Abstract

In this paper, we consider a multi-drawing urn model with random addition. At each
discrete time step, we draw a sample of \( m \) balls. According to the composition of the drawn
colors, we return the balls together with a random number of balls depending on two discrete
random variables \( X \) and \( Y \) with finite means and variances. Via the stochastic approximation
algorithm, we give limit theorems describing the asymptotic behavior of white balls.

Keywords: unbalanced urn, martingale, stochastic algorithm, central limit theorem.

1 Introduction

The classical Pólya urn was introduced by Pólya and Eggenberger [2] describing contagious diseases.
The first model is as follows: An urn contains balls of two colors at the start, white and black. At
each step, one picks a ball randomly and returns it to the urn with a ball of the same color.

Afterward this model was generalized and it has become a simple tool to describe several models
such finance, clinical trials (see [3], [8]), biology (see [15]), computer sciences, internet (see [6], [18]),
etc.

Recently, H. Mahmoud, M.R. Chen, C.Z Wei, M. kuba and H. Sulzbach [9, 10, 11, 12, 13, 14],
have focused on the multidrawing urn. Instead of picking a ball, one picks a sample of \( m \) balls
\((m \geq 1)\), say \( l \) white and \( m - l \) black balls. the pick is returned back to the urn together with
\( a_{m-l} \) white and \( b_l \) black balls, where \( a_l \) and \( b_l, 0 \leq l \leq m \) are integers. At first, they treated two
particular cases when \( \{a_{m-l} = c \times l \) and \( b_{m-l} = c \times (m-l)\} \) and when \( \{a_{m-l} = c \times (m-l) \) and
\( b_{m-l} = c \times l\}, \) where \( c \) is a positive constant. By different methods as martingales and moment
methods, the authors described the asymptotic behavior of the urn composition. When considering the general case and in order to ensure the existence of a martingale, they supposed that $W_n$, the number of white balls in the urn after $n$ draws, satisfies the affinity condition i.e, there exists two deterministic sequences $(\alpha_n)$ and $(\beta_n)$ such that, for all $n \geq 0$, $\mathbb{E}[W_{n+1} | \mathcal{F}_n] = \alpha_n W_n + \beta_n$. Under this condition, the authors focused on small and large index urns. Later, the affinity condition was removed in the work of C. Mailler, N. Lasmer and S. Olfa [1], they generalized this model and looked at the case of more than two colors.

In the present paper, we deal with an unbalanced urn model, which was not been sufficiently addressed in the literature. It was mainly dealt with in the works of R. Aguech [16], S. Janson [19] and H. Renlund [1, 5]. In [16] and [19], the authors dealt with model with a simple pick, whereas in [1, 5] the author considered a model with two picks and, under some conditions, they described the asymptotic behavior of the urn composition.

In this paper, we aim to give a generalization of a recent work [17]. We deal with an unbalanced urn model with random addition. We consider an urn containing two different colors white and blue. We suppose that the urn is non empty at time 0. Let denote by $W_n$ (resp $B_n$) the number of white balls (resp blue balls) and by $T_n$ the total number of balls in the urn at time $n$. Let $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ be strictly positive sequences of independent identically distributed discrete random variables with finite means and variances. The model we study is defined as follows: At a discrete time, we pick out a sample of $m$ balls from the urn (we suppose that $T_0 = W_0 + B_0 \geq m$) and according to the composition of the sample, we return the balls with $Q_n(\xi_n, m-\xi_n)$ balls, where $Q_n$ is a $2 \times 2$ matrix depending on the variables $X_n$ and $Y_n$ and $\xi_n$ is the number of white balls in the $n^{th}$ sample.

Let $(\mathcal{F}_n)_{n \geq 0}$ be the $\sigma$-field generated by the first $n$ draws. We summarize the evolution of the urn by the recurrence

$$
\begin{pmatrix} W_n \\
B_n
\end{pmatrix} \overset{D}{=} \begin{pmatrix} W_{n-1} \\
B_{n-1}
\end{pmatrix} + Q_n \begin{pmatrix} \xi_n \\
m - \xi_n
\end{pmatrix}.
$$

(1)

Note that, with these notations, we have

$$
\mathbb{P}[\xi_n = k | \mathcal{F}_{n-1}] = \frac{\binom{W_{n-1}}{k} \binom{B_{n-1}}{m-k}}{\binom{T_{n-1}}{m}}.
$$

The paper is organized as follows. In Section 2, we give the main results of the paper. In the first paragraph of Section 3 we develop Theorem 1 [4] and apply it to our urn model. The rest of this section is devoted to the prove the theorems.

**Notation:** For a random variable $R$, we denote by $\mu_R = \mathbb{E}(R)$ and $\sigma^2_R = \mathbb{V}ar(R)$. Note that $\mu_X, \mu_Y, \sigma^2_X$ and $\sigma^2_Y$ are finite.
2 Main Results

Theorem 1. Consider the urn model evolving by the matrix $Q_n = \begin{pmatrix} 0 & X_n \\ X_n & 0 \end{pmatrix}$. We have the following results:

1. 
   \[ T_n \overset{a.s.}{=} m \mu n + o(\sqrt{n} \ln(n)^\delta), \]
   \[ W_n \overset{a.s.}{=} \frac{m \mu X}{2} n + o(\sqrt{n} \ln(n)^\delta) \quad \text{and} \quad B_n \overset{a.s.}{=} \frac{m \mu X}{2} n + o(\sqrt{n} \ln(n)^\delta); \quad \delta > \frac{1}{2}. \]

2. 
   \[ \frac{W_n - \frac{1}{2} T_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{m(\sigma_X^2 + \mu_X^2)}{12}\right). \]

3. 
   \[ \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{m(\sigma_X^2 + \mu_X^2) + m^2\sigma_X^2}{12}\right). \]

Theorem 2. Consider the urn model evolving by the matrix $Q_n = \begin{pmatrix} X_n & 0 \\ 0 & X_n \end{pmatrix}$. There exists a positive random variable $\tilde{W}_\infty$, such that

\[ T_n \overset{a.s.}{=} m \mu n + o(\sqrt{n} \ln(n)^\delta), \quad W_n \overset{a.s.}{=} \tilde{W}_\infty n + o(n) \quad \text{and} \quad B_n \overset{a.s.}{=} (m \mu - \tilde{W}_\infty)n + o(n). \]

Remark: The random variable $\tilde{W}_\infty$ is absolutely continuous whenever $X$ is bounded.

Theorem 3. Consider the urn model evolving by the matrix $Q_n = \begin{pmatrix} 0 & X_n \\ Y_n & 0 \end{pmatrix}$. Let 

\[ z := \frac{\sqrt{\mu Y}}{\sqrt{\mu_X} + \sqrt{\mu_Y}}, \]

we have the following results:

1. 
   \[ T_n \overset{a.s.}{=} m \sqrt{\mu_X \mu_Y} n + o(n), \]
   \[ W_n \overset{a.s.}{=} m \sqrt{\mu_X \mu_Y} z n + o(n) \quad \text{and} \quad B_n \overset{a.s.}{=} m \sqrt{\mu_X \mu_Y}(1 - z) n + o(n). \]

2. 
   \[ \frac{W_n - z T_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{G(z)}{3}\right), \]

where,

\[ G(x) = \sum_{i=0}^{4} a_i x^i, \]

with

\[ a_0 = m^2(\sigma_X^2 + \mu_X^2), \quad a_1 = m(1 - 2m)(\sigma_X^2 + \mu_X^2), \]
\[ a_2 = 3m(m - 1)(\sigma_X^2 + \mu_X^2) - 2m(m - 1)\mu_X \mu_Y, \quad a_3 = m\mathbb{E}(X - Y)^2 - 2(m^2 - m)(\sigma_X^2 + \mu_X^2 - \mu_X \mu_Y) \]
\[ \text{and} \quad a_4 = m(m - 1)\mathbb{E}(X - Y)^2. \]
Theorem 4. Consider the urn evolving by the matrix \( Q_n = \begin{pmatrix} X_n & 0 \\ 0 & Y_n \end{pmatrix} \). We have the following results:

1. If \( \mu_X > \mu_Y \),
   \[
   T_n \overset{a.s.}{=} m \mu_X n + o(n), \quad W_n \overset{a.s.}{=} m \mu_X n + o(n) \quad \text{and} \quad B_n \overset{a.s.}{=} B_\infty n^\rho + o(n^\rho),
   \]
   where \( \rho = \frac{\mu_Y}{\mu_X} \) and \( B_\infty \) is a positive random variable.

2. If \( \mu_X = \mu_Y \),
   \[
   T_n \overset{a.s.}{=} m \mu_X n + o(n), \quad W_n \overset{a.s.}{=} W_\infty n + o(n) \quad \text{and} \quad B_n \overset{a.s.}{=} (\mu_X m - W_\infty) n + o(n),
   \]
   where \( W_\infty \) is a positive random variable.

Remark: The case when \( \mu_X < \mu_Y \) is obtained by interchanging the colors.

Example: Let \( m = 1 \), this particular case was studied by R. Aguech [16]. Using martingales and branching processes, R. Aguech proved the following results:

If \( \mu_X > \mu_Y \),
   \[
   W_n = \mu_X n + o(n), \quad B_n = D n^\rho \quad \text{and} \quad T_n = \mu_X n + o(n),
   \]
   where \( D \) is a positive random variable.

If \( \mu_X = \mu_Y \),
   \[
   W_n = \mu_X \frac{W}{W + B} n + o(n) \quad \text{and} \quad B_n = \mu_X \frac{B}{W + B} n + o(n),
   \]
   where \( W \) and \( B \) are positive random variables obtained by embedding some martingales in continuous time.

3 Proofs

The stochastic algorithm approximation plays a crucial role in the proofs in order to describe the asymptotic composition of the urn. As many versions of the stochastic algorithm exist in the literature (see [?] for example), we adapt the version of H. Renlund in [4 5].

3.1 A basic tool: Stochastic approximation

Definition 1. A stochastic approximation algorithm \( (U_n)_{n \geq 0} \) is a stochastic process taking values in \([0, 1]\) and adapted to a filtration \( \mathcal{F}_n \) that satisfies

\[
U_{n+1} - U_n = \gamma_{n+1} \left( f(U_n) + \Delta M_{n+1} \right),
\]

where \( (\gamma_n)_{n \geq 1} \) and \( (\Delta_n)_{n \geq 1} \) are two \( \mathcal{F}_n \)-measurable sequences of random variables, \( f \) is a function from \([0, 1]\) onto \( \mathbb{R} \) and the following conditions hold almost surely.

(i) \( \frac{\underline{\gamma}}{n} \leq \gamma_n \leq \frac{\overline{\gamma}}{n} \),
(ii) $|\Delta M_n| \leq K_u$,

(iii) $|f(U_n)| \leq K_f$,

(iv) $\mathbb{E}[\gamma_n+1 \Delta M_{n+1} | \mathcal{F}_n] \leq K_e \gamma_n^2$,

where the constants $c_l, c_u, K_u, K_f,$ and $K_e$ are positive real numbers.

**Definition 2.** Let $Q_f = \{ x; f(x) = 0 \}$. A zero $p \in Q_f$ will be called stable if there exists a neighborhood $\mathcal{N}_p$ of $p$ such that $f(x)(x - p) < 0$ whenever $x \in \mathcal{N}_p \setminus \{p\}$. If $f$ is differentiable, then $f'(p)$ is sufficient to determine that $p$ is stable.

**Theorem 5 (I).** Let $U_n$ be a stochastic algorithm defined in Equation (12). If $f$ is continuous, then $\lim_{n \to +\infty} U_n$ exists almost surely and is in $Q_f$. Furthermore, if $p$ is a stable zero, then $\mathbb{P}\left( U_n \longrightarrow p \right) > 0$.

**Remark:** The conclusion of Theorem 5 holds if we replace the condition (ii) in Definition 1 by the following condition $\mathbb{E}[\Delta M_{n+1}^2 | \mathcal{F}_n] \leq K_u$.

**Proof of Theorem 5.** For the convenience of the reader, we adapt the proof of Theorem 5 and we show that, under the new condition (ii) $\mathbb{E}[\Delta M_{n+1}^2 | \mathcal{F}_n] \leq K_u$, the conclusion remains true. In fact, the following lemmas are useful.

**Lemma 1.** Let $V_n = \sum_{i=1}^{n} \gamma_i \Delta M_i$. Then, $V_n$ converges almost surely.

**Proof.** Set $A_i = \gamma_i \Delta M_i$ and $\bar{A}_i = \mathbb{E}[A_i | \mathcal{F}_{i-1}]$. Define the martingale $C_n = \sum_{i=1}^{n} (A_i - \bar{A}_i)$, then

$$
\mathbb{E}(C_n^2) \leq \sum_{i=1}^{n} \mathbb{E}(A_i^2) = \sum_{i=1}^{n} \mathbb{E}(\gamma_i^2 \Delta M_i^2) \\
\leq \sum_{i=1}^{n} c_u^2 \mathbb{E}(\Delta M_i^2),
$$

if there exists some positive constant $K_u$ such that $\mathbb{E}[\Delta M_{n+1}^2 | \mathcal{F}_n] \leq K_u$, we conclude that $C_n$ is an $L^2$- martingale and thus converges almost surely.

Next, since

$$
\sum_{i \geq 1} |\bar{A}_i| \leq \sum_{i \geq 1} \frac{c_u^2}{(i - 1)^2} K_i < +\infty,
$$

the series $\sum_{i \geq 1} A_i$ must also converges almost surely.

**Lemma 2.** Let $U_\infty$ be the set of accumulation point of $\{U_n\}$ and $Q_f = \{ x; f(x) = 0 \}$ be the zeros of $f$. Suppose $f$ is continuous. Then,

$$
\mathbb{P}\left( U_\infty \subseteq Q_f \right) = 1.
$$

**Proof.** See [4]
Next, we prove the main result of the theorem. If \( \lim_{n \to +\infty} U_n \) does not exist, we can find two rational numbers in the open interval \( \liminf_{n \to +\infty} U_n, \limsup_{n \to +\infty} U_n \).

Let \( p < q \) be two arbitrary different rational numbers. If we can show that
\[
\mathbb{P}\left( \{\liminf_{n \to +\infty} U_n \leq p\} \cap \{\limsup_{n \to +\infty} U_n \geq q\} \right) = 0,
\]
then, the existence of the limit will be established and the claim of the theorem follows from Lemma 2.

For this reason, we need to distinguish two different cases whether or not \( p \) and \( q \) are in the same connected component of \( Q_f \).

**Case 1:** \( p \) and \( q \) are not in the same connected component of \( Q_f \).

See the proof in [4].

**Case 2:** \( p \) and \( q \) are in the connected component of \( Q_f \).

Let \( p \) and \( q \) be two arbitrary rational numbers such that \( p \) and \( q \) are in the same connected component of \( Q_f \). Assume that \( \liminf_{n \to +\infty} U_n \leq p \) and fix an arbitrary \( \varepsilon \) in such a way that \( 0 \leq \varepsilon \leq q - p \).

We aim to show that \( \limsup_{n \to +\infty} U_n \leq q \) i.e, it is sufficient to show that \( \limsup_{n \to +\infty} U_n \leq p + \varepsilon \).

In view of Lemma 1, we have \( V_n = \sum_{i=1}^{n} \gamma_i \Delta M_i \) converges a.s, then, for a stochastic \( N_1 > 0 \), for \( n, m > N_1 \) we have \( |W_n - W_m| < \frac{\varepsilon}{4} \) and \( \gamma_n \Delta M_n \leq \frac{\varepsilon}{4} \).

Let \( N = \max\left(\frac{4K_f}{\varepsilon}, N_1\right) \). By assumption, there is some stochastic \( n > N \) such that \( U_n - p < \frac{\varepsilon}{2} \).

Let
\[
\tau_1 = \inf\{k \geq n; U_k \geq p\} \quad \text{and} \quad \sigma_1 = \inf\{k > \tau_1; U_k < p\},
\]
and define, for \( n \geq 1 \),
\[
\tau_{n+1} = \inf\{k > \sigma_n; U_k \geq p\} \quad \sigma_{n+1} = \inf\{k > \tau_n; U_k < p\}.
\]

Now, for all \( k \) we have
\[
U_{\tau_k} = U_{\tau_{k-1}} + \gamma_{\tau_k-1}(f(U_{\tau_{k-1}}) + \Delta M_{\tau_k}).
\]
Recall that \( \gamma_{\tau_k-1}f(X_{\tau_k-1}) \leq \frac{K_f}{\tau_k-1} \leq \frac{K_f}{n} \), for \( n \geq N \geq \frac{4K_f}{\varepsilon} \) we have \( \gamma_{\tau_k-1}f(X_{\tau_k-1}) < \frac{\varepsilon}{4} \). It follows,
\[
\gamma_{\tau_k-1}(f(U_{\tau_k-1}) + \Delta M_{\tau_k}) \leq \frac{K_f}{n} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\]

Note that \( f(x) = 0 \) when \( x \in [p, q] \) (\( p \) and \( q \) are in \( Q_f \)). For \( j \) such that \( \tau_k + j - 1 \) is a time before the exit time of the interval \( [p, q] \), we have
\[
U_{\tau_k+j} = X_{\tau_k} + W_{\tau_k+j} - W_{\tau_k}.
\]
As \( |W_{\tau_k+j} - W_{\tau_k}| < \frac{\varepsilon}{4} \), we have \( U_{\tau_k+j} \leq p + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \leq p + \varepsilon \), the process will never exceed \( p + \varepsilon \) before \( \sigma_{k+1} \). We conclude that \( \sup_{k \geq n} U_k \leq p + \varepsilon \).

To establish that the limit is to a stable point, we refer the reader to [4] to see a detailed proof.

\( \square \)
Theorem 6 \[5\]. Let \((U_n)_{n \geq 0}\) satisfying Equation (12) such that \(\lim_{n \to +\infty} U_n = U^*\). Let \(\hat{\gamma}_n := n\gamma_n f(U_{n-1})\) where \(f(x) = \frac{-f(x)}{x-U^*}\). Assume that \(\hat{\gamma}_n\) converges almost surely to some limit \(\hat{\gamma}\). Then, if \(\hat{\gamma} > \frac{1}{2}\) and if \(E[(n\gamma_n \Delta M_n)^2|\mathcal{F}_{n-1}] \to \sigma^2 > 0\), then
\[
\sqrt{n}(U_n - U^*) \to \mathcal{N}\left(0, \frac{\sigma^2}{2\hat{\gamma} - 1}\right).
\]

3.2 Proof of the main results

Proof of Theorem \[1\]. Consider the urn model defined in Equation (1) with 
\[
Q_n = \begin{pmatrix} 0 \\ X_n \end{pmatrix}
\]
We have the following recursions:
\[
W_{n+1} = W_n + X_{n+1}(m - \xi_{n+1}) \quad \text{and} \quad T_{n+1} = T_n + mX_{n+1}.
\]

Proof of claim 1

Lemma 3. Let \(Z_n = \frac{W_n}{T_n}\) be the proportion of white balls in the urn after \(n\) draws. Then, \(Z_n\) satisfies the stochastic approximation algorithm defined by (12) with \(\gamma_n = \frac{1}{T_n}\), \(f(x) = \mu_X m(1 - 2x)\) and \(\Delta M_{n+1} = X_{n+1}(m - \xi_{n+1} - mZ_n) - \mu m(1 - Z_n)\).

Proof. We need to check the conditions of definition \[1\]

(i) Recall that \(T_n = T_0 + m \sum_{i=1}^n X_i\), with \((X_i)_{i \geq 1}\) are iid random variables. It follows, by Rajchman strong law of large numbers, that
\[
T_n \overset{a.s.}{=} \mu_X mn + o(\sqrt{n} \ln(n)\delta), \quad \delta \geq \frac{1}{2},
\]

it follows that \(\frac{1}{m\mu_X + 1} \leq \frac{1}{T_n} \leq \frac{2}{m\mu_X n}\), then, \(c_l = \frac{1}{m\mu_X + 1}\) and \(c_u = \frac{2}{m\mu_X n}\),

(ii) \(E[\Delta M^2_{n+1}|\mathcal{F}_n] \leq 6m^2 + m)E(X^2) + 9m^2\mu^2 = K_u\),

(iii) \(|f(Z_n)| = m\mu_X|1 - 2Z_n| \leq 3m\mu_X = K_f\),

(iv) \(E(\gamma_{n+1}\Delta M_{n+1}|\mathcal{F}_n) \leq \frac{1}{T_n}E(\Delta M_{n+1}|\mathcal{F}_n) = 0 = K_e\).

Proposition 1. The proportion of white balls in the urn after \(n\) draws, \(Z_n\), converges almost surely to \(\frac{1}{2}\).

Proof of Proposition \[7\]. Since the process \(Z_n\) satisfies the stochastic approximation algorithm defined by Equation (12), we apply Theorem \[5\]. As the function \(f\) is continuous we conclude that \(Z_n\) converges almost surely to \(\frac{1}{2}\): the unique stable zero of the function \(f\).
We apply the previous results to the urn composition. As we can write \( \frac{W_n}{n} \overset{a.s.}{=} \left(\frac{1}{2} + o(1)\right)\left(\mu_X m + o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right)\right) \), then this corollary follows:

**Corollary 1.** The number of white balls in the urn after \( n \) draws, \( W_n \), satisfies for \( n \) large enough

\[
W_n \overset{a.s.}{=} \frac{\mu_X m n}{2} + o(\sqrt{n} \ln(n)^\delta), \quad \delta > \frac{1}{2}.
\]

**Proof of claim 2** We aim to apply Theorem \( \ref{thm:claim2} \). For this reason, we need to find this limits:

\[
\lim_{n \to \infty} \mathbb{E}\left[\left(\frac{n}{T_n}\right)^2 \Delta M_{n+1}^2 | \mathcal{F}_n\right] \quad \text{and} \quad \lim_{n \to \infty} -\frac{n}{T_n} f'(Z_n).
\]

We have

\[
\mathbb{E}[\Delta M_{n+1}^2 | \mathcal{F}_n] = \mathbb{E}(X_n^2)\mathbb{E}[(m - \xi_{n+1} - mZ_n)^2 | \mathcal{F}_n] + \mu_X^2 \mathbb{E}[(m - 2mZ_n)^2 | \mathcal{F}_n]
\]

\[
-2\mu_X^2 \mathbb{E}[(m - \xi_{n+1} - mZ_n)(m - 2mZ_n) | \mathcal{F}_n]
\]

\[
= (\sigma_X^2 + \mu_X^2) \left[ m^2 - 4m^2 Z_n + 4m^2 Z_n^2 + mZ_n(1 - Z_n) \frac{T_n - m}{T_n - 1} \right] - \mu_X^2 \left[ m^2 + 4m^2 Z_n^2 - 4m^2 Z_n \right].
\]

As \( n \) tends to infinity, we have \( Z_n \overset{a.s.}{\to} \frac{1}{2} \) and \( \frac{T_n - m}{T_n - 1} \overset{a.s.}{\to} 1 \). Then,

\[
\lim_{n \to \infty} \mathbb{E}[\Delta M_{n+1}^2 | \mathcal{F}_n] \overset{a.s.}{=} (\sigma_X^2 + \mu_X^2) \frac{m}{4} \quad \text{and} \quad \lim_{n \to \infty} -\frac{n}{T_n} f'(Z_n) \overset{a.s.}{=} 2.
\]

According to Theorem \( \ref{thm:claim2} \), \( \sqrt{n}(Z_n - \frac{1}{2}) \) converges in distribution to \( \mathcal{N}(0, \frac{\sigma_X^4 + \mu_X^4}{12\mu_X m}) \). Finally, by writing \( \left( \frac{W_n - \frac{1}{2} T_n}{\sqrt{n}} \right) = \sqrt{n}(Z_n - \frac{1}{2}) \frac{T_n}{\sqrt{n}} \), we conclude using Slutsky theorem.

**Proof of claim 3** To prove this claim, we follow the proof of Lemma 3 and Theorem 2 in \( \ref{thm:claim3} \). Using the same methods, we show in a first step that the variables \( (X_n(m - \xi_n))_{n \geq 0} \) are \( \alpha \)-mixing variables with a strong mixing coefficient \( \alpha(n) = o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right) \), \( \delta > \frac{1}{2} \). To conclude, we adapt the Bernstein method. Consider the same notation as in Theorem 2 in \( \ref{thm:claim3} \), and define \( S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{\xi}_i \)

where \( \tilde{\xi}_i = X_i(m - \xi_i) - \mu_X(m - \mathbb{E}(\xi_i)) \). At first, we need to estimate the variance of \( W_n \).

**Proposition 2.** The variance of \( W_n \) satisfies

\[
\mathbb{V}ar(W_n) = \frac{m(\sigma_X^2 + \mu_X^2) + m^2 \sigma_X^2}{12} n + o(\sqrt{n} \ln(n)^\delta), \quad \delta > \frac{1}{2}.
\]

**Proof of Proposition 2** Recall that the number of white balls in the urn satisfies Equation \( \ref{eq:claim2} \), then

\[
\mathbb{V}ar(W_{n+1}) = \mathbb{V}ar(W_n) + \mathbb{V}ar(X_n(m - \xi_n)) + 2 \mathbb{C}ov(W_{n-1}, X_n(m - \xi_n)).
\]

We have \( \mathbb{V}ar(X_n(m - \xi_n)) = (\sigma_X^2 + \mu_X^2) \left( \mathbb{V}ar(mZ_{n-1}) + \mathbb{E}\left( mZ_{n-1}(1 - Z_{n-1}) \frac{T_{n-1} - m}{T_{n-1} - 1} \right) \right) + \sigma_X^2 \mathbb{E}(m - \xi_n)^2. \)
Using Equation (14) and the fact that \( Z_n \overset{a.s.}{\to} 1 \), we obtain
\[
\text{Var}(W_{n+1}) = \left(1 - \frac{2}{n} + o\left(\frac{\ln(n)^\delta}{n^2}\right)\right)\text{Var}(W_n) + \frac{m(\sigma_X^2 + \mu_X^2) + m^2\sigma_X^2}{4} + o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right)
\]
\[= a_n \text{Var}(W_n) + b_n,
\]
where \( a_n = \left(1 - \frac{2}{n} + o\left(\frac{\ln(n)^\delta}{n^2}\right)\right) \) and \( b_n = \frac{m(\sigma_X^2 + \mu_X^2) + m^2\sigma_X^2}{4} + o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right) \).

Thus,
\[
\text{Var}(W_n) = \left(\prod_{k=1}^{n} a_k\right)\left(\text{Var}(W_0) + \sum_{k=0}^{n-1} \frac{b_k}{\prod_{j=0}^{k} a_j}\right).
\]

There exists a constant \( a \) such that
\[
\prod_{k=1}^{n} a_k = e^a \left(1 + o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right)\right),
\]
which leads to
\[
\text{Var}(W_n) = \frac{m(\sigma_X^2 + \mu_X^2) + m^2\sigma_X^2}{12} n + o(\sqrt{n\ln(n)}), \quad \delta > \frac{1}{2}.
\]

Recall that we follow the proof of Theorem 2 in [17], using Equation (15), we conclude that
\[
\frac{W_n - \mathbb{E}(W_n)}{\sqrt{n}} \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{m(\sigma_X^2 + \mu_X^2) + m^2\sigma_X^2}{12}\right).
\]

**Proof of Theorem 2.** Consider the urn model defined in (1) with \( Q_n = \begin{pmatrix} X_n & 0 \\ 0 & X_n \end{pmatrix} \). The following recurrences hold:
\[
W_{n+1} = W_n + X_{n+1} \xi_{n+1} \quad \text{and} \quad T_{n+1} = T_n + mX_{n+1}.
\]
As \( T_n \) is a sum of iid random variables then \( T_n \) satisfies the following
\[
T_n \overset{a.s.}{=} \frac{\mu X m}{2} n + o(\sqrt{n\ln(n)}).
\]

The processes \( \tilde{M}_n = \prod_{k=1}^{n-1} \left(\frac{T_k}{T_k + m\mu_X}\right)W_n \) and \( \tilde{N}_n = \prod_{k=1}^{n-1} \left(\frac{T_k}{T_k + m\mu_X}\right)B_n \) are two \( \mathcal{F}_n \) positive martingales. In view of (18), we have \( \prod_{k=1}^{n-1} \left(\frac{T_k}{T_k + m\mu_X}\right) \overset{a.s.}{=} \frac{e^\gamma}{n} \left(1 + o\left(\frac{\ln(n)^\delta}{\sqrt{n}}\right)\right) \) for a positive constant \( \gamma \).

Thus, there exists nonnegative random variables \( \tilde{W}_\infty \) and \( \tilde{B}_\infty \) such that \( \tilde{W}_\infty + \tilde{B}_\infty \overset{a.s.}{=} m\mu_X \) and
\[
\frac{W_n}{n} \overset{a.s.}{\to} \tilde{W}_\infty, \quad \text{and} \quad \frac{B_n}{n} \overset{a.s.}{\to} \tilde{B}_\infty.
\]

**Example:** In the original Pólya urn model [2], when \( m = 1 \) and \( X = C \) (deterministic), the random variable \( \tilde{W}_\infty/C \) has a Beta\( (\frac{B_n}{C}, \frac{W_n}{C}) \) distribution [7, 19]. Whereas, M.R. Chen and M.
Kuba [10] considered the case when $X = C$ (non random) and $m > 1$. They gave moments of all orders of $W_n$ and proved that $\hat{W}_\infty$ cannot be an ordinary Beta distribution.

**Remark:** Suppose that the random variable $X$ has moments of all orders, let $m_k = E(X^k)$, for $k \geq 1$. We have, almost surely, $W_n \leq T_n$ then, by Minskowski inequality, we obtain $\mathbb{E}(W_n^{2k}) \leq (mn)2k \mathbb{E}(X^{2k})$. Using Carleman’s condition we conclude that, if $\sum_{k \geq 1} \mu_{2k} = \infty$, then the random variable $\hat{W}_\infty$ is determined by its moments. Unfortunately, till now we still unable to give exact expressions of moments of all orders of $W_n$. But, we can characterize the distribution of $\hat{W}_\infty$ in the case when the variable $X$ is bounded.

**Lemma 4.** Assume that $X$ is a bounded random variable, then, for fixed $W_0, B_0$ and $m$ the random variable $\hat{W}_\infty$ is absolutely continuous.

The proof that $\hat{W}_\infty$ is absolutely continuous is very close to that of Theorem 4.2 in [11]. We give the main proposition to make the proof clearer.

**Proposition 3.** Let $\Omega_\ell$ be a sequence of increasing events such that $\mathbb{P}(\cup_{\ell \geq 0} \Omega_\ell) = 1$. If there exists nonnegative Borel measurable function $\{f_\ell\}_{\ell \geq 1}$ such that $\mathbb{P}(\Omega_\ell \cap \hat{W}_\infty^{-1}(B)) = \int_B f_\ell(x)dx$ for all Borel sets $B$, then, $f = \lim_{\ell \to +\infty} f_\ell$ exists almost everywhere and $f$ is the density of $\hat{W}_\infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that there exists a constant $A$ such that, we have almost surely, $X \leq A$.

**Lemma 5.** Define the events

$$\Omega_\ell := \{W_\ell \geq mA \text{ and } B_\ell \geq mA\},$$

then, $(\Omega_\ell)_{\ell \geq 0}$ is a sequence of increasing events, moreover we have $\mathbb{P}(\cup_{\ell \geq 0} \Omega_\ell) = 1$.

Next, we just need to show that the restriction of $\hat{W}_\infty$ on $\Omega_{\ell,j} = \{\omega; W_\ell(\omega) = j\}$ has a density for each $j$, with $Am \leq j \leq T_{\ell-1}$. Let $(p_c)_{c \in \text{supp}(X)}$ the distribution of $X$.

**Lemma 6.** For a fixed $\ell > 0$, there exists a positive constant $\kappa$, such that, for every $c \in \text{supp}(X)$, $n \geq \ell + 1$, $Am \leq j \leq T_{\ell-1}$ and $k \leq Am(n+1)$, we have

$$\sum_{i=0}^{m} \mathbb{P}(W_{n+1} = j + k | W_n = j + k - ci) \leq p_c(1 - \frac{1}{n} + \frac{\kappa}{n^2}). \quad (19)$$

**Proof.** According to Lemma 4.1 [11], for $Am \leq j \leq T_{\ell-1}$, $n \geq \ell$ and $k \leq Am(n+1)$, the following holds:

$$\sum_{i=0}^{m} \binom{j + c(k - i)}{i} \binom{T_n - j - c(k - i)}{m - i} = \frac{T_n^m}{m!} + \frac{(1 - m - 2c)T_n^{m-1}}{2(m-1)!} + ..., \quad (20)$$

which is a polynomial in $T_n$ of degree $m$ with coefficients depending on $W_0, B_0, m$ and $c$ only.
Let \( u_{n,k}(c) = \sum_{i=0}^{m} \mathbb{P}(W_{n+1} = j + k | W_n = j + k - ic) \). Applying Equation (20) to our model we have
\[
\begin{align*}
  u_{n,k}(c) &= p_c \sum_{i=0}^{m} \binom{j + k}{i} \binom{T_n - j - k}{m - i} \binom{T_n}{m}^{-1} \\
  &= p_c \left( \frac{T_n}{m} \right)^{-1} \left( \frac{T_n^m}{m!} + \frac{(1-m-2c)T_n^{m-1} + \ldots}{(m-1)!} \right) \left( \frac{T_n^m}{m!} + \frac{(1-m)T_n^{m-1} + \ldots}{2(m-1)!} \right)^{-1} \\
  &= p_c \left( 1 - \frac{1}{n} + O\left( \frac{1}{n^2} \right) \right). 
\end{align*}
\]

Later, we will limit the proof by mentioning the main differences with Lemma 4.1 [11]. For a fixed \( \ell \) and \( n \geq \ell + 1 \), we denote by \( v_{n,j} = \max_{0 \leq k \leq Am} \mathbb{P}(W_{\ell+n} = j + k | W_\ell = j) \). We have the following inequality:
\[
\begin{align*}
v_{n+1,j} &\leq \max_{0 \leq k \leq Am(n+1)} \left\{ \sum_{i=0}^{m} \sum_{c \in \text{supp}(X)} \mathbb{P}(W_{\ell+n+1} = j + k | W_{\ell+n} = j + k - ci) \right\} \\
&\leq \max_{0 \leq k \leq Am(n+1)} \left\{ \sum_{i=0}^{m} \sum_{c \in \text{supp}(X)} \mathbb{P}(W_{\ell+n+1} = j + k | W_{\ell+n} = j + k - ci) \times \mathbb{P}(W_{\ell+n} = j + k - ci | W_\ell = j) \right\} \\
&\leq \max_{0 \leq k \leq Am(n+1)} \sum_{i=0}^{m} \sum_{c \in \text{supp}(X)} \mathbb{P}(W_{\ell+n+1} = j + k | W_{\ell+n} = j + k - ci) \\
&\quad \times \max_{0 \leq k \leq Am} \mathbb{P}(W_{\ell+n} = j + \tilde{k} | W_\ell = j) \\
&\leq \sum_{c \in \text{supp}(X)} p_c \left( 1 - \frac{1}{n+l} + \frac{\kappa}{(n+l)^2} \right) v_{n,j} \\
&= \left( 1 - \frac{1}{n+l} + \frac{\kappa}{(n+l)^2} \right) v_{n,j}.
\end{align*}
\]

This implies that there exists some positive constant \( C(\ell) \), depending on \( \ell \) only, such that, for a fixed \( \ell \) and for all \( n \geq \ell + 1 \), we get
\[
\max_{0 \leq k \leq Am(n-l)} \mathbb{P}(W_n = j + k | W_l = j) \leq \prod_{i=\ell}^{n} \left( 1 - \frac{1}{i} + \frac{\kappa}{i^2} \right) \leq \frac{C(\ell)}{n}. \tag{22}
\]

The rest of the proof follows.

\[ \square \]

**Proof of Theorem 3**. Consider the urn model evolving by the matrix \( Q_n = \begin{pmatrix} 0 & X_n \\ Y_n & 0 \end{pmatrix} \). According to Equation (11), we have the following recursions:
\[
W_{n+1} = W_n + X_{n+1}(m - \xi_{n+1}) \quad \text{and} \quad T_{n+1} = T_n + mX_{n+1} + \xi_{n+1}(Y_{n+1} - X_{n+1}). \tag{23}
\]
Lemma 7. The proportion of white balls after \( n \) draws, \( Z_n \), satisfies the stochastic algorithm defined by (22), where \( f(x) = m(\mu_Y - \mu_X)x^2 - 2\mu_Xmx + \mu_Xm \), \( \gamma_n = \frac{1}{T_n} \) and \( \Delta M_{n+1} = D_{n+1} - \mathbb{E}[D_{n+1} | \mathcal{F}_n] \), with \( D_{n+1} = \xi_{n+1}(Z_n(X_{n+1} - Y_{n+1}) - X_{n+1}) + mX_{n+1} \).

Proof. We check the conditions of Definition 1, indeed,

(i) recall that \( T_n = T_0 + m \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \xi_i(Y_i - X_i) \), then \( \mathbb{E} \left[ \frac{T_n}{n} \right] \leq \mathbb{E} \left[ \frac{T_0}{n} + \frac{m}{n} \sum_{i=1}^{n} X_i + \frac{m}{n} \sum_{i=1}^{n} |Y_i - X_i| \right] \). By the strong law of large numbers we have \( \frac{T_n}{n} \leq m(\mu_X + \mu_Y - X_1) + 1 \). On the other hand, we have \( T_n \geq \min \left( X_i, Y_i \right) \). Indeed, the following bound holds

\[
\frac{1}{(m(\mu_X + \mu_Y - X_1) + 1)n} \leq \frac{1}{T_n} \leq \frac{1}{\min \left( X_i, Y_i \right)n},
\]

then \( c_l = \frac{1}{(m(\mu_X + \mu_Y - X_1) + 1)n} \) and \( c_u = \frac{1}{\min \left( X_i, Y_i \right)n} \).

(ii) \( \mathbb{E}[\Delta M_{n+1} | \mathcal{F}_n] \leq (\mu_X - \mu_Y)^2 + 3\mu_X(m + m^2) + m^2\mu_X^2 + 2m^2\mu_X\mu_Y + m^2(|\mu_X - \mu_Y| + 3\mu_X) = K_u \),

(iii) \( |f(Z_n)| \leq m(\mu_Y - \mu_X + 3\mu_X) = K_f \),

(iv) \( \mathbb{E}[\frac{1}{T_{n+1}} \Delta M_{n+1} | \mathcal{F}_n] \leq \frac{1}{T_n} \mathbb{E}[\Delta M_{n+1} | \mathcal{F}_n] = 0 \)

\( \square \)

Proposition 4. The proportion of white balls in the urn after \( n \) draws, \( Z_n \), satisfies as \( n \) tends to infinity

\[
Z_n \xrightarrow{a.s.} z := \frac{\sqrt{\mu_X}}{\sqrt{\mu_X} + \sqrt{\mu_Y}}.
\]

Proof. The proportion of white balls in the urn satisfies the stochastic approximation algorithm defined in (12). As the function \( f \) is continuous, by Theorem 5, the process \( Z_n \) converges almost surely to \( z = \frac{\sqrt{\mu_X}}{\sqrt{\mu_X} + \sqrt{\mu_Y}} \), the unique zero of \( f \) with negative derivative.

\( \square \)

Next, we give an estimate of \( T_n \), the total number of balls in the urn after \( n \) draws, in order to describe the asymptotic of the urn composition. By Equation (23), we have

\[
\frac{T_n}{n} = \frac{T_0}{n} + \frac{m}{n} \sum_{i=1}^{n} X_i + \frac{m(\mu_Y - \mu_X)}{n} \sum_{i=1}^{n} Z_{i-1} + \frac{1}{n} \sum_{i=1}^{n} \left[ \xi_i(Y_i - X_i) - \mathbb{E}[\xi_i(Y_i - X_i) | \mathcal{F}_{i-1}] \right].
\]

Since \( (X_i)_{i \geq 1} \) are iid random variables, then by the strong law of large numbers we have \( \sum_{i=1}^{n} Z_{i-1} \) converges almost surely, as \( n \) tends to infinity, to \( z \). Finally, we prove that last term in the right side tends to zero, as \( n \) tends to infinity. In fact, let \( G_n = \sum_{i=1}^{n} \left[ \xi_i(Y_i - X_i) - \mathbb{E}[\xi_i(Y_i - X_i) | \mathcal{F}_{i-1}] \right] \), then \( (G_n, \mathcal{F}_n) \) is a martingale difference sequence such that

\[
\frac{\mathbb{E}[\Delta G_n | \mathcal{F}_{i-1}]}{n} \xrightarrow{a.s.} 0.
\]
where $\nabla G_n = G_n - G_{n-1} = \xi_n(Y_n - X_n) - \mathbb{E}[\xi_n(Y_n - X_n)|\mathcal{F}_{n-1}]$ and $< G >_n$ denotes the quadratic variation of the martingale.

By a simple computation, we have the almost sure convergence of $\mathbb{E}[\nabla G_i^2|\mathcal{F}_{i-1}]$ to $(mz(1-z) + m^2z^2)(\sigma_Y^2 + \sigma_X^2)$. Therefore, Cesáro lemma ensures that, $\frac{\langle G \rangle_n}{n}$ converges to $(mz(1-z) + m^2z^2)(\sigma_Y^2 + \sigma_X^2)$ and $\frac{\langle G \rangle_n}{n} \overset{a.s.}{\longrightarrow} 0$. Thus, for $n$ large enough we have

$$\frac{T_n}{n} \overset{a.s.}{\longrightarrow} m\sqrt{\mu_X}\sqrt{\mu_Y}. \quad (25)$$

In view of Equation (25), we describe the asymptotic behavior of the urn composition after $n$ draws. One can write $W_n = W_0 + \sum_{i=1}^{n} \xi_i(Y_i - X_i)$ and $\frac{B_n}{n} = \frac{B_0}{n} + \sum_{i=1}^{n} \xi_i(Y_i - X_i)$, using Equations (24) and Slutsky theorem, we have, as $n$ tends to infinity, $\frac{W_n}{n} \overset{a.s.}{\longrightarrow} m\sqrt{\mu_X}\sqrt{\mu_Y}z$ and $\frac{B_n}{n} \overset{a.s.}{\longrightarrow} m\sqrt{\mu_X}\sqrt{\mu_Y}(1-z)$.

**Proof of claim 2**

Later, we aim to apply Theorem 6. In our model, we have $\gamma_n = \frac{1}{T_n}$, then we need to control the following asymptotic behaviors

$$\lim_{n \to +\infty} \mathbb{E}\left[\left(\frac{n}{T_n}\right)^2 \Delta M_{n+1}^2 \mathbb{I}_n \right] \quad \text{and} \quad \lim_{n \to +\infty} -\frac{n}{T_n} f'(Z_n).$$

In fact, recall that $\frac{n}{T_n}$ converges almost surely to $\frac{1}{m\sqrt{\mu_X}\sqrt{\mu_Y}}$ and $\mathbb{E}[\Delta M_{n+1}^2|\mathcal{F}_n] = \mathbb{E}[D_{n+1}^2|\mathcal{F}_n] + \mathbb{E}[D_{n+1}|\mathcal{F}_n]^2$. Since $\mathbb{E}[D_{n+1}|\mathcal{F}_n] \overset{a.s.}{\to} 0$, we have,

$$\mathbb{E}[D_{n+1}^2|\mathcal{F}_n] = \mathbb{E}\left[Z_n^2(X_{n+1} - Y_{n+1})^2 - 2Z_n X_{n+1} + X_{n+1}|\mathcal{F}_n\right] \mathbb{E}[\xi_{n+1}^2|\mathcal{F}_n] + m^2 \mathbb{E}(X^2) + 2m^2 \left(Z_n^2(\mathbb{E}(X^2) - \mu_X\mu_Y) - Z_n \mathbb{E}(X^2)\right).$$

Using the fact that $\mathbb{E}[\xi_{n+1}^2|\mathcal{F}_n] = mZ_n(1 - Z_n)\frac{T_{n-1} - m^2Z_n^2}{n} + \frac{m^2Z_n^2}{n}$ and that $Z_n$ converges almost surely to $z$, we conclude that $\mathbb{E}[D_{n+1}|\mathcal{F}_n] \overset{a.s.}{\to} 0$. Applying Theorem 6, we obtain the following

$$\sqrt{n}(Z_n - z) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{G(z)}{3m^2\mu_X\mu_Y}\right). \quad (26)$$

But, we can write $\frac{W_n - zT_n}{\sqrt{n}} = \sqrt{n}\left(\frac{W_n}{T_n} - z\right)$. Thus, it is enough to use Slutsky theorem to conclude the proof.

**Proof of Theorem [7]**

Consider the urn model defined in (1) with $Q_n = \begin{pmatrix} X_n & 0 \\ 0 & Y_n \end{pmatrix}$. The process of the urn satisfies the following recursions:

$$W_{n+1} = W_n + X_{n+1}\xi_{n+1} \quad \text{and} \quad T_{n+1} = T_n + mY_{n+1} + \xi_{n+1}(X_{n+1} - Y_{n+1}). \quad (27)$$

**Lemma 8.** If $\mu_X \neq \mu_Y$, the proportion of white balls in the urn after $n$ draws satisfies the stochastic algorithm defined by (22) where $\gamma_n = \frac{1}{T_{n-1}}$, $f(x) = m(\mu_Y - \mu_X)x(x - 1)$ and $\Delta M_{n+1} = D_{n+1} - \mathbb{E}[D_{n+1}|\mathcal{F}_n]$ with $D_{n+1} = \xi_{n+1}(Z_n(Y_{n+1} - X_{n+1}) + X_{n+1}) - mZ_nY_{n+1}$.

**Proof.** We check that, if $\mu_X \neq \mu_Y$, the conditions of definition (1) hold. Indeed,
Proposition 5. The proportion of white balls in the urn after n draws, Z_n, satisfies almost surely

\[ \lim_{n \to \infty} Z_n = \begin{cases} 
1, & \text{if } \mu_X > \mu_Y; \\
0, & \text{if } \mu_X < \mu_Y; \\
\hat{Z}_\infty, & \text{if } \mu_X = \mu_Y,
\end{cases} \]

where \( \hat{Z}_\infty \) is a positive random variable.

Proof of Proposition 5. Recall that, if \( \mu_X \neq \mu_Y \), \( Z_n \) satisfies the stochastic algorithm defined in Lemma 8. As the function \( f \) is continuous, by Theorem 6 we conclude that \( Z_n \) converges almost surely to the stable zero of the function \( h \) with a negative derivative, which is 1 if \( \mu_X > \mu_Y \) and 0 if \( \mu_X < \mu_Y \).

In the case when \( \mu_X = \mu_Y \), we have \( Z_{n+1} = Z_n + \frac{P_{n+1}}{T_{n+1}} \), where \( P_{n+1} = X_{n+1} \xi_{n+1} - Z_n (mY_{n+1} + \xi_{n+1}(X_{n+1} - Y_{n+1})) \). Since \( \mathbb{E}[P_{n+1} | \mathcal{F}_n] = 0 \), then \( Z_n \) is a positive martingale which converges almost surely to a positive random variable \( \hat{Z}_\infty \).

As a consequence, we have

Corollary 2. The total number of balls in the urn, \( T_n \), satisfies as \( n \) tends to infinity

if \( \mu_X \geq \mu_Y \)

\[ \frac{T_n}{n} \xrightarrow{a.s.} m\mu_X. \]

Proof. In fact, let \( M_n = \sum_{i=1}^{n} \xi_i (X_i - Y_i) - \mathbb{E}[\xi_i (X_i - Y_i) | \mathcal{F}_{i-1}] \), we have

\[ \frac{T_n}{n} = \frac{T_0}{n} + \frac{m}{n} \sum_{i=1}^{n} Y_i + \frac{1}{n} \sum_{i=1}^{n} \xi_i (X_i - Y_i) \]

\[ = \frac{T_0}{n} + \frac{m}{n} \sum_{i=1}^{n} Y_i + \frac{m(\mu_X - \mu_Y)}{n} \sum_{i=1}^{n} Z_{i-1} + \frac{M_n}{n}. \]

As it was proved in the previous theorem, we show that, as \( n \) tends to infinity, we have \( \frac{M_n}{n} \xrightarrow{a.s.} 0 \). Recall that, if \( \mu_X > \mu_X \), \( Z_n \) converges almost surely to 1. Then, using Cesáro lemma, we obtain the limits requested. If \( \mu_X = \mu_Y \), we have \( \frac{1}{n} \sum_{i=1}^{n} Y_i \) converges to \( \mu_Y \). \( \square \)
Using the results above, the convergence of the normalized number of white balls follows immediately. Indeed, if $\mu_X > \mu_Y$, we have, as $n$ tends to infinity,

$$\frac{W_n}{n} = \frac{W_n T_n}{T_n n} \xrightarrow{a.s.} m\mu_X,$$

Let $G_n = \left( \prod_{i=1}^{n-1} \left(1 + \frac{m\mu_X}{T_i}\right) \right)^{-1} B_n$, then $(G_n, F_n)$ is a positive martingale. There exists a positive number $A$ such that $\prod_{i=1}^{n-1} \left(1 + \frac{m\mu_Y}{T_i}\right) \simeq An^\rho$. Then, as $n$ tends to infinity we have

$$\frac{B_n}{n^\rho} \xrightarrow{a.s.} B_\infty,$$

where $B_\infty$ is a positive random variable.

If $\mu_X = \mu_Y$, the sequences $\left( \prod_{i=1}^{n-1} \left(1 + \frac{m\mu_X}{T_i}\right) \right)^{-1} W_n$ and $\left( \prod_{i=1}^{n-1} \left(1 + \frac{m\mu_Y}{T_i}\right) \right)^{-1} B_n$ are $F_n$ martingales such that $\left( \prod_{i=1}^{n-1} \left(1 + \frac{m\mu_X}{T_i}\right) \right)^{-1} \simeq B_n$, where $B > 0$, then, as $n$ tends to infinity, we have

$$\frac{W_n}{n} \xrightarrow{a.s.} W_\infty \quad \text{and} \quad \frac{B_n}{n} \xrightarrow{a.s.} \tilde{B}_\infty,$$

where $W_\infty$ and $\tilde{B}_\infty$ are positive random variables satisfying $\tilde{B}_\infty = m\mu_X - W_\infty$.

**Remark:** The case when $\mu_X < \mu_Y$ is obtained by interchanging the colors. In fact we have the following results:

$$T_n \xrightarrow{a.s.} m\mu_Y n + o(n), \quad W_n = \bar{W}_\infty n^\sigma + o(n) \quad \text{and} \quad B_n = m\mu_Y n + o(n),$$

where $\bar{W}_\infty$ is a positive random variable and $\sigma = \frac{\mu_X}{\mu_Y}$.

**References**

[1] C. Mailler, N. Lasmer and O. Selmi. (2017). Multiple drawing multi-color urns by stochastic approximation.(to appear).

[2] F. Eggenberger and G. Pólya. (1923). Über die statistik verkeletter vorge. Zeitschrift für Angewandte Mathematik und Mechanic, 1:279-289.

[3] G. Pagés and S. Laruelle. (2015). Randomized urns models revisited using stochastic approximation. Annals of Applied Probability, (23)4: 1409-1436.

[4] H. Renlund. (2010). Generalized Polya urns via stochastic approximation. arxiv: 1002.3716v1.

[5] H. Renlund. (2011). Limit theorem for Stochastic approxiamtion algorithm. arxiv:1102.4741v1.

[6] H. Mahmoud. (2004). Random spouts as internet model and Pólya processes. Actainformat-ica, 41: 1-18.
[7] K.B. Athreya and P.E. Ney. (1972). Branching Processes. Springer-Verlag, Berlin.

[8] L.J. Wei. (1978). An application of an urn model to the design of sequential controlled clinical trials. Journal of American Statistics Association, (73), 363:559-563.

[9] M. Kuba, H. Mahmoud and A. Panholzer. (2013). Analysis of a generalized Friedman’s urn with multiple drawings, Discrete Applied Mathematics, 161, 2968-2984.

[10] M. R Chen and M. Kuba. (2013). On generalized Polya urn models, Theory of probability and its application, 40, 1169-1186.

[11] M. R Chen and C. Z Wei. (2005). A new urn model. Applied Probability, (42)4, 964-976.

[12] M. Kuba and H. Sulzbach. (2016). On martingale tail sums in affine two-color urn models with multiple drawings. arXiv:1509.09053.

[13] M. Kuba and H. Mahmoud. (2016). Two-colour balanced affine urn models with multiple drawings I: Central limit theorem. arxiv:1503.09069.

[14] M. Kuba and H. Mahmoud. (2016). Two-colour balanced affine urn models with multiple drawings II: large-index and triangular urns. arxiv: 1509.09053.

[15] N.L. Johnson, S. Kotz. (1977). Urn models and their application. Johnson Wiley.

[16] R. Aguech. (2009). Limit Theorems for Random Triangular Urns Schemes. Journal of Applied Probability, 46(3), 827-843.

[17] R. Aguech, N. Lasmer, O. Selmi. (2017). A generalized urn model with multiple drawing and random addition. (to appear.)

[18] R.N. Goldman. (1985). Polya’s urn model and computer aided geometry design. SIAM Journal on Algebraic Discrete Methods, 6(1), 1-28.

[19] S. Janson. (2006). Limit theorems of triangular urn schemes. Probability Theory and Related Fields, 134(3), 417-452.