SOME REMARKS ON CMV
MATRICES AND DRESSING ORBITS

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Abstract. The CMV matrices are the unitary analogs of Jacobi matrices. In the
finite case, it is well-known that the set of Jacobi matrices with a fixed trace is nothing
but a coadjoint orbit of the lower triangular group. In this note, we will give the
analog of this result for the CMV matrices. En route, we also discuss the Hamiltonian
formulation of the Lax equations for the defocusing Ablowitz-Ladik hierarchy.

1. Introduction.

A major development in the theory of orthogonal polynomials on the unit circle
(OPUC) is the introduction of the so-called CMV matrices by Cantero, Moral and
Valázquez in 2003 [CMV]. CMV matrices are the unitary analogs of Jacobi matrices
(which arise in the theory of orthogonal polynomials on the real line (OPRL)) and
as clearly demonstrated in the two-volume monograph of Simon [S], they provide a
powerful tool in OPUC. It is well-known that Jacobi matrices are associated with
one of the most celebrated examples of integrable systems, namely, the Toda lattice
and much has been written about the subject. (See, for example, the monograph
[T] and the references therein.) In view of this, it is natural to ask if there is
an integrable system related to the CMV matrices and OPUC in the same way
that Toda relates to Jacobi matrices and OPRL. The answer to this question was
obtained in the recent work of Nenciu [N1],[N2], who showed that the sought-after
integrable system is the defocusing Ablowitz-Ladik (AL) system (a.k.a. defocusing
discrete nonlinear Schrödinger equation) [AL]. More specifically, by making use of
the connection between Ablowitz-Ladik and OPUC, the author in [N1],[N2] has
successfully derived the Lax pair formulation for the nonlinear equation.

Returning to the Jacobi operators, it is well-known that in the finite case, the
collection of such operators with a fixed trace is a coadjoint orbit of the lower
triangular group, when the dual of its Lie algebra is identified with the real sym-
metric matrices. (See, for example, [A], [K] and [R].) In the case of the finite CMV
matrices, it is natural to ask if there is an analogous Poisson geometric meaning.
In this short note, we will provide an answer to this question and en route, we will discuss the Hamiltonian formulation of the Lax equations for the defocusing AL hierarchy. As the reader will see in Section 2, we have a coboundary Poisson Lie group \( \mathfrak{g}^\mathbb{R}, \{ \cdot, \cdot \}_J \) (in the sense of Drinfeld [D]) containing the unitary group \( U(n) \) as a Poisson Lie subgroup. Also, there exists a special CMV matrix \( x^f \) with \( \theta \)-factorization \( x^e \langle x^o \rangle \) (we are following the terminology of [S] here). If \( \mathcal{L}_{x^f} \) and \( \mathcal{L}_{x^o} \) are the dressing orbits of the dual group \( \mathfrak{g}^\mathbb{R}_J \) through \( x^e \) and \( x^o \) respectively, then our main result is the following: the factors \( g^e \) and \( g^0 \) in the (unique) \( \theta \)-factorization of a CMV matrix \( g(\alpha) \) are respectively elements of \( \mathcal{L}_{x^e} \) and \( \mathcal{L}_{x^o} \); moreover, the set of CMV matrices is the image of the symplectic leaf \( \mathcal{L}_{x^e} \times \mathcal{L}_{x^o} \) of \( U(n) \times U(n) \) (equipped with the product structure) under the Poisson automorphism \( m : \mathcal{L}_{x^e} \times \mathcal{L}_{x^o} \rightarrow \{ \text{CMV matrices} \} \) where \( m : U(n) \times U(n) \rightarrow U(n) \) is the multiplication map of the group \( U(n) \).

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2. CMV matrices and dressing orbits.

Let \( \mathfrak{g}^\mathbb{R} \) be \( GL(n, \mathbb{C}) \) considered as a real Lie group, and let \( K \) and \( B \) be respectively the unitary group \( U(n) \) and the lower triangular group with positive diagonal entries. It is well-known that \( \mathfrak{g}^\mathbb{R} \) admits the Iwasawa decomposition
\[
\mathfrak{g}^\mathbb{R} = KB
\] (2.1)
which means every \( g \in \mathfrak{g}^\mathbb{R} \) admits a factorization
\[
g = g_+ g_-^{-1}
\] (2.2)
for unique \( g_+ \in K \) and \( g_- \in B \). (We shall henceforth use the notation \( g_+ \) and \( g_- \) with this interpretation.) On the Lie algebra level, we have
\[
\mathfrak{g}^\mathbb{R} = \mathfrak{k} \oplus \mathfrak{b},
\] (2.3)
where \( \mathfrak{g}^\mathbb{R}, \mathfrak{k} \) and \( \mathfrak{b} \) are respectively the Lie algebras of \( \mathfrak{g}^\mathbb{R}, K \) and \( B \). We shall denote by \( \Pi_\mathfrak{k} \) and \( \Pi_\mathfrak{b} \) the projection maps onto \( \mathfrak{k} \) and \( \mathfrak{b} \) relative to the splitting in (2.3) and set
\[
J = \Pi_\mathfrak{k} - \Pi_\mathfrak{b}.
\] (2.4)
From classical r-matrix theory \([STS1],[STS2]\), \(J : g^R \rightarrow g^R\) is a solution of the modified Yang-Baxter equation (mYBE). Consequently, we can equip \(g^R\) with the \(J\)-bracket
\[
[X,Y]_J = \frac{1}{2}([JX,Y] + [X,JY]).
\] (2.5)

In what follows, we shall denote the vector space \(g^R\) equipped with the \(J\)-bracket by \(g^R_J\). Indeed, it is easy to check from (2.5) that \(g^R_J = \mathfrak{k} \oplus \mathfrak{b}\) (Lie algebra anti-direct sum).

Note that explicitly, the projection maps are given by the formulas
\[
\Pi_k X = X_+ - (X_+_+)\,^*, \quad \Pi_b X = X_- + X_+ + (X_+)\,^*,
\] (2.6)
where \(X_+, X_0\) and \(X_-\) are the strict upper, diagonal and strict lower parts of \(X \in g^R\) and we will make use of these in the sequel. As non-degenerate invariant pairing on \(g^R\), we take
\[
(X,Y) = \text{Im} \, \text{tr} (XY).
\] (2.7)

This choice is critical for what we have in mind and with respect to \((\cdot, \cdot)\), we now define the right and left gradients of a smooth function \(\varphi\) on \(G^R\) by
\[
(D\varphi(X), X) = \left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tX} g), \quad (D'\varphi(X), X) = \left. \frac{d}{dt} \right|_{t=0} \varphi(g e^{tX}), \quad X \in g^R.
\] (2.8)

**Proposition 2.1.** (a) \(J\) is skew-symmetric relative to the pairing \((\cdot, \cdot)\). Consequently, \((G^R, \{\cdot, \cdot\}_J)\) is a coboundary Poisson Lie group with tangent Lie bialgebra \((g^R, g^R_J)\) where \{\cdot, \cdot\}_J is the Sklyanin bracket
\[
\{\varphi, \psi\}_J(g) = (J(D'\varphi(g)), D'\psi(g)) - (J(D\varphi(g)), D\psi(g)).
\] (2.9)

Moreover, \(K = U(n)\) is a Poisson Lie subgroup of \((G^R, \{\cdot, \cdot\}_J)\).

(b) The Hamiltonian equation of motion generated by \(\varphi \in C^\infty(G^R)\) is given by
\[
\dot{g} = g(\Pi_t(D'\varphi(g))) - (\Pi_t(D\varphi(g))) g.
\] (2.10)

In particular, for the Hamiltonian \(H_k(g) = \frac{1}{\alpha} \text{Re} \, \text{tr} \, g^k\), the corresponding equation of motion is
\[
\dot{g} = g (ig_+^k + i(g_+^k)^*) - (ig_+^k + i(g_+^k)^*) g.
\] (2.11)

(c) The underlying group of the Poisson group \(G^R_J\) dual to \((G^R, \{\cdot, \cdot\}_J)\) consists of \(G\) equipped with the multiplication
\[
g \ast h \equiv g_+ h g_+^{-1}.
\] (2.12)
(d) Equip \(G^\mathbb{R} \times G^\mathbb{R}\) with the product structure. Then the Hamiltonian equations of motion generated by \(\tilde{H}_k(g_1, g_2) = \frac{1}{k} \text{Re} \text{tr} (g_1 g_2)^k\) are given by
\[
\begin{align*}
\dot{g}_1 &= g_1 (\Pi_k (i(g_2 g_1)^k)) - (\Pi_k (i(g_1 g_2)^k)) g_1, \\
\dot{g}_2 &= g_2 (\Pi_k (i(g_1 g_2)^k)) - (\Pi_k (i(g_2 g_1)^k)) g_2.
\end{align*}
\] (2.13)
Moreover, the monodromy matrix \(g = g_1 g_2\) satisfies (2.11).

Proof. (a) Since \(\mathfrak{k}\) is a real form of \(\mathfrak{gl}(n, \mathbb{C})\), it follows that \(\text{tr}(XY) \in \mathbb{R}\) for \(X, Y \in \mathfrak{k}\). Consequently, \(\mathfrak{k}\) is an isotropic subalgebra of \(\mathfrak{g}^\mathbb{R}\) relative to \((\cdot, \cdot)\), i.e., \((\mathfrak{k}, \mathfrak{k}) = 0\). On the other hand, \(\mathfrak{b}\) is also an isotropic subalgebra of \(\mathfrak{g}^\mathbb{R}\) relative to \((\cdot, \cdot)\) because the diagonal entries of the elements in \(\mathfrak{b}\) are real. Combining these two facts, it follows that \(J\) is skew-symmetric relative to \((\cdot, \cdot)\). The rest of the assertion concerning \((G^\mathbb{R}, \{\cdot, \cdot\}_J)\) then follows from standard results in [STS2]. Finally, in order to show that \(K\) is a Poisson Lie subgroup, it suffices to check that \(K\) is a Poisson submanifold of \((G^\mathbb{R}, \{\cdot, \cdot\}_J)\). We shall leave the simple verification to the reader.
(b) The calculation is standard. Note that in deriving (2.11), we have made use of the formula \(D'H_k(g) = DH_k(g) = ig^k\) and the explicit expression for \(\Pi_k\) in (2.6).
(c), (d) We shall leave the verification to the reader. \(\square\)

The equations in (2.11) together with the fact that \(K = \text{U}(n)\) is a Poisson Lie subgroup of \((G^\mathbb{R}, \{\cdot, \cdot\}_J)\) means that the restriction of these equations to \(K\) are Hamiltonian with respect to the induced structure on \(K\). Moreover, eqn(4.13) in [N1] is a special case of (2.11) above if we take \(g\) to be a finite CMV matrix. Since the CMV matrix is a very special unitary matrix (see Definition 2.2 below), we ask the question if the collection of such matrices has a natural Poisson geometric meaning which would allow such a restriction to happen. Before we turn to answer this question, let us recall the definition of a finite CMV matrix [N1],[N2] in a form which is suitable for our purpose here. To simplify the language, we shall drop the term “finite” from now on.

We begin with some notations. Let \(\mathbb{D}\) be the open unit disk \(\{z \in \mathbb{C} | |z| < 1\}\) and let \(\partial \mathbb{D}\) be its boundary. Given an \((n-1)\)-tuple \(\alpha = (\alpha_0, \cdots, \alpha_{n-2}) \in \mathbb{D}^{n-1}\), we define unitary matrices
\[
\theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad \rho_j = (1 - |\alpha_j|^2)^{\frac{1}{2}}, \quad j = 0, \cdots, n-2 \quad (2.14)
\]
\[
\theta_{n-1} = -1. \quad (2.15)
\]
Definition 2.2. The CMV matrix associated with an \((n-1)\)-tuple \(\alpha = (\alpha_0, \cdots, \alpha_{n-2}) \in \mathbb{D}^{n-1}\) is the penta-diagonal unitary matrix given by
\[
g(\alpha) = g^e \left( \{\alpha_{2j}\}_{j=0}^{\frac{n-2}{2}} \right) g^0 \left( \{\alpha_{2j+1}\}_{j=0}^{\frac{n-3}{2}} \right) \tag{2.16}
\]
where
\[
g^e \left( \{\alpha_{2j}\}_{j=0}^{\frac{n-2}{2}} \right) = \text{diag} \left( \theta_0, \theta_2, \cdots, \theta_{2\left[\frac{n-4}{2}\right]} \right) \tag{2.17}
\]
and
\[
g^0 \left( \{\alpha_{2j+1}\}_{j=0}^{\frac{n-3}{2}} \right) = \text{diag} \left( 1, \theta_1, \theta_3, \cdots, \theta_{2\left[\frac{n-2}{2}\right]+1} \right). \tag{2.18}
\]

Remark 2.3 (a) It follows from the above definition that the \((2j, 2j+2)\) and the \((2j+1, 2j-1)\) entries of a CMV matrix are zero for any \(j\). The factorization in (2.16) above is called the \(\theta\)-factorization following the terminology in [S]. We have more to say on this below.

(b) In the original definition of the CMV matrix in [N1], [N2], there is an extra parameter \(\alpha_{n-1} \in \partial \mathbb{D}\) involved. But subsequently, the author restricts her attention to \(\alpha_{n-1} = -1\). The reader will see that this assumption is natural from our point of view.

(c) Given a nontrivial probability measure \(d\mu\) on \(\partial \mathbb{D}\), Cantero, Moral and Valázquez [CMV] produces an orthonormal basis of \(L^2(\partial \mathbb{D}, d\mu)\) by applying Gram-Schmidt to \(1, z, z^{-1}, z^2, z^{-2}, \cdots\). The matrix representation of the operator \(f(z) \mapsto zf(z)\) in \(L^2(\partial \mathbb{D}, d\mu)\) in this basis is an infinite CMV matrix. The finite case which we consider here corresponds to a trivial probability measure \(d\mu\) supported at \(n\) points and the \(\alpha_j\)'s are the Verblunsky coefficients which appear in Szegő recursion [S].

In general, we shall denote by \(g^e\) any \(n \times n\) block diagonal matrix with \(2 \times 2\) blocks on the main diagonal of the form
\[
\begin{pmatrix}
\bar{\alpha} & \rho \\
\rho & -\alpha
\end{pmatrix}, \quad \rho = (1 - |\alpha|^2)^{\frac{1}{2}}, \quad \alpha \in \mathbb{D} \tag{2.19}
\]
except when \(n\) is odd, the last block is the number \(-1\). We shall denote the collection of such matrices by \(\mathcal{T}^e\). Similary, we shall denote by \(g^o\) any \(n \times n\) block diagonal matrix which begins with the \(1 \times 1\) block equal to 1 followed by \(2 \times 2\) blocks of the form in (2.19) except when \(n\) is even, the last block is the number \(-1\). We shall use the symbol \(\mathcal{T}^o\) to denote the collection of such matrices. Clearly, for given \(g^e \in \mathcal{T}^e\) and \(g^o \in \mathcal{T}^o\), there exists unique \(\underline{\alpha} = (\alpha_0, \cdots, \alpha_{n-2}) \in \mathbb{D}^{n-1}\) such that
\[
g^e g^o = g(\underline{\alpha}). \tag{2.20}
\]
Indeed, more is true, namely, it is straightforward to verify that the map

\[ m \mid T^e \times T^o : T^e \times T^o \rightarrow \{ \text{CMV matrices} \} \]

\[ (g^e, g^o) \mapsto g^e g^o \] (2.21)

is a diffeomorphism, where \( m : K \times K \rightarrow K \) is the multiplication map of the group \( K \).

In order to understand the Poisson geometric meaning of the collection of CMV matrices, we appeal to the following result in the theory of Poisson Lie groups: the symplectic leaves of a Poisson Lie group are given by the orbits of so-called dressing actions [STS2],[LW]. Indeed, it follows from Theorem 13 of [STS2] (which applies to the coboundary case) that the symplectic leaf of \((G^R, \{\cdot, \cdot\}_J)\) passing through \( x \in G^R \) is given by

\[ \mathcal{L}_x = \left\{ g^{-1} x (x^{-1} g x)_+ \mid g \in G^R_J \right\}. \] (2.22)

In analogy with Example 4.2.7 in [S], we introduce the following special CMV matrix

\[ x_f = x^e_f x^o_f \] (2.23)

corresponding to

\[ \underline{\alpha} = (0, 0, \cdots, 0). \] (2.24)

In other words,

\[ x^e_f = \text{diag}(w^*, w^*, \cdots) \] (2.25)

and

\[ x^o_f = \text{diag}(1, w^*, w^*, \cdots) \] (2.26)

where

\[ w^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (2.27)

We now come to the main result of this work.

**Theorem 2.4.** (a) \( \mathcal{L}_x^e = T^e \).

(b) \( \mathcal{L}_x^o = T^o \).

(c) The product \( \mathcal{L}_x^e \times \mathcal{L}_x^o \) is a symplectic leaf of \( K \times K \) equipped with the product structure. Moreover, the collection of CMV matrices is the image of \( \mathcal{L}_x^e \times \mathcal{L}_x^o \) under the Poisson automorphism \( m \mid \mathcal{L}_x^e \times \mathcal{L}_x^o : \mathcal{L}_x^e \times \mathcal{L}_x^o \rightarrow \{ \text{CMV matrices} \} \) where \( m : K \times K \rightarrow K \) is the multiplication map of the group \( K \).
Proof. (a) Take an arbitrary element

\[ a = g_+^{-1} x_f^e ((x_f^e)^{-1} g x_f^e)_+ \]
\[ = g_-^{-1} x_f^e ((x_f^e)^{-1} g x_f^e)_- \]  
(2.28)

in the dressing orbit through \( x_f^e \). We first consider the case where \( n \) is even. From the first line of (2.28), it is clear that \( a \) is unitary. On the other hand, since \( g_- \) is lower triangular, it follows from the second line of (2.28) that \( a \) is block lower triangular with \( 2 \times 2 \) blocks on the diagonal. Moreover, from the fact that the diagonal entries of \( g_- \) are positive, it is easy to see that each of the \( 2 \times 2 \) blocks on the main diagonal has the following properties: (i) the entry in the upper right hand corner is positive, (ii) the determinant is negative (since \( \det(w^*) = -1 \)). Consequently, the matrix \((a^*)^{-1}\) is upper block triangular with diagonal blocks having the same properties. But \( a = (a^*)^{-1} \), so it follows that \( a \) must be block diagonal, i.e.,

\[ a = diag \left( \phi_0, \phi_2, \cdots, \phi_{2[n-1]} \right) \]  
(2.29)

where for each \( j \), \( \phi_{2j} \) is a unitary \( 2 \times 2 \) matrix with a positive entry in the upper right hand corner and whose determinant is \(-1\). Consequently, \( \phi_{2j} \) must be of the form

\[ \phi_{2j} = \begin{pmatrix} \bar{\alpha}_{2j} & \rho_{2j} \\ \rho_{2j} & -\alpha_{2j} \end{pmatrix} \]  
(2.30)

for some \( \alpha_{2j} \in \mathbb{D} \), where \( \rho_{2j} = (1 - |\alpha_{2j}|^2)^{\frac{1}{2}} \). Hence we have shown that \( \mathcal{L}_{x_f^e} \subset \mathcal{T}^e \).

Conversely, take an arbitrary element

\[ g^e = diag \left( \theta_0, \theta_2, \cdots, \theta_{2[n-1]} \right) \]  
(2.31)

in \( \mathcal{T}^e \) where \( \theta_{2j} \) is of the form given in (2.14). Define a block diagonal matrix

\[ g = diag \left( l_0, l_2, \cdots, l_{2[n-1]} \right) \]  
(2.32)

such that

\[ l_{2j} = \begin{pmatrix} \rho_{2j} & 0 \\ -\alpha_{2j} & 1 \end{pmatrix}, \quad j = 0, \cdots, \left[ \frac{n-1}{2} \right]. \]  
(2.33)

Clearly, \( g \) is lower triangular so that

\[ g_+^{-1} x_f^e ((x_f^e)^{-1} g x_f^e)_+ \]
\[ = (gx_f^e)_+ \]
\[ = (diag(l_0 w^*, l_2 w^*, \cdots))_+. \]  
(2.34)
But from the definition of $l_{2j}$, we find that $l_{2j}w^* \text{ admits the factorization }

\begin{equation}
l_{2j}w^* = \theta_{2j} \begin{pmatrix} \rho_{2j} & 0 \\ -\alpha_{2j} & 1 \end{pmatrix}.
\end{equation}

Hence it follows that $g_+^{-1}x_{f}^+(x_{f}^+)^{-1}gx_{f}^+ = g^e$. Consequently, we have the reverse inclusion $T^e \subset L_{x_f}$ as well. When $n$ is odd, everything goes through the same as before except that for each of the matrices in (2.29),(2.31)-(2.32), the last block is a $1 \times 1$ block. We shall leave the easy detail to the reader.

(b) The argument is similar to (a).

(c) This is clear from the definition of the CMV matrices. \hfill \Box

**Corollary 2.5.** Let $g^e$, $g^o$ have their usual meaning and let $g(\alpha) = g^e g^0$ and $\tilde{g}(\alpha) = g^0 g^e$ where $\alpha \in \mathbb{D}$ is uniquely determined by $g^e$, $g^o$. Then the equations

\begin{align*}
\dot{g}^e &= g^e (\Pi_t (ig(\alpha)^k)) - (\Pi_t (i\tilde{g}(\alpha)^k))g^e, \\
\dot{g}^o &= g^o (\Pi_t (ig(\alpha)^k)) - (\Pi_t (i\tilde{g}(\alpha)^k))g^o
\end{align*}

are the Hamiltonian equations of motion on the symplectic manifold $L_{x_f} \times L_{x_f}$ generated by the Hamiltonian $\tilde{H}_k(g^e, g^o) = \frac{1}{k} \text{Re} \text{ tr} (g(\alpha))^k$. Moreover, under the Hamiltonian flow defined by (2.36), $g(\alpha)$ evolves according to

\begin{equation}
\dot{g}(\alpha) = g(\alpha) (\Pi_t (ig(\alpha)^k)) - (\Pi_t (i\tilde{g}(\alpha)^k))g(\alpha).
\end{equation}

**Proof.** This is a consequence of the above theorem and Proposition 2.1 (d). \hfill \Box

**Remark 2.6** (a) It follows from the r-matrix formulation that the equations (2.36)-(2.37) can be solved via factorization problems. Actually, the same remark also holds true for the infinite case in [N1],[N2] as one can extend the Iwasawa decomposition to the group of bounded invertible operators on $l^2(\mathbb{Z}^+)$ (cf. [DLT]).

(b) The above corollary suggests that in a sense, it seems more natural to consider (2.36). Whether this is so from the point of view of OPUC remains to be seen.
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