Existence of Bound States in Continuous $0 < D < \infty$ Dimensions

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March 27, 2022

ABSTRACT

In modern fundamental theories there is consideration of higher dimensions, often in the context of what can be written as a Schrödinger equation. Thus, the energetics of bound states in different dimensions is of interest. By considering the quantum square well in continuous $D$ dimensions, it is shown that there is always a bound state for $0 < D \leq 2$. This binding is complete for $D \to 0$ and exponentially small for $D \to 2^-$. For $D > 2$, a finite-sized well is always needed for there to be a bound state. This size grows like $D^2$ as $D$ gets large. By adding the proper angular momentum tail a volcano, zero-energy, bound state can be obtained.

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1 Introduction

It is a well-known property of 1-dimensional quantum mechanics [1]-[3] that any (even
infinitesimally small) potential well, that is bounded above by and asymptotes to zero
energy, supports a negative-energy bound state. A proof by demonstration can be given
by noting that any finite well can be bounded from above by a square well, and the
square well can be explicitly solved to demonstrate a bound state.

A more sophisticated demonstration can use a Gaussian trial wave function to demon-
strate, by the Rayleigh-Ritz variational method, that there must be a bound state. This
particular proof fails for higher dimensions. Since a square well of radius $a$ and depth
$V_0$ in 3-dimensions must have $V_0a^2 > (\pi/2)^2$ for there to be a bound state, it can be
tempting to state that this ends the discussion, but not quite [4]. [Except where noted,
in this paper we use units $(\hbar^2/2m) = 1$.]

What if the potential goes above zero? If, in addition, it asymptotes above zero, then
one needs a finite-sized potential to have a bound state. For example, the Morse and
Rosen-Morse potentials

$$V_M(z) = A_0 \left[ e^{-2z/a} - e^{-z/a} \right], \quad V_{RM}(z) = C_0 [1 + \tanh z/a] - \frac{U_0}{\cosh^2 z/a}$$

only have (negative-energy) bound states if

$$A_0a^2 < 1/2, \quad C_0a^2 < \left[ 1 + 2U_0a^2 \right] - \left[ 1 + 4U_0a^2 \right]^{1/2}.$$  \(1\)

What if a potential becomes positive at places but asymptotes to zero? This turns
out to be of current interest in theories of higher dimensions [5]-[12]. In these theories
one can be trying to discover if volcano-shaped potentials have zero-energy bound states
in what amounts to a 1-dimensional Schrödinger-equation [13]. An example would be
the volcano potential

$$V(z) = -\frac{1}{2} + \frac{19}{4} \frac{z^2}{[1 + z^2]^2}.$$  \(3\)
But it can be demonstrated that the potential of Eq. (3) does not have a bound state by considering the potential (also shown in Figure 1)

\[ V(z) = -\left(\sqrt{5} - \frac{1}{2}\right) + \frac{19}{4} \frac{z^2}{(1 + z^2)^2}. \]  

(4)

This is a supersymmetric potential \[14\] of the form

\[ V(z) = [W'(z)]^2 - W''(z). \]  

(5)

\[ \psi_0(z) = N \exp[-W(z)], \]  

(6)

\[ W(z) = \left[\frac{\sqrt{5}}{2} - \frac{1}{4}\right] \ln \left(1 + z^2\right). \]  

(7)

Supersymmetry means that the potential of Eq. (4) must have a zero-energy bound state \[15\]. Therefore, since the potential of Eq. (4) is everywhere below that of Eq. (3), the potential (3) can not have a bound state \[16\].

Figure 1: The thin line shows the potential of Eq. (3). The thick curve shows the potential of Eq. (4), which has a zero-energy bound state.

There is also now the complementary idea that higher dimensions might grow out of lower dimensions (instead of \textit{vice versa}) \[17, 18\]. All these ideas stimulate the question of
just what condition is necessary to have a bound state in arbitrary *continuous* dimensions. Here we will study this by using the analytically solvable square well potential. This may provide physical insight into the concept of how compactifying/expanding dimensions arise.

2 The D-dimensional Schrödinger equation

We start with the integer $D$-dimensional, radial, Schrödinger equation \([19]-[21]\)

\[
E R_D(z) = \left[ -\frac{d^2}{dz^2} - \frac{(D-1)}{z} \frac{d}{dz} + \frac{l(l+D-2)}{z^2} + V(z) \right] R_D(z),
\]

with volume element $r^{D-1}dr$. The angular momentum factor $l(l+D-2)$ comes from the $D$-dimensional spherical harmonics \([19]\). It is often useful to substitute

\[
R_D(z) \equiv \frac{\chi_D(z)}{z^{(D-1)/2}}
\]

into Eq. (8). This transforms the $D$-dimensional radial Schrödinger equation into an effective $1$-dimensional Schrödinger equation

\[
E\chi_D(z) = \left[ -\frac{d^2}{dz^2} + U_D(z) \right] \chi_D(z),
\]

where $U$ is an effective potential

\[
U_D(z) = \frac{(D-1)(D-3)}{4z^2} + \frac{l(l+D-2)}{z^2} + V(z)
\]

\[
= \left( l + \frac{D-3}{2} \right) \left( l + \frac{D-1}{2} \right) + V(z).
\]

Now consider the radial bound-state problem in *continuous* dimensions $0 < D < \infty$ \([22]\). To do this, we use the square well:

\[
V(z) = \begin{cases} 
-V_0, & |z| < a, \\
0, & |z| > a.
\end{cases}
\]
We are looking for the lowest energy eigenstate, so we can set \( l = 0 \). Using the notation
\[
v = V_0 a^2, \quad \epsilon = -E_0 a^2 = |E_0| a^2, \quad z = ya,
\]
the internal and external Schrödinger equations are
\[
0 = \left[ -\frac{d^2}{dy^2} - \frac{D - 1}{y} \frac{d}{dy} - (v - \epsilon) \right] R_I, \tag{15}
\]
\[
0 = \left[ -\frac{d^2}{dy^2} - \frac{D - 1}{y} \frac{d}{dy} + (\epsilon) \right] R_E. \tag{16}
\]

The bound solutions to these equations are those that are finite at the origin with zero slope there, are normalizable, and are valid for all \( D > 0 \). They are \[23\]
\[
R_I \propto \frac{J_{(D-2)/2}(y\sqrt{v-\epsilon})}{(y\sqrt{v-\epsilon})^{(D-2)/2}} \propto \frac{j_{(D-3)/2}(y\sqrt{v-\epsilon})}{(y\sqrt{v-\epsilon})^{(D-3)/2}}, \tag{17}
\]
\[
R_E \propto \frac{K_{(D-2)/2}(y\sqrt{\epsilon})}{(y\sqrt{\epsilon})^{(D-2)/2}} \propto \frac{k_{(D-3)/2}(y\sqrt{\epsilon})}{(y\sqrt{\epsilon})^{(D-3)/2}}, \tag{18}
\]
where \( J \) and \( j \) are Bessel and spherical Bessel functions and \( K \) and \( k \) are modified Bessel and modified spherical Bessel functions.

To obtain the bound, ground-state solution one needs to find the value of \( \epsilon \) such that the internal \((y \leq 1 \text{ or } z \leq a)\) solutions, \( R_I \), and external \((y \geq 1 \text{ or } z \geq a)\) solutions, \( R_E \), satisfy the \( y = 1 \) (or \( z = a \)) boundary condition
\[
\lim_{y \to 1} \left\{ \frac{d}{dy} \frac{R_I(y\sqrt{v-\epsilon})}{R_I(y\sqrt{v-\epsilon})} \right\} = \lim_{y \to 1} \left\{ \frac{d}{dy} \frac{R_E(y\sqrt{\epsilon})}{R_E(y\sqrt{\epsilon})} \right\}. \tag{19}
\]

Eq. (19) is equivalent to
\[
\sqrt{v-\epsilon}J_{(D-2)/2}(\sqrt{v-\epsilon}) = \frac{\sqrt{\epsilon}K'_{(D-2)/2}(\sqrt{\epsilon})}{K_{(D-2)/2}(\sqrt{\epsilon})} \tag{20}
\]
where “prime” denotes derivative with respect to the argument. (Eq. (19) can also be written in terms of the spherical Bessel functions.) Bessel-function recursion relations transform Eq. (20) to
\[
\frac{\sqrt{v-\epsilon}J_{D/2}(\sqrt{v-\epsilon})}{J_{(D-2)/2}(\sqrt{v-\epsilon})} = \frac{\sqrt{\epsilon}K_{D/2}(\sqrt{\epsilon})}{K'_{(D-2)/2}(\sqrt{\epsilon})} \tag{21}
\]
3 Ground-state eigenenergies

3.1 Binding for dimensions $0 < D < 2$

Here we are looking for the size of the binding energy $\epsilon_D(v)$ as $v(>\epsilon_D) \to 0$. We can use either analytic approximations or numerical methods to evaluate the Bessel functions in Eqs. (13)-(21). To begin, one finds that

$$\lim_{v \to 0} \epsilon_D \to 0_+ \sim v.$$  

(22)

Examples are

$$\epsilon_{0.01}(v = 0.1) = 0.0986, \quad \epsilon_{0.001}(v = 0.1) = 0.0998.$$  

(23)

That is, as $D \to 0_+$ for small $v$, the particle becomes totally bound, with no quantum ground-state energy. Actually, for $D \to 0_+$ this tight binding holds for general well sizes, e.g.,

$$\epsilon_{0.01}(v = 1) = 0.995, \quad \epsilon_{0.001}(v = 1) = 0.999.$$  

(24)

As $D$ increases, the binding becomes less tight. When one reaches $D = 1$ one has the standard 1-dimensional solution (which in spherical harmonic notation involves $j_{-1}$ and $k_{-1}$)

$$\sqrt{\epsilon_1} = \sqrt{v - \epsilon_1} \tan \left( \sqrt{v - \epsilon_1} \right), \quad \epsilon_1 \sim v^2 \left[ 1 - \frac{4}{3}v + \ldots \right].$$  

(25)

In particular,

$$\epsilon_{D=1}(v = 0.1) = 0.0088.$$  

(26)

As $0 < D < 2$ continues to rise, for a given small $v$ the binding energy gets smaller and smaller, varying as a power

$$\epsilon_D(v) \propto (v)^{\alpha(D)/(2-D)},$$  

(27)

where $\alpha$ is a decreasing function of $D$. (See Figure 2.) Finally, $\epsilon_D$ reaches a limit as $D \to 2_-$:

$$\epsilon_{1.99999}(v = 0.1) \approx \epsilon_{1.999999}(v = 0.1) \approx 8.8 \times 10^{-18}.$$  

(28)
This is 21 orders of magnitude smaller than the value of $\epsilon_1(v = 0.1)$ for $D = 1$ given in Eq. 28. (This limit will be explained in the next subsection.) But it is to be emphasized that the entire restricted regime $0 < 2 < D$ always has a bound state, no matter how small is $v$.

![Figure 2: log$_{10} [\epsilon_D]$ is plotted as a function of $D$ for $v = 0.1$](image)

### 3.2 Binding for dimension $D = 2$

When $D = 2$, the interior and exterior wave functions are proportional to the particularly elegant functions $J_0$ and $K_0$, which can be evaluated especially easily for the small arguments $\sqrt{v - \epsilon}$ and $\sqrt{\epsilon}$. When these are used in Eq. (19), one finds

$$\lim_{v \to 0} \epsilon_D = 2 \sim \exp \left[ -\frac{4}{(v + v^2/8)} + 2(\ln 2 - \gamma) \right],$$

where $\gamma$ is Euler’s constant. Substituting in $v = 0.1$ gives

$$\epsilon_D = 2(v = 0.1) = 8.8 \times 10^{-18},$$

in agreement with our limiting result of Eq. 28.

That is, the limit $D \to 2_-$ is continuous, and this limit yields an exponentially small binding. This type of result for $D = 2$ is actually known [1], although not as commonly
as one might think. This result does not violate the general Rayleigh-Ritz principle [24].

One simply has to use something like an exponentially varying trial wave function of the form \( \exp[-(r - r_0)^\alpha] \). Further, this result has been given as an intuitive explanation of why superconductivity works. The small attraction still gives a bound state on the 2-dimensional Fermi surface.

3.3 Condition for binding when \( D > 2 \)

For \( D > 2 \), there must always be a finite-sized \( v > v_D \) for there to be a bound state. The way to determine \( v_D \) is to solve the boundary equation (19) for \( \epsilon \to 0 \). When this is done, the right-hand side reduces to \( \left[ -(D - 2) \right] \) and the left-hand side reduces to \( \left[ -(D - 2)/2 + \sqrt{v_D} \left( \{d/d \sqrt{v_D} \} J_{(D-2)/2}(\sqrt{v_D}) \right) / J_{(D-2)/2}(\sqrt{v_D}) \right] \). In other words, the problem amounts to finding the first zero of

\[
0 = \left[ \sqrt{v_D} J_D(\sqrt{v_D}) - (D - 2)J_{(D-2)/2}(\sqrt{v_D}) \right].
\]

(31)

As \( D \to 2+ \), one finds

\[
\lim_{\delta \to 0+} v_{2+\delta} \sim 4 \left( \frac{D - 2}{2} \right) = 2\delta.
\]

(32)

For example, \( v_{D=2.02} = 0.0402 \).

For odd-integer \( D \), the boundary equation can be written in terms of integer-order spherical Bessel functions. These functions are in terms of powers and trigonometric functions. Although they become more complicated for higher integers, the lower-order equations are simple:

\[
D = 3: \quad 0 = \cot \sqrt{v_3}, \quad v > v_3 = \frac{\pi^2}{4} \sim 2.5,
\]

(33)

note that \( \sqrt{v_3} = \pi/2 \) is one unit of phase space,

\[
D = 5: \quad 0 = \sin \sqrt{v_5}, \quad v > v_5 = 4\frac{\pi^2}{4} \sim 9.9
\]

(34)

\[
D = 7: \quad 0 = \tan \sqrt{v_7} - \sqrt{v_7}, \quad v > v_7 \sim 8.2 \frac{\pi^2}{4} \sim 20.
\]

(35)
Thinking in units of $\nu = (D - 2)/2$, one sees that $v_D$ is becoming quadratic in this variable. Indeed, as $D$ becomes large, one finds

$$\lim_{D \to \infty} v_D \sim \left(\frac{D - 2}{2}\right)^2.$$ 

(36)

An example is $v_{1000} \sim 2.63 \times 10^5$. The transition from the limit of Eq. (32) to the limit of Eq. (36) can be seen in Figure 3.

Figure 3: A plot of $\log_{10}[v_D]$ as a function of $\log_{10}[(D - 2)^2/4]$. The long-dash curve is the limit curve of Eq. (32) and the short-dash curve is the limit curve of Eq. (36).

This all is consistent with the previous results; the higher the dimension the less is the binding. It also agrees with known results in higher, integer $D = N$ dimensions. For example, the ground-state energies of the confining harmonic oscillator and the infinitely deep hydrogen atom are (in ordinary units),

$$E_0^{HO} = \frac{N}{2} \hbar \omega, \quad E_0^{HA} = -\frac{me^4}{2\hbar^2} \frac{1}{[1 + (N - 3)/2]^2},$$

(37)

respectively [20, 25].
4 Square-well volcanos

Now we can return to the beginning and ask, “Under what conditions will one have a zero-energy bound (ground) state if one adds an exterior angular-momentum barrier to the square well?”

\[
V(z) = \begin{cases} 
-V_0, & |z| < a, \\
+b^2/z^2, & |z| > a.
\end{cases}
\] (38)

The interior Schrödinger equation, its solution, and the boundary condition will be the same as in Eqs. (15), (17), and (19)-(21), except that everywhere \((v - \epsilon) \to v\). The exterior Schrödinger equation is now

\[
0 = \left[ -\frac{d^2}{dy^2} - \frac{D - 1}{y} \frac{d}{dy} + \frac{b^2}{y^2} \right] R_E.
\] (39)

The solution and boundary condition are

\[
R_E(y) \propto y^{-s}, \quad \lim_{y \to 1} \left\{ \frac{d}{dy} \left[ \frac{R_E(y)}{R_E(y)} \right] \right\} = -s,
\] (40)

where

\[
b^2 = s(s + 1) - s(D - 1).
\] (41)

Normalizability of the wave function and positivity of Eq. (41) mean

\[
S > \frac{D}{2}, \quad s > D - 2.
\] (42)

Using these new boundary conditions one finds, analogously to Eq. (21), that for the existence of a zero-energy bound state one needs

\[
s = \sqrt{v} J_{D/2}(\sqrt{v}) - \frac{(D - 2)}{2}.
\] (43)

Figure 4 shows the solution for \(s\) as a function of \(D \leq 5\) and \(v \leq 25\). For low \(\{D, v\}\) there is a widening infinite ridge as one goes to higher \(v > D\), followed by an infinite valley. For higher \(v\) a new (wider) infinite ridge/valley sequence begins. (Further sequences, not shown, start for yet higher \(v\).)
Figure 4: The solution of Eq. (43), which is the condition for a zero-energy state. The solution $s$ is plotted as a function of $D$ and $v$. [Finite numerical cell size causes the narrow, infinite peak/valley in $s$, near the $D - v$ origin, to appear finite in places.]

To better understand this result, in Figure 4 we show two physically illuminating $D$ slices, $D = 1$ and $D = 3$. For $D = 1$ one needs $s > 1/2$, so that the zero energy state is normalizable and hence a bound “ground state.” This is the asymptotic result as $v \to 0$. As $v$ gets larger the height of the tail must get larger to force the ground state to remain at zero energy. Finally, at the depth of the ground state for an infinite well, $v = \pi^2/4$, the tail is infinitely strong, making it an effective infinite well. For still larger $v$ the solution becomes discontinuously unphysical, rising from negative infinity. The solution eventually becomes positive (but not bound) since at first $s < 1/2$. When $s > 1/2$ one has a new bound solution, which is an excited state. This can be verified when the previous pattern is repeated (with a wider peak). There is a second discontinuous peak/valley
jump at $v = \pi^2/4$, the energy of the first (even) excited state for the infinite square well. Finally, as expected, the $D = 3$ slice shows no bound-state zero-energy solution for small $v$. It is only when $s > 1/2$ that a solution exists. The wider peak/valley discontinuity is at the energy of the 3-dimensional infinite-well ground state, $v = \pi^2$.

Figure 5: Plots of $s$ vs. $v$ for the cases $D = 1$ and $D = 3$.

5 Discussion

If one goes back to Eq. (12) one sees that for $l = 0$ it is the dimensions $1 < D < 3$ that have an “effective attractive” angular-momentum barrier for the $\chi$ equation (10). Among this set of dimensions, $D = 2$ is the only integer dimension [26]. The other $D$ with this property have power-law binding ($1 < D < 2$) or need a finite $v_D$ to have binding ($2 < D < 3$). $D = 2$ is truly the boundary case for quantum binding [27]. Further, $1 < D < 3$ is a transition region in general. If one looks at Figure 4, one sees that it is for $D > 1$ that the binding energy starts to go steeply towards zero. Also, if one looks at Figure 3 it is for $D > 3$ [or log$_{10} \{(D - 2)^2/4\} > (-0.60)$] that $v_D$ goes over to the more rapidly rising quadratic function of $D$.

It is to be noted that how one handles the spherical harmonics in continuous dimensions is a separate interesting question. Specifically, the boundary conditions will be related to questions of statistics, as in the fractional quantum Hall effect. One also can consider the introduction of spin and/or explicit relativistic binding. These are topics
for further investigation.

Here the energetics of bound states in continuous radial dimensions has been investigated. This could shed light on the energetics of changing dimensions in the physical universe.

**Acknowledgements**

I first thank Csaba Csáki Joshua Erlich, who interested me in the bound-state problem as it relates to fundamental theories of extra dimensions. Wolfgang Schleich raised the question about the odd behavior of the 2-dimensional bound-state problem. It was these two stimuli that initiated this work. I also am very grateful for the comments and suggestions of Kurt Gottfried, André Martin, Eugen Merzbacher, and Jean-Marc Richard. This work was supported by the US DOE.

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