A novel method of ranking intuitionistic fuzzy numbers using value and \( \theta \) multiple of ambiguity at flexibility parameters

Rituparna Chutia

Accepted: 31 July 2021 / Published online: 14 August 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
In this paper, a novel method of ordering intuitionistic fuzzy numbers, combining the ‘value’ and the ‘ambiguity’ of an intuitionistic fuzzy number, is developed. The value and the ambiguity are calculated at \((\alpha, \beta)\)-levels, rather than calculating for the whole range of integration. These levels are termed as flexibility parameters, which allows a decision-maker to take a decision at levels of decision-making. In many studies, the reasonable properties of ranking intuitionistic fuzzy numbers were never tested. However, in this study, every effort is made to investigate the reasonable properties thoroughly. Furthermore, ordering of intuitionistic fuzzy numbers by existing methods relies heavily on intuition and the geometry of intuitionistic fuzzy numbers. The proposed method fully complies with the reasonable properties of ranking intuitionistic fuzzy numbers, as well as the coherent intuition and geometry of the intuitionistic fuzzy numbers. In addition, newer properties are being developed in this study. This demonstrates the novelty of the proposed method. A few numerical examples are also discussed demonstrating the proposed method. Lastly, the proposed method has been successfully applied to a risk analysis problem.

Keywords  Intuitionistic fuzzy number · Ranking · Value · Ambiguity · Flexibility parameter

1 Introduction
The information involved in problems of decision-making, operation research, risk analysis, and so forth may be vague; in such situations, fuzzy set theory is very essential. Most of the time, such vague information is often expressed as linguistic terms, which in turn can be expressed as fuzzy numbers. Furthermore, in those decision-making processes, ordering of fuzzy numbers is essential for making a proper decision. The concept of ordering fuzzy quantities is possible due to the idea of ranking fuzzy numbers. This situation of ordering fuzzy quantities was first overcome by the pioneering works through developing a method using maximizing set of Jain (1976), Jain (1977) and Dubois and Prade (1980).

Atanassov (1986) in 1986 proposed the concept of intuitionistic fuzzy sets to cover the absence or incomplete information in a fuzzy set. The intuitionistic fuzzy set is the generalization of fuzzy set by adding the concept of non-membership degree to the elements in the fuzzy set. This generalization of fuzzy sets to intuitionistic fuzzy sets adds more abundant and flexible information as compared to fuzzy sets. Atanassov (1989, 1994, 2000) are some works that have abundant concepts on pioneering works on intuitionistic fuzzy set. Li (2008) initiated the concept of intuitionistic fuzzy number (IFN) as a quantifier of an ill-known quantity that holds information about the membership and non-membership of an element simultaneously. As such, its applications have been evident in numerous fields of decision-making, risk analysis, optimization problems, etc. Generally, in such problems selection among the alternatives is an indispensable task. Hence, ordering of the IFNs is very essential for a proper and approximate decision.

For years, the ordering of IFNs got little attention among the researchers as compared to the ordering of fuzzy numbers. However, in recent years, ordering of IFNs is gearing up; as such, several definitions of ordering IFNs are being forwarded. Furthermore, most of the existing methods are based on the geometry and the coherent intuition, and a little attention is drawn toward fulfilling the reasonable properties of ranking IFNs. Some works on ranking IFNs are Das and Guha (2016), Grzegorzewski (2003), Kumar and Kaur (2013), Mitchell (2004), Nehi (2010), Salahshour et al. (2012), Seikh et al. (2012), Wan and Li (2013), Zhang and Xu
(2012) and Chutia and Saikia (2018). Here, a brief review of a few existing methods of ordering IFNs is as follows. Interpreting each IFN as an ensemble of ordinary fuzzy numbers, a statistical viewpoint method was developed by Mitchell (2004) for ranking IFNs. Wang and Zhang (2009) transformed the ranking IFNs into the ranking of interval numbers and used in ranking the IFNs. The method seems obscure among the researchers, as the resulting index is interval numbers. Value and ambiguity were incorporated in ranking IFNs in several attempts by Das and De (2016), Dubey and Mehra (2011), Zeng et al. (2014) and many more. Nayagam et al. (2018) performed a thorough review of some existing methods of ranking IFNs, pointing out the limitations of various methods, and developed a new improved value index of ranking IFNs. However, the images of IFNs and the reasonable properties were never incorporated in their study. Furthermore, the ratio of the value index with the ambiguity index to rank triangular IFNs was developed by Li (2010) and applied to multi-attribute decision-making problems. Later, the limitations and of Li’s approach due to nonlinearity was removed by De and Das (2012). Nan et al. (2010) used membership function average indices to rank IFNs. The notion of minimum possibility variance coefficients in ranking IFNs was introduced by Wan (2013). Rezvani (2013) again used values and ambiguities of the membership degree and the non-membership degree of ordering IFNs and defined the value-index and the ambiguity-index. Nayagam et al. (2016) defined eight different classes of IFNs and defined different ranking indices for these classes. In a recent study, Chutia and Saikia (2018) studied the ordering of IFNs based on the concept of ‘value’ and ‘ambiguity’ calculated at various $(\alpha, \beta)$-levels of decision-making. However, limitations in ordering the images of IFNs when ‘values’ are equal are mentioned in the same. Equivalently, Dutta and Saikia (2021) developed a method of ranking IFNs based on the notion of ‘value’ multiplied with mean and ‘ambiguity’ multiplied with mean through the index of optimism following Chutia and Saikia (2018). However, the images of the IFNs were not incorporated in that study. Furthermore, the reasonable properties of ordering IFNs were also not verified. Very recently, Darehmiraki (2019) introduced an ordering method for IFNs through the concept of areas calculated on the left side at various $(\alpha, \beta)$-levels of decision-making.

The quantities ‘value’ and ‘ambiguity’ are very popular tools in ranking methods under fuzzy theory. The ranking of fuzzy numbers was efficiently handled using these quantities (Chutia and Chutia 2017; Chutia 2017, 2021a). Also, ranking of IFNs by using these quantities is efficiently carried out in various studies (Li 2014; Chutia and Saikia 2018; Li 2010; Li et al. 2010). Apart from these, these quantities are also used as tools of ranking numbers under various generalization of fuzzy theory (Chutia and Saikia 2020; Chutia 2021b; Deli and Şubaş 2017; Deli 2019).

The above literature review admits that the existing studies rarely rank the images and validate the reasonable properties of ranking IFNs. Hence, in this study, these points will be incorporated. Hence, in this current study, the objectives are as follows. The main objective of this work is to develop a method of ordering IFNs. Generally, there exist numerous methods of ordering IFNs. Most of the existing methods generally consider the coherent intuition and the geometry of the IFNs to validate the method. Only in a few cases, the reasonable properties of Wang and Kerre (2001a) are considered as base properties to validate the method. In this paper, a novel method of ordering IFNs, combining the notions of ‘value’ and ‘ambiguity,’ is developed. These quantities are very popular tools in the ranking of IFNs. However, in this work, these quantities are combined differently, that is through an ambiguity inclusion-exclusion function, $\theta$. This function decides whether to include or exclude the ambiguity in the ranking process. Further, the ‘flexibility parameter’ allows the decision-makers to take a decision at levels $(\alpha, \beta)$ of the IFNs. Inclusion of the flexibility parameter in evaluating the value and the ambiguity incorporates the idea of evaluating these quantities for higher $\alpha$ and lower $\beta$, instead of the whole range of integration. The method overcomes all the limitations of the existing methods. Furthermore, newer properties are also suggested in this work. One of the interesting features of the present method is that it successfully and consistently ranks the IFNs and their corresponding images, which were never attempted in any of the existing studies. Apart from this, the method also ranks IFNs that are symmetric about the y-axis.

Apart from this section, the rest of the paper is organized as follows. Section 2 introduces the basic definition of IFN and related definitions essential for the discussion. Section 3 discusses the definitions and theorems of the proposed ranking method. Furthermore, the rationality validation of the proposed ranking method is discussed by proving the Wang and Kerre’s (2001a) reasonable properties on ranking fuzzy quantities. Apart from that, newer properties are also discussed. In Sect. 4, some numerical examples are discussed that highlight some main features of the proposed method. Finally, in Sect. 6, conclusions and the main features of the proposed method are highlighted.

### 2 Preliminaries

In this section, a brief review of related definitions and notations of IFN is being done, which are essential for further study.

**Definition 2.1** (Nayagam et al. 2016) An IFN, $\bar{A}$, in the set of real numbers, $\mathbb{R}$, with membership function and non-
A novel method of ranking intuitionistic fuzzy numbers using value and \( \theta \) multiple of...

As like the fuzzy number, the cut sets of IFN can also be defined for the membership and the non-membership functions. These sets are defined by Atanassov (1999), Li (2010) as follows:

An \((\alpha, \beta)\)-cut set of an IFN, \( \tilde{a} \), is a crisp subset of \( \mathbb{R} \), which is defined as \( \tilde{a}_{\alpha, \beta} = \{ x | \mu_{\tilde{a}}(x) \geq \alpha, v_{\tilde{a}}(x) \leq \beta \} \), where \( 0 \leq \alpha + \beta \leq 1 \), \( \mu_{\tilde{a}} \) and \( v_{\tilde{a}} \) are the membership and the non-membership function of \( \tilde{a} \), respectively. An \( \alpha \)-cut set of an IFN, \( \tilde{a} \), is a crisp subset of \( \mathbb{R} \), which is defined as \( \tilde{a}_{\alpha} = \{ x | \mu_{\tilde{a}}(x) \geq \alpha \} \) where \( 0 \leq \alpha \leq 1 \), \( \mu_{\tilde{a}} \) is the membership function of \( \tilde{a} \). It is clear that \( \tilde{a}_{\alpha} \) represents a closed interval, denoted by \( \tilde{a}_{\alpha} = [L_{\tilde{a}}^\mu(\alpha), R_{\tilde{a}}^\mu(\alpha)] \), which can be calculated using the membership function (Eq. 3) as

\[
\tilde{a}_{\alpha} = [L_{\tilde{a}}^\mu(\alpha), R_{\tilde{a}}^\mu(\alpha)] = [\alpha + \mu(x_0 - a), b - \beta(b - y_0)].
\]

A \( \beta \)-cut set of an IFN, \( \tilde{a} \), is a crisp subset of \( \mathbb{R} \), which is defined as \( \tilde{a}_{\beta} = \{ x | v_{\tilde{a}}(x) \leq \beta \} \), where \( 0 \leq \beta \leq 1 \), \( v_{\tilde{a}} \) is the non-membership function of \( \tilde{a} \). It is clear that \( \tilde{a}_{\beta} \) represents a closed interval, denoted by \( \tilde{a}_{\beta} = [L_{\tilde{a}}^v(\beta), R_{\tilde{a}}^v(\beta)] \), which can be calculated using the non-membership function (Eq. 4) as

\[
\tilde{a}_{\beta} = [L_{\tilde{a}}^v(\beta), R_{\tilde{a}}^v(\beta)] = [x_0 + \beta(c - x_0), y_0 + \beta(d - y_0)].
\]

The notations, fuzzy centers of the membership and the non-membership function are defined as

\[
C_{\tilde{a}}^\mu(\alpha) = \frac{L_{\tilde{a}}^\mu(\alpha) + R_{\tilde{a}}^\mu(\alpha)}{2} \quad \text{and} \quad C_{\tilde{a}}^v(\beta) = \frac{L_{\tilde{a}}^v(\beta) + R_{\tilde{a}}^v(\beta)}{2}
\]

for all \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

The support of an IFN, \( \tilde{a} \), for the membership function is defined as \( \text{supp}(\mu_{\tilde{a}}) = \{ x | \mu_{\tilde{a}}(x) \geq 0 \} \), and the support of the non-membership function is defined as \( \text{supp}(v_{\tilde{a}}) = \{ x | v_{\tilde{a}}(x) \leq 1 \} \). Furthermore, the following notations are used in this study \( L_{\tilde{a}}^\mu(0) = \text{inf} \text{supp}(\mu_{\tilde{a}}) \), \( R_{\tilde{a}}^\mu(0) = \text{sup} \text{supp}(\mu_{\tilde{a}}) \), \( L_{\tilde{a}}^v(1) = \text{inf} \text{supp}(v_{\tilde{a}}) \), \( R_{\tilde{a}}^v(1) = \text{sup} \text{supp}(v_{\tilde{a}}) \).

The image of an IFN, \( \tilde{a} = (\mu_{\tilde{a}}, v_{\tilde{a}}) \), is an IFN given by \( -\tilde{a} = (-\mu_{\tilde{a}}, -v_{\tilde{a}}) \). Thus, if \( \tilde{a}_{\alpha} = [L_{\tilde{a}}^\mu(\alpha), R_{\tilde{a}}^\mu(\alpha)] \) and \( \tilde{a}_{\beta} = [L_{\tilde{a}}^v(\beta), R_{\tilde{a}}^v(\beta)] \) be the \( \alpha \)-cut and \( \beta \)-cut sets of \( \tilde{a} \), respectively, then the \( \alpha \)-cut and \( \beta \)-cut sets of \( -\tilde{a} \) are \( -\tilde{a}_{\alpha} = [-R_{\tilde{a}}^\mu(\alpha), -L_{\tilde{a}}^\mu(\alpha)] \) and \( -\tilde{a}_{\beta} = [-R_{\tilde{a}}^v(\beta), -L_{\tilde{a}}^v(\beta)] \), respectively. An IFN, \( \tilde{a} \), is called symmetric about y-axis if \( -L_{\tilde{a}}^\mu(\alpha) = R_{\tilde{a}}^\mu(\alpha) \) and \( -L_{\tilde{a}}^v(\beta) = R_{\tilde{a}}^v(\beta) \). Furthermore, an IFN, \( \tilde{a} \), is called a crisp-valued IFN, if \( \tilde{a} = (a, a) \), that is \( \tilde{a} = a \in \mathbb{R} \).

The arithmetic operations on IFN are defined by Li (2008), Li (2010) while developing a methodology of ranking IFNs. Furthermore, these arithmetic operations are...
The proposed method of ranking IFN extensively studied by Chakraborty et al. (2015) using different methodology, namely \((\alpha, \beta)\)-cut method, vertex method and extension principle method. The \((\alpha, \beta)\)-cut method of arithmetic operations is adopted in this research work. Let \(\tilde{a} = \langle (a, x_0, y_0, b), (c, x_0, y_0, d) \rangle\) and \(\tilde{b} = \langle (p, m_0, n_0, q), (r, m_0, n_0, s) \rangle\) be two IFNs and \(\lambda \in \mathbb{R} - \{0\}\). Let the \(\alpha\)-cut and \(\beta\)-cut sets of \(\tilde{a}\) and \(\tilde{b}\) be \(\tilde{a}_\alpha = \left[ L^{\mu}_a(\alpha), R^\mu_a(\alpha) \right], \tilde{a}_\beta = \left[ L^\nu_a(\beta), R^\nu_a(\beta) \right]\) and \(\tilde{b}_\alpha = \left[ L^\mu_b(\alpha), R^\mu_b(\alpha) \right], \tilde{b}_\beta = \left[ L^\nu_b(\beta), R^\nu_b(\beta) \right]\), respectively, then the arithmetic operations are defined as

\[
\begin{align*}
\tilde{a} + \tilde{b} &= \left[ L^\mu_a(\alpha) + L^\mu_b(\alpha), R^\mu_a(\alpha) + R^\mu_b(\alpha) \right], \\
\tilde{a} + \tilde{b}_\beta &= \left[ L^\nu_a(\beta) + L^\nu_b(\beta), R^\nu_a(\beta) + R^\nu_b(\beta) \right], \\
\tilde{a} - \tilde{b} &= \left[ L^\mu_a(\alpha) - R^\mu_b(\alpha), R^\mu_a(\alpha) - L^\mu_b(\alpha) \right], \\
\tilde{a} - \tilde{b}_\beta &= \left[ L^\nu_a(\beta) - R^\nu_b(\beta), R^\nu_a(\beta) - L^\nu_b(\beta) \right],
\end{align*}
\]

Eventually, these arithmetic operations on the \((\alpha, \beta)\)-cut are calculated to obtain the following expressions, which were clearly described by Chakraborty et al. (2015).

\[
\begin{align*}
\tilde{a} + \tilde{b} &= \langle a + b, x_0 + m_0, y_0 + n_0, d + r \rangle, \\
\tilde{a} - \tilde{b} &= \langle a - b, x_0 - m_0, y_0 - n_0, d - r \rangle.
\end{align*}
\]

The collection of the IFNs that follows the above-defined arithmetic operations with bounded support and convex IFNs is denoted by the set \(\mathcal{I} \mathcal{F}\).

### 3.1 Definitions and notions related to the proposed method

In this subsection, the basic definitions that are essential to develop the current method are forwarded. Furthermore, a few propositions are also forwarded, which basically help in discussing the properties of the current method.

**Definition 3.1** Let \(\tilde{a} \in \mathcal{I} \mathcal{F}\) with membership and non-membership functions denoted as \(\mu_{\tilde{a}}(x)\) and \(\nu_{\tilde{a}}(x)\), respectively. Let the \(\alpha\)-cut sets and the \(\beta\)-cut sets of membership and non-membership functions of \(\tilde{a}\) be \(\tilde{a}_\alpha = \left[ L^\mu_a(\alpha), R^\mu_a(\alpha) \right]\) and \(\tilde{a}_\beta = \left[ L^\nu_a(\beta), R^\nu_a(\beta) \right]\), respectively. Then, value, \(V_{\alpha, \beta}(\tilde{a})\), and ambiguity, \(A_{\alpha, \beta}(\tilde{a})\), of \(\tilde{a}\) at decision level higher than \(\alpha\) and lower than \(\beta\) are defined as

\[
\begin{align*}
V_{\alpha, \beta}(\tilde{a}) &= \int_0^1 f(r)(R^\mu_a(r) + L^\mu_a(r))dr \\
&+ \int_0^\beta g(r)(R^\nu_a(r) + L^\nu_a(r))dr, \\
A_{\alpha, \beta}(\tilde{a}) &= \int_0^1 f(r)(R^\mu_a(r) - L^\mu_a(r))dr \\
&+ \int_0^\beta g(r)(R^\nu_a(r) - L^\nu_a(r))dr
\end{align*}
\]

where the function, \(f(\alpha)\), is non-negative and non-decreasing function on the interval \([0, 1]\) with \(f(0) = 0\) and \(\int_0^1 f(\alpha)d\alpha = \frac{1}{\tau}\); the function, \(g(\beta)\), is a non-negative and non-increasing function on the interval \([0, 1]\) with \(g(1) = 0\) and \(\int_0^1 g(\beta)d\beta = \frac{1}{\nu}\).

**Proposition 3.1** Let \(\tilde{a}, \tilde{b} \in \mathcal{I} \mathcal{F}\), then

\[
V_{\alpha, \beta}(\tilde{a} + \tilde{b}) = V_{\alpha, \beta}(\tilde{a}) + V_{\alpha, \beta}(\tilde{b}),
\]

and

\[
V_{\alpha, \beta}(\tilde{a} - \tilde{b}) = V_{\alpha, \beta}(\tilde{a}) - V_{\alpha, \beta}(\tilde{b}).
\]

**Proof** Let \(\tilde{a}, \tilde{b} \in \mathcal{I} \mathcal{F}\) with \(\alpha\)-cut sets and the \(\beta\)-cut sets of membership and non-membership functions of \(\tilde{a}\), \(\tilde{b}\) be \(\tilde{a}_\alpha = \left[ L^\mu_a(\alpha), R^\mu_a(\alpha) \right], \tilde{a}_\beta = \left[ L^\nu_a(\beta), R^\nu_a(\beta) \right]\) and \(\tilde{a}_\beta = \left[ L^\nu_b(\beta), R^\nu_b(\beta) \right]\), respectively, then it follows that

\[
\begin{align*}
V_{\alpha, \beta}(\tilde{a} + \tilde{b}) &= \int_0^1 f(r)(R^\mu_a(r) + R^\mu_b(r) + L^\mu_a(r) + L^\mu_b(r))dr \\
&+ \int_0^\beta g(r)(R^\nu_a(r) + R^\nu_b(r) + L^\nu_a(r) + L^\nu_b(r))dr \\
&= \int_0^1 f(r)(R^\mu_a(r) + L^\mu_a(r))dr + \int_0^\beta g(r)(R^\nu_a(r) + L^\nu_a(r))dr
\end{align*}
\]
A novel method of ranking intuitionistic fuzzy numbers using value and \( \theta \) multiple of...
+ \int_0^\beta g(r)(-mL_\alpha^v(r) + mR_\alpha^v(r))dr
\]
\[= m \left[ \int_\alpha^1 f(r)(R_\alpha^b(r) - L_\alpha^b(r))dr \right]
\[+ \int_0^\beta g(r)(R_\alpha^v(r) - L_\alpha^v(r))dr \]
\[= m \mathcal{A}_{\alpha,\beta}(\tilde{a}). \]

Proposition 3.4 Let \( \tilde{a} \in \mathcal{IF} \) and \( -\tilde{a} \in \mathcal{IF} \) be its image, then \( \mathcal{V}_{\alpha,\beta}(-\tilde{a}) = -\mathcal{V}_{\alpha,\beta}(\tilde{a}) \) and \( \mathcal{A}_{\alpha,\beta}(-\tilde{a}) = \mathcal{A}_{\alpha,\beta}(\tilde{a}) \).

Proof The proof follows immediately from Proposition 3.3, taking into account \( k = -1 \) in its proof.

Proposition 3.5 Let \( \tilde{a}, \tilde{b} \in \mathcal{IF} \) such that \( \inf \sup \mu_{\tilde{a}} > \sup \sup \mu_{\tilde{b}} \) and \( \inf \sup \nu_{\tilde{a}} > \sup \sup \nu_{\tilde{b}} \), then \( \mathcal{V}_{\alpha,\beta}(\tilde{a}) > \mathcal{V}_{\alpha,\beta}(\tilde{b}) \).

Proof Let \( \tilde{a}, \tilde{b} \in \mathcal{IF} \) with \( \alpha \)-cut sets and the \( \beta \)-cut sets of membership and non-membership functions of \( \tilde{a} \), \( \tilde{b} \) be
\[\tilde{a}_\alpha = [L_\alpha^\mu(\alpha), R_\alpha^\mu(\alpha)], \tilde{b}_\alpha = [L_\alpha^\mu(\alpha), R_\alpha^\mu(\alpha)]\] and
\[\tilde{a}_\beta = [L_\beta^\nu(\beta), R_\beta^\nu(\beta)], \tilde{b}_\beta = [L_\beta^\nu(\beta), R_\beta^\nu(\beta)],\] respectively. Now, if \( \inf \sup \mu_{\tilde{a}} > \sup \sup \mu_{\tilde{b}} \) and \( \inf \sup \nu_{\tilde{a}} > \sup \sup \nu_{\tilde{b}} \), then \( L_\alpha^\mu(\alpha) > R_\beta^\nu(\beta) \) and \( L_\alpha^\mu(\alpha) > R_\beta^\nu(\beta) \). Thus, it implies that \( R_\alpha^\mu(\alpha) > L_\beta^\nu(\beta) \) and \( L_\alpha^\mu(\alpha) > R_\beta^\nu(\beta) \). Hence, it follows immediately that
\[R_\alpha^\mu(\alpha) + L_\alpha^\mu(\alpha) > R_\beta^\nu(\beta) + L_\beta^\nu(\beta) \]
\[\implies \int_\alpha^1 f(r)(R_\alpha^b(r) + L_\alpha^b(r))dr \]
\[> \int_\alpha^1 f(r)(R_\alpha^b(r) + L_\alpha^b(r))dr \quad (14)\]
and
\[R_\alpha^\mu(\alpha) + L_\alpha^\mu(\alpha) > R_\beta^\nu(\beta) + L_\beta^\nu(\beta) \]
\[\implies \int_0^\beta g(r)(R_\alpha^v(r) + L_\alpha^v(r))dr \]
\[> \int_0^\beta g(r)(R_\alpha^v(r) + L_\alpha^v(r))dr \quad (15)\]

Summing the inequalities 14 and 15, the result follows immediately.

Proposition 3.6 If \( \tilde{a} \in \mathcal{IF} \) such that it is symmetric about the \( y \)-axis, then \( \mathcal{V}_{\alpha,\beta}(\tilde{a}) = 0 \).

Proof Let \( \tilde{a} \in \mathcal{IF} \), then the \( \alpha \)-cut sets and the \( \beta \)-cut sets of membership and non-membership functions of \( \tilde{a} \) be
\[\tilde{a}_\alpha = [L_\alpha^\mu(\alpha), R_\alpha^\mu(\alpha)] \] and \( \tilde{a}_\beta = [L_\beta^\nu(\beta), R_\beta^\nu(\beta)] \), respectively. Since, \( \tilde{a} \) is symmetric about the \( y \)-axis, it follows that
\[-L_\alpha^\mu(\alpha) = R_\alpha^\mu(\alpha) \] and \( -L_\beta^\nu(\beta) = R_\beta^\nu(\beta) \). Thus, it is evident that \( \mathcal{V}_{\alpha,\beta}(\tilde{a}) = 0 \). □

Proposition 3.7 For an arbitrary IFN \( \tilde{a} \in \mathcal{IF} \), \( \mathcal{A}_{\alpha,\beta} \geq 0 \). Further, \( \mathcal{A}_{\alpha,\beta} = 0 \) if \( \tilde{a} \in \mathcal{IF} \) is a crisp-valued IFN.

Proof Let \( \tilde{a} \in \mathcal{IF} \), then the \( \alpha \)-cut sets and the \( \beta \)-cut sets of membership and non-membership functions of \( \tilde{a} \) be \( \tilde{a}_\alpha = [L_\alpha^\mu(\alpha), R_\alpha^\mu(\alpha)] \) and \( \tilde{a}_\beta = [L_\beta^\nu(\beta), R_\beta^\nu(\beta)] \), respectively. As \( R_\alpha^\mu(\alpha) - L_\alpha^\mu(\alpha) \geq 0 \) and \( R_\beta^\nu(\beta) - L_\beta^\nu(\beta) \geq 0 \), it follows that
\[\int_\alpha^1 f(r)(R_\alpha^\mu(\alpha) - L_\alpha^\mu(\alpha))dr + \int_0^\beta g(r)(R_\beta^\nu(\beta) - L_\beta^\nu(\beta))dr \geq 0.\]
Hence, the result \( \mathcal{A}_{\alpha,\beta} \geq 0 \). Further, if \( \tilde{a} \in \mathcal{IF} \) is a crisp-valued IFN, then immediately it follows that \( \mathcal{A}_{\alpha,\beta} = 0 \). □

Corollary 3.1 If \( \tilde{a} = ((a, x_0, y_0), (c, x_0, y_0), d)) \), then for \( 0 \leq a < 1 \) and \( 0 < \beta < 1 \)
\[\mathcal{V}_{\alpha,\beta}(\tilde{a}) = \frac{1}{2} (a + b)(1 - \alpha^2) + \frac{1}{2} (a + b)(1 - \beta^2) \]
\[+ (x_0 + y_0) \left( \beta - \beta^2 + \frac{1}{3} \beta^3 \right) + (c + d) \left( \frac{1}{2} \beta^2 - \frac{1}{3} \beta^3 \right) \]
and
\[\mathcal{A}_{\alpha,\beta}(\tilde{a}) = \frac{1}{2} (b - a)(1 - \alpha^2) - \frac{1}{2} (x_0 - y_0 - a + b)(1 - \alpha^3)\]
\[+ (y_0 - x_0) \left( \beta - \beta^2 + \frac{1}{3} \beta^3 \right) + (d - c) \left( \frac{1}{2} \beta^2 - \frac{1}{3} \beta^3 \right). \]

Proof Let \( \tilde{a} = ((a, x_0, y_0), (c, x_0, y_0), d)) \) be an IFN with membership and non-membership functions denoted by \( \mu_{\tilde{a}}(x) \) and \( \nu_{\tilde{a}}(x) \) and as given in Eqs. 3 and 4, respectively. Let the \( \alpha \)-cut sets and the \( \beta \)-cut sets of the membership and non-membership functions of \( \tilde{a} \) be given by Eqs. 6 and 7, respectively. Choosing \( f(\alpha) \) and \( g(\beta) \) as \( f(\alpha) = \alpha \) and \( g(\beta) = 1 - \beta \), respectively. Then, value, \( \mathcal{V}_{\alpha,\beta}(\tilde{a}) \), and ambiguity, \( \mathcal{A}_{\alpha,\beta}(\tilde{a}) \), of \( \tilde{a} \) at decision level higher than \( \alpha \) and lower than \( \beta \) are obtained as
\[\mathcal{V}_{\alpha,\beta}(\tilde{a}) = \int_\alpha^1 f(r)(L_\alpha^\mu(r) + R_\alpha^\mu(r))dr \]
\[+ \int_0^\beta g(r)(L_\alpha^\mu(r) + R_\alpha^\mu(r))dr \]
\[= \int_\alpha^1 r[a + r(x_0 - a) + b - r(b - y_0)]dr \]
\[+ \int_0^\beta (1 - r)[x_0 + r(c - x_0) + y_0 + r(d - y_0)]dr \]
\[= \frac{1}{2} (a + b)(1 - \alpha^2) + \frac{1}{3} (x_0 + y_0 - a - b)(1 - \alpha^3)\]
\[+ (x_0 + y_0) \left( \beta - \beta^2 + \frac{1}{3} \beta^3 \right). \]
A novel method of ranking intuitionistic fuzzy numbers using value and $\theta$ multiple of...

\[ + (c + d) \left( \frac{1}{2} \beta^2 - \frac{1}{3} \beta^3 \right), \]

and

\[
A_{\alpha, \beta}(\tilde{a}) = \int_{\alpha}^{\beta} f(r)(R^\mu_a(r) - L^\mu_a(r))dr \\
+ \int_{\alpha}^{\beta} g(r)(R^\nu_a(r) - L^\nu_a(r))dr \\
= \int_{\alpha}^{\beta} r[\beta - r(b - y_0) - a - r(x_0 - a)]dr \\
+ \int_{\alpha}^{\beta} (1 - r)[y_0 + r(d - y_0) - x_0 - r(c - x_0)]dr \\
= \frac{1}{2}(b - a)(1 - \alpha^2) - \frac{1}{3}(x_0 - y_0 - a + b)(1 - \alpha^3) \\
+ (y_0 - x_0)\left( \beta - \beta^2 + \frac{1}{3} \beta^3 \right) \\
+ (d - c)\left( \frac{1}{2} \beta^2 - \frac{1}{3} \beta^3 \right),
\]

respectively. \(\square\)

3.2 The proposed method

In this subsection, a novel method of ranking IFNs is developed based on the notion of ‘value’ and ‘ambiguity’ of an IFN. The value and ambiguity of an IFN are as defined in Definition 3.1. Although these quantities were used in ranking IFNs, yet the methods developed previously have some limitations. Hence, an attempt has been made to overcome those limitations in this current study. The proposed method is discussed below.

Let $\tilde{a}, \tilde{b} \in \mathcal{IF}$ with $\alpha$-cut sets and the $\beta$-cut sets of membership and non-membership functions of $\tilde{a}, \tilde{b}$ be $\tilde{a}_\alpha = [L^\mu_a(\alpha), R^\mu_a(\alpha)], \tilde{b}_\alpha = [L^\mu_b(\alpha), R^\mu_b(\alpha)]$ and $\tilde{a}_\beta = [L^\nu_b(\beta), R^\nu_b(\beta)], \tilde{b}_\beta = [L^\nu_a(\beta), R^\nu_a(\beta)],$ respectively. Let $V_{\alpha, \beta}(\tilde{a}), V_{\alpha, \beta}(\tilde{b})$ and $A_{\alpha, \beta}(\tilde{a}), A_{\alpha, \beta}(\tilde{b})$ be the values and ambiguities of $\tilde{a}, \tilde{b},$ respectively, at decision level higher than $\alpha$ ($0 \leq \alpha < 1$) and lower than $\beta$ ($0 < \beta \leq 1$) as defined in Definition 3.1. Then the ranking index, $R_{\alpha, \beta},$ for decision levels higher than $\alpha$ ($0 \leq \alpha < 1$) and lower than $\beta$ ($0 < \beta \leq 1$) is defined as

\[ R_{\alpha, \beta}(\tilde{a}, \theta) = V_{\alpha, \beta}(\tilde{a}) + \theta A_{\alpha, \beta}(\tilde{a}), \tag{18} \]

where $\theta : \mathcal{F} \to \{0, -1, 1\}$ such that

\[
\theta = \begin{cases} 
0, & \text{if } V_{\alpha, \beta}(\tilde{a}) \neq V_{\alpha, \beta}(\tilde{b}) \\
-1, & \text{if } V_{\alpha, \beta}(\tilde{a}) = V_{\alpha, \beta}(\tilde{b}) \text{ and } t_0 \geq 0 \\
1, & \text{if } V_{\alpha, \beta}(\tilde{a}) = V_{\alpha, \beta}(\tilde{b}) \text{ and } t_0 < 0.
\end{cases}
\]

where $t_0 = C^\mu_a$ or $t_0 = C^\nu_a$ or $t_0 = C^\mu_b$ or $t_0 = C^\nu_b$ are the fuzzy centres of $\alpha$-level and $\beta$-level sets as defined in Eq. 8.

The ordering of IFNs, $\tilde{a}, \tilde{b} \in \mathcal{IF},$ based on the ranking index, $R_{\alpha, \beta},$ for decision levels higher than $\alpha$ ($0 \leq \alpha < 1$) and lower than $\beta$ ($0 < \beta \leq 1$) is defined by relations $\succ, \prec$ and $\asymp$ as;

- $\tilde{a} \succ \tilde{b}$ if, and only if, $R_{\alpha, \beta}(\tilde{a}, \theta) > R_{\alpha, \beta}(\tilde{b}, \theta);$  
- $\tilde{a} \prec \tilde{b}$ if, and only if, $R_{\alpha, \beta}(\tilde{a}, \theta) < R_{\alpha, \beta}(\tilde{b}, \theta);$  
- $\tilde{a} \asymp \tilde{b}$ if, and only if, $R_{\alpha, \beta}(\tilde{a}, \theta) = R_{\alpha, \beta}(\tilde{b}, \theta).$

The order relations $\succeq$ and $\preceq$ are formulated as

- $\tilde{a} \succeq \tilde{b}$ if, and only if, $\tilde{a} \succ \tilde{b}$ or $\tilde{a} \asymp \tilde{b}.$
- $\tilde{a} \preceq \tilde{b}$ if, and only if, $\tilde{a} \prec \tilde{b}$ or $\tilde{a} \asymp \tilde{b}.$

A pseudo-code of the proposed method, which will help in better understanding the steps of the method, is as follows:

Let $\tilde{a}, \tilde{b}$ be IFNs.
Evaluate $V_{ij}(\cdot)$ and $A_{ij}(\cdot)$’s.

\[ \text{for } i = 0(0.1)1 \text{ do} \]
\[ \quad \text{for } j = 0(0.1)1 \text{ do} \]
\[ \quad \quad \text{if } V_{ij}(\tilde{a}) \neq V_{ij}(\tilde{b}) \text{ then} \]
\[ \quad \quad \quad \theta \leftarrow 0 \]
\[ \quad \quad \text{else if } V_{ij}(\tilde{a}) = V_{ij}(\tilde{b}) \text{ then} \]
\[ \quad \quad \quad \text{if } t_0 \geq 0 \text{ then} \]
\[ \quad \quad \quad \quad \theta \leftarrow -1 \]
\[ \quad \quad \quad \text{else} \]
\[ \quad \quad \quad \quad \theta \leftarrow 1 \]
\[ \quad \quad \text{end if} \]
\[ \quad \text{end if} \]
\[ \quad R_{ij}(\tilde{a}, \theta) \leftarrow V_{ij}(\tilde{a}) + \theta A_{ij}(\tilde{a}) \]
\[ \quad R_{ij}(\tilde{b}, \theta) \leftarrow V_{ij}(\tilde{b}) + \theta A_{ij}(\tilde{b}) \]
\[ \text{end for} \]
\[ \text{end for} \]

\[ \text{for } i = 0(0.1)1 \text{ do} \]
\[ \quad \text{for } j = 0(0.1)1 \text{ do} \]
\[ \quad \quad \text{if } R_{ij}(\tilde{a}, \theta) > R_{ij}(\tilde{b}, \theta) \text{ then} \]
\[ \quad \quad \quad \tilde{a} \succ \tilde{b} \]
\[ \quad \quad \text{else if } R_{ij}(\tilde{a}, \theta) < R_{ij}(\tilde{b}, \theta) \text{ then} \]
\[ \quad \quad \quad \tilde{a} \prec \tilde{b} \]
\[ \quad \quad \text{else} \]
\[ \quad \quad \quad \tilde{a} \asymp \tilde{b} \]
\[ \quad \quad \text{end if} \]
\[ \quad \text{end for} \]
\[ \text{end for} \]

Next, a few theorems related to the ranking index, $R_{\alpha, \beta},$ are being discussed, which will be helpful in developing and discussing the properties of the current method of ordering IFNs.
Thus, the results follow as

$$R_{\alpha,\beta}(\bar{a} + \bar{b}, \theta) = \alpha, \beta(\bar{a}(\bar{a}) + \beta(\bar{b})).$$

Hence, it follows that

$$R_{\alpha,\beta}(\bar{a} - \bar{b}, \theta) = \alpha, \beta(\bar{a}(\bar{a}) - \beta(\bar{b})).$$

**Proof** Let $\bar{a}, \bar{b} \in \mathcal{IF}$, then it follows from Propositions 3.1 and 3.2 that

$$V_{\alpha,\beta}(\bar{a} + \bar{b}) = V_{\alpha,\beta}(\bar{a}) + V_{\alpha,\beta}(\bar{b})$$

and

$$A_{\alpha,\beta}(\bar{a} + \bar{b}) = A_{\alpha,\beta}(\bar{a}) + A_{\alpha,\beta}(\bar{b}).$$

Thus, the results follow as

$$R_{\alpha,\beta}(\bar{a} + \bar{b}, \theta) = V_{\alpha,\beta}(\bar{a} + \bar{b}) + \alpha, \beta(\bar{a} + \bar{b})
= (V_{\alpha,\beta}(\bar{a}) + \alpha, \beta(\bar{b})) + \alpha, \beta(\bar{a} + \bar{b})
= \alpha, \beta(\bar{a} + \alpha, \beta(\bar{a}) + \beta(\bar{b}))
= \alpha, \beta(\bar{a} + \beta(\bar{b})).$$

Eventually, it is true that $R_{\alpha,\beta}(\bar{a} - \bar{b}, \theta) = R_{\alpha,\beta}(\bar{a} + (\bar{b}, \theta) = R_{\alpha,\beta}(\bar{a}, \theta) + R_{\alpha,\beta}(\bar{b}, \theta).$

**Theorem 3.2** If $\bar{a} = a$, such that $\bar{a}$ is a crisp-valued IFN, then $V_{\alpha,\beta}(\bar{a}) = a(1 - \alpha^2) + 2a \left[ \beta - \frac{1}{2} \beta^2 \right]$ for $0 \leq \alpha < 1$ and $0 < \beta \leq 1.$

**Proof** The proof is trivial.

**Theorem 3.3** If $\bar{a}, \bar{b} \in \mathbb{R} - \{0\}$ be two crisp-valued IFNs such that $\bar{a} = a$ and $\bar{b} = b$, then $\bar{a} \geq \bar{b}$ if, and only if, $a \geq b.$

**Proof** Let $a \geq b$, then $a(1 - \alpha^2) + 2a \left[ \beta - \frac{1}{2} \beta^2 \right] \geq b(1 - \alpha^2) + 2b \left[ \beta - \frac{1}{2} \beta^2 \right]$ for $0 \leq \alpha < 1$ and $0 < \beta \leq 1.$ Thus, $V_{\alpha,\beta}(\bar{a}) \geq V_{\alpha,\beta}(\bar{b})$, which eventually leads to $\tilde{\alpha} \geq \tilde{b}.$ Conversely, let $\bar{a} \geq \bar{b}$, then $V_{\alpha,\beta}(\bar{a}) \geq V_{\alpha,\beta}(\bar{b})$. Thus,

$$a(1 - \alpha^2) + 2a \left[ \beta - \frac{1}{2} \beta^2 \right] \geq b(1 - \alpha^2) + 2b \left[ \beta - \frac{1}{2} \beta^2 \right].$$

Eventually, it leads to the result $a \geq b.$

**Corollary 3.2** If $\bar{a} = 0$, then $V_{\alpha,\beta}(\bar{a}) = A_{\alpha,\beta}(\bar{a}) = 0$ for $0 \leq \alpha, \beta \leq 1.$

**Proof** The proof is trivial.

### 3.3 Properties and validation of the proposed method

In this subsection, the properties of the present method are stated and proved. Furthermore, some reasonable properties of Wang and Kerre (2001a,b) are also stated and proved to check the rationality validation of the proposed ranking method. Furthermore, newer properties that the present method follows are also stated and proved.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathcal{IF}$, then the order relation $\succeq$ satisfies the following properties.

$A_1$: $\tilde{a} \succeq \tilde{a}$

$A_2$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{a} \preceq \tilde{b}$, then $\tilde{a} \sim \tilde{b}$.

$A_3$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \preceq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_4$: If $\inf \sup(\mu_{\tilde{a}}) > \sup \inf(\mu_{\tilde{b}})$ and $\inf \sup(\nu_{\tilde{a}}) > \sup \inf(\nu_{\tilde{b}})$, then $\tilde{a} \succeq \tilde{b}$.

$A_5$: Let $\mathcal{F}$ and $\mathcal{F}_r$ be two arbitrary finite sets of fuzzy quantities in which $R_{\alpha}$ can be applied and $\tilde{a}$ and $\tilde{b}$ are in $\mathcal{F} \cap \mathcal{F}_r$, then the ranking order $\tilde{a} \succeq \tilde{b}$ by $R_{\alpha}$ on $\mathcal{F}_r$ if, and only if, $\tilde{a} \succeq \tilde{b}$ by $R_{\alpha}$ on $\mathcal{F}$.

$A_6$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{c}$.

$A_7$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{c}$.

$A_8$: If $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{c}$, then $\tilde{a} \succeq \tilde{b}$.

$A_9$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_10$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{11}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{12}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{13}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{14}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{15}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{16}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{17}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{18}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{19}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{20}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{21}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{22}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{23}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{24}$: If $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} \succeq \tilde{b}$.

$A_{25}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

$A_{26}$: If $\tilde{a} \succeq \tilde{b}$ and $\tilde{b} \succeq \tilde{c}$, then $\tilde{a} \succeq \tilde{c}$.

Hereafter, these reasonable properties of the proposed method are examined and proved by the theorems stated below.
Theorem 3.4 Let \( \tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{IF} \), then the relations \( > \) and \( \sim \) satisfy the axioms for the order relations given below:

1. \( \tilde{a} \geq \tilde{a} \) (reflexivity).
2. If \( \tilde{a} > \tilde{b} \) and \( \tilde{b} > \tilde{c} \), then \( \tilde{a} > \tilde{c} \) (transitivity). The same holds for the order relation \( \geq \).
3. \( \tilde{a} > \tilde{b} \) or \( \tilde{b} > \tilde{a} \) (law of trichotomy).
4. \( \tilde{a} = \tilde{b} \) if and only if \( \tilde{a} \sim \tilde{b} \).

Proof The proof of the above statements is as follows.

(1) The proofs of this statement is followed immediately.

(2) Here, \( \tilde{a} > \tilde{b} \) happens, either for \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \) or \( R_{a,b}(\tilde{a}, \pm 1) > R_{a,b}(\tilde{b}, \pm 1) \). Further, \( \tilde{b} > \tilde{c} \) happens, either for \( R_{a,b}(\tilde{b}, 0) > R_{\alpha,\beta}(\tilde{c}, 0) \) or \( R_{a,b}(\tilde{b}, \pm 1) > R_{\alpha,\beta}(\tilde{c}, \pm 1) \). Taking into account these situation, the following four cases arise.

Case 1: Consider that \( \tilde{a} > \tilde{b} \) and \( \tilde{b} > \tilde{c} \) happens for \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \) and \( R_{a,b}(\tilde{b}, \pm 1) > R_{a,b}(\tilde{c}, \pm 1) \). Eventually, \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \). Hence, the result.

Case 2: Consider that \( \tilde{a} > \tilde{b} \) and \( \tilde{b} > \tilde{c} \) happens for \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \) and \( R_{a,b}(\tilde{b}, \pm 1) > R_{a,b}(\tilde{c}, \pm 1) \). Eventually, \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \). Hence, the result.

Case 3: Consider that \( \tilde{a} > \tilde{b} \) and \( \tilde{b} > \tilde{c} \) happens for \( R_{a,b}(\tilde{a}, \pm 1) > R_{a,b}(\tilde{b}, \pm 1) \) and \( R_{a,b}(\tilde{b}, \pm 1) > R_{a,b}(\tilde{c}, \pm 1) \). Eventually, \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \). Hence, the result.

Case 4: Consider that \( \tilde{a} > \tilde{b} \) and \( \tilde{b} > \tilde{c} \) happens for \( R_{a,b}(\tilde{a}, \pm 1) > R_{a,b}(\tilde{b}, \pm 1) \) and \( R_{a,b}(\tilde{b}, \pm 1) > R_{a,b}(\tilde{c}, \pm 1) \). Eventually, \( R_{a,b}(\tilde{a}, 0) > R_{a,b}(\tilde{b}, 0) \). Hence, the result.

(3) This statement is followed immediately, as the order relations \( > \) and \( \sim \) particularly based on order relation \( > \) and \( \geq \) of real numbers.

(4) If \( \tilde{a} = \tilde{b} \), then \( R_{a,b}(\tilde{a}, \theta) = R_{a,b}(\tilde{b}, \theta) \). Thus, the statement is followed.

\[ \square \]

Theorem 3.5 Let \( \tilde{a}, \tilde{b} \in \mathcal{IF} \) such that \( \inf\sup(\mu_{\tilde{a}}) > \sup\sup(\mu_{\tilde{b}}) \) and \( \inf\sup(\nu_{\tilde{a}}) > \sup\sup(\nu_{\tilde{b}}) \), then \( \tilde{a} \geq \tilde{b} \).

Proof Let, \( \inf\sup(\mu_{\tilde{a}}) > \sup\sup(\mu_{\tilde{b}}) \) and \( \inf\sup(\nu_{\tilde{a}}) > \sup\sup(\nu_{\tilde{b}}) \), then by Proposition 3.5 \( V_{a,b}(\tilde{a}) > V_{a,b}(\tilde{b}) \) for all \( 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \). Hence, \( \theta = 0 \), which follows that \( \mu > v \), in fact by definition of \( \geq, \tilde{a} \geq \tilde{b} \).

\[ \square \]
Proof Let $\tilde{a} \geq \tilde{b}$, then $R_{\alpha, \beta}(\tilde{a}, \theta) \geq R_{\alpha, \beta}(\tilde{b}, \theta)$. Let $k > 0$, then using Proposition 3.3, it follows that

$$R_{\alpha, \beta}(k\tilde{a}, \theta) = R_{\alpha, \beta}(k\tilde{a}) + \theta A_{\alpha, \beta}(k\tilde{a})$$

$$= k \left[ R_{\alpha, \beta}(\tilde{a}) + \theta A_{\alpha, \beta}(\tilde{a}) \right]$$

$$= kR_{\alpha, \beta}(\tilde{a}, \theta).$$

Thus, when $\tilde{a} \geq \tilde{b}$, it follows that $R_{\alpha, \beta}(\tilde{a}, \theta) + \theta A_{\alpha, \beta}(\tilde{a}) \geq R_{\alpha, \beta}(\tilde{b}, \theta) + \theta A_{\alpha, \beta}(\tilde{b})$. Equivalently, it follows that $k[R_{\alpha, \beta}(\tilde{a}) + \theta A_{\alpha, \beta}(\tilde{a})] \geq k \left[ R_{\alpha, \beta}(\tilde{b}) + \theta A_{\alpha, \beta}(\tilde{b}) \right]$, which can be trivially expressed as $R_{\alpha, \beta}(k\tilde{a}) + \theta A_{\alpha, \beta}(k\tilde{a}) \geq R_{\alpha, \beta}(k\tilde{b}) + \theta A_{\alpha, \beta}(k\tilde{b})$ by using the results from Proposition 3.3. Thus, the result, $k\tilde{a} \geq k\tilde{b}$, follows immediately.

Let $k < 0$, assume $k = -m < 0$, then the following cases arise.

Case 1: Let $\tilde{a} \geq \tilde{b}$ for $\theta = 0$, then $R_{\alpha, \beta}(\tilde{a}) \neq R_{\alpha, \beta}(\tilde{b})$. Further, $R_{\alpha, \beta}(\tilde{a}, \theta) \geq R_{\alpha, \beta}(\tilde{b}, \theta)$ for $R_{\alpha, \beta}(\tilde{a}) + 0 \cdot A_{\alpha, \beta}(\tilde{a}) \geq R_{\alpha, \beta}(\tilde{b}) + 0 \cdot A_{\alpha, \beta}(\tilde{b})$. Clearly, $V_{\alpha, \beta}(-m\tilde{a}) \neq V_{\alpha, \beta}(-m\tilde{b})$ and $V_{\alpha, \beta}(-m\tilde{a}) \leq V_{\alpha, \beta}(-m\tilde{b})$ as $V_{\alpha, \beta}(\tilde{a}) + 0 \cdot A_{\alpha, \beta}(\tilde{a}) \geq V_{\alpha, \beta}(\tilde{b}) + 0 \cdot A_{\alpha, \beta}(\tilde{b})$. Thus, $R_{\alpha, \beta}(-m\tilde{a}, \theta) \geq R_{\alpha, \beta}(-m\tilde{b}, \theta)$. Hence, the result.

Case 2: Let $-m\tilde{a} \geq -m\tilde{b}$ for $\theta = \pm 1$, then $V_{\alpha, \beta}(-m\tilde{a}) = V_{\alpha, \beta}(-m\tilde{b})$. Further, $R_{\alpha, \beta}(-m\tilde{a}, \theta) \geq R_{\alpha, \beta}(-m\tilde{b}, \theta)$ for $V_{\alpha, \beta}(-m\tilde{a}) + 0 \cdot A_{\alpha, \beta}(-m\tilde{a}) \geq V_{\alpha, \beta}(-m\tilde{b}) + 0 \cdot A_{\alpha, \beta}(-m\tilde{b})$. Clearly, $V_{\alpha, \beta}(-m\tilde{a}) \neq V_{\alpha, \beta}(-m\tilde{b})$ and $V_{\alpha, \beta}(-m\tilde{a}) \geq V_{\alpha, \beta}(-m\tilde{b})$ as $V_{\alpha, \beta}(\tilde{a}) + 0 \cdot A_{\alpha, \beta}(\tilde{a}) \leq V_{\alpha, \beta}(\tilde{b}) + 0 \cdot A_{\alpha, \beta}(\tilde{b})$. Thus, $R_{\alpha, \beta}(-m\tilde{a}, \theta) \geq R_{\alpha, \beta}(-m\tilde{b}, \theta)$. Hence, the result.

Theorem 3.13 Let $\tilde{a}, \tilde{b} \in \mathcal{I}(\mathcal{F})$ and $k \in \mathbb{R} - \{0\}$. If $\tilde{a} \geq \tilde{b}$, then $\tilde{a} > \tilde{b}$ if $k > 0$, and $\tilde{a} < \tilde{b}$ if $k < 0$.

Proof The proof is very trivial by taking into account ‘>’ in the proof of Theorem 3.11.

Theorem 3.14 Let $\tilde{a}, \tilde{b} \in \mathcal{I}(\mathcal{F})$ and $k \in \mathbb{R} - \{0\}$. If $\tilde{k} \tilde{a} > \tilde{k} \tilde{b}$, then $\tilde{a} > \tilde{b}$ if $k > 0$, and $\tilde{a} < \tilde{b}$ if $k < 0$.

Proof The proof is very trivial by taking in account ‘>’ in the proof of Theorem 3.12.

Theorem 3.15 Let $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{I}(\mathcal{F})$. If $\tilde{a} \geq \tilde{b}$, then $\tilde{a} - \tilde{c} \geq \tilde{b} - \tilde{c}$.

Proof To prove this theorem, a claim has to be made on invariance of $\theta$ in ordering $\tilde{a}, \tilde{b}$ and $\tilde{a} - \tilde{c}, \tilde{b} - \tilde{c}$. The claim is as follows:

Claim: The value of $\theta$ in ordering $\tilde{a}$ and $\tilde{b}$ is invariant in ordering $\tilde{a} - \tilde{c}$ and $\tilde{b} - \tilde{c}$. The proof of the claim is as follows: Let $\theta = 0$ in ordering $\tilde{a}$ and $\tilde{b}$. Hence, $V_{\alpha, \beta}(\tilde{a}) \neq V_{\alpha, \beta}(\tilde{b})$. Taking into account the proof of Theorem 3.15 and the definition of $\geq$, the result follows immediately.
A novel method of ranking intuitionistic fuzzy numbers using value and \( \theta \) multiple of…

**Proof** Here, \( \tilde{a} > \tilde{b} \) happens, either for \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \) or \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, \pm 1) \). Further, \( \tilde{c} > \tilde{d} \) happens, either for \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, 0) \) or \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, \pm 1) \). Taking into account these situation, the following four cases arise.

Case 1: Let \( \tilde{a} > \tilde{b} \) and \( \tilde{c} > \tilde{d} \) for \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \) and \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, 0) \), respectively. Eventually, \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \). Hence, it follows that \( \tilde{a} + \tilde{c} > \tilde{b} + \tilde{d} \).

Case 2: Let \( \tilde{a} > \tilde{b} \) and \( \tilde{c} > \tilde{d} \) for \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \) and \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, 0) \), respectively. Evidently, \( V_{a,\beta}(\tilde{a}) > V_{a,\beta}(\tilde{b}) \) and \( V_{a,\beta}(-\tilde{c}) = V_{a,\beta}(-\tilde{d}) \), which implies \( V_{a,\beta}(\tilde{a} + \tilde{c}) > V_{a,\beta}(\tilde{b} + \tilde{d}) \). Eventually, \( R_{a,\beta}(\tilde{a} + \tilde{c}, 0) > R_{a,\beta}(\tilde{b} + \tilde{d}, 0) \). Hence, it follows that \( \tilde{a} + \tilde{c} > \tilde{b} + \tilde{d} \).

Case 3: Let \( \tilde{a} > \tilde{b} \) and \( \tilde{c} > \tilde{d} \) for \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \) and \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, 0) \), respectively. Evidently, \( V_{a,\beta}(\tilde{a}) > V_{a,\beta}(\tilde{b}) \) and \( V_{a,\beta}(\tilde{c}) > V_{a,\beta}(\tilde{d}) \), which implies \( V_{a,\beta}(\tilde{a} + \tilde{c}) > V_{a,\beta}(\tilde{b} + \tilde{d}) \). Eventually, \( R_{a,\beta}(\tilde{a} + \tilde{c}, 0) > R_{a,\beta}(\tilde{b} + \tilde{d}, 0) \). Hence, it follows that \( \tilde{a} + \tilde{c} > \tilde{b} + \tilde{d} \).

Case 4: Let \( \tilde{a} > \tilde{b} \) and \( \tilde{c} > \tilde{d} \) for \( R_{a,\beta}(\tilde{a}, 0) > R_{a,\beta}(\tilde{b}, 0) \) and \( R_{a,\beta}(\tilde{c}, 0) > R_{a,\beta}(\tilde{d}, 0) \), respectively. Eventually, \( R_{a,\beta}(\tilde{a} + \tilde{c}, 0) > R_{a,\beta}(\tilde{b} + \tilde{d}, 0) \). Hence, it follows that \( \tilde{a} + \tilde{c} > \tilde{b} + \tilde{d} \).

\[ \Box \]

**Theorem 3.22** Let \( \tilde{a}, \tilde{b} \in \mathcal{F} \) and symmetric about y-axis; if \( \tilde{a} > \tilde{b} \), then \(-\tilde{a} > -\tilde{b}\).

**Proof** Taking into account the proof of Theorem 3.20 and the definition of \( \succeq \), the result follows immediately. \( \Box \)

**4 Comparative numerical study**

In this section, a thorough comparative study is done to highlight the performance of the current method of ranking IFNs. The numerical study is conducted considering a few sets of IFNs. The numerical study depicts the outperformance of the current method over the existing methods.

**Example 4.1** Let \( \tilde{a} = (1, 3, 3, 5), (1, 3, 3, 5) \) and \( \tilde{b} = (2, 3, 3, 4), (2, 3, 3, 4) \) be two IFNs. The ordering of the IFNs by various methods is depicted in Table 1. Ye (2011) fails to distinguish these distinct IFNs, as well as their corresponding images, as the ordering of the IFNs and their corresponding images are \( \tilde{a} \sim \tilde{b} \) and \( -\tilde{a} \sim -\tilde{b} \), respectively. Nayagam et al. (2016) ordering is illogical and depicts inconsistency in ordering the IFNs and their corresponding images, as the ordering of the IFNs and their corresponding images are \( \tilde{a} \succ \tilde{b} \) and \( -\tilde{a} \prec -\tilde{b} \), respectively. Chutia and Saikia’s (2018) limitation is depicted in ordering this pair of IFNs as \( \tilde{a} \sim \tilde{b} \) and \( -\tilde{a} \sim -\tilde{b} \), respectively. Darehmiraki (2019) ordering is based on the difference of the areas on the left side of the membership function and non-membership function. This method depicts indistinguishable criteria for this distinct pair of IFNs as \( \tilde{a} \sim \tilde{b} \) and \( -\tilde{a} \sim -\tilde{b} \) at all levels of decision-making, except for the decision levels \((0.9, 0.1), (0.5, 0.5)\) and \((0.1, 0.9)\). However, the ordering of the IFNs and their corresponding images are consistent by the current method as \( \tilde{a} \prec \tilde{b} \) and \( -\tilde{a} \succ -\tilde{b} \) at all levels of decision-making. Hence, the current method is logical and consistent.

**Example 4.2** Let \( \tilde{a} = (1, 4, 4, 5), (1, 4, 4, 5) \) and \( \tilde{b} = (2, 3, 3, 6), (2, 3, 3, 6) \) be two IFNs. The ordering of the IFNs by various methods is depicted in Table 2. Ye (2011) fails to distinguish these distinct IFNs, as well as their corresponding images, as the ordering of the IFNs and their corresponding images are \( \tilde{a} \sim \tilde{b} \) and \( -\tilde{a} \sim -\tilde{b} \), respectively.
respectively. Nayagam et al. (2016) ordering is illogical and depicts inconsistency in ordering the IFNs and their corresponding images, as the ordering of the IFNs and their corresponding images are $\tilde{a} \succ \tilde{b}$ and $-\tilde{a} \succ -\tilde{b}$, respectively. Chutia and Saikia’s (2018) ordering is based on the quantity ‘value’, hence it ranks consistently the IFNs and their corresponding images in the order $\tilde{a} \prec \tilde{b}$ and $-\tilde{a} \prec -\tilde{b}$, respectively, for all levels of decision-making. Darehmiraki (2019) ordering is based on the difference of the areas of the membership function and non-membership function, hence the ordering of the IFNs is $\tilde{a} \sim \tilde{b}$; and their corresponding images is $-\tilde{a} \sim -\tilde{b}$ at decision levels $(\alpha, \beta) = (0.9, 0.1), (0.5, 0.5), (0.1, 0.9)$. However, at other decision levels Darehmiraki (2019) method depicts consistency since it ranks the IFNs as $\tilde{a} \prec \tilde{b}$; and their corresponding images as $-\tilde{a} \prec -\tilde{b}$. Ordering of the IFNs is consistent and efficient in ranking the IFNs and their corresponding images by the current approach and tally with the results of Darehmiraki (2019), except at decision levels $(\alpha, \beta) = (0.9, 0.1), (0.5, 0.5), (0.1, 0.9)$. Hence, the current method is logical and consistent.

**Example 4.3** Consider the crisp-valued IFNs $\tilde{a} = 1$ and $\tilde{b} = 2$. The ordering of the IFNs by various methods is depicted in Table 3. The ordering by Ye (2011) of the IFNs and the corresponding images are $\tilde{a} \prec \tilde{b}$ and $-\tilde{a} \prec -\tilde{b}$, which is logical. Moreover, Nayagam et al. (2016) ordering of the IFNs and the corresponding images are $\tilde{a} \prec \tilde{b}$ and $-\tilde{a} \prec -\tilde{b}$, which depict inconsistency. Darehmiraki (2019) preference at $(\alpha, \beta) = (0.9, 0.1), (0.5, 0.5)$ and $(0.1, 0.9)$ for the IFNs and their corresponding images are $\tilde{a} \sim \tilde{b}$ and $-\tilde{a} \sim -\tilde{b}$.

| Methods | $\tilde{a}$ | $\tilde{b}$ | $-\tilde{a}$ | $-\tilde{b}$ | Decision result |
|---------|------------|------------|-------------|-------------|-----------------|
| Ye (2011) | 3.5000 | 3.5000 | -3.5000 | -3.5000 | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
| Chutia and Saikia (2018) | 3.0000 | 3.0000 | -3.0000 | -3.0000 | $\tilde{a} \prec \tilde{b}$, $-\tilde{a} \prec -\tilde{b}$ |
| Nayagam et al. (2016) | 10.000 | 9.2500 | 10.000 | 9.2500 | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |

| Methods | $\tilde{a}$ | $\tilde{b}$ | $-\tilde{a}$ | $-\tilde{b}$ | Decision result |
|---------|------------|------------|-------------|-------------|-----------------|
| Ye (2011) | 3.5000 | 3.5000 | -3.5000 | -3.5000 | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
| Chutia and Saikia (2018) | 3.0000 | 3.0000 | -3.0000 | -3.0000 | $\tilde{a} \prec \tilde{b}$, $-\tilde{a} \prec -\tilde{b}$ |
| Nayagam et al. (2016) | 10.000 | 9.2500 | 10.000 | 9.2500 | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
respectively. At other preference levels, this method depicts mixed and inconsistent decision. Thus, this approach fails to rank crisp-valued IFNs. Chutia and Saikia’s (2018) ranking is based on the ‘values’ calculated at various levels of α and β; hence, this approach depicts consistency in ranking the IFNs and their corresponding images, as the ordering are \( \tilde{a} < \tilde{b} \) and \(-\tilde{a} > -\tilde{b}\). In the current method \( V_{\alpha,\beta}(\tilde{a}) \neq V_{\alpha,\beta}(\tilde{b}) \); hence, \( \theta = 0 \), which suggest the decision-maker to make preference based on ’value.’ Hence, the current method consistently ranks the IFNs and their corresponding images by preferring \( \tilde{b} \) to \( \tilde{a} \) and \(-\tilde{a} \) to \(-\tilde{b}\), respectively. 

**Example 4.4** Consider the IFNs \( \tilde{a} = (\langle -1, 0, 0, 1 \rangle, \langle -2, 0, 0, 2 \rangle) \) and \( \tilde{b} = (\langle -3, 0, 0, 3 \rangle, \langle -3, 0, 0, 3 \rangle) \), which are symmetric about the y-axis. As discussed in Theorem 3.20 and 3.22, the ordering of the IFNs and their corresponding images are \( \tilde{a} > \tilde{b} \) and \(-\tilde{a} > -\tilde{b}\), respectively. The ordering by Chutia and Saikia (2018) tallies with the current approach. However, Ye (2011) and Darehmiraki (2019) fail to distinguish this distinct pair of IFNs. Further, Nayagam et al. (2016) ordering is illogical as the preference is given to the IFN \( \tilde{b} \), which has more ambiguity than \( \tilde{a} \). A comparison of the decision made by the various methods is depicted in Table 4.

### 5 Application of the proposed method in risk analysis

The proposed ranking method for IFNs has been demonstrated, in this section, as a supplementary tool for the risk analysis problem. Schmucke (1984) first discussed the risk analysis problem in a fuzzy environment using the parameters’ probability of failure and severity of loss. It was claimed that because these parameters are imprecise, expressing them as linguistic terms such as high, low, medium, and so on are much more justified. These parameters, in turn, can be expressed as fuzzy numbers. There is a lot of research that uses these parameters as fuzzy numbers in risk analysis. Some such types of studies in risk analysis problems using different types of fuzzy numbers are Zhang (1986), Chen (1996), Chen and Chen (2008, 2009), Wei and Chen (2009), Chen et al. (2012), Zhu and Xu (2012), De (2020, 2018), Bhattacharya and De (2020), De and Beg (2016), De and Mahata (2019), Uluçay et al. (2019), Mondal et al. (2019) and Ye (2011).

### 5.1 Fuzzy risk analysis

Fuzzy risk analysis problems invite linguistic variables as parameters, which can then be expressed as IFNs. A fuzzy risk analysis in a production system is presented here. The proposed method of ranking IFNs is used to order the risk in the production systems. As a result, the highest risk to the production systems can be identified during decision-making.

Consider \( n \) production systems \( C_i, 1 \leq i \leq n \). Each production system consisting of \( p \) sub-components \( A_{ik}, 1 \leq k \leq p \). The sub-component \( A_{ik} \) is assessed by two parameters probability of failure \( R_{ik} \) and severity of loss \( W_{ik} \), which are linguistic terms where \( 1 \leq k \leq p \) and \( 1 \leq i \leq n \). The structure of risk analysis under a fuzzy environment is depicted by the logical diagram shown in Fig. 1 (Schmucke 1984). The following steps are necessary for a logical risk analysis under a fuzzy environment.

**Step (1)** For each sub-component \( A_{ik} \), consider the probability of failure \( R_{ik} \) and severity of loss \( W_{ik} \) in

### Table 3 Ranking of IFNs in Examples 4.3

| Methods                          | \( \tilde{a} \)   | \( \tilde{b} \)   | \( -\tilde{a} \) | \( -\tilde{b} \) | Decision result                      |
|----------------------------------|------------------|------------------|-----------------|-----------------|-------------------------------------|
| Ye (2011)                        | 1.0000           | 2.0000           | -1.0000         | -2.0000         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| Chutia and Saikia (2018)         |                  |                  |                 |                 |                                      |
| \( (\alpha, \beta) = (0.9, 0.1) \) | 0.7600           | 1.5200           | -0.7600         | -1.5200         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| \( (\alpha, \beta) = (0.5, 0.5) \) | 3.0000           | 6.0000           | -3.0000         | -6.0000         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| \( (\alpha, \beta) = (0.1, 0.9) \) | 3.9600           | 7.9200           | -3.9600         | -7.9200         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| Nayagam et al. (2016)            | 1.0000           | 4.0000           | 1.0000          | 4.0000          | \( \tilde{a} < \tilde{b}, -\tilde{a} < -\tilde{b} \) |
| Darehmiraki (2019)               |                  |                  |                 |                 |                                      |
| \( (\alpha, \beta) = (0.9, 0.1) \) | 0.0000           | 0.0000           | 0.0000          | 0.0000          | \( \tilde{a} \sim \tilde{b}, -\tilde{a} \sim -\tilde{b} \) |
| \( (\alpha, \beta) = (0.5, 0.5) \) | 0.0000           | 0.0000           | 0.0000          | 0.0000          | \( \tilde{a} \sim \tilde{b}, -\tilde{a} \sim -\tilde{b} \) |
| \( (\alpha, \beta) = (0.1, 0.9) \) | 0.0000           | 0.0000           | 0.0000          | 0.0000          | \( \tilde{a} \sim \tilde{b}, -\tilde{a} \sim -\tilde{b} \) |
| Current method                   |                  |                  |                 |                 |                                      |
| \( (\alpha, \beta) = (0.9, 0.1) \) | 0.3800           | 0.7600           | -0.3800         | -0.7600         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| \( (\alpha, \beta) = (0.5, 0.5) \) | 1.5000           | 3.0000           | -1.5000         | -3.0000         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
| \( (\alpha, \beta) = (0.1, 0.9) \) | 1.9800           | 3.9600           | -1.9800         | -3.9600         | \( \tilde{a} < \tilde{b}, -\tilde{a} > -\tilde{b} \) |
linguistic terms such as low, medium, high, etc. where 1 ≤ k ≤ p and 1 ≤ i ≤ n, n is the number of production systems and p is the number of sub-components in each of the production systems.

Step (2) Obtain the total risk $\tilde{R}_i$ of the production system $C_i$ integrating $\tilde{R}_k$ and $\tilde{W}_{ik}$ of each sub-component $A_{ik}$ using the fuzzy weighted mean method. The total risk $\tilde{R}_i$ is given as

$$\tilde{R}_i = \frac{\sum_{k=1}^{p} \tilde{W}_{ik} \otimes \tilde{R}_k}{\sum_{k=1}^{p} \tilde{W}_{ik}}$$

(19)

Step (3) Obtain the values, $V_{a,\beta}(\tilde{R}_i)$, and the ambiguities, $A_{a,\beta}(\tilde{R}_i)$, at decision levels higher than $a$ and lower than $\beta$ for each of the production systems.

Step (4) Rank the $\tilde{R}_i$’s using the proposed ranking method. The largest $\mathcal{R}_{a,\beta}(\tilde{R}_i, \theta)$ for decision levels higher than $a$ and lower than $\beta$ will have high risk of probability of failure.

5.2 A case study

In recent years, poultry farming has played a significant role in raising people’s living standards by alleviating poverty and creating job opportunities. India has made tremendous strides in broiler production over the last few decades. Assam is a state in India’s north-eastern region that has seen little industrialization. As a result, unemployment has become a major issue in recent years, particularly in rural Assam. To evaluate the risk in poultry farming under different constraints, a case study of fuzzy risk analysis on poultry farming in rural Assam was conducted (Chutia and Saikia 2018) (Table 6).

Consider three farmers $C_i$, $i = 1, 2, 3$ start independently poultry farms restricted to different constraints, such that farmer $C_1$ is financially sound and experienced, $C_2$ has minimum capital and has no financial support from government and $C_3$ is financially sound whereas inexperienced. Eight different sub-components $A_{ik}$, $k = 1, 2, \ldots, 8$ are identified, which play a major role in poultry farming. Namely, $A_{i1}$: availability of land, $A_{i2}$: financial support, $A_{i3}$: availability of expert laborer, $A_{i4}$: availability of clean water, $A_{i5}$: transportation, $A_{i6}$: availability of electricity, $A_{i7}$: food

Table 4 Ranking of IFNs in Examples 4.4

| Methods                        | $\tilde{a}$ | $\tilde{b}$ | $-\tilde{a}$ | $-\tilde{b}$ | Decision result |
|--------------------------------|-------------|-------------|--------------|--------------|----------------|
| Ye (2011)                      | 0.0000      | 0.0000      | 0.0000       | 0.0000       | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
| Chutia and Saikia (2018)       | (0.9, 0.1)  | 0.0560      | 0.0933       | 0.0560       | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
|                               | (0.5, 0.5)  | 1.0000      | 1.6667       | 1.0000       | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
|                               | (0.1, 0.9)  | 1.9440      | 3.2400       | 1.9440       | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
| Nayagam et al. (2016)          | 0.5000      | 1.5000      | 0.5000       | 1.5000       | $\tilde{a} < \tilde{b}$, $-\tilde{a} < -\tilde{b}$ |
| Darehmiraki (2019)             | (0.9, 0.1)  | 0.0000      | 0.0000       | 0.0000       | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
|                               | (0.5, 0.5)  | 0.0000      | 0.0000       | 0.0000       | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
|                               | (0.1, 0.9)  | 0.0000      | 0.0000       | 0.0000       | $\tilde{a} \sim \tilde{b}$, $-\tilde{a} \sim -\tilde{b}$ |
| Current method                 | (0.9, 0.1)  | -0.1183     | -0.2273      | -0.1183      | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
|                               | (0.5, 0.5)  | -0.7917     | -1.4167      | -0.7917      | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
|                               | (0.1, 0.9)  | -1.3050     | -2.2860      | -1.3050      | $\tilde{a} > \tilde{b}$, $-\tilde{a} > -\tilde{b}$ |
Table 5  A 9-member linguistic term set (Ye 2011)

| Linguistic term | IFNs                                                        |
|-----------------|-------------------------------------------------------------|
| Absolutely-low  | ⟨(0.00, 0.00, 0.00, 0.00), (0.00, 0.00, 0.00, 0.00)⟩          |
| Very-low        | ⟨(0.00, 0.00, 0.02, 0.07), (0.00, 0.00, 0.02, 0.07)⟩          |
| Low             | ⟨(0.04, 0.10, 0.18, 0.23), (0.04, 0.10, 0.18, 0.23)⟩          |
| Fairly-low      | ⟨(0.17, 0.22, 0.36, 0.42), (0.17, 0.22, 0.36, 0.42)⟩          |
| Medium          | ⟨(0.32, 0.41, 0.58, 0.65), (0.32, 0.41, 0.58, 0.65)⟩          |
| Fairly-high     | ⟨(0.58, 0.63, 0.80, 0.86), (0.58, 0.63, 0.80, 0.86)⟩          |
| High            | ⟨(0.72, 0.78, 0.92, 0.97), (0.72, 0.78, 0.92, 0.97)⟩          |
| Very-high       | ⟨(0.93, 0.98, 1.00, 1.00), (0.93, 0.98, 1.00, 1.00)⟩          |
| Absolutely-high | ⟨(1.00, 1.00, 1.00, 1.00), (1.00, 1.00, 1.00, 1.00)⟩          |

supply and \(A_{i8}\): good poultry baby (Chutia 2017). These sub-components are the major factors that should be considered for a productive result in a poultry farm. As previously discussed, the probabilistic values of the sub-components \(A_{11}, A_{12}, \ldots, A_{i8}\) are imprecise, leading to their classification as IFNs. The sub-components are evaluated using two parameters: failure probability and severity of loss. The linguistic terms assigned are depicted in Table 6. The corresponding IFNs of the linguistic variables are depicted in Table 5 (Ye 2011).

As a result, under such conditions, which farmer will be at the greatest risk of failure? In such cases, the proposed method can be successfully applied to determine the highest risk of failure in production systems.

The total risk \(\tilde{R}_i\) of probability of failure for the farmers \(C_i\) where \(1 \leq i \leq 3\) are obtained by using Eq. 19 and taking the parameters from Table 6. Hence, the total risk \(\tilde{R}_i\) for each of the farmers \(C_i\) where \(1 \leq i \leq 3\) are given by

\[
\tilde{R}_1 = \frac{\sum_{k=1}^{8} \tilde{W}_{1k} \otimes \tilde{R}_{1k}}{\sum_{k=1}^{8} \tilde{W}_{1k}} = \langle(0.0057, 0.0234, 0.1466, 0.4163),
(0.0057, 0.0234, 0.1466, 0.4163)\rangle,
\]

\[
\tilde{R}_2 = \frac{\sum_{k=1}^{8} \tilde{W}_{2k} \otimes \tilde{R}_{2k}}{\sum_{k=1}^{8} \tilde{W}_{2k}} = \langle(0.1935, 0.2603, 0.5293, 0.7067),
(0.1935, 0.2603, 0.5293, 0.7067)\rangle,
\]

\[
\tilde{R}_3 = \frac{\sum_{k=1}^{8} \tilde{W}_{3k} \otimes \tilde{R}_{3k}}{\sum_{k=1}^{8} \tilde{W}_{3k}}
\]

Table 6  Linguistic terms of \(\tilde{R}_{ik}\) and \(\tilde{W}_{ik}\) for the sub-components \(A_{ik}\)

| Farmer \(C_i\) | Sub-component \(A_{ik}\) | Linguistic value of \(\tilde{R}_{ik}\) | Linguistic value of \(\tilde{W}_{ik}\) |
|----------------|--------------------------|-------------------------------------|-------------------------------------|
| \(C_1\)       | \(A_{11}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{12}\)               | Absolutely-low                      | Very-low                            |
|                | \(A_{13}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{14}\)               | Absolutely-low                      | Fairly-low                          |
|                | \(A_{15}\)               | Low                                 | Fairly-low                          |
|                | \(A_{16}\)               | Low                                 | Very-low                            |
|                | \(A_{17}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{18}\)               | Very-low                            | Low                                 |
| \(C_2\)       | \(A_{21}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{22}\)               | High                                | Fairly-high                         |
|                | \(A_{23}\)               | Fairly-high                         | High                                |
|                | \(A_{24}\)               | Very-low                            | Fairly-low                          |
|                | \(A_{25}\)               | Low                                 | Fairly-low                          |
|                | \(A_{26}\)               | Low                                 | High                                |
|                | \(A_{27}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{28}\)               | Very-low                            | Fairly-high                         |
|                | \(A_{31}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{32}\)               | Absolutely-low                      | Very-low                            |
|                | \(A_{33}\)               | High                                | Fairly-high                         |
|                | \(A_{34}\)               | Very-low                            | Fairly-low                          |
|                | \(A_{35}\)               | Low                                 | Fairly-low                          |
|                | \(A_{36}\)               | Low                                 | Medium                              |
|                | \(A_{37}\)               | Very-low                            | Absolutely-low                     |
|                | \(A_{38}\)               | Very-low                            | Fairly-high                         |
ous levels of decision-making are identical to the methods seen that the ranking order by the proposed method at various cases, the proposed method for ranking IFNs is an extremely useful tool in the decision-making process. The ranking indices obtained with the proposed method are shown in Table 7. As depicted in the table, it is intuitive that the ordering of the IFNs that are symmetric about the y-axis and their corresponding images are the same. In fact, this intuition is very clearly stated through property and discussed through Theorems 3.20 and 3.22. An attractive feature is that the current method allows the decision-maker to make a decision at various \((\alpha, \beta)\)-levels of decision-making.

The numerical examples are also evident that the current method is superior to the existing methods, as it very efficiently handles IFNs of various types. Apart from these numerical examples, the reasonable properties of Wang and Kerre (2001a) are proved under the current study. This is a very important phenomenon, which depicts the novelty of the current study. Apart from these reasonable properties, newer properties are stated and proved under the current study. One of the newer properties is that it discusses the consistency in ordering the IFNs and their corresponding images.

One of the limitations is that the property,
\[
A_7, \quad \text{of Wang and Kerre (2001a) is not obeyed by the current method as } V_{\alpha, \beta}(\tilde{a}) \neq V_{\alpha, \beta}(\tilde{a})V_{\alpha, \beta}(\tilde{b}) \text{ and } A_{\alpha, \beta}(\tilde{a}) \neq A_{\alpha, \beta}(\tilde{a})A_{\alpha, \beta}(\tilde{b}) \text{ for IFNs } \tilde{a} \text{ and } \tilde{b}. \text{ Hence, there is a scope for future study in this line. Another limitation is that the proposed method cannot rank more than two IFNs at a time, as this approach allows pairwise comparison. Hence, to compare more than two IFNs, one can use the property } A_3.
\]

Table 7: Ranking indices of total risk \(\tilde{R}_i\) obtained for the farmers \(C_i\)

| Methods                | \(R_1\)  | \(R_2\)  | \(R_3\)  | Decision result |
|------------------------|----------|----------|----------|-----------------|
| Ye (2011)              | 0.1480   | 0.4224   | 0.3515   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| Chutia and Saikia (2018) |          |          |          |                 |
| \((\alpha, \beta) = (0.9, 0.1)\) | 0.0693   | 0.3040   | 0.2421   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| \((\alpha, \beta) = (0.5, 0.5)\) | 0.3390   | 1.2000   | 0.9935   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| \((\alpha, \beta) = (0.1, 0.9)\) | 0.4998   | 1.6351   | 1.3419   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| Nayagam et al. (2016)  | 0.0397   | 0.2167   | 0.1596   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| Darehmiraki (2019)     |          |          |          |                 |
| \((\alpha, \beta) = (0.9, 0.2)\) | -0.0208  | -0.0806  | -0.0652  | \(\tilde{R}_1 \succ \tilde{R}_3 \succ \tilde{R}_2\) |
| \((\alpha, \beta) = (0.5, 0.4)\) | 0.0234   | 0.0839   | 0.0696   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| \((\alpha, \beta) = (0.2, 0.9)\) | -0.0384  | -0.0884  | -0.0754  | \(\tilde{R}_1 \succ \tilde{R}_3 \succ \tilde{R}_2\) |
| Current method         |          |          |          |                 |
| \((\alpha, \beta) = (0.9, 0.1)\) | 0.0460   | 0.1560   | 0.1277   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| \((\alpha, \beta) = (0.5, 0.5)\) | 0.2063   | 0.6268   | 0.5181   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |
| \((\alpha, \beta) = (0.1, 0.9)\) | 0.2919   | 0.8360   | 0.6953   | \(\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1\) |

Incorporating the flexibility parameter introduces the concept of evaluating the value and ambiguity for higher \(\alpha\) and lower \(\beta\), rather than the entire range of integration. The present method efficiently handles any arbitrary IFNs. Generally, in the existing methods of ranking IFNs, the images were never considered. In this study, the consistency in ranking the IFNs and their corresponding images is a successful and unique attempt. Another conclusion that can be drawn from intuition is that the ordering of the IFNs that are symmetric about the y-axis and their corresponding images are the same. In fact, this intuition is very clearly stated through property and discussed through Theorems 3.20 and 3.22. One of the limitations is that the property, \(A_7\), of Wang and Kerre (2001a) is not obeyed by the current method as \(V_{\alpha, \beta}(\tilde{a}) \neq V_{\alpha, \beta}(\tilde{a})V_{\alpha, \beta}(\tilde{b})\) and \(A_{\alpha, \beta}(\tilde{a}) \neq A_{\alpha, \beta}(\tilde{a})A_{\alpha, \beta}(\tilde{b})\) for IFNs \(\tilde{a}\) and \(\tilde{b}\). Hence, there is a scope for future study in this line. Another limitation is that the proposed method cannot rank more than two IFNs at a time, as this approach allows pairwise comparison. Hence, to compare more than two IFNs, one can use the property \(A_3\).

6 Discussions and conclusions

In this paper, a novel method of ranking IFNs is being proposed based on the notion of value and ambiguity at various \((\alpha, \beta)\)-levels of decision-making. These quantities are combined uniquely, using an ambiguity inclusion–exclusion function, \(\theta\). This function determines whether the ambiguity should be included or excluded in the ranking process. Furthermore, the ‘flexibility parameter’ enables a decision-maker to make decisions at levels \((\alpha, \beta)\) of the IFNs. respectively. Because the risk obtained is IFNs, one cannot simply look into it and speak about the highest risk. In such cases, the proposed method for ranking IFNs is an extremely useful tool in the decision-making process. The ranking indices obtained with the proposed method are shown in Table 7. A comparative analysis with some existing methods can be done from Table 7. As depicted in the table, it is seen that the ranking order by the proposed method at various levels of decision-making are identical to the methods by Ye (2011), Nayagam et al. (2016) and Chutia and Saikia (2018). Farmers’ risks are ranked as \(\tilde{R}_2 > \tilde{R}_3 > \tilde{R}_1\), resulting in a risk-based ordering of farmers as \(C_2 > C_3 > C_1\). Intuitively, the risk at different levels of decision is justified because the farmer \(C_2\) with the least capital and no financial assistance from the government faces the greatest risk. However, Darehmiraki (2019) produces quite a different ordering of the risk.
Acknowledgements I appreciatively acknowledge the time and expertise devoted to reviewing papers by the Associate Editor, the members of the editorial board, and the anonymous referees, which helped in improving the paper.

Declarations

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this paper.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

Informed consent This article does not contain any studies with human participants, hence no informed consent is not declared.

References

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96
Atanassov KT (1989) More on intuitionistic fuzzy sets. Fuzzy Sets Syst 33(1):37–45
Atanassov KT (1994) New operations defined over the intuitionistic fuzzy sets. Fuzzy Sets Syst 61(2):137–142
Atanassov KT (1999) Intuitionistic fuzzy sets. Intuitionistic fuzzy Sets. Springer, Heidelberg, pp 1–137
Atanassov KT (2000) Two theorems for intuitionistic fuzzy sets. Fuzzy Sets Syst 110(2):267–269
Bhattacharya K, De S (2020) Decision making under intuitionistic fuzzy metric distances. Ann Optim Theory Pract 3(2):49–64
Chakraborty D, Jana DK, Roy TK (2015) Arithmetic operations on generalized intuitionistic fuzzy number and its applications to transportation problem. OPSEARCH 52:431–471
Chen SM (1996) New methods for subjective mental workload assessment and fuzzy risk analysis. Cybern Syst 27(5):449–472
Chen S-J, Chen S-M (2008) Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers. Comput Math Appl 55(8):1670–1685
Chen S-M, Chen J-H (2009) Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. Expert Syst Appl, 36(3, Part 2):6833–6842
Chen SM, Munif A, Chen GS, Liu HC, Kuo BC (2012) Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and right heights. Expert Syst Appl 39(7):6320–6334
Chutia R (2017) Ranking of fuzzy numbers by using value and angle in the epsilon-deviation degree method. Appl Soft Comput 60:706–721
Chutia R (2021) Ranking interval type-2 fuzzy number on a novel value-ambiguity ranking index and its application in risk analysis. Soft Comput 25(13):8177–8196
Chutia R (2021) Ranking of Z-numbers based on value and ambiguity at levels of decision making. Int J Intell Syst 36(1):313–331
Chutia R, Chutia B (2017) A new method of ranking parametric form of fuzzy numbers using value and ambiguity. Appl Soft Comput 52:1154–1168
Chutia R, Saikia S (2018) Ranking intuitionistic fuzzy numbers at levels of decision-making and its application. Expert Syst Appl 35(5):e12292
Chutia R, Saikia S (2020) Ranking of interval type-2 fuzzy numbers using value and ambiguity. In: 2020 international conference on computational performance evaluation (ComPE). Shillong, India, pp 305–310
Darehmiraki M (2019) A novel parametric ranking method for intuitionistic fuzzy numbers. Iran J Fuzzy Syst 16(1):129–143
Das D, De P (2016) Ranking of intuitionistic fuzzy numbers by new distance measure. J Intell Fuzzy Syst 30(2):1099–1107
Das S, Guha D (2016) A centroid-based ranking method of trapezoidal intuitionistic fuzzy numbers and its application to MCDM problems. Fuzzy Inf Eng 8(1):41–74
De SK (2018) Triangular dense fuzzy lock sets. Soft Comput 22(21):7243–7254
De SK (2020) On degree of fuzziness and fuzzy decision making. Cybern Syst 51(5):600–614
De SK, Beg I (2016) Triangular dense fuzzy sets and new defuzzification methods. J Intell Fuzzy Syst 31(1):469–477
De PK, Das D (2012) Ranking of trapezoidal intuitionistic fuzzy numbers. In: 2012 12th international conference on intelligent systems design and applications (ISDA), Kochi, India, pp 184–188
De SK, Mahata GC (2019) A comprehensive study of an economic order quantity model under fuzzy monsoon demand. Sadhana-Acad P Eng S, 44(4):1–12
Delgado M, Vila M, Voxman W (1998) On a canonical representation of fuzzy numbers. Fuzzy Sets Syst 93(1):125–135
Deli İ (2019) A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis. Soft Comput 23(23):12529–12545
Deli İ, Şubaş Y (2017) A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. Int J Mach Learn Cybern 8(4):1309–1322
Dubey D, Mehra A (2011) Linear programming with triangular intuitionistic fuzzy number. In: Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology. Atlantis Press, Aix-les-Bains, France, pp 563–569
Dubois D, Prade H (1980) Fuzzy sets and systems: theory and applications. Academic Press Inc, Orlando
Dutta P, Saikia B (2021) Arithmetic operations on normal semi elliptic intuitionistic fuzzy numbers and their application in decision-making. Granul Comput 6:163–179
Grzegorzewski P (2003) Distances and orderings in a family of intuitionistic fuzzy numbers. In: EUSFLAT conference, Zittau, Germany, pp 223–227
Jain R (1976) Decision making in the presence of fuzzy variables. IEEE Trans Syst Man Cybern Syst 6(10):698–703
Jain R (1977) A procedure for multiple-aspect decision making using fuzzy sets. Int J Syst Sci 8(1):1–7
Kumar A, Kaur M (2013) A ranking approach for intuitionistic fuzzy numbers and its application to MADM problems. Int J Intell Syst 30(2):381–396
Li D-F (2014) Decision and game theory in management with intuitionistic fuzzy numbers and its application to decision making. Granul Comput 6:163–179
Mitchell HB (2004) Ranking-intuitionistic fuzzy numbers. Int J Approx Reason 36(3, Part 2):6833–6842
Mondal SP, Goswami A, Kumar De S (2019) Nonlinear triangular intuitionistic fuzzy number and its application in decision-making. Granul Comput 6:163–179
Orthogonal analysis on printed circuit board assembly. Microelectron Reliab 52:1154–1168
De SK (2018) Triangular dense fuzzy lock sets. Soft Comput 22(21):7243–7254
De SK (2020) On degree of fuzziness and fuzzy decision making. Cybern Syst 51(5):600–614
De SK, Beg I (2016) Triangular dense fuzzy sets and new defuzzification methods. J Intell Fuzzy Syst 31(1):469–477
De PK, Das D (2012) Ranking of trapezoidal intuitionistic fuzzy numbers. In: 2012 12th international conference on intelligent systems design and applications (ISDA), Kochi, India, pp 184–188
De SK, Mahata GC (2019) A comprehensive study of an economic order quantity model under fuzzy monsoon demand. Sadhana-Acad P Eng S, 44(4):1–12
Delgado M, Vila M, Voxman W (1998) On a canonical representation of fuzzy numbers. Fuzzy Sets Syst 93(1):125–135
Deli İ (2019) A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis. Soft Comput 23(23):12529–12545
Deli İ, Şubaş Y (2017) A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. Int J Mach Learn Cybern 8(4):1309–1322
Dubey D, Mehra A (2011) Linear programming with triangular intuitionistic fuzzy number. In: Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology. Atlantis Press, Aix-les-Bains, France, pp 563–569
Dubois D, Prade H (1980) Fuzzy sets and systems: theory and applications. Academic Press Inc, Orlando
Dutta P, Saikia B (2021) Arithmetic operations on normal semi elliptic intuitionistic fuzzy numbers and their application in decision-making. Granul Comput 6:163–179
Grzegorzewski P (2003) Distances and orderings in a family of intuitionistic fuzzy numbers. In: EUSFLAT conference, Zittau, Germany, pp 223–227
Jain R (1976) Decision making in the presence of fuzzy variables. IEEE Trans Syst Man Cybern Syst 6(10):698–703
Jain R (1977) A procedure for multiple-aspect decision making using fuzzy sets. Int J Syst Sci 8(1):1–7
Kumar A, Kaur M (2013) A ranking approach for intuitionistic fuzzy numbers and its application. J Appl Res Technol 11(3):381–396
Li D-F (2008) A note on “using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly. Microelectron Reliab 48(10):1741
Li D-F (2010) A ratio ranking method of triangular intuitionistic fuzzy numbers and its applications to MADM problems. Comput Math Appl 60(6):1557–1570
Li D-F (2014) Decision and game theory in management with intuitionistic fuzzy sets, vol 308. Springer, Berlin
Li DF, Nan JX, Zhang MJ (2010) A ranking method of triangular intuitionistic fuzzy numbers and its application to decision making. Int J Comput Intell Syst 3(5):522–530
Mitchell HB (2004) Ranking-intuitionistic fuzzy numbers. Int J Uncertain Fuzz 12(03):377–386
Nan J-X, Li D-F, Zhang M-J (2010) A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers. Int J Comput Intell Syst 3(3):280–289
Nayagam LGV, Jeevaraj S, Dhanasekaran P (2016) A linear ordering on the class of trapezoidal intuitionistic fuzzy numbers. Expert Syst Appl 60:269–279
Nayagam VLG, Jeevaraj S, Dhanasekaran P (2018) An improved ranking method for comparing trapezoidal intuitionistic fuzzy numbers and its applications to multicriteria decision making. Neural Comput Appl 30:671–682
Nehi HM (2010) A new ranking method for intuitionistic fuzzy numbers. Int J Fuzzy Syst 12(1):80–86
Rezvani S (2013) Ranking method of trapezoidal intuitionistic fuzzy numbers. Ann Fuzzy Math Inform 5(3):515–523
Salahshour S, Shekari G, Hakimzadeh A (2012) A novel approach for ranking triangular intuitionistic fuzzy numbers. AWER Procedia Inf Technol 1:442–446
Schmucke KJ (1984) Fuzzy sets: natural language computations, and risk analysis. University of Michigan Computer Science Press, Michigan
Seikh MR, Nayak PK, Pal M (2012) Generalized triangular fuzzy numbers in intuitionistic fuzzy environment. Int J Eng Res Dev 5(1):08–13
Ulucay V, Deli I, Sahin M (2019) Intuitionistic trapezoidal fuzzy multi-numbers and its application to multi-criteria decision-making problems. Complex Intell Syst 5(1):65–78
Wan S-P (2013) Multi-attribute decision making method based on possibility variance coefficient of triangular intuitionistic fuzzy numbers. Int J Uncertain Fuzz 21(02):223–243
Wan S-P, Li D-F (2013) Possibility mean and variance based method for multi-attribute decision making with triangular intuitionistic fuzzy numbers. J Intell Fuzzy Syst 24(4):743–754
Wang X, Kerre EE (2001) Reasonable properties for the ordering of fuzzy quantities (I). Fuzzy Sets Syst 118(3):375–385
Wang X, Kerre EE (2001) Reasonable properties for the ordering of fuzzy quantities (II). Fuzzy Sets Syst 118(3):387–405
Wang J, Zhang Z (2009) Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number. Control Decis 24(2):226–230
Wei SH, Chen SM (2009) A new approach for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers. Expert Syst Appl 36(1):589–598
Ye J (2011) Expected value method for intuitionistic trapezoidal fuzzy multicriteria decision-making problems. Expert Syst Appl 38(9):11730–11734
Zeng X-T, Li D-F, Yu G-F (2014) A value and ambiguity-based ranking method of trapezoidal intuitionistic fuzzy numbers and application to decision making. Sci World J
Zhang W-R (1986) Knowledge representation using linguistic fuzzy relations. University of South Carolina, Columbia
Zhang X, Xu Z (2012) A new method for ranking intuitionistic fuzzy values and its application in multi-attribute decision making. Fuzzy Optim Decis Mak 11(2):135–146
Zhu L-S, Xu R-N (2012) Fuzzy risks analysis based on similarity measures of generalized fuzzy numbers. Springer, Berlin, pp 569–587

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.