Consensus Capacity of Noisy Broadcast Channels

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Abstract

We study communication with consensus over a broadcast channel - the receivers reliably decode the sender’s message when the sender is honest, and their decoder outputs agree even if the sender acts maliciously. We characterize the broadcast channels which permit this byzantine consensus and determine their capacity.

I. INTRODUCTION

We consider communication with consensus over a broadcast channel\(^1\). We require the following:

(i) When the sender is honest, the receivers reliably decode the sender’s message.
(ii) Even if the sender acts maliciously, the receivers’ decoder outputs agree.

In the latter case, no correct decoding is demanded; indeed a malicious sender need not have a message in mind while crafting its attack. The problem may be thought of as a common message transmission problem [3] over broadcast channels with the additional stipulation of consensus among receivers even when the sender deviates.

We are motivated by the byzantine generals problem [4, pg. 384] [5], where a commanding general (sender node) wants to communicate a message to a set of lieutenant generals (other nodes) such that, (i) if the commander is honest, all honest lieutenants agree on the commander’s message, and (ii) all honest lieutenants agree on the same message even if the commander is malicious. It is well-known that with three nodes (a commander and two lieutenants), this is impossible to achieve when the nodes communicate over private pairwise communication channels [4]–[6]. For the general case, the impossibility holds when at least one-third of the nodes may act maliciously. There has been a renewed interest in this problem because of its applications to blockchains [7]. We address the following questions:

Which broadcast channels allow byzantine consensus?

And when consensus is possible, what is the capacity of communication with consensus?

There is an extensive literature in information theory on communication in the presence of external adversaries, both passive [8], [9] and active [10]–[13] (also see surveys [14]–[16]). More closely related to the present work are those on communication when the users are byzantine [17]–[24]. Our setup can also be thought of as one in the line of works in cryptography which use stochastic resources (channels and sources) not controlled by the users to realize cryptographic tasks such as privacy amplification [25]–[33], oblivious transfer (and secure computation, in general) [34]–[37], and commitment [38]–[40] with information theoretic security. Here we realize (byzantine-robust) “reliable noiseless broadcast” from a noisy broadcast channel. Unlike byzantine consensus protocols over private links, which typically require several rounds of interaction (leading to poor throughput in applications), ours is a one-way setup (like [41], [42]), where the only means of communication is via the broadcast channel and we exploit the non-trivial correlations in the output signals induced by the channel to achieve consensus.

We show that communication with consensus is possible only when the broadcast channel has embedded in it a natural “common channel” whose output both receivers can unambiguously determine from their own channel outputs. Interestingly, in general, the consensus capacity may be larger than the point-to-point capacity of the common channel, i.e., while decoding, the receivers may make use of parts of their output signals on which they may not have consensus provided there are some parts (namely, the common channel output) on which they can agree. The (non-byzantine) common message capacity is a natural upper bound on the consensus capacity. It turns out this bound may also be loose in general (see Figure 3).

II. SETUP AND PRELIMINARIES

Notation: See [43] for definitions of information theoretic quantities such as mutual information. We mostly follow the notation from there. We employ the method of types in some of our proofs for which we adopt the notation from [44].

Consider a two-receiver memoryless broadcast channel \(W_{YZ|X}\) from a sender (Alice) with input alphabet \(\mathcal{X}\) to receivers, Bob and Carol, resp., with output alphabets \(\mathcal{Y}\) and \(\mathcal{Z}\), resp. We consider finite alphabets. An \((n,K)\) consensus code consists of:

(i) an encoder: \(f : [1 : K] \to \mathcal{X}^n\), and
(ii) decoders: \(g_B : \mathcal{Y}^n \to [1 : K] \cup \{\perp\}\) & \(g_C : \mathcal{Z}^n \to [1 : K] \cup \{\perp\}\).

\(^1\)We use the term broadcast channel in the sense it is used in network information theory [1], [2], where it refers to a potentially noisy channel with a single sender and multiple receivers. In cryptography, the term generally refers to the noiseless special case of this.
We show an example where $A$ (i.e., $2^{-n}P$) is also no smaller than that for average error probability. Thus the consensus capacity is agnostic to the choice of maximal or average error capacities for point-to-point channels, it is clear that given an $(n, 2^nR)$ consensus code, we can construct an $(n, 2^nR/2)$ consensus code with $\lambda$ no larger than $2\lambda'$ by discarding half the codewords with the worse $\lambda_m$'s (and replacing decoder outputs which map to discarded codewords by $\perp$). Hence, the consensus capacity for maximal error probability criterion is also no smaller than that for average error probability. Thus the consensus capacity is agnostic to the choice of maximal or average error probability in its definitions.

Remark 1. Notice that the definition above demands $-\log(P_e^{(n)}) = O(n)$. It turns out that the capacity remains unchanged even if this is relaxed to $P_e^{(n)} = o(1/n)$, the condition under which we prove our converse. Surprisingly, it turns out that a converse cannot be shown if this is further relaxed to $P_e^{(n)} = o(1)$. In Appendix A we show an example where $C_{\text{Byz}} = 0$, but a positive rate is achievable with $P_e^{(n)} = o(1/n^{3\varepsilon-\epsilon})$, for any $\epsilon > 0$.

A. Some remarks on the definitions

a) Average error probability: We may also define a notion of “average” error probability $P_{\text{avg}}$ as the maximum of $\eta$ and $\lambda'$ defined below:

$$\lambda' = \frac{1}{K} \sum_{m=1}^{K} \lambda_m.$$  

We now argue that the capacities remain unchanged if we replace maximal error probability with average error probability in their definitions. Clearly, $\lambda' \leq \lambda$ and hence $P_{\text{avg}} \leq P_{\text{max}}$. Thus, the consensus capacity for the average error criterion is no smaller than that for maximal error probability. Along the lines of the standard expurgation argument connecting maximal and average error capacities for point-to-point channels, it is clear that given an $(n, 2^nR)$ consensus code, we can construct an $(n, 2^nR/2)$ consensus code with $\lambda$ no larger than $2\lambda'$ by discarding half the codewords with the worse $\lambda_m$’s (and replacing decoder outputs which map to discarded codewords by $\perp$). Hence, the consensus capacity for maximal error probability criterion is also no smaller than that for average error probability. Thus the consensus capacity is agnostic to the choice of maximal or average error probability in its definitions.

Fig. 1: For any input $x^n$, the outputs of $g_B$ and $g_C$ agree. If $x^n$ is the codeword for some message $m$, then both output $m$. The rate of the encoder is $\log(K)/n$. The encoder and the decoders are deterministic; we comment on this and other choices we make in setting up the problem in Section II-A.

a) Error probability: An error occurs when either

(i) the outputs of the decoders do not match (i.e., $g_B(Y^n) \neq g_C(Z^n)$) irrespective of what the sender transmitted; or

(ii) if the sender transmitted the codeword $f(m)$ corresponding to a message $m \in [1 : K]$ and the output of at least one of the decoders does not match the message $m$.

We will refer to a sender whose transmission is not from the codebook as a malicious sender. We define

$$\lambda_m = 1 - \Pr(g_B(Y^n) = g_C(Z^n) = m | f(m)), \ m \in [1 : K]$$

$$\eta_{x^n} = \Pr(g_B(Y^n) \neq g_C(Z^n) | x^n), \ x^n \in X^n,$$

where we use the shorthand notation $\Pr(. | x^n)$ to denote $\Pr(. | X^n = x^n)$. Let

$$\lambda = \max_{m \in [1 : K]} \lambda_m,$$
$$\eta = \max_{x^n \in X^n} \eta_{x^n}.$$  

The probability of error of the code $(f, g_B, g_C)$ is defined as:

$$P_e = \max(\lambda, \eta). \ \ \ \ \ (1)$$

We write $P_e^{(n)}$ when we want to explicitly show the dependence on the block length $n$.

b) Achievable rates, capacity: We say rate $R$ is achievable with consensus if there is an $\epsilon > 0$ such that for all sufficiently large $n$ there is an $(n, [2^nR])$ consensus code with $P_e^{(n)} \leq 2^{-n\epsilon}$ (we suppress the floor function in the sequel). The consensus capacity $C_{\text{Byz}}$ is the supremum of all rates achievable with consensus.

We write $P_e^{(n)}$ when we want to explicitly show the dependence on the block length $n$. $P_e^{(n)} = o(1/n)$. It turns out that the capacity remains unchanged even if this is relaxed to $P_e^{(n)} = o(1/n)$, the condition under which we prove our converse. Surprisingly, it turns out that a converse cannot be shown if this is further relaxed to $P_e^{(n)} = o(1)$. In Appendix A we show an example where $C_{\text{Byz}} = 0$, but a positive rate is achievable with $P_e^{(n)} = o(1/n^{3\varepsilon-\epsilon})$, for any $\epsilon > 0$.

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b) Randomization: Allowing for common randomness shared by the sender and both receivers does not change the capacities\textsuperscript{3}. This also implies that private randomization by the sender does not alter the capacities (since turning the private randomness at the sender into common randomness by providing it to both the decoders cannot decrease the capacity).

The presence of randomness shared by the decoders (or more generally, samples of correlated sources at the decoders independent of the channel) and unknown to the sender can be absorbed in the model as an additional component in the channel outputs \(Y\) and \(Z\) independent of the input and the rest of the channel outputs; so we do not introduce separate notation for this. Our results will show that this additional shared randomness has no effect on the consensus capacity \(C_{\text{Byz}}\) (see Remark \textsuperscript{4}). However, see Appendix A for the role common randomness shared by the decoders and unknown to the decoder may play in communication with consensus.

B. Common channel

The common channel of a broadcast channel will play a vital role in the characterization of its consensus capacity.

**Definition 1.** The characteristic graph of a channel \(W_{Y|Z|X}\) is the bipartite graph \(G_W = (\mathcal{N}, \mathcal{E})\), with vertex set \(\mathcal{N} = \mathcal{Y} \cup \mathcal{Z}\) and edge set \(\mathcal{E} = \{(y, z) : W(y, z|x) > 0 \text{ for some } x \in \mathcal{X}\}\). Let \(\mathcal{V}\) be such that \(G_v = (\mathcal{N}_v, \mathcal{E}_v), v \in \mathcal{V}\) are the distinct connected components\textsuperscript{5} of the characteristic graph \(G_W\). The common channel \(W_{V|X}\) of \(W_{Y|Z|X}\) is a point-to-point channel with input alphabet \(\mathcal{X}\) and output alphabet \(\mathcal{V}\) such that

\[
W_{V|X}(v|x) = \sum_{\{y,z\} \in \mathcal{E}_v} W_{Y|Z|X}(y, z|x), \quad v \in \mathcal{V}, x \in \mathcal{X}.
\]

Also define, for \((x, y, z, v) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{V}\),

\[
W_{Y|Z|X|V}(y, z|x, v) = \begin{cases} 
W_{Y|Z|X}(y, z|x), & \{y, z\} \in \mathcal{E}_v \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly both receivers can infer the common channel output. Specifically, the functions \(\phi_1 : \mathcal{Y} \to \mathcal{V}\) and \(\phi_2 : \mathcal{Z} \to \mathcal{V}\) which map their arguments to the index of the connected component to which they belong are such that \(\phi_1(Y) = \phi_2(Z) = V\) irrespective of the channel input symbol \(x\). Hence, \(C_{\text{p-to-p}}(W_{V|X}) = \max_{P_X} I(X; V)\) is a lower bound on \(C_{\text{Byz}}\). As mentioned, a natural upper bound is the (non-byzantine) common message capacity,

\[
C_{\text{com-msg}}(W_{Y|Z|X}) = \max_{P_X} \min(I(X; Y), I(X; Z)).
\]

Hence,

\[
C_{\text{p-to-p}}(W_{V|X}) \leq C_{\text{Byz}}(W_{Y|Z|X}) \leq C_{\text{com-msg}}(W_{Y|Z|X}).
\]

Our main result (Theorem \textsuperscript{4}) will imply that \(C_{\text{Byz}}(W_{Y|Z|X}) > 0\) if and only if \(C_{\text{p-to-p}}(W_{V|X}) > 0\) and that, in general, the inequalities above may be loose (also see Figure \textsuperscript{3}). We note in passing that, when \(|\mathcal{X}| = 1\), the common channel reduces to the common random variable (of the pair \(Y, Z\)) related to the notion of common information of Gács and Körner [45].

**Example 1** (Two-step binary erasure broadcast channel). Let \(p \in [0, 1], q \in (0, 1]\) and \(\mathcal{X} = \{0, 1\}, \mathcal{Y} = \mathcal{Z} = \{0, 1, \bar{0}, \bar{1}, e\}\). See Figure \textsuperscript{2}.

\[
W_{Y|Z|X}(y, z|x) = \begin{cases} 
1 - p, & y = z = x, x \in \mathcal{X} \\
pQ(y|x)Q(z|x), & y, z \in \{\bar{0}, \bar{1}, e\}, x \in \mathcal{X},
\end{cases}
\]

where \(Q(e|x) = 1 - Q(\bar{e}|x) = q, x \in \mathcal{X}\) with a slight abuse notation to denote \(\bar{0}\) (\(\bar{1}\), resp.) by \(\bar{x}\) when \(x = 0\) (1, resp.). From Figure \textsuperscript{2}, the common channel can be seen to be \(W_{V|X}(\Delta|x) = 1 - W_{V|X}(x|x) = p\) where the common channel output alphabet is \(\mathcal{V} = \{0, 1, \Delta\}\). Note that \(q = 0\) above amounts to the noiseless channel (with an additional Bernoulli-\(p\) common random variable output independent of the input). Clearly all capacities are \(1\) here. We do not include this straightforward case in our parametrization so that the discussion below can be kept general.

\textsuperscript{3}Notice that a malicious sender may choose its transmission depending on the realization of the common randomness. Hence the probability of error when the sender and the receivers share common randomness is the weighted average of probabilities of error (of the deterministic codes) under the different possible realizations of common randomness. Thus, there is a deterministic code whose probability of error is no worse than that of a code with common randomness.

\textsuperscript{4}A connected component of a graph is an induced subgraph in which every pair of vertices is connected by a path and which is not connected to any vertices in the rest of the graph. Without loss of generality, we assume that each letter in \(Y(Z, \text{resp.})\) receive positive probability under \(W_{Y|X}(W_{Z|X}, \text{resp.})\) for some input letter so that none of the connected components consist of a single vertex.
Fig. 2: Two-step binary erasure broadcast channel. The channel erases in two steps – with probability $1-p$, both receivers (simultaneously) receive the input symbol unerased; with the remaining probability $p$, the input symbol is passed further through independent binary erasure channels which erase with probability $q$ and whose unerased output symbols acquire a $\tilde{}$. The characteristic graph has three connected components (unless $p=1$ when there is only one connected component). The common channel is a binary erasure channel with erasure probability $p$ and erasure symbol $\Delta$.

III. Consensus Capacity of the Two-Step Binary Erasure Broadcast Channel

We first illustrate some of the key ideas behind the proof of our capacity theorem (Theorem 4) by considering the special case of the two-step binary erasure broadcast channel. Notice that the marginal channel to each receiver is, effectively, a binary erasure channel (BEC) with erasure probability $pq$, and hence $C_{\text{com-msg}} = 1 - pq$, which is also the point-to-point capacity of the marginal channels; the uniform probability $P_X$ simultaneously maximizes $I(X;Y)$ and $I(X;Z)$, and hence (4). For $p < 1$, we show that $C_{\text{Byz}} = 1 - pq = C_{\text{com-msg}} > C_{p-to-p}(W_{V|X}) = 1 - q$. If $p = 1$, the common channel is trivial and we show that $C_{\text{Byz}} = 0$.

A. Converse: $C_{\text{Byz}} = 0$ if $p = 1$

With $p = 1$, the channel is the independent binary erasure broadcast channel $W_{Y,Z|X}(y,z|x) = Q(y|x)Q(z|x)$, where $Q(e|x) = 1 - Q(\tilde{x}|x) = q$. Let $q \in (0,1)$ as the case of $q = 1$ is obvious. Note that the characteristic graph has a single connected component. Consider an $(n, 2^{nR})$ consensus code $(f, g_B, g_C)$ with error probability $P_e$. The key ingredient will be the following claim which states that changing the channel input vector at one location ($k$-th, say) should only produce a small effect on the decisions of the decoders.

**Claim 1.** Suppose $k \in [1:n]$, $m \in [1:2^{nR}]$, and $\tilde{x}_1, \ldots, \tilde{x}_{k-1}, \tilde{x}_k, x_{k+1}, \ldots, x_n \in \mathcal{X}$. Let $A_m$ be the event $(g_B(Y^n) = g_C(Z^n) = m)$.

$$\Pr(A_m|X^n = (x_1^{k-1}, x_k, x_{k+1}^n)) - \Pr(A_m|X^n = (\tilde{x}_1^{k-1}, \tilde{x}_k, x_{k+1}^n)) \leq P_e \rho,$$

where $\rho = \frac{5}{\min(q^2, (1-q)^2)}$.

We prove the claim later, the intuition follows: As the channel is memoryless, any change in the decisions of the decoders must be based on the channel outputs $Y_k$ and $Z_k$, resp. (i.e., outputs at the location with the change in input). Since the...
characteristic graph has a single connected component, the decoders cannot extract a non-trivial common part from $Y_k, Z_k$ and, as their decisions must agree with high probability for any input, the effect on their decisions must be small.

To complete the proof, consider two distinct messages $m, \hat{m}$ with codewords $x^n = f(m)$ and $\hat{x}^n = f(\hat{m})$. Summing (5) over $k = 1, \ldots, n$,

$$\Pr(A_m | X^n = x^n) - \Pr(A_m | X^n = \hat{x}^n) \leq nP_e \rho.$$ 

Since $\Pr(A_m | X^n = x^n) = \Pr(g_B(Y^n) = g_C(Z^n) = m | X^n = x^n) \geq 1 - P_e$,

$$\Pr(g_B(Y^n) = g_C(Z^n) = m | X^n = \hat{x}^n) \geq 1 - P_e (1 + n \rho).$$

Thus, $P_e \geq \lambda_\rho \geq 1 - P_e (1 + n \rho)$ and hence $P_e \geq 1/(2 + n \rho)$. Since the definition of $C_{Byz}$ requires $P_e$ to decay faster than this as $n \to \infty$, $C_{Byz} = 0$.

**Proof of Claim 1.** Consider the random variables $\tilde{Y}_1^k, \tilde{Z}_1^k, Y_k^n, Z_k^n$ jointly distributed as

$$p_{\tilde{Y}_1^k, \tilde{Z}_1^k, Y_k^n, Z_k^n}(\tilde{y}_1^k, \tilde{z}_1^k, y_k^n, z_k^n) = \prod_{i=1}^{k} W_{YZ|X}(\tilde{y}_i, \tilde{z}_i | x_i) \prod_{i=k}^{n} W_{YZ|X}(y_i, z_i | x_i).$$

(6)

Notice that we are defining a coupling where $(\tilde{Y}_1^{k-1}, Y_k^n, Z_k^{k-1})$ and $(\tilde{Y}_1^{k-1}, Y_k^n, Z_k^{k+1})$ have the same distributions as $(Y^n, Z^n)$ in the first and second terms, resp. of (5). Let $S = (\tilde{Y}_1^{k-1}, Y_k^n), T = (\tilde{Z}_1^{k-1}, Z_k^{k+1})$. For $s = (y_1^{k-1}, y_k^n), t = (z_1^{k-1}, z_k^{k+1})$, abusing notation, we will write $g_B(s, y)$ to mean $g_B((y_1^{k-1}, y_k^n, y_{k+1}^{k+1}))$. Similarly, we will also use $g_C(t, z)$. We have (by (1)), for $x \in \mathcal{X}$,

$$P_e \geq \Pr(\text{decoders disagree}|X^n = (x_1^k, x, x_k^{k+1}))$$

$$= \sum_{y, z} W_{YZ|X}(y, z | x) \Pr(g_B(S, y) \neq g_C(T, z)).$$

Hence, for every edge $(y, z) \in E$ in the characteristic graph (i.e., $W_{YZ|X}(y, z | x)$ for some $x$),

$$\Pr(g_B(S, y) \neq g_C(T, z)) \leq \frac{P_e}{\min_{(x, y, z) : W_{YZ|X}(y, z | x) > 0} W_{YZ|X}(y, z | x)}$$

$$= \frac{P_e}{\min(q^2, (1 - q)^2)}.$$ 

(7)

Consider the event $E$ in which the decoder outputs do not depend on the $k$-th element of their channel output vectors,

$$E = \left( \bigcup_{(y, z) \in E} \{g_B(S, y), g_C(T, z) \} \right) = 1.$$ 

Since the characteristic graph of the channel is connected and has a spanning tree with 5 edges, from (7), we may conclude using a union bound that

$$\Pr(E) \geq 1 - \frac{5P_e}{\min(q^2, (1 - q)^2)} = 1 - P_e \rho.$$ 

(8)

i.e., under the distribution of $(S, T)$, with probability at least $1 - P_e \rho$, the decoder outputs do not depend on the $k$-th element of their channel output vectors. Then,

$$\Pr(g_B(S, Y_k) = g_C(T, Z_k) = g_B(S, \hat{Y}_k) = g_C(T, \hat{Z}_k))$$

$$= \sum_{y, z, y, \hat{z}} W_{YZ|X}(y, z | x_k) W_{YZ|X}(\hat{y}, \hat{z} | \hat{x}_k) \Pr(g_B(S, \hat{y}) = g_C(T, \hat{z}) = g_B(S, y) = g_C(T, z))$$

$$\geq \sum_{y, z, y, \hat{z}} W_{YZ|X}(y, z | x_k) W_{YZ|X}(\hat{y}, \hat{z} | \hat{x}_k) \Pr(E)$$

$$\geq 1 - P_e \rho.$$


Hence, 
\[
\Pr(g_B(S, Y_k) = g_C(T, Z_k) = m) - \Pr(g_B(S, \tilde{Y}_k) = g_C(T, \tilde{Z}_k) = m) \leq P_\epsilon \rho.
\]

**Remark 2.** Below we strengthen the converse to show that even a single bit cannot be communicated with consensus over this channel with \( P_\epsilon^{(n)} \to 0 \). This also means that for this channel the converse does not require the more restrictive \( P_\epsilon^{(n)} = o(1/n) \). However, as mentioned in Remark 1, in general, such a requirement is necessary and our proof of the converse of Theorem 4 generalizes the proof idea above.

We will show that for the independent binary erasure broadcast channel with erasure probability \( q > 0 \), there exists \( \epsilon > 0 \) such that \( P_\epsilon^{(n)} \geq \epsilon \) for any \((n,2)\) consensus code, \( n \in \mathbb{N} \). Consider an \((n,2)\) consensus code \((f,g_B,g_C)\), with codewords \( f(1) = (x_1, \ldots, x_n) =: x^n \) and \( f(2) = (\hat{x}_1, \ldots, \hat{x}_n) := \hat{x}^n \). We have
\[
\begin{align*}
\Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = \hat{x}^n) & \geq 1 - P_\epsilon, \\
\Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = x^n) & \leq 1 - \Pr(g_B(Y^n) = g_C(Z^n) = 1|X^n = x^n) \leq P_\epsilon.
\end{align*}
\]

Furthermore, by Claim 1, for all \( k \in [1:n] \), \( m \in \{1, 2\} \),
\[
\Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = (\hat{x}^{k-1}, x^n_k)) - \Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = (x^{k}, x^n_{k-1})) \leq P_\epsilon \rho
\]
i.e., \( \Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = (\hat{x}^{k}, x^n_{k-1})) \) is at most \( P_\epsilon \) for \( k = 0 \) (by (10)); changes by at most \( P_\epsilon \rho \) at each step as \( k \) increases from 0 to \( n \) in steps of 1 (by (11)); and is atleast \( 1 - P_\epsilon \) at \( k = n \) (by (9)). Hence, there must be a \( k \in [1:n] \) such that
\[
\frac{1}{2} - P_\epsilon \rho \leq \Pr(g_B(Y^n) = g_C(Z^n) = 2|X^n = (\hat{x}_1, x^n_{k+1})) \leq \frac{1}{2} + P_\epsilon \rho.
\]

For this \( k \), fix \( X^n = \tilde{x}^n := (\hat{x}_1, x^n_{k+1}) \). Then \((Y^n, Z^n)\) have the following joint distribution: \((Y_i, Z_i)\) are independent over \( i = 1, \ldots, n \), with the joint distribution of \((Y_i, Z_i)\) given by
\[
P_{Y_i, Z_i}(y, z) = \begin{cases} 
q^2, & (y, z) = (e, e) \\
(1 - q)^2, & (y, z) = (\tilde{x}_i, \tilde{x}_i) \\
q(1 - q), & (y, z) \in \{(e, \tilde{x}_i), (\tilde{x}_i, e)\} \\
0, & \text{otherwise}.
\end{cases}
\]

For all \( i \in [1:n] \), \( P_{Y_i, Z_i} \) has zero Gács-Körner common information [45] (i.e., their maximum correlation [46]-[48] is less than unity). Hence, by a result of Witsenhausen [49], there exists \( \epsilon' > 0 \) (which depends only on \( q \)) such that for all deterministic functions\( ^3 \) (specifically, \( g_B \) and \( g_C \)), (12) holds only if \( P_\epsilon \rho \geq \epsilon' \). Thus, for arbitrarily small \( P_\epsilon \), \((n,2)\) consensus codes do not exist for any \( n \in \mathbb{N} \).

Indeed, using this argument, we can prove such an impossibility for any channel that satisfies the following properties:

1) The characteristic graph has a single connected component (it is easy to see that our proof of Claim 1 made use of only this property of the channel); and

2) For each of its input symbols \( x \in \mathcal{X} \), the joint distribution \( W_{Y,Z|X}(\cdot, \cdot|x) \) induced at the output by the channel has zero common information (so that the impossibility in [49] applies).

**B. Achievability:** \( C_{Byz} = 1 - pq \) if \( p < 1 \)

We introduce some notation and describe our decoder before giving the intuition behind our scheme. For \( x^n \in \mathcal{X}^n \) and \( v^n \in \mathcal{Y}^n \), we write \( x^n \triangleright v^n \) if \( v^n \) is an “erased” version of \( x^n \), i.e., if \( v_i \in \{x_i, \Delta\}, i \in [1:n] \). Similarly, \( x^n \blacktriangleright y^n \) if \( y_i \in \{x_i, \tilde{x}_i, e\}, i \in [1:n] \). Let \( d(x^n, \tilde{x}^n) := \frac{1}{m}(d_{\text{Hamming}}(x^n, \tilde{x}^n)) \) be the relative distance between \( x^n, \tilde{x}^n \in \mathcal{X}^n \).

Let \( \delta > 0 \). For an encoder \( f \) of rate \( R \), the decoder outputs \( g_B(y^n) = m \) if it is the unique \( m \in [1:2^nR] \) such that

1) \( f(m) \triangleright \phi_1(y^n), \) where \( \phi_1(y^n) := (\phi_1(y_i))_{i \in [1:n]} \).

2) there is a \( \tilde{x}^n \) such that \( d(f(m), \tilde{x}^n) < \delta \) and \( \tilde{x}^n \triangleright y^n \).

\( g_B(y^n) = \perp \) if no such unique \( m \) exists. \( g_C \) is similarly defined (with \( \phi_2 \) in lieu of \( \phi_1 \)). The first decoding condition requires the codeword to match the bits left unerased by the common channel; we denote this by \( m \triangleright y^n \). The second condition, denoted by \( m \blacktriangleright y^n \), requires an “explaining” vector \( \tilde{x}^n \) which is \( \delta \)-close to the codeword and matches the bits left unerased in \( y^n \).

\(^3\)Witsenhausen [49] considers functions which make a binary decision. Here, we may view the decoders \( g_B,g_C \) as making a binar decision returning either the symbol 2 or a symbol from \( \{1, \perp\} \).
The intuition behind our coding scheme is as follows: Since the first decoding condition above only depends on the common channel output, both decoders will make the same decision on this. However, if they were to rely only on this condition, they cannot achieve rates above the common channel capacity $1 - q$. Instead, if they were to use the decoding condition $f(m) \rightarrow y^n$ (resp., $f(m) \rightarrow z^n$) which (with an erasure code) can achieve all rates below $1 - pq$ in the non-byzantine setting, there is a simple attack for the byzantine sender – send $f(m)$ with one of the bits flipped. A receiver for which this bit is erased by the channel may accept $m$ while one for which this bit is left unerased will reject $m$; since there is a finite probability $2pq(1 - q)$ that this bit is erased for exactly one of the receivers, with non-vanishing probability they may disagree. The second decoding condition above circumvents this by tolerating some errors. A malicious sender may still try to get the receivers to disagree by sending a vector which is close to a codeword (but is not the codeword itself) is sent, there is a significant probability that the message corresponding to that codeword is rejected; but this rejection (based on the first decoding condition) is carried out by both the receivers simultaneously so that their decisions still agree (see case (iii) below).

Turning to the formal proof, for $x^n \in \mathcal{X}^n$, define the event

$$B_{x^n} = (\exists m \in [1 : 2^nR] : d(f(m), x^n) \geq \delta, m \cap Y^n, m \cap Y^n).$$

**Claim 2.** Let $R < 1 - pq$. There are positive $\delta, \epsilon$ such that, for sufficiently large $n$, there is an encoder $f : [1 : 2^nR] \rightarrow \mathcal{X}^n$ with $d(f(m), f(m')) \geq 2\delta$ for every pair $m \neq m'$ and

$$\Pr(B_{x^n} | x^n) \leq 2^{-n\epsilon}, \text{ for all } x^n \in \mathcal{X}^n.$$  \hspace{1cm} (13)

To see that this implies the theorem, let us consider three possibilities for the transmitted vector $x^n$:

- **Case (i):** $x^n = f(m)$ for some $m$. Then, $m \cap Y^n$ and $m \cap Y^n$. Moreover, for all $m' \neq m$, the encoder in Claim 2 has $d(f(m), f(m')) \geq 2\delta$. Hence, $\Pr(g_B(Y^n) \neq m(f(m)) = \Pr(\exists m' \neq m \text{ s.t. } m'n \cap Y^n, m'n \cap Y^n | f(m)) = \Pr(B_{f(m)} | f(m)) \leq 2^{-n\epsilon}$, where the last inequality follows from (13). Similarly, $\Pr(g_C(Z^n) \neq m(f(m)) \leq 2^{-n\epsilon}$. By a union bound, $\Pr(g_B(Y^n) = g_C(Z^n) = m(f(m)) \geq 1 - 2^{-n\epsilon+1}$.

- **Case (ii):** $d(f(m), x^n) \geq \delta$ for all $m \in [1 : 2^nR]$. Then, $\Pr(g_B(Y^n) \neq \perp | x^n) \leq \Pr(B_{x^n} | x^n) \leq 2^{-n\epsilon}$ where the last step is from (13). Hence, by a union bound $\Pr(g_B(Y^n) = g_C(Z^n) = \perp | x^n) \geq 1 - 2^{-n\epsilon+1}$.

- **Case (iii):** there is an $m$ such that $d(f(m), x^n) < \delta$, but $x^n \neq f(m)$. Since $d(f(m), x^n) < \delta$, by triangle inequality, $d(f(m'), x^n) \geq d(f(m'), f(m)) - d(f(m), x^n) > 2\delta - \delta = \delta$ for all $m' \neq m$. Hence, $\Pr(g_B(Y^n) \notin \{m, \perp\} | x^n) \leq \Pr(B_{x^n} | x^n) \leq 2^{-n\epsilon}$, where the last step follows from (13). By the union bound, $\Pr(g_B(Y^n), g_C(Z^n) \in \{m, \perp\} | x^n) \geq 1 - 2^{-n\epsilon+1}$.

We will argue, that for this $x^n$ and under the event $(g_B(Y^n), g_C(Z^n) \in \{m, \perp\})$, the decoder outputs must match which will complete the proof. $m \cap Y^n$ as $x^n$ may serve as the explaining vector $\tilde{x}^n$ since $d(f(m), x^n) < \delta$ and $x^n \rightarrow Y^n$. Similarly, $m \cap Z^n$. Hence, the second decoding condition for message $m$ is met for both decoders. Since $\phi_1(Y) = \phi_2(Z)$, either the first condition for message $m$ is met or not met together for both decoders. Hence, $\Pr(g_B(Y^n) = g_C(Z^n) | x^n) \geq 1 - 2^{-n\epsilon+1}$.

**Proof of Claim 2.** We use the method of proof and follow the notation from [44]. Let $P$ be the uniform type$^6$ on $\mathcal{X}$, $P(0) = P(1) = 1/2$, and $H_2$ denote the binary entropy function.

**Lemma 3.** For $\delta < 1/4$, $0 < \epsilon \leq R \leq 1 - H_2(2\delta) - \epsilon$ and sufficiently large $n$, there exists an encoder $f : [1 : 2^nR] \rightarrow \{0, 1\}^n$ whose codewords $f(m), m \in [1 : 2^nR]$ are of type $P$ such that

$$d(f(m), f(m')) \geq 2\delta \text{ for all } m \neq m',$$

and for every joint type $P_{X'X} \in \mathcal{P}^n(\mathcal{X} \times \mathcal{X})$ and $x^n \in \mathcal{X}^n$,

$$\{m : f(m) \in T_{X'|X}(x^n)\} \leq 2^n\left(R - I(X';X) + \epsilon\right).$$  \hspace{1cm} (14)

We can show the above lemma using a random coding argument. The first property is similar to the Gilbert-Varshamov bound and gives a minimum distance guarantee (also see [50, Problem 10.1(c)]). The second property is similar to [44, (V.10)]. The lemma follows from Lemma 10 where we take $U = \mathcal{X}$ and $P$ to be the uniform type, i.e., $P(0) = P(1) = 0.5$. For $P := \{P_{X'X} \in \mathcal{P}^n(\mathcal{X} \times \mathcal{X}) : P_X = P_{X'}, = P, \Pr( X \neq X') < 2\delta, \min_{P_{X'X} \in P} I(X';X') = \min_{P_{X'X} \in P} 1 - H_2(\Pr(X \neq X')) \leq 1 - H_2(2\delta)\}$ where is last inequality follows from $\delta < 1/4$. Choose $\delta > 0$ sufficiently small so that $\delta < 1/4, 2(H_2(\delta) + \delta) < (1 - pq - R)$, and $R < 1 - H_2(2\delta)$. Further, choose sufficiently small $\epsilon > 0$ so that $\epsilon \leq R \leq 1 - H_2(2\delta) - \epsilon$ and consider the codebook from Lemma 3. Let $D$ be the set of all joint types $P_{X'X',Y} \in \mathcal{P}^n(\mathcal{X} \times \mathcal{X} \times \mathcal{Y})$ such that

(i) $X \triangleright \phi_1(Y)$, \hspace{1cm} (ii) $\Pr(X' \neq X) \geq \delta$, and

$^6$For simplicity, we assume $n$ is even; the case of odd $n$ is easily handled by perturbing $P$ (or leaving unused, say, the last bit).
(iii) \( X' \triangleright \phi_1(Y), \exists P_{X|X'Y} \) s.t. \( \Pr (X' \neq \bar{X}) < \delta, \bar{X} \triangleright Y \).

Consider the definition of \( B_n \). Suppose, for channel input \( x^n \), output \( Y^n \) and \( m \in [1 : 2^nR] \) are such that \( d(f(m), x^n) \geq \delta \) and \( m \triangleright Y^n, m \triangleright Y^n \). Then, the joint type of \((x^n, f(m), Y^n)\) belongs to \( \mathcal{D} \), i.e., \((x^n, f(m), Y^n) \in \mathcal{T}_{X \triangleright Y}^n \) for some \( P_{X|Y} \in \mathcal{D} \), since (i) \( x^n \triangleright \phi_1(Y) \), (ii) \( d(f(m), x^n) \geq \delta \), and (iii) \( m \triangleright Y^n \). Further, let \( \mathcal{D}_{\text{typical}} \) be the set of joint types \( P_{X|Y} \) s.t. \( P_Y(e) \leq \rho q + \sqrt{\epsilon} \) and \( \Pr(\phi_1(Y) = |X = x) = \sum_{y \in [0, 1]} P_{|Y|X}(y|x) \leq p + 2\sqrt{\epsilon/\delta} \) for all \( x \in X \) for which \( P_X(x) \geq \delta /4 \). Here, we further restrict \( \epsilon > 0 \) to be small enough so that \( \rho q + \sqrt{\epsilon}, p + 2\sqrt{\epsilon/\delta} < 1 \) (recall, \( p < 1 \)). By a union bound over \( \mathcal{D} \),

\[
\Pr (B_n|x^n) \leq \sum_{P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}}} \Pr (\exists m : (x^n, f(m), Y^n) \in \mathcal{T}_{X \triangleright Y}^n )
\]

\[
+ \sum_{P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}}} \Pr ((x^n, Y^n) \in \mathcal{T}_{X \triangleright Y}^n ).
\]

The second term can be upper bounded by \( 3(n + 1)2^{2-2\epsilon n} \) using Sanov’s theorem. To see this, let \( J := 1_{\phi_1(Y) = \Delta} \) be the indicator random variable of \((\phi_1(Y) = \Delta) \). Let \( \tilde{J} := 1_{(Y = e)} \) and \( J^n = (1_{(\phi_1(Y) = \Delta)})_{i=1}^n \), \( \tilde{J}^n = (1_{(Y = e)})_{i=1}^n \). Let \( \tilde{P} \) be the type of \( x^n \).

\[
\sum_{P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}}} \Pr ((x^n, Y^n) \in \mathcal{T}_{X \triangleright Y}^n |x^n)
\]

\[
\leq \sum_{x \in P(x) \geq \delta/4} \sum_{P_{j|x}: P_{j|x}(1|x)} \Pr (J^n \in \mathcal{T}_{j|x}^n (x^n)) + \sum_{P_{j|x}: P_{j|x}(1|x) > p+2\sqrt{\epsilon/\delta}} \Pr (\tilde{J}^n \in \mathcal{T}_{\tilde{j}}^n)
\]

\[
\leq 2(n + 1)2^{2-2\epsilon n} + (n + 1)2^{2-2\epsilon n} \leq 3(n + 1)2^{2-2\epsilon n}.
\]

The second inequality follows from Sanov’s theorem and Pinsker’s inequality. Specifically, suppose \( x \in X \) is such that \( P(x) \geq \delta /4 \). Then, for \( \sum_{P_{j|x}: P_{j|x}(1|x) > p+2\sqrt{\epsilon/\delta}} \Pr (J^n \in \mathcal{T}_{j|x}^n (x^n)) \), the exponent of the upper bound from Sanov’s theorem is

\[
-(n \tilde{P}(x))D \left( p + 2\sqrt{\epsilon/\delta} \mid \frac{p}{\epsilon} \right) \leq -n \tilde{P}(x) \frac{8\epsilon}{\delta \ln 2} \leq -2\epsilon,
\]

where the first inequality follows from Pinsker’s inequality. For the first term of the upper bound on \( \Pr (B_n|x^n) \) above,

\[
\sum_{P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}}} \Pr (\exists m : (x^n, f(m), Y^n) \in \mathcal{T}_{X \triangleright Y}^n )
\]

\[
= \sum_{P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}}} \sum_{m, f(m) \in \mathcal{M}} \Pr (y^n|X^n, f(m)) W(y^n|x^n)
\]

\[
\leq 2^n \left( |X| + 1 \right) W(y^n|x^n)
\]

where (a) follows from (14), the fact that \( |\mathcal{T}_{Y|X}^n (x^n, f(m))| \leq 2^{nH(Y|X^n)} \), and \( W(y^n|x^n) \) is the same for each \( y^n \in \mathcal{T}_{Y|X}^n (x^n, f(m)) \) and is upper bounded by \( 1/|\mathcal{T}_{Y|X}^n (x^n)| \) which in turn is upper bounded by \( 2^{-nH(Y|X^n)+\epsilon} \) for sufficiently large \( n \). For \( P_{X|Y} \in \mathcal{D} \cap \mathcal{D}_{\text{typical}} \), let \( \zeta_{X \triangleright Y} := 2^n(1-{n-I(X';X)} + \epsilon - I(Y;X'|X)) \). We argue that \( \zeta_{X \triangleright Y} \leq 2^{2-2\epsilon n} \) for sufficiently small \( \epsilon > 0 \).

Case (i): \( R \leq I(X';X) \). Recall that \( J := 1_{\phi_1(Y) = \Delta} \). Then \( I(Y';X') \geq I(J;X'|X) \). We show that \( I(J;X'|X) \geq 4\epsilon \). To see the intuition behind the argument, suppose \( I(J;X'|X) = 0 \). Since \( X \triangleright \phi_1(Y) \) and \( X' \triangleright \phi_1(Y) \), we have \( X = X' \) whenever \( J = 0 \). Also, since \( P_{X|Y} \in \mathcal{D}_{\text{typical}} \), for \( x \) s.t. \( P_X(x) \geq \delta /4 \), we have \( P_{X|Y}(x) \leq p + 2\sqrt{\epsilon/\delta} \) which is strictly smaller than 1 (by the further restriction on \( \epsilon \) we imposed); hence for such \( x \), \( P_{X|Y}(0|x) > 0 \). Together with our supposition that \( I(J;X'|X) = 0 \), this implies that for such \( x \), \( P_{X|X,J}(x|x, 0) = 1 \). Thus, \( \Pr (X' \neq X) \leq \sum_{x: P_X(x) \leq \delta /4} P_X(x) \leq \delta /2 \) which contradicts \( \Pr (X' \neq X) \geq \delta \). This intuition can be extended to obtain a contradiction for
\( I(J; X'|X) \leq 4\epsilon \) for a sufficiently small choice of \( \epsilon > 0 \). To see this, suppose \( I(J; X'|X) \leq 4\epsilon \). By Pinsker’s inequality, this implies that

\[
\sum_{j, x', x} P_{JX}(j, x) |P_{X'|JX}(x'|j, x) - P_{X'|X}(x'|x)| \leq \frac{8\epsilon \ln 2}{\epsilon}.
\]

As argued above, for \( x \in \mathcal{X} \) such that \( P_X(x) \geq \delta/4 \), \( P_{X'|JX}(x|0, x) = 1 \). Hence, for such \( x \), \( P_{X'|X}(x|x) \geq 1 - \frac{\sqrt{8\epsilon \ln 2}}{P_{JX}(0, x)} \).

Thus,

\[
\Pr(X' \neq X) = 1 - \sum_x P_{X|X'}(x, x)
\leq \sum_{x: P_X(x) \geq \delta/4} \frac{\sqrt{8\epsilon \ln 2}}{P_{JX}(0|x)} + \sum_{x: P_X(x) < \delta/4} \frac{\delta}{4}
\leq \frac{2\sqrt{8\epsilon \ln 2}}{1-p} + \frac{\delta}{2},
\]

where in the last step we used the fact that for \( x \) s.t. \( P_X(x) \geq \delta/4 \), \( P_{JX}(0|x) = 1 - P_{JX}(1|x) \geq 1 - p - 2\sqrt{\epsilon/\delta} \). This contradicts \( \Pr(X \neq X') \geq \delta \) if \( \epsilon > 0 \) is chosen sufficiently small such that \( \frac{2\sqrt{8\epsilon \ln 2}}{1-p} + \frac{\delta}{2} \) (recall, \( p < 1 \)). Thus, \( I(J; X'|X) \geq 4\epsilon \) which implies that \( \zeta_{XX'|Y} \leq 2^{-2n\epsilon} \).

Case (ii): \( R > I(X'; Y) \). Here, \(|R - I(X'|X)'| - I(Y; X'|X) = R - I(X'; XY) \leq R - I(X'; Y) \). We will show that \( R \leq I(X'; Y) - 4\epsilon \) to obtain \( \zeta_{XX'|Y} \leq 2^{-2n\epsilon} \). To see the intuition, suppose the conditional distribution \( P_{X'|XX|Y} \) is such that \( \bar{X} = X' \) (instead of just \( \Pr(\bar{X} \neq X') \leq \delta \)). Since \( \bar{X} \succ Y \), this implies \( X' \succ Y \). Hence, \( I(X'; Y) = H(X') - H(X'|Y) = 1 - \Pr(Y = e) H(\bar{X}|Y = e) \geq 1 - \Pr(Y = e) \geq 1 - pq - \sqrt{\epsilon} \), where in the second equality we used \( X' \succ Y \) and the fact that \( P_X \) is uniform since all codewords are of uniform type. For \( \epsilon > 0 \) small enough such that \( 4\epsilon + \sqrt{\epsilon} < 1 - pq - R \) (recall, \( R < 1 - pq \)), \( R \geq I(X'; Y) - 4\epsilon \). We can extend this argument to the case when \( \Pr(X' \neq \bar{X}) < \delta \). To see this, note that

\[
H(X'|Y) \leq H(X'|\bar{X}|Y) = H(\bar{X}|Y) + H(X'|\bar{X}Y) \leq H(\bar{X}|Y) + H(X'|\bar{X}).
\]

Hence,

\[
I(X'; Y) = H(X') - H(X'|Y)
\]

\[
\overset{(a)}{=} 1 - H(X'|Y)
\]

\[
\overset{(b)}{\geq} 1 - H(\bar{X}|Y) - H(X'|\bar{X})
\]

\[
\overset{(c)}{\geq} 1 - \Pr(Y = e) H(\bar{X}|Y = e) - H(X'|\bar{X})
\]

\[
\overset{(d)}{=} 1 - \Pr(Y = e) - H(X'|\bar{X})
\]

\[
\overset{(e)}{\geq} 1 - \Pr(Y = e) - (H_2(\delta) - \delta)
\]

\[
\overset{(f)}{\geq} 1 - pq - \sqrt{\epsilon} - H_2(\delta) - \delta,
\]

where (a) follows from the fact that \( P_X \) is uniform, (b) follows from (15), (c) from \( \bar{X} \succ Y \), (d) from the fact that the entropy of a binary random variable is at most 1, (e) from \( \Pr(\bar{X} \neq X') \leq \delta \) and Fano’s inequality, and (f) from \( P_Y(e) \leq pq + \sqrt{\epsilon} \). Choosing \( \epsilon > 0 \) sufficiently small such that \( \sqrt{\epsilon} + 4\epsilon + H(\delta) + \delta \leq 1 - pq - R \) (recall, we chose \( \delta > 0 \) such that \( 2H_2(\delta) + \delta \leq 1 - pq - R \)), we can conclude that \( R \leq I(X'; Y) - 4\epsilon \). Thus,

\[
\Pr(B_n | x^n) \leq 3(n + 1)^2 2^{-2n\epsilon} + |D| 2^{-2n\epsilon} \leq 2^{-n\epsilon}
\]

for sufficiently large \( n \) since \( |D| \leq |P^n(\mathcal{X} \times \mathcal{X} \times \mathcal{Y})| \) is polynomial in \( n \).

IV. Consensus Capacity of Broadcast Channels

The proof ideas from Section III generalize to allow a characterization of the consensus capacity of all broadcast channels. To write it down, define the effective input alphabet \( \mathcal{U} \subseteq \mathcal{X} \) as follows: recall the common channel \( W_{V|X} \) (Definition 1); consider the convex polytope \( S \) which is the convex hull of the \(|V|\)-dimensional vectors \( \{W_{V|X}(.|x), x \in \mathcal{X}\} \). Each \( u \in \mathcal{U} \) is such that \( W_{V|X}(.|u) \) is a distinct vertex of \( S \); if more than one such \( x \in \mathcal{X} \) correspond to the same vertex, one among them
is arbitrarily chosen to represent that vertex in $\mathcal{U}$. In other words, let $\mathcal{U} \subseteq \mathcal{X}$ be a smallest-sized set such that there is a conditional distribution $\tilde{P}_{\mathcal{U}|X}$ satisfying

$$W_{V|X}(v|x) = \sum_{u \in \mathcal{U}} \tilde{P}_{\mathcal{U}|X}(u|x)W_{V|X}(v|u),$$

for all $v \in \mathcal{V}, x \in \mathcal{X}$. Thus, the cardinality of $\mathcal{U}$ is the number of vertices of the polytope S. While the choice of $\tilde{P}_{\mathcal{U}|X}$ and $\mathcal{U}$ may not be unique, we choose one among the valid ones for the rest of the discussion (also see Remark 5 below).

**Theorem 4.** The consensus capacity of $W_{Y,Z|X}$ is

$$C_{Byz} = \max_{P_U} \min_{P_{X|U}:P_{X|U}(x|u) > 0} \min_{W_{V|X}} (I(U;Y), I(U;Z)),$$

where the maximization is over p.m.f.s $P_U$ over $\mathcal{U}$ and the mutual informations are evaluated under $P_{U,Y,Z}(u,x,y,z) = P_U(u)P_{X|U}(x|u)W_{Y,Z|X}(y,z|x)$.

We prove this in Section V. The expression for capacity can be interpreted as follows: Unlike the expression (4) for $C_{com-msg}$, the input distribution $P_U$ avoids using letters $x$ with $W_{V|X}(\cdot|x)$ which are not vertices of $\mathcal{S}$. A byzantine sender can attack by replacing such an $x$ by sending a letter from $\mathcal{U}$ picked according to an appropriate distribution that induces the same common channel output distribution. The minimization in the capacity expression represents a similar attack where, for each letter $u$, the sender randomly chooses among the letters $x$ which correspond to the same vertex in $\mathcal{S}$ as $u$. It turns out that both these attacks cannot be detected by the receivers in a manner which permits consensus. In Fig. 3, $C_{Byz} < C_{com-msg}$.

**Remark 3.** Notice that when the common channel has capacity $C_{p-to-p}(W_{V|X}) = 0$, $\mathcal{U}$ is a singleton set and $C_{Byz} = 0$. Since $C_{Byz} \geq C_{p-to-p}(W_{V|X})$ (a fact which can also be verified from Theorem 4 using $V = \phi_1(Y) = \phi_2(Z)$), $C_{Byz} > 0$ if and only if $C_{p-to-p}(W_{V|X}) > 0$.

**Remark 4.** It is easy to see from Theorem 4 that $C_{Byz}$ remains unchanged if the receivers are provided additional correlated randomness unknown to the sender which they can use to coordinate their actions (augment the channel outputs to $(Y, S)$ and $(Z, T)$ where $(S, T)$ is independent of $(X, Y, Z)$ and notice that $I(U; Y; S) = I(U; Y)$ and $I(U; Z, T) = I(U; Z)$). Recall that the converse of Theorem 4 is shown under $P^n_U = o(1/n)$ (see Remark 1). However, we show in Appendix A that for the example of Section III-A which has $C_{Byz} = 0$ (in fact, even in a stronger sense; see Remark 2), when common randomness unknown to the sender is available to the decoders, a positive rate can be achieved with $P_x^n(\epsilon) = o(n^{-\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$.

**Remark 5.** It is clear from (17) that if there was a choice in selecting $\mathcal{U}$, the capacity does not depend on this choice – the minimization is over $P_{X|U}$ such that, for each $u$, $P_{X|U}(\cdot|u)$ has support (only) over all letters $x \in \mathcal{X}$ which correspond to the same vertex of the polytope $\mathcal{S}$ as $u$. 

---

**Fig. 3:** An example to show that $C_{Byz}$ could be strictly in between the point-to-point capacity of the common channel and the common message capacity of the broadcast channel. For all values of $p$ except $p = 0.5$, $C_{Byz} = 1$. For $p = 0.5$, when the common channel is trivial, $C_{Byz} = 0$. Here $\mathcal{U} = \{0, 1\} \subseteq \mathcal{X}$ (except for $p = 0.5$ when $\mathcal{U}$ is a singleton).
Example 2 (Capacity of the channel in Figure 3). Let \( p \in [0, 0.5] \). Consider the following channel with \( \mathcal{X} = \{0, e, 1\} \), \( \mathcal{Y} = \mathcal{Z} = \{a, b, c, d\} \)

\[
W_{YZ|X}(y, z|x) = \begin{cases} 
1 - p, & (x, y, z) \in \{(0, a, a), (1, d, d)\} \\
p, & (x, y, z) \in \{(0, c, c), (1, b, b)\} \\
1/2, & (x, y, z) \in \{(e, a, b), (e, c, d)\}
\end{cases}
\]

The characteristic graph of \( W_{YZ|X} \) in Figure 3 has two connected components: \( G_0 \) on the vertices \( \{a, b\} \cup \{a, \} \) and \( G_1 \) on \( \{c, d\} \cup \{c, d\} \). Denote the connected components by \( \mathcal{V} = \{0, 1\} \). The common channel is \( W_{V|X} \) with \( W_{V|X}(1-x|x) = 1 - W_{V|X}(x|x) = p \) for \( x \in \{0, 1\} \), and \( W_{V|X}(0, e) = W_{V|X}(1, e) = 1/2 \). Hence, the capacity of the common channel is that of a binary symmetric channel with cross over probability \( p \)

\[
C_{p\text{-to-}}(W_{V|X}) = 1 - H(p).
\]

The consensus capacity of this channel is computed as follows: For \( p \neq 0.5 \), the vertices of the convex hull of \( \{W_{V|X}(.|x), x \in \{0, 1, e\}\} \) correspond to the symbols 0 and 1 of \( \mathcal{X} \), i.e., \( \mathcal{U} = \{0, 1\} \), since

\[
W_{V|X}(v|e) = \frac{1}{2}W_{V|X}(v|0) + \frac{1}{2}W_{V|X}(v|1), v \in \mathcal{V}.
\]

For \( u \in \{0, 1\}\), \( W_{V|X}(.|x) = W_{U|U}(.|u) \) if and only if \( x = u \). Hence, the consensus capacity is (17) evaluated under \( P_{UVZ}(u, y, z) = P_{U}(u)W_{YZ|X}(y, z|u) \).

\[
C_{\text{Byz}} = \max_{P_U} \min(I(U; Y), I(U; Z))
\]

\[
= \max_{P_U} H(U)
\]

\[
= 1,
\]

where (a) follows from the fact that \( H(U|Y) = H(U|Z) = 0 \) under \( P_{UVZ}(u, y, z) = P_{U}(u)W_{YZ|X}(y, z|u) \). For \( p = 0.5 \), the vectors \( W_{V|X}(.|x) \) are identical for \( x \in \mathcal{X} \) and the polytope collapses to a point. Then, \( \mathcal{U} \) is singleton and \( C_{\text{Byz}} = 0 \).

Finally, we compute the common message capacity

\[
C_{\text{com-msg}} = \max_{P_X} \min(I(X; Y), I(X; Z)).
\]

Suppose \( P_X \) is a maximizer. By symmetry, \( P'_X(x) = P_X(1-x) \) for \( x \in \{0, 1\} \) and \( P'_X(e) = P_X(e) \) also achieves the maximum. Furthermore, since \( I(X; Y) \) and \( I(X; Z) \) are concave functions of \( P_X \) and so is \( \min(I(X; Y), I(X; Z)) \), any convex combination (specifically, the uniform convex combination) of \( P_X \) and \( P'_X \) also achieves the maximum. Hence, without loss of generality, we make take the maximizing \( P_X \) to be of the form \( P_X(0) = P_X(1) = 1/2 \), \( P_X(e) = q \), where \( q \in [0, 1] \). With the mutual informations evaluated under this,

\[
C_{\text{com-msg}} = \max_{q \in [0, 1]} \min(I(X; Y), I(X; Z))
\]

\[
= \max_{q \in [0, 1]} H(1X = 1) + P(X \neq 1)I(X; Y|X \neq 1)
\]

\[
= \max_{q \in [0, 1]} H(1X = 1) + P(X \neq 1)I(X; Y|X \neq 1)
\]

\[
= \max_{q \in [0, 1]} H_2 \left( \frac{1 - q}{2} \right) + \frac{1 + q}{2} \left( H(Y|X \neq 1) - \sum_{x \in \{0, e\}} P(X = x|X \neq 1)H(Y|X = x) \right)
\]

\[
= \max_{q \in [0, 1]} H_2 \left( \frac{1 - q}{2} \right) + \frac{1 + q}{2} \left( H_2 \left( \frac{p(1 - q) + q}{1 + q} \right) \right)
\]
where (a) follows from the fact that \( I(X; Y) = I(X; Z) \) when \( P_X \) is of the form we are working with, (b) from the fact that the indicator function \( 1_{Y \in \{b, d\}} \) is a function of \( Y \), and, (c) follows from the fact that \( (Y \in \{b, d\}) \) if and only if \( (X = 1) \).

We evaluate this expression numerically and compare with the other capacities in Figure 3.

V. PROOF OF THEOREM 4

Before proving the converse (in Section V-A) and achievability (in Section V-B), we prove a technical lemma.

Lemma 5. Let \( u \in \mathcal{U} \) and \( \lambda_x, x \in \mathcal{X} \) be a p.m.f., i.e., \( \lambda_x \geq 0, x \in \mathcal{X} \) and \( \sum_{x \in \mathcal{X}} \lambda_x = 1 \). Suppose
\[
\sum_{x \in \mathcal{X}} \lambda_x W_{V|X}(v|x) = W_{V|X}(v|u), \quad \text{for all } v \in \mathcal{V}.
\]

Then \( \lambda_x > 0 \) only if \( W_{V|X}(v|x) = W_{V|X}(v|u) \) for all \( v \in \mathcal{V} \).

Proof of Lemma 5. Recall that \( u \in \mathcal{U} \) corresponds to a vertex of the convex polytope \( Q \) which is the convex hull of the \( |V| \)-dimensional vectors \( \{W_{V|X}(.,x), x \in \mathcal{X}\} \), i.e., \( W_{V|X}(.,|x) \) is a vertex of the polytope. Let \( \mathcal{X}_u = \{x \in \mathcal{X} : W_{V|X}(.,|x) = W_{V|X}(.,|u)\} \), i.e., the set of all letters in \( \mathcal{X} \) which correspond to the same vertex as \( u \). Since \( W_{V|X}(.,|u) \) is a vertex of the polytope, there cannot exist a p.m.f. \( \eta_x, x \in \mathcal{X} - \mathcal{X}_u \) such that
\[
W_{V|X}(.,|u) = \sum_{x \in \mathcal{X} - \mathcal{X}_u} \eta_x W_{V|X}(.,|x). \quad (18)
\]

To prove the lemma, suppose for the sake of contradiction that there is a p.m.f. \( \lambda_x \geq 0, x \in \mathcal{X} \) such that
\[
\sum_{x \in \mathcal{X}} \lambda_x W_{V|X}(.,|x) = W_{V|X}(.,|u),
\]

and
\[
\Lambda := \sum_{x \in \mathcal{X} - \mathcal{X}_u} \lambda_x > 0.
\]

Then,
\[
\sum_{x \in \mathcal{X} - \mathcal{X}_u} \lambda_x W_{V|X}(.,|x) = W_{V|X}(.,|u) - \sum_{x \in \mathcal{X}_u} \lambda_x W_{V|X}(.,|x)
\]
\[
\overset{(a)}{=} \left(1 - \sum_{x \in \mathcal{X}_u} \lambda_x\right) W_{V|X}(.,|u)
\]
\[
= \left( \sum_{x \in \mathcal{X} - \mathcal{X}_u} \lambda_x \right) W_{V|X}(.,|u)
\]
\[
= \Lambda W_{V|X}(.,|u),
\]
where (a) follows from the definition of \( \mathcal{X}_u \) above. Now consider the p.m.f., \( \eta_x, x \in \mathcal{X} - \mathcal{X}_u \),
\[
\eta_x = \frac{\lambda_x}{\Lambda}
\]

Then,
\[
\sum_{x \in \mathcal{X} - \mathcal{X}_u} \eta_x W_{V|X}(.,|x) = W_{V|X}(.,|u),
\]
which contradicts the fact that there is no p.m.f. \( \eta_x, x \in \mathcal{X} - \mathcal{X}_u \) which satisfies (18).
A. Converse of Theorem 4

Consider the following claim which we will prove later.

Claim 6.

\[ C_{\text{Byz}} \leq \max_{P_X} \min_{P_{X'|X} \in \mathcal{P}_{X'|X}} \min(I(X;Y), I(X;Z)) \]

(19)

where \( \mathcal{P}_{X'|X} \) is the set of all \( P_{X'|X} \) such that

\[ \sum_{x' \in X'} P_{X'|X}(x'|x)W_{V|X}(v|x') = W_{V|X}(v|x) \]

(20)

for all \( v \in V, x \in X \), and the mutual information is evaluated under the joint distribution \( p_{X,X'|Y,Z}(x,x',y,z) = p_X(x)p_{X'|X}(x'|x)W_{Y,Z|X}(y,z|x') \).

We now argue that the maximization in (19) may be restricted to \( P_X \) which have support only over \( U \) and, hence,

\[ C_{\text{Byz}} \leq \max_{P_X} \min_{P_{X'|U} \in \mathcal{P}_{X'|U}} \min(I(U;Y), I(U;Z)) \]

(21)

where \( \mathcal{P}_{X'|U} \) is defined analogously to (20). Given any \( P_X \), consider \( \bar{P}_{X|U} \) defined by \( \bar{P}_{X|U}(x,u) = P_X(x)\bar{P}_{U|X}(u|x) \), where \( \bar{P}_{U|X} \) is as in (16). We will show that under the induced marginal distribution \( \bar{P}_U \)

\[
\min_{P_{X'|U} \in \mathcal{P}_{X'|U}} \min(I(U;Y), I(U;Z)) \geq \min_{P_{X'|X} \in \mathcal{P}_{X'|X}} \min(I(X;Y), I(X;Z)),
\]

where the RHS is evaluated with \( P_X \). To do this, suppose \( \bar{P}_{X'|U} \in \mathcal{P}_{X'|U} \) is a minimizer for the LHS. We need only show that there is a \( P_{X'|X} \in \mathcal{P}_{X'|X} \) such that \( I(X;Y) \) (resp., \( I(X;Z) \)) is no larger than \( I(U;Y) \) (resp., \( I(U;Z) \)) of the LHS. Consider the \( P_{X|X,Y,Z} \) defined by

\[
P_{X|X,Y,Z}(x,u,x',y,x) = P_X(x)\bar{P}_{U|X}(u|x)\bar{P}_{X'|U}(x'|u)W_{Y,Z|X}(y,z|x')
\]

(22)

which induces the following \( P_{X'|X} \)

\[
P_{X'|X}(x'|x) = \sum_u \bar{P}_{U|X}(u|x)P_{X'|U}(x'|u), x, x' \in X.
\]

We have \( P_{X'|X} \in \mathcal{P}_{X'|X} \) since, for all \( v, x, \)

\[
\sum_{x'} P_{X'|X}(x'|x)W_{V|X}(v|x') = \sum_{x'} \left( \sum_u \bar{P}_{U|X}(u|x)\bar{P}_{X'|U}(x'|u) \right) W_{V|X}(v|x')
\]

\[
= \sum_u \bar{P}_{U|X}(u|x) \left( \sum_{x'} \bar{P}_{X'|U}(x'|u)W_{V|X}(v|x') \right)
\]

\[
\overset{(a)}{=} \sum_u \bar{P}_{U|X}(u|x)W_{V|X}(v|u)
\]

\[
\overset{(b)}{=} W_{V|X}(v|x),
\]

where (a) follows from \( \bar{P}_{X'|U} \in \mathcal{P}_{X'|U} \) and (b) from (16). Further, since \( X - U - (Y,Z) \) is a Markov chain under (22), we have \( I(X;Y) \leq I(U;Y) \) and \( I(X;Z) \leq I(U;Z) \). Thus, we have shown (21). The converse now follows from observing that if \( P_{X'|U} \in \mathcal{P}_{X'|U} \), then, by Lemma 5, \( P_{X'|U}(x'|u) > 0 \) only if \( W_{V|X}(v|x) = W_{V|X}(v|u) \). This follows from Lemma 5.

It only remains to prove Claim 6. Suppose \( (f,g,b,c) \) is an \( (n,2^{nR}) \) byzantine agreement code with error probability \( P_e \). Consider any \( P_{X'|X} \in \mathcal{P}_{X'|X} \). We shall argue that for a uniformly chosen message \( M \), if the sender passes the codeword \( X^n = f(M) \) through the discrete memoryless channel (DMC) \( P_{X'|X} \) and sends the output \( X^n \) through the broadcast channel \( W_{Y,Z|X} \), the decoders acting on the output vectors \( Y^n \) and \( Z^n \) must output the message \( M \) with sufficiently high probability. Specifically, we will show the following:
Claim 7. Under the above experiment,
\[ \Pr(g_B(Y^n) = g_C(Z^n) = M) \geq 1 - (n + 1)\rho P_e, \]
where
\[ \rho = \frac{|Y| + |Z| - 1}{\min(x,y,z) : W_{Y,Z|X}(y,z|x) > 0 \ W_{Y|Z,X}(y,z|x)}. \]  

Assuming Claim 7 for the moment, suppose \( R \) is achievable, i.e., there is a sequence of \( (n, 2^{nR}) \) byzantine agreement codes such that \( P_e^{(n)} = o(1/n) \). Then, under the above experiment, \( \Pr(g_B(Y^n) \neq M) \) and \( \Pr(g_C(Z^n) \neq M) \), which are upper bounded by \( (n + 1)\rho P_e^{(n)} \), approach 0 as \( n \to \infty \). Note that here we make use of the requirement that the error probability \( P_e^{(n)} \) must fall super-linearly in the blocklength \( n \). Then, using Fano’s inequality and following standard single-letterization steps (e.g., [43, Sec. 7.9]),
\[ R \leq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i) + R \Pr(g_B(Y^n) \neq M) + \frac{1}{n} \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i) + (n + 1)R \rho P_e^{(n)} + \frac{1}{n}. \]
Similarly,
\[ R \leq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Z_i) + (n + 1)R \rho P_e^{(n)} + \frac{1}{n}. \]
Since \( P_e^{(n)} = o(1/n) \), for any \( \epsilon > 0 \), there is a sufficiently large \( n \) such that
\[ R \leq \min \left( \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i), \frac{1}{n} \sum_{i=1}^{n} I(X_i; Z_i) \right) + \epsilon \\
\leq \min(I(X; Y), I(X; Z)) + \epsilon, \]
where we used Jensen’s inequality in the last step whose RHS is evaluated under the joint distribution
\[ P_{X^n,Y^n,Z^n}(x^n, y^n, z^n) = P_X(x)P_{X^n|X}(x^n|x)W_{Y^n|Z,X}(y^n,z^n|x^n) \]
with \( P_X(x) = \frac{1}{n} \sum_{i=1}^{n} \Pr(X_i = x) \) and \( X^n = f(M) \) where \( M \) is a uniformly chosen message. Since this holds for any choice of \( P_{X^n|X} \in \mathcal{P}_{X^n|X} \), Claim 6 follows.

We now prove Claim 7 to complete the proof of Claim 6. It will suffice to show that for each message \( m \in \{1, \ldots, 2^{nR}\} \), under the above experiment where the codeword \( f(m) \) is first sent over a DMC \( P_{X^n|X} \in \mathcal{P}_{X^n|X} \) and its output sent over \( W_{Y^n|Z,X} \) to produce \( \hat{Y}^n, \hat{Z}^n \),
\[ \Pr(g_B(\hat{Y}^n) = g_C(\hat{Z}^n) = m) \geq 1 - (n + 1)\rho P_e. \]  
Let \( f(m) = x^n = (x_1, x_2, \ldots, x_n) \). Let \( Y^n, Z^n \) be the outputs of the channel when \( x^n \) is transmitted, i.e.,
\[ P_{Y^n,Z^n}(y^n,z^n) = \prod_{i=1}^{n} W_{Y_i,Z_i|X_i}(y_i,z_i|x_i). \]
We are given (by (1))
\[ \Pr((g_B(Y^n) = g_C(Z^n) = m) \geq 1 - P_e. \]  
Let \( \hat{X}^n \) denote the output resulting from sending \( x^n \) over the DMC \( P_{X^n|X} \in \mathcal{P}_{X^n|X} \), and let \( \hat{Y}^n, \hat{Z}^n \) denote the channel outputs from sending this \( \hat{X}^n \) over \( W_{Y^n|Z,X} \), i.e.,
\[ P_{\hat{X}^n,\hat{Y}^n,\hat{Z}^n}(\hat{x}^n, \hat{y}^n, \hat{z}^n) = \prod_{i=1}^{n} P_{X_i|X}(\hat{x}_i|x_i)W_{Y_i,Z_i|X}(\hat{y}_i, \hat{z}_i|x_i). \]
We show (24) by showing the following for \( \epsilon > 0 \) and \( k \in [1 : n] \):
If
\[ \Pr(g_B((\hat{Y}_1, \ldots, \hat{Y}_{k-1}, Y_k, \ldots, Y_n)) = \\
g_c((\hat{Z}_1, \ldots, \hat{Z}_{k-1}, Z_k, \ldots, Z_n)) = m) \geq 1 - \epsilon, \]
then
\[ \Pr(g_B((\hat{Y}_1, \ldots, \hat{Y}_k, Y_{k+1}, \ldots, Y_n)) = \\
g_c((\hat{Z}_1, \ldots, \hat{Z}_k, Z_{k+1}, \ldots, Z_n)) = m) \geq 1 - \epsilon - P_\epsilon \rho. \]

Since (26) implies (28) for \( k = 1 \) with \( \epsilon = P_\epsilon \), applying these recursively for \( k = 1, \ldots, n \) will give (24) (since \( \rho \geq 1 \)). Note that the joint distribution of the random variables in (28) is
\[ p_{\hat{Y}_1, \ldots, \hat{Y}_k, \hat{Z}_1, \ldots, \hat{Z}_k, Y_{k+1}, \ldots, Y_n}^{k-1}(\hat{y}_1^{k-1}, y_k^n, \hat{z}_1^{k-1}, z_k^n) = \prod_{i=1}^{k-1} \left( \sum_{\hat{z}_i} P_{X|X}(\hat{x}_i|x_i)W_{YZ|X}(\hat{y}_i, \hat{z}_i|\hat{x}_i) \right) \prod_{i=k}^{n} W_{YZ|X}(y_i, z_i|x_i), \]
while that of the random variables in (29) is
\[ p_{\hat{Y}_1, \ldots, \hat{Y}_k, \hat{Z}_1, \ldots, \hat{Z}_k, Y_{k+1}, \ldots, Y_n}^{k}(\hat{y}_1^{k}, y_k^n, \hat{z}_1^{k}, z_k^{k+1}) = \prod_{i=1}^{k} \left( \sum_{\hat{z}_i} P_{X|X}(\hat{x}_i|x_i)W_{YZ|X}(\hat{y}_i, \hat{z}_i|\hat{x}_i) \right) \prod_{i=k+1}^{n} W_{YZ|X}(y_i, z_i|x_i). \]

To show that (28) implies (29), we define the following joint distribution of these random variables
\[ p_{\hat{Y}_k, \hat{Z}_k, Y_k, Z_k}^{k}(\hat{y}_k, \hat{z}_k, y_k, z_k) = \prod_{i=1}^{k} \left( \sum_{\hat{x}_i} P_{X|X}(\hat{x}_i|x_i)W_{YZ|X}(\hat{y}_i, \hat{z}_i|\hat{x}_i) \right) W_{YZ|X}(y_k, z_k|x_k), \]
where the coupling \( P_{\hat{Y}_k, \hat{Z}_k, Y_k, Z_k} \) is given by
\[ P_{\hat{Y}_k, \hat{Z}_k, Y_k, Z_k}(\hat{y}_k, \hat{z}_k, y_k, z_k) = \sum_{\hat{x}_k} P_{X|X}(\hat{x}_k|x_k) \sum_{v} W_{V|X}(v|\hat{x}_k) W_{YZ|XV}(\hat{y}_k, \hat{z}_k|x_k, v), \]
i.e., under the coupling, both \((\hat{Y}_k, \hat{Z}_k)\) and \((Y_k, Z_k)\) have the same common channel output. We first demonstrate that this is a valid coupling by verifying that it has the correct marginals for \((\hat{Y}_k, \hat{Z}_k)\) and \((Y_k, Z_k)\).
\[ P_{\hat{Y}_k, \hat{Z}_k}(\hat{y}_k, \hat{z}_k) = \sum_{\hat{x}_k} P_{X|X}(\hat{x}_k|x_k) \sum_{v} W_{V|X}(v|\hat{x}_k)W_{YZ|XV}(\hat{y}_k, \hat{z}_k|x_k, v) \]
\[ = \sum_{\hat{x}_k} P_{X|X}(\hat{x}_k|x_k)W_{YZ|X}(\hat{y}_k, \hat{z}_k|x_k), \]
which matches (31); we used Definition 1 in the last step above. To verify that the marginals of \((Y_k, Z_k)\) in the coupling match that in (30),
\[ P_{Y_k, Z_k}(y_k, z_k) \]
\[
= \sum_{\hat{x}_k} P_{X'|X}(\hat{x}_k|x_k) \sum_v W_{V|X}(v|\hat{x}_k) W_{Y,Z|X V}(y_k, z_k|x_k, v)
\]
\[
= \sum_v \left( \sum_{\hat{x}_k} P_{X'|X}(\hat{x}_k|x_k) W_{V|X}(v|\hat{x}_k) \right) W_{Y,Z|X V}(y_k, z_k|x_k, v)
\]
\[
= (a) \sum_v W_{V|X}(v|x_k) W_{Y,Z|X V}(y_k, z_k|x_k, v)
\]
\[
= (b) W_{Y,Z|X}(y_k, z_k|x_k),
\]
where (a) follows from (20) since \( P_{X'|X} \in P_{X'|X} \) and (b) from Definition 1. Now (29) will follow from (28) if we show that under the joint distribution of (32),

\[
Pr \left( g_B((\hat{Y}_1, \ldots, \hat{Y}_{k-1}, Y_k, Y_{k+1}, \ldots, Y_n)) = g_C((\hat{Z}_1, \ldots, \hat{Z}_{k-1}, Z_k, Z_{k+1}, \ldots, Z_n)) \right)
\]
\[
g_B((\hat{Y}_1, \ldots, \hat{Y}_k, Y_{k+1}, \ldots, Y_n)) = g_C((\hat{Z}_1, \ldots, \hat{Z}_k, Z_{k+1}, \ldots, Z_n)) \geq 1 - P_e \rho.
\]

**Lemma 8.** Let \( P_{ST} \) be a joint distribution over \( S \times T, \delta > 0 \), and \( \psi_B \) and \( \psi_C \) be functions defined on \( Y \times S \) and \( Z \times T \), respectively. Under the joint distribution \( P_{STY,Z|X}(s, t, y, z|x) = P_{ST}(s, t) W_{Y,Z|X}(y, z|x) \), suppose for every \( x \in \mathcal{X} \),

\[
Pr(\psi_B(Y, S) \neq \psi_C(Z, T)|X = x) \leq \delta.
\]

Then, for every \( v \in \mathcal{V} \),

\[
Pr \left( \bigcup_{v} \{ \psi_B(y, s), \psi_C(z, t) \} = 1 \right) \geq 1 - \delta \rho,
\]

where the probability is over \( (S, T) \sim P_{ST} \) and \( \rho \) is given by (23).

To show (34), we invoke Lemma 8 with \( S = (\hat{Y}_k, \ldots, \hat{Y}_{k+1}, Y_{k+1}, \ldots, Y_n), T = (\hat{Z}_k, \ldots, \hat{Z}_{k+1}, Z_{k+1}, \ldots, Z_n), \psi_B(y, (y_k, y_{k+1})), \psi_C(z, (\hat{z}_k, \hat{z}_{k+1})), \) and \( \psi_C(z, (\hat{z}_k, \hat{z}_{k+1})) = g_C((\hat{z}_k, \hat{z}_{k+1})). \)

With \( \delta = P_e \), (35) follows from the fact the decoder outputs must agree with probability at least \( 1 - P_e \) for all inputs and, specifically, the input \((\hat{X}_1, \ldots, \hat{X}_{k-1}, x, x_{k+1}, \ldots, x_n)\), i.e.,

\[
Pr(\psi_B(Y, S) \neq \psi_C(Z, T)|X = x)
\]
\[
= (a) \sum_{\hat{x}_1, \ldots, \hat{x}_{k-1}} \left( \prod_{i=1}^{k-1} P_{X'|X}(\hat{x}_i|x_i) \right) Pr(g_B(Y^n) \neq g_C(Z^n)|X^n = (\hat{x}_1, \ldots, \hat{x}_{k-1}, x, x^n)
\]
\[
\leq (b) \sum_{\hat{x}_1, \ldots, \hat{x}_{k-1}} \left( \prod_{i=1}^{k-1} P_{X'|X}(\hat{x}_i|x_i) \right) P_e
\]
\[
= P_e,
\]

where the probability in the RHS of (a) is over the memoryless channel \( W_{Y,Z|X} \) and (b) follows from (1). We need to show (34) which translates to

\[
Pr \left( g_B(Y_k, S) = \psi_C(Z_k, T) \right)
\]
\[
= \psi_B(\hat{Y}_k, S) = \psi_C(\hat{Z}_k, T) \geq 1 - P_e \rho,
\]

(37)
where \((S, T)\) is independent of \((\hat{Y}_k, \hat{Z}_k, Y_k, Z_k)\) with \(P_{\hat{Y}_k, \hat{Z}_k, Y_k, Z_k}\) given by (33). Let us define the events \(A_{y_k, \hat{z}_k, y_k, \hat{z}_k} = \{\psi_B(y_k, S) = \psi_C(\hat{z}_k, T) = \psi_B(\hat{y}_k, S) = \psi_C(\hat{z}_k, T)\}\).

\[
\Pr(\psi_B(Y_k, S) = \psi_C(Z_k, T) = \psi_B(\hat{Y}_k, S) = \psi_C(\hat{Z}_k, T)) \\
\overset{(a)}{=} \sum_{\hat{y}_k, \hat{z}_k, y_k, z_k} P_{\hat{Y}_k, \hat{Z}_k, Y_k, Z_k}(\hat{y}_k, \hat{z}_k, y_k, z_k) \Pr(A_{y_k, \hat{z}_k, y_k, \hat{z}_k}) \\
\overset{(b)}{=} \sum_{\hat{y}_k, \hat{z}_k, y_k, z_k} \sum_{\hat{x}_k, v} P_{X|X}(\hat{x}_k|x_k) W_{V|X}(v|\hat{x}_k) \\
W_{YZ|XV}(y_k, z_k|x_k, v) W_{YZ|XV}(\hat{y}_k, \hat{z}_k|\hat{x}_k, v) \\
\Pr(A_{y_k, \hat{z}_k, y_k, \hat{z}_k}) \\
\overset{(c)}{=} \sum_{\hat{x}_k, v} P_{X|X}(\hat{x}_k|x_k) W_{V|X}(v|\hat{x}_k) \\
\sum_{(y_k, z_k) \in E_v} W_{YZ|XV}(y_k, z_k|x_k, v) W_{YZ|XV}(\hat{y}_k, \hat{z}_k|\hat{x}_k, v) \\
\Pr(A_{y_k, \hat{z}_k, y_k, \hat{z}_k}) \\
\overset{(d)}{=} \sum_{\hat{x}_k, v} P_{X|X}(\hat{x}_k|x_k) W_{V|X}(v|\hat{x}_k) \\
\sum_{(y_k, z_k) \in E_v} W_{YZ|XV}(y_k, z_k|x_k, v) W_{YZ|XV}(\hat{y}_k, \hat{z}_k|\hat{x}_k, v) \\
(1 - P_e \rho) \\
= 1 - P_e \rho,
\]

where (a) follows from the independence of \((S, T)\) and \((\hat{Y}_k, \hat{Z}_k, Y_k, Z_k)\) (notice that the probability of \(\Pr(A_{y_k, \hat{z}_k, y_k, \hat{z}_k})\) is over the distribution of \((S, T)\)), (b) from (33), and (c) from the fact that \(W_{YZ|XV}(y, z|x, v) > 0\) only if the edge \(\{y, z\}\) lies in the edge set \(E_v\) of the connected component \(G_v\) corresponding to the common channel output letter \(v\). Inequality (d) follows from Lemma 8 which, as discussed, we may invoke with \(\delta = P_e\). By (36), we may conclude that \(\Pr(A_{y_k, \hat{z}_k, y_k, \hat{z}_k}) \geq 1 - P_e \rho\) since under the event in (36) all \(\{y, z\} \in E_v\) result in the same output.

**Proof of Lemma 8.** For \(x \in X',\) by (35),

\[
\delta \geq \Pr(\psi_B(Y, S) \neq \psi_C(Z, T)|X = x) \\
\overset{(a)}{=} \sum_{y, z} W_{YZ|X}(y, z|x) \Pr(\psi_B(S, y) \neq \psi_C(T, z)|X = x) \\
\overset{(b)}{=} \sum_{y, z} W_{YZ|X}(y, z|x) \Pr(\psi_B(S, y) \neq \psi_C(T, z)),
\]

where (a) and (b) use the description of \(P_{STY|Z,X}\) given in (36). Hence, for every edge \(\{y, z\} \in E\) in the characteristic graph (i.e., \(W_{YZ|X}(y, z|x) > 0\) for some \(x\)),

\[
\Pr(\psi_B(S, y) \neq \psi_C(T, z)) \leq \frac{\delta}{\min_{(x, y, z):W_{YZ|X}(y, z|x) > 0} W_{YZ|X}(y, z|x)}. \tag{38}
\]

For every \(v \in V,\) since its corresponding connected component \(G_v(N_v, E_v')\) is connected, it has a spanning tree, say, \(G'_v(N_v, E'_v), E'_v \subseteq E_v\).

\[
\Pr \left( \bigcup_{\{y, z\} \in E_v} \{g_B(S, y), g_C(T, z)\} = 1 \right)
\]
\[
\begin{align*}
(a) & \quad \Pr \left( \bigcup_{\{y,z\}\in E'_n} \{ g_B(S,y), g_C(T,z) \} = 1 \right) \\
(b) & \quad \geq 1 - \sum_{\{y,z\}\in E'_n} \Pr (g_B(S,y) \neq g_C(T,z)) \\
(c) & \quad \geq 1 - \frac{\delta |E'|}{\min_{(x,y,z) \in W_{YZ|X}(y,z|x)} W_{YZ|X}(y,z|x)} \\
(d) & \quad \geq 1 - \delta \rho.
\end{align*}
\]

Here, (a) follows from \(G'_v\) being a spanning tree of \(G_v\); (b) follows from a union bound; (c) follows from (38); and, (d) follows from \(G'_v\) being a tree and hence \(|E'_{n}| = N_v - 1 \leq |X| + |Y| - 1\). This concludes the proof.

\section{Achievability of Theorem 4}

We use the following notation for this section:

**Notation:** Random variables are denoted by capital letters like \(X, X', U, Y, \) etc. The corresponding alphabets are denoted by calligraphic letters in the same format, for example, the random variables \(X\) and \(X'\) have alphabet \(X\). Its \(n\)–product set is denoted by \(X^n\). We use bold faced letters to denote \(n\)–length vectors, for example \(x\) denotes a vector in \(X^n\) and \(X\) denotes a random vector in \(X^n\). For an alphabet \(X\), let \(\mathcal{P}^n(X')\) denote the set of all empirical distributions of \(n\) length strings from \(X^n\). For a random variable \(X\), we denote its distribution by \(P_X\) and use the notation \(X \sim P_X\) to indicate this. If \(P_X \in \mathcal{P}^n(X)\), we use \(T_{X}^n\) to denote the set of all sequences with relative frequencies specified by \(P_X\). For \(x \in X\), the statement \(x \in T_{X}^n\) means that \(x\) has the empirical distribution \(P_X\). When \(P_X\) is not already defined, we write \(x \in T_{X}^n\) to implicitly define \(P_X\) as the empirical distribution of \(x\). Similarly, for \((x, y) \in X^n \times Y^n\), we write \((x, y) \in T_{X,Y}^n\) to define \(P_{XY}\) as the joint empirical distribution the vectors \(x\) and \(y\). For a broadcast channel \(W_{Y|X}\), we denote its marginal channels to the receivers by \(W_{Y|X}\) and \(W_{Z|X}\) respectively. For a channel \(W\), its \(n\)-fold product (memoryless use) is denoted by \(W^n\). For any number \(a\), we will use \(\exp a\) to denote \(2^a\) and \(\log a\) to denote \(\log_2 a\). For any set \(S\), we use \(S^c\) to denote its complement. For \(s, s' \in S\) we define

\[
1_{\{s = s'\}} = \begin{cases} 
1 & \text{if } s = s' \\
0, & \text{otherwise.}
\end{cases}
\]

We will first state some properties of joint types from [50, Chapter 2] which will be useful in the later analysis. Let \(X\) and \(Y\) be two jointly distributed random variables such that \(P_{XY} \in \mathcal{P}^n(X \times Y)\). For \(x \in T_{X}^n\), a distribution \(Q\) on \(X\) and a discrete memoryless channel \(U\) from \(X\) to \(Y\),

\[
|\mathcal{P}^n(X)| \leq (n + 1)^{|X|} \quad (39)
\]

\[
(n + 1)^{-|X|} \exp (nH(X)) \leq |T_{X}^n| \leq \exp (nH(X)) \quad (40)
\]

\[
(n + 1)^{-|X||Y|} \exp (nH(Y|X)) \leq |T_{Y|X}^n| \leq \exp (nH(Y|X)) \quad (41)
\]

\[
(n + 1)^{-|X|} \exp \{-nD(P_X||Q)\} \leq \sum_{x' \in T_{X}^n} Q^n(x') \leq \exp \{-nD(P_X||Q)\} \quad (42)
\]

\[
\sum_{y \in T_{Y|X}^n(x)} U^n(y|x) \leq \exp \{-nD(P_{XY}||P_XU)\} \quad (43)
\]

We first generalize the notion of relative distance between binary strings that was used in Section III-B.

**Definition 2.** Let \((u, x) \in U^n \times X^n\) with empirical distribution \(P_{UX}\), i.e., \((u, x) \in T_{UX}^n\). We define \(d(u, x)\) and \(d(P_{UX})\) as follows:

For \(\tilde{P}_{U|X}\) given by (16), let \(P_{UX\tilde{U}}(u, x, \tilde{u}) = P_{UX}(u, x)\tilde{P}_{U|X}(\tilde{u}|x)\) for \(u, \tilde{u} \in U\) and \(x \in X\). Then,

\[
d(u, x) = d(P_{UX}) := P(U \neq \tilde{U}). \quad (44)
\]

For \(u, u' \in U^n\), by Definition 2, \(d(u, u') = d(u', u) = P(U \neq U')\) under the joint type \(P_{UU'}\) given by \((u, u') \in T_{UU}^n\). To see this, by (49), \(\tilde{P}_{U|X}(\tilde{u}|u') = 1_{\{\tilde{u} = u\}'}, \tilde{u}, u' \in U\). Hence, \(P_{UU'|X}(u, u, \tilde{u}) = P_{UU'}(u, u')1_{\{\tilde{u} = u\}'}. \) Thus, \(\tilde{U} = U'\) and

\[
d(u, u') = d(P_{UU'}) = P(U \neq \tilde{U}) = P(U \neq U') = d(u', u). \quad (45)
\]

Next, we will show that for \((u, x, u'), (u, x) + d(u', x) \geq d(u, u')\). Let \((u, u', x) \in T_{UU'|X}^n\). Let \(P_{UU'|X}(u, u', x, \tilde{u}) = P_{UU'|X}(u, u', \tilde{u})\tilde{P}_{U|X}(\tilde{u}|x)\) for \(u, u', \tilde{u} \in U\) and \(x \in X\). Then

\[
d(u, x) + d(u', x) = P(U \neq \tilde{U}) + P(U' \neq \tilde{U})
\]
implies that for any $u' \in \mathcal{U}$ and $P_{U|U'}$,
\[
\sum_{v \in \mathcal{V}} \sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \geq (1 - P_{U|U'}(u'|u')) \gamma.
\]
This holds with equality if $P_{U|U'}(u'|u') = 1$.

**Proof.** Consider any channel $W_{V|Z|X}$ with $|\mathcal{U}| \geq 2$, i.e. $C_{\text{p-to-p}}(W_{V|X}) > 0$. Lemma 5 implies that for any $u' \in \mathcal{U}$ and conditional distribution $P_{U|U'}$ mapping symbols in $\mathcal{U}$ to symbols in $\mathcal{U}$, if
\[
\sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) = W_{V|X}(v|u')
\]
then $P_{U|U'}(u'|u') = 1_{\{u=u'\}}$. This also implies that
\[
\tilde{P}_{U|X}(u|u') = 1_{\{u=u'\}}, \ u, u' \in \mathcal{U}.
\]
In other words, for any channel with $C_{\text{p-to-p}}(W_{V|X}) > 0$ (i.e. $|\mathcal{U}| \geq 2$), there exists $\gamma > 0$ such that
\[
\min_{u' \in \mathcal{U} \setminus \{u\}} \min_{u \in \mathcal{U} \setminus \{u'\}} \sum_{v \in \mathcal{V}} \left| \sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \right| = \gamma
\]
Consider $u' \in \mathcal{U}$ and $P_{U|U'}$ such that $P_{U|U'}(u'|u') \neq 1$,
\[
\sum_{v \in \mathcal{V}} \left| \sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \right|
= \sum_{v \in \mathcal{V}} \left| \sum_{u \in \mathcal{U} \setminus \{u'\}} P_{U|U'}(u|u') W_{V|X}(v|u) - (1 - P_{U|U'}(u'|u')) W_{V|X}(v|u') \right|
\leq (a) (1 - P_{U|U'}(u'|u')) \left| \sum_{v \in \mathcal{V}} \sum_{u \in \mathcal{U} \setminus \{u'\}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \right|
\geq (b) (1 - P_{U|U'}(u'|u')) \gamma
\]
where in (a), we defined $P_{U|U'}(u|u') := P_{U|U'}(u'|u')/(1 - P_{U|U'}(u'|u'))$, $u \in \mathcal{U} \setminus \{u'\}$, and (b) follows from (50). Hence, for any $u' \in \mathcal{U}$ and $P_{U|U'}$,
\[
\sum_{v \in \mathcal{V}} \left| \sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \right| \geq (1 - P_{U|U'}(u'|u')) \gamma
\]
which holds with equality if $P_{U|U'}(u'|u') = 1$. \hfill \Box

The next lemma uses a random coding argument to generate a codebook which will be used to show the achievability.

**Lemma 10.** For any $\epsilon > 0$, $n \geq n_0(\epsilon)$, $K \geq \exp \left( n \epsilon \right)$, $\alpha > 0$, type $P \in \mathcal{P}^n(\mathcal{U})$ satisfying $\min_u P(u) \geq \alpha$, $\delta > 0$ and for $R := \log K/n$, $\mathcal{P} := \{P_{U|U'} \in \mathcal{P}^n(\mathcal{U} \times \mathcal{U}) : P_U = P_{U'}, P_{U \neq U'} < 2\delta \}$ such that $\epsilon \leq R \leq \min_{P_{U|U'} \in \mathcal{P}} I(U; U') - \epsilon$, there exists an encoder $f : [1 : K] \rightarrow \mathcal{U}^n$, whose codewords $u_i = f(i)$, $i \in [1 : K]$ are of type $P$ such that
\[
d(u_i, u_j) \geq 2\delta \quad \text{for all } i \neq j, i, j \in [1 : K],
\]
and for every joint type $P_{U,X} \in \mathcal{P}^n(\mathcal{U} \times \mathcal{X})$ and $x \in \mathcal{X}^n$ satisfying $P_U = P$ and $x \in \mathcal{T}_X^n$,
\[
|\{i \in [1 : K] : (u_i, x) \in \mathcal{T}_{U,X}^n\}| \leq \exp \left( n \left( |R - I(U; X)|^+ + \epsilon \right) \right),
\]
where $\exp(a)$ denotes $2^a$.

**Proof.** We use a random coding argument to show the existence of a codebook satisfying properties (52) and (53). Let $\mathcal{T}_n$ denote the type class of $P$. We generate $\exp(nR)$ (recall that log and exp are with respect to base 2. In particular $\exp(nR) = 2^{nR}$) independent random codewords $U_1, U_2, \ldots, U_K$, each distributed uniformly on $\mathcal{T}_n$. 

\[
\sum_{v \in \mathcal{V}} \sum_{u \in \mathcal{U}} P_{U|U'}(u|u') W_{V|X}(v|u) - W_{V|X}(v|u') \geq (1 - P_{U|U'}(u'|u')) \gamma.
\]
We will show that the probability that statement (52) or statement (53) for any fixed joint type \( P_{U \times X} \in \mathcal{P}^n (U \times \mathcal{X}) \) and \( x \in \mathcal{X}^n \), do not hold falls doubly exponentially in \( n \). Since \( |\mathcal{X}^n| \) grows only exponentially in \( n \) and \( |\mathcal{P}^n (U \times \mathcal{X})| \) polynomially in \( n \), a union bound will imply the existence of a codebook satisfying properties (52) and (53). We will use the concentration result [13, Lemma A1], which we restate below for quick reference.

**Lemma 11.** Let \( Z_1, \ldots, Z_K \) be arbitrary random variables, and let \( \zeta_i (Z_1, \ldots, Z_i) \) be arbitrary with \( 0 \leq \zeta_i \leq 1, i = 1, 2, \ldots, K \). Then the condition

\[
\mathbb{E} [\zeta_i (Z_1, \ldots, Z_i) | Z_1, \ldots, Z_{i-1}] \leq a \ a.s., \ i = 1, 2, \ldots, K
\]

implies that

\[
P \left\{ \frac{1}{K} \sum_{i=1}^{K} \zeta_i (Z_1, \ldots, Z_i) > t \right\} \leq \exp \left\{ -K \left( t - a \log e \right) \right\}.
\]

We will first analyze (53). Let \( \mathcal{E}_1 \) be the event

\[
\mathcal{E}_1 = \left\{ \left| \{ i \in [1 : K] : (U_i, x) \in \mathcal{T}^n_{U \times X} \} \right| > \exp \left( n \left( |R - I(U; X)|^+ + \epsilon \right) \right) \right\}.
\]

Suppose \((U_1, \ldots, U_K)\) are the random variables \((Z_1, \ldots, Z_K)\) in Lemma 11. Let

\[
\zeta_i (U_1, \ldots, U_i) := \begin{cases} 1, & \text{if } U_i \in \mathcal{T}^n_{U \times X}(x), \\ 0, & \text{otherwise.} \end{cases}
\]

Then,

\[
\mathbb{E} [\zeta_i (U_1, \ldots, U_i) | U_1, \ldots, U_{i-1}] = P(U_i \in \mathcal{T}^n_{U \times X}(x)) = \frac{|\mathcal{T}^n_{U \times X}(x)|}{|\mathcal{T}^n_U|} \leq \frac{\exp \{ nH(U|X) \}}{(n + 1)^{|U|} \exp \{ nH(U) \}} = (n + 1)^{|U|} \exp \{ -nI(U; X) \} =: a,
\]

where \((a)\) uses (40) and (41). Let \( t_1 = \frac{1}{K} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \). Then,

\[
P(\mathcal{E}_1) = P \left\{ \frac{1}{K} \sum_{i=1}^{K} \zeta_i (U_1, \ldots, U_i) > t_1 \right\}
\]

\[
\leq \exp \left\{ -K \left( t_1 - a \log e \right) \right\} \leq \exp \left\{ -K \left( t_1 - a \log e \right) \right\}
\]

\[
\leq \exp \left\{ - \left( \frac{1}{2} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \right) \right\}
\]

\[
\leq \exp \left\{ - \left( \frac{1}{2} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \right) \right\}
\]

\[
\leq \exp \left\{ - \left( \frac{1}{2} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \right) \right\}
\]

\[
\leq \exp \left\{ - \left( \frac{1}{2} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \right) \right\}
\]

\[
\leq \exp \left\{ - \left( \frac{1}{2} \exp \{ n (|R - I(U; X)|^+ + \epsilon) \} \right) \right\}
\]

where \((a)\) uses (55), \((b)\) follows by noting that \( K = \exp \{ nR \} \) and \((c)\) holds for large enough \( n \) such that \( (n + 1)^{|U|} \log e \leq \exp \{ n \epsilon \} / 2 \). To analyze (52), let \( \mathcal{P} := \{ P_{UU'} \in \mathcal{P}^n (U \times U) : P_U = P_{U}, = P, d(P_{UU'}, < 2\delta) \} \). Note that

\[
P \left( \frac{1}{K} \left| \{ i \in [1 : K] : d(U_i, U_j) < 2\delta \right\} \right) \geq \exp \left( - \frac{n \epsilon}{2} \right)
\]

\[
P \left( \frac{1}{K} \left| \{ i \in [1 : K] : (U_i, U_j) \in \mathcal{T}^n_{UU'} \right\} \right) \geq \exp \left( - \frac{n \epsilon}{2} \right)
\]

We will show that

\[
P \left( \frac{1}{K} \left| \{ i \in [1 : K] : (U_i, U_j) \in \mathcal{T}^n_{UU'} \right\} \right) \geq \exp \left( - \frac{n \epsilon}{2} \right)
\]

\[
\leq 2 \exp \left( \frac{1}{4} \exp \left( \frac{n \epsilon}{2} \right) \right)
\]
and use an expurgation argument to complete the proof. To show (57), we first note that
\[
P \left( \frac{1}{K} \{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j \neq i \text{ and } P_{UU} \in \mathcal{P} \} \right) \geq \exp \left( -\frac{n \epsilon}{2} \right)
\]
\[
\leq P \left( \frac{1}{K} \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j < i \text{ and } P_{UU} \in \mathcal{P} \right\} \right)
\]
\[
+ \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j > i \text{ and } P_{UU} \in \mathcal{P} \right\} \right) \geq \exp \left( -\frac{n \epsilon}{2} \right)
\]
\[
\leq P \left( \frac{1}{K} \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j < i \text{ and } P_{UU} \in \mathcal{P} \right\} \right) \geq \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right)
\]
\[
+ \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j > i \text{ and } P_{UU} \in \mathcal{P} \right\} \right) \geq \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right)
\]
\[
\leq 2P \left( \frac{1}{K} \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j < i \text{ and } P_{UU} \in \mathcal{P} \right\} \right) \geq \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right)
\]
where (a) follows from a union bound and (b) uses the symmetry of the random codebook. Thus, we only need to analyze (58). For any \(u_1, \ldots, u_{i-1}\), let
\[
\xi_i(u_1, \ldots, u_i) := \begin{cases} 1, & \text{if } u_i \in \cup_{j<i}T_{UU}^n(u_j), \text{ for some } P_{UU} \in \mathcal{P}; \\ 0, & \text{otherwise}. \end{cases}
\]
We will now apply Lemma 11. Suppose \((U_1, \ldots, U_K)\) are the random variables \((Z_1, \ldots, Z_K)\) and \(\xi_i, i \in [1 : K]\) correspond to the functions \( \xi_i \), \( i \in [1 : K] \). Then,
\[
E[\xi_i(U_1, \ldots, U_i)|U_1, \ldots, U_{i-1}] = (u_1, \ldots, u_{i-1})
\]
\[
\leq \sum_{P_{UU} \in \mathcal{P}} \sum_{j<i} P \left( \left\{ U_i \in T_{UU}^n(u_j) \right\} \right)
\]
\[
\leq \sum_{P_{UU} \in \mathcal{P}} \left| T_{UU}^n(u_j) \right| \frac{\exp \{ nH(U') \}}{(n+1)^{\left| U \right|} \exp \{ nH(U) \}}
\]
\[
\leq \sum_{P_{UU} \in \mathcal{P}} (n+1)^{\left| U \right|} \exp \{ n(R - I(U; U')) \}
\]
where (a) follows from \(U_{i\perp}(U_1, \ldots, U_{i-1})\) and (b) follows from (40) and (41). Suppose
\[
R - \min_{P_{UU} \in \mathcal{P}} I(U; U') < -3\epsilon/4,
\]
then
\[
E[\xi_i(U_1, \ldots, U_i)|U_1, \ldots, U_{i-1}] \leq |\mathcal{P}|(n+1)^{|X|} \exp \{ n(-3\epsilon/4) \} = a'.
\]
Let \(t' = \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right)\). Then from Lemma 11,
\[
P \left( \frac{1}{K} \left\{ i \in [1 : K] : (U_i, U_j) \in T_{UU}^n \text{ for some } j \in [1 : K], j < i \text{ and } P_{UU} \in \mathcal{P} \right\} \right) \geq \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right)
\]
\[
= P \left\{ \frac{1}{K} \sum_{i=1}^{K} \xi_i(U_1, \ldots, U_i) > t' \right\}
\]
\[
\leq \exp \left\{ -K(t' - a' \log e) \right\}
\]
\[
= \exp \left\{ -K \left( \frac{1}{2} \exp \left( -\frac{n \epsilon}{2} \right) - |\mathcal{P}|(n+1)^{|X|} \exp \left( -\frac{3n \epsilon}{4} \right) \log e \right) \right\}
\]
We invoke Lemma 61 (b) holds for sufficiently large \( n \) and (c) uses \( K = \exp(nR) \geq \exp(ne) \). This and (58) imply (57).

By using (56), (57) and taking union bound over \( P_{UX} \in \mathcal{P}^n(U \times X) \) and \( x \in X^n \) (recall that \( |\mathcal{P}^n(U \times X)| \) and \( |X^n| \) grow only polynomially in \( n \)), we can conclude that there exists a codebook of rate \( R \) such that \( \epsilon \leq R \leq \min_{P_{UX} \in \mathcal{P}} I(U;U') - 3\epsilon/4 \), whose codewords are of type \( P \)

\[
\frac{1}{K} | \{ i \in [1 : K] : d(u_i, u_j) \leq 2\delta \text{ for some } j \in [1 : K], j \neq i \} | < \exp \left( -\frac{n\epsilon}{2} \right).
\tag{61}
\]

and for every joint type \( P_{UX} \in \mathcal{P}^n(U \times X) \) and \( x \in X^n \) satisfying \( P_U = P \) and \( x \in T^n_X \),

\[
\{ i \in [1 : K] : (u_i, x) \in T^n_{UX} \} \leq \exp \left( n \left( |R - I(U;X)|^+ + \epsilon \right) \right).
\tag{62}
\]

In order to obtain (52) from (61), we expurgate \( \exp \left( -\frac{n\epsilon}{2} \right) \) fraction of codewords to obtain \( d(u_i, u_j) > 2\delta \) for every pair of distinct codewords \( u_i, u_j \). The new rate is

\[
R' = \frac{\log (K - K \exp(-n\epsilon/2))}{n} = \frac{\log (\exp(nR)(1 - \exp(-n\epsilon/2)))}{n}
\geq R - \frac{\log(1 - \exp(-n\epsilon/2))}{n}
\geq R - \epsilon/4 \text{ for sufficiently large } n.
\tag{63}
\]

Let \( n_0 \) be such that (56), (60) and (63) hold. Since, rate \( R \) satisfies (59), we have shown the existence of a codebook of rate \( R' \) such that \( \epsilon \leq R' \leq \min_{P_{UX} \in \mathcal{P}} I(U;U') - \epsilon \) and it satisfies (52) and (53).

\[ \square \]

Proof of achievability. From Remark 3, we note that consensus capacity is positive only if \( C_{p-to-p}(W_{V|X}) > 0 \), i.e. \( |U| \geq 2 \). So, we consider a channel \( W_{Y|Z,X} \) with \( |U| \geq 2 \). Consider any rate \( R_0 > 0 \) such that

\[
R_0 < \max_{P_U} \min_{P_{UX} : P_{UX}(x|u) > 0} \min \left( I(U;Y), I(U;Z) \right).
\]

For sufficiently small \( \gamma > 0 \), there exist \( \alpha > 0 \), \( R \) and \( n_1(\alpha) \) such that for every \( n \geq n_1(\alpha) \), there exists \( P_U \in \mathcal{P}^n(U) \) satisfying \( \min_{u \in U} P_U(u) \geq \alpha \) and

\[
R_0 \leq R < \min_{P_{UX} : P_{UX}(x|u) > 0} \min \left( I(U;Y), I(U;Z) \right) - \gamma,
\tag{64}
\]

where the mutual information is evaluated under \( P_{UXYZ}(u,x,y,z) = P_U(u)P_{X|U}(x|u)W_{Y|Z,X}(y,z|x) \). There exist sufficiently small \( \epsilon, \delta > 0 \) such that for all \( n \geq n_1(\alpha) \) and \( P_U \in \mathcal{P}^n(U) \) satisfying (64), \( \min_{P_{UX} \in \mathcal{P}} I(U;U') - \epsilon \geq R \geq \epsilon \) where \( \mathcal{P} = \{ P_{U} \in \mathcal{P}^n(U \times U) : P_U' = P_U, P_U(U \neq U') < 2\delta \} \). We invoke Lemma 10 with \( \epsilon, \delta, \alpha \) as chosen and \( n \geq \max \{ n_1(\alpha), n_0(\epsilon) \} \), \( P = P_U \in \mathcal{P}^n(U) \) satisfying (64), to get an encoder \( f : [1 : 2^{nR}] \rightarrow U^n \). Let \( u(m) = f(m) \), \( m \in [1 : 2^{nR}] \) denote the codewords. For \( y \in Y^n \), we use \( \phi_1(y) \) to denote \( (\phi_1(y_i))_{i \in [1 : n]} \).

For \( x \in X^n \), \( u \in U^n \), \( v \in Y^n \) and \( y \in Y^n \), we write

\[
(x,v) \triangleright y, \quad \text{if for } (x,v) \in T^n_{X,Y}, D(P_{XV}||P_{XW_{Y|X}}) \leq 3\epsilon, \quad \text{and}
\]

\[
(u,x) \triangleright y, \quad \text{if for } (u,x,y) \in T^n_{UXY}, D(P_{UXY}||P_{UXW_{Y|X}}) \leq 3\epsilon.
\tag{65}
\tag{66}
\]

For an encoder \( f \) of rate \( R \), the decoder outputs \( g_B(y) = m \) if it is the unique \( m \in [1 : 2^{nR}] \) such that

1) \( f(m) \triangleright \phi_1(y) \),

\footnote{Note that for \( \delta < 1/4 \), by Fano’s inequality \( I(U;U') \geq H(U) - H(2\delta) - 2\delta \log |U|. \) Thus, there is a sufficiently small choice of \( \delta > 0 \) to so that (64) holds if \( \epsilon < \gamma/2. \)
2) there is an $\bar{x} \in \mathcal{X}^n$ such that $d(f(m), \bar{x}) < \delta$ and $(f(m), \bar{x}) \triangleright y$.

g_b(y) = \perp$ if no such unique $m$ exists. $g_c$ is similarly defined (with $\phi_2, W_{Z|X}$ in lieu of $\phi_1, W_{Y|X}$, respectively.). The first decoding condition requires the codeword to be consistent with the common channel’s output; we denote this by $m \triangleright y$. The second condition, denoted by $m \triangleright y$, requires an “explaining” vector $\tilde{x}$ which is $\delta$-close to the codeword and is consistent with the output $y$. For $x \in \mathcal{X}^n$, define the event

$$B_x = (\exists m \in [1 : 2^{nR}] : d(f(m), x) \geq \delta, \triangleright m \triangleright Y, m \triangleright Y).$$

**Claim 12.** For sufficiently small $\delta > 0$, there exists a sufficiently small $\epsilon > 0$ such that for large enough $n$, for every $x \in \mathcal{X}^n$,

$$Pr(x \triangleright \phi_1(Y)|x) \geq 1 - 2^{-\alpha n},$$

Pr($B_x | x \leq 2^{-\alpha n}$, and

(68) if there exists $m \in [1 : 2^{nR}]$ such that $d(f(m), x) < \delta$,

$$Pr(m \triangleright Y|x) \geq 1 - 2^{-\alpha n}.$$ (69)

To see that this implies the theorem, let us consider three possibilities for the transmitted vector $x$: (i) $x = f(m)$ for some $m$, (ii) $d(f(m), x) \geq \delta$ for all $m \in [1 : 2^{nR}]$, and (iii) there is an $m$ such that $d(f(m), x) < \delta$, but $x \neq f(m)$.

Case (i): $x = f(m)$ for some $m$. For all $m' \neq m$, the encoder in Lemma 10 has $d(f(m), f(m')) \geq 2\delta$. Hence,

$$Pr(g_b(Y) \neq m|f(m)) = Pr(m \triangleright Y \cup m \triangleright Y \cup \exists m' \neq m s.t. m' \triangleright Y, m' \triangleright Y)|f(m))$$

$$\leq Pr((f(m) \triangleright \phi_1(Y)|f(m)) + Pr(m \triangleright Y|f(m)) + Pr(B_{f(m)}|f(m))$$

$$\leq 2^{-\alpha n} + 2^{-\alpha n} + 2^{-\alpha n} \leq 2^{-\alpha n + \log 3},$$

where the upper bounds on the probabilities follow from (67)-(69). Similarly, $Pr(g_c(Z) \neq m|f(m)) \leq 2^{-\alpha n + \log 3}$. By a union bound,

$$Pr(g_b(Y) = g_c(Z) = m|f(m)) \geq 1 - 2^{-\alpha n + \log 6}.$$

Case (ii): $d(f(m), x) \geq \delta$ for all $m \in [1 : 2^{nR}]$. In this case, $Pr(g_b(Y) \neq \perp |x) \leq Pr(B_x | x \leq 2^{-\alpha n}$ where the last step is from (68). Similarly, $Pr(g_c(Z) \neq \perp |x \leq 2^{-\alpha n}$. Hence, by a union bound $Pr(g_b(Y) = g_c(Z) = \perp |x) \geq 1 - 2^{-\alpha n + 1}$.

Case (iii): There is an $m$ such that $d(f(m), x) < \delta$, but $x \neq f(m)$. Then, for all $m' \neq m$, by (46), $d(f(m'), x) \geq d(f(m'), f(m)) - d(f(m), x) \geq 2\delta - \delta = \delta$. Hence, $Pr(g_b(Y) \neq \perp m, \perp |x) \leq Pr(B_x | x \leq 2^{-\alpha n}$, where the last step follows from (68). By a union bound, $Pr(g_b(Y), g_c(Z) \in \{m, \perp \}|x) \geq 1 - 2^{-\alpha n + 1}$. Further, $Pr(m \triangleright Y|x) \geq 1 - 2^{-\alpha n}$ by (69).

Similarly, $Pr(m \triangleright Z|x) \geq 1 - 2^{-\alpha n}$. Hence, $Pr(m \triangleright Y, m \triangleright Z|x) \geq 1 - 2^{-\alpha n + 1}$ by a union bound. Since $\phi_1(Y) = \phi_2(Z)$, either the first condition for message $m$ is met or not met together for both decoders. Hence,

$$Pr(g_b(Y) = g_c(Z)|x)$$

$$\geq Pr(m \triangleright Y, m \triangleright Z, (g_b(Y), g_c(Z) \in \{m, \perp \})|x)$$

$$\geq 1 - 2^{-\alpha n + 2}$$

where the last step involves taking a union bound.

It only remains to prove Claim 12.

**Proof of Claim 12.** We will first show (67).

$$Pr((x \triangleright \phi_1(Y)) \cup x)$$

$$= Pr((x, \phi_1(Y)) \in T_{XY}^n, D(P_{XY}||P_X W_{Y|X}) > 3\epsilon|x)$$

$$\leq \sum_{P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}): D(P_{XY}||P_X W_{Y|X}) > 3\epsilon} Pr((x, \phi_1(Y)) \in T_{XY}^n|x)$$

$$= \sum_{P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}): D(P_{XY}||P_X W_{Y|X}) > 3\epsilon} W^n_{V|X}(v|x)$$

$$\leq \sum_{P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}): D(P_{XY}||P_X W_{Y|X}) > 3\epsilon} \exp (-nD(P_{XY}||P_X W_{Y|X}))$$

$$\leq |\mathcal{P}(\mathcal{X} \times \mathcal{Y})| \exp (-3n\epsilon)$$
such that the joint type $P$ and there exists $P$, which will show (75) holds for sufficiently large $n$. This shows (67).

Next, we show (69). Let $m \in [1 : 2^n R]$ and $x \in \mathcal{X}^n$ such that $d(f(m), x) < \delta$. Note that, since $d(f(m), x) < \delta$, $\Pr(m \uparrow Y \uparrow | x) \geq \Pr((f(m), x) \uparrow Y | x)$ (as $x$ may serve as the explaining vector $x$ under the event $(f(m), x) \uparrow y$). We will show that

$$\Pr((f(m), x) \uparrow Y)^c | x) \leq 2^{-n \epsilon}$$

which will show (69).

\[
\begin{align*}
\Pr((f(m), x) \uparrow Y)^c | x) &= \sum_{P \in P^n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y}) : (f(m), x) \in T^n_{UXY}} \Pr((f(m), x, Y) \in T^n_{UXY}) \\
&= \sum_{P \in P^n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y}) : (f(m), x) \in T^n_{UXY}} \sum_{y \in T^n_{U|X}(f(m), x)} W_{Y|X}(y | x) \\
&\leq \sum_{P \in P^n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y}) : (f(m), x) \in T^n_{UXY}} \exp(-3n \epsilon) \\
&\leq |P^n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})| \exp(-3n \epsilon) \\
&\leq \exp(-2n \epsilon) \leq \exp(-n \epsilon).
\end{align*}
\]

where (a) follows from (43) and (b) holds for sufficiently large $n$.

Now, we are left to show (68). Recall that

$$B_x = (\exists m \in [1 : 2^n R] : d(f(m), x) \geq \delta, m \uparrow Y, m \uparrow Y).$$

\[
\begin{align*}
\Pr(B_x | x) &= \Pr((x \uparrow \phi_1(Y))^c | x) + \Pr((x \uparrow \phi_1(Y)) \cap B_x | x) \\
&\leq 2^{-2n \epsilon} + \Pr((x \uparrow \phi_1(Y)) \cap B_x | x) \tag{71}
\end{align*}
\]

where (a) follows from (70). Let $D$ be the set of joint types $P_{UXY} \in P^n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ satisfying

\[
\begin{align*}
d(P_{UX}) &\geq \delta, \tag{72} \\
D(P_{X|Y} \| P_X W_{|X}) &\leq 3 \epsilon, \tag{73} \\
D(P_{U|X} \| P_U W_{|X}) &\leq 3 \epsilon, \tag{74}
\end{align*}
\]

and there exists $P_{X|UX}$ such that

\[
\begin{align*}
d(P_{UX}) &\leq \delta, \text{ and} \tag{75} \\
D(P_{UX} \| P_U W_{|X}) &\leq 3 \epsilon \tag{76}
\end{align*}
\]

where $W_{Y|X}(y | x) := W_{Y|X}(y | x)$ for all $x, y$. Under the events $\{x \uparrow \phi_1(Y)\}$ and $B_x$, there is an $m \in [1 : 2^n R]$ and $x \in \mathcal{X}^n$ such that the joint type $P_{UXY}$ defined by $(f(m), x, y, x) \in T^n_{UXY}$ satisfies (72)-(76). In particular, $(x \uparrow \phi_1(Y))$ implies (73), $d(f(m), x) \geq \delta, m \uparrow Y$ and $m \uparrow Y$ in $B_x$ implies (72), (74) and (75)-(76) respectively. Hence,

\[
\{x \uparrow \phi_1(Y)\} \cap B_x \subseteq \{3P_{UXY} \in D, m \in [1 : 2^n R] \text{ such that } (f(m), x, Y) \in T^n_{UXY}\}. \tag{77}
\]

Thus,

$$\Pr((x \uparrow \phi_1(Y)) \cap B_x | x)$$
By Pinsker's inequality, this implies that

\[
\Pr((x, f(m), Y) \in T^n_{UV}(x)) = \sum_{x \in \mathcal{T}_n^D} \Pr((x, f(m), Y) \in T^n_{UV}(x)) \leq \sum_{x \in \mathcal{T}_n^D} \sum_{m \in [1:2^nR]} \Pr((x, f(m), Y) \in T^n_{UV}(x)) \\
= \sum_{x \in \mathcal{T}_n^D} \sum_{m \in [1:2^nR]} \sum_{y \in \mathcal{T}_n^D} \Pr((x, f(m), Y) \in T^n_{UV}(x)) \\
\leq \sum_{x \in \mathcal{T}_n^D} \sum_{m \in [1:2^nR]} \sum_{y \in \mathcal{T}_n^D} \exp(-n(I(Y; U|X) - \epsilon)) \\
\leq \sum_{x \in \mathcal{T}_n^D} \exp\left(n \left(|R - I(U; X)|^+ + \epsilon\right)\right) \exp(-n(I(Y; U|X) - \epsilon)) \\
= \sum_{x \in \mathcal{T}_n^D} \exp\left(n \left(|R - I(U; X)|^+ - I(Y; U|X) + 2\epsilon\right)\right),
\]

(78)

where (a) is obtained by taking union bound over \(P_{UXY} \in \mathcal{D}, m \in [1:2^nR]\) (see (77)), (b) follows by the following argument: for each \(y \in \mathcal{T}_n^{\phi_1(X),\phi_1(Y)}(x, u), W_{Y|X}(y|x)\) is the same and is hence upper bounded by \(1/|\mathcal{T}_n^{\phi_1(X)}(x)|\). By (41), \(1/|\mathcal{T}_n^{\phi_1(X)}(x)| \leq (n + 1)^{|X|}|Y| \exp(-nH(Y|X)) \leq \exp(n(-H(Y|X) + \epsilon))\) for sufficiently large \(n\). Also, \(|\mathcal{T}_n^{\phi_1(X)}(x, u)| \leq \exp(nH(Y|XU))\) by (41). The inequality (c) follows from (53).

For each \(P_{UXY} \in \mathcal{D}\), let

\[\zeta_{UXY} := \exp\left(n \left(|R - I(U; X)|^+ - I(Y; U|X) + 2\epsilon\right)\right)\]

(79)

For each \(P_{UXY} \in \mathcal{D}\), if we can show that \(\zeta_{UXY} \leq 2^{-2ne}\), then, from (71) and (78),

\[
\Pr(B_\epsilon|X) \leq 2^{-2ne} + \Pr(\{x : \phi_1(Y) \in B_\epsilon|X\}) \\
\leq 2^{-2ne} + |\mathcal{D}|2^{-2ne} \\
\leq 2^{-2ne}
\]

for sufficiently large \(n\). Thus, we would have shown (68). To this end, fix a distribution \(P_{UXY} \in \mathcal{D}\). We consider two possibilities.

Case (i) \(R \leq I(U; X)\):

Then,

\[\zeta_{UXY} = \exp(n(-I(Y; U|X) + 2\epsilon))\]

Note that \(I(Y; U|X) \geq I(\phi_1(Y); U|X)\). Let \(V := \phi_1(Y)\). We will show that \(I(V; U|X) > 4\epsilon\). This would imply that \(\zeta_{UXY} \leq \exp(-2ne)\). For the sake of contradiction, assume \(I(V; U|X) \leq 4\epsilon\). Using this and (73),

\[
7\epsilon \geq I(V; U|X) + D(P_{UXV}\|W_{V|X|x}) \\
= \sum_{u,x,v} P_{UXV}(u,x,v) \log\left(\frac{P_{UXV}(u,x,v)}{P_X(x)P_{UXV}(u|x)}\right) + \log\left(\frac{P_X(x)W_{V|X|v|x}|x}}{P_X(x)W_{V|X}|v|x}|x}\right) \\
= D(P_{UXV}\|P_{UXV}\|W_{V|X}x).\]

By Pinsker’s inequality, this implies that

\[
\sqrt{14\epsilon \ln 2} \geq \sum_{u,x,v} \left|P_{UXV}(u,x,v) - P_{UXV}(u,x)W_{V|X}(v|x)|x\right| \\
\geq \sum_{u,x,v} \left|P_{UXV}(u,x,v) - P_{UXV}(u,x) \sum_{\tilde{u} \in U} P_{UXV}(\tilde{u}|x)W_{V|X}(v|x)\right| \\
\geq \sum_{u,v} \left|P_{UXV}(u,v) - \sum_{x,\tilde{u}} P_{UXV}(u,x)P_{UXV}(\tilde{u}|x)W_{V|X}(v|x)\right|
\]

(80)

where (a) follows from (16) with \(P_{UXV}(u|x) := \tilde{P}_{UXV}(u|x)\) for all \(u, x\). Define

\[
P_{UX}(u, \tilde{u}) := \sum_x P_{UX}(u,x)P_{UXV}(\tilde{u}|x)
\]

(81)
for all \( u, \tilde{u} \in U \). Then, (80) gives
\[
\sqrt{14 \epsilon \ln 2} \geq \sum_{u,v} \left| P_{U|V}(u,v) - \sum_{\tilde{u}} P_{U|\tilde{U}}(u, \tilde{u})W_{V|X}(v|\tilde{u}) \right|.
\] (82)

Next, applying Pinsker’s inequality to (74) gives
\[
\sum_{u,v} \left| P_{U|V}(u,v) - P_{U}(u)W_{V|X}(v|u) \right| \leq \sqrt{6 \epsilon \ln 2}.
\] (83)

From (82) and (83),
\[
\sqrt{14 \epsilon \ln 2} + \sqrt{6 \epsilon \ln 2} \geq \sum_{u,v} \left| \sum_{\tilde{u}} P_{U|\tilde{U}}(u, \tilde{u})W_{V|X}(v|\tilde{u}) - P_{U}(u)W_{V|X}(v|u) \right|
= \sum_{u,v} \left| \sum_{\tilde{u}} P_{U|\tilde{U}}(\tilde{u}|u)W_{V|X}(v|\tilde{u}) - W_{V|X}(v|u) \right|
\geq \alpha \sum_{u,v} \left| \sum_{\tilde{u}} P_{U|\tilde{U}}(\tilde{u}|u)W_{V|X}(v|\tilde{u}) - W_{V|X}(v|u) \right|
\]
where (a) follows by recalling that \( P_{U}(u) \geq \alpha, u \in U \). This implies that for every \( u' \in U \),
\[
\sum_{v \in V} \left| \sum_{\tilde{u} \in U} P_{U|\tilde{U}}(\tilde{u}|u')W_{V|X}(v|\tilde{u}) - W_{V|X}(v|u') \right| \leq \frac{1}{\alpha} \left( \sqrt{2 \epsilon \ln 2 (\sqrt{7} + \sqrt{3})} \right).
\] (84)

Also, from (81) and Definition 2 (equation (44)), \( d(P_{U|X}) = d(P_{U|\tilde{U}}) \). Thus, using (72),
\[
P(U \neq \tilde{U}) = \sum_{u, \tilde{u}, u \neq \tilde{u}} P_{U|\tilde{U}}(u, \tilde{u}) \geq \delta.
\]
This implies that
\[
1 - \delta \geq \sum_{u} P_{U|\tilde{U}}(u, u) = \sum_{u} P_{U}(u)P_{U|u}(u|u)
\geq \min_{u' \in U} P_{U|\tilde{U}}(u'|u').
\]
Thus, there exists \( u' \in U \) for which
\[
P_{U|\tilde{U}}(u'|u') \leq 1 - \delta.
\] (85)

From Claim 9 and (85), there exists \( u' \in U \) for which
\[
\sum_{v \in V} \left| \sum_{u \in U} P_{U|V}(u|u')W_{V|X}(v|u) - W_{V|X}(v|u') \right| \geq \delta \gamma.
\] (86)
This contradicts (84) for \( \epsilon \) small enough such that (recall that \( \gamma > 0 \) in Claim 9)
\[
\delta > \frac{1}{\gamma \alpha} \left( \sqrt{2 \epsilon \ln 2 (\sqrt{7} + \sqrt{3})} \right).
\] (87)
Thus, \( I(Y; U|X) \geq 4 \epsilon \). This implies that for every \( P_{U|X} \in \mathcal{D} \) with \( I(U; X) \geq R \),
\[
\zeta_{U|XY} \leq \exp(-2\epsilon R).
\] (88)

*Case (ii):* We consider distributions \( P_{U|XY} \in \mathcal{D} \) with \( I(U; X) < R \). In this case, (79) is
\[
\zeta_{U|XY} = \exp \left( n \left( R - I(U; X) - I(Y; U|X) + 2\epsilon \right) \right)
= \exp \left( n \left( R - I(U; XY) + 2\epsilon \right) \right)
= \exp \left( n \left( R - I(U; Y) + 2\epsilon - I(U; X|Y) \right) \right)
\] (89)
(90)
We will show that \( I(U; Y) - R > 4 \epsilon \) for sufficiently small \( \delta > 0, \epsilon > 0 \). This would imply that \( \zeta_{U|XY} \leq \exp(-2\epsilon R) \). Since \( P_{U|XY} \in \mathcal{D} \), there exists conditional distribution \( P_{X|U|XY} \) satisfying (75) and (76) (see definition of \( \mathcal{D}, (72)-(76) \)). We consider this \( P_{X|U|XY} \). Using (76) and Pinsker’s inequality,
\[
\sqrt{6 \epsilon \ln 2} \geq \sum_{u,y} \left| P_{U|Y}(u,y) - \sum_{x} P_{U}(u)P_{X|U}(\bar{x}|u)W_{Y|X}(y|x) \right|
\]
\[
\begin{align*}
&\sum_{u,y} P_{UY}(u, y) - \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) W_{V|X}(v|\bar{x}) W_{Y|XV}(y|\bar{x}, v) \\
&\geq \sum_{u,y} P_{UY}(u, y) - \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) \left( \sum_{\tilde{u}} \tilde{P}_{U}(\tilde{u}|\bar{x}) W_{V|X}(v|\tilde{u}) \right) W_{Y|XV}(y|\bar{x}, v) \\
&= \sum_{u,y} \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) \tilde{P}_{U}(\bar{x}|u) W_{V|X}(v|u) W_{Y|XV}(y|\bar{x}, v) \\
&\geq \sum_{u,y} \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) \tilde{P}_{U}(\bar{x}|u) W_{V|X}(v|u) W_{Y|XV}(y|\bar{x}, v) - d(P_{UX}) \\
&\geq \sum_{u,y} \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) \tilde{P}_{U}(\bar{x}|u) W_{V|X}(v|u) W_{Y|XV}(y|\bar{x}, v) - \delta
\end{align*}
\]

where (a) follows from (3), (b) follows from (16), (c) follows from Definition 2, i.e., \(d(P_{UX}) = \sum_{u,\bar{x},\tilde{u} \neq u} P_{U}(u) P_{X|U}(\bar{x}|u) \tilde{P}_{U}(\tilde{u}|\bar{x})\) and (d) follows from (75). Thus,

\[
\sum_{u,y} \left| P_{UY}(u, y) - \sum_{\bar{x},v} P_{U}(u) P_{X|U}(\bar{x}|u) \tilde{P}_{U}(\bar{x}|u) W_{V|X}(v|u) W_{Y|XV}(y|\bar{x}, v) \right| \leq \sqrt{6e \ln 2 + \delta}. 
\]

(91)

For \(u \in \mathcal{U}\), let \(\mathcal{X}_u := \{x \in \mathcal{X} : W_{V|X}(v|x) = W_{V|X}(v|u), \forall v \in \mathcal{V}\}\). Using Lemma 5 and (16), we will show that for \(u \in \mathcal{U}\)

\[
\tilde{P}_{U|X}(u|x) = 1 \text{ if and only if } x \in \mathcal{X}_u.
\]

(92)

For \(u \in \mathcal{U}\), suppose \(x \in \mathcal{X}_u\), then by (16),

\[
\sum_{u'} \tilde{P}_{U|X}(u'|x) W_{V|X}(v|u') = W_{V|X}(v|x) = W_{V|X}(v|u).
\]

Using Lemma 5, we conclude that \(\tilde{P}_{U|X}(u|x) = 1\). Now suppose, \(\tilde{P}_{U|X}(u|x) = 1\) for \(x \in \mathcal{X}\). Then, by (16), \(W_{V|X}(v|x) = W_{V|X}(v|x)\) which implies that \(x \in \mathcal{X}_u\). Thus,

\[
\eta := \min_{u \in \mathcal{U}} \min_{x \not\in \mathcal{X}_u} \left( 1 - \tilde{P}_{U|X}(u|x) \right) > 0.
\]

(93)

This implies that

\[
\begin{align*}
d(P_{UX}) &= \sum_{u} \sum_{\bar{x}} \sum_{u' \neq u} P_{UX}(u, \bar{x}) \tilde{P}_{U|X}(u'|\bar{x}) \\
&= \sum_{u} \sum_{\bar{x} \not\in \mathcal{X}_u} \sum_{u' \neq u} P_{UX}(u, \bar{x}) \tilde{P}_{U|X}(u'|\bar{x}) \\
&= \sum_{u} \sum_{\bar{x} \not\in \mathcal{X}_u} P_{UX}(u, \bar{x}) \left( 1 - \tilde{P}_{U|X}(u'|\bar{x}) \right) \\
&\geq \sum_{u} \sum_{\bar{x} \not\in \mathcal{X}_u} P_{UX}(u, \bar{x}) \eta
\end{align*}
\]

where (a) follows by noting that \(\tilde{P}_{U|X}(u'|\bar{x}) = 0\) for \(\bar{x} \not\in \mathcal{X}_u\) and \(u' \neq u\) (see (92)) and (b) follows from (93). Let

\[
t := \sum_{u} \sum_{\bar{x} \not\in \mathcal{X}_u} P_{UX}(u, \bar{x}).
\]

(94)
We see that \( t \leq d(P_{U|X})/\eta \leq \delta/\eta \) (see (75)). This implies that
\[
\sum_{u} \sum_{x \in X_u} P_{U,X}(u,x) = 1 - \sum_{u} \sum_{x \notin X_u} P_{U,X}(u,x) = 1 - t \geq 1 - \delta/\eta.
\]
Hence, \( 1 - t > 0 \) for sufficiently small \( \delta > 0 \). For such \( \delta \), we can define a joint distribution \( P_{U,X} \) as
\[
P_{U,X}(u,x) = P_U(u)P_{X|\emptyset}(x|u) := \begin{cases} \frac{1}{1 - t} P_{U,X}(u,x), & \text{if } x \in X_u, \\ 0, & \text{otherwise.} \end{cases}
\]
Thus, proceeding from (91),
\[
\sqrt{6} \ln 2 + \delta \geq \sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|u) W_{Y|XV}(y|x,v) \right|
\]
\[
= \sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|u) W_{Y|XV}(y|x,v) \right|
\]
\[
- \sum_{x \notin X_u} \sum_{u,y} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|u) W_{Y|XV}(y|x,v)
\]
\[
\geq \sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x \in X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|u) W_{Y|XV}(y|x,v) \right|
\]
\[
- \sum_{u,y} \sum_{x \notin X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x)
\]
\[
\geq \sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x \in X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|u) W_{Y|XV}(y|x,v) \right|
\]
\[
- \left( \sum_{u,y} \sum_{x \in X_u} t P_{U,X}(u,x) W_{V|X}(v|u) W_{Y|XV}(y|x,v) \right) - t
\]
\[
= \sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x \in X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|x) W_{Y|XV}(y|x,v) \right| - 2t
\]
\[
\geq \left( \sum_{u,y} P_{U,Y}(u,y) - \sum_{x \notin X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|x) W_{Y|XV}(y|x,v) \right) - 2t
\]
\[
\geq (c) \sum_{u,y} P_{U,Y}(u,y) - \sum_{x \notin X_u} P_{U,X}(u,x) \tilde{P}_{U|X}(u|x) W_{V|X}(v|x) W_{Y|XV}(y|x,v) \right| - 2t
\]
where (a) follows from (94), (b) uses (92) and (96), and (c) follows from (3). Thus,
\[
\sum_{u,y} \left| P_{U,Y}(u,y) - \sum_{x} P_U(u) P_{X|\emptyset}(x|u) W_{Y|XV}(y|x) \right| \leq 2t + \sqrt{6} \ln 2 + \delta
\]
\[
= \delta(2/\eta + 1) + \sqrt{6} \ln 2.
\]
From (87), recall that \( \epsilon \) satisfies
\[
\sqrt{6} \ln 2 < \frac{\sqrt{3} \delta \gamma \alpha}{\sqrt{t + \gamma}}.
\]
(97)
With this,
\[
\sum_{u,y} \left| P_{UY}(u,y) - \sum_{x} P_U(u) P_{X|U}(\bar{x}|u) W_{Y|X}(y|x) \right| \leq \delta \left( \frac{2}{\eta} + 1 + \frac{\sqrt{3} \gamma \alpha}{\sqrt{7} + \sqrt{3}} \right). \tag{98}
\]
Further
\[
\sum_{u,y} \left| \sum_{x} P_U(u) P_{X|U}(\bar{x}|u) W_{Y|X}(y|x) - \sum_{x} P_U(u) P_{X|U}(\bar{x}|u) W_{Y|X}(y|x) \right|
= \sum_{u,y} \left| \sum_{x} P_X(\bar{x}|u) W_{Y|X}(y|x) (P_U(u) - P_U(\bar{u})) \right|
= \sum_{u} \sum_{y} \sum_{x} P_{X|U}(\bar{x}|u) W_{Y|X}(y|x) \left| P_U(u) - P_U(\bar{u}) \right|
= \sum_{u} \left| P_U(u) - P_U(\bar{u}) \right|
\leq \delta \left( \frac{2}{\eta} + 1 + \frac{\sqrt{3} \gamma \alpha}{\sqrt{7} + \sqrt{3}} \right), \tag{99}
\]
where \((a)\) follows from (98). From (98) and (99), we obtain
\[
\sum_{u,y} \left| P_{UY}(u,y) - \sum_{x} P_U(u) P_{X|U}(\bar{x}|u) W_{Y|X}(y|x) \right| \leq 2\delta \left( \frac{2}{\eta} + 1 + \frac{\sqrt{3} \gamma \alpha}{\sqrt{7} + \sqrt{3}} \right). \tag{100}
\]
Let \(P_{UY}(u,y) = \sum_{x} P_U(u) P_{X|U}(x|u) W_{Y|X}(y|x), u \in U, y \in Y\) where \(P_{X|U}(x|u) > 0\) only if \(x \in X_u\) (see (96)). With this and (64), we see that
\[
I(U;\hat{Y}) \geq \min_{\left. p_{X|U}; p_{X|U}(x|u) > 0 \right\} \text{only if } W_{V|X}(v|x) = W_{V|X}(v|u) \forall v} \min \left( I(U;Y'), I(U;Z') \right) > R, \tag{101}
\]
where the mutual information in the RHS of (101) is evaluated under \(P_{UXY';Z'}(u,x,y,z) = P_U(u) P_{X|U}(x|u) W_{Y|X}(y,x) W_{Z|X}(z|x)\). Since mutual information \(I(U;Y)\) is continuous in \(P_{UY}\), from (100) and (101), we can choose sufficiently small \(\delta > 0\) such that
\[
I(U;Y) - R > 0. \tag{102}
\]
Further, we choose \(\epsilon > 0\) small enough such that \(I(U;Y) - R \geq 4\epsilon\) and (87) holds. From (90), this implies that
\[
\zeta_{U;XY} \leq \exp(-2n\epsilon).
\]
\[\Box\]
\[\Box\]

VI. DISCUSSION

While we considered the two-receiver broadcast channel, the results readily generalize to more than two receivers. For instance, for the three-receiver broadcast channel, the common channel may be defined analogous to Definition 1 via a characteristic tripartite hypergraph whose hyperedges are the triples of symbols which occur together at the channel outputs of the three receivers with positive probability for some channel input symbol. The connected components of this hypergraph are the output symbols of the common channel. The definition of the effective input alphabet remains unchanged. The capacity expression in (17) is modified so that the inner minimum is of the three mutual information quantities corresponding to the three receivers instead of two. The proofs of converse and achievability can be verified to generalize with no significant changes required.

As discussed in Remarks 1 and 4, the rate of communication with consensus that can be achieved is sensitive to how fast the error probability \(P_e(n)\) is required to decay with the blocklength \(n\). Here, we studied the most natural regime where the error probability decays exponentially, i.e., \(-\log(P_e(n)) = \Omega(n)\), and found that the capacity remains unchanged as long as \(P_e(n)\) is required to decay at least inverse linearly, i.e., \(P_e(n) = o(1/n)\). Understanding the behaviour of capacity in regimes where this is further relaxed (for instance, to simply \(P_e(n) \to 0\)) would be of interest. The example from Appendix A shows that in these regimes, the presence of common or correlated randomness among the receivers which is unknown to the sender has an effect on the capacity.
The curious case of $P_e^{(n)}$ approaching 0 slower than $o(1/n)$

The converse of Theorem 4 in Section V made use of the requirement that $P_e^{(n)} = o(1/n)$. We will see that a converse cannot be shown if this is further relaxed to $P_e^{(n)} = o(1)$. For this, we show an example where $C_{Byz} = 0$, but a positive rate is achievable with $P_e^{(n)} = o(1/n^{2-\epsilon})$, for any $\epsilon > 0$. The example is in fact the independent binary erasure broadcast channel (i.e., the two-step binary erasure broadcast channel with $p = 1$) of Section III-A, but with additional common randomness shared by the decoders which is unknown to the sender. Without the additional common randomness, even two messages cannot be communicated over this channel with $P_e^{(n)} \rightarrow 0$ (see Remark 2). We also know that the presence of such randomness does not affect $C_{Byz}$ which requires $P_e^{(n)} = o(1/n)$ (see Remark 3). However, we will show that the availability of common randomness among the receivers unknown to the sender facilitates communication with consensus at non-zero rates over the independent binary erasure channel with $P_e^{(n)} = o(1/n^{2-\epsilon})$ for any $\epsilon > 0$.

The independent binary erasure broadcast channel with common randomness is $W_{(YS)|(ZS)} = W_{Y|X}W_{Z|X}W_{S|X}$, where the channels $W_{Y|X}$ and $W_{Z|X}$ are identical binary erasure channels (BEC) with erasure probability $0 < q < 1$ and $W_{S|X}$ is a completely noisy channel (i.e., $x$). Theorem 13. Let $\tilde{\Theta}$. Claim 14. Let $P_e^{(n)} = o(1/n^{2-\epsilon})$. For all $n \geq 1$, there exists $\delta > 0$ such that for every $n$ there is an $f : [1 : 2^n] \rightarrow \{0, 1\}$ such that $d_H(f(m), f(m')) > n(2q + \delta)$ for all distinct $m, m' \in [1 : 2^n]$.

We will employ the code in Claim 14 with the decoders $g_B$ and $g_C$ described below: For $x^n \in \{0, 1\}$ and $y^n \in \{0, 1\}$, we write $x^n \rightarrow y^n$ if $y_i \in \{x_i, e\}$ for all $i \in [1 : n]$. Let $\ell$ be the integer such that $\frac{4\ell}{2^n} < 2^\ell \leq \frac{4\ell}{2^n}$ for all the $\ell$ bits, and return the integer whose binary representation is given by these $\ell$ bits. Hence, $h$ maps a uniform distribution over $\{0, 1\}$ to a uniform distribution over $[1 : 2^n]$.

Let $y^n, z^n \in \{0, 1\}$ and $s^n \in \{0, 1\}$. The decoder output $g_B(y^n, s^n) = m$ if there is a unique $m \in [1 : 2^n]$ such that $\exists m$ s.t. $\tilde{x}^n \rightarrow y^n$ and $d_H(\tilde{x}^n, f(m)) \leq h(s^n)$.

If no such unique $m$ exists, $g_B(y^n, s^n) = \perp$. Decoder output $g_C(z^n, s^n)$ is similarly defined.

Let $x^n$ be the string sent over the channel, and $(Y^n, S^n)$ and $(Z^n, S^n)$ be the random variables corresponding to the strings received by Bob and Carol, respectively. Define the event

$$
B_{x^n} = \left( \exists m \in [1 : 2^n] \text{ s.t. } d_H(f(m), x^n) > n\left(q + \delta \frac{2^n}{2}\right) \text{ and } \exists \tilde{x}^n \rightarrow Y^n \text{ s.t. } d_H(\tilde{x}^n, f(m)) \leq h(S^n) \right).
$$

We will first prove the theorem assuming the following claim.

Claim 15. When $k = \frac{1}{32}$, $P(B_{x^n}|x^n) \leq e^{-\frac{n\delta^2}{4}}$.

Consider three possibilities for the input $x^n$: (i) $x^n = f(m)$ for some $m$, (ii) $d_H(x^n, f(m)) > n\left(q + \delta \frac{2^n}{2}\right)$ for all $m \in [1 : 2^n]$, and (iii) $0 < d_H(x^n, f(m)) \leq n\left(q + \delta \frac{2^n}{2}\right)$ for some $m$.

Case (i): We have $x^n \rightarrow y^n$ and $d_H(x^n, f(m)) = 0$. Moreover, for all $m' \neq m$, $d_H(x^n, f(m')) = d_H(f(m), f(m')) > n(2q + \delta)$. Hence, by Claim 15,

$$
P(g_B(Y^n, S^n) \neq m|x^n) = P(\exists m' \neq m, \tilde{x}^n \rightarrow Y^n \text{ s.t. } d_H(\tilde{x}^n, f(m')) \leq h(s^n)|x^n) \leq P(B_{x^n}|x^n) \leq e^{-\frac{n\delta^2}{4}}.
$$

Similarly, $P(g_C(Z^n, S^n) \neq m|x^n) \leq e^{-\frac{n\delta^2}{4}}$. By a union bound, $\lambda_m = 2e^{-\frac{n\delta^2}{4}}$.

Case (ii): Since $d_H(x^n, f(m)) > n\left(q + \delta \frac{2^n}{2}\right)$ for all $m \in [1 : 2^n]$, by Claim 15,

$$
P(g_B(Y^n, S^n) \neq \perp|x^n) \leq P(\exists m \in [1 : 2^n], \tilde{x}^n \rightarrow y^n \text{ s.t. } d_H(\tilde{x}^n, f(m)) \leq h(S^n)|x^n) \leq P(B_{x^n}|x^n) \leq e^{-\frac{n\delta^2}{4}}.
$$

We refer to Section V for the proof of Claim 15.
Similarly, \( P(g_B(Y^n, S^n)) \neq \perp x^n) \leq e^{-\frac{\delta}{2} x^n}. \) By a union bound, \( P(g_B(Y^n, S^n) = g_C(Z^n, S^n) = \perp x^n) \geq 1 - 2e^{-\frac{\delta}{2} x^n}. \)

Case (iii): Define \( \tilde{X}_B \) such that \( \tilde{X}_B(i) \) (the coordinate \( i \) of \( \tilde{X}_B \)) is \( Y_i \) if \( Y_i \neq e \) and \( f_i(m) \) (the coordinate \( i \) of \( f(m) \)), otherwise. By definition, \( \tilde{X}_B \Rightarrow Y^n. \) Moreover, for all \( \tilde{x}^n \Rightarrow Y^n, \)

\[
d_H(\tilde{x}^n, f(m)) = |\{i : Y_i \notin \{f_i(m), e\}\}| + |\{i : \tilde{x}_i \neq f_i(m), Y_i = e\}| \geq |\{i : Y_i \notin \{f_i(m), e\}\}| = d_H(\tilde{X}_B^m, f(m)) \tag{104}
\]

Similarly, we define \( \tilde{X}_C \) with \( Z^n \) in lieu of \( Y^n. \)

Suppose, for all \( m' \neq m, \) there exists no \( \tilde{x}^n \) such that \( d_H(\tilde{x}^n, f(m')) \leq h(S^n) \) and \( \tilde{x}^n \Rightarrow Y^n \) or \( \tilde{x}^n \Rightarrow Z^n. \) Then, since \( \tilde{X}_B \Rightarrow Y^n \) and \( \tilde{X}_C \Rightarrow Z^n, \) the decoders agree on \( m \) if both \( d_H(\tilde{X}_B^m, f(m)) \) and \( d_H(\tilde{X}_C^m, f(m)) \) are at most \( h(S^n), \) and, by \( (104), \) they agree on \( \perp \) if both are more than \( h(S^n). \) Defining \( \mu = (1-q)d_H(x^n, f(m)), \)

\[
P(g_B(Y^n, S^n) = g_C(Z^n, S^n)|x^n)
\]

\[
\geq P(\exists m' \neq m, \tilde{x}^n \text{ s.t. } d_H(\tilde{x}^n, f(m')) \leq h(S^n) \text{ and } (\tilde{x}^n \Rightarrow Y^n \text{ or } \tilde{x}^n \Rightarrow Z^n, d_H(\tilde{X}_B^m, f(m)) \in [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon], \quad d_H(\tilde{X}_C^m, f(m)) \in [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon] \quad \text{and} \quad h(S^n) \notin [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon]|x^n). \tag{105}
\]

For any \( m' \neq m, \) by the triangle inequality,

\[
d_H(x^n, f(m')) \geq d_H(f(m), f(m')) - d_H(x^n, f(m)) > n(2q + \epsilon) - n(q + \delta/2) = n(q + \delta/2).
\]

Hence, \( P(\exists m' \neq m, \tilde{x}^n \Rightarrow Y^n \text{ s.t. } d_H(\tilde{x}^n, f(m')) \leq h(S^n)|x^n) = P(\mathcal{B}_{x^n}|x^n) \leq e^{-\frac{\alpha}{8} x^n} \) by Claim 15. By a union bound,

\[
P(\exists m' \neq m, \tilde{x}^n \Rightarrow Y^n \text{ s.t. } d_H(\tilde{x}^n, f(m')) \leq h(S^n) \text{ and } (\tilde{x}^n \Rightarrow Y^n \text{ or } \tilde{x}^n \Rightarrow Z^n)|x^n) \leq e^{-\frac{\alpha}{8} x^n}. \tag{106}
\]

For all \( i \in [1 : 2^n], Y_i = e \) independently with probability \( q. \) Hence, \( d_H(\tilde{X}_B^m, f(m)) = |\{i : Y_i \notin \{f_i(m), e\}\}| = |\{i : f_i(m) \neq x_i \text{ s.t. } Y_i = e\}| \) is a binomial distribution with mean \( \mu = (1-q)d_H(x^n, f(m)) \) and success probability \( (1-q). \) For any \( \epsilon > 0, \) by the Chernoff bound,

\[
P(d_H(\tilde{X}_B^m, f(m)) \notin [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon]|x^n) = P(d_H(\tilde{X}_B^m, f(m)) - \mu > n^{\frac{1}{4}} + \epsilon|x^n) \leq e^{-\frac{\epsilon}{2}((\frac{1}{4} + \epsilon)^2) \leq 2e^{-\frac{\epsilon}{2} n^{\frac{1}{4}}}. \]

The final inequality used the bound \( \mu \leq n. \) By a union bound,

\[
P(d_H(\tilde{X}_B^m, f(m)) \notin [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon] \text{ or } d_H(\tilde{X}_C^m, f(m)) \notin [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon]|x^n) \leq 4e^{-\frac{\epsilon}{2} n^{\frac{1}{4}}}. \tag{107}
\]

Using \( (106) \) and \( (107), \) we union bound the LHS of \( (105) \) as

\[
P(g_B(Y^n, S^n) = g_C(Z^n, S^n)|x^n) \geq 1 - 2e^{-\frac{\alpha}{8} x^n} - 4e^{-\frac{\epsilon}{2} n^{\frac{1}{4}}} - P(h(S^n) \notin [\mu - n^{\frac{1}{4}} + \epsilon, \mu + n^{\frac{1}{4}} + \epsilon])
\]

\[
\geq 1 - 2e^{-\frac{\alpha}{8} x^n} - 4e^{-\frac{\epsilon}{2} n^{\frac{1}{4}}} + 2 \frac{n^{\frac{1}{4}} + \epsilon}{2} \geq 1 - \frac{5}{\epsilon} n^{-\frac{1}{4} + \epsilon}. \tag{108}
\]

Here, (a) used the fact that \( h(S^n) \) is distributed uniformly over \([1 : 2^n]\) independent of \( x^n; \) and (b) used the bounds \( 2^k > \frac{n^{\frac{1}{4}}}{\epsilon} \)

and \( 1/\epsilon < \epsilon \geq 2e^{-\frac{\alpha}{8} x^n} + 4e^{-\frac{\epsilon}{2} n^{\frac{1}{4}}} \) for sufficiently large \( n. \) We conclude the proof by proving Claims 15 and 14.

**Proof of Claim 15.** Size of \( \{i : Y_i = e\} \) is distributed according to the binomial distribution with mean \( nq \) and success probability \( q. \) By the Chernoff bound, there exists a constant \( k \) such that

\[
P\left(|\{i : Y_i = e\}| \leq n \left(q + \frac{\delta}{4}\right)\right) = P\left(|\{i : Y_i = e\}| \leq nq \left(1 + \frac{\delta}{4q}\right)\right) \geq 1 - e^{-\frac{nq(k^2 + \delta^2)}{16q^2}} \geq 1 - e^{-\frac{n\delta^2}{4q^2}}. \tag{108}
\]

In the final inequality, we used \((2 + \frac{\delta}{4})^2 \leq 48\) for sufficiently small \( \delta. \) Conditioned on this event, for all \( m' \) such that \( d_H(x^n, f(m')) > n(q + \delta/2), \) and \( \tilde{x}^n \Rightarrow Y^n, \)

\[
d_H(\tilde{x}^n, f(m')) \geq |\{i : Y_i \notin \{f_i(m'), e\}\}| = |\{i : x_i \neq f_i(m')\}| - |\{i : Y_i = e\}| > n(q + \frac{\delta}{2}) - n \left(q + \frac{\delta}{4}\right) > \frac{n\delta}{4}.
\]

Claim now follows from the fact that \( h(S^n) \leq 2^k \leq \frac{n}{\delta}. \)

**Proof of Claim 14.** By the Gilbert-Varshamov bound [51, Theorem 4.2.1], for every \( 0 < \gamma < \frac{1}{2}, \) there exists a linear code with rate \( 1 - H(\gamma) \) and relative distance \( \gamma \) (Hamming distance more than \( n\gamma \) between any two codes). Choose \( \delta > 0 \) small enough that \( 2q + \delta < \frac{3}{4} \) and \( 1 - H(2q + \delta) > R. \) Then, for any \( n, \) there is a \( f : [1 : 2^n] \to \{0, 1\}^n \) such that the hamming distance \( d_{\text{Hamming}}(f(m), f(m')) > n(2q + \delta) \) for all distinct \( m, m' \in [1 : 2^n]. \) The claim follows.

This concludes the proof of the theorem.
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