Chemical Principle and PDE of Variational Electrodynamics

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Abstract
We formulate the Chemical Principle condition of variational electrodynamics to select globally-bounded two-body orbits possessing vanishing far-fields. For periodic orbits further satisfying some mild conditions at breaking points we construct a synchronization function in an infinite-dimensional Hilbert space. The partial differential equation (PDE) naturally associated with the Chemical Principle is in the Sobolev space \( W^{2,2}(\mathbb{R}^3) \) and takes the form of a Fredholm-Schroedinger PDE problem with spin-orbit terms. We construct a perturbation scheme about an orbit with a high-frequency standing shock controlled by Weierstrass-Erdmann collisions-at-a-distance which trap the proton at the center position while the atomic far-fields vanish by destructive interference. Last, we formulate the condition to vanish the far-fields with the perturbative scheme and discuss some simple estimates of orbital magnitudes.

Keywords: variational electrodynamics; functional analysis; neutral differential-delay equations; state-dependent delay. (*) This is arXiv:1606.07646v17.

1. Introduction
A. Significance of the problem
Schroedinger’s equation for the hydrogen atom [1] is a partial differential equation (PDE) defined on an infinite-dimensional space. On the other hand, physics derives this PDE from a mechanical problem with the Coulomb force, which is an ordinary differential equation (ODE) on a finite-dimensional space. Here we derive a PDE for the electromagnetic two-body problem of variational electrodynamics [2–4] starting from a physically sensible infinite-dimensional problem. To name a few, some reasons our approach had to wait so long were:

- By the 1930’s there was still no equation of motion for electrodynamics and the equations that came after [2, 5–7] are infinite-dimensional problems in the case of two-body motion.

- The two-body problem of variational electrodynamics [2, 7] involves four state-dependent delays of neutral type and four velocity-dependent denominators[5, 7], a completely different and much harder problem than the Coulomb-mechanical ODE which is often compared to.
Only after 1962 the no-interaction theorem [8] exposed the severe limitations of the ODE quantization program.

Only after the 1970’s the differential-delay equations with state-dependent delays started to be researched and understood as infinite-dimensional problems [2, 9–13] (for an extensive list of references see also [14]).

B. What is this paper about

The Liénard-Wiechert vector fields of a point charge are vector functions \( \vec{V} : (t, \mathbf{x}) \in \mathbb{R}^4 \to \mathbb{R}^3 \) which split in a transversal vector field with modulus decreasing as \( 1/r_{\text{cone}}(t, \mathbf{x}) \) plus a reminder vector field with modulus decreasing as \( 1/r_{\text{cone}}^2(t, \mathbf{x}) \), where \( r_{\text{cone}}(t, \mathbf{x}) : \mathbb{R}^4 \to \mathbb{R} \) is the distance in light-cone from the point charge[15]. The former electromagnetic vector fields are separated by their different asymptotic behavior and are henceforth called the near-field and the far-field, respectively. An example of a globally-bounded two-body orbit possessing globally-bounded far-fields that do not vanish asymptotically is the \( C^\infty \) circular orbit [16, 17].

The Chemical Principle is the condition to select globally-bounded orbits of the time-reversible equations of motion of variational electrodynamics[2, 7] further possessing electric and magnetic far-fields that are bounded almost everywhere and vanish asymptotically by destructive interference. The Chemical Principle problem is the infinite-dimensional \( 2\frac{1}{2} \)-body problem that studies a two-body orbit by placing a (test) third charge to suffer its far-fields.

As we expect orbits with vanishing far-fields to involve velocity discontinuities [18], we develop the functional-analytic ingredients to formulate the infinite-dimensional problem in PDE form. We introduce the orbital properties to achieve distributional synchronization and associate the Chemical Principle problem with a natural PDE defined on a Hilbert space where only second derivatives are discontinuous (i.e., the Sobolev space \( W^{2, 2}(B \subset \mathbb{R}^3) \)). The natural PDE is further extended here to a Fredholm-Schroedinger PDE on the Hilbert space \( W^{2, 2}(\mathbb{R}^3) \).

The Chemical Principle condition is introduced and used here in three different ways:

1. We show that the longitudinal part of the Chemical Principle condition is equivalent to an elliptic PDE along globally-bounded orbits.
2. Again along globally-bounded orbits, the asymptotic limit of the Chemical Principle condition yields a delay condition here called the quasi-semiflow equation.
3. The quasi-semiflow equation is used to eliminate the advanced and the retarded accelerations of the other particle from each equation of motion. The resulting equation of motion involving a single acceleration is used in a perturbative scheme with the proton trapped at the center by Weierstrass-Erdmann collisions-at-a-distance[2].

The technique of distributional synchronization introduced here defines a natural parameter for the Fredholm-Schroedinger PDE and includes the spin-orbit terms. We also derive some necessary conditions to have vanishing far-fields which are imposed by the perturbation theory to obtain some simple estimates of magnitudes.
C. How the paper is divided

This paper is divided into several small sections, as follows: In §2-A we introduce the light-cone condition and associated delay functions, prove Lemma 2.1 about the existence and uniqueness of the delay functions and Lemma 2.3 about a lower bound for the delay functions. In §2-B we calculate the time derivative, spatial derivative and Laplacian derivative of the delay functions for piecewise differentiable trajectories. In §2-C we introduce the necessary properties for distributional synchronization when the velocities are discontinuous. In §2-D we discuss the conditions for our perturbation theory and orbital approximations. We also include Lemma 2.4 about singular denominators along two-leg rotationally-symmetric orbits. In §2-E we classify two-body collisions-at-a-distance and study rotationally-symmetric two-leg collisions. In §2-F we introduce the synchronization function and prove three distributional results: the musical Lemma 2.5 and its converse (Lemma 2.6) for synchronization of breaking points. We also prove Lemma 2.7 about the continuity of the gradient and time derivative of the delay functions and Lemma 2.8 about regular distributions associated with globally-bounded orbits. In §3-A we introduce the Chemical Principle problem starting from an expression for the Liénard-Wiechert fields of point charges. We then formulate the condition to vanish the two-body far-fields acting on a third test charge. In §3-B we derive the quasi-semiflow condition along musical orbits as a consequence of the Chemical Principle. The quasi-semiflow condition is then used in §3-C to eliminate the dependence on the retarded and advanced accelerations of the other charge from each charge’s equation of motion. The electronic equation of motion without the protonic accelerations is studied in §3-D, while the protonic equation of motion without the electronic accelerations is studied in §3-D. We finish §3 by stating the consistency conditions of the perturbation scheme in §3-F and discussing some rough estimates in §3-G. In §4 we derive the Chemical Principle PDE. We start by explaining the natural normed space for such PDE and in §4-A we derive a preliminary PDE from the longitudinal part of the Chemical Principle condition. In §4-B we use the preliminary PDE to construct an educated PDE that can be put in a normed space with second-derivative-only discontinuities, i.e., $W^{2,2}(B)$ where $B$ is a ball containing the globally-bounded orbit. Still in §4-B we extend the Chemical PDE from $W^{2,2}(B)$ to $W^{2,2}(\mathbb{R}^3)$ and formulate a complex PDE. In §4-C we discuss the simplifications and normalizations applicable to the Chemical Principle PDE. Finally, in §4-D we perform the maximum simplification possible in order to leave the PDE still in $W^{2,2}(\mathbb{R}^3)$, which yields a forced linear PDE with spin-orbit terms, here called the Fredholm-Schroedinger PDE problem. In §5-A we summarize our results and in §5-B put the discussions and conclusion. In Appendix 7 we review the Wheeler-Feynman equations of motion and the Weierstrass-Erdmann corner conditions for reference in the paper and in Appendix 8 we put some useful Lemmas and derive several identities and asymptotic expansions of the delay functions which are used to express the electromagnetic fields using the partial derivatives of the delay functions throughout the paper.

2. Delay functions and distributional synchronization

A. The light-cone condition and associated delay functions

We use a unit system where $c \equiv 1$ and the electronic charge and mass are $e_1 \equiv -1$ and $m_1 \equiv 1$ while the protonic charge and mass are $e_2 = 1$ and $m_2 = M_p$, respectively.
The accepted value for the protonic mass is about $M_p \approx 1837$ in our unit system but our calculations are made with an arbitrary protonic mass. We work on the ambient space $\mathbb{R} \times \mathbb{R}^3$ where every point has a time $t$ obtained by Einstein synchronization of clocks and spatial coordinates $x \in \mathbb{R}^3 \equiv (x, y, z)$. Henceforth a globally-bounded trajectory is a continuous function $x_j(t) : \mathbb{R} \to \mathbb{R}^3$ with a bounded image. We reserve the name orbit for a set made of one trajectory for each particle further satisfying the Wheeler-Feynman equations of motion and the Weierstrass-Erdmann corner conditions derived in Ref. [2] and reviewed in Appendix 7. The sub-index $j$ is henceforth used to distinguish the particles and we shall use equivalently either $j = 1$ or $j = \epsilon$ to denote the electronic quantities and either $j = 2$ or $j = p$ to denote the protonic quantities.

State-dependent delays appear in variational electrodynamics because of the light-cone condition. Even though the light-cone construction has a meaning for any chase the particles and we shall use equivalently either the Feynman equations of motion and the Weierstrass-Erdmann corner conditions derived in

For a globally-bounded sub-luminal trajectory $x_j(t)$ we can define the following two light-cone conditions: (i) the advanced light-cone condition is the time for a light signal emitted at point $(t, x)$ to intersect the trajectory $x_j(t)$ at the later time $t^+_j(t, x) \equiv t + \phi^+_j(t, x)$ and (2) the retarded light-cone condition is the time for a signal emitted by the trajectory at an earlier time $t^-_j(t, x) \equiv t - \phi^-_j(t, x)$ to arrive at point $(t, x)$. These former deviating arguments define light-cone maps $t^\pm_j(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by

$$
t^\pm_j(t, x) \equiv t \pm \phi^\pm_j(t, x),
$$

where the delay functions $\phi^\pm_j(t, x)$ are defined implicitly by

$$
\phi^\pm_j(t, x) \equiv |x - x_j(t^\pm)| = |t - t^\pm_j|,
$$

as the Euclidean norm of the spatial separation in $\mathbb{R}^3$, thus defining continuous functions $\phi^\pm_j(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. Here we study globally-bounded continuous and piecewise-differentiable sub-luminal trajectories possessing velocity discontinuities on a countable set of points. For these, one solution to (2) is the accumulation point of the bounded iterative series \{t^n\} defined by $t^+_1 \equiv t$ for $n = 1$ and recursively for $n > 1$ by $t^\pm_{n+1} \equiv t \pm |x - x_j(t^\pm_n)|$. In the following we prove that (2) has a unique solution along sub-luminal orbits.

**Lemma 2.1.** For a globally-bounded sub-luminal trajectory $x_j(t)$ the advanced and retarded light-cone conditions (2) have unique solutions $t^\pm_j(t, x)$.

**Proof.** We start with the future light-cone of a given globally-bounded sub-luminal trajectory $x_j(s)$, defined by Eq. (2) with the plus sign. Our proof by contradiction assumes there are two different future-time-solutions to Eq. (2) at a given $(t, x)$, i.e., $t_a = t + |x - x_j(t_a)|$ and $t_b = t + |x - x_j(t_b)|$. We have the inequalities

$$
t_b - t_a = |x - x_j(t_b)| - |x - x_j(t_a)| \leq |x_j(t_b) - x_j(t_a)|
$$

$$
\leq \int_{t_a}^{t_b} |\dot{x}_j(s)| ds = \int_{t_a}^{t_b} |\dot{x}_j(s)| ds \leq t_b - t_a.
$$

Finally, we have
The inequality sign holds on the right-hand side of (4) when $t_b - t_a \neq 0$ because the velocity satisfies (1) almost everywhere. In view of the left-hand side of (4), the only sensible alternative is $t_b - t_a = 0$ and thus the solution must be unique. The proof of uniqueness of solutions of (2) for the past light-cone is analogous.

**Lemma 2.2.** The condition $\phi^\pm_j(t, x) = 0$ has the unique solution $x = x_j(t)$ which implies $t_j^\pm(t, x) = t$.

**Proof.** The functions $\phi^\pm_j(t, x) \geq 0$ and $t_j^\pm(t, x) = t \pm \phi^\pm_j(t, x)$ defined respectively by Eqs. (3) and (2) are continuous because the trajectories are continuous. For either the plus or the minus case the condition $\phi^\pm_j(t, x) = |x - x_j(t)| = 0$ has the unique solution $x = x_j(t)$, which implies $t_j^\pm(t, x) = t$. □

The following lower bound for $\phi^\pm_j(t, x)$ is useful.

**Lemma 2.3.** For a continuous trajectory $x_j(t)$ the delay functions $\phi^\pm_j(t, x)$ satisfy the lower-bound condition $\phi^\pm_j(t, x) \geq |x - x_j(t)|/2$.

**Proof.** Starting from definition (3) and using the triangular inequality we have

$$\phi_j(t, x) = |x - x_j(t \pm \phi_j)| \geq |x - x_j(t)| - |x_j(t) - x_j(t \pm \phi_j)|$$

where the last inequality holds when $x_j(t)$ possesses a sub-luminal velocity defined almost everywhere inside either $[t, t + \phi_j]$ or $[t - \phi_j, t]$. The proof is completed by passing the last $\phi_j(t, x)$ to the left-hand side and dividing by two for both types of sub-luminal orbits. □

**B. Derivatives of the delay functions**

Starting from the implicit definition (3) and assuming the globally-bounded trajectory $x_j(t)$ possesses a velocity at time $t_j^\pm(t, x) = t \pm \phi_j^\pm(t, x)$, the gradient and time-derivative of $\phi^\pm_j(t, x)$ evaluate to

$$\nabla \phi^\pm_j(t, x) = \frac{\dot{n}_j^\pm}{(1 \pm \hat{n}_j^\pm \cdot v_j^\pm)},$$

$$\frac{\partial \phi^\pm_j}{\partial t}(t, x) = -\frac{\dot{n}_j^\pm \cdot v_j^\pm}{(1 \pm \hat{n}_j^\pm \cdot v_j^\pm)} = \pm \left(|\nabla \phi^\pm_j| - 1\right),$$

where $v_j^\pm \equiv v_j(t_j^\pm)$ and $\hat{n}_j^\pm(t, x)$ stands for a vector field of unitary modulus defined by

$$\hat{n}_j^\pm(t, x) \equiv \frac{x - x_j^\pm}{|x - x_j^\pm|}.$$  

(8)

To simplify the notation, the Cartesian $x$-component of the vector field $\hat{n}_j^\pm(t, x)$ defined by Eq. (8) is henceforth denoted without either the over-hat or the indices $\pm$ or $j$, i.e.,

$$n_x = \frac{x - x_j}{|x - x_j|}.$$  

(9)
with analogous definitions for the \( n_y \) and \( n_z \) components. The spatial derivatives of \( n_x \) are

\[
\begin{align*}
\frac{\partial n_x}{\partial x} &= \frac{1}{\phi_j} - \frac{n_x^2 \pm n_x v_{jx}}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)}, \\
\frac{\partial n_x}{\partial y} &= -\frac{n_x n_y \pm n_x v_{jy}}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)}, \\
\frac{\partial n_x}{\partial z} &= -\frac{n_x n_z \pm n_x v_{jz}}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)}.
\end{align*}
\] (10)

The other six combinations of derivatives are obtained by permuting the indices \( x, y \) and \( z \) in Eqs. (10), (11) and (12). For example, using Eq. (10) and its \( y \) and \( z \) versions we have

\[
\vec{\nabla} \cdot \mathbf{n}_j^\pm = \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} + \frac{\partial n_z}{\partial z} = \frac{2}{\phi_j^2}.
\] (13)

At points where \( \mathbf{x}_j(t_j) \) possesses a derivative \( \mathbf{v}_j \), the time-derivative of \( \mathbf{n}_j \) can be evaluated from (8), yielding

\[
\frac{\partial \mathbf{n}_j}{\partial t} = -\frac{\mathbf{v}_j}{\phi_j} + \frac{\mathbf{\dot{n}}_j \cdot \mathbf{v}_j}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)} (\mathbf{n}_j \pm \mathbf{v}_j) = \frac{(\mathbf{n}_j \cdot \mathbf{v}_j) \mathbf{n}_j - \mathbf{v}_j}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)}.
\] (14)

At points where \( \mathbf{x}_j(t_j) \) possesses an acceleration defined at \( t_j^\pm \), i.e., \( \mathbf{a}_j \equiv d^2 \mathbf{x}_j(t)/dt^2 |_{t_j^\pm} \) exists, the second time-derivative of \( \phi_j(t, \mathbf{x}) \) can be evaluated from the last term of the right-hand side of (7) using (14), yielding

\[
\frac{\partial^2 \phi_j}{\partial t^2} = \frac{|\mathbf{\dot{n}}_j \times \mathbf{v}_j|^2}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^3} - \frac{\mathbf{n}_j \cdot \mathbf{a}_j}{(1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^3},
\] (15)

and the second derivatives \( \frac{\partial^2 \phi_j}{\partial x^2} \) and \( \frac{\partial^2 \phi_j}{\partial x \partial t} \) can be evaluated from the \( x \)-component of (6), yielding

\[
\begin{align*}
\frac{\partial^2 \phi_j}{\partial x^2} &= \left( \frac{1}{1 \pm \mathbf{n}_j \cdot \mathbf{v}_j} \right) \frac{\partial n_x}{\partial x} \pm n_x \frac{\partial^2 \phi_j}{\partial x \partial t}, \\
\frac{\partial^2 \phi_j}{\partial x \partial t} &= \left( \frac{1}{1 \pm \mathbf{n}_j \cdot \mathbf{v}_j} \right) \frac{\partial n_x}{\partial t} \pm n_x \frac{\partial^2 \phi_j}{\partial t^2}.
\end{align*}
\] (16)

Again for orbits possessing two piecewise-defined derivatives, the mixed spatial derivatives of \( \phi_j(t, \mathbf{x}) \) are obtained starting from Eqs. (10), (11) and (12) to calculate the gradient of \( \mathbf{n}_j \cdot \mathbf{v}_j \), yielding

\[
\vec{\nabla} (\mathbf{n}_j \cdot \mathbf{v}_j) = \frac{\mathbf{v}_j}{\phi_j} \pm \left( \frac{\mathbf{n}_j \cdot \mathbf{a}_j}{(1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)} - \frac{(\mathbf{v}_j \pm \mathbf{n}_j \cdot \mathbf{v}_j)}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)} \right) \mathbf{n}_j,
\] (18)

which can be used together with (6) to evaluate the partial derivative \( \frac{\partial^2 \phi_j}{\partial y \partial x} = \frac{\partial^2 \phi_j}{\partial x \partial y} \) with the symmetric formula

\[
\frac{\partial^2 \phi_j}{\partial y \partial x} = - \left( \frac{\mathbf{n}_j \cdot \mathbf{a}_j}{(1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^3} \right) n_x n_y + \frac{1}{\phi_j (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^2} \left( \frac{n_x n_y \mathbf{v}_j^2}{(1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^2} + (n_x v_y + n_y v_x) \right).
\] (19)
Some additional relations involving the delay functions and their first and second derivatives are obtained in Appendix 8.

C. Orbital properties for distributional synchronization

The most general orbits of variational electrodynamics involve absolutely continuous trajectories possessing velocity discontinuities of bounded variation and a Lebesgue-integrable acceleration [2]. Here we consider only orbits whose trajectories possess two derivatives defined piecewise. The delay functions \( \phi_j(t, x) \) must be defined as distributions in order to accommodate the following three situations: (a) the velocity is discontinuous on a countable set and thus only the distributional version of the gradient (6) is sensible, (b) when the velocity reaches the speed of light, the gradient (6) has a divergent denominator and (c) the second derivatives of \( \phi_j \) include singularities of type \( 1/r \), which belong to \( L^2_{loc}(\mathbb{R}^3) \) as found by substituting (10) into (16) and also in formula (19). We distinguish three classes of periodic orbits: (i) the globally continuous velocities have a modulus smaller than the speed of light everywhere, henceforth type-(i) orbits, (ii) the velocities are lesser than the light speed wherever defined and discontinuous on a countable set of breaking points, henceforth type-(ii) orbits and (iii) the velocities are globally continuous and both particles reach the speed of light at a central collision point, henceforth regular collisional or type-(iii) orbits, e.g. the orbits studied numerically in [19].

The presence of velocity discontinuities is an obstacle for the existence of a regular distribution in \( W^{2,2}(\mathcal{B}) \) because of a delta-function distribution coming from the piecewise integration by parts necessary to re-arrange the distributional derivative. For type-(ii) orbits we need some further properties in order to construct a regular distribution. These are henceforth called musical properties, i.e.,

\( (P1) \) Both trajectories are periodic and have the same integer number \( 2N_p \) of breaking points inside each period \( T \).

\( (P2) \) There exists a \( \mu \in \mathbb{R} \) and a \( C^2 \) vector function \( U_e(t) \) which is globally bounded and has the property that the image trajectory \( X_\mu(t) \) defined by

\[ X_\mu(t) \equiv -U_e(t) + \mu x_e(t), \quad \forall t \in \mathbb{R}, \quad (20) \]

is sub-luminal, i.e.,

\[ |dX_\mu/dt| = |\dot{U}_e - \mu \dot{x}_e| < 1, \quad (21) \]

with the possible exception of a countable set of points. Additionally, the protonic breaking points are in a one-to-one correspondence with the electronic breaking points according to

\[ x_p(t^*_\kappa) = X_\mu(t^*_\kappa + \tau), \quad (22) \]

where \( (t_p, t_e) = \{(t^*_\kappa, t^*_\kappa + \tau), \kappa = 1, \ldots, 2N_p \} \) and either \( \tau = 0, \forall \kappa \) or \( \tau = (T/2), \forall \kappa \).

\( (P3) \) At the breaking points \( t^*_\kappa \) the velocities jump discontinuously while satisfying

\[ v^\ell_p(t^*_\kappa) = -\dot{U}_e(t^*_\kappa + \tau) + \mu v^\ell_e(t^*_\kappa + \tau) \equiv V^\ell_\mu(t^*_\kappa + \tau), \quad (23) \]

\[ v^r_p(t^*_\kappa) = -\dot{U}_e(t^*_\kappa + \tau) + \mu v^r_e(t^*_\kappa + \tau) \equiv V^r_\mu(t^*_\kappa + \tau), \quad (24) \]

where \( V_\mu \equiv dX_\mu/dt \) and superscripts \( \ell \) and \( r \) indicate the velocity on the right-hand side and on the left-hand side of the breaking point, respectively.
The type-(ii) orbits that further satisfy properties $P_1$, $P_2$ and $P_3$ are henceforth called *musical orbits*. The one-to-one correspondence (22) takes a protonic breaking point $i_\kappa$ into its mirror electronic breaking point $I_\mu(i_\kappa)$.

D. Singular denominators and high-frequency standing shock

Condition (21) puts a limit on the modulus of the electronic velocity when $|\mu|$ is large and $|\dot{\mathbf{U}}_e| \ll 1$. Here we assume that both velocities are near-luminal at all times and $\mu \simeq -1$, which is the simplest scenario to analyse. The orbit illustrated in Fig. 1 is henceforth called a rotationally symmetric two-leg orbit with a standing shock [7]. Along the former orbit the normal component of each charge’s near-luminal velocity flips sign discontinuously with a fast frequency while each trajectory oscillates longitudinally with the same small amplitude $\tau_{\text{fast}}$ about the respective circular trajectory. We define the inter-particle normal by evaluating (8) at $x_i(t)$, i.e.,

$$\hat{n}^\pm_{ji}(t) \equiv \hat{n}^\pm_{ji}(t, x_i(t)) = \frac{x_i(t) - x^\pm_{ji}}{|x_i(t) - x^\pm_{ji}|},$$ (25)

and we define the angular momentum of the velocity by

$$\vec{\ell}^\pm_{ji}(t_i) \equiv \hat{n}^\pm_{ji} \times v_i(t_i).$$ (26)

The conditions of our perturbation theory about trapped-proton chemical orbits are:

- (c1) The period $T$ is much larger than the maximum particle separation along light-cones, i.e.,

$$\max \{r_p\} \ll T.$$ (27)

- (c2) We assume that the limit $\max \{|x_p(t)|, |x_e(t)|\} \to 0$ becomes a good approximation along symmetrical two-leg orbits when $M_p$ is large.

- (c3) We assume that both velocities rebound up and down in the short time $\tau_{\text{fast}} \ll \max \{r_p\}$, as illustrated in Fig. 1 and we further assume that we have

$$\max |\vec{\ell}^\pm_{ji}(t)| \ll 1.$$ (28)

Figure 1 illustrates a musical orbit involving rotationally symmetric two-leg collisions which have the property that along either trajectory the past and future light-cone points are equivalent by a rotation. At any breaking point the velocities reverse direction discontinuously until the next breaking point. The qualitative details illustrated in Fig. 1 are as follows:

- The protonic breaking point (solid red) and their respective mirror electronic breaking points (solid blue) are radially opposed.

- The particle positions at the same time would be on the same side of the circle, and thus Eq. (22) holds with $\tau = T/2$. 

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Figure 1: Illustrated is a rotationally symmetric planar orbit with a standing shock created by the periodic repetition of an inner and an outer two-leg collision, each satisfying (23) and (24) with the same negative $\mu$. The electronic trajectory is the solid blue line while the protonic trajectory is the solid green line. The dashed black lines indicate the past and future separations in light-cone from each protonic breaking point (solid red dots) to the respective electronic breaking points (solid black dots). The dashed orange lines indicate the separations in light-cone from each electronic breaking point to the past and future protonic breaking points. The mirror electronic breaking point related to each (red) protonic breaking point by (22) is the solid blue dot at the end of each solid black line. The solid black arrows on the electronic trajectory illustrate the ascending and descending segments of the piecewise flight, which define the fast quasi-period $\tau_{\text{fast}} \ll r_p$. Trajectories and magnitudes are not on a realistic scale, for visualization purposes only.
• The velocity flip on the mirror breaking points (blue dots) must be parallel and with the opposite sign relative to the respective protonic breaking points (red dots) to make \( \mu = -\mu^* = \Delta v_{p}^{red}/\Delta v_{e}^{blue} \). For that reason, we chose the mirror of an inner protonic breaking point to be an inner electronic breaking point and the mirror of an outer protonic breaking point to be an outer electronic breaking point (and vice-versa).

The particle trajectories must satisfy the Wheeler-Feynman equations of motion (128), which involve the same two legs of the Weierstrass-Erdmann corner conditions (135) and (136). Since the flipping velocities are supposed to be quasi-luminal, the denominator (134) introduced by (58) and (59) on the future leg of the equation of motion (128) for particle \( i \) is singular (near-zero) when particle \( j = 3 - i \) is diving into particle \( i \) while the denominator of (58) and (59) on the past leg is singular when particle \( j = 3 - i \) is moving away from particle \( i \).

**Lemma 2.4.** For a rotationally symmetric orbit in the limit when \( |\hat{n}_{pe} \cdot v_{e}| \to 1 \) and \( |\hat{n}_{ep} \cdot v_{p}| \to 1 \), a sufficient condition for a singular denominator at all times in either equation of motion or in either of the Weierstrass-Erdmann conditions is

\[
2r_{inn} = n_{inn}T_{fast}, \quad (29) \\
2r_{out} = n_{out}T_{fast}, \quad (30)
\]

where \( T_{fast} \) is the quasi-period illustrated in Fig. 1, \( n_{inn} < n_{out} \) are integers and \( r_{inn} < r_{out} \) are the separations in light-cone at the inner and outer two-leg collisions, respectively.

**Proof.** Let us start from the protonic equation of motion (127) and protonic Weierstrass-Erdmann conditions (135) and (136), which all involve the same velocity denominators. At an inner breaking point for the protonic equation of motion the inner electronic-velocity-discontinuity-points happen simultaneously and the electron starts moving up on both legs until the next (upper) protonic breaking point when the velocities are discontinuous again and the electron starts diving into the proton on both legs. Since the future and past denominators (134) involve opposite signs, there must be one singular denominator at all times if the normal components of the electronic velocities in the past and future light-cone points have the same sign on both legs at all times. The proof for the electronic equation of motion and electronic Weierstrass-Erdmann conditions is analogous.

**E. Collisions-at-a-distance**

As reviewed in Appendix 7, at breaking points the velocity \( v_{i} \) of charge \( i \) is discontinuous and the Weierstrass-Erdmann conditions for a minimizer, (135) and (136), require at least one discontinuity of charge \( j \)’s velocity \( v_{j}^{\pm} \), either at the future light-cone point or at the past light-cone point, or at both. Henceforth, whenever only one of the two light-cone velocities is discontinuous, the breaking point is called a **one-leg-collision**, otherwise the breaking point is called a **two-leg collision**.

In order to express the continuity of the partial momenta (135) and the continuity of the partial energies (136) we introduce two continuous functions to avoid denominators involving \( \pm \) signs due to definition (25): (1) \( \hat{n}_{i}^{\circ} \) is defined as the normal pointing from the charge in the future along the past light-cone of particle \( i \) and (2) \( \hat{n}_{i}^{\circ} \) is defined as
the normal pointing from the charge in the future along the future light-cone of particle $i$. Naturally, the other charge across either of the light-cones is in the past and its normal (25) is reversed, compensating for the sign change in (134) and yielding a version of (134) that involves a positive sign for both past and future denominators, a simplification. We further define

$$
\Delta v_i = v_i' - v_i, \quad (31)
$$
$$
v_i = \frac{1}{2}(v_i' + v_i'), \quad (32)
$$
$$
M'_{i,\ell} = \frac{m_i}{\sqrt{1 - |v_i'|^2}}, \quad (33)
$$
$$
\mathcal{M}_i = \frac{1}{2} \left( \frac{m_i}{\sqrt{1 - |v_i'|^2}} + \frac{m_i}{\sqrt{1 - |v_i|^2}} \right), \quad (34)
$$
$$
\Delta \mathcal{M}_i = \left( \frac{m_i}{\sqrt{1 - |v_i'|^2}} - \frac{m_i}{\sqrt{1 - |v_i|^2}} \right), \quad (35)
$$
$$
r_{ij} = l^b(t, x_i(t)), \quad (36)
$$
$$
\mathcal{W}_j^+ = r_{ij}^+ (1 + \hat{n}_{ij}^b \cdot v_j^+)(1 + \hat{n}_{ij}^b \cdot v_j^+), \quad (37)
$$
$$
\mathcal{W}_j^- = r_{ij}^- (1 + \hat{n}_{ij}^b \cdot v_j^-)(1 + \hat{n}_{ij}^b \cdot v_j^-), \quad (38)
$$
$$
\mathcal{W}_j = \frac{1}{2} (\mathcal{W}_j^+ + \mathcal{W}_j^-). \quad (39)
$$

where again the superscripts $\ell$ and $r$ indicate the left-hand side and the right-hand side of the discontinuity point, respectively. The particle indices are defined by $(i, j) \in \{(\epsilon, p), (p, \epsilon)\}$. For two-leg collisions we further modify the upper-indices of the $\pm$ to denote quantities at the left-hand side or at the right-hand side of either a future or a past light-cone-point. At breaking points the condition of continuity of the partial momenta (135) and the condition of continuity of the partial energies (136) yield the following equations for $(i, j) \in \{(\epsilon, p), (p, \epsilon)\}$

$$
M'_{i,\ell} v_i'^\ell - M'_{i,\ell} v_i = \frac{1}{2r_{ji}^-} \left( \frac{v_j^-}{1 + \hat{n}_{ji}^b \cdot v_j^-} - \frac{v_j^-}{1 + \hat{n}_{ji}^b \cdot v_j^-} \right) + \frac{1}{2r_{ji}^+} \left( \frac{v_j^+}{1 + \hat{n}_{ji}^b \cdot v_j^+} - \frac{v_j^+}{1 + \hat{n}_{ji}^b \cdot v_j^+} \right)
$$
$$
= \left( \Delta v_i^+ + \hat{n}_{ji}^b \times (v_j^+ \times v_j^-) \right) + \left( \Delta v_i^- + \hat{n}_{ji}^b \times (v_j^- \times v_j^-) \right),
$$

$$
\Delta \mathcal{M}_i = \frac{1}{2r_{ji}^-} \left( \frac{1}{1 + \hat{n}_{ji}^b \cdot v_j^-} - \frac{1}{1 + \hat{n}_{ji}^b \cdot v_j^-} \right) + \frac{1}{2r_{ji}^+} \left( \frac{1}{1 + \hat{n}_{ji}^b \cdot v_j^+} - \frac{1}{1 + \hat{n}_{ji}^b \cdot v_j^+} \right)
$$
$$
= \left( \hat{n}_{ji}^b \cdot v_j^+ - \hat{n}_{ji}^b \cdot v_j^- \right) + \left( \hat{n}_{ji}^b \cdot v_j^- - \hat{n}_{ji}^b \cdot v_j^- \right).
$$

The rotational symmetry implies $\mathcal{W}_j^+ = \mathcal{W}_j^- = \mathcal{W}_j = r_{ji}(1 - (\hat{n}_{ji}^b \cdot v_j)^2)$ and we further use (31), (32), (33) and (34) to write $v_i'^\ell = v_i \pm \frac{1}{2}\Delta v_i$ and $M'_{i,\ell} = M_i \pm \frac{1}{2}\Delta \mathcal{M}_i$, which
substituted into the left-hand side of (40) yields

$$\mathcal{M} \Delta \nu_i + \Delta \mathcal{M} \nu_i = \left( \frac{\Delta \nu_j^+ + \Delta \nu_j^-}{2 \mathcal{W}_j} \right) + \left( \hat{n}_{ji} \times (\nu_j^+ + \nu_j'^-) + \hat{n}_{ji} \times (\nu_j^- + \nu_j'^+) \right) \mathcal{W}_j.$$

(42)

The continuity of the partial energy (41) is satisfied by the symmetry along rotationally symmetric two-leg orbits and thus $\Delta \mathcal{M}_i = 0$ for $i \in \{e, p\}$, which further implies that the modulus of each jumping velocity is continuous across collisions by (35). The partial momentum continuity condition (42) in the limit when $|\nu_j \times \nu_j'| = 2|\bar{e}_{ij}| \to 0$ yields

$$\mathcal{M}_e \Delta \nu_e - \left( \frac{\Delta \nu_e^+ + \Delta \nu_e^-}{2 \mathcal{W}_e} \right) = O\left( \frac{\bar{e}_{ep}}{\mathcal{W}_p} \right),$$

(43)

$$\mathcal{M}_p \Delta \nu_p - \left( \frac{\Delta \nu_p^+ + \Delta \nu_p^-}{2 \mathcal{W}_p} \right) = O\left( \frac{\bar{e}_{ep}}{\mathcal{W}_e} \right),$$

(44)

where $e$ and $p$ are image points of each other. Assuming that (27) holds we can approximate the second term on the left-hand side of Eq. (43) by the velocity jump $\Delta \nu_p$ at the image of the electronic point, i.e., $(\Delta \nu_e^+ + \Delta \nu_e^-)/2 \simeq -\Delta \nu_p$. Analogously, when (27) holds the second term on the left-hand side of Eq. (44) can be approximated by the velocity jump at the image of the protonic point, i.e., $(\Delta \nu_p^+ + \Delta \nu_p^-)/2 \simeq -\Delta \nu_e$. Under the former approximations a nontrivial solution in the limit when $|\bar{e}_{ep}|/(\mathcal{W}_e \mathcal{M}_e) \to 0$ requires a vanishing determinant on the left-hand sides of (43) and (44), i.e., $\mathcal{M}_e \mathcal{M}_p \mathcal{W}_e \mathcal{W}_p = 1$, and assuming $r_{\text{in}} \simeq r_{\text{out}} \equiv r$ we have

$$\mu_2 = \frac{\Delta \nu_p}{\Delta \nu_e} = -\frac{1}{\sqrt{\mathcal{M}_p}} \frac{(1 - \nu_p^2)}{1 - (\bar{n}_{ep} \cdot \nu_p)} \frac{(1 - (\bar{n}_{pe} \cdot \nu_e))}{1 - (\bar{n}_{ep} \cdot \nu_e)},$$

(45)

$$r = \frac{1}{\sqrt{\mathcal{M}_p}} \frac{(1 - \nu_p^2)}{1 - (\bar{n}_{ep} \cdot \nu_p)} \frac{(1 - \nu_e^2)}{1 - (\bar{n}_{ep} \cdot \nu_e)},$$

(46)

where we have used $\bar{n}_i = r(1 - (\bar{n}_{ij} \cdot \nu_i)^2)$. Substituting $\mu = -\mu_2 = 1$ into Eq. (45) we have

$$M_p^{2/3} = \frac{(1 - \nu_p^2)}{1 - (\bar{n}_{ep} \cdot \nu_p)^2} \frac{(1 - (\bar{n}_{pe} \cdot \nu_e)^2)}{1 - (\bar{n}_{ep} \cdot \nu_e)^2}.$$  

(47)

Substituting (155) into Eq. (47) in the limit when $|\bar{e}_{ep}| \to 0$ and $|\bar{e}_{ep}| \to 0$ gives the limiting ratio involving the quasi-luminal velocities along the orbit with a high-frequency standing shock illustrated in Fig. 1, i.e.,

$$\frac{(1 - \nu_p^2)}{(1 - \nu_p^2)} = \frac{1}{M_p^{2/3}}.$$

(48)
F. Synchronization function and distributional synchronization

Along musical orbits we further use the image trajectory (20) to construct a synchronization function with an implicit state-dependent definition analogous to the definition of the delay functions (3), i.e.,

$$\phi^{\pm}_\mu(t, x) \equiv |x - X_\mu(t \pm \phi^{\pm}_\mu)|,$$

from where we define the corresponding synchronization times by

$$t^{\pm}_\mu(t, x) \equiv t \pm \phi^{\pm}_\mu(t, x).$$

Notice that since the image trajectory $X_\mu(t)$ is sub-luminal, existence and uniqueness of the synchronization function (49) is granted by Lemma 2.1. A lower bound for the synchronization function is obtained by replacing $x_j(t)$ by $X_\mu(t)$ in Lemma 2.3, yielding

$$\phi^{\pm}_\mu(t, x) \geq \frac{|x - X_\mu(t)|}{2}.$$  \hspace{1cm} (51)

Substituting $v_j$ by $V_\mu \equiv \frac{dX_\mu}{dt}$ into formulas (6) and (7) we obtain the gradient and time derivative of the synchronization function (49), i.e.,

$$\vec{\nabla}\phi^{\pm}_\mu(t, x) = \hat{n}^{\pm}_\mu \left( 1 \pm \hat{n}^{\pm}_\mu \cdot V_\mu \right),$$

$$\frac{\partial \phi^{\pm}_\mu}{\partial t}(t, x) = -\hat{n}^{\pm}_\mu \cdot \frac{V_\mu}{(1 + \hat{n}^{\pm}_\mu \cdot V_\mu)} = \pm \left( |\vec{\nabla}\phi^{\pm}_\mu| - 1 \right),$$

where $\hat{n}^{\pm}_\mu$ is obtained from (8) by replacing $x_j$ with $X_\mu$, i.e.,

$$\hat{n}^{\pm}_\mu(t, x) \equiv \frac{x - X_\mu(t^{\pm}_\mu)}{\phi^{\pm}_\mu(t, x)}.$$  \hspace{1cm} (54)

Next we show that the times $t^{\pm}_\mu(t, x)$ defined by (50) are synchronized with the protonic breaking points $t^\pm_p(t, x)$ along musical orbits.

**Lemma 2.5.** Let $(t, x)$ be such that $t^* \equiv t^\pm_p(t, x)$ as defined by (2) is the time of a protonic breaking point. Then $t^{**} = t^* + \tau = t^\pm_p(t + \tau, x)$ as defined by (50) is the time of an electronic breaking point and $\hat{n}^{\pm}_p(t, x) = \hat{n}^{\pm}_\mu(t + \tau, x)$.

**Proof.** We start with the plus sign and fix $(t, x)$ in order for $t^* = t^\pm_p(t, x)$ to be the time of a protonic breaking point. According to the musical property (22) we have that $t^{**} \equiv t^* + \tau$ is the time of an electronic breaking point. Using definition (2), orbital property (22) and definition (50) for we have

$$t^{**} = t + |x - x_p(t^*)| + \tau = t + \tau + |x - X_\mu(t^{**})| = t^\pm_p(t + \tau, x),$$

where in the last equality we have used definition (50) with the plus sign. The result that $\hat{n}^{\pm}_p(t, x) = \hat{n}^{\pm}_\mu(t + \tau, x)$ follows from property (22) together with definitions (8) and (54). The proof for the minus sign is analogous.

The converse can be shown in the same way by exchanging indices $p$ and $\mu$, i.e.,
Lemma 2.6. Let $(t, \mathbf{x})$ be such that $t^{**} \equiv t^\pm_\mu(t, \mathbf{x})$ as defined by (50) is the time of an electronic breaking point. Then $t^* = -\tau + t^{**} = t^\pm_\mu(t - \tau, \mathbf{x})$ as defined by (2) is the time of a protonic breaking point.

Our next result is about the continuity of the first derivatives of the linear combination $\Delta^\pm(t, \mathbf{x}) \equiv \phi^\pm(t, \mathbf{x}) - \phi^\pm_\mu(t + \tau, \mathbf{x})$.

Lemma 2.7. The functions $\Delta^\pm(t, \mathbf{x}) \equiv \phi^\pm(t, \mathbf{x}) - \phi^\pm_\mu(t + \tau, \mathbf{x})$ possess a continuous derivative and a continuous gradient and a continuous time-derivative if (23) and (24) hold.

Proof. 1. Using (6) and (52), the gradient of $\Delta^\pm(t, \mathbf{x}) \equiv \phi^\pm(t, \mathbf{x}) - \phi^\pm_\mu(t + \tau, \mathbf{x})$ is

$$\nabla \Delta^\pm(t, \mathbf{x}) = \frac{\mathbf{n}^\pm_p(t)}{1 + \mathbf{n}^\pm_p \cdot \mathbf{v}^\pm_p} - \frac{\mathbf{n}^\pm_\mu(t)}{1 + \mathbf{n}^\pm_\mu \cdot \mathbf{v}^\pm_\mu}, \quad (55)$$

Our proof relies on Lemma 2.5 that for any fixed position $\mathbf{x}$ each gradient is a function only of time and the discontinuities happen only at the critical times $t = t^\pm(\mathbf{x})$ when $t^{**} = t^\pm_\mu(t^\pm(\mathbf{x}) + \tau, \mathbf{x}) = t^\pm_\mu(t^\pm(\mathbf{x}) + \tau, \mathbf{x})$ at which times we have $\mathbf{n}_p(t, \mathbf{x}) = \mathbf{n}_\mu(t^{**}, \mathbf{x})$ (also by Lemma 2.5). Conditions (23) and (24) ensure that (55) vanishes either from the left-hand side or from the right-hand side as $t$ crosses each critical time $t^\pm(\mathbf{x})$, and therefore (55) is continuous at breaking points. At all other times both terms on the right-hand side of (55) are continuous and thus $\nabla \Delta^\pm(t, \mathbf{x})$ is continuous.

2. The proof that $\partial_t \Delta^\pm(t, \mathbf{x})$ is continuous uses (7) and (53) to express $\partial_t \Delta^\pm(t, \mathbf{x})$ as

$$\partial_t \Delta^\pm(t, \mathbf{x}) = \frac{\mathbf{n}^\pm_p(t)}{1 + \mathbf{n}^\pm_p \cdot \mathbf{v}^\pm_p} - \frac{\mathbf{n}^\pm_\mu(t)}{1 + \mathbf{n}^\pm_\mu \cdot \mathbf{v}^\pm_\mu}. \quad (56)$$

Again the proof is completed by noticing that for any $\mathbf{x}$ and at the critical times $t = t^\pm(\mathbf{x})$ such that $t^* = t^\pm(t^\pm(\mathbf{x}), \mathbf{x})$ we have by Lemma 2.5 that $\mathbf{n}_p(t^\pm(\mathbf{x}), \mathbf{x}) = \mathbf{n}_\mu(t^\pm(\mathbf{x}) + \tau, \mathbf{x})$ and again conditions (23) and (24) ensure that the time-derivative (56) vanishes either from the left-hand side or from the right-hand side as $t$ crosses each critical point $t^\pm(\mathbf{x})$.

The singularities of the second derivatives (16) and (19) happen when $\phi^\pm(t, \mathbf{x}) = 0$, which are orbital points by Lemma 2.2. Along musical orbits the singularities of the $\phi^\pm(t, \mathbf{x})$ are therefore inside a sphere $B(|\mathbf{x}| < r_3) \subset \mathbb{R}^3$, as used below.

Lemma 2.8. Let $\Delta^\pm(t, \mathbf{x}) \equiv \phi^\pm(t, \mathbf{x}) - \phi^\pm_\mu(t + \tau, \mathbf{x})$ be the linear combination with continuous first derivatives of Lemma 2.7 and $B(|\mathbf{x}| < r_3) \subset \mathbb{R}^3$ an open ball containing the globally-bounded musical orbit. The functions $\Delta^\pm(t, \mathbf{x})$ belong to $\mathcal{W}^{2,2}(B)$ when (23) and (24) hold.

Proof. 1. The functions $\Delta^\pm(t, \mathbf{x})$ are locally integrable and thus define distributions $\mathcal{D}^\pm$ on $B$. Since the gradients of the $\Delta^\pm(t, \mathbf{x})$ are continuous (by Lemma 2.7) the second distributional derivative of each $\mathcal{D}^\pm$ integrated piecewise by parts has a vanishing boundary term and is dominated by the sum of the Sobolev norms of $\phi^\pm(t, \mathbf{x})$ and $\phi^\pm_\mu(t + \tau, \mathbf{x})$. Otherwise the integration by parts generates a singular distribution.
2. The Sobolev norm of $\Delta^\pm$ is defined by
\[
\|\Delta^\pm\|_{W^{2,1}(\mathcal{B})} \equiv \left( \sum_{|k| \leq 2} \int_{\mathcal{B}} (D^k \Delta^\pm)^2 d^3x \right)^{1/2},
\]
which involves a sum of integrals over $\mathcal{B}$ of squared partial derivatives for all multi-
indices $k$ satisfying $|k| \leq 2$ [20], where squared derivatives diverge at the most as
$1/(\phi_p \phi_q)$ by Eqs. (6), (16) and (19).

3. Chemical Principle infinite-dimensional problem

A. Chemical Principle problem postulated from the far-fields

To formulate the Chemical Principle infinite-dimensional problem we start from the
Liénard-Wiechert vector-fields of a point charge [15], i.e.,
\[
E_j^\pm(t, x) \equiv e_j \left(1 - |v_j^\pm|^2\right) \frac{[\nabla \phi_j^\pm]^3}{\phi_j^2} \left(\hat{n}_j^\pm \pm v_j^\pm\right) + e_j \frac{\hat{n}_j^\pm \times (\hat{n}_j^\pm \times J_j^\pm)}{\phi_j},
\]
\[
B_j^\pm(t, x) \equiv -e_j \left(1 - |v_j^\pm|^2\right) \frac{[\nabla \phi_j^\pm]^3}{\phi_j^2} \left(\hat{n}_j^\pm \times v_j^\pm\right) \pm e_j \frac{\hat{n}_j^\pm \times J_j^\pm}{\phi_j},
\]
where
\[
J_j^\pm(t, x) \equiv \frac{1}{(1 \pm \hat{n}_j \cdot v_j)^2} a_j(t) \mp \hat{n}_j \cdot a_j
\]
\[
= \frac{1}{(1 \pm \hat{n}_j \cdot v_j)^2} \left(\hat{n}_j \mp (v_j \times a_j)\right).
\]
The second vector-field on the right-hand side of (58) and (59) is henceforth the far-field
component. The far-field of the globally-bounded orbit is defined by
\[
E_{j, far}^\pm(t, x) \equiv \sum_{j=e,p} \frac{\epsilon_j}{\phi_j} \hat{n}_j^\pm \times (\hat{n}_j^\pm \times J_j^\pm) \rightarrow \frac{1}{r} \hat{r} \times \left(\sum_{j=e,p} \epsilon_j \frac{\hat{n}_j^\pm \times J_j^\pm}{\phi_j} \right),
\]
\[
B_{j, far}^\pm(t, x) \equiv \pm \sum_{j=e,p} \frac{\epsilon_j}{\phi_j} \hat{n}_j^\pm \times J_j^\pm \rightarrow \pm \frac{1}{r} \hat{r} \times \left(\sum_{j=e,p} \epsilon_j \frac{J_j^\pm}{\phi_j} \right),
\]
where $J_j^\pm(t, x)$ is defined by (60) and (61) for $j \in \{e, p\}$. On the last terms of the right-hand sides of (62) and (63) we have used the asymptotic limits (145) and (146), i.e.,

$$\mathbf{n}_j^\pm(t, x) \rightarrow \hat{r} = x/|x|$$

and

$$1/\phi_j^\pm(t, x) \rightarrow 1/|x| \equiv 1/r$$

to obtain the far-fields of the globally-bounded orbit when $|x| \rightarrow \infty$.

Unlike the absorber hypothesis used by Wheeler and Feynman [5] as an asymptotic condition, here we formulate a condition to be enforced at every point $(t, x) \in \mathbb{R}^3$ from sets of zero measure and which complete specification includes the asymptotic condition as well. Substituting $(e_p, e_s) = (1, -1)$ into (62) and (63) we find that the far-fields (62) and (63) are determined at every point $(t, x) \in \mathbb{R}^3$ by

$$J_{fa}^\pm(t, x) = \sum_{j=e, p} e_j J_j^\pm(t, x) = J_p^\pm(t, x) - J_e^\pm(t, x).$$

(64)

Given that $J_p^\pm(t, x)$ and $J_e^\pm(t, x)$ are bounded along musical orbits by Eq. (60), the decomposition of $J_{fa}^\pm(t, x)$ in a component longitudinal to the protonic normal $\mathbf{n}_p^\pm(t, x)$ plus a component transversal to the protonic normal $\mathbf{n}_p^\pm(t, x)$ can be writ as

$$J_{fa}^\pm(t, x) \equiv J_p^\pm(t, x) - J_e^\pm(t, x) = \mathbf{x}_p^\pm(t, x) \hat{n}_p^\pm + \hat{r} \times \mathbf{K}_N^\pm(t, x).$$

(65)

with

$$\mathbf{x}_p^\pm(t, x), \mathbf{K}_N^\pm(t, x) \in L^\infty(\mathbb{R}^3).$$

(66)

Condition (66) is automatically satisfied along musical orbits and ensures that the far-field vector-functions $\mathbf{E}_{fa}^\pm(t, x)$ and $\mathbf{B}_{fa}^\pm(t, x)$ belong to the same $W^{1, 2}(B)$ of Lemma 2.8. The most general planar orbit perpendicular to the $\hat{z}$ direction is given in cylindrical coordinates by

$$\mathbf{K}_N^\pm(t, x) = \mathbf{x}_p^\pm - \frac{z \mathbf{x}_p^\pm}{\rho} \hat{\phi} + \mathbf{x}_z^\pm \hat{z},$$

(67)

with

$$\mathbf{x}_p^\pm(t, x), \mathbf{x}_z^\pm(t, x) \in L^\infty(\mathbb{R}^3).$$

(68)

Substituting (67) into (65) we obtain

$$J_{fa}^\pm(t, x) \equiv J_p^\pm(t, x) - J_e^\pm(t, x) = \frac{r}{\rho} \mathbf{x}_p^\pm(t, x) \hat{\rho} + \left( \frac{z \mathbf{X}_p^\pm - \rho \mathbf{X}_z^\pm}{r} \right) \hat{\phi},$$

(69)

$$\hat{r} \times J_{fa}^\pm(t, x) = \frac{r}{\rho} \mathbf{x}_p^\pm(t, x) \hat{r} \times \hat{\rho} \hat{\phi} + \left( \frac{z \mathbf{X}_p^\pm - \rho \mathbf{X}_z^\pm}{r} \right) \hat{r} \times \hat{\phi},$$

(70)

$$\hat{r} \times \mathbf{K}_N^\pm = \left( \frac{z \mathbf{X}_p^\pm - \rho \mathbf{X}_z^\pm}{r} \right) \hat{\rho} \hat{\phi} + \frac{z \mathbf{X}_z^\pm}{\rho r} \hat{\phi} - \frac{z \mathbf{X}_z^\pm}{r} \hat{z}.$$ 

(71)

According to (69), (70) and (71), a sufficient condition for the asymptotic vanishing of the far-fields (62) and (63) of a globally-bounded planar orbit is

$$\lim_{|x| \rightarrow \infty} |\mathbf{x}_p^\pm(t, x)| = 0,$$

(72)

$$\lim_{|x| \rightarrow \infty} |\hat{r} \times \mathbf{K}_N^\pm(t, x)| = 0.$$ 

(73)

Equation (65) together with (66) and the asymptotic conditions (72) and (73) are henceforth called the Chemical Principle condition.
B. Quasi-semiflow condition

At distant points away from the globally-bounded orbit and along a direction \( \hat{\mathbf{r}} \equiv \mathbf{x}/|\mathbf{x}| \), we can eliminate \( t \) and \( r = |\mathbf{x}| \) from Eqs. (2) and (3) in favour of \( t_e \) and \( t_p \) using the asymptotic form (147), yielding

\[
t_p \pm \hat{\mathbf{r}} \cdot \mathbf{x}_p(t_p) = t_e \pm \hat{\mathbf{r}} \cdot \mathbf{x}_e(t_e). \tag{74}
\]

On the other hand, Eqs (72) and (73) imply the asymptotic vanishing of the right-hand side of (65) and after using (61) to re-write the asymptotic form of (65) and introducing the arbitrary unit direction \( \hat{\mathbf{n}}_\infty \equiv \pm \hat{\mathbf{r}} \) we obtain

\[
\frac{a_j(t_p) - \hat{\mathbf{n}}_\infty \times (\mathbf{v}_p(t_p) \times a_j(t_p))}{(1 + \hat{\mathbf{n}}_\infty \cdot \mathbf{v}_p(t_p))^3} = \frac{a_j(t_e) - \hat{\mathbf{n}}_\infty \times (\mathbf{v}_e(t_e) \times a_j(t_e))}{(1 + \hat{\mathbf{n}}_\infty \cdot \mathbf{v}_e(t_e))^3}, \tag{75}
\]

while Eq. (74) expressed in terms of \( \hat{\mathbf{n}}_\infty \equiv \pm \hat{\mathbf{r}} \) becomes

\[
t_e + \hat{\mathbf{n}}_\infty \cdot \mathbf{x}_e(t_e) = t_p + \hat{\mathbf{n}}_\infty \cdot \mathbf{x}_p(t_p) \to t_p. \tag{76}
\]

In Eq. (76), the last term on the right-hand side is the limit when \( \max(|\mathbf{x}_p(t)|) \to 0 \) along the arbitrary direction \( \hat{\mathbf{n}}_\infty \). Equation (75) for \( t_p \) and \( t_e \) related by (76) is the asymptotic consequence of the Chemical Principle henceforth called the quasi-semiflow condition. Notice that (75) is still of neutral type and it should hold piecewise along the arbitrary direction \( \hat{\mathbf{n}}_\infty \) when \( t_e \in \mathbb{R} \) and \( t_p \in \mathbb{R} \) satisfy (76). The chemical orbits must satisfy the quasi-semiflow differential-delay condition (75) and the Wheeler-Feynman equations of motion (127), and thus we must find a common solution of (75) and (127) to have asymptotically-vanishing far-fields.

C. Equation of motion without the retarded and advanced accelerations

Condition (75) provides a natural way to eliminate the retarded and advanced accelerations of particle \( j \) across the light-cones from the right-hand side of (128), thus yielding a piecewise-defined differential-delay equation involving only the acceleration of particle \( i \) evaluated at the present time, as follows. Using (75) with \( \hat{\mathbf{n}}_\infty \equiv \pm \hat{\mathbf{n}}_{ji}^\pm \) to eliminate \( a_j^\pm \) from the right-hand side of (58), the electric field (58) of particle \( j \) at the position \( \mathbf{x}_j(t_i) \) of particle \( i \) can be expressed as

\[
\mathbf{E}_j^\pm = e_j \left( \frac{1 - |\mathbf{v}_j^\pm|^2}{r_j^\pm (\mathbf{d}_j^\pm)^3} (\hat{\mathbf{n}}_{ji}^\pm \pm \mathbf{v}_j^\pm) \right) + \frac{\hat{\mathbf{n}}_{ji}^\pm \times (\mathbf{d}_j^\pm \times \mathbf{a}_j)}{r_j^\pm (\mathbf{d}_j^\pm)^3} \pm \frac{\hat{\mathbf{n}}_{ji}^\pm \times (\mathbf{v}_j \times \mathbf{a}_j)}{r_j^\pm (\mathbf{d}_j^\pm)^3}, \tag{77}
\]

with

\[
d_j^\pm(t_i) \equiv (1 \pm \hat{\mathbf{n}}_{ji}^\pm \cdot \mathbf{v}_j), \tag{78}
\]

where we have used \( \mathbf{d}_j^\pm \) and \( r_j^\pm \) as given by (134) and (36). Notice that the right-hand side of (77) depends only on \( \mathbf{x}_i, \mathbf{v}_i \) and \( \mathbf{a}_i \) evaluated at the same present time \( t_i \) of the left-hand side of Eq. (128). Analogously, the magnetic field (59) produced by particle \( j \) at the position of particle \( i \) obtained from (77) using \( \mathbf{B}_j^\pm = \mp \hat{\mathbf{n}}_{ji}^\pm \times \mathbf{E}_j^\pm \) is

\[
\mathbf{B}_j^\pm = -e_j \frac{1 - |\mathbf{v}_j^\pm|^2}{r_j^\pm (\mathbf{d}_j^\pm)^3} (\hat{\mathbf{n}}_{ji}^\pm \cdot \mathbf{a}_i) \ell_j^\pm - e_j \frac{\hat{\mathbf{n}}_{ji}^\pm \cdot \mathbf{a}_i \ell_j^\pm}{r_j^\pm (\mathbf{d}_j^\pm)^3} + e_j \frac{\ell_j^\pm}{r_j^\pm (\mathbf{d}_j^\pm)^2}. \tag{79}
\]
when max $|x_p(t)| \to 0$.

The limiting protonic equation of motion has some extra terms over the correspondent equations obtained by exchanging $e$ and $p$ in Eq. (81) because $\tilde{n}_p^\pm \neq \tilde{n}_e^\pm$ even in the limit when $\max |x_p(t)| \to 0$. As the analogue of the instantaneous direction $\tilde{n}_p^\pm(t_p)$ used in §3-D we introduce the instantaneous direction $\tilde{n}_p^\pm(t_p)$ and deviations $\delta_p^\pm(t_p)$ by

$$\tilde{n}_p^\pm(t_p) \equiv -x_p(t_p)/|x_p(t_p)|,$$

$$\delta_p^\pm \equiv \tilde{n}_p^\pm - \tilde{n}_e^\pm.$$

Proof. In the limit when $\max |x_p(t)| \to 0$ the protonic normals coincide and we have $\tilde{n}_p^\pm(\tau_j) = \tilde{n}_e^\pm(\tau_j) = x_e(\tau_j)/|x_e(\tau_j)|$ and thus $\tilde{\ell}_p^\pm = \tilde{\ell}_e^\pm$. The terms of order $\max |x_p(t)|$ added to the right-hand side of (81) are only indicated by $O(\max |x_p(t)|)$.
where \( \hat{n}_p^\pm \) is defined by (25). In the limit when \( \max |x_p(t)| \to 0 \) and assuming that \( \tilde{e}_\infty(t_\varepsilon) \) is almost constant along the orbit we have

\[
\tilde{e}_p^\varepsilon(t_p) \equiv \frac{x_\varepsilon(t_p)}{|x_\varepsilon(t_p)|} - \frac{x_\varepsilon(t_p \pm r_\varepsilon)}{|x_\varepsilon(t_p \pm r_\varepsilon)|} \simeq \pm \hat{n}_p^\varepsilon \times \vec{e}_\varepsilon, \tag{85}
\]

where the constant vector \( \vec{e}_\varepsilon \equiv \langle \tilde{e}_\infty(t_\varepsilon) \rangle \) is the time average of \( \tilde{e}_\infty(t_\varepsilon) \). Equation (85) is a first approximation in powers of \( \vec{e}_\varepsilon \). To express the analogue of Eq. (81) we introduce the corresponding instantaneous vector products with \( \hat{n}_p^\varepsilon(t_p) \) by

\[
\tilde{L}_p(t_p) \equiv \hat{n}_p^\varepsilon(t_p) \times v_p(t_p), \tag{86}
\]

\[
\vec{L}_p(t_p) \equiv \hat{n}_p^\varepsilon(t_p) \times \vec{a}_p(t_p). \tag{87}
\]

In the following we assume that the particles rotate along the orbit illustrated in Fig. 1 with a protonic radius much smaller than the electronic radius and therefore \( |\tilde{L}_p| \ll |\vec{L}_{\infty}| \).

**Theorem 3.2.** In the limit when \( \max |x_p(t)| \to 0 \) and \( |\tilde{L}_p| \ll |\vec{L}_{\infty}| \) and the CP Eqs. (65) and (66) hold, the normal protonic acceleration satisfies

\[
\left( \frac{M_p}{\sqrt{1 - v_p^2}} - \frac{M_p^2 \ell_p^2}{(1 - v_p^2)^{3/2}} + \ell_p^2 c_{\kappa_2}^2 \right) \hat{n}_p^\varepsilon \cdot \vec{a}_p + \hat{d}_{p}^{\kappa_2} \vec{e}_\varepsilon \cdot \vec{L}_p + O(|\tilde{L}_p|) = -f_p(0, x^+, x^-, v^+, v^-) - \ell_p^2 \hat{a}_p(0, x^+, x^-, v^+, v^-) + O(\max |x_p(t)|), \tag{88}
\]

where \( \hat{d}_{p}^{\kappa_4}(t_p) \) and \( c_{\kappa_4}^2(t_p) \) are defined by (149) and (150), respectively and the state-dependent near-field functions \( \hat{d}_p(x_p, x^+, x^-, v^+, v^-) \) and \( f_p(x_p, x^+, x^-, v^+, v^-) \) are defined by (153) and (154). The terms of order \( \max |x_p(t)| \) added to the right-hand side of (88) are only indicated by \( O(\max |x_p(t)|) \) and the local terms of \( O(|\tilde{L}_p|) \) are indicated on the left-hand side of (88).

**Proof.** Taking the scalar product of \( \hat{n}_p^\varepsilon \) with the protonic equation of motion (128) and using (84) and (85) to re-arrange yields

\[
\frac{M_p}{\sqrt{1 - v_p^2}} \hat{n}_p^\varepsilon \cdot \vec{a}_p = \sum \pm c_p d_p^\varepsilon \hat{n}_p^\varepsilon \cdot E^\varepsilon + \sum \pm c_p d_p^\varepsilon v_p \cdot E^\varepsilon - \tilde{e}_\varepsilon \left( \sum \pm c_p d_p^\varepsilon \frac{B^\varepsilon}{2} \right) + \hat{n}_p^\varepsilon \cdot \vec{a}_p \sum \pm c_p v_p \cdot E^\varepsilon \equiv 0, \tag{89}
\]

where we have used \( B_p^\varepsilon = \pm \hat{n}_p^\varepsilon \times E_p^\varepsilon \) to re-arrange and substituted \( m_p = M_p \). The last term of the second line of (89) is calculated using Lemma 7.1 and after using the approximation that \( (\hat{n}_p^\varepsilon \cdot v_p)(v_p \cdot \vec{a}_p) \simeq \hat{n}_p^\varepsilon \cdot \vec{a}_p \) and passing to the left-hand side of (88) it becomes the second term in the parenthesis multiplying \( \hat{n}_p^\varepsilon \cdot \vec{a}_p \). The first term of the second line of (89) is evaluated using (77) with \( j = \varepsilon \) and yields the \( \ell_p^2 \hat{a}_p \) term on the right-hand side of (88) and the last two terms on the left-hand side of (88), plus smaller terms and \( O(|\tilde{L}_p|) \) terms.
F. Enforcing the Chemical Principle via the quasi-semiflow Eq. (75)

A necessary consequence of the Chemical Principle along the rotationally-symmetric planar orbits of §3-D in the limit when $|x_p(t)| \rightarrow 0$ is obtained when (75) and (76) are satisfied in the light-cone condition, i.e., when $\hat{n}^{\infty} = \pm \hat{n}_{pe}$ and $t_p = t_e = \pm r$. Along a planar orbit the former yields two conditions for each sign,

\[
\frac{\hat{n}_{pe} \cdot \mathbf{a}_p(t_p)}{(d_{pe}^{\pm})^3} \bigg|_{t_e, t_p = t_e \pm r} = \pm \frac{\hat{n}_{pe} \cdot \mathbf{a}_e(t_e)}{(d_{pe}^{\pm})^3} \bigg|_{t_e, t_p = t_e \pm r},
\]

where we have used Lemma 8.1 and $d_{pe}^{\pm}(t_e)$ and $d_{pe}^{\pm}(t_p)$ are defined by (78) and (134). Equation (90) enforces (75) with $\hat{n}^{\infty} = \hat{n}_{pe}$ along the direction parallel to $\hat{n}_{pe}$, while (91) enforces (75) with $\hat{n}^{\infty} = \hat{n}_{pe}$ along the planar component perpendicular to $\hat{n}_{pe}$. Notice in Eq. (91) that $\vec{\ell}_{pe}$ and $\vec{\ell}_{ap}$ are defined by (26) and (80) using protonic normals that are different even in the limit when $\max |x_p(t)| \rightarrow 0$ (i.e., $\hat{n}_{pe}^+ \neq \hat{n}_{pe}^-$).

G. Predictions of the perturbation scheme and estimates of magnitudes

The unit system used here has $m_e \equiv 1$, $e^2/c \equiv 1$ and the Bohr radius of atomic physics is

\[
r_B \equiv \hbar^2 \simeq M_p^{4/3}.
\]

Equation (90) requires both normal accelerations to have the same sign. Keeping in mind that the right-hand sides of (81) and (88) do not change sign by Corollary 8.1, the simplest way to accomplish (90) along the orbit of Fig. 1 is when the electronic longitudinal mass is positive on the left-hand side of (81) while the protonic longitudinal mass is negative on the left-hand side of (88). Assuming that Lemma 2.4 holds, a positive longitudinal electronic mass in (81) requires

\[
r > \frac{2\ell_{pe}^2 \sqrt{1 - v_e^2}}{(1 - (\hat{n}_{pe} \cdot v_e)^2)^{3/2}} \simeq \frac{2\ell_{pe}^2}{(1 - v_e^2)^{3/2}},
\]

where a negative longitudinal protonic mass in (88) requires

\[
\ell_{pe}^2 > M_p^{2/3}(1 - v_e^2),
\]

where we have used condition (48) to eliminate $(1 - v_e^2)$ in favour of $(1 - v_e^2)$. Combining (93) and (94) we obtain

\[
r_B > \frac{2M_p^{2/3}}{\sqrt{1 - v_e^2}} > 2M_p^{2/3},
\]

in agreement with (92).
4. Natural PDE and infinite-dimensional normed space

A. Longitudinal part of the Chemical Principle condition and natural PDE

Along musical orbits a natural PDE follows from the longitudinal part of (65). Using Eq. (60), the scalar product of (65) with $\hat{n}_p(t, x)$ yields

$$\frac{\hat{n}_p \cdot a_p}{(1 \pm \hat{n}_p \cdot v_p)^3} = \frac{\hat{n}_p \cdot a_v}{(1 \pm \hat{n}_v \cdot v_v)^3} \pm \frac{\hat{n}_p \cdot \hat{n}_v \cdot (v_v \times a_v)}{(1 \pm \hat{n}_v \cdot v_v)^3} + \mathcal{R}_L^\pm (t, x),$$  

(96)

and after using (143) and a vector identity for the second term on the right-hand side of (96) we obtain

$$\Delta^2 \phi_p^{\pm} = -\frac{\hat{n}_p \cdot a_v}{(1 \pm \hat{n}_v \cdot v_v)^3} \bigg|_{t_\pm(t, x)} + \frac{\hat{n}_v \cdot (v_v \times a_v)}{(1 \pm \hat{n}_v \cdot v_v)^3} \bigg|_{t_\pm(t, x)} - \mathcal{R}_L^\pm (t, x)$$

\[+ \frac{\left| \nabla \phi_p \right|}{\phi_p} \left( 2 + \left| \nabla \phi_p \right|^2 \hat{n}_p \times v_p \right) \bigg|_{t_{\pm}^2(t, x)}, \]

(97)

where the evolution time is indicated under the respective bar on each term of the right-hand side of (97). Equation (97) involves the Laplacian derivative of a function whose partial derivatives of first order are discontinuous. Since derivative discontinuities are unavoidable along chemical orbits [18], the best one can hope for is a normed space where only the second derivatives are discontinuous, as the $\Delta^2$ of Lemma 2.8.

B. Chemical Principle PDE(s) in $\mathcal{W}^{2,2}(B)$ and in $\mathcal{W}^{2,2}(\mathbb{R}^3)$

Here we derive the second-derivative-only discontinuity problem, which is a PDE in $\mathcal{W}^{2,2}(B)$. According to Lemma 2.8 the difference $\Delta^2(t, x) = \phi_p^{\pm}(t, x) - \hat{\phi}_p^{\pm}(t + \tau, x)$ generates a regular distribution in $\mathcal{W}^{2,2}(B)$ whose Laplacian derivative is obtained by evaluating $\Delta^2 \phi_p^{\pm}(t + \tau)$ with (144) and subtracting it from (97), yielding

$$\Delta^2 \Delta^\pm = -\frac{\hat{n}_p \cdot a_v}{(1 \pm \hat{n}_v \cdot v_v)^3} \bigg|_{t_\pm(t, x)} + \frac{\hat{n}_v \cdot (v_v \times a_v)}{(1 \pm \hat{n}_v \cdot v_v)^3} \bigg|_{t_\pm(t, x)} - \mathcal{R}_L^\pm (t, x)$$

\[+ \frac{\left| \nabla \phi_p \right|}{\phi_p} \left( 2 + \left| \nabla \phi_p \right|^2 \hat{n}_p \times v_p \right) \bigg|_{t_{\pm}^2(t, x)} - \frac{\left| \nabla \phi_p \right|}{\phi_p} \left( 2 + \left| \nabla \phi_p \right|^2 \hat{n}_p \times \mathbf{V}_\mu \right) \bigg|_{t_{\pm}^2(t, x)}, \]

(98)

where $\mathbf{V}_\mu \equiv d\mathbf{X}_\mu/dt$ and $a_\mu \equiv d^2\mathbf{X}_\mu/dt^2$. Notice on the right-hand side of Eq. (98) that protonic quantities are evaluated at $t_\pm^\pm(t, x)$ while electronic quantities are evaluated at two different times, i.e., the electronic terms related to $\phi_p^{\pm}(t, x)$ are evaluated at $t^\pm_\mu(t + \tau, x)$ while the remaining electronic quantities are evaluated at $t^\pm_\mu(t, x)$. Equation (98) is our first Chemical Principle PDE involving only second-derivative discontinuities. It is important to keep the following in mind:
1. The right-hand side of (98) contains bounded terms (the first two lines) and singular terms (the third and fourth lines). According to (66), the longitudinal components $\mathcal{X}_L^\pm(t, x)$ are bounded functions that could be chosen to simplify the bounded part of (98). On the other hand, the singular terms of (98) could not be removed by any choice of a $\mathcal{X}_L^\pm(t, x) \in L^\infty (\mathbb{R}^3)$. These singularities absolutely require either $W^{2,2}(\mathcal{B})$ or $W^{2,2}(\mathbb{R})$.

2. For $|x| > \max(|x_p(t)|, |X_\mu(t)|)$ the oscillatory functions $\Delta^\pm(t, x)$ can be approximated using (146), yielding

$$\Delta^\pm(t, x) \simeq \hat{r} \cdot X_\mu(t_{p\mu}^\pm) - \hat{r} \cdot x_p(t_{p\mu}^\pm).$$

3. In the limit when $\max\{|x_p(t)|, |X_\mu(t)|\} \to 0$ we have the limiting forms

$$t_{p\mu}^\pm(t, x) \to (t \pm |x|),$$

$$t_{p\mu}^\pm(t + \tau, x) \to (t + \tau \pm |x|),$$

which are explicitly synchronized since $t_{p\mu}^\pm = t_{p\mu}^\pm + \tau$. On the other hand the times $t_{p\mu}^\pm(t, x)$ do not assume any simple limiting form and fortunately these appear as argument only of the bounded terms of the right-hand side of (98).

4. Equation (98) has an extra reminder that fits as a distribution in $W^{2,2}(\mathcal{B})$, i.e.,

$$\Delta^2 \Delta^\pm(t, x) = \mathcal{X}_L^\pm(t, x) + \frac{(Q_p^\pm(t, x) - Q_\mu^\pm(t, x))}{r} + \left(\frac{1}{\phi_p} - \frac{1}{r}\right)Q_p^\pm(t, x) - \left(\frac{1}{\phi_\mu} - \frac{1}{r}\right)Q_\mu^\pm(t, x),$$

where $\mathcal{X}_L^\pm(t, x) \in L^\infty (\mathbb{R}^3)$ and

$$Q_p^\pm(t, x) = 2|\nabla \phi_p^\pm| + |\nabla \phi_p^\pm|^2 |\hat{n}_p \times \mathbf{v}_p|^2,$$

$$Q_\mu^\pm(t, x) = 2|\nabla \phi_\mu^\pm| + |\nabla \phi_\mu^\pm|^2 |\hat{n}_\mu \times \mathbf{V}_\mu|^2.$$

In the limit when max\{$|x_p(t)|, |X_\mu(t)|\} \to 0$ we can approximate the $1/\phi_p^\pm$ and $1/\phi_\mu^\pm$ singularities on the right-hand side of (102) by $1/r$ while the right-hand side of (99) is a periodic function of $t_{p\mu}^\pm \simeq \tau + t_{p\mu}^\pm \simeq t + \tau \pm |x|$ by (100) and (101). Using the above approximations, Eq. (102) becomes the simpler PDE

$$\Delta^2 \Delta^\pm(t, x) = \mathcal{X}_L^\pm(t, x) + \frac{(Q_p^\pm(t, x) - Q_\mu^\pm(t, x))}{r}.$$
where $w$ is real, $\alpha$ and $\beta$ are complex numbers, $i \equiv \sqrt{-1}$ and again $r \equiv |x|$. From (106) and (107) we further define the square-normalizable complex functions $\Psi : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ and $\Psi^\dagger : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ by

\[
\Psi(t, x) \equiv \mathcal{P}(\varphi) \exp(-qr), \tag{108}
\]

\[
\Psi^\dagger(t, x) \equiv \mathcal{P}(\varphi^\dagger) \exp(-qr), \tag{109}
\]

where $q > 0$ is real and $\mathcal{P}(\varphi)$ is a quasi-polynomial on the variable $\varphi$. The function $\Psi(t, x)$ defined by Eq. (108) inherits a Laplacian derivative and a continuous gradient almost everywhere because $\varphi(t, x)$ possesses these properties by Lemma 2.7. The second-derivative of (108) and (109) introduces again the same $1/r$ singularity because $\Delta^2 r = 2/r$. Our next Lemma shows that $\Psi(t, x) \in \mathcal{W}^{2,2}(\mathbb{R}^3)$ along musical orbits.

**Lemma 4.1.** The function $\Psi(t, x)$ defined by (108) belongs to $\mathcal{W}^{2,2}(\mathbb{R}^3)$ for musical orbits satisfying (23) and (24).

**Proof.** 1. The derivatives of $\Psi(t, x)$ are easily evaluated from (108). The first derivative respect to $x$ is

\[
\frac{\partial \Psi}{\partial x} = \left(-q \hat{r}_x + \mathcal{P}'(\varphi) \frac{\partial \varphi}{\partial x}\right) \exp(-qr), \tag{110}
\]

where $\hat{r}_x \equiv x/r$ and analogous expressions hold for the partial derivatives respect to $y$ and the $z$. The second derivatives respect to $xx$ and $xy$ are

\[
\frac{\partial^2 \Psi}{\partial x^2} = \left(q^2 \hat{r}_x^2 - q \frac{\partial \hat{r}_x}{\partial x}\right) \mathcal{P}(\varphi) + \left(\frac{\partial \varphi}{\partial x}\right)^2 \mathcal{P}'(\varphi) \exp(-qr)
\]

\[+ \mathcal{P}'(\varphi) \left(2 q \hat{r}_x \frac{\partial \varphi}{\partial x} - 2 q \hat{r}_x \hat{r}_x \frac{\partial \varphi}{\partial x}\right) \exp(-qr), \tag{111}
\]

\[
\frac{\partial^2 \Psi}{\partial x \partial y} = \left(q^2 \hat{r}_x \hat{r}_y - q \frac{\partial \hat{r}_x}{\partial y}\right) \mathcal{P}(\varphi) + \left(\frac{\partial \varphi}{\partial y}\right)^2 \mathcal{P}'(\varphi) \exp(-qr)
\]

\[+ \mathcal{P}'(\varphi) \left(2 q \hat{r}_x \frac{\partial \varphi}{\partial y} - q \hat{r}_x \hat{r}_y \frac{\partial \varphi}{\partial x} + \hat{r}_y \frac{\partial \varphi}{\partial x}\right) \exp(-qr), \tag{112}
\]

where $\frac{\partial \hat{r}_x}{\partial y} \equiv \frac{(1-x^2)}{r} + \frac{x y}{r}$ and $\frac{\partial \hat{r}_y}{\partial y} \equiv -\frac{\hat{r}_x}{r}$. The other partial derivatives are obtained by permuting $x$, $y$ and $z$ in the above formulas.

2. We generalize the Sobolev norm for complex functions by

\[
||\Psi||_{\mathcal{W}^{2,2}(\mathbb{R}^3)} \equiv \left(\sum_{|k| \leq 2} \int_{\mathbb{R}^3} (D^k\Psi)(D^k\Psi)^*d^3x\right)^{1/2}, \tag{113}
\]

where the upper star indicates complex conjugation and $k$ is the multi-index of the partial derivative [20]. As discussed under (99), $\Delta^\dagger(t, x)$ is bounded along musical orbits and the gradient of $\varphi(t, x)$ is

\[
\nabla \varphi = \alpha \exp(-i\omega r) \nabla \Delta^\dagger(t, x) + \beta \exp(i\omega r) \nabla \Delta^\dagger(t, x)
\]

\[= -i\omega \left(\alpha \exp(-i\omega r) \Delta^\dagger(t, x) - \beta \exp(i\omega r) \Delta^\dagger(t, x)\right), \tag{114}
\]
where $\hat{r} \equiv \frac{\hat{r}}{|\hat{r}|}$. Notice on (114) that along musical orbits the modulus $|\hat{\nabla}\varphi|$ is bounded and the second derivatives $\frac{\partial^2 \varphi}{\partial x^2 y^2}$ and $\frac{\partial^2 \varphi}{\partial x^2 z^2}$ are a sum of terms which diverge at the most as $1/r$ or $1/\phi_j$ according to Eqs. (6), (16) and (19) and therefore are finitely square-integrable. The former is sufficient for the decreasing exponential factor on (108), (110), (111) and (112) to dominate the quasi-polynomials and ensure the integrability in $\mathbb{R}^3$ of $|\Psi(t, x)|^2$ and its squared derivatives up to order two as defined by (113), proving that $\Psi(t, x) \in H^2 \equiv W^{2,2}(\mathbb{R}^3)$. The details about integrating the divergencies in $\mathbb{R}^3$ are the same outlined in the proof of Lemma 2.8.

\[ \square \]

C. A normal form for the Chemical Principle PDE

In order to simplify (105) we define the set $\mathcal{A}(\omega) \subset W^{2,2}(\mathbb{R}^3)$ of all complex functions $\Psi(t, x)$ of the form (108). Let $\hat{e}(\omega)$ be a musical orbit satisfying (23) and (24) and reals $\omega$ and $q$ defined in (106) and (108) respectively. The definition of $\mathcal{A}(\omega)$ it is enough for our purposes here but it could be further enlarged to quasi-polynomials of $\varphi(t, x)$ and $r = |x|$. We define the Schroedinger linear operator $S: \Psi(t, x) \in \mathcal{A}(\omega) \subset W^{2,2}(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ by

\[
S(\Psi(t, x)) \equiv \frac{1}{2q} \Delta^2 \Psi + \frac{1}{r} \Psi + i(\nu + \frac{\omega}{q})\partial_t \Psi,
\]

where $\nu$ is real and again $r \equiv |x|$ and $i \equiv \sqrt{-1}$. Notice on the right-hand side of (115) that the following quantities belong to $L^2(\mathbb{R}^3)$: (a) $\partial_t \Psi$ is a continuous function because $\partial_t \Delta^2$ is continuous by Lemma 2.7 and moreover $\partial_t \Psi \in L^2(\mathbb{R}^3)$ because it is a quasi-polynomial times a decreasing exponential and (b) the second term on the right-hand side of (115), (namely $\frac{1}{r}$), belongs to $L^2(\mathbb{R}^3)$ because the integration element $d^4 x = r^2 d\Omega$ of $\mathbb{R}^3$ cancels the factor $1/r^2$ and the remaining $|\Psi(t, x)|^2$ is integrable because the decreasing exponential dominates any quasi-polynomial. Finally, to evaluate the action of $S$ on $\mathcal{A}(\omega) \cap W^{2,2}(\mathbb{R}^3)$ we need the Laplacian derivative and the time derivative of $\Psi(t, x)$ from (108), (110), (111) and (112), i.e.,

\[
\begin{align*}
\Delta^2 \Psi(t, x) &= \left( q^2 - \frac{2q}{r} \right) P + (\Delta^2 P - 2q \hat{r} \cdot \hat{\nabla} P) \exp(-qr), \\
\partial_t \Psi(t, x) &= \partial_t P \exp(-qr).
\end{align*}
\]

We henceforth assume $P(\varphi(t, x)) = \varphi(t, x)$ and in the following we use (115) to simplify (105). Substituting $P(\varphi) = \varphi(t, x)$ and Eqs. (116) and (117) into Eq. (115) and using (105) yields

\[
S(\Psi(t, x)) = i\nu \partial_t \Psi + \left( \frac{G}{2q} + \frac{(q^2 - \omega^2)\varphi}{2q} + i\omega \varphi \hat{r} \cdot \hat{\nabla} \varphi \right) \exp(-qr)
- \frac{i\alpha \omega}{q} \left( (\hat{r} \cdot \hat{n}^+ - 1) \hat{\nabla} \phi^+ - (\hat{r} \cdot \hat{n}^- - 1) \hat{\nabla} \phi^- \right) \exp(-i\omega r - qr)
+ \frac{i\beta \omega}{q} \left( (\hat{r} \cdot \hat{n}^- - 1) \hat{\nabla} \phi^- - (\hat{r} \cdot \hat{n}^+ - 1) \hat{\nabla} \phi^+ \right) \exp(i\omega r - qr)
+ \frac{1}{qr} \left( \partial_t \varphi \hat{r} - i\omega \varphi \hat{r} + \varepsilon \right) \exp(-qr),
\]

(118)
where we have defined

\[
G \equiv \alpha \exp(-i\omega r)X_+^+(t, x) + \beta \exp(i\omega r)X_-^-(t, x),
\]

\[
\varepsilon \equiv \frac{\alpha}{2} \exp(-i\omega r) \left( |\hat{\nabla}\phi^+_\mu|^2 \hat{n}_\mu \times \hat{v}_\mu - |\hat{\nabla}\phi^-_\mu|^2 \hat{n}_\mu \times \hat{V}_\mu |^2 \right) + \frac{\beta}{2} \exp(i\omega r) \left( |\hat{\nabla}\phi^-_\mu|^2 \hat{n}_\mu \times \hat{v}_\mu - |\hat{\nabla}\phi^+_\mu|^2 \hat{n}_\mu \times \hat{V}_\mu |^2 \right).
\]

Notice that the complex function \( \varepsilon(t, x) \) on the right-hand side of (118) comes from each second term on the right-hand sides of (103) and (104) while the first terms on the right-hand side of (103) and (104) generated the term proportional to \( \partial_t \varphi^\dagger \) of the last line on the right-hand side of (118) by use of (7) and (53).

**D. Fredholm-Schroedinger PDE associated with the Chemical Principle**

Along a musical orbit the functions \( \varphi, \varphi^\dagger \) and \( \partial_t \varphi^\dagger \) and \( X^\pm_\mu(t, x) \in L^\infty(\mathbb{R}^3) \) and thus the fourth line on the right-hand side of (118) contains a singularity which could not be canceled by any \( X^\pm_\mu(t, x) \in L^\infty(\mathbb{R}^3) \). Therefore we do not derive a linear Schroedinger equation with a pure \( 1/r \) term but rather a Fredholm-Schroedinger PDE with a spin-orbit term, i.e.,

\[
\frac{1}{2g} \Delta^2 \Psi + \frac{1}{r} \Psi + iv \partial_t \Psi = \frac{1}{qr} \left( \partial_t \varphi^\dagger - i\omega \varphi^\dagger + \varepsilon \right) \exp(-qr) + \mathcal{R} \exp(-qr),
\]

where the reminder \( \mathcal{R} \) is defined by

\[
\mathcal{R} \equiv \left( \frac{1}{\phi_r} - \frac{1}{r} \right) \mathcal{R}_\varphi - \left( \frac{1}{\phi_\mu} - \frac{1}{r} \right) \mathcal{R}_\mu + \mathcal{R}_K(K^+_L, K^-_L),
\]

with

\[
\mathcal{R}_\varphi \equiv \frac{\alpha}{2g} \exp(-i\omega r)Q^+_\varphi + \frac{\beta}{2g} \exp(i\omega r)Q^-_\varphi,
\]

\[
\mathcal{R}_\mu \equiv \frac{\alpha}{2g} \exp(-i\omega r)Q^+_\mu + \frac{\beta}{2g} \exp(i\omega r)Q^-_\mu,
\]

and \( Q^\pm_\varphi \) and \( Q^\pm_\mu \) defined in (103) and (104). The last term on the right-hand side of (122) includes the first three lines of the right-hand side of (118) in the square-integrable combination \( \mathcal{R}_K(K^+_L, K^-_L) \exp(-qr) \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \).

In order to have \( S(\Psi) = 0 \) in \( A_{(a)} \cap W^{2,2}(\mathbb{R}^3) \), the pre-factor of the singularity would have to vanish identically along the orbit, i.e.,

\[
\partial_t \varphi^\dagger - i\omega \varphi^\dagger + \varepsilon(t, x) = 0,
\]

a simplification we do not have in general. Alternatively, (125) can be used to find optimal values for \( \alpha, \beta \) and \( \omega \) in order to minimize the pre-factor. Equation (125) is easy to approximate along musical orbits in the limit when \( \max \{|x_\varphi(t)|, |x_\mu(t)|\} \to 0 \) because it involves only the distributionally synchronized times \( t^\pm_\mu \simeq t^\pm_\varphi = t + \tau \pm |x| \).
defined by (100) and (101). We can rewrite (121) as a two-component (complex) linear
PDE with a nonlinear term belonging to $W^{2,2}(\mathbb{R}^3)$, i.e.,
\[
\frac{1}{2q} \Delta^2 \Psi + \frac{1}{r} \Psi + i\nu \partial_t \Psi - \frac{1}{qr} (\partial_t \Psi \dagger - i\nu \Psi \dagger) = \frac{c}{qr} \exp(-qr) + R \exp(-qr). \tag{126}
\]

The value of $1/q$ is twice the coefficient of the Laplacian derivative in Eq. (126) and sets
the scale for the PDE. We henceforth use it as another name for the Bohr radius, i.e.,
$\frac{1}{q} \equiv r_B$.

5. Summary, discussions and conclusion

A. Summary of results
- In a nutshell, the Chemical Principle uses the (restricted) three-body problem
to classify orbits of the two-body problem. Unlike linear stability analysis, the
Chemical Principle implements a defensive stability of the orbit against crashing
into third particles and boundaries by selecting orbits that will not disturb a third
charge or boundaries located anywhere outside its near-field region.
- We have associated the longitudinal component of the Chemical Principle condition
(65) with a PDE that takes the form of a complex Fredholm-Schroedinger equation
with a spin-orbit term plus a nonlinear forcing.
- We found a use for the light-cone maps outside the trajectories after expressing the
Liénard-Wiechert far-fields in terms of Laplacian derivatives using (143).
- We introduced a perturbation theory to enforce the Chemical Principle condition
along orbits with a standing shock. The resulting trajectories involve piecewise-
quasi-straight-light motions with many rebounds while the trapped proton has
a large acceleration and the protonic far-fields cancel the electronic far-fields by
destructive interference at infinity.

B. Discussions and Conclusion
- The two-body motion satisfying the Chemical Principle would be impossible with-
out Weierstrass-Erdmann rebounds and moreover its velocity is not in the range
of small perturbations of an orbit of Newtonian mechanics with the Coulomb po-
tential.
- Newton’s third law limits the protonic to electronic acceleration ratio along Coulomb-
ODE orbits to $1/M_p$, and thus our derivation is quite different from Nelson’s deriva-
tion of a linear Schroedinger equation from Newtonian mechanics [23]. Moreover,
we start from the variational generalization of Wheeler-Feynman’s electrodynam-
ics [2, 7] rather than from the neighborhood of an orbit of a Hamiltonian ODE
[23]. Our derivation is limited to singularities of type $1/|x - x_o(t)|$, which appear
naturally in the second derivative of the delay functions and the Liénard-Wiechert
potentials. Fortunately the $1/|x - x_o(t)|$ singularity is enough for atomic physics,
chemistry and variational electrodynamics [2].

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• The failure of the famous Wheeler-Feynman 20th-century quantization program [3, 4] had at least two causes: (i) the misleading analogy with finite-dimensional Hamiltonian ODE’s and energy formulas for conserved quantities of Newtonian few-body problems, which inconsistency was harder to detect before the no-interaction theorem [8] and (ii) Wheeler and Feynman were working before the full understanding that delay equations are infinite-dimensional problems [11, 9, 12, 13, 2] and the mathematics to deal with such problems was not out yet, e.g. the theory of distributions, the theory of infinite-dimensional normed spaces and the theory of semi-flows, just to name a few.

• So far our predictions with the Chemical Principle agree with the hydrogenoid magnitudes with an error of the same order of the orbital magnitudes and spectral lines calculated by its precursor, the generalized absorber condition tested for hydrogenoid atoms along orbits with standing high-frequency shocks in Ref. [7]. A comparison of variational electrodynamics with the modern spectroscopic precision should wait for the understanding of the relation between the Rydberg-Ritz Principle and the Chemical Principle [7]. The frequency necessary to disarm the high-frequency standing modes is an ultra-violet frequency, in accordance with present-day physics understanding (present-day physics calls it high-energy because of the Rydberg-Ritz Principle).

• In 1963, Schild [16] tried to construct an electromagnetic model for the neutron using quantization rules with no success. We notice that the Chemical Principle can offer an alternative model for the neutron.

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7. Appendix: the Weierstrass-Erdmann corner conditions

A. Wheeler-Feynman equations of motion and Weierstrass-Erdmann corner conditions

The conditions for a type-(ii) minimizer of variational electrodynamics[7, 2] are

- a. To satisfy the Wheeler-Feynman equations of motion on the \( \widehat{C}^2 \) segments, which by use of (25) and \( B_j^\pm = \mp \hat{n}_j^\pm \times E_j^\pm \) [15] are expressible as

\[
\frac{m_i \dot{a}_i(t)}{\sqrt{1 - v_i^2(t)}} = \epsilon_i [E_j - (v_i \cdot E_j) v_i + v_i \times B_j] \quad (127)
\]

\[
= \frac{\epsilon_i}{2} \sum_{\pm} (1 \pm \hat{n}_j^\pm \cdot v_i) E_j^\pm - \frac{\epsilon_i}{2} \sum_{\pm} (v_i \cdot E_j^\pm)(v_i \pm \hat{n}_j^\pm), \quad (128)
\]

in agreement with equation (38) of Ref. [7], equation (2.2) of Ref. [17] and equation (23) of Ref. [24]. On the right-hand side of (127) and (128) are the fields of particle
\( j \equiv (3 - i) \) at the position \( \mathbf{x}_i(t) \), i.e.,

\[
E_j \equiv \frac{1}{2} \left( E^+_j(t, \mathbf{x}_i(t)) + E^-_j(t, \mathbf{x}_i(t)) \right) , \quad \text{(129)}
\]

\[
B_j \equiv \frac{1}{2} \left( B^+_j(t, \mathbf{x}_i(t)) + B^-_j(t, \mathbf{x}_i(t)) \right) , \quad \text{(130)}
\]

evaluated by the semi-sum of the Liénard-Wiechert fields (58) and (59). The deviating arguments on the right-hand side of Eq. (127) involve two maps from particle \( i \)'s time into a time along particle \( j \)'s trajectory: (1) the \textit{retarded} map taking time \( t_i \equiv t \) to the time \( t_j^- (t) \) along trajectory \( j \) when particle \( j \)'s trajectory intersects the boundary of the past light-cone of \( (t, \mathbf{x}_i(t)) \) and (2) the \textit{advanced} map taking the time \( t_i \equiv t \) to the time \( t_j^+ (t) \) along trajectory \( j \) when particle \( j \)'s trajectory intersects the boundary of the future light-cone of \( (t, \mathbf{x}_i(t)) \), i.e.,

\[
t \rightarrow t_j^- (t) \equiv t_j^- (t, \mathbf{x}_i(t)), \quad \text{(131)}
\]

\[
t \rightarrow t_j^+ (t) \equiv t_j^+ (t, \mathbf{x}_i(t)), \quad \text{(132)}
\]

where the \( t_j^\pm (t, \mathbf{x}) \) are defined by (2). The Liénard-Wiechert fields (58) and (59) include denominators which depend on the arbitrary direction \( \hat{n}_j^\pm (t, \mathbf{x}) \) via the inverse of the modulus of (6), i.e.,

\[
(1 \pm \hat{n}_j^\pm \cdot \mathbf{v}_j^\pm) = \frac{1}{|\nabla \phi_j^\pm (t, \mathbf{x})|} , \quad \text{(133)}
\]
henceforth Liénard-Wiechert \textit{directional denominators}. Along the trajectories these directional denominators on the fields on the right-hand side of (127) define a function of time obtained by evaluating (133) at \((t, \mathbf{x}_i(t))\) and using (25), yielding

\[
D_j^\pm (t) \equiv 1 / |\nabla \phi_j^\pm (t, \mathbf{x}_i(t))| = 1 \pm \hat{n}_j^\pm \cdot \mathbf{v}_j^\pm , \quad \text{(134)}
\]

here called \textit{velocity denominators}.

\[\bullet\] b. At breaking points a type-(ii) minimizer of variational electrodynamics should further satisfy the Weierstrass-Erdmann continuity conditions for the partial momenta, \( P_i^r = P_i^\ell \), and the continuity of the partial energies \( E_i^r = E_i^\ell \), where superscripts \( r \) and \( \ell \) indicate the quantity on the left-hand side and on the right-hand side of the discontinuity point, respectively, and

\[
P_i^{(r, \ell)} \equiv \frac{m_i \mathbf{v}_i}{\sqrt{1 - \mathbf{v}_i^2}} - \left( \frac{\mathbf{v}_j^-}{2r_j^- D_j^- (t)} + \frac{\mathbf{v}_j^+}{2r_j^+ D_j^+ (t)} \right)_{r, \ell} , \quad \text{(135)}
\]

\[
E_i^{(r, \ell)} \equiv \frac{m_i}{\sqrt{1 - \mathbf{v}_i^2}} - \left( \frac{1}{2r_j^- D_j^- (t)} + \frac{1}{2r_j^+ D_j^+ (t)} \right)_{r, \ell} , \quad \text{(136)}
\]

where \( r_j^\pm \) are the (continuous) inter-particle distances (36). Again, the indices \( r \) and \( \ell \) used as superscripts and subscripts on (135) and (136) indicate the left-hand side or the right-hand side of the respective (past or future) breaking point.
The following simple result is useful.

Lemma 7.1. Inside each piecewise segment where the acceleration \( a_i(t) \) is defined we have

\[
m_i \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v_i^2(t)}} \right) = v_i \cdot E_j = \sum_{\pm} e_i \frac{v_i \cdot E_{j\pm}}{2}. \tag{137}
\]

Proof. After taking the scalar product of Eq. (127) with \( v_i \), dividing both sides by \((1 - v_i^2)\) and re-arranging we have the result. \( \square \)

Finally, the many-body problem of variational electrodynamics has an equation of motion for a particle of charge \( e_3 \) and mass \( m_3 \) acted upon by the field of the others, i.e.,

\[
m_3 \frac{d}{dt} \left( \frac{v_3}{\sqrt{1 - v_3^2}} \right) = e_3[E(t, x_3(t)) + v_3 \times B(t, x_3(t))], \tag{138}
\]

where \( x_3(t) \) and \( v_3(t) \) are the position and velocity of the third particle[7, 6, 15]. The fields on the right-hand side of Eq. (138) are created by all other charges but \( e_3 \) and Eq. (138) is supposed to hold almost-everywhere [2]. Equation (138) is actually the first of three variational conditions for a minimizer of the respective many-body problem, the other two being the Weierstrass-Erdmann corner conditions at velocity discontinuity points [2, 22]. The vector functions of time \( E(t, x_3(t)) \) and \( B(t, x_3(t)) \) on the right-hand side of (138) are the semi-sums of vector fields (58) and (59) along the trajectory \( x_3(t) \), i.e.,

\[
E(t, x_3(t)) = \frac{1}{2} \sum_{j \neq 3} \left( E^+_{j}(t, x_3(t)) + E^-_{j}(t, x_3(t)) \right), \tag{139}
\]

\[
B(t, x_3(t)) = \frac{1}{2} \sum_{j \neq 3} \left( B^+_{j}(t, x_3(t)) + B^-_{j}(t, x_3(t)) \right). \tag{140}
\]

The sign \( \sum_{j \neq 3} \) on the right-hand side of (139) and (140) is a reminder that the fields of charge \( e_3 \) do not contribute to its equation of motion [2]. For the restricted three-body problem of the Chemical Principle problem the right-hand sides of (139) and (140) include the fields of the hydrogen atom only, i.e., the electronic and the protonic fields.

8. Appendix: Usable expressions, definitions and limits

In this Appendix we derive some useful expressions involving the partial derivatives of the delay functions and some identities, Lemmas and asymptotic limits that are used throughout the paper. For brevity of notation we drop the upper indices (±) indicating the advanced and the retarded delay functions \( \phi^+_{ij}(t, x) \) and \( \phi^-_{ij}(t, x) \), an economy of notation often used here when there is no risk of ambiguity. Using (15) together with the \( y \) and \( z \) versions of (16) and (17) to yield the second derivatives respect to \( y \) and \( z \) we obtain

\[
\Delta^2 \phi_j - \partial^2_t \phi_j = \frac{2}{\phi_j(1 \pm \hat{n}_j \cdot v_j)}, \tag{141}
\]
where \( \Delta^2 \phi_j \equiv \partial_x^2 \phi_j + \partial_y^2 \phi_j + \partial_z^2 \phi_j \) is the Laplacian derivative of \( \phi_j(t, x) \) and we have used (14) to cancel the term

\[
n_x \partial_x n_x + n_y \partial_y n_y + n_z \partial_z n_z = 0. \tag{142}
\]

Substituting (15) into (141) yields the Laplacian derivative of \( \phi_j(t, x) \) at points where \( x_j(t_j) \) possesses two derivatives, i.e.,

\[
\Delta^2 \phi_j = -\frac{\hat{n}_j \cdot \mathbf{a}_j}{(1 \pm \hat{n}_j \cdot \mathbf{v}_j)^3} + \frac{|\hat{n}_j \times \mathbf{v}_j|^2}{\phi_j(1 \pm \hat{n}_j \cdot \mathbf{v}_j)^3} + \frac{2}{\phi_j(1 \pm \hat{n}_j \cdot \mathbf{v}_j)}, \tag{143}
\]

where we have used the identity \(|\hat{n}_j \times \mathbf{v}_j|^2 = \mathbf{v}_j^2 - (\hat{n}_j \cdot \mathbf{v}_j)^2\). The Laplacian derivative of the synchronization function is calculated from Eq. (143) substituting \( \mathbf{v}_j(t_j) \) by \( \mathbf{v}_\mu \equiv d\mathbf{X}_\mu/dt \) and \( \mathbf{a}_j(t_j) \) by \( \mathbf{a}_\mu \equiv d^2 \mathbf{X}_\mu/dt^2 \), yielding

\[
\Delta^2 \phi_\mu = -|\nabla \phi_\mu|^3 \hat{n}_\mu \cdot \mathbf{a}_\mu + \frac{\nabla \phi_\mu}{|\nabla \phi_\mu|^3}|\hat{n}_\mu \times \mathbf{v}_\mu|^2 + \frac{2|\nabla \phi_\mu|^3}{\phi_\mu}. \tag{144}
\]

The asymptotic expansions of the \( \phi_j^\pm(t, x) \) and \( \hat{n}_j^\pm(t, x) \) defined by (3) and (8) involve the non-analytic function \( x_j(t_j^\pm(t, x)) \), i.e.,

\[
\hat{n}_j^\pm(t, x) = (1 + \frac{\hat{r} \cdot x_j(t_j^\pm)}{r}) \hat{r} - \frac{x_j(t_j^\pm)}{r} + O\left(\frac{1}{r^2}\right), \tag{145}
\]

\[
\phi_j^\pm(t, x) = r - \hat{r} \cdot x_j(t_j^\pm) + \frac{|\hat{r} \times x_j(t_j^\pm)|^2}{2r} + O\left(\frac{1}{r^2}\right), \tag{146}
\]

\[
t_j^\pm(t, x) = t \pm r \mp \hat{r} \cdot x_j(t_j^\pm) \pm \frac{3 \hat{r} \cdot x_j(t_j^\pm)|^2 - |x_j|^2}{2r^3} + O\left(\frac{1}{r^4}\right), \tag{147}
\]

where \( r \equiv |x| \) and \( \hat{r} \equiv x/r \). Since the non-analytic continuous functions \( x_j(t, x) \) are never expanded, Eqs. (145), (146) and (148) hold at velocity discontinuity points as well.

The normal electronic equation of motion obtained in Theorem 3.1 involves functions of \( t_j \) constructed from sums of inverse powers of the real functions \( r_j^\pm(t_j) \equiv 1 \mp \hat{n}_j^\pm \cdot \mathbf{v}_j^\pm \) and \( d_j^\pm(t) \equiv 1 \mp \hat{n}_j^\pm \cdot \mathbf{v}_i(t) \) with exponents defined by positive integers \((s, q)\), i.e.,

\[
sh_i^\pm(t) \equiv \frac{1}{2} \left( \frac{1}{(r_j^\pm)^s(d_j^\pm)^q} - \frac{1}{(r_j^\pm)^s(d_j^\pm)^q} \right), \tag{149}
\]

\[
ch_i^\pm(t) \equiv \frac{1}{2} \left( \frac{1}{(r_j^\pm)^s(d_j^\pm)^q} + \frac{1}{(r_j^\pm)^s(d_j^\pm)^q} \right), \tag{150}
\]

\[
sh_i^\pm(t) \equiv \frac{1}{2} \left( \frac{(1 - |v_j^+|^2)}{(r_j^\pm)^s(d_j^\pm)^q} - \frac{(1 - |v_j^-|^2)}{(r_j^\pm)^s(d_j^\pm)^q} \right), \tag{151}
\]

\[
ch_i^\pm(t) \equiv \frac{1}{2} \left( \frac{(1 - |v_j^+|^2)}{(r_j^\pm)^s(d_j^\pm)^q} + \frac{(1 - |v_j^-|^2)}{(r_j^\pm)^s(d_j^\pm)^q} \right). \tag{152}
\]
Proof. For Lemma 8.2. \( \hat{a} \) where

\[
\mathcal{H}_i = \frac{1}{2} \left( \frac{d_i^+(1 - |v_j^+|^2)}{(r_{ji}^+)^2(d_{ji}^+)^3} + \frac{d_i^-(1 - |v_j^-|^2)}{(r_{ji}^-)^2(d_{ji}^-)^3} \right),
\]

(153)

\[
\mathcal{F}_i = \frac{1}{2} \left( \frac{d_i^+(1 - |v_j^+|^2)(1 - v_i \cdot v_j^+ + \hat{n}_i^+ \cdot v_j^+ - \hat{n}_i^- \cdot v_j^-)}{(r_{ji}^+)^2(d_{ji}^+)^3} \right) + \frac{1}{2} \left( \frac{d_i^-(1 - |v_j^-|^2)(1 - v_i \cdot v_j^- - \hat{n}_i^- \cdot v_j^- + \hat{n}_i^+ \cdot v_j^+)}{(r_{ji}^-)^2(d_{ji}^-)^3} \right).
\]

(154)

Finally, some useful results about denominators, starting from a geometric identity

\[
1 - (\hat{n}_i \cdot v_j)^2 = 1 - v_j^2 + |\hat{n}_i \times v_j|^2.
\]

(155)

Notice that when \( x_p(t) = 0 \) we have \( \hat{n}_p = x_i(t_i)/|x_i| \) and Eq. (155) yields

\[
1 - (\hat{n}_p \cdot v_j)^2 = 1 - v_j^2 + |\hat{r}_{ji}|^2.
\]

(156)

**Lemma 8.1.** For \( v \) and \( a \in \mathbb{R}^3 \) the vector product \( v \times a \) can be expressed as

\[
v \times a = (\hat{n} \times v) \times (\hat{n} \times a) + \hat{n} \cdot v(\hat{n} \times a) - \hat{n} \cdot a(\hat{n} \times v),
\]

(157)

where \( \hat{n} \in \mathbb{R}^3 \) is an arbitrary vector satisfying \( |\hat{n}| = 1 \).

**Proof.** Using an orthogonal basis with the \( \hat{x} \)-axis along the unit vector \( \hat{n} \) we have \( \hat{n} \times v = (0, -v_{nz}, v_{ny}) \) and \( \hat{n} \times a = (0, -a_{nz}, a_{ny}) \). Evaluating \( v \times a \) and using the former basis we have the result. \( \square \)

**Lemma 8.2.** For \( v \) and \( a \in \mathbb{R}^3 \) the scalar product \( v \cdot a \) can be expressed as

\[
v \cdot a = (\hat{n} \cdot v)(\hat{n} \cdot a) + |\hat{n} \times v| \cdot (\hat{n} \times a),
\]

(158)

where \( \hat{n} \in \mathbb{R}^3 \) is an arbitrary vector satisfying \( |\hat{n}| = 1 \).

**Proof.** The result follows by using the identity \( v = (\hat{n} \cdot v)\hat{n} - \hat{n} \times (\hat{n} \times v) \) on the left-hand side of (158). \( \square \)

**Lemma 8.3.** For \( v_i \in \mathbb{R}^3 \) and \( v_p \in \mathbb{R}^3 \) we have the equality

\[
(1 - v_i^2)(1 - v_p^2) = (1 - v_i \cdot v_p)^2 + |v_i \times v_p|^2 - |v_i - v_p|^2.
\]

(159)

**Proof.** Using the identity \( v_i^2 + v_p^2 = |v_i - v_p|^2 + 2v_i \cdot v_p \) we obtain

\[
(1 - v_i^2)(1 - v_p^2) = 1 - v_i^2 - v_p^2 + v_i^2 v_p^2 = (1 - 2v_i \cdot v_p + (v_i \cdot v_p)^2) + v_i^2 v_p^2 - (v_i \cdot v_p)^2 - |v_i - v_p|^2
\]

\[
= (1 - v_i \cdot v_p)^2 + |v_i \times v_p|^2 - |v_i - v_p|^2.
\]

(160)

where on the last equality we have used the identity \( v_i^2 v_p^2 - (v_i \cdot v_p)^2 = |v_i \times v_p|^2 \) and completed a square to have the result. \( \square \)
Corollary 8.1. In the limit when $|\vec{L}_{vp}| \ll |\vec{L}_{ve}|$, $|\vec{L}_{ve}| \to 0$ and (48) holds and assuming Lemma 2.4 holds we have

$$\mathcal{F}_e = \frac{2}{r^2 M^2 p^{2/3}},$$

(161)

$$\mathcal{F}_p = \frac{2 M^2 p^{2/3}}{r^2}.$$  

(162)

Proof. In the limit when $|\vec{L}_{ve}|$ and $|\vec{L}_{vp}|$ are sufficiently small and assuming there is a singular denominator at all times granted by Lemma 2.4, it follows from definition (154) that

$$\mathcal{F}_e \to \frac{2(1 - v_e^2)}{r^2(1 - v_p^2)} \left(1 - \vec{v}_e \cdot \vec{v}_p + (|\vec{v}_e| - |\vec{v}_p|)\right),$$

(163)

$$\mathcal{F}_p \to \frac{2(1 - v_p^2)}{r^2(1 - v_e^2)} \left(1 - \vec{v}_e \cdot \vec{v}_p - (|\vec{v}_e| - |\vec{v}_p|)\right).$$

(164)

On the other hand, using (160) and (48) in the limit when $|\vec{L}_{vp}| \ll |\vec{L}_{ve}|$ and $|\vec{L}_{ve}| \to 0$ we have

$$1 - \vec{v}_e \cdot \vec{v}_p \to \frac{1}{2} \left(1 + \frac{1}{M^2 p^{2/3}}\right)(1 - v_p^2),$$

(165)

$$|\vec{v}_e| - |\vec{v}_p| \to \frac{1}{2} \left(1 - \frac{1}{M^2 p^{2/3}}\right)(1 - v_p^2).$$

(166)

Substituting (165) and (166) into (163) and (164) we have the result. □

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