LAGRANGIAN CONSTANT CYCLE SUBVARIETIES IN LAGRANGIAN FIBRATIONS

by

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Résumé. — We show that the image of a dominant meromorphic map from an irreducible compact Calabi-Yau manifold in a wider sense $X$ whose general fiber is of dimension strictly between 0 and $\dim X$ is rationally connected. Using this result, we construct for any hyper-Kähler manifold $X$ admitting a Lagrangian fibration a Lagrangian constant cycle subvariety $\Sigma_H$ in $X$ for every divisor class $H$ whose restriction to some smooth Lagrangian fiber is ample. We also show that up to a scalar multiple, the class of a zero-cycle supported on $\Sigma_H$ in $\text{CH}_0(X)$ does not depend neither on $H$ nor on the Lagrangian fibration (provided $b_2(X) \geq 8$).

1 Introduction

This note is devoted to the construction of some subvarieties $Y$ in a projective hyper-Kähler manifold $X$ admitting a Lagrangian fibration such that every point in $Y$ is rationally equivalent to each other in $X$. These subvarieties, called constant cycle subvarieties in [12], depend on the choices of a Lagrangian fibration and a divisor class $H \in \text{Pic}(X)$ whose restriction to some smooth Lagrangian fiber is ample, thus the image of the Gysin map $\text{CH}_0(Y) \to \text{CH}_0(X)$ depends a priori on these choices as well. The result of this note is motivated by the conjectural picture on the splitting property of the conjectural Bloch-Beilinson filtration of projective hyper-Kähler manifolds due to Beauville [1] and Voisin [26] as we will explain below.

Note that the study of constant cycle subvarieties in hyper-Kähler manifolds was initiated by Huybrechts in the case of K3 surfaces [12]. Our motivation for studying these subvarieties comes rather from the attempt to generalize the Beauville-Voisin canonical zero-cycle of a projective K3 surface [2] to higher dimensional cases. For a K3 surface $S$, recall that there are at least two ways to characterize the canonical zero-cycle $o_S$:

i) $o_S$ is the degree-one generator of the image of the intersection product [2]

$\sim : \text{CH}^1(S) \otimes \text{CH}^1(S) \to \text{CH}^2(S)$;

ii) $o_S$ is the class of any point supported on a constant cycle curve in $S$ [24]. More generally, for any $n$-dimensional subvariety of $S^{[n]}$ parameterizing a family of zero-cycles of constant class $z \in \text{CH}_0(S)$, $z$ is proportional to $o_S$.

Each characterization gives a priori different generalization of $o_S$. The first one is related to Beauville’s conjecture on the weak splitting property of the Chow ring of projective hyper-Kähler manifolds [1]:

\[ \ldots \]
Conjecture 1.1 (Beauville [1]). — Let $X$ be a projective hyper-Kähler manifold. The restriction of the cycle class map $\text{CH}^*(X)_Q := \text{CH}^*(X) \otimes_{\mathbb{Z}} Q \to H^*(X, Q)$ to the $Q$-sub-algebra generated by divisor classes is injective.

The reader is referred to [1, 23, 10, 17, 26] for recent developments of this conjecture. In particular, since $H^{4n}(X, Q) = Q$, Beauville’s conjecture contains as a sub-conjecture the fact that the intersection of any $2n$ divisor classes in $\text{CH}^*(X)_Q$ is proportional to the same degree one zero-cycle $o_X \in \text{CH}^{2n}(X)_Q$ where $2n$ is the dimension of $X$, which generalizes property i) of $o_S$.

The generalization of property ii) is formulated in [26 Conjectures 0.4 and 0.8]: for $0 \leq i \leq n$, let $S_i \text{CH}_0(X)$ denote the subgroup of $\text{CH}_0(X)$ generated by the classes of points whose rational orbit is of dimension $\geq i$. One hopes that this decreasing rational orbit filtration $S_i \text{CH}_0(X)$ would define a splitting of the conjectural Bloch-Beilinson filtration $F_{BB}^*$ in the sense that the inclusion $S_i \text{CH}_0(X) \hookrightarrow \text{CH}_0(X)$ induces an isomorphism

$$S_i \text{CH}_0(X) \xrightarrow{\sim} \text{CH}_0(X)/F_{BB}^{2n-2i+1} \text{CH}_0(X).$$

Using the axioms of the Bloch-Beilinson conjecture, the surjectivity of the above map is proved in [26] to be a consequence of the following conjecture:

Conjecture 1.2 ([26]). — Let $X$ be a projective hyper-Kähler manifold of dimension $2n$. The dimension of the set of points in $X$ whose rational orbit has dimension $\geq i$ is $2n - i$.

We refer to [26] for more details on Voisin’s circle of ideas for studying the splitting property of the Bloch-Beilinson filtration on $\text{CH}_0(X)$. When $i = n$, Conjecture [12] is equivalent to the existence of constant cycle subvarieties of $X$ of dimension $n$ (which are necessarily Lagrangian, by Roitman-Mumford’s theorem) and one would expect to recover the conjectural canonical zero-cycle for any projective hyper-Kähler manifolds $X$ by taking the class of a point in any of these constant cycle Lagrangian subvarieties, hence the second generalization of $o_S$.

In general it is difficult to construct Lagrangian constant cycle subvarieties. However, if $X$ admits a Lagrangian fibration $\pi : X \to B$, we prove in Section 3 the following theorem, which proves in particular Conjecture [12] in the case $i = n$ for every Lagrangian fibration.

Theorem 1.3. — Let $X$ be a projective hyper-Kähler manifold admitting a Lagrangian fibration $\pi : X \to B$. For each divisor class $H \in \text{Pic}(X)$ whose restriction to some smooth Lagrangian fiber is ample, there exists a Lagrangian constant cycle subvariety $\Sigma_{n, H} \subset X$ all of whose points are rationally equivalent to a multiple of $H^n : [F]$ in $\text{CH}_0(X)$.

The fact that $[F] \in \text{CH}^0(X)$ is independent of $F$ is a direct consequence of the following general result.

Theorem 1.4. — Let $X$ be a Calabi-Yau manifold and $f : X \to B$ a dominant meromorphic map over a Kähler base $B$. If $0 < \dim B < \dim X$, then $B$ is rationally connected.

Here a Calabi-Yau manifold is an irreducible (in the sense of Riemannian geometry) compact Kähler manifold with finite dimensional group and trivial canonical bundle. The Riemannian holonomy group of a Calabi-Yau manifold associated to its Kähler metric is either $\text{SU}(n)$ or $\text{Sp}(n)$. Hyper-Kähler manifolds and Calabi-Yau manifolds in the strict sense are examples of Calabi-Yau manifolds. In the case where $X$ is a projective hyper-Kähler manifold, Theorem 1.4 is to compare with Matsushita’s result [14 Theorem 2], saying that if $X \to B$ is a surjective morphism over a normal base $B$ such that $0 < \dim B < \dim X$, 1. Precisely, let $z \in X$ and $O_z$ be the set of points in $X$ which are rationally equivalent to $z$, $O_z$ is called the rational orbit of $z$ and is a countable union of Zariski closed subset of $X$; we define $\dim O_z$ to be the supremum of the dimension of all irreducible components of $O_z$.
then \( B \) is a \( \mathbb{Q} \)-factorial klt Fano variety of dimension \( \frac{1}{2} \dim X \) with Picard number 1. Note also that in the case where \( X \) is projective and \( f : X \to B \) is a surjective morphism over a normal \( \mathbb{Q} \)-Gorenstein variety \( B \) without the assumption that \( \dim B < \dim X \), either \( K_B \) is numerically trivial or \( B \) is uniruled \cite[Corollary 2]{13}. Theorem 1.4 also improves and gives a new proof of the main result of \cite[Corollary 1.3]{13}. This result will allow to rephrase Theorem 1.3 replacing the fiber \( F \) by the cycle \( L^n \in \text{CH}^n(X) \). Furthermore, under the mild assumption that a very general projective deformation of the Lagrangian fibration \( \pi : X \to B \) with \( p(X) \geq 3 \) satisfies Matsushita’s conjecture \cite{23} (for instance when \( b_2(X) \geq 8 \) \cite{20}), Theorem 1.4 allows us to define for such a variety \( X \) a canonical zero-cycle \( \alpha_X \in \text{CH}_0(X) \) by taking the class of a point supported on any Lagrangian constant cycle subvariety \( \Sigma_{n, \Omega} \) defined above, whose class is also proportional to a product of \( 2n \) divisors:

**Theorem 1.5.** — i) Let \( X \) be a projective hyper-Kähler manifold admitting a Lagrangian fibration of dimension \( 2n \) given by a line bundle \( L \) with \( q(L) = 0 \). Let \( H \) be any divisor class on \( X \); then the zero-cycle \( L^n \cdot H^n = 0 \) is proportional to the class of a point \( x \in X \) which belongs to a Lagrangian constant cycle subvariety. 

ii) If a very general projective deformation of \( X \) preserving the Lagrangian fibration with \( p(X) \geq 3 \) satisfies Matsushita’s conjecture (in particular if \( b_2(X) \geq 8 \) \cite{20}), then the class of a point in \( \Sigma_{n, \Omega} \) modulo rational equivalence is independent of the divisor class \( H \).

iii) Under the same hypothesis as in ii), the class of a point in \( \Sigma_{n, \Omega} \) modulo rational equivalence is independent of the Lagrangian fibration.

## 2 Base variety of rationally fibered Calabi-Yau manifolds

We will prove Theorem 1.4 in this section.

**Proof of Theorem 1.4** — Up to a bimeromorphic modification, we suppose that \( B \) is smooth. If \( B \) has a non-trivial holomorphic 2-form \( \alpha \), then \( 2 \leq \dim B < \dim X \) and \( f^* \alpha \neq 0 \) is degenerated, contradicting the Calabi-Yau assumption. Thus \( B \) is projective.

By Graber-Harris-Starr’s theorem \cite{11}, it suffices to show that if \( B \) satisfies the condition in Theorem 1.4, then \( B \) is uniruled. Indeed, suppose that \( B \) is not rationally connected, and let \( B \to B' \) be the MRC-fibration of \( B \), then the composition map \( X \to B \to B' \) is dominant with \( 0 < \dim B' < \dim X \). So \( B' \) would be uniruled, contradicting \cite{11} Corollary 1.4.

Now suppose \( B \) is not uniruled. By \cite{4}, the canonical class \( c_1(K_B) \) is pseudo-effective; let \( T \) be a closed positive current of bidegree \( (1,1) \) on \( B \) representing \( c_1(K_B) \). This means that the class \( c_1(K_B) \in H^2(B, \mathbb{R}) \) is a limit of effective divisor classes. Let \( X \to \tilde{X} \) be a resolution of \( f : X \to B \) with \( \tilde{X} \) smooth. It is standard to show that \( p_\ast q^\ast c_1(K_B) \) is a pseudo-effective class; indeed, this follows from the fact that \( p \cdot q^\ast \) maps effective divisor classes to effective divisor classes.

Since \( q : \tilde{X} \to B \) is surjective, the induced map \( q^\ast K_B \to \Omega^\Omega_X \) is non-zero where \( k = \dim B \). As \( X \) is smooth, this map determines a non-zero morphism \( L \to \Omega^\Omega_X \) where \( L \) is a line bundle such that \( c_1(L) = p_\ast q^\ast c_1(K_B) \).

Let \( \omega \) be a Kähler form in \( X \). Since the Riemannian holonomy group \( \text{Hol}(X) \) of \( X \) is either \( \text{SU}(n) \) or \( \text{Sp}(n/2) \) where \( n = \dim X \), there exists a Kähler-Einstein metric on \( T_X \) whose corresponding Kähler form is cohomologous to \( \omega \) \cite{27}, which further implies that \( \Omega^\Omega_X \) is \( \omega \)-polystable by Donaldson-Uhlenbeck-Yau.

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2. We say that a Lagrangian fibration \( \pi : X \to B \) satisfies Matsushita’s conjecture if either \( \pi : X \to B \) is isotrivial or the induced moduli map \( B \to \mathcal{M}_B \) to some suitable moduli space of abelian varieties is generically injective.
theorem \[8,19\]. Precisely, \( \Omega_X^k = E_1 \oplus \cdots \oplus E_m \) where \( E_i \) is defined as the parallel transport of each summand in the decomposition of the \( \text{Hol}(X) \)-module \( \Omega_{X|\mathbb{R}}^k \) into irreducible components over any \( x \in X \). Each \( E_i \) is \( \omega \)-stable of slope \( \mu_\omega(E) = 0 \).

The following result can be found in \[5\] §13, n° 1 and 3] or in \[6\] Chapter VI.3.

**Lemma 2.1.** —

i) If \( \text{Hol}(X) = \text{SU}(n) \), then the \( \text{Hol}(X) \)-module \( \Omega_{X|\mathbb{R}}^k \) is irreducible.

ii) If \( \text{Hol}(X) = \text{Sp}(n/2) \), the decomposition of \( \Omega_{X|\mathbb{R}}^k \) into irreducible \( \text{Sp}(n/2) \)-sub-modules is described as follows:

\[
\Omega_{X|\mathbb{R}}^k = \bigoplus_{k \geq k-2r \geq 0} \eta^r \wedge p^{k-2r},
\]

where \( p^{k-2r} \) are irreducible \( \text{Sp}(n/2) \)-sub-modules of \( \Omega_{X|\mathbb{R}}^{k-2r} \). Moreover,

\[
\dim_{\mathbb{C}} p^{k-2r} = \binom{2n}{k-2r} - \binom{2n}{k-2r-2}.
\]

By virtue of the above lemma, if \( \text{Hol}(X) = \text{Sp}(n/2) \) and \( k \) is odd or \( \text{Hol}(X) = \text{SU}(n) \), then \( \dim E_i > 1 = \text{rank}(L) \) for all \( i \) (since \( 0 < k < n \)). As \( c_1(L) \) is pseudo-effective (so \( \mu_\omega(L) \geq 0 = \mu_\omega(E)_i \)) and \( E_i \) is stable, there is no non-trivial morphism from \( L \) to \( E_i \) for all \( i \), contradicting the non-vanishing of \( L \to \Omega_X^k \). Finally if \( \text{Hol}(X) = \text{Sp}(n/2) \) and if \( k \) is even, then \( m \geq 2 \) and there exists exactly one \( i \) such that \( \text{rank}(E_i) = 1 \). Moreover, \( E_i \cong \mathcal{O}_X \) and \( E_i \hookrightarrow \Omega_X^k \) is given by the multiplication by \( \eta^{k/2} \). We deduce that if \( U \) is a Zariski open subset of \( X \) restricted to which \( f \) is well-defined, then locally the pullback under \( f|_U \) of a non-zero holomorphic \( k \)-form \( \alpha \) on \( f(U) \) is proportional to \( \eta^{k/2} \), which contradicts the fact that \( \eta \) is non-degenerate.

\[ \square \]

As an immediate consequence,

**Corollary 2.2.** — The class of a fiber \( F \) in a Lagrangian fibration modulo rational equivalence is independent of \( F \).

In particular, there exists \( \mu \in \mathbb{Z} \setminus \{0\} \) such that \( L^n = \mu F \) in \( \text{CH}^\bullet(X) \).

**Remark 2.3.** — So far we have been interested in Calabi-Yau manifolds. As for complex tori, which are also Ricci-flat varieties, it is known that the image of a complex torus under a holomorphic map is always a product of projective spaces and a complex torus \[7\].

## 3 Construction of constant cycles subvarieties on Lagrangian fibrations

Let \( X \) be a variety.

**Definition 3.1.** — A subvariety \( Y \) of \( X \) is called constant cycle subvariety if every point in \( Y \) is rationally equivalent in \( X \) to each other.

**Lemma 3.2.** — Let \( Y \subset X \) be a connected subvariety. If the image of the Gysin map \( i_* : \text{CH}_0(Y)_Q \to \text{CH}_0(X)_Q \) is generated by an element \( \alpha_Y \) in \( \text{CH}_0(X)_Q \), then \( Y \) is a constant cycle subvariety. In this case, we say that \( Y \) is represented by the zero-cycle \( \alpha_Y \).

**Proof.** — It suffices to show that if every point supported on \( Y \) is torsion in \( \text{CH}_0(X) \), then \( Y \) is a constant cycle subvariety. Let

\[
\alpha : Y \hookrightarrow X \to \text{Alb}(X)
\]
be the composition of the inclusion map $Y \hookrightarrow X$ with the Albanese map $X \to \text{Alb}(X)$. If the image of $i_* : \text{CH}_0(Y) \to \text{CH}_0(X)$ consists of torsion classes, then by Roitman’s theorem [18] the map $\alpha$ factorizes through $\text{CH}_0(X)_{\text{tors}} \cong \text{Alb}(X)_{\text{tors}} \hookrightarrow \text{Alb}(X)$ via the cycle class map $Y \to \text{CH}_0(X)_{\text{tors}}$. As $Y$ is connected, $\alpha$ is constant, hence $Y \to \text{CH}_0(X)_{\text{tors}} \hookrightarrow \text{CH}_0(X)$ is constant. □

**Remark 3.3.** — A variety $X$ is called $\text{CH}_0$-trivial if the degree map $\deg : \text{CH}_0(X) \to \mathbb{Z}$ is injective. Obvious examples of constant cycle subvarieties are provided by subvarieties in a $\text{CH}_0$-trivial variety and any $\text{CH}_0$-trivial subvariety in a variety. In particular, rationally connected subvarieties are interesting examples of constant cycle subvarieties from a deformation-theoretic point of view: since rational connectedness is an open and closed property, these subvarieties remain constant cycle as long as they survive under deformations of the ambient variety in which they embed. Note that in general, the property of being constant cycle is not stable under deformation of the embedding of a subvariety. As a counter-example, take an irrational constant cycle curve $C$ inside a K3 surface $S$ (such as examples constructed in [12]). The deformation of $C \to S$ covers a Zariski open subset of $S$, whereas $\text{CH}_0(S)$ is highly non trivial [16].

**Remark 3.4.** — The main result of N. Fakhruddin in [9] implies that a general hypersurface in a projective space with large degree has no constant cycle subvarieties.

The property of being constant cycle for a subvariety is *birational* in the following sense:

**Lemma 3.5.** — A subvariety $Y$ of $X$ is constant cycle if and only if there exists a Zariski open subset $U$ of $Y$ such that all points in $U$ are rationally equivalent in $X$.

**Proof.** — This follows from the well-known fact that every zero-cycle in $Y$ is rationally equivalent to a zero-cycle supported in $U$. □

Now we restrict ourselves to constant cycle subvarieties on projective hyper-Kähler manifolds. Let $X$ be a projective hyper-Kähler manifold of dimension $2n$ and let $\eta$ be a holomorphic symplectic 2-form on $X$. The following result is a direct consequence of Mumford-Roitman’s theorem [22, Proposition 10.24]:

**Proposition-Definition 3.6.** — If $Y$ is a constant cycle subvariety of $X$, then $Y$ is isotropic for $\eta$. In particular, $\dim Y \leq n$. If $\dim Y = n$, then $Y$ is called a Lagrangian constant cycle subvariety.

The rest of Section 3 is devoted to the proof of Theorem 1.3 and Theorem 1.5.

**Proof of Theorem 1.3.** — First we prove the following

**Lemma 3.7.** — Let $A$ be an abelian variety of dimension $g$ and $h$ an ample divisor of $A$. There exist a finite number of points $x \in A$ such that $h^g = D[x]$ in $\text{CH}_0(A)$ where $D$ is the degree of $h^g$, and the set of these points is nonempty.

**Proof.** — For finiteness, let $x_0$ be any point in $A$ and $\text{alb} : A \to \text{Alb}(A)$ be the Albanese map of $A$ with respect to $x_0$. Since $A$ is an abelian variety, its Albanese map is an isomorphism. Recall that alb factorizes through the Deligne cycle class map $\alpha : \text{CH}_0(A)_{\text{hom}} \to \text{Alb}(A)$, where $\text{CH}_0(A)_{\text{hom}}$ denotes the subgroup of $\text{CH}_0(A)$ homologous to zero and the morphism $\text{Alb}(A) \to \text{CH}_0(A)_{\text{hom}}$ is given by $x \mapsto [x] - [x_0]$. If $h^g = D[x]$ in $\text{CH}_0(A)$, then

$$D \cdot x = D \cdot \text{alb}(x) = \alpha(D[x] - D[x_0]) = \alpha(h^g - D[x_0])$$

in $A$. Hence there are at most $D^{2g}$ points $x \in A$ such that $D[x] = h^g$ in $\text{CH}_0(A)$.

For existence, first of all we remark that if $x' \in A$ is a point such that $(t'_h)^g = D[x']$ in $\text{CH}_0(A)$ for some translation map $t'_h$, then $h^g = D[x + a]$. Since $h$ is ample, there exists a translation map $t_a$ such that $t^2_a h$ is
symmetric. So by Poincaré’s formula [3 Corollary 16.5.7], there exists \( x' \in A \) such that \( (t'_s h)^\eta = D[x'] \) in \( \text{CH}_0(A) \). Thus \( (t'_s h)^\eta = D[x'] + t \) in \( \text{CH}_0(A) \) for some torsion element \( t \in \text{CH}_0(A)_{\text{hom}} \). Let \( t' \) be a \( D \)-division point of \( t \). Since

\[
\alpha \left(D[x + t'] - D[x_0]\right) = D \cdot x + \alpha(t) = \alpha \left(t' + D[x] - D[x_0]\right) ,
\]

one gets \( D[x + t'] = t' + D[x] = (t'_s h)^\eta \), therefore \( h^\eta = D[x + t' + a] \).

\[ \square \]

Let \( U \subset B \) be a Zariski open subset of \( B \) parametrizing smooth fibers of \( \pi \) such that \( H_{[\nu^{-1}(0)]} \) is ample for any \( b \in U \). Set \( X_U := \pi^{-1}(U) \). By a standard argument (see for example the proof of [22 Theorem 10.19]), there exist countably many relative Hilbert schemes \( H_i \) of points of length 1 over \( U \), which can be considered as subvarieties of \( X_U \), parametrizing the data of a point \( t \) in \( U \) and a point \( x \in X_t \) such that \( D[x_1] = H_{[x_t]}^r \) in \( \text{CH}_0(X_t) \).

Let \( p : H_i \to U \) be the natural projection. Since the \( X_t \)'s are abelian varieties, \( p \) is finite and dominant by Lemma [3,7] so there exists an irreducible component \( M \) of \( H_i^r \) such that \( p_M \) is finite and dominant as well. By construction, viewing \( H_i \) as a subvariety of \( X_U \), \( M \) is Zariski locally closed in \( X \) of dimension \( n \); we define \( \Sigma_{n, M} \) as the closure of \( M \) in \( X \), which is also of dimension \( n \). Finally, for every \( x \in M \), let \( j : X_t \to X \) be the inclusion of the fiber of \( \pi \) containing \( x \), then \( D[x] = (j'^r H)^n \) in \( \text{CH}_0(X_t) \) thus

\[
D[x] = H^n \cdot [F] = D' \cdot H^n \cdot L^n \tag{3.1}
\]

for some \( D' \in \mathbb{Z}[0] \) in \( \text{CH}_0(X) \), where the last equality follows from Corollary [2.2]. Hence \( M \) is a Zariski open subset of \( \Sigma_{n, M} \) whose points are rationally equivalent to a scalar multiple of \( H^n \cdot L^n \). We conclude by Lemma [3,5] that \( \Sigma_{n, M} \) is a constant cycle subvariety of \( X \).

\[ \square \]

Before we start proving of Theorem [1.5], let us recall the following result of Matsushita and Voisin which will be useful later. Let \( \pi : X \to B \) be a Lagrangian fibration and let \( L \) be the pullback of an ample divisor class from the base. Let \( j : F \to X \) be the inclusion map of a smooth Lagrangian fiber in \( X \).

**Lemma 3.8 (Matsushita [15] + Voisin [21]).** — If \( j' : H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q}) \) denotes the restriction map and \( \mu_{[F]} : H^2(X, \mathbb{Q}) \to H^{2n+2}(X, \mathbb{Q}) \) the cup product against \([F]\), then

\[
\ker \mu_{[F]} = \ker j' = \ker q(L, \cdot)
\]

where \( q : \text{Sym}^2 H^2(X, \mathbb{Q}) \to \mathbb{Q} \) is the Beauville-Bogomolov-Fujiki form associated to \( X \). In particular, the image of \( j' : H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q}) \) is of rank one.

**Proof.** — The first equality is exactly the statement of [21 Lemme 1.5]. For the second equality, by [15 Lemma 2.2], we have the inclusion \( \ker q(L, \cdot) \subset \ker j' \). It follows that

\[
b_1(X) - 1 = \dim_\mathbb{Q} \ker q(L, \cdot) \leq \dim_\mathbb{Q} \ker j' \leq b_2(X) - 1
\]

where the last inequality results from the non-vanishing of \( j' \) (on any ample divisor class). Hence \( \ker q(L, \cdot) = \ker j' \) for dimensional reason.

\[ \square \]

**Proof of Theorem 1.5.** — Let \( \pi : X \to B \) be a Lagrangian fibration on a polarized hyper-Kähler manifold \((X, H)\) and let \( D \in \text{Pic}(X) \). If \( D_F \) is cohomologous to 0 on \( F \), then \( D^n \cdot L^n = 0 \) in \( \text{CH}_0(X) \) by [25 Theorem 0.9]. If \( D_F \neq 0 \) in \( H^2(F, \mathbb{Q}) \), then since \( j' : H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q}) \) is of rank one by Lemma [3.8], either \( D_F \) or \(-D_F \) is ample on \( F \). Suppose without loss of generality that \( D_F \) is ample on \( F \), then by (3.1), there exists \( d \in \mathbb{Z} \setminus \{0\} \)
such that $D^n \cdot F = d[x]$ in $\text{CH}_0(X)$ for any point $x$ in the Lagrangian constant cycle subvariety $\Sigma_{n,D}$. Since $D^n \cdot L^n$ is non-trivially proportional to $D^n \cdot F$ in $\text{CH}_0(X)$ by Corollary 2.2 (this proves $i$).

To prove $ii$), let $H_1$ and $H_2$ be two divisor classes such that $H_1$ is ample and $H_2$ satisfies the assumption of Theorem 1.3 By Lemma 3.8, for a smooth fiber $F_b := \pi^{-1}(b)$, there exist $\alpha, \beta \in \mathbb{Z}[0]$ such that $(\alpha H_1 - \beta H_2)_{F_b}$ is cohomologous to 0 on $F_b$. It follows that $(\alpha H_1 - \beta H_2)_{F_b}$ is cohomologous to 0 on $F_b$ for all $b \in U$ where $U \subset B$ is the smooth locus of $\pi$. This implies by Lemma 3.3 that the product $[F_b] \cdot (\alpha H_1 - \beta H_2)$, hence $L^n \cdot (\alpha H_1 - \beta H_2)$, is cohomologous to 0 on $X$. Since a very general deformation of $X$ preserving the Lagrangian fibration and $H_1, H_2$ satisfies Matsushita’s conjecture, we can apply [25 Theorem 0.9] so that $\alpha H_1 \cdot L^n = \beta H_2 \cdot L^n$ in $\text{CH}^1(X) \otimes \mathbb{Q}$. It follows that

$$a^n H_1^n \cdot L^n - \beta^n H_2^n \cdot L^n = (\alpha H_1 - \beta H_2) \cdot \left( \sum_{i=0}^{n-1} a^i \beta^{n-1-i} H_1^i \cdot H_2^{n-1-i} \right) \cdot L^n = 0 \text{ in } \text{CH}_0(X).$$

Since the zero-cycles supported on the constant cycle subvarieties $\Sigma_{n,H_1}$ and $\Sigma_{n,H_2}$ constructed above are proportional to $H_1^n \cdot L^n$ and $H_2^n \cdot L^n$ in $\text{CH}_0(X)$ respectively, the second statement of Theorem 1.3 follows.

Now we prove $iii$). Let $\pi : X \to B$ and $\pi' : X \to B'$ be two Lagrangian fibrations and let $L := \pi'^{-1}(\mathcal{O}(1))$ and $L' := \pi'^{-1}(\mathcal{O}(1))$. We have to show that $H^n \cdot L^n$ is proportional to $H^n \cdot L'^n$ in $\text{CH}_0(X)$. We also assume that $L$ and $L'$ are not proportional, otherwise the proof is finished. By the same argument as in the proof of $ii$), there exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that $\lambda_1 H^n \cdot L^n = L'^n$. It suffices to show that $\deg (L'^n \cdot L^n) \neq 0$ to conclude. We have $\lambda_1 \neq 0$ and only if $\deg (L'^n \cdot L^n) \neq 0$ if and only if $q(L, L') \neq 0$ by Lemma 3.8 hence it suffices to show that $q(L, L') \neq 0$. This follows from the fact that the restriction of $q$ to $\text{NS}(X)_\mathbb{Q}$ is of signature $(1, 1 - \rho(X))$, so the restriction of $q$ to the two-dimensional subspace generated by $L$ and $L'$ cannot be zero. Since $q(L, L) = q(L', L') = 0$, this implies that $q(L, L') \neq 0$.

\[\square\]

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