The Three Faces of $U(3)$

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Abstract

$U(n)$ is a semi-direct product group that is characterized by non-trivial homomorphisms mapping $U(1)$ into the automorphism group of $SU(n)$. For $U(3)$, there are three non-trivial homomorphisms that induce three separate defining representations. In a toy model of $U(3)$ Yang-Mills (endowed with a suitable inner product) coupled to massive fermions, this renders three distinct covariant derivatives acting on a single matter field. By employing a mod3 permutation of the vector space carrying the defining representation induced by a “large” gauge transformation, the three covariant derivatives and one matter field can alternatively be expressed as a single covariant derivative acting on three distinct species of matter fields possessing the same $U(3)$ quantum numbers. One can interpret this as three species of matter fields in the defining representation.

1 Introduction

In this note we consider a toy model of $U(3)$ Yang-Mills coupled to massive fermionic matter fields. Off hand it seems $U(3)$ is an untenable symmetry group for constructing a gauge field theory. After all, a tenant of standard gauge theory says the most general symmetry group must be a direct product of semi-simple and $U(1)$ groups. (see e.g. [1])

From where comes the tenant? For a physically acceptable gauge field theory, one must start with a compact real group $G$ and impose a positive-definite, $Ad$-invariant, real bilinear form on the gauge symmetry Lie algebra $\mathfrak{g}$. And it is well-known that the Lie algebra of a compact real group decomposes into a direct sum of semisimple $\mathfrak{s}_i$ and $\mathfrak{u}(1)_j$ factors $\bigoplus_{i,j} \mathfrak{s}_i \oplus \mathfrak{u}(1)_j$ if and only if the Killing form on $\mathfrak{g}$ is non-degenerate and hence negative-definite (see e.g. [2]).

Because $U(3)$ is not semi-simple, its Killing form is degenerate. But a Killing inner product is not the only possibility. It happens that $U(3)$ is a connected, compact real group. Being compact, it is endowed with at least one bi-invariant metric [3,4], and in fact one can formulate on $\mathfrak{u}(3)$ a two-parameter class of positive-definite, $Ad$-invariant, real bilinear forms. Hence, it is possible to construct a consistent gauge theory with $U(3)$ gauge symmetry without the Killing inner product.

Notably, unlike $SU(3) \times U(1)$ where the gauge field associated with $U(1)$ completely decouples from the rest, all of the $U(3)$ gauge fields will mutually interact as a true $U(3)$ symmetry dictates. In fact, we have $U(3) = SU(3) \times U(1)$ as a semi-direct product, and an element $u(3) \in U(3)$ can be factored as $u(3) = s(3)u(1)$ with $u(1) \in U(1)$ and $s(3) \in SU(3)$. The semi-direct product $SU(3) \times U(1)$ is
characterized by a (not necessarily unique) homomorphism $\varphi : U(1) \to Aut SU(3)$ where $Aut SU(3)$ is the automorphism group of $SU(3)$. \[5-7\] In particular, in the defining representation, said homomorphism induces a (not necessarily unique) representation $\varrho : U(1) \to L_B(C^3)$ where $L_B(C^3)$ denotes the set of linear bounded matrix operators on $C^3$. \[5–7\] Now, in the defining representation there are three non-trivial ways to represent the $U(1)$ factor in $L_B(C^3)$; with $e^{i\theta}$ in one of the diagonal entries, 1 in the other two diagonal entries, and 0 in all off-diagonal entries. Then an element of $U(3)$ represented in $L_B(C^3)$ can be written $\rho_r(u(3)) = \rho_r(su(3))\rho_r(u(1))$ where $\rho_r : U(3) \to L_B(C^3)$ is an extension of $\varrho_r$ and $r \in \{1, 2, 3\}$.

There is no reason to favor one particular representation over another, so when constructing a gauge field theory coupled to fermions in the defining representation the most general Lagrangian contains the standard Yang-Mills term $-\frac{1}{2} F \cdot F$ and fermion terms $\sum_r \overline{\Psi} D^{(r)} \Psi$ summed over the three representations $\rho_r$. Consider permuting some chosen basis of $C^3$ with some unitary permutation matrix in $L_B(C^3)$. There are two classes of such permutations: one class induces “small” gauge transformations and the other induces “large” gauge transformations. Of course, the small gauge transformations represent a redundant state description in the quantum version. In contrast, the large gauge transformations represent the non-abelian analog of charge conjugation, and they effect a genuine matter field re-characterization: They essentially permute $r$ up to non-trivial phases. Accordingly, the $U(3)$ symmetry allows the fermion contribution $\sum_r \overline{\Psi} D^{(r)} \Psi$ to be rewritten with the covariant derivative in a single representation as $\sum_r \overline{\Psi}^{(r)} D^{(r)} \Psi^{(r)}$ where $\Psi^{(r)}$ are three different species of fermion matter fields — each species a $U(3)$ triplet characterized by three quantum numbers coming from the action of the Cartan subalgebra.

This is our main result: The most general $U(3)$ gauge invariant Lagrangian for fermions in a chosen defining representation includes precisely three species of matter fields relative to an imbedding $U(1) \hookrightarrow SU(3) \rtimes U(1)$. We make no claim here that $U(3)$ models QCD phenomenology\[8\] Our purpose is to point out the viability of semi-direct product groups in gauge field theories and to highlight the emanating effect of multiple defining representations: The three types of matter fields coming from $U(3)$ may or may not be a phenomenological red herring.

Of course, the occurrence of three generations in particle physics is still a mystery, and there have been attempts to explain the “three” using a variety of mechanisms. Most notable perhaps are preon models \[8-11\], and super string models \[12, 13\]. But there are also models based on non-anomalous discrete $R$-symmetry \[14\], extra dimensions with anomaly cancellation \[15\], and the anthropic principle \[16\].

2 $U(3)$ Toy Model

2.1 The inner product

$U(3)$ is neither simple nor semisimple, and its Killing form is only semi-definite. So the first order of business is to construct a suitable inner product on $u(3)$.

Proposition 2.1 The Killing form of $U(3)$ is given by $K(X, Y) = 6\text{tr}(X \cdot Y) - 2\text{tr}(X)\text{tr}(Y)$ and is negative semi-definite for all skew-Hermitian $X, Y \in u(3)$.\[1\] In which case the $U(1)$ subgroup would have nothing to do with electromagnetism.
Proof: The Lie algebra brackets are \([u_{ab}, u_{cd}] = \delta_{bc} u_{ad} - \delta_{ad} u_{bc}\) where \(u_{ab} \in \mathfrak{u}(3)\) are a chosen skew-Hermitian basis with \(a, b, c, d \in \{1, 2, 3\}\). From these brackets it follows that the adjoint map is given by \(\text{ad}_X(u_{ab}) = \sum_c x_c u_{cb} - x_b u_{ac}\) with \(X = \sum_{a,b} x_{ab} u_{ab}\) and \(x_{ab} \in \mathbb{R}\). Hence,

\[
\text{ad}_X \circ \text{ad}_Y(u_{ab}) = \sum_{c,d} (x_c y_d u_{db} - x_b y_d u_{dc} + x_b y_d u_{ad} - x_c y_d u_{cd})
\]

\[
= \sum_c (x_c y_{ac} - x_b y_{aa} \delta_{cb} + x_b y_{cb} - x_c y_{bb} \delta_{ac}) u_{ab}
\]
implies

\[
K(X, Y) := \text{tr}(\text{ad}_X \circ \text{ad}_Y) = \sum_{a,b,c} (x_c y_{ac} - x_b y_{aa} \delta_{cb} + x_b y_{cb} - x_c y_{bb} \delta_{ac})
\]

\[
= 3\text{tr}(X \cdot Y) - \text{tr}(X)\text{tr}(Y) + 3\text{tr}(X \cdot Y) - \text{tr}(X)\text{tr}(Y).
\]

The center of \(\mathfrak{u}(3)\) is \(\text{span}_\mathbb{R}\{1\}\), and it is easy to see that \(K(i1, X) = 0\) for all \(X \in \mathfrak{u}(3)\). Negativity follows from the skew-hermiticity of \(X, Y\). □

This suggests to define a bilinear inner product on the Lie algebra \(\mathfrak{u}(3)\) in the defining representation \(\rho : U(3) \rightarrow L_B(\mathbb{C}^3)\) by

\[
\langle A_\alpha, A_\beta \rangle := -\frac{1}{6} \left[ 6 \text{tr}(A_\alpha A_\beta^\dagger) - 2 \left( 1 - \frac{g_1^2}{g_2^2} \right) \text{tr}(A_\alpha)\text{tr}(A_\beta^\dagger) \right]
\]

where the basis elements \(\{A_\alpha\} = \{\rho'(u_{ab})\}\) are \(3 \times 3\) skew-Hermitian matrices with \(\alpha \in \{1, \ldots, 9\}\) and the parameters \(g_1, g_2 \in \mathbb{R}\) obey \(0 < g_2^2 < g_1^2\). It is clearly positive-definite, \(Ad\)-invariant, and real. For a triangular decomposition of the basis \(\{A_\alpha\}\) denoted by \(\{S^+_a, H_a\}\) with \(a \in \{1, 2, 3\}\), the structure constants associated with the brackets \([S^+_a, H_a]\) differ from those associated with the Killing form. These structure constants, which are functions of \((g_1, g_2)\), characterize quantum numbers of non-neutral gauge bosons, and eigenvalues of \(\{H_a\}\) characterize quantum numbers of matter fields.

### 2.2 Semidirect structure of \(U(3)\)

Mathematically, it is fruitful to view \(U(3)\) as an extension of a group \(H \cong U(1)\) by a normal subgroup \(N \cong SU(3) \rtimes U(3)\). This is represented by the short exact sequence

\[
1 \rightarrow N \xrightarrow{f} U(3) \xrightarrow{\pi} H \rightarrow 1.
\]

If there exists an injective homomorphism \(s : H \rightarrow U(3)\) such that \(\pi \circ s = \text{id}_H\), then the extension is a semidirect product \(N \rtimes H\). In this case, \(U(3)\) can be regarded as a principle bundle with base \(H\), structure group \(N\), and global section(s) \(s : H \rightarrow U(3)\). A choice of section corresponds to a choice of coset representative. Then \(s(H) \cong U(1) \subset U(3)\) yields a unique decomposition \(U(3) = SU(3)U(1)\) with \(SU(3) \cap U(1) = \{\text{id}\}\), and \(s\) induces a homomorphism \(\tilde{s} : s(H) \cong U(1) \rightarrow \text{Aut} N\). These observations are demonstrated by the following theorem;
**Theorem 2.2** (8) Let $1 \to N \xrightarrow{f} U(3) \xrightarrow{\pi} H \to 1$ be a short exact sequence equipped with an injective homomorphism $s : H \to U(3)$ such that $\pi \circ s = id_H$. Then there exists a homomorphism $\varphi : H \to \text{Aut} N$ and an isomorphism $\theta : U(3) \to N \rtimes_{\varphi} H$.

**Proof:** For $h \in H$ and $n \in N$,

$$\pi(s(h)f(n)s(h^{-1})) = \pi \circ s(h) \pi \circ f(n) \pi \circ s(h^{-1}) = h \cdot id_H \cdot h^{-1}. \tag{5}$$

Since $f$ is injective and $\text{im} f = \ker \pi$, then $s(h)f(n)s(h^{-1}) = f(n')$ for some unique $n'(n, h) \in N$ that depends on $(n, h)$. It is convenient to write $\varphi_h(\cdot) \equiv n'(\cdot, h)$ so that $\varphi_h : N \to N$. Note that $\varphi_h(id_N) = id_N$ for all $h \in H$ since $s$ is a homomorphism.

**Lemma 2.3** The function $\varphi_h \in \text{Aut} N$.

**Proof:** First, $f(\varphi_{id}(n)) = f(n)$ implies $\varphi_{id}(n) = n$ for all $n \in N$. Next, for $n_1, n_2 \in N$,

$$s(h)f(n_1)f(n_2)s(h^{-1}) = s(h)f(n_1)s(h^{-1})s(h)f(n_2)s(h^{-1}) = f(\varphi_h(n_1)\varphi_h(n_2)) \tag{6}$$

where we used $s$ is a homomorphism. On the other hand, from the definition of $\varphi_h$, we have $s(h)f(n_1n_2)s(h^{-1}) = f(\varphi_h(n_1n_2))$. Injective $f$ then implies $\varphi_h(n_1n_2) = \varphi_h(n_1)\varphi_h(n_2)$. \(\square\)

Let $\varphi : H \to \text{Aut} N$ by $h \mapsto \varphi_h$.

**Lemma 2.4** $\varphi : H \to \text{Aut} N$ is a homomorphism.

**Proof:** For $h_1, h_2 \in H$,

$$s(h_1)s(h_2)f(n)s(h_2)^{-1}s(h_1)^{-1} = s(h_1)f(\varphi_h(n))s(h_1)^{-1} = f(\varphi_{h_1}(\varphi_h(n))) \tag{7}$$

On the other hand, $s(h_1)s(h_2)f(n)s(h_2)^{-1}s(h_1)^{-1} = s(h_1h_2)f(n)s(h_1h_2)^{-1} = f(\varphi_{h_1h_2}(n))$ since $s$ is a homomorphism. Again injective $f$ implies $\varphi_{h_1h_2} = \varphi_{h_1} \circ \varphi_{h_2}$. \(\square\)

It follows that $\varphi$ defines a group operation on $N \rtimes_{\varphi} H$ by $(n_1, h_1)(n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1h_2)$ if the inverse is defined by $(n, h)^{-1} := (\varphi_{h^{-1}}(n^{-1}), h^{-1})$ for all $(n, h) \in N \rtimes_{\varphi} H$.

Finally, let $\theta^{-1} : N \rtimes_{\varphi} H \to U(3)$ by $(n, h) \mapsto f(n)s(h)$. Then

$$\theta^{-1}((n_1, h_1)(n_2, h_2)) = \theta^{-1}(n_1\varphi_{h_1}(n_2), h_1h_2)$$

$$= f(n_1)(s(h_1)f(n_2)s(h_1^{-1})s(h_1)s(h_2)$$

$$= f(n_1)s(h_1)f(n_2)s(h_2)$$

$$= \theta^{-1}(n_1, h_1)\theta^{-1}(n_2, h_2). \tag{8}$$

Since the decomposition $U(3) = NH$ is unique (which we won’t bother to prove), the homomorphism $\theta^{-1}$ is bijective. One can go on to show that the semidirect product reduces to a direct product if and only if $H \triangleleft U(3)$; in which case $N$ and $H$ commute and $\varphi$ is trivial. \(\square\)
Observe the homomorphism $\tilde{\varphi} : s(H) \cong H \in N \rtimes_{\varphi} H \to \text{Aut } N$ induced by $s$ is given by

$$\tilde{\varphi}_{s(h)}(n, id_H) = s(h)(n, id_H)s(h^{-1}) = [(id_N, h)(n, id_H)](id_N, h^{-1}) = [(\varphi_h(n), h)](id_N, h^{-1}) = (\varphi_h(n), id_H).$$

In this sense, $\tilde{\varphi}$ induced by the section $s$ coincides with $\varphi$. It is important to note that there may be multiple homomorphisms $\varphi$ and hence multiple sections $s$ that render a semidirect product. Physically, a non-trivial $\varphi$ corresponds to a direct interaction between the gauge fields of the respective subgroups.

In particular, for the matrix group $U(3)$ as a semidirect product, there exist three such non-trivial sections;

$$s : H \to \begin{cases} \text{diag}(e^{i\omega}, 1, 1) \\ \text{diag}(1, e^{i\omega}, 1) \\ \text{diag}(1, 1, e^{i\omega}) \end{cases}$$

where $\omega \in \mathbb{R}$. Each section gives rise to a different conjugation of $SU(3)$ by $s(h)$, and each of these induces a different representation $\varrho_r : H \to L_B(\mathbb{C}^3)$ where $r \in \{1, 2, 3\}$. These can then be extended to three defining representations $\rho_r : U(3) \to L_B(\mathbb{C}^3)$.

### 2.3 Lagrangian matter field term

Given the existence of a suitable inner product and three representations, constructing the model is rather elementary. The decisive step is to insist that all allowed defining representations be included in the Lagrangian;

**Postulate 2.5** The matter field portion of the Lagrangian of a gauge field theory must include all allowed defining representations.

Accordingly, in our toy model of Yang-Mills coupled to a massive matter field in the defining representations, the gauge field term uses the chosen inner product $\frac{1}{2} \langle F, F \rangle$ with $F \in u(3)$ and the matter field term will be $\sum_r i\bar{\Psi} D_B^{(r)} \Psi$ where we have (unconventionally) included the bare mass parameter in the covariant derivative $D_B^{(r)}$. In momentum space, the matrix representation of the covariant derivative is $[D_B^{(r)}] = (\frac{i}{\partial} + m_B^{(r)})[1] + A_{\mu}^{(r)}[i\Lambda^{(r)}]$ with gauge fields $A_{\mu}^a$, and $\{\Lambda^{(r)}\}$ a basis of $u(3)$ in the $r$-defining representation.

In the quantum version of this model, each $D_B^{(r)}$ will give rise to different vertex factors in the Feynman rules and hence *ostensibly different* renormalizations of the gauge fields, matter fields, and $r$-dependent mass parameters. The renormalized matter field term is then $\sum_r i\bar{\Psi} D_R^{(r)} \Psi$ where $[D_R^{(r)}] = (i\bar{\psi} + m_B^{(r)})[1] + A_{\mu}^{(r)}[i\Lambda^{(r)}]$. In effect, through renormalization, the quantum theory distinguishes the classically isomorphic vector spaces carrying the defining representations even when $m_B^{(r)} = \frac{1}{3} m_B$ where $m_B := \sum_r m_B^{(r)}$. Notably, if there exists any bare mass degeneracy among the defining representations, the quantum version will remove the degeneracy (assuming different renormalizations for different $r$).
We can make use of the $U(3)$ symmetry to re-characterize the matter field Lagrangian. There exists a class of elements in $U(3)$ of the form

$$P(x) := \begin{pmatrix} 0 & 0 & e^{i\theta_1(x)} \\ e^{i\theta_2(x)} & 0 & 0 \\ 0 & e^{i\theta_3(x)} & 0 \end{pmatrix}$$ (11)

with $\theta_1(x), \theta_2(x), \theta_3(x) \in \mathbb{R}$. The adjoint action of $P(x)$ on the Lie algebra $\mathfrak{u}(3)$ leaves the normal subalgebra $\mathfrak{su}(3)$ invariant, but it cyclically permutes the generators of the $s(H)$ matrices

$$\text{diag}(i\omega, 0, 0) \xrightarrow{Ad_P} \text{diag}(0, i\omega, 0) \xrightarrow{Ad_P} \text{diag}(0, 0, i\omega) \xrightarrow{Ad_P} \text{diag}(i\omega, 0, 0).$$ (12)

Similarly, $P^{-1}(x) = P^1(x)$ permutes in the reverse direction. Crucially, $P^3 = e^{i(\theta_1+\theta_2+\theta_3)}\text{diag}(1, 1, 1)$. We claim that $\theta_1(x) + \theta_2(x) + \theta_3(x) = \pm (2n)\pi$ with $n \in \mathbb{N}$ induces small gauge transformations while $\theta_1(x) + \theta_2(x) + \theta_3(x) = \pm (2n+1)\pi$ induces large gauge transformations: The latter cannot be reached by a gauge transformation homotopic to the identity because $\det P = -1^2$. It then follows from $\text{tr} \log P = i\pi(2k+1)$ that $\log P$ in this case involves a combination of Cartan generators (not present in the small permutation case) that contributes a multivalued mod 3 phase to matter field configurations, and it transforms between three physically distinct classes of gauge field configurations that survive gauge fixing in the quantized theory.

Given $P$ we have $\mathcal{P}^{(2)} = P \mathcal{P}^{(1)} P^{-1}$ and $\mathcal{P}^{(3)} = P^2 \mathcal{P}^{(1)} P^{-2}$. Define the fields $\Psi^{(r)} := P^{r-1} \Psi$. Clearly $P$ cyclically permutes the components of $\Psi$ up to phases. Consequently, when $P^3 = -1$ we can write $\sum_r \overline{\mathcal{P}}^{(r)} B \Psi = \sum_r \overline{\Psi}^{(r)} B \Psi^{(r)}$. In the quantum version, if the $\theta_i(x)$ are not all equal, the $SU(3)$-identical $\Psi^{(r)}$ are physically distinct fields with inequivalent masses (again, assuming different renormalizations for different $r$). Hence we claim

**Claim 2.6** Given postulate [2.5] matter fields with $U(3)$ gauge symmetry necessarily come in three species due to the existence of large gauge transformations that realize mod 3 permutations of the basis in a defining representation.

This perspective can be turned around: One can view fermions in the defining representation as a single field, and different fermion species are just a manifestation of the three faces of $U(3)$.

### 3 Summary and Outlook

Our analysis started with the observation that $U(1)$ gauge symmetry can be incorporated into gauge field theories via semi-direct products and not simply as direct products. In particular, for $U(3)$ the construction of the semi-direct product is not unique; it comes in three versions. We argued these three versions can be interpreted as three species of matter fields. The interpretation relies on including all three versions of the semi-direct product in the Lagrangian, the large-gauge-transformation status of certain permutation operators, and the identification $\Psi^{(r)} := P^{r-1} \Psi$.

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\(^{2}\)To see this use the identity in three dimensions $\det A = 1/6 \left( (\text{tr} A)^3 - 3 \text{tr} A \text{tr} (A^2) + 2 \text{tr} (A^3) \right)$ and put $A \to U(x)$ with $U(x) = 1 + i\sigma^\alpha(x) A_\alpha + O(\sigma^2)$ an infinitesimal gauge transformation. To first order in $\sigma$, find that $\det U(x) > 0$.

\(^{3}\)We did not consider $U(2)$ as a replacement for $SU(2) \times U(1)$, but off hand the same mechanism would appear to apply and it should be studied in the context of spontaneous symmetry breaking.
From here, it is natural to wonder if there could be realistic strong-force phenomenology coming from gauged $U(3)$. Long ago Fischbach, et. al. [17] proposed the symmetry group $SU(3)_C \times U(1)_B$ with $SU(3)_C$ being the color symmetry of QCD and $U(1)_B$ coupling to baryon number, but it was effectively falsified by experiment [18]. However, the gauge-field interactions for $U(3)$ differ considerably from the $SU(3) \times U(1)$ case. All of the gauge fields associated with the Cartan subalgebra of $U(3)$ take part in both gauge and matter field interactions. So if there is somehow any vestige of a long-range charge carrier coming from $U(3)$, it will couple to both gauge and matter field mass-energy and therefore have a chance of being consistent with gravity — which was the downfall of $SU(3)_C \times U(1)_B$. Less clear and more imperative is whether $U(3)$ can somehow agree with QCD.

There are reasons to suspect there might be some kind of charge-carrying abelian gauge field beyond the Standard Model. Along these lines, many models incorporate a “dark photon” that interacts with a hidden matter field sector but may or may not interact with the Standard Model sector. The dark photon literature is quite extensive: For a review see [19] and references therein. The idea of appending a hypercolor symmetry group $SU(3)_H \times U(1)_H$ to the minimal supersymmetric $SU(5)_{GUT}$ was studied in [20–23]. The extra factor group resolves some shortcomings of the model, and it can be viewed as a D3 – D7 brane system in type IIB supergravity. The semi-direct product group $(SU(3)_C \times SU(2)_L) \times U(1)_Y$ and anomaly cancellation were used by [24] to put constraints on matter field hypercharge. As evidenced by the literature, it is possible to construct interesting models with extra $U(1)$ factors. We hope this note together with these studies will spur further investigation.

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4 To remind, the $U(1)$ subgroup is not the electromagnetic gauge symmetry.

5 The authors of [22] and [23] often write $U(3)_H$ but they put $U(3)_H \equiv SU(3)_H \times U(1)_H$ in [22] and remark that $U(3)_H \simeq SU(3)_H \times U(1)_H$ in [23]. In both papers they use $SU(3)_H \times U(1)_H$ in calculations.
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