SOME NOTES ON THE EQUIVALENCE OF FIRST-ORDER RIGIDITY IN VARIOUS GEOMETRIES

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ABSTRACT. These pages serve two purposes. First, they are notes to accompany the talk Hyperbolic and projective geometry in constraint programming for CAD by Walter Whiteley at the János Bolyai Conference on Hyperbolic Geometry, 8–12 July 2002, in Budapest, Hungary. Second, they sketch results that will be included in a forthcoming paper that will present the equivalence of the first-order rigidity theories of bar-and-joint frameworks in various geometries, including Euclidean, hyperbolic and spherical geometry. The bulk of the theory is outlined here, with remarks and comments alluding to other results that will make the final version of the paper.

1. Introduction

In this paper, we explore the connections among the theories of first-order rigidity of bar and joint frameworks (and associated structures) in various metric geometries extracted from the underlying projective space of dimension $n$, or $\mathbb{R}^{n+1}$. The standard examples include Euclidean space, elliptical (or spherical) space, hyperbolic space, and a metric on the exterior of hyperbolic space.

In his book, Pogorelov explored more general issues of uniqueness, and local uniqueness of realizations in these standard spaces, with some first-order correspondences as corollaries [11]. We will take the opposite tack – beginning directly with the first-order theory, in this paper. We believe this presents a more transparent and accessible starting point for the correspondences. In a second paper, we will use the additional technique of ‘averaging’ in combination with the first-order results to transfer results about pairs of objects with identical distance constraints in one space to corresponding pairs in a second space.

Like Pogorelov (and perhaps for related reasons) we will begin with the correspondence between the theory in elliptical or spherical space and the theory in Euclidean space (§4). This correspondence of configurations is direct – using gnomic projection (or central projection) from the upper half sphere to the corresponding Euclidean space. This correspondence between spherical frameworks and their central projections into the plane is also embedded in previous studies of frameworks in dimension $d$ and their one point cones into dimension $d + 1$ [18].

With a firm grounding for the first-order rigidity in spherical space, it is simpler to work from the spherical $n$-space to the other metrics extracted from the underlying $\mathbb{R}^{n+1}$ (§5). The correspondence works for any metric of the form $(p, q) = \sum_{i=1}^{n+1} a_i p_i q_i$, $a_i \neq 0$, in addition to

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the special case of Euclidean space (with \( a_{n+1} = 0 \)). It has a particularly simple form, for selected normalizations of the rays as points in the space, such as \( \langle p, p \rangle = \pm 1 \), which is the form we present.

Having examined the theory of first-order motions, we pause to present the motions as the solutions to a matrix equation \( R_X(G, p)x = 0 \) for the metric space \( X (\S 6) \). In this setting, we have the equivalent theory of static rigidity working with the row space and row dependences (the self-stresses) of these matrices, instead of the column dependences (the motions). The correspondence is immediate, but it takes a particular nice form for the ‘projective’ models in Euclidean space of the standard metrics. In this setting, the rigidity correspondence is a simple matrix multiplication:

\[
R_X(G, p)[T_{XY}] = R_Y(G, p)
\]

for the same underlying configuration \( p \), where \([T_{XY}]\) is a block diagonal matrix with a block entry for each vertex, based on how the sense of ‘perpendicular’ is twisted at that location from one metric to the other. As a consequence of this simple correspondence of matrices, we see that row dependencies (the static self-stresses) are completely unchanged by the switch in metric. As a biproduct of this static correspondence, there is a correspondence for the first-order rigidity of the structures with inequalities, the tensegrity frameworks, which are well understood as a combination of first-order theory and self-stresses of the appropriate signs for the edges with pre-assigned inequality constraints.

As this shared underlying statics hints, there is a shared underlying projective theory of statics (and associated first-order kinematics) [4].

We will not present that theory here but we note the projective invariance, in all the metrics, of the first-order and static theories (§7). There are various extensions that follow from this underlying projective theory, such as inclusion of ‘vertices at infinity’ in Euclidean space [4], and the possibility that polarity has a role to play (see below).

As an application of these correspondences, we consider a classical theory of rigidity for polyhedra – the theorems of Cauchy, Alexandrov, and the associated theory of Andreev. This theory provides theorems about the first-order rigidity of convex polyhedra and convex polytopes with either rigid faces, or 2-faces triangulated with bars and joints in dimensions \( d \geq 3 \), in Euclidean space. Since the basic concepts of convexity transfer among the metrics (if we remove the equator on the sphere, or the corresponding line at infinity in Euclidean space), this first-order and static theory immediately transfers to identical theorems in the other metric spaces (§7). There are some first-order extensions of Cauchy’s Theorem to versions of local convexity, which will automatically extend to the various metrics and on through to hyperplanes and angles, giving additional generalizations. Moreover, this theory for hyperplanes and angles will be projectively invariant, if we are careful with the transfer of concepts such as ‘convexity’ through the projective transformations.

In hyperbolic space, there is a correspondence between rigidity of ‘bar-and-joint frameworks’ with vertices and distance constraints in the exterior hyperbolic space (or ideal points) and planes and angle constraints in the interior hyperbolic space. We present this correspondence directly, although it can be viewed as a polarity about the absolute. With this correspondence, the first-order Cauchy theory in exterior hyperbolic space gives a first-order theory for planes and angles in hyperbolic space. This result turns out to be a generalization of the first-order version of Andreev’s Theorem. In this setting, the constraint that angles be
less than $\pi/2$ disappears and the angles have the full range of angles in a convex polyhedron ($< \pi$).

Moreover, as this hints, there is a correspondence, via spherical polarity, which connects the first-order Cauchy Theorem in the spherical or elliptic space with an Andreev style first-order theorem for planes and angles of a simple convex polytope in elliptical geometry (§none). The effect of polarity in Euclidean space is drastically different. It has an interesting, and distinctive interpretations in dimensions $d = 2$ and $d = 3$ [22, 23].

The general problem of characterizing which graphs have some (almost all) realizations in $d$-space as first-order rigid frameworks is hard for dimensions $d \geq 3$. With these correspondences, we realize that this problem is identical in all the metric spaces and we will not get additional leverage by comparing first-order behaviour under the various metrics.

On the other hand, in general geometric constraint programming in fields such as CAD, there is an interest in more general systems of geometric objects and general constraints. For example, circles of variable radii with angles of intersection as constraints are in interest in CAD. As people familiar with hyperbolic geometry may realize, these are equivalent, both a first-order and at all orders, to planes and angles in hyperbolic 3-space. The correspondence presented here provides the final step in the correspondence between circles and angles in the plane and points and distances in Euclidean 3-space [13].

The basic first-order correspondence among metrics should extend to differentiable surfaces from these discrete structures. The major difference here is that static rigidity and first-order rigidity are distinct concepts in the this world which corresponds to infinite matrices. Still the correspondence should apply to both theories, and all the metrics.

2. First-Order Rigidity in $\mathbb{E}^n$

2.1. Euclidean $n$-space. Let $\mathbb{E}^n$ denote the set of vectors in $\mathbb{R}^{n+1}$ with $x_{n+1} = 1$,

$$\mathbb{E}^n = \{x \in \mathbb{R}^{n+1} \mid e \cdot x = 1\},$$

where $e = (0, 0, \ldots, 1) \in \mathbb{R}^{n+1}$. An $m$-plane of $\mathbb{E}^n$ is the intersection of $\mathbb{E}^n$ with an $(m + 1)$-subspace of $\mathbb{R}^{n+1}$. The distance between $x, y \in \mathbb{E}^n$ is $d_{E}(x, y) = |x - y| = \sqrt{\sum_{i}(x_i - y_i)^2}$.

2.2. Frameworks and rigidity in $\mathbb{E}^n$. A graph $G = (V, E)$ consists of a finite vertex set $V = \{1, 2, \ldots, v\}$ and an edge set $E$, where $E$ is a collection of unordered pairs of vertices. A bar-and-joint framework $G(p)$ in $\mathbb{E}^n$ is a graph $G$ together with a map $p : V \rightarrow \mathbb{E}^n$. Let $p_i$ denote $p(i)$.

A motion of the framework $G(p)$ is a continuous family of functions $p(t) : V \rightarrow \mathbb{E}^n$ with $p(0) = p$ such that for $\{i, j\} \in E$, $d_{E}(p_i(t), p_j(t)) = c_{ij}$, where $c_{ij}$ is a constant, for all $t$. A framework is rigid if all motions are trivial: for each $t$, there is a rigid motion $A_t$ of $\mathbb{E}^n$, such that $A_t(p_i) = p_i(t)$, for all $i \in V$.

2.3. Motivation for first-order rigidity. Suppose $p(t)$ is a motion of the framework $G(p)$ in $\mathbb{E}^n$ differentiable at $t = 0$. Since $d_{E}(p_i(t), p_j(t)) = c_{ij}$ for each $\{i, j\} \in E$, the derivative of $p(t)$ must satisfy

$$(p_i - p_j) \cdot (p_i'(0) - p_j'(0)) = 0,$$

where $x \cdot y$ denotes the Euclidean inner product of the vectors $x$ and $y$. Since the framework lies in $\mathbb{E}^n$ during the motion ($p_k(t) \in \mathbb{E}^n$ for all $k \in V$), $p_k(t)$ satisfies $e \cdot p_k(t) = 0$ for all
\[ k \in V. \text{ Hence its derivative satisfies,} \]
\[ e \cdot p_i'(0) = 0 \]
for each \( i \in V. \) This motivates the following definition.

2.4. **First-order rigidity in \( \mathbb{E}^n. \)** A **first-order motion** of the framework \( G(p) \) in \( \mathbb{E}^n \) is a map \( u : V \to \mathbb{R}^{n+1} \) satisfying, for each \( \{i,j\} \in E \) and \( k \in V, \)
\[
(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{and} \quad e \cdot u_k = 0,
\]
where \( u_i \) denotes \( u(i). \)

![Figure 1](image)

**Figure 1.** \( u \) is a first-order motion if \( (p_i - p_j) \cdot u_i = (p_i - p_j) \cdot u_j \) for all edges \( \{i,j\}. \) That is, the projection of \( u_i \) onto \( p_i - p_j \) must equal the projection of \( u_j \) onto \( p_i - p_j. \)

A **trivial first-order motion** of \( \mathbb{E}^n \) is a map \( u : \mathbb{E}^n \to \mathbb{R}^{n+1} \) satisfying
\[
(x - y) \cdot (u(x) - u(y)) = 0 \quad \text{and} \quad e \cdot u(z) = 0,
\]
for all \( x, y \) and \( z \) in \( \mathbb{E}^n. \) \( G(p) \) is **first-order rigid** in \( \mathbb{E}^n \) if all the first-order motions of the framework \( G(p) \) are restrictions of trivial first-order motions of \( \mathbb{E}^n. \)

2.5. **Remark.** Any rigid motion of \( \mathbb{E}^n \) yeilds a trivial first-order motion of a given framework: the isometry restricts to a motion of the framework whose derivative satisfies the equations in (1).

2.6. **Remark.** First-order rigidity is a good indicator of rigidity: first-order rigidity implies rigidity, but not conversely.

3. **First-Order Rigidity in \( S^m_+ \)**

3.1. **Spherical \( n \)-Space.** Let \( S^n_+ \) denote the upper hemisphere of the unit sphere in \( \mathbb{R}^{n+1}, \)
\[
S^n_+ = \{ x \in \mathbb{R}^{n+1} \mid x \cdot x = 1, \, e \cdot x > 0 \}.
\]
An \( m \)-plane of \( S^n_+ \) is the intersection of \( S^n_+ \) with an \((m + 1)\)-subspace of \( \mathbb{R}^{n+1}. \) The distance between two points \( x, y \in S^n_+ \) is given by the angle subtended by the vectors \( x \) and \( y, \)
\[
d_{S^n_+}(x,y) = \arccos(x \cdot y).
\]

3.2. **Frameworks and rigidity in \( S^n_+ \).** A **bar-and-joint framework** \( G(p) \) in \( S^n_+ \) is a graph \( G \) together with a map \( p : V \to S^n_+. \) A **motion** of the framework \( G(p) \) in \( S^n_+ \) is a continuous family of functions \( p(t) : V \to S^n_+ \) with \( p(0) = p \) such that for \( \{i,j\} \in E, \)
\[
d_{S^n_+}(p_i(t), p_j(t)) = c_{ij}, \text{ where } c_{ij} \text{ is a constant, for all } t. \] A framework is **rigid** if all motions are **trivial:** for each \( t, \) there is a rigid motion \( A_t \) of \( S^n_+ \), such that \( A_t(p_i) = p_i(t), \) for all \( i \in V. \)
3.3. **Motivation for first-order rigidity in \( \mathbb{S}^n_+ \).** To extend the definitions of first-order motion and first-order rigidity to frameworks in \( \mathbb{S}^n_+ \), mimic the motivation presented in section 2.3. If \( p(t) \) is a motion of a framework \( G(p) \) in \( \mathbb{S}^n_+ \), then for all \( t \) and \( \{i, j\} \in E \),
\[
d_{\mathbb{S}^n_+}(p_i(t) \cdot p_j(t)) = c_{ij},
\]
where \( c_{ij} \) is constant for all \( \{i, j\} \in E \), and for all \( t \) and \( k \in V \),
\[
p_k(t) \cdot p_k(t) = 1.
\]
Equivalently, for all \( t \), \( \{i, j\} \in E \) and \( k \in V \),
\[
q_i(t) \cdot q_j(t) = \cos c_{ij},
\]
\[
p_k(t) \cdot p_k(t) = 1.
\]
If the motion \( p(t) \) is differentiable at \( t = 0 \), then \( p(t) \) must satisfy,
\[
p_i(t) \cdot p_j(t) + p_i'(t) \cdot p_j = 0,
\]
\[
p_k(t) \cdot p_k'(t) = 0.
\]
This leads to the following definition.

3.4. **First-Order Rigidity in \( \mathbb{S}^n_+ \).** A **first-order motion** of the framework \( G(p) \) in \( \mathbb{S}^n_+ \) is a map \( u: V \rightarrow \mathbb{R}^{n+1} \) satisfying, for each \( \{i, j\} \in E \) and for each \( k \in V \),
\[
(2) \quad p_i \cdot u_j + p_j \cdot u_i = 0 \quad \text{and} \quad p_k \cdot u_k = 0.
\]

A **trivial first-order motion** of \( \mathbb{S}^n_+ \) is a map \( u: \mathbb{S}^n_+ \rightarrow \mathbb{R}^{n+1} \) satisfying
\[
x \cdot u(y) + y \cdot u(x) = 0 \quad \text{and} \quad z \cdot u(z) = 0,
\]
for all \( x, y \) and \( z \) in \( \mathbb{E}^n \). The framework \( G(p) \) is **first-order rigid** in \( \mathbb{S}^n_+ \) if all first-order motions of \( G(p) \) are restrictions of trivial first-order motions.

3.5. **Remark.** Note that the equations in (2) are equivalent to the following conditions,
\[
(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{and} \quad p_k \cdot u_k = 0,
\]
which are similar to the equations defining first-order rigidity in \( \mathbb{E}^n \).

3.6. **Remark.** If \( G(p) \) is a bar-and-joint framework in \( \mathbb{S}^n_+ \), then the graph obtained from \( G \) by adjoining a new vertex with edges incident with all vertices of \( G \), together with the map \( \hat{p}: V \cup \{v + 1\} \rightarrow \mathbb{E}^{n+1} \) given by
\[
\hat{p}(i) = \begin{cases} 
p(i) & \text{if } i \neq v + 1 \\
0 & \text{if } i = v + 1 \end{cases},
\]
is first-order rigid in \( \mathbb{E}^{n+1} \) iff \( G(p) \) is first-order rigid in \( \mathbb{S}^{n+1}_+ \). That is, frameworks in \( \mathbb{S}^n_+ \) can be modeled by the cone on the same framework in \( \mathbb{E}^{n+1} \).

4. **Equivalence of First-Order Rigidity in \( \mathbb{S}^n_+ \) and \( \mathbb{E}^n \).**

This section presents two maps, a map carrying a framework \( G(p) \) in \( \mathbb{S}^n_+ \) into a framework \( G(q) \) in \( \mathbb{E}^n \), and a map carrying the first-order motions of \( G(p) \) into first-order motions of \( G(q) \). The latter map carries trivial first-order motions of \( \mathbb{S}^n_+ \) to trivial first-order motions of \( \mathbb{E}^n \), yielding the result \( G(p) \) is first-order rigid iff \( G(q) \) is first-order rigid.
4.1. **Mapping frameworks and first-order motions.** If $G(p)$ is a framework in $\mathbb{S}_+^n$, then $G(\psi \circ p)$ is a framework in $\mathbb{E}^n$, where $\psi : \mathbb{S}^n \to \mathbb{E}^n$ is given by $\psi(x) = x/(e \cdot x)$. The inverse of $\psi$ is given by $\psi^{-1}(x) = x/\sqrt{x \cdot x}$.

**Figure 2.** Mapping first-order motions of a framework in $\mathbb{S}_+^n$ to first-order motions of a framework in $\mathbb{E}^n$.

If $u$ is a first-order motion of the framework $G(p)$ in $\mathbb{S}_+^n$, let $\varphi$ denote the map

$$\varphi : u_i \mapsto \frac{1}{e \cdot p_i} (u_i - (u_i \cdot e) e).$$

If $G(q)$ is a framework in $\mathbb{E}^n$ with first-order motion $v$, then $\varphi^{-1}$ is given by

$$\varphi^{-1} : v_i \mapsto \frac{1}{\sqrt{q_i \cdot q_i}} (v_i - (v_i \cdot q_i) e).$$

Observe that $\varphi$ and $\varphi^{-1}$ map into the appropriate tangent spaces: $\psi^{-1}(q_i) \cdot \varphi^{-1}(v_i) = 0$ and $\varphi(u_i) \cdot e = 0$.

4.2. **Theorem.** $u$ is a first-order motion of the framework $G(p)$ in $\mathbb{S}_+^n$ iff $\varphi \circ u$ is a first-order motion of the framework $G(\psi \circ p)$ in $\mathbb{E}^n$. Moreover, $u$ is a trivial first-order motion iff $\varphi \circ u \circ \psi^{-1}$ is a trivial first-order motion.

**Pf.** Note that

$$\begin{aligned}
(\psi(p_i) - \psi(p_j)) \cdot (\varphi(u_i) - \varphi(u_j)) &= \frac{p_i \cdot u_i}{(e \cdot p_i)^2} - \frac{p_i \cdot u_j + p_j \cdot u_i}{(e \cdot p_i)(e \cdot p_j)} + \frac{p_j \cdot u_j}{(e \cdot p_j)^2}.
\end{aligned}$$
If $u$ is a first-order motion of $G(p)$, then $u_i \cdot p_i = 0$ for all $i \in V$, and $p_i \cdot u_j + p_j \cdot u_i = 0$ for all $\{i, j\} \in E$. By (3), $(\psi(p_i) - \psi(p_j)) \cdot (\varphi(u_i) - \varphi(u_j)) = 0$ for all $\{i, j\} \in E$. The definition of $\varphi$ ensures that $\varphi(u_i) \cdot e = 0$. Therefore, $\varphi \circ u$ is a first-order motion of $G(\psi \circ p)$.

Conversely, suppose $\varphi \circ u$ is a first-order motion of $G(\psi \circ p)$. Then for all $\{i, j\} \in E$, $(\psi(p_i) - \psi(p_j)) \cdot (\varphi(u_i) - \varphi(u_j)) = 0$. The observation at the end of the 4.1 gives that $p_i \cdot u_j + p_j \cdot u_i = 0$. So $u$ is a first-order motion of $G(p)$.

Suppose $u$ is a trivial first-order motion. Then $x \cdot u(x) = 0$ for all $x \in S^n_+$ and $x \cdot u(y) + y \cdot u(x) = 0$ for all $x, y \in S^n_+$. Let $v : \mathbb{E}^n \to \mathbb{R}^{n+1}$ denote the composition $\phi \circ u \circ \psi^{-1}$. If $\hat{x}, \hat{y} \in \mathbb{E}^n$ with $x$ denoting $\psi^{-1}(\hat{x})$ and $y$ denoting $\psi^{-1}(\hat{y})$, then (3) gives

$$(\hat{x} - \hat{y}) \cdot (v(\hat{x}) - v(\hat{y})) = \frac{x \cdot u(x)}{(e \cdot x)^2} - \frac{x \cdot u(y) + y \cdot u(x)}{(e \cdot x)(e \cdot y)} + \frac{y \cdot u(y)}{(e \cdot y)^2} = 0.$$  

So $v$ is a trivial first-order motion. The converse follows similarly.

**Corollary.** $G(p)$ is first-order rigid in $S^n_+$ iff $G(\psi \circ p)$ is first-order rigid in $\mathbb{E}^n$.

4.3. **Remark.** $S^n_+$ versus $S^n$: Given a discrete framework, there exists a rotation of the $n$-sphere such that no vertex of the framework lies on the equator of the sphere. Therefore, we need not restrict our frameworks to a hemisphere.

5. **Equivalence of First-Order Rigidity in Other Geometries.**

5.1. **Geometries.** For $x, y \in \mathbb{R}^{n+1}$, let $\langle x, y \rangle_k$ denote the function

$$\langle x, y \rangle_k = x_1 y_1 + \cdots + x_{n-k+1} y_{n-k+1} - x_{n-k+2} y_{n-k+2} - \cdots - x_{n+1} y_{n+1},$$

and let $X^n_{c,k}$ denote the set,

$$X^n_{c,k} = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_k = c, x_{n+1} > 0\},$$

for some constant $c \neq 0$ and $k \in \mathbb{N}$. We write $X^n$ to simplify notation, if $c$ and $k$ are understood. If $k = 1$ and $c = -1$, then $X^n$ is hyperbolic space, $\mathbb{H}^n$. If $k = 1$ and $c = 1$, then $X^n$ is exterior hyperbolic space, $\mathbb{D}^n$. Spherical space $S^n_+$ is the case $k = 0, c = 1$. Note that $\mathbb{E}^n \neq X^n$ for any choice of $c$ and $k$.

5.2. **Remark.** In more generality we can replace $\langle x, y \rangle_k$ with

$$\langle x, y \rangle = a_1 x_1 y_1 + \cdots + a_{n+1} x_{n+1} y_{n+1},$$

where $a_i \neq 0$ for all $i$, with the exception for Euclidean space: $a_1 = a_2 = \cdots = a_n = 1$ and $a_{n+1} = 0$.

5.3. **First-order rigidity in $X^n$.** A metric $d_X$ can be placed on $X^n$ so that $d_X(x, y)$ is a function of $\langle x, y \rangle_k$. A sufficient condition for the distance $d_X(x, y)$ remaining constant is the requirement $\langle x, y \rangle_k$ remain constant. Therefore, the same analysis motivates the following extensions of the definitions of first-order rigidity to $X^n$.

A bar-and-joint framework $G(p)$ in $X^n$ is a graph $G$ together with a map $p : V \to X^n$. A first-order motion of the framework $G(p)$ in $X^n$ is a map $u : V \to \mathbb{R}^{n+1}$ satisfying for each $\{i, j\} \in E$,

$$\langle p_i, u_j \rangle_k + \langle p_j, u_i \rangle_k = 0,$$

where $\langle p_i, u_j \rangle_k$ and $\langle p_j, u_i \rangle_k$ are defined as above.
and for each \( i \in V \),
\[
\langle p_i, u_i \rangle_k = 0.
\]

A trivial first-order motion of \( X^n \) is a map \( u : X^n \to \mathbb{R}^{n+1} \) satisfying
\[
\langle x, u(y) \rangle_k + \langle y, u(x) \rangle_k = 0 \quad \text{and} \quad \langle z, u(z) \rangle_k = 0
\]
for all \( x, y, z \in X^n \). \( G(p) \) is first-order rigid in \( X^n \) if all first-order motions of \( G(p) \) are the restrictions of trivial first-order motions of \( X^n \).

5.4. \( X^n \) and \( \mathbb{E}^n \). In section 4 we established the equivalence between first-order rigidity in \( \mathbb{E}^n \) and first-order rigidity in \( S^n_+ \). We need only demonstrate the equivalence holds between the first-order rigidity theories of \( X^n \) and \( S^n_+ \).

5.5. \( X^n \) and \( S^n_+ \). Let \( \psi_{S^+} : X^n \to S^n_+ \) denote the map \( x \mapsto x / \sqrt{x \cdot x} \), and let \( \varphi_{S^+} \) denote the map
\[
\varphi_{S^+} : u_i \mapsto \frac{J_k(u_i)}{\sqrt{p_i \cdot p_i}},
\]
where \( J_k(x) = (x_1, \ldots, x_{n-k+1}, -x_{n-k+2}, \ldots, -x_{n+1}) \).

![Figure 3. Mapping a bar-and-joint framework from the spherical plane \( S^2_+ \) into the hyperbolic plane \( \mathbb{H}^2 \).](image)

5.6. Theorem. \( G(p) \) is first-order rigid in \( X^n \) iff \( G(\psi_{S^+} \circ p) \) is first-order rigid in \( S^n_+ \).

\textit{Pf.} Since, \( \langle x, y \rangle_k = x \cdot J_k(y) \) we have
\[
\left( \psi_{S^+}(p_i) - \psi_{S^+}(p_j) \right) \cdot \left( \varphi_{S^+}(u_i) - \varphi_{S^+}(u_j) \right) = \frac{\langle p_i, u_i \rangle_k}{p_i \cdot p_i} - \frac{\langle p_i, u_j \rangle_k + \langle p_j, u_i \rangle_k}{\sqrt{p_i \cdot p_i \cdot p_j \cdot p_j}} + \frac{\langle p_j, u_j \rangle_k}{p_j \cdot p_j}.
\]
As in the proof of Theorem 4.2, the above equation and the definitions of \( \psi_{S^+} \) and \( \varphi_{S^+} \) give that \( \varphi_{S^+} \circ u \) is a first-order motion of \( G(\psi_{S^+} \circ p) \) iff \( u \) is a first-order motion of \( G(p) \).

It is clear that trivial motions of \( S^n_+ \) map to trivial motions of \( X^n \). However, a trivial motion of \( X^n \) maps onto a “trivial motion” of a proper subset of \( S^n_+ \). The following fact finishes of this proof.
Figure 4. Mapping first-order motions of a framework in $S^n_+$ to first-order motions of a framework in $\mathbb{H}^n$.

**Fact.** Given a first-order motion $u$ of $K_{n+1}$, the complete graph on $n + 1$ vertices in $\mathbb{E}^n$, there exists a unique trivial first-order motion of $\mathbb{E}^n$ extending $u$.

(This result and the equivalence of the first-order theories of $\mathbb{E}^n$ and $S^n_+$ give the corresponding result for $S^n_+$, which was needed to finish the proof of the proceeding theorem.)

5.7. **Remark.** There is no obstruction to defining a framework with vertices in $\mathbb{H}^n$ and $\mathbb{D}^n$: the equations defining first-order motions provide formal constraints between these vertices, although the geometric interpretations of these constraints may not be obvious. In general, the theorem holds for frameworks with vertices on the surface $\langle x, x \rangle_k = \pm 1$, but not with vertices on $\langle x, x \rangle_k = 0$.

6. **The Rigidity Matrix**

6.1. **Projective models of $X^n$.** The projective model of $X^n$ is the subset of $\mathbb{E}^n$ obtained by projecting from the origin the points of $X^n$ onto $\mathbb{E}^n$,

$$\left\{ \frac{1}{e \cdot x} x \mid x \in X^n \right\} \subset \mathbb{E}^n.$$

The projective model of hyperbolic $n$-space $\mathbb{H}^n$ is the interior of the unit $n$-ball $B^n$ of $\mathbb{E}^n$ and the projective model of exterior hyperbolic $n$-space $\mathbb{D}^n$ is the exterior of $B^n$. The unit $(n - 1)$-sphere $S^{n-1}$ is the *absolute*, the points at infinity of hyperbolic geometry. Spherical $n$-space is model projectively by $\mathbb{E}^n$. 
Since we are now restricting our attention to points in $\mathbb{E}^n$, we identify $\mathbb{E}^n$ with $\mathbb{R}^n$ and write $PX^n$ to denote the projective model of $X^n$ as a subset of $\mathbb{R}^n$. Distance in $PX^n$ is calculated by normalizing the points into $X^n$ and applying the definition of distance in $X^n$. For example, the distance between points $x$ and $y$ in $P_{S^n}$ (so $x, y \in \mathbb{R}^n$) is

$$d_{P_{S^n}}(x, y) = \arccos \left( \frac{1 + x \cdot y}{\sqrt{1 + x \cdot x} \sqrt{1 + y \cdot y}} \right),$$

and for points $x$ and $y$ in $P_{H^n}$,

$$d_{P_{H^n}}(x, y) = \text{arccosh} \left( \frac{1 - x \cdot y}{\sqrt{1 - x \cdot x} \sqrt{1 - y \cdot y}} \right).$$

6.2. The rigidity matrix of a framework. A first-order motion $u : V \to \mathbb{R}^n$ of the framework $G(p)$ in $\mathbb{R}^n$, satisfies

$$(p_i - p_j) \cdot (u_i - u_j) = 0.$$
This system of homogeneous linear equations, indexed by the edges of $G$, induces a linear transformation with matrix $R_E(G, p)$, called the *rigidity matrix* of $G(p)$,

$$
R_E(G, p) = \{i, j\} \begin{pmatrix}
    i & \cdots & j \\
    \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots \\
    \end{pmatrix}
$$

The kernel of $R_E(G, p)$ is precisely the space of first-order motions of $G(p)$. A first-order motion $u : V \to \mathbb{R}^n$ of the framework $G(p)$ in $P\mathbb{H}^n$, $P\mathbb{D}^n$ or $P\mathbb{S}^n$ satisfies

$$(k_{ij} + k_{ji}) \cdot (u_i + u_j) = 0,$$

where $k_{ij}$ is

$$
k_{ij} = \begin{cases} 
    \left(\frac{1-p_i \cdot p_j}{1-p_i \cdot p_i}\right) p_i - p_j, & \text{for } P\mathbb{H}^n \text{ or } P\mathbb{D}^n \\
    \left(\frac{1+p_i \cdot p_j}{1+p_i \cdot p_i}\right) p_i - p_j, & \text{for } P\mathbb{S}_+^n
\end{cases}.$$

The matrix of the linear transformation induced by this system of linear equations is the *rigidity matrix* $R_X(G, p)$ of $G(p)$,

$$
R_X(G, p) = \{i, j\} \begin{pmatrix}
    i & \cdots & j \\
    \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots \\
    \end{pmatrix}
$$

Note that $k_{ij}$ depends on $X$.

6.3. **Transforming rigidity matrices.** Let $T_K(G, p)$ denote the matrix

$$
T_K(G, p) = \begin{bmatrix}
    T_{p_1} & 0 & 0 & 0 \\
    0 & T_{p_2} & 0 & 0 \\
    0 & 0 & \vdots & 0 \\
    0 & 0 & 0 & T_{p_v}
\end{bmatrix},
$$

where $T_{p_k} = I + K(p_k^{(i)} p_k^{(j)})$ ($I$ is the $n \times n$ identity matrix and $(p_k^{(i)} p_k^{(j)})$ is the $n \times n$ matrix with $p_k^{(i)} p_k^{(j)}$ as entry $(i, j)$, where $p_k^{(i)}$ is the $i$-th component of $p_k$). For example, for $n = 3$ and $p_k = (x_1, x_2, x_3)$,

$$
T_{p_k} = \begin{bmatrix}
    1 + K x_1^2 & K x_1 x_2 & K x_1 x_3 \\
    K x_1 x_2 & 1 + K x_2^2 & K x_2 x_3 \\
    K x_1 x_3 & K x_2 x_3 & 1 + K x_3^2
\end{bmatrix}.
$$

**Theorem.** Let $G(p)$ be a framework with $p \in \mathbb{R}^n$. Then

1. $T_K(G, p)$ satisfies

$$
R_{P\mathbb{H}} \times T_{-1}(G, p) = R_E(G, p) \quad \text{and} \quad R_{P\mathbb{S}^n} \times T_1(G, p) = R_E(G, p);
$$

2. $G(p)$ is first-order rigid in $P\mathbb{S}^n$ iff $G(p)$ is first-order rigid in $P\mathbb{E}^n$;

3. $G(p)$ is first-order rigid in $P\mathbb{H}^n \cup P\mathbb{D}^n$ iff $G(p)$ is first-order rigid in $P\mathbb{E}^n$ and $p_i \cdot p_i \neq 1$ for all $i \in V$ (no vertex is on the absolute).
Figure 6. A visual summary of the equivalence of first-order rigidity in the projective models of hyperbolic geometry $H$, spherical geometry $S$ and Euclidean geometry $E$. Here $T_{SE}$ denotes the linear transformation $T_1(G,p)$ defined in the text, $T_{ES}$ the inverse of $T_{SE}$.

Proof. (1) Since $T_{p_i}$ multiplies only the columns corresponding to vertex $i$, we need only verify $k_{ij} \times T_{p_i} = p_i - p_j$. This is a straightforward calculation,

$$k_{ij} \times (\text{column } \ell \text{ of } T_{p_i})$$

$$= \left( \frac{1 + K(p_i \cdot p_j)}{1 + K(p_i \cdot p_i)} \right) p_i - p_j \cdot (e_\ell + Kp_{i}^{(\ell)}p_i)$$

$$= \left( \frac{1 + K(p_i \cdot p_j)}{1 + K(p_i \cdot p_i)} \right) (p_i \cdot e_\ell + Kp_{i}^{(\ell)}(p_i \cdot p_i)) - \left( p_{j}^{(\ell)} + Kp_{i}^{(\ell)}(p_j \cdot p_i) \right)$$

$$= \left( \frac{1 + K(p_i \cdot p_j)}{1 + K(p_i \cdot p_i)} \right) \left( 1 + K(p_i \cdot p_i) \right) p_i^{(\ell)} - \left( p_{j}^{(\ell)} + Kp_{i}^{(\ell)}(p_j \cdot p_i) \right)$$

$$= \left( 1 + K(p_i \cdot p_j) \right) p_i^{(\ell)} - \left( p_{j}^{(\ell)} + Kp_{i}^{(\ell)}(p_j \cdot p_i) \right)$$

$$= p_i^{(\ell)} + Kp_{i}^{(\ell)}(p_i \cdot p_j) - p_{j}^{(\ell)} - Kp_{i}^{(\ell)}(p_j \cdot p_i)$$

$$= p_i^{(\ell)} - p_{j}^{(\ell)},$$

which is column $\ell$ of $p_i - p_j$. 12
(2), (3): Since the determinant of \( T_K(G,p) \) is the product \( \prod_{i=1}^n \det(T_{p_i}) \) and

\[
\det(T_{p_i}) = 1 + K(p_i \cdot p_i),
\]

the dimension of the vector space of first-order motions of \( G(p) \) is the same in each geometry iff \( 1 + K(p_i \cdot p_i) \neq 0 \) for all \( i \in V \).

6.4. Remark. It is well-known that the rank of the rigidity matrix, and thus first-order rigidity, of a framework in \( \mathbb{E}^n \) is invariant under projective transformations of \( \mathbb{E}^n \). Due to the equivalence of first-order theories, the same is true of frameworks in \( X^n \). (In fact, there exists an underlying projective theory.) Intuitively at least, this projective invariance suggests the equivalences presented in this paper since all the geometries discuss can be obtained from projective geometry by choosing an appropriate set of transformations.

![Underlying Projective Theory](image)

- Invariance under projective transformations.
- Points at infinity in Euclidean metric.
- Polarity has some interpretations.

**Figure 7.** A visual summary of the underlying projective theory: hyperbolic space \( H \), Euclidean space \( E \) and spherical space \( S \) can be realized as subgeometries of projective geometry.

7. The First-Order Uniqueness Theorems of Andreev and Cauchy-Dehn

An immediate consequence of the equivalence of these first-order rigidity theories is the ability to transfer results between the theories.
7.1. The Cauchy-Dehn Theorem. The Cauchy-Dehn theorem for polytopes in $\mathbb{R}^n$, $n \geq 3$, states that a convex, triangulated polyhedron in $\mathbb{R}^n$, $n \geq 3$, is first-order rigid. Before the generalization of this theorem can be stated, convexity in $\mathbb{R}^n$ needs to be defined. A set $S \subset \mathbb{R}^n$ is convex if, for any line $L$ of $\mathbb{R}^n$, $L \cap S$ is connected. Therefore, $S \subset \mathbb{R}^n$ is convex iff $\psi_E(S) \subset \mathbb{E}^n$ is convex.

Theorem. (Cauchy-Dehn) A convex, triangulated polytope $P$ in $\mathbb{R}^n$, $n \geq 3$, is first-order rigid.

7.2. A first-order version of Andreev’s uniqueness theorem. If $p$ denotes a point of $\mathbb{D}^n$, then the set of points $x$ in $\mathbb{R}^{n+1}$ satisfying $\langle p, x \rangle_1 = 0$ (orthogonal in the hyperbolic sense) defines a unique hyperplane of $\mathbb{R}^{n+1}$ through the origin. Therefore, to each point of $p$, there corresponds a unique hyperplane of $\mathbb{H}^n$,

$$P = \{ x \in \mathbb{H}^n \mid \langle p, x \rangle_1 = 0 \},$$

and conversely.

If $q$ is another point of $\mathbb{D}^n$ with $Q$ the corresponding hyperplane of $\mathbb{H}^n$, the angle of intersection of the hyperplanes $P$ and $Q$ is defined to be $\arccos(\langle p, q \rangle_1)$. So equations (1) and (5) defining a first-order motion $u$ of a framework $G(p)$ in $\mathbb{D}^n$,

$$\langle p_i, u_j \rangle_k + \langle p_j, u_i \rangle_k = 0 \quad \text{and} \quad \langle p_i, u_i \rangle_k = 0,$$

are precisely the conditions defining a “first-order motion” of a collection of planes under angle constraints (a bar-and-joint framework is merely a collection of points under distance constraints). Polyhedra with fixed dihedral angles are examples of such objects.

Under this point-plane correspondence of $\mathbb{D}^n$ and $\mathbb{H}^n$, the Cauchy-Dehn theorem for $\mathbb{D}^n$ gives a first-order version of Andreev’s uniqueness theorem. Indeed, a simple, convex polytope in $\mathbb{H}^n$ is a triangulated, convex polytope in $\mathbb{D}^n$. We use stiff to denote the analogous definition of first-order rigid.

Theorem. (Andreev) If $M$ is a simple, convex polytope in $\mathbb{H}^n$, $n \geq 3$, then $M$ is stiff.

7.3. Remark. The usual hypothesis of Andreev’s theorem requires the polytope $M$ to have dihedral angles not exceeding $\pi/2$. This supposition implies $M$ is simple.

7.4. Remark. The point-plane correspondence described above is known as polarity. There is a version of this result for the sphere that requires a better discussion of polarity on the sphere.

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