ASYMPTOTICS OF PERTURBED SOLITON FOR DAVEY–STEWARTSON II EQUATION

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It is shown that, under a small perturbation of lump (soliton) for Davey–Stewartson (DS-II) equation, the scattering data gain the nonsoliton structure. As a result, the solution has the form of Fourier type integral. Asymptotic analysis shows that, in spite of dispersion, the principal term of the asymptotic expansion for the solution has the solitary wave form up to large time.

0. Introduction. The Davey–Stewartson II (DS-II) equation, describing the interaction of the gravitational and capillary waves on a surface of liquid [1,2], is the example of the 2+1-dimensional equation integrable by the inverse scattering transform (IST) [3-7]. One of the most important advantage of IST is that the problem of the structure for the solution of the original equation is reduced to study spectral data for associated scattering problem.

The presence of solitary waves in the solution of the DS-II equation is determined by the existence of poles of the solution for the scattering problem [6,7]. Since, the problems of soliton stability and of variation of their parameters are reduced to study dependence of spectral data with respect to perturbation. For 1+1-dimensional integrable equations, it is known [8-16], that a small perturbation of the potential for the scattering problem implies a regular variation of spectral data and, hence, small changes of soliton parameters.

Investigation of solutions for 2+1-dimensional nonlinear equations shows that there is a more rich collection of forms of evolution. In first, the 1-dimensional solitons of the Kadomtsev–Petviashvili equation and of the 2-dimensional nonlinear Schrödinger equation are unstable with respect to transversal perturbations [4], [5],
In this work the problem of perturbation for the 2-dimensional soliton of the DS-II equation is considered. It would be naturally to expect results similar to results for perturbation of solitons for the 1-dimensional integrable equations. However, the structure of the perturbed solution proves to be essentially different. For construction of the asymptotic solution, we use the IST formalism. It is shown, that under perturbation of the 1-soliton initial data, the pole of the solution, for scattering problem, disappears. That is, the scattering data gain the nonsoliton structure and the soliton is unstable with respect to perturbation of the initial data, with IST point of view[25]. On the other hand, the such disappearance of the pole essentially affects to the asymptotics of the continuous component of the scattering data, which was absent in the unperturbed case. In turn, the constructed asymptotic expansion of the scattering data is used for the construction of the asymptotics for the solution of the DS-II equation. It is remarkable, that, in spite of the nonsoliton structure, the solution conserves the soliton form for the principal term of the asymptotics up to large time.

1. Setting of a problem. The DS-II equation system is considered in the form [7]:

\begin{equation}
\begin{aligned}
i \partial_t q + 2(\partial_x^2 + \partial_y^2)q + (g + \overline{g})q &= 0, \\
\partial_z g &= \partial_z |q|^2,
\end{aligned}
\end{equation}

where \( z = x + iy \), and the overbar represents complex conjugation.

The scattering problem for (1.1) has the following form [3-7]:

\begin{equation}
\begin{aligned}
\left( \begin{array}{cc}
\partial_z & 0 \\
0 & \partial_x
\end{array} \right) \phi &= \left( \begin{array}{cc}
0 & \frac{2}{\pi} \\
-\frac{2}{\pi} & 0
\end{array} \right) \phi, \\
E(-kz)\phi(k, z, t) &\to \left( \begin{array}{c}
1 \\
0
\end{array} \right), \quad |z| \to \infty,
\end{aligned}
\end{equation}

where \( E(kz) = \text{diag}(\exp(kz), \exp(\overline{kz})) \). Hereafter, the dependence of functions on \( \overline{z} \) is omitted.

The scattering data \( \mathcal{L}(t) \) of the problem (1.2), (1.3) consist of the continuous and discrete parts. The first of them is defined by the equality

\begin{equation}
b(k, t) = \frac{i}{4\pi} \int \int_{\mathcal{C}} d\zeta \wedge d\overline{\zeta} q^{(1)} \phi^{(1)} \exp(-k\zeta),
\end{equation}
where \( dz \wedge d\bar{z} = 2i \, dx \, dy \) and \( \phi^{(1)} \) is the corresponding component of the vector \( \phi \).

The discrete part \( \mathcal{L}^d \) of the scattering data consists of poles of \( \phi \) and of some their characterizing constants. This part corresponds to the existence of the solitons and their parameters.

The case, when the continuous component vanishes, corresponds to the pure soliton solution. The elementary example is the 1-soliton solution ([7]):

\[
q(z, t) = q_0(z, t) = 2 |\nu|^{2} \exp\left\{ k_0 z - \bar{k}_0 \bar{z} + 2i(k_0^2 + \bar{k}_0^2)t \right\},
\]

\[
g(z, t) = \frac{-4(z + 4ik_0t + \mu)^2}{(|z + 4ik_0t + \mu|^2 + |\nu|^2)^2},
\]

corresponding to the initial data

\[
q_0(z, 0) = q_0(z) = \frac{2\nu}{|z + \mu|^2 + |\nu|^2} \exp\{k_0 z - \bar{k}_0 \bar{z}\},
\]

and the vanishing boundary conditions for the function \( g \) as \( |z| \to \infty \).

The discrete part of the scattering data is defined by frequencies, amplitudes and phases of solitons. For the 1-soliton solution, it has the following form:

\[
\mathcal{L}^d = \{k_0, \mu, \nu\}
\]

If \( \mathcal{L}^d = \emptyset \), then the solution is called as nonsoliton.

In the paper the asymptotics of the solution \( q = q_\varepsilon \) for the system (1.1) with the initial data

\[
q_\varepsilon(z, 0) = q_\varepsilon(z) = q_0(z) + \varepsilon q_1(z),
\]

where \( q_1 \) is a smooth function with a finite support and \( \varepsilon \) is a small positive parameter, is constructed.

2. Solution asymptotics of scattering problem for \( t = 0 \). Denote by \( \phi_\varepsilon \) the solution of (1.2), (1.3) for \( q = q_\varepsilon \). The asymptotics of \( \psi_\varepsilon(k, z) = \phi_\varepsilon(k, z, 0) \) is constructed in the form

\[
\psi_\varepsilon(k, z) = \psi_0(k, z) + \varepsilon \psi_1(k, z) + \ldots.
\]

The problem (1.2), (1.3) is equivalent to the system of the integral equations ([7])

\[
(I - G[q, k])\phi = E_1,
\]
where

\[
G[q,k] \phi = \frac{i}{4 \pi} \int \int_{\mathbb{C}} d\zeta \wedge d\zeta \times
\]

\[
\left( \begin{array}{cc}
0 & \frac{\tilde{\eta}(\zeta,t) \exp\left\{ k(z-\zeta) \right\}}{z-\zeta} \\
-\frac{\tilde{\eta}(\zeta,t) \exp\left\{ k(z-\zeta) \right\}}{z-\zeta} & 0
\end{array} \right) \phi(k,\zeta,t),
\]

\]

$I$ is the unit matrix, and $E_1(kz)$ is the first column of the matrix $E(kz)$.

Substituting (2.1) and (1.8) into (2.2), we obtain the equations for $\psi_0$ and $\psi_1$:

\[
(I - G[q_0,k])\psi_0 = E_1,
\]

(2.3)

\[
(I - G[q_0,k])\psi_1 = G[q_1,k] \psi_0,
\]

(2.4)

The equation (2.3) has the explicit solution [7]:

\[
\psi_0(k,z) = E_1(kz) - \frac{\exp\left\{ (k - k_0)z \right\}}{k - k_0} A_1(z),
\]

(2.5)

\[
A_1(z) = \frac{1}{|z + \mu|^2 + |\nu|^2} \left( \frac{(z + \mu) \exp\{z k_0\}}{\nu \exp\{z k_0\}} \right).
\]

The vectors $A_1$ and $A_2 = \sigma A_1$, where

\[
\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

are the pair of the eigenfunctions for the equation:

\[
(I - G[q_0,k])W = F,
\]

(2.6)

as $k = k_0$.

For the vectors $f$ and $w$, we define the sesquilinear form

\[
(f,w)_{q} = \int \int_{\mathbb{C}} dz \wedge d\bar{z} \left( f^{(1)} \bar{w}^{(1)} + q f^{(2)} \bar{w}^{(2)} \right),
\]

(2.7)

where $f^{(i)}$, $w^{(i)}$ are the components of $f$, $w$.

For the solvability of the equation (2.6) as $k = k_0$ it is necessary to carry out the following condition:

\[
(F, C_i)_{q_0} = 0, \quad i = 1, 2,
\]

(2.8)
where

\[ C_1(z) = \frac{1}{|z + \mu|^2 + |\nu|^2} \left( (z + \mu) \exp\{-zk_0\} \right), \quad C_2 = \sigma C_1 \]

are the eigenfunctions of the equation, which is formal adjoint for (2.6) with respect to the sesquilinear form (2.7). Hence, for the existence of the pole more than the first order for the solution of (2.4), the sufficient condition is to carry out one of the following inequalities:

\[ (G[q_1, k_0] \lim_{k \to k_0} ((k - k_0)\psi_0), C_i)_{q_0} \neq 0, \quad i = 1, 2. \]

Simple transformations imply that the latter requirement is equivalent to one of the inequality:

\[ Q_i = (A_1, C_i)_{q_1} \neq 0, \quad i = 1, 2. \]

The constants \( Q_i \) depend on the perturbation \( q_1 \). We shall consider the function \( q_1 \) such that \( Q_1 \neq 0 \). This perturbation we shall call the nondegenerate perturbation. In this case the order of the pole for the correction terms of (2.1) increases. It implies that this series is asymptotic as \( |k - k_0| > C\varepsilon^\gamma \) (for any \( 0 \leq \gamma < 1 \), but become ineligible at the neighborhood of \( k = k_0 \). By this reason, for \( k \) close to \( k_0 \), we construct the asymptotics in the other form:

(2.9)

\[ \psi^\varepsilon(k, z) = E(kz) \left( \varepsilon^{-1} V_{-1}(\kappa, z) + V_0(\kappa, z) + \ldots \right), \]

where \( \kappa = (k - k_0)\varepsilon^{-1} \).

Substituting (2.9) and (2.1) into (2.2), we obtain the following equations:

(2.10)

\[ (I - G[q_0, k_0])V_{-1} = 0, \]

\[ (I - G[q_0, k_0])V_0 = I + G[q_1, k_0]V_{-1} + \kappa \partial_k G[q_0, k_0]V_{-1} + \]

\[ + \kappa \partial_k G[q_0, k_0]V_{-1}, \]

(2.11)

where

\[ G[q, k] = \frac{i}{4\pi} \int \int \zeta \times d\zeta \left( \begin{array}{cc} 0 & q(\zeta, t) \exp(\kappa \zeta - k\zeta) \\ -\frac{q(\zeta, t) \exp(\kappa \zeta - k\zeta)}{z - \zeta} & 0 \end{array} \right). \]

One can see, that the function

(2.12)

\[ V_{-1}(\kappa, z) = E(-k_0 z)A(z)\alpha(\kappa), \]
where $A$ is the matrix with the column $A_1$ and $A_2$, is the solution of the homogeneous equation (2.10) for any vector $\alpha(\varepsilon)$.

On the other hand, the solvability condition (2.8) for (2.6) implies that the solvability condition for the equation

$$(I - G[q_0, k_0])W = F$$

has the form

$$(E_i(k_0 z)F, C_i)_{q_0} = 0, \quad i = 1, 2.$$ 

The vector $\alpha$ is determined from the solvability condition of (2.11). Simple transformations and calculations give the following quantities for the components of $\alpha$:

(2.13) 

$$\alpha^{(1)}(\varepsilon) = -\frac{4\pi i(Q_2 + 4\pi i\varepsilon)}{|Q_1|^2 + |Q_2 + 4\pi i\varepsilon|^2},$$

$$\alpha^{(2)}(\varepsilon) = \frac{4\pi iQ_1}{|Q_1|^2 + |Q_2 + 4\pi i\varepsilon|^2}.$$

Thus, for the nondegenerate perturbation ($Q_1 \neq 0$), the asymptotic solution of (1.2), (1.3) has the form (2.1), (2.5) as $|k - k_0| > C\varepsilon^\gamma$ and the form (2.9), (2.12), (2.13) as $(k - k_0) = O(\varepsilon^\delta)$ for any $C > 0$, $0 \leq \gamma < 1$, $0 < \delta \leq 1$. Therefore, for the nondegenerate perturbation of soliton initial data, the combined solution of the scattering problem has no a pole at the neighborhood of $k_0$.

3. Asymptotics of scattering data. The time evolution for the continuous component of the scattering data is defined by the equality ([7]):

$$b(k, t) = b(k, 0) \exp\{2it(k^2 + \mathbf{r}^2)\}.$$ 

So, for the calculation of its asymptotics, it is sufficient to construct the asymptotics of $b^\varepsilon(k) = b(k, 0)$. Due to (2.5), (2.9) and (1.4) the asymptotics of $b^\varepsilon$ has the composite form:

(3.1) 

$$b^\varepsilon(k) = \varepsilon b_1(k) + \ldots, \quad |k - k_0| > C\varepsilon^\gamma, \quad 0 \leq \gamma < 1, \quad C > 0,$$

(3.2) 

$$b^\varepsilon(k) = \varepsilon^{-1}B_{-1}\left(\frac{k - k_0}{\varepsilon}\right) + B_0\left(\frac{k - k_0}{\varepsilon}\right) + \ldots, \quad k - k_0 = O(\varepsilon^\delta), \quad 0 < \delta \leq 1.$$
Moreover, the formula (1.4) and the asymptotics of $\psi_1^\varepsilon$ imply the equality

$$b_1(k) = -\frac{i}{4\pi} \int_C dz \wedge d\bar{z} \frac{q_0(z)}{q(z)} \psi_{1}^{(1)}(k, z) \exp\{-kz\} + q_i(z) \psi_{0}^{(1)}(k, z) \exp\{-kz\}$$

The latter formula is not constructible because, for $\psi_1$, the clear expression is not obtained. For $b_1$, we can obtain the more effective formula.

Direct calculation implies that the vector

$$\psi_0^\ast(k, z) = E_1(-kz) - \frac{\exp\{(-k + k_0)z\}}{k - k_0} C_1(z)$$

is the solution of the equation

$$(I - G^\ast) \psi_0^\ast = E_1(-kz),$$

where $G^\ast$ is the operator formally adjoint for $G$ with respect to the sesquilinear form (2.7).

Computing value of the sesquilinear form $(\psi_1, \psi_0^\ast)_{q_0}$, due to the equation (2.4) after simple transformations we obtain that

$$(\psi_1, E_1(-kz))_{q_0} + (\psi_0, E_1(-kz))_{q_1} = (\psi_0, \psi_0^\ast)_{q_1},$$

The equalities (3.3) and (3.4) imply that

$$b_1 = \frac{i}{4\pi} (\psi_0, \psi_0^\ast)_{q_1}$$

The equality (3.5) is similar to the formula for the variation of the scattering data for the nonsoliton solution of the DS-II equation from [26].

For $k - k_0 = O(\varepsilon^\delta)$, substituting (2.9) (1.4), we obtain that

$$B_{-1}(\varepsilon) \sim -\frac{i}{4\pi} \int_C dz \wedge d\bar{z} \frac{q_0}{q(z)} \psi_{1}^{(1)} \exp\{kz\}$$

Further, substituting (1.6) and (2.12) in (3.6), we come to the equality

$$B_{-1}(\varepsilon) = -\bar{\alpha}^{(2)}(\varepsilon).$$
4. Asymptotics of solution for inverse problem in nonsoliton case. In the nonsoliton case the inverse scattering problem (usually called the D-problem [7]), for the vector $u$ with the components

\begin{equation}
(4.1) \quad u^{(1)} = \phi^{(1)} \exp\{-kz\}, \quad u^{(2)} = -\overline{\phi^{(2)}} \exp\{-kz\},
\end{equation}

has the form

\begin{equation}
(4.2) \quad \begin{pmatrix} \partial_k & 0 \\ 0 & \partial_k \end{pmatrix} u = \begin{pmatrix} 0 & -b(k,0) \exp\{-is\} \\ b(k,0) \exp\{is\} & 0 \end{pmatrix} u,
\end{equation}

\begin{equation}
\quad u \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |k| \to \infty,
\end{equation}

where $s = 2t(k^2 + \overline{k}^2) - i(kz - \overline{k}z)$.

For $b(k,0) = b^\varepsilon(k)$, the solution asymptotics of the problem (4.2) is constructed in the form:

\begin{equation}
(4.3) \quad u^\varepsilon = u_0 + \varepsilon u_1 + \ldots.
\end{equation}

The asymptotics of $b^\varepsilon$ was constructed in the preceding section. It has the composite form. The typical variable for the principle term of the asymptotics reads as $\varkappa = (k - k_0)\varepsilon^{-1}$. Therefore, it is naturally to come in (4.2) the variable of the same scale: $l = 4\pi(k - k_0)\varepsilon^{-1}$.

Substitute (4.3), (3.2) in (4.2) and pass to the variable $l$. As a result, we obtain the problems for $u_j$. The problem for the main term of the asymptotics of the expansion (4.3) has the form:

\begin{equation}
\begin{pmatrix} \partial_l & 0 \\ 0 & \partial_l \end{pmatrix} u_0 = \frac{1}{4\pi} \begin{pmatrix} 0 & -B_{-1} \exp\{-is\} \\ B_{-1} \exp\{is\} & 0 \end{pmatrix} u_0,
\end{equation}

\begin{equation}
\quad u_0 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |l| \to \infty.
\end{equation}

Due to passing to the variable $l$, the phase $s$ of the exponential contains both the terms depending on the "fast" parameters $z, t$, and the "slow" parameters $\varepsilon t, \varepsilon z$. It is naturally, the dependence on the "fast" parameters to separate from the "slow" parameters. So, denote:

\begin{equation}
S_0 = 2t(k_0^2 + \overline{k}_0^2) - i(k_0z - \overline{k}_0z),
\end{equation}

\begin{equation}
\theta = \frac{\varepsilon}{4\pi} (4ik_0t + z), \quad \tau = \frac{\varepsilon t}{8\pi}.
\end{equation}
After these exchanging, the problem for $u_0$ take the form:

$$
\left( \frac{\partial}{\partial t} 0 \right) u_0 = \begin{pmatrix} 0 & W_0 \exp\{-i\varepsilon \tau (l^2 + \bar{l}^2)\} \\
-W_0 \exp\{i\varepsilon \tau (l^2 + \bar{l}^2)\} & 0 \end{pmatrix} u_0,
$$

$$
u_0 \rightarrow \begin{pmatrix} 1 \\
0 \end{pmatrix}, \quad |l| \rightarrow \infty.
$$

(4.4)

Here

$$W_0 = \frac{iQ_1 \exp\{l\theta - \bar{l}\bar{\theta}\}}{|\Omega_1|^2 + |\Omega_2 + l|^2},$$

$$\Omega_1 = iQ_1 \exp\{-i\theta_0\}, \quad \Omega_2 = iQ_2.$$

In system (4.4) the index of exponential, depending on $\tau$, will be considered as the perturbation. Extract two first terms in the Taylor expansion of this exponential and the summand of the order $\varepsilon$ carry over the equation for the next correction. As a result of the such approximation, we obtain the problem for $u_0$:

$$
\left( \frac{\partial}{\partial t} 0 \right) u_0 = \begin{pmatrix} 0 & W_0 \\
-W_0 & 0 \end{pmatrix} u_0,
$$

$$
u_0 \rightarrow \begin{pmatrix} 1 \\
0 \end{pmatrix}, \quad |l| \rightarrow \infty.
$$

(4.5)

For $\theta \neq 0$, the solution of this problem has the form:

$$u_0(l, \theta, \Omega_1, \Omega_2) = \begin{pmatrix} 1 \\
0 \end{pmatrix} - \frac{1}{\theta} \frac{|l + \Omega_2\bar{\theta}|}{|\Omega_1\exp\{l\theta - \bar{l}\bar{\theta}\}|} \frac{1}{|\Omega_1|^2 + |\Omega_2 + l|^2}.$$

For $\theta \neq 0$, the solution of the problem for the correction $u_1$ exists and is unique. However, $u_1$ has the pole of the second order with respect to $\theta$ as $\theta = 0$. The expansion (4.3) is ineligible at the neighborhood of $\theta = 0$ by the same reasons as the expansion (2.1) is ineligible at the neighborhood of $k_0$.

At a small neighborhood of $\theta = 0$, the asymptotic solution of the problem (4.2) has the another form:

$$u_\varepsilon = \varepsilon^{-1} U_{-1} + U_0 + \ldots
$$

(4.5)

Substitute (4.5), (3.2) in (4.2). As a result, we obtain the problem for $U_{-1}$:

$$
\left( \frac{\partial}{\partial t} 0 \right) U_{-1} = \begin{pmatrix} 0 & \bar{\omega}_0 \\
-\omega_0 & 0 \end{pmatrix} U_{-1},
$$

$$
U_{-1} \rightarrow 0, \quad |l| \rightarrow \infty.
$$

(4.6)
where $\omega_0 = \frac{\Omega_0}{|\Omega_1|^2 + |\Omega_2|^2 + l^2}$.

The system (4.6) has two linearly independent solutions decreasing as $|l| \to \infty$:

$$A_1 = \left( \frac{\Omega_2 + l}{\Omega_1} \right) \frac{1}{|l + \Omega_2|^2 + |\Omega_1|^2}, \quad A_2 = \left( \frac{-\Omega_1}{-(\Omega_2 + l)} \right) \frac{1}{|l + \Omega_2|^2 + |\Omega_1|^2}.$$ 

Hence, the general solution of (4.6) decreasing at infinity reads as follows:

$$U_{-1} = A \beta(\theta),$$

where $A$ is the matrix with the column $A_j$ and $\beta(\theta)$ is an arbitrary vector.

For calculation $\beta$, we consider the problem for $U_0$. Taking in consideration the term carrying over the previous step, it has the form:

$$\begin{align*}
\begin{pmatrix}
\partial_l \\
0
\end{pmatrix} U_0 &= \begin{pmatrix}
0 & \omega_0 \\
-\omega_0 & 0
\end{pmatrix} U_0 + \begin{pmatrix}
0 & \omega_1 \\
-\omega_1 & 0
\end{pmatrix} U_{-1} + \\
&\quad + \begin{pmatrix}
0 \\
-\omega_0 \left[(lZ - \overline{lZ}) + i\tau(l^2 + l^2)\right]
\end{pmatrix}
\begin{pmatrix}
\omega_0 \left[\overline{lZ} - lZ\right] - i\tau(l^2 + l^2)
\end{pmatrix}
U_{-1},
\end{align*}$$

$$U_0 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |l| \to \infty,$$

where $\omega_1 = -\frac{\beta_0}{4\pi} \exp\{iS_0\}$, $Z = \frac{1}{4\pi}(4i\kappa_0 t + z)$. The system of the integral equations corresponding to (4.7) reads as follows:

$$\begin{align*}
(I - H[\omega_0, l]) U_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + H[\omega_1, l]U_{-1} + H[f, l]U_{-1},
\end{align*}$$

where

$$H[h, l]w = -\frac{i}{2\pi} \int \int_{\mathbb{C}} dm \wedge d\overline{m} \left( \frac{0}{m - l} \right) \frac{\tau(m)}{m - l} w(m),$$

$$f = [(lZ - \overline{lZ}) + i\tau(l^2 + l^2)]\omega_0.$$ 

The solvability condition for (4.8) has the form:

$$\begin{align*}
(B_1, F)_{\omega_0} &= 0, \quad (B_2, F)_{\omega_0} = 0,
\end{align*}$$

where $F$ is the right hand side in (4.8),

$$B_1 = \left( \frac{\Omega_2 + l}{\Omega_1} \right) \frac{1}{|l + \Omega_2|^2 + |\Omega_1|^2}, \quad B_2 = \sigma B_1.$$
and the integration in the sesquilinear form is realized with respect to \( l \). Note that \((B_1, A_2)_{\omega_0} = (B_2, A_1)_{\omega_0} = 0\).

The solvability condition (4.9) implies the system of the equations for \( \beta \):

\[
\begin{align*}
M_1\overline{\beta^{(1)}} + (M_2 + 2i\pi Z - \pi \tau \Omega_2)\overline{\beta^{(2)}} &= 0, \\
(M_2 - 2i\pi \bar{Z} - \pi \tau \Omega_2)\overline{\beta^{(1)}} - M_1\overline{\beta^{(2)}} &= -2i\pi.
\end{align*}
\]

Here

\[
M_1 = (B_1, A_1)_{\omega_1}, \quad M_2 = (B_2, A_1)_{\omega_1},
\]

and the integration in the sesquilinear form is realized with respect to \( l \). From (4.10), the formulae for the components of \( \beta \) are deduced:

\[
\begin{align*}
\beta^{(1)} &= \frac{2i\pi (M_2 - 2\pi Z - \pi \tau \Omega_2)}{|M_1|^2 + |M_2 + 2\pi Z - \pi \tau \Omega_2|^2}, \\
\beta^{(2)} &= \frac{-2i\pi M_1}{|M_1|^2 + |M_2 + 2\pi Z - \pi \tau \Omega_2|^2}.
\end{align*}
\]

Taking in consideration the dependence on \( S_0 \) for \( \omega_1 \) and the explicit form of the vectors \( B_{1,2} \), we correct \( M_{1,2} \):

\[
M_1 = l_1 \exp\{-iS_0\}, \quad \text{where} \quad l_1 = \text{const}, \quad M_2 = \text{const}.
\]

Thus, the asymptotical solution of (4.2) has the form (4.3) as \( \theta > \varepsilon \delta \), \( 0 \leq \delta < 1 \), and the form (4.5) as \( \theta \leq \varepsilon \gamma \), \( 0 < \gamma \leq 1 \).

5. Asymptotics of solution for DS-II equation. The solution of the DS-II equation can be constructed by the following formula:

\[
q(z, t) = \frac{i}{\pi} \int \int_C dp \wedge \overline{\sigma \theta(p, t)} u^{(1)}(z, t, p) \exp\{is\}.
\]

Using the constructed asymptotics of the scattering data (3.1), (3.2) and the asymptotical representation (4.3), (4.7) of \( u^{(1)} \), we shall construct the principle term of the formal asymptotics for \( q_\varepsilon \). The structure of the asymptotical expansion for \( u^{(1)} \) implies, that the asymptotics of \( q_\varepsilon \) is different for different \( \theta \).

If \( \theta > O(\varepsilon^\delta) \) \( (\delta < 1) \), then, for the calculation of \( q_\varepsilon \) we use the formulae (3.1), (3.2) and (4.3). The analize of the asymptotics for the integral (5.1) with respect to \( \varepsilon \) implies that

\[
q_\varepsilon = O(\varepsilon^{1-\delta}), \quad \text{as} \quad \theta > O(\varepsilon^\delta), \quad \delta < 1.
\]
If $\theta \leq \varepsilon^\delta$, than, for the calculation of $q_\varepsilon$, we substitute (3.1), (3.2), (4.7) in (5.1). As a result we obtain that

$$q_\varepsilon(z, t) \sim \frac{i}{16\pi^3} \int d\ell \wedge dB_{-1}(\ell) U_{-1}^{(1)}(\ell, z, t) \exp\{iS_0\} + O(\varepsilon).$$

The latter formula implies:

$$q_\varepsilon(z, t) \sim -\frac{iM_1}{|M_1|^2 + |M_2 + \frac{1}{2}(4k_0 t + z) - \pi\tau\Omega_2|^2}.$$

The constants $l_1$ and $M_2$ (see (4.11)) can be determined putting $t = 0$ in (5.3). Due to (5.3) and (1.6) we obtain that

$$l_1 = -\frac{i\nu}{2}, \quad M_2 = \frac{\mu}{2}.$$

The comparising of the parameters for the perturbed solution (5.3) and the pure soliton solution (1.7) shows that the frequency $k_0$ of the soliton is invariable under the perturbation of the initial data and its phase shift is modulated by time in the form:

$$\mu_\varepsilon(\tau) = \mu - 2\pi\tau\Omega_2,$$

Thus, for $0 < t < \varepsilon^{-1} \text{Const}$, the asymptotical solution of (1.1), (1.8) reads as follows:

$$q_\varepsilon(z, t) \sim \frac{27}{|z + 4ik_0 t + \mu_\varepsilon|^2 + |\nu|^2} \exp\{k_0z - k_0\overline{z} + 2i(k_0^2 + \overline{k_0})t\},$$

$$g_\varepsilon(z, t) \sim -\frac{4(z + 4ik_0 t + \mu_\varepsilon)^2}{(|z + 4ik_0 t + \mu_\varepsilon|^2 + |\nu|^2)^2}.$$ 

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