Introduction: Originally proposed as a means to evade the Coleman-Mandula theorem and unify internal and space-time symmetries, supersymmetry (SUSY) has been an active area of research due to its potential in solving the hierarchy and cosmological constant problems and other puzzles in high-energy physics [1–5]. Bosons and fermions related by SUSY transformations, referred to as superpartners, have the same mass [5]; SUSY breaking lifts this degeneracy proportionally to the breaking scale. Despite extensive searches at high energies, conclusive experimental evidence for SUSY is yet to be found.

The last 30 years have witnessed active SUSY-related research in solid-state systems, including the tricritical Ising model [6], the boundary of topological insulators and superconductors [7–15], the bulk of semimetals [16, 17], high-$T_c$ superconductors [18], the Josephson-junction array [19] and various other model systems [20–24], as well as in the cold atom system [25]. It has been argued that, even though the corresponding microscopic models do not exhibit it, SUSY emerges macroscopically at continuous quantum phase transitions of such solid-state systems. The gapless nature of the continuous phase transition implies that the resulting superpartners are massless.

In this letter we present the first example of emergent SUSY at the first-order quantum phase transition (FOQPT) of a solid-state system. It can be realized by tuning only one parameter and the corresponding superpartners are massive. Specifically, we consider a 2+1D Majorana field coupled to an Ising field, and we derive the macroscopic phase diagram (Fig. 1) through the one-loop renormalization group (RG) analysis. In Fig. 1, there are three phases with different Ising ordering, separated by three FOQPT lines and a quantum critical point. The FOQPT lines can be reached by tuning one parameter as they are in a two-dimensional parameter space, and have emergent SUSY with gapped Majorana and Ising fields serving as the massive superpartners. Interestingly, the emergent SUSY is always accompanied by a topological phase transition of the Majorana field even though its mass does not vanish. Finally, we propose an experimental realization of the emergent SUSY based on the time-reversal (TR) invariant topological superconductor (TSC).

Emergent SUSY in a massive Majorana-Ising system: We consider a 2+1D action that describes a Majorana fermion field with the mass $m_f$ and the order parameter $\langle \phi \rangle$. The gray dot at $(\eta_1, \eta_2, \eta_3)$ in the phase diagram (Fig. 1) is the quantum critical point discussed in Ref. [10]. The Ising order parameter $\langle \phi \rangle$ (silver arrows) and the dispersion of the Majorana fermion field with the mass $m_f$ (the red and blue curves) are schematically shown for all the phases. The Majorana field has zero mass at the gray dotted line in the phase I. The orange dashed lines $\eta_1$ and $\eta_2$ are parameterized as $(r^*/g^*)^2, (m^*/g^*)^2 = (1, -1 + t_1)$ and $(-1, t_2)$, respectively, where $t_1 \in [-0.3, 0.3]$. The dashed line $\eta_3$ is given by tuning the magnetic field in the experimental setup.
fermion $\gamma$ interacting with an Ising field $\phi$ and explore the long-distance behavior of such action with one-loop RG analysis. The 2+1D action is

$$S = \int d^4x \left[ \frac{1}{2} \gamma^T (\partial_\tau - iv_f \alpha \cdot \nabla + m_\sigma \gamma) + \frac{1}{2} g \gamma^T \sigma \gamma + \frac{1}{2} \tilde{g} \phi (-\partial^2 - \tilde{v}_g^2 \nabla^2 + r) \phi + \frac{1}{3!} b \phi^3 + \frac{1}{4!} u \phi^4 \right], \quad (1)$$

where $x = (\tau, \mathbf{x})$ with the imaginary time $\tau$, and $\alpha = (\sigma_+, \sigma_-)$ with the Pauli matrices $\sigma_i$. Without loss of generality, we choose $v_f > 0$ by rotating the index of the Majorana field [26], $g \geq 0$ by flipping the sign of $\phi$, and $u > 0$ to make the bosonic potential bounded below. Eq. (1) has rotational invariance along $\alpha$ and, for a uniform vacuum expectation value (VEV) of $\phi$, it is the most general rotationally invariant action to $\phi \gamma^T \sigma \gamma$ and $\phi^4$ order [26]. The action (1) is not invariant under the TR transformations, $\gamma_{\alpha} \rightarrow -\gamma_{\alpha}$ and $\phi_{\alpha} \rightarrow -\phi_{\alpha}$, unless $m, b = 0$. The TR-invariant case was analyzed in [10], where it was shown that SUSY with massless superpartners emerges by tuning the parameter $r$ to the continuous phase transition point $r = 0$. Below we demonstrate that SUSY can also emerge for the full action (1) with non-zero $m$ and $b$ by tuning the ratio $r/m^2$. To this end we compute the one-loop RG coefficients using dimensional regularization (DR) in $d = 4 - \epsilon$ dimensions [16, 26] and study the flow of various parameters, including $v_b$, $v_f$, $g$, $u$, $m$ and $b$. We assume that the parameter $r$ is tunable as a way to control the ratio $r/m^2$ (another way is to tune $m$); this is reinforced by the fact that its flow is unstable. [26]

The RG equations decouple in sectors which can be studied sequentially, starting with the completely decoupled one of the bosonic and fermionic velocities:

$$\frac{dv_b}{dl} = \frac{\tilde{g}^2 (v_b^2 - v_f^2)}{32\pi^2 v_b v_f}, \quad \frac{dv_f}{dl} = \frac{\tilde{g}^2 (v_b - v_f)}{6\pi^2 v_b (v_b + v_f)^2}, \quad (2)$$

where $l$ parametrizes the scaling $(\tau, \mathbf{x}) \rightarrow \epsilon^l (\tau, \mathbf{x})$, and $\tilde{g} = \tilde{g} e^{-r/2}$ with $\tilde{g}$ independent of $l$ and having the energy unit, see e.g. [27]. Structure of perturbation theory implies that the dimensionful parameters $m$, $b$ and $r$ cannot appear in Eq. (2) in DR; thus the velocities flow stably to $v_f = v_b \equiv v$ for any non-zero $g$, as in the TR invariant case with $r = 0$ [10]. By rescaling the spacial coordinate as $\mathbf{x} \rightarrow \epsilon \mathbf{x}$, we can choose $v_f = v_b = 1$ to study the RG flows of other parameters.

We next consider the RG equations of $g$ and $u$,

$$\frac{dg}{dl} = \frac{\tilde{g}}{2} \left( \frac{\epsilon}{2} - \frac{7 \epsilon^2}{32\pi^2} \right),$$

$$\frac{du}{dl} = \frac{12\tilde{g}^4 - 25\tilde{g}^2 u - 3u^2}{16\pi^2} + \tilde{u} \epsilon, \quad (3)$$

where $\tilde{u} = u \tilde{u}^e$. Similar to the velocities, the RG equations of $g$ and $u$ are the same as the TR invariant case at $r = 0$, since they are dimensionless in the absence of the dimensional regulator [10]. Thereby, $g$ stably flows to a non-zero value $g^* = \sqrt{16\pi^2 \tilde{g}^e / \tilde{u}}$, while $u$ stably flows to $u^* = 3(g^*)^2$.

The RG equations for $m$ and $\tilde{b} = b \tilde{u}^{-r/2}$ are

$$\frac{dm}{dl} = \frac{3g^2 m}{16\pi^2},$$

$$\frac{d\tilde{b}}{dl} = \tilde{b}(1 + \epsilon) + \frac{(3g^2 - 6\tilde{u}) + 24g^2 m}{32\pi^2} \cdot (4)$$

For $g = g^*$, the mass anomalous dimension is $3\epsilon/7 < 1$. Thus, $m$ is relevant and flows to a value $m^*$ determined by either the whole system size or the temperature. In contrast to the conventional Ising model without TR-breaking term [28], the $\phi^3$ term is included in Eq. (1) since it can be generated by the TR-breaking $m$ term as shown below. For a non-zero $m$, it is useful to consider the flow of $b/(mg)$ at the fixed point of the flow of $u$, i.e. $u = 3g^2$. It is given by

$$\frac{db}{dl} = -\frac{b}{4\pi^2} \frac{b}{mg} - 3. \quad (5)$$

This indicates that $b$ flows stably to $b^* = 3m^* g^*$, and thus $b$ can be driven away from zero by a non-zero $m$ as long as $g \neq 0$. The RG flow of $(u, b)$, stably flowing to $(u^*, b^*) = (3(g^*)^2, 3m^* g^*)$, is also verified by numerically plotting the RG flows of $u/g^2$ and $b/(gm)$ in Fig. 2a. In summary, one-loop RG analysis shows that at the low-energy-long-distance limit, the action $S$ flows to

$$S^* = \int d^4x \left[ \frac{1}{2} \gamma^T (i\partial_\tau \alpha^\mu + m^* \sigma_\mu) \gamma + \frac{1}{2} g^* \gamma^T \sigma \gamma + \frac{1}{2} \tilde{g}^* \phi (-\partial^2 + r^*) \phi + \frac{1}{2} \tilde{g}^2 m^* \phi^3 + \frac{1}{8} (g^*)^2 \phi^4 \right], \quad (6)$$

where the summation over repeated index is implied, $r^*$ is the macroscopic value of $r$, $\mu = 0, 1, 2, \partial_\tau = (i\partial_\tau, \nabla)$, $\alpha^\mu = (-\mathbf{l}, -\alpha)$ and $\sigma^2 = \sigma^2 + \nabla^2$.

Before deriving the macroscopic phase diagram from Eq. (6), we first demonstrate the existence of emergent SUSY. Let us tune $r^*$ in the effective action $S^*$ to the value $r^* = (m^*)^2$ while keeping the VEV of the boson field at $\langle \phi \rangle = 0$. (We discuss in the next section the case of $\langle \phi \rangle \neq 0$.) Eq. (6) becomes the “real” version of the 2+1D Wess-Zumino SUSY model. Indeed, it is invariant under the infinitesimal SUSY transformation: $\delta_\xi \phi = \xi^T \sigma_\gamma \gamma$ and $\delta_\xi \gamma = \gamma \sigma_\alpha \alpha^\mu \xi (i\partial_\mu \phi) + \xi (-m^* \phi - g^* \phi^3 / 2)$, where $\xi = (\xi_1, \xi_2)^T$ are the constant Grassmann-valued parameters of the SUSY transformation [5, 26]. This shows that SUSY with massive superpartners can emerge macroscopically in a Majorana-Ising system by tuning the parameter $r^*/(m^*)^2$.

Topological FOQPT: In this section, we demonstrate that SUSY with massive superpartners emerges on all the FOQPT lines in the $r^* - m^*$ phase diagram (Fig. 1). Furthermore, we show that the FOQPT with emergent SUSY is topological in the sense that the Majorana field undergoes an unusual topological phase transition, which leads to experimentally testable phenomena. To see this,
FIG. 2. (a) shows the RG flow of \( b / (mg) \) and \( u / g^2 \) in Eq. (3) and Eq. (4) with \( \bar{g} = \sqrt{16\pi^2/7} \) and the black dot at (3,3). (b) and (c) show the fermion mass \( m_f \) (red lines) and the VEV of the boson field \( \langle \phi \rangle \) (blue lines) when tuning the parameters along the lines \( \eta_1 \) and \( \eta_2 \) in Fig. 1, respectively. (d) schematically depicts a 1+1D chiral Majorana mode propagating along the mass domain wall of massive Majorana fields.

we first derive the bosonic VEV \( \langle \phi \rangle \) in Eq. (6), which stands for the macroscopic magnetic ordering, by searching for the global minimum of the bosonic part of \( S^* \). We neglect the possibility that a new vacuum with lower energy appears after including all orders of quantum correction [27]. Owing to the negative sign in front of \( \partial^2 \), \( \langle \phi \rangle \) must be uniform in \( (\tau, \varphi) \). By minimizing the bosonic potential \( V(\phi) = r^* \phi^4 / 2 + g^* \phi^6 / 2 + (g^*)^2 \phi^4 / 8 \), we found a non-magnetic phase (I) and two magnetic phases (II,III) with opposite values of \( \langle \phi \rangle \) (See Fig. 1):

\[
\begin{align*}
    \text{I}: r^* > (m^*)^2, \langle \phi \rangle &= 0 \\
    \text{II}: r^* < (m^*)^2, m^* > 0, \langle \phi \rangle &= -3m^* - \sqrt{9(m^*)^2 - 8r^*} / 2g^* \\
    \text{III}: r^* < (m^*)^2, m^* < 0, \langle \phi \rangle &= -3m^* + \sqrt{9(m^*)^2 - 8r^*} / 2g^*.
\end{align*}
\]

The phases I and II (III) are separated by the transition line \( r^* = (m^*)^2 \) with \( m^* > 0 \) (\( m^* < 0 \)) as depicted by the line 1 (2) in Fig. 1, while the Phases II and III are separated by the line \( m^* = 0 \) and \( r^* > 0 \) (the line 3 in Fig. 1).

All the three transition lines are FQOPT lines with emergent SUSY. To show this, we consider a path across the phase transition line 1 or 2, such as the path \( \eta_1 \) in Fig. 1. As shown by the blue line in Fig. 2b, \( \langle \phi \rangle \) vanishes on the phase I side, but approaches \( \langle \phi \rangle = -2m^*/g^* \) as \( r^* = (m^*)^2 + 0^- \) (on the phase II or III side). Therefore, lines 1 and 2 are FQOPT lines with two degenerate vacua \( \langle \phi \rangle = 0 \) and \( -2m^*/g^* \), where the emergent SUSY was demonstrated above for the former vacuum. Around the latter vacuum, the action for the boson fluctuation \( \delta \phi = \phi + 2m^*/g^* \) and the fermion \( \gamma \) has the same form as Eq. (6) with the replacement \( m^* \rightarrow -m^* \) and \( r^* \rightarrow (m^*)^2 \) [26], which also exhibits SUSY. Then, SUSY emerges along the FQOPT lines 1 and 2 in either of the two vacua. Similarly, the line 3 is also a FQOPT line with two degenerate vacua \( \langle \phi \rangle = \pm \sqrt{-2r^*} / g^* \) as shown by the blue line in Fig. 2c. Around either of the two vacua, the corresponding action for the boson fluctuation \( \delta \phi = \phi - (\pm \sqrt{-2r^*} / g^*) \) and the fermion \( \gamma \) can be obtained from Eq. (6) by replacements \( m^* \rightarrow \pm \sqrt{-2r^*} \) and \( r^* \rightarrow -2r^* \), and thus possesses SUSY. We conclude that SUSY exists for all three FQOPT lines in Fig. 1 around any of the degenerate vacua.

The three FQOPT lines are also accompanied by a topological phase transition for Majorana fields. From Eq. (6), the Majorana fermion mass to the leading order is given by \( m_f = m^* + g^* \langle \phi \rangle \). As shown by the red lines in Fig. 2b and 2c, \( m_f \) changes suddenly between \( \pm m^* \) across the lines 1 and 2, and between \( \pm \sqrt{-2r^*} \) across the line 3. The sign flip of the fermion mass signals a topological phase transition, as the fermion mass domain wall can trap a 1+1D chiral Majorana mode (see Fig. 2d). [29] More importantly, the non-zero \( m_f \) across the transition indicates the fermion gap does not close, thus representing a unique topological phase transition without gap-closing owing to its first-order transition nature. Although a similar scenario has been discussed in the literature [30–32], our case is special because the uncharged mass amplitude \( |m_f| \) across the transition is required by the emergent SUSY. A suitable rewrite of the bosonic potential \( V(\phi) = \frac{1}{4\pi^2u}[\frac{1}{2}(r - \frac{1}{4}m^*)^2 + (\phi + \frac{2\pi}{u})^2] \) exposes this feature: the FQOPT at lines 1 and 2 now occurs at \( r = b^2/(3u) \) and \( b \neq 0 \), where \( \langle \phi \rangle \) changes between 0 and \( -2b/u \), and \( m_f \) jumps between \( m \) and \( -2bg/u \). Therefore, the relation \( bg = um \) imposed by SUSY is essential to maintain the uncharged magnitude of \( m_f \) and flip its sign across the transition.

Experimental Setup: In this section, we demonstrate that, by tuning an external magnetic field \( B \), the emergent SUSY at FQOPT can be realized on the surface of a TR-invariant TSC with surface magnetic doping, as shown in Fig. 3a. The action of the TR-invariant TSC reads [33–35]

\[
S_0 = \int \frac{d\tau dk^3}{(2\pi)^3} \left[ \psi_{\tau,k}(\partial_\tau + h(k))\psi_{\tau,k} + \frac{1}{2}\psi_{\tau,k}\Delta_k \psi_{\tau,-k}^T + \frac{1}{2}\psi_{\tau,-k}^T\Delta_k^T \psi_{\tau,k} \right] (8)
\]

where \( \psi_{\tau,k} = (\psi_{\tau,k,\uparrow}, \psi_{\tau,k,\downarrow}) \) is the Grassman field for the electron with the momentum \( k = (k_x, k_y, k_z) \), \( h(k) = k^2 / \mu - \mu \) with the chemical potential \( \mu \), and \( \Delta_k = \Delta_{\nu}(k \cdot s)i\gamma_y \) with the p-wave pairing \( \Delta_\nu > 0 \) and the Pauli
FIG. 3. (a) shows the experimental setup of TR-invariant TSC with surface magnetic doping to observe the emergent SUSY. An external magnetic field $B$ along $z$ is applied as a tuning parameter. (b) shows the fermion mass $m_f$ (red line) and the total surface magnetization $\langle M \rangle$ (blue line) versus the magnetic field $B$ along the path $\eta_3$ in Fig. 1. Here $B = B_c$ is where the FOQPT with emergent SUSY happens and $m_c \approx 1.8(g^*)^2$.

matrices $s_i$ for spin. Eq. (8) may be used to describe the Ce-based heavy fermion SCs and half-Heusler SCs, and its superfluid version has been realized in B phase of He-3 [36–38]. For $\mu/\mu_0 > 0$, one can solve Eq. (8) with an open boundary condition at $z = 0$ and obtain gapless Majorana modes $\gamma$ on the surface [26, 33–35]. The magnetic doping at the surface of TSC can introduce the surface out-of-plane Ising magnetism $M$, which couples to the surface electrons. Together with the applied magnetic field coupled to both the electron spin and $M$, we find in [26] that the total action describing the interplay between surface Majorana modes and Ising magnetism $M$ reads

$$S_E = \int d^2x \left[ \frac{1}{2} \gamma^T (\partial_x - i v_f \mathbf{\alpha} \cdot \nabla) \gamma + \frac{1}{2} g M \gamma^T \sigma_y \gamma + \frac{1}{2} M (\partial_x^2 - v_b^2) \nabla^2 + r_0 \right] M + \frac{1}{4!} u M^4 + \frac{1}{2} \mu_B B \gamma^T \sigma_y \gamma + a_1 B M + \frac{a_2}{3!} B M^3 \right].$$

(9)

Compared with the action Eq. (1), we find that the external magnetic field induces an extra $M$-linear term in Eq. (9), which generates a non-zero magnetization $\phi_0$ with $\phi_0$ satisfying $a_1 B + r_0 \phi_0 + a_2 B \phi_0^2/2 + u \phi_0^3/3! = 0$ and $\phi_0(B \to 0) = 0$. Therefore, we should consider the magnetic fluctuation defined by $\phi = M - \phi_0$, and the resulting macroscopic action for $\gamma$ and $\phi$ takes the same form as Eq. (6) with $m^* = \mu_B B + g^* \phi_0$, $r^* = r_0^* + a_2 B \phi_0^2 + 3(g^* \phi_0)^2/2$ and $a_2^* = 3g^* \mu_B$ according to the RG flows Eq. (2)-(4). Although $\mu_B$, $r_0$ and $a_1$ do not have a classical $B$-dependence, their macroscopic values $\mu_B^*$, $r_0^*$ and $a_1^*$ can, in general, depend on $B$. In the following, we neglect the $B$-dependence of $\mu_B^* \neq 0$, $r_0^*$ and $a_1^*$ for simplicity, and consider the case where $r_0^* > 0$, i.e. the net surface magnetization is zero without the applied magnetic field. In this case, as along as $a_1^*/\mu_B^* \leq r_0^*/(3g^*)$, we can tune $B$ to a non-zero critical value $B_c$ to realize $r^* = (m^*)^2$ and emergent SUSY [26]. To demonstrate this possibility, we choose the values of parameters as $r_0^*/(g^*)^4 = 0.2$, $a_1^*/g^* = -1$, $\mu_B^*(g^*)^2 = 1$ and $B \geq 0$. As $B$ increases to the critical value $B_c \approx 1.1(g^*)^4$, the FOQPT line 1 in Fig. 1 is reached along the path $\eta_3$. A signature of emergent SUSY at FOQPT line 1 is that the fermion mass $m_f = m^* + g^*(\phi)$ should have unchanged magnitude and flip sign across the transition, verified by the red line in Fig. 3b. The unchanged amplitude of $|m_f|$ can be confirmed by local density of states measurement with scanning tunneling microscopy, and the sign flip can be tested by checking the resulting topological phase transition, i.e. the appearance or disappearance of chiral 1+1D domain-wall fermion. Moreover, the total surface magnetization $\langle M \rangle = \phi_0 + \langle \phi \rangle$ has a sudden change (see the blue line in Fig. 3b) as an evidence of FOQPT, which can be measured by superconducting quantum interference devices.

Conclusion and Discussion: In conclusion, SUSY with massive superpartners can emerge at the topological FOQPT occurring on the surface of a TR-invariant TSC in a tunable external magnetic field. Although the emergent SUSY with massive superpartners was proposed in a cold-atom system with spontaneous symmetry breaking [25], similar to that at line 3 in Fig. 1, that proposal requires to tune more than one parameter and does not discuss the relation between SUSY and the topological FOQPT.

Finally, we discuss the validity of one-loop approximation for the emergent SUSY at FOQPT. Near the SUSY hyper-surface ($u \sim g^2 \sim \epsilon$, $b \sim \mu g$, $v_f \sim v_b \sim 1$ and $r \sim m^2$), the $L$-loop contribution ($L > 1$) to the RG equation is of order $[\ln(m/\mu)]^{L-1} g^2$, which means the one-loop approximation is valid as long as $\ln(m/\mu) < 1/g^2 \sim 1/\epsilon \sim \ln(\Lambda/\mu)$ with $\Lambda$ the ultraviolet energy cutoff [26]. Such condition leads to a critical scale $l_c$ above which the relevant $m$ satisfies $\ln(m/\mu) \sim \ln(\Lambda/\mu)$ and the RG flow might be altered by the higher-loop contribution. However, since $l_c$ increases as the initial $m$ at $l = 0$ mass decreases, the deviation from SUSY hyper-surface at $l \approx l_c$ should be negligible for a small initial $m$, which may lead to a clear SUSY signature in numerical simulations. (See more details in [26]). Our work also helps shed light upon the emergent symmetry of a FOQPT [39].

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In this section, we show that Eq. (1) of the main text is the most general rotationally invariant action to $\phi \gamma^T \sigma_0 \gamma$ and $\phi^4$ order if assuming $\phi$ has uniform classical vacuum.

The rotation transformation along $z$ is defined as $\gamma^T T_{\tau} x \rightarrow \gamma^T T_{\tau, R_\theta} x e^{-i \sigma_y \theta / 2}$ and $\phi_{\tau, x} \rightarrow \phi_{\tau, R_\theta x}$, where $R_\theta$ is the
rotational matrix along $z$ for angle $\theta$ counterclockwise. According to the transformation, the Pauli matrices and the derivatives in the action can be classified according to their angular momentum $L_z$, as shown in Tab.I. Based on Tab.I, the most general action without derivatives is

$$S_0 = \int d\tau d^{d-1}x \left[ \sum_{n \geq 0} g_n \phi^n (\gamma^T \sigma_y \gamma) + \sum_{n \geq 1} u_n \phi^n \right],$$

(A1)

where Hermiticity requires $g_n, u_n \in \mathbb{R}$, and the Hermitian conjugate transformations of the fields are $\gamma^T_{\tau,x} = \gamma^T_{-\tau,x}$ and $\phi^T_{\tau,x} = \phi_{-\tau,x}$ since $\tau$ is imaginary time. In $S_0$, $u_1$ is the linear $\phi$ term, $g_0, u_2$ are the mass terms, and all other terms are the on-site interaction terms. The most general kinetic term of $\gamma$ with leading order derivatives is

$$S_1 = \int d\tau d^{d-1}x \gamma^T \left[ A \partial_\tau + i B_1 (\sigma_z + i \sigma_x) (\partial_x - i \partial_y) + i B_1' (\sigma_z - i \sigma_x) (\partial_x + i \partial_y) \right] \gamma,$$

(A2)

where $A \in \mathbb{R}$. The most general kinetic term of $\phi$ with leading order derivatives is

$$S_2 = \int d\tau d^{d-1}x \left[ -C_0 \phi^2 - C_1 \nabla^2 \right] \phi,$$

(A3)

where $C_0, C_1 \in \mathbb{R}$ and there are no first order derivatives since $[\phi_{\tau,x}, \phi_{\tau',x}] = 0$. Then, $S' = S_0 + S_1 + S_2$ is the most general rotationally invariant action if the kinetic term only contains the leading order derivatives and the on-site interaction terms have no derivatives.

Next we show how to derive Eq. (1) of the main text from $S'$. Since we want the classical vacuum of $\phi$ to be uniform, we assume $C_0 > 0, C_1 \geq 0$. We also assume $A \neq 0$ which is typically true for a legitimate fermion action. By changing integration variable $\tau \rightarrow \text{sgn}(A) \tau$ and defining $B_1 = -\frac{1}{2} |A| v_f e^{i \theta_B}, g_n = \bar{g}_n |A| (C_0)^{n/2}, u_n = \bar{u}_n C_0^{n/2}$,

$$C_1 = v_f C_0, \gamma^T_{\text{sgn}(A)\tau,x} = \frac{1}{2 |A|} \gamma^T_{\tau,x} e^{-i \sigma_y \theta_B/2} \text{ and } \phi_{\text{sgn}(A)\tau,x} = \phi_{\tau,x} / \sqrt{2 C_0}. S' \text{ becomes}$$

$$S' = \int d\tau d^{d-1}x \left[ \frac{1}{2} \gamma^T (\partial_\tau - i v_f \alpha \cdot \nabla) \gamma - \frac{1}{2} (\phi^2 + v_f^2 \nabla^2) \phi + \sum_{n \geq 0} g_n \frac{1}{2 n + 1} \phi^n \gamma^T \sigma_y \gamma + \sum_{n \geq 1} u_n \frac{1}{2 n + 2} \phi^n \right],$$

(A4)

where $v_f \geq 0$ can be chosen by adjusting $\theta_B$, and $\gamma'$ and $\phi'$ are renamed as $\gamma$ and $\phi$ since they are integrated over in the partition function. The renaming of $\gamma'$ and $\phi'$ would not cause any physical confusion since $\gamma'$ and $\phi'$ behave the same as $\gamma$ and $\phi$ under rotation, TR and hermitian conjugate:

$$\gamma^T_{\tau,x} = \sqrt{2 |A|} \gamma^T_{\text{sgn}(A)\tau,x} e^{i \sigma_y \theta_B/2} S_{\text{O(2)}} \frac{1}{2 |A|} \gamma^T_{\tau,Rx} e^{-i \sigma_y \theta_B/2} = \gamma^T_{\tau,Rx} e^{-i \sigma_y \theta_B/2},$$

$$\gamma^T_{\tau,x} = \sqrt{2 |A|} \gamma^T_{\text{sgn}(A)\tau,x} e^{i \sigma_y \theta_B/2} TR \sqrt{2 |A|} \gamma^T_{\tau,Rx} e^{i \sigma_y \theta_B/2} = \gamma^T_{\tau,Rx} e^{-i \sigma_y \theta_B/2},$$

$$(\gamma^T_{\tau,x})^\dagger = \sqrt{2 |A|} e^{-i \sigma_y \theta_B/2} \gamma_{\text{sgn}(A)\tau,x} = \gamma^*_{\tau,x},$$

and obvious for $\phi'$ and $\phi$ due to their simple relation $\phi'_{\tau,x} = \sqrt{2 C_0} \phi_{\text{sgn}(A)\tau,x}$. The $\bar{u}_1$ term in Eq. (A4) serves to guarantee the vacuum expectation value of $\phi$ to be zero in the non-magnetic phase, and thus can be omitted. In this case, if we only keep the on-site interaction terms up to the $\phi^3 \sigma_y \gamma$ and $\phi^4$ order and define $\bar{g}_0 = m, \bar{g}_1 = \sqrt{2} g, \bar{u}_2 = r, \bar{u}_3 = \sqrt{2} b/3$ and $\bar{u}_4 = u/3!$, Eq. (A4) is the same as Eq. (1) in the main text. Therefore, Eq. (1) in the main text is the most general rotational invariant action if (i) only keeping leading order derivatives in the kinetic terms, (ii) neglecting derivatives in the on-site interaction, (iii) only keeping terms to $\phi \gamma^T \sigma_y \gamma$ and $\phi^4$ order for the on-site interaction, and (iv) assuming the classical vacuum of $\phi$ is uniform.

At last, we show Eq. (1) in the main text is in the most general rotationally invariant form if only keeping terms to $\phi \gamma^T \sigma_y \gamma$ and $\phi^4$ order for the on-site interaction and assuming the classical vacuum of $\phi$ is uniform, which is the statement at the beginning of this section. It means that we need to argue why higher-order derivatives in the kinetic energy terms and derivatives in the on-site interaction can be neglected. The argument will be done by dimension analysis. The dimensions of fields are $[\gamma] = (d - 1)/2$ and $[\phi] = (d - 2)/2$. As a result, we have $[v_f] = [v_h] = 0$, $[\bar{u}_n] = (2 - n) d/2 + n$ and $[\bar{g}_n] = nd/2 + n + 1$. The zero dimension of $v_f$ and $v_h$ means any higher-order derivatives in the kinetic terms of $\gamma$ and $\phi$ are irrelevant, and thus can be neglected. Now we discuss the on-site interaction. Since we perform RG analysis in $d = 4 - \epsilon$, we may analyze the dimension of the interaction for $d = 4$, resulting that $[\bar{u}_n] = 4 - n$ and $[\bar{g}_n] = 1 - n$. It means, keeping $\phi \gamma^T \sigma_y \gamma$ and $\phi^4$ order is equivalent to neglect all the irrelevant on-site interaction terms at $d = 4$. The remaining on-site interaction terms include $h, g$ and $u$. $g$ and $u$ are marginal, and thus adding derivatives to the two terms would make them irrelevant. $[b] = 1$ for $d = 4$ and thus allows one derivative. However, this derivative would be a total derivative since $(\partial_x \phi) \phi^2 = (\partial_x \phi^3)/3$ and can be neglected. Therefore, the derivatives in the on-site interaction can be neglected if only keeping terms to the $\phi \gamma^T \sigma_y \gamma$ and $\phi^4$ order. The statement at the beginning of this section is proven.
Appendix B: Details For RG Equations

In this section, we derive Eq. (2)-(4) of the main text. We first derive the Callan-Symanzik equation for $N$-point function, and then show the RG equations to the one-loop order.

1. Callan-Symanzik Equation

The regularization scheme chosen here is the dimensional regularization with $d = 4 - \epsilon$. For a generic dimension $d$, the dimensions of the fields, velocities, masses and interaction couplings are $[\gamma] = (d - 1)/2$, $[\phi] = (d - 2)/2$, $[v_b] = [v_f] = 0$, $[m] = [r] = 2$, $[g] = (4 - d)/2$, $[b] = (6 - d)/2$ and $[u] = 4 - d$. For $d = 4$, $[g] = [u] = 0$ and $[b] = 1$. In order to keep the dimensions of the couplings the same as $d = 4$ in the $d = 4 - \epsilon$ scheme, we introduce a parameter $\mu$ with $[\mu] = 1$ by doing the transformation

$$g \to g\mu^{\epsilon/2}, \quad b \to b\mu^{\epsilon/2} \quad \text{and} \quad u \to u\mu^{\epsilon}.$$  \hfill (B1)

In addition, in order to include the quantum corrections, we should introduce $Z$ and $Y$ factors to the action, and the action becomes

$$S = \int d^4 x \left\{ \frac{1}{2} \gamma^T [\nabla, \partial_x + \bar{v}_f (-i \alpha \cdot \nabla)] + Z_m m \sigma_g \gamma + \frac{1}{2} Z_g g \mu^{\epsilon/2} \phi \gamma^T \sigma_g \gamma + \frac{1}{2} \phi (-Z_\phi \partial_x^2 - Z_{v_b} v_b^2 \nabla^2 + Z_r r) \phi + Y \phi \\
+ \frac{1}{3!} Z_b b \mu^{3/2} \mu^3 + \frac{1}{4!} Z_u u \mu \mu^4 \right\},$$  \hfill (B2)

where the partition function is $Z = \int D\gamma D\phi e^{-S}$, $X = (x, x)$, the linear $\phi$ term is now included explicitly to deal with the quantum corrections and $Y$ is chosen to make $\langle \phi \rangle = 0$ for no magnetic ordering. Here, $Z$ and $Y$ factors are chosen to only cancel the divergent part of the quantum corrections ($MS$ scheme[27]). Since the divergence of the quantum corrections is given by $1/e^n$ with $n$ positive integer, the $Z$ factors must have the form

$$\ln(Z_a) = \sum_{n=1}^{+\infty} A^{(n)}_a \frac{1}{e^n}$$  \hfill (B3)

with $a = \gamma, \phi, v_f, v_b, m, r, g, b, u$ and “ln” function formally defined as $\ln(1 + x) = \sum_{n=1}^{+\infty} (-1)^{n-1} x^n$. Now, we derive the expressions $\beta$ and $\Gamma$ functions, where we use $\Gamma$ instead of commonly used $\gamma$ to label the Gamma functions since the latter is reserved for the Majorana field. Since $\mu$ is not physical, the physical action as well as any physical observable should not depend on $\mu$. By defining

$$\phi^{(0)} = Z^{1/2}_\phi \phi, \quad \gamma^{(0)} = Z^{1/2}_\gamma \gamma, \quad g^{(0)} = \mu^{\epsilon/2} Z_g g Z^{-1/2}_\phi Z^{-1}_\gamma, \quad b^{(0)} = Z_b Z^{-3/2}_\phi Z^{\epsilon/2}_\phi, \quad u^{(0)} = \mu^{\epsilon} Z_u u Z^{-2}_\phi,$$

$$v_f^{(0)} = Z_{v_f} v_f Z^{-1}_\gamma, \quad (v_b^{(0)})^2 = Z_{v_b} v_b^2 Z^{-1}_\phi, \quad m^{(0)} = Z_m m Z^{-1}_\gamma, \quad r^{(0)} = Z_r r Z^{-1}_\phi \quad \text{and} \quad Y^{(0)} = Z^{-1/2}_\phi Y,$$  \hfill (B4)

the partition function becomes $Z = C_0 \int D\gamma^{(0)} D\phi^{(0)} e^{-S}$ with $S$ becomes

$$S = \int d^4 x \left\{ \frac{1}{2} (\gamma^{(0)})^T [\partial_x + v_f^{(0)} (-i \alpha \cdot \nabla)] + m^{(0)} \sigma_g \gamma^{(0)} + \frac{1}{2} g^{(0)} (\phi^{(0)}) (\gamma^{(0)})^T \sigma_g \gamma^{(0)} + \frac{1}{2} \phi^{(0)} (-\partial_x^2 - (v_b^{(0)})^2 \nabla^2 + r^{(0)}) \phi^{(0)} \\
+ Y^{(0)} \phi^{(0)} + \frac{1}{3!} b^{(0)} (\phi^{(0)})^3 + \frac{1}{4!} u^{(0)} (\phi^{(0)})^4 \right\}.$$  \hfill (B5)

In this way, we obtain a partition function with an action that is independent of $\mu$. Since $\mu$ is not physical, $S$ in the above equation should be the physical action that gives the physical observables. It means the fields $\phi^{(0)}$ and $\gamma^{(0)}$ and the parameters $w^{(0)}$’s in the above equation are physical and independent of $\mu$, where $w = v_f, v_b, r, m, g, b, u$. As
a result, we have

\[ \frac{d}{d \ln \mu} \ln(\mu'^{1/2} Z_g Z_{\phi}^{-1/2} Z_{\gamma}^{-1}) = 0 \iff \frac{d g}{d \ln \mu} = -\frac{\epsilon}{2} g + \beta_g \text{ with } \beta_g = gD(A^{(1)}_g - \frac{1}{2} A^{(1)}_\phi - A^{(1)}_\gamma) \]

\[ \frac{d}{d \ln \mu} \ln(\mu'^{1/2} Z_b Z_{\phi}^{3/2}) = 0 \iff \frac{d b}{d \ln \mu} = -\frac{\epsilon}{2} b + \beta_b \text{ with } \beta_b = bD(A^{(1)}_b - \frac{3}{2} A^{(1)}_\phi) \]

\[ \frac{d}{d \ln \mu} \ln(Z_u u Z_{\phi}^{-2}) = 0 \iff \frac{d u}{d \ln \mu} = -\epsilon u + \beta_u \text{ with } \beta_u = uD(A^{(1)}_u - 2A^{(1)}_\phi) \]

\[ \frac{d}{d \ln \mu} \ln(Z_v v Z_{\phi}^{-1}) = 0 \iff \frac{1}{v} \frac{d v}{d \ln \mu} = \Gamma_v \text{ with } \Gamma_v = D(A^{(1)}_v - A^{(1)}_\gamma) \]

\[ \frac{d}{d \ln \mu} \ln(Z_m m Z_{\phi}^{-1}) = 0 \iff \frac{1}{m} \frac{d m}{d \ln \mu} = \Gamma_m \text{ with } \Gamma_m = D(A^{(1)}_m - A^{(1)}_\gamma) \]

\[ \frac{d}{d \ln \mu} \ln(Z_r r Z_{\phi}^{-1}) = 0 \iff \frac{1}{r} \frac{d r}{d \ln \mu} = \Gamma_r \text{ with } \Gamma_r = D(A^{(1)}_r - A^{(1)}_\phi) \]

\[ \Gamma_\phi = \frac{1}{2} \frac{d}{d \ln \mu} \ln(Z_\phi) = -\frac{1}{2} D(A^{(1)}_\phi) \]

\[ \Gamma_\gamma = \frac{1}{2} \frac{d}{d \ln \mu} \ln(Z_\gamma) = -\frac{1}{2} D(A^{(1)}_\gamma) \]    

where \( D = \frac{\nu}{2}\partial_\gamma + \frac{b}{2}\partial_b + u\partial_u \) and the condition that \( \beta \) and \( \Gamma \) functions are finite at \( \epsilon \to 0 \) is used. Now we derive the Callan-Symanzik equation. The physical \( N \)-point function is defined as

\[ F^{(0)}_{N_\phi,N_\gamma}(X_n,w^{(0)}) = \left\langle \prod_{n_1=1}^{N_\phi} \phi_{X_{n_1}}^{(0)} \prod_{n_2=1}^{N_\gamma} \gamma_{X_{n_2}^{+n_2}}^{(0)} \right\rangle \]  

with \( X_n \) indicating all its coordinate dependence. The above equation is related with the \( N \)-point function of \( \gamma \) and \( \phi \) by Eq. (B4):

\[ F_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) = \left\langle \prod_{n_1=1}^{N_\phi} \phi_{X_{n_1}} \prod_{n_2=1}^{N_\gamma} \gamma_{X_{n_2}^{n_2}} \right\rangle = Z_{\phi}^{-N_\phi/2} Z_{\gamma}^{-N_\gamma/2} F^{(0)}_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) \]  

Combining the fact that \( F^{(0)}_{N_\phi,N_\gamma} \) is independent of \( \bar{\mu} \) and Eq. (B6), we arrive at the Callan-Symanzik equation with respect to \( \bar{\mu} \):

\[ \left( \frac{\partial}{\partial \ln \mu} + N_\phi \Gamma_\phi + N_\gamma \Gamma_\gamma + \beta_\gamma \partial_\gamma + \beta_u \partial_u + \beta_b \partial_b + v_1 \Gamma_v \partial_v + v_b \Gamma_{vb} \partial_{vb} + m \Gamma_m \partial_m + r \Gamma_r \partial_r - \epsilon D \right) F_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) = 0 \]  

But we want the Callan-Symanzik equation with respect to the physical scale instead of the non-physical \( \bar{\mu} \). To do so, consider the \( N \)-point function with scaled coordinates \( F_{N_\phi,N_\gamma}(tX_n,w,\bar{\mu}) \). By defining \( \gamma_X = \gamma_X t^{\langle \gamma \rangle} \), \( \phi_X = \phi_X t^{\langle \phi \rangle} \), \( w' = wt^{[w]} \), \( Y' = Yt^{[Y]} \) and \( \bar{\mu}' = \bar{\mu}t^{[\bar{\mu}]} \), we have

\[ F_{N_\phi,N_\gamma}(tX_n,w,\bar{\mu}) = \tau^{-N_\phi \langle \phi \rangle - N_\gamma \gamma} F_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) \]  

Differentiating the above equation by \( t \) at \( t \to 1 \) and using Eq. (B9), we have

\[ X_n \partial_X \partial_{X_n} F_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) = \left[ -N_\phi \left( \frac{d-2}{2} + \Gamma_\phi \right) - N_\gamma \left( \frac{d-1}{2} + \Gamma_\gamma \right) + \left( \frac{\epsilon}{2} g - \beta_\gamma \right) \partial_\gamma + \left( \frac{\epsilon}{2} b + \beta_b \right) \partial_b + \left( \epsilon u - \beta_u \right) \partial_u \right. \]

\[ + \left. (-v_1 \Gamma_v) \partial_v + (-v_b \Gamma_{vb}) \partial_{vb} + (1 - \Gamma_m) m \partial_m + (2 - \Gamma_r) r \partial_r \right] F_{N_\phi,N_\gamma}(X_n,w,\bar{\mu}) \]  

The meaning of the above equation can be better illustrated in the integrated form:

\[ F_{N_\phi,N_\gamma}(c' X_n,w(0),\bar{\mu}) = \left[ \zeta_\phi(l) \right]^{N_\phi} \left[ \zeta_\gamma(l) \right]^{N_\gamma} F_{N_\phi,N_\gamma}(X_n,w(l),\bar{\mu}) \]  

Please note that there might be some typographical errors in the derivations, particularly in the equations involving logarithmic terms and derivatives.
\[
\frac{dg}{dl} = \frac{\epsilon}{2} g - \beta_g, \quad \frac{db}{dl} = (\frac{\epsilon}{2} + 1) b - \beta_b, \quad \frac{du}{dl} = \epsilon u - \beta_u, \quad \frac{dv_f}{dl} = -v_f \Gamma_{v_f}, \quad \frac{dv_b}{dl} = -v_b \Gamma_{v_b},
\]
\[
\frac{dm}{dl} = (1 - \Gamma_m)m, \quad \frac{dr}{dl} = (2 - \Gamma_r)r, \quad \frac{d\zeta}{dl} = -\left(\frac{d - 2}{2} + \Gamma_{\zeta}\right)\zeta, \quad \frac{d\gamma}{dl} = -\left(\frac{d - 1}{2} + \Gamma_{\gamma}\right)\gamma,
\]
(13)
and \(\zeta(0) = \zeta(0) = 1\). Eq. (12) indicates that the \(N\)-point function at a larger scale \(e^l X_n\) is the same as the \(N\)-point function at \(X_n\) with scaled fields \(\phi, \gamma\) and parameters \(w\) according to Eq. (13). It means, we can define an action \(S^l\) that is the same as \(S\) except that the parameters \(w\) in \(S^l\) can scale with \(l\) according to Eq. (13). In this case, the \(N\)-point function generated by \(S^l\) at \(X_n\) and the \(N\)-point function given by \(S\) at \(e^l X_n\) just deviate from each other by a factor \[\left[\zeta(l)\right]^{N_\phi}[\zeta(l)]^{N_\gamma}\]. Then, \(S_l\) can be viewed as an effective action of \(S\) at a larger scale \(e^l\). And Eq. (13) is called the RG equations. For the purpose of finding SUSY in this work, we only care about \(v_f/v_b, b/(gm)\) and \(u/g^2\) since \(r/m^2\) is assumed to be tunable. Therefore, in the following, we will derive the RG equations of \(v_f, v_b, m, g, b, u\) to the one-loop order.

2. One-Loop RG Equations

In order to obtain Eq. (2)-(4) in the main text, we need to find the corresponding \(\beta\) and \(\Gamma\) functions in Eq. (13). According to Eq. (6) and Eq. (3), it is equivalent to deriving the expression of the \(Z\) factors in Eq. (2). Since the \(Z\) factors are given by the quantum corrections, we need to evaluate the loop diagrams. For convenience, we separate Eq. (2) into two parts \(S = S_c + S_{ct}\) with

\[
S_c = \int d^d X \left\{ \frac{1}{2} \partial_\tau v_f \left( -i\alpha \cdot \nabla \right) + m \sigma_y \gamma + \frac{1}{2} g \mu^{t/2} \phi \gamma^T \sigma_y \gamma \right\}
\]

\[
\frac{1}{4!} b \mu^{t/2} \phi^3 + \frac{1}{4!} u \mu^{t/4}\phi^4 \right\}
\]
(14)

\[
S_{ct} = \int d^d X \left\{ \frac{1}{2} \gamma \left[ (Z_{\phi} - 1) \partial_\tau + (Z_{v_f} - 1) v_f \left( -i\alpha \cdot \nabla \right) \right] + m \sigma_y \gamma + \frac{1}{2} (Z_{\phi} - 1) g \mu^{t/2} \phi \gamma^T \sigma_y \gamma \right\}
\]

\[
\frac{1}{2} \left[ (Z_{\phi} - 1) \partial_\tau^2 + (Z_{v_b} - 1) v_b^2 \nabla^2 + (Z_{\gamma} - 1) r \right] \phi + \frac{1}{4!} (Z_b - 1) b \mu^{t/2} \phi^3 + \frac{1}{4!} (Z_u - 1) u \mu^{t/4} \phi^4 \right\}
\]
(15)
Here \(S_c\) is Eq. (2) without quantum corrections and \(S_{ct}\) is called the counter-terms. The \(Y\) term in Eq. (15) does not need to be included explicitly in the diagrams since it just cancel all the tadpole graphs. According to Eq. (14), the fermion propagator is

\[
G_{\gamma}(k) = (i\omega - m \sigma_y - v_f k \cdot \alpha)^{-1}
\]
(16)
and the boson propagator reads

\[
G_{\phi}(q) = \left( \nu^2 + v_b^2 q^2 + r \right)^{-1}
\]
(17)
Here \(k = (\omega, k)\) and \(q = (\nu, q)\). Equipped with Eq. (14)-(17), we can evaluate the loop diagrams. For simplicity, we only consider the one-loop diagrams, which together with the corresponding counter-terms are shown in Fig. 4.

The one-loop contribution to the fermion self-energy \(\Sigma(k)\) is given by Fig. 4 a and b, which reads

\[
\Sigma(k) = g^2 \mu \int \frac{d^d q}{(2\pi)^d} \sigma_g \gamma G_{\gamma}(k - q) \sigma_g G_{\phi}(q) \left[ (Z_{\phi} - 1) \gamma + (Z_{v_f} - 1) \gamma \right] + O(\epsilon^0)
\]
(18)
where we only keep the divergent terms since \(Z\) factors are chosen to only cancel the divergent part.

The one-loop contribution to the boson self-energy \(\Pi(q)\) is given by Fig. 4 c, d, e and f, which reads

\[
\Pi(q) = \frac{b^2 \mu^c}{2} \int \frac{d^d q_1}{(2\pi)^d} G_{\phi}(q_1 - q) G_{\phi}(q_1) - \frac{g^2 \mu^c}{2} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}[G_{\gamma}(k) \sigma_g G_{\gamma}(k + q) \sigma_g]}{V^2} - \frac{u \mu^c}{2} \int \frac{d^d q_1}{(2\pi)^d} G_{\phi}(q_1)
\]

\[
- \left[ (Z_{\phi} - 1) \nu^2 + (Z_{v_b} - 1) v_b^2 q^2 + (Z_{\gamma} - 1) r \right] = \frac{b^2}{16 \pi^2 \nu_b^4} \frac{g^2 \left( 6 m^2 + \nu^2 + v_b^2 q^2 \right)}{16 \pi^2 \nu_f^2 \epsilon} + \frac{r u}{16 \pi^2 \nu_b^4} \epsilon
\]
(19)
FIG. 4. One-loop Feynman diagrams and the corresponding counter-terms for the $Z$ factors in Eq. (B2). The solid (dashed) line is the fermion (boson) propagator in Eq. (B16) (Eq. (B17)). The cross maker stands for the counter-term in Eq. (B15).

The fermion-boson coupling term $\Gamma^{\phi \gamma \gamma}$ has the one-loop correction given by Fig. 4 g, h and i, which reads

$$F^{\phi \gamma \gamma} = -g^3 \frac{\bar{\mu}^{3\epsilon/2}}{2} \int \frac{d^d q}{(2\pi)^d} \left[ \sigma_y G_{\gamma}(q') \sigma_y G_{\gamma}(q') \sigma_y \right] G_\phi(q') - g\bar{\mu}^{\epsilon/2}(Z_g - 1)\sigma_y + O(\epsilon^0)$$

$$= g^3 \frac{\bar{\mu}^{3\epsilon/2}}{2} \int d^d q \frac{\bar{\mu}^{3\epsilon/2}}{4\pi^2} \epsilon (v_b + v_f) \sigma_y - g(Z_g - 1)\sigma_y + O(\epsilon^0) ,$$

(B20)

where the contribution of Fig. 4h is not explicitly included since it is not divergent for $d = 4 - \epsilon$.

For the $\phi^3$ term, the one-loop contribution is given by Fig. 4 j, k, l and m, and reads

$$F^{\phi^3} = -g^3 \frac{\bar{\mu}^{3\epsilon/2}}{2} \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left[ \sigma_y G_{\gamma}(k) \sigma_y G_{\gamma}(k) \sigma_y G_{\gamma}(k) \right] + \frac{3}{2} ub\bar{\mu}^{3\epsilon/2} \int \frac{d^d q}{(2\pi)^d} G_\phi(q)^2 - (Z_b - 1)b\bar{\mu}^{\epsilon/2} + O(\epsilon^0)$$

$$= -\frac{3g^3m}{4\pi^2v_f^4} \epsilon + \frac{3bu}{16\pi^2v_b^4} \epsilon - (Z_b - 1)b + O(\epsilon^0) ,$$

(B21)

where Fig. 4l is not divergent and included in $O(\epsilon^0)$.

Fig. 4 n, o, p, q and r give the one-loop contribution of the $\phi^4$ term. However, only Fig. 4 n, o and r are divergent.
and contribute to the RG equations, which gives
\[
F_{\phi^4} = \frac{3u^2}{2} \bar{\mu}^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} G_\phi(q)^2 - 3g^4 \bar{\mu}^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \text{Tr}[G\gamma(k)\sigma_y G\gamma(k)\sigma_y G\gamma(k)\sigma_y] - (Z_u - 1)u\bar{\mu} + O(\epsilon^0) \\
= \frac{3u^2}{16\pi^2 v_b^3} \frac{1}{\epsilon} - 3g^4 \frac{1}{4\pi^2 v_f^3} \epsilon - (Z_u - 1)u + O(\epsilon^0) .
\] (B22)

According to Eq. (B18)-(B22) and Eq. (B3), we have the expressions of \( A_a^{(1)} \)'s as
\[
A_{\gamma}^{(1)} = -\frac{g^2}{4\pi^2 v_b(v_b + v_f)^2} \\
A_{v_f}^{(1)} = -\frac{g^2(2v_b + v_f)}{12\pi^2 v_b v_f(v_b + v_f)^2} \\
A_m^{(1)} = \frac{g^2}{4\pi^2 v_b v_f(v_b + v_f)} \\
A_{v_b}^{(1)} = -\frac{g^2}{16\pi^2 v_f^2} \\
A_g^{(1)} = \frac{g^2}{4\pi^2 v_b v_f(v_b + v_f)} \\
A_b^{(1)} = \frac{3\left( u - \frac{4g^2 m}{m_f} \right)}{16\pi^2} \\
A_u^{(1)} = \frac{3\left( u^2 v_f^4 - 4g^4 v_b^3 \right)}{16\pi^2 u v_b^3 v_f^2} \\
A_r^{(1)} = \frac{-6m^2 g^2 v_b^3 + (ur + b^2) v_f^3}{16\pi^2 v_b^3 v_f^3 r} .
\] (B23)

Plug the above equation into Eq. (B6), we can get the \( \beta \) and \( \Gamma \) functions and then we can derive the RG equations according to Eq. (B13). Due to the transformation Eq. (B1), \( g, u \) and \( b \) in the obtained RG equations should be replaced by \( \tilde{g}, \tilde{u} \) and \( \tilde{b} \) according to the convention defined in the main text. As a result, we can get Eq. (2)-(4) in the main text.

Now we show that \( r \) does not have any stable flow. According to Eq. (B23), the RG equation of \( r \) for \( v_f = v_b = 1 \) reads
\[
\frac{dr}{dl} = -\frac{b^2 + g^2 (r - 6m^2) + r(u - 32\pi^2)}{16\pi^2} .
\] (B24)

Although \( r \) is relevant since its dimension is \( 2 - (u + g^2)/(16\pi^2) = 2 - 4\epsilon/7 > 0 \) for \( u^* = 3(g^*)^2 \) and \( g^* = \sqrt{16\pi^2\epsilon/7} \), it might have stable flow if combined with the other independent relevant parameter \( m \). To check that, we consider the RG flow of \( r/m^2 \), which reads
\[
\frac{d}{dl} \left( \frac{r}{m^2} \right) = -\frac{u - 5g^2}{16\pi^2} \left( \frac{r}{m^2} \right) - \frac{b^2 - 6g^2 m^2}{16\pi^2 m^2} .
\] (B25)

Again, the dimension of \( r/m^2 \) is \( -(u - 5g^2)/(16\pi^2) = 2(g^*)^2/(16\pi^2) > 0 \), and thus \( r/m^2 \) has no stable flow.

3. Higher-loop Contribution and finite-scale nature of Emergent SUSY at FOQPT

Unlike the continuous phase transition, the validity of perturbation theory in series of loops is not obvious in our case due to the existence of relevant \( m \), \( r \) and \( b \), which might make the higher-loop terms not small. In this part, we discuss the validity of neglecting the higher-loop terms and the related finite-scale nature of emergent SUSY at first-order quantum phase transition(FOQPT). Here we still replace the \( \tilde{g}, \tilde{u}, \tilde{b} \) in the main text by \( g, u, b \) as above.
As shown above, the RG equations are obtained from the counter-terms. The counter-terms are determined by the graphs with non-negative superficial degree of divergence \( D = 4L - 2I_\phi - I_\gamma \), where \( L \) is the number of loops in the graph, and \( I_\phi \) and \( I_\gamma \) are the numbers of internal bosonic and fermionic propagators, respectively.[27] To estimate the order of higher-loop terms, we need to know the structure of a generic connected graph. As one end of the external propagator and the two ends of the internal propagator are connected to vertexes, we have \( E_\phi + 2I_\phi = V_\phi + 3V_b + 4V_u \) and \( E_\gamma + 2I_\gamma = 2V_g \), where \( E_\phi \) and \( E_\gamma \) are the numbers of external bosonic and fermionic propagators, respectively, and \( V_g, V_b, \) and \( V_u \) are number of \( g, b \) and \( u \) vertexes, respectively. Furthermore, the momentum conservation gives \( L = I_\phi + I_\gamma - (V_\phi + V_b + V_u) + 1 \). From the above relations, we have

\[
L = -\frac{E_\phi - E_\gamma + V_\phi + V_b + 2V_u + 2}{2}, \quad D = 4 - E_\phi - \frac{3E_\gamma}{2} - V_b. \tag{B26}
\]

Equipped with those relations, we next address the possible issues brought by the three relevant parameters by estimating the order of the \( L \)-loop contribution to the counter-term near the SUSY hyper-surface, i.e. \( u \sim g^2, b \sim gm, r \sim m^2 \) (as \( r/m^2 \) is tunable) and \( v_f \sim v_b \sim 1 \).

Since only \( D \geq 0 \) graphs contribute to counter-terms, high powers of \( b \) do not exist in the RG equations as they can make \( D \) negative according to Eq. (B26). Therefore, the relevant \( b \) cannot cause any divergence of the series. On the other hand, \( D \geq 0 \) requires \( 2E_\phi + 3E_\gamma \leq 8 \), leading to the following six combinations: \( (E_\phi, E_\gamma) = (1, 0), (2, 0), (3, 0), (4, 0), (0, 2) \) or \( (1, 2) \). Here we use the fact that \( E_\gamma \) can only be even. Clearly, those six combinations correspond exactly to the counter-terms that we include in Eq. (B2), verifying the renormalizability of our theory. Among the six combinations, we only need to care about the last five ones since the \( Y \) term with \( (E_\phi, E_\gamma) = (1, 0) \) does not contribute to the RG equation. In the following, we estimate the order of the counter-terms for the five combinations, namely the \( Z \) factors. A generic connected graph has the form \( g^{v_b} b^{v_u} u^{v_\phi} F(k) \) with the loop-integral part \( F(k) \) having dimension \( |F(k)| = D \). Near the SUSY hyper-surface, only \( m \) and \( k \) can carry dimensions, and thus the graph is approximately \( g^{2L-2+2E_\phi+E_\gamma} \sum C_n m^{D+4V_b+4V_u} k^n \), where \( C_n \) is dimensionless and given by the loop integral. Based on this estimation, all the \( Z \) factors approximately have the form \( 1 + \sum L_{i=1}^\infty C_L^{i} g^i \) where the dimensionless \( C_L \) has the form \( \sum_{i=1}^L C_L^{i} / \epsilon \) and only depends on \( \ln(\mu/m) \). Since only \( C_L^{1} \) contributes to the RG equations, we only need to care about whether \( \sum_{i=1}^{\infty} C_L^{(i+1)} (g^2)^i \) is well defined. As \( m \) is relevant, \( |m| \gg \mu \) is only true at a relatively large scale, and thus the \( C_L^{1} \) is tunable. Then, the \( L \)-loop term in the series of interest is of the order \( \sim [\ln(\mu/m)]^{L-1} (g^2)^L \). Therefore, the higher-loop contributions to the RG equations are negligible if \( \ln(\mu/m) \ll 1/g^2 \sim 1/\epsilon \sim \ln(\Lambda/\mu) \), where \( \Lambda \) is the ultraviolet energy cut-off of the model. Clearly, in the assumption that the ultraviolet energy cut-off is much larger than any dimensionful parameter of the model, the higher-loop terms can be neglected and the one-loop result is trustworthy. In the following, we discuss how this constraint affects the scaling property of the emergent SUSY.

| \( \ln(m/\mu) \) | \( \mu/g \) | \( b_1 \) | \( \mu/g \) | \( v_f \) | \( m/\mu \) | \( l_c \) |
|----------------|------|------|------|------|------|------|
| 0              | 2.5  | 2.5  | 2.9778| 2.9778| 100.746|
| 2              | 2.2  | 2.2  | 2.9616| 2.9616| 100.746|
| 20             | 2    | 2    | 2.99297| 2.99297| 100.746|
| 40             | 2    | 2    | 2.89208| 2.89208| 60.5685|

**TABLE II.** The finite scaling range for the emergent SUSY at the first-order quantum phase transition (FOQPT). Parameters with lower index \( i \) and \( j \) are the values at \( l = 0 \) and \( l = l_c \), respectively. The results are given by numerically solving the one-loop RG equations in the main text for \( \epsilon = 10^{-3}, g_i = 1 \) and \( v_f = v_b = 1 \). \( l_c \) is given by \( \ln(m/\mu) = 0.1(1/\epsilon) \), \( (u/g^2, b/(mg)) = (3, 3) \) is the SUSY point.

As the mass \( m \) is relevant, \( m \) increases as the scale parameter \( \Lambda \) increases from 0. Since the validity of the one-loop approximation requires the mass to be much smaller than \( \Lambda \), the one-loop RG equations would not hold if the scale parameter \( l \) becomes larger than a critical value \( l_c \) where the mass becomes comparable with \( \Lambda \). It means that the stable RG flow to the SUSY hyper-surface indicated by the one-loop RG equations might only happen in a finite range of the scale, i.e. \( l \in [0, l_c] \), since terms with more loops will be significant for \( l \geq l_c \) and might lead to the deviation of the RG flow away from the SUSY point. As a result, for a smaller initial mass \( m_i \) (the mass at \( l = 0 \)), we will expect a larger \( l_c \) so that the RG flow can approach closer to the SUSY point (Let us always assume the RG flow is stopped when \( l \sim l_c \)). To see this, let us choose \( \ln(\Lambda/\mu) \sim \epsilon = 10^{-3}, g_i = 1 \) and \( v_f = v_b = 1 \) for convenience, and we define that the one-loop RG equations are valid only if \( \ln(m/\mu) \leq 0.1(1/\epsilon) \), i.e. \( \ln(m/\mu) = 100 \). With those choices
and the definition, we numerically solve the RG equations listed in the main text for \((u_i/g_i^2, b_i/(g_im_i)) = (2, 2)\) and \(\ln(m_i/\mu) = 0, 20\) and 40. As shown in the last three rows of Tab. II, the resulting critical scale \(l_c\) is \(l_c \approx 100, 80\) and 60, respectively, and the deviation from the SUSY hyper-surface at \(l = l_c\) is smaller when the initial mass is smaller. On the other hand, if the initial mass (and the \(g_i\)) is fixed, \(l_c\) would stay unchanged as \(u_i/g_i^2\) and \(b_i/(g_im_i)\) change. In this case, if the system is initially closer to the SUSY hyper-surface, the SUSY signature would be more clear at \(l = l_c\), as shown in the first three rows of Tab. II.

In conclusion, the validity of the one-loop RG equations requires the mass to be much smaller than the ultraviolet energy cutoff, implying that the stable flow to the SUSY hyper-surface might only hold in a finite range of scale \((l \in [0, l_c])\). If the initial mass and the initial deviation from the SUSY hyper-surface are smaller, the SUSY signature at \(l \approx l_c\) would be more clear, maybe even “exact” within the numerical error in a numerical simulation. Such finite-scale nature distinguishes the emergent SUSY at FOQPT from that at the continuous quantum phase transition, where the latter has no limit on the scaling and becomes exact when \(l\) goes to infinity.

Appendix C: First-Order Phase Transitions and Emergent SUSY

In this section, we first show details on determining the phases of Eq. (6) in the main text and then discuss the SUSY. For convenience, we drop the “*” of all parameters in Eq. (6) in the main text.

1. First-Order Phase Transition

\(\langle \phi \rangle\) is given by the global minimum of bosonic part of Eq. (6) in the main text, which reads

\[
S_b^\phi = \int d\tau d^{d-1}x \left[ \frac{1}{2} \phi (-\partial^2 + r)\phi + \frac{1}{2} g m \phi^3 + \frac{1}{8} g^2 \phi^4 \right].
\] (C1)

Since \(\langle \phi \rangle\) indicates the macroscopic magnetic ordering and must be real, we should impose \(\phi^*_{\tau, x} = \phi_{\tau, x}\) when solving for the global minimum of \(S_b^\phi\). \(\phi^*_{\tau, x}\) can be splitted into two parts \(S_T^\phi = S_T + S_V\) with

\[
S_T = \int d\tau d^{d-1}x \frac{1}{2} \phi (-\partial^2)\phi \text{ and } S_V = \int d\tau d^{d-1}x \ V(\phi).
\] (C2)

Here \(V(\phi) = \frac{1}{2} r \phi^2 + \frac{1}{2} g m \phi^3 + \frac{1}{8} g^2 \phi^4\). By defining \(\phi_{\tau, x}^* = \frac{1}{\beta V} \sum_{\omega, k} e^{-i \omega \tau + i k x} \phi_{\omega, k}\) and \(\phi^*_{\omega, k} = \phi_{-\omega, -k}\), \(S_T\) can be re-written as

\[
S_T = \frac{1}{\beta V} \sum_{\omega, k} (\omega^2 + k^2)|\phi_{\omega, k}|^2.
\] (C3)

It means \(S_T \geq 0\) and \(S_T = 0\) holds only if \(\phi\) is uniform in \((\tau, x)\). On the other hand, for any non-uniform field \(\phi_{\tau, x}\), we can always find a uniform field \(\phi^0\) such that \(S_V[\phi^0] \leq S_V[\phi]\). (Simply, one can pick one position \((\tau_0, x_0)\) such that \(V(\phi_{\tau, x}) \geq V(\phi_{\tau_0, x_0})\) holds for any \((\tau, x)\) and define the uniform field as \(\phi^0 = \phi_{\tau_0, x_0}\)). Therefore, the global minimum of \(S_b^\phi\) must be uniform given \(\phi^*_{\tau, x} = \phi_{\tau, x}\). Therefore, we only need to minimize the bosonic potential \(V(\phi)\) to obtain the minimum field. The extrema of \(V(\phi)\) are given by

\[
\frac{dV(\phi)}{d\phi} = \frac{1}{2} \phi (g^2 \phi^2 + 3g m \phi + 2r) = 0.
\] (C4)

In this case, if \(9m^2 < 8r\), there is only one extremum that is \(\phi_0 = 0\), and if \(9m^2 \geq 8r\), the extrema are

\[
\phi_0 = 0 \text{ and } \phi_\pm = \frac{-3m \pm \sqrt{9m^2 - 8r}}{2g}.
\] (C5)

Clearly, \(\phi_\pm \neq 0\) as long as \(m \neq 0\) and they exist. The values of \(V(\phi)\) at the three extrema are

\[
V(\phi_0) = 0 \text{ and } V(\phi_\pm) = \frac{1}{16} \phi_\pm^2 (2g m \phi_\pm + 4r).
\] (C6)

Now we discuss the global minimum of \(V(\phi)\). As mentioned before, if \(9m^2 < 8r\), \(\phi_0\) is only one extremum and thereby is the global minimum. If \(9m^2/8 \geq r > m^2\), we have \(4r - 3m^2 > m^2 > |m\sqrt{9m^2 - 8r}|\), which gives
2gm\phi_\pm + 4r = 4r - 3m^2 \pm m\sqrt{9m^2 - 8r} > 0 and thus \( V(\phi_\pm) > 0 \). Therefore, \( \phi_0 = 0 \) is the global minimum if \( r > m^2 \). For \( r < m^2 \) and \( m > 0 \), we have

\[
2gm\phi_- + 4r < 2m^2 - m\sqrt{9m^2 - 8r} < 0,
\]
which gives \( V(\phi_-) < 0 \). \( m > 0 \) results in \( 2gm\phi_- + 4r < 2gm\phi_+ + 4r \) and \( \phi_-^2 > \phi_+^2 \). In this case, if \( 2gm\phi_+ + 4r \geq 0 \), \( V(\phi_+) > V(\phi_-) \). And if \( 2gm\phi_+ - 4r < 0, \phi_-^2 > \phi_+^2 \) gives \( V(\phi_+) > V(\phi_-) \). Then, we have \( V(\phi_-) < V(\phi_+) \). Therefore, \( \phi_- \) is the global minimum if \( r < m^2 \) and \( m < 0 \). Since \( V(\phi) \) is invariant under \( m \rightarrow -m \) and \( \phi \rightarrow -\phi \), the global minimum is at \( \phi_+ \) if \( r < m^2 \) and \( m > 0 \). Therefore, \( r = m^2, m \neq 0 \) and \( r < 0, m = 0 \) are where the first order phase transition happens. In the following, we show the form of the action at the first order phase transition.

At \( r = m^2 \) with \( m \neq 0 \), there are two degenerate vacua: (i) \( \langle \phi \rangle = 0 \), and (ii) \( \langle \phi \rangle = -2m/g \). Around the first vacuum, the form of the action is just \( S^* \) with \( r = m^2 \). Around the second vacuum, we need to use the fluctuation \( \delta \phi = \phi - (-2m/g) \) in \( S^* \), and the resulted action for \( \gamma \) and \( \delta \phi \) reads

\[
S^* = \int d\tau d^{d-1}x \left\{ \frac{1}{2} \gamma^T (i\partial_\mu \alpha^\mu - m\sigma_y)\gamma + \frac{1}{2} g\delta \phi \gamma^T \sigma_y \gamma + \frac{1}{2} \delta \phi (\partial^2 + m^2)\delta \phi - \frac{gm}{2} \delta \phi^3 + \frac{1}{8} g^4 \delta \phi^4 \right\}.
\]

At \( r < 0 \) and \( m = 0 \), \( S^* \) reads

\[
S^* = \int d\tau d^{d-1}x \left\{ \frac{1}{2} \gamma^T (i\partial_\mu \alpha^\mu \pm \sqrt{-2r})\gamma + \frac{1}{2} g\delta \phi \gamma^T \sigma_y \gamma + \frac{1}{2} \delta \phi (\partial^2 - 2r)\delta \phi + \frac{-g\sqrt{-2r}}{2} \delta \phi^3 + \frac{1}{8} g^4 \delta \phi^4 \right\}.
\]

Since \( r < 0 \), the boson part has spontaneous symmetry breaking (SSB) and the new vacua are at \( \phi = \pm \sqrt{-2r}/g \). Around the new vacua \( \phi = \delta \phi \pm \sqrt{-2r}/g \), we have

\[
S^* = \int d\tau d^{d-1}x \left\{ \frac{1}{2} \gamma^T (i\partial_\mu \alpha^\mu \pm \sqrt{-2r})\gamma + \frac{1}{2} g\delta \phi \gamma^T \sigma_y \gamma + \frac{1}{2} \delta \phi (\partial^2 - 2r)\delta \phi + \frac{\pm g\sqrt{-2r}}{2} \delta \phi^3 + \frac{1}{8} g^4 \delta \phi^4 \right\}.
\]

2. Emergent SUSY

Since the fields considered here are one two-component Majorana field and one Ising field, there should be two supercharges \( Q_a \) with \( a = 1, 2 \) instead of four for \( N = 1 \) Wess-Zumino model. [5] The supercharges satisfy

\[
\{ Q_{a1}, Q_{a2} \} = 2\alpha^\mu_{a1a2} P_\mu ,
\]

where \( \alpha^\mu = \sigma_y (\alpha^\mu)^T \sigma_y \) and \( P_\mu \) is the energy-momentum operator that gives \( [P_\mu, \varphi] = i\partial_\mu \varphi \) for any field operator \( \varphi \). Here the metric is chosen as \( (-,+,+) \). The infinitesimal SUSY transformation is defined as

\[
\delta \xi \varphi = -i[\xi^T \sigma_y Q, \varphi] ,
\]

which gives

\[
(\delta \xi \eta - \delta \eta \xi) \phi = 2i\eta^T \alpha^\mu \xi \partial_\mu \varphi .
\]

Here \( \xi, \eta \) are two two-component Grassmann numbers. Eq. (C13) shows the closure of the SUSY algebra.

Now, we show the SUSY of \( S^* \) at \( r = m^2 \) with vacuum \( \langle \phi \rangle = 0 \). \( S^* \) can be re-written as

\[
S^* = S_{\gamma,0} + S_{\gamma,\phi} + S_{\phi,0} + S_{\phi,3} + S_{\phi,4} ,
\]

where

\[
S_{\gamma,0} = \int d\tau d^{d-1}x \frac{1}{2} \gamma^T (i\partial_\mu \alpha^\mu + m\sigma_y)\gamma
\]

\[
S_{\gamma,\phi} = \int d\tau d^{d-1}x \frac{1}{2} g\delta \phi \gamma^T \sigma_y \gamma
\]

\[
S_{\phi,0} = \int d\tau d^{d-1}x \frac{1}{2} \delta \phi (\partial^2 + m^2)\phi
\]

\[
S_{\phi,3} = \int d\tau d^{d-1}x \frac{1}{2} g\delta \phi^3
\]

\[
S_{\phi,4} = \int d\tau d^{d-1}x \frac{1}{8} g^4 \delta \phi^4 .
\]
The SUSY transformation in this case can be defined as
\[ \delta_{\xi} \phi = \xi^T \sigma_y \gamma, \quad \delta_{\xi} \gamma = \sigma_y \alpha^\mu (i \partial_\mu) \xi \phi + \xi (-m \phi - g \phi^2 / 2). \] (C16)

To demonstrate \( S^* \) is SUSY invariant, we can first act \( \delta_{\xi} \) on \( S_{\gamma,0} \) and get
\[ \delta_{\xi} S_{\gamma,0} = -\delta_{\xi} (S_{\phi,0} + \frac{1}{3} S_{\phi,3}) + \int d\tau d^{d-1}x \frac{-g \phi^2}{2} \xi^T i \partial_\mu \alpha^\mu \gamma. \] (C17)

Then act \( \delta_{\xi} \) on \( S_{\gamma,0} \) and get
\[ \delta_{\xi} S_{\gamma,0} = -\delta_{\xi} (S_{\phi,0} + \frac{2}{3} S_{\phi,3} + S_{\phi,4}) + \int d\tau d^{d-1}x \frac{g \phi^2}{2} \xi^T i \partial_\mu \alpha^\mu \gamma. \] (C18)

As a result, we have
\[ \delta_{\xi} (S_{\gamma,0} + S_{\gamma,0}) = -\delta_{\xi} (S_{\phi,0} + S_{\phi,3} + S_{\phi,4}) \Leftrightarrow \delta_{\xi} S^* = 0. \] (C19)

The defined SUSY transformation must be close, i.e. satisfying Eq. (C13). It is true for \( \phi \), i.e.
\[ (\delta_{\xi} \delta_{\eta} - \delta_{\eta} \delta_{\xi}) \phi = 2i \eta^T \alpha^\mu \partial_\mu \phi. \] (C20)

For \( \gamma \), we have
\[ (\delta_{\xi} \delta_{\eta} - \delta_{\eta} \delta_{\xi}) \gamma = 2i \eta^T \alpha^\mu \partial_\mu \gamma - (\eta \xi^T - \xi \eta^T)(i \alpha^\mu \partial_\mu + m \sigma_y + g \phi \sigma_y) \gamma, \] (C21)

where \( \eta \xi^T - \xi \eta^T = \sum_{\alpha, \mu} \eta^T \alpha^\mu \xi \alpha^\mu \) and \( \sigma_y \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu \sigma_y = 2 \sigma_y \delta^\nu \mu \) are used. Clearly, the closure of the algebra for \( \gamma \) requires the equation of motion of \( \gamma \), which is \( (i \alpha^\mu \partial_\mu + m \sigma_y + g \phi \sigma_y) \gamma = 0 \). We call the algebra is close through the equation of motion. The requirement of the equation of motion is because we integrate out the auxiliary field.[5]

Around the other vacuum at \( r = m^2 \), the action takes the from Eq. (C8). In this case, the action is the same as Eq. (C14) after replacing \( \phi \) and \( m \) in Eq. (C14) by \( \delta \phi \) and \( -m \), respectively. Therefore, Eq. (C8) is SUSY invariant if also performing the replacement for the SUSY transformation.

At \( m = 0 \) and \( r < 0 \), the action Eq. (C10) is the same as Eq. (C14) if replacing \( \phi \) and \( m \) in Eq. (C14) by \( \delta \phi \) and \( \pm \sqrt{-2r} \), respectively. Therefore, \( S^* \) has SUSY at \( r < 0 \) and \( m = 0 \) around either of the two vacua \( \langle \phi \rangle = \pm \sqrt{-2r} / g. \)

**Appendix D: Details on the Experimental Setup**

In this section, we derive the gapped surface Majorana modes from Eq. (8) in the main text and discuss the realization of SUSY.

Recall Eq. (8) in the main text
\[ S_0 = \frac{1}{V} \int d\tau \sum_k \left[ \bar{\psi}_{\tau,k} \partial_\tau \psi_{\tau,k} + \bar{\psi}_{\tau,k} \hat{h}(k) \psi_{\tau,k} + \frac{1}{2} \bar{\psi}_{\tau,k} \Delta_k (\bar{\psi}_{\tau,-k})^T + \frac{1}{2} \bar{\psi}_{\tau,-k} \Delta_k^T \psi_{\tau,k} \right]. \] (D1)

To solve for the surface modes, we consider a semi-infinite configuration \( z < 0 \) with open boundary condition at \( z = 0 \). Then, we address the \( z \) direction in the real space with \( \bar{\psi}_{\tau,k} = \int dz e^{ikz} \bar{\psi}_{\tau,k,z} \), and the above action becomes
\[ S = \frac{1}{2S_0} \sum_{k_\|} \int d\tau \int_{-\infty}^0 dz \bar{\psi}_{\tau,k,\|,z} (\partial_\tau + h_{BDG}(k_\|, -i \partial_z)) \psi_{\tau,k,\|,z}, \] (D2)

where
\[ h_{BDG}(k_\|, -i \partial_z) = h_0(k_\|^2, -i \partial_z) + h_1(k_\|), \] (D3)

\[ h_0(k_\|^2, -i \partial_z) = \tau_z (-\frac{\partial^2}{2m_0} - \bar{\mu}) + \Delta_p (-i \partial_z) \tau_z s_x, \quad h_1(k_\|) = \Delta_p (-\tau_z s_z) k_x + \Delta_p \bar{k}_y (-\tau_y), \] (D4)
\[ \mu = \mu - k^2/(2m_0) \], \( \tau_i \)'s are Pauli matrices for the particle-hole index and \( \tilde{\Psi}_{\tau, k_{a,j}} = (\tilde{\psi}_{\tau, k_{a,j}}, \tilde{\psi}_{-\tau, k_{a,j}}^T) \). In the following, we first solve for the zero modes of \( h_0 \) and then add \( h_1 \) as a perturbation since \( k \) is small. The zero mode equation for \( h_0 \) reads

\[ h_0(k_{a,j}^2, -i\partial_z) \Phi_z = 0 \Leftrightarrow \left[ -\tilde{\mu} - \partial^2 \over 2m_0 + \Delta_p \tau_y s_x \partial_z \right] \Phi_z = 0 . \quad (D5) \]

Define four orthonormal vectors \( \xi_{a,j} \)'s as

\[
\begin{align*}
\xi_{+,1} &= (1, 1, i, i)^T/\sqrt{2} \\
\xi_{+,2} &= (1, -1, -i, i)^T/\sqrt{2} \\
\xi_{-,1} &= (1, 1, -i, -i)^T/\sqrt{2} \\
\xi_{-,2} &= (1, -1, i, -i)^T/\sqrt{2} ,
\end{align*}
\]

(D6)

where \( a = \pm, j = 1, 2 \) and \( \tau_y s_x \xi_{a,j} = a \xi_{a,j} \). The wave function can be re-expressed as \( \Phi_z = \sum_{a,j} f_{a,j}(z) \xi_{a,j} \), and Eq. (D5) is equivalent to

\[ (\partial_z^2 - 2m_0 \Lambda_p \partial_z + 2m_0 \mu) f_{a,j}(z) = 0 \quad (D7) \]

for all \( a = \pm \) and \( j = 1, 2 \) with boundary condition \( f_{a,j}(0) = f_{a,j}(-\infty) = 0 \). Without loss of generality, we choose \( \Delta_p m_0 > 0 \). In this case, we only have solution for \( a = + \) and \( 2m_0 \mu - k^2_i > 0 \), which is \( f_{+,j}(k_{a,j}^2, z) = C_0 e^ {m_0 \Delta_p z} \sinh(z \sqrt{\Delta_p^2 m_0^2 - 2m_0 \mu}) \) with \( C_0 \) the normalization constant that makes \( f_+ \) real. The wave function of the zero modes in general has the form

\[ \Phi_{k_{a,j},z} = f_+(k_{a,j}^2, z) \sum_{j=1,2} C_j \xi_{+,j} . \quad (D8) \]

Clearly, there are two independent zero modes, of which the wavefunction can be chosen as

\[
\begin{align*}
\Phi_{1,k_{a,j},z} &= \frac{f_+(k_{a,j}^2, z)}{\sqrt{2}} \left( e^{-i\pi/4} \xi_{+,1} + e^{i\pi/4} \xi_{+,2} \right) \\
\Phi_{2,k_{a,j},z} &= \frac{f_+(k_{a,j}^2, z)}{\sqrt{2}} \left( -e^{-i\pi/4} \xi_{+,1} - e^{i\pi/4} \xi_{+,2} \right) .
\end{align*}
\]

To get the low-energy effective model, we define \( \tilde{\gamma}_{\tau,k_{a,i}} = \int_{-\infty}^{0} dz \Psi_{k_{a,i},z} \Phi_{i,k_{a,j},z} \). Using \( \Psi_{\tau,k_{a,i}} = \sum_i \tilde{\gamma}_{\tau,k_{a,i}} \Phi_{i,k_{a,i},z} + \ldots \) with “…” the high energy contribution, we can project \( h_1 \) to the zero modes and get

\[ S_{\text{eff}} = \frac{1}{2S_0} \int d\tau \sum_{k_a} \tilde{\gamma}_{\tau,k_{a,i}} [\partial_\tau + \Delta_p (k_x \sigma_z + k_y \sigma_x)] \tilde{\gamma}_{\tau,k_{a,i}} . \quad (D10) \]

where \( \tilde{\gamma}_{\tau,k_{a,i}} = (\tilde{\gamma}_{\tau,k_{a,1}}, \tilde{\gamma}_{\tau,k_{a,2}}) \) and the terms related with the high-energy modes are neglected here. Define \( \gamma_{\tau,k_{a,i}} = \int d^2 x e^{-ik_{a,i} \cdot x} \gamma_{\tau,x} \). Due to \( \tau_x \Phi_{i,k_{a,j},z}^{\dagger} = \Phi_{i,k_{a,j},z} \), \( \tau_y s_x \Phi_{i,k_{a,j},z}^{\dagger} = \sum_j \Phi_{j,k_{a,j},z}^{\dagger} (i \sigma_y)_{ij} \), \( \Psi_{T,\tau,-k_{a,i}}^T = \Psi_{\tau,k_{a,-i}} \tau_x \) and \( \Psi_{\tau,k_{a,i}}^T \tau_y \rightarrow \gamma_{T,\tau,x}^T \rightarrow \gamma_{T,\tau,x}^T (i \sigma_y) \). As a result, the above action becomes

\[ S_{\text{eff}} = \frac{1}{2} \int d\tau d^2 x \gamma_{T,\tau,x}^T [\partial_\tau + \Delta_p (-i \partial_x \sigma_z - i \partial_y \sigma_x)] \gamma_{T,\tau,x} . \quad (D11) \]

Replacing \( \Delta_p \) by \( v_f \) in the above action, we can get fermionic part of the Eq. (1) in the main text. Note that we neglect the high-energy modes, and thus \( \Delta_p \approx v_f \) holds only to the leading order.

The surface magnetic doping of TSC can be phenomenologically described by the standard Ginzburg-Landau free energy of Ising magnetism, \( S_M = \int d^4 x \left[ \frac{1}{2} M (-\partial^2_t - v^2 \nabla^2 + r_0) M + \frac{1}{4} u M^4 \right] \), where \( M_{x,y} \) is the order parameter of surface Ising magnetism along \( z \). \( M \) is coupled to electrons at the surface through the exchange interaction \( S_{\text{ex}} = \int d\tau \int d^3 r M_{x,y} g_M(z) \bar{\psi}_{\tau,s_y} \psi_{\tau,x} \) with \( g_M(z) \) localized at the surface. The matrix form of the exchange interaction in the BdG bases is \( \tau_z s_z/2 \), of which the projection to the Majorana modes is \( \sigma_y/2 \). And thus the Ising coupling after the surface projection takes the form \( \frac{1}{2} g \int d^3 x M \gamma^T T \sigma_y \gamma \) with \( M \rightarrow M \) and \( g = \int dz f_+(0) \frac{1}{2} g M \). Here we neglect the momentum dependence in \( f_+(k_{a,j}^2, z) \) as \( k_i \) is small. Now the total action should read

\[ S = \int d^4 x \left[ \frac{1}{2} \gamma^T (\partial_\tau - iv_f \alpha \cdot \nabla) \gamma + \frac{1}{2} \bar{\gamma} \bar{\gamma} + \frac{1}{2} M (-\partial^2_t - v^2 \nabla^2 + r_0) M + \frac{1}{4} u M^4 \right] , \quad (D12) \]
which has TR symmetry. Furthermore, a magnetic field \( B \) along \( z \) is applied and coupled to both electron spin and Ising magnetism on the surface through the Zeeman-type action: \( S_B = \int d\tau d^3r g_s(z) B \psi_{\tau,r} s_z \psi_{\tau,r} + \int d^3x(a_1 BM + \frac{a_2}{\mu} BM^3) \) with \( g_s(z) \) localized at the surface. Here, we neglect the orbital effect as all fields are charge neutral, and add an \( M^3 \) term since it is allowed by symmetry and can be generated at the quantum level. After projecting the \( g_s \) term to the surface in the same way as the Ising coupling, we can include \( S_B \) into the total action, and get Eq. (9) in the main text with \( \mu_B = \int dz f_+(0, z)^2 g_s(z) \). Next, we show how we cancel the \( M \)-linear term.

To cancel the \( M \)-linear term in the action, we define \( \phi \) and \( \phi_0 \) as mentioned in the main text. The resulted action for \( \gamma \) and \( \phi \) has the form of Eq. (1) in the main text with \( m = \mu_B B + g \phi_0 \), \( r = r_0 + a_2 B \phi_0 + u \phi_0^2/2 \) and \( b = a_2 B + u \phi_0 \). As mentioned in the main text, the RG equations further lead the action to the form of Eq. (6) in the main text with \( m^* = \mu_B B + g \phi_0^* \), \( r^* = r_0^* + 3g \mu_B \phi_0^* + 3(g \phi_0^*)^2/2 \) and \( a_2^* = 3g \mu_B^* \). Here * stands for the macroscopic value instead of complex conjugate and \( \phi_0^* \) satisfies

\[
a_1^* B + r_0^* \phi_0^* + 3g \mu_B^* B (\phi_0^*)^2/2 + (g^*)^2 (\phi_0^*)^3/2 = 0 .
\] (D13)

At last, we demonstrate the condition for the non-zero \( B_c \) in the case where \( r_0^* > 0 \) and \( \mu_B^*, a_1^* \neq 0 \). As mentioned in the main text, we neglect the \( B \)-dependence in \( r_0^* \), \( \mu_B^* \) and \( a_1^* \) for simplicity. Due to \( r_0^* > 0 \), \( \phi_0(B \rightarrow 0) = 0 \) is guaranteed in Eq. (D13), so we do not need to consider it separately. At \( B = B_c \), we have \( r^* = (m^*)^2 \), which gives

\[
r_0^* + g \mu_B^* B_c \phi_0^* + (g^* \phi_0^*)^2/2 - (\mu_B^* B_c)^2 = 0 .
\] (D14)

Combining the above equation with Eq. (D13), we have

\[
\frac{(g^* (\phi_0^*))^2 + 2r_0^*}{(2a_1^* + 3g \mu_B^* (\phi_0^*))^2} (4(a_1^*)^2 - 4 \mu_B^* (\phi_0^*)^2(\mu_B^* B_c^2 - 2a_1^* g^*) + (g^*)^2 (\mu_B^* (\phi_0^*)^4) = 0
\]

\[
B_c = \frac{2r_0^* \phi_0^* + (g^*)^2 (\phi_0^*)^3}{2a_1^* + 3g \mu_B^* (\phi_0^*)^2}.
\] (D15)

The non-zero \( B_c \) exists in the case where \( 2a_1^* + 3g \mu_B^* (\phi_0^*)^2 \neq 0 \) and \( 4(a_1^*)^2 - 4 \mu_B^* (\phi_0^*)^2(\mu_B^* B_c^2 - 2a_1^* g^*) + (g^*)^2 (\mu_B^* (\phi_0^*)^4) = 0 \) has non-zero solutions for \( \phi_0^* \). The two conditions can be first simplified as \( 3r_0^*/g^* \neq a_1^*/\mu_B^* \), \( a_1^*/\mu_B^* \leq r_0^*/(2g^*) \) and \( (a_1^*/\mu_B^* - r_0^*/(3g^*))(a_1^*/\mu_B^* - r_0^*/g^*) \geq 0 \). Combining the three conditions together, we eventually have \( a_1^*/\mu_B^* \leq r_0^*/(3g^*) \) as the condition for the existence of non-zero \( B_c \).