We analyze quantum phase transitions in a system of optical lattice bosons coupled to an array of atomic quantum dots, or pseudospins-1/2. The system parallels the Bose-Hubbard model with a single difference of the direct tunneling between the lattice sites being replaced by an assisted tunneling via coupling to the atomic quantum dots. We calculate the phase diagram of the combined system, numerically within the Gutzwiller ansatz and analytically using the mean-field decoupling approximation. The result of the assisted Bose-Hubbard model is that the Mott-superfluid transition still takes place, however, the Mott lobes strongly depend on the system parameters such as the detuning. One can even reverse the usual hierarchy of the lobes with the first lobe becoming the smallest. The phase transition in the bosonic subsystem is accompanied by a magnetization rotation in the pseudospin subsystem with the tilting angle being an effective order parameter. When direct tunneling is taken into account, the Mott lobes can be made disappear and the bosonic subsystem becomes superfluid throughout.

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I. INTRODUCTION

The celebrated Bose-Hubbard model (BHM) describes a system of interacting bosons with hopping between nearest neighbor sites\(^1\). This model exhibits a quantum phase transition from the incompressible Mott insulator state to the compressible superfluid state as a result of the competition between the kinetic energy and the on-site repulsion. Once it was understood that such model could be realized in optical lattices of cold bosons in a controlled way\(^2\) it did not take much time to experimentally observe the Mott-superfluid quantum phase transition\(^3,4\).

After the experimental realization in optical lattices, the model became extremely popular, having been explored and studied under various conditions and modifications. The topic of non-standard Bose-Hubbard models became a separate research branch motivated by further experimental advances\(^5\). Various additional terms (e.g. density-induced tunneling, three-body interactions, dipolar interactions, interactions with fermions) and features of bosons (e.g. spin) were taken into account. Unsurprisingly, these resulted in modifications of the typical Bose-Hubbard model phase diagram, with new phases appearing and Mott lobes being modified in size and shape\(^5\).

In this work we study another non-standard Bose-Hubbard model where the usual, direct tunneling between the sites is suppressed and instead an assisted tunneling due to the coupling to atomic quantum dots (AQDs), which are described as pseudospin-1/2, is taken into account. Assisted tunneling, hopping, coupling via different excitations or particles are not rare in solid state physics and condensed matter with many examples ranging from, for instance, phonon-assisted tunneling in semiconductors\(^6\) and electron chains\(^7\) to the celebrated Anderson model where conduction electrons of a metal hybridize with d-electrons of magnetic impurities\(^8,9\).

In our setup AQDs are represented by hard-core bosons coupled to repulsive bosons of different atomic species in the optical lattice by laser-induced Raman transitions (Fig. 1). Two AQDs coupled via one bosonic reservoir were considered previously and it was shown how entanglement could be generated between two AQDs, or two pseudospins due to the induced effective interaction\(^10,11\). On the other hand, an AQD can induce Josephson tunneling between two condensates coupled only through the artificial impurity\(^12\). A quantum order-disorder phase transition was shown to take place in a system of independent AQDs coupled to a single Bose-Einstein condensate\(^13\). Raman-assisted hopping was also used to realize the so-called anyon Hubbard model with a rich ground state physics\(^14\). The studies point towards a possibility of quantum phase transitions within our coupled system.

We refer to our model as assisted Bose-Hubbard model...
(ABHM) and treat it within the Gutzwiller ansatz\[13\] which allows calculation of the superfluid order parameter and pseudospin components, depending on chosen chemical potential and interaction. We also perform an analytical calculation of the phase boundary based on the mean-field decoupling approximation\[13\]. Both approaches are in perfect agreement with each other. We find that the Mott-superfluid transition does take place in the bosonic subsystem. However, the Mott-superfluid phase boundary strongly depends on the system parameters to the extent that the usual Mott lobe hierarchy with the first lobe being the largest can be reversed. If direct hopping is included, we show it contributes to the reduction of the lobes with their disappearance for large enough values of the direct hopping. The Mott-superfluid transition is accompanied by the pseudospin rotation from $z$- to $x$-axis, so that the tilting angle with $z$-axis becomes finite, once the bosonic subsystem is superfluid. We study in detail how the phase transitions can be modified and manipulated in our model.

II. THE MODEL

We consider spinless bosons coupled to an array of atomic quantum dots as shown in Fig. 1. Fig. 1 exemplifies a one-dimensional system, whereas the extension to higher dimensions is trivial. The general Hamiltonian of the system is the following

$$H = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

$$+ T \left[ \sum_i (\hat{a}_i^\dagger \hat{\sigma}_z^i + h.c.) + \sum_i (\hat{a}_i^\dagger \hat{\sigma}_-^i + h.c.) \right]$$

$$- (\delta + \mu) \sum_i \frac{1 + \hat{\sigma}_z^i}{2}. \quad (1)$$

The first three terms constitute the standard Bose-Hubbard model. Here $\hat{a}_i^\dagger, \hat{a}_i$ are creation and annihilation operators of bosons on site $i$, $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$, $J$ is the direct hopping between nearest neighbors, $U$ is repulsive interaction and $\mu$ is chemical potential.

The terms proportional to $T$ describe the laser-induced hopping between the lattice sites and the atomic quantum dots. The quantum dots are described by operators $\hat{b}_i^\dagger, \hat{b}_i$ which represent bosons of another hyperfine species than the bosons in the optical lattice, trapped in a narrow potential with an effective on-site repulsion $U_b \to \infty$. Laser-induced Raman transitions can couple the atomic quantum dots to the lattice bosons. $T$ is then proportional to the Rabi frequency $\Omega_R$ of the Raman transition between the two degenerate atomic states of atoms in the lattice and the AQD

$$T = \hbar \Omega_R \int d^3r \psi_i^k(r) \psi_b^k(r), \quad k \in i, i - 1. \quad (2)$$

This kind of coupling was first introduced in Ref.\[17\].

In the limit of $U_b \to \infty$ we can map the bosonic opera-
tors on the dots onto pseudospin-1/2 operators $\hat{b} \rightarrow \hat{\sigma}_-\hat{b}$, $\hat{b}^\dagger \rightarrow \hat{\sigma}_+\hat{b}$ as it was originally done in Ref.\cite{Ref} Here $\hat{\sigma}_+ = (\hat{\sigma}_x + i\hat{\sigma}_y)/2$, $\hat{\sigma}_- = (\hat{\sigma}_x - i\hat{\sigma}_y)/2$, where $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are Pauli matrices. The wave function $\psi_i(r)$ describes bosons on the $i$-site of the optical lattice, and $\psi_b^r$ is the wave function of the boson in the atomic quantum dot. We assume that the quantum dot wave-function overlaps only with wave-functions $\psi_i(r)$ of the nearest neighbor sites. In this way, there is no direct interaction between the pseudospins, but only an induced one. The parameter $\delta$ describes the detuning necessary to suppress spontaneous emission at the AQD sites.

A. Gutzwiller ansatz approach

We now aim to describe the quantum phase transitions in the ABHM for various detunings using the Gutzwiller ansatz. The version of Gutzwiller ansatz for a system of bosons in a lattice\cite{Ref} is a product of Fock states

$$|\Psi\rangle_{BH} = \prod_i^{N} \left(\sum_{n=0}^{\infty} f_n^{(i)}|n\rangle_i\right), \quad (3)$$

where $N$ is the total number of lattice sites and $f_n^{(i)}$ are coefficients of a site $j$ with a $n$ number of bosons. Since just two states describe the state of a quantum dot, we can assign spin-up and spin-down notations for them, so that one AQB is described by the state

$$a_0 |\uparrow\rangle + a_1 |\downarrow\rangle. \quad (4)$$

The state of the coupled system is then

$$|\Psi\rangle = \prod_i^{N} \left(\sum_{n=0}^{\infty} f_n^{(i)}|n\rangle_i \otimes (a_0^{(i)}|\uparrow\rangle_i + a_1^{(i)}|\downarrow\rangle_i)\right), \quad (5)$$

where $i$ numbers the lattice sites (see Fig. 1). All the coefficients are normalized:

$$\sum_n |f_n^{(i)}|^2 = 1, \quad |a_0^{(i)}|^2 + |a_1^{(i)}|^2 = 1. \quad (6)$$

We assume that the coefficients $f_n$ are real, however, we need $a_0$ and $a_1$ to be complex. We can now approximate the lattice sites with as many as seven states, depending on the chemical potential. Evaluating the Hamiltonian locally we get

$$\langle \Psi|H|\Psi\rangle = -2J \left[\sum_{n=0}^{n_0+2} \sqrt{n+1}f_n f_{n+1}\right]^2 + \frac{U}{2} \left[\sum_{n=0}^{n_0+3} f_n^2(n^2 - n)\right] + 2T[a_0^*a_1 + a_0a_1^*\sum_{n=0}^{n_0+2} \sqrt{n+1}f_n f_{n+1}] - \mu [a_0^2 + \sum_{n=0}^{n_0+3} n f_n^2] - \delta[a_0]^2, \quad (7)$$

where $n_0$ is the average occupation of optical lattice site per lobe. In this way we get the local energy in terms of the coefficients of the Gutzwiller ansatz. In order to reduce the number of parameters, we divide everything through $U$ and introduce $\tilde{\mu} = \mu/U$, $\tilde{\delta} = \delta/U$, $\tilde{T} = T/U$ and $\tilde{J} = J/U$. The local order parameters can be expressed in terms of the coefficients as follows

$$\varphi = \langle \hat{a}_i \rangle = \sum_{n=0}^{n_0+2} \sqrt{n+1}f_n f_{n+1},$$

$$s_x = \frac{1}{2}\langle \hat{\sigma}_x \rangle = \frac{1}{2}(a_0^*a_1 + a_1^*a_0). \quad (8)$$

Note that as we have two-dimensional spins, $s_y = 0$ and $s_z = \sqrt{1/4 - s_x^2}$.

The phase diagram will be obtained by numerical minimization of the local energy w.r.t. the normalized coefficients for fixed values of $\delta, \tilde{\mu}, \tilde{T}$ and $\tilde{J}$ from which the order parameters will follow.

B. Mean field description

We can also calculate the phase boundary using a simple decoupling approximation\cite{Ref}. Namely, we assume the mean-field decoupling in the terms proportional to $T$ and $J$

$$\hat{a}_i^\dagger\hat{\sigma}_- \approx s_x a_i^\dagger + \varphi a_i,$$

$$\hat{a}_i^\dagger\hat{a}_j \approx \varphi(a_i^\dagger + a_j) - \varphi^2,$$  

which amounts to first order contributions in fluctuations in the order parameters. This kind of decoupling allows

FIG. 3: First five lobes of the Bose-Hubbard model phase boundary with assisted hopping only (direct hopping $J = 0$) for different detuning $\delta$. Inside the lobes the system is in the Mott state ($\varphi = 0$ and $s_x = 0$), outside the lobes the system is superfluid so that $\varphi \neq 0$ and $s_x \neq 0$.\[\text{FIG. 3: First five lobes of the Bose-Hubbard model phase boundary with assisted hopping only (direct hopping $J = 0$) for different detuning $\delta$. Inside the lobes the system is in the Mott state ($\varphi = 0$ and $s_x = 0$), outside the lobes the system is superfluid so that $\varphi \neq 0$ and $s_x \neq 0$.}
us to deal with the effective single site Hamiltonian

\[ H_i = \frac{U}{2} n_i (n_i - 1) - \mu n_i - (\delta + \mu) \frac{n_i}{2} + z J \varphi^2 + (z T s_x - z J \varphi) (\hat{a}_i^\dagger \hat{a}_i ) + z T \varphi \hat{a}_i^\dagger \hat{a}_i - 2z T \varphi s_x. \]

Here we introduced \( z = 2d \) the number of nearest neighbors in terms of the spatial dimension \( d \).

Treating the single site Hamiltonian with second-order perturbation theory, we expand the ground state energy per site to second order in \( \varphi \) and \( s_x \) (in the following we will omit the site index \( i \)).

Minimising the unperturbed single site Hamiltonian gives the ground state energy as

\[ E_n^{(0)} = \begin{cases} \frac{U}{2} n(n-1) - \mu n - (\mu + \delta) & \text{if } \mu + \delta \geq 0, \\ \frac{U}{2} n(n-1) - \mu n & \text{if } \mu + \delta \leq 0. \end{cases} \]

The chemical potential should also satisfy the inequality \( U(n-1) < \mu < Un \). For \( \mu + \delta > 0 \) all AQDs are occupied, whereas for \( \mu + \delta < 0 \) all AQDs are empty in the ground state. For \( \mu + \delta = 0 \) the two energies coincide as the AQD is in the superposition state of being occupied and empty, we refer to this condition as the "degeneracy line" in our plots.

First order corrections to the energy are zero, as expected, whereas the second-order correction contains the following contributions

\[ E_n^{(2)} = z^2 T^2 F s_x^2 - 2z T (1 + z J F) \varphi s_x + (z T^2 - z J F, \varphi^2. \]

where

\[ F = \frac{n+1}{\mu - Un} + \frac{n}{U(n-1) - \mu}. \]

and the absolute value of \( |\mu + \delta| \) accounts for both cases of Eq. (11).

The first derivative test of (12) gives the critical point being \((\varphi, s_x) = (0, 0)\) as expected. The Hessian matrix of second derivatives reads

\[ \begin{pmatrix} 2 z^2 T^2 F & -2z T (1 + z J F) \\ -2z T (1 + z J F) & 2z (J (1 + z J F) - \frac{z T^2}{|\mu + \delta|}) \end{pmatrix}. \]

The phase boundary is defined by the condition requiring the determinant of the Hessian matrix to be zero for a zero eigenvalue since this is where the determinant changes sign. For \( T = 0 \) the boundary condition for the BHM follows

\[ \tilde{J}_{BH} = -\frac{1}{2z} = \frac{1}{z} \frac{(\mu - n)(n - 1 - \mu)}{(\mu + 1)}. \]

For \( J = 0 \) we get the condition for the ABHM

\[ \tilde{T}_{ABH} = \frac{1}{z} \sqrt{-\frac{|\tilde{\mu} + \tilde{\delta}|}{F}} = \frac{1}{z} \left( \frac{|\tilde{\mu} + \tilde{\delta}| (\mu - n)(n-1 - \mu)}{\tilde{\mu} + 1} \right)^{1/2}. \]

The general phase boundary condition can be written as

\[ \tilde{T} = \frac{1}{z} \sqrt{-\frac{|\tilde{\mu} + \tilde{\delta}| (1 + z F, \tilde{J})}{F}}, \]

or, equivalently, as

\[ \tilde{J} = -\frac{1}{z F} - \frac{z T^2}{|\tilde{\mu} + \tilde{\delta}|} \]

which reduces to (10) for \( J = 0 \) and to (15) for \( T = 0 \). In the limit of large \( \mu \), \( J_{BH} \rightarrow 0 \) while \( T_{ABH} \rightarrow (|\tilde{\mu} - n|) (n - 1 - \mu) \) for \( \mu = 1 \) shown in Fig. 3. The square root is a second order polynomial with periodic boundary conditions and a maximum value of 1/2. For the one-dimensional case plotted, the Mott lobes tend towards becoming symmetric with a maximum value of \( T_c = 1/4 \) in the large \( \mu \) limit (also need \( \tilde{\mu} + \tilde{\delta} > 0 \)). Furthermore one should take into account that for large occupancies the Bose-Einstein condensation will take place, hence the numerical and analytical results are not valid in this regime.

III. RESULTS

A. Phase diagrams for different detuning \( \delta \)

All results presented in the section are for \( z = 2 \), i.e. for one-dimensional case. For higher dimensions the results are trivially rescalable. We start with comparing the standard phase diagram for BHM with the phase diagram for our model with \( J = 0 \) and \( \delta = 0 \), so that only assisted hopping is present in the system. The results are shown in Fig. 2. We notice several interesting features. First of all, the Mott-superfluid transition does not take place in our model with the Mott lobes being in general larger than the BHM ones, which is expected since the assisted hopping is effectively weaker than direct hopping. Secondly, the usual lobe hierarchy, at least for the first several lobes, is reversed, with the first lobe being the smallest. Thirdly in the large \( \mu \) limit, the lobes tend towards a fixed value of \( T_c = 1/4 \) while in BHM due to the costly onsite interaction for large densities \( J_c = 0 \). Lobes are broader in our case and the first lobe is slanted upwards.

To understand what is going on in the pseudospin subsystem we plot \( s_z \) versus \( \tilde{\mu} \) and \( \tilde{T} \) in Fig. 2 lowest panel. We see that pseudospin behavior is correlated with the superfluid order parameter. Inside the Mott lobes \( s_z = 0 \) and \( s_z = 1/2 \), whereas outside the lobes \( s_z \) is always smaller than 1/2, and \( s_z \) acquires a nonzero value. It means that when superfluidity in the bosonic subsystem is established, it effectively rotates the pseudospin from being aligned along the \( z \)-axis to being aligned along the \( z \)-axis. Hence, the tilting angle of the spin with the \( z \)-axis plays the role of an effective order parameter in the pseudospin subsystem.
FIG. 4: Phase diagram of the ABHM for negative detuning ($\tilde{\delta} = -1$ and $\tilde{\delta} = -1.5$). Inside the lobes the superfluid order parameter $\varphi = 0$, whereas outside the lobes $\varphi$ is finite.

We now explore how the detuning $\delta$ influences this behavior. In Fig. 3 we show the first five lobes of the ABHM phase diagram for various values of $\delta$. For small detuning $\tilde{\delta} < 1$ the first lobe is the smallest. For $\tilde{\delta} = 1$ the lobes are of the same size as follows from (16). With increasing $\delta$ the usual Mott lobe hierarchy is restored, although lobes are much larger compared to those in BHM in Fig. 2 and moreover for large $\mu$ they do not vanish.

When detuning is negative, the "degeneracy line" $\mu + \delta = 0$ is shifted upwards to positive $\tilde{\mu}$-s revealing the following effect: if the degeneracy falls in between the lobes, it will "push" them apart resulting in their slanting (see Fig. 4 upper panel). When the degeneracy is inside a lobe, it will split it into two smaller lobes (see Fig. 4 lower panel) each having a different ground state energy.

In the pseudospin subsystem $s_x$ component of the pseudospin will have a maximum value along the degeneracy line, because $s_z = 0$ for $\mu + \delta = 0$ (see also Fig. 6). This explains the red "cones" in Fig. 5 and a part of such a cone in the phase diagram for $s_x$ in Fig. 2 (lowest panel). The lowest panel of Fig. 5 displays $s_z$, which

FIG. 5: Phase diagram of the AQD subsystem for negative detuning $\tilde{\delta} = -1$ and $\tilde{\delta} = -1.5$. The size of $s_x$ is shown for $\tilde{\mu} = \mu/U$ versus $\tilde{T} = T/U$. Violet indicates zero $s_x$ inside the Mott lobes. $s_x$ is maximum along the energy degeneracy line $\mu + \delta = 0$. The lowest panel shows $s_z$ which changes sign as it crosses the "degeneracy line". $s_z$ has its maximum value $|s_z| = 0.5$ inside the Mott lobes.
FIG. 6: The z-component of the pseudospin $s_z$ versus $\tilde{\mu} = \mu/U$ for various values of $T$ and fixed $\delta = -1.5$.

changes its sign, effectively flips while crossing the "degeneracy line". In Fig. 6 we reveal how it happens with increasing $T$. The flat regions in Fig. 6 correspond to the Mott lobes when $s_z$ is exactly $1/2$ or $-1/2$ depending on the value of chemical potential.

## B. The effect of direct hopping $J$

We now explore the effect of $J$ on the phase diagram of ABHM. One can see from Eq. (17) that $J$ suppresses $T$ and therefore reduces the Mott lobes. It follows from Eq. (17) that the minimum value of $J$ for which mottness will be completely suppressed is equal to $J_c$, the values for the Mott lobe tips in the standard BHM. The gradual disappearance of the Mott lobes is demonstrated in Fig. 7 for $\delta = 1$ and two different values of $J$: $\tilde{J} = 0.02$ and $\tilde{J} = 0.04$. The lobes become narrower and eventually disappear starting from higher lying lobes. Interestingly, when the third lobe completely disappears for $\tilde{J} = 0.04$, there is still a "shadow" of it in the system manifested in the area of a decreased superfluid order parameter.

The behavior of pseudospins remains similar to that in Figs. 2 and 5 so we do not show it here.

In Fig. 8 we show how the tips of the Mott lobes of BHM (denoted by $J_c$) are modified by the coupling $T$ to AQDs for a fixed $\delta = 0.1$. The effect of the additional coupling is most pronounced for the first lobe, one can also see how the general trend of the first lobe being the largest reverses for larger $T$. This behavior shows that direct hopping is effectively enhanced due to the assisted coupling $T$ and therefore promotes superfluidity.

## IV. CONCLUSIONS

We have considered quantum phase transitions in a system of lattice bosons coupled to atomic quantum dots, or qubits. The Mott-superfluid transition takes place...
within the bosonic subsystem even if the direct hopping between the sites is absent. The Mott lobes of ABHM are in general larger than those of BHM as the assisted coupling is effectively weaker. The lobes hierarchy, shape and sizes are strongly influenced by the detuning \( \delta \), because it effectively changes the chemical potential. For finite \( \delta \) the lobes become even bigger, the system favors the mottness state. For finite but negative detuning the degeneracy line of the ground state energy is shifted upwards resulting in the splitting of the corresponding Mott lobes into two smaller ones. The pseudospin subsystem is in tune with the bosons in that it can be described by the spin order parameter, in particular \( s_z \), which is zero inside the Mott lobes and finite outside. It is maximum along the energy degeneracy line, whereas \( s_z \) component demonstrates a spin flip while crossing the line.

The system can become superfluid throughout if we include large enough direct hopping \( J \). We have demonstrated that a) lobes can be manipulated in the coupled system and with right parameter adjustments are made to disappear, b) there is a spin rotation associated with the Mott-superfluid transition in the AQD subsystem and spin-flip across the energy degeneracy line. Hence the finite superfluid parameter serves as an external magnetic field for the pseudospin subsystem.

Our model demonstrates that taking into account an additional hopping can drastically change the phase diagram of the system, so that the system can become superfluid throughout if desired. It is interesting if similar ideas could be applied to superconducting systems, although the task becomes incomparably more difficult.

Another exciting aspect of the model is that AQDs constitute qubits, which we show to be easily manipulated if coupled to a superfluid. Extending this with an additional qubit chain may facilitate a realization of additional ground state degeneracies and further degeneracy line from the second detuning. With additional degeneracy lines, it may be possible to split a single lobe into more than two or create alternating lobes, each with their own ground state energy.

This may have interesting applications in quantum computation. In the future, it would be useful to properly consider spin correlations which are not possible in our single site approach.

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