Quantum Localization near Bifurcations in Classically Chaotic Systems

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Abstract. We show that strongly localized wave functions occur around classical bifurcations. Near a saddle node bifurcation the scaling of the inverse participation ratio on Planck’s constant and the dependence on the parameter is governed by an Airy function. Analytical estimates are supported by numerical calculations for the quantum kicked rotor.

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Many mechanisms of localization in dynamical systems have been discussed. The saturation in the spreading of a wave packet in the quantum kicked rotor can be connected to Anderson localization [1, 2]. Other effects, such as trapping due to cantori [3, 4] or in elliptic islands, have a classical origin. The scarring phenomenon, an increase in intensity near classical periodic orbits, seems to be intermediate [5, 6]. Scars are most prominent on orbits that are weakly unstable or marginally stable, as in the stadium billiard. This suggests that near a classical bifurcation, where the semiclassical amplitudes diverge, they should be most prominent. As we will show here this is indeed the case: the classical slowing down in phase space near a bifurcation causes an enhanced eigenfunction amplitude, but there is also a quantum interference between two orbits that brings in quantum modulations. Specifically, we here study the localization of the wave functions in the quantized standard map near saddle node bifurcations of period one orbits.

The quantum kicked rotor is described by a finite unitary matrix which in the momentum plane wave representation is given by

\[ U_{nm} = \frac{1}{N} \sum_{l=0}^{N-1} e^{-ik\cos(\frac{2\pi l}{N})} e^{-i\frac{2\pi}{N}(n-m)} e^{-i\frac{\tau}{2}m^2}. \] (1)

The dimension \( N \) of the momentum basis is related to Planck’s constant by \( \hbar = 4\pi/N \). The parameters \( k \) and \( \tau \) in (1) are connected to \( \hbar \) and the classical kicking strength parameter, \( K \), by \( k = K/\tau \) and \( \tau = 4\pi/N \), where the latter choice ensures the absence of dynamical localization. In the classical limit the quantum kicked rotor corresponds to the standard map defined on a torus,

\[ p_{n+1} = p_n + K \sin \vartheta_n \mod 4\pi, \quad \vartheta_{n+1} = \vartheta_n + p_{n+1} \mod 2\pi. \] (2)

The classical standard map depends on \( K \) only. For \( K > 0 \) the phase space is mixed and at \( K \approx 0.93996... \) the last KAM torus breaks up. For \( K > 5 \) the system is
essentially fully mixing with exponentially small stable islands. However, even for sufficiently large \( K \) values the classical phase space is very complex. Of particular interest here are parameter values near a saddle node bifurcation where pairs of periodic orbits are created or destroyed. Such bifurcations carry large semiclassical weights since the second derivative of the action vanishes and the semiclassical amplitude diverges. The effect of such bifurcations on the spectral statistics has been much studied \[8\]. Here we focus on the wave functions.

The simplest examples of saddle node bifurcations arise when \( K \) passes through \( K^* = 4\pi \). The two pairs of stable and unstable period one orbits are created. Fig. 1 shows the phase space structure for a \( K \)-value slightly above the bifurcation, \( K/4\pi = 1.0216 \).

![Fig. 1 Classical phase space near the bifurcation for \( K/4\pi = 1.0216 \). The bifurcation takes place at \( (\theta_0 = \pi/2, p_0 = 2\pi) \).](image)

The quantity we use to detect localization in wave functions is the average inverse participation ratio of the wave functions, i.e. the fourth moment of their components \( c_{\alpha i} = \langle i|\alpha \rangle \) of the \( \alpha \)'s eigenstate in some orthonormal basis \( |i\rangle \) summed over all eigenstates and normalized by the value expected within random matrix theory for the orthogonal ensemble \[11, 12\],

\[
L = \frac{1}{3} \sum_\alpha \sum_i |c_{\alpha i}|^4. \tag{3}
\]

Thus, if all wave functions are ergodically spread over all basis states, \( L = 1 \). Localized states will show an \( L > 1 \). This is a global quantity that is related to the return probability which can be calculated in the following way. The transition probability from state \( i \) to state \( j \) after \( m \) kicks is \( P_{ij}(m) = |\text{Tr}(A_{ij}U^m)|^2 \), where \( A_{ij} \) is a projector \( A_{ij} = |j\rangle\langle i| \) from the \( i \)-th to the \( j \)-th basis state. The localization measure \( L \) is nothing but the average long time return probability, \( L = 1/3 \sum_i P_{ii} \), where \( P_{ii} = \lim_{M \to \infty} \sum_{m=1}^M P_{ii}(m) \) is the return probability to state \( |i\rangle \).

The quantity \( L \) can also be estimated using the semiclassical procedures proposed in \[9\] for matrix elements. Specifically, let \( A_{ii} = |i\rangle\langle i| \) be the projector on the \( i \)-th basis state. Then \( \rho_{A_{ii}} = \sum_\alpha |\langle i|\alpha \rangle|^2 \delta(\varphi - \varphi_\alpha) \) is the matrix element weighted density of states. The semiclassical expression for \( \rho_{A_{ii}} \) has two terms, \( \rho_{A_{ii}} = \rho_0(A_{ii}) + \rho_{osc} \).
where the first term is the smooth phase space average and comes from the Thomas-Fermi density of states, whereas the latter part can be expressed semiclassically as a sum over periodic orbits. The weight of each periodic orbit $p$ also depends on the integral of the observable along the points of the orbit. The fourth moment that enters the expression for $L$ is a measure of the fluctuations of the matrix elements

$$\langle |\alpha|A_{ii}|\alpha\rangle = |\langle i|\alpha\rangle|^2 = |c_{\alpha i}|^2$$

around their average and can hence be calculated from the square of the sum over periodic orbits. Within the diagonal approximation we therefore obtain

$$\langle |c_{\alpha i}|^4\rangle_\alpha - \langle |c_{\alpha i}|^2\rangle_\alpha^2 \simeq \sum_p |w_p|^2 |A_p^{(i)}|^2$$

(4)

where the $w_p$ contain the stability information about the orbit and the $A_p^{(i)}$ the contribution from the observable [9, 13]. When all orbits are uniformly hyperbolic the random matrix value is recovered. However, in our situation near a bifurcation we get diverging amplitudes $w_p$ which, when expressed in a uniform approximation, are replaced by an Airy function. For the fixed points near $K^*/4\pi = \text{integer}$, we can write

$$\rho_{osc} = \sum_p t_p + t_{bif} \quad \text{where the sum extends over all orbits except the ones near the bifurcation. The weights $t_p$ are given in a conventional form [\text{13}] whereas the bifurcating orbit contribution is $t_{bif} \approx \tilde{A}_{bif} N^{1/6}A_{\text{Ai}}[N^{2/3}(K-K^*)]$. Therefore the fluctuations are expected to show a contribution which is a square of an Airy function}

$$L - 1 \simeq N^{1/3}A_{\text{Ai}}^2[hN^{2/3}(K-K^*)].$$

(5)

In Fig. 2 we present the variation of $L$ as a function of $K$ around the saddle–node bifurcation of period–one orbits at $K^*/4\pi = 1$ for different values of $N$. For $K < K^*$ the random matrix behavior is recovered with increasing $N$. Beyond $K^*$ Airy–type oscillations appear. Both the $K$ and the $L$ values have been rescaled according to the $N$–scaling in [7]. After rescaling the first peaks for different $N$ the curves fall on top of each other. Higher than second moments of the eigenvector components show an even enhanced effect provided the distribution function of $|c_{\alpha i}|^2$ develops a tail due to the presence of localized states.
As an illustration in Fig. 3 we show the state with the largest inverse participation ratio at the value of $K/4\pi = 1.0216$ for $N = 401$. This state is localized at $p = 2\pi$. In $\vartheta$-representation the state is indeed localized around $\vartheta = \pi/2$ and $\vartheta = 3\pi/2$, right at the pair of orbits that is born in the saddle-node bifurcation. Due to the uncertainty principle its spatial extent is larger than in $p$-representation.

![Probability density](image)

**Fig. 3** A localized state at $K = 1.0216$ for $N = 401$ in $p$- (left panel) and $\vartheta$-representation (right panel). The constant line indicates the mean value expected from random matrix theory.

In summary, we have analyzed the localization of quantum eigenstates in the vicinity of bifurcating orbits. The scaling and uniformization predictions based on the semiclassical theory describe the variations of global quantities like the average inverse participation number of the states. We would like to stress that the properties of the fixed points (e.g. through local Lyapunov exponent as in Hellers scar theory) do not explain the localization since the effect of the bifurcation is seen over a larger scale in the parameter $K$ than expected from the appearance of the stable island and the local escape time calculated from the Lyapunov exponent. Such a theory could also not explain the interference pattern seen in the inverse participation ratio.

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