Naked Singularities in Higher Dimensional Szekeres space-time

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In this paper we study the quasi-spherical gravitational collapse of \((n+2)\)-dimensional Szekeres space-time. The nature of the central shell focusing singularity so formed is analyzed by studying both the radial null and time-like geodesic originated from it. We follow the approach of Barve et al to analyze the null geodesic and find naked singularity in different situations.

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I. INTRODUCTION

In recent years, there is an exhaustive study of gravitational collapse of inhomogeneous spherical dust \([1-4]\). It is mainly concentrated to central shell focusing singularity. The behaviour of the singularity (black hole or naked singularity) has been analyzed mainly by the studying of outgoing radial null geodesic and the strength is measured using due to Tipler \([5]\). It has been shown that appearance of naked singularity strongly depends on the initial data \([6-10]\).

In contrast, there is very little information about singularity formation and structure in non-spherical collapse. Here horizon will be formed if and only if the hoop conjecture \([11]\) is satisfied. Most of the studied so far in non-spherical collapse are by numerical analysis and for some definite shape of the gravitating mass \([12-16]\).

However, the study of gravitational collapse for quasi-spherical Szekeres space-time \([17]\) was started by Szekeres himself \([18]\) long ago. Afterwards, it was studied further by Joshi et al \([19]\) and extensively by Goncalves \([20]\). Recently, we have obtained solutions for \((n+2)\)-dimensional Szekeres space-time with perfect fluid (or dust) as the matter content \([21]\). Subsequently, we have also analyzed the gravitational collapse for the above dust solution to examine the local behaviour of the singularity so formed \([22]\).

In this paper, we extend our study for global characteristic of the singularity by studying both null and time like geodesic originated from the singularity using the formalism of Barve et al \([23]\). The paper is organized as follows: In section II we have written the basic equations and regularity conditions. In section III and IV, we have studied the radial null and time-like geodesics originated from the singularity. We have studied the local visibility of the singularity for radial null and time-like geodesics due to collapse in section V. Finally the paper ends with a short discussion.

II. BASIC EQUATIONS AND REGULARITY CONDITIONS

The metric ansatz for \((n+2)D\) Szekeres space-time is given by \([21]\)
\[ ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^{n} dx_i^2 \]  

(1)

where \( \alpha \) and \( \beta \) are functions of all the \( (n+2) \) space-time co-ordinates with expression [21] (where \( \beta' \neq 0 \))

\[ e^\alpha = R' + R \nu' \]  

(2)

and

\[ e^\beta = R(t, r) e^{\nu'(r, x_1, ..., x_n)} \]  

(3)

Here \( R \) satisfied the differential equation

\[ \dot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}} \]  

(4)

and the expression for \( \nu \) is

\[ e^{-\nu} = A(r) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r) x_i + C(r) \]  

(5)

with the restriction

\[ \sum_{i=1}^{n} B_i^2 - 4AC = f(r) - 1 \]  

(6)

for the arbitrary functions \( A(r), B_i(r) \) and \( C(r) \). Also in the expression (4), \( f(r) \) and \( F(r) \) are arbitrary functions of \( r \) alone. But due to complexity of the problem we shall restrict ourselves to marginally bound case only (i.e., \( f(r) = 0 \)). In this case equation (4) can easily integrated to give

\[ R = \left[ r^{\frac{n+1}{2}} - \frac{n + 1}{2} \sqrt{f(r)} \ t \right]^{\frac{1}{n+1}} \]  

(7)

and the energy density for the matter field

\[ \rho(t, r, x_1, ..., x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{R^n(R' + R\nu')} \]  

(8)

takes the form

\[ \rho_i(r, x_1, ..., x_n) = \rho(0, r, x_1, ..., x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{r^n(1 + r\nu')} \]  

(9)

initially at \( t = 0 \), where we have chosen the scaling of \( R \) as

\[ R(0, r) = r. \]

Now assuming that the collapse starts from a regular initial hypersurface so we take the following series forms for \( F(r), \rho_i(r) \) and \( \nu'(r) \) [21]

\[ F(r) = \sum_{j=0}^{\infty} F_j r^{n+j+1}, \]  

\[ \rho_i(r) = \sum_{j=0}^{\infty} \rho_j r^j \]  

and

\[ \nu'(r) = \sum_{j=-1}^{\infty} \nu_j r^j, \quad (\nu_{-1} \geq -1) \]

where the coefficients are related as

\[ \rho_0 = \frac{n(n+1)}{2} F_0, \quad \rho_1 = \frac{n}{2} \left( n + 1 + \frac{1}{1+\nu_{-1}} \right) F_1, \]

\[ \rho_2 = \frac{n}{2} \left[ \left( n + 1 + \frac{2}{1+\nu_{-1}} \right) F_2 - \frac{F_1\nu_0}{(1+\nu_{-1})^2} \right], \]

\[ \rho_3 = \frac{n}{2} \left[ \left( n + 1 + \frac{3}{1+\nu_{-1}} \right) F_3 - \frac{2F_2\nu_0}{(1+\nu_{-1})^2} - \frac{(1+\nu_{-1})\nu_1 - \nu_0^2}{(1+\nu_{-1})^3} F_1 \right], \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

OR

\[ \rho_0 = \frac{n}{2} \left[ \frac{F_1}{\nu_0} + (n+1)F_0 \right], \quad \rho_1 = \frac{n}{2} \left[ \frac{2F_2}{\nu_0} + \left\{ (n+1) - \frac{\nu_1}{\nu_0^2} \right\} F_1 \right], \]

\[ \rho_2 = \frac{n}{2} \left[ \frac{3F_3}{\nu_0} + \left\{ (n+1) - \frac{2\nu_1}{\nu_0^2} \right\} F_2 + \left( \frac{\nu_1^2}{\nu_0^4} - \frac{\nu_2}{\nu_0^3} \right) F_1 \right], \]

\[ \rho_3 = \frac{n}{2} \left[ \frac{4F_4}{\nu_0} + \left\{ (n+1) - \frac{3\nu_1}{\nu_0^2} \right\} F_3 + 2 \left( \frac{\nu_1^2}{\nu_0^4} - \frac{\nu_2}{\nu_0^3} \right) F_2 + \left( \frac{2\nu_1\nu_2}{\nu_0^3} - \frac{\nu_3}{\nu_0^2} - \frac{\nu_0^2}{\nu_0^4} \right) F_1 \right], \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

according as \( \nu_{-1} > -1 \) or \( \nu_{-1} >= -1 \).

The singularity curve \( t = t_s(r) \) for the shell focusing singularity is characterized by

\[ R(t_s(r), r) = 0 \]

and we have
\[ t_s(r) = 2^{\frac{n+1}{n+1}} \sqrt{F(r)} \]  

(15)

where \( t_0 = \frac{2}{(n+1)F_0} \) is the time for the central shell focusing singularity. However, near \( r = 0 \), above singularity curve can be approximately written as

\[ t_s(r) = t_0 - \frac{F_m}{(n+1)F_0^{3/2}} r^m \]  

(16)

where \( m \geq 1 \) is an integer and \( F_m \) is the first non-vanishing term beyond \( F_0 \).

### III. RADIAL NULL GEODESICS

The equation of the outgoing radial null geodesic (ORNG) which passes through the central singularity in the past is taken as (near \( r = 0 \))

\[ t_{ORNG} = t_0 + a \xi, \]  

(17)

to leading order in \( t-r \) plane with \( a > 0, \xi > 0 \). Now to visualize the singularity the time \( t \) in the geodesic (17) should be less than \( t_s(r) \) in equation (16). Hence comparing these times we have the restrictions

\[ \xi \geq m \quad \text{and} \quad a < -\frac{F_m}{(n+1)F_0^{3/2}} \]  

(18)

Further, for the metric (1) an outgoing radial null geodesic should satisfy

\[ \frac{dt}{dr} = R' + R \nu' \]  

(19)

We shall now examine the feasibility of the null geodesic starting from the singularity with the above restrictions for the following two cases namely, (i) \( \xi > m \) and (ii) \( \xi = m \).

When \( \xi > m \) then near \( r = 0 \) the solution for \( R \) in (7) simplifies to

\[ R = \left(\frac{F_m}{2F_0}\right)^{\frac{2}{n+1}} r^{\frac{2m}{n+1}+1} \]  

(20)

Now combining (17) and (20) in equation (19) we get (upto leading order in \( r \))

\[ a \xi r^{\xi-1} = \left(\nu_{-1} + 1 + \frac{2m}{n+1}\right) \left(\frac{F_m}{2F_0}\right)^{\frac{2}{n+1}} r^{\frac{2m}{n+1}} \]  

(21)

which implies

\[ \xi = 1 + \frac{2m}{n+1} \quad \text{and} \quad a = \frac{\left(\nu_{-1} + 1 + \frac{2m}{n+1}\right)}{(1 + \frac{2m}{n+1})} \left(\frac{F_m}{2F_0}\right)^{2/(n+1)} \]  

(22)

We note that if \( \nu_{-1} = 0 \) then the above results are same as TBL model [9], so we restrict ourselves to \( \nu_{-1} \neq 0 \). As \( \xi > m \) so from (22)
Since \( m \) is an integer so we have the following possibilities:

\[
\begin{align*}
  n = 2 \ (4D \ space-time) & : \quad m = 1, 2 \quad , \xi = \frac{1}{4}, \frac{7}{4} \\
  n \geq 3 \ (5D \ or \ higher \ dim. \ space-time) & : \quad m = 1 \quad , \xi = 1 + \frac{2}{n+1}.
\end{align*}
\]

Now for \( \nu_{-1} > -1 \), if we assume that the initial density gradient falls off rapidly to zero at the centre then we must have \( \rho_1 = 0, \rho_2 < 0 \) i.e., \( F_1 = 0, F_2 < 0 \). Hence the least value of \( m \) should be 2. Thus for a self consistent outgoing radial null geodesic starting from the singularity we must have \( n = 2 \) only. So the singularity will be naked only in four dimension. However, if we relax the restriction on the initial density and simply assume that the initial density gradually decreases as we go away from the centre (i.e., \( \rho_1 < 0 \)) then \( F_1 < 0 \) and \( m = 1 \) is the possible solution. Therefore, naked singularity is possible in any dimension. However for \( \nu_{-1} = -1 \), the assumption about the initial density gradient (\( \rho_1 = 0, \rho_2 < 0 \)) does not implying \( F_1 = 0 \). Hence here \( m = 1 \) is a possible solution for all dimension with \( F_1 < 0 \). Therefore naked singularity may occur in any dimension for this choice of the parameter \( \nu_{-1} \).

Now we shall study the second case namely \( \xi = m \). Here also for \( \nu_{-1} = 0 \) we have identical results as in TBL model [9]. For \( \nu_{-1} \neq 0 \), using the same procedure as above we get

\[
m = \frac{n + 1}{n - 1}
\]

and

\[
a = -\frac{1}{m} \left( -\frac{n + 1}{2} \sqrt{F_0} a - \frac{F_m}{2F_0} \right)^{\frac{1}{n}} \left[ (\nu_{-1} + m) \frac{F_m}{2F_0} + \frac{1}{2} (n + 1)(\nu_{-1} + 1) \sqrt{F_0} a \right]
\]

Hence integral solution for \( m \) is possible only for \( n = 2 \) and 3. So consistent outgoing radial null geodesic through the central shell focusing singularity is possible only upto five dimension i.e., we cannot have naked singularity for higher dimensional space-time greater than five. It is to be noted that this conclusion is not affected by the value of the parameter \( \nu_{-1} \).

Now we shall examine whether the restriction (18) for \( a \) is consistent with the expression \( a \) in equation (23). In fact equation (23) takes the form

\[
2^{\frac{n+1}{2}} bm = -[(n + 1)b - \zeta]^{-\frac{n}{2}} \left[ (\nu_{-1} + m)\zeta + (n + 1)(\nu_{-1} + 1)b \right]
\]

with the transformation

\[
a = bF_0^{-\frac{n-1}{2}}, \quad F_m = \zeta F_0^{\frac{n+1}{2}}.
\]

Since equation (24) is a real valued equation of \( b \), so we must have

\[(n + 1)b + \zeta < 0,
\]

which using (25) gives us the restriction on \( a \) in equation (18). Hence the geodesic (17) will have consistent solution for \( a \) and \( m \). So the above conclusion regarding the formation of naked singularity is justified. Further, introducing the variable \( \phi \) by the relation

\[
\phi = -(n + 1)b - \zeta
\]
In Figs.1 and 2, we have shown the variation of $\phi$ ($0 < \phi < -\zeta$) for the variations of $\nu_{-1} (> -1)$ and $\zeta (< 0)$ for four and five dimensions (i.e., $n = 2$ and 3) respectively. However for $\nu_{-1} = -1$, the dependence of $\phi$ over $\zeta$ has been presented in Figs.3 and 4 for $n = 2$ and 3 respectively.

We have seen from equation (24)

$$4\phi^{n-1}(\zeta + \phi)^{n+1} = [2\zeta - (n - 1)(\nu_{-1} + 1)\phi]^{n+1}$$

with the restriction $0 < \phi < -\zeta$. 

(27)
Finally, we investigate whether there is only one null geodesic emanating from the singularity (i.e., when singularity is visible for an infinitesimal time) or it is possible to have an entire family of geodesics through the singularity (this occurs when singularity is visible for an infinite time). So we write the equation for the outgoing radial null geodesics to next order as

\[ t = t_0 + ar^\xi + hr^\xi + \sigma \]  

(28)

where as before \( a, \xi, h, \sigma \) are all positive. Then proceeding in the same way we have

(i) when \( \xi > m \), \( a \) and \( \xi \) have the same expressions as in equation (22) while

\[ \sigma = \frac{m(1-n)}{1+n} + 1 \]  

(29)

and

\[ h = \frac{a}{\xi + \sigma} (-\nu_{-1} - \sigma) \sqrt{F_0} \left( \frac{F_m}{2F_0} \right)^{\frac{1+n}{1+n}} \]  

(30)

We note that since here \( m \) can have values 1 and 2 for \( n = 2 \) and \( m = 1 \) for \( n \geq 3 \) so \( \sigma \) is always positive. But for \( h \) to be positive definite, \( \nu_{-1} \) is restricted within the range \(-1 \leq \nu_{-1} \leq -\sigma \) i.e., \( \nu_{-1} \) is negative definite.

(ii) for \( \xi = m \), \( a \) and \( \xi \) have the same expressions as in equation (23) and \( \sigma \) is obtained from the equation

\[ m + \sigma = \frac{2\pi n}{\pi + 1} b(n+1) [-\zeta - (n+1)b]^{\frac{n}{\pi + 1}} - 2\pi n \nu_{-1} F_0^{-1/2} [-\zeta - (n+1)b]^{1/n} \]  

(31)

with expressions for \( b \) and \( \zeta \) from equation (25), while the other constant \( h \) is totally arbitrary. Thus it is possible to have an entire family of outgoing null geodesics terminated in the past at the singularity, provided \( \sigma \) obtained from equation (31) is positive definite.

**IV. RADIAL TIME-LIKE GEODESICS**

We shall now investigate whether it is possible to have any outgoing time-like geodesic originated from the singularity. For simplicity of calculation we shall consider only outgoing radial time-like geodesic (ORTG). Let us denote by \( K^a = \frac{dx^a}{d\tau} \), a unit tangent vector field to a ORTG with \( \tau \) an affine parameter along the geodesic. Hence from the geodesic equation we have [20]

\[ K^t (K^r)' + 2\dot{a}K^t K^r + K^r (K^r)' + a' (K^r)^2 = 0 \]  

(32)

with

\[ K^t = \pm \sqrt{1 + c^{2n} (K^r)^2} . \]  

(33)

In order to satisfy the above two equations the simplest choice for \((K^r, K^t)\) is

\[ K^t = \pm 1, \quad K^r = 0 . \]  

(34)

and so we have the solution

\[ t - t_0 = \pm (\tau - \tau_0), \quad r = r_0 = \text{constant}. \]  

(35)
Here + (or −) sign corresponds to ORTG (or ingoing RTG) and \( \tau_0 \) is the proper time at which radial time-like geodesic passes through the central singularity.

Now similar to null geodesic let us choose the radial time-like geodesic near the singularity to be of the form (to leading order)

\[
t_{ORTG}(r) = t_0 + c r^p
\]

with \( c \) and \( p \) as positive constants. Further, consistent with equations (34) and (35) we assume

\[
K^r(t, r) = A(t - t_0)^\lambda r^\delta
\]

where \( A (> 0), \lambda \) and \( \delta \) are constants. Hence we have

\[
k^r(t_{ORTG}, r) = A r^q, \quad q = \lambda p + \delta.
\]

Also from the geodesic equation (36) using (38) we get

\[
\frac{dt_{ORTG}}{dr} = c p r^{p-1} = \frac{K^t}{K^r} = \sqrt{A^{-2} r^{-2q} + (R' + R\nu')^2}
\]

Using the solution for \( R \) near the singularity in equation (39) and equating equal powers of \( r \) we have

(i) \( p = 1 - q, \quad c = \frac{1}{A(1-q)} \) if \( -\frac{2m}{n+1} < q < 1 \).

(ii) \( p = 1 + \frac{2m}{n+1}, \quad c = \frac{(\nu_{-1} + 1 + \frac{2m}{n+1})}{(1 + \frac{2m}{n+1}) \left( \frac{F}{F_0} \right)^{\frac{n}{n+1}}} \) if \( q < -\frac{2m}{n+1} \).

(iii) \( p = 1 + \frac{2m}{n+1}, \quad c = \frac{1}{p} \left[ A^{-2} + \left( \frac{F}{F_0} \right)^{\frac{n}{n+1}} \right]^{1/2} \) if \( q = -\frac{2m}{n+1} \).

Therefore equation (36) has consistent solution for \( c \) and \( p \) depending on the parameters involved. So it is possible to have outgoing radial time-like geodesic originated from the singularity.

**V. LOCAL VISIBILITY**

We have shown the existence of outgoing both null and time-like geodesics which were originated in the past from the singularity. We shall now examine whether these geodesics are visible to any non-space like observer. Now for visibility of the singularity (local nakedness) any future directed geodesics (null or time-like) starting from the singularity should be outside the domain of dependence of any trapped surfaces both before and at the time of formation of apparent horizon (AH) which is the outer boundary of the trapped surface. Now if \( t_{ah}(r) \) is the time of formation of apparent horizon so we must have [8,9,18]

\[
R(t_{ah}(r), r) = -1
\]

i.e.,

\[
R^{n-1}(t_{ah}(r), r) = F(r)
\]

The apparent horizon is given by the curve

\[
t_{ah}(r) = t_0 - \frac{1}{(n+1)F_0^{3/2}}(F_m r^m + O(r^{m+1})) - \frac{2}{n+1}F_0^{\frac{n-1}{2n-1}} \left( r^{\frac{n+1}{n-1}} + O(r^{\frac{n+1}{n-1}+1}) \right)
\]
As the apparent horizon and the singularity curve form at the same time at \( r = 0 \), so the visibility of the singularity is determined by the relative slopes of the curve \( t_{ah}(r) \) and the curve for outgoing radial non-space like geodesics (denoted by \( t_{ORG}(r) \)). Hence the necessary and sufficient condition for singularity to be at least locally naked is that

\[
\lim_{r \to 0} \left\{ \left( \frac{dt_{ah}}{dr} \right) \bigg/ \left( \frac{dt_{ORG}}{dr} \right) \right\} > 1 \quad (42)
\]

Now we shall examine this condition for both ORNG and ORTG separately.

### A. Outgoing Radial Null Geodesic

In this case the ratio of the slopes is

\[
\left( \frac{dt_{ah}}{dt_{ORG}} \right) \bigg/ \left( \frac{dr}{dr} \right) = -\frac{1}{a\xi(n+1)F_0^{3/2}} \left( mF_mr^{m-\xi} + O(r^{m-\xi+1}) \right)
-\frac{2}{a\xi(n-1)}F_0^{\frac{n+1}{2}} \left( r^{\frac{n+1}{2} - \xi} + O(r^{\frac{n+1}{2} - \xi+1}) \right) \quad (43)
\]

So we have the following possibilities:

(i) If \( m > \frac{n+1}{n-1} \) then the above ratio of the slope is always negative, hence the condition is violated. So we always get black hole.

(ii) If \( m < \frac{n+1}{n-1} \) then the above ratio of the slopes approaches to \(+\infty\) as \( r \to 0^+ \) for \( m < \xi \) and while the limit will be finite and greater than unity for \( m = \xi \) provided \( a < -\frac{m}{(n+1)k}F_0^{3/2} \). Thus the singularity will be locally naked for any dimension if \( m = 1 \) and \( a < -\frac{1}{(n+1)k}F_0^{3/2} \), while naked singularity appears only for four dimension if \( m = 2 \) and \( a < -\frac{2}{3k}F_0^{3/2} \).

(iii) If \( m = \frac{n+1}{n-1} \) then we have shown that naked singularity appears only up to five dimension and the singularity will be locally visible for

\[ F_m < -2F_0^{\frac{n+1}{2}}, \quad m < \xi \]

or

\[ F_m < -2F_0^{\frac{n+1}{2}} - a(n+1)F_0^{3/2}, \quad m = \xi. \]

### B. Outgoing Radial Time-like Geodesic

In this case the ratio of the slopes is

\[
\left( \frac{dt_{ah}}{dt_{ORTG}} \right) \bigg/ \left( \frac{dr}{dr} \right) = -\frac{1}{c\pi(n+1)F_0^{3/2}} \left( mF_mr^{m-p} + O(r^{m-p+1}) \right)
-\frac{2}{c\pi(n-1)}F_0^{\frac{n+1}{2}} \left( r^{\frac{n+1}{2} - p} + O(r^{\frac{n+1}{2} - p+1}) \right) \quad (44)
\]
We shall now study the following possibilities for local visibility:

(i) If \( m > \frac{n+1}{n} \) then as before only black hole appears.

(ii) If \( m < \frac{n+1}{n} \) we have the conclusion as in ORNG except that instead of restricting \( a \), we have here \( c < -\frac{m}{(n+1)p} F^2 \).

(iii) Similar is the situation for \( m = \frac{n+1}{n} \).

Hence the conditions for local visibility of naked singularity is consistent for both time-like and null geodesic.

VI. DISCUSSION AND CONCLUDING REMARKS

In this paper we have studied quasi-spherical gravitational collapse in \((n + 2)\)-D Szekeres space-time by studying both the null and time-like geodesics originated from the central shell focusing singularity. Following the approach of Barve et al [23] for null geodesic we have shown that if we choose the parameter \( \nu \) to be greater than \(-1\), then it is possible to have a class of outgoing radial null geodesic for four and five dimensions only with the restriction that initial density falls off rapidly to the centre. However for \( \nu = -1 \), naked singularity is possible in all dimensions irrespective of the assumption on the initial density. On the other hand, for time-like geodesic the results are very similar to the study of null geodesic. Regarding local visibility we have consistent results for both ORNG and ORTG with some restrictions on the parameters involved in the equation of the geodesics. Therefore we conclude that it is possible to have at least local naked singularity for the given higher dimensional non-spherical space-time.

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