A new method of solving quartic and higher degree diophantine equations

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Abstract

In this paper we present a new method of solving certain quartic and higher degree homogeneous polynomial diophantine equations in four variables. The method can also be extended to solve simultaneous homogeneous polynomial diophantine equations, in five or more variables, with one of the equations being of degree \( \geq 4 \). We show that, under certain conditions, the method yields an arbitrarily large number of integer solutions of such diophantine equations and diophantine systems, some examples being a sextic equation in four variables, a tenth degree equation in six variables, and two simultaneous equations of degrees four and six in six variables. The method of solving these homogeneous equations also simultaneously yields arbitrarily many rational solutions of certain related nonhomogeneous equations of high degree. In contrast to existing methods, we obtain the arbitrarily large number of solutions without finding a parametric solution of the equations under consideration and without relating the solutions to rational points on an elliptic curve of positive rank. It appears from the examples given in the paper that there may exist projective varieties on which there are an arbitrarily large number of integer points and on which a curve of genus 0 or 1 does not exist.

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1 Introduction

This paper is concerned with quartic and higher degree homogeneous polynomial diophantine equations of the type,

\[
f(x_1, x_2, x_3, x_4) = 0, \tag{1.1}
\]
where \( f(x_1, x_2, x_3, x_4) \) is a form, with integer coefficients, in the 4 variables \( x_1, x_2, x_3, x_4 \), as well as with simultaneous diophantine equations of the type,

\[
f_r(x_1, x_2, \ldots, x_n) = 0, \quad r = 1, 2, \ldots, k;
\]

where \( f_r(x_i) \) are forms, with integer coefficients, in the \( n \) independent variables \( x_1, x_2, \ldots, x_n \) where \( n \geq 5, k < n - 2 \) and the degree of at least one of the forms \( f_r(x_i) \) is \( \geq 4 \).

If there exist infinitely many integer solutions of Eq. (1.1) or of the diophantine system (1.2), till now there are primarily just two ways of obtaining these infinitely many solutions: either we find a solution in terms of one or more independent, arbitrary integer parameters, or we find a solution in terms of two parameters \( X \) and \( Y \) where the ordered pair \( (X, Y) \) represents a rational point on an elliptic curve defined over \( \mathbb{Q} \), and such that the elliptic curve has rank \( \geq 1 \). Since the elliptic curve has positive rank, we can find infinitely many rational points on the elliptic curve by using the group law, and thus obtain infinitely many rational solutions of Eq. (1.1) or of the diophantine system (1.2). On appropriate scaling, these infinitely many rational solutions yield infinitely many primitive integer solutions of Eq. (1.1) or of the simultaneous equations (1.2).

In this paper we first show that when Eq. (1.1) satisfies certain properties, we may obtain a sequence of integer solutions of quartic and higher degree equations of type (1.1) by a new iterative method. In contrast to existing methods, the method described in this paper neither involves finding a parametric solution nor does it require relating the integer solutions to rational points on an elliptic curve of positive rank. In fact, we show how to construct equations of arbitrarily high degree with the desired properties, and give examples of equations for which we obtain an infinite sequence of integer solutions. While the octic and higher degree equations, that we could solve by the new method, are such that there is always a curve of genus 0 or 1 on the surface defined by the equation under consideration, we show that there exist quartic and sextic equations of type (1.1) with infinitely many integer solutions and for which it seems that there is no particular reason that there should necessarily exist a curve of genus 0 or 1 on the surface defined by the equation.

As in the case of the single diophantine equation (1.1), when the diophantine system (1.2) possesses certain properties, an extension of the new method may be applied to obtain a sequence of integer solutions of (1.2). We show how to construct such diophantine systems in which one of the equations is of arbitrarily high degree. This leads to an example of a diophantine equation of arbitrarily high degree \( 2d \) in \( 2d \) variables for which we obtain an arbitrarily long sequence of integer solutions. While this sequence is expected to consist of infinitely many distinct integer solutions, this remains to
be proved.

We give a couple of illustrative examples of diophantine systems for which an arbitrarily large number of integer solutions may be obtained by the method described in this paper. For instance, we obtain arbitrarily many solutions of a diophantine system consisting of one equation of degree 10 in 8 variables and two linear equations. This leads to an example of a single diophantine equation of degree 10 in 6 variables as well as to an example of a pair of simultaneous equations of degrees four and six in 6 variables, and for both of these examples, we obtain an arbitrarily large number of integer solutions. It appears that there exist diophantine systems with an arbitrarily large number of integer solutions and for which it is unlikely that there exists a curve of genus 0 or 1 on the projective variety defined by the simultaneous equations comprising the diophantine system.

As a related result, when we obtain an arbitrarily large number of integer solutions of an equation of type (1.1) or a diophantine system of type (1.2), we also simultaneously get a nonhomogeneous equation of high degree with an arbitrarily large number of rational solutions.

In Section 2 we give an overview of the new method of solving diophantine equations of type (1.1). In Sections 3, 4 and 5, we show how to construct examples of quartic, sextic and higher degree equations that may be solved by this method. In Section 6 we give an overview of an extension of the method for solving simultaneous diophantine equations in several variables, and show how to construct diophantine systems that may be solved by this method. In Sections 7 and 8, we give examples of such diophantine systems with an arbitrarily large number of integer solutions. We conclude the paper with some open problems and remarks regarding such diophantine equations.

2 An overview of the new method

In this section we describe, in general terms, the process of generating a sequence of integer solutions of certain quartic and higher degree homogeneous diophantine equations in four variables.

2.1

We now consider the diophantine equation,

\[ f(x_1, x_2, x_3, x_4) = 0. \]  

(2.1)

where \( f(x_1, x_2, x_3, x_4) \) is a quaternary form of degree \( d \geq 4 \) in four independent variables \( x_1, x_2, x_3, x_4 \). Since Eq. (2.1) is homogeneous, if \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) is any rational solution of (2.1), then \( (k\alpha_1, k\alpha_2, k\alpha_3, k\alpha_4), k \in \mathbb{Q} \setminus \{0\} \) is...
also a rational solution of (2.1). All such solutions will be considered equivalent, and they all represent the same point on the surface (2.1). When we refer to more than one solution of (2.1), we mean solutions that are not equivalent to each other. It is clear that any rational solution of Eq. (2.1) yields, on appropriate scaling, a solution in integers. Thus, it suffices to find rational solutions of (2.1).

If there exist integers \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), not all 0, such that \( f(\alpha_i) = 0 \), we will refer to the quadruple \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) both as a solution of Eq. (2.1) and as a point on the surface defined by Eq. (2.1).

We assume that Eq. (2.1) satisfies the following properties:

\( D_1 \) : At least one integer solution \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) of the diophantine equation (2.1) is known such that both \( \alpha_1, \alpha_2 \) are not simultaneously 0 and similarly, both \( \alpha_3, \alpha_4 \) are not simultaneously 0.

\( D_2 \) : The form \( f(x_i) \) satisfies the condition \( f(x_1, x_2, x_3, x_4) = f(-x_1, x_2, x_3, x_4) \).

\( D_3 \) : There exist bilinear forms \( B_1(m_1, m_2, x_1, x_2) \) and \( B_2(m_1, m_2, x_1, x_2) \) with rational coefficients satisfying the following conditions:

(i) for all rational numerical values of \( x_1 \) and \( x_2 \) such that \( (x_1, x_2) \neq (0, 0) \), the forms \( B_j(m_1, m_2, x_1, x_2), \ j = 1, 2, \) are linearly independent linear forms in the variables \( m_1, m_2 \); and for all rational numerical values of \( m_1 \) and \( m_2 \) such that \( (m_1, m_2) \neq (0, 0) \), the forms \( B_j(m_1, m_2, x_1, x_2), \ j = 1, 2, \) are linearly independent linear forms in the variables \( x_1, x_2 \);

(ii) on substituting

\[
\begin{align*}
x_3 &= B_1(m_1, m_2, x_1, x_2), \\
x_4 &= B_2(m_1, m_2, x_1, x_2),
\end{align*}
\]

in Eq. (2.1), we get

\[
\psi(x_1, x_2)\{\phi_0(m_1, m_2)x_1^2 + \phi_1(m_1, m_2)x_1x_2 + \phi_2(m_1, m_2)x_2^2\} = 0, \tag{2.3}
\]

where \( \psi(x_1, x_2) \) is a binary form of degree \( d - 2 \) in the variables \( x_1, x_2 \) such that the equation \( \psi(x_1, x_2) = 0 \) has no rational solutions, and \( \phi_0(m_1, m_2), \phi_1(m_1, m_2), \phi_2(m_1, m_2) \) are polynomials that do not vanish simultaneously for any rational values of \( m_1 \) and \( m_2 \) such that \( (m_1, m_2) \neq (0, 0) \).

We will show in Section 2.2 that when Eq. (2.1) satisfies the above three properties, it is possible to generate a sequence of rational solutions of Eq. (2.1) starting from the known solution.

We note that it is fairly straightforward to determine all those solutions of Eq. (2.1) in which both \( x_1, x_2 \) are 0 or both \( x_3, x_4 \) are 0 since in these cases, Eq. (2.1) reduces to a homogeneous equation in just two variables. We will exclude all such solutions of Eq. (2.1) since the method described below is based on the substitution given by (2.2) and if both \( x_1, x_2 \) are 0, we get
the trivial result $x_3 = 0, x_4 = 0$, and similarly, a trivial situation also arises from any solution in which both $x_3, x_4$ are 0. Thus, henceforth, whenever we refer to a solution of Eq. (2.1) or, a point on the surface defined by Eq. (2.1), we mean a solution (or a point) for which both $x_1, x_2$ are not simultaneously 0 and also both $x_3, x_4$ are not simultaneously 0.

2.2

Given any rational point $P = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ on a surface of type (2.1) which satisfies the properties $D_1, D_2, D_3$, mentioned in Section 2.1 above, we can, in general, find two new rational points on the surface (2.1). The first of these is simply the reflection of $P$, denoted by $R(P)$ and defined by $R(P) = (-\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. In view of property $D_2$, the point $R(P)$ lies on the surface (2.1). A second rational point on the surface (2.1) may be obtained as described below.

We note that corresponding to each solution $(x_1, x_2, x_3, x_4) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of Eq. (2.1), we can find rational numerical values of the parameters $m_1, m_2$ by solving the two linear relations,

$$
\alpha_3 = B_1(m_1, m_2, \alpha_1, \alpha_2),
$$

$$
\alpha_4 = B_2(m_1, m_2, \alpha_1, \alpha_2),
$$

obtained by writing $x_i = \alpha_i$ in the relations (2.2). We also note that all equivalent solutions $(k\alpha_1, k\alpha_2, k\alpha_3, k\alpha_4)$ yield the same pair of values for $m_1$ and $m_2$ which are such that $(m_1, m_2) \neq (0, 0)$. With these values of $m_1, m_2$, we make the substitution (2.2) in Eq. (2.1) and reduce it to Eq. (2.3). Since $\psi(x_1, x_2) \neq 0$, Eq. (2.3) reduces to the following quadratic equation in $x_1$ and $x_2$:

$$
\phi_0(m_1, m_2)x_1^2 + \phi_1(m_1, m_2)x_1x_2 + \phi_2(m_1, m_2)x_2^2 = 0.
$$

In view of property $D_3$, we note that the coefficients of $x_1^2, x_1x_2$ and $x_2^2$ in Eq. (2.5) cannot vanish simultaneously. Since $(x_1, x_2, x_3, x_4) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is already a known solution of (2.1), Eq. (2.5) is necessarily solvable and one of its solutions, corresponding to the known solution, is given by $(x_1, x_2) = (\alpha_1, \alpha_2)$. At the same time, (2.5) has a second solution which satisfies the condition $(x_1, x_2) \neq (0, 0)$. With these values of $x_1$ and $x_2$, we get a new point on the surface Eq. (2.1) by using the relations (2.2). Since $(m_1, m_2) \neq (0, 0)$, it now follows from condition (i) of property $D_3$ that the new point just obtained also satisfies the condition that its last two coordinates are not simultaneously 0.

Any two solutions of Eq. (2.1) obtained by using the substitution (2.2) and solving the resulting quadratic equation (2.5) will be referred to as conjugate
solutions, and the points on the surface (2.1) corresponding to conjugate solutions will be referred to as conjugate points. The conjugate of a point \( P \) will be denoted by \( C(P) \).

Thus, starting from a known rational point \( P \) which satisfies the condition that its first two coordinates are not simultaneously 0 and its last two coordinates are also not simultaneously 0, we can find two new rational points \( R(P) \) and \( C(P) \) both of which satisfy the condition that their first two coordinates are not simultaneously 0 and their last two coordinates are also not simultaneously 0.

On the two new rational points just obtained, we can again perform either of the two operations of finding the reflection or the conjugate (written briefly as the \( R \) and \( C \) operations respectively), and in this manner, we can continue to perform these two operations any number of times and in any order. At each step, the rational points thus generated satisfy the condition that their first two coordinates are not simultaneously 0 and their last two coordinates are also not simultaneously 0, and hence we can actually execute the \( R \) and \( C \) operations any number of times.

As long as it is clear in which order the \( R \) and \( C \) operations are to be performed, we may drop the parentheses — for instance, we may write the point \( R(C(C(R(C(P))))) \) simply as \( RCCRC(P) \). We may similarly also insert parentheses as long as there is no ambiguity.

It readily follows from the definitions of \( C(P) \) and \( R(P) \) that

\[
C^2(P) = C(C(P)) = P \quad \text{and} \quad R^2(P) = R(R(P)) = P.
\]

Therefore new rational points on the surface (2.1) can be generated only if we perform the operations of finding the reflection and the conjugate of a point alternately.

The combined operation of first finding the conjugate \( C(P) \) of a point \( P \), and then finding the reflection of the point \( C(P) \) will be referred to as the \( RC \) operation, and the point thus obtained will be written as \( RC(P) \). We can repeat this \( RC \) operation on the point \( RC(P) \) to obtain a new point, denoted by \( (RC)^2(P) \), and we may continue this process any number of times. The point obtained by performing the \( RC \) operation \( k \) times will be denoted by \( (RC)^k(P) \). We thus obtain a sequence of rational points on the surface (2.1) given by,

\[
P, \quad RC(P), \quad (RC)^2(P), \quad (RC)^3(P), \ldots, \quad (RC)^k(P), \ldots.
\] (2.6)

Similarly we define the \( CR \) operation as the combined operation of first taking the reflection of a point \( P \) on the surface (2.1) and then finding its conjugate, and the point thus obtained is denoted by \( CR(P) \). By repeating the \( CR \) operation \( k \) times, we obtain the point \( (CR)^k(P) \), and we thus get
the sequence of rational points on the surface (2.1) given by,

\[ P, \ CR(P), \ (CR)^2(P), \ (CR)^3(P), \ldots, \ (CR)^k(P), \ldots, \] \hspace{1cm} (2.7)

For convenience, we define \((RC)^0(P) = P\) and \((CR)^0(P) = P\). It follows from the definitions of the operations of finding the reflection and the conjugate of rational points on the surface (2.1) that if \((RC)P = Q\) then \(P = (CR)Q\). Further, if \((RC)^k(P) = Q\) then, for any positive integer \(h < k\), we have \((RC)^{k-h}P = (CR)^h(Q)\). Similarly, it follows that if \((CR)^k(P) = Q\) then, for any positive integer \(h < k\), we have \((CR)^{k-h}(P) = (RC)^h(Q)\). We note here that, in general, \(CR(P) \neq RC(P)\). In Section 3.2, we will see a specific example illustrating this fact.

We note that there may exist certain points \(P\) on the surface (2.1) such that the conjugate point of \(P\) coincides with the point \(P\). Such points satisfy the condition \(C(P) = P\) and will be called self-conjugate points. Similarly, there may exist points \(P\) on the surface (2.1) such that the reflection of \(P\) coincides with the point \(P\). Such points that remain invariant under reflection satisfy the condition \(R(P) = P\) and, when the definition of reflection is clear from the context, we will simply refer to these points as invariant points.

We will denote the individual coordinates of a point \(P\) by \(x_i(P)\), \(i = 1, 2, 3, 4\). Similarly, we will denote the individual coordinates of the points \(R(P)\), \(C(P)\) and \(RC(P)\) by \(x_i(RP)\), \(x_i(CP)\) and \(x_i(RCP)\), \(i = 1, 2, 3, 4\) respectively.

2.3 With respect to both the sequences of points (2.6) and (2.7), there exists the possibility that for some positive integer \(k\), the \((k+1)\)th point of the sequence is the same as either the initial point \(P\) or another point that occurs earlier in the sequence.

For the sequence (2.6), this will happen if either \((RC)^k(P) = P\) or \((RC)^j(P) = (CR)^j(P)\) where \(1 \leq j < k\). In the latter case, it is readily seen that \((RC)^{k-j}(P) = P\). Thus, in either case, there exists an integer \(m\) such that \((RC)^m(P) = P\). We then say that \(P\) is a point of finite order and the least positive integer \(m\) such that \((RC)^m(P) = P\) will be called the order of \(P\) with respect to the \(RC\) operation. If \(P\) is a point of finite order, the sequence (2.6) will only generate finitely many points on the surface (2.1).

If there is no integer \(m\) such that \((RC)^m(P) = P\), we say that \(P\) is a point of infinite order with respect to the \(RC\) operation and the sequence of points (2.6) gives infinitely many distinct rational points on the surface (2.1).

We similarly define the order of a point \(P\) with respect to the \(CR\) operation as the least positive integer \(m\) such that \((CR)^m(P) = P\). We note that
when \((RC)^m(P) = P\), then \((CR)^m(P) = (CR)^m(RC)^m(P) = P\), and similarly when \((CR)^m(P) = P\), then \((RC)^m(P) = P\). It follows that the order of any point \(P\) with respect to the \(RC\) operation is the same as the order of \(P\) with respect to the \(CR\) operation. We may thus simply write that the order of a point \(P\) is \(m\) without referring to the \(RC\) or the \(CR\) operation.

2.4

For any specific surface defined by an equation (2.1) satisfying the three properties \(D_1, D_2, D_3\) mentioned in Section 2.1, we may obtain a sequence of rational points (2.6) or (2.7), and we may then wish to establish that this is indeed an infinite sequence of distinct rational points on the surface (2.1). We could prove this possibly by using the method of induction. The following lemma gives an alternative method for proving the existence of an infinite sequence of rational, and hence integer, points on the surface (2.1).

Lemma 1. Let there exist an equation of type (2.1) satisfying the properties \(D_1, D_2, D_3\). If, on the surface defined by Eq. (2.1), the number of rational self-conjugate points be \(n_1\) and the number of rational invariant points be \(n_2\), then there are infinitely many integer points on the surface (2.1) if \(n_1 + n_2\) is an odd integer.

Proof. Starting from each rational self-conjugate or invariant point \(P\), we will repeatedly apply the \(RC\) operation to generate sequences of rational points of the type (2.6). We will show that if the initial point \(P\) is of finite order, the total number of self-conjugate points and invariant points in each such sequence is exactly 2. We will also show that any two of the sequences either consist of exactly the same points or they have no points in common. Thus the total number of self-conjugate points and invariant points in all such sequences put together is an even number. Since \(n_1 + n_2\) is odd, there remains at least one self-conjugate point or one invariant point that must necessarily be of infinite order. It follows that there are infinitely many rational, and hence integer, points on the surface (2.1).

We will now prove that in each sequence obtained by starting from a rational self-conjugate or an invariant point of finite order, the total number of self-conjugate points and invariant points is exactly 2. If the initial point \(P\) is of order 1, then \(RC(P) = P\) so that \(C(P) = R(P) = P\), and while the sequence (2.6) reduces to just the single point \(P\), this point is counted once as a self-conjugate point and once as an invariant point. Thus the total number of self-conjugate points and invariant points in the sequence is to be taken as 2. We also note that such a point \(P\) is also included twice in the number \(n_1 + n_2\), once as a self-conjugate point and once as an invariant point.
Next let $P$ be a rational self-conjugate point of finite order $m \geq 2$ so that $C(P) = P$ and

$$(RC)^m(P) = P. \quad (2.8)$$

Starting with the point $P$ and repeatedly applying the $RC$ operation, we get a sequence of $m$ distinct rational points,

$$(P, (RC)(P), (RC)^2(P), \ldots, (RC)^{m-1}(P)). \quad (2.9)$$

We will now determine the points of the sequence (2.9) that are self-conjugate or invariant.

If $(RC)^h(P), 1 \leq h \leq m - 1$ is a self-conjugate point, then

$$(C(RC)^h(P) = (RC)^h(P),$$

or,

$$(CR)^hC(RC)^h(P) = (CR)^h(RC)^h(P),$$

or,

$$C(RC)^{2h}(P) = P,$$

or,

$$CC(RC)^{2h}(P) = C(P),$$

or,

$$(RC)^{2h}(P) = P.$$

We also note that when $m$ is even, it follows from (2.8) that $(RC)^{m/2}(P) = (CR)^{m/2}(P)$, and hence,

$$C(RC)^{m/2}(P) = C(CR)^{m/2}(P) = CC(RC)^{m/2-1}RC(P) = (RC)^{m/2}(P).$$

It follows that when $m$ is odd, the sequence (2.9) contains exactly one self-conjugate point, namely the initial point $P$, and when $m$ is even, it contains exactly two self-conjugate points, namely the point $P$ and the point $(RC)^{m/2}(P)$, which is distinct from $P$.

If $(RC)^h(P), 1 \leq h \leq m - 1$ is an invariant point, then

$$R(RC)^h(P) = (RC)^h(P),$$

or,

$$R(RC)^h(P) = R(CR)^{h-1}C(P),$$

or,

$$R(RC)^h(P) = R(CR)^{h-1}(P), \text{ since } C(P) = P,$$

or,

$$R(RC)^{2h-1}(P) = P.$$
It follows that when \( m \) is odd, the sequence (2.9) contains exactly one invariant point distinct from \( P \), namely \((RC)^{(m+1)/2}(P)\), but when \( m \) is even, it does not contain any invariant points.

Thus, whether \( m \) is odd or even, the total number of self-conjugate points and invariant points in the sequence of points (2.9), obtained by starting from a rational self-conjugate point of finite order, is exactly 2 (including the initial self-conjugate point \( P \)).

Next, let \( P \) be a rational invariant point of finite order \( m \geq 2 \), so that \( R(P) = P \) and the relation (2.8) is also satisfied. We will now determine the points of the sequence (2.9) that are self-conjugate or invariant.

If \((RC)^h(P), 1 \leq h \leq m - 1\), is a self-conjugate point, then

\[
C(R)C(R)^h(P) = (RC)^{h+1}(P),
\]

or

\[
C(R)^{2h}(P) = P,
\]

or

\[
R(RC)^{2h}(P) = R(P),
\]

or

\[
(R)^{2h+1}(P) = P, \quad \text{since } R(P) = P.
\]

We also note that when \( m \) is odd, it follows from (2.8) that \((RC)^{(m-1)/2}(P) = (CR)^{(m+1)/2}(P)\), and hence,

\[
C(R)^{(m-1)/2}(P) = C(CR)^{(m+1)/2}(P) = (RC)^{(m-1)/2}(P).
\]

It follows that when \( m \) is is odd, the sequence (2.9) contains exactly one self-conjugate point, namely \((RC)^{(m-1)/2}(P)\), which is distinct from \( P \) but when \( m \) is even, it does not contain any self-conjugate points.

If \((RC)^h(P), 1 \leq h \leq m - 1\) is a point that remains invariant under reflection, then

\[
R(RC)^h(P) = (RC)^{h+1}(P),
\]

or

\[
R(RC)^h(P) = (RC)^hR(P), \quad \text{since } R(P) = P,
\]

or

\[
R(RC)^h(P) = R(CR)^h(P),
\]

or

\[
(R)^{h+1}(P) = (CR)^h(P),
\]

or

\[
(R)^{2h}(P) = P,
\]

We also note that when \( m \) is even, it follows from (2.8) that \((RC)^m/2P = (CR)^m/2(P)\), and hence,

\[
R(RC)^m/2P = R(CR)^m/2(P) = (RC)^m/2R(P) = (RC)^m/2(P).
\]

It follows that when \( m \) is odd, the sequence (2.9) contains exactly one invariant point, namely the initial point \( P \), and when \( m \) is even, it contains
exactly two invariant points, namely the point \( P \) and the point \((RC)^{m/2}P\) which is distinct from \( P \).

Thus, whether \( m \) is odd or even, the total number of self-conjugate points and invariant points in the sequence of points \((2.9)\), obtained by starting from a rational invariant point of finite order, is also exactly 2 (including the initial invariant point \( P \)).

We have thus shown that in all the sequences of type \((2.9)\), obtained by starting from a rational self-conjugate or an invariant point of finite order, the total number of self-conjugate points and invariant points is exactly 2. We will now show that two such sequences either consist of exactly the same points or they have no points in common.

If any two sequences of the type \((2.9)\), obtained by starting from two distinct rational points \( P \) and \( P_1 \), of finite orders \( m \) and \( m_1 \) respectively, have a point in common, we must have \((RC)^h(P) = (RC)^{h_1}(P_1)\) for some integers \( h \) and \( h_1 \). There exists an integer \( \lambda \) such that \( \lambda m + h \geq h_1 \) and it follows from the relation \((RC)^h(P) = (RC)^{h_1}(P_1)\) that \((RC)^{\lambda m + h - h_1}(P) = P_1\). It now follows that all points of the second sequence are included in the first sequence. Similarly, there exists an integer \( \lambda_1 \) such that \( \lambda_1 m_1 + h_1 \geq h \) and it follows from the relation \((RC)^h(P) = (RC)^{h_1}(P_1)\) that \((RC)^{\lambda_1 m_1 + h_1 - h}(P_1) = P\). It now follows that all points of the first sequence are included in the second sequence, and hence the two sequences consist of exactly the same points.

Thus the total number of self-conjugate points and invariant points in all the distinct sequences of type \((2.9)\), obtained by starting from all the rational self-conjugate or invariant points of finite order, is an even integer. Since the total number of rational self-conjugate points and invariant points is an odd integer, at least one of these points cannot be of finite order. Starting from such a point that is not of finite order, we obtain an infinite number of rational, and hence integer, points on the surface \((2.1)\). This proves the lemma.

\(\square\)

2.5

We have, for the sake of simplicity, stipulated in property \( D_2 \) that the form \( f(x_i) \) should satisfy the condition \( f(x_1, x_2, x_3, x_4) = f(-x_1, x_2, x_3, x_4) \). In fact, our method will work as long as the form \( f(x_i) \) possesses a certain symmetry so that when we know one solution of Eq. \((2.1)\), we immediately get another solution using the symmetry. For instance, the form \( f(x_i) \) may satisfy the condition \( f(x_1, x_2, x_3, x_4) = f(x_2, x_1, x_3, x_4) \) or \( f(x_1, x_2, x_3, x_4) = f(x_3, x_4, x_1, x_2) \). A more interesting diophantine equation is

\[
F(x_1, x_2) + 4F(x_3, x_4) = 0, \quad (2.10)
\]
where \( F(x_1, x_2) \) is a binary quartic form, and we note that whenever a point \( P \) given by \((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) lies on the surface (2.10), another point on the surface (2.10) is given by \((2\alpha_3, 2\alpha_4, \alpha_1, \alpha_2)\). Thus, in this case, we define \( R(P) = (2\alpha_3, 2\alpha_4, \alpha_1, \alpha_2) \). If Eq. (2.10) also satisfies the properties \( D_1 \) and \( D_3 \), we can obtain a sequence of points on the surface (2.10) by the method described above.

### 2.6

In Sections 3, 4 and 5, we will show that there exist infinitely many examples of quartic and higher degree diophantine equations of type (2.1) which have infinitely many solutions. These infinitely many solutions are obtained by reducing (2.1) to an equation of type (2.5). Now Eq. (2.5) is a quadratic equation in the variables \( x_1 \) and \( x_2 \), and it has a rational solution if and only if its discriminant \( \phi_1^2(m_1, m_2) - 4\phi_0(m_1, m_2)\phi_2(m_1, m_2) \) becomes a perfect square. It readily follows that when Eq. (2.1) has infinitely many solutions, the nonhomogeneous polynomial diophantine equation in the variables \( m_1, m_2 \) and \( z \) given by

\[
\phi_1^2(m_1, m_2) - 4\phi_0(m_1, m_2)\phi_2(m_1, m_2) = z^2, \tag{2.11}
\]

also simultaneously has infinitely many solutions in rational numbers.

### 3  Quadratic diophantine equations

In Section 3.1 we will construct a very general quartic equation that can be solved by the method described in Section 2. In Sections 3.2 and 3.3, we discuss some specific quartic diophantine equations and prove that there exist infinitely many quartic diophantine equations for which we can obtain an arbitrarily large number of solutions by applying this method.

#### 3.1  A general quartic equation

Consider the quadratic forms in the variables \( x_1, x_2, x_3, x_4 \), given by,

\[
Q(x_1, x_2) = x_1^2 + px_1x_2 + qx_2^2,
Q_1(x_1, x_2, x_3, x_4) = x_1x_3 - qx_2x_4,
Q_2(x_1, x_2, x_3, x_4) = x_1x_4 + x_2x_3 + px_2x_4,
\tag{3.1}
\]

where \( p, q \) are arbitrary parameters. If we write

\[
x_3 = m_1x_1 + (pm_1 + qm_2)x_2, \quad x_4 = m_2x_1 - m_1x_2, \tag{3.2}
\]
where $m_1$ and $m_2$ are arbitrary parameters, then we have the identities,

$$Q(x_3, x_4) = Q(m_1, m_2)Q(x_1, x_2),$$
$$Q_1(x_1, x_2, x_3, x_4) = m_1Q(x_1, x_2),$$
$$Q_2(x_1, x_2, x_3, x_4) = m_2Q(x_1, x_2).$$

(3.3)

We now consider the quartic equation,

$$F(x_1, x_2, x_3, x_4) = 0,$$

(3.4)

where $F(x_1, x_2, x_3, x_4)$ is the quartic form defined by,

$$F(x_1, x_2, x_3, x_4) = a_1Q(x_1, x_2)Q(-x_1, x_2)$$
$$+ a_2Q_1(x_1)Q_1(-x_1, x_2, x_3, x_4) + a_3Q_2(x_1)Q_2(-x_1, x_2, x_3, x_4)$$
$$+ a_4\{Q(x_1, x_2)Q_1(-x_1, x_2, x_3, x_4) + Q(-x_1, x_2)Q_1(x_1)\}$$
$$+ a_5\{Q(x_1, x_2)Q_2(-x_1, x_2, x_3, x_4) + Q(-x_1, x_2)Q_2(x_1)\}$$
$$+ (a_6x_1^2 + a_7x_2^2 + a_8x_2x_3 + a_9x_2x_4 + a_{10}x_3^2 + a_{11}x_3x_4 + a_{12}x_4^2)Q(x_3, x_4),$$

(3.5)

with $a_j$, $j = 1, 2, \ldots, 12$, being arbitrary integer coefficients.

It is readily seen that $F(x_1, x_2, x_3, x_4) = F(-x_1, x_2, x_3, x_4)$. Thus, the form $F(x_1, x_2, x_3, x_4)$ satisfies the property $D_2$. Further, if in the form $F(x_i)$, we substitute the values of $x_3, x_4$ given by (3.2), it is readily observed, in view of the relations (3.3), that Eq. (3.4) reduces to an equation of type (2.3) where $\psi(x_1, x_2)$ is, in fact, given by $Q(x_1, x_2)$.

The parameters $p$, $q$ and the coefficients $a_j$ can easily be chosen such that the property $D_1$ and the remaining conditions stipulated in property $D_3$ are satisfied.

Thus, we can find a sequence of rational solutions of Eq. (3.4) starting from a known solution. In the next two subsections, we consider a couple of special cases of Eq. (3.4).

3.2

In Eq. (3.4), we take

$$p = 3, \quad q = -1, \quad a_1 = 1, \quad a_2 = -2h, \quad a_3 = -h,$$
$$a_4 = -h, \quad a_5 = -h, \quad a_6 = -h, \quad a_7 = 4h, \quad a_8 = -h,$$
$$a_9 = -h, \quad a_{10} = 1, \quad a_{11} = 3, \quad a_{12} = -1,$$
where \( h \) is an arbitrary integer parameter, and we thus get the following equation:

\[
x_1^4 - 11x_1^2x_2^2 + x_2^4 + x_3^4 + 6x_3^2x_4 + 7x_3x_4 - 6x_3x_4^2 + x_4^4
\]

\[
+ h(4x_1^2x_2x_3 - 2x_1^2x_2x_4 + x_1x_3^2 - 3x_1x_3x_4 + 2x_1x_4^2
\]

\[
+ 2x_2x_3 + 8x_3^2x_4 + 3x_2^2x_3^2 + 6x_2x_3x_4 - 15x_2x_4^2
\]

\[
- 2x_1^3x_3 - 4x_1x_2x_4 - 2x_2x_3^2 + x_2x_4^3 = 0. \quad (3.6)
\]

It is easily verified that for all values of \( h \), a solution of Eq. (3.6) is given by \((x_1, x_2, x_3, x_4) = (1, 1, 1, 1)\).

On substituting the values of \( x_3, x_4 \) given by

\[
x_3 = m_1x_1 + (3m_1 - m_2)x_2, \quad x_4 = m_2x_1 - m_1x_2, \quad (3.7)
\]

Eq. (3.6) reduces, after removing the irreducible factor \( x_1^2 + 3x_1x_2 - x_2^2 \), to the following equation:

\[
\{(m_1^2 - 3m_1m_2 + 2m_2^2)h + m_1^4 + 6m_1m_2 + 7m_1m_2^2 - 6m_1m_2^2 + m_4 + 1\}x_1^2
\]

\[
- \{(m_1^3 + 3m_1^2m_2 + 2m_1m_2^3 - m_2 - 6m_2 + 6m_1m_2^2 + 3m_2 - 4m_1 + 2m_2)h
\]

\[
- 3m_1^4 - 18m_1^3m_2 - 18m_1^2m_2^3 - 18m_1m_2^3 - 3m_2^4 + 3]\}x_1x_2
\]

\[
- \{(2m_1^3 + 5m_1^2m_2 + 5m_1m_2^3 + m_2^3 - 6m_1^2 - 12m_1m_2 + 3m_2 - 2m_1 + 2m_2)h
\]

\[
+ m_1^4 + 6m_1^3m_2 + 7m_1^2m_2^2 - 6m_1m_2^3 + m_4 + 1\}x_2^2 = 0. \quad (3.8)
\]

It is now readily verified that, for any arbitrary value of \( h \), Eq. (3.6) satisfies all the three properties \( D_1, D_2 \) and \( D_3 \) mentioned in Section 2.1.

Now, starting from the point \( P_0 = (1, 1, 1, 1) \) and repeatedly applying the RC operation, we obtain a sequence of points \( P_0, P_1, P_2, \ldots, P_n \) on the surface (3.3). The coordinates of these points are rational functions of \( h \). We will show that given an arbitrary positive integer \( n \), however large, there exists an integer value of \( h \) such that these \( n \) points are distinct and rational.

If the points \( P_0, P_1, P_2, \ldots, P_n \) are not distinct, \( P_0 \) must be a point of finite order not exceeding \( n \). The point \( P_0 \) will be of finite order \( r < n \) only if the following conditions are satisfied:

\[
x_1(P_r) = x_2(P_r) = x_3(P_r) = x_4(P_r) \neq 0, \quad (3.9)
\]

where, as already noted, the coordinates \( x_i(P_r), i = 1, 2, 3, 4 \), are rational functions of \( h \). We now show that these conditions cannot be satisfied identically for all values of \( h \).

We consider the case \( h = 0 \) when Eq. (3.6) may be written as,

\[
x_1^4 - 11x_1^2x_2^2 + x_2^4 + x_3^4 + 6x_3^2x_4 + 7x_3x_4^2 - 6x_3x_4^3 + x_4^4 = 0. \quad (3.10)
\]
while Eq. (3.8) reduces to the following equation:

\[
\begin{align*}
(m_1^4 + 6m_1^3m_2 + 7m_1^2m_2^2 - 6m_1m_2^3 + m_2^4 + 1)x_1^2 \\
+ 3(m_1^2 + 3m_1m_2 - m_2^2 - 1)(m_1^2 + 3m_1m_2 - m_2^2 + 1)x_1x_2 \\
- (m_1^4 + 6m_1^3m_2 + 7m_1^2m_2^2 - 6m_1m_2^3 + m_2^4 + 1)x_2^2 = 0. \tag{3.11}
\end{align*}
\]

Let a rational point \( P \) on the surface \((3.10)\) be given by \((3^t, 3^t, \alpha, \beta)\) where \(\alpha, \beta\) are integers such that \(\alpha \equiv 1 \pmod{3}, \beta \equiv 1 \pmod{3}\), and \(t\) is a nonnegative integer. The values of \(m_1, m_2\) corresponding to the point \( P \) are given by

\[
m_1 = (\alpha + \beta)/3^{t+1}, \quad m_2 = (\alpha + 4\beta)/3^{t+1}. \tag{3.12}
\]

With these values of \(m_1, m_2\), we now work out the conjugate \( C(P) \) of the point \( P \). Eq. (3.11) gives the values of the ratios \( x_1/x_2 \) for a point \( P \) and its conjugate \( C(P) \). It follows from (3.11) that

\[
\frac{x_1(P)}{x_2(P)} \cdot \frac{x_1(CP)}{x_2(CP)} = -1.
\]

Since \(x_1(P) = x_2(P) = 3^t\), we get \(-x_1(CP) = x_2(CP) = m\) where \(m\) is an arbitrary rational number, and we obtain the values of \(x_3(CP), x_4(CP)\) using the relations (3.7). We thus obtain the point \( C(P) \), and on taking the reflection of this point, we obtain the point \( RC(P) \) which is given by \((3^{t+1}, 3^{t+1}, \alpha', \beta')\) where \(\alpha' = 2(\alpha + \beta) - 3\alpha, \beta' = 2(\alpha + \beta) + 3\beta\). Thus, we get \(\alpha' \equiv 1 \pmod{3}, \beta' \equiv 1 \pmod{3}\) and it now follows by induction that the sequence of points on the surface \((3.10)\), obtained by starting from \((1, 1, 1, 1)\) as the initial point \( P_0 \), consists of infinitely many distinct rational points. The first four points of the sequence obtained in this manner are \(P_0 = (1, 1, 1, 1), P_1 = (3, 3, 1, 7), P_2 = (3^2, 3^2, 13, 37), P_3 = (3^3, 3^3, 61, 211)\).

Reverting to the original equation (3.6), we can now conclude that the conditions (3.9) cannot be identically satisfied for all values of \(h\) since they are not satisfied in the special case \(h = 0\). The conditions (3.9) are, in fact, polynomial equations in \(h\) and they can be satisfied by, at most, a finite number of integer values of \(h\). Excluding these values of \(h\), we can still assign infinitely many integer values to \(h\), and thus obtain infinitely many surfaces \((3.6)\) on which there are at least \(n\) distinct rational points where \(n\) is any arbitrarily chosen positive integer.

When \(h = 7\), the first four points of the sequence of points found on the surface \((3.6)\) starting from the point \( P_0 \) are as follows:

\[(1, 1, 1, 1), \quad (13, -31, 97, 196), \]
\[(2244925401, 1768375579, -3244635281, 5477857719), \]

15
Further, while $RC(P_0) = (-130, 31, 97, 196)$, we note that $CR(P_0) = (15, 8, 8, -15)$. This illustrates the fact that, in general, $RC(P) \neq CR(P)$ as was stated in Section 2.2.

We have obtained infinitely many integer solutions of Eq. (3.10) of the type $(3^k, 3^k, \alpha, \beta)$. These solutions satisfy the condition $x_1 = x_2$. We note that the complete solution of (3.10) satisfying the additional condition $x_1 = x_2$ is readily obtained and is given by

$$x_1 = x_2 = r^2 - 8rs + 3s^2, \quad x_3 = r^2 - 6rs + 21s^2, \quad x_4 = 2r^2 - 6s^2, \quad (3.13)$$

and

$$x_1 = x_2 = -r^2 + rs + 3s^2, \quad x_3 = -r^2 + 6rs - 6s^2, \quad x_4 = r^2 + 3s^2, \quad (3.14)$$

where $r$ and $s$ are arbitrary parameters. Thus Eq. (3.10) is, by itself, not of much intrinsic interest except that the infinite sequence of points $P_0, P_1, P_2, \ldots$, found on the surface (3.10) has been used to prove that there exist infinitely many values of $h$ for which the diophantine equation (3.6) has an arbitrarily large number of solutions.

We note that for any arbitrary value of $h$, the surface defined by Eq. (3.6) has singularities at the points $(0, 0, -3 + \sqrt{13}, 2)$ and $(0, 0, -3 - \sqrt{13}, 2)$.

We give below the nonhomogeneous equation obtained by equating the discriminant of Eq. (3.8) to a perfect square when $h = 7$:

$$13m_1^8 + 156m_1^7m_2 + 650m_1^6m_2^2 + 936m_1^5m_2^3 - 273m_1^4m_2^4 - 936m_1^3m_2^5 + 650m_1^2m_2^6 - 156m_1m_2^7 + 13m_2^8 + 14m_1^7 + 56m_1^6m_2 - 294m_1^5m_2^2 - 1554m_1^4m_2^3 - 966m_1^3m_2^4 + 1806m_1^2m_2^5 - 644m_1m_2^6 + 70m_2^7 + 161m_1^6 + 392m_1^5m_2 - 2254m_1^4m_2^2 - 4606m_1^3m_2^3 + 4046m_1^2m_2^4 - 952m_1m_2^5 + 63m_2^6 - 84m_1^5 - 1428m_1^4m_2 - 1722m_1^3m_2^2 + 6384m_1^2m_2^3 - 1596m_1^2m_2^4 - 42m_1^5 - 186m_1^4 - 3784m_1^3m_2^2 + 5222m_1^2m_2^3 - 3860m_1m_2^4 + 1411m_1^4 + 2058m_3^3 - 2436m_1^2m_2 + 336m_1m_2^2 - 210m_2^3 + 392m_1^2 - 952m_1m_2 + 462m_2^2 - 224m_1 + 28m_2 + 13 = z^2. \quad (3.15)$$

We have given above the first four solutions of Eq. (3.6) when $h = 7$, obtained by starting from the point $P_0$, and corresponding to these solutions
of Eq. \((3.6)\), we can readily find rational solutions of Eq. \((3.15)\). The values of \((m_1, m_2, z)\) for the first three solutions of Eq. \((3.15)\) are as follows:

\[
(2/3, 5/3, 66), \quad (-2178/1283, -1415/1283, 249877066/1646089),
\]

\[
\begin{pmatrix}
1872911657224140 \\
10773306028890859
\end{pmatrix}
\begin{pmatrix}
27763349640930281 \\
10773306028890859
\end{pmatrix}
\begin{pmatrix}
3239413319832561746630798860799374 \\
116064122792136130054389733757881
\end{pmatrix}.
\]

The next solution involves integers consisting of more than 64 digits and is hence omitted.

### 3.3

As a second example, in Eq. \((3.4)\) we take,

\[
p = 1, \quad q = 3, \quad a_1 = 6, \quad a_2 = 6, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 6, \quad a_6 = -3, \quad a_7 = -105, \quad a_8 = 12, \\
a_9 = 6, \quad a_{10} = 3, \quad a_{11} = 3, \quad a_{12} = -3,
\]

when we get the quartic equation,

\[
6x_1^4 + (30x_2^2 + 10x_2x_3 - 6x_2x_4 - 9x_3^2 - 3x_3x_4 - 9x_4^2)x_1^2 \\
+ 54x_2^4 + 36x_3^2x_2 + 18x_3^2x_4 - 105x_2^2x_3^2 - 105x_2^2x_3x_4 \\
- 261x_2^2x_4^2 + 12x_2x_3^3 + 18x_2x_3^2x_4 + 42x_2x_3x_4^2 + 18x_2x_4^3 \\
+ 3x_3^4 + 6x_3^2x_4 + 9x_3^2x_4^2 + 6x_3x_4^3 - 9x_4^4 = 0. \quad (3.16)
\]

It is easily verified that \((x_1, x_2, x_3, x_4) = (0, 1, 1, 0)\) is a solution of Eq. \((3.16)\).

On substituting

\[
x_3 = m_1x_1 + (m_1 + 3m_2)x_2, \quad x_4 = m_2x_1 - m_1x_2,
\]

Eq. \((3.16)\) reduces, after removing factor \((x_1^2 + x_1x_2 + 3x_2^2)\), to

\[
(3m_1^4 + 6m_1^3m_2 + 9m_1^2m_2^2 + 6m_1m_2^3 - 9m_2^4 - 9m_1^3 - 3m_1m_2 - 9m_2^2 + 6)x_1^2 \\
+ (3m_1^4 + 30m_2m_1^3 + 45m_1^2m_2^2 + 90m_1m_2^3 + 27m_2^4 + 12m_1^3 + 18m_2^3m_2 \\
+ 42m_1m_2^2 + 18m_2^3 - 6m_1^2 - 36m_1m_2 + 10m_1 - 6m_2 - 6)x_1x_2 - (3m_1^4 - 6m_1^3m_2 \\
- 27m_1^2m_2^2 - 54m_1m_2^3 - 81m_2^4 - 6m_1^3 - 42m_1^2m_2 - 54m_1m_2^2 - 108m_2^3 \\
+ 87m_1^2 + 105m_1m_2 + 315m_2^2 - 6m_1 - 36m_2 - 18)x_2^2 = 0. \quad (3.18)
\]
It is now readily verified that Eq. (3.16) satisfies all the three properties $D_1$, $D_2$ and $D_3$ mentioned in Section 2.1. Thus we may generate a sequence of rational points on the surface (3.16) taking $(0, 1, 1, 0)$ as the initial point.

We will now show that the total number of all rational self-conjugate points and rational invariant points on the surface (3.16) is an odd integer, and then apply Lemma 1 to prove that there are infinitely many integer solutions of Eq. (3.16).

The discriminant $d(m_1, m_2)$ of Eq. (3.18) is given by

\[ d(m_1, m_2) = 45m_1^8 + 180m_1^7m_2 + 810m_1^6m_2^2 + 1800m_1^5m_2^3 + 4095m_1^4m_2^4 
+ 5400m_1^3m_2^5 + 7290m_1^2m_2^6 + 4860m_1m_2^7 + 3645m_2^8 + 180m_2^9m_2 
+ 540m_1^5m_2^2 + 2160m_1^4m_2^3 + 3420m_1^3m_2^4 + 6480m_1^2m_2^5 + 4860m_1m_2^6 
+ 4860m_2^7 + 1044m_3^8 + 3384m_2^5m_2 + 9000m_1^4m_2^2 + 13536m_1^3m_2^3 
+ 10872m_1^2m_2^4 + 6264m_1m_2^5 - 8100m_2^6 + 60m_2^7 + 492m_1^4m_2^2 + 324m_1^3m_2^3 
+ 2628m_1^2m_2^4 + 756m_1m_2^5 + 1860m_2^6 - 3036m_1^4m_2^2 - 5112m_1^3m_2^3 - 15648m_1^2m_2^4 
- 10512m_1m_2^5 - 13176m_2^6 - 192m_3^8 - 504m_2^7m_2 - 720m_1m_2^6 - 1512m_2^7 
+ 2908m_2^6 + 3048m_1m_2 + 8244m_2^2 - 264m_1 - 792m_2 - 396.\]

If there exist any rational self-conjugate points on the surface (3.16), there must exist rational numbers $m_1$ and $m_2$ such that Eq. (3.18) has two coincident roots and hence $d(m_1, m_2)$ must be 0. As $m_1$ and $m_2$ are rational numbers, we write $m_1 = n_1/n_0$, $m_2 = n_2/n_0$ where $n_1$, $n_2$ and $n_0$ are integers such that $n_0 \neq 0$ and $\text{gcd}(n_0, n_1, n_2) = 1$, and now, on equating $d(m_1, m_2)$ to 0, we get an equation that may be written as follows:

\[ 396n_0^8 + (264n_1 + 792n_2)n_0^7 - (2908n_1^2 + 3048n_1n_2 + 8244n_2^2)n_0^6 
+ (192n_1^3 + 504n_1^2n_2 + 720n_1n_2^2 + 1512n_2^3)n_0^5 
+ (3036n_1^4 + 5112n_1^3n_2 + 15648n_1^2n_2^2 + 10512n_1n_2^3 + 13176n_2^4)n_0^4 
- 12(n_1^5 + n_1n_2 + 3n_2^2)(5n_1^3 + 36n_1^2n_2 - 24n_1n_2^2 + 135n_2^3)n_0^3 
- 36(29n_1^7 + 36n_1^6n_2 - 25n_2^5)(n_1^2 + n_1n_2 + 3n_2^2)n_0^2 
- 180(n_1^8 + n_1n_2 + 3n_2^3)n_0n_2 - 45(n_1^2 + n_1n_2 + 3n_2^2)^4 = 0.\]

We note that in Eq. (3.20), all terms except the last are even integers, hence $(n_1^2 + n_1n_2 + 3n_2^2)$ is an even integer and it readily follows that both $n_1$ and $n_2$ must be even integers. On substituting $n_1 = 2n_3$ and $n_2 = 2n_4$ in
Eq. (3.20), we get the following equation:

\[
99n_0^6 + (132n_3 + 396n_4)n_0^5 - (2908n_3^2 + 3048n_3n_4 + 8244n_4^2)n_0^4 \\
+ (384n_3^2 + 1008n_3n_4 + 1440n_3n_4^2 + 3024n_4^3)n_0^3 \\
+ (12144n_3^4 + 20448n_3^3n_4 + 62592n_3^2n_4^2 + 42048n_3n_4^3 + 52704n_4^4)n_0^2 \\
- 96(n_3^2 + n_3n_4 + 3n_4^2)(5n_3^3 + 36n_3^2n_4 - 24n_3n_4^2 + 135n_4^3)n_0 \\
- 576(29n_3^4 + 36n_3n_4 - 25n_4^2)(n_3^2 + n_3n_4 + 3n_4^2)n_0^2 \\
- 5760n_4(n_3^2 + n_3n_4 + 3n_4^2)^3n_0 - 2880(n_3^2 + n_3n_4 + 3n_4^2)^4 = 0. \tag{3.21}
\]

We note that in Eq. (3.21), all terms except the first are even integers, hence \(n_0\) must be an even integer. This is a contradiction since \(\gcd(n_0, n_1, n_2) = 1\). Hence the equation \(d(m_1, m_2) = 0\) has no rational solutions. It follows that there are no rational self-conjugate points on the surface (3.16).

Next we determine the number of rational invariant points on the surface (3.16). These points are precisely the rational points on (3.16) satisfying the conditions \(x_1 = 0\) and \(x_2 \neq 0\). On substituting \(x_1 = 0\) in (3.16), we get the condition,

\[
18x_2^4 + 12x_2^3x_3 + 6x_2^3x_4 - 35x_2^2x_3^2 - 35x_2^2x_3x_4 \\
- 87x_2^2x_4^2 + 4x_2x_3^3 + 6x_2x_3^2x_4 + 14x_2x_3x_4^2 + 6x_2x_4^3 \\
+ x_3^4 + 2x_3^3x_4 + 3x_3^2x_4^2 + 2x_3x_4^3 - 3x_4^4 = 0. \tag{3.22}
\]

Eq. (3.22) represents a curve of genus 3 in projective space and hence it has a finite number of integer solutions. It is readily verified that if \((x_2, x_3, x_4) = (\alpha_2, \alpha_3, \alpha_4)\) is any integer solution of (3.22) with \(\alpha_2 \neq 0\) and \(\alpha_4 \neq 0\), then another distinct integer solution of (3.22) is given by \((x_2, x_3, x_4) = (\alpha_2, \alpha_3 + \alpha_4, -\alpha_4)\). Thus, integer solutions of (3.22) can be paired off except for those solutions in which \(x_4 = 0\). On substituting \(x_4 = 0\) in (3.22), we get

\[
(x_2 - x_3)(18x_2^3 + 30x_2^2x_3 - 5x_2x_3^2 - x_3^3) = 0, \tag{3.23}
\]

and so we get just one rational solution of (3.22) with \(x_4 = 0\), namely \((x_2, x_3, x_4) = (1, 1, 0)\). This yields the single rational invariant point \((0, 1, 1, 0)\). As the other invariant points occur in pairs, this shows that there are an odd number of rational invariant points on the surface (3.16).

As there are no self-conjugate points on the surface (3.16), it follows that the total number of self-conjugate points and invariant points is an odd integer. It now follows from Lemma 1 that there are infinitely many rational points on the surface (3.16).

We now obtain a sequence of rational points on the surface (3.16) by repeatedly applying the \(RC\) operation starting from the point \((0, 1, 1, 0)\). The first four points of the sequence are

\[(0, 1, 1, 0), \quad (−63, 44, 44, 21),\]
and

\[
( -147609072097506717422185174080259362584795541167553561, \\
4022653635947814403673938708520812105643977659529244, \\
111318341489401190419048903211940940726579266647898028, \\
-77464352589959936491092062638791236314929817761076995).
\]

Each solution of the above sequence yields a solution of the nonhomogeneous diophantine equation,

\[ d(m_1, m_2) = z^2, \quad (3.24) \]

where \( d(m_1, m_2) \) is given by (3.19). The values of \((m_1, m_2, z)\) for the first three solutions of Eq. (3.24), corresponding to the respective solutions of Eq. (3.16), are given by

\[
(0, 1/3, 7), \quad (-1848/2335, 1537/7005, 122166401/5452225), \\
732016714554891233832 \quad 858029179234358903561 \\
925123942970233269817 \quad 2775371828910699809451 \\
14804259873811552782039788426057859064445551 \quad 855854309856791419406657915087787523213489.
\]

The surface defined by the quartic equation (3.16) has no singularities. It is a K3 surface whose arithmetic genus and geometric genus are both 1. These computations were done using the online MAGMA calculator [6].

## 4 Sextic diophantine equations

As in the case of quartic equations, we will first construct, in Section 4.1, a very general sextic equation that can be solved by the method described in Section 2. In Sections 4.2 and 4.3, we will discuss specific numerical examples of sextic equations.

### 4.1 A general sextic equation

We first define one quadratic and two cubic forms in the variables \(x_1, x_2, x_3, x_4\), as follows:

\[
Q(x_1, x_2) = x_1^2 + px_1x_2 + qx_2^2, \\
C_1(x_1, x_2, x_3, x_4) = x_1^2x_3 + qx_2^2x_3 + pqx_2x_4, \\
C_2(x_1, x_2, x_3, x_4) = x_1^2x_4 - px_2^2x_3 - (p^2 - q)x_2^2x_4, \quad (4.1)
\]
where \( p, q \) are arbitrary parameters. If we write
\[
x_3 = m_1 x_1 + (p m_1 + q m_2) x_2, \quad x_4 = m_2 x_1 - m_1 x_2, \tag{4.2}
\]
where \( m_1 \) and \( m_2 \) are arbitrary parameters, we have the following identities:
\[
\begin{align*}
Q(x_3, x_4) &= Q(m_1, m_2)Q(x_1, x_2), \\
C_1(x_1, x_2, x_3, x_4) &= (m_1 x_1 + m_2 q x_2)Q(x_1, x_2), \\
C_2(x_1, x_2, x_3, x_4) &= ((m_2 x_1 - (m_1 + pm_2) x_2)Q(x_1, x_2). \\
\end{align*}
\tag{4.3}
\]

Let us now consider the equation,
\[
S(x_1, x_2, x_3, x_4) = 0, \tag{4.4}
\]
where \( S(x_1, x_2, x_3, x_4) \) is a sextic form defined by,
\[
\begin{align*}
S(x_1, x_2, x_3, x_4) &= a_1 C_1^2(x_i) + a_2 C_1(x_i) C_2(x_i) + a_3 C_2^2(x_i) \\
&+ Q(x_3, x_4) [x_2 \{ a_4 C_1(x_i) + a_5 C_2(x_i) \} + x_4 \{ a_6 C_1(x_i) + a_7 C_2(x_i) \}] \\
&+ (a_8 x_1^2 + a_9 x_2^2 + a_{10} x_2 x_3 + a_{11} x_2 x_4 + a_{12} x_3^2 + a_{13} x_3 x_4 + a_{14} x_4^2) Q^2(x_3, x_4), \\
\end{align*}
\tag{4.5}
\]
with \( a_j, j = 1, 2, \ldots, 14 \), being arbitrary integer coefficients.

We note that \( x_1 \) occurs only in even degrees in the forms \( Q(x_1, x_2), C_1(x_i) \) and \( C_2(x_i) \), and it is easily observed that this is also true for the form \( S(x_i) \). It follows that \( S(x_1, x_2, x_3, x_4) = S(\alpha_1, x_2, x_3, x_4) \). Thus, the form \( S(x_1, x_2, x_3, x_4) \) satisfies the property \( D_2 \). Further, in view of the relations \( \text{(4.3)} \), it is also easily observed that when we substitute the values of \( x_3, x_4 \) given by \( \text{(4.2)} \) in Eq. \( \text{(4.4)} \), the sextic equation \( \text{(4.4)} \) reduces to an equation of type \( \text{(2.3)} \) in which \( \psi(x_1, x_2) = Q^2(x_1, x_2) \).

The parameters \( p, q \) and the coefficients \( a_j \) can easily be chosen such that the property \( D_1 \) and the remaining conditions stipulated in property \( D_3 \) are satisfied. Thus, starting from a known solution of Eq. \( \text{(4.4)} \), we can find a sequence of rational points on the surface \( \text{(4.4)} \).

We note that Eq. \( \text{(4.4)} \) does not have any terms of the type \( x_1^{\beta} x_2^{\gamma}, j = 0, 1, \ldots, 6 \) and accordingly, the surface represented by \( \text{(4.4)} \) has singularities at \((\alpha_1, \alpha_2, 0, 0)\) for any arbitrary values of \( \alpha_1, \alpha_2 \). We would naturally prefer to construct sextic equations in which all the terms \( x_i^{\beta}, i = 1, 2, 3, 4, \) are present but all efforts to do so were futile.

In the next two subsections, we discuss special cases of Eq. \( \text{(4.4)} \).
4.2

In Eq. (3.4), we take

\[
p = 1, \quad q = -1, \quad a_1 = 3, \quad a_2 = -1, \quad a_3 = 5, \quad a_4 = 44957, \quad a_5 = 6, \quad a_6 = 1, \quad a_7 = 2939, \quad a_8 = 29654, \quad a_9 = 13121, \quad a_{10} = 2, \quad a_{11} = -25057, \quad a_{12} = -7856, \quad a_{13} = -8176, \quad a_{14} = 891,
\]

when we get the following sextic equation:

\[
(3x_3^2 - x_3 x_4 + 5x_2^2)x_1^4 - (5x_2^2 x_3^2 + 13x_2^2 x_3 x_4 + 19x_2^2 x_4^2 - 44957 x_2 x_3^3 - 44963x_2x_3^3 + 44951x_2 x_3 x_4^2 + 6x_2 x_4^3 - 29654x_3^4 - 59309x_3^2x_4^2 + 26714x_3^2x_4^2 + 56370x_2 x_3^3 - 26715x_4^4)x_1^2 + 7x_2 x_3 x_4^2 + 23x_2^3 x_3 x_4 \\
+ 21x_2^4 x_4^2 - 44963x_2^2 x_3^2 - 89932x_2 x_3^2 x_4 - 6x_2 x_3 x_4^2 + 44969x_3^2 x_4^2 + 13112x_2^2 x_3 x_4^2 - 21940x_2 x_3^2 x_4^2 - 29181x_3^2 x_3 x_4^2 \\
+ 19000x_2^2 x_3^4 + 2x_2 x_3^3 - 25053x_2 x_3^3 x_4 - 50116x_2 x_3^2 x_4^2 + 25053x_2 x_3 x_4^3 \\
+ 50116x_2 x_3 x_4 - 25057 x_2 x_4^5 - 7856x_3^6 - 23888x_3^5 x_4 - 7605 x_3^4 x_4^2 \\
+ 25670x_3^4 x_4^2 + 7605 x_3^3 x_4^3 - 9958 x_3 x_4^5 + 891 x_4^6 = 0. \quad (4.6)
\]

It is easily verified that Eq. (4.6) satisfies the three properties \(D_1, D_2\) and \(D_3\), and a solution of Eq. (4.6) is given by \((0, 1, 0, -1)\).

Now, starting from the point \(P = (0, 1, 0, -1)\) and repeatedly applying the \(RC\) operation, we obtain a sequence of points \(P, (RC)P, (RC)^2P, \ldots\), on the surface \(4.6\). It turns out that \(P\) is a point of order 11, and we get just eleven distinct rational points as follows:

\[
P = (0, 1, 0, -1), \quad (RC)P = (1, 1, -1, -2), \quad (RC)^2P = (-1, 1, -1, -2), \quad (RC)^3P = (-1, 0, 1, 3), \quad (RC)^4P = (-1, 2, 3, -1), \quad (RC)^5P = (-3, 2, 7, -5), \quad (RC)^6P = (1, 2, 3, -1), \quad (RC)^7P = (1, 0, 1, 3), \quad (RC)^8P = (2, 1, -4, -7), \quad (RC)^9P = (0, 1, 0, -1) = P.
\]

It would be recalled that there cannot be a point of order 11 on an elliptic curve (see Mazur’s theorem \([7, p. 57]\)). It follows from the above example that the \(RC\) operation of generating a sequence of rational points on a surface defined by an equation of type \((2.1)\) is unrelated to the process of finding a sequence of rational points on an elliptic curve by repeated addition of a known rational point.
4.3

As a second example, in Eq. (4.4) we take,

\[
\begin{align*}
p &= 1, \quad q = 3, \quad a_1 = 44, \quad a_2 = -36, \\
a_3 &= -36, \quad a_4 = 120, \quad a_5 = -72, \quad a_6 = 0, \\
a_7 &= 0, \quad a_8 = 0, \quad a_9 = -9, \quad a_{10} = -972, \\
a_{11} &= -486, \quad a_{12} = 9, \quad a_{13} = 9, \quad a_{14} = 9,
\end{align*}
\]

when we get the sextic equation,

\[
(44x_3^2 - 36x_3x_4 + 36x_4^2)x_1^4 + (300x_3^2x_3^2 + 12x_3x_4)
+
36x_2x_4^2 + 120x_2x_3^3 + 48x_4^2x_3^2 + 288x_3x_4^2x_3^2 - 216x_3x_4^3)x_1^2
+
540x_3^4x_3^2 + 324x_4^4x_4^2 + 432x_3^3x_3^3
+
648x_3^3x_4 + 1512x_2x_3x_4^2 + 648x_2^3x_4^3 - 9x_2^4x_3^2
-
18x_2^2x_3^3x_4 - 63x_2^3x_4^2 - 54x_2^2x_3x_4^3 - 81x_2^4x_3^4
-
972x_2x_3^5 - 2430x_2^4x_4 - 7776x_2^5x_4^2 - 9234x_2^6x_4^3
-
11664x_2x_3x_4^4 - 4374x_2^5x_4^5 + 9x_2^6 + 27x_3^5x_4 + 90x_3^4x_4^2
+
135x_3^3x_4^3 + 198x_3^2x_4^4 + 135x_3x_4^5 + 81x_4^6 = 0. \quad (4.7)
\]

It is easily verified that Eq. (4.7) satisfies the three properties \(D_1, D_2\) and \(D_3\), and a solution of Eq. (4.7) is given by \((0, 1, 1, 0)\). Thus we may generate a sequence of rational points on the surface (4.7) taking \((0, 1, 1, 0)\) as the initial solution.

On substituting

\[
x_3 = m_1x_1 + (m_1 + 3m_2)x_2, \quad x_4 = m_2x_1 - m_1x_2, \quad (4.8)
\]

Eq. (4.7) reduces, after removing the irreducible factor \((x_1^2 + x_1x_2 + 3x_2^2)^2\), to

\[
(9m_1^6 + 27m_1^5m_2 + 90m_1^4m_2^2 + 135m_1^3m_2^3 + 198m_1^2m_2^4
+
135m_1m_2^5 + 81m_2^6 + 44m_1^2 - 36m_1m_2 + 36m_2^2)x_1^2
+
(9m_1^6 + 63m_1^5m_2 + 180m_1^4m_2^2 + 423m_1^3m_2^3 + 540m_1^2m_2^4
+
567m_1m_2^5 + 243m_2^6 - 972m_1^5 - 2430m_1^4m_2 - 7776m_1^3m_2^2
-
9234m_1^2m_2^3 - 11664m_1m_2^4 - 4374m_2^5 + 120m_1^3 + 48m_2^6
+
288m_1m_2^5 - 216m_2^6 + 36m_1^2 + 228m_1m_2 - 180m_2^2)x_1x_2
+
(9m_1^6 + 45m_1^5m_2 + 198m_1^4m_2^2 + 405m_1^3m_2^3 + 810m_1^2m_2^4
+
729m_1m_2^5 + 729m_2^6 - 486m_1^5 - 3888m_1^4m_2 - 9234m_1^3m_2^2
-23328m_1^2m_2^3 - 21870m_1m_2^4 - 26244m_2^5 - 9m_1^4 - 18m_1^2m_2
-63m_1^2m_2^2 - 54m_1m_2^3 - 81m_1^2 + 72m_1^4 + 504m_1^2m_2
+
648m_1m_2^5 + 1296m_2^6 + 36m_1^2 + 180m_1m_2 + 540m_2^2)x_2^2 = 0. \quad (4.9)
\]
We will now show that the total number of all the rational self-conjugate points and the rational invariant points on the surface (1.7) is an odd integer, and then apply Lemma 1 to prove that there are infinitely many integer solutions of Eq. (1.7).

If there exist any rational self-conjugate points on the surface (1.7), there must exist rational numbers $m_1$ and $m_2$ such that Eq. (1.9) has two coincident roots. Thus, the discriminant of Eq. (1.9) must be 0. The discriminant of Eq. (1.9) is given by $-9(m_1^2 + m_1m_2 + 3m_2^2)^2d(m_1, m_2)$ where

$$d(m_1, m_2) = 27m_1^8 + 108m_1^7m_2 + 486m_1^6m_2^2 + 1080m_1^5m_2^3$$
$$+ 2457m_1^4m_2^4 + 3240m_1^3m_2^5 + 4374m_1^2m_2^6 + 2916m_1m_2^7$$
$$+ 2187m_2^8 - 2916m_1^6m_2 - 8748m_1^5m_2^2 - 34992m_1^4m_2^3$$
$$- 55404m_1^3m_2^4 - 104976m_1^2m_2^5 - 78732m_1m_2^6 - 78732m_2^7$$
$$- 105012m_1^6 - 315036m_1^5m_2 - 971388m_1^4m_2^2 - 141716m_1^3m_2^3$$
$$- 1759140m_1^2m_2^4 - 1102788m_1m_2^5 - 236520m_2^6 + 48m_1^5$$
$$+ 1008m_1^4m_2 + 3120m_1^3m_2^2 + 7056m_1^2m_2^3 + 8208m_1m_2^4$$
$$+ 6480m_2^5 + 26168m_1^4 + 23760m_1^3m_2 + 69576m_1^2m_2^2$$
$$- 13104m_1m_2^3 - 18792m_2^4 - 1728m_1^3 + 3888m_1^2m_2$$
$$+ 24624m_1m_2^2 - 66096m_2^3 - 1776m_2^4 + 2064m_1m_2$$
$$- 720m_2^2 + 448m_1 + 1344m_2 + 560. \quad (4.10)$$

Since $(m_1^2 + m_1m_2 + 3m_2^2)^2 \neq 0$, we must have $d(m_1, m_2) = 0$. As $m_1$ and $m_2$ are rational numbers, we write $m_1 = n_1/n_0$, $m_2 = n_2/n_0$ where $n_1$, $n_2$ and $n_0$ are integers such that $n_0 \neq 0$ and $\gcd(n_0, n_1, n_2) = 1$, and now, the condition $d(m_1, m_2) = 0$ may be written as follows:

$$560n_0^8 + 448(n_1 + 3n_2)n_0^7 - 48(37n_1^2 - 43n_1n_2 + 15n_2^2)n_0^6$$
$$- 432(4n_1^3 - 9n_1^2n_2 - 57n_1n_2^2 + 153n_2^3)n_0^5$$
$$+ 8(3271n_1^4 + 2970n_1^3n_2 + 8697n_1^2n_2^2 - 1638n_1n_2^3 - 2349n_2^4)n_0^4$$
$$+ 48(n_1^2 + n_1n_2 + 3n_2^2)(n_1^3 + 20n_1n_2^2 + 42n_1^2n_2 + 45n_2^3)n_0^3$$
$$- 36(2917n_1^7 + 2917n_1n_2 + 730n_2^2)(n_1^2 + n_1n_2 + 3n_2^2)^2n_0^2$$
$$- 2916n_2(n_1^2 + n_1n_2 + 3n_2^2)^3n_0 + 27(n_1^2 + n_1n_2 + 3n_2^2)^4 = 0. \quad (4.11)$$

We note that in Eq. (4.11), all terms except the last are even integers, hence $(n_1^2 + n_1n_2 + 3n_2^2)$ must be an even integer and it readily follows that both $n_1$ and $n_2$ must be even integers. On substituting $n_1 = 2n_3$ and $n_2 = 2n_4$
in Eq. (4.11), we get the following equation:

\[35n_0^4 + 56(n_3 + 3n_4)n_0^7 - 12(37n_3^2 - 43n_3n_4 + 15n_4^2)n_0^6
- 216(4n_3^3 - 9n_3^2n_4 - 57n_3n_4^2 + 153n_4^3)n_0^5
+ 8(3271n_3^4 - 897n_3^3n_4 + 1638n_3n_4^3 - 2349n_4^4)n_0^4
+ 96(n_3^2 + n_3n_4 + 3n_4^2)(n_3^3 + 20n_3^2n_4 + 42n_3n_4^2 + 45n_4^3)n_0^3
- 144(2917n_3^4 + 2917n_3n_4 + 730n_4^2)(n_3^3 + n_3n_4 + 3n_4^2)^2n_0^2
- 23328n_4(n_3^2 + n_3n_4 + 3n_4^2)^3n_0 + 432(n_3 + n_3n_4 + 3n_4^2)^4 = 0. \] (4.12)

We note that in Eq. (4.12), all terms except the first are even integers, hence \(n_0\) must be an even integer. This is a contradiction since \(\gcd(n_0, n_1, n_2) = 1\). Hence Eq. (4.11) has no integer solutions. It follows that there are no rational self-conjugate points on the surface (4.7).

Next we determine the number of rational invariant points on the surface (4.7). These points are precisely the rational points on (4.7) satisfying the condition,\n
\[60x_2^4x_3^2 + 60x_2^4x_3x_4 + 36x_2^4x_4^2 + 48x_2^3x_3^3 + 72x_2^3x_3^2x_4 + 168x_2^3x_3x_4^2
+ 72x_2^3x_3^4 - x_2^3x_4^3 - 2x_2^2x_3^2x_4^2 - 7x_2^2x_3x_4^3 - 6x_2^2x_3^3x_4 - 9x_2^2x_3^4 - 108x_2x_3^5
- 270x_2x_3^4x_4 - 856x_2x_3^3x_4^2 - 1026x_2x_3^2x_4^3 - 1296x_2x_3x_4^4 - 486x_2x_4^5
+ x_3^6 + 3x_3^5x_4 + 10x_3^4x_4^2 + 15x_3^3x_4^3 + 22x_3^2x_4^4 + 15x_3x_4^5 + 9x_4^6 = 0. \] (4.13)

Eq. (4.13) represents a curve of genus 5 in projective space and hence it has a finite number of integer solutions. It is readily verified that if \((x_2, x_3, x_4) = (\alpha_2, \alpha_3, \alpha_4)\) is any integer solution of (4.13) with \(\alpha_2 \neq 0\) and \(\alpha_4 \neq 0\), then another distinct integer solution of (4.13) is given by \((x_2, x_3, x_4) = (\alpha_2, \alpha_3 + \alpha_4, -\alpha_4)\). Thus, integer solutions of (4.13) can be paired off except for those solutions in which \(x_4 = 0\). On substituting \(x_4 = 0\) in (4.13), we get

\[x_3^2(x_2 - x_3)(60x_2^3 + 108x_2^2x_3 + 107x_2x_3^2 - x_3^3) = 0. \] (4.14)

We note that we cannot take \(x_3 = 0\) as a solution of Eq. (4.14) since both \(x_3\) and \(x_4\) cannot be simultaneously 0. Thus, we get just one solution of (4.13) with \(x_2 \neq 0\) and \(x_4 = 0\), namely \((x_2, x_3, x_4) = (1, 1, 0)\). This yields the single rational invariant point \((0, 1, 1, 0)\) on the surface (4.7). As the other rational invariant points occur in pairs, this shows that there are an odd number of rational invariant points on the surface (4.7).

As there are no self-conjugate points on the surface (4.7), it follows that the total number of rational self-conjugate points and rational invariant points is an odd integer. It now follows from Lemma 1 that there are infinitely many rational points on the surface (4.7).
We now obtain a sequence of rational points on the surface \((4.7)\) by repeatedly applying the \(RC\) operation starting from the point \((0, 1, 1, 0)\). The first three points of the sequence are

\[
(0, 1, 1, 0), \quad (-411, 37, 37, 137), \quad (3112824595430551806, 686796656401231307, -183526740019270303, 1115958479906433472).
\]

The coordinates of the next point of the sequence are given by integers consisting of 112 and 113 digits, and are accordingly omitted.

The nonhomogeneous diophantine equation obtained by equating the discriminant of Eq. \((4.9)\) to 0 reduces to the equation,

\[
d(m_1, m_2) + z^2 = 0. \tag{4.15}
\]

The values of \((m_1, m_2, z)\) for the first three solutions of Eq. \((4.15)\), corresponding to the three rational points on the surface \((4.7)\) obtained above, are given by

\[
(0, 1/3, 137/3), \quad (-10138/52607, -16623/52607, 50623360597/2767496449) \quad \text{and} \quad \\
(0, 137/3, 1/3), \quad (-16623/52607, -10138/52607, 52607/2767496449) \quad \text{and} \quad \\
(0, 37/3, 137/3), \quad (686796656401231307, -183526740019270303, 1115958479906433472).
\]

5 Octic and Higher Degree Equations

We now show how to construct a diophantine equation, of arbitrarily high even degree, that can be solved by the method described in Section 2. We will use the quadratic forms \(Q(x_1, x_2), Q_j(x_1, x_2, x_3, x_4), j = 1, 2\) and the quartic form \(F(x_1, x_2, x_3, x_4)\) defined by \((3.1)\) and \((3.5)\) to construct a quaternary form \(G(x_1, x_2, x_3, x_4)\) of degree \(2d\) where \(d\) is an arbitrary integer.

Let \(Q_{j,3}(x_1, x_2, x_3, x_4), j = 0, 1, \ldots, d-1\) be arbitrary quadratic forms in the variables \(x_1, x_2, x_3, x_4\) such that \(x_1\) occurs only in degree 2 in each of
these forms. Let

\[ G(x_1, x_2, x_3, x_4) = \sum_{j=0}^{d-1} Q^{d-1-j}(x_1, x_2)Q^j(x_3, x_4)Q_{j+3}(x_i) \]

\[ + \left\{ \sum_{j=0}^{d-2} b_j Q^{d-2-j}(x_1, x_2)Q^j(x_3, x_4) \right\} F(x_1, x_2, x_3, x_4), \tag{5.1} \]

where \( b_j, j = 0, 1, \ldots, d-2 \) are arbitrary integers.

We now consider the diophantine equation,

\[ G(x_1, x_2, x_3, x_4) = 0. \tag{5.2} \]

If we substitute the values of \( x_3 \) and \( x_4 \) given by (3.2) in Eq. (5.2), it follows from the relations (3.3) and (3.5) that Eq. (5.2) reduces to an equation of type (2.3) where \( \psi \) is an arbitrary parameter. We also note that if we take \( p = 0 \), then \( x_1 \) occurs only in even degrees in the forms \( Q(x_1, x_2), Q_{j+3}(x_i) \) and \( F(x_i) \), and therefore \( G(x_1, x_2, x_3, x_4) = G(-x_1, x_2, x_3, x_4) \). Finally, we can readily choose the forms \( Q_{j+3}(x_i) \), the integers \( b_j \) and the arbitrary integer coefficients of the form \( F(x_1, x_2, x_3, x_4) \) such that Eq. (5.2) has a solution in integers and the remaining conditions stipulated in property \( \text{D}_3 \) are satisfied.

Thus we can construct a diophantine equation, of arbitrary degree 2d, satisfying all the three properties \( \text{D}_1, \text{D}_2 \) and \( \text{D}_3 \), and we can obtain a sequence of rational solutions starting from a known solution.

As a numerical example, in (5.1), we take \( d = 5, p = 0, q = 2, Q_3(x_i) = x_1^2 + x_2^2, Q_4(x_i) = 2x_1^2 + 3x_2^2, Q_5(x_i) = Q_6(x_i) = 0, Q_7(x_i) = -(x_3^2 + 2x_4^2) \), and the coefficients \( a_j \) of the form \( F(x_1, x_2, x_3, x_4) \) as follows:

\[ a_1 = 1, \quad a_2 = -1, \quad a_3 = -2, \quad a_4 = h, \]
\[ a_5 = h, \quad a_6 = -1, \quad a_7 = -2, \quad a_8 = 2h, \]
\[ a_9 = 0, \quad a_{10} = -1, \quad a_{11} = 0, \quad a_{12} = -2, \]

where \( h \) is an arbitrary parameter.

We thus get the following equation of degree 10:

\[ (x_1^2 + 2x_2^2)^4(x_1^2 + x_2^2) + (x_1^2 + 2x_2^2)^3(x_3^2 + 2x_4^2)(2x_1^2 + 3x_2^2) \]
\[ - (x_3^2 + 2x_4^2)^5 + (x_1^2 + 2x_2^2)^3\{x_1^4 + 4x_1^2x_2^2 + 4x_2^4 \}
\[ - 4x_2^2x_3^2 - 8x_2^2x_4^2 - x_3^4 - 4x_3^2x_4^2 - 4x_4^4 \]
\[ + 2hx_2(x_1^2x_3 - 2x_1^2x_4 + 2x_2^2x_3 - 4x_2^2x_4 + x_3^3 + 2x_3x_4^2) \} = 0. \tag{5.3} \]

It is readily verified that Eq. (5.3) satisfies the three properties \( \text{D}_1, \text{D}_2 \) and \( \text{D}_3 \) and for any arbitrary value of \( h, (x_1, x_2, x_3, x_4) = (1, 1, 1, 1) \) is a solution of Eq. (5.3).
We may now repeatedly apply the RC operation and find a sequence of rational points on the surface \([5.3]\). The coordinates of these points are rational functions of \(h\). As in the case of Eq. \([3.6]\), we will now prove that given an arbitrary positive integer \(n\), however large, there exists an integer value of \(h\) such that these \(n\) points are distinct. The proof is similar to the one given in case of Eq. \([3.6]\).

When \(h = 0\), Eq. \([5.3]\) reduces to the following equation:

\[
(x_1^2 + 2x_2^2)^4(x_1^2 + x_2^2) + (x_1^2 + 2x_2^2)(x_3^2 + 2x_1^2)(2x_1^2 + 3x_2^2) \\
- (x_3^2 + 2x_2^2)^5 + (x_1^2 + 2x_2^2)(x_1^4 + 4x_1^2x_2^2 + 4x_2^4) \\
- 4x_2^2x_3 - 8x_2^2x_4 - x_4 - 4x_2^2x_4 - 4x_2^4) = 0. \quad (5.4)
\]

On substituting

\[
x_3 = m_1x_1 + 2m_2x_2, \quad x_4 = m_2x_1 - m_1x_2, \quad (5.5)
\]

Eq. \([5.4]\) reduces, after removing the factor \((x_1^2 + 2x_2^2)^4\) to the following equation:

\[
(m_1^2 + 2m_2^2 + 1)(m_1^8 + 8m_1^4m_2^2 + 24m_1^4m_2^4 + 32m_1^2m_2^6 + 16m_2^8) \\
- m_1^6 - 6m_1^4m_2^2 - 12m_1^2m_2^6 - 8m_2^6 + m_1^4 + 4m_1^2m_2^2 + 4m_2^4 - 2)x_1^2 \\
+ (2m_1^{10} + 20m_1^8m_2^2 + 80m_1^6m_2^4 + 160m_1^4m_2^6 + 160m_1^2m_2^8 \\
+ 64m_2^{10} + 2m_1^4 + 8m_1^2m_2^6 + 8m_2^4 + m_1^2 + 2m_2^2 - 3)x_2^2 = 0. \quad (5.6)
\]

Let a rational point \(P\) on the surface \([5.4]\) be given by \((3^t, 3^t, \alpha, \beta, )\) where \(\alpha, \beta\) are integers such that \(\alpha \equiv 1 \pmod 3\), \(\beta \equiv 1 \pmod 3\), and \(t\) is a nonnegative integer. The values of \(m_1, m_2\) corresponding to the point \(P\) are given by

\[
m_1 = (\alpha - 2\beta)/3^{t+1}, \quad m_2 = (\alpha + \beta)/3^{t+1}. \quad (5.7)
\]

With these values of \(m_1, m_2\), we now work out the conjugate \(C(P)\) of the point \(P\). It follows from \([5.6]\) that

\[
\frac{x_1(P)}{x_2(P)} + \frac{x_1(C(P))}{x_2(C(P))} = 0.
\]

Since \(x_1(P) = x_2(P) = 3^k\), we get \(-x_1(C(P)) = x_2(C(P)) = k\) where \(k\) is an arbitrary integer, and we obtain the values of \(x_3(C(P))\, x_4(C(P))\) using the relations \([5.5]\). We thus obtain the point \(C(P)\), and on taking the reflection of this point, we obtain the point \(RC(P)\) which is given by \((3^{t+1}, 3^{t+1}, \alpha', \beta')\) where \(\alpha' \equiv -1 \pmod 3\), \(\beta' \equiv -1 \pmod 3\). On applying the \(RC\) operation once again, we get the point \((RC)^2(P)\) which is given by \((3^{t+2}, 3^{t+2}, \alpha'', \beta'')\) where \(\alpha'' \equiv 1 \pmod 3\), \(\beta'' \equiv 1 \pmod 3\) and it now follows by induction that the sequence of points on the surface \([5.4]\), obtained by starting from \((1, 1, 1, 1)\) as
the initial point $P_0$, consists of infinitely many distinct rational points. The first four points of the sequence obtained in this manner are $P_0 = (1, 1, 1, 1)$, $P_1 = (3, 3, 5, -1)$, $P_2 = (3^2, 3^2, 1, -11)$, $P_3 = (3^3, 3^3, -43, -13)$.

Now reverting to the original equation, an argument similar to the one given in case of Eq. (3.6) establishes that for any given arbitrary value of $n$, however large, there exist infinitely many integer values of $h$ such that Eq. (5.3) has at least $n$ distinct rational solutions.

We note if the repeated application of the $RC$ operation yields an infinite sequence of rational points on a surface defined by an equation of type (5.2) where we take $p = 0$, there will always be a curve of genus 0 or 1. It may, however, be noted that efforts to construct such equations of degree $\geq 8$ led only to such surfaces on which there is a curve of genus 0 or 1.
6 Overview of an extension of the new method

6.1

We now describe an extension of the method discussed in Section 2 and show how it can be used to solve certain single diophantine equations as well as diophantine systems consisting of several simultaneous equations of the type,

\[ f_r(x_1, x_2, \ldots, x_n) = 0, \quad r = 1, 2, \ldots, k, \]  

(6.1)

where \( n \geq 5 \), \( k < n - 2 \) and \( f_r(x_i) \) are forms, with integer coefficients, in the \( n \) independent variables \( x_1, x_2, \ldots, x_n \), and at least one of the forms \( f_r(x_i) \) is of degree \( d \geq 4 \).

We denote by \( V \) the set of all rational solutions of the Eqs (6.1) and assume that the following properties, analogous to the three properties mentioned in Section 2, are satisfied:

\( D'_1 \) : At least one solution \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of Eqs. (6.1) is known such that \((\alpha_1, \alpha_2) \neq (0, 0)\).

\( D'_2 \) : There exists a bijective mapping \( R : V \to V \) distinct from the identity mapping.

\( D'_3 \) : It is possible to set up \( n - k - 1 \) auxiliary equations

\[ \psi_j(x_1, x_2, \ldots, x_n, m_1, \ldots, m_{n-k}) = 0, \quad j = 1, 2, \ldots, n - k - 1, \]  

(6.2)

such that

(i) for rational numerical values of \( x_1, x_2, \ldots, x_n \) such that \((x_1, x_2) \neq (0, 0)\), Eqs. (6.2) can be solved to obtain rational numerical values, not all 0, of the variables \( m_1, \ldots, m_{n-k-1} \);

(ii) Eqs. (6.2) can be solved together with the equations \( f_r(x_1, x_2, \ldots, x_n) = 0, \quad r = 1, 2, \ldots, k - 1 \) to obtain the values of \( x_3, x_4, \ldots, x_n \) in terms of \( x_1, x_2, m_1, m_2, \ldots, m_{n-k-1} \) and on substituting these values, the last equation \( f_k(x_1, x_2, \ldots, x_n) = 0 \), reduces to an equation of the type

\[ \phi_0(m_i)x_1^2 + \phi_1(m_i)x_1x_2 + \phi_2(m_i)x_2^2 = 0, \]  

(6.3)

where \( \phi_j(m_i), \quad j = 0, 1, 2 \) are polynomials in \( m_1, m_2, \ldots, m_{n-k-1} \) such that the simultaneous equations \( \phi_j(m_i) = 0, \quad j = 0, 1, 2 \) do not have any rational solutions.

When the three properties \( D'_1, D'_2, D'_3 \) are satisfied, we can obtain a sequence of rational solutions of the simultaneous equations (6.1).

The method of solving Eqs. (6.1) is similar to that of solving Eq. (2.1). Given any point \( P \) on the projective variety defined by Eqs. (6.1), we can, in general, obtain two new points on the variety — the point \( R(P) \) and the
conjugate point \( C(P) \) and we can, as before, perform the \( R \) and \( C \) operations repeatedly to get new points on the variety. While \( R(P) \) is immediately obtained from the definition of the mapping \( R \), we will describe below how to find the conjugate of a given point \( P \). The properties of the \( R \) and \( C \) operations and the definition of the order of a point \( P \) discussed in Section 2 are also applicable in the case of simultaneous equations in several variables.

Starting from a known rational point \( P \), with coordinates \((\alpha_1, \alpha_2, \ldots, \alpha_n)\), on the variety \( (6.1) \), we find the numerical values of \( m_1, m_2, \ldots, m_{n-k-1} \) corresponding to the point \( P \) by solving the \( n-k-1 \) equations,

\[
\psi_j(\alpha_1, \alpha_2, \ldots, \alpha_n, m_1, \ldots, m_{n-k-1}) = 0, \quad j = 1, 2, \ldots, n-k-1, \quad (6.4)
\]

and with these values of \( m_i \) (which are not all 0), the quadratic equation \( (6.3) \) is necessarily solvable — one solution corresponding to the known point \( P \) is \( (x_1, x_2) = (\alpha_1, \alpha_2) \), while the second solution gives the \( x_1 \) and \( x_2 \) coordinates of the conjugate point \( C(P) \). These values of \( x_1 \) and \( x_2 \) together with the values of \( m_1, m_2, \ldots, m_{n-k-1} \), yield the remaining coordinates \( x_3, x_4, \ldots, x_n \) of the point \( C(P) \). We thus obtain the conjugate point \( C(P) \).

We can now obtain the point \( RC(P) \), and on repeatedly applying the \( RC \) operation, we obtain a sequence of rational points,

\[
P, \quad RC(P), \quad (RC)^2(P), \quad (RC)^3(P), \ldots, \quad (RC)^k(P), \ldots, \quad (6.5)
\]
on the variety defined by \( (6.1) \).

We note that when there are several mappings \( R : V \rightarrow V \) as happens in the case of symmetric diophantine systems, by using these mappings separately or in combination with each other, we can, starting from a known rational point \( P \), generate several sequences of rational points on the variety defined by \( (6.1) \).

We have stipulated in property \( D'_3 \) that, for arbitrary numerical values of \( x_i \) with \( (x_1, x_2) \neq (0, 0) \), the auxiliary equations \( (6.2) \) should yield a nonzero solution for \( m_i \) and also that the simultaneous equations \( \phi_j(m_i) = 0, j = 0, 1, 2 \) should not have any rational solutions. We now note that we can generate the sequence of rational points \( (6.5) \) even when a weaker form of property \( D'_3 \) is satisfied by Eqs. \( (6.1) \). It is sufficient for the \( RC \) operation to be executed successfully at every stage if we choose the initial point \( P \) and the auxiliary equations in such a manner that the point \( P \) and the rational points of the sequence \( (6.5) \) successively generated at each stage are such that when we substitute the values of the coordinates \( x_i \) of these points in the auxiliary equations \( (6.2) \), these equations can be solved to obtain rational values, not all 0, of the variables \( m_i \), and with these values of \( m_i \), the three coefficients \( \phi_j(m_i) \), \( j = 0, 1, 2 \), of the resulting equation \( (6.3) \) do not vanish simultaneously.
We note that even if we cannot prove in a certain case that the three coefficients \( \phi_j(m_i), j = 0, 1, 2 \) in Eq. (6.3), do not vanish simultaneously, we may still be able to generate a sequence of rational points on the variety (6.1). However, if there actually exist rational values of \( m_i, i = 1, 2, \ldots, n - k - 1 \), such that the three coefficients \( \phi_j(m_i), j = 0, 1, 2 \) of Eq. (6.3) vanish simultaneously, Eq. (6.3) is identically satisfied with these values of \( m_i \), and we get a parametric solution of Eqs. (6.1). Further, if we generate a sequence (6.5) of rational points on the variety (6.1) and at some stage, we get a rational point which yields these values of \( m_i \), the sequence terminates.

As we are interested in constructing varieties that do not have a parametric solution and on which we can find an arbitrarily large number of rational points, it will be important to prove that there are no rational values of \( m_i \) for which the three coefficients \( \phi_j(m_i), j = 0, 1, 2 \), of Eq. (6.3) vanish simultaneously.

If we can successfully generate an infinite sequence of distinct rational points on a given projective variety defined by Eqs. (6.1) by the method described above, we also simultaneously obtain infinitely many rational solutions of Eq. (6.3), and hence also of the diophantine equation

\[
\phi_1^2(m_i) - 4\phi_0(m_i)\phi_2(m_i) = z^2, \quad (6.6)
\]

since Eq. (6.6) must be satisfied for Eq. (6.3) to have rational solutions.

As an example of a diophantine system of type (6.1), we mention the system of equations,

\[
x_1 + x_2 + x_3 = x_4 + x_5 + x_6, \quad (6.7)
\]
\[
x_1^3 + x_2^3 + x_3^3 = x_4^3 + x_5^3 + x_6^3, \quad (6.8)
\]
\[
x_1^4 + x_2^4 + x_3^4 = x_4^4 + x_5^4 + x_6^4, \quad (6.9)
\]

for which a parametric solution is already known [3, pp. 305-6]. It follows from Eqs. (6.7) and (6.8) that

\[
(x_1 + x_2 - x_6)^3 - (x_1^3 + x_2^3 - x_6^3) = (x_4 + x_5 - x_3)^3 - (x_4^3 + x_5^3 - x_3^3),
\]

or

\[
3(x_1 + x_2)(x_1 - x_6)(x_2 - x_6) = 3(x_4 + x_5)(x_4 - x_3)(x_5 - x_3),
\]

and hence, if we write two auxiliary equations,

\[
x_1 - x_6 + m_1(x_3 - x_4) = 0, \quad (6.11)
\]
\[
x_2 - x_6 + m_2(x_3 - x_5) = 0, \quad (6.12)
\]

and solve the four equations (6.7), (6.8), (6.11), (6.12) for \( x_3, x_4, x_5, x_6 \), and substitute these values in Eq. (6.9), we can reduce our diophantine system to an equation of type (6.3). We note that in view of the symmetry of Eqs. (6.7), (6.8) and (6.9), there are several choices for the mapping \( R \). Thus starting
from the parametric solution of the simultaneous equations (6.7), (6.8) and (6.9) given in [3], we can obtain several sequences of parametric solutions of this diophantine system.

Sequences of solutions for several other solvable diophantine systems, such as $x_r^1 + x_r^2 + x_r^3 = x_r^4 + x_r^5 + x_r^6$, $r = 1, 5$ and $x_r^1 + x_r^2 + x_r^3 = x_r^4 + x_r^5 + x_r^6$, $r = 2, 6$, may be obtained in a similar manner. As parametric solutions of these diophantine systems are already known (see, for instance, [1], [2], [4], [5]), we will not obtain any new solutions of these systems. These examples are mentioned here just to illustrate the applicability of the general method to various diophantine systems.

We will now focus on constructing a projective variety (6.1) on which we can find an arbitrarily large number of rational points and on which the existence of a curve of genus 0 or 1 is not certain. In the next subsection we will construct projective varieties defined by equations of type (6.1) in which one of the forms $f_r(x_i)$ is of arbitrarily high degree while the other forms $f_r(x_i)$ are all linear forms.

6.2

It will be convenient to consider projective varieties in $2n$ independent variables. We will accordingly consider varieties defined by equations of the type

$$f(x_1, x_2, \ldots, x_{2n}) = 0,$$

$$L_j(x_1, x_2, \ldots, x_{2n}) = 0, \quad j = 1, 2, \ldots, n - 2,$$

where $f(x_i)$ is a form of degree $d \geq 4$ and $L_j(x_i)$, $j = 1, 2, \ldots, n - 2$, are $n - 2$ linear forms in the $2n$ variables $x_1, x_2, \ldots, x_{2n}$.

We will illustrate how we can construct systems of simultaneous equations (6.13) and (6.14) satisfying the three properties $D_1', D_2', D_3'$ with the first equation (6.13) being of arbitrarily high degree.

We first obtain a composition of forms identity of the type,

$$\psi(x_1, x_2, \ldots, x_n)\psi(m_1, m_2, \ldots, m_n) = \psi(x_{n+1}, x_{n+2}, \ldots, x_{2n})$$

where $n \geq 3$ and $\psi(x_1, x_2, \ldots, x_n)$ is a form in $n$ variables $x_1, x_2, \ldots, x_n$, and the values of the variables $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ are given in terms of bilinear forms $B_j(m_i, x_i), j = 1, 2, \ldots, n$ in the variables $m_1, m_2, \ldots, m_n$ and $x_1, x_2, \ldots, x_n$.

One way of obtaining an identity of type (6.15) is by using algebraic integers belonging to the field $\mathbb{Q}(\rho)$ where $\rho$ is a root of the equation,

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$
with $p_1, p_2, \ldots, p_n$ being arbitrary rational integers such that Eq. (6.16) is irreducible. We will denote the norm of an algebraic integer $ξ$ by $N(ξ)$. If $α, β, γ$ are three nonzero integers of the field $Q(ρ)$ given by

$$
α = x_1 + x_2ρ + x_3ρ^2 + \cdots + x_nρ^{n-1},
β = m_1 + m_2ρ + m_3ρ^2 + \cdots + m_nρ^{n-1},
γ = x_{n+1} + x_{n+2}ρ + x_{n+3}ρ^2 + \cdots + x_{2n}ρ^{n-1},
$$

(6.17)
such that $αβ = γ$, then the values of $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ are given by

$$
x_{n+j} = B_j(m_1, m_2, \ldots, m_n, x_1, x_2, \ldots, x_n), \ j = 1, 2, \ldots, n,
$$

(6.18)
where $B_j(m_i, x_i), \ j = 1, 2, \ldots, n$, are bilinear forms in the variables $m_1, m_2, \ldots, m_n$ and $x_1, x_2, \ldots, x_n$ and the relation $N(α)N(β) = N(γ)$ yields a composition of forms identity of the type (6.15) in which $ψ(x_1, x_2, \ldots, x_n) = N(α)$ is a form of degree $n$.

Next, for some positive integer $s$, we obtain $s$ identities of the type,

$$
φ_j(x_1, x_2, \ldots, x_{2n}) = ζ_j(m_1, m_2, \ldots, m_n)ψ(x_1, x_2, \ldots, x_n),
$$

(6.19)
where $φ_j(x_1, x_2, \ldots, x_{2n}), \ j = 1, 2, \ldots, s$, are $s$ forms of degree $n$ in the $2n$ variables $x_1, x_2, \ldots, x_{2n}$ while $ζ_j(m_i), \ j = 1, 2, \ldots, s$, are polynomials in $m_1, m_2, \ldots, m_n$ and the identities are true when the values of $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ are given by (6.18).

We can readily obtain $n$ identities of type (6.19) by considering (6.18) as $n$ linear equations in the variables $m_1, m_2, \ldots, m_n$ and solving these equations for $m_1, m_2, \ldots, m_n$, when we get $m_j = φ_j(x_i)/ψ(x_1, x_2, \ldots, x_n), \ j = 1, 2, \ldots, n$, where $φ_j(x_i)$ are forms of degree $n$ in the $2n$ variables $x_1, x_2, \ldots, x_{2n}$ and hence we immediately get $n$ identities of the type (6.19). More identities of type (6.19) can also be obtained.

We now construct Eq. (6.13) in which we take the form $f(x_1, x_2, \ldots, x_{2n})$ as follows:

$$
f(x_1, x_2, \ldots, x_{2n}) = ψ(x_1, x_2, \ldots, x_n)Q_1(x_i) + ψ(x_{n+1}, \ldots, x_{2n})Q_2(x_i)
+ \sum_{j=1}^{s} φ_j(x_1, x_2, \ldots, x_{2n})Q_{j+2}(x_i),
$$

(6.20)
where $Q_j(x_i), \ j = 1, 2, \ldots, s + 2$ are arbitrary quadratic forms in the variables $x_1, x_2, \ldots, x_{2n}$. We note that the form $f(x_1, x_2, \ldots, x_{2n})$ is of degree $n + 2$ where $n$ is any arbitrary positive integer greater than 2.

We will take Eqs. (6.18) as our $n$ auxiliary equations. When we substitute the values of $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ given by (6.18) in Eq. (6.13), in view of the identities (6.15) and (6.19), we find that $ψ(x_1, x_2, \ldots, x_n)$ is a factor
of \( f(x_1, x_2, \ldots, x_n) \). We note that \( \psi(x_1, x_2, \ldots, x_n) \), being the norm of a nonzero algebraic integer, cannot be 0, and hence, on removing this factor, Eq. (6.13) reduces to

\[
Q_1(x_i) + \psi(m_1, m_2, \ldots, m_n)Q_2(x_i) + \sum_{j=1}^{s} \zeta_j(m_1, m_2, \ldots, m_n)Q_{j+2}(x_i) = 0.
\]  
(6.21)

We will reduce this equation further after we have chosen the linear equations (6.14).

We will now illustrate how we can choose the form \( \psi(x_1, x_2, \ldots, x_n) \) and the quadratic forms \( Q_j(x_i) \) in (6.20), as well as the linear equations (6.14) such that the diophantine system given by Eqs. (6.13) and (6.14) satisfies the properties \( D'_1, D'_2 \) and \( D'_3 \). As a simple example, we take the \( n-2 \) linear equations (6.14) as

\[
x_j = 0, \quad j = 3, 4, \ldots, n,
\]  
(6.22)

and the quadratic forms \( Q_j(x_i) \) as follows:

\[
Q_1(x_i) = \sum_{i=1}^{n} a_i x_i^2, \quad Q_2(x_i) = \sum_{i=n+1}^{2n} a_i x_i^2,
\]

\[
Q_j(x_i) = 0, \quad j = 3, 4, \ldots, s + 2,
\]

where the coefficients \( a_i \) are arbitrary integers such that \( a_1 < 0 \) and \( a_i > 0, \quad i = 2, 3, \ldots, 2n. \)

On substituting \( x_j = 0, \quad j = 3, 4, \ldots, n \) in the form \( \psi(x_1, x_2, \ldots, x_n) \), we get,

\[
\psi(x_1, x_2, 0, 0, \ldots, 0) = x_1^n + p_1 x_1^{n-1} x_2 + p_2 x_1^{n-2} x_2^2 + \cdots + p_n x_2^n.
\]  
(6.24)

We will take \( n \) as even and choose the integers \( p_j \) such that \( p_j = 0 \) for all odd values of \( j \) so that

\[
\psi(x_1, x_2, 0, 0, \ldots, 0) = \psi(-x_1, x_2, 0, 0, \ldots, 0).
\]

Now (6.13) may be written as

\[
\psi(x_1, x_2, 0, 0, \ldots, 0)(a_1 x_1^2 + a_2 x_2^2) + \psi(x_{n+1}, \ldots, x_{2n}) \left( \sum_{i=n+1}^{2n} a_i x_i^2 \right) = 0.
\]  
(6.25)

It follows that if \( (\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \) is a solution of our diophantine system, then \( (-\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \) is also a solution. Thus, we may define \( R \) as the reflection mapping given by

\[
R(\alpha_1, \alpha_2, \ldots, \alpha_{2n}) = (-\alpha_1, \alpha_2, \ldots, \alpha_{2n}),
\]  
(6.26)
and now our diophantine system satisfies the property $D_2$.

We have already used the auxiliary equations (6.13) to reduce Eq. (6.13) to Eq. (6.21). We now solve Eqs. (6.18) and the linear equations (6.22) for $x_3, x_4, \ldots, x_{2n}$ and on substituting the values thus obtained in Eq. (6.21), we get

$$a_1 x_1^2 + a_2 x_2^2 + \psi(m_1, m_2, \ldots, m_n)Q'_2(x_1, x_2) = 0,$$  \hspace{1cm} (6.27)

where we note that $Q'_2(x_1, x_2)$ is necessarily a positive definite form in the variables $x_1, x_2$.

We have now reduced Eq. (6.13) to an equation of type (6.3). We will show that Eq. (6.27) cannot be identically 0 for any rational numerical values of $m_1, m_2, \ldots, m_n$ and all values of $x_1$ and $x_2$. If this were to happen when $m_i = \mu_i$, $i = 1, 2, \ldots, n$, and $\psi(\mu_1, \mu_2, \ldots, \mu_n) > 0$, on taking $(x_1, x_2) = (0, 1)$, the left-hand side of Eq. (6.27) becomes positive, while if $\psi(\mu_1, \mu_2, \ldots, \mu_n) < 0$, on taking $(x_1, x_2) = (1, 0)$, the left-hand side of Eq. (6.27) becomes negative, and hence, in either case, we have a contradiction.

We have thus reduced our diophantine system to an equation of type (6.3) in which the coefficients of $x_1^2, x_1 x_2$ and $x_2^2$ do not vanish simultaneously for any rational numerical values of $m_1, m_2, \ldots, m_n$. Thus, our diophantine system satisfies condition (ii) of property $D'_3$.

We now choose the arbitrary integers $a_i$ such that Eqs. (6.13) and (6.14) have a solution $(\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ where $(\alpha_1, \alpha_2) \neq (0, 0)$ and also $(\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{2n}) \neq (0, 0, \ldots, 0)$.

Now starting from the known point $P = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ on the variety defined by Eqs. (6.13) and (6.14), we first find the values of $m_1, m_2, \ldots, m_n$ by solving the equations

$$\alpha_{n+j} = B_j(m_i, \alpha_i), \hspace{1cm} j = 1, 2, \ldots, n. \hspace{1cm} (6.28)$$

We note that for rational numerical values of $x_1, x_2, \ldots, x_n$, not all 0, the linear forms $B_j(m_i, x_i), \hspace{1cm} j = 1, 2, \ldots, n$, are linearly independent since $N(\alpha) \neq 0$. Thus, Eqs. (6.28) are solvable, and since $(\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{2n}) \neq (0, 0, \ldots, 0)$, we obtain a solution in which the values of $m_1, m_2, \ldots, m_n$ are not all 0. With these values of $m_i$, Eq. (6.27) has two solutions corresponding to the point $P$ and its conjugate $C(P)$. We thus get a solution for the $x_1$ and $x_2$ coordinates of the point $C(P)$ such that $(x_1, x_2) \neq (0, 0)$. For all points on the variety, we have $x_j = 0$, $j = 3, 4, \ldots, x_n$, and now we use the relations (6.18) to get the remaining coordinates of the point $C(P)$. We now note that for rational numerical values of $m_1, m_2, \ldots, m_n$, not all 0, the linear forms $B_j(m_i, x_i), \hspace{1cm} j = 1, 2, \ldots, n$, are linearly independent since $N(\beta) \neq 0$. Since $(x_1, x_2) \neq (0, 0)$ and the values of $m_i$ are also not all 0, it follows that the coordinates $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ of the point $C(P)$ cannot be simultaneously
0. Now on taking the reflection of \( C(P) \), we get the point \( RC(P) \) which is such that its \( x_1, x_2 \) coordinates are not simultaneously 0 and the last \( n \) coordinates, \( x_{n+1}, x_{n+2}, \ldots, x_{2n} \), are also not simultaneously 0.

Thus when we take a point \( P \) whose \( x_1, x_2 \) coordinates are not simultaneously 0 and whose last \( n \) coordinates are also not simultaneously 0, the auxiliary equations (6.18) can be solved to obtain rational numerical values, not all 0, of \( m_1, m_2, \ldots, m_n \) and on applying the \( RC \) operation, we get the point \( RC(P) \) whose coordinates satisfy the same conditions. We can therefore repeatedly apply the \( RC \) operation, and the rational points successively generated at each stage satisfy the same conditions as the coordinates of the initial point \( P \), and thus the weaker form of condition (i) of property \( D'_3 \) is satisfied.

Thus starting from the point \( P \), we may apply the \( RC \) operation any number of times to obtain a sequence of rational points,

\[
P, \ RC(P), \ (RC)^2(P), \ (RC)^3(P), \ldots, \ (RC)^k(P), \ldots
\]

on the variety defined by Eqs. (6.13) and (6.14).

Each of these points represents a solution not only of Eqs. (6.13) and (6.14) but also of the single diophantine equation of degree \( n + 2 \) in \( n + 2 \) variables obtained by eliminating the \( n - 2 \) variables \( x_3, x_4, \ldots, x_n \) from Eqs. (6.13) and (6.14). This diophantine equation is given by Eq. (6.25).

Thus, for any arbitrary even integer \( n \) however large, we have constructed a projective variety defined by an equation of degree \( n + 2 \) in \( n + 2 \) variables such that starting from a known point \( P \) on the variety, we can generate an arbitrarily long sequence of rational points on the variety. It is expected that, in general, this sequence consists of infinitely many distinct rational points.

As a numerical example, taking \( \rho \) as a root of the equation \( x^6 + 2 = 0 \),
we get a composition of forms identity of type Eq. (6.13) in which

\[
\psi(x_1, x_2, \ldots, x_6) = x_1^6 + (12 x_2 x_6 + 12 x_3 x_5 + 6 x_4^2) x_1^4 \\
- (12 x_2^2 x_5 + 24 x_2 x_3 x_4 + 4 x_5^3 - 24 x_3 x_6^2 - 48 x_2 x_5 x_6 - 8 x_3^2) x_1^3 \\
+ (12 x_2^3 x_4 + 18 x_2^2 x_3^2 + 36 x_2 x_3 x_6^2 - 72 x_2 x_4 x_5^2 - 72 x_2^2 x_4 x_6) \\
+ 36 x_2^3 x_5^2 + 12 x_4^4 + 48 x_4 x_3^3 + 72 x_5^2 x_6^2 x_1^2 \\
- (12 x_2^2 x_3 + 48 x_2 x_5 x_6 - 72 x_2^2 x_5 x_6 - 48 x_3 x_4 x_6) \\
+ 48 x_2 x_3 x_4^3 - 96 x_2 x_3 x_5^3 + 96 x_2 x_5 x_6^2 + 24 x_3 x_5 - 24 x_3 x_4^2 \\
+ 144 x_4 x_2 x_3 x_5^2 - 48 x_3 x_5^3 - 96 x_4 x_5 x_6^2 + 48 x_4 x_5^3 - 96 x_5 x_6^3) x_1 \\
+ 2 x_5^6 + 24 x_2 x_4 x_6 + 12 x_2^2 x_5^2 - 24 x_2 x_3 x_6^2 - 48 x_3 x_4 x_5 x_6 \\
- 8 x_3 x_5^3 + 16 x_4 x_3 x_6^2 + 24 x_2 x_3 x_5^2 + 36 x_2 x_3 x_4^2 \\
- 144 x_2 x_3 x_5 x_6^2 + 72 x_2 x_3 x_5^2 x_6 + 24 x_2 x_4 x_6 - 24 x_2 x_3 x_4 \\
+ 144 x_2 x_3 x_5 x_6^2 - 96 x_2 x_3 x_4 x_5^2 - 48 x_2 x_4 x_6 + 48 x_2 x_3 x_4^2 \\
+ 96 x_2 x_3 x_4 x_5^2 - 96 x_2 x_5 x_6^2 + 4 x_5 + 24 x_4 x_5^2 - 96 x_3 x_4 x_5 x_6 \\
- 16 x_2 x_5^3 + 48 x_3 x_4 x_6^2 + 72 x_2 x_5 x_6^2 + 48 x_3 x_5^3 \\
- 48 x_3 x_4 x_5 - 192 x_3 x_4 x_5 x_6 + 96 x_3 x_5 x_6^2 + 8 x_6 \\
- 32 x_2 x_5^3 + 144 x_2 x_5 x_6^2 - 96 x_4 x_5 x_6^2 + 16 x_6^3 + 32 x_6^6. \tag{6.29}
\]

As described above, we construct the following diophantine equation of degree 8 in the 8 variables \(x_1, x_2, x_7, x_8, \ldots, x_{12}\):

\[
(x_1^6 + 2 x_2^6)(-x_1^6 + 3 x_2^3) = \psi(x_7, x_8, \ldots, x_{12}) (x_7^2 + x_8^2 + x_9^2 + x_{10}^2 + x_{11}^2 + 3 x_{12}^2). \tag{6.30}
\]

Starting from the point \(P\) given by \((x_1, x_2, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) = (1, 1, 0, 0, 0, 0, 0, 1)\), we get an arbitrarily long sequence of rational points on the variety \((6.30)\). The next two points \(RC(P)\) and \((RC)^2 P\) of this sequence are given by \((-19, 109, -60, 60, -60, 60, -60, 79)\) and

\[
(-521379318716593857204393, 556588462311221678035287, \\
-163345550361845074758840, 174763856538766110586320, \\
-109258836892219115576040, 485050791706620297477120, \\
1670808317491786482902760, 5483216674456163821019527).
\]

The next point \((RC)^3 P\) involves integers consisting of 251 digits and is therefore omitted. There seems to be no doubt that the above sequence of rational points on the variety \((6.30)\) contains infinitely many distinct rational points but this remains to be proved.

In the next two sections, we give examples of projective varieties whose construction is a little more involved, and we prove that there exists an arbitrarily long sequence of distinct rational points on these varieties.
As a natural extension of our earlier notation, we will denote the individual coordinates of the points \( P, C(P), R(P) \) and \( RC(P) \) by \( x_i(P), x_i(CP), x_i(RP) \) and \( x_i(RCP), i = 1, 2, \ldots, 2n \), respectively.

### 7 Examples based on composition of quartic forms

If the quartic form \( F(x_1, x_2, x_3, x_4) \) in four variables \( x_1, x_2, x_3, x_4 \) is defined by

\[
F(x_1, x_2, x_3, x_4) = x_1^4 - 2px_1^3x_3 + px_1^2x_2^2 - (2p^2 - 4q)x_1^2x_2x_4
+ (p^2 + 2q)x_2^3 + p(p^2 - 3q)x_1^2x_4 - 4qx_1^2x_3x_4 + 4px_1x_2x_3x_4
- 2pqx_1x_3 - 2q(p^2 - 2q)x_1x_3x_4^2 + qx_2^4 - 2pqx_2^3x_4 + pqx_2^2x_3^2
+ q(p^2 + 2q)x_2^2x_4^2 - 4q^2x_2x_3x_4^2 + 2pq^2x_2x_4^3 + q^2x_3^4 + pq^2x_3^2x_4^2 + q^3x_4^4,
\]

we have the identity

\[
F(x_1, x_2, x_3, x_4)F(m_1, m_2, m_3, m_4) = F(x_5, x_6, x_7, x_8),
\]

where the values of \( x_5, x_6, x_7, x_8 \) are given by the relations,

\[
\begin{align*}
x_5 &= m_1x_1 - qm_4x_2 - qm_3x_3 - q(m_2 - pm_4)x_4, \\
x_6 &= m_2x_1 + m_1x_2 - qm_4x_3 - qm_3x_4, \\
x_7 &= m_3x_1 + (m_2 - pm_4)x_2 + (m_1 - pm_3)x_3 \\
&- \{pm_2 - (p^2 - q)m_4\}x_4, \\
x_8 &= m_4x_1 + m_3x_2 + (m_2 - pm_4)x_3 + (m_1 - pm_3)x_4,
\end{align*}
\]

with \( p, q \) being arbitrary rational parameters.

The identity (7.2) may be obtained, as indicated in Section 6.2, by using norms of algebraic integers belonging to the field \( \mathbb{Q}(\rho) \) where \( \rho \) is a root of the equation,

\[
x^4 + px^2 + q = 0,
\]

where \( p, q \) are arbitrary rational integers such that Eq. (7.4) is irreducible.

As mentioned in Section 6.2, we solve the four equations (7.3) for \( m_1, m_2, m_3, m_4 \) and thus obtain the following identities:

\[
\begin{align*}
G_1(x_i) &= m_1F(x_1, x_2, x_3, x_4), & \quad G_2(x_i) &= m_2F(x_1, x_2, x_3, x_4), \\
G_3(x_i) &= m_3F(x_1, x_2, x_3, x_4), & \quad G_4(x_i) &= m_4F(x_1, x_2, x_3, x_4),
\end{align*}
\]
where $G_j(x_i), j = 1, 2, 3, 4$, are quartic forms in the 8 variables $x_1, x_2, \ldots, x_8$, given by

$$G_1(x_i) = x_1^3 + (qx_2x_8 - 2px_3x_5 + qx_3x_7 + qx_4x_6 - pqx_4x_8)x_1^2 + (px_2^2x_5 - qx_2^2x_7 - 2qx_2x_3x_6 - 2p^2x_2x_4x_5 + 2qx_2x_4x_5 + 2pqx_2x_4x_7 + p^2x_3^2x_5 + qx_2^2x_5 - pqx_3^2x_7 + 2q^2x_3x_4x_8 + p^3x_4^2x_5 - 2pqx_4^2x_5 - p^2qx_4^2x_7 + q^2x_4^2x_7)x_1 + q(x_4^2x_6 - x_2^2x_3x_5 - 2px_2^2x_4x_6 + qx_2^2x_4x_8 + px_3^2x_6 - qx_2x_3^2x_8 + 2px_2x_3x_4x_5 - 2qx_2x_3x_4x_7 + p^2x_2x_4^2x_6 + qx_2x_4^2x_6 - pqx_2^2x_4x_8 - px_3^3x_5 + qx_3^3x_7 - qx_4^3x_4x_6 - p^2x_3^2x_5^2 + qx_3x_4^2x_7 - pqx_3^2x_4x_7 - pqx_4^2x_6 + q^2x_4^2x_8),$$

(7.6)

$$G_2(x_i) = x_1^3 + (px_2^2x_6 - qx_2^2x_8 + 2px_2x_3x_5 - 2qx_2x_3x_7 - 2p^2x_2x_4x_6 + 2qx_2x_4x_6 + 2pqx_2x_4x_7 + p^2x_3^2x_6 + qx_2^2x_6 - pqx_3^2x_7) + p^3x_4^2x_5 - 2pqx_4^2x_5 - p^2qx_4^2x_7 + q^2x_4^2x_7)x_1 + p^2x_2^2x_7 - 2pqx_2^2x_7 + qx_2^2x_7 - pqx_3^2x_6 + q^2x_3^2x_8 + pqx_3^2x_4x_5 - q^2x_4^2x_7 + q^3x_4^2x_6 - p^2qx_3^2x_6 + pq^2x_3^2x_8 + p^2x_4^2x_7 + p^2x_4^2x_6 - p^2x_4^2x_7 - pqx_4^2x_6 + q^2x_4^2x_8 + p^2x_4^2x_5 - q^2x_4^2x_5 - pq^2x_4^2x_7,$$

(7.7)

$$G_3(x_i) = x_1^3 + (x_2x_6 - px_2x_8 + x_3x_5 + px_3x_7 - px_4x_6 + p^2x_3x_8 + qx_4x_8)x_1^2 + (x_2^2x_5 - 2qx_2x_3x_8 - 2px_2x_4x_5 + 2px_2x_4x_7 + px_3^2x_5 + qx_2^2x_7 - 2qx_3x_4x_6 + 2pqx_3x_4x_8 + p^2x_4^2x_5 - 2pqx_4^2x_5 - p^2qx_4^2x_7 + q^2x_4^2x_7)x_1 + q(-x_3^2x_8 + x_2^2x_3x_7 + x_2^2x_4x_6 + px_3^2x_4x_8 - x_2^2x_4x_6 - 2x_2x_3x_4x_5 + px_3x_4^2x_6 - qx_2x_4^2x_8 - x_3^2x_5 + x_3^2x_4x_6 + px_3^2x_4x_5 + qx_3^3x_7 + qx_4^3x_4x_6),$$

(7.8)

$$G_4(x_i) = x_1^3 + (x_27 + x_3x_6 + px_3x_5 + x_4x_6 - px_4x_7)x_1^2 + (x_2^2x_5) + 2x_2x_3x_5 + 2px_2x_4x_6 + 2q^2x_2x_4x_8 + px_3^2x_6 + qx_2^2x_5 - 2qx_3x_4x_7 + p^2x_4^2x_6 - qx_4^2x_6 - pqx_4^2x_8)x_1 + p^2x_3^2x_5 + pqx_2x_4^2x_7 - pqx_3^2x_4^2x_6 + pqx_4^3x_5 + q^2x_3x_4^2x_8 - q^2x_4^2x_7 + 2pqx_2x_3x_4x_6 - qx_2x_4^2x_5 - q^4x_2x_4x_5 - q^2x_3x_4x_5 - x_3^2x_5.)$$

(7.9)
Using the forms $G_j(x_i)$, $j = 1, 2, 3, 4$, we construct a form $H(x_i)$ defined as follows:

$$
H(x_i) = (-2x_5x_6 + 3x_6^2 + 2px_6x_7 - 3px_5x_8 - 2qx_7x_8 + 3qx_8^2)G_1(x_i) + (2x_5^2 - 3x_5x_6 - 2px_5x_7 + 3px_5x_8 + 2qx_7^2 - 3qx_7x_8)G_2(x_i) + 2q(x_5x_8 - x_6x_7)G_3(x_i) - 3q(x_5x_8 - x_6x_7)G_4(x_i). \tag{7.10}
$$

We now write $p = h + 3$, $q = h + 4$ where $h$ is an arbitrary integer, and consider the diophantine system given by the following three simultaneous equations:

$$
F^2(x_1, x_2, x_3, x_4)(-x_1^2 + 3x_2^2 + 31x_4^2)
+ h\{F(x_1, x_2, x_3, x_4) + F(x_5, x_6, x_7, x_8)\}H(x_i)
- F^2(x_5, x_6, x_7, x_8)\{x_5 - 2x_7 + 2x_8 + h(x_6 - x_8)\}^2 = 0, \tag{7.11}
$$

$$
x_3 = 0, \tag{7.12}
$$

$$
x_4 = x_2. \tag{7.13}
$$

We note that Eq. (7.11) is of degree 10 in 8 variables.

It is readily verified that

$$
F(x_1, x_2, 0, x_4) = F(-x_1, x_2, 0, x_4), \tag{7.14}
$$

$$
H(x_1, x_2, 0, x_4, \ldots, x_8) = H(-x_1, x_2, 0, x_4, \ldots, x_8). \tag{7.15}
$$

It now follows that if any point $P$ represented by the coordinates $(x_1, x_2, 0, x_4, x_5, x_6, x_7, x_8)$ lies on the variety defined by Eqs. (7.11), (7.12) and (7.13), then the reflection of $P$ given by $R(P) = (-x_1, x_2, 0, x_4, x_5, x_6, x_7, x_8)$ also lies on that variety. Thus, our diophantine system satisfies the property $D_2$.

We will choose the four equations given by (7.3) as our auxiliary equations. In view of the general proof given in Section 6.2 in the context of the auxiliary equations (6.18), it follows that with our choice of auxiliary equations, if we take our initial point $P$ such that the $x_1, x_2$ coordinates of $P$ are simultaneously not 0 and the last four coordinates of $P$ are also not simultaneously 0, when we repeatedly apply the $RC$ operation, we will successively generate rational points whose coordinates satisfy the same conditions as the coordinates of the point $P$, and hence the weaker form of condition (i) of property $D_3'$ is satisfied.

Next, we note that in view of the relations (7.2), (7.5) and (7.10), when we substitute the values of $x_5, x_6, x_7, x_8$ given by (7.3) in Eq. (7.11), then $F^2(x_1, x_2, x_3, x_4)$ can be factored out, and on further substituting the values of $x_3, x_4$ given by Eqs. (7.12) and (7.13), Eq. (7.11) finally reduces to a quadratic equation of type (6.3) in which the coefficient of $x_1^2$ is as follows:

$$
1 + F^2(m_1, m_2, m_3, m_4)\{m_1 - 2m_3 + 2m_4 + h(m_2 - m_4)\}^2
$$

41
We omit the coefficients of \( x_1x_2 \) and \( x_2^2 \) as they are cumbersome to write. Since the coefficient of \( x_1^2 \) is necessarily positive for all rational values of \( m_i \) and \( h \), it follows that our diophantine system completely satisfies condition (ii) of property \( D'_3 \).

It is readily verified that for any arbitrary value of \( h \), a solution of Eqs. (7.11), (7.12) and (7.13) is given by \((3, 1, 0, 1, 3, 1, 0, 1)\). Thus our diophantine system also satisfies the property \( D'_1 \).

We will now show that for infinitely many integer values of \( h \), the diophantine system given by equations (7.11), (7.12) and (7.13) has an arbitrarily large number of integer solutions.

We first consider the special case of the projective variety defined by Eqs. (7.11), (7.12) and (7.13) when \( h = 0 \). Now Eq. (7.11) reduces on making the substitutions (7.3), removing the factor \( F^2(x_1, x_2, x_3, x_4) \) and then substituting the values of \( x_3, x_4 \) given by (7.12) and (7.13), to the following quadratic equation:

\[
-x_1^2 + 34x_2^2 = \phi^2(m_1, m_2, m_3, m_4)(x_1 + 2x_2)^2, \tag{7.16}
\]

where

\[
\phi(m_1, m_2, m_3, m_4) = (m_1^2 + m_1m_2 - 3m_1m_3 - 5m_1m_4 + 2m_2^2 + 2m_2m_3
- 6m_2m_4 + 4m_3^2 + 4m_3m_4 + 8m_4^2)(m_1^2 - m_1m_2 - 3m_1m_3 + 5m_1m_4
+ 2m_2^2 - 2m_2m_3 - 6m_2m_4 + 4m_3^2 - 4m_3m_4 + 8m_4^2)(m_1 - 2m_3 + 2m_4).
\tag{7.17}
\]

Now Eq. (7.16) may be written as,

\[
\{\phi^2(m_i) + 1\}x_1^2 + 4\phi^2(m_i)x_1x_2 + 4\{\phi^2(m_i) - 34\}x_2^2 = 0. \tag{7.18}
\]

We note that for rational numerical values of \( m_1, m_2, m_3, m_4 \), the coefficients of \( x_1^2 \) and \( x_2^2 \) cannot vanish.

When we start with the rational point \( P_0 = (3, 1, 0, 1, 3, 1, 0, 1) \) and repeatedly apply the \( RC \) operation, at each successive step we will get values of \( m_1, m_2, m_3, m_4 \) such that Eq. (7.16) has rational solutions for \( x_1 \) and \( x_2 \). As there are no rational values of \( x_1, x_2 \) for which the left-hand side of Eq. (7.16) can become 0, it follows that the values of \( m_1, m_2, m_3, m_4 \) that we get at each successive step are such that \( \phi(m_1, m_2, m_3, m_4) \neq 0 \).

If \( P \) is any rational point on the projective variety under consideration, it follows from Eq. (7.16) that

\[
\frac{x_1(P)}{x_2(P)} + \frac{x_1(CP)}{x_2(CP)} = -\frac{4\phi^2(m_i)}{\phi^2(m_i) + 1}. \tag{7.19}
\]

We thus get,

\[
\frac{x_1(CP)}{x_2(CP)} = -\frac{x_1(P)}{x_2(P)} - \frac{4\phi^2(m_i)}{\phi^2(m_i) + 1}. \tag{7.20}
\]

42
and, on taking the reflection of the conjugate point \( C(P) \), we find that,

\[
\frac{x_1(RCP)}{x_2(RCP)} = \frac{x_1(P)}{x_2(P)} + \frac{4\phi^2(m_i)}{\phi^2(m_i) + 1}. \tag{7.21}
\]

Since \( \phi(m_i) \neq 0 \), it follows that

\[
\frac{x_1(RCP)}{x_2(RCP)} > \frac{x_1(P)}{x_2(P)}. \tag{7.22}
\]

Thus, starting from the point \( P_0 \) and repeatedly applying the \( RC \) operation, we get a sequence of rational points such that the ratios \( x_1(P_j)/x_2(P_j) \) pertaining to the successive points \( P_j, j = 0, 1, 2, \ldots \), form a strictly increasing monotonic sequence. We thus get an infinite sequence of rational points in the special case when \( h = 0 \).

The first four points of the sequence of rational points obtained in the special case \( h = 0 \) are \((3, 1, 0, 1, 3, 1, 0, 1)\), \((5, 1, 0, 1, -5, 1, 0, 1)\),

\[(489, 87, 0, 87, 841, -353, 0, -353)\]

and

\[(228105, 39465, 0, 39465, -769129, 369377, 0, 369377)\].

We now revert to the projective variety defined by Eqs. (7.11), (7.12) and (7.13) when \( h \) is an arbitrary integer. Starting from the rational point \( P_0 \), we may apply the \( RC \) operation \( n \) times where \( n \) is any arbitrary positive integer howsoever large, and thus obtain a sequence of rational points \( P_0, P_1, P_2, \ldots, P_n \) on the variety such that the coordinates of these points are given by rational functions of \( h \). Now, following the same argument as was used in Section 3.2, it is established that there are infinitely many integer values of \( h \) such that there are an arbitrarily large number of distinct rational points on the variety defined by the simultaneous equations (7.11), (7.12) and (7.13).

We can readily eliminate the variables \( x_3 \) and \( x_4 \) from Eqs. (7.11), (7.12) and (7.13). The resulting equation, which is of degree 10 in 6 variables \( x_1, x_2, x_5, x_6, x_7, x_8 \), has an arbitrarily large number of integer solutions for infinitely many integer values of \( h \). As this tenth degree equation is too cumbersome to write, we do not give it explicitly.

For any arbitrary value of \( h \), we note that the ratio

\[
\frac{F(x_5, x_6, x_7, x_8)}{F(x_1, x_2, x_3, x_4)} = F(m_1, m_2, m_3, m_4), \tag{7.23}
\]

is the same for any point \( P \) and its conjugate \( C(P) \) (since the values of \( m_1, m_2, m_3, m_4 \) are the same for both \( P \) and \( C(P) \)). Further, since \( x_3 = 0 \)
for all solutions of the simultaneous equations (7.11), (7.12) and (7.13), and
\[ F(x_1, x_2, 0, x_4) = F(-x_1, x_2, 0, x_4), \]
the above ratio is also the same for any rational point \( P \) on the variety defined by Eqs. (7.11), (7.12) and (7.13) and its reflection \( R(P) \).

It follows that the ratio \( F(x_5, x_6, x_7, x_8)/F(x_1, x_2, x_3, x_4) \) is the same for all the points of the infinite sequence of rational points \( P_0, P_1, P_2, \ldots \). For the point \( P_0 \), this ratio is 1, and hence all the points of the aforementioned infinite sequence also satisfy the equation,

\[ F(x_1, x_2, x_3, x_4) = F(x_5, x_6, x_7, x_8). \]  \( 7.24 \)

Using this relation, we can reduce Eq. (7.11) to the following sextic equation:

\[
F(x_1, x_2, x_3, x_4) \left\{ -x_1^2 + 3x_2^2 + 31x_4^2 \\
- (x_5 - 2x_7 + 2x_8 + h(x_6 - x_8))^2 \right\} + 2hH(x_i) = 0, \]  \( 7.25 \)

It follows that there are infinitely many values of \( h \) for which the diophantine system consisting of the four simultaneous equations (7.12), (7.13), (7.24) and (7.25) has an arbitrarily large number of integer solutions. Using the values of \( x_3 \) and \( x_4 \) given by (7.12) and (7.13), we can readily eliminate these two variables and reduce the aforementioned diophantine system to just a pair of simultaneous equations, one of degree 4 and one of degree six, in the six variables \( x_1, x_2, x_5, x_6, x_7, x_8 \). As the sextic equation is very cumbersome to write, we do not give these two equations explicitly. When \( h \neq 0 \), if we eliminate \( x_1 \) from these two equations, we get an equation of degree 8 in 5 variables and the possibility of finding an elliptic curve on the projective variety defined by these equations for any arbitrary integer value of \( h \) seems rather remote.

It, therefore, appears that there exist nonzero integer values of \( h \) for which the projective varieties defined by Eqs. (7.11), (7.12), (7.13), as well as by Eqs. (7.12), (7.13), (7.24) and (7.25) have an arbitrarily large number of rational points but it is unlikely that there is a curve of genus 0 or 1 on either of the two varieties.

When \( h = 1 \), the first four solutions of the simultaneous equations (7.12), (7.13), (7.24) and (7.25), obtained as described above, are as follows:

\[ (3, 1, 0, 1, 3, 1, 0, 1), \quad (7, 1, 0, 1, -7, 1, 0, 1), \]
\[ (2734239, 3306073, 0, 3306073, 13666439, -2392627, 4558960, -3695187), \]
and
\begin{align*}
&\left(-3163872323529222681410246909549439930487861, \\
&294781492568707072189641105966791102577, \\
&0, 294781492568707072189641105966791102577, \\
&-292141098083040479066980131428699532093439, \\
&520113356714484706819641105966791102577, \\
&89674865037670929120182583747883623109040, \\
&2508328560483795657430399046587193165809837\right) .
\end{align*}

8 Examples based on composition of cubic forms

If the cubic form \( C(x_1, x_2, x_3) \) in three variables \( x_1, x_2, x_3 \) is defined by
\begin{align*}
C(x_1, x_2, x_3) &= x_1^3 - px_1^2 x_2 + (p^2 - 2q)x_1^2 x_3 + qx_1 x_2^2 - (pq - 3r)x_1 x_2 x_3 \\
&\quad - (2pr - q^2)x_1 x_2^2 - rx_2^3 + prx_2^2 x_3 - qrx_2 x_3^2 + r^2 x_3^3, \quad (8.1)
\end{align*}
we have the identity,
\begin{align*}
C(x_1, x_2, x_3)C(m_1, m_2, m_3) &= C(x_4, x_5, x_6), \quad (8.2)
\end{align*}
where the values of \( x_4, x_5, x_6 \) are given by the relations,
\begin{align*}
x_4 &= m_1 x_1 - rm_3 x_2 - (rm_2 - prm_3)x_3, \\
x_5 &= m_2 x_1 + (m_1 - qm_3)x_2 - (qm_2 - (pq - r)m_3)x_3, \\
x_6 &= m_3 x_1 + (m_2 - pm_3)x_2 + (m_1 + pm_2 - (p^2 - q)m_3)x_3, \quad (8.3)
\end{align*}
with \( p, q, r \) being arbitrary rational parameters.

The identity (8.2) may be obtained, as indicated in Section 6.2, by using norms of algebraic integers belonging to the field \( \mathbb{Q}(\rho) \) where \( \rho \) is a root of the equation,
\begin{align*}
x^3 + px^2 + qx + r &= 0, \quad (8.4)
\end{align*}
where \( p, q, r \) are arbitrary rational integers such that Eq. (8.4) is irreducible.

We can now solve Eqs. (8.3) for \( m_1, m_2, m_3 \) to obtain three identities,
\begin{align*}
C_j(x_1, x_2, \ldots, x_6) &= m_jC(x_1, x_2, x_3), \quad j = 1, 2, 3, \quad (8.5)
\end{align*}
where \( C_j(x_i), j = 1, 2, 3, \) are cubic forms in the six variables \( x_1, x_2, \ldots, x_6 \) and the identities (8.5) are naturally satisfied when the values of \( x_4, x_5, x_6 \) are given by the relations (8.3). We can then construct the quintic form,
\begin{align*}
f(x_1, x_2, \ldots, x_6) &= C(x_1, x_2, x_3)Q_1(x_i) + C(x_4, x_5, x_6)Q_2(x_i) \\
&\quad + \sum_{j=1}^{3} C_j(x_i)Q_{j+2}(x_i). \quad (8.6)
\end{align*}
Lemma 2. If accordingly we need to modify our definition of reflection.

We now prove a lemma that leads to a new definition of reflection.

Lemma 2. If \(a\) and \(b\) are arbitrary rational numbers such that \(a \neq \pm b\), the quadratic form \(Q(x_1, x_2)\) and the two cubic forms \(C_4(x_1, x_2)\) and \(C_5(x_1, x_2)\) defined by

\[
\begin{align*}
Q(x_1, x_2) &= (a - b)^2 x_1^2 + 2(a - b)^2 x_1 x_2 + 4(a^2 + ab + b^2) x_2^2, \\
C_4(x_1, x_2) &= x_1^3 - \frac{12(a^2 + ab + b^2)}{(a - b)^2} x_1 x_2^2 - \frac{8(a^2 + ab + b^2)}{(a - b)^2} x_2^3, \\
C_5(x_1, x_2) &= x_1^2 x_2 + 2x_1 x_2^2 - \frac{4ab}{(a - b)^2} x_2^3,
\end{align*}
\]

remain unchanged when \(x_1\) is replaced by

\[-ax_1/(a + b) - \left\{2(a^2 + ab + b^2)\right\} x_2/(a^2 - b^2)\]

and \(x_2\) is replaced by

\[(a - b)x_1/\{2(a + b)\} - bx_2/(a + b).\]

Proof. The lemma is readily verified by direct computation. \(\square\)

We now consider the simultaneous diophantine equations,

\[
k_1 C(x_1, x_2, x_3) Q(x_1, x_2) = k_2 C(x_4, x_5, x_6) Q_1(x_4, x_5, x_6), \quad (8.8)
\]

\[
x_3 = 0, \quad (8.9)
\]

where \(k_1, k_2\) are arbitrary nonzero integers, \(Q_1(x_4, x_5, x_6)\) is an arbitrary quadratic form in the variables \(x_4, x_5, x_6\), and the form \(C(x_1, x_2, x_3)\) is defined by \((8.1)\) where we take the values of the parameters \(p, q, r\) in the cubic
form \(C(x_1, x_2, x_3)\) as follows:

\[
\begin{align*}
p &= h, \\
q &= -2\{(h + 6)a^2 - (2h - 6)ab + (h + 6)b^2\}/(a - b)^2, \quad (8.10) \\
r &= 4\{2a^2 - (h - 2)ab + 2b^2\}/(a - b)^2,
\end{align*}
\]

with \(a, b\) and \(h\) being arbitrary rational parameters such that \(a \neq \pm b\).

With these values of \(p, q, r\), we get

\[
C(x_1, x_2, 0) = C_4(x_1, x_2) - hC_5(x_1, x_2),
\]

and so, on using the linear equation (8.9), we may write Eq. (8.8) as follows:

\[
k_1\{C_4(x_1, x_2) - hC_5(x_1, x_2)\}Q(x_1, x_2) = k_2C(x_4, x_5, x_6)Q_1(x_4, x_5, x_6). \quad (8.11)
\]

If an arbitrary rational point \(P\) on the variety defined by Eqs. (8.8) and (8.9) is given by \((x_1, x_2, 0, x_4, x_5, x_6)\), instead of our usual definition of reflection, we now define the reflection of \(P\) as follows:

\[
R(P) = (-\frac{ax_1}{a + b} - \frac{2(a^2 + ab + b^2)x_2}{a^2 - b^2}, \frac{(a - b)x_1}{2(a + b)} - \frac{bx_2}{a + b}), 0, x_4, x_5, x_6). \quad (8.12)
\]

It follows from Lemma 2 and Eq. (8.11) that the point \(R(P)\) lies on the variety defined by Eqs. (8.8) and (8.9). Thus, the diophantine system given by Eqs. (8.8) and (8.9) satisfies the property \(D_2'\).

We will choose the three equations given by Eq. (8.3) as our auxiliary equations. Here again, in view of the general proof given in Section 6.2 in the context of the auxiliary equations (6.18), it follows that with our choice of auxiliary equations, if we take our initial point \(P\) such that the \(x_1, x_2\) coordinates of \(P\) are simultaneously not 0 and the last three coordinates of \(P\) are also not simultaneously 0, when we repeatedly apply the \(RC\) operation, we will successively generate rational points whose coordinates satisfy the same conditions as the coordinates of the point \(P\), and hence the weaker form of condition (i) of property \(D_3'\) is satisfied.

In view of the identity (8.2), on substituting the values of \(x_4, x_5, x_6\) given by (8.3) in Eq. (8.8), we can factor out \(C(x_1, x_2, x_3)\) and thus reduce Eq. (8.8) to a quadratic equation in \(x_1\) and \(x_2\). We can readily choose the parameters \(a, b, h, k_1, k_2\) and the quadratic form \(Q_1(x_4, x_5, x_6)\) such that property \(D_1'\) is also satisfied.

As a specific example, we take \(a = 1, b = -2, k_1 = 8, k_2 = 39\), when we get the values of \(p, q, r\) from Eq. (8.10) as

\[
p = h, \quad q = -2h - 4, \quad r = 8h/9 + 8/3. \quad (8.13)
\]
and we take the quadratic form $Q_1(x_4, x_5, x_6)$ as given by

$$Q_1(x_4, x_5, x_6) = (x_4 + x_5 + x_6)^2 + (h + 3)(49x_4^2 - 36x_5^2 + x_6^2),$$  \hspace{1cm} (8.14)

and now on using the relation (8.9), Eq. (8.8) reduces to the following equation:

$$72(9x_1^3 - 9hx_1^2x_2 - 18(h + 2)x_1x_2^2 - 8(h + 3)x_2^3)(3x_1^4 + 6x_1x_2 + 4x_2^2)$$
$$= 13(81x_1^3 - 81hx_1^2x_5 + 81(h^2 + 4h + 8)x_1x_5 - 162(h + 2)x_5x_6^2$$
$$+ (162h^2 + 540h + 648)x_4x_5x_6 + (180h^2 + 864h + 1296)x_4x_6^2$$
$$- 72(h + 3)x_5^3 + 72h(h + 3)x_5^2x_6 + 144(h + 3)(h + 2)x_5x_6^2$$
$$+ 64(h + 3)^2x_6^3\{(x_4 + x_5 + x_6)^2 + (h + 3)(49x_4^2 - 36x_5^2 + x_6^2)}.$$  \hspace{1cm} (8.15)

It is readily verified that, for all values of $h$, the point $P_0$ given by $(1, 1, 0, 6, -7, 0)$ lies on the variety defined by Eqs. (8.9) and (8.15).

On substituting the values of $x_4, x_5, x_6$ given by (8.3) where we take $x_3 = 0$ and the values of $p, q, r$ as given by (8.13), Eq. (8.15) reduces, on removing the factor $9x_1^3 - 9hx_1^2x_2 - 18(h + 2)x_1x_2^2 - 8(h + 3)x_2^3$, to the following quadratic equation:

$$\{1053\psi_0(m_i)\psi_3(m_i) + 157464\}x_1^2 + \{234\psi_1(m_i)\psi_3(m_i) + 314928\}x_1x_2$$
$$+ \{1053\psi_0(m_i)\psi_3(m_i) + 13(h + 3)\psi_2(m_i)\psi_3(m_i) + 209952\}x_2^2 = 0,$$  \hspace{1cm} (8.16)

where

$$\psi_0(m_i) = -(49h + 148)m_1^2 - 2m_1m_2 + 2m_1m_3 + (36h + 107)m_2^2$$
$$- 2m_2m_3 - (h + 4)m_3^2,$$

$$\psi_1(m_i) = -9m_1^2 + (324h + 954)m_1m_2 + (392h^2 + 2351h + 3507)m_1m_3$$
$$- 9m_2^2 + (648h^2 + 3230h + 3840)m_2m_3 + (9h^2 + 26h - 12)m_3^2,$$

$$\psi_2(m_i) = 6885m_1^2 + (11664h + 23310)m_1m_3 - 2997m_2^2$$
$$+ (162h - 18)m_2m_3 + (8447h^2 + 27839h + 18492)m_3^2,$$

$$\psi_3(m_i) = 81m_1^3 - 81hm_1^2m_2 + 81(h^2 + 4h + 8)m_1m_3^2$$
$$- 162(h + 2)m_1m_2^2 + (162h^2 + 540h + 648)m_1m_2m_3 + (180h^2$$
$$+ 864h + 1296)m_1m_3^2 - 72(h + 3)m_2^3 + 72(h + 3)hm_2m_3$$
$$+ 144(h + 3)(h + 2)m_2m_3^2 + 64(h + 3)^2m_3^3.$$  \hspace{1cm} (8.17)

We will now show that for infinitely many integer values of $h$, there are an arbitrarily large number of rational points on the projective variety defined by the simultaneous equations (8.9) and (8.15).

We first consider the special case when $h = -3$. It is readily seen that when $h = -3$, the coefficients of $x_1^2$ and $x_2^2$ in Eq. (8.16) differ by a constant
For these three equations to be satisfied, it is clear that $\psi_\alpha$ respectively. The point $(RC)_P(3, RC)$, we may apply the $RC$ operation repeatedly to obtain a sequence of rational points $(RC)_P, (RC)^2P, \ldots$, all of which lie on the variety. The points $(RC)_P$ and $(RC)^2P$ of the sequence are given by

$$(-54, 135, 0, -162\alpha_4, 144\alpha_4 - 90\alpha_5 - 72\alpha_6, -36\alpha_4 + 36\alpha_5 + 18\alpha_6)$$

and

$$(24, 24, 0, -288\alpha_4, 481\alpha_4 + 50\alpha_5 + 52\alpha_6, -169\alpha_4 - 26\alpha_5 - 28\alpha_6)$$

respectively. The point $(RC)^2_P$ may be written equivalently as $(1, 1, 0, \alpha'_4, \alpha'_5, \alpha'_6)$ where

$$\alpha'_4 = -12\alpha_4, \alpha'_5 = (481\alpha_4 + 50\alpha_5 + 52\alpha_6)/24, \alpha'_6 = -(169\alpha_4 + 26\alpha_5 + 28\alpha_6)/24.$$

Since $|\alpha'_4| > |\alpha_4|$, it follows that in the special case when $h = -3$, starting from the known point $P_0$ given by $(1, 1, 0, 6, -7, 0)$ and repeatedly applying the $RC$ operation, we get an infinite sequence of distinct rational points on the variety defined by Eq. (8.9) and Eq. (8.15). The first four points of this sequence are as follows: $(1, 1, 0, 6, -7, 0)$, $(-18, 45, 0, -324, 498, -156)$, $(3, 3, 0, -216, 317, -104)$ and $(-18, 45, 0, 3888, -5794, 1924)$.

We now revert to the projective variety defined by Eq. (8.9) and Eq. (8.15) when $h$ is an arbitrary integer. We will first show that there do not exist values of $m_i$ for which Eq. (8.16) is identically satisfied for all values of $x_1, x_2$. Accordingly, we consider the following three equations obtained by equating to 0 the coefficients of $x_1^2, x_1x_2$ and $x_2^2$ in Eq. (8.16):

$$1053\psi_0(m_i)\psi_3(m_i) + 157464 = 0, \quad (8.18)$$
$$234\psi_1(m_i)\psi_3(m_i) + 314928 = 0, \quad (8.19)$$
$$1053\psi_0(m_i)\psi_3(m_i) + 13(h + 3)\psi_2(m_i)\psi_3(m_i) + 209952 = 0 \quad (8.20)$$

For these three equations to be satisfied, it is clear that $\psi_3(m_i)$ cannot be 0. Now multiplying Eq. (8.18) by 2 and subtracting Eq. (8.19), we get, on removing the factors $(h + 3)\psi_3(m_i),$

$$441m_1^2 + 324m_1m_2 + (392h + 1175)m_1m_3 - 324m_2^2$$
$$+ (648h + 1286)m_2m_3 + (9h + 8)m_3^2 = 0. \quad (8.21)$$
Similarly, on multiplying Eq. (8.18) by 4 and Eq. (8.20) by 3, and taking the difference, we get,

\[
(8208h + 24651)m_1^2 + 54m_1m_2 + (11664h^2 + 58302h + 69984)m_1m_3
- (3969h + 11880)m_2^2 + 18h(9h + 26)m_2m_3
+ (8447h^3 + 53180h^2 + 102036h + 55584)m_3^2 = 0. \tag{8.22}
\]

On eliminating \( h \) between Eqs. (8.21) and (8.22), we get,

\[
391465164951m_1^6 + (1055469390948m_2 + 951172925787m_3)m_1^5
+ (397900362636m_2^2 + 80307380180m_2m_3 + 700636785345m_3^2)m_1^4
- (1072269163872m_2^3 + 2176139543544m_2m_3 + 1060172093268m_3^2)m_1^3
+ 84790818585m_3^3m_1^2 - (856912684464m_4^4 + 1942353345168m_2^2m_3
+ 23317207521m_2m_3^2 + 118110680960m_2m_3^3 + 31160975585m_3^4)m_1^2
+ (19716865258m_5^5 + 1012778109936m_2m_3^6 + 82067816188m_3^2m_2^3)
+ 196155128069m_2^2m_3^3 + 64601546074m_2m_3^4 + 1729745180m_3^5)m_1
+ 241665851520m_5^6 + 931403339136m_2m_3^7 + 1368728626356m_3^2m_2^3
+ 976907330410m_2^3m_3^4 + 77122752312m_2^2m_3^5
+ 1429695768m_3^6 - 708224m_3^6 = 0. \tag{8.23}
\]

Now Eq. (8.23) represents a curve of genus 4 and it has only finitely many solutions for \( m_1, m_2, m_3 \) and it follows from Eq. (8.21) that there are only finitely many values of \( h \) for which there exist rational values of \( m_1, m_2, m_3 \). Thus when \( h \) is arbitrary, we cannot find values of \( m_i \) such that the three coefficients of Eq. (8.16) vanish simultaneously. We have thus shown that condition (ii) of property \( D'_3 \) is satisfied.

Thus when \( h \) is arbitrary, starting from the rational point \( P_0 \), we can apply the \( RC \) operation \( n \) times where \( n \) is any arbitrary positive integer however large, and obtain a sequence of rational points \( P_0, P_1, P_2 \ldots P_n \) on the variety defined by Eq. (8.9) and Eq. (8.15). The coordinates of these points are rational functions of \( h \). Now following the same argument as was used in Section 3.2, it follows that there exist infinitely many integer values of \( h \) for which there are an arbitrarily large number of rational points on the projective variety defined by Eqs. (8.9) and (8.15).

Since Eq. (8.15) does not contain \( x_3 \), we have, in effect, shown that for infinitely many values of \( h \), the quintic equation in five variables, given by (8.15), has an arbitrarily large number of solutions.

For any arbitrary value of \( h \), we note that the ratio

\[
\frac{C(x_4, x_5, x_6)}{C(x_1, x_2, x_3)} = C(m_1, m_2, m_3), \tag{8.24}
\]
is the same for any point $P$ and its conjugate $C(P)$ (since the values of $m_1$, $m_2$, $m_3$ are the same for both $P$ and $C(P)$). Further, for all rational points $P$ on the variety defined by Eqs. (8.8) and (8.9), we have $x_3(P) = 0$, and hence the values of both $C(x_1, x_2, x_3)$ and $C(x_4, x_5, x_6)$ are the same for any point $P$ and its reflection $R(P)$. Thus, the value of the ratio $C(x_4, x_5, x_6)/C(x_1, x_2, x_3)$ is the same for all points of the sequence $P_0, P_1, P_2, \ldots$. For the point $P_0$, this ratio is 8, and hence all points of the aforementioned sequence also satisfy the equation

$$8C(x_1, x_2, x_3) = C(x_4, x_5, x_6). \tag{8.25}$$

Eq. (8.25) reduces, on using Eq. (8.9), to the following equation:

$$
\begin{align*}
648x_1^3 - 648hx_1^2x_2 - 1296(h+2)x_1x_2^2 - 576(h+3)x_2^3 - 81x_4^3 \\
+ 81hx_4^2x_5 - 81(h^2 + 4h + 8)x_4^2x_6 + 162(h+2)x_4x_5^2 \\
- 54(3h^2 + 10h + 12)x_4x_5x_6 - 36(5h^2 + 24h + 36)x_4x_6^2 + 72(h+3)x_5^3 \\
- 72h(h+3)x_5^2x_6 - 144(h+2)(h+3)x_5x_6^2 - 64(h+3)^2x_6^3 = 0. \tag{8.26}
\end{align*}
$$

We have thus shown that for infinitely many values of $h$, the simultaneous diophantine equations (8.15) and (8.26) have an arbitrarily large number of solutions. When $h = 1$, the first three solutions of the simultaneous equations (8.15) and (8.26) are given by $(1, 1, 0, 6, -7, 0)$,

$$(-368765338, 605494801, 0, -297321236, -366427558, 715340340)$$

and

$$(47588476550214311358184089744533539572969269, \quad -1704177113488256267304814143821106318848779, \quad 0, 89001945291963614347958963002771027507707960, \quad -104068631557829167473628000790179775910721261, \quad 40516749860848893002398908535963291149411960).$$

It follows from Eqs. (8.24) and (8.25) that the values of $m_1$, $m_2$, $m_3$ corresponding to all the rational points $P_0, P_1, P_2, \ldots$, satisfy the condition $C(m_1, m_2, m_3) = 8$. This condition may be written as follows:

$$\psi_3(m_i) = 648. \tag{8.27}$$

Further, the values of $m_1$, $m_2$, $m_3$ corresponding to all the rational points of the infinite sequence, $P_0, P_1, P_2, \ldots$, must also satisfy the condition that the discriminant of Eq. (8.16) is a perfect square. On using the relation (8.27), this condition may be written as follows:

$$-13689\psi_0^2(m_i) - 169(h+3)\psi_0(m_i)\psi_2(m_i) + 169\psi_1^2(m_i) \\
-7371\psi_0(m_i) + 702\psi_1(m_i) - 39(h+3)\psi_2(m_i) - 243 = z^2, \tag{8.28}$$

51
where \( z \) is some rational number.

It follows that there exist infinitely many integer values of \( h \) for which the simultaneous equations (8.27) and (8.28) have an arbitrarily large number of rational solutions for \( m_1, m_2, m_3 \) and \( z \). When \( h = 1 \), the first two solutions of Eqs. (8.27) and (8.28), corresponding to the first two solutions of Eqs. (8.15) and (8.26), are given below:

\[
(m_1, m_2, m_3, z) = \left( \frac{50}{43}, 0, -\frac{117}{86}, \frac{13756545}{86} \right),
\]

\[
m_1 = -\frac{5977631151469496370601034}{1446700126228932448001123},
\]

\[
m_2 = \frac{367813193903754671081930}{1446700126228932448001123},
\]

\[
m_3 = \frac{12237047979702532325384490}{1446700126228932448001123},
\]

\[
z = \frac{105463580688578364176884811517}{1446700126228932448001123}.
\]

The next solution of Eqs. (8.27) and (8.28), corresponding to the third solution of Eqs. (8.15) and (8.26), involves integers with more than 132 digits and is omitted.

9 Concluding remarks

We have shown in this paper that there exist quartic and sextic surfaces, defined by equations of type (3.4) and (4.4) respectively, on which we can find infinitely many integer points by a new iterative method. It is, however, not easy to determine whether the aforesaid method can be applied to obtain integer solutions of a specific equation of degree 4 or 6 in four variables. It would be of interest to find criteria by which it is possible to decide whether the method of this paper is applicable to a given equation.

We have also shown that there exist projective varieties, defined by equations in several variables, on which there are an arbitrarily large number of integer points. In all such cases, we were able to find these integer points without finding a curve of genus 0 or 1 on the projective variety under consideration.

While we have constructed a few examples of surfaces and projective varieties defined by high degree equations and on which there are an arbitrarily large number of integer points, it appears that more general examples of such surfaces and projective varieties can be constructed.

The crucial question is whether or not there exists a curve of genus 0 or 1 on the surfaces and projective varieties on which we have found an arbitrarily large number of integer points. It would be of considerable interest if it could be proved that a curve of genus 0 or 1 does not lie on these surfaces or projective varieties. It also needs to be determined whether these surfaces and varieties are of sufficiently general type.
In the light of the examples already given in this paper and in view of the various possibilities that arise from the general method described in Sections 2 and 6, it appears that there may exist projective varieties, defined by high degree equations, on which there are an arbitrarily large number of integer points and on which a curve of genus 0 or 1 does not exist.

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