Fuzzy Non-Trivial Gauge Configurations

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Abstract

In this talk we will report on few results of discrete physics on the fuzzy sphere. In particular non-trivial field configurations such as monopoles and solitons are constructed on fuzzy $S^2$ using the language of K-theory, i.e. projectors. As we will show, these configurations are intrinsically finite dimensional matrix models. The corresponding monopole charges and soliton winding numbers are also found using the formalism of noncommutative geometry and cyclic cohomology.

Fuzzy physics is aimed to be an alternative method to approach discrete physics. Problems of lattice physics especially those with topological roots are all avoided on fuzzy spaces. For example, chiral anomaly, Fermion doubling and the discretization of non-trivial topological field configurations were all formulated consistently on the fuzzy sphere [see [1] and the extensive list of references therein]. The paradigm of fuzzy physics is “discretization by quantization”, namely given a space, we treat it as a phase space and then quantize it. This requires the existence of a symplectic structure on this space. One such class of spaces which admit symplectic forms are the co-adjoint orbits, for example $\mathbb{CP}^1 = S^2$, $\mathbb{CP}^2$, $\mathbb{CP}^3$ and so on. Their quantization to obtain their fuzzy counterparts is done explicitly in [2, 1]. Here we will only summarize the important results for $S^2$ which are needed for the purpose of this paper.

1 Fuzzy $S^2$

Fuzzy $S^2$ or $S^2_F$ is the algebra $A = Mat_{2l+1}$ of $(2l+1) \times (2l+1)$ matrices which is generated by the operators $n^F_i$, $i = 1, 2, 3$, which are defined by

$$n^F_i = \frac{L_i}{\sqrt{l(l+1)}}. \quad (1)$$

$L_i$’s satisfy $[L_i, L_j] = i\epsilon_{ijk}L_k$ and $\sum_{i=1}^3 L_i^2 = l(l+1)$ respectively, where $l$ is a positive integer. In other words, $L_i$’s are the generators of the IRR $l$ of
A general element \( \hat{f} \) of \( \mathcal{A} \) admits an expansion, in terms of \( n_i^F \)'s, of the form
\[
\hat{f}(\vec{n}^F) = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} n_1^F \ldots n_k^F,
\]
which will terminate by the nature of the operators \( n_i^F \)'s. The continuum limit is defined by \( l \rightarrow \infty \). In such a limit the fuzzy coordinates \( n_i^F \)'s tend, by definition, to the commutative coordinates \( n_i \)'s. By inspection the commutators of the fuzzy coordinates among each others vanish at \( l \rightarrow \infty \), but from the Casimir equation above we must have \( \sum_{i=1}^{3} n_i^2 = 1 \). Furthermore, the noncommutative algebra at this limit becomes the commutative algebra of functions on continuum \( S^2 \), namely \( \mathcal{A} \rightarrow \mathcal{A} \), where a general element \( f \) of \( \mathcal{A} \) will admit the expansion
\[
f(\vec{n}) = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} n_1 \ldots n_k.
\]

Viewing \( S^2 \) as a submanifold of \( \mathbb{R}^3 \), one can check the following basic identity
\[
D_2 = D_3|_{r=\rho} + \frac{i\gamma^3}{\rho}. \tag{2}
\]
\( \gamma^a = \sigma_a, \ a = 1, 2, 3 \), are the flat gamma matrices in 3-dimensions. \( D_2, D_3 \) are the Dirac operators on \( S^2 \) and \( \mathbb{R}^3 \) respectively. \( D_3|_{r=\rho} \) is the restriction of the Dirac operator on \( \mathbb{R}^3 \) to the sphere \( r = \rho \), where \( \rho \) is the radius of the sphere, namely \( \sum_{a=1}^{3} x_a^2 = \rho^2 \) for any \( \vec{x} \in S^2 \). The Clifford algebra on \( S^2 \) is two dimensional and therefore at each point \( \vec{n} = \vec{x}/\rho \) one has only two independents gamma matrices, they can be taken to be \( \gamma^1 \) and \( \gamma^2 \). \( \gamma^3 \) should then be identified with the chirality operator \( \gamma = \vec{\sigma}.\vec{n} \) on \( S^2 \).

Next by using the canonical Dirac operator \( D_3 = -i\sigma_a \partial_a \) in (4) one can derive the two following equivalent expressions for the Dirac operator \( D_2 \) on \( S^2 \):
\[
D_{2g} = \frac{1}{\rho}(\vec{\sigma} \vec{L} + 1),
\]
\[
D_{2w} = -\frac{1}{\rho} \epsilon_{ijk} \sigma_i n_j \vec{J}_k. \tag{3}
\]
\( \vec{L}_k = -i\epsilon_{kij} x_i \partial_j \) is the orbital angular momentum and \( \vec{J}_k = \vec{L}_k + \frac{\sigma_k}{2} \) is the total angular momentum. \( g \) and \( w \) in (3) stands for Grosse-Klimčík-Prešnajder and Watamuras Dirac operators respectively. It is not difficult to check that \( D_{2w} = i\gamma D_{2g} = D_3|_{r=\rho} + \frac{i\gamma^3}{\rho} \), which means that \( D_{2w} \) and \( D_{2g} \) are related by a unitary transformation and therefore are equivalent. The spectrum of these Dirac operators is trivially derived to be given by \( \pm \frac{1}{\rho}(j + \frac{1}{2}) \) where \( j \) is the eigenvalue of \( \vec{J} \), i.e. \( \vec{J}^2 = j(j + 1) \) and \( j = 1/2, 3/2, \ldots \).
The fuzzy versions of the Dirac operators (3) are taken to be

\[
D_{2g} = \frac{1}{\rho}(\bar{\sigma}.ad\bar{L} + 1)
\]

\[
D_{2w} = \frac{1}{\rho}\epsilon_{ijk}\sigma_i n_j^F L_k^R.
\]

\(ad\bar{L} = \bar{L}^L - \bar{L}^R\) is the fuzzy derivation which annihilates the identity matrix in \(A\) as the classical derivation \(\bar{L}\) annihilates the constant function in \(A\). \(\bar{L}^L\) and \(-\bar{L}^R\) are the generators of the IRR \(l\) of \(SU(2)\) which act on the left and on the right of the algebra \(A\) respectively, i.e \(\bar{L}^L f = \bar{L} f\) and \(-\bar{L}^R f = -f\bar{L}\) for any \(f \in A\). From this definition one can see that \(Ad\bar{L}\) provide the generators of the adjoint action of \(SU(2)\) on \(A\), namely \(Ad\bar{L}(f) = [\bar{L}, f]\) for any \(f \in A\).

These two fuzzy Dirac operators are not unitarily equivalent anymore. This can be checked by computing their spectra. The spectrum of \(D_{2g}\) is exactly that of the continuum only cut-off at the top total angular momentum \(j = 2l + \frac{1}{2}\). In other words the spectrum of \(D_{2g}\) is equal to \(\{\pm \frac{1}{\rho}(j + \frac{1}{2}), j = \frac{1}{2}, \frac{3}{2}, ... 2l - \frac{1}{2}\}\) and \(D_{2g}(j) = \frac{1}{\rho}(j + \frac{1}{2})\) for \(j = 2l + \frac{1}{2}\). The spectrum of \(D_{2w}\) is, however, highly deformed as compared to the continuum spectrum especially for large values of \(j\). It is given by \(D_{2w}(j) = \pm \frac{1}{\rho}(j + \frac{1}{2})\sqrt{1 + \frac{(j+1/2)^2}{4l(l+1)}}\). From these results it is obvious that \(D_{2g}\) is superior to \(D_{2w}\) as an approximation to the continuum.

In the same way one can find the fuzzy chirality operator \(\Gamma\) by the simple replacement \(\bar{n} \rightarrow \bar{n}^F\) in \(\gamma = \bar{\sigma}\bar{n}\) and insisting on the result to have the following properties: 1)\(\Gamma^2 = 1\), \(\Gamma^+ = \Gamma\) and \([\Gamma, f] = 0\) for all \(f \in A\). One then finds

\[
\Gamma = \frac{1}{l + \frac{1}{2}}(-\bar{\sigma}\bar{L}^R + \frac{1}{2}).
\]

Interestingly enough this fuzzy chirality operator anticommutes with \(D_{2w}\) and not with \(D_{2g}\) so \(D_{2w}\) is a better approximation to the continuum than \(D_{2g}\) from this respect. This is also clear from the spectra above, in the spectrum of \(D_{2g}\) the top angular momentum is not paired to anything and therefore \(D_{2g}\) does not admit a chirality operator.

## 2 Fuzzy Non-Trivial Gauge Configurations
2.1 Classical Monopoles

Monopoles are one of the most fundamental non trivial configurations in field theory. The wave functions of a particle of charge $q$ in the field of a monopole $p$, which is at rest at $r = 0$, are known to be given by the expansion

$$\psi^{(N)}(r, g) = \sum_{j,m} c^j_m(r) \langle j, m | D^{(j)}(g) | j, -\frac{N}{2} \rangle,$$

where $D^{(j)} : g \rightarrow D^{(j)}(g)$ is the $j$ IRR of $g \in SU(2)$. The integer $N$ is related to $q$ and $p$ by the Dirac quantization condition: $N = \frac{qp}{2\pi}$. $r$ is the radial coordinate of the relative position $\vec{x}$ of the system, the angular variables of $\vec{x}$ are defined through the element $g \in SU(2)$ by $\vec{r}.\vec{n} = g\tau_3g^{-1}$, $\vec{n} = \vec{x}/r$.

It is also a known result that the precise mathematical structure underlying this physical system is that of a $U(1)$ principal fiber bundle $SU(2) \rightarrow S^2$. In other words for a fixed $r = \rho$, the particle $q$ moves on a sphere $S^2$ and its wave functions (6) are precisely elements of $S(S^2, SU(2))$, namely sections of a $U(1)$ bundle over $S^2$. They have the equivariance property

$$\psi^{(N)}(\rho, ge^{i\theta} \frac{\vec{n}}{2}) = e^{-i\theta \frac{N}{2}} \psi^{(N)}(\rho, g),$$

i.e they are not really functions on $S^2$ but rather functions on $SU(2)$ because they clearly depend on the specific point on the $U(1)$ fiber. In this paper, we will only consider the case $N = \pm 1$. The case $|N| \neq 1$ being similar and is treated in great detail in [1, 5].

An alternative description of monopoles can be given in terms of K-theory and projective modules. It is based on the Serre-Swan’s theorem which states that there is a complete equivalence between vector bundles over a compact manifold $M$ and projective modules over the algebra $C(M)$ of smooth functions on $M$. Projective modules are constructed from $C(M)^n = C(M) \otimes C^n$ where $n$ is some integer by the application of a certain projector $p$ in $\mathcal{M}_n(C(M))$, i.e the algebra of $n \times n$ matrices with entries in $C(M)$.

In our case $M = S^2$ and $C(M) = A \equiv$ the algebra of smooth functions on $S^2$. For a monopole system with winding number $N = \pm 1$, the appropriate projective module will be constructed from $A^2 = A \otimes C^2$. It is $\mathcal{P}^{(\pm 1)}A^2$ where $\mathcal{P}^{(\pm 1)}$ is the projector

$$\mathcal{P}^{(\pm 1)} = \frac{1 \pm \vec{r}.\vec{n}}{2}.$$
It is clearly an element of $\mathcal{M}_2(\mathcal{A})$ and satisfies $\mathcal{P}^{(\pm 1)^2} = \mathcal{P}^{(\pm 1)}$ and $\mathcal{P}^{(\pm 1)^+} = \mathcal{P}^{(\pm 1)}$. $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$ describes a monopole system with $N = \pm 1$ as one can directly check by computing its Chern character as follows

$$\pm 1 = \frac{1}{2\pi i} \int Tr \mathcal{P}^{(\pm 1)} d\mathcal{P}^{(\pm 1)} \wedge d\mathcal{P}^{(\pm 1)}.$$  \hfill (9)

On the contrary to the space of sections $\mathcal{S}(\mathbb{S}^2, SU(2))$, elements of $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$ are by construction invariant under the action $g \rightarrow g exp(i\theta \frac{1}{2})$. The other advantage of $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$ as compared to $\mathcal{S}(\mathbb{S}^2, SU(2))$ is the fact that its fuzzification is much more straightforward.

### 2.2 On The Equivalence of $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$ and $\mathcal{S}(\mathbb{S}^2, SU(2))$

Before we start the fuzzification of $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$, let us first comment on the relation between the wave functions $\psi^{(\pm 1)}$ given in equation (1) and those belonging to $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$. The projector $\mathcal{P}^{(\pm 1)}$ can be rewritten as $\mathcal{P}^{(\pm 1)} = D(\frac{1}{2}) \frac{1+\gamma_5}{2} D(\frac{3}{2}) (g)$ where $D(\frac{1}{2}) : g \rightarrow D(\frac{1}{2}) (g) = g$ is the $\frac{1}{2}$ IRR of $SU(2)$. Hence $\mathcal{P}^{(\pm 1)} D(\frac{1}{2}) (g) \pm \rightarrow D(\frac{1}{2}) (g) \pm = \frac{1}{2} \pm \tau_3 \pm \rightarrow \pm \pm \rightarrow \pm \pm >$. In the same way one can show that $\mathcal{P}^{(\pm 1)} D(\frac{3}{2}) (g) \mp \rightarrow = 0$. This last result means that

$$\mathcal{P}^{(\pm 1)} = D(\frac{1}{2}) (\pm) \pm < \pm D(\frac{1}{2})^+ (\pm)$$  \hfill (10)

$< \pm D(\frac{1}{2})^+ (\pm) \psi$ defines a map from $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$ into $\mathcal{S}(\mathbb{S}^2, SU(2))$ as follows

$$< \pm D(\frac{1}{2})^+ (\pm) \rightarrow < \pm D(\frac{1}{2})^+ (\pm) \psi = \psi^{(\pm 1)} (\rho, g).$$  \hfill (11)

$< \pm D(\frac{1}{2})^+ (\pm) \psi$ has the correct transformation law [3] under $g \rightarrow g exp(i\theta \frac{1}{2})$ as one can check by using the basic equivariance property

$$D(\frac{1}{2}) (ge^{i\theta \frac{1}{2}}) \pm \rightarrow e^{i\theta \frac{1}{2}} D(\frac{1}{2}) (g) \pm >.$$  \hfill (12)

In the same way $D(\frac{1}{2}) (g) \pm >$ defines a map, $\mathcal{S}(\mathbb{S}^2, SU(2)) \rightarrow \mathcal{P}^{(\pm 1)} \mathcal{A}^2$, which takes the wave functions $\psi^{(\pm 1)}$ to the two components elements $\psi^{(\pm 1)} D(\frac{1}{2}) (g) \pm >$ of $\mathcal{P}^{(\pm 1)} \mathcal{A}^2$. Under $g \rightarrow g exp(i\theta \frac{1}{2})$, the two phases coming from $\psi^{(\pm 1)}$ and $D(\frac{1}{2}) (g) \pm >$ cancel exactly so that their product is a function over $\mathbb{S}^2$. 

2.3 Fuzzy Monopoles

Towards fuzzification one rewrites the winding number \( \langle 9 \rangle \) in the form

\[
\pm 1 = -\frac{1}{4\pi} \int d(\cos \theta) \wedge d\phi \; \text{Tr} \; \gamma \mathcal{P}^{(\pm 1)} [\mathcal{D}, \mathcal{P}^{(\pm 1)}] [\mathcal{D}, \mathcal{P}^{(\pm 1)}] (\vec{n})
\]

\[= -Tr_{\omega} \left( \frac{1}{|D|^2} \gamma \mathcal{P}^{(\pm 1)} [\mathcal{D}, \mathcal{P}^{(\pm 1)}] [\mathcal{D}, \mathcal{P}^{(\pm 1)}] \right). \tag{13}
\]

The first line is trivial to show starting from \( \langle 9 \rangle \), whereas the second line is essentially Connes trace theorem \( \langle 7 \rangle \). \( |D| \) = positive square root of \( \mathcal{D}^\dagger \mathcal{D} \) while \( Tr_{\omega} \) is the Dixmier trace \( \langle 7, 9, 10 \rangle \). In the fuzzy setting, this Dixmier trace will be replaced by the ordinary trace because the algebra of functions on fuzzy \( S_F^2 \) is finite dimensional.

\( \mathcal{D} \) in \( \langle 13 \rangle \) is either \( \mathcal{D}_{2g} \) or \( \mathcal{D}_{2w} \) which are given in equation \( \langle 8 \rangle \). They both give the same answer \( \pm 1 \). The fuzzy analogues of \( \mathcal{D}_{2g} \) and \( \mathcal{D}_{2w} \) are respectively \( D_{2g} \) and \( D_{2w} \) given by equation \( \langle 4 \rangle \). These latter operators were shown to be different and therefore one has to decide which one should we take as our fuzzy Dirac operator. \( D_{2g} \) does not admit as it stands a chirality operator and therefore its use in the computation of winding numbers requires more care which is done in \( \langle 1, 6 \rangle \). \( D_{2w} \) admits the fuzzy chirality operator \( \langle 8 \rangle \) which will be used instead of the continuum chirality \( \gamma = \vec{\sigma}.\vec{n} \). However \( D_{2w} \) has a zero eigenvalue for \( j = 2l + \frac{1}{2} \) so it must be regularized for its inverse in \( \langle 13 \rangle \) to make sense. This will be understood but not done explicitly in this paper, a careful treatment is given in \( \langle 1, 5 \rangle \).

Finally the projector \( \mathcal{P}^{(\pm 1)} \) will be replaced by a fuzzy projector \( p^{(\pm 1)} \) which we will now find. We proceed like we did in finding the chirality operator \( \Gamma \), we replace \( \vec{n} \) in \( \langle 8 \rangle \) by \( \vec{n}^F = \vec{L} / \sqrt{l(l+1)} \) and insist on the result to have the properties \( p^{(\pm 1)}_{12} = p^{(\pm 1)}_1 \) and \( p^{(\pm 1)}_{1+} = p^{(\pm 1)}_1 \). We also require this projector to commute with the chirality operator \( \Gamma \), the answer for winding number \( N = \pm 1 \) turns out to be

\[
p^{(1)} = \frac{1}{2} + \frac{1}{2l+1} \left( \vec{\tau}.\vec{L} + \frac{1}{2} \right).
\]

This can be rewritten in the following useful form

\[
p^{(1)} = \frac{\vec{K}^{(1)} + (l - \frac{1}{2})(l + \frac{1}{2})}{(l + \frac{1}{2})(l + \frac{3}{2}) - (l - \frac{1}{2})(l + \frac{1}{2})}, \tag{14}
\]

where \( \vec{K}^{(1)} = \vec{L} + \frac{\vec{\tau}}{2} \). This allows us to see immediately that \( p^{(1)} \) is the projector on the subspace with the maximum eigenvalue \( l + \frac{1}{2} \). Similarly,
the projector \( p^{(-1)} \) will correspond to the subspace with minimum eigenvalue \( l - \frac{1}{2} \), namely
\[
p^{(-1)} = \frac{\tilde{K}^{(1)2} - (l + \frac{1}{2})(l + \frac{3}{2})}{(l - \frac{1}{2})(l + \frac{1}{2}) - (l + \frac{1}{2})(l + \frac{3}{2})}.
\] (15)

By construction (14) as well as (15) have the correct continuum limit (8), and they are in the algebra \( M_2(A) \) where \( A \) is the fuzzy algebra on fuzzy \( S^2_\mathbb{F} \), i.e. \( 2(2l + 1) \times 2(2l + 1) \) matrices. Fuzzy monopoles with winding number \( \pm 1 \) are then described by the projective modules \( p^{(\pm 1)} A^2 \).

If one include spin, then \( A^2 \) should be enlarged to \( A^4 \). It is on this space that the Dirac operator \( D_{2w} \) as well as the chirality operator \( \Gamma \) are acting. In the fuzzy the left and right actions of the algebra \( A \) on \( A \) are not the same. The left action is generated by \( L_i^L \) whereas the right action is generated by \( -L_i^R \) so that we are effectively working with the algebra \( A^L \otimes A^R \). A representation \( \Pi \) of this algebra is provided by \( \Pi(\alpha) = \alpha \otimes 1_{2 \times 2} \) for any \( \alpha \in A^L \otimes A^R \). It acts on the Hilbert space \( A^4 \oplus A^4 \).

With all these considerations, one might as well think that one must naively replace \( \text{Tr} \omega \rightarrow \text{Tr}, \gamma \rightarrow \Gamma, D \rightarrow D_{2w} \) and \( P^{(\pm 1)} \rightarrow p^{(\pm 1)} \) in (13) to get its fuzzy version. This is not totally correct since the correct discrete version of (13) turns out to be
\[
c(\pm 1) = -\text{Tr} \epsilon P^{(\pm 1)} [F_{2w}, P^{(\pm 1)}] [F_{2w}, P^{(\pm 1)}],
\] (16)

with
\[
F_{2w} = \begin{pmatrix} 0 & D_{2w} \\ D_{2w} & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}.
\] (17)

and
\[
P^{(\pm 1)} = \begin{pmatrix} \frac{1 + \Gamma}{2} p^{(\pm)} & 0 \\ 0 & \frac{1 - \Gamma}{2} p^{(\pm)} \end{pmatrix}.
\] (18)

[For a complete proof see [1] or [5].] For \( p^{(+1)} \) one finds that \( c(+1) = +1 + [2(2l + 1) + 1] \) while for \( p^{(-1)} \) we find \( c(-1) = -1 + [2(2l) + 1] \). They are both wrong if compared to (13)!

The correct answer is obtained by recognizing that \( c(\pm 1) \) is nothing but the index of the operator
\[
\hat{c}^{(+)} = \frac{1 - \Gamma}{2} p^{(\pm)} D_{2w} p^{(\pm)} \left( 1 + \Gamma \right) \frac{1}{2}.
\] (19)
This index counts the number of zero modes of $\hat{f}^{(+)}$. The proof starts by remarking that, by construction, only the matrix elements $\langle p^{(+)} U_- | \hat{f}^{(+)} | p^{(+)} U_+ \rangle$ where $U_\pm = \frac{1 \pm \Gamma}{2} A^4$, exist and therefore $\hat{f}^{(+)}$ is a mapping from $\hat{V}_+ = p^{(+)} U_+$ to $\hat{V}_- = p^{(+)} U_-$. Hence $\text{Index} \hat{f}^{(+)} = \dim \hat{V}_+ - \dim \hat{V}_-.$

Since one can write the chirality operator $\Gamma$ in the form $\Gamma = \frac{1}{l + 1/2} \left[ j(j + 1) - (l + 1/2)^2 \right]$ where $j$ is the eigenvalue of $(-\vec{L}^R + \frac{\vec{\sigma}}{2})^2$, $j = l \pm 1/2$ for which $\Gamma|_{j = l \pm 1/2} = \pm 1$ defines the subspace $U_\pm$ with dimension $2(l \pm 1/2) + 1$. On the other hand, for $p^{(+)}$ which projects down to the subspace with maximum eigenvalue $k_{\text{max}} = l + \frac{1}{2}$ of the operator $\vec{K}^{(+)} = \vec{L} + \frac{\vec{\sigma}}{2}$, $\hat{V}_\pm$ has dimension $[2(l \pm 1/2) + 1][2(l + 1/2) + 1]$ and so the index is $\text{Index} \hat{f}^{(+)} = c^{(+)} = 2(2l + 2)$. This result signals the existence of zero modes of the operator $\hat{f}^{(+)}$. Indeed for $\Gamma = +1$ one must couple $l + \frac{1}{2}$ to $l + \frac{1}{2}$ and obtain $j = 2l + 1, 2l, ..., 0$, whereas for $\Gamma = -1$ we couple $l + \frac{1}{2}$ to $l - \frac{1}{2}$ and obtain $j = 2l, ..., 1$. $j$ here denotes the total angular momentum $\vec{J} = \vec{L} - \vec{L}^R + \frac{\vec{\sigma}}{2} + \frac{\vec{\tau}}{2}$. Clearly the eigenvalues $j^{(+)} = 2l + 1$ and 0 in $\hat{V}_+$ are not paired to anything. The extra piece in $c^{(+)}$ is therefore exactly equal to the number of the top zero modes, namely $2j^{(+)} + 1 = 2(2l + 1) + 1$. These modes do not exist in the continuum and therefore they are of no physical relevance and must be projected out. This can be achieved by replacing the projector $p^{(+)}$ by a corrected projector $\pi^{(+)} = p^{(+)}[1 - \pi^{(j^{(+))}}]$ where $\pi^{(j^{(+))}}$ projects out the top eigenvalue $j^{(+)}$, it can be easily written down explicitly. Putting $\pi^{(+)}$ in (16) gives exactly $c^{(+)} = +1$ which is the correct answer.

The same analysis goes for $p^{(-)}$. Indeed if we replace it by the corrected projector $\pi^{(-)} = p^{(-)}[1 - \pi^{(j^{(-))}}]$ where $\pi^{(j^{(-))}}$ projects out the top eigenvalue $j^{(-)} = 2l$, then equation (16) will give exactly $c^{(-)} = -1$ which is what we want.

### 3 Conclusion

It was shown in this article that topological quantities can be precisely and strictly defined in the discrete setting by using the methods of noncommutative geometry and fuzzy physics.

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