Random cascade models of multifractality: real-space renormalization and travelling waves

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Abstract. Random multifractals occur in particular at critical points of disordered systems. For Anderson localization transitions, Mirlin and Evers (2000 Phys. Rev. B 62 7920) have proposed the following scenario: (a) the inverse participation ratios (IPR) $Y_q(L)$ display the following fluctuations between the disordered samples of linear size $L$: with respect to the typical value $Y_q^{typ}(L) = e^{\text{ln}Y_q(L)} \sim L^{-\tau_{typ}(q)}$ that involve the typical multifractal spectrum $\tau_{typ}(q)$, the rescaled variable $y = Y_q(L)/Y_q^{typ}(L)$ is distributed with a scale-invariant distribution presenting the power-law tail $1/y^{1+\beta_q}$, so the disorder-averaged IPR $\bar{Y}_q(L) \sim L^{-\tau_{av}(q)}$ have multifractal exponents $\tau_{av}(q)$ that differ from the typical ones $\tau_{typ}(q)$ whenever $\beta_q < 1$; (b) the tail exponents $\beta_q$ and the multifractal exponents are related by the relation $\beta_q\tau_{typ}(q) = \tau_{av}(q/\beta_q)$. Here we show that this scenario can be understood by considering the real-space renormalization equations satisfied by the IPR. For the simplest multifractals described in terms of random cascades, these renormalization equations are formally similar to the recursion relations for disordered models defined on Cayley trees and they admit travelling-wave solutions for the variable $(\text{ln}Y_q)$ in the effective time $t_{eff} = \text{ln}L$: the exponent $\tau_{typ}(q)$ represents the velocity, whereas the tail exponent $\beta_q$ represents the usual exponential decay of the travelling-wave tail. In addition, we obtain that the relation in (b) above can be obtained as a self-consistency condition from the self-similarity of the multifractal spectrum at all scales. Our conclusion is thus that the Mirlin–Evers scenario should apply to random critical points of other types, and even to random multifractals occurring in other fields.
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1. **Introduction**

To explain the motivations for the present work, we first need to recall how the ideas of multifractality, on one hand, and the idea of travelling waves, on the other hand, have turned out to play a role in the field of disordered systems.

1.1. **Multifractality and critical disordered systems in finite dimensions**

Multifractality is a notion that first appeared in fluid dynamics to characterize the statistical properties of turbulence (see the book [1] and references therein). Among the various areas where the multifractal formalism has then turned out to be relevant (see for instance [2]–[8] and references therein), the case of critical points in the presence of frozen
disorder is of particular interest. The multifractal character of critical eigenfunctions for quantum Anderson localization transitions has been the subject of very detailed studies (see the reviews [9,10] and references therein). More generally, multifractality of order parameters and correlation functions is expected to be a generic property of random critical points whenever disorder is relevant: it has been found in particular in disordered classical spin models like random ferromagnets [11]–[17], spin glasses and random field spin systems [18]–[20], as well as in disordered polymer models like the directed polymers in random media [21] and disordered wetting models [22]. The only exceptions to these multifractal behaviours seem to be the ‘multiscaling’ behaviours [23], which are even stronger than multifractality, that have been found for some critical correlation functions in disordered quantum spin chains governed by ‘infinite disorder fixed points’ [24].

1.2. Travelling waves and disordered systems

Localized waves that propagate by keeping a fixed shape were first discovered in fluid dynamics in 1834 by J S Russell who wrote: ‘that singular and beautiful phenomenon which I have called the Wave of Translation’ [25]. The name ‘Wave of Translation’ has not survived, but the idea has flourished under other names. ‘Solitary waves’ or ‘solitons’ have been found in many areas of physics where non-linear equations of motion occur (see the book [26] and references therein). ‘Travelling waves’ also appear in particular in the context of front propagation into unstable states [27]: in many cases called ‘pulled fronts’, the velocity is actually determined by the exponentially small tail invading the unstable state, and can be thus determined by a linear analysis in the tail region [27,28]. It turns out that travelling waves of this type also appear in the field of disordered systems, but with of course different meanings for the space and time variables with respect to usual spatio-temporal waves. It is useful to distinguish three cases:

(i) In disordered models defined on Cayley trees, it is the probability \( P_L(A) \) of some observable \( A \) that propagates without deformation in the effective ‘time’ corresponding to the length \( L \) along the tree:

\[
  t_{\text{eff}} = L.
\]  

This property was discovered by Derrida and Spohn [29] for the specific example of the directed polymer in a random medium, where the observable \( A \) of interest is the free energy, and was then found in various other statistical physics models [30]. This travelling-wave propagation of probability distributions has also been found in quantum models defined on Cayley trees, in particular in the Anderson localization problem [31]–[35] and in some superfluid–insulator transitions [36]. The conclusion is thus that the recursion relations that can be written for observables of disordered models defined on trees naturally lead to the travelling-wave propagation of the corresponding probability distributions. This property is not limited to the discrete Cayley trees, but actually still holds for continuously branching trees [29].

(ii) For some two-dimensional disordered models, travelling waves in the effective time given by the logarithm of the system size \( L \)

\[
  t_{\text{eff}} = \ln L
\]

have been found, first for Dirac fermions in a random magnetic field by Chamon, Mudry and Wen [37,38], and then for disordered \( XY \) models by Carpentier and
Le Doussal [39], who have derived non-linear renormalization equations that admit travelling-wave solutions. This approach was then used to study other two-dimensional related models [40], as well as the problem of a particle in a logarithmically correlated random Gaussian potential in finite dimensions. For random energy models of this type with logarithmically correlated potentials, many recent developments can be found in [41]–[43].

(iii) For finite dimensional Anderson localization models exactly at criticality, where the eigenfunctions become multifractal (see the reviews [9, 10] or section 2.1 below), Evers and Mirlin [44] have proposed that the probability distribution $P_L(\ln Y_q)$ of the logarithm of the inverse participation ratios $Y_q$ (which are the order parameters of the Anderson transition) propagate as travelling waves in the effective time $t_{eff} = \ln L$ given again by the logarithm of the system size $L$. This property has been checked numerically in various Anderson localization models in various dimensions [44, 45], and has been recently obtained by a functional renormalization method in dimension $d = 2 + \epsilon$ by Foster et al [46].

In summary, travelling-wave propagation of probability distributions has been found (i) in most disordered models defined on Cayley trees, (ii) in some specific two-dimensional disordered models and for a particle in a logarithmically correlated random potential, and (iii) at Anderson localization transitions in finite dimension.

1.3. Multifractality and travelling waves

We believe that the case (iii) described above, concerning Anderson localization transitions, should actually apply to all random critical points in any finite dimension that are characterized by multifractal properties. For the directed polymer in a random medium of dimension $1 + 3$, we have indeed found numerically the presence of travelling waves at criticality [21]. More generally, we propose that besides random critical points, random multifractal measures are also generically related to the travelling-wave propagation, in the effective time $t = \ln L$, of the probability distributions of the IPR associated with the multifractal measure. Since the case of arbitrary multifractals is clearly beyond the scope of this paper, we will restrict our analysis here to the simplest multifractal measures, namely the random cascade models that have been much studied in the context of turbulence [1]. These models satisfy simple real-space renormalization equations that are formally similar to the recursion relations for disordered models defined on Cayley trees (see case (i) described above). This interpretation thus allows us to understand the presence of travelling waves at criticality in finite dimensions, as a consequence of a hierarchical real-space renormalization procedure on an appropriate tree structure in the renormalization scale $\ln L$.

The paper is organized as follows. In section 2, we recall the multifractal notation and the Evers–Mirlin scenario concerning the travelling waves that occur at Anderson localization transitions. In section 3, we write real-space renormalization equations for describing how the IPR evolve upon coarse-graining. In section 4, we show that for random cascade models, these renormalization equations for the probability distributions of the IPR admit travelling-wave solutions. In section 5, we obtain that the self-similarity of the multifractal spectrum at all scales imposes the Mirlin–Evers relation $\beta_q \tau_{typ}(q) = \tau_{av}(q \beta_q)$ that relates the tail exponents $\beta_q$ of the travelling wave to the typical and disorder-averaged multifractal spectra $(\tau_{typ}(q), \tau_{av}(q))$. Our conclusions are summarized in section 6.
2. A reminder on the travelling waves at Anderson localization transitions

2.1. A reminder on typical and averaged multifractal spectra

At Anderson localization transitions, it is convenient to associate with a critical eigenstate \( \psi_L(\vec{r}) \) defined on a volume \( L^d \) the measure
\[
\mu_L(\vec{r}) = |\psi_L(\vec{r})|^2
\]
normalized to
\[
\int_{L^d} d^d\vec{r} \mu_L(\vec{r}) = 1.\tag{4}
\]
The inverse participation ratios (IPR) of arbitrary order \( q \) are defined by
\[
Y_q(L) \equiv \int_{L^d} d^d\vec{r} \mu_L^q(\vec{r}).\tag{5}
\]
As a consequence of the normalization of equation (4), one has the identity
\[
Y_q(L) = 1.\tag{6}
\]
The localization/delocalization transition can be characterized by the asymptotic behaviour in the limit \( L \to \infty \) of the \( Y_q(L) \). In the localized phase, these moments \( Y_q(L) \) converge to finite values \( Y_q^{\text{loc}}(L = \infty) > 0 \).

In the delocalized phase, the decay of the moments follows the scaling
\[
Y_q^{\text{deloc}}(L) \propto L^{-(q-1)d}.\tag{7}
\]
Exactly at criticality, the typical decay of the \( Y_q(L) \) defines a series of generalized exponents \( \tau_{\text{typ}}(q) = (q-1)D_{\text{typ}}(q) \):
\[
Y_q^{\text{typ}}(L) \equiv e^{\ln Y_q(L)} \propto L^{-\tau_{\text{typ}}(q)} = L^{-(q-1)D_{\text{typ}}(q)}\tag{8}
\]
where the notation \( \bar{A} \) denotes the average of the observable \( A \) over the disordered samples.

The notion of multifractality corresponds to the case where \( D_{\text{typ}}(q) \) depends on \( q \), whereas monofractality corresponds to \( D_{\text{typ}}(q) = \text{cst} \) as in equation (7). The exponents \( D_{\text{typ}}(q) \) represent generalized dimensions [2]: \( D_{\text{typ}}(0) \) represents the dimension of the support of the measure—here it is simply given by the space dimension \( D_{\text{typ}}(0) = d \); \( D_{\text{typ}}(1) \) is usually called the information dimension [2], because it describes the behaviour of the ‘information’ entropy
\[
s_L \equiv -\sum_{\vec{r}} \mu_L(\vec{r}) \ln \mu_L(\vec{r}) = -\partial_q Y_q(L)|_{q=1} \propto D_{\text{typ}}(1) \ln L.\tag{9}
\]
Finally \( D_{\text{typ}}(2) \) is called the correlation dimension [2] and describes the decay of
\[
Y_2^{\text{typ}}(L) \equiv e^{\ln Y_2(L)} \propto L^{-D_{\text{typ}}(2)}.\tag{10}
\]
In the multifractal formalism, one also introduces the singularity spectrum \( f_{\text{typ}}(\alpha) \) defined as follows: in a sample of size \( L^d \), the number \( N_L(\alpha) \) of points \( \vec{r} \) where the weight \( |\psi_L(\vec{r})|^2 \) scales as \( L^{-\alpha} \) behaves typically as
\[
N_L^{\text{typ}}(\alpha) \propto L^{f_{\text{typ}}(\alpha)}.	ag{11}
\]
The saddle-point calculus in $\alpha$ of the IPR
\[ Y_q^{\text{typ}}(L) \simeq \int d\alpha L^{f_{\text{typ}}(\alpha)} L^{-q\alpha} \] (12)
yields the usual Legendre transform formula:
\[ -\tau_{\text{typ}}(q) = \max_{\alpha} [f_{\text{typ}}(\alpha) - q\alpha]. \] (13)

Following [2], many authors consider that the singularity spectrum has a meaning only for $f_{\text{typ}}(\alpha) \geq 0$ [47]–[50]. However, when multifractality arises in random systems, disorder-averaged values may involve generalized exponents [51]–[54] other than the typical values (see equation (8)), and it is thus useful to introduce another series of generalized exponents $\tau_{\text{av}}(q) = (q-1)D_{\text{av}}(q)$ [44]:
\[ Y_q(L) \propto L^{-\tau_{\text{av}}(q)} = L^{-(q-1)D_{\text{av}}(q)}. \] (14)

For these disorder-averaged values, the corresponding singularity spectrum $f_{\text{av}}(\alpha)$ may become negative, $f_{\text{av}}(\alpha) < 0$ [44], [51]–[55], for describing rare events (cf equation (11)). The difference between the two generalized exponent sets $D_{\text{typ}}(q)$ and $D_{\text{av}}(q)$ associated with typical and averaged values has as its origin the broad distribution of the IPR over the samples [44,45], as we now describe.

2.2. Probability distributions of the IPR $Y_q(L)$ over the samples

The scenario proposed by Evers and Mirlin [44,45] in the context of quantum localization models is as follows: the probability distribution of the logarithm of the inverse participation ratios of equation (5) becomes scale invariant around its typical value [44,45], i.e.,
\[ \ln Y_q(L) = \ln Y_q(L) + u_q = \ln Y_q^{\text{typ}}(L) + u_q \] (15)
where $u_q$ remains a random variable of order $O(1)$ in the limit $L \to \infty$. According to [44] the probability distribution $G_L(u_q)$ generically develops an exponential tail
\[ G_L(u_q) \propto e^{-\beta_q u_q}. \] (16)

As a consequence, the ratio $y_q = Y_q(L)/Y_q^{\text{typ}}(L) = e^{u_q}$ with respect to the typical value $Y_q^{\text{typ}}(L) = e^{\ln Y_q(L)}$ presents the power-law decay
\[ \Pi \left( y_q \equiv \frac{Y_q(L)}{Y_q^{\text{typ}}(L)} \right) \propto \frac{1}{y_q^{1+\beta_q}}. \] (17)

The conclusions of [44,45] are then as follows:

(i) The typical singularity spectrum has meaning only for $f_{\text{typ}}(\alpha) \geq 0$, i.e. it usually exists only on a finite interval $[\alpha_+, \alpha_-]$, where the termination points $\alpha_{\pm}$ satisfy $f_{\text{typ}}(\alpha_-) = f_{\text{typ}}(\alpha_+)$. Denoting by $q_{\pm}$ the corresponding values of $q$ under the Legendre transformation of equation (13), one obtains the linear behaviours outside the interval $[q_-, q_+]$:
\[ \tau_{\text{typ}}(q) = q\alpha_- \quad \text{for } q < q_- \] (18)
\[ \tau_{\text{typ}}(q) = q\alpha_+ \quad \text{for } q > q_. \] (19)
(ii) The disorder-averaged singularity spectrum $f_{av}(\alpha)$ has a meaning outside this interval where $f_{av}(\alpha) < 0$ and describes the probabilities of rare events. In the region where both exist, they are expected to coincide:

$$f_{av}(\alpha) = f_{typ}(\alpha) \quad \text{if } \alpha_+ \leq \alpha \leq \alpha_-.$$  

Equivalently, the disorder-averaged exponents $\tau_{av}(q)$ are expected to coincide with the typical exponents $\tau_{typ}(q)$ on the interval $[q_-, q_+]$:

$$\tau_{av}(q) = \tau_{typ}(q) \quad \text{if } q_- \leq q \leq q_+$$  

whereas $\tau_{av}(q)$ will be different from equation (19) outside the interval $[q_-, q_+]$.

(iii) From the point of view of the tail exponents $\beta_q$ of equation (17), this means that

$$\beta_q > 1 \quad \text{if } q_- < q < q_+$$  

$$\beta_q < 1 \quad \text{if } q < q_- \text{ or } q > q_+$$  

$$\beta_{q_\pm} = 1.$$  

(iv) Finally, Mirlin and Evers [44] have derived for special cases the following relation:

$$\beta_q \tau_{typ}(q) = \tau_{av}(q/\beta_q)$$

and they have conjectured its generic validity [10,44].

This scenario has been tested numerically for various Anderson transitions (see the review [10] and the more recent work [55]). We have found previously that this scenario also applies to the transition of the directed polymer in dimension $1 + 3$ [21]. In the following, we justify this scenario via a real-space renormalization analysis.

3. Real-space renormalization analysis

In critical phenomena, it is well known that critical properties are stable under coarse-graining. This explains their universal character (independence with respect to microscopic details) and why renormalization is an appropriate framework. Similarly, for random critical points, the multifractal spectrum is expected to be stable under coarse-graining [9]. It is thus natural to consider how the inverse participation ratios (IPR) evolve upon coarse-graining.

3.1. Evolution of the IPR upon coarse-graining

To go from the microscopic scale $l = 1$ to the macroscopic scale $l = L$ of the whole system, it is convenient to introduce intermediate scales $l_m$ regularly placed on a logarithmic scale as follows. For definiteness, we consider a discrete system with $L = b^M$ and introduce the intermediate scales

$$l_m = b^m \quad \text{with } m = 0, 1, \ldots, M.$$  

Then the whole volume $L^d = l_M^d$ can be decomposed into

$$K = l_d^d$$

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subvolumes of sizes $l^d_{M-1}$, and the process can be recursively iterated: each volume of size $l^d_m = Kl^d_m$ is made of $K$ volumes of sizes $l^d_m$, denoted here by an index $i = 1, 2, \ldots K$. We consider a (non-normalized) positive field $\mu(\vec{r})$, and associate with each volume $(i)$ of size $l^d_m$ the integrals of $\mu^q(\vec{r})$ over the corresponding volume:

$$Z_q^{(i)}(m) \equiv \int_{l^d_m} d^d r \mu^q(\vec{r}).$$

(28)

The corresponding IPR are then defined by the ratios

$$Y_q^{(i)}(m) \equiv \frac{Z_q^{(i)}(m)}{[Z_1^{(i)}(m)]^q}.$$  

(29)

Upon coarse-graining, the integrals $Z_q$ are simply additive for any $q$:

$$Z_q(m+1) = \sum_{i=1}^K Z_q^{(i)}(m)$$

(30)

whereas the IPR satisfy the recursions

$$Y_q(m+1) = \sum_{i=1}^K [w_i(m)]^q Y_q^{(i)}(m)$$

(31)

where the coefficients are given by the weights

$$w_i(m) \equiv \frac{Z_q^{(i)}(m)}{\sum_{j=1}^K Z_q^{(j)}(m)}$$

(32)

that represent the ratios of the normalization in the volume $(i)$ of size $l^d_m$ with respect to the normalization of the volume of size $l^d_{m+1}$. By construction one has the following constraint:

$$\sum_{i=1}^K w_i(m) = 1.$$  

(33)

For instance for Anderson localization models where the additive positive field $\mu(\vec{r})$ is given by equation (3), one expects that at sufficiently large scale:

(i) in the delocalized phase, the $K$ weights $w_i$ all converge towards the same value $1/K$, so the system becomes asymptotically homogeneous at sufficiently large scales;

(ii) in the localized phase, one single weight converge to 1, whereas all $(K-1)$ other weights converge to zero;

(iii) exactly at criticality, the $K$ weights remain finite in contrast to case (ii), but they remain distributed with a non-trivial distribution, in contrast to case (i).

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3.2. The notion of random cascade models

Since the general case where the weights are correlated among generations is more difficult to analyse, we will consider from now on the much simpler case where the weights of different generations are uncorrelated, and where the non-negative weights \( w_1, \ldots, w_K \) corresponding to an elementary coarse-graining step in equation (31) are drawn with some fixed probability distribution \( Q^*_b(w_1, \ldots, w_K) \), independent of the generation \( m \), symmetric in its \( K \) arguments, and satisfying only the normalization constraint of equation (33):

\[
Q^*_b(w_1, \ldots, w_K) = R^*_b(w_1, \ldots, w_K) \delta \left( \sum_{i=1}^{K} w_i(m) - 1 \right).
\] (34)

This type of random cascade model has been much studied in the context of turbulence for describing the spatial distribution of energy dissipation (see the book [1] and references therein). In this context, the weights \( w_i \) are usually called ‘cascade generators’ or ‘multipliers’ or ‘breakdown coefficients’. Various forms have been proposed over the years, in particular the log-normal [56], bimodal [57], log-stable [58,59], log-Poisson [60]–[62], and log-infinitely divisible ones [63]: see [64] for a comparative test of these various cascade generators.

However besides the specific form of the distribution of these weights, the important hypothesis is of course the independence of the weights of different generations. For turbulence, the validity of this hypothesis is discussed in [65,66]. For random critical points, this hypothesis is not expected to be valid, but one expects instead some Markovian structure [67]. Nevertheless, since random cascade models are clearly the simplest multifractal models, it is important to understand in detail their properties. In the following, we show that travelling waves appear very naturally in random cascade models, and that their properties agree with the Mirlin–Evers scenario concerning Anderson transitions.

4. Travelling-wave analysis for random cascade models

In the present section, we analyse the real-space renormalization equation (31) for the case of random cascade models characterized by some fixed distribution \( Q^*_b(w_1, \ldots, w_K) \) (see equation (34)). The renormalization equation (31) is then analogous to the recursion equations for disordered models defined on a Cayley tree of branching number \( K \), and it is thus natural to obtain travelling-wave propagation of probability distribution as for the directed polymer model [29] mentioned in the introduction. The fact that renormalization of multifractals in finite dimensions involves a hierarchical structure analogous to the Cayley tree has already been stressed in various contexts [37]–[40], [68].

4.1. The travelling-wave ansatz for the inverse participation ratios (IPR)

We look for solutions of the real-space renormalization equation (31) with the following ansatz:

\[
Y_q(m) = e^{-\nu_q m} y_q
\] (35)

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where \( v_q \) is a constant depending on \( q \), and where \( y_q \) is a random variable whose distribution does not depend upon the generation \( m \). On a logarithmic scale, this corresponds to the travelling-wave form

\[
\ln Y_q(m) = -v_q m + \ln y_q
\]

where \( v_q \) represents the velocity with respect to the variable \( m = (\ln l_m)/(\ln b) \). The velocity \( v_q \) is thus directly related to the typical exponent \( \tau_{\text{typ}}(q) \) of equation (8):

\[
v_q = (\ln b) \tau_{\text{typ}}(q).
\]  

With the ansatz of equation (35), the renormalization equation (31) becomes

\[
y_q = e^{v_q} \sum_{i=1}^{K} [w_i]^q y_q^{(i)}
\]  

for the reduced random variables \( y_q \). The probability distribution \( P_q^*(y_q) \) should be stable by this iteration:

\[
P_q^*(y_q) = \int dw_1 \cdots dw_K Q_q^*(w_1, \ldots, w_K) \int dy_q^{(1)} P_q^*(y_q^{(1)}) \cdots \int dy_q^{(K)} P_q^*(y_q^{(K)}) \delta
\]

\[
\times \left( y_q - e^{v_q} \sum_{i=1}^{K} [w_i]^q y_q^{(i)} \right).
\]  

\[\text{(39)}\]

### 4.2. Tail analysis

A generic property of multiplicative stochastic processes is that they lead to probability distributions presenting power-law tails [69]–[74]. Here for the specific case of random multiplicative cascade models, one expects also that the stable distribution \( P_q^*(y_q) \) solution of equation (39) will present a power-law tail in \( y_q \):

\[
P_q^*(y_q) \sim \frac{A}{y_q^{1+\beta_q}}
\]

\[\text{(40)}\]

where the exponent \( \beta_q \) is not fixed for the moment, and satisfies only the condition \( \beta_q > 0 \) for having a normalizable probability distribution. In the travelling-wave language of equation (36), this power-law tail is equivalent to the exponential tail \( e^{-\delta_q u_q} \) for the variable \( u_q = \ln Y_q(m) + m v_q = \ln y_q \). So this corresponds exactly to the standard exponential tail analysis of travelling fronts [27, 28].

The stability of the power-law tail of equation (40) in the region \( y_q \to +\infty \) means that at leading order, only one of the \( K \) variables \( y_q^{(i)} \) in equation (39) becomes large with a probability also given by the tail of equation (40): after the introduction of a factor \( K \), for choosing one of the \( K \) variables \( y_q^{(i)} \), we may assume the choice \( i = 1 \) leading to

\[
\frac{A}{y_q^{1+\beta_q}} \simeq K \int dw_1 \cdots dw_K Q_b^*(w_1, \ldots, w_K) \int dy_q^{(1)} P_q^*(y_q^{(1)}) \cdots \int dy_q^{(K)} P_q^*(y_q^{(K)})
\]

\[
\times \int dy_q^{(1)} \frac{A}{y_q^{(1)+\beta_q}} \delta \left( y_q - e^{v_q} [w_1]^q y_q^{(1)} \right)
\]

\[
\simeq \frac{A}{y_q^{1+\beta_q}} K \int dw_1 \cdots dw_K Q_b^*(w_1, \ldots, w_K) w_1^{q\beta_q} e^{v_q \beta_q}.
\]  

\[\text{(41)}\]
One thus obtains the following compatibility equation:

\[ 1 = Ke^{v_q \beta} w_i^{q \beta} \]  

(42)

in terms of the partial moment of the joint probability distribution \( Q_i^*(w_1, \ldots, w_K) \) of equation (34):

\[ \overline{w_i^p} = \int dw_1 dw_2 \cdots dw_K Q_i^*(w_1, \ldots, w_K)w_i^p. \]  

(43)

Equation (42) means that each mode \( \beta \) is characterized by the velocity \( v_q(\beta) \) given by

\[ v_q(\beta) = -\frac{1}{\beta} \ln \left( Kw_i^{q \beta} \right). \]  

(44)

### 4.3. Selection of the tail exponent \( \beta_q \) and of the velocity \( v_q \) of the travelling wave

In the field of travelling waves, the selection of the tail exponent \( \beta \) of equation (40) and that of the corresponding velocity \( v(\beta) \) of equation (36) usually depend on the form of the initial condition [27]–[29]. In our present case, the initial condition is completely localized, \( Y_q(m = 0) = 1 \), at the lowest scale \( l_{m=0} = 1 \). One then expects the solution that will be dynamically selected [27]–[29] to correspond to the tail exponent \( \beta_q^{\text{selec}} \) and to the velocity \( v_q^{\text{selec}} = v_q(\beta_q^{\text{selec}}) \) determined by the following extremization criterion:

\[ 0 = \left[ \partial_\beta v_q(\beta) \right]_{\beta = \beta_q^{\text{selec}}} = \left[ \frac{1}{\beta^2} \ln \left( Kw_i^{q \beta} \right) - \frac{1}{\beta^2} \partial_\beta \ln \left( w_i^{q \beta} \right) \right]_{\beta = \beta_q^{\text{selec}}}. \]  

(45)

In summary, for any given distribution \( Q_i^*(w_1, \ldots, w_K) \) that defines a random cascade model (see equation (34)), the properties of the travelling waves can be obtained as follows: one has to compute the partial moment of equation (43), and to solve equation (45) in order to determine the tail exponent \( \beta_q \) and the velocity \( v_q \) that are dynamically selected. The typical multifractal exponents \( \tau_{\text{typ}}(q) \) are then obtained from the selected velocities by means of equation (37):

\[ \tau_{\text{typ}}(q) = \frac{v_q^{\text{selec}}}{\ln b}. \]  

(46)

### 5. Relations between the tail exponents and the multifractal exponents

As explained in the previous section, the selection criterion of equation (45) is usually the final outcome of a travelling-wave analysis. However here in our real-space renormalization framework, we still have some freedom in the choice of the rescaling factor \( b \) introduced in equation (26). Of course, the final multifractal exponents \( \tau_{\text{typ}}(q), \tau_{\text{av}}(q) \) should not depend on the choice of the coarse-graining scale \( b \). In the present section, we use this freedom to consider the case of large \( b \), and we show that some self-consistency conditions arise.

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5.1. Self-consistency conditions when the rescaling factor $b$ is large

When the rescaling factor $b$ introduced in equation (26) is large, the number $K = b^d$ of buildings blocks (see equation (27)) of a single renormalization step also becomes large. Then from the self-similarity of the multifractal spectrum at all scales [9], one obtains that the elementary weights $w_i$ should themselves follow the multifractal statistics, i.e. their associated IPR

$$ \mathcal{Y}_p(b) \equiv \sum_{i=1}^{K=b^d} w_i^p $$

should have for typical scalings

$$ \mathcal{Y}_p^{\text{typ}}(b) \propto b^{-\tau_{\text{typ}}(p)} $$

and for averaged scalings

$$ \overline{\mathcal{Y}}_p(b) = K \mathcal{w}_i^p \propto b^{-\tau_{\text{av}}(p)} $$

in terms of the typical and averaged multifractal exponents $\tau_{\text{typ}}(p)$ and $\tau_{\text{av}}(p)$ introduced in equations (8) and (14). The velocity of equation (44) then reads at leading order for large $b$

$$ v_q(\beta) = -\frac{1}{\beta} \ln \left( b^{-\tau_{\text{av}}(p=q\beta)} \right) = \frac{\tau_{\text{av}}(p=q\beta)}{\beta} \ln b $$

Taking into account equation (46), we thus obtain the following consistency relation:

$$ \tau_{\text{typ}}(q) = \frac{v_{q(\beta)}^{\text{selec}}}{\ln b} = \frac{\tau_{\text{av}}(p=q\beta_{q(\beta)}^{\text{selec}})}{\beta_{q(\beta)}^{\text{selec}}} $$

that relates the typical exponent $\tau_{\text{typ}}(q)$ to the selected tail exponent $\beta_{q(\beta)}^{\text{selec}}$ and to the averaged multifractal exponent $\tau_{\text{av}}(p=q\beta_{q(\beta)}^{\text{selec}})$. As recalled around equation (25), the relation of equation (51) has already been derived for special cases by Mirlin and Evers [44] and has been conjectured to be general [10]. Our present derivation from a self-consistency condition of the travelling-wave analysis thus favours the general validity of this formula.

The relations of equation (51) introduce non-trivial constraints on the multifractal exponents, which are not always compatible with the usual selection criterion of equation (45) based on the extremization of the velocity. Let us first describe an explicit case before returning to the general case.

5.2. An example of Gaussian multifractality

The simplest multifractal spectrum corresponds to the following Gaussian form for the disorder-averaged multifractal spectrum:

$$ \tau_{\text{av}}^{\text{Gauss}}(q) = d(q-1) \left( 1 - \frac{q}{q_c^{2}} \right) $$

In particular, this Gaussian forms appears in various models in the weak multifractality regime [10], in particular in perturbation theory for $d = 2 + \epsilon$ [46].
5.2.1. Consequences of the self-consistency equation (51). In the region where $\tau_{\text{typ}}^{Gauss}(q) = \tau_{\text{av}}^{Gauss}(q)$ corresponding to $\beta_q > 1$, equation (51) yields [10]

$$\beta_q = \frac{q_c^2}{q^2}. \quad (53)$$

This solution is consistent for $\beta_q > 1$, i.e. in the interval

$$-q_c < q < +q_c. \quad (54)$$

In the region where $\tau_{\text{typ}}^{Gauss}(q) \neq \tau_{\text{av}}^{Gauss}(q)$ corresponding to $\beta_q < 1$, the typical exponents vary linearly with $q$ [10, 46]:

$$\tau_{\text{typ}}^{Gauss}(q) = qd \left( 1 - \frac{\text{sgn}(q)}{q_c} \right)^2 \quad \text{for } |q| > q_c. \quad (55)$$

Equation (51) then yields [10, 46]

$$\beta_q = \frac{q_c}{|q|}. \quad (56)$$

5.2.2. Analysis via the selection criterion of equation (51). From equation (52), we obtain the velocity as a function of the tail exponent $\beta$ using equation (50):

$$v_q(\beta) = \frac{\tau_{\text{av}}(p = q\beta)}{\beta} \ln b = \frac{\ln b}{\beta} \ln(\beta q - 1) \left( 1 - \frac{\beta q}{q_c^2} \right). \quad (57)$$

Its derivative

$$\partial_\beta v_q(\beta) = d(\ln b) \left[ \frac{1}{\beta^2} - \frac{q^2}{q_c^2} \right] \quad (58)$$

yields the following solution, $\beta_{q_{\text{selec}}} > 0$, for the selection criterion of equation (45):

$$\beta_{q_{\text{selec}}} = \frac{q_c}{|q|} \quad (59)$$

and the corresponding selected velocity reads

$$v_{q_{\text{selec}}} = v_q(\beta_{q_{\text{selec}}}) = (\ln b) dq \left( 1 - \frac{\text{sgn}(q)}{q_c} \right)^2. \quad (60)$$

The typical multifractal exponent reads (equation (46))

$$\tau_{\text{typ}}(q) = \frac{v_{q_{\text{selec}}}}{\ln b} = dq \left( 1 - \frac{\text{sgn}(q)}{q_c} \right)^2, \quad (61)$$

i.e. one recovers the correct result only in the linear regions of the typical spectrum, where it is different from the disorder-averaged spectrum. This seems to indicate that in the region where $\tau_{\text{typ}}(q) = \tau_{\text{av}}(q)$, this additional constraint $\tau_{\text{typ}}(q) = \tau_{\text{av}}(q)$ completely fixes the tail exponent and the velocity at values that do not satisfy the usual selection criterion of equation (51). This phenomenon seems generic even beyond the Gaussian case, as we now explain.

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5.3. The general case: competition with the usual velocity selection

For the general case, the selection condition of equation (45) for $\beta_{q}^{\text{selec}}$ becomes

$$0 = [q \tau'_{\text{av}}(q\beta) - \tau_{\text{av}}(q\beta)]_{\beta_{q}^{\text{selec}}} \quad (62)$$

In terms of the solution $p^*$ of the following equation:

$$0 = [p \tau'_{\text{av}}(p) - \tau_{\text{av}}(p)]_{p=p^*} \quad (63)$$

the selected tail exponent reads

$$\beta_{q}^{\text{selec}} = \left| \frac{p^*}{q} \right| \quad (64)$$

and the corresponding velocity is given by

$$v_{q}^{\text{selec}} = v_{q}(\beta_{q}^{\text{selec}}) = \frac{\tau_{\text{av}}(q\beta_{q}^{\text{selec}})}{\beta_{q}^{\text{selec}}} \ln b. \quad (65)$$

Equation (37) yields the following typical exponent:

$$\tau_{\text{typ}}(q) = \frac{v_{q}^{\text{selec}}}{\ln b} = \frac{\tau_{\text{av}}(q\beta_{q}^{\text{selec}})}{\beta_{q}^{\text{selec}}} = q \frac{\tau_{\text{av}}(|p^*| \text{sgn}(q))}{|p^*| \text{sgn}(q)} \quad (66)$$

which is linear in $q$.

In summary, as in the Gaussian case, we obtain that:

(i) the usual selection criterion of equation (45) is compatible with the self-consistency equation (51) only in the region where the typical and averaged spectra differ: $\tau_{\text{typ}}(q) \neq \tau_{\text{av}}(q)$ where $\beta_q < 1$;

(ii) in the region where $\tau_{\text{typ}}(q) = \tau_{\text{av}}(q)$ and $\beta_q > 1$, the self-consistency equation (51) indicates that the selected tail exponent is the solution $\beta_{q}^{\text{selec}} > 1$ of

$$\beta_{q}^{\text{selec}} \tau_{\text{av}}(q) = \tau_{\text{av}}(q\beta_{q}^{\text{selec}}) \quad (67)$$

(the other trivial solution of this equation being $\beta = 1$), and does not correspond to the usual selection criterion of equation (45).

6. Conclusions and perspectives

In this paper, we have proposed that the Mirlin–Evers scenario concerning the probability distributions of inverse participation ratios (IPR) at Anderson transitions can be better understood by considering the real-space renormalization equations satisfied by the IPR upon coarse-graining. For the simplest multifractal models, namely the random cascade models, we have shown that these renormalization equations are formally similar to the recursion equations for disordered models defined on Cayley trees, and that, as a consequence, they admit travelling-wave solutions of the type known as ‘pulled fronts’, where the velocity is actually determined by a linear analysis in the exponentially small tail region. Finally, we have obtained that the self-similarity of the multifractal spectrum at all scales imposes the Mirlin–Evers relation of equation (25) that relates the tail exponents $\beta_q$ of the travelling waves to the typical and disorder-averaged multifractal
spectra \((\tau_{\text{typ}}(q), \tau_{\text{av}}(q))\). This shows that random cascade models already capture many properties that have been previously found at Anderson transitions. It would be thus interesting in the future to show that the travelling-wave solutions of the renormalization equations persist beyond random cascade models, since at random critical points, one expects the strict statistical independence of the weights of different generations not to be valid, but one expects instead some Markovian structure \([67]\). Further work is needed to formulate correctly appropriate models of Markovian cascades.

Nevertheless, since random cascade models are clearly the simplest multifractal models, and since their properties are very similar to the properties found previously for the more complex case of Anderson transitions, we believe that our present real-space renormalization analysis is in favour of a wide validity of the Mirlin–Evers scenario beyond Anderson transitions:

(i) The first generalization concerns all random critical points in finite dimension, where the local order parameter and the correlation functions generically display multifractal statistics (see the discussion in section 1.1). For instance, we have found numerically the presence of travelling waves at criticality for the directed polymer in a random medium of dimension 1 + 3 \([21]\). For other many-body random phase transitions like classical disordered spin models, these travelling-wave properties should also appear after an appropriate translation (see \([67]\) for such a translation between multifractality at Anderson transitions and in classical disordered spin models). It seems that for two-dimensional disordered Potts models where multifractality has been much studied \([11]–[17], [67]\) only the disorder-averaged multifractal spectrum has been considered up to now. It would thus be very interesting to study numerically the probability distribution over the disordered samples of the appropriate observables in order to test the travelling-wave scenario and to measure the corresponding tail exponents \(\beta_q\). Some probability distributions over the disordered samples of extensive observables like the susceptibility have already been measured for some random critical points in \([75]–[78]\), but a quantitative analysis of the tails remains to be done.

(ii) Besides phase transitions in disordered systems, a further generalization concerns other areas of physics where multifractality occurs. We believe that the present real-space renormalization analysis on the multifractal measure retains its validity, provided the notion of fluctuations between different realizations of the multifractal cascade has a physical meaning. Indeed, in disordered systems, each disordered sample is characterized by a given realization of the multifractal cascade, so the fluctuations between cascade realizations correspond to the sample-to-sample fluctuations, which play a major role in the understanding of disordered systems. In other fields where multifractality occurs, one should first clarify the physical meaning of a given realization of the cascade to see whether it is of interest to distinguish between typical exponents and disorder-averaged exponents, and to introduce probability distributions over cascade realizations.

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