Abstract. In this paper we consider the equiform motion of a sphere in Euclidean space $E^7$. We study and analyze the corresponding kinematic three-dimensional surface under the hypothesis that its scalar curvature $K$ is constant. Under this assumption, we prove that $|K| < 2$.

1. Introduction

An equiform transformation in the $n$-dimensional Euclidean space $E^n$ is an affine transformation whose linear part is composed of an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $x \in E^n$ according to the rule

$$x \mapsto sa + d, \quad A \in SO(n), \quad s \in R - \{0\}, \quad d \in E^n.$$ 

The number $s$ is called the scaling factor. An equiform motion is defined if the parameters of (1.1), including $s$, are given as functions of a time parameter $t$. Then a smooth one-parameter equiform motion moves a point $x$ via $x(t) = s(t)A(t)x(t) + d(t)$. The kinematic corresponding to this transformation group is called equiform kinematic, see [2, 4].

Under the assumption of the constancy of the scalar curvature, kinematic surfaces obtained by the motion of a circle have been obtained in [1]. In a similar context, one can consider hypersurfaces in space forms generated by one-parameter family of spheres and having constant curvature: [3, 5, 6, 7].

In this paper we consider the equiform motions of a sphere $k_0$ in $E^n$. The point paths of the sphere generate a 3-dimensional surface, containing the positions of the starting sphere $k_0$. The first order properties of these surfaces for the points of these spheres have been studied for arbitrary dimensions $n \geq 3$ [1]. We restrict our
considerations to dimension \( n = 7 \) because, at any moment, the infinitesimal transformations of the motion map the points of the sphere \( \mathbb{k}_0 \) to the velocity vectors, whose end points will form an affine image of \( \mathbb{k}_0 \) (in general a sphere) that span a subspace \( W \) of \( E^n \) with \( n \leq 7 \) [8].

On other hand, in the case of cyclic surface foliated by circle, for fixed \( t \), we have a circle in a fixed frame and its image is an ellipse in moving frame. Then, we need at least space of dimension 5 (2 for the circle, 2 for an ellipse and one dimension for skew) [1]. In the present paper, for fixed \( t \), we have a sphere in a fixed frame and its image is an ellipsoid in moving frame. Then, we need at least space of dimension 7 (3 for the sphere, 3 for an ellipsoid and one dimension for skew), see [8]).

Let \( x(\theta, \phi) \) be a parameterization of \( \mathbb{k}_0 \) and let \( X(t, \theta, \phi) \) be the resultant 3-surface by the equiform motion. We consider a certain position of the moving space given by \( t = 0 \), and we would like to obtain information about the motion at least during a certain period around \( t = 0 \) if we know its characteristics for one instant. Then we restrict our study to the properties of the motion for the limit case \( t \to 0 \). A first choice is then approximate \( X(t, \theta, \phi) \) by the first derivative of the trajectories. Solliman, et al. studied 3-dimensional surfaces in \( E^7 \) generated by equiform motions of a sphere proving that, in general, they are contained in a canal hypersurface [8].

The purpose of this paper is to describe the kinematic surfaces obtained by the motion of a sphere and whose scalar curvature \( K \) is constant. As a consequence of our results, we prove:

A kinematic three-dimensional surface obtained by the equiform motion of a sphere and with constant scalar curvature \( K \) satisfies

\[ |K| < 2. \]

Moreover, we show the description of the motion of such 3-surface by giving the equations that determine the kinematic geometry.

2. The representation of a kinematic surface

In two copies \( \sum^0, \sum \) of Euclidean 7-space \( E^7 \), we consider a unit sphere \( \mathbb{k}_0 \) centered at the origin of the 3-space \( \varepsilon_0 = [x_1 x_2 x_3] \) and represented by

\[
\mathbf{x}(\theta, \phi) = \begin{pmatrix} \cos(\theta) \cos(\phi) & \sin(\theta) \cos(\phi) & \sin(\phi) & 0 & 0 & 0 & 0 \end{pmatrix}^T, \quad \theta \in [0, 2\pi], \quad \phi \in [0, \pi].
\]

Under a one-parameter equiform motion of moving space \( \sum^0 \) with respect to a fixed space \( \sum \) the general representation of the motion of this surface in \( E^7 \) is given by

\[
X(t, \theta, \phi) = s(t)A(t)x(\theta, \phi) + d(t), \quad t \in I \subset \mathbb{R}.
\]

Here \( d(t) = \{b_i(t)\}_i \) : \( i = 1, 2, \ldots, 7 \) describes the position of the origin of \( \sum^0 \) at time \( t \), \( A(t) = \{a_{ij}(t)\}_i,j \) : \( i, j = 1, 2, \ldots, 7 \) is an orthogonal matrix and \( s(t) \) provides the scaling factor of the moving system. With \( s = \text{const.} \neq 0 \) (sufficient to set \( s = 1 \)), we have an ordinary Euclidean rigid body motion. For varying \( t \) and fixed \( x(\theta, \phi) \), equation (2.1) gives a parametric representation of the surface.
of the motion is given by

\[ X(t, \theta, \phi) = \left[ s(0)A(0) + t\left( \dot{s}(0)A(0) + s(0)\dot{A}(0) \right) \right] x(\theta, \phi) + d(0) + td(0), \]

where \( \cdot \) denotes differentiation with respect to the time \( t \). Assuming that the moving frames \( \sum^0 \) and \( \sum \) coincide at the zero position \( (t = 0) \), we have

\[ A(0) = I, \quad s(0) = 1, \quad \text{and} \quad d(0) = 0. \]

Thus we have

\[ X(t, \theta, \phi) = \left[ I + t\left( s' + \Omega \right) \right] x(\theta, \phi) + td', \]

where \( \Omega = \dot{A}(0) = (\omega_i), i = 1, 2, \ldots, 21 \) is a skew symmetric matrix, \( s' = \dot{s}(0) \), \( d' = \ddot{d}(0) \) and all values of \( s, b \) and their derivatives are computed at \( t = 0 \). With respect to these frames, the representation of the motion up to the first order is

\[
\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix} = \begin{pmatrix} 1 + s't & \omega_1 t & \omega_2 t & \omega_3 t & \omega_4 t & \omega_5 t & \omega_6 t \\ -\omega_1 t & 1 + s't & \omega_7 t & \omega_8 t & \omega_9 t & \omega_{10} t & \omega_{11} t \\ -\omega_2 t & -\omega_7 t & 1 + s't & \omega_{12} t & \omega_{13} t & \omega_{14} t & \omega_{15} t \\ -\omega_3 t & -\omega_8 t & -\omega_{12} t & 1 + s't & \omega_{16} t & \omega_{17} t & \omega_{18} t \\ -\omega_4 t & -\omega_9 t & -\omega_{13} t & -\omega_{16} t & 1 + s't & \omega_{19} t & \omega_{20} t \\ -\omega_5 t & \omega_{10} t & -\omega_{14} t & -\omega_{17} t & -\omega_{19} t & 1 + s't & \omega_{21} t \\ -\omega_6 t & -\omega_{11} t & -\omega_{15} t & -\omega_{18} t & -\omega_{20} t & -\omega_{21} t & 1 + s't \end{pmatrix} 
\]

or in the equivalent form

\[
\begin{pmatrix} X_1 \\ X_2 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix} = t \begin{pmatrix} b'_1 \\ b'_2 \\ b'_{14} + \cos(\theta) \cos(\phi) \\ b'_{15} \\ b'_6 \\ b'_{17} \end{pmatrix} + \begin{pmatrix} \omega_1 t \\ 1 + s't \end{pmatrix} 
\]

\[
\begin{pmatrix} \omega_2 t \\ \omega_7 t \\ \omega_{12} t \\ -\omega_{13} t \\ -\omega_{14} t \\ -\omega_{15} t \end{pmatrix} \cdot \sin(\phi) + \begin{pmatrix} \omega_8 t \\ -\omega_{10} t \\ \omega_{11} t \end{pmatrix} \cdot \sin(\theta) \cos(\phi) \]

\[
= t \bar{B} + \cos(\theta) \cos(\phi) \bar{a}_0 + \sin(\theta) \cos(\phi) \bar{a}_1 + \sin(\phi) \bar{a}_2. 
\]
For any fixed $t$ in equation (2.3), we generally get an ellipsoid for $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$ centered at the point $t(b'_1, b'_2, b'_3, b'_4, b'_5, b'_6)$. The latter ellipsoid turns to a 2-dimensional sphere if $\vec{a}_0$, $\vec{a}_1$, and $\vec{a}_2$ form an orthonormal basis. This gives the following conditions:

\[(2.5) \quad \sum_{i=2}^{6} \omega_i \omega_{i+5} = \omega_1 \omega_7 - \sum_{i=3}^{6} \omega_i \omega_{i+2} = \omega_1 \omega_2 + \sum_{i=8}^{11} \omega_i \omega_{i+4} = 0,\]

\[(2.6) \quad \sum_{i=2}^{6} \omega_i^2 = \sum_{i=7}^{11} \omega_i^2, \quad \omega_1^2 + \sum_{i=3}^{6} \omega_i^2 = \omega_2^2 + \sum_{i=12}^{15} \omega_i^2.\]

3. Scalar curvature of the kinematic surface

In this section we shall compute the scalar curvature of 3-surfaces in $E^7$ generated by equiform motions of a sphere which satisfies the conditions (2.5)-(2.6). The tangents to the parametric curves $t = \text{const.}$, $\theta = \text{const.}$, and $\phi = \text{const.}$ at the zero position are

$$X_t = [s'I + \Omega]x + d', \quad X_{\theta} = [I + (s'I + \Omega)t]x_{\theta}, \quad X_{\phi} = [I + (s'I + \Omega)t]x_{\phi}. $$

The coordinate functions of the first fundamental form of $X(t, \theta, \phi)$ are

$$g_{11} = X_t^T X_t, \quad g_{12} = X_t^T X_{\theta}, \quad g_{13} = X_t^T X_{\phi}, \quad g_{22} = X_{\theta}^T X_{\theta}, \quad g_{23} = X_{\theta}^T X_{\phi}, \quad g_{33} = X_{\phi}^T X_{\phi}. $$

Under the conditions (2.5)-(2.6), we obtain

$$g_{11} = \gamma + \alpha_5 \cos(2\phi) + \alpha_3 \sin(\phi) + 2 \cos(\phi) \left[ \cos(\phi) \left( \alpha_4 \cos(2\theta) + \alpha_1 \sin(2\theta) \right) + \sin(\theta) \left( \alpha_7 + \alpha_2 \sin(\phi) \right) + \cos(\theta) \left( \alpha_6 - 2\alpha_3 \sin(\phi) \right) \right],$$

$$g_{12} = \cos(\phi) \left[ 2t \cos(\phi) \left( \alpha_4 \cos(2\theta) - \alpha_4 \sin(2\theta) \right) - \omega_1 \cos(\phi) \right] - \sin(\theta) \left[ t \left( \alpha_6 - 2\alpha_3 \sin(\phi) \right) + b'_1 + \omega_2 \sin(\phi) \right] + \cos(\theta) \left[ t \left( \alpha_7 + 2\alpha_2 \sin(\phi) \right) + b'_2 + \omega_7 \sin(\phi) \right],$$

$$g_{13} = 2t \cos(\phi) \left( \alpha_2 \sin(\theta) - \alpha_3 \cos(\theta) \right) - t \sin(2\phi) \left( \alpha_5 + \alpha_4 \cos(2\theta) \right) + \alpha_1 \sin(2\theta) - \sin(\phi) \left[ \left( b'_1 + t \alpha_6 \right) \cos(\theta) + \left( b'_2 + t \alpha_7 \right) \sin(\theta) \right],$$

$$g_{22} = \cos^2(\phi) \left[ 1 + 2t s' + 2t^2 \left( \delta - \alpha_4 \cos(2\theta) - \alpha_6 \sin(2\theta) \right) \right],$$

$$g_{23} = t^2 \left[ 2 \cos^2(\phi) \left( \alpha_4 \cos(2\theta) + \alpha_3 \sin(\theta) \right) + \sin(2\phi) \left( \alpha_4 \sin(2\theta) - \alpha_1 \cos(2\theta) \right) \right],$$

$$g_{33} = 1 + 2t s' + t^2 \left[ \gamma - \alpha_5 \cos(2\phi) + 2 \sin^2(\phi) \left( \alpha_4 \cos(2\theta) + \alpha_1 \sin(2\theta) \right) + 2 \sin(2\phi) \left( \alpha_3 \cos(\theta) - \alpha_2 \sin(\theta) \right) \right].$$
where

\[
\begin{align*}
\alpha_1 &= \frac{1}{2} \left[ \sum_{i=2}^{6} \omega_i \omega_{i+5} \right], \\
\alpha_2 &= \frac{1}{2} \left[ \omega_1 \omega_2 + \sum_{i=8}^{11} \omega_i \omega_{i+4} \right], \\
\alpha_3 &= \frac{1}{2} \left[ \omega_1 \omega_7 - \sum_{i=3}^{11} \omega_i \omega_{i+9} \right], \\
\alpha_4 &= \frac{1}{4} \sum_{i=2}^{6} \left( \omega_i^2 - \omega_{i+5}^2 \right), \\
\alpha_5 &= \frac{1}{4} \left( \omega_i^2 \right), \\
\alpha_6 &= \left( \omega_1^2 \right), \\
\alpha_7 &= \left( \omega_2^2 \right), \\
\alpha_8 &= \left( \omega_4^2 \right), \\
\beta &= \sum_{i=4}^{11} b_i^2, \\
\gamma &= \beta + s'^2 + \frac{1}{4} \left[ 2(\omega_1^2 + \omega_2^2 + \omega_3^2) + \sum_{i=2}^{15} \omega_i^2 + \sum_{i=12}^{15} \omega_i^2 \right], \\
\delta &= \frac{1}{4} \left[ 2(s'^2 + \omega_1^2) + \sum_{i=2}^{11} \omega_i^2 \right].
\end{align*}
\]

The conditions (2.5)-(2.6) lead to the following relations

\[\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \gamma = \beta + 2\delta.\]

In order to calculate the scalar curvature, we need to compute the Christoffel symbols of the second kind, which are defined as

\[(\text{3.1})\quad \Gamma_{ij}^l = \frac{1}{2g^{lm}} \left[ \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{im}}{\partial x^j} \right],\]

where \(i, j, l\) are indices that take the values 1, 2, 3, \(x_1 = t, x_2 = \theta, x_3 = \phi\), and \(g^{lm}\) is the inverse matrix of \((g_{ij})\). Then the scalar curvature of the surface \(X(t, \theta, \phi)\) is

\[K(t, \theta, \phi) = g^{ij} \left[ \frac{\partial \Gamma_{ij}^l}{\partial x_l} - \Gamma_{il}^m \Gamma_{jm}^l - \Gamma_{jl}^m \Gamma_{im}^l \right].\]

At the zero position \((t = 0)\), the scalar curvature of \(X(t, \theta, \phi)\) is given by

\[(\text{3.2})\quad K = K(0, \theta, \phi) = \frac{P \left( \cos(n_1 \theta \pm m_1 \phi), \sin(n_1 \theta \pm m_1 \phi) \right)}{Q \left( \cos(n_2 \theta \pm m_2 \phi), \sin(n_2 \theta \pm m_2 \phi) \right)}.
\]

This quotient writes then as

\[(\text{3.3})\quad P \left( \cos(n_1 \theta \pm m_1 \phi), \sin(n_1 \theta \pm m_1 \phi) \right) - Q \left( \cos(n_2 \theta \pm m_2 \phi), \sin(n_2 \theta \pm m_2 \phi) \right) = 0.
\]

The assumption on the constancy of the scalar curvature \(K\) implies that equation (3.3) is a linear combination of the functions \(\cos(n \theta \pm m \phi), \sin(n \theta \pm m \phi)\). Because these functions are linearly independent, the corresponding coefficients must vanish. Throughout this work, we have employed the Mathematica program in order to compute the explicit expressions of these coefficients.
Assumption 3.1. Without loss of generality, we assume that the two conditions (2.5)-(2.6) are satisfied and there are no translation motions in the plane which contain the starting sphere, i.e.,
\[ b_1' = b_2' = b_3' = 0. \]

3.1. Kinematic surfaces with zero scalar curvature. We assume that \( K = 0. \) From the expression (3.2), we have
\[
P\left( \cos(n_1 \theta \pm m_1 \phi) \sin(n_1 \theta \pm m_1 \phi) \right) = \sum_{i=0}^{12} \sum_{j=-12}^{12} \left( A_{i,j} \cos(i \theta + j \phi) + B_{i,j} \sin(i \theta + j \phi) \right) = 0.
\]

In this case, a straightforward computation shows that the coefficients of \( \cos(12 \phi), \cos(6 \theta + 12 \phi) \) and \( \sin(6 \theta + 12 \phi) \) are
\[
A_{0,12} = \frac{3}{8192} \left[ 16 \omega_1^6 - 120 \omega_1^4 \left( \omega_1^2 + \omega_2^2 \right) + 9 \omega_1^2 \left( \omega_2^2 + \omega_3^2 \right)^2 - 5 \left( \omega_2^2 + \omega_3^2 \right)^3 \right],
\]
\[
A_{6,12} = \frac{3}{32768} \left[ \omega_2^6 - 15 \omega_2^4 \omega_3^2 + 15 \omega_2^2 \omega_3^4 - \omega_3^6 \right],
\]
\[
B_{6,12} = \frac{3}{16384} \omega_2 \omega_7 \left( 3 \omega_2 - \omega_3^2 \right) \left( \omega_2^2 - 3 \omega_3^2 \right).
\]

By solving the three equations \( A_{0,12} = 0, A_{6,12} = 0, \) and \( B_{6,12} = 0, \) we get
\[ \omega_1 = \omega_2 = \omega_7 = 0. \]

Then
\[
B_{0,9} = \frac{3}{256} \alpha_8 \left[ \alpha_8^2 - 6 \left( \alpha_6^2 + \alpha_3^2 \right) \right],
\]
\[
A_{3,9} = \frac{3}{256} \alpha_6 \left( 3 \alpha_7^2 - \alpha_3^2 \right),
\]
\[
B_{3,9} = \frac{3}{256} \alpha_7 \left( \alpha_7^2 - 3 \alpha_3^2 \right).
\]

The three equations \( B_{0,9} = 0, A_{3,9} = 0, \) and \( B_{3,9} = 0 \) imply
\[ \alpha_6 = \alpha_7 = \alpha_8 = 0. \]

From these values, equation \( A_{0,6} = 0 \) leads to
\[
\left( \beta + 2 \delta \right) \left( \beta + s^2 - 2 \delta \right) = 0.
\]

It is worth to point out that the quantities \( \beta \) and \( \delta \) are positive and thus we obtain the following condition
\[ \delta = \frac{1}{2} \left( \beta + s^2 \right). \]

At this time, the explicit computations of coefficients imply that all \( A_{i,j} \) and \( B_{i,j} \) are equal zero. So, we have the following:
Theorem 3.1. A kinematic 3-surface in $\mathbb{E}^7$ foliated by spheres and with zero constant scalar curvature satisfies

$$\omega_1 = \omega_2 = \omega_7 = 0,$$

$$\sum_{i=4}^{7} b_i \omega_{i-1} = \sum_{i=4}^{7} b_i \omega_{i+4} = \sum_{i=4}^{7} b_i \omega_{i+8} = 0,$$

$$\omega_2^2 = \sum_{i=4}^{7} b_i^2.$$

3.2. Kinematic surfaces with non-zero constant scalar curvature. We assume that the kinematic 3-surface has constant scalar curvature $K \neq 0$. From (3.1), we have

$$P\left(\cos(n_1 \theta \pm m_1 \phi), \sin(n_1 \theta \pm m_1 \phi)\right) - KQ\left(\cos(n_2 \theta \pm m_2 \phi), \sin(n_2 \theta \pm m_2 \phi)\right) = \sum_{j=-12}^{12} \sum_{i=6}^{12} \left(A_{i,j} \cos(i \theta + j \phi) + B_{i,j} \sin(i \theta + j \phi)\right) = 0.$$

In this case, a straightforward computation shows that the coefficients of $\cos(12\phi)$, $\cos(12\theta + 6\phi)$ and $\sin(12\theta + 6\phi)$ are

$$A_{0,12} = \frac{1}{16384} (K + 6) \left[16 \omega_1^6 - 120 \omega_1^4 \left(\omega_2^2 + \omega_7^2\right) + 90 \omega_1^2 \left(\omega_2^2 + \omega_7^2\right)^2 - 5 \left(\omega_2^2 + \omega_7^2\right)^3\right],$$

$$A_{6,12} = -\frac{1}{65536} (K + 6) \left[\omega_2^6 - 15 \omega_2^4 \omega_7^2 + 15 \omega_2^2 \omega_7^4 - \omega_7^6\right],$$

$$B_{6,12} = \frac{1}{32768} \omega_2 \omega_7 (K + 6) \left(3 \omega_2^2 - \omega_7^2\right) \left(\omega_2^2 - 3 \omega_7^2\right).$$

We consider the three equations $A_{0,12} = 0$, $A_{6,12} = 0$, and $B_{6,12} = 0$. From here, we discuss two possibilities: $K = -6$ and $\omega_1 = \omega_2 = \omega_7 = 0$.

1. Case $K = -6$. A computation of coefficients yields

$$A_{5,11} = \frac{1}{2048} \left[\alpha_6 \left(\omega_2^4 - 6 \omega_2^2 \omega_7^2 + \omega_7^4\right) + 4 \alpha_7 \omega_2 \omega_7 \left(\omega_2^2 - \omega_7^2\right)\right] = 0,$$

$$B_{5,11} = \frac{1}{2048} \left[\alpha_7 \left(\omega_2^4 - 6 \omega_2^2 \omega_7^2 + \omega_7^4\right) + 4 \alpha_6 \omega_2 \omega_7 \left(\omega_2^2 - \omega_7^2\right)\right] = 0.$$

We consider two cases: $\alpha_6 = \alpha_7 = 0$ and $\omega_2 = \omega_7 = 0$.

Case (1): We assume $\alpha_6 = \alpha_7 = 0$. The computation of coefficients leads to

$$B_{0,11} = \frac{1}{2048} \alpha_8 \left[8 \omega_1^4 - 24 \omega_1^2 \left(\omega_2^2 + \omega_7^2\right) + 3 \left(\omega_2^2 + \omega_7^2\right)^2\right] = 0,$$

$$A_{4,11} = \frac{1}{512} \alpha_8 \omega_2 \omega_7 \left(\omega_2^2 - \omega_7^2\right) = 0,$$

$$B_{4,11} = \frac{1}{2048} \alpha_8 \left(\omega_2^4 - 6 \omega_2^2 \omega_7^2 + \omega_7^4\right) = 0,$$

which implies two subcases: $\alpha_8 = 0$ and $\omega_1 = \omega_2 = \omega_7 = 0$. 
Subcase (1.1): If $\alpha_8 = 0$, then we have

$$A_{0,10} = \frac{1}{1024} \left[ 8 \omega_1^2 - \omega_1^2 \left( \omega_2^2 + \omega_3^2 \right) + 3 \left( \omega_2^2 + \omega_3^2 \right) \right] \left( \beta + s^2 + 6 \delta - \omega_1^2 - \omega_2^2 - \omega_3^2 \right)$$

$$A_{4,10} = \frac{1}{2048} \omega_2 \omega_7 \left( \omega_4^2 - 6 \omega_2^2 \omega_3^2 + \omega_4^2 \right) \left( \beta + s^2 + 6 \delta - \omega_1^2 - \omega_2^2 - \omega_3^2 \right)$$

$$B_{4,10} = \frac{1}{512} \omega_2 \omega_7 \left( \omega_4^2 - \omega_3^2 \right) \left( \beta + s^2 + 6 \delta - \omega_1^2 - \omega_2^2 - \omega_3^2 \right).$$

The last term in the above three equations is not zero because

$$\beta + s^2 + 6 \delta - \omega_1^2 - \omega_2^2 - \omega_3^2 = 7 \sum_{i=4}^7 b_i^2 + \omega_2^2 + \sum_{i=8}^{11} \omega_i^2 + 2 \left[ 2 s^2 + \omega_4^2 + \sum_{i=6}^6 \omega_i^2 \right] > 0.$$

The three equations $A_{0,10} = 0$, $A_{4,10} = 0$, and $B_{4,10} = 0$ lead $\omega_1 = \omega_2 = \omega_7 = 0$. Now, the coefficient $A_{0,6}$ must equal zero, that is,

$$\left( \beta + 2 \delta \right)^2 \left( 2 \beta + 8 \delta - s^2 \right) = 0,$$

contradiction.

Subcase (1.2): If $\omega_1 = \omega_2 = \omega_7 = 0$ and $\alpha_8 \neq 0$, the equation $B_{0,9} = 0$ implies that $\alpha_8 = 0$: contradiction.

Case (2): If $\omega_2 = \omega_7 = 0$ and $\alpha_6, \alpha_7 \neq 0$, the computation of coefficients yields

$$A_{5,11} = \frac{1}{2048} \alpha_6 \omega_4^4 = 0,$$

$$B_{5,11} = \frac{1}{2048} \alpha_7 \omega_4^4 = 0.$$

Because $\alpha_6 \neq 0$ and $\alpha_7 \neq 0$, we conclude $\omega_1 = 0$. New computations give

$$A_{3,9} = \frac{9}{256} \alpha_6 \left( \alpha_6^2 - 3 \alpha_7^2 \right),$$

$$B_{3,9} = \frac{9}{256} \alpha_7 \left( 3 \alpha_6^2 - \alpha_7^2 \right).$$

By solving the equations $A_{3,9} = 0$ and $B_{3,9} = 0$, we get $\alpha_6 = \alpha_7 = 0$: contradiction.

Corollary 3.1. There are no kinematic 3-surfaces in $E^7$ foliated by spheres and with scalar curvature $K$ equal $-6$.

2) Case $\omega_1 = \omega_2 = \omega_7 = 0$ and $K \neq -6$.

A computation of the coefficients yields

$$B_{0,9} = \frac{1}{256} \alpha_8 \left[ \alpha_8^2 - 6 \left( \alpha_6^2 + \alpha_7^2 \right) \right] \left( 2K + 3 \right) = 0,$$

$$A_{3,9} = \frac{1}{256} \alpha_6 \left( 3 \alpha_7^2 - \alpha_6^2 \right) \left( 2K + 3 \right) = 0,$$

$$B_{3,9} = \frac{1}{256} \alpha_7 \left( \alpha_7^2 - 3 \alpha_6^2 \right) \left( 2K + 3 \right) = 0.$$
which gives two cases: $K = -\frac{3}{2}$ or $\alpha_7 = \alpha_6 = \alpha_8 = 0$.

**Case (1):** Assume $K = -\frac{3}{2}$. Now, we obtain

\[
A_{0,8} = \frac{1}{64} \left[ \alpha_8^2 - \left( \alpha_6^2 - \alpha_7^2 \right) \right] \left( 6\delta - \beta - 2s'^2 \right),
\]
\[
A_{2,8} = \frac{1}{64} \left( \alpha_6^2 - \alpha_7^2 \right) \left( 6\delta - \beta - 2s'^2 \right),
\]
\[
B_{2,8} = \frac{1}{32} \alpha_7 \alpha_6 \left( 6\delta - \beta - 2s'^2 \right).
\]

Solving the three equations $A_{0,8} = 0$, $A_{2,8} = 0$, and $B_{2,8} = 0$, we find two cases: $\delta = \frac{\beta + 2s'^2}{6}$ and $\alpha_6 = \alpha_7 = \alpha_8 = 0$.

**Case (1.1):** If $\delta = \frac{\beta + 2s'^2}{6}$, we obtain

\[
A_{0,4} = \frac{1}{72} \left( 2\beta + s'^2 \right) \left[ 4(2\beta + s'^2)^2 - 9 \left( \alpha_8^2 + 4(\alpha_6^2 + \alpha_7^2) \right) \right],
\]

which leads the following condition

\[
4(2\beta + s'^2)^2 = 9 \left( \alpha_8^2 + 4(\alpha_6^2 + \alpha_7^2) \right).
\]

At this point, all coefficients $A_{i,j}$ and $B_{i,j}$ are equal to zero.

**Case (1.2):** Assume $\alpha_6 = \alpha_7 = \alpha_8 = 0$ and $\delta \neq \frac{\beta + 2s'^2}{6}$. The coefficient $A_{0,6}$ is $A_{0,6} = \frac{1}{32} \left( \beta + 2\delta \right) \left[ 14\delta - \beta - 4s'^2 \right]$. From $A_{0,6} = 0$, we conclude

\[
\delta = \frac{\beta + 4s'^2}{14}.
\]

As a consequence, all coefficients $A_{i,j}$ and $B_{i,j}$, $i = 1, 2, ..., 12$, $j = -12, ..., 12$ are zero.

From the above reasonings, it follows the next:

**Theorem 3.2.** A kinematic 3-surface in $E^7$ foliated by spheres and with $K = -\frac{3}{2}$ satisfies $\omega_1 = \omega_2 = \omega_7 = 0$ and one of the following pairs of equations:

\[
s'^2 + 3 \sum_{i=3}^{6} \omega_i^2 = \sum_{i=4}^{7} b_i'^2,
\]

\[
4 \left[ s'^2 + 2 \sum_{i=4}^{7} b_i'^2 \right] = 9 \left[ \left( \sum_{i=4}^{7} b_i' \omega_{i+8} \right)^2 + 4 \left( \sum_{i=4}^{7} b_i' \omega_{i-1} \right)^2 + 4 \left( \sum_{i=4}^{7} b_i' \omega_{i+4} \right)^2 \right],
\]

or

\[
\sum_{i=4}^{7} b_i' \omega_{i-1} = \sum_{i=4}^{7} b_i' \omega_{i+4} = \sum_{i=4}^{7} b_i' \omega_{i+8} = 0,
\]

\[
3 s'^2 + 7 \sum_{i=3}^{6} \omega_i^2 = \sum_{i=4}^{7} b_i'^2.
\]
Case (2): Assume \( \alpha_6 = \alpha_7 = \alpha_8 = 0 \) and \( K \neq -\frac{3}{2} \). In this case we obtain
\[
A_{0,6} = \frac{1}{16} \left( \beta + 2\delta \right)^2 \left[ K \left( \beta + 2\delta \right) - 2 \left( 2\delta - \beta - s'^2 \right) \right] = 0,
\]
which yields
\[
K = \frac{2(2\delta - \beta - s'^2)}{\beta + 2\delta}.
\]
From here, all coefficients \( A_{i,j} \) and \( B_{i,j} \) are equal zero. So, we have the following:

**Theorem 3.3.** A kinematic 3-surface in \( E^7 \) foliated by spheres and with constant scalar curvature \( K \) satisfies
\[
K = 2 \left[ \sum_{i=3}^{6} \omega_i^2 - \sum_{i=4}^{7} b_i'^2 \right] \frac{\beta + 2\delta}{s'^2 + \sum_{i=3}^{6} \omega_i^2 + \sum_{i=4}^{7} b_i'^2}
\]
satisfies
\[
\omega_1 = \omega_2 = \omega_7 = 0,
\]
\[
\sum_{i=4}^{7} b_i' \omega_{i-1} = \sum_{i=4}^{7} b_i' \omega_{i+4} = \sum_{i=4}^{7} b_i' \omega_{i+8} = 0.
\]
From the expression (3.4), we can write the quantity \( \beta \) in the form
\[
\beta = \frac{2 \left( 2 - K \right) \delta - s'^2}{K + 2}.
\]
As \( \beta \) is positive, we have two cases:

(a) Case \( K + 2 < 0 \) and \( \left( 2 - K \right) \delta - s'^2 < 0 \). This implies
\[
K < -2, \text{ and } K > \frac{2\delta - s'^2}{\delta} = \frac{2 \sum_{i=3}^{6} \omega_i^2}{s'^2 + \sum_{i=3}^{6} \omega_i^2} > 0,
\]
which is a contradiction.

(b) Case \( K + 2 > 0 \) and \( \left( 2 - K \right) \delta - s'^2 > 0 \). This gives the following condition for \( K \):
\[
-2 < K < \frac{2\delta - s'^2}{\delta} = \frac{2 \sum_{i=3}^{6} \omega_i^2}{s'^2 + \sum_{i=3}^{6} \omega_i^2} < 2.
\]
As consequence of Theorems 3.2 and 3.3, we have the next statement, which was established in the Introduction.

**Corollary 3.2.** A kinematic three-dimensional surface in \( E^7 \) obtained by the equiform motion of a sphere and with constant scalar curvature \( K \) satisfies \( |K| < 2 \).
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