Witt groups and torsion Picard groups of smooth real curves

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Introduction

The Witt group of smooth real projective curves was first computed by Knebusch in [Kn]. If the curve is not complete but still smooth, the Witt group is also studied in [Kn] but not explicitly calculated. However, for some precise examples of smooth affine curves, we may find explicit calculations ([Knu] and [Ay-O]). In this paper, the Witt group of a general smooth curve is explicitly calculated in terms of topological and geometrical invariants of the curve. My method is strongly inspired by Sujatha’s calculation of the Witt group of a smooth projective real surface and uses a comparison theorem between the graded Witt group and the tale cohomology groups established in [Mo].

In the second part of the paper, we are interested in the torsion subgroup of the Picard group (denoted by $\text{Pic}_{\text{tors}}(X)$) of a smooth geometrically connected (non complete) curve $X$ over a real closed field $R$. Let $C$ be the algebraic closure of $R$ and $X_C := X \times_{\text{Spec} R} \text{Spec} C$. We compute $\text{Pic}_{\text{tors}}(X)$ and $\text{Pic}_{\text{tors}}(X_C)$ using the Kummer exact sequence for tale cohomology. These calculations depend on a new invariant $\eta(X) \in \mathbb{N}$ (resp. $\eta(X_C)$) which we introduce in this note. We study relations between $\eta(X)$, $\eta(X_C)$ and the level and Pythagoras number of curves using new results of Huisman and Mah [Hu-Ma].

The last part is devoted to the study of smooth affine hyperelliptic curves. For such curves we calculate the Witt group and the torsion Picard group determining the invariant $\eta$.

1 Preliminaries

Let $k$ be a field. By a variety over $k$ we mean a reduced separated scheme of finite type over $\text{Spec} k$. A curve over $k$ is a variety of dimension 1.
1.1 Witt group and Etale cohomology

Let $X$ be a smooth connected curve over a real closed field $R$ and $R(X)$ denote its function field. Let $W(X)$ and $W(R(X))$ denote respectively the Witt ring of $X$ and $R(X)$. We denote the set of codimension one points of $X$ by $X^{(1)}$. Since $X$ is smooth, an element $x \in X^{(1)}$ gives a second residue homomorphism $\partial_x$ defined on $W(R(X))$ with value in the Witt group $W(k(x))$ of the residue field at $x$. Thus one obtains an exact sequence:

$$0 \to W(X) \to W(R(X)) \xrightarrow{\partial=(\partial_x)} \bigoplus_{x \in X^{(1)}} W(k(x))$$

Let $I(R(X))$ be the ideal of even rank forms and $I^n(R(X))$ denote its powers for $n \geq 0$ ($I^0(R(X)) = W(R(X))$). Recall that $I^n(R(X))$ is additively generated by the set of $n$-fold Pfister forms, i.e. forms isometric to forms of the type

$$<< a_1, \ldots, a_n >> := << 1, a_1 > \otimes \ldots \otimes 1, a_n >>, a_i \in R(X)^*$$

For $n \geq 0$ we write $I^n(X) = I^n(R(X)) \cap W(X)$. Thus the previous exact sequence induces the following exact sequence:

$$0 \to I^n(X) \to I^n(R(X)) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} I^{n-1}(k(x)) \quad (1)$$

Let $X(R)$ denote the set of $R$-rational points of $X$ and $\text{Cont}(X(R), \mathbb{Z})$ denote the set of continuous maps from $X(R)$ into $\mathbb{Z}$. Thus $\text{Cont}(X(R), \mathbb{Z}) \simeq \mathbb{Z}^s$ with $s$ denoting the number of semi-algebraic connected components of $X(R)$. Since $W(X)$ injects into $W(R(X))$, the total signature homomorphism $\Lambda : W(X) \to \text{Cont}(X(R), \mathbb{Z})$ has kernel precisely the torsion subgroup $W_t(X)$. For $n \geq 0$ we set $I^n_t(X) = I^n(X) \cap W_t(X)$.

Let $\mathcal{H}^n$ denote the sheaf associated to the presheaf $U \mapsto H^1_{et}(U)$ where for any scheme $Y$ over a field of characteristic $\neq 2$ $H^1_{et}(Y)$ denotes the tale cohomology group $H^1_{et}(Y, \mathbb{Z}/2)$. Recall that there is an exact sequence $[BO]$, Th. 4.2:

$$0 \to H^0(X, \mathcal{H}^n) \to H^n(R(X)) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} H^{n-1}(k(x)) \quad (2)$$

where $H^0(X, \mathcal{H}^n)$ denotes the group of global sections of the sheaf $\mathcal{H}^n$. Let $H^0_t(X, \mathcal{H}^n) := \{ \alpha \in H^0(X, \mathcal{H}^n) \mid \alpha \cup (-1)^k = 0 \text{ for some } k \}$ where $(-1)$ is the non trivial element in $H^1(R) = R^*/R^{*2}$. Using the exact sequences (1) and (2) (see [MF]), for every $n \geq 0$ we have a well defined homomorphism

$$e_n : I^n(X) \to H^0(X, \mathcal{H}^n)$$

with kernel $I^{n+1}(X)$. We denote by $e'_n : I^n(X)/I^{n+1}(X) \to H^0(X, \mathcal{H}^n)$ the corresponding injective map.

The following theorem gives an affirmative answer to a global version of a question on quadratic forms raised by Milnor.
Theorem 1.1 \cite{Ma}, \cite{Su}

Let $X$ be a smooth integral curve over a real closed field $R$. Then

$$e' = (e'_n) : \bigoplus_{n \geq 0} I^n(X)/I^{n+1}(X) \to \bigoplus_{n \geq 0} H^0(X, \mathcal{H}^n)$$

gives an isomorphism between the graded Witt group and the graded unramified cohomology group. Moreover the restriction to the torsion part

$$e' = (e'_n) : \bigoplus_{n \geq 0} I^t_n(X)/I^{n+1}_t(X) \to \bigoplus_{n \geq 0} H^0_t(X, \mathcal{H}^n)$$

is also an isomorphism.

1.2 Complexification of real varieties

We recall the definition of the level.

Definition 1.2 The level of a commutative ring with units $A$ is the smallest integer $n$ such that $-1$ is a sum on $n$ squares in $A$. If $X$ is a variety over a real closed field $R$, the level of $X$ is the level of the $R$-algebra $\mathcal{O}(X)$ where $\mathcal{O}$ is the structure sheaf of $X$.

Let $X$ be a smooth connected variety over a real closed field $R$. We always write $C = R(\sqrt{-1})$ for the algebraic closure of $R$ and $X_C := X \times_{\text{Spec } R} \text{Spec } C$. We denote the canonical morphism $X_C \to X$ by $\pi$. We need the following lemma concerning complexification of real varieties.

Lemma 1.3 Let $X$ be a smooth connected variety over a real closed field $R$. If $X_C$ is not connected then $X_C$ is a disjoint union $X_C = Y \bigsqcup Y'$ where both restrictions $\pi|_Y : Y \to X$ and $\pi|_{Y'} : Y' \to X$ are isomorphisms. In particular, we have $X(R) = \emptyset$. Moreover $X_C$ is not connected if and only if the level of $R(X)$ is 1 if and only if the level of $X$ is 1.

Proof: The first part of the lemma is \cite{CT-S}, Lem. 1.1, the statement follows from the fact that $\pi$ is finite and tame. If $X_C$ is not connected, since $\pi|_Y : Y \to X$ is an isomorphism, obviously $-1 \in R(X)^{\ast 2}$. If $-1 \in R(X)^{\ast 2}$ then $R(X)(\sqrt{-1})$ is a product of two fields and $X_C$ is not connected.

We prove now the last part of the lemma. Using Kummer exact sequence (see next section) for the field $R(X)$ and $X$, we have a commutative diagramm with exact lines

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{O}(X)^{\ast}/\mathcal{O}(X)^{\ast 2} & \to & H^1(X) & \to & \text{Pic}_2(X) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R(X)^{\ast}/R(X)^{\ast 2} & \to & H^1(R(X)) & \to & 0 & \to & 0
\end{array}
$$

Since $X$ is smooth $H^1(X) = H^0(X, \mathcal{H}^1)$ and by a previous exact sequence the map $H^1(X) \to H^1(R(X))$ is injective. By snake lemma the map $\mathcal{O}(X)^{\ast}/\mathcal{O}(X)^{\ast 2} \to R(X)^{\ast}/R(X)^{\ast 2}$ is also injective and the proof is done. \hfill \Box
Remark 1.4 The last equivalence in the previous lemma is not valid in the singular case. Consider the affine curve \( X \) with coordinates ring \( \mathbb{R}[x, y]/y^2 + (1 + x^2)^2 \). Then \(-1\) is clearly a square in \( \mathbb{R}(X) \) but it is well known that the level of \( X \) is 3.

By a well-known theorem of Pfister, if \( X \) is a smooth connected curve over a real closed field \( R \) with \( X(R) = \emptyset \) then the level of \( R(X) \) is 1 or 2. Moreover, if \( X \) is not complete and geometrically connected then \( X \) is affine and the level of \( X \) is \( \leq 3 \) [Ma]. One gets the following consequence

Corollary 1.5 Let \( X \) be a smooth connected curve over a real closed field \( R \) with \( X(R) = \emptyset \). Then \( X_C \) is connected if and only if the level of \( R(X) \) is 2. If \( X \) is not complete then \( X_C \) is connected if and only if the level of \( X \) is 2 or 3.

2 Witt group of real curves

2.1 Etale cohomology groups of real curves

For any smooth variety \( X \) over a field \( k \) of characteristic \( \neq 2 \), there is an exact sequence of etale sheaves:

\[
0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 0
\]

where \( \mathbb{G}_m \) is the sheaf of units and \( \mu_n \) the sheaf of \( n^{th} \) roots of unity (\( \mu_2 \) is isomorphic to \( \mathbb{Z}/2 \)). From the previous exact sequence one gets the exact sequences

\[
0 \rightarrow \mathcal{O}(X)^*/\mathcal{O}(X)^{\mu_n} \rightarrow H^1_{et}(X, \mu_n) \rightarrow Pic_n(X) \rightarrow 0 \tag{3}
\]

\[
0 \rightarrow Pic(X)/n \rightarrow H^2_{et}(X, \mu_n) \rightarrow Br_n(X) \rightarrow 0 \tag{4}
\]

where \( Pic_n(X) \) is the \( n \)-torsion subgroup of the Picard group of \( X \) and \( Br_n(X) \) is the \( n \)-torsion subgroup of the cohomological Brauer group of \( X \).

Let \( X \) be a smooth connected real curve over a real closed field \( R \). The sets \( X(R) \) and \( X(C) = X_C(C) \) (the set of \( C \)-rational points) are semi-algebraic spaces over \( R \). The cohomology of these sets considered here is always the semi-algebraic cohomology. For \( R = \mathbb{R} \) and \( C = \mathbb{C} \), it coincides with classical cohomology. There is an exact sequence of elements

\[
0 \rightarrow \mathbb{Z}/2 \rightarrow \pi_*(\mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \rightarrow 0
\]

This sequence gives rise to the long exact sequence of etale cohomology groups

\[
\ldots H^i(X) \xrightarrow{res} H^i(X_C) \xrightarrow{cor} H^i(X) \xrightarrow{(-1)^i} H^{i+1}(X) \ldots \tag{5}
\]

where the boundary maps from \( H^i(X) \) to \( H^{i+1}(X) \) are cup-products by \( (-1) \in H^1(X) \).
2.1.1 The case of complete real curves

Let $X$ be a smooth connected curve over a real closed field $R$ and assume $X/R$ is complete. We may easily calculate the tale cohomology groups $H^i(X)$, $i \geq 0$, using the exact sequences (3), (4) and computations in \[ \text{CT-} \text{S} \] of the torsion and the cotorsion of $Pic(X)$. Let $q := \dim_R H^1(X, \mathcal{O}_X)$ and $s$ denote the number of semi-algebraic connected components of $X(R)$. Then $Pic(X)_{tors} \simeq (\mathbb{Q}/\mathbb{Z})^q \oplus (\mathbb{Z}/2)^{s-1}$ if $X(R) \neq \emptyset$, $Pic(X)_{tors} \simeq (\mathbb{Q}/\mathbb{Z})^q$ if $X(R) = \emptyset$ \[ \text{CT-} \text{S} \], Th. 1.6]. Moreover, $Pic(X)/2 \simeq (\mathbb{Z}/2)^s$ if $X(R) \neq \emptyset$, $Pic(X)/2 \simeq (\mathbb{Z}/2)$ if $X(R) = \emptyset$ \[ \text{CT-} \text{S} \], Th. 1.3]; and by a theorem of Witt $Br(X) \simeq (\mathbb{Z}/2)^s$. For $i > 3$ $H^i(X) \simeq H^0(X(R), \mathbb{Z}/2) \oplus H^1(X(R), \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{2s}$ \[ \text{CT-} \text{S} \], Th. 2.3.1]. Thus we obtain

**Proposition 2.1** Let $X$ be a smooth geometrically connected curve over a real closed field $R$, with $X/R$ complete. Let $g$ denote the genus of $X$. Thus

(i) If $X(R) \neq \emptyset$ then $H^0(X) = \mathbb{Z}/2$, $H^1(X) = (\mathbb{Z}/2)^{g+s}$, $H^i(X) = (\mathbb{Z}/2)^{2s}$ for $i \geq 2$.

(ii) If $X(R) = \emptyset$ then $H^0(X) = \mathbb{Z}/2$, $H^1(X) = (\mathbb{Z}/2)^{g+1}$, $H^2(X) = (\mathbb{Z}/2)$, $H^i(X) = 0$ for $i \geq 3$.

We deal now with the case $X_C$ not connected.

**Proposition 2.2** Let $X$ be a smooth connected curve over a real closed field $R$ such that $X_C$ is not connected and $X/R$ is complete. Let $g$ denote the genus of $Y$ ($Y$ is given by Lemma 1.3]. Thus $H^0(X) = \mathbb{Z}/2$, $H^1(X) = (\mathbb{Z}/2)^{2g}$, $H^2(X) = (\mathbb{Z}/2)$, $H^i(X) = 0$ for $i \geq 3$.

**Proof:** By Lemma 1.3 $X_C = Y \coprod Y'$ and the restrictions of $\pi$ to $Y$ and $Y'$ are isomorphisms. Thus $X$ is a smooth complete connected curve over $C$ of genus $g$. Then $dim_R H^1(X, \mathcal{O}_X) = 2g$. The group $\mathcal{O}(X)^*/\mathcal{O}(X)^{\ast 2}$ is trivial since $\pi : Y \to X$ is an isomorphism and $Y$ is complete over $C$. Using the exact sequence (3) we get $H^1(X) = (\mathbb{Z}/2)^{2g}$. The other tale cohomology groups could be deduced from the previous remarks. We may calculate $H^1(X)$ in a different way. The exact sequence (5) gives

$$0 \to H^0(X) \to H^0(X_C) \to H^0(X) \to H^1(X)$$

$$\to H^1(X_C) \to H^1(X) \to H^2(X) \to H^2(X_C) \to H^2(X) \to 0$$

We already know $H^0(X)$, $H^2(X)$. Let $Pic^0(Y)$ denote the kernel of the degree map defined on $Pic(Y)$, then we have an exact sequence

$$0 \to Pic^0(Y) \to Pic(Y) \to \mathbb{Z} \to 0$$

It is well known that $Pic^0(Y)$ is a divisible group, hence the previous exact sequence splits since a divisible group is an injective $\mathbb{Z}$–module. Moreover $Pic^0(Y)_{tors} = (\mathbb{Q}/\mathbb{Z})^{2g}$ and $Br(Y) = 0$. Using (3) and (4) one gets $H^1(Y) = H^1(Y') = Pic_2(Y) = (\mathbb{Z}/2)^{2g}$ and $H^2(Y) = H^2(Y') = Pic(Y)/2 = \mathbb{Z}/2$. Counting dimensions in the previous exact sequence (5) we obtain $H^1(X) = (\mathbb{Z}/2)^{2g}$. \[ \square \]
2.1.2 The case of non complete real curves

Let \( X \) be a smooth connected curve over a real closed field. In this section we assume that \( X/R \) is not complete. By Nagata’s embedding theorem [Na] and resolution of singularities one can realize \( X \) as an open and dense subvariety of a complete smooth variety \( \bar{X} \) over \( R \). Let \( Z := \bar{X} \setminus X \), then \( Z \) consists of \( r \) real points \( \{P_1, \ldots, P_r\} \) and \( c \) complex points \( \{Q_1, \ldots, Q_c\} \) with the notation that a closed point \( P \) is real (resp. complex) if the residue field at \( P \) is \( R \) (resp. C). Let \( Z_C \) denote \( \bar{X}_C \setminus X_C \), then \( Z_C \) consists of \( r + 2c \) closed points.

Let \( s \) denote the number of semi-algebraic connected components of \( X(R) \) and \( t \) the number of such components which are complete (if \( R = \mathbb{R} \) it means compact). Then \( X(R) \) is topologically a disjoint sum of \( t \) circles and \( r \) open intervals i.e. \( s = t + r \). If \( X_C \) is connected then \( g \) will denote the genus of \( \bar{X}_C \) (or \( \bar{X} \)). If \( X_C \) is not connected then \( g \) will denote the genus of \( Y \) (see Lemma [L3], \( X_C = Y \coprod Y' \)). We will keep this notations all along this paper also in the complete case i.e. \( X = \bar{X} \).

**Proposition 2.3** Let \( X \) be a smooth geometrically connected curve over a real closed field \( R \) such that \( X/R \) is not complete. Thus

(i) If \( X(R) \neq \emptyset \) then \( H^0(X) = \mathbb{Z}/2 \), \( H^1(X) = (\mathbb{Z}/2)^{g+c+s} \), \( H^i(X) = (\mathbb{Z}/2)^{s+t} \) for \( i \geq 2 \).

(ii) If \( X(R) = \emptyset \) then \( H^0(X) = \mathbb{Z}/2 \), \( H^1(X) = (\mathbb{Z}/2)^{g+c} \), \( H^i(X) = 0 \) for \( i \geq 2 \).

**Proof:** By [CT-S, Th. 1.3] \( Pic(X) = D(X) \oplus (\mathbb{Z}/2)^t \) with \( D(X) \) a divisible subgroup, thus \( Pic(X)/2 \simeq (\mathbb{Z}/2)^t \). Using Witt Theorem on Brauer groups and (4), we may calculate \( H^2(X) \). For \( i \geq 3 \) \( H^i(X) \simeq H^0(X(R), \mathbb{Z}/2) \oplus H^1(X(R), \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{s+t} \).

Hence we are reduced to calculate \( H^1(X) \). We first calculate \( H^1(X_C) \). There is an exact sequence

\[
0 \to H^1(\bar{X}_C) \to H^1(X_C) \to H^0(Z_C) \to H^2(\bar{X}_C) \to H^2(X_C) = 0 \tag{6}
\]

which is part of the Gysin sequence [M, p. 244]. We have \( H^2(X_C) = 0 \) using (4) since \( Pic(X_C) \) is a divisible group and \( Br(X_C) = 0 \). Since \( Pic^0(\bar{X}_C) \) is a divisible group, using the split exact sequence

\[
0 \to Pic^0(\bar{X}_C) \to Pic(\bar{X}_C) \xrightarrow{deg} \mathbb{Z} \to 0
\]

and (4), we have \( H^2(\bar{X}_C) \simeq (\mathbb{Z}/2) \). Since \( H^1(\bar{X}_C) \simeq (\mathbb{Z}/2)^g \) and \( H^0(Z_C) = (\mathbb{Z}/2)^{r+2c} \), counting dimensions in (6), one obtains

\[
H^1(X_C) \simeq (\mathbb{Z}/2)^{2g+2c+r-1}
\]

Now, using the exact sequence

\[
0 \to H^0(X) \to H^0(X_C) \to H^0(X) \to H^1(X)
\]

\[
\to H^1(X_C) \to H^1(X) \to H^2(X) \to H^2(X_C) = 0
\]

one gets \( H^1(X) = (\mathbb{Z}/2)^{g+c+s} \) if \( X(R) \neq \emptyset \), and \( H^1(X) = (\mathbb{Z}/2)^{g+c} \) if \( X(R) = \emptyset \). \( \Box \)
Proposition 2.4 Let $X$ be a smooth connected curve over a real closed field $R$ such that $X_C$ is not connected and $X/R$ is not complete. Thus $H^0(X) = \mathbb{Z}/2$, $H^1(X) = (\mathbb{Z}/2)^{2g+c-1}$, $H^i(X) = 0$ for $i \geq 3$.

Proof: For $H^i(X)$, $i \geq 3$, the proof of the previous proposition works. We have a decomposition $X_C = Y \coprod Y'$ as in Lemma 1.3. The closed subset $Z_C$ of $X_C$ consists of $2c$ closed points. Again by Lemma 1.3, $\bar{X}_C = Y \coprod \bar{Y}'$ with $\bar{Y}, \bar{Y}'$ complete over $C$. Obviously $\bar{Y} \setminus Y$ consists of $c$ closed points of $Z_C$. Using the exact sequence (6) for $Y, \bar{Y}$ and $\bar{Y} \setminus Y$, we obtain $H^1(Y) \simeq (\mathbb{Z}/2)^{2g+c-1}$. Thus

$$H^1(X_C) \simeq (\mathbb{Z}/2)^{4g+2c-2}$$

Then counting dimensions in (5)

$$0 \to H^0(X) \to H^0(X_C) \to H^0(X) \to H^1(X) \to H^1(X_C) \to H^1(X) \to H^2(X) = 0$$

we have

$$H^1(X) \simeq (\mathbb{Z}/2)^{2g+c-1}$$

The result is compatible with the fact $X$ and $Y$ are isomorphic via $\pi$ (Lemma 1.3).

\[\square\]

2.2 Separation of real connected components

Let $X$ be a smooth connected curve over a real closed field. Let $\text{Cont}(X(R), \mathbb{Z}/2)$ be the set of continuous map from $X(R)$ into $\mathbb{Z}/2$. For every $n$ there is a map

$$h_n : H^0(X, \mathcal{H}^n) \to \text{Cont}(X(R), \mathbb{Z}/2)$$

For $\alpha \in H^0(X, \mathcal{H}^n)$, $p \in X(R)$, $h_n(\alpha)(p)$ is the image of $\alpha$ in $H^n(k(p)) \simeq \mathbb{Z}/2$. This map was studied in [CT-Pa]. It is well known that

$$H^0(X, \mathcal{H}^n) = \ker(h_n)$$

The following result will be very useful in this note.

Lemma 2.5 Let $X$ be a smooth connected curve over a real closed field. Then the map

$$h_1 : H^0(X, \mathcal{H}^1) \to \text{Cont}(X(R), \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^s$$

is surjective

Proof: We may assume that $X(R) \neq \emptyset$. Let $C_1, \ldots, C_s$ denote the semi-algebraic connected components of $X(R)$. By a theorem of Knebush [Kn], there exist $q_1, \ldots, q_s \in W(X)$ such that the signature of $q_i$, denoted by $\tilde{q}_i := \Lambda(q_i)$, is 2 on $C_i$ and 0 outside. Clearly one gets $\tilde{q}_i \in I(X)$ and $\tilde{q}_i \notin I^2(X)$ since its signature is not divisible by 4. In [Md] the author has defined homomorphisms

$$\text{sign}_n : I^n(X)/I^{n+1}(X) \to \text{Cont}(X(R), 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}) \simeq (\mathbb{Z}/2)^s$$
Let \( q \in I^n(X) \) and \( \bar{q} \) denote the class of \( q \) in \( I^n(X)/I^{n+1}(X) \), then \( sign_n(\bar{q}) = (1/2^n)\bar{q} \) mod 2. We may prove that \( sign_1 \) is surjective using the classes of the \( q_i \) in \( I(X)/I^2(X) \). Moreover the following diagram is commutative

\[
\begin{array}{c}
I(X)/I^2(X) \xrightarrow{sign_1} (\mathbb{Z}/2)^s \\
\downarrow e'_1 \parallel \hspace{1cm} \downarrow e'_1 \\
H^0(X, \mathcal{H}^1) \xrightarrow{h_1} (\mathbb{Z}/2)^s
\end{array}
\]

Since \( e'_1 \) is an isomorphism, one gets the result. \( \square \)

### 2.3 Some topological remarks

In this section we will assume that \( R = \mathbb{R} \).

In the proof of Propositions 2.3 and 2.4, we calculate \( H^1(X_C) \) with the Gysin sequence in tale cohomology. The following lemma is a topological verification of this computation.

**Lemma 2.6** Let \( Y \) be a smooth connected curve over \( \mathbb{C} \). Assume \( Y \) is not complete. Let \( \bar{Y} \) be the smooth completion of \( Y \), \( k \) be the number of closed points in \( \bar{Y} \setminus Y \) and \( g \) denote the genus of \( \bar{Y} \). Then \( H^1(Y) \simeq (\mathbb{Z}/2)^{2g+k-1} \)

**Proof:** With the previous notations \( \bar{Y}(\mathbb{C}) \) has a structure of a compact 2-manifold, more precisely a sphere with \( g \) handles. By a comparison theorem [Mi, Th. 3.12, p.117], for any finite abelian group \( M \) we have \( H^i_0(Y, M) \approx H^i(\bar{Y}(\mathbb{C}), M) \) and \( H^i_{et}(\bar{Y}, M) \approx H^i(\bar{Y}(\mathbb{C}), M) \). We write \( \bar{Y}(\mathbb{C}) \setminus Y(\mathbb{C}) := \{ P_1, \ldots, P_k \} \). Let \( U := \bigsqcup_{i=1}^{k} U_i \) where \( U_i \) is a small closed ball of \( \bar{Y}(\mathbb{C}) \) centered at \( P_i \). Then \( Y(\mathbb{C}) \cap U \simeq \bigsqcup_{i=1}^{k} S^1 \). Using Mayer-Vietoris exact sequence,

\[
0 \to H^0(\bar{Y}(\mathbb{C}), \mathbb{Z}) \to H^0(Y(\mathbb{C}), \mathbb{Z}) \oplus H^0(U, \mathbb{Z}) \to H^0(Y(\mathbb{C}) \cap U, \mathbb{Z}) \to
\]

\[
H^1(\bar{Y}(\mathbb{C}), \mathbb{Z}) \to H^1(Y(\mathbb{C}), \mathbb{Z}) \oplus H^1(U, \mathbb{Z}) \to
\]

\[
H^1(Y(\mathbb{C}) \cap U, \mathbb{Z}) \to H^2(\bar{Y}(\mathbb{C}), \mathbb{Z}) \to H^2(Y(\mathbb{C}), \mathbb{Z}) \oplus H^2(U, \mathbb{Z}) = 0
\]

which gives

\[
0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}^k \to \mathbb{Z}^k \to \mathbb{Z}^{2g} \to
\]

\[
H^1(Y(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}^k \to \mathbb{Z} \to 0
\]

we find \( H^1(Y(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z}^{2g+k-1} \). Universal-coefficient formula and comparison theorem give the result. \( \square \)

Let \( X \) be a smooth connected curve over \( \mathbb{R} \). Let \( G = Gal(\mathbb{C}/\mathbb{R}) \) and consider the space \( X(\mathbb{C}) \) equipped with the continuous action of \( G \). Then the quotient space \( X(\mathbb{C})/G \) is a 2-manifold. Let \( \beta : X(\mathbb{C}) \to X(\mathbb{C})/G \) denote the quotient map. Let \( \mathcal{A} \) be a \( G \)-sheaf of group on \( X(\mathbb{C}) \), then the equivariant cohomology groups \( H^i(X(\mathbb{C}); G, \mathcal{A}) \) are defined in [Gr, Ch. 5]. Let consider the following spectral sequence converging to the equivariant cohomology groups

\[
E^{p,q}_2 = H^p(X(\mathbb{C})/G, \mathcal{H}^q(G, \mathcal{A})) \Rightarrow E^{p+q} = H^{p+q}(X(\mathbb{C}); G, \mathcal{A})
\]
where $\mathcal{H}^q(G, \mathcal{A})$ is the sheaf on $X(\mathbb{C})/G$ associated to the presheaf

$$U \mapsto H^q(\beta^{-1}(U); G, \mathcal{A})$$

For the sheaf $\mathcal{A} = \mathbb{Z}/2$, we have [N, 1,23; 1-24]

$$E_{2}^{p,q} = H^p(X(\mathbb{R})/G, \mathbb{Z}/2) \text{ if } q > 0$$

and $H^n(X) \simeq H^n(X(\mathbb{C}); G, \mathbb{Z}/2)$. Then the spectral sequence $E_{2}^{p,q}$ for $\mathbb{Z}/2$ consists of the following

$$\begin{array}{ccc}
E_2^{0,2} = H^0(X(\mathbb{R}), \mathbb{Z}/2) & E_2^{1,2} = H^1(X(\mathbb{R}), \mathbb{Z}/2) & E_2^{2,2} = 0 \\
E_2^{0,1} = H^0(X(\mathbb{R}), \mathbb{Z}/2) & E_2^{1,1} = H^1(X(\mathbb{R}), \mathbb{Z}/2) & E_2^{2,1} = 0 \\
E_2^{0,0} = H^0(X(\mathbb{C})/G, \mathbb{Z}/2) & E_2^{1,0} = H^1(X(\mathbb{C})/G, \mathbb{Z}/2) & E_2^{2,0} = H^2(X(\mathbb{C})/G, \mathbb{Z}/2)
\end{array}$$

We have clearly

$$E_{2}^{0,0} = E_{\infty}^{0,0} = E^{0}$$

Moreover $E_{2}^{1,0} = E_{\infty}^{1,0}$ and $E_{2}^{0,1} = \ker(d_{2}^{0,1} : E_{2}^{0,1} \to E_{2}^{2,0})$. Hence the filtration

$$0 \subseteq E_{1}^{1} \subseteq E_{2}^{1}$$

is given by $E_{1}^{1} = E_{\infty}^{1,0} = H^1(X(\mathbb{C})/G, \mathbb{Z}/2)$ and and $E_{1}^{1}/E_{1}^{1} = E_{\infty}^{0,1} = \ker(d_{2}^{0,1} : H^0(X(\mathbb{R}), \mathbb{Z}/2) \to H^2(X(\mathbb{C})/G, \mathbb{Z}/2))$. We have obviously the following exact sequence which is the five terms exact sequence of low degree

$$0 \to H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \to H^1(X) \xrightarrow{\epsilon} H^0(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{d_{2}^{0,1}} H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \quad (6)$$

where $\epsilon$ is the edge map. In the Bloch-Ogus spectral sequence $H^p(X, \mathcal{H}^q) \Rightarrow H^{p+q}(X)$, we have $H^p(X, \mathcal{H}^q) = 0$ if $p > q$ ($X$ is smooth). Thus the edge map $\epsilon' : H^1(X) \to H^0(X, \mathcal{H}^1)$ is an isomorphism. The following diagramm is commutative [N, Rem. 1.8]

$$\begin{array}{ccc}
H^1(X) & \simeq & H^0(X(\mathbb{C}); G, \mathbb{Z}/2) \\
\downarrow \epsilon' & & \downarrow \epsilon \\
H^0(X, \mathcal{H}^1) & \overset{h_1}{\to} & H^0(X(\mathbb{R}), \mathbb{Z}/2)
\end{array}$$

where $h_1$ is the map defined previously. Using (6) and the fact that $h_1$ is surjective (Lemma 2.3), we obtain

$$\dim_{\mathbb{Z}/2}(H^1(X(\mathbb{C})/G, \mathbb{Z}/2)) = \dim_{\mathbb{Z}/2}(\ker(h_1)) = H^0_i(X, \mathcal{H}^1) = \dim_{\mathbb{Z}/2}(H^1(X)) - s$$

Since $\epsilon$ is surjective in the exact sequence (6), the differential $d_{2}^{0,1} : H^0(X(\mathbb{R}), \mathbb{Z}/2) \to H^2(X(\mathbb{C})/G, \mathbb{Z}/2)$ vanishes. Consequently $E_{2}^{2,0} = E_{\infty}^{2,0} = H^2(X(\mathbb{C})/G, \mathbb{Z}/2)$. Moreover $E_{2}^{1,1} = E_{1}^{1,1} = H^1(X(\mathbb{R}), \mathbb{Z}/2)$ and $E_{2}^{0,2} = E_{\infty}^{0,2} = H^0(X(\mathbb{R}), \mathbb{Z}/2)$. We have a filtration

$$0 \subseteq E_{2}^{2} \subseteq E_{1}^{2} \subseteq E_{0}^{2} = H^2(X(\mathbb{C})/G, \mathbb{Z}/2)$$

with $E_{2}^{2} = E_{\infty}^{2,0} = H^2(X(\mathbb{C})/G, \mathbb{Z}/2)$, $E_{1}^{2} = E_{\infty}^{1,1} = H^1(X(\mathbb{R}), \mathbb{Z}/2)$, $E_{0}^{2} = E_{\infty}^{0,2} = H^0(X(\mathbb{R}), \mathbb{Z}/2)$. Consequently we have

$$\dim_{\mathbb{Z}/2}(H^2(X(\mathbb{C})/G, \mathbb{Z}/2)) = \dim_{\mathbb{Z}/2}(H^2(X)) - s - t$$

We sum up the previous results.
Proposition 2.7 Let $X$ be a smooth complete connected curve over $\mathbb{R}$. Thus,

(i) If $X(\mathbb{R}) \neq \emptyset$ then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^g$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq 0$.

(ii) If $X(\mathbb{R}) = \emptyset$ and $X_\mathbb{C}$ is connected then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{g+1}$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$.

(iii) If $X(\mathbb{R}) = \emptyset$ and $X_\mathbb{C}$ is not connected then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^g$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$.

Proposition 2.8 Let $X$ be a smooth non complete connected curve over $\mathbb{R}$. Thus

(i) If $X(\mathbb{R}) \neq \emptyset$ then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{g+c}$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq 0$.

(ii) If $X(\mathbb{R}) = \emptyset$ and $X_\mathbb{C}$ is connected then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{g+c}$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq 0$.

(iii) If $X(\mathbb{R}) = \emptyset$ and $X_\mathbb{C}$ is not connected then $H^0(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, $H^1(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{g+c-1}$, $H^2(X(\mathbb{C})/G, \mathbb{Z}/2) \simeq 0$.

Remark 2.9 In fact one could generalize the previous calculation over any real closed field $R$. Let $j : X_{el} \to X_b$ be the morphism of site introduced in [21]. Then the Leray spectral sequence for $j^*$

$$H^p(X_b, R^q j_* \mathbb{Z}/2) \Rightarrow H^{p+q}(X)$$

together with a comparison theorem, give the exact sequence (6) over $R$ [21, 20-3-1, p.237].

2.4 Witt groups of real curves

Theorem 2.10 Let $X$ be a smooth connected curve over a real closed field $R$. Let $l$ denote the level of $R(X)$ and $u := \dim_{\mathbb{Z}/2}(H^1(X))$.

(i) If $X(R) \neq \emptyset$ then $W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{u-s}$.

(ii) If $X(R) = \emptyset$ and $l = 2$ then $W(X) \simeq \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{u-1}$

(iii) If $X(R) = \emptyset$ and $l = 1$ then $W(X) \simeq (\mathbb{Z}/2)^{u+1}$.

Proof: Assume $X(R) \neq \emptyset$. We have $W(X) = W_t(X) \oplus \mathbb{Z}^s$ since $W_t(X) = I_t(X)$ is the kernel of the total signature homomorphism $\Lambda$. Since $I_t^2(X) = 0$, $I_t(X)$ is a group of exponent 2. By Theorem [21], $I_t(X) \simeq H^0_t(X, \mathcal{H}^1) = \ker(h_1)$. Since $h_1$ is surjective (Lemma [22]) and $H^1(X) \simeq H^0(X, \mathcal{H}^1)$, the proof is done.

Assume $X(R) = \emptyset$. Thus $W(X) = W_t(X)$ and $I_t^2(X) = 0$. Hence $W(X)$ is a group of exponent 4 and we have to determine $|W(X)|$ and $|2W(X)|$. By Theorem
the rank mod 2 homomorphism $e'_1 : W(X)/I(X) \to \mathbb{Z}/2$ is an isomorphism and also the discriminant homomorphism $e'_0 : W(X)/I(X) \to \mathbb{Z}/2$ is an isomorphism and also the discriminant homomorphism $e'_1 : W(X)/I(X) \to H^0(X, \mathcal{H}^1) \simeq H^1(X)$. Consequently $|W(X)| = |W(X)/I(X)||I(X)| = 2^{u+1}$. To determine $|2W(X)|$ we look at the following exact sequence:

$$0 \to N \to W(X)/I(X) \overset{<1>}{\to} 2W(X) \to 0$$

Since $W(R(X))/I(R(X)) = W(X)/I(X) = \mathbb{Z}/2$, the only non-zero element is the class of $<1>$. If $N \neq 0$ then $<1,1> = 0$ in $2W(X) \subseteq I(X)$. Since we have injections $I(X) \hookrightarrow I(R(X)), H^1(X) \hookrightarrow H^1(R(X))$ and $H^1(R(X)) \simeq R(X)/R(X)^{+2}$, we get $e_1(<1>) = -1 \in R(X)^{+2}$ i.e $l = 1$. Conversely, if $l = 1$ then $N \neq 0$. So $N \simeq \mathbb{Z}/2$. Consequently $2W(X) = 0$ and

$$W(X) \simeq (\mathbb{Z}/2)^{u+1}$$

If $N = 0$ i.e $l = 2$ then $|2W(X)| = |W(X)/I(X)| = 2$ and

$$W(X) \simeq \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{u-1}$$

We finally obtain the explicit calculation of the Witt group of a smooth connected real curve using results of the previous section.

**Theorem 2.11** Let $X$ be a smooth complete connected curve over a real closed field $R$.

(i) If $X(R) \neq \emptyset$ then $W(X) \simeq \mathbb{Z}^g \oplus (\mathbb{Z}/2)^g$.

(ii) If $X(R) = \emptyset$ and $X_C$ is connected then $W(X) \simeq \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g$

(iii) If $X(R) = \emptyset$ and $X_C$ is not connected then $W(X) \simeq (\mathbb{Z}/2)^{2g-1}$.

In the case $X_C$ is not connected, we obtain the same result as [Knu, cor. 2.1.7, p. 475] concerning the Witt group of a smooth projective curve over $\mathbb{C}$.

**Theorem 2.12** Let $X$ be a smooth connected curve over a real closed field $R$. Assume $X/R$ is not complete.

(i) If $X(R) \neq \emptyset$ then $W(X) \simeq \mathbb{Z}^g \oplus (\mathbb{Z}/2)^{g+c}$.

(ii) If $X(R) = \emptyset$ and $X_C$ is connected then $W(X) \simeq \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{g+c-1}$

(ii) If $X(R) = \emptyset$ and $X_C$ is not connected then $W(X) \simeq (\mathbb{Z}/2)^{2g-1}$.
3 Torsion Picard groups of curves

3.1 Torsion Picard groups of real curves

All along this section $X$ will be a smooth geometrically connected curve over a real closed field $R$. So $g$ will denote the genus of $\bar{X}_C$. We keep the notations of the previous section. If $X$ is complete, it was shown in [CT-S] using Roitman’s theorem and a trace argument that

$$\text{Pic}^\text{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^s-1$$

if $X(R) \neq \emptyset$ and

$$\text{Pic}^\text{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^g$$

if $X(R) = \emptyset$.

In this section we will assume that $X$ is not complete. By [CT-S, Th. 1.3],

$$\text{Pic}(X) \simeq D(X) \oplus (\mathbb{Z}/2)^l$$

where $D(X)$ is a divisible group. Recall that $Z := \bar{X} \setminus X$, and $Z$ consists of $r$ real points $\{P_1, \ldots, P_r\}$ and $c$ complex points $\{Q_1, \ldots, Q_c\}$. Let us denote $U_n(X) := \mathcal{O}(X)^*/\mathcal{O}(X)^n$ the group of units modulo $n$. Let $\text{Jac}(\bar{X})$ denote the Jacobian variety of $\bar{X}$, recall that we have an injective map

$$S : \text{Pic}^0(\bar{X}) \rightarrow \text{Jac}(\bar{X})(R)$$

which is surjective if $\bar{X}(R) \neq \emptyset$.

We will now associate an integer $\eta(X)$ to the curve $X$ as follows. We denote by $\text{Div}(X)$ (resp. $\text{Div}(\bar{X})$) the group of divisors on $X$ (resp. $\bar{X}$) which is the free abelian group on the closed points of $X$ (resp. $\bar{X}$). We denote by $\text{Div}_{\text{rat}}(X)$ (resp. $\text{Div}_{\text{rat}}(\bar{X})$) the subgroup of divisors rationally equivalent to 0 i.e the subgroup of principal divisors. We have well defined homomorphisms $\text{div} : R(X) \rightarrow \text{Div}_{\text{rat}}(X)$ and $\text{div} : R(\bar{X}) \rightarrow \text{Div}_{\text{rat}}(\bar{X})$. If $D = \sum P n_P$ is a divisor, $\text{supp}(D)$ is the set of all points $P$ with $n_P \neq 0$. Let $CH_0(Z)$ be the group of 0-cycles on $Z$ modulo rational equivalence, then clearly $CH_0(Z)$ is just the free abelian group on the closed points of $Z$. Let $A_0(Z)$ the free subgroup of $CH_0(Z)$ which consists of divisors of degree 0. Observe that for a complex point $Q$, the degree of the associated divisor $Q$ is 2. In the following we will also keep the same notations for $X_C, \bar{X}_C, Z_C := \bar{X}_C \setminus X_C$. Let $i : Z \hookrightarrow \bar{X}$ and $j : X \hookrightarrow \bar{X}$ be the inclusions. By [Fu, Prop. 1.8, Ch. 1], we have an exact sequence

$$CH_0(Z) \xrightarrow{i_*} \text{Pic}(\bar{X}) \xrightarrow{j^*} \text{Pic}(X) \rightarrow 0$$

Let $B(Z) := \text{Ker}(i_*)$ then $B(Z)$ is a free subgroup of $A_0(Z)$ of rank $m$. We set

$$\eta(X) := m$$

We may see that $\eta(X) \leq r+c-1$. Let $D_1, \ldots, D_{\eta(X)}$ be a basis of $B(Z)$. Since $i_*(D_j) = 0$ in $\text{Pic}^0(\bar{X})$, then $i_*(D_j) = \text{div}(f_j)$ with $f_j \in R(\bar{X}) = R(X)$ and by the following lemma $f_j \in \mathcal{O}(X)^*$. 

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Lemma 3.1 The following sequence

\[ 0 \to \mathcal{O}(X)^* \to R(X)^* \overset{\text{div}}{\to} \text{Div}_{\text{rat}}(X) \]

is exact i.e \( f \in R(X)^* \) lies in \( \mathcal{O}(X)^* \) if and only if \( \text{supp}(\text{div}(f)) \subseteq \mathbb{Z} \).

Proof: Let \( G = \text{Gal}(C/R) = \{1, \sigma\} \). We have a short exact sequence

\[ 0 \to \mathcal{O}(X_C)^* \to C(X)^* \overset{\text{div}}{\to} \text{Div}_{\text{rat}}(X_C) \to 0 \]

which induces a long exact sequence in Galois cohomology

\[ 0 \to (\mathcal{O}(X_C)^*)^G \to (C(X)^*)^G \overset{\text{div}}{\to} (\text{Div}_{\text{rat}}(X_C))^G \to H^1(G, \mathcal{O}(X_C)^*) \to \ldots \]

Since \( (C(X)^*)^G = R(X)^* \) we get \( (\mathcal{O}(X_C)^*)^G = \mathcal{O}(X)^* \). Moreover, \( \pi : X_C \to X \) induces a flat pull-back (see [Fu]) \( \pi^* : \text{Div}(X) \to \text{Div}(X_C) \) which is an injection and respects rational equivalence. We obtain an injection \( \pi^* : \text{Div}_{\text{rat}}(X) \to (\text{Div}_{\text{rat}}(X_C))^G \). The statement follows now easily. \( \Box \)

Let \( \{f_1, \ldots, f_\eta(X)\} \) be the set of units in \( \mathcal{O}(X) \) which we create with \( D_1, \ldots, D_{\eta(X)} \).

In fact, we may complete the previous exact sequence in the following way:

\[ 0 \to R^* = \mathcal{O}(\tilde{X})^* \to \mathcal{O}(X)^* \overset{\varphi}{\to} \text{CH}_0(Z) \overset{i}{\to} \text{Pic}(\tilde{X}) \overset{j^*}{\to} \text{Pic}(X) \to 0 \]

where \( \varphi \) is the composition of \( \mathcal{O}(X)^* \hookrightarrow R(X) = R(\tilde{X}) \) and \( R(\tilde{X}) \overset{\text{div}}{\to} \text{Div}(\tilde{X}) \).

Proposition 3.2 With the previous notations and \( n > 0 \), we have

(i) \( U_n(X) \simeq (\mathbb{Z}/n)^{\eta(X)} \oplus (\mathbb{Z}/2) \) if \( n \) is even.

(ii) \( U_n(X) \simeq (\mathbb{Z}/n)^{\eta(X)} \) if \( n \) is odd.

Proof: We set \( m := \eta(X) \). We fix a basis \( D_1, \ldots, D_m \) of \( B(Z) \) and we get the associated \( f_i \in \mathcal{O}(X)^* \) for \( i = 1, \ldots, m \). We first claim that any \( f \in \mathcal{O}(X)^* \) can be written uniquely as a product \( f = af_1^{n_1} \cdots f_m^{n_m} \) with \( n_j \in \mathbb{Z} \) and \( a \in R^* \). We look at \( f \) as a rational function on \( \tilde{X} \), then \( \text{div}(f) = D \in \text{Div}(\tilde{X}) \) is an integral combination of points in \( \{P_1, \ldots, P_r, Q_1, \ldots, Q_c\} \) and the degree of \( D \) is zero. The divisor \( D \) is in fact in \( \text{CH}_0(Z) \) and clearly \( i_*(D) = 0 \). Hence \( D \) can be written uniquely as an integral combination of \( D_1, \ldots, D_m \). Thus we have the claim.

The classes of \( f_1, \ldots, f_m \) (and the constant function \(-1\) if \( n \) is even) in \( U_n(X) \) generate this group. Since \( D_1, \ldots, D_m \) is a basis of \( B(Z) \), we may prove that the order of \( f_1, \ldots, f_m \) is exactly \( n \) in \( U_n(X) \) and \( m \) (resp. \( m + 1 \)) is the the minimal number of generators in \( U_n(X) \) if \( n \) is odd (resp. \( n \) is even). The statement follows now easily. \( \Box \)

We are now able to calculate the torsion Picard group of a non complete curve.
**Theorem 3.3** Let $X$ be a smooth geometrically connected curve over a real closed field $R$. Assume $X/R$ is not complete. Then

$$\text{Pic}_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{g+r+c-\eta(X)-1} \oplus (\mathbb{Z}/2)^t$$

with $t = 0, r = 0$ if $X(R) = \emptyset$.

**Proof:**

We have $\text{Pic}_{\text{tors}}(X) \simeq D(X)_{\text{tors}} \oplus (\mathbb{Z}/2)^t$ and for any $n > 0$ an exact sequence

$$0 \to U_n(X) \to H^{1}_{\text{et}}(X, \mu_n) \to \text{Pic}_n(X) \to 0 \quad (3)$$

By [51, p. 226] $H^{1}_{\text{et}}(X, \mu_n) \simeq (\mathbb{Z}/n)^{g+c+r-1} \oplus (\mathbb{Z}/2)^{t+1}$, if $n$ is odd one has to drop all summands $\mathbb{Z}/2$ (For $n = 2$ it gives the result of Theorem 2.3). For $n > 1$, we denote by $D_n(X)$ the $n$-torsion subgroup of $D(X)$. Using Proposition 3.2 and the exact sequence (3), the sequence

$$0 \to (\mathbb{Z}/n)^{\eta(X)} \oplus (\mathbb{Z}/2) \to (\mathbb{Z}/n)^{g+c+r-1} \oplus (\mathbb{Z}/2)^{t+1} \to D_n(X) \oplus (\mathbb{Z}/2)^t \to 0$$

is exact if $n$ is even, and

$$0 \to (\mathbb{Z}/n)^{\eta(X)} \to (\mathbb{Z}/n)^{g+c+r-1} \to D_n(X) \to 0$$

is exact if $n$ is odd. Then for any prime number $p > 1$ we get

$$D_p(X) \simeq (\mathbb{Z}/p)^{g+c+r-\eta(X)-1}$$

By a structure theorem on divisible groups $D(X)_{\text{tors}}$ is a direct sum of some quasicyclic $p$-groups $\mathbb{Z}(p^\infty)$ for some primes $p$. The group $\mathbb{Z}(p^\infty)$ could be seen as the $p$-primary component of $\mathbb{Q}/\mathbb{Z}$. The result on the $p$-torsion part of $D(X)$ implies that, for any prime $p$, we have exactly $g+c+r-\eta(X)$ copies of $\mathbb{Z}(p^\infty)$ in the decomposition of $D(X)_{\text{tors}}$ as a direct sum of quasicyclic groups. Thus the proof is done. \hfill \Box

### 3.2 Torsion Picard group of complex curves

All along this section $R$ will be a real closed field and $C = R(\sqrt{-1})$.

Let $Y$ be a smooth connected curve over $C$. If $Y$ is complete then $\text{Pic}_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{2g}$, so we will assume that $Y$ is not complete. Let $\tilde{Y}$ denote the smooth completion of $Y$ and assume that $\tilde{Y} \setminus Y$ consists on $k$ closed points. In this section, using Lemma 1.3, we deal with the case of smooth real curves of level 1. We keep the notations we used for real curves, in particular for $\eta(Y), U_n(Y)$ and $g$ will denote the genus of $\tilde{Y}$.

**Proposition 3.4** For $Y$ satisfying the previous conditions,

$$\text{Pic}_{\text{tors}}(Y) \simeq (\mathbb{Q}/\mathbb{Z})^{2g+k-\eta(Y)-1}$$
Proof: By a topological computation, we may prove that \( H^1(Y(C), \mathbb{Z}) = \mathbb{Z}^{2g+k-1} \) and then \( H^1(Y(C), \mathbb{Z}/n) = (\mathbb{Z}/n)^{2g+k-1} \) for any \( n > 0 \). By a comparison Theorem of Huber [Hub] and since \( C \) is algebraically closed,

\[
H^1_{et}(Y, \mu_n) = (\mathbb{Z}/n)^{2g+k-1}
\]

for any \( n > 0 \). Arguing as for real curves, for any \( n > 0 \), \( U_n(Y) \simeq (\mathbb{Z}/n)^{\eta(Y)} \), thus we get an exact sequence

\[
0 \to (\mathbb{Z}/n)^{\eta(Y)} \to (\mathbb{Z}/n)^{2g+k-1} \to \text{Pic}_n(Y) \to 0
\]

If \( B \) is an abelian group, \( \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z}) \) is the Pontrjagin dual of \( B \). Suppose that \( B \) is finite, then \( B \simeq \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \). According to the above remark, we get another exact sequence

\[
0 \to \text{Pic}_n(Y) \to (\mathbb{Z}/n)^{2g+k-1} \to (\mathbb{Z}/n)^{\eta(Y)} \to 0
\]

which is a split exact sequence of \( \mathbb{Z}/n \)-modules. Hence

\[
\text{Pic}_n(Y) \simeq (\mathbb{Z}/n)^{2g+k-\eta(Y)-1}
\]

for any \( n > 0 \). Since \( \text{Pic}(Y) \) is a divisible group, using a structure theorem on divisible groups, the statement follows. \( \square \)

3.3 Relations between \( \eta(X), \eta(X_C) \) and the level of curves

Let \( X \) be a smooth geometrically connected curve over a real closed field \( R \). We assume that \( X \) is not complete. We denote by \( Z_C = \bar{X}_C \setminus X_C \), \( Z = \bar{X} \setminus X \), \( B(Z) \) the free group \( \text{Ker}(\text{CH}_0(Z) \to \text{Pic}(\bar{X})) \), \( B(Z_C) \) the free group \( \text{Ker}(\text{CH}_0(Z_C) \to \text{Pic}(\bar{X}_C)) \). We recall that \( \eta(X) \) (resp. \( \eta(X_C) \)) is the rank of \( B(Z) \) (resp. \( B(Z_C) \)).

We recall that have an injection \( \pi^*: \text{Div}(\bar{X}) \to \text{Div}(\bar{X}_C) \) (see the proof of Lemma 3.1): For \( D \in \text{Div}(\bar{X}) \) we replace a complex point \( Q \in \text{Supp}(D) \) by the sum \( Q' + Q' \) where \( Q', Q' \) are the two complex points of \( \bar{X}_C \) lying over \( Q \).

**Proposition 3.5** We always have \( \eta(X_C) \geq \eta(X) \), moreover \( 0 \leq \eta(X) \leq c + r - 1 \) and \( 0 \leq \eta(X_C) \leq r + 2c - 1 \). If \( c = 0 \) (i.e. we have only real points at “infinity”) then \( \eta(X) = \eta(X_C) \).

**Proof:** The first assertion is clear. If \( c = 0 \) then \( r \geq 1 \) since \( X \) is not complete. Since \( \text{Pic}(\bar{X}) \) injects into \( \text{Pic}(\bar{X}_C)^G \), the class of a divisor \( D \in \text{Div}(\bar{X}) \subseteq \text{Div}(\bar{X}_C) \) is zero in \( \text{Pic}(\bar{X}) \) if only if its class is zero in \( \text{Pic}(\bar{X}_C) \). It is the case, in particular, if \( \text{supp}(D) \) is contained in \( \{P_1, \ldots, P_r\} = X \setminus X = X_C \setminus X_C \), which gives the proof. \( \square \)

**Remark 3.6** If \( r + c = 1 \) then trivially \( \eta(X) = 0 \).
Let $G = Gal(C/R) = \{1, \sigma\}$. We denote by $B(Z_c)^-$ (resp. $B(Z')^+$) the subgroup of $B(Z_c)$ which elements are anti-invariant (resp. invariant) by $\sigma$ i.e. $B(Z_c)^- = Ker(B(Z_c) \overset{1\to}{\longrightarrow} B(Z_c))$ (resp. $B(Z_c)^+ = Ker(B(Z_c) \overset{1\to}{\longrightarrow} B(Z_c))$). Then $B(Z_c)^-$ (resp. $B(Z_c)^+$) is a free group and we denote by $\eta^-(X_c)$ (resp. $\eta^+(X_c)$) its rank. Clearly one gets $B(Z_c)^+ = B(Z)$ and $\eta^+(X_c) = \eta(X)$. Moreover, $B(Z_c)$ could be seen as a subgroup of $\text{Div}_{rat}(\bar{X}_c)$ and $B(Z_c)^-$ is a subgroup of $\text{Div}_{rat}(\bar{X}_c)^- = Ker(\text{Div}_{rat}(\bar{X}_c) \overset{1\to}{\longrightarrow} \text{Div}_{rat}(\bar{X}_c))$. Then we obtain a well defined map

$$\phi : B(Z_c)^- \to H^1(G, \text{Div}_{rat}(\bar{X}_c))$$

which to $D$ associates the class of $D \in \text{Div}_{rat}(\bar{X}_c)$ in the Galois cohomology group $H^1(G, \text{Div}_{rat}(\bar{X}_c))$.

If $X(R) = \emptyset$, the level of $X$ is 2 or 3. We reformulate the result of [Hu-Ma] in terms of our new invariants.

**Proposition 3.7** Assume $X(R) = \emptyset$. If the level of $X$ is 2 then $\eta(X_c) \geq \eta^-(X_c) > 0$ and $\eta(X_c) > \eta(X)$. More precisely the level of $X$ is 2 if and only if the map $\phi$ is non zero.

**Proof:** The level of $X$ is 2 if and only there exists $f \in \mathcal{O}(X_c)^*$ such that $N(f) = f \bar{f} = -1$. Assume such $f$ exists and let $D := \text{Div}(f) \in \text{Div}(\bar{X}_c)$. Clearly $D \in B(Z_c)$ and since $D + \bar{D} = 0$ one gets $D \in B(Z_c)^-$. This proves that $\eta^-(X_c) > 0$.

Conversely assume $D$ lies in $B(Z_c)^-$. It corresponds to the divisor of a non trivial element $f_D$ in $\mathcal{O}(X_c)^* \subseteq C(X)$. Moreover $\text{div}(f_D \bar{f_D}) = D + \bar{D} = 0$, so $N(f_D) \in R^*$ and we may assume that it is 1 or $-1$. We have shown that there is a one-to-one mapping between $B(Z_c)^-$ and the set of functions $f \in \mathcal{O}(X_c)^*$ such that $N(f) = f \bar{f} = \pm 1$ modulo $\{z \in C|zz = 1\}$. Hence the level of $X$ is 2 if and only if there exists $D \in B(Z_c)^-$ such that $N(f_D) = -1$. Since $H^1(G, C(X)^*) = 0$ by Hilbert’s Theorem 90, $N(f_D) = 1$ if only if $f_D = g/\bar{g}$ for $g \in C(X)^*$ if only if $\phi(D) = 0$.

We assume now that the level of $X$ is 2 and $\eta(X) = \eta(X_c)$. Since $\eta(X) = \eta^+(X_c)$, it means that every $D \in B(Z_c)^-$ is invariant by $\sigma$ since it can be written as an integral combination of elements in $B(Z_c)^+$. But then $B(Z_c)^-$ is trivial and $\eta^-(X_c) = 0$, contradiction. \[\Box\]

**Remark 3.8** If $X$ has only one complex point at infinity and $X(R) = \emptyset$, such curves are called maximal in [Hu-Ma]. Then $Z_c = \{Q, \bar{Q}\}$ and $\eta(X_c) = \eta^-(X_c) = 1$ if and only the class of $n(Q - \bar{Q})$ is zero in $\text{Pic}^0(X_c)$ for a $n \in \mathbb{N} \setminus \{0\}$. We could see in [Hu-Ma] that there exists maximal curves with $\eta(X_c) = 1$ but the level of $X$ is not 2.

### 3.4 Pythagoras number of curves

We always assume that $X$ is not complete. For a commutative ring with units $A$, we denote by $\sum A^2$ (resp. $\sum_{i=1}^n A^2$) the set of sums of squares (resp. $n$ squares) in $A$, $p(A) = \inf\{n \in \mathbb{N} | \sum A^2 = \sum_{i=1}^n A^2\}$ or $\infty$ the classical Pythagoras number of $A$ and $p_\ast(A) = \inf\{n \in \mathbb{N} | A^\ast \cap \sum A^2 = \sum_{i=1}^n A^2\}$ or $\infty$. 

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Assume \( X(R) = \emptyset \). Let \( l := \text{level}(X) \). Since \( l \) is finite and every element in \( \mathcal{O}(X) \) can be written as a difference of two squares and is finally a sum of squares, we get:

\[
2 \leq l \leq p_*(\mathcal{O}(X)) \leq p(\mathcal{O}(X)) \leq l + 1 \leq 4
\]

Now we would like to know some conditions for which \( l = p_*(\mathcal{O}(X)) = 2 \)

Remark that for the field \( R(X) \) the equality holds by a result of Pfister.

**Proposition 3.9** Assume \( X(R) = \emptyset \). We have \( l = p_*(\mathcal{O}(X)) = 2 \) if and only if the map

\[
\pi^* : \text{Pic}(X) \to (\text{Pic}(X_C))^G
\]

is surjective.

**Proof:** Consider the short exact sequence

\[
0 \to \mathcal{O}(X_C)^* \to C(X)^* \xrightarrow{\text{div}} \text{Div}_{\text{rat}}(X_C) \to 0
\]

which induces a long exact sequence

\[
\ldots H^1(G, C(X)^*) \to H^1(G, \text{Div}_{\text{rat}}(X_C)) \to H^2(G, \mathcal{O}(X_C)^*) \to H^2(G, C(X)^*)
\]

Now \( H^1(G, C(X)^*) = 0 \) by Hilbert’s Theorem 90 and one gets an exact sequence

\[
0 \to H^1(G, \text{Div}_{\text{rat}}(X_C)) \xrightarrow{\alpha} \mathcal{O}(X)^*/N(\mathcal{O}(X_C)^*) \xrightarrow{\beta} R(X)^*/N(C(X)^*)
\]

with \( N \) denoting the norm. We have to understand the boundary map \( \alpha \). An element of \( H^1(G, \text{Div}_{\text{rat}}(X_C)) \) could be seen as the class of \( D = \text{div}(f_D) \) (denoted by \([D]\)) with \( f_D \in C(X)^* \) and \( \bar{D} = -D \). Then \( \alpha([D]) \) corresponds to the class of \( N(f_D) \in \mathcal{O}(X)^* \) modulo \( N(\mathcal{O}(X_C)^*) \). We could see that \( \alpha \) is well defined.

By a well known Theorem of Pfister, the Pfister form \(< < 1 >>\) is universal over \( R(X) \), hence \( R(X)^*/N(C(X)^*) = 0 \) and \( \alpha \) is an isomorphism.

We claim now that \( H^1(G, \text{Div}_{\text{rat}}(X_C)) \) is the cokernel of \( \pi^* : \text{Pic}(X) \to (\text{Pic}(X_C))^G \) (we don’t have to assume that \( X(R) = \emptyset \)), and the proof will be done. The short exact sequence of \( G \)-modules

\[
0 \to \text{Div}_{\text{rat}}(X_C) \to \text{Div}(X_C) \to \text{Pic}(X_C) \to 0
\]

gives a long exact sequence

\[
\ldots \to \text{Div}(X) \to (\text{Pic}(X_C))^G \to H^1(G, \text{Div}_{\text{rat}}(X_C)) \to H^1(G, \text{Div}(X_C)) \to \ldots
\]

Since \( H^1(G, \text{Div}(X_C)) = 0 \), the statement follows easily. \( \square \)
Example 3.10 Let $X$ be the affine plane curve given by the equation $x^2 + y^2 + 1 = 0$. Then $p_*(\mathcal{O}(X)) = 2$ since $p(\mathcal{O}(X)) = 2$ [CDLR, Th. 3.7]. Therefore $\pi^*$ is surjective.

Proposition 3.11 We assume $X(R) \neq \emptyset$. We have $p_*(\mathcal{O}(X)) = 2$ if $\pi^* : Pic(X) \rightarrow (Pic(X_\eta))^G$ is surjective. Conversely $\pi^*$ is surjective if $p_*(\mathcal{O}(X)) = 2$ and every $f \in \mathcal{O}(X)^*$ which is positive on $X(R)$ is a sum of squares in $\mathcal{O}(X)$.

Proof: The proof is straightforward using the exact sequence

$$0 \rightarrow H^1(G, Div_{rat}(X_\eta)) \rightarrow \mathcal{O}(X)^*/N(\mathcal{O}(X_\eta)^*) \rightarrow R(X)^*/N(C(X)^*)$$

since every positive function $f \in R(X)$ on $Spec, R(X)$ is a sum of two squares in $R(X)$.

Example 3.12 Let $X$ be the affine plane curve given by the equation $x^2 + y^2 - 1 = 0$. Since $p(\mathcal{O}(X)) = 2$ [CDLR, Th. 3.7], it follows that $p_*(\mathcal{O}(X)) = 2$. Moreover every positive function in $\mathcal{O}(X)$ is a sum of squares [S2, Prop. 2.17], hence $\pi^*$ is surjective.

4 Witt groups and torsion Picard groups of smooth affine plane curves

4.1 Conics

In order to have an easy application of the previous results, we calculate the Witt group and the torsion Picard group of a smooth curve in the real plane given by the zero set of $P(x, y) \in \mathbb{R}[x, y]$ with $deg(P) \leq 2$. In any case we have $g = 0$.

Up to an isomorphism over $\mathbb{R}$, we are reduced to deal with the following cases:

1) The case of an ellipse: $P(x, y) = x^2 + y^2 - 1$. Then $r = 0$, $c = 1$, $\eta(X) = 0$, $s = t = 1$. We have $W(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ and $Pic_{tors}(X) \simeq \mathbb{Z}/2$ (compare with [Ay-Oj]).

2) The case of a parabola: $P(x, y) = x^2 + y$. Then $r = 1$, $c = 0$, $\eta(X) = 0$, $s = 1$, $t = 0$. We have $W(X) \simeq \mathbb{Z}$ and $Pic_{tors}(X) \simeq 0$.

3) The case of an hyperbola: $P(x, y) = x^2 - y^2 - 1$. Then $r = 2$, $c = 0$, $s = 2$, $t = 0$, $\eta(X) = 1$ since the function $f = (x - y) \in \mathcal{O}(X)^*$ and the divisor of the corresponding rational function $f = (X - Y)/Z \in \mathbb{R}(X)$ has a support contained in the set of the two infinity points where $X$ is the smooth projective curve given by the equation $X^2 - Y^2 - Z^2 = 0$. We have $W(X) \simeq \mathbb{Z}$ and $Pic_{tors}(X) \simeq 0$ (compare with [Kni]).

4) The case of an imaginary ellipse: $P(x, y) = x^2 + y^2 + 1$. Then $r = 0$, $c = 1$, $\eta(X) = 0$, $s = t = 0$. We have $W(X) \simeq \mathbb{Z}/4$ and $Pic_{tors}(X) \simeq 0$.

5) The case of a line: $P(x, y) = x$. Then $r = 1$, $c = 0$, $\eta(X) = 0$, $s = 1$. We have $W(X) \simeq \mathbb{Z}$ and $Pic_{tors}(X) \simeq 0$.

6) The non geometrically connected case: $P(x, y) = x^2 + 1$. Then $r = 0$, $c = 1$ after a blowing-up at infinity of the singular projective curve $X^2 + Z^2 = 0$, $s = t = 0$. We have $W(X) \simeq \mathbb{Z}/2$ and $Pic_{tors}(X) \simeq 0$. 

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4.2 Hyperelliptic curves

We study here smooth geometrically connected affine curves in the real plane given by an equation $y^2 + P(x) = 0$ with $P(x) \in \mathbb{R}[X]$ non constant and square free ($X$ is smooth). Up to an isomorphism we are reduced to curves with equations $y^2 + P(x) = 0$ and $y^2 - P(x) = 0$ with $P(x)$ monic.

4.2.1 Curves with equations $y^2 + P(x) = 0$, $P$ monic

Let $X$ be the plane curve with equation $y^2 + P(x) = 0$. We denote by $d$ the degree of $P$ and $k$ the number of real roots of $P$.

First assume that $d$ is even, $d := 2d'$. After some blowings-up at infinity of the singular projective curve associated to $X$, we see that $\overline{X} \setminus X$ consists on one complex point, it means that $r = 0$, $c = 1$, $\eta(X) = 0$. Moreover by Hurwitz formula $g = d' - 1$. The number of real roots of $P$ is even ($k = 2k'$) and $s = t = k'$. Then

\[
W(X) \simeq \mathbb{Z}^{k'} \oplus (\mathbb{Z}/2)^{d'} \text{ if } k > 0
\]

\[
W(X) \simeq \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{d'-1} \text{ if } k = 0
\]

Moreover

\[
Pic_{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{d'-1} \oplus (\mathbb{Z}/2)^{k'}
\]

Assume $d$ is odd, $d := 2d' + 1$. Thus $\overline{X} \setminus X$ is a real point, we have $r = 1$, $c = 0$, $\eta(X) = 0$. Moreover $g = d'$ and $k := 2k' + 1$ is odd. We have $t = k'$ and $s = k' + 1$. Then

\[
W(X) \simeq \mathbb{Z}^{k'+1} \oplus (\mathbb{Z}/2)^{d'}
\]

and

\[
Pic_{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{d'} \oplus (\mathbb{Z}/2)^{k'}
\]

4.2.2 Curves with equations $y^2 - P(x) = 0$, $P$ monic

Let $X$ be the plane curve with equation $y^2 - P(x) = 0$. We denote by $d$ the degree of $P$ and $k$ the number of real roots of $P$.

First assume that $d$ is even, $d := 2d'$. We have 2 real points in $\overline{X} \setminus X$, hence $r = 2$, $c = 0$, $\eta(X) = 0$ or 1. We also get $g = d' - 1$, $k = 2k'$. If $k > 0$, then $s = k' + 1$, $t = k' - 1$. If $k = 0$ then $t = 0$, $s = 2$. Then

\[
W(X) \simeq \mathbb{Z}^{k'+1} \oplus (\mathbb{Z}/2)^{d'-1} \text{ if } k > 0
\]

\[
W(X) \simeq \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{d'-1} \text{ if } k = 0
\]

Moreover

\[
Pic_{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{d'-\eta(X)} \oplus (\mathbb{Z}/2)^{k'-1} \text{ if } k > 0
\]

and

\[
Pic_{tors}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{d'-\eta(X)} \text{ if } k = 0
\]
Assume $d := 2d' + 1$ is odd. Then $r = 1$, $c = 0$, $\eta(X) = 0$, $k = 2k' + 1$, $g = d'$, $s = k' + 1$, $t = k'$. We get

$$W(X) \simeq \mathbb{Z}^{k' + 1} \oplus \mathbb{Z}/2^{d'}$$

and

$$\text{Pic}_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{d'} \oplus (\mathbb{Z}/2)^{k'}$$

### 4.2.3 Some remarks

Let $X$ be the affine curve with equation $y^2 - P(x)$, with $P$ monic, and $X'$ the curve with equation $y^2 + P(x)$.

If the degree $d$ of $P$ is odd, then we remark that we have obtained the same results for $X$ and $X'$. This is not surprising: by the isomorphism over $\mathbb{R}$ $(x, y) \mapsto (-x, y)$, $X$ is isomorphic to the curve with equation $y^2 + Q(x)$ with $Q(x) = -P(-x)$ monic and $Q$ has exactly the same degree and the same number of real roots than $P$.

In the remainder of this section we assume that the degree of $P$ is even, $d = 2d' = 2g + 2$. We would like to know when $\eta(X) = 0$ or $1$. We have $X \setminus X = \{P_1, P_2\} = \bar{X}_C \setminus X_C$ and $\bar{X}_C \setminus X_C' = \{Q, \bar{Q}\}$ with $P_1, P_2$ real points and $Q$ a complex point.

**Proposition 4.1** Under the conditions stated above, we have $\eta(X) = \eta(X_C) = \eta(X'_C)$.

**Proof:** We have $\eta(X) = \eta(X_C)$ by Proposition 3.3. The curves $X'_C$ and $X_C$ are isomorphic over $\mathbb{C}$ by $f : (x, y) \mapsto (x, iy)$. Hence $\text{Pic}(X'_C) \simeq \text{Pic}(X'_C)$. Since they have the same number of points at infinity, Proposition 3.4 shows that $\eta(X'_C) = \eta(X_C)$. $\square$

Consequently $\eta(X) = 1$ if and only if $P_1 - P_2$ is a torsion point in the jacobian $\text{Jac}(\bar{X})(\mathbb{R}) = (\text{Jac}(X_C)(\mathbb{C}))^G$ if and only if $Q - \bar{Q}$ is a torsion point in $\text{Jac}(X_C)(\mathbb{C})$. More precisely, if these points are torsion points in their Jacobian, they should have the same order since the isomorphism $f : X'_C \to X'_C; (x, y) \mapsto (x, iy)$ induces an isomorphism between $\text{Jac}(X'_C)$ and $\text{Jac}(X_C)$ and the image of $Q - \bar{Q}$ is $P_1 - P_2$ interchanging $P_1$ and $P_2$ if necessary ([Hu-Ma, Lem. 2.5]).

We parametrize the set of genus $g$ hyperelliptic affine curves like $X$ by $\mathbb{R}^{2g+2}$ (the set of coefficients of $P$). Let $M_{g,1}$ be the subset of curves with $\eta(X) = 1$. The following proposition may be proved in much the same ways as [Hu-Ma, Lem. 2.4, Prop. 2.6].

**Proposition 4.2** The set $M_{g,1}$ has measure 0 in $\mathbb{R}^{2g+2}$.

We will see that $M_{1,1} \neq \emptyset$ in the next section.

### 4.2.4 Quartics

Let $X$ be the smooth geometrically connected affine curve of equation $y^2 - P(x) = 0$ with $P$ monic of degree 4. We know that $g = 1$ and we want to know when $\eta(X) = 0$ or $1$. Let $k = 2k'$ be the number of real roots of $P$ and we set $\bar{X} \setminus X = \{P_1, P_2\}$ as previously. Then we distinguish 3 cases: $k = 0$, $k = 2$, $k = 4$. In this section, we use ideas of [Hu-Ma] that we adapt to our problem.
Assume $k = 0$. Then $X(\mathbb{R})$ has two connected components and also $\bar{X}(\mathbb{R})$; moreover $P_1$ and $P_2$ are in two different connected components of $X(\mathbb{R})$. Therefore $Jac(X)(\mathbb{R}) \simeq (\mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}/2$ and $p = P_1 - P_2$ is in the non neutral component of $Jac(\bar{X})(\mathbb{R})$ by \cite[Lem. 2.6]{[S2]}. Consequently $\eta(X) = 1$ if and only if $p$ is a torsion point of even order in $Jac(\bar{X})(\mathbb{R})$. We make an explicit calculation. Up to an isomorphism we may assume that $P(x) = ((x+b)^2 + a^2)((x+b)^2 + c^2)$ with $a, b, c \in \mathbb{R}$ and $a, c > 0$. We get a Weierstrass equation $u^2 = (v + 4b^2)(v - (c + a)^2)$ Then $p$ has coordinates $(0, 2b(c^2 - a^2))$ and we have 3 points of order 2: $p_1 = (-4b^2, 0), p_2 = ((c - a)^2, 0), p_3 = ((a + c)^2, 0)$. We see that $p, p_1, p_2$ are on the non neutral component. Thus

**Proposition 4.3** Under the above conditions:

$$Pic_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})$$

if and only if $(2n - 1)p = p_1$ or $(2n - 1)p = p_2$ or $2np = p_3$ for $n \in \mathbb{N} \setminus \{0\}$. Else

$$Pic_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^2$$

When $\eta(X) = 1$, using the duplication formula, each case $(2n - 1)p = p_1$ or $(2n - 1)p = p_2$ or $2np = p_3$ is equivalent to a polynomial equation in $a, b, c$ defined over $\mathbb{Q}$. For example, $p = p_1$ if and only if $b = 0, p = p_2$ if and only if $c = a$ and $2p = p_3$ if and only if $-16b^2ac + c^4 - 2a^2c^2 + a^4 = 0$.

Assume $k = 2$. Then $X(\mathbb{R})$ has two connected components but $\bar{X}(\mathbb{R})$ has only one. We get $Jac(\bar{X})(\mathbb{R}) \simeq \mathbb{R}/\mathbb{Z}$ and $p = P_1 - P_2$ is in the neutral component of $Jac(\bar{X})(\mathbb{R})$. Up to an isomorphism we may assume that $P(x) = ((x+b)^2 + a^2)((x-b)^2 - c^2)$ with $a, b, c \in \mathbb{R}$ and $a, c > 0$. We get a Weierstrass equation $u^2 = (v + 4b^2)(v^2 - 2(c^2 - a^2)v + (c^2 + a^2)^2)$ Then $p$ has coordinates $(0, 2b(c^2 + a^2))$ and we have only 1 point of order 2: $p_1 = (-4b^2, 0)$.

**Proposition 4.4** Under the above conditions:

$$Pic_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})$$

if and only if $np = p_1$ or $2np = -p$ for $n \in \mathbb{N} \setminus \{0\}$. Else

$$Pic_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^2$$

For example $p = p_1$ if and only if $b = 0, 2p = p_1$ if and only if

$$-64c^4b^4 + 128a^2b^4c^2 - 64a^4b^4 - a^8 - c^8 + 16b^2c^6 - 16b^2a^6 - 4a^6c^2 - 6a^4c^4 - 4a^2c^6 + 16b^2c^4a^2 - 16b^2a^4c^2 = 0$$

and $2p = -p$ if and only if

$$a^8 + c^8 - 16b^2c^6 + 16b^2a^6 + 4a^6c^2 + 6a^4c^4 + 4a^2c^6 - 16b^2c^4a^2 + 16b^2a^4c^2 - 256a^2b^4c^2 = 0$$
Assume \( k = 4 \). Then \( X(\mathbb{R}) \) has 3 connected components but \( \tilde{X}(\mathbb{R}) \) has only 2. We get \( \text{Jac}(\tilde{X})(\mathbb{R}) \simeq (\mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}/2 \) and \( p = P_1 - P_2 \) is in the neutral component of \( \text{Jac}(\tilde{X})(\mathbb{R}) \) since \( P_1 \) and \( P_2 \) lie to the same connected component of \( \tilde{X}(\mathbb{R}) \). Up to an isomorphism we may assume that \( P(x) = ((x+b)^2 - a^2)((x-b)^2 - c^2) \) with \( a, b, c \in \mathbb{R} \) and \( a, c > 0 \). We get a Weierstrass equation \( u^2 = (v + 4b^2)(v + (c-a)^2)(v + (c+a)^2) \) Then \( p \) has coordinates \((0, 2b(c^2 - a^2))\) and we have 3 points of order 2: \( p_1 = (-4b^2, 0), p_2 = (-c^2, 0), p_3 = -(c-a)^2, 0) \). We see that \( p, p_3 \) are on the neutral component if \( 4b^2 > (c-a)^2 \) and \( p, p_2 \) are on the neutral component if \( 4b^2 < (c-a)^2 \). We have to remark that \( 2b \neq \pm(c-a) \) since the discriminant of \( P \) is \( 0 \) (the curve is smooth).

**Proposition 4.5** Under the above conditions:

\[
\text{Pic}_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/2)
\]

if and only \( np = p_3 \) or \( 2np = -p \) in the case \( 4b^2 > (c-a)^2 \), and \( np = p_1 \) or \( 2np = -p \) in the case \( 4b^2 < (c-a)^2 \), for a \( n \in \mathbb{N} \setminus \{0\} \). Else

\[
\text{Pic}_{\text{tors}}(X) \simeq (\mathbb{Q}/\mathbb{Z})^2 \oplus (\mathbb{Z}/2)
\]

Assume \( 4b^2 > (c-a)^2 \), then for example, \( p = p_3 \) if and only if \( a = c, 2p = -p \) if and only if

\[
16c^4b^2a^2 + 16c^2a^4b^2 + 256b^4c^2a^2 + c^8 + a^8
\]

\[
-4c^6a^2 + 6c^4a^4 - 16c^6b^2 - 4c^2a^6 - 16a^6b^2 = 0
\]

Assume \( 4b^2 > (c-a)^2 \) then \( p = p_1 \) if and only if \( b = 0 \).

Following [Hu-Ma], if we assume that \( P \) admits a factorization \( P(x) = ((x + b)^2 + a^2)((x - b)^2 + c^2) \) in the case \( k = 0 \), \( P(x) = ((x + b)^2 + a^2)((x - b)^2 - c^2) \) in the case \( k = 2 \), \( P(x) = ((x + b)^2 - a^2)((x - b)^2 - c^2) \) in the case \( k = 4 \); with \( a, b, c \in \mathbb{Q} \) and \( a, c > 0 \). Then \( p \) and all the points denoted by \( p_1, p_2, p_3 \) are rational points. Now we used a famous theorem of Mazur which asserts that the torsion subgroup of \( \text{Jac}(\tilde{X})(\mathbb{Q}) \) is either \( \mathbb{Z}/n\mathbb{Z} \) for \( n = 1, 2, 3, \ldots, 10, 12 \) or \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) for \( n = 2, 4, 6, 8 \). We see that if \( k = 0 \) or \( k = 4 \), \( \text{Jac}(\tilde{X})(\mathbb{Q}))_{\text{tors}} \) is of the second type since it has already 3 distinct elements of order 2. If \( k = 2 \), \( \text{Jac}(\tilde{X})(\mathbb{Q}))_{\text{tors}} \) is clearly of the first type with \( n \) even. Therefore we obtain a finite number of conditions to assert that \( \eta(X) = 1 \).

**Proposition 4.6** We assume that \( P \) admit a rational factorization. Then,

(i) If \( k = 0 \) then \( \eta(X) = 1 \) if and only if \( (2n - 1)p = p_1 \) or \( (2n - 1)p = p_2 \) or \( 2np = p_3 \) for \( 0 < n \leq 2 \) (6 cases).

(ii) If \( k = 2 \) then \( \eta(X) = 1 \) if and only if \( np = p_1 \) for \( n = 1, \ldots, 6 \) or \( 2np = -p \) for \( n = 1, 2, 3, 4 \) (10 cases).
(iii) If $k = 4$ and $4b^2 > (c-a)^2$ then $\eta(X) = 1$ if and only if $np = p_3$ for $n = 1, \ldots, 4$ (4 cases) ($2np = -p$ is not allowed).

(ii) If $k = 4$ and $4b^2 < (c-a)^2$ then $\eta(X) = 1$ if and only if $np = p_1$ for $n = 1, \ldots, 4$ (4 cases).

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