Exact Solution of Kemmer Equation for Coulomb Potential

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Abstract

This article illustrates the bound states of Kemmer equation for spin-1 particles. The asymptotic, exact and Coulomb field solutions are obtained by using action principle. In the conclusion the energy spectrum of spin-1 particles moving in a Coulomb potential compared with the energy spectrum of spin-0 and spin-1/2 particles.

I. INTRODUCTION

In the present article we analyzed the problem of massive, charged particles of spin-1 in a Coulomb potential using action principle. The quantum mechanics of charged, massive, spin-1 bosons in the presence of an external field, was studied for many different situations using different techniques [1,2,3,4]. These works especially included an investigation of the solution of the equation in the presence of a magnetic field. These techniques are far more difficult to employ when the anomalous magnetic moment coupling terms are included into the equations of motion.

Relativistic spin-1 particle was at first described by Kemmer in 1939 [5]. Kemmer equation is a Dirac type equation, involves matrices obeying a different scheme of commutation rules. It is reviewed because of the interest in the quark-antiquark bound state problem. Corben and Schwinger were the first to present a general theory for massive particles of spin-1 in an external field and arbitrary magnetic moment [6]. They generalized the equations of Proca and studied the motion of such particles in an external Coulomb field. Their work provides the basis for many other studies.

This article is the first that illustrates the exact solution of Kemmer equation for massive spin-1 particle in the presence of a Coulomb potential. The method used here is a generally accepted method to solve the relativistic particle equations. The Lagrangian density of Kemmer Hamiltonian was separated into radial and angular parts by using the spin space rotation operators of $SO(3)$ group in the action principle and then the radial equations of motion are obtained by using Euler-Lagrange equations of motion. Finally the asymptotic, exact and Coulomb field solutions of these equations are discussed.

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II. RADIAL LAGRANGIAN

The Dirac-like relativistic Kemmer equation for spin-0 and spin-1 particle of mass \( m \) is

\[(\beta^\mu \pi_\mu - m) \Psi (x) = 0\]  \( (1) \)

where the 16 × 16 Kemmer matrices \( \beta^\mu \) (\( \mu = 0, 1, 2, 3 \)) satisfy the commutation relation

\[\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^\mu\nu \beta^\lambda + g^\lambda\nu \beta^\mu \]  \( (2) \)

and represented as

\[\beta^\mu = \gamma^\mu \otimes I + I \otimes \gamma^\mu \]  \( (3) \)

with \( \gamma^\mu \) usual Dirac matrices. The dynamical state \( \Psi (x) \) is a 16-component and includes both spin-0 and spin-1 particles. \( \pi_\mu \) is 4-vector electromagnetic momentum. After a basic step Eq.(1) can be written in the form

\[\left[ (\gamma^0 \otimes I + I \otimes \gamma^0) \pi_0 - (\overrightarrow{\alpha} \otimes \gamma^0 + \gamma^0 \otimes \overrightarrow{\alpha}) \cdot \overrightarrow{\pi} - m\gamma^0 \otimes \gamma^0 \right] \Psi (x) = 0 . \]  \( (4) \)

where \( \overrightarrow{\alpha} \) are 4 × 4 matrices and given in terms of 2 × 2 Pauli matrices as

\[\overrightarrow{\alpha} = \gamma^0 \overrightarrow{\gamma} = \begin{pmatrix} 0 & -\sigma \otimes \sigma \otimes \sigma \\ \sigma \otimes \sigma \otimes \sigma & 0 \end{pmatrix} \]  \( (5) \)

The Lagrangian density of this Hamiltonian is

\[L (x) = \Psi^\dagger \left[ (\gamma^0 \otimes I + I \otimes \gamma^0) \pi_0 - (\overrightarrow{\alpha} \otimes \gamma^0 + \gamma^0 \otimes \overrightarrow{\alpha}) \cdot \overrightarrow{\pi} - m\gamma^0 \otimes \gamma^0 \right] \Psi (x) . \]  \( (6) \)

This form must be separated into radial and angular parts to obtain the radial equations. The approach of separation used here is a technique employed by Barut and Ünal [7], which introduces a rotation aligning the \( z \)-axis of the coordinate system with the radial direction \( \hat{r} \). In the identical-two-fermion system spin space the rotation operator is

\[S_R = e^{\pm i(\sigma^1 - \theta \sigma^3)} \]  \( (7) \)

where \( \sigma^1 \) is the first component and \( \sigma^3 \) is the third component of Pauli matrices.

This operator preserves the wavefunction resulting

\[\Psi \left( \overrightarrow{r} \right) \rightarrow \Psi \left( \overrightarrow{r} \right) = S_R \begin{pmatrix} -a \left( r, \theta, \varphi \right) \\ ib \left( r, \theta, \varphi \right) \\ ic \left( r, \theta, \varphi \right) \\ d \left( r, \theta, \varphi \right) \end{pmatrix} \]  \( (8) \)

where \( a, b, c, d \) are 4-component spinors. Hence, Eq.(6) takes the following form:

\[L \left( \overrightarrow{r} \right) = \left( -a^\dagger, -ib^\dagger, -ic^\dagger, d^\dagger \right) S_R^{-1} \left[ (\gamma^0 \otimes I + I \otimes \gamma^0) \right] \pi_0 - \]

\[\left( a^\dagger, ib^\dagger, ic^\dagger, d^\dagger \right) \]  \( (9) \)
\[
\left( \mathbf{\alpha} \otimes \gamma^0 + \gamma^0 \otimes \mathbf{\alpha} \right) \cdot \mathbf{\pi} - m\gamma^0 \otimes \gamma^0 \right] S_R \begin{pmatrix} -a \\ ib \\ ic \\ d \end{pmatrix}.
\]

\[9\]

The 3-space rotation has the following effect in the spin space:

\[S_R^{-1} \begin{pmatrix} \alpha^\theta \\ \alpha^\varphi \\ \alpha^r \end{pmatrix} S_R = \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix}. \]

\[10\]

Therefore, in the square paranthesis of the Lagrangian \(\alpha_r, \alpha_\theta, \alpha_\varphi\) are rotated into \(\alpha_1, \alpha_2, \alpha_3\) respectively. Then Lagrangian density can be rewritten as

\[
L\left(\mathbf{r}^\dagger\right) = \left( -a^\dagger, -ib^\dagger, -ic^\dagger, d^\dagger \right) \left\{ \left( \gamma^0 \otimes I + I \otimes \gamma^0 \right) \pi_0 + i(\gamma^0 \otimes \alpha^3 + \alpha^3 \otimes \gamma^0) \partial_r + \frac{1}{r}(\gamma^0 \otimes \alpha^+ (\sigma^- - \partial_-) + \gamma^0 \otimes \alpha^- (\partial_+ - \sigma^+) + \alpha^+ (\sigma^- - \partial_-) \otimes \gamma^0
\]

\[+ \alpha^- (\partial_+ - \sigma^+) \otimes \gamma^0) \right\} - m\gamma^0 \otimes \gamma^0 \right] \begin{pmatrix} -a \\ ib \\ ic \\ d \end{pmatrix}.
\]

\[11\]

The \(\partial_+\) and \(\partial_-\) operators in Lagrangian are helicity lowering and raising operators, respectively and given by

\[
\partial_+ = \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{1}{2} \sigma^3 \cot \theta \\
\partial_- = -\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{1}{2} \sigma^3 \cot \theta
\]

\[12\]

To evaluate the matrix products in Lagrangian we first define new 4 × 4 quantities in bloch form:

\[
\mathbf{\sigma}_1 = \mathbf{\sigma} \otimes I \\
\mathbf{\sigma}_2 = I \otimes \mathbf{\sigma}
\]

\[13\]

where I is a 2 × 2 unit matrix, and

\[
\sigma^+ = \frac{1}{2} (\sigma^1 + i\sigma^2) \\
\sigma^- = \frac{1}{2} (\sigma^1 - i\sigma^2)
\]

\[14\]

Then we can define 16 × 16 matrices as follows:
\[ \overrightarrow{\alpha_{(1)}} = \overrightarrow{\alpha} \otimes I \]
\[ \overrightarrow{\alpha_{(2)}} = I \otimes \overrightarrow{\alpha} \]
\[ \gamma_{(1)} = \gamma^\mu \otimes I \]
\[ \gamma_{(2)} = I \otimes \gamma^\mu \] (15)

where I is now 4 \times 4 unit matrix. In terms of these we can write the following matrices
\[ \alpha_{(1)}^\pm = \frac{1}{2} (\alpha_{(1)}^1 \pm i \alpha_{(1)}^2) \]
\[ \alpha_{(2)}^\pm = \frac{1}{2} (\alpha_{(2)}^1 \pm i \alpha_{(2)}^2) \] (16)

Straightforward algebra allows us to rewrite the Lagrangian in the form:
\[ L(\overrightarrow{r}) = \{2\pi_0 a^\dagger a - 2\pi_0 d^\dagger d + \langle a^\dagger \sigma_3^2 b + a^\dagger \sigma_3^1 c - b^\dagger \sigma_2^3 c - b^\dagger \sigma_2^3 d \right\} \]
\[ -c^+ \sigma_1^3 a - c^+ \sigma_2^3 d + d^+ \sigma_1^3 b + d^+ \sigma_2^3 c \left( \partial_\tau + \frac{1}{r} \right) + \frac{i}{2r} \left[ a^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 b - \right] a^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 c - b^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 a + b^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 d + c^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 a - c^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 d - d^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 b + d^+ \left( \overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2 \right)^3 c - \frac{1}{r} \left[ -a^\dagger \partial_+ \sigma_2^- b - \right. \]
\[ -a^\dagger \partial_+ \sigma_2^- c + b^\dagger \partial_+ \sigma_2^- a + b^\dagger \partial_+ \sigma_1^- d + c^\dagger \partial_+ \sigma_1^- a + c^\dagger \partial_+ \sigma_2^- d - d^\dagger \partial_+ \sigma_1^- b - d^\dagger \partial_+ \sigma_2^- c \]
\[ + \frac{1}{r} \left[ -a^\dagger \partial_- \sigma_2^+ b - a^\dagger \partial_- \sigma_1^+ c + b^\dagger \partial_- \sigma_2^+ a + b^\dagger \partial_- \sigma_1^+ d + c^\dagger \partial_- \sigma_1^+ a + c^\dagger \partial_- \sigma_2^+ d \right. \]
\[ -d^\dagger \partial_- \sigma_1^+ b - d^\dagger \partial_- \sigma_2^+ c \right] - ma^\dagger a + mb^\dagger b + mc^\dagger c - md^\dagger d \} \] (17)

where everything is 16-dimensional. One can verify that the square of total angular momentum in spherical coordinates
\[ J^2 = \{ \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{\cot^2 \theta}{4} \} + J^2_r \] (18)

can be written in terms of \( \partial_+ \) and \( \partial_- \) operators as
\[ J^2 = \frac{1}{2} [\partial_+^\dagger \partial_+ + \partial_-^\dagger \partial_-] + J^2_r \] (19)

This allows us to write the 4-component a,b,c,d bispinors in terms of angular functions:
and the effect of helicity raising and helicity lowering operators on $D$ operator: 

$$\partial_- D^{j}_{\lambda, m} = \sqrt{j (j + 1)} D^{j}_{\lambda+1, m}$$

$$\partial_+ D^{j}_{\lambda, m} = \sqrt{j (j + 1)} D^{j}_{\lambda-1, m}$$

Next by calculating both sides separately in Lagrangian it is seen that

$$\sigma^+_k \partial_- D = \sqrt{j (j + 1)} D^j \sigma^+_k$$

and

$$\sigma^-_k \partial_+ D = \sqrt{j (j + 1)} D^j \sigma^-_k$$

for $k = 1, 2$.

For Lagrangian density the action is given in the form:

$$A = \int dt \ r^2 d\vec{r} \sum_{j m} \Psi^\dagger (r, t) D^{j \dagger}_{\lambda, m} (\vec{r}) \left[i \frac{\partial}{\partial t} - H\right] D^j_{\lambda, m} (\vec{r}) \Psi (r, t)$$
where
\[ d\tilde{r} = \sin \theta d\theta d\varphi. \] (29)

We can carry out all the angular integrations with the aid of Eq.(24), thereby obtaining an effective radial Lagrangian:

\[ L(\vec{r}) = 4\pi^2 \sum_{j,m} (2j + 1) r [a^+, b^+, c^+, d^+] \begin{pmatrix} (2\pi_0 - m) & T_2 & T_1 & 0 \\ -T_2 & m & 0 & -T_1 \\ -T_1 & 0 & m & -T_2 \\ 0 & T_1 & T_2 & -(2\pi_0 + m) \end{pmatrix} \]

\[ \times r \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \] (30)

where

\[ T_1 = \sigma^3_1 \partial_r - \frac{i\sqrt{j(j+1)}}{r} \sigma^2_1 - \frac{i}{2r} (\vec{\sigma}_1 \times \vec{\sigma}_2)^3, \] (31)

\[ T_2 = \sigma^3_2 \partial_r - \frac{i\sqrt{j(j+1)}}{r} \sigma^2_2 + \frac{i}{2r} (\vec{\sigma}_1 \times \vec{\sigma}_2)^3. \] (32)

By using the internal transformation matrices which are given as follows

\[ \pi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \pi^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & +1 & 1 & 0 \\ +1 & 0 & 0 & 1 \end{pmatrix} \] (33)

we define new quantities in terms of linear combinations of the original spinor components:

\[ \pi \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} r = \frac{r}{\sqrt{2}} \begin{pmatrix} a + d \\ b + c \\ c - b \\ d - a \end{pmatrix} = \begin{pmatrix} \Psi_A \\ \Psi_B \\ \Psi_C \\ \Psi_D \end{pmatrix} \] (34)

\[ r \begin{pmatrix} a^+ \\ b^+ \\ c^+ \\ d^+ \end{pmatrix} \pi^\dagger = \frac{r}{\sqrt{2}} \left( \begin{pmatrix} a^+ + d^+ \\ b^+ + c^+ \\ c^+ - b^+ \\ d^+ - a^+ \end{pmatrix} \right) = \begin{pmatrix} \Psi_A^\dagger \\ \Psi_B^\dagger \\ \Psi_C^\dagger \\ \Psi_D^\dagger \end{pmatrix} \] (35)

Since, \( \pi\pi^\dagger = \pi^\dagger\pi = 1 \) we can rewrite the Lagrangian in the form
\[ L \left( \vec{r}^2 \right) = 4\pi^2 \sum_{j,m} (2j + 1) \ r \ [a^i, b^i, c^i, d^i] \ \pi^i \pi \left( \begin{array}{cccc}
(2\pi_0 - m) & T_2 & T_1 & 0 \\
-T_2 & m & 0 & -T_1 \\
-T_1 & 0 & m & -T_2 \\
0 & T_1 & T_2 & -(2\pi_0 + m) \\
\end{array} \right) \times \pi^i r \left( \begin{array}{c}
a \\
b \\
c \\
d \end{array} \right) \] (36)

Then, with the aid of identities given in Eqs.(34), (35) we obtain the radial Lagrangian in the form:

\[ L \left( \vec{r}^2 \right) = 4\pi^2 \sum_{j,m} (2j + 1) \ \left\{ \Psi_A^\dagger [-m \Psi_A + (T_1 + T_2) \Psi_B - 2\pi_0 \Psi_D] + \Psi_B^\dagger [m \Psi_B - (T_1 - T_2) \Psi_A] + \Psi_C^\dagger [m \Psi_C + (T_1 - T_2) \Psi_D] + \Psi_D^\dagger [-2\pi_0 \Psi_A + (T_2 - T_1) \Psi_C - m \Psi_D] \right\} \] (37)

### III. RADIAL EQUATIONS

The Euler-Lagrange equations of motion are

\[ \frac{\partial L}{\partial q^i} - \partial_r \left( \frac{\partial L}{\partial (\partial_r q^i)} \right) = 0 \] (38)

where \( q = \Psi_A, \Psi_B, \Psi_C, \Psi_D \).

These yield four sets of 4-component equations, ultimately a set of sixteen equations:

\[ m\Psi_A - (T_1 + T_2) \Psi_B + 2\pi_0 \Psi_D = 0 \] (39)
\[ m\Psi_B - (T_1 + T_2) \Psi_A = 0 \] (40)
\[ m\Psi_C + (T_1 - T_2) \Psi_D = 0 \] (41)
\[ m\Psi_D + (T_1 - T_2) \Psi_C + 2\pi_0 \Psi_A = 0 \] (42)

The explicit forms of \((T_1 + T_2)\) and \((T_1 - T_2)\) are given as

\[ T_1 + T_2 = \left( \begin{array}{cccc}
2\partial_r & -\frac{\Lambda}{r} & -\frac{\Lambda}{r} & 0 \\
\frac{\Lambda}{r} & 0 & 0 & -\frac{\Lambda}{r} \\
0 & \frac{\Lambda}{r} & 0 & -2\partial_r \\
0 & 0 & \frac{\Lambda}{r} & -\frac{\Lambda}{r} \\
\end{array} \right) \], \quad T_1 - T_2 = \left( \begin{array}{cccc}
0 & \frac{\Lambda}{r} & -\frac{\Lambda}{r} & 0 \\
-\frac{\Lambda}{r} & 2\partial_r & \frac{2}{r} & -\frac{\Lambda}{r} \\
\frac{\Lambda}{r} & -\frac{2}{r} & -2\partial_r & -\frac{\Lambda}{r} \\
0 & \frac{\Lambda}{r} & -\frac{\Lambda}{r} & 0 \end{array} \right) \] (43)

where \( \Lambda = \sqrt{j(j + 1)} \).

Then, by defining \( \Psi_A, \Psi_B, \Psi_C, \Psi_D \) spinors as

\[
\Psi_A = \left( \begin{array}{c}
\Psi_{A1} \\
\Psi_{A2} \\
\Psi_{A3} \\
\Psi_{A4} \end{array} \right), \quad \Psi_B = \left( \begin{array}{c}
\Psi_{B1} \\
\Psi_{B2} \\
\Psi_{B3} \\
\Psi_{B4} \end{array} \right), \quad \Psi_C = \left( \begin{array}{c}
\Psi_{C1} \\
\Psi_{C2} \\
\Psi_{C3} \\
\Psi_{C4} \end{array} \right), \quad \Psi_D = \left( \begin{array}{c}
\Psi_{D1} \\
\Psi_{D2} \\
\Psi_{D3} \\
\Psi_{D4} \end{array} \right)
\]

we obtain sixteen radial equations from Eqs.(39), (12):
\[ m\psi_{A1} - 2\partial_r \psi_{B1} + \frac{\Lambda}{r} \psi_{B2} + \frac{\Lambda}{r} \psi_{B3} + 2\pi_0 \psi_{D1} = 0 \] (44)
\[ m\psi_{A2} - \frac{\Lambda}{r} \psi_{B1} + \frac{\Lambda}{r} \psi_{B4} + 2\pi_0 \psi_{D2} = 0 \] (45)
\[ m\psi_{A3} - \frac{\Lambda}{r} \psi_{B1} + \frac{\Lambda}{r} \psi_{B4} + 2\pi_0 \psi_{D3} = 0 \] (46)
\[ m\psi_{A4} - \frac{\Lambda}{r} \psi_{B2} - \frac{\Lambda}{r} \psi_{B3} + 2\partial_r \psi_{B4} + 2\pi_0 \psi_{D4} = 0 \] (47)
\[ m\psi_{B1} - 2\partial_r \psi_{A1} + \frac{\Lambda}{r} \psi_{A2} + \frac{\Lambda}{r} \psi_{A3} = 0 \] (48)
\[ m\psi_{B2} - \frac{\Lambda}{r} \psi_{A1} + \frac{\Lambda}{r} \psi_{A4} = 0 \] (49)
\[ m\psi_{B3} - \frac{\Lambda}{r} \psi_{A1} + \frac{\Lambda}{r} \psi_{A4} = 0 \] (50)
\[ m\psi_{B4} - \frac{\Lambda}{r} \psi_{A2} - \frac{\Lambda}{r} \psi_{A3} + 2\partial_r \psi_{A4} = 0 \] (51)
\[ m\psi_{C1} + \frac{\Lambda}{r} \psi_{D2} - \frac{\Lambda}{r} \psi_{D3} = 0 \] (52)
\[ m\psi_{C2} - \frac{\Lambda}{r} \psi_{D1} + 2\partial_r \psi_{D2} + \frac{\Lambda}{r} \psi_{D3} - \frac{\Lambda}{r} \psi_{D4} = 0 \] (53)
\[ m\psi_{C3} + \frac{\Lambda}{r} \psi_{D1} - 2\partial_r \psi_{D3} - \frac{2}{r} \psi_{D2} + \frac{\Lambda}{r} \psi_{D4} = 0 \] (54)
\[ m\psi_{C4} + \frac{\Lambda}{r} \psi_{D2} - \frac{\Lambda}{r} \psi_{D3} = 0 \] (55)
\[ m\psi_{D1} + \frac{\Lambda}{r} \psi_{C2} - \frac{\Lambda}{r} \psi_{C3} + 2\pi_0 \psi_{A1} = 0 \] (56)
\[ m\psi_{D2} - \frac{\Lambda}{r} \psi_{C1} + 2\partial_r \psi_{C2} + \frac{2}{r} \psi_{C3} - \frac{\Lambda}{r} \psi_{C4} + 2\pi_0 \psi_{A2} = 0 \] (57)
\[ m\psi_{D3} + \frac{\Lambda}{r} \psi_{C1} - 2\partial_r \psi_{C3} - \frac{2}{r} \psi_{C2} + \frac{\Lambda}{r} \psi_{C4} + 2\pi_0 \psi_{A3} = 0 \] (58)
\[ m\psi_{D4} + \frac{\Lambda}{r} \psi_{C2} - \frac{\Lambda}{r} \psi_{C3} + 2\pi_0 \psi_{A4} = 0 \] (59)

In these equations to eliminate the equations that correspond to spin-0 we must analysis the spin orientations of spinor wavefunctions. The spinor wavefunction of the system must be written in terms of upper and lower components for particle system:

\[
\Psi = \begin{pmatrix} L_1 \\ S_1 \end{pmatrix} \otimes \begin{pmatrix} L_2 \\ S_2 \end{pmatrix} = \begin{pmatrix} L_1 L_2 \\ L_1 S_2 \\ S_1 L_2 \\ S_1 S_2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\] (60)

where \( a, b, c, d \) spinors are given in the form

\[
a = \begin{pmatrix} L_{1+} \\ L_{1-} \end{pmatrix} \otimes \begin{pmatrix} L_{2+} \\ L_{2-} \end{pmatrix} = \begin{pmatrix} L_{1+} L_{2+} \\ L_{1+} L_{2-} \\ L_{1-} L_{2+} \\ L_{1-} L_{2-} \end{pmatrix} = \begin{pmatrix} a_+ \\ a_0 \\ a_5 \\ a_- \end{pmatrix}
\] (61)

8
We can see the following equalities:

\[ b = \begin{pmatrix} L_{1+} \\ L_{1-} \end{pmatrix} \otimes \begin{pmatrix} S_{2+} \\ S_{2-} \end{pmatrix} = \begin{pmatrix} L_{1+}S_{2+} \\ L_{1-}S_{2+} \\ L_{1+}S_{2-} \\ L_{1-}S_{2-} \end{pmatrix} = \begin{pmatrix} b_+ \\ b_0 \\ b_0 \end{pmatrix} \]  

(62)

\[ c = \begin{pmatrix} S_{1+} \\ S_{1-} \end{pmatrix} \otimes \begin{pmatrix} L_{2+} \\ L_{2-} \end{pmatrix} = \begin{pmatrix} S_{1+}L_{2+} \\ S_{1+}L_{2-} \\ S_{1-}L_{2+} \\ S_{1-}L_{2-} \end{pmatrix} = \begin{pmatrix} c_+ \\ c_0 \\ c_0 \\ c_- \end{pmatrix} \]  

(63)

\[ d = \begin{pmatrix} S_{2+} \\ S_{2-} \end{pmatrix} \otimes \begin{pmatrix} S_{2+} \\ S_{2-} \end{pmatrix} = \begin{pmatrix} S_{1+}S_{2+} \\ S_{1+}S_{2-} \\ S_{1-}S_{2+} \\ S_{1-}S_{2-} \end{pmatrix} = \begin{pmatrix} d_+ \\ d_0 \\ d_0 \\ d_- \end{pmatrix} \]  

(64)

Since our system is a single-particle system we can omit indices 1 and 2. Hence, from Eqs. (61), (74) it is seen that

\[ a_0 = a_0, \quad d_0 = d_0 \]  

(65)

and from Eqs. (62), (63)

\[ b_0 = b_0, \quad b_+ = c_+ \quad c_0 = c_0, \quad b_- = c_- \]  

(66)

Then, the new forms of \( \Psi_A, \Psi_B, \Psi_C, \Psi_D \) spinors are

\[ \Psi_A = \frac{r}{\sqrt{2}} \begin{pmatrix} a_+ + d_+ \\ a_0 + d_0 \\ a_0 + d_0 \\ a_+ + d_0 \end{pmatrix} = \begin{pmatrix} \Psi_{A1} \\ \Psi_{A2} \\ \Psi_{A3} \\ \Psi_{A4} \end{pmatrix}, \quad \Psi_B = \frac{r}{\sqrt{2}} \begin{pmatrix} b_+ + c_+ \\ b_0 + c_0 \\ b_0 + c_0 \\ b_+ + c_0 \end{pmatrix} = \begin{pmatrix} \Psi_{B1} \\ \Psi_{B2} \\ \Psi_{B3} \\ \Psi_{B4} \end{pmatrix} \]  

(67)

\[ \Psi_C = \frac{r}{\sqrt{2}} \begin{pmatrix} c_+ - b_+ \\ c_0 - b_0 \\ c_0 - b_0 \\ c_+ - b_+ \end{pmatrix} = \begin{pmatrix} \Psi_{C1} \\ \Psi_{C2} \\ \Psi_{C3} \\ \Psi_{C4} \end{pmatrix}, \quad \Psi_D = \frac{r}{\sqrt{2}} \begin{pmatrix} d_+ - a_+ \\ d_0 - a_0 \\ d_0 - a_0 \\ d_+ - a_+ \end{pmatrix} = \begin{pmatrix} \Psi_{D1} \\ \Psi_{D2} \\ \Psi_{D3} \\ \Psi_{D4} \end{pmatrix} \]  

(68)

We can see the following equalities:

\[ \Psi_{A2} = \Psi_{A3}, \quad \Psi_{B2} = \Psi_{B3} \]
\[ \Psi_{D2} = \Psi_{D3}, \quad \Psi_{C2} = -\Psi_{C3} \]
\[ \Psi_{C1} = \Psi_{C4} = 0 \]  

(69)

When we use these equalities in Eqs. \([44 - 59]\) we find that some of the solutions are identical. By eliminating one of the identical equations, we obtain 10 equations for spin-1 particle:

\[ m\Psi_{A1} - 2\partial_r \Psi_{B1} + \frac{\Lambda}{r}\Psi_{B2} + 2\pi_0 \Psi_{D1} = 0 \]  

(70)

\[ m\Psi_{A2} - \frac{\Lambda}{r}\Psi_{B1} + \frac{\Lambda}{r}\Psi_{B4} + 2\pi_0 \Psi_{D2} = 0 \]  

(71)
\[ m\Psi_A - 2 \frac{\Lambda}{r} \Psi_B + 2 \partial_r \Psi_B + 2\pi_0 \Psi_D = 0 \]  \hspace{1cm} (72)

\[ m\Psi_B - 2 \partial_r \Psi_A + 2 \frac{\Lambda}{r} \Psi_A = 0 \]  \hspace{1cm} (73)

\[ m\Psi_B - \frac{\Lambda}{r} \Psi_A + \frac{\Lambda}{r} \Psi_A = 0 \]  \hspace{1cm} (74)

\[ m\Psi_B - 2 \frac{\Lambda}{r} \Psi_A + 2 \partial_r \Psi_A = 0 \]  \hspace{1cm} (75)

\[ m\Psi_C - \frac{\Lambda}{r} \Psi_D + 2 \partial_r \Psi_D = 0 \]  \hspace{1cm} (76)

\[ m\Psi_D + 2 \frac{\Lambda}{r} \Psi_C + 2\pi_0 \Psi_A = 0 \]  \hspace{1cm} (77)

\[ m\Psi_C + 2 \partial_r \Psi_D = 0 \]  \hspace{1cm} (78)

\[ m\Psi_D + 2 \frac{\Lambda}{r} \Psi_C + 2\pi_0 \Psi_A = 0 \]  \hspace{1cm} (79)

IV. ANALYSIS OF THE RADIAL EQUATIONS

A. Asymptotic Solutions

If we write the radial equations of spin-1 particle in the limit \( r \to \infty \), we obtain the following equations:

\[ m\Psi_A - 2 \partial_r \Psi_B + +2\pi_0 \Psi_D = 0 \]  \hspace{1cm} (80)

\[ m\Psi_B + 2\pi_0 \Psi_D = 0 \]  \hspace{1cm} (81)

\[ m\Psi_A + 2 \partial_r \Psi_B + 2\pi_0 \Psi_D = 0 \]  \hspace{1cm} (82)

\[ m\Psi_B - 2 \partial_r \Psi_A = 0 \]  \hspace{1cm} (83)

\[ m\Psi_B = 0 \]  \hspace{1cm} (84)

\[ m\Psi_B + 2 \partial_r \Psi_A = 0 \]  \hspace{1cm} (85)

\[ m\Psi_C + 2 \partial_r \Psi_D = 0 \]  \hspace{1cm} (86)

\[ m\Psi_D + +2\pi_0 \Psi_A = 0 \]  \hspace{1cm} (87)

\[ m\Psi_D + 2 \partial_r \Psi_C + 2\pi_0 \Psi_A = 0 \]  \hspace{1cm} (88)

\[ m\Psi_D + +2\pi_0 \Psi_A = 0 \]  \hspace{1cm} (89)

The solutions of these equations are as follows:

\[ \Psi_A = A_1 e^{ikr} + B_1 e^{-ikr} \]  \hspace{1cm} (90)

\[ \Psi_A = A_2 e^{ikr} + B_2 e^{-ikr} \]  \hspace{1cm} (91)

\[ \Psi_A = A_3 e^{ikr} + B_3 e^{-ikr} \]  \hspace{1cm} (92)

\[ \Psi_A = A_4 e^{ikr} + B_4 e^{-ikr} \]  \hspace{1cm} (93)

\[ \Psi_A = A_5 e^{ikr} + B_5 e^{-ikr} \]  \hspace{1cm} (94)

\[ \Psi_A = A_6 e^{ikr} + B_6 e^{-ikr} \]  \hspace{1cm} (95)
\[ \Psi_{C2} = A_7 e^{ikr} + B_7 e^{-ikr} \]  
\[ \Psi_{B1} = A_8 e^{ikr} + B_8 e^{-ikr} \]  
\[ \Psi_{B4} = A_9 e^{ikr} + B_9 e^{-ikr} \]  

where

\[ k^2 = E^2 - \frac{m^2}{4} \]  

**B. Exact Solutions**

For free case the exact solutions of spin-1 particle are found in the form

\[ \Psi_{A1} (\rho) = \frac{1}{2} \left[ A \rho J (\rho) + A_1 e^{ip} + B_1 e^{-ip} \right] \]  
\[ \Psi_{A4} (\rho) = \frac{1}{2} \left[ -A \rho J (\rho) + A_1 e^{ip} + B_1 e^{-ip} \right] \]  
\[ \Psi_{B1} (\rho) = \frac{1}{2} \left[ D [J (\rho) + \frac{1}{2} \rho (J_{-1} (\rho) - J_{-1} (\rho))] + E e^{ip} + F e^{-ip} \right] \]  
\[ \Psi_{B4} (\rho) = \frac{1}{2} \left[ -D [J (\rho) + \frac{1}{2} \rho (J_{-1} (\rho) - J_{-1} (\rho))] + E e^{ip} + F e^{-ip} \right] \]  
\[ \Psi_{D1} (\rho) = \frac{1}{2} \left[ C \rho J (\rho) + G e^{ip} + H e^{-ip} \right] \]  
\[ \Psi_{D4} (\rho) = \frac{1}{2} \left[ -C \rho J (\rho) + G e^{ip} + H e^{-ip} \right] \]  
\[ \Psi_{B2} = \frac{\Lambda k}{km} [\Psi_{A1} (\rho) - \Psi_{A4} (\rho)] \]

where \( \rho = kr \).

**C. Coulomb Field Solutions**

For Coulomb field

\[ \pi_0 = P_0 - e \Phi = E + \frac{Ze^2}{r}. \]  

If it is used in spin-1 equations we obtain the following expreses for \( \Psi_A, \Psi_B, \Psi_C, \Psi_D \) spinors:

\[ \Psi_{A1} (\rho) = \frac{1}{2} \left[ AR_{n,l} (\rho) + CR_{n,l} (\rho) \right] \]  
\[ \Psi_{A4} (\rho) = \frac{1}{2} \left[ -AR_{n,l} (\rho) + CR_{n,l} (\rho) \right] \]  
\[ \Psi_{D1} (\rho) = -\frac{1}{m} \left( E + \frac{k \alpha}{\rho} \right) \left[ AR_{n,l} (\rho) + CR_{n,l} (\rho) \right] \]  
\[ \Psi_{D4} (\rho) = \frac{1}{m} \left( E + \frac{k \alpha}{\rho} \right) \left[ AR_{n,l} (\rho) - CR_{n,l} (\rho) \right] \]
\[
\begin{align*}
\Psi_{B1}(\rho) &= \frac{k}{m} \partial_\rho \left[ AR_{n,l}(\rho) + CR_{n,l}(\rho) \right] \\
\Psi_{B4}(\rho) &= \frac{k}{m} \partial_\rho \left[ AR_{n,l}(\rho) - CR_{n,l}(\rho) \right] \\
\Psi_{B2} &= 1 \rho BR_{n,l}(\rho)
\end{align*}
\]

where
\[
\alpha = e^2, \quad n = \frac{2\alpha}{k} E
\]

and
\[
R_{n,l}(\rho) = e^{\pm i\rho} [\pm 2i\rho]^{\frac{1}{2}} \left(1 + \sqrt{\frac{l(l+1)}{4}}\right) {1 \choose F_1} \left(\frac{1}{2} + \frac{1}{4} \sqrt{l(l+1)} \mp \frac{in}{2}, 1 + \frac{1}{2} \sqrt{l(l+1)}, \pm 2i\rho\right)\]

is confluent hypergeometric function.

**V. DISCUSSION**

In relativistic dynamics the exact solutions of the wave equation are very important because of the understanding of physics that can be brought by such solutions. These solutions are valuable tools in determining the radiative contributions to the energy.

Exact solutions of the Kemmer equation with potential interaction are very rare. There has been a great deal of interest in the Kemmer equation for a constant magnetic field but, on the other hand, we first solved exactly the Kemmer equation for relativistic Coulomb potential. Since Kemmer equation is a two-body Dirac-like equation it includes both spin-0 and spin-1 particles.16 solutions we obtained include both spin-0 and spin-1 particles. The identical solutions of these 16 equations represent the spin-0 particle. By eliminating one of the identical equations, we obtained 10 equations for spin-1 particle.

The asymptotic solutions of spin-1 particle are found in terms of periodic functions. These are important in physics in order to leave the components of spinors finite. The free solutions of spin-1 particle are found in terms of Bessel’s functions. In these solutions the \( \Psi_{A1}(\rho), \Psi_{B1}(\rho), \Psi_{D1}(\rho) \) and \( \Psi_{A4}(\rho), \Psi_{B4}(\rho), \Psi_{D4}(\rho) \) can be interpreted as helicity states of the particle and anti-particle, respectively.

For Coulomb potential the solutions are obtained in terms of confluent hypergeometric functions:

\[
R_{n,l}(\rho) = e^{\pm i\rho} [\pm 2i\rho]^{\frac{1}{2}} \left(1 + \sqrt{\frac{l(l+1)}{4}}\right) {1 \choose F_1} \left(\frac{1}{2} + \frac{1}{4} \sqrt{l(l+1)} \mp \frac{in}{2}, 1 + \frac{1}{2} \sqrt{l(l+1)}, \pm 2i\rho\right)
\]

where \( l(l+1) = j(j+1) - \alpha^2 \).

Here \( _1 F_1 (a, b; \pm 2i\rho) \) denotes the degenerate hypergeometric function. For large \( \rho \) this functions behaves as
Demanding that $F$ be normalizable, that is $\int_0^\infty dr \ |F|^2 = 1$ implies that $[\Gamma (a)]^{-1}$ must vanish. This is the desired quantization condition \[9\]. For our solution the quantization condition is

$$\frac{1}{2} \left( 1 + \sqrt{\frac{j (j+1) - \alpha^2}{4}} \right) \mp \frac{i \alpha E}{\sqrt{E^2 - m^2/4}} = -n$$

which leads to

$$E = \frac{\pm (m/2)}{\left[ 1 + \sqrt{\frac{4\alpha^2}{n + \frac{1}{2}(1+\frac{1}{2})\sqrt{(j+\frac{1}{2})^2 - \alpha^2 + 1/4}]} \right]^{1/2}}$$

Hence, by using the Coulomb field solutions of Kemmer equation we can predict the energy levels of spin-1 particles. This spectra is similar to the energy spectrum of spin-$\frac{1}{2}$ particle moving in Coulomb potential which is given by

$$E = \frac{\pm m}{\left[ 1 + \sqrt{\frac{\alpha^2}{n - (j+\frac{1}{2}) + \sqrt{(j+\frac{1}{2})^2 - \alpha^2}]} \right]^{1/2}}$$

In a previous work \[10\], the bound states of a spinless charged particle in the Coulomb field has been obtained. The radial wave functions of spin-0 particle for Coulomb interaction was obtained in terms of associated Laguerre polynomials:

$$F_{nJ} (\rho) = N e^{-\rho/2} \rho^{l+1} \mathcal{L}_{n'}^{l+1} (\rho)$$

where $n' = \lambda - l - 1$, $\lambda = n - J + l$ and $N$ is a normalization constant. The eigen-energy of spin-0 particle for Coulomb potential is

$$E = \frac{m}{\sqrt{1 + \frac{\alpha^2}{(n-J+1)^2}}}$$

From the energy spectrums of spin-1 and spin-0 particles one can see the first order contribution $(j \alpha^2)$ to the energy that comes from the spin-Coulomb field interaction by expanding the $\left(n + \sqrt{(j + \frac{1}{2}) - \alpha^2}\right)$ quantum number in series in the case of spin-1. It can also be seen how these contributions depend on spin by considering the spin-$\frac{1}{2}$ case.

The exact solutions of spin-1 particle for Coulomb potential is also important in QED. In QED the vacuum polarization in Coulomb field is calculated using Coulomb wave functions \[11\]. The solutions obtained here can be used for this purpose.
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