FRAME RELATED OPERATORS FOR WOVEN FRAMES

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Abstract. A new notion in frame theory has been introduced recently under the name woven-weaving frames by Bemrose et. al. In the studying of frames, some operators like analysis, synthesis, Gram and frame operator play the central role. In this paper, for the first time, we introduce and define these operators for woven-weaving frames and review some properties of them. In continuation, we investigate the effect of different types of operators on the woven frames and their bounds. Also, we provide some conditions that shows sum of woven frames are also woven frames. Finally, we apply these properties with an example.

1. Introduction

The theory of frames plays an important role in signal processing because of their resilience to quantization [15], resilience to additive noise, as well as their numerical stability of reconstruction and greater freedom to capture signal characteristics. Also frames have been used in sampling theory to oversampled perfect reconstruction filter banks, system modeling, neural networks and quantum measurements [12]. New applications in image processing, robust transmission over the internet and wireless [16], coding and communication [23] were given.

Discrete frames in Hilbert spaces has been introduced by Duffin and Schaeffer [11] and popularized by Daubechies, Grossmann and Meyer [10]. A discrete frame is a countable family of elements in a separable Hilbert space which allows stable and not necessarily unique decompositions of arbitrary elements in an expansion of frame elements.

The last two decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence which include bounded quasi-projectors [13], fusion frames [5], pseudo-frames [19], oblique frames [8], g-frames [24], continuous frames [21], K-frames [14], fractional calculus [17, 18], Hilbert-Schmidt frames [22] and etc.

In one of the direction of applications of frames in signal processing, a new concept of woven-weaving frames in a separable Hilbert space introduced by
Bemrose et. al. [2, 6]. From the perspective of its introducers, woven frames has potential applications in wireless sensor networks that require distributed data and signal processing. In the field of mathematics of weaving frames, Arefijamaal [1], Vashisht, Deepshikha and Garg [25, 9, 26, 27, 28] have done more meritorious studies. By the concepts of weaving, we introduce related operators for weaving and woven frames, and investigate properties of this operators. Also, we study some features of woven frames in Hilbert spaces.

This paper is organized as follows: Section 2 contains the basic definitions of frames and woven frames. Also, we bring some properties of this type of frames that are useful for our study. In section 3, we introduce analysis, synthesis and frame operators of woven and weaving frames and study some properties of these operators. Also some behaviour of woven frames and their sum in Hilbert spaces will be studied in Section 3.

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is the indexing set such that can be finite or infinite countable set. Also, $[m]$ is the natural numbers set $\{1, 2, ..., m\}$.

2. Frames and Woven Frames

As a preliminarily of frames, at the first, we mention some prerequisites of discrete frames and woven-weaving frames in this section.

2.1. Discrete frame. We recall definition and some properties of frames that we need in throughout of the paper. For a comprehensive treatment of frame theory, we refer to the excellent textbook by Christensen [7].

**Definition 2.1.** A countable sequence of elements $\{f_i\}_{i \in \mathbb{I}}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $0 < A, B < \infty$ such that:

$$A\|f\|^2 \leq \sum_{i \in \mathbb{I}} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (2.1)$$

The numbers $A$ and $B$ are called frame bounds. The frame $\{f_i\}_{i \in \mathbb{I}}$ is called tight frame, if $A = B$ and is called Parseval frame if $A = B = 1$. Also the sequence $\{f_i\}_{i \in \mathbb{I}}$ is called Bessel sequence, if the upper inequality in (2.1) holds.

For a frame in $\mathcal{H}$, we define the mapping:

$$U : \mathcal{H} \rightarrow \ell^2(\mathbb{I}), \quad U(f) = \{\langle f, f_i \rangle\}_{i \in \mathbb{I}}.$$

The operator $U$ is usually called the analysis operator. The adjoint operator of $U$ is given by:

$$T : \ell^2(\mathbb{I}) \rightarrow \mathcal{H}, \quad T\{c_i\} = \sum_{i \in \mathbb{I}} c_i f_i,$$
and is called the synthesis operator. By composing $U$ and $T$, we obtain the frame operator:

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad S(f) = TU(f) = \sum_{i \in \mathbb{I}} \langle f, f_i \rangle f_i.$$  

The operator $S$ is positive, self-adjoint and invertible and every $f \in \mathcal{H}$ can be represented as:

$$f = \sum_{i \in \mathbb{I}} \langle f, S^{-1}f_i \rangle f_i = \sum_{i \in \mathbb{I}} \langle f, f_i \rangle S^{-1}f_i.$$ 

The family $\{S^{-1}f_i\}_{i \in \mathbb{I}}$ is said the canonical dual frame of $\{f_i\}_{i \in \mathbb{I}}$.

### 2.2. Woven frames.

Woven frames in Hilbert spaces were introduced by Bemrose et al. [2, 4, 6] in 2015. After that, Vashisht, Deepshikha, Arefijamaal and etc. have done more studies over the past few years [1, 9, 26, 27]. In the following, we briefly mention definition of woven frames by presenting an example.

**Definition 2.2.** Let $F = \{f_{ij}\}_{i \in \mathbb{I}}$ for $j \in [m]$ is a family of frames for the separable Hilbert space $\mathcal{H}$. If there exist universal constants $C$ and $D$, such that for every partition $\{\sigma_j\}_{j \in [m]}$ of $\mathbb{I}$ and for every $j \in [m]$, the family $F_j = \{f_{ij}\}_{i \in \sigma_j}$ is a frame for $\mathcal{H}$ with bounds $C$ and $D$, then $F$ is said a woven frames. For every $j \in [m]$, the frames $F_j = \{f_{ij}\}_{i \in \sigma_j}$ are said a weaving frame.

The constants $C$ and $D$ are called the lower and upper woven frame bounds. If $C = D$, then $F = \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is said a tight woven frame and if for every $j \in [m]$, the family $F_j = \{f_{ij}\}_{i \in \sigma_j}$ is a Bessel sequence, then the family $F = \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is said a Bessel woven.

In the continuation, we dissect woven and weaving frames for $m = 2$.

**Example 2.3.** Let $\{e_i\}_{i=1}^{2}$ be an orthonormal basis for the two dimensional vector space $V = \text{span} \{e_i\}_{i=1}^{2}$ with inner product and suppose that $F$ and $G$ are the sets:

$$F = \{2e_1, 2e_2 - e_1, 3e_2\}, \quad G = \{2e_1, 2e_1 + e_2, 2e_2\}.$$ 

Since both of $F$ and $G$ span the space $V$, then those are frames. For obtaining their bounds, we have:

$$\sum_{i=1}^{3} |\langle f, f_i \rangle|^2 = |\langle f, 2e_1 \rangle|^2 + |\langle f, 2e_2 - e_1 \rangle|^2 + |\langle f, 3e_2 \rangle|^2,$$

then

$$4\|f\|^2 \leq \sum_{i=1}^{3} |\langle f, f_i \rangle|^2 \leq 17\|f\|^2,$$

thus $F = \{f_i\}_{i=1}^{3}$ is a frame for $V$ with lower bound 4 and upper bound 17. Similarly, $G = \{g_i\}_{i=1}^{3}$ is a frame for $V$ with frame bounds 4 and 9. The
sets \( F \) and \( G \) are woven frames for \( V \). For example, if \( \sigma_1 = \{1, 2\} \), then for every \( f \in V \), we have
\[
4 \|f\|^2 \leq \sum_{i \in \sigma_1} |\langle f, f_i \rangle|^2 + \sum_{i \in \sigma_1^c} |\langle f, g_i \rangle|^2 \leq 12 \|f\|^2.
\]
So \( \{f_i\}_{i \in \sigma_1} \cup \{g_i\}_{i \in \sigma_1^c} \) is a weaving frame with bounds \( C_1 = 4 \) and \( D_1 = 12 \). Now, if we take:
\[
C = \min \{ C_i ; 1 \leq i \leq 8 \}, \quad D = \max \{ D_i ; 1 \leq i \leq 8 \},
\]
then \( F \) and \( G \) form a woven frames for \( V \) with universal bounds \( C \) and \( D \).

### 3. Related Operators for Woven and Weaving Frames

In this section, for first time, we introduce analysis, synthesis and frame operators of weaving and woven frames. As a prerequisite for this operators, we define the following space.

For each family of subspaces \( \left\{ (\ell^2(\mathbb{I}))_j \right\}_{j \in [m]} \) of \( \ell^2(\mathbb{I}) \), we have
\[
(\ell^2(\mathbb{I}))_j = \left\{ \{c_{ij}\}_{i \in \sigma_j} \mid c_{ij} \in \mathbb{C}, \sigma_j \subset \mathbb{I}, \sum_{i \in \sigma_j} |c_i|^2 < \infty \right\}, \quad \forall j \in [m].
\]
We define the space:
\[
\left( \sum_{j \in [m]} \bigoplus (\ell^2(\mathbb{I}))_j \right)_{\ell^2} = \left\{ \{c_{ij}\}_{i \in \mathbb{I}, j \in [m]} \mid \{c_{ij}\}_{i \in \mathbb{I}} \in (\ell^2(\mathbb{I}))_j, \forall j \in [m] \right\},
\]
with the inner product
\[
\left\langle \{c_{ij}\}_{i \in \mathbb{I}, j \in [m]} : \{c'_{ij}\}_{i \in \mathbb{I}, j \in [m]} \right\rangle = \sum_{i \in \mathbb{I}, j \in [m]} |c_{ij}^* c'_{ij}|,
\]
it is easy to show that this space is a Hilbert space.

**Theorem 3.1.** The family \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is a Bessel woven if and only if the operator
\[
T_F : \left( \sum_{j \in [m]} \bigoplus (\ell^2(\mathbb{I}))_j \right)_{\ell^2} \rightarrow \mathcal{H}, \quad T_F \{c_{ij}\}_{i \in \mathbb{I}, j \in [m]} = \sum_{i \in \mathbb{I}, j \in [m]} c_{ij} f_{ij}
\]
is well defined, linear and bounded.

**Proof.** Let \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) be a Bessel woven. Then for every fix \( j \in [m] \) and \( \sigma_j \subset \mathbb{I} \), the family \( \{f_{ij}\}_{i \in \sigma_j} \) is a Bessel sequence with Bessel bound
If we show the synthesis operator of \( \{ f_{ij} \}_{i \in \sigma_j} \) with \( T_{\sigma_j} \), thus for every \( \{ c_{ij} \}_{i \in [1, j] \in [m]} \in \left( \sum_{j \in [m]} \bigoplus (\ell^2(\mathbb{I}))_{j} \right)_{t_2} \), we have:

\[
\| T_F \{ c_{ij} \} \|^2 = \left\| \sum_{i \in [1, j] \in [m]} c_{ij} f_{ij} \right\|^2
\]

\[
= \left\| \sum_{i \in \mathbb{I}} c_{i1} f_{i1} + \sum_{i \in \mathbb{I}} c_{i2} f_{i2} + \ldots + \sum_{i \in \mathbb{I}} c_{im} f_{im} \right\|^2
\]

\[
\leq 2 \left\| T_{\sigma_1} \{ c_{i1} \}_{i \in \sigma_1} \right\|^2 + 2 \left\| T_{\sigma_2} \{ c_{i2} \}_{i \in \sigma_2} \right\|^2 + \ldots + 2 \left\| T_{\sigma_m} \{ c_{im} \}_{i \in \sigma_m} \right\|^2
\]

\[
\leq 2D_1 \left\| \{ c_{i1} \}_{i \in \sigma_1} \right\|^2 + 2D_2 \left\| \{ c_{i2} \}_{i \in \sigma_2} \right\|^2 + \ldots + 2D_m \left\| \{ c_{im} \}_{i \in \sigma_m} \right\|^2
\]

\[
\leq 2mD \left\| \{ c_{ik} \}_{i \in \sigma_k} \right\|^2
\]

\[
\leq 2mD \left\| \{ c_{ij} \}_{i \in [1, j] \in [m]} \right\|^2
\]

where

\[
D = \max \{ D_j ; \ 1 \leq j \leq m \}, \quad \left\| \{ c_{ik} \}_{i \in \sigma_k} \right\| = \max \left\{ \left\| \{ c_{ij} \}_{i \in \sigma_j} \right\| , 1 \leq j \leq m \right\}.
\]

This calculation shows that \( T_F \) is bounded and well defined. It is clear that \( T_F \) is linear.

Conversely, suppose \( T_F \) is well defined, linear and bounded with bound \( D \). For every \( f \in \mathcal{H} \), we have:

\[
\left\langle T_F \{ c_{ij} \}_{i \in [1, j] \in [m]}, f \right\rangle = \left\langle \sum_{i \in [1, j] \in [m]} c_{ij} f_{ij}, f \right\rangle = \left\langle \{ c_{ij} \}_{i \in [1, j] \in [m]}, \{ \langle f, f_{ij} \rangle \}_{i \in [1, j] \in [m]} \right\rangle,
\]

therefore

\[
T_F^* f = \{ \langle f, f_{ij} \rangle \}_{i \in [1, j] \in [m]},
\]

and

\[
\sum_{i \in [1, j] \in [m]} |\langle f, f_{ij} \rangle|^2 = \| T_F^* f \|^2
\]

\[
\leq \| T_F^* \|^2 \| f \|^2
\]

\[
= \| T_F \|^2 \| f \|^2
\]

\[
\leq D \| f \|^2.
\]

So the family \( \{ f_{ij} \}_{i \in [1, j] \in [m]} \) is a Bessel woven. 

If we only need to know that \( \{ f_{ij} \}_{i \in [1, j] \in [m]} \) is a Bessel woven and the value of the Bessel bound is irrelevant, we need to check that the operator \( T_F \) is well defined or not.
Corollary 3.2. If $\{f_{ij}\}_{i \in I, j \in [m]}$ is a family in $H$ and the series $\sum_{i \in I, j \in [m]} c_{ij} f_{ij}$ is convergent for all $\{c_{ij}\}_{i \in I, j \in [m]} \in \left( \sum_{j \in [m]} \bigoplus (\ell^2(I))_j \right)_2$. Then the sequence $\{f_{ij}\}_{i \in I, j \in [m]}$ is a Bessel woven.

The Bessel woven condition:

$$\sum_{i \in I, j \in [m]} |\langle f, f_{ij} \rangle|^2 \leq D \|f\|^2, \quad \forall f \in H,$$

remains the same, regardless of how the elements $\{f_{ij}\}_{i \in I, j \in [m]}$ are numbered.

Corollary 3.3. Suppose that the family $\{f_{ij}\}_{i \in I, j \in [m]}$ is a Bessel woven for $H$. Then, the series $\sum_{i \in I, j \in [m]} c_{ij} f_{ij}$ converges unconditionally for all $\{c_{ij}\}_{i \in I, j \in [m]} \in \left( \sum_{j \in [m]} \bigoplus (\ell^2(I))_j \right)_2$.

Proof. Since $\{f_{ij}\}_{i \in I, j \in [m]}$ is a Bessel woven, then by Theorem 3.1, [7], we have:

$$\sum_{i \in I, j \in [m]} |\langle f, f_{ij} \rangle|^2 \leq D \|f\|^2, \quad \forall f \in H,$$

which is equivalent to

$$\|T_F\{c_{ij}\}\|_2 \leq D \sum_{i \in I, j \in [m]} |c_{ij}|^2, \quad \forall \{c_{ij}\}_{i \in I, j \in [m]} \in \left( \sum_{j \in [m]} \bigoplus (\ell^2(I))_j \right)_2.$$

Therefore the series $\sum_{i \in I, j \in [m]} c_{ij} f_{ij}$ is convergent, so there exists $f \in H$ such that

$$\left\| \sum_{i \in I, j \in [m]} c_{ij} f_{ij} - f \right\| \leq \frac{\varepsilon}{2}.$$

Now, we show that the series $\sum_{i \in I, j \in [m]} c_{ij} f_{ij}$ converges unconditionally. By Cauchy-Schwarz inequality, we have:

$$\|T_F\{c_{ij}\}_{i \in I, j \in [m]}\|^2 = \sup_{\|g\|=1} \left| \left\langle \sum_{i \in I, j \in [m]} c_{ij} f_{ij}, g \right\rangle \right|.$$

$$\leq \sup_{\|g\|=1} \sum_{i \in I, j \in [m]} |c_{ij} \langle f_{ij}, g \rangle|$$

$$\leq \left( \sum_{i \in I, j \in [m]} |c_{ij}|^2 \right)^{\frac{1}{2}} \sup_{\|g\|=1} \left( \sum_{i \in I, j \in [m]} |\langle g, f_{ij} \rangle|^2 \right)^{\frac{1}{2}}.$$

Since $\{f_{ij}\}_{i \in I, j \in [m]}$ is Bessel sequence, then $\sum_{i \in I, j \in [m]} |\langle g, f_{ij} \rangle|^2$ is finite. Then for every $\varepsilon > 0$, there exists $M \in \mathbb{I}$

$$\sum_{i=M+1, j \in [m]} |\langle g, f_{ij} \rangle|^2 < \frac{\varepsilon}{2}. $$
Then
\[ \left\| \sum_{i=M+1,j \in [m]}^{\infty} c_{ij} f_{ij} \right\| < \frac{\varepsilon}{2}. \]

So we have
\[ \left\| \sum_{i=1,j \in [m]}^{M} c_{ij} f_{ij} - f \right\| \leq \left\| \sum_{i=M+1,j \in [m]}^{\infty} c_{ij} f_{ij} - f \right\| \leq \frac{\varepsilon}{2}. \]

Therefore
\[ \left\| \sum_{k=1,j \in [m]}^{M} c_{ik,j} f_{ik,j} - f \right\| < \varepsilon, \quad \forall \varepsilon > 0. \]

If we suppose \( F = \{1, 2, \ldots, M\} \), we have
\[ \left\| \sum_{i \in F,j \in [m]} c_{ij} f_{ij} - f \right\| < \varepsilon, \quad \varepsilon > 0. \]

So by Lemma 2.1.1, [7], the series \( \sum_{i \in [m]} c_{ij} f_{ij} \) converges unconditionally.

Like frames and its extensions, we can characterize a woven frame in term of its woven frame operator.

**Definition 3.4.** Let \( F = \{f_{ij}\}_{i \in [m]} \) be a woven Bessel. Then for every partition \( \{\sigma_j\}_{j \in [m]} \), the family \( F_j = \{f_{ij}\}_{i \in \sigma_j} \) for \( j \in [m] \) is a Bessel sequence. Therefore, we define the analysis operator of \( F_j \) by
\[ U_{\sigma_j} : H \rightarrow (\ell^2(\mathbb{I}))_j, \quad U_{\sigma_j}(f) = \{\langle f, f_{ij} \rangle\}_{i \in \sigma_j}, \quad \forall j \in [m], f \in H, \]
then \( \text{Ran} \left( U_{\sigma_j} \right) \subseteq (\ell^2(\mathbb{I}))_j \subseteq \ell^2(\mathbb{I}) \). The adjoint of \( U_{\sigma_j} \) is called the synthesis operator and in this paper, we denote \( T_{\sigma_j} = U_{\sigma_j}^* \). By elementary calculation, for every \( j \in [m] \), we have:
\[ T_{\sigma_j} : (\ell^2(\mathbb{I}))_j \rightarrow H, \quad T_{\sigma_j} \{c_{ij}\}_i = \sum_{i \in \sigma_j} c_{ij} f_{ij}, \quad \forall \{c_{ij}\}_i \in (\ell^2(\mathbb{I}))_j. \]

The frame operator of a weaving Bessel is obtained by combination of analysis and synthesis operators. For every \( f \in H \) and \( j \in [m] \):
\[ S_{\sigma_j} f = T_{\sigma_j} U_{\sigma_j} f = T_{\sigma_j} \{\langle f, f_{ij} \rangle\}_{i \in \sigma_j} = \sum_{i \in \sigma_j} \langle f, f_{ij} \rangle f_{ij}, \]

The operator \( S_{\sigma_j} \) is bounded, self-adjoint and invertible. We call the family \( \{S_{\sigma_j}^{-1} f_{ij}\}_{i \in \sigma_j} \) standard dual weaving frame of \( F_j \). Now, we define the
analysis and synthesis operators for the Bessel woven $F = \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$:

$$U_F : \mathcal{H} \rightarrow \left( \sum_{j \in [m]} \bigoplus (\ell^2(\mathbb{l}))_j \right)_{\ell^2}$$

and

$$T_F : \left( \sum_{j \in [m]} \bigoplus (\ell^2(\mathbb{l}))_j \right)_{\ell^2} \rightarrow \mathcal{H}, \quad T_F \{c_{ij}\}_{i \in \mathbb{I}, j \in [m]} = \sum_{i \in \mathbb{I}, j \in [m]} c_{ij} f_{ij}.$$ 

The operators $U_F$ and $T_F$ are well defined and bounded analysis and synthesis operators, respectively. Also, by combination of $U_F$ and $T_F$, the woven frame operator $S_F$, for all $f \in \mathcal{H}$, is defined by

$$S_F : \mathcal{H} \rightarrow \mathcal{H}, \quad S_F f = T_F U_F f = \sum_{i \in \mathbb{I}, j \in [m]} \langle f, f_{ij} \rangle f_{ij}.$$ 

The operator $S_F$ is bounded, linear and self-adjoint operator. Also every $f \in \mathcal{H}$ can be represented as

$$f = \sum_{i \in \mathbb{I}, j \in [m]} \langle f, S_F^{-1} f_{ij} \rangle f_{ij} = \sum_{i \in \mathbb{I}, j \in [m]} \langle f, f_{ij} \rangle S_F^{-1} f_{ij}.$$ 

The family $\{S_F^{-1} f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is said the standard dual woven of $F$.

### 3.1. Main results

In this subsection, we explain some properties of operators of woven frames and Bessel wovens. Also, we bring some conditions that under those, the sum of Bessel wovens shall be woven frames.

In the next theorem, we demonstrate that the woven frames are equivalent to boundedness of woven frame operator.

**Theorem 3.5.** Let $\{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ be finite family of Bessel sequences in $\mathcal{H}$. Then the following conditions are equivalent:

(i) $\{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is woven frames with universal woven frame bounds $C$ and $D$.

(ii) for the operator $S_F f = \sum_{i \in \mathbb{I}, j \in [m]} \langle f, f_{ij} \rangle f_{ij}$, we have $CI_{\mathcal{H}} \leq S_F \leq DI_{\mathcal{H}}$.

**Proof.** $i \Rightarrow ii$: By statement of (i), we have for every $f \in \mathbb{I}$

$$\langle S_F f, f \rangle = \left( \sum_{i \in \mathbb{I}, j \in [m]} \langle f, f_{ij} \rangle f_{ij}, f \right) = \sum_{i \in \mathbb{I}, j \in [m]} |\langle f, f_{ij} \rangle|^2,$$

then we obtain

$$C \|f\|^2 \leq \langle S_F f, f \rangle \leq D \|f\|^2, \quad \forall f \in \mathcal{H},$$

or equivalently

$$CI_{\mathcal{H}} \leq S_F \leq DI_{\mathcal{H}}.$$
\( \Rightarrow \text{i}: \) Let \( U_F \) be the analysis operator associated with \( \{ f_{ij} \}_{i \in I, j \in [m]} \). By the fact that \( \| S_F \| = \| T_F U_F \| = \| U_F \| \), for all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I, j \in [m]} |\langle f, f_{ij} \rangle|^2 = \| U_F f \|^2 \\
\leq \| U_F \|^2 \| f \|^2 \\
= \| S_F \| \| f \|^2 \\
\leq D \| f \|^2 .
\]

So the upper woven bound is established. For lower weaving bound, for all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I, j \in [m]} |\langle f, f_{ij} \rangle|^2 = \langle S_F f, f \rangle \\
= \langle S_{\frac{1}{2}} f, S_{\frac{1}{2}}^2 f \rangle \\
= \| S_{\frac{1}{2}} f \|^2 \\
\geq C \| f \|^2 .
\]

\( \Box \)

The next result shows that we can constitute tight woven frames from every woven frames by weaving operators.

**Theorem 3.6.** Let \( F = \{ f_{ij} \}_{i \in I, j \in [m]} \) be woven frame for \( \mathcal{H} \) with universal woven bounds \( C \) and \( D \) and the woven frame operator \( S_F \). If we define the positive square root of \( S_F^{-1} \) with \( S_F^{-\frac{1}{2}} \), then \( \{ S_F^{-\frac{1}{2}} f_{ij} \}_{i \in I, j \in [m]} \) is tight woven frame and for all \( f \in \mathcal{H} \), we have:

\[
f = \sum_{i \in I, j \in [m]} \langle f, S_F^{-\frac{1}{2}} f_{ij} \rangle S_F^{-\frac{1}{2}} f_{ij}.
\]

**Proof.** Since the sequence \( \{ f_{ij} \}_{i \in I, j \in [m]} \) is a woven frame with bounds \( C \) and \( D \), we can write \( CI_{\mathcal{H}} \leq S_F \leq DI_{\mathcal{H}} \), thus \( D^{-1} I_{\mathcal{H}} \leq S_F^{-1} \leq C^{-1} I_{\mathcal{H}} \). Now, by definition of woven frame operator of \( F = \{ f_{ij} \}_{i \in I, j \in [m]} \), we have for every \( f \in \mathcal{H} \):

\[
S_F f = \sum_{i \in I, j \in [m]} \langle f, f_{ij} \rangle f_{ij}.
\]

By substitution of \( S_F^{-\frac{1}{2}} f \), we have:

\[
S_F^{-\frac{1}{2}} f = \sum_{i \in I, j \in [m]} \langle S_F^{-\frac{1}{2}} f, f_{ij} \rangle f_{ij},
\]
therefore

\[ f = S_F^{-\frac{1}{2}} \left( S_F^\frac{1}{2} f \right) = S_F^{-\frac{1}{2}} \left( \sum_{i \in I, j \in [m]} \left< f, S_F^{-\frac{1}{2}} f_{ij} \right> f_{ij} \right) = \sum_{i \in I, j \in [m]} \left< f, S_F^{-\frac{1}{2}} f_{ij} \right> S_F^{-\frac{1}{2}} f_{ij}. \]

Thus we obtain

\[ \|f\|^2 = \left< \sum_{i \in I, j \in [m]} \left< f, S_F^{-\frac{1}{2}} f_{ij} \right> S_F^{-\frac{1}{2}} f_{ij}, f \right> = \sum_{i \in I, j \in [m]} \left| \left< f, S_F^{-\frac{1}{2}} f_{ij} \right> \right|^2. \]

So \( \left\{ S_F^{-\frac{1}{2}} f_{ij} \right\}_{i \in I, j \in [m]} \) is tight woven frames with universal bound 1. \( \square \)

In the following theorem, we investigate the effect of a bounded and invertible operator on woven frames.

**Theorem 3.7.** Let \( F = \{f_{ij}\}_{i \in I, j \in [m]} \) be a woven frame for \( \mathcal{H} \) with woven frame operator \( S_F \) and universal bounds \( C \) and \( D \) and \( E : \mathcal{H} \to \mathcal{H} \) be a bounded operator. Then the operator \( E \) is invertible if and only if \( \{Ef_{ij}\}_{i \in I, j \in [m]} \) is woven frame for \( \mathcal{H} \). In this case, the universal bounds for \( F \) are \( C \|E^{-1}\|^{-2}, D \|E\|^{-2} \) and the woven frame operator \( ES_F E^* \).

**Proof.** Let \( E \) be invertible, since \( \{f_{ij}\}_{i \in I, j \in [m]} \) is a woven frames for \( \mathcal{H} \), with bounds \( C \) and \( D \), so the boundedness of \( T \) verifies the upper bound:

\[ \sum_{i \in I, j \in [m]} |\langle f, Ef_{ij} \rangle|^2 \leq D \|E^* f\|^2 \leq D \|E^*\|^2 \|f\|^2 = D \|E\|^2 \|f\|^2. \]
For lower bound, we assume that $g \in \mathcal{H}$. Since $E$ is surjective, there exists $f \in \mathcal{H}$ such that $Ef = g$. Therefore
\[
\|g\|^2 = \|Ef\|^2 \\
= \|(EE^{-1})^* E\|^2 \\
= \|(E^{-1})^* E^* Ef\|^2 \\
\leq \|E^{-1}\|^2 \|E^* Ef\|^2 \\
\leq \frac{\|E^{-1}\|^2}{C} \sum_{i \in \mathbb{I}, j \in [m]} |\langle E^* Ef, f_{ij} \rangle|^2 \\
= \frac{\|E^{-1}\|^2}{C} \sum_{i \in \mathbb{I}, j \in [m]} |\langle Ef, Ef_{ij} \rangle|^2,
\]
so we have:
\[
C \|E^{-1}\|^{-2} \|g\|^2 \leq \sum_{i \in \mathbb{I}, j \in [m]} |\langle g, Ef_{ij} \rangle|^2.
\]
This calculations provide the desired results.

Conversely, If $\{Ef_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is a woven frames for $\mathcal{H}$ then its woven frame operator is invertible
\[
\sum_{i \in \mathbb{I}, j \in [m]} \langle f, Ef_{ij} \rangle Ef_{ij} = E \left( \sum_{i \in \mathbb{I}, j \in [m]} \langle f, Ef_{ij} \rangle f_{ij} \right) = ES_F E^* f,
\]
which implies that $E$ is invertible. \hfill \Box

From the previous theorem, we can obtain the next result.

**Theorem 3.8.** Let $F = \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ be woven frame with universal woven bounds $C$ and $D$ for $\mathcal{H}$ and woven frame operator $S_F$. Then for every $\sigma_j \subset \mathbb{I}$, $j \in [m], we have:

(i) The sequence $\{S_{\sigma_j} f_{ij}\}_{i \in \sigma_j}$ for every $j \in [m]$ is a weaving frame i.e. the family $\{S_F f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is woven frames for $\mathcal{H}$, with universal lower and upper woven bounds $C^3$ and $D^3$, respectively.

(ii) The sequence $\{S_{\sigma_j}^{-1} f_{ij}\}_{i \in \sigma_j}$ for every $j \in [m]$ is a weaving frame i.e. the family $\{S_F^{-1} f_{ij}\}_{i \in \mathbb{I}, j \in [m]}$ is a woven frames for $\mathcal{H}$, with universal lower and upper woven bounds $\frac{C}{D^2}$ and $\frac{D}{C^2}$, respectively.

**Proof.** By Theorem 3.7, if we put $E = S_{\sigma_j}, S_F$ and $E = S_{\sigma_j}^{-1}, S_F^{-1}$, then provide the results.

In the next theorem, we give some conditions that under those, sum of wovens are woven also.
Theorem 3.9. Let \( F = \{f_{ij}\}_{i \in I, j \in [m]} \) and \( G = \{g_{ij}\}_{i \in I, j \in [m]} \) be woven Bessel sequences in \( \mathcal{H} \), with analysis operators \( U_F, U_G \) and woven frame operators \( S_F, S_G \), respectively. Also, let \( E_1, E_2 : \mathcal{H} \rightarrow \mathcal{H} \) be bounded operators. Then the family \( \{E_1 f_{ij} + E_2 g_{ij}\}_{i \in I, j \in [m]} \) is a woven frames for \( \mathcal{H} \) if and only if the operator \( U_F E_1^* + U_G E_2^* \) is an invertible operator on \( \mathcal{H} \).

Proof. The family \( \{E_1 f_{ij} + E_2 g_{ij}\}_{i \in I, j \in [m]} \) is a woven frames if and only if its analysis operator \( U \) is invertible on \( \mathcal{H} \), by Theorem 4.1 in [3]. Therefore
\[
U f = \{\langle f, E_1 f_{ij} + E_2 g_{ij} \rangle\}_{i \in I, j \in [m]}
= \{\langle f, E_1 f_{ij} \rangle + \langle f, E_2 g_{ij} \rangle\}_{i \in I, j \in [m]}
= \{\langle E_1^* f, f_{ij} \rangle\}_{i \in I, j \in [m]} + \{\langle E_2^* f, g_{ij} \rangle\}_{i \in I, j \in [m]}
= U_F E_1^* f + U_G E_2^* f,
\]
so the result is obtained. \( \square \)

Now, by this theorem, we can immediately get the following corollary.

Corollary 3.10. By the hypotheses of Theorem 3.9, if \( U_F E_1^* + U_G E_2^* \) is an invertible operator on \( \mathcal{H} \), then the operator
\[
S = E_1 S_F E_1^* + E_2 S_G E_2^* + E_1 U_F^* U_G E_2^* + E_2 U_F^* U_F E_1^*,
\]
is a positive operator.

Proof. By Theorem 3.9, the family of sequences \( \{E_1 f_{ij} + E_2 g_{ij}\}_{i \in I, j \in [m]} \) is a woven frames with the analysis operator \( U_F E_1^* + U_G E_2^* \), for which its woven frame operator is positive:
\[
S f = (U_F E_1^* f + U_G E_2^* f)^* (U_F E_1^* f + U_G E_2^* f)
= (E_1 U_F^* f + E_2 U_G^* f) (U_F E_1^* f + U_G E_2^* f)
= E_1 S_F E_1^* f + E_1 U_F^* U_G E_2^* f + E_2 U_G^* U_F E_1^* f + E_2 S_G E_2^* f.
\]
\( \square \)

In the following theorem, we check frame bounds conditions on a dense subset in Hilbert space:

Theorem 3.11. Suppose \( \{f_{ij}\}_{i \in I, j \in [m]} \) is a woven frames for \( \mathcal{W} \) with universal woven bounds \( C \) and \( D \), where \( \mathcal{W} \) is a dense subspace of \( \mathcal{H} \). Then \( \{f_{ij}\}_{i \in I, j \in [m]} \) is a woven frames for \( \mathcal{H} \), with universal bound \( C \) and \( D \).

Proof. Since \( \{f_{ij}\}_{i \in I, j \in [m]} \) is woven frames for \( \mathcal{W} \), then
\[
C \|f\|^2 \leq \sum_{i \in I, j \in [m]} |\langle f, f_{ij} \rangle|^2 \leq D \|f\|^2, \quad \forall f \in \mathcal{W}.
\]
(3.1)
For upper bound, by contradiction, suppose that there exists \( g \in \mathcal{H} \), such that:
\[
\sum_{i \in I, j \in [m]} |\langle g, f_{ij} \rangle|^2 \geq D \|f\|^2.
\]
By Corollary 3.3, the series \( \sum_{i \in \mathbb{I}, j \in [m]} |\langle g, f_{ij} \rangle|^2 \) is not convergent unconditionally. Then there exists a finite set \( M \subset \mathbb{I} \) where
\[
\sum_{i \in M, j \in [m]} |\langle g, f_{ij} \rangle|^2 \geq D \|f\|^2, \quad g \in \mathcal{H}.
\]
Since \( W \) is a dense subset in \( \mathcal{H} \), so there exist \( h \in W \) such that
\[
\sum_{i \in M, j \in [m]} |\langle h, f_{ij} \rangle|^2 \geq D \|f\|^2, \quad h \in \mathcal{H},
\]
then the sequence \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is not Bessel in \( W \) and this is a contradiction.

So the family \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is a Bessel woven in \( \mathcal{H} \). Now for lower frame bound, by (3.1), we have for every \( f \in W \)
\[
C \|f\|^2 \leq \sum_{i \in \mathbb{I}, j \in [m]} |\langle f, f_{ij} \rangle|^2 = \|U_F f\|^2.
\]
Then \( U_F \) is a bounded operator in \( W \) for which is dense in \( \mathcal{H} \). Therefore the statement of (3.2) holds for all \( f \in \mathcal{H} \). □

In the next theorem, from every woven frame in Hilbert space \( \mathcal{H} \), we constitute woven frame for smaller spaces, by using orthogonal projection.

**Theorem 3.12.** Suppose \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is woven frame for Hilbert space \( \mathcal{H} \), with universal bounds \( C \) and \( D \) and \( P \) denotes the orthogonal projection onto a closed subspace \( W \) of \( \mathcal{H} \). Then \( \{P f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is a woven frames for \( W \) with the same universal bounds.

**Proof.** Because \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is a woven frames for \( \mathcal{H} \), with universal bounds \( C \) and \( D \), so we have:
\[
C \|f\|^2 \leq \sum_{i \in \mathbb{I}, j \in [m]} |\langle f, f_{ij} \rangle|^2 \leq D \|f\|^2, \quad \forall f \in \mathcal{H}.
\]
Since we have \( P f = f \), for all \( f \in W \) and \( P f = 0 \), for all \( f \in W^\perp \), then for every \( f \in W \) we can write
\[
\sum_{i \in \mathbb{I}, j \in [m]} |\langle f, f_{ij} \rangle|^2 = \sum_{i \in \mathbb{I}, j \in [m]} |\langle P f, f_{ij} \rangle|^2 = \sum_{i \in \mathbb{I}, j \in [m]} |\langle f, Pf_{ij} \rangle|^2.
\]
Therefore \( \{P f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is woven frames for \( W \) with bounds \( C \) and \( D \). □

The following corollary follows from Theorem 3.12.

**Corollary 3.13.** Let \( \{f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) be a woven frame for Hilbert space \( \mathcal{H} \) and \( V, W \) are closed subspaces of \( \mathcal{H} \) such that \( V \cap W \neq \emptyset \) and \( P \) denotes the orthogonal projection of \( \mathcal{H} \) onto \( V \cap W \). Then \( \{P f_{ij}\}_{i \in \mathbb{I}, j \in [m]} \) is a woven frame for \( V \cap W \).
Application:
In this section, we provide an example of woven frames in the Euclidean space $\mathbb{R}^3$, then from this woven frame, we constitute a woven frame for smaller subspace of $\mathbb{R}^3$, by Theorem 3.12 and Corollary 3.13.

**Example 3.14.** Let $\{e_i\}_{i=1}^3$ be the standard orthonormal basis for $\mathbb{R}^3$. Suppose there exist constants $\alpha > 0$ and $\beta = \frac{1}{\sqrt{1+\alpha^2}}$. Also $G, Q$ are the sets

\[
G = \{g_i\}_{i=1}^6 = \{\beta e_1, \alpha \beta e_1, \beta e_2, \alpha \beta e_2, \beta e_3, \alpha \beta e_3, \}
\]

and:

\[
Q = \{q_i\}_{i=1}^6 = \{\alpha e_1, \beta e_1, \alpha e_2, \beta e_2, \alpha e_3, \beta e_3, \}
\]

$G$ is a Parseval frame:

\[
\sum_{i=1}^6 |\langle f, g_i \rangle|^2 = |\langle f, \beta e_1 \rangle|^2 + |\langle f, \alpha \beta e_1 \rangle|^2 + |\langle f, \beta e_2 \rangle|^2 + |\langle f, \alpha \beta e_2 \rangle|^2 + |\langle f, \beta e_3 \rangle|^2 + |\langle f, \alpha \beta e_3 \rangle|^2
\]

\[
= \beta^2 (1 + \alpha^2) \sum_{i=1}^3 |\langle f, e_i \rangle|^2
\]

\[
= \|f\|^2.
\]

Similarly $Q$ is a Parseval frame, for which $G, Q$ are woven frames, since every $\sigma \subset \{1, 2, 3, 4, 5, 6\}$ gives a spanning set. For example, we calculate for two weaving: Let $\sigma_1 = \{2, 4, 6\}$. Then for every $f \in \mathbb{R}^3$, we have:

\[
\sum_{i \in \sigma_1} |\langle f, g_i \rangle|^2 + \sum_{i \in \sigma_1^c} |\langle f, q_i \rangle|^2 = \alpha^2 \beta^2 \sum_{i=1}^3 |\langle f, e_i \rangle|^2
\]

\[
= \alpha^2 \beta^2 \|f\|^2
\]

\[
= \frac{\alpha^2}{1 + \alpha^2} \|f\|^2.
\]

So this weaving is tight frame with bounds $C_{\sigma_1} = D_{\sigma_1} = \frac{\alpha^2}{1 + \alpha^2}$. Also for $\sigma_2 = \{1, 2\}$, we have:

\[
\sum_{i \in \sigma_2} |\langle f, g_i \rangle|^2 + \sum_{i \in \sigma_2^c} |\langle f, q_i \rangle|^2 = |\langle f, \beta e_1 \rangle|^2 + |\langle f, \alpha \beta e_1 \rangle|^2 + |\langle f, \beta e_2 \rangle|^2 + |\langle f, \alpha \beta e_2 \rangle|^2 + \alpha^2 \beta^2 \sum_{i=1}^3 |\langle f, e_i \rangle|^2
\]

\[
= (\alpha^2 \beta^2 + \beta^2) \sum_{i=1}^3 |\langle f, e_i \rangle|^2
\]

\[
= \|f\|^2,
\]

thus this weaving is Parseval frame and $C_{\sigma_2} = D_{\sigma_2} = 1$. Now, If we have:

\[
C = \min \{C_{\sigma_j} \mid s.t \ j \in [6]\}
\]
and

\[ D = \max \left\{ D_j, \quad s.t \quad j \in [64] \right\}, \]

therefore \( G, Q \) are woven frames with universal bounds \( C \) and \( D \). Now, let \( V_1 = \overline{\text{span}} \{ e_{3i}, e_{3i+1} \} \) and \( P \) denotes the orthogonal projection from \( \mathbb{R}^3 \) onto \( V_1 \). Then by Theorem 3.12, \( \{ Pg_i \}_{i=1}^6 \) and \( \{ Pq_i \}_{i=1}^6 \) are woven frames for \( V_1 \).

Also, suppose \( V_2 = \overline{\text{span}} \{ e_{3i+1}, e_{3i+2} \} \). Then \( V_1 \cap V_2 = \overline{\text{span}} \{ e_{3i+1} \} \). Let \( P' \) be an orthonormal projection of \( \mathbb{R}^3 \) onto \( \overline{\text{span}} \{ e_{3i+1} \} \). Thus by Corollary 3.13, \( \{ P'g_i \}_{i=1}^6 \) and \( \{ P'q_i \}_{i=1}^6 \) are woven frames for \( V_1 \cap V_2 \) with same bounds \( C \) and \( D \).

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