GREEN’S CANONICAL SYZYGY CONJECTURE FOR RIBBONS

ANAND DEOPURKAR

ABSTRACT. We verify the analogue for ribbons of Green’s canonical syzygy conjecture, formulated by Bayer and Eisenbud. Our proof uses the results of Voisin and Hirschowitz–Ramanan on Green’s conjecture for general smooth curves.

1. INTRODUCTION

Let $C$ be a smooth projective curve of genus $g \geq 2$ over a field $k$ of characteristic zero. In 1984, Green made the following conjecture.

**Conjecture 1.1** (Green’s conjecture [Gre84]). The Koszul cohomology group $K_{p,2}(C, \omega_C)$ vanishes if and only if $p$ is smaller than the Clifford index of $C$.

The statement of the conjecture generalizes classical results of Noether and Petri about the degrees of the generators of the canonical ideal of $C$. It relates the shape of the minimal free resolution of the canonical ideal of $C$ with the presence of linear series on $C$.

In 1995, Bayer and Eisenbud laid out an approach to prove a generic version of Green’s conjecture using ribbons—double structures on $\mathbb{P}^1$ obtained by appropriately gluing $\text{Spec} k[s, \varepsilon]/\varepsilon^2$ and $\text{Spec} k[t, \eta]/\eta^2$. They extended the definition of Clifford index to ribbons and made an analogous conjecture.

**Conjecture 1.2** (Green’s conjecture for ribbons [BE95]). Let $C$ be a ribbon of arithmetic genus $g$. Then $K_{p,2}(C, \omega_C)$ vanishes if and only if $p$ is smaller than the (ribbon) Clifford index of $C$.

Green’s conjecture for ribbons implies Green’s conjecture for a general curve of every Clifford index, by semi-continuity and the smoothability results of Fong [Fon93].

Voisin made a breakthrough by proving Green’s conjecture for general curves [Voi02, Voi05]. Combined with prior work of Hirschowitz and Ramanan [HR98], her results imply that the conjecture holds for every smooth curve of odd genus and maximum Clifford index. There has been remarkable progress on the conjecture since then, thanks to the work of Aprodu and Farkas. They have proved that it holds for general curves of every gonality, and most notably, for every curve lying on a K3 surface [Apr05, AFT11, AFT12].

In this paper, we show that the results of Voisin and Hirschowitz–Ramanan imply Green’s conjecture for ribbons.

**Theorem 1.3.** Green’s conjecture (Conjecture 1.2) holds for every ribbon.

We thus get another proof of the following.
Corollary 1.4. Green’s conjecture (Conjecture 1.1) holds for a non-empty open subset of curves of a given genus and Clifford index.

The main ideas in our proof are already present in the works of Voisin and Aprodu. The use of ribbons, however, highlights the crux of the argument and circumvents many of the delicate surrounding issues. We must admit that our proof does not address the original motivation of Bayer and Eisenbud, as it hinges on already knowing the statement for general smooth curves. It would be wonderful to have an independent argument. Nonetheless, any proof is better than no proof.

Knowing that Green’s conjecture holds for ribbons is important for another reason, motivated by recent progress in the log minimal model program for $\overline{M}_g$. For a canonical curve $C \subset \mathbb{P}^{g-1}$ with $K_{p,2}(C, \omega_C) = 0$, we can define a point in a Grassmannian called the $p$th syzygy point that encodes the vector space of $p$th syzygies among the generators of the homogeneous ideal of $C$ (see [DFS] for details). The GIT quotients of the loci of $p$th syzygy points are expected to lead to the canonical model of $\overline{M}_g$. We expect that the $p$th syzygy point of a general ribbon will be GIT semi-stable. The vanishing of $K_{p,2}$ shows that the $p$th syzygy point is at least well-defined!

The paper is organized as follows. Section 2 contains basic results about ribbons and their deformations. Section 3 proves Theorem 1.3 for ribbons of odd genus and maximum Clifford index. Section 4 extends the result to all ribbons. Without loss of generality, we take $k$ to be algebraically closed (of characteristic 0). All schemes and stacks are locally of finite type over $k$.

2. Ribbons

We quickly review the theory of ribbons from [BE95]. Let $D$ be a reduced and connected scheme over $k$. A ribbon over $D$ is a scheme $C$ with an isomorphism $D \to C_{\text{red}}$ such that the ideal $I$ of $D \subset C$ satisfies $I^2 = 0$ and is locally free of rank 1 when considered as a sheaf on $D$. A ribbon over $\mathbb{P}^1$ is called a rational ribbon. All our ribbons will be rational, so we drop this adjective.

A ribbon $C$ of arithmetic genus $g$ gives an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^1}(-g-1) \to \Omega^1_C|_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1} \to 0.
\]

We have $\Omega^1_C|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-b-2)$ for some integers $a$ and $b$ with $0 \leq a \leq b \leq g-1$ and $a + b = g-1$. The Clifford index $\text{Cliff}(C)$ of $C$ is defined to be the integer $a$. We say that $C$ is hyperelliptic if the following equivalent conditions hold: (1) the inclusion $\mathbb{P}^1 \subset C$ admits a retraction $C \to \mathbb{P}^1$, (2) $\text{Cliff}(C) = 0$, (3) the sequence in (1) is split.

A ribbon of arithmetic genus $g$ may be described explicitly as obtained from gluing $U_1 = \text{Spec} \ k[s, e]/e^2$ and $U_2 = \text{Spec} \ k[t, \eta]/\eta^2$ by isomorphisms

\[
e = t^{-g-1}\eta
\]

\[
s^{-1} = t + F(t)\eta,
\]

on $U_1 \cap U_2$, where $F(t)$ lies in $k[t, t^{-1}]/(k[t] + t^{-g+1}k[t^{-1}])$. The element $F(t)$ corresponds to the class of the extension (1) in $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-g-1))$. In particular,
$F(t) = 0$ if and only if $C$ is hyperelliptic. Two ribbons $C_1$ and $C_2$ are isomorphic by a map that restricts to the identity on the underlying $\mathbb{P}^1$ if and only if $F_1(t) = cF_2(t)$ for some $c \in k^*$. Let $x \in C$ be a closed point, $\beta : C' \to C$ the blow up at $x$, and $E \subset C'$ the exceptional divisor. Then $C'$ is a ribbon of arithmetic genus $g - 1$ and $\omega_{C'} = \beta^* \omega_C(-E)$. Blowing up is related to the Clifford index as follows.

**Proposition 2.1** (See [BE95, Corollary 2.5]). The Clifford index $\text{Cliff}(C)$ is the smallest number $k$ such that there exists a sequence

$$C_k \to \cdots \to C_1 \to C_0 = C$$

where $C_{i+1} \to C_i$ is a blow-up at a closed point and $C_k$ is hyperelliptic.

The usual Clifford index of smooth curves and the Clifford index for ribbons obey semi-continuity.

**Theorem 2.2** (See [EG95, Theorem 2.1]). Let $C \to \Delta$ be a proper flat family over a DVR $\Delta$ where the geometric general fiber $C_\Delta$ is smooth and the special fiber $C_0$ is a ribbon. Then $\text{Cliff}(C_\Delta) \leq \text{Cliff}(C_0)$.

All ribbons of a given genus and Clifford index are related.

**Theorem 2.3** (See [BE95, § 8]). Let $c \geq 1$. There exists a surface $X_{g,c} \subset \mathbb{P}^8$ such that all canonically embedded ribbons of genus $g$ and Clifford index $c$ are hyperplane sections of $X_{g,c}$. In particular, the graded betti numbers $\dim K_{p,q}(C, \omega_C)$ depend only on the genus $g$ and the Clifford index $c$ of $C$.

We need basic results about the deformation theory of ribbons.

**Proposition 2.4.** A ribbon $C$ has a smooth versal deformation space.

**Proof.** For non-hyperelliptic ribbons, this follows from [BE95, Theorem 6.1]. Here is another proof that works for all ribbons, using deformation theory. Denote by $T^i$ the deformation-obstruction functors of Lichtenbaum and Schlessinger for $i = 0, 1, 2$ (see [Har10, § 1.3]). Let $T^i_i$ be the sheaf $T^i(C/k, \mathcal{O}_C)$. Suppose $A \to A'$ is a surjection of local Artin $k$-algebras with kernel $k$. Then the obstructions to lifting a deformation of $C$ over $\text{Spec} A'$ to a deformation over $\text{Spec} A$ lie successively in $H^0(C, T^2_C), H^1(C, T^1_C)$, and $H^2(C, T^0_C)$. Since $C$ has dimension $1$, we have $H^2(C, T^0_C) = 0$. Since $C$ is a local complete intersection, we have $T^2_C = 0$. To compute $T^1_C$, we do a simple local computation. Consider $U = \text{Spec} k[s, e]/\epsilon^2$. Set $S = k[s, e], J = e^2S$, and $R = S/J$. The cotangent complex for $U/k$ is given by

$$L_* : 0 \to J/J^2 \xrightarrow{d} \Omega_{S/k} \otimes_R R.$$ 

By definition, $T^1_U \subset H^1(\text{Hom}_R(L_*), R)$. Let $e$ be a generator of the free $R$ module $J/J^2$. Note that $\Omega_{S/k} \otimes_R R = R(ds, de)$ and $d(e) = e\epsilon e$. Denote by $\partial_s, \partial_e$, and $e^*$ the dual generators of $ds, de$, and $e$, respectively. Then

$$T^1_U = \text{coker}(d^*: R(\partial_s, \partial_e) \to R(e^*))$$,
where \( d^*(\partial_0) = 0 \) and \( d^*(\partial_0) = e e^* \). Therefore, we get \( T^1_U = R/e(e^*) = k[s](e^*) \). By a similar computation on \( V = \text{Spec} k[t, \eta]/\eta^2 \) and gluing, we get \( T^1_C = \mathcal{O}_P(2g + 2) \). Therefore, \( H^1(C, T^1_C) = 0 \) and hence \( C \) is unobstructed.

**Proposition 2.5.** Let \((U, 0)\) be a versal deformation space of \( C \) and \( C_U \to U \) a versal family. Let \( N \subset U \) be the closed subset consisting of \( u \in U \) where \( C_u \) has worse than nodal singularities. Then the codimension of \( N \) in \( U \) at 0 is at least 2.

**Proof.** Note that all ribbons degenerate isotrivially to the hyperelliptic ribbon. Indeed, in the gluing description \( (\ref{2}) \), we may replace \( F(t) \) by \( cF(t) \) and take \( c \) to 0. Therefore, it suffices to prove the proposition for the hyperelliptic ribbon. Furthermore, since \((U, 0)\) is irreducible, it suffices to exhibit an irreducible pointed scheme \((T, 0)\) and a map \( \phi : (T, 0) \to (U, 0) \) such that the codimension of \( \phi^{-1}(N) \) in \( T \) at 0 is at least 2. Said differently, it suffices to construct a family of curves \( C_T \to T \) such that \( C_0 \) is a hyperelliptic ribbon and the locus of \( t \in T \) such that \( C_t \) has worse than nodal singularities has codimension at least 2. Take \( T = \mathbb{A}^{2g+3} = \mathbb{A}(a_0, \ldots, a_{2g+2}) \) with \( 0 = (0, \ldots, 0) \) and let \( C_T \to T \) be the family of hyperelliptic curves defined affine locally by

\[
y^2 = a_{2g+2}x^{2g+2} + \cdots + a_1x + a_0.
\]

The locus of worse than nodal curves corresponds to set of \((a_0, \ldots, a_{2g+2})\) for which the polynomial \( a_{2g+2}x^{2g+2} + \cdots + a_1x + a_0 \) has a zero of multiplicity at least 3. It is easy to see that this locus has codimension 2. \( \square \)

### 3. Proof for Ribbons of Odd Genus and Maximum Clifford Index

Let \( X \) be a projective scheme and \( L \) an invertible sheaf on \( X \). Set \( V = H^0(X, L) \). Recall that the Koszul cohomology group \( K_{p,q}(X, L) \) is defined as the middle cohomology group in

\[
\wedge^{p+1} V \otimes H^0(X, L^{q-1}) \to \wedge^p V \otimes H^0(X, L^q) \to \wedge^{p-1} V \otimes H^0(X, L^{q+1}).
\]

Let \( g = 2k + 1 \) with \( k \geq 1 \). Let \( \mathcal{U} \) be the (non-separated) stack of all curves \( C \) that satisfy the following conditions:

1. \( C \) is Gorenstein of arithmetic genus \( g \) and \( h^0(C, \mathcal{O}_C) = 1 \).
2. \( \omega_C \) is very ample and embeds \( C \) as an arithmetically Gorenstein subscheme in \( \mathbb{P}^{g-1} \).
3. \( C \) is a versal deformation space of \( C \) is smooth.

All three are open conditions and hence \( \mathcal{U} \) is an open substack of the stack of all curves constructed, for example, in \[Hal10\]. The first two conditions are more important than the third; the third is there just to avoid any problems about divisor theory. Denote by \( \mathcal{U}^{\text{nodal}} \subset \mathcal{U} \) the open substack parametrizing curves with at worst nodes as singularities. Then \( \mathcal{U}^{\text{nodal}} \subset \overline{M}_g \). In fact, \( \mathcal{U}^{\text{nodal}} \) is precisely \( \overline{M}_g^{\text{va}} \), the stack of stable curves with a very ample dualizing sheaf; it is studied in \[Apr05\]. Note that \( M_g \cap \overline{M}_g^{\text{va}} \) is the complement in \( M_g \) of the hyperelliptic locus; we denote it by \( M_g^{\text{nh}} \). Also note that all non-hyperelliptic ribbons satisfy the three conditions—the first is clear; the second is \[BE95, \text{Theorem 5.3}] \; and the third is \[Proposition 2.4].
Let $\pi: C \to U$ be the universal curve. Set $V = \pi_*(\omega_C)$. Then $V$ is locally free of rank $g$ on $U$. Consider the Koszul complex
\[ K_* : \wedge^{k+1} V \xrightarrow{\delta_1} \wedge^k V \otimes V \xrightarrow{\delta_2} \wedge^{k-1} V \otimes \pi_* (\omega_2^2) \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_k} \wedge^0 V \otimes \pi_* (\omega_k^k). \]

Define $E$ and $F$ by
\[ E = \text{coker} \delta_1, \]
\[ F = \text{ker} \delta_3. \]

**Proposition 3.1.** $E$ and $F$ are locally free on $U$ of the same rank.

**Proof.** $E$ is clearly locally free. Consider a point $[C] \in U$. Since the homogeneous coordinate ring of the canonical embedding of $C$ is Gorenstein, we have the duality
\[ K_{p,q}(C, \omega_C) \cong K_{g-2-p,-q}(C, \omega_C)^*. \]

In particular, we have $K_{p,q}(C, \omega_C) = 0$ for $p \geq 4$ and for $p = 3$ and $q < g-2$. Therefore, the complex $K_*$ is exact from the third place onward: $\ker(\delta_{q+1}) = \text{im}(\delta_q)$ for all $q \geq 3$. Since $K_*$ is a finite complex of locally free sheaves, it follows that $F = \ker \delta_3$ is locally free.

It is easy to see that the rank of both $E$ and $F$ is $\frac{2g!}{(k-1)(k+1)}$. \hfill $\square$

The map $\delta_2$ in $K_*$ induces a map $\delta: E \to F$. Observe that
\[ \text{coker} \delta|_{[C]} = K_{k-1,2}(C, \omega_C). \]

Let $D \subset U$ be defined by the vanishing of $\text{det}(\delta)$.

Let $D_{k+1} \subset U$ be the closure of the locus of $(k+1)$-gonal curves in $M_g^{\text{nh}}$.

**Proposition 3.2.** Let $C$ be a non-hyperelliptic ribbon of odd genus $2k + 1$. We have the equality $D_{k+1} = D$ in an open subset containing $[C] \in U$.

**Proof.** The results of Voisin [Vo] and Hirschowitz–Ramanan [HR] give $D_{k+1} = D$ on $M_g^{\text{nh}}$. Consider the open inclusion $M_g^{\text{nh}} \subset U^{\text{nodal}}$. Note that $U^{\text{nodal}} \setminus M_g^{\text{nh}}$ has one divisorial component $\Delta_0$ whose general fiber corresponds to an irreducible nodal curve. In [Vo], Voisin shows that $K_{k-1,2}(C, \omega_C) = 0$ for every $C$ in a linear series on a K3 surface, which includes irreducible nodal curves. Therefore $D$ does not contain $\Delta_0$ as a component. Since $D_{k+1}$ is the closure of a divisor on $M_g^{\text{nh}}$, it does not contain $\Delta_0$ as a component either. Therefore, the equality of divisors $D = D_{k+1}$ holds in codimension 1 on $U^{\text{nodal}}$ and hence on all of $U^{\text{nodal}}$. The same reasoning (Proposition 2.5) implies that it holds around every point in $U$ corresponding to a ribbon. \hfill $\square$

**Corollary 3.3.** For a ribbon $C$ of genus $2k + 1$ with the maximum Clifford index $k$, we have $K_{k-1,2}(C, \omega_C) = 0$.

**Proof.** Note that the smooth curves parametrized by $D_{k+1}$ have Clifford index at most $k - 1$. By upper semi-continuity (Theorem 2.2), we see that $[C] \notin D_{k+1}$, and hence $[C] \notin D$. By (3), this is equivalent to $K_{k-1,2}(C, \omega_C) = 0$. \hfill $\square$
4. Proof for all ribbons

We now deduce Green’s conjecture for all ribbons from [Corollary 3].

**Lemma 4.1.** Let $C$ be a ribbon and $\beta: C' \to C$ the blow up at a closed point. Then we have an inclusion $K_{k,1}(C', \omega_{C'}) \subseteq K_{k,1}(C, \omega_C)$.

*Proof.* This follows by the same argument as in [Voi02, Corollary 1]. We reproduce the details for completeness.

Since $\omega_{C'} = \beta^* \omega_C(-E)$ where $E$ is the exceptional divisor of $\beta$, we have inclusions $H^0(C', \omega^l_{C'}) \to H^0(C, \omega^l_C)$ for $l \geq 0$.

Set $V' = H^0(C', \omega_{C'})$ and $V = H^0(C, \omega_C)$. Consider the Koszul complexes

$$\begin{align*}
\wedge^{k+1}V' &\xrightarrow{\delta'} \wedge^kV' \otimes V' \xrightarrow{j_1} \wedge^{k-1}V' \otimes H^0(C', \omega^2_{C'}) \to \ldots \\
\wedge^{k+1}V &\xrightarrow{\delta} \wedge^kV \otimes V \xrightarrow{j_2} \wedge^{k-1}V \otimes H^0(C, \omega^2_C) \to \ldots
\end{align*}$$

For any vector space $U$, the Koszul differential $\delta: \wedge^{k+1}U \to \wedge^kU \otimes U$ has a retract $\wedge: \wedge^kU \otimes U \to \wedge^{k+1}U$ given by the wedge product. Precisely, the two are related by $\wedge \circ \delta = (k + 1) \text{id}$.

Suppose $\alpha' \in \wedge^kV' \otimes V'$ is such that $j_2(\alpha') = \delta(\beta)$ for some $\beta \in \wedge^{k+1}V$. Then

$$\wedge \circ j_2(\alpha') = \wedge \circ \delta(\beta) = (k + 1)\beta.$$ 

It is easy to see that $\wedge \circ j_2(\alpha') = j_1 \circ \wedge(\alpha')$. Set $\beta' = \wedge(\alpha')/(k + 1)$.

Then

$$j_2(\delta'(\beta')) = \delta(j_1(\beta')) = \delta(\beta') = j_2(\alpha').$$

Since $j_2$ is injective, we get $\delta'(\beta') = \alpha'$. In other words, any element of $\wedge^kV' \otimes V'$ that becomes a coboundary in $\wedge^kV \otimes V$ is already a coboundary. Therefore the map on cohomology $K_{k,1}(C', \omega_{C'}) \to K_{k,1}(C, \omega_C)$ is injective.

**Theorem 4.2.** Let $C$ be a ribbon of genus $g$ and Clifford index $c$. Then $K_{p,2}(C, \omega_C) = 0$ if and only if $p < c$.

*Proof.* The “only if” direction is the ‘easy’ direction; it is the content of [BE95, Corollary 7.3].

We now prove the other direction. For $c = 0$, there is nothing to prove. Henceforth, we assume $c \geq 1$. In particular, $C$ is non-hyperelliptic. Recall the following:

1. $K_{p,q}(C, \omega_C) = 0$ implies $K_{p',q}(C, \omega_C) = 0$ for all $p' \leq p$,
2. we have the duality $K_{p,q}(C, \omega_C) = K_{g-2-p,3-q}(C, \omega_C)^\ast$.

Since all canonically embedded ribbons of genus $g$ and Clifford index $c$ have the same graded betti numbers (Theorem 2.3), it suffices to prove the theorem for one such ribbon. That is, we must show that $K_{c-1,1}(C, \omega_C) = 0$ for one ribbon of genus $g$ and Clifford index $c$. 


By definition of the Clifford index, we have \(2c \leq g - 1\). Write \(g = 2k + 1 - i\) and \(c = k - i\) for integers \(k\) and \(i\) with \(k \geq 1\) and \(k \geq i \geq 0\). Let \(\overline{C}\) be a ribbon of genus \(2k + 1\) and maximum Clifford index \(k\). By Proposition 2.1, there is a sequence of blow-ups

\[C_k \to \cdots \to C_i \to \cdots \to C_0 = \overline{C},\]

where \(C_k\) is hyperelliptic. In this sequence, \(C = C_i\) is a ribbon of genus \(g = 2k + 1 - i\) and Clifford index \(c = k - i\). By Corollary 3, we know that \(K_{k-1,2}(C, \omega_C) = 0\), which gives \(K_{k,1}(C, \omega_C) = 0\) by duality. By repeated applications of Lemma 4.1, we deduce that \(K_{k,1}(C, \omega_C) = 0\). Note that \(g - 2 - k = k - i - 1 = c - 1\). Therefore, we get \(K_{c-1,2}(C, \omega_C) = 0\) by duality. The proof of the theorem is thus complete. □

REFERENCES

[AF11] Marian Aprodu and Gavril Farkas, Green’s conjecture for curves on arbitrary K3 surfaces, Compos. Math. 147 (2011), no. 3, 839–851.

[AF12] _____, Green’s conjecture for general covers, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 211–226.

[Apr05] Marian Aprodu, Remarks on syzygies of d-gonal curves, Math. Res. Lett. 12 (2005), no. 2-3, 387–400.

[BE95] Dave Bayer and David Eisenbud, Ribbons and their canonical embeddings, Transactions of the AMS 347 (1995), no. 3, 719–756.

[DFS] Anand Deopurkar, Maksym Fedorchuk, and David Swinarski, Toward GIT stability of syzygies of canonical curves, Algebraic Geometry (To appear).

[EG95] David Eisenbud and Mark Green, Clifford indices of ribbons, Trans. Amer. Math. Soc. 347 (1995), no. 3, 757–765.

[Fon93] Lung-Ying Fong, Rational ribbons and deformation of hyperelliptic curves, J. Algebraic Geom. 2 (1993), no. 2, 295–307.

[Gre84] Mark L. Green, Koszul cohomology and the geometry of projective varieties, J. Differential Geom. 19 (1984), no. 1, 125–171.

[Hal10] Jack Hall, Moduli of Singular Curves, arXiv:1011.6007 [math.AG] (2010).

[Har10] Robin Hartshorne, Deformation theory, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010.

[HR98] André Hirschowitz and Sundararaman Ramanan, New evidence for Green’s conjecture on syzygies of canonical curves, Ann. Sci. École Norm. Sup. (4) 31 (1998), no. 2, 145–152.

[Voi02] Claire Voisin, Green’s generic syzygy conjecture for curves of even genus lying on a K3 surface, J. Eur. Math. Soc. (JEMS) 4 (2002), no. 4, 363–404.

[Voi05] _____, Green’s canonical syzygy conjecture for generic curves of odd genus, Compos. Math. 141 (2005), no. 5, 1163–1190.