 = \frac{4(n-1)^2(n+1)^3(n^2-4)(2n+1)}{n} p^2 x^{2n+1} \]

\[ + \frac{32(n-2)(n-1)^3(n+1)^4}{n^2} p^3 \left[ (4n^4 + 41n^3 + 86n^2 + 58n + 12)x^{3n+2} - (n+2)(n+3)(n+4)(2n+1)x^{2n+3} \right]. \hspace{1cm} (A2) \]

\[ K_3 = n(n+1)x - 2(n+1)^2(n-2)(n-1)(2n+1)px^{n+2} \]

\[ - \frac{2(n-2)(n-1)^2(n+1)^3}{n} p^2 \left[ (8n^3 + 69n^2 + 66n + 16)x^{2n+3} - 8(n+1)(n+2)(2n+1)x^{n+4} \right] \]

\[ - \frac{8(n-2)(n-1)^3(n+1)^4}{3n^2} p^3 \left[ 48(n+3)^2(n+1)(n+2)(2n+1)x^{n+6} \right. \]

\[ -6(n+1)(32n^4 + 399n^3 + 1005n^2 + 828n + 208)x^{2n+5} \]

\[ +(142n^5 + 1643n^4 + 5222n^3 + 5878n^2 + 2700n + 432)x^{3n+4} \]. \hspace{1cm} (A3)

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The same equation can be obtained by linearizing the general master equation which has been mentioned by the end of the previous section.