A Circle Chart of the Rectangular Window Transformation of Fourier

Nikolay Parfentyev\textsuperscript{1,*} and Stepan Trukhanov\textsuperscript{2}

\textsuperscript{1}All Russian Institute of Optical Physic Measurements (VNIIOFI), Moscow,
\textsuperscript{2}Department of Physics, Moscow State University of Civil Engineering, Moscow

*Corresponding author email: ProtonMail@raz20vod.com

Abstract. Restoration of the original time signal at the reverse window transformation of Fourier can be carried out, both on the real and imaginary part of the Fourier spectrum of direct window transformation. Unlike the classic Fourier transformation in window transformation, information about the signal phase becomes redundant. Either component of the direct signal conversion Fourier contains complete information about the original time signal. The amount of squares of both components are equal to each other. This should ensure that the component types are matched at the entrance and exit of the reverse conversion block. The inconsistency of the component at the in and exit of the reverse conversion gives a time signal with the rearranged frequency components. The window direct Fourier conversion of the zero phase harmonic signal and phase $\pi/2$ gives a single-polar function with amplitude 1 and a bipolar with 0.724611 amplitudes, which are independent of integration time, corresponding to the maximum filling of the integration window. At the same time, the ratio of stripes of unipolar and bipolar functions is 1.05, which normalizes the amplitude of the bipolar delta function to a magnitude of $1/\sqrt{2}$.

1. Introduction

Fourier's transformation is the theoretical basis for building numerous tools and instruments of modern technology. This transformation is one of the most powerful tools of modern engineering, and serves as a field for countless mathematical methods with an immense number of applications. The most interesting thing for technology and engineering is the window transformation of Fourier, and in particular, the transformation with a rectangular window, because it contains the real conditions of the experiment. The reverse transformation of Fourier, which allows to restore the original time signal on its frequency spectrum, is also well known. This paper discusses the details of The Fourier's practical reverse transformation, which is generally not given sufficient attention in the literature [1-5].

2. Computer Simulation of Fourier's Cyclical Transformation

The analysis of The Fourier's transformation, both direct and reverse, was carried out using a simple computer model by LabView 8.0, where the time signal was represented by three harmonics of different frequencies and amplitudes. At the same time, two harmonics are represented by cos, the third harmonica - sinus. The direct transformation of Fourier gives this signal gives the two components of the Fourier integral real and imaginary, the nature of the changes of which on the spectrum is represented on Figure 1.
As you would expect, the final time of integration is the reason that in the spectrum, in addition to the main frequencies, there are overtones, with frequencies different from the main tone, on an odd number (equal to 1/T, where T-time integration. In the imaginary component it consists of two parts of different signs and vice versa. When the harmonica is presented in the form of two peaks, the amplitude of the harmonica spectrum at the base tone is zero. Obviously, with the increasing integration time, the peaks become delta functions of the appropriate kind. In the case where the harmonica is represented by a combination of two variable peaks as the time increases, they will strive for a sign-variable delta function.

The result of Fourier's reverse conversion is shown in the Figure 2. The left part of the picture represents the overlap of three graphs: 1) the original time signal - solid line, 2) the result of the restoration of the time signal on the real component of the Fourier conversion function - the dotted line, 3) the result of the signal recovery on the imaginary component - the point line. It's easy to see that both signal recovery results almost completely reproduce the original time signal.

Since each of the components of the direct Fourier transform contains complete information about the initial time signal, it is natural that the sum of the squared amplitudes of the imaginary and real components are equal to each other.

\[
\int \text{Re}(F(\omega))^2 d\omega = \int \text{Im}(F(\omega))^2 d\omega
\]

Figure 1 contains digital windows with integration results.

Figure 2. Result of Reverse Signal Conversion

On the right part of the picture is the result of a reverse Fourier conversion for a case where the types of component change at the in and exit of the reverse converter: the continuous curve corresponds to the actual part of the converter's output when the imaginary component of the direct function is input, and the dotted one is reversed (point curve). In the second
case, the result should be taken with a reverse sign. It is obvious that the resulting temporary function has a valid direct conversion component consistent with the imaginary component of the original signal, and the imaginary component corresponds to the actual component of the original signal.

Figure 2 Result of Reverse Signal Conversion
On the right part of the picture is the result of a reverse Fourier conversion for a case where the types of component change at the in and exit of the reverse converter:

the continuous curve corresponds to the actual part of the converter's output when the imaginary component of the direct function is input, and the dotted one is reversed (point curve). In the second case, the result should be taken with a reverse sign. It is obvious that the resulting temporary function has a valid direct conversion component consistent with the imaginary component of the original signal, and the imaginary component corresponds to the actual component of the original signal.

3. Fourier Window Conversion Diagram
The Fourier window conversion cycle results allow us to draw up a cycle scheme for the fourier for the time signal set by the actual numbers. Figure 3 shows a diagram of this transformation, where the dotted lines correspond to imaginary components and solid ones are valid.

![Image of a time signal circular conversion diagram.](image)

Figure 3 A time signal circular conversion diagram.

Note that the full recovery of the \( f(t) \) time signal can be done in two ways:
- on the real component of the return conversion of Fourier, in the entrance of which is submitted the real component of the direct conversion function \( \text{Re}(F(\omega)) \),
- on the imaginary component of the return conversion Fourier, on the input of which is submitted an imaginary component of the direct conversion function \( \text{Im}(F(\omega)) \), If the component at the intake and output of the converter is given a temporary function \( f(t)^* \) with rearranged frequency components.

4. Direct Window Harmonic Function Conversion
It should be noted that the equivalence of the imaginary and real components of the direct Fourier transform takes place only with a finite lower limit of integration. For a classical mathematical form with both infinite limits, both components are required to restore the original time signal, since each of them contains an independent delta function for any frequency.

With a finite window width, a harmonic function can be represented in each component:
1) a unipolar function, when the phases of the harmonic and the converting function coincide;
2) a bipolar function, in the case when these phases differ by \( \pi/2 \). In the case of any phase difference, the transformation of the harmonic function in each of the components includes both a bipolar and a unipolar function.
The spectral amplitude of the bipolar function is 0 at the frequency of the transform function. In the case where the frequencies of the transforming and main functions are close, beats appear that contain a low-frequency component, the half-period of which coincides with the integration interval. The maximum filling of the integration window is corresponding to the positive maximum of the bipolar spectral function - 0.724611.

Figure 4. Extreme bipolar function.

Since the amplitude of bipolar function does not depend on the integration interval, it seems that this situation violates the law of conservation of energy. In the case of a limit tending to infinity, the bipolar function becomes a bipolar delta function. Since the delta function is a work of amplitude on the width of the spectral band, it can be argued that the relative width of the double polar function is a corresponding number of times less than the width of the unipolar.

From the law of energy conservation, it follows that the band of multipolar function is less unipolar. The relationship of bands can be calculating as

$$\frac{\Delta \omega_1}{\Delta \omega_2} = \frac{1}{2 \times 0.724611^2} = 1.05$$

Naturally, this attitude also does not depend on the integration time.

As a result, the amplitude of the multipolar delta function acquires a value of 1/2, and the total amount of energy is 1, i.e. the same as that of a unipolar one.

5. Conclusions

A pie chart proposed for the double Fourier transform. A paradoxical fact has been discovered - the restoration of the initial time signal can be performed on the base of any component of the direct transformation, following the rules of unity of the transformed component for direct and inverse transforms. Violation of this rule leads to the appearance of a temporary signal containing rearranged frequency components. As a result of studying the transformation of harmonic Fourier functions, it turned out that the spectral amplitude of the bipolar function does not depend on the integration interval and is 0.724611. Nevertheless, the product of the square of the amplitude of each of the branches of the bipolar function by the frequency band is 1/2.

Reference

[1] Müller, Meinard. The Fourier Transform in a Nutshell — Springer, 2015. — C. In Fundamentals of Music Processing, Section 2.1, p. 40—56. — ISBN 978-3-319-21944-8. — doi:10.1007/978-3-319-21945-5

[2] Sergei V. Fedorenko. The inverse cyclotomic Discrete Fourier Transform algorithm. Proceedings of Sixteenth International Workshop on Algebraic and Combinatorial Coding Theory at Svetlogorsk, Russia, September 02-08, 2018
[3] L. Ma, Yongle Wu, Z. Zhuang, Y. Liu A novel real-time Fourier and inverse Fourier transforming system based on non-uniform coupled-line phaser International Journal of Electronics and Communications Volume 94, September 2018, Pages 102-108

[4] M. Dobroka, H. Szegedi, P. Vass Inversion-Based Fourier Transform as a New Tool for Noise Rejection DOI:10.5772/66338 February 2017

[5] V. Sukhov, A. Stoytchev Generalizing the inverse FFT off the unit circle. Scientific Reports volume 9, Article number: 14443 (2019)