Holomorphic Functions on an Algebraic Group
Invariant under Zariski-dense Subgroups

by
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1. Introduction

Let $G$ be a reductive complex linear-algebraic group, $\Gamma$ a subgroup, $\mathcal{O}(G)^\Gamma$ the algebra of $\Gamma$-invariant holomorphic functions on $G$. It is known [1] that $\mathcal{O}(G)^\Gamma = \mathbb{C}$ if $\Gamma$ is dense in $G$ with respect to the algebraic Zariski-topology.

We are interested in similar results for non-reductive groups. If $G$ is a complex linear-algebraic group with $G/G'$ non-reductive (where $G'$ denotes the commutator group), then there exists a surjective group morphism $\tau : G \to (\mathbb{C}, +)$ and $\Gamma = \tau^{-1}(\mathbb{Z})$ is a Zariski-dense subgroup of $G$ with $\mathcal{O}(G)^\Gamma \simeq \mathcal{O}(\mathbb{C}^*) \neq \mathbb{C}$. Hence we are led to the question whether the following two properties are equivalent:

Let $G$ be a connected complex linear-algebraic group.
(i) $G/G'$ is reductive.
(ii) $\mathcal{O}(G)^\Gamma = \mathbb{C}$ for every Zariski-dense subgroup $\Gamma$.

The above argument gave us (ii) $\Rightarrow$ (i) and the result of Barth and Otte [1] implies the equivalence of (i) and (ii) for $G$ reductive.

We will prove that (i) and (ii) are likewise equivalent in the following two cases:
a) $G$ is solvable.
b) The adjoint representation of $S$ on $\text{Lie}(U)$ has no zero weight, where $S$ denotes a maximal connected semisimple subgroup of $G$ and $U$ the unipotent radical of $G$.

Case b) is equivalent to each of the following two conditions
b') $G/G'$ is reductive and the semisimple elements are dense in $G'$.
b'') $G/G'$ is reductive and $N_{G'}(T)/T$ is finite, where $T$ is a maximal torus in $G'$ and $N_{G'}(T)$ denotes the normalizer of $T$ in $G'$.

For instance, if we take $G$ to be a semi-direct product $SL_2(\mathbb{C}) \ltimes \rho(\mathbb{C}^n, +)$ with $\rho : SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ irreducible, then $G$ fulfills the condition of case b) if and only if $n$ is an even number.

The proof for case a) is based on the usual solvable group methods and the structure theorem on holomorphically separable solvmanifolds by Huckleberry and E. Oeljeklaus.

The proof for case b) relies on the discussion of semisimple elements of infinite order in such a $\Gamma$. For this reason we conclude the paper with an example of Margulis which implies that $G = SL_2(\mathbb{C}) \ltimes \rho(\mathbb{C}^3, +)$ ($\rho$ irreducible) admits a Zariski-dense discrete subgroup $\Gamma$ such that no element of $\Gamma$ is semisimple. Thus condition b) is really needed in order to
find semisimple elements in Zariski-dense subgroups. In consequence, our method does not work for the example of Margulis. However, this only means that we can not prove $O(G)^T = C$ for Margulis’ example. We have no knowledge whether there actually exist non-constant holomorphic functions in this case.

Finally we discuss invariant meromorphic and plurisubharmonic functions on certain groups.

2. Solvable groups

Here we will discuss solvable groups. First we will develope some auxiliary lemmata.

Lemma 1. Let $G$ be a connected complex linear-algebraic group such that $G/G'$ is reductive.

Then $[G, G'] = G'$.

Proof. By taking the appropriate quotient, we may assume $[G, G'] = \{e\}$. We have to show that this implies $G' = \{e\}$. Now $[G, G'] = \{e\}$ means that $G'$ is central, hence $Ad(G)$ factors through $G/G'$. But $G/G'$ is reductive and acts trivially (by conjugation) on both $G/G'$ and $G'$. Due to complete reducibility of representations of reductive groups it follows that $Ad(G)$ is trivial, i.e. $G$ is abelian, i.e. $G' = \{e\}$. \qed

Lemma 2. Let $G$ be a connected complex linear-algebraic group, $H \subset G'$ a connected complex Lie subgroup which is normal in $G'$.

Then $H$ is algebraic.

Proof. Let $U$ denote the unipotent radical of $G'$. Then $G'/U$ is semisimple. Now $A = (H \cap U)^0$ is algebraic, because every connected complex Lie subgroup of a unipotent group is algebraic. Normality of $H$ implies that $H/(H \cap U)$ is semisimple. Hence $H/A$ is semisimple, too. It follows that $H/A$ is an algebraic subgroup of $G'/A$. Thus $H$ has to be algebraic. \qed

Lemma 3. Let $G$ be a connected topological group, $H$ a normal subgroup, such that $H \cap G'$ is totally disconnected.

Then $H$ is central.

Proof. For each $h \in H$ the set $S_h = \{ghg^{-1}h^{-1} : g \in G\}$ is both totally disconnected and connected and therefore reduces to $\{e\}$. \qed

Lemma 4. Let $G$ be a connected complex linear-algebraic group, $A \subset G'$ a complex Lie subgroup which is normal in $G$ and Zariski-dense in $G'$. Assume moreover that $[G, G'] = G'$.

Then $A = G'$. 2
Proof. The connected component $A^0$ of $A$ is algebraic (Lemma 2). Thus $G/A^0$ is again algebraic. Moreover $G'/A^0$ is the commutator group of $G/A^0$. Therefore, by replacing $G$ with $G/A^0$ we may assume that $A$ is totally disconnected. But totally disconnected normal subgroups of connected Lie groups are central (Lemma 3). Since $A$ is Zariski-dense in $G'$, and $[G, G'] = G'$, this may occur only for $A = G' = \{e\}$. Thus $A = G'$.

**Theorem 1.** Let $G$ be a connected solvable complex linear-algebraic group, $\Gamma$ a subgroup which is dense in the algebraic Zariski-topology. Assume that $G/G'$ is reductive.

Then $O(G)^\Gamma = C$.

Proof. Let $G/\Gamma \to G/H$ denote the holomorphic reduction, i.e.

$$H = \{g \in G : f(g) = f(e) \forall f \in O(G)^\Gamma\}.$$  

Now $\Gamma \subset H$ normalizes $H^0$. Since $\Gamma$ is Zariski-dense in $G$ and the normalizer of a connected Lie subgroup is necessarily algebraic, it follows that $H^0$ is normal in $G$. Let $A = H^0 \cap G'$. This is again a closed normal subgroup in $G$. By a result of Huckleberry and E. Oeljeklaus [3] $H/H^0$ is almost nilpotent (i.e. admits a subgroup of finite index which is nilpotent). Let $\Gamma_0$ be a subgroup of finite index in $\Gamma$ with $\Gamma_0/(\Gamma_0 \cap H^0)$ nilpotent. By definition this means there exists a number $k$ such that $C^k\Gamma_0 \subset H^0$ where $C^k$ denotes the central series. Now $[G, G'] = G'$ implies $C^kG = G'$ for all $k \geq 1$. Therefore $C^k\Gamma_0$ is Zariski-dense in $G'$. It follows that $A = H^0 \cap G'$ is a closed normal Lie subgroup of $G$ which is Zariski-dense in $G'$. By the preceding lemma it follows that $A = G'$, i.e. $G' \subset H$. Now $G/G'$ is assumed to be reductive. Thus the statement of the theorem now follows from the result for reductive groups ([1]).

3. GROUPS WITH MANY SEMISIMPLE ELEMENTS

Here we will prove the following theorem.

**Theorem 2.** Let $G$ be a connected complex linear-algebraic group. Assume that $G/G'$ is reductive and that furthermore one (hence all) of the following equivalent conditions is fulfilled.

1. $G'$ contains a dense open subset $\Omega$ such that each element in $\Omega$ is semisimple.
2. For any maximal torus $T$ in $G'$ the quotient $N_{G'}(T)/T$ is finite.
3. Let $S$ denote a maximal connected semisimple subgroup of $G$ and $U$ the unipotent radical of $G$. Let $\rho : S \to GL(\text{Lie} U)$ denote the representation obtained by restriction from the adjoint representation $\text{Ad} : G \to GL(\text{Lie} G)$. The condition is that all weights of $\rho$ are non-zero.

Under these assumptions $O(G)^\Gamma = C$ for any Zariski-dense subgroup $\Gamma \subset G$.  

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Examples.
(a) Let $G$ be a reductive group. Then $G/G'$ is reductive and $G'$ semisimple, hence $N_{G'}(T)/T$ finite for any maximal torus $T \subset G'$. Therefore this theorem is a generalization of the result of Barth and Otte [1] on reductive groups.

(b) Let $G$ be a parabolic subgroup of a semisimple group $S$. $G/G'$ is obviously reductive. Furthermore a maximal torus $T$ in $G$ is already a maximal torus in $S$. Hence $N_S(T)/T$ is finite. Consequently $N_G(T)/T$ is finite and $G$ fulfills the assumptions of the theorem.

(c) Let $G$ be a semi-direct product of $SL_2(\mathbb{C})$ with a unipotent group $U \simeq \mathbb{C}^n$ induced by an irreducible representation $\xi : SL_2(\mathbb{C}) \to GL(U)$. Then $G$ fulfills the assumptions of the theorem if and only if $n$ is even.

Now we will demonstrate that (1), (2) and (3) are indeed equivalent. The equivalence of (2) and (3) is rather obvious from standard results on algebraic groups. For the equivalence of (1) and (2) we need some elementary facts on semisimple elements in a connected algebraic group $G$. Let $G_s$ denote the set of all semisimple elements in $G$ and $T$ be a maximal torus in $G$. Now $g \in G_s$ iff $g$ is conjugate to an element in $T$. It follows that $G_s$ is the image of the map $\zeta : G \times T \to G_s$ given by $\zeta(g, t) = gtg^{-1}$. In particular $G_s$ is a constructible set. Now a torus contains only countably many algebraic subgroups, hence a generic element $h \in T$ generates a Zariski-dense subgroup of $T$. It follows that for a generic element $h \in T$ the assumption $g \in G$ with $ghg^{-1} \in T$ implies $gTg^{-1} = T$. From this it follows that a generic fiber of $\zeta$ has the dimension $\dim N_G(T)$. Therefore the dimension of $G_s = \text{Image}(\zeta)$ equals $\dim G - \dim N_G(T)$. Thus we obtained the following lemma, which implies the equivalence of (1) and (2).

**Lemma 5.** Let $G$ be a connected linear-algebraic group, $T$ a maximal torus and $G_s$ the set of semisimple elements in $T$.

Then $G_s$ is dense in $G$ if and only if $\dim N_G(T) = \dim T$.

Next we state some easy consequences of the assumptions of Theorem 2.

**Lemma 6.** Let $G$ be an algebraic group fulfilling the assumptions of Theorem 2 and $\tau : G \to H$ a surjective morphism of algebraic groups.

Then $H$ likewise fulfills the assumptions of Theorem 2.

**Proof.** Surjectivity of $\tau$ gives a surjective morphism of algebraic groups from $G/G'$ onto $H/H'$. Therefore $H/H'$ is reductive. The surjectivity of $\tau$ furthermore implies $\tau(G') = H'$. Since morphisms of algebraic groups map semisimple elements to semisimple elements, it follows that $H$ fulfills condition (1). □
Lemma 7. Let $G$ be an algebraic group fulfilling the assumptions of Theorem 2. Then
the center $Z$ of $G$ must be reductive.

Proof. Condition (2) implies that $(Z \cap G')^0$ is contained in a maximal torus of $G'$. Since $G/G'$ is reductive, this implies that $Z$ is reductive. \hfill \Box

The following lemma illuminates why semisimple elements are important for our purposes.

Lemma 8. Let $G$ be a complex linear-algebraic group, $g \in G$ an element of infinite order, $\Gamma$ the subgroup generated by $g$ and $H$ the Zariski-closure of $\Gamma$.

Then $Z = H/\Gamma$ is a Cousin group (hence in particular $O(Z) = C$) if $g$ is semisimple; but $Z$ is biholomorphic to some $(C^*)^n$ (hence holomorphically separable) if $g$ is not semisimple.

Proof. Note that $\bar{\Gamma} = H$ implies $H = H^0 \Gamma$. Hence $H/\Gamma = H^0/(H^0 \cap \Gamma)$ is connected. If $g$ is semisimple, the Zariski-closure of $\Gamma$ is reductive and the statement follows from [1]. If $g$ is not semisimple then $H \simeq (C^*)^{n-1} \times C$ for some $n \geq 1$ and $g$ is not contained in the maximal torus of $H$. This implies $H/\Gamma \simeq (C^*)^n$. \hfill \Box

Lemma 9. Let $G$ be a connected real Lie group, $\Gamma$ a subgroup such that each element $\gamma \in \Gamma$ is of finite order.

Then $\Gamma$ is almost abelian and relatively compact in $G$.

Proof. If $G$ is abelian, then $G \simeq R^k \times (S^1)^n$. In this case $\Gamma \subset (S^1)^n$ and the statement is immediate.

Now let us assume that $G$ may be embedded into a complex linear-algebraic group $\tilde{G}$. Let $H$ denote the (complex-algebraic) Zariski-closure of $\Gamma$ in $\tilde{G}$. By the theorem of Tits [8] $\Gamma$ is almost solvable, hence $H^0$ is solvable. Now the commutator group of $H^0$ is unipotent and therefore contains no non-trivial element of finite order. Hence $\Gamma \cap H^0$ is abelian, which completes the proof for this case, since we discussed already the abelian case.

Finally let us discuss the general case. By the above considerations $Ad(\Gamma_0)$ is contained in an abelian connected compact subgroup $K$ of $Ad(G)$ for some subgroup $\Gamma_0$ of finite index in $\Gamma$. Now $N = (Ad)^{-1}(K)$ is a central extension $1 \rightarrow Z \rightarrow N \rightarrow K \rightarrow 1.$ (where $Z$ is the center of $G$). But complete reducibility of the representations of compact groups implies that this sequence splits on the Lie algebra level. Hence $N$ is abelian and we can complete the proof as before. \hfill \Box

Lemma 10. Let $G$ be a complex linear-algebraic group, $\Gamma$ a Zariski-dense subgroup.
Then $\Gamma$ contains a finitely generated subgroup $\Gamma_0$ such that the Zariski-closure of $\Gamma_0$ contains $G'$.

**Proof.** Consider all finitely generated subgroups of $\Gamma$ and their Zariski-closure in $G$. There is one such group $\Gamma_0$ for which the dimension of the Zariski-closure $A$ is maximal. Clearly $A$ must contain the connected component of the Zariski-closure for any finitely generated subgroup of $\Gamma$. This implies that $A^0$ is normal in $G$. Furthermore maximality implies that the group $\Gamma/A^0$ contains no element of infinite order. Hence $\Gamma/A^0$ is almost abelian, which implies $G' \subset A$ ($\Gamma/A^0$ is Zariski-dense in $G/A^0$).

Caveat: There is no hope for $A = G$, e.g. take $G = \mathbb{C}^*$ and let $\Gamma$ denote the subgroup which consists of all roots of unity.

Theorem 2 follows by induction on $\dim(G)$ using the following lemma.

**Lemma 11.** Let $G$ be a positive-dimensional complex linear-algebraic group fulfilling the assumptions of Theorem 2, $\Gamma$ a Zariski-dense subgroup.

Then there exists a positive-dimensional normal algebraic subgroup $A$ with $\mathcal{O}(G)^\Gamma \subset \mathcal{O}(G)^A$.

**Proof.** If $G$ is abelian, the assumptions imply that $G$ is reductive and $\mathcal{O}(G)^\Gamma = \mathbb{C}$.

Otherwise let $H = \{g : f(g) = f(e) \forall f \in \mathcal{O}(G)^\Gamma\}$. Now $\Gamma \subset H$, hence $\Gamma$ normalizes $H^0$. The normalizer of a connected Lie subgroup is algebraic, thus $H^0$ is a normal subgroup of $G$. It follows that $(H \cap G')^0$ is a normal algebraic subgroup (Lemma 2). This completes the proof unless $H \cap G'$ is discrete.

This leaves the case where $(H \cap G')$ is discrete. Then $H^0$ is contained in the center $Z$ of $G$ (Lemma 3). Let $A$ denote the Zariski-closure of $H^0$. The center is reductive (Lemma 7). It follows that for each $Z$-orbit every $H^0$-invariant functions is already $A$-invariant.

Therefore we can restrict to the case where $H$ is discrete. Now $\Gamma$ is discrete and contains a subgroup $\Gamma_0$ which is finitely generated and whose Zariski-closure contains $G'$. By a theorem of Selberg $\Gamma_0$ contains a subgroup of finite index $\Gamma_1$ which is torsion-free. Now let $\Gamma_2 = \Gamma_1 \cap G'$. Then being Zariski-dense, $\Gamma_2$ must contain a semisimple element of infinite order. Using Lemma 8, this yields a contradiction to the assumption that $H$ is discrete. $\square$

4. An example

At a first glance, it seems to be obvious that a Zariski-dense subgroup should contain enough elements of infinite order to generate a subgroup which is still Zariski-dense. However, one has to careful.

**Lemma 12.** Let $G = \mathbb{C}^* \ltimes \mathbb{C}$ with group law $(\lambda, z) \cdot (\mu, w) = (\lambda \mu, z + \lambda w)$ and $\Gamma$ the subgroup generated by the elements $a_n = (e^{2\pi i/n}, 0)$ ($n \in \mathbb{N}$) and $a_0 = (1, 1)$. Then
\[ \gamma \in G' = \{ (1, x) : x \in \mathbb{C} \} \] for any element \( \gamma \in \Gamma \) of infinite order, although \( \Gamma \) is Zariski-dense in \( G \).

**Proof.** It is clear that \( \Gamma \) is Zariski-dense in \( G \). The other assertion follows from the fact that any element in \( G \) is either unipotent or semisimple. Hence every element \( g \in G \setminus G' \) is conjugate to an element in \( \mathbb{C}^* \times \{0\}. \qed

5. Margulis’ example

We will use an example of Margulis to demonstrate the following.

**Proposition 1.** There exists a discrete Zariski-dense subgroup \( \Gamma \) in \( G = \text{SL}_2(\mathbb{C}) \ltimes \rho \) \((\mathbb{C}^3,+)) \) with \( \rho \) irreducible such that \( \Gamma \) contains no semisimple element.

Thus the condition \( G/G' \) reductive is not sufficient to guarantee the existence of semisimple elements in Zariski-dense subgroups.

Margulis [5,M2] constructed his example in order to prove that there exist free non-commutative groups acting on \( \mathbb{R}^n \) properly discontinuous and by affin-linear transformations, thereby contradicting a conjecture of Milnor [7].

We will now start with the description of Margulis’ example. Let \( B \) denote the bilinear form on \( \mathbb{R}^3 \) given by \( B(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 \), \( W = \{ x \in \mathbb{R}^3 : B(x) = 0 \} \) the zero cone and \( W^+ = \{ x \in W : x_3 > 0 \} \) the positive part. Let \( S = \{ x \in W^+ : |x| = 1 \} \). Let \( H \) be the connected component of the isometry group \( O(2,1) \) of \( B \). (As a Lie group \( H \) is isomorphic to \( \text{PSL}_2(\mathbb{R}) \).) Let \( G_\mathbb{R} = H \ltimes (\mathbb{R}^3,+) \) the group of affine-linear transformations on \( \mathbb{R}^3 \) whose linear part is in \( H \).

The following is easy to verify: Let \( x^+, x^- \) to different vectors in \( S \). Then there exists a unique vector \( x^0 \) such that \( B(x_0, x^0) = 1 \), \( B(x^+, x^0) = 0 = B(x^-, x^0) \) and \( x^0, x^-, x^+ \) form a positively oriented basis of the vector space \( \mathbb{R}^3 \). Furthermore for any \( \lambda \in ]0,1[ \) there is an element \( g \in H \) (depending on \( x^+, x^- \in S \) and \( \lambda \)) defined as follows:

\[ g : ax^0 + bx^- + cx^+ \mapsto ax^0 + \frac{b}{\lambda}x^- + \lambda cx^+ \]

Conversely any non-trivial diagonalizable element \( g \in H \) is given in such a way and \( x^+, x^- \) and \( \lambda \) are uniquely determined by \( g \).

The result of Margulis is the following:

Let \( x^+, x^-, \tilde{x}^+, \tilde{x}^- \) four different points in \( S \), \( \lambda, \tilde{\lambda} \in ]0,1[ \), \( v, \tilde{v} \in \mathbb{R}^3 \) such that \( v, x^- \) resp. \( \tilde{v}, \tilde{x}^- \) forms a positively oriented basis of \( \mathbb{R}^3 \). Let \( h, \tilde{h} \in H \) be the elements corresponding to \( x^-, x^+, \lambda \) resp. \( \tilde{x}^-, \tilde{x}^+, \tilde{\lambda} \) and \( g, \tilde{g} \in G_\mathbb{R} = H \ltimes \mathbb{R}^3 \) given by \( g = (h,v), \tilde{g} = (\tilde{h},\tilde{v}) \).
Then there exists a number $N = N(g, \tilde{g})$ such that the elements $g^N$, $\tilde{g}^N$ generate a (non-commutative) free discrete subgroup $\Gamma \subset G_\mathbb{R}$ such that the action on $\mathbb{R}^3$ is properly discontinuous and free.

Now an element $g \in G_\mathbb{R}$ is conjugate to an element in $H$ if and only if $g(w) = w$ for some $w \in \mathbb{R}^3$. Hence no element in $\Gamma$ is conjugate to an element in $H$. In particular no element in semisimple. Furthermore it is clear that $\Gamma$ is Zariski-dense in the complexification $G = SL_2(\mathbb{C}) \ltimes \mathbb{C}^3$ of $G_\mathbb{R}$.

6. Meromorphic functions

**Proposition 2.** Let $G$ be a connected complex linear-algebraic group with $G = G'$ and an open subset $\Omega$ such that each element in $\Omega$ is semisimple. Let $\Gamma$ be a Zariski-dense subgroup.

Then any $\Gamma$-invariant plurisubharmonic or meromorphic function is constant and there exist no $\Gamma$-invariant hypersurface.

*Proof.* We may assume that $\Gamma$ is closed (in the Hausdorff topology). Since $G = G'$, it follows that $H^0$ is a normal algebraic subgroup for each Zariski-dense subgroup $H$. Therefore we may assume that $\Gamma$ is discrete and furthermore it suffices (by induction on $\dim(G)$ to demonstrate that the functions resp. hypersurfaces are invariant under a positive-dimensional subgroup. Now $G = G'$ implies that $\Gamma$ admits a finitely generated subgroup $\Gamma_0$ which is still Zariski-dense. By the theorem of Selberg $\Gamma_0$ admits a subgroup of finite index $\Gamma_1$ which is torsion-free. Thus $\Gamma_1$ contains a semisimple element of infinite order $\gamma$ which generates a subgroup $I$ whose Zariski-closure $\bar{I}$ is a torus. $G = G'$ implies that this torus is contained in a connected semisimple subgroup $S$ of $G$. Now known results on subgroups in semisimple groups [4][2] imply that the functions resp. hypersurfaces are invariant under $\bar{I}$, which is positive-dimensional. 

For this result it is essential to require $G = G'$ and not only $G/G'$ reductive.

**Lemma 13.** Let $G = \mathbb{C}^* \times \mathbb{C}^*$ and $\Gamma \simeq \mathbb{Z}$ a (possibly Zariski-dense) discrete subgroup.

Then $G$ admits $\Gamma$-invariant non-constant plurisubharmonic and meromorphic functions.

*Proof.* $G/\Gamma \simeq \mathbb{C}^2/\Lambda$ with $\Lambda \simeq \mathbb{Z}^3$. Let $V = \langle \Lambda \rangle_{\mathbb{R}}$ the real subvector space of $\mathbb{C}^2$ spanned by $\Lambda$ and $t : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{V} \simeq \mathbb{R}$ a $\mathbb{R}$-linear map. Then $t^2$ yields a $\Gamma$-invariant plurisubharmonic function on $G$.

Let $L = V \cap iV$ and $\gamma \in \Lambda \setminus L$. Let $H = \langle \gamma \rangle_{\mathbb{C}}$. Then $H \neq L$, hence $H + L = \mathbb{C}^2$. It follows that the $H$-orbits in $G/\Gamma$ are closed and induce a fibration $G/\Gamma \to G/H\Gamma$ onto a one-dimensional torus. One-dimensional tori are projective and therefore admit non-constant meromorphic functions. 

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