SOLUTIONS OF FRACTIONAL SCHRODINGER EQUATION WITH SIGN-CHANGING NONLINEARITY

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Abstract. We look for a solution to a nonlinear, fractional Schrödinger equation
\[ (-\Delta)^{\alpha/2} u + V(x)u = f(x, u) - \Gamma(x)|u|^{q-2}u \] on \( \mathbb{R}^N \),
where potential \( V \) is coercive or \( V = V_{\text{per}} + V_{\text{loc}} \) is a sum of periodic in \( x \) potential \( V_{\text{per}} \) and localized potential \( V_{\text{loc}} \), \( \Gamma \in L^\infty(\mathbb{R}^N) \) is periodic in \( x \), \( \Gamma(x) \geq 0 \) for a.e. \( x \in \mathbb{R}^N \) and \( 2 < q < 2^*_\alpha \). If \( f \) has the subcritical growth, but higher than \( \Gamma(x)|u|^{q-2}u \), then we find a ground state solution being a minimizer on the Nehari manifold. Moreover we show that if \( f \) is odd in \( u \) and \( V \) is periodic, this equation admits infinitely many solutions, which are pairwise geometrically distinct. Finally, we obtain the existence result in the case of coercive potential \( V \).

MSC 2010: Primary: 35Q55; Secondary: 35A15, 35J20, 58E05

Keywords: ground state, variational methods, Nehari manifold, fractional Schrödinger equation, periodic and localized potentials, coercive potential, sign-changing nonlinearity.

INTRODUCTION

We consider the following nonlinear, fractional Schrödinger equation
\[ (-\Delta)^{\alpha/2} u + V(x)u = f(x, u) - \Gamma(x)|u|^{q-2}u \] on \( \mathbb{R}^N \), \( \alpha \in (0, 2], N > \alpha \),
with \( u \in H^{\alpha/2}(\mathbb{R}^N) \), which appears in different areas of mathematical physics. Recently, the fractional Schrödinger equation has been introduced to describe the propagation dynamics of wave packets in the presence of the harmonic potential and also of the free particle (see [33,34]). In [23] has been proposed an optical realization of this equation, based on transverse light dynamics in aspherical optical cavities. The case of a linear potential is also a fundamental problem in quantum mechanics that can be treated and solved analytically (see [16,26]). Such an equation was also studied in the quantum scattering problem (see [17]).

The fractional Laplacian \( (-\Delta)^{\alpha/2} \) of a function \( \psi : \mathbb{R}^N \to \mathbb{R} \) is defined by the Fourier transform by the formula
\[ \mathcal{F}
\left((-\Delta)^{\alpha/2} \psi\right)(\xi) := |\xi|^\alpha \hat{\psi}(\xi), \]
where
\[ \mathcal{F} \psi(\xi) := \hat{\psi}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} \psi(x) \, dx \]
denote the usual Fourier transform. When \( \psi : \mathbb{R}^N \to \mathbb{R} \) is smooth enough, it can be defined by the principal value of the singular integral
\[
(-\Delta)^{\alpha/2} \psi(x) = c_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{\psi(x) - \psi(y)}{|x - y|^{N+\alpha}} \, dy,
\]
where \( c_{N,\alpha} \) is some normalization constant. It is known, that \((-\Delta)^{\alpha/2}\) reduces to \(-\Delta\) as \( \alpha \to 2^- \) – see [12]. In this paper we identify \((-\Delta)^{\alpha/2}\) with the classical Laplace operator \(-\Delta\) when \( \alpha = 2 \). By the very definition we observe that the fractional Laplacian is non-local (see [8, 12]).

For \( 0 < \alpha < 2 \), let us remind the definition of the fractional Sobolev space:
\[
H^{\alpha/2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} |u(x)|^2 \, dx < \infty \right\}.
\]
It is a Hilbert space endowed with the norm
\[
u \mapsto \sqrt{\int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} |u|^2 \, dx}.
\]

The equation (1.1) describes the behaviour of so-called standing wave solutions \( \Phi(x,t) = u(x)e^{-i\omega t} \) of the following time-dependent fractional Schrödinger equation
\[
i \frac{\partial \Phi}{\partial t} = (-\Delta)^{\alpha/2} \Phi + (V(x) + \omega)\Phi - g(x, |\Phi|).
\]
Such an equation was introduced by Laskin by expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths (see [20, 21]). The time-dependent equation has been recently studied by A. Liemert and A. Kienle ([22]) with the linear potential \( V(x) = \beta x \).

The nonlinear term \( f \) satisfies the following conditions:

(F1) \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is measurable, \( \mathbb{Z}^N \)-periodic in \( x \in \mathbb{R}^N \) and continuous in \( u \in \mathbb{R} \) for a.e. \( x \in \mathbb{R}^N \), and there are \( c > 0 \) and \( 2 < q < p < 2^*_\alpha := \frac{2N}{N-\alpha} \) such that
\[
|f(x,u)| \leq c(1 + |u|^{p-1}) \text{ for all } u \in \mathbb{R}, \ x \in \mathbb{R}^N,
\]
(F2) \( f(x,u) = o(|u|) \) uniformly in \( x \) as \( |u| \to 0^+ \),
(F3) \( F(x,u)/|u|^q \to \infty \) uniformly in \( x \) as \( |u| \to \infty \), where \( F(x,u) = \int_0^u f(x,s) \, ds \) is the primitive of \( f \) with respect to \( u \),
(F4) \( u \mapsto f(x,u)/|u|^q \) is strictly increasing on \((-\infty,0)\) and \((0,\infty)\).
We impose on \( \Gamma \) the following condition:

\[(\Gamma) \ \Gamma \in L^\infty(\mathbb{R}^N) \text{ is } \mathbb{Z}^N\text{-periodic in } x \in \mathbb{R}^N, \Gamma(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^N.\]

Note that the nonlinearity \((x, u) \mapsto f(x, u) - \Gamma(x)|u|^{q-2}u\) does not satisfy the Ambrosetti-Rabinowitz type condition. In fact it may be sign-changing, for example – consider \(f(x, u) = |u|^{p-1}u\) and \(\Gamma \equiv 1\), where \(2 < q < p < 2^*_\alpha\).

We assume that the potential \( V \) satisfies

\[(V_\alpha 1) \ V = V_{\text{per}} + V_{\text{loc}}, \text{ where } V_{\text{per}} \in L^\infty(\mathbb{R}^N) \text{ is } \mathbb{Z}^N\text{-periodic in } x \in \mathbb{R}^N \text{ and } V_{\text{loc}} \in L^\infty(\mathbb{R}^N) \text{ is such that } \lim_{|x| \to \infty} V_{\text{loc}}(x) = 0; \] moreover

\[
\begin{cases}
V_0 := \text{ess inf}_{x \in \mathbb{R}^N} V(x) > 0 & \text{for } 0 < \alpha < 2, \\
\inf \sigma (-\Delta + V(x)) > 0 & \text{for } \alpha = 2,
\end{cases}
\]

or

\[(V_\alpha 2) \ V \in C(\mathbb{R}^N, \mathbb{R}) \text{ is such that } \lim_{|x| \to \infty} V(x) = \infty \text{ and } V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0.\]

We consider a Hilbert space

\[
E^{\alpha/2} := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{\alpha} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty \right\}
\]
endowed with the following norm

\[
\|u\|^2 := \int_{\mathbb{R}^N} |\xi|^{\alpha} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)|u|^2 dx
\]
and the scalar product

\[
\langle u, v \rangle := \int_{\mathbb{R}^N} |\xi|^{\alpha} \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi + \int_{\mathbb{R}^N} V(x)u(x)v(x) dx.
\]

Our goal is to find a ground state of the energy functional \( \mathcal{J} : E^{\alpha/2} \to \mathbb{R} \) of class \( C^1 \) given by

\[
\mathcal{J}(u) := \begin{cases}
\frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{\alpha} |\hat{u}(\xi)|^2 d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx - \mathcal{I}(u), & \text{for } 0 < \alpha < 2, \\
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2 dx - \mathcal{I}(u), & \text{for } \alpha = 2,
\end{cases}
\]
where

\[
\mathcal{I}(u) := \int_{\mathbb{R}^N} \left( F(x, u(x)) - \frac{1}{q} \Gamma(x)|u(x)|^q \right) dx
\]
i.e. we look for a critical point being a minimizer of $\mathcal{J}$ on the Nehari manifold

$$\mathcal{N} := \{ u \in E^{\alpha/2} \setminus \{0\} : \mathcal{J}'(u)(u) = 0 \}.$$  

Obviously $\mathcal{N}$ contains all nontrivial critical points, hence a ground state is the least energy solution.

Note that if $\Gamma \not\equiv 0$, then the nonlinear part of the energy functional

$$\int_{\mathbb{R}^N} \left( F(x, u) - \frac{1}{q} \Gamma(x) |u|^q \right) dx$$

is sign-changing, moreover $u \mapsto (f(x, u) - \Gamma(x) |u|^{q-2}u) / |u|^2$ is no longer increasing on $(-\infty, 0)$ and $(0, \infty)$.

The classical Schrödinger equation (the case $\alpha = 2$) have been studied by many authors; see for instance [1, 7, 11, 19, 24, 25] and references therein. The fractional case have been also widely investigated in [2, 6, 9, 14, 18, 27]; see also references therein.

The existence of nontrivial solutions was obtained by S. Secchi in [28] for subcritical $f \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R})$ satisfying the Ambrosetti-Rabinowitsch type condition $0 < \mu F(x, u) < uf(x, u)$ for $\mu > 2$ and coercive potential $V \in \mathcal{C}^1(\mathbb{R}^N)$. In [28] was also introduced the Nehari manifold method with the classical monotonicity condition: $t \mapsto t^{-1}uf(x, tu)$ is increasing on $(0, \infty)$. M. Cheng proved in [10] that (1.1) has a nontrivial solution for the subcritical nonlinearity $f(x, u) = |u|^{p-1}u + \omega u$ and coercive potential $V(x) > 1$ for a.e. $x \in \mathbb{R}^N$. He showed also that there is a ground state solution being minimizer on the Nehari manifold for $0 < \omega < \lambda$, where $\lambda = \inf \sigma(\mathcal{A})$ and $\sigma(\mathcal{A})$ is the spectrum of the self-adjoint operator $\mathcal{A} := (-\Delta)^{\alpha/2} + V(x)$ on $L^2(\mathbb{R}^N)$. For $f(x, u) = |u|^{p-1}$, where $p$ is a subcritical exponent, there is a positive and spherically symmetric solution (see [13]). The uniqueness of ground states $Q = Q(|x|) \geq 0$ of equation $(-\Delta)^{\alpha/2}Q + Q - Q^{q+1} = 0$ in $\mathbb{R}$ was obtained by R.L. Frank and E. Lenzmann in [15]. Recently, S. Secchi proved the existence of radially symmetric solution of $(-\Delta)^{\alpha/2}u + V(x)u = g(u)$ for $g$ which does not satisfy the Ambrosetti-Rabinowitsch condition ( [29]). Such a result was known before for $\alpha = 2$ and constant potentials $V$ ( [4]). In our case the nonlinear term depends on $x$, does not satisfy the Ambrosetti-Rabinowitsch condition and the classical monotonicity condition is violated.

When $\alpha = 2$, the existence of ground states was obtained in [5] with the assumption $(V_1)$. Using the abstract setting provided in [5] we are able to extend this result for $0 < \alpha < 2$ and $q > 2$. In fact, we state the following result.

**Theorem 1.1.** Let $\alpha \in (0, 2]$. Suppose that $(V_1)$, $(F1)$–$(F4)$ are satisfied and $V_{\text{loc}} \equiv 0$ or $V_{\text{loc}}(x) < 0$ for a.e. $x \in \mathbb{R}^N$. Then (1.1) has a ground state, i.e. there is a nontrivial critical point $u$ of $\mathcal{J}$ such that $\mathcal{J}(u) = \inf_{\mathcal{N}} \mathcal{J}$. 

For $V_{\text{loc}} \equiv 0$ we provide a multiplicity result, which is new also in the case $\alpha = 2$.

Suppose that $u$ is a solution of (1.1) and $k \in \mathbb{Z}^N$, observe that $u(\cdot - k)$ is also the solution, provided by $V_{\text{loc}} \equiv 0$. Therefore all elements of the orbit

$$\mathcal{O}(u) := \{u(\cdot - k) : k \in \mathbb{Z}^N\}$$

of $u$ under the $\mathbb{Z}^N$-action are solutions. Thus, we define that $u_1$ and $u_2$ are geometrically distinct if their orbits satisfy $\mathcal{O}(u_1) \cap \mathcal{O}(u_2) = \emptyset$.

**Theorem 1.2.** Let $\alpha \in (0, 2]$. Suppose that $(V_{\alpha}1)$, $(\Gamma)$ $(F1)$–$(F4)$ are satisfied, $V_{\text{loc}} \equiv 0$ and suppose that $f$ is odd in $u$. Then (1.1) admits infinitely many pairs $\pm u$ of geometrically distinct solutions.

Moreover, we investigate the existence of solutions when the potential is coercive.

**Theorem 1.3.** Let $\alpha \in (0, 2]$. Suppose that $(V_{\alpha}2)$, $(\Gamma)$ $(F1)$–$(F4)$ are satisfied. Then (1.1) has a ground state, i.e. there is a nontrivial critical point $u$ of $\mathcal{J}$ such that $\mathcal{J}(u) = \inf_{\mathcal{N}} \mathcal{J}$.

Observe that this result does not require such a regularity of the potential and the nonlinear term as [28][Theorem 3.1] – we do not need the differentiability of the right side of (1.1) and the potential. Note that the coercive potential has been studied in [32], however our nonlinearity is not of the form $f(u) + h(x)$, where $f$ satisfies the Ambrosetti-Rabinowitz condition.

## 2. Preliminary facts

Suppose that $E$ is an Hilbert space with respect to the norm $\| \cdot \|$. Let us consider a functional $\mathcal{J} : E \to \mathbb{R}$ of the general form

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \mathcal{I}(u),$$

where $\mathcal{I} : E \to \mathbb{R}$ is of $C^1$-class. Let us recall a critical point theorem from [5], which is based on the approach of [31] and [3].

**Theorem 2.1** ([5]). Suppose that the following conditions hold:

- $(J1)$ there is $r > 0$ such that $a := \inf_{\|u\|=r} \mathcal{J}(u) > \mathcal{J}(0) = 0$;
- $(J2)$ there is $q \geq 2$ such that $\mathcal{I}(t_n u_n)/t_n^q \to \infty$ for any $t_n \to \infty$ and $u_n \to u \neq 0$ as $n \to \infty$;
- $(J3)$ for $t \in (0, \infty) \setminus \{1\}$ and $u \in \mathcal{N}$

$$\frac{t^2 - 1}{2} \mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u) < 0;$$

- $(J4)$ $\mathcal{J}$ is coercive on $\mathcal{N}$.
Then \( \inf_{\mathcal{N}} J > 0 \) and there exists a bounded minimizing sequence for \( J \) on \( \mathcal{N} \), i.e. there is a sequence \((u_n) \subset \mathcal{N} \) such that \( J(u_n) \to \inf_{\mathcal{N}} J \) and \( J'(u_n) \to 0 \).

This setting allows us to find a bounded minimizing sequence for \( J \) on \( \mathcal{N} \). We only need to check whether (J1)–(J4) are satisfied – we follow arguments from [5], we provide details below.

**Lemma 2.2.** Let \( E = E^{\alpha/2} \) and take

\[
\mathcal{I}(u) = \int_{\mathbb{R}^N} \left( F(x, u(x)) - \frac{1}{q} \Gamma(x)|u(x)|^q \right) \, dx.
\]

Suppose that (\( \Gamma \)), (F1)–(F4) are satisfied. Suppose also that (V\( \alpha,1 \)) or (V\( \alpha,2 \)) hold. Then (J1)–(J4) hold.

**Proof.** (J1) Fix \( \varepsilon > 0 \). Observe that (F1) and (F2) implies that \( F(x, u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \) for some \( C_\varepsilon > 0 \). Therefore

\[
\int_{\mathbb{R}^N} F(x, u) \, dx - \int_{\mathbb{R}^N} \frac{1}{q} \Gamma(x)|u|^q \, dx \leq \int_{\mathbb{R}^N} F(x, u) \, dx \leq C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p),
\]

for some constant \( C > 0 \) provided by the Sobolev embedding theorem. Thus there is \( r > 0 \) such that

\[
\int_{\mathbb{R}^N} F(x, u) \, dx - \int_{\mathbb{R}^N} \frac{1}{q} \Gamma(x)|u|^q \, dx \leq \frac{1}{4}\|u\|^2
\]

for \( \|u\| \leq r \). Therefore

\[
J(u) \geq \frac{1}{4}\|u\|^2 = \frac{1}{4}r^2 > 0
\]

for \( \|u\| = r \).

(J2) By (F3) and Fatou’s lemma we get

\[
\mathcal{I}(t_n u_n)/t_n^q = \int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{t_n^q} \, dx - \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|u_n|^q \, dx \to \infty.
\]

(J3) Fix \( u \in \mathcal{N} \) and consider

\[
\psi(t) = \frac{t^2 - 1}{2}\mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u)
\]

for \( t \geq 0 \). Then \( \psi(1) = 0 \) and

\[
\frac{d\psi(t)}{dt} = \int_{\mathbb{R}^N} tf(x, u)u - f(x, tu)u \, dx + (t^{q-1} - t) \int_{\mathbb{R}^N} \Gamma(x)|u|^q \, dx.
\]

Since \( u \in \mathcal{N} \)

\[
\int_{\mathbb{R}^N} f(x, u)u \, dx - \int_{\mathbb{R}^N} \Gamma(x)|u|^q \, dx = \|u\|^2 > 0.
\]
Therefore, for \( t > 1 \) we have
\[
\frac{d\psi(t)}{dt} < \int_{\mathbb{R}^N} t^{q-1} f(x, u) u - f(x, tu) u \, dx = t^{q-1} \int_{\mathbb{R}^N} f(x, u) u - \frac{f(x, tu) u}{t^{q-1}} \, dx < 0,
\]
by (F4). Similarly \( \frac{d\psi(t)}{dt} > 0 \) for \( t < 1 \). Therefore \( \psi(t) < \psi(1) = 0 \) for \( t \neq 1 \), i.e.
\[
\frac{t^2 - 1}{2} \mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u) < 0.
\]

(J4) Let \((u_n) \subset \mathcal{N}\) be a sequence such that \(\|u_n\| \to \infty\) as \(n \to \infty\). (F3) implies that
\[
f(x, u) u = q \int_0^u \frac{f(x, s)}{s^{q-1}} s^{q-1} \, ds \geq q \int_0^u \frac{f(x, s)}{s^{q-1}} s^{q-1} \, ds = qF(x, u)
\]
for \(u \geq 0\) and similarly \(f(x, u) u \geq qF(x, u)\) for \(u < 0\). Therefore
\[
\mathcal{J}(u_n) = \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} F(x, u_n) \, dx + \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|u_n|^q \, dx = \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2 + \int_{\mathbb{R}^N} \frac{1}{q}f(x, u_n)u_n - F(x, u_n) \, dx \geq \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2 \to \infty
\]
as \(n \to \infty\), since \(q > 2\).

3. Palais-Smale sequences decomposition

The main theorem in this section generalizes the decomposition result form [5][Theorem 4.1]. We consider the functional \(\mathcal{J} : H^{\alpha/2}(\mathbb{R}^N) \to \mathbb{R}\) of the form
\[
\mathcal{J}(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} G(x, u) \, dx.
\]
We assume that \((V, 1)\) holds and therefore we may consider the following norm
\[
\|u\|^2 = \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)|u|^2 \, dx.
\]
We suppose that \(G(x, u) = \int_0^u g(x, s) \, ds\), where \(g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}\) satisfies:

\(G1\) \(g(\cdot, u)\) is measurable and \(\mathbb{Z}^N\)-periodic in \(x \in \mathbb{R}^N\), \(g(x, \cdot)\) is continuous in \(u \in \mathbb{R}\) for a.e. \(x \in \mathbb{R}^N\);

\(G2\) \(g(x, u) = o(|u|)\) as \(|u| \to 0^+\) uniformly in \(x \in \mathbb{R}^N\);

\(G3\) there exists \(2 < r < 2_\alpha^*\) such that \(\lim_{|u| \to \infty} g(x, u)/|u|^{r-1} = 0\) uniformly in \(x \in \mathbb{R}^N\);

\(G4\) for each \(a < b\) there is a constant \(c > 0\) such that \(|g(x, u)| \leq c\) for a.e. \(x \in \mathbb{R}^N\) and \(a \leq u \leq b\).
We will also denote
\[ \mathcal{J}_{\text{per}}(u) = \mathcal{J}(u) - \int_{\mathbb{R}^N} V_{\text{loc}}(x)|u|^2 dx. \]

**Theorem 3.1.** Suppose that (G1)–(G4) and (V,1) hold. Let \((u_n)\) be a bounded Palais-Smale sequence for \(\mathcal{J}\). Then passing to a subsequence of \((u_n)\), there is an integer \(\ell > 0\) and sequences \((y_k^n) \subset \mathbb{Z}^N\), \(w_k \in H^{\alpha/2}(\mathbb{R}^N)\), \(k = 1, \ldots, \ell\) such that:

(a) \(u_n \rightharpoonup u_0\) and \(\mathcal{J}'(u_0) = 0\);
(b) \(|y_k^n| \to \infty\) and \(|y_k^n - y_{k'}^n| \to \infty\) for \(k \neq k'\);
(c) \(w_k \not= 0\) and \(\mathcal{J}'_{\text{per}}(w_k) = 0\) for each \(1 \leq k \leq \ell\);
(d) \(u_n - u_0 - \sum_{k=1}^{\ell} w_k(\cdot - y_k^n) \to 0\) in \(H^{\alpha/2}(\mathbb{R}^N)\) as \(n \to \infty\);
(e) \(\mathcal{J}(u_n) \to \mathcal{J}(u_0) + \sum_{k=1}^{\ell} \mathcal{J}_{\text{per}}(w_k)\).

**Remark 3.2.** Note that (G2)–(G4) imply that for every \(\varepsilon > 0\) there is \(C_{\varepsilon} > 0\) such that
\[ |g(x,u)| \leq \varepsilon |u| + C_{\varepsilon}|u|^{r-1} \]
for any \(u \in \mathbb{R}\) and a.e. \(x \in \mathbb{R}^N\).

Proof of Theorem 3.1 is, in fact, the reformulation of the proof of [5][Theorem 4.1]. We use [28][Lemma 2.4] instead of classical Lions’ lemma and we observe that the classical norm on \(H^{\alpha/2}(\mathbb{R}^N)\)
\[ \|u\|_{H^{\alpha/2}}^2 = \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} |u(x)|^2 dx \]
is \(\mathbb{Z}^N\)-invariant. Then the remaining arguments in proof of [5][Theorem 4.1] are the same.

4. **Existence of solutions for the sum of periodic and localized potential**

**Lemma 4.1.** Let \(\alpha \in (0,2)\) and denote by
\[ \|u\|_{H^{\alpha/2}}^2 = \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} |u(x)|^2 dx \]
the classical norm on the fractional Sobolev space \(H^{\alpha/2}(\mathbb{R}^N)\). Suppose that (V,1) hold. Then the following norm
\[ \|u\|^2 = \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx \]
is equivalent to \(\|\cdot\|_{H^{\alpha/2}}\).

**Proof.** Indeed - observe that
\[ \|u\|^2 \leq \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + |V|_\infty \int_{\mathbb{R}^N} |u(x)|^2 dx \leq \max\{1, |V|_\infty\} \|u\|_{H^{\alpha/2}}^2 \]
and
\[ \|u\|^2 \geq \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi + V_0 \int_{\mathbb{R}^N} |u(x)|^2 \, dx \geq \min\{1, V_0\} \|u\|^2_{H^{\alpha/2}}. \]

\[ \square \]

**Remark 4.2.** The norm equivalence is also true for \( \alpha = 2 \).

The above lemma implies that \( E^{\alpha/2} \) coincides with \( H^{\alpha/2}(\mathbb{R}^N) \). Thus our functional \( J : H^{\alpha/2}(\mathbb{R}^N) \to \mathbb{R} \) has the form
\[ J(u) = \frac{1}{2} \|u\|^2 - \mathcal{I}(u), \]
where
\[ \mathcal{I}(u) = \int_{\mathbb{R}^N} \left( F(x, u) - \frac{1}{q} \Gamma(x)|u|^q \right) \, dx. \]
The Nehari manifold is given by
\[ \mathcal{N} = \{ u \in H^{\alpha/2}(\mathbb{R}^N) \setminus \{0\} : \|u\|^2 = \mathcal{I}'(u)(u) \}. \]

Now we are ready to prove our first result.

**Proof of Theorem 1.1.**

By Theorem 2.1 we find a bounded minimizing sequence on \( \mathcal{N} \), i.e. sequence \( (u_n) \subset \mathcal{N} \) such that \( J(u_n) \to c := \inf_{\mathcal{N}} J > 0 \) and \( J'(u_n) \to 0 \). By Theorem 3.1 we have that
\[ J(u_n) \to J(u_0) + \sum_{k=1}^\ell J_{\text{per}}(w^k), \]
where \( w^k \) are critical points of the periodic part of the functional \( J \), i.e. critical points of
\[ J_{\text{per}}(u) = J(u) - \int_{\mathbb{R}^N} V_{\text{loc}}(x)u^2 \, dx. \]
Suppose that \( V_{\text{loc}} \equiv 0 \), i.e. \( J = J_{\text{per}} \). If \( u_0 = 0 \), we have
\[ c \leftarrow J(u_n) \to \sum_{k=1}^\ell J_{\text{per}}(w^k) \geq \ell c \]
and therefore \( \ell = 1 \) and \( w^1 \neq 0 \) is a ground state. If \( u_0 \neq 0 \) we have
\[ c \leftarrow J(u_n) \to J(u_0) + \sum_{k=1}^\ell J_{\text{per}}(w^k) \geq (\ell + 1)c \]
and therefore \( \ell = 0 \) and \( J(u_n) \to J(u_0) = c \), so \( u_0 \) is a ground state.

Suppose that \( V_{\text{loc}}(x) < 0 \) for a.e. \( x \in \mathbb{R}^N \). Then \( \inf_{\mathcal{N}_{\text{per}}} J_{\text{per}} = c_{\text{per}} > c \), where
\[ \mathcal{N}_{\text{per}} = \{ u \neq 0 : J_{\text{per}}'(u)(u) = 0 \}. \]
Suppose that $u_0 = 0$. Therefore
\[ c \leftarrow J(u_n) \to \sum_{k=1}^{\ell} J_{\text{per}}(w^k) \geq \ell c_{\text{per}} > \ell c \]
and $\ell = 0$. Thus $J(u_n) \to 0 = c$ – a contradiction. Therefore $u_0 \neq 0$ and observe that
\[ c \leftarrow J(u_n) \to J(u_0) + \sum_{k=1}^{\ell} J_{\text{per}}(w^k) \geq c + \ell c_{\text{per}}, \]
and $\ell = 0$. It means that $J(u_n) \to J(u_0) = c$ and $u_0$ is a ground state.

5. Multiplicity result for the periodic potential

Put $c = \inf_{N} J > 0$ and $\beta = \inf_{N} \| u \| > 0$. Theorem 1.1 provide that $c$ is attained at some function in $N$. By $\tau_k$ we denote the $\mathbb{Z}^N$-action on $H^{\alpha/2}(\mathbb{R}^N)$, i.e.
\[ \tau_k u = u(\cdot - k). \]

Obviously, $\tau_k \tau_{-k} u = u$.

**Lemma 5.1.** There holds
\[ \langle \tau_k u, v \rangle = \langle u, \tau_{-k} v \rangle \]
for every $u, v \in H^{\alpha/2}(\mathbb{R}^N)$ and $k \in \mathbb{Z}^N$.

**Proof.** For $\alpha = 2$ the observation is trivial. Let $\alpha < 2$. Note that
\[ F(\tau_k u)(\xi) = e^{-2\pi i \xi \cdot k} F(u)(\xi) \]
for every $k \in \mathbb{Z}^N$ and $\xi \in \mathbb{R}^N$. Thus
\[ F(\tau_k u) \cdot \overline{F(v)} = e^{-2\pi i \xi \cdot k} F(u) \overline{F(v)} = F(u) \cdot e^{-2\pi i \xi \cdot (-k)} F(v) = F(u) \cdot \overline{F(\tau_{-k} v)}. \]
Therefore
\[ \int_{\mathbb{R}^N} |\xi|^{\alpha} \overline{\tau_k u(\xi) \overline{v(\xi)}} d\xi = \int_{\mathbb{R}^N} |\xi|^{\alpha} \hat{u}(\xi) \overline{\tau_{-k} v(\xi)} d\xi. \]

Obviously, using a change of variables $x \mapsto x + k$
\[ \int_{\mathbb{R}^N} V(x)(\tau_k u) v dx = \int_{\mathbb{R}^N} V(x) u(\tau_k v) dx \]
and we conclude. \qed
Remark 5.2. For given \( k \in \mathbb{Z}^N \), let us consider \( \tau_k \) as an operator \( \tau_k : H^{\alpha/2}(\mathbb{R}^N) \to H^{\alpha/2}(\mathbb{R}^N) \). Then obviously \( \tau_k \) is linear. Moreover
\[
\|\tau_k u\| = \|u\|,
\]
thus \( \tau_k \) is a bounded operator and \( \|\tau_k\| = 1 \). Thus we may consider an adjoint operator \( \tau_k^* : H^{\alpha/2}(\mathbb{R}^N) \to H^{\alpha/2}(\mathbb{R}^N) \). The above lemma says that
\[
\tau_k^* = \tau_{-k}.
\]
Moreover \( \tau_k \) is an isomorphism and \( \tau_k^{-1} = \tau_k = \tau_k^* \). Thus \( \tau_k \) is an orthogonal operator.

Lemma 5.3. Let \( \alpha \in (0, 2] \). The functional \( J \) is \( \mathbb{Z}^N \)-invariant.

Proof. Let us start with the trivial observation that if \( u \in H^{\alpha/2}(\mathbb{R}^N) \), and \( w \in \mathcal{O}(u) \), then by Lemma 5.1
\[
\|u\| = \|w\|.
\]
Indeed \( -w = \tau_k u \) for some \( k \in \mathbb{Z}^N \). Then
\[
\|w\|^2 = \langle w, w \rangle = \langle \tau_k u, \tau_k u \rangle = \langle u, \tau_{-k} \tau_k u \rangle = \langle u, u \rangle = \|u\|^2.
\]
Then
\[
J(w) = \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}^N} F(x, w) \, dx + \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|w|^q \, dx.
\]
Changing variables in the integrals \( x \mapsto x + k \) and the \( \mathbb{Z}^N \)-periodicity of \( F \) and \( \Gamma \) in \( x \) gives
\[
\int_{\mathbb{R}^N} F(x, w) \, dx = \int_{\mathbb{R}^N} F(x, u) \, dx, \quad \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|w|^q \, dx = \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|u|^q \, dx.
\]
Therefore \( J(w) = J(u) \).

\[ \square \]

Lemma 5.4. \( \mathcal{N} \) is \( \mathbb{Z}^N \)-invariant.

Proof. Suppose that \( u \in \mathcal{N} \). Then
\[
J'(\tau_k u)(\tau_k u) = \|\tau_k u\|^2 - \int_{\mathbb{R}^N} f(x, \tau_k u) \tau_k u \, dx + \int_{\mathbb{R}^N} \Gamma(x)|\tau_k u|^q \, dx.
\]
By the \( \mathbb{Z}^N \)-periodicity of \( f \) and \( \Gamma \) we have
\[
\int_{\mathbb{R}^N} f(x, \tau_k u) \tau_k u \, dx = \int_{\mathbb{R}^N} f(x, u) u \, dx, \quad \int_{\mathbb{R}^N} \Gamma(x)|\tau_k u|^q \, dx = \int_{\mathbb{R}^N} \Gamma(x)|u|^q \, dx.
\]
Moreover, by Lemma 5.1 \( \|\tau_k u\| = \|u\| \) and finally \( J'(\tau_k u)(\tau_k u) = 0 \), which shows that \( \mathcal{O}(u) \subset \mathcal{N} \).

\[ \square \]

Remark 5.5. Lemma 5.1 implies that the unit sphere \( S^1 \) is \( \mathbb{Z}^N \)-invariant.
Recall that for each $u \in H^{\alpha/2}(\mathbb{R}^N)$ there is unique number $t(u) > 0$ such that $t(u)u \in \mathcal{N}$ and moreover the function $m : S^1 \to \mathcal{N}$ given by $m(u) = t(u)u$ is a homeomorphism (see [5]). The inverse $m^{-1} : \mathcal{N} \to S^1$ is given by $m^{-1}(u) = u/\|u\|$. 

**Lemma 5.6. Functions:**

- $m : S^1 \to \mathcal{N}$,
- $m^{-1} : \mathcal{N} \to S^1$,
- $\nabla J : H^{\alpha/2}(\mathbb{R}^N) \to H^{\alpha/2}(\mathbb{R}^N)$,
- $\nabla (J \circ m) : S^1 \to H^{\alpha/2}(\mathbb{R}^N)$

are $\mathbb{Z}^N$-equivariant.

**Proof.**

- **Equivariance of $m$.**
  Take $u \in S^1$, since $\|u\| = \|\tau_k u\|$, we have $\tau_k u \in S^1$. There is unique number $t = t(u) > 0$ such that $m(u) = t(u)u \in \mathcal{N}$. We claim that $t(u)\tau_k u \in \mathcal{N}$. Indeed
  $$t(u)\tau_k u = \tau_k (t(u)u) = \tau_k m(u) \in \mathcal{N},$$
  by Lemma 5.4. Thus
  $$m(\tau_k u) = t(u)\tau_k u = \tau_k m(u).$$

- **Equivariance of $m^{-1}$.**
  Let $u \in \mathcal{N}$. By Lemma 5.4 we have that $\tau_k u \in \mathcal{N}$. Observe that
  $$m^{-1}(\tau_k u) = \frac{\tau_k u}{\|\tau_k u\|} = \frac{\tau_k u}{\|u\|} = \tau_k \left( \frac{u}{\|u\|} \right) = \tau_k m^{-1}(u).$$

- **Equivariance of $\nabla J$.**
  Take $u, v \in H^{\alpha/2}(\mathbb{R}^N)$. Then
  $$\langle \nabla J(\tau_k u), v \rangle = J'(\tau_k u)(v)$$
  $$= \langle \tau_k u, v \rangle - \int_{\mathbb{R}^N} f(x, \tau_k u) v \, dx + \int_{\mathbb{R}^N} \Gamma(x)|\tau_k u|^{q-2}(\tau_k u)v \, dx$$
  $$= \langle u, \tau_{-k} v \rangle - \int_{\mathbb{R}^N} f(x, u)\tau_{-k} v \, dx + \int_{\mathbb{R}^N} \Gamma(x)|u|^{q-2}u(\tau_{-k} v) \, dx$$
  $$= \langle \nabla J(u), \tau_{-k} v \rangle = \langle \tau_k \nabla J(u), v \rangle$$
  Therefore $\nabla J(\tau_k u) = \tau_k \nabla J(u)$ for every $u \in H^{\alpha/2}(\mathbb{R}^N)$.

- **Equivariance of $\nabla (J \circ m)$.**
  For $u \in S^1$ and $z \in T_u S^1$, by the abstract setting in [5] we know that
  $$(J \circ m)'(u)(z) = \|m(u)\|J'(m(u))(z).$$
So take \( u \in S^1 \) and \( z \in T_{\tau_k}S^1 \). Then \( \tau_{-k}z \in T_u S^1 \). Therefore
\[
\langle \nabla(J \circ m)(\tau_k u), z \rangle = (J \circ m)'(\tau_k u)(z) = \|m(\tau_k u)\|J'(m(\tau_k u))(z) \\
= \|\tau_k m(u)\|J'(\tau_k m(u))(z) = \|m(u)\|\langle \nabla J(\tau_k m(u)), z \rangle \\
= \|m(u)\|\langle \tau_k \nabla J(m(u)), z \rangle = \|m(u)\|\langle \nabla J(m(u)), \tau_{-k} z \rangle \\
= (J \circ m)'(u)(\tau_{-k}z) = \langle \nabla(J \circ m)(u), \tau_{-k}z \rangle = \langle \tau_k \nabla(J \circ m)(u), z \rangle.
\]
Thus \( \tau_k \nabla(J \circ m)(u) = \nabla(J \circ m)(\tau_k u) \) for every \( u \in S^1 \).

\[\square\]

**Lemma 5.7.** Function \( m^{-1} : N \to S^1 \) is Lipschitz continuous.

**Proof.** Let \( u, v \in N \). Observe that
\[
\|m^{-1}(u) - m^{-1}(v)\| = \left\| \frac{u - v}{\|u\|} + \frac{v\|v\| - v\|u\|}{\|u\| \cdot \|v\|} \right\| \leq \left\| \frac{u - v}{\|u\|} \right\| + \frac{\|v\| - \|u\|}{\|u\| \cdot \|v\|} \leq 2\frac{\|u - v\|}{\|u\|} \leq \frac{2}{\beta}\|u - v\|.
\]

\[\square\]

To show Theorem 1.2 we employ the method introduced by A. Szulkin and T. Weth in [31]. Put \( \mathcal{C} = \{ u \in S^1 : (J \circ m)'(u) = 0 \} \). Choose a set \( \mathcal{F} \subset \mathcal{C} \) such that \( \mathcal{F} = -\mathcal{F} \) and for each orbit \( O(w) \) there is unique representative \( v \in \mathcal{F} \). To show Theorem 1.3 we need to show that \( \mathcal{F} \) is infinite. Suppose by a contradiction that
\[
\mathcal{F} \text{ is finite.}
\]

Put
\[
\kappa = \inf \{ \|v - w\| : v, w \in \mathcal{C}, v \neq w \}
\]
and note that \( \kappa > 0 \) — see [31][Lemma 2.13].

The following lemma has the crucial role in the proof of the multiplicity result and originally has been proven in [31][Lemma 2.14]. Since the nonlinear term is sign-changing we need a slight modification of the proof.

**Lemma 5.8.** Let \( d \geq c \). If \( (v^1_n), (v^2_n) \subset S^1 \) are two Palais-Smale sequences for \( J \circ m \) such that \( J(m(v^i_n)) \leq d, i = 1, 2 \), then
\[
\|v^1_n - v^2_n\| \to 0
\]
or
\[
\limsup_{n \to \infty} \|v^1_n - v^2_n\| \geq \rho(d) > 0
\]
where \( \rho(d) \) depends only on \( d \), but not on the particular choice of sequences.
Proof. We have that $u_i^* = m(v_i^*)$, $i = 1, 2$, are Palais-Smale sequences for $\mathcal{J}$. Moreover they are bounded in $H^{\alpha/2}(\mathbb{R}^N)$, since $\mathcal{J}$ is coercive on $\mathcal{N}$. Therefore $(u_1^*)$ and $(u_2^*)$ are bounded in $L^2(\mathbb{R}^N)$, say $|u_1^*|_2 + |u_2^*|_2 \leq M$ for some $M > 0$.

- **Case 1:** Assume that $|u_1^* - u_2^*|_p \to 0$. Fix $\varepsilon > 0$. Then, by (F1), (F2) we have

$$
\|u_1^* - u_2^*\|^2 = \mathcal{J}'(u_1^*)(u_1^* - u_2^*) - \mathcal{J}'(u_2^*)(u_1^* - u_2^*)
+ \int_{\mathbb{R}^N} [f(x, u_1^*) - f(x, u_2^*)](u_1^* - u_2^*) \, dx
- \int_{\mathbb{R}^N} \Gamma(x) \left[ |u_1^*|^{q-2} u_1^* - |u_2^*|^{q-2} u_2^* \right] (u_1^* - u_2^*) \, dx
\leq \varepsilon \|u_1^* - u_2^*\|
+ \int_{\mathbb{R}^N} \left[ \varepsilon (|u_1^*| + |u_2^*|) + C_\varepsilon (|u_1^*|^{p-1} + |u_2^*|^{p-1}) \right] |u_1^* - u_2^*| \, dx
- \int_{\mathbb{R}^N} \Gamma(x) \left[ |u_1^*|^{q-2} u_1^* - |u_2^*|^{q-2} u_2^* \right] (u_1^* - u_2^*) \, dx
\leq (1 + C_0)\varepsilon \|u_1^* - u_2^*\| + D_\varepsilon |u_1^* - u_2^*|_p + C_1 |\Gamma|_\infty |u_1^* - u_2^*|^q
$$

for each $n \geq n_\varepsilon$ and some constants $C_0, C_1, D_\varepsilon > 0$. By our assumption we have that

$$D_\varepsilon |u_1^* - u_2^*|_p \to 0.$$

Observe that

$$C_1 |\Gamma|_\infty |u_1^* - u_2^*|^q \leq C_1 |\Gamma|_\infty |u_1^* - u_2^*|_{2p} |u_1^* - u_2^*|_{p} \leq (1 - \theta q) |u_1^* - u_2^*|_{p} \to 0,$$

where $\theta \in (0, 1)$ is such that

$$\frac{1}{q} = \frac{1}{2} + \frac{1 - \theta}{p}.$$

Thus

$$C_1 |\Gamma|_\infty |u_1^* - u_2^*|^q \leq C_1 |\Gamma|_\infty M^{\theta q} |u_1^* - u_2^*|_{p} \to 0.$$

Finally

$$\lim_{n \to \infty} \sup_{n} \|u_1^* - u_2^*\|^2 \leq \lim_{n \to \infty} \sup_{n} \|(1 + C_0)\varepsilon \|u_1^* - u_2^*\|
+ \lim_{n \to \infty} D_\varepsilon |u_1^* - u_2^*|_p
+ \lim_{n \to \infty} \sup_{n} C_1 |\Gamma|_\infty |u_1^* - u_2^*|^q
= (1 + C_0)\varepsilon \lim_{n \to \infty} \sup_{n} \|u_1^* - u_2^*\|$$

for every $\varepsilon > 0$. Therefore $\lim_{n \to \infty} \|u_1^* - u_2^*\| = 0$. Finally

$$\|u_1^* - u_2^*\| = \|m^{-1}(u_1^*) - m^{-1}(u_2^*)\| \leq L \|u_1^* - u_2^*\| \to 0,$$

where $L > 0$ is a Lipschitz constant for $m^{-1}$. 
• **Case 2:** Assume that \(|u_n^1 - u_n^2|_p \neq 0\). By the Lions lemma there are \(y_n \in \mathbb{R}^N\) such that
\[
\int_{B(y_n,1)} |u_n^1 - u_n^2|^2 \, dx = \max_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n^1 - u_n^2|^2 \, dx \geq \varepsilon
\]
for some \(\varepsilon > 0\). In view of Lemma 5.6 we can assume that \((y_n)\) is bounded. Therefore, up to a subsequence we have
\[
\begin{align*}
\lim u_n^1 & \rightarrow u^1, \\
\lim u_n^2 & \rightarrow u^2
\end{align*}
\]
where \(u^1 \neq u^2\) and \(J'(u^1) = J'(u^2) = 0\), and
\[
\|u_n^1\| \rightarrow \alpha^1, \quad \|u_n^2\| \rightarrow \alpha^2,
\]
where \(\beta \leq \alpha^i \leq \nu(d) = \sup\{\|u\| : u \in \mathcal{N}, J(u) \leq d\}, i = 1, 2\). Suppose that \(u^1 \neq 0\) and \(u^2 \neq 0\). Therefore \(u^i \in \mathcal{N}\) for \(i = 1, 2\). Moreover
\[
v^i = m^{-1}(u^i) \in S^1, \quad i = 1, 2
\]
and \(v^1 \neq v^2\). Then
\[
\liminf_{n \to \infty} \|v_n^1 - v_n^2\| = \liminf_{n \to \infty} \left\| \frac{u_n^1}{\|u_n^1\|} - \frac{u_n^2}{\|u_n^2\|} \right\| \geq \left\| \frac{u_n^1}{\alpha^1} - \frac{u_n^2}{\alpha^2} \right\| = \|\beta_1 v_1 - \beta_2 v_2\|,
\]
where \(\beta_i = \frac{\|u\|_{\alpha^i}}{\alpha^i} \geq \frac{\beta}{\nu(d)}, i = 1, 2\). Of course \(\|v^1\| = \|v^2\| = 1\). Therefore
\[
\liminf_{n \to \infty} \|v_n^1 - v_n^2\| \geq \|\beta_1 v^1 - \beta_2 v^2\| \geq \min_{i=1,2} \{\beta_i\} \|v^1 - v^2\| \geq \frac{\beta \kappa}{\nu(d)}.
\]
If \(u^2 = 0\), then \(u^1 \neq u^2 = 0\). Therefore
\[
\liminf_{n \to \infty} \|v_n^1 - v_n^2\| = \liminf_{n \to \infty} \left\| \frac{u_n^1}{\|u_n^1\|} - \frac{u_n^2}{\|u_n^2\|} \right\| \geq \left\| \frac{u_n^1}{\alpha^1} - \frac{u_n^2}{\alpha^2} \right\| = \left\| \frac{u_n^1}{\alpha^1} \right\| \geq \frac{\beta}{\nu(d)}.
\]
The case \(u^1 = 0\) is similar, the proof is completed.

\[\square\]

**Proof of Theorem 1.2.** The unit sphere \(S^1 \subset H^{\alpha/2}(\mathbb{R}^N)\) is a Finsler \(C^{1,1}\)-manifold and by [30][Lemma II.3.9], \(\mathcal{J} \circ m : S^1 \rightarrow \mathbb{R}\) admits a pseudo-gradient vector field. The obtained discreteness od Palais-Smale sequences (Lemma 5.8) allows us to repeat the proof of Lemma 2.15, Lemma 2.16 and Theorem 1.2 from [31] in our case. We show that for every \(k \in \mathbb{N}\) there is \(u \in S^1\) such that \((\mathcal{J} \circ m)'(u) = 0\) and \(\mathcal{J}(m(u)) = c_k\), where
\[
c_k = \inf\{d \in \mathbb{R} : \gamma \left(\{v \in S^1 : \mathcal{J}(m(v)) \leq d\}\right) \geq k\}
\]
is the Lusternik-Schnirelmann value and \(\gamma\) denote the Krasnoselkii genus (see [30]). Moreover \(c_k < c_{k+1}\), thus we get a contradiction.

\[\square\]
6. Existence of solutions for the coercive potential

We will need the following form of the Sobolev-Gagliardo-Nirenberg inequality.

**Lemma 6.1** (Proposition II.3 in [28]). Let \( r > 1 \). Then there is a positive constant \( C > 0 \), such that for every function \( u \in H^{\alpha/2}(\mathbb{R}^N) \) there hold
\[
|u|^{r+1} \leq C \|u\|_{H^{\alpha/2}}^{(r-1)N/\alpha} |u|^{r+1} \|u\|_{2}^{r+1 - (r-1)N/\alpha}.
\]

In [10] it was proven that \( E^{\alpha/2} \) is compactly embedded into \( L^r(\mathbb{R}^N) \) when \( N > \alpha \) and \( 2 \leq r < 2^*_\alpha \). However we will show that the method introduced by S. Secchi in [28] may be applied in this case.

**Proof of Theorem 1.3.**

In view of Theorem 2.1, we obtain a bounded minimizing sequence \( (u_n) \subset \mathcal{N} \), i.e.
\[
\mathcal{J}(u_n) \to \inf_{\mathcal{N}} \mathcal{J} =: c, \quad \mathcal{J}'(u_n) \to 0.
\]

Therefore we may suppose that \( u_n \rightharpoonup u_0 \) in \( E^{\alpha/2} \) and \( u_n \to u_0 \) in \( L^r_{\text{loc}}(\mathbb{R}^N) \) for \( 1 \leq r < 2^*_\alpha \). It is a classical argument that \( u_0 \) is a weak solution to our problem. We only need to check whether \( u_0 \neq 0 \). To show this, observe that for \( n \geq n_0 \)
\[
\frac{c}{2} \leq \mathcal{J}(u_n) - \frac{1}{2} \mathcal{J}'(u_n)(u_n)
= \frac{1}{2} \int_{\mathbb{R}^N} (f(x, u_n)u_n - 2F(x, u_n)) \, dx - \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \Gamma(x)|u|^q \, dx
\leq \frac{1}{2} \int_{\mathbb{R}^N} (f(x, u_n)u_n - 2F(x, u_n)) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} f(x, u_n)u_n \, dx
\leq \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon|u_n|^2 + C_\varepsilon|u_n|^p) \, dx,
\]
where \( n_0 \geq 1 \) is large enough. Therefore
\[
\frac{c}{2} \leq \frac{\varepsilon}{2} |u_n|^2 + \frac{C_\varepsilon}{2} |u_n|^p.
\]

By the Sobolev-Gagliardo-Nirenberg inequality (Lemma 6.1) we obtain
\[
\frac{c}{2} \leq \frac{\varepsilon}{2} |u_n|^2 + \frac{C_\varepsilon}{2} \|u_n\|_{H^{\alpha/2}(\mathbb{R}^N)}^{(p-2)N/\alpha} \cdot |u_n|^p_{2^{(p-2)N/\alpha}}.
\]

Observe that the boundedness of \( (u_n) \) in \( E^{\alpha/2} \) with respect to the norm \( \| \cdot \| \) implies the boundedness of \( (u_n) \) in \( H^{\alpha/2}(\mathbb{R}^N) \) with respect to the classical norm \( \| \cdot \|_{H^{\alpha/2}(\mathbb{R}^N)} \). Therefore \( \|u_n\|_{H^{\alpha/2}(\mathbb{R}^N)} \leq D \) for some \( D > 0 \). Thus
\[
\frac{c}{2} \leq \frac{\varepsilon}{2} |u_n|^2 + \frac{C_\varepsilon}{2} D^{(p-2)N/\alpha} \cdot |u_n|^p_{2^{(p-2)N/\alpha}}.
\]
Denote $\hat{C}_\varepsilon = \frac{C \cdot C_\varepsilon^p}{2} D^{(p-2)N}$. Then

$$\frac{c}{2} \leq \frac{\varepsilon}{2} |u_n|^2 + \hat{C}_\varepsilon |u_n|^2 \frac{p \cdot (p-2)N}{\alpha}.$$ 

Take any $\varepsilon \leq \frac{c}{2 (\sup_n \|u_n\|)^2}$. Then

$$\frac{c}{2} \leq \frac{c |u_n|^2}{4 (\sup_n \|u_n\|)^2} + \hat{C}_\varepsilon |u_n|^2 \frac{p \cdot (p-2)N}{\alpha}.$$ 

Of course

$$\frac{|u_n|^2}{(\sup_n \|u_n\|)^2} \leq 1$$

and therefore

$$\frac{c}{2} \leq \frac{c}{4} + \hat{C}_\varepsilon |u_n|^2 \frac{(p-2)N}{\alpha}.$$ 

Finally

$$\frac{c}{4} \leq \hat{C}_\varepsilon |u_n|^2 \frac{(p-2)N}{\alpha}.$$ 

Therefore

$$\ln \frac{c}{4} \leq \left( p - \frac{(p-2)N}{\alpha} \right) \ln (C_1 |u_n|),$$

where $C_1 = \hat{C}_\varepsilon \frac{1}{\alpha \cdot (p-2)N}$. Thus

$$\frac{\alpha}{\alpha p - (p-2)N} \ln \frac{c}{4} \leq \ln (C_1 |u_n|).$$

And finally

$$|u_n|^2 \geq \left( \frac{1}{C_1} \exp \left( \frac{\alpha}{\alpha p - (p-2)N} \ln \frac{c}{4} \right) \right)^2 =: \tilde{c} > 0.$$ 

Take any $R > 0$ and observe that

$$|u_n|^2 = \int_{B(0,R)} |u_n|^2 \, dx + \int_{\mathbb{R}^N \setminus B(0,R)} |u_n|^2 \, dx.$$ 

Assume by a contradiction that $u_0 = 0$, i.e. $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Then for every $R > 0$ there is $n_0$ such that for $n \geq n_0$ we have

$$\int_{B(0,R)} |u_n|^2 \, dx \leq \frac{\tilde{c}}{2}.$$ 

Then

$$\int_{\mathbb{R}^N \setminus B(0,R)} |u_n|^2 \, dx \geq \frac{\tilde{c}}{2}.$$ 

On the other hand

$$\frac{\tilde{c}}{2} \leq \int_{\mathbb{R}^N \setminus B(0,R)} |u_n|^2 \, dx = \int_{\mathbb{R}^N \setminus B(0,R)} \frac{V(x) |u_n|^2}{V(x)} \, dx \leq \frac{1}{\inf_{|x| \geq R} V(x)} \int_{\mathbb{R}^N \setminus B(0,R)} V(x) |u_n|^2 \, dx$$

$$\leq \frac{\|u_n\|^2}{\inf_{|x| \geq R} V(x)} \leq \sup_n \frac{\|u_n\|^2}{\inf_{|x| \geq R} V(x)}.$$
Taking $R > 0$ big enough we obtain a contradiction, since $V(x) \to \infty$ as $|x| \to \infty$. Therefore $u_0 \neq 0$, which completes the proof.

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