On a Functional Differential Equation of Determinantal Type

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Abstract

We solve the functional equations

\[
\begin{vmatrix}
1 & 1 & 1 \\
f(x) & f(y) & f(z) \\
f'(x) & f'(y) & f'(z)
\end{vmatrix} = 0,
\begin{vmatrix}
1 & 1 & 1 \\
f(x) & g(y) & h(z) \\
f'(x) & g'(y) & h'(z)
\end{vmatrix} = 0,
\]

for suitable functions \(f\), \(g\) and \(h\) subject to \(x + y + z = 0\). These equations essentially characterise the Weierstrass \(\wp\)-function and its degenerations.

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1 Introduction

The purpose of the following note is to present a simple and direct proof of

**Theorem 1** Let $f$ be a three-times differentiable function satisfying the functional equation

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ f'(x) & f'(y) & f'(z) \end{vmatrix} = 0, \quad x + y + z = 0. \quad (1)$$

Up to the manifest invariance

$$f(x) \to \alpha f(\delta x) + \beta,$$

the solutions of (1) are one of $f(x) = \wp(x + d)$, $f(x) = e^x$ or $f(x) = x$. Here $\wp$ is the Weierstrass $\wp$-function and $3d$ is a lattice point of the $\wp$-function.

In fact our approach gives a simple proof of

**Theorem 2** Let $f$, $g$ and $h$ be three-times differentiable functions satisfying the functional equation

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(y) & h(z) \\ f'(x) & g'(y) & h'(z) \end{vmatrix} = 0, \quad x + y + z = 0. \quad (2)$$

Up to the manifest invariance

$$f(x) \to \alpha f(\delta x + \gamma_1) + \beta, \quad g(x) \to \alpha g(\delta x + \gamma_2) + \beta, \quad h(x) \to \alpha h(\delta x + \gamma_3) + \beta,$$

where $\gamma_1 + \gamma_2 + \gamma_3 = 0$, the nonconstant solutions of (2) are given by $f(x) = g(x) = h(x) = e^x$, $x$, or $\wp(x)$. If (say) $h(z)$ is a constant then either

1. One of the functions $f(x)$ or $g(y)$ is the same constant as $h(z)$, in which case the remaining function is arbitrary, or

2. $f(x) = g(x) = e^x$.

**Remarks:** (i) In fact the exponential and linear function solutions satisfy (1) and (2) without the constraint $x + y + z = 0$.

(ii) The theorems immediately give the general analytic solutions to the same functional equations viewed as functions of a complex variable, showing that the solutions are in fact meromorphic.
The arbitrary constant $\delta$ in the invariance of (1) is accommodated in the Weierstrass $\wp$-function solution by the homogeneity relation $\wp(tx; t^{-4}g_2, t^{-6}g_3) = t^{-2}\wp(x; g_2, g_3)$.

Weierstrass has shown [5] that any meromorphic function possessing an algebraic addition formula is either an elliptic function or is of the form $R(z)$ or $R(e^{\lambda z})$, where $R$ is a rational function. A priori the functional equation (1) is distinct from assuming $f$ possesses an algebraic addition formula.

Interest in (1) arises from a question of mathematical physics: What one-dimensional quantum mechanical models with pair-wise interactions have ground states of product type? In addressing this question [3, 4] the functional equation

$$F(x)F(y) + F(y)F(z) + F(z)F(x) = G(x) + G(y) + G(z), \quad (3)$$

appears (with $x + y + z = 0$), where on physical grounds $F(x)$ is taken to be odd. By applying $(\partial_x - \partial_y)\partial_x \partial_y$ to this equation we obtain (1) with $f(x) = F'(x)$ and similarly (1) may be integrated to yield (3). The models that arise in the solution of this question include the Calogero-Moser-Sutherland models. They are rich in interesting mathematics involving representation theory, harmonic analysis and special functions (see for example [4]); the classical analogues of the models yield completely integrable Hamiltonian systems. Now the solutions that yield these models were obtained assuming the function $F(x)$, or equivalently $f(x)$, to be meromorphic with at least one pole. With such an assumption it is easy to show that $f(x) = \wp(x + d)$ is the general solution, and further requiring $f(x)$ to be even dictates $d = 0$. Although one can in fact show that there are no (nonconstant) even entire solutions to (1), so answering the physical question, the general solution to (1) appeared difficult to obtain. Indeed, the functional equation (2) was introduced in [2] as a means to understand (1) but the proof of the main theorem of [2] fails to result in a direct proof of Theorem 1. Here we present a simple and direct proof of (1) that also allows a new and simpler proof of (2).

2 Proofs

The strategy of our proof is first to isolate a necessary condition for nonconstant solutions of (1) and (2) in the form of a differential equation. The solutions of this differential equation are given in terms of the Weierstrass $\wp$-function or one of its degenerations, and the second step is to determine the various free parameters that arise in the solution. This will prove the theorems for nonconstant functions $f$, $g$ and $h$. Finally we consider the case where some of these functions are constant.

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1We are grateful to V.M. Buchstaber for informing us on this matter.
While the approach to solving a functional equation via an associated differential equation(s) is standard \[1\], the simplicity of our proof depends on one rather nonobvious step that we wish to highlight in advance. First, by taking (2) and various of its derivatives we obtain an equation of the form $F(f(x), g(y)) = 0$ involving $f(x)$, $g(y)$ and their derivatives. Viewing $x$ (say $x = x_0$) as fixed we have in general a nonlinear ordinary differential equation for $g(y)$ which may in principle be solved. While such a solution satisfies $F(f(x_0), g(y)) = 0$, it needn’t in general satisfy $F(f(x), g(y)) = 0$, the various derivatives $\partial^k_x F(f(x), g(y))|_{x_0} = 0$ (supposing they exist) yielding further differential equations for $g(y)$. These further equations give us restrictions on the allowed functions $g(y)$ and between them one may eliminate various derivatives of $g$ appearing. For example, given $F(f(x_0), g(y)) = 0$ and $\partial_x F(f(x), g(y))|_{x_0} = 0$ one could choose to eliminate the highest derivative of $g$ appearing; similarly one can use further partial derivatives to eliminate additional derivatives of $g$. If we suppose that $F(f(x), g(y)) = 0$ determines $g(y)$ then the equation $F(f(x_0), g(y)) = 0$, together with an appropriate number of further derivatives, also determines $g(y)$. The nonobvious step in our proof is to provide a functional $F(f(x), g(y)) = 0$ that alone readily gives $g(y)$. We will remark in the course of the proof when this is done, and note the several perhaps surprising simplifications that follow.

Lemma 1 Let $f$, $g$ and $h$ be three-times differentiable, nonconstant functions that satisfy (4). Then each satisfies the (same) differential equation

$$w'(x)^2 = p_3 w(x)^3 + p_2 w(x)^2 + p_1 w(x) + p_0.$$  \hspace{1cm} (4)

Proof: We begin by deriving several algebraic consequences of assuming that the (nonconstant) functions $f$, $g$ and $h$ of (2) are $N$-times differentiable. Let $1 \leq k, l, s \leq N$. The algebraic identities we obtain yield a large supply of functional equations involving only the functions $f(x)$, $g(y)$ and their derivatives. We will obtain (4) by eliminating an appropriate derivative. A minimum choice of $N = 3$ will arise in the proof.

Set $\partial = \partial_y - \partial_x$ and let

$$a_k = \partial^{k-1}(g(y) - f(x)), \quad b_k = \partial^k(g(y) + f(x)), \quad c_k = \partial^k(g(y) f(x)).$$ \hspace{1cm} (5)

Then differentiation of (2) yields $N$ equations,

$$a_k h'(z) - b_k h(z) + c_k = 0.$$

Comparing any two of these equations shows

$$(a_k b_l - a_l b_k) h(z) = a_k c_l - a_l c_k, \quad (a_k b_l - a_l b_k) h'(z) = b_k c_l - b_l c_k.$$ \hspace{1cm} (6)
while comparison of any three yields
\[
\begin{vmatrix}
  a_k & a_l & a_s \\
  b_k & b_l & b_s \\
  c_k & c_l & c_s \\
\end{vmatrix} = 0. \tag{7}
\]

Consider \( z = -x - y \) as a function of \( x \) and \( y \). Differentiating the first of equations (6) with respect to \( y \) say, and comparing with the second equation results in
\[
\begin{vmatrix}
  a_k & a_l & a_k \partial_y a_l - a_l \partial_y a_k \\
  b_k & b_l & b_k \partial_y b_l - a_l \partial_y b_k \\
  c_k & c_l & a_k \partial_y c_l - a_l \partial_y c_k + b_k c_l - b_l c_k \\
\end{vmatrix} = 0. \tag{8}
\]

Similar expressions result upon differentiation with respect \( x \).

Observe that at this stage (7) and (8) and their linear combinations provide us with many functional equations involving only the functions \( f(x) \), \( g(y) \) and their derivatives. In particular we may eliminate various combinations of derivatives between them. For example, by taking \( k = 1, l = 2 \) and \( s = 3 \) the linear combination \( (8) - a_1 \times (7) \),
\[
\begin{vmatrix}
  a_1 & a_2 & a_1 g''(y) - a_2 g'(y) - a_1 a_3 \\
  b_1 & b_2 & a_1 g'''(y) - a_2 g''(y) - a_1 b_3 \\
  c_1 & c_2 & a_1 \partial_y c_2 - a_2 \partial_y c_1 - a_1 c_3 + (b_1 c_2 - b_2 c_3) \\
\end{vmatrix} = 0, \tag{9}
\]
cancels each of the \( g'''(y) \) derivatives appearing in the third column. This expression, quadratic in \( g''(y) \), factorises to give
\[
0 = \left( f'(x) g'(y) - g'(y)^2 - f(x) g''(y) + g(y) g'''(y) \right) \times
\left( 3f'(x)^3 - 3f'(x)g'(y)^2 - 4f(x)f''(x) + 4g(y)f'(x) + 4f(x)f''(x)g''(x) + g(y)^2 f^{(3)}(x) \right)
- 2f(x)f''(x)g''(y) + 2g(y)f'(x)g''(y) + f(x)^2 f^{(3)}(x) - 2f(x)g(y)f^{(3)}(x). \tag{10}
\]

This elimination of the \( g'''(y) \) derivatives gives us the equation \( F(f(x), g(y)) = 0 \) we choose to work with. The factorisation we encounter is one of the simplifications we drew attention to earlier. Of course we could have taken the \( k = 1, l = 2 \) and \( s = 3 \) form of (8) as our equation \( F(f(x), g(y)) = 0 \). It appears that this, together with one further partial derivative, \( \partial_x F(f(x), g(y)) = 0 \) is sufficient to determine \( g(y) \), but at the expense of far greater work. In particular there is no similar factorisation to that we encountered above. The proof presented was devised to circumvent the difficulties of this latter route.
Now the first term appearing on the right of (10) may be written as
\[(f(x) - g(y))^2 \frac{d}{dy} \left( \frac{f'(x) - g'(y)}{f(x) - g(y)} \right),\]
while the second term may be expressed as
\[
\frac{(f(x) - g(y))^4}{g'(y)} \frac{d}{dy} \left( \frac{f''(x) (f'(x)^2 - g'(y)^2)}{(f(x) - g(y))^3} \right) - \frac{2 f'(x) f''(x)}{(f(x) - g(y))^2} + \frac{f^{(3)}(x)}{f(x) - g(y)}.
\]

Our assumption of nonconstant solutions means that \(f(x) \neq g(y)\) and \(g'(y) \neq 0\). Thus the vanishing of (10) means either
\[
\frac{f'(x) - g'(y)}{f(x) - g(y)} = C_1(x)
\]
or
\[
\frac{f'(x) (f'(x)^2 - g'(y)^2)}{(f(x) - g(y))^3} - \frac{2 f'(x) f''(x)}{(f(x) - g(y))^2} + \frac{f^{(3)}(x)}{f(x) - g(y)} = C_2(x),
\]
according to whether the first or second terms in (10) vanish. Therefore, assuming only third derivatives exist, we have shown (after rearranging) that either
\[
g'(y) = l_1 g(y) + l_0, \quad \text{or} \quad g'(y)^2 = p_3 g(y)^3 + p_2 g(y)^2 + p_1 g(y) + p_0. \tag{11}
\]
Both are cases of (4). A consequence of (10) is that derivatives to all orders exist for solutions of either of these differential equations. The (analytic) solutions of these differential equations may be expressed (generically) in terms of the exponential and Weierstrass \(\wp\)-function, and indeed the exponential solution of the linear differential equation corresponds to a degeneration of the \(\wp\)-function differential equation. Further, the identical argument but upon interchanging the role of \(y\) and \(x\) shows that \(f(x)\) is also a \(\wp\)-function or a degeneration. We have however yet to establish that \(f\) and \(g\) satisfy the same differential equation. Before turning to this there is one point that needs to be clarified.

In principle it is possible for the function \(g(y)\) giving the vanishing of (10) to be a solution of one of the differential equations (11) in one domain and satisfy the other differential equation outside of it, with \(g(y)\) and its first three derivatives matching at any boundary. Such a possibility does not arise in our problem. One can readily show that matching a solution \(g_1(y)\) of the first differential equation (11) with a solution \(g_2(y)\) of the second differential equation at a point \(y_0\), and requiring the first four derivatives to agree at this point, entails \(g_1(y) \equiv \)}
$g_2(y)$. Thus the solutions to the vanishing of (10) being envisaged perforce have discontinuous fourth derivative at such points $y_0$. Now for our problem, the coefficients $l_i$ and $p_i$ of the differential equations (11) are in fact functions of $x$, given explicitly below. We will shortly see that it is this aspect of our problem that rules out the functions $g(y)$ envisaged in this paragraph.

We now establish that the functions $f$ and $g$ satisfy the same differential equation by determining the coefficients $l_i$ and $p_i$ appearing above in two different ways. By directly differentiating (11) one obtains (for example) that

$$l_1 = \frac{g''(y)}{g'(y)}, \quad p_3 = \frac{g'(y)g^{(4)}(y) - g''(y)g^{(3)}(y)}{3g'(y)^2}.$$  

(The remaining coefficients will be listed below.) Alternately the coefficients may be determined by evaluating the functions $C_{1,2}(x)$ arising upon integration. These functions may be determined in a variety of ways. If (10) vanishes, so does the derivative with respect to $x$ of the right-hand side of the equation. Substituting either of the expressions for $g''(y)$ corresponding to the two terms of (10) into this derivative equation yield equations for $g'(y)$. Comparison with the equations defining $C_{1,2}(x)$ show (with a little work)

$$C_1(x) = \frac{f''(x)}{f'(x)}, \quad C_2(x) = \frac{f'(x)f^{(4)}(x) - f''(x)f^{(3)}(x)}{3f'(x)^2}.$$  

(The same expressions arise by applying L’Hospital’s rule to the equations defining $C_{1,2}(x)$.) The coefficients $l_i$ and $p_i$ are then determined to be

$$l_0 = \frac{f'(x)^2 - f(x)f''(x)}{f'(x)}, \quad l_1 = \frac{f''(x)}{f'(x)}.$$  

$$p_0 = f'(x)^2 - 2f(x)f''(x) + \frac{f(x)^2f^{(3)}(x)}{f'(x)} - \frac{f(x)^3}{3f'(x)^3}(f'(x)f^{(4)}(x) - f''(x)f^{(3)}(x)),$$

$$p_1 = 2f''(x) - \frac{2f(x)f^{(3)}(x)}{f'(x)} + \frac{f(x)^2}{f'(x)^3}(f'(x)f^{(4)}(x) - f''(x)f^{(3)}(x)),$$

$$p_2 = \frac{f^{(3)}(x)}{f'(x)} - \frac{f(x)}{f'(x)^3}(f'(x)f^{(4)}(x) - f''(x)f^{(3)}(x)),$$

$$p_3 = \frac{f'(x)f^{(4)}(x) - f''(x)f^{(3)}(x)}{3f'(x)^3}.$$  

(12)
Thus, for example, we have shown that

\[ l_1 = \frac{g''(y)}{g'(y)} = \frac{f''(x)}{f'(x)}, \]

and so \( l_1 \) is in fact a constant. Similarly \( p_3 \) is seen above to be a constant. Further, these constants are determined by expressions symmetric in the interchange of \( f \) and \( g \) and the same is true of the remaining coefficients. This symmetry means that \( f \) and \( g \) will satisfy the same differential equation. Also, if we had chosen to work with \( f \) and \( h \) the above argument shows that they satisfy the same differential equation. Therefore, each of \( f, g \) and \( h \) satisfy the same differential equation. Finally observe that the coefficient \( p_3 \), which is constant, involves the fourth derivative of the functions satisfying (II) for arbitrary position. This expression shows one cannot construct a constant \( p_3 \) from two (nonconstant) solutions of (II) whose first three derivatives agree and whose fourth derivatives differ. This rules out having solutions to (II) that satisfy the first equation on one region and the second equation elsewhere, and having discontinuous fourth derivative on the boundaries.

\[ \square \]

Corollary 1 A nonconstant solution \( f(x) \) of (II) satisfies the differential equation (1).

Remarks: (i) If one was happy to assume the five-times differentiability of the function \( f(x) \) satisfying (II) then this result may be obtained very quickly. Let \( f = g \) in the lemma. We need proceed no further than (8). Simplification arises because we have only the one function and its derivatives appearing throughout and we may further let \( y = x \) yielding (nonlinear) differential equations satisfied by \( f(x) \). Thus, for example, taking \( k = 2, l = 4 \) and upon setting \( y = x \) in (8) we obtain

\[ 0 = (f')^6 \left( \frac{f''}{f'} \right)' \left( \frac{1}{f'} \left( \frac{f''}{f'} \right)' \right)' \tag{13} \]

(This in fact corresponds to first nonzero term appearing in a Taylor series expansion around \( y = x + \epsilon \) in the \( k = 1, l = 2 \) expressions of the lemma.) The two final two factors here correspond to the two terms appearing in (II); they may be straightforwardly integrated to yield the corollary.

(ii) Another (again not straightforward) route to solving (I) is first to express this functional equation in a rather different manner. Parameterise \( x + y + z = 0 \) by \( x = \xi + \eta, y = \xi - \eta \) and \( z = -2\xi \). Then with \( \Delta \) denoting the standard central difference operator (defined by \( \Delta f(\xi) = f(\xi + \eta) - f(\xi - \eta) \)) and \( \mu \) denoting the
standard average operator (with \( \mu f(\xi) = (f(\xi + \eta) + f(\xi - \eta))/2 \)) then (1) may be written

\[
\Delta f(\xi) \mu f'(\xi) - \Delta f'(\xi) \mu f(\xi) = \Delta f(\xi) F'(z) - \Delta f'(\xi) F(z).
\]

Here we have set \( F(z) = f(-2\xi) \) and \( F'(z) = -\frac{d}{d\xi} F(z)/2 \) and we may view this as an equation in \( \eta \) with \( \xi \) fixed. Expanding this equation as a series in \( \eta \) leads to an infinite set of relations which are first order in \( F(z) \),

\[
a_k F'(z) - b_k F(z) + c_k = 0.
\]

These equations are analogous to those involving \( a_i, b_i \) and \( c_i \) in the lemma. Here the coefficients are functions of \( f(\xi) \) and its derivatives. The lowest order coefficients are

\[
a_1 = f', \quad b_1 = f'', \quad c_1 = ff'' - f'^2
a_2 = f''', \quad b_2 = f''', \quad c_2 = -4f'f''' + ff''' + 3f'''.
\]

Elimination of \( F(z) \) amongst these is also sufficient to obtain (13).

At this stage we have found a necessary condition for nonconstant functions satisfying (2) and (1): the functions must satisfy (4) and so (7) are of the form

\[
\alpha \wp(\delta x + \gamma) + \beta, \quad \alpha e^{\delta x} + \beta, \quad \alpha x + \beta.
\]

As remarked upon in the introduction, the exponential and linear solutions clearly satisfy the functional equations without the restriction \( x + y + z = 0 \) and we need only discuss the first solution here involving the \( \wp \)-function. The noted invariances of the functional equations mean we need only determine whether any restrictions must be placed on the translation parameter \( \gamma \) unspecified by the differential equation.

**Lemma 2** The function \( f(x) = \wp(x + \gamma) \) satisfies (4) provided \( 3\gamma \) is a lattice point of the \( \wp \)-function.

Similarly, the functions \( f(x) = \wp(x + \gamma_1), g(y) = \wp(y + \gamma_2), h(z) = \wp(z + \gamma_3) \) satisfy (3) provided \( \gamma_1 + \gamma_2 + \gamma_3 \) is a lattice point of the \( \wp \)-function and these may be chosen so that \( \gamma_1 + \gamma_2 + \gamma_3 = 0 \).

**Proof:** The result follows from the following identity (7)

\[
\begin{vmatrix}
1 & 1 & 1 \\
\wp(a) & \wp(b) & \wp(c) \\
\wp'(a) & \wp'(b) & \wp'(c)
\end{vmatrix} = 2\frac{\sigma(a + b + c) \sigma(a - b) \sigma(b - c) \sigma(c - a)}{\sigma(a)^3 \sigma(b)^3 \sigma(c)^3},
\]

where \( \sigma \) is the Weierstrass sigma function that vanishes at the lattice points of the \( \wp \)-function. Letting \( a = x + \gamma, b = y + \gamma, c = z + \gamma \) we see that
\(f(x) = \varphi(x + \gamma)\) satisfies (1) provided \(3\gamma\) is a lattice point as stated. Similarly we see the functions \(f, g\) and \(h\) given in the lemma satisfy (2) provided \(\gamma_1 + \gamma_2 + \gamma_3\) is again a lattice point. Suppose now \(\gamma_1 + \gamma_2 + \gamma_3 \equiv L\) is such a lattice point, and set \(\gamma'_3 = -(\gamma_1 + \gamma_2)\). The periodicity of the \(\varphi\)-function means

\[
h(z) = \varphi(z + \gamma_3) = \varphi(z + \gamma_3 - L) = \varphi(z + \gamma'_3)
\]

and so we may choose the solutions of (2) so that

\[
\gamma_1 + \gamma_2 + \gamma_3 = 0.
\]

\(\Box\)

It remains to discuss the case when at least one of the functions in (2) is a constant which we may take to be \(h(z)\).

**Lemma 3** Let \(f, g, h\) satisfy (3). If \(h(z)\) is a constant then either

1. One of the functions \(f(x)\) or \(g(y)\) is the same constant as \(h(z)\), in which case the remaining function is arbitrary, or

2. up to the invariance of (2), \(f(x) = g(x) = e^x\).

**Proof:** Using the invariance of (2), we may suppose without loss of generality that \(h(z) = 0\). Then (for all \(x, y\))

\[
0 = \left| \begin{array}{cc}
    f(x) & g(y) \\
    f'(x) & g'(y)
  \end{array} \right| = (\partial_y - \partial_x) f(x) g(y). \tag{14}
\]

If neither of \(f(x)\) or \(g(y)\) vanish identically then \(f(x) = e^{\delta x + \epsilon_1}\) and \(g(y) = e^{\delta y + \epsilon_2}\), with \(\epsilon_{1,2}\) arbitrary and \(\delta\) possibly zero (giving the constant solutions). Using the invariance of (2) we may choose \(f(x) = g(x) = e^x\). Finally if (say) \(f(x) = 0\) we see that \(g(y)\) is arbitrary. \(\Box\)

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