Schrödinger Equation for Particle with Friction

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ABSTRACT

A new quantum mechanical wave equation describing a particle with frictional forces is derived. It depends on a parameter $\alpha$ whose range is determined by the coefficient of friction $\gamma$, that is, $0 \leq \alpha \leq \gamma$. For one extreme value of this parameter, $\alpha = 0$, we recover Kostin’s equation. For the other extreme value, $\alpha = \gamma$, we obtain an equation in which friction manifests in “magnetic” type terms. It further exhibits breakdown of translational invariance, manifesting through a symmetry breaking parameter $\beta$, as well as localized stationary states in the absence of external potentials. Other physical properties of this new class of equations are also discussed.
It is known that not all classical dynamical systems can be successfully quantized. Apart from problems such as nonrenormalizability, topological obstructions, anomalies etc., there is a whole class of systems for which there is no available method of quantization, namely non-hamiltonian systems. The most common examples are systems with dissipation, although these are by no means the only ones. The Bargmann-Wigner higher spin field equations [1] and field theories of fundamental anyons [2] are notable cases too. The “canonical” example of a non-hamiltonian system is, nevertheless, the case of a particle on the line with friction. Its equation of motion is
\[ m\ddot{x} + \gamma \dot{x} + \frac{dV}{dx} = 0 \] (1)
where overdot denotes time derivative and \( V(x) \) is the potential of the particle. There is no local action functional which produces (1) as the equation of motion, and therefore no canonical structure and no quantum mechanics. This model (and its higher dimensional versions), apart from its interest as a phenomenological description of dissipation due to interactions, also has recently found applications in string theory [3].

There are two points of view one could adopt on the previous problem. One is to consider it as a fundamental physical question as to whether and how such systems can be quantized. The other is to consider such dissipative systems as phenomenological descriptions of more fundamental (hamiltonian) systems interacting with a many degrees of freedom reservoir, in which the dissipation is a macroscopic manifestation of these interactions. The question, then, is to find an adequate quantum mechanical description of these systems without explicitly involving the degrees of freedom of the reservoir. The first question is more fundamental and mathematically intriguing, while the second is more physical.

There have been several attempts to this problem, which can be classified into two similar categories as above: those who start with an initial expanded system including the reservoir and then “integrate out” the extra degrees of freedom [4-8], and those who start on the outset with a modified quantization scheme which
classically reduces to the desired dynamics [9,10]. The first approach is clearly more physically motivated. It has, however, some disadvantages. For instance, coupling a particle to a string of oscillators to reproduce dissipation and then integrating them out, leads to a state where the fourth moment of the particle position diverges, that is

$$< x^{2n} > = \infty \text{ for } n \geq 2$$

(2)

This is because the zero-point motion of the infinite set of oscillators perturbs the particle in a substantial way. Obviously this is an unsatisfactory feature which would hopefully be absent in a fundamental quantization procedure of dissipative systems. It is also not clear that the answers obtained are in general model-independent.

Using the second approach, Kostin [9] has proposed a modified Schrödinger equation for the model (1) in which the friction is reproduced through a wavefunction-dependent potential. This has some unusual and rather controversial properties: it violates the superposition principle, and has stationary states in which the energy does not dissipate. (This equation was rederived by Yasue [11] using Nelson’s stochastic quantization scheme [12].) Although the above properties are not necessarily fatal (we know that the quantum mechanics of these systems must be radically different from the standard one), probably because of them Kostin’s equation has not been used very extensively.

The purpose of this paper is to present an alternative Schrödinger equation for this process. In fact, we will derive a family of equations which contains Kostin’s as a special case. Physical properties of these new equations, in comparison with Kostin’s, will also be discussed.

The main new feature of our approach consists of allowing a wavefunction-dependent first-derivative term in Schrödinger’s equation, that is

$$i \dot{\psi} = -\frac{1}{2}d^2 \psi + Wd\psi + (U + V)\psi$$

(3)
where $\psi$ is the wavefunction and $W, U$ are potentials explicitly depending on $\psi$ in addition to $x, t$. (From now on we adopt the shorthand $d$ for the $x$-derivative and put $m = \hbar = 1$.) Such a first-derivative term in the Schrödinger equation is very natural for an equation of motion with first-order terms in time derivatives. In the case of magnetic forces, e.g., which are also first-order in time derivatives, such terms are present, arising from the gauge potential. In fact, performing a $\psi$-dependent wavefunction redefinition, we can always bring (3) to the form

$$i\dot{\psi} = \Phi^*(-\frac{1}{2}d^2 + U + V)\Phi\psi = -\frac{1}{2}D^2\psi + (U + V)\psi$$

(4)

for some new $U$, where

$$\Phi = e^{i\phi(\psi,x,t)}$$

(5)

is now a wavefunction-dependent phase and

$$D = d + iF, \quad F = d\phi.$$  

(6)

The advantage of (4) over (3) is that it has manifestly unitary time evolution for real $U$ and $F$. This means, in particular, that the normalization

$$N = \int x\psi^*\psi$$

(7)

is preserved in time. Defining the expectation value of $x$

$$<x> = \int x\psi^*\psi$$

(8)

and using (4) we get

$$<\dot{x}> = \int x\psi^*\Phi^*(-i\phi)\Phi\psi = -\frac{i}{2}\int x\psi^*D\psi - (D\psi)^*\psi$$

(9)

$$<\ddot{x}> = \int x\psi^*(-dV - dU + F)\psi$$

(10)

In the above we integrated by parts, assuming good behavior of $\psi$ at infinity. A
sufficient condition for (4) to have the desired classical limit is that the equation
of motion (1) hold at the expectation value level. Thus we must have

\[
< \ddot{x} > = -\gamma < \dot{x} > - < dV >
\]  

(11)

and therefore

\[
\int_x \psi^*(-dU + \dot{F})\psi = i\frac{\gamma}{2} \int_x \psi^*D\psi - (D\psi)^*\psi
\]  

(12)

Writing \(\psi\) in terms of its phase and logarithmic modulus

\[
\psi = e^{\rho+i\theta}
\]  

(13)

we perform the time derivative in (12) using (4). Adopting the notation where
subscripts denote partial derivative with respect to \(\rho, \theta, x, t\), we obtain after some
calculation

\[
\int_x \psi^*D\psi - (D\psi)^*\psi = \int_x -\frac{i}{4}d\rho F_{\rho} [\psi^*D\psi - (D\psi)^*\psi]
\]

\[
+ \frac{1}{4} F_{\theta} [\psi^*D^2\psi + (D^2\psi)^*\psi - 4(U + V)\psi^*\psi]
\]

\[
+ \frac{1}{2}dU_{\rho}\psi^*\psi + \frac{i}{2}U_{\theta} [\psi^*D\psi - (D\psi)^*\psi]
\]

\[
+ U_{\theta} F\psi^*\psi + (F_t - U_x)\psi^*\psi
\]  

(14)

The above equation must hold for arbitrary (well-behaved) \(\psi\).

We shall choose not to have explicit time dependence in the terms of our
equation, so that we do not spoil time invariance, and put \(F_t = 0\). We will also
not allow \(F\) and \(U\) to be nonlocal expressions of \(\psi\) or functions of derivatives of
\(\psi\), because then (4) would become a higher-order and/or nonlocal equation. We
notice, then, that the second term in the right hand side of (14) is second order
in \(x\)-derivatives. Therefore, since the left-hand side is first order in derivatives,
this term must vanish. Similarly, if we want the first term to be first order in
derivatives, $F_ρ$ should no more contain $ψ$. So we must have

$$F_{ρρ} = F_θ = 0 , \text{ or } F = ρc(x) + c_1(x) \quad (15)$$

The $ψ$-independent term $c_1(x)$ can always be gauged away by redefining the phase
of $ψ$, so we will omit it.

The relevant terms that will reproduce friction are, now, the first and fourth
terms, so we must have

$$-\frac{i}{4} dF_ρ + \frac{i}{2} U_θ = \frac{i}{2} γ \quad (16)$$

and substituting (15) we obtain

$$U = θ(γ + \frac{1}{2} c'(x)) + ε(ρ, x) \quad (17)$$

Finally, the remaining terms in (14) must be a total derivative, so that they do not
contribute to the integral, that is

$$e^{2ρ}\left(\frac{1}{2}dU_ρ + U_θF - U_x\right) = dA \quad (18)$$

for some $A$. Using (17) and (15) we get

$$e^{2ρ}\left[\frac{1}{2}ε_ρdρ + \frac{1}{2}ε_ρx + (γ + \frac{1}{2}c')ρc - \frac{1}{2}θc'' - ε_x\right] = A_ρdρ + A_θdθ + A_x \quad (19)$$

Since there is no $dθ$ term, $A$ must be independent of $θ$ and thus we must have

$$c''(x) = 0 , \text{ so } c(x) = -2α(x - β) \quad (20)$$

(the choice of factors is for later convenience). Again, β can be shifted away with
a shift in the coordinate $x$ and shall be temporarily omitted. So $F$ becomes

$$
F = -2\alpha x \rho 
$$

For (19) to be a perfect differential in $\rho$ and $x$, then, we have the consistency condition

$$
A_{\rho x} = \left[ e^{2\rho} \frac{1}{2} \epsilon_{\rho \rho} \right]_x = \left[ e^{2\rho} \left( \frac{1}{2} \epsilon_{\rho x} - (\gamma - \alpha) 2\alpha x \rho - \epsilon_x \right) \right]_\rho = A_{x \rho} 
$$

which determines the eventual form of $\epsilon$ and $U$ as

$$
\epsilon(\rho, x) = -\alpha (\gamma - \alpha) (\rho + \frac{1}{2}) x^2 + \epsilon(\rho) 
$$

$$
U = (\gamma - \alpha) \theta - \alpha (\gamma - \alpha) (\rho + \frac{1}{2}) x^2 + \epsilon(\rho) 
$$

Equations (21) and (23) constitute the solution of (11) and determine the Schrödinger equation of the particle.

The above solutions depend on two arbitrary parameters, $\alpha$ and $\beta$ (the latter one hidden in the choice of origin), as well as an arbitrary function of $\psi^* \psi$, $\epsilon(\rho)$. This last arbitrariness is present also in the ordinary case without the friction term, since such a term does not alter the expectation value of the equations of motion. Although its significance is not quite clear, we shall choose to omit it, since it does not seem to be relevant to the problem of incorporating friction. The full-blown Schrödinger equation for the particle with friction, restoring $\beta$, becomes thus

$$
i \dot{\psi} = -\frac{1}{2} \left( \frac{d - i \alpha (x - \beta) \ln(\psi^* \psi)}{\psi^* \psi} \right)^2 \psi + V\psi 
- \frac{i}{2} (\gamma - \alpha) \ln \frac{\psi}{\psi^*} \psi - \frac{1}{2} \alpha (\gamma - \alpha) \left( 1 + \ln(\psi^* \psi) \right) (x - \beta)^2 \psi
$$

The parameter $\alpha$ is interesting. Apparently, it can take any real value. Notice, however, that for any normalizable wavefunction, $\psi$ must vanish fast enough at
\( x \to \pm \infty \) and thus \( \ln \psi^* \psi \to -\infty \). Therefore, the second term in the potential \( U \) will be badly behaved if \( \alpha (\gamma - \alpha) < 0 \) and the above equation will be unstable. To avoid this, we must have

\[
0 \leq \alpha \leq \gamma
\]

(25)

Therefore, the range of \( \alpha \) is restricted by the magnitude of \( \gamma \). In particular, for \( \gamma = 0 \) this parameter is absent, which shows that it is specific to the friction problem. Also, the parameter \( \beta \) becomes irrelevant when \( \alpha = 0 \) and thus when \( \gamma = 0 \), which shows that it also is a characteristic of the friction. For the extreme value \( \alpha = 0 \) we recover Kostin’s equation. For the other extreme value \( \alpha = \gamma \) we obtain an equation without additional potentials but with a “magnetic” (gauge) term.

Indeed, we can view \( F \) as the spatial component of a gauge field and \( U \) as the time component of the same field. Notice, then, that the quantity \( \dot{F} - dU \) which appears in (12) is the field strength of the above gauge potential. One could worry, then, that since the above equations for all values of \( \alpha \) reproduce the same equation (12), they are gauge equivalent. This, however, is not the case. Although it is true that all equations have the same expectation value of the field strength (the integral in (12)), they differ locally by total derivative terms. Therefore, although they all have the same classical limit, they describe different quantum mechanics. In fact, if we wanted to perform a gauge transformation so as to gauge away the spatial component \( F \) and trade it for a potential term, then we would end up with a potential which would contain nonlocal terms in the wavefunction. This nonlocality is, however, an artifact of the temporal gauge \( F = 0 \), since in our gauge (24) is perfectly local. This nonlocality is, perhaps, the reason why our equation was not previously discovered in Kostin’s approach.

A remarkable property of (24) is that, in spite of its nonlinearity, it is still invariant under rescaling of \( \psi \). This means, in particular, that physics does not
depend on the normalization of the wavefunction. Specifically, the transformation
\[ \psi \rightarrow \exp \left[ i\theta_0 e^{-\gamma t} + i\alpha \rho_0 (x - \beta)^2 + \rho_0 \right] \psi \] (26)
with \( t_0, \rho_0 \) constants, leaves (24) invariant and constitutes the generalization of the original complex rescaling transformation of the linear Schrödinger equation for the case of nonzero friction. Notice that, for \( \alpha \neq \gamma \), it explicitly involves time and for \( \alpha \neq 0 \) it explicitly involves the space coordinate.

The original Kostin’s equation exhibited stationary nondissipating states. In fact, it is easy to see that all the energy eigenstates of the hamiltonian without friction remain stationary states for Kostin’s equation. The dissipation, then, appears only upon mixing these states due to the nonlinear nature of the equation. It is also easy to see that the special case \( \alpha = \gamma \) of (24) also exhibits the same stationary states. It is less clear, however, whether (24) for arbitrary \( \alpha \) will have such states. A stationary state has the form
\[ \psi = e^{i\theta(x,t)} \chi(x). \] (27)

Therefore, the gauge potential term \( F \) becomes now independent of time and can be gauged away through the redefinition
\[ \theta = \varphi + \int_{y=x_o}^{x} \alpha y \ln(\chi^* \chi) \] (28)
where \( x_o \) is an irrelevant constant (we have shifted \( \beta \) back to zero for convenience). Using, now, (24) we find that the phase \( \varphi \) will be independent of \( x \) while \( \chi \) will satisfy the nonlinear eigenvalue equation
\[ \left[ -\frac{1}{2} d^2 + V + \alpha(\gamma - \alpha) \left\{ \int_{y=x_o}^{x} y \ln(\chi^* \chi) - \frac{1}{2} \left( 1 + \ln(\chi^* \chi) \right) x^2 \right\} - \frac{i}{2} (\gamma - \alpha) \ln \frac{\chi^*}{\chi} \right] \chi = E \chi \] (29)
with the phase satisfying
\[ -\dot{\varphi} = E + (\gamma - \alpha)\varphi \quad (30) \]

The eigenvalue \( E \) in general changes upon rescaling of \( \chi \) and thus does not correspond to the energy of the particle. Also, although in the frictionless case \( \chi \) can always be chosen real, this is not necessarily true any more in the present nonlinear case.

It is clear that for \( \alpha = 0 \) or \( \gamma \) (29) is identical to the eigenvalue problem of the frictionless Hamiltonian and we recover all energy eigenstates as stationary states. For arbitrary \( \alpha \) we cannot, at the moment, make a general statement on the existence of such states, although it is clear that if they exist they will be different from the ordinary ones. This is physically appealing, since we expect friction to alter the properties of the states of the system. In fact, the value of \( \alpha \) may be fixed by further examining this issue.

We can examine some qualitative properties of (29). Notice that, if the wavefunction \( \chi \) is real and falls off exponentially at infinity with some power law in \( x \), that is,
\[ \ln \chi \rightarrow -\kappa |x|^n \quad \text{for} \quad |x| \rightarrow \infty , \quad \kappa, n > 0 \quad (31) \]
then the \( \chi \)-dependent potentials in (29) will behave as
\[ \frac{\alpha(\gamma - \alpha)\kappa n}{n + 2} |x|^{n+2} \quad (32) \]
confirming that the above equation is stable for \( 0 < \alpha < \gamma \) in this case. It is tempting to speculate that the physically relevant value of \( \alpha \) is the one achieving maximal stability, that is, \( \alpha = \frac{1}{2}\gamma \). To further check this, we will examine the case of an external harmonic potential at \( x_o \):
\[ V(x) = \frac{1}{2} \omega^2 (x - x_o)^2 \quad (33) \]
and treat the friction perturbatively, that is, assume \( \gamma \ll \omega \). Starting with the harmonic oscillator ground state as the zeroth-order wavefunction and plugging
it into the left-hand side of (29) we obtain a first-order effective potential

\[ V_1(x) = \frac{1}{2} \tilde{\omega}^2 (x - \tilde{x}_o)^2 + \frac{1}{4} \alpha (\gamma - \alpha) \omega x^4 + \text{constant} \]  \hspace{1cm} (34)

where

\[ \tilde{\omega}^2 = \omega^2 - \alpha (\gamma - \alpha) \quad \text{and} \quad \tilde{x}_o = \frac{\omega^2}{\tilde{\omega}^2} x_o. \]  \hspace{1cm} (35)

We see that the first-order effect of friction is a shift of the origin of the harmonic potential (of which we will say more later) as well as a change of its strength, in addition to the appearance of anharmonic terms. If we choose now \( \alpha = \frac{1}{2} \gamma \) we get

\[ \tilde{\omega}^2 = \omega^2 - \left( \frac{\gamma}{2} \right)^2. \]  \hspace{1cm} (36)

This is exactly the shift in the frequency of damped oscillations of an oscillator with friction, further corroborating this choice. The anharmonic terms account for the quantum mechanical effects of the friction and ensure the stability of the problem even for the overdamped case \( \gamma > 2\omega \) (in this case, of course, perturbation theory is not valid).

Remarkably, when the external potential is zero we can find an exact stationary state for the problem. It is easy to check that

\[ \chi = C \exp \left[ -\frac{\alpha (\gamma - \alpha)}{12} x^4 \right] \]  \hspace{1cm} (37)

with \( C \) a constant, is a solution of (29). Thus, for \( \alpha \neq 0, \gamma \), the particle has a localized stationary state in the absence of external potentials, due entirely to friction. Even more remarkably, the particle exhibits spontaneous violation of translation invariance. Remember that the origin \( x = 0 \) is fixed by the requirement that the term \( \beta \) in (20) vanish. The wavefunction (37) then is centered around this special point. Choosing a different value for \( \beta \) would place this state at the point \( x = \beta \). The parameter \( \beta \) then is a symmetry breaking parameter for the problem, due entirely to friction (remember that \( \beta \) becomes irrelevant when \( \gamma = 0 \)).
A similar effect is also evident in the first-order potential (34) in the problem with the harmonic potential, manifesting in the shift of the origin of the harmonic forces. If this point coincided with the point \( x = \beta \), there would be no such shift. It is not clear, of course, that this effect would survive higher-order in \( \gamma \) effects. At the classical limit, for instance, such a shift cannot occur and thus it must be washed out by nonlinear effects.

It seems odd that there should be spontaneous symmetry breaking in a finite degrees of freedom problem. It is not so surprising, however, in the picture where friction is reproduced by coupling the particle to a continuous infinity of harmonic oscillator degrees of freedom, whose frequency extends all the way down to zero. It is conceivable that an appropriate choice of such a system exhibits spontaneous symmetry breaking. Clearly this point deserves further investigation. We find it curious, however, that our direct approach to the problem should invoke such “memories” of a possible infinite dimensional description.

A property of Kostin’s equation was that it involved the phase \( \theta \) of the wavefunction which is defined only modulo \( 2\pi \). This is inconsequential in most cases, since this only amounts to a time-dependent redefinition of the phase of the solutions (see (26)). It is problematic, however, on spaces with topologically nontrivial loops, since, in general, there is no way of defining a single-valued phase on such spaces. For instance, Kostin’s equation cannot deal with a particle on the circle. (This is related to the fact that a smooth decay of the energy is incompatible with momentum quantization on the circle.) Our equation with \( \alpha = \gamma \), on the other hand, does not involve the phase. It exhibits nevertheless a similar problem since it is not explicitly translation invariant, due to the gauge field \( F \). (This is related to the parameter \( \beta \), as explained above.) It is not clear, then, how it can manifest the correct periodicity on the circle. The case of general \( \alpha \) presents a mixture of the two problems.

Finally, we point out that the nonlinearity of the above equations is not unfamiliar. Viewed as field equations, it simply means that they represent interacting
theories which, in the second quantized picture, have particle creation and annihilation. This is very much in line of the description of friction as an interaction process with a reservoir of other particles. It should be noted, however, that there is no action which gives (24) as a lagrangian equation of motion for any $\alpha$. The non-hamiltonian nature of the problem emerges, then, in the next level of quantization. Therefore, the second quantization of the above theories is an open issue.

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