ON GROMOV-WITTEN THEORY OF ROOT GERBES

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Abstract. This research announcement discusses our results on Gromov-Witten theory of root gerbes. A complete calculation of genus 0 Gromov-Witten theory of $\mu_r$-root gerbes over a smooth base scheme is obtained by a direct analysis of virtual fundamental classes. Our result verifies the genus 0 part of the so-called decomposition conjecture which compares Gromov-Witten theory of étale gerbes with that of the bases. We also verify this conjecture in all genera for toric gerbes over toric Deligne-Mumford stacks.

1. Introduction

Orbifold Gromov-Witten theory, constructed in symplectic category by Chen-Ruan [10] and in algebraic category by Abramovich, Graber and Vistoli [1], [2], has been an area of active research in recent years. Calculations of orbifold Gromov-Witten invariants in examples present numerous new challenges, see [13], [11], [24], and [5] for examples.

Étale gerbes over a smooth base are examples of smooth Deligne-Mumford stacks. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack and $G$ a finite group scheme over $\mathcal{X}$. Intuitively one can think of a $G$-banded gerbe over $\mathcal{X}$ as a fibre bundle over $\mathcal{X}$ with fibre the classifying stack $B G$. A detailed definition of gerbes can be found in, for example, [17], [7], [14]. Our main goal is to compute Gromov-Witten theory of $G$-banded gerbes.

For trivial $G$-banded gerbes $\mathcal{X} \times B G$, the computation of their Gromov-Witten invariants is handled as a special case of a general product formula for orbifold Gromov-Witten invariants of product stacks $\mathcal{X} \times \mathcal{Y}$. We will not discuss this here, see [3] for more details.

Root gerbes provide an interesting class of non-trivial gerbes. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack. Let $\mathcal{L} \rightarrow \mathcal{X}$ be a line bundle over $\mathcal{X}$ and $r$ a positive integer. The stack $\sqrt[1/r]{\mathcal{L}/\mathcal{X}}$ of $r$-th roots of $\mathcal{L}$ is a smooth Deligne-Mumford stack, and is a $\mu_r$-gerbe over $\mathcal{X}$. Our study of Gromov-Witten theory of root gerbes is aided by the so-called decomposition conjecture [18] in physics, which states in mathematical terms that Gromov-Witten theory of a root gerbe $\sqrt[1/r]{\mathcal{L}/\mathcal{X}}$ over $\mathcal{X}$ should be equivalent to Gromov-Witten theory of the disjoint union of $r$ copies of $\mathcal{X}$ after a change of variables.

In this note, we present results on computations of genus 0 Gromov-Witten theory of root gerbes over smooth varieties. Our results verify the decomposition conjecture in genus 0 for these root gerbes. We also discuss the case of toric gerbes [2], for which

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1 One can also formulate the decomposition conjecture for arbitrary $G$-gerbes (which is more general than a $G$-banded gerbe). The conjecture states that Gromov-Witten theory of the $G$-gerbe is equivalent to certain twist of the Gromov-Witten theory of some étale cover of the base. Details of the conjecture in this generality will be discussed elsewhere.

2 Note that these gerbes are iterated root gerbes over toric Deligne-Mumford stacks [15], [20].
we verify the decomposition conjecture in all genera by applying some sophisticated techniques in toric Gromov-Witten theory. Detailed proofs of results discussed in this note will be given in [4].

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2. Results on root gerbes

2.1. Root gerbes. Let $X$ be a proper smooth Deligne-Mumford stack, and $L$ a line bundle over $X$. Let $r > 0$ be an integer. Recall [2], [8] that the stack $\sqrt{L/X}$ of $r$-th roots of $L$ is defined to be the $X$-groupoid whose objects over an $X$-scheme $f : Y \to X$ are pairs $(M, \varphi)$, with $M$ a line bundle over $Y$ and $\varphi : M^\otimes r \to f^*L$ an isomorphism. An arrow from $(M, \varphi)$ to $(N, \psi)$ lying over a $X$-morphism $h : (Y, f) \to (Z, g)$ is an isomorphism $\rho : M \to h^*N$ such that $\varphi = (h^*\psi) \circ \rho^\otimes r$:

$$
\begin{array}{ccc}
M^\otimes r & \xrightarrow{\rho^\otimes r} & h^*N^\otimes r \\
\varphi & \downarrow & h^*\psi \\
\Downarrow f^*L & \cong & h^*g^*L.
\end{array}
$$

Alternatively, the stack $\sqrt{L/X}$ may be presented as the quotient stack $[L^*/C^*]$, where $L^*$ is the complement of the zero section inside the total space of $L$, and $C^*$ acts via $\lambda \cdot z := \lambda^rz$. Clearly $\sqrt{L/X}$ is a smooth Deligne-Mumford stack. It is also easy to see that $\sqrt{L/X} \to X$ is a $\mu_r$-banded gerbe. See [2] and [8] for more discussions.

In this note we are primarily concerned with the case that the base $X$ is a smooth projective variety $\mathbb{F} X$.

By definition the Chen-Ruan orbifold cohomology groups (see [9], [1] for more details) are cohomology groups of the inertia stacks. Since the root gerbe $\sqrt{L/X}$ is naturally banded by $\mu_r$, its inertia stack $I\sqrt{L/X}$ is a disjoint union of $r\mu_r$-gerbes over $X$. Thus the Chen-Ruan orbifold cohomology with rational coefficients $H^*_{CR}(\sqrt{L/X}, \mathbb{Q})$ is isomorphic to a direct sum of $r$ copies of $H^*(X, \mathbb{Q})$.

2.2. Moduli of stable maps. One of our results is the construction of moduli stack of twisted stable maps. Let $\beta \in H_2(X, \mathbb{Z})$ and let $\mathcal{K}_{0,n}(\sqrt{L/X}, \beta)$ be the moduli stack of genus zero, $n$-point twisted stable maps to $\sqrt{L/X}$ of degree $\beta$ in the sense of Abramovich-Graber-Vistoli [2]. Given $[f : \mathcal{C} \to \sqrt{L/X}] \in \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)$, the stack structures at marked points of $\mathcal{C}$ determine an $n$-tuple $\vec{g} = (g_1, \cdots, g_n)$ of elements in $\mu_r$. We may write $g_i = e^{2\pi i m_i/r_i}$, where $0 \leq m_i \leq r_i - 1$ and $(m_i, r_i) = 1$. By Riemann-Roch for twisted curves, $\vec{g}$ satisfies the condition

$$
\prod_{1 \leq i \leq n} g_i = e^{2\pi i k/r}, \text{ where } k = \int_\beta c_1(L).
$$

Such $n$-tuples $\vec{g}$ are called admissible vectors.

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3In this case, the stack $\sqrt{L/X}$ is a toric stack bundle introduced in [19].
Let
\[ \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \subset \mathcal{K}_{0,n}(\sqrt{L/X}, \beta) \]
be the locus which parametrizes stable maps with admissible vector \( \bar{g} \). Post-composition with the natural map \( \sqrt{L/X} \to X \) gives a map
\[ p : \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \to \overline{\mathcal{M}}_{0,n}(X, \beta), \]
where \( \overline{\mathcal{M}}_{0,n}(X, \beta) \) is the moduli stack of genus zero, \( n \)-point degree \( \beta \) stable maps to \( X \). We prove a structure result for the moduli stack \( \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \). More precisely, we obtain a construction of \( \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \) as follows.

1. construct a stack \( P_{\bar{g}}^{\bar{g}} \) over \( \overline{\mathcal{M}}_{0,n}(X, \beta) \) as a category fibered in groupoids whose objects are certain morphisms of logarithmic structures \([22],[23]\). This construction, which can be seen as a generalization of the root construction, introduces additional automorphisms along the locus in \( \overline{\mathcal{M}}_{0,n}(X, \beta) \) parametrizing maps with singular domains;
2. show that \( \mathcal{K}_{0,n}(\mathfrak{g}, \beta)^{\bar{g}} \) is a \( \mu_r \)-banded gerbe over \( P_{\bar{g}}^{\bar{g}} \) and therefore \( p \) factors through \( P_{\bar{g}}^{\bar{g}} \).

The \( \mu_r \)-gerbe in step (2) is the root gerbe of a line bundle which we can explicitly write down. Our result thus generalizes a result of Bayer and Cadman \([5]\) on the moduli stack of twisted stable maps to the classifying stack \( \mathcal{B}_{\mu_r} \).

Our strategy of computing genus 0 Gromov-Witten invariants of \( \sqrt{L/X} \) is to relate them with invariants of \( X \). To do so, we carry out a comparison with respect to the map \( p \) between the perfect relative obstruction theories on \( \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \) and \( \overline{\mathcal{M}}_{0,n}(X, \beta) \) and deduce from there the following push-forward formula for virtual fundamental classes:

**Theorem 2.1.**

\[ p_*[\mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}}]^{\text{vir}} = \frac{1}{r}[\overline{\mathcal{M}}_{0,n}(X, \beta)]^{\text{vir}}. \]

### 2.3. Gromov-Witten invariants.

Let \( \pi : \mathfrak{g} = \sqrt{L/X} \to X \) be a \( \mu_r \)-root gerbe. Then the inertia stack admits the following decomposition
\[ I_{\mathfrak{g}} = \bigsqcup_{g \in \mu_r} \mathfrak{g}_g, \]
where \( \mathfrak{g}_g \) is a root gerbe isomorphic to \( \mathfrak{g} \). Let \( \pi_g : \mathfrak{g}_g \to X \) be the induced morphism. On each component there is an isomorphism between the rational cohomology groups
\[ \pi_g^* : H^*(X, \mathbb{Q}) \xrightarrow{\cong} H^*(\mathfrak{g}_g, \mathbb{Q}). \]

Let \( \bar{g} = (g_1, ..., g_n) \) be an admissible vector. There are evaluation maps
\[ ev_i : \mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}} \to \mathfrak{g}_{g_i}^{\text{rig}}, \]
where \( \mathfrak{g}_{g_i}^{\text{rig}} \) is a component of the *rigidified inertia stack* \( I_{\mathfrak{g}}^{\text{rig}} = \cup_{g \in \mu_r} \mathfrak{g}_g^{\text{rig}} \) (see \([2]\), Section 4.4 for the definition). Although the evaluation maps \( ev_i \) do not take values in \( I_{\mathfrak{g}} \), as explained in \([2]\), Section 6.1.3, one can still define a pull-back map at cohomology level,
\[ ev_i^* : H^*(\mathfrak{g}_g, \mathbb{Q}) \to H^*(\mathcal{K}_{0,n}(\sqrt{L/X}, \beta)^{\bar{g}}, \mathbb{Q}). \]
Given $\delta_i \in H^*(\mathcal{Y}_{g_i}, \mathbb{Q})$ for $1 \leq i \leq n$ and integers $k_i \geq 0, 1 \leq i \leq n$, one can define descendant orbifold Gromov-Witten invariants

$$
\langle \delta_1 \psi_1^{k_1}, \ldots, \delta_n \psi_n^{k_n} \rangle_{0, n, \beta}^\mathcal{Y} := \int_{[\mathcal{X}_0,n,(\mathcal{Y}/\mathcal{X})_{\beta}]} \prod_{i=1}^n ev_i^*(\delta_i) \psi_1^{k_1},
$$

where $\overline{\psi}_i$ are the pullback of the first Chern classes of the tautological line bundles over $\mathcal{M}_{g,n}(\mathcal{X}, \beta)$. See [2] for more discussion on descendant classes.

For classes $\delta_i \in H^*(\mathcal{Y}_{g_i}, \mathbb{Q})$, set $\overline{\delta}_i = (\pi^*_{g_i})^{-1}(\delta_i)$. Descendant Gromov-Witten invariants $\langle \delta_1 \psi_1^{k_1}, \ldots, \delta_n \psi_n^{k_n} \rangle^\mathcal{Y}_{0, n, \beta}$ of $\mathcal{X}$ are similarly defined. Theorem 2.1 implies

**Theorem 2.2.**

$$
\langle \delta_1 \psi_1^{k_1}, \ldots, \delta_n \psi_n^{k_n} \rangle_{0, n, \beta}^{\mathcal{Y}} = \frac{1}{\mu} \langle \overline{\delta}_1 \overline{\psi}_1^{k_1}, \ldots, \overline{\delta}_n \overline{\psi}_n^{k_n} \rangle_{0, n, \beta}^\mathcal{X}.
$$

Moreover, if $\overline{g}$ is not admissible, then the Gromov-Witten invariants vanish.

In the following we use complex number $\mathbb{C}$ as coefficients for the cohomology. For $\overline{\rho} \in H^*(\mathcal{X}, \mathbb{C})$ and an irreducible representation $\rho$ of $\mu_r$, we define

$$
\overline{\alpha}_\rho := \frac{1}{\mu} \sum_{\gamma \in \mu_r} \chi_\rho(g^{-1}) \pi_\gamma(\overline{\rho}),
$$

where $\chi_\rho$ is the character of $\rho$. The map $(\overline{\rho}, \rho) \mapsto \overline{\alpha}_\rho$ clearly defines an additive isomorphism

$$
\bigoplus_{[\rho] \in \overline{\mu}_r} H^*(\mathcal{X})_{[\rho]} \simeq H^*(\mathcal{Y}, \mathbb{C}),
$$

where $\overline{\mu}_r$ is the set of isomorphism classes of irreducible representations of $\mu_r$, and for $[\rho] \in \overline{\mu}_r$ we define $H^*(\mathcal{X})_{[\rho]} := H^*(\mathcal{X}, \mathbb{C})$.

Theorem 2.2 together with orthogonality relations of characters of $\mu_r$ implies the following

**Theorem 2.3.**

$$
\langle \overline{\alpha}_1^{\rho_1} \overline{\psi}_1^{k_1}, \ldots, \overline{\alpha}_n^{\rho_n} \overline{\psi}_n^{k_n} \rangle^\mathcal{Y}_{0, n, \beta} = \begin{cases} \frac{1}{\mu} \langle \overline{\alpha}_1 \overline{\psi}_1^{k_1}, \ldots, \overline{\alpha}_n \overline{\psi}_n^{k_n} \rangle^\mathcal{X}_{0, n, \beta} \chi_\rho \left( \exp \left( \frac{\text{-}2\pi i \int_{\mathcal{X}} c_1(\mathcal{X})}{\mu} \right) \right) & \text{if } \rho_1 = \rho_2 = \ldots = \rho_n =: \rho, \\ 0 & \text{otherwise}. \end{cases}
$$

We may reformulate this in terms of generating functions. Let

$$
\{ \overline{\phi}_i | 1 \leq i \leq \text{rank} H^*(\mathcal{X}, \mathbb{C}) \} \subset H^*(\mathcal{X}, \mathbb{C})
$$

be an additive basis. According to the discussion above, the set

$$
\{ \overline{\phi}_i | 1 \leq i \leq \text{rank} H^*(\mathcal{X}, \mathbb{C}), [\rho] \in \overline{\mu}_r \}
$$

is an additive basis of $H^*(\mathcal{Y}, \mathbb{C})$. Recall that the genus 0 descendant potential of $\mathcal{Y}$ is defined to be

$$
\mathcal{G}_\mathcal{Y}^0(\{ t_{i\rho,j} \}_{1 \leq i \leq \text{rank} H^*(\mathcal{X}, \mathbb{C}), \rho \in \overline{\mu}_r, j \geq 0 }; Q) := \sum_{n \geq 0, \beta \in H_2(\mathcal{X}, \mathbb{Z})} \frac{Q^\beta}{n!} \prod_{i=1}^n t_{i\rho,k,i} \prod_{k=1}^n \prod_{i_1, \ldots, i_n} \phi_{i_1, \ldots, i_n}^{\rho_1, \ldots, \rho_n} \cdot j_{i_1, \ldots, i_n}^{j_1, \ldots, j_n}.
$$

The descendant potential $\mathcal{G}_\mathcal{Y}^0$ is a formal power series in variables $t_{i\rho,j}, 1 \leq i \leq \text{rank} H^*(\mathcal{X}, \mathbb{C}), \rho \in \overline{\mu}_r, j \geq 0$ with coefficients in the Novikov ring $\mathbb{Q}[\overline{\mathcal{N}}(\mathcal{X})]$, where
\( \overline{NE}(X) \) is the effective Mori cone of the coarse moduli space of \( \mathcal{G} \). Here \( Q^\beta \) are formal variables labeled by classes \( \beta \in \overline{NE}(X) \). See e.g. \([24]\) for more discussion on descendant potentials for orbifold Gromov-Witten theory.

Similarly the genus 0 descendant potential of \( X \) is defined to be

\[
\mathcal{F}_X^0 \left( \{ t_{i,j} \}_{1 \leq i \leq \text{rank} \, H^*(X, \mathbb{C}), j \geq 0}; Q \right) := \sum_{n \in \mathbb{N}} \frac{Q^n}{n!} \prod_{k=1}^{n} t_{ik} \left( \prod_{j=0}^{n} \frac{\phi_{ik}^{-1}}{\partial x_{ik}^{j}} \right) \chi_{ij}.
\]

\( \mathcal{F}_X^0 \) is a formal power series in variables \( t_{i,j}, 1 \leq i \leq \text{rank} \, H^*(X, \mathbb{C}), j \geq 0 \) with coefficients in \( \mathbb{Q}[\overline{NE}(X)] \) and \( Q^\beta \) is (again) a formal variable. Using Theorem 2.3 we prove

**Theorem 2.4.**

\[
\mathcal{F}_X^0 \left( \{ t_{i,j} \}_{1 \leq i \leq \text{rank} \, H^*(X, \mathbb{C}), \rho \in \mathcal{C}}; j \geq 0}; Q \right) = \frac{1}{r^2} \sum_{|\rho| \in \mathcal{C}} \mathcal{H}_X^0 \left( \{ t_{i,j} \}_{1 \leq i \leq \text{rank} \, H^*(X, \mathbb{C}), j \geq 0}; Q_{\rho} \right),
\]

where \( Q_{\rho} \) is defined by the following rule:

\[
Q_{\rho}^\beta := Q^\beta \chi_{\rho} \left( \exp \left( \frac{-2\pi i \int_{\beta} c_1(\mathbb{L})}{r} \right) \right),
\]

and \( \chi_{\rho} \) is the character associated to the representation \( \rho \).

Theorem 2.4 confirms the decomposition conjecture for genus 0 Gromov-Witten theory of \( \mathcal{G} \).

**Remark 2.5.**

(1) If \( X \) has generically semi-simple quantum cohomology, then an application of Givental’s formula \([16]\) shows that Theorem 2.4 implies the decomposition conjecture for \( \mathcal{G} \) in all genera.

(2) By definition a gerbe over \( X \) is *essentially trivial* if it becomes trivial after contracted product with the trivial \( O_X \)-gerbe. For a finite abelian group \( G \), any essentially trivial \( G \)-banded gerbe can be obtained as a fiber product over the base of root gerbes. All the results presented here for root gerbes can be easily extended to this more general class of gerbes.

### 3. Results on toric gerbes

Toric gerbes over toric orbifolds are toric Deligne-Mumford stacks in the sense of Borisov-Chen-Smith \([3]\). Any toric Deligne-Mumford stack can be constructed by taking a sequence of root of line bundles on the toric orbifolds, see \([20], [15]\).

A toric Deligne-Mumford stack is defined in terms of a stacky fan \( \Sigma = (N, \Sigma, \beta) \), where \( N \) is a finitely generated abelian group, \( \Sigma \subset N_{\mathbb{Q}} = N \otimes \mathbb{Q} \) is a simplicial fan and \( \beta : \mathbb{Z}^n \to N \) is a map determined by the elements \( \{ b_1, \cdots, b_n \} \) in \( N \). By assumption, \( \beta \) has finite cokernel and the images of \( b_i \)'s under the natural map \( N \to N_{\mathbb{Q}} \) generate the simplicial fan \( \Sigma \). The toric Deligne-Mumford stack \( \mathfrak{X}(\Sigma) \) associated to \( \Sigma \) is defined to be the quotient stack \( [Z/G] \), where \( Z \) is the open subvariety \( \mathbb{C}^n \setminus \bigvee (J_\Sigma) \), \( J_\Sigma \) is the irrelevant ideal of the fan, and \( G \) is the product of an algebraic torus and a finite abelian group. The \( G \)-action on \( Z \) is given by a group homomorphism \( \alpha : G \to (\mathbb{C}^*)^n \), where \( \alpha \) is obtained by applying the functor \( \text{Hom}_Z(\cdot, \mathbb{C}^*) \) to the Gale dual \( \beta^\vee : \mathbb{Z}^n \to N^\vee \) of \( \beta \) and \( G = \text{Hom}_Z(N^\vee, \mathbb{C}^*) \).
Every stacky fan $\Sigma$ has an underlying \textit{reduced} stacky fan $\Sigma_{\text{red}} = (\overline{\Sigma}, \Sigma, \overline{\beta})$, where $\overline{\Sigma} := N/N_{\text{tor}}$, $\overline{\beta} : \mathbb{Z}^n \to \overline{\Sigma}$ is the natural projection given by the vectors $\{\overline{b}_1, \ldots, \overline{b}_n\} \subseteq \overline{N}$. With these data one gets a toric Deligne-Mumford stack $X(\Sigma_{\text{red}}) = [\mathbb{Z}/\overline{\beta}]$, where $\overline{G} = \text{Hom}_{\mathbb{Z}}(\overline{\Sigma}^\vee, \mathcal{C}^*)$ and $\overline{\Sigma}^\vee$ is the Gale dual $\beta^\vee : \mathbb{Z}^n \to \overline{\Sigma}^\vee$ of the map $\overline{\beta}$. The stack $X(\Sigma_{\text{red}})$ is a toric orbifold\footnote{I.e. the generic stabilizer is trivial.}, and can be obtained by rigidifying $X(\Sigma)$. We assume that $X(\Sigma)$ and $X(\Sigma_{\text{red}})$ are semi-projective (see e.g. \cite{20} for definition).

### 3.1. Orbifold cohomology

The Chen-Ruan orbifold cohomology ring of a toric Deligne-Mumford stack has been computed\footnote{Strictly speaking what’s computed in \cite{6} is the orbifold Chow ring. The computation for Chen-Ruan orbifold cohomology ring is identical.} in \cite{6}. We recall the answer. Let $M = N^*$ be the dual of $N$. Let $\mathbb{C}[N]^{\Sigma}$ be the group ring of $N$, i.e. $\mathbb{C}[N]^{\Sigma} := \bigoplus_{c \in \Sigma} \mathbb{C}y^c$, $y$ is the formal variable. Define the following multiplication

\begin{equation}
y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1 + c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{\tau}_1, \overline{\tau}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

Let $I(\Sigma)$ be the ideal in $\mathbb{C}[N]^{\Sigma}$ generated by the elements $\sum_{i=1}^n \theta(b_i)y^{b_i}$, $\theta \in M$. Then there is an isomorphism of $\mathbb{Q}$-graded algebras:

$$H^*_\text{CR}(X(\Sigma), \mathbb{C}) \cong \frac{\mathbb{C}[N]^{\Sigma}}{I(\Sigma)}.$$
Theorem 3.1. The map
\[ \bigoplus_{[\rho] \in N_{\text{tor}}} H^*_\text{CR}(\mathcal{X}(\Sigma_{\text{red}}))_{[\rho]} \longrightarrow H^*_\text{CR}(\mathcal{X}(\Sigma), \mathbb{C}), \quad y^{e,\rho}_\rho \mapsto y^{e,\rho}, \]
is an isomorphism of \( \mathbb{Q} \)-graded algebras.

Theorem 3.1 is easily deduced as a corollary of the calculations in [6].

3.2. Gromov-Witten theory. A detailed discussion of the basics of orbifold Gromov-Witten theory can be found in [21]. We obtain a comparison of Gromov-Witten theory of \( \mathcal{X}(\Sigma) \) and \( \mathcal{X}(\Sigma_{\text{red}}) \). Our result is most conveniently stated in terms of the total descendant potential, which is the generating function of all descendant Gromov-Witten invariants. The main tool is a detailed calculation of Gromov-Witten invariants of toric stacks [12].

**Theorem 3.2.** The total descendant potential of \( \mathcal{X}(\Sigma) \) is a sum of \( |N_{\text{tor}}| \) copies (indexed by \( \tilde{N}_{\text{tor}} \)) of the total descendant potential of \( \mathcal{X}(\Sigma_{\text{red}}) \), under the following change of variables:

1. the cohomology variables are changed according to the isomorphism in Theorem [7].
2. the Novikov variables in the descendant potential of \( \mathcal{X}(\Sigma_{\text{red}}) \) indexed by \([\rho] \in \tilde{N}_{\text{tor}}\) are rescaled as follows: \( Q^d \mapsto Q^d \chi_\rho(\sum_{i=1}^n a_i \alpha_i) \) for \( d = \sum_{i=1}^n a_i e_i \) in the Mori cone \( \overline{NE}(\mathcal{X}(\Sigma_{\text{red}})) \subset \text{Ker}(\hat{\beta}) \otimes \mathbb{Z} \subset \mathbb{R}^n \);
3. the genus variable \( h \) in the total descendant potential of \( \mathcal{X}(\Sigma_{\text{red}}) \) is rescaled by \( 1/|N_{\text{tor}}| \).

This Theorem verifies the decomposition conjecture for the Gromov-Witten theory of the gerbe \( \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma_{\text{red}}) \). See [3] for more details.

**Remark 3.3.**

(1) Our results in fact are valid more generally for toric gerbes over toric Deligne-Mumford stacks which are not necessarily orbifolds. Details will be given in [3].

(2) When the base is a \( \mathbb{P}^1 \)-stack with at most two cyclic stack points, our results have also been proven by P. Johnson [21] by a completely different method.

**Example 3.4.** We illustrate part of Theorem 3.2 concerning quantum cohomology rings in the example \( \mathbb{P}(4, 6) \rightarrow \mathbb{P}(2, 3) \), which is the \( \mu_2 \)-gerbe obtained as the stack of square roots of \( \mathcal{O}_{\mathbb{P}(2,3)}(1) \).

In [2] the quantum cohomology rings of \( \mathbb{P}(4, 6) \) and \( \mathbb{P}(2, 3) \) (and more generally all weighted projective lines) are computed:

\[
QH^*_\text{orb}(\mathbb{P}(2, 3), \mathbb{C}) \simeq \mathbb{C}[q][x, y]/(xy - q, 2x^2 - 3y^3),
\]

\[
QH^*_\text{orb}(\mathbb{P}(4, 6), \mathbb{C}) \simeq \mathbb{C}[q][u, v, \xi]/(uv - q \xi, 2u^2 \xi - 3v^3, \xi^2 - 1).
\]

For \( i = 0, 1 \) let \( QH^*_\text{orb}(\mathbb{P}(2, 3), \mathbb{C})_i \) be a copy of \( QH^*_\text{orb}(\mathbb{P}(2, 3), \mathbb{C}) \) with generators \( x_i, y_i \) and \( q \) rescaled by \( (-1)^i \):

\[
QH^*_\text{orb}(\mathbb{P}(2, 3), \mathbb{C})_i = \mathbb{C}[q][x_i, y_i]/(x_i y_i - (-1)^i q, 2x_i^2 - 3y_i^3).
\]

Let \( 1_0 := \frac{1}{2}(1 + \xi), 1_1 := \frac{1}{2}(1 - \xi) \) and \( u_i := (-1)^i u_1, v_i := (-1)^i v_1 \). Then it is easy to check that the additive basis \( \{ 1_i, u_i, v_i, v_i^2 | i = 0, 1 \} \) determines an isomorphism of
algebras:

\[ QH^*_{orb}(\mathbb{P}(4,6), \mathbb{C}) \simeq QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_0 \oplus QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_1, \]

\[ 1_i \mapsto 1 \in QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C}^*_i), u_i \mapsto x_i, v_i \mapsto y_i. \]

For instance,

\[ 1_0 1_1 = \frac{1}{4} (1 - \xi^2) = 0, \quad 1_0 1_0 = \frac{1}{2} (1 + \xi) = 1_0, \quad 1_1 1_1 = 1_1, \]

\[ u_0 v_1 = 0, \quad u_1 v_0 = 0, \]

\[ u_0 v_0 = \frac{1}{2} (uv + uv\xi) = uv 1_0 = q\xi 1_0 = q 1_0, \]

\[ u_1 v_1 = \frac{1}{2} (uv - uv\xi) = uv 1_1 = q\xi 1_1 = q 1_1, \]

\[ 2u_i^2 = 2u_i^2 1_i = 3v_i^3 \xi 1_i = 3v_i^3 (-1)^i 1_i = 3v_i^3, \quad i = 0, 1. \]

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