Sigma involutions associated with parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$

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**ABSTRACT**

An irreducible module for the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ is said to be of $\sigma$-type if an automorphism of the fusion algebra of $K(\mathfrak{sl}_2, k)$ of order $k$ is trivial on it. For any integer $k \geq 3$, we show that there exists an automorphism of order 2 of the subalgebra of the fusion algebra of $K(\mathfrak{sl}_2, k)(\theta)$ spanned by the irreducible direct summands of $\sigma$-type irreducible $K(\mathfrak{sl}_2, k)$-modules, where $\theta$ is an involution of $K(\mathfrak{sl}_2, k)$. We discuss some examples of such an automorphism as well.

**1. Introduction**

Symmetry in the fusion algebra of a vertex operator algebra is an important subject in the representation theory of vertex operator algebras. The subalgebra of the fusion algebra on which the symmetry is trivial may admit another symmetry. Our main concern is the symmetry in such a subalgebra which appears if we consider the subalgebra as a subalgebra of the fusion algebra of an orbifold of the original vertex operator algebra.

The fusion algebra of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ associated with $\mathfrak{sl}_2$ and an integer $k \geq 2$ has an automorphism of order $k$ (see for example [1–3]). An irreducible $K(\mathfrak{sl}_2, k)$-module is said to be of $\sigma$-type if the automorphism is trivial on it. If $k \geq 3$, then the automorphism group of $K(\mathfrak{sl}_2, k)$ is of order 2, and there is a unique involution $\theta$ [4,5]. Any $\sigma$-type irreducible $K(\mathfrak{sl}_2, k)$-module is $\theta$-stable, and it is a direct sum of two irreducible modules for the fixed point subalgebra $K(\mathfrak{sl}_2, k)(\theta)$ (see Section 2.4 and [6,7]).

In this paper, we show that there exists an automorphism of order 2 of the subalgebra of the fusion algebra of $K(\mathfrak{sl}_2, k)(\theta)$ spanned by the irreducible direct summands of $\sigma$-type irreducible $K(\mathfrak{sl}_2, k)$-modules. We call such an automorphism a $\sigma$-involution. We develop
a general theory of $\sigma$-involutions. Interesting examples of $\sigma$-involutions appear as automorphisms of certain lattice vertex operator algebras. We discuss the relationship between those $\sigma$-involutions and isometries of the underlying lattices.

The parafermion vertex operator algebra $K(sl_2, k)$ is by definition the commutant of the Heisenberg vertex operator algebra generated by the Cartan subalgebra of $sl_2$ in the simple affine vertex operator algebra associated with the affine Kac-Moody algebra $sl_2$ at an integral level $k \geq 2$; see [1,4,5,8,9] and Section 2.4 for basic properties of $K(sl_2, k)$. It is a simple, self-dual, rational, and $G_2$-cofinite vertex operator algebra of CFT-type with central charge $2(k - 1)/(k + 2)$. The irreducible $K(sl_2, k)$-modules are denoted by $M^{ij}$ with $0 \leq i \leq k$ and $0 \leq j < k$, where $j$ is considered to be an integer modulo $k$. We have $M^{ij} \cong M^{k - ij - i}$, and $M^{ij}$, $0 \leq j < i \leq k$, form a complete set of representatives of the equivalence classes of irreducible $K(sl_2, k)$-modules. We write $M^0$ for $K(sl_2, k) = M^{k,0} \cong M^{0,0}$. Among those irreducible $M^0$-modules, $M^{2ij}$, $0 \leq j \leq [k/2]$, are of $\sigma$-type, where $[k/2]$ is the largest integer which does not exceed $k/2$. In fact, the automorphism of order $k$ of the fusion algebra of $M^0$ is trivial on $M^{ij}$ if and only if $i = 2j$ for some $0 \leq j \leq [k/2]$ (Theorem 3.1).

Assume that $k \geq 3$, and let $M^{0,+}$ and $M^{0,-}$ be the eigenspaces for $\theta$ in $M^0$ with eigenvalues 1 and $-1$, respectively. Thus $M^{0,+} = K(sl_2, k)^{(\theta)}$. The irreducible $M^{0,+}$-modules and the fusion product among them are known [6,7]. Using the results, we see that the $\sigma$-type irreducible $M^0$-module $M^{2ij}$ is a direct sum of two irreducible $M^{0,+}$-modules $(M^{2ij})^\epsilon$ for $\epsilon = 0, 1$.

$$M^{2ij} = (M^{2ij})^0 \oplus (M^{2ij})^1,$$

where the top level of $(M^{2ij})^0$ agrees with the top level of $M^{2ij}$. Moreover, it follows from the fusion product among $(M^{2ij})^\epsilon$ for $0 \leq j \leq [k/2]$ and $\epsilon \in \{0, 1\}$ (Theorem 3.4 and Remark 3.5) that

$$(M^{2ij})^\epsilon \mapsto (-1)^{j+\epsilon}(M^{2ij})^\epsilon$$

gives rise to an automorphism of order 2 of the subalgebra of the fusion algebra of $M^{0,+}$ spanned by $(M^{2ij})^\epsilon$ for $0 \leq j \leq [k/2]$ and $\epsilon \in \{0, 1\}$ (Theorem 3.6). Therefore, for a vertex operator algebra $V$ containing a vertex operator subalgebra $W \cong K(sl_2, k)$ such that $V$ is a direct sum of $\sigma$-type irreducible $W$-modules, we can define an automorphism $\sigma_W$ of the vertex operator algebra $V$ by multiplying the elements of the irreducible direct summands isomorphic to $(M^{2ij})^\epsilon$ by $(-1)^{j+\epsilon}$ (Theorem 3.7). We call $\sigma_W$ the $\sigma$-involution of $V$ associated with $W$.

A lattice vertex operator algebra $V_{\sqrt{2}A_{k-1}}$ associated with $\sqrt{2}$ times a root lattice of type $A_{k-1}$ contains a vertex operator subalgebra $W \cong K(sl_2, k)$ [10]. It is shown that the $\sigma$-involution associated with $W$ coincides with the lift $\theta$ of the $-1$-isometry of $\sqrt{2}A_{k-1}$ as an automorphism of $V_{\sqrt{2}A_{k-1}}$ (Theorem 4.3). We also study $\sigma$-involutions of a lattice vertex operator algebra $V_L$ for a positive definite even lattice $L$ containing a sublattice $N \cong \sqrt{2}A_{k-1}$. We show that if $N$ is RSSD in $L$, that is, $2L \subset N + \text{Ann}_L(N)$, then $V_L$ is a direct sum of $\sigma$-type irreducible $W$-modules [(1) of Theorem 4.5]. Assume further that $L$ has no element of square norm 2. Then the $\sigma$-involution $\sigma_W$ corresponds to the RSSD involution $t_N$ of $L$ associated with $N$ [(2) of Theorem 4.5]. As to the notion of RSSD involutions, see Section 2.1 and [11,12].

Let $\nu$ be a fixed point free isometry of $\sqrt{2}A_{k-1}$ of order $k$ which corresponds to a Coxeter element of the Weyl group of the root system of type $A_{k-1}$. Let $\nu \in \text{Aut}(V_{\sqrt{2}A_{k-1}})$ be
a lift of ν. We determine the centralizer of ̂ν and the normalizer of ⟨̂ν⟩ in Aut\( \left( V_{\sqrt{A_{k-1}}} \right) \) (Theorem 5.1), and discuss their structures from a point of view of σ-involutio ns (Proposition 5.3). Furthermore, we show that the automorphism group of the fixed point subalgebra \( V_{\sqrt{A_{k-1}}} \) is isomorphic to the quotient group of the normalizer of ⟨̂ν⟩ in Aut\( \left( V_{\sqrt{A_{k-1}}} \right) \) by ⟨̂ν⟩ provided that k is an odd prime (Theorem 6.2).

The notion of RSSD involutions was introduced in [11] as a generalization of reflections. We present an example of lattice vertex operator algebras in which the RSSD involutions of the lattice corresponding to σ-involutio ns are in fact reflections associated with roots, and they generate the Weyl group of the root system (Theorem 7.6).

In the case \( k = 3 \), σ-involutio ns were previously introduced in [13], and their properti es together with interesting examples were studied in [14]. In this paper, we consider σ-involutio ns associated with \( K(\mathfrak{sl}_2, k) \) for an arbitrary integer \( k \geq 3 \). We extend and refine the arguments in [14] by using a slightly different method to develop the theory of σ-involutio ns for a general k. The calculation concerning Virasoro vectors of central charge \( 4/5 \) in Griess algebras plays a role in [14], while our method is based on the representation theory of the orbifold \( K(\mathfrak{sl}_2, k) \) \( ^{(\theta)} \) [6,7], and a realization of the parafermion vertex operator algebra \( K(\mathfrak{sl}_2, k) \) in the lattice vertex operator algebra \( V_{\sqrt{A_{k-1}}} \).

σ-involutio ns not related to \( V_{\sqrt{A_{k-1}}} \) are of special interest as they would give extra symmetries. In the case \( k = 3 \), such a σ-involution of \( V_{K_{12}} \) was studied [14, Section 5.5], where \( K_{12} \) is the Coxeter-Todd lattice of rank 12. We show that such a σ-involution also exists in the case \( k = 5 \); see Section 7.3. In fact, the σ-involutio n is related to the parafermion vertex operator algebra \( K(\mathfrak{sl}_2, 5) \) contained in \( U_{5A} \), where \( U_{5A} \) denotes the vertex operator algebra \( U \) constructed in [15,16] for the 5A case.

This paper is organized as follows. Section 2 is devoted to preliminaries. We recall the notion of RSSD involutions. Moreover, we review a central extension of a positive definite even lattice by a group of order 2 and the automorphism group of a lattice vertex operator algebra. We collect basic properties of the parafermion vertex operator algebra \( K(\mathfrak{sl}_2, k) \) as well. In Section 3, we recall a \( \mathbb{Z}_k \) symmetry in the fusion algebra of \( K(\mathfrak{sl}_2, k) \), and introduce a σ-involutio n associated with \( K(\mathfrak{sl}_2, k) \). In Section 4, we discuss some σ-involutio ns of the lattice vertex operator algebra \( V_{\sqrt{A_{k-1}}} \), and show how a σ-involutio n is related to an RSSD involuti on. In Section 5, we study the centralizer of a lift \( ̂ν \) of a fixed point free isometry \( ν \) of the lattice \( A_{k-1} \) of order \( k \) in Aut\( (V_{\sqrt{A_{k-1}}} ) \). In Section 6, we determine the automorphism group of the fixed point subalgebra of the vertex operator algebra \( V_{\sqrt{A_{k-1}}} \) by \( ̂ν \) in the case where \( k \) is an odd prime. In Section 7, we provide some examples of σ-involutio ns of certain lattice vertex operator algebras. We review the irreducible modules and fusion rules for \( U_{5A} \) in Appendix.

2. Preliminaries

Let \( (L, \langle \cdot, \cdot \rangle) \) be a positive definite integral lattice, that is, \( L \) is a free \( \mathbb{Z} \)-module of finite rank equipped with a positive definite symmetric \( \mathbb{Z} \)-bilinear form \( \langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z} \). We set \( L(n) = \{ \alpha \in L \mid \langle \alpha, \alpha \rangle = n \} \) for a positive integer \( n \). An element \( \alpha \in L(2) \) is called a root. If \( \langle \alpha, \alpha \rangle \in 2\mathbb{Z} \) for any \( \alpha \in L \), then \( L \) is said to be even. The isometry group \( O(L) \) of \( L \) is the group of automorphisms of the free \( \mathbb{Z} \)-module \( L \) preserving \( \langle \cdot, \cdot \rangle \),
that is,

\[ O(L) = \{ g \in \text{Aut}(L) \mid (g\alpha, g\beta) = (\alpha, \beta) \text{ for } \alpha, \beta \in L \} . \]

Let \( \mathbb{Q}L \) be the \( \mathbb{Q} \)-vector space spanned by \( L \), which may be identified with \( \mathbb{Q} \otimes \mathbb{Z} L \). We extend \( \langle \cdot, \cdot \rangle \) to \( \mathbb{Q}L \times \mathbb{Q}L \to \mathbb{Q} \) by \( \mathbb{Q} \)-linearly. The dual lattice \( \{ \alpha \in \mathbb{Q}L \mid \langle \alpha, L \rangle \subseteq \mathbb{Z} \} \) of \( L \) is denoted by \( L^* \).

### 2.1. RSSD involutions

We recall the notion of RSSD involutions, which was introduced in [11, Section 2.6] as a generalization of reflections. Let \( L \) be a positive definite integral lattice. For a sublattice \( A \) of \( L \), set

\[ \text{Ann}_L(A) = \{ \alpha \in L \mid \langle \alpha, A \rangle = 0 \} . \]

Then \( \text{rank} A + \text{rank}(\text{Ann}_L(A)) = \text{rank} L \) and

\[ A \oplus \text{Ann}_L(A) \subset L \subset L^* \subset A^* \oplus \text{Ann}_L(A)^* \subset \mathbb{Q}L \tag{1} \]

with \( \mathbb{Q}L = \mathbb{Q}A \oplus \mathbb{Q}\text{Ann}_L(A) \). In fact, \( \mathbb{Q}\text{Ann}_L(A) = (\mathbb{Q}A)^\perp \) is the orthogonal complement of \( \mathbb{Q}A \) in \( \mathbb{Q}L \) with respect to \( \langle \cdot, \cdot \rangle \). Define an orthogonal transformation \( t_A \) on \( \mathbb{Q}L \) by

\[ t_A = \begin{cases} -1 & \text{on } \mathbb{Q}A, \\ 1 & \text{on } \mathbb{Q}\text{Ann}_L(A). \end{cases} \]

A sublattice \( A \) of \( L \) is said to be relatively semiselfdual (RSSD) in \( L \) if \( 2L \subset A + \text{Ann}_L(A) \) [11, Definition 2.6.7] (see also [12, Definition 2.5]). We cite Proposition 2.6.8 of [11] as follows.

**Proposition 2.1:** If \( A \) is RSSD in \( L \), then \( t_A(L) = L \).

Indeed, each \( \alpha \in L \) is uniquely written as \( \alpha = a + b \) with \( a \in A^* \) and \( b \in \text{Ann}_L(A)^* \), and \( t_A(\alpha) = -a + b \). If \( 2\alpha = 2a + 2b \in A + \text{Ann}_L(A) \), then \( 2a \in A \), so \( t_A(\alpha) \in L \).

For an RSSD sublattice \( A \) of \( L \), the isometry \( t_A \in O(L) \) of the lattice \( L \) is called an RSSD involution [11, Definition 2.6.7]. Let

\[ \text{pr}_{A^*} : A^* \oplus \text{Ann}_L(A)^* \to A^* \]

be the projection. Then \( \text{pr}_{A^*}(\alpha) = a \) for \( \alpha = a + b \in L \) with \( a \in A^* \) and \( b \in \text{Ann}_L(A)^* \). Hence the next lemma holds.

**Lemma 2.2:** \( 2\text{pr}_{A^*}(L) \subset A \) if and only if \( A \) is RSSD in \( L \).

**Remark 2.3:** Let \( A \) be a sublattice of a positive definite even lattice \( L \), and set \( B = \text{Ann}_L(A) \). Then the vertex operator algebra \( V_L \) associated with \( L \) is a simple current extension of \( V_{A \oplus B} = V_A \otimes V_B \).
2.2. \( \hat{L} \) and its automorphisms

We review a central extension \( \hat{L} \) of a lattice \( L \) by a group of order 2; see for example [17, Section 3.8]. Let \((L, \langle \cdot, \cdot \rangle)\) be a positive definite even lattice. Set \( \bar{L} = L/2L \), which is a vector space over \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) with dimension rank\( L \). A map \( q : \bar{L} \rightarrow \mathbb{Z}_2 \) is called a quadratic form if the associated map \( b : L \times L \rightarrow \mathbb{Z}_2 \) defined by

\[
b(x, y) = q(x + y) + q(x) + q(y)
\]

is bilinear. This condition implies that \( b \) is alternating. We also have

\[
q(a_1 v_1 + \cdots + a_m v_m) = \sum_{i=1}^{m} a_i q(v_i) + \sum_{1 \leq i < j \leq m} a_i a_j q(v_i, v_j)
\]

for \( a_i \in \mathbb{Z}_2 \) and \( v_i \in L \) by (2).

If \( \mu : \bar{L} \rightarrow \mathbb{Z}_2 \) is a linear map, then \( q + \mu \) is also a quadratic form with the same associated bilinear form. For an alternating bilinear form \( b \) on \( L \), there is a quadratic form on \( L \) whose associated bilinear form is \( b \). Such a quadratic form is uniquely determined by \( b \) up to \( \text{Hom}(\bar{L}, \mathbb{Z}_2) \). On the other hand, for a quadratic form \( q \) on \( \bar{L} \), there is a bilinear form \( \epsilon \) on \( L \) satisfying \( q(x) = \epsilon(x, x) \) for \( x \in \bar{L} \). Such a bilinear form \( \epsilon \) is uniquely determined by \( q \) up to an alternating bilinear form on \( \bar{L} \). For \( \alpha, \beta \in L \), let \( \bar{\alpha} = \alpha + 2L \) and \( \bar{\beta} = \beta + 2L \in \bar{L} \). If \( b : L \times L \rightarrow \mathbb{Z}_2 \) is a \( \mathbb{Z} \)-bilinear map, then the radical of \( b \) contains \( 2L \). So \( b \) induces a bilinear form \( \bar{L} \times \bar{L} \rightarrow \mathbb{Z}_2 \); \( (\bar{\alpha}, \bar{\beta}) \mapsto b(\alpha, \beta) \). For simplicity of notation, we also write \( b \) for the induced bilinear form. Then \( b(\bar{\alpha}, \bar{\beta}) = b(\alpha, \beta) \). Similarly, we sometimes do not distinguish \( \text{Hom}(L, \mathbb{Z}_2) \) and \( \text{Hom}(\bar{L}, \mathbb{Z}_2) \).

Consider a map \( q : \bar{L} \rightarrow \mathbb{Z}_2 \) defined by

\[
q(\bar{\alpha}) = \frac{1}{2} \langle \alpha, \alpha \rangle + 2\mathbb{Z} \quad \text{for } \alpha \in L.
\]

The map \( q \) is a quadratic form whose associated bilinear form \( b \) is given by \( b(\bar{\alpha}, \bar{\beta}) = \langle \alpha, \beta \rangle + 2\mathbb{Z} \). Hence there is a bilinear form \( \epsilon : \bar{L} \times \bar{L} \rightarrow \mathbb{Z}_2 \) satisfying

\[
\epsilon(\bar{\alpha}, \bar{\alpha}) = \frac{1}{2} \langle \alpha, \alpha \rangle + 2\mathbb{Z} \quad \text{for } \alpha \in L.
\]

Such a bilinear form \( \epsilon \) is uniquely determined up to an alternating bilinear form on \( \bar{L} \). The condition (4) implies that

\[
\epsilon(\bar{\alpha}, \bar{\beta}) + \epsilon(\bar{\beta}, \bar{\alpha}) = \langle \alpha, \beta \rangle + 2\mathbb{Z} \quad \text{for } \alpha, \beta \in L.
\]

We fix a bilinear form \( \epsilon : \bar{L} \times \bar{L} \rightarrow \mathbb{Z}_2 \) satisfying (4). Let

\[
1 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{L} \longrightarrow L \longrightarrow 0
\]

be a central extension of the additive group \( L \) by a group \( \langle \kappa \rangle \) of order 2 with 2-cocyle \( \epsilon \). Each element of \( \hat{L} \) is uniquely written as \( e^{\alpha} \kappa^a \) for \( \alpha \in L \) and \( a \in \mathbb{Z}_2 \), and the product of \( e^{\alpha} \kappa^a \) and \( e^{\beta} \kappa^b \) is

\[
e^{\alpha} \kappa^a e^{\beta} \kappa^b = e^{\alpha + \beta} \kappa^{\epsilon(\bar{\alpha}, \bar{\beta}) + a + b}.
\]
Let \( \varepsilon' : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}_2 \) be another bilinear form satisfying (4). Then there is a quadratic form \( \xi \) on \( \mathcal{L} \) such that
\[
\xi(\alpha + \beta) + \xi(\alpha) + \xi(\beta) = \varepsilon(\alpha, \beta) + \varepsilon'(\alpha, \beta) \quad \text{for } \alpha, \beta \in \mathcal{L}.
\]
Set \( e^\alpha = e^{\alpha \xi(\alpha)} \). Then the multiplication in \( \hat{\mathcal{L}} \) is written as
\[
e^{\alpha \xi(\alpha)} e^{\beta \xi(\beta)} = e^{\alpha + \beta + \varepsilon'(\alpha, \beta) + a + b}.
\]
For \( g \in O(\mathcal{L}) \), define a bilinear form \( \varepsilon^g : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}_2 \) by \( \varepsilon^g(\alpha, \beta) = \varepsilon(g\alpha, g\beta) \). Then \( \varepsilon^g \) also satisfies (4). Let
\[
b_g = \varepsilon + \varepsilon^g : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}_2; \quad b_g(\alpha, \beta) = \varepsilon(\alpha, \beta) + \varepsilon(g\alpha, g\beta).
\]
Then \( b_g \) is an alternating bilinear form. Hence there is a quadratic form \( \eta \) on \( \mathcal{L} \) with associated bilinear form \( b_g \);
\[
\eta(\alpha + \beta) + \eta(\alpha) + \eta(\beta) = b_g(\alpha, \beta) \quad \text{for } \alpha, \beta \in \mathcal{L}.
\]
Define a map \( \hat{g} : \hat{\mathcal{L}} \to \hat{\mathcal{L}} \) by
\[
\hat{g}(e^{\alpha \xi(\alpha)}) = e^{g^\alpha \xi(g\alpha) + a}.
\]
Then \( \hat{g} \) is an automorphism of the group \( \hat{\mathcal{L}} \). The automorphism \( \hat{g} \) depends on the choice of \( \eta \) satisfying (8). The inverse of \( \hat{g} \) is given by
\[
\hat{g}^{-1}(e^{\alpha \xi(\alpha)}) = e^{g^{-1} \xi(g^{-1} \alpha) + a}.
\]
For \( f \in O(\mathcal{L}) \), let \( \xi \) be a quadratic form on \( \mathcal{L} \) with associated bilinear form \( b_f = \varepsilon + \varepsilon^f \), and define a map \( \hat{f} : \hat{\mathcal{L}} \to \mathcal{L} \) by \( \hat{f}(e^{\alpha \xi(\alpha)}) = e^{f^\alpha \xi(f\alpha) + a} \). Then
\[
\hat{f} \hat{g}(e^{\alpha \xi(\alpha)}) = e^{g^f \xi(g\alpha) + \xi(g\alpha)}.
\]
Note that the map \( \eta + \xi^g : \mathcal{L} \to \mathbb{Z}_2; \alpha \mapsto \eta(\alpha) + \xi(g\alpha) \) is a quadratic form with associated bilinear form \( b_{fg} = \varepsilon + \varepsilon^g \).

Let \( \eta : \mathcal{L} \to \mathbb{Z}_2 \) be a map. Then a map \( \phi : \hat{\mathcal{L}} \to \hat{\mathcal{L}} \) defined by \( \phi(e^{\alpha \xi(\alpha)}) = e^{g^\alpha \xi(g\alpha) + a} \) is an automorphism of the group \( \hat{\mathcal{L}} \) if and only if \( \eta \) is a quadratic form with associated bilinear form \( b_g \). In such a case, \( \phi \) is called a lift of \( g \).

Denote by \( O(\hat{\mathcal{L}}) \) the group of automorphisms \( \phi \) of \( \hat{\mathcal{L}} \) such that \( \phi \) is a lift of some \( g \in O(\mathcal{L}) \). Let \( \varphi : O(\hat{\mathcal{L}}) \to O(\mathcal{L}) \) be a group homomorphism defined by \( \varphi(\phi) = g \). Then we have an exact sequence
\[
1 \longrightarrow \text{Hom}(\mathcal{L}, \mathbb{Z}_2) \longrightarrow O(\hat{\mathcal{L}}) \xrightarrow{\varphi} O(\mathcal{L}) \longrightarrow 1 \tag{10}
\]
of groups, where \( \text{Hom}(\mathcal{L}, \mathbb{Z}_2) \to O(\hat{\mathcal{L}}); \lambda \mapsto \hat{\lambda} \) is defined by \( \hat{\lambda}(e^{\alpha \xi(\alpha)}) = e^{\alpha \xi(\lambda \alpha) + a} \); see [18, Section 2.4], [17, Section 3.8, Proposition 6], and [19, Proposition 5.4.1].
by (3). Since \( b_g = \varepsilon + \varepsilon^g \), we have
\[
\sum_{0 \leq i, j \leq n-1} b_g(g^i\alpha, g^j\alpha) = \sum_{1 \leq j \leq n-1} (\alpha, g^j\alpha) + 2\mathbb{Z} \tag{12}
\]
by (5). Moreover, \( (\alpha, g^i\alpha) = (\alpha, g^{n-j}\alpha) \) as \( g^n\alpha = \alpha \). Therefore,
\[
\eta(\overline{\alpha} + \overline{g\alpha} + \cdots + \overline{g^{n-1}\alpha}) = \begin{cases} 
\sum_{i=0}^{n-1} \eta(g^i\alpha) & \text{if } n \text{ is odd}, \\
\sum_{i=0}^{n-1} \eta(g^i\alpha) + (\alpha, g^{n/2}\alpha) + 2\mathbb{Z} & \text{if } n \text{ is even}.
\end{cases}
\]
Note that \( \hat{g}^n(e^\delta) = e^{\delta n}\alpha \kappa^\delta \) with \( \delta = \sum_{i=0}^{n-1} \eta(g^i\alpha) \); see [20, Lemma 12.1] and [21, Propositions 7.1 and 7.2].

When \( g \) is fixed point free on \( L \) of order \( n \), we have \( (1 + g + \cdots + g^{n-1})\alpha = 0 \) and then \( \sum_{1 \leq j \leq n-1} (\alpha, g^j\alpha) = -(\alpha, \alpha) \in 2\mathbb{Z} \). Therefore, \( \sum_{i=0}^{n-1} \eta(g^i\alpha) = 0 \) by (11) and (12). Thus the following proposition holds.

**Proposition 2.4:** Suppose \( g \in O(L) \) is fixed point free on \( L \) of order an integer \( n \geq 2 \). Then the order of \( \hat{g} \in O(\hat{L}) \) is also \( n \).

The next simple lemma will be used later.

**Lemma 2.5:** Let \( g : L \rightarrow L \) be an automorphism of a finite rank free \( \mathbb{Z} \)-module \( L \) of order an odd integer \( p \geq 3 \). Assume that \( g \) is fixed point free on \( L \). Then the map
\[
\text{Hom}(L, \mathbb{Z}_2) \rightarrow \text{Hom}(\overline{L}, \mathbb{Z}_2); \quad \mu \mapsto \mu + \mu^g
\]
is a \( \mathbb{Z}_2 \)-linear isomorphism, where \( \mu^g \) is defined by \( \mu^g(\overline{\alpha}) = \mu(\overline{g\alpha}) \) for \( \overline{\alpha} = \alpha + 2L \in \overline{L} \).

**Proof:** Suppose \( \mu + \mu^g = 0 \). Then \( \mu((1 + g + \cdots + g^{p-1})\alpha) = p\mu(\overline{\alpha}) \). Since \( g \) is fixed point free of order \( p \), we have \( (1 + g + \cdots + g^{p-1})\alpha = 0 \). Thus \( \mu(\overline{\alpha}) = 0 \) as \( p \) is an odd integer. Hence the kernel of the linear map \( \mu \mapsto \mu + \mu^g \) is trivial, so the assertion holds. \( \blacksquare \)

We consider the relationship between the normalizer \( N_{O(L)}(\langle g \rangle) \) of \( \langle g \rangle \) (resp. the centralizer \( C_{O(L)}(g) \) of \( g \)) in \( O(L) \) and the normalizer \( N_{O(\hat{L})}(\langle \hat{g} \rangle) \) of \( \langle \hat{g} \rangle \) (resp. the centralizer \( C_{O(\hat{L})}(\hat{g}) \) of \( \hat{g} \)) in \( O(\hat{L}) \).

**Proposition 2.6:** Assume that \( g \in O(L) \) is fixed point free on \( L \) of order an odd integer \( p \geq 3 \). Let \( \hat{g} \in O(\hat{L}) \) be a lift of \( g \).

1. For each \( f \in N_{O(L)}(\langle g \rangle) \), there is a unique lift \( \hat{f} \) of \( f \) in \( N_{O(\hat{L})}(\langle \hat{g} \rangle) \), and the restriction of \( \varphi \) in (10) to \( N_{O(L)}(\langle g \rangle) \) is an isomorphism from \( N_{O(\hat{L})}(\langle \hat{g} \rangle) \) to \( N_{O(L)}(\langle g \rangle) \).
2. For each \( f \in C_{O(L)}(g) \), there is a unique lift \( \hat{f} \) of \( f \) in \( C_{O(\hat{L})}(\hat{g}) \), and the restriction of \( \varphi \) in (10) to \( C_{O(L)}(g) \) is an isomorphism from \( C_{O(\hat{L})}(\hat{g}) \) to \( C_{O(L)}(g) \).

**Proof:** Let \( \eta \) be a quadratic form on \( \overline{L} \) with associated bilinear form \( b_g = \varepsilon + \varepsilon^g \) such that \( \hat{g}(e^\delta\kappa^a) = e^{\delta\alpha\kappa^a(n(\overline{\alpha}) + a)} \). Let \( f \in N_{O(L)}(\langle g \rangle) \). Then \( f^{-1}gf = g^m \) for some \( 1 \leq m \leq p - 1 \),
where \( m \) and \( p \) are coprime. Let \( \hat{f} \in O(\hat{L}) \) be a lift of \( f \), and let \( \xi \) be a quadratic form on \( \bar{L} \) with associated bilinear form \( b_\xi = \varepsilon + \varepsilon\hat{f} \) such that \( \hat{f}(e^\alpha k^a) = e^{\alpha} k^{\xi(\alpha) + a} \).

Set \( h = \hat{g}^m \). Since \( \hat{g}^m \) is a lift of \( h \), we have \( \hat{g}^m(e^\alpha k^a) = e^{\alpha} h \xi(\alpha) + a \) for some quadratic form \( \xi \) on \( \bar{L} \) with associated bilinear form \( b_\xi = \varepsilon + \varepsilon h \). We also have \( \hat{f}^{-1}(e^\alpha k^a) = e^{\alpha} h \xi(\alpha) + a \) for some \( \alpha \in \text{Hom}(\bar{L}, \mathbb{Z}_2) \) by (10), where \( \eta \) and \( \xi \) are defined by \( \eta = \xi(\alpha) + a \) and \( \xi h = \xi(\alpha) \), respectively. There is a unique \( \mu \in \text{Hom}(\bar{L}, \mathbb{Z}_2) \) such that \( \lambda = \mu + \mu h \) by Lemma 2.5, where \( \mu h \) is defined by \( \mu h(\alpha) = \eta(\alpha) \) and \( \xi h(\alpha) = \xi(\alpha) \), respectively. The assertion (2) follows from the above argument with \( m = 1 \).

If \( g \) is the \(-1\)-isometry of \( L \), then \( g \) is fixed point free of order 2. In this case, we have \( b_g = 0 \). Define \( \theta \in O(\hat{L}) \) by

\[
\theta(e^\alpha k^a) = e^{-\alpha} k^a
\]

for \( \alpha \in L \) and \( a \in \mathbb{Z}_2 \). Then \( \theta \) is a lift of \(-1\) with order 2. Moreover, \( \phi \theta = \theta \phi \) for any \( \phi \in O(\hat{L}) \). For any fixed point free \( f \in O(L) \) of order an odd integer \( p \geq 3 \), we have \( -f \) is fixed point free on \( L \) of order \( 2p \), and \( f \theta \) is a lift of \(-f\).

### 2.3. Automorphisms of \( V_L \)

Let \( (L, \{ \cdot, \cdot \}) \) be a positive definite even lattice. The structure of the automorphism group \( \text{Aut}(V_L) \) of the vertex operator algebra \( V_L \) associated with \( L \) was studied in [18, Section 2]. Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L \). We extend \( \{ \cdot, \cdot \} \) to \( \mathfrak{h} \times \mathfrak{h} \to \mathbb{C} \) by \( \mathbb{C} \)-linearly. Let \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \) be the corresponding affine Lie algebra. Let \( M(1) \) be the \( \mathfrak{h} \)-module induced from an \( \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \)-module \( \mathbb{C} \), where \( \mathfrak{h} \otimes \mathbb{C}[t] \) acts trivially on \( \mathbb{C} \) and \( c \) acts as 1 on \( \mathbb{C} \). We denote by \( h(n) \) the action of \( h \otimes t^n \) on \( M(1) \) for \( h \in \mathfrak{h} \).

Let \( \mathbb{C}[L]_e = \mathbb{C}[\hat{L}]/(\kappa + 1) \mathbb{C}[\hat{L}] \) be the quotient algebra of the group algebra \( \mathbb{C}[\hat{L}] \) of \( \hat{L} \) by the ideal generated by \( \kappa + 1 \). We use the same symbol \( e^\alpha \) to denote the image of \( e^\alpha \in L \) in \( \mathbb{C}[L]_e \). Then \( \hat{g} \in O(\hat{L}) \) defined in (9) acts on \( \mathbb{C}[L]_e \) by \( \hat{g}(e^\alpha) = (-1)^{\eta(\alpha)} e^{\alpha} \).

The vertex operator algebra \( V_L \) is defined to be \( V_L = M(1) \otimes \mathbb{C}[L]_e \) as a vector space (see [19, Chapter 8] and [22, Section 6.4]). Any \( \hat{g} \in O(\hat{L}) \) acts on \( V_L \) by

\[
\hat{g}(h_1(-n_1) \cdots h_r(-n_r) \otimes e^\alpha) = g(h_1(-n_1) \cdots g(h_r)(-n_r) \otimes \hat{g}(e^\alpha))
\]

for \( h_i \in \mathfrak{h}, n_i > 0, \) and \( \alpha \in L \). Set

\[
N(V_L) = \{ \exp(v(0)) \mid v \in (V_L)_1 \},
\]

which is a normal subgroup of \( \text{Aut}(V_L) \) as \( \phi \exp(v(0)) \phi^{-1} = \exp((\phi v)(0)) \) for \( \phi \in \text{Aut}(V_L) \).

It is known [18, Theorem 2.1] that

\[
\text{Aut}(V_L) = N(V_L)O(\hat{L}) \quad \text{with} \quad N(V_L) \cap O(\hat{L}) \supset \text{Hom}(L, \mathbb{Z}_2).
\]

Now, assume that \( L(2) = \emptyset \), where \( L(2) = \{ \alpha \in L \mid \{\alpha, \alpha\} = 2 \} \). Then \( (V_L)_1 = \{ h(-1) \mid h \in \mathfrak{h} \} \), and

\[
N(V_L) = \{ \exp(h(0)) \mid h \in \mathfrak{h} \} \quad \text{with} \quad N(V_L) \cap O(\hat{L}) = \text{Hom}(L, \mathbb{Z}_2).
\]

(13)
Thus $\varphi : O(L) \rightarrow O(L)$ in (10) can be extended to $\varphi : \text{Aut}(V_L) \rightarrow O(L)$ with kernel $\text{Ker} \varphi = N(V_L)$, and we obtain an exact sequence

$$1 \longrightarrow N(V_L) \longrightarrow \text{Aut}(V_L) \overset{\varphi}{\longrightarrow} O(L) \longrightarrow 1$$

of groups [14, Remark 5.14]. The automorphism $\exp(h(0))$ acts on $V_L$ by

$$\exp(h(0))(h_1(-n_1) \cdots h_r(-n_r) \otimes e^\gamma) = \exp((h, \alpha))h_1(-n_1) \cdots h_r(-n_r) \otimes e^\gamma.$$

The next theorem is a refinement of [14, Theorem 5.15].

**Theorem 2.7:** Let $L$ be a positive definite even lattice with $L(2) = \emptyset$. Let $g \in O(L)$ be fixed point free on $L$ of order an odd integer $p \geq 3$, and let $\hat{g} \in O(L)$ be a lift of $g$.

1. $C_{\text{Aut}(V_L)}(\hat{g}) = C_{N(V_L)}(\hat{g}) : C_{O(L)}(\hat{g})$ is a split extension of $C_{N(V_L)}(\hat{g})$ by $C_{O(L)}(\hat{g})$.
2. $N_{\text{Aut}(V_L)}(\hat{g}) = N_{N(V_L)}(\hat{g})$ is a split extension of $C_{N(V_L)}(\hat{g})$ by $N_{O(L)}(\hat{g})$.
3. $C_{N(V_L)}(\hat{g}) = \{\exp(h(0)) \mid h \in 2\pi \sqrt{-1}(1-g)L^*\}$.
4. $C_{N(V_L)}(\hat{g}) \cong \text{Hom}(L/(1-g)L, \mathbb{Z}_p)$, where the left-hand side is a multiplicative group and the right-hand side is an additive group. The isomorphism is given by assigning $\exp(2\pi \sqrt{-1} \gamma(0)) \in C_{N(V_L)}(\hat{g})$ with $\gamma \in ((1-g)L)^*$ to the $\mathbb{Z}$-module homomorphism

$$L/(1-g)L \rightarrow \mathbb{Z}_p; \quad \alpha + (1-g)L \mapsto p(\gamma, \alpha + (1-g)L) + p\mathbb{Z}$$

for $\alpha \in L$.

**Proof:** The assertion (1) follows from (2) of Proposition 2.6, (13), and (14).

Let $\phi \in N_{\text{Aut}(V_L)}(\hat{g})$. Then $\phi \hat{g} \phi^{-1} = \hat{g}^m$ for some $1 \leq m \leq p-1$, where $m$ and $p$ are coprime. Set $f = \varphi(\phi)$. Then $\hat{f} \hat{g} \hat{f}^{-1} = \hat{g}^m$. As shown in the proof of Proposition 2.6, there is a unique lift $\hat{f}$ of $f$ satisfying $\hat{f} \hat{g} \hat{f}^{-1} = \hat{g}^m$. Then $\hat{f} \in N_{O(L)}(\hat{g})$ and $\hat{f}^{-1} \phi \in C_{\text{Aut}(V_L)}(\hat{g})$, so we have $\phi \in C_{N(V_L)}(\hat{g})N_{O(L)}(\hat{g})$ by the assertion (1). Thus the assertion (2) holds by (1) of Proposition 2.6.

For $h \in h$, we have $\hat{g} \exp(h(0)) \hat{g}^{-1} = \exp((gh)(0))$. Hence $\exp(h(0)) \in C_{N(V_L)}(\hat{g})$ if and only if $(gh - h, \alpha) \in 2\pi \sqrt{-1}\mathbb{Z}$ for all $\alpha \in L$. Since $g \in O(L)$, the condition on $h$ is equivalent to that $(h, (1-g)\alpha) \in 2\pi \sqrt{-1}\mathbb{Z}$ for all $\alpha \in L$. Thus the assertion (3) holds.

For $\alpha \in L$, we have $\alpha \equiv g^i\alpha \pmod{(1-g)L}$, $1 \leq i \leq p-1$. Since $g$ is fixed point free of order $p$, we also have $1 + g + \cdots + g^{p-1} = 0$ on $L$. Hence $p\alpha \equiv 0 \pmod{(1-g)L}$, so $L \supset (1-g)L \supset pL$. Thus $L^* \subset ((1-g)L)^* \subset \frac{1}{p}L^*$.

For $\gamma \in L^*$, define $\chi_\gamma \in \text{Hom}(L, \mathbb{Z}_p)$ by $\chi_\gamma(\alpha) = (\gamma, \alpha) + p\mathbb{Z}$. Then the map $\chi : L^* \rightarrow \text{Hom}(L, \mathbb{Z}_p)$ is a surjective homomorphism of $\mathbb{Z}$-modules. The kernel of $\chi$ is $pL^*$. Moreover, $\text{Ker} \chi_{\gamma} \supset (1-g)L$ if and only if $\gamma \in p((1-g)L)^*$. Thus $\gamma \mapsto \chi_{\gamma}$ gives an isomorphism

$$((1-g)L)^*/L^* \rightarrow \text{Hom}(L/(1-g)L, \mathbb{Z}_p)$$

of additive groups. Since $\gamma \mapsto \exp(2\pi \sqrt{-1} \gamma(0))$ induces an isomorphism

$$((1-g)L)^*/L^* \rightarrow C_{N(V_L)}(\hat{g})$$

from an additive group to a multiplicative group, the assertion (4) holds.
Remark 2.8: The assumption that the order of $g$ is odd is not necessary for the assertions (3) and (4) of Theorem 2.7.

If $p$ is an odd prime in the above theorem, then rank $L$ is divisible by $p - 1$, and $L/(1 - g)L \cong p^m$ is elementary abelian of order $p^m$ by [23, Lemma A.1], where $m = \text{rank } L/(p - 1)$. Hence the following corollary holds by Proposition 2.6.

Corollary 2.9: Assume that $p$ is an odd prime in the above theorem. Then $C_{\text{Aut}(V_L)}(\mathcal{\hat{g}}) \cong p^m : C_{O(L)}(g)$ and $N_{\text{Aut}(V_L)}(\mathcal{\hat{g}}) \cong p^m : N_{O(L)}(g)$, where $m = \text{rank } L/(p - 1)$.

2.4. Parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$

We collect basic properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ associated with $\mathfrak{sl}_2$ and a positive integer $k$; see [1,4,5,8,9] for details. If $k = 1$, then $K(\mathfrak{sl}_2, k)$ reduces to the trivial vertex operator algebra $\mathbb{C}1$. We assume that $k \geq 2$.

(1) $M^0 = K(\mathfrak{sl}_2, k)$ is a simple, self-dual, rational, and $C_2$-cofinite vertex operator algebra of CFT-type with central charge $2(k - 1)/(k + 2)$.

(2) $K(\mathfrak{sl}_2, 2) \cong L(1/2, 0)$ is the simple Virasoro vertex operator algebra of central charge $1/2$. If $k \geq 3$, there is, up to a scalar multiple, a unique primary vector $W^3$ of weight 3, and $K(\mathfrak{sl}_2, k)$ is generated by the vector $W^3$ as a vertex operator algebra.

(3) $M^{ij}, 0 \leq i \leq k, 0 \leq j < k$, are irreducible $M^0$-modules with

$$M^{ij} \cong M^{k-i-j-i},$$

and $M^{ij}, 0 \leq j < i \leq k$, form a complete set of representatives of the equivalence classes of irreducible $M^0$-modules. The index $j$ is considered to be modulo $k$. Note that $M^0 = M^{k,0} = M^{0,0}$.

(4) The top level of $M^{ij}$ is one dimensional with the conformal weight

$$h(M^{ij}) = \frac{1}{2k(k + 2)}(k(i - 2j) - (i - 2j)^2 + 2k(i - j + 1)j)$$

for $0 \leq j \leq i \leq k$.

(5) The fusion product of irreducible $M^0$-modules is given by

$$M^{i_1j_1} \boxtimes_{M^0} M^{i_2j_2} = \sum_{r \in R(i_1, i_2)} M^{r(2j_1-i_1+2j_2-i_2-r)/2},$$

where $R(i_1, i_2)$ is the set of integers $r$ such that

$$|i_1 - i_2| \leq r \leq \min\{i_1 + i_2, 2k - i_1 - i_2\}, \quad i_1 + i_2 + r \in 2\mathbb{Z}.$$

Let $M^j = M^{k,j}$ for $0 \leq j \leq k - 1$, which are the simple current $M^0$-modules. In fact,

$$M^0 \boxtimes_{M^0} M^{ij} = M^{ij+1}.$$

(6) If $k \geq 3$, the automorphism group $\text{Aut}(M^0)$ of $M^0$ is generated by an involution $\theta$ such that $\theta(W^3) = -W^3$, and

$$M^{ij} \circ \theta \cong M^{i,j-i}.$$
Now, assume that $k \geq 3$. An irreducible $M^0$-module $M^{ij}$ is said to be $\theta$-stable if $M^{ij} \circ \theta \cong M^{ij}$. We see from (15) and (19) that $M^{ij}$ is $\theta$-stable if and only if $i = 2j$ for $0 \leq j \leq \lfloor k/2 \rfloor$, or $k$ is even and $(i, j) = (k/2, 0)$, where $\lfloor k/2 \rfloor$ is the largest integer which does not exceed $k/2$. The irreducible $M^0$-modules $M^{2j}$, $0 \leq j \leq \lfloor k/2 \rfloor$, are said to be of $\sigma$-type. An $M^0$-module is said to be of $\sigma$-type if it is a direct sum of irreducible $M^0$-modules of $\sigma$-type.

3. $\sigma$-involutions

In this section, we introduce a $\sigma$-involution associated with the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$.

3.1. Automorphism $\tau_W$

First of all, we recall a $\mathbb{Z}_k$ symmetry in the fusion algebra of $M^0 = K(\mathfrak{sl}_2, k)$ for $k \geq 2$ [3]. For $0 \leq i \leq k$ and $0 \leq j < k$, let $0 \leq l < 2k$ be such that $l \equiv i - 2j$ (mod $2k$), and set

$$\tilde{M}^{ij} = M^{ij}.$$

Then (17) can be written as

$$\tilde{M}^{i_1, l_1} \boxtimes_{M^0} \tilde{M}^{i_2, l_2} = \sum_{r \in R(i_1, i_2)} \tilde{M}^{r, l_1 + l_2}. \quad (20)$$

Moreover,

$$\tilde{M}^{i, l} \cong \tilde{M}^{k-i, k+l}, \quad (21)$$

$$\tilde{M}^{i, l} \circ \theta \cong \tilde{M}^{i, -l} \quad (22)$$

by (15) and (19), respectively.

The following theorem is known (see [24, Section 4] and [3]).

**Theorem 3.1:** Let $k \geq 2$ be an integer. Then a map defined by

$$\tilde{M}^{i, l} \mapsto \zeta_k^l \tilde{M}^{i, l} \quad \text{for } 0 \leq i \leq k \text{ and } 0 \leq l < 2k \text{ with } i \equiv l \pmod{2}$$

is compatible with (20) and (21), and it induces an automorphism of the fusion algebra of $M^0$ of order $k$, where $M^0 = K(\mathfrak{sl}_2, k)$ and $\zeta_k = \exp(2\pi \sqrt{-1}/k)$.

The next lemma is a consequence of (21).

**Lemma 3.2:** $\tilde{M}^{i, l}$ for $0 \leq i \leq k$ and $0 \leq l < k$ with $i \equiv l \pmod{2}$ form a complete set of representatives of the equivalence classes of irreducible $M^0$-modules.

Let $(V, Y, 1, \omega)$ be a vertex operator algebra, and assume that it contains a vertex operator subalgebra $W$ isomorphic to $K(\mathfrak{sl}_2, k)$. Then $V$ is a direct sum of irreducible $W$-modules by [25, Theorem 4.5]. For $0 \leq l < k$, we denote by $V_W[l]$ the sum of all irreducible $W$-submodules of $V$ isomorphic to $\tilde{M}^{i, l}$ for some $0 \leq i \leq k$ with $i \equiv l \pmod{2}$. Then

$$V = \bigoplus_{l=0}^{k-1} V_W[l] \quad (23)$$
by Lemma 3.2. In view of (21), we may consider $l$ for $V_W[l]$ to be modulo $k$. Then (20) implies that

$$Y(u, x)v \in V_W[l_1 + l_2] [[x, x^{-1}]]$$

for $u \in V_W[l_1]$ and $v \in V_W[l_2]$. The following theorem holds by (23) and (24).

**Theorem 3.3:** Let $V$ be a vertex operator algebra containing a vertex operator subalgebra $W \cong K(sl_2, k)$ for an integer $k \geq 2$. Then a linear map $\tau_W$ on $V$ defined by

$$\tau_W(v) = \zeta_l^j v \quad \text{for } v \in V_W[l], \ 0 \leq l < k$$

is an automorphism of the vertex operator algebra $V$, where $\zeta_k = \exp(2\pi \sqrt{-1}/k)$.

The fixed point subalgebra $V(\tau_W)$ of $V$ by the automorphism $\tau_W$ is $V_W[0]$. Note that $V$ is a $\sigma$-type $W$-module if and only if $V = V_W[0]$.

### 3.2. $M^{2ij}$ as $K(sl_2, k)(\theta)$-module

Let $k \geq 3$. Then $\text{Aut}(K(sl_2, k))$ is generated by an involution $\theta$. The fixed point subalgebra $K(sl_2, k)(\theta)$ of $K(sl_2, k)$ by the involution $\theta$ is simple, self-dual, rational, $C_2$-cofinite, and of CFT-type by [26,27]. The representation theory of $K(sl_2, k)(\theta)$ was studied in detail [6,7]. In fact, the irreducible modules and their highest weight vectors are obtained in [6], and the fusion rules are determined in [7]. We use those results for $\sigma$-type irreducible $K(sl_2, k)$-modules.

Let $M^{0,+}$ and $M^{0,-}$ be the eigenspaces with eigenvalues 1 and $-1$ for $\theta$ in $M^0 = K(sl_2, k)$, respectively. Thus $M^{0,+} = K(sl_2, k)(\theta)$, and $M^{0,-}$ is the irreducible $M^{0,+}$-module generated by the weight 3 primary vector $W^3$. We see from [6, Proposition 3.14] that for $0 \leq j \leq [k/2]$, a $\sigma$-type irreducible $M^0$-module $M^{2ij}$ is a direct sum

$$M^{2ij} = (M^{2ij})^0 \oplus (M^{2ij})^1$$

of two irreducible $M^{0,+}$-modules, where the conformal weight of $(M^{2ij})^0$ is

$$h((M^{2ij})^0) = h(M^{2ij}) = \frac{j(j + 1)}{k + 2},$$

and $h((M^{2ij})^1) - h((M^{2ij})^0) = 3, 1, \text{or } 2$ according as $j = 0, 1 \leq j < k/2$, or $k$ is even and $j = k/2$, respectively. Note that $(M^{0,0})^0 = M^{0,+}$ and $(M^{0,0})^1 = M^{0,-}$.

The following theorem is taken from Theorem 5.1 of [7].

**Theorem 3.4:** Let $k \geq 3$ be an integer.

1. For $0 \leq j \leq [k/2]$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$,

   $$\ (M^{0,0})^{\epsilon_1} \bigotimes_{M^{0,+}} (M^{2ij})^{\epsilon_2} = (M^{2ij})^{\epsilon_1 + \epsilon_2},$$

   where $\epsilon_1 + \epsilon_2$ is considered to be modulo 2.
If \( k \geq 4 \), then
\[
(M^{2,1})^0 \boxtimes_{M^{0,+}} (M^{2j})^0 = (M^{2(j-1),j-1})^0 + (M^{2j})^1 + (M^{2(j+1),j+1})^0
\]
for \( 1 \leq j \leq \lfloor k/2 \rfloor - 1 \).

For \( j = \lfloor k/2 \rfloor \),
\[
(M^{2,1})^0 \boxtimes_{M^{0,+}} (M^{2j})^0 = (M^{2(j-1),j-1})^0 + (M^{2j})^1
\]
if \( k \) is odd, and
\[
(M^{2,1})^0 \boxtimes_{M^{0,+}} (M^{k,k/2})^0 = (M^{k-2,k/2-1})^0
\]
if \( k \) is even.

**Remark 3.5:** Since the fusion product of irreducible \( M^{0,+} \)-modules is commutative and associative [28, Theorem 3.7], the above equations (25)–(28) determine the fusion product among \((M^{2j})^\epsilon\) for all \( 0 \leq j \leq \lfloor k/2 \rfloor \) and \( \epsilon \in \{0, 1\} \).

In the case \( k = 2 \), the automorphism \( \theta \) is trivial, and the \( \sigma \)-type irreducible modules for \( K(sl_2, 2) \cong L(1/2, 0) \) are \( M^0 \) and \( M^{2,1} \cong L(1/2, 1/2) \). Thus (28) corresponds to the fusion product \( L(1/2, 1/2) \boxtimes_{M^0} L(1/2, 1/2) = L(1/2, 0) \).

The next theorem follows from Theorem 3.4 and Remark 3.5.

**Theorem 3.6:** Let \( k \geq 3 \) be an integer. Then a map defined by
\[
(M^{2j})^\epsilon \mapsto (-1)^{j+\epsilon}(M^{2j})^\epsilon \quad \text{for } 0 \leq j \leq \lfloor k/2 \rfloor \text{ and } \epsilon \in \{0, 1\}
\]
is compatible with (25)–(28), and it induces an automorphism of order 2 of the subalgebra of the fusion algebra of \( M^{0,+} \) spanned by \((M^{2j})^\epsilon\) for \( 0 \leq j \leq \lfloor k/2 \rfloor \) and \( \epsilon \in \{0, 1\} \), where \( M^{0,+} = K(sl_2, k)(\theta) \).

Let \( V \) be a vertex operator algebra containing a vertex operator subalgebra \( W \cong K(sl_2, k) \). Suppose \( V^{(\tau W)} = V \), that is, \( V \) is of \( \sigma \)-type as a \( W \)-module. Denote by \( V_{W^+}[j, \epsilon] \) the sum of all irreducible \( M^{0,+} \)-submodules of \( V \) isomorphic to \((M^{2j})^\epsilon\). Then
\[
V = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \bigoplus_{\epsilon \in \{0,1\}} V_{W^+}[j, \epsilon].
\]

The following theorem is a consequence of Theorem 3.6 and (29).

**Theorem 3.7:** Let \( V \) be a vertex operator algebra containing a vertex operator subalgebra \( W \cong K(sl_2, k) \) for an integer \( k \geq 3 \). Assume that \( V \) is of \( \sigma \)-type as a \( W \)-module. Then a linear map \( \sigma_W \) on \( V \) defined by
\[
\sigma_W(v) = (-1)^{j+\epsilon}v \quad \text{for } v \in V_{W^+}[j, \epsilon], \ 0 \leq j \leq \lfloor k/2 \rfloor, \ \epsilon \in \{0, 1\}
\]
is an automorphism of the vertex operator algebra \( V \) of order 2.

We call \( \sigma_W \) the \( \sigma \)-involution of \( V \) associated with \( W \). We say \( W \cong K(sl_2, k) \) is a \( \sigma \)-type parafermion vertex operator subalgebra of \( V \) or of \( \sigma \)-type in \( V \) if \( V \) is of \( \sigma \)-type as a \( W \)-module.
4. \(\sigma\)-involutions of \(V_{\sqrt{2}A_{k-1}}\)

In this section, we study \(\sigma\)-involutions of the lattice vertex operator algebra \(V_{\sqrt{2}A_{k-1}}\) for an integer \(k \geq 3\). Furthermore, we obtain a sufficient condition on a positive definite even lattice \(L\) containing \(\sqrt{2}A_{k-1}\) for which \(V_L\) possesses a \(\sigma\)-involution, and show how a \(\sigma\)-involution is related to an RSSD involution.

We use the notation in Sections 3–5 of [2]. Let \(L^{(k)} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k\) with \(\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}\), and set \(\gamma_k = \alpha_1 + \cdots + \alpha_k\). Let

\[ N = \{ \alpha \in L^{(k)} \mid \langle \alpha, \gamma_k \rangle = 0 \}. \]

Then \(N \cong \sqrt{2}A_{k-1}\). Let \(\beta_i = \alpha_i - \alpha_{i+1}\), so \(\{\beta_1, \ldots, \beta_{k-1}\}\) is a \(\mathbb{Z}\)-basis of \(N\).

Since \(\langle \alpha, \alpha \rangle \in 4\mathbb{Z}\) for \(\alpha \in N\), we may take a \(\mathbb{Z}\)-bilinear map \(\varepsilon : N \times N \to \mathbb{Z}_2\) satisfying (4) for \(N\) in place of \(L\) to be \(\varepsilon = 0\). Then the central extension (6) of \(N\) in place of \(L\) splits. So the twisted group algebra \(\mathbb{C}[N]_{\varepsilon}\) is isomorphic to the ordinary group algebra \(\mathbb{C}[N]\), and \(V_N = M(1) \otimes \mathbb{C}[N]\) as a vector space. Moreover, we may take a quadratic form \(\eta\) satisfying (8) for \(N\) to be \(\eta = 0\), so that \(\text{Aut}(V_N) = N(V_N) : O(N)\) is a split extension of \(N(V_N)\) by \(O(N)\).

The vertex operator algebra \(V_N\) contains a subalgebra

\[ T \cong L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0) \otimes M^0, \]

where \(L(c_m, 0)\) is a simple Virasoro vertex operator algebra of central charge

\[ c_m = 1 - \frac{6}{(m + 2)(m + 3)}, \]

and \(M^0 = K(\mathfrak{sl}_2, k)\) [10]. In fact, \(M^0\) is the commutant of \(L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0)\) in \(V_N\). The lift \(\theta \in \text{Aut}(V_N)\) of the \(-1\)-isometry of \(N\) leaves \(M^0\) invariant, and its restriction to \(M^0\) generates \(\text{Aut}(M^0)\).

The vertex operator algebra \(V_N\) decomposes into a direct sum of irreducible \(T\)-modules as follows:

\[ V_N = \bigoplus_{0 \leq i_s \leq s} \bigoplus_{\substack{i_s \equiv 0 \mod 2 \atop 1 \leq s \leq k}} L(c_1, h_{i_1+1,i_2+1}^1) \otimes \cdots \otimes L(c_{k-1}, h_{i_{k-1}+1,i_k+1}^{k-1}) \otimes M^{i_s,i_k/2}, \]

where

\[ h_{r,s}^m = \frac{(r(m + 3) - s(m + 2))^2 - 1}{4(m + 2)(m + 3)} \]

for \(1 \leq r \leq m + 1\) and \(1 \leq s \leq m + 2\), and \(L(c, h)\) is an irreducible highest weight \(L(c, 0)\)-module with the highest weight \(h\) (see [29] and [2, Theorem 5.2]). Since \(M^{i_s,i_k/2}\) is a \(\sigma\)-type irreducible \(M^0\)-module, the next lemma holds.

**Lemma 4.1:** \(V_N\) is of \(\sigma\)-type as an \(M^0\)-module.

By the above lemma, we can consider the \(\sigma\)-involution \(\sigma_{M^0}\) of \(V_N\) associated with \(M^0\) as in Theorem 3.7. The next lemma will be used to relate the \(\sigma\)-involution \(\sigma_{M^0}\) and the involution \(\theta\).
Lemma 4.2: (1) The weight one subspace \((V_N)_1\) of \(V_N\) agrees with the weight one subspace \(M(1)_1\) of \(M(1)\), which is \(k-1\) dimensional.

(2) \((V_N)_1\) is spanned by the top levels of
\[
L(c_1, 0) \otimes \cdots \otimes L(c_{p-1}, 0) \otimes L(c_p, h^p_{1,3}) \otimes L(c_{p+1}, h^{p+1}_{3,3}) \otimes \cdots \otimes L(c_{k-1}, h^{k-1}_{3,3})
\otimes (M^{2,1})^0
\]
for \(1 \leq p \leq k - 1\).

Proof: Since \(N\) is a rank \(k-1\) lattice with minimal square norm 4, the assertion (1) holds. Let \(1 \leq p \leq k - 1\), and set \(i_1 = \cdots = i_p = 0\) and \(i_{p+1} = \cdots = i_k = 2\). Then \(h^m_{1,1} = 0\) for \(1 \leq m \leq p - 1\), \(h^p_{1,3} = (p+1)/(p+3)\), and \(h^m_{3,3} = 2/(m+2)(m+3)\) for \(p+1 \leq m \leq k - 1\). Hence we have
\[
h^p_{1,3} + h^{p+1}_{3,3} + \cdots + h^{k-1}_{3,3} = \frac{k}{k+2}.
\]
The top level of \((M^{2,1})^0\) agrees with the top level of \(M^{2,1}\), and it is one dimensional with weight \(2/(k+2)\). Therefore, the top level of (30) is one dimensional with weight one for any \(1 \leq p \leq k - 1\). Thus the assertion (2) holds.

By a similar argument as in the proof of [14, Lemma 5.4], we obtain the next theorem.

Theorem 4.3: Let \(k \geq 3\) be an integer. Then \(\sigma_{M^0} = \theta\) as automorphisms of \(V_N\), where \(N = \sqrt{2}A_{k-1}\) and \(\theta\) is the lift of the \(-1\)-isometry of \(N\).

Proof: We see from Theorem 3.7 and Lemma 4.2 that \(\sigma_{M^0}\) acts as \(-1\) on \((V_N)_1 = M(1)_1\). The automorphism \(\theta\) also acts as \(-1\) on \(M(1)_1\). Thus \(\sigma_{M^0}\theta\) is the identity on \(M(1)_1\), so \(\sigma_{M^0}\theta = \exp(\beta(0))\) for some \(\beta \in \mathbb{C} \otimes \mathbb{Z} N\) by [18, Lemma 2.5]. The conformal vector of \(M^0\) is described in (4.4) of [5] as
\[
\frac{1}{4k(k+2)} \sum_{1 \leq p, q \leq k} (\alpha_p - \alpha_q)(-1)^2 1 + \frac{1}{k+2} \sum_{1 \leq p, q \leq k} e^{\alpha_p - \alpha_q}.
\]
Note that \(\exp(\beta(0))\) fixes \((\alpha_p - \alpha_q)(-1)^2 1\) and multiplies \(e^{\alpha_p - \alpha_q}\) by \(\exp((\beta, \alpha_p - \alpha_q))\). Since both \(\sigma_{M^0}\) and \(\theta\) fix the conformal vector of \(M^0\), we have that \(\exp((\beta, \alpha_p - \alpha_q)) = 1\) for all \(1 \leq p, q \leq k, p \neq q\). Thus \(\exp(\beta(0)) = 1\), and \(\sigma_{M^0} = \theta\).

We discuss a condition on a rational lattice \(L\) such that \(N \subset L \subset N^*\) for which the \(V_N\)-module \(V_L\) is of \(\sigma\)-type as an \(M^0\)-module. Let
\[
\lambda_i = \frac{1}{2k} k - \frac{1}{2} \alpha_i, \quad 1 \leq i \leq k.
\]
Then \(2\lambda_i \equiv 2\lambda_k \pmod{N}\), \(\lambda_1 + \cdots + \lambda_k = 0\), and \(\{\lambda_2, \ldots, \lambda_k\}\) is a \(\mathbb{Z}\)-basis of \(N^*\) [2, Lemma 4.2]. Let
\[
N(j, a) = N - \sum_{i=1}^{k} a_i \lambda_i + 2j \lambda_k
\]
for $0 \leq j < k$ and $a = (a_1, \ldots, a_k) \in \{0, 1 \}^k$. Since $2k\lambda_k \in N$, we may consider $j$ to be modulo $k$. Any coset of $N$ in $N^*$ is of the form $N(j, a)$ for some $j$ and $a$. However, $j$ and $a$ are not uniquely determined [2, Lemma 4.3].

The irreducible $V_N$-module $V_{N(j,a)}$ decomposes into a direct sum of irreducible $T$-modules as follows:

$$V_{N(j,a)} = \bigoplus_{0 \leq i_s \leq s} L(c_1, h_{i_1+1,i_2+1}) \otimes \cdots \otimes L(c_{k-1}, h_{i_{k-1}+1,i_{k}+1}) \otimes M^{i_k j + (i_k - b_k)/2},$$

where $b_s = \sum_{i=1}^s a_i [2, Theorem 5.2]$. We denote by $\text{wt}(a)$ the number of non-zero entries $a_i$ of $a = (a_1, \ldots, a_k)$. Then $b_k = \text{wt}(a)$, and the irreducible $M^0$-module $M^{i_k j + (i_k - b_k)/2}$ is of $\sigma$-type if and only if $i_k$ is even and $2j = \text{wt}(a)$. In fact, $M^{i_k j + (i_k - b_k)/2} = \tilde{M}^{i_k, b_k - 2j}$ in the notation of Section 3.1. Thus $V_{N(j,a)}$ is of $\sigma$-type as an $M^0$-module if and only if $2j = \text{wt}(a)$.

Let $X = 1/2 N$. Then $N^*/X \cong \mathbb{Z}_2$. For $N \subset L \subset N^*$, we have $2L \subset N$ if and only if $L \subset X$. We show that $X$ is the union of $N(j,a)$ for all $0 \leq j < k$ and $a \in \{0, 1 \}^k$ such that $2j = \text{wt}(a)$. Indeed, let $0 \leq j < k$ and $a = (a_1, \ldots, a_k) \in \{0, 1 \}^k$ be such that $2j = \text{wt}(a)$. Take $1 \leq i_1 < i_2 < \cdots < i_{2j} \leq k$ so that $a_i = 1$ if $i \in \{i_1, \ldots, i_{2j}\}$. For $1 \leq r < s \leq k$, we have

$$2\lambda_k - \lambda_r - \lambda_s = \frac{1}{2}(\alpha_r - \alpha_k) + \frac{1}{2}(\alpha_s - \alpha_k)$$

$$= \frac{1}{2}(\alpha_r - \alpha_s) \quad (\text{mod } N)$$

by (31) as $\alpha_s - \alpha_k \in N$. Since $\alpha_r - \alpha_s = \beta_r + \cdots + \beta_{s-1}$, it follows that

$$2j\lambda_k - \sum_{i=1}^k a_i\lambda_i = \sum_{p=1}^j (2\lambda_k - \lambda_{i_{2p-1}} - \lambda_{i_{2p}})$$

$$= \sum_{p=1}^j \frac{1}{2}(\beta_{i_{2p-1}} + \beta_{i_{2p-1}+1} + \cdots + \beta_{i_{2p}-1}) \quad (\text{mod } N).$$

Hence $N(j,a) \subset X$.

Next, we need to show that any element of $X$ belongs to $N(j,a)$ for some $0 \leq j < k$ and $a \in \{0, 1 \}^k$ such that $2j = \text{wt}(a)$. For $a = (a_1, \ldots, a_k)$ with $a_r = a_{r+1} = 1$ and $a_i = 0$, $i \neq r, r+1$, we have $\beta_r / 2 \in N(1,a)$. In this case, $N(1,a)$ satisfies the condition $2j = \text{wt}(a)$ with $j = 1$. For the sum of cosets of $N$ in $N^*$, we have

$$N(j,a) + N(j', a') = N(j + j' - (\text{wt}(a) + \text{wt}(a') - \text{wt}(a + a'))/2, a + a'),$$

where $a + a'$ is the sum of $a$ and $a'$ as elements of $(\mathbb{Z}_2)^k$, that is, the symmetric difference as subsets of $\{0, 1 \}^k$; see Section 4 of [2]. Let

$$j'' = j + j' - (\text{wt}(a) + \text{wt}(a') - \text{wt}(a + a'))/2,$$

and let $a'' = a + a'$. If $2j = \text{wt}(a)$ and $2j' = \text{wt}(a')$, then $2j'' = \text{wt}(a'')$. Thus for any $d_1, \ldots, d_{k-1} \in \{0, 1 \}$, $d_1\beta_1/2 + \cdots + d_{k-1}\beta_{k-1}/2$ belongs to some $N(j,a)$ such that $2j = \text{wt}(a)$ as desired.
By the above argument, we obtain the following proposition.

**Proposition 4.4:** Let $L$ be a rational lattice such that $N \subset L \subset N^*$, where $N = \sqrt{2}A_{k-1}$ with $k \geq 3$ an integer. Then the $V_N$-module $V_L$ is a $\sigma$-type $M^0$-module if and only if $2L \subset N$.

We consider a positive definite even lattice containing $N$ as an RSSD sublattice.

**Theorem 4.5:** Let $L$ be a positive definite even lattice containing a sublattice $N = \sqrt{2}A_{k-1}$ with $k \geq 3$ an integer. Suppose $N$ is RSSD in $L$.

1. $V_L$ is a $\sigma$-type $M^0$-module.
2. If $L(2) = \emptyset$, then $\varphi(\sigma_{M^0}) = t_N$, where $\varphi : \text{Aut}(V_L) \to O(L)$ is as in (14), $\sigma_{M^0}$ is the $\sigma$-involution of $V_L$ associated with $M^0$ as in Theorem 3.7, and $t_N$ is the RSSD involution of $L$ associated with $N$.

**Proof:** Set $B = \text{Ann}_L(N)$. Then rank $N + \text{rank } B = \text{rank } L$, and $V_N \otimes V_B$ is a vertex operator subalgebra of $V_L$; see Remark 2.3. We have $L \subset L^* \subset N^* \oplus B^*$ as in (1). Let $\alpha \in L$. Then $\alpha = a + b$ for some $a \in N^*$ and $b \in B^*$. Hence $V_{N+B+\alpha} = V_{N+a} \otimes V_{B+b}$ as $V_N \otimes V_B$-modules. Since $N$ is RSSD in $L$, we have $2a \in N$ by Lemma 2.2. Thus $V_{N+a}$ is a $\sigma$-type $M^0$-module by Proposition 4.4. This proves the assertion (1).

Assume that $L(2) = \emptyset$. Theorem 4.3 implies that $\varphi(\sigma_{M^0}) = -1$ on $N$. Moreover, $\varphi(\sigma_{M^0})$ is 1 on $B$ as $V_N \otimes V_B \subset V_L$. Hence the assertion (2) holds. 

5. **Centralizer of $\hat{\nu}$ in $\text{Aut}(V_{\sqrt{2}A_{k-1}})$**

We keep the notation in Section 4. Thus $k \geq 3$ is an integer, $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ for $1 \leq i, j \leq k$, $\beta_i = \alpha_i - \alpha_{i+1}$, and $N = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_{k-1} \cong \sqrt{2}A_{k-1}$. For convenience, we set $\beta_k = \alpha_k - \alpha_1$. We regard the indices of $\alpha$ and $\beta$ as elements of $\mathbb{Z}_k$. The vertex operator algebra $V_N = M(1) \otimes \mathbb{C}[N]$ is spanned by

$$\beta_{i_1}(-n_1) \cdots \beta_{i_r}(-n_r) \otimes e^{\alpha}$$

for $r \geq 0$, $1 \leq i_s \leq k-1$, $n_s > 0$, and $\alpha \in N$. Moreover,

$$\text{Aut}(V_N) = N(V_N) : O(N)$$

is a split extension of $N(V_N)$ by $O(N)$.

Let $\nu \in O(N)$ be an isometry of $N$ induced by a cyclic permutation

$$\nu : \alpha_1 \mapsto \alpha_2 \mapsto \cdots \mapsto \alpha_k \mapsto \alpha_1$$

of order $k$. The isometry $\nu$ is fixed point free on $N$. In fact, $\nu$ corresponds to a Coxeter element of the Weyl group of the root system of type $A_{k-1}$. Let $\hat{\nu} \in \text{Aut}(V_N)$ be a lift of $\nu$. Then $\hat{\nu}$ transforms the vector of $V_N$ in (33) as

$$\hat{\nu}(\beta_{i_1}(-n_1) \cdots \beta_{i_r}(-n_r) \otimes e^{\alpha}) = \nu(\beta_{i_1})(-n_1) \cdots \nu(\beta_{i_r})(-n_r) \otimes e^{\nu \alpha}.$$

The automorphism $\hat{\nu}$ is the identity on $M^0$ by the definition of $M^0$ in [5, Section 4], so it commutes with the $\sigma$-involution $\sigma_{M^0}$ of $V_N$ associated with $M^0$. 

When we consider \(O(N)\) to be a subgroup of \(\text{Aut}(V_N)\), we write \(v\) and \(-1\) for the automorphisms \(\hat{v}\) and \(\theta\) of \(V_N\), respectively. The centralizer \(C_{\text{Aut}(V_N)}(\hat{v})\) of \(\hat{v}\) in \(\text{Aut}(V_N)\) is

\[
C_{\text{Aut}(V_N)}(\hat{v}) = C_{\text{N}(V_N)}(\hat{v}) : C_{O(N)}(v)
\]

by (34). Moreover, \(O(N) = \langle -1 \rangle \times \text{Sym}_k\), and \(C_{O(N)}(v) = \langle -1 \rangle \times \langle v \rangle\), where a symmetric group \(\text{Sym}_k\) of degree \(k\) is the Weyl group of the root system of type \(A_{k-1}\). We also have

\[
C_{\text{N}(V_N)}(\hat{v}) = \{ \exp(h(0)) | h \in 2\pi \sqrt{-1}((1 - v)N)^* \}
\]

\[
\cong \text{Hom}(N/(1 - v)N, \mathbb{Z}_k)
\]

by Theorem 2.7 and Remark 2.8.

Now, \(|N/(1 - v)N| = k\). Indeed, \(1 + v + \cdots + v^{k-1} = 0\) on \(N\) as \(v\) is fixed point free on \(N\) of order \(k\). Since \(v^i \beta_1 = \beta_{1+i}\) for \(0 \leq i \leq k - 2\), the minimal polynomial of \(v\) on \(N\) is \(1 + x + \cdots + x^{k-1}\). This implies that \(|N/(1 - v)N| = k\); see the proof of [23, Lemma A.1].

Let \(\rho\) be the Weyl vector of the root system of type \(A_{k-1}\), that is, \(\rho\) is the half-sum of positive roots. Then \(\langle \rho, \beta_i \rangle = \sqrt{2}\) for \(1 \leq i \leq k - 1\), and

\[
\langle \rho, (1 - v)\beta_i \rangle / \sqrt{2} = \begin{cases} 0 & \text{if } 1 \leq i \leq k - 2, \\ k & \text{if } i = k - 1. \end{cases}
\]

Thus \(\rho / \sqrt{2}k \in ((1 - v)N)^*\). Since \(|N/(1 - v)N| = k\) implies \(|((1 - v)N)^*/N^*| = k\), we see that \(\rho / \sqrt{2}k + N^*\) generates \(((1 - v)N)^*/N^*\).

Define \(\psi \in N(V_N)\) by

\[
\psi = \exp(2\pi \sqrt{-1}\rho(0)/\sqrt{2}k).
\]

Then \(C_{\text{N}(V_N)}(\hat{v}) = \langle \psi \rangle\) is a cyclic group of order \(k\), and

\[
C_{\text{Aut}(V_N)}(\hat{v}) = \langle \psi \rangle : (\langle \theta \rangle \times \langle \hat{v} \rangle).
\]

The automorphism \(\psi\) acts on the vector in (33) as \(\xi_k^{\langle \rho, \alpha \rangle / \sqrt{2}}\), where \(\xi_k = \exp(2\pi \sqrt{-1}/k)\). Moreover, \(\theta\) and \(\psi\) generate a subgroup isomorphic to a dihedral group \(\text{Dih}_{2k}\) of order \(2k\) as \(\theta \psi \theta = \psi^{-1}\).

As for the normalizer of \(\langle \hat{v} \rangle\) in \(\text{Aut}(V_N)\), we have

\[
N_{\text{Aut}(V_N)}(\langle \hat{v} \rangle) = C_{\text{N}(V_N)}(\hat{v}) : C_{O(N)}(\langle v \rangle)
\]

by (34). The normalizer of \(\langle v \rangle\) in \(\text{Sym}_k\) is a split extension of \(\langle v \rangle\) by the multiplicative group \(\mathbb{Z}_k^\times\) consisting of invertible elements in \(\mathbb{Z}_k\). Therefore, the following theorem holds.

**Theorem 5.1:** Let \(k \geq 3\) be an integer, and let \(\theta\), \(\hat{v}\), and \(\psi\) be as above. Then the centralizer of \(\hat{v}\) and the normalizer of \(\langle \hat{v} \rangle\) in \(\text{Aut}(V_{\sqrt{2}A_{k-1}})\) are as follows.

1. \(C_{\text{Aut}(V_{\sqrt{2}A_{k-1}})}(\langle \hat{v} \rangle) = \langle \psi \rangle : (\langle \theta \rangle \times (\langle \hat{v} \rangle))\) with \(\langle \psi \rangle : \langle \theta \rangle \cong \text{Dih}_{2k}\).
2. \(N_{\text{Aut}(V_{\sqrt{2}A_{k-1}})}(\langle \hat{v} \rangle) = \langle \psi \rangle : (\langle \theta \rangle \times (\langle \hat{v} \rangle : \mathbb{Z}_k^\times))\).
In order to describe the action of $\mathbb{Z}_k^\times$ on $\langle \psi \rangle$, we consider the root lattice $A_{k-1}$ inside $\mathbb{R}^k$. Let $\varepsilon_1, \ldots, \varepsilon_k$ be the unit vectors in $\mathbb{R}^k$. Then $\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq k - 1$, form the set of simple roots of $A_{k-1}$, and the Weyl vector $\rho$ is

$$\rho = \frac{1}{2} \sum_{i=1}^{k} (k + 1 - 2i) \varepsilon_i.$$ 

Let $r_i$ be the isometry of $\mathbb{R}^k$ induced by the transposition of $\varepsilon_i$ and $\varepsilon_{i+1}$. Then the isometry $v \in O(N)$ corresponds to $\lambda = r_1 \cdots r_{k-1}$. For convenience, we regard the index of $\varepsilon$ as an element of $\mathbb{Z}_k$. For $s \in \mathbb{Z}_k^\times$, let $\tau_s$ be a permutation on $\{\varepsilon_1, \ldots, \varepsilon_k\}$ defined by $\tau_s(\varepsilon_i) = \varepsilon_{si}$, where $si$ is the product of $s$ and $i$ in $\mathbb{Z}_k$. Then $\tau_s \lambda \tau_s^{-1} = \lambda^s$ as $\lambda(\varepsilon_i) = \varepsilon_{i+1}$. The normalizer of $\langle \lambda \rangle$ in $\text{Sym}_k$ is $N_{\text{Sym}_k}(\langle \lambda \rangle) = \langle \lambda \rangle : H$, where $H = \{ \tau_s | s \in \mathbb{Z}_k^\times \} \cong \mathbb{Z}_k^\times$.

We have $\langle \rho, \varepsilon_i - \varepsilon_{i+1} \rangle = 1$ and $\langle \tau_s \rho, \varepsilon_i - \varepsilon_{i+1} \rangle = s^{-1}$. Hence $\hat{\tau}_s \psi \hat{\tau}_s^{-1} = \psi^{s^{-1}}$, where $\hat{\tau}_s$ is a lift of the isometry of $N = \sqrt{2}A_{k-1}$ induced by $\tau_s$. In particular, $\hat{\tau}_{k-1} \psi \hat{\tau}_{k-1} = \psi^{-1}$. Thus $\hat{\tau}_{k-1} \theta$ centralizes $\psi$.

By the above argument, we obtain the following corollary.

**Corollary 5.2:** $N_{\text{Aut}(V_N \circlearrowleft A_{k-1})}(\langle \hat{\psi} \rangle)/\langle \hat{\psi} \rangle \cong \langle \psi \rangle : \langle \theta \rangle$ is a dihedral group and $\langle \psi \rangle : \mathbb{Z}_k^\times$ is a Frobenius group.

Recall that $\sigma_{M^0} = \theta$ in $\text{Aut}(V_N)$ by Theorem 4.3. Since $\psi^{2i} \theta = \psi^i \theta \psi^{-i}$, the next proposition holds.

**Proposition 5.3:** $\psi^{2i} \sigma_{M^0} = \psi^i \sigma_{M^0} \psi^{-i}$ is the $\sigma$-involution associated with $\psi^i(M^0)$. There are $k$ or $k/2$ such $\sigma$-involutions for $0 \leq i \leq k - 1$ or $0 \leq i \leq k/2 - 1$ according as $k$ is odd or even.

### 6. Automorphism group of $V_N \circlearrowleft \sqrt{2}A_{p-1}$

Let $p$ be an odd prime. We keep the notation in Sections 4 and 5 with $k = p$. In this section, we determine the automorphism group $\text{Aut}(V_N^{(\hat{\psi})})$ of the fixed point subalgebra $V_N^{(\hat{\psi})}$ of $V_N$ by $\hat{\psi}$, where $N = \sqrt{2}A_{p-1}$ and $\hat{\psi}$ is a lift of the fixed point free isometry $v$ of $N$ of order $p$. Since $V_N$ is simple, self-dual, rational, $C_2$-cofinite, and of CFT-type, the fixed point subalgebra $V_N^{(\hat{\psi})}$ is also simple, self-dual, rational, $C_2$-cofinite, and of CFT-type by [26,27]. Thus every irreducible $V_N^{(\hat{\psi})}$-module appears in an irreducible $\hat{\psi}$-twisted $V_N$-module for some $0 \leq i \leq p - 1$ by [30, Theorem 3.3].

We first discuss irreducible $V_N^{(\hat{\psi})}$-modules contained in an irreducible $\hat{\psi}$-twisted $V_N$-module. Irreducible twisted modules for a lattice vertex operator algebra were constructed explicitly [31–33]. Following [31, (4.17)] (see also [34, Remark 3.1]), set

$$h^{(i,v)} = \{ h \in \mathfrak{h} | vh = \xi_p^{-i} h \}, \quad 0 \leq i \leq p - 1,$$

where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} N$ and $\xi_p = \exp(2\pi \sqrt{-1}/p)$. Then $h^{(0,v)} = 0$, and $\dim h^{(i,v)} = 1$ for $1 \leq i \leq p - 1$. Let $\hat{\mathfrak{h}}[v]$ be the $v$-twisted affine Lie algebra as in (4.3)–(4.5) of [32], and let $S[v]$
be the induced $\hat{\mathfrak{h}}[v]$-module as in (4.9) of [32]. Define a $v$-invariant alternating $\mathbb{Z}$-bilinear map $c^v : N \times N \to \mathbb{Z}_{2p}$ by

$$c^v(\alpha, \beta) = 2 \sum_{i=1}^{p-1} (iv^i(\alpha), \beta) + 2p\mathbb{Z}$$

for $\alpha, \beta \in N$ [32, Remark 2.2]. We denote the radical of $c^v$ by $R_N^v$.

Let $\hat{N}_v$ be a central extension of $N$ by a cyclic group $\langle \kappa_{2p} \rangle$ of order $2p$ with the commutator map $c^v$. Then the irreducible $\hat{v}$-twisted $V_N$-module constructed in [32,33] is of the form

$$V_N^{T,\hat{v}} = S[v] \otimes T,$$

where $T$ is an irreducible $\hat{N}_v$-module (see [32, (4.25)], [33, (7.6)]). There are $|R_N^v/(1 - v)N|$ inequivalent irreducible $\hat{v}$-twisted $V_N$-modules of such a form [33, Proposition 6.2].

It follows from [35, Lemma 3.2] that $R_N^v = N \cap (1 - v)N^*$. Since $2\lambda_i = \gamma_p/p - \alpha_i$ by (31) with $k = p$, we have $-\beta_i = (1 - v)(2\lambda_i) \in (1 - v)N^*$. Thus $R_N^v = N$. Moreover, $|N/(1 - v)N| = p$ by [23, Lemma A.1], so there are $p$ inequivalent irreducible $\hat{v}$-twisted $V_N$-modules of the form (35). Among the irreducible $V_N$-modules $V_N^{(j,a)}$, there are exactly $p$ $\hat{v}$-stable ones, namely, $V_N^{(j,(0,\ldots,0))}$ for $0 \leq j \leq p - 1$. Thus any irreducible $\hat{v}$-twisted $V_N$-module is isomorphic to $V_N^{T,\hat{v}}$ for some $T$ by [34, Theorem 10.2]. The conformal weight of $V_N^{T,\hat{v}}$ is

$$h(V_N^{T,\hat{v}}) = \frac{1}{4p^2} \sum_{i=1}^{p-1} i(p - i) \dim \mathfrak{h}^{(i,v)}$$

$$= \frac{(p - 1)(p + 1)}{24p}$$

by [32, (6.28)], which is not an integer.

Since $\hat{v}$ has prime order, the above argument for $\hat{v}$ can be applied to any $\hat{v}^i, 1 \leq i \leq p - 1$. Therefore, we have that the conformal weight of any irreducible $\hat{v}^i$-twisted $V_N$-module is not an integer for $1 \leq i \leq p - 1$.

Next, we discuss irreducible $V_N^{(\hat{v})}$-modules contained in an irreducible untwisted $V_N$-module $V_N^{(j,a)}$. Note that the conformal weight of any irreducible $V_N^{(\hat{v})}$-module except for $V_N^{(\hat{v})}$ is positive. If $V_N^{(j,a)}$ is not $\hat{v}$-stable, then it is an irreducible $V_N^{(\hat{v})}$-module. In this case, the quantum dimension of $V_N^{(j,a)}$ as a $V_N^{(\hat{v})}$-module is greater than 1, so $V_N^{(j,a)}$ is not a simple current $V_N^{(\hat{v})}$-module by [36, Proposition 4.17].

We see from [2, Theorem A.1] that the conformal weight of $V_N^{(j,(0,\ldots,0))}$ is $j(p - j)/p$ for $0 \leq j \leq p - 1$, which is not an integer unless $j = 0$ as $p$ is a prime. If $j = 0$, then $N(j,(0,\ldots,0)) = N$. Let

$$V_N(i) = \{v \in V_N \mid \hat{v}v = \zeta_p^{-i}v\}, \quad 0 \leq i \leq p - 1.$$ 

Then $V_N(0) = V_N^{(\hat{v})}$, and the weight one subspace of $V_N(i)$ corresponds to $\mathfrak{h}^{(i,v)}$. Moreover, $V_N(i), 0 \leq i \leq p - 1$, are simple current $V_N^{(\hat{v})}$-modules by [36, Theorem 6.3].
By the above argument, we obtain the following lemma.

**Lemma 6.1:** Among the irreducible $V_N^{\hat{\nu}}$-modules, only $V_N(i)$, $1 \leq i \leq p - 1$, are simple currents with non-zero weight one subspace.

Let $g$ be an automorphism of $V_N^{\hat{\nu}}$. Then for each $1 \leq i \leq p - 1$, Lemma 6.1 implies that $V_N(i) \circ g = V_N(j)$ for some $1 \leq j \leq p - 1$. Hence $g$ can be extended to an automorphism of the vertex operator algebra $V_N$ by [37, Theorem 2.1] as $V_N = \bigoplus_{i=0}^{p-1} V_N(i)$ is a $\mathbb{Z}_p$-graded simple current extension of $V_N^{\hat{\nu}}$. Moreover, the next theorem holds by [37, Corollary 2.2] and Corollary 5.2.

**Theorem 6.2:** Let $p$ be an odd prime. Then $\text{Aut}(V_N^{\hat{\nu}}) \cong N_{\text{Aut}(V_N)}((\hat{\nu}))/\langle \hat{\nu} \rangle$, which has the shape $p : (2 \times (p - 1))$ with $p : 2$ a dihedral group of order $2p$, and $p : (p - 1)$ a Frobenius group of order $p(p - 1)$.

### 7. Examples

In this section, we discuss $\sigma$-involutions of certain lattice vertex operator algebras. Those examples illustrate the relationship between $\sigma$-involutions and RSSD involutions. We also deal with $\sigma$-involutions not related to $V_{\sqrt{2}A_{k-1}}$.

#### 7.1. $\sigma$-involutions of $V_{A_{p-1} \otimes R}$

Let $p \geq 3$ be an integer, and let $R$ be a root lattice of type $A$, $D$, or $E$ of rank $n$. We study $\sigma$-involutions of a vertex operator algebra $V_{A_{p-1} \otimes R}$ associated with the tensor product $A_{p-1} \otimes R$ of $A_{p-1}$ and $R$. The tensor product of two lattices $(A, \langle \cdot, \cdot \rangle_A)$ and $(B, \langle \cdot, \cdot \rangle_B)$ is by definition the tensor product $A \otimes_{\mathbb{Z}} B$ of $\mathbb{Z}$-modules $A$ and $B$ equipped with a symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle \alpha \otimes \beta, \alpha' \otimes \beta' \rangle = \langle \alpha, \alpha' \rangle_A \cdot \langle \beta, \beta' \rangle_B$$

for $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$. For simplicity of notation, we denote $A \otimes_{\mathbb{Z}} B$ by $A \otimes B$.

Let $\epsilon_1, \ldots, \epsilon_p$ be the unit vectors in $\mathbb{R}^p$, and set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq p - 1$, and set $\alpha_0 = \epsilon_p - \epsilon_1$. We also take the set $\{\beta_1, \ldots, \beta_n\}$ of simple roots of $R$. Then $\alpha_i \otimes \beta_j$, $1 \leq i \leq p - 1, 1 \leq j \leq n$, form a $\mathbb{Z}$-basis of $A_{p-1} \otimes R$. Note that the symbols $\alpha_i$ and $\beta_j$ here are different from those used in Sections 4–6.

Set $A_\beta = A_{p-1} \otimes \mathbb{Z}\beta$ for $\beta \in R$, and set $A_R = A_{p-1} \otimes R$. Then $A_\beta \cong \sqrt{2}A_{p-1}$ for $\beta \in R(2)$, where $R(2) = \{\beta \in R | \langle \beta, \beta \rangle = 2\}$. The lattice $A_R$ is positive definite and even. We slightly generalize [12, Lemma 3.3] as follows.

**Lemma 7.1:**

1. $\langle x, x \rangle \geq 4$ for $0 \neq x \in A_R$.
2. $A_R(4) = \{\alpha \otimes \beta | \alpha \in A_{p-1}(2), \beta \in R(2)\}$.

**Proof:** We have $A_R \subset \bigoplus_{i=1}^p \mathbb{Z}\epsilon_i \otimes R$, and $\langle \epsilon_i \otimes x, \epsilon_j \otimes y \rangle = \delta_{ij}\langle x, y \rangle$ for $x, y \in R$. Let $0 \neq x \in A_R$. Then $x = \sum_{i=1}^p \epsilon_i \otimes b_i$ for some $b_i \in R$, and $\langle x, x \rangle = \sum_{i=1}^p \langle b_i, b_i \rangle$. Let $i_1, \ldots, i_s$ be
Lemma 7.2: If \( \langle x, x \rangle = 4 \). Then \( s = 2 \) and \( \langle b_{i_1}, b_{i_2} \rangle = \langle b_{i_2}, b_{i_1} \rangle = 2 \). Since \( x \in A_R \), we have \( x = (\varepsilon_{i_1} - \varepsilon_{i_2}) \otimes b_{i_1} \). Thus the assertion (2) holds.

By a similar argument as in the proof of [14, Proposition 5.24], we obtain the next lemma.

Lemma 7.2: If \( \beta \in R(2) \), then \( A_\beta \) is RSSD in \( A_R \). Moreover, \( t_{A_\beta} = 1 \otimes r_\beta \) in \( O(A_R) \), where \( t_{A_\beta} \) is the RSSD involution associated with \( A_\beta \), and \( r_\beta : x \mapsto x - \langle x, \beta \rangle \beta \) is the reflection on \( R \) associated with \( \beta \).

Proof: Let \( \beta \in R(2) \). Then we can choose the set \( \{\beta_1, \ldots, \beta_n\} \) of simple roots of \( R \) containing \( \beta \), say, \( \beta = \beta_i \). Since \( R \) is a root lattice of type \( A, D, \) or \( E \), we have \( \langle \beta, \beta_j \rangle = 0 \) or \(-1\) for \( j \neq i \). If \( \langle \beta, \beta_j \rangle = 0 \), then \( A_{\beta_j} \subset \text{Ann}_{A_R}(A_\beta) \). If \( \langle \beta, \beta_j \rangle = -1 \), then \( \langle \beta, \beta + 2\beta_j \rangle = 0 \), so \( A_{2\beta_j} \subset A_\beta + \text{Ann}_{A_R}(A_\beta) \). Thus \( A_\beta \) is RSSD in \( A_R \).

Both \( t_{A_\beta} \) and \( 1 \otimes r_\beta \) are \(-1\) on \( A_\beta \), and \( 1 \) on \( A_\beta \) for \( \beta_j \) such that \( \langle \beta, \beta_j \rangle = 0 \). If \( \langle \beta, \beta_j \rangle = -1 \), then we consider

\[
\alpha \otimes \beta_j = \alpha \otimes \left( -\frac{1}{2} \beta \right) + \alpha \otimes \frac{1}{2} (\beta + 2\beta_j)
\]

for \( \alpha \in A_{p-1} \). The first term on the right-hand side of the above equation belongs to \( \mathbb{Q}A_\beta \), and the second term belongs to \( \mathbb{Q}\text{Ann}_{A_R}(A_\beta) \), so we have

\[
t_{A_\beta}(\alpha \otimes \beta_j) = -\alpha \otimes \left( -\frac{1}{2} \beta \right) + \alpha \otimes \frac{1}{2} (\beta + 2\beta_j)
\]

\[
= \alpha \otimes (\beta + \beta_j),
\]

which agrees with \((1 \otimes r_\beta)(\alpha \otimes \beta_j)\). Therefore, \( t_{A_\beta} = 1 \otimes r_\beta \) on \( A_R \).

Let \( \beta \in R(2) \). Then \( A_\beta \cong \sqrt{2}A_{p-1} \), so the vertex operator algebra \( V_{A_\beta} \) contains a subalgebra \( W_\beta \cong K(\mathfrak{sl}_2, \rho) \) which corresponds to \( M^0 \subset V_N \) in the notation of Section 4. The vertex operator algebra \( V_{A_R} \) is a \( \sigma \)-type \( W_\beta \)-module by (1) of Theorem 4.5 and Lemma 7.2. Let \( \sigma_{W_\beta} \in \text{Aut}(V_{A_\beta}) \) be the \( \sigma \)-involution associated with \( W_\beta \) as in Theorem 3.7. Since \( A_R(2) = \emptyset \) by Lemma 7.1, we have an exact sequence

\[
1 \longrightarrow N(V_{A_R}) \longrightarrow \text{Aut}(V_{A_\beta}) \longrightarrow O(A_R) \longrightarrow 1
\]

of groups as in (14). Moreover,

\[
\varphi(\sigma_{W_\beta}) = 1 \otimes r_\beta \quad (36)
\]

by (2) of Theorem 4.5 and Lemma 7.2.

Now, we assume that \( p \) is an odd prime. Let \( \nu \in O(A_{p-1}) \) be an isometry of \( A_{p-1} \) induced by

\[
\nu : \varepsilon_1 \mapsto \varepsilon_2 \mapsto \cdots \mapsto \varepsilon_p \mapsto \varepsilon_1.
\]

Then \( \nu(\alpha_i) = \alpha_{i+1} \) for \( 1 \leq i \leq p-2 \), and \( \nu(\alpha_{p-1}) = \alpha_0 \). The isometry \( \nu \) is fixed point free on \( A_{p-1} \) of order \( p \). We consider \( \nu \otimes 1 \in O(A_R) ; \alpha \otimes \beta \mapsto (\nu \alpha) \otimes \beta \), which is fixed point
free on \( \mathcal{A}_R \) of order \( p \). For simplicity of notation, we also denote \( \nu \otimes 1 \) by \( \nu \). Then the restriction of \( \nu \in O(\mathcal{A}_R) \) to \( \mathcal{A}_R^{\beta} \) for \( \beta \in R(2) \) agrees with \( \nu \in O(N) \) in Section 5.

Lemma 7.3: \((1 - \nu)\mathcal{A}_R = ((1 - \nu)\mathcal{A}_{p-1}) \otimes R, and \mathcal{A}_R/(1 - \nu)\mathcal{A}_R \cong p^n \) is elementary abelian of order \( p^n \).

Proof: Since \((1 - \nu)(\alpha \otimes \beta) = ((1 - \nu)\alpha) \otimes \beta \), the first assertion holds. The second assertion follows from [23, Lemma A.1] as \( p \) is an odd prime. 

Let \( \hat{\nu} \in \text{Aut}(V_{A_p}) \) be a lift of \( \nu \). In order to deal with the restriction of \( \hat{\nu} \) to \( V_{A_\gamma} \) for \( \gamma \in R(2) \), we recall the definition of \( \hat{\nu} \) in Section 2.2. For simplicity, we denote \( \mathcal{A}_R \) by \( L \). Let \( \varepsilon : \tilde{L} \times \tilde{L} \rightarrow \mathbb{Z}_2 \) be a bilinear form satisfying (4), and let \( \eta : \tilde{L} \rightarrow \mathbb{Z}_2 \) be a quadratic form satisfying
\[
\eta(\tilde{\alpha} + \tilde{\beta}) + \eta(\tilde{\alpha}) + \eta(\tilde{\beta}) = \varepsilon(\tilde{\alpha}, \tilde{\beta}) + \varepsilon(\tilde{\nu} \tilde{\alpha}, \tilde{\nu} \tilde{\beta}) \quad \text{for } \alpha, \beta \in L
\]
as in (8), where \( \tilde{L} = L/2L, \tilde{\alpha} = \alpha + 2L, \) and \( \tilde{\beta} = \beta + 2L \). Then \( e^{a_\alpha} e^{a_\beta} = e^{a_\alpha + a_\beta} + e^{a_\alpha} e^{a_\beta} + e^{a_\alpha} e^{a_\beta} + e^{a_\alpha + a_\beta} \)
in \( \tilde{L} \) as in (7), and \( \hat{\nu} \in O(\tilde{L}) \) is given by \( \hat{\nu}(e^{a_\alpha}) = e^{\nu a_\alpha} e^{a_{\tilde{\gamma}}} \), as in (9).

Since \( \langle \alpha, \alpha \rangle \in 4\mathbb{Z} \) for \( \alpha \in A_\gamma \), the restriction of \( \varepsilon \) to \( \overline{A}_\gamma \times \overline{A}_\gamma \) is an alternating bilinear form, where \( \overline{A}_\gamma = A_\gamma + 2L \). Note that \( 2A_\gamma \subset A_\gamma \cap 2L \). Thus there is a quadratic form \( \xi : \overline{A}_\gamma \rightarrow \mathbb{Z}_2 \) such that
\[
\xi(\tilde{\alpha} + \tilde{\beta}) + \xi(\tilde{\alpha}) + \xi(\tilde{\beta}) = \varepsilon(\tilde{\alpha}, \tilde{\beta}) \quad \text{for } \alpha, \beta \in A_\gamma.
\]
By a similar argument as in the proof of Lemma 2.5, we see that \( \text{Hom}(\overline{A}_\gamma, \mathbb{Z}_2) \rightarrow \text{Hom}(A_\gamma, \mathbb{Z}_2); \mu \mapsto \mu + \mu^\nu \) is a \( \mathbb{Z}_2 \)-linear isomorphism, where \( \mu^\nu(\tilde{\alpha}) = \mu(\tilde{\nu} \tilde{\alpha}) \). Hence there is a unique \( \mu \in \text{Hom}(\overline{A}_\gamma, \mathbb{Z}_2) \) such that \( \eta = \xi + \xi^\nu + \mu + \mu^\nu \) on \( \overline{A}_\gamma \), where \( \xi^\nu(\tilde{\alpha}) = \xi(\tilde{\nu} \tilde{\alpha}) \). Let \( e^{a_\alpha} = e^{a_\alpha}(\xi + \xi^\nu) \) for \( \alpha \in A_\gamma \). Then \( e^{a_\alpha} e^{a_\beta} = e^{a_\alpha + a_\beta} \), and \( \hat{\nu}(e^{a_\alpha}) = e^{\nu a_\alpha} \). Therefore, by changing the section \( \alpha \mapsto e^{a_\alpha} \) with \( \alpha \mapsto e^{a_\alpha} \) for \( \alpha \in A_\gamma \), we can regard the restriction of \( \hat{\nu} \) to \( V_{A_\gamma} \), the automorphism \( \hat{\nu} \) of \( V_N \) discussed in Section 5.

Let \( \rho \) be the Weyl vector of \( A_{p-1} \) with respect to the set of simple roots \( \{ \alpha_1, \ldots, \alpha_{p-1} \} \). Then \( \langle \rho, \alpha_i \rangle = 1 \) for \( 1 \leq i \leq p - 1 \), so it follows that
\[
\langle \rho, (1 - \nu)\alpha_i \rangle = \begin{cases} 0 & \text{if } 1 \leq i \leq p - 2, \\ p & \text{if } i = p - 1. \end{cases}
\]
Hence \( \rho \otimes R \subset ((1 - \nu)\mathcal{A}_R)^* \).

Lemma 7.4: For \( \gamma \in R \), we have \( \rho \otimes \gamma \in (p(\mathcal{A}_R))^* \) if and only if \( \gamma \in R \cap pR^* \).

Proof: If \( \gamma \in R \cap pR^* \), then \( \langle \rho \otimes \gamma, \alpha \otimes \beta \rangle = \langle \rho, \alpha \rangle \langle \gamma, \beta \rangle \in p\mathbb{Z} \) for \( \alpha \in A_{p-1} \) and \( \beta \in R \), so \( \rho \otimes \gamma \in p(\mathcal{A}_R)^* \). Conversely, assume that \( \rho \otimes \gamma \in p(\mathcal{A}_R)^* \). Then \( \langle \gamma, \beta \rangle = \langle \rho \otimes \gamma, \alpha_i \otimes \beta \rangle \in p\mathbb{Z} \) for \( \beta \in R \), so \( \gamma \in R \cap pR^* \). 

Lemma 7.5: \( |(R \cap pR^*)/pR| = p \) if \( R = A_n \) with \( n + 1 \equiv 0 \) (mod \( p \)) or \( R = E_6 \) with \( p = 3 \). Otherwise, \( R \cap pR^* = pR \).
**Proof:** If \(|R^*/R|\) is coprime to \(p\), then \(R \cap pR^* = pR\). Since \(p\) is an odd prime, \(|R^*/R|\) is divisible by \(p\) only if \(R = A_n\) with \(n + 1 \equiv 0 \pmod{p}\) or \(R = E_6\) with \(p = 3\). In fact, \(R^*/R\) is a cyclic group of order \(n + 1\) if \(R = A_n\), and a cyclic group of order 3 if \(R = E_6\). Thus the assertion holds.

For \(\beta \in R\), define \(\psi_\beta \in N(V_{A_R})\) by

\[
\psi_\beta = \exp(2\pi \sqrt{-1}(\rho \otimes \beta)(0)/p).
\]

Then \(\psi_\beta \in C_N(V_{A_R})(\hat{\nu})\) by (3) of Theorem 2.7. A map \(R \rightarrow C_{N(V_{A_R})}(\hat{\nu})\); \(\beta \mapsto \psi_\beta\) is a homomorphism from an additive group to a multiplicative group. Its kernel is \(R \cap pR^*\) by Lemma 7.4. Thus the image \(\{\psi_\beta \mid \beta \in R\}\) of the homomorphism is isomorphic to \(R/(R \cap pR^*)\).

Let \(\beta \in R(2)\). It follows from (36) that \(\sigma_{W_\beta} \psi_\beta \sigma_{W_\beta} = \psi_\beta^{-1}\). Thus \(\sigma_{W_\beta}\) and \(\psi_\beta\) generate a dihedral group \(\text{Dih}_{2p}\) of order \(2p\). Note that \(\psi_\beta^i \sigma_{W_\beta} = \psi_\beta^i \sigma_{W_\beta} \psi_\beta^{-i}\) is the \(\sigma\)-involution associated with \(\psi_\beta^i(W_\beta)\) for \(0 \leq i \leq p - 1\).

For the set of simple roots \(\{\beta_1, \ldots, \beta_n\}\) of \(R\), let \(W_l = W_{\beta_l}, \sigma_l = \sigma_{W_l},\) and \(\psi_l = \psi_{\beta_l}\) for \(1 \leq l \leq n\). The following theorem is a generalization of [14, Theorem 5.28].

**Theorem 7.6:** Let \(p\) be an odd prime, and let \(R, A_R, \hat{\nu}, W_l, \sigma_l,\) and \(\psi_l\) for \(1 \leq l \leq n\) be as above. Then the subgroup of \(C_{\text{Aut}(V_{A_R})}(\hat{\nu})\) generated by \(\sigma\)-involutions \(\psi_1^i \sigma_l\) for \(i = 0, 1\) and \(1 \leq l \leq n\) is isomorphic to

1. \(p^{n-1} : \text{Weyl}(R)\) if \(R = A_n\) with \(n + 1 \equiv 0 \pmod{p}\) or \(R = E_6\) with \(p = 3\),
2. \(p^n : \text{Weyl}(R)\) otherwise,

where \(n = \text{rank } R\) and \(\text{Weyl}(R)\) is the Weyl group of the root system of \(R\).

**Proof:** We have

\[
C_{\text{Aut}(V_{A_R})}(\hat{\nu}) \cong C_{N(V_{A_R})}(\hat{\nu}) : C_{\text{O}(A_R)}(\nu)
\]

by Proposition 2.6 and Theorem 2.7. Since \(\psi_l, 1 \leq l \leq n\), generate the subgroup \(\{\psi_\beta \mid \beta \in R\}\) of \(C_{N(V_{A_R})}(\hat{\nu})\) isomorphic to \(R/(R \cap pR^*)\), and since \(\psi(\psi_1^i \sigma_l) = 1 \otimes r_{\beta_l}\) by (36), the assertion holds by Lemma 7.5.

**7.2. \(\sigma\)-involutions of \(V_{L_C}\)**

We consider \(\sigma\)-involutions of a lattice vertex operator algebra \(V_{L_C}\), where \(L_C\) is a positive definite even lattice constructed in [35] by using a certain code \(C\). First, we recall the description of \(L_C\) in [35, Section 4]. A lattice of more general form \(L_{C \times D}\) was studied in [35]. Since we only deal with the case \(D = \{0\}\), we simply write \(L_C\) for \(L_{C \times \{0\}}\).
Let \( p \geq 3 \) be an odd integer. Let \( \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{p-1} \), and \( N = \sqrt{2}A_{p-1} \) be as in Section 4 with \( k = p \). Thus \( \langle \alpha_i, \alpha_j \rangle = 2\delta_{ij} \), and \( \beta_i = \alpha_i - \alpha_{i+1}, 1 \leq i \leq p-1 \), form a \( \mathbb{Z} \)-basis of \( N \). Set

\[
\beta_u = \frac{1}{2} \sum_{i=1}^{p-1} u_i \beta_i \in N^*
\]

for \( u = (u_1, \ldots, u_{p-1}) \in \mathbb{Z}^{p-1} \). Then

\[
\langle \beta_u, \beta_v \rangle = \frac{1}{2} uAv^t \in \mathbb{Z}
\]

for \( u, v \in \mathbb{Z}^{p-1} \), where \( A \) is a \( (p-1) \times (p-1) \) matrix with \( (i, j) \) entry \( \langle \beta_i, \beta_j \rangle / 2 \), that is, the Cartan matrix of type \( A_{p-1} \). Note that \( \langle \beta_u, \beta_u \rangle \in \mathbb{Z} \). Set

\[
L(u) = N + \beta_u.
\]

Since \( L(u) = L(v) \) if and only if \( u \equiv v \pmod{2\mathbb{Z}^{p-1}} \), we may write \( L(\overline{u}) \) for \( L(u) \), where \( \overline{u} = (\overline{u}_1, \ldots, \overline{u}_{p-1}) \in \mathbb{Z}_2^{p-1} \) with \( \overline{u}_i = u_i + 2\mathbb{Z} \).

We have \( uAv^t \equiv u'Av'^t \pmod{2\mathbb{Z}} \) if \( u \equiv u' \) and \( v \equiv v' \pmod{2\mathbb{Z}^{p-1}} \), so we can define an inner product \( \overline{u} \cdot \overline{v} \) on \( \mathbb{Z}_2^{p-1} \) by

\[
\overline{u} \cdot \overline{v} = uAv^t + 2\mathbb{Z} \in \mathbb{Z}_2.
\]

Since the determinant of the matrix \( A \) is \( \det A = p \not\equiv 2 \pmod{4} \), the inner product (37) on \( \mathbb{Z}_2^{p-1} \) is non-degenerate.

For \( \alpha, \beta \in N^* \) with \( \alpha - \beta \in N \), we have \( \langle \alpha, \alpha \rangle \equiv \langle \beta, \beta \rangle \pmod{2\mathbb{Z}} \). Define \( w(\overline{u}) \in \mathbb{Z} \) by

\[
w(\overline{u}) = \min \{ \langle x, x \rangle \mid x \in L(\overline{u}) \} \in \mathbb{Z}.
\]

Then \( w(\overline{u}) \equiv \langle \beta_u, \beta_u \rangle \pmod{2\mathbb{Z}} \). We also define a map \( q : \mathbb{Z}_2^{p-1} \to \mathbb{Z}_2 \) by

\[
q(\overline{u}) = w(\overline{u}) + 2\mathbb{Z} = \frac{1}{2} uAu^t + 2\mathbb{Z}.
\]

Then \( q(\overline{u} + \overline{v}) + q(\overline{u}) + q(\overline{v}) = \overline{u} \cdot \overline{v} \), that is, \( q \) is a quadratic form on \( \mathbb{Z}_2^{p-1} \) with associated bilinear form \( \overline{u} \cdot \overline{v} \). We denote by \( K \) the vector space \( \mathbb{Z}_2^{p-1} \) over \( \mathbb{Z}_2 \) equipped with the inner product \( \overline{u} \cdot \overline{v} \) defined in (37) and the quadratic form \( q(\overline{u}) \) defined in (39).

Let \( \alpha \) be a positive integer. For \( u = (u_1, \ldots, u_d) \in (\mathbb{Z}^{p-1})^d \), set

\[
\beta(u) = (\beta_{u_1}, \ldots, \beta_{u_d}) \in (N^*)^d,
\]

where \( (N^*)^d \) is an orthogonal sum of \( d \) copies of \( N^* \). Moreover, set

\[
L(u) = N^d + \beta(u).
\]

We also write \( L(\overline{u}) \) for \( L(u) \), where \( \overline{u} = (\overline{u}_1, \ldots, \overline{u}_d) \in K^d \).
We extend the non-degenerate inner product $\overline{u} \cdot \overline{v}$ on $\mathcal{K}$ defined in (37) to a non-degenerate inner product on $\mathcal{K}^d$ by

$$\overline{u} \cdot \overline{v} = \sum_{i=1}^{d} u_i \cdot v_i$$

for $\overline{u} = (u_1, \ldots, u_d)$, $\overline{v} = (v_1, \ldots, v_d) \in \mathcal{K}^d$. Then $\overline{u} \cdot \overline{v} = 2\langle \beta(u), \beta(v) \rangle + 2\mathbb{Z}$. Likewise, we extend the map $w : \mathcal{K} \to \mathbb{Z}$ defined in (38) to a map $\mathcal{K}^d \to \mathbb{Z}$ by

$$w(\overline{u}) = \sum_{i=1}^{d} w(u_i).$$

Then $w(\overline{u}) = \min\{\langle x, x \rangle | x \in L(\overline{u})\}$, and $w(\overline{u}) \equiv \langle \beta(u), \beta(u) \rangle \pmod{2\mathbb{Z}}$. We also define $q : \mathcal{K}^d \to \mathbb{Z}_2$ by

$$q(\overline{u}) = w(\overline{u}) + 2\mathbb{Z},$$

which is a quadratic form on $\mathcal{K}^d$ with associated bilinear form $\overline{u} \cdot \overline{v}$.

For a $\mathbb{Z}$-submodule $\mathcal{C}$ of $\mathcal{K}^d$, set

$$L_\mathcal{C} = \bigcup_{\overline{u} \in \mathcal{C}} L(\overline{u}).$$

Then $L_\mathcal{C}$ is a sublattice of $(\mathbb{N}^*)^d$ containing $\mathbb{N}^d$. We have the following lemma; see Corollary 4.4 and Proposition 4.5 of [35] in more general form.

**Lemma 7.7:**

1. $L_\mathcal{C}$ is integral if and only if $\mathcal{C}$ is self-orthogonal with respect to the inner product defined in (40).
2. $L_\mathcal{C}$ is even if and only if $\mathcal{C}$ is totally isotropic with respect to the quadratic form $q$ defined in (42).

As for $L_\mathcal{C}(2) = \{\alpha \in L_\mathcal{C} | \langle \alpha, \alpha \rangle = 2\}$, we have $L_\mathcal{C}(2) = \emptyset$ if and only if $w(\overline{u}) \neq 2$ for any $\overline{u} \in \mathcal{C}$. For example, if the number of non-zero entries $u_i$ of $\overline{u} = (u_1, \ldots, u_d)$ is at least 4 for any non-zero $\overline{u} \in \mathcal{C}$, then $L_\mathcal{C}(2) = \emptyset$.

Now, assume that $p$ is an odd prime. Let $\nu$ be as in Section 5 with $k = p$. We extend $\nu$ to an isometry of $(\mathbb{N}^*)^d$ diagonally, that is, $\nu(x_1, \ldots, x_d) = (\nu x_1, \ldots, \nu x_d)$ for $x_i \in \mathbb{N}^*$. Then $\nu$ induces a fixed point free action on $\mathcal{K}^d$ which preserves the inner product and the maps $w$ and $q$ defined in (40)–(42), respectively.

Suppose $\mathcal{C}$ is a $\nu$-invariant $\mathbb{Z}$-submodule of $\mathcal{K}^d$ such that $w(\overline{u}) \in 2\mathbb{Z}$ for any $\overline{u} \in \mathcal{C}$ and such that $w(\overline{u}) \geq 4$ if $\overline{u} \neq 0$. Then $L_\mathcal{C}$ is a positive definite even lattice with $L_\mathcal{C}(2) = \emptyset$. Moreover, $L_\mathcal{C}$ is invariant under $\nu$, and the restriction of $\nu$ to $L_\mathcal{C}$ is a fixed point free isometry of $L_\mathcal{C}$ of order $p$. Let $\hat{\nu} \in \text{Aut}(V_{L_\mathcal{C}})$ be a lift of $\nu$. Then the following lemma holds by Corollary 2.9.

**Lemma 7.8:** $\text{C}_{\text{Aut}(V_{L_\mathcal{C}})}(\hat{\nu}) \cong p^d : \text{C}_{\text{O}(L_\mathcal{C})}(\nu)$.

We denote by $N_l$ the $l$th direct summand of $\mathbb{N}^d$, so $N_l \cong \sqrt{2} A_{p-1}$. 


Lemma 7.9: \( N_l \) is RSSD in \( L_C \) for \( 1 \leq l \leq d \).

Proof: Since \( 2L_C \subset N^d \subset N_l + \text{Ann}_L(N_l) \), the assertion holds.

Let \( W_l \cong K(sl_2, p) \) be a subalgebra of the vertex operator algebra \( V_{N_l} \) which corresponds to \( M^0 \subset V_N \) in the notation of Section 4. Then \( V_{L_C} \) is a \( \sigma \)-type \( W_l \)-module by (1) of Theorem 4.5 and Lemma 7.9. Let \( \sigma_l = \sigma_{W_l} \in \text{Aut}(V_{L_C}) \) be the \( \sigma \)-involution associated with \( W_l \) as in Theorem 3.7. Then \( \varphi(\sigma_l) = t_{N_l} \) by (2) of Theorem 4.5, where \( t_{N_l} \in O(L_C) \) is the RSSD involution associated with \( N_l \).

As mentioned in Section 7.1, we can regard the restriction of \( \hat{\nu} \in \text{Aut}(V_{L_C}) \) to \( V_{N_l} \) as the automorphism \( \hat{\nu} \) of \( V_N \) discussed in Section 5. Recall the vector \( \rho \) considered in Section 5. Since \( \langle \rho, \beta_i \rangle = \sqrt{2} \) for \( 1 \leq i \leq p-1 \), we have \( \langle \sqrt{2} \rho, (1 - \nu) \beta_i / 2 \rangle = 0 \) or \( p \) according as \( 1 \leq i \leq p - 2 \) or \( i = p - 1 \). Thus

\[
\sqrt{2} \rho \in p \left( (1 - \nu) \left( \frac{1}{2} \mathbb{Z} \beta_1 + \cdots + \frac{1}{2} \mathbb{Z} \beta_{p-1} \right) \right)^*.
\]

(43)

We denote by \( \rho_l \in N_l \) the vector corresponding to \( \rho \in N \). Then \( \sqrt{2} \rho_l \in p((1 - \nu) L_C)^* \) by (43). Let

\[
\psi_l = \exp(2\pi \sqrt{-1}(\sqrt{2} \rho_l)(0)/p)
\]

for \( 1 \leq l \leq d \). Then \( \psi_l \in C_{N(V_{L_C})}(\hat{\nu}) \) by Theorem 2.7. Since \( N_l \) is orthogonal to \( N_{l'} \) for \( l \neq l' \), we have that \( \psi_l, 1 \leq l \leq d \), generate \( C_{N(V_{L_C})}(\hat{\nu}) \cong p^d \).

Since \( \sigma_l \psi_l \sigma_l = \psi_l^{-1} \), we see that \( \sigma_l \) and \( \psi_l \) generate a dihedral group \( \text{Dih}_{2p} \) of order \( 2p \), and that \( \psi_l^{2i} \sigma_l = \psi_l^i \sigma_l \psi_l^{-i} \) is the \( \sigma \)-involution associated with \( \psi_l(W_l) \) for \( 0 \leq i \leq p - 1 \). Therefore, the following theorem holds.

Theorem 7.10: Let \( p \) be an odd prime, and let \( C, L_C, \hat{\nu}, W_l, \sigma_l, \) and \( \psi_l \) for \( 1 \leq l \leq d \) be as above. Then the subgroup of \( C_{\text{Aut}(V_{L_C})}(\hat{\nu}) \) generated by \( \sigma \)-involution \( \psi_l \sigma_l \) for \( i = 0, 1 \) and \( 1 \leq l \leq d \) is isomorphic to \( (\text{Dih}_{2p})^d \).

A possible way to obtain more \( \sigma \)-involution is the use of the action of \( O(L_C) \). Let \( g \in O(L_C) \). Then \( g(N_l) \cong \sqrt{2}A_{p-1} \) is RSSD in \( L_C \) by Lemma 7.9. So a vertex operator subalgebra \( W \) of \( V_{g(N_l)} \) corresponding to \( M^0 \subset V_N \) is a \( \sigma \)-type parafermion vertex operator subalgebra of \( V_{L_C} \) by Theorem 4.5. Hence we can consider the \( \sigma \)-involution \( \sigma_W \) of \( V_{L_C} \) associated with \( W \). Moreover, \( \varphi(\sigma_W) = t_{g(N_l)} \in O(L_C) \) is the RSSD involution associated with \( g(N_l) \). Since \( h t_{g(N_l)} h^{-1} = t_{g(N_l)} \) for \( h \in O(L_C) \), the group \( \langle t_{g(N_l)} \mid g \in O(L_C) \rangle \) generated by \( t_{g(N_l)} \) for \( g \in O(L_C) \) is a normal subgroup of \( O(L_C) \).

Assume that \( g \in N_{O(L_C)}((\nu)) \). Then \( g(N_l) \) is \( \nu \)-invariant, and \( t_{g(N_l)} \) commutes with \( \nu \). Thus \( \langle t_{g(N_l)} \mid g \in N_{O(L_C)}((\nu)) \rangle \) is a normal subgroup of \( C_{O(L_C)}((\nu)) \). In this case, we also have \( W \subset V_{L_C}(\hat{\nu}) \). We will discuss such an example in Section 7.3 below.

7.3. Non-standard \( \sigma \)-involution and \( \text{Aut}(V_{L_C}(\hat{\nu})) \)

As shown in Theorem 4.5, one can associate a \( \sigma \)-type parafermion vertex operator subalgebra \( K(sl_2, p) \subset V_L \) for an integer \( p \geq 3 \) with an RSSD sublattice \( \sqrt{2}A_{p-1} \) of a positive
definite even lattice $L$. For the orbifold $V_{L_C}^{(5)}$ of a lattice vertex operator algebra $V_{L_C}$ discussed in Section 7.2, it may also contain some special $\sigma$-type parafermion vertex operator algebras which are not obtained from RSSD sublattices isometric to $\sqrt{2}A_{p-1}$. In [14], a special case in which $p = 3$ and $L_C \cong K_{12}$, the Coxeter-Todd lattice of rank 12, has been studied in detail. It turns out that a special $\sigma$-type parafermion vertex operator subalgebra $K(sl_2, 3) \cong W_3(4/5)$ is related to some extra automorphism of the orbifold $V_{K_{12}}^{(5)}$ which cannot be extended to an automorphism of $V_{K_{12}}$. In this section, we discuss another special case in which $p = 5$ and $L_C$ is a coinvariant sublattice of the Leech lattice $\Lambda$ associated with a $5B$ element of $O(\Lambda) = CO_0$.

### 7.3.1. Rank 16 lattice $L_C$

From now on, let $p = 5$. We use the notation in Section 7.2 with $p = 5$ and $d = 4$. Thus $N = \text{span}_\mathbb{Z}\{\beta_1, \beta_2, \beta_3, \beta_4\} \cong \sqrt{2}A_4$ and $K \cong \mathbb{Z}_2^4$. The fixed point free isometry $v$ of order 5 acts as $v\beta_i = \beta_{i+1}$, where $i$ is considered to be modulo 5 and $\beta_0 = -(\beta_1 + \beta_2 + \beta_3 + \beta_4)$. The isometry $v$ also acts on $K$ as

$$(i_1, i_2, i_3, i_4) \mapsto (i_4, i_1 + i_4, i_2 + i_4, i_3 + i_4).$$

Set

$$\lambda = \frac{1}{5}(\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4),$$

(44)

which is $2\lambda_k$ with $k = 5$ in the notation of (31). Then $\lambda \in N^*$ and $\lambda + N$ generates an order 5 subgroup of $N^*/N$. Note that $\langle \lambda, \beta_i \rangle = 0$ for $i = 1, 2, 3$, $\langle \lambda, \beta_4 \rangle = 2$, and $\langle \lambda, \lambda \rangle = 8/5$. We also have $v^i\lambda = \lambda + \beta_0 + \cdots + \beta_{i-1}$ for $1 \leq i \leq 4$. In particular, the coset $\lambda + N$ is fixed by $v$. We extend $v$ to an isometry of $(N^*)^4$ diagonally.

Now, consider a $\mathbb{Z}$-submodule $C$ of $K^4$ generated by

$$\begin{align*}
&[(1, 0, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0)], \\
&[(0, 1, 0, 0), (0, 1, 0, 0), (0, 1, 0, 0), (0, 1, 0, 0)], \\
&[(0, 0, 1, 0), (0, 0, 1, 0), (0, 0, 1, 0), (0, 0, 1, 0)], \\
&[(0, 0, 0, 1), (0, 0, 0, 1), (0, 0, 0, 1), (0, 0, 0, 1)], \\
&[(1, 0, 0, 0), (0, 0, 1, 1), (1, 0, 1, 1), (0, 0, 0, 0)], \\
&[(0, 1, 0, 0), (1, 1, 1, 0), (1, 0, 1, 0), (0, 0, 0, 0)], \\
&[(0, 0, 0, 1), (1, 1, 0, 0), (1, 1, 0, 0), (0, 0, 0, 0)], \\
&[(0, 0, 0, 0), (1, 1, 0, 0), (1, 1, 0, 0), (0, 0, 0, 0)].
\end{align*}$$

Let $c_1, \ldots, c_8$ be those eight elements of $C$ from the top to the bottom in order. Then $v$ acts on $C$ as

$$
\begin{align*}
&c_1 \mapsto c_2 \mapsto c_3 \mapsto c_4 \mapsto c_1 + c_2 + c_3 + c_4 \mapsto c_1, \\
&c_5 \mapsto c_6 \mapsto c_7 \mapsto c_8 \mapsto c_5 + c_6 + c_7 + c_8 \mapsto c_5.
\end{align*}
$$

Thus $C$ is $v$-invariant, so the lattice $L_C$ is invariant under $v$. We have $|C| = 2^8$, and $C$ is self-dual with respect to the inner product on $K^4$ defined as in (40). Hence the dual lattice of
$L_C$ is $(L_C)^* = (L_C \times a)^* = L_C \times \mathbb{Z}_2^3$ by [35, Proposition 4.3], and $(L_C)^*/L_C \cong 5^4$. By a direct calculation, we can verify that $w(c) \in \{0, 4, 6, 8\}$ for $c \in C$, where $w(c)$ is defined as in (41) for $c \in K^4$. The number of $c \in C$ with given value of $w(c)$ is as follows.

\[
\begin{array}{cccc}
\text{Type} & \text{I} & \text{II} & \text{III} & \text{IV} \\
5 & 5 & 60 & 60
\end{array}
\]  

More precisely, the five elements of either type I or type II belong to a single $\langle \nu \rangle$-orbit, while the 60 elements of either type III or type IV are divided into 12 $\langle \nu \rangle$-orbits. We denote by $\mathcal{K}_l$ the $l$th direct summand of $K^4$ for $1 \leq l \leq 4$, and consider the action of an alternating group $\text{Alt}_4$ of degree 4 on the set $\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$ of those direct summands. This action induces an action of $\text{Alt}_4$ on $C$. In fact, the $\text{Alt}_4$-orbit containing

\[
[(1, 0, 0, 0), (0, 0, 1, 1), (0, 0, 1, 1), (0, 0, 0, 0)]
\]

has 12 elements of $C$, which form a complete set of representatives of the 12 $\langle \nu \rangle$-orbits of the type III elements. Since the action of $\text{Alt}_4$ commutes with the action of $\nu$, the 60 type III elements comprise an orbit under the action of $\langle \nu \rangle \times \text{Alt}_4$. Likewise, the 60 type IV elements comprise an orbit under the action of $\langle \nu \rangle \times \text{Alt}_4$. Note that $\text{Alt}_4$ acts trivially on the type I and the type II elements.

Let $N_l$ be the $l$th direct summand of $N^4$. Then $N^4$ is an orthogonal sum of $N_l$ for $1 \leq l \leq 4$. We consider the action of $\text{Alt}_4$ on the set $\{N_1^4, N_2^4, N_3^4, N_4^4\}$. Since $C$ is invariant under the action of $\text{Alt}_4$ on $K^4$, this action induces an action of $\text{Alt}_4$ on the lattice $L_C$. Thus the isometry group $O(L_C)$ of the lattice $L_C$ contains $\langle \nu \rangle \times \text{Alt}_4$.

**Remark 7.12:** Recall the notation $\beta(u) \in (N^*)^d$ for $u \in (\mathbb{Z}^{p-1})^d$ in Section 7.2. Let $c \in C$ with $w(c) = 4$. We consider $c$ to be an element of $\{0, 1\}^{16}$, and define $\beta(c)$ as $\beta(u)$ for
$u = c$ with $p = 5$ and $d = 4$. Then $\beta(c) \in L_C$ and $\langle \beta(c), \beta(c) \rangle = 4$. Denote by $A(\beta(c))$ the sublattice of $L_C$ spanned by $v^i(\beta(c))$ for $0 \leq i \leq 3$. Since $v$ is fixed point free of order 5, we have $1 + v + v^2 + v^3 + v^4 = 0$ on $L_C$. Thus $A(\beta(c))$ is the $v$-invariant sublattice generated by $\beta(c)$. We have $(\beta(c), v(\beta(c))) = -2, 0, -2, 0$ according as $c$ is of type I, II, III, or IV, respectively. Therefore, $A(\beta(c)) \cong \sqrt{2}A_4$ if $c$ is of type I, II, or III, and $A(\beta(c)) \cong A_4(1)$ if $c$ is of type IV by [12, Lemma D.20], where $A_4(1)$ is a rank $4$ lattice having the Gram matrix

$$
\begin{pmatrix}
4 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{pmatrix}.
$$

### 7.3.2. Dihedral $\sqrt{2}E_8$-pairs

We show that $L_C$ can be realized as a sum of two sublattices isometric to $\sqrt{2}E_8$ [12]. Let $\gamma = \beta_1$ and $\delta = \beta_3 + \beta_4$. Then $\langle \gamma, \gamma \rangle = \langle \delta, \delta \rangle = 4$ and $\langle \gamma, \delta \rangle = 0$. Let $F = (\mathbb{Z} \gamma + \mathbb{Z} \delta)^4 \subset N^4$. Then $F \cong \sqrt{2}A_4^8$. Define

$$M = \text{span}_{\mathbb{Z}} \left\{ F, \frac{1}{2}(\gamma, \gamma, \gamma, \gamma), \frac{1}{2}(\delta, \delta, \delta, \delta), \frac{1}{2}(\gamma, \delta, \gamma + \delta, 0), \frac{1}{2}(\delta, \gamma + \delta, \gamma, 0) \right\},$$

and $M' = v^2(M)$. We can express the generators of $M/F \cong \mathbb{Z}_2^4$ as elements of $\mathbb{Z}_2^8$ in the following manner.

$$\frac{1}{2}(\gamma, \gamma, \gamma, \gamma) + F \leftrightarrow (1, 1, 1, 1, 0, 0, 0, 0),$$

$$\frac{1}{2}(\delta, \delta, \delta, \delta) + F \leftrightarrow (0, 0, 0, 0, 1, 1, 1, 1),$$

$$\frac{1}{2}(\gamma, \delta, \gamma + \delta, 0) + F \leftrightarrow (1, 0, 1, 0, 0, 1, 1, 0),$$

$$\frac{1}{2}(\delta, \gamma + \delta, \gamma, 0) + F \leftrightarrow (0, 1, 1, 0, 1, 1, 0, 0).$$

Since $\langle \gamma/2, \gamma/2 \rangle = \langle \delta/2, \delta/2 \rangle = 1$, the symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on the lattice $(N^4)^4$ induces the standard inner product on $2F^* / F \cong \mathbb{Z}_2^8$, and $M/F$ defines a self-dual $\mathbb{Z}_2$-code with minimal weight $4$ in $\mathbb{Z}_2^8$. That is, $M/F$ is isomorphic to the $[8, 4, 4]$ extended Hamming code. Therefore, $M$ is isometric to $\sqrt{2}E_8$.

Note that $F + v^2(F) = N^4$. Using the elements $c_1, \ldots, c_8$ of $C$, we have

$$\frac{1}{2}(\gamma, \gamma, \gamma, \gamma) + N^4 = c_1, \quad \frac{1}{2}(\delta, \delta, \delta, \delta) + N^4 = c_3 + c_4,$$

$$\frac{1}{2}(\gamma, \delta, \gamma + \delta, 0) + N^4 = c_5, \quad \frac{1}{2}(\delta, \gamma + \delta, \gamma, 0) + N^4 = c_7 + c_8.$$

Since $c_1, c_3 + c_4, c_5, c_7 + c_8, v^2(c_1), v^2(c_3 + c_4), v^2(c_5)$, and $v^2(c_7 + c_8)$ generate $C$, it follows that $L_C = M + M'$. Thus the next lemma holds.

**Lemma 7.13:** $L_C = M + M'$ with $M \cong M' \cong \sqrt{2}E_8$ and $M \cap M' = 0.$
Uniqueness of such a lattice as $L_C$ is known. In fact, the following theorem holds [23, Theorem 4.3] (see also [12, Corollary 7.17]).

**Theorem 7.14:** There is, up to isometry, a unique positive definite even lattice $Q$ of rank 16 such that $Q^*/Q \cong 5^4$ and $Q(2) = \emptyset$.

**Remark 7.15:** The lattice $Q$ described in the above theorem was studied in [12,23] as a sum of two lattices isometric to $\sqrt{2}E_8$. It is isometric to $\text{DIH}_{10}(16)$ in the notation of [12, Section 7]. The lattice $Q$ is also isometric to the coinvariant sublattice $\Lambda^g = \text{Ann}_\Lambda(\Lambda^g)$ of the Leech lattice $\Lambda$ associated with an isometry $g$ of class 5B in $O(\Lambda) = \text{Co}_0$, where $\Lambda^g = \{\alpha \in \Lambda \mid g\alpha = \alpha\}$ [23, Theorem 1.6 and Corollary 1.7]; see also Section 5.3 of [38].

By Lemma 7.11, we may take $Q = L_C$. Let

$$\lambda_1 = (\lambda, 0, 0, 0), \quad \lambda_2 = (0, \lambda, 0, 0), \quad \lambda_3 = (0, 0, \lambda, 0), \quad \lambda_4 = (0, 0, 0, \lambda),$$

where $\lambda$ is defined as in (44). Then $\lambda_i + L_C$ for $1 \leq i \leq 4$ generate the discriminant group $\mathcal{D}(L_C) = (L_C)^*/L_C$ of $L_C$.

For $\alpha \in (L_C)^*$, let $\overline{\alpha} = \alpha + L_C$. Since $\langle \alpha + x, \beta + y \rangle \in \langle \alpha, \beta \rangle + \mathbb{Z}$ for $\alpha, \beta \in (L_C)^*$ and $x, y \in L_C$, we can define a map $f : \mathcal{D}(L_C) \times \mathcal{D}(L_C) \to \mathbb{Z}_5$ by

$$f(\overline{\alpha}, \overline{\beta}) = 5\langle \alpha, \beta \rangle + 5\mathbb{Z} \quad \text{for} \quad \alpha, \beta \in (L_C)^*,$$

which is a non-degenerate symmetric bilinear form of plus type on $\mathcal{D}(L_C)$. In fact, $\langle \lambda_i, \lambda_j \rangle = 8\delta_{ij}$. Let $q : \mathcal{D}(L_C) \to \mathbb{Z}_5$ be a map defined by

$$q(\overline{\alpha}) = \frac{5}{2}\langle \alpha, \alpha \rangle + 5\mathbb{Z} \quad \text{for} \quad \alpha \in (L_C)^*.$$

Then $q$ is a non-singular quadratic form on $\mathcal{D}(L_C)$ whose associated bilinear form is $f$. We have $q(\overline{\lambda}_i) = 4$, and $q(2\overline{\lambda}_i) = 1$ in $\mathbb{Z}_5$. The isometry group of $(\mathcal{D}(L_C), q)$ is the general orthogonal group $O(\mathcal{D}(L_C)) = GO(\mathcal{D}(L_C), q) \cong GO_4^+(5)$. Note that $GO_4^+(5)$ is isomorphic to $(\text{SL}_2(5) \circ \text{SL}_2(5)).2^2$, an extension of the central product $\text{SL}_2(5) \circ \text{SL}_2(5)$ by $2^2$, which has three subgroups of index 2.

The isometry group $O(Q)$ of the lattice $Q = L_C$ was also studied in [23]. We recall some of the properties of $O(Q)$ from Theorem 1.5 and Corollary B.3 of [23]. Let $D$ be the kernel of the action of $O(Q)$ on $\mathcal{D}(Q)$.

**Theorem 7.16:** Let $Q$ be as in Theorem 7.14.

1. There exists an embedding $O(Q) \to \text{Frob}(20) \times GO_4^+(5)$ such that the image has index 2 and contains neither direct factor, where $\text{Frob}(20)$ is a Frobenius group of order 20. The intersection of the image with $\text{Frob}(20)$ is $D$, and the intersection of the image with $GO_4^+(5)$ has the shape $(\text{SL}_2(5) \circ \text{SL}_2(5)) : 2$.

2. The action of $O(Q)$ on $\mathcal{D}(Q)$ induces $GO(\mathcal{D}(Q)) \cong GO_4^+(5)$, and the kernel $D$ of the action is a dihedral group of order 10.

Since $2M^* \subset M$ and $2(M')^* \subset M'$, both $M$ and $M'$ are RSSD sublattices of $L_C$. 

Lemma 7.17: (1) The dihedral group $D$ is generated by $t_M$ and $t'_M$, where $t_M$ and $t'_M$ are the RSSD involutions of $L_C$ associated with $M$ and $M'$, respectively. Moreover, $\nu = t_M t'_M$ as elements of $O(L_C)$.

(2) $\langle \nu \rangle$ is a normal subgroup of $O(L_C)$.

(3) The centralizer $C_{O(L_C)}(\nu)$ of $\nu$ in $O(L_C)$ is isomorphic to $\langle \nu \rangle \times ((SL_2(5) \circ SL_2(5)) : 2)$.

Proof: Since $2M^* \subset M$, it follows from [23, Lemma A.5] that $t_M$ acts trivially on $D(L_C)$; see also [23, Notation 1.3]. Thus $t_M \in D$ by (2) of Theorem 7.16 with $Q = L_C$. Likewise, we have $t'_M \in D$, so the dihedral group $D$ is generated by $t_M$ and $t'_M$.

Since $\lambda - \nu \lambda \in N$, the isometry $\nu$ also acts trivially on $D(L_C)$. Thus $\nu \in D$, and $\langle \nu \rangle = [D, D]$, the derived subgroup of $D$, is a normal subgroup of $O(L_C)$. In particular, $t_M$ inverts $\nu$. Since $M' = \nu^2(M)$ implies $t_{M'} = \nu^2 t_M \nu^{-2}$, we have $t_M t_{M'} = t_M \nu^2 t_M \nu^{-2} = \nu$. Thus the assertions (1) and (2) hold. The assertion (3) follows from the assertion (1) and Theorem 7.16. ■

Recall that $N_l$ is the $l$th direct summand of $N^4$ for $1 \leq l \leq 4$. Let

$$A = \{g(N_l) \mid g \in O(L_C)\}.$$ 

Since $\langle \nu \rangle$ is a normal subgroup of $O(L_C)$, any $A \in A$ is $\nu$-invariant. We have $N_l \in A$ for $1 \leq l \leq 4$ by the action of $\text{Alt}_4 \subset O(L_C)$.

Lemma 7.18: Let $A \in A$.

(1) $A$ is a direct summand of a sublattice of $L_C$ isometric to an orthogonal sum of four members of $A$.

(2) $A$ is RSSD in $L_C$.

Proof: Since $N^4$ is an orthogonal sum of $N_l$ for $1 \leq l \leq 4$, the assertion (1) holds. The assertion (2) follows from Lemma 7.9. ■

Lemma 7.19: Let $A \in A$. Then the RSSD involution $t_A \in O(L_C)$ associated with $A$ acts on $D(L_C)$ as a reflection which maps $\lambda_A + L_C$ to its negative, where $\lambda_A$ is an element of $A^*$ corresponding to $\lambda \in N^*$ defined in (44) with $q(\lambda_A + L_C) = 4$ a square element of $\mathbb{Z}_5$.

Proof: By Lemma 7.18, $A$ is a direct summand of an orthogonal sum of four members of $A$. Then $t_A$ acts as $-1$ on $A$ and $1$ on the other three summands orthogonal to $A$. That means it acts on $D(L_C)$ as a reflection associated with $\lambda_A + L_C$. ■

Lemma 7.20: The subgroup $\langle t_A \mid A \in A \rangle$ of $C_{O(L_C)}(\nu)$ generated by the RSSD involutions $t_A$ associated with $A \in A$ is isomorphic to an index 2 subgroup of $GO_4^+(5)$ having the shape $O^+(16) \circ SL_2(5)$.

Proof: For $g \in O(L_C)$, we have $gt_A g^{-1} = t_{g(A)}$. Thus $\langle t_A \mid A \in A \rangle$ is a normal subgroup of $O(L_C)$. Since $t_A$ commutes with $\nu$, the assertion follows from Theorem 7.16, Lemmas 7.17 and 7.19. ■
For any $A \in \mathcal{A}$, let $W_A \cong K(\mathfrak{sl}_2, 5)$ be a vertex operator subalgebra of $V_A$ corresponding to $M^0 \subset V_N$ in the notation of Section 4. Then $W_A$ is a $\sigma$-type parafermion vertex operator subalgebra of $V_{Lc}$ by (1) of Theorem 4.5 as $A$ is RSSD in $L_c$ by Lemma 7.18. Moreover, $\varphi(\sigma_{W_A}) = t_A$ by (2) of Theorem 4.5, where $\varphi : \text{Aut}(V_{Lc}) \to O(L_c)$ is as in (14), and $\sigma_{W_A}$ is the $\sigma$-involution of $V_{Lc}$ associated with $W_A$.

Let $\hat{\nu} \in \text{Aut}(V_{Lc})$ be a lift of $\nu$. It acts trivially on $W_A$, so $W_A \subset V_{Lc}^{(\hat{\nu})}$. Let $\rho_A$ be an element of $(1/\sqrt{2})A$ corresponding to $\rho \in (1/\sqrt{2})N$ defined in Section 5 with $k = 5$. Then $\sqrt{2}\rho_A \in 5((1 - \nu)L_c)^{\ast}$ by (43) with $p = 5$. Let

$$\psi_A = \exp(2\pi \sqrt{-1}(\sqrt{2}\rho_A)(0)/5)$$

for $A \in \mathcal{A}$. Then $\psi_A \in C_N(V_{Lc}^{(\hat{\nu})})(\hat{\nu})$ by Theorem 2.7. Hence $\psi_A(W_A) \subset V_{Lc}^{(\hat{\nu})}$ for $0 \leq i \leq 4$ are also $\sigma$-type parafermion vertex operator subalgebras of $V_{Lc}$. We consider the $\sigma$-involutions of $V_{Lc}^{(\hat{\nu})}$ associated with $\psi_A(W_A)$'s. Let

$$\mathcal{W} = \{\psi_A^{i}(W_A) \mid A \in \mathcal{A}, i = 0, 1\}.$$

**Proposition 7.21:** Let $H = \langle \sigma_W \mid W \in \mathcal{W} \rangle$ be the subgroup of $\text{Aut}(V_{Lc}^{(\hat{\nu})})$ generated by the $\sigma$-involutions $\sigma_W$ associated with $W \in \mathcal{W}$. Then

$$H \cong 5^4 : ((\text{SL}_2(5) \circ \text{SL}_2(5)) : 2) \cong C_{\text{Aut}(V_{Lc})}(\hat{\nu})/\langle \hat{\nu} \rangle.$$

**Proof:** Since $\varphi(\sigma_{W_A}) = t_A$ for $A \in \mathcal{A}$, the assertion follows from Corollary 2.9, Theorem 7.10, (3) of Lemma 7.17, and Lemma 7.20.

### 7.3.3. Ising vectors and $U_{SA}$

We show that there exist some $\sigma$-type parafermion vertex operator subalgebras of $V_{Lc}^{(\hat{\nu})}$ which are not contained in $\mathcal{W}$. We first review a few facts about Ising vectors and vertex operator algebras generated by two Ising vectors.

A weight 2 vector $e \in V_2$ of a vertex operator algebra $V$ is called an Ising vector if the vertex operator subalgebra $\text{VOA}(e)$ generated by $e$ is isomorphic to the simple Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$. Given an Ising vector $e$, one can decompose $V$ as

$$V = V_e[0] \oplus V_e[1/2] \oplus V_e[1/16],$$

where $V_e[h]$ is the sum of all irreducible $\text{VOA}(e)$-submodules of $V$ isomorphic to $L(1/2, h)$ for $h = 0, 1/2, 1/16$. Then the linear map $\tau_e$ on $V$ defined by

$$\tau_e = \begin{cases} 
1 & \text{on } V_e[0] \oplus V_e[1/2], \\
-1 & \text{on } V_e[1/16]
\end{cases}$$

is an automorphism of the vertex operator algebra $V$. Note that $\tau_e$ agrees with $\tau_W$ defined in Theorem 3.3 in the case $k = 2$.

Vertex operator algebras $\text{VOA}(e, f)$ generated by two Ising vectors $e$ and $f$ have been studied. In [15,16], nine such vertex operator algebras were constructed as subalgebras of $V_{\sqrt{2}E_8}$, which are denoted as $U_{nx}$ for $nX = 1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, and 3C$. 
Uniqueness of \( \text{VOA}(e,f) \) with \(|\tau_e \tau_f| = 5\) was established in [39] as a subalgebra of a vertex operator algebra over \( \mathbb{R} \) of CFT type with trivial weight 1 subspace. We apply Theorem 3.12 of [39] to a vertex operator algebra over \( \mathbb{C} \) in the following form. Note that \(|\tau_e \tau_f| = 5\) if and only if \((e,f) = 3/2^9\) by [40].

**Theorem 7.22:** Let \( V \) be a vertex operator algebra of CFT type with trivial weight 1 subspace \( V_1 = 0 \). Suppose \( V \) possesses a positive definite invariant hermitian form \((\cdot, \cdot)\) with \((1,1) = 1\). Let \( e, f \in V_2 \) be Ising vectors such that the product \( \tau_e \tau_f \) of the involutions \( \tau_e \) and \( \tau_f \) associated with \( e \) and \( f \) has order 5. Then the vertex operator subalgebra \( \text{VOA}(e,f) \) generated by \( e \) and \( f \) is isomorphic to \( U_{5A} \).

The properties of \( U_{5A} \) were studied in [15,16]. In particular, we have \( U_{5A}^{\langle \tau_e \tau_f \rangle} \cong K(\mathfrak{sl}_2, 5) \otimes K(\mathfrak{sl}_2, 5) \) [15, (3.62)].

Recall from Lemma 7.13 that \( L_C = M + M' \) is a sum of two sublattices \( M \) and \( M' \) isometric to \( \sqrt{2}E_8 \) with \( M \cap M' = 0 \). Since both \( M \) and \( M' \) are doubly even lattices, we can pick a 2-cocycle \( \varepsilon \) on \( L_C \) such that \( \varepsilon \) is trivial on \( M \) and \( M' \). For instance, we may define a \( \mathbb{Z} \)-bilinear map \( \varepsilon : L_C \times L_C \to \mathbb{Z}_2 \) by

\[
\varepsilon(\alpha, \beta) = \langle \alpha_2, \beta_1 \rangle + 2\mathbb{Z}
\]

for \( \alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2 \) with \( \alpha_1, \beta_1 \in M \) and \( \alpha_2, \beta_2 \in M' \). Then \( \varepsilon(M,M) = \varepsilon(M',M') = 0 \), and

\[
\varepsilon(\alpha, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle + 2\mathbb{Z} \quad \text{for} \quad \alpha \in L_C.
\]

Thus \( \varepsilon(\alpha, \beta) + \varepsilon(\beta, \alpha) = \langle \alpha, \beta \rangle + 2\mathbb{Z} \).

As in [16, (3.12)], we may define two Ising vectors \( e_M \) and \( e_{M'} \) in \( V_{L_C} \) as follows.

\[
e_M = \frac{1}{16} \omega_M + \frac{1}{32} \sum_{\alpha \in M(4)} e^\alpha,
\]

\[
e_{M'} = \frac{1}{16} \omega_{M'} + \frac{1}{32} \sum_{\alpha \in M'(4)} e^\alpha,
\]

where \( \omega_M \) and \( \omega_{M'} \) are the conformal vectors of \( V_M \) and \( V_{M'} \), respectively. Then \( \varphi(\tau_{e_M}) = t_M \) and \( \varphi(\tau_{e_{M'}}) = t_{M'} \) by [16, Lemma 4.1], where \( \varphi : \text{Aut}(V_{L_C}) \to O(L_C) \) is a group homomorphism defined as in (14). Therefore, \( \varphi(\tau_{e_M} \tau_{e_{M'}}) = t_M t_{M'} \). That is,

\[
\varphi(\tau_{e_M} \tau_{e_{M'}}) = v \tag{48}
\]

by Lemma 7.17. This implies that \(|\tau_{e_M} \tau_{e_{M'}}| = 5\).

Since \( V_{L_C} \) is a lattice vertex operator algebra, it possesses a positive definite invariant hermitian form. We consider the fixed point subalgebra \( V_{L_C}^{(\theta)} \) of \( V_{L_C} \) by the lift \( \theta \) of the \(-1\)-isometry of the lattice \( L_C \). The weight 1 subspace of \( V_{L_C}^{(\theta)} \) is trivial as \( L_C(2) = \emptyset \). Note that both \( \omega_M \) and \( \omega_{M'} \) belong to \( V_{L_C}^{(\theta)} \). Thus we can apply Theorem 7.22 to \( V_{L_C}^{(\theta)}, e_M, \) and \( e_{M'} \) to obtain the following lemma.
Lemma 7.23: The vertex operator subalgebra\( \text{VOA}(e_M, e_{M'}) \) of \( V_{L_C}^{(\theta)} \) generated by \( e_M \) and \( e_{M'} \) is isomorphic to \( U_{SA} \). Moreover,

\[
\text{VOA}(e_M, e_{M'})^{(\tau_M \tau_{M'})} \cong K(sl_2, 5) \otimes K(sl_2, 5).
\]

Remark 7.24: Since \( L_C = Q \cong \Lambda_g \subset \Lambda \) by Remark 7.15, we have \( \text{VOA}(e_M, e_{M'}) \subset V_{L_C}^{(\theta)} \subset V_{A}^{(\theta)} \). Thus \( \text{VOA}(e_M, e_{M'}) \subset V^2 \), the Moonshine vertex operator algebra.

Since \( \tau_{e_M} \tau_{e_{M'}} \in \text{Aut}(V_{L_C}) \) is a lift of \( \nu \) by (48), we may take \( \hat{\nu} \) to be \( \tau_{e_M} \tau_{e_{M'}} \). Using the notation in Section 2.4, (B.23) of [15] can be written as

\[
U_{SA} \cong \bigoplus_{j=0}^{4} M^j \otimes M^{2j},
\]

that is, \( U_{SA} \) is a \( \mathbb{Z}_5 \)-graded simple current extension of \( M^0 \otimes M^0 = K(sl_2, 5) \otimes K(sl_2, 5) \).

Denote by \( W_1 \) and \( W_2 \) the first and the second tensor factor of \( \text{VOA}(e_M, e_{M'})^{(\hat{\nu})} \cong K(sl_2, 5) \otimes K(sl_2, 5) \). For \( i = 1, 2 \), one can associate an automorphism \( \tau_{W_i} \) of \( V_{L_C} \) with \( W_i \) by Theorem 3.3. Then \( \tau_{W_1} \) (resp. \( \tau_{W_2} \)) acts on \( M^j \otimes M^{2j} \) as \( \zeta_j^3 \) (resp. \( \zeta_j^1 \)), where \( \zeta_j = \exp(2\pi \sqrt{-1}/5) \). Hence \( \langle \hat{\nu} \rangle = \langle \tau_{W_1} \rangle = \langle \tau_{W_2} \rangle \) as subgroups of \( \text{Aut}(\text{VOA}(e_M, e_{M'})) \). Then the actions of \( \langle \hat{\nu} \rangle \), \( \langle \tau_{W_1} \rangle \), and \( \langle \tau_{W_2} \rangle \) on the \( \text{VOA}(e_M, e_{M'}) \)-module \( V_{L_C} \) agree with each other. In particular, \( V_{L_C}^{(\hat{\nu})} \) is of \( \sigma \)-type as a \( W_j \)-module for \( i = 1, 2 \). That is, \( W_i \) is a \( \sigma \)-type parafermion vertex operator subalgebra of \( V_{L_C}^{(\hat{\nu})} \); however, it is not of \( \sigma \)-type in \( V_{L_C} \). In fact, \( W_i \) is not of \( \sigma \)-type even in \( \text{VOA}(e_M, e_{M'}) \). Therefore, the corresponding \( \sigma \)-involutions \( \sigma_{W_1} \) and \( \sigma_{W_2} \) are only defined on \( V_{L_C}^{(\hat{\nu})} \), and they cannot be extended to automorphisms of \( V_{L_C} \).

It is known [41, Theorem 5.11] that the automorphism group \( \text{Aut}(V_{L_C}^{(\hat{\nu})}) \) of \( V_{L_C}^{(\hat{\nu})} \) is isomorphic to an index 2 subgroup of the general orthogonal group \( G\text{O}_6^+ (5) \) different from \( S\text{O}_6^+ (5) \). We have the following theorem.

Theorem 7.25: The automorphism group \( \text{Aut}(V_{L_C}^{(\hat{\nu})}) \) of the vertex operator algebra \( V_{L_C}^{(\hat{\nu})} \) is generated by the \( \sigma \)-involutions \( \sigma_W \) for \( W \in \mathcal{W} \cup \{ W_1, W_2 \} \).

Proof: We slightly modify the proof of [41, Theorem 5.11] with \( Q = L_C \) and \( \hat{g} = \hat{\nu} \). Let \( G \) be the subgroup of \( \text{Aut}(V_{L_C}^{(\hat{\nu})}) \) generated by \( \sigma_W \) for \( W \in \mathcal{W} \cup \{ W_1, W_2 \} \). Let \( N \) and \( C \) be as in the proof of [41, Theorem 5.11]. Actually, the group \( C \) agrees with \( H = \langle \sigma_W \mid W \in \mathcal{W} \rangle \) defined in Proposition 7.21. Recall that \( \sigma_W \) for \( W \in \mathcal{W} \) are automorphisms of \( V_{L_C} \), while \( \sigma_{W_1} \) and \( \sigma_{W_2} \) are only defined on \( V_{L_C}^{(\hat{\nu})} \), and they cannot be extended to \( V_{L_C} \). Thus \( G \) is strictly larger than \( H \), and not contained in \( N \). Then \( G \) contains the derived subgroup \( \Omega_6^+ (5) \) of \( G\text{O}_6^+ (5) \). Since \( G \) contains a reflection, it is an index 2 subgroup of \( G\text{O}_6^+ (5) \) different from \( S\text{O}_6^+ (5) \). Thus \( G = \text{Aut}(V_{L_C}^{(\hat{\nu})}) \) as desired.

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Appendix. Representations of $U_{5A}$

As in Section 7.3, let $U_{5A}$ be the vertex operator algebra $U$ constructed in [15,16] for the 5A case. The irreducible modules for $U_{5A}$ were classified [15, Theorem 3.19], and the fusion rules were determined [42, Theorem 5.3]. In those papers, $U_{5A}$ is studied as an extension of a tensor product $L(1/2,0) \otimes L(25/28,0) \otimes L(25/28,0)$ of three Virasoro vertex operator algebras. In this appendix, we consider $U_{5A}$ as an extension of $K(sl_2,5) \otimes K(sl_2,5)$, and review the irreducible modules and the fusion rules for $U_{5A}$. Actually, $U_{5A}$ is a $Z_5$-code vertex operator algebra [2, Section 10]. Thus the irreducible $U_{5A}$-modules are known in a more general context [2, Section 8]. We discuss the fusion product of irreducible $U_{5A}$-modules by using the induction functor as well.

Recall from (49) that

$$U_{5A} \cong \bigoplus_{j=0}^{4} M^j \otimes M^{2j}$$

is a $Z_5$-graded simple current extension of $M^0 \otimes M^0 = K(sl_2,5) \otimes K(sl_2,5)$. The vertex operator algebra structure on $U_{5A}$ which extends the $M^0 \otimes M^0$-module structure is unique [43, Proposition 5.3]. Since $M^0 = K(sl_2,5)$ is simple, self-dual, rational, $C_2$-cofinite, and of CFT-type, $U_{5A}$ is also simple, self-dual, rational, $C_2$-cofinite, and of CFT-type by [44, Theorem 2.14]. Note that the contragredient module of $M^j \otimes M^{2j}$ is $M^{-j} \otimes M^{-2j}$.

For a vertex operator algebra $V$, we denote by $\text{Irr}(V)$ (resp. $\text{Irr}(V)_{sc}$) the set of equivalence classes of irreducible $V$-modules (resp. simple current $V$-modules). We consider a map $b_V : \text{Irr}(V)_{sc} \times \text{Irr}(V) \to \mathbb{Q}/\mathbb{Z}$ defined by

$$b_V(A,X) = h(A \boxtimes_V X) - h(A) - h(X) + \mathbb{Z}$$

for $A \in \text{Irr}(V)_{sc}$ and $X \in \text{Irr}(V)$, where $h(X)$ is the conformal weight of $X$. It is known [2, (8.1)] that

$$b_{M^0}(M^p, M^{ij}) = \frac{p(i - 2j)}{5} + \mathbb{Z}$$

for $0 \leq i \leq 5$, $0 \leq j \leq 4$, and $0 \leq p \leq 4$. Since

$$(M^p \otimes M^q) \boxtimes_{M^0 \otimes M^0} (M^{i_1 j_1} \otimes M^{i_2 j_2}) = (M^p \boxtimes_{M^0} M^{i_1 j_1}) \otimes (M^q \boxtimes_{M^0} M^{i_2 j_2}),$$

it follows that

$$b_{M^0 \otimes M^0}(M^p \otimes M^q, M^{i_1 j_1} \otimes M^{i_2 j_2}) = b_{M^0}(M^p, M^{i_1 j_1}) + b_{M^0}(M^q, M^{i_2 j_2})$$

$$= \frac{1}{5}(p(i_1 - 2j_1) + q(i_2 - 2j_2)) + \mathbb{Z}$$

for $0 \leq i_1, i_2 \leq 5, 0 \leq j_1, j_2 \leq 4$, and $0 \leq p, q \leq 4$.

There are 225 inequivalent irreducible $M^0 \otimes M^0$-modules $M^{i_1 j_1} \otimes M^{i_2 j_2}$ for $0 \leq j_1 < i_1 \leq 5$ and $0 \leq j_2 < i_2 \leq 5$. Among them, we can verify that there are exactly 45 $M^{i_1 j_1} \otimes M^{i_2 j_2}$ for which

$$b_{M^0 \otimes M^0}(M^p \otimes M^{2p}, M^{i_1 j_1} \otimes M^{i_2 j_2}) = 0$$

for $0 \leq p \leq 4$. We denote by $\text{Irr}^0(M^0 \otimes M^0)$ the set of those 45 irreducible $M^0 \otimes M^0$-modules.

Let $U^0 = U_{5A} \cong \bigoplus_{j=0}^{4} M^j \otimes M^{2j}$. We consider

$$U^0 \boxtimes_{M^0 \otimes M^0} X = \bigoplus_{j=0}^{4} (M^j \otimes M^{2j}) \boxtimes_{M^0 \otimes M^0} X$$

for $X \in \text{Irr}^0(M^0 \otimes M^0)$. For simplicity of notation, we write $[i_1, j_1; i_2, j_2]$ for $M^{i_1 j_1} \otimes M^{i_2 j_2}$. We also write $U^0 \boxtimes X$ for $U^0 \boxtimes_{M^0 \otimes M^0} X$. Let

$$U^1 = U^0 \boxtimes [5,0;4,2], \quad U^2 = U^0 \boxtimes [5,0;2,1],$$

$$U^3 = U^0 \boxtimes [2,0;5,3], \quad U^4 = U^0 \boxtimes [2,0;4,0],$$

where $U^0 \boxtimes [a,b]$ denotes the $[a,b]$-component of $U^0 \boxtimes X$. These modules are irreducible and self-dual.

In addition, each $U_i$ has a second, non-self-dual component, which we call $U_i^\perp$ and denote by $U_i^\perp \boxtimes [a,b]$. The components $U_i^\perp$ are related to the irreducible $M^0 \otimes M^0$-module $M^{i_1 j_1} \otimes M^{i_2 j_2}$ by

$$U_i^\perp \boxtimes [a,b] = U_i \boxtimes [a,b] \oplus U_i \boxtimes [a,b].$$

The vector space $U_i^\perp \boxtimes [a,b]$ is spanned by elements of the form $U_i \boxtimes [a,b] \otimes \sum_{j=0}^{4} U_j \boxtimes [a,b]$.

Finally, we consider the tensor product of $U_i$ and $U_j$ for $i \neq j$. For example, $U_i \boxtimes U_j$ is spanned by elements of the form $U_i \boxtimes U_j \otimes \sum_{k=0}^{4} U_k \boxtimes U_k$.

In summary, the representation theory of $U_{5A}$ is quite rich and interesting, and there are many interesting connections to other areas of mathematics.
\[
U^5 = U^0 \boxtimes [2, 0; 3, 2], \quad U^6 = U^0 \boxtimes [1, 0; 5, 4],
U^7 = U^0 \boxtimes [1, 0; 4, 1], \quad U^8 = U^0 \boxtimes [1, 0; 2, 0].
\]

Then by (15), (18), and (A1), we have

\[
U^0 = [5, 0; 5, 0] + [5, 1; 5, 2] + [5, 2; 5, 4] + [5, 3; 5, 1] + [5, 4; 5, 3],
U^1 = [5, 0; 4, 2] + [5, 1; 1, 0] + [5, 2; 4, 1] + [5, 3; 4, 3] + [5, 4; 4, 0],
U^2 = [5, 0; 2, 1] + [5, 1; 3, 1] + [5, 2; 2, 0] + [5, 3; 3, 0] + [5, 4; 3, 2],
U^3 = [2, 0; 5, 3] + [2, 1; 5, 0] + [3, 0; 5, 2] + [3, 1; 5, 4] + [3, 2; 5, 1],
U^4 = [2, 0; 4, 0] + [2, 1; 4, 2] + [3, 0; 1, 0] + [3, 1; 4, 1] + [3, 2; 4, 3],
U^5 = [2, 0; 3, 2] + [2, 1; 2, 1] + [3, 0; 3, 1] + [3, 1; 2, 0] + [3, 2; 3, 0],
U^6 = [1, 0; 5, 4] + [4, 0; 5, 1] + [4, 1; 5, 3] + [4, 2; 5, 0] + [4, 3; 5, 2],
U^7 = [1, 0; 4, 1] + [4, 0; 4, 3] + [4, 1; 4, 0] + [4, 2; 4, 2] + [4, 3; 1, 0],
U^8 = [1, 0; 2, 0] + [4, 0; 3, 0] + [4, 1; 3, 2] + [4, 2; 2, 1] + [4, 3; 3, 1]
\]

as \(M^0 \otimes M^0\)-modules. The 45 irreducible \(M^0 \otimes M^0\)-modules which appear on the right-hand side of (A2) are exactly the 45 members of \(\text{Irr}^0(M^0 \otimes M^0)\).

We have the following theorem by Theorems 2.14 and 3.2 of [44]; see also Theorem 2.2 and Proposition 2.3 of [2].

**Theorem A.1:** There are exactly nine inequivalent irreducible \(U_{SA}\)-modules, which are \(U^i, 0 \leq i \leq 8\).

The top level of \(M^{ij}\) is one dimensional, and its weight \(h(M^{ij})\) is given by (16). Since

\[
h(M^{i_1j_1} \otimes M^{i_2j_2}) = h(M^{i_1j_1}) + h(M^{i_2j_2}),
\]
we can calculate the conformal weight of each irreducible direct summand on the right-hand side of (A2). Then we see that the weight and the dimension of the top level of \(U^i, 0 \leq i \leq 8\), are as in Table A1.

Next, we discuss the fusion product \(U^i \boxtimes U^j\). Let \(C(M^0 \otimes M^0)\) be the category of \(M^0 \otimes M^0\)-modules. We consider two full subcategories of \(C(M^0 \otimes M^0)\), namely, \(\text{Rep}^0 U^0\) and \(C^0(M^0 \otimes M^0)\), where \(\text{Rep}^0 U^0\) is the braided tensor category of \(U^0\)-modules, and \(C^0(M^0 \otimes M^0)\) is the \(\mathbb{C}\)-linear additive braided monoidal category with simple objects being the members of \(\text{Irr}^0(M^0 \otimes M^0)\). By Theorem 2.67 of [45], a functor defined by

\[
F : C^0(M^0 \otimes M^0) \rightarrow \text{Rep}^0 U^0; \quad X \mapsto U^0 \boxtimes_{M^0 \otimes M^0} X
\]
is a braided tensor functor. Hence

\[
(U^0 \boxtimes_{M^0 \otimes M^0} X) \boxtimes_{U^0} (U^0 \boxtimes_{M^0 \otimes M^0} Y) = U^0 \boxtimes_{M^0 \otimes M^0} (X \boxtimes_{M^0 \otimes M^0} Y)
\]
for \(X, Y \in \text{Irr}^0(M^0 \otimes M^0)\). Since

\[
(M^{i_1j_1} \otimes M^{i_2j_2}) \boxtimes_{M^0 \otimes M^0} (M^{i_1'j_1'} \otimes M^{i_2'j_2'}) = (M^{i_1j_1} \boxtimes_{M^0} M^{i_1'j_1'}) \otimes (M^{i_2j_2} \boxtimes_{M^0} M^{i_2'j_2'}),
\]
we can calculate the fusion product \((U^0 \boxtimes_{M^0 \otimes M^0} X) \boxtimes_{U^0} (U^0 \boxtimes_{M^0 \otimes M^0} Y)\) by (17) and (A3). In fact, we obtain the following.

---

**Table A1.**

|   | \(U^0\) | \(U^1\) | \(U^2\) | \(U^3\) | \(U^4\) | \(U^5\) | \(U^6\) | \(U^7\) | \(U^8\) |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Weight | 0  | 6/7  | 2/7  | 2/7  | 1/7  | 4/7  | 6/7  | 5/7  | 1/7  |
| Dimension | 1  | 3  | 1  | 1  | 2  | 5  | 3  | 4  | 2  |
Theorem A.2: The fusion product of irreducible $U_{5A}$-modules is as follows, where $i \boxtimes j = k_1 + \cdots + k_r$, implies $U^i \boxtimes U^j = U^{k_1} + \cdots + U^{k_r}$.

\begin{align*}
0 \boxtimes i &= i & \text{for } 0 \leq i \leq 8, \\
1 \boxtimes 1 &= 0 + 2, & 1 \boxtimes 2 &= 1 + 2, & 1 \boxtimes 3 &= 4, \\
1 \boxtimes 4 &= 3 + 5, & 1 \boxtimes 5 &= 4 + 5, \\
1 \boxtimes 6 &= 7, & 1 \boxtimes 7 &= 6 + 8, & 1 \boxtimes 8 &= 7 + 8, \\
2 \boxtimes 2 &= 0 + 1 + 2, & 2 \boxtimes 3 &= 5, \\
2 \boxtimes 4 &= 4 + 5, & 2 \boxtimes 5 &= 3 + 4 + 5, \\
2 \boxtimes 6 &= 8, & 2 \boxtimes 7 &= 7 + 8, & 2 \boxtimes 8 &= 6 + 7 + 8, \\
3 \boxtimes 3 &= 0 + 3 + 6, & 3 \boxtimes 4 &= 1 + 4 + 7, \\
3 \boxtimes 5 &= 2 + 5 + 8, & 3 \boxtimes 6 &= 3 + 6, \\
3 \boxtimes 7 &= 4 + 7, & 3 \boxtimes 8 &= 5 + 8, \\
4 \boxtimes 4 &= 0 + 2 + 3 + 5 + 6 + 8, & 4 \boxtimes 5 &= 1 + 2 + 4 + 5 + 7 + 8, \\
4 \boxtimes 6 &= 4 + 7, \\
4 \boxtimes 7 &= 3 + 5 + 6 + 8, \\
4 \boxtimes 8 &= 4 + 5 + 7 + 8, \\
5 \boxtimes 5 &= 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8, & 5 \boxtimes 6 &= 5 + 8, \\
5 \boxtimes 7 &= 4 + 5 + 7 + 8, \\
5 \boxtimes 8 &= 3 + 4 + 5 + 6 + 7 + 8, \\
6 \boxtimes 6 &= 0 + 3, & 6 \boxtimes 7 &= 1 + 4, & 6 \boxtimes 8 &= 2 + 5, \\
7 \boxtimes 7 &= 0 + 2 + 3 + 5, & 7 \boxtimes 8 &= 1 + 2 + 4 + 5, \\
8 \boxtimes 8 &= 0 + 1 + 2 + 3 + 4 + 5.
\end{align*}