On $(\lambda, \mu, \gamma)$-derivations of BiHom-Lie algebras

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Abstract

In this paper, we generalize the results about generalized derivations of Lie algebras to the case of BiHom-Lie algebras. In particular we give the classification of generalized derivations of Heisenberg BiHom-Lie algebras. The definition of the generalized derivation depends on some parameters $(\lambda, \mu, \gamma) \in \mathbb{C}^3$. In particular for $(\lambda, \mu, \gamma) = (1, 1, 1)$, we obtain classical concept of derivation of BiHom-Lie algebra and for $(\lambda, \mu, \gamma) = (1, 1, 0)$ we obtain the centroid of BiHom-Lie algebra. We give classifications of 2-dimensional BiHom-Lie algebra, centroids and derivations of 2-dimensional BiHom-Lie algebras.

1 Introduction

The investigations of various quantum deformations or $q$-deformations of Lie algebras began a period of rapid expansion in 1980’s stimulated by introduction of quantum groups motivated by applications to the quantum Yang-Baxter equation, quantum inverse scattering methods and constructions of the quantum deformations of universal enveloping algebras of semi-simple Lie algebras. Various $q$-deformed Lie algebras have appeared in physical contexts such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory in the context of deformations of infinite-dimensional algebras, primarily the Heisenberg algebras, oscillator algebras and Witt and Virasoro algebras. In [37, 40, 43, 44, 46, 58, 59, 75, 77], it was in particular discovered that in these $q$-deformations of Witt and Visaroro algebras and some related algebras, some interesting $q$-deformations of Jacobi identities, extending Jacobi identity for Lie algebras, are satisfied. This has been one of the initial motivations for

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the development of general quasi-deformations and discretizations of Lie algebras of vector fields using more general $\sigma$-derivations (twisted derivations) in \[56\].

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov \[56\], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general $\sigma$-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the $q$-deformed Jacobi identities observed for the $q$-deformed algebras in physics, along with $q$-deformed versions of homological algebra and discrete modifications of differential calculi. Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras and more general quasi-Lie algebras and color quasi-Lie algebras where introduced first in \[70,71,98\]. Quasi-Lie algebras and color quasi-Lie algebras encompass within the same algebraic framework the quasi-deformations and discretizations of Lie algebras of vector fields by $\sigma$-derivations obeying twisted Leibniz rule, and the well-known generalizations of Lie algebras such as color Lie algebras, the natural generalizations of Lie algebras and Lie superalgebras. In quasi-Lie algebras, the skew-symmetry and the Jacobi identity are twisted by deforming twisting linear maps, with the Jacobi identity in quasi-Lie and quasi-Hom-Lie algebras in general containing six twisted triple bracket terms. In Hom-Lie algebras, the bilinear product satisfies the non-twisted skew-symmetry property as in Lie algebras, and the Hom-Lie algebras Jacobi identity has three terms twisted by a single linear map, reducing to the Lie algebras Jacobi identity when the twisting linear map is the identity map. Hom-Lie admissible algebras have been considered first in \[83\], where in particular the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras. Since the pioneering works \[57,89,72,84\], Hom-algebra structures expanded into a popular area with increasing number of publications in various directions. Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, Hom-algebra extensions of cohomological structures and representations, formal deformations of Hom-associative and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-Hopf algebras \[10,34,48,69,73,84,86,94,95,102,104\]. Hom-Lie algebras, Hom-Lie superalgebras and color Hom-Lie algebras and their $n$-ary generalizations have been further investigated in various aspects for example in \[1,7,10,23,26,29,32,34,36,55,65,67,68,81,83,91,92,94,100,102,106,108\]. In \[33\], Hom-algebras has been considered from a category theory point of view, constructing a category on which algebras would be Hom-algebras. A generalization of this approach led to the discovery of BiHom-algebras in \[54\], called BiHom-algebras because the defining identities are twisted by two morphisms instead of only one for Hom-algebras. BiHom-Frobenius algebras and double constructions have been investigated in \[57\].

Derivations and generalized derivations of different algebraic structures are an important subject of study in algebra and diverse areas. They appear in many fields of Mathematics and Physics. In particular, they appear in representation theory and co-
homology theory among other areas. They have various applications relating algebra to geometry and allow the construction of new algebraic structures. There are many generalizations of derivations, for example, Leibniz derivations and $\delta$-derivations of prime Lie and Malcev algebras and related $n$-ary algebras structures [43, 49, 52]. The properties and structure of generalized derivations algebras of a Lie algebra and their subalgebras and quasi-derivation algebras were systematically studied in [74], where it was proved for example that the quasi-derivation algebra of a Lie algebra can be embedded into the derivation algebra of a larger Lie algebra. Derivations and generalized derivations of $n$-ary algebras were considered in [90, 101] and it was demonstrated substantial differences in structures and properties of derivations on Lie algebras and on $n$-ary Lie algebras for $n > 2$. Generalized derivations of Lie superalgebras and Hom-Leibniz algebras have been considered in [107, 111]. Generalized derivations of Lie color algebras and $n$-ary (color) algebras have been studied in [44, 60, 62, 64]. Generalized derivations of Lie triple systems have been considered in [109]. Generalized derivations of various kinds can be viewed as a generalization of $\delta$-derivation. Quasi-Hom-Lie and Hom-Lie structures for $\sigma$-derivations and $(\sigma, \tau)$-derivations have been considered in [43, 50]. Graded $q$-differential algebra and applications to semi-commutative Galois Extensions and Reduced Quantum Plane and $q$-connection was studied in [110]. Generalized $N$-complexes coming from twisted derivations where considered in [73].

Generalizations of derivations in connection with extensions and enveloping algebras of Hom-Lie color algebras and Hom-Lie superalgebras have been considered in [13, 43, 28]. Generalized derivations of multiplicative $n$-ary Hom-$\Omega$ color algebras have been studied in [31]. Derivations, $L$-modules, $L$-comodules and Hom-Lie quasi-bialgebras have been considered in [20, 25]. In [66], constructions of $n$-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been investigated. Generalized Derivations of $n$-BiHom-Lie algebras have been studied in [30]. Color Hom-algebra structures associated to Rota-Baxter operators have been considered in context of Hom-dendriform color algebras in [27]. Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $O$-operator systems and their connections with (tri)dendriform systems and pre-Lie algebras have been considered in [78-80]. Generalisations of derivations are important for Hom-Gerstenhaber algebras, Hom-Lie algebroids and Hom-Lie-Rinehart algebras and Hom-Poisson homology [88].

It is well known that for a derivation $d$ of Lie algebra $L$, is just an endomorphisms on $L$ such that

$$d ([x, y]) = [d(x), y] + [x, d(y)] \quad (1)$$

for all $x, y \in L$. There were several non-equivalent ways generalizing this definition, for example:

1) The mapping $d \in \text{End}(L)$ is called a generalized derivation of $L$ if there exist elements $d', d'' \in \text{End}(L)$ such that,

$$[d(x), y] + [x, d'(y)] = d'' ([x, y]) \quad (2)$$

3
for all \( x, y \in L \), and we call \( d \in \text{End}(L) \) a quasiderivation of \( L \) if there exists \( d' \in \text{End}(L) \) such that

\[
[d(x), y] + [x, d(y)] = d'([x, y]).
\]

(3)

The centroid of \( L \) denoted as \( \Gamma(L) \) is defined by

\[
\Gamma(L) = \{ d \in \text{End}(L) \mid d([x, y]) = [d(x), y] = [x, d(y)] \}, \forall x, y \in L \}.
\]

(4)

(see for example [53]).

2) Given an arbitrary \( \delta \in \mathbb{K} \), a \( \delta \)-derivation of a Lie algebra \( L \) is defined to be a \( \mathbb{K} \)-linear mapping \( d : L \rightarrow L \) satisfying the identity

\[
d([x, y]) = \delta [d(x), y] + \delta [x, d(y)]
\]

(5)

(see for example [50]). Observe that, any linear mapping in the centroid \( \Gamma(L) \) is a \( \frac{1}{2} \)-derivation of \( L \).

3) We call a linear operator \( d \in \text{End}(L) \) an \( (\alpha, \beta, \gamma) \)-derivation of \( L \) if there exist \( \alpha, \beta, \gamma \in \mathbb{K} \) such that for all \( x, y \in L \)

\[
\alpha d([x, y]) = \beta [d(x), y] + \gamma [x, d(y)].
\]

(6)

(See for example [89]). Observe that, any linear mapping in the centroid \( \Gamma(L) \) is a \( (1, 1, 0) \)-derivation of \( L \).

In [94], the notion of \( \alpha^k \)-derivation of Hom-Lie algebra, a generalization of derivation of Lie algebras [1], is considered. In [110] the authors extend the definition of type [1] of a generalized derivation of Lie algebras to Hom-Lie algebras. The definition of type [1] is extended to the BiHom-Lie case in [2]. In this article, we aim to discuss the version of generalized derivations of BiHom-Lie algebras.

The paper is organized as follows. In Section 2, we recall some basic definitions and facts needed later for considerations and results in this article. In Section 3, we introduce \( (\lambda, \mu, \gamma) \)-\( \alpha^k \beta^l \)-derivations and show their pertinent properties. Also, we classify the possible values of \( \lambda, \mu, \gamma \in \mathbb{C} \) for a space \( \text{Der}^{\lambda,\mu,\gamma}_{\alpha^k \beta^l}(G) \) of \( (\lambda, \mu, \gamma) \)-\( \alpha^k \beta^l \)-derivations of regular BiHom-Lie algebra \( G \). The previous classification is applied to Heisenberg BiHom-Lie algebra case. Next, we analyze each one of the following cases: \( \text{Der}^{\delta,0,0}_{\alpha^k \beta^l}(G) \) with \( \delta \in \{0, 1\} \), \( \text{Der}^{\delta,1,0}_{\alpha^k \beta^l}(G), \text{Der}^{\delta,1,1}_{\alpha^k \beta^l}(G), \text{Der}^{1,1,1}_{\alpha^k \beta^l}(G), \text{Der}^{0,1,1}_{\alpha^k \beta^l}(G) \). In Section 4, we give a method to determine whether two different 2-dimensional multiplicative BiHom-Lie algebras are isomorphic or not, and then we obtain a complete classification of 2-dimensional multiplicative BiHom-Lie algebras up to isomorphism. In Section 5, we deal with the problem of description of centroids and derivations of 2-dimensional BiHom Lie algebras. Here we provide algorithms to find centroids and derivations by using an algebra software.
2 Definitions and Preliminary Results

Definition 2.1 ([42,44]). A BiHom-Lie algebra over a field $\mathbb{K}$ is a 4-tuple $(L, [\cdot, \cdot], \alpha, \beta)$, where $L$ is a $\mathbb{K}$-linear space, $\alpha: L \to L$, $\beta: L \to L$ and $[,] : L \times L \to L$ are linear maps, satisfying the following conditions, for all $x, y, z \in L$:

\begin{align*}
\alpha \circ \beta &= \beta \circ \alpha, \quad (\text{skew-symmetry}) \\
[\beta(y), \alpha(x)] &= - [\beta(x), \alpha(y)] \\
[\beta^2(y), [\beta(z), \alpha(x)]] &= 0 \quad (\text{BiHom-Jacobi identity}).
\end{align*}

A BiHom-Lie algebra is called a multiplicative BiHom-Lie algebra if for any $x, y \in L$,

$$
\alpha([x, y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x, y]) = [\beta(x), \beta(y)].
$$

A BiHom-Lie algebra is called a regular BiHom-Lie algebra if $\alpha, \beta$ are bijective maps.

In general for $n$-dimensional case in terms of structure constants we have:

\begin{align*}
[e_i, e_j] &= \sum_{s=1}^{n} C_{ij}^s e_s, \\
\alpha(e_j) &= \sum_{s=1}^{n} a_{sj} e_s \quad \text{and} \quad \beta(e_j) = \sum_{s=1}^{n} b_{sj} e_s.
\end{align*}

Substituting (11) in the skew-symmetry identity (8) yields

$$
\sum_{1 \leq p, q \leq n} (b_{pi} a_{qj} + b_{pj} a_{qi}) C_{pq}^s = 0.
$$

Substituting (11) in the BiHom-Jacobi identity (9) yields

$$
\sum_{1 \leq p, q, s, t, s' \leq n} (b_{s'k} b_{qj} a_{sk} + b_{s'j} b_{qk} a_{si} + b_{s'k} b_{qi} a_{sj}) b_{ps} C_{qs}^l C_{pl}^{s'} = 0.
$$

Substituting (11) in the multiplicativity conditions (10) yields

\begin{align*}
\sum_{1 \leq k \leq n} C_{ij}^k a_{sk} &= \sum_{1 \leq p, q \leq n} a_{ps} a_{qj} C_{pq}^s, \\
\sum_{1 \leq k \leq n} C_{ij}^k b_{sk} &= \sum_{1 \leq p, q \leq n} b_{pi} b_{qj} C_{pq}^s
\end{align*}

for all $i, j, k \in \{1, \ldots, n\}$.

Definition 2.2. A morphism $f: (L, [\cdot, \cdot], \alpha, \beta) \to (L', [\cdot, \cdot]', \alpha', \beta')$ of BiHom-Lie algebras is a linear map $f: L \to L'$ such that $\alpha' \circ f = f \circ \alpha$, $\beta' \circ f = f \circ \beta$ and

$$
f([x, y]) = [f(x), f(y)]', \quad \forall x, y \in L.
$$

In particular, BiHom-Lie algebras $(L, [\cdot, \cdot], \alpha, \beta)$ and $(L', [\cdot, \cdot]', \alpha', \beta')$ are isomorphic if $f$ is an isomorphism map.
Let \((L, [\cdot, \cdot], \alpha, \beta)\) be \(n\)-dimensional BiHom-Lie algebra with ordered basis \((e_1, \ldots, e_n)\) and \(L'\) be \(n\)-dimensional vector spaces with ordered basis \((e'_1, \ldots, e'_n)\). Let \(f: L \to L'\) be an isomorphism map. Let \(\alpha' = f \alpha f^{-1}\) and \(\beta' = f \beta f^{-1}\). We set with respect to a basis \((e'_1, \ldots, e'_n)\):

\[
f(e_j) = \sum_{i=1}^{n} f_{ij} e'_i,
\]

Condition \((15)\) translates to the following equation

\[
\sum_{k=1}^{n} \sum_{1 \leq p, q \leq n} f_{pi} f_{iq} C_{pq}^{ij} = \sum_{1 \leq p, q \leq n} f_{pi} f_{iq} C_{pq}^{ij}, \quad i, j, s \in \{1, \ldots, n\}.
\]

Then, if the previous condition satisfied, \(L'\) is a BiHom-Lie algebra isomorphic to \(L\).

**Definition 2.3 (242).** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie algebra. A subspace \(h\) of \(L\) is called a BiHom-Lie subalgebra of \((L, [\cdot, \cdot], \alpha, \beta)\) if \(\alpha(h) \subset h, \beta(h) \subset h\) and \([h, h] \subset h\). In particular, a BiHom-Lie subalgebra \(h\) is said to be an ideal of \((L, [\cdot, \cdot], \alpha, \beta)\) if \([h, L] \subset h\) and \([L, h] \subset h\).

If \(I\) is an ideal of \((L, [\cdot, \cdot], \alpha, \beta)\), then \((L/I, [\cdot, \cdot], \alpha|_I, \beta|_I)\), where \([x, y] = [x, y], \) for all \(x, y \in L/I\) and \(\alpha, \beta: L/I \to L/I\) naturally induced by \(\alpha\) and \(\beta\), inherits a BiHom-Lie algebra structure, which is named quotient BiHom-Lie algebra.

In the following, we give some examples and applications of ideals of BiHom-Lie algebras.

**Proposition 2.4.** If \((L, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra, then \(I = \ker \alpha + \ker \beta\) is an ideal of \(L\).

**Proof.** By \((7)\) we get \(\alpha(I) \subset I\) and \(\beta(I) \subset I\). By \((10)\) we obtain \([I, L] \subset I\). \(\Box\)

**Remark 2.5.** If \(L\) is finite-dimensional and \(\alpha\) (or \(\beta\)) is diagonalizable, then there exist a subspace \(G\) such that \(L = I \oplus G\) and \((G, [\cdot, \cdot], \alpha_G|_G, \beta_G|_G)\) is a regular BiHom-Lie algebra.

**Definition 2.6.** Given a complex BiHom-Lie multiplicative algebra \(L\), the center of \(L\) is given by \(C(L) = \{x \in L \mid [x, y] = 0 \quad \forall y \in L\}\). The descending central series of a BiHom-Lie algebra \(L\) is given by the ideals

\[
L^0 = L; \quad L^k = [L, L^{k-1}], \quad k \geq 1.
\]

\(L\) is called nilpotent if \(L^n = \{0\}\) for some \(n \in \mathbb{N}\). If \(L^{n-1} \neq \{0\}\), then \(L\) is said to be \(n\)-step nilpotent BiHom-Lie algebra. The derived series of a BiHom-Lie algebra \(L\) is given by the ideals \(L^{(0)} = L, L^{(k)} = [L^{(k-1)}, L^{(k-1)}], k \geq 1\). \(L\) is called solvable if \(L^{(n)} = \{0\}\) for some \(n \in \mathbb{N}\). If \(L^{(n-1)} \neq \{0\}\), then \(L\) is said to be \(n\)-step solvable BiHom-Lie algebra.
Remark 2.7. The center \( C(L) \) of \( L \) is not necessarily an ideal of \( L \). If \( \alpha \) and \( \beta \) are surjective then \( C(L) \) is an ideal of \( L \).

Definition 2.8. Let \((L,[\cdot,\cdot],\alpha,\beta)\) be a BiHom-Lie algebra. \((L,[\cdot,\cdot],\alpha,\beta)\) is called a simple BiHom-Lie algebra if \((L,[\cdot,\cdot],\alpha,\beta)\) has no proper ideals and is not abelian. \((L,[\cdot,\cdot],\alpha,\beta)\) is called a semisimple BiHom-Lie algebra if \( L \) is a direct sum of certain ideals.

Proposition 2.9 (\([34]\)). Let \((L,[\cdot,\cdot])\) be an ordinary Lie algebra over a field \( \mathbb{K} \) and let \( \alpha, \beta : L \to L \) two commuting linear maps such that \( \alpha([a,b]) = [\alpha(a),\alpha(b)] \) and \( \beta([a,b]) = [\beta(a),\beta(b)] \), for all \( a, b \in L \). Define the linear map \([\cdot,\cdot] : L \times L \to L\), \([a,b] = [\alpha(a),\beta(b)] \), for all \( a, b \in L \). Then \( L(\alpha,\beta) := (L,[\cdot,\cdot],\alpha,\beta) \) is a BiHom-Lie algebra, called the Yau twist of \((L,[\cdot,\cdot])\).

Example 2.1 (Heisenberg BiHom-Lie algebras). Let \((X,Y,Z)\) be a basis of a Heisenberg Lie algebra \((h_1,[\cdot,\cdot])\) such that
\[
[X,Y] = Z, \quad [X,Z] = [Y,Z] = 0.
\]
Let \(\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \) be the matrix of a linear map \( h_1 \to h_1 \) and let \(\begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix} \)
the matrix of a linear map \( h_1 \to h_1 \) relative to a basis \((X,Y,Z)\) of \( h_1 \). The Yau twist \( H(\alpha,\beta) \) of \( h_1 \) is called Heisenberg BiHom-Lie algebra. The bracket of Heisenberg BiHom-Lie algebra \((h_1,[\cdot,\cdot],\alpha,\beta)\) is given by \([X,Y] = bZ\), \([Y,X] = -yZ\) and the other brother bracket are 0. For \( m > 1 \), we define Heisenberg BiHom-Lie algebra \((h_m,[\cdot,\cdot],\alpha,\beta)\) by
\[
[X_i,Y_i] = \frac{b}{y_i}Z, \quad [Y_i,X_i] = -\frac{a}{b_i}Z \quad \forall i \in \{1,\ldots,m\}
\]
and the other brackets are 0;
\[
\alpha = \text{diag}(b_1,\ldots,b_m,\frac{a}{b_1},\ldots,\frac{a}{b_m},a),
\]
\[
\beta = \text{diag}(y_1,\ldots,y_m,\frac{x}{y_1},\ldots,\frac{x}{y_m},x).
\]

Proposition 2.10 (\([34]\)). Let \((G,[\cdot,\cdot],\alpha,\beta)\) be a regular BiHom-Lie algebra. Define the bilinear map \([\cdot,\cdot] : G \times G \to G\) by
\[
[x,y] = [\alpha^{-1}(x),\beta^{-1}(y)],
\]
for all \( x, y \in G \). Then \((G,[\cdot,\cdot])\) is a Lie algebra, which we call it the induced Lie algebra of \((L,[\cdot,\cdot],\alpha,\beta)\).

Proposition 2.11 (\([34]\)). The induced Lie algebra of the multiplicative simple BiHom-Lie algebra is semisimple. There exist simple ideal \( L_1 \) and an integer \( m \neq 2 \) such that
\[
L = L_1 \oplus \alpha(L_1) \oplus \cdots \oplus \alpha^{m-1}(L_1) = L_1 \oplus \beta(L_1) \oplus \cdots \oplus \beta^{m-1}(L_1).
\]
Proposition 2.12. Any finite-dimensional multiplicative simple BiHom-Lie algebra is regular.

Proof. The statement holds since $\ker \alpha$ and $\ker \beta$ are ideals of the simple BiHom-Lie algebra $(L, [~,~], \alpha, \beta)$.

\[ \square \]

Proposition 2.13. A regular multiplicative BiHom-Lie algebra $(G, [~,~], \alpha, \beta)$ is nilpotent if and only if the induced Lie algebra $(G, [~,~])$ is nilpotent.

Proposition 2.14. A regular multiplicative BiHom-Lie algebra $(G, [~,~], \alpha, \beta)$ is solvable if and only if the induced Lie algebra $(G, [~,~])$ is solvable.

Definition 2.15. A BiHom-Lie algebra $L$ is said to be decomposable if it can be decomposed into the direct sum of two or more nonzero ideals. We say $L$ is indecomposable if it is not decomposable.

Proposition 2.16. Every decomposable 2-dimensional multiplicative BiHom-Lie algebra $L$ is nonregular and it satisfies $L = [L, L] \oplus C(L)$.

In this work, $(L, [~,~], \alpha, \beta)$ denotes a multiplicative BiHom–Lie algebra (over field $\mathbb{C}$), $I = \ker \alpha + \ker \beta$, $G$ a BiHom-Lie subalgebra of $L$ satisfying $L = I \oplus G$ (if it exists), and $\Omega = \{ f \in \text{End}(L) \mid f \circ \alpha = \alpha \circ f, f \circ \beta = \beta \circ f \}$.

3 Generalized derivations of BiHom-Lie algebras

Definition 3.1 (42). For any integer $k, l$, a linear map $D: L \to L$ is called an $\alpha^k \beta^l$-derivation of the BiHom-Lie algebra $(L, [~,~], \alpha, \beta)$, if $D \in \Omega$ and

\[ D([x, y]) = [D(x), \alpha^k \beta^l(y)] + [\alpha^k \beta^l(x), D(y)], \]

for all $x, y \in L$. The set of all $\alpha^k \beta^l$-derivations of a BiHom-Lie algebra $(L, [~,~], \alpha, \beta)$ is denoted by $\text{Der}_{\alpha^k \beta^l}(L)$, and we denote by $\text{Der}(L)$ the vector space spanned by the set \{ $d \in \text{Der}_{\alpha^k \beta^l}(L) \mid k, l \in \mathbb{N}$ \}.

Definition 3.2. Let $(L, [~,~], \alpha, \beta)$ be a BiHom–Lie algebra and $\lambda, \mu, \gamma$ elements of $\mathbb{C}$. A linear map $d \in \Omega$ is a generalized $\alpha^k \beta^l$-derivation or a $(\lambda, \mu, \gamma)$-$\alpha^k \beta^l$-derivation of $L$ if for all $x, y \in L$ we have

\[ \lambda d([x, y]) = \mu [d(x), \alpha^k \beta^l(y)] + \gamma [\alpha^k \beta^l(x), d(y)]. \]

We denote the set of all $(\lambda, \mu, \gamma)$-$\alpha^k \beta^l$-derivations by $\text{Der}_{\alpha^k \beta^l}^{(\lambda, \mu, \gamma)}(L)$ and $\text{Der}_{\alpha^k \beta^l}^{(\lambda, \mu, \gamma)}(L)$ the vector space spanned by \{ $d \in \text{Der}_{\alpha^k \beta^l}^{(\lambda, \mu, \gamma)}(L) \mid k, l \in \mathbb{N}$ \}.

Lemma 3.3. For any $D \in \text{Der}_{\alpha^k \beta^l}^{(\lambda, \mu, \gamma)}(L)$ and $D' \in \text{Der}_{\alpha^k \beta^l}^{(\lambda', \mu', \gamma')}\ (L)$, their usual commutator defined by

\[ [D, D'] = D \circ D' - D' \circ D, \]

satisfies $[D, D']' \in \text{Der}_{\alpha^k \beta^l}^{(\lambda, \mu, \gamma)}(L)$. 

\[ (19) \]
Proof. The proof is similar to the one of $\text{Der}(L)$ in [32, Lemma 3.2].

Let us now classify the possible values of $\lambda, \mu, \gamma \in \mathbb{C}$ for a linear map $d: G \to G$ to be a $(\lambda, \mu, \gamma)$-$\alpha^k\beta^l$-derivation of $G$.

**Lemma 3.4.** Let $(G, [\cdot, \cdot], \alpha, \beta)$ be a BiHom–Lie algebra such that the maps $\alpha$ and $\beta$ are surjective. Let $\lambda, \mu, \gamma$ be elements of $\mathbb{C}$.

1) If $\lambda \neq 0$ and $\mu^2 \neq \gamma^2$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(0, 0, 1, 0)}(G)$.

2) If $\lambda \neq 0$, $\mu \neq 0$ and $\gamma = -\mu$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(1, 0, 0)}(G) \cap \text{Der}_{\alpha^k\beta^l}^{(0, 1, -1)}(G) = \text{Der}_{\alpha^k\beta^l}^{(1, 1, -1)}(G)$.

3) If $\lambda \neq 0$, $\mu = \gamma$ and $\mu \neq 0$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(\frac{0}{\mu^2}, 1, 1)}(G)$.

4) If $\lambda \neq 0$, $\mu = \gamma = 0$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(1, 0, 0)}(G)$.

5) If $\lambda = 0$ and $\mu^2 \neq \gamma^2$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(0, 1, 0)}(G)$.

6) If $\lambda = 0$, $\mu \neq 0$ and $\mu = \gamma$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(0, 1, 1)}(G)$.

7) If $\lambda = 0$ and $\mu = -\gamma$. Then $\text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G) = \text{Der}_{\alpha^k\beta^l}^{(0, 1, -1)}(G)$.

Proof. Let $x, y \in G$. Since $\alpha$ and $\beta$ are surjective, there exists $a, b \in G$ such that $x = \beta(a)$, $y = \alpha(b)$. Suppose any $\lambda, \mu, \gamma \in \mathbb{C}$ are given. Then for $d \in \text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G)$ and arbitrary $a, b \in G$ we have

\[
\lambda d([\beta(a), \alpha(b)]) = \mu [d(\beta(a)), \alpha^k \beta^l(\alpha(b))] + \gamma [\alpha^k \beta^l(\beta(a)), d(\alpha(b))]
\]

Thus, using $d \circ \alpha = \alpha \circ d$, $d \circ \beta = \beta \circ d$, $\alpha \circ \beta = \beta \circ \alpha$ and [3], we have

\[
\lambda d([\beta(a), \alpha(b)]) = \mu [\beta(d(a)), \alpha^k \beta^l+1(\beta(b))] + \gamma [\alpha^k \beta^l+1(a), \alpha(d(b))]
\]

\[
\lambda d([\beta(b), \alpha(a)]) = -\mu [\alpha^k \beta^l+1(a), \alpha(d(b))] - \gamma [\beta(d(a)), \alpha^k \beta^l+1(b)]
\]

By summing the two previous equalities we obtain

\[
0 = (\mu - \gamma) \left( [\beta(d(a)), \alpha^k \beta^l+1(b)] - [\alpha^k \beta^l+1(a), \alpha(d(b))] \right).
\]

So, $(\mu - \gamma) ([d(x), \alpha^k \beta^l(y)] - [\alpha^k \beta^l(x), d(y)]) = 0$. Therefore, for $\mu \neq \gamma$, $[d(x), \alpha^k \beta^l(y)] = [\alpha^k \beta^l(x), d(y)]$. Hence, applying $d \in \text{Der}_{\alpha^k\beta^l}^{(\lambda, \mu, \gamma)}(G)$ yields

\[
\lambda d([x, y]) = (\mu + \gamma)[d(x), \alpha^k \beta^l(y)].
\]

The rest of the proof is easily deduced. 

"
Theorem 3.5. Let \((G, [-, -], \alpha, \beta)\) be a BiHom–Lie algebra such that the maps \(\alpha\) and \(\beta\) are surjective. For any \(\lambda, \mu, \gamma \in \mathbb{C}\) there exists \(\delta \in \mathbb{C}\) such that the subspace \(\text{Der}_{\alpha, \beta}^{(6, \mu, \gamma)}(G)\) is equal to one of the four following subspaces:

1) \(\text{Der}_{\alpha, \beta}^{(0, 0, 0)}(G)\);
2) \(\text{Der}_{\alpha, \beta}^{(1, 0, 0)}(G)\);
3) \(\text{Der}_{\alpha, \beta}^{(6, 1, 0)}(G)\);
4) \(\text{Der}_{\alpha, \beta}^{(6, 1, 1)}(G)\);
5) \(\text{Der}_{\alpha, \beta}^{(1, 1, -1)}(G)\);
6) \(\text{Der}_{\alpha, \beta}^{(0, 1, -1)}(G)\).

Example 3.1. Let \(H\) be a 3-dimensional Heisenberg BiHom-Lie algebra (see Ex. 2.1).

\[
\text{Der}_{\alpha, \beta}^{(1, 0, 0)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \middle| d_1, d_2, d_3 \in \mathbb{C} \right\};
\]

\[
\text{Der}_{\alpha, \beta}^{(6, 1, 0)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & \frac{\alpha_x x}{\beta_y y} & 0 \\ 0 & 0 & d_1 \frac{\alpha_x x}{\beta_y y} \end{pmatrix} \middle| d_1 \in \mathbb{C} \right\};
\]

\[
\text{Der}_{\alpha, \beta}^{(6, 1, 1)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_2 + d_1 \frac{\alpha_x x}{\beta_y y} \end{pmatrix} \middle| d_1, d_2 \in \mathbb{C} \right\};
\]

\[
\text{Der}_{\alpha, \beta}^{(0, 1, 1)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \middle| d_1, d_2, d_3 \in \mathbb{C} \right\};
\]

\[
\text{Der}_{\alpha, \beta}^{(1, 1, -1)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 \frac{\alpha_x x}{\beta_y y} & 0 \\ 0 & 0 & d_3 \end{pmatrix} \middle| d_1 \in \mathbb{C} \right\};
\]

\[
\text{Der}_{\alpha, \beta}^{(0, 1, -1)}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 \frac{\alpha_x x}{\beta_y y} & 0 \\ 0 & 0 & d_3 \end{pmatrix} \middle| d_1, d_3 \in \mathbb{C} \right\}.
\]

Next proposition allows to extend some results from [17] to BiHom-Lie case.

Proposition 3.6. Any \((\lambda, \mu, \gamma)-\alpha^0 \beta^0\)-derivation of regular multiplicative BiHom-Lie algebra \((G, [-, -], \alpha, \beta)\) is a \((\lambda, \mu, \gamma)\)-derivation of induced Lie algebra \((G, [-, -])\).

Now we will discuss in detail the possible Theorem 3.5 for a finite-dimensional BiHom-Lie algebra \(L\), and we give the connection between the generalized derivation of the type studied in [2] and the generalized derivations of the type studied in this work.

1) \(\text{Der}_{\alpha, \beta}^{(0, 0, 0)}(L) = \Omega\). We have \(\Omega\) is a Lie algebra, where Lie bracket is given by [19].
2) \( \text{Der}_{\alpha^k \beta^l}^{(1,0,0)}(L) = \{ d \in \Omega \mid d(L^2) = 0 \} \) and therefore its dimension is

\[
\dim \text{Der}_{\alpha^k \beta^l}^{(1,0,0)}(L) = \text{codim}L^2 \dim L.
\]

If the BiHom-Lie algebra \( L \) is simple, then \( \text{Der}_{\alpha^0 \beta^0}^{(1,0,0)}(L) = \{0\} \).

3) \( \text{Der}_{\alpha^k \beta^l}^{(0,1,0)}(L) \):

(a) If \( \delta = 0 \), then \( \text{Der}_{\alpha^k \beta^l}^{(0,1,0)}(L) = \{ d \in \Omega \mid d(L) \subset C(\alpha^k \beta^l(L)) \} \), where

\[
C(\alpha^k \beta^l(L)) = \{ x \in L \mid [x, y] = 0; \forall y \in \alpha^k \beta^l(L) \},
\]

is the centralizer of \( \alpha^k \beta^l(L) \). Therefore,

\[
\dim \text{Der}_{\alpha^k \beta^l}^{(0,1,0)}(L) = \dim L \dim C(\alpha^k \beta^l(L)).
\]

If the BiHom-Lie algebra \( L \) is simple, then \( \text{Der}_{\alpha^0 \beta^0}^{(0,1,0)}(L) = \{0\} \).

(b) If \( \delta = 1 \), then \( \text{Der}_{\alpha^k \beta^l}^{(1,1,0)}(L) \) is the \( \alpha^k \beta^l \)-centroid of \( L \) denoted \( \Gamma_{\alpha^k \beta^l}(L) \). We denote by \( \Gamma(L) \) the vector space spanned by the set \( \{ d \in \Gamma_{\alpha^k \beta^l}(L) \mid k, l \in \mathbb{N} \} \).

(i) If \( \phi \in \Gamma_{\alpha^k \beta^l}(G) \) and \( d \in \text{Der}_{\alpha^k \beta^l}(G) \), then \( \phi \circ d \) is a \( \alpha^{k+s} \beta^{r+t} \)-derivation of \( G \).

(ii) \( \Gamma_{\alpha^k \beta^l}(L) \cap \text{Der}_{\alpha^k \beta^l}(L) = \text{CDer}_{\alpha^k \beta^l}(G) \), where \( \text{CDer}_{\alpha^k \beta^l}(G) \) is the set of \( \alpha^k \beta^l \)-central derivations defined by

\[
\text{CDer}_{\alpha^k \beta^l}(G) = \text{Der}_{\alpha^k \beta^l}^{(1,0,0)}(G) \bigcap \text{Der}_{\alpha^k \beta^l}^{(0,1,0)}(G).
\]

(iii) For any \( d \in \text{Der}_{\alpha^k \beta^l}(G) \) and \( \phi \in \Gamma_{\alpha^l \beta^t}(G) \) one has

- The composition \( d \circ \phi \) is in \( \Gamma_{\alpha^{k+t} \beta^{l+s}}(G) \) if and only if \( \phi \circ d \) is a central derivation of \( L \);

- The composition \( d \circ \phi \) is a \( \alpha^{k+t} \beta^{l+s} \)-derivation of \( G \) if and only if \( [d, \phi] = 0 \) is a \( \alpha^{k+t} \beta^{l+s} \)-central derivation of \( G \). (see [9] for the Leibniz case and [3] for the associative algebras case).

Suppose that \( L \) admits a generalized derivation \( D \in \text{Der}_{\alpha^0 \beta^0}^{(1,1,0)}(L) \). If \( \lambda \in \sigma(D) \) is an eigenvalue of \( D \), then the corresponding generalized eigenspace \( L_\lambda \) is an ideal of \( L \). Moreover, the generalized eigenspace decomposition \( L = \oplus_{\lambda \in \sigma(D)} L_\lambda \) is given in terms of ideals of \( L \). Suppose that the BiHom-Lie algebra \( L \) is simple, then \( \text{Der}_{\alpha^0 \beta^0}^{(1,1,0)}(L) \) is the one–dimensional BiHom-Lie algebra containing multiples of the identity operator.

(c) For \( \delta \notin \{0, 1\} \). Suppose that \( G \) is non-abelian. Then, by Proposition 2.13Proposition 3.6 and Proposition 2.19, the following statements are equivalent:
(i) $G$ admits an invertible generalized derivation $D \in \text{Der}_{\alpha^0, \beta^0}^{(\delta, 1, 0)}(G)$.

(ii) $G$ is at most a 2-step nilpotent BiHom-Lie algebra.

(iii) $G$ admits an invertible semisimple generalized derivation $D \in \text{Der}_{\alpha^0, \beta^0}^{(\delta, 1, 0)}(G)$ with minimal polynomial $q(x) = (x - \delta^{-1})(x - 1)$.

4) $\text{Der}_{\alpha^k, \beta^l}^{(\delta, 1, 1)}(L)$:

(a) For $\delta = 0$ we have a Lie algebra

$$\text{Der}^{(0, 1, 1)}(L) = \{d \in \Omega \mid \exists k, l \in \mathbb{N} : [d(x), \alpha^k \beta^l(y)] = -[\alpha^k \beta^l(x), d(y)], \forall x, y \in L\}.$$ 

If the BiHom-Lie algebra $L$ is simple, then by Proposition 2.11, Proposition 3.6 and [47, Corollary 2.10, Corollary 2.11], we have $\text{Der}^{(0, 1, 1)}(\delta)(L) = \{0\}$. If the simple BiHom-Lie algebra $L$ admits an invertible generalized derivation $D \in \text{Der}_{\alpha^0, \beta^0}^{(0, 1, 1)}(L)$, then $L$ is solvable.

(b) For $\delta = 1$. We get the Lie algebra of derivations of $L$:

$$\text{Der}_{\alpha^k, \beta^l}^{(1, 1, 1)}(L) = \text{Der}_{\alpha^k, \beta^l}(L) \text{ and } (\text{Der}(L), [\cdot, \cdot]) \text{ is a Lie algebra } (\text{Der}(L) \text{ the vector space spanned by } \{d \in \text{Der}_{\alpha^k, \beta^l}(L) \mid k, l \in \mathbb{N}\}).$$

(c) For $\delta \notin \{-1, 0, 1, 2\}$. When $G$ admits an invertible semisimple generalized derivation $D \in \text{Der}_{\alpha^k, \beta^l}^{(\delta, 1, 1)}(G)$ by Proposition 2.14, Proposition 3.6 and [47, Proposition 2.8], $G$ is at most a 3-step solvable BiHom-Lie algebra. When the invertible semisimple generalized $D$ has only two different eigenvalues, by Proposition 2.13, Proposition 3.6 and [47, Lemma 2.2], $G$ is at most a 2-step nilpotent BiHom-Lie algebra.

5) $\text{Der}_{\alpha^k, \beta^l}^{(1, 1, -1)}(G)$: We have

$$\text{Der}_{\alpha^k, \beta^l}^{(1, 1, -1)}(G) = \text{Der}_{\alpha^k, \beta^l}^{(0, 1, -1)}(G) \cap \text{Der}_{\alpha^k, \beta^l}^{(1, 0, 0)}(G)$$

$$= \left\{ d \in \Omega \mid d([x, y]) = 0 = [d(x), \alpha^k \beta^l(y)] = [\alpha^k \beta^l(x), d(y)] \right\}.$$ 

Then $\text{Der}_{\alpha^k, \beta^l}^{(1, 1, -1)}(G)$ is the set of $\alpha^k \beta^l$-central derivations of $G$. Define the bilinear map

$$\mu: \Omega \times \Omega \to \Omega, \mu(f, g) = \frac{1}{2} (f \circ g + g \circ f).$$

Then $(\text{CDer}(G), \mu)$ is a Jordan algebra.

6) $\text{Der}_{\alpha^k, \beta^l}^{(0, 1, -1)}(L)$: We have

$$\text{Der}_{\alpha^k, \beta^l}^{(0, 1, -1)}(G) = \left\{ d \in \Omega \mid [d(x), \alpha^k \beta^l(y)] = [\alpha^k \beta^l(x), d(y)] \right\}.$$
Then $\text{Der}_{\alpha^k \beta^l}(L)$ is called $\alpha^k \beta^l$-quasi-centroid of $L$ and denoted $QC_{\alpha^k \beta^l}(L)$. With the bilinear map $\mu$ defined in (20), we have that $(QC_{\alpha^k \beta^l}(G), \mu)$ is a Jordan algebra.

We end this section with a construction of a BiHom-Lie algebra from an extension of a Lie algebra $L$ by a $(b, a, a)$-derivation of $L$.

**Proposition 3.7.** Let $(L, [\cdot, \cdot])$ be a Lie algebra and $D \in \text{End}(L)$ be a non-zero $(b, a, a)$-derivation and let $\alpha : L \oplus CD \to L \oplus CD$ and $\beta : L \oplus CD \to L \oplus CD$ defined respectively by $\alpha(x + \lambda D) = x + \lambda aD$ and $\beta(x + \lambda D) = x + \lambda b; x \in L, \lambda \in \mathbb{C}$. Let Define the bilinear map $[\cdot, \cdot] : L \oplus CD \times L \oplus CD \to L \oplus CD$, $[x + \lambda d, y + \mu d] = [x, y] - \mu bd(x) + \lambda ad(y)$. Then $(L \oplus CD, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra.

4 Classification of multiplicative 2-dimensional BiHom-Lie algebras

In this section, we aim to classify 2-dimensional non-trivial BiHom-Lie algebras. An $n$-dimensional multiplicative BiHom-Lie algebra is identified to its structure constants with respect to a fixed basis. It turns out that the axioms of multiplicative BiHom-Lie algebra structure translate to a system of polynomial equations that define the algebraic variety of $n$-dimensional multiplicative BiHom-Lie algebra which is embedded into $\mathbb{K}^{n^3 + 2n^2}$. The classification requires to solve this algebraic system. The calculations are handled using a computer algebra system. For $n = 2$, we include in the following an outline of the computation.

1. Solving (17), we obtain the following solutions:

   - **1.1.** $\alpha = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$;
   - **1.2.** $\alpha = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} x \\ 1 \\ x \end{pmatrix}$;
   - **1.3.** $\alpha = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} x \\ z \\ 0 \end{pmatrix}$.

2. For each solution in **1.** we provide a list of non-trivial 2-dimensional multiplicative BiHom-Lie algebras. We solve the system of equations (12), (13) and (14) such that
   - $a_{12} = a_{21} = 0, b_{12} = b_{21} = 0$, for **1.1**
   - $a_{12} = a_{21} = 0, a_{22} = a_{11}, b_{21} = 0, b_{12} = 1, b_{22} = b_{11}$, for **1.2**
   - $a_{12} = 0, a_{11} = 1 , a_{22} = a_{11}, b_{21} = 0, b_{22} = b_{11}$, for **1.3**

3. Fix a BiHom-Lie algebra $L$ in **2** and solving the equation (18) such that $C_{\beta}^k$ are the structure constants corresponding to $L$ and $f_{12} = f_{21} = 0$ (resp. $f_{11} = f_{22} = 0$) if $[L, L] \neq < e_2 >$ (resp. $[L, L] = < e_2 >$).
Therefore, we get the following result.

**Proposition 4.1.** Every $2$-dimensional multiplicative BiHom-Lie algebra is isomorphic to one of the following non-isomorphic BiHom-Lie algebras: each algebra is denoted by $L^i_j$ where $i$ is related to the couple $(\alpha, \beta)$, $j$ is the number.

| $L^1_1$ | $e_1, e_1 = e_1$, | $e_1, e_2 = e_1$, | $e_2, e_1 = z_1 e_1$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = 0$, | $\alpha (e_2) = be_2$, | $\beta (e_1) = 0$, | $\beta (e_2) = ye_2$. |

| $L^1_2$ | $e_1, e_1 = e_1$, | $e_1, e_2 = 0$, | $e_2, e_1 = e_1$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = 0$, | $\alpha (e_2) = be_2$, | $\beta (e_1) = 0$, | $\beta (e_2) = ye_2$. |

| $L^1_3$ | $e_1, e_1 = e_1$, | $e_1, e_2 = 0$, | $e_2, e_1 = e_1$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = 0$, | $\alpha (e_2) = be_2$, | $\beta (e_1) = 0$, | $\beta (e_2) = ye_2$. |

| $L^1_4$ | $e_1, e_1 = e_1$, | $e_1, e_2 = 0$, | $e_2, e_1 = e_1$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = a e_1$, | $\alpha (e_2) = e_2$, | $\beta (e_1) = 0$, | $\beta (e_2) = ye_2$. |

| $L^1_5$ | $e_1, e_1 = e_1$, | $e_1, e_2 = e_1$, | $e_2, e_1 = 0$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = e_1$, | $\alpha (e_2) = e_2$, | $\beta (e_1) = 0$, | $\beta (e_2) = ye_2$. |

| $L^1_6$ | $e_1, e_1 = e_1$, | $e_1, e_2 = 0$, | $e_2, e_1 = e_1$, | $e_2, e_2 = 0$, |
|---------|-------------------|-------------------|-------------------|-------------------|
| $\alpha (e_1) = 0$, | $\alpha (e_2) = be_2$, | $\beta (e_1) = e_1$, | $\beta (e_2) = e_2$. |
\[
L_1^7: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0, \\
\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.
\]

\[
L_1^8: [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = \frac{-x}{a}e_1, \quad [e_2, e_2] = 0, \\
\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.
\]

\[
L_1^9: [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_2, \\
\alpha(e_1) = e_1, \quad \alpha(e_2) = 0, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.
\]

\[
L_1^{10}: [e_1, e_1] = 0, \quad [e_1, e_2] = e_1 + e_2, \quad [e_2, e_1] = -e_1 - e_2, \quad [e_2, e_2] = 0, \\
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.
\]

\[
L_1^{11}: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1, \\
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.
\]

\[
L_2^{11}: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0, \\
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.
\]

\[
L_3^{11}: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1, \\
\alpha_3e_1 = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.
\]

\[
L_1^{12}: [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = -e_1, \\
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_1 + e_2.
\]

\[
L_1^{13}: [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1e_1, \quad [e_2, e_2] = t_1e_1, \\
\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.
\]

\[
L_2^{13}: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = t_1e_1, \\
\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.
\]

\[
L_3^{13}: [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1, \\
\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.
\]
\[ L_1^{14} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1, \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1. \]

\[ L_1^{15} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = t_1e_1, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2. \]

\[ L_1^{16} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2. \]

\[ L_1^{17} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = (1-z)e_1, \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2. \]

**Corollary 4.2.** Every decomposable 2-dimensional multiplicative BiHom-Lie algebra \( L \) is isomorphic to one of these 4 algebras: \( L_1^1, L_1^2, L_1^3, L_1^4 \).

**Remark 4.3.** \( \langle e_1 + e_2 \rangle \) is an ideal of BiHom-Lie algebra \( L_1^{10} \). For the others BiHom-Lie algebras \( \langle e_1 \rangle \) is an ideal of \( L_1^1 \). Hence, every 2-dimensional multiplicative BiHom-Lie algebra is not simple.

## 5 Centroids and derivations of 2-dimensional multiplicative BiHom-Lie algebras

Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a \( n \)-dimensional multiplicative BiHom-Lie algebra. Let
\[ \alpha^r \beta^l(e_j) = \sum_{k=1}^{n} m_{kj} e_k. \]

An element \( d \) of \( \text{Der}_{\alpha^r \beta^l}^{(\delta, \mu, \gamma)}(L) \), being a linear transformation of the vector space \( L \), is represented in a matrix form \((d_{ij})_{1 \leq i, j \leq n}\) corresponding to \( d(e_j) = \sum_{k=1}^{n} d_{kj} e_k \), for \( j = 1, \ldots, n \). According to the definition of the \((\delta, \mu, \gamma)\)-\( \alpha^r \beta^l \)-derivation the entries \( d_{ij} \) of the matrix \((d_{ij})_{1 \leq i, j \leq n}\) must satisfy the following systems \( S \) of equations:
\[ \sum_{k=1}^{n} d_{ik} a_{kj} = \sum_{k=1}^{n} a_{ik} d_{kj}; \quad \sum_{k=1}^{n} d_{ik} b_{kj} = \sum_{k=1}^{n} b_{ik} d_{kj}; \]
\[ \delta \sum_{k=1}^{n} c^k_{ij} d_{sk} - \mu \sum_{k=1}^{n} \sum_{l=1}^{n} d_{kl} m_{ij} c^k_{sl} - \gamma \sum_{k=1}^{n} \sum_{l=1}^{n} d_{lj} m_{ki} c^k_{kl} = 0, \]

where \((a_{ij})_{1 \leq i, j \leq n}\) is the matrix of \( \alpha \), \((b_{ij})_{1 \leq i, j \leq n}\) is the matrix of \( \beta \) and \((c^k_{ij})\) are the structure constants of \( L \). First, let us give the following definitions:
Definition 5.1. A BiHom-Lie algebra is called characteristically nilpotent (denoted by CN) if the Lie algebra $\text{Der}_{\alpha_0\beta_0}(L)$ is nilpotent.

Definition 5.2. Let $L$ be an indecomposable BiHom-Lie algebra. We say $L$ is small if $\Gamma_{\alpha_0\beta_0}(L)$ is generated by central derivation and the scalars. The centroid of a decomposable BiHom-Lie algebra is small if the centroids of each indecomposable factor are small.

Now we apply the algorithms mention in the previous paragraph to centroid and derivation of 2-dimensional complex BiHom-Lie algebras. To find the centroids and derivations of 2-dimensional complex BiHom-Lie algebras we use the classification results from the previous section. The results are given in the following theorem. Moreover, we give the type of $\Gamma_{\alpha_0\beta_0}(L^1_j)$ and $\text{Der}_{\alpha_0\beta_0}(L^1_j)$ if $(r, l) = (0, 0)$.

Theorem 5.3.

$L_1^1: [e_1, e_1] = e_1, [e_1, e_2] = e_1, [e_2, e_1] = z_1 e_1, [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \alpha(e_2) = be_2, \beta(e_1) = 0, \beta(e_2) = ye_2.$

| $\alpha'\beta'$ | $\Gamma_{\alpha'\beta'}(L_1^1)$ Type of $\Gamma_{\alpha_0\beta_0}(L_1)$ | $\text{Der}_{\alpha'\beta'}(L_1^1)$ | CN |
|-----------------|------------------------------------------|-----------------|-----|
| $(r, l) = (0, 0)$ | $z_1 = 0$ | Not small | $(0 0)$ | Yes |
| $(r, l) = (0, 0)$ | $z_1 \neq 0$ | Small | $(0 0)$ | Yes |
| $(r, l) \neq (0, 0)$ | $0 0$ | $(0 d_2)$ | |

$L_1^2: [e_1, e_1] = e_1, [e_1, e_2] = 0, [e_2, e_1] = e_1, [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \alpha(e_2) = be_2, \beta(e_1) = 0, \beta(e_2) = ye_2.$

| $\alpha'\beta'$ | $\Gamma_{\alpha'\beta'}(L_2^1)$ Type of $\Gamma_{\alpha_0\beta_0}(L_1)$ | $\text{Der}_{\alpha'\beta'}(L_2^1)$ | CN |
|-----------------|------------------------------------------|-----------------|-----|
| $(r, l) = (0, 0)$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ | Small | $(0 0)$ | Yes |
| $(r, l) \neq (0, 0)$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | $(0 d_2)$ | |

$L_1^3: [e_1, e_1] = e_1, [e_1, e_2] = 0, [e_2, e_1] = 0, [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \alpha(e_2) = be_2, \beta(e_1) = 0, \beta(e_2) = ye_2.$
| $\alpha r\beta l$ | $\Gamma_{\alpha\beta}(L_3^1)$ | Type of $\Gamma_{\alpha\beta}(L_3^1)$ | $\text{Der}_{\alpha\beta}(L_3^1)$ | $\text{CN}$ |
|-----------------|---------------------|-------------------------------|---------------------|-----|
| $(r, l) = (0, 0)$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$ | Not small | $\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$ | Yes |
| $(r, l) \neq (0, 0)$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$ | |

$L_4^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = 0, \quad \alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$

| $(r, l) = (0, 0)$ | $z_1 = 0$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$ | Not small | $\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$ | Yes |
| $(r, l) = (0, 0)$ | $z_1 \neq 0$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$ | Small | $\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$ | Yes |
| $(r, l) \neq (0, 0)$ | $z_1 = 0; \quad b^r y^l = 1$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ | |
| $(r, l) \neq (0, 0)$ | $b^r y^l \neq 1$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$ | |
| $(r, l) \neq (0, 0)$ | $z_1 \neq 0; \quad b^r y^l = 1$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$ | |

$L_5^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0, \quad \alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$

| $(r, l) = (0, 0)$ | $\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$ | Small | $\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$ | Yes |
| $(r, l) \neq (0, 0)$ | $b^r y^l = 1$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ | |
| $(r, l) \neq (0, 0)$ | $b^r y^l \neq 1$ | $\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$ | | $\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$ | |
\[ L_1^2 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2. \]

| \( l = 0 \) | \( \Gamma_{\alpha^r \beta^l}(L_1^2) \) | Type of | \( \Gamma_{\alpha^r \beta^l}(L_1^2) \) | Der \( \alpha^r \beta^l \) \( (L_1^2) \) | CN |
|---|---|---|---|---|---|
| \[ \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \] Not small | \( \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix} \) | Yes |

\[ L_1^3 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \]
\[ \alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2. \]

| \( l = 0 \) | \( \Gamma_{\alpha^r \beta^l}(L_1^3) \) | Type of | \( \Gamma_{\alpha^r \beta^l}(L_1^3) \) | Der \( \alpha^r \beta^l \) \( (L_1^3) \) | CN |
|---|---|---|---|---|---|
| \[ \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \] Not small | \( \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix} \) | Yes |

\[ L_1^4 : [e_1, e_1] = e_1, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2. \]

| \( l = 0 \) | \( \Gamma_{\alpha^r \beta^l}(L_1^4) \) | Type of | \( \Gamma_{\alpha^r \beta^l}(L_1^4) \) | Der \( \alpha^r \beta^l \) \( (L_1^4) \) | CN |
|---|---|---|---|---|---|
| \[ \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \] Not small | \( \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix} \) | Yes |

\[ L_1^5 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ye_2. \]
### $L_1^6$:

$[e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$

$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$

### $L_1^7$:

$[e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$

$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = xe_1, \quad x \neq 0 \quad \beta(e_2) = e_2.$

### $L_1^8$:

$[e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -\frac{x}{a}e_1, \quad [e_2, e_2] = 0,$

$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.$
\( L_9^n : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_2, \)
\( \alpha(e_1) = e_1, \quad \alpha(e_2) = 0, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_2. \)

|                  | \( \Gamma_{\alpha' \beta'}(L_9^n) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_9^n) \) | \( \text{Der}_{\alpha' \beta'}(L_9^n) \) | CN               |
|------------------|--------------------------------------|-----------------------------------------------|--------------------------------------|------------------|
| \( r = 0, l = 0 \) | \( \begin{pmatrix} e_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Not small                                    | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes              |
| \( r = 0, l \neq 0 \) | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) |                                               | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) |                 |
| \( r \neq 0 \)    | \( \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix} \) |                                               | \( \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix} \) |                 |

\( L_1^{10} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1 + e_2, \quad [e_2, e_1] = -e_1 - e_2, \quad [e_2, e_2] = 0, \)
\( \alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2. \)

| \( \Gamma_{\alpha' \beta'}(L_1^{10}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_1^{10}) \) | \( \text{Der}_{\alpha' \beta'}(L_1^{10}) \) | CN |
|-------------------------------------|-----------------------------------------------|--------------------------------------|----|
| \( \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Small                                          | \( \begin{pmatrix} d_1 & d_2 \\ d_1 & d_2 \end{pmatrix} \) | No |

\( L_1^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1. \)
\( \alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1. \)

|                  | \( \Gamma_{\alpha' \beta'}(L_1^{11}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_1^{11}) \) | \( \text{Der}_{\alpha' \beta'}(L_1^{11}) \) | CN |
|------------------|--------------------------------------|-----------------------------------------------|--------------------------------------|----|
| \( l = 0 \)     | \( \begin{pmatrix} e_1 & c_2 \\ 0 & c_1 \end{pmatrix} \) | Not small                                    | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( l \geq 1 \)  | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) |                                               | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) |     |

\( L_2^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0. \)
\( \alpha_3(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1. \)

|                  | \( \Gamma_{\alpha' \beta'}(L_2^{11}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_2^{11}) \) | \( \text{Der}_{\alpha' \beta'}(L_2^{11}) \) | CN |
|------------------|--------------------------------------|-----------------------------------------------|--------------------------------------|----|
| \( l = 0 \)     | \( \begin{pmatrix} e_1 & c_2 \\ 0 & c_1 \end{pmatrix} \) | Not small                                    | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( l \geq 1 \)  | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) |                                               | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) |     |
\[ L_{3}^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1. \]
\[ \alpha_{34}(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1. \]

| \( l = 0 \) | \( \frac{\alpha}{\beta}(L_{3}^{11}) \) | Type of \( \frac{\alpha}{\beta}(L_{3}^{11}) \) | \( \text{Der}_{\alpha, \beta}(L_{3}^{11}) \) | CN |
|---|---|---|---|---|
| \( \begin{pmatrix} e_1 & e_2 \\ 0 & c_1 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | |

\[ L_{1}^{12} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = -e_1. \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_1 + e_2. \]

| \( l = 0 \) | \( \frac{\alpha}{\beta}(L_{1}^{12}) \) | Type of \( \frac{\alpha}{\beta}(L_{1}^{12}) \) | \( \text{Der}_{\alpha, \beta}(L_{1}^{12}) \) | CN |
|---|---|---|---|---|
| \( \begin{pmatrix} e_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( \begin{pmatrix} c_1 & t c_1 \\ 0 & c_1 \end{pmatrix} \) | |

\[ L_{1}^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = t_1 e_1, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = z e_1. \]

| \( r = l = 0 \) | \( z_1 = -1 \) | \( \frac{\alpha}{\beta}(L_{1}^{13}) \) | Type of \( \frac{\alpha}{\beta}(L_{1}^{13}) \) | \( \text{Der}_{\alpha, \beta}(L_{1}^{13}) \) | CN |
|---|---|---|---|---|---|
| \( \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( r = l = 0 \) | \( z_1 = 0 \) | \( \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( r = l = 0 \) | \( z_1 \neq -1 \) | \( \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \((r, l) \in \{(0, 1), (1, 0)\}\) | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | |
| \( r > 1, l > 1 \) | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | |

\[ L_{2}^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = t_1 e_1, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = z e_1. \]
| \( r = l = 0 \) | \( \Gamma_{\alpha' \beta'}(L_{13}^{13}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_{13}^{13}) \) | \( \text{Der}_{\alpha' \beta'}(L_{13}^{13}) \) | \( \text{CN} \) |
|---|---|---|---|---|
| \( (r, l) \in \{(0, 1), (1, 0)\} \) | \( \begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix} \) | Not small | \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | Yes |
| \( r > 1, l > 1 \) | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | |

\( L_{13}^{13} : [e_1, e_1] = 0, \ [e_1, e_2] = 0, \ [e_2, e_1] = 0, \ [e_2, e_2] = e_1, \)
\( \alpha(e_1) = 0, \ \alpha(e_2) = e_1, \ \beta(e_1) = 0, \ \beta(e_2) = ze_1. \)

| \( r = l = 0 \) | \( \Gamma_{\alpha' \beta'}(L_{3}^{13}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_{3}^{13}) \) | \( \text{Der}_{\alpha' \beta'}(L_{3}^{13}) \) | \( \text{CN} \) |
|---|---|---|---|---|
| \( (r, l) \neq (0, 0) \) | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |

\( L_{14}^{14} : [e_1, e_1] = 0, \ [e_1, e_2] = 0, \ [e_2, e_1] = 0, \ [e_2, e_2] = e_1, \)
\( \alpha(e_1) = e_1, \ \alpha(e_2) = e_1 + e_2, \ \beta(e_1) = 0, \ \beta(e_2) = ze_1. \)

| \( r = l = 0 \) | \( \Gamma_{\alpha' \beta'}(L_{1}^{14}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_{1}^{14}) \) | \( \text{Der}_{\alpha' \beta'}(L_{1}^{14}) \) | \( \text{CN} \) |
|---|---|---|---|---|
| \( l \geq 1 \) | \( \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} \) | Not small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |

\( L_{15}^{15} : [e_1, e_1] = 0, \ [e_1, e_2] = e_1, \ [e_2, e_1] = 0, \ [e_2, e_2] = t_1e_1, \)
\( \alpha(e_1) = 0, \ \alpha(e_2) = e_1, \ \beta(e_1) = e_1, \ \beta(e_2) = e_2. \)

| \( r = 0, l \geq 0 \) | \( \Gamma_{\alpha' \beta'}(L_{1}^{15}) \) | Type of \( \Gamma_{\alpha'' \beta''}(L_{1}^{15}) \) | \( \text{Der}_{\alpha' \beta'}(L_{1}^{15}) \) | \( \text{CN} \) |
|---|---|---|---|---|
| \( r \geq 1 \) | \( \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} \) | Small | \( \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} \) | Yes |

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\[ L_{16}^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1, \]
\[ \alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2. \]

\[
\begin{array}{|c|c|c|c|}
\hline
 & \Gamma_{\alpha^r \beta^l}(L_{16}^1) & Type of & Der_{\alpha^r \beta^l}(L_{16}^1) & CN \\
\hline
r = 0, l \geq 0 & \begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix} & Small & \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} & Yes \\
\hline
r \geq 1 & \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix} & \\
\hline
\end{array}
\]

\[ L_{17}^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = (1 - z)e_1, \]
\[ \alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2. \]

Corollary 5.4. The following statements hold.

(i) The dimensions of the centroids of 2-dimensional BiHom-Lie Algebras vary between one and two.

(ii) Every 2-dimensional multiplicative BiHom-Lie algebra have a small centroid if and only if it isomorphic to one of the following BiHom-Lie algebras:

\[ L_1^1(z_1)(z_1 \neq 0), \quad L_2^1, \quad L_3^1(z_1)(z_1 \neq 0), \quad L_5^1, \]
\[ L_8^1, \quad L_{10}^1, \quad L_{11}^1, \quad L_{12}^1, \quad L_{13}^1, \quad L_{15}^1, \quad L_{16}^1, \quad L_{17}^1. \]

(iii) The dimensions of the derivations of 2-dimensional BiHom-Lie algebras vary between zero and two.

(iv) Every 2-dimensional multiplicative BiHom-Lie algebra is characteristically nilpotent if and only if it is not isomorphic to \( L_{10}^1 \).

References

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