To the radiation of ultra-relativistic plasma

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Abstract. The energy of electromagnetic radiation (the black body radiation) is generalized for the system containing an ultra-relativistic fully ionized plasma. It is shown that the radiation energy depends monotonically on one dimensionless characteristic parameter, which includes particle density and plasma temperature.

According to the Planck law [1, 2] the spectral energy distribution of the black radiation in volume $V$ is equal

$$dE(\omega) = \frac{V}{\pi^2 c^3} \frac{h\omega^3 d\omega}{\exp(h\omega/T) - 1},$$

(1)

where $c$ is the velocity of light. This formula implies that there is equilibrium between photons and matter at temperature $T$. Interaction between the photons and matter should be small enough as not to disturb ideality of the photon gas. The net absorption and creation of photons by matter in the volume $V$ are absent, these processes are in a balance. Also, interaction between the photons themselves is extremely weak. At the same time, to reach equilibrium in the photon gas, a weak photon-matter interaction is necessary. These conditions are satisfied quite well in gases for all frequencies that lie far away from the frequencies of the absorption lines of matter.

The Planck law has been verified experimentally, e.g., by pumping laser radiation into the empty (or filled with a rarefied gas) cavity through a small hole and then observing the spectral distribution of the radiation emitted through the hole. The spectral distribution of the emitted radiation is the Planck distribution with high accuracy.

The Planck radiation is also confirmed by experimental discovery of the cosmic background radiation by Penzias and Wilson [3], which has been predicted theoretically by Gamow [4].

The presence of the fully ionized non-relativistic plasma leads to the distortion of the Planck distribution of radiation in the region of frequency $\omega \leq \Omega_p$, where $\Omega_p = (4\pi n_e e^2/m)^{1/2}$ is the electron plasma frequency [5]. The thermodynamic functions of such radiation are the functions of the dimensionless parameter $a_0 = h\Omega_p/T$ (temperature $T$ is in energy units). When $h\Omega_p/T \ll 1$, the thermodynamic functions for radiation closely coincide with the ones obtained with the Planck distribution. However, in the case when $h\Omega_p/T \geq 1$ the situation dramatically changes. The free energy $F(a_0)$ as function of the parameter $a_0$ monotonically decreases and possesses the exponential asymptotic for large values of $a_0$ [5].

In this paper we consider the case of ultra-relativistic plasma and calculate energy $E$ of the stationary non-Planck radiation. As we show, in the case under consideration this energy is a monotonically decreasing function of the characteristic parameter $a \equiv h\tilde{\Omega}/T = \ldots$
(4πε^2 n_e^2 c^2 (1 + z)/3T^3)^{1/2}$, which appears instead the parameter $a_0$ characterized the non-relativistic case. Therefore, the energy $E$ is always lower than the Planck radiation energy for the same temperature.

Let us consider a fully ionized plasma in some volume $V$. We suppose that plasma itself is in the equilibrium state and has the same temperature $T$ as the surrounding matter. The distortion of the Planck radiation caused by existence in plasma a different dispersion for electromagnetic field than that in vacuum, where $\omega = ck$.

For the ultra-relativistic temperatures, when $T \gg mc^2$. Here, for simplicity, we consider the case of ultra-relativistic electrons and nuclei, when $T \gg mc^2$. Therefore, $m$ is the mass of the nuclei, e.g., proton for hydrogen.

For the case under consideration the spectrum of the weakly damping transversal mode is different from one for the non-relativistic plasma $\omega = \sqrt{\Omega_p^2 + c^2 k^2}$ and depends on temperature. The ultra-relativistic spectrum of the transversal mode is determined by the dielectric function

$$\varepsilon^{tr}(\omega, k) = 1 + \frac{\pi(e^2 n_e + \sum_i z_i^2 e^2 n_i)c}{\omega k T} \left\{ -\frac{2\omega}{ck} + \left(1 - \frac{\omega^2}{c^2 k^2}\right) \ln \left|\frac{ck - \omega}{ck + \omega}\right| \right\}, \quad \omega > 0, k > 0, \quad (2)$$

where the charge $z_i e$ and the density $n_i$ are related to the ions or nuclei. Below we will use the model of a one sort of ions with a charge $z$ and accept the quasi-neutrality condition. In this case, after introducing the characteristic frequency $\tilde{\Omega}^2 = 4\pi^2 n_e e^2 (1 + z)/3T$, Eq. (2) reads

$$\varepsilon^{tr}(\omega, k) = 1 + \frac{3\tilde{\Omega}^2}{4\omega c k} \left\{ \frac{2\omega}{ck} + \left(1 - \frac{\omega^2}{c^2 k^2}\right) \ln \left|\frac{ck - \omega}{ck + \omega}\right| \right\}. \quad (3)$$

The imagine part of the collisionless $\varepsilon^{tr}$ equals zero for $\omega > ck$ and small for $\omega < ck$. We also neglect the influence of collisions for ultra-high temperatures. Then the spectrum is defined by the standard equation

$$k^2 = \frac{\omega^2}{c^2} \varepsilon^{tr}(\omega, k). \quad (4)$$

In general, the expression for the spectrum cannot be found from (3) and (4) in analytical form. Let us consider the limiting cases

1) $\zeta \equiv ck/\omega \ll 1$:

$$\varepsilon^{tr}(\omega, k) = 1 + \frac{3\tilde{\Omega}^2}{4\omega^2 \zeta} \left\{ -\frac{2}{\zeta} + \left(1 - \frac{1}{\zeta^2}\right) \ln \left|\frac{1 - \zeta}{1 + \zeta}\right| \right\} \simeq 1 - \frac{3\tilde{\Omega}^2}{4\omega^2 \zeta} \left\{ \frac{4\zeta^3}{3} + \frac{4\zeta^5}{15} + \frac{4\zeta^7}{35} + \ldots \right\}. \quad (5)$$

The dispersion relation for the case under consideration reads

$$\zeta^2 = 1 - \frac{\tilde{\Omega}^2 \zeta^2}{k^2 c^2} \left\{ 1 + \frac{\zeta^2}{5} + \frac{3\zeta^4}{35} + \frac{6\zeta^6}{21} \right\}. \quad (6)$$

Rewrite (6) introducing $\epsilon = k^2 c^2/\tilde{\Omega}^2 \ll 1$

$$\zeta^4 + 5(1 + \epsilon)\zeta^2 - 5\epsilon = 0. \quad (7)$$

We neglect the terms of higher order and find the solution to order of $\epsilon^2$

$$\zeta^2 = \epsilon(1 - \epsilon) - \frac{2\epsilon^2}{5}, \quad \text{or} \quad \omega^2 = \tilde{\Omega}^2 + \frac{7}{5} k^2 c^2 \approx \tilde{\Omega}^2 \quad (8)$$
2) $\zeta \equiv ck/\omega \gg 1$

$$\varepsilon^{tr}(\omega, k) \approx 1 + \frac{3\tilde{\Omega}^2 \zeta}{4k^2c^2} \left\{ -\frac{4}{\zeta} - \frac{4}{3\zeta^3} \right\}. \quad (9)$$

The dispersion relation with the necessary accuracy is

$$\omega^2 = k^2c^2 + 3\tilde{\Omega}^2, \quad kc \gg \tilde{\Omega}. \quad (10)$$

This case cannot be realized.

3) $|\zeta| \to 1$

$$\varepsilon^{tr}(\omega, k) \approx 1 - \frac{3\tilde{\Omega}^2}{2k^2c^2}. \quad (11)$$

The dispersion relation with the necessary accuracy is

$$\omega^2 = k^2c^2 + 3\tilde{\Omega}^2, \quad kc \gg \tilde{\Omega}. \quad (12)$$

To calculate the free energy of radiation $F$ (or the respective potential $\Omega = F$, since $\mu = 0$) we can use the general representation

$$F = \frac{VT}{\pi^2} \int_0^{\infty} dk k^2 \ln[1 - \exp(-\hbar \omega(k)/T)], \quad (13)$$

where we should find $\omega(k)$ from the dispersion relation (4).

Let us rewrite the dispersion relation (4) and the free energy (13) introducing the integration variable $\chi = hck/T$. We also use the variables $\varpi = \hbar \omega/T$ and $a = h\tilde{\Omega}/T$ ($\zeta = ck/\omega \equiv \chi/\varpi$)

$$\frac{\chi^2}{\varpi^2} = 1 + \frac{3a^2}{4\varpi^2} \left\{ -\frac{2\varpi}{\chi} + \left( 1 - \frac{\varpi^2}{\chi^2} \right) \ln \left| \chi/\varpi - 1 \right| \right\}. \quad (14)$$

$$F = \frac{VT^4}{h^3c^3\pi^2} \int_0^{\infty} d\chi \chi^2 \ln[1 - \exp(-\varpi(\chi;a))] \equiv F_0\phi(h\tilde{\Omega}/T), \quad (15)$$

where the function $\phi(h\tilde{\Omega}/T) > 0$ describes deviation from the free energy $F_0$ of the Planck radiation

$$\phi(a) = -\frac{45}{\pi^4} \int_0^{\infty} d\chi \chi^2 \ln[1 - \exp(-\varpi(\chi;a))], \quad F_0 \equiv -\frac{V\pi^2T^4}{45h^3c^3}. \quad (16)$$

Here $\varpi = \varpi(\chi;a)$ is the solution of the dispersion relation (3). We have note that the relations (13), (15) are general and, therefore, valid also for the free energy of radiation which is determined by some other dispersion relation with the solution $\varpi'(\chi;a)$ and characteristic value of frequency $\tilde{\Omega}'$ (e.g., for the case of non-relativistic plasma, where $\tilde{\Omega}'$ equals to the plasma frequency $\Omega_p$ [5]).

Taking into account that in the case under consideration $a \sim \tilde{\Omega}/T \sim 1/T^{3/2}$ the radiation energy $E = F - T \left( \frac{dF}{dT} \right)_V$, as follows from (15), can be represented in the form

$$E = E_0 \left[ \phi(a) - \frac{a}{2} \frac{\partial \phi(a)}{\partial a} \right]. \quad (17)$$
Let us try to calculate analytically the asymptotic behavior of energy $e = E/V$ for small $a$. For this purpose we expand $\phi(a)$ at small $a$. For $a = 0$ we have solution (4) $\varpi = \chi$ and, therefore, for a small $a \ll 1$ as it follows from (12)

$$\varpi \simeq \chi(1 + \frac{3a^2}{4\chi^2}).$$

(18)

After simple calculation one can arrive at the asymptotical expression for $\phi(a \ll 1)$

$$\phi(a \ll 1) = 1 + \frac{45a^2}{8\pi^2}.$$ (19)

Therefore, for a small $a$ in the considered approximation the radiation energy for plasma, coincides with the Boltzmann law value $e_0$

$$\frac{e}{e_0} \simeq \left[ 1 + \frac{45}{8\pi^2}a^2 \right] = 1,$$

$$e_0 \equiv \frac{\pi^2T^4}{15\hbar^3c^3}.$$ (20)

This means we have take into account the terms of order $\tilde{a}^4$ in the expansion $\phi(a \ll 1)$ over $a$. However, in the coefficient at the term $\sim a^4$ diverges, therefore, the respective series on $a^2$ is asymptotic. To find the relation $e/e_0$ the numerical calculation is necessary. The results for the cases of purely numerical calculations (when all calculations are numerically) and partly analytical calculation (when the partial derivatives for the integral transformations were found analytically) are shown on figure 1. The radiation energy $e/e_0$ is a monotonic function of $a$ and for all values of $a \neq 0$ lies lower than the Planck energy.

For the case of classical non-relativistic plasma the respective relation similar to (17) is

$$E = E_0 \left[ \phi'(a_0) - \frac{a_0}{6} \frac{\partial \phi'(a_0)}{\partial a_0} \right].$$ (21)

Since in this case $\phi'(a_0) = 1 - \lambda a_0^2$, where $a_0^2 = \hbar \Omega_p/T$ and $\lambda = 5/(2\pi^2) > 0$ the radiation energy is always smaller than $E_0$ [5].
It is possible to show that the spectral distribution in the ultra-relativistic (as in the case of Maxwellian plasma \[5\]) possesses a gap for small values of frequency. In the case of Maxwellian plasma this gap equals \(\Omega_p\). The explicit form of the spectral distribution with a gap has been recently reconsidered on the basis of the Green function approach \[9\], which completely takes into account not only frequency, but also the spatial dispersion of the dielectric function. The results of this paper can be used for the laboratory and astrophysical applications. The deviations from Planck distribution can be checked also by the use of laser radiation sources (see, e.g., \[10, 11\]).

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