MINIMAL GRID DIAGRAMS OF THE PRIME ALTERNATING KNOTS WITH 12 CROSSINGS

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Abstract. In this article, we give a list of minimal grid diagrams of the 12 crossing prime alternating knots. This is a continuation of the work in [4].

1. Introduction

There are 1288 prime alternating knots with 12 crossings [5]. We wrote a series of computer programs on Maple to find their minimal grid diagrams. A grid diagram is a knot diagram with finitely many horizontal segments and the same number of vertical segments such that the vertical segments cross over the horizontal segments at all crossings. See Figure 1.

Imagine there is a vertical axis behind a grid diagram of a knot $K$ with $n$ vertical segments and $n$ horizontal segments. For each horizontal segment take the point on the axis at the same level to form a triangle and then replace the horizontal segment with the other two sides of the triangle. This results a polygonal embedding of $K$ with $3n$ edges such that each of the $n$ half planes determined by the $n$ vertical segments and the axis contains a single arc of the knot. Namely, we have an arc presentation of $K$ in $n$ arcs [2]. The minimal number of vertical segments in all grid diagrams of a knot is called the arc index. According to Theorem 1, the minimal grid diagrams of the 12 crossing knots have 14 vertical segments.

**Theorem 1** (Bae-Park [1]). If $K$ is an alternating knot, then the arc index of $K$ is the minimal crossing number plus two.

For each of the mentioned 1288 knots, we pursued the following steps:

1. Using the DT notation from [5], the regions divided by the knot diagram are described.
2. Considering the knot diagram as a planar graph, a spanning tree is generated whose contraction leads to an arc presentation of the knot.
3. The arc presentation is converted to a grid diagram in various forms such as 2d and 3d graphics, latex picture commands and a sequence of 3d coordinates of vertices as a polygonal knot.
4. Using Knotplot [6], a DT notation for the grid diagram is obtained.
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(5) Using Knotscape [7], the grid diagram is confirmed to be the same as the original knot up to taking mirror images.

2. An example

We give a detailed description of the steps (1)–(3) using the knot 12a1. The diagram in Figure 2(a) shows a minimal diagram of the knot 12a1 obtained from [5] with edge labels compatible with the DT notation [4, 8, 10, 14, 2, 16, 20, 6, 22, 12, 24, 18] in the following sense. At a crossing incident to the four edges with labels $i, i+1, j, j+1$, modulo 24, we may assume that $i$ and $j+1$ are odd and $i+1$ and $j$ are even. Then $i+1$ appears in the $(j/2+1)$-st (modulo 12) place in the DT notation. We orient the knot in the direction as the labels increase. The diagram divides the plane into 14 regions, namely, $[2, 8], [6, 14], [12, -19], [17, -22], [-21, -23, -17], [1, -8, 3], [4, -9, -1], [10, -15, -5], [-18, 23, -20, -12], [22, 18, -11, 16], [-24, 21, -16, -10, -4], [9, 5, -14, 7, -2], [15, 11, 19, 13, -6], and [20, 24, -3, -7, -13] which are named by their oriented boundary edges which are compatible with their neighbors. For example, $[20, 24, -3, -7, -13]$ is the unbounded region whose boundary is oriented clockwise. The numbers with minus signs indicate that the corresponding edges are oriented in the opposite direction of the knot. All other regions are oriented counterclockwise.

The thickened edges of Figure 2(a) form a spanning tree of the diagram. This tree is rooted at the crossing between the edges 12 and 13. From the root, it grows as the following sequence of oriented edges indicates.

$(-) - 12, -11, -18, -15, -10, -22, 23, 6, -9, -4, 2$

The knot diagram in Figure 2(b) is obtained from that of Figure 2(a) by a plane isotopy. The edges are labeled in the same manner except that the one labeled with 3 is divided into two parts. The thirteen arcs inside the circle are placed at distinct horizontal levels with heights as marked at their ends near the circle. At the root of the tree the undercrossing arc has the height of 7 and the overcrossing arc 8. As one adds the edges of the tree in the order given by $(-)$, new crossings appear one by one. At each new crossing, if a new arc undercrosses, then a lower height is
given and if overcrosses then a higher height. The heights of the arcs in the circle decide the heights of the two ends of the thirteen edges outside the circle. Suppose that the circle is the projection of the cylinder $x^2 + y^2 = 1$ onto the $xy$-plane. If we collapse the cylinder to the $z$-axis, then the endpoints of the edges outside the circle have endpoints on the $z$-axis. Except the edge labeled 3, the other twelve edges can be moved to be contained in distinct vertical half planes along the $z$-axis. We bring the center of the edge labeled 3 to the $z$-axis at a new height, highest as in the figure or lowest. Then the two parts can be placed in two new half planes. The edge labeled 3 is handled in this special way because it is located at the extension of the last edge of the spanning tree. From the point on the circle between the edges 13 and 20, we read the heights of the endpoints in pairs counterclockwise. If the two ends on an edge are not adjacent, we read them at the * marks. In this way we obtained

$$(**): \quad \begin{array}{c}
[7, 11], [6, 8], [10, 13], [9, 11], [4, 10], [12, 14], [7, 13],
[1, 12], [4, 14], [3, 9], [2, 5], [3, 6], [1, 5], [2, 8]
\end{array}$$

This sequence of pairs determines the grid diagram of Figure 1. The first vertical segment spans the interval [7, 11], the second spans [6, 8], and so on. Any cyclic permutation of $$(**)$$ gives another minimal grid diagram of the knot 12a1.

The idea behind this construction is originally due to Bae and Park [1], modified by the authors using the spanning tree [3]. The spanning tree $$\begin{array}{c}
\end{array}$$ is constructed so that the edges outside the circle have endpoints not interleaved among others except the one divided into two parts. Interested readers are encouraged to read [1] or [3].

3. Minimal grid diagrams of 12 crossing prime alternating knots
| 12a344 | 12a345 | 12a346 | 12a347 | 12a348 | 12a349 | 12a350 |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
| 12a351 | 12a352 | 12a353 | 12a354 | 12a355 | 12a356 | 12a357 |
|        |        |        |        |        |        |        |
| 12a358 | 12a359 | 12a360 | 12a361 | 12a362 | 12a363 | 12a364 |
|        |        |        |        |        |        |        |
| 12a365 | 12a366 | 12a367 | 12a368 | 12a369 | 12a370 | 12a371 |
|        |        |        |        |        |        |        |
| 12a372 | 12a373 | 12a374 | 12a375 | 12a376 | 12a377 | 12a378 |
|        |        |        |        |        |        |        |
| 12a379 | 12a380 | 12a381 | 12a382 | 12a383 | 12a384 | 12a385 |
|        |        |        |        |        |        |        |
| 12a386 | 12a387 | 12a388 | 12a389 | 12a390 | 12a391 | 12a392 |
|        |        |        |        |        |        |        |
| 12a393 | 12a394 | 12a395 | 12a396 | 12a397 | 12a398 | 12a399 |
|        |        |        |        |        |        |        |
| 12a400 | 12a401 | 12a402 | 12a403 | 12a404 | 12a405 | 12a406 |
|        |        |        |        |        |        |        |
| 12a407 | 12a408 | 12a409 | 12a410 | 12a411 | 12a412 | 12a413 |
|        |        |        |        |        |        |        |
| 12a414 | 12a415 | 12a416 | 12a417 | 12a418 | 12a419 | 12a420 |
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References

[1] Yongju Bae and Chan-Young Park, An upper bound of arc index of links, Math. Proc. Camb. Phil. Soc. 129 (2000) 491–500.
[2] Peter R. Cromwell, Embedding knots and links in an open book I: Basic properties, Topology Appl. 64 (1995) 37–58.
[3] Gyo Taek Jin and Hwa Jeong Lee, Prime knots whose arc index is smaller than the crossing number, J. Knot Theory Ramifications, 21(10) (2012), 1250103, 33 pp.
[4] Gyo Taek Jin and Hwa Jeong Lee, Minimal grid diagrams of 11 crossing prime alternating knots, J. Knot Theory Ramifications, to appear. https://doi.org/10.1142/S0218216520500765
[5] Charles Livingston and Allison H. Moore, KnotInfo: Table of Knot Invariants, https://knotinfo.math.indiana.edu, November 20, 2020.
[6] Knotplot, http://knotplot.com/
[7] Knotscape, http://www.math.utk.edu/~morwen/knotscape.html

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