On the Decomposition of Generalized Semiautomata

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Abstract
Semiautomata are abstractions of electronic devices that are deterministic finite-state machines having inputs but no outputs. Generalized semiautomata are obtained from stochastic semiautomata by dropping the restrictions imposed by probability. It is well-known that each stochastic semiautomaton can be decomposed into a sequential product of a dependent source and deterministic semiautomaton making partly use of the celebrated theorem of Birkhoff-von Neumann. It will be shown that each generalized semiautomaton can be partitioned into a sequential product of a generalized dependent source and a deterministic semiautomaton.

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1 Introduction
The theory of discrete stochastic systems has been initiated by the work of Shannon [14] and von Neumann [10]. While Shannon has considered memory-less communication channels and their generalization by introducing states, von Neumann has studied the synthesis of reliable systems from unreliable components. The fundamental work of Rabin and Scott [12] about deterministic finite-state automata has led to two generalizations. First, the generalization of transition functions to conditional distributions studied by Carlyle [3] and Starke [15]. This in turn yields a generalization of discrete-time Markov chains in which the chains are governed by more than one

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transition probability matrix. Second, the generalization of regular sets by introducing stochastic automata as described by Rabin \[11\].

By the work of Turakainen \[16\], stochastic acceptors can be viewed equivalently as generalized automata in which the "probability" is neglected. This leads to a more accessible approach to stochastic automata \[5\].

On the other hand, the class of nondeterministic automata \[13\] can be generalized to monoidal automata, where the input alphabet corresponds to an arbitrary monoid instead of a free monoid \[8, 9, 17\]. This leads to the class of monoidal automata whose languages are closed under a smaller set of operations when compared with regular languages.

A first step into the study of automata theory are semiautomata which are abstractions of electronic devices that are deterministic finite-state machines having inputs but no outputs \[7, 9\]. Generalized semiautomata are obtained from stochastic semiautomata by dropping the restrictions imposed by probability \[5, 16\]. It is well-known that each stochastic automaton can be decomposed into a sequential product of a dependent source and deterministic semiautomaton \[2\]. This result makes use in part of the celebrated theorem of Birkhoff-von Neumann that each doubly stochastic matrix can be represented as a convex combination of permutation matrices. In this paper, it will be shown that each generalized semiautomaton can be partitioned into a sequential product of a generalized dependent source and a deterministic semiautomaton.

**Notation.** Let $X$ be a set. The set of all mappings on $X$, $T(X) = \{f \mid f : X \to X\}$, forms a monoid under function composition $(fg)(x) = g(f(x))$, $x \in X$, and the identity function $\text{id}_X : X \to X : x \mapsto x$ is the identity element. The monoid $T(X)$ is called the full transformation monoid of $X$.

## 2 Semiautomata

Semiautomata are abstractions of electronic devices which are deterministic finite-state machines having input but no output \[7, 9\].

A (deterministic) semiautomaton (SA) is a triple

\[
A = (S, \Sigma, \{\delta_x \mid x \in \Sigma\})
\]

where

- $S$ is the non-empty finite set of states,
- $\Sigma$ is the set of input symbols,
- $\delta_x : S \to S$ is a (partial) mapping for each $x \in \Sigma$.

Let $\Sigma^*$ denote the free monoid over the alphabet $\Sigma$. By the universal property of free monoids \[4, 9\], the mapping $\delta : \Sigma \to T(S) : x \mapsto \delta_x$ extends
uniquely to a monoid homomorphism \( \delta : \Sigma^* \to T(S) : u \mapsto \delta_u \) such that for each word \( u = x_1 \ldots x_k \in \Sigma^* \),

\[
\delta_u = \delta_{x_1} \cdots \delta_{x_k}
\]

and particularly \( \delta_e = \text{id}_S \). The mapping \( \delta \) is called the transition function of \( A \). Its image \( T(A) = \{ \delta_u \mid u \in \Sigma^* \} \) is a submonoid of the full transformation monoid \( T(S) \) generated by \( \{ \delta_x \mid x \in \Sigma \} \). The semiautomaton \( A \) is also denoted by \( A = (S, M, \delta) \) or \( A = (S^A, M^A, \delta^A) \).

A semiautomaton \( A = (S, \Sigma, \delta) \) serves as a skeleton of a deterministic finite-state machine that is exactly in one state at a time. If the semiautomaton \( A \) is in state \( s \) and reads the word \( u \in \Sigma^* \), it transits into the state \( s' = \delta_u(s) \).

**Example 1.** Consider the semiautomaton \( A = (S, \Sigma, \delta) \) with state set \( S = \{1, 2, 3\} \), input alphabet \( \Sigma = \{x, y\} \), and transition function \( \delta \) given by the automaton graph in Fig. 1. The associated transformation monoid is generated by the transformations

\[
\delta_x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \delta_y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}.
\]

We have

\[
\delta_{xx} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \delta_{xy} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix},
\]

\[
\delta_{yx} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \delta_{yy} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}.
\]

Hence, the transformation monoid \( T(A) \) is given by \( \{ \text{id}_S, \delta_x, \delta_y, \delta_{xy} \} \).

![Figure 1: Semiautomaton.](image)

### 3 Generalized Semiautomata

Stochastic automata are a generalization of non-deterministic finite state automata [5]. Generalized automata can be obtained from stochastic automata by dropping the restrictions imposed by probability [5, 16-17].
A generalized semiautomaton (GSA) is a triple
\[ A = (S, \Sigma, \{Q_x \mid x \in \Sigma\}) , \]
where
- \( S \) is the non-empty finite set of states,
- \( \Sigma \) is the input alphabet, and
- \( Q \) is a collection of \( n \times n \) nonnegative matrices \( Q_x, x \in \Sigma \), where \( n \) is the number of states.

In view of the universal property of free monoids \[4, 9\], the mapping \( Q : \Sigma \to \mathbb{R}^{n \times n} : x \mapsto Q_x \) extends uniquely to a monoid homomorphism \( Q : \Sigma^* \to \mathbb{R}^{n \times n} \) such that for each word \( u = x_1 \ldots x_k \in \Sigma^* \),
\[ Q_u = Q_{x_1} \cdots Q_{x_k} \tag{2} \]
and particularly \( Q_\epsilon = I_n \) is the \( n \times n \) identity matrix. The mapping \( Q \) is called the transition function of \( A \). Its image \( T(A) = \{ Q_u \mid u \in \Sigma^* \} \) is a submonoid of the full transformation monoid \( T(S) \) generated by \( \{Q_x \mid x \in \Sigma\} \). The generalized semiautomaton \( A \) is also denoted by \( A = (S, \Sigma, Q) \) or \( A = (S^A, \Sigma^A, Q^A) \).

The state set \( S = \{s_1, \ldots, s_n\} \) can be viewed as the standard basis for the Euclidean vector space \( \mathbb{R}^n \), where \( s_i \) is the basis vector whose \( i \)th coordinate is 1 and all others are 0. In this way, the \((i, j)\)th entry of the matrix \( Q_u = (s^{(u)}_{ij}) \) is given by \( s^{(u)}_{ij} = s^T_i Q_u s_j \).

**Proposition 3.1.** Each deterministic semiautomaton is a generalized automaton.

**Proof.** Let \( A = (S, \Sigma, \delta) \) be a deterministic semiautomaton and let \( S = \{s_1, \ldots, s_n\} \). Define the generalized semiautomaton \( B = (S, \Sigma, Q) \), where for each \( x \in \Sigma \), the \((i, j)\)th entry of \( Q_x \) is 1 if \( \delta_x(s_i) = s_j \) and otherwise 0. Then the mapping \( T(A) \to T(B) : \delta_u \mapsto Q_u \) is a monoid isomorphism. \( \square \)

A generalized semiautomaton \( A = (S, \Sigma, P) \) is called stochastic if the matrices \( P_x, x \in \Sigma \), are stochastic, i.e., \( P_x \) is a matrix of nonnegative real numbers such that each row sum is equal to 1. The product of stochastic matrices is again a stochastic matrix and so the transition monoid \( T(A) \) consists of the stochastic matrices \( P_u, u \in \Sigma^* \). In particular, the \((i, j)\)th element \( p(s_j \mid u, s_i) \) of the matrix \( P_u \) is the transition probability that the automaton enters state \( s_j \) when started in state \( s_i \) and reading the word \( u \).

**Example 2.** Let \( m \geq 2 \) be an integer. Put \( \Sigma = \{0, \ldots, m - 1\} \). The stochastic semiautomaton \( A = (\{s_1, s_2\}, \Sigma, P) \) given by
\[ P_x = \frac{1}{m} \begin{pmatrix} m - x & x \\ m - x - 1 & x + 1 \end{pmatrix}, \quad x \in \Sigma, \]
is called $m$-adic semiautomaton. For each word $u = x_1 \ldots x_k \in \Sigma^*$,

\[
P_u = \frac{1}{m^k} \begin{pmatrix} m^k - w_k & w_k \\ m^k - w_k - 1 & w_k + 1 \end{pmatrix},
\]

where $w_k = x_k m^{k-1} + \ldots + x_2 m + x_1$ and the entry $\frac{1}{m^k} w_k$ corresponds in the $m$-adic representation to $0.x_k \ldots x_1$.

A generalized semiautomaton $A = (S, \Sigma, D)$ is called doubly stochastic if the matrices $D_x, x \in \Sigma$, are doubly stochastic, i.e., $D_x$ is a matrix of nonnegative real numbers such that each row and column sum is equal to 1. The product of doubly stochastic matrices is again a doubly stochastic matrix and so the transition monoid $T(A)$ consists of the doubly stochastic matrices $D_u, u \in \Sigma^*$.

4 Decomposition of Generalized Semiautomata

The objective is to decompose each generalized semiautomata into a sequential product of a generalized dependent source and a deterministic semiautomaton. The corresponding result for stochastic semiautomata has been proved by Bukharaev [2].

A generalized dependent source is a triple

\[
\Gamma = (\Sigma, \Xi, \{\gamma(z \mid x) \mid x \in \Sigma, z \in \Xi\}),
\]

where $\Sigma$ and $\Xi$ are alphabets and $\gamma : \Sigma \times \Xi \to \mathbb{R}_{\geq 0} : (x, z) \to \gamma(z \mid x)$ is a mapping which is extended recursively to $\Sigma^* \times \Xi^*$ as follows:

- $\gamma(\epsilon \mid \epsilon) = 1$,
- $\gamma(v \mid u) = 0$ for all $u \in \Sigma^*$ and $v \in \Xi^*$ with $|u| \neq |v|$, and
- $\gamma(zv \mid xu) = \gamma(x \mid z)\gamma(u \mid v)$ for all $x \in \Sigma, u \in \Sigma^*, z \in \Xi$ and $v \in \Xi^*$.

A generalized dependent source $\Gamma$ is also denoted by $\Gamma = (\Sigma, \Xi, \gamma)$.

In particular, a dependent source is a generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where $\Sigma$ and $\Xi$ are alphabets and for each $x \in \Sigma$, $\gamma(\cdot \mid x)$ defines a (conditional) probability measure on $\Xi$. This measure can be extended for each $u \in \Sigma^*$ to a (conditional) probability measure $\gamma(\cdot \mid u)$ on $\Xi^*$ along the same lines as above. Note that a dependent source can be viewed as a stochastic input-output automaton with a single state [2, 5].

The sequential product of generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$ and generalized semiautomaton $B = (S, \Xi, Q^B)$ defines a generalized semiautomaton $A = (S, \Sigma, Q^A)$ such that for all $x \in \Sigma$,

\[
Q^A_x = \sum_{z \in \Xi} \gamma(z \mid x) \cdot Q^B_z.
\]
By induction, for all $u \in \Sigma^*$,

$$Q^A_u = \sum_{v \in \Xi^* \Sigma} \gamma(v \mid u) \cdot Q^B_v. \tag{4}$$

A permutation matrix $P$ is a square binary matrix which has exactly one entry of 1 in each row and each column and 0’s elsewhere. By the Birkhoff-von Neumann theorem [6], for each $n \times n$ doubly stochastic matrix $P$ there exist real numbers $\alpha_1, \ldots, \alpha_N \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and permutation matrices $P_1, \ldots, P_N$ such that

$$P = \alpha_1 P_1 + \ldots + \alpha_N P_N. \tag{5}$$

This representation is also known as Birkhoff-von Neumann decomposition. Such a representation of a doubly stochastic matrix as a convex combination of permutation matrices may not be unique. By the Marcus-Ree Theorem [1], $N \leq n^2 - 2n + 2$ for dense matrices.

A square matrix $P$ is called deterministic if it has exactly one entry of 1 in each row and 0’s elsewhere. In particular, each permutation matrix is deterministic. For each $n \times n$ stochastic matrix $P$ there exist real numbers $\alpha_1, \ldots, \alpha_N \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and deterministic matrices $P_1, \ldots, P_N$ such that

$$P = \alpha_1 P_1 + \ldots + \alpha_N P_N. \tag{6}$$

Such a representation of a stochastic matrix as a convex combination of deterministic matrices may not be unique.

A square matrix $P$ is called semideterministic if in each nonzero row there is exactly one entry of 1 and 0’s elsewhere. In particular, each deterministic matrix is semideterministic.

**Proposition 4.1.** For each nonnegative square matrix $A$, there exist real numbers $\alpha_1, \ldots, \alpha_N \geq 0$ and semideterministic matrices $P_1, \ldots, P_N$ such that

$$A = \alpha_1 P_1 + \ldots + \alpha_N P_N. \tag{7}$$

**Proof.** For each nonnegative square matrix $P = (p_{ij})$ let $p_{i, \pi(i)}$ be a minimal nonzero entry in row $i$. Consider the semideterministic matrix $D = (d_{ij})$ with $d_{i, \pi(i)} = 1$ for each $i$ and $d_{ij} = 0$ otherwise. Moreover, put $m(P) = \min\{p_{ij} \mid p_{ij} \neq 0\}$. Then $P - m(P)D$ is a nonnegative matrix with at least one more zero entry than $P$. Iterating this step a finite number $N$ of times gives a sequence $(P_k)_{1 \leq k \leq N}$ of nonnegative matrices and a sequence $(D_k)_{1 \leq k \leq N}$ of semideterministic matrices such that $P_1 = A$, $P_{k+1} = P_k - m(P_k)D_k$ for $1 \leq k \leq N$, and $P_{N+1} = 0$. This yields the decomposition of $A$ as a linear combination of semideterministic matrices $A = \sum_{k=1}^N m(P_k)D_k$.\[\square\]
For doubly stochastic and stochastic matrices, the proof is similar.

Example 3. Consider the nonnegative matrix

\[
A = \begin{pmatrix} 2 & 4 & 6 \\ 2 & 2 & 8 \\ 3 & 3 & 6 \end{pmatrix}.
\]

A sequence of reductions showing the selected entries at each step is

\[
\begin{align*}
\begin{pmatrix} 2 & 4 & 6 \\ 2 & 2 & 8 \\ 3 & 3 & 6 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 4 & 6 \\ 0 & 1 & 8 \\ 0 & 2 & 6 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{pmatrix}, \\
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

yields the decomposition

\[
A = 2 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

\[\blacksquare\]

Theorem 4.2. Each generalized semi-automaton \(A = (S, \Sigma, Q)\) can be represented as a sequential product of a generalized dependent source \(\Gamma = (\Sigma, \Xi, \gamma)\) and a semideterministic semi-automaton \(B = (S, \Xi, \delta)\).

In particular, each stochastic (or strongly stochastic) semi-automaton \(A = (S, \Sigma, P)\) can be represented as a sequential product of a dependent source \(\Gamma = (\Sigma, \Xi, \gamma)\) and a deterministic (or permutation) semi-automaton \(B = (S, \Xi, \delta)\).

Proof. Let \(\{D_1, \ldots, D_N\}\) denote the collection of \(n \times n\) semideterministic matrices. Put \(\Xi = \{1, \ldots, N\}\) and for each \(x \in \Sigma\), write \(Q_x\) as a conical combination of semideterministic matrices

\[
Q_x = \sum_{z \in \Xi} \alpha(z, x) D_z.
\]

This defines the generalized dependent source \(\Gamma = (\Sigma, \Xi, \gamma)\), where for each \(x \in \Sigma\) and \(z \in \Xi\),

\[
\gamma(z \mid x) = \alpha(z, x),
\]

and the deterministic automaton \(B = (S, \Xi, \delta)\), where for each \(z \in \Xi\), the transition \(\delta_z : S \to S\) is given by the matrix \(D_z\) as in the proof of Prop. 3.1.

Then we obtain for each \(x \in \Sigma\),

\[
Q_x = \sum_{z \in \Xi} \gamma(z \mid x) Q_z^B.
\]

The second part is clear from the above remarks. \[\square\]
Example 4. Consider the generalized semiautomaton

\[ A = (\{s_1, s_2\}, \{x_1, x_2\}, \{Q_{x_1}, Q_{x_2}\}), \]

where

\[ Q_{x_1} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_{x_2} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}. \]

Then

\[ Q_{x_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

and

\[ Q_{x_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]

Put \( \Xi = \{z_1, \ldots, z_5\} \) and

\[ D_{z_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_{z_3} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]

\[ D_{z_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{z_5} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]

Then

\[ Q_{x_1} = D_{z_1} + D_{z_2} + 3D_{z_3} \quad \text{and} \quad Q_{x_2} = D_{z_4} + 2D_{z_5}. \]

This gives the state transition table of the deterministic semiautomaton

\[ B = (S, \Xi, \delta), \]

where

\[
\begin{array}{c|cccccc}
\delta^B & z_1 & z_2 & z_3 & z_4 & z_5 \\
\hline
s_1 & s_1 & s_1 & s_2 & s_1 & s_2 \\
 s_2 & s_1 & s_2 & s_2 & s_2 & s_2 \\
\end{array}
\]

and the transitions of the generalized dependent source \( \Gamma = (\Sigma, \Xi, \gamma) \), where

\[
\begin{array}{c|cccccc}
\gamma & z_1 & z_2 & z_3 & z_4 & z_5 \\
\hline
x_1 & 1 & 1 & 3 & 0 & 0 \\
x_1 & 0 & 0 & 0 & 1 & 2 \\
\end{array}
\]

Example 5. Reconsider the \( m \)-adic semiautomaton \( A = (\{s_1, s_2\}, \Sigma, P) \).

For each \( x \in \Sigma \),

\[ P_x = \frac{m - x - 1}{m} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{x}{m} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]

Put \( \Xi = \{z_1, z_2, z_3\} \) and

\[ D_{z_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{z_3} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]
Then for each $x \in \Sigma$,

$$P_x = \frac{m-x-1}{m}D_{z1} + \frac{1}{m}D_{z2} + \frac{x}{m}D_{z3}.$$ 

This provides the state transition table of the deterministic semiautomaton $B = (S, \Xi, \delta)$, where

$$\begin{array}{c|ccc}
\delta^B & z_1 & z_2 & z_3 \\
\hline
s_1 & s_1 & s_1 & s_2 \\
s_2 & s_1 & s_2 & s_2 \\
\end{array}$$

and the transitions of the dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where for each $x \in \Sigma$,

$$\begin{array}{c|ccc}
\gamma & z_1 & z_2 & z_3 \\
\hline
x & \frac{m-x-1}{m} & \frac{1}{m} & \frac{x}{m} \\
\end{array}$$

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