NEW HKT MANIFOLDS ARISING FROM QUATERNIONIC REPRESENTATIONS

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Abstract. We give a procedure for constructing an $8n$-dimensional HKT Lie algebra starting from a $4n$-dimensional one by using a quaternionic representation of the latter. The strong (respectively, weak, hyper-Kähler, balanced) condition is preserved by our construction. As an application of our results we obtain a new compact HKT manifold with holonomy in $SL(n,\mathbb{H})$ which is not a nilmanifold. We find in addition new compact strong HKT manifolds.

We also show that every Kähler Lie algebra equipped with a flat, torsion-free complex connection gives rise to an HKT Lie algebra. We apply this method to two distinguished $4$-dimensional Kähler Lie algebras, thereby obtaining two conformally balanced HKT metrics in dimension $8$.

Both techniques prove to be an effective tool for giving the explicit expression of the corresponding HKT metrics.

1. Introduction

A hyper-Hermitian structure on a $4n$-dimensional manifold $M$ is given by a hypercomplex structure $\{J_\alpha\}, \alpha = 1, 2, 3$ and a Riemannian metric $g$ compatible with $J_\alpha$, for any $\alpha$. The hyper-Hermitian manifold $(M, \{J_\alpha\}, g)$ is said to be hyperkähler with torsion (HKT) if there exists a hyper-Hermitian connection $\nabla^B$ whose torsion tensor is a $3$-form, i.e. if

$$\nabla^B g = 0, \quad \nabla^B J_\alpha = 0, \quad \alpha = 1, 2, 3, \quad c(X, Y, Z) = g(X, T^B(Y, Z))$$

is a $3$–form, where $T^B$ is the torsion of $\nabla^B$. An HKT structure is called strong or weak according to the fact that the $3$-form $c$ is closed or not. HKT geometry is a natural generalization of hyper-Kähler geometry, since when $c = 0$ the connection $\nabla^B$ coincides with the Levi-Civita connection.

On any Hermitian manifold there exists a unique Hermitian connection whose torsion tensor is a $3$-form. Such a connection is called in Hermitian geometry the Bismut connection $\nabla^B$ or KT connection. In the case of an HKT manifold the three Bismut connections associated to the Hermitian structures $(J_\alpha, g)$ coincide and this connection is said to be an HKT connection. In terms of the associated Kähler forms

$$\omega_\alpha(X, Y) = g(J_\alpha X, Y), \quad \alpha = 1, 2, 3,$$

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the HKT condition becomes equivalent to
\begin{equation}
J_1 d\omega_1 = J_2 d\omega_2 = J_3 d\omega_3,
\end{equation}
or to \(\bar{\partial}_J (\omega_2 - i\omega_3) = 0\) \cite{18}. It has been shown in \cite{27} that if \((M, \{ J_\alpha \}, g)\) is almost hyper-Hermitian, then condition (1) implies the integrability of \(J_\alpha, \alpha = 1, 2, 3\).

Hyper-Hermitian connections with totally skew-symmetric torsion are present in many branches of theoretical and mathematical physics. For instance, the previous connections appear on supersymmetric sigma models with Wess-Zumino term \cite{16, 21, 22} and in supergravity theory \cite{17, 31}.

The first examples of strong HKT structures were found on reductive Lie groups with compact semisimple factor by using the hypercomplex structure constructed by Joyce in \cite{24} (see for instance \cite{18, 33}). These Lie groups are also examples of generalized hyper-Kähler manifolds. This type of structures was introduced by Gates, Hull and Röcek in \cite{16} and studied in more detail in \cite{9}.

In \cite{2} it was proved that, as in the hyper-Kähler case, locally any HKT metric admits an HKT potential and in \cite{19} HKT reduction has been studied in order to construct new examples. Non-homogeneous examples of compact HKT manifolds have been constructed in \cite{36} by considering the total space of a hyperholomorphic bundle over an HKT manifold.

Some geometrical and topological properties have been studied. For instance, a simple characterisation of HKT geometry in terms of the intrinsic torsion of the \(Sp(n)Sp(1)\)-structure was obtained in \cite{27}. A version of Hodge theory for HKT manifolds has been given in \cite{37} by exploiting a remarkable analogy between the de Rham complex of a Kähler manifold and the Dolbeault complex of an HKT manifold. More recently, in \cite{38} balanced HKT metrics are studied, showing that the HKT metrics are precisely the quaternionic Calabi-Yau metrics defined in terms of the quaternionic Monge-Ampère equation. Moreover, by \cite{38} a balanced HKT manifold has Obata connection with holonomy in \(SL(n, \mathbb{H})\).

In 4 dimensions any hyper-Hermitian manifold is HKT, but in higher dimensions this is no longer true. In fact, there exist hypercomplex manifolds of dimension \(\geq 8\) which do not admit any HKT metric compatible with the hypercomplex structure \cite{13, 8}. These manifolds are nilmanifolds, i.e. they are compact quotients of nilpotent Lie groups by co-compact discrete subgroups. Strong KT geometry on 6-dimensional nilmanifolds has been studied in \cite{12, 34}.

It is known that on a Lie algebra \(\mathfrak{g}\), a hyper-Hermitian structure with an abelian hypercomplex structure \(\{ J_\alpha \}\), i.e. such that
\[ [J_\alpha X, J_\alpha Y] = [X, Y], \]
for any \(X, Y \in \mathfrak{g}\) and \(\alpha = 1, 2, 3\), is always weak HKT, so many HKT manifolds can be constructed by considering abelian hypercomplex structures on solvable Lie algebras \cite{15}.

HKT geometry on nilmanifolds was investigated in \cite{13}. Subsequently in \cite{8}, it has been shown that a hyper-Hermitian structure on a nilmanifold is HKT if and only if the hypercomplex structure is abelian. In this work we provide a counterexample to this result for HKT solvmanifolds (§3.1). HKT structures on nilmanifolds are always weak, and by
the holonomy of the Obata connection is contained in $SL(n, \mathbb{H})$, since for complex nilmanifolds the canonical bundle is trivial as a holomorphic line bundle. As pointed out in [38], the only known examples of compact HKT manifolds with holonomy in $SL(n, \mathbb{H})$ are those nilmanifolds admitting abelian hypercomplex structures. In the present paper we obtain a new compact HKT manifold with holonomy in $SL(n, \mathbb{H})$ which is not a nilmanifold (§3.1). The existence of strong HKT structures on solvmanifolds is an open problem.

It has been shown in [4] that given a Hermitian or hyper-Hermitian Lie algebra, its tangent Lie algebra with respect to a suitable connection admits a natural hyper-Hermitian structure. In the paper we consider both cases separately and show that the tangent Lie algebra of an HKT Lie algebra may admit an HKT structure. In this way we can construct a family of new compact strong HKT manifolds, compact quotients of Lie groups which are neither reductive nor solvable (Theorem 5.1).

The construction is made starting from a $4n$-dimensional HKT Lie algebra $\mathfrak{g}$ with a flat hyper-Hermitian connection (Theorem 3.2). This connection turns out to be a quaternionic representation of $\mathfrak{g}$ on $\mathbb{H}^q$. The weak, strong, hyper-Kähler or balanced condition is preserved by the construction. This result is generalised in Theorem 3.3 starting from any quaternionic representation of $\mathfrak{g}$ on $\mathbb{H}^q$, for arbitrary $q$, and it can be viewed as an application of the general results obtained in [26] for hyperholomorphic bundles over HKT manifolds.

We apply the previous procedure to the known strong HKT reductive Lie algebras. One of the compact strong HKT manifolds is the product of $S^1$ by the 7-dimensional manifold with a weakly integrable generalized $G_2$-structure found in [14]. Starting with an abelian hypercomplex structure on $\mathfrak{g}$ our construction does not give an abelian hypercomplex structure on the new Lie algebra, unless the quaternionic representation is trivial (Theorem 3.1).

On the other hand, for the hyper-Hermitian Lie algebras constructed in [4] starting from a Hermitian Lie algebra with a flat and torsion-free complex connection, we prove that if the hyper-Hermitian structure is HKT, then the starting Hermitian Lie algebra is necessarily Kähler (Theorem 5.1). The hyper-Hermitian Lie algebras obtained in this way have flat Obata connection. As an application of this result we find two 8-dimensional Lie algebras with a conformally balanced weak HKT structure, so in particular these metrics are not hyper-Kähler.

2. Preliminaries

Let $M$ be a $2n$-dimensional manifold. We recall that an almost complex structure $J$ on $M$ is integrable if and only if the Nijenhuis tensor

$$N(X, Y) = J([X, Y] - [JX, JY]) - ([JX, Y] + [X, JY])$$

vanishes for all vector fields $X, Y$. In this case $J$ is called a complex structure on $M$.

A Riemannian metric $g$ on a complex manifold $(M, J)$ is said to be Hermitian if it is compatible with $J$, i.e. if $g(JX, JY) = g(X, Y)$, for any $X, Y$. We recall from [8] that on
any Hermitian manifold \((M, J, g)\) there exists a unique connection \(\nabla^B\), called the Bismut connection, such that
\[
\nabla^B J = \nabla^B g = 0
\]
and its torsion tensor
\[
c(X, Y, Z) = g(X, T^B(Y, Z))
\]
is totally skew-symmetric, where \(T^B\) is the torsion of \(\nabla^B\). The geometry associated to the Bismut connection is called KT geometry and when \(c = 0\) it coincides with the usual Kähler geometry. A Hermitian metric \(g\) on a complex manifold \(M\) is called balanced if the Lee form \(\theta = Jd^*\omega\) vanishes or equivalently if \(d^*\omega = 0\), where \(d^*\) is the adjoint of \(d\) with respect to \(g\) and \(\omega\) is the associated Kähler form. Moreover, by [23] if \(\{e_1, \ldots, e_{2n}\}\) is an orthonormal basis of the tangent space \(T_pM\), then
\[
(2) \quad \theta_p(v) = -\frac{1}{2} \sum_{i=1}^{2n} c(Jv, e_i, Je_i),
\]
for any tangent vector \(v \in T_pM\).

A hyper-Hermitian structure \(\{\{J_\alpha\}, g\}\) on a \(4n\)-dimensional manifold \(M\) is given by a hypercomplex structure, i.e. a triple of complex structures \(\{J_\alpha\}_{\alpha=1,2,3}\) satisfying the quaternion relations
\[
J^2_\alpha = -\text{id}, \quad \alpha = 1, 2, 3, \quad J_1J_2 = -J_2J_1 = J_3,
\]
and by a Riemannian metric \(g\) compatible with \(J_\alpha\), for any \(\alpha\). Given a hypercomplex structure \(\{J_\alpha\}\) on \(M\) there exists a unique torsion-free connection \(\nabla^O\), called the Obata connection, such that \(\nabla^O J_\alpha = 0\), for \(\alpha = 1, 2, 3\) (cf. [28]).

If on a hyper-Hermitian manifold \((M, \{J_\alpha\}, g)\) there exists a hyper-Hermitian connection such that its torsion tensor \(c\) is a 3-form, then the manifold \(M\) is called hyper-Kähler with torsion (HKT). This is equivalent to the fact that the three Bismut connections associated to each Hermitian structure \((J_\alpha, g)\) coincide. If \(c = 0\) the Bismut connection is equal to both the Levi-Civita connection and the Obata connection, since the manifold is hyper-Kähler. HKT structures are called strong or weak depending on whether the torsion \(c\) is closed or not. In the case of an HKT manifold the Lee forms associated to the Hermitian structures \((J_\alpha, g), \alpha = 1, 2, 3\), coincide (see [23]). Moreover, in general on an HKT manifold the Ricci tensor of the Bismut connection is not symmetric; by [23] it is symmetric if and only if the torsion 3-form \(c\) is co-closed, i.e. if and only if \(d^*c = 0\).

A Riemannian manifold \((M, g)\) is called generalized hyper-Kähler or a \((4, 4)\)-manifold if it has a pair of strong HKT structures \(\{J^+_\alpha\}, g\) and \(\{J^-\alpha\}, g\) for which \(c^- = -c^+\), where \(c^{\pm}\) denotes the torsion 3-form of the hyper-Hermitian connection associated to the HKT structure \(\{J^\pm\alpha\}, g\). These manifolds have been introduced in [16] and further studied in [20, 9].

In this paper we consider Lie algebras endowed with HKT structures which induce left-invariant HKT structures on the corresponding simply connected Lie groups.
Let $g$ be a Lie algebra with an (integrable) complex structure $J$ and an inner product $g$ compatible with $J$. If the associated Kähler form $\omega(X,Y) = g(JX,Y)$ satisfies $d\omega = 0$, where
\[
d\omega(X,Y,Z) = -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y),
\]
for any $X,Y,Z \in g$, the Hermitian Lie algebra $(g,J,g)$ is Kähler. Equivalently, $(g,J,g)$ is Kähler if and only if $\nabla^g J = 0$, where $\nabla^g$ is the Levi-Civita connection of $g$, which can be computed by
\[
2g(\nabla^g_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y),
\]
for any $X,Y,Z \in g$.

Given a hyper-Hermitian structure $(\{J_\alpha\},g)$ on $g$, when the associated Kähler forms $\omega_\alpha$ are closed, the hyper-Hermitian Lie algebra $(g,\{J_\alpha\},g)$ is hyper-Kähler. This is equivalent to the condition $\nabla^g J_\alpha = 0$, $\alpha = 1,2,3$. We point out that a Lie group with a left-invariant hyper-Kähler structure is necessarily flat since a hyper-Kähler metric is Ricci flat and in the homogeneous case, Ricci flatness implies flatness (see [1]). A characterisation of hyper-Kähler Lie groups has been carried out in [3] in order to obtain new complete hyper-Kähler metrics.

For a Lie group $G$ with a left-invariant hyper-Hermitian structure $(\{J_\alpha\},g)$, it was shown in [13] that $(\{J_\alpha\},g)$ is HKT if and only if
\[
g([J_\alpha X,J_\alpha Y],Z) + g([J_\alpha Y,J_\alpha Z],X) + g([J_\alpha Z,J_\alpha X],Y) \\
= g([J_\beta X,J_\beta Y],Z) + g([J_\beta Y,J_\beta Z],X) + g([J_\beta Z,J_\beta X],Y),
\]
for any $X,Y,Z$ in the Lie algebra $g$ of $G$ and for any $\alpha, \beta \in \{1,2,3\}$. The Bismut connection $\nabla^B$ on $G$ is given by the following equation
\[
g(\nabla^B_X Y, Z) = \frac{1}{2 \alpha} \{g([X,Y] - [J_\alpha X,J_\alpha Y], Z) \\
- g([Y,Z] + [J_\alpha Y,J_\alpha Z], X) + g([Z,X] - [J_\alpha Z,J_\alpha X], Y),
\]
for any $X,Y,Z \in g$ and $\alpha \in \{1,2,3\}$.

3. Construction of HKT structures on tangent Lie algebras

Let $\{J_\alpha\}$ be a hypercomplex structure on a Lie algebra $g$ and assume that $D$ is a flat connection on $g$ such that $DJ_\alpha = 0$, for any $\alpha = 1,2,3$. In other words,
\[
D : g \to \mathfrak{gl}(n,\mathbb{H})
\]
is a Lie algebra homomorphism, where $\dim g = 4n$. Consider the tangent Lie algebra $T_D g := g \ltimes_D g$ with the following Lie bracket:
\[
[(X_1,X_2),(Y_1,Y_2)] = ([X_1,Y_1], D_{X_1}Y_2 - D_{Y_1}X_2)
\]
and hypercomplex structure:
\[
\bar{J}_1(X_1,X_2) = (J_1X_1,J_1X_2), \quad \bar{J}_2(X_1,X_2) = (J_2X_2,J_2X_1), \quad \bar{J}_3 = \bar{J}_1\bar{J}_2.
\]
Since $D$ is flat, the Lie bracket (8) on $T_D g$ satisfies the Jacobi identity. The integrability of the complex structure $\tilde{J}_\alpha$ with respect to $D$ follows from the fact that $J_\alpha$ is integrable and parallel with respect to $D$ (see [4, Proposition 3.3]). We show next that the Obata connection $\nabla^O$ for any $\alpha$ of Lemma 3.1 is given in terms of the Obata connection $\nabla^O$ of $\{\tilde{J}_\alpha\}$ and the flat connection $D$. 

**Lemma 3.1.** The Obata connection $\nabla^O$ of the hypercomplex Lie algebra $(T_D g, \{\tilde{J}_\alpha\})$ is related to the Obata connection $\nabla^O$ of $(g, \{J_\alpha\})$ and the flat connection $D$ by

\[
\tilde{\nabla}^O_{(X_1,X_2)}(Y_1,Y_2) = (\nabla^O_{X_1}Y_1, D_{X_1}Y_2),
\]

for any $(X_1,X_2), (Y_1,Y_2) \in T_D g$. Therefore, $\tilde{\nabla}^O$ and $\nabla^O$ have the same infinitesimal holonomy. In particular, $\tilde{\nabla}^O$ is flat if any only if $\nabla^O$ is flat.

**Proof.** The connection $\tilde{\nabla}^O$ defined by (8) is torsion-free and satisfies $\tilde{\nabla}^O\tilde{J}_\alpha = 0$ for $\alpha = 1, 2, 3$, therefore $\tilde{\nabla}^O$ is the Obata connection corresponding to $\{\tilde{J}_\alpha\}$. Moreover, if $R^O$ and $\tilde{R}^O$ denote the curvature tensors of $\nabla^O$ and $\tilde{\nabla}^O$, respectively, we have the following relation:

\[
R^O_{(X_1,X_2),(Y_1,Y_2)}(Z_1,Z_2) = \left[\tilde{\nabla}^O_{(X_1,X_2)}, \tilde{\nabla}^O_{(Y_1,Y_2)}\right](Z_1,Z_2) - \nabla^O_{[(X_1,X_2),(Y_1,Y_2)]}(Z_1,Z_2) = (\tilde{R}^O_{X_1,Y_1}Z_1, 0).
\]

We study next a necessary and sufficient condition for $\{\tilde{J}_\alpha\}$ to be an abelian hypercomplex structure, since it is known that such structures on a Lie algebra give rise to weak HKT structures, see for instance [18, 15, 6, 11]. In fact, we show that a hypercomplex structure $\{J_\alpha\}$ on $g$ induces an abelian hypercomplex structure on $T_D g$ if and only if $\{J_\alpha\}$ is abelian and $D = 0$. In other words, this construction preserves abelianness of the hypercomplex structure if and only if $T_D g$ is a trivial central extension of $g$.

**Theorem 3.1.** Let $\{J_\alpha\}$ be a hypercomplex structure on a 4n-dimensional Lie algebra $g$, $D$ a flat connection such that $D_{J_\alpha} = 0$, for any $\alpha$, and $\{\tilde{J}_\alpha\}$ the induced hypercomplex structure on $T_D g$ given by (8). Then $\{\tilde{J}_\alpha\}$ is abelian if and only if $\{J_\alpha\}$ is abelian on $g$ and $D = 0$, i.e. $T_D g = g \oplus \mathbb{H}^n$.

**Proof.** The hypercomplex structure $\{\tilde{J}_\alpha\}$ on $T_D g$ is abelian if and only if $\{J_\alpha\}$ is abelian on $g$ and

\[
[(\tilde{J}_\alpha(X,0), \tilde{J}_\alpha(0,Y)) = [(X,0), (0,Y)], \quad X, Y \in g.
\]

The above equation is equivalent to the condition

\[
D_X = D_{J_\alpha X} J_\alpha,
\]

for any $\alpha = 1, 2, 3$ and $X \in g$. Let $(\alpha, \beta, \gamma)$ be a cyclic permutation of $(1, 2, 3)$. Equation (8) implies that

\[
J_\alpha D_{J_\alpha X} = D_{J_\alpha X} J_\alpha = D_{J_\beta X} J_\beta = J_\beta D_{J_\beta X},
\]
for any \( X \in \mathfrak{g} \), therefore,
\[
-D_{J_\alpha}X = J^\alpha_\alpha D_{J_\alpha}X = J_\alpha J_\beta D_{J_\beta}X = J_\alpha D_{J_\alpha(J_\beta X)} = J_\alpha D_{J_\beta X} = -J_\alpha D_X.
\]

Hence,
\[
D_X = -J_\alpha D_{J_\alpha}X = -D_{J_\alpha X}J_\alpha,
\]
which together with (3) implies that \( D_X = 0 \) for any \( X \in \mathfrak{g} \).

Starting with an abelian hypercomplex structure on \( \mathfrak{g} \) it is possible to get a non-vanishing flat connection \( D \) such that \( T_D \mathfrak{g} \) carries an HKT structure. For instance, this can be done for the 4-dimensional Lie algebra \( \text{aff}(\mathbb{C}) \) with its abelian hypercomplex structure (see [4]). Therefore, in view of Theorem 3.1 the induced hypercomplex structure on the tangent Lie algebra \( T_D \text{aff}(\mathbb{C}) \) will be non-abelian.

We consider next a hyper-Hermitian Lie algebra \( (\mathfrak{g}, \{J_\alpha\}, g) \) with a flat connection \( D \) such that \( DJ_\alpha = 0 \), for any \( \alpha = 1, 2, 3 \). Let \( \{\tilde{J}_\alpha\} \) be the hypercomplex structure on \( T_D \mathfrak{g} \) given by (2) and let \( \tilde{g} \) be the inner product on \( T_D \mathfrak{g} \) induced by \( g \) such that \( (\mathfrak{g}, 0) \) and \( (0, \mathfrak{g}) \) are orthogonal. Then \( \tilde{g} \) is compatible with all complex structures \( \tilde{J}_\alpha \), that is, \( (T_D \mathfrak{g}, \{\tilde{J}_\alpha\}, \tilde{g}) \) is a hyper-Hermitian Lie algebra. The following result gives a necessary and sufficient condition for \( (T_D \mathfrak{g}, \{\tilde{J}_\alpha\}, \tilde{g}) \) to be HKT.

**Theorem 3.2.** Let \( (\mathfrak{g}, \{J_\alpha\}, g) \) be a hyper-Hermitian Lie algebra and \( D \) a flat connection such that \( DJ_\alpha = 0 \), for any \( \alpha \). Then \( (T_D \mathfrak{g}, \{\tilde{J}_\alpha\}, \tilde{g}) \) is HKT if and only if \( (\mathfrak{g}, \{J_\alpha\}, g) \) is HKT and \( Dg = 0 \).

**Proof.** We first prove that the HKT condition is equivalent to the fact that \( (\mathfrak{g}, \{J_\alpha\}, g) \) is HKT and the operator \( DJ_\alpha X - DJ_\beta X \beta \) is symmetric with respect to \( g \), for any \( \alpha \) and \( \beta \).

By using (4) we have that \( (T_D \mathfrak{g}, \{\tilde{J}_\alpha\}, \tilde{g}) \) is HKT if and only if
\[
\tilde{g}([\tilde{J}_\alpha(X_1, X_2), \tilde{J}_\alpha(Y_1, Y_2)], [Z_1, Z_2]) + \tilde{g}([\tilde{J}_\alpha(Y_1, Y_2), \tilde{J}_\alpha Z], [X_1, X_2])
\]
\[
+ \tilde{g}([\tilde{J}_\beta(X_1, X_2), \tilde{J}_\beta(Y_1, Y_2)], [Z_1, Z_2]) - \tilde{g}([\tilde{J}_\beta(Y_1, Y_2), \tilde{J}_\beta Z], [X_1, X_2]) = 0,
\]
for any \( \alpha \) and \( \beta \), \( X_i, Y_i, Z_i \in \mathfrak{g}, i = 1, 2 \). The first component of the previous expression vanishes if and only if \( (\{J_\alpha\}, g) \) is HKT on \( \mathfrak{g} \). The vanishing of the second component with \( X_2 = 0 \) and \( Y_1 = 0 \) yields the following
\[
g(DJ_\alpha X_1 J_\alpha Y_2 - DJ_\beta X_1 J_\beta Y_2, Z_2) = g(Y_2, DJ_\alpha X_1 J_\alpha Z_2 - DJ_\beta X_1 J_\beta Z_2),
\]
for any \( X_1, Y_2, Z_2 \). Conversely, if \( DJ_\alpha X_1 J_\alpha - DJ_\beta X_1 J_\beta \) is symmetric with respect to \( g \), the second component of (4) is zero by direct computation.

Therefore, from the relation
\[
(D_{J_\alpha X}g)(J_\alpha Y, W) = g(D_{J_\alpha X}J_\alpha Y, W) = g(D_{J_\alpha X}J_\alpha W, Y), \quad X, Y, W \in \mathfrak{g}.
\]
we get that the operator $D_{J_\alpha X} J_\alpha - D_{J_\beta X} J_\beta$ is symmetric with respect to $g$ if and only if
\[ (D_{J_\alpha X} g)(J_\alpha Y, W) = (D_{J_\beta X} g)(J_\beta Y, W), \]
for any $X, Y, W \in \mathfrak{g}$.

By (12), if $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$, we have
\[ (D_X g)(Y, W) = (D_{J_\alpha (J_\alpha X)} g)(J_\alpha J_\alpha Y, W) = (D_{J_\beta (J_\alpha X)} g)(J_\beta J_\alpha Y, W) = (D_{J_\gamma X} g)(J_\gamma Y, W). \]

Thus
\[ g(D_X Y, W) + g(Y, D_X W) = g(D_{J_\gamma X} J_\gamma Y, W) - g(Y, J_\gamma D_{J_\gamma X} W), \]
for any $X, Y, W \in \mathfrak{g}$ and $\gamma = 1, 2, 3$.

The above equality is equivalent to the condition that the adjoint $(D_X - D_{J_\gamma X} J_\gamma)^*$ of the operator $D_X - D_{J_\gamma X} J_\gamma$ with respect to $g$ coincides with $-D_X - D_{J_\gamma X} J_\gamma$. Then
\[ (D_{J_\gamma X} J_\gamma)^* - D_{J_\gamma X} J_\gamma = D_X + (D_X)^*. \]
Since the left-hand side is skew-symmetric and the right one is symmetric, in particular we get that $D_X = -(D_X)^*$ and thus the theorem follows.

In the case of the Obata connection $D = \nabla^O$ we get the following result:

**Corollary 3.1.** Let $(\mathfrak{g}, \{J_\alpha\}, g)$ be a hyper-Hermitian Lie algebra with flat Obata connection $\nabla^O$. Then $(T_C \mathfrak{g}, \{\tilde{J}_\alpha\}, \tilde{g})$ is HKT if and only if $g$ is hyper-Kähler.

**Proof.** Since $\nabla^O$ is torsion free, $\nabla^O$ is metric if and only it coincides with the Levi-Civita connection. \qed

We will apply Theorem 3.2 in the next sections in order to construct new strong and weak HKT manifolds.

The following result shows that the properties of being hyper-Kähler, HKT strong (resp. weak) and balanced are preserved by our construction.

**Proposition 3.1.** Let $(\mathfrak{g}, \{J_\alpha\}, g)$ be an HKT Lie algebra and $D$ a flat connection such that $Dg = 0$ and $DJ_\alpha = 0$, for any $\alpha$. Then:

(i) The HKT structure $(\{\tilde{J}_\alpha\}, \tilde{g})$ on $T_D \mathfrak{g}$ is hyper-Kähler if and only if $(\{J_\alpha\}, g)$ is hyper-Kähler on $\mathfrak{g}$.

(ii) $(\{\tilde{J}_\alpha\}, \tilde{g})$ is strong (respectively weak) if and only if $(\{J_\alpha\}, g)$ is strong (respectively weak).

(iii) $(\{\tilde{J}_\alpha\}, \tilde{g})$ is balanced if and only if $(\{J_\alpha\}, g)$ is balanced.

**Proof.** It follows from (8) in Lemma 3.2 that the Obata connection $\tilde{\nabla}^O$ corresponding to $\{\tilde{J}_\alpha\}$ satisfies $\tilde{\nabla}^O \tilde{g} = 0$ if and only if $\nabla^O g = 0$, that is, $\tilde{g}$ is hyper-Kähler if and only if $g$ is hyper-Kähler.
In order to show (ii) and (iii), by a direct computation we have that the Bismut connection \( \tilde{\nabla}^B \) of the new HKT structure \( \{ \tilde{J}_\alpha, \tilde{g} \} \) on \( T_D \mathfrak{g} \) is related to the Bismut connection \( \nabla^B \) of the HKT structure \( \{ J_\alpha, g \} \) on \( \mathfrak{g} \) by
\[
\tilde{g}(\tilde{\nabla}^B_{X_1,X_2}(Y_1,Y_2),(Z_1,Z_2)) = g(\nabla^B_{X_1}Y_1,Z_1) + g(D_{X_1}Y_2,Z_2),
\]
for any \( X_i, Y_i, Z_i \in \mathfrak{g}, \ i = 1,2,3. \) Therefore, if we denote by \( \tilde{c} \) and \( c \) the torsions of the HKT structures on \( T_D \mathfrak{g} \) and \( \mathfrak{g} \), respectively, we obtain
\[
\tilde{c}((X_1,X_2),(Y_1,Y_2),(Z_1,Z_2)) = c(X_1,Y_1,Z_1),
\]
and
\[
d\tilde{c}((X_1,X_2),(Y_1,Y_2),(Z_1,Z_2),(W_1,W_2)) = dc(X_1,Y_1,Z_1,W_1).
\]
This shows that the strong (respectively weak) condition is preserved.

For (iii), we may use as orthonormal basis of \( (T_D \mathfrak{g}, \tilde{g}) \) the basis \( \{(e_i,0),(0,e_i),i = 1,\ldots,4n\} \), where \( \{e_1,\ldots,e_{4n}\} \) is an orthonormal basis of \( (\mathfrak{g},g) \). Then, by (2) we get that the Lee form \( \tilde{\theta} \) of the new HKT structure on \( T_D \mathfrak{g} \) is given by \( \tilde{\theta} = \theta \circ p \), where \( \theta \) is the Lee form of the old HKT structure on \( \mathfrak{g} \) and \( p : T_D \mathfrak{g} \to \mathfrak{g} \) is the orthogonal projection. Indeed:
\[
\tilde{\theta}((X_1,X_2)) = -\frac{1}{2} \sum_{i=1}^{4n} \tilde{c}((J_1X_1,J_1X_2),(e_i,0),(J_1e_i,0))
- \frac{1}{2} \sum_{i=1}^{4n} \tilde{c}((J_1X_1,J_1X_2),(0,e_i),(0,J_1e_i))
= -\frac{1}{2} \sum_{i=1}^{4n} \theta((J_1X_1,e_i,J_1e_i)) = \theta(X_1),
\]
for any \( (X_1,X_2) \in T_D \mathfrak{g} \).

\[\square\]

**Remark 3.1.** The construction of HKT structures on tangent Lie algebras given in Theorem 3.2 can be iterated, since if one considers on the HKT Lie algebra \( (T_D \mathfrak{g}, \{ \tilde{J}_\alpha \}, \tilde{g}) \) the connection \( \tilde{D} \) defined by
\[
\tilde{D}_{(X_1,X_2)}(Y_1,Y_2) = (D_{X_1}Y_1,D_{X_1}Y_2),
\]
\( \tilde{D} \) is hyper-Hermitian and flat, and therefore \( T_D(T_D \mathfrak{g}) \) admits an HKT structure (compare with [3]). Moreover, starting with a balanced 4n-dimensional HKT Lie algebra \( (\mathfrak{g}, \{ J_\alpha \}, g) \) (see, for instance, §3.2), Proposition 3.2 implies that the successive tangent HKT Lie algebras are balanced. Since balanced HKT structures have holonomy in \( SL(m,\mathbb{H}) \), then the HKT structures on the simply connected Lie groups corresponding to the tangent Lie algebras and on any of their compact quotients (provided these exist) have holonomy in \( SL(m,\mathbb{H}) \) for some \( m \geq n \).

Let \( (\mathfrak{g}, \{ J_\alpha \}, g) \) be a hyper-Hermitian 4n-dimensional Lie algebra and \( D \) a flat connection on \( \mathfrak{g} \). The following conditions are equivalent:

1. \( Dg = 0 \) and \( DJ_\alpha = 0, \ \alpha = 1,2,3; \)
2. for any \( X \in \mathfrak{g}, D_X \) is skew-symmetric with respect to \( g \) and commutes with \( J_\alpha \) for each \( \alpha; \)
(3) the map $D : \mathfrak{g} \to \mathfrak{sp}(n)$ is a Lie algebra homomorphism.

Therefore, Theorem 3.2 can be rephrased as follows:

**Corollary 3.2.** Let $(\mathfrak{g}, \{J_{\alpha}\}, g)$ be a hyper-Hermitian $4n$-dimensional Lie algebra and $D$ a flat connection such that $DJ_{\alpha} = 0$, for any $\alpha$. The hyper-Hermitian structure $(\{\tilde{J}_{\alpha}\}, \tilde{g})$ on the tangent algebra $T_D \mathfrak{g}$ is HKT if and only if $(\mathfrak{g}, \{J_{\alpha}\}, g)$ is HKT and $D : \mathfrak{g} \to \mathfrak{sp}(n)$ is a Lie algebra homomorphism.

**Remark 3.2.** If $G$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, there is a one-to-one correspondence between Lie algebra homomorphisms $D : \mathfrak{g} \to \mathfrak{sp}(n)$ and quaternionic unitary representations of $G$ on $\mathbb{H}^n$.

The previous construction can be generalised by replacing the flat connection $D$ on $\mathfrak{g}$ by a quaternionic representation of $\mathfrak{g}$ on $\mathbb{H}^q$. In fact, given a $4n$-dimensional Lie algebra $\mathfrak{g}$ and a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(q, \mathbb{H}),$$

instead of considering the tangent Lie algebra of $\mathfrak{g}$, one can define on $T_{\rho} \mathfrak{g} := \mathfrak{g} \rtimes_{\rho} \mathbb{H}^q$ the following Lie bracket:

$$[(X, V), (Y, W)] = ([X, Y], \rho(X)W - \rho(Y)V),$$

for any $X, Y \in \mathfrak{g}$ and $V, W \in \mathbb{H}^q$.

Given a hypercomplex structure $\{J_{\alpha}\}$ on $\mathfrak{g}$ we define $\{\tilde{J}_{\alpha}\}$ on $T_{\rho} \mathfrak{g}$ as follows:

$$\tilde{J}_{\alpha}(X, V) = (J_{\alpha}X, i_{\alpha}V),$$

with $i_1 = i$, $i_2 = j$, $i_3 = k$, where $i_{\alpha}V$ denotes left quaternionic multiplication by $i_{\alpha}$ on $\mathbb{H}^q$. The integrability of $\tilde{J}_{\alpha}$ follows from [1, Proposition 3.3].

If $q = n$, the Lie algebra $T_{\rho} \mathfrak{g}$ coincides with the tangent Lie algebra $T_D \mathfrak{g}$ associated to the connection $D = \rho$.

If $(\mathfrak{g}, \{J_{\alpha}\}, g)$ is hyper-Hermitian, the Lie algebra $T_{\rho} \mathfrak{g}$ becomes hyper-Hermitian with the hypercomplex structure given by (14) and the inner product $\tilde{g}$ induced by $g$ and the natural inner product on $\mathbb{H}^q$ such that $\mathfrak{g}$ is orthogonal to $\mathbb{H}^q$.

**Remark 3.3.** We point out that Corollary 3.2 goes through with the obvious changes for the Lie algebra $T_{\rho} \mathfrak{g}$, and it turns out that the HKT structure $(\{\tilde{J}_{\alpha}\}, \tilde{g})$ on $T_{\rho} \mathfrak{g}$ is strong (respectively, weak, hyper-Kähler, balanced) if and only if $(\{J_{\alpha}\}, g)$ is strong (respectively, weak, hyper-Kähler, balanced). The analogue of Theorem 3.1 is also true for $T_{\rho} \mathfrak{g}$. In fact, given a hypercomplex structure $\{J_{\alpha}\}$ on $\mathfrak{g}$, consider the induced hypercomplex structure $\{\tilde{J}_{\alpha}\}$ on $T_{\rho} \mathfrak{g}$ as in (14). An analogous proof to that of Theorem 3.1 gives that $\{\tilde{J}_{\alpha}\}$ is abelian on $T_{\rho} \mathfrak{g}$ if and only if $\{J_{\alpha}\}$ is abelian on $\mathfrak{g}$ and $\rho = 0$, i.e. $T_{\rho} \mathfrak{g} = \mathfrak{g} \oplus \mathbb{H}^q$. 
Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\pi : G \to GL(q, \mathbb{H})$ a quaternionic representation of $G$ on $\mathbb{H}^q$. The following result is a generalisation of Theorem 3.2 and its proof follows by using analogous arguments, replacing the connection $D$ with the representation $\rho = (d\pi)_e$ of $\mathfrak{g}$. This may be regarded as the left-invariant counterpart of [36, Theorem 7.2], where it has been shown that the natural metric on the total space of a hyperholomorphic bundle over an HKT manifold is HKT.

Theorem 3.3. Let $(\{ J_\alpha \}, g)$ be a left invariant hyper-Hermitian structure on a connected Lie group $G$ and $\pi : G \to GL(q, \mathbb{H})$ a quaternionic representation. Then $(T(\pi) G, \{ \tilde{J}_\alpha \}, \tilde{g})$ is HKT if and only if $(G, \{ J_\alpha \}, g)$ is HKT and $\pi$ is unitary, that is, $\pi(G)$ is contained in $Sp(q)$.

Moreover, the new HKT structure $(\{ \tilde{J}_\alpha \}, \tilde{g})$ is strong (respectively weak, hyper-Kähler, balanced) if and only if $(\{ J_\alpha \}, g)$ is strong (respectively weak, hyper-Kähler, balanced).

We show next that Theorem 3.3 is a useful tool for constructing new examples of compact HKT manifolds with holonomy in $SL(n, \mathbb{H})$. In fact, we obtain as an application of the results in this section a new compact HKT manifold which is balanced, therefore its holonomy is contained in $SL(3, \mathbb{H})$ [38, Remark 4.9]. We point out that this manifold is the first known example of a compact 12-dimensional HKT manifold with holonomy in $SL(n, \mathbb{H})$ which is not a nilmanifold (see [38, Remark 4.4]).

3.1. A balanced HKT solvmanifold. Let $(\mathfrak{g}, \{ J_\alpha \}, g)$ be the 8-dimensional HKT Lie algebra with basis $\{ e_1, \ldots, e_8 \}$ and dual basis $\{ e^1, \ldots, e^8 \}$ of $\mathfrak{g}^*$, such that the Lie bracket is given as follows:

$$[e_5, e_k] = e_{k-4}, \quad k = 6, 7, 8,$$
$$[e_6, e_8] = e_3,$$
$$[e_6, e_7] = -e_4,$$
$$[e_7, e_8] = -e_2,$$

with inner product

$$g = \sum_{i=1}^{8} (e^i)^2,$$

and hypercomplex structure

$$J_1 e_1 = e_2, \quad J_1 e_3 = e_4, \quad J_1 e_5 = e_6, \quad J_1 e_7 = e_8, \quad J_1^2 = -id,$$
$$J_2 e_1 = e_3, \quad J_2 e_2 = -e_4, \quad J_2 e_5 = e_7, \quad J_2 e_6 = -e_8, \quad J_2^2 = -id,$$
$$J_3 e_1 = e_4, \quad J_3 e_2 = e_3, \quad J_3 e_5 = e_8, \quad J_3 e_6 = e_7, \quad J_3^2 = -id.$$

We observe that span $\{ e_2, \ldots, e_8 \}$ with the inner product induced by $g$ is a 2-step nilpotent Lie algebra of Heisenberg type (see [26]) and $\{ J_\alpha \}$ is an abelian hypercomplex structure on $\mathfrak{g}$. Since the inner product $g$ is hyper-Hermitian, it follows from [15] that $(\mathfrak{g}, \{ J_\alpha \}, g)$
is an HKT Lie algebra. Moreover, [3, Proposition 4.11] implies that \( g \) is balanced. We fix a basis \( \{f_1, \ldots, f_4\} \) of \( \mathbb{R}^4 \) and consider the endomorphism \( \rho : g \to \text{End}(\mathbb{R}^4) \) defined by:

\[
\rho(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(e_k) = 0, \ k = 2, \ldots, 8.
\]

We extend \( \{J_\alpha\} \) to a hypercomplex structure \( \{\tilde{J}_\alpha\} \) on \( T_\rho g \) as follows:

\[
\tilde{J}_1 f_1 = f_2, \quad \tilde{J}_1 f_3 = f_4, \quad \tilde{J}_2^2 = -\text{id}, \\
\tilde{J}_2 f_1 = f_3, \quad \tilde{J}_2 f_2 = -f_4, \quad \tilde{J}_3^2 = -\text{id}, \\
\tilde{J}_3 f_1 = f_4, \quad \tilde{J}_3 f_2 = f_3.
\]

Let \( \tilde{g} \) be the inner product on \( T_\rho g \) obtained by extending \( g \) in the obvious way. It follows that \( \rho : g \to \mathfrak{sp}(1) \) is a Lie algebra homomorphism, therefore, \( (T_\rho g, \{\tilde{J}_\alpha\}, \tilde{g}) \) is a 12-dimensional balanced HKT Lie algebra (see Remark 3.3). Let \( S = N \times \mathbb{R}^4 \) be the simply connected solvable Lie group with Lie algebra \( T_\rho g \), where \( N \) is the simply connected 2-step nilpotent Lie group with Lie algebra \( g \). It is well known that \( N \) admits a lattice \( \Gamma_1 \) \([20, 32]\), which can be chosen to be compatible with \( \{J_\alpha\} \), and there exists a lattice \( \Gamma_2 \) in \( \mathbb{R}^4 \) such that the action of \( N \) on \( \mathbb{R}^4 \) is compatible with \( \Gamma_2 \). Therefore, setting \( \Gamma := \Gamma_1 \times \Gamma_2 \), \( \Gamma \) is a lattice in \( S \) and \( \Gamma \backslash S \) is a solvmanifold carrying a balanced HKT structure, hence it has holonomy in \( SL(3, \mathbb{H}) \). This manifold is an \( S^1 \)-bundle over a 2-step nilmanifold. We point out that \( \{\tilde{J}_\alpha\} \) is not abelian on \( T_\rho g \), therefore this provides a counterexample to the analogue of [3, Theorem 4.6] for HKT solvmanifolds, since \( \tilde{g} \) is HKT but the corresponding hypercomplex structure is not abelian.

4. Weak HKT structures on 8-dimensional tangent Lie algebras

In this section we apply Corollary 3.2 to all 4-dimensional non-reductive Lie algebras \( g \) admitting hypercomplex structures to obtain HKT structures on the corresponding tangent Lie algebras \( T_X g \). We will proceed as follows. Given an HKT structure \( \{J_\alpha, g\} \) on \( g \), we will consider three endomorphisms \( \{J'_1, J'_2, J'_3\} \) which form a basis of \( \mathfrak{sp}(1) \) and we will provide an explicit homomorphism \( D : g \to \mathfrak{sp}(1) \) by expressing \( DX \) as a linear combination of \( \{J'_\alpha\} \) for any \( X \) in a basis of \( g \).

Given a 4-dimensional Lie algebra \( g \) with basis \( \{e_1, \ldots, e_4\} \), let \( \{e^1, \ldots, e^4\} \) be the basis of \( g^* \) dual to \( \{e_1, \ldots, e_4\} \). By [3] a non-abelian 4-dimensional real Lie algebra admitting a hypercomplex structure is isomorphic to one of the following Lie algebras:

1. \( \mathfrak{sp}(1) \oplus \mathfrak{u}(1) \);
2. \( \mathfrak{aff}(\mathbb{C}) = (-e^{13} + e^{24}, -e^{23} - e^{14}, 0, 0) \);
3. \( (0, -e^{12}, -e^{13}, -e^{14}) \);
4. \( (0, -\frac{1}{2}e^{12}, -\frac{1}{2}e^{13}, -e^{23} - e^{14}) \),
where for instance \((-e^{13} + e^{24}, -e^{23} - e^{14}, 0, 0)\) denotes the Lie algebra with non-zero Lie brackets

\[
\begin{align*}
[e_1, e_3] &= e_1 = -[e_2, e_4], \\
[e_2, e_3] &= e_2 = [e_1, e_4].
\end{align*}
\]

The Lie algebra \(\mathfrak{sp}(1) \oplus \mathfrak{u}(1)\) is the only reductive one in the above list and admits by \([18]\) a strong HKT structure, that we will consider in the next section in order to construct new strong HKT examples.

The Lie algebra (2) is the only 4-dimensional one admitting an abelian hypercomplex structure. Moreover, any hypercomplex structure on \(\mathfrak{aff}(\mathbb{C})\) is abelian \([3]\). Take, for instance:

\[
\begin{align*}
J_1 e_1 &= -e_4, & J_1 e_2 &= e_3, \\
J_2 e_1 &= e_2, & J_2 e_3 &= -e_4.
\end{align*}
\]

If we consider the inner product

\[
g = \sum_{i=1}^{4}(e^i)^2,
\]

it turns out that \((\{J_\alpha\}, g)\) defines a weak HKT structure on \(\mathfrak{aff}(\mathbb{C})\). There is a Lie algebra homomorphism \(D : \mathfrak{aff}(\mathbb{C}) \to \mathfrak{sp}(1)\) defined as follows

\[
\begin{align*}
D_{e_1} &= D_{e_2} = 0, \\
D_{e_3} &= a_1 J_1' + a_2 J_2' + a_3 J_3', \\
D_{e_4} &= bD_{e_3},
\end{align*}
\]

where \(a_i, i = 1, 2, 3, b\) are real numbers and the endomorphisms \(J_\alpha'\) are given by

\[
\begin{align*}
J_1' e_1 &= e_4, & J_1' e_2 &= e_3, & J_1'^2 &= -\text{id}, \\
J_2' e_1 &= -e_2, & J_2' e_3 &= -e_4, & J_2'^2 &= -\text{id}, \\
J_3' e_1 &= -e_3, & J_3' e_2 &= e_4, & J_3'^2 &= -\text{id}.
\end{align*}
\]

Corollary \([3, 2]\) implies that the induced hyper-Hermitian structure \((\{\tilde{J}_\alpha\}, \tilde{g})\) is weak HKT on \(T_D \mathfrak{aff}(\mathbb{C})\) and the hypercomplex structure \(\{\tilde{J}_\alpha\}\) is not abelian (see Theorem \([3, 1]\)).

For the Lie algebra (3) we can consider the weak HKT structure \((\{J_\alpha\}, g)\) defined by \((15)\) and \((16)\). In this case, we have the following homomorphism \(D:\)

\[
\begin{align*}
D_{e_1} &= a_1 J_1' + a_2 J_2' + a_3 J_3', \\
D_{e_2} &= D_{e_3} = D_{e_4} = 0,
\end{align*}
\]

where \(a_i, i = 1, 2, 3\) are real numbers and the endomorphisms \(J_\alpha'\) are as in \((17)\).

For the last Lie algebra (4) one takes the hyper-Hermitian structure \((\{J_\alpha\}, g)\) with

\[
\begin{align*}
J_1 e_1 &= e_4, & J_1 e_2 &= -e_3, \\
J_2 e_1 &= \frac{\sqrt{2}}{2} e_2, & J_2 e_4 &= \frac{\sqrt{2}}{2} e_3,
\end{align*}
\]

and \(g\) given by

\[
g = (e^1)^2 + (e^4)^2 + 2 \left((e^2)^2 + (e^3)^2\right).
\]
Let $D$ be the homomorphism defined in (18) with the following endomorphisms $J'_\alpha$:

\begin{align*}
J'_1 e_1 &= -e_4, & J'_1 e_2 &= -e_3, & J'_1 = -\text{id} \\
J'_2 e_1 &= -\frac{\sqrt{2}}{2} e_2, & J'_2 e_4 &= \frac{\sqrt{2}}{2} e_3, & J'_2 = -\text{id}, \\
J'_3 e_1 &= \frac{\sqrt{2}}{2} e_3, & J'_3 e_4 &= \frac{\sqrt{2}}{2} e_2, & J'_3 = -\text{id}.
\end{align*}

The induced hyper-Hermitian structure $(\{\tilde{J}_\alpha\}, \tilde{g})$ on the tangent Lie algebra is weak HKT. We point out that the Obata connection $\nabla^O$ corresponding to $(g, \{J_\alpha\})$ has holonomy $\mathfrak{sl}(1, \mathbb{H})$, therefore, $\text{hol}(\nabla^O) = \mathfrak{sl}(1, \mathbb{H})$ (Lemma 3.1). In fact, the Obata connection $\nabla^O$ has been calculated in [3]:

\begin{align*}
\nabla^O_{e_1} &= \frac{3}{4} \text{id}, & \nabla^O_{e_2} &= -\frac{\sqrt{2}}{4} J'_2, & \nabla^O_{e_3} &= \frac{\sqrt{2}}{4} J'_3, & \nabla^O_{e_4} &= \frac{1}{4} J'_1,
\end{align*}

hence, $\mathfrak{hol}(\nabla^O) = \text{span}\{J'_\alpha\} \cong \mathfrak{sl}(1, \mathbb{H})$. In particular, the canonical bundles of the corresponding simply connected Lie groups (with respect to $J_\alpha$ and $\tilde{J}_\alpha$, respectively, for any $\alpha$) are holomorphically trivial (see [37]). However, we observe that $g$ (hence $\tilde{g}$) is not balanced.

5. New compact strong HKT manifolds

In [24] Joyce showed that for any compact semisimple Lie algebra $\mathfrak{s}$ there exists a non-negative integer $k$ such that $\mathfrak{s} \oplus k \mathfrak{u}(1)$ admits a hypercomplex structure $\{J_\alpha\}$, constructed by using the decomposition of $\mathfrak{s}$ in terms of certain $\mathfrak{sp}(1)$ subalgebras. By [23, 18] there exists a compatible hyper-Hermitian inner product $g$ such that its restriction to $\mathfrak{s}$ is equal to the opposite of the Killing-Cartan form. Moreover, the hyper-Hermitian structure $(\{J_\alpha\}, g)$ is strong HKT and induces a left-invariant HKT structure on each Lie group $G$ with Lie algebra $\mathfrak{s} \oplus k \mathfrak{u}(1)$.

In this section we apply Theorem 3.2 to obtain HKT structures on non-trivial extensions

\[ T_D(\mathfrak{s} \oplus k \mathfrak{u}(1)) = (\mathfrak{s} \oplus k \mathfrak{u}(1)) \ltimes_D \mathbb{H}^n, \]

where $4n = \dim \mathfrak{s} + k$ and $D$ is an appropriate flat hyper-Hermitian connection on $\mathfrak{s} \oplus k \mathfrak{u}(1)$. For instance, one could consider $D = \nabla^O$, the Obata connection, but $\nabla^O$ is not hyper-Hermitian, since this would imply that $g$ is hyper-Kähler on $\mathfrak{s} \oplus k \mathfrak{u}(1)$, which is impossible. If we consider $D = \nabla^B$, the Bismut connection associated to $(\{J_\alpha\}, g)$, $D$ is hyper-Hermitian and flat, but in this case it is identically zero and therefore $T_{\nabla^B}(\mathfrak{s} \oplus k \mathfrak{u}(1))$ is a trivial central extension of $\mathfrak{s} \oplus k \mathfrak{u}(1)$.

In order to get non-trivial extensions we will investigate the existence of other flat hyper-Hermitian connections on $\mathfrak{s} \oplus k \mathfrak{u}(1)$. The simplest case is the 4-dimensional reductive Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$.

Starting with the standard strong HKT structure $(\{J_\alpha\}, g)$ on the Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ considered in [18], we will show that $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ admits a flat hyper-Hermitian
connection $D$, essentially defined by the projection homomorphism from $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ to $\mathfrak{sp}(1)$.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ with non zero Lie brackets

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$ 

Consider the hyper-Hermitian structure given by the hypercomplex structure

$$D_{e_1} e_1 = -e_4, \quad D_{e_1} e_2 = e_3, \quad D_{e_1} e_3 = -e_1, \quad D_{e_1} e_4 = e_2.$$ 

and by the inner product $g$ such that the basis $\{e_1, \ldots, e_4\}$ is orthonormal. It follows by [18] that $(\{J_\alpha\}, g)$ is strong HKT.

By [3] the centralizer of $\{J_1, J_2\}$ in $\text{End}(\mathfrak{sp}(1) \oplus \mathfrak{u}(1))$, which is isomorphic to $\mathfrak{gl}(1, \mathbb{H})$, is spanned by the identity $\text{id}$ and by the following endomorphisms:

$$J'_1 e_1 = e_4, \quad J'_1 e_2 = e_3, \quad J'_1 e_3 = -e_1, \quad J'_1 e_4 = e_2, \quad J'_1^2 = -\text{id},$$

$$J'_2 e_1 = -e_3, \quad J'_2 e_2 = e_4, \quad J'_2 e_3 = e_1, \quad J'_2 e_4 = -e_2, \quad J'_2^2 = -\text{id}.$$

Therefore, the connection $D$ on $\mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ defined by

$$D_{e_1} = \frac{1}{2} J'_1, \quad D_{e_2} = \frac{1}{2} J'_2, \quad D_{e_3} = \frac{1}{2} J'_3, \quad D_{e_4} = 0,$$

is flat and hyper-Hermitian with respect to $g$.

In view of Corollary 3.2 the connection $D$ corresponds to the projection

$$\mathfrak{sp}(1) \oplus \mathbb{R} e_4 \to \mathfrak{sp}(1).$$

In this way we can apply Theorem 3.3 in order to get a new 8-dimensional Lie algebra $T_D(\mathfrak{sp}(1) \oplus \mathfrak{u}(1))$ with structure equations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2,$$

$$[e_1, e_8] = -[e_2, e_7] = [e_3, e_6] = -\frac{1}{2} e_5,$$

$$[e_1, e_7] = [e_2, e_8] = -[e_3, e_5] = -\frac{1}{2} e_6,$$

$$[e_1, e_6] = -[e_2, e_5] = -[e_3, e_8] = \frac{1}{2} e_7,$$

$$[e_1, e_5] = [e_2, e_6] = -[e_3, e_7] = \frac{1}{2} e_8.$$ 

The induced hypercomplex structure $\{\tilde{J}_\alpha\}$ on $T_D(\mathfrak{sp}(1) \oplus \mathfrak{u}(1))$ (see (7)) and the inner product $\tilde{g}$ give a left-invariant strong HKT structure $\{\tilde{J}_\alpha^+\}, \tilde{g}$ on the simply connected Lie group with Lie algebra $T_D g$. Such a Lie group is a product of $\mathbb{R}$ by the 7-dimensional Lie group $SU(2) \ltimes \mathbb{R}^4$. This 7-dimensional Lie group was already considered in [14], where it was shown that it admits a compact quotient $M^7$. Therefore, we obtain a compact 8-dimensional manifold $M^7 \times S^1$ admitting a strong HKT structure with flat Obata connection.
Remark 5.1. In view of Remark 3.4, the previous construction can be iterated in higher dimensions and this allows to find new compact examples of dimension $2^n + 2$, for any $n \geq 1$.

More in general, we can consider the compact semisimple Lie algebra $\mathfrak{sp}(l)$. By [24, 33, 18] it follows that there exists a strong HKT structure $(\{J_\alpha\}, g)$ on the direct sum $\mathfrak{sp}(l) \oplus \mathfrak{u}(1)$.

For any $l \geq 1$, it is possible to construct a homomorphism

$$\mathfrak{sp}(l) \oplus \mathfrak{u}(1) \rightarrow \mathfrak{sp} \left( \frac{l(l+1)}{2} \right)$$

given by

$$(X, Y) \in \mathfrak{sp}(l) \oplus \mathfrak{u}(1) \mapsto i(X),$$

where the inclusion $i : \mathfrak{sp}(l) \hookrightarrow \mathfrak{sp} \left( \frac{l(l+1)}{2} \right)$ is a natural injective homomorphism. This generalises the case $l = 1$ considered previously (see (19)), where $i$ was the identity of $\mathfrak{sp}(1)$.

Therefore, applying Corollary 3.2, we have the following:

Proposition 5.1. For any $l \geq 1$, the Lie algebra $\mathfrak{sp}(l) \oplus \mathfrak{u}(1)$ with the strong HKT structure $(\{J_\alpha\}, g)$ admits a flat hyper-Hermitian connection $D$ and the induced hyper-Hermitian structure on the tangent algebra $T_D(\mathfrak{sp}(l) \oplus \mathfrak{u}(1))$ is strong HKT.

Consider the following strong HKT reductive Lie algebras $\mathfrak{g}$ (see [33]):

$$\begin{align*}
\mathfrak{su}(2l+1), \ (l > 1), & \quad \mathfrak{su}(2l) \oplus \mathfrak{u}(1), \quad \mathfrak{so}(2l+1) \oplus \mathfrak{u}(1), \ (l > 3), \\
\mathfrak{so}(4l) \oplus 2l \mathfrak{u}(1), & \quad \mathfrak{so}(4l+2) \oplus (2l-1) \mathfrak{u}(1).
\end{align*}$$

It is possible to find a Lie algebra homomorphism $D$ between $\mathfrak{g}$ and $\mathfrak{sp}(n)$, where $4n = \dim \mathfrak{g}$, given by an inclusion of the Levi subalgebra of $\mathfrak{g}$ in $\mathfrak{sp}(n)$. Therefore, applying Corollary 3.2, we get new strong HKT structures on $T_D \mathfrak{g}$. Note that the Lie algebra $T_D \mathfrak{g}$ has a Levi-decomposition with a compact Levi subalgebra and an abelian radical. Since $D$ is the inclusion, for any Lie algebra $\mathfrak{g}$ in the list (20), there exists a connected Lie group $K$ with Lie algebra $T_D \mathfrak{g}$ such that the Levi factor of $K$ is compact. We observe that $K$ is a subgroup of the isometry group $E(4n)$ of the Euclidean $4n$-dimensional space. Therefore, the existence of lattices, which in this case are crystallographic groups, is guaranteed by Bieberbach’s theorem [7]. Moreover, if one fixes $n$, $E(4n)$ has only a finite number of lattices up to isomorphism, hence, so does $K$. In conclusion we have the following:

Theorem 5.1. Let $K$ be a connected Lie subgroup of $E(4n)$ with Lie algebra $T_D \mathfrak{g}$, where $\mathfrak{g}$ is isomorphic to one of the Lie algebras in the list (20). Then $K$ admits a left-invariant strong HKT structure which induces a strong HKT structure on any compact quotient of $K$.  


Remark 5.2. When \( g = \mathfrak{so}(7) \oplus \mathfrak{su}(1) \) or \( \mathfrak{su}(3) \), there exists no Lie algebra homomorphism \( D : \mathfrak{su}(3) \to \mathfrak{sp}(2) \) or \( D : \mathfrak{so}(7) \oplus \mathfrak{su}(1) \to \mathfrak{sp}(6) \). This follows by computing the dimension of the irreducible fundamental representations of \( SU(3) \) and \( SO(7) \). However, in both cases there exists a positive integer \( q \) and a Lie algebra homomorphism \( \rho : g \to \mathfrak{sp}(q) \). Applying Theorem 3.1, we obtain new strong HKT Lie algebras \( T_{\rho} g \).

6. HKT structures associated to Hermitian Lie algebras

Starting with a Hermitian Lie algebra \( (g, J, g) \) with a flat torsion-free connection \( D \) which is complex, i.e. such that \( DJ = 0 \), we can construct a hyper-Hermitian Lie algebra. The previous torsion-free connection is then complex-flat in the terminology of [25]. Next, we prove that the new hyper-Hermitian Lie algebra is HKT if and only if \( (g, J, g) \) is Kähler.

Consider as in Section 3 the tangent Lie algebra \( T_{D} g := g \ltimes_{D} g \) with the following Lie bracket:

\[
[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], D_{X_1} Y_2 - D_{Y_1} X_2)
\]

and the following hypercomplex structure (see [4, Corollary 4.3]):

\[
(21) \quad J_1(X_1, X_2) = (JX_1, -JX_2), \quad J_2(X_1, X_2) = (X_2, -X_1), \quad J_3 = J_1 J_2.
\]

If one considers the inner product \( \tilde{g} \) induced by \( g \) such that \( (g, 0) \) is orthogonal to \( (0, g) \), the following theorem shows that \( \{J_{a}\}, \tilde{g}\) is HKT on \( T_{D} g \) if and only if \( (J, g) \) is Kähler on \( g \).

Theorem 6.1. Let \( (g, J, g) \) be a Hermitian Lie algebra and \( D \) a flat torsion-free complex connection on \( g \). Then \( (T_{D} g, \{J_{a}\}, \tilde{g}) \) is HKT if and only if \( (g, J, g) \) is Kähler.

Proof. By [3] \( (T_{D} g, \{J_{a}\}, \tilde{g}) \) is HKT if and only if

\[
\tilde{g}([[J_1(X_1, X_2), J_1(Y_1, Y_2)], (Z_1, Z_2))] + \tilde{g}([[J_1(Y_1, Y_2), J_1(Z_1, Z_2)], (X_1, X_2)])
\]

\[
+ \tilde{g}([[J_1(Z_1, Z_2), J_1(X_1, X_2)], (Y_1, Y_2)]) = \tilde{g}([[J_3(X_1, X_2), J_3(Y_1, Y_2)], (Z_1, Z_2)])
\]

\[
+ \tilde{g}([[J_3(Y_1, Y_2), J_3(Z_1, Z_2)], (X_1, X_2)]) + \tilde{g}([[J_3(Z_1, Z_2), J_3(X_1, X_2)], (Y_1, Y_2))],
\]

for any \( X_i, Y_i, Z_i \in g, i = 1, 2 \). One has

\[
\tilde{g}([[J_1(X_1, X_2), J_1(Y_1, Y_2)], (Z_1, Z_2))] + \tilde{g}([[J_1(Y_1, Y_2), J_1(Z_1, Z_2)], (X_1, X_2))
\]

\[
+ \tilde{g}([[J_1(Z_1, Z_2), J_1(X_1, X_2)], (Y_1, Y_2)]) = g([JX_1, JY_1], Z_1) + g([JY_1, JZ_1], X_1)
\]

\[
+ g([JZ_1, JX_1], Y_1) - g(D_{JX_1} JY_2, Z_2) - g(D_{JY_1} JZ_2, X_2) - g(D_{JZ_1} JX_2, Y_2)
\]

\[
+ g(D_{JY_1} JX_2, Z_2) + g(D_{JZ_1} JY_2, X_2) + g(D_{JX_1} JZ_2, Y_2).
\]
On the other hand,
\[
\begin{align*}
\tilde{g}(\{ J_3(X_1, X_2), J_3(Y_1, Y_2), (Z_1, Z_2) \}) + \tilde{g}(\{ J_3(Y_1, Y_2), J_3(Z_1, Z_2) \}) + (X_1, X_2)) \\
+ \tilde{g}(\{ J_3(Z_1, Z_2), J_3(X_1, X_2) \}) = g(\{ JX_2, JY_2, Z_1 \}) + g(\{ JY_2, JZ_2, X_1 \}) \\
+ g(\{ JZ_2, JX_2, Y_1 \}) - g(D_{JX_2} JZ_1, Y_2) - g(D_{JY_2} JX_1, Z_2) - g(D_{JZ_2} JY_1, X_2) \\
+ g(D_{JX_2} JZ_1, X_2) + g(D_{JZ_2} JX_1, Y_2) + g(D_{JX_2} JY_1, Z_2).
\end{align*}
\] (23)

Subtracting the right-hand sides of the equations (22) and (23), and using the fact that $D$ is torsion free, we get
\[
d\omega(JX_1, JY_1, JZ_1) - d\omega(JX_2, JY_2, JZ_1) - d\omega(JX_1, JY_2, JZ_2) - d\omega(JY_1, JZ_2, JX_2),
\]
where $\omega$ is the Kähler form of $(J, g)$.

If $(\{ J_\alpha \}, \tilde{g})$ is HKT, the above expression vanishes for any $X_i, Y_i, Z_i \in g$. In particular, setting $X_2 = Y_2 = 0$, we obtain that $\omega$ is closed, that is, $(g, J, g)$ is Kähler. Conversely, if $d\omega = 0$, the hyper-Hermitian structure $(\{ J_\alpha \}, \tilde{g})$ is HKT.

\textbf{Remark 6.1.} We recall from [1], Corollary 4.3 that the Obata connection $\tilde{\nabla}^O$ of the HKT Lie algebra $(T_D g, \{ J_\alpha \}, \tilde{g})$ is given by:
\[
\tilde{\nabla}^O_{(X_1, X_2)}(Y_1, Y_2) = (D_{X_1}Y_1, D_{X_2}Y_2).
\]

and therefore is flat.

Theorem 6.1 can be applied to any of the 4-dimensional Kähler Lie algebras classified in [30]. Next, we illustrate our construction in two particularly interesting examples.

\textbf{Example 6.1.} Let $g = \mathbb{R}e_1 \oplus \mathfrak{e}(2)$ be a trivial central extension of $\mathfrak{e}(2)$, the Lie algebra of the isometry group of the Euclidean plane. We fix a basis $\{ e_2, e_3, e_4 \}$ of $\mathfrak{e}(2)$ with non-zero Lie brackets:
\[
[e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3,
\]
and an inner product $g$ on $g$ given by:
\[
g = \sum_{i=1}^{4} (e_i)^2.
\]

Let $J$ be the following complex structure on $g$:
\[
J e_1 = e_2, \quad J e_3 = e_4.
\]

It turns out that $(g, J, g)$ is Kähler. We define a flat torsion-free connection $D$ on $g$ as follows:
\[
D_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_{e_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_{e_3} = D_{e_4} = 0.
\]

Since $D$ also satisfies $DJ = 0$, Theorem 6.1 implies that $(T_D g, \{ J_\alpha \}, \tilde{g})$ is HKT.
We show next that the previous HKT structure is weak by computing the torsion 3-form $c$, which turns out to be non-closed.

The hypercomplex structure $\{J_\alpha\}$ on $T_D g$ is given by

\begin{align*}
J_1 e_1 &= e_2, & J_1 e_3 &= e_4, & J_1 e_5 &= -e_6, & J_1 e_7 &= -e_8, \\
J_2 e_1 &= -e_5, & J_2 e_2 &= -e_6, & J_2 e_3 &= -e_7, & J_2 e_4 &= -e_8,
\end{align*}

and the Kähler form of $(J_1, \tilde{g})$ is

$$\omega_1 = e^{12} + e^{34} - e^{56} - e^{78}.$$ 

Then the torsion $c$ of the HKT structure and its exterior derivative $dc$ are given by

\begin{align*}
c &= -J_1 d\omega_1 = 2e^{256}, & dc &= -4e^{1256},
\end{align*}

hence $c$ is not closed. It turns out that $c$ is co-closed, or, equivalently, the Ricci tensor of the Bismut connection is symmetric and that the metric $\tilde{g}$ is not balanced since $d(\omega_1 \wedge \omega_1 \wedge \omega_1) \neq 0$. However, the Lee form $\theta = 2e^1$ is closed, therefore $\tilde{g}$ is conformally balanced on the corresponding simply connected 8-dimensional solvable Lie group.

**Example 6.2.** It is well known that the complex hyperbolic space $SU(2,1)/S(U(2) \times U(1))$ with its standard Kähler structure admits a simply connected solvable Lie group $G$ acting simply transitively by holomorphic isometries. The 4-dimensional Lie algebra $g$ of $G$ has a basis $e_1, \ldots, e_4$ with non-zero Lie brackets:

\begin{align*}
[e_1, e_4] &= -\frac{1}{2} e_1, & [e_2, e_4] &= -\frac{1}{2} e_2, & [e_1, e_2] &= e_3, & [e_3, e_4] &= -e_3.
\end{align*}

The Kähler structure on $SU(2,1)/S(U(2) \times U(1))$ induces a complex structure $J$ and an inner product $g$ on $g$ given by:

\begin{align*}
Je_1 &= e_2, & Je_3 &= -e_4, & g &= \sum_{i=1}^{4} (e^i)^2.
\end{align*}

Therefore, $(g, J, g)$ is Kähler non-flat. Consider the flat torsion-free connection $D$ on $g$ defined by $D_{e_3} = 0$ and:

\begin{align*}
D_{e_1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, & D_{e_2} &= \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, & D_{e_4} &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}

Since $DJ = 0$, Theorem 5.1 implies that $(T_D g, \{J_\alpha\}, \tilde{g})$ is HKT, where $\{J_\alpha\}$ is defined as in (21). The Kähler form of $(J_1, \tilde{g})$ is given by:

$$\omega_1 = e^{12} - e^{34} - e^{56} + e^{78},$$

with corresponding torsion 3-form $c$:

$$c = -J_1 d\omega_1 = -\frac{1}{2} e^{268} - \frac{1}{2} e^{158} + 2 e^{378} + \frac{1}{2} e^{167} - \frac{1}{2} e^{257} - e^{356}.$$
It can be checked that $c$ is not closed (hence the HKT structure is weak) and the metric $\tilde{g}$ is not balanced. However, the Lee form $\theta = -3e^4$ is closed, hence exact on the corresponding simply connected 8-dimensional solvable Lie group $T_DG$. Therefore, the left invariant metric induced by $\tilde{g}$ on $T_DG$ is conformally balanced.

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