Stability analysis of a smoking behavior model

S M Lestari¹, Y Yulida¹*, A S Lestia², M A Karim¹

¹ Mathematics Department, Universitas Lambung Mangkurat, Banjarbaru, Indonesia
² Statistics Department, Universitas Lambung Mangkurat, Banjarbaru, Indonesia

*Email: y_yulida@ulm.ac.id

Abstract. This research discussed the mathematical model of smoking behavior. The model will be analogous to an epidemic model which will be divided into several compartments/groups. This research aimed to explain the formation of a mathematical model of smoking behavior, to investigate the equilibrium point, the value of the basic reproduction number, to analyze the stability of the model, then to determine and interpret the numerical solutions using the fourth-order Runge-Kutta method. By the results of this research, a mathematical model of smoking behavior which consists of three compartments, namely the population of non-smokers, smokers and ex-smokers, was obtained. Based on the model formed the smoke-free equilibrium point and the smoker equilibrium point, then the basic reproduction number was also obtained using the next generation matrix. Furthermore, the result of the stability analysis of the smoker-free population was asymptotically stable provided that the basic reproduction number is less than one, while the population was asymptotically stable provided that the basic reproduction number is greater than one. The simulation of the model was presented to support the explanation of the stability analysis of the model using the fourth-order Runge-Kutta method based on the parameters that met the requirements of the stability analysis.

1. Introduction
Tobacco is one of the most dangerous killers in the world. In 2015, the Indonesian Health Development Research Agency showed that there were more than 230,000 deaths due to the consumption of tobacco products each year [1]. Globocan stated that of the total cancer deaths in Indonesia, lung cancer was the first cause of death, which was 12.6% [2]. Based on data from the Rumah Sakit Umum Pusat Persahabatan Jakarta, which is the National referral hospital for lung disease, it was found that 87% of lung cancer cases were related to smoking.
Tirtosastro stated that cigarettes contain about 4,800 chemicals with the main components namely tar, nicotine, and carbon monoxide [3]. Soleh in 2017 said that Nicotine can cause pleasant feelings that make smokers addicted to continue smoking. The more a person smokes, the higher content of the tar, nicotine, and carbon monoxide in his body which can cause various kinds of diseases in the body's organs, including several types of cancer-related to the respiratory tract to the lungs, bladder, disorders of pregnancy, and also diseases related to blood vessels such as heart disease and stroke [4].
The mathematical model of smoking behavior was first introduced by Castillo-Garsow et al in 1997 which categorizes the human population into three classes of individuals, namely non-smokers, smokers, and ex-smokers [5]. Sharomi et al in 2008 have developed a model by dividing two classes of individual ex-smokers [6]. Then in 2019 Hammadi et al have developed a model by dividing the population into three classes of individuals and representing them with a system of nonlinear differential equations [7]. Then in 2020 Simamarta et al reviewed the model that had been developed by Hammadi et al [8]. In this
study, the author discusses the formation of a mathematical model of smoking behavior and then proceeds to investigate the equilibrium point, determine stability and perform numerical simulations from the analysis that has been obtained.

2. Method
The research method used in the study was carried out in several stages, namely conducting model formation, model analysis and determining model solutions. The first step is to form a model, determine the equilibrium point of the model [9] and the basic reproduction number using the next generation matrix method ([10,11]). The second step is to analyze stability through linearization ([11–15]). The third step is to determine the model solution using the fourth-order Runge-Kutta [16].

3. Smoking Behavior Model

3.1. Model formulation
The following is a diagram of the spread of smoking behavior

\[ \begin{align*}
\[ P \quad \alpha NS \quad S \quad \beta S \quad E \]
\end{align*} \]

Figure 1. Mathematical Model flow Diagram smoking behavior model.

In this model, the human population is divided into three different classes, namely the class of non-smokers, smokers, and ex-smokers. The number of human population at time \( t \) is denoted by \( N(t) \), while the non-smokers, smokers and ex-smokers are denoted by \( P(t) \), \( S(t) \) and \( E(t) \) respectively. Furthermore, \( N(t) \) can be expressed as \( N(t) = P(t) + S(t) + E(t) \). The assumptions used in the model are:
- New individuals due to birth will enter the class of non-smokers, with a constant natural birth rate
- Non-smokers can become Smokers, because of the interaction between smokers and non-smokers
- Individual Smokers are assumed to be able to quit smoking with their own awareness.
- The death rate of each population class is different. The death rate in the non-smoker population is lower than the death rate for ex-smokers and the death rate in the ex-smoker population is lower than the death rate for smokers.

The mathematical model is given as follows

\[
\begin{align*}
\frac{dP}{dt} &= \pi - \alpha PS - \mu_1 P \\
\frac{dS}{dt} &= \alpha PS - \beta S - \mu_2 S \\
\frac{dE}{dt} &= \beta S - \mu_3 E
\end{align*}
\]

(1)

where

- \( \pi \) : Rate of population birth
- \( \alpha \) : The rate of transmission of smoking habits due to the interaction of individuals who have never smoked with individual smokers.
- \( \beta \) : Rate of smokers quitting smoking habits.
- \( \mu_1 \) : Natural death rate of the non-smoking population.
- \( \mu_2 \) : The death rate of the smoker population.
\( \mu_3 \): The death rate of the ex-smoker population.

3.2. Analysis of the Model

In this model analysis the first step is to determine the equilibrium points. To determine the equilibrium of the system of Equations (1) is by creating the system:

\[
\frac{dp}{dt} = 0, \quad \frac{ds}{dt} = 0 \quad \text{and} \quad \frac{dE}{dt} = 0
\]

Equation (1) can be written as

\[
\pi - \alpha PS - \mu_1 P = 0 \quad (2)
\]

\[
\alpha PS - \beta S - \mu_2 S = 0 \quad (3)
\]

\[
\beta S - \mu_3 E = 0 \quad (4)
\]

Based on Equations (2), (3) and (4), two equilibrium points are obtained, namely

- The smoker-free equilibrium \( T_1 (\frac{\pi}{\mu_1}, 0, 0) \)
- The Smoker-present equilibrium \( T_2 (\frac{\pi}{\mu_1 R_0}, (\frac{\mu_1 R_0 - 1}{\alpha}), (\frac{\beta R_0 (R_0 - 1)}{\alpha \mu_3})) \)

where basic reproduction number \( R_0 = \frac{\alpha \pi}{\mu_1 (β + \mu_2)} \).

In epidemiology, the rate of spread of a disease in a model is called the basic reproduction number \( R_0 \). The basic reproductive number is the average number of secondary cases that occur due to a primary case in an entirely susceptible population.

In this model, if \( R_0 < 1 \) then the spread of smoking habits gradually decreases and the population will be free from the smoking population and if \( R_0 > 1 \) then the spread of smoking habits will continue to increase and the smoking population will exist in the population. The basic reproduction number is determined using the next generation matrix method, as follows.

Equation (1) we view into two parts, namely

\[
\frac{dS}{dt} = \mathcal{A} - \mathcal{B}
\]

where \( \mathcal{A} = [\alpha PS], \mathcal{B} = [\beta S + \mu_2 S] \) and \( \left( \frac{dp}{dt} \right) \bigg|_{\mathcal{E}} = \left( \frac{\pi - \alpha PS - \mu_1 P}{\beta S - \mu_3 E} \right) \).

Equation (5) is linearized using the Taylor series around the smoker-free equilibrium, we get

\[
A = \left( \frac{\partial \mathcal{A}}{\partial S} \right) |_{T_1} = [\alpha P]
\]

and

\[
B = \left( \frac{\partial \mathcal{B}}{\partial S} \right) |_{T_2} = [\beta + \mu_2]
\]

and

\[
B^{-1} = \left[ \frac{1}{\beta + \mu_2} \right]
\]

From Equations (6) and (7), the next generation matrix is obtained as follows:

\[
\mathbf{W} = AB^{-1} = \left[ \frac{\alpha \pi}{\mu_1 (\beta + \mu_2)} \right]
\]

The basic reproduction number is the dominant eigenvalue of the \( \mathbf{W} \) matrix (8), namely
We provide Theorem 1 for using $R_0$ in analyzing the stability of smoker-free equilibrium.

**Theorem 1.** Consider the system of (1) with the smoker-free equilibrium $T_1$ and $R_0$ given by (9). If $R_0 < 1$ then $T_1$ is locally asymptotically stable, but unstable if the $R_0 > 1$.

**Proof.** Jacobian’s matrix on the smoker-free equilibrium $T_1\left(\frac{\pi}{\mu_1},0,0\right)$ can be given by:

$$J(T_1) = \begin{bmatrix}
-\mu_1 & -\frac{\alpha \pi}{\mu_1} & 0 \\
0 & \frac{\alpha \pi}{\mu_1} - \beta - \mu_2 & 0 \\
0 & \beta & -\mu_3
\end{bmatrix}$$

$J(T_1)$’s matrix gets a bunch of eigen value, they are $\lambda_1 = -\mu_1$, $\lambda_2 = (R_0 - 1)(\beta + \mu_2)$, and $\lambda_3 = -\mu_3$. All eigenvalues are negative if $R_0 < 1$. Thus, the smoker-free equilibrium is locally asymptotically stable if $R_0 < 1$, but unstable if the $R_0 > 1$.

We refer Theorem 2 to use the basic reproduction number $R_0$ in analyze the equilibrium stability of smoker-present equilibrium.

**Theorem 2.** Consider the system of (1) with the smoker-present equilibrium $T_2\left(\frac{\pi}{\mu_1 R_0},\left(\frac{\mu_1 R_0 - 1}{\alpha}\right),\frac{\beta \mu_4 (R_0 - 1)}{\alpha \mu_3}\right)$ can be given by:

$$J(T_2) = \begin{bmatrix}
-\alpha \left(\frac{\mu_1 (R_0 - 1)}{\alpha} \right) - \mu_1 & -\alpha \left(\frac{\pi}{\mu_1 R_0}\right) & 0 \\
\alpha \left(\frac{\mu_1 (R_0 - 1)}{\alpha} \right) & \frac{\pi}{\mu_1 R_0} - \beta - \mu_2 & 0 \\
0 & \beta & -\mu_3
\end{bmatrix}$$

$J(T_2)$’s matrix gets a bunch of eigen value, they are $\lambda_4 = -\mu_1$ and other fill up the quadratic polynomial $a_0 \lambda^2 + a_1 \lambda + a_2 = 0$

where:

- $a_0 = 1$
- $a_1 = \mu_1 R_0 - \frac{\alpha \pi}{\mu_1 R_0} + \beta + \mu_2 = \mu_1 R_0 > 0$
- $a_2 = \mu_1 R_0 (\beta + \mu_2) - \alpha \pi + \alpha \pi \mu_1 (R_0 - 1)$

By using the Routh-Hurwitz theorem [17], all eigenvalues of $J(T_2)$ are negative real parts if $R_0 < 1$. Thus, the smoking-free equilibrium is locally asymptotically stable if $R_0 < 1$, but unstable if the $R_0 > 1$.

### 4. Numerical Analysis Simulation

In this section, we use an explicit fourth-order Runge-Kutta (RK4) to numerically determine Equation (1) of the smoking behavior model with the given initial conditions.
4.1. Numerical Simulation at Smoker-Free Equilibrium.
We present numerical simulations to illustrate the solution of the proposed model at Smoker-free equilibrium and Smoker-present equilibrium. Initial conditions of Smoker-free simulation are $P(0) = 100$, $S(0) = 40$, $E(0) = 10$. The parameters on fig.2 are: $\pi = 5$, $\alpha = 0.003$, $\beta = 0.8$, $\mu_1 = 0.05$, $\mu_2 = 0.078$, $\mu_3 = 0.06$ and $R_0 = 0.0341 < 1$. Furthermore, the numerical solution Equation (1) at the smoke-free equilibrium for 200 iterations is presented in Table 1 below.

| Iterasi | Iterasi | P   | S   | E   |
|---------|---------|-----|-----|-----|
| 1       | 0.5     | 94.994 | 29.8418 | 23.365 |
| 20      | 10      | 88.091 | 0.0852  | 37.591 |
| 40      | 20      | 92.753 | 0.0002  | 20.698 |
| 60      | 30      | 95.604 | 0      | 11.359 |
| 80      | 40      | 97.334 | 0      | 6.234 |
| 100     | 50      | 98.383 | 0      | 3.421 |
| 120     | 60      | 99.019 | 0      | 1.877 |
| 140     | 70      | 99.405 | 0      | 1.030 |
| 160     | 80      | 99.639 | 0      | 0.565 |
| 180     | 90      | 99.781 | 0      | 0.310 |
| 200     | 100     | 99.867 | 0      | 0.170 |
| 240     | 120     | 99.951 | 0      | 0.051 |
| 280     | 140     | 99.982 | 0      | 0.015 |
| 320     | 160     | 99.993 | 0      | 0.005 |
| 360     | 180     | 99.998 | 0      | 0.001 |
| 400     | 200     | 99.999 | 0      | 0.000 |

The result of numerical analysis of stability of the smoker-free equilibrium which is illustrated in the Figure 2 below:

![Figure 2. Dynamics of smoking behavior model for smoker-free equilibrium](image-url)
Based on Table 1 and Figure 2, for an increasing time, the non-smoker population converges towards $99,999 \approx 100$, while the smokers and ex-smokers population towards zero. This indicates that the smoker-free equilibrium $T_1(100; 0; 0)$ is asymptotically stable when $R_0 = 0.0341 < 1$.

### 4.2. Numerical Simulation at Smoker-present equilibrium.

In this section, we present some numerical simulations to describe solution of the proposed model at Smoker-free equilibrium and Smoker-present equilibrium. Initial conditions of Smoker-free simulation are $P(0) = 100$, $S(0) = 40$, $E(0) = 10$. The parameters on fig.2 are : $\pi = 10, \alpha = 0.003, \beta = 0.8, \mu_1 = 0.05, \mu_2 = 0.078, \mu_3 = 0.06$ dan $R_0 = 6.8337 > 1$. Furthermore, the numerical solution Equation (1) at the smoker-present equilibrium for 200 iterations is presented in Table 2 below.

**Table 2. Numerical Solution at smoker-present equilibrium**

| Iterasi | $t$  | $P$       | $S$       | $E$       |
|---------|------|-----------|-----------|-----------|
| 1       | 0.5  | 43,1080   | 74,0683   | 33,1351   |
| 15      | 7.5  | 28,6922   | 5,5436    | 120,4623  |
| 20      | 10   | 34,9481   | 7,0286    | 114,8911  |
| 40      | 20   | 28,7644   | 9,2466    | 124,1413  |
| 60      | 30   | 29,0666   | 9,8076    | 126,2218  |
| 80      | 40   | 29,2801   | 9,7403    | 127,6703  |
| 100     | 50   | 29,2735   | 9,7204    | 128,5700  |
| 120     | 60   | 29,2664   | 9,7222    | 129,0552  |
| 140     | 70   | 29,2664   | 9,7229    | 129,3178  |
| 160     | 80   | 29,2667   | 9,7229    | 129,4622  |
| 180     | 90   | 29,2667   | 9,7229    | 129,5416  |
| 200     | 100  | 29,2667   | 9,7229    | 129,5851  |
| 220     | 110  | 29,2667   | 9,7229    | 129,6090  |
| 240     | 120  | 29,2667   | 9,7229    | 129,6221  |
| 260     | 130  | 29,2667   | 9,7229    | 129,6293  |
| 280     | 140  | 29,2667   | 9,7229    | 129,6333  |
| 300     | 150  | 29,2667   | 9,7229    | 129,6354  |
| 320     | 160  | 29,2667   | 9,7229    | 129,6366  |
| 340     | 170  | 29,2667   | 9,7229    | 129,6373  |
| 360     | 180  | 29,2667   | 9,7229    | 129,6376  |
| 380     | 190  | 29,2667   | 9,7229    | 129,6378  |
| 400     | 200  | 29,2667   | 9,7229    | 129,6378  |

The result of numerical analysis of stability of the smoker-present equilibrium which is illustrated in the figure 3 below:
Figure 3. Dynamics of smoking behavior model for smoker-present equilibrium

Based on Table 2 and Figure 3, for an increasing time, the non-smoking population converges towards $29,2667 \approx 29$, the smoking population converges towards $9,7229 \approx 10$ and the ex-smokers population towards $129,6378 \approx 130$. This indicates that the smoker-free equilibrium $T_2(29; 10; 130)$ is asymptotically stable when $R_0 = 6,8337 > 1$.

5. Conclusion

This paper discusses the mathematical model of smoking behavior. The proposed model has two equilibria, namely the smoker-free equilibrium and the smoker-present equilibrium. We find that the properties of the proposed model are completely determined by the basic reproduction number of the model.

6. References

[1] Kemkes.go.id 2019 Jangan Biarkan Rokok Merenggut Napas Kita Biro Komun. dan Pelayanan Masyarakat, Kementer. Kesehat. RI
[2] P2p.kemkes.go.id 2018 Penyakit Kanker di Indonesia Berada Pada Urutan 8 di Asia Tenggara dan Urutan 23 di Asia Kementer. Kesehat. Republik Indonesia.
[3] Tirtosastro S and Murdiyati A S 2010 Kandungan Kimia Tembakau dan Rokok Bul. Tanam. Tembakau, Serat Miny. Ind. 2 pp 33–43
[4] Soleh M and Szmita D 2017 Model Matematika Jumlah Perokok Dengan Dinamika akar kuadrat dan Faktor Migrasi Seminar Nasional Teknologi Informasi, komunikasi dan Industri (SNTIKI) 9
[5] Castillo-Garsow C, Jordan-Salivia G and Rodriguez-Herrera A 1997 Mathematical Models for the Dynamics of Tobacco Use, Recovery, and Relapse Public Health 84 pp 543–547
[6] Sharomi O and Gumel A B 2008 Curtailing smoking dynamics: a mathematical modeling approach Appl. Math. Comput. 195 475–499
[7] Hammadi I J 2019 Analisis pada model matematika perubahan perilaku merokok dengan program python (Universitas Gadjah Mada)
[8] Simamarta C M D, Susyanto N, Hammadi I J and Rahmaditya C 2020 A Mathematical model of smoking behavior in Indonesia with density-dependent death rate J.Math.Sci 2020 118–125
[9] Perko L 2002 Differential Equation and Dynamics Systems (New York: Springer-Verlag)
[10] Driessche P and Watmough J 2002 Reproduction Numbers and Sub-threshold Endemic Equilibria for Compartmental Models of Disease Transmission Math. Biosci. 180 29–48
[11] Martcheva M 2015 An Introduction to Mathematical Epidemiology (US: Springer)
[12] Bellomo N and Preziosi L 1995 Modelling mathematical Methods and Scientific Computation (Florida: CRC Press)
[13] Brauer F and Castillo-Chavez C 2010 Mathematical Models in Population Biology and Epidemiology (New York: Springer)
[14] Cheng D and Al E 2010 Linearization of Nonlinear Systems (Beijing: Berlin Heidelberg: Science Press Beijing and Springer-Verlag)
[15] Taylor M R and Bhathawala P H 2012 Linearization of Nonlinear Differential Equation by Taylor’s Series Expansion and Use of Jacobian Linearization Process Int. J. Theor. Appl. Sci. 4 36–48
[16] Mulyono 2016 Prosiding Konferensi Nasional Penelitian Matematika dan Pembelajarannya Kajian Sejumlah Metode untuk Mencari Solusi Numerik Persamaan Diferensial (Surakarta: Universitas Muhammadiyah) pp 971–980