Intern. Journ. Pure and Appl. Math., 55, N4, (2009), 7-11.
A singular integral equation for electromagnetic wave scattering

A G Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract
A 3D singular integral equation is derived for electromagnetic wave scattering by bodies of arbitrary shape. Its numerical solution by a projection method is outlined.

MSC: 78A40, 78A45, 45E99
Key words: electromagnetic waves; scattering theory; integral equations

1 Introduction
Consider the following scattering problem. An incident electromagnetic field \((E_0, H_0)\) is scattered by a bounded region \(D\), filled with a material with parameters \((\epsilon, \sigma, \mu_0)\). The exterior region \(D'\) is a homogeneous region with parameters \((\epsilon_0, \sigma = 0, \mu_0)\). Consider for simplicity the case when \(\epsilon = const\) in \(D\), \(\sigma = const \geq 0\) in \(D\). Let \(\epsilon' = \epsilon + \frac{i\omega}{\omega}\). The governing equations in \(\mathbb{R}^3\) are

\[
\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon' E.
\]

At the boundary \(S\) of \(D\) one has

\[
[N, E^+] = [N, E^-],
\]

and

\[
N \cdot \epsilon' E^+ = N \cdot \epsilon_0 E^-,
\]

where \(N\) is the unit normal to \(S\), pointing into \(D'\), \(E^+(E^-)\) is the limiting value of \(E\) on \(S\) from inside (outside) \(S\), \([N, E]\) is the cross product, and \(E \cdot N\) is the dot product of two vectors.
Let

\[ k^2 = \omega^2 \varepsilon_0 \mu_0, \quad K^2 = \omega^2 \varepsilon' \mu_0, \quad K^2 = \begin{cases} k^2, & \text{in } D', \\ K^2, & \text{in } D. \end{cases} \]  

(4)

Equations (1) imply

\[ \nabla \times \nabla \times E - K^2 E = 0, \quad H = \frac{\nabla \times E}{i \omega \mu_0} \quad \text{in } \mathbb{R}^3. \]  

(5)

Therefore, it is sufficient to find \( E \) satisfying the first equation (5), boundary conditions (2), (3), and the radiation condition

\[ E = E_0 + V; \quad V_r - ikV = o \left( \frac{1}{r} \right), \quad r := |x| \to \infty. \]  

(6)

Equation (5) for \( E \) can be written as

\[ LE := \nabla \times \nabla \times E - k^2 E = pE; \quad p := p(x) = K^2 - k^2 = \begin{cases} 0, & \text{in } D', \\ K^2 - k^2, & \text{in } D. \end{cases} \]  

(7)

The incident field \( E_0 \) solves equation (7) with \( p = 0 \).

Let \( \delta(x) \) denote the delta-function and \( \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \)

Let \( G = G_{ij}(x) \) solve the problem:

\[ LG = \delta(x)\delta_{ij}, \quad G_r - ikG = o \left( \frac{1}{r} \right), \quad r \to \infty. \]  

(8)

Then the solution to (7)-(6) solves the integral equation

\[ E = E_0 + \int_{\mathbb{R}^3} G(x - y)p(y)E(y)dy. \]  

(9)

The kernel \( G(x) = G(|x|) \) is symmetric, \( G_{ij} = G_{ji} \), see formula (15). Let us prove

**Lemma 1.1.** There is at most one solution to (9) satisfying (2) and (3).

**Proof.** If there are two solutions then their difference \( E \) solves the homogeneous equation (9) and satisfies (2) and (3). Thus, \( E \) solves (7), (8), (2) and (3). Therefore, \( E \) and \( H = \frac{\nabla \times E}{i \omega \mu_0} \) solve equations (1) and satisfy condition (2), (3) and (6). It is known (see e.g., [2]) that this implies \( E = H = 0 \).

Lemma 1.1 is proved.  

\( \square \)
Lemma 1.2. If $E$ solves equation (9), then it satisfies (1), (2), (3) and (7). Therefore, (9) has at most one solution.

Proof. Applying operator $L$ to (9) one obtains equation (7). The integral in (9) is the term $V$ in (6). It satisfies the radiation condition because $G$ does. Equation (7) is equivalent to (5). Equation (5) together with the formula $H = \frac{\nabla \times E}{i \omega \mu_0}$ yield both equations (1). Conditions (2) and (3) are consequences of equations (1). Therefore, every solution to (9) is in one-to-one correspondence with the solution to equations (1). This correspondence is given by the formulas $E = E$, $H = \frac{\nabla \times E}{i \omega \mu_0}$. By Lemma 1.1 equation (9) has at most one solution satisfying (2) and (3). We have proved that every solution to (9) satisfies (2) and (3). Therefore, (9) has at most one solution. Lemma 1.2 is proved.

Lemma 1.3. Equation (9) has a unique solution.

Proof. Uniqueness of the solution to (9) is proved in Lemma 1.2. Existence of it follows from the existence of the solution to the scattering problem and the fact, established in the proof of Lemma 1.2, that a solution to (9) solves equation (5) and satisfies the radiation condition (6) and conditions (2), (3). Lemma 1.3 is proved.

From lemmas 1-3 the following result follows

Theorem 1.4. Equation (9) has a unique solution $E$. This solution $E$ generates the solution to the scattering problem by the formula $E = E$, $H = \frac{\nabla \times E}{i \omega \mu_0}$.

In Section 2 we construct the Green’s function $G$.

2 Construction of $G$

Let us look for $G$ of the form

$$G(x) = \int_{\mathbb{R}^3} e^{i \xi \cdot x} \tilde{G}(\xi) d\xi, \quad \tilde{G}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i \xi \cdot x} G(x) dx.$$  \tag{10}

Take the Fourier transform of (8) and get

$$- [\xi, \xi, \tilde{G}] - k^2 \tilde{G} = \frac{1}{(2\pi)^3} I, \quad I_{ij} = \delta_{ij},$$  \tag{11}
where \([a, b]\) is the cross product of two vectors, and \(a \cdot b\) is their dot product. This implies

\[- \xi \cdot \tilde{G} + (\xi^2 - k^2) \tilde{G} = \frac{1}{(2\pi)^3} I. \tag{12}\]

From (12) one finds

\[\xi \cdot \tilde{G} = -\frac{\xi}{(2\pi)^3 k^2}. \tag{13}\]

Thus,

\[\tilde{G}_{ij} = \frac{\delta_{ij}}{(2\pi)^3(\xi^2 - k^2)} - \frac{\xi_i \xi_j}{(2\pi)^3 k^2(\xi^2 - k^2)}. \tag{14}\]

Taking the inverse Fourier transform and using the radiation condition (6), one gets

\[G_{ij}(x) = g(x) \delta_{ij} + \frac{1}{k^2} \partial_{ij}g(x); \quad g = \frac{e^{ik|x|}}{4\pi|x|}, \quad \partial_i := \frac{\partial}{\partial x_i}. \tag{15}\]

From (15) and (9) one gets:

\[E_i(x) = E_{0i}(x) + (K^2 - k^2) \int_D g(x, y) E_i(y) dy + \frac{K^2 - k^2}{k^2} \partial_i \int_D \frac{\partial g(x, y)}{\partial x_j} E_j(y) dy, \quad 1 \leq i \leq 3, \tag{16}\]

where summation over the repeated indices is understood. Equation (16) is a vector singular integral equation. The operator

\[TE = (K^2 - k^2) \int_D g(x, y) E(y) dy\]

is compact in \(L^2(D)\). Let

\[\gamma := \frac{K^2 - k^2}{k^2}, \quad QE = \nabla \int_D \nabla_x g(x, y) \cdot E(y) dy. \tag{17}\]

Then equation (16) can be written as

\[E = E_0 + TE + \gamma QE. \tag{18}\]

This is equation (9).

Numerically one can solve equation (16) (or (18)) by a projection method. For example, let \(\{\phi_j(x)\}\) be a basis of \(L^2(D)\) and \(\phi_j \in H^1_0(D)\), where \(H^1_0(D)\) is the closure of \(C_0^{\infty}(D)\) functions in the norm of the Sobolev space \(H^1(D)\).
Multiply equation (16) by $\overline{\phi_m}$ (the bar stands for the complex conjugate), integrate over $D$ and then the third term by parts, to get:

$$E_{im} = E_{0im} + (K^2 - k^2) \int_D \int_D dx \overline{\phi_m(x)} g(x,y) \sum_{m'=1}^{M} E_{im'} \phi_{m'}(y)$$

$$- \gamma \int_D dx \frac{\partial \overline{\phi_m(x)}}{\partial x_i} \int_D \frac{\partial g(x,y)}{\partial x_j} \sum_{m'=1}^{M} E_{jm'} \phi_{m'}(y) dy, \quad 1 \leq m \leq M, \quad 1 \leq i \leq 3.$$  \hspace{1cm} (19)

This is a linear algebraic system for finding the coefficients:

$$E_{im}^{(M)} := E_{im} := \int_D E_i(x) \overline{\phi_m(x)} dx.$$  \hspace{1cm} (20)

The number $M$ determines the accuracy of the approximate solution $E(x)$. One has

$$\lim_{M \to \infty} \|E^{(M)}(x) - E(x)\|_{L^2(D)} = 0.$$  \hspace{1cm} (21)

References

[1] S. Mikhlin, S. Prössdorf, Singular integra; operators, Springer-Verlag, Berlin, 1986.

[2] C. Müller, Foundations of the mathematical theory of electromagnetic waves, Springer-Verlag, Berlin, 1969.