VISUALIZING OVERTWISTED DISCS IN OPEN BOOKS

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ABSTRACT. We give an alternative proof of a theorem of Honda-Kazez-Matić that every non-right-veering open book supports an overtwisted contact structure. We also study two types of examples that show how overtwisted discs are embedded relative to right-veering open books.

1. INTRODUCTION

In [15], we have introduced open book foliations and their basic machinery by using that of braid foliations [2, 3, 4, 5, 6, 7, 8, 9] and showed applications of open book foliations including a self-linking number formula of general closed braids. In [16] we study the geometric structure of a 3-manifold by using open book foliations. In this note we study more applications of open book foliations. We will assume the readers are familiar with the definition and basic machinery of open book foliations in [15].

One of the features of open book foliations is that one can visualize how surfaces are embedded with respect to general open books. In this paper we use this feature to illustrate overtwisted discs and give constructive methods to detect overtwisted contact structures.

We first give an alternative proof of a tightness criterion theorem by Honda, Kazez and Matić [14]: If an open book is not right-veering then it supports an overtwisted contact structure.

The converse does not hold: In fact, Honda, Kazez and Matić [14] show that if a contact structure ξ is supported by a non-right veering open book (S, φ), by applying positive stabilizations to (S, φ) one can find a right-veering open book (Ŝ, ̂φ) that supports ξ. We concretely visualize an overtwisted disc relative to the right-veering (Ŝ, ̂φ).

Lastly, we give an infinite family of open books that are right-veering and non-destabilizable but compatible with overtwisted contact structures. This negatively answers a question of Honda, Kazez and Matić [14]. Our family generalizes the previously known examples by Lekili [19] and Lisca [20], but our proof of overtwistedness is more direct.

2. OVERTWISTED DISCS IN NON-RIGHT-VEERING OPEN BOOKS

Recall that an overtwisted disc is an embedded disc in a contact 3-manifold whose boundary is a limit cycle in the characteristic foliation of the disc. In particular, every overtwisted
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disc has Legendrian boundary. In the framework of open book foliations a transverse overtwisted disc plays a corresponding role:

**Definition 2.1.** [13] Def 4.1] Let \( D \subset M_{(S, \phi)} \) be an oriented disc whose boundary is positively braided (i.e., a transverse knot) with respect to the open book \((S, \phi)\). If the following are satisfied \( D \) is called a transverse overtwisted disc:

1. \( G_{-} \) is a connected tree with no fake vertices,
2. \( G_{+} \) is homeomorphic to \( S^1 \),
3. \( \mathcal{F}_{\phi}(D) \) contains no \( c \)-circles,

(Terminologies like \( G_{-}, G_{+}, \) fake vertices and \( c \)-circles are defined in §2.1 of [15].)

In [13] we show that the manifold \( M_{(S, \phi)} \) contains a transverse overtwisted disc if and only if the contact manifold \((M_{(S, \phi)}, \xi_{(S, \phi)})\) contains an overtwisted disc. Hence from now on, we may not distinguish a transverse overtwisted disc and a usual overtwisted disc, and often call a transverse overtwisted disc simply an overtwisted disc.

Next we review the notion of right-veering mapping classes then reprove Honda Kazez Matić’s tightness criterion in Theorem 2.4.

**Definition 2.2.** [14] Let \( \gamma, \gamma' \) be oriented properly embedded arcs in the surface \( S \) that start from the same point \( * \in \partial S \). Suppose, after some isotopy relative to the endpoints, \( \gamma \) and \( \gamma' \) realize the minimal geometric intersection number. We say that \( \gamma' \) lies strictly on the right side of \( \gamma \) if around the common starting point \( * \) the curve \( \gamma' \) strictly lies on the right side of \( \gamma \). In such case we denote \( \gamma > \gamma' \).

**Definition 2.3.** [14] Definition 2.1] Let \( C \) be a boundary component of \( S \). We say that \( \phi \in \text{Aut}(S, \partial S) \) is right-veering with respect to \( C \) if \( \gamma \geq \phi(\gamma) \) holds for any isotopy classes \( \gamma \) of properly embedded curves which start at a point on \( C \). We say that the diffeomorphism \( \phi \) (or the open book \( (S, \phi) \)) is right-veering if \( \phi \) is right-veering with respect to all the boundary components of \( S \). In particular, the identity \( \text{id} \in \text{Aut}(S, \partial S) \) is right-veering.

The following theorem gives a characterization of open books supporting tight contact structures.

**Theorem 2.4.** [14] Theorem 1.1] If \( \phi \) is not right-veering then \((S, \phi)\) supports an overtwisted contact structure.

**Remark 2.5.** Conversely, Honda, Kazez and Matić also prove that given an overtwisted contact structure \( \xi \) there exists a non-right-veering open book \((S, \phi)\) supporting \( \xi \) in [14] p.444 where Eliashberg’s classification of overtwisted contact structures [10] plays an important role. By Eliashberg’s classification, an overtwisted contact structure admits an open book which is a negative stabilization of some open book. Clearly such an open book has a non-right-veering monodromy. Therefore, a contact structure \( \xi \) is tight if and only if every open book supporting \( \xi \) is right-veering.

**Proof.** If \( \phi \) is not right-veering, then there exists a properly embedded oriented arc \( \alpha \subset S \) such that \( \phi(\alpha) > \alpha \). By [14] Lemma 5.2, there exists a sequence of properly embedded oriented arcs \( \alpha_0, \ldots, \alpha_k \) such that

\[ \ldots \]
(i) \( \alpha_0, \ldots, \alpha_k \) have the same initial point, \( n \in \partial S \),
(ii) \( \phi(\alpha) = \alpha_0 > \cdots > \alpha_k = \alpha \),
(iii) consecutive \( \alpha_i \) and \( \alpha_{i+1} \) have disjoint interiors and distinct terminal points \( p_i, p_{i+1} \).
Since \( \phi(\alpha_k) = \alpha_0 \) and \( \phi = id \) near the bindings we have \( p_0 = p_k \). We may assume that:
(iv) the terminal points \( p_0, p_1, \ldots, p_{k-1} \in \partial S \) are mutually distinct.

Let \( \beta_i \) (resp. \( \tilde{\beta}_i \)) be a sub-arc of \( \alpha_i \) whose endpoints are \( p_i \) and a point very close to \( p_i \) (resp. \( n \)). See Figure 1. We orient \( \alpha_i \) against the parametrization, i.e., the positive direction of \( \alpha_i \) is from \( p_i \) to \( n \). The orientation of \( \alpha_i \) induces those of \( \beta_i \) and \( \tilde{\beta}_i \). We define sets of oriented arcs for \( i = 0, \ldots, k \):
\[
A_i = \beta_0 \cup \cdots \cup \beta_{i-1} \cup \alpha_i \cup \beta_{i+1} \cup \cdots \cup \beta_k.
\]

Let \( t_i = \frac{i}{k} \in [0,1] \). In the following, we construct an oriented surface \( D_i \) properly embedded in the product region \( S \times [t_i, t_{i+1}] \), where \( i = 0, \ldots, k-1 \), such that \( D_i \cap S_{t_i} = -A_i \) and \( D_i \cap S_{t_{i+1}} = A_{i+1} \).

The surface \( D_i \) consists of \( k \) connected components; one non-product region and \( k-1 \) product regions defined by \( \beta_j \times [t_i, t_{i+1}] \) where \( j \neq i, i+1 \) and \( 0 \leq j \leq k \).

In the open book foliation of \( D_i \) the point \( n \) becomes a negative elliptic point, the points \( p_0, \ldots, p_{k-1} \) become positive elliptic points, and the arc \( \beta_j \times \{ t \} \) becomes an a-arc in the page \( S_t \). (See Prop. 2.2 of [15] for the definition of a-arcs.) The non-product component of \( D_i \) is defined by the movie presentation as sketched in Figure 2. It is a saddle shape surface with a positive hyperbolic point \( h_i \).

Now we glue \( D_i \) and \( D_{i+1} \) along \( A_{i+1} \subset S_{t_{i+1}} \) (\( i = 0, \ldots, k-2 \)) and obtain a surface \( D_0 \cup \cdots \cup D_{k-1} \subset S \times [0,1] \) whose oriented boundary is \( (-A_0) \cup A_k \). Since arcs \( \beta_1, \ldots, \beta_k \) are very close to \( \partial S \) and \( \phi = id \) near \( \partial S \), we have \( A_0 = \phi(A_k) \). So in the manifold \( M(S,\phi) \) we can identify \( A_0 \) and \( A_k \) and obtain a surface which we denote by \( D \).

The topological type of \( D \) is the disc and its open book foliation \( \mathcal{F}_{ob}(D) \) is depicted in Figure 3. Clearly our \( D \) is a transverse overtwisted disc. \( \square \)

**Remark 2.6.** Honda, Kazez and Matić’s proof and our proof are based on the same combinatorial lemma [14, Lemma 5.2] in order to use the assumption of right-veeringness. Our proof is more elementary and different from the original one that is written in the language of convex surface theory:

In [14], they prove the existence of a bypass, half of an overtwisted disc, by applying [14, Lemma 5.2] and the right-to-life principal [12, Lemma 2.9] [13, Proposition 2.2] which
involves the Legendrian realization principle and Eliashberg’s classification of tight contact structures on the 3-ball.

On the other hand, we use [14, Lemma 5.2] to explicitly construct a chain of positive elliptic and hyperbolic points surrounding the center negative elliptic point of an overtwisted disc. Hence our proof concretely visualizes an overtwisted disc.

3. OVERTWISTED DISCS IN RIGHT-VEERING OPEN BOOKS

The converse of Theorem 2.4 does not hold in general. Honda, Kazez and Matić [14] show that every contact structure is supported by a right-veering open book:
Their argument is the following: Given a contact structure \((M, \xi)\) choose a compatible open book \((S, \phi)\). For a boundary component \(C\) of \(S\) take two boundary-parallel arcs \(a_C\) and \(b_C\) such that the geometric intersection number \(i(a_C, b_C) = 2\). Apply positive stabilizations to \((S, \phi)\) along \(a_C\) and \(b_C\) for all the boundary components \(C\) on which \(\phi\) is non-right-veering. The new open book \((\hat{S}, \hat{\phi})\) is now right-veering, see [14] Prop.6.1, and supports the same contact structure \(\xi\).

Suppose that we start from a non-right-veering open book \((S, \phi)\), hence \(\xi\) is overtwisted. In the following we concretely describe how is an overtwisted disc embedded with respect to the stabilized right-veering open book \((\hat{S}, \hat{\phi})\).

**Example 3.1** (Overtwisted discs in Honda-Kazez-Matić’s stabilizations). Let \((S, \phi)\) be a not-right-veering open book. By the proof of Theorem 2.4 one can construct an overtwisted disc, \(D\), in \(M(\hat{S}, \hat{\phi})\). The open book foliation of \(D\) has a unique negative elliptic point, say \(n\), that lies on the binding component \(C \subset \partial S\), and \(k\) positive elliptic points \(p_1, \ldots, p_k\) and \(k\) positive hyperbolic points \(h_1, \ldots, h_k\). Let \(S_{t_i}\) be the singular fiber that contains \(h_i\). We may assume

\[
0 < t_1 < t_2 < \cdots < t_k < \frac{1}{2} < 1.
\]

For \(t \in [0, 1)\) let \(b_t \in S_t \cap D\) be the b-arc (cf. Prop.2.2 of [15]) that ends at the point \(n\).

Now we apply Honda-Kazez-Matić’s stabilization to get a right-veering open book \((\hat{S}, \hat{\phi})\).

The monodromy \(\hat{\phi}\) satisfies \(\hat{\phi} = T_\beta \circ T_\alpha \circ \phi\), where \(\alpha\) and \(\beta\) are core circles of the annuli plumbed along \(a_C\) and \(b_C\) and \(T_\alpha, T_\beta\) are positive Dehn twists along \(\alpha, \beta\).

We will construct an overtwisted disc \(\hat{D}\) by giving a movie presentation relative to the open book \((\hat{S}, \hat{\phi})\). For sake of simplicity we assume that:

- \(\phi\) is non-right-veering only along \(C\).
- \(p_k \in \partial S \setminus C\).

In the general case a construction of \(\hat{D}\) is similar but more complicated. It is obtained as an application of arguments in [17].

Choose stabilization arcs \(a_C\) and \(b_C\) such that \(i(\alpha, b_{1/2}) = 1\) and \(i(\beta, b_{1/2}) = 0\) as shown in Figure 4(a). Such arcs can always be found by the assumptions above.

To the region \(\{\hat{S}_t | t \in [0, \frac{1}{2}]\}\) add a continuous family of b-arcs that are co-cores of the annuli plumbed along \(a_C\), see Figure 4(a). We denote the positive and negative elliptic points which are the endpoints of the newly added b-arcs by \(p'\) and \(n'\). Except this family of b-arcs, the movie presentation of \(\hat{D}\) in the interval \([0, \frac{1}{2}]\) is the same as that of \(D\). Let \(\hat{b}_t := b_t\).

In the interval \([\frac{1}{2}, 1)\), \(\hat{D}\) is described as in the passage of \((b) \rightarrow (c) \rightarrow (d)\) of Figure 4. We form one negative and one positive hyperbolic points \(h_-\) and \(h_+\) as in \((b)\) and \((c)\), respectively. The describing arcs of \(h_-\) and \(h_+\) are parallel to \(\alpha\). We have \(\hat{b}_1 = T_{\alpha}^{-1}(b_1)\). Note that \(\hat{\phi}(\hat{b}_1) = (T_\beta \circ T_\alpha \circ \phi)(\hat{b}_1) = T_\beta \circ T_\alpha(T_\alpha^{-1}b_0) = T_\beta(b_0) = b_0 = \hat{b}_0\), moreover \(\hat{\phi}(\hat{D} \cap S_1) = \hat{D} \cap S_0\) so this movie presentation indeed defines an overtwisted disc in \(M(\hat{S}, \hat{\phi})\).
The open book foliations of the overtwisted discs $D$ and $\hat{D}$ are depicted in Figure 5, where $F_{\text{ob}}(\hat{D})$ is obtained by inserting two $bb$-tiles of opposite signs into the shaded region of $F_{\text{ob}}(D)$ that is bounded by $b_1$ and $b_1$.

Strictly speaking, the disc $\hat{D}$ is not a transverse overtwisted disc since the condition (3) of Definition 2.1 is not satisfied. However, applying the same technique we use in the alternative proof in [15] of the Bennequin-Eliashberg inequality [11, 11], the condition (3) will be satisfied. Thus we can regard $\hat{D}$ as an overtwisted disc.

4. Generalization of Lekili and Lisca’s examples

In [14, Question 6.2] Honda, Kazez and Matić ask whether a right-veering and non-destabilizable open book always supports a tight contact structure. Lekili [19] and Lisca [20] negatively answer the question by constructing examples. They study open book decompositions of 3-manifolds whose tight contact structures are well-studied and classified (in [19, Poincaré homology 3-spheres, and in [20, lens spaces). In both constructions the most technical points are showing that their open books indeed support overtwisted
contact structures. Advanced tools such as Ozsváth-Szabó’s Heegaard Floer invariants and properties of planar open books enable them to overcome the difficulty.

We generalize Lekili and Lisca’s examples in Theorem 4.1 below. Our proof of overtwistedness is direct and does not require any knowledge of classification of tight contact structures of ambient manifolds or Ozsváth-Szabó’s invariants.

**Theorem 4.1.** Let $S$ be a 2-sphere with four holes. Let $a, b, c, d, e$ be simple closed curves on $S$ as shown in Figure 6. Let $\Phi_{h,i,k} = T^h_a T^i_b T^c_c T^{k-1}_e$ where $T^x_x (x = a, b, c, d, e)$ denotes the right-handed Dehn twist along $x$. Then for all $h, i, k \geq 1$, $\Phi_{h,i,k}$ is right-veering and the open book $(S, \Phi_{h,i,k})$ is non-destabilizable and supports an overtwisted contact structure.

**Remark 4.2.** Lekili’s examples [19, Theorem 1.2 and Remark 4.1] are $\Phi_{2,i,1} (i \leq 5)$, and Lisca’s examples [20, Theorem 1.1] are $\Phi_{h,1,l} (h, l > 0)$.

**Proof.** We owe [19] [20] the proof that $\Phi_{h,i,k}$ is right-veering and non-destabilizable. Hence we only show that $(S, \Phi_{h,i,k})$ supports an overtwisted contact structure. We define a transverse overtwisted disc $D$ in the open book $(S, \Phi_{h,i,k})$ by the movie presentation as in Sketches (1)–(7) of Figure 7. For example, Sketch (1) depicts the page $S_0$ with the set of
arcs $D \cap S_0$. On $\partial S$ there are two negative elliptic points $n_1$, $n_2$, and $(2k+3)$ positive elliptic points $p_1, \ldots, p_{2k+3}$. The movie presentation shows that $\mathcal{F}_{ob}(D)$ contains two negative hyperbolic points and $2k+3$ positive hyperbolic points. Note that $\Phi_{h,i,k}(D \cap S_1) = D \cap S_0$.

The corresponding open book foliation $\mathcal{F}_{ob}(D)$ is depicted in Figure 8. We can verify that $D$ meets the conditions in Definition 2.1 of transverse overtwisted discs.

\[\Phi_{h,i,k}(D \cap S_1) = D \cap S_0.\]

**Remark 4.3.** After the announcement of this example (on December 26, 2011), Kazez and Roberts [18] found more counterexamples to the conjecture of Honda, Kazez and Matić.

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• The describing arc joining the $b$-arc emanating from $n_1$ and the $a$-arc from $p_4$ (or $p_{2i+2}$ in the $i$-th iteration) represents a positive hyperbolic point.
• Iterate the Steps (2) and (3) for $k$ times.

• The shaded boxes labeled $h, 2k$ contain parallel $h, 2k$ arcs.
• The edges out of the shaded boxes are weighted as indicated.
• The describing arc joins the $b$-arc from $n_2$ to $p_{2k+1}$ and the $a$-arc emanating from $p_{2k+3}$ and it represents a positive hyperbolic point.

• The describing arc joins the $b$-arc from $n_1$ to $p_{2k+2}$ and the $a$-arc emanating from $p_1$ and it represents a positive hyperbolic point.
The describing arc joins the $b$-arc from $n_2$ to $p_{2k+3}$ and the $a$-arc emanating from $p_2$ and it represents a positive hyperbolic point.

- The leaves in the page $S_1$. It satisfies $\Phi_{h,i,k}(D \cap S_1) = D \cap S_0$.

Figure 7. Movie presentation of a transverse overtwisted disc in $(S, \Phi_{h,i,k})$.

Figure 8. The open book foliation $\mathcal{F}_{ob}(D)$. 
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