On Girsanov’s transform for backward stochastic differential equations

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Abstract. By using a simple observation that the density processes appearing in Itô’s martingale representation theorem are invariant under the change of measures, we establish a non-linear version of the Cameron-Martin formula for solutions of a class of systems of quasi-linear parabolic equations with non-linear terms of quadratic growth. We also construct a local stochastic flow and establish a Bismut type formula for such system of quasi-linear PDEs. Gradient estimates are obtained in terms of the probability representation of the solution. Another interesting aspect indicated in the paper is the connection between the non-linear Cameron-Martin formula and a class of forward-backward stochastic differential equations (FBSDEs).

Key words. Brownian motion, backward SDE, SDE, non-linear equations

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1 Introduction

The goal of the article is to present a unified approach for Girsanov’s techniques of changing probability measures used in the recent literature on backward stochastic differential equations (BSDE). The framework we formulate has an advantage allowing us to bring together seemingly different sorts of results on BSDE, forward-backward stochastic differential equations (FBSDE), and function equations on a probability space.

Consider the following system of quasi-linear parabolic equations

\[
\frac{\partial}{\partial t} u^i = \frac{1}{2} \Delta u^i + \sum_{j=1}^{d} f^j(u, \nabla u) \frac{\partial u^i}{\partial x^j} \quad \text{in } \mathbb{R}^d
\]

\(i = 1, \ldots, m\), where the drift vector field involves a solution \(u\) and its total derivative \(\nabla u = (\frac{\partial u^i}{\partial x^j})\) through \(f = (f^j)_{j \leq d}\). In the interesting cases, \(f^j : \mathbb{R}^d \times \mathbb{R}^{m \times d} \to \mathbb{R} (j = 1, \ldots, d)\) are Lipschitz continuous but unbounded, and therefore the non-linear term (often called the convection term) appearing in (1.1) is of quadratic growth. This is a kind of non-linear feature which appears in many physical PDEs, see for example [5] and [11]. If \(m = d = 3\) and \(f^j(u) = u^j\), then (1.1) is a modification of the Navier-Stokes equations with the pressure term and the divergence-free condition dropped altogether, while the same non-linear convection term has been retained. This kind of PDEs has been used as simplified models for phenomena such as turbulence flows. Due to the special structure of the system (1.1), the maximum principle applies to \(|u(x, t)|^2\), thus a bounded solution exists as long as the initial data \(u(x, 0)\) is regular and bounded, which makes a distinctive difference from the Navier-Stokes equations. According to Theorem 7.1 on page 596 in [8], if the initial data \(u_0\) is smooth and bounded with bounded derivatives, then a bounded, smooth solution \(u\) to the initial value problem of (1.1) exists for all time. Our main interest in this article is to establish probabilistic representations for the solution \(u\) by applying Girsanov’s theorem to BSDEs.

To this end, it will be a good idea to look at Peng’s non-linear Feynmann-Kac formula (see [15]) for general quasi-linear parabolic equations with Lipschitz non-linear term. The main idea in [14] and [15] can be described as following. Thanks to Ito’s calculus, Brownian motion \(B = (B^1, \ldots, B^d)\) may be considered as “coordinates” on the space of continuous paths equipped with the Wiener measure, and it is a much tested idea that one may “read out” solutions of quasi-linear partial differential equations in terms of \(B\).
Suppose \( u(x, t) = (u^1(x, t), \cdots, u^n(x, t)) \) is a sufficiently smooth solution to the Cauchy problem of the system of quasi-linear parabolic equations

\[
\left( \frac{1}{2} \Delta - \frac{\partial}{\partial t} \right) u^i + h^i(u, \nabla u) = 0 \quad \text{in } \mathbb{R}^d \times [0, \infty)
\]

with the initial data \( u^i(x, 0) = u^i_0(x) \). Applying Itô’s formula to \( Y^i_t = u^i(x + B_t, T - t) \) (and set \( Z_{ij}^i = \frac{\partial u^i}{\partial x^j}(x + B_t, T - t) \)) to obtain

\[
Y^i_T - Y^i_t = \sum_{j=1}^d \int_t^T \frac{\partial u^i}{\partial x^j}(x + B_s, T - s)dB^j_s \\
+ \int_t^T \left( \frac{1}{2} \Delta - \frac{\partial}{\partial s} \right) u^i(x + B_s, T - s) ds \\
= \sum_{j=1}^d \int_t^T Z_{ij}^i dB^j_s - \int_t^T h^i(Y_s, Z_s) ds
\]

for \( t \in [0, T] \). The pair \((Y, Z)\) is a solution to the stochastic differential equations

\[
dY^i_t = -h^i(Y_t, Z_t)dt + \sum_{j=1}^d Z_{ij}^i dB^j_t
\]

(1.2)
on \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with the terminal data \( Y^i_T = u^i_0(x + B_T) \).

Peng [15] and Pardoux-Peng [14] made two crucial observations: firstly the process \( Z_t \) may be recovered from the Itô representation of the martingale \( S^i_t = S^i_0 + \int_0^t \sum_{j=1}^d Z_{ij}^i dB^j_s \). Secondly \( S \) is indeed the martingale part of the continuous semimartingale \( Y \), therefore (1.2) is a closed system and may be solved by specifying a terminal value \( Y_T \). It was proved in Pardoux-Peng [14] that if \( h^i \) are global Lipschitz continuous, then (1.2) may be solved as long as \( Y_T \in L^2(\Omega, \mathcal{F}_T, P) \).

In the case that \( h^i(y, z) \) has a special form such as \( \sum_{j=1}^d f^i_j(y, z) z^j \), one is able to solve (1.2) by firstly solve the trivial BSDE \( dY^i_t = \sum_{j=1}^d Z_{ij}^i dB^j_t \) then change the probability measure, even for non-Lipschitz non-linear term \( h \) as long as \( f \) is Lipschitz continuous. We are thus able to extend the Cameron-Martin formula to a class of systems of quasi-linear parabolic equations with quadratic growth. The non-linear version of the Cameron-Martin formula, which may be considered as our contribution of this article, is of independent interest.
Suppose that the initial data $u_0$ is Lipschitz continuous and bounded. Let $(\mathcal{F}_t)_{t \geq 0}$ be Brownian filtration, i.e. the filtration over $W^d$ of all continuous paths in $\mathbb{R}^d$ generated by the coordinate process $\{B_t : t \geq 0\}$, augmented by the Wiener measure. Let $T > 0$ and $x \in \mathbb{R}^d$ be fixed but arbitrary. For each $\xi = (\xi^1, \ldots, \xi^m)$ where $\xi^i \in L^\infty(W^d, \mathcal{F}_T, P^x)$. Define

$$\tilde{B}(\xi)_t = B_t - \int_0^t f(Y(\xi)_s, Z(\xi)_s)ds$$

where $Y(\xi)_t = E^P(\xi|\mathcal{F}_t)$ are bounded martingales and $Z(\xi) = (Z(\xi)^{i,j})$ are determined by Itô’s martingale representation:

$$\xi^i - E^P(\xi^i|\mathcal{F}_0) = \sum_{j=1}^d \int_0^T Z(\xi)^{i,j}_s dB^j_s.$$

We prove that there is a unique $\xi \in L^\infty(W^d, \mathcal{F}_T, P^x)$ such that $\xi = u_0(\tilde{B}(\xi)_T)$, and $u(x, T) = E^P(Y(\xi)_0)$, which can be considered as the non-linear version of the Cameron-Martin formula. More information about the solution $u$ may be obtained as we can represent the derivative $\frac{\partial u}{\partial x}$ in terms of the process $(Z(\xi)^{i,j})_{t \leq T}$, see Theorem 3.1 below. The non-linear version of the Cameron-Martin formula may be reformulated in terms of FBSDE as well, see (2.4) and Corollary 2.3.

The main reason we are interested in representations of solutions to physical PDEs in terms of Brownian motion or in general in terms of Itô’s diffusions, lies in the fact that it is then possible to employ probabilistic methods such as Itô’s calculus, Malliavin’s calculus of variations and path integration method to the study of non-linear PDEs. We demonstrate this point by deriving explicit gradient estimates for solutions of a class of systems of quasi-linear parabolic equations.

Finally we would like to point out that the type of BSDEs such as (1.2) with quadratic growth non-linear terms has been well studied in Kobylianski [7]. Her work has been extended and generalized substantially in Briand and Hu [3] and [2]. BSDEs with quadratic growth driven by martingales are solved by Morlais [13] and Tevzadze [16]. However their methods can be applied to scalar BSDEs, not systems in general.
2 Girsanov’s theorem, martingale representation and BSDE

Let $B = (B^1, \cdots, B^d)$ be a Brownian motion in $\mathbb{R}^d$ started at 0 on a complete probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $B$ augmented by probability zero sets in $\mathcal{F}$ and $\mathcal{F}_\infty = \sigma\{\mathcal{F}_t : t \geq 0\}$. Since we are interested in $\mathcal{F}_\infty$-measurable random variables only, for simplicity, we assume that $\mathcal{F} = \mathcal{F}_\infty$. According to Itô’s martingale representation theorem, any martingale $S$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is continuous and has a unique representation in terms of Itô’s integration

$$S_t - S_0 = \sum_{j=1}^{d} \int_0^t D_B(S)^j_s dB_s^j$$

where $D_B(S)^j$ are predictable processes, called the density processes of $S$ with respect to Brownian motion $B$. Since we will deal with several equivalent measures on $(\Omega, \mathcal{F})$ at the same time, it is desirable to have labels associated with notations which involve a probability measure. We will follow this convention if confusions may arise. Therefore, $E^P$ and $E^P\{\cdot | \mathcal{F}_t\}$ denote the expectation and conditional expectation with respect to $P$ respectively.

Let $Q$ be a probability measure on $(\Omega, \mathcal{F})$ equivalent to $P$, whose Radon-Nikodym’s derivative with respect to $P$ restricted on $\mathcal{F}_t$ is denoted by $R_t$, that is, $\frac{dQ}{dP}|_{\mathcal{F}_t} = R_t$ for $t \geq 0$. Then $R$ is a positive martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $E^P(R_t) = 1$. Girsanov’s theorem \cite{Girsanov} establishes a correspondence between local $P$-martingales and local $Q$-martingales. If $X = (X_t)_{t \geq 0}$ is a local martingale under probability $P$, then $X$ is a continuous semi-martingale under $Q$ and $\tilde{X}_t = X_t - \int_0^t \frac{1}{R_s} d\langle X, R \rangle_s$ is a local martingale under $Q$, where the bracket process $\langle X, R \rangle_t$ is defined under the probability $P$, which is however invariant under the change of equivalent probability measure.

It is convenient to formulate the Girsanov’s transform in terms of exponential martingales. The exponential martingale $\mathcal{E}(N)_t = \exp\{N_t - \frac{1}{2} \langle N \rangle_t\}$ of a continuous local martingale $N$ (under $P$) is the unique solution to the stochastic exponential equation

$$\mathcal{E}(N)_t = 1 + \int_0^t \mathcal{E}(N)_s dN_s.$$ 

Up to an initial data $R_0$, the Radon-Nikodym derivative $R = (R_t)_{t \geq 0}$ of $Q$ with respect to the measure $P$ must be an exponential martingale of some
continuous local martingale \( N \), so that \( R_t = R_0 \exp \{ N_t - \frac{1}{2} \langle N \rangle_t \} \). Then \( \tilde{X} = X - \langle X, N \rangle \) for any \( P \)-local martingale \( X \).

According to Lévy’s theorem, \( \tilde{B} = (\tilde{B}^1, \ldots, \tilde{B}^d) \) is a Brownian motion under \( Q \) which is \( (\mathcal{F}_t)_{t \geq 0} \)-adapted, but the natural filtration \( (\tilde{\mathcal{F}}_t)_{t \geq 0} \) of \( \tilde{B} \) may not coincide with \( (\mathcal{F}_t)_{t \geq 0} \). It can happen that \( \tilde{\mathcal{F}}_t \) is strictly smaller than \( \mathcal{F}_t \) for some \( t \). But, nevertheless any \( (\mathcal{F}_t) \)-martingale under \( Q \) may be represented as Itô’s integral of \( (\mathcal{F}_t) \)-predictable processes against \( \tilde{B} \). The starting point of our approach is the following elementary fact.

**Lemma 2.1** For any local \( P \)-martingale \( X \) (with respect to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) \( D_B(X) = D_B(\tilde{X}) \). That is, the density process appearing in Itô’s martingale representation is invariant under the change of equivalent probability measures.

The lemma follows directly from the definition of density processes and the Girsanov’s theorem, so its proof is left to the reader. With the help of this simple fact, we can deal with the question of how a backward stochastic differential equation is transformed under change of equivalent probability measures.

**Theorem 2.2** Suppose that \( g \) is global Lipschitz continuous, \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), and \( (Y, S) \) is the unique solution pair to BSDE

\[
dY_t = -g(t, Y_t, D_B(S)_t)dt + dS_t, \quad Y_T = \xi
\]

on \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \). Let \( \tilde{S} = S - \langle S, N \rangle \). Then \( (Y, \tilde{S}) \) solves

\[
dY_t = -g(t, Y_t, D_B(\tilde{S})_t)dt + \sum_{j=1}^d D_B(\tilde{S})_t^j D_B(N)_t^j dt + d\tilde{S}_t
\]

on \( (\Omega, \mathcal{F}, \mathcal{F}_t, Q) \), \( Y_T = \xi \).

**Proof.** Since \( (Y, S) \) is a solution to (2.1), \( Y^i \) are continuous semimartingale with martingale parts \( S^j \). By Lemma 2.1, \( D_B(S^i) = D_B(\tilde{S}^i) \), so that

\[
\tilde{S}^i_t - \tilde{S}^i_0 = \sum_{j=1}^d \int_0^t D_B(\tilde{S}^i)_{s}^j d\tilde{B}^j_s = \sum_{j=1}^d \int_0^t D_B(S^i)_{s}^j d\tilde{B}^j_s
\]
and

\[ \tilde{S}_t^i = S_t^i - \sum_{j=1}^{d} \int_0^t D_{\tilde{B}}(\tilde{S}_s^i)_{j} D_B(N)_s^j ds. \]

Therefore

\[ Y_T^i - Y_t^i = -\int_t^T g^i(s, Y_s, D_{\tilde{B}}(\tilde{S})_s) ds \]

\[ + \sum_{j=1}^{d} \int_t^T D_{\tilde{B}}(\tilde{S}_s^i)_{j} D_B(N)_s^j ds + \tilde{S}_T^i - \tilde{S}_t^i \quad (2.3) \]

which is valid under the probability \( Q \). That is, the pair \((Y, \tilde{S})\) solves BSDE (2.2) on \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\), with terminal values \( Y_T^i = \xi^i \). \( \blacksquare \)

The most interesting case is the following special choice of \( N \).

**Corollary 2.3** Under the same assumptions as in the previous theorem, and

\[ N_t = \sum_{j=1}^{d} \int_0^t f^j(Y_s, D_B(S)_s) dB^j_t \]

where \( f : \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}^d \) is Borel measurable. Then \((Y, \tilde{S})\) solves the BSDE

\[ dY_t = -g(t, Y_t, D_{\tilde{B}}(\tilde{S})_t) dt + \sum_{j=1}^{d} D_{\tilde{B}}(\tilde{S}_t)_{j} f^j(Y_t, D_B(S)_t) dt + d\tilde{S}_t \]

with the terminal value \( Y_T = \xi \) on \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\).

One may reformulate Corollary 2.3 in terms of forward-backward stochastic differential equations. Observe that, with the choice of \( N \) made in Corollary 2.3, \( X = x + \tilde{B} \) is the solution to the stochastic differential equation

\[ dX_t = -f(Y_s, Z_t) dt + dB_t, \quad X_t = x \]

while \((Y, Z = D_B(S))\) is the solution of the BSDE

\[ dY_t = -g(t, Y_t, Z_t) dt + Z_t dB_t, \quad Y_T = u_0(X_T), \]
thus \((X, Y, Z)\) is a solution to the following forward-backward stochastic differential equations (FBSDEs)

\[
\begin{align*}
  dX_t &= -f(Y_s, Z_t) dt + dB_t, \\
  dY_t &= -g(t, Y_t, Z_t) dt + Z_t dB_t, \\
  X_t &= x, \\ Y_T &= u_0(X_T).
\end{align*}
\] (2.4)

FBSDEs such as (2.4) have been studied by various authors, and are well presented in the research monograph [10]. By utilizing the fundamental apriori estimates established in [8], it has been proved in [9] that FBSDE (2.4) has a unique solution such that \(Y\) is bounded, as long as \(u_0\) is bounded and Lipschitz, and if \(f\) and \(g\) are Lipschitz continuous.

**Corollary 2.4** Let \(f\) and \(g\) be Lipschitz continuous, \(u_0\) be bounded and Lipschitz continuous. Let \((X, Y, Z)\) be the unique solution of (2.4) such that \(Y\) is bounded. Define \(Q\) on \((\Omega, \mathcal{F}_T, \mathbb{Q})\) by

\[
\frac{dQ}{dP}\bigg|_{\mathcal{F}_T} = \exp\left\{ \int_0^T f(Y_s, Z_s).dB_s - \frac{1}{2} \int_0^T |f(Y_s, Z_s)|^2 ds \right\}.
\]

Then \((Y, \tilde{Z})\) is the unique solution (such that \(Y\) is bounded) to

\[
\begin{align*}
  dY_t &= \left[-g(t, Y_t, \tilde{Z}_t) + \sum_{j=1}^d f^j(Y_t, \tilde{Z}_t) \tilde{Z}_t^j \right] dt + \sum_{j=1}^d \tilde{Z}_t^j d\tilde{B}_t^j, \\
  Y_T &= u_0(X_T).
\end{align*}
\] (2.5)

**3 Cameron-Martin formula for PDEs**

In this section we apply the machinery developed in the previous section to the initial value problem of the following quasi-linear parabolic system

\[
\frac{\partial u^i}{\partial t} + \sum_{j=1}^d f^j(u, \nabla u) \frac{\partial u^i}{\partial x^j} = \frac{1}{2} \Delta u^i + g^i(u, \nabla u) \quad \text{in } \mathbb{R}^d
\] (3.1)

for \(i = 1, \cdots, m\), where \(f^j\) and \(g^i\) are global Lipschitz continuous in their arguments, together with the initial data \(u(x, \cdot) = u_0(x)\) which is bounded and global Lipschitz continuous. According to Theorem 7.1 in [8], there is
a unique bounded solution $u(x, t)$ to the initial value problem of (3.1). By Itô’s formula, $Y_t = u(x + B_t, T - t)$ and $Z_t = \nabla u(x + B_t, T - t)$ solve the BSDE

$$
dY_t^i = \left[ -g^i(Y_t, Z_t) + \sum_{j=1}^{d} f^j(Y_t, Z_t)Z_t^{i,j} \right] dt + \sum_{j=1}^{d} Z_t^{i,j} dB_t^j \quad (3.2)
$$

with the terminal value $Y_T = u_0(x + B_T)$ over the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, which is indeed the unique bounded solution of BSDE (3.2).

On the other hand, since $g^i$ are global Lipschitz, according to Pardoux and Peng [14], for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ there is a unique solution pair $(Y, Z)$ to the backward stochastic differential equation:

$$
dY_t = -g(Y_t, Z_t)dt + \sum_{j=1}^{d} Z_t^j dB_t^j, \quad Y_T = \xi \quad (3.3)
$$

on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, which is the system derived from (3.2) with the quadratic non-linear term dropped. The solution $(Y, Z)$ depends on the terminal value $\xi$, so it is denoted by $Y(\xi)$ and $Z(\xi)$ respectively. For $x \in \mathbb{R}^d$ and $T > 0$, let

$$
N(\xi)_t = \sum_{j=1}^{d} \int_0^t f^j(Y(\xi)_s, Z(\xi)_s) dB_s^j \quad (3.4)
$$

and

$$
\tilde{B}(\xi)_t = B_t - \int_0^t f(Y(\xi)_s, Z(\xi)_s) ds. \quad (3.5)
$$

**Theorem 3.1** Let $u$ be a classical solution of (3.1) on $[0, T] \times \mathbb{R}^d$ such that $u(0, \cdot) = u_0(\cdot)$. Suppose that $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ is the solution to the function equation

$$
\xi = u_0(x + \tilde{B}(\xi)_T),
$$

then

$$
u(x + \tilde{B}(\xi)_t, T - t) = Y(\xi)_t
$$

and

$$
\nabla u(x + \tilde{B}(\xi)_t, T - t) = Z(\xi)_t
$$

for all $t \leq T$ almost surely.
Proof. Define an equivalent probability measure $Q$ by $\frac{dQ}{dP}|_{\mathcal{F}_t} = \mathcal{E}(N(\xi))_t$. Then $\tilde{B}(\xi)$ is a Brownian motion (up to time $T$) under $Q$. Let $\tilde{S} = S(\xi) - \langle N(\xi), S(\xi) \rangle$. According to Corollary 2.3, $(Y(\xi), \tilde{S})$ solves the following BSDE

$$dY_t = \sum_{j=1}^d D\tilde{B}(\tilde{S})_t f^j(Y_t, D\tilde{B}(\tilde{S})_t)dt - g(t, Y_t, D\tilde{B}(\tilde{S})_t)dt + d\tilde{S}_t$$

with terminal value $Y_T = \varphi(x + \tilde{B}(\xi)_T)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. This system is exactly the BSDE that $Y_t = u(x + \tilde{B}(\xi)_t, T - t)$ should satisfy, and the conclusions follow immediately.

By Theorem 3.1, in order to provide a probabilistic representation for (3.1), the problem is reduced to solve the function equation

$$\xi = u_0 \left( x + B_T - \int_0^T f(Y(\xi)_s, Z(\xi)_s) ds \right) := \phi(\xi). \quad (3.6)$$

The following local existence of the solutions to (3.6) is elementary.

**Proposition 3.2** Let $C_{u_0}$ and $C_f$ be the Lipschitz constants for $u_0$ and $f$ respectively, and

$$\tau = \sqrt{\frac{1}{8C_{u_0}^2 C_f^2} + 1} - 1.$$

If $T \leq \tau$, then (3.6) admits a unique fixed point $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$.

**Proof.** For $\xi$ and $\eta$ in $L^2(\Omega, \mathcal{F}_T, P)$,

$$|\phi(\xi) - \phi(\eta)| \leq C_{u_0} \int_0^T |f(Y(\xi)_s, Z(\xi)_s) - f(Y(\eta)_s, Z(\eta)_s)|ds$$

$$\leq C_{u_0} C_f \int_0^T |Y(\xi - \eta)_s| + |Z(\xi - \eta)_s|ds$$

and furthermore,

$$E \left[ |\phi(\xi) - \phi(\eta)|^2 \right] \leq C_{u_0}^2 C_f^2 E \left[ \left( \int_0^T |Y(\xi - \eta)_s| + |Z(\xi - \eta)_s|ds \right)^2 \right]$$

$$\leq 2C_{u_0}^2 C_f^2 T \left[ \int_0^T (E|Y(\xi - \eta)_s|^2 + E|Z(\alpha - \beta)_s|^2)ds \right].$$
Now, for any \( \eta \in L^2(\Omega, F_t, P) \),

\[
\|Y(\eta)_s\|_2^2 = E[|E(\eta|F_s)|^2] \leq \|\eta\|_2^2,
\]

and

\[
E \left[ \int_0^t |Z(\eta)_s|^2 ds \right] = E \left[ |Y(\eta)_t - Y(\eta)_0|^2 \right] 
\leq 2\|\eta\|_2^2,
\]

so that

\[
E \left[ |\phi(\xi) - \phi(\eta)|^2 \right] \leq 2C_u^2 C_f^2 T \left[ \int_0^T \|\xi - \eta\|_2^2 ds + 2\|\xi - \eta\|_2^2 \right] 
= 2C_u^2 C_f^2 T(T + 2)\|\xi - \eta\|_2^2.
\]

Hence

\[
||\phi(\xi) - \phi(\eta)||_2 \leq \sqrt{2}C_u C_f \sqrt{T(T + 2)}||\xi - \eta||_2
\]

and the claim follows from a simple application of the fixed point theorem.

\[\blacksquare\]

**Theorem 3.3** Under the previous assumptions, the function equation (3.6) has a unique solution.

**Proof.** We try to extend the local solution constructed in the previous proposition to the case \( T > \tau \). To do this we need a uniform gradient estimate of the solutions to PDE (3.1): there exists a constant \( C_u \) such that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |\nabla u|(x, t) \leq C_u.
\]

For the proof see for example [8] and [4]. Based on such constant \( C_u \) we define

\[
\tau' = \sqrt{\frac{1}{8C_u^2 C_f^2}} + 1 - 1.
\]

Now we construct the random variable \( \xi' \) on the time interval \([\tau, \tau + \tau']\). Consider the following function equation:

\[
\xi = u \left( x + B'_{\tau + \tau'} - \int_\tau^{\tau + \tau'} f(s, Y(\xi)_s, Z(\xi)_s) ds, \tau \right) := \phi'(\xi). \tag{3.7}
\]
where $B'$ is a Brownian motion on $[\tau, \tau + \tau']$ defined by $B'_s = B_s - B_\tau$ for $s \in [\tau, \tau + \tau']$. Analogous to the proof of Proposition 3.2, (3.7) admits a unique fixed point $\xi' \in L^\infty(\Omega, \mathcal{F}_{\tau+\tau'}, P) \subset L^2(\Omega, \mathcal{F}_{\tau+\tau'}, P)$. Based on such $\xi'$, we have the following representation formulae on $[\tau, \tau + \tau']$:

$u(x + \tilde{B}(\xi')_t, 2\tau + \tau' - t) = Y(\xi')_t$

and

$\nabla u(x + \tilde{B}(\xi')_t, 2\tau + \tau' - t) = Z(\xi')_t$

for $t \in [\tau, \tau + \tau']$, where

$\tilde{B}(\xi')_t = B'_t - \int_\tau^t f(u, Y(\xi')_u, Z(\xi')_u)du.$

We then move to the next interval $[\tau + \tau', \tau + 2\tau']$ and repeat the above procedure until we touch $T$.

\section{Some applications}

In this section we establish some explicit gradient estimates for the solution of (3.1) by using the representation theorem 3.1 to demonstrate the usefulness of non-linear Cameron-Martin’s formula. Further applications will appear in a separate paper.

Let us retain the notations and assumptions established in the previous section. Let $T > 0$ be fixed, $u = (u^1, \ldots, u^m)$ be the unique solution to the initial value problem (3.1) with initial data $u_0$, where $f$ (which determines the nature of the quadratic non-linear term) depends only on $y$, i.e. $f(y, z)$ does not depend on $z$ and $g = 0$. That is, $u$ is the solution to the initial value problem of the following system of quasi-linear parabolic equations

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^d f^j(u) \frac{\partial u^i}{\partial x^j} = \frac{1}{2} \Delta u^i \quad \text{in } \mathbb{R}^d$$

for $i = 1, \ldots, m$.

Let $\xi$ be the solution to the function equation (3.6) established in Theorem 3.3.

Let $Q$ be the equivalent measure with density process $\frac{dQ}{dP} |_{\mathcal{F}_t} = R_t = \mathcal{E}(N(\xi))_t$, where

$N(\xi)_t = \int_0^t f(Y(\xi))_.dB_.$. 

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Theorem 4.1 1) Let \( p \in [1, 2) \) and \((P_t)_{t \geq 0}\) the heat semi-group, i.e. \( P_t = e^{\frac{t}{2} \Delta} \). Then

\[
\int_0^t P_s |\nabla u_i|^p (x, T - s) \, ds \leq d^{1 - \frac{p}{2}} (E^P \langle Y(\xi)^i \rangle_t)^\frac{p}{2} \left( \int_0^t E^P R_s^{\frac{p}{2}} \, ds \right)^{1 - \frac{p}{2}}
\]

for any \( t \leq T, \, i = 1, \cdots, m \).

2) We have

\[
|\nabla u_i|^2 (x, T) \leq \lim_{t \to 0} \frac{1}{t} E^P \langle Y(\xi)^i \rangle_t
\]

for any \( t \leq T, \, i = 1, \cdots, m \).

Proof. By Theorem 3.1

\[
Y(\xi)^i_t - Y(\xi)^i_0 = \sum_{j=1}^d \int_0^t \frac{\partial u_i}{\partial x_j} (x + \bar{B}(\xi)_s, T - s) dB_s^j
\]

so that

\[
\langle Y(\xi)^i \rangle_t = \int_0^t |\nabla u_i|^2 (x + \bar{B}(\xi)_s, T - s) \, ds
\]

and therefore

\[
E^P \int_0^t |\nabla u_i|^2 (x + \bar{B}(\xi)_s, t - s) \, ds = E^P \langle Y(\xi)^i \rangle_t.
\]

On the other hand, for \( 1 \leq p < 2 \) we have

\[
E^Q \left\{ \int_0^t |\nabla u_i|^p (x + \bar{B}(\xi)_s, T - s) \, ds \right\}
\]

\[
= \int_0^t E^Q \left\{ |\nabla u_i|^p (x + \bar{B}(\xi)_s, T - s) \right\} \, ds
\]

\[
\leq \int_0^t \left( E^Q \left\{ \frac{1}{R_s} |\nabla u_i|^2 (x + \bar{B}(\xi)_s, T - s) \right\} \right)^\frac{p}{2} \left( E^Q \left( R_s^{\frac{p}{2}} \right) \right)^{1 - \frac{p}{2}} \, ds
\]

\[
= \int_0^t \left( E^P \left\{ |\nabla u_i|^2 (x + \bar{B}(\xi)_s, T - s) \right\} \right)^\frac{p}{2} \left( E^P \left( R_s^{\frac{p}{2}} \right) \right)^{1 - \frac{p}{2}} \, ds
\]
Since \( \tilde{B}(\xi) \) is a Brownian motion under \( Q \), so that

\[
E^Q \int_0^t |\nabla u|^p (x + \tilde{B}(\xi)_s, T - s) ds
\]

\[
= \int_0^t P_s (|\nabla u|^p (\cdot, T - s)) (x) ds
\]

\[
\leq \int_0^t \left( E^P |\nabla u|^2 (x + \tilde{B}(\xi)_s, T - s) \right)^{\frac{p}{2}} \left( E^P R_s^{\frac{2}{2-p}} \right)^{1-\frac{p}{2}} ds
\]

\[
\leq d^{1-\frac{p}{2}} \left( \int_0^t E^P |\nabla u|^2 (x + \tilde{B}(\xi)_s, T - s) ds \right)^{\frac{p}{2}} \left( \int_0^t E^P R_s^{\frac{2}{2-p}} ds \right)^{1-\frac{p}{2}}
\]

which is the first estimate. To prove the second one, we write the previous estimate as

\[
\frac{1}{t} \int_0^t P_s |\nabla u|^p (x, T - s) ds
\]

\[
\leq d^{1-\frac{p}{2}} \left( \frac{1}{t} E^P \langle Y(\xi)^i \rangle_t \right)^{\frac{p}{2}} \left( \frac{1}{t} \int_0^t E^P R_s^{\frac{2}{2-p}} ds \right)^{1-\frac{p}{2}}.
\]

Letting \( t \to 0 \) one obtains that

\[
\sqrt{\frac{1}{t} E^P \langle Y(\xi)^i \rangle_t} \leq d^{1-\frac{p}{2}} \lim_{t \to 0} \sqrt{\frac{1}{t} E^P \langle Y(\xi)^i \rangle_t}.
\]

then letting \( p \uparrow 2 \) we obtain 2).

**Lemma 4.2** Then for any \( p \in [1, 2) \)

\[
E^P \left( R_t^{\frac{2}{2-p}} \right) \leq \exp \left\{ \frac{p}{(2-p)^2} t \max_{|y| \leq |u_0|} |f(y)|^2 \right\}.
\]

**Proof.** Let

\[
H_t = \exp \left[ \frac{2}{(2-p)} \int_0^t f(Y(\xi)_s) dB_s - \frac{2}{(2-p)^2} \int_0^t |f(Y(\xi)_s)|^2 ds \right]
\]

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which is exponential martingale, so that $\mathbb{E}^P(H_t) = 1$. Then

$$R_t^{2-p} = H_t \exp \left[ p \frac{t}{(2-p)^2} \int_0^t |f(Y(s))|^2 ds \right]$$

$$\leq H_t \exp \left[ p \frac{t}{(2-p)^2} \max_{|y| \leq |u_0|_{\infty}} |f(y)|^2 \right]$$

which yields the claim. ■

**Corollary 4.3** We have

$$\int_0^t P_s |\nabla u_i|^p(x, T-s)ds \leq d^{1-p} |u_i^0|_{p\infty} \exp \left\{ p \frac{t}{2(2-p)} \max_{|y| \leq |u_0|_{\infty}} |f(y)|^2 \right\}$$

(4.2)

for any $p \in [1, 2)$, and $i = 1, \cdots, m$, where $P_s = e^{s\Delta}$ the heat semigroup.

**Proof.** Observe that

$$E^P(Y(\xi_i)_s) = E^P \left[ E^P(\xi_i|F_s) - E^P(\xi_i|F_0) \right]^2$$

$$= E^P \left[ E^P(\xi_i|F_s)^2 - E^P(\xi_i|F_0)^2 \right]$$

$$\leq ||\xi_i||^2$$

$$\leq |u_i^0|_{\infty}$$

and therefore, the item 1) in Theorem 4.1 together with Lemma 4.2 yields the gradient estimate (4.2). ■

## 5 Non-linear flow associated with quasi-linear PDEs

In this section we construct a non-linear stochastic flow associated with the quasi-linear system (4.1).

Assume that $f^j : \mathbb{R}^m \to \mathbb{R}^d$ ($j = 1, \cdots, d$) are differentiable with bounded 1st, 2nd and 3rd derivatives, and the initial value $u^i(x, 0) = u_i^0(x)$ ($i = 1, \cdots, m$) are bounded, differentiable with 1st and 2nd bounded derivatives:

$$|\nabla^{k-1} u_0| \leq C_0, \quad |\nabla^k f| \leq C_1 \quad \text{for} \quad k = 1, 2, 3$$

for some non-negative constants $C_0$ and $C_1$, here $\nabla^k$ denote the $k$-th derivative in space variables. Of course $\nabla^0 u_0 = u_0$. In general, we apply $\nabla$ to mean
the total derivative operator in space variables. For example, \( \nabla u \) means \( \frac{\partial u}{\partial x^j} \) but does not include the derivative in time parameter \( t \).

We are going to construct a continuous adapted process \( \xi = (\xi_t) \) with values in the function space \( C^2_b(\mathbb{R}^d, \mathbb{R}^m) \) such that \( \nabla^j u(\cdot, t) = E^P(\nabla^j \xi_t) \) at least for small \( t \). The spirit in devising such a formula is quite similar to those initiated in the seminal works Bismut [1] and Malliavin [12].

Consider the function space \( L^\infty(\Omega; C([0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^m))) \). If

\[
\xi \in L^\infty(\Omega; C([0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^m)))
\]

then for any \( \omega, \xi(\omega) \in C([0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^m)) \), so that \( \xi_t(\omega) \in C^2_b(\mathbb{R}^d, \mathbb{R}^m) \) and \( t \to \xi_t(\omega) \) is continuous from \( [0, T] \) to \( C^2_b(\mathbb{R}^d, \mathbb{R}^m) \), and \( x \to \xi_t(\omega, x) \) has continuous and bounded first and second derivatives. Let \( H_T \) denote the space of all functions \( \xi \) in \( L^\infty(\Omega; C([0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^m))) \) such that \( \xi = (\xi_t)_{t \in [0, T]} \) is adapted. \( H_T \) is equipped with the \( L^\infty \)-norm, namely

\[
||\xi|| = \text{ess sup} \sup_{\omega \in \Omega} \sup_{t \in [0, T]} \sum_{j=0}^2 \sup_{x \in \mathbb{R}^d} |\nabla^j \xi_t(x, \omega)|
\]

where, as we have explained,

\[
\nabla \xi_t(x, \omega) = \left( \frac{\partial \xi^i}{\partial x^j}(x, \omega, t) \right)_{i=1, \ldots, m, j=1, \ldots, d}
\]

etc. \( H_T \) is a Banach space under \( || \cdot || \).

In this section, if \( \zeta = (\zeta^i) \), where \( \zeta^i \in L^2(\Omega, \mathcal{F}, P) \), then we define \( Y(\zeta)_s = E^P(\xi(\zeta)|\mathcal{F}_s) \) and \( Z(\zeta) = D_B(Y(\zeta)) \). Therefore, for any \( t > 0 \), if \( \zeta^i \in L^2(\Omega, \mathcal{F}_t, P) \), then \( (Y(\zeta)_s, Z(\zeta)_s) \) is the unique solution of the BSDE

\[
dY(\zeta)_s = Z(\zeta)_s dB_s, \quad Y(\zeta)_t = \zeta.
\]

It is easy to see that, if \( \xi \in H_T \), then \( Y(\nabla^k \xi_t) = \nabla^k Y(\xi_t) \) and \( Z(\nabla^k \xi_t) = \nabla^k Z(\xi_t) \) for any \( t \leq T \) and \( k = 0, 1, 2 \). This follows from the fact that the mappings \( \zeta \to Y(\zeta) \) and \( \zeta \to Z(\zeta) \) are both affine.

Let \( \xi \in H_T \). Then, according to the non-linear Cameron-Martin formula, for any fixed \( t \leq T \), we define a probability \( Q_{t,x} \) on \( (\Omega, \mathcal{F}_t) \) by

\[
\frac{dQ_{t,x}}{dP}|_{\mathcal{F}_t} = \exp \left[ \int_0^t f(Y(\xi_t(\cdot, x))_s) dB_s - \frac{1}{2} \int_0^t |f|^2(Y(\xi_t(\cdot, x))_s) ds \right].
\]
We will omit the argument \( \cdot \) (a sample point) and the space variable \( x \) if no confusion may arise. Under \( Q_{t,x} \), \( \tilde{B}(\xi_t)_s = B_s - \int_0^s f(Y(\xi_t)_r) \, dr \) \((0 \leq s \leq t)\) is Brownian motion up to time \( t \). Define
\[
X(\xi)_t = x + \tilde{B}(\xi)_t = x + B_t - \int_0^t f(Y(\xi)_s) \, ds \quad \text{for } t \leq T.
\]

According to Theorem 3.1, we want to find a fixed point \( \xi \in H_T : \xi = u_0(X(\xi)) \). To this end we define \( \Phi(\xi) = u_0(X(\xi)_t) \) for any \( \xi \in H_T \). Then
\[
\nabla \Phi(\xi)_t = \nabla u_0(X(\xi)_t) \nabla X(\xi)_t
\]
and
\[
\nabla^2 \Phi(\xi)_t = \nabla^2 u_0(X(\xi)_t)(\nabla X(\xi)_t, \nabla X(\xi)_t) + \nabla u_0(X(\xi)_t) \nabla^2 X(\xi)_t.
\]

**Lemma 5.1**

1) For any \( T > 0 \) and \( \xi \in H_T \)
\[
\|\Phi(\xi)\| \leq C_0 (1 + d)^2 + C_0 C_1 T \{(2d + 1) + (1 + C_1 T)\|\xi\|\} \|\xi\|.
\]

2) If \( K = 2C_0 (1 + d)^2 \) and
\[
T \leq \frac{1}{2 \sqrt{C_0 C_1} \sqrt{d + 1/2} + C_0 (1 + C_1) (1 + d)^2} \wedge 1,
\]
then \( \|\Phi(\xi)\| \leq K \) as long as \( \xi \in H_T \) and \( \|\xi\| \leq K \).

**Proof.** By definition for any \( t \leq T \) and any \( x \) (but the argument \( x \) is suppressed from the notations for simplicity, and \( \| \cdot \|_\infty \) denotes the essential supremum norm)
\[
\nabla X(\xi)_t = I_{\mathbb{R}^d} - \int_0^t \nabla f(Y(\xi)_s) Y(\nabla \xi)_s ds
\]
and
\[
\nabla^2 X(\xi)_t = \int_0^t \nabla^2 f(Y(\xi)_s) (Y(\nabla \xi)_s, Y(\nabla \xi)_s) ds
\]
\[
- \int_0^t \nabla f(Y(\xi)_s) Y(\nabla^2 \xi)_s ds.
\]
From these equations and the fact that the conditional expectation is a contraction on $L^\infty(\Omega, \mathcal{F}_t, P)$, one can easily to see the following estimates:

$$|\nabla X(\xi)_t| \leq d + C_1 t |\nabla \xi_t|_\infty,$$

and

$$|\nabla^2 X(\xi)_t| \leq C_1 t |\nabla \xi_t|^2 + C_1 t |\nabla^2 \xi_t|.$$

Therefore

$$|\Phi(\xi)| \leq C_0,$$

$$|\nabla \Phi(\xi)_t| \leq C_0 |\nabla X(\xi)_t| \leq C_0 \{d + C_1 t |\nabla \xi_t|_\infty\}$$

and

$$|\nabla^2 \Phi(\xi)_t| \leq C_0 |\nabla X(\xi)_t|^2 + C_0 |\nabla^2 X(\xi)_t| \leq C_0 \{d + C_1 t |\nabla \xi_t|_\infty\}^2 + C_0 C_1 t \left[ |\nabla \xi_t|^2 + |\nabla^2 \xi_t|_\infty \right]$$

which yield the required estimates. 

**Lemma 5.2** Let $T > 0$. There is positive constant depending only on $C_0, C_1$ and $d$ such that

$$||\Phi(\xi) - \Phi(\eta)|| \leq K_0 C_1 T (1 + T) (1 + ||\xi|| + ||\eta||) ||\xi - \eta||$$

for any $\xi, \eta \in H_T$.

**Proof.** By a simple computation, we have

$$|X(\xi)_t - X(\eta)_t| \leq \left| \int_0^t f(Y(\xi_t - \eta_t)_s) ds \right| \leq t C_1 |\xi_t - \eta_t|_\infty,$$

$$|\nabla X(\xi)_t - \nabla X(\eta)_t| \leq \left| \int_0^t (\nabla f(Y(\xi_t)_s) - \nabla f(Y(\eta_t)_s)) Y(\nabla \xi_t)_s ds \right|$$

$$+ \left| \int_0^t \nabla f(Y(\eta_t)_s) Y(\nabla(\xi_t - \eta_t)) ds \right| \leq C_1 \int_0^t Y(\xi_t - \eta_t)_s ||Y(\nabla \xi_t)_s| ds$$

$$+ C_1 \int_0^t |Y(\nabla(\xi_t - \eta_t))_s| ds \leq C_1 t |\xi_t - \eta_t| |\nabla \xi_t| + C_1 t |\nabla \xi_t - \nabla \eta_t|.$$
and, similarly
\[
|\nabla^2 X(\xi)_t - \nabla^2 X(\eta)_t| \leq C_1 t |\xi_t - \eta_t| |\nabla \xi_t|^2 \\
+ C_1 t (|\nabla \xi_t| + |\nabla \eta_t|) |\nabla \xi_t - \nabla \eta_t| \\
+ C_1 t |\xi_t - \eta_t| |\nabla^2 \xi_t| + C_1 t |\nabla^2 \xi_t - \nabla^2 \eta_t|
\]
and the estimate follows easily from these inequalities.

\[
\Phi(\xi) = \Phi(\eta) = u_0(X(\xi))
\]

Now we are in a position to establish a Bismut type formula (see Bismut \cite{[1]} for the linear case) for the solution of quasi-linear system (4.1).

**Theorem 5.3** There is \( T > 0 \) depending only on \( d, C_0 \) and \( C_2 \), so that there is a unique fixed point \( \xi \) of \( \Phi \) in \( H_T \). Moreover

\[
\nabla^j u(x, t) = E^P(\nabla^j \xi_t(\cdot, x)), \quad j = 0, 1, 2.
\]

and

\[
u u(x + \tilde{B}(\xi)_t, t - s) = Y(\xi)_s \quad \text{for all } s \leq t \leq T \text{ a.e.}
\]

**Proof.** According to Theorem \[3.1\] for \( t \leq T \) we have \( u(x, t) = E^P(\xi_t(\cdot, x)|F_0) \). Taking expectation we obtain \( u(x, t) = E^P(\xi_t(\cdot, x)) \). Since \( \xi_t(\omega, \cdot) \in C^2_b(\mathbb{R}^d, \mathbb{R}^m) \) so we may take derivatives under integration to obtain \[5.1\].

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