All noncommutative spaces of $\kappa$-Poincaré geodesics

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Received 15 August 2022, revised 3 October 2022
Accepted for publication 12 October 2022
Published 10 November 2022

Abstract
Noncommutative spaces of geodesics provide an alternative way of introducing noncommutative relativistic kinematics endowed with quantum group symmetry. In this paper we present explicitly the seven noncommutative spaces of time-, space- and light-like geodesics that can be constructed from the time-, space- and light-versions of the $\kappa$-Poincaré quantum symmetry in $(3+1)$ dimensions. Remarkably enough, only for the light-like (or null-plane) $\kappa$-Poincaré deformation the three types of noncommutative spaces of geodesics can be constructed, while for the time-like and space-like deformations both the quantum time-like and space-like geodesics can be defined, but not the light-like one. This obstruction comes from the constraint imposed by the coisotropy condition for the corresponding deformation with respect to the isotropy subalgebra associated to the given space of geodesics, since all these quantum spaces are constructed as quantizations of the corresponding classical coisotropic Poisson homogeneous spaces. The known quantum space of geodesics on the light cone is given by a five-dimensional homogeneous quadratic algebra, and the six noncommutative spaces of time-like and space-like geodesics are explicitly obtained as six-dimensional nonlinear algebras. Five out of these six spaces are here presented for the first time, and Darboux generators for all of them are found, thus showing that the quantum deformation parameter $\kappa^{-1}$ plays exactly the same algebraic role on quantum geodesics as the Planck constant $\hbar$ plays in the usual phase space description of quantum mechanics.

Keywords: quantum groups, noncommutative spaces, Minkowski spacetime, worldlines, light-like geodesics, kappa-deformation, Poincaré group

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1. Introduction

It is well-known that quantum groups [1–3] provide Hopf algebra deformations of relativistic symmetries that generate noncommutative spacetimes which are covariant under the corresponding quantum kinematical groups of transformations. Then, provided that the quantum deformation parameter is assumed to be related to the Planck scale [4], it is natural to think of quantum Poincaré groups as effective (flat spacetime) symmetries in a quantum gravity context, since it is widely assumed that noncommutative spacetimes should arise in the interplay between gravity and quantum effects at such scale [5–11]. Amongst all possible quantum deformations of the Poincaré symmetry, the so-called \( \kappa \)-Poincaré quantum groups and their associated noncommutative \( \kappa \)-Minkowski spacetimes have focused most of the attention in this direction (see [12–25] and references therein). It is also worth stressing that although the original \( \kappa \)-Poincaré deformation is of time-like nature [12–14] (the quantum deformation parameter \( \kappa \) is algebraically and thus dimensionally linked to the time translation generator \( P_0 \)), both the space-like [18] and the light-like (or null-plane) [19, 26–28] \( \kappa \)-Poincaré quantum groups were soon introduced and provided physically different models.

Indeed, classical Minkowski spacetime can be constructed as a homogeneous space of the Poincaré group \( M = G/H \) where \( H \) (the isotropy subgroup) is the Lorentz group \( SO(3, 1) \). From this perspective, \( \kappa \)-Minkowski spacetimes (time-, space- and light-like ones) can be constructed as the quantization of the Poisson homogeneous space \((M, \pi)\) that arises when \( M \) is endowed
with the coisotropic Poisson homogeneous structure $\pi$ (see [29–32]) generated by the corresponding classical $r$-matrix (either time-, space- or light-like) that generates the $\kappa$-deformation (see [33, 34] for a detailed explanation in the time-like and light-like cases, respectively).

It is also well-known [35–37], as it will be explained in detail in section 3, that the three spaces of geodesics on Minkowski spacetime can also be constructed as homogeneous spaces given as the coset spaces of the Poincaré group with respect to the isotropy subgroup for each type of geodesics. Therefore, the quantization of the coisotropic Poisson homogeneous structures induced on them by the three $\kappa$-Poincaré $r$-matrices would give rise, in principle, to nine different quantum spaces of geodesics. In this way, the highly nontrivial problem dealing with the definition of the quantum geodesics associated to the $\kappa$-Minkowski noncommutative spacetimes can be solved and, obviously, these novel quantum spaces of geodesics will be, by construction, covariant under the suitable (co)action of the $\kappa$-Poincaré quantum group that quantizes the Poisson–Lie Poincaré structure under consideration.

However, this fully constructive approach has not been considered in the literature until very recently. In particular, in [33] the (six-dimensional (6D)) noncommutative space of time-like geodesics associated to the time-like $\kappa$-Poincaré quantum group was constructed, and some phenomenological implications of this model for the associated ‘fuzzy’ worldlines of free massive particles were further analysed in [38]. Moreover, in [34], the relevant problem concerning the construction of the (5D) space of quantum geodesics on the light cone was fully solved for the light-like (or null-plane) $\kappa$-deformation of the Poincaré group.

Surprisingly enough, in [34] it was also found that both the time-like and the space-like $\kappa$-deformations cannot be used to provide coisotropic Poisson homogeneous structures on the space of geodesics on the light cone, and therefore the two corresponding quantum spaces of light-like geodesics cannot be constructed. This was discovered to be an outstanding property of the space of geodesics on the light cone, since it can also be proven that the three types of $\kappa$-deformations allow the construction of coisotropic Poisson homogeneous spaces for both the time-like and the space-like spaces of geodesics. Therefore, from the nine possible noncommutative spaces of geodesics coming from the three $\kappa$-Poincaré deformations and the three types of geodesics on Minkowski spacetime, two of them are excluded due to the coisotropy constraint.

Since only two of the remaining seven spaces have been deduced so far in [33, 34], the aim of this paper is to construct explicitly the five unknown quantum spaces of $\kappa$-Poincaré geodesics and to present a global overview of all these seven novel noncommutative structures in a common setting. We think that this comprehensive view will be helpful in order to explore their mathematical interrelations as well as their possible physical features as kinematical models at the Planck scale, that we plan to study in the sequel.

The structure of the paper is as follows. In the next two sections, we provide all the essential ingredients of the formalism that will be used in the paper. We start in section 2 by recalling the $(3 + 1)$D Poincaré Lie algebra $\mathfrak{g}$ expressed in the usual kinematical basis and we introduce the so called null-plane basis [39]. The $r$-matrices underlying the three classes of $\kappa$-Poincaré deformations (see [24] and references therein) are given in a covariant way, and their Lie bialgebra structures $(\mathfrak{g}, \delta)$ are explicitly computed. In section 3, we introduce the $(3 + 1)$D Minkowskian spacetime as well as the 6D time-like, 6D space-like and 5D light-like spaces of geodesics as homogeneous spaces $G/H$ of the Poincaré group $G$, where $H$ is the appropriate isotropy subgroup for each space. Then, it is checked for each space whether each of the Lie bialgebra structures associated to the three $\kappa$-deformations fulfills the coisotropy condition $\delta(h) \subset \mathfrak{h} \wedge \mathfrak{g}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup $H$. This allows us to identify which deformations provide a coisotropic Poisson homogeneous structure on a given $G/H$ that will be suitable for quantization. Also, the first-order of all the noncommutative spaces that satisfy
the coisotropy condition are derived from the associated \(\kappa\)-Poincaré Lie bialgebra structures. Finally, the generic approach to the construction of a noncommutative space as the quantization of a coisotropic Poisson homogeneous space is illustrated in detail by recovering through such a procedure the three well-known \(\kappa\)-Minkowski spacetimes arising from the time-, space- and light-like \(\kappa\)-deformations.

This formalism is applied to the construction of all the quantum spaces of geodesics in section 4 where, in particular, sections 4.1 and 4.2 contain the explicit 6D noncommutative spaces of time- and space-like geodesics, respectively, which arise from the three types of \(\kappa\)-Poincaré quantum groups. We stress that although the noncommutative space of time-like geodesics induced from the time-like deformation has already been obtained in [33], the other five noncommutative spaces of geodesics here presented are novel ones. It is worth mentioning that all these quantum spaces are no longer of Lie-algebraic type, which means that the first-order structures obtained in section 3 by the Lie bialgebra structures have to be completed in a substantially non-trivial way by including many higher order contributions. Furthermore, Darboux-type coordinates are found for suitable submanifolds in the six coisotropic Poisson homogeneous structures, which implies that for each of these 6D quantum spaces of geodesics there exist appropriate generators for which the defining noncommutative algebra is given by canonical Heisenberg–Weyl relations where the quantum deformation parameter \(\kappa^{-1}\) replaces the Planck constant \(\hbar\). Indeed, this means that the phenomenological analysis performed in [38] could be extended to the other five new spaces by following a similar procedure. For the sake of completeness, we sketch in 4.3 the basic features of the 5D noncommutative space of light-like geodesics introduced in [34], which arises from the light-like \(\kappa\)-Poincaré deformation. Finally, some remarks and open problems are presented in a closing section.

2. \(\kappa\)-Poincaré Lie bialgebras

Let us consider the \((3 + 1)\)D Poincaré Lie algebra \(\mathfrak{g} = \mathfrak{iso}(3, 1) \equiv \mathfrak{so}(3, 1) \ltimes \mathbb{R}^4\) in the usual kinematical basis \(\{P_0, P_a, K_a, J_a\} (a = 1, 2, 3)\) spanned, in this order, by the generators of time translation, space translations, boosts and rotations. The commutation rules for \(\mathfrak{g}\) are given by

\[
\begin{align*}
[J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, \\
[J_a, P_0] &= 0, & [K_a, P_0] &= P_a, & [K_a, P_b] &= \delta_{ab} P_0, \\
[K_a, K_b] &= -\epsilon_{abc} J_c, & [P_\mu, P_\nu] &= 0,
\end{align*}
\]

(2.1)

where \(a, b, c = 1, 2, 3, \mu, \nu = 0, 1, 2, 3\), and the speed of light is set to \(c = 1\). Hereafter sum over repeated indices will be assumed, unless otherwise stated.

In the following it will be useful to consider also the so-called ‘null-plane basis’ \(\{P_\pm, P_i, E_i, F_i, K_3, L_3\} (i = 1, 2)\) [39], associated with the boost generator \(K_3\) in such a way that the Poincaré generators can be casted into three different classes according to the adjoint action of \(K_3\)

\[
[K_3, X] = \gamma X, \quad X \in \mathfrak{g},
\]

(2.2)

where the parameter \(\gamma\) is called the ‘goodness’ of the generator \(X\) [39]. Explicitly, the null-plane
the so-called space-like are obtained from a classical associated to a given deformation is determined by the cocommutator map of them are generated by classical generators \[34\] are defined as:

\[
\begin{align*}
\gamma &= +1: \quad P_+ = \frac{1}{2}(P_0 + P_3), \quad E_1 = \frac{1}{2}(K_1 - J_2), \quad E_2 = \frac{1}{2}(K_2 + J_1). \\
\gamma &= 0: \quad K_3, \quad L_3 = -J_3, \quad P_1, \quad P_2. \\
\gamma &= -1: \quad P_- = P_0 - P_3, \quad F_1 = K_1 + J_2, \quad F_2 = K_2 - J_1,
\end{align*}
\]

from which the commutation rules of the Poincaré algebra are written as:

\[
\begin{align*}
[L_3, E_i] &= -\epsilon_{i\beta} F_{j}, \\
[L_3, P_\pm] &= 0, \\
[K_3, P_i] &= 0, \\
[E_i, P_\pm] &= 0, \\
[F_i, P_\pm] &= \delta_{ij} P_\mp, \\
[E_i, F_j] &= \delta_{ij} K_3 + \epsilon_{i\beta} L_3,
\end{align*}
\]

where \(i, j = 1, 2\) and \(\alpha, \beta = \pm, 1, 2\).

We recall that all possible quantum deformations of the Poincaré algebra are coboundary ones \[40–42\] (except for the very particular \((1 + 1)D\) case \[26, 41, 43\]), which means that all of them are generated by classical \(r\)-matrices. The coboundary Lie bialgebra structure \((g, \delta)\) associated to a given deformation is determined by the cocommutator map \(\delta : g \to g \wedge g\), which is obtained from a classical \(r\)-matrix \(r \in g \wedge g\) in the form

\[
\delta(X) = [X \otimes 1 + 1 \otimes X, r], \quad \forall \ X \in g.
\]

Among all possible \(r\)-matrices for the Poincaré algebra \[40\], we shall focus in this paper on the so-called \(\kappa\)-Poincaré \(r\)-matrices, where \(\kappa^{-1}\) plays the role of the quantum deformation parameter. These \(r\)-matrices can be written in covariant notation as \[24, 40\]

\[
r = v^\mu M_{\mu\nu} \wedge P_\nu, \tag{2.6}
\]

where \(v^\mu\) are the components of a Minkowskian four-vector \(v = (v^0, v^1, v^2, v^3)\) and \(M_{\mu\nu}\) denote a set of generators of the Lorentz group, which are related to the kinematical basis \(2.1\) in the form

\[
M_{0\omega} \equiv K_\omega, \quad M_{12} = J_3, \quad M_{23} = J_1, \quad M_{31} = J_2. \tag{2.7}
\]

Consequently, according to the time-, space- and light-like nature of the four-vector \(v\), three different classes of classical \(r\)-matrices \[24\] arise, each of them determining a different type of \(\kappa\)-Poincaré deformation. We display in table 1 a representative of each equivalence class of \(r\)-matrices determined by a choice of four-vector \(v\) \[34\], which are written in the covariant notation \(2.6\), in the kinematical basis \(2.1\) and in the null-plane one \(2.4\).

From the explicit expressions presented in table 1, it becomes clear that both time- and space-like \(\kappa\)-deformations are naturally adapted to the kinematical basis \(2.1\), while the light-like \(\kappa\)-deformation can be better understood in the null-plane basis \(2.4\). Moreover, let us note
characterized by the primitive (i.e., with nondeformed coproduct) generator of the coproduct of the associated quantum Poincaré that, as expected

\[ \delta \]

The corresponding cocomutator map \( \delta \) coming from (2.5) is explicitly written in table 2 in the most adapted basis for each deformation, thus providing the three classes of \( \kappa \)-Poincaré Lie bialgebras (\( g, \delta \)). As it is well-known, each of them provides the first-order deformation in \( \kappa^{-1} \) of the coproduct of the associated quantum Poincaré algebra.

From either tables 1 or 2, it comes out that the time-, space- and light-like deformation is characterized by the primitive (i.e., with nondeformed coproduct) generator \( P_0, P_3 \) and \( P_+ \), respectively. In particular, \( r_{\text{time}} \) corresponds to the well-known \( \kappa \)-Poincaré algebra [12–17] for which the deformation parameter \( \kappa \) has dimensions of \( \text{time}^{-1} \) (recall that \( c = 1 \)). The second Lie bialgebra, determined by \( r_{\text{space}} \), underlies the \( q \)-Poincaré algebra obtained in [18] (cf type 1. (a) with \( z = 1/\kappa \)), with \( \kappa \) having dimensions of \( \text{length}^{-1} \). Both \( r_{\text{time}} \) and \( r_{\text{space}} \) lead to quasitriangular (or standard) deformations of the Poincaré algebra since they are solutions of the modified classical Yang–Baxter equation (with non-vanishing Schouten bracket). The third Lie bialgebra structure, coming from \( r_{\text{light}} \), provides the null-plane quantum Poincaré algebra introduced in [19, 27, 28] (where \( z = 1/\kappa \)), which is a triangular (or nonstandard) quantum deformation with vanishing Schouten bracket. Therefore, despite its apparent formal similarity, the third \( r \)-matrix \( r_{\text{light}} \) (and therefore its Lie bialgebra) is completely different from the other two from a kinematical viewpoint, and this fact will be essential as far as the construction of the corresponding noncommutative spaces of geodesics is concerned.

### 3. Homogeneous spaces, the coisotropy condition and quantization

The \((3 + 1)\)D Poincaré group \( G = \text{ISO}(3,1) \) with Lie algebra \( g = \text{iso}(3,1) \) allows the construction of several \( D \)D homogeneous spaces which can be expressed in a generic form as a left
can identify the tangent space at every point

\[\cosetspace\]

where \(Poincar\)\(´e\) group, namely the Minkowski spacetime and the three types of spaces of geodesics, 

\[\ell M\text{ on }M\]

We consider in this paper the four most relevant homogeneous spaces coming from the \(\ell\)-group, namely the Minkowskispace and the three types of spaces of geodesics, 

\[\ell M\text{ is the }\ell\text{-group associated with the translation generators of }M\text{.}
\]

We leave a point on \(M\) invariant, which is taken as the origin \(O\) of the homogeneous space, thus playing the role of ‘rotations’ around \(O\), while the \(\ell\) generators belonging to \(t = \text{span}\{T_1, \ldots, T_\ell\}\) move \(O\) along \(\ell\) basic directions, thus behaving as translations on \(M\). When appropriately defined, the local coordinates \(t^{i_1, \ldots, i_\ell}\) of the Poincaré group associated with the translation generators of \(t\) descend to \(\ell\) coordinates on \(M\).

We consider in this paper the four most relevant homogeneous spaces coming from the Poincaré group, namely the Minkowski spacetime and the three types of spaces of geodesics,

\[\text{Table 2. The three classes of }\kappa\text{-Poincaré Lie bialgebras } (g, \delta) \text{ obtained from the } r\text{-matrices given in table 1, via the relation (2.5), in the kinematical basis (2.1) for the time- and space-like deformations, and in the null-plane basis (2.4) for the light-like deformation. The index } a = 1, 2, 3 \text{ while } i = 1, 2.\]

| Class        | Time-like \(\kappa\)-Poincaré Lie bialgebra from \(r_{\text{time}}\) | Space-like \(\kappa\)-Poincaré Lie bialgebra from \(r_{\text{space}}\) | Light-like \(\kappa\)-Poincaré Lie bialgebra from \(r_{\text{light}}\) |
|--------------|------------------------------------------------|------------------------------------------------|------------------------------------------------|
| \(\delta(P_0) = (J_0) = 0\) | \(\delta(P_a) = \frac{1}{2} P_a \wedge P_0\) | \(\delta(K_1) = \frac{1}{2}(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2)\) | \(\delta(T_1) = \frac{1}{2}(T_1 \wedge P_0 + E_1 \wedge P_1 + E_2 \wedge P_2)\) |
| \(\delta(K_2) = \frac{1}{2}(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3)\) | \(\delta(K_3) = \frac{1}{2}(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1)\) | \(\delta(T_2) = \frac{1}{2}(T_2 \wedge P_0 + E_2 \wedge P_1 + E_3 \wedge P_2)\) | \(\delta(T_3) = \frac{1}{2}(T_3 \wedge P_0 + E_3 \wedge P_1 + E_4 \wedge P_2)\) |

\[\cosetspace\]

\[M^\ell = G/H\]  

(3.1)

where \(M^\ell\) is the \(\ell\)-manifold and \(H\) is the \((10 - \ell)\)-D isotropy subgroup with Lie algebra \(h\). We can identify the tangent space at every point \(m = gH \in M^\ell, g \in G\), with the translation sector:

\[T_m(M^\ell) = T_{gH}(G/H) \simeq g/h \simeq \mathfrak{t} = \text{span}\{T_1, \ldots, T_\ell\}.\]

(3.2)

In fact, at a Lie algebra level, such construction comes from the Cartan decomposition of the Poincaré algebra \(g\), as a vector space, given by the sum of two subspaces

\[\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.\]

(3.3)

Hence the generators of the isotropy subalgebra \(h\) leave a point on \(M^\ell\) invariant, which is taken as the origin \(O\) of the homogeneous space, thus playing the role of ‘rotations’ around \(O\), while the \(\ell\) generators belonging to \(\mathfrak{t} = \text{span}\{T_1, \ldots, T_\ell\}\) move \(O\) along \(\ell\) basic directions, thus behaving as translations on \(M^\ell\). When appropriately defined, the local coordinates \(t^{i_1, \ldots, i_\ell}\) of the Poincaré group associated with the translation generators of \(\mathfrak{t}\) descend to \(\ell\) coordinates on \(M^\ell\).
which are explicitly defined as follows:

- The $(3 + 1)$D Minkowski spacetime $\mathcal{M} = G/H_{st}$
  \[ g = t_{st} \oplus h_{st}, \quad t_{st} = \text{span}\{P_0, P_a\}, \quad h_{st} = \text{span}\{K_a, J_a\} = so(3, 1). \]

- The 6D space of time-like lines $\mathcal{W}_{tl} = G/H_{tl}$
  \[ g = t_{tl} \oplus h_{tl}, \quad t_{tl} = \text{span}\{P_a, K_a\}, \quad h_{tl} = \text{span}\{P_0, J_a\} = \mathbb{R} \oplus so(3). \]

- The 6D space of space-like lines $\mathcal{W}_{sl} = G/H_{sl}$
  \[ g = t_{sl} \oplus h_{sl}, \quad t_{sl} = \text{span}\{P_0, P_i, K_3, J_i\}, \quad h_{sl} = \text{span}\{P_3, K_i, J_3\} = \mathbb{R} \oplus so(2, 1). \]

- The 5D space of light-like lines $\mathcal{L} = G/H_{ll}$
  \[ g = t_{ll} \oplus h_{ll}, \quad t_{ll} = \text{span}\{P_-, P_i, F_i\}, \quad h_{ll} = \text{span}\{P_+, E_i, K_3, L_3\}. \]

where $a = 1, 2, 3, i = 1, 2$ and the notation ‘st’, ‘tl’, ‘sl’ and ‘ll’ means, in this order, spacetime, time-like, space-like and light-like.

### 3.1. The coisotropy condition

The method that we propose in order to construct quantum group invariant noncommutative analogues of the aforementioned homogeneous spaces $\mathcal{M}$ consists in quantizing the unique coisotropic Poisson homogeneous structure $\pi$ on $\mathcal{M}$ that is covariant under the Poisson–Lie Poincaré group that is defined by the $\kappa$-matrix underlying the chosen quantum deformation. The quantization of such Poisson-noncommutative structure on the classical space $\mathcal{M}$ will then provide the defining relations for the quantum space that is covariant under the given quantum deformation. However, as it was discussed in detail in \[29–32, 44\], the construction of such Poisson structure $\pi$ is only guaranteed whenever the so-called coisotropy condition for the cocommutator $\delta$ with respect to the isotropy subalgebra $h$ of $H$ holds, namely

\[ \delta(h) \subset h \wedge g. \]  

(3.5)

In the particular case when

\[ \delta(h) \subset h \wedge h, \]  

(3.6)

the condition is obviously fulfilled, but in such a way that the isotropy subalgebra $h$ is also a sub-Lie bialgebra $(h, \delta|_h)$ of $(g, \delta)$. This means that, after quantization, the isotropy subgroup $H$ will be promoted to a quantum subgroup.

The verification (or not) of the required coisotropy condition (3.5) for the three classes of $\kappa$-Poincaré Lie bialgebras and for all the homogeneous spaces described in (3.4) can be straightforwardly obtained from the explicit expressions of the cocommutator map $\delta$ given in table 2 and from the definition of the corresponding isotropy subalgebras $h$.

These results are summarized in table 3 showing that the three classes of $\kappa$-Poincaré algebras can be used to provide a noncommutative $\kappa$-Minkowskian spacetime since the coisotropy condition is always fulfilled. However, we stress that the only quantum $\kappa$-Poincaré algebra that enables the construction of the noncommutative counterpart of the four homogeneous spaces (3.4) is just the light-like (or null-plane) quantum Poincaré algebra, since both the (usual) time-like and the space-like $\kappa$-Poincaré deformations do not always satisfy the coisotropy condition, which precludes the construction of their associated light-like quantum spaces of geodesics.
It is worth stressing that the first-order in the local coordinates of the noncommutative spaces

\[ \text{Table 3.} \quad \text{[34] Coisotropy condition (3.5) for the three } \kappa \text{-Poincaré Lie bialgebras given in table 2 with respect to the four different isotropy subalgebras (3.4) that ensures the existence } \check{\square} \text{ (or not } \check{\times} \text{) of a noncommutative Minkowskian spacetime (st) and a noncommutative space of time-like (tl) and light-like (ll) geodesics.} \]

| \( \mathfrak{h}_a \) | \( \mathfrak{h}_b \) | \( \mathfrak{h}_c \) | \( \mathfrak{h}_d \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \delta(\mathfrak{h}_a) \subset \mathfrak{h}_a \wedge g^\vee \) | \( \delta(\mathfrak{h}_b) = 0 \vee \) | \( \delta(\mathfrak{h}_c) \subset \mathfrak{h}_c \wedge g^\vee \) | \( \delta(\mathfrak{h}_d) \subset \mathfrak{h}_d \wedge g^\vee \) |
| \( \delta(\mathfrak{h}_a) \subset \mathfrak{h}_a \wedge g^\vee \) | \( \delta(\mathfrak{h}_b) = 0 \vee \) | \( \delta(\mathfrak{h}_c) = 0 \vee \) | \( \delta(\mathfrak{h}_d) \subset \mathfrak{h}_d \wedge g^\vee \) |
| \( \delta(\mathfrak{h}_a) \subset \mathfrak{h}_a \wedge g^\vee \) | \( \delta(\mathfrak{h}_b) = 0 \vee \) | \( \delta(\mathfrak{h}_c) \subset \mathfrak{h}_c \wedge g^\vee \) | \( \delta(\mathfrak{h}_d) \subset \mathfrak{h}_a \wedge g^\vee \) |

3.2. From coisotropic Lie bialgebras to first-order noncommutative spaces

It is worth stressing that the first-order in the local coordinates of the noncommutative spaces \( M'_\kappa \) can be deduced directly from the cocommutators written in table 2 by means of the dual map \( \delta^\vee : g^\vee \otimes g^\vee \to g^\vee \), which is a Lie bracket on the dual Poincaré algebra \( g^\vee \).

In our case, let us denote the quantum or noncommutative translation coordinates by \( \hat{t}^1, \ldots, \hat{t}^l \) corresponding to the classical Poincaré group local coordinates \( t^1, \ldots, t^l \) of the translation sector \( t^1 = \{T_1, \ldots, T_l\} \).

The duality between the generators \( t^1 \) and the quantum coordinates \( \hat{t}^1, \ldots, \hat{t}^l \) is determined by a canonical pairing given by the bilinear form

\[ \langle \hat{t}^j, T_k \rangle = \delta^j_k, \quad \forall \ j, k. \] (3.7)

Let us consider a given comultiplication \( \delta \) in table 2. If the coisotropy condition (3.5) is satisfied, then the quantum coordinates \( \hat{t}^1, \ldots, \hat{t}^l \) close a Lie algebra which is just the annihilator \( \mathfrak{h}_\perp \) of \( \mathfrak{h} \) on the dual Poincaré algebra \( g^\vee \), and therefore this defines the noncommutative space

\[ \mathfrak{h}_\perp = M'_\kappa \] (3.8)

at the first-order in such quantum coordinates (see [32] for details). In fact, it is worth stressing that this is just the meaning of the coisotropy condition in algebraic terms: when (3.5) is not fulfilled, the first-order commutation rules given by dualizing the Lie bialgebra comultiplication are such that the generators of the space under consideration do not close on a subalgebra since they include other dual generators that correspond to transformations which do not correspond to translations on the chosen space.

In particular, if we denote by \( (\hat{\xi}^\mu, \hat{\zeta}^a, \hat{\theta}^3) \) \( (\mu = 0, 1, 2, 3; a = 1, 2, 3) \) the quantum coordinates dual to the kinematical generators \( (P_\mu, K_a, J_3) \) (2.1), and by \( (\hat{\xi}^\alpha, \hat{\zeta}^a, \hat{\theta}^i, \hat{\zeta}^3) \) \( (\alpha = \pm 1, 2; i = 1, 2) \) those dual to the generators in the null-plane basis \( (P_\mu, E_i, F_i, K_3, L_3) \) (2.4), then the corresponding first-order quantum spaces (3.8) corresponding to the classical homogenous spaces (3.4) will be:

- The \( (3 + 1) \)D \( \kappa \)-Minkowski spacetime \( M_\kappa = \mathfrak{h}_\perp = \{\hat{t}^\mu\} \).
- The 6D \( \kappa \)-space of time-like lines \( V_{\text{tl},\kappa} = \mathfrak{h}_\perp = \{\hat{\xi}^\mu, \hat{\zeta}^a\} \).
- The 6D \( \kappa \)-space of space-like lines \( V_{\text{sl},\kappa} = \mathfrak{h}_\perp = \{\hat{\xi}^\alpha, \hat{\zeta}^a, \hat{\theta}^i\} \).
- The 5D \( \kappa \)-space of light-like lines \( L_\kappa = \mathfrak{h}_\perp = \{\hat{\xi}^\pm, \hat{\theta}^i\} \).

\[ (3.9) \]
The explicit expressions of these four first-order noncommutative spaces are displayed in table 4 for each $\kappa$-deformation. The three noncommutative Minkowski spacetimes $\mathcal{M}_\kappa$ appear in the first line and turn out to have no higher order contributions when the full quantum space is computed, as we will see in the next section (see [15] for $r_{\text{time}}$, [44] for $r_{\text{space}}$, and [27] for $r_{\text{light}}$, as well as [24]). Thus, they are noncommutative spacetimes of Lie-algebraic type and, furthermore, all of them are isomorphic as Lie algebras (although their physical interpretation is different).

As we will see in section 4, this will be no longer the case for the quantum spaces of geodesics, which will be defined in all the cases by nonlinear relations in terms of the local coordinates on the appropriate parameterization of the Poincaré group. We recall that among the seven possible complete noncommutative spaces of geodesics that are allowed by the coisotropy condition, only two of them have been constructed so far, namely $\mathcal{W}_{\kappa}$ from $r_{\text{time}}$ in [33] and $\mathcal{L}_\kappa$ from $r_{\text{light}}$ in [34]. Surprisingly enough, we realize that although such two noncommutative spaces are in fact commutative ones at the first-order in the quantum coordinates (they are Abelian algebras in table 4), we will see that they will be defined as noncommutative algebras when contributions at all orders in the quantum coordinates are considered. Namely, the full quantum space $\mathcal{W}_{\kappa}$ from $r_{\text{time}}$ involves cumbersome expressions with hyperbolic trigonometric functions, meanwhile the complete $\mathcal{L}_\kappa$ from $r_{\text{light}}$ is defined by quadratic relations.

3.3. From coisotropic Poisson homogeneous spaces to noncommutative spaces

We recall that coboundary Lie bialgebras $(\mathfrak{g}, \delta)$ are the tangent counterpart of coboundary Poisson–Lie groups $(G, \Pi)$ [2], where the Poisson structure $\Pi$ on $G$ is given by the so-called Sklyanin bracket

$$\{f_1, f_2\} = \epsilon^{ij}(X^L_i f_1 X^L_j f_2 - X^R_i f_1 X^R_j f_2), \quad f_1, f_2 \in \mathcal{C}(G),$$

such that $X^L_i$ and $X^R_i$ are left- and right-invariant vector fields defined by

$$X^L_i f(g) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tY_i}g), \quad X^R_i f(g) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tY_i}g), \quad$$

where $f \in \mathcal{C}(G)$, $g \in G$ and $Y_i \in \mathfrak{g}$. The quantization (as a Hopf algebra) of the Poisson–Lie group $(G, \Pi)$ is just the corresponding quantum group.

A Poisson homogeneous space $(G/H, \pi)$ for a Poisson–Lie group $(G, \Pi)$ is the classical homogeneous space $G/H$ (like (3.4)) endowed with a Poisson structure $\pi$ which is covariant under the action of the Poisson–Lie group $(G, \Pi)$. In the case of a coisotropic Poisson homogeneous space (so $\mathfrak{h}$ satisfies (3.5)), the Poisson structure $\pi$ on $G/H$ is straightforwardly derived through canonical projection from the Poisson–Lie structure II on $G$. Noncommutative spaces can be finally obtained as quantizations of coisotropic Poisson homogeneous spaces.

The steps of the procedure to construct all allowed $\kappa$-noncommutative spaces associated with Poincaré homogeneous spaces (3.4), as shown in table 3 (i.e. ten cases), are summarized as follows:

- Consider a faithful representation $\rho$ of the Poincaré algebra $\mathfrak{g}$.
- Obtain by exponentiation a generic element of the Poincaré group $G$ with the appropriate order. This means that for the generic left coset space $M' = G/H$ (3.1) the corresponding Poincaré group element $G_{M'}$ must be constructed in the form
Table 4. Non-vanishing commutation relations that define the first-order noncommutative spaces $h^+ \equiv M'_\kappa$ corresponding to the four Poincaré homogeneous spaces (3.4) obtained from the three classes of $\kappa$-Poincaré Lie bialgebras $(g, \delta)$ given in table 2 in agreement with the coisotropy condition shown in table 3. The index $a = 1, 2, 3$ while $i = 1, 2$.

| Space    | $r_{\text{time}}$ | $r_{\text{space}}$ | $r_{\text{light}}$ |
|----------|-------------------|---------------------|---------------------|
| $M_\kappa$ | $[\hat{\kappa}^a, \hat{\kappa}^0] = \frac{1}{n} \hat{\kappa}^a$ | $[\hat{\kappa}^0, \hat{\kappa}^3] = \frac{1}{n} \hat{\kappa}^0$ | $[\hat{\kappa}^-, \hat{\kappa}^+] = \frac{1}{n} \hat{\kappa}^-$ |
| $W_{1\kappa}$ | $0$ | $[\hat{\xi}^i, \hat{\xi}^3] = \frac{1}{n} \hat{\xi}^i$ | $[\hat{\xi}^3, \hat{\xi}^0] = \frac{1}{n} \hat{\xi}^3$ |
| $W_{1\kappa}$ | $[\hat{\theta}^1, \hat{\theta}^2] = \frac{1}{n} \hat{\xi}^3$ | $[\hat{\theta}^2, \hat{\theta}^1] = -\frac{1}{2} \hat{\xi}^3$ | $[\hat{\theta}^1, \hat{\theta}^2] = \frac{1}{n} \hat{\xi}^3$ |
| $L_{\kappa}$ | $\times$ | $\times$ | $0$ |
where $T_1, \ldots, T_\ell$ are the translation generators on $M'$ (3.2) and $H$ is the $(10-\ell)$D isotropy subgroup.

- Calculate the corresponding left- and right-invariant vector fields (3.11) from $G_{M'}$ (3.12).
- Compute the Poisson brackets among the local translation coordinates $(t^1, \ldots, t^\ell)$ by applying the Sklyanin bracket (3.10) from a classical $r$-matrix given in table 1. The resulting expressions define the coisotropic Poisson homogeneous space.
- Finally, quantize the Poisson homogeneous space thus obtaining the noncommutative space in terms of the quantum coordinates $\hat{t}^1, \ldots, \hat{t}^\ell$.

As an instructive warming-up application of this methodology, in the next section we provide the explicit derivation of the three well-known $\kappa$-Minkowskian spacetimes.

3.4. The construction of $\kappa$-Minkowski spacetimes

Let us now apply the above approach to the $(3+1)$D Minkowski spacetime $\mathcal{M} = G/H_{st}$ (3.4). We consider the faithful representation $\rho: g \rightarrow \text{End}(\mathbb{R}^5)$ for a generic element $X \in g$ given, in the kinematical basis (2.1), by

$$\rho(X) = x^\mu \rho(P_\mu) + \xi^a \rho(K_a) + \theta^i \rho(J_i) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\
x^1 & 0 & -\theta^3 & \theta^2 & \theta^1 \\
x^2 & \xi^2 & 0 & -\theta^1 & \theta^3 \\
x^3 & -\theta^2 & \theta^3 & 0 & 0
\end{pmatrix},$$

and the corresponding exponential map provides a 5D representation of the Poincaré group $G$. According to (3.12), we construct an element of the Poincaré group $G$ in the form

$$G_M = \exp(x^0 \rho(P_0))\exp(x^1 \rho(P_1))\exp(x^2 \rho(P_2))\exp(x^3 \rho(P_3)) H_{st},$$

where the Lorentz subgroup $H_{st} = \text{SO}(3,1)$ is parameterized by

$$H_{st} = \exp(\xi_1 \rho(K_1)) \exp(\xi_2 \rho(K_2)) \exp(\xi_3 \rho(K_3)) \exp(\theta^1 \rho(J_1)) \times \exp(\theta^2 \rho(J_2)) \exp(\theta^3 \rho(J_3)).$$

From this, we compute the corresponding left- and right-invariant vector fields (3.11). The Sklyanin bracket (3.10) for $r_{\text{time}}$ and $r_{\text{space}}$, expressed in the kinematical basis given in table 1, gives rise in this case to linear Poisson brackets for the classical coordinates $x^\mu$. Therefore these Poisson brackets can be directly quantized, thus defining the Lie-algebraic $\kappa$-Minkowskian spacetimes [15, 24, 44] with quantum coordinates $\hat{x}^\mu$ which are shown in table 4.

In the null-plane basis (2.4) we consider the following representation $\rho: g \rightarrow \text{End}(\mathbb{R}^5)$ for a generic element $X \in g$ [27]:
\[ \rho(X) = x^+ \rho(P^+) + x^- \rho(P^-) + x^i \rho(P_i) + e^j \rho(E_j) + f^j \rho(F_j) + \xi^3 \rho(K_3) + \phi^j \rho(L_j) \]

\[ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x^+ + x^- & 0 & \frac{1}{2}e^1 + f^1 & \frac{1}{2}e^2 + f^2 & \xi^3 \\ x^1 & \frac{1}{2}e^1 + f^1 & 0 & \phi^3 & -\frac{1}{2}e^1 + f^1 \\ x^2 & \frac{1}{2}e^2 + f^2 & -\phi^3 & 0 & -\frac{1}{2}e^2 + f^2 \\ \frac{1}{2}x^+ - x^- & \xi^3 & \frac{1}{2}e^1 - f^1 & \frac{1}{2}e^2 - f^2 & 0 \end{pmatrix}. \quad (3.16) \]

And the Poincaré group element is obtained as

\[ G_M = \exp(x^+ \rho(P^+))\exp(x^1 \rho(P_1))\exp(x^2 \rho(P_2))\exp(x^- \rho(P^-))H_a, \quad (3.17) \]

where \( H_a \) is the Lorentz subgroup in the null-plane basis. Note that the relations between the spacetime coordinates in the representation (3.13) and those in (3.16) read

\[ x^0 = \frac{1}{2}x^+ + x^-, \quad x^3 = \frac{1}{2}x^+ - x^-, \quad x^+ = x^0 + x^3, \quad x^- = \frac{1}{2}(x^0 - x^3), \quad (3.18) \]

keeping \( x^1 \) and \( x^2 \).

Again, if we now calculate the corresponding invariant vector fields (3.11) and next compute the Sklyanin bracket (3.10) for \( r_{\text{light}} \) written in the null-plane basis in table 1, then we obtain linear Poisson brackets for the classical coordinates \( x^\alpha (\alpha = \pm, 1, 2) \), whose trivial quantization gives rise to the light-like or null-plane \( \kappa \)-Minkowskian spacetime [24, 27] with quantum coordinates \( \hat{x}^\alpha \) given in table 4.

### 4. Noncommutative spaces of geodesics

#### 4.1. Quantum time-like geodesics

This section contains the main results of this paper, namely the complete noncommutative spaces of time- and space-like geodesics (corresponding to the second and third columns of table 3). For the sake of clarity, we firstly summarize (see [33] for a detailed discussion) the construction of the homogeneous space of time-like geodesics, i.e. the geodesics followed by massive free particles on Minkowski spacetime. In order to do that we use the second decomposition of the Poincaré Lie algebra given in (3.4), and we use a parameterization of the Poincaré Lie group (3.12) adapted to this coset space \( \mathcal{W}_a = G/H_a \) in the kinematical basis (2.1). In particular, we parameterize the Poincaré Lie group from the 5D matrix representation (3.13) as
\[ G_{\text{tl}} = \exp(\eta_1 \rho(K_1)) \exp(y_1 \rho(P_1)) \exp(\eta_2 \rho(K_2)) \exp(y_2 \rho(P_2)) \times \exp(\eta_3 \rho(K_3)) \exp(y_3 \rho(P_3)) H_\text{tl}, \] (4.1)

where \( H_\text{tl} \) is the stabilizer of the worldline corresponding to a massive particle at rest at the origin of \( \mathcal{M} \), namely

\[ H_\text{tl} = \exp(\phi_1 \rho(J_1)) \exp(\phi_2 \rho(J_2)) \exp(\phi_3 \rho(J_3)) \exp(y_0 \rho(P_0)). \] (4.2)

In this way \((y^a, \eta^a)\) provide a set of coordinates on the 6D space \( \mathcal{W}_{\text{tl}} \). At the level of the Lie group, the time-like geodesic parameterization (4.1) and the spacetime parameterization (3.14) are related by

\[ x^\mu = x^\mu(y^a, \eta^a), \quad \xi^a = \eta^a, \quad \theta^a = \phi^a, \] (4.3)

where

\[
\begin{align*}
x^0 &= y^1 \sinh \eta_1 + \cosh \eta_1 \left(y^2 \sinh \eta_2 \cosh \eta_2 + y^3 \sinh \eta_3 \right), \\
x^1 &= y^1 \cosh \eta_1 + \sinh \eta_1 \left(y^2 \sinh \eta_2 \cosh \eta_2 + y^3 \sinh \eta_3 \right), \\
x^2 &= y^2 \cosh \eta_2 + \sinh \eta_2 \left(y^0 \cosh \eta_0 \cosh \eta_3 + y^3 \sinh \eta_3 \right), \\
x^3 &= y^3 \cosh \eta_3,
\end{align*}
\] (4.4)

so involving the six coordinates on the space of time-like geodesics plus the ‘extra’ coordinate \( y^0 \).

In the following, we present the three noncommutative spaces of time-like geodesics defined by the \( \kappa \)-Poincaré family of \( r \)-matrices given in table 1 by applying the very same procedure to the one described in section 3.4 to construct the three \( \kappa \)-Minkowski spacetimes. Thus we compute the left- and right-invariant vector fields (3.11) from \( G_{\text{tl}} \) (4.1) and obtain the Poisson–Lie structure on the Poincaré group associated to a given Lie bialgebra by means of the Sklyanin bracket (3.10). Then we project the Poisson–Lie structure to the coset space \( \mathcal{W}_{\text{tl}} \) in order to get the coisotropic Poisson structure of time-like worldlines and, finally, we quantize the Poisson homogeneous space to obtain the algebra of ‘quantum geodesic’ observables \( \mathcal{W}_{\text{tl},\kappa} \).

### 4.1.1 From the time-like \( \kappa \)-deformation

Here we sketch the essential results given in [33].

If we consider the well-known time-like \( \kappa \)-Poincaré \( r \)-matrix [15] \( r_{\text{time}} \) and we follow the above-mentioned procedure, then by projecting the Sklyanin bracket to the homogeneous space coordinates we get a coisotropic structure for the classical space of time-like geodesics which can be straightforwardly quantized, since no ordering problems appear. In this way, the
quantum space of time-like geodesics \( \mathcal{W}_{\text{tl}} \) reads:

\[
[\hat{y}_1, \hat{y}_2]_t = \frac{1}{\kappa} \left( \hat{y}_2 \sinh \hat{\eta}_1 - \hat{y}_1 \tanh \hat{\eta}_2 \cosh \hat{\eta}_3 \right),
\]

\[
[\hat{y}_1, \hat{y}_3]_t = \frac{1}{\kappa} \left( \hat{y}_3 \sinh \hat{\eta}_1 - \hat{y}_1 \tanh \hat{\eta}_3 \cosh \hat{\eta}_2 \right),
\]

\[
[\hat{y}_2, \hat{y}_3]_t = \frac{1}{\kappa} \left( \cosh \hat{\eta}_1 \cosh \hat{\eta}_2 \cosh \hat{\eta}_3 - 1 \right),
\]

\[
[\hat{\eta}_1, \hat{\eta}_1]_t = \frac{1}{\kappa} \left( \cosh \hat{\eta}_1 \cosh \hat{\eta}_2 \cosh \hat{\eta}_3 - 1 \right),\]

\[
[\hat{\eta}_2, \hat{\eta}_2]_t = \frac{1}{\kappa} \left( \cosh \hat{\eta}_1 \cosh \hat{\eta}_2 \cosh \hat{\eta}_3 - 1 \right),
\]

\[
[\hat{\eta}_3, \hat{\eta}_3]_t = \frac{1}{\kappa} \left( \cosh \hat{\eta}_1 \cosh \hat{\eta}_2 \cosh \hat{\eta}_3 - 1 \right),\]

\]

\[
[\hat{\eta}_a, \hat{\eta}_b]_t = 0, \quad \forall \ a, b, \quad [\hat{y}_a, \hat{\eta}_b]_t = 0, \quad a \neq b. \quad (4.6)
\]

It is remarkable that the coisotropic Poisson structure given by the Poisson brackets identical to (4.5) is found to be symplectic almost everywhere, i.e. it is symplectic in the 6D smooth submanifold where \((\eta_1, \eta_2, \eta_3) \neq (0, 0, 0)\). In fact, the new coordinates

\[
q_1 = \frac{y_1 \cosh \eta_2 \cosh \eta_3}{\cosh \eta_1 \cosh \eta_2 \cosh \eta_3 - 1},
\]

\[
q_2 = \frac{y_2 \cosh \eta_3}{\cosh \eta_1 \cosh \eta_2 \cosh \eta_3 - 1},
\]

\[
q_3 = \frac{y_3}{\cosh \eta_1 \cosh \eta_2 \cosh \eta_3 - 1},
\]

\[
p^a = \eta^a,
\]

can be considered as the Darboux coordinates on such submanifold, since their Poisson brackets read

\[
\{q^a, q^b\} = \{p^a, p^b\} = 0, \quad \{q^a, p^b\} = \frac{1}{\kappa} \delta_{ab}. \quad (4.8)
\]

Therefore, the quantum counterpart of (4.7) leads to the usual noncommutative phase space algebra of quantum mechanics where the deformation parameter \(\kappa^{-1}\) replaces the Planck constant \(\hbar\). It is worth mentioning that, as it was shown in [38], the behavior of the probability distributions at the \((\eta_1, \eta_2, \eta_3) = (0, 0, 0)\) submanifold do not present any divergency.
4.1.2. From the space-like $\kappa$-deformation. Similarly to the previous case, we can repeat the same procedure to obtain the Poisson homogenous structure on the space of time-like geodesics associated to the second $r$-matrix $r_{\text{space}}$ from table 1. The Poisson structure so obtained is evidently different from the one obtained before, but it has clear formal analogies: it admits a trivial quantization (no ordering problems arise again), and it is almost everywhere symplectic. In particular, the non-vanishing brackets (already quantized) defining $\mathcal{W}_{\text{Di,space}}$ from $r_{\text{space}}$ read

$$
\begin{align*}
[\hat{y}^1, \hat{y}^2]_{s} &= -\frac{1}{\kappa} \hat{y}^1 \tanh \hat{\eta}^2 \tanh \hat{\eta}^3, \\
[\hat{y}^1, \hat{y}^3]_{s} &= \frac{1}{\kappa} \hat{y}^1 \cosh \hat{\eta}^1, \\
[\hat{y}^2, \hat{y}^3]_{s} &= \frac{1}{\kappa} \hat{y}^2 \cosh \hat{\eta}^2, \\
[\hat{y}^1, \hat{\eta}^1]_{s} &= -\frac{1}{\kappa} \tanh \hat{\eta}^3 \cosh \hat{\eta}^2, \\
[\hat{y}^2, \hat{\eta}^2]_{s} &= -\frac{1}{\kappa} \tanh \hat{\eta}^3, \\
[\hat{y}^3, \hat{\eta}^3]_{s} &= -\frac{1}{\kappa} \sinh \hat{\eta}^3.
\end{align*}
$$

(4.9)

The classical Darboux coordinates (which now are only defined in the submanifold $\eta^1 \neq 0$) take the form

$$
\begin{align*}
q^1 &= -\frac{y^1 \cosh \eta^2 \cosh \eta^3}{\sinh \eta^3}, \\
q^2 &= -\frac{y^2 \cosh \eta^3}{\sinh \eta^3}, \\
q^3 &= -\frac{y^3}{\sinh \eta^3}, \\
p^a &= \eta^a,
\end{align*}
$$

(4.10)

and fulfill again the canonical Poisson brackets (4.8).

4.1.3. From the light-like $\kappa$-deformation. In section 2 it was already discussed that the light-like $\kappa$-Poincaré $r$-matrix $r_{\text{light}}$ in table 1 is obtained as the sum of the time-like and space-like ones (2.8). Thus, its associated Lie bialgebra and Poisson–Lie structure are given by the linear superposition of the two structures that we have just found. Again, it is immediate to realize that no ordering problems appear and the full quantization can be performed directly, giving
rise to a quantum space $\mathcal{W}_{\text{id},\kappa}$ defined by

$$
[\hat{y}^1, \hat{y}^2]_l = \frac{1}{\kappa} \left( \hat{y}^2 \sinh \hat{y}^1 - \frac{\hat{y}^1 \tanh \hat{y}^2 (\sinh \hat{y}^3 + 1)}{\cosh \hat{y}^3} \right),
$$

$$
[\hat{y}^1, \hat{y}^3]_l = \frac{1}{\kappa} \left( \hat{y}^3 \sinh \hat{y}^1 - \frac{\hat{y}^1 (\sinh \hat{y}^3 - 1)}{\cosh \hat{y}^3} \right),
$$

$$
[\hat{y}^2, \hat{y}^3]_l = \frac{1}{\kappa} \left( \hat{y}^3 \cosh \hat{y}^1 \sinh \hat{y}^2 - \frac{\hat{y}^2 (\sinh \hat{y}^3 - 1)}{\cosh \hat{y}^3} \right),
$$

$$
[\hat{y}^1, \hat{\eta}^1]_l = \frac{1}{\kappa} \left( \cosh \hat{\eta}^1 \cosh \hat{y}^2 \cosh \hat{y}^3 - \sinh \hat{\eta}^1 \hat{y}^3 - 1 \right),
$$

$$
[\hat{y}^2, \hat{\eta}^1]_l = \frac{1}{\kappa} \left( \cosh \hat{\eta}^1 \cosh \hat{y}^2 \cosh \hat{y}^3 - \sinh \hat{\eta}^1 \hat{y}^3 - 1 \right),
$$

$$
[\hat{y}^3, \hat{\eta}^1]_l = \frac{1}{\kappa} \left( \cosh \hat{\eta}^1 \cosh \hat{y}^2 \cosh \hat{y}^3 - \sinh \hat{\eta}^1 \hat{y}^3 - 1 \right).
$$

(4.11)

The Darboux coordinates in this case are found to be

$$
q^1_l = \frac{y^1 \cosh \eta^2 \cosh \eta^3}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - \sinh \eta^1 \eta^3 - 1},
$$

$$
q^2_l = \frac{y^2 \cosh \eta^3}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - \sinh \eta^1 \eta^3 - 1},
$$

$$
q^3_l = \frac{y^3}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - \sinh \eta^1 \eta^3 - 1},
$$

\[\rho^l = \eta^l,\]

(4.12)

satisfying the canonical Poisson brackets (4.8).

We stress that under linearization, that is, by considering the first-order in the quantum coordinates ($\hat{y}^a$, $\hat{\eta}^a$) of the three quantum spaces $\mathcal{W}_{\text{id},\kappa}$ (4.5), (4.9) and (4.11), we recover the first-order noncommutative spaces given in the second row in table 4, provided that, only at this first-order, the quantum coordinates are given by $\hat{x}^a \equiv y^a$ and $\hat{\xi}^a \equiv \eta^a$ (see (4.3) and (4.4)). In this respect, also observe that if we compute the above Poisson brackets from the three $r$-matrices for the classical spacetime coordinates $x^a$ given by (4.4) we recover the three linear $\kappa$-Minkowski spacetimes shown in the first row of table 4. This makes evident that the full quantum spaces (4.5), (4.9) and (4.11) are defined by strongly non-linear relations that can only be envisaged once the complete classical coisotropic Poisson structures are computed.

It is also worth noticing that the three sets of Darboux coordinates $q^a_l$ (4.7), $q^a_r$ (4.10) and $q^a_l$ (4.12) are connected through the simple relation

$$
\frac{1}{q^1_l} = \frac{1}{q^1_r} + \frac{1}{q^1_l},
$$

(4.13)

while Darboux momenta coincide for the three cases $\rho^l = \eta^l$. Observe also that in none of the three cases the definition of the Darboux coordinates $q^a_l$ can be linearized in terms of the local coordinates on the given space, which is consistent with the fact that the first-order noncommutative spaces given in table 4 are not isomorphic as Lie algebras to the three Heisenberg–Weyl algebras quantizing (4.8).
4.2. Quantum space-like geodesics

Let us now describe the noncommutative spaces of geodesics corresponding to the three families of space-like $\kappa$-Poincaré deformations from table 1. Similarly to the previous section, our first task is to parameterize the Poincaré Lie group in such a manner that we obtain a set of coordinates that descend to the appropriate quotient space $\mathcal{W}_d = G/H_d$ (3.4). Such a parameterization is given by

\[
G_{\mathcal{W}_d} = \exp(\pi^1 \rho(J_2)) \exp(u^1 \rho(P_1)) \exp(\pi^2 \rho(J_1)) \exp(u^2 \rho(P_2)) \exp(\pi^0 \rho(K_3)) \exp(u^0 \rho(P_0)) H_d,
\]

where now the isotropy subgroup $H_d$ of a space-like line takes the form

\[
H_d = \exp(v^2 \rho(K_2)) \exp(v^1 \rho(K_1)) \exp(v^3 \rho(J_3)) \exp(u^3 \rho(P_3)).
\]

Therefore $(u^r, u^s, u^l, \pi^0, \pi^1, \pi^2)$ define a set of coordinates on the 6D space $\mathcal{W}_d$ (see (3.12)). It is useful to observe that, at the level of the Lie group, the four ‘spacetime coordinates’ $x^\mu$ in (3.14) can be expressed in terms of the seven ‘space-like geodesic coordinates’ $(u^r, \pi^0, \pi^1, \pi^2)$ in (4.14) as

\[
\begin{align*}
    x^0 &= u^0 \cosh \pi^0 + u^3 \sinh \pi^0, \\
    x^1 &= u^1 \cos \pi^1 + u^2 \sin \pi^1 \sin \pi^2 + \sin \pi^1 \cos \pi^2 (u^0 \sinh \pi^0 + u^3 \cosh \pi^0), \\
    x^2 &= u^2 \cos \pi^2 - \sin \pi^2 (u^0 \sinh \pi^0 + u^3 \cosh \pi^0), \\
    x^3 &= \cos \pi^1 \cos \pi^2 (u^0 \sinh \pi^0 + u^3 \cosh \pi^0) - u^1 \sin \pi^1 + u^2 \cos \pi^1 \sin \pi^2.
\end{align*}
\]

Note that, similarly to the time-like case, these relations do not only involve the six coordinates on the space of space-like geodesics $\mathcal{W}_d$, but also the ‘extra’ one $u^3$; in fact the linear approximation of (4.16) gives that $\dot{x}^0 = u^0$. The remaining translation coordinates $(\pi^0, \pi^1, \pi^2)$ in (4.14) correspond, at this first-order, to $(\xi^3, \theta^2, \theta^3)$ in (3.14).

In the rest of this section, we present the explicit noncommutative algebra of quantum space-like geodesics, obtained as the quantization of the three Poisson homogeneous structures on $\mathcal{W}_d$ which, in turn, is constructed by using the canonical projection $G_{\mathcal{W}_d} \to \mathcal{W}_d$ and the fact that all these Poisson structures are coisotropic with respect to this quotient.

The three noncommutative algebras of operators share a number of characteristics, which make them formally similar to the structures presented in the previous section.Remarkably, all of them share the same subset of vanishing commutators

\[
[\hat{x}^m, \hat{x}^n]_k = 0, \quad m, n = 0, 1, 2, \quad [\hat{u}^m, \hat{x}^n]_k = 0, \quad m \neq n,
\]

where the label $k \in \{t, s, l\}$ refers to the time-, space, and light-like $r$-matrix considered as in the previous section. These relations will be essential in order to avoid any ordering ambiguity under quantization.

4.2.1. From the time-like $\kappa$-deformation. We start from the $r$-matrix $r_{\text{time}}$ of table 1 and after quantization of its coisotropic Poisson homogeneous structure we obtain that the quantum
space of geodesics $\mathcal{W}_{\lambda,\kappa}$ is defined by the following non-vanishing commutators

\[
\begin{align*}
[\hat{u}^0, \hat{u}^1]_t &= -\frac{1}{\kappa} \hat{u}^1 \cosh \pi^0, \\
[\hat{u}^0, \hat{u}^2]_t &= -\frac{1}{\kappa} \hat{u}^2 \cosh \pi^0, \\
[\hat{u}^1, \hat{u}^2]_t &= \frac{1}{\kappa} \hat{u}^1 \tanh \pi^0 \tan \pi^2, \\
[\hat{u}^0, \pi^0]_t &= -\frac{1}{\kappa} \sinh \pi^0, \\
[\hat{u}^1, \pi^1]_t &= \frac{1}{\kappa} \tanh \pi^0, \\
[\hat{u}^2, \pi^2]_t &= -\frac{1}{\kappa} \tan \pi^0,
\end{align*}
\]

(4.18)

The corresponding Poisson homogeneous structure can be expressed in canonical form on the 6D submanifold $\pi^0 \neq 0$ by means of the following change to Darboux-type coordinates

\[
\begin{align*}
q^0_t &= -\frac{u^0}{\sinh \pi^0}, \\
q^1_t &= \frac{u^1 \cosh \pi^0 \cos \pi^2}{\sinh \pi^0}, \\
q^2_t &= -\frac{u^2 \cosh \pi^0}{\sinh \pi^0}, \\
p^m &= \pi^m,
\end{align*}
\]

(4.19)

such that a $\kappa$-canonical Poisson structure is obtained:

\[
\begin{align*}
\{q^0_t, q^1_t\} &= \{p^m, p^n\} = 0, & \{q^m_t, p^n\} &= \frac{1}{\kappa} \delta_{mn}, & m, n = 0, 1, 2.
\end{align*}
\]

(4.20)

4.2.2. From the space-like $\kappa$-deformation. From the $\kappa$-Poincaré $r$-matrix $r_{\text{space}}$ we obtain, after quantization, that the noncommutative algebra of space-like geodesics $\mathcal{W}_{\lambda,\kappa}$ is defined by the non-vanishing brackets

\[
\begin{align*}
[\hat{u}^0, \hat{u}^1]_s &= -\frac{1}{\kappa} (\hat{u}^0 \sin \pi^1 - \hat{u}^1 \tan \pi^0), \\
[\hat{u}^0, \hat{u}^2]_s &= -\frac{1}{\kappa} (\hat{u}^0 \cos \pi^1 \sin \pi^2 + \hat{u}^2 \tan \pi^0), \\
[\hat{u}^1, \hat{u}^2]_s &= \frac{1}{\kappa} \left( \hat{u}^1 \tan \pi^2 \right. \\
&\left. + \hat{u}^2 \sin \pi^1 \right), \\
[\hat{u}^0, \pi^0]_s &= \frac{1}{\kappa} (\cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1), \\
[\hat{u}^1, \pi^1]_s &= -\frac{1}{\kappa} \left( \cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1 \right), \\
[\hat{u}^2, \pi^2]_s &= \frac{1}{\kappa} \left( \cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1 \right).
\end{align*}
\]

(4.21)
In this case the Poisson homogeneous structure is non-degenerate on the 6D submanifold \((\pi^0, \pi^1, \pi^2) \neq (0, 0, 0)\), and the Darboux-type coordinates are given by

\[
\begin{align*}
q^0_s &= \frac{u^0}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1}, \\
q^1_s &= -\frac{u^1 \cosh \pi^0 \cos \pi^2}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1}, \\
q^2_s &= \frac{u^2 \cosh \pi^0}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - 1}, \\
\rho^m &= \pi^m,
\end{align*}
\]

(4.22)

verifying again the canonical Poisson brackets (4.20).

4.2.3. From the light-like \(\kappa\)-deformation. Finally, for the \(r\)-matrix \(r_{\text{light}}\) we obtain a Lie bialgebra and therefore, a Poisson homogeneous structure, that is the linear superposition of the ones associated with the time- and space-like deformations that we have just presented. Upon quantization, the non-vanishing brackets defining the quantum space \(\mathcal{W}_{\text{ld, r}}\) read

\[
\begin{align*}
[\hat{u}^0, \hat{u}^1]_l &= -\frac{1}{\kappa} \left( \hat{u}^0 \sin \hat{\pi}^1 - \hat{u}^1 (\sinh \hat{\pi}^0 - 1) \right), \\
[\hat{u}^0, \hat{u}^2]_l &= \frac{1}{\kappa} \left( \hat{u}^0 \cos \hat{\pi}^1 \sin \hat{\pi}^2 + \hat{u}^2 (\sinh \hat{\pi}^0 - 1) \right), \\
[\hat{u}^1, \hat{u}^2]_l &= \frac{1}{\kappa} \left( \hat{u}^1 (\sin \hat{\pi}^0 + 1) \frac{\cosh \hat{\pi}^0}{\sin \hat{\pi}^1} + \hat{u}^2 \sin \hat{\pi}^1 \right), \\
[\hat{u}^0, \hat{\pi}^0]_l &= \frac{1}{\kappa} (\cosh \hat{\pi}^0 \cos \hat{\pi}^1 \cos \hat{\pi}^2 - \sinh \hat{\pi}^0 - 1), \\
[\hat{u}^1, \hat{\pi}^0]_l &= -\frac{1}{\kappa} \left( \cosh \hat{\pi}^0 \cos \hat{\pi}^1 \cos \hat{\pi}^2 - \sinh \hat{\pi}^0 - 1 \right), \\
[\hat{u}^2, \hat{\pi}^0]_l &= \frac{1}{\kappa} \left( \cosh \hat{\pi}^0 \cos \hat{\pi}^1 \cos \hat{\pi}^2 - \sinh \hat{\pi}^0 - 1 \right).
\end{align*}
\]

(4.23)

Similarly to the previous case, this Poisson homogeneous structure is symplectic when restricted to the 6D submanifold \((\pi^0, \pi^1, \pi^2) \neq (0, 0, 0)\), with Darboux coordinates given by

\[
\begin{align*}
q^0_l &= \frac{u^0}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - \sinh \pi^0 - 1}, \\
q^1_l &= -\frac{u^1 \cosh \pi^0 \cos \pi^2}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - \sinh \pi^0 - 1}, \\
q^2_l &= \frac{u^2 \cosh \pi^0}{\cosh \pi^0 \cos \pi^1 \cos \pi^2 - \sinh \pi^0 - 1}, \\
\rho^m &= \pi^m,
\end{align*}
\]

(4.24)

fulfilling once more (4.20).

If we consider the first-order in the quantum coordinates \((\hat{u}^m, \hat{\pi}^m)\) \((m = 0, 1, 2)\) in the three quantum spaces \(\mathcal{W}_{\text{ld, r}}\) (4.18), (4.21) and (4.23), we recover, as expected, the first-order non-commutative spaces given in the third row in table 4, by taking into account that in such linear
approximation \((\hat{\theta}^m, \hat{\xi}^l, \hat{\theta}^2, \hat{\theta}^1)\) coincide with \((\hat{u}^m, \hat{\pi}^0, \hat{\pi}^1, \hat{\pi}^2)\), respectively. In addition, when the above Poisson structures from the three \(r\)-matrices are calculated for the classical spacetime coordinates \(x^o\) when expressed as the functions (4.16), the Poisson version of the three linear \(\kappa\)-Minkowski spacetimes shown in the first row of table 4 is recovered, thus showing the complete self-consistency of this approach.

It is interesting to note that a similar relation to (4.13) is satisfied among the Darboux coordinates (4.19), (4.22) and (4.24) for the three different \(\kappa\)-Poincaré Poisson homogeneous structures on the space of space-like geodesics, namely

\[
\frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2}, \quad m = 0, 1, 2.
\]  

(4.25)

4.3. Quantum light-like geodesics

For the sake of completeness, we summarize here the defining relations for the 5D noncommutative space of light-like geodesics \(L_\kappa\) defined by the classical \(r\)-matrix \(r_{\text{light}}\) in table 1, which has recently been constructed and analysed in [34]. We insist on the fact that this is the only light-like space of geodesics that can be constructed, since \(r_{\text{time}}\) and \(r_{\text{space}}\) do not provide a coisotropic Poisson homogeneous structure amenable to be quantized.

In this case the Poincaré group representation is constructed in the null-plane basis (3.16) and reads

\[
G_\kappa = \exp\left(y^- \rho(P_-)\right) \exp\left(f^1 \rho(F_1)\right) \exp\left(f^2 \rho(F_2)\right) \exp\left(y^1 \rho(P_1)\right) \times \exp\left(y^2 \rho(P_2)\right) H_{ll},
\]  

(4.26)

where \(H_{ll}\) is the isotropy subgroup of a light-like line, which in this case reads (see [34] for details)

\[
H_{ll} = \exp\left(y^+ \rho(P_+)\right) \exp\left(e^1 \rho(E_1)\right) \exp\left(e^2 \rho(E_2)\right) \exp\left(\xi^3 \rho(K_3)\right) \times \exp\left(\delta^3 \rho(L_3)\right).
\]  

(4.27)

Then the generators \(\{P_-, P_j, F_i\}\) \((i, j = 1, 2)\) give rise to translations on the 5D homogeneous space \(L = G/H_{ll}\) (3.4) which is thus parameterized in terms of the coordinates \((\gamma^-, \gamma^i, \gamma^j)\). Then the coisotropic Poisson homogeneous structure can be straightforwardly computed and quantized, giving rise to the following commutation rules that define the quantum space of \(\kappa\)-Poincaré light-like geodesics \(L_\kappa\):

\[
[\hat{\gamma}^-, \hat{\gamma}^i] = -\frac{2}{\kappa} \hat{f}^i \hat{\gamma}^-, \quad [\hat{\gamma}^i, \hat{\gamma}^j] = \frac{2}{\kappa} \left(\hat{f}^1 \hat{\gamma}^2 - \hat{f}^2 \hat{\gamma}^1\right),
\]

\[
[\hat{\gamma}^i, \hat{\gamma}^j] = \frac{1}{\kappa} \delta_{ij} \left((\hat{f}^1)^2 + (\hat{f}^2)^2\right), \quad [\hat{\gamma}^-, \hat{\gamma}^i] = [\hat{f}^1, \hat{\gamma}^j] = 0, \quad i, j = 1, 2.
\]  

(4.28)

Note that all these commutators are given by homogeneous quadratic expressions, and therefore their linearization is zero, as table 4 predicts. In fact, this is an illustrative instance of the fact that, in some cases, the linear approximation to quantum spaces given by the dual of the cocommutator map provides a very limited amount of information. Finally it is worth mentioning that, as it was shown in [34], on a given irreducible representation fixed by a value of the Casimir operator \(\hat{C}\) of the algebra (4.28), a nonlinear change of basis transforms (4.28) into
\[ [\hat{q}^i, \hat{p}^j] = \frac{1}{\kappa} \delta_{ij}, \quad [\hat{y}^-, \hat{p}^j] = 0, \quad [\hat{y}^-, \hat{q}^i] = -\frac{2C}{\kappa} \hat{p}^i, \quad i = 1, 2. \] (4.29)

Therefore, the (5D) algebra \( L_{\kappa} \) can be interpreted as a non-central extension, generated by \( \hat{y}^- \), of a direct sum of two Heisenberg–Weyl algebras \((\hat{q}^i, \hat{p}^j)\), while all the (6D) noncommutative spaces of time- and space-like lines previously constructed can be written as the direct sum of three Heisenberg–Weyl algebras.

5. Concluding remarks

This paper has been focused on showing that the construction of noncommutative spaces of geodesics with quantum group symmetry can be explicitly performed by quantizing the coisotropic Poisson homogeneous spaces of geodesics that can be associated to each quantum deformation via its underlying classical \( r \)-matrix. Here we have explicitly considered the quantum geodesics on Minkowski spacetime arising from the \( \kappa \)-Poincaré quantum group, but the method is fully applicable to any other quantum deformation of the Poincaré group (for instance, to those ones endowed with a quantum Lorentz subgroup [45]), as well as to the spaces of geodesics and quantum deformations of other kinematical groups.

In this setting, the coisotropy condition for the Lie bialgebra associated to a given quantum deformation with respect to the isotropy subgroup of the chosen space of geodesics turns out to be the key property that allows the construction to be performed. When this condition is analyzed for the three possible \( \kappa \)-deformations of the Poincaré group, the first remarkable finding is the fact that the light-like or null-plane quantum Poincaré deformation is the only one that enables the construction of the three (time-, space- and light-like) quantum spaces of geodesics, since the time-like and space-like \( \kappa \)-deformations do not fulfill the coisotropy condition for the space of geodesics on the light cone.

Therefore, we have explicitly presented the seven possible \( \kappa \)-Poincaré coisotropic Poisson homogeneous spaces of geodesics and their quantization, by emphasizing among them the five which are new ones, and by sketching under a common framework the other two. From a mathematical viewpoint, we stress that this construction can be performed provided that for each space of geodesics a careful choice of the parameterization of the Poincaré group leading to suitable coordinates on the homogeneous space is found, and for the case of the light-like deformation the so called null-plane basis of the Poincaré algebra is the natural one to be used. In addition, we have also emphasized the fact that the seven quantum spaces of geodesics are all of them defined by nonlinear algebras which are non-trivial higher order generalizations of the first-order noncommutative spaces that can be directly obtained from the dual Lie bialgebra structure.

The results here presented open a number of questions which are worth to be faced in the near future. For instance, the finding that Darboux-type coordinates can be given within the spaces of geodesics paves the way to the analysis, following [38, 46], of the fuzzy properties of all the types of quantum worldlines here introduced as well as the fuzziness of the events that can be defined from them. In particular, this would be specially interesting for the three quantum spaces of worldlines that can be obtained from the null-plane deformation, since in this case the quantum space of light-like geodesics should be obtained as a common ‘boundary’ of the time- and space-like spaces. Moreover, the construction of the spaces of noncommutative (anti-)deSitter geodesics which are covariant under the already known \( \kappa \)-(A)dS quantum groups (see [47–50]) could be achieved by following the very same approach here presented,
and in this way the role of a nonvanishing cosmological constant could be analysed in this novel noncommutative geometric setting. Work on all these lines is in progress.

Acknowledgments

This work has been partially supported by Agencia Estatal de Investigación (Spain) under Grant PID2019-106802GB-I00/AEI/10.13039/501100011033, by the Regional Government of Castilla y León (Junta de Castilla y León, Spain) and by the Spanish Ministry of Science and Innovation MICIN and the European Union NextGenerationEU/PRTR. The authors would like to acknowledge the contribution of the European Cooperation in Science and Technology COST Action CA18108.

Data availability statement

No new data were created or analysed in this study.

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References

[1] Drinfel’d V G 1987 Quantum groups Proc. Int. Congr. Math. (Berkeley 1986) ed A Gleason (Providence, RI: American Mathematical Society) pp 798–820
[2] Chari V and Pressley A 1994 A Guide to Quantum Groups (Cambridge: Cambridge University Press)
[3] Majid S 1995 Foundations of Quantum Group Theory (Cambridge: Cambridge University Press)
[4] Majid S 1988 Hopf algebras for physics at the Planck scale Class. Quantum Grav. 5 1587–606
[5] Snyder H 1947 Quantized space-time Phys. Rev. 71 38–41
[6] Doplicher S, Fredenhagen K and Roberts J E 1994 Spacetime quantization induced by classical gravity Phys. Lett. B 331 39–44
[7] Garay L J 1995 Quantum gravity and minimum length Int. J. Mod. Phys. A 10 145–65
[8] Szabo R J 2003 Quantum field theory on noncommutative spaces Phys. Rep. 378 207–99
[9] Amelino-Camelia G 2013 Quantum-spacetime phenomenology Living Rev. Relativ. 16 5
[10] Hossenfelder S 2013 Minimal length scale scenarios for quantum gravity Living Rev. Relativ. 16 2
[11] Addazi A et al 2022 Quantum gravity phenomenology at the dawn of the multi-messenger era—a review Prog. Part. Nucl. Phys. 125 103948
[12] Lukierski J, Ruegg H, Nowicki A and Tolstoy V N 1991 q-deformation of Poincaré algebra Phys. Lett. B 264 331–8
[13] Giller S, Kosinski P, Majewski M, Maslanka P and Kunz J 1992 More about the q-deformed Poincaré algebra Phys. Lett. B 286 57–62
[14] Lukierski J, Nowicki A and Ruegg H 1992 New quantum Poincaré algebra and κ-deformed field theory Phys. Lett. B 293 344–52
[15] Maslanka P 1993 The n-dimensional κ-Poincaré algebra and group J. Phys. A: Math. Gen. 26 L1251–3
[16] Majid S and Ruegg H 1994 Bicrossproduct structure of κ-Poincare group and non-commutative geometry Phys. Lett. B 334 348–54
[17] Zakrzewski S 1994 Quantum Poincare group related to the $\kappa$-Poincare algebra J. Phys. A: Math. Gen. 27 2075–82
[18] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1994 Four-dimensional quantum affine algebras and space-time $q$-symmetries J. Math. Phys. 35 4928–40
[19] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1995 A new null-plane quantum Poincare algebra Phys. Lett. B 351 137–45
[20] Ballesteros A, Bruno N R and Herranz F J 2003 A non-commutative Minkowskian spacetime from a quantum AdS algebra Phys. Lett. B 574 276–82
[21] Lukierski J and Woronowicz M 2006 New Lie-algebraic and quadratic deformations of Minkowski space from twisted Poincaré symmetries Phys. Lett. B 633 116–24

Borowiec A and Pachol A 2009 $\kappa$-Minkowski spacetime as the result of Jordanian twist deformation Phys. Rev. D 79 045012

Gubitosi G and Mercati F 2013 Relative locality in $\kappa$-Poincaré Class. Quantum Grav. 30 145002

Borowiec A and Pachol A 2014 $\kappa$-deformations and extended $\kappa$-Minkowski spacetimes Sigma 10 107

Ballesteros A, Gubitosi G and Mercati F 2021 Interplay between spacetime curvature, speed of light and quantum deformations of relativistic symmetries Symmetry 13 2099

Ballesteros A, Herranz F J, Olmo M A, Perena C M and Santander M 1995 Non-standard quantum $\left(1 + \frac{1}{1}\right)$ Poincare group: a $T$-matrix approach J. Phys. A: Math. Gen. 28 7113–25

Ballesteros A, Herranz F J and Pereña C M 1997 Null-plane quantum universal $R$-matrix Phys. Lett. B 391 71–7

Arratia O, Herranz F J and Olmo M A 1998 Bicrossproduct structure of the null-plane quantum Poincare algebra J. Phys. A: Math. Gen. 31 L1–7

Lu J 1990 Multiplicative and affine Poisson structures on Lie groups PhD Thesis

Ciccoli N and Gavarini F 2006 A quantum duality principle for coisotropic subgroups and Poisson quotients Adv. Math. 199 104–35

Ballesteros A, Meusburger C and Naranjo P 2017 AdS Poisson homogeneous spaces and Drinfel’d doubles J. Phys. A: Math. Theor. 50 395202

Ballesteros A, Gutierrez-Sagredo I and Mercati F 2021 Coisotropic Lie bialgebras and complementary dual Poisson homogeneous spaces J. Phys. A: Math. Theor. 54 315203

Ballesteros A, Gutierrez-Sagredo I and Herranz F J 2019 Noncommutative spacetimes of worldlines Phys. Lett. B 792 175–81

Ballesteros A, Gutierrez-Sagredo I and Herranz F J 2022 The noncommutative space of light-like worldlines Phys. Lett. B 829 137120

Low R J 1989 The geometry of the space of null geodesics J. Math. Phys. 30 809–11

Beem J K and Parker P E 1991 The space of geodesics Geom. Dedicata 38 87–99

Herranz F J and Santander M 1998 Homogeneous phase spaces: the Cayley–Klein framework Geometría y Física. Memorias la Real Acad. Ciencias vol XXXII ed J F Cariñena, E Martinez and M F Rañada (Madrid) pp 59–84

Ballesteros A, Gubitosi G, Gutierrez-Sagredo I and Mercati F 2021 Fuzzy worldlines with $\kappa$-Poincaré symmetries J. High Energy Phys. JHEP12(2021)080

Leutwyler H and Stern J 1978 Relativistic dynamics on a null plane Ann. Phys., NY 112 94–164

Zakrzewski S 1997 Poisson structures on the Poincaré group Commun. Math. Phys. 185 285–311

Zakrzewski S 1995 Poisson Poincaré groups Quantum Groups, Formalism and Applications ed J Lukierski, Z Popowicz and J Sobczyk (Warsaw: Polish Scientific Publishers PWN) pp 433–9

Podleś P and Woronowicz S L 1996 On the classification of quantum Poincaré groups Commun. Math. Phys. 178 61–82

Vaksman L L and Korogodsky L I 1989 The algebra of bounded functions on the quantum group of motions of the plane and $q$-analogues of Bessel functions Sov. Math. Dokl. 39 173–7

Gutierrez-Sagredo I and Herranz F J 2021 Cayley–Klein Lie bialgebras: noncommutative spaces, Drinfel’d doubles and kinematical applications Symmetry 13 1249

Ballesteros A, Gutierrez-Sagredo I and Herranz F J 2022 Noncommutative (AdS and Minkowski spacetimes from quantum Lorentz subgroups Class. Quantum Grav. 39 015018

Lizzi F, Manfredonia M, Mercati F and Poulain T 2019 Localization and reference frames in $\kappa$-Minkowski spacetime Phys. Rev. D 99 085003
[47] Ballesteros A, Herranz F J, Olmo M A and Santander M 1994 Quantum (2 + 1) kinematical algebras: a global approach J. Phys. A: Math. Gen. 27 1283–97
[48] Ballesteros A, Herranz F J, Musso F and Naranjo P 2017 The κ-(A)dS quantum algebra in (3 + 1) dimensions Phys. Lett. B 766 205–11
[49] Ballesteros A, Gubitosi G, Gutierrez-Sagredo I and Herranz F J 2017 Curved momentum spaces from quantum (anti-)de Sitter groups in (3 + 1) dimensions Phy. Rev. D 97 106024
[50] Ballesteros A, Gutierrez-Sagredo I and Herranz F J 2019 The κ-(A)dS noncommutative spacetime Phys. Lett. B 796 93–101