THE FERMAT PRINCIPLE IN GENERAL RELATIVITY
AND APPLICATIONS*

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ABSTRACT

In this paper we use a general version of Fermat’s principle for light rays in General Relativity and a curve shortening method to write the Morse relations for light rays joining an event with a smooth timelike curve in a Lorentzian manifold with boundary. As a physical meaning, one can apply the Morse relations to have a mathematical description of the gravitational lens effect in a very general context.

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INTRODUCTION

The Fermat’s Principle in Classical Optics states that the trajectory of a light ray from a source $A$ to a target $B$ is such that it is a minimizer, or better, a stationary curve for the travel time among all the paths joining the points $A$ and $B$. This variational principle can be extended in the context of General Relativity where the trajectory of a light ray under the action of the gravitational field in vacuum is given by a null geodesic in a Lorentzian manifold modelling the space–time generated by a gravitational mass distribution.

A formulation of the Fermat’s principle is given once the following data are determined:

1. a set of trial curves joining the light source and the observer;
2. a functional that associates to each trial curve a real number, which has to be related to a measurement of the time passed from the instant at which the photon departed from the light source to the instant at which the photon arrives to the observer.

A mathematical proof of the Fermat’s principle consists in proving that the trajectory of a light ray is characterized as a stationary point of the time functional in the set of trial curves. The geodesics in a semi-Riemannian manifold are characterized as solutions of differential equations, and the local theory of the light rays can be developed in terms of systems of differential equations in $\mathbb{R}^n$. However, the variational approach has the advantage of providing techniques for proving global existence results, and also for producing several kinds of estimates on the number of solutions, given in terms of the topology of the space of trial curves. To this aim in this paper we prove the Morse Relations for light rays, that will be presented in details in Section 1. We now proceed to a general discussion of the mathematical problem, its physical applications and a presentation of the new results that will be proven in this paper. We fix a Lorentzian manifold $(M, g)$ that is the mathematical model of our relativistic spacetime, and we assume that $M$ is endowed with a time orientation given by the choice of a continuous timelike vector field $W$ on $M$. Such assumption is indeed very mild; namely, given any Lorentzian manifold, there exists always a two-fold covering $\tilde{M}$ of $M$ that admits a time orientation (cf. [19]), and clearly there is a two-to-one correspondence between the geodesics in $\tilde{M}$ and those on $M$. If we want to study the light rays emitted by some source at a given time in the past, represented by an event $p$ of $M$, and reaching an observer sometimes during its life, whose worldline is given by a timelike curve $\gamma$ in $M$, then we need to determine all the lightlike future pointing geodesics joining $p$ and $\gamma$ in $M$. We are assuming here that both the source and the receivers are pointlike, i.e. they have dimensions which are neglectible with respect to their distance; a variational principle for light rays between a spatially extended source and a spatially extended receiver may be found in [23]. In analogy with the principle in classical optics, the set of trial curves is chosen to be the set of all possible future pointing trajectories joining the source with the observer, and that are run at the speed of light. This amounts to saying that a trial curve is a curve whose tangent vector is everywhere in the light cone, and it belongs to the same half light cone as the vector field $W$. The choice of the regularity to impose on the trial curves and, most of all, the choice of the functional to be extremized.
are rather delicate questions, which have deep consequences for the mathematical theory to be developed. The first relativistic formulation of the principle, valid in the case of a static spacetime, is due to Weyl (see [33]); the validity of the general relativistic Fermat’s principle was successively extended to the case of stationary spacetimes by Levi–Civita (see [15]). For conformally stationary spacetimes, an alternative formulation of the principle is given in [6]. The first attempt to extend the Fermat’s principle beyond the (conformally) stationary case is due to Uhlenbeck (see [31]), who considered a Lorentzian manifold diffeomorphic to a space-time splitting $\mathcal{M}_0 \times \mathbb{R}$ and a time dependent metric which is diagonal with respect to this product. The variational principle proven in [31] employs the time functional given by the the projection onto the second factor calculated at the final point of each trial curve. Such functional does not depend on the parameterization of the trial curve as, for instance, the length functional for curves in a Riemannian manifold, and this lack of rigidity makes it a difficult task to obtain results of existence and multiplicity of critical points. For this reason, in order to prove the classical Morse relations the author employs an action functional whose Lagrangian function depends quadratically by the velocities. This kind of functional has a strict relationship with the energy functional for Riemannian geodesics, obtained by removing the square root inside the integral that defines the length. The same variational principle was used in [8] to obtain Morse relations for light rays on orthogonal splitting Lorentzian manifolds, using an infinite dimensional setting and covering some gaps that occur in [31]. Such a variational principle was extended in [1] to stably causal Lorentzian manifolds and applied in [9,10] to obtain multiplicity results and Morse Relations for light rays joining an event $p$ with a timelike curve $\gamma$ in the presence of a smooth convex boundary. A very general version of the principle, valid in all spacetimes, was given recently by Kovner (see [14]), who introduced the so called arrival time functional with respect to the observer $\gamma$, defined on the space of piecewise smooth lightlike curves joining $p$ and $\gamma$. Such functional is given by fixing any (future pointing) parameterization of $\gamma$, and assigning to each trial curve the value of the the parameter of $\gamma$ at the arrival point. Any two future pointing parameterizations of $\gamma$ differ by an order preserving diffeomorphism between two intervals of the real line; it is an easy observation that the stationary points of the arrival time functional do not indeed depend on the choice of the parameterization of $\gamma$. A rigorous mathematical proof of the Kovner’s claim was given in [21]. However, the proof in [21] needs the assumption that the critical points have nonzero derivative everywhere. An alternative variational principle on the space of lightlike curves $z$ with a suitable prescribed parameterization and satisfying $\dot{z}(s) \neq 0$ for all $s$, can be found in [22]. However, using this approach it is not possible to obtain Morse Relations, as it will be clear from the discussion presented in Appendix B. In reference [10] the reader will find a more detailed presentation of the different versions of the Relativistic Fermat Principle and some examples and applications to the multiple image effect (the so called ”gravitational lens effect”). In a paper published in 1979, Walsh, Carlswell and Weymann (see [32]) discussed the possibility that the double quasar 0957+561 would be a good candidate for a gravitational lens effect. Such a name refers to the phenomena occurring when a multiple image of some stellar object is observed. The multiple image effects are due to the deflection of the light in
presence of a gravitational field. We refer e.g. to [27,28] for a detailed physical description of the gravitational lens effect and many physical examples. The version of the relativistic Fermat principle introduced by Kovner allows also to treat nonstationary situations such as a gravitational wave sweeping over a gravitational lensing situation. More details can be found in [5].

Some natural questions arise in the study of the gravitational lensing effect; for instance, it is tried to understand under which circumstances a multiple imaging of a distant source can occur, and, in this case, how many images of the source can be seen. In mathematical terms, these questions can be answered by giving conditions on the topology and the metric of the spacetime that guarantee a multiplicity of lightlike geodesics between \( p \) and \( \gamma \) lying inside an open set \( \Lambda \), that represent the region of the universe where to localize the description for any gravitational lens. As already mentioned, a technique for investigating these issues is provided by the Morse Theory, which is a well established mathematical theory that relates the critical points of a smooth functional with the topology of the underlying space.

The main purpose of this paper is to develop an infinite dimensional Morse theory under minimal assumptions on the global structure of the spacetime and on the timelike curve \( \gamma \). This in particular allows us to extend the results in [6,8,10,31] concerning Morse Relations. Most of all, we want to push the results beyond the compactness assumption of global hyperbolicity made in [8,31]; we also generalize the results of [10] in the following directions:

- we do not assume the stable causality of the Lorentzian manifold \((\mathcal{M},g)\), that will only be assumed to be time orientable;
- we do not assume any regularity for the boundary \(\partial\Lambda\) of the region \(\Lambda\);
- we do not assume that \(\gamma\) is embedded as a closed subset of \(\Lambda \subseteq \mathcal{M}\).

Observe in particular that the second generalization above allows to extend the results also to light rays moving on a region of the universe exterior to a static blackhole (see [13]). For the functional framework, we will employ Kovner’s arrival time functional, denoted by \(\tau\); observe that the definition of \(\tau\) does not require the existence of a global time function on \(\Lambda\), which was a crucial assumption in [9,10].

For a correct physical interpretation of our results, all the relevant information about the light rays joining \( p \) and \( \gamma \) must be encoded the open subset \(\Lambda\). For this reason, if \(\Lambda \neq \mathcal{M}\), we assume the following convexity property of \(\Lambda\):
every lightlike geodesic starting from any event in $\Lambda$
and moving outside $\overline{\Lambda}$ does not come back in $\Lambda$. \hfill (*)

Note that assumption (\*) is not strictly necessary to develop our theory. As a matter of facts, we will use a more general assumption: the light convexity of the boundary of $\Lambda$ (cf. (3) in Sect. 2). Observe also that in the Minkowski space time a set $\Lambda_0 \times \mathbb{R}$ satisfies condition (\*) precisely when $\Lambda_0$ is convex. Other simple examples of spacetimes satisfying (\*) are the regions outside the event horizon of the Schwarzschild and Reissner-Nordström spacetimes (see [16]). Our Morse relations are given for future pointing lightlike geodesics; we remark here that there is also a time-reversed version of the Fermat’s principle. Namely, $p$ can be interpreted as a pointlike receiver at a particular instant of time and $\gamma$ as the worldline of a pointlike light source, in which case one is interested in determining the past pointing light rays from $p$ to $\gamma$. Clearly, the results proven in the paper are still valid in the past pointing case. From a mathematical point of view, the case of past pointing light rays it is completely analogous, and it will not be treated explicitly in this paper. In the next section we will give a formal statement of our results, and we will present arguments to show that our assumptions cannot be weakened to obtain a Morse Theory.

The reader is referred to classical books as [2,13,19] for the main notions and properties in Lorentzian geometry. Finally, we remark that alternative approaches to the study of the Morse theory for light rays are available; for instance, in references [24,25,26] the author applies the Morse Theory in a time independent quasi-Newtonian setting.

1. STATEMENT OF THE RESULTS AND DISCUSSION ABOUT THE ASSUMPTIONS

Let $(\mathcal{M},g)$ be a smooth Lorentz manifold, $\Lambda$ an open connected subset of $\mathcal{M}$, $p \in \Lambda$, $\gamma:[\alpha,\beta] \rightarrow \Lambda$ a smooth timelike curve such that $p \notin \gamma([\alpha,\beta])$. Here and in the rest of the paper we will often set $\langle \cdot,\cdot \rangle \equiv g(z)[\cdot,\cdot]$. We assume that $(\mathcal{M},g)$ is time orientable. This means that there exists a smooth vector field $W$ on $\mathcal{M}$ such that $\langle W(z),W(z) \rangle < 0$ for any $z \in \mathcal{M}$. With respect to the orientation $W$ we assume that

$$\gamma \text{ is future pointing},$$

namely

$$\langle \dot{\gamma}(s),W(\gamma(s)) \rangle < 0 \quad \forall s \in [\alpha,\beta].$$

Since we want to study future pointing light rays joining $p$ and $\gamma$ in $\Lambda$, we only shall consider past and future relatively to $\Lambda$. More precisely given two points $q_1$ and $q_2$ in $\Lambda$, we say that $q_2$ is in the future of $q_1$ (in symbols $q_2 \in J^+(q_1,\Lambda)$) if there exists a piecewise smooth curve $y:[0,1] \rightarrow \Lambda$ such that $\langle \dot{y},\dot{y} \rangle \leq 0$ (i.e. $y$ is a causal curve), $\langle \dot{y},W(y) \rangle < 0$
(i.e. \( y \) is future pointing), \( y(0) = q_1, y(1) = q_2 \). In general if \( A \subset \Lambda \) the future of \( A \) (in \( \Lambda \)) is the set
\[
J^+(A, \Lambda) = \bigcup_{a \in A} J^+(a, \Lambda),
\]
while the past of \( A \) (in \( \Lambda \)) is the set
\[
J^-(A, \Lambda) = \{ q \in \Lambda : A \cap J^+(q, \Lambda) \neq \emptyset \}.
\]
To have future pointing lightlike curves joining \( p \) and \( \gamma \) in \( \Lambda \) clearly we need the following assumptions:

(2) \textit{there exists} \( q_+ \in \gamma(] \alpha, \beta [) \cap J^+(p, \Lambda) \).

Moreover a light-convexity assumption on the closure \( \overline{\Lambda} \) of the open subset \( \Lambda \) is needed:

(3) \( \overline{\Lambda} \) \textit{is light convex, i.e. all the lightlike geodesics in} \( \Lambda \cup \partial \Lambda \) \textit{are enterely contained in} \( \Lambda \).

Here \( \partial \Lambda \) is the topological boundary of \( \Lambda \). Finally, to be able to define the arrival time functional we need

(4) \( \gamma : ] \alpha, \beta [ \to \Lambda \) \textit{is injective}.

By (4), on the space of the curves joining \( p \) and \( \gamma \) on the interval \([0,1]\), it is well defined the arrival time functional
\[
\tau(z) = \gamma^{-1}(z(1)). \tag{1.1}
\]
The following assumptions says that \( \tau \) is bounded from below on the set of the future pointing lightlike curves joining \( p \) and \( \gamma \).

(5) \textit{there exists} \( q_- \in \gamma(] \alpha, \beta [) \setminus J^+(p, \Lambda) \).

Since we do not require that \( \partial \Lambda \) is smooth we are not able to use the same penalizing argument as in [9,10] to overcome the difficulties due to the presence of the boundary. To develop a Morse Theory for the arrival time functional we should need a flow which is strictly decreasing far from the critical points. Since the boundary is not smooth a convenient approach is a shortening method. Assumption (3) is necessary because we need a flow that it is invariant with respect to the lightlike curves with image in \( \Lambda \). Note that to develop a Morse Theory the presence of the boundary is a difficulty to bypass because we want to treat with "free" critical points lying in \( \overline{\Lambda} \). Note also that, even if the boundary would be smooth, a shortening method seems technically more simple that the penalized techniques used in [8,9]. Moreover, since \( \tau \) is invariant by reparameterizations, as well as the space of future pointing light-like curves joining \( p \) and \( \gamma \), a shortening approach seems to be a good help to overcome also this kind of difficulty.
To use the shortening method we need an assumption assuring the existence of minimizers in $\Lambda$ between events and timelike curves. For this reason we need also the following assumption:

\begin{equation}
\text{(6) there exists a smooth timelike vector field } W \text{ in } M \text{ having the following properties:}
\end{equation}

\begin{enumerate}
\item $\gamma$ is an integral curve of $W$ (namely $\dot{\gamma} = W(\gamma)$ for any $s \in ]\alpha, \beta[$),
\item for any $q \in J^+(p, \Lambda) \cup J^-(\gamma(\alpha, \beta), \Lambda)$ if $\gamma_q$ is the maximal integral curve of $W$ such that $\gamma_q(0) = q$, there is $\overline{q} \in [\text{Im } \gamma_q \cap \Lambda] \setminus J^+(p, \Lambda)$.
\end{enumerate}

Here $\text{Im } \gamma_q$ denotes the image of the curve $\gamma_q$. Note that (6) is certainly satisfied if $\Lambda$ is invariant with respect to the flow of $W$.

Morse Theory gives an algebraic relation (in terms of formal series) between the critical points of a suitable functional (in our case the arrival time functional) and the topology of the space where the functional is defined. At this point there are two chances: to introduce a Sobolev space of lightlike curves or to use broken lightlike geodesics as space of trial curves. To state Morse relations we prefer here to use the second choice since it is not required the use of any auxiliary (Riemann) structure. Nevertheless in section 2 we shall give an infinite dimensional formulation of the Fermat Principle using Sobolev spaces. Indeed, even if we shall use a shortening procedure it is more convenient an infinite dimensional approach to study the arrival time functional nearby its critical points. This is due to the fact that here it is hard to try to reduce (as e.g. in [8,31]) the study the functional $\tau$ on a space of curves joining two given points. For this reason we are not able to adapt to our case the Milnor finite dimensional approximation scheme (cf. [18]) nearby critical points.

Now set

$$
B^+_{p,\gamma}(\Lambda) = \left\{ z : [0,1] \to \Lambda : z \text{ is a } C^2 \text{ piecewise curve such that } \\
z(0) = p, \ z(1) \in \gamma(\alpha, \beta] \text{ and, on any interval } [a,b] \subset ]\alpha, \beta[ \text{ where } z \text{ is of class } C^2, \\
z \text{ is a constant or a future pointing light-like geodesic} \right\}.
$$

(1.2)

We point out that a curve $z \in B^+_{p,\gamma}(\Lambda)$ may be constant on some interval $[a,b] \subset [0,1]$ (and therefore $z|_{[a,b]}$ is not a light-like geodesic). Nevertheless the topological structure of the problem is carried on by the space $B^+_{p,\gamma}(\Lambda)$, instead of the space $B^+_{p,\gamma}(\Lambda)$ of the broken lightlike geodesics (without subintervals where $z$ is constant). A simple example in appendix B shows that Morse Relations can not be written using $B^+_{p,\gamma}(\Lambda)$. In section 4 we shall prove the homotopy equivalence between $B^+_{p,\gamma}(\Lambda)$ (endowed with the uniform topology) and the Sobolev spaces of future pointing, lightlike, $H^{1,r}$-curves joining $p$ and $\gamma$ ($r \in [1, +\infty]$). Using the space $B^+_{p,\gamma}(\Lambda)$ we can state our last assumptions. In section 2 we shall prove that it is equivalent to that one formulated in [9,10]. For any $c \in ]\alpha, \beta[$ (cf. (4)) we denote by $\tau^c$ the $c$-sublevel of the functional $\tau$ in $B^+_{p,\gamma}(\Lambda)$:

\begin{equation}
\tau^c = \left\{ z \in B^+_{p,\gamma}(\Lambda) : \tau(z) \leq c \right\}
\end{equation}

(1.3)
Definition 1.1. Fix \( c > \alpha \). We say that \( \mathcal{B}^{+}_{p,\gamma}(\Lambda) \) is \( c \)-precompact if any sequence \( \{z_n : n \in \mathbb{N}\} \subset \tau^c \) has a subsequence uniformly convergent in \( \overline{\Lambda} \), up to reparameterizations. We say that \( \tau \) is pseudo-coercive in \( \mathcal{B}^{+}_{p,\gamma}(\Lambda) \), if \( \mathcal{B}^{+}_{p,\gamma}(\Lambda) \) is \( c \)-precompact for any \( c \in ]\alpha,\beta[ \).

Note that by assumption (5) there exists \( \hat{s} > \alpha \) such that \( \tau^c \) is the empty set for any \( c \in ]\alpha,\hat{s}[ \). Whenever \( \Lambda = \mathcal{M} \), pseudo–coercivity coincides with the global hyperbolicit y of the set of the events in the future of \( p \) (cf. Proposition 2.13 and [9]).

Before stating our main result, we recall some definitions.

Definition 1.2. Let \( (\mathcal{M}, \langle \cdot, \cdot \rangle) \) be a Lorentzian manifold, and \( z: [0,1] \rightarrow \mathcal{M} \) be a geodesic. A smooth vector field \( \zeta \) along \( z \) is called Jacobi field if it satisfies the equation

\[
D_s^2 \zeta + R(\zeta, \dot{z}) \dot{z} = 0 ,
\]

where \( R \) is the curvature tensor of the metric \( \langle \cdot, \cdot \rangle \) (cf. [2]). A point \( z(s), \ s \in ]0,1[ \) is said to be conjugate to \( z(0) \) along \( z \) if there exists a nonvanishing Jacobi field \( \zeta \) along \( z|_{[0,s]} \) such that

\[
\zeta(0) = \zeta(s) = 0 .
\]

The multiplicity of the conjugate point \( z(s) \) is the maximal number of linearly independent Jacobi fields satisfying (1.5).

By (1.4) the set of the Jacobi fields is a vector space of dimension \( 2\dim \mathcal{M} \). Hence the multiplicity of a conjugate point is finite and, by (1.5), is at most \( \dim \mathcal{M} \) (actually it is at most \( \dim \mathcal{M} - 1 \) because \( \zeta(s) = s\dot{z}(s) \) is a Jacobi field which is null only at \( s = 0 \)).

Definition 1.3. The index \( \mu(z) \) is the number of conjugate points \( z(s), \ s \in ]0,1[ \) to \( z(0) \), counted with their multiplicity.

It is well known that the index of a lightlike geodesic is finite (see [2]).

Definition 1.4. Let \( p \) be a point and \( \gamma \) a timelike curve on a Lorentzian manifold \( (\mathcal{M}, g) \). Then \( p \) and \( \gamma \) are said nonconjugate by lightlike geodesics if for any lightlike geodesic \( z: [0,1] \rightarrow \mathcal{M} \) joining \( p \) and \( \gamma \), \( z(1) \) is nonconjugate to \( p \) along \( z \).

It is well known that such a condition is true except for a residual set of pairs \( (p,\gamma) \). For some results where \( p \) and \( \gamma \) are conjugate see [7]. Let \( X \) be a topological space and \( K \) a field. For any \( l \in \mathbb{N} \) let \( H_l(X; K) \) be the \( l \)-th homology group of \( X \) with coefficients in \( K \). Since \( K \) is a field, then \( H_l(X; K) \) is a vector space whose dimension \( \beta_l(X; K) \) (eventually \( +\infty \)) is called the \( l \)-th Betti number of \( X \) (with coefficients in \( K \)). The Poincaré polynomial \( \mathcal{P}(X; K) \) is defined as the following formal series:

\[
\mathcal{P}(X; K)(\kappa) = \sum_{l \in \mathbb{N}} \beta_l(X; K) \kappa^l .
\]
Let $\mathcal{G}^+_{p,\gamma}(\Lambda)$ be the set of the future pointing lightlike geodesics joining $p$ and $\gamma$ and having image contained in $\Lambda$. The main result of this paper is the following theorem.

**Theorem 1.5.** Let $\Lambda$, $p$, $\gamma$ satisfying (1)-(6). Assume that the following assumptions hold true:

1. $p$ and $\gamma$ are nonconjugate;
2. $\tau$ is pseudo-coercive on $\mathcal{B}^+_{p,\gamma}(\Lambda)$.

Then for any field $\mathbf{K}$ there exists a formal series $S(\kappa)$ with coefficients in $\mathbb{N} \cup \{+\infty\}$, such that

$$\sum_{z \in \mathcal{G}^+_{p,\gamma}(\Lambda)} \kappa^\mu(z) = \mathcal{P}(\mathcal{B}^+_{p,\gamma}(\Lambda); \mathbf{K})(\kappa) + (1 + \kappa)S(\kappa). \quad (1.6)$$

The same result holds for the lightlike geodesics joining $p$ and $\gamma$ in the past of $p$, under an obvious modification of the assumptions.

**Remark 1.6.** Observe that the Betti numbers $\beta_l(X; \mathbf{K})$ (and the coefficient of the formal series $S(\kappa)$ in (1.6)) depend in a substantial way on the choice of the field $\mathbf{K}$. On the other hand, the left hand side of the equality (1.6) does not depend on $\mathbf{K}$, hence one can obtain more information on $\mathcal{G}^+_{p,\gamma}(\Lambda)$ by letting the coefficient field $\mathbf{K}$ arbitrary in (1.6). For example, a result of Serre, where the choice of $\mathbf{K}$ is essential (cf. [27]), is used to prove part (b) of Theorem 1.11.

**Remark 1.7.** Note that assumption $L_2$ cannot be removed. Indeed, let $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_0)$ be a Riemannian manifold such that there exist two points $p_1$, $p_2 \in \mathcal{M}_0$ which are not joined by any geodesic for the metric $\langle \cdot, \cdot \rangle_0$. Consider the (static) Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$, where $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is given by

$$\langle \zeta, \zeta \rangle = \langle \xi, \xi \rangle_0 - \theta^2,$$

for any $z = (x, t) \in \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \theta) \in T_z \mathcal{M}$. Let $\Lambda = \mathcal{M}$. Consider the point $p = (p_1, 0)$ and the timelike curve $\gamma(s) = (p_2, s)$. Clearly assumption (1)-(6) are satisfied, but not $L_2$. Theorem 1.5 does not hold for $p$ and $\gamma$, since there are no lightlike geodesics joining $p$ and $\gamma$, while $\mathcal{P}(\mathcal{B}^+_{p,\gamma}(\Lambda); \mathbf{K})(\kappa) \neq 0$ for any field $\mathbf{K}$.

**Remark 1.8.** Let $c_l$ be the number of the future pointing lightlike geodesics joining $p$ and $\gamma$ having index $q$. Then (1.6) can be written in the following way:

$$\sum_{l=0}^{\infty} c_l \kappa^l = \sum_{l=0}^{\infty} \beta_l(\mathcal{B}^+_{p,\gamma}(\Lambda); \mathbf{K}) \kappa^l + (1 + \kappa)S(\kappa). \quad (1.7)$$
From (1.7) we deduce that a certain number of future pointing light rays joining $p$ and $\gamma$ are obtained according to the topology of $B^+_{p, \gamma}(\Lambda)$. In particular, setting $\kappa = 1$ in (1.7), we have the following estimate on the number $\text{card}(G^+_{p, \gamma}(\Lambda))$ of the light rays joining $p$ and $\gamma$:

$$\text{card}(G^+_{p, \gamma}(\Lambda)) = \sum_{l=0}^{\infty} \beta_l(B^+_{p, \gamma}(\Lambda); I K) + 2S(1).$$

Since $S(1)$ is nonnegative we also get the classical Morse inequalities

$$c_l \geq \beta_l(B^+_{p, \gamma}(\Lambda); I K), \quad \forall l \in \mathbb{N}.$$  \hfill (1.9)

An example of the influence of the topology of $B^+_{p, \gamma}(\Lambda)$ on the number of future pointing, lightlike geodesics between $p$ and $\gamma$ is given by the next Theorem.

**Theorem 1.9.** Under the assumptions of Theorem 1.5 we have: (a) If $B^+_{p, \gamma}(\Lambda)$ is contractible the number $\text{card}(G^+_{p, \gamma}(\Lambda))$ is infinite or odd; (b) if $B^+_{p, \gamma}(\Lambda)$ is not contractible, there exist at least two future pointing light rays joining $p$ and $\gamma$. (We recall that a topological space is said to be contractible if it is homotopically equivalent to a point.)

Actually, the topology of $B^+_{p, \gamma}(\Lambda)$ is in general not known for arbitrary Lorentzian manifold. More information can be obtained if its topology can be related to the topology of the manifold $\Lambda$. Let $\Omega(\Lambda)$ be the based loop space of all the continuous curves $z : [0, 1] \rightarrow \Lambda$ such that $z(0) = z(1) = \bar{z}$. Since $\Lambda$ is connected, $\Omega(\Lambda)$ does not depend on $\bar{z}$. We equip $\Omega(\Lambda)$ with the uniform topology. Since the Poincaré polynomial is a homotopical invariant, we have the following result as an immediate consequence of Theorem 1.5.

**Theorem 1.10.** Besides the assumptions of Theorem 1.5, assume also

L$_3$) $B^+_{p, \gamma}(\Lambda)$ has the same homotopy type of the based loop space $\Omega(\Lambda)$.

Then for any field $K$ there exists a formal series $S(\kappa)$ with coefficients in $\mathbb{N} \cup \{+\infty\}$, such that

$$\sum_{z \in G^+_{p, \gamma}(\Lambda)} r^\mu(z) = P(\Omega(\Lambda), I K)(\kappa) + (1 + \kappa)S(\kappa).$$

(1.11)

In appendix A we give a general condition assuring that L$_3$) is satisfied. Thanks to the results proved in section 4 and appendix A we see that assumption L$_3$) is certainly satisfied if $(\Lambda, \langle \cdot, \cdot \rangle)$ is conformally stationary (for instance if $(\Lambda, \langle \cdot, \cdot \rangle)$ is a Robertson-Walker space-time).

**Theorem 1.11.** Under the assumptions of Theorem 1.10 we have:

(a) If $\Lambda$ is contractible, then the number of the future pointing lightlike geodesics joining $p$ with $\gamma$ and with image in $\Lambda$ is infinite or odd;
(b) if $\Lambda$ is noncontractible, then the number of the future pointing lightlike geodesics joining $p$ with $\gamma$ and with image in $\Lambda$ is infinite.

Conditions assuring the finiteness of the images can be found in [12]. To write Morse Relations for light rays using the arrival time functional $\tau$ it does not seem that the pseudocoercivity assumption can be weakened. Indeed it is necessary to have that the sublevels $\tau^c$ of the arrival time functional are complete with respect to a suitable metric and the Palais-Smale sequences are precompact (with respect to such a metric). Note that the sequences in Definition 1.1 are allowed to reach $\partial \Lambda$. The light convexity of $\Lambda$ will guarantee the existence of minimizers with image entirely included in $\Lambda$.

Morse Relations are proved regarding lightlike geodesics as critical points of the functional $\tau$. They are written using the geometric index $\mu$ instead of the Morse index thanks to Theorem 4.14. Its proof is based on a different approach to the Index Theorem for lightlike geodesics, with respect to that one of [2], where the Index Theorem is proved on a quotient space of the admissible variations. For the proof of Theorem 4.14 we have to choose a suitable manifold where the critical points of $\tau$ are lightlike geodesics.

2. MINIMIZERS FOR THE ARRIVAL TIME ON SOBOLEV CURVES SPACES

We begin the section introducing the Sobolev spaces $H^{1,r}([0,1],\Lambda)$ with $r \in [1, +\infty]$. This can be rapidly done in the following way.

Let $W$ be the smooth timelike vector field on $M$ whose existence is assumed in (1). The manifold $M$ can be equipped by a natural Riemannian structure setting

$$\langle \zeta, \zeta \rangle_R = \langle \zeta, \zeta \rangle - \frac{2(W(z), \zeta)^2}{\langle W(z), W(z) \rangle}.$$  \hfill (2.1)

The Riemannian metric (2.1) can be used to introduce a Riemann distance on $\Lambda$ that we shall denote by $d_R$. Such a distance allows us introduce the space of the absolutely continuous curves between $[0,1]$ and $\Lambda$ (denoted by $AC([0,1],\Lambda)$). Finally for any $r \in [1, +\infty]$ we set

$$H^{1,r}([0,1],\Lambda) = \left\{ z \in AC([0,1],\Lambda) : \int_0^1 (\langle \dot{z}, \dot{z} \rangle_R)^{r/2} ds < +\infty \right\}$$  \hfill (2.2)

while

$$H^{1,\infty}([0,1],\Lambda) = \left\{ z \in AC([0,1],\Lambda) : \sup \left\{ \langle \dot{z}(s), \dot{z}(s) \rangle_R : s \in [0,1] \right\} < +\infty \right\}.$$  \hfill (2.2a)

Using local coordinates and the Palais definition of Sobolev manifolds (cf. [20]) we see that the spaces defined above do not depend on the choice of $W$. Moreover we set

$$\Omega^{1,r}_{p,\gamma}(\Lambda) = \left\{ z \in H^{1,r}([0,1],\Lambda) : z(0) = p, \quad z(1) \in \gamma(\alpha,\beta) \right\}$$  \hfill (2.2b)
For any absolutely continuous curve \( z \) we can extend the classical definition of causal curve saying that \( z \) is a causal curve if \( \langle \dot{z}(s), \dot{z}(s) \rangle \leq 0 \) almost everywhere (a.e.). Moreover we say that a causal curve is future pointing if

\[
\langle \dot{z}(s), W(z(s)) \rangle < 0 \quad \text{for almost every } s \text{ such that } \dot{z}(s) \neq 0.
\]

It is possible to prove that the above notions are equivalent to that ones given in [13] for continuous curves, whenever we deal with absolutely continuous curves. This can be done using Proposition 2.2. To develop a Morse theory for light rays the following spaces will be also used:

\[
\mathcal{L}_{p, \gamma}^{+, r}(\Lambda) = \left\{ z \in \Omega^{1, r}_{p, \gamma}([0, 1], \Lambda) : \langle \dot{z}, \dot{z} \rangle = 0 \text{ a.e. and } z \text{ is future pointing} \right\}.
\] (2.3)

More in general for any event \( p^* \) and any future pointing, injective, timelike curve \( \gamma^* : ]\alpha, \beta[ \rightarrow \Lambda \) we shall use the following notation:

\[
\mathcal{L}_{p^*, \gamma^*}^{+, r}([a, b], \Lambda) = \left\{ z \in \Omega_{p^*, \gamma^*}^{1, r}([a, b], \Lambda) : \langle \dot{z}, \dot{z} \rangle = 0 \text{ a.e., } z \text{ is future pointing, } z(a) = p^* \text{ and } z(b) \in \gamma^*[\alpha, \beta[ \right\}.
\]

We shall denote by the same symbol \( \tau \) the functional defined by \( \gamma^{-1}(z(b)) \) on the space \( \mathcal{L}_{p^*, \gamma^*}^{+, r}([a, b], \Lambda) \).

**Remark 2.1.** Note that the above spaces are not smooth manifold: a tangent spaces is not well defined on the curves \( z \) such that \( \dot{z}(s) = 0 \) on a subset of \([0, 1]\) having positive Lebesgue measure. For this reason if we want to deal with smooth manifolds we need to use an approximation of \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \) by suitable smooth manifolds and to study apriori estimates for the limit process (cf. [9,10]). In this paper we shall work directly on \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \) showing first that assumptions (1)–(6) and pseudocoercivity allow to find smooth minimizers in \( \Lambda \) which are lightlike geodesics. This is a further motivation to choose a shortening procedure for the arrival time functional, since it permits to bypass the nonsmoothness of the spaces defined by (2.3). It will be possible to write Morse Relations using the Poincaré polynomial of \( \mathcal{B}_{p, \gamma}^{+, r}(\Lambda) \) because we shall prove in Sect. 3 that it is homotopically equivalent to \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \).

The next result is a local version of the relativistic Fermat Principle and it is the first step for the shortening procedure.

**Proposition 2.2.** Let \( q \in M \). Then there exists \( \rho(q) > 0 \) having the following property.

For any \( \gamma^* \) timelike injective curve and integral curve of \( Y \) such that:

- \( 0 < d_R(q, Im\gamma^*) \leq \rho(q) \),
- \( Im\gamma^* \cap J^+(q, M) \neq \emptyset \),
- \( Im\gamma^* \setminus J^+(q, M) \neq \emptyset \),
- \( \text{there exists a unique future pointing light-like geodesic joining } q \text{ and } \gamma^* \text{ and minimizing the arrival time on } \mathcal{L}_{q, \gamma^*}^{+, r}(M) \).

The proof can be e.g. obtained as a limit process of timelike problems using the results of [11]. Anyway here we shall give a variational proof working directly in the lightlike case.
The proof is quite different from the classical geometrical one (cf. [13]). The following
Remarks will be used for the proof of Proposition 2.2 and other results in the present paper.

**Remark 2.3.** For any \( z \in \Lambda \) there exists a neighborhood \( U_z \) of \( z \) and a coordinate system
\( \varphi = (x_1, \ldots, x_{N-1}, t) \) \( (N = \text{dim} \Lambda) \) on \( U_z \) such that \( W = \frac{\partial}{\partial t} \) and \( U_z = \Sigma \times [a, b] \) where \( \Sigma \) is a spacelike hypersurface parameterized by \( x_1, \ldots, x_{N-1} \). Moreover, in the coordinate \( x = (x_1, \ldots, x_{N-1}) \) and \( t \in [a, b] \), the metric \( g \) is given by
\[
g(x, t)[(\xi, \theta), (\xi, \theta)] = \langle \alpha(x, t) \xi, \xi \rangle_0 + 2\langle \delta(x, t), \xi \rangle_0 \theta - \beta(x, t) \theta^2,
\]
where \( \langle \cdot, \cdot \rangle \in T_{(x, \frac{a+b}{2})} \Sigma \times \mathbb{R}, \langle \cdot, \cdot \rangle_0 \) is the restriction of \( g \) to \( \Sigma \), \( \alpha \) is a smooth, symmetric positive definite operator, \( \delta \) is a smooth vector field on \( \Sigma \) and \( \beta \) is a smooth positive real function. Note that \( \langle \cdot, \cdot \rangle_0 \) is a Riemannian metric on \( \Sigma \). Indeed it is sufficient to choose \( \Sigma = \{ (x, t) : t = \frac{a+b}{2} \} \), \( \alpha(x, t), \delta(x, t) \) and \( \beta(x, t) \) such that
\[
\langle \alpha(x, t) \xi_1, \xi_2 \rangle_0 = g(x, t)[\xi_1, \xi_2] \text{ for any } \xi_1, \xi_2 \in T_{(x, \frac{a+b}{2})} \Sigma,
\]
and \( \beta(x, t) = -g(x, t)[W, W] \).

**Remark 2.4.** We will assume henceforth that \( W \) is renormalized in such a way that:
\[
\langle W(z), W(z) \rangle = -1 \text{ for any } z \in M.
\]
In particular, \( \langle \gamma, \dot{\gamma} \rangle \equiv -1 \), and so the parameter of \( \gamma \) can be interpreted as proper time. Therefore, in the coordinate systems \( (x_1, \ldots, x_{N-1}, t) \) with \( \frac{\partial}{\partial t} = W \) (cf. Remark 2.3), \( z = (x, t) \in L^+_{p, \gamma}(\Lambda) \) if and only if
\[
\dot{t} = \langle \delta, \dot{x} \rangle_0 + \sqrt{\langle \delta, \dot{x} \rangle_0^2 + \langle \alpha \dot{x}, \dot{x} \rangle_0} \tag{2.4}
\]
because
\[
\beta(x, t) = -\langle W(z), W(z) \rangle \equiv 1.
\]
Moreover, in such a coordinate system, any integral curve of \( W \) can be written as
\[
s \mapsto (\bar{x}, s),
\]
for some \( \bar{x} \in \Sigma \).

In order to prove Proposition 2.2, the following preliminary results are needed.

**Lemma 2.5.** Under the assumptions of Proposition 2.2, if \( p(q) \) is sufficiently small, there exists a minimizer of \( \tau \) on \( L^+_{p, \gamma}(M) \) having \( H^{1, \infty} \) regularity.
Proof. By Remarks 2.3–2.4 and the assumptions of Proposition (2.2), if \( \rho(q) \) is sufficiently small, we can consider a sufficiently small neighborhood \( U \) of \( q \) such that \( U = V \times I \), where \( V \) is an open neighborhood contained in \( \Sigma \), \( I = [-\lambda_0, \lambda_0[ \), \( q = (q_0, 0) \in V \times I \), the curve \( \gamma_s(t) = (q_s, t) \) is defined in \( [-\lambda_0, \lambda_0[ \) and \( d_R(q_0, q_s) \to 0 \) whenever \( d_R(q_0, \text{Im} \gamma_s) \to 0 \). If \( z \in L_{p, \gamma}^+ \) and takes its values in \( U \), then \( z(s) = (x(s), t(s)) \), \( x(0) = q_0 \), \( x(1) = q_s \) and \( t(s) \) satisfies the Cauchy problem

\[
\begin{aligned}
\dot{t} &= \langle \delta(x, t), \dot{x} \rangle_0 + \sqrt{\langle \alpha(x, t) \dot{x}, \dot{x} \rangle_0 + \langle \delta(x, t), \dot{x} \rangle_0^2} \\
\dot{t}(0) &= 0 
\end{aligned}
\] (2.5)

Moreover

\[
\tau(z) = t_x(1) = \int_0^1 \langle \delta(x, t_x), \dot{x} \rangle_0 ds + \int_0^1 \sqrt{\langle \alpha(x, t_x) \dot{x}, \dot{x} \rangle_0 + \langle \delta(x, t_x), \dot{x} \rangle_0^2} ds,
\]

where \( t_x \) is the solution of the Cauchy problem above. Using a minimal Riemannian geodesic between \( q_0 \) and \( q_s \) in \( V \) shows that

\[
\inf_{\mathcal{L}_{q, \gamma}^+} \tau \to 0 \quad \text{as } d_R(q, q_s) \to 0.
\]

We show now the existence of a minimizer for \( \tau \). Consider a minimizing sequence \( (z_m) \) for \( \tau \) and set \( z_m = (x_m, t_m) \). The coercivity of \( \alpha \) shows that the Riemann length of \( x_m \) is bounded. Then by (2.5) we deduce that \( \dot{t}_m \) is bounded in \( L^1 \).

Let \( \varphi_m \) be the solution of

\[
\begin{aligned}
\dot{\varphi}_m &= -\langle z_m(\varphi_m), W(z_m(\varphi_m)) \rangle + 1 \\
\varphi_m(0) &= 0
\end{aligned}
\]

and set

\[
\psi_m(s) = \varphi_m(\lambda_m s),
\]

where \( \lambda_m \) is such that \( \varphi_m(\lambda_m) = 1 \). Such a number exists since \( \varphi_m > 0 \) and we can assume that \( z_m \) is in \( L^\infty \) (choosing a minimizing sequence of curves of class \( C^1 \)), so that \( \varphi_m \) is far from 0.

Then

\[
\dot{\psi}_m = \lambda_m \dot{\varphi}_m(\lambda_m s) = \frac{\lambda_m}{-\langle z_m(\psi_m), W(z_m(\psi_m)) \rangle + 1}
\]

hence

\[
\lambda_m = \dot{\psi}_m [-\langle z_m(\psi_m), W(z_m(\psi_m)) \rangle + 1]
\]

and integrating over \( [a, b] \) gives

\[
\int_a^b \dot{\psi}_m [-\langle z_m(\psi_m), W(z_m(\psi_m)) \rangle + 1] ds = \lambda_m (b - a).
\]
Since
\[ \int_a^b |\dot{z}_m| \, ds \] is uniformly bounded,
there exists a positive constant $M$ independent on $m \in \mathbb{N}$ such that
\[ \left| \int_a^b \langle \dot{z}_m, W(z_m) \rangle \, ds \right| \leq M. \]

Therefore the sequence $(\lambda_m)_{m \in \mathbb{N}}$ is bounded. Now set $y_m = z_m(\psi_m)$. Then
\[ \langle \dot{y}_m, W(y_m) \rangle = \psi_m \langle \dot{z}_m, W(z_m) \rangle = -\frac{\lambda_m}{-\langle \dot{z}_m, W(z_m) \rangle + 1} \langle \dot{z}_m, W(z_m) \rangle, \]
and we deduce the existence of a positive constant $c_1$ such that
\[ |\langle \dot{y}_m, W(y_m) \rangle| \leq c_1, \]
from which we deduce the existence of a positive constant $c_2$ such that
\[ \|\dot{y}_m\|_{L^\infty} \leq c_2. \]

So we have a sequence of curves $z_m = (x_m, t_m)$ such that, up to a reparameterization (and replacing $y_m$ with $z_m$),

$(x_m, t_m)$ uniformly converges to a curve $z = (x, t)$;
the sequence $(\dot{x}_m)$ is equibounded in $L^\infty$;
the sequence $(x_m)$ weakly converges to $\dot{x}$ in $L^2$;

In particular the sequence $(\dot{x}_m)$ weakly converges to $\dot{x}$ in $L^1$ and then by well known properties of the weak convergence (cf. [3] for instance),
\[ \lim_{m \to \infty} \int_0^1 \sqrt{\langle \alpha(x, t)\dot{x}_m, \dot{x}_m \rangle_0 + \langle \delta(x, t)\dot{x}_m, \dot{x}_m \rangle_0^2} \, ds \leq \lim_{m \to \infty} \int_0^1 \sqrt{\langle \alpha(x, t)\dot{x}_m, \dot{x}_m \rangle_0 + \langle \delta(x, t)\dot{x}_m, \dot{x}_m \rangle_0^2} \, ds. \]

Moreover, we clearly have
\[ \left| \int_a^b \left( \sqrt{\langle \alpha(x, t)\dot{x}_m, \dot{x}_m \rangle_0 + \langle \delta(x, t)\dot{x}_m, \dot{x}_m \rangle_0^2} - \sqrt{\langle \alpha(x_m, t_m)\dot{x}_m, \dot{x}_m \rangle_0 + \langle \delta(x_m, t_m)\dot{x}_m, \dot{x}_m \rangle_0^2} \right) \, ds \right| \leq \int_a^b (|\langle (\alpha(x_m, t_m) - \alpha(x, t))\dot{x}_m, \dot{x}_m \rangle_0| + |\langle \delta(x_m, t_m) + \delta(x, t), \dot{x}_m \rangle_0 \langle \delta(x_m, t_m) - \delta(x, t), \dot{x}_m \rangle_0 \rangle^{1/2} \, ds 
\[ \longrightarrow 0 \text{ as } m \to +\infty. \]
Finally, by the uniform convergence of \((x_m, t_m)\) to \((x, t)\),
\[
\int_a^b \langle \delta(x_m, t_m), \dot{x}_m \rangle_0 \to \int_a^b \langle \delta(x, t), \dot{x} \rangle_0.
\]
It follows that for any \(s_1 < s_2\),
\[
\liminf_{m \to \infty} t_m(s_2) - t_m(s_1) = \liminf_{m \to \infty} \left( \int_{s_1}^{s_2} \langle \delta(x_m, t_m), \dot{x}_m \rangle_0 ds + \int_{s_1}^{s_2} \sqrt{\langle \alpha(x_m, t_m)\dot{x}_m, \dot{x}_m \rangle_0 + \langle \delta(x_m, t_m), \dot{x}_m \rangle^2_0} ds \right)
\geq \left( \int_{s_1}^{s_2} \langle \delta(x, t), \dot{x} \rangle_0 ds + \int_{s_1}^{s_2} \sqrt{\langle \alpha(x, t)\dot{x}, \dot{x} \rangle_0 + \langle \delta(x, t), \dot{x} \rangle^2_0} ds \right).
\]
So we have obtained:
\[
\begin{align*}
\dot{t} &\geq \langle \delta(x, t), \dot{x} \rangle_0 + \sqrt{\langle \alpha(x, t)\dot{x}, \dot{x} \rangle_0 + \langle \delta(x, t), \dot{x} \rangle^2_0} \quad \text{and} \\
t(0) &= 0,
\end{align*}
\]
Let \(t_x\) be the solution of the Cauchy problem assigned above relatively to the curve \(x\). Comparison theorems for ordinary differential equations show that \(t_x \leq t\). Hence, \((x, t_x)\) is a minimizer for \(\tau\).

Finally, our (reparameterized) minimizing sequence \(z_m\) satisfies:
\[
\begin{align*}
\langle \dot{z}_m, W(z_m) \rangle &\text{ is bounded and} \\
\dot{z}_m &\text{ is weakly convergent to } \dot{z} \text{ in } L^2.
\end{align*}
\]
Therefore \(\langle \dot{z}, W(z) \rangle\) is bounded and the minimizer \(z\) is therefore in \(H^{1,\infty}\).

**Lemma 2.6.** Assume \(q \notin \gamma^\ast(\alpha, \beta]\). Let \(z\) be a minimizer of \(\tau\) on \(L^{1,\infty}_{q, \gamma^\ast}(\mathcal{M})\) such that \(z \in H^{1,\infty}([0,1], \mathcal{M})\). Then there exists a curve \(y \in L^{1,\infty}_{q, \gamma^\ast}\) such that \(y\) minimizes \(\tau\) and
\[
\inf \left\{ \|\dot{y}(s)\|_R : s \in [0,1] \setminus N \right\} > 0,
\]
where \(N\) is a subset of \([0,1]\) having null Lebesgue measure. Moreover,
\[
y([0,1]) = z([0,1]).
\]

**Proof.** For any \(\delta > 0\), let \(\varphi_\delta\) be the solution of the Cauchy problem
\[
\begin{align*}
\dot{\varphi}_\delta &= -\langle \dot{z}(\varphi_\delta), W(z(\varphi_\delta)) \rangle + \delta \\
\varphi_\delta(0) &= 0.
\end{align*}
\]

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Moreover, put $\psi_\delta(s) = \varphi_\delta(\lambda_\delta s)$, where $\lambda_\delta$ is such that $\varphi_\delta(\lambda_\delta) = 1$. Such a number exists since $\dot{\varphi}_\delta > 0$ and it is far from zero because $\langle \dot{z}, W(z) \rangle$ is in $L^\infty$.

Now

$$\dot{\psi}_\delta = \frac{\lambda_\delta}{-\langle \dot{z}(\psi_\delta), W(z(\psi_\delta)) \rangle + \delta}.$$ 

Then, setting $y = z(\psi_\delta)$, we have:

$$\langle \dot{y}, W(y) \rangle = \dot{\psi} \langle \dot{z}, W \rangle = \frac{\lambda_\delta}{-\langle \dot{z}(\psi_\delta), W(z(\psi_\delta)) \rangle + \delta} \langle \dot{z}, W \rangle,$$

while

$$\lambda_\delta = (-\langle \dot{z}(\psi_\delta), W(z(\psi_\delta)) \rangle + \delta) \ddot{\psi}_\delta = \int_0^1 (-\langle \dot{z}(\psi_\delta), W(z(\psi_\delta)) \rangle + \delta) \dot{\psi}_\delta ds \leq c_0$$

where $c_0$ is a positive constant, because $\dot{z}$ is in $L^\infty$.

Note that

$$\lambda_\delta = \int_0^1 -\langle \dot{z}, W(z) \rangle ds + \delta \rightarrow \int_0^1 -\langle \dot{z}, W(z) \rangle ds \equiv \lambda_0 \quad \text{as } \delta \rightarrow 0.$$ 

Since $y_\delta = z(\psi_\delta)$ is equibounded in $L^\infty$ (because $\dot{\psi}_\delta$ is equibounded and $\psi_\delta(0) = 0$ we have (unless to consider $\delta_m \rightarrow 0$) the existence of $y_0$ such that:

$$y_\delta \rightarrow y_0 \quad \text{uniformly and } \dot{y}_\delta \rightarrow \dot{y}_0 \quad \text{weakly in } L^2.$$ 

Setting $y_0 = (x_0, t_0)$, arguing as in the proof of Lemma 2.5 we see that $t_0 = t_*$ where $t_*$ is the solution of (2.5) with $x$ replaced by $x_0$. Then $\langle \dot{y}_0, \dot{y}_0 \rangle = 0$ almost everywhere. Moreover, since $\dot{y}_\delta \rightarrow \dot{y}_0$ weakly in $L^2$ we have

$$\langle \dot{y}_\delta, W(y_\delta) \rightarrow \langle \dot{y}_0, W(y_0) \rangle \quad \text{weakly in } L^2,$$

while

$$-\langle \dot{y}_\delta, W(y_\delta) \rangle \geq 0 \quad a.e.$$ 

and, by (2.7)

$$-\langle \dot{y}_\delta, W(y_\delta) \rangle \leq \lambda_\delta \quad a.e.$$ 

Then

$$0 \leq -\langle \dot{y}_0, W(y_0) \rangle \leq \lambda_0 \quad a.e.$$ 

where $\lambda_0 = \lim_{\delta \rightarrow 0} \lambda_\delta$. But $\lambda_0 = -\int_0^1 \langle \dot{y}_0, W(y_0) \rangle$, hence $\langle \dot{y}_0, W(y_0) \rangle = \lambda_0$ a.e. This implies that $\lambda_0 \neq 0$ (because $q \notin \gamma_\ast([\alpha, \beta])$) and therefore (2.6) follows with $y = y_0$. Finally, by the definition of $y_\delta$ and the continuity of $y$ we deduce that $y$ and $z$ have the same image. \qed

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Lemma 2.7. Let \( z \) be a curve in \( \mathcal{L}^{+,∞}_{p,γ} \) satisfying (2.6). Then there exists a neighborhood \( \mathcal{V} \) of \( z \) in \( H^{1,∞}([0,1],Λ) \) such that \( \mathcal{V} \cap \mathcal{L}^{+,∞}_{p,γ} \) is a \( C^1 \)–manifold and for any \( z \in \mathcal{V} \) its tangent space is given by

\[
T_z(\mathcal{L}^{+,∞}_{p,γ}) = \left\{ \zeta \in H^{1,∞}([0,1],TΛ) : \zeta(s) \in T_z(\zeta(s))Λ \text{ for any } s, \right. \\
\zeta(0) = 0, \, \zeta(1) \parallel \dot{γ}(z) = 0 \text{ a.e.} \}
\]

Here \( TΛ \) denotes the tangent bundle of \( Λ \) and \( D_s \) the covariant derivative along \( z \).

Proof. Consider the map \( ψ : \Omega^{1,∞}_{p,γ}(Λ) \to L^{∞}([0,1],\mathbb{R}) \) such that

\[
ψ(z) = \sqrt{2}(\dot{z},W(z)) + \sqrt{\dot{z} \cdot \dot{z} + 2(\dot{z},W(z))^2} = \sqrt{2}(\dot{z},W(z)) + \sqrt{(\dot{z} \cdot \dot{z})R}.
\]

Note that \( ψ^{-1}(0) = \mathcal{L}^{+,∞}_{p,γ} \). The set \( \Omega^{1,∞}_{p,γ}(Λ) \) is a manifold and, for any \( z \in \Omega^{1,∞}_{p,γ} \) its tangent space is

\[
T_z\Omega^{1,∞}_{p,γ} = \left\{ \zeta \in H^{1,∞}([0,1],TΛ) : \zeta(s) \in T_z(\zeta(s))Λ \text{ for any } s \in [0,1], \right. \\
\zeta(0) = 0, \, \zeta(1) \parallel \dot{ζ}(γ(z)) \}
\]

By the above formula we immediately deduce that \(ψ\) is of class \( C^1 \) in a neighborhood of \( z \). We claim that, \( ∀z \in \mathcal{V} \cap \mathcal{L}^{+,∞}_{p,γ} \), the differential \( dψ(z)[·] \) is surjective.

Indeed let \( U_z \) be the parallel transport of \( \dot{γ}(τ(z)) \) along \( z \), i.e. the solution of the Cauchy problem

\[
\begin{align*}
D_zU_z &= 0 \\
U_z(1) &= \dot{γ}(τ(z))
\end{align*}
\]

Then for any \( φ \in L^{∞}([0,1],\mathbb{R}) \) it is easy to show the existence of \( λ \in H^{1,∞}([0,1],\mathbb{R}) \) such that \( λ(0) = 0 \),

\[
dψ(z)[λU_z] = φ
\]

and the kernel of \( dψ(z) \) splits. Then \( \mathcal{V} \cap ψ^{-1}(0) \) is a manifold whose tangent space at \( z \) is the kernel of \( dψ(z) \) in \( T_z\Omega^{1,∞}_{p,γ} \).

Remark 2.8. The functional \( τ \) is differentiable on \( \Omega^{1,∞}_{p,γ} \). Since \( τ \) is defined by setting \( γ(τ(z)) = z(1) \), its differential satisfies

\[
\dot{γ}(τ(z))dτ(z)[ζ] = \zeta(1)
\]

and therefore

\[
dτ(z)[ζ] = 0,
\]

if and only if

\[
\zeta(1) = 0 \text{ for any } ζ \in T_z(\mathcal{L}^{+,∞}_{p,γ}).
\]
Now let $V$ be a smooth vector field along $z$ such that $V(0) = V(1) = 0$ and $U_z$ the parallel transport of $\gamma(\tau(z))$ along $z$. If $z$ satisfies (2.6) put
\[
\lambda(s) = \int_0^s \langle D_r V, \frac{\dot{z}}{\langle U_z, \dot{z} \rangle} \rangle dr.
\]
Then $V - \lambda U_z \in T_z(\mathcal{L}_{p,\gamma}^+)$ and, if $z$ is a critical point of $\tau$ on $\mathcal{L}_{p,\gamma}^+$ it is
\[
\tau(z)[V - \lambda U_z] = 0 \text{ for any } V
\]
and therefore
\[
0 = \lambda(1) = \int_0^1 \langle D_s V, \frac{\dot{z}}{\langle U_z, \dot{z} \rangle} \rangle ds \text{ for any } V.
\]
Conversely if $z$ satisfies the above condition, $\zeta(1) = 0$ for any $V$, hence $z$ is a critical point of $\tau$.

The following Theorem is the Relativistic Fermat principle proved in $H^{1,\infty}$ (cf. also [21]).

**Theorem 2.9 (Fermat principle)** Let $z \in \mathcal{L}_{p,\gamma}^{+\infty}$ such that (2.6) is satisfied. Then, $z$ is a critical point of $\tau$ if and only if it is a reparameterization of a $C^2$-geodesic in $\mathcal{L}_{p,\gamma}^{+\infty}$.

**Proof.** If $z$ is a critical point of $\tau$ (satisfying (2.6)) it is
\[
\int_0^1 \langle D_s V, \frac{\dot{z}}{\langle U_z, \dot{z} \rangle} \rangle dr = 0
\]
for any $V$ along $z$ such that $V(0) = V(1) = 0$. If $y$ is a reparameterization of $z$ such that $\langle U_w, \dot{w} \rangle$ is constant we deduce $\int_0^1 \langle D_s V, \dot{y} \rangle = 0$ for any $V$ and therefore $y$ is a smooth geodesic. Conversely is $z$ is a reparameterization of a $C^2$-geodesic, it is $D_s (\mu' \dot{z}) = 0$ for some $\mu \in H^{1,\infty}([0,1], \mathbb{R})$. Then $\langle U_z, \mu' \dot{z} \rangle$ is constant, so
\[
\int_0^1 \langle D_s V, \frac{\dot{z}}{\langle U_z, \dot{z} \rangle} \rangle dr = 0
\]
for any $V$ such that $V(0) = V(1) = 0$. Then, by Remark 2.8 we deduce that $z$ is a critical point of $\tau$. \(\square\)

We are finally ready to prove Proposition 2.12.

**Proof of Proposition 2.2.** By Lemmas 2.5–2.6 and Theorem 2.9 there exists a future pointing lightlike geodesic $w$ joining $q$ and $\gamma_s$, minimizing $\tau$ on $\mathcal{L}_{q,s}^+$. The uniqueness of $w$ is a consequence of the local invertibility of the exponential map (cf. [19]).

**Remark 2.10.** Fix $z = \mathcal{L}_{p,\gamma}^{+\infty}(\Lambda)$ and $[a_1, a_2] \subset [0,1]$. Let $\gamma_2$ be the integral curve of $W$ such that $\gamma_2(0) = z(a_2)$. Since it is future pointing, a simple contradiction argument shows
that, whenever \( d_R(z(a_1), z(a_2)) \to 0 \), the infimum of \( \tau \) on \( L^+_{z(a_1), \gamma_2}([a_1, a_2], \Lambda) \) tends to 0, and if \( z(a_1) = z(a_2) \) the infimum is 0.

**Remark 2.11.** Since \( \tau \) and \( L^+_{q, \gamma_2}^{+r}(\Lambda) \) are invariant by reparameterizations, it is clear that there are nonsmooth minimizers. Note that, among the minimizers there are also curves having null derivatives in subsets of \([0,1]\) with positive Lebesgue measure.

Now we shall prove the equivalence between Definition 1.1 and the corresponding definition given in [9,10] (where the pseudocoercivity of \( \tau \) is given in \( L^+_{p, \gamma}^{+r}(\Lambda) \)).

**Lemma 2.12.** Let \( z \in L^+_{p, \gamma}^{+1}(\Lambda) \). Then there exists \( z_n \) in \( B^+_{p, \gamma}(\Lambda) \) such that \( z_n \to z \) uniformly.

**Proof.** We apply can Proposition 2.2 to obtain the existence of a minimizer in the space \( L^+_{z(a_1), \gamma_2}([a_1, a_2], \Lambda) \) where \( \gamma_2 \) is the integral curve of \( w \) such that \( \gamma_2(0) = z_2 \). Since \( z \) is fixed, if \( a_2 - a_1 \) is sufficiently small, \( \int_{a_1}^{a_2} \sqrt{\langle \dot{z}, \dot{z} \rangle_R} \, ds \) is small and also the length (with respect to the Riemann structure \((2.1)\)) of the geodesic minimizing \( \tau \) is small. Then, choosing a suitable partition of the interval \([0,1]\) allows to construct a broken geodesic \( \hat{z} \) such that the distance between \( z \) and \( \hat{z} \) with respect to the \( H^{1,1} \)-norm is arbitrarily small. Therefore, the uniform distance can be made as small as we want and we are done. 

By Lemma 2.12 it follows immediately the following

**Proposition 2.13.** For any \( r \in [1, +\infty] \), \( \tau \) is pseudo coercive on \( L^+_{p, \gamma}^{+r}(\Lambda) \) if and only if it is pseudo coercive on \( B^+_{p, \gamma}(\Lambda) \).

For any \( z \in H^{1,1}([0,1], \Lambda) \) denote by \( \ell(z) \) its length induced by the Riemann structure \((2.1)\).

**Lemma 2.14.** Assume \( \tau \) pseudocoercive on \( B^+_{p, \gamma}(\Lambda) \). Then, for any \( c \in [\alpha, \beta[ \) there exists \( D(c) > 0 \) such that

\[
\tau(z) \leq c \Rightarrow \ell(z) \leq D(c) \text{ for any } z \in L^+_{p, \gamma}^{+r}(\Lambda).
\]

**Proof.** Assume by contradiction the existence of a sequence \( z_n \) in \( L^+_{p, \gamma}^{+r}(\Lambda) \) such that \( \tau(z_n) \leq c \in [\alpha, \beta[ \) and \( \ell(z_n) \to +\infty \). Since \( \tau \) and \( \ell \) are invariant by reparameterizations, by the pseudocoercivity of \( \tau \) (and Proposition 2.13) we can assume that \( z_n \) has a subsequence (that we shall continue to denote by \( z_n \)) uniformly convergent to a continuous curve \( z : [0,1] \to \bar{\Lambda} \). Since \( z([0,1]) \) is compact, it can be covered by a finite number of open neighborhoods \( U_i \) in \( \mathcal{M} \) such that, on any \( U_i \), the metric \( g \) has the form

\[
ds^2 = \langle \alpha(x,t)\xi, \xi \rangle_0 + 2\langle \delta(x,t), \xi \rangle_0 \theta - \theta^2
\]
(cf. Remarks 2.3 and 2.4). If \( |a_n^i, b_n^i| \) is an interval such that \( z_n(s) \in U_i \) for any \( s \in |a_n^i, b_n^i| \) and \( z_n = (x_n, t_n) \) we have

\[
t_n(b_n^i) - t_n(a_n^i) = \int_{a_n^i}^{b_n^i} t_n ds = \int_{a_n^i}^{b_n^i} \left( (\delta, \dot{x}_n)_0 + \sqrt{(\delta, \dot{x}_n^2)_0 + (\alpha x_n, \dot{x}_n)_0} \right) ds.
\]

Since any \( \bar{U}_i \) is compact and the number of \( U_i \) is finite we deduce that \( \int_{a_n^i}^{b_n^i} \sqrt{(\dot{x}_n, \dot{x}_n)_0} ds \) (and consequently \( \int_{a_n^i}^{b_n^i} |\dot{t}_n| ds \)) is bounded independently by \( n \) and \( i \). Finally, since any \( z_n \) is included in \( \bigcup_i U_i \) and \( \dot{z}_n = (\dot{x}_n, \dot{t}_n) \) on any \( U_i \), we deduce that \( \int_{a_n^i}^{b_n^i} \|\dot{z}_n\|_R ds \) is bounded independently by \( n \) and \( i \). Then \( \ell(z_n) \) is bounded getting a contradiction. \( \square \)

Lemma 2.14 will be used, together with the following proposition, to construct the shortening flow for \( \tau \).

**Proposition 2.15.** Let \( W, \Lambda \) and \( \gamma : \alpha, \beta \rightarrow \Lambda \) be satisfying (1)–(6) of Sect. 1. Assume that \( \tau \) is pseudo coercive on \( B^{p,\gamma}_w(\Lambda) \) and fix \( c \in ]\alpha, \beta[ \). Then there exists \( \rho_\ast(c) > 0 \) satisfying the following property. Let \( z \in \tau^c \cap L^{\gamma,\tau}_w(\Lambda) \), \( [a, b] \subset [0, 1], \ z_1, z_2 \in z([0, 1]) \) with \( z_1 = z(a), \ z_2 = z(b) \) and \( d_R(z_1, z_2) \leq \rho_\ast(c) \). Let \( \gamma_i : \alpha_i, \beta_i^+ \rightarrow \Lambda(i = 1, 2) \) be the maximal integral curve (in \( \Lambda \) ) of

\[
\begin{cases}
\dot{\eta} = W(\eta) \\
\eta(0) = z_i, \ i = 1, 2.
\end{cases}
\]

Moreover, for any \( \dot{z}_1 \in \gamma([\alpha_1, 0]) \) with \( d_R(\dot{z}_1, z_1) \leq \rho_\ast(c) \) there exists a unique future pointing light–like geodesic \( \Gamma \) such that \( \Gamma(a) = \dot{z}_1, \ \Gamma(b) \) is in the image of \( \gamma_2, \ \Gamma(s) \in \Lambda \) for any \( s \in [a, b] \) and

\[
\tau(\Gamma) = \inf \left\{ \tau(y) : y \in L^{\gamma_1, \gamma_2}_w([a, b], \Lambda) \right\}
\]

**Proof.** By pseudo coercivity it is immediate to check the existence of \( K \), compact subset of \( \bar{\Lambda} \), such that

\[ z \in \tau^c \Rightarrow z([0, 1]) \subset K \tag{2.10} \]

Now take a finite family \( U_1, \ldots, U_m \) of open subsets of \( M \) covering \( K \) and such that any \( U_i \) is compact and \( U_i \) satisfies the properties of Remarks 2.3 and 2.4. By Proposition 2.2 and Remark 2.10 if \( \rho_\ast(c) \) is sufficiently small there exists a minimizer \( w \) in \( L^{\gamma_1, \gamma_2}_w([a, b], U_i) \) for some \( i = 1, \ldots, m \). (Note that, by assumption (6), \( w(b) \in \Lambda \). Since \( \bigcup_{i=1}^m \bar{U}_i \) is compact, by the local invertibility of the exponential map (and the minimality of \( \tau(w) \) ) we see that, if \( \rho_\ast(c) \) is sufficiently small, the minimizing geodesic is unique. Then we have just to prove
that the minimizer is included in \( \Lambda \). If \( z_1 = z_2 \) this is obvious. Then suppose that \( z_1 \neq z_2 \).

If \( \rho_*(c) \) is sufficiently small and \( d_R(z_1, z_2) \leq \rho_*(c) \), \( z_1, z_2 \in U_i \cap \Lambda \) (for some \( i \)). Then, using Proposition 2.2 and choosing \( \rho_*(c) \) sufficiently small, we can construct two continuous maps \( \theta_1, \theta_2 : [0, 1] \to U_i \cap \Lambda \) having the following properties:

- for any \( \lambda \in [0, 1] \), \( \theta_2(\lambda) \) is in the future of \( \theta_1(\lambda) \)
- \( \theta_1(0) = \hat{z}_1 \) \( \theta_2(0) = z_2 \)
- \( \theta_1(\lambda) \neq \theta_2(\lambda) \) for any \( \lambda \neq 1 \).
- \( \theta_1(1) = \theta_2(1) \).
- for any \( \lambda \in [0, 1] \) there exists a unique minimizer of \( \tau \) on \( L_{\theta_1(\lambda), \gamma_2(\lambda)}([a, b], U_i) \) where \( \gamma_2(\lambda) \) is the maximal integral curve of \( W \) such that \( \gamma_2(\lambda)(0) = \theta_2(\lambda) \).

Now set

\[
A = \left\{ \lambda \in [0, 1] : \text{the light--like or constant geodesic minimizing} \ \tau \ \text{on} \ L_{\theta_1(\lambda), \gamma_2(\lambda)}([a, b], U_i) \ \text{does not intersect} \ \partial \Lambda \right\}.
\]

Since \( \theta_1(1) = \theta_2(1) \in \Lambda, 1 \in A \). Take

\[
\lambda_0 \equiv \inf A \geq 0.
\]

By the definition of \( \lambda_0 \), there exists \( \lambda_n \to \lambda_0^+ \) and a sequence \( w_n \) of lightlike geodesic minimizing \( \tau \) on \( L_{\theta_1(\lambda_n), \gamma_2(\lambda_n)}([a, b], U_i) \) such that \( w_n([0, 1]) \subset \Lambda \). Unless to consider a subsequence, by (2.4) we obtain the existence of a lightlike geodesic \( w \) such that

\[
w_n \to w \ \text{with respect to the} \ C^2 \ - \text{norm}.
\]

\[
w(a) = \theta_1(\lambda_0), \ w(b) \in \gamma_2^{-1}(\lambda_0) \ (\gamma_2 \subset \Lambda)
\]

\[
w([a, b]) \subset \bar{\Lambda}.
\]

If \( w([a, b]) \subset \Lambda \), then \( \lambda_0 = 0 \) and we are done. If \( w([a, b]) \cap \partial \Lambda \neq \emptyset \), since \( \theta_1(\lambda_0) \neq \theta_2(\lambda_0) \) are in \( \Lambda \) we get a contradiction with the light--convexity of \( \bar{\Lambda} \).

3. HOMOTOPICAL EQUIVALENCE BETWEEN \( L_{\rho, \gamma}^+(\Lambda) \) AND \( B_{\rho, \gamma}^+(\Lambda) \)

In this section (under assumptions (1)--(6)) we shall introduce a shortening flow and we shall use it to prove that \( L_{\rho, \gamma}^+(\Lambda) \) and \( B_{\rho, \gamma}^+(\Lambda) \) are homotopically equivalent for any \( r \in [1, +\infty] \). This flow will also be used to get the deformation of the sublevels of \( \tau \) for the Morse Theory, whenever we are far from lightlike geodesics. Indeed far from lightlike geodesics, \( \tau \) will be strictly decreasing with a speed uniformly far from zero. In other words \( \tau \) will verify the Palais-Smale compactness condition along the flow. To construct the shortening flow we shall use the same ideas in [18] adapting them to our case. Note that
here we can not use the finite dimensional approach nearby critical curves (used in [18] for Riemannian geodesics) because we are not working with fixed endpoints. So the shortening approach will be used only far from geodesics. The shortening procedure can be introduced in the following way. Fix $c > \inf \{ \tau(z), z \in \mathcal{L}^{+,r}_{p,\gamma}(\Lambda) \}$. Consider $D(c)$ as in Lemma 2.14, $\rho_*(c)$ as in Proposition 2.15 and take $N = N(c)$ such that

$$\frac{D(c)}{N} < \rho_*(c).$$

Choose a partition $\{0 = s_0 < s_1 \ldots s_{N-1} < s_N = 1\}$ of $[0,1]$ such that for any $i \in \{1, \ldots N\}$,

$$s_i - s_{i-1} = \frac{1}{N}.$$  

For any $z \in \tau^c \cap \mathcal{L}^{+,r}_{p,\gamma}(\Lambda)$, choose $N+1$ points $z_0, z_1, \ldots, z_N$ on $z([0,1])$ such that $z(0) = p$, $z_N = z(1)$ and $d_R(z_i, z_{i-1}) = l(z)/N$, for any $i \in \{1, \ldots N\}$, where $l(z)$ denotes the length of $z$ with respect to the Riemannian structure (2.1) (see Figure 1). Denote by $\gamma_i$ ($i = 1, \ldots, N$) the maximal integral curve of $W$ such that $\gamma_i(0) = z_i$ (see Figure 2).

Observe that $\gamma_N(s) = (s + \tau(z))$ for all $s$. Let $w_1$ be the lightlike geodesic minimizing $\tau$ on $\mathcal{L}^{+,r}_{p,\gamma}(\{s_0, s_1\}, \Lambda)$ (recall that $z_0 = p$ and $s_0 = 0$), $w_2$ the lightlike geodesic minimizing $\tau$ on $\mathcal{L}^{+,r}_{w_1(s_1),\gamma_2}(\{s_1, s_2\}, \Lambda)$, and so on (see Figure 3). In Figures 3, 4 and 5 the points $w_i(s_i)$ are denoted by $\overline{w}_i$. Note that the number $N$ can be chosen big enough in order that $d_R(w_i(s_i), z_{i+1}) \leq \rho_*(c)$, for any $i = 1, \ldots, N - 1$ and for any $z \in \tau^c$.

**Remark 3.1.** Let $K = K(c)$ be a compact subset of $\bar{\Lambda}$ as in (2.10). By compactness, $K(c)$ can be covered by a finite family $(U_j)$ satisfying Remark 2.3. Moreover, $N$ can be chosen so large that $z([s_{i-1}, s_i])$ and the minimizer of $\tau$ on $\mathcal{L}^{+,r}_{w_{i-1}(s_{i-1}),\gamma_i}(\{s_{i-1}, s_i\}, \Lambda)$ are contained in some $U_j$.

The Lorentzian metric on $U_j$ is described as

$$(\zeta, \zeta) = \langle \alpha_j(x,t)\xi, \xi \rangle_0 + 2\langle \delta_j(x,t), \xi \rangle_0 \theta - \theta^2$$  

(3.1)

(cf. Remark 2.3), where $\alpha_j(x,t)$ is a positive linear operator, $\delta_j(x,t)$ is smooth vector field, $z = (x,t) \in U_j$ and $\zeta = (\xi, \theta) \in T_zM$. With the notation above, for any future pointing curve $z$ with image contained in some $U_j$, the condition $\langle \dot{z}, \dot{z} \rangle = 0$ holds if and only if

$$t = \langle \delta_j(x,t), \dot{x} \rangle_0 + \frac{\sqrt{\langle \alpha_j(x,t), \dot{x} \rangle_0 + \langle \delta_j(x,t), \dot{x} \rangle_0^2}}{2}$$

(3.2)

Moreover, any $\gamma_i$ is an integral curve of $W$, so, in $U_j$, it has the form $s \mapsto (x_j, t_j + s)$, if $z_j = (x_j, t_j)$. Note that $\mathcal{L}^{+,r}_{p,\gamma_1}(\{s_0, s_1\}, \Lambda)$ is nonempty, since it contains the restriction $z|_{[s_0, s_1]}$. Now, using elementary comparison theorems for ordinary differential equations and the metric (3.1) on $U_j$ allow to deduce that also any space $\mathcal{L}^{+,r}_{w_{i-1}(s_{i-1}),\gamma_i,c}(\{s_{i-1}, s_i\}, \Lambda)$ is nonempty for any $i \in \{2, \ldots N\}$.  

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Note also that, if \( \eta_1 \) is the curve defined by setting \( \eta_1([s_{i-1}, s_i]) = w_i \), then \( \tau(\eta_1) \leq \tau(z) \leq c \) (always by comparison theorems in O.D.E.). In particular \( \eta_1([0,1]) \) is contained in \( K(c) \).

**Remark 3.2** A second curve \( \eta_2 \) will be constructed in the following way starting from \( \eta_1 \). On any minimizer \( w_i \) \( (i = 1, \ldots, N) \) consider the point \( m_i \) such that \( d(w_i(s_{i-1}), m_i) = d(m_i, w(s_i)) \).

For \( i = 1, \ldots, N \), we denote by \( \lambda_i \) the maximal integral curve of \( W \) such that \( \lambda_i(0) = m_i \); moreover, we set \( \lambda_{N+1}(s) = \gamma(s + \tau(\eta_1)) \) (see Figure 4).

Consider now the following subdivision of the interval \([0,1]\). Let \( \sigma_0 = 0 \), \( \sigma_1 = \frac{1}{2N} \), \( \sigma_j = \frac{2j-1}{2N} \) for \( j = 2, \ldots, N \), and \( \sigma_{N+1} = 1 \).

Denote by \( u_1 \) the minimizer of \( \tau \) on \( \mathcal{L}_{p, \lambda_1}^+([\sigma_0, \sigma_1], \Lambda) \), by \( u_2 \) the minimizer of \( \tau \) on \( \mathcal{L}_{u_1(\sigma_1), \lambda_2}^+([\sigma_1, \sigma_2], \Lambda) \) and so, inductively, we denote by \( u_j \) the minimizer of \( \tau \) in \( \mathcal{L}_{u_{j-1}(\sigma_{j-1}), \lambda_j, \epsilon}^+([\sigma_{j-1}, \sigma_j], \Lambda) \), \( j = 2, \ldots, N + 1 \). Finally, (see Figure 5) we denote by \( \eta_2 \) the curve such that

\[
\eta_2|_{[\sigma_{j-1}, \sigma_j]} = u_j.
\]

Using again comparison theorems in ordinary differential equations one proves that \( \tau(\eta_2) \leq \tau(\eta_1) \).

The continuous flow \( \eta(\sigma, z) \) can be constructed as follows. Fix \( \sigma \in [0,1] \) and consider for instance the interval \([s_0, s_1]\). We choose \( \eta(\sigma, z)|_{[s_0, s_1]} \) as follows. Set \( p = (x_0, 0) \) and \( \gamma_1(s) = (x_1, t_1+s) \) (in some neighborhood \( U_1 \) as in Remark 3.1). Since \( z(s) = (x(s), t(s)) \), the curve \( x(s) \) joins \( x_0 \) with \( x_1 \).

Let \( y(\sigma) \) be the minimizer of the functional

\[
y \mapsto \int_{s_0}^{s_1} \langle \delta_i(y, t_y), \hat{y}\rangle_0 ds + \int_{s_0}^{s_1} \sqrt{\langle \alpha_i(y, t_y), \hat{y}\rangle_0 + \langle \delta_i(y, t_y), \hat{y}\rangle_0^2} ds
\]

with boundary conditions \( y(0) = x_0 \) and \( y(s_1) = x(\sigma s_1) \), where \( t_y \) is the solution of (3.2) with \( t_y(0) = 0 \) in the interval \([0, \sigma s_1]\). Denote by \( \hat{y}(\sigma) \) the extension of \( y(\sigma) \) to \([s_0, s_1]\) taking \( \hat{y}(s) = x(s) \) for \( s \in [\sigma s_1, s_1] \). Finally, denote by \( \hat{t}_y \) the corresponding solution of (3.2) in the interval \([s_0, s_1]\). The curve \( (\hat{y}(\sigma), \hat{t}_y(\sigma)) \) will be \( \eta(\sigma, z) \) in the interval \([s_0, s_1]\).

In the same way we can construct \( \eta(\sigma, z) \) on the other intervals \([s_{i-1}, s_i]\). Note that, by construction, \( \eta(1, z) = \eta_1 \). Similarly, we can extend the flow \( \eta \) to a map defined on \([0, 2] \times \tau^\epsilon \) in such a way that \( \eta(2, z) = \eta_2 \). Now, we iterate the shortening argument above, replacing the original curve \( z \) with the curve \( \eta_2 \). Successively we apply the above construction, starting from \( \eta_2 \). By induction we obtain a flow \( \eta(\sigma, z) \), defined on \( \mathbb{R}^+ \times \tau^\epsilon \). Since \( \tau(\eta(\sigma, z)) \leq \tau(z) \) for any \( \sigma \) and for any \( z \), using \( \eta \) we immediately deduce
Lemma 3.3. Fix \( r \in [1, +\infty] \). For any \( c \in ]\alpha, \beta[ \), \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \cap \tau^{c} \) is homotopically equivalent to \( \mathcal{B}_{p, \gamma}^{+} \cap \tau^{c} \).

Moreover choosing a suitable continuous map \( \rho_{*}(c) \ (c \in ]\alpha, \beta[ ) \) and arguing as in section 9 of [8] we can also obtain

Proposition 3.4. \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \) is homotopically equivalent to \( \mathcal{B}_{p, \gamma}^{+}(\Lambda) \).

Suppose that \( \tau(\eta_1) = \tau(\eta_2) \) and consider the situation is a single interval \([\sigma_j, \sigma_{j+1}]\). Since \( \tau(\eta_1) = \tau(\eta_2) \) simple comparison theorems in O.D.E. show that \( \eta_1 \) is a minimizer on the interval \([\sigma_j, \sigma_{j+1}]\). Suppose that it consists of two (nonconstant) lightlike geodesics. If it is not a lightlike geodesic, by the above construction it has a discontinuity at \( s_{j+1} = \frac{\sigma_{j+1} + \sigma_j}{2} \). Denote by \( U_{\eta_1} \) the parallel transport of \( \dot{\gamma}(\tau(\eta_1)) \) along the curve \( \eta_1 \). Since \( \eta_1 \) is a minimizer satisfying (2.6), by Lemma 2.7 and Remark 2.8 it is

\[
\int_{\sigma_j}^{\sigma_{j+1}} \frac{\langle Ds, \dot{\eta}_1 \rangle}{\langle U_{\eta_1}, \dot{\eta}_1 \rangle} ds = 0
\]

for any \( C^\infty \)-vector field along \( \eta_1 \) such that \( V(0) = 0, V(1) = 0 \). In particular \( \frac{\dot{\eta}_1}{\langle U_{\eta_1}, \dot{\eta}_1 \rangle} \) is a \( C^1 \) curve. Therefore \( \eta_1(s_{j+1}^{-}) = \eta_1(s_{j+1}^{+}) \), so \( \eta_1 \) is the image of a future pointing lightlike geodesic in the interval \([\sigma_j, \sigma_{j+1}]\).

Then, whenever we are far from lightlike geodesics and there are not intervals where \( \eta_1 \) is a constant, \( \tau(\eta_2) < \tau(\eta_1) \). If \( \eta_1 \) possesses some interval where it is a constant it is possible to construct a ”localized” flow where \( \tau \) is strictly decreasing ignoring such intervals and using the above construction of the flow in a small neighborhood of \( \eta_1 \).

Finally compactness arguments similar to the ones used for the shortening method for Riemannian geodesics (cf. [18]), allows to obtain the analogous of the classical deformation results (cf e.g. [4,17]) for the functional \( \tau \) on \( \mathcal{L}_{p, \gamma}^{+, r}(\Lambda) \).

Since, to obtain Morse Relations, we shall work with respect to the \( H^{1,2} \) structure, we give the statements of the deformation results only for \( r = 2 \).

Proposition 3.5. Let \( c \) be a regular value for \( \tau \) on \( \mathcal{L}_{p, \gamma}^{+, 2} \) (namely \( \tau^{-1}(\{c\}) \) does not contain geodesics).

Then, there exists a positive number \( \delta = \delta(c) \) and a continuous map \( H \in C^0([0,1] \times \tau^{c+\delta}, \tau^{c+\delta}) \), such that:

(a) \( H(0, z) = z \), for every \( z \in \tau^{c+\delta} \);
(b) \( H(1, \tau^{c+\delta}) \subseteq \tau^{c-\delta} \);
(c) \( H(\sigma, z) \in \tau^{c-\delta} \), for any \( \sigma \in [0,1] \) and \( z \in \tau^{c-\delta} \).

Proposition 3.6 Let \( K_c \) be the set of lightlike geodesics on \( \tau^{-1}(\{c\}) \cap \mathcal{L}_{p, \gamma}^{+, 2} \). Then for any open neighborhood \( U \) of \( K_c \), there exists a positive number \( \delta = \delta(U, c) \) and a homotopy \( H \in C^0([0,1] \times \tau^{c+\delta}, \tau^{c+\delta}) \), such that:
(a) $H(0, z) = z$, for any $z \in \tau^{c+\delta}$.
(b) $H(1, \tau^{c+\delta} \setminus U) \subset \tau^{c-\delta}$;
(c) $H(\sigma, z) \in \tau^{c-\delta}$, for every $\sigma \in [0,1]$ and $z \in \tau^{c-\delta}$.

**Remark 3.7.** There are two main differences between the shortening method described above and the classical shortening method for Riemannian geodesics. In our case, we locally minimize a functional which is not given in an integral form. Secondly, we minimize the functional in the space of curves joining a point with a curve, and not two fixed points.

**Remark 3.8.** The flow used in proving Propositions 3.5 and 3.6 are just what we need for a Lusternik–Schnirelmann theory. Then, without using the nondegeneracy assumption of Theorem 1.5 we can obtained the existence of at least $\text{cat}(B^+_{p,\gamma}(\Lambda))$ future pointing light–like geodesics in $B^+_{p,\gamma}(\Lambda)$. (Here $\text{cat}(X)$ denotes the minimal number of contractible subsets of $X$ covering it). Moreover if $\text{cat}(B^+_{p,\gamma}(\Lambda)) = +\infty$ there is a sequence $z_n$ of future pointing light–like geodesics in $B^+_{p,\gamma}(\Lambda)$ such that $\tau(z_n) \to \beta$.

### 4. ON THE BEHAVIOUR OF $\tau$ NEARBY LIGHTLIKE GEODESICS

To develop a Morse Theory we shall use the space $L^+_{\gamma}(\Lambda)$ (that contains $B^+_{\gamma}(\Lambda)$), because $H^{1,2}([0,1],\Lambda)$ is an Hilbert manifold (endowed with its natural metric). Since $L^+_{\gamma}(\Lambda)$ is invariant by reparameterization as well as $\tau$, it will be useful to consider equivalence classes of curves or, better, to single out one parameterization. This will be done on an open neighborhood $N_w$ of $w([0,1])$ for any lightlike geodesic $w$.

Towards this goal consider the parallel vector field $U_w$ along $w$ of $\dot{\gamma}(\tau(w))$ (which is a time-like vector). Since $\gamma$ is an integral curve of $W$, by Remark 3.4 it is $\langle U_w, U_w \rangle \equiv -1$. Now, by the pseudocoercivity of $\tau$ it follows that $w$ does not have self-intersection, so its image is a submanifold and $U_w$ can be extended to a smooth vector field $Y$ on $\Lambda$ such that

$$\langle Y(z), \dot{Y}(z) \rangle = -1 \text{ for any } z \in \Lambda, \quad (4.1)$$

$$D_w(s) Y(w(s)) = 0 \text{ for any } s \in [0,1], \quad (4.2)$$

(cf. [30]). Using the vector field $Y$ we define the following space:

$$Q^+_{\gamma}(\Lambda) = \left\{ z \in L^+_{\gamma}(\Lambda) : \langle Y(z), \dot{z} \rangle = \int_0^1 \langle Y(z), \dot{z} \rangle ds \ \text{a.e.} \right\}. \quad (4.3)$$

Note that by (4.2), the geodesic $w$ is in $Q^+_{\gamma}(\Lambda)$ and

$$\int_0^1 \langle Y(w), \dot{w} \rangle ds < 0.$$

The space $Q^+_{\gamma}(\Lambda)$ will be used to study Morse Theory nearby the critical point $w$ for the functional $\tau$. The first step in this direction is to prove that $Q^+_{\gamma}(\Lambda)$ is a $C^1$-manifold.
Remark 4.1. It is well known that the space \( \Omega^{+,2}_{p,\gamma}(\Lambda) \) defined by (2.2) is a \( C^\infty \)-manifold and its tangent space at any \( z \) is given by

\[
T_z \Omega^{+,2}_{p,\gamma}(\Lambda) = \left\{ \zeta \in H^{1,2}([0,1], \mathcal{T}\Lambda) : 
\zeta(s) \in T_z \Lambda \text{ for any } s, \zeta(0) = 0, \zeta(1) \text{ is parallel to } \dot{\gamma}(\tau(z)) \right\}
\]

where \( \mathcal{T}\Lambda \) denotes the tangent bundle of \( \Lambda \).

Remark 4.2. Consider the map

\[
\phi : \Omega^{+,2}_{p,\gamma}(\Lambda) \rightarrow \left\{ h \in L^2([0,1], \mathbb{R}) : \int_0^1 h \, ds = 0 \right\}
\]

defined as

\[
\phi(z) = \langle Y(z), \dot{z} \rangle - \int_0^1 \langle Y(z), \dot{z} \rangle \, ds. \quad (4.4)
\]

It is a standard computation to prove that \( \phi \) is of class \( C^\infty \) and its differential satisfies:

\[
d\phi(z)[\zeta] = \langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{z} \rangle + 
- \int_0^1 (\langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{z} \rangle) \, ds \quad (4.5)
\]

where \( D \) is the Levi-Civita connection relatively to the Lorentzian structure \( g \).

Proposition 4.3. The space

\[
\mathcal{P}^{+,2}_{p,\gamma}(\Lambda) = \left\{ z \in \Omega^{+,2}_{p,\gamma}(\Lambda) : \langle Y(z), \dot{z} \rangle = \int_0^1 \langle Y(z), \dot{z} \rangle \, ds < 0 \right\}, \quad (4.6)
\]

is a manifold whose tangent space is given by

\[
T_z \mathcal{P}^{+,2}_{p,\gamma}(\Lambda) = \left\{ \zeta \in T_z \Omega^{+,2}_{p,\gamma}(\Lambda) : \langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{z} \rangle = 
\int_0^1 (\langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{z} \rangle) \, ds \right\} \quad (4.7)
\]

Proof. Consider the map \( \phi \) defined by (4.4). By (4.5) and the Implicit Function Theorem it is sufficient to prove that for any \( h \in L^2([0,1], \mathbb{R}) \) such that \( \int_0^1 h \, ds = 0 \) there exists \( \zeta \in T_z \Omega^{+,2}_{p,\gamma}(\Lambda) \) such that

\[
\langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{z} \rangle = h. \quad (4.8)
\]

Choose \( \zeta = \mu Y \) with \( \mu(0) = 0 \). Then \( \zeta \) satisfies (4.8) if and only if \( \mu \) satisfies the Cauchy problem

\[
\begin{cases}
-\dot{\mu} + \mu(D_Y Y, \dot{z}) = h, \\
\mu(0) = 0,
\end{cases}
\]

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because, by (4.1), \( \langle Y, Y \rangle \equiv -1 \). Such a problem has a (unique) solution in \( H^{1,2}([0, 1], \mathbb{R}) \) and we are done. \( \square \)

**Remark 4.4.** For any \( z \in \mathcal{P}^{+,2}_p(\Lambda) \), \( \| \dot{z} \|_R^2 \) is uniformly far from 0. (Here \( \| \cdot \|_R \) is the norm induced by the Riemann structure (2.1)). Indeed if \( z \in \mathcal{P}^{+,2}_p(\Lambda) \) it is

\[
0 < -\int_0^1 \langle Y(z), \dot{z} \rangle ds = -\langle Y(z), \dot{z} \rangle \leq \| \langle Y(z), \dot{z} \rangle \|_R \leq \| Y(z) \|_R \| \dot{z} \|_R.
\]

**Lemma 4.5.** Let \( \psi: \mathcal{P}^{+,2}_p(\Lambda) \rightarrow L^2([0, 1], \mathbb{R}) \) defined as

\[
\psi(z) = \sqrt{2} \langle \dot{z}, W(z) \rangle + \sqrt{\langle \dot{z}, \dot{z} \rangle_R} \tag{4.9}
\]

Then \( \psi \) is of class \( C^1 \) and, for any \( \zeta \in \mathcal{P}^{+,2}_p(\Lambda) \),

\[
\frac{1}{\sqrt{\langle \dot{z}, \dot{z} \rangle_R}} (\langle D_s \zeta, \dot{z} \rangle + 2 \langle \zeta, W(z) \rangle (\langle D_s \zeta, W(z) \rangle + \langle D \zeta W, \dot{z} \rangle)) \tag{4.10}
\]

Note that, by Remark 4.4, since \( \zeta \in H^{1,2} \) it is \( d\psi(z)[\zeta] \in L^2 \).

**Proof.** Standard computations show that the differential of the map

\[
\psi_1(z) = \sqrt{2} \langle \dot{z}, W(z) \rangle
\]

along the direction \( \zeta \) is given by

\[
d\psi_1(z)[\zeta] = \sqrt{2} (\langle D_s \zeta, W(z) \rangle + \langle D \zeta W, \dot{z} \rangle).
\]

To evaluate the differential of the map

\[
\psi_2(z) = \sqrt{\langle \dot{z}, \dot{z} \rangle_R}
\]

(at an instant \( s_0 \)) we can assume that (in a neighborhood of \( z(s_0) \) we are in \( \mathbb{R}^n \) and

\[
\langle \zeta, \zeta \rangle_R = \langle L(z)[\zeta], \zeta \rangle_E
\]

where \( \langle \cdot, \cdot \rangle_E \) is the Euclidean scalar product of \( \mathbb{R}^n \) and \( L(z) \) is a smooth positive definite linear operator. Using such a position the vector fields in the tangent space at any curve \( z \) (on a suitable interval \( [s_0 - \delta, s_0 + \delta] \) ) will be \( H^{1,2} \)-vector fields (defined on \( [s_0 - \delta, s_0 + \delta] \)) with values in \( \mathbb{R}^n \).

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Suppose that $\zeta$ is of class $C^1$. Then, by Remark 4.4, $\dot{z} + \lambda \dot{\zeta}$ is uniformly far from 0 for any $\lambda$ sufficiently small, and

$$d\psi_2(z)[\zeta] = \lim_{\lambda \to 0} \frac{1}{\lambda} (\langle L(z + \lambda \zeta)[\dot{z} + \lambda \dot{\zeta}], \dot{z} + \lambda \dot{\zeta} \rangle_E - \sqrt{\langle L(z)[\dot{z}], \dot{z} \rangle_E}).$$

Therefore there exists $\theta = \theta(\lambda, s) \in [0, 1]$ such that

$$d\psi_2(z)[\zeta] = \lim_{\lambda \to 0} \frac{\langle dL(z + \lambda \theta \zeta)[\zeta], \dot{z} \rangle_E + 2\langle L(z + \lambda \theta \zeta)[\dot{\zeta}], \dot{z} + \lambda \dot{\zeta} \rangle_E}{2\sqrt{\langle L(z + \lambda \theta \zeta)[\dot{z}], \dot{z} \rangle_E}}$$

where the limit is done with respect to the $L^2$ norm.

Then, by the Lebesgue Convergence Theorem we obtain

$$d\psi_2(z)[\zeta] = \frac{\langle dL(z)[\dot{z}], \dot{z} \rangle_E + 2\langle L(z)[\dot{\zeta}], \dot{z} \rangle_E}{2\sqrt{\langle L(z)[\dot{z}], \dot{z} \rangle_E}}$$

which is a map in $L^2([s_0 - \delta, s_0 + \delta])$ by Remark 4.4.

Since

$$\langle L(z)[\dot{z}], \dot{z} \rangle_E = \langle \dot{z}, \dot{z} \rangle_R = \langle \dot{z}, \dot{z} \rangle + 2\langle W(z), \dot{z} \rangle^2$$

and

$$\langle dL(z)[\dot{z}], \dot{z} \rangle_E + 2\langle L(z)[\dot{\zeta}], \dot{z} \rangle_E = d(\langle L(z)[\dot{z}], \dot{z} \rangle_E)[\zeta],$$

we obtain

$$d\psi_2(z)[\zeta] = \frac{1}{\sqrt{\langle \dot{z}, \dot{z} \rangle_R}} \langle (D_s \zeta, \dot{z}) + 2(\dot{z}, W(z))(\langle D_s \zeta, W(z) \rangle + \langle D_s W, \dot{z} \rangle \rangle).$$  \hspace{1cm} (4.11)

Now consider $\zeta \in H^{1,2}([s_0 - \delta, s_0 + \delta], \mathbb{R}^n)$ and $\zeta_1$ of class $C^1$. Then (in local coordinates) for any $\lambda$ sufficiently small with respect to $\|\dot{\zeta}_1\|_{L^\infty}$, there exist $C_1, C_2 > 0$ such that

$$\left| \frac{\psi_2(z + \lambda \zeta_1) - \psi_2(z + \lambda \zeta)}{\lambda} \right| = \frac{1}{\lambda} \left| \frac{\langle L(z + \lambda \zeta_1)[\dot{z} + \lambda \dot{\zeta}_1], \dot{z} + \lambda \dot{\zeta}_1 \rangle_E - \langle L(z + \lambda \zeta)[\dot{z} + \lambda \dot{\zeta}], \dot{z} + \lambda \dot{\zeta} \rangle_E}{\sqrt{\langle L(z + \lambda \zeta_1)[\dot{z} + \lambda \dot{\zeta}_1], \dot{z} + \lambda \dot{\zeta}_1 \rangle_E + \sqrt{\langle L(z + \lambda \zeta)[\dot{z} + \lambda \dot{\zeta}], \dot{z} + \lambda \dot{\zeta} \rangle_E}} \right|$$

$$\leq \frac{C_1 \|\zeta - \zeta_1\|_{L^\infty}}{\sqrt{\langle L(z + \lambda \zeta_1)[\dot{z} + \lambda \dot{\zeta}_1], \dot{z} + \lambda \dot{\zeta}_1 \rangle_E}} + \frac{C_2 \|\dot{\zeta} + \lambda \dot{\zeta}_1\|_{L^1}}{\sqrt{\langle L(z + \lambda \zeta_1)[\dot{z} + \lambda \dot{\zeta}_1], \dot{z} + \lambda \dot{\zeta}_1 \rangle_E}} \langle \dot{z} + \lambda \dot{\zeta}_1 \rangle$$

where $\|\cdot\|_E$ is the Euclidean norm in $\mathbb{R}^n$. Then there exists $C > 0$ such that

$$\left| \frac{\psi_2(z + \lambda \zeta_1) - \psi_2(z + \lambda \zeta)}{\lambda} \right| \leq C(\|\zeta_1 - \zeta\|_{L^\infty} \|\dot{\zeta}_1\|_E + \|\dot{\zeta}_1 - \dot{\zeta}\|_E).$$
Therefore, since \( \dot{z} \) is in \( L^2 \), \( \| \dot{\zeta} \|_{L^2} \) is bounded (according to the \( L^2 \)-norm of \( \zeta \)) and \( \zeta(0) = \dot{\zeta}(0) \) we deduce the existence of a constant \( C_0 \) such that for any \( \lambda \leq 1 \),

\[
\frac{1}{\lambda} \| \psi_2(z + \lambda \zeta) - \psi_2(z + \zeta) \|_{L^2} \leq C_0 \| \zeta \|_{H^{1,2}} \tag{4.12}
\]

Now, by the Lebesgue Convergence Theorem and Remark 4.4 it is not difficult to see that the linear operator \( d\psi_2(z) \) is continuous as linear map from \( T_z \mathcal{P}_{\gamma}^{+2}(\Lambda) \) to \( L^2([0,1], \mathbb{R}) \). Then for any \( \epsilon > 0 \) there exists \( \vartheta \in [0, \epsilon] \) such that

\[
\| \zeta_1 - \zeta \|_{H^{1,2}} \leq \vartheta \implies \| d\psi_2(z)[\zeta] - d\psi_2(z)[\zeta_1] \|_{L^2} < \epsilon. \tag{4.13}
\]

Moreover always fixing \( \zeta_1 \) such that \( \| \zeta_1 - \zeta \|_{H^{1,2}} \leq \vartheta \), for any \( \lambda \leq 1 \) we have

\[
\frac{1}{\lambda} \| \psi_2(z + \lambda \zeta) - \psi_2(z + \zeta) \|_{L^2} \leq C_0 \vartheta \leq C_0 \epsilon. \tag{4.14}
\]

Finally

\[
\frac{\psi_2(z + \lambda \zeta) - \psi_2(z)}{\lambda} - d\psi_2(z)[\zeta] = \frac{\psi_2(z + \lambda \zeta) - \psi_2(z)}{\lambda} - d\psi_2(z)[\zeta_1] + \frac{\psi_2(z + \lambda \zeta) - \psi_2(z + \lambda \zeta_1)}{\lambda} + d\psi_2(z)[\zeta_1] - d\psi_2(z)[\zeta].
\]

Since \( \psi_2 \) is differentiable at \( z \) along the direction \( \zeta_1 \), for any \( \lambda \) sufficiently small it is

\[
\left\| \frac{\psi_2(z + \lambda \zeta) - \psi_2(z)}{\lambda} - d\psi_2(z)[\zeta_1] \right\|_{L^2} \leq \epsilon,
\]

so, combining (4.13) and (4.14) gives the existence of \( \hat{\lambda} \) such that \( |\lambda| \leq \hat{\lambda} \) implies

\[
\left\| \frac{\psi_2(z + \lambda \zeta) - \psi_2(z)}{\lambda} - d\psi_2(z)[\zeta] \right\|_{L^2} \leq \epsilon + C_0 \epsilon + \epsilon,
\]

proving that (4.11) is satisfied for any \( \zeta \in H^{1,2} \).

The continuity of \( d\psi_2(\cdot) \) (which is a consequence of Remark 4.4 and Lebesgue Theorem) says that \( \psi_2 \) is of class \( C^1 \) in \( \mathcal{P}_{\gamma}^{+2}(\Lambda) \) and its differential is given by (4.11). \( \square \)

**Remark 4.6.** If \( z \in \mathcal{Q}_{p,\gamma}^{+2}(\Lambda) \) then \( \dot{z} \neq 0 \) almost everywhere and it is lightlike. Since \( \langle Y(z), \dot{z} \rangle \) is negative and \( Y \) is time like, there exists a positive constant \( \nu_z \) such that

\[
-\langle Y(z), \dot{z} \rangle \geq \nu_z \| Y(z) \|_R \| \dot{z} \|_R.
\]

Moreover \( \langle Y(z), \dot{z} \rangle \) is constant, therefore \( \| \dot{z} \|_R \) is uniformly bounded.

Now we can finally prove that \( \mathcal{Q}_{p,\gamma}^{+2}(\Lambda) \) is a manifold in a neighborhood of \( w \).
Proposition 4.7. There exists an open neighborhood $O_w$ of the geodesic $w$ in $\Omega_{p,\gamma}^{+,2}(\Lambda)$ such that $Q_{p,\gamma}^{+,2}(\Lambda) \cap O_w$ is a manifold of class $C^1$ and, for any $z \in Q_{p,\gamma}^{+,2}(\Lambda) \cap O_w$, 

\[ T_z(Q_{p,\gamma}^{+,2}(\Lambda) \cap O_w) = \left\{ \zeta \in \Omega_{p,\gamma}^{+,2}(\Lambda) : \langle D_z\zeta', \dot{z} \rangle = 0 \text{ a.e., and} \right\} \]

\[ \langle Y, D_z\zeta \rangle + \langle D_\zeta Y, \dot{z} \rangle = \int_0^1 (\langle Y, D_\zeta \zeta \rangle + \langle D_\zeta Y, \dot{z} \rangle) \, ds \text{ a.e.,} \]

Proof. Let $\psi : P_{p,\gamma}^{+,2}(\Lambda) \to L^2([0,1] \mathbb{R})$ be the $C^1$-map given by (4.9). By (4.10) its differential at any point $z \in Q_{p,\gamma}^{+,2}(\Lambda)$ is 

\[ d\psi(z)[\zeta] = \frac{\langle D_s\zeta, \dot{z} \rangle}{\sqrt{\langle \dot{z}, \dot{z} \rangle_{\mathbb{R}}}}. \]

Since our result is of local nature (in a neighborhood of the geodesic $w$), by (4.7) it suffices to show that, for any $h \in L^2([0,1] \mathbb{R})$, there exists $\zeta \in T_z \Omega_{p,\gamma}^{+,2}(\Lambda)$ such that 

\[ \left\{ \begin{array}{l}
\langle Y(w), D_s\zeta \rangle + \langle D_z Y, \dot{w} \rangle = \int_0^1 (\langle Y, D_\zeta \zeta \rangle + \langle D_\zeta Y, \dot{w} \rangle) \, ds \\
\frac{\langle D_s\zeta, \dot{w} \rangle}{\sqrt{\langle \dot{w}, \dot{w} \rangle_{\mathbb{R}}}} = h.
\end{array} \right. \]

Choose $\zeta(s) = \mu(s)Y(w(s)) + \lambda(s)\dot{w}(s)$. Since $\langle Y, Y \rangle \equiv -1$, denoting by $c_w$ the real constant $\langle Y, \dot{w} \rangle$, it will be sufficient to verify the existence of two real functions $\mu$ and $\lambda$ such that $\mu(0) = 0$, $\lambda(0) = \lambda(1) = 0$ and 

\[ \left\{ \begin{array}{l}
\mu' + \mu Y + \lambda c_w \text{ is constant,} \\
\mu' c_w = h \sqrt{\langle \dot{w}, \dot{w} \rangle_{\mathbb{R}}},
\end{array} \right. \]

and this can be done choosing 

\[ \mu(s) = \frac{1}{c_w} \int_0^s h \sqrt{\langle \dot{w}, \dot{w} \rangle_{\mathbb{R}}} \, dr, \]

and 

\[ c_w \lambda' = c + \frac{1}{c_w} h \sqrt{\langle \dot{w}, \dot{w} \rangle_{\mathbb{R}}} - \frac{Y \dot{w} c_w}{c_w} \int_0^s h \sqrt{\langle \dot{w}, \dot{w} \rangle_{\mathbb{R}}} \, dr, \]

where (integrating both terms of the above equality) the constant $c$ can be choose so that $\lambda(1) = \lambda(0) = 0$. □

Remark 4.8. To describe the tangent space $T_z(Q_{p,\gamma}^{+,2}(\Lambda) \cap O_w)$ we can operate in the following way. Take $\zeta \in T_z P_{p,\gamma}^{+,2}(\Lambda)$ and choose $\mu \in H^{1,2}([0,1], \mathbb{R})$ such that $\mu(0) = 0$ and 

\[ \langle D_\zeta [\zeta - \mu Y] \rangle, \dot{z} \rangle = 0, \]

namely, $\mu$ has to satisfy 

\[ \left\{ \begin{array}{l}
\langle D_\zeta \zeta, \dot{z} \rangle - \mu' c_w - \mu \langle D_\zeta Y, \dot{z} \rangle = 0, \\
\mu(0) = 0,
\end{array} \right. \]
where $c_z \equiv \langle Y, \dot{z} \rangle$ is a negative constant. Then $\mu$ is given by

$$
\mu(s) = \int_0^s \frac{\langle D_s \zeta, \dot{z} \rangle}{c_z} e^{\exp(-\int_r^s \frac{\langle D_s Y, \dot{z} \rangle}{c_z} d\sigma) dr},
$$

and, by (4.16),

$$
d\tau(z)[\zeta - \mu Y] = -\langle \dot{\gamma}(\tau(z)), \zeta(1) \rangle + \langle \dot{\gamma}(\tau(z)), Y(z(1)) \rangle \mu(1)
$$

where

$$
\mu(1) = \int_0^1 \frac{\langle D_s \zeta, \dot{z} \rangle}{c_z} e^{\exp(-\int_r^1 \frac{\langle D_s Y, \dot{z} \rangle}{c_z} d\sigma) dr}.
$$

Then, using Remark 4.4 and the same technique of the proof of Lemma 4.5 allows to deduce that $\tau$ is of class $C^2$ on $Q_{p,\gamma}(\Lambda) \cap O_w$.

In sections 2 we show the existence (for any timelike curve sufficiently closed to a fixed event) of minimizing lightlike geodesics for the functional $\tau$. Now we need a sort of converse of the above principle.

**Proposition 4.9.** Any future pointing lightlike geodesic $w$ is a critical point of $\tau$ on $Q_{p,\gamma}^{+2}(\Lambda) \cap O_w$.

**Proof.** Take $V \in T_w \Omega_{p,\gamma}^{+2}(\Lambda)$ such that $V(0) = V(1) = 0$ and $\mu \in H^{1,2}([0,1], \mathbb{R})$ such that $\mu(0) = 0$ and

$$
\langle D_w[V - \mu Y], \dot{w} \rangle = 0.
$$

(4.17)

By (4.16), Proposition 4.7 and Remark 4.8 it will be sufficient to prove that

$$
0 = d\tau(w)[V - \mu Y] = \mu(1) \langle Y(w(1)), \dot{\gamma}(\tau(w)) \rangle
$$

(4.18)

Now by (4.17)

$$
\mu(1) = \int_0^1 \frac{\langle D_{\dot{w}} V, \dot{w} \rangle}{\langle Y, \dot{w} \rangle} ds
$$

because $D_{\dot{w}} = 0$. Then $\mu(1) = 0$ for any $V$ because $\langle Y, \dot{w} \rangle$ is constant ($\neq 0$) and $w$ is a geodesic. □

**Remark 4.10.** The above proof shows also that, for any $\zeta \in T_w (Q_{p,\gamma}^{+2}(\Lambda) \cap O_w$, since $w$ is a geodesic it is $\zeta(1) = 0$.

**Remark 4.11.** Let $w$ be a future pointing lightlike geodesic. The same computations in [22] allows to prove that, for any $\zeta \in T_w Q_{p,\gamma}^{+2}(\Lambda)$, the Hessian of $\tau$ along the direction $\zeta$ is given by

$$
H^\tau(w)[\zeta, \zeta] = -\frac{1}{\langle \dot{\gamma}(\tau(w)), \dot{w}(1) \rangle} \int_0^1 (\langle D_s \zeta, D_s \zeta \rangle - \langle R(\zeta, \dot{w}), \zeta \rangle) ds.
$$

(4.19)
Now we equip \( T_wQ^{+,2}_{p,\gamma}(\Lambda) \) with the Hilbert structure

\[
\langle \zeta_1, \zeta_2 \rangle = \int_0^1 \langle D^Y_s \zeta_1, D^Y_s \zeta_2 \rangle_Y ds, \tag{4.20}
\]

where \( \langle \zeta, \zeta \rangle_Y = \langle \zeta, \zeta \rangle - 2 (Y(w), \zeta)^2 \) (which is equivalent to (2.1)) and \( D^Y_s \) is the covariant derivative with respect to \( \langle \zeta, \zeta \rangle_Y \).

**Proposition 4.12.** The linear map associated to the quadratic form (4.19) on \( T_wQ^{+,2}_{p,\gamma}(\Lambda) \) is a compact perturbation of the identity with respect to the Hilbert structure (4.20).

**Proof.** It is

\[
\langle D_s \zeta, D_s \zeta \rangle = \langle D_s \zeta, D_s \zeta \rangle + 2 (D_s \zeta, Y(w))^2 - 2 \langle D_s \zeta, Y(w) \rangle^2 = \langle D_s \zeta, D_s \zeta \rangle_Y - 2 \langle D_s \zeta, Y(w) \rangle^2. \tag{4.21}
\]

Now there exists a bilinear map \( \Gamma \) defined on the vector fields on \( \Lambda \) such that

\[ D_s \zeta = D^Y_s \zeta + \Gamma(w)[\zeta, \zeta]. \]

Moreover, by (4.15)

\[
\langle D_s \zeta, Y(w) \rangle = -\langle D \zeta, \dot{w} \rangle + \int_0^1 (\langle D_s \zeta, Y(w) \rangle + \langle D \zeta, \dot{w} \rangle) ds \tag{4.22}
\]

while, by Remark 4.10

\[
\int_0^1 \langle D_s \zeta, Y(w) \rangle = -\int_0^1 \langle \zeta, D_s Y \rangle. \tag{4.23}
\]

Since \( H^{1,2} \) is compactly embedded in \( L^\infty \), combining (4.21), (4.22) and (4.23) gives the proof. \( \square \)

Denote now by \( m(w, \tau) \) the maximal dimension of a subspace of \( T_w(Q^{+,2}_{p,\gamma}(\Lambda) \cap O_w) \) where the restriction of \( H^\tau(w)[\cdot, \cdot] \) is negative definite. It is called the Morse Index of the quadratic form \( H^\tau(w)[\cdot, \cdot] \). The following Index Theorem holds:

**Theorem 4.13.** Let \( w \) be a geodesic in \( Q^{+,2}_{p,\gamma}(\Lambda) \). Then

\[ m(w, \tau) = \mu(w). \]

To prove Theorem 4.13 some preliminary results are needed.

**Lemma 4.14.** Let \( \zeta \) be a Jacobi field along \( w \) such that \( \zeta(0) = 0, \zeta(1) = 0 \). Then \( \zeta \in T_wQ^{+,2}_{p,\gamma}(\Lambda) \).
Proof. Let $\zeta$ be a Jacobi field along $w$ with $\zeta(0) = 0$ and $\zeta(1) = 0$. It is immediately checked that, $\langle w, D_s \zeta \rangle \equiv 0$. Therefore we have just to prove that the function

$$\varphi(s) = \langle D_s \zeta, Y(w) \rangle + \langle \zeta, D_s Y \rangle$$

is constant. Since $D_s Y \equiv 0$ and $w$ is a geodesic

$$\varphi'(s) = \langle D_s^2 \zeta, Y(w) \rangle + \langle \zeta, D_s D_s Y \rangle.$$

Now, since $\zeta$ is a Jacobi field and

$$D_s D_s Y = D_s D_s + R(\dot{w}, \zeta) Y$$

(cf [2]), it is

$$\varphi'(s) = -(R(\zeta, \dot{w}) \dot{w}, Y(w)) + \langle D_s D_s Y + R(\dot{w}, \zeta) Y, \dot{w} \rangle = -\langle R(\zeta, \dot{w}) \dot{w}, Y(w) \rangle + \langle R(\dot{w}, \zeta) Y, \dot{w} \rangle = 0$$

because of the symmetry properties of $R$ (cf. [2]). \qed

An integration by parts shows immediately that the following Lemma holds.

**Lemma 4.15.** If $\zeta$ is a Jacobi field along $w$ such that $\zeta(0) = 0$ and $\zeta(1) = 0$, then

$$H^\tau(w)[\zeta, \zeta_1] = 0 \text{ for any } \zeta_1 \in T_w Q^+_p, \gamma(\Lambda) \quad (4.24)$$

**Lemma 4.16.** Let $\zeta \in T_w Q^+_p, \gamma(\Lambda)$ such that $(4.24)$ is satisfied. Then $\zeta$ is a $C^2$-Jacobi field along $w$ such that $\zeta(0) = 0$ and $\zeta(1) = 0$.

**Proof.** If $\zeta \in T_w Q^+_p, \gamma(\Lambda)$, $\zeta(0) = 0$ and by Remark 4.9, $\zeta(1) = 0$ because $w$ is a geodesic. Then, assuming that $(4.24)$ holds, we have to verify that $\zeta$ is of class $C^2$ and it satisfies (1.4).

Let $V$ be a $C^\infty$-vector field along $w$ such that $V(0) = 0$, $V(1) = 0$. Set $c_w = \langle Y(w), \dot{w} \rangle$ (which is a non zero constant) and choose

$$\mu(s) = \int_0^s \frac{1}{c_w} \langle D_s V, \dot{w} \rangle.$$

Now let $\lambda$ be the unique real map such that

$$\langle D_s V, Y(w) \rangle + \langle \dot{w}, D_s Y \rangle - \mu'(Y(w), Y(w)) - \mu(\dot{w}, D_s Y) - \lambda'c_w = \text{const}$$

$$\lambda(0) = \lambda(1) = 0.$$
A straightforward computation shows that \( \zeta_1 = V - \mu Y(w) - \lambda \dot{w} \in T_w Q^{+,2}_p(\Lambda) \). Therefore by (4.24) we have

\[
\int_0^1 \langle D_s \zeta, D_s V - \mu' Y(w) - \lambda' \dot{w} \rangle - \langle R(\zeta, \dot{w}) \dot{w}, V - \mu Y(w) - \lambda \dot{w} \rangle = 0
\]

for any \( C^\infty \) vector field \( V \) along \( w \). Now \( \langle D_s \zeta, \dot{w} \rangle \equiv 0 \) and \( \langle R(\zeta, \dot{w}) \dot{w}, \dot{w} \rangle \equiv 0 \). This allows us to deduce immediately that \( \zeta \) is of class \( C^2 \). So, to obtain (1.4), it suffices to prove that, for any \( V \),

\[
\int_0^1 -\mu' \langle D_s \zeta, Y(w) \rangle + \mu \langle R(\zeta, \dot{w}) \dot{w}, Y(w) \rangle = 0 \tag{4.25}
\]

which is equivalent to

\[
\int_0^1 -\langle D_s V, \dot{w} \rangle \langle D_s \zeta, Y(w) \rangle + \langle V, \dot{w} \rangle \langle R(\zeta, \dot{w}) \dot{w}, Y(w) \rangle = 0
\]

Then an integration by parts shows that (4.25) is equivalent to

\[
\frac{d}{ds} \left( \langle D_s \zeta, Y(w) \rangle \dot{w} \right) + \langle R(\zeta, \dot{w}) \dot{w}, Y(w) \rangle = 0.
\]

Now, \( w \) is a geodesic while, by (4.15),

\[
\langle D_s \zeta, Y(w) \rangle = \left( \int_0^1 \langle D_s \zeta, Y(w) \rangle + \langle D \zeta Y, \dot{w} \rangle \right) - \langle D \zeta Y, \dot{w} \rangle
\]

therefore (4.25) is equivalent to

\[
-\frac{d}{ds} (\langle D \zeta Y, \dot{w} \rangle) + \langle R(\zeta, \dot{w}) \dot{w}, Y(w) \rangle = 0.
\]

But

\[
D_s D \zeta Y = D \zeta D_s Y + R(\dot{w}, \zeta) Y,
\]

\( w \) is a geodesic and \( D_s Y \equiv 0 \), therefore (4.25) follows by the symmetry properties of \( R \).

Proof of Theorem 4.13. We prove a generalization of the Morse Index Theorem for Riemannian geodesics (cf. for instance [16,17]) to lightlike geodesics. For any \( \sigma \in [0, 1] \) set

\[
T_\sigma = \left\{ \zeta \in H^{1,2}([0, \sigma]), T \Lambda : \zeta(s) \in T_{w(s)} \text{ for any } s \in [0, \sigma], \right. \\
\zeta(0) = 0, \zeta(\sigma) = 0, \\
\langle Y, D_s \zeta \rangle + \langle D \zeta Y, \dot{w} \rangle \equiv c_\zeta \text{ a.e. in } [0, \sigma] \\
\langle D_s \zeta, \dot{w} \rangle = 0 \text{ a.e. in } [0, \sigma] \right\}.
\]

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Note that \( T_1 = T_w \mathcal{Q}^{+,-2}(\Lambda) \). Set, for any \( \zeta \in T_\sigma \)
\[
H_\sigma([\zeta,\zeta]) = \int_0^\sigma (\langle D_s \zeta, D_s \zeta \rangle - \langle R(\zeta, \dot{\omega}) \dot{\omega}, \zeta \rangle) ds.
\]
Note that
\[
H^+(w)[\cdot,\cdot] = \frac{-1}{\langle \dot{\gamma}(\tau(w)), \dot{w}(1) \rangle} H_1(\cdot,\cdot),
\]
while \(-\langle \dot{\gamma}(\tau(w)), \dot{w}(1) \rangle > 0\) because
\[
\dot{\gamma}(\tau(w)) = Y(\gamma(\tau(w))) = Y(w(1))
\]
and \( \langle Y(w(s)), \dot{w}(s) \rangle \) is constant.

Therefore, denoting by \( i(H_1) \) the Morse Index of the quadratic form \( H_1 \) we have to prove that
\[
i(H_1) = \mu(w)
\]
(4.26)

On \( T_\sigma \) we define the Hilbert structure
\[
\langle \zeta, \zeta \rangle_\sigma = \int_0^\sigma \langle D_Y \zeta, D_Y \zeta \rangle_Y dr.
\]
(4.27)

Denote by \( L_\sigma \) the linear operator in \( T_\sigma \) such that
\[
\langle L_\sigma \zeta_1, \zeta_2 \rangle_\sigma = H_\sigma([\zeta_1, \zeta_2]).
\]
(4.28)

By Proposition 4.12 we see that
\[
L_\sigma = I_\sigma - K_\sigma
\]
where \( I_\sigma \) is the identity on \( T_\sigma \) and \( K_\sigma : T_\sigma \to T_\sigma \) is a compact operator. We shall denote by \( w_\sigma \) the geodesic \( w_{\mid[0,s]} \). It is well known that \( T_\sigma \) has the following orthogonal decomposition consisting of eigenspaces of \( L_\sigma \)
\[
T_\sigma = H_\sigma^+ \oplus H_\sigma^0 \oplus H_\sigma^-
\]
where (4.28) is positive definite on \( H_\sigma^+ \), definite negative on \( H_\sigma^- \) and \( H_\sigma^0 = \ker L_\sigma \).

For any \( \sigma \in ]0,1[ \) let \( \lambda_1(\sigma) \geq ... \geq \lambda_k(\sigma) \) be the eigenvalues of the compact operator \( K_\sigma \). Any eigenvalues is here repeated according with its (finite) multiplicity. By Lemmas 4.15-4.17 \( w(\sigma) \) is conjugate to \( w(0) \) with multiplicity equal to \( m \) if and only if there exists \( k \in \mathbb{N} \) such that
\[
\lambda_k(\sigma) < 1, \quad \lambda_{k+1}(\sigma),...,\lambda_{k+m}(\sigma) = 1, \quad \text{and} \quad \lambda_{k+m+1}(\sigma) > 1.
\]

Since \( K_\sigma \) is compact the eigenvalues of \( K_\sigma \) are characterized by the well known Poincaré Formula:
\[
\lambda_k(\sigma) = \max_{\dim V=k} \left[ \min_{\zeta \in V, \|\zeta\|_\sigma=1} \langle (K_\sigma \zeta, \zeta) \rangle_\sigma \right]
\]
(4.29)
where \(|\cdot|_\sigma = \langle \cdot, \cdot \rangle_\sigma^{1/2}\). Using (4.29) it is not difficult to prove that any \(\lambda_k\) is a continuous function of \(\sigma\). Moreover using (4.21)-(4.23) it is easy to show that there exists \(\sigma_0 > 0\) such that, for any \(\sigma \in [0, \sigma_0]\), \(H_\sigma\) is positive definite on \(T_\sigma\). Then it will be sufficient to show that any \(\lambda_k\) is strictly increasing. (4.30)

Fix \(k \in \mathbb{N}\) and \(0 \leq \sigma_1 < \sigma_2 \leq 1\). By (4.29) there exists a subspace \(V\) of \(T_{\sigma_1}\) having dimension \(k\) such that

\[
\lambda_k(\sigma_1) = \min_{\zeta \in V: |\zeta|_{\sigma_1} = 1} \langle (K_{\sigma_1} \zeta, \zeta)_{\sigma_1} \rangle.
\] (4.31)

For any \(\zeta \in V\) set \(c_\zeta \equiv \langle D_s \zeta, Y(w) \rangle + \langle \dot{w}, D_\zeta Y \rangle\) and \(c_w \equiv \langle \dot{w}, Y(w) \rangle\). Take the vector field

\[A_\zeta = \zeta + \lambda \dot{w}\]

where \(\lambda = \frac{c_\zeta}{c_w}\) and \(\lambda(\sigma_1) = 0\). Since \(\zeta \in T_{\sigma_1}\), \(w\) is a lightlike geodesic and \(D_s Y \equiv 0\) it is

\[
\langle D_s A_\zeta, \dot{w} \rangle \equiv 0,
\] (4.32)

\[
\langle D_s A_\zeta, Y(w) \rangle + \langle \dot{w}, D_\zeta Y \rangle \equiv 0,
\] (4.33)

and

\[A_\zeta(\sigma_1) = 0, \ A_\zeta(0) = \lambda(0) \dot{w}(0).\] (4.34)

Now denote by \(\hat{A}_\zeta\) the extension to \([0, \sigma_2]\) of \(A_\sigma\) obtained by setting \(\hat{A}_\zeta = 0\) on \([\sigma_1, \sigma_2]\).

Note that, for any \(\zeta \in V\)

\[
\langle K_{\sigma_2} \hat{A}_\zeta, \hat{A}_\zeta \rangle_{\sigma_2} = \langle K_{\sigma_1} \zeta, \zeta \rangle_{\sigma_1}.
\]

Now set

\[B_\zeta = \hat{A}_\zeta + \mu \dot{w}\]

where \(\mu\) satisfies

\[
\mu' = \text{const}, \ \mu(0) = -\lambda(0), \ \mu(\sigma_2) = 0.
\] (4.35)

Since \(\hat{A}_\zeta\) satisfies (4.32)-(4.34) (with \(A_\zeta\) replaced by \(\hat{A}_\zeta\)), thanks to (4.35) we deduce that \(B_\zeta \in T_{\sigma_2}\). Note that the map

\[B : V \longrightarrow T_{\sigma_2}\]

is a linear and injective. Then the space

\[V_*= \{ B_\zeta : \zeta \in V \}\]

is a subspace of \(T_{\sigma_2}\) having dimension \(k\). Moreover, by our construction,

\[\langle K_{\sigma_2} B_\zeta, B_\zeta \rangle_{\sigma_2} = \langle K_{\sigma_1} \zeta, \zeta \rangle_{\sigma_1}\]

for any \(\zeta \in V\).

Then, by (4.29) and (4.31)

\[
\lambda_k(\sigma_1) = \min_{\zeta \in V_*, |\zeta|_{\sigma_1} = 1} \langle (K_{\sigma_1} \zeta, \zeta)_{\sigma_1} \rangle = \min_{\zeta \in V_*, |\zeta|_{\sigma_2} = 1} \langle (K_{\sigma_2} \zeta, \zeta)_{\sigma_2} \rangle \leq \lambda_k(\sigma_2).
\] (4.36)

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To conclude the proof assume by contradiction that

\[ \lambda \equiv \lambda_k(\sigma_1) = \lambda_k(\sigma_2). \]

By the spectral properties of \( K_{\sigma_2} \), \( T_{\sigma_2} \) admits the orthogonal decomposition

\[ T_{\sigma_2} = H^- \oplus H^0 \oplus H^+, \]

such that \( \lambda I_{\sigma_2} - K_{\sigma_2} \) is negative definite on \( H^- \), positive definite on \( H^+ \) and \( H^0 = \text{Ker}(\lambda I_{\sigma_2} - K_{\sigma_2}) \).

We claim that

\[ V_* \cap (H^0 \oplus H^+) \neq \{0\}. \quad (4.38) \]

Indeed if \( \zeta = \zeta_0 + \zeta_+ \in V_* \cap (H^0 \oplus H^+) \), where \( \zeta_0 \in H^0 \) and \( \zeta_+ \in H^+ \), if \( \zeta_+ \neq 0 \) and \( |\zeta|_{\sigma_2} = 1 \) it is

\[ \langle K_{\sigma_2} \zeta_0, \zeta_0 \rangle_{\sigma_2} + \langle K_{\sigma_2} \zeta_+, \zeta_+ \rangle_{\sigma_2} = \lambda \langle \zeta_0, \zeta_0 \rangle_{\sigma_2} + \lambda \langle \zeta_+, \zeta_+ \rangle_{\sigma_2} \leq \lambda \langle \zeta_0, \zeta_0 \rangle_{\sigma_2} \]

in contradiction with (4.36) and (4.37) because \( \zeta \in V_* \). Then \( \zeta_+ = 0 \) and

\[ \zeta = \zeta_0 \in \text{Ker}(\lambda I_{\sigma_2} - K_{\sigma_2}). \]

Then the same proof of Lemma 4.16 allows to deduce that \( \zeta \) is of class \( C^2 \). Since \( \mu \dot{w} \) is of class \( C^2 \) then \( \hat{A}_\zeta \) is of class \( C^2 \), and by the construction of \( \hat{A}_\zeta \) we deduce that

\[ D_s(A_\zeta)(\sigma_1) = 0. \]

Since \( \zeta \) is a Jacobi field in \([0, \sigma_1]\), \( w \) is a geodesic and \( \lambda' \) is constant, \( A_\zeta = \zeta + \lambda \dot{w} \) satisfies (1.4) with initial condition \( A_\zeta(\sigma_1) = 0 \) and \( D_s(A_\zeta)(\sigma_1) = 0 \). Then by the uniqueness of the Cauchy problem it is \( A_\zeta \equiv 0 \). Since \( \zeta(0) = 0 \), then \( \lambda(0) = 0 \) and therefore \( \lambda \equiv 0 \). This imply that \( \mu \equiv 0 \) and \( B_\sigma \equiv 0 \) proving (4.38). Then the orthogonal projection of \( V_* \) on \( H^- \) has dimension \( n \) and

\[ \lambda_k(\sigma_1) = \lambda \leq \min_{\zeta \in V_* \cap |\zeta|_{\sigma_2} = 1} \langle K_{\sigma_2} \zeta, \zeta \rangle_{\sigma_2} \leq \lambda_k(\sigma_2) \]

proving (4.30) and concluding the proof of Theorem 4.13. □

5. PROOFS OF THEOREMS 1.5, 1.9 AND 1.11

Now we are finally ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** Whenever we are far from the geodesics in \( L_{p, \gamma}^+(\Lambda) \), we can use the shortening flow at section 3 to obtain a flow where \( \tau \) is strictly decreasing. Nearby any
geodesic we can construct an homotopy equivalence between $\mathcal{L}_{p,\gamma}^{+,\frac{2}{r}}(\Lambda)$ and $\mathcal{Q}_{p,\gamma}^{+,\frac{2}{r}}(\Lambda)$ simply by a convex combination between the identity in $H^{1,2}([0,1],\mathbb{R})$ and the parameterization described by the condition

$$\langle \dot{z}, Y(z) \rangle = \int_0^1 \langle \dot{z}, Y(z) \rangle ds \text{ a.e.}$$

Then, we can use the shortening flow for $\tau$ far from geodesics and, thanks to Proposition 4.7, Remark 4.8, and Proposition 4.9, we can use the classical Morse Theory (cf. e.g. [4,17]) to describe the topology nearby a geodesic. In this way we obtain

$$\sum_{w \in G_{p,\gamma}^+(\Lambda)} \lambda^{m(w,\tau)} = \mathcal{P}(\mathcal{L}_{p,\gamma}^{+,\frac{2}{r}}(\Lambda))(\kappa) + (1 + \kappa)S(\kappa)$$

where $S$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

Since $\mathcal{B}_{p,\gamma}^+(\Lambda)$ is homotopically equivalent to $\mathcal{L}_{p,\gamma}^{+,\frac{2}{r}}(\Lambda)$ (cf. Proposition 3.4), applying Theorem 4.13 gives the proof of (1.6). □

**Proof of Theorem 1.9.** Assume that $\mathcal{B}_{p,\gamma}^+(\Lambda)$ is contractible. Then the Poincaré polynomial of $\mathcal{B}_{p,\gamma}^+(\Lambda)$ with respect to any field $\mathbb{K}$ is given by

$$\mathcal{P}(\mathcal{B}_{p,\gamma}^+(\Lambda), \mathbb{K})(\kappa) = 1$$

Let $\mathcal{G}_{p,\gamma}^+(\Lambda)$ be the set of future pointing lightlike geodesics joining $p$ with $\gamma$. Formula (1.6) gives

$$\text{card}\mathcal{G}_{p,\gamma}^+(\Lambda) = 1 + 2S(1).$$

Then $\text{card}\mathcal{G}_{p,\gamma}^+(\Lambda)$ is odd or infinite, according if $S(1)$ is finite or infinite. If $\mathcal{B}_{p,\gamma}^+(\Lambda)$ is not contractible, $\text{cat}\mathcal{B}_{p,\gamma}^+(\Lambda) \geq 2$ so the conclusion follows by Remark 3.9. □

**Proof of Theorem 1.11.** If $\Lambda$ is contractible, then $\mathcal{L}_{p,\gamma}^+(\Lambda)$ is contractible. Then by assumption $L_3$), $\mathcal{B}_{p,\gamma}^+(\Lambda)$ is contractible and the proof follows by the first part of Theorem 1.9.

If $\Lambda$ is not contractible by $L_3$) and a result of [29], for a suitable $\mathbb{K}$ $\mathcal{P}(\mathcal{B}_{p,\gamma}^+(\Lambda))(\kappa)$ has infinitely many coefficients different from zero and the conclusion follows by formula (1.6). □

**APPENDIX A. ON THE TOPOLOGY OF $\mathcal{B}_{p,\gamma}^+(\Lambda)$.**

Under light–convexity and pseudo–coercivity of $\tau$ we have seen (in section 3) that $\mathcal{B}_{p,\gamma}^+(\Lambda)$ is homotopically equivalent to $\mathcal{L}_{p,\gamma}^{+,\frac{2}{r}}(\Lambda)$ (for any $r \in [1, +\infty]$).

In this appendix we shall give a general condition assuring that $\mathcal{L}_{p,\gamma}^{+,\frac{1}{r}}(\Lambda)$ is homeomorphic to $\Omega_{p,\gamma}^{1,1}(\Lambda)$. Then, using standard technique one see that $\Omega_{p,\gamma}^{1,1}(\Lambda)$ is homotopically equivalent to the based loop space of $\Lambda$. 39
Proposition A.1 Suppose that there exists a smooth hypersurface \( \Lambda_0 \) in \( \Lambda \) and a smooth time-like vector field \( Y \) in \( \Lambda \) such that

1) \( Y \) is complete in \( \Lambda \)
2) \( \Lambda = \{ \eta(\sigma,y) : y \in \Lambda_0, \ \sigma \in \mathbb{R}, \ \dot{\eta} = Y(\eta), \ \eta(0) = y \} \)
3) for any integral curve \( \eta \) of \( Y \) there exist a unique \( s \in \mathbb{R} \) such that \( \eta(s) \in \Lambda_0 \)
4) \( \gamma : \mathbb{R} \to \Lambda \) is an integral curve of \( Y \) with \( \gamma(0) \in \Lambda_0 \)
5) \( p \in \Lambda_0 \) and \( p \neq \gamma(0) \).
6) The Cauchy problem

\[
\begin{cases}
\sigma' = -\frac{1}{(Y,Y)} \left( \langle Y, \eta_y[y] \rangle + \frac{1}{2} \sqrt{\langle Y, \eta_y[y] \rangle^2 - \langle Y, Y \rangle \langle \eta_y[y], \eta_y[y] \rangle} \right) \\
\sigma(0) = 0
\end{cases}
\] (A.1)

can be solved in the interval \([0,1]\) for any \( y \in H^{1,1}([0,1],\Lambda) \) such that \( y(0) = p \) and \( y(1) = \gamma(0) \).

Then \( L^{+,1}_p,\gamma(\Lambda) \) is homeomorphic to \( \Omega^{1,1}_{p,\gamma}(\Lambda) \).

(Here \( Y = Y(\eta(\sigma,y)) \) and \( \eta_y \) denotes the derivative of \( \eta \) with respect to the second variable).

Proof. Take \( z(s) = \eta(\sigma(s),y(s)) \). Suppose \( y \in H^{1,1}([0,1],\Lambda) \), \( y(0) = p \) and \( y(1) = \gamma(0) \). If \( \sigma \) satisfies 6) a straightforward computation show that the curve \( z(s) \) is in \( L^{+,1}_p,\gamma(\Lambda) \). Conversely if \( z \in L^{+,1}_p,\gamma(\Lambda) \) can be projected on \( \Lambda_0 \) using the integral curve of \( Y \) (cf. assumption 3). Since (A.1) has a unique solution we are done. \( \square \)

APPENDIX B. MORSE RELATIONS ON THE SPACE OF THE PIECEWISE LIGHTLIKE GEODESICS.

In this appendix we show by a simple example that we can not write Morse Relations using the topology of the piecewise (non null) lightlike geodesics (endowed with the topology of the uniform convergence). This space, as in section 1, will be denoted by \( \hat{B}^{+,1}_{p,\gamma}(\Lambda) \).

On the space \( \mathbb{R}^2 \times \mathbb{R} \) we consider the flat Minkowski metric

\[ ds^2 = dx_1^2 + dx_2^2 - dt^2. \]

Take \( p = (y_0,0) \) and \( \gamma(s) = (y_1,s) \). It is immediate to verify that \( \hat{B}^{+,1}_{p,\gamma}(\Lambda) \) is homeomorphic to the space \( C_{y_0,y_1} \) of the piecewise non null geodesics in \( \mathbb{R}^2 \) (with respect to the Euclidean metric) joining \( y_0 \) with \( y_1 \) endowed with the uniform topology. Considering the positions on the unit circle assumed by the unit speed of any broken geodesic is not difficult to show that \( C_{y_0,y_1} \) has infinitely many connect components. Then, if Morse Relations hold, one should obtain the existence of infinitely many geodesics joining \( p \) and \( \gamma \) and this is clearly false.

Analogously one see that in the (2+1)-dimensional Minkowski spacetime the infinite dimensional space where the relativistic Fermat Principle is proved, has infinitely many connect components. Then also in this case it is not possible to write the Morse Relations.
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