KdV Surfaces

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Abstract

We consider 2-surfaces arising from the Korteweg de Vries (KdV) equation. The surfaces corresponding to KdV are in a three dimensional Minkowski space. They contain a family of quadratic Wein- garten and Willmore-like surfaces. We show that a subset of KdV surfaces can be obtained from a variational principle where the Lagrange function is a polynomial function of the Gaussian and mean curvatures. We finally give a method for constructing the surfaces explicitly, i.e., finding their parametrizations or finding their position vectors.
1 Introduction

2-surfaces in $\mathbb{R}^3$ have some special subclasses, like surfaces of constant Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, developable surfaces, Bianchi surfaces, surfaces where the inverse of the mean curvature is harmonic and the Willmore surfaces. These surfaces arise in many different branches of sciences; in particular, in various parts of theoretical physics (string theory, general theory of relativity), biology and differential geometry [1], [2].

Examples of some of these surfaces like Bianchi surfaces, surfaces where the inverse of the mean curvature is harmonic [3], and the Willmore surfaces [4], [5] are very rare. The main reason is the difficulty of solving corresponding differential equations. For this purpose, some indirect methods [6]-[15] have been developed for the construction of 2- surfaces in $\mathbb{R}^3$ and in three dimensional Minkowskian geometries ($M_3$). Among these methods, soliton surface technique is very effective. In this method, one mainly uses the deformations of the Lax equations of the integrable equations. This way, it is possible to construct families of surfaces corresponding to some integrable equations like sine Gordon, Korteweg de Vries (KdV) equation, modified Korteweg de Vries (mKdV) equation and Nonlinear Schrödinger (NLS) equation [6]-[10], belonging to the afore mentioned subclasses of 2-surfaces in a three dimensional flat geometry. In particular, using the symmetries of the integrable equations and their Lax equation, we arrive at classes of 2-surfaces. There are many attempts in this direction and examples of new two surfaces. On the other hand, there are some 2-surfaces which do not have many explicit examples and could not be generated by the solitonic techniques.

Other examples of surfaces are the minimal surfaces [16], surfaces with constant mean curvature, Willmore surfaces [4], [5] and surfaces solving the shape equation [17]-[22]. All these surfaces come from a variational principle where the Lagrange function is a polynomial of degree two in the mean curvature of the surface. There are more general surfaces solving the Euler-Lagrange equations corresponding to more general Lagrange functions of the mean and Gaussian curvatures of the surface [20]-[22].

In this work, by use of the deformation of Lax equations, we generate some new Weingarten and Willmore-like surfaces. For this purpose, we use the KdV equation and its Lax representation in $sl(2,\mathbb{R})$ algebra (surfaces in $M_3$). In Section 3, we study the variation of a functional where the Lagrange
function is a function of the mean and Gaussian curvatures. Following [20]-[22], we give the corresponding Euler-Lagrange equations. Solutions of these equations define a family of surfaces extremizing the functional we started with. In Section 4, we give the surfaces corresponding to the KdV equation. These surfaces contain quadratic Weingarten and Willmore-like surfaces. We show that KdV surfaces contain also a subclass of surfaces which extremize families of functionals. For all these surfaces, we find all possible functionals where the Euler-Lagrange equations are exactly solved.

Using the method deformation of Lax equations, we can obtain the fundamental forms, Gauss and mean curvatures of the surfaces. A parametrization or the position vector of these surfaces can not be obtained directly. Deformation technique does not produce the surfaces explicitly. This method reduces the construction of the surfaces to linear equations (the Lax equations). The solutions of these linear equations give directly the position vectors of the corresponding surfaces. In the last section we give such a method. This method gives the position vectors of the KdV surfaces explicitly.

2 Deformation of soliton equations

Surfaces corresponding to integrable equations are called integrable surfaces and a connection formula, relating integrable equations to surfaces, was first established by Sym [6], [7]. Here, we shall give a brief introduction (following our previous work [10]) of the recent status of the subject and also give some new results. Below $M_2$ and $M_3$ are two and three dimensional pseudo-Riemannian geometries, respectively.

Let $F : \mathcal{U} \rightarrow M_3$ be an immersion of a domain $\mathcal{U} \in M_2$ into $M_3$. Let $(x, t) \in \mathcal{U}$. The surface $F(x, t)$ is uniquely defined up to rigid motions by the first and the second fundamental forms. Let $N(x, t)$ be the normal vector field defined at each point of the surface $F(x, t)$. Then the triple $\{F_x, F_t, N\}$ define a basis of $T_p(S)$, where $S$ is the surface parameterized by $F(x, t)$ and $p$ is a point in $S$, $p \in S$. The motion of the basis on $S$ is characterized by the Gauss-Weingarten (GW) equations. The compatibility condition of these equations are the well-known Gauss-Mainardi-Codazzi (GMC) equations. The GMC equations are coupled nonlinear partial differential equations for the coefficients $g_{ij}(x, t)$ and $h_{ij}(x, t)$ of the first and the second fundamental forms, respectively. For certain particular surfaces, these equations reduce to a sin-
gle or to a system of integrable equations. The correspondence between the
GMC equations and the integrable equations has been studied extensively,
see for example [10].

Let us first give the connection between the integrable equations with a
surface in $M_3$:

**Theorem 1** (Fokas-Gelfand [8]) Let $U(x, t; \lambda), V(x, t; \lambda), A(x, t; \lambda), B(x, t; \lambda)$
take values in an algebra $\mathcal{G}$ and let them be differentiable functions of $x, t$ and
$\lambda$ in some neighborhood of $M_2 \times \mathbb{R}$. Assume that these functions satisfy

$$U_t - V_x + [U, V] = 0,$$

and

$$A_t - B_x + [A, V] + [U, B] = 0.$$  

Define $\Phi(x, t; \lambda)$ in a group $G$ and suppose that $F(x, t; \lambda)$ takes values in the
algebra $\mathcal{G}$ by the equations

$$\Phi_x = U \Phi, \quad \Phi_t = V \Phi,$$

and

$$F_x = \Phi^{-1} A \Phi, \quad F_t = \Phi^{-1} B \Phi.$$

Then for each $\lambda$, $F(x, t; \lambda)$ defines a 2-dimensional surface in $\mathbb{R}^3$,

$$y_j = F_j(x, t; \lambda), \quad j = 1, 2, 3, \quad F = \sum_{k=1}^{3} F_k e_k,$$

where $e_k$, $(k = 1, 2, 3)$ form a basis of $\mathcal{G}$. The first and the second fundamental
forms of $S$ are

$$(ds)^2 \equiv g_{ij} dx^i dx^j = < A, A > dx^2 + 2 < A, B > dx dt + < B, B > dt^2,$$

$$(ds_{II})^2 \equiv h_{ij} dx^i dx^j = < A_x + [A, U], C > dx^2 + 2 < A_t + [A, V], C > dx dt + < B_t + [B, V], C > dt^2,$$

where $i, j = 1, 2$, $x^1 = x$ and $x^2 = t$, $< A, B > = -\frac{1}{2} \text{trace}(AB)$, $[A, B] = AB - BA$, $||A|| = \sqrt{< A, A >}$, and $C = \frac{[A, B]}{||[A, B]||}$. A frame on this surface
$S$, is

$$\Phi^{-1} A \Phi, \quad \Phi^{-1} B \Phi, \quad \Phi^{-1} C \Phi.$$
The Gauss and the mean curvatures of $S$ are given by $K = \det(g^{-1} h)$, $H = \frac{1}{2}\text{trace}(g^{-1} h)$.

From now on, subscripts $x$ and $t$ denote the derivatives of the objects with respect to $x$ and $t$, respectively. Subscript $nx$ means $n$ times $x$ derivative, where $n$ is a positive integer. Given $U$ and $V$, finding $A$ and $B$ from the equation $A_t - B_x + [A, V] + [U, B] = 0$ is in general a difficult task. However, there are some deformations which provide $A$ and $B$ directly. As an example of such deformations, we shall make use of the $\lambda$ parameter deformations [6],[7]:

$$A = \frac{\partial U}{\partial \lambda}, \quad B = \frac{\partial V}{\partial \lambda}, \quad F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}.$$ 

For the KdV equation the group $G$ is $SL(2, R)$ and the algebra $G$ is $sl(2, R)$ with the base $2 \times 2$ matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

It is clear from the construction that, this technique does not provide the explicit form of the desired surfaces. To determine these surfaces, we need the position vector $F$ which is given as $F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}$. This requires that the Lax equations $\Phi_x = U\Phi$, $\Phi_t = V\Phi$ must be solved exactly. This means that for a given solution of the nonlinear equation (KdV) we have to solve the corresponding Lax equations. Hence, the deformation technique is a kind of linearization of the construction of some surfaces.

3 Surfaces from a variational principle

Let $H$ and $K$ be the mean and the Gaussian curvatures of a 2-surface $S$ (either in $M_3$ (three dimensional Minkowski space) or in $\mathbb{R}^3$) then we have the following definition.

**Definition 2** Let $S$ be a 2-surface with its Gaussian ($K$) and mean ($H$) curvatures. A functional $\mathcal{F}$ is defined by
\[ \mathcal{F} = \int_S \mathcal{E}(H, K) dA + p \int_V dV \]  

where \( \mathcal{E} \) is some function of \( H \) and \( K \), \( p \) is a constant and \( V \) is the volume enclosed within the surface \( S \). For open surfaces, we let \( p = 0 \).

The following proposition gives the first variation of the functional \( \mathcal{F} \).

**Proposition 3** Let \( \mathcal{E} \) be a twice differentiable function of \( H \) and \( K \). Then the Euler-Lagrange equation for \( \mathcal{F} \) reduces to\[ \left( \nabla^2 + 4H^2 - 2K \right) \frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \nabla + 2KH) \frac{\partial \mathcal{E}}{\partial K} - 4HE + 2p = 0. \]  

Here, and from now on, \( \nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) \) and \( \nabla \cdot \nabla = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} h^{ij} \frac{\partial}{\partial x^j} \right) \); \( g = \det (g_{ij}) \), \( g^{ij} \) and \( h^{ij} \) are inverse components of the first and second fundamental forms; \( x^i = (x, t) \) and we assume Einstein’s summation convention on repeated indices over their ranges.

**Example 1.** We have some examples:

i) Minimal surfaces: \( \mathcal{E} = 1 \), \( p = 0 \).

ii) Constant mean curvature surfaces: \( \mathcal{E} = 1 \).

iii) Linear Weingarten surfaces: \( \mathcal{E} = aH + b \), where \( a \) and \( b \) are some constants.

iv) Willmore surfaces: \( \mathcal{E} = H^2 \), [4], [5].

v) Surfaces solving the shape equation: \( \mathcal{E} = (H-c)^2 \), where \( c \) is a constant, [17]-[22].

**Definition 4** Surfaces obtained from the solutions of the equation\[ \nabla^2 H + aH^3 + bHK = 0, \]  

where \( a \) and \( b \) are arbitrary constants, are called Willmore-like surfaces.
Remark 1 If \( a \neq 2 \) and \( b \neq -2 \), then these surfaces do not arise from a variational problem. The case \( a = -b = 2 \) corresponds to the Willmore surfaces.

For compact 2-surfaces, the constant \( p \) may be different than zero, but for noncompact surfaces we assume it to be zero. For such cases, we require asymptotic conditions, where \( H \) goes to a constant value and \( K \) goes to zero asymptotically. This requires that the KdV equation must have solutions decaying rapidly to zero at \( |x| \to \pm \infty \). We know that the soliton solutions satisfy this condition. For this purpose, we shall use the Euler-Lagrange equations (4) for surfaces obtained by KdV equation and look for solutions (surfaces) of these equations.

4 KdV Surfaces

In this section, we investigate some surfaces arising from the KdV equation. KdV surfaces are embedded in a three dimensional Minkowski space with signature +1. We can use our surface generating technique introduced in Theorem 1.

4.1 KdV surfaces from deformations of symmetries

Proposition 5 Let \( u(x,t) \) satisfy

\[
  u_t = \frac{1}{4} u_{3x} + \frac{3}{2} uu_x,
\]

and the corresponding \( sl(2,\mathbb{R}) \) valued Lax pair \( U, V \) be

\[
  U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix},
\]

\[
  V = \begin{pmatrix} -\frac{1}{4} u_x & -\frac{1}{2} u_x \\ \frac{1}{2}(2\lambda + u) (\lambda - u) & \frac{1}{4} u_x \end{pmatrix},
\]

where \( \lambda \) is a constant. The corresponding matrices of \( U \) and \( V \) are

\[
  A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix},
\]
where \( A = \mu (\partial U / \partial \lambda) \), \( B = \mu (\partial V / \partial \lambda) \) and \( \mu \) is a constant.

The surface \( S \), generated by \( U, V, A \) and \( B \), has the following first and second fundamental forms \((i, j = 1, 2)\)

\[
(ds_I)^2 = g_{ij} dx^i dx^j = -\mu^2 dx dt - \frac{\mu^2}{2} (4\lambda - u) dt^2,
\]

\[
(ds_{II})^2 = h_{ij} dx^i dx^j = -\mu dx^2 - \mu (2\lambda + u) dx dt - \frac{\mu}{4} (u_{2x} + (u + 2\lambda)^2) dt^2,
\]

with the corresponding Gaussian and mean curvatures

\[
K = -\frac{u_{2x}}{\mu^2}, \quad H = -\frac{2(\lambda - u)}{\mu},
\]

where \( x^1 = x, \ x^2 = t \).

We shall now consider the travelling wave solutions of the KdV equation. This means that \( u_t + u_x/c = 0 \), where \( c \neq 0 \) is a constant. Combining this with the KdV equation (6), we get

**Proposition 6** Let \( S \) be the surface obtained in Proposition 5 and \( u \) satisfy

\[
u_{2x} = -3u^2 - \frac{4}{c} u + 4\beta.
\]

Then \( S \) is a quadratic Weingarten surface satisfying the relation

\[
4\mu^2 c K - (\mu H + 2\lambda) [3c (\mu H + 2\lambda) + 8] + 16c \beta = 0,
\]

where \( c \neq 0 \) and \( \beta \) are constants.

**Proposition 7** The surface \( S \) defined in Proposition 5 is a Willmore-like surface, i.e. Gaussian and mean curvatures satisfy the equation

\[
\nabla^2 H + a H^3 + b H K = 0,
\]

(14)
where $\nabla^2$ is the Laplace-Beltrami operator on the surface. Here, we used the travelling wave solution of KdV equation and its consequence $u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma$, where

$$b = -1, a = -\frac{7}{4},$$

$$\beta = \frac{28\lambda \alpha - 16\alpha^2 - 21\lambda^2}{20},$$

$$\gamma = \frac{16\alpha^3 - 56\lambda \alpha^2 + 70\alpha \lambda^2 - 28\lambda^3}{5}.$$  \hspace{1cm} (15)

$\alpha = -1/c$, $(c \neq 0)$, $\lambda$ and $c$ are arbitrary constants.

**Proposition 8** By using the travelling wave solution of the KdV equation and Proposition 5, one can show that the mean curvature of the KdV surface $S$ satisfies a more general differential equation

$$\nabla^2 H = \frac{1}{2\mu^3} \left[5\mu^3 H^3 + 2\mu^2(3\lambda - 2\alpha)H^2 + 4\mu(-9\lambda^2 + 12\alpha \lambda - 8\alpha^2 - 12\beta)H - 56\lambda^3 + 112\lambda^2 \alpha - 64\alpha^2 \lambda + 32\lambda \beta - 64\alpha \beta - 16\gamma \right].$$ \hspace{1cm} (18)

4.2 KdV surfaces from a variational principle

**Proposition 9** KdV surface $S$ defined in Proposition 5 satisfies the generalized shape equation (4) for some $\mathcal{E}$, which is a polynomial of $H$ and $K$. Let $\text{deg}(\mathcal{E}) = N$, then

i) for $N = 3$:

$$\mathcal{E} = a_1 H^3 + a_2 H^2 + a_3 H + a_4 + a_5 K + a_6 KH,$$

$$a_1 = -8165p\mu^4 / 3 \left(54944\alpha \gamma - 553728\lambda \alpha^3 - 2558912\lambda^3 \alpha + 1420444\lambda \alpha \beta + 3338368\lambda^2 \alpha^2 - 416080\lambda^2 \beta - 19456\lambda \gamma + 407552\lambda^4 + 95427\beta^2 + 94848\alpha^2 \beta \right),$$
\[ a_2 = \frac{6a_1(14533\lambda - 12312\alpha)}{8165\mu}, \]
\[ a_3 = -\frac{9a_1(72064\alpha^2 + 187636\lambda^2 - 331168\lambda\alpha - 108813\beta)}{8165\mu^2}, \]
\[ a_4 = a_1 \left( 523816\lambda^3 - 2214672\lambda^2\alpha - 415296\alpha^3 + 2503776\lambda\alpha^2 + 1018557\alpha\beta - 304368\lambda\beta + 27600\gamma \right)/8165\mu^3, \]
\[ a_6 = -\frac{972a_1}{355}, \]

where \( \mu \neq 0, \ p \neq 0, \lambda, \alpha = -1/c, \ (c \neq 0), \beta, \gamma \) and \( a_5 \) are arbitrary constants, but \( \lambda, \alpha, \beta \) and \( \gamma \) can not be zero at the same time.

ii) for \( N = 4 \):
\[ E = a_1H^4 + a_2H^3 + a_3H^2 + a_4H + a_5 + a_6K + a_7KH + a_8KH^2, \]
\[ a_2 = \left( \left[ -303319296\lambda^2\gamma - 502986720\gamma\beta - 10818223680\lambda^3\beta - 4217392512\beta\alpha^3 - 22696872768\lambda\alpha^2\beta + 3385990152\lambda\beta^2 - 4991385168\alpha\beta^2 - 1902053376\alpha^2\gamma - 12134983680\lambda\alpha^3 - 36770221056\lambda^2\alpha^2 - 6024089088\lambda\alpha^4 - 2436418760\lambda^4\alpha + 31860385824\lambda^2\alpha^2\beta + 1157755008\lambda\alpha^3\beta + 5069893632\lambda^5 \right] a_1 - 9250945\ p\mu^5 \right)/\left(3399\mu \left[ -553728\lambda\alpha^3 + 3338368\lambda^2\alpha^2 - 416080\lambda^2\beta + 95427\beta^2 + 94848\alpha^2\beta - 2558912\lambda^3\alpha - 19456\lambda\gamma + 407552\lambda^4 + 1420444\alpha\beta + 54944\alpha\gamma \right] \right), \]
\[ a_3 = -6\left[ -14533\lambda + 12312\alpha \right] 1133\mu a_2 + \left[ 63409476\lambda^2 + 66459936\alpha^2 + 40215820\beta - 107133888\lambda\alpha \right] a_1 \left( 9250945\mu^2 \right), \]
\[ a_4 = \left( -\left[ -108813\beta + 187636\lambda^2 - 331168\lambda\alpha + 72064\alpha^2 \right] 10197\mu a_2 + \left[ 2257768448\lambda^3 + 359782560\gamma - 1687519464\lambda\beta + 602232192\alpha^3 - 6679109376\lambda^2\alpha + 4236473088\lambda\alpha^2 + 4082502096\alpha\beta \right] a_1 \right)/\left( 9250945\mu^3 \right), \]
\[ a_5 = -\left[ -523816\lambda^3 + 415296\alpha^3 + 304368\lambda\beta - 27600\gamma - \right]. \]
\[1018557 \alpha \beta - 2503776 \lambda \alpha^2 + 2214672 \lambda^2 \alpha \left[1133 \mu a_2 + \left[ -757178400 \alpha \gamma + 2345255472 \lambda^4 + 8043002568 \lambda \alpha \beta - 150602272 \alpha^4 - 3033745920 \lambda \alpha^3 - 7014105936 \alpha^2 \beta - 1846341216 \lambda^2 \beta + 7872480 \beta^2 + 543057600 \lambda \gamma + 10787593728 \lambda^2 \alpha^2 - 866258752 \lambda^3 \alpha \right] a_1 \right] \right) \left( 9250945 \mu^4 \right),
\]
\[a_7 = \frac{12 \left( [-517728 \alpha + 670472 \lambda] a_1 - 91773 a_2 \mu \right)}{402215 \mu},
\]
\[a_8 = \frac{-2280 a_1}{1133},
\]
where \( \mu \neq 0, \lambda, \alpha = -1/c, (c \neq 0), \beta, \gamma, a_1, a_6 \) and \( p \) are arbitrary constants, but \( \lambda, \alpha, \beta \) and \( \gamma \) can not be zero at the same time.

Here we used the travelling wave solution \( u_t = \alpha u_x, (\alpha = -1/c) \) of the KdV equation and its consequences \( u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma, v_{2x} = -3u^2 + 4\alpha u + 4\beta, u_{4x} = -6u_x^2 + (4\alpha - 6u)u_{2x} \).

**Remark 2** The KdV surface satisfies the Euler-Lagrange equation (4) for the Lagrangian with degree \( N \)
\[
\mathcal{E} = \sum_{n=1}^{N} \left( \sum_{k=1}^{N-n} a_{kn} H^{2k+1} \right) K^n + \sum_{l=0}^{N} b_l H^l + eK, \quad N = 3, 4, 5, ...
\]
Here \( a_{kn}, b_l \) and \( e \) are constants. Some of the constants can be written in terms of others.

## 5 Derivation of the surfaces

In the previous sections, we found possible surfaces satisfying certain equations, without giving the \( F \) functions explicitly. In this section, we shall find the position vector \( \overrightarrow{y} = \left( y_1(x,t), y_2(x,t), y_3(x,t) \right) \) of the corresponding KdV surfaces. To determine \( \overrightarrow{y} \), we use the equations
\[
F_x = \Phi^{-1} A \Phi, \quad F_t = \Phi^{-1} B \Phi,
\]
(20)
where \( F = \vec{\sigma} \cdot \vec{y} \). Hence, we need the \( 2 \times 2 \) matrix \( \Phi \) solving the Lax equation for the given function \( u(x, t) \). Our method of constructing the position vector \( \vec{y} \) of integrable surfaces consists of the following steps:

i) Finding a solution \( u = u(x, t) \) of the KdV equation with a given symmetry: Here, we consider travelling wave solutions \( u_t = -u_x/c \). By using this assumption we get

\[
    u_x^2 = -2u^3 - \frac{4}{c}u^2 + 8\beta u + 2\gamma, \tag{21}
\]

where \( c \neq 0, \beta \) and \( \gamma \) are arbitrary constants.

ii) Finding solution of the Lax equation

\[
    \Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{22}
\]

for given \( U \) and \( V \):

In our case, corresponding \( sl(2, R) \) valued \( U, V \) of the KdV equation are given in (7) and (8). Consider the \( 2 \times 2 \) matrix \( \Phi \)

\[
    \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{23}
\]

By using \( \Phi \) and \( U \), we can write \( \Phi_x = U\Phi \) in matrix form as

\[
    \begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ (\lambda - u)\Phi_{11} & (\lambda - u)\Phi_{12} \end{pmatrix}. \tag{24}
\]

By using \( (\Phi_{11})_x = \Phi_{21} \) and \( (\Phi_{21})_x = (\lambda - u)\Phi_{11} \), we have

\[
    (\Phi_{11})_{xx} - (\lambda - u)\Phi_{11} = 0. \tag{25}
\]

Similarly, we have an equation for \( \Phi_{12} \) as

\[
    (\Phi_{12})_{xx} - (\lambda - u)\Phi_{12} = 0. \tag{26}
\]

By solving (25) and (26) we determine the explicit \( x \)-dependence of \( \Phi_{11}, \Phi_{12} \) and also \( \Phi_{21}, \Phi_{22} \). By using \( \Phi_t = V\Phi \), we get

\[
    (\Phi_{11})_t = -\frac{1}{4}u_x\Phi_{11} + (\frac{1}{2}u + \lambda)\Phi_{21}, \tag{27}
\]

\[
    (\Phi_{21})_t = \left[ -\frac{1}{4}u_{2x} + \frac{1}{2} (2\lambda + u) (\lambda - u) \right] \Phi_{11} + \frac{1}{4}u_x\Phi_{21}, \tag{28}
\]
\[ (\Phi_{12})_t = -\frac{1}{4} u_x \Phi_{12} + \left( \frac{1}{2} u + \lambda \right) \Phi_{22}, \]  
(29)

\[ (\Phi_{22})_t = \left[ -\frac{1}{4} u_{2x} + \frac{1}{2} (2 \lambda + u) (\lambda - u) \right] \Phi_{12} + \frac{1}{4} u_x \Phi_{22}. \]  
(30)

By solving these equations, we determine the explicit \( t \)-dependence of \( \Phi_{11}, \Phi_{21}, \Phi_{12} \) and \( \Phi_{22} \). Hence we complete finding the solution \( \Phi \) of the Lax equation.

\textbf{iii) Finding } \( F \):

If \( \Phi \) depends on \( \lambda \) explicitly, \( F \) can be found directly from

\[ F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 e_1 + y_2 e_2 + y_3 e_3. \]  
(31)

If \( \Phi \) has been found for a fixed value of \( \lambda \), we use \( F_x = \Phi^{-1} A \Phi, \]  
\( F_t = \Phi^{-1} B \Phi \)

to find \( F = y_1 e_1 + y_2 e_2 + y_3 e_3 \). For our case, \( A \) and \( B \) are given in (9) and (10) which are the corresponding matrices of \( U \) and \( V \). Integrating the equations

\[ F_x = \Phi^{-1} A \Phi, \]  
\[ F_t = \Phi^{-1} B \Phi, \]  
(32)

we get \( F \). By writing \( F \) as a linear combination of \( e_1, e_2 \) and \( e_3 \), and collecting the coefficients of \( e_i \) (\( i = 1, 2, 3 \)), we get the components of the vector \( \vec{y} \).

\textbf{Example 2.} Let \( u = u_0 = \frac{2}{3} (\alpha \pm \sqrt{\alpha^2 + 3\beta}) \) be the constant solution of the integrated form \( u_x^2 + 2u^3 - 4\alpha u^2 - 8\beta u - 2\gamma = 0 \) of the KdV equation \( u_t = \frac{1}{4} u_{3x} + \frac{2}{3} uu_x \), where \( \alpha = -1/c, c \neq 0 \). By denoting \( \lambda - u_0 = m^2 \), we find the solutions of (25) and (26) as

\[ \Phi_{11} = A_1(t)e^{mx} + B_1(t)e^{-mx}, \]  
(33)

\[ \Phi_{12} = A_2(t)e^{mx} + B_2(t)e^{-mx}, \]  
(34)

and

\[ \Phi_{21} = (\Phi_{11})_x = m \left[ A_1(t)e^{mx} - B_1(t)e^{-mx} \right], \]  
(35)

\[ \Phi_{22} = (\Phi_{12})_x = m \left[ A_2(t)e^{mx} - B_2(t)e^{-mx} \right]. \]  
(36)

Since \( u \) is constant,

\[ (\Phi_{11})_t = (\frac{1}{2} u_0 + \lambda) \Phi_{21}, \]  
\[ (\Phi_{21})_t = \left[ \frac{1}{2} (2 \lambda + u_0) m \right] \Phi_{11}, \]  
(37)
\[ (\Phi_{12})_t = (\frac{1}{2} u_0 + \lambda)\Phi_{22}, \quad (\Phi_{22})_t = \left[ \frac{1}{2} (2\lambda + u_0) m \right] \Phi_{12}. \quad (38) \]

Denoting \( \frac{1}{2} (2\lambda + u_0) = n \) and using (33), (34), (35) and (36) in the equations (37) and (38), we find

\[ A_1(t) = C_1 e^{nmt}, \quad B_1(t) = D_1 e^{-nmt}, \quad (39) \]
\[ A_2(t) = C_2 e^{nmt}, \quad B_2(t) = D_2 e^{-nmt}, \quad (40) \]

where \( C_1, C_2, D_1 \) and \( D_2 \) are arbitrary constants. Thus \( \Phi \) has the following form

\[
\Phi = \begin{pmatrix}
C_1 e^{m(nt+x)} + D_1 e^{-m(nt+x)} & C_2 e^{m(nt+x)} + D_2 e^{-m(nt+x)} \\
m(C_1 e^{m(nt+x)} - D_1 e^{-m(nt+x)}) & m(C_2 e^{m(nt+x)} - D_2 e^{-m(nt+x)})
\end{pmatrix}.
\]

(41)

Remark 3 \( \det(\Phi) = 2m(C_2D_1 - C_1D_2) = \text{constant, in Example 2.} \)

Since \( \Phi \) depends on \( \lambda \) explicitly in (41), we can write \( F \) directly as

\[ F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 e_1 + y_2 e_2 + y_3 e_3, \quad (42) \]

where \( e_1, e_2, e_3 \) are basis elements of \( \mathfrak{sl}(2, R) \) and

\[ y_1 = -\left( \frac{D_1 C_2 + C_1 D_2}{D_1 C_2 - C_1 D_2} \right) \frac{(4\lambda - u)t + x}{2\sqrt{\lambda - u}}, \quad (43) \]
\[ y_2 = \left( \frac{D_1 C_1 - D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u)t + x}{2\sqrt{\lambda - u}}, \quad (44) \]
\[ y_3 = -\left( \frac{D_1 C_1 + D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u)t + x}{2\sqrt{\lambda - u}}. \quad (45) \]

Thus we find the position vector \( \vec{y} = (y_1(x, t), y_2(x, t), y_3(x, t)) \), where \( y_1, y_2 \) and \( y_3 \) are functions given as (43), (44) and (45), respectively. This solution corresponds to a plane in \( \mathbb{R}^3 \).

Example 3. Let \( u = 2k^2 c^2 \text{sech}^2 k(t - cx) \) be a solution of the KdV equation, where \( k^2 = -1/c^3 \). By denoting \( k(t - cx) = \xi \), we find the solutions of (25) and (26) as

\[ \Phi_{11} = A_1(t) \text{sech} \xi + B_1(t) \text{[sinh} \xi + \xi \text{sech} \xi], \quad (46) \]
\[ \Phi_{12} = A_2(t) \text{sech } \xi + B_2(t) [\sinh \xi + \xi \text{ sech } \xi], \quad (47) \]

and

\[ \Phi_{21} = (\Phi_{11})_x = kc \ A_1(t) \text{ sech } \xi \tanh \xi + kc \ B_1(t) [\xi \text{ sech } \xi \tanh \xi - \cosh \xi - \text{sech } \xi], \quad (48) \]

\[ \Phi_{22} = (\Phi_{12})_x = kc \ A_2(t) \text{ sech } \xi \tanh \xi + kc \ B_2(t) [\xi \text{ sech } \xi \tanh \xi - \cosh \xi - \text{sech } \xi], \quad (49) \]

for \( \lambda = k^2 c^2 \). By using these functions and considering (27), (29), (28), (30) with \( u_x = 4k^3 \ c^3 \text{ sech}^2 \xi \tanh \xi, \ u_{2x} = 4k^3 \ c^3 (2\text{sech}^2 \xi \tanh^2 \xi - \text{sech}^4 \xi) \), we get

\[ B_1(t) = B_1 \quad \text{and} \quad A_1(t) = 2B_1 kt + C_1, \quad (50) \]

\[ B_2(t) = B_2 \quad \text{and} \quad A_2(t) = 2B_2 kt + C_2, \quad (51) \]

where \( B_1, B_2, C_1 \) and \( C_2 \) are arbitrary constants. Thus components of \( \Phi \) are

\[ \Phi_{11} = B_1 \left( 2kt \text{ sech } \xi + \sinh \xi + \xi \text{ sech } \xi \right) + C_1 \text{ sech } \xi, \quad (52) \]

\[ \Phi_{12} = B_2 \left( 2kt \text{ sech } \xi + \sinh \xi + \xi \text{ sech } \xi \right) + C_2 \text{ sech } \xi, \quad (53) \]

\[ \Phi_{21} = kc \left[ B_1 \left( 2kt \text{ sech } \xi \tanh \xi - \cosh \xi - \text{sech } \xi + \xi \text{ sech } \xi \tanh \xi \right) \right. \]

\[ + C_1 \text{ sech } \xi \tanh \xi \], \quad (54) \]

\[ \Phi_{22} = kc \left[ B_2 \left( 2kt \text{ sech } \xi \tanh \xi - \cosh \xi - \text{sech } \xi + \xi \text{ sech } \xi \tanh \xi \right) \right. \]

\[ + C_2 \text{ sech } \xi \tanh \xi \]. \quad (55) \]

**Remark 4** \( \det(\Phi) = 2kc(C_2 B_1 - C_1 B_2) = \text{constant, in Example 3}. \)

Since \( \Phi \) is determined for fixed value of \( \lambda \), where components are given as (52), (53), (55) and (56), we obtain \( F_x \) and \( F_t \). They read

\[ F_x = \Phi^{-1} A \Phi = \begin{pmatrix} F_{x11}^{11} & F_{x12}^{12} \\ F_{x21}^{21} & F_{x22}^{22} \end{pmatrix}, \quad F_t = \Phi^{-1} B \Phi = \begin{pmatrix} F_{t11}^{11} & F_{t12}^{12} \\ F_{t21}^{21} & F_{t22}^{22} \end{pmatrix}. \quad (56) \]
where

\[ F_{x}^{11} = -\frac{\mu}{2k c \cosh^2 \xi (C_2 B_1 - C_1 B_2)} \left[ B_1 B_2 \left( 2\xi \sinh \xi \cosh \xi \right) + 4kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4k \xi + \xi^2 + 4k^2 t^2 - \cosh^2 \xi \right] + \left( B_1 C_2 + B_2 C_1 \right) \left( \sinh \xi \cosh \xi + 2kt + \xi \right) + C_1 C_2, \]  

(57)

\[ F_{x}^{12} = -\frac{\mu}{2k c \cosh^2 \xi (C_2 B_1 - C_1 B_2)} \left[ B_2^2 \left( 2\xi \sinh \xi \cosh \xi \right) + 4kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4k \xi + \xi^2 + 4k^2 t^2 - \cosh^2 \xi \right] + 2B_2 C_2 \left( \sinh \xi \cosh \xi + 2kt + \xi \right) + C_2^2, \]  

(58)

\[ F_{x}^{21} = -\frac{\mu}{2k c \cosh^2 \xi (C_2 B_1 - C_1 B_2)} \left[ B_2^2 \left( 2\xi \sinh \xi \cosh \xi \right) + 4kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4k \xi + \xi^2 + 4k^2 t^2 - \cosh^2 \xi \right] + 2B_1 C_1 \left( \sinh \xi \cosh \xi + 2kt + \xi \right) + C_1^2, \]  

(59)

\[ F_{x}^{22} = -F_{x}^{11}, \]  

(60)

\[ F_{t}^{11} = -\frac{\mu}{2k c \cosh^2 \xi (C_2 B_1 - C_1 B_2)} \left[ B_1 B_2 \left( 6\xi \sinh \xi \cosh \xi \right) + 12kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4k \xi + \xi^2 + 5 \cosh^2 \xi \right] + \left( B_1 C_2 + B_2 C_1 \right) \left( 3 \sinh \xi \cosh \xi + 2kt + \xi \right) + C_1 C_2, \]  

(61)

\[ F_{t}^{12} = -\frac{\mu}{2k c \cosh^2 \xi (C_2 B_1 - C_1 B_2)} \left[ B_2^2 \left( 6\xi \sinh \xi \cosh \xi \right) + 12kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4k \xi + \xi^2 + 5 \cosh^2 \xi \right] + 2B_2 C_2 \left( 3 \sinh \xi \cosh \xi + 2kt + \xi \right) + C_2^2, \]  

(62)
\[ F_{21}^t = -\frac{\mu}{2k_c \cosh^2 \xi \left(C_2B_1 - C_1B_2\right)} \left[B_1^2 \left(6\xi \sinh \xi \cosh \xiight)ight. \\
+ 12kt \sinh \xi \cosh \xi + \cosh^4 \xi + 4kt \xi + 4k^2 t^2 + \xi^2 - 5 \cosh^2 \xi \right] \\
+ 2B_1 C_1 \left(3 \sinh \xi \cosh \xi + 2kt + \xi\right) + C_1^2, \] 

\[ F_{22}^t = -F_{11}^t. \] 

By solving these equations, we get the position vector of the surface through the function \( F \) corresponding to the KdV equation with non-constant solution as

\[ F = y_1 e_1 + y_2 e_2 + y_3 e_3, \] 

where \( e_1, e_2, e_3 \) are basis elements of \( \mathfrak{sl}(2, \mathbb{R}) \) and

\[ y_1 = 2\zeta_1 \left(B_1B_2\zeta_2 + \zeta_3(C_1B_2 + C_2B_1) + 16c^3C_1C_2\right), \]

\[ y_2 = \zeta_1 \left(\zeta_2(B_2^2 - B_1^2) + 2\zeta_3(B_1C_1 - B_2C_2) - 16c^3(C_1^2 - C_2^2)\right), \]

\[ y_3 = \zeta_1 \left(\zeta_2(B_1^2 + B_2^2) + 2\zeta_3(B_1C_1 + B_2C_2) + 16c^3(C_1^2 + C_2^2)\right), \]

\[ \zeta_1 = \frac{\mu}{32c^2(B_1C_2 - B_2C_1)(1 + e^{2\xi})}, \]

\[ \zeta_2 = -8(1 - e^{2\xi})(3t - cx)^2 + 4kc^3(9t - cx)(1 + e^{2\xi}) \]

\[ + c^3(1 - e^{4\xi}) - 2c^3 \sinh 2\xi, \]

\[ \zeta_3 = 8kc^3(3t - cx) \left(1 - e^{2\xi}\right). \]

Thus we find the \( \vec{\mathbf{y}} = (y_1(x, t), y_2(x, t), y_3(x, t)) \) vector, where \( y_1, y_2 \) and \( y_3 \) are given as (66), (67) and (68), respectively.

### 6 Conclusion

In this work, we considered two families of surfaces, the Willmore-like surfaces and the surfaces derivable from a variational principle. Willmore-like surfaces, except for some particular values of the parameters, do not arise from a variational problem. To construct these two families of surfaces, we introduced a two step procedure. The first step is to use the method of deformation of the Lax equations corresponding to nonlinear partial differential equations. Any surface obtained through this method is called \textit{integrable}. At
this step, it is possible to find the first and second fundamental forms, the Gaussian and mean curvatures of these surfaces. In the second step of our approach, we determine the explicit locations, i.e., the position vectors, of these surfaces by solving the corresponding Lax equations of some integrable equations. As an application we used the KdV equation and its Lax equation. Corresponding to these equations, we have found several families of Willmore-like surfaces and a hierarchy of surfaces arising from a variational problem, where the Lagrange function is a polynomial of the Gaussian and mean curvatures of these surfaces.

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