STRONGLY CLEAN MATRIX RINGS OVER COMMUTATIVE RINGS

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Abstract. A ring $R$ is called strongly clean if every element of $R$ is the sum of a unit and an idempotent that commute. By SRC factorization, Borooah, Diesl, and Dorsey [3] completely determined when $M_n(R)$ over a commutative local ring $R$ is strongly clean. We generalize the notion of SRC factorization to commutative rings, prove that commutative $n$-SRC rings ($n \geq 2$) are precisely the commutative local rings over which $M_n(R)$ is strongly clean, and characterize strong cleanness of matrices over commutative projective-free rings having ULP. The strongly $\pi$-regular property (hence, strongly clean property) of $M_n(C(X,\mathbb{C}))$ with $X$ a P-space relative to $\mathbb{C}$ is also obtained where $C(X,\mathbb{C})$ is the ring of complex valued continuous functions.

Key Words: strongly clean ring, matrix ring, commutative ring, strongly $\pi$-regular ring, ring of complex valued continuous functions.

2000 Mathematics Subject Classification: Primary 16U99, 13A99; Secondary 26A99.

1. Introduction

Let $R$ be an associative ring with identity and $U(R)$ denote the set of units of $R$. An element $a \in R$ is called strongly clean if $a = e + u$ for some $e^2 = e$ and $u \in U(R)$ such that $eu = ue$ and the ring $R$ is a strongly clean ring if every element of $R$ is strongly clean [17].

Clearly, local rings are strongly clean. An element $a \in R$ is strongly $\pi$-regular if both chains $Ra \supset Ra^2 \supset \cdots$ and $aR \supset a^2 R \supset \cdots$ terminate. $R$ is strongly $\pi$-regular if every element of $R$ is strongly $\pi$-regular [2]. Strongly $\pi$-regular elements are strongly clean [17]. Hence, strongly $\pi$-regular rings are strongly clean [4] [17]. The authors of [1] and [5] proved independently that for a topological space $X$, $C(X)$ is a strongly clean ring iff $X$ is strongly zero-dimensional. We proved that $C(X,\mathbb{C})$ is strongly clean iff $X$ is strongly zero-dimensional [9]. So $C(X)$ and $C(X,\mathbb{C})$ with $X$ strongly zero-dimensional are strongly clean. In his foundational paper [17], Nicholson asked if the matrix ring over a strongly clean ring is strongly clean. Wang and Chen [18] answered this question negatively. Then a natural question arose: When is the matrix ring over a strongly clean ring strongly clean? For local rings, Chen, Yang, and Zhou [6] characterized when the $2 \times 2$ matrix ring $M_2(R)$ over a commutative local ring $R$ is strongly clean; Li [13] characterized...
when a single $2 \times 2$ matrix over a commutative local ring is strongly clean; Borooah, Diesl, and Dorsey [3] characterized when the matrix ring $M_n(R)$ over a commutative local ring $R$ is strongly clean; and recently, Yang and Zhou [20] characterized when the $2 \times 2$ matrix ring $M_2(R)$ over a local ring $R$ is strongly clean. For strongly $\pi$-regular rings, Yang and Zhou [19] proved that the matrix rings over some strongly $\pi$-regular rings are strongly clean. A completely regular space $X$ is called a P-space (relative to $\mathbb{R}$) if every prime ideal in $C(X)$ is maximal [10, p.63]. In [9], we found that the matrix ring over $C(X)$ with $X$ a P-space relative to $\mathbb{R}$ is strongly $\pi$-regular (hence, strongly clean).

In this paper, we continue the study of when a matrix ring is strongly clean. The authors of [3] defined SRC factorization for a commutative local ring. They proved that for a commutative local ring $R$, $M_n(R)$ is strongly clean iff $R$ is an $n$-SRC ring and they showed that a matrix ring over a Henselian ring is strongly clean. The theory of SRC factorization is a useful tool for judging strong cleanness of matrix rings over commutative local rings. However, the theory is constraint to commutative local rings. In Section 2, we generalize this definition to commutative rings (Definition 1), get a sufficient but not necessary condition for a matrix ring over a commutative ring to be strongly clean (Theorem 5 and Example 13), and characterize an $n$-SRC ring (Theorem 14). After reading an earlier version of this paper (arXiv:0803.2176v1), Alex Diesl and Tom Dorsey improved upon our results, and we thank them for giving their permission to include their results here. Specifically, Propositions 8, 10, and 13 are due to Diesl and Dorsey, as are Remarks 7(2) and 9, and Example 15 (generalizing our observation for $n = 2$). Also, their Lemma 3 refines our Corollary 4, and the proof we give of Corollary 4 is due to them. In Section 3, we study the strong cleanness of matrices over the class of commutative projective-free rings having ULP (see Definitions 11 and 20). The class of commutative projective-free rings having ULP includes commutative local rings, PID (principal ideal domains), polynomial rings with finitely many indeterminates over a PID (Quillen-Suslin Theorem), and etc.. We characterize when a single matrix over this class of rings is strongly clean (Theorems 25 and 28). These results can help us to find all strongly clean matrices over $R$ even if $M_n(R)$ is not strongly clean. In Section 4, we find new classes of strongly clean matrix rings— matrix rings over $C(X, \mathbb{C})$ are strongly $\pi$-regular (hence, strongly clean) when $X$ is a P-space relative to $\mathbb{C}$ (see the definition in Section 4).

Throughout the paper, when $R[t]$ is a UFD (unique factorization domain), we let $\text{gcd}(h(t), g(t))$ be the greatest common divisor of the polynomials $h(t), g(t) \in R[t]$. If $R$ is a field, we require $\text{gcd}(h(t), g(t))$ to be the monic greatest common divisor of the polynomials $h(t), g(t) \in R[t]$. The symbol $\text{Max}(R)$ denotes the maximal spectrum of a commutative ring $R$, $J(R)$ denotes the Jacobson radical, and $\mathbb{N}$ denotes the set of positive integers.

2. Strong cleanness of $M_n(R)$ over a commutative ring $R$

The authors of [3] defined SR factorization and SRC factorization: Let $R$ be a commutative local ring. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an SR factorization if $h_0(t)$ and $h_1(t)$ are monic and $h_0(0)$ and $h_1(1) \in U(R)$. The ring $R$ is an $n$-SR ring if every monic polynomial of degree $n$ in $R[t]$ has an SR factorization. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an SRC factorization if it is an SR factorization and $\text{gcd}(\overline{h_0(t)}, \overline{h_1(t)}) = 1$ in the PID $\overline{R}[t] = \frac{\overline{R}}{J(\overline{R})}[t]$. The ring $R$ is an $n$-SRC ring if every monic polynomial of degree
n in $R[t]$ has an SRC factorization. $R$ is an SRC ring if it is an n-SRC ring for every $n \in \mathbb{N}$. They proved that the matrix ring $M_n(R)$ is strongly clean iff $R$ is an n-SRC ring.

Recall that, for a commutative ring $R$, a pair of polynomials $(f_0(t), f_1(t))$ in $R[t]$ is unimodular if $f_0(t)R[t] + f_1(t)R[t] = R[t]$ or equivalently, $f_0(t)h_0(t) + f_1(t)h_1(t) = 1$ with some $h_0(t)$ and $h_1(t)$ in $R[t]$. For a commutative local ring $R$ and monic polynomials $f_0(t)$ and $f_1(t)$ in $R[t]$, $\gcd (\overline{f_0}(t), \overline{f_1}(t)) = 1$ iff $(\overline{f_0}(t), \overline{f_1}(t))$ is unimodular in $R[t]$ iff $(f_0(t), f_1(t))$ is unimodular in $R[t]$. 0 and 1 are the only idempotents of local rings. So we generalize above definition to commutative rings.

**Definition 1.** Let $R$ be a commutative ring and let $f(t) \in R[t]$ be a monic polynomial. A factorization $f(t) = f_0(t)f_1(t)$ in $R[t]$ is called an SR factorization if $f_i(t)$ is monic in $R[t]$ and $f_i(e_i) \in U(R)$ with idempotents $e_0 \neq e_1 \in R$ ($i = 0, 1$). The factorization $f(t) = f_0(t)f_1(t)$ is called an SRC factorization if, in addition, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. The ring $R$ is called an n-SR (resp., n-SRC) ring if every monic polynomial of degree $n$ has an SR (resp., SRC) factorization.

**Theorem 2.** Let $R$ be a commutative ring. Then $R$ is strongly clean iff $R$ is a 1-SR ring iff $R$ is a 1-SRC ring.

**Proof.** Suppose that $R$ is strongly clean. Let $f(t) = t + a \in R[t]$. Write $-a = e + u$ where $e^2 = e \in R$, $u \in U(R)$, and $eu = ue$. So $f(e) = -u \in U(R)$. Hence, $f(t) = f_0(t)f_1(t)$ with $f_0(t) = t + a$ and $f_1(t) = 1$ is an SR factorization. Obviously, this is also an SRC factorization.

Suppose that $R$ is a 1-SR ring. Let $a \in R$. Then $f(t) = t - a$ has an SR factorization in $R[t]$. It must be that $f(t) = f_0(t)$ or $f(t) = f_1(t)$. So there exists $e^2 = e \in R$ such that $f(e) = e - a \in U(R)$. Thus, $a$ is strongly clean.

**Lemma 3.** Let $R$ be a commutative ring and let $A \in M_n(R)$. Let $f \in R[t]$ be a monic polynomial for which $f(A) = 0$ (e.g. the characteristic polynomial $\chi_A$ of $A$, by the Cayley-Hamilton Theorem [14]). If $f(e)$ is a unit for some idempotent $e \in R$, then $A$ is strongly clean.

**Proof.** Let $e$ be such an idempotent. We claim that $A - eI$ is a unit. Using long division, write $f(t) = (t - e)g(t) + f(e)$. Then, $0 = f(A) = (A - eI)g(A) + f(e)I$. Then, $(A - eI)g(A)f(e)^{-1} = I$, and we conclude (since the two operators involved commute) that $A - eI$ is invertible. Since $eI$ is a central idempotent of $M_n(R)$, we conclude that $A = (A - eI) + eI$ is strongly clean.

Note that the first few lines of work are not needed when $f = \chi_A$, since then $f(e) = \det(eI - A)$, which shows immediately that $eI - A$ is invertible.

**Corollary 4.** Let $R$ be a commutative ring and let $A \in M_n(R)$. If $f = \chi_A$ has an n-SRC factorization, then $A$ is strongly clean.

**Proof.** By hypothesis, there exist monic polynomials $f_0, f_1 \in R[t]$ such that $f = f_0f_1$ and $(f_0, f_1)$ is unimodular, and idempotents $e_0, e_1$ for which $f_0(e_0), f_1(e_1)$ are units. Find $g_0, g_1$ such that $g_0g_0 + f_1g_1 = 1$. By [3 Lemma 11], $\ker(f_0(A)) \oplus \ker(f_1(A)) = R^n$. It is clear that both $\ker(f_0(A))$ and $\ker(f_1(A))$ are $A$-invariant. Now, $A|_{\ker(f_0(A))}$ satisfies the polynomial $f_0$ and $A|_{\ker(f_1(A))}$ satisfies the polynomial $f_1$. By Lemma [3] $A|_{\ker(f_0(A))}$ and $A|_{\ker(f_1(A))}$ are strongly clean. It follows from [17] that $A$ is strongly clean. Indeed, let $\varphi \in \text{End}_R(R^n)$ be the projection of $R^n$ onto $\ker(f_0(A))$, relative to the direct sum.
$R^n = \ker(f_0(A)) \oplus \ker(f_1(A))$. Then, $A\varphi = \varphi A$ and $\varphi A$ and $(1 - \varphi)A$ are strongly clean in $\varphi M_n(R)\varphi$ and $(1 - \varphi)M_n(R)(1 - \varphi)$, respectively. \hfill $\square$

**Theorem 5.** If $R$ is an $n$-SRC ring, then $M_n(R)$ is strongly clean.

**Proof.** For any matrix $A \in M_n(R)$, the characteristic polynomial, $\chi_A(t)$, of $A$ has an $n$-SRC factorization. So $A$ is strongly clean by Corollary 4. That is, $M_n(R)$ is strongly clean. \hfill $\square$

**Remark 6** (On Theorem 5). Being an $n$-SRC ring is not necessary for the matrix ring $M_n(R)$ to be strongly clean (see Example 15).

**Remark 7** (On Definition 1). In Corollary 4 there is no restriction that $e_0 \neq e_1$, but in Definition 1 we require $e_0 \neq e_1$. Allowing the idempotents to agree does not really gain anything, since given an $n$-SRC factorization $f = f_0f_1$ with $e_0 = e_1$ and $f(e_0) \in U(R)$, $f = f \cdot 1$ is an $n$-SRC factorization with respect to $e_0$ and any other idempotent.

2). Logically, allowing idempotents other than 0 and 1 to appear in Definition 1 is not as much of a generalization as we might think. But it can simplify computation. Recall [17, Proposition 2]: If $\{e_1, e_2, \cdots, e_n\}$ is a set of complete orthogonal central idempotents, then $R = \bigoplus_{i=1}^{n} e_iR = \bigoplus_{i=1}^{n} e_iRe_i$, and $R$ is strongly clean iff $e_iRe_i$ is strongly clean for $i = 1, \cdots, n$. Observe that, for any idempotent $e \in R$ (with $R$ commutative) and $g(t) \in R[t]$, $g(e) = eg(1) + (1 - e)g(0)$, and moreover, that $eg(1) = eg(e)$. In particular, $g(e)$ is a unit in $R$ iff $eg(1) = eg(e)$ is a unit in the corner ring $eR$ and $(1 - e)g(0) = ((1 - e)g)(0)$ is a unit in the corner ring $(1 - e)R$. Thus, allowing two idempotents $e_1$ and $e_2$ for the polynomials $f_0(t)$ and $f_1(t)$ in an SR factorization $f = f_0f_1$, look at the associated four term direct sum decomposition corresponding to $e_0e_1 + e_0(1 - e_1) + (1 - e_0)1 = 1$. We get a sum of $f$: $f = f_0f_1 = e_0e_1f_0f_1 + e_0(1 - e_1)f_0f_1 + (1 - e_0)e_1f_0f_1 + (1 - e_0)(1 - e_1)f_0f_1$. $e_0e_1f(t)$ and $e_0e_1g(t)$ are units at the identity of $e_0e_1R$. $(1 - e_0)(1 - e_1)f_0(t)$ and $(1 - e_0)(1 - e_1)f_1(t)$ are units at 0 of $(1 - e_0)(1 - e_1)R$. In the other two factors, one of $f_0$ and $f_1$ (multiplied with corresponding identity of the corner rings) is a unit at the corresponding identity and the other is a unit at 0. So each component of $f_0$ and $f_1$ has an SR factorization corresponding to the trivial idempotents 0 and “1” of the corresponding corner rings.

3). We still call the factorization an SR (SRC) factorization as in [3] because Definition 1 is essentially the same as that in [3] when we deal with the strong cleanness of matrix rings $M_n(R)$ with $n \geq 4$ (see Proposition 8 and Proposition 13 below) although Definition 1 is really a generalization as Example 9 shows.

**Proposition 8.** Let $R$ be an $n$-SR ring for some $n \geq 4$. Then $R$ is local.

**Proof.** Suppose $e \in R$ is a nontrivial idempotent. Thus, $R$ is a nontrivial direct product, say $R = R_1 \times R_2$, of rings. Consider the monic polynomial $h(t) = (t^{n-1}(t-1), t^{n-2}(t-1)^2) \in R[t]=R_1[t] \times R_2[t]$. Suppose that $h = fg$ is an $n$-SRC factorization. Write $f = (f_1, f_2)$ and $g = (g_1, g_2)$. Clearly, $f_1g_1$ is an $n$-SR factorization of $t^{n-1}(t-1)$ in $R_1[t]$ and $f_2g_2$ is an $n$-SRC factorization of $t^{n-2}(t-1)^2$ in $R_2[t]$.

Now, more generally, suppose that $fg$ is an $n$-SRC factorization of $t^k(t-1)^{n-k}$, over an arbitrary nonzero commutative ring $R$. The same is then true passing to a quotient $F = R/m$, where $m$ is a maximal ideal. But $F$ is a field, so $F[t]$ is a UFD, and it follows that the image of the monic polynomial $f$ (resp. $g$) must be $t^i(t-1)^j$ for some $i$ and...
Proposition 12. For a projective-free ring, we have the following result.

\[ \text{Projective finite is called I-finite.} \]

Definition 11. \[ A \text{ ring } R \text{ has no nontrivial idempotents.} \]

Returning to our previous situation, \( f_1 \) must have degree either \( n-1 \) or 1, whereas \( f_2 \) must have degree either \( n-2 \) or 2. Since \( 1, 2, n-1, n-2 \) are all distinct (since \( n \geq 4 \)), we conclude that \( f \) cannot be monic, and hence \( h \) has no \( n \)-SR factorization. We conclude that every idempotent of \( R \) is trivial.

It remains to show that \( R \) is local. Observe that the property \( n \)-SR passes to quotient rings. In particular, if \( R \) has \( n \)-SR, where \( n \geq 4 \), every quotient of \( R \) also has no nontrivial idempotents. Thus, suppose that \( R \) has two distinct maximal ideals \( m_1 \) and \( m_2 \). By the Chinese Remainder Theorem, \( R/(m_1 \cap m_2) \cong R/m_1 \times R/m_2 \), which clearly has nontrivial idempotents. We conclude that \( m_1 = m_2 \). It follows that \( R \) has a unique maximal ideal, so \( R \) is local, as desired.

Remark 9. The hypothesis that \( n \geq 4 \) in Proposition \( 8 \) is, in fact, necessary. One can show that \( \mathbb{C} \times \mathbb{C} \) is an \( n \)-SR ring for \( n = 2, 3 \). Other examples include \( R \times R \) where \( R \) is quadratically or cubically closed fields or complete local rings with closed fields as quotients.

Proposition 10. Let \( n = 2 \) or 3 and let \( R \) be a \( n \)-SR ring. If \( R[t] \) has an irreducible monic polynomial of degree \( n \), then \( R \) has only the trivial idempotents.

Proof. Let \( f \in R[t] \) be irreducible, monic, and degree \( n \). Suppose that \( e \in R \) is a nontrivial idempotent: regard \( R \) as the direct product of \( eR \) and \( (1-e)R \). It follows that either \( ef(t) \) or \( (1-e)f(t) \) is an irreducible polynomial, since otherwise, both factorizations must be into monic polynomials of degree \( 1 \) and \( n-1 \), respectively, and we can piece these together to factor \( f \) as a product of a monic degree 1 and degree \( n-1 \) polynomial. Without loss of generality, suppose \( g(t) = ef(t) \) is irreducible in \( eR[t] \). Consider the monic polynomial \( f' = (g(t), t^{n-1}(t-1)) \in R[t] \). Any \( n \)-SR factorization of \( f' \) must have first coordinate either degree 0 or \( n \), since \( g \) is irreducible. On the other hand, the second coordinate, as in the proof of Proposition \( 8 \) must have degree \( 1 \) or \( n-1 \), and it follows as in that proof, since \( 0, 1, n-1, n \) are all distinct, that \( f' \) has no \( n \)-SR factorization. We conclude from this contradiction that \( R \) has no nontrivial idempotents. \( \square \)

Definition 11. \[ \text{A ring } R \text{ is called projective-free if every finitely generated projective } R\text{-module is free of unique rank.} \]

Camillo and Yu \([5]\) proved that \( R \) is semiperfect if \( R \) is I-finite and clean (A ring \( R \) is called I-finite if \( R \) does not have an infinite set of non-zero orthogonal idempotents). For a projective-free ring, we have the following result.

Proposition 12. Let \( R \) be a projective-free ring. Then the following are equivalent:

1. \( R \) is a strongly clean ring.
2. \( R \) is a clean ring.
3. \( R \) is a local ring.
4. \( R \) is an exchange ring.
5. \( R \) is a semiperfect ring.

If, in addition, \( R \) is commutative, then the above are equivalent to the following:

6. \( R \) is a 1-SR ring.
7. \( R \) is a 1-SRC ring.
Proof. “(3) ⇒ (1) ⇒ (2)” This is clear.
“(2) ⇒ (4)” This is a well-known result in
“(4) ⇒ (3)” We prove $R$ has only 0 and 1 as its idempotents. Suppose $e^2 = e \in R$. Then $R = Re \oplus R(1 - e)$. Since $R$ is projective-free, we get $Re = 0$ or $R(1 - e) = 0$. So $e = 0$ or $e = 1$. Now let $r \notin U(R)$. Then because $R$ is an exchange ring, there exists $e^2 = e$ such that $e \in Rr$ and $1 - e \in R(1 - r)$. That is, $1 \in Rr$ or $1 \in R(1 - r)$. But $r \notin U(R)$, so $1 \in R(1 - r)$. Similarly, $1 \in (1 - r)R$. So $1 - r \notin U(R)$. Therefore, $R$ is local.
“(3) ⇒ (5)” This is clear.
“(5) ⇒ (2)” This is a result of [5].
“(1) ⇔ (6) ⇔ (7)” This is Theorem [2] □

We have not determined whether the rings in Proposition [10] must be local under the hypothesis. However, the SRC hypothesis forces locality for $n \geq 2$, as the next proposition shows.

Proposition 13. Let $R$ be a $n$-SRC ring for some $n \geq 2$. Then $R$ is local.

Proof. By Theorem [5] $M_n(R)$ is strongly clean. Since it is known that strong cleanness passes to corners, $R$ must therefore be a strongly clean ring. It will therefore suffice to show that $R$ has no nontrivial idempotents, since a ring with only trivial idempotents is strongly clean iff it is local by Proposition [12]. The result now follows from Theorem [8] for $n \geq 4$. However, we give a different, elementary argument, that works for all $n \geq 2$, rather than handing only the cases $n = 2$ and $n = 3$ separately. Suppose that $e \in R$ is a nontrivial idempotent. Consider the polynomial $f(t) = t^n - et \in R[t]$. Since $R$ is an $n$-SRC ring, there is a factorization $f = f_0(t)f_1(t)$ of $f(t)$ into monic polynomials such that $(f_0(t), f_1(t))$ is unimodular and such that there are idempotents $e_0, e_1 \in R$ such that $f_0(e_0), f_1(e_1)$ are units in $R$. We claim that such a factorization cannot exist.

A trivial factorization cannot occur, since if $g^2 = g \in R$, then $f(g) = g(1 - e)$ cannot be a unit (since it annihilates $e \neq 0$). Thus, $f_0, f_1$ are unimodular monic polynomials, and each has degree at least 1. It will therefore suffice to show that $f$ does not have a nontrivial factorization as a product of a pair of unimodular monic polynomials. Indeed, let $f = f_0f_1$ be such a factorization. Since $e$ is not a unit, $e \in m$ for some maximal ideal $m$. Since the images of $f_0$ and $f_1$ are unimodular in $(R/m)[t]$, we may assume that $R$ is a field and that $f(t) = t^n$. But $R[t]$ is then a UFD, in which case $f_0$ and $f_1$ must be, up to units, each power of $t$, but this forces $f_0R[t] + f_1R[t] \subseteq tR[t]$, since $f_0$ and $f_1$ were monic polynomials with degree at least 1. This is a contradiction, and we conclude that the original strongly clean ring $R$ has only the trivial idempotents, and hence is local. □

Now we immediately get the following result.

Theorem 14. Let $R$ be a commutative ring and $(n \geq 2)$. Then the following are equivalent:

1. $R$ is an $n$-SRC ring.
2. $R$ is a local $n$-SRC ring.
3. $R$ is local and $M_n(R)$ is strongly clean.

Proof. “(1) ⇔ (2)” By Proposition [13].
“(2) ⇒ (3)” By Theorem [5].
“(3) ⇒ (2)” By [8] Corollary 15. □
But for a commutative ring $R$, being an $n$-SRC ring is not a necessary condition for $M_n(R)$ to be strongly clean.

**Example 15.** Let $R$ be a Boolean ring with more than 2 elements. Then $R$ is not an $n$-SRC ring for $n \geq 2$ because Boolean rings other than $\mathbb{Z}/2\mathbb{Z}$ can not be SR rings by Proposition 17A since $t^{(n-1)}(t-1)+1$ is always irreducible for $n \geq 2$. But $M_n(R)$ is strongly clean for any positive integer $n$.

We define SRC factorizations and SRC-rings on commutative rings and it is clear about commutative SRC-rings by Theorem 14 now. In fact, they can be defined on non-commutative rings. For example, the $(\ast)$-factorization and $(\ast\ast)$-factorization used to characterize strong cleanness of $M_2(R)$ over a local ring $R$ (need not be commutative) is essentially the SR and SRC factorization for non-commutative case 19. However, for the non-commutative case, we know very little.

3. **Strong cleanness of matrices over projective-free rings having ULP**

Section 2 shows that the theory of SRC factorization can not give us new classes of strongly clean matrix rings except the local ones. However, it can help us to find all strongly clean matrices over projective-free rings having ULP (see Definition 20) even though the matrix ring is not strongly clean. This is the topic of Section 3.

A matrix $A \in M_n(R)$ is called singular if $A$ is non-invertible and nonsingular if $A$ is invertible. Here, we give a more detailed definition related to singularity of a matrix.

**Definition 16.** A singular matrix $A \in M_n(R)$ is called purely singular if $I - A$ is nonsingular or semi-purely singular if $I - A$ is singular. A nonsingular matrix $A \in M_n(R)$ is called purely nonsingular if $I - A$ is nonsingular or semi-purely nonsingular if $I - A$ is singular.

Every matrix belongs to exactly one of the above four types. All types of matrices are strongly clean except purely singular ones. So we have the following lemma.

**Lemma 17.** The matrix ring $M_n(R)$ is strongly clean if and only if its purely singular matrices are strongly clean.

**Lemma 18.** [20] Let $R$ be a projective-free ring. Then a purely singular matrix $T \in M_n(R)$ is strongly clean iff $T$ is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $T_0$ is semi-purely nonsingular and $T_1$ is semi-purely singular.

By this lemma, we get a necessary condition for a matrix to be strongly clean when $R$ is commutative projective-free.

**Corollary 19.** Let $R$ be a commutative projective-free ring. If $T \in M_n(R)$ is strongly clean, then $\chi_T(t)$ has an $n$-SR factorization.

**Proof.** If $T$ is nonsingular, then $\chi_T(t) = \det(tI - T) = f_0(t)f_1(t) = \chi_T(t) \cdot 1$ with $f_0(t) = \chi_T(t)$, $f_1(t) = 1$, $e_0 = 0$, and $e_1 = 1$ is an $n$-SR factorization. If $T$ is semi-purely singular, then $\chi_T(t) = \det(tI - T) = f_0(t)f_1(t) = 1 \cdot \chi_T(t)$ with $f_0(t) = 1$, $f_1(t) = \chi_T(t)$, $e_0 = 0$, and $e_1 = 1$ is an $n$-SR factorization. If $T$ is purely singular, then, by Lemma 18 $T$ is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $T_0$ is semi-purely nonsingular and $T_1$ is semi-purely singular. So $\chi_T(t) = \chi_{T_0}(t) \cdot \chi_{T_1}(t)$ with $f_0(t) = \chi_{T_0}(t)$, $f_1(t) = \chi_{T_1}(t)$, $e_0 = 0$, and $e_1 = 1$ is an $n$-SR factorization. \qed
Definition 20. A commutative ring $R$ is said to have the unimodular lifting property (ULP for short) if, for any pair $(f_0(t), f_1(t))$ of monic polynomials in $R[t]$, the unimodularity of $(\overline{f_0(t)}, \overline{f_1(t)})$ in $\frac{R}{m}[t]$ for all $m \in \text{Max}(R)$ implies the unimodularity of $(f_0(t), f_1(t))$ in $R[t]$.

A ring $R$ is semilocal if $R/J(R)$ is semisimple. A commutative ring is semilocal iff it has finitely many maximal ideals.

Proposition 21. Commutative semilocal rings have ULP.

Proof. Let $R$ be a commutative semilocal ring. Then $R$ has finitely many maximal ideals, say $m_1, \ldots, m_n$. Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $(\overline{f_0(t)}, \overline{f_1(t)})$ be unimodular in $\frac{R}{m_k}[t]$ for $k = 1, 2, \ldots, n$. Since $\overline{f_0(t)} \frac{R}{m_k}[t] + \overline{f_1(t)} \frac{R}{m_k}[t] = \frac{R}{m_k}[t]$, we get $f_0(t)R[t] + f_1(t)R[t] + m_k[t] = R[t]$. Hence, $f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t) = 1$ for some $a_k(t), b_k(t) \in R[t]$ and $c_k(t) \in m_k[t]$. Therefore,

$$1 = \prod_{k=1}^n (f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t)) = f_0(t)a'(t) + f_1(t)b'(t) + c'(t)$$

for some $a'(t), b'(t) \in R[t]$ and $c'(t) \in J(R)[t]$. Thus, $R[t] = f_0(t)R[t] + f_1(t)R[t] + c'(t)R[t] = f_0(t)R[t] + f_1(t)R[t] + J(R)[t]$. Notice that $\frac{R[t]}{f_0(t)R[t] + f_1(t)R[t]}$ is a finitely generated $R$-module and $J(R) = f_0(t)R[t] + f_1(t)R[t] = \frac{R[t]}{f_0(t)R[t] + f_1(t)R[t]}$. So, $f_0(t)R[t] + f_1(t)R[t] = R[t]$ by Nakayama Lemma. Therefore, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. \hfill $\Box$

Corollary 22. Commutative local rings have ULP.

Proposition 23. Every UFD has ULP.

Proof. Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $(\overline{f_0(t)}, \overline{f_1(t)})$ be unimodular in $\frac{R}{m}[t]$ for every $m \in \text{Max}(R)$. Then $\gcd(\overline{f_0(t)}, \overline{f_1(t)}) = 1$ in $\frac{R}{m}[t]$. We want to prove that $\gcd(f_0(t), f_1(t))$ is a unit in $R[t]$. Suppose $\gcd(f_0(t), f_1(t))$ is not a unit.

Case 1. $\gcd(f_0(t), f_1(t)) = m \in R$ but $m \notin U(R)$.

Then there exists $m_0 \in \text{Max}(R)$ such that $m \in m_0$. So $\gcd(\overline{f_0(t)}, \overline{f_1(t)}) = \overline{m} = 0$ in $\frac{R}{m_0}[t]$. This is a contradiction.

Case 2. $\gcd(f_0(t), f_1(t)) = g(t) \in R[t]$ with $\deg(g(t)) \geq 1$ in $R[t]$.

Then for any $m \in \text{Max}(R)$, $\gcd(\overline{f_0(t)}, \overline{f_1(t)}) \neq 1$ in $\frac{R}{m}[t]$ because the coefficient of the leading item of $g(t)$ is a unit.

Hence, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. \hfill $\Box$

Given a monic polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$, the matrix

$$C_f = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & 0 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\
0 & 0 & 0 & \cdots & 1 & -a_{n-1}
\end{pmatrix}$$

is called the companion matrix of $f(t)$.

Lemma 24. \cite{12} Theorem VII.4.3 Let $F$ be a field and $f(t)$ be a monic polynomial in $F[t]$. Then $f(t)$ is the characteristic and minimal polynomial of the companion matrix $C_f$.

Theorem 25. Let $R$ be a commutative ring having ULP and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Then the companion matrix $C_f$ is strongly clean iff $\chi_{C_f}(t) = f(t)$ has an $n$-SRC factorization.
STRONGLY CLEAN MATRIX RINGS OVER COMMUTATIVE RINGS

A topological space $X$ is said to be completely regular if whenever $F$ is a closed set and $x$ is a point in its complement, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f[F] = \{0\}$. Let $C(X)$ (resp., $C(X, \mathbb{C})$) denote the ring of all real (resp., complex) valued continuous functions from a completely regular Hausdorff space $X$ to the real number field $\mathbb{R}$ (resp., complex number field $\mathbb{C}$). For a function $f \in C(X)$ (or $C(X, \mathbb{C})$), the set $z(f) = \{x \in X : f(x) = 0\}$ is called the zero-set of $f$. An open set $U \subseteq X$ is called functionally open if the complement $X \setminus U$ is a zero-set. A topological space $X$ is called strongly zero-dimensional if $X$ is a completely regular Hausdorff space and every finite functionally open cover $\{U_i\}_{i=1}^k$ of the space $X$ has a finite open refinement $\{V_j\}_{j=1}^m$ such that $V_i \cap V_j = \emptyset$ for any $i \neq j$ [8]. A completely regular space $X$ is called a $P$-space relative to $\mathbb{C}$ if every prime ideal in $C(X, \mathbb{C})$ is maximal.

**Proof.** “$\Leftarrow$”. By Corollary 1

“$\Rightarrow$”. The argument of Corollary 19 shows that if $T$ is not purely singular, then $\chi_T(t)$ has a trivial SRC factorization, that is, one of the factors is 1 and the other is $\chi_T(t)$ itself. So we can assume $C_f$ is purely singular. Then by Lemma 18 there exists $P \in M_n(R)$ such that $P^{-1}C_fP = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ with $T_0$ being $k \times k$ semi-purely nonsingular matrix and $T_1$ being $(n-k) \times (n-k)$ semi-purely singular matrix where $0 < k < n$. Then for every maximal ideal $m$ in $R$, $C_f = C_f \in M_n(R)$ has $\mathcal{F}(t) \in \mathbb{R}[t]$ as the characteristic and minimal polynomial by Lemma 24. So $\mathcal{F}(t) = \chi_C(t) = \chi_{T_0}(t) \cdot \chi_{T_1}(t) = \det(tI_k - T_0) \cdot \det(tI_{n-k} - T_1)$. If $\gcd(\det(tI_k - T_0), \det(tI_{n-k} - T_0)) = 1$, then the minimal polynomial of $C_f$ is $\chi_C(t) = \det(tI_k - T_0) \cdot \det(tI_{n-k} - T_1)$ which has degree less than $\deg(\chi_f) = \deg(f)$. This is a contradiction. So $f_0(t) = \det(tI - T_0)$, $f_1(t) = \det(tI - T_1)$, $e_i = i$, and $f_i(e_i) \in U(R)$ ($i = 0, 1$) give an $n$-SRC factorization for $\chi_{C_f}(t) = f(t)$.

**Corollary 26.** Let $R$ be a commutative ring having ULP and let $f(t) \in R[t]$ be a monic polynomial of degree $\deg(f(t)) = n$. Then the following are equivalent:

1. For all $A \in M_n(R)$ with $\chi_A(t) = f(t)$, $A$ is strongly clean.
2. The companion matrix $C_f$ is strongly clean.
3. $f(t)$ has an n-SRC factorization.

**Proof.** “(1) $\Rightarrow$ (2)” is clear. “(2) $\Rightarrow$ (3)” is Theorem 25. “(3) $\Rightarrow$ (1)” is Corollary 4.

**Question 27.** Does every commutative projective-free ring have ULP?

**Theorem 28.** Let $R$ be a commutative projective-free ring. Then a purely singular matrix $A \in M_n(R)$ is strongly clean iff $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ with $e_i = i$ ($i = 0, 1$) and $A$ is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$.

**Proof.** “$\Rightarrow$”. By Lemma 18, $A$ is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $T_0$ is semi-purely nonsingular and $T_1$ is semi-purely singular. By Corollary 19, $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ where $\chi_{T_0}(t) = f_0(t)$, $\chi_{T_1}(t) = f_1(t)$, $e_i = i$, $f_i(e_i) \in U(R)$ ($i = 0, 1$).

“$\Leftarrow$”. By Corollary 26, $T_0$ and $T_1$ are strongly clean because $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$ have trivial SRC factorizations. So $A$ is strongly clean because the strongly clean property is invariant under similarity.

4. **Strong cleanliness of $M_n(C(X, \mathbb{C}))$ with $X$ a P-space relative to $\mathbb{C}$**
Matrix rings over $C(X)$ with $X$ a P-space relative to $\mathbb{R}$ are strongly $\pi$-regular [9]. In this section, we prove the similar results for $C(X, \mathbb{C})$ with $X$ a Hausdorff P-space relative to $\mathbb{C}$. First, we give some notions. For an ideal $I \subseteq C(X, \mathbb{C})$, $z[I] = \{z(f) : f \in I\}$. An ideal $I \subseteq C(X, \mathbb{C})$ is a $z$-ideal if $z(g) \in z[I]$ implies $g \in I$. Let $S$ be a ring and $R$ be a subring of $S$ such that they share the same identity. The ring $S$ is called a finite extension of $R$ if $S$, as an $R$-module, is generated by a finite set $X$ of generators.

**Theorem 29.** Let $X$ be a Hausdorff P-space relative to $\mathbb{C}$. Then $R = C(X, \mathbb{C})$ is strongly regular. Hence, every finite extension of $R$ is strongly $\pi$-regular. In particular, $\mathbb{M}_n(R)$ is strongly $\pi$-regular.

**Proof.** Suppose $X$ is a P-space relative to $\mathbb{C}$. For $p \in X$, set $O_p = \{f \in R : z(f) \text{ is a neighborhood of } p\}$ and $M_p = \{f \in R : f(p) = 0\}$. Then $M_p$ is a maximal ideal and $O_p$ is a $z$-ideal in $R$ with $O_p \subseteq M_p$.

Let $A_p$ be the family of all zero-sets containing a given point $p$. Then $A_p$ is the unique $z$-ultrafilter converging to $p$ [10, p.47]. For any ideal $I$ in $R$, $z[I]$ is a $z$-filter and if $I$ is a maximal ideal, then $z[I]$ is a $z$-ultrafilter. Thus $z[O_p] \subseteq z[M_p] = A_p$. So $M_p$ is the only maximal ideal that contains $O_p$. Notice that $z(f^n) = z(f)$ for any $n \in \mathbb{N}$. If $I$ is a $z$-ideal and $f^n \in I$ then $z(f) = z(f^n) \in z[I]$ implies $f \in I$. So $I$ is a radical ideal, that is, $I$ is an intersection of prime ideals containing $I$. Hence, $O_p$ is an intersection of prime ideals. Since $M_p$ is the only maximal ideal that contains $O_p$, $O_p \neq M_p$ implies $O_p$ is contained in a prime ideal that is not maximal. However, every prime ideal is maximal if $X$ is a P-space relative to $\mathbb{C}$. Hence, $O_p = M_p$.

Let $p$ be any point in $z(f)$. Then $f(p) = 0$ implies $f \in M_p = O_p$. Hence, $z(f)$ is open, that is, every zero-set is clopen. Suppose $I$ is an ideal of $R$ and $z(f) \in z[I]$, then $z(f) = z(g)$ for some $g \in I$. Define $h : X \to \mathbb{C}$ by $h(x) = 0$ if $x \in z(f)$ and $h(x) = \frac{f(x)}{g(x)}$ if $x \not\in z(f)$. Then $h \in R$ and $f = gh$. Thus, $f \in I$, so $I$ is a $z$-ideal. Hence, every ideal in $R$ is a $z$-ideal. So every ideal is a radical ideal.

Since $f$ and $f^2$ belong to the same prime ideals, $(f) = \bigcap_{f \in \mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{f^2 \in \mathfrak{p} \text{ prime}} \mathfrak{p} = (f^2)$. So $f = f^2 f_0$ for some $f_0 \in R$. So $R$ is strongly regular. Hence, by [11, Corollary 4], every finite extension of $R$ is strongly $\pi$-regular. In particular, $\mathbb{M}_n(R)$ is strongly $\pi$-regular since $\mathbb{M}_n(R)$ is the finite extension of $R$. □

**Corollary 30.** Let $X$ be a P-space relative to $\mathbb{C}$ and $G$ be a locally finite group and let $R = C(X, \mathbb{C})$. Then $\mathbb{M}_n((RG)[|X|])$ and $\mathbb{M}_n \left(\frac{(RG)[z]}{(x^2)}\right)$ are strongly clean. In particular, $\mathbb{M}_n(R)$ is strongly clean.

**Proof.** By Theorem 29 and [19, Corollary 3.2]. □

**Corollary 31.** If $X$ is a discrete space, then $\mathbb{M}_n(C(X, \mathbb{C}))$ is strongly $\pi$-regular (hence, strongly clean).

**Proof.** Every discrete space is a P-space relative to $\mathbb{C}$. □

**Acknowledgments**

The first author and the second were partially supported by NSERC of Canada and the Initial Grant of Harbin Institute of Technology respectively.
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