Adelic Formulas for Gamma and Beta Functions of One-Class Quadratic Fields:
Applications to 4-Particle Scattering String Amplitudes

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Abstract

Regularized adelic formulas for gamma and beta functions for arbitrary quasicharacters (either ramified or not) and in an arbitrary field of algebraic numbers are concretized as applied to one-class quadratic fields (and to the field of rational numbers). Applications to 4-tachyon tree string amplitudes, to the Veneziano (open strings) and Virasoro (closed strings) amplitudes as well as to massless 4-particle amplitudes of the Ramond–Neveu–Schwarz superstring and a heterotic string are discussed. Certain relations between different superstring amplitudes are established.

1 INTRODUCTION

Here, it is relevant to note that the ideas and methods developed by the great Russian mathematician and physicist Nikolai Nikolaevich Bogolyubov (see his selected works [1, 2]) are widely used in the construction of \( p \)-adic mathematical physics. The present work is devoted to the regularization of divergent adelic products for string and superstring 4-particle amplitudes and in many aspects follows the Bogolyubov ideas. In fact, the theory of strings and superstrings has evolved from the theory of dispersion relations in quantum field theory to whose development a fundamental contribution was made by Bogolyubov.

In the last decade, an interest in \( p \)-adic numbers has considerably increased. These numbers were discovered by the German mathematician K. Hensel [3] about 100 years ago. Until recently, they have not found any significant applications although constituted an essential part of number theory and theory of representations [4–8]. Unexpected applications of these numbers have been associated with the discovery of a non-Archimedean structure of space–time at supersmall, the so-called Planck, distances (of order \( 10^{-33} \) cm) in quantum gravitation and theory of strings. Therefore, one cannot use real numbers with Archimedean structure as space–time coordinates at such small distances; one needs other, non-Archimedean number fields. An attempt to preserve rational numbers as physical observables in appropriate non-Archimedean fields leads us to the unambiguous conclusion that the fields of \( p \)-adic numbers \( \mathbb{Q}_p \), where \( p = 2, 3, 5, \ldots \) are prime numbers, can be chosen as such number fields (the Ostrovskii theorem!) [4]. Thus, we arrive at the problem of constructing \( p \)-adic mathematical physics. By now, a considerable progress has been made in this direction (see [9, 10], where the motivation, the history of the problem, and extensive bibliography were presented).

The construction of adelic formulas relating \( N \)-particle scattering amplitudes to appropriate \( p \)-adic amplitudes constitutes a separate field of \( p \)-adic mathematical physics. The first such formulas (without regularization) were proposed by Freund and Witten [12] and, independently, by Volovich [13] in 1987 as applied to 4-tachyon string amplitudes. Mathematically, the problem

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is reduced to the construction of adelic formulas for local gamma and beta functions of algebraic number fields. In [14, 15], such regularized adelic formulas were constructed in arbitrary fields of algebraic numbers for principal (unramified) quasicharacters and in [16, 17], for ramified ones. In the latter case, these formulas are valid under the condition that the quasicharacters of the appropriate group of ideles are trivial (i.e., are equal to 1) on the multiplicative group of the original field of algebraic numbers [7, 17]. This condition yields explicit relations between the parameters of local characters, similar to the formulas for a field of rational numbers. These relations for one-class quadratic fields were obtained in my work [18] (see Section 2). In Section 3, we present regularized adelic formulas for gamma and beta functions as applied to one-class quadratic fields (and to the field of rational numbers). As compared with [16, 17], here we remove the technical requirements that the ranks of local characters should be identical at all finite points (see Remark 5). In Section 4, we apply the formulas obtained to 4-tachyon tree string amplitudes: to the Veneziano (open strings) and Virasoro (closed strings) amplitudes as well as to the massless 4-particle scattering amplitude of the Ramond–Neveu–Schwarz superstring and massless scattering amplitudes of four charged particles of a heterotic string. We establish certain relations between different superstring amplitudes. We also discuss the problem of other possible applications of adelic formulas for ramified quasicharacters.

2 IDELES AND THEIR QUASICHARACTERS FOR ONE-CLASS QUADRATIC FIELDS

First, recall the necessary information from the theory of adeles (and ideles) as applied to the quadratic fields \( \mathbb{Q}(\sqrt{d}) \), where \( d \in \mathbb{Z}, d \neq 0, 1 \), is squares free [7, 8, 18, 19]. For this purpose, we first introduce the following notation.

Denote by \( P_d \) and \( S_d \) the sets of prime numbers \( p \) such that \( d \in \mathbb{Q}_p^{\times 2} \) or \( d \not\in \mathbb{Q}_p^{\times 2} \), respectively.

Let \( A_d \) be the adele ring and \( A_d^\times \) be the idele group of the field \( \mathbb{Q}(\sqrt{d}) \). An arbitrary adele \( X \in A_d \) is expressed as

\[
X = (X_\infty, X_2, X_3, \ldots, X_p, \ldots),
\]

where the symbol \( X_p \) determines the following points of the ring \( A_d \):

\[
X_\infty = \begin{cases} 
  z \in \mathbb{C}, & d < 0, \\
  \{(x, x') \in \mathbb{R}^2, & d > 0; 
\end{cases}
\]

\[
X_p = \begin{cases} 
  (x_p, x'_p) \in \mathbb{Q}_p^2, & p \in P_d, \\
  x_p + \sqrt{d}x'_p \in \mathbb{Q}_p(\sqrt{d}), & p \in S_d;
\end{cases}
\]

in this case, for all \( p \) starting from a certain value, \( (x_p, x'_p) \in \mathbb{Z}_p^2 \), \( p \in P_d \), and \( x_p \in \mathbb{Z}_p(\sqrt{d}) \), \( p \in S_d \). Here, \( \mathbb{Z}, \mathbb{Z}_p, \) and \( \mathbb{Z}_p(\sqrt{d}) \) are the rings of integers of the fields \( \mathbb{Q}, \mathbb{Q}_p, \) and \( \mathbb{Q}_p(\sqrt{d}) \), respectively. The group of ideles \( A_d^\times \) is defined similarly. Ideles are invertible adeles.

An arbitrary (multiplicative) quasicharacter \( \Theta \) of the group \( A_d^\times \) on the idele \( X \) is represented as [7, 11, 18]

\[
\Theta(X; \alpha) = \Theta_\infty(X_\infty) \prod_{p \in P_d} \theta_p(x_p)\theta'_p(x'_p)\left|x_p\right|^{i\alpha_p}\left|x'_p\right|^{i\alpha'_p} \prod_{p \in S_d} \theta_p(X_p)|X_pX_p|^{i\alpha_p}|X|^\alpha, \quad \alpha \in \mathbb{C},
\]

(2.1)

where \( X_p = x_p - \sqrt{d}x'_p \), \( X_pX_p = x_p^2 - dx'_p^2 \), and

\[
\Theta_\infty(X_\infty) = \begin{cases} 
  z^{\nu}(z\overline{z})^{-\nu/2}, & \nu \in \mathbb{Z}, d < 0, \\
  \text{sgn}^\nu x \cdot \text{sgn}^{\nu'} x'|^\alpha, & \nu, \nu' \in F_2, a \in \mathbb{R}, d > 0.
\end{cases}
\]

(2.2)

Here, \(|X|\) denotes the normalization of the idele \( X \) and \( F_2 \) is a field of integers modulo 2.
Suppose that the local characters $\theta_p$ and $\theta'_p$ in (2.1) are normalized by the conditions
\[ \theta_p(p) = 1, \quad \theta'_p(p) = 1, \quad p \in P_d; \quad \theta_p(q) = 1, \quad p \in S_d, \tag{2.3} \]
where $q = q_p$ is the module over the field $\mathbb{Q}_{p}(\sqrt{d})$, $p \in S_d$.

In (2.1), only a finite number of characters $\theta_p$ and $\theta'_p$ are nontrivial (ramified). Denote the ranks of these characters by $\rho_p = \rho(\theta_p)$ and $\rho'_p = \rho(\theta'_p)$, respectively. Denote by $F$ the set of finite points for which the rank of the local character is equal to 0 (unramified points); the set of other finite points is denoted by $R$ (ramified points).

The triviality of the quasicharacter $\Theta$ on the principal idele $X = I_x$ generated by the number $x \in \mathbb{Q}^\times(\sqrt{d})$ is expressed as
\[ \Theta(I_x; \alpha) = \Theta(x) = 1, \quad x \in \mathbb{Q}^\times(\sqrt{d}). \tag{2.4} \]

Condition (2.4) implies that, actually, the quasicharacter $\Theta$ is defined over the factor group $A_d^\times / \mathbb{Q}^\times(\sqrt{d})$, the idele class group over the field $\mathbb{Q}(\sqrt{d})$ [7].

Denote the norm of the leading ideal $J$ of the quasicharacter $\Theta$ by $N$ [7]:
\[ N = N(J) = \prod_{p \in P_d \cap R} p^{\rho_p + \rho'_p} \prod_{p \in S_d \cap R} q_p^{\rho_p}. \tag{2.5} \]

A standard additive character $\chi$ of the adele ring $A_d$ is given by
\[ \chi(X) = \chi_{\infty}(X_{\infty}) \prod_{p \in P_d} \chi_p(x_p + x'_p) \prod_{p \in S_d} \chi_p(X_p + \mathcal{X}_p), \quad X \in A_d, \tag{2.6} \]
where
\[
\chi_{\infty}(X_{\infty}) = \begin{cases} \exp[-2\pi i (z + \overline{z})/d, & d < 0, \\ \exp[-2\pi i (x + x')/d, & d > 0; \end{cases}
\]
\[ \chi_p(x) = \exp(2\pi i x_p), \quad x \in \mathbb{Q}_p, \]
where $\{x\}_p$ is the fractional part of the $p$-adic number $x$.

Denote by $r_p$ the rank of the local character $\chi_p(X + \mathcal{X}), X \in \mathbb{Q}_p(\sqrt{d}), p \in S_d$. Only a finite number of these ranks are different from 0.

Denote by $D$ the discriminant of the field $\mathbb{Q}(\sqrt{d})$:
\[ D = \begin{cases} d, & d \equiv 1 \pmod{4}, \\ 4d, & d \equiv 2, 3 \pmod{4}, \end{cases} \quad |D| = \prod_{p \in S_d} q_p^{\rho_p}. \tag{2.7} \]

The unities $\epsilon$ of the quadratic fields $\mathbb{Q}(\sqrt{d})$ are [4] $\epsilon = \pm 1$ and, in addition,
\[ \epsilon = \pm i \quad \text{for } d = -1, \]
\[ \epsilon = \pm e^{\pm \pi/3} = \pm(1 \pm \sqrt{-3})/2 \quad \text{for } d = -3, \]
\[ \epsilon = \pm \Omega^\gamma, \quad \gamma \in \mathbb{Z}, \quad \text{for } d > 0, \]
where $\Omega$ is the primary unity of the field $\mathbb{Q}(\sqrt{d}), N(\Omega) = \Omega\overline{\Omega} = \pm 1, \Omega > 0$.

All one-class quadratic fields $\mathbb{Q}(\sqrt{d})$ are well known [4]: there exist nine imaginary fields of this type with $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$ and an infinite number of real fields with $d = p, \ p \equiv 3 \pmod{4}$; $d = 2p, \ p \equiv 3 \pmod{4}$; and $d = pp', \ p, p' \equiv 3 \pmod{4}$ ($p$ and $p'$ are prime numbers).

For one-class fields, the group of classes of divisors is trivial; i.e., all these divisors belong to the ring of integers (to the maximum order), and the prime factor decomposition in this field is unique [4].
Proposition (see [4, 18]). All prime divisors \( p \equiv p_p \) and \( \overline{p} \equiv \overline{p}_p \) of one-class quadratic fields \( \mathbb{Q}(\sqrt{d}) \) and their norms \( q \equiv q_p \) are classified under the following cases:

(a) \( p \) divides \( D_p, p \in S_d, \quad p = cp^2, \quad q = p; \) \hfill (2.8)

(b) \( p \) does not divide \( D_p, p \in S_d, \quad p = p, \quad q = p^2; \) \hfill (2.9)

(c') \( p \) does not divide \( D_p, p \in P_d, D = 4d, \quad p, \overline{p} = a \pm \sqrt{db}, \quad q = p, \) \hfill (2.10)
where \( a > 0 \) and \( b > 0 \) are integer solutions to the Diophantine equation

\[ p = |x^2 - d y^2|; \] \hfill (2.11)

(c'') \( p \) does not divide \( D_p, p \in P_d, D = d, \quad p, \overline{p} = a + \frac{b}{2} \pm \sqrt{\frac{b}{2}}, \quad q = p, \) \hfill (2.12)
where \( a > 0 \) and \( b \neq 0 \) are integer solutions to the Diophantine equation

\[ p = x^2 + xy + \frac{1 - d}{4} y^2 \] \hfill (2.13)

(for \( d = -7 \) and \( p = 2 \), the solution is \( a = 0, \ b = 1 \).)

Here, in cases (c') and (c''), the divisors \( p \) and \( \overline{p} \) can be chosen so that the conditions

\[ |p|_p = 1/p, \quad |\overline{p}|_p = 1 \] \hfill (2.14)

are fulfilled.

Remark 1. For one-class quadratic fields, the Diophantine equations (2.11) and (2.13) are solvable for any prime \( p \).

The following two theorems give the necessary and sufficient conditions for the triviality of the quasicharacter of the idele group of a field on the multiplicative group of this field.

Theorem 1 (see [8]). The quasicharacter

\[ \Theta(X; \alpha) = \text{sgn}^\nu x_{\infty} \prod_{p=2}^\infty \theta_p(x_p)|x_p|^{i \alpha_p}X|^{\alpha}, \quad \nu \in F_2, \quad \alpha \in \mathbb{C}, \] \hfill (2.15)

\((\theta_p(p) = 1)\) of the group \( A^\times \) of ideles \( X = (x_\infty, x_2, \ldots, x_p, \ldots) \) of the field \( \mathbb{Q} \) is trivial on the group \( \mathbb{Q}^\times \) if and only if the following conditions hold:

\[ \theta(-1) = 1, \quad \theta(p) = p^{i \alpha_p}, \quad p = 2, 3, \ldots, \] \hfill (2.16)

where \( \theta \) is a character of the group \( A^\times \) of the form

\[ \theta(X) = \text{sgn}^\nu x_{\infty} \prod_{p=2}^\infty \theta_p(x_p). \] \hfill (2.17)

Example of a ramified character of the group \( A^\times / \mathbb{Q}^\times \):

\[ \theta(X) = \text{sgn} x_\infty \theta_2(x_2) \prod_{p=3}^\infty |x_p|^{i \alpha_p}, \quad \theta_2(-1) = -1, \quad \theta_2(p) = p^{i \alpha_p}, \quad p = 3, 5, \ldots. \]

Theorem 2 (see [18]). The quasicharacter \( \Theta \) of the idele group \( A^\times_d \) of the one-class quadratic field \( \mathbb{Q}(\sqrt{d}) \) is trivial on the group \( \mathbb{Q}^\times(\sqrt{d}) \) if and only if the following conditions hold:

\[ \theta(-1) = 1 \quad \text{for all admissible } d; \] \hfill (2.18)

\[ \theta(i) = 1 \quad \text{for } d = -1; \] \hfill (2.19)

\[ \theta(i^{i \alpha}) = 1 \quad \text{for } d = -3; \] \hfill (2.20)

\[ \theta(\Omega) = \Omega^{-i \alpha} \quad \text{for all admissible } d > 0; \] \hfill (2.21)

\[ p^{i \alpha_p} = \theta(p)|p|^{i \alpha_{ab}(d)}, \quad p^{i \alpha_{\overline{p}}} = \theta(\overline{p})|\overline{p}|^{i \alpha_{a\overline{b}}(d)}, \quad p \in P_d; \] \hfill (2.22)

\[ p^{i \alpha_p} = \theta(p)|p|^{i \alpha_{ab}(d)}, \quad p \text{ divides } D, \quad p \in S_d; \] \hfill (2.23)

\[ p^{2 i \alpha_p} = \theta(p)p^{i \alpha_{ab}(d)}, \quad p \text{ does not divide } D, \quad p \in S_d; \] \hfill (2.24)
where the character \( \theta \) denotes a character of the idele group \( A_d^\times \),

\[
\theta(X) = \theta_\infty(X_\infty) \prod_{p \in P_d \cap R} \theta_p(x_p) \theta'_p(x'_p) \prod_{p \in S_d \cap \R} \theta_p(X_p),
\]

(2.25)

\[
\theta_\infty(X_\infty) = \begin{cases}
  z^\nu(z\overline{z})^{-\nu/2}, & \nu \in \Z, \ d < 0, \\
  \text{sgn}^\nu x \cdot \text{sgn}^{\nu'} x', & \nu, \nu' \in F_2, \ d > 0
\end{cases}
\]

(2.26)

\( (h(d) = 1 \text{ for } d > 0, \ h(d) = 0 \text{ for } d < 0). \)

**Remark 2.** Equation (2.21) ambiguously determines the real number \( a \):

\[
a = -\frac{\text{Arg} \, \theta(\Omega)}{\ln \Omega} + \frac{2\pi}{\ln \Omega} Z.
\]

Therefore, the real numbers \( \alpha_p \) and \( \alpha'_p \) are determined by equalities (2.22)–(2.24) ambiguously. However, in what follows, we will need only the exponential functions \( p^{i\nu_p} \) and \( p^{i\nu'_p}, \ p \in P_d, \) and \( q^{i\nu}, \ p \in S_d \), of these numbers; for \( d > 0 \), these exponential functions are determined uniquely.

**Remark 3.** For the principal quasicharacter \( |X|^\alpha, \alpha \in \C, \) conditions (2.18)–(2.24) hold trivially by virtue of the known Artin multiplication formula \( |I_x| = 1, \ x \in \Q_p \sqrt{d} \).

### 3 ADELIC FORMULAS FOR GAMMA AND BETA FUNCTIONS OF ONE-CLASS QUADRATIC FIELDS AND A FIELD OF RATIONAL NUMBERS

Taking into account equality (2.18), we can represent the regularized adelic formulas of one-class quadratic fields \( \Q(\sqrt{d}) \) for gamma functions of the quasicharacter \( \Theta(X; \alpha) \) (defined by (2.1)) of the idele group \( A_d^\times / \Q^\times(\sqrt{d}) \) as [16, 17]

\[
\theta(-1) = \text{reg} \left[ \prod_{p=2}^{\infty} \Gamma(\alpha + i\alpha_p; \theta_p) \prod_{p \in P_d \cap F} \Gamma(\alpha + i\alpha'_p; \theta'_p) \right] \begin{cases}
  \Gamma_{\omega}(\alpha; \nu), & d < 0, \\
  \Gamma_{\infty}(\alpha + i\nu) \Gamma_{\infty}(\alpha; \nu'), & d > 0
\end{cases}
\]

(3.1)

where the character \( \theta(X) \) is defined by (2.25) and

\[
\Gamma_{\omega}(\alpha; \nu) = \int_{\C} z^{\nu}(z\overline{z})^{-\nu/2+\alpha-1} \exp[-2\pi i(z + \overline{z})] |dz \wedge d\overline{z}|
\]

\[
= i^{\nu/2}(2\pi)^{-2\nu} \Gamma\left(\alpha + \frac{\nu}{2}\right) \Gamma\left(\alpha - \frac{\nu}{2}\right) \sin \frac{\pi}{2}(2\alpha - \nu), \quad \nu \in \Z,
\]

(3.2)

is the gamma function of quasicharacter \( z^\nu(z\overline{z})^{-\nu/2+\alpha} \) of the field \( \C \);

\[
\Gamma_{\infty}(\alpha; \nu) = \int_{\R} \text{sgn}^\nu x |x|^\alpha \exp(-2\pi i \nu) \, dx = 2i^{-\nu} (2\pi)^{-\alpha} \Gamma(\alpha) \cos \frac{\pi}{2}(\alpha - \nu), \quad \nu \in F_2,
\]

(3.3)

is the gamma function of quasicharacter \( \text{sgn}^\nu x |x|^\alpha \) of the field \( \R \) (\( \Gamma(\alpha) \) is the Euler gamma function);

\[
\Gamma(\alpha; \theta) = q^{-\alpha} \int_{\R} \theta(x) |x|^\alpha \chi(x) \, dx = \begin{cases}
  q^{(\alpha-1)/2} \Gamma_q(\alpha), & p \in F, \\
  \kappa(\theta) q^{(\alpha-1)/2(r+p)}, & p \in R
\end{cases}
\]

(3.4)

\[
\Gamma_q(\alpha) = G_q(q^\alpha), \quad \alpha \neq \frac{2\pi i}{\ln q} Z, \quad G_q(x) = \frac{1-x/q}{1-x^{-1}},
\]

(3.4')

is the reduced gamma function of a local p-field with the module \( q \) and

\[
\kappa(\theta) = q^{\alpha/2} \int_{|x|=1} \theta(x) \chi(\pi^{-r-p}x) \, dx, \quad |\kappa(\theta)| = 1.
\]
By Theorem 2 (see Section 2), equalities (2.18)–(2.24) are valid for one-class quadratic fields. Using these equalities, we eliminate the numbers $\alpha_0$ and $\alpha'_0$ from the gamma functions in adelic formulas (3.1). As a result, we obtain the adelic formulas

$$\theta(-1)\mathcal{X}(N|D|)^{1/2-\alpha} = \text{reg} \prod_{p \in F} \Gamma_q(\alpha + i\alpha_p) \prod_{p \in P_d \cap F} \Gamma_p(\alpha + i\alpha'_p) \left\{ \begin{array}{ll}
\Gamma_{\omega}(\alpha; \nu), & d < 0, \\
\Gamma_{\infty}(\alpha + i\alpha, \nu)\Gamma_{\infty}(\alpha; \nu), & d > 0,
\end{array} \right. \quad (3.5)$$

where the numbers $N$ and $D$ are defined by formulas (2.5) and (2.7),

$$\mathcal{X} = \prod_{p = 2}^{\infty} \mathcal{X}(\theta_p)q^{-i\alpha_p(p+\rho_p)} \prod_{p \in P_d \cap R} \mathcal{X}(\theta'_p)p^{-i\alpha'_p}, \quad |\mathcal{X}| = 1, \quad (3.6)$$

where the quantities $q_p^{-i\alpha_p}$ and $p^{-i\alpha'_p}$ are determined by equalities (2.22)–(2.24), while $\Gamma_q(\alpha + i\alpha_p)$ and $\Gamma_p(\alpha + i\alpha'_p)$ are determined by the formulas

$$\Gamma_q(\alpha + i\alpha_p) = G_p[p^\rho\theta(p)p^{i\beta(\alpha, \nu)}], \quad (3.7)$$

$$\Gamma_p(\alpha + i\alpha'_p) = G_p[p^\rho\theta(p)p^{i\beta(\alpha, \nu)}], \quad p \in P_d \cap F, \quad (3.8)$$

$$\Gamma_q(\alpha + i\alpha_p) = G_q[q^\rho\theta(p)p^{i\beta(\alpha, \nu)}], \quad p \in S_d \cap F. \quad (3.9)$$

(In (3.6), we assume that $\mathcal{X}(\theta_p) = 1$ for $p \in F$.)

Suppose that, in addition to the quasicharacter $\Theta(X; \alpha)$, another quasicharacter $\Pi(Y; \beta)$ of the idele group $A_\mathbb{Q}^\times / \mathbb{Q}^\times (\sqrt{D})$ is defined,

$$\Pi(Y; \beta) = \pi_{\infty}(Y_{\infty}) \prod_{p \in P_d} \pi_p(Y_p) \prod_{p \in S_d} \pi_p(YP) \prod_{p \in F} \pi_p(Y_p) \pi_p(YP) |Y|^{-\beta}, \quad \beta \in \mathbb{C}, \quad (3.10)$$

where $b, \beta_p, \text{ and } \beta'_p$ are real numbers and

$$\pi_{\infty}(Y_{\infty}) = \begin{cases} 
\zeta^{\mu}(\zeta)^{-\mu/2}, & \mu \in \mathbb{Z}, \ d < 0, \\
\text{sgn}^\mu y \cdot \text{sgn}^\mu y' |y|^{\beta}, & \mu, \mu' \in F_2, \ d > 0.
\end{cases} \quad (3.11)$$

Let us construct, by formulas (2.25) and (2.26), the character $\pi(Y)$ corresponding to the quasicharacter $\Pi$ and then the character $\sigma(Z) = \bar{\theta}(Z) \pi(Z)$, so that

$$\theta \pi \sigma = 1, \quad \sigma_p = \bar{\theta}_p \pi_p, \quad p = 2, 3, \ldots, \quad \sigma'_p = \bar{\theta}_p \pi_p, \quad p \in P_d. \quad (3.12)$$

Denote by $R, R', R''$ and $F, F', F''$ the sets of ramified and unramified points of the characters $\theta, \pi, \text{ and } \sigma$, respectively.

The beta functions of local fields are defined as the following products of three gamma functions:

$$B_\omega(\alpha, \nu; \beta; \gamma, \eta, \mu) = \Gamma_{\omega}(\alpha; \nu) \Gamma_{\omega}(\beta; \mu) \Gamma_{\omega}(\gamma; \eta), \quad \beta + \gamma = 1, \quad \nu + \mu + \eta = 0, \quad \nu, \mu, \eta \in \mathbb{Z}, \quad (3.13)$$

for the quasicharacters $\zeta(\zeta^{-2+\alpha})$ and $\zeta(\zeta^{-2+\beta})$, $\nu, \mu \in \mathbb{Z}$, of the field $\mathbb{C}$;

$$B_{\infty}(\alpha, \nu; \beta; \gamma, \eta) = \Gamma_{\infty}(\alpha; \nu) \Gamma_{\infty}(\beta; \mu) \Gamma_{\infty}(\gamma; \eta), \quad \beta + \gamma = 1, \quad \nu + \mu + \eta = 0, \quad \nu, \mu, \eta \in F_2, \quad (3.14)$$

for the quasicharacters $\text{sgn}^\mu x |x|^\alpha$ and $\text{sgn}^\mu y |y|^\beta$, $\nu, \mu \in F_2$, of the field $\mathbb{R}$; and

$$B(\alpha, \beta; \pi; \gamma, \sigma) = \Gamma(\alpha; \theta) \Gamma(\beta; \pi) \Gamma(\gamma; \sigma), \quad \alpha + \beta + \gamma = 1, \quad \theta \pi \sigma = 1, \quad (3.15)$$
for the local quasicharacters $|x|^\alpha \theta(x)$ and $|y|^\beta \pi(y)$ of the $p$-fields.

In particular, for the principal quasicharacters $|x|^\alpha_p, |y|^\beta_p, p \in P_d \cap F$, $|X|^\alpha_p, |Y|^\beta_p, p \in S_d \cap F$, of the $p$-fields with the module $q = q_p$, the beta function is defined as

$$B_q(\alpha, \beta, \gamma) = \Gamma_q(\alpha) \Gamma_q(\beta) \Gamma_q(\gamma).$$  \hspace{1cm} (3.16)

By virtue of (3.1) and (3.13)–(3.15), the local beta functions defined by the quasicharacters $\Theta(X; \alpha)$ and $\Pi(Y; \beta)$ of the idele group $\mathbb{A}_e^\times / \mathbb{Q}^\times(\sqrt{d})$ satisfy the following adelic equality (for $\alpha + \beta + \gamma = 1$):

$$1 = \text{reg} \left[ \prod_{p=2}^{\infty} B(\alpha + i\alpha_p, \theta_p; \beta + i\beta_p, \pi_p; \gamma + i\gamma_p, \sigma_p) \right] \times \prod_{p \in P_d \cap (F \cup F^{\prime \prime})} B(\alpha + i\alpha_p\prime, \theta_p\prime; \beta + i\beta_p\prime, \pi_p\prime; \gamma + i\gamma_p\prime, \sigma_p\prime)$$

$$= B_\infty(\alpha, \beta, \gamma, \alpha_p, \beta_p, \gamma_p, \alpha_p\prime, \beta_p\prime, \gamma_p\prime)$$

$$= G_p[p^\alpha \theta(p)p^{ih(d)\alpha}], G_p[p^\beta \pi(p)p^{ih(d)\beta}], G_p[p^\gamma \sigma(p)p^{ih(d)\gamma}], \quad p \in P_d \cap F, \text{ (3.17)}$$

$$= G_q[q^\alpha \theta(p)q^{ih(d)\alpha}], G_q[q^\beta \pi(p)q^{ih(d)\beta}], G_q[q^\gamma \sigma(p)q^{ih(d)\gamma}], \quad p \in S_d \cap F. \text{ (3.18)}$$

Note that, for $d > 0$, the following equalities hold in (3.17) in accordance with the formulas (2.21) and (3.12):

$$\Omega^{ia} = \overline{\sigma}(\Omega), \quad \Omega^{ib} = \pi(\Omega), \quad \Omega^{ic} = \overline{\pi}(\Omega) = \theta(\Omega) \pi(\Omega); \text{ (3.21)}$$

therefore, $a + b + c = 0$ in this case as well.

For the principal quasicharacters $|X|^\alpha$ and $|Y|^\beta$ ($\alpha = \mu = \eta = 0$, $\nu = \mu' = \eta' = 0$, $\alpha_p = \beta_p = \gamma_p = 0$, $\alpha_p\prime = \beta_p\prime = \gamma_p\prime = 0$, $\kappa = 1$, $N(J) = 1$, and $R$ is an empty set), the adelic formula (3.17) for an arbitrary quadratic field $\mathbb{Q}(\sqrt{d})$ is rewritten as [19]

$$\sqrt{|D|} = \text{reg} \left[ \prod_{p \in P_d} B_p^2(\alpha, \beta, \gamma) \prod_{p \in S_d} B_q(\alpha, \beta, \gamma) \right] \begin{cases} B_\omega(\alpha, \beta, \gamma), & \alpha + \beta + \gamma = 1, \ d < 0, \\ B_\infty^2(\alpha, \beta, \gamma), & \alpha + \beta + \gamma = 1, \ d > 0, \end{cases} \text{ (3.22)}$$

where

$$B_\infty(\alpha, \beta, \gamma) = B_\infty(\alpha, \beta, \gamma), \text{ (3.23)}$$

$$B_\omega(\alpha, \beta, \gamma) = B_\omega(\alpha, \beta, \gamma). \text{ (3.23')}$$
In particular, for the Gauss field \( \mathbb{Q}(\sqrt{-1}) \), this formula is represented as

\[
B_{\omega}(\alpha, \beta, \gamma)B_{2}(\alpha, \beta, \gamma) \gg \prod_{p=1(4)} B_{2}^{2}(\alpha, \beta, \gamma) \prod_{p=5(4)} B_{p^2}(\alpha, \beta, \gamma) \gg 2, \quad \alpha + \beta + \gamma = 1.
\] (3.24)

For the field \( \mathbb{Q} \) of rational numbers, we proceed similarly and in a more simple manner.

By Theorem 1 (see Section 2), the adelic formula for the gamma functions of the field \( \mathbb{Q} \) that are determined by the quasicharacter \( \Theta(X; \alpha) \) of the form (2.15) is given by [16, 17] (cf. (3.5))

\[
\Gamma_{\infty}(\alpha; \nu) \gg \prod_{p \in F} \Gamma_{p}(\alpha + i\alpha_{p}) = \theta(-1) \prod_{p \in R} \xi_{p} \theta_{p}^{-\alpha_{p}} N_{1/2}^{1-\alpha},
\] (3.25)

where

\[
\Gamma_{p}(\alpha + i\alpha_{p}) = \frac{1 - p^{\alpha-1}}{1 - p^{-\alpha}} \theta_{p}, \quad p \in F.
\] (3.26)

(The character \( \theta(X) \) is defined by (2.17).)

For the quasicharacters \( \Theta(X; \alpha) \) and \( \Pi(Y; \beta) \) of the form (2.15), the adelic formula for the beta functions of the field \( \mathbb{Q} \) is given by [16, 17]

\[
B_{\infty}(\alpha, \nu; \beta, \mu; \gamma, \eta) \gg \prod_{p \in F \cap F' \cap F''} B_{p}(\alpha + i\alpha_{p}, \beta + i\beta_{p}, \gamma + i\gamma_{p}) = \xi(N N' N'')^{1/2-\alpha},
\] (3.27)

where the characters \( \theta \) and \( \pi \) are constructed by the quasicharacters \( \Theta \) and \( \Pi \) by formula (2.17), \( \Theta \pi \sigma = 1; \rho_{p}, \rho'_{p}, \rho''_{p} \) and \( N, N', N'' \) are the ranks and the norms of the leading ideals of the characters \( \theta, \pi, \) and \( \sigma \) of the field \( \mathbb{Q} \), respectively; and

\[
\xi = \prod_{p \in R} \xi_{p} \theta_{p}^{-\alpha_{p}} \rho_{p} \prod_{p \in F'} \xi_{p} \pi_{p}^{-\beta_{p} \rho_{p}} \prod_{p \in F''} \xi_{p} \sigma_{p}^{-\gamma_{p} \rho''_{p}}, \quad |\xi| = 1.
\] (3.28)

By virtue of (2.16), (3.16), and (3.26),

\[
B_{p}(\alpha + \alpha_{p}, \beta + \beta_{p}, \gamma + \gamma_{p}) = \frac{1 - p^{\alpha-1}}{1 - p^{-\alpha}} \theta_{p}^{-\alpha_{p} \rho_{p}} (1 - p^{-\beta} \pi_{p}^{-\beta_{p} \rho'_{p}}) (1 - p^{-\gamma} \sigma_{p}^{-\gamma_{p} \rho''_{p}}), \quad p \in F,
\] (3.29)

in formula (3.27).

For the principal quasicharacters \( |X|^\alpha \) and \( |Y|^\beta \) (\( \nu = \mu = \eta = 0, \alpha_{p} = \beta_{p} = \gamma_{p} = 0, \) and \( N = 1 \)) formula (3.27) takes the form [20]

\[
B_{\infty}(\alpha, \beta, \gamma) \gg \prod_{p=2}^{\infty} B_{p}(\alpha, \beta, \gamma) = 1, \quad \alpha + \beta + \gamma = 1.
\] (3.30)

To sum up, we present the following theorems.

**Theorem 3** (see [17]). Suppose that the quasicharacters \( \Theta \) and \( \Pi \) of the form (2.15) are given on the idele class group of the field \( \mathbb{Q} \), and let \( \theta, \pi, \) and \( \sigma, \theta \pi \sigma = 1, \) be the characters, of the form (2.17), of the idele group of the field \( \mathbb{Q} \). Then, the adelic formulas (3.25), (3.27) are valid for local gamma and beta functions.

**Theorem 4.** Suppose that the quasicharacters \( \Theta \) and \( \Pi \) of the form (2.1) are given on the idele class group of the one-class field \( \mathbb{Q}(\sqrt{d}) \), and let \( \theta, \pi, \) and \( \sigma, \theta \pi \sigma = 1, \) be the characters, of the form (2.25), of the idele group of the field \( \mathbb{Q}(\sqrt{d}) \). Then, the adelic formulas (3.5) and (3.17) are valid for local gamma and beta functions.
Remark 4. The adelic formulas (3.1) and (3.17) for the gamma function $\Gamma_{\mathcal{A}_d}(\alpha; \ldots)$ and the beta function $B_{\mathcal{A}_d}(\alpha, \beta; \gamma; \ldots)$ of the adelic ring $\mathcal{A}_d$ imply that $\Gamma_{\mathcal{A}_d}(\alpha; \ldots) = \theta(-1)$ and $B_{\mathcal{A}_d}(\alpha, \beta; \gamma; \ldots) = 1$, $\alpha + \beta + \gamma = 1$. Similar expressions are valid for the field $\mathcal{Q}$.

Remark 5. In [17, 16], I established the adelic formulas under the condition that the ranks of the local characters $\theta_p, \pi_p, \sigma_p$, $p = 2, 3, \ldots$, and $\theta'_p, \pi'_p, \sigma'_p$, $p \in P_d$, are identical:

$$\rho(\theta_p) = \rho(\pi_p) = \rho(\sigma_p), \quad p = 2, 3, \ldots; \quad \rho(\theta'_p) = \rho(\pi'_p) = \rho(\sigma'_p), \quad p \in P_d.$$ 

In the present work, this requirement is removed at the expense of complicating the adelic formulas.

Example of characters $\theta, \pi$, and $\sigma = \bar{\theta}\pi$ with identical ranks for $p = 5$: $x_5 = 2^k$, $a \in \mathbb{Z}_5^\times$, $|1 - a|_5 < 1$.

$$\theta_5(x_5) = \pi_5(x_5) = \exp(2\pi ik/5), \quad k = 0, 1, 2, 3; \quad \rho(\theta_5) = \rho(\pi_5) = \rho(\bar{\theta}_5) = 1.$$ 

4 APPLICATION TO 4-PARTICLE STRING AMPLITUDES

Suppose that $s, t, u$ are the Mandelstam variables for a 4-particle scattering process with the momenta $(p_1, p_2, p_3, p_4)$, $p_1 + p_2 + p_3 + p_4 = 0$, $p_i^2 = m_i^2$ in an $n$-dimensional space $\mathbb{R}^n$ with the Minkowskian metric $p^2 = (p^1)^2 - (p^1)^2 - \cdots - (p^n)^2$, so that

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2, \quad s + t + u = \sum m_i^2. \quad (4.1)$$

In the general case of 4-particle scattering processes with the Regge trajectory $\alpha(s) = \alpha + \alpha's$, we define the generalized crossing-symmetric Veneziano $V$ and Virasoro $W$ amplitudes and their $p$-adic analogues $V_p$ and $W_p$ by the following formulas ($s + t + u = \sum m_i^2$) [21]:

$$V(s, t, u) = B_\infty(-\alpha - \alpha's, -\alpha - \alpha't, -\alpha - \alpha'u); \quad (4.2)$$
$$V_p(s, t, u) = B_q(-\alpha - \alpha's, -\alpha - \alpha't, -\alpha - \alpha'u); \quad (4.3)$$
$$W(s, t, u) = B_\omega(-\alpha/2 - \alpha'/2s, -\alpha/2 - \alpha'/2t, -\alpha/2 - \alpha'/2u); \quad (4.4)$$
$$W_p(s, t, u) = B_q(-\alpha/2 - \alpha'/2s, -\alpha/2 - \alpha'/2t, -\alpha/2 - \alpha'/2u). \quad (4.5)$$

The slope $\alpha'$ and the intercept $\alpha$ of the trajectory $\alpha(s) = \alpha + \alpha's$ must satisfy the following condition:

$$3\alpha + \alpha'\sum m_i^2 = \left\{ \begin{array}{ll} -1 & \text{for amplitude } V, \\ -2 & \text{for amplitude } W. \end{array} \right. \quad (4.6)$$

By virtue of (3.21), the amplitudes (4.2)–(4.5) satisfy the following adelic relations (provided that $s + t + u = \sum m_i^2$):

$$V^2(s, t, u) \text{ reg } \prod_{p \in P_d} V^2_p(s, t, u) \prod_{p \in S_d} V_p(s, t, u) = \sqrt{|D|}, \quad d > 0; \quad (4.7)$$
$$W(s, t, u) \text{ reg } \prod_{p \in P_d} W^2_p(s, t, u) \prod_{p \in S_d} W_p(s, t, u) = \sqrt{|D|}, \quad d < 0. \quad (4.8)$$

Formulas (4.7), (4.8) are well known [13, 19]. They provide a decomposition of generalized Veneziano $V$ ($d > 0$) and Virasoro $W$ ($d < 0$) amplitudes into the infinite product of the inverses $V_p^{-1}$ and $W_p^{-1}$ of the $p$-adic amplitudes $V_p$ and $W_p$, respectively.

Let us present certain important particular cases of the generalized amplitudes introduced. These are, first of all, 4-tachyon crossing-symmetric string amplitudes for tree orientable diagrams—the Veneziano and Virasoro amplitudes [11–13, 21–25].
The Veneziano amplitude for open strings in $\mathbb{R}^{26}$.

There exist only one tree orientable diagram that is conformally equivalent to a unit circle with four punctured points on its boundary. This diagram corresponds to the crossing-symmetric Veneziano amplitude (for $\alpha = 1$, $\alpha' = 1/2$, $m_2^2 = -2$, see formula (4.2))

$$V(s, t, u) = B_{\infty}(-1 - s/2, -1 - t/2, -1 - u/2), \quad s + t + u = -8. \quad (4.9)$$

The amplitudes $V_p$ are determined similarly by formula (4.3).

The Virasoro amplitude for closed strings in $\mathbb{R}^{26}$.

There exists only one tree orientable diagram that is conformally equivalent to a unit sphere with four punctured points. This diagram corresponds to the crossing-symmetric Virasoro amplitude (for $\alpha = 2$, $\alpha' = 1/4$, $m_2^2 = -8$, see formula (4.4))

$$W(s, t, u) = B_{\infty}(-1 - s/8, -1 - t/8, -1 - u/8), \quad s + t + u = -32. \quad (4.10)$$

The amplitudes $W_p$ are determined similarly by formula (4.5).

The Veneziano and Virasoro amplitudes are generalized to the case of ramified quasicharacters by the formulas (under conditions (4.1) and (4.6))

$$V_{\nu\mu\eta}(s, t, u) = B_{\infty}(-\alpha - \alpha' s, v; -\alpha - \alpha' t, \mu; -\alpha - \alpha' u, \eta), \quad \nu + \mu + \eta = 0, \quad \nu, \mu, \eta \in F_2, \quad d > 0; \quad (4.11)$$

$$W_{\nu\mu\eta}(s, t, u) = B_{\infty}(-\alpha - \alpha' s, v; -\alpha - \alpha' t, \mu; -\alpha - \alpha' u, \eta), \quad \nu + \mu + \eta = 0, \quad \nu, \mu, \eta \in Z, \quad d < 0. \quad (4.12)$$

These amplitudes are crossing symmetric with respect to the permutations of the variables $(s, v)$, $(t, \mu)$, and $(u, \eta)$ and satisfy the adelic formulas of Section 3 (for one-class quadratic fields and a field of rational numbers).

The massless 4-particle amplitude of the Ramond–Neveu–Schwarz superstring in $\mathbb{R}^{10}$.

This amplitude is proportional to [11, 21]

$$A_{\infty}(s, t, u) = \Gamma_{\infty}(-s/2; 1)\Gamma_{\infty}(-t/2; 1)\Gamma_{\infty}(-u/2; 1), \quad s + t + u = 0. \quad (4.13)$$

The corresponding simple adelic formula for $\nu = \mu = \eta = 1$ and $\rho_p = \rho(\theta_p) = \rho(\pi_p) = \rho(\sigma_p)$, $p \in R = R' = R''$, is rewritten as (see (3.25))

$$A_{\infty}(s, t, u) \operatorname{reg} \prod_{p \in P} \Gamma_p(-s/2 + i\alpha_p)\Gamma_p(-t/2 + i\beta_p)\Gamma_p(-u/2 + i\gamma_p) = \kappa N\sqrt{N}, \quad s + t + u = 0. \quad (4.14)$$

where $\kappa$ is defined by (3.28); the functions $\Gamma_p(-s/2 + i\alpha_p), \ldots$ are calculated by (3.26), and the quantities $p^{-i\alpha_p}, \ldots$, by formulas of the type (2.16).

Another adelic formula for the amplitude $A_{\infty}(s, t, u)$ was proposed in [26].

Massless amplitudes for four charged particles of a heterotic superstring in $\mathbb{R}^{10}$.

There are four basic types of such amplitudes; namely [21],

$$A_{\infty}^{(k)}(s, t, u) = B_{\infty}(-1 - s/8 - S/2, -1 - t/8 - T/2, -1 - u/8 - U/2), \quad (4.15)$$

where $s + t + u = 0$ and $S + T + U = -8$; here, $k = 1$ corresponds to the set of indices $(S = -8, T = 0, \text{and } U = 0); \ k = 2, \text{to } (S = -6, T = -2, \text{and } U = 0); \ k = 3, \text{to}$
(S = −4, T = −4, and U = 0); and k = 4, to (S = −4, T = −2, U = −2). The other amplitudes are obtained by the permutation of indices S, T, and U, S + T + U = 0 (that assume the values 0, −2, −4, −6, and −8).

The amplitudes \( A^{(k)}_{\infty}(s, t, u) \), \( k = 1, 2, 3, 4 \), are the beta functions of the field \( \mathbb{R} \) (which are not crossing symmetric). Therefore, they satisfy any adelic formula from Section 3.

Another type of adelic formulas for these amplitudes is obtained if we represent them as the products of the three gamma functions

\[
\Gamma_{\infty}(-s/8; \nu), \quad \Gamma_{\infty}(-t/8; \mu), \quad \Gamma_{\infty}(-u/8; \eta),
\]

and apply the adelic formula (3.25) to each.

For the calculations, we first apply formulas (3.3):

\[
S = −8, \quad \Gamma_{\infty}(-1−s/8+4) = -(16\pi)^{-3}i(16−s)(8−s)s\Gamma_{\infty}(-s/8; 1),
\]

\[
S = −6, \quad \Gamma_{\infty}(-1−s/8+3) = (16\pi)^{-2}(8−s)s\Gamma_{\infty}(-s/8; 0),
\]

\[
S = −4, \quad \Gamma_{\infty}(-1−s/8+2) = (16\pi)^{-1}i\Gamma_{\infty}(-s/8; 1),
\]

\[
S = −2, \quad \Gamma_{\infty}(-1−s/8+1) = \Gamma_{\infty}(-s/8; 0),
\]

\[
S = 0, \quad \Gamma_{\infty}(-1−s/8+0) = \frac{16\pi i}{8+s}\Gamma_{\infty}(-s/8; 1),
\]

and then formulas (3.23). As a result, we obtain

\[
A^{(1)}_{\infty}(s, t, u) = -\frac{i}{16\pi} \frac{(16−s)(8−s)s}{(8+t)(8+u)} \Gamma_{\infty}(-s/8; 1)\Gamma_{\infty}(-t/8; 1)\Gamma_{\infty}(-u/8; 1),
\]

\[
A^{(2)}_{\infty}(s, t, u) = \frac{i}{16\pi} \frac{(8−s)s}{8+u} \Gamma_{\infty}(-s/8; 0)\Gamma_{\infty}(-t/8; 0)\Gamma_{\infty}(-u/8; 1),
\]

\[
A^{(3)}_{\infty}(s, t, u) = -\frac{i}{16\pi} \frac{u}{8+u} \Gamma_{\infty}(-s/8; 1)\Gamma_{\infty}(-t/8; 1)\Gamma_{\infty}(-u/8; 1),
\]

\[
A^{(4)}_{\infty}(s, t, u) = \frac{i}{16\pi} \frac{u}{8+u} \Gamma_{\infty}(-s/8; 1)\Gamma_{\infty}(-t/8; 0)\Gamma_{\infty}(-u/8; 0).
\]

Formulas (4.21)–(4.24) and (4.13) show that the amplitudes \( A^{(1)}_{\infty}(s, t, u) \) and \( A^{(2)}_{\infty}(s, t, u) \) are proportional to the massless superstring amplitude \( A_{\infty}(s/4, t/4, u/4) \).

Thus, we established a relation between the scattering amplitudes in the theory of a heterotic string and the Ramond–Neveu–Schwarz superstring.

**Remark 6.** A function of the form (see (4.13), (4.21)–(4.24))

\[
B'_{\infty}(\alpha, \nu; \beta, \mu; \gamma, \eta) = \Gamma_{\infty}(\alpha; \nu)\Gamma_{\infty}(\beta; \mu)\Gamma_{\infty}(\gamma; \eta),
\]

\[
\alpha + \beta + \gamma = 0, \quad \nu + \mu + \eta = 1, \quad \nu, \mu, \eta \in F_{2},
\]

is not a beta function of the type \( B_{\infty} \) that was considered in Section 3. It as if complements \( B_{\infty} \).

We denote this function by \( B'_{\infty} \) and call a primed beta function. As we have seen, it describes the superstring amplitudes. The integral representation of this function is given by

\[
B'_{\infty}(\alpha, \nu; \beta, \mu; \gamma, \eta) = -\pi i \left( \text{sgn}^\nu x |x|^{\alpha-1} \ast (\text{sgn}^\mu x |x|^{\beta-1}) \ast (\text{sgn}^\eta x |x|^{\gamma-1}) \right)_{x=1},
\]

where \( \ast \) denotes convolution. We will dwell on this point in a different time and different place.

**The amplitudes \( V_{\nu\mu\eta} \).**

In total, there exist four solutions to the equation \( \nu + \mu + \eta = 0, \nu, \mu, \eta \in F_{2} \): 000, 110, 101, and 011. Therefore, by (4.11), there exist four different amplitudes \( V_{\nu\mu\eta} \). One of them,
$V_{000} = V$, is the Veneziano amplitude, and the rest three quantities $V_{110}, V_{101}$, and $V_{011}$ form a 3-vector in which each component is expressed in terms of another, for example,

$$V_{101}(s,t,u) = V_{110}(s,u,t) = V_{011}(t,s,u);$$

and all of them are expressed in terms of the Veneziano amplitude $V$, for example

$$V_{110}(s,t,u) = \frac{1 + \alpha + \alpha' t}{\alpha + \alpha' s} V\left(s - \frac{1}{\alpha'}, t + \frac{1}{\alpha'}, u\right), \quad (4.25)$$

by virtue of the easily verifiable relation

$$B_\infty(\alpha, 1; \beta, -1; \gamma, 0) = \frac{\beta - 1}{\alpha} B_\infty(\alpha + 1, \beta - 1, \gamma), \quad \alpha + \beta + \gamma = 1.$$

The amplitudes $W_{\nu\mu\eta}$.

There exist an infinite number of amplitudes $W_{\nu\mu\eta}$ (4.12), exactly as many as there are integer solutions to the equation $\nu + \mu + \eta = 0$, $\nu, \mu, \eta \in \mathbb{Z}$: $W_{000} = W$ is the Virasoro amplitude; the quantities $W_{\nu\mu\eta}, W_{\nu\eta\mu},$ and $W_{\eta\mu\nu}, 2\nu + \eta = 0, \nu \neq \eta$, form a 3-vector; and the quantities $W_{\nu\mu\eta}, W_{\nu\eta\mu}, \ldots, W_{\eta\mu\nu}, \nu + \mu + \eta = 0, \nu \neq \mu \neq \eta \neq \nu$, form a 6-vector. In each of these groups of vectors, each component is expressed in terms of another, for example,

$$W_{\mu\nu\eta}(s,t,u) = W_{\nu\mu\eta}(t,s,u) = \ldots = W_{\eta\mu\nu}(u,s,t).$$

The physical sense of these amplitudes is not yet clear.

Everything that was said in this section concerning one-class quadratic fields may also apply to any one-class fields of algebraic numbers. The appropriate adelic formulas were obtained in [14–17].

ACKNOWLEDGMENTS

I am grateful to I.V. Volovich for fruitful discussions.

This work was supported by the Russian Foundation for Basic Research (project no. 96-01-01008) and the Program “Leading Scientific Schools of the Russian Federation” (project no. 96-15-96131).

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