Abstract: Let \( H \) be a finite dimensional \( C^\ast \)-Hopf algebra and \( A \) the observable algebra of Hopf spin models. For some coaction of the Drinfeld double \( D(H) \) on \( A \), the crossed product \( A \rtimes \hat{D}(H) \) can define the field algebra \( \mathcal{F} \) of Hopf spin models. In the paper, we study \( C^\ast \)-basic construction for the inclusion \( A \subseteq \mathcal{F} \) on Hopf spin models. To achieve this, we define the action \( \alpha : D(H) \times \mathcal{F} \to \mathcal{F} \), and then construct the resulting crossed product \( \mathcal{F} \rtimes D(H) \), which is isomorphic \( A \otimes \text{End}(\hat{D}(H)) \).

Furthermore, we prove that the \( C^\ast \)-basic construction for \( A \subseteq \mathcal{F} \) is consistent to \( \mathcal{F} \rtimes D(H) \), which yields that the \( C^\ast \)-basic constructions for the inclusion \( A \subseteq \mathcal{F} \) is independent of the choice of the coaction of \( D(H) \) on \( A \).

Keywords: Hopf algebras; observable algebras; basic construction; crossed product

MSC: 46L05; 16S35

1. Introduction

Quantum chains on one-dimensional lattice have several interesting features, such as the emergence of braid group statistics, integrability, quantum symmetry, and so on, which are impossible in higher dimensions. \( G \)-spin quantum chain [1], as the simplest example of lattice field theories, exhibits quantum symmetry described by the quantum double of a finite group \( G \), which generalizes “order \times \text{disorder}” symmetry of the lattice Ising models. This research has been extended to \( G \)-spin models determined by a normal subgroup \( N [2] \), where we construct a field algebra on which the symmetry algebra \( D(N; G) \) can act and the observable algebra \( A_N \) arises as the \( D(N; G) \)-invariant subalgebra of the field algebra, and then prove a one-to-one correspondence between the minimal central projectors of \( D(N; G) \) (which induce the inequivalent representations of \( D(N; G) \)) and the inequivalent representations of the observable algebra. Subsequently, we show that the observable algebra can be described as the infinite iterated crossed product [3].

In [4], Nill and Szlachányi generalized the group algebra of a finite group \( G \) to arbitrary finite dimensional \( C^\ast \)-Hopf algebras \( H \), and also proved that such “Hopf spin models” always have \( D(H) \) as a universal localized cosymmetry. This means that, under the assumption of a Haag dual vacuum representation (i.e., absence of spontaneous symmetry breaking), the full superselection structure of these models is precisely created by the irreducible representations of \( D(H) \).

In [5], V.F.R. Jones introduced an index for subfactors and he found his celebrated polynomial invariant for knots by using the subfactor theory in [6]. Later, the Jones index theory has been developed by Pimsner and Popa [7], Kosaki [8], Watatani [9], and so on. For example, Pimsner and Popa [7] showed that the Jones index is finite if and only if there exists a Pimsner–Popa basis. Kosaki [8] investigated the index for a conditional expectation of an arbitrary factor onto a subfactor, where he
used the spatial theory [10] and the theory on operator-valued weights [11]. Inspired by previous work, Watatani [9] considered the index for a conditional expectation on a C*-algebra, and established a link between transfer in the K-theory of C*-algebras [9,12] and the multiplication map. Based on the quasi-basis, Watatani developed a C*-version of basic construction. As we all know, the basic construction for type II₁ factors has a specific form, since a factor of type II₁ admits a faithful trace that is a state of this kind for which the Gelfand–Naimark–Segal construction (which yields a correspondence between states and cyclic ∗-representations of C*-algebras) may be performed, while the C*-version of basic construction does not have this form.

The paper studies the C*-basic construction for a pair of C*-algebras on Hopf spin models for a finite dimensional C*-Hopf algebra, and is organized as follows.

In Section 2, we review the definition and properties about Hopf spin models which include the quantum double D(H), the observable algebra A, and the field algebra F. In addition, we briefly describe the C*-basic construction.

In Section 3, we define an action of D(H) on the field algebra F, and then construct the crossed product C*-algebra of F by D(H), which is denoted by F × D(H). Moreover, let E be the canonical conditional expectation from the field algebra F onto the observable algebra A; then, there is an isomorphism of the crossed product F × D(H) onto the C*-basic construction (F, ε_A)C*.

2. Preliminaries

Let H be a finite dimensional C*-Hopf algebra with its comultiplication Δ, counit ε and antipode S. We use Sweedler’s notation Δ(a) = a(1) ⊗ a(2) for any a ∈ H. The map Δ(n) : H → H⊗(n+1) is the composition of n coproducts given by Δ(n)(a) = a(1) ⊗ a(2) ⊗ ... ⊗ a(n+1), since the coproduct is coassociativity, where H⊗n means the n-fold tensor product of H. We have (Sa(1))a(2) = a(1)S(a(2)) = ε(a) for any a ∈ H and S² = id. In addition, there is the (normalized) Haar integral h in H, which satisfies ah = ha = ε(a)h for all a ∈ H and ε(h) = 1. In addition, such an h is unique and h² = h = h* = S(h).

Throughout this paper, H denotes a finite dimensional C*-Hopf algebra, H<sup>op</sup> taking the opposite coproduct in H and Ĥ the dual C*-Hopf algebra of H. The readers can refer to the standard Hopf algebra theory and notation in [13,14]. We denote elements of H and Ĥ by a, b, ..., and ϕ, ψ, ..., respectively. The canonical pairing ⟨,⟩ : H × Ĥ → C is given by ⟨a, ϕ⟩ → ⟨a, ϕ⟩. There are Hopf module left and right actions of H on Ĥ given by Sweedler’s arrows:

\[ a → ϕ = ϕ(1)⟨a, ϕ(2)⟩, \]
\[ ϕ ↘ a = ⟨ϕ(1), ϕ(2)⟩. \]

**Definition 1.** The quantum double D(H) of H is the bicrossed product of H and H<sup>op</sup>.

For ϕ ⊗ a, ψ ⊗ b ∈ D(H), the Hopf ∗-algebra structures are given by

\[
(ϕ ⊗ a)(ψ ⊗ b) = ϕ(a_1) → ψ ← S⁻¹(a_3)) ⊗ a(2)b, \quad \text{(multiplication)}
\]
\[
Δ(ϕ ⊗ a) = (ϕ(2) ⊗ a(1)) ⊗ (ϕ(1) ⊗ a(2)), \quad \text{(coproduct)}
\]
\[
ε(ϕ ⊗ a) = ε(ϕ)ε(a), \quad \text{(counit)}
\]
\[
S(ϕ ⊗ a) = S(a_1) → S(ϕ) ← a(1)) ⊗ S(a(2)), \quad \text{(antipode)}
\]
\[
(ϕ ⊗ a)^* = (a_1^*) → ϕ* ← S(a_3^*) ⊗ a_2^*, \quad \text{(**-operation)}
\]

Moreover, D(H) is a finite-dimensional Hopf C*-algebra with the unique Haar integral E.

Following Nill and Szlachányi [4], we shall recall some definitions about Hopf spin models, such as the observable algebra, the field algebra, and so on.
Now, we consider the set of integers \(\mathbb{Z}\), which can be regarded as the set of elements of one-dimensional lattice: the odd integers represent links, the even ones represent lattice sites. There are a copy of \(H\) on each lattice site and a copy of its dual \(\hat{H}\) on each link. When \(H\) and \(\hat{H}\) act on each other in the "natural way", nontrivial commutation relations hold only between adjacent links and sites. Hence, the link-site algebra form the crossed products \(\mathcal{W}(\hat{H}) = \hat{H} \rtimes H\) ("Weyl algebras" [15]), and the site-link ones form \(\mathcal{W}(H) = H \rtimes \hat{H}\). If \(H = CG\) for some finite group \(G\), then Hopf spin models reduce to the \(G\)-spin models [1].

**Definition 2.** The quasilocal observable algebra \(A_{\text{loc}}\) of Hopf spin models is a unital associative \(*\)-algebra generated by \(\{A_2(a), A_{2i+1}(\varphi) : a \in H, \varphi \in \hat{H}, i \in \mathbb{Z}\}\) subject to

\[
\begin{align*}
A_2(a)A_2(b) &= BA, \\ A_i \in A_i, B \in A_j, |i - j| \geq 2, \\
A_{2i+1}(\varphi)A_2(a) &= A_{2i}(a(1))A_2(\varphi(2))A_{2i+1}(\varphi(1)), \\
A_2(a)A_{2i-1}(\varphi) &= A_{2i-1}(\varphi(1))A_2(a(1))A_{2i-1}(\varphi(2)).
\end{align*}
\]

By \(A_{n,m}\), we denote the unital \(*\)-subalgebra which is generated by the \(C^*\)-algebras \(A_i, n < i < m\).

Using the \(C^*\)-inductive limit [4,16], \(A_{\text{loc}}\) become a \(C^*\)-algebra \(A\). In general, we call \(A\) the observable algebra of Hopf spin models.

There is a coaction \(\rho\) of \(D(H)\) on the observable algebra \(A\) given by the following proposition.

**Proposition 1.** [4] For \(|I| = 2\), define \(\rho_I : A_i \rightarrow A_i \otimes D(H)\) by

\[
\begin{align*}
\rho_{2i+1}(A_{2i+1}(a)A_{2i+1}(\varphi)) &= A_{2i}(a(1))A_{2i+1}(\varphi(2)) \otimes (\varphi(2) \otimes a(2)), \\
\rho_{2i-1}(A_{2i-1}(\varphi)A_2(a)) &= A_{2i-1}(\varphi(1))A_2(a(1)) \otimes (\varphi(2) \otimes a(1)),
\end{align*}
\]

where \(a \in H, \varphi \in \hat{H}\). Then,

1. \(\rho_{i,i+1}\) provides a coaction of \(D(H)\) on \(A_{i,i+1}\) with respect to \(\Delta\) (if \(i\) is even) or \(\Delta^{op}\) (if \(i\) is odd) on \(D(H)\);
2. \(\rho_{i,i+1}\) can be extended to a coaction of \(D(H)\) on \(A\).

Every comodule algebra action \(\rho : A \rightarrow A \otimes D(H)\) uniquely determines a module algebra action of the dual \(\hat{D}(H)\) on \(A\) given by: for any \(\xi \in \hat{D}(H), A \in A,\)

\[
\rho_{\xi} : A \rightarrow A \\
A \mapsto (\text{id}_A \otimes \xi)(\rho(A)).
\]

The following relations can be easily proved:

\[
\begin{align*}
\rho_{\xi}(AB) &= \rho_{\xi}(A)\rho_{\xi}(B), \\
\rho_{\xi}(1_A) &= \varepsilon(\xi)1_A, \\
\rho_{\xi}(A) &= (\rho_{S(\xi)}(A))^*, \\
\rho_{\xi}\rho_{\eta} &= \rho_{\xi\eta}, \\
\rho_{\varepsilon} &= \text{id}_A.
\end{align*}
\]

**Definition 3.** On the Hopf spin models, the field algebra \(F\) is defined as the crossed product \(A \rtimes \hat{D}(H)\) with respect to the coaction \(\rho\) of \(D(H)\) on \(A\) defined in Proposition 1.
In the following, we recall the definition of the C*-basic construction for a pair of C*-algebras $A \subseteq B$.

Let $\Gamma \colon B \to A$ be a positive faithful conditional expectation. Then, $B_A$ is a pre-Hilbert module over $A$ with an $A$-valued inner product $\langle x, y \rangle = \Gamma(x^*y)$ for any $x, y \in B$. By $B$, we denote the completion of $B_A$, then $B$ is a Hilbert C*-module over $A$. Let $L_A(B)$ be the set of all (right) $A$-module homomorphisms on $B$ with adjoints. Then, $L_A(B)$ is a C*-algebra with the usual operator norm.

Every $b \in B$ is viewed as a left multiplication operator $\lambda(b)$ in $L_A(B)$. Again, the map $\gamma_A \colon B_A \to B_A$ given by $\gamma_A(\cdot) = \Gamma(\cdot)$ is bounded and thus can be extended to a bounded linear operator on $B$. For convenience, we still use the same symbol $\gamma_A$. Then, $\gamma_A$ is a projection in $L_A(B)$, which is called the Jones projection, i.e., $\gamma_A = \gamma_A^2 = \gamma_A^*$.

Then, the C*-basic construction $\langle B, \gamma_A \rangle_{C^*}$ is defined as the closure of the linear span $\{\lambda(x)\gamma_A \lambda(y) \in L_A(B) : x, y \in B\}$.

**Lemma 1.** The *-homomorphism $\lambda \colon B \to L_A(B)$ and the projection $\gamma_A$ satisfy the following covariant relation:

1. $\gamma_A \lambda(b) \gamma_A = \lambda(\lambda(b)) \gamma_A$ for any $b \in B$;
2. If $b \in B$, then $b \in A$ if and only if $\gamma_A \lambda(b) = \lambda(b) \gamma_A$.

3. The C*-Basic Construction on Hopf Spin Models

In this section, we define an action $\alpha$ of $D(H)$ on the field algebra $F$ which is compatible with the algebraic structure of $F$. Under this action, the fixed point algebra is consistent with the observable algebra $A$. We then construct the crossed product C*-algebra $F \rtimes D(H)$ by way of a Hopf module left action $\alpha$ which extends $F \equiv F \rtimes 1_{D(H)}$, and prove that the C*-basic construction $\langle F, \epsilon_A \rangle_{C^*}$ is isomorphic to the crossed product $F \rtimes D(H)$.

The map $\alpha \colon D(H) \times F \to F$ given on the generating elements of $F$ as

$$\alpha(X \otimes (A, \xi)) = (A, \xi_{(1)}) \langle X, \xi_{(2)} \rangle$$

for any $X \in D(H)$, $(A, \xi) \in F$, can be bilinearly extended both in $D(H)$ and $F$. We often use the notation $X(A, \xi)$ to denote $\alpha(X \otimes (A, \xi))$.

**Proposition 2.** The map $\alpha$ defines an action of $D(H)$ on $F$, which means that $\alpha$ is a bilinear map satisfying the relations:

$$1_{(D(H))}(A, \xi) = (A, \xi),$$
$$X(1_A, 1_{D(H)}) = \epsilon(X)(A, 1_{D(H)}),$$
$$(XY)(A, \xi) = X(Y(A, \xi)),$$
$$X((A, \xi)(B, \eta)) = \sum_{(X)} X_{(1)}(A, \xi)X_{(2)}(B, \eta),$$
$$X(A, \xi)^* = (S(X^*)(A, \xi))^*,$$

for any $X, Y \in D(H), (A, \xi), (B, \eta) \in F$. 


Proof. Now, we prove the last two equalities, while the others are quite elementary and are omitted. As for the fourth equality, we compute

$$X((A, \xi)(B, \eta)) = X(A \rho_{\xi(1)}(B), \xi(2) \eta)$$

$$= (A \rho_{\xi(1)}(B), (\xi(2) \eta)(1)) \langle X, (\xi(2) \eta)(2) \rangle$$

$$= (A \rho_{\xi(1)}(B), \xi(2) \eta(1)) \langle X, \xi(3) \eta(2) \rangle$$

$$= (A \rho_{\xi(1)}(B), \xi(2) \eta(1)) \langle X(1), \xi(3) \rangle \langle X(2), \eta(2) \rangle$$

$$= (A, \xi(1))(B, \eta(1)) \langle X(1), \xi(2) \rangle \langle X(2), \eta(2) \rangle$$

$$= X(1)(A, \xi)X(2)(B, \eta).$$

Here, we used the dual pairing property. To prove the fifth equation, we can calculate

$$X(A, \xi)^* = X(\rho_{\xi(1)}(A^*), \xi(2)^*)$$

$$= (\rho_{\xi(1)}(A^*), \xi(2)^*) \langle X, \xi(3)^* \rangle$$

$$= (\rho_{\xi(1)}(A^*), \xi(2)^*) \langle S(X^*), \xi(3) \rangle$$

$$= (A, \xi(1))^* \langle S(X^*), \xi(2) \rangle$$

$$= ((A, \xi(1))^* \langle S(X^*), \xi(2) \rangle)^*$$

$$= (S(X^*)(A, \xi))^*,$$

where we used the fact that the coproduct is a *-algebra homomorphism and the definition of *-operation in the dual of a Hopf C*-algebra. □

We will consider the $D(H)$-invariant subalgebra of $F$ in the following. Let

$$F^{D(H)} = \{(A, \xi) \in F: X(A, \xi) = \epsilon(X)(A, \xi), \forall X \in D(H)\}.$$

One can verify that $F^{D(H)}$ is a $C^*$-subalgebra of the field algebra $F$, which is consistent with the observable algebra $A$. Furthermore,

$$F^{D(H)} = \{(A, \xi) \in F: E(A, \xi) = (A, \xi)\}.$$

Indeed, for $(A, \xi) \in F$,

$$E(A, \xi) = \epsilon(E)(A, \xi) = (A, \xi),$$

and

$$X(A, \xi) = X(E(A, \xi)) = (XE)(A, \xi) = \epsilon(X)E(A, \xi) = \epsilon(X)(A, \xi).$$

Remark 1. Since the field algebra $F$ is a $D(H)$-module algebra and $D(H)$ is semisimple, then $F$ can be decomposed into a direct sum

$$F = \bigoplus_{r \in [D(H)]} F^r, \quad F^r = M^r(F)$$

using the properties of minimal central idempotents $M^r$ in $D(H)$, where $r$ denotes the equivalence class of irreducible representations of $D(H)$. On the other hand, $(\epsilon, F)$ is an irreducible representation of $D(H)$, which corresponds to the $D(H)$-module. Since

$$F^\xi \triangleq \{(A, \xi) \in F: X(A, \xi) = \epsilon(X)(A, \xi), X \in D(H)\}$$
is a $D(H)$-module, the observable algebra $A = F^e$ corresponding to the trivial representation $e$.

From Proposition 2, we can construct the crossed product $C^*$-algebra $F \rtimes_A D(H)$ (or mostly, simply as $F \rtimes D(H)$, when the action is understood), whose vector space is $F \otimes D(H)$ (by $(A, \xi) \otimes X$, we denote $(A, \xi, X)$ for simplicity) and multiplication is given by

$$(A, \xi, X)(B, \eta, Y) = ((A, \xi)X(1)(B, \eta), X(2)Y) = (A\rho^{e(1)}_\xi(B, \xi(2)\eta(1), X(2)Y)X(1), \eta(2)).$$

The $*$-algebra structure on $F \rtimes D(H)$ is given by

$$(A, \xi, X)^* = (1_A, 1_{D(H)}^\epsilon, X^*)(\rho^{e\ast(1)}_\xi(A^\ast)\xi(2), 1_{D(H)}^\epsilon).$$

This is an algebra with unit $(1_A, 1_{D(H)}, 1_{D(H)})$ and there are natural $*$-inclusions of algebras $F \subseteq F \rtimes D(H)$ given by $(A, \xi) \mapsto (A, \xi, 1_{D(H)})$ and $D(H) \subseteq F \rtimes D(H)$ given by $X \mapsto (1_A, 1_{D(H)}^\epsilon, X)$.

**Proposition 3.** (1) The element $(1_A, 1_{D(H)}^\epsilon, E)$ is a self-adjoint idempotent element that is

$$(1_A, 1_{D(H)}^\epsilon, E)^2 = (1_A, 1_{D(H)}^\epsilon, E)^* = (1_A, 1_{D(H)}^\epsilon, E).$$

(2) The element $(A, \xi, 1_{D(H)})$ in $F \rtimes D(H)$ satisfies the following covariant relation

$$(1_A, 1_{D(H)}^\epsilon, E)(A, \xi, 1_{D(H)})(1_A, 1_{D(H)}^\epsilon, E) = (E(A, \xi), 1_{D(H)})(1_A, 1_{D(H)}^\epsilon, E).$$

**Proof.** (1) We have

$$(1_A, 1_{D(H)}^\epsilon, E)^2 = (\rho_{1_{D(H)}^\epsilon}(1_{D(H)}^\epsilon, E)(E(1), 1_{D(H)}^\epsilon))$$

$$= (1_A, 1_{D(H)}^\epsilon, E(1)E(2))$$

$$= (1_A, 1_{D(H)}^\epsilon, E),$$

and

$$(1_A, 1_{D(H)}^\epsilon, E)^* = (1_A, 1_{D(H)}^\epsilon, E^*)(\rho_{1_{D(H)}^\epsilon}(1_A, 1_{D(H)}^\epsilon, 1_{D(H)}))$$

$$= (1_A, 1_{D(H)}^\epsilon, E^*)(E(1), 1_{D(H)}^\epsilon)$$

$$= (1_A, 1_{D(H)}^\epsilon, E^*)$$

$$= (1_A, 1_{D(H)}^\epsilon, E),$$

where we used the identity $\rho_{1_{D(H)}^\epsilon} = \text{id}_A$. Hence, $(1_A, 1_{D(H)}^\epsilon, E)$ is a self-adjoint idempotent element.
The C∗-Proof.

By the basic constructions bounded A-module map on F can be extended uniquely to the Hilbert A-module F, we denote Lₐ(F) by the C*-algebra of norm bounded right A-module endomorphisms of F with adjoints, and have

\[ \langle F, e_A \rangle_{C^*} = \overline{\text{span}\{ \lambda(A, \xi)e_A\lambda(B, \eta) \in Lₐ(F): (A, \xi), (B, \eta) \in F \}} \].

The following theorem is our main result of this paper, which is devoted to characterizing the basic constructions \( \langle F, e_A \rangle_{C^*} \) associated with the inclusion \( A \subseteq F \).

**Theorem 1.** The C*-basic construction \( \langle F, e_A \rangle_{C^*} \) is C*-isomorphic to the crossed product \( F \rtimes D(H) \), that is,

\[ \langle F, e_A \rangle_{C^*} \cong F \rtimes D(H). \]

**Proof.** Since \( E: F \to A \) is a conditional expectation of index-finite type, it follows from [17] that

\[ F \rtimes D(H) = \overline{\text{span}\{ (A, \xi)E(B, \eta): (A, \xi), (B, \eta) \in F \}} \]

Let

\[ \Phi: \langle F, e_A \rangle_{C^*} \to F \rtimes D(H) \]

be a map given by

\[ \Phi(\sum_i \lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \sum_i (A_i, \xi_i)E(B_i, \eta_i). \]

Then, \( \Phi \) is well-defined. In fact, suppose that \( \sum_i \lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i) = 0 \), then, for any \( (C, \zeta) \in F \),

\[ \sum_i \lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)(C, \zeta) = 0. \]

By definition, we have \( \sum_i \lambda(A_i, \xi_i)e_A\lambda((B_i, \eta_i)(C, \zeta)) = 0 \). It follows from Lemma 1 that \( \sum_i (A_i, \xi_i)E((B_i, \eta_i)(C, \zeta))E = 0 \), that is,

\[ \sum_i (A_i, \xi_i)E(B_i, \eta_i)(C, \zeta)E = 0, \]

and then, for any \( (D, \zeta) \in F \),

\[ \sum_i (A_i, \xi_i)E(B_i, \eta_i)(C, \zeta)E(D, \zeta) = 0. \]

In particular,

\[ \langle \sum_i (A_i, \xi_i)E(B_i, \eta_i) \rangle (\sum_i (A_i, \xi_i)E(B_i, \eta_i))^* = 0. \]
Hence, $\sum_i (A_i, \xi_i) E(B_i, \eta_i) = 0$.

In order to show that $\Phi$ is an algebra $*$-homomorphism, we can compute

$$
\Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda((B_i, \eta_i)(A_i, \xi_i))e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(E((B_i, \eta_i)(A_i, \xi_i))\lambda(A_i, \xi_i))e_A\lambda(B_i, \eta_i)) = (\langle F, \hat{A}\rangle) \cdot 1_{D(H)}(\alpha, 1_{D(H)} \cdot E)(B_i, \eta_i, 1_{D(H)}) = \cdot 1_{D(H)}(\alpha, 1_{D(H)} \cdot E)(B_i, \eta_i, 1_{D(H)}) = \cdot 1_{D(H)}(\alpha, 1_{D(H)} \cdot E)(B_i, \eta_i, 1_{D(H)}) \cdot 1_{D(H)}(\alpha, 1_{D(H)} \cdot E)(B_i, \eta_i, 1_{D(H)}) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)) = \Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)),
$$

for any $(A_i, \xi_i), (A_j, \xi_j), (B_i, \eta_i), (B_j, \eta_j) \in F$. Here, we used Lemma 1(1) and Proposition 3 (2). Moreover,

$$
\Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i))^* = \Phi(\lambda(B_i, \eta_i)^*e_A\lambda(A_i, \xi_i)^*) = (B_i, \eta_i)^*E(A_i, \xi_i^*) = ((A_i, \xi_i)E(B_i, \eta_i))^* = (\Phi(\lambda(A_i, \xi_i)e_A\lambda(B_i, \eta_i)))^*,
$$

since $E$ is a self-adjoint idempotent element (see Proposition 3). Hence, $\Phi$ is a $C^*$-homomorphism. By Theorem 2.1.7 in [18], $\Phi$ is norm-decreasing.

Again, $\Phi$ is bijective. Then, the open mapping theorem shows that $\Phi$ is $C^*$-isomorphic. Hence, the $C^*$-basic construction $\langle F, e_A\rangle_{C^*}$ is $C^*$-isomorphic to the crossed product $F \rtimes D(H)$. □

**Remark 2.** (1) By the Takesaki duality theorem [19], $F \rtimes D(H) = \langle A \rtimes \hat{D}(H) \rangle \times D(H)$ is isomorphic to $\mathcal{A} \otimes \text{End}(D(H))$, which yields that the $C^*$-basic constructions for the inclusion $\mathcal{A} \subseteq F$ is independent of the choice of the field algebra $\mathcal{A}$. Moreover, since the field algebra $F$ is constructed as a crossed product of $A$ with the action of $D(H)$, and any right $D(H)$-coaction on $\mathcal{A}$ can bring about a natural left $D(H)$-action on $\mathcal{A}$, then the $C^*$-basic constructions for the inclusion $\mathcal{A} \subseteq F$ do not depend on the choice of the right $D(H)$-coaction on $\mathcal{A}$, which is defined in Proposition 1.

(2) Let $\mathcal{A} = (F_0) \subseteq F_1 \subseteq F_2 \subseteq \cdots$ be the tower of the Jones type basic construction associated with the inclusion $\mathcal{A} \subseteq F$ with $[F : A] = |H|^2 1_F$, where $F_{n+2} = \langle F_{n+1}, e_{n+2} \rangle$ is the basic construction associated with the inclusion $F_n \subseteq F_{n+1}$ for any $n \geq 0$. By induction, we can get the inclusion tower

$$
A \subseteq F \subseteq F \rtimes D(H) \subseteq F \rtimes D(H) \rtimes \hat{D}(H) \subseteq F \rtimes D(H) \rtimes \hat{D}(H) \rtimes D(H) \subseteq \cdots
$$

(3) Theorem 1 shows that $C^*$-basic construction $\langle F, e_A\rangle_{C^*}$ is essentially the crossed product, which improves and perfects the theory of Hopf spin models. Let $H$ be the group algebra $\mathbb{C}G$ of some finite group $G$, and it follows from Theorem 1 that $C^*$-basic construction on G-spin models is also the crossed product of the field algebra of G-spin models by the quantum double $D(G)$.

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