An ultrawide-zero-frequency bandgap metamaterial with negative moment of inertia and stiffness

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Keywords: metamaterials, negative effective moment of inertia, negative effective stiffness, coupled bandgap, ultrawide-zero-frequency bandgap

Abstract

Metamaterials have demonstrated great potential for controlling wave propagation since they are flexibly adjustable. A new one-dimensional metamaterial model with both a negative effective moment of inertia and negative effective stiffness is proposed. A negative effective moment of inertia and negative effective stiffness can be achieved by adjusting the structural parameters in certain frequency ranges. Bandgaps in the low-frequency range with exponential wave attenuation can be generated in the metamaterial. A flat band is obtained that couples two Bragg bandgaps to achieve a wide bandgap in the low-frequency range, where the effective moment of inertia and effective stiffness are both infinite. A zero-frequency bandgap can be achieved by adjusting the structural parameters. Quick attenuation of wave is observed in the zero-frequency ranges with single-negative parameters. Furthermore, an ultrawide-zero-frequency bandgap is obtained by optimizing the structural parameters of the system. In addition, it is easy to switch between the Bragg and locally resonant bandgaps. This new metamaterial can be applied to ultralow-frequency-vibration isolation.

1. Introduction

Metamaterials, with superior properties, are artificial structures composed of periodic materials or periodic structures [1–3]. The bandgaps of acoustic waves or elastic waves can be generated in metamaterials. In recent decades, it was found that metamaterials can exhibit properties such as negative masses [4–7], negative bulk modulus [8, 9], negative refraction phenomena [10, 11], topological effects [12, 13] and invisibility cloaking [14, 15]. These characteristics can be used to develop and manufacture new acoustic functional devices and to manipulate the propagation of acoustic waves or elastic waves.

Negative effective material properties have been investigated widely in the design of metamaterials. In 2000, Liu et al. introduced negative effective mass density in phononic crystals with local resonance by embedding heavy balls coated with soft silicon rubber in epoxy [16]. Since then, a large number of metamaterial models have been proposed to achieve negative effective parameters by designing unit cells [8, 17, 18].

Mass-spring systems can also exhibit negative effective parameters due to Bragg scattering and local resonance mechanisms in different frequency ranges. In 2007, Milton and Willis introduced negative effective mass in a mass-in-mass acoustic metamaterial model. They confirmed the existence of single-negative or double-negative properties [19]. Liu et al. investigated an elastic model with double-negative parameters constructed by a chiral mass-spring unit [20]. Wang presents a new representative cell of metamaterials in an effort to provide a comprehensive model for generating negative mass or negative modulus [21]. In 2018, Oh et al. proposed a spin-harnessed metamaterial and obtained a Bragg bandgap from zero frequency [22]. In 2019, Wang et al. designed a metamaterial rod with resonators containing negative-stiffness mechanisms for generating low-frequency bandgaps [23]. Then, Zhang et al.
explored the dark state, zero-index and topology in acoustic metamaterials with negative mass and negative coupling [24]. Recently, Bormashenko et al investigated negative effective mass metamaterials based on electromechanical coupling exploiting plasma oscillations of free electron gas [25]. Li et al proposed a 1-D metamaterial chain with a heterogeneous resonator to generate merged broad bandgaps by parameter adjustment [26]. Although, a number of studies involving negative mass or negative modulus have been reported. Limited by the structure, most of those systems are single-negative systems. Furthermore, few studies have considered the coupling of translational movement and rotational movement.

On the other hand, metamaterials with low-frequency-coupled bandgaps constitute an interesting field. In 2018, Chang et al designed a 1D mass-spring system with a substructure. They also found that the coupled Bragg resonant bandgap is much wider than the local resonant single bandgap [27]. Krushynska et al proposed ‘accordion-like’ meta-structures to obtain extremely wide bandgaps and uniform wave attenuation at low frequencies [28]. In 2019, Zhou et al investigated a high-static-low-dynamic-stiffness resonator with an inertial amplification mechanism, which is able to create a much lower band gap than a pure high-static-low-dynamic-stiffness resonator [29]. In 2020, Gao and Wang proposed two different metamaterials with hybrid unit cells to achieve a wider coupled bandgap [30]. To the best of our knowledge, little work has been devoted to structures with ultrawide-zero-frequency bandgaps because they are difficult to design.

In the present study, our primary objective is to obtain a wider and lower bandgap by designing a periodic metamaterial with a negative moment of inertia and stiffness. To that end, this paper is organized as follows. In section 2, a chiral coupled one-dimensional metamaterial with translational and rotational motion is investigated. The concept of a negative effective moment of inertia is proposed. In section 3, the dispersion relation, effective parameters and transmittance of the one-dimensional periodic system are analyzed. In section 4, wave propagation of optimized periodic systems is investigated. Finally, some conclusions are drawn in section 5.

2. Metamaterial model

2.1. Model and methods
The one-dimensional periodic structure in figure 1 is considered. The system is formed by a periodic structure, with the representative cell consisting of one central rigid disc, one rigid lever and linear elastic springs attached to the discs, as shown in figure 1(a). The circular discs, whose mass and moment of inertia are $m$ and $J_1$, respectively, can rotate freely. The moment of inertia of levers are $J_2$. The middle of the lever is hinged at point $O$, and $O_1$ is connected to $O_2$ of the disc by spring $k_2$. The distance from $O$ to $O_1$, $R$, is equal
to the distance from $O_2$ to the center of the disc. Additionally, the center of the disc is connected to the fixed point $O$ by spring $k_1$. The lattice constant of the system is $L$. In this periodic system, both the rigid disc and lever are supported by a rigid rod. This rigid rod is fixed when the longitudinal wave propagates along the periodic system.

It is assumed that the rotational angles of both the lever and the disc are small. Additionally, the damping and gravity effects are ignored. In the current model, the disc is limited to rotary motion and to the distance from point $O$ where $\alpha$ denotes the angular displacement of the disc

\[
\begin{align*}
\ddot{u}_n &= -2k_1u_n - k_2 (u_n - \alpha_n R + \theta_n R) - k_2 (u_n + \alpha_{n+1} R - \theta_n R), \quad (1a) \\
\dot{\theta}_n &= -k_2 R (u_n - \alpha_n R + \theta_n R) + k_2 R (u_n + \alpha_{n+1} R - \theta_n R), \quad (1b) \\
J_2 \ddot{\alpha}_n &= -k_2 R (u_{n-1} + \alpha_n R - \theta_{n-1} R) + k_2 R (u_n - \alpha_n R + \theta_n R). \quad (1c)
\end{align*}
\]

where $u_n$ denotes the displacement of the disc in the $n$th unit cell, $\theta_n$ is the angular displacement of the disc and $\alpha_n$ is the angular displacement of the lever. Based on Floquet–Bloch theory, the harmonic wave solution of the one-dimensional lattice system can be expressed as

\[
\begin{align*}
u_n &= \tilde{u}_n e^{i(-mqL + \omega t)}, \quad (2a) \\
\theta_n &= \tilde{\theta}_n e^{i(-mqL + \omega t)}, \quad (2b) \\
\alpha_n &= \tilde{\alpha}_n e^{i(-mqL + \omega t)}, \quad (2c)
\end{align*}
\]

where $\tilde{u}_n$, $\tilde{\theta}_n$ and $\tilde{\alpha}_n$ are the amplitudes of $u_n$, $\theta_n$ and $\alpha_n$, respectively. In equation (2), $q$ and $\omega$ are the wavenumber and circular frequency, respectively. Substituting equation (2) into equation (1), and the dynamic equation can be rewritten as

\[
\begin{align*}
-m\omega^2 \tilde{u}_n &= -2k_1 \tilde{u}_n - k_2 \left( \tilde{u}_n - \tilde{\alpha}_n R + \tilde{\theta}_n R \right) - k_2 \left( \tilde{u}_n + \tilde{\alpha}_n R - \tilde{\theta}_n R \right), \quad (3a) \\
-J_1 \omega^2 \tilde{\theta}_n &= -k_2 R \left( \tilde{u}_n - \tilde{\alpha}_n R + \tilde{\theta}_n R \right) + k_2 R \left( \tilde{u}_n + \tilde{\alpha}_n R - \tilde{\theta}_n R \right), \quad (3b) \\
-J_2 \omega^2 \tilde{\alpha}_n &= -k_2 R \left( \tilde{u}_n e^{iqL} + \tilde{\alpha}_n R - \tilde{\theta}_n R e^{iqL} \right) + k_2 R \left( \tilde{u}_n - \tilde{\alpha}_n R + \tilde{\theta}_n R \right). \quad (3c)
\end{align*}
\]

If equation (3) has a nontrivial solution, the dispersion relation of this one-dimensional system can be obtained as

\[
\begin{vmatrix}
2 (k_1 + k_2) - m\omega^2 & 0 & -k_2 R (1 - e^{-iqL}) \\
0 & J_1 \omega^2 - 2k_2 R^2 & k_2 R^2 (1 + e^{-iqL}) \\
-k_2 R (e^{iqL} - 1) & k_2 R^2 (e^{iqL} + 1) & J_2 \omega^2 - 2k_2 R^2
\end{vmatrix} = 0. \quad (4)
\]

From equations (1a) and (1b), one can obtain

\[
\begin{align*}
u_n &= \frac{k_2 R}{2 (k_1 + k_2) - m\omega^2} \left( \alpha_n - \alpha_{n+1} \right), \quad (5a) \\
\theta_n &= \frac{k_2 R^2}{2k_2 R^2 - J_1 \omega^2} \left( \alpha_n + \alpha_{n+1} \right). \quad (5b)
\end{align*}
\]

Substituting equation (5) into (1c), one obtains

\[
J_2 \omega^2 \alpha_n = k_{eff} R^2 (2\alpha_n - \alpha_{n-1} - \alpha_{n+1}), \quad (6)
\]

where
Figure 2. Negative moment of inertia and negative stiffness ranges for six cases.

\[ J_{\text{eff}} = J_2 \left(1 - \frac{2\omega_3^2}{\omega_2^2} + \frac{4}{\left(2 - \frac{\omega_3^2}{\omega_2^2}\right)\omega_2^2}\right), \quad (7a) \]

\[ k_{\text{eff}} = k_2 \left(\frac{1}{2 - \frac{\omega_3^2}{\omega_2^2}} - \frac{1}{2 \left(\frac{\omega_3^2}{\omega_2^2} + 1\right) - \frac{\omega_3^2}{\omega_2^2}}\right). \quad (7b) \]

\( J_{\text{eff}} \) and \( k_{\text{eff}} \) are the effective moment of inertia and the effective stiffness of the system, respectively. In equation (7), \( \omega_1, \omega_2, \omega_3, \) and \( \omega_4 \) can be given by

\[ \omega_1^2 = \frac{k_1}{m}, \quad \omega_2^2 = \frac{k_2}{m}, \quad \omega_3^2 = \frac{k_2R^2}{J_1}, \quad \omega_4^2 = \frac{k_2R^2}{J_2}. \quad (8) \]

It is clearly shown by equation (7) that the effective moment of inertia is a function of \( \omega_3 \) and \( \omega_4 \), which are controlled by the rotational motion of the discs and levers. The effective stiffness is a function of \( \omega_1, \omega_2 \) and \( \omega_3 \), which are controlled by the translational motion and rotational motion of the discs. Equation (6) can be further reduced to

\[ \omega^2 = 4\frac{k_{\text{eff}}R^2 \sin^2 \frac{\theta}{2}}{J_{\text{eff}}}. \quad (9) \]

The dispersion curves can be obtained with equation (9).

The transfer matrix method is used to analyze the transmittance properties of the metamaterial. There are \( N \) unit cells in a one-dimensional periodic system. Thus, the recursive relation can be obtained as

\[ (2k_{\text{eff}}R^2 - J_{\text{eff}}\omega^2) \alpha_n = k_{\text{eff}}R^2 \left(\alpha_{n-1} + \alpha_{n+1}\right), \quad n = 1, 2, 3, 4, \ldots, N - 1. \quad (10) \]
The dynamic behavior of the system is controlled by three parameters $2.2$. Negative effective moment of inertia and negative stiffness

The angular displacement of the $n$th effective unit cell is $\alpha_n$. The transmittance of the system is defined as $T = \prod_{n=1}^{N} T_n$. Equation (10) yields

$$T_n = \frac{\alpha_n}{\alpha_{n-1}} = -k_{\text{eff}}R^2 \eta - k_{\text{eff}}R^2(2 - T_{n+1}) + J_{\text{eff}}\omega^2, n = 2, 3, 4, \ldots, N, \quad T_{N+1} = 1.$$ (11)

### 2.2. Negative effective moment of inertia and negative stiffness

The dynamic behavior of the system is controlled by three parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$, defined as

$$\lambda_1 = \frac{\omega_1^2}{\omega_2^2} = \frac{k_1}{k_2}, \quad \lambda_2 = \frac{\omega_3^2}{\omega_4^2} = \frac{f_1}{mR^2}, \quad \lambda_3 = \frac{\omega_5^2}{\omega_6^2} = \frac{f_2}{f_1}.$$ (12a, 12b, 12c)

The equation (7a) can be rewritten as

$$J_{\text{eff}} = J_1 \frac{f_1}{f_2},$$ (13)

where the functions $f_1$ and $f_2$ are given by

$$f_1 \left( \frac{\omega}{\omega_4} \right) = 2\lambda_3 - \frac{\omega^2}{\omega_4^2}, \quad f_2 \left( \frac{\omega}{\omega_4} \right) = 2 + 2\lambda_3 - \frac{\omega^2}{\omega_4^2}.$$ (14a, 14b)

Equation (7b) can be rewritten as

$$k_{\text{eff}} = k_2 \frac{g_1}{g_2 g_3},$$ (15)

where the functions $g_1$, $g_2$ and $g_3$ are given by

$$g_1 \left( \frac{\omega}{\omega_4} \right) = \lambda_3 \left(2\lambda_1 \lambda_2 \lambda_3 + (\lambda_2 - 1) \frac{\omega^2}{\omega_4^2}\right), \quad g_2 \left( \frac{\omega}{\omega_4} \right) = \frac{\omega^2}{\omega_4^2} - 2\lambda_3, \quad g_3 \left( \frac{\omega}{\omega_4} \right) = \frac{\omega^2}{\omega_4^2} - 2\lambda_2 \lambda_3 (\lambda_1 + 1).$$ (16a, 16b, 16c)

The ranges for the negative moment of inertia and stiffness are controlled by five positive roots of $f_1 = 0, f_2 = 0, g_1 = 0, g_2 = 0$ and $g_3 = 0$. These roots are assumed to be $\xi_1, \xi_2, \eta_1, \eta_2$ and $\eta_3$, respectively. These ranges of frequencies are illustrated in figure 2 for six different cases, $\xi_1 < \eta_1 < \eta_2 < \xi_2, \xi_1 < \eta_1 < \eta_2 < \xi_2 < \xi_1, \eta_2 < \xi_1 < \eta_1 < \eta_2 < \eta_1, \eta_2 < \xi_1 < \eta_1 < \eta_2 < \eta_1, \eta_2 < \xi_1 < \eta_1 < \eta_2 < \eta_1$, and $\xi_1 < \eta_1 < \eta_2 < \xi_2$.
Figure 4. Frequency ranges for negative effective parameters for (a) $\lambda_1 = 0.5, \lambda_2 = 1$, (b) $\lambda_1 = 1.5, \lambda_2 = 1$.

Figure 5. Frequency ranges for negative effective parameters for (a) $\lambda_1 = 1, \lambda_2 = 0.2$, (b) $\lambda_1 = 1, \lambda_2 = 0.8$, (c) $\lambda_1 = 1, \lambda_2 = 1.5$.

$\eta_2 (\xi_1) < \xi_2 \leq \eta_3$, as given in appendix A. The red areas, blue areas and gray areas correspond to the frequency ranges with a negative moment of inertia, negative stiffness, and double-negative, respectively.

Figures 3–5 show the effects of $\lambda_1, \lambda_2$ and $\lambda_3$ on the dimensionless frequency, respectively. It can be observed that for a given $\omega$, single-negative or double-negative behavior can be achieved by adjusting $\lambda_1, \lambda_2$ and $\lambda_3$. A single-negative of the effective parameters corresponds to a locally resonant bandgap. A double-negative of the effective parameters corresponds to a Bragg bandgap or passband with a negative slope.

Figure 3 shows the frequency ranges of the negative moment of inertia and negative stiffness for different $\lambda_1$ with $\lambda_2$ and $\lambda_3$. In figure 3, the pink areas and blue areas represent the frequency ranges with negative moment of inertia and negative stiffness, respectively. The serial numbers 1⃝, 2⃝, 3⃝, 4⃝, 5⃝ and 6⃝ correspond to the six cases in figure 2. Figure 3 shows the variation in the boundary for the frequency ranges of the negative moment of inertia and negative stiffness with $\lambda_1$ for $\lambda_2 = 0.6, \lambda_3 = 1$ and $\lambda_2 = 1.5, \lambda_3 = 1$. Figure 3(a) shows cases 1⃝, 2⃝, 3⃝, and 4⃝, and figure 3(b) shows cases 5⃝ and 6⃝.
When $\lambda_1 = 0.5$, $\lambda_3 = 1$ and $\lambda_1 = 1.5$, $\lambda_3 = 1$, the effect of $\lambda_2$ on the distribution of the frequency ranges with negative parameters is shown in figure 4. Figure 4 shows that the negative stiffness includes two blue areas when $0 < \lambda_2 < 1$. In contrast, the negative stiffness includes only one area when $\lambda_2 > 1$. Figure 4(a) shows cases 1⃝, 2⃝, 3⃝, 5⃝ and 6⃝, and figure 4(b) shows cases 1⃝, 2⃝, 3⃝, 4⃝ and 6⃝. The effect of the structural parameter $\lambda_3$ is shown in figure 5. Figure 5(a) corresponds to case 1⃝. It can be observed in figure 5(a) that the moment of inertia associated with pink areas always generates double-negativity when $\lambda_1 = 1$ and $\lambda_2 = 0.2$. Figure 5(b) shows the effects of $\lambda_3$ on the frequency ranges of negativity when $\lambda_1 = 1$ and $\lambda_2 = 0.8$ and corresponds to cases 2⃝, 3⃝ and 4⃝. Figure 5(c) shows the effects of $\lambda_3$ on the frequency ranges of negativity when $\lambda_1 = 1$ and $\lambda_2 = 1.5$ and corresponds to cases 5⃝ and 6⃝.

3. Dispersion characteristics of waves

In this section, the dispersion characteristics of waves in the proposed metamaterial are investigated. Figure 6 shows the influence of $\lambda_1$, $\lambda_2$ and $\lambda_3$ on the bandgap behavior of the system. The colored areas represent the range of the bandgaps, and the blank areas represent the range of the passbands. According to figure 6(a), the middle passband is a flat band at point A, and the upper passband is a flat band at point B. The upper bandgap disappears at point C. Point A joins the lower bandgap and upper bandgap so that the two bandgaps merge into a single wider bandgap. A similar phenomenon can be observed in figure 6(b). The lower bandgap and the upper bandgap are coupled into one bandgap at point D, and the upper bandgap disappears at point F. The upper passband is a flat band at point E. According to figure 6(c), the upper boundary of the lower bandgap increases much more gradually than the lower boundary, and the
width of the bandgap noticeably increases with increasing $\lambda_3$. The width of the upper bandgap decreases initially but then increases with increasing $\lambda_3$. The upper bandgap disappears at point $G$.

Figure 7 shows the dispersion, the effective moment of inertia, the inverse effective stiffness and the transmittance of the metamaterial when $\lambda_1 = 0.5$, $\lambda_2 = 0.4$ and $\lambda_3 = 1$. The five nondimensional frequencies $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$, and $\eta_3$ in the bands correspond to the cases $J_{\text{eff}} = \infty$, $J_{\text{eff}} = 0$, $k_{\text{eff}}^{-1} = \infty$, $k_{\text{eff}}^{-1} = 0$, and $k_{\text{eff}}^{-1} = 0$, respectively. These frequencies are obtained in appendix A. To display the process of wave attenuation, the transmittance of a system with 10 unit cells is shown in figure 7(d). The propagation of waves in the periodic system is attenuated in the bandgap, but the waves can propagate when they are out of the bandgaps. In figures 9(a) and (d), the blue areas, gray areas and deep yellow areas correspond to the bandgap induced by the single-negative properties, the double-positive properties and the double-negative properties, respectively. The white areas and light yellow areas correspond to the passbands induced by the double-positive properties and the double-negative properties, respectively. In figures 9(b) and (c), the white areas correspond to the positive moment of inertia or stiffness, and the blue areas correspond to the negative moment of inertia or stiffness. The dispersion curve is obtained by solving equations (4) or (9), and the results are the same. As shown in figure 7(a), there are three passbands in the simplest Brillouin zone since each unit cell has three degrees of freedom. The slopes of the middle and bottom bands are positive. Therefore, the directions of the group velocity and phase velocity are the same, which means that the metamaterial has both a positive moment of inertia and positive stiffness in this frequency range. In
contrast, the slope of the upper band is negative, which means that in this frequency range, the material has both a negative moment of inertia and a negative stiffness. The direction of the group velocity is negative, and the phase velocity is positive. Note that the bottom bandgap consists of two parts. One part is close to the bottom band (gray areas), which has double-positive properties and is Bragg bandgap. The other part is close to the middle band and is caused by local resonance (blue areas), corresponding to single-negative properties. The upper bandgap is caused by Bragg scattering and is double negative. Therefore, the material will exhibit different properties under different frequency ranges.

3.1. Ultrawide bandgap

Figure 8 shows the dispersion characteristics of the model with $\lambda_1 = 0.5$, $\lambda_2 = 2/3$ and $\lambda_3 = 1$. When $\eta_1 = \eta_2 = \eta_3$, the middle band becomes a flat band. This means that a wider bandgap is obtained. This corresponds to point D in figure 6(b). From figure 8, it can be seen that two bandgaps can be coupled to form a bandgap by adjusting $\lambda_2$, and the waves attenuate quickly. The transmittance only has a single sharp resonance peak at $\xi_1$, which is very hard to detect from the outside incident wave. As shown in figure 8, the bandgap presents double-negative (deep yellow areas) and double-positive properties (gray areas) when higher and lower than this flat-band frequency. These two bandgaps are combined into a wider Bragg
bandgap by the flat-band. This flat-band frequency can be ignored. Therefore, the bandgap can be used to control the transmittance characteristics of the wave over a wide frequency range. Similarly, this phenomenon can be found when $\lambda_1 = 1.5$, $\lambda_2 = 0.4$ and $\lambda_3 = 1$. This corresponds to point A in figure 6(a).

3.2. Zero-frequency bandgap
Figure 9 illustrates that the bottom band disappears completely when $\lambda_1 = 0$, and the other parameters are the same as those in figure 7. This figure corresponds to $\lambda_1 = 0$ in figure 6(a). The three passbands become two passbands under the condition of $k_1 = 0$.

According to equations (15) and (16a), $\lambda_1 = 0$ corresponds to the effective stiffness equal to zero at $\omega/\omega_4 = 0$. As seen from figures 9(a) and (c), a zero-frequency bandgap is obtained from 0 to $\eta_3$. It indicates that this structure more effectively restrains wave propagation in the ultralow-frequency range. In this frequency range, the system presents a positive effective moment of inertia and negative effective stiffness. The zero-frequency bandgap is caused by local resonance. Additionally, another bandgap with both a negative effective moment of inertia and negative effective stiffness is obtained within the deep yellow areas. According to figure 9(a), the slope of the bottom band is positive, and the metamaterial has both a positive moment of inertia and positive stiffness. This means that the wave speed is positive and that the wave goes forward in the frequency range of the white region. The slope of the upper band is negative, and the material has both negative inertia and stiffness in the frequency range of the light yellow areas.

4. Ultrawide-zero-frequency bandgap
To obtain a wider and lower bandgap, the structural parameters need to be further optimized. According to section 3.1, the condition for producing flat bands is $\eta_1 = \eta_2 = \eta_3$. From section 3.2, the bottom band

![Figure 11. Effect of $\lambda_2$ variation on the bandgap behavior of the system. (a) The effect of $\lambda_2$ variation on the imaginary part of the wavenumber when $\lambda_3 = 0.5$, (b) the effect of $\lambda_2$ variation on the imaginary part of the wavenumber when $\lambda_3 = 1$, and (c) the effect of $\lambda_2$ variation on the imaginary part of the wavenumber when $\lambda_3 = 1.5$.](image)
Figure 12. Dispersion characteristics of the model with $\lambda_2 = 1$ and $\lambda_3 = 1$. (a) The dispersion relation of the model with an ultrawide-zero-frequency bandgap, and (b) the transmittance spectrum of the system composed of 10 units.

Figure 13. Dispersion characteristics of the model with $\lambda_2 \to 1$ and $\lambda_3 = 1$. (a) The dispersion relation, (b) the effective moment of inertia, (c) the inverse of the effective stiffness, and (d) the transmittance spectrum of the system composed of 10 units.

disappears when $\lambda_1 = 0$. If both conditions are satisfied, an ultrawide-zero-frequency bandgap may be achieved. As shown in figure 10, this model turns into a new system without $k_1$.

Figure 11 shows the influence of $\lambda_2$ on the imaginary part of the wavenumber. As seen from figure 11, these two bandgaps are combined into a wider bandgap when $\lambda_2 = 1$. Note that the value of $\lambda_3$ does not affect the position of the coupling bandgap. The properties of the bandgaps are marked. The colored areas represent the ranges of the bandgaps, and the blank areas represent the ranges of the passbands.

Figure 12 shows the dispersion characteristics of the model with $\lambda_2 = 1$ and $\lambda_3 = 1$. As shown in figure 12(a), all the passbands are flat bands. The zero-frequency bandgap and the upper bandgap can be coupled to obtain an ultrawide-zero-frequency bandgap. According to figure 12, no waves can be propagated in the system. In other words, $\lambda_2 = 1$ makes the effective stiffness always equal to zero. This means that the connection between the effective rigid levers has disappeared. Therefore, this system can block any frequency.

Figure 13 shows the dispersion characteristics of the model with $\lambda_2 \to 1$ and $\lambda_3 = 1$. The bottom bandgap is the locally resonant bandgap with single-negative properties. The upper bandgap is double negative and is the Bragg bandgap. In contrast, figure 14 shows the dispersion characteristics of the model with $\lambda_2 \to 1$ and $\lambda_3 = 1$. The bottom bandgap is double-positive and is the Bragg bandgap. The upper bandgap is single-negative and is the locally resonant bandgap. Obviously, $\lambda_2 = 1$ is a switch point between the Bragg bandgap and locally resonant bandgap.
5. Conclusion

In this paper, a one-dimensional model system with three degrees of freedom is studied. The effective moment of inertia and the effective stiffness can illustrate the properties of bandgaps, such as double-positive, single-negative or double-negative. This paper provides a detailed analysis of the negative effective moment of inertia, negative stiffness and overlapping negative moment of inertia and negative stiffness. The two Bragg bandgaps can be coupled to obtain an ultrawide bandgap when \( \eta_1 = \eta_2 = \eta_3 \). Furthermore, a zero-frequency bandgap can be achieved when \( k_1 = 0 \). The slope of all the passbands becomes zero when these two conditions are met at the same time. Therefore, an ultrawide-zero-frequency bandgap is obtained. This result is important because the material can block all waves with frequencies lower than a certain value. The proposed model to form an ultrawide-zero-frequency bandgap is expected to become a new method for constructing low-frequency-wave-controlling devices.

Acknowledgments

This work is supported in part by The National Science Fund for Distinguished Young Scholars under Grants No. 11925205, in part by the National Natural Science Foundation of China under Grants Nos. 11632003 and 51921003, and by the Research Project of State Key Laboratory of Mechanics and Control of Mechanical Structures (NUAA).

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. The relationships of the five roots \( \xi_1, \xi_2, \eta_1, \eta_2 \) and \( \eta_3 \)

The frequency ranges with a negative effective moment of inertia are discussed in this appendix. Two positive roots of \( f_1 \cdot f_2 = 0 \), \( \xi_1 \) and \( \xi_2 \) are

\[
\xi_1 = \sqrt{2\lambda_3}, \quad \xi_2 = \sqrt{2 + 2\lambda_3}.
\]  

(A.1)

The following relations can be readily obtained

\[
\xi_1 < \xi_2.
\]  

(A.2)

The frequency ranges for the negative moment of inertia can be determined as

\[
\xi_1 < \frac{\omega}{\omega_4} < \xi_2.
\]  

(A.3)
For the determination of the frequency ranges for negative effective stiffness, the number of positive roots of \( g_1 \cdot g_2 \cdot g_3 = 0 \) contains two cases.

**Case 1.** \( 0 < \lambda_2 < 1 \).

The positive roots of \( g_1 \cdot g_2 \cdot g_3 = 0 \) are

\[
\eta_1 = \sqrt{2\lambda_3 \sqrt{\frac{\lambda_1 \lambda_2}{(1 - \lambda_2)}}}, \quad \eta_2 = \sqrt{2\lambda_3 \sqrt{\lambda_2 (\lambda_1 + 1)}}. \tag{A.4}
\]

For \( 0 < \lambda_2 \leq \frac{1}{1+\lambda_1}, \frac{\lambda_1 \lambda_2}{(1-\lambda_2)} \leq \frac{\lambda_1 \lambda_2}{(1+\lambda_1)} = \lambda_2 (\lambda_1 + 1) \leq 1 \), and this means that

\[
\eta_1 \leq \eta_3 \leq \eta_2. \tag{A.5}
\]

For \( \frac{1}{1+\lambda_1} < \lambda_2 < 1, 1 = \frac{1}{1+\lambda_1} (\lambda_1 + 1) < \lambda_2 (\lambda_1 + 1) = \frac{\lambda_1 \lambda_2}{(1+\lambda_1)} < \frac{\lambda_1 \lambda_2}{(1-\lambda_2)}, \) and the relations of the three roots can be determined as

\[
\eta_2 < \eta_3 < \eta_1. \tag{A.6}
\]

The frequency ranges for negative stiffness can be determined to be

\[
\min (\eta_1, \eta_2) < \frac{\omega}{\omega_4} < \eta_3 \quad \text{and} \quad \frac{\omega}{\omega_4} > \max (\eta_1, \eta_2). \tag{A.7}
\]

**Case 2.** \( \lambda_2 \geq 1 \).

The positive roots of \( g_1 \cdot g_2 \cdot g_3 = 0 \) are

\[
\eta_2 = \sqrt{2\lambda_3 \sqrt{\lambda_2 (\lambda_1 + 1)}}. \tag{A.8}
\]

Obviously,

\[
\eta_2 \leq \eta_3 \tag{A.9}
\]

The frequency range for negative stiffness is

\[
\eta_2 < \frac{\omega}{\omega_4} < \eta_3. \tag{A.10}
\]

Then, the relationships of the five roots \( \xi_1, \xi_2, \eta_1, \eta_2, \) and \( \eta_3 \) should be discussed.

**Case (1).** \( 0 < \lambda_2 \leq \frac{1}{1+\lambda_1} \).

The relations can be determined from equations (A.2) and (A.5)

\[
\eta_1 \leq \eta_1 < \eta_2 (\xi_1) < \xi_2. \tag{A.11}
\]

**Case (2).** \( \frac{1}{1+\lambda_1} < \lambda_2 \leq \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3 + 1} \).

This yields \( \lambda_2 (\lambda_1 \lambda_3 + \lambda_3 + 1) \leq 1 + \lambda_3 \). Thus, \( \frac{\lambda_1 \lambda_2}{(1-\lambda_2)} \leq 1 + \lambda_3 \), which leads to

\[
\eta_1 \leq \xi_2. \tag{A.12}
\]

According to equations (A.6) and (A.12), the relations for the five roots are

\[
\eta_2 (\xi_1) < \eta_3 < \eta_1 \leq \xi_2. \tag{A.13}
\]

**Case (3).** \( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3 + 1} < \lambda_2 \leq \min \left( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3}, 1 \right) \).

When \( \lambda_1 \lambda_3 \geq 1 \), \( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3} \leq 1 \). In contrast, when \( \lambda_1 \lambda_3 < 1 \), \( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3} > 1 \). For \( \lambda_1 \lambda_3 \geq 1 \) and \( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3} < \lambda_2 < \left( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3} \right)^{-1}, \lambda_2 (\lambda_1 \lambda_3 + \lambda_3) < \lambda_3 + 1 \), which indicates that

\[
\eta_3 < \xi_2. \tag{A.14}
\]

The following result is obtained by referring to the derivation process of (A.12),

\[
\xi_2 < \eta_1. \tag{A.15}
\]

Based on equations (A.6), (A.14) and (A.15), the following relation can be obtained

\[
\eta_2 (\xi_1) < \eta_3 < \xi_2 < \eta_1. \tag{A.16}
\]
For $\lambda_1 \lambda_3 < 1$ and $\frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3 + 1} < \lambda_2 < 1$, 
\[ \frac{\lambda_3 + 1}{\lambda_1 \lambda_3 + \lambda_3 + 1} < \lambda_2 < 1 < \frac{\lambda_3 + 1}{\lambda_1 \lambda_3 + \lambda_3}. \] (A.17)

Therefore, the same result as equations (A.14) and (A.15) can be obtained, and the relations of the five roots are the same as equation (A.16).

Case 4. $\lambda_1 \lambda_3 > 1$ and $\lambda_2 < 1$.
Through reference to the derivation process of equation (A.14), the following result is obtained:
\[ \xi_2 \leq \eta_3. \] (A.18)

Based on equations (A.2), (A.6) and (A.18), the five roots satisfy
\[ \eta_2 (\xi_1) < \xi_2 \leq \eta_3 < \eta_1. \] (A.19)

Case 5. $\lambda_1 \lambda_3 < 1$ and $1 \leq \lambda_2 < \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3}$.
Through reference to the derivation process of equation (A.14), the same result is achieved of
\[ \eta_3 < \xi_2. \] (A.20)

The following relation can be obtained from equations (A.9) and (A.20)
\[ \eta_2 (\xi_1) < \eta_3 < \xi_2. \] (A.21)

Case 6. $\lambda_2 \geq \max \left( \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3}, 1 \right)$.
Through reference to the derivation process of equation (A.14), similar results can be obtained when $\lambda_1 \lambda_3 \leq 1$ and $\lambda_2 \geq \frac{\lambda_1 + 1}{\lambda_1 \lambda_3 + \lambda_3} \geq 1$
\[ \xi_2 \leq \eta_3. \] (A.22)

Based on equations (A.2) and (A.22), the four roots satisfy
\[ \eta_2 (\xi_1) < \xi_2 \leq \eta_3. \] (A.23)

For $\lambda_1 \lambda_3 > 1$ and $\lambda_2 > 1$,
\[ \lambda_2 > 1 > \frac{\lambda_3 + 1}{\lambda_1 \lambda_3 + \lambda_3}. \] (A.24)

Through reference to the derivation process of equation (A.14), the opposite result is derived
\[ \xi_2 < \eta_3. \] (A.25)

The following relation is derived with equations (A.2) and (A.25)
\[ \eta_2 (\xi_1) < \xi_2 < \eta_3. \] (A.26)

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