Tomography of quantum states of the universe and cosmological dynamics

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Abstract. It has been proven that conventional quantum mechanics can be reformulated
without using the concept of wave function or matrix density. Quantum states can be
described in terms of a standard positive probability distribution. Such probability distributions
(tomograms) satisfy classical-like equations, which substitute the Schrödinger equation. We
consider quantum cosmology in the probability distribution approach. We derive the tomogram
corresponding to the Wheeler-de Witt equation. We also use the circumstance that cosmological
evolution equations for a homogeneous and isotropic universe can be converted into very
simple classical equations to derive a quantum cosmological model which links the tomogram
reconstructed from observations to the initial conditions of the universe.

1. Introduction

Recently it was proven that conventional quantum mechanics can be described without using
the notions of wave function or matrix density[1]. Quantum states are alternatively reconstructed
by fair probability functions, obtained by the Radon transform of the Von Neumann density
operator in the form of Wigner functions.

These probability functions, which are called tomograms, satisfy the following properties

1) the set of tomograms $W(X, \mu, \nu)$, for all the values of $\mu$ and $\nu$, describes completely the
quantum state of a system i.e. it gives the same information of the corresponding wave function
$\psi(x)$, they follow a classical-like evolution equation.

2) If the states in classical statistical mechanics are represented by the function $f(q, p)$, one
can introduce the corresponding classical tomograms by the transform

$$W_{\text{class.}}(X, \mu, \nu) = \int f(q, p)\delta(X - \mu q - \nu p)dqdp.$$

3) Classical and quantum tomograms are observables and for a given physical system, one
can compare directly the outcomes of the classical and quantum theories.

In quantum cosmology, the states of the universe obey the Wheeler-DeWitt equations. The
solutions of this equation are called wave functions of the universe.

One of the tasks for quantum cosmology is to obtain a fundamental law for the initial
quantum state of the universe [2]. The two main proposals, the Hartle-Hawking no boundary and
Vilenkin’s *tunneling from nothing* where formulated in order to give reasonable initial conditions for the observed classical universe. Alternatively one can try to study the initial conditions of the universe from the phenomenology of the universe.

In this paper we apply, in a heuristic way, the tomographic approach to quantum cosmology. First we show the construction of tomograms applying the Radon transform to the Wigner function, then we derive the relationship between tomograms and the wave function. Next we find the equation for the tomograms of the universe corresponding to the Wheeler-DeWitt equation [3].

Using the tomographic approach we can also describe the evolution of the universe (quantum transitions) by means of standard real positive transition probabilities [4]. Observing that the cosmological equations for a flat universe can be cast in the form of a generic harmonic oscillator, we assume that the quantum behavior of the universe can be described by the quantum behavior of the harmonic oscillator. We show that present tomograms of the universe are related to the initial ones through positive transition probability functions. This suggests that it is possible, in principle, to trace back the initial quantum state of the universe, by reconstructing from observations the tomograms of the present universe.

### 2. Quantum Mechanics in tomographic representation

In order to write the Wheeler-DeWitt equation in tomographic form we review some properties of the modified Radon transform of Wigner function [5]. The Wigner function is expressed in terms of density matrix of the universe \( \rho \) in the form (\( \hbar = 1 \))

\[
W(q, p) = \int \rho \left( q + \frac{u}{2}, q - \frac{u}{2} \right) e^{-ipu} du
\]

The Radon transform of the Wigner function in the modified form is the integral transform of the form

\[
W(X, \mu, \nu) = \int W(q, p)e^{ik(X - \mu q - \nu p)} \frac{dkdqdp}{(2\pi)^2}
\]

Here \( X, \mu, \nu \) are real numbers. The Wigner function can be found using the inverse Radon relation

\[
W(q, p) = \frac{1}{2\pi} \int e^{i(X - \mu q - \nu p)}W(X, \mu, \nu)dXd\mu d\nu.
\]

The standard Radon transform is obtained from the two above by taking \( \mu = \cos \varphi, \nu = \sin \varphi \).

One can see that the tomograms are given as a marginal distribution since

\[
W(X, \mu, \nu) = \int W(q, p)\delta(X - \mu q - \nu p) \frac{dqdp}{2\pi}
\]

It is clear that

\[
\int W(X, \mu, \nu)dX = 1,
\]

since the Wigner function is normalized

\[
\int W(q, p) \frac{dqdp}{2\pi} = 1
\]

for normalized wave functions. Tomograms satisfy the property
\[ W(\lambda X, \lambda \mu, \lambda \nu, t) = |\lambda|^{-1} W(X, \mu, \nu, t). \]  

(7)

Formulae (4) – (6) are valid for arbitrary density matrices, both for pure and mixed states. For pure states of the universe, the tomograms can be expressed directly in terms of the wave function of the universe using,

\[ W(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \psi(y)e^{i\frac{\mu}{\nu} y^{2} - \frac{i}{\nu} X y} dy \right|^2. \]  

(8)

3. Wheeler-DeWitt equation for tomograms

Let us consider a one dimensional Wheeler-DeWitt equation for a FLRW universe with cosmological constant and no material sources,

\[ \frac{1}{2} \left\{ \frac{1}{a^p} \frac{d}{da} a^p \frac{d}{da} - a^2 + \Lambda a^4 \right\} \psi(a) = 0. \]  

(9)

Here \(0 \leq a < +\infty\), is in the classical theory the expansion factor and \(p\) is an index introduced to take into account the ambiguity of operator ordering. The Radon transform discussed in previous sections makes sense only for variables that take values from \(-\infty\) to \(+\infty\), so we make the change of variables \(a = \exp x\) and the Wheeler-DeWitt equation becomes \([3]\)

\[ \frac{1}{2} \left\{ \exp(-2x) \frac{d^2}{dx^2} + (p - 1) \exp(-2x) \frac{d}{dx} \right. \left. - 2U(x) \right\} \Psi(x) = 0, \]  

(10)

where \(U(x) = (\exp(2x) - \Lambda \exp(4x))/2\).

The corresponding tomographic equation is

\[ \left\{ \text{Im} \left[ \exp \left( 2 \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + i\nu \frac{\partial}{\partial X} \right) \left( \frac{1}{2} \mu \frac{\partial}{\partial X} - i \left( \frac{\partial}{\partial \mu} \right)^{-1} \frac{\partial}{\partial \nu} \right)^2 \right] \right. \]

\[ + (p - 1) \text{Im} \left[ \exp \left( 2 \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + i\nu \frac{\partial}{\partial X} \right) \left( \frac{1}{2} \mu \frac{\partial}{\partial X} - i \left( \frac{\partial}{\partial \mu} \right)^{-1} \frac{\partial}{\partial \nu} \right) \right] \]

\[ - 2 \text{Im} \left[ \exp \left( - 2 \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + i\nu \frac{\partial}{\partial X} \right) - \Lambda \exp \left( - 4 \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right. \right. \]

\[ \left. \left. + 2i\nu \frac{\partial}{\partial X} \right) \right\} W(X, \mu, \nu) = 0, \]  

(11)

where the operator \((\partial/\partial X)^{-1}\) is defined by the relation

\[ \left( \frac{\partial}{\partial X} \right)^{-1} \int f(y)e^{iyX} dy = \int \frac{f(y)}{(iy)} e^{iyX} dy. \]  

(12)

But instead of solving equation (11), we exploit the properties of cosmological equations to the describe the evolution of the cosmological quantum states in terms of a simple quantum mechanical system.
4. Cosmology

Classically a homogeneous and isotropic universe is described by one of the following metrics

\[ ds^2 = -c^2dt^2 + \frac{a^2}{1-kr^2} \left( dr^2 + r^2d\theta^2 + r^2\sin^2\theta \right) d\phi^2 \]  

(13)

where the three conditions \( k > 0 \), \( k = 0 \) and \( k < 0 \) are related to a closed universe, a flat universe and an open universe respectively.

The cosmological equations with a perfect fluid and a cosmological constant \( \Lambda \)

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{2}{3}\Lambda \quad \text{and} \quad \frac{\dot{a}^2}{a^2} + k = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \]  

(14)

where \( P \) and \( \rho \) are respectively the pressure and the energy-momentum density of a perfect fluid, which satisfy the equation of state \( P = (\gamma - 1)\rho \).

It is easy to show that equations (14) for a cosmological fluid, with \( \Lambda \neq 0 \), with the change of variables

\[ z = a^\sigma \]  

(15)

where

\[ \chi = \frac{3}{2}\gamma - 1 \quad \text{and} \quad \sigma = (1 + \chi)^{-1} = \frac{2}{3\gamma} \]

can be transformed into the following equation

\[ \ddot{z} = \frac{\Lambda\gamma}{2\sigma}z + \frac{k\chi}{\sigma}z^{2\sigma-1} \]  

(16)

or

\[ \ddot{z} = \frac{\Lambda}{3}z + k \left( 1 - \frac{2}{3\gamma} \right) z^{1-(4/3\gamma)}. \]  

(17)

Therefore a flat universe with a fluid and a cosmological constant can be regarded dynamically as a harmonic oscillator (anti de Sitter universe), a free particle (Einstein-de Sitter universe) and a repulsive harmonic oscillator (de Sitter universe).

5. The transition probability functions

The tomographic map can be used not only for the description of the universe state by probability distributions [4], but also to describe the evolution of the universe (quantum transitions) by means of standard real positive transition probabilities. The transition probability \( \Pi(X, \mu, \nu, t, X', \mu', \nu', t_0) \) is the propagator which gives the tomogram of the universe \( W(X, \mu, \nu, t) \), if the tomogram at the initial time \( t_0 \) is known, in the form

\[ W(X, \mu, \nu, t) = \int \Pi(X, \mu, \nu, t, X', \mu', \nu', t_0)W(X', \mu', \nu', t_0)dX'd\mu'd\nu'. \]  

(18)

The positive transition probability describing the evolution of the universe has the obvious nonlinear properties used in classical probability theory, namely

\[ \Pi(X_3, \mu_3, \nu_3, t_3, X_1, \mu_1, \nu_1, t_1) = \int \Pi(X_3, \mu_3, \nu_3, t_3, X_2, \mu_2, \nu_2, t_2) \times \Pi(X_2, \mu_2, \nu_2, t_2, X_1, \mu_1, \nu_1, t_1)dX_2d\mu_2d\nu_2. \]  

(19)

which follows from the associativity property of the evolution maps. Relation (19) implies that the state of the universe evolves from the initial one to the final one through all intermediate states. The remarkable fact is that this quantum evolution of the universe state can described completely with the standard positive transition probabilities like in classical dynamics.
6. Evolution of the quantum universe in the oscillator model framework

As we have shown the equation for the universe evolution can be cast in the form of an oscillator equation if \( k = 0 \). The oscillator has the frequency \( \omega^2 = \Lambda/3 \). For \( \Lambda = 0 \) one has the model of free motion. For \( \Lambda < 0 \) one has the model of an inverted oscillator and for \( \Lambda > 0 \) one has the standard oscillator as solution of the equation (16).

Since the problem of gravity quantization is not established with complete rigor, we assume below that the quantum behavior of the universe in the framework of the considered minisuperspace model, is described by the quantum behavior of the oscillator. Though the connection (15) of the expansion factor \( a(t) \) with the classical observable \( z \) which obeys to oscillator motion provides constraints on the ranging domain of this variable, we assume in the quantum picture of the variable to lie on the real line \( \mathbb{R} \). In such approach we apply the tomographic probability representation, developed in the previous sections, to the quantum states of the universe in the framework of the oscillator model. For the considered model the general equation for the universe tomogram evolution takes the simple form of a first order differential equation

\[
\frac{\partial W(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial W(X, \mu, \nu, t)}{\partial \nu} + \omega^2 \nu W(X, \mu, \nu, t) = 0,
\]

(20)

\[
\frac{\partial W}{\partial t} - \mu \frac{\partial W}{\partial \nu} + \omega^2 \nu \frac{\partial W}{\partial \mu} = 0.
\]

(21)

Analogously for the propagator of the tomographic equation for the universe in the framework of the oscillator model one has

\[
\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + \omega^2 \nu \frac{\partial \Pi}{\partial \mu} = \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t).
\]

(22)

The solution to this equation can be found to be in the case \( \Lambda < 0 \)

\[
\Pi^{osc}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu \cos \omega t + \omega \nu \sin \omega t)
\]

\[
\times \delta \left( \nu' - \nu \cos \omega t - \frac{\mu}{\omega} \sin \omega t \right).
\]

(23)

In the limit \( \Lambda = 0 \) (free motion) the equation for the tomogram (21) becomes

\[
\frac{\partial W(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial W(X, \mu, \nu, t)}{\partial \nu} = 0.
\]

(24)

The corresponding propagator solution reads

\[
\Pi^{free}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu)\delta(\nu' - \nu - \mu t).
\]

(25)

Finally for the case \( \Lambda > 0 \) the propagator has the form corresponding to a repulsive oscillator

\[
\Pi^{rep}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu \cosh \omega t - \omega \nu \sinh \omega t)
\]

\[
\times \delta \left( \nu' - \nu \cosh \omega t - \frac{\mu}{\omega} \sinh \omega t \right).
\]

(26)

Thus the universe evolution can be described in the oscillator model of minisuperspace for \( \Lambda > 0 \), \( \Lambda = 0 \) and \( \Lambda < 0 \) by means of the standard transition probabilities expressed as propagators \( \Pi^{osc} \), \( \Pi^{free} \) and \( \Pi^{rep} \) respectively.
7. Discussion and conclusions

As tomograms are observables quantities they have to be reconstructed directly from the cosmological observations, therefore a consistent definition of tomogram of the universe is necessary, but it lies in the physical interpretation we give to it. In quantum cosmology, the tomographic representation contains the same interpretative problems given by the wave function of the universe.

Having a good definition for the cosmological tomograms is important, because equation (18) suggests that, in principle, it is possible to reconstruct the initial conditions from the knowledge of the tomograms of the present universe.

More realistically, according to Hartle [2], it would be possible to investigate quantum cosmology phenomenologically by asking for the constraints placed by present observations on the initial conditions of the universe. In other words one should ask that tomograms are consistent with the observed features of the universe. In the specific cases studied in this work, the evolutionary equations for classical and quantum tomograms are the same. The quantum and classical solutions differ in the choice of initial conditions. Quantum initial conditions must satisfy the uncertainty principle and can not be arbitrary as the classical initial conditions. Then it is possible to see if the evolution of a universe with quantum initial conditions is more natural than one with classical initial conditions.

Finally, this approach appears to be well-suited to analyze the transition of universes from a quantum state to a classical one [6].

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