Quasiconformal properties of $Q_{p,0}$ curves and Dirichlet-type curves

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Abstract

Let $\Gamma$ be a closed Jordan curve, and $f$ the conformal mapping that sends the unit disc $\mathbb{D}$ onto the interior domain of $\Gamma$. If $\log f'$ belongs to the Dirichlet space $\mathcal{D}$, we call $\Gamma$ a Weil-Petersson curve. The purpose of this note is to extend recent results, obtained by G. Cui and Ch. Bishop in the case of Weil-Petersson curves, to the case when $\log f'$ belongs to either some $Q_{p,0}$, space, for $0 < p \leq 1$, or to some weighted-Dirichlet space contained in $\mathcal{D}$. More precisely, we will characterize the quasiconformal extensions of $f$, and describe some of the geometric properties of $\Gamma$, that arise in this context.

Introduction

A closed Jordan curve $\Gamma$ is called a Weil-Petersson curve if the conformal map $f$ from the unit disc $\mathbb{D}$ onto the interior domain of $\Gamma$ satisfies that $\log f'$ belongs to the Dirichlet space $\mathcal{D}$, that is,

$$
\int_{\mathbb{D}} |(\log f')'|^2 dA(z) = \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 dA(z) < \infty.
$$

Weil-Petersson curves have drawn much attention recently, we refer the reader to the monograph by Ch. Bishop and the references therein, where numerous characterizations of these type of curves are provided in different settings, including geometric function theory and Teichmüller theory.

Motivated by some of the questions mentioned in this monograph, we consider in this note...

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conformal mappings $f$ on the unit disc such that $\log f'$ belongs to certain spaces of analytic functions closely related to the Dirichlet space, namely: Dirichlet-type spaces that we will denote by $D_{\log p}$, $p \geq 0$, where we integrate against a weight that blows up logarithmically when we approach the boundary of the unit disc, and the $Q_{p,0}$ spaces, $0 < p \leq 1$.

We say that an analytic function $g$ defined in $\mathbb{D}$ is in $D_{\log p}$, $p \geq 0$ if
\[
\int_{\mathbb{D}} |g'|^2 \left( \log \frac{1}{1-|z|} \right)^p < \infty.
\]
These spaces are contained in the Dirichlet space $D$, which corresponds to $p = 0$.

An analytic function $g$ on $\mathbb{D}$ is said to be in $Q_{p,0}$, $p > 0$ if the measure $d\mu = |g'(z)|^2 (1 - |z|^2)^p dA(z)$ is a $p$-vanishing Carleson measure on $\mathbb{D}$, that is,
\[
\lim_{|I| \to 0} \frac{1}{|I|^p} \int_{Q_I} |g'(z)|^2 (1 - |z|^2)^p dA(z) = 0,
\]
where $I$ is an arc on $T = \partial \mathbb{D}$, and $Q_I$ is the corresponding Carleson box $Q_I = \{ z \in \mathbb{D}; 1 - |I| \leq |z| < 1, z/|z| \in I \} \subset \mathbb{D}$.

If $p > 1$, the space $Q_{p,0}$ coincides with the little Bloch space $B_0$, and for $p = 1, Q_{1,0} = VMOA$, the space of analytic functions on $\mathbb{D}$ with vanishing mean oscillation. In fact, for $0 < p < 1$, $D \subset Q_{p,0} \subset VMOA \subset B_0$.

Our first results are related to quasiconformal extensions of the conformal mappings. Recall that a global homeomorphism on the plane $\rho$ is called quasiconformal if it preserves orientation, belongs to the Sobolev class $W^{1,2}_{loc}(\mathbb{C})$, and satisfies the Beltrami equation $\partial \rho - \mu \overline{\partial} \rho = 0$, where $\mu$ is a measurable function, called the complex dilatation, such that $||\mu||_{\infty} < 1$.

**Theorem 1.** Let $f : \mathbb{D} \to \Omega$ be a conformal map of the unit disc $\mathbb{D}$ onto a bounded domain $\Omega$ whose boundary is a Jordan curve $\Gamma$. For any $p \geq 0$, if $\log f' \in D_{\log p}$, then $f$ can be extended to a global quasiconformal mapping with complex dilatation $\mu$ such that
\[
\int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\mu(z)|^2}{(|z|^2 - 1)^2} \left( \log \frac{|z|}{|z| - 1} \right)^p dA(z) < \infty. \tag{1}
\]
On the other hand, if there exists a quasiconformal extension of $f$ to the whole plane with complex dilatation $\mu$ satisfying (1), and such that $||\mu||_{\infty} < 1/2$ in a neighborhood of the boundary of the unit disc, then $\log f' \in D_{\log p}$.

**Remark.** Note that if $\log f' \in D_{\log p}$, for $p > 2$, then $f'$ is Dini smooth in $\overline{\mathbb{D}}$. In particular, $\Gamma$ is a Dini-smooth curve. For the sake of completeness, we will give the proof of this remark in the next section.
In the context of $Q_{p,0}$ spaces, we obtain the following:

**Theorem 2.** Let $f : \mathbb{D} \rightarrow \Omega$ be the conformal map of the unit disc $\mathbb{D}$ onto a bounded domain $\Omega$ whose boundary is a Jordan curve $\Gamma$. If $\log f' \in Q_{p,0}$, $0 < p \leq 1$, then $d\nu = \frac{|\mu(z)|^2}{(|z|-1)^{2-p}} \, dA(z)$ is a $p$-vanishing Carleson measure in $\mathbb{C}\setminus\mathbb{D}$, that is, for any arc $I \in \mathbb{T}$,

$$
\lim_{I \to 0} \frac{1}{|I|^p} \int_{\tilde{Q}_I} \frac{|\mu(z)|^2}{(|z|-1)^{2-p}} \, dA(z) = 0, 
$$

(2)

where $\tilde{Q}_I$ is the Carleson box $\tilde{Q}_I = \{z \in \mathbb{C}\setminus\mathbb{D}; 1 < |z| \leq 1 + |I|, z/|z| \in I\} \subset \mathbb{C}\setminus\mathbb{D}$ associated to $I$.

Conversely, if there exists a quasiconformal extension of $f$ to the whole plane with complex dilatation $\mu$ satisfying (2), and such that $||\mu||_{\infty} < (1+p)/2$ in a neighborhood of the boundary of the unit disc, then $\log f' \in Q_{p,0}$.

Let us just point out that the assumptions on the $||\mu||_{\infty}$ norm are rather technical, nevertheless they are needed in our proofs. It is is very likely that the conclusions still hold without this extra assumption.

Particular instances of these results are already known, namely the Dirichlet space, that corresponds to $p = 0$ in Theorem 1, and the $VMOA$ space that corresponds to $p = 1$ in Theorem 2. The Dirichlet result is due to G. Cui ([6]), the proof involves the Schwarzian derivative of $f$, and in fact, the result is obtained without using the extra assumption $||\mu||_{\infty} < 1/2$. Although we will follow a different approach, let us mention here that characterizations of the Schwarzian derivative of $f$ when $\log f' \in Q_p$ or $Q_{p,0}$ can be found in [10] and [11]. Our proofs will follow closely the argument presented by E. Dyn’kin in ([7], sect 4) where the $VMOA$ case is established.

In order to contextualize these results, let us consider the following extension of the conformal mapping $f$, that we also denote by $f$:

$$
f(z) = f(1/\bar{z}) + f'(1/\bar{z})(z - 1/\bar{z}), \quad |z| > 1.
$$

(3)

An immediate computation shows that,

$$
\mu(z) = \frac{\partial_z f(z)}{\partial_{\bar{z}} f(z)} = -\frac{1}{z^2} \left( z - \frac{1}{\bar{z}} \right) \frac{f''(1/\bar{z})}{f'(1/\bar{z})}, \quad |z| > 1.
$$

Hence, if we define

$$
\eta(z) = (1 - |z|) \left| \frac{f''(z)}{f'(z)} \right|, \quad z \in \mathbb{D},
$$

it holds that for $|z| > 1$,

$$
|\mu(z)| = \frac{|z| + 1}{|z|^2} \eta(1/\bar{z}).
$$

(4)
In general, the mapping (3) is not homeomorphic, but Becker and Pommerenke [5] proved that (3) is indeed a quasiconformal extension of $f$ to a neighborhood of $T = \partial\mathbb{D}$ if $f(\mathbb{D})$ is a Jordan domain and $\limsup_{|z| \to 1^-} \eta(z) < 1$. Thus, in view of (4), the statements in Theorem 1 and Theorem 2 are not surprising.

The corresponding result when $\log f' \in B_0$, i.e. $\eta(z) = (1 - |z|) \frac{|f''(z)/f'(z)|}{|z|} \to 0$ as $|z| \to 1$, is that the extension (3) provides a quasiconformal extension of $f$, with complex dilatation $\mu$, such that $\mu(z) \to 0$ as $|z| \to 1 + 0$. It is known that the converse holds as well, meaning that if such a quasiconformal extension exists (not necessarily the one given in (3)), then $\log f' \in B_0$.

Given a space $X$ of analytic functions in $\mathbb{D}$, we say that a Jordan curve $\Gamma$ is a $X$-curve if the logarithmic derivative of the conformal mapping $f$ sending $\mathbb{D}$ onto the interior of $\Gamma$ is in $X$. In the second part of this note, we will study the geometric properties of the curves $\Gamma$ associated to the spaces of functions we have been considering.

Classical results [13] state that the $B_0$-curves are the so called asymptotically conformal curves, which are characterized as follows:

$$\sup_{z \in \Gamma(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1 \text{ as } |z_1 - z_2| \to 0,$$

where $\Gamma(z_1, z_2)$ denotes the subarc of $\Gamma$ joining $z_1$ and $z_2$ with smaller diameter. In the VMOA setting, it holds that $\Gamma$ is a VMOA-curve if and only if $\Gamma$ is an asymptotically smooth curve, that is for all $z_1, z_2 \in \Gamma$

$$\frac{\ell_\Gamma(z_1, z_2)}{|z_1 - z_2|} \to 1 \text{ as } |z_1 - z_2| \to 0,$$

where $\ell_\Gamma(z_1, z_2)$ is the length along the curve of $\Gamma(z_1, z_2)$. We refer the reader to ([5], [7], [12], [13]), where the history and different proofs of these results can be found.

The first attempt to geometrically characterize Dirichlet-curves, known as Weil-Petterson curves (WP), appeared in [8], where the authors state that if $\Gamma$ is a WP curve, then

$$\int_{\Gamma} \int_{\Gamma} \frac{\ell_\Gamma(z_1, z_2) - |z_1 - z_2|}{|z_1 - z_2|^3} |dz_1||dz_2| < \infty. \quad (5)$$

Although there is an error in the proof given in [8], Ch. Bishop proves in [3] that the statement is correct. Moreover, among many other results, he shows, using techniques that involve the beta numbers, that the expression in (5) characterizes WP curves.

The beta numbers were introduced by P. Jones in order to study the rectifiability properties of sets $E \subset \mathbb{R}^2$. Given a dyadic square $Q \subset \mathbb{R}^2$, the beta number $\beta_E(Q)$ associated to the
set $E$ is defined as
\[
\beta(Q) = \beta_E(Q) = \frac{1}{\text{diam}(Q)} \inf_{L} \sup \{ \text{dist}(z, L) : z \in 3Q \cap E \},
\]
where the infimum is over all lines $L$ that hit $3Q$.

Ch. Bishop proves in [3] a new version of the traveling salesman theorem, which states that for any Jordan arc with endpoints $z_1$ and $z_2$,
\[
\ell_\Gamma(z_1, z_2) - |z_1 - z_2| \simeq \sum_Q \beta^2_\Gamma(Q) \text{diam}(Q),
\]
where the sum is taken over all dyadic squares $Q \subset \mathbb{R}^2$. As a consequence, it is shown that the condition $\sum_Q \beta^2_\Gamma(Q) < \infty$, is equivalent to [3], and therefore provides a characterization of WP curves in terms of the $\beta$-numbers.

For $Q_{p,0}$ spaces $0 < p < 1$, the only result involving the geometry of the curves known to the author, is due to J. Pau and J.A. Peláez ([10], Th. 4), where they show that $\Gamma$ is a $Q_{p,0}$-curve if
\[
\int_0^1 \frac{\epsilon^2(t)}{t^{2-p}} dt < \infty,
\]
where $\epsilon(t) = \sup_{|z_1 - z_2| \leq t} \sup_{z \in \Gamma(z_1, z_2)} \left( \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} - 1 \right)^{1/2}$.

We present in the next theorem a result on the other direction. For that, we need to consider a different version of the beta numbers. Let $\Gamma$ be a Jordan curve, and $\gamma \subset \Gamma$ be an arc in $\Gamma$ with endpoints $w_1, w_2$. We define
\[
\beta(\gamma) = \frac{1}{|w_1 - w_2|} \max \{ \text{dist}(w, L) : w \in \gamma \},
\]
where $L$ is the line passing through $w_1$ and $w_2$.

**Theorem 3.** (i) If a Jordan curve $\Gamma$ is a $Q_{p,0}$-curve, $0 < p \leq 1$, then for any dyadic arc $J \subset \mathbb{T}$, we have
\[
\frac{1}{|J|^p} \sum_{I \subset J} \beta^2(\gamma_I) |I|^p \to 0 \text{ as } |J| \to 0,
\]
where the sum is over all the dyadic intervals $I$ in $\mathbb{T}$ such that $I \subset J$, and $\gamma_I = f(I)$.

(ii) If a Jordan curve $\Gamma$ is a $D_{\log_p}$-curve, $p \geq 0$, then
\[
\sum_I \beta^2(\gamma_I) \left( \log \frac{1}{|I|} \right)^p < \infty,
\]
where the sum is over all the dyadic intervals $I$ in $\mathbb{T}$.
Next, for any arc \( I \in \mathbb{T} \), with endpoints \( z_1, z_2 \), we define the quantity
\[
\Delta(I) = \frac{\ell_\Gamma(w_1, w_2) - |w_1 - w_2|}{|w_1 - w_2|}
\]
where \( w_i = f(z_i), i = 1, 2 \), that is, the image of \( z_1, z_2 \) under the conformal map \( f \) that sends the unit disc onto the interior of \( \Gamma \). Applying the previous theorem and (6), we obtain the following corollary:

**Corollary 1.** (i) If \( \Gamma \) is a \( Q_{p,0} \)-curve, \( 0 < p < 1 \), then for any dyadic arc \( J \in \mathbb{T} \)
\[
\frac{1}{|J|^p} \sum_I \Delta(I)|I|^p \rightarrow 0 \quad \text{as} \quad |J| \rightarrow 0,
\]
where the sum is over all the dyadic intervals \( I \subset J \).

(ii) Analogously, if \( \Gamma \) is a \( D_{\log p} \)-curve, \( p \geq 0 \), then
\[
\sum_I \Delta(I) \left( \log \frac{1}{|I|} \right)^p < \infty,
\]
where the sum is over all the dyadic intervals \( I \) in \( \mathbb{T} \).

Observe that in (i), we do not obtain the result for \( p = 1 \).

These expressions are the corresponding discrete versions of (5) for the spaces we are considering. Note that in this case, the sum is over dyadic arcs on \( \mathbb{T} \), or equivalently, over a dyadic partition on the curve \( \Gamma \), but with respect to harmonic measure instead of arclength. We will comment more on this at the end of section 2.

Throughout this paper, the letter \( c \) denotes a constant that may change at different occurrences. The notation \( A \lesssim B \) \( (A \gtrsim B) \) means that there is a constant \( c \) such that \( A \leq cA \) \( (A \geq cB) \). Also, as usual, we denote by \( \mathbb{T} \) the boundary of the unit disc, and by \( B(z, R) \) the ball of radius \( R \) centered at the point \( z \in \mathbb{C} \).

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1 **Quasiconformal characterizations**

*Proof of the Remark.* We begin this section by showing that if \( \log f' \in D_{\log p} \), for \( p > 2 \), then \( f' \) is Dini smooth in \( \overline{D} \).
For any point \( z \in \mathbb{D} \) with \( 1 - |z| < 1/10 \), consider the ball \( B_z = B \left( z, \frac{1}{10} (1 - |z|) \right) \). Note that if \( w \in B_z \), then \( \log \left( \frac{1}{1 - |w|} \right) > \log \left( \frac{2}{3(1 - |z|)} \right) > \frac{1}{2} \log \left( \frac{1}{1 - |z|} \right) \).

Since \( \left| \frac{f''}{f'} \right| \) is subharmonic, for any \( z \in \mathbb{D} \) such that \( 1 - |z| < 1/10 \), it holds that

\[
\left| \frac{f''(z)}{f'(z)} \right|^2 \leq \frac{1}{(1 - |z|)^2} \int_{B_z} \left| \frac{f''(w)}{f'(w)} \right|^2 \, dA(w)
\]

\[
\leq \left( \frac{\log \left( \frac{1}{1 - |z|} \right) - p}{(1 - |z|)^2} \right) \int_{B_z} \left| \frac{f''(w)}{f'(w)} \right|^2 \left( \log \frac{1}{1 - |w|} \right)^p \, dA(w).
\]

The last integral is bounded by the integral in \( \mathbb{D} \). Therefore, if \( 1 - |z| < 1/10 \),

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{(1 - |z|)} \left( \log \frac{1}{1 - |z|} \right)^{p/2}.
\]

This implies that when \( p > 2 \), the integral

\[
\int_{9/10}^1 \left| \frac{f''(r z)}{f'(r z)} \right| \, dr \leq \int_{9/10}^1 \frac{1}{(1 - r) \left( \log \left( \frac{1}{1 - r} \right) \right)^{p/2}} \, dr < \infty
\]

converges uniformly in \( z \in \mathbb{T} \), and therefore \( \log f' \) is continuous in \( \overline{\mathbb{D}} \). In particular, both \( f' \) and \( 1/f \) are continuous in \( \overline{\mathbb{D}} \). Thus, by (9) we have that \( |f''(z)| \leq \frac{1}{(1 - |z|) \left( \log \frac{1}{1 - |z|} \right)^{p/2}} \), which by well-known classical results implies that \( f' \) is Dini-continuous in \( \mathbb{T} \).

\[ \square \]

Next, let us recall the statement in Theorem [1]. We have to show the equivalence of the conditions \( \log f' \in D_{\log} \) and

\[
\int_{\mathbb{C} \setminus \mathbb{D}} \left( \frac{|\mu(z)|^2}{(|z|^2 - 1)^2} \right) \left( \log \frac{|z|}{|z| - 1} \right)^p \, dA(z) < \infty, \ p \geq 0.
\]

**Proof of Theorem 1.** Firstly, let us assume that \( \log f' \in D_{\log}, \ p \geq 0 \), that is,

\[
\int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 \left( \log \frac{1}{1 - |z|} \right)^p \, dA(z) < \infty.
\]

Since \( D_{\log} \subset B_0 \), it follows that \( \eta(z) = (1 - |z|) \left| \frac{f''(z)}{f'(z)} \right| \to 0 \) as \( |z| \to 1 - 0 \). Therefore, by the results in [3] mentioned in the Introduction, the expression in (9) defines a quasiconformal extension of \( f \) to a neighborhood \( U \) of \( \mathbb{T} \), and from [4] we deduce that (10) holds on \( U \).

In order to obtain a global quasiconformal mapping, we apply the following theorem (see
If \( f : G \to G' \) is a \( K \)-quasiconformal map and \( E \subset G \) is a compact set, then there exists a \( K' \)-quasiconformal map in the whole plane that coincides with \( f \) in \( E \), and with \( K' \) depending only on \( K, G \) and \( E \).

Setting \( E = \{ z : |z| \leq R_0 \} \), for some \( R_0 > 1 \), the above result provides a global quasiconformal extension of the conformal mapping \( f \) whose complex dilatation \( \mu \) satisfies (10).

To prove the converse, let us assume that (11) holds. Since we are studying the boundary behavior of the map, we can assume without lost of generality that the complex dilation \( \mu \) vanishes outside some neighborhood of \( T \), and that \( ||\mu||_\infty \neq 0 \).

The following estimate established by E. Dyn’kin in ([7], Th. 1) plays a fundamental role in our proof: If \( ||\mu||_\infty = k < 1 \)

\[
\left| \frac{f''(z)}{f'(z)} \right| \lesssim (1 - |z|)^{-k} \int_{1-|z|}^{\infty} \frac{\omega(z, t)}{t^{2-k}} \ dt, \quad |z| < 1,
\]

(11)

where

\[
\omega^2(z, t) = \frac{1}{\pi t^2} \int_{|w-z| \leq t} |\mu(w)|^2 \ dA(w).
\]

Let \( 0 < a < 1 \) to be determined later on. By applying the Cauchy-Schwarz inequality, we obtain

\[
\int_{1-|z|}^{\infty} \frac{\omega(z, t)}{t^{2-k}} \ dt \leq \left( \int_{1-|z|}^{\infty} \frac{\omega^2(z, t)}{t^{2a(2-k)}} \ dt \right)^{1/2} \left( \int_{1-|z|}^{\infty} \frac{1}{t^{2(1-a)(2-k)}} \ dt \right)^{1/2}.
\]

The right-hand side integral is finite, provided that \( 2(1-a)(2-k) > 1 \), i.e.

\[
a < \frac{3 - 2k}{2(2-k)},
\]

in which case

\[
\int_{1-|z|}^{\infty} \frac{1}{t^{2(1-a)(2-k)}} \ dt \simeq \frac{1}{(1 - |z|)^{2(1-a)(2-k)-1}}.
\]

These estimates, together with (11) and Fubini’s theorem, imply that

\[
\int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \left( \log \frac{1}{1 - |z|} \right)^p \ dA(z)
\]

\[
\lesssim \int_D \left( \log \frac{1}{1 - |z|} \right)^p \left( \int_{1-|z|}^{\infty} \frac{1}{t^{2a(2-k)+2}} \left( \int_{|w-z| \leq t} |\mu(w)|^2 \ dA(w) \right) \ dt \right) \ dA(z)
\]

\[
\lesssim \int_{C \setminus D} |\mu(w)|^2 \left( \int_{|w| \leq 1} \frac{1}{t^{2a(2-k)+2}} \left( \int_{B(w,t) \cap D} \left( \log \frac{1}{1 - |z|} \right)^p \ dA(z) \right) \ dt \right) \ dA(w).
\]

(12)
Since \( \mu \) has compact support, only the behaviour in a small neighborhood of \( T \) matters. For \( w \in \mathbb{C} \setminus \mathbb{D} \) and \( t > |w| - 1 \) small enough, setting \( \tilde{w} = w/|w| \), it holds that \( B(w, t) \cap \mathbb{D} \subset B(\tilde{w}, t) \cap \mathbb{D} \). Therefore, for any \( 0 < s < 1 \),

\[
\int_{B(w, t) \cap \mathbb{D}} \frac{1}{1 - |z|} \left( \log \frac{1}{1 - |z|} \right)^p dA(z) \lesssim t^{2-s}(\log(1/t))^p, \tag{13}
\]

this is because if \( Q = \{(x, y) ; |x| < l, 0 < y < l\} \) \( p \geq 0 \), \( 0 < s < 1 \), a simple calculation yields,

\[
\int_Q |y|^s(\log(1/y))^p dxdy \simeq l^{2-s}(\log(1/l))^p. \tag{14}
\]

The condition \( s = 2k + 2(1-a)(2-k) - 1 < 1 \) holds if and only if \( a > 1/(2-k) \). Since \( k < 1/2 \), we can choose \( 0 < a < 1 \) so that \( \frac{1}{2-k} < a < \frac{3-2k}{2(2-k)} \). Thus, we conclude by (12) and (13) that

\[
\int_D \left| \frac{f''(z)}{f'(z)} \right|^2 \left( \log \frac{1}{1 - |z|} \right)^p dA(z) \lesssim \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\mu(w)|^2}{(|w| - 1)^2} \left( \log \frac{1}{|w| - 1} \right)^p dA(w),
\]

which yields the conclusion, since \( \mu \) vanishes outside a small neighborhood of \( T \). \( \square \)

In order to prove Theorem 2, we need to show the equivalence of the conditions \( \log f' \in Q_{p,0} \), \( 0 < p \leq 1 \), and the dilatation coefficient \( \mu \) satisfying:

\[
\lim_{t \to 0} \frac{1}{|t|^p} \int_{Q_t} \frac{|\mu(z)|^2}{(|z| - 1)^2 - p} dA(z) = 0. \tag{14}
\]

**Proof of Theorem 2.** Since \( Q_{p,0} \subset B_0 \), we proceed as in the previous case to show that if \( \log f' \in Q_{p,0} \), \( 0 < p \leq 1 \), then the expression in (3) provides a global quasiconformal extension of the conformal mapping \( f \) with \( \mu \) as in (14).

To prove the converse, we will again apply the estimate in (11).

Writing \( t^{2-k} = t^{a(2-k)} \ t^{(1-a)(2-k)} \) for some \( 0 < a < 1 \) to be fixed later on, and applying the Cauchy-Schwarz inequality, we obtain

\[
\int_{1-|z|}^\infty \frac{\omega(z, t)}{t^{2-k}} \ dt \leq \left( \int_{1-|z|}^\infty \frac{\omega^2(z, t)}{t^{2a(2-k)}} \ dt \right)^{1/2} \left( \int_{1-|z|}^\infty \frac{1}{t^{2(1-a)(2-k)}} \ dt \right)^{1/2}. \tag{15}
\]

By choosing \( a \) so that \( 2(1-a)(2-k) > 1 \), that is, \( a < \frac{3-2k}{2(2-k)} \), we obtain

\[
\int_{1-|z|}^\infty \frac{1}{t^{2(1-a)(2-k)}} \ dt \simeq \frac{1}{(1 - |z|)^{2(1-a)(2-k)-1}}. \tag{16}
\]
Let $\tilde{Q} = \tilde{Q}_I \subset \mathbb{C} \setminus \mathbb{D}$ be the Carleson box associated to an arc $I \subset \mathbb{T}$. Denote by $Q$ the Carleson box in $\mathbb{D}$ which is symmetric to $\tilde{Q}$, that is $Q = \{re^{it} : e^{it} \in I, 1 - |I| < r \leq 1\}$.

By [11], [15], [16], and Fubini's theorem,

$$
\int_Q \frac{|f''(z)|^2}{f'(z)^2} (1 - |z|)^p \, dA(z)
\leq \int_Q \left( \int_{1 - |z|}^{1 + 2(1 - a)(2 - k) - 1 - p} \frac{1}{t^{2 + 2a(2 - k)}} \left( \int_{|w - z| \leq t} |\mu(w)|^2 \, dA(w) \right) \, dt \right) \, dA(z)
\leq \int_{\mathbb{C} \setminus \mathbb{D}} |\mu(w)|^2 \left( \int_{\{z,t : z \in Q, |w - z| \leq t\}} \frac{1}{(1 - |z|)^{2k + 2(1 - a)(2 - k) - 1 - p}} \frac{1}{t^{2 + 2a(2 - k)}} \, dt \, dA(z) \right) \, dA(w)
= \int_{\{w \notin \tilde{Q}_N\}} + \int_{\{w \notin \tilde{Q}_N\}} \mathcal{E} = I_1 + I_2,
$$
where $\tilde{Q}_n = 2^n \tilde{Q}; n \geq 1$. The letter $N \geq 2$ denotes an integer that will be fixed later on.

If $w \notin \tilde{Q}_N$, and $z \in Q$, we have $|w - z| \simeq |w - \xi_I|$, where $\xi_I$ is the middle point of $I$. Set $s = 2k + 2(1 - a)(2 - k) - 1 - p$. Provided that $s < 1$, by integrating on $t$, the inner integral in (17) can be expressed as

$$
\int_Q \left( \int_{1 - |z|}^{1 + 2a(2 - k)} \frac{1}{|w - z|^{1 - 2a(2 - k)}} \, dA(z) \right) \simeq \frac{1}{|w - \xi_I|^{1 + 2a(2 - k)}} |I|^{2 - s}.
$$

Therefore,

$$
I_2 \lesssim |I|^{2 - s} \int_{w \notin \tilde{Q}_N} \frac{|\mu(w)|^2}{|w - \xi_I|^{1 + 2a(2 - k)}} \, dA(w)
= |I|^{2 - s} \sum_{n \geq N} \int_{Q_{n+1} \setminus Q_n} \frac{|\mu(w)|^2}{|w - \xi_I|^{1 + 2a(2 - k)}} \, dA(w)
\lesssim |I|^{2 - s} \sum_{n \geq N} \frac{1}{(2^n |I|)^{1 + 2a(2 - k)}} \int_{Q_{n+1}} |\mu(w)|^2 \, dA(w)
\lesssim \frac{1}{|I|^{2 - p}} \sum_{n \geq N} \frac{1}{(2^n |I|)^{2 - p}} \int_{Q_{n+1}} |\mu(w)|^2 \, dA(w)
\lesssim \sum_{n \geq N} \frac{1}{(2^n)^{2a(2 - k) - 1 + p}} \int_{Q_{n+1}} |\mu(w)|^2 \, dA(w).
$$

Since the last integral is bounded by $c \, (\text{diam}(\tilde{Q}_{n+1}))^p \simeq (2^n |I|)^p$, by imposing the condition $2a(2 - k) - 1 > 0$, we get

$$
I_2 \lesssim |I|^p \sum_{n \geq N} \frac{1}{(2^n)^{2a(2 - k) - 1}} \leq \epsilon |I|^p,
$$
for any $\epsilon > 0$ as close to 0 as we wish, by choosing some fixed $N$ big enough.
On the other hand, if \( w \in \tilde{Q}_N \), the integral \( I_1 \) in (18) can be written as

\[
I_1 \simeq \int_{\tilde{Q}_N} |\mu(w)|^2 \left( \int_{|w|-1}^{\infty} \frac{1}{t^{2+2a(2-k)}} \left( \int_{B(w,t) \cap Q} \frac{1}{(1-|z|)^s} dA(z) \right) dt \right) dA(w).
\]

Since \( s < 1 \),

\[
\int_{B(w,t) \cap Q} \frac{1}{(1-|z|)^s} dA(z) \lesssim t^{2-s}.
\]

Thus, by (21) and (22), we conclude that

\[
I_1 \lesssim \int_{\tilde{Q}_N} |\mu(w)|^2 \left( \int_{|w|-1}^{\infty} \frac{1}{t^{3-p}} dt \right) dw \lesssim \int_{\tilde{Q}_N} \frac{|\mu(w)|^2}{(|w|-1)^{2-p}} dA(w) = o(|I|^p),
\]

since \( \mu \) satisfies (14).

The last estimate (22), together with (19) and (18), complete the proof that \( \log f' \in Q_{p,0} \), as long as there exists \( 0 < a < 1 \) satisfying

\[
s = 2k + 2(1-a)(2-k) - 1 - p < 1, \quad a < \frac{3-2k}{2(2-k)} \quad \text{and} \quad 2a(2-k) - 1 > 0.
\]

An immediate computation shows that these conditions hold for some \( 0 < a < 1 \), provided that \( 2 - p < 3 - 2k \), or equivalently \( k < (1+p)/2 \).

\[\square\]

## 2 Geometric properties

Before proceeding to the proof of Theorem 3, let us introduce some notation. For each dyadic interval \( I \subset \mathbb{T} \), we denote by \( Q_I \subset \mathbb{D} \) its corresponding Carleson box and by \( W_I = T(Q_I) \) the top of \( Q_I \), that is \( W_I = \{ re^{it} \in Q_I; e^{it} \in I, |I|/2 < r < |I| \} \). Also denote by \( \mathcal{W} \) the collection of all \( W_I \) associated to the dyadic intervals \( I \subset \mathbb{T} \). For any \( W \in \mathcal{W} \), we define \( \eta(W) = \sup_{z \in W} \frac{|f'(z)|}{2(1-|z|)} \).

Let \( z_I \in W_I = T(Q_I) \in \mathcal{W} \), and suppose \( z \in 2Q_I \). Let \( W_I = W_0, ..., W_N \) be the list of the tops of Carleson boxes hit by the hyperbolic geodesic from \( z_I \) to \( z \). We will call this chain of \( W \)'s associated to \( z \), \( C(z) \). The following result due to Ch. Bishop, that we state here as a proposition, will be our main tool. The proof can be found in ([3], sect 4) under the normalization \( f'(z_I) = 1 \). Nevertheless, for the sake of completeness, we will include the details in the Appendix.

**Proposition 1.** Let \( \Gamma \) be an asymptotically conformal curve, and \( f \) the conformal map from \( \mathbb{D} \) onto the interior of \( \Gamma \). Let \( z_I \in W_I = T(Q_I) \in \mathcal{W} \), and suppose \( z \in 2Q_I \). Then

\[
f(z) - f(z_I) = f'(z_I)(z - z_I) + O \left( \text{diam}(f(W_I)) \sum_{W \in C(z)} \eta(W) \left( \frac{\text{diam}(W)}{\text{diam}(W_I)} \right)^{\alpha} \right).
\]

(23)
for some $0 < \alpha < 1$. Moreover, $\alpha$ can be as close to 1 as we need by choosing $\text{diam}(W_I)$ small enough.

**Proof of Theorem 3.**

(i) Since $\Gamma$ is a $Q_p$ curve, $0 < p \leq 1$, for a dyadic arc $I \subset \mathbb{T}$, it holds that

$$\lim_{|I| \to 0} \frac{1}{|I|^p} \int_{Q_I} \left| \frac{f''(z)}{f'(z)} \right|^2 (1 - |z|)^p dA(z) = 0. \quad (24)$$

On the other hand, note that for any $z \in W \in \mathcal{W}$, $1 - |z| \simeq \text{diam}(W)$. Moreover, due to the subharmonicity of the function $|f''/f'|$, we obtain

$$\eta^2(W) \lesssim \frac{1}{(\text{diam}(W))^2} \int_{cW} \left| \frac{f''(z)}{f'(z)} \right|^2 (1 - |z|)^2 dA(z) \simeq \int_{cW} \left| \frac{f''(z)}{f'(z)} \right|^2 dA(z),$$

for some constant $c > 1$.

Therefore, for any dyadic box $Q$,

$$\sum_{W \in Q} \eta(W)^2 \text{diam}(W)^p \lesssim \sum_{W \in Q} \int_{cW} \left| \frac{f''(z)}{f'(z)} \right|^2 dA(z) \text{diam}(W)^p \lesssim \int_{cQ} \left| \frac{f''(z)}{f'(z)} \right|^2 (1 - |z|)^p dA(z).$$

The expression (24) can then be written as

$$\frac{1}{|I|^p} \sum_{W \subset Q_I} \eta(W)^2 \text{diam}(W)^p \to 0 \text{ as } |I| \to 0. \quad (25)$$

The equation in (23) shows that $f$ is almost linear on the segment $2I$, meaning that the arc of $\Gamma$, $\gamma_I = f(I)$, deviates from a straight line at most

$$\alpha(W_I) = \sup_{z \in 2I} \left( \frac{\text{diam}(f(W_I))}{\text{diam}(f(z))} \sum_{W \in C(z)} \eta(W) \left( \frac{\text{diam}(W)}{\text{diam}(W_I)} \right)^{\alpha} \right).$$

Note that by the circular distortion theorem for quasiconformal mappings, and by Koebe’s theorem, $\text{diam}(f(W_I)) \simeq |I||f'(z_I)| \simeq \text{diam}(f(I))$. In particular, for each dyadic interval $I \subset \mathbb{T}$, and $W_I = T(Q_I) \in \mathcal{W}$, we can choose a boundary point $z \in 2I$ so that, the corresponding sum along its chain $C(z)$ is comparable to $\alpha(W_I)$. We call this chain $C(W_I)$.

Since $\gamma_I = f(I)$ is a quasiarc, $\text{diam}(f(I)) \simeq |w_1 - w_2|$, where $w_1, w_2$ are the endpoints of $\gamma_I$. Thus, by the definition of $\beta(\gamma_I)$ given in (7) we get

$$\beta(\gamma_I) \lesssim \frac{\alpha(W_I)}{\text{diam}(\gamma_I)} \lesssim \sum_{W' \in C(W_I)} \eta(W') \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^{\alpha}. \quad (26)$$
Choose $1/2 < \alpha < 1$ and $s > 0$ so that $p < s < 2\alpha$. By the Cauchy-Schwarz inequality,

$$
\beta(\gamma_I) \leq \left( \sum_{W' \in C(W_I)} \eta^2(W') \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^s \right)^{1/2} \left( \sum_{W' \in C(W_I)} \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^{2\alpha-s} \right)^{1/2}.
$$

Note that the last sum is a geometric sum, which is finite since $2\alpha - s > 0$. Therefore, given a dyadic arc $J \subset T$ of small enough diameter,

$$
\frac{1}{|J|^p} \sum_{I: I \text{ dyadic} \subset J} \beta^2(\gamma_I) |I|^p \leq \frac{1}{|J|^p} \sum_{I: I \text{ dyadic} \subset J} \sum_{W' \in C(W_I)} \eta^2(W') \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^s |I|^p
\leq \frac{1}{|J|^p} \sum_{W' \in C(W_I)} \eta^2(W') (\text{diam}(W'))^s \sum_{\{I: W' \in C(W_I)\}} \frac{1}{(\text{diam}(W_I))^{s-p}}.
\tag{27}
$$

To estimate the last sum, observe that $W' = T(Q_{I'})$ for some dyadic interval $I'$. If $I$ is such that $W' \in C(W_I)$, then $I' \subset cI$ for some constant $c > 0$. Since $s > p$, we get that the last sum, which is a geometric sum, can be bounded by a multiple of $\frac{1}{(\text{diam}(W_I))^{s-p}}$. Thus, by (27) and (25), we deduce that

$$
\frac{1}{|J|^p} \sum_{I: I \text{ dyadic} \subset J} \beta^2(\gamma_I) |I|^p \leq \frac{1}{|J|^p} \sum_{W' \in C(W_I)} \eta^2(W') (\text{diam}(W'))^s \rightarrow 0 \text{ as } |J| \rightarrow 0,
$$

as we wanted to show.

(ii) The proof follows a similar argument as in (i). The condition for $D_{\log_p}$-curves, $p \geq 0$, can be written now as:

$$
\sum_{W \subset \mathbb{D}} \eta(W)^2 (\log (1/ \text{diam}(W)))^p < \infty.
\tag{28}
$$

Moreover, applying the Cauchy-Schwarz inequality in (26), we get

$$
\beta(\gamma_I) \leq \left( \sum_{W' \in C(W_I)} \eta^2(W') \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^\alpha \right)^{1/2} \left( \sum_{W' \in C(W_I)} \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^\alpha \right)^{1/2}.
$$

Since $\alpha > 0$, the last sum converges. Thus,

$$
\sum_{I: I \text{ dyadic} \subset T} \beta^2(\gamma_I) (\log(1/|I|))^p \leq \sum_{I: I \text{ dyadic} \subset T} \sum_{W' \in C(W_I)} \eta^2(W') \left( \frac{\text{diam}(W')}{\text{diam}(W_I)} \right)^\alpha (\log(1/|I|))^p
\leq \sum_{W \subset \mathbb{D}} \eta^2(W') (\text{diam}(W'))^\alpha \sum_{\{I: W' \in C(W_I)\}} \frac{1}{(\text{diam}(W_I))^{\alpha}} (\log(1/|I|))^p.
$$
Since 0 < \alpha < 1, p \geq 0, and diam(W_I) \simeq |I|, the last sum is bounded by a multiple of 
\frac{1}{(\text{diam}(W'))^p}(\log(1/\text{diam}(W')))^p, which yields 
\[
\sum_{I: I \text{ dyadic} \subset T} \beta^2(\gamma_I) (\log(1/|I|))^p \lesssim \sum_{W' \subset \mathbb{D}} \eta^2(W')(\log(1/\text{diam}(W')))^p < \infty.
\]
because of \[(28)\].

**Remark 1:** Since \( f \) is quasiconformal in \( \mathbb{C} \), there is \( 0 < \alpha < 1 \), so that 
\[ \left( \frac{|I|}{\text{diam}(f(I))} \right)^{1/\alpha} \leq \frac{\text{diam}(f(I))}{\text{diam}(f(z))} \right)^\alpha \] (see \[4\], corollary 3.10.4).
Therefore, we obtain that 
\[ \sum_{I: I \text{ dyadic} \subset T} \beta^2(\gamma_I) (\log(1/|I|))^p < \infty \] is equivalent to 
\[
\sum_{I: I \text{ dyadic} \subset T} \beta^2(\gamma_I) \left( \log \left( \frac{1}{\text{diam}(\gamma_I)} \right) \right)^p < \infty.
\] \( \quad \) (29)

\[ \square \]

**Remark 2:** We will assume that the reader is familiar with the concept of multi-resolution families, and the relations between the different definitions of beta-numbers (see, Appendix B in \[4\] for an excellent exposition).
It is is obvious that the results in Theorem 3 still hold if we consider any translation of the dyadic intervals, in particular the 1/3-translation. With this "1/3-trick", one generates a multi-resolution family on \( T \) which is sent by the conformal map to a family of arcs in \( \Gamma \).
Let us denote this family by \( \tilde{F} \). It turns out that \( \tilde{F} \) is a MR family when \( \Gamma \) is a chord-arc curve. Applying now Lemma B.1 in \[4\] to the expression \( \tilde{2} \), we obtain that:

If \( \log f' \in D_{\log p}, p \geq 0 \), then 
\[
\sum_{Q} \beta_T^2(Q) \left( \log \left( \frac{1}{\text{diam}(Q)} \right) \right)^p < \infty,
\]
where the sum is taken over all dyadic squares \( Q \subset \mathbb{R}^2 \).

**Proof of Corollary 1.** (i) In \[4\], Ch. Bishop proves and discusses equivalent formulations of the traveling salesman theorem expressed in terms of different multi-resolution families, (see, Appendix B in \[4\]). In our setting, given a dyadic arc \( I \subset T \) with endpoints \( z_1, z_2 \) and \( w_i = f(z_i), i = 1, 2, \) the traveling salesman theorem states that 
\[
\ell_T(w_1, w_2) - |w_1 - w_2| \simeq \sum_{I' \subset F_I} \beta^2(\gamma_{I'}) \text{diam}(\gamma_{I'}),
\]
where $\gamma_I = f(I)$, and $\mathcal{F}_I$ is the MR family generated by the $1/3$ translates of the dyadic intervals contained in $I$.

Recall that by definition,

$$\Delta(I) = \frac{\ell_I(w_1, w_2) - |w_1 - w_2|}{|w_1 - w_2|}.$$ 

Since $\text{diam}(f(I)) \simeq |w_1 - w_2|$, we get that

$$1 \frac{1}{|J|^p} \sum_{\text{dyadic} \subset J} \Delta(I) |I|^p \lesssim \frac{1}{|J|^p} \sum_{\text{dyadic} \subset J} \sum_{I' \subset \mathcal{F}_I} \beta^2(\gamma_I') \frac{\text{diam}(f(I'))}{\text{diam}(f(I))} |I|^p.$$ 

Since the curve $\Gamma$ is asymptotically conformal, there is $0 < \alpha < 1$ as close to 1 as we wish, such that for any arc $I' \subset I \subset T$

$$\frac{\text{diam}(f(I'))}{\text{diam}(f(I))} \lesssim \left( \frac{|I'|}{|I|} \right)^\alpha.$$ 

Therefore,

$$1 \frac{1}{|J|^p} \sum_{\text{dyadic} \subset J} \Delta(I) |I|^p \lesssim \frac{1}{|J|^p} \sum_{I' \subset \mathcal{F}_I} \beta^2(\gamma_I') |I'|^\alpha \sum_{I' \subset \mathcal{F}_I} |I|^{p-\alpha}.$$ 

The last sum is a geometric sum. Choosing $1 > \alpha > p$, this geometric sum is bounded by a multiple of $|I'|^{p-\alpha}$. Hence, applying Theorem 3, we get

$$1 \frac{1}{|J|^p} \sum_{\text{dyadic} \subset J} \Delta(I) |I|^p \lesssim \frac{1}{|J|^p} \sum_{I' \subset \mathcal{F}_I} \beta^2(\gamma_I') |I'|^p \rightarrow 0 \text{ as } |J| \rightarrow 0.$$ 

(ii) Proceeding as in the previous case, we get

$$\sum_{\text{dyadic} \subset T} \Delta(I)(\log(1/|I|))^p \lesssim \sum_{\text{dyadic} \subset T} \sum_{I' \subset \mathcal{F}_I} \beta^2(\gamma_I') \left( \frac{|I'|}{|I|} \right)^\alpha (\log(1/|I|))^p 
\lesssim \sum_{I' \subset \mathcal{F}_I} \beta^2(\gamma_I') |I'|^\alpha \sum_{I' \subset \mathcal{F}_I} \frac{1}{|I|^{\alpha}} (\log(1/|I|))^p 
\lesssim \sum_{I' \text{dyadic} \subset T} \beta^2(\gamma_I')(\log(1/|I'|))^p < \infty,$$ 

because of Theorem 3 (ii).

\begin{flushright}
\textcircled{□}
\end{flushright}

### 3 Appendix

In this section we include a sketch of the proof of Proposition 1. We will follow the argument in [3].
Suppose \( z_I \in W_I = T(Q_I) \in \mathcal{W} \), and suppose \( z \in 2Q_I \). Let \( W_I = W_0, ..., W_N \) be the list of the tops of Carleson boxes hit by the hyperbolic geodesic from \( z_I \) to \( z \). Then

\[
| \log f'(z) - \log f'(z_I) | = \left| \int_{z_I}^{z} \frac{f''(\xi)}{f'(\xi)} \, d\xi \right| \lesssim \sum_{n=0}^{N} \eta(W_n).
\]

Thus,

\[
\left| \frac{f'(z)}{f'(z_I)} - 1 \right| = \left| e^{\log f'(z) - \log f'(z_I)} - 1 \right| \leq e^{\log f'(z) - \log f'(z_I)} - 1 \leq e^{\sum_{n=0}^{N} \eta(W_n)} - 1. \tag{30}
\]

Let us assume first that \( \sum_{n=0}^{N} \eta(W_n) \leq 1 \), and let \( 0 < \alpha < 1 \). Since \( e^x - 1 \lesssim x \) for \( x \in [0, 1] \), we obtain by (30) that

\[
|f(z) - f(z_I) - (z - z_I)f'(z_I)| = \left| \int_{z_I}^{z} f'(\xi) - f'(z_I) \, d\xi \right| \leq \sum_{k} \int_{z_k}^{z_{k+1}} |f'(\xi) - f'(z_I)| \, |d\xi|
\]

\[
\lesssim |f'(z_I)| \sum_{k} \text{diam}(W_k) \sum_{n=0}^{k} \eta(W_n) \lesssim |f'(z_I)| \sum_{n} \eta(W_n) \sum_{k=0}^{k=n} \text{diam}(W_k)
\]

\[
\lesssim |f'(z_I)| \sum_{n} \eta(W_n) \text{diam}(W_n) \lesssim |f'(z_I)| \text{diam}(W_0) \sum_{n} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^{\alpha}
\]

\[
\approx \text{diam}(f(W_I)) \sum_{W \in C(z)} \eta(W) \left( \frac{\text{diam}(W)}{\text{diam}(W_I)} \right)^{\alpha}. \tag{31}
\]

We have used that fact that the \( \text{diam}(W_k) \) decrease geometrically, and hence the sum is dominated by a multiple of its largest term.

Assume next that \( \sum_{n=0}^{N} \eta(W_n) > 1 \). We can then choose \( M \) along the hyperbolic geodesic such that \( \sum_{n=0}^{M} \eta(W_n) \leq 1 \) and \( \sum_{n=0}^{M+1} \eta(W_n) \geq 1 \). On one hand, by (31),

\[
|f(z_M) - f(z_I) - (z_M - z_I)f'(z_I)| \lesssim \text{diam}(f(W_I)) \sum_{n=0}^{M} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^{\alpha}. \tag{32}
\]

On the other hand, since \( \log f' \in B_0 \), for all sufficiently small Carleson boxes, the conformal map \( f \) restricted to such a Carleson box \( 2Q_I \) has a \( K \)-quasiconformal extension to the reflection across \( 2I \), with \( K \) close to 1. By Mori’s theorem we have that

\[
|f(z) - f(z_M)| \lesssim C |f(z) - f(z_I)| \left| \frac{z - z_M}{z - z_I} \right|^\alpha.
\]

for some \( C = C(\alpha) < \infty \), where we may take \( \alpha < 1 \) as close to 1 as we wish.
Therefore,

\[ |f(z) - f(z_M)| \lesssim \text{diam}(f(W_0)) \left( \frac{\text{diam}(W_M)}{\text{diam}(W_0)} \right)^\alpha \]

\[ \leq \text{diam}(f(W_0)) \left( \sum_{n=0}^{M} \eta(W_n) \right) \left( \frac{\text{diam}(W_M)}{\text{diam}(W_0)} \right)^\alpha \]

\[ \lesssim \text{diam}(f(W_0)) \sum_{n=0}^{M} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^\alpha, \tag{33} \]

since \( \text{diam}(W_n) \geq \text{diam}(W_M) \) for all \( n \leq M \). Similarly,

\[ |z - z_M| = |z - z_I| \left| \frac{z - z_M}{z - z_I} \right| \leq |z - z_I| \left| \frac{z - z_M}{z - z_I} \right|^\alpha \]

\[ \leq \text{diam}(W_0) \sum_{n=0}^{M} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^\alpha. \]

By Koebe’s theorem,

\[ |(z - z_M)f'(z_I)| \lesssim \text{diam}(f(W_0)) \sum_{n=0}^{M} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^\alpha. \tag{34} \]

The result follows from (32), (33) and (34), since

\[ |f(z) - f(z_I) - (z - z_I)f'(z_I)| \]

\[ \leq |f(z) - f(z_M) - (z - z_M)f'(z_I)| + |f(z_M) - f(z_I) - (z_M - z_I)f'(z_I)| \]

\[ \leq |f(z) - f(z_M)| + |(z - z_M)f'(z_I)| + |f(z_M) - f(z_I) - (z_M - z_I)f'(z_I)| \]

\[ \lesssim \text{diam}(f(W_0)) \sum_{n=0}^{M} \eta(W_n) \left( \frac{\text{diam}(W_n)}{\text{diam}(W_0)} \right)^\alpha \]

\[ \lesssim \text{diam}(f(W_I)) \sum_{W \in C(z)} \eta(W) \left( \frac{\text{diam}(W)}{\text{diam}(W_I)} \right)^\alpha. \]

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