Beta-representations of 0 and Pisot numbers

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RÉSUMÉ. Soient $\beta$ un nombre réel supérieur à 1, $d$ un entier positif, et soit

$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, \; z_i \in \{-d, \ldots, d\}\}$$

l’ensemble des mots infinis sur l’alphabet $\{-d, \ldots, d\} \subset \mathbb{Z}$ ayant la valeur 0 en base $\beta$. En utilisant un résultat de Feng sur le spectre de $\beta$ nous prouvons que si $Z_{\beta,[\beta]-1}$ est reconnaissable par un automate de Büchi fini, alors $\beta$ doit être un nombre de Pisot. En conséquence de résultats antérieurs, $Z_{\beta,d}$ est reconnaissable par un automate de Büchi fini pour tout entier positif $d$ si et seulement si $Z_{\beta,d}$ est reconnaissable par un automate de Büchi fini pour un $d \geq [\beta] - 1$. Ces conditions sont équivalentes à ce que $\beta$ soit un nombre de Pisot. La borne $[\beta] - 1$ ne peut pas être réduite.

Abstract. Let $\beta > 1$, $d$ a positive integer, and

$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, \; z_i \in \{-d, \ldots, d\}\}$$

be the set of infinite words having value 0 in base $\beta$ on the alphabet $\{-d, \ldots, d\} \subset \mathbb{Z}$. Based on a result of Feng on spectra of numbers, we prove that if the set $Z_{\beta,[\beta]-1}$ is recognizable by a finite Büchi automaton then $\beta$ must be a Pisot number. As a consequence of previous results, the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer $d$ if and only if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq [\beta] - 1$. These conditions are equivalent to the fact that $\beta$ is a Pisot number. The bound $[\beta] - 1$ cannot be further reduced.

1. Introduction

Let $\beta$ be a real number $> 1$. The so-called beta-numeration has been introduced by Rényi in [13], and since then there are been many works in this domain, in connection with number theory, dynamical systems, and automata theory, see the survey [10] or more recent [14] for instance.

2010 Mathematics Subject Classification. 11K16, 68Q45.
Mots-clefs. Pisot number, automaton.
By a greedy algorithm each number of the interval \([0, 1]\) is given a \(\beta\)-expansion, which is an infinite word on a canonical alphabet of non-negative integers. When \(\beta\) is an integer, we obtain the classical numeration systems. When \(\beta\) is not an integer, a number \(x\) may have different \(\beta\)-representations. The \(\beta\)-expansion obtained by the greedy algorithm is the greatest in the lexicographic ordering of all the \(\beta\)-representations of \(x\). The question of converting a \(\beta\)-representation into another one is equivalent to the study of the \(\beta\)-representations of \(0\). We focus on the question of the recognizability by a finite automaton of the set of \(\beta\)-representations of \(0\).

Let \(d\) be a positive integer, and let
\[
Z_{\beta,d} = \{ z_1 z_2 \cdots : \sum_{i \geq 1} z_i \beta^{-i} = 0, \ z_i \in \{-d, \ldots, d\}\}
\]
be the set of infinite words having value 0 in base \(\beta\) on the alphabet \(\{-d, \ldots, d\} \subset \mathbb{Z}\).

The following result has been formulated in [10]:

**Theorem 1.1.** Let \(\beta > 1\). The following conditions are equivalent:

1. the set \(Z_{\beta,d}\) is recognizable by a finite Büchi automaton for every integer \(d\),
2. the set \(Z_{\beta,d}\) is recognizable by a finite Büchi automaton for one integer \(d \geq \lceil \beta \rceil\),
3. \(\beta\) is a Pisot number.

(3) implies (1) is proved in [8], (1) implies (3) is proved in [2] and (2) implies (1) is proved in [9].

Recently Feng answered an open question raised by Erdős, Joó and Komornik [6], see also [1], on accumulation points of the set
\[
Y_{d}(\beta) = \{ \sum_{k=0}^{n} a_k \beta^k : n \in \mathbb{N}, a_k \in \{-d, \ldots, d\}\}.
\]

**Theorem 1.2 ([7]).** Let \(\beta > 1\). Then \(Y_{d}(\beta)\) is dense in \(\mathbb{R}\) if and only if \(\beta < d + 1\) and \(\beta\) is not a Pisot number.

Remark that the problem studied by Feng is closely related to the following one. For \(\beta > 1\) let
\[
X_{d}(\beta) = \{ \sum_{k=0}^{n} a_k \beta^k : n \in \mathbb{N}, a_k \in \{0, \ldots, d\}\}.
\]
Since \(X_{d}(\beta)\) is discrete its elements can be arranged into an increasing sequence
\[
0 = x_0(\beta, d) < x_1(\beta, d) < \cdots
\]
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Denote $\ell_d(\beta) = \liminf_{n \to \infty} (x_{k+1}(\beta, d) - x_k(\beta, d))$. Feng has obtained the following corollary: $\ell_d(\beta) = 0$ if and only if $\beta < d + 1$ and $\beta$ is not a Pisot number.

Previously in [5] Bugeaud had shown, using (1) implies (3) of Theorem 1.1, that if $\beta$ is not a Pisot number then there exists an integer $d$ such that $\ell_d(\beta) = 0$.

In the present note we use Feng’s theorem to prove the conjecture stated in [10]:

If the set $Z_{[\beta]-1}$ is recognizable by a finite Büchi automaton then $\beta$ is a Pisot number.

Moreover we obtain a simpler proof of the implication (2) $\Rightarrow$ (3) of Theorem 1.1. Note that the value $d = \lceil \beta \rceil - 1$ is the best possible as $Z_{\beta,d}$ is reduced to $\{0^\omega\}$ if $d < \lceil \beta \rceil - 1$.

To summarize, this gives the following result:

The set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer $d$ if and only if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$ if and only if $\beta$ is a Pisot number.

Normalization in base $\beta$ is the function which maps a $\beta$-representation on the canonical alphabet $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ of a number $x \in [0,1]$ onto the greedy $\beta$-expansion of $x$. Since the set of greedy $\beta$-expansions of the elements of $[0,1]$ is recognizable by a finite Büchi automaton when $\beta$ is a Pisot number, see [4], the following result holds true:

Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if $\beta$ is a Pisot number.

We end this paper by considering finite $\beta$-representations of 0. We recall an old result of [8]:

Let $\beta$ be a complex number. The set $W_{\beta,d}$ of finite $\beta$-representations of 0 is recognizable by a finite automaton for every $d$ if and only if $\beta$ is an algebraic number with no conjugate of modulus 1.

In view of previous results and examples we conclude with a conjecture:

Let $\beta > 1$ be an algebraic number such that $W_{\beta,[\beta]-1} \neq \{0^\omega\}$. If $\beta$ has a conjugate of modulus 1 then $W_{\beta,[\beta]-1}$ is not recognizable by a finite automaton.

2. Preliminaries

2.1. Words and automata. Let $A$ be a finite alphabet. A finite word $w$ on $A$ is a finite concatenation of letters from $A$, $w = w_1 \cdots w_n$ with $w_i$ in $A$. The set of all finite words over $A$ is denoted by $A^*$. An infinite word $w$ on $A$ is an infinite concatenation of letters from $A$, $w = w_1 w_2 \cdots$ with $w_i$
in $A$. The set of all infinite words over $A$ is denoted by $A^\infty$. The infinite concatenation $uwv \cdots$ is noted $w^\omega$. If $w = uv$, $u$ is a prefix of $w$.

An automaton $A = (A, Q, I, T)$ over $A$ is a directed graph labeled by letters of the alphabet $A$, with a denumerable set $Q$ of vertices called states. $I \subseteq Q$ is the set of initial states, and $T \subseteq Q$ is the set of terminal states. The automaton is said to be finite if the set of states $Q$ is finite.

A finite path of $A$ is successful if it starts in $I$ and terminates in $T$. The set of finite words recognized by $A$ is the set of labels of its successful finite paths.

An infinite path of $A$ is successful if it starts in $I$ and goes infinitely often through $T$. The set of infinite words recognized by $A$ is the set of labels of its successful infinite paths. An automaton used to recognize infinite words in this sense is called a Büchi automaton.

2.2. Beta-numeration. We now recall some definitions and results on the so-called beta-numeration, see [10] or [14] for a survey. Let $\beta > 1$ be a real number. Any real number $x \in [0, 1]$ can be represented by a greedy algorithm as $x = \sum_{i=1}^{+\infty} x_i\beta^{-i}$ with $x_i$ in the canonical alphabet $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ for all $i \geq 1$. The greedy sequence $(x_i)_{i \geq 1}$ corresponding to a given real number $x$ is the greatest in the lexicographical order, and is said to be the $\beta$-expansion of $x$, see [13]. It is denoted by $d_\beta(x) = (x_i)_{i \geq 1}$.

When the expansion ends in infinitely many 0’s, it is said finite, and the 0’s are omitted. The $\beta$-expansion of 1 is denoted $d_\beta(1) = (t_i)_{i \geq 1}$.

An infinite (resp. finite) $\beta$-representation of 0 on an alphabet $\{-d, \ldots, d\}$ is an infinite (resp. finite) sequence $z_1z_2\cdots$ of letters from this alphabet such that $\sum_{i \geq 1} z_i\beta^{-i} = 0$. If $d < \lceil \beta \rceil - 1$, then 0 has only the trivial $\beta$-representation on the alphabet $\{-d, \ldots, d\}$. On the other hand, $(-1)t_1t_2\cdots$ is a nontrivial $\beta$-representation of 0 on the alphabet $\{-\lceil \beta \rceil + 1, \ldots, \lceil \beta \rceil - 1\}$, and thus we consider only alphabets $\{-d, \ldots, d\}$ with $d \geq \lceil \beta \rceil - 1$.

Note that, if $Z_\beta,d$ is recognizable by a finite Büchi automaton, then, for every $c < d$, $Z_\beta,c = Z_\beta,d \cap \{-c, \ldots, c\}^\infty$ is recognizable by a finite Büchi automaton as well.

We briefly recall the construction of the (not necessarily finite) Büchi automaton recognizing $Z_{\beta,d}$, see [8] and [10]:

- the set of states is $Q_d \subset \{\sum_{k=0}^{n} a_k\beta^k \mid n \in \mathbb{N}, a_k \in \{-d, \ldots, d\}\} \cap [-\frac{\beta-1}{\beta-1}, \frac{\beta-1}{\beta-1}]
- for $s, t \in Q_d, a \in \{-d, \ldots, d\}$ there is an edge $s \xrightarrow{a} t \iff t = \beta s + a$
- the initial state is 0
- all states are terminal.
Example 2.1. Take $\beta = \varphi = \frac{1 + \sqrt{5}}{2}$ the Golden Ratio. It is a Pisot number, with $d_\varphi(1) = 11$. A finite Büchi automaton recognizing $Z_{\varphi,1}$ is designed in Figure 1. The initial state is 0, and all the states are terminal. The signed digit $(-1)$ is denoted $\bar{1}$.

![Figure 1. Finite Büchi automaton recognizing $Z_{\varphi,1}$ for $\varphi = \frac{1 + \sqrt{5}}{2}$.](image)

Notation: In the sequel $y_{m-1} \cdots y_0 \cdot y_{-1} y_{-2} \cdots$ denotes the numerical value $y_{m-1} \beta^{m-1} + \cdots + y_0 + y_{-1} \beta^{-1} + y_{-2} \beta^{-2} + \cdots$.

2.3. Numbers. A number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic is a Parry number. It is a simple Parry number if $d_\beta(1)$ is finite.

A *Pisot number* is an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1. Every Pisot number is a Parry number, see [3] and [15]. The converse is not true, see for instance Example 4.4 below.

A *Salem number* is an algebraic integer greater than 1 such that all its Galois conjugates have modulus $\leq 1$ with at least one conjugate with modulus 1.

3. Infinite representations

3.1. Main result. We answer a conjecture raised in [10] and obtain the following result.

**Theorem 3.1.** Let $\beta > 1$. The following conditions are equivalent:

1. The set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer $d$,
2. The set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$,
3. $\beta$ is a Pisot number.

It will be a consequence of the result which follows.

**Proposition 3.2.** Let $d \geq \lceil \beta \rceil - 1$. If $\beta$ is not a Pisot number then the set $Z_{\beta,d}$ is not recognizable by a finite Büchi automaton.
Proof. Since \( \beta \) is not a Pisot number and \( d \geq \lfloor \beta \rfloor - 1 \), by Feng \cite{Feng2009}, the set

\[
Y_d(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, \ a_k \in \{-d, \ldots, d\} \right\}
\]

is dense in \( \mathbb{R} \). In particular there exists a sequence \((r_n)_{n \in \mathbb{N}}\) of elements of \( Y_d(\beta) \) such that \( r_n \neq r_m \) for all \( n \neq m \), and \( \lim_{n \to \infty} r_n = 0 \). As \( r_n \) belongs to \( Y_d(\beta) \), it can be written as \( r_n = \sum_{k=0}^{\ell_n} a^{(n)}_k \beta^k \) where \( a^{(n)}_0, a^{(n)}_1, \ldots, a^{(n)}_{\ell_n} \) are in \( \{-d, \ldots, d\} \), \( a^{(n)}_{\ell_n} \neq 0 \) and \( \ell_n \) is minimal.

Clearly the number

\[(3.1) \quad 0.a^{(n)}_{\ell_n} a^{(n)}_{\ell_n-1} \cdots a^{(n)}_0 = \frac{r_n}{\beta^{\ell_n}} \to 0 \]

as \( n \) tends to \( \infty \).

Let us fix \( k \) in \( \mathbb{N} \). Since there exist finitely many words of length \( k \) on \( \{-d, \ldots, d\} \), there exists a word \( S_k \) of length \( k \) which is a prefix of infinitely many elements of the sequence \((a^{(n)}_{\ell_n} a^{(n)}_{\ell_n-1} \cdots a^{(n)}_0) \omega_n\). Set \( S_k = x_1 \cdots x_k \).

As \( a^{(n)}_{\ell_n} \neq 0 \), \( x_1 \neq 0 \) as well.

Since infinitely many \( a^{(n)}_{\ell_n} a^{(n)}_{\ell_n-1} \cdots a^{(n)}_0) \omega_n \) have \( S_k \) as a prefix, there exists a letter \( x_{k+1} \) such that \( S_{k+1} = x_1 \cdots x_{k}x_{k+1} \) is a prefix of infinitely many \( a^{(n)}_{\ell_n} a^{(n)}_{\ell_n-1} \cdots a^{(n)}_0) \omega_n \). We continue in this manner and we get an infinite string \( x_1 x_2 \cdots \). To formulate an important property of it we need the following notion.

**Definition 3.3.** Let \( z_1 z_2 \cdots \) be a \( \beta \)-representation of 0 on \( \{-d, \ldots, d\} \). It is said to be **rigid** if \( 0.z_1 z_2 \cdots z_j \neq 0.0 z_2' \cdots z_j' \) for all \( j \geq 2 \) and for all \( z_2' \cdots z_j' \in \{-d, \ldots, d\}^* \).

**Claim 1** \( x_1 x_2 \cdots \) is a rigid \( \beta \)-representation of 0.

First we prove that it is indeed a \( \beta \)-representation of 0: by the construction, for every \( n \in \mathbb{N} \) there exist infinitely many \( N \)'s such that the common prefix of \( x_1 x_2 \cdots \) and \( a^{(N)}_{\ell_N} a^{(N)}_{\ell_N-1} \cdots a^{(N)}_0) \omega \) is longer than \( n \). Thus

\[
|0.x_1 x_2 \cdots 0.a^{(N)}_{\ell_N} a^{(N)}_{\ell_N-1} \cdots a^{(N)}_0| \leq \frac{1}{\beta^n} \frac{2d}{\beta - 1}.
\]

Therefore, by (3.1),

\[
|0.x_1 x_2 \cdots | \leq |0.x_1 x_2 \cdots 0.a^{(N)}_{\ell_N} a^{(N)}_{\ell_N-1} \cdots a^{(N)}_0| + |0.a^{(N)}_{\ell_N} a^{(N)}_{\ell_N-1} \cdots a^{(N)}_0| \leq \frac{1}{\beta^n} \frac{2d}{\beta - 1} + \frac{r_N}{\beta^{\ell_N}} \to 0
\]

and thus \( x_1 x_2 \cdots \) is a \( \beta \)-representation of 0.

Second, we prove that this representation is rigid. Suppose the opposite, that is to say that there exist some index \( j \) and a word \( x_2' \cdots x_j' \)
indices do not coincide. Let us suppose that there exist \( k < n \) such that \( r \). This implies that there is some contradiction with the choice of \( \ell N \) of \( x_1x_2\cdots x_j = 0.0x_2'\cdots x_j' \). We show that the elements of the sequence \((0.x_{n+1}x_{n+2}\cdots)n\) with distinct indices do not coincide. Let us suppose that there exist \( k < n \) such that \( 0.x_{k+1}x_{k+2}\cdots = 0.x_{n+1}x_{n+2}\cdots \). Then

\[-x_1x_2\cdots x_k \cdot = -x_1x_2\cdots x_n.\]

This implies that there is some \( r \) such that

\[ r = a_{\ell N}^{(N)} a_{\ell N-1}^{(N)} \cdots a_0^{(N)} \cdot = x_1x_2\cdots x_n a_{\ell N}^{(N)} a_{\ell N-n}^{(N)} \cdots a_0^{(N)} \cdot = 0^n \cdot k x_1 x_2 \cdots x_k a_{\ell N}^{(N)} a_{\ell N-n}^{(N)} \cdots a_0^{(N)} \cdot \]

a contradiction with the choice of \( \ell N \).

Define polynomials \( P_n(X) = \sum_{k=0}^{n} x_{n-k} X^k \). Then the division by \((X - \beta)\) gives \( P_n(X) = Q_n(X)(X - \beta) + s_n \), and

\[ P_n(\beta) = s_n = x_1 \beta^{n-1} + \cdots + x_{n-1} \beta + x_n = -0.x_{n+1}x_{n+2} \cdots \]

Consequently to Claim 2, the set of remainders of the division by \((X - \beta)\) is infinite. By Proposition 3.1 in [8] the set \( Z_{\beta,d} \) is not recognizable by a finite Büchi automaton. □

Remark 3.4. The fact that, if \( \beta \) is not a Pisot number, then the set \( Z_{\beta,d} \) is not recognizable by a finite Büchi automaton for any \( d \geq [\beta] \) was already settled in Theorem 1.1, but the proof given in Proposition 3.2 is more direct than the original one.

3.2. Normalization. Normalization in base \( \beta \) is the function which maps a \( \beta \)-representation on the canonical alphabet \( A_\beta = \{0, \ldots, [\beta] - 1\} \) of a number \( x \in [0, 1) \) onto the greedy \( \beta \)-expansion of \( x \). From the Büchi automaton \( Z \) recognizing the set of representations of 0 on the alphabet \( \{-[\beta] + 1, \ldots, [\beta] - 1\} \), one constructs a Büchi automaton (a converter) \( C \) on the alphabet \( A_\beta \times A_\beta \) that recognizes the set of couples on \( A_\beta \) that have the same value in base \( \beta \), as follows:

\[ s \xrightarrow{(a,b)} t \text{ in } C \iff s \xrightarrow{a-b} t \text{ in } Z, \]

see [10] for details. Obviously \( C \) is finite if and only if \( Z \) is finite.

Then we take the intersection of the set of second components with the set of greedy \( \beta \)-expansions of the elements of \([0, 1]\), which is recognizable
by a finite Büchi automaton when \( \beta \) is a Pisot number, see [4]. Thus the following result holds true.

**Corollary 3.5.** Normalization in base \( \beta > 1 \) is computable by a finite Büchi automaton on the alphabet \( A_\beta \times A_\beta \) if and only if \( \beta \) is a Pisot number.

### 3.3. F-Numbers

In [11] Lau defined for \( 1 < \beta < 2 \) the following notion, that we extend to any \( \beta > 1 \).

**Definition 3.6.** \( \beta > 1 \) is said to be a F-number if

\[
L_{\lceil \beta \rceil - 1}(\beta) = Y_{\lceil \beta \rceil - 1}(\beta) \cap \left[ -\frac{\lceil \beta \rceil - 1}{\beta - 1}, \frac{\lceil \beta \rceil - 1}{\beta - 1} \right]
\]

is finite.

Feng proved in [7] that if \( 1 < \beta < 2 \) is a F-number then \( \beta \) is a Pisot number. This property extends to any \( \beta > 1 \). Another way of proving it consists in realizing that the set of states \( Q_{\lceil \beta \rceil - 1} \) of the automaton for \( Z_{\beta, \lceil \beta \rceil - 1} \) is included into \( L_{\lceil \beta \rceil - 1}(\beta) \), and we get

**Corollary 3.7.** \( \beta \) is a F-number if and only if \( \beta \) is a Pisot number.

### 4. Finite representations

One can ask: what are the results in case we consider only finite representations of numbers. The situation is quite different. Let \( \beta \) be any complex number and let

\[
W_{\beta,d} = \{ z_1z_2\cdots z_n \mid n \geq 1, \sum_{i=1}^{n} z_i\beta^{-i} = 0, z_i \in \{-d, \ldots, d\} \}
\]

be the set of finite representations of 0 on \( \{-d, \ldots, d\} \).

**Theorem 4.1 ([8]).** Let \( \beta \) be a complex number. The set \( W_{\beta,d} \) is recognizable by a finite automaton for every \( d \) if and only if \( \beta \) is an algebraic number with no conjugate of modulus 1.

**Remark 4.2.** The construction of the automaton recognizing \( Z_{\beta,d} \) presented in Section 2.2 can be extended to the case that \( \beta \) is a complex number, \( |\beta| > 1 \), with set of states \( Q_d \subset \{ \sum_{k=0}^{n} a_k\beta^k \mid n \in \mathbb{N}, a_k \in \{-d, \ldots, d\}, |\sum_{k=0}^{n} a_k\beta^k| \leq \frac{d}{|\beta|-1} \} \).

The automaton recognizing \( W_{\beta,d} \) is the subautomaton formed by the finite paths starting and ending in state 0.

**Example 4.3.** Take \( \beta = \varphi = \frac{1+\sqrt{5}}{2} \) the Golden Ratio. A finite automaton recognizing \( W_{\varphi,1} \) is designed in Figure 2. The initial state is 0, which is the only terminal state.
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From Remark 4.2 we know that if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton then $W_{\beta,d}$ is recognizable by a finite automaton, but the converse is false, as shown by the following example.

**Example 4.4 ([10]).** Let $\beta$ be the root $> 1$ of the polynomial $X^4 - 2X^3 - 2X^2 - 2$. Then $d_\beta(1) = 2202$ and $\beta$ is a simple Parry number which is not a Pisot number and has no root of modulus 1. The set $W_{\beta,2}$ is recognizable by a finite automaton but $Z_{\beta,2}$ is not recognizable by a finite Büchi automaton.

The following gives an example of a Salem number which is a Parry number such that the set of representations of 0 is not recognizable by a finite automaton.

**Example 4.5 ([12]).** Let $\beta$ be the root $> 1$ of $X^4 - 2X^3 + X^2 - 2X + 1$. $\beta$ is a Salem number, and $d_\beta(1) = 1(1100)^\omega$. It has been proved by an adhoc construction that $W_{\beta,1}$ is not recognizable by a finite automaton.

In the case that $\beta > 1$ we have from [9] that $W_{\beta,d}$ is recognizable by a finite automaton for every $d$ if and only if $W_{\beta,d}$ is recognizable by a finite automaton for one $d \geq \lceil \beta \rceil$. So we raise the following conjecture.

**Conjecture 4.6.** Let $\beta > 1$ be an algebraic number such that $W_{\beta,\lceil \beta \rceil - 1} \neq \{0^\omega\}$. If $\beta$ has a conjugate of modulus 1 then $W_{\beta,\lceil \beta \rceil - 1}$ is not recognizable by a finite automaton.

**Acknowledgements**

Ch.F. acknowledges support of ANR/FWF project “FAN”, grant ANR-12-IS01-0002. E.P. acknowledges financial support by the Czech Science Foundation grant GACR 13-03538S.

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