Bilinear form, soliton, breather, hybrid and periodic-wave solutions for a (3+1)-dimensional Korteweg–de Vries equation in a fluid

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Abstract Fluids are studied in such disciplines as atmospheric science, oceanography and astrophysics. In this paper, we investigate a (3+1)-dimensional Korteweg–de Vries equation in a fluid. Bilinear form and \( N \)-soliton solutions are obtained, where \( N \) is a positive integer. Via the \( N \)-soliton solutions, we derive the higher-order breather solutions. We observe the interaction between the two perpendicular first-order breathers on the \( x-y \) and \( x-z \) planes and the interaction between the periodic line wave and the first-order breather on the \( y-z \) plane, where \( x \), \( y \) and \( z \) are the independent variables in the equation. We discuss the effects of \( \alpha \), \( \beta \), \( \gamma \) and \( \delta \) on the amplitude of the second-order breather, where \( \alpha \), \( \beta \), \( \gamma \) and \( \delta \) are the constant coefficients in the equation: Amplitude of the second-order breather decreases as \( \alpha \) increases; amplitude of the second-order breather increases as \( \beta \) increases; amplitude of the second-order breather keeps invariant as \( \gamma \) or \( \delta \) increases. Via the \( N \)-soliton solutions, hybrid solutions comprising the breathers and solitons are derived. Based on the Riemann theta function, we obtain the periodic-wave solutions, and find that the periodic-wave solutions approach to the one-soliton solutions under a limiting condition.

Keywords Fluid · (3+1)-dimensional Korteweg–de Vries equation · Bilinear form · Soliton solutions · Breather solutions · Hybrid solutions · Periodic-wave solutions

1 Introduction

Fluids have been studied in such disciplines as atmospheric science, oceanography and astrophysics [1–6]. Analytic solutions for the nonlinear evolution equations (NLEEs) such as the soliton, breather, rogue-wave and periodic-wave solutions have been applied in nonlinear optics, fluid mechanics and plasma physics [7–23]. Hybrid solutions among the nonlinear waves have been studied [24–28]. Methods have been proposed to construct the analytic solutions, such as the bilinear method, Lie group analysis, Bäcklund transformation and Darboux transformation [29–37]. For instance, the multiple soliton solutions have been derived for a (3+1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation via the bilinear method [38]. Breather solutions have been investigated in a higher-order nonlinear Schrödinger equation based on the Darboux transformation [39]. Periodic-wave solutions have been studied in a (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation via the Riemann theta function [40].
Korteweg–de Vries (KdV) equation in a fluid,
\[ u_{ty} + u_{xxx} + \alpha (u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} + \delta u_{zy} = 0, \]  
(1)
where \( u \) is a real differential function depending on the independent variables \( x, y, z \) and \( t \), \( \alpha, \beta, \gamma \) and \( \delta \) are the real constants, and the subscripts indicate the partial derivatives. Equation (1) has been reduced to some special cases in the fluids, as follows:

- When \( \alpha = 3 \) and \( \beta = \gamma = \delta = 0 \), through the reduction \( \partial_x = \partial_y \), Eq. (1) has been simplified as the (1+1)-dimensional KdV equation [42],
\[ U_t + 6UU_x + U_{xxx} = 0, \]  
(2)
which can describe certain phenomena in plasmas and fluids, such as the ion-acoustic solitons in a plasma and the shallow water waves under gravity [42–44], where \( U(x, t) \) is the wave function depending on the scaled space variable \( x \) and time variable \( t \);
- When \( \alpha = -3 \) and \( \beta = \gamma = \delta = 0 \), Eq. (1) has been degenerated to the (2+1)-dimensional BLMP equation [41,45],
\[ u_{yy} + u_{xxxx} - 3 (u_y u_x)_x = 0. \]  
(3)
Equation (3) has been applied as a (2+1)-dimensional model for the interaction of the Riemann wave along the \( y \) axis and a long wave along the \( x \) axis [45];
- When \( \alpha = 3, \beta = \gamma = 1 \) and \( \delta = 0 \), Eq. (1) has been reduced into a (2+1)-dimensional extended BLMP equation for an incompressible fluid [46],
\[ u_{ty} + u_{yyyy} + 3 (u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} = 0. \]  
(4)
Equation (1) has passed the Painlevé test, and one-soliton and two-soliton solutions have been derived [41].

However, to our knowledge, bilinear form, \( N \)-soliton, breather, hybrid and periodic-wave solutions for Eq. (1) have not been considered, where \( N \) is a positive integer. In Sect. 2, bilinear form and \( N \)-soliton solutions for Eq. (1) will be studied. In Sect. 3, the higher-order breather solutions for Eq. (1) will be obtained. In Sect. 4, hybrid solutions comprising the solitons and breathers for Eq. (1) will be derived. In Sect. 5, periodic-wave solutions and their asymptotic behaviors for Eq. (1) will be investigated. Our conclusions will be given in Sect. 6.

2 Bilinear form and \( N \)-soliton solutions for Eq. (1)

Via the variable transformation [41]
\[ u = \frac{6}{\alpha} (\ln f)_x, \]  
(5)
Eq. (1) can be converted into the following form:
\[ f(2f_{xy}, f_{xxx} - f_{yy} - 3f_{xx} x) - f_{xx}(f_y + f_{yy} - 3f_{xx} x) = 0. \]  
(6)

where \( f \) is a real differentiable function of \( x, y, z \) and \( t \). Equation (6) can be integrated with respect to \( x \), with the integral constant equal to zero, and the bilinear form for Eq. (1) is derived as
\[ (D_x D_y + D_y^3 D_x^2 + \beta D_x^2 + \gamma D_y^2 + \delta D_y D_y) f \cdot \tilde{f} = 0, \]  
(7)
where \( D_x, D_y, D_z \) and \( D_t \) are the bilinear differential operators defined by [29]
\[ D^l_x D^m_y D^h_z D^p_t F \cdot G = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^h \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^p F(x, y, z, t) G(x', y', z', t'), \]
with \( F \) being a differentiable function of \( x, y, z \) and \( t \), \( G \) being a differentiable function of the formal variable \( x', y', z' \) and \( t' \), while \( l, m, h \) and \( p \) being the non-negative integers.

In order to derive the \( N \)-soliton solutions for Eq. (1), we assume the following expression:
\[ f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^N f_N, \]  
(8)
where \( \varepsilon \) is a real constant and \( f_\varepsilon \)’s \( (\varepsilon = 1, 2, 3, \ldots, N) \) are the real functions of \( x, y, z \) and \( t \). Substituting Expansion (8) into Bilinear Form (7) and making the coefficients of \( \varepsilon^\varepsilon \)’s vanish, the \( N \)-soliton solutions for Eq. (1) can be derived as
$$u = \frac{6}{\alpha} (\ln f)_x,$$

$$f = \sum_{\mu=0,1}^{N} \exp \left[ \sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leq i < j}^{N} \mu_i \mu_j \ln (A_{ij}) \right],$$

where

$$\xi_t = a_t(x + b_t y + c_i z + d_i t) + \xi_t^0,$$

d$ = - a_t^2 - b_t^{-1} \beta - b_t \gamma - c_i \delta,$

$$C_{ij} = \frac{C_{ij}}{M_{ij}},$$

$$M_{ij} = 3a_t^2 b_t^2 b_j + 3a_t^2 b_t b_j^2 - 3a_t a_j b_t b_j (b_t + b_j) - (b_t - b_j)^2 \beta,$$

$$A_{ij} = \frac{C_{ij}}{M_{ij}},$$

$$C_{ij} = 3a_t^2 b_t^2 b_j + 3a_t^2 b_t b_j^2 - 3a_t a_j b_t b_j (b_t + b_j) - (b_t - b_j)^2 \beta,$$

$$A_{ij} > 0, b_t \neq 0 \text{ and } M_{ij} \neq 0, i, j = 1, 2, \ldots, N,$$

$$\xi^0_t \text{'s are the complex constants, } \sum_{\mu=0,1}^{N} \text{ denotes a summation over all the possible combinations of } \mu_1 = 0, 1, \mu_2 = 0, 1, \text{ and } \mu_N = 0, 1,$$

$$\sum_{1 \leq i < j}^{N} \text{ indicates a summation over all possible pairs } (i, j) \text{ under the condition } 1 \leq t < j \leq N.$$

3 The higher-order breather solutions for Eq. (1)

In this section, motivated by Ref. [47], we construct the higher-order breather solutions for Eq. (1) with certain values of the parameters $a_t$'s, $b_t$'s, $c_i$'s and $\xi^0_t$'s in $N$-Soliton Solutions (9) with $N$ being an even integer. We assume that

$$N = 2T, \quad a_r = a_{T+r} = a_r + i a_r,$$

$$b_r = b_{T+r} = b_r + i b_r,$$

$$c_r = c_{T+r} = c_r + i c_r, \quad \xi^0_r = \xi^0_{T+r} = \xi^0_T + i \xi^0_T,$$

where $a_r$'s, $b_r$'s, $c_r$'s and $\xi^0_r$'s are all the real constants, $*$ represents the complex conjugate, $T$ is a positive integer, $r = 1, 2, \ldots, T$ and $i^2 = -1$. The $T$th-order breather solutions for Eq. (1) are given as

$$u = \frac{6}{\alpha} (\ln f)_x,$$

$$f = \sum_{\mu=0,1}^{2T} \exp \left[ \sum_{j=1}^{2T} \mu_j \xi_j + \sum_{1 \leq i < j}^{2T} \mu_i \mu_j \ln (A_{ij}) \right],$$

where

$$\xi_r = \xi^0_{T+r} = \xi_{r+1} + i \xi_{r+2},$$

$$\xi_{r+1} = a_{r+1} x + (a_{r+1} b_{r+1} - a_r b_r) y$$

$$+ (a_{r+1} c_{r+1} - a_r c_r) z - [a_{r+1}^2 - 3a_{r+1} a_r] b_{T+1}^{-1} (a_{r+1} b_{T+1} + a_r b_r) \beta - (a_{r+1} b_{T+1} - a_r b_r) \gamma - (a_{r+1} c_{r+1} - a_r c_r) \delta t + \xi^0_{r+1},$$

$$\xi_{r+2} = a_{r+2} x + (a_{r+2} b_{r+2} + a_r b_r) y$$

$$+ (a_{r+2} c_{r+2} + a_r c_r) z + [a_{r+2}^2 - 3a_{r+2} a_r] b_{T+1}^{-1} (a_{r+2} b_{T+1} - a_r b_r) \beta$$

$$+ (a_{r+2} b_{T+1} + a_r b_r) \gamma + (a_{r+2} c_{r+2} + a_r c_r) \delta t + \xi^0_{r+2},$$

$$A_{ij} = \frac{C_{ij}^*}{M_{ij}},$$

$$C_{ij} = 3a_t^2 b_t^2 b_j + 3a_t^2 b_t b_j^2 - 3a_t a_j b_t b_j (b_t + b_j)$$

$$- (b_t - b_j)^2 \beta, \quad M_{ij} = 3a_t^2 b_t^2 b_j + 3a_t^2 b_t b_j^2 + 3a_t a_j b_t b_j (b_t + b_j)$$

$$- (b_t - b_j)^2 \beta, \quad A_{r,T+r} > 1.$$
The second-order breather via Solutions (11) with $T = 2$, $\alpha = \beta = \gamma = \delta = 1$, $a_1 = a_1^* = -1 - 2i$, $b_1 = b_1^* = c_1 = c_1^* = -\frac{1}{2} - \frac{i}{4}$, $a_2 = a_2^* = 2i$, $b_2 = b_2^* = -1 + i$, $c_2 = c_2^* = -\frac{1}{2} + 2i$, $\xi_0 = \xi_0^* = \xi_0^2 = \xi_0^4 = 0$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$

\[
\frac{2\pi}{a_1^*c_1^*+a_2^*c_2^*} \text{ in the } z \text{ direction. The locations of the first-order breathers are related to } \xi_{11}. \text{ When } a_{12} = b_{12} = c_{12} = \xi_{12}^0 = 0 \text{ in Solutions (12), the first-order breather solutions can be degenerated to the one-soliton solutions for Eq. (1).}

When supposing $T = 2$ in Solutions (11), we can obtain the second-order breather solutions for Eq. (1). The second-order breathers depict the interactions between the two first-order breathers. As discussed on the first-order breathers, we can construct the second-order breathers comprising different first-order breathers. For example, the second-order breathers consisting of the two perpendicular first-order breathers can be constructed on the $x - y$ plane under the condi-

\[
(\text{a}_1) \quad t = -\frac{1}{3} \\
(\text{a}_2) \quad t = 0 \\
(\text{a}_3) \quad t = \frac{1}{3}
\]

\[
(\text{b}_1) \quad t = -\frac{1}{3} \\
(\text{b}_2) \quad t = 0 \\
(\text{b}_3) \quad t = \frac{1}{3}
\]

\[
(\text{c}_1) \quad t = -\frac{1}{3} \\
(\text{c}_2) \quad t = 0 \\
(\text{c}_3) \quad t = \frac{1}{3}
\]
Bilinear form, soliton, breather, hybrid and periodic-wave solutions

We observe the interaction between the two perpendicular first-order breathers on the $x-y$ and $x-z$ planes, as presented in Figs. 1a and b. In Fig. 1c, since $\alpha_{11} \neq 0$, $\alpha_{11}b_{11} - \alpha_{12}b_{12} = 0$ and $\alpha_{11}c_{11} - \alpha_{12}c_{12} = 0$, $\xi_{11}$ is not related to $y$ and $z$. One of the first-order breathers is reduced into the periodic line wave. When $t = -\frac{1}{3}$, we observe the first-order breather, as shown in Fig. 1c.1. As $t$ goes by, we can find that the periodic line wave appears and interacts with the first-order breather, as shown in Fig. 1c.2. When $t = \frac{1}{3}$, we find that the amplitude of the periodic line wave decreases and the amplitude of the first-order breather keeps unchanged, as shown in Fig. 1c.3.

Furthermore, we investigate the influence of the coefficients in Eq. (1) on the amplitudes of the second-order breathers. Comparing Fig. 2 with Fig. 1, when $\alpha$ increases, we find that the amplitude of the second-order breather decreases. Comparing Fig. 3 with Fig. 1, when $\beta$ increases, we observe that the amplitude of the second-order breather increases. Comparing Figs. 4 and 5 with Fig. 1, we find that when $\nu$ or $\delta$ increases, the amplitude of the second-order breather keeps invariant. Under the condition $\frac{a_{11}}{a_{21}} = \frac{\alpha_{11}b_{11} - \alpha_{12}b_{12}}{\alpha_{21}b_{21} - \alpha_{22}b_{22}} = \frac{\alpha_{11}c_{11} - \alpha_{12}c_{12}}{\alpha_{21}c_{21} - \alpha_{22}c_{22}}$, we can derive the interaction between the two parallel first-order breathers, as shown in Fig. 6.

4 Hybrid solutions comprising the solitons and breathers for Eq. (1)

Hybrid solutions consisting of the $\Theta$ solitons and $\Lambda$ breathers for Eq. (1) are derived with the following conditions in Solutions (9):

$$N = 2\Lambda + \Theta, \quad a_{v} = a_{A_{11}}^{s} + i\alpha_{12}, b_{v} = b_{A_{11}}^{s} + i\alpha_{12},$$

$$c_{v} = c_{A_{11}}^{s} + i\alpha_{12}, \quad \xi_{v}^{0} = \xi_{A_{11}}^{s} + i\alpha_{12},$$

$$a_{s} = a_{s_{1}}, \quad b_{s} = b_{s_{1}}, c_{s} = c_{s_{1}}, \quad \xi_{s}^{0} = \xi_{s_{1}},$$

$$(v = 1, 2, \ldots, \Lambda, \quad s = 2\Lambda + 1, 2\Lambda + 2, \ldots, N)$$

where $\Theta$ and $\Lambda$ are the positive integers, $a_{s_{1}}$’s, $b_{s_{1}}$’s, $c_{s_{1}}$’s and $\xi_{s_{1}}^{0}$’s are the real constants.
For example, the hybrid solutions comprising the first-order breather and one kink-type soliton are obtained from Solutions (9) with the parameters as

\[ N = 3, \quad \Lambda = 1, \quad \Theta = 1, \quad a_1 = a_2^* = a_{11} + ia_{12}, \]
\[ b_1 = b_2^* = b_{11} + ib_{12}, \]
\[ c_1 = c_2^* = c_{11} + ic_{12}, \quad \xi_1^0 = \xi_2^0 = \xi_{11}^0 + i\xi_{12}^0, \]
\[ a_3 = a_{31}, \quad b_3 = b_{31}, c_3 = c_{31}, \quad \xi_3^0 = \xi_{31}^0. \] (14)

Under the condition \( a_{11} \neq a_{31} \), we observe that the first-order breather is parallel to the one kink-type soliton, as shown in Fig. 7. When \( t = 0 \), the first-order breather interacts with the kink-type soliton. After the interaction, the first-order breather and the kink-type soliton keep their velocities and shapes unchanged, meaning that the interaction is elastic.

When \( N = 4, \quad \Lambda = 1 \) and \( \Theta = 2 \) via Conditions (13), the interaction among the first-order breather and two kink-type solitons is shown in Fig. 8.

5 Periodic-wave solutions and asymptotic behaviors for Eq. (1)

5.1 Periodic-wave solutions for Eq. (1)

In this section, motivated by Ref. [40], the Riemann theta function will be applied to construct the periodic-wave solutions for Eq. (1). We introduce the one-Riemann theta function as [40]

\[ f = \psi(\zeta_1, \mu_1) = \sum_{\rho = -\infty}^{+\infty} e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \zeta_1}, \] (15)

where \( \zeta_1 = Q_1x + B_1y + W_1z + R_1t + \epsilon_1, \) \( \mu_1 \) is a pure imaginary number and \( \text{Im}(\mu_1) > 0, \) \( \rho \) is an integer, and \( Q_1, B_1, W_1, R_1 \) and \( \epsilon_1 \) are the real constants. Substituting Transformation (5) and Expression (15) into Eq. (1), we obtain

\[ \vartheta(D_x, D_y, D_z, D_t) \psi(\zeta_1, \mu_1) \cdot \psi(\zeta_1, \mu_1) = \left( D_t D_x + D_x^2 D_y + \beta D_x^2 + \gamma D_y^2 + \delta D_z D_y + c \right) \]
Bilinear form, soliton, breather, hybrid and periodic-wave solutions

\[(a) \quad t = -15 \]

\[(b) \quad t = -15 \]

\[(c) \quad t = -15 \]

\[(a_1) \quad t = 0 \]

\[(a_2) \quad t = 0 \]

\[(a_3) \quad t = 15 \]

\[(b_1) \quad t = 0 \]

\[(b_2) \quad t = 0 \]

\[(b_3) \quad t = 15 \]

\[(c_1) \quad t = 0 \]

\[(c_2) \quad t = 0 \]

\[(c_3) \quad t = 15 \]

Fig. 6  Interaction between the two parallel first-order breathers via Solutions (11) with \( T = 2, \alpha = \beta = \gamma = \delta = 1, a_1 = a_3^* = -\frac{1}{2}, b_1 = b_3^* = -\frac{1}{8} - i, c_1 = c_3^* = -\frac{5}{8} - \frac{7}{8}i, a_2 = a_4^* = 1, \)
\( b_2 = b_4^* = -\frac{1}{4} - \frac{3}{4}i, c_2 = c_4^* = -\frac{5}{4} - \frac{3}{4}i, \xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = 0 \) and \( (a) \ z = 0; (b) \ y = 0; (c) \ x = 0 \)

\[ \psi(\xi_1, \mu_1) \cdot \psi(\xi_1, \mu_1) = 0, \quad (16) \]

where \( \vartheta(D_x, D_y, D_z, D_t) \) is a polynomial about \( D_x, D_y, D_z \) and \( D_t \), and \( c \) is an integration constant. Applying Expression (15) into Expression (16), we get the results, as follows:

\[ \vartheta(D_x, D_y, D_z, D_t) \psi(\xi_1, \mu_1) \cdot \psi(\xi_1, \mu_1) = \sum_{\rho=-\infty}^{+\infty} \sum_{\rho=-\infty}^{+\infty} \vartheta(D_x, D_y, D_z, D_t) e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \xi_1} \cdot e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \xi_1} \]
Fig. 7 Interaction between the first-order breather and one kink-type soliton via Solutions (9) under Conditions (14) with $\alpha = 3, \beta = \gamma = \delta = 1, a_1 = a_2^* = 1 + 2i, b_1 = b_2^* = 1 + i, c_1 = c_2^* = 1 - \frac{1}{2}i, a_3 = c_3 = 2, b_3 = -1$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$
Fig. 8 Interaction among the first-order breather and two kink-type solitons via Solutions (9) under Conditions (13) with \( \alpha = a_4 = c_4 = 3, \beta = \gamma = \delta = b_3 = c_3 = 1, a_1 = a_2^z = 1 + 2i, b_1 = b_2^z = 1 + i, c_1 = c_2^z = 1 + \frac{1}{2}i, a_3 = 2, b_4 = 2 \) and (a) \( z = 0; \) (b) \( y = 0; \) (c) \( x = 0 \).
\[
\sum_{q=\infty}^{+\infty} \sum_{\rho=-\infty}^{+\infty} \tilde{\vartheta} \left[ 2i\pi (\rho - \varrho) Q_1, 2i\pi (\rho - \varrho) B_1, 2i\pi (\rho - \varrho) W_1, 2i\pi (\rho - \varrho) R_1 \right] e^{\pi i (\varrho^2 + \rho^2) \mu_1 + 2\pi i (\rho + \varrho) \xi_1} = \sum_{q'=\infty}^{+\infty} \tilde{\vartheta} (q') e^{2\pi i \varrho' \xi_1}, \quad \varrho' = \varrho + \rho, \quad (17)
\]

where
\[
\tilde{\vartheta} (q') = \sum_{\rho=-\infty}^{+\infty} \tilde{\vartheta} \left[ 2i\pi (2\rho - q') Q_1, 2i\pi (2\rho - q') B_1, 2i\pi (2\rho - q') W_1, 2i\pi (2\rho - q') R_1 \right] e^{\pi i (\varrho^2 + (\rho - \varrho)^2) \mu_1}
\]

Expression (18) suggests that \(\tilde{\vartheta} (q')\) for \(q' \in \mathbb{Z}\) are completely dominated by \(\tilde{\vartheta} (0)\) and \(\tilde{\vartheta} (1)\), where \(\mathbb{Z}\) represents the set of integers. When \(\tilde{\vartheta} (0) = \tilde{\vartheta} (1) = 0\), we have \(\vartheta (D_1, D_1, D_2, D_2) \psi (\zeta_1, \mu_1) \cdot \psi (\xi_1, \mu_1) = 0\).

Through Expression (18), \(\tilde{\vartheta} (0)\) and \(\tilde{\vartheta} (1)\) can be derived as
\[
\tilde{\vartheta} (0) = \sum_{\rho=-\infty}^{+\infty} \vartheta \left[ 4\rho \pi i Q_1, 4\rho \pi i B_1, 4\rho \pi i W_1, 4\rho \pi i R_1 \right] e^{\pi i (\rho^2 + \rho^2) \mu_1}
\]

\[
\tilde{\vartheta} (1) = \sum_{\rho=-\infty}^{+\infty} \vartheta \left[ 2i\pi (2\rho - 1) Q_1, 2i\pi (2\rho - 1) W_1, 2i\pi (2\rho - 1) R_1 \right] e^{\pi i (2\rho^2 - 2\rho + 1) \mu_1}
\]

Equations (19) can be converted into the following linear system about \(R_1\) and \(c\), i.e.,
\[
\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} R_1 \\ c \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \quad (21)
\]

Solving System (21), the periodic-wave solutions for Eq. (1) can be derived as
\[
u = \frac{6}{\alpha} \ln \psi (\zeta_1, \mu_1), \quad R_1 = \frac{p_{22} q_1 - p_{21} q_2}{p_{11} p_{22} - p_{12} p_{21}}, \quad c = \frac{p_{21} q_1 - p_{11} q_2}{p_{12} p_{21} - p_{11} p_{22}}. \quad (22)
\]

5.2 Asymptotic behaviors of periodic-wave solutions for Eq. (1)

In this section, we will study the asymptotic behaviors of Periodic-Wave Solutions (22). Expressions (20) can be expanded as
\[
p_{11} = -32\pi^2 B_1 \left( \Delta^2 + 4\Delta^4 + \cdots + \rho^2 \Delta^2 \rho^2 + \cdots \right),
\]
\[p_{12} = 1 + 2 \left( \Delta^2 + \Delta^8 + \cdots + \Delta^{2\rho^2} + \cdots \right),\]

\[p_{21} = -8\pi^2 B_1 \left( \Delta + 9 \Delta^5 + \cdots + (2\rho - 1)^2 \Delta^2 + \cdots \right),\]

\[p_{22} = 2 \left( \Delta^1 + \Delta^5 + \cdots + \Delta^{2\rho^2 - 2\rho + 1} + \cdots \right),\]

\[q_1 = 2 \left( -256\pi^4 Q_1^3 B_1 + 16\beta \pi^2 Q_1^2 + 16\gamma \pi^2 B_1^2 + 16\pi^2 B_1 W_1 + c \right) + 16\gamma \pi^2 B_1^2 + 16\delta \pi^2 B_1 W_1 + c \),\]

\[\kappa_5 = 2 \left( -1296\pi^4 Q_1^3 B_1 + 36\beta \pi^2 Q_1^2 + 36\gamma \pi^2 B_1^2 + 36\delta \pi^2 B_1 W_1 + c \right).\]

Then, \( R_1 \) and \( c \) in System (21) can be rewritten as

\[
\begin{aligned}
\left( \begin{array}{c}
R_1 \\
\end{array} \right) &= \Psi_0 + \Psi_1 \Delta + \Psi_2 \Delta^2 + \cdots, \\
\Psi_0 &= \left( \frac{2\gamma [1-\gamma(2)]}{8\pi^2 B_1 \gamma_1(1)} \right), \\
\Psi_1 &= \left( \frac{2\gamma [1-\gamma(2)]}{8\pi^2 B_1 \gamma_1(1)} \right), \\
\Psi_2 &= \left( \frac{2\gamma [1-\gamma(2)]}{8\pi^2 B_1 \gamma_1(1)} \right), \\
\end{aligned}
\]

\[
\Psi_3 = 0, \cdots.
\]

Substituting Expressions (27) into Expressions (24) and taking \( \Delta \to 0 \), we obtain

\[
\begin{aligned}
c &\to 0, \\
R_1 &\to -16\pi^4 Q_1^3 B_1 + 4\beta \pi^2 Q_1^2 + 4\gamma \pi^2 B_1^2 + 4\delta \pi^2 B_1 W_1 + \text{terms with higher powers of } \Delta. \\
\end{aligned}
\]

Assuming that

\[
\begin{aligned}
Q_1 &= \frac{a_1}{2i\pi}, \\
B_1 &= \frac{a_1 b_1}{2i\pi}, \\
W_1 &= \frac{a_1 c_1}{2i\pi}, \\
\end{aligned}
\]

we have

\[
\begin{aligned}
2i\pi \xi_1 &= 2i\pi \left( Q_1 x + B_1 y + W_1 z - 16\pi^4 Q_1^3 B_1 + 4\beta \pi^2 Q_1^2 + 4\gamma \pi^2 B_1^2 + 4\delta \pi^2 B_1 W_1 + \frac{t}{l + \epsilon_1} \right) \\
&\quad \left( a_1 (x + b_1 y + c_1 z + d_1 t) + \xi^0_1 - i\mu_1 \xi_1 - i\mu_1. \\
\right)
\end{aligned}
\]

Substituting Expression (30) into Expression (15), we further derive that

\[
\psi (\xi_1, \mu_1) = \sum_{\rho = -\infty}^{\infty} e^{i\rho^2 \mu_1 + 2\pi i \rho \xi_1}
\]

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According to the above discussion, we find that Periodic-Wave Solutions (22) approach to the one-soliton solutions with the limiting condition $\Delta \to 0$.

6 Conclusions

Fluids have been studied in such disciplines as atmospheric science, oceanography and astrophysics. In this paper, we have studied a (3+1)-dimensional KdV equation in a fluid, i.e., Eq. (1). Via Transformation (5) and the bilinear method, Bilinear Form (7) and N-Soliton Solutions (9) for Eq. (1) have been obtained, where $N$ is a positive integer. Based on $N$-Soliton Solutions (9), we have obtained the $T$th-order breather solutions for Eq. (1), i.e., Solutions (11), where $T$ is a positive integer. Hybrid solutions comprising the solitons and breathers for Eq. (1) have been constructed via $N$-Soliton Solutions (9) under Conditions (13). Via the Riemann theta function, Periodic-Wave Solutions (22) for Eq. (1) have been obtained.

Via Solutions (11), we have observed the interaction between the two perpendicular first-order breathers on the $x - y$ and $x - z$ planes and the interaction between the periodic line wave and the first-order breather on the $y - z$ plane, as shown in Fig. 1. Furthermore, we have discussed the effects of $\alpha$, $\beta$, $\gamma$ and $\delta$ on the amplitude of the second-order breather, where $\alpha$, $\beta$, $\gamma$ and $\delta$ are the constant coefficients in Eq. (1): Amplitude of the second-order breather decreases as $\alpha$ increases, as shown in Fig. 2; amplitude of the second-order breather increases as $\beta$ increases, as shown in Fig. 3; amplitude of the second-order breather keeps invariant as $\gamma$ or $\delta$ increases, as shown in Figs. 4 and 5. We have observed the interaction between the two parallel first-order breathers, as shown in Fig. 6. We have seen that the first-order breather interacts with one kink-type soliton and the interaction is elastic, as shown in Fig. 7. Interaction among the first-order breather and two kink-type solitons has been shown in Fig. 8. With the discussion of the asymptotic behaviors of Periodic-Wave Solutions (22), we have found that under a certain limit process, Periodic-Wave Solutions (22) approach to the one-soliton solutions.

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Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

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