Special solutions for Ricci flow equation in 2D using the linearization approach

Stefan Adrian Carstea *
Mihai Visinescu †

Department of Theoretical Physics,
National Institute of Physics and Nuclear Engineering,
Magurele, P.O.Box MG-6, RO-077125 Bucharest, Romania

Abstract

The 2D Ricci flow equation in the conformal gauge is studied using the linearization approach. Using a non-linear substitution of logarithmic type, the emergent quadratic equation is split in various ways. New special solutions involving arbitrary functions are presented. Some special reductions are also discussed.

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In the last time the Ricci flow equations become a major tool for addressing a variety of problems in physics [1, 2, 3] and mathematics [4]. The Ricci flows are second order non-linear parabolic differential equations for the components of the metric $g_{\mu\nu}$ of an $n$-dimensional Riemannian manifold which are driven by the Ricci curvature tensor $R_{\mu\nu}$:

$$\frac{\partial}{\partial t} g_{\mu\nu} = -R_{\mu\nu}. \quad (1)$$

This equation describes geometric deformations of the metric $g_{\mu\nu}$ with parameter $t$. In particular the Ricci flow equations on two dimensional manifolds have attracted considerable attention on the physical literature in connection with two-dimensional black hole geometry, exact solutions of the renormalization group equations that describe the decay of singularities in non-compact spaces, etc. [1].

*E-mail: acarst@theory.nipne.ro
†E-mail: mvisin@theory.nipne.ro
As an attempt to quantize gravity, it is interesting to investigate quantum field theory in a curved space-time background. Solving relativistic field equations in (3+1)-dimensional curved space-time is generally a difficult process. An alternative approach is to consider lower-dimensional space-times models where exact solutions may be obtained. It is now long time since lower-dimensional gravity proved to exhibit many of the qualitative features of (3+1)-dimensional general relativity, high dimensional black holes, cosmological models and branes [5, 6].

The purpose of this paper is to present new explicit solutions for 2D Ricci flow equation using a linearization approach.

In two dimensions it is useful to consider a local system of conformally flat coordinates in which the metric has the form

\[ ds^2 = \frac{1}{2} e^{\Phi(x,y,t)} (dx^2 + dy^2) = 2 e^{\Phi(z_+,z_-;t)} dz_+ dz_- \]

using Cartesian coordinates \( x, y \) or the complex conjugate variables \( 2z_{\pm} = y \pm ix \).

Having in mind that the only non-vanishing component of the Ricci tensor is

\[ R_{+-} = -\partial_+ \partial_- \Phi(z_+,z_-;t) \]

the Ricci flow equation (1) becomes

\[ \frac{\partial}{\partial t} e^{\Phi(z_+,z_-;t)} = \partial_+ \partial_- \Phi(z_+,z_-;t). \] (2)

In what follows we shall use the substitution

\[ v(z_+,z_-;t) = e^{\Phi(z_+,z_-;t)} \]

writing eq. (2) in the form

\[ (v)_t = (\ln v)_{z_+ z_-}. \] (3)

This equation has been studied in detail from the algebraic point of view in [3]. It is considered as a ”continual” version of the general Toda-type equation for a given Lie algebra. Also in [3] it is presented a formal power series solution by expanding the path-ordered exponentials. Although the proposed general solution provides a formal complete solution for (3), its form is quite intricate and difficult to handle.

Our approach to the equation is somewhat different. We use a direct non-linear substitution, then split the resulting nonlinear equation. This results in a class of special solutions. Moreover we consider that the equation (3) is rather in the class of linearizable systems than Lax-pair solvable ones.

Our supposition comes from the following observations:

1. Starting with (3) and using the substitution \( v = \varphi_+ \), after integrating once with respect to \( z_+ \) we get

\[ \varphi_t \varphi_+ = \varphi_{z_+} z_+ + C \varphi_+ \] (4)
where \( C \) should be a function of \( z_- \) and \( t \), but for the moment it is considered constant. Now making the substitution \( \varphi = \ln F \) we will end up with the following quadratic equation:

\[
F_t F_{z_+} - F_{z_+ z_-} F + F_{z_+} F_{z_-} - C F F_{z_+} = 0. \tag{5}
\]

This equation has the following multi-shock-like solution:

\[
F = 1 + e^{\eta_1} + e^{\eta_2} + \ldots + e^{\eta_N} \tag{6}
\]

where \( \eta_i = k_i z_+ - C(z_- - t) \), for any \( k_i \) and positive integer \( N \). Of course the solution (6) is not a general solution because (5) is not a bilinear Hirota form (it is not gauge-invariant \( \S \)) and \( C \) is just a constant. But the lacking of interaction between exponentials is characteristic to linearizable systems (like Burgers, Liouville, etc.) \( \S \).

2. Equation (3) does not pass the Painlevé test. Usually Painlevé test is considered an integrability detector. Of course it is not infallible. There are many equations which do not pass the Painlevé test but still are completely integrable. These are either in the class of Hamiltonian systems with separable Hamilton-Jacobi equation, or are again some linearizable systems \( \S \).

Accordingly we are going to seek an underlying linear (or solvable) system for (3). Let us remark that eq. (3) is symmetric in variables \( z_+ \) and \( z_- \) and, consequently, in the rest of the paper, the role of these variables can be interchanged.

Let us assume that \( C \) is no longer a constant, but a free function of \( z_- \) and \( t \). In this case, defining

\[
\varphi = \psi + \int_{-\infty}^{t} C(z_-, t) dt
\]

then (4) will have the form

\[
\psi_{z_+} \psi_t = \psi_{z_+ z_-}. \tag{7}
\]

Using the same nonlinear substitution \( \psi = \ln F \) we get

\[
F_{z_+}(F_t + F_{z_-}) = F F_{z_+ z_-}. \tag{8}
\]

Equation (8) can be split in some linear or nonlinear solvable equations. Of course, all the possibilities we are going to analyze will give only special solutions and not general ones.

First of all, we shall split eq. (8) into a system of linear equations. Here we list the possibilities:
• **Linearization Ia**

\[
F_{z_+ z_-} = 0 \\
F_t + F_{z_-} = 0
\]

with the general solution

\[
F(z_+, z_-; t) = f(z_+) + g(t - z_-)
\]

where \(f, g\) are arbitrary functions. The solution of (8) is

\[
v(z_+, z_-; t) = \frac{f'(z_+)}{f(z_+) + g(t - z_-)}
\]

which is in fact the generalization of the multi-shock solution (6).

• **Linearization Ib**

\[
F_{z_+} = F_{z_+ z_-} \\
F_t + F_{z_-} = F.
\]

This system is equivalent with the previous one by means of the transformation:

\[
F(z_+, z_-; t) \rightarrow e^{z_-} F(z_+, z_-; t)
\]

and \(v(z_+, z_-; t)\) is invariant.

• **Linearization Ic**

\[
F_t + F_{z_+} = F_{z_+ z_-} \\
F_{z_+} = F
\]

with the general solution

\[
F(z_+, z_-; t) = h(t - z_-) e^{z_+ + z_-}
\]

for any arbitrary function \(h\). Unfortunately this gives a trivial solution for (3), namely \(v = 1\).

The next attempt is to split eq. (8) in a solvable system of nonlinear equations. Like in the linearization of the type I, we have the possibilities:

• **Linearization IIa**

\[
F_{z_+} = F^{\alpha}, \forall \alpha, \\
F^{\alpha - 1}(F_t + F_{z_-}) = F.
\]

The advantage of this splitting is that the first equation of the system (11) is a Bernoulli one with the solution

\[
F(z_+, z_-; t) = \{(1 - \alpha)[z_+ + h(z_-, t)]\}^\frac{1}{\alpha - 1}
\]
where \( h(z_-, t) \) is an arbitrary function. Introducing this expression in the second equation of the system one finds a linear equation for \( h(z_-, t) \):

\[
h_t + (1 - \alpha) h_{z-} = 0.
\]

In this way, the general solution for the system (11) is

\[
F(z_+, z_-; t) = \{ (1 - \alpha)[z_+ + Cz_+(t - \frac{z_-}{1 - \alpha})] \}^{\frac{1}{1 + \alpha}}
\]

which gives

\[
v(z_+, z_-; t) = \frac{1 + C(t - \frac{z_-}{1 - \alpha})}{z_+ + Cz_+(t - \frac{z_-}{1 - \alpha})} = \frac{1}{z_+}.
\] (12)

Accordingly this nonlinear splitting gives a stationary solution, i.e. independent of the parameter of deformation \( t \).

- **Linearization IIb**

\[
\begin{align*}
F_{z+} &= F^\alpha F_{z+} \\
F_t + F_{z-} &= F^{\alpha + 1}.
\end{align*}
\] (13)

From the first equation of the system we get

\[
F_{z-} = \frac{1}{\alpha + 1} F^{\alpha + 1} + \beta(z_-, t)
\]

with \( \beta \) an arbitrary function. Introducing in the second equation we get

\[
F_t + F_{z-} = (\alpha + 1)(F_{z-} - \beta)
\]

having the solution

\[
F(z_+, z_-; t) = -\frac{\alpha + 1}{\alpha} \int^{z_-} \beta(\xi, z_- + \alpha t - \xi) d\xi + h(z_+, z_- + \alpha t).
\]

Now solving the first equation for \( h \) in the case of a general function \( \beta \) is a difficult task. In any case, for \( \beta = 0 \) the system can be solved having the solution

\[
v(z_+, z_-; t) = \frac{f'(z_+)}{f(z_+) - \alpha(1 + \alpha) z_- + \alpha t}
\] (14)

We remark that the nonlinear splitting of the bilinear form gives solutions which are less general than (9).

A different approach to eq. (8) can be done using a special combination of the variables \( z_+ \) and \( z_- \) as a new independent variable. For example the well known solutions of cigar-type, or rational type, discussed in \( \text{[3]} \) can be obtained by introducing the following substitution:

\[
z_+ + z_- = \xi.
\] (15)
Then (3) becomes

\[ v_t = (\ln v) \xi \xi. \]

This equation has been extensively studied by Rosenau in [10] where, using the "addition property", he found cigar-type and rational type solutions.

Of course one can consider other symmetric combinations of \( z_+ \) and \( z_- \) to give new independent variables. The general method would imply the Lie-point symmetries (to find all the similarity reductions) but we are not going to do this here. Rather we will consider the following substitution:

\[ z_+ z_- = \xi. \]  

Using the same machinery we end up with the following bilinear equation:

\[ F_\xi(F_t + \xi F_\xi) = \xi F F_\xi \xi \]  

and, of course

\[ v(z_+, z_-; t) = \partial_{\xi} \ln F. \]

As in the linearization I, the following alternatives are obvious:

- **Linearization IIIa**
  
  Now, we can split (17) in the same way as before:

\[ F_t + \xi F_\xi = 0 \]

\[ F_\xi \xi = 0 \]

with the general solution

\[ F(\xi, t) = a \xi e^{-t} + b \]

where \( a, b \) are constants. In this case

\[ v(z_+, z_-; t) = \frac{1}{z_+ z_- + \frac{b}{a} e^t}. \]  

- **Linearization IIIb**

\[ F_\xi = \xi F_\xi \xi \]

\[ F_t + \xi F_\xi = F. \]

From the first equation of this system we have

\[ F(\xi, t) = \frac{1}{2} \xi^2 h(t) + \mu(t) \]

and from the second \( h(t) = ae^{-t}, \mu(t) = ce^t \) with \( a, c \) constants. Accordingly

\[ v(z_+, z_-; t) = \frac{z_+ z_-}{\frac{1}{2}(z_+ z_-)^2 + \frac{c}{a} e^{2t}}. \]
The next three splittings give trivial or stationary solutions:

- **Linearization IIIc**
  \[ F_{\xi} = F_{\xi\xi} \]
  \[ F_t + \xi F_\xi = \xi F. \]
  We have \( F = ae^{\xi} \) which leads to
  \[ v(z_+, z_-; t) = 1. \] (20)

- **Linearization IIId**
  \[ F_{\xi} = \xi F \]
  \[ F_t + \xi F_\xi = F_{\xi\xi} \]
  with \( F(\xi, t) = ae^{\frac{1}{2}\xi^2 + t} \) and we get
  \[ v(z_+, z_-; t) = z_+z_- . \] (21)

Finally, let us consider the following splittings of eq. (17) in solvable systems of nonlinear equations:

- **Linearization IVa**
  \[ F_{\xi} = F^\alpha \]
  \[ F^{\alpha-1}(F_t + \xi F_\xi) = \xi F_{\xi\xi} . \]
  From the first equation of the system we get
  \[ F = [(1 - \alpha)(\xi + g(t))]^{1/\alpha} , \]
  but if we introduce it in the second equation we have \( g' + 1 = \alpha z \) which gives \( g' = -1 \) and \( \alpha = 0 \) and consequently \( F \) is trivial.

- **Linearization IVb**
  Another incompatible splitting would be (\( \alpha \) and \( \beta \) are constants):
  \[ F_{\xi} = \xi^\beta F^\alpha \]
  \[ \xi^{\beta-1}F^{\alpha-1}(F_t + \xi F_\xi) = F_{\xi\xi} \]
  which has no solution.

- **Linearization IVc**
  Another choice could be
  \[ F_{\xi\xi} = FF_\xi \]
  \[ F_t + \xi F_\xi = \xi F^2 . \]
From the first equation of the above system we have

$$F = h(t) \tan \left[ \frac{1}{2} (\xi h(t) + g(t)) \right]$$

with $h(t)$ and $g(t)$ arbitrary functions of $t$. From the second equation we obtain differential equations for $h$ and $g$ which are not compatible. Therefore we haven’t any solution in this case.

To conclude, the linearization approach represents an efficient procedure to generate solutions of the Ricci flow equation (3). We remark that the solution (9) is the most general and represents the largest linearizable sector.

The exhaustive study of the Lie symmetries and similarity reductions performed on the quadratic equation (8) will be the subject of forthcoming work [11].

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