Spectral Efficiency Bounds for Interference-Limited SVD-MIMO Cellular Communication Systems

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Abstract—The ergodic spectral efficiency (SE) in interference-limited multiple-input multiple-output (MIMO) downlink cellular systems is characterized based on stochastic geometry. A single user is served by using singular value decomposition precoding and combining. By approximating the expectations of the channel eigenvalues, we derive upper and lower bounds on the ergodic SE. The obtained upper bound is the best possible system-level performance of any MIMO strategy in non-cooperative cellular networks. We validate our analytical results through simulation. We also conjecture that there exists the optimal number of streams being proportional to the pathloss exponent.

Index Terms—MIMO, downlink, SVD, ergodic spectral efficiency, the quarter circle law, Poisson point process.

I. INTRODUCTION

The performance of MIMO systems has been extensively investigated over a few decades [1]–[3]. The prior work analyzed the MIMO systems in terms of the deterministic signal-to-noise ratio (SNR) and a SNR-based simulation study was universal [2]. Under this approach, the results only hold for particular user locations, having difficulty to provide a system-level view incorporating many possible user locations.

To resolve the limitation, the system-level analysis has been performed in a tractable network model where the locations of the base stations (BSs) are modeled by using a homogeneous Poisson point process (PPP). Leveraging this Poisson network model, the signal-to-interference-plus-noise ratio was characterized in a single-input single-output downlink cellular system [4], [5]. Extending [4] to the MIMO broadcast channel [6], the coverage probability and rate were derived by using a zero-forcing (ZF) receiver with inter cell interference (ICI) cancellation. Further in [7], the optimal number of antennas for ICI cancellation when using partial ZF receivers was derived along with analytical expressions for coverage and rate distribution. In [8], the performance of the downlink multi-antenna heterogeneous cellular network was studied for ZF precoding. Cooperation gain under ZF beamforming [9] were investigated for MIMO networks, which provides the optimal loading factor (∼ 0.6) to maximize per-BS ergodic sum rate. Formulating ICI as an infinite sum of independent and conditionally distributed Gaussian random variables, average rate. Formulating ICI as an infinite sum of independent and conditionally distributed Gaussian random variables, average rate. Formulating ICI as an infinite sum of independent and conditionally distributed Gaussian random variables, average rate. Formulating ICI as an infinite sum of independent and conditionally distributed Gaussian random variables, average rate. Formulating ICI as an infinite sum of independent and conditionally distributed Gaussian random variables, average rate.

A common limitation in prior work [6]–[10] is the use of ZF precoder and/or equalizer due to analytical tractability. In a MIMO system, however, the ZF is suboptimal as it cannot extract the power gain from channels. In a single-user MIMO system, the optimal strategy is singular value decomposition (SVD) based precoding and combining. Thus, we analyze the system-level performance of an SVD-MIMO system.

In this paper, we characterize the ergodic spectral efficiency (SE) of MIMO downlink cellular systems in which a single user with multiple antennas is served by using SVD precoding and combining. The major difficulty of the analysis has been the characterization of eigenvalues and the ICI, coupled with each other in MIMO channels. To resolve such challenge, we use two key techniques: (i) the quarter circle law [12] to characterize eigenvalues, and (ii) 2-D homogeneous PPP [13] to model the MIMO networks. Leveraging these techniques, we approximate the expectation of each eigenvalue and derive the bounds of the ergodic SE of SVD-MIMO channels by decoupling the eigenvalue power gain from the ICI. Since we use the optimal precoder and combiner, our result serves as an upper bound on the system-level performance of any MIMO strategy in a non-cooperative cellular network under the assumptions of equal power allocation and no interference cancellation. This is not the case in the prior work [6]–[11]. We also observe that there exists the optimal number of streams that maximizes the ergodic SE and it is proportional to the pathloss exponent.

Notation: A is a matrix and a is a column vector. $A^H$, $A^\dagger$ and $A^{-1}$ denote conjugate transpose, transpose and inverse, respectively. $A^{(n)}$ represents a first $(n \times n)$ sub-matrix, and $a^{(n)}$ is a first $(n \times 1)$ sub-vector $- (1 \times n)$ for a row vector. $CN(\mu, \sigma^2)$ is a complex Gaussian distribution with mean $\mu$ and variance $\sigma^2$.

II. SYSTEM MODEL

We consider a single-user downlink cellular network model in which BSs are distributed according to a homogeneous PPP, $\Phi = \{d_i, i \in \mathbb{N}\}$ of intensity $\lambda$. Users are also distributed as an independent homogeneous PPP, $\Phi_U = \{u_i, i \in \mathbb{N}\}$. Each BS’s coverage area is presented as a Voronoi cell yielding minimum pathloss. Each BS and user are equipped with $N$ antennas.

We assume that each BS transmits $m \leq N$ streams to its associated user with equal power allocation. Denoting $H_i \in \mathbb{C}^{N \times N}$ as the channel matrix between the user at $u_i$ and the associated BS at $d_i$, its SVD is represented as $H_i = U_i \Lambda_i V_i^H$. The matrices $U_i$ and $V_i$ are $N \times N$ unitary matrices and $\Lambda_i \in \mathbb{R}^{N \times N}$ is a diagonal matrix of singular values $\sigma_i$; $\Lambda_i = \text{diag}(\sigma_1, \cdots, \sigma_N)$ where $\sigma_1 \geq \cdots \geq \sigma_N$.
\[ \cdots \geq \sigma_N. \]

Assuming the perfect channel state information at transmitters and at receivers, the BS at \( d_i \) transmits symbols \( s_i = [s_{i,1}, s_{i,2}, \cdots, s_{i,m}, 0, \cdots, 0]^\top \in \mathbb{C}^{m \times 1} \), with \( \mathbb{E}[s_{i,k}] = 1/m \) to its associated user at \( u_i \), through a precoding matrix \( \tilde{V}_i \in \mathbb{C}^{N \times N} \). Then, the received signals go through \( \tilde{U}_i^H \in \mathbb{C}^{N \times N} \).

Per Slivnyak’s theorem \cite{14}, we consider a typical mobile user at the origin \( u_1 = 0 \). Noting that \( s_1 \) has \( m \) non-zero entries, the received signal \( y(m) \in \mathbb{C}^{m \times 1} \) is given by

\[ y(m) = \|d_1\|^{-\frac{\alpha}{2}} \tilde{A}_1^{(m)} s_1^{(m)} + \sum_{i=2}^{\infty} \|d_i\|^{-\frac{\alpha}{2}} (\tilde{H}_i^H)^{(m)} s_i^{(m)} + \tilde{n}^{(m)} \]

with \( \tilde{H}_i^H = \tilde{U}_i^H \tilde{H}_i V_i \), where \( \tilde{H}_i \) is the channel matrix between the BS at \( d_i \) and the typical user. \( \tilde{A}_1 = \text{diag}(\sigma_1, \cdots, \sigma_N) \) and \( \tilde{n} = \tilde{U}_i^H n \) where \( n \sim \mathcal{C}\mathcal{N}(0, \nu I_N) \) is the additive white Gaussian noise. The pathloss exponent is considered as \( \alpha > 2 \). We assume Rayleigh fading, i.e., all the channel coefficients follow the IID \( \mathcal{C}\mathcal{N}(0,1) \). Since \( \tilde{U}_i^H \) and \( V_i \) are unitary, the matrix \( \tilde{H}_i \) is also an IID Rayleigh fading channel matrix and \( \tilde{n} \sim \mathcal{C}\mathcal{N}(0, \nu I_N) \).

From \eqref{eq:1}, the \( k \)th received signal becomes

\[ y_k = \|d_1\|^{-\frac{\alpha}{2}} \sigma_k s_{1,k} + \sum_{i=2}^{\infty} \|d_i\|^{-\frac{\alpha}{2}} (\tilde{H}_{i,k}^H)^{(m)} s_{i,k} + \tilde{n}_k \]

where \( \tilde{H}_{i,k}^H \) is the \( k \)th row vector of \( \tilde{H}_i^H \). We assume that the noise is negligible \cite{14}. The signal-to-interference ratio (SIR) of \( y_k \) is expressed as

\[ \text{SIR}_k = \frac{\|d_1\|^{-\alpha} \sigma_k^2 q_{i,k}}{\sum_{i=2}^{\infty} \|d_i\|^{-\alpha} q_{i,k}} \]

where \( q_{i,k} = \|\tilde{H}_{i,k}^H(s_{i,k})^2\|^\frac{\alpha}{2} \). Since \( q_{i,k} \) is the sum of \( m \) exponential random variables, \( q_{i,k} \) follows Chi-squared distribution with \( 2m \) degree-of-freedom \( \chi^2_{2m} \).

### III. MAIN RESULTS

In this section, we derive the upper and lower bounds of the ergodic SE for \eqref{eq:1} without the noise. The ergodic SE of the typical user is expressed as

\[ r(N, m, \alpha, \lambda) = \mathbb{E} \left[ \sum_{k=1}^{m} \log_2(1 + \text{SIR}_k) \right]. \]

#### A. Eigenvalue Characterization

To characterize eigenvalues, we exploit the asymptotic distribution of eigenvalues instead of the non-asymptotic distribution due to its exceptionally high complexity \cite{15}. From the quarter circle law, the PDF and CDF of the eigenvalue \( X \) of \( H/\sqrt{N} \) \cite{11}, where \( H \) is the \( N \times N \) channel matrix whose entries are distributed as \( \mathcal{C}\mathcal{N}(0,1) \), are given as

\[ f_X(x) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}}, \quad \text{for } 0 < x \leq 4 \]

\[ F_X(x) = \frac{1}{2\pi} \left( \pi + x \sqrt{\frac{4}{x} - 1} - 2 \tan^{-1} \left( \frac{x - 2}{x - 4} \sqrt{\frac{4}{x} - 1} \right) \right). \]

and the PDF and CDF of \( Y = \ln X \) are derived as

\[ g_Y(y) = \frac{1}{\pi} e^y \sqrt{\frac{1}{e^y} - \frac{1}{4}}, \quad \text{for } -\infty < y \leq \ln 4 \]

\[ G_Y(y) = \frac{1}{2\pi} \left\{ \pi + e^y \sqrt{4e^{-y} - 1} + 2 \tan^{-1} \left( \frac{e^{-y}(e^y - 2)}{4e^{-y} - 1} \right) \right\}. \]

We derive Proposition 1 using \eqref{eq:5} and \eqref{eq:6} to approximate the expectation of each eigenvalue \( \sigma^2_i \). Proposition 1 is used to derive the upper and lower bounds of the ergodic SE.

**Proposition 1.** The expectation of the \( i \)-th eigenvalue \( \sigma^2_i \) of an \( N \times N \) matrix, whose entries are IID zero-mean complex random variables with unit variance, is approximated by

\[ \mathbb{E} \left[ \sigma^2_i \right] \approx \frac{N}{4\pi} \left[ -a_i(a_i - 2) \sqrt{a_i - 1} + a_i - 1 \right] \]

\[ + \ln \left( \frac{a_i - 2}{a_i - 1} \right) \]

\[ \leq \mathbb{E} \left[ \sigma^2_i \right] \leq \mathbb{E} \left[ \sigma^2_i \right] \]

where \( a_i = F_X^{-1}(1 - i/N) \), and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \).

**Proof:** See Appendix A.

Using Proposition 1, we can approximate the expectation of \( \sigma^2_i \) only with the parameters \( i \) and \( N \). The numerical validity of Proposition 1 is demonstrated in Section IV.

#### B. Upper and Lower Bounds of Ergodic Spectral Efficiency

Lemma 1 in \cite{15} is used to convert the ergodic SE to an integral form. Leveraging Proposition 1 and Lemma 1, we derive Theorem 1 which is the main result in this paper.

**Theorem 1.** The ergodic spectral efficiency of a typical mobile user in the MIMO cellular model \eqref{eq:1} is bounded by

\[ r(N, m, \alpha) \leq \log_2 e \int_0^{\infty} \frac{1}{\pi} \frac{m - \sum_{k=1}^{m} e^{-z U_{N,k}}}{z^2 F_1 \left( m, -\frac{2}{\alpha}, 1 - \frac{2}{\alpha}, -z \right) } \text{ dz} \]

\[ r(N, m, \alpha) \geq \log_2 e \int_0^{\infty} \frac{1}{\pi} \frac{m - \sum_{k=1}^{m} e^{-z U_{N,k}}}{z^2 F_1 \left( m, -\frac{2}{\alpha}, 1 - \frac{2}{\alpha}, -z \right) } \text{ dz} \]

with

\[ U_{N,k} = \frac{N^2}{4\pi} \left[ -a_k(a_k - 2) \sqrt{a_k - 1} + a_k - 1 \right] \]

\[ + \ln \left( \frac{a_k - 2}{a_k - 1} \right) \]

where \( a_k = F_X^{-1}(1 - k/N) \), and

\[ L_{N,k} = \frac{N}{\pi} \int_{b_k}^{\infty} y e^y \sqrt{\frac{1}{e^y} - \frac{1}{4}} \text{ dy} + \ln N \]

where \( b_k = F_X^{-1}(1 - k/N) \). The function \( F_1(\cdot, \cdot, \cdot) \) is the Gauss-hypergeometric function defined as

\[ F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 (1 - t)^{c-b-1} (e^y - 1)^{a-1} \text{ dy}. \]
Fig. 1. The simulation and approximation results of the expectations of $\sigma^2_i/N$ for the number of antennas $N \in \{4, 8, 16, 32\}$.

Proof: See Appendix B

Under assumptions of a Poisson network model, equal power allocation and no ICI cancellation, the derived upper bound shows the best possible system-level performance of any MIMO non-cooperative cellular network since it is the characterization of the optimal MIMO transceiver technique with respect to ergodic SE.

The proposed bounds provide insight into the optimal number of streams. The ICI term, $2F_1 (m, -\frac{2}{\alpha}, 1 - \frac{2}{\alpha}, -z)$, increases as the number of streams $m$ increases, so the individual SIR of each stream decreases. Also, the multiplexing gain increases as $m$ increases. This implies a trade-off between multiplexing gain and SIR gain with respect to $m$. Thus, it is expected that there exists the optimal number of streams $m^*$ that maximizes the ergodic SE. In the next section, this intuition is validated.

IV. NUMERICAL RESULTS

In this section, we provide the simulation results. Fig. 1 shows the expectation of $\sigma^2_i/N$ for an $N \times N$ channel matrix with $N \in \{4, 8, 16, 32\}$. It illustrates that the approximations closely match with the simulation results even for small $N$.

The obtained bounds and our intuition regarding the optimal number of streams $m^*$ are also validated. Fig. 2(a) shows analytical bounds and simulation results for a pathloss exponent $\alpha \in \{3, 4, 5\}$ and the number of antennas $N \in \{2, 4, \cdots, 16\}$ with $m = N$. In the figure, gaps are reasonably tight; the average gap between the bounds is about 0.24 bps/Hz in Fig. 2(a). Hence, the obtained bounds can estimate the true ergodic SE within the error of 0.24 bps/Hz. It is noticeable that the ergodic SE scales almost linearly with $N$ due to the multiplexing gain and the gap between the bounds narrows with increasing $N \in \{2, 4, \cdots, 16\}$. The ergodic SE also increases as the pathloss exponent $\alpha$ increases since the ICI diminishes faster than the desired signal with the larger $\alpha$.

Fig. 2(b) shows the analytical bounds and simulation results with $N = 8$ and the different number of streams $m$ for $\alpha \in \{3, 4, 5\}$. As observed, the bounds closely match to the simulation results. The curves also confirms the intuition that there exists the optimal point of $m$, beyond which sending more streams degrades the ergodic SE. Specifically, it is more efficient not to use an additional stream with the small eigenvalues due to the trade-off between multiplexing gain and SIR gain as explained from Theorem 1. Moreover, the optimal $m$ is proportional to the pathloss exponent $\alpha$; the ICI term $2F_1 (m, -\frac{2}{\alpha}, 1 - \frac{2}{\alpha}, -z)$ in the derived bounds decreases as the pathloss exponent $\alpha$ increases, resulting the increase of the SIR. Thereupon, as $\alpha$ becomes larger, the multiplexing gain becomes more desirable than the power gain in attempt to maximize the ergodic SE. This leads to the increase of $m^*$.

ICI cancellation using the remaining $N-m$ receive antennas always decreases interference power, which results in the increase of the SIR. Consequently, employing the interference cancellation would lightly lift the ergodic SE curves in Fig. 2(b) except that ergodic SE would remain unchanged at $m = N$, and hence there still exists an optimal value for $m$. 

Fig. 2. The ergodic SE of the analytical bounds and simulation results with (a) the different number of antennas $N$ and the number of streams $m = N$, and (b) the different number of streams $m$ and $N = 8$ antennas for a pathloss exponent $\alpha \in \{3, 4, 5\}$. Here, $m^*$ represents the optimal number of streams for simulation results.
V. CONCLUSION

For a point-to-point SVD-MIMO downlink system, this paper derives upper and lower bounds on ergodic spectral efficiency by modeling a cellular network as a homogeneous Poisson point process and approximating the expectation of the channel eigenvalues. The upper bound on ergodic spectral efficiency applies to any MIMO strategy in a non-cooperative cellular network with equal power allocation across streams and no interference cancellation. We conjecture that there exists an optimal number of streams that maximizes the ergodic spectral efficiency, which is proportional to the pathloss exponent. Incorporating interference cancellation in SVD-MIMO analysis would be desirable for future work.

APPENDIX A

PROOF OF PROPOSITION 1

Let $X$ be the eigenvalue of $H\sqrt{N}$ with the PDF of (5). Since singular values of an IID sub-Gaussian square matrix can take a finite interval values $[17]$ and the empirical distribution of $\sqrt{X}$ tends to circular distribution $[18]$, we divide the PDF in (5) into $N$ non-overlapping regions $A_i$, $i = 1, \cdots, N$, where each region has a probability of 1/N. Considering that $X_i = \sigma_i^2/N$ has the domain of $A_i = (a_i, a_{i-1}]$ with $a_i = F^{-1}(1-i/N)$, $E[X_i]$ can be approximated as

$$E[X_i] \simeq E[X|X \in A_i] = \int_{a_i}^{a_{i-1}} x f_X(x)dx. \quad (12)$$

The PDF of $X$ given $X \in A_i$ is represented as

$$\lim_{\Delta x \to 0} f_X(x|X \in A_i) \Delta x = \lim_{\Delta x \to 0} \frac{Pr(X \in [x, x + \Delta x]|X \in A_i)}{Pr(X \in A_i)}. \quad (13)$$

By the definition of $A_i$, $Pr(X \in A_i) = 1/N$ and

$$Pr(X \in [x, x + \Delta x]|X \in A_i) = \int_{x}^{x+\Delta x} f_X^{(i)}(x)dx \quad (14)$$

where $f_X^{(i)}(x) = f_X(x)$ if $x \in A_i$, and $f_X^{(i)}(x) = 0$ otherwise. Hence, (13) becomes

$$f_X(x|X \in A_i) = \lim_{\Delta x \to 0} \frac{f_X(x)\Delta x}{\int_{x}^{x+\Delta x} f_X^{(i)}(x)dx} = N f_X^{(i)}(x). \quad (15)$$

Finally, we put (15) into (12) with $E[\sigma_i^2] = N E[X_i]$. $lacksquare$

APPENDIX B

PROOF OF THEOREM 1

(Upper bound) Replacing $\text{SIR}_k$ in [4] with [3], we have the ergodic SE expressed as

$$r(N, m, \alpha, \lambda) = E \left[ \sum_{k=1}^{m} \log_2 \left( 1 + \frac{\|d_i\|^2 \sigma_i^2}{\sum_{i=2}^{\infty} \|d_i\|^2 \sigma_i^2 q_k} \right) \right].$$

(a) $E \left[ \log_2 \left( 1 + \frac{\|d_i\|^2 \sigma_i^2}{\sum_{i=2}^{\infty} \|d_i\|^2 \sigma_i^2 q_k} \right) \right] \simeq E \left[ e^{-\alpha \|d_i\|^2 \sigma_i^2 q_k} \right]$, $\mathcal{M}_i(z) = E \left[ e^{-z \|d_i\|^2 \sigma_i^2 q_k} \right] \tag{17}$

where (a) is from Jensen’s inequality and (b) comes from Lemma 1 in [16]. Under Proposition 1, $\mathcal{M}_{su}(z)$ becomes $\mathcal{M}_{su}(z) = E \left[ e^{-z \|d_i\|^2 \sigma_i^2 q_k} \right] \simeq e^{-z L_{N,k}}$, where $L_{N,k}$ is defined in Theorem 1. The Laplace transform of the ICI $\mathcal{M}_I(z)$ is derived by closely following the Appendix D in [19] as $\mathcal{M}_I(z) = 1/2F_1(m, -\frac{z}{\sigma}, 1 - \frac{z}{\sigma}, -z)$. This completes the proof for the upper bound.

(Lower bound) With $\mathcal{M}_{sl}(z) = E \left[ e^{-z \|d_i\|^2 \sigma_i^2 q_k} \right]$ and $\mathcal{M}_I(z)$ in (17), the ergodic SE (3) is lower bounded by

$$r(N, m, \alpha, \lambda) \geq E \left[ \log_2 \left( 1 + \frac{\|d_i\|^2 \sigma_i^2 q_k}{\sum_{i=2}^{\infty} \|d_i\|^2 \sigma_i^2 q_k} \right) \right] \tag{18}$$

The inequality (c) is from Jensen’s inequality and (d) is from Lemma 1 in [16]. The expectation $E[\ln \sigma_i^2]$ can be derived by following the similar steps in the proof of Proposition 1; let $Y_k = \ln(\sigma_i^2/N)$, then $E[Y_k]$ can be approximated by using (7) and (8), instead of (5) and (6) as $E[Y_k] \simeq \frac{1}{2\sqrt{\pi}} \int_{b_{k-1}}^{b_k} y e^{y^2/2} e^{-1/\sqrt{y}} dy$, where $b_k = G^{-1}(1-k/N)$. Since $E[\ln \sigma_i^2] = E[Y_k] + \ln N$, $\mathcal{M}_{sl}(z)$ becomes $\mathcal{M}_{sl}(z) \simeq e^{-z L_{N,k}}$, where $L_{N,k}$ is defined in Theorem 1. We omit $\lambda$ in the ergodic SE $r(\cdot)$ as the derived bounds are not a function of $\lambda$. $lacksquare$

REFERENCES

[1] D. Tse and P. Viswanath, Fundamentals of wireless communication. Cambridge university press, 2005.
[2] A. J. Paulraj, D. A. Gore, R. U. Nabar, and H. Bölcskei, “An overview of MIMO communications-a key to gigabit wireless,” Proc. of the IEEE, vol. 92, no. 2, pp. 198–218, Feb. 2004.
[3] A. Goldsmith, S. A. Jafar, N. Flandal, and S. Vishwanath, “Capacity limits of MIMO channels,” IEEE J. on Sel. Areas in Comm., vol. 21, no. 5, pp. 684–702, Jun. 2003.
[4] J. G. Andrews, F. Baccelli, and R. K. Ganti, “A tractable approach to coverage and rate in cellular networks,” IEEE Trans. on Wireless Comm., vol. 59, no. 11, pp. 3122–3134, Nov. 2011.
[5] M. Di Renzo, A. Guidotti, and G. E. Corazza, “Average rate of downlink heterogeneous cellular networks over generalized fading channels: A stochastic geometry approach,” IEEE Trans. on Comm., vol. 61, no. 7, pp. 3050–3071, May. 2013.
[6] S. T. Veetil, K. Kuchi, A. K. Krishnaswamy, and R. K. Ganti, “Coverage and rate in cellular networks with multi-user spatial multiplexing,” in IEEE Int. Conf. on Comm., Jun. 2013, pp. 5855–5859.
[7] S. T. Veetil, K. Kuchi, and R. K. Ganti, “Performance of PZF and MMSE receivers in cellular networks with multi-user spatial multiplexing,” IEEE Trans. on Wireless Comm., vol. 14, no. 9, pp. 4867–4878, 2015.
[8] H. S. Dhillon, M. Kountouris, and J. G. Andrews, “Downlink MIMO HetNets: Modeling, ordering results and performance analysis,” IEEE Trans. on Wireless Comm., vol. 12, no. 10, pp. 5208–5222, Oct. 2013.
[9] K. Hosseini, W. Yu, and S. R. Adve, “A stochastic analysis of network MIMO systems,” IEEE Trans. on Signal Processing, Aug. 2016.
[10] W. Lu and M. Di Renzo, “Stochastic geometry analysis of multi-user MIMO cellular networks using zero-focusing precoding,” in IEEE Int. Conf. on Comm., Jun. 2015, pp. 1477–1482.
[11] M. Di Renzo and W. Lu, “Stochastic geometry modeling and performance evaluation of MIMO cellular networks using the equivalent-in-distribution (EID)-based approach,” IEEE Trans. on Comm., vol. 63, no. 3, pp. 977–996, Jan. 2015.
[12] V. A. Marčenko and L. A. Pastur, “Distribution of eigenvalues for some sets of random matrices,” Mathematics of the USSR-Sbornik, vol. 1, no. 4, p. 197, 1967.
[13] S. N. Chiu, D. Stoyan, W. S. Kendall, and J. Mecke, Stochastic geometry and its applications. John Wiley & Sons, 2013.
[14] F. Baccelli and B. Blaszczyszyn, *Stochastic geometry and wireless networks*. Now Publishers Inc, 2009, vol. 1.

[15] E. Wigner, “Distribution laws for the roots of a random Hermitian matrix,” *Statistical Theories of Spectra: fluctuations*, pp. 446–461, 1965.

[16] K. A. Hamdi, “A useful lemma for capacity analysis of fading interference channels,” *IEEE Trans. on Comm.*, vol. 58, no. 2, Feb. 2010.

[17] F. Wei, “Upper bound for intermediate singular values of random matrices,” *J. Math. Anal. Appl.*, Aug. 2016.

[18] C. Bordenave and D. Chafai, “Around the circular law,” *Probability Surveys*, vol. 9, Jan. 2012.

[19] J. Park, N. Lee, J. G. Andrews, and R. W. Heath Jr, “On the optimal feedback rate in interference-limited multi-antenna cellular systems,” *IEEE Trans. on Wireless Comm.*, Aug. 2015.