An Exact Formula to Describe the Amplification Process in a Photomultiplier Tube

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Abstract

An analytical function is derived that exactly describes the amplification process due to a series of discrete, Poisson-like amplifications like those in a photo multiplier tube (PMT). A numerical recipe is provided that implements this function as a computer program. It is shown how the program can be used as the core-element of a faster, simplified routine to fit PMT spectra with high efficiency. The functionality of the method is demonstrated by fitting both, Monte Carlo generated and measured PMT spectra.


1 Introduction

In September 1999, the LHCb RICH group tested Hamamatsu’s 64-Multi-anode Photo Multiplier Tubes as a possible photodetector choice for the LHCb RICH detector [RIC00]. During the data analysis, the need for an accurate model of the output of a PMT arose that could be fitted to the measured pulse height spectra, mainly in order to have a precise estimate of the signal lost below the threshold cut. In order to perform a fit to the spectra, an analytical function is needed that can be calculated reasonably quickly by a computer.

Such a function is derived here. First (section 2), an analytical expression is derived that describes the output of a PMT. The starting assumption is that the number of photoelectrons per event, as well as the number of secondary electrons caused by each primary electron at each stage of the dynode chain, are well described by Poisson distributions. Furthermore it is shown how this expression can be adapted to avoid some of the numerical problems arising in the original expression, so that it can be calculated by a computer. A complete numerical recipe is given and a FORTRAN implementation of the program is listed in appendix A. This expression can of course be used to calculate any “snowball” like effect described by a series of Poisson distributions.

In section 2 it is described how the exact expression derived in the first part can be used as the central element of a faster, approximate function, and how the number of parameters can be reduced making reasonable assumptions, so that fitting a large number of spectra in a finite time becomes feasible. This is then adapted to describe the digitised output of laboratory read-out electronics, rather than the number of electrons at the end of a dynode chain.

This approximate function is used in section 4 of the paper to fit Monte Carlo generated spectra as well as real data, demonstrating the validity of the method.

2 An Analytical Function

2.1 The Electron Probability Distribution

In the following, an expression for the number of photoelectrons at the end of a dynode chain of a PMT is derived. The number of incident photons, and hence of photoelectrons produced in the cathode, is assumed to follow a Poisson distribution. This is appropriate for the testbeam data where PMTs were used to detect Cherenkov photons generated by a particle traversing a dielectric. With a mean number of photoelectrons produced in the cathode of $\lambda_1$, the probability to find $k_1$ photoelectrons arriving at the first dynode is:

$$P(k_1) = e^{-\lambda_1} \frac{\lambda_1^{k_1}}{k_1!}. \quad (1)$$

The probability to find $k_2$ electrons after the first dynode is the sum over all values for $k_1$ of the probabilities $P(k_1)$, each multiplied by the probability that the dynode returns $k_2$ electrons given that $k_1$ arrive:

$$P(k_2) = \sum_{k_1=0}^{\infty} P(k_1) \cdot P(k_2|k_1). \quad (2)$$

Each of the $k_1$ electrons produces a Poisson-distributed response from the dynode with mean $\lambda_2$ where $\lambda_2$ is the gain at the $1^{st}$ dynode; all $k_1$ electrons together produce a response distributed according to the convolution of $k_1$ Poisson distributions, each with mean $\lambda_2$. This results in a single Poisson distribution with mean $\lambda_2 \cdot k_1$:

$$P(k_2|k_1) = e^{-\lambda_2 k_1} \frac{(\lambda_2 k_1)^{k_2}}{k_2!}. \quad (3)$$

Hence the probability to find $k_2$ electrons after the first dynode is given by:

$$P(k_2) = \sum_{k_1=0}^{\infty} P(k_1) \cdot e^{-\lambda_2 k_1} \frac{(\lambda_2 k_1)^{k_2}}{k_2!}. \quad (4)$$

Inserting the right–hand side of equation 4 for $P(k_1)$ yields, after manipulation:

$$P(k_2) = e^{-\lambda_1} \frac{\lambda_2^{k_2}}{k_2!} \sum_{k_1=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_2})^{k_1}}{k_1!} \cdot k_1^{k_2}. \quad (5)$$
Generalising this for \( n - 1 \) dynodes yields:

\[
P(k_n) = e^{-\lambda_n} \frac{\lambda_n^{k_n}}{k_n!} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{n-1}=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_2})^{k_1} (\lambda_2 e^{-\lambda_3})^{k_2} \cdots (\lambda_{n-2} e^{-\lambda_{n-1}} k_{n-2})^{k_{n-2}}}{k_1! k_2! \cdots k_{n-1}! (\lambda_{n-1} e^{-\lambda_n})^{k_{n-1}}}.\]

(6)

Each term in equation (6) is of the form of an exponential series, i.e. \( a e^x \), except for the last term with the summation parameter \( k_{n-1} \), which appears as \( \frac{\lambda_n}{k_{n-1}} \). This term can be expressed in terms of the \( k_n \)th derivative of \( e^{y k_{n-1}} \) with respect to the new variable \( y \) at \( y = 0 \):

\[
P(k_n) = e^{-\lambda_n} \frac{\lambda_n^{k_n}}{k_n!} \frac{d^{k_n}}{dy^{k_n}} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{n-1}=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_2})^{k_1} (\lambda_2 e^{-\lambda_3})^{k_2} \cdots (\lambda_{n-2} e^{-\lambda_{n-1}} k_{n-2})^{k_{n-2}}}{k_1! k_2! \cdots k_{n-1}! (\lambda_{n-1} e^{-\lambda_n})^{k_{n-1}}}.\]

(9)

Now each summation can be carried out in turn, starting with that over \( k_{n-1} \):

\[
\sum_{k_{n-1}=0}^{\infty} \frac{(\lambda_{n-1} e^{-\lambda_n} k_{n-2} e^y)^{k_{n-1}}}{k_{n-1}!} = \exp(\lambda_{n-1} e^{-\lambda_n} k_{n-2} e^y) = (\exp(\lambda_{n-1} e^{-\lambda_n} e^y))^{k_{n-2}},
\]

(10)

then over \( k_{n-2} \):

\[
k_{n-1}^{k_{n-1}} = \frac{d^{k_n}}{dy^{k_n}} e^{y k_{n-1}} \Big|_{y=0}.
\]

(7)

Now the last term in equation (9) can be written as

\[
\frac{(\lambda_{n-1} e^{-\lambda_n} k_{n-2})^{k_{n-1}}}{k_{n-1}!} \frac{d^{k_n}}{dy^{k_n}} \left( \frac{(\lambda_{n-1} e^{-\lambda_n} k_{n-2} e^y)^{k_{n-1}}}{k_{n-1}!} \right) \Big|_{y=0}.
\]

(8)

and so on. After performing all these summations, the probability of finding \( k_n \) electrons after \( n - 1 \) dynodes, with gains \( \lambda_2, \ldots, \lambda_n \), starting off with an average of \( \lambda_1 \) photo electrons arriving at the first dynode, is given by:
This gives a recursive formula for the first derivatives:

\[ f_i^{(m)} = \sum_{j=0}^{m-1} \binom{m-1}{j} f_i^{(j)} f_{i+1}^{(m-j)}, \quad \text{with } f_j^{(0)} = f_j \forall j \in \mathbb{N}. \]  

With this expression, equation (14) can finally be calculated, by starting with \( f_n(0) = 1 \) and calculating \( f_i^{(m)} \) subsequently for all values \( i = n, n-1, \ldots, 1 \) and all values \( m = 0, 1, \ldots, k_n \).

### 2.3 Numerical Difficulties

While the previous section gives a valid algorithm on how to calculate \( P(k_n) \) using equation (14) and the recursive formula (17), it turns out that the finite precision of a normal computer will only allow calculations to be performed for rather small values of \( k_n \) before some numbers become either too large or too small to be stored straightforwardly in the computer memory. This problem is addressed in the following discussion.

#### The factor \( \frac{\lambda_n^{k_n}}{k_n!} \)

For any reasonably large number of dynodes, where the mean number of electrons coming off the last dynode, and therefore the interesting values for \( k_n \), is typically in the thousands or even millions, \( e^{-\lambda_1 \frac{\lambda_n^{k_n}}{k_n!}} \) quickly becomes very small, while \( f_1^{(k_n)}(y) \bigg|_{y=0} \) grows to extremely large values.

In order to calculate \( P(k_n) \) for such values of \( k_n \), it is necessary to absorb the small factor \( \frac{\lambda_n^{k_n}}{k_n!} \) into the \( f_i^{(m)} \). This can be done by replacing \( y \) in equation (12) with \( py \) and introducing a compensating factor \( \left( \frac{1}{p} \right)^{k_n} \):

\[ P(k_n) = e^{-\lambda_1 \frac{\lambda_n^{k_n}}{k_n!}} \left( \frac{1}{p} \right)^{k_n} \frac{d^{k_n}}{dy^{k_n}} f_1(py) \bigg|_{y=0}. \]
Choosing \( p \) such that \( p^k = \frac{\lambda_n}{k_n!} \) changes equation \( 14 \) to

\[
P(k_n) = e^{-\lambda_n} \left. \frac{d^k}{dy^k} f_1(py) \right|_{y=0}
\]

with \( p^k = \frac{\lambda_n}{k_n!} \). (19)

Defining

\[
f^*_{k_n,i} = \left. \frac{d^m}{dy^m} f_1(p_k y) \right|_{y=0}
\]

with \( p_k = \frac{\lambda_n}{(k_n!)^{1/m}} \),

\[
P(k_n) = e^{-\lambda_n} f^*_{k_n,1}.
\]

The recursive formula established for calculating \( f^{(k_n)}_1 \) remains essentially the same for \( f^{(k_n,1)} \):

\[
f^{*(k_n)}_{k_n,i} = \sum_{j=0}^{k_n-1} \binom{k_n-1}{j} f^{*(j)}_{k_n,i} f^{*(k_n-j)}_{k_n,i+1}
\]

with \( f^{*(m)}_{k_n,n} = p_k^m \) and \( p_k = \frac{\lambda_n}{(k_n!)^{1/m}} \),

with one additional complication. In the original algorithm, when calculating \( f^{(k_n)}_1 \) using the recursive formula \( 14 \) all values for \( f^m_1 \) with \( m < k_n \) calculated in the previous iterations\(^1\) could be used in the recursive formula for the current iteration. Now, for calculating \( f^{(k_n,1)} \) all values \( f^{(m)}_{k_n,i} \) with \( m < k_n, i \leq n \) have to be re-calculated at each iteration, because at each iteration the value for \( p \) in equation \( 22 \) changes.

To calculate \( f^{*(k_n,1)} \), from equation \( 22 \) the values for

\[
f^{*(m)}_{k_n,i}, m < k_n
\]

are needed. These can be calculated using only the values for \( f^{*(m)}_{k_n-1,i} \) which have been calculated one iteration earlier:

\[
f^{*(m)}_{k_n,i} = f^{*(m)}_{k_n-1,i} \left( \frac{p_k}{p_{k-1}} \right)^m
\]

\(^1\)where \( P(0), \ldots, P(k_n-1) \) were calculated

so the values for \( f^{*(k_n)}_{k_n,i} \) need to be stored only for one iteration.

The binomial factor When calculating \( f^{*(k_n)}_{k_n,i} \), using the recursive formula \( 22 \) the factor \( (k_n-1)^{-1} \) in

\[
f^{*(k_n)}_{k_n,i} = \sum_{j=0}^{k_n-1} \binom{k_n-1}{j} f^{*(j)}_{k_n,i} f^{*(k_n-j)}_{k_n,i+1}
\]

can get very large for large values of \( k_n \), while the corresponding values for \( f^{*(j)}_{k_n,i} f^{*(k_n-j)}_{k_n,i+1} \) can get very small. To avoid the associated numerical problems, one can define the arrays \( u^{(j)}_{k_n,i} \) and \( v^{(j)}_{k_n,i} \) that ‘absorb’ the binomial factor, such that equation \( 22 \) becomes:

\[
f^{*(k_n)}_{k_n,i} = \sum_{j=0}^{k_n-1} u^{(j)}_{k_n,i} v^{(k_n-j)}_{k_n,i+1},
\]

where

\[
u^{(j)}_{k_n,i} = \sqrt{\frac{(k_n-1)!}{j! (k_n-j-1)!}} f^{*(j)}_{k_n,i},
\]

\( j < k_n \).

2.4 Combining Results

At each iteration \( k_n \), before calculating \( f^{*(k_n)}_{k_n,i} \) using equation \( 23 \) the values for \( u^{(j)}_{k_n,i} \) and \( v^{(j)}_{k_n,i} \), \( j < k_n \), are calculated from their values in the previous iteration:

\[
u^{(j)}_{k_n,i} = \left( \frac{p_{k_n}}{p_{k_n-1}} \right)^j \sqrt{\frac{(k_n-1)!}{(k_n-1-j)!}} u^{(j)}_{k_n-1,i},
\]

\( j < k_n \). (26)
These results are then used to calculate:

\[
f_{k_n,i}^{(k_n)} = \sum_{j=0}^{k_n-1} u_{k_n,i}^{(j)} \lambda_n^{(k_n-j)}
\]

and

\[
u_{k_n,i}^{(k_n)} = v_{k_n,i}^{(k_n)} = f_{k_n,i}^{(k_n)}
\]

starting from

\[
u_{k_n,0}^{(k_n)} = v_{k_n,0}^{(k_n)} = f_{k_n,0}^{(k_n)} = \frac{\lambda_n^{k_n}}{k_n!}
\]

and

\[
u_{0,i}^{(0)} = v_{0,i}^{(0)} = f_{0,i}^{(0)} = f_i
\]

where the \(f_i\) are defined by equation \(13\).

2.5 The Complete Numerical Recipe

Using the above formulae, the problem of calculating the probability distribution of finding \(k_n\) electrons at the end of a PMT with \(n - 1\) dynodes can be solved by a computer. A FORTRAN implementation is listed in Appendix A. The program takes as its input the array \(\lambda[n]\), with dimension \(n\), which contains the average number of photo electrons arriving at the first dynode \(\lambda[1]\) and the gain at each of the \(n - 1\) dynodes, \(\lambda[2], \ldots, \lambda[n]\). The program fills the array \(P[\text{max}]\) with the probabilities \(P[k]\) to find \(k\) electrons at the end of the dynode chain for all values \(k \leq \text{max}\). The parameter \(\text{max}\) is also passed to the program.

The values for \(u_{k,i}^{(j)}, v_{k,i}^{(j)}\) needed in the recursive formulae, are stored in two two-dimensional arrays, where one dimension is taken by the index \(i = 1, \ldots, n\), and the other by the index \(j = 0, \ldots, \text{max}\). As the values for \(u_{k,i}^{(j)}, v_{k,i}^{(j)}\) are needed only for one value of \(k\) at a time, the arrays do not need to be three-dimensional; the values for \(u_{k,i}^{(j)}, v_{k,i}^{(j)}\) needed at the iteration calculating \(P[k]\) replace those from the previous iteration, \(u_{k-1,i}^{(j)}, v_{k-1,i}^{(j)}\).

The steps to calculate \(P[k], k = 0, \ldots, \text{max}\) are:

1. Initialise program, test whether input is sensible, for example if the overall gain is larger than 0. Calculate all values for \(\left(\frac{p_1}{p_{j-1}}\right)^j, j \leq \text{max}\) and store them in an array \(p_{\text{rec}}[j], j = 1, \ldots, \text{max}\) for later use.

2. Start with calculating the probability to find zero electrons: \(k = 0\)

3. Calculate \(u_{0,i}^{(0)} = v_{0,i}^{(0)} = f_i\) for \(i = n, n-1, \ldots, 1\), as defined by equation \(13\)

4. Store the result in the array: \(P[0] = e^{-\lambda_i}u_{0,1}\)

5. Increment \(k\) by 1. If \(k > \text{max}\), stop program.

6. Calculate \(u_{k,n}^{(k)} = v_{k,n}^{(k)} = \frac{\lambda_i^k}{k_i!}\)

7. Calculate \(u_{k,i}^{(j)}, v_{k,i}^{(j)}\) for \(j < k\) and \(i = n, \ldots, 1\) from \(u_{k-1,i}^{(j)}, v_{k-1,i}^{(j)}\) according to equation \(26\) using the values of \(p_{\text{rec}}[k]\) calculated in step 1.

8. Calculate \(u_{k,i}^{(k)} = v_{k,i}^{(k)}\) for all values of \(i < n\) using the recursive formula \(27\). Let the outer loop go over the index \(i\), starting with \(i = n - 1\) and decremented it by 1 until \(i = 1\), and the inner loop over the summation index \(j\), starting with \(j = 0\) and incrementing \(j\) by 1 until \(j = k - 1\).

9. Store result: \(P[k] = e^{-\lambda_i}u_{k,1}\)

10. Goto step 5

3 Fitting PMT Spectra

3.1 Increasing Speed by Approximating \(P(k_n)\)

When fitting PMT-pulse-height spectra, speed is a major problem. The number of operations needed to calculate \(P(k_n)\) using the recursive formula in equation \(17\) is

\[
N_{\text{steps}} \approx \sum_{i=0}^{k_n-1} \sum_{j=0}^{i} n j \sim k_n^3,
\]

which becomes prohibitive for a typical PMT with a gain of \(\sim 10^5\) and higher. Therefore, for fitting the spectra, only the exact distribution after the first \(m\) dynodes is calculated and then scaled by the gain of the remaining dynodes, \(g_{\text{eff}} = (g_{n+1}g_{n+2} \cdot \cdot \cdot g_{n-1})\). When scaling the output of the exact distribution calculated for the first \(m\)
dynodes, $P_{\text{exact}}(k_{m+1})$, to the final distribution, the result is convoluted with a Gaussian of width $\sigma_{\text{scale}}$, taking into account the additional spread in the distribution at each remaining dynode:

$$\sigma_{\text{scale}} = \sqrt{k_{m+1}\sigma_0} \quad (32)$$

with:

$$\sigma_0 = (g_{m+1}g_{m+2}\cdots g_{n-1})^\frac{1}{n-m}.$$  \quad (33)

So the approximated function, $P_\sim(k_n)$ is

$$P_\sim(k_n) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(j\cdot g_{m+1} - k_n)^2}{2\sigma_0^2}} P(j). \quad (34)$$

In practice the sum only needs to be calculated for values of $j \cdot g_{\text{left}}$ that are a few $\sigma_{\text{scale}}$ around $k_n$.

### 3.2 Reducing the Number of Parameters

$P(k_n)$ depends on $n$ parameters: the gain of each dynode and the number of photoelectrons produced in the cathode. For the case of the 12-dynode PMT, there are 13 parameters. It is possible, however, to reduce this number to two:

1. the mean number of photoelectrons produced in the photo cathode

2. the gain at the first dynode.

Using

$$g \propto V^\alpha, \quad (35)$$

where $V$ is the voltage difference over which the electron is accelerated, the gain at the other dynodes can be calculated from the gain at the first dynode. The parameter $\alpha$ has values typically between 0.7 and 0.8 [Ham00]; in the following, $\alpha = 0.75$ is used.

### 3.3 Adapting the Function to Fit Measured Data

In practice, spectra are not measured in numbers of photoelectrons, but in ADC counts digitised by the readout electronics. The function describing the spectra needs to relate the ADC counts, $k_{\text{adc}}$, to the number of electrons at the end of the dynode chain, $k_n$. This requires two parameters: the offset, or pedestal mean, $p_0$, and the conversion factor, $c_n$ of $k_n$ to ADC counts. The resulting function is convoluted with a Gaussian of width $\sigma$ to take into account electronics noise:

$$F_{\text{cont}}(k_{\text{adc}}) = \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{j^2}{2\sigma^2}} \right) \ast \left( P((k_{\text{adc}} - p_0) / c_n) \cdot c_n \right), \quad (36)$$

where $\ast$ is the convolution operator. $F_{\text{cont}}$ treats $k_{\text{adc}}$ as a continuous variable, with a one-to-one relation between $k_{\text{adc}}$ and $k_n$; in fact the readout electronics deliver only integer-value ADC counts, integrating over the corresponding pulse heights. Thus the final function for describing ADC spectra is:

$$F(k_{\text{adc}}) = \int_{k_{\text{adc}} = -0.5}^{k_{\text{adc}} = +0.5} F_{\text{cont}}(k_{\text{adc}}') \cdot dk_{\text{adc}}'. \quad (37)$$

### 4 Example Fits

The fits are performed as binned log-likelihood fits: for each 1-ADC-count wide bin $k_{\text{adc}}$, containing $n_i$ events, the binomial probability of having $n_i$ "successes" in $N_{\text{all}}$ trials is calculated, where $N_{\text{all}}$ is the total number of events. The probability of an individual “success” is given by $F(k_{\text{adc}})$.

The probability distribution for the number of electrons after the fourth dynode is calculated without approximation. Then the function is scaled, approximating the additional spread due to the remaining dynodes with a Gaussian, as described in the previous section.
Table 1: Voltage distribution in 12-dynode PMT, normalised to the voltage between dynodes 3 and 4.

| voltage dynode number | Cathode | 1 | 2 | 2 | 1 | 1 | 1 | ... | 1 | 1 | 1 | 2 |

Figure 1: MC–generated PMT ADC–spectrum, from 100k events, with $\lambda_1 = 0.15$. The fit is superimposed.

![Figure 1](image1.png)

Table 2: Monte Carlo input compared with mean and RMS of the results from fits to 128 simulated spectra, with $\lambda_1 = 0.15$

| MC input | Mean fit result $\pm$ RMS spread |
|----------|----------------------------------|
| $\lambda_1$ | 0.150 $\pm$ 0.0013 |
| $g_1$ | 5.000 $\pm$ 0.058 |
| $p_0$ | 100.00 $\pm$ 0.0038 |
| $\sigma$ | 1.0000 $\pm$ 0.0027 |
| $c_n$ | $3.20 \cdot 10^{-4}$ $\pm$ 0.0046 $\cdot 10^{-4}$ |

Figure 2: MC–generated PMT ADC–spectrum, from 100k events, with $\lambda_1 = 3$. The fit is superimposed.

![Figure 2](image2.png)

Table 3: Monte Carlo input compared with mean and RMS of the results from fits to 128 simulated spectra (representing 2 64–channel MaPMT's), with $\lambda_1 = 3$

| MC input | Mean fit result $\pm$ RMS spread |
|----------|----------------------------------|
| $\lambda_1$ | 3.000 $\pm$ 0.022 |
| $g_1$ | 5.000 $\pm$ 0.017 |
| $p_0$ | 100.000 $\pm$ 0.021 |
| $\sigma$ | 1.000 $\pm$ 0.016 |
| $c_n$ | $6.4 \cdot 10^{-4}$ $\pm$ $(6.45 \pm 0.17) \cdot 10^{-4}$ |

4.1 MC–Generated Spectra

The validity of the method has first been established on Monte Carlo simulated data. The Monte Carlo program simulates the output of a PMT pixel. The gain at the first dynode is $g_1 = 5$ and the gains at the other dynodes are calculated from $g \propto V^{\alpha}$ with $\alpha = 0.75$. The values for $V$ are given in table 1.

The fit function is applied to two sets of 128 simulations with $10^5$ events each, one set with 0.15 photoelectrons per event, one with 3.0 photoelectrons per event. A spectrum from each set is shown in figures 1 and 2.

The fits are performed varying the gain of only one dynode and calculating the gains at the other dynodes using the same value for $\alpha$ as in the Monte Carlo program that generated the spectrum. The fit results agree very well with the input values, as shown in tables 2 and 3. To test the sensitivity of the fit result on the exact knowledge of $\alpha$, the fit to the spectrum in figure 1 is repeated assuming dif-
Table 4: Monte Carlo input compared with fit–result for the MC–generated pulse height spectrum shown in figure 1, using different assumptions in the fit.

| Parameter | MC input | Fit result α = 0.75 | Fit result α = 0.5 | Fit result α = 1 | Fit result: 3 indep. dyn’s |
|-----------|----------|---------------------|-------------------|-----------------|-----------------------------|
| $\alpha$  | 0.75     | 0.1490              | 0.1491            | 0.1489          | 0.1492±0.0013               |
| $g_1$     | 5.00     | 5.039               | 4.852             | 5.291           | 4.74±0.44                   |
| $g_2, g_3, g_{12}$ | $g_1 \cdot \left(\frac{2}{7}\right)^\alpha = 3.69$ | $g_1 \cdot \left(\frac{2}{7}\right)^\alpha = 2.19$ | | | 4.51±1.35 |
| $p_0$     | 100.00   | 100.000             | 100.000           | 100.000         | 100.000±0.003               |
| $\sigma$  | 1.0000   | 1.0028              | 1.0029            | 1.0027          | 1.0028±0.0025               |
| $c_n$     | $3.20 \cdot 10^{-4}$ | $2.90 \cdot 10^{-4}$ | $0.373 \cdot 10^{-4}$ | $19.7 \cdot 10^{-4}$ | $(4.37 \pm 0.98) \cdot 10^{-4}$ |

Different values for this parameter in the fit–function: $\alpha = 0.5$ and $\alpha = 1.0$. The results are given in table 4. Another fit was performed that does not use the formula $g \propto V^\alpha$. Here it is only assumed that dynodes with the same accelerating voltage have the same gain. Instead of one gain, three gains need to be fitted, one for each accelerating voltage. The fits are performed using the function minimisation and error analysis package MINUIT [Jam94]. The results from this fit, with error–estimates provided by MINUIT, are given in the last column of table 4.

Comparing the results for the different assumptions shows that they have little impact on the the fitted value for the number of photoelectrons and the gain at the first dynode. Most of the error introduced by an incorrect estimate of the parameter $\alpha$ is absorbed into the ratio of ADC–counts to electrons, $c_n$, while the values for $\lambda_1$ and $g_1$ come out close to the input values.

4.2 Application to Testbeam Data

The fit method has been applied to spectra obtained from a prototype RICH detector, incorporating an array of nine 64–channel Hamamatsu PMTs and operated in a CERN testbeam [RIC00]. Fits were performed to estimate the signal loss at the first dynode and below the threshold cut.

Figure 3 shows an example of such a fit to a spectrum obtained in the testbeam. The fit describes

Figure 3: Data from 6k events in black, with fit superimposed. The dashed line indicates the single photoelectron contribution. The signal loss refers to the fraction of photoelectrons lost below the threshold cut; both the total fraction of photoelectrons, and the fraction of single photoelectron events lost below the cut is given. These numbers do not include the loss at the first dynode due to photoelectrons producing zero secondary electrons.

Table 5: Result of fit applied to testbeam data

| Parameter | Fit result |
|-----------|------------|
| $\lambda_1$ | 0.107±0.005 |
| $g_1$     | 3.60±0.20  |
| $p_0$     | 43.06±0.01 |
| $\sigma$  | 0.724±0.008|
| $c_n$     | $(61±30) \cdot 10^{-4}$ |
4.3 Background

Apart from the Gaussian noise taken into account here, various other sources of background, such as electrons released due to the photoelectric effect in the first dynode, thermal electrons from the photocathode or the dynodes, genuine photoelectrons missing the first dynode, etc, can contribute to a PMT pulse height spectrum. A detailed discussion of such background is beyond the scope of this paper. However, any type of background that originates from within the dynode chain can be naturally accommodated in the fit method described here, since this background undergoes the same type of amplification process as the signal. To illustrate this, a spectrum has been generated with a Monte Carlo program, assuming a signal of 1.6 photoelectrons per event and a background of 0.16 photoelectrons per event due to the photoelectric effect in the first dynode (see [CZ01] for a fit to real data showing this kind of background, using a different method). The function to fit this spectrum is obtained by convoluting the background–free function $P(k)$ with another function $P_{bg}(k)$. $P_{bg}(k)$ is identical to $P(k)$ except that the amplification due to the first dynode is missing and that the number of photoelectrons per event hitting the second dynode, $\lambda_{bg}$, is a new free parameter. In the example given here, $P$ and $P_{bg}(k)$ are calculated to give the exact distributions for signal and background respectively after the fourth dynode; then the two distributions are convoluted and the result is scaled according to equation 34. The generated spectrum and the fit result are shown in figure 4 the fit function is shown again in figure 5 showing the non–pedestal and the single photoelectron contributions separately.

5 Summary

An analytical formula for the the probability distribution of the number of electrons at the end of a dynode chain, or any “snowball” like process described by a series of Poisson distributions, is de-
The formula describes the amplification process at all stages exactly, in particular without approximating Poisson distributions with Gaussians. It is evaluated as a function of the number of photoelectrons coming from the cathode and the gains at each dynode. The initially found formula is adapted to reduce numerical problems due to the multiplication of very large numbers with very small ones. A numerical recipe is given that implements that function.

It is shown how the function can be used as the core element of an approximated, but faster algorithm, that calculates the exact distribution for the first few dynodes and then scales the result according to the gain at the remaining dynodes, approximating the additional spread at those dynodes with a Gaussian. The number of dynodes for which the distribution is calculated exactly is not limited in principle and can be adjusted according to the precision required, and the computing time available. It is also shown how to modify the function to describe ADC–spectra obtained from read–out electronics, rather than directly the number of electrons at the end of a dynode chain.

This fast algorithm is then used to fit Monte Carlo generated ADC–spectra. In the fit function, the electron distribution after the first four out of twelve dynodes is calculated exactly. The fit results reproduce the MC–input values well. The dependence of the fit result on the assumptions made to reduce the number of fit–parameters is investigated. These results show that the fitted value for the number of photoelectrons per event is very weakly dependent on the different assumptions considered here, and the fitted gain on the first dynode also does not depend strongly on them. Real data from a multi–anode PMT used in the 1999 LHCb–RICH testbeam are fitted, and shown to be described well by the function. Finally it is illustrated how the fit function can be modified further to accommodate background from within the dynode chain, using the example of the photoelectric effect in the first dynode.

Acknowledgements

I wish to thank the LHCb RICH group, and in particular the colleagues involved in the 1999 LHCb–RICH testbeam. Special thanks go to James Libby, David Websdale and Guy Wilkinson for many helpful suggestions.

\section{FORTRAN Routine to Calculate $P(k_n)$}

\begin{verbatim}
SUBROUTINE DYNODE_CHAIN(OUT, MAX, LAMBDA, DYNODES)
IMPLICIT NONE
* This program takes as its input the maximum number of electrons at
* the end of the dynode chain, for which it should calculate $P(k_n)$.
* MAX, the average number of photo-electrons hitting the first
* dynode, LAMBDA(1), the gains at each dynode, LAMBDA(2), ...
* LAMBDA(DYNODES) and the dimension of the array LAMBDA.
* DYNODES. It calls the routine MAKE_P_RATIO, which is listed at the
* end of this file.
* The output is put into the array OUT(MAX), where the probability
* to find $k_n < \text{MAX}$ electrons at the end of the dynode chain is
* given by OUT(k_n).
* Written by Jonas Rademacker.
*
* INTEGER MAX, DYNODES
* DOUBLE PRECISION OUT(0:MAX), LAMBDA(DYNODES)
* INTEGER ABS_MAX, MAX_DYN
* PARAMETER(ABS_MAX=50001,MAX_DYN=13)
* INTEGER IX,IT,N,J, K
* To avoid having to define a limit on the number $k_n$ that can be
* calculated, one could create these arrays outside the program and
* pass them on.
* DOUBLE PRECISION $P(\text{MAX,DYN})$ ! corresponds to $f^{\text{star}}$ in the text
* DOUBLE PRECISION $U(1:MAX,DYN)$, $V(1:MAX,DYN)$, $\text{ABS}_\text{MAX}$
* DOUBLE PRECISION FASTNULL
* PARAMETER (FASTNULL=1.d-300)
* DOUBLE PRECISION MEAN
* DOUBLE PRECISION $P_{\text{ratio}}(\text{ABS}_\text{MAX})$, $F_{\text{factor}}$, $U_{\text{factor}}$, $V_{\text{factor}}$
* INTEGER MAX_OLD
* SAVE MAX_OLD
* DATA MAX_OLD/-9999/
* SAVE P_ratio
* -- Some initialisations and tests --
* DO IX=1,MIN(ABS_MAX,MAX),1
* OUT(IX)=0.d0
* ENDDO
* IF(ABS_MAX.LT.MAX)THEN
* RETURN
* ENDIF
* MEAN = 1.D0
* DO IX=1,DYNODES,1
* MEAN = MEAN*LAMBDA(IX)
* ENDDO
* IF(MAX.LE.0.d0)THEN
* OUT(0)=1.d0
* RETURN
* ENDIF
* MEAN = 1.D0
* DO IX=1,DYNODES,1
* MEAN = MEAN+LAMBDA(IX)
* ENDDO
* IF(MAX.LE.0.d0)THEN
* OUT(0)=1.d0
* RETURN
* ENDIF
* -- make and save the factors $P_{\text{ratio}}(k)<p_{k}/p_{k-1}>^{k}$ --
* IF(ABS_MAX.GT.MAX)THEN
* MAX_OLD=MAX
* CALL MAKE_P_RATIO(P_ratio,MAX)
* ENDIF
* -- Calculate the probability to see zero electrons (k,n=0) --
* $f(\text{DYNODES})=1.d0$
\end{verbatim}
\begin{verbatim}
U(DYNODES,0)=F(DYNODES)
V(DYNODES,0)=F(DYNODES)
DO IX=DYNODES-1,1,-1
   X(IX) = LAMBDA(IX)*DEXP(-LAMBDA(IX+1))
   U(IX,0) = X(IX)*F(IX+1)
   V(IX,0) = F(IX)
ENDDO
OUT(0)=DEXP(-LAMBDA(1))*F(1) ! <---- save the result

* -- Calculate the probabilities for k_n=1,...,MAX electrons --

DO K=1,MAX,+1
   * . calculate f_k
   IF(F(DYNODES).LT.FASTNULL)THEN
      F(DYNODES)=0.d0
   ELSE
      F(DYNODES)=F(DYNODES) * LAMBDA(DYNODES)/DBLE(K)
   ENDIF
   U(DYNODES,K)=F(DYNODES)
   V(DYNODES,K)=F(DYNODES)
   * . re-calculate U and V from previous iteration:
   DO J=0,K-1,+1
      F_FACTOR=P_ratio(K)**(DBLE(J)/DBLE(K))
      IF(K-1-J.GT.0)THEN
         U_FACTOR=DSQRT(DBLE(K-1)/DBLE(K-1-J))*F_FACTOR
      ELSE
         U_FACTOR=F_FACTOR
      ENDIF
      V_FACTOR=DSQRT(DBLE(K-1)/DBLE(K-J))*F_FACTOR
      DO I=DYNODES,1,-1
         U(I,J)=U(I,J)*U_FACTOR
         V(I,J)=V(I,J)*V_FACTOR
      ENDDO
   ENDDO
   * . apply the recursive formula to get f^{k}_i
   DO I=DYNODES-1,1,-1
      F(I)=0.d0
      DO J=0,K-1
         F(I)=F(I)+U(I,K-1-J)*X(I)*V(I+1,J+1)
      ENDDO
      U(I,K)=F(I)
      V(I,K)=F(I)
   ENDDO

* . calculate P(k):
OUT(K)=DEXP(-LAMBDA(1))*F(1) ! <---- save the result
ENDDO
RETURN
END

**

SUBROUTINE MAKE_P_RATIO(P_ratio,MAX)
IMPLICIT NONE
INTEGER MAX
DOUBLE PRECISION P_ratio(MAX)

INTEGER APPROX_FROM
PARAMETER(APPROX_FROM=25)
DOUBLE PRECISION PI, E
PARAMETER(PI=3.1415927d0, E=2.718281828d0)

PARAMETER(NFAC=1.D0)
P_ratio(1)=1.d0
DO N=2,MIN(APPROX_FROM-1,MAX),+1
   NFAC=NFAC*DBLE(N-1)
   P_ratio(N)=(NFAC**(1.d0/DBLE(N-1)))/DBLE(N)
ENDDO
DO N=APPROX_FROM,MAX,+1
   P_ratio(N)=
      (2.D0*PI*DBLE(N-1))**(1.D0/(2.D0*DBLE(N-1)))*DBLE(N-1)/(E*DBLE(N))*
      (1.d0+1.d0/DBLE(12*(N-1)) +
      1.d0/DBLE(288*(N-1)**2))**(1.D0/DBLE(N-1))
ENDDO
RETURN
END

**

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\end{verbatim}