A PARTITION TEMPERLEY–LIEB ALGEBRA
(WORK IN PROGRESS)

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Abstract. We introduce a generalization of the Temperley–Lieb algebra. This generalization is defined by adding certain relations to the algebra of braids and ties. A specialization of this last algebra corresponds to one small Ramified Partition algebra, this fact is the motivation for the name of our generalization.

Introduction

The Temperley–Lieb algebra appears originally in Statistical Mechanics as well as in Knot theory, quantum groups and subfactors of von Neumann algebras. This algebra was discovered by Temperley and Lieb by building transfer matrices[15]. Further, this algebra was rediscovered by V. Jones[8] who used it in the construction of his polynomial invariant for knots known as the Jones polynomial[9].

From a purely algebraic point of view, the Temperley–Lieb algebra is a quotient of the Iwahori–Hecke algebra by the two–sided ideal generated by the Steinberg elements $h_{ij}$ associated to $h_i$'s, where $|i - j| = 1$ and $h_i$'s denote the usual generators of the Iwahori–Hecke algebra, view p. 35[5]. In other words, the Temperley–Lieb algebra can be defined by the usual presentation of the Iwahori–Hecke algebra but by adding the relations $h_{ij} = 0$, for all $|i - j| = 1$. Using this point of view, there are several generalizations of the Temperley–Lieb algebra, e.g. see [4, 6]. This paper proposes a generalization of the Temperley–Lieb algebra by adding relations of Steinberg types to the algebra of braid and ties[1, 14].

The algebra of braid and ties $E_n(u)$, where $u$ is a parameter and $n$ denotes a positive integer, can be regarded as a generalization of the Hecke algebra and recently E. O. Banjo proved that $E_n(1)$ is isomorphic to a small ramified partition algebra, see Theorem 4.2[2]. The possible connexion of the $E_n(u)$ and the Partition algebras [10, 13] was speculated first by S. Ryom–Hansen[14]. The algebra $E_n(u)$ is defined by two sets of generators and relations. One set of generators $T_1, \ldots, T_{n-1}$ reflects the braid generators of the Yokonuma–Hecke algebra[17, 16, 3] of type A and the other set of generators $E_1, \ldots, E_{n-1}$ reflects the behavior of the monoid $P_n$ associated to the set partitions of $\{1, \ldots, n\}$. Thus,
\[ \mathcal{E}_n(u) \] also can be thought as a \( u \)-deformation of an amalgam among the symmetric group on \( n \) symbols and \( P_n \).

In short, in this paper we define and study the Partition Temperley–Lieb algebra, denoted \( \text{PTL}_n(u) \), which is defined by adding to the presentation of \( \mathcal{E}_n(u) \) mentioned above the following relations

\[ E_i E_j T_{ij} = 0 \quad \text{for all} \quad |i - j| = 1 \]

where \( T_{ij} \) is the Steinberg element associated to the \( T_i \)'s.

This work is organized as follows. In Section 1 we fix notation s and we recall the definition of the Jimbo representation. In Section 2 we recall the definition of the algebra \( \mathcal{E}_n(u) \), we have included also some results from [14] which are used in the paper. In Section 3 we construct a non–faithful tensor representation of the algebra \( \mathcal{E}_n(u) \) which is used in Section 4 for the definition of our Partition Temperley–Lieb algebra \( \text{PTL}_n(u) \). The Section 5 shows two presentations of the \( \text{PTL}_n(u) \). By using one of these presentations we constructed a span linear set of \( \text{PTL}_n(u) \) which is conjectured that is a basis for the Partition Temperley–Lieb algebra. Finally, based on a conjecture that the algebra \( \mathcal{E}_n(u) \) supports a Markov trace, we prove in Section 7 under which condition this trace could pass to \( \text{PTL}_n(u) \).

1. Preliminaries

Along the paper algebra means unital associative algebra, with unity 1, over the field of rational function \( K := \mathbb{C}(\sqrt{u}) \) in the variable \( \sqrt{u} \). Consequently, we put \( u = (\sqrt{u})^2 \).

Let \( H_n = H_n(u) \) be the Iwahori–Hecke algebra of type \( A \), that is, the algebra presented by generators \( 1, h_1, \ldots, h_{n-1} \) subject to braid relations among the \( h_i \)'s and the quadratic relation \( h_i^2 = u + (u - 1)h_i \), for all \( i \).

We shall recall the Jimbo representation of the Hecke algebra. Set \( V \) the \( K \)-vector space with basis \( \{ v_1, v_2 \} \). Denotes by \( \mathbf{J} \) the endomorphism of \( V \otimes V \) defined through the mapping

\[
\begin{align*}
\mathbf{J}(v_i \otimes v_j) &= -v_i \otimes v_j \quad \text{for} \quad i = j \\
\mathbf{J}(v_1 \otimes v_2) &= (u - 1) v_1 \otimes v_2 + \sqrt{u} v_2 \otimes v_1 \\
\mathbf{J}(v_2 \otimes v_1) &= \sqrt{u} v_1 \otimes v_2.
\end{align*}
\]

The Jimbo representation of \( H_n \) in \( V^\otimes n \) is defined by mapping \( h_i \mapsto \mathbf{J}_i \), where \( \mathbf{J}_i \) acts as the identity, with exception of the factors \( i \) and \( i + 1 \), where acts by \( \mathbf{J} \).

**Proposition 1.1.** The kernel of the Jimbo representation is the two–sided ideal generated by \( h_{ij} \), where \( |i - j| = 1 \) and

\[ h_{ij} := 1 + h_i + h_j + h_i h_j + h_j h_i + h_i h_j h_i. \]

It is well known that the Temperley–Lieb algebra can be defined as the quotient of the Iwahori–Hecke algebra by the Kernel of Jimbo representation. Thus, the Temperley–Lieb algebra can be defined by adding extra non–redundant relations to the above presentations of the Hecke algebra. More precisely, we have the following definition.
Definition 1.2. The Temperley–Lieb algebra $TL_n = TL_n(u)$ is the algebra generated by $1, h_1, \ldots, h_{n-1}$ subject to the following relations:

$$h_i^2 = u + (u - 1)h_i \quad \text{for all } i \quad (1.1)$$
$$h_i h_j = h_j h_i \quad \text{for } |i - j| > 1 \quad (1.2)$$
$$h_i h_j h_i = h_j h_i h_j \quad \text{for } |i - j| = 1 \quad (1.3)$$
$$h_{ij} = 0 \quad \text{for } |i - j| = 1. \quad (1.4)$$

It is well known that the dimension of $TL_n$ is the $n$th Catalan number $C_n := \frac{1}{n+1} \binom{2n}{n}$ [8] and that $TL_n$ has a presentation (reduced) with idempotents generators. Indeed, making $f_i := \frac{1}{1 + u}(1 + h_i)$ we have the following proposition.

Proposition 1.3. $TL_n$ can be presented by generators $1, f_1, \ldots, f_{n-1}$ satisfying the following relations

$$f_i^2 = f_i \quad \text{for all } i \quad (1.5)$$
$$f_i f_j = f_j f_i \quad \text{for } |i - j| > 1 \quad (1.6)$$
$$f_i f_j f_i = \frac{u}{(1 + u)^2} f_i \quad \text{for } |i - j| = 1. \quad (1.7)$$

By virtue Proposition 1.1, the Jimbo representation of the Iwahori–Hecke algebra defines a representation of the Temperley–Lieb algebra. In terms of the generators $f_i$’s, this representation, called also the Jimbo representation, acts on $V^\otimes n$ by mapping $f_i \mapsto F_i$. The endomorphism $F_i$ acts as the identity, with exception of the factors $i$ and $i+1$, where acts by $F \in \text{End}(V^\otimes 2)$,

$$F(v_i \otimes v_j) = 0 \quad \text{for } i = j$$
$$F(v_1 \otimes v_2) = (u + 1)^{-1}(u v_1 \otimes v_2 + \sqrt{u} v_2 \otimes v_1)$$
$$F(v_2 \otimes v_1) = (u + 1)^{-1}(\sqrt{u} v_1 \otimes v_2 + v_2 \otimes v_1).$$

2. The algebra of braids and ties

Let $n$ be the poset $\{1, \ldots, n\}$. A partition of $n$ is a collection of pairwise disjoint subposets (called parts) whose union is equal to $n$. We shall denote $P_n$ the set formed by the partitions of $n$. The cardinal $b_n$ of $P_n$ is known as the $n$th Bell number.

Let $I \in P_n$, an arc $i \prec j$ of $I$ is an ordered pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that

(1) $i < j$
(2) $i$ and $j$ are in the same part of $I$
(3) if $k$ is in the same part as $i$ and $i < k \leq j$, then $k = j$

In other words the arcs are pairs of adjacent elements in each part of $I$. Therefore the elements of $P_n$ can be represented by a graph with vertices $\{1, \ldots, n\}$ and whose edge
connecting the vertices $i$ and $j$ if and only if $i \sim j$ is an arc of $I$. For example, for $n = 3$ we have

$$\{\{1, 2\}, \{3\}\} \quad \text{is represented by} \quad \bullet \quad \bullet \quad \bullet$$

and so on.

The set $P_n$ can be regarded naturally as a poset, where the partial order $\preceq$, is defined by: $I = (I_1, \ldots, I_k) \preceq J = (J_1, \ldots, J_l)$ if and only if each $J_i$ is a union of certain $I_i$’s. By using $\preceq$ we give to $P_n$ a structure of commutative monoid by defining the product $I \ast J$, of $I$ with $J$, as the minimum element of the poset $P_n$ containing $I$ and $J$. Clearly the unity is $\{\{1\}, \{2\}, \ldots, \{n\}\}$ which is represented by $\bullet \quad \bullet \quad \cdots \quad \bullet$. The monoid $P_n$ is generated by the unity and the elements:

$$\bullet \quad \cdots \quad \bullet \quad \cdots \quad \bullet \quad \text{for all } 1 \leq i \leq n$$

The Hasse diagram for $P_3$ is:

And we have, for example:

$$\bullet \quad \bullet \quad \ast \quad \bullet \quad \bullet \quad \text{for } I = \{1, 2, 3\} \quad \text{and} \quad J = \{1, 2, 3\}$$

As usual we denote $S_n$ the symmetric group on symbols and we denote $s_i$ the transposition $(i, i + 1)$.

For $I = \{I_1, \ldots, I_m\} \in P_n$ and $w \in S_n$ we define $wI = \{wI_1, \ldots, wI_m\}$, where $wI_i$ is the subposet of $n$ obtained by applying $w$ to the elements of $I_i$. 
Definition 2.1. We denote $\mathcal{E}_n = \mathcal{E}_n(u)$ the algebra generated by $1, T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ satisfying the following relations:

\[
\begin{align*}
T_i T_j &= T_j T_i \quad \text{for } |i - j| > 1 \\
T_i T_j T_i &= T_j T_i T_i \quad \text{for } |i - j| = 1 \\
T_i^2 &= 1 + (u - 1) E_i (1 + T_i) \quad \text{for all } i \\
E_i E_j &= E_j E_i \quad \text{for all } i, j \\
E_i^2 &= E_i \quad \text{for all } i \\
E_i T_j &= T_j E_i \quad \text{for } |i - j| > 1 \\
E_i T_i &= T_i E_i \quad \text{for all } i \\
E_i E_i T_i &= T_i E_i E_j = E_j T_i E_j \quad \text{for } |i - j| = 1 \\
E_i T_j T_i &= T_j T_i E_j \quad \text{for } |i - j| = 1.
\end{align*}
\]

If $w = s_{i_1} \cdots s_{i_k} \in S_n$ is reduced form for $w$, we write $T_w := T_{i_1} \cdots T_{i_k}$ (this is a possible debt to a well known result of H. Matsumoto).

For $i < j$, we define $E_{ij}$ as

\[ E_{ij} = \begin{cases} 
E_i & \quad \text{for } j = i + 1 \\
T_i \cdots T_{j-2} E_{j-1} T_{j-2}^{-1} \cdots T_i^{-1} & \quad \text{otherwise}
\end{cases} \]

For any $J = \{i_1, i_2, \ldots, i_k\}$ subposet of $n$ we define $E_J = 1$ if $k = 1$ and

\[ E_J := E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{k-1} i_k} \quad \text{for } k > 1 \]

Note that $E_{\{i,j\}} = E_{ij}$. Also we note that in Lemma 4[1] it is proved that $E_J$ can be computed as

\[ E_J = \prod_{j \in J, j \neq i_0} E_{i_0 j} \quad (i_0 := \min\{i; i \in J\}) \]

For $I = \{I_1, \ldots, I_m\} \in \mathcal{P}_n$, we define $E_I$ as

\[ E_I = \prod_k E_{I_k} \]

The Corollary 2[1] implies the following proposition.

**Proposition 2.2.** The mapping $E_i \mapsto \bullet \cdots \bullet \cdots \bullet$ defines a monoid isomorphism between the monoid generated by $1, E_1, \ldots, E_{n-1}$ and $\mathcal{P}_n$.

**Proposition 2.3** (Corollary 1[1]). For $I \in \mathcal{P}_n$ and $w \in S_n$, we have

\[ T_w E_I T_w^{-1} = E_{wI}. \]

**Theorem 2.4** (Corollary 3[1]). The set $\mathcal{S}_n := \{E_I T_w; w \in S_n, I \in \mathcal{P}_n\}$ is a linear basis of $\mathcal{E}_n$. Hence the dimension of $\mathcal{E}_n$ is $b_n n!$. 

3. A TENSORIAL REPRESENTATION FOR \( \mathcal{E}_n \)

In this section we will define a tensorial representation for \( \mathcal{E}_n \). This representation is nothing more than a variation of that constructed by S. Ryom–Hansen in Section 3[14]. We note that, contrary to the representation constructed by Ryom–Hansen, our variation is a non–faithful representation. This fact is the key point in order to define the Partition Temperley–Lieb algebra as a quotient of \( \mathcal{E}_n \).

Let \( V \) be the \( K \)–vector space with basis \( \{v^r_i; 1 \leq i, r \leq n\} \), we define the endomorphisms \( E \) and \( T \) of \( V^{\otimes 2} \) through the following mapping,

\[
E(v^r_i \otimes v^s_j) := \begin{cases} 
0 & \text{for } r \neq s \\
v^r_i \otimes v^s_j & \text{for } r = s 
\end{cases}
\]

\[
T(v^r_i \otimes v^s_j) := \begin{cases} 
-v^j_i \otimes v^r_j & \text{for } r \neq s \\
v^r_i \otimes v^j_s & \text{for } r = s, i = j \\
(u-1) v^r_i \otimes v^s_j + \sqrt{u} v^s_j \otimes v^r_i & \text{for } r = s, i < j \\
\sqrt{u} v^r_i \otimes v^s_j & \text{for } r = s, i > j 
\end{cases}
\]

Define now, \( E_i \) (respectively \( T_i \)) as the endomorphism of \( V^{\otimes n} \) that acts as the identity with exception on the factors \( i \) and \( i + 1 \) where acts by \( E \) (respectively \( T \)).

**Theorem 3.1.** The mapping \( E_i \mapsto E_i, T_i \mapsto T_i \) defines a representation \( J_n \) of \( \mathcal{E}_n \) in \( V^{\otimes n} \).

**Proof.** The proof uses the same strategy as Theorem 1[14]. We only need to check that the operators \( E_i \) and \( T_i \) satisfy the respective relations \( (2.1)–(2.9) \). The relations \( (2.1), (2.4)–(2.7) \) clearly hold.

To check relation \( (2.3) \) it is enough to take \( n = 2 \). Evaluating the relation in \( v^r_i \otimes v^s_j \) with \( r = s \), the relation becomes the Hecke quadratic relation. In the case \( r \neq s \), the operator \( E(1 + T) \) acts as zero and \( T^2 \) as the identity, hence the relation holds.

To check the remaining of the relations, without loss of generality, we can suppose \( n = 3 \). Also we observe that it is enough to check the relations in question on the basis elements \( x = v^r_i \otimes v^s_j \otimes v^t_k \). By simplicity we shall introduce the following notation: whenever we have two repetitions in the upper indices in the basis elements, we omit the two repeated upper indices and we replace the remaining indices by a prime, e.g. \( v^r_i \otimes v^s_j \otimes v^t_k \) is written simply as \( v_i \otimes v'_j \otimes v_k \). Then when we have two repetitions in the upper indices we shall distinguish three forms of elements:

\[
v^r_i \otimes v^s_j \otimes v_k \quad v_i \otimes v^r_j \otimes v_k \quad v_i \otimes v_j \otimes v'_k
\]  

(3.1)

Further, in these elements we can suppose that the lower indices are 1 or 2 since \( T \) acts according the order in the pair formed by lower indices. Now, the action of \( T \) on primed and unprimed elements is, up to sign, a transposition, so we can suppose that the lower index of the primed factor is always 1. Therefore, the elements in the form as \( (3.1) \) can be reduced to consider the following cases:

\[
v^r_1 \otimes v^s_1 \otimes v_1 \quad v^r_1 \otimes v^r_1 \otimes v_1 \quad v_1 \otimes v^r_1 \otimes v'_1 \\
v^r_1 \otimes v^r_1 \otimes v_2 \quad v^r_1 \otimes v^r_1 \otimes v_2 \quad v_1 \otimes v^r_2 \otimes v'_1 \\
v^r_1 \otimes v^r_1 \otimes v_1 \quad v^r_2 \otimes v^r_1 \otimes v_1 \quad v_2 \otimes v^r_1 \otimes v'_1 \\
v^r_1 \otimes v^r_2 \otimes v_2 \quad v^r_2 \otimes v^r_2 \otimes v_2 \quad v_2 \otimes v^r_2 \otimes v'_2
\]  

(3.2)
The checking of (2.8) and (2.9) are similar and routine. Thus we shall check only the first relation of (2.8). If all upper indices in \( x \) are distinct the operator \( E_iE_j \) acts as zero and as the identity if all upper indices are equal. Hence \( E_1E_2T_1 \) and \( T_1E_1E_2 \) coincide on such \( x \)'s. Now it is easy to check the relation whenever \( x \) is an element of \((3.2)\) whose unprimed factor has equal lower indices. The checking on the other elements of \((3.2)\) results from a direct computation, e.g., for \( x = v_1 \otimes v_2 \otimes v'_1 \) we have

\[
E_1E_2T_1(x) = (u - 1)E_1E_2(x) + \sqrt{u}E_1E_2(v_2 \otimes v_1 \otimes v'_1) = 0 = T_1E_1E_2(x)
\]

Finally we will check the relation (2.2). If in the basis elements the upper indices are all equal we are in the situation of Jimbo representation \( J \). If all upper indices are different the action becomes, up to sign, in the permutation action on the factors of the basis elements. Therefore, it only remains to check that (2.2) is true when one evaluates on the elements of \((3.2)\). Now, it is easy to see that the evaluation of both sides of (2.2) on the elements of \((3.2)\) whose unprimed factors are equal is \(-\sigma_{13}\), where \(\sigma_{13} \) permutes the the first with the third factor in the tensor product. The check of (2.2) on the remaining elements of \((3.2)\) is all similar for all. We shall do, as a representative case, the case \( x = v'_1 \otimes v_1 \otimes v_2 \):

\[
T_2T_1T_2(x) = (u - 1)T_2T_1(v'_1 \otimes v_1 \otimes v_2) + \sqrt{u}T_2T_1(v'_1 \otimes v_2 \otimes v_1) = -(u - 1)T_2(v_1 \otimes v'_1 \otimes v_2) - \sqrt{u}T_2(v_2 \otimes v'_1 \otimes v_1) = (u - 1)(v_1 \otimes v_2 \otimes v'_1) + \sqrt{u}(v_2 \otimes v_1 \otimes v'_1) = T_1(v_1 \otimes v_2 \otimes v'_1) = -T_1T_2(v_1 \otimes v'_1 \otimes v_2) = T_1T_2T_1(x).
\]

\( \square \)

4. The PTL algebra

We want to define a generalization of Temperley–Lieb algebra by using the algebra \(\mathcal{E}_n\), we shall call this generalization the Partition Temperley–Lieb algebra which is denoted \(PTL_n\). A first natural attempt of definition \(PTL_n\) is as the algebra that results by adding to defining relations of \(\mathcal{E}_n\) the relations \(T_{ij} = 0\), where \(T_{ij}\) are the Steinberg elements \(T_{ij}\)'s associated to the \(T_i\)'s,

\[
T_{ij} := 1 + T_i + T_j + T_iT_j + T_iT_j + T_iT_j \text{ where } |i - j| = 1
\]

As in the classical case we want that the Jimbo representation \( J \) of \(\mathcal{E}_n\) passes to \(PTL_n\), hence the \(T_{ij}\)’s must be killed by \( J \). But unfortunately this does not happen. In fact, for \( n = 3 \) and by taking \( x = v_1 \otimes v_2 \otimes v'_1 \), we have

\[
T_1x = (u - 1)v_1 \otimes v_2 \otimes v'_1 + \sqrt{u}v_2 \otimes v_1 \otimes v'_1 \quad T_2x = -v_1 \otimes v'_1 \otimes v_2
\]

\[
T_2T_1x = -(u - 1)v_1 \otimes v'_1 \otimes v_2 - \sqrt{u}v_2 \otimes v'_1 \otimes v_1 \quad T_1T_2x = v'_1 \otimes v_1 \otimes v_2
\]

\[
T_1T_2T_1x = (u - 1)v'_1 \otimes v_1 \otimes v_2 + \sqrt{u}v'_1 \otimes v_2 \otimes v_1
\]

Then

\[
(JT_{12})x = u v_1 \otimes v_2 \otimes v'_1 - u v_1 \otimes v'_1 \otimes v_2 + \sqrt{u}v_2 \otimes v_1 \otimes v'_1 - \sqrt{u}v'_1 \otimes v_1 + u v'_1 \otimes v_1 \otimes v_2 + \sqrt{u}v'_1 \otimes v_2 \otimes v_1
\]
Therefore \( J \) does not kill \( T_{12} \).

Having in mind the above discussion we make the following definition.

**Definition 4.1.** The Partition Temperley–Lieb algebra \( PTL_n = PTL_n(u) \) is defined by adding to the defining presentation of \( \mathcal{E}_n \) the relations:

\[
E_i E_j T_{i,j} = 0 \quad \text{for all} \quad |i - j| = 1. \tag{4.1}
\]

Clearly, from (2.8) we have that \( E_i E_j T_{i,j} = 0 \) is equivalent to \( T_{i,j} E_i E_j = 0 \).

**Remark 4.2.** Notice that by taking \( E_i = 1 \) the algebra \( PTL_n \) coincides with the classical Temperley–Lieb algebra. Also, we note that the defining relations of \( PTL_n \) hold when \( T_i \) is replaced by the generators \( h_i \) of the Temperley–Lieb algebra and \( E_i \) is replaced by 1, thus the mapping \( E_i \mapsto 1 \) and \( T_i \mapsto h_i \) defines an algebra homomorphism from \( PTL_n \) onto \( TL_n \).

**Theorem 4.3.** The Jimbo representation \( J_n \) of \( \mathcal{E}_n \) factors through the algebra \( PTL_n \).

**Proof.** Without loss of generality we can suppose that \( n = 3 \). Thus, we must prove that \( J_3(E_1E_2T_{12}) = 0 \). Now, keeping the notations used during the proof of Theorem 3.1 to prove the theorem it is enough to see that \( J_3(E_1E_2T_{12}) \) kill the basis elements \( x = v_i^* \otimes v_j^* \otimes v_k^* \). If all upper indices in \( x \) are equal, \( J_3 \) is the Jimbo representation of the Hecke algebra, so \( J_3(T_{12}) \) kill \( x \); hence \( J_3(E_1E_2T_{12}) \) kill \( x \) too. If the upper indices of \( x \) are not all equal, we have that \( x \) is killed by \( E_1 \) or \( E_2 \), hence \( J_3(E_1E_2T_{12})(x) = 0 \). \( \square \)

We are going to prove now that the set of relations (4.1) can be reduced to only one. To do this we need to introduce the following element \( \Gamma \),

\[
\Gamma := T_1 T_2 \cdots T_{n-1}
\]

**Lemma 4.4.** For all \( 1 \leq i, j \leq n - 1 \) we have:

1. \( T_i = \Gamma^{i-1} T_i \Gamma^{-(i-1)} \)
2. \( T_{i,i+1} = \Gamma^{i-1} T_{i,2} \Gamma^{-(i-1)} \)
3. \( E_i = \Gamma^{i-1} E_i \Gamma^{-(i-1)} \)
4. \( T_{i+1} \Gamma^{i-1} = \Gamma^{i-1} T_2 \)
5. \( E_{(i,i+2)} = \Gamma^{i-1} E_{(1,3)} \Gamma^{-(i-1)} \)

**Proof.** The statement (1) results from an inductive argument on \( i \) and the braid relations of \( T_i \)'s. The statement (2) is a result applying (1). The proof of statement (3) is analogous to the proof of (1), that is: an argument inductive on \( i \) and using the relation (2.6). The statement (4) is clear, since (1). Finally, we have:

\[
\Gamma^{i-1} E_{(1,3)} \Gamma^{-(i-1)} = \Gamma^{i-1} T_2 E_i T_2^{-1} \Gamma^{-(i-1)} = \Gamma^{i-1} T_2 (\Gamma^{-(i-1)} E_i \Gamma^{(i-1)} ) T_2^{-1} \Gamma^{-(i-1)} = T_{i+1} E_i T_{i+1}^{-1}
\]

Thus, the statement (5) is proved. \( \square \)

**Proposition 4.5.** The relation \( E_1E_2T_{1,2} = 0 \) implies the relations \( E_i E_j T_{i,j} = 0 \), for all \( |i - j| = 1 \).
Proof. We can suppose $j = i + 1$, since $T_{ij} = T_{ji}$ and $E_i$ and $E_j$ commute. From the statements (1) and (3) Lemma 4.4 we have:

$$E_i E_{i+1} T_{i,i+1} = (\Gamma_i^{-1} E_i \Gamma_i^{-1}) (\Gamma_i^{-1} E_i \Gamma_i^{-1}) = \Gamma_i^{-1} E_i E_i \Gamma_i^{-1} T_{i,1} \Gamma_i^{-1} (\Gamma_i^{-1} E_i E_i \Gamma_i^{-1})$$

Hence the proof follows. \qed

Corollary 4.6. The Partition Temperley–Lieb algebra $PTL_n$ can be regarded as the quotient of $E_n$ by the two–sided ideal generated by $E_1 E_2 T_{12}$.

5. Others presentations for $PTL_n$

In order to have more comfortable notations we shall introduce the following element $\delta$,

$$\delta := \frac{1 - u}{1 + u} \in K$$

5.1. Having in mind the definition of the idempotents generators $f_i$ of the Temperley–Lieb algebra, it is natural to consider the following definition.

$$F_i := \frac{1}{u + 1} (1 + T_i) \quad (1 \leq i \leq n - 1)$$

It is obvious that $F_i$ commute with $E_i$ (and $T_i$) and that they form a set of generators for the algebra $PTL_n$, but notice that the $F_i$’s are not idempotents. In fact, from (2.3) we have

$$F_i^2 = \frac{1}{(u + 1)^2} (1 + 2T_i + 1 + (u - 1)E_i + (u - 1)E_i T_i)$$

then

$$F_i^2 = (1 + \delta) F_i - \delta E_i F_i$$

We have the following proposition

Theorem 5.1. $PTL_n$ can be presented by the generators $1, E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}$ subject to the relations (2.4), (2.5) together with the following relations

$$F_i^2 = (1 + \delta) F_i - \delta E_i F_i \quad (5.1)$$

$$F_i F_j = F_j F_i \quad \text{for all } |i - j| > 1 \quad (5.2)$$

$$F_i E_j = E_j F_i \quad \text{for all } |i - j| > 1 \quad (5.3)$$

$$E_i F_i = F_i E_i \quad (5.4)$$

and for all $|i - j| = 1$:

$$E_i E_j F_i = F_i E_i E_j = E_j F_i E_j + \frac{1}{u + 1} (E_i E_j - E_j) \quad (5.5)$$

$$E_i F_j F_i = F_j F_i E_j + \frac{1}{u + 1} [(E_i - E_j) F_j + F_i (E_i - E_j)] - \frac{1}{(u + 1)^2} (E_i - E_j) \quad (5.6)$$

$$F_i F_j F_i = \frac{1}{(u + 1)^2} (F_i - (1 - u) E_i F_i) \quad (5.7)$$
Proof. It is easy to check that (2.1) (respectively (2.6)) is equivalent to (5.2) (respectively (5.3)), so having in mind the previous discussion to the theorem, it only remains to prove that the relations (5.5)–(5.7) hold and that relations (2.8), (2.9), (4.1) and (2.2) can be deduced from the relations of the theorem.

We have that $T_i = (u+1)F_i - 1$. Now replacing this expression of $T_i$ in (2.8) (respectively (2.9)) it is a routine to check that (2.8) becomes (5.5) (respectively (5.6)).

We have to check that relation (4.1) is equivalent to relation (5.7). We have

$$T_iT_j = ((u+1)F_i - 1)((u+1)F_j - 1) = (u+1)^2F_iF_j - (u+1)F_i - (u+1)F_j + 1$$

then

$$T_iT_jT_i = (u+1)^3F_iF_jF_i - (u+1)^2F_i^2 - (u+1)^2F_jF_i + (u+1)F_i$$

$$- (u+1)^2F_iF_j + (u+1)F_i + (u+1)F_j - 1$$

Therefore, by using (5.1), we deduce

$$T_iT_jT_i = (u+1)^3F_iF_jF_i + (1-u^2)E_iF_i$$

$$- (u+1)^2F_iF_j + (u+1)F_i + (u+1)F_j - 1$$

Now, substituting each summand of $T_{ij}$ by its expression in term of $F_i$'s one obtains

$$T_{ij} = (u+1)^3F_iF_jF_i + (1-u^2)E_iF_i - (u+1)F_i$$

Hence (4.1) is equivalent (5.7).

Finally notice that (5.7) implies (2.2), since the above expression of $T_iT_jT_i$ in terms of $F_i$'s tells us that (2.2) is equivalent to

$$(u+1)^2F_iF_jF_i + (1-u)E_iF_i + F_j = (u+1)^2F_iF_jF_i + (1-u)E_iF_i + F_i$$

Thus the proof is concluded. \qed

5.2. In this subsection we shall show a presentation of $\text{PTL}_n$ by idempotent generators. For $1 \leq i < j \leq n-1$, we define

$$L_i := \frac{1}{1+u} (T_i + 1) (\alpha + (1-\alpha)E_i) \quad \text{where} \quad \alpha := \frac{1+u}{2}$$

notice that

$$L_i = \frac{1}{2} (T_i + \delta T_i E_i + \delta E_i + 1) = \frac{1}{2} (1 + T_i)(1 + \delta E_i)$$

(5.8)

Also we have

$$L_i = \frac{u+1}{2} F_i + \frac{1-u}{2} E_i F_i$$

(5.9)

It is clear that $L_i$ commute with $E_i$, $T_i$ and $F_i$. We have the following useful lemma.

**Lemma 5.2.** For all $i$ we have:

1. $L_i^2 = L_i$
2. $(1+u)E_i L_i = E_i(1+T_i)$
3. $T_i = 2L_i + (u-1)E_i L_i - 1$
4. $E_i L_i = E_i F_i$
5. $F_i = (1+\delta)L_i - \delta E_i L_i$. 
Proof. We have:

\[ L_i^2 = 4^{-1}(1 + T_i)^2(1 + \delta E_i)^2 = 4^{-1}(2(1 + T_i) + (u - 1)E_i(1 + T_i))(1 + (2\delta + \delta^2)E_i) \]

then

\[
L_i^2 &= 4^{-1}(1 + T_i)(2 + (u - 1)E_i)(1 + (2\delta + \delta^2)E_i) \\
&= 4^{-1}(1 + T_i)(2 + (2\delta + \delta^2) + (u - 1) + (u - 1)(2\delta + \delta^2))E_i \\
&= 4^{-1}(1 + T_i)(2 + (2\delta + \delta^2)(1 + u) + u - 1)E_i \\
&= 4^{-1}(1 + T_i)(2 + 2\delta E_i) = L_i.
\]

The second assertion follows by multiplying the formula of \( L_i \) by \( E_i \). To prove the third assertion, we bring first \( E_iT_i \) from the second assertion and then we substitute this expression of \( E_iT_i \) in (5.8), thus the third assertion follows. The fourth assertion results by multiplying (5.9) by \( E_i \). The fifth assertion result directly from (4) and (5.9). □

**Theorem 5.3.** PTL\(_n\) can be presented by the generators \( 1, E_1, \ldots, E_{n-1}, L_1, \ldots, L_{n-1} \) subject to the relations (2.4), (2.5) together with the following relations

\[
L_i^2 = L_i \tag{5.10}
\]

\[
L_iL_j = L_jL_i \quad \text{for all} \quad |i - j| > 1 \tag{5.11}
\]

\[
L_iE_j = E_jL_i \quad \text{for all} \quad |i - j| > 1 \tag{5.12}
\]

\[
L_iE_i = E_iL_i \tag{5.13}
\]

and for all \( |i - j| = 1 \):

\[
E_iE_jL_i = L_iE_iE_j = E_jL_iE_j + 2^{-1}(E_iE_j - E_j) \tag{5.14}
\]

\[
4L_iL_jE_i + 2E_j(L_j + L_i) + E_i = 4E_jL_iL_j + 2(L_i + L_j)E_i + E_j \tag{5.15}
\]

\[
8L_iL_jL_i + 4(u - 1)\left[L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_j\right] + (u - 1)^2(u + 5)E_iE_jL_iL_jL_i = 2L_i + 3(u - 1)L_i + (u - 1)^2E_iE_jL_i \tag{5.16}
\]

Proof. We will use the presentation of Theorem 5.1. From (5)Lemma 5.2 follows that PTL\(_n\) is generated by \( 1, E_i \)'s and \( L_i \)'s. Checking that (5.11)–(5.16) are equivalent, respectively, to (5.10)–(5.15) is a straight forward and just a routine, so we leave the computation to the reader. Thus, to finish the proof it only remains to check that (5.16) is equivalent to (5.17).

We have

\[
F_iF_j = ((1 + \delta)L_i - \delta E_iL_i)((1 + \delta)L_j - \delta E_jL_j)
\]

\[
= (1 + \delta)^2L_iL_j - \delta(1 + \delta)L_iE_jL_j - \delta(1 + \delta)L_iL_jE_i + \delta^2E_iL_iE_jL_j
\]

Hence

\[
F_iF_jF_i = (1 + \delta)^3L_iL_jL_i - \delta(1 + \delta)^2[L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i]
\]

\[
+ \delta^2(1 + \delta)L_iE_jL_iL_i + \delta^2(1 + \delta)L_iE_jL_iL_i + \delta^2(1 + \delta)L_iL_jE_iL_i
\]

\[
- \delta^3E_iL_iE_jL_iL_iE_iL_i
\]
Using now (2.4), (2.5), (5.13) and (5.14) we get
\[
F_iF_jF_i = (1 + \delta)^3 L_iL_jL_i - \delta(1 + \delta)^2 [L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i]
\]
\[
+ (2\delta^2(1 + \delta) - 3\delta^3)E_iE_jL_jL_i + 2\delta(1 + \delta)E_iL_jE_iL_i
\]
Now applying on the last term the relation (5.13) and using later (5.14), we get
\[
E_iL_iL_jE_iL_i = L_i(E_iL_jE_i)L_i, \text{ so}
\]
\[
E_iL_iL_jE_iL_i = L_i \left[ E_iE_jL_j - \frac{1}{2}(E_iE_j - E_i) \right] L_i
\]
\[
= E_iE_jL_iL_jL_i - \frac{1}{2} E_iE_jL_i^2 + \frac{1}{2} L_iE_iL_i \quad \text{(by using (5.14))}
\]
\[
= E_iE_jL_iL_jL_i - \frac{1}{2} E_iE_jL_i + \frac{1}{2} E_iL_i \quad \text{(by using (5.10) and (5.13))}
\]
Then
\[
F_iF_jF_i = (1 + \delta)^3 L_iL_jL_i - \delta(1 + \delta)^2 [L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i]
\]
\[
+ (2\delta^2 + 3\delta^2)E_iE_jL_iL_jL_i - \delta^2(1 + \delta) \left[ \frac{1}{2} E_iE_jL_i - \frac{1}{2} E_iL_i \right]
\]
On the other side, from (4) Lemma 5.2, we have
\[
F_i + (u - 1)E_iF_i = (1 + \delta)L_i - \delta E_iL_i + (u - 1)E_iL_i = (1 + \delta)L_i - (u + 2)\delta E_iL_i
\]
Therefore, the relation (5.16) is equivalent to
\[
(1 + \delta)^3 L_iL_jL_i - \delta(1 + \delta)^2 [L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i] + (2\delta^3 + 3\delta^2)E_iE_jL_iL_jL_i
\]
\[
= \frac{1}{(u + 1)^2} \left[ (1 + \delta)L_i - (u + 2)\delta E_iL_i \right] + \delta^2(1 + \delta) \left[ \frac{1}{2} E_iE_jL_i - \frac{1}{2} E_iL_i \right]
\]
which is reduced, after multiplication by \((u + 1)^2\), to
\[
\frac{8}{(u + 1)} L_iL_jL_i - 4\delta [L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i] + (1 - u)^2(2\delta + 3)E_iE_jL_iL_jL_i
\]
\[
= (1 + \delta)L_i - (u + 2)\delta E_iL_i + (1 - u)^2(1 + \delta) \left[ \frac{1}{2} E_iE_jL_i - \frac{1}{2} E_iL_i \right]
\]
or equivalently
\[
\frac{8}{(u + 1)} L_iL_jL_i - 4\delta [L_iE_jL_jL_i + E_iL_iL_jL_i + L_iL_jE_iL_i] + (1 - u)^2(2\delta + 3)E_iE_jL_iL_jL_i
\]
\[
= (1 + \delta)L_i - 3\delta E_iL_i + (1 - u)\delta E_iE_jL_i
\]
Multiplying this last equation by \(u + 1\) we obtain (5.10). \(\square\)

**Remark 5.4.** By taking \(E_i = 1\) the elements \(L_i\)'s become \(f_i\)'s and the Theorem 5.3 and Theorem 5.1 become Theorem 1.3.
6. A linear basis for $\text{PTL}_n$

By using essentially Theorems 5.1 and 4.3 we shall construct a linear basis of $\text{PTL}_n$. Further we use also the following lemmas.

**Lemma 6.1.** For all $i, j$ such that $|i - j| = 1$, we have:

1. $F_i E_j = T_i E_j T_i^{-1} F_i + \frac{1}{w+1} (E_j - T_i E_j T_i^{-1})$
2. $E_j F_i = F_i T_i E_j T_i^{-1} + \frac{1}{w+1} (E_j - T_i E_j T_i^{-1})$

**Proof.** It is enough to expand $F_i$ in both side of the equality. □

**Lemma 6.2.** Any word in $1, F_1, \ldots, F_{n-1}, E_1, \ldots, E_{n-1}$ can be expressed as a $K$–linear combination of words in $E_i$’s and $F_i$’s having at most one $F_{n-1}$, $E_{n-1}$, or $F_{n-1}E_{n-1}$.

**Proof.** It is a consequence of Proposition 1[1] and the fact that $F_i$ is a linear expression of 1 and $T_i$. □

**Definition 6.3.** A word in $F_1, \ldots, F_{n-1}$ is called $F$–reduced (or simply reduced) if and only if has the form

$$ (F_i \cdots F_{j_1})(F_{i_2} \cdots F_{j_2}) \cdots (F_{i_k} \cdots F_{j_k}) $$

(6.1)

where $0 \leq k \leq n - 1$ and

\[
1 \leq i_1 < i_2 < \cdots < i_k \leq n - 1 \\
1 \leq j_1 < j_2 < \cdots < j_k \leq n - 1 \\
i_1 \geq i_2, i_2 \geq j_2, \ldots, i_k \geq j_k
\]

**Proposition 6.4.** Any word in $1, E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}$ may be written as $K$–linear combination of words in the form $E_i F$, where $i \in \mathcal{P}_n$ and $F$ is $F$–reduced.

**Proof.** We have adapted the proof of Proposition 2.8[5]. We will use induction on $n$. The assertion is clearly valid for $n = 2$. We assume now that the proposition is valid for $n$. Let $W$ a word in $1, E_1, \ldots, E_n, F_1, \ldots, F_n$. By using Lemma 6.1 we can move the $E_i$’s appearing in $W$ to the front position, obtaining in this way that $W$ is a linear combination of words in the form $E_i W'$, where $W'$ is a word in $1, F_1, \ldots, F_n$. Thus, to prove the proposition it is enough to show that $W'$ is a linear combination of words in the form desired. Now, if $W'$ does not contain $F_n$ then we are done. If $W'$ contains $F_n$, according to Lemma 6.2 we have that $W'$ is a linear combination of words in the form

$$ W_1 R_n W_2 $$

where $R_n = E_n$, $F_n$ or $E_n F_n$ and $W_i$ are words in $1, E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}$. If $R_n = E_n$, according to Lemma 6.1 we can move $R_n$ to the front position and then using the induction hypothesis we are done. Suppose $R_n = F_n$, we note that by induction hypothesis $W_2$ is a linear combination of words in the form

$$ E_j V(F_n F_{n-1} \cdots F_{j_k}) $$

where now $J \in \mathcal{P}_{n-1}$, $V$ is a word reduced in $1, F_1, \ldots, F_{n-2}$ (notice that $F_n F_{n-1} \cdots F_{j_k}$ could be empty). Hence $W'$ is a linear combination of words of the form

$$ W_1 F_n E_j V(F_n F_{n-1} \cdots F_{j_k}) $$
Now, \( F_n E_I = E_{s_n} F_n \), so using \([5.1]\) and \([5.3]\) follows that \( W' \) can be written as a linear combination \((1 + \delta)N_1 - \delta N_2 \) with \( N_1 := E_{J'} V'(E_n F_{n-1} \ldots F_{j_k}) \) and \( N_2 := E_{J'} V'(E_n F_{n-1} \ldots F_{j_k}) \), where \( J' \in P_n \) and \( V' \) is a word in \( 1, F_1, \ldots F_{n-1} \). Again we note that in \( N_2, E_n \) can move to the front position, so \( N_2 \) is in fact in the form of \( N_1 \). Therefore, \( W' \) is a linear combination of words in the form

\[ E_{J'} V'(E_n F_{n-1} \ldots F_{j_k}) \]

where \( J' \in P_n \) and \( V' \) is a word in \( 1, F_1, \ldots F_{n-1} \). Applying the induction hypothesis, on \( V' \), we deduce that \( W' \) is a linear combination of words in the form \( E_I F \), where \( I \in P_n \) and \( F \) has the form

\[ F = (F_{i_1} \cdots F_{j_i})(F_{i_2} \cdots F_{j_2}) \cdots (F_{i_k} \cdots F_{j_k}) \]

with \( i \)'s increasing and \( i_l \geq j_l \), for all \( 1 \leq l \leq k \). Thus it remains to prove that in \( F \)'s the \( j \)'s can be taken increasing. Suppose \( j_1 \geq j_2 \), so

\[ F = (F_{i_1} \cdots F_{j_1+1})(F_{i_2} \cdots (F_{j_1} F_{j_1+1} F_{j_1}) \cdots F_{j_2}) \cdots (F_{i_k} \cdots F_{j_k}) \]

Then, by using \([5.7]\), we have \( F = (u + 1)^{-2} F_1 - (u + 1)^{-1} \delta F_2 \), where

\[ F_1 := (F_{i_1} \cdots F_{j_1+1})(F_{i_2} \cdots F_{i_2} \cdots F_{j_2}) \cdots (F_{i_k} \cdots F_{j_k}) \]

and

\[ F_2 := (F_{i_1} \cdots F_{j_1+1})(F_{i_2} \cdots (E_{j_1} F_{j_1}) \cdots F_{i_2} \cdots F_{j_k}) \]

Clearly (applying Lemma \([6.1]\)), \( E_{j_1} \) in \( F_2 \) can be moved to the front position. Therefore, by using an inductive argument we deduce that \( F \) can be expressed as a linear combination in the desired form. Hence \( W' \) can be written in the desired form. Thus, the proof is concluded.

\[ \square \]

**Conjecture 6.5.** The set formed by the elements \( E_I F \), where \( I \in P_n \) and \( F \) is reduced, is a linear basis of \( PTL_n \). Hence the dimension of \( PTL_n \) is \( b_n C_n \).

7. Markov trace

For \( d \) a positive integer we denote \( Y_{d,n} = Y_{d,n}(u) \) the Yokonuma–Hecke algebra, i.e. the algebra presented by braid generators \( g_1, \ldots, g_{n-1} \) together with the framing generators \( t_1, \ldots, t_n \) which satisfies the following defining relations: braids relation (of type \( A \)) among the \( g_i \)'s, \( t_i t_j = t_j t_i \), \( g_i t_j = t_{s(i,j)} g_i \) and

\[ g_i^2 = 1 + (u - 1) e_i (1 + g_i) \]

where \( e_i \) is defined as

\[ e_i := \frac{1}{d} \sum_{s=1}^{d} t_i^s t_i^{-s} \]

**Proposition 7.1.** We have a natural algebra morphism \( \psi : \mathcal{E}_n \mapsto Y_{d,n} \) defined through the mapping \( T_i \mapsto g_i \) and \( E_i \mapsto e_i \).

**Proof.** According to Lemma 2.1\([12]\) the defining relations of \( \mathcal{E}_n \) are satisfied by changing \( T_i \) by \( g_i \) and \( E_i \) by \( e_i \). Hence the proof follows. \[ \square \]
Theorem 7.2 (See [11]). Let $z, x_1, \ldots, x_{d-1}$ be in $\mathbb{C}$. There exists a unique family of linear map $\{\text{tr}_n\}_n$ on inductive limit associated to the family $\{Y_{d,n}\}_n$ with values in $\mathbb{C}$ satisfying the rules:

$$\text{tr}_n(ab) = \text{tr}_n(ba)$$
$$\text{tr}_n(1) = 1$$
$$\text{tr}_{n+1}(a g_n) = z \text{tr}_n(a) \quad \text{for} \quad a \in Y_{d,n}$$
$$\text{tr}_{n+1}(a t_{n+1}^m) = x_m \text{tr}_n(a) \quad \text{for} \quad a \in Y_{d,n}, 1 \leq m \leq d - 1.$$ 

It is natural to consider the composition $\text{tr}_n \circ \psi$ which defines a Markov trace on $\mathcal{E}_n$. This supports the following conjecture.

Conjecture 7.3. [Aicardi, Juyumaya] The algebra $\mathcal{E}_n$ supports a Markov trace. I.e. for all $n \in \mathbb{N}$ we have a unique linear map $\rho_n : \mathcal{E}_n \rightarrow K(A, B)$ such that for all $x, y \in \mathcal{E}_n$, we have:

1. $\rho_n(1) = 1$
2. $\rho_n(xy) = \rho_n(yx)$
3. $\rho_{n+1}(x T_n) = \rho_{n+1}(x E_n T_n) = A \rho_n(x)$
4. $\rho_{n+1}(x E_n) = B \rho_n(x)$

where $A$ and $B$ are parameters.

Example 7.4. According to the rule (3)Conjecture[7.3] of $\rho$ we have, $\rho(E_1 T_1^2 T_1) = A \rho(E_1 T_1^2)$. Now, $E_1 T_1^2 = E_1 (1 + (u - 1) E_1 (1 + T_1)) = u E_1 + (u - 1) E_1 T_1$. So

$$\rho(E_1 T_1^2 T_1) = A (u B + (u - 1) A) = u A B + (u - 1) A^2.$$ 

Assuming that Conjecture[7.3] is true, we are going to study when the Markov trace $\rho_n$ passes to $\text{PTL}_n$. According to Corollary[4.6] studying the factorization of $\rho_n$ to $\text{PTL}_n$ is reduced to studying the values of $\rho_n$ on the two–sided ideal generated by $E_i E_j T_{12}$. For this study we need the following lemmas.

Lemma 7.5. (1) $T_1 T_{12} = [1 + (u - 1) E_1] T_{12}$

2. $T_2 T_{12} = [1 + (u - 1) E_2] T_{12}$
3. $T_1 T_2 T_{12} = [1 + (u - 1) E_1 + (u - 1) E_{1,3} + (u - 1)^2 E_1 E_2] T_{12}$
4. $T_2 T_1 T_{12} = [1 + (u - 1) E_2 + (u - 1) E_{1,3} + (u - 1)^2 E_1 E_2] T_{12}$
5. $T_1 T_2 T_{12} = [1 + (u - 1) E_1 + E_2 + E_{1,3}] + (u - 1)^2 (u + 2) E_1 E_2] T_{12}$

Proof. The proof of the statements results by expanding the left side and then using the defining relations of $\mathcal{E}_n$. As example we shall check the first statement:

$$T_1 T_{12} = T_1 + T_1^2 + T_1 T_2 + T_1^2 T_2 + T_1 T_2 T_1 + T_1^2 T_2 T_1$$
$$= T_1 + 1 + (u - 1) E_1 + (u - 1) E_1 T_1 + T_1 T_2 + T_1 T_2 + (u - 1) E_1 T_2 + (u - 1) E_1 T_2 + T_1 T_2 T_1 + T_1 T_2 T_1$$
$$+ (u - 1) E_1 T_2 T_1 + (u - 1) E_1 T_2 T_1 = T_{12} + (u - 1) E_1 T_{12}.$$ 

Lemma 7.6. (1) $\rho_3(T_{12}) = (u + 1) A^2 + 3 A + (u - 1) AB + 1$

2. $\rho_3(E_{(1,2,3)} T_{12}) = (u + 1) A^2 + (u + 2) A B + B^2$
3. $\rho_3(E_i T_{12}) = (u + 1) A^2 + (u + 1) A B + A + B$, for all $I \in \mathcal{P}(3)$ of cardinal 2.
Proof. The proof is only a routine of computations. We shall prove, as an example, the third claim. Suppose $I = \{\{1, 2\}, \{3\}\}$, hence $E_I = E_1$. Then, by linearity and using the example above we have

$$\rho_3(E_1T_{12}) = B + A + AB + A^2 + A^2 + uAB + (u - 1)A^2$$

Hence we have proved the claim. \hfill \Box

Theorem 7.7. The Markov trace $\rho_n : \mathcal{E}_n \rightarrow K(A, B)$ passes to PTL$_n$ if only if $A = -B$ or $A = -B/(1 + u)$.

Proof. From Corollary 4.6 we have that $\rho_n$ pass to PTL$_n$ if only if $\rho_n(xE_1E_2T_{12}y) = 0$, for all $x, y \in \mathcal{E}_n$. Now, by linearity and trace properties of $\rho_n$ follows that it is enough to study the conditions to have $\rho_n(xE_1E_2T_{12}) = 0$, for all $x$ in a linear basis of $\mathcal{E}_n$. We consider now the basis $S_n$ of $\mathcal{E}_n$, see Theorem 2.4. Using the rules that define $\rho_n$ we deduce that the computation of $\rho_n(xE_1E_2T_{12})$, for $x \in S_n$, results in a $K(A, B)$-linear combination of $\rho_3(zE_1E_2T_{12})$ with $z \in S_3$. Now, $z$ is of the form $E_1T_w$, with $w \in S_3$ and $I \in \mathcal{P}(3)$; since $T_w$ commutes with $E_1E_2$ having in mind the Lemma 7.6 and the fact that $E_1E_2$ is the maxim element of $\mathcal{P}(3)$, we obtain that $zE_1E_2T_{12}$ is a $K$-scalar multiple of $E_1E_2T_{12}$. Therefore, $\rho_n(xE_1E_2T_{12}y) = 0$, for all $x, y \in \mathcal{E}_n$ is equivalent to have $\rho_3(E_1E_2T_{12}) = 0$. Now, from (2)Lemma 7.6 we have $\rho(E_1E_2T_{12}) = 0$ is equivalent to $(u + 1)A^2 + (u + 2)AB + B^2 = 0$, then $A = -B$ or $A = -B/(1 + u)$. \hfill \Box

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